BUDT 758P: Decision Analytics
Solutions to Assignment 3

Problem 1

Part *a*: Time on the general looms is more valuable since that constraint has a more negative shadow price (-0.651 compared to -0.574 for the regular looms). Consequently, increasing machine time for general looms reduces total cost more per hour.

Part *b*: Types 1, 3 and 11 are the biggest drivers of cost, as they have the largest shadow prices (0.8, 0.8, and 0.7845 respectively).

Part *c*: The constraints have different signs: demand constraints require you to produce a minimum amount, while the machine time constraints cap the production capacity you have available. You are minimizing cost, so the demand constraints have positive shadow prices (increasing demand increases cost), while machine time constraints are the opposite (increasing machine time may reduce cost, since it allows you to meet more demand using in-hour production as opposed to outsourcing).

Part *d*: The reduced cost for fabric type 9 is 0.09145. The cost of outsourcing would have to be reduced by this amount before you would prefer to outsource this type.

Problem 2

The primal LP is a minimization problem with 3 constraints and 4 variables. Thus, the dual is a maximization problem with 4 constraints and 3 variables. By following the rules, we obtain:

$$\max 12y_1 + 8y_2 + 24y_3$$

subject to

$$\begin{aligned} 6y_1 + 2y_2 + 4y_3 &\leq 10 \\ 2y_1 + 2y_2 + 12y_3 &\leq 8 \\ -y_1 + y_2 &\leq 0 \\ -y_2 + y_3 &\leq 0 \\ y_1, y_2, y_3 &\geq 0 \end{aligned}$$

Overall this is similar to the "outsourcing" example from class: this LP can be viewed from the point of view of a contractor who can supply the required amounts of high-, medium-, and low-grade paper (12, 8, and 4 tons, respectively). The decision variables y_1, y_2, y_3 represent prices per ton of each grade of paper. The contractor offers these prices to the recycling company, so $12y_1 + 8y_2 + 4y_3$ is the total dollar amount of the deal. The contractor wishes to maximize revenue earned.

The first two constraints ensure that the offer is fair: the cost of outsourcing to the contractor should be lower for the recycling company than the cost of producing an equivalent amount of paper in-house using their two plants. For example, running the first plant for one day costs \$10K and produces 6, 2, and 4 tons of each paper grade. The total price offered by the contractor for the same combination of paper is $6y_1 + 2y_2 + 4y_3$, so this should be lower than $10K if the offer is to be accepted.

The last two constraints ensure that $y_3 \le y_2 \le y_1$. Thus, the price for low-grade paper should be lower than for medium-grade paper, and that price in turn should be lower than for high-grade paper. This reflects the fact that paper grades can be reclassified: since high-grade paper is the most "flexible" (you can reclassify it to medium-grade, and then reclassify that to low-grade), it should be worth more than the other types (likewise, medium-grade is worth more than low-grade for the same reason).

Problem 3

Part *a*: Let x_{ij} be the amount invested in *i*- month investment vehicles at the start of month *j*. The value of *i* can be 1, 3 or 7, whereas the value of *j* can be 1, 2, …, 12.

Our objective is to maximize

$$0.005(x_{1,1} + x_{1,2} + x_{1,3} + ... + x_{1,12}) + 0.02(x_{3,1} + x_{3,2} + x_{3,3} + ... + x_{3,12})$$
$$+ 0.07(x_{7,1} + x_{7,2} + x_{7,3} + ... + x_{7,12})$$

subject to

$$-x_{1,1} - x_{3,1} - x_{7,1} = -3500$$
$$x_{1,1} - x_{1,2} - x_{3,2} - x_{7,2} = 410$$
$$x_{1,2} - x_{1,3} - x_{3,3} - x_{7,3} = -115$$
$$x_{1,3} + x_{3,1} - x_{1,4} - x_{3,4} - x_{7,4} = -330$$
$$x_{1,4} + x_{3,2} - x_{1,5} - x_{3,5} - x_{7,5} = -1245$$
$$x_{1,5} + x_{3,3} - x_{1,6} - x_{3,6} - x_{7,6} = -850$$
$$x_{1,6} + x_{3,4} - x_{1,7} - x_{3,7} - x_{7,7} = 600$$
$$x_{1,7} + x_{3,5} + x_{7,1} - x_{1,8} - x_{3,8} - x_{7,8} = -150$$
$$x_{1,8} + x_{3,6} + x_{7,2} - x_{1,9} - x_{3,9} - x_{7,9} = -475$$
$$x_{1,9} + x_{3,7} + x_{7,3} - x_{1,10} - x_{3,10} - x_{7,10} = -780$$
$$x_{1,10} + x_{3,8} + x_{7,4} - x_{1,11} - x_{3,11} - x_{7,11} = 260$$
$$x_{1,11} + x_{3,9} + x_{7,5} - x_{1,12} - x_{3,12} - x_{7,12} = 530$$
$$x_{ij} \ge 0 \quad \textit{for all } i, j$$

Each constraint represents a month, January through December. The left-hand side of each constraint gives the money coming in from investments that mature at the beginning of that month, minus the money that we invest during that month. The right-hand side gives the predicted expenses minus cash coming in ($2450 per month in salary, plus an extra $3800 in January only).

