UNIT-1 RANDOM VARIABLES:-

Introduction to probability - Definition of random Variable - Discrete and Continuous random Variables - Probability Mass and Probability density functions - Mathematical expectation and Variance - Moments - Moment generating functions.

Introduction to probability :-

Random Experiment:-

An experiment whose outcome or result can be predicted with certainity is called a deterministic experiment.

Although all possible outcomes of an experiment may be known in advance, the outcome of a particular performance of the experiment cannot be predicted owing to a number of unknown causes. Such an experiment is called a random experiment.

Eg: Tossing a coin, rolling a dice.

Definition:

Let S be the sample space (the set of all possible outcomes which are assumed equally likely) and A be an event (a subset of S consisting of possible outcomes) associated with a random experiment. Let $n(S)$ and $n(A)$ be the number of elements of S and A. Then the probability of event A occurring, denoted as $P(A)$ is defined by

$$P(A) = \frac{n(A)}{n(S)} = \frac{\text{Number of cases favourable to } A}{\text{Exhaustive number of cases in } S}.$$

Axiomatic Definition of Probability:-

Let S be the sample space and A be an event associated with a random experiment. Then the probability of the event A, denoted by $P(A)$, is defined as a real number satisfying the following axioms.

(i) $0 \le P(A) \le 1$

(ii) $P(S) = 1$

(iii) If A and B are mutually exclusive events,

$$P(A \cup B) = P(A) + P(B)$$

(iv) If $A_1, A_2 \ldots A_n$ are a set of mutually exclusive events,

$$P(A_1 \cup A_2 \cup \ldots \cup A_n \ldots) = P(A_1) + P(A_2) + \cdots + P(A_n) + \cdots$$

Theorem: If A and B are any 2 events, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Conditional Probability:

The conditional probability of an event B, assuming that the event A has happened is denoted by $P(B/A)$ and defined as

$$P(B/A) = \frac{P(A \cap B)}{P(A)}, \text{ provided } P(A) \neq 0.$$

For example, when a fair die is tossed, the conditional probability of getting '1' given that an odd number has been obtained.

Independent events:

A set of events is said to be independent if the occurrence of any one of them does not depend on the occurrence or non-occurrence of the others.

$$P(A \cap B) = P(A)P(B)$$

Random Variables:-

A random Variable is a function that assigns a real number $X(s)$ to every element $s \in S$, where S is the Sample Space Corresponding to a random Experiment E.

i.e) $X : S \to R$

Random Variable

Discrete Random Variable   Continuous Random Variable.

Discrete Random Variable:-

If X is random Variable (RV) which can take a finite number or Countably infinite number of values, X is called a discrete RV.

Probability mass function:- (PMF)

If X is a discrete random variable which takes the values $x_1, x_2 \ldots x_n$ such that $P(X = x_i) = p_i$ then p_i is called the probability function provided p_i $(i = 1, 2, 3 \ldots)$ Satisfy the following Conditions

(i) $p_i \geq 0$, for all i and

(ii) $\sum_i p_i = 1$.

Continuous Random Variable :-

If X is an Random variable which can take all values (ie infinite number of values) in an interval, then X is called a Continuous RV.

Probability Density Function :- (PDF)

If X is a continuous RV such that

$$P\left\{x - \frac{1}{2}dx \le X \le x + \frac{1}{2}dx\right\} = f(x)\,dx$$

then $f(x)$ is called the probability density function of X Provided $f(x)$ satisfies following conditions

(i) $f(x) \geq 0$ for all x

(ii) $$\int_{-\infty}^{\infty} f(x)\,dx = 1.$$

Cumulative Distribution Function :- (CDF)

If X is an RV, discrete or continuous then $P(X \le x)$ is called the cumulative distribution function of X and denoted as $F(x)$

If X is discrete, $$F(x) = \sum_{\substack{j \\ x_j \le x}} P_j$$

If X is continuous,

$$F(x) = P(-\infty < X \le x) = \int_{-\infty}^{x} f(x)\,dx.$$

Note:-

(i) If $F(x)$ is the c.d.f of a Continuous random Variable x, then the p.d.f of x, $f(x)$ is given by

$$f(x) = \frac{d}{dx}[F(x)]$$

(ii) If x is a discrete random Variable with P.m.f $P(x_i)$ and $F(x)$ is the c.d.f of x, then

$$P(x_i) = F(x_i) - F(x_{i-1})$$

Properties of CDF :-

(i) $F(x)$ is ~~an~~ non-decreasing function. If $a<b$, then $F(a) \le F(b)$

(ii) $0 \le F(x) \le 1$

(iii) $\lim_{x \to \infty} F(x) = 1$

(iv) $\lim_{x \to -\infty} F(x) = 0$

(v) If F is the distribution function of the random Variable x and if $a<b$, then

$$P(a < x \le b) = F(b) - F(a)$$

Problems:- (PMF)

1. If the random variable X takes the values 1, 2, 3, 4 such that $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4)$. Find the probability distribution and cumulative distribution function of X.

