Assignment-4

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(MA6.102) Probability and Random Processes.

problem - 1:

Given $f_X(x) = \begin{cases} p\lambda e^{-\lambda x} & \text{if } x \geq 0 \\ (1-p)\lambda e^{\lambda x} & \text{if } x < 0 \end{cases}$

$$E[X] = \int_0^{\infty} x p\lambda e^{-\lambda x} dx + \int_{-\infty}^{0} x(1-p)\lambda e^{\lambda x} dx$$

$$= p \cdot \int_0^{\infty} x\lambda e^{-\lambda x} dx + (1-p)\int_{-\infty}^{0} x \cdot \lambda e^{\lambda x} dx$$

$\boxed{y = -x}$

$$= p \cdot \frac{1}{\lambda} + (1-p)\int_{\infty}^{0} -y \cdot \lambda e^{-\lambda y}(-dy)$$

$$= p \cdot \frac{1}{\lambda} - (1-p) \cdot \int_0^{\infty} y\lambda e^{-\lambda y} dy$$

$$= \frac{p}{\lambda} - \frac{(1-p)}{\lambda} = \frac{2p-1}{\lambda}$$

$$\boxed{\therefore E[X] = \frac{2p-1}{\lambda}}$$

$$E[X^2] = \int_0^{\infty} x^2 \cdot p\lambda e^{-\lambda x} dx + \int_{-\infty}^{0} x^2 (1-p)\lambda e^{\lambda x} dx$$

$$= p\int_0^{\infty} x^2 \lambda e^{-\lambda x} dx + (1-p)\int_{-\infty}^{0} x^2 \lambda e^{\lambda x} dx$$

$(y = -x)$

$$= p \cdot \frac{2}{\lambda^2} + (1-p)\int_{\infty}^{0} y^2 \lambda e^{-\lambda y}(-dy)$$

$$= p\left(\frac{2}{\lambda^2}\right) + (1-p)\frac{2}{\lambda^2} = \frac{\cancel{2p} + 2 - \cancel{2p}}{\lambda^2} = \frac{2}{\lambda^2}$$

$$\Rightarrow E[x^2] = \frac{2}{\lambda^2}$$

$$\Rightarrow Var(X) = E[X^2] - E^2[X]$$

$$= \frac{2}{\lambda^2} - \left(\frac{2p-1}{\lambda}\right)^2$$

$$= \frac{2-(2p-1)^2}{\lambda^2} = \frac{2-4p^2-1+4p}{\lambda^2}$$

$$\boxed{Var(X) = \frac{1+4P(1-P)}{\lambda^2}}$$

$$\Rightarrow \int_0^\infty x \lambda e^{-\lambda x}\,dx$$

$$= \lambda\left[x \int e^{-\lambda x}\,dx - \int 1\cdot\left(\int e^{-\lambda x}\,dx\right)dx\right]_0^\infty$$

$$= \lambda\left[x\cdot\frac{e^{-\lambda x}}{-\lambda} + \int \frac{e^{-\lambda x}}{\lambda}\,dx\right]$$

$$= \left[-x e^{-\lambda x} + \frac{e^{-\lambda x}}{-\lambda}\right]_0^\infty$$

$$\boxed{= 0 + \left[\frac{e^{-\lambda x}}{-\lambda}\right]_0^\infty = \frac{1}{\lambda}}$$

$$\cancel{E(X^2)} \int_0^\infty x^2 \lambda e^{-\lambda x}\,dx$$

$$= \lambda\left[x^2 \int e^{-\lambda x}\,dx - \int 2x\left(\int e^{-\lambda x}\,dx\right)dx\right]_0^\infty$$

$$= \cancel{\lambda}\left[x^2\cdot\frac{e^{-\lambda x}}{-\cancel{\lambda}} + 2\int x\,\frac{e^{-\lambda x}}{\cancel{\lambda}}\,dx\right]_0^\infty$$

$$= 0 + \frac{2}{\lambda}\cdot\frac{1}{\lambda} = \frac{2}{\lambda^2}$$

$$\boxed{\begin{aligned} E[X] &= \frac{2P-1}{\lambda} \\ Var(X) &= \frac{1+4P(1-P)}{\lambda^2} \end{aligned}}$$

Problem - 2 : (For continuous case)

i) X and Y be continuous R.V.

with joint CDF $F_{XY}(x,y)$

and marginal CDF $F_X(x)$, $F_Y(y)$

independence $\Rightarrow$

$$P(X \in A, Y \in B) = P(X \in A) \cdot P(Y \in B)$$

take $A = (-\infty, x]$, $B = (-\infty, y]$

$$\Rightarrow F_{XY}(x,y) = P(X \leq x, Y \leq y)$$

$$= P(X \in A, Y \in B) = P(X \in A) \cdot P(Y \in B)$$

$$= P(X \leq x) \cdot P(Y \leq y) = F_X(x) \cdot F_Y(y)$$

$$\Rightarrow \boxed{F_{XY}(x,y) = F_X(x) \cdot F_Y(y)}$$

$\Leftarrow$ if $F_{XY}(x,y) = F_X(x) \cdot F_Y(y)$

we know $F_{XY}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(x,y)\, dx\, dy$

