Assignment - 5

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2025122010

(MA6102) Probability and Random Processes

Problem 1 :-

Given $f_{X,Y}(x,y) = cx(y-x)e^{-y} \quad 0 \le x < y < \infty$

To find c

$$\int_{y=0}^{\infty}\int_{x=0}^{y} cx(y-x)e^{-y}\,dx\,dy = 1$$

$$\Rightarrow \quad c\cdot\int_{y=0}^{\infty} e^{-y}\int_{x=0}^{y}(xy-x^2)\,dx\,dy = 1$$

$$c\cdot\int_{y=0}^{\infty} e^{-y}\left[\frac{x^2y}{2}-\frac{x^3}{3}\right]_0^y dy = 1$$

$$c\cdot\int_{y=0}^{\infty} e^{-y}\left(\frac{y^3}{2}-\frac{y^3}{3}\right)dy = 1$$

$$c\cdot\int_0^{\infty}\frac{1}{6}\cdot y^3\cdot e^{-y}\,dy = 1$$

$$\frac{c}{6}\cdot\int_0^{\infty} y^3\cdot e^{-y}\,dy = 1$$

$$\frac{c}{\cancel{6}}\cdot\frac{\cancel{6}}{\,} = 1 \quad \Rightarrow \quad \boxed{c=1}$$

$$f_Y(y) = \int_{x=0}^{y} x(y-x)e^{-y}\,dx = \left[\left(\frac{yx^2}{2}-\frac{x^3}{3}\right)\cdot e^{-y}\right]_0^y$$

$$= \frac{y^3}{6}e^{-y}$$

$$\boxed{\therefore\ f_Y(y) = \frac{1}{6}\cdot y^3 e^{-y} \quad y \ge 0}$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$= \frac{x(y-x)\cdot \cancel{e^{-y}}}{\frac{y^3}{6}\cancel{e^{-y}}} = \frac{6x(y-x)}{y^3}$$

$$\boxed{\therefore f_{X|Y}(x|y) = \frac{6x(y-x)}{y^3} \quad 0 \le x \le y}$$

$$E(X|Y=y) = \int_{x=0}^{y} x \cdot f_{X|Y}(x|y)\, dx$$

$$= \int_0^y \frac{6x(y-x)}{y^3} \cdot x \, dx$$

$$= \int_0^y \left(\frac{6x^2}{y^2} - \frac{6x^3}{y^3}\right) dx$$

$$= \left[\frac{6x^3}{3y^2} - \frac{6x^4}{4y^3}\right]_0^y$$

$$= \frac{\cancel{6}^{2} y}{\cancel{3}} - \frac{\cancel{6}^{3} y}{\cancel{4}_{2}}$$

$$= 2y - \frac{3}{2}y = \frac{y}{2}$$

$$\boxed{\therefore E(X|Y=y) = \frac{y}{2}}$$

problem-2r

Given X_1, X_2 are i.i.d

$$f_X(x) = \frac{1}{x^2} \quad \text{for } x > 1$$

and $Y_1 = X_1 + X_2$; $Y_2 = \frac{X_2}{X_1 + X_2}$

$\Rightarrow$ $\boxed{x_2 = y_2 \cdot y_1}$; $\boxed{x_1 = y_1 - y_2 y_1}$

$g(x_1, x_2) = x_1 + x_2$; $h(x_1, x_2) = \frac{x_2}{x_1 + x_2}$

$\frac{\partial g}{\partial x_1} = 1$; $\frac{\partial g}{\partial x_2} = 1$

$\frac{\partial h}{\partial x_1} = \frac{-x_2}{(x_1+x_2)^2}$; $\frac{\partial h}{\partial x_2} = \frac{x_1}{(x_1+x_2)^2}$

$$\cancel{f_{XY}}\ f_{Y_1 Y_2}(y_1, y_2) = \frac{f_{X_1 X_2}(y_1(1-y_2), y_1 y_2)}{\begin{vmatrix} 1 & 1 \\ \frac{-x_2}{(x_1+x_2)^2} & \frac{x_1}{(x_1+x_2)^2} \end{vmatrix}}$$

