Assignment-6

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Abstract—**In this document, we find the value of k such that the equation represents a pair of straight lines.**

Download all latex-tikz codes from

https://github.com/poojah15/EE5609_AI20MTECH14003/tree/master/Assignment_6

1 Problem Statement

Find the value of k such that $6x^2 + 11xy - 10y^2 + x + 31y + k = 0$ represent pairs of straight lines.

2 Theory

The general equation of second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{2.0.1}$$

and can be expressed as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.2}$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{2.0.3}$$

$$\mathbf{u}^T = \begin{pmatrix} d & e \end{pmatrix} \tag{2.0.4}$$

Let the pair of straight lines be given by

$$\mathbf{n}_1^T\mathbf{x} = c_1 \tag{2.0.5}$$

$$\mathbf{n}_2^T\mathbf{x} = c_2 \tag{2.0.6}$$

Equating their product with (2.0.2), we get

$$(\mathbf{n}_1^T\mathbf{x} - c_1)(\mathbf{n}_2^T\mathbf{x} - c_2) = \mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.7}$$

$$\implies \mathbf{n}_1 * \mathbf{n}_2 = \{a, 2b, c\} \tag{2.0.8}$$

$$c_2\mathbf{n}_1 + c_1\mathbf{n}_2 = -2\mathbf{u} \tag{2.0.9}$$

$$c_1c_2 = f \tag{2.0.10}$$

The slopes of lines are given by the roots of the polynomial

$$cm^2 + 2bm + a = 0 \tag{2.0.11}$$

$$\implies m_i = \frac{-b \pm \sqrt{-\left|\mathbf{V}\right|}}{c} \tag{2.0.12}$$

and

$$\mathbf{n}_i = k_i\begin{pmatrix} -m_i \\ 1 \end{pmatrix}, \quad i = 1, 2. \tag{2.0.13}$$

From (2.0.9),

$$\begin{pmatrix} \mathbf{n}_1 & \mathbf{n}_2 \end{pmatrix}\begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \tag{2.0.14}$$

3 Solution

Given,

$$6x^2 + 11xy - 10y^2 + x + 31y + k = 0 \tag{3.0.1}$$

Substituting the coefficients of equation (3.0.1) in (2.0.3), (2.0.4) and (2.0.11), we get

$$\mathbf{V} = \begin{pmatrix} 6 & \frac{11}{2} \\ \frac{11}{2} & -10 \end{pmatrix} \tag{3.0.2}$$

$$\mathbf{u}^T = \begin{pmatrix} \frac{1}{2} & \frac{31}{2} \end{pmatrix} \tag{3.0.3}$$

$$-10m^2 + 11m + 6 = 0 \tag{3.0.4}$$

Solving for m by using (2.0.12), we get

$$m = \frac{\frac{-11}{2} \pm \frac{19}{2}}{-10} \tag{3.0.5}$$

$$\implies m_1 = \frac{-2}{5}, m_2 = \frac{3}{2} \tag{3.0.6}$$

From (2.0.13), we have

$$\mathbf{n}_1 = k_1\begin{pmatrix} \frac{2}{5} \\ 1 \end{pmatrix} \tag{3.0.7}$$

$$\mathbf{n}_2 = k_2\begin{pmatrix} \frac{-3}{2} \\ 1 \end{pmatrix} \tag{3.0.8}$$

By substituting (3.0.7) and (3.0.8) in (2.0.8), we get

$$k_1 \begin{pmatrix} \frac{2}{5} \\ 1 \end{pmatrix} * k_2 \begin{pmatrix} \frac{-3}{2} \\ 1 \end{pmatrix} = \{6, 11, -10\} \tag{3.0.9}$$

$$\implies k_1 k_2 = -10 \tag{3.0.10}$$

By inspection, we get the values, $k_1 = 5, k_2 = -2$. Substituting the values of k_1 and k_2 in (3.0.7) and (3.0.8) respectively, we get

$$\mathbf{n}_1 = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \tag{3.0.11}$$

$$\mathbf{n}_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \tag{3.0.12}$$

Substituting (3.0.11) and (3.0.12) in (2.0.14), we get

$$\begin{pmatrix} 2 & 3 \\ 5 & -2 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2 \begin{pmatrix} \frac{1}{2} \\ \frac{31}{2} \end{pmatrix} \tag{3.0.13}$$

$$\implies 2c_2 + 3c_1 = -1 \tag{3.0.14}$$

$$5c_2 - 2c_1 = -31 \tag{3.0.15}$$

Solving the above equations, we get

$$c_1 = 3, c_2 = -5 \tag{3.0.16}$$

$\therefore$ From (2.0.10), we get

$$f = -15 \tag{3.0.17}$$

$$i.e., \boxed{k = -15} \tag{3.0.18}$$

Hence the solution.

4 Graphical Illustration

From (3.0.1), let us consider

$$6x^2 + 11xy - 10y^2 \tag{4.0.1}$$

$$\implies 6x^2 - 4xy + 15xy - 10y^2 \tag{4.0.2}$$

$$\implies 2x(3x - 2y) + 5y(3x - 2y) \tag{4.0.3}$$

$$\implies (3x - 2y)(2x + 5y) \tag{4.0.4}$$

Equation (4.0.4) can be written as,

$$(3x - 2y + p)(2x + 5y + q) = 0 \tag{4.0.5}$$

$$\implies 6x^2 + 11xy - 10y^2 + (3q + 2p)x + (-2q + 5p)y = 0 \tag{4.0.6}$$

Equating (3.0.1) and (4.0.6), we get

$$3q + 2p = 1 \tag{4.0.7}$$

$$-2q + 5p = 31 \tag{4.0.8}$$

Solving for p and q we obtain,

$$p = 5 \tag{4.0.9}$$

$$q = -3 \tag{4.0.10}$$

Substituting the values of p and q in (4.0.5), we obtain

$$(3x - 2y + 5)(2x + 5y - 3) = 0 \tag{4.0.11}$$

Hence (4.0.11) represents equation of two straight lines. Graphically,

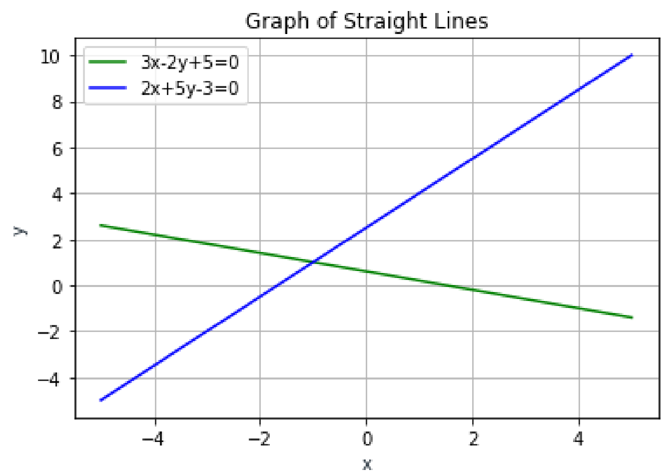

Fig. 1: Plot of two straight lines.