Sol:-

Let $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4) = k$

$\Rightarrow P(X=1) = \frac{k}{2}, \; P(X=2) = \frac{k}{3}, \; P(X=3) = k,$

$P(X=4) = \frac{k}{5}$

$\because \sum_{i=1}^{4} P(X=x_i) = 1$

$\Rightarrow \frac{k}{2} + \frac{k}{3} + k + \frac{k}{5} = 1 \Rightarrow \frac{15k+10k+30k+6k}{30} = 1$

$\Rightarrow \frac{61}{30}k = 1 \Rightarrow \boxed{k = \frac{30}{61}}$

$\therefore$ The Probability distribution of X is

X :	1	2	3	4
P(X) :	$\frac{15}{61}$	$\frac{10}{61}$	$\frac{30}{61}$	$\frac{6}{61}$

CDF $\quad P(X \le x)$

X :	1	2	3	4
P(X) :	$15/61$	$10/61$	$30/61$	$6/61$
F(x) :	$15/61$	$25/61$	$55/61$	1

(2). A discrete random variable X has the probability function given below:

X:	0	1	2	3	4	5	6	7
P(X):	0	K	2K	2K	3K	K^2	$2K^2$	$7K^2+K$

Find (i) the value of K

(ii) $P(X<6)$, $P(X\geq 6)$ and $P(0<X<4)$

(iii) The distribution function of X.

Sol:

W.K.T $\sum_i P(x_i) = 1$

$$\sum_{i=0}^{7} P(X=x_i) = 1$$

$$0+K+2K+2K+3K+K^2+2K^2+7K^2+K = 1$$

$$10K^2+9K = 1 \Rightarrow 10K^2+9K-1 = 0$$

-10 ; 10, -1

$$10K^2+10K-K-1 = 0$$

$$10K(K+1)-(K+1) = 0$$

$$(10K-1)(K+1) = 0$$

$K = 1/10$, $K = -1$ (not possible)

$$\boxed{K = \frac{1}{10}}$$

(ii) $P(X<6) = P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)+P(X=5)$

$= 0+K+2K+2K+3K+K^2$

$$= k^2 + 8k = \frac{1}{100} + \frac{8}{10} = \frac{81}{100}$$

$$P(X \geq 6) = 1 - P(X < 6)$$

$$= 1 - \frac{81}{100} = \frac{19}{100}$$

$$P(0 < X < 4) = P(X=1) + P(X=2) + P(X=3)$$

$$= k + 2k + 2k = 5k = \frac{5}{10}$$

$$P(0 < X < 4) = \tfrac{1}{2}$$

(iii)

X :	0	1	2	3	4	5	6	7
$P(x)$:	0	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{1}{100}$	$\frac{2}{100}$	$\frac{17}{100}$
$F(x)$:	0	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{5}{10}$	$\frac{8}{10}$	$\frac{81}{100}$	$\frac{83}{100}$	1

③. A random Variable X has the following Probability distribution :

x :	-2	-1	0	1	2	3
$P(x)$:	0.1	k	0.2	$2k$	0.3	$3k$

a) Find k b) Evaluate $P(X < 2)$ and $P(|x| < 2)$

c) Find the c.d.f of x.

(4). A discrete R.V has the following Probability distribution

x:	0	1	2	3	4	5	6	7	8
$P(x)$:	a	$3a$	$5a$	$7a$	$9a$	$11a$	$13a$	$15a$	$17a$

Find the value of K (ii) $P(X<3)$ (iii) cdf of x.

(5). If the Probability mass function of a R.V X is given by $P(X=r) = Kr^3$ $r=1,2,3,4$
Find the Value of K (ii) $P\left(\frac{1}{2}<X<\frac{1}{5} \,/\, X>1\right)$ and (iii) cdf of X.

