Assignment-12

Pooja H
AI20MTECH14003

***Abstract*—This document, explains the concept of subspaces.**

Download all latex-tikz codes from

https://github.com/poojah15/EE5609_AI20MTECH14003/tree/master/Assignment_12

1 Problem Statement

Let $\mathbf{W}_1$ and $\mathbf{W}_2$ be subspaces of a vector space $\mathbf{V}$ such that the set-theoretic union of $\mathbf{W}_1$ and $\mathbf{W}_2$ is also a subspace. Prove that one of the spaces $\mathbf{W}_i$ is contained in the other.

2 Solution

Given $\mathbf{W}_1 \cup \mathbf{W}_2$ is a subspace, we need to prove that

$$\mathbf{W}_1 \subseteq \mathbf{W}_2 \quad or \quad \mathbf{W}_2 \subseteq \mathbf{W}_1 \tag{2.0.1}$$

Let us assume that

$$\mathbf{W}_1 \nsubseteq \mathbf{W}_2 \tag{2.0.2}$$

We need to show that

$$\mathbf{W}_2 \subseteq \mathbf{W}_1 \tag{2.0.3}$$

i.e., the generators of $\mathbf{W}_2$ are in $\mathbf{W}_1$. Consider a vector, $\mathbf{w}_1 \in \mathbf{W}_1 \backslash \mathbf{W}_2$ and a vector $\mathbf{w}_2 \in \mathbf{W}_2$. Since $\mathbf{W}_1 \cup \mathbf{W}_2$ is a subspace,

$$\mathbf{w}_1 + \mathbf{w}_2 \in \mathbf{W}_1 \cup \mathbf{W}_2 \tag{2.0.4}$$

$$\implies \mathbf{w}_1 + \mathbf{w}_2 \in \mathbf{W}_1 \quad or \tag{2.0.5}$$

$$\mathbf{w}_1 + \mathbf{w}_2 \in \mathbf{W}_2 \tag{2.0.6}$$

But, $\mathbf{w}_1 + \mathbf{w}_2 \notin \mathbf{W}_2$ because for some vector $-\mathbf{w}_2 \in \mathbf{W}_2$,

$$(\mathbf{w}_1 + \mathbf{w}_2) - \mathbf{w}_2 = \mathbf{w}_1 \notin \mathbf{W}_2 \tag{2.0.7}$$

Hence it must be that, $\mathbf{w}_1 + \mathbf{w}_2 \in \mathbf{W}_1$ because for some vector $-\mathbf{w}_1 \in \mathbf{W}_1$,

$$(\mathbf{w}_1 + \mathbf{w}_2) - \mathbf{w}_1 = w_2 \in \mathbf{W}_1 \tag{2.0.8}$$

Thus, we have shown that every vector $\mathbf{w}_2$ in $\mathbf{W}_2$ is also in $\mathbf{W}_1$. Hence, $\mathbf{W}_2 \subseteq \mathbf{W}_1$