Q1

Let $0 \leq h(X) \leq M$ and $0 \leq a < M$. Define

$$Y = M - h(X) \geq 0.$$

Applying Markov's inequality with $t = M - a > 0$ gives

$$\mathbb{P}(Y \geq M - a) \leq \frac{\mathbb{E}[Y]}{M - a} = \frac{M - \mathbb{E}[h(X)]}{M - a}.$$

Since

$$Y \geq M - a \iff h(X) \leq a,$$

we obtain

$$\mathbb{P}(h(X) \leq a) \leq \frac{M - \mathbb{E}[h(X)]}{M - a}.$$

Therefore,

$$\mathbb{P}(h(X) \geq a) = 1 - \mathbb{P}(h(X) < a) \geq 1 - \mathbb{P}(h(X) \leq a) = \frac{\mathbb{E}[h(X)] - a}{M - a}.$$

$$\mathbb{P}(h(X) \geq a) \geq \frac{\mathbb{E}[h(X)] - a}{M - a}. \tag{1}$$

Q2 We consider the probability space $(\Omega, \mathcal{F}, P)$ where:

- $\Omega = [0, 1]$
- $\mathcal{F}$ is the Borel σ-algebra on $[0, 1]$
- P is the uniform probability measure: $P([a, b]) = b - a$ for all $0 \leq a \leq b \leq 1$

$$X_n(\omega) = \frac{n}{n+1}\omega + (1-\omega)^n, \quad \omega \in [0, 1]$$
$$X(\omega) = \omega, \quad \omega \in [0, 1]$$

We want to show that X_n converges to X almost surely as $n \to \infty$.

Definition 1. *A sequence of random variables (X_n) converges almost surely to X if:*

$$P\left(\{\omega \in [0, 1] : \lim_{n\to\infty} X_n(\omega) = X(\omega)\}\right) = 1$$

That is, for almost every $\omega \in [0, 1]$, we have $X_n(\omega) \to X(\omega)$ as $n \to \infty$.

Fix $\omega \in [0, 1]$ and examine $X_n(\omega)$:

$$X_n(\omega) = \frac{n}{n+1}\omega + (1-\omega)^n$$

Check the behavior as $n \to \infty$ by considering different cases:
Case 1: $\omega = 0$

$$X_n(0) = \frac{n}{n+1} \cdot 0 + (1-0)^n = 0 + 1^n = 1 \quad \text{for all } n$$

Thus:
$$\lim_{n\to\infty} X_n(0) = 1$$
But $X(0) = 0$, so:
$$\lim_{n\to\infty} X_n(0) = 1 \neq 0 = X(0)$$
Conclusion: At $\omega = 0$, X_n does NOT converge to X.
Case 2: $\omega = 1$

$$X_n(1) = \frac{n}{n+1} \cdot 1 + (1-1)^n = \frac{n}{n+1} + 0^n = \frac{n}{n+1}$$

Thus:
$$\lim_{n\to\infty} X_n(1) = \lim_{n\to\infty} \frac{n}{n+1} = 1$$
And $X(1) = 1$, so:
$$\lim_{n\to\infty} X_n(1) = 1 = X(1)$$
Conclusion: At $\omega = 1$, X_n converges to X.
Case 3: $0 < \omega < 1$ For $0 < \omega < 1$, we have:

- $\frac{n}{n+1}\omega \to 1 \cdot \omega = \omega$ as $n \to \infty$
- $(1-\omega)^n \to 0$ as $n \to \infty$ since $|1-\omega| < 1$

Therefore:

$$\lim_{n\to\infty} X_n(\omega) = \omega + 0 = \omega = X(\omega)$$

Conclusion: For all $\omega \in (0,1)$, X_n converges to X.

From the case analysis above:

- For $\omega = 0$: $\lim_{n\to\infty} X_n(\omega) \neq X(\omega)$
- For $\omega \in (0,1]$: $\lim_{n\to\infty} X_n(\omega) = X(\omega)$

Thus, the set where X_n does NOT converge to X is:

$$\{\omega \in [0,1] : \lim_{n\to\infty} X_n(\omega) \neq X(\omega)\} = \{0\}$$

The set where X_n converges to X is:

$$\{\omega \in [0,1] : \lim_{n\to\infty} X_n(\omega) = X(\omega)\} = (0,1]$$

The probability measure is uniform on $[0,1]$, so:

$$P(\{0\}) = 0$$

$$P((0,1]) = P([0,1]) - P(\{0\}) = 1 - 0 = 1$$

Therefore:

$$P\left(\{\omega \in [0,1] : \lim_{n\to\infty} X_n(\omega) = X(\omega)\}\right) = P((0,1]) = 1$$

We have shown that:

$$P\left(\{\omega \in [0,1] : \lim_{n\to\infty} X_n(\omega) = X(\omega)\}\right) = 1$$

which means:

$$\boxed{X_n \to X \text{ almost surely}}$$

The only point of non-convergence is $\omega = 0$, which has probability zero under the uniform distribution on $[0,1]$.