(6). The Probability function of an infinite discrete distribution is given by $P(X=j) = \frac{1}{2^j}$ $(j=1,2\ldots)$

(i) check whether it is a Pmf (ii) $P(X$ is even$)$
(iii) $P(X \geq 5)$ and (iv) $P(X$ is divisible by $3)$

Problems: (PDF)

1. If the density function of a Continuous RV is given by

$$f(x) = \begin{cases} ax, & 0 \le x \le 1 \\ a, & 1 \le x \le 2 \\ 3a - ax, & 2 \le x \le 3 \\ 0, & \text{elsewhere} \end{cases}$$

(i) find the Value of a (ii) Find the cdf of X

(iii) $P(X \le 1.5)$

Sol:

Since $f(x)$ is a pdf

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

(i) $$\int_0^3 f(x)\,dx = 1$$

$$\Rightarrow \int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^3 (3a - ax)\,dx = 1$$

$$\Rightarrow a\left[\frac{x^2}{2}\right]_0^1 + a\left[x\right]_1^2 + a\left[3x - \frac{x^2}{2}\right]_2^3 = 1$$

$$\Rightarrow a\left[\frac{1}{2} - 0\right] + a\left[2 - 1\right] + a\left[\left(9 - \frac{9}{2}\right) - (6 - 2)\right] = 1$$

$$\Rightarrow \frac{a}{2} + a + a\left[\frac{9}{2} - 4\right] = 1$$

$$\Rightarrow \frac{a}{2} + a + \frac{a}{2} = 1 \Rightarrow 2a = 1$$

$$\Rightarrow \boxed{a = \frac{1}{2}}$$

(ii) $F(x) = P(X \leq x)$

$F(x) = 0$, when $x < 0$

$$F(x) = \int_0^x \frac{x}{2}\,dx = \frac{1}{2}\left(\frac{x^2}{2}\right)_0^x = \frac{x^2}{4}, \quad 0 \leq x < 1$$

$$F(x) = \int_0^1 \frac{x}{2}\,dx + \int_1^x \frac{1}{2}\,dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\right]_0^1 + \frac{1}{2}[x]_1^x$$

$$= \frac{1}{4} + \frac{1}{2}[x-1] = \frac{1}{4} + \frac{x}{2} - \frac{1}{2}$$

$$= \frac{x}{2} - \frac{1}{4}, \quad 1 \leq x < 2$$

$$F(x) = \int_0^1 \frac{x}{2}\,dx + \int_1^2 \frac{1}{2}\,dx + \int_2^x \left(\frac{3}{2} - \frac{x}{2}\right)dx$$

$$= \left[\frac{x^2}{4}\right]_0^1 + \left[\frac{x}{2}\right]_1^2 + \left[\frac{3}{2}x - \frac{x^2}{4}\right]_2^x$$

$$= \frac{1}{4} + 1 - \frac{1}{2} + \left(\frac{3x}{2} - \frac{x^2}{4}\right) - (3-1)$$

$$= \frac{1}{4} + 1 - \frac{1}{2} + \frac{3x}{2} - \frac{x^2}{4} - 2$$

$$= \frac{3x}{2} - \frac{x^2}{4} - \frac{5}{4}, \quad 2 \leq x < 3$$

$F(x) = 1$, $\quad x \geq 3$.

(iii) $P[X \not> 1.5] = P[0 \le x \le 1) + P(1 \le x \le 1.5)$

$$= \int_0^1 \frac{x}{2}\,dx + \int_1^{1.5} \frac{1}{2}\,dx$$

$$= \left(\frac{x^2}{4}\right)_0^1 + \left(\frac{x}{2}\right)_1^{1.5}$$

$$= \frac{1}{4} + \frac{1.5}{2} - \frac{1}{2} = \frac{1}{4} + \frac{0.5}{2}$$

$$= \frac{2.}{4} = \frac{1}{2} \;//.$$

(2). A Continuous R.V X that can assume any value between $x=2$ and $x=5$ has the density function given by $f(x) = k(1+x)$, Find $P(x<4)$ and $P(3<x<4)$.

(3). A Continuous R.V x has the pdf

$$f(x) = \begin{cases} \frac{k}{1+x^2}, & -\infty < x < \infty \\ 0, & \text{otherwise} \end{cases}$$

Find (i) K (ii) cdf of x and (iii) $P(x \geq 0)$.

(4). A random variable X has the distribution function

$$F(x) = \begin{cases} 0, & x \le 1 \\ K(x-1)^4, & 1 < x \le 3 \\ 1 & \text{if } x > 3 \end{cases}$$

Find $P(2 < x < 3)$.

Discrete and Continuous R.V ~~to~~ problems:-

① Two dice are rolled at once. Find the probability distribution of the sum of the numbers on them.

②. Let X denote the number of tails in a single toss of 4 fair coins. Determine $P(X<2)$ and cdf of X.

③. Verify whether the function $P(X=x) = \frac{x^2+1}{81}$, $x = 0, 1, 2, 3$ is pmf.

④. Let X be a continuous random variable with pdf $f(x) = Kx(1-x)$, $0 \leq x \leq 1$ find k and determine a number 'b' such that $P(X \leq b) = P(X \geq b)$.

⑤. Verify whether the functions are probability density functions.

