

Assignment-18

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Abstract—In this document, we explore the properties of eigenvalues of similar matrices.

Download all latex-tikz codes from

[https://github.com/poojah15/
EE5609_AI20MTECH14003/tree/master/
Assignment_18](https://github.com/poojah15/EE5609_AI20MTECH14003/tree/master/Assignment_18)

1 PROBLEM STATEMENT

Let \mathbf{A} and \mathbf{B} be $n \times n$ matrices over \mathbf{C} . Then,

- 1) \mathbf{AB} and \mathbf{BA} always have the same set of eigenvalues.
- 2) If \mathbf{AB} and \mathbf{BA} have the same set of eigenvalues then $\mathbf{AB} = \mathbf{BA}$
- 3) If \mathbf{A}^{-1} exists, then \mathbf{AB} and \mathbf{BA} are similar
- 4) The rank of \mathbf{AB} is always the same as the rank of \mathbf{BA} .

2 ANSWERS FOR DIFFERENT CASES

<p>AB and BA always have the same set of eigenvalues.</p>	<p>True.</p> <p>Let λ be an eigenvalue of AB, and x be a corresponding eigenvector. Then</p> $\mathbf{ABx} = \lambda \mathbf{x}$ <p>Left-multiplying by B:</p> $\mathbf{B(AB)x} = \mathbf{B(\lambda x)}$ $(\mathbf{BA})\mathbf{Bx} = \lambda(\mathbf{Bx}) \text{ (by associativity of multiplication)}$ <p>$\Rightarrow \lambda$ is an eigenvalue of BA with Bx as the corresponding eigenvector, assuming Bx is not a null vector.</p> <p>If Bx is null, then B is singular, so that both AB and BA are singular, and $\lambda = 0$. Since both the products are singular, 0 is an eigenvalue of both.</p> <p>Example: Let</p> $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$ <p>Then</p> $\mathbf{AB} = \begin{pmatrix} 2 & 0 \\ 0 & 12 \end{pmatrix}, \mathbf{BA} = \begin{pmatrix} 2 & 0 \\ 0 & 12 \end{pmatrix}$ <p>Since AB and BA results with the same characteristic equation, $\lambda^2 - 14\lambda + 24$ they will have same set of eigenvalues that is $\lambda_1 = 2, \lambda_2 = 12$</p>
<p>If AB and BA have the same set of eigenvalues then AB = BA</p>	<p>False.</p> <p>Counter example: Let</p> $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ <p>then</p> $\mathbf{AB} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{BA} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ <p>\Rightarrow Similar eigenvalues, but AB \neq BA</p>

<p>If \mathbf{A}^{-1} exists, then \mathbf{AB} and \mathbf{BA} are similar</p>	<p>True.</p> <p>Given that \mathbf{A}^{-1} exists and hence, $\mathbf{AB} = \mathbf{A}^{-1}(\mathbf{AB})\mathbf{A} = (\mathbf{A}^{-1}\mathbf{A})\mathbf{BA} = \mathbf{BA}.$ Hence, $\mathbf{AB} \simeq \mathbf{BA}$</p> <p>Example: Let</p> $\mathbf{A} = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$ <p>then</p> $\begin{aligned} \mathbf{AB} &= \begin{pmatrix} 16 & 22 \\ 9 & 13 \end{pmatrix} = \mathbf{A}^{-1}(\mathbf{AB})\mathbf{A} \\ &= \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 16 & 22 \\ 9 & 13 \end{pmatrix} \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 7 & 18 \\ 8 & 22 \end{pmatrix} \\ &= \mathbf{BA} \end{aligned}$
<p>The rank of \mathbf{AB} is always the same as the rank of \mathbf{BA}.</p>	<p>False.</p> <p>Counter example: Let</p> $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ <p>then</p> $\mathbf{AB} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{BA} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ <p>From the above \mathbf{AB} and \mathbf{BA}, it is noted that the $\text{rank}(\mathbf{AB}) = 0$ and $\text{rank}(\mathbf{BA}) = 1$. Hence the rank of \mathbf{AB} need not always be same as rank of \mathbf{BA}.</p>