Assignment-9

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Abstract—In this document, we check whether the given lines intersect. If they don't, then find the closest points using SVD.

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1 PROBLEM STATEMENT

If the lines

$$\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}, \quad (1.0.1)$$

$$\frac{x-3}{3k} = \frac{y-1}{1} = \frac{z-6}{-5}, \quad (1.0.2)$$

find the value of k

2 SOLUTION

In the given problem,

$$\mathbf{A}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{m}_1 = \begin{pmatrix} -3 \\ 2k \\ 2 \end{pmatrix}, \mathbf{A}_2 = \begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 3k \\ 1 \\ -5 \end{pmatrix} \quad (2.0.1)$$

To find the value of k, let's assume that the given lines are perpendicular to each other. Then the dot product of their direction vectors should be 0. i.e.,

$$\mathbf{m}_1 \mathbf{m}_2 = 0 \quad (2.0.2)$$

$$\implies \begin{pmatrix} -3 \\ 2k \\ 2 \end{pmatrix} \begin{pmatrix} 3k \\ 1 \\ -5 \end{pmatrix} = 0 \quad (2.0.3)$$

$$\implies k = -\frac{10}{7} \quad (2.0.4)$$

The lines will intersect if

$$\mathbf{A}_1 + \lambda_1 \mathbf{m}_1 = \mathbf{A}_2 + \lambda_2 \mathbf{m}_2 \quad (2.0.5)$$

$$\implies \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} -3 \\ 2k \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3k \\ 1 \\ -5 \end{pmatrix} \quad (2.0.6)$$

$$\implies \lambda_1 \begin{pmatrix} -3 \\ 2k \\ 2 \end{pmatrix} - \lambda_2 \begin{pmatrix} 3k \\ 1 \\ -5 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad (2.0.7)$$

$$\implies \begin{pmatrix} -3 & 3k \\ 2k & 1 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \quad (2.0.8)$$

$$\implies \begin{pmatrix} -3 & -\frac{30}{7} \\ -\frac{20}{7} & 1 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \quad (2.0.9)$$

Row reducing the augmented matrix,

$$\begin{pmatrix} -3 & -\frac{30}{7} & 2 \\ -\frac{20}{7} & 1 & -1 \\ 2 & -5 & 3 \end{pmatrix} \xrightarrow[R_2 \leftarrow R_2 + \frac{20}{7}R_1]{R_1 \leftarrow -\frac{R_1}{3}} \begin{pmatrix} 1 & \frac{10}{7} & -\frac{2}{3} \\ 0 & \frac{249}{49} & -\frac{61}{21} \\ 2 & -5 & 3 \end{pmatrix} \quad (2.0.10)$$

$$\xrightarrow[R_2 \leftarrow \frac{49}{249}R_2]{R_3 \leftarrow R_3 - 2R_1} \begin{pmatrix} 1 & \frac{10}{7} & -\frac{2}{3} \\ 0 & \frac{249}{49} & -\frac{61}{21} \\ 2 & -5 & 3 \end{pmatrix} \quad (2.0.11)$$

$$\xrightarrow[R_2 \leftarrow \frac{49}{249}R_2]{R_3 \leftarrow R_3 - 2R_1} \begin{pmatrix} 1 & \frac{10}{7} & -\frac{2}{3} \\ 0 & 1 & -\frac{427}{747} \\ 0 & -\frac{55}{7} & \frac{13}{3} \end{pmatrix} \quad (2.0.12)$$

$$\xrightarrow[R_3 \leftarrow -\frac{747}{118}R_3]{R_3 \leftarrow R_3 + \frac{55}{7}R_2} \begin{pmatrix} 1 & \frac{10}{7} & -\frac{2}{3} \\ 0 & 1 & -\frac{427}{747} \\ 0 & 0 & 1 \end{pmatrix} \quad (2.0.13)$$

$$\xrightarrow[R_1 \leftarrow R_1 + \frac{2}{3}R_3 - \frac{10}{7}R_2]{R_2 \leftarrow R_2 + \frac{427}{47}R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (2.0.14)$$