(i) $f(x) = Ke^{-Kx}$, $x \geq 0$, $K > 0$

(ii) $f(x) = \frac{2}{27} x e^{-x^2/4}$, $0 \leq x \leq \infty$

Mathematical Expectation and Variance :-

Mean: If X is a random variable, then the expected value or the mean value of X is defined as

$$E[X] = \begin{cases} \sum_i x_i \, p(x_i), & \text{if } X \text{ is discrete RV} \\ \int_{-\infty}^{\infty} x f(x) \, dx, & \text{if } X \text{ is Continuous RV} \end{cases}$$

and is denoted by μ (or) $\overline{X}$

Variance: Variance characterizes the variability in the distributions. Since two distributions with same mean can still have different dispersion of data about their means

$$Var(X) = E(X^2) - [E(X)]^2 = \sigma^2$$

where

$$E(X^2) = \begin{cases} \sum_i x_i^2 \, p(x_i), & \text{if } x \text{ is discrete} \\ \int_{-\infty}^{\infty} x^2 f(x) \, dx, & \text{if } x \text{ is Continuous} \end{cases}$$

Some important results on expectation :-

(i) $E[K] = K$

(ii) $E[X+K] = E[X] + K$

(iii) $E[aX \pm b] = aE[X] \pm b$

(iv) $Var(K) = 0$

(v) $Var(KX) = K^2 Var(X)$

(vi) $Var(X+K) = Var(X)$

(vii) $Var(aX+b) = a^2 Var(X)$.

①. A random variable X has the following distribution:

X :	1	2	3	4	5	6
$P(X=x)$:	$1/36$	$3/36$	$5/36$	$7/36$	$9/36$	$11/36$

Find mean and Variance of the distribution.

Sol:

$$\text{Mean} = E(x) = \Sigma x p(x)$$

$$= \frac{1}{36} + \frac{6}{36} + \frac{15}{36} + \frac{28}{36} + \frac{45}{36} + \frac{66}{36}$$

$$= \frac{161}{36} = 4.47$$

$$\boxed{\mu = 4.47}$$

$$E(x^2) = \Sigma x^2 p(x)$$

$$= \frac{1}{36} + 4\left(\frac{3}{36}\right) + 9\left(\frac{5}{36}\right) + 16\left(\frac{7}{36}\right) + 25\left(\frac{9}{36}\right) + 36\left(\frac{11}{36}\right)$$

$$= \frac{1}{36} + \frac{12}{36} + \frac{45}{36} + \frac{112}{36} + \frac{225}{36} + \frac{396}{36}$$

$$= 21.97$$

$$Var(x) = E(x^2) - [E(x)]^2$$

$$= 21.97 - (4.47)^2$$

$$= 1.99$$ //

(2). The probability distribution of a random variable X is given below. Find (i) E(X), (ii) Var(X) (iii) E(2X−3) and (iv) Var(2X−3)

X :	−2	−1	0	1	2
$P(X=x)$:	0.2	0.1	0.3	0.3	0.1

Sol:

$$E(X) = \Sigma x p(x) = -0.4 - 0.1 + 0 + 0.3 + 0.2 = 0$$

$$E(x^2) = \Sigma x^2 p(x) = 0.8 + 0.1 + 0 + 0.3 + 0.4 = 1.6$$

$$Var(x) = E(x^2) - (E(x))^2 = 1.6 - 0 = 1.6$$

$$E(2X-3) = 2E(X) - 3 = 2(0) - 3 = -3$$

$$Var(2X-3) = 4\,Var(X) = 4(1.6) = 6.4$$ //

(3). If X has the distribution function

$$F(x) = \begin{cases} 0 & x < 1 \\ 1/3, & 1 \le x < 4 \\ 1/2, & 4 \le x < 6 \\ 5/6, & 6 \le x < 10 \\ 1, & x \geq 10 \end{cases}$$

(1) Find the probability distribution of X.

(2) $P(2 < x < 6)$

(3) Mean of X

(4) Variance of X.

Sol:

Given :-

X :	1	4	6	10
F(x) :	1/3	1/2	5/6	1
p(x) :	1/3	1/6	1/3	1/6

(2) $P(2 < x < 6) = P(X = 4) = 1/6$ //

(3) $E[X] = \Sigma x p(x) = 1/3 + 4/6 + 6/3 + 10/6$

$$= \frac{1}{3} + \frac{2}{3} + 2 + \frac{5}{3}$$

$$= \frac{8}{3} + 2 = \frac{14}{3}$$

(4) $$Var(x) = E[x^2] - (E[x])^2$$

$$E[x^2] = \sum x^2 p(x) = \frac{1}{3} + \frac{16}{6} + \frac{36}{3} + \frac{100}{6}$$

$$= \frac{1}{3} + \frac{8}{3} + 12 + \frac{50}{3}$$

$$= \frac{59}{3} + 12 = \frac{95}{3}$$

$$Var(x) = \frac{95}{3} - \left(\frac{14}{3}\right)^2 = \frac{95}{3} - \frac{196}{9} = \frac{285-196}{9} = \frac{89}{9}$$ //

(4). When a die is thrown, X denotes the number that turns up. Find $E(X)$, $E(X^2)$ and $Var(X)$.

