Assignment-18

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AI20MTECH14003

***Abstract*—In this document, we explore the properties of eigenvalues of non-diagonalizable matrices.**

Download all latex-tikz codes from

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https://github.com/poojah15/
    EE5609_AI20MTECH14003/tree/master/
    Assignment_18
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1 Problem Statement

Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices over $\mathbf{C}$. Then,

1) $\mathbf{AB}$ and $\mathbf{BA}$ always have the same set of eigenvalues.
2) If $\mathbf{AB}$ and $\mathbf{BA}$ have the same set of eigenvalues then $\mathbf{AB} = \mathbf{BA}$
3) If $\mathbf{A}^{-1}$ exists, then $\mathbf{AB}$ and $\mathbf{BA}$ are similar
4) The rank of $\mathbf{AB}$ is always the same as the rank of $\mathbf{BA}$.

2 Answers for different cases

AB and **BA** always have the same set of eigenvalues.	**True**. Let λ be an eigenvalue of **AB**, and **x** be a corresponding eigenvector. Then $\mathbf{ABx}=\lambda\mathbf{x}$ Left-multiplying by **B**: $\mathbf{B}(\mathbf{AB})\mathbf{x} = \mathbf{B}(\lambda\mathbf{x})$ $(\mathbf{BA})\mathbf{Bx} = \lambda(\mathbf{Bx})$ (by associativity of multiplication) $\Longrightarrow$ λ is an eigenvalue of **BA** with **Bx** as the corresponding eigenvector, assuming **Bx** is not a null vector. If **Bx** is null, then **B** is singular, so that both **AB** and **BA** are singular, and $\lambda = 0$. Since both the products are singular, 0 is an eigenvalue of both. Example: Let $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix}$ Then $\mathbf{AB} = \begin{pmatrix} 2 & -2 \\ 4 & -4 \end{pmatrix}, \mathbf{BA} = \begin{pmatrix} 0 & -2 \\ 0 & -2 \end{pmatrix}$ Since **AB** and **BA** results with the same characteristic equation, $\lambda^2 + 2\lambda = 0$ they will have same set of eigenvalues that is $\lambda_1 = 0, \lambda_2 = -2$
If **AB** and **BA** have the same set of eigenvalues then **AB** = **BA**	**False**. Counter example: Let $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix}$ then $\mathbf{AB} = \begin{pmatrix} 2 & -2 \\ 4 & -4 \end{pmatrix}, \mathbf{BA} = \begin{pmatrix} 0 & -2 \\ 0 & -2 \end{pmatrix}$ $\Longrightarrow$ Same eigenvalues ($\lambda_1 = 0, \lambda_2 = -2$), but $\mathbf{AB} \neq \mathbf{BA}$

If $\mathbf{A}^{-1}$ exists, then $\mathbf{AB}$ and $\mathbf{BA}$ are similar	**True**. Given that $\mathbf{A}^{-1}$ exists and hence, $$\mathbf{AB} = \mathbf{A}^{-1}(\mathbf{AB})\mathbf{A} = (\mathbf{A}^{-1}\mathbf{A})\mathbf{BA} = \mathbf{BA}.$$ Hence, $\mathbf{AB} \simeq \mathbf{BA}$ Example: Let $$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix}$$ then $$\begin{aligned} \mathbf{AB} &= \begin{pmatrix} 2 & -2 \\ 4 & -4 \end{pmatrix} = \mathbf{A}^{-1}(\mathbf{AB})\mathbf{A} \\ &= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & -2 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -2 \\ 0 & -2 \end{pmatrix} \\ &= \mathbf{BA} \end{aligned}$$
The rank of $\mathbf{AB}$ is always the same as the rank of $\mathbf{BA}$.	**False**. Counter example: Let $$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$ then $$\mathbf{AB} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \mathbf{BA} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ From the above $\mathbf{AB}$ and $\mathbf{BA}$, it is noted that the rank($\mathbf{AB}$) = 2 and rank($\mathbf{BA}$)=1. Hence the rank of $\mathbf{AB}$ need not always be same as rank of $\mathbf{BA}$.