The above matrix has *rank* = 3. Hence, the lines do not intersect which implies that the given lines

are skew lines. To find the closest points using SVD, consider the equation (2.0.9) which can be expressed as

$$\mathbf{Mx} = \mathbf{b} \qquad (2.0.15)$$

By singular value decomposition $\mathbf{M}$ can be expressed as

$$\mathbf{M} = \mathbf{USV}^T \qquad (2.0.16)$$

where the columns of $\mathbf{V}$ are the eigenvectors of $\mathbf{M}^T\mathbf{M}$, the columns of $\mathbf{U}$ are the eigenvectors of $\mathbf{MM}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} \frac{1037}{49} & 0 \\ 0 & \frac{2174}{49} \end{pmatrix} \qquad (2.0.17)$$

$$\mathbf{MM}^T = \begin{pmatrix} \frac{1341}{49} & \frac{30}{7} & \frac{108}{7} \\ \frac{30}{7} & \frac{449}{49} & -\frac{75}{7} \\ \frac{108}{7} & -\frac{75}{7} & 29 \end{pmatrix} \qquad (2.0.18)$$

2.1 To get $\mathbf{V}$ and $\mathbf{S}$

The characteristic equation of $\mathbf{M}^T\mathbf{M}$ is obtained by evaluating the determinant

$$\begin{vmatrix} \frac{1037}{49} - \lambda & 0 \\ 0 & \frac{2174}{49} - \lambda \end{vmatrix} = 0 \quad (2.1.1)$$

$$\implies \lambda^2 - \frac{286699}{637}\lambda + \left[\frac{1037 \times 2174}{49^2}\right] = 0 \quad (2.1.2)$$

The eigenvalues are the roots of equation 2.1.2 is given by

$$\lambda_{11} = \frac{2174}{49} \qquad (2.1.3)$$

$$\lambda_{12} = \frac{1037}{49} \qquad (2.1.4)$$

The corresponding eigen vectors are,

$$\mathbf{u}_{11} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad (2.1.5)$$

$$\mathbf{u}_{12} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad (2.1.6)$$

$$\therefore \mathbf{V} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad (2.1.7)$$

$\mathbf{S}$ is given by

$$\mathbf{S} = \begin{pmatrix} \frac{\sqrt{2174}}{7} & 0 \\ 0 & \frac{\sqrt{1037}}{7} \\ 0 & 0 \end{pmatrix} \qquad (2.1.8)$$

2.2 To get $\mathbf{U}$

The characteristic equation of $\mathbf{MM}^T$ is obtained by evaluating the determinant

$$\begin{vmatrix} \frac{1341}{49} - \lambda & \frac{30}{7} & \frac{108}{7} \\ \frac{30}{7} & \frac{449}{49} - \lambda & -\frac{75}{7} \\ \frac{108}{7} & -\frac{75}{7} & 29 - \lambda \end{vmatrix} = 0 \quad (2.2.1)$$

$$\implies -\lambda^3 + \frac{3211}{49}\lambda^2 - \frac{2254438}{2401}\lambda = 0 \quad (2.2.2)$$

The eigenvalues are the roots of equation 2.2.2 is given by

$$\lambda_{21} = \frac{2174}{49} \qquad (2.2.3)$$

$$\lambda_{22} = \frac{1037}{49} \qquad (2.2.4)$$

$$\lambda_{23} = 0 \qquad (2.2.5)$$

The corresponding eigen vectors are ,

$$\mathbf{u}_{21} = \begin{pmatrix} -\frac{6}{7} \\ \frac{1}{5} \\ -1 \end{pmatrix}, \mathbf{u}_{22} = \begin{pmatrix} -\frac{3}{2} \\ -\frac{10}{7} \\ 1 \end{pmatrix}, \mathbf{u}_{23} = \begin{pmatrix} -\frac{602}{747} \\ \frac{384}{249} \\ 1 \end{pmatrix} \qquad (2.2.6)$$

Normalizing the eigen vectors,

$$\|\mathbf{u}_{21}\| = \sqrt{\left(\frac{-6}{7}\right)^2 + \left(\frac{1}{5}\right)^2 + 1} = \frac{\sqrt{2174}}{35} \quad (2.2.7)$$

$$\implies \mathbf{u}_{21} = \begin{pmatrix} -\frac{210}{7\sqrt{2174}} \\ \frac{35}{5\sqrt{2176}} \\ -\frac{35}{\sqrt{2174}} \end{pmatrix} \qquad (2.2.8)$$

$$\|\mathbf{u}_{22}\| = \sqrt{\left(\frac{-3}{2}\right)^2 + \left(\frac{-10}{7}\right)^2 + 1} = \frac{\sqrt{1037}}{14}$$

$$(2.2.9)$$

$$\implies \mathbf{u}_{22} = \begin{pmatrix} -\frac{42}{2\sqrt{1037}} \\ -\frac{20}{\sqrt{1037}} \\ \frac{14}{\sqrt{1037}} \end{pmatrix} \qquad (2.2.10)$$