(5). If $Var(X) = 4$, find $Var(3X+8)$ where X is a random Variable.

(6). Let X be a random Variable with $E(X)=1$ and $E[X(X-1)] = 4$. Find $Var\,X$ and $Var(2-3X)$, $Var\left(\frac{X}{2}\right)$.

(7). The cdf of a random Variable X is $F(x) = 1-(1+x)e^{-x}$, $x>0$. Find the pdf of X. Also find mean and Variance of X.

① The density function of a random Variable X is given by $f(x) = kx(2-x)$, $0 \le x \le 2$. Find k, mean and Variance of the distribution.

Sol:

Since $f(x)$ is pdf

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{0}^{2} kx(2-x)\,dx = 1$$

$$k\int_{0}^{2} (2x - x^2)\,dx = 1$$

$$k\left[\cancel{2}\frac{x^2}{\cancel{2}} - \frac{x^3}{3}\right]_0^2 = 1$$

$$k\left[(4 - 8/3) - (0-0)\right] = 1$$

$$k\left[\frac{4}{3}\right] = 1 \Rightarrow \boxed{k = \frac{3}{4}}$$

Mean:

$$E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_{0}^{2} x\,kx(2-x)\,dx$$

$$= \frac{3}{4}\int_{0}^{2} (2x^2 - x^3)\,dx$$

$$= \frac{3}{4}\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2$$

$$= \frac{3}{4}\left[\left(\frac{16}{3} - \frac{16}{4}\right) - (0-0)\right]$$

$$= \frac{3}{4}\left[\frac{16}{3} - 4\right] = \frac{3}{4}\left(\frac{4}{3}\right) = 1$$

$$\boxed{\text{Mean} = 1}$$

Variance:

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \int_0^2 x^2 k\, x(2-x)\, dx$$

$$= \frac{3}{4}\int_0^2 (2x^3 - x^4)\, dx$$

$$= \frac{3}{4}\left[\frac{2x^4}{4} - \frac{x^5}{5}\right]_0^2$$

$$= \frac{3}{4}\left[\frac{32}{4} - \frac{32}{5}\right] = \frac{3}{4}\left[8 - \frac{32}{5}\right]$$

$$= \frac{3}{4}\left[\frac{8}{5}\right] = \frac{6}{5}$$

$$\text{Var}(x) = E(x^2) - (E(x))^2$$

$$= \frac{6}{5} - 1$$

$$\boxed{\text{Variance} = 1/5}$$

Note: $\int_0^\infty e^{-x} x^{n-1} dx = \Gamma n = (n-1)!$

7. The cdf of a random variable X is $F(x) = 1-(1+x)e^{-x}, \; x>0$. Find the pdf of X.

Also find mean and variance of X.

Sol:

$$f(x) = \frac{d}{dx} F(x)$$

$$= \frac{d}{dx}\{1 - e^{-x} - xe^{-x}\}$$

$$= 0 - e^{-x}(-1) - [xe^{-x}(-1) + e^{-x}(1)]$$

$$= \cancel{e^{-x}} + xe^{-x} - \cancel{e^{-x}}$$

$$f(x) = xe^{-x}, \quad x > 0.$$

Mean:

$$E(x) = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_0^\infty x \cdot xe^{-x} dx$$

$$= \int_0^\infty x^2 e^{-x} dx$$

$$\mu = \int_0^\infty x^{3-1} e^{-x} dx = \Gamma 3 = 2! = 2$$

(or) $\int_0^\infty x^2 e^{-x}\, dx$

$\int u\, dv = uv - u'v_1 + u''v_2 - \ldots$

$u = x^2$ $\qquad \int dv = \int e^{-x}\, dx$

$u' = 2x$ $\qquad v = e^{-x}/-1$

$u'' = 2$ $\qquad v_1 = e^{-x}$

$u''' = 0$ $\qquad v_2 = \frac{e^{-x}}{-1}$

$$E(X) = \left[-x^2 e^{-x} - 2x e^{-x} - 2 e^{-x} \right]_0^\infty$$

$$= \left[(0-0-0) - (0-0-2) \right]$$

$\boxed{\text{Mean} = 2}$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \int_0^\infty x^2 . x e^{-x}\, dx$$

$$= \int_0^\infty x^3 e^{-x}\, dx$$

$$= \int_0^\infty x^{4-1} e^{-x}\, dx = \Gamma 4 = 3! = 6.$$

$$Var(x) = E(x^2) - (E(x))^2 = 6 - 4 = 2$$ //.

