

Assignment-6

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Abstract—In this document, we find the value of k such that the equation represents a pair of straight lines.

Download all latex-tikz codes from

https://github.com/poojah15/EE5609_AI20MTECH14003/tree/master/Assignment_6

1 PROBLEM STATEMENT

Find the value of k such that $6x^2 + 11xy - 10y^2 + x + 31y + k = 0$ represent pairs of straight lines.

2 THEORY

The general equation of second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (2.0.1)$$

and can be expressed as

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.0.2)$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad (2.0.3)$$

$$\mathbf{u}^T = \begin{pmatrix} d & e \end{pmatrix} \quad (2.0.4)$$

Let the pair of straight lines be given by

$$\mathbf{n}_1^T \mathbf{x} = c_1 \quad (2.0.5)$$

$$\mathbf{n}_2^T \mathbf{x} = c_2 \quad (2.0.6)$$

Equating their product with (2.0.2), we get

$$\begin{aligned} & (\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) \\ &= \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \end{aligned} \quad (2.0.7)$$

$$\Rightarrow \mathbf{n}_1 * \mathbf{n}_2 = \{a, 2b, c\} \quad (2.0.8)$$

$$c_2 \mathbf{n}_1 + c_1 \mathbf{n}_2 = -2\mathbf{u} \quad (2.0.9)$$

$$c_1 c_2 = f \quad (2.0.10)$$

The slopes of lines are given by the roots of the polynomial

$$cm^2 + 2bm + a = 0 \quad (2.0.11)$$

$$\Rightarrow m_i = \frac{-b \pm \sqrt{-|\mathbf{V}|}}{c} \quad (2.0.12)$$

and

$$\mathbf{n}_i = k_i \begin{pmatrix} -m_i \\ 1 \end{pmatrix}, \quad i = 1, 2. \quad (2.0.13)$$

From (2.0.9),

$$\begin{pmatrix} \mathbf{n}_1 & \mathbf{n}_2 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \quad (2.0.14)$$

3 SOLUTION

Given,

$$6x^2 + 11xy - 10y^2 + x + 31y + k = 0 \quad (3.0.1)$$

Substituting the coefficients of equation (3.0.1) in (2.0.3), (2.0.4) and (2.0.11), we get

$$\mathbf{V} = \begin{pmatrix} 6 & \frac{11}{2} \\ \frac{11}{2} & -10 \end{pmatrix} \quad (3.0.2)$$

$$\mathbf{u}^T = \begin{pmatrix} \frac{1}{2} & \frac{31}{2} \end{pmatrix} \quad (3.0.3)$$

$$-10m^2 + 11m + 6 = 0 \quad (3.0.4)$$

Solving for m by using (2.0.12), we get

$$m = \frac{\frac{-11}{2} \pm \frac{19}{2}}{-10} \quad (3.0.5)$$

$$\Rightarrow m_1 = \frac{-2}{5}, m_2 = \frac{3}{2} \quad (3.0.6)$$

From (2.0.13), we have

$$\mathbf{n}_1 = k_1 \begin{pmatrix} \frac{2}{5} \\ 1 \end{pmatrix} \quad (3.0.7)$$

$$\mathbf{n}_2 = k_2 \begin{pmatrix} \frac{-3}{2} \\ 1 \end{pmatrix} \quad (3.0.8)$$

By substituting (3.0.7) and (3.0.8) in (2.0.8), we get

$$k_1 \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} * k_2 \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} = \{6, 11, -10\} \quad (3.0.9)$$

$$\implies k_1 k_2 = -10 \quad (3.0.10)$$

By inspection, we get the values, $k_1 = 5, k_2 = -2$. Substituting the values of k_1 and k_2 in (3.0.7) and (3.0.8) respectively, we get

$$\mathbf{n}_1 = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \quad (3.0.11)$$

$$\mathbf{n}_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \quad (3.0.12)$$

From (2.0.8), we know that,

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (3.0.13)$$

Using Teoplitz matrix representation, the convolution of \mathbf{n}_1 with \mathbf{n}_2 , is as follows:

$$\begin{pmatrix} 2 & 0 & 5 \\ 5 & 2 & 0 \\ 0 & 5 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 11 \\ -10 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (3.0.14)$$

Hence, \mathbf{n}_1 and \mathbf{n}_2 satisfies (2.0.8). Substituting (3.0.11) and (3.0.12) in (2.0.14), we get

$$\begin{pmatrix} 2 & 3 \\ 5 & -2 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2 \begin{pmatrix} \frac{1}{2} \\ \frac{31}{2} \end{pmatrix} \quad (3.0.15)$$

The augmented matrix is,

$$\begin{pmatrix} 2 & 3 & -1 \\ 5 & -2 & -31 \end{pmatrix} \xrightarrow[R_2 \leftarrow R_2 - 5R_1]{R_1 \leftarrow \frac{R_1}{2}} \begin{pmatrix} 1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & -\frac{19}{2} & -\frac{57}{2} \end{pmatrix} \quad (3.0.16)$$

$$\xrightarrow[R_1 \leftarrow R_1 - \frac{3}{2}R_2]{R_2 \leftarrow \frac{R_2}{-19/2}} \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \end{pmatrix} \quad (3.0.17)$$

Hence we obtain,

$$c_1 = 3, c_2 = -5 \quad (3.0.18)$$

\therefore From (2.0.10), we get

$$f = -15 \quad (3.0.19)$$

$$i.e., \boxed{k = -15} \quad (3.0.20)$$

Hence the solution. Graphically,

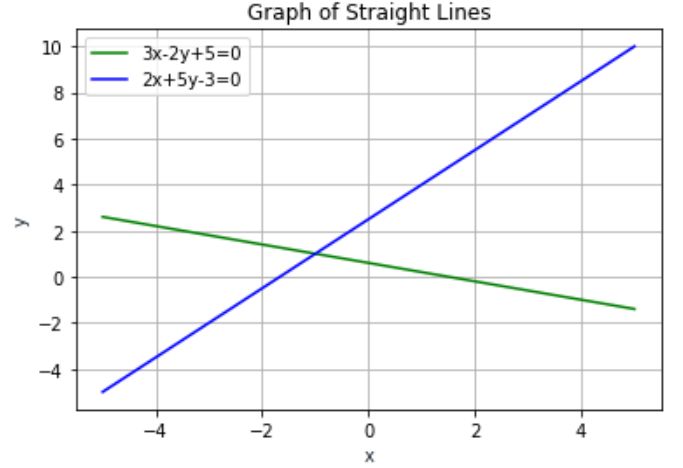


Fig. 1: Plot of two straight lines.