Solution to Problem 3

Let S_n be the number of smokers in a random sample of size n and define the estimator

$$M_n = \frac{S_n}{n}.$$

Then $\mathbb{E}[M_n] = f$ and

$$\mathrm{Var}(M_n) = \mathrm{Var}\left(\frac{S_n}{n}\right) = \frac{1}{n^2}\,\mathrm{Var}(S_n) = \frac{1}{n^2}\cdot nf(1-f) = \frac{f(1-f)}{n}.$$

Since $f(1-f) \le \frac{1}{4}$ for any $f \in [0,1]$, we have the uniform bound

$$\mathrm{Var}(M_n) \le \frac{1}{4n}.$$

By Chebyshev's inequality,

$$\mathbb{P}(|M_n - f| \ge \varepsilon) \le \frac{\mathrm{Var}(M_n)}{\varepsilon^2} \le \frac{1}{4n\varepsilon^2}.$$

Requiring this to be at most δ gives

$$\frac{1}{4n\varepsilon^2} \le \delta \quad\Longrightarrow\quad n \ge \frac{1}{4\varepsilon^2\delta}.$$

Thus the smallest sample size (using this bound) can be taken as

$$n = \left\lceil \frac{1}{4\varepsilon^2\delta} \right\rceil.$$

(a) If ε is reduced to $\frac{2}{3}$ of its original value, i.e. $\varepsilon' = \frac{2}{3}\varepsilon$, then

$$n' = \left\lceil \frac{1}{4(\varepsilon')^2\delta} \right\rceil = \left\lceil \frac{1}{4\left(\frac{2}{3}\varepsilon\right)^2\delta} \right\rceil = \left\lceil \frac{1}{4\cdot\frac{4}{9}\varepsilon^2\delta} \right\rceil = \left\lceil \frac{9}{4}\cdot\frac{1}{4\varepsilon^2\delta} \right\rceil = \left\lceil \frac{9}{4}n \right\rceil.$$

So the recommended n increases by a factor of $(3/2)^2 = 9/4$ (approximately $2.25\times$).

(b) If δ is reduced to $\frac{3}{5}$ of its original value, i.e. $\delta' = \frac{3}{5}\delta$, then

$$n' = \left\lceil \frac{1}{4\varepsilon^2\delta'} \right\rceil = \left\lceil \frac{1}{4\varepsilon^2\cdot\frac{3}{5}\delta} \right\rceil = \left\lceil \frac{5}{3}\cdot\frac{1}{4\varepsilon^2\delta} \right\rceil = \left\lceil \frac{5}{3}n \right\rceil.$$

So the recommended n increases by a factor of $5/3$ (approximately $1.6667\times$).

Problem 4

Suppose that X_n converges almost surely to X and Y_n converges almost surely to Y. Show that $X_n + Y_n$ converges almost surely to $X + Y$. Does the corresponding result also hold for convergence in probability and convergence in distribution?

Solution:

Almost Sure Convergence

We are given $X_n \xrightarrow{a.s.} X$ and $Y_n \xrightarrow{a.s.} Y$. By definition, this means the set of outcomes where the sequences do not converge to their limits has probability 0.

Let Z_X be the set of outcomes where X_n does not converge to X:

$$Z_X = \left\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) \neq X(\omega)\right\}$$

We are given $P(Z_X) = 0$.

Let Z_Y be the event where Y_n does not converge to Y:

$$Z_Y = \left\{\omega \in \Omega : \lim_{n\to\infty} Y_n(\omega) \neq Y(\omega)\right\}$$

We are given $P(Z_Y) = 0$.

We are interested in the event Z_{X+Y}, where the sum $X_n + Y_n$ does not converge to $X + Y$:

$$Z_{X+Y} = \left\{\omega \in \Omega : \lim_{n\to\infty} (X_n(\omega) + Y_n(\omega)) \neq X(\omega) + Y(\omega)\right\}$$

From the properties of limits of real numbers, if the sum $\lim(X_n + Y_n)$ fails to equal $X + Y$, it must be because at least one of the original sequences failed to converge to its respective limit. Therefore, any outcome ω in Z_{X+Y} must also be in Z_X or in Z_Y.

