Chapter: 3.

Electromagnetic Waves

Plane Electromagnetic Waves in Free space!

The Maxwell's Equations are

$$\left.\begin{aligned} \text{div } D &= \nabla\cdot D = \rho \\ \text{div } B &= \nabla\cdot B = 0 \\ \text{curl } E &= -\partial B/\partial t \\ \text{curl } H &= J + \frac{\partial D}{\partial t} \end{aligned}\right\} \text{ and } \begin{aligned} B &= \mu H \\ D &= \epsilon E \\ J &= \sigma E \end{aligned} \quad \text{——} \; (1)$$

The characteristics of free space are:

$$\boxed{\rho = 0 \;, \quad \sigma = 0 \;, \quad \mu = \mu_0 \text{ and } \epsilon = \epsilon_0} \quad \text{——} \; (2)$$

$\therefore$ By using # (2) results, the equation (1) can be reduce to

$$\left.\begin{aligned} \text{div } E &= 0 \quad \text{——(a)} \\ \text{div } H &= 0 \quad \text{——(b)} \\ \text{curl } E &= -\mu_0 \frac{\partial H}{\partial t} \quad \text{——(c)} \\ \text{curl } H &= \epsilon_0 \frac{\partial E}{\partial t} \quad \text{——(d)} \end{aligned}\right\} \quad \text{——} \; (3)$$

Now Taking curl of eqn 3(c), we get

$$\text{curl}\cdot\text{curl } E = -\mu_0 \frac{\partial}{\partial t}(\text{curl } H)$$

substituting curl H from # 3(d), we get

$$\text{curl curl } E = -\mu_0 \frac{\partial}{\partial t}\left(\epsilon_0 \cdot \frac{\partial E}{\partial t}\right)$$

$$\text{curl curl } E = -\mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2} \quad \text{—— (4)}$$

Now we know that,

$$\boxed{\text{curl curl } E = \text{grad div } E - \nabla^2 E} \quad [\text{Vector Identity}]$$

$$\therefore \quad \text{curl curl } E = -\nabla^2 E \quad [\because \text{div } E = 0 \text{ from } \# \, 3(a)]$$

$\therefore$ On substituting this result in # (4), becomes

$$-\nabla^2 E = -\mu_0 \epsilon_0 \partial^2 E / \partial t^2$$

$\nabla(\nabla \cdot E)$

$$\therefore \quad \boxed{\nabla^2 E - \mu_0 \epsilon_0 \partial^2 E / \partial t^2 = 0} \quad \text{—— (5)}$$

Now taking curl of equation 3(d)

$$\text{curl curl } H = \epsilon_0 \frac{\partial}{\partial t}(\text{curl } E)$$

Substituting curl E from equation [3(c)], we get

$$\text{curl curl } H = \epsilon_0 \frac{\partial}{\partial t}\left(-\mu_0 \frac{\partial H}{\partial t}\right) = -\mu_0 \epsilon_0 \partial^2 H / \partial t^2 \quad \text{—— (6)}$$

Again Using identity $\boxed{\text{curl curl } H = \text{grad div } H - \nabla^2 H}$ and with div $H = 0$ from [3(b)], we get

$$\text{curl curl } H = -\nabla^2 H$$

$\therefore$ On substituting this value in equation (6) becomes

$$\boxed{\nabla^2 H - \mu_0 \epsilon_0 \partial^2 H / \partial t^2 = 0} \quad \text{—— (7)}$$

Equation (5) and (7) represents wave equations governing electromagnetic fields E and H in free space.

This equation resembles with the general wave equation

$$\boxed{\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}} \quad \text{— (8)}$$

$v \rightarrow$ velocity of wave.

Comparing (7) and (8), the field vectors E and H are propagated in free space as waves at speed equal to

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}}, \quad \epsilon_0 = 8.542 \times 10^{-12}\ \text{farad/m.}$$

$c = 3 \times 10^8$ m/sec. — speed of light.

$$\left.\begin{aligned} \nabla^2 E - \frac{1}{c^2}\partial^2 E/\partial t^2 &= 0 \\ \nabla^2 H - \frac{1}{c}\partial^2 H/\partial t^2 &= 0 \\ \nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} &= 0 \end{aligned}\right\} \text{— (9)}$$

Now let us find the solution of above equations for a plane electromagnetic wave.

A plane wave is defined as a wave whose amplitude is same at any point in a plane $\perp$ to a specified direction.

The plane waves solutions of above equations can be written as

$$E(r,t) = E_0\, e^{ik\cdot r - i\omega t} \quad \text{— (10)}$$

$$H(r,t) = H_0\, e^{ik\cdot r - i\omega t} \quad \text{— (11)}$$

where E_0, H_0 are complex amplitude which are constant in space and time, while k is a wave propagation vector.

$$K = kn = \frac{2\pi n}{\lambda} = \frac{2\pi\nu}{c}\cdot n = \frac{\omega}{c}\cdot n \quad \text{— (12)}$$

where n is a unit vector in the direction of wave propagation

Now in order to apply the conditions $\nabla\cdot E = 0$, and $\nabla\cdot H = 0$, let us find $\nabla\cdot E$ and $\nabla\cdot H$.

$$\nabla\cdot E = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot E_0 e^{ik\cdot r - i\omega t}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left[(\hat{i}E_{0x} + \hat{j}E_{0y} + \hat{k}E_{0z})e^{i(k_x x + k_y y + k_z\cdot z) - i\omega t}\right]$$

Since $[k\cdot r = \hat{i}k_x + \hat{j}k_y + \hat{k}z) = [k_x x + k_y y + k_z\cdot z]$

$$\nabla\cdot E = \left(E_{0x}\, ik_x + E_{0y}\, ik_y + E_{0z}\,\hat{i}k_z\right] e^{i(k_x x + k_y y + k_z\cdot z)}$$

$$= i(\hat{i}k_x + \hat{j}k_y + \hat{k}k_z)\cdot(\hat{i}E_{0x} + \hat{j}E_{0y} + \hat{k}E_{0z})e^{ik\cdot r - i\omega t}$$

$$= ik\cdot E_0 e^{ik\cdot r - i\omega t} = ik\cdot E$$

Similarly $\nabla\cdot H = ik\cdot H$

$\therefore$ $\nabla\cdot E = 0$ and $\nabla\cdot H = 0$ demand that

$$k\cdot E = 0 \text{ and } k\cdot H = 0$$

This means electromagnetic vectors E and H are both $\perp$ to the direction of propagation vector ~~k~~ K. This shows that electromagnetic waves are Transverse in nature

Summary For electromagnetic waves in free space:

1. In free space EM Waves travels with the speed of light.
2. The electromagnetic field vectors E and H are mutually perpendicular and they are also $\perp$ to the direction of propagation of EM Waves. Hence EM waves are Transverse in nature.
3. The field vectors E and H are in same phase.