$F_X(x) = \int_{-\infty}^{x} f_X(x)\, dx$; $F_Y(y) = \int_{-\infty}^{y} f_Y(y)\, dy$

$$f_{XY}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x,y)$$

$$= \frac{\partial^2}{\partial x \partial y} \left(F_X(x) \cdot F_Y(y) \right)$$

$$= \frac{\partial}{\partial x} \left(F_X(x) \cdot \frac{\partial (F_Y(y))}{\partial y} \right)$$

$$= \frac{\partial F_X(x)}{\partial (x)} \cdot \frac{\partial F_Y(y)}{\partial y} = f_X(x) \cdot f_Y(y)$$

$$\boxed{\therefore f_{XY}(x,y) = f_X(x) \cdot f_Y(y)} \quad \forall x, y$$

$$\boxed{\therefore X, Y \text{ independent} \Leftrightarrow F_{XY}(x,y) = F_X(x) \cdot F_Y(y)}$$

(ii) For discrete case.

(i) independence $\Rightarrow$ ~~$P(X=a, Y=b) = P_X(a) \cdot P_Y(b)$~~

$$P(X=a, Y=b) = P_X(a) \cdot P_Y(b)$$

$$\Rightarrow F_{XY}(x,y) = \sum_{a \leq x} \sum_{b \leq y} P_X(a) \cdot P_Y(b)$$

$$= \sum_{a \leq x} P_X(a) \cdot \sum_{b \leq y} P_Y(b) = F_X(x) \cdot F_Y(y)$$

independence $\Rightarrow$

$$\Rightarrow \boxed{F_{XY}(x,y) = F_X(x) \cdot F_Y(y)}$$

(ii) if $F_{XY}(x,y) = F_X(x) \cdot F_Y(y)$ then

we know

$$P(X=u, Y=v) = F_{XY}(u,v) - F_{XY}(u-1,v) - F_{XY}(u,v-1) + F_{XY}(u-1,v-1)$$

$$\Rightarrow P(X=u, Y=v) = F_X(u) \cdot F_Y(v) - F_X(u-1) \cdot F_Y(v) - F_X(u) \cdot F_Y(v-1) + F_X(u-1) \cdot F_Y(v-1)$$

$$= [F_X(u) - F_X(u-1)] \cdot [F_Y(v) - F_Y(v-1)]$$

$$= P_X(u) \cdot P_Y(v)$$

$$\boxed{\therefore X \text{ and } Y \text{ independent} \iff F_{XY}(x,y) = F_X(x) \cdot F_Y(y) \quad \forall x,y}$$

↳ For both discrete and continuous.

Problem - 3 :

$X, Y \sim \text{Uniform}[a, b]$, independent

$$f_{XY}(x,y) = f_X(x) \cdot f_Y(y) = \frac{1}{b-a} \cdot \frac{1}{b-a} = \frac{1}{(b-a)^2}$$

$$D = |X - Y|$$

$$E[|X-Y|] = \int_a^b \int_a^b |x-y| \, f_{X,Y}(x,y) \, dx \, dy$$

$$= \int_a^b \int_a^b \frac{1}{(b-a)^2} \cdot |x-y| \, dx \, dy$$

$$= \frac{2}{(b-a)^2} \int_a^b \int_a^x (x-y) \, dx \, dy$$

$$= \frac{2}{(b-a)^2} \int_a^b \int_a^x (x-y) \, dy \, dx = \frac{2}{(b-a)^2} \int_a^b \left[xy - \frac{y^2}{2} \right]_a^x dx$$

$$= \frac{2}{(b-a)^2} \int_a^b \left(\cancel{x^2} \, x^2 - \frac{x^2}{2} - ax + \frac{a^2}{2} \right) dx$$

$$= \frac{\cancel{2}}{(b-a)^2} \int_a^b \frac{(x-a)^2}{\cancel{2}} \, dx = \frac{1}{(b-a)^2} \times \frac{(b-a)^3}{3}$$

$$= \frac{b-a}{3}$$

$$\boxed{\therefore E[|X-Y|] = \frac{b-a}{3}}$$

$\hookrightarrow$ expected distance b/w two points

Problem-4 :