$$\|\mathbf{u}_{23}\| = \sqrt{\left(\frac{-602}{747}\right)^2 + \left(\frac{384}{249}\right)^2 + 1} = \frac{\sqrt{4027743}}{1000}$$

$$(2.2.11)$$

$$\implies \mathbf{u}_{23} = \begin{pmatrix} -\frac{602000}{747\sqrt{4027743}} \\ \frac{384000}{249\sqrt{4027743}} \\ \frac{1000}{\sqrt{4027743}} \end{pmatrix} \qquad (2.2.12)$$

$$\mathbf{U} = \begin{pmatrix} \frac{-210}{7\sqrt{2174}} & \frac{-42}{2\sqrt{1037}} & \frac{-602000}{747\sqrt{4027743}} \\ \frac{35}{5\sqrt{2174}} & \frac{-20}{\sqrt{1037}} & \frac{384000}{249\sqrt{4027743}} \\ \frac{-35}{\sqrt{2174}} & \frac{14}{\sqrt{1037}} & \frac{1000}{\sqrt{4027743}} \end{pmatrix} \qquad (2.2.13)$$

2.3 To get $\mathbf{x}$

Using (2.0.16) we rewrite $\mathbf{M}$ as follows,

$$\begin{pmatrix} -3 & -\frac{30}{7} \\ -\frac{20}{7} & 1 \\ 2 & -5 \end{pmatrix} = \begin{pmatrix} \frac{-210}{7\sqrt{2174}} & \frac{-42}{2\sqrt{1037}} & \frac{-602000}{747\sqrt{4027743}} \\ \frac{35}{5\sqrt{2174}} & \frac{-20}{\sqrt{1037}} & \frac{384000}{249\sqrt{4027743}} \\ \frac{-35}{\sqrt{2174}} & \frac{14}{\sqrt{1037}} & \frac{1000}{\sqrt{4027743}} \end{pmatrix}$$
$$\begin{pmatrix} \frac{\sqrt{2174}}{7} & 0 \\ 0 & \frac{\sqrt{1037}}{7} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^T \quad (2.3.1)$$

By substituting the equation (2.0.16) in equation (2.0.15) we get

$$\mathbf{USV}^T\mathbf{x} = \mathbf{b} \qquad (2.3.2)$$

$$\implies \mathbf{x} = \mathbf{VS}_+\mathbf{U}^T\mathbf{b} \qquad (2.3.3)$$

where $\mathbf{S}_+$ is Moore-Penrose Pseudo-Inverse of $\mathbf{S}$

$$\mathbf{S}_+ = \begin{pmatrix} \frac{7}{\sqrt{2174}} & 0 & 0 \\ 0 & \frac{7}{\sqrt{1037}} & 0 \end{pmatrix} \qquad (2.3.4)$$

From (2.3.3) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -3.6889 \\ 0.6210 \\ -0.0767 \end{pmatrix} \qquad (2.3.5)$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -0.55382 \\ 0.13500 \end{pmatrix} \qquad (2.3.6)$$

$$\mathbf{x} = \mathbf{VS}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} 0.13500 \\ -0.55382 \end{pmatrix} \qquad (2.3.7)$$

2.4 Verification of $\mathbf{x}$

Verifying the solution of (2.3.7) using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \qquad (2.4.1)$$

Evaluating the R.H.S in (2.4.1) we get,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} \frac{20}{7} \\ -\frac{172}{7} \end{pmatrix} \qquad (2.4.2)$$

$$\implies \begin{pmatrix} \frac{1037}{49} & 0 \\ 0 & \frac{2174}{49} \end{pmatrix}\mathbf{x} = \begin{pmatrix} \frac{20}{7} \\ -\frac{172}{7} \end{pmatrix} \qquad (2.4.3)$$

Solving the augmented matrix of (2.4.3) we get,

$$\begin{pmatrix} \frac{1037}{49} & 0 & \frac{20}{7} \\ 0 & \frac{2174}{49} & -\frac{172}{7} \end{pmatrix} \xleftarrow[R_2 \leftarrow \frac{49}{2174}R_2]{R_1 \leftarrow \frac{49}{1037}R_1} \begin{pmatrix} 1 & 0 & \frac{140}{1037} \\ 0 & 1 & -\frac{602}{1087} \end{pmatrix} \quad (2.4.4)$$

Hence, Solution of (2.4.1) is given by,

$$\mathbf{x} = \begin{pmatrix} 0.13500 \\ -0.55382 \end{pmatrix} \qquad (2.4.5)$$

Comparing results of $\mathbf{x}$ from (2.3.7) and (2.4.5) we conclude that the solution is verified.