③. For the following density function $f(x) = \lambda e^{-\lambda x}$, $x \geq 0$, $\lambda > 0$. find mean and variance

Sol:

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

$$\text{Mean} = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_0^{\infty} x \lambda e^{-\lambda x}\, dx.$$

$$= \lambda \int_0^{\infty} e^{-\lambda x} x\, dx.$$

$$\int_0^{\infty} e^{-ax} x^{n-1}\, dx = \frac{\Gamma n}{a^n}$$

$$= \lambda \int_0^{\infty} e^{-\lambda x} x^{2-1}\, dx$$

$$= \frac{\lambda \Gamma 2}{\lambda^n} = \frac{\lambda 1!}{\lambda^n} = \lambda^{1-n}$$

$$\boxed{\text{Mean} = \lambda^{1-n}}$$

$$E[x^2] = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \int_0^{\infty} x^2 \lambda e^{-\lambda x}\, dx$$

$$= \lambda \int_0^{\infty} e^{-\lambda x} x^2\, dx$$

$$= \lambda \int_0^\infty \bar{e}^{\lambda x} x^{3-1} dx = \frac{\lambda \sqrt{3}}{\lambda^n} = \frac{2!\,\lambda}{\lambda^n}$$

$$= 2\lambda^{1-n}.$$

$$Var(x) = E(x^2) - [E(x)]^2$$

$$= 2\lambda^{1-n} - (\lambda^{1-n})^2$$

$$Var(x) = 2\lambda^{1-n} - \lambda^{2-2n} \quad //$$

Practice problems :- (both discrete and Continuous).

1. The probability function of an infinite discrete distribution is given by $P[X=j] = \frac{1}{2^j}$, $j = 1, 2 \ldots \infty$ Find the Mean and Variance of the distribution.

Ans:- Mean = 2, Variance = 2.

2. Suppose X is discrete random variable such that $P(X=0) = 1 - P(X=1)$ and $E(X) = 3\,Var(X)$. Find $P(X=0)$ and $P(X=1)$.

Ans: $1/3$, $2/3$

3. A continuous random variable has pdf $f(x) = \begin{cases} a+bx, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$

If the mean is $1/2$ then find a, b. Also find $Var(x)$.

Ans: $a=1, b=0$, $Var = 1/12$.

(4). If $f(x) = \begin{cases} x e^{-x^2/2}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

find mean and variance of the distribution.

Moments - Moment Generating Functions (MGF) and their properties :-

Moments :- [Discrete case]

Let X be discrete RV taking the values $x_1, x_2, \ldots x_n$ with probability mass function $p_1, p_2 \ldots p_n$ respectively then the r^{th} moment about the origin is

$$\mu_r' \text{ (about the origin)} = E[x^r] = \sum_{i=1}^{n} x_i^r p_i$$

$$\mu_r' \text{ (about any point } x=A) = E[(x-A)^r] = \sum_{i=1}^{n} (x_i - A)^r p_i$$

$$\mu_r \text{ (about mean)} = \sum_{i=1}^{n} (x_i - \bar{x})^r p_i = E[(x-\bar{x})^r]$$

In particular,

$$\mu_1' = \sum_{i=1}^{n} x_i p_i = \text{Mean} = \mu_1$$

$$\mu_2' = \sum_{i=1}^{n} x_i^2 p_i$$

$$\mu_2 = \mu_2' - (\mu_1')^2$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$

Continuous Case :-

If X is a Continuous RV with pdf $f(x)$ defined in the interval (a, b) then

$$\mu_r' = \int_a^b x^r f(x)\, dx = E[x^r]$$

$$\mu_r' = \int_a^b [x-A]^r f(x)\, dx = E[(x-A)^r]$$

$$\mu_r = \int_a^b [x-\bar{x}]^r f(x)\, dx = E[(x-\bar{x})^r]$$

Moment Generating Functions :- (MGF)

Moment generating function of a random Variable X about the origin is defined by

$$M_x(t) = E[e^{tx}] = \begin{cases} \sum e^{tx} p(x), & x \text{ is discrete} \\ \int_{-\infty}^{\infty} e^{tx} f(x)\, dx, & x \text{ is Continuous} \end{cases}$$

Note :-

$$M_x(t) = E[e^{tx}] = E\left[1 + \frac{tx}{1!} + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + \cdots\right]$$

$$= 1 + t\,E[x] + \frac{t^2}{2!} E[x^2] + \cdots + \frac{t^r}{r!} E[x^r]$$

$$= 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \dots + \frac{t^r}{r!}\mu_r' + \dots$$

Note:

$$\mu_r' = \frac{d^r}{dt^r}\left[M_X(t)\right]_{t=0}$$

1. Find the moment generating function of the RV X whose probability function

$P(X=x) = \frac{1}{2^x}, \quad x = 1, 2, \ldots$ Hence find its mean.