This implies $Z_{X+Y} \subseteq Z_X \cup Z_Y$, and thus:

$$0 \leq P(Z_{X+Y}) \leq P(Z_X \cup Z_Y) \leq P(Z_X) + P(Z_Y)$$

Substituting the given values:

$$P(Z_{X+Y}) \leq 0 + 0 = 0$$

By definition, this means $X_n + Y_n \xrightarrow{a.s.} X + Y$.

Convergence in Probability

We are given $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$. For any $\epsilon > 0$, we need to show that:

$$\lim_{n\to\infty} P\left(|(X_n + Y_n) - (X + Y)| > \epsilon\right) = 0$$

First, by the triangle inequality:

$$|(X_n + Y_n) - (X + Y)| \leq |X_n - X| + |Y_n - Y|$$

If the sum $|X_n - X| + |Y_n - Y|$ is greater than ϵ, then at least one of the terms must be greater than $\epsilon/2$. This allows us to relate the events:

$$\{|(X_n + Y_n) - (X + Y)| > \epsilon\} \subseteq \left\{|X_n - X| > \frac{\epsilon}{2}\right\} \cup \left\{|Y_n - Y| > \frac{\epsilon}{2}\right\}$$

Therefore,

$$P\left(|(X_n + Y_n) - (X + Y)| > \epsilon\right) \leq P\left(|X_n - X| > \frac{\epsilon}{2}\right) + P\left(|Y_n - Y| > \frac{\epsilon}{2}\right)$$

Now, we take the limit as $n \to \infty$. By our premises, $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$, which means both terms on the right-hand side go to 0, and thus we get:

$$\lim_{n\to\infty} P\left(|(X_n + Y_n) - (X + Y)| > \epsilon\right) \leq 0 + 0$$

The limit must be 0. Thus, $X_n + Y_n \xrightarrow{P} X + Y$.

Convergence in Distribution

No, the result does not hold. Convergence in distribution only describes the convergence of the marginal CDFs (F_{X_n} and F_{Y_n}) and does not preserve the **joint distribution** of (X_n, Y_n), which is required to find the distribution of the sum $X_n + Y_n$. We can show this via the construction of a counter-example.

Let X be a single Bernoulli(1/2) random variable, and define the sequences X_n and Y_n as:

$$X_n = X, \quad \text{for all } n$$
$$Y_n = 1 - X, \quad \text{for all } n$$

X_n has a Bernoulli(1/2) distribution for all n. This sequence of distributions trivially converges to the Bernoulli(1/2) distribution of X. So, $X_n \xrightarrow{d} X$.

Notice that, Y_n also has a Bernoulli(1/2) distribution for all n. Let Y be a variable with a Bernoulli(1/2) distribution. The sequence F_{Y_n} trivially converges to F_Y. So, $Y_n \xrightarrow{d} Y$.

Now, to check if the sum converges, look at the limit of the sum, say $Z_n = X_n + Y_n$.

$$Z_n = X_n + Y_n = X + (1 - X) = 1 \quad \text{for all } n$$

Z_n is a constant random variable 1 for all n. Hence, this sequence converges in distribution to a constant variable.

Now, look at the sum of the limits, say $Z = X + Y$.

Here, $X \sim$ Bernoulli(1/2) and $Y \sim$ Bernoulli(1/2). Since the premises do not define a joint distribution for the limits, we check against the case where X and Y are independent.

The distribution of Z (a sum of two independent Bernoulli variables) is Binomial(2, 1/2): The limit of Z_n converges in distribution to the constant 1. The distribution of the (independent) sum of limits is $P(Z = 0) = 1/4, P(Z = 1) = 1/2, P(Z = 2) = 1/4$. Since both the distributions are not the same, the statement does not hold.

If interested: Check out Slutsky's Theorem.

Q5

Let $\{N_t,\ t \geq 0\}$ be a Poisson process with rate $\lambda > 0$. For $0 \leq t_1 \leq t_2$ we want the joint pmf of (N_{t_1}, N_{t_2}).