(a) we know

$$E[X|A] = \sum_x x \cdot P(X = x | A)$$

$$\Rightarrow E[X|A] = \sum_x x \cdot \frac{P(X = x \cap A)}{P(A)}$$

$$= \sum_x x \frac{P(X = x, A)}{P(A)}$$

$$= \frac{1}{P(A)} \cdot \sum_x x \cdot P(X = x, A)$$

$$E[X \cdot 1_A] = \sum_x x \cdot P(X = x, A)$$

$$\Rightarrow \boxed{E[X|A] = \frac{E[1_A X]}{P(A)}}$$

$1_A \to$ indicator R.V of event A

(b)

$$X = D_1 + D_2$$

$$E[X|A_i] = E[D_1 + D_2 | D_1 = i]$$

$$= E[D_1 | D_1 = i] + E[D_2 | A_i]$$

$$= i + E[D_2]$$

$$E[D_2] = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6}$$

$$= 3.5$$

$$\Rightarrow \boxed{E[X|A_i] = i + 3.5}$$

$$\boxed{\therefore E[X|A_i] = i + 3.5} \quad \text{for } i \in [1:6]$$

Problem 5 :

Given $F_X(x) = \begin{cases} 1 - \frac{a^3}{x^3} & \text{if } x \geq a \\ 0 & \text{if } x < a \end{cases}$

$$\boxed{\text{PDF } f_X(x) = \frac{d}{dx} F_X(x)}$$

$$\Rightarrow \boxed{f_X(x) = \begin{cases} \frac{3a^3}{x^4} & \text{if } x \geq a \\ 0 & \text{if } x < a \end{cases}}$$

$$E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx = \int_{a}^{\infty} x \cdot \frac{3a^3}{x^4}\, dx$$

$$= \int_a^{\infty} \frac{3a^3}{x^3}\, dx = 3a^3 \int_a^{\infty} x^{-3}\, dx$$

$$= 3a^3 \left[\frac{x^{-2}}{-2}\right]_a^{\infty} = \frac{3a^3}{2a^2} = \frac{3a}{2}$$

$$\Rightarrow \boxed{E[X] = \frac{3a}{2}}$$

$$E[X^2] = \int_a^{\infty} x^2 \cdot \frac{3a^3}{x^4}\, dx = \int_a^{\infty} \frac{3a^3}{x^2}\, dx$$

$$= 3a^3 \cdot \int_a^{\infty} \frac{1}{x^2}\, dx = 3a^3 \left[-x^{-1}\right]_a^{\infty}$$

$$= \frac{3a^3}{a} = 3a^2$$

$$\Rightarrow \boxed{E[X^2] = 3a^2}$$

$$\therefore \operatorname{Var}(X) = E[X^2] - E^2[X] = 3a^2 - \left(\frac{3a}{2}\right)^2$$

$$= 3a^2 - \frac{9a^2}{4}$$

$$= \frac{3a^2}{4}$$

$$\boxed{\therefore \operatorname{Var}(X) = \frac{3a^2}{4} \; ; \; E[X] = \frac{3a}{2}}$$

Problem - 6 :

Given
$$f_{XY}(x,y) = c(y^2-x^2)e^{-y}$$
$$0 < y < \infty$$
$$-y \le x \le y$$

$$\Rightarrow \int_0^\infty \int_{-y}^{y} c(y^2-x^2)e^{-y}\,dx\,dy = 1$$

$$\Rightarrow c \cdot \int_0^\infty e^{-y}\left[\int_{-y}^{y}(y^2-x^2)\,dx\right]dy$$

$$\int_{-y}^{y}(y^2-x^2)\,dx = y^2(y+y) - \left[\frac{x^3}{3}\right]_{-y}^{y}$$
$$= 2y^3 - \left[\frac{y^3}{3}+\frac{y^3}{3}\right] = 2y^3 - \frac{2y^3}{3}$$
$$= \frac{4y^3}{3}$$

Now $\int_0^\infty e^{-y}\cdot\frac{4y^3}{3}\,dy = \frac{4}{3}\int_0^\infty y^3\cdot e^{-y}\,dy$

$$= \frac{4}{3}\times 6 = 8$$

$$\Rightarrow c\cdot 8 = 1 \Rightarrow \boxed{c = \frac{1}{8}}$$

$$\int_0^\infty y^3\cdot e^{-y}\,dy = \left[-y^3\cdot e^{-y}\right]_0^\infty + \int_0^\infty 3y^2e^{-y}\,dy$$

$$= 3\int_0^\infty y^2e^{-y}\,dy = 3\left[\left[-y^2e^{-y}\right]_0^\infty + \int_0^\infty 2y\,e^{-y}\right]$$