Sol:

$$MGF = M_X(t) = E[e^{tx}]$$

$$= \sum e^{tx} p(x)$$

$$= \sum_{x=1}^{\infty} e^{tx} \frac{1}{2^x}$$

$$= \left[\frac{e^t}{2} + \frac{e^{2t}}{2^2} + \frac{e^{3t}}{2^3} + \frac{e^{4t}}{2^4} + \cdots \right]$$

$$= \frac{e^t}{2}\left[1 + \frac{e^t}{2} + \frac{e^{2t}}{2^2} + \frac{e^{3t}}{2^3} + \cdots \right]$$

$$= \frac{e^t}{2}\left[1 - \frac{e^t}{2}\right]^{-1}$$

$$= \frac{e^t}{2}\left[\frac{2-e^t}{2}\right]^{-1} = \frac{e^t}{\cancel{2}} \cdot \frac{\cancel{2}}{2-e^t}$$

$$M_X(t) = \frac{e^t}{2-e^t}$$

$$\text{Mean} = \mu_1' = \frac{d}{dt}\left[M_X(t)\right]_{t=0}$$

$$= \frac{d}{dt}\left[\frac{e^t}{2-e^t}\right]_{t=0}$$

$$= \left[\frac{(2-e^t)e^t - e^t(-e^t)}{(2-e^t)^2} \right]_{t=0}$$

$$= \frac{(2-1)(1) - 1(-1)}{(2-1)^2} = \frac{1+1}{1} = 2.$$

$$\boxed{\text{Mean} = 2}$$

(2). A random variable X has density function given by

$$f(x) = \begin{cases} \frac{1}{K}, & 0 < x < K \\ 0 & \text{otherwise} \end{cases}$$

Find (i) MGF, (ii) r^{th} moment (iii) Mean and (iv) Variance.

Sol:

$$MGF = M_X(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx.$$

$$= \int_0^k e^{tx} \frac{1}{K}\, dx$$

$$= \frac{1}{K}\left[\frac{e^{tx}}{t}\right]_0^k$$

$$= \frac{1}{K}\left[\frac{e^{kt}}{t} - \frac{1}{t}\right]$$

$$M_X(t) = \frac{1}{Kt}\left[e^{kt} - 1\right]$$

$$= \frac{1}{kt}\left[\cancel{1} + \frac{kt}{1!} + \frac{(kt)^2}{2!} + \frac{(kt)^3}{3!} + \dots \cancel{kt} - \cancel{1}\right]$$

$$= 1 + \frac{kt}{2!} + \frac{(kt)^2}{3!} + \dots \frac{(kt)^r}{(r+1)!} + \dots$$

(ii) r^{th} moment:

$$\mu_r' = E[x^r] = \int_{-\infty}^{\infty} x^r f(x)\, dx.$$

$$= \int_0^k x^r \frac{1}{k}\, dx$$

$$= \frac{1}{k}\left[\frac{x^{r+1}}{r+1}\right]_0^k = \frac{1}{k}\left[\frac{k^{r+1}}{r+1}\right]$$

$$= \frac{k^r}{r+1}$$

(or) Co-efficient of $\frac{t^r}{r!} = \mu_r' = \frac{k^r}{r+1}$

$$\text{Mean} = \mu_1' = \text{Co-efficient of } \frac{t}{1!} = \frac{k}{2!} = \frac{k}{2}.$$

$$\mu_2' = \text{Coefficient of } \frac{t^2}{2!} = \frac{k^2}{3!} = \frac{k^2}{3}.$$

$$\text{Variance} = \mu_2' - \mu_1^2$$

$$= \frac{k^2}{3} - \left(\frac{k}{2}\right)^2 = \frac{k^2}{3} - \frac{k^2}{4} = \frac{k^2}{12}$$

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③. Let X be a RV with pdf $f(x) = \begin{cases} \frac{1}{3} e^{-x/3}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$

Find MGF, mean and Variance.

UNIVERSITY QUESTIONS

UNIT - 1

PART - A

1. A Continuous random Variable X that can assume any values between $x=2$ and $x=5$ has a density function given by $f(x) = K(1+x)$. Find $P(X<4)$.