By the independent increments property the random variables N_{t_1} and $N_{t_2} - N_{t_1}$ are independent. Thus for integers $x, y \geq 0$,

$$P\big(N_{t_1} = x,\ N_{t_2} = y\big) = \begin{cases} 0, & y < x, \\ P\big(N_{t_1} = x\big)\ P\big(N_{t_2} - N_{t_1} = y - x\big), & y \geq x. \end{cases}$$

Now use that $N_{t_1} \sim \text{Poisson}(\lambda t_1)$ and $N_{t_2} - N_{t_1} \sim \text{Poisson}(\lambda(t_2 - t_1))$. For $y \geq x$,

$$\begin{aligned} P\big(N_{t_1} = x,\ N_{t_2} = y\big) &= \frac{e^{-\lambda t_1} (\lambda t_1)^x}{x!} \cdot \frac{e^{-\lambda(t_2 - t_1)} \big(\lambda(t_2 - t_1)\big)^{\,y-x}}{(y-x)!} \\ &= e^{-\lambda t_2}\, \frac{\lambda^y\, t_1^x\, (t_2 - t_1)^{\,y-x}}{x!\,(y-x)!}. \end{aligned}$$

So the joint pmf is

$$P\big(N_{t_1} = x,\ N_{t_2} = y\big) = \begin{cases} 0, & y < x, \\ e^{-\lambda t_2}\, \dfrac{\lambda^y\, t_1^x\, (t_2 - t_1)^{\,y-x}}{x!\,(y-x)!}, & y \geq x, \end{cases}$$

for integers $x, y \geq 0$.

Problem 6

A process is WSS if it satisfies two conditions:

1. The mean function $\mu_X(t) = \mathbb{E}[X_t]$ is constant.
2. The autocorrelation function $R_X(t_1, t_2) = \mathbb{E}[X_{t_1} X_{t_2}]$ depends only on the time lag $\tau = t_1 - t_2$.

Keep in mind that if X and Y are independent random variables, then for functions g and h, the random variables $g(X)$ and $h(Y)$ are also independent. We compute the mean $\mu_X(t)$:

$$\begin{aligned}\mu_X(t) &= \mathbb{E}[X_t] = \mathbb{E}[A\cos(\omega_c t + \Theta)] \\ &= \mathbb{E}[A] \cdot \mathbb{E}[\cos(\omega_c t + \Theta)] \quad \text{(Since } A \text{ and } \Theta \text{ are independent)}\end{aligned}$$

Now, we compute the expectation of the cosine term. Since $\Theta \sim U[0, 2\pi]$, its PDF is $f_\Theta(\theta) = \frac{1}{2\pi}$.

$$\begin{aligned}\mathbb{E}[\cos(\omega_c t + \Theta)] &= \int_0^{2\pi} \cos(\omega_c t + \theta) \frac{1}{2\pi}\, d\theta \\ &= \frac{1}{2\pi} \left[\sin(\omega_c t + \theta)\right]_0^{2\pi} \\ &= \frac{1}{2\pi} \left(\sin(\omega_c t + 2\pi) - \sin(\omega_c t + 0)\right) \\ &= \frac{1}{2\pi} \left(\sin(\omega_c t) - \sin(\omega_c t)\right) = 0\end{aligned}$$

Substituting this back into the mean function:

$$\mu_X(t) = \mathbb{E}[A] \cdot 0 = 0$$

The mean is constant.

Next, we compute $R_X(t_1, t_2) = \mathbb{E}[X_{t_1} X_{t_2}]$.

$$\begin{aligned}R_X(t_1, t_2) &= \mathbb{E}\left[(A\cos(\omega_c t_1 + \Theta))(A\cos(\omega_c t_2 + \Theta))\right] \\ &= \mathbb{E}\left[A^2 \cos(\omega_c t_1 + \Theta)\cos(\omega_c t_2 + \Theta)\right] \\ &= \mathbb{E}[A^2] \cdot \mathbb{E}\left[\cos(\omega_c t_1 + \Theta)\cos(\omega_c t_2 + \Theta)\right]\end{aligned}$$

We use the following identity

$$\cos(\alpha)\cos(\beta) = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$$

The expression simplifies as follows

$$\begin{aligned}&\mathbb{E}\left[\cos(\omega_c t_1 + \Theta)\cos(\omega_c t_2 + \Theta)\right] \\ &= \mathbb{E}\left[\frac{1}{2}\left(\cos(\omega_c(t_1 - t_2)) + \cos(\omega_c(t_1 + t_2) + 2\Theta)\right)\right] \\ &= \frac{1}{2}\mathbb{E}\left[\cos(\omega_c(t_1 - t_2))\right] + \frac{1}{2}\mathbb{E}\left[\cos(\omega_c(t_1 + t_2) + 2\Theta)\right]\end{aligned}$$

The first term is a constant (dependent only on the time lag). Upon expanding the second term into an integral and solving, we get the following result.

$$\mathbb{E}[\dots] = \frac{1}{2}\cos(\omega_c(t_1 - t_2)) + 0 = \frac{1}{2}\cos(\omega_c(t_1 - t_2))$$

Finally, we get the full autocorrelation function:

$$R_X(t_1, t_2) = \mathbb{E}[A^2] \cdot \frac{1}{2}\cos(\omega_c(t_1 - t_2))$$

Let $\tau = t_1 - t_2$. The function is:

$$R_X(\tau) = \frac{\mathbb{E}[A^2]}{2}\cos(\omega_c \tau)$$

This function depends only on the time lag τ.
Since the mean is constant and the autocorrelation depends only on τ, X_t is **WSS**.