For the objective coefficients, we multiply the nominal return by the fraction of the year that we hold the investment. For example,

$$0.06 \times \tfrac{1}{12} = 0.005 .$$

The optimal value is $844.60. Multiple solutions are possible; one example is as follows:

	1 Month	3 Months	7 Months
Jan	410	3090	0
Feb	0	0	0
Mar	115	0	0
Apr	0	0	3535
May	0	0	1245
Jun	600	250	0
Jul	0	0	0
Aug	0	150	0
Sep	0	725	0
Oct	780	0	0
Nov	4205	0	0
Dec	0	0	5645

Part *b*: The sensitivity report shows that the right-hand side value of the first constraint can be increased by $3090 before this constraint becomes non-binding (which would cause the problem to become infeasible, since we have equality constraints). Since the right-hand side constraint subtracts the cash coming in, increasing this quantity by $3090 means that we would have $3090 less cash coming in at the beginning of the year. This means that we only need to invest

$$3800 - 3090 = 710$$

from the starting capital.

Part *c*: We formulate the dual LP as follows. There are twelve decision variables $y_1, \ldots, y_{12}$, each corresponding to a month of the year. The values of all of these decision variables are *unrestricted*, since the primal problem has only equality constraints.

The dual problem has $3 \times 12 = 36$ constraints, one for each variable of the original problem. The first set of 12 constraints is given by

$$\begin{aligned}
&y_2 - y_1 \geq 0.005 \\
&y_3 - y_2 \geq 0.005 \\
&y_4 - y_3 \geq 0.005 \\
&\ldots \\
&y_{12} - y_{11} \geq 0.005 \\
&- y_{12} \geq 0.005
\end{aligned}$$

and corresponds to 1-month investments made in each month. The second set is

$$y_4 - y_1 \geq 0.02$$
$$y_5 - y_2 \geq 0.02$$
$$\ldots$$
$$y_{12} - y_9 \geq 0.02$$
$$-y_{10} \geq 0.02$$
$$-y_{11} \geq 0.02$$
$$-y_{12} \geq 0.02$$

and matches 3-month investments in each month. The final set of 12 constraints is

$$y_8 - y_1 \geq 0.07$$
$$y_9 - y_2 \geq 0.07$$
$$\ldots$$
$$y_{12} - y_5 \geq 0.07$$
$$-y_6 \geq 0.07$$
$$-y_7 \geq 0.07$$
$$\ldots$$
$$-y_{12} \geq 0.07$$

and matches 7-month investments made in each month. The objective is to minimize

$$-3500y_1 + 410y_2 - 115y_3 - 330y_4 - 1245y_5 - 850y_6 + 600y_7 - 150y_8$$
$$-475y_9 - 780y_{10} + 260y_{11} + 530y_{12}$$

Looking at the sensitivity report for the primal, we see that the optimal values of the dual variables are all *negative*. This makes sense, as the right-hand side of each primal constraint represents expenses minus cash coming in, so increasing this quantity would mean that expenses were higher and less money was available for investing.

The dual problem can be interpreted as a way to find "the penalty of not investing." That is, the dual variable y_j represents the opportunity cost of not investing $1 in month *j*. In particular, the very last constraint tells us that

$$y_{12} \leq -0.07.$$

Because we are only budgeting for one year, we do not consider any expenses beyond December. So, every spare dollar in December should be invested into 7-month securities. The only reason not to invest in long-term securities is to have enough cash on hand to meet short-term expenses. However, if we do not consider such expenses beyond December, then we should always invest in 7-month securities at the end of the year. Thus, the opportunity cost of not investing $1 in 7-month securities in December is at least $0.07 (i.e. the penalty is less than -0.07), as that would be the return on that dollar that we would have received if we had invested.

The earlier we are in the year, the higher the penalty of not investing. For example, \$1 invested in January can be more useful than \$1 invested in December, since we would be able to reinvest that dollar multiple times (for example, by reinvesting that same dollar in 1-month securities every month). Our constraints reflect this. For example,

$$y_{12} - y_{11} \geq 0.005$$

can be rewritten as

$$y_{11} \leq y_{12} - 0.005 .$$

The opportunity cost of not investing \$1 in November is at least \$0.005 more than the cost of not investing \$1 in December. This is because we have the option of investing that dollar for one month (for a net gain of \$0.005) and getting it back again by December. Similarly,

$$y_9 \leq y_{12} - 0.02$$

because we could invest \$1 in 3-month securities in September and get it back by December. If we do not invest the dollar in September, we pass up the chance to make that extra \$0.02.

The objective function of the dual weighs the penalties by the cash flows in different months. We are penalized more heavily in months where we have spare cash (negative objective coefficients), since that is where we have available dollars. The dual LP finds the opportunity cost that reflects the individual's specific cash flows.

(**Note:** Depending on how you set up the primal, we might need to flip the signs in the dual. It is possible to have a formulation where the dual variables are positive, but the dual objective function has the signs flipped. In this case the dual variables would have the same interpretation.)