Assignment-11

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***Abstract*—In this document, we find the basis for the space V**

Download all latex-tikz codes from

https://github.com/poojah15/EE5609_AI20MTECH14003/tree/master/Assignment_11

1 Problem Statement

Let $\mathbf{V}$ be the space of 2×2 matrices over $\mathbf{F}$. Find a basis $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4\}$ for $\mathbf{V}$ such that $\mathbf{A}_j^2 = \mathbf{A}_j$ for each j

2 Solution

Every 2×2 matrix may be written as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.1}$$

This shows that

$$\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4\} = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\} \tag{2.0.2}$$

can be the basis for the space $\mathbf{V}$ of all 2×2 matrices. However $\mathbf{A}_2$ and $\mathbf{A}_3$ doesn't satisfy the property of $\mathbf{A}^2 = \mathbf{A}$. Consider b = 0 and c = 0, then the matrix

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \tag{2.0.3}$$

can't be a basis as it is the linear combination of $\mathbf{A}_1$ and $\mathbf{A}_4$. Hence either b or c or both must be non zero. Hence,

$$\mathbf{A}_2 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \tag{2.0.4}$$

$$\mathbf{A}_3 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \tag{2.0.5}$$

Here, $\mathbf{A}_2^2 = \mathbf{A}_2$ and $\mathbf{A}_3^2 = \mathbf{A}_3$. Therefore the basis can be

$$\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4\} = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\} \tag{2.0.6}$$

$\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4\}$ forms the basis, iff they are linearly independent and the linear combination of them span the space $\mathbf{V}$. To show that they are linearly independent, we show that the equation has a trivial solution.

$$a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.0.7}$$

$$\implies a + b = 0 \tag{2.0.8}$$

$$b = 0 \tag{2.0.9}$$

$$c = 0 \tag{2.0.10}$$

$$c + d = 0 \tag{2.0.11}$$

The corresponding matrix form is $\mathbf{Ax} = 0$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.0.12}$$

Row reducing the augmented matrix,

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \xleftrightarrow[R_4 \leftarrow R_4 - R_3]{R_2 \longleftrightarrow R_3} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{2.0.13}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{2.0.14}$$

Therefore, a = b = c = d = 0. Hence the matrices are linearly independent. To show that the linear combination of $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4\}$ span the space $\mathbf{V}$, consider an arbitrary matrix,

$$\begin{pmatrix} w & x \\ y & z \end{pmatrix} \tag{2.0.15}$$

Compute a, b, c, d such that

$$\begin{pmatrix} w & x \\ y & z \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.16}$$

$$= \begin{pmatrix} a+b & c \\ b & c+d \end{pmatrix} \tag{2.0.17}$$

Equating the entries, this produces system of linear equations,

$$a+b = w, y = b, x = c, z = c+d \tag{2.0.18}$$
$$\implies a = w - y \tag{2.0.19}$$
$$b = y \tag{2.0.20}$$
$$c = x \tag{2.0.21}$$
$$d = z - x \tag{2.0.22}$$

In particular, there exists atleast one solution regardless of the values of w, x, y, z. For example, consider the following matrix,

$$\begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ -2 & 7 \end{pmatrix} \tag{2.0.23}$$

Here, $a = 5, b = -2, c = 4, d = 3$. Using (2.0.16), we get

$$5\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + 4\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + 3\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ -2 & 7 \end{pmatrix} \tag{2.0.24}$$

Hence $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$ forms the basis for the given space $\mathbf{V}$.