2. Check whether the following is a probability density function or not.

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0,\ \lambda > 0 \\ 0, & \text{elsewhere} \end{cases}$$

3. If a random Variable has the MGF given by $M_X(t) = \frac{2}{2-t}$. Determine the Variance of X.

4. Let the random Variable X denote the sum obtained in rolling a pair of fair die. Determine the probability mass function of X.

5. The CDF of a Continuous random Variable is given by

$$F(x) = \begin{cases} 0, & x < 0 \\ 1-e^{-x/5}, & 0 \leq x < \infty \end{cases}$$

Find the pdf and mean of X.

6. A continuous random variable X has probability density function $f(x) = \begin{cases} 3x^2, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$

Find k such that $P(X > k) = 0.5$.

7. If the pdf of a random variable X is $f(x) = \frac{x}{2}$ in $0 \le x \le 2$. Find $P(X > 1.5 / X > 1)$.

8. A random variable X has cdf

$$F_X(x) = \begin{cases} 0; & x < 1 \\ \frac{1}{2}(x-1); & 1 \le x < 3 \\ 1; & x \geq 3 \end{cases}$$

Find the pdf of X and the expected value of X.

9. If a random variable X has the MGF $M_X(t) = \frac{3}{3-t}$, Find the standard deviation of X.

10. Check whether $f(x) = \frac{1}{4} x e^{-x/2}$ for $0 < x < \infty$ can be a probability density function of X.

PART – B

(1). If the random Variable X takes the values 1, 2, 3 and 4 such that

$$2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4)$$ then

find the probability distribution and cummulative distribution function of X.

(2). A random Variable X has the following probability function

X:	0	1	2	3	4	5	6	7
$P(x)$:	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

(i) Find the value of k

(ii) Evaluate $P(X<6)$, $P(X \geq 6)$

(iii) If $P(X \leq c) > \frac{1}{2}$ Find the minimum value of c.

(3). A random Variable X has the following probability distribution

X:	-2	-1	0	1	2	3
$P(X=x)$:	0.1	k	0.2	$2k$	0.3	$3k$

Find k, Evaluat $P(X<2)$, $P(-2<X<2)$, CDF of X and mean of X.

(4). The probability function of an infinite discrete distribution is given by $P(X=j) = \frac{1}{2^j}$, $j = 1, 2 \ldots \infty$

Verify that the total probability is 1. Find ~~the Mean and~~ $P(X$ is even$)$, $P(X \geq 5)$ and $P(X$ is divisible by $3)$.

(5). The distribution function of a random variable X is given by $F(x) = 1 - (1+x)e^{-x}$, $x \geq 0$. Find the density function, mean and variance of X.

(6). If the density function of X is

$$f(x) = \begin{cases} ce^{-2x}, & 0 < x < \infty \\ 0, & \text{otherwise} \end{cases}$$

, find c. Also find $P(X > 2)$.

(7). If the probability density of X is given by

$$f(x) = \begin{cases} 2(1-x), & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

find its r^{th} moment about origin. Hence evaluate $E\left[(2x+1)^2\right]$

(8) Find MGF Corresponding to the distribution

$$f(\theta) = \begin{cases} \frac{1}{2} e^{-\theta/2}, & \theta > 0 \\ 0, & \text{otherwise} \end{cases}$$

and hence find its mean and Variance.

(9). Find the MGF of the random Variable X having the probability density function

$$f(x) = \begin{cases} \frac{x}{4} e^{-x/2}, & x > 0 \\ 0, & \text{elsewhere} \end{cases}$$

Also deduce the first four moments about the origin.

(10). A Continuous R.V X has the pdf

$$f(x) = \begin{cases} \frac{k}{1+x^2}, & -\infty < x < \infty \\ 0, & \text{elsewhere} \end{cases}$$

Find the Value of k, cdf of X and $P(X \geq 0)$.

(11). The probability density function of a random variable X is given by

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ K(2-x), & 1 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

Find the value of K, $P(0.2 < x < 1.2)$

$P[0.5 < x < 1.5 / x \geq 1]$ and cdf of X.

(12). The probability mass function of a random variable X is defined as $P(X=0) = 3c^2$, $P(X=1) = 4c - 10c^2$, $P(X=2) = 5c - 1$ where $c > 0$ and $P(X=r) = 0$ if $r \neq 0, 1, 2$

Find (i) the value of c

(ii) $P(0 < x < 2 / x > 0)$

(iii) cdf of x.

(iv) The largest value of X for which $F(x) < 1/2$.

(13). A random Variable X has pdf $f(x) = kx^2 e^{-x}, x > 0$

Find the r^{th} moment about origin and hence find the mean and Variance.

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