$$= 6\int_0^\infty ye^{-y}\,dy$$

$$= 6\left(\left[-e^{-y}\right]_0^\infty + \int_0^\infty e^{-y}\right)$$

$$= 6$$

$$\Rightarrow \boxed{c = \frac{1}{8}}$$

ii) marginal PDF $f_Y(y)$

$$f_Y(y) = \int_{-y}^{y} f_{XY}(x,y)\,dx = \int_{-y}^{y} \frac{1}{8}(y^2 - x^2)e^{-y}\,dx$$

$$= \frac{e^{-y}}{8}\int_{-y}^{y}(y^2 - x^2)\,dx = \frac{e^{-y}}{\cancel{8}_2} \cdot \frac{\cancel{4}y^3}{3}$$

marginal $f_Y(y)$

$$\boxed{\therefore f_Y(y) = \frac{1}{6}y^3 e^{-y},\ y \geq 0}$$

ii) marginal PDF ~~of~~ $f_X(x)$

$$-y \leq x \leq y \Rightarrow y \geq |x|$$

$$f_X(x) = \int_{y=|x|}^{\infty} f_{XY}(x,y)\,dy$$

$$= \int_{|x|}^{\infty} \frac{1}{8}(y^2 - x^2)e^{-y}\,dy$$

$$= \frac{1}{8}\left[\int_{|x|}^{\infty} y^2 e^{-y}\,dy - x^2\int_{|x|}^{\infty} e^{-y}\,dy\right]$$

$$= \frac{1}{8}\left[\left[-y^2 \cdot e^{-y}\right]_{|x|}^{\infty} + \int_{|x|}^{\infty} e^{-y} \cdot 2y + x^2\left[e^{-y}\right]_{|x|}^{\infty}\right]$$

$$= \frac{1}{8}\left[e^{-|x|}\left(|x|^2 + 2|x| - x^2\right) + x^2 e^{|x|}\right]$$

$$= \frac{1}{8}\left[e^{-|x|}\left(|x|^2 + 2|x| + 2 - x^2\right)\right]$$

marginal f_X

$$= \frac{1}{8}\left[e^{-|x|}\left(2|x| + 2\right)\right] = \frac{|x|+1}{4} \cdot e^{-|x|}$$

$$\Rightarrow \boxed{f_X(x) = \frac{|x|+1}{4} \cdot e^{-|x|},\ -\infty < x < \infty}$$

Problem - 7.

Given X_1, X_2, X_3 be independent Continuous R.V.

with common PDF $f_X(a)$

$$P(X_1 < X_2 < X_3) = \iiint_{a_1 < a_2 < a_3} f_{X_1 X_2 X_3}(a_1, a_2, a_3)\, da_1\, da_2\, da_3$$

$$= \int_{-\infty}^{\infty} \int_{a_1}^{\infty} \int_{a_2}^{\infty} f_X(a_1) \cdot f_X(a_2) \cdot f_X(a_3)\, da_1\, da_2\, da_3$$

$$= \int_{-\infty}^{\infty} \int_{a_1}^{\infty} \underbrace{\int_{a_2}^{\infty} f_X(a_3)\, da_3}_{1 - F_X(a_2)} \cdot f_X(a_1) \cdot f_X(a_2)\, da_1\, da_2$$

$$= \int_{-\infty}^{\infty} \int_{a_1}^{\infty} f_X(a_1) \cdot f_X(a_2) \left(1 - F_X(a_2)\right) da_2\, da_1$$

$$= \int_{-\infty}^{\infty} f_X(a_1) \cdot \left[\int_{a_1}^{\infty} f_X(a_2) \left(1 - F_X(a_2)\right) da_2 \right] da_1$$

$$\Rightarrow \int_{a_1}^{\infty} f_X(a_2) \left(1 - F_X(a_2)\right) da_2$$

let $t = F_X(a_2)$

$dt = f_X(a_2)\, da_2$

$$\Rightarrow \int_{t = F_X(a_1)}^{1} (1 - t)\, dt = \left[t - \frac{t^2}{2} \right]_{F_X(a_1)}^{1}$$

$$= \frac{1}{2} - \left(F_X(a_1) - \frac{F_X^2(a_1)}{2} \right)$$

$$= \frac{\left(1 - F_X(a_1)\right)^2}{2}$$

$$\Rightarrow P(X_1 < X_2 < X_3) = \int_{-\infty}^{\infty} f_X(a_1) \cdot \frac{(1 - F_X(a_1))^2}{2} \, da_1$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} (1 - F_X(a_1))^2 f_X(a_1) \, da_1$$

$$t = F_X(a_1) \Rightarrow dt = f_X(a_1) \, da_1$$

$$\frac{1}{2} \int_0^1 (1-t)^2 \, dt = \frac{1}{2} \left[\frac{(1-t)^3}{3} \right]_0^1 = \frac{1}{6}$$

$$\boxed{\therefore P(X_1 < X_2 < X_3) = \frac{1}{6}}$$