—

A process is SSS if for any $k \geq 1$, any set of time points $(t_1, \ldots, t_k)$, and any time shift h, the joint distribution of $(X_{t_1}, \ldots, X_{t_k})$ is identical to the joint distribution of $(X_{t_1+h}, \ldots, X_{t_k+h})$.

Let's examine the time-shifted vector $\mathbf{X}_{t+h}$:

$$\begin{aligned}
\mathbf{X}_{t+h} &= (X_{t_1+h}, \ldots, X_{t_k+h}) \\
&= (A\cos(\omega_c(t_1 + h) + \Theta), \ldots, A\cos(\omega_c(t_k + h) + \Theta)) \\
&= (A\cos(\omega_c t_1 + \omega_c h + \Theta), \ldots, A\cos(\omega_c t_k + \omega_c h + \Theta))
\end{aligned}$$

Let's define a new random variable $\Phi = (\omega_c h + \Theta) \pmod{2\pi}$. Since Θ is uniformly distributed over $[0, 2\pi]$, adding any constant $\omega_c h$ and taking modulo 2π results in a new random variable Φ that is *also* uniformly distributed over $[0, 2\pi]$. So, Φ and Θ have the exact same distribution (they are identically distributed). Since A and Θ are independent, A and Φ are also independent.

We can now write the shifted vector $\mathbf{X}_{t+h}$ in terms of Φ:

$$\mathbf{X}_{t+h} = (A\cos(\omega_c t_1 + \Phi), \ldots, A\cos(\omega_c t_k + \Phi))$$

Compare this to the original vector $\mathbf{X}_t$:

$$\mathbf{X}_t = (A\cos(\omega_c t_1 + \Theta), \ldots, A\cos(\omega_c t_k + \Theta))$$

Since A is the same in both expressions, and Θ and Φ are identically distributed and independent of A, the two vectors $\mathbf{X}_t$ and $\mathbf{X}_{t+h}$ must have the same joint probability distribution.

This holds true for any k and any h. Thus, this is an SSS process!

Problem 7. Consider a WSS process X_t with autocorrelation $R_X(\tau) = e^{-a|\tau|}$, where $a > 0$, for all $\tau \in \mathbb{R}$. Find the power spectral density of X_t.

Solution:

$$\begin{aligned}
S_X(\omega) &= \mathcal{F}\{R_X(\tau)\} \\
&= \int_{-\infty}^{\infty} R_X(\tau) e^{-j\omega\tau} d\tau \\
&= \int_{-\infty}^{\infty} e^{-a|\tau|} e^{-j\omega\tau} d\tau \\
&= \int_{-\infty}^{0} e^{-a(-\tau)} e^{-j\omega\tau} d\tau + \int_{0}^{\infty} e^{-a\tau} e^{-j\omega\tau} d\tau \\
&= \int_{-\infty}^{0} e^{(a-j\omega)\tau} d\tau + \int_{0}^{\infty} e^{-(a+j\omega)\tau} d\tau \\
&= \left[\frac{e^{(a-j\omega)\tau}}{a-j\omega}\right]_{-\infty}^{0} + \left[\frac{e^{-(a+j\omega)\tau}}{-(a+j\omega)}\right]_{0}^{\infty} \\
&= \left(\frac{e^0}{a-j\omega} - \lim_{\tau\to-\infty} \frac{e^{(a-j\omega)\tau}}{a-j\omega}\right) + \left(\lim_{\tau\to\infty} \frac{e^{-(a+j\omega)\tau}}{-(a+j\omega)} - \frac{e^0}{-(a+j\omega)}\right) \\
&= \left(\frac{1}{a-j\omega} - 0\right) + \left(0 - \frac{1}{-(a+j\omega)}\right) \quad \text{(since } a > 0\text{)} \\
&= \frac{1}{a-j\omega} + \frac{1}{a+j\omega} \\
&= \frac{(a+j\omega)+(a-j\omega)}{(a-j\omega)(a+j\omega)} \\
&= \frac{2a}{a^2-(j\omega)^2} \\
&= \frac{2a}{a^2+\omega^2}
\end{aligned}$$

$$S_X(\omega) = \frac{2a}{a^2+\omega^2}$$