$x_1 = y_1 - y_1 y_2$
$x_2 = y_1 y_2$

$$\Rightarrow f_{Y_1, Y_2}(y_1, y_2) = \frac{\frac{1}{y_1^2 (1-y_2)^2} \times \frac{1}{y_1^2 y_2^2}}{\begin{vmatrix} 1 & 1 \\ \frac{-x_2}{(x_1+x_2)^2} & \frac{x_1}{(x_1+x_2)^2} \end{vmatrix}}$$

$$= \frac{1}{y_1^4 \cdot (1-y_2)^2 \cdot y_2^2} \times \left(\frac{(x_1+x_2)^2}{x_1+x_2} \right|_{x_1 = y_1 - y_1 y_2,\ x_2 = y_1 y_2}$$

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$$f_{Y_1 Y_2}(y_1, y_2) = \frac{1}{y_1{}^{4}\,{}^{3} y_2^2 (1-y_2)^2} \cdot y_1 \qquad \left(\because x_1 + x_2 = y_1\right)$$

$$\boxed{f_{Y_1,Y_2}(y_1;y_2) = \frac{1}{y_1^3 \cdot y_2^2 (1-y_2)^2}}$$

For domain

$$x_1 = y_1(1-y_2) > 1 \; ; \; x_2 = y_1 y_2 > 1$$

$$Y_2 = \frac{X_2}{X_1 + X_2}$$

$$\boxed{\therefore \; y_2 \in (0, 1)}$$

$\Rightarrow \; y_1(1-y_2) > 1$ and $y_1 y_2 > 1$

$$\Rightarrow \boxed{y_1 > \max\left(\frac{1}{1-y_2}, \frac{1}{y_2}\right)}$$

$$\boxed{\therefore \; f_{Y_1 Y_2}(y_1, y_2) = \begin{cases} \frac{1}{y_1^3 y_2^2 (1-y_2)^2}, & y_1 > \max\left(\frac{1}{y_2}, \frac{1}{1-y_2}\right) \text{ and } 0 < y_2 < 1 \\ 0, & \text{o.w} \end{cases}}$$

Problem-3:

Given $f_X(x) = \lambda e^{-\lambda x}$ $\quad x > 0$

$Y_1 = X_1 + X_2$ and $Y_2 = \frac{X_1}{X_2}$

$\Rightarrow y_1 = x_1 + x_2$; $y_2 = \frac{x_1}{x_2}$ $\Big|$ $g(x_1, x_2) = x_1 + x_2$, $h(x_1, x_2) = \frac{x_1}{x_2}$

$\Rightarrow \boxed{x_1 = \frac{y_1 y_2}{y_2+1} ,\quad x_2 = \frac{y_1}{1+y_2}}$

$$f_{Y_1 Y_2}(y_1, y_2) = \frac{f_{X_1 X_2}(x_1, x_2)}{\left| \begin{matrix} \frac{\partial g(x_1,x_2)}{\partial x_1} & \frac{\partial g(x_1,x_2)}{\partial x_2} \\ \frac{\partial h(x_1,x_2)}{\partial x_1} & \frac{\partial h(x_1,x_2)}{\partial x_2} \end{matrix} \right|} \Bigg|_{x_1 = \frac{y_1 y_2}{y_2+1},\ x_2 = \frac{y_1}{y_2+1}}$$

$$\Rightarrow f_{Y_1 Y_2}(y_1, y_2) = \frac{f_{X_1 X_2}\left(\frac{y_1 y_2}{y_2+1}, \frac{y_1}{y_2+1}\right)}{\left| \begin{matrix} 1 & 1 \\ \frac{1}{x_2} & -\frac{x_1}{x_2^2} \end{matrix} \right|} \Bigg|_{x_1 = \frac{y_1 y_2}{y_2+1},\ x_2 = \frac{y_1}{y_2+1}}$$

$$= \frac{\lambda^2 e^{-\lambda\left(\frac{y_1 y_2 + y_1}{y_2+1}\right)}}{\frac{x_1}{x_2^2} + \frac{1}{x_2}} \Bigg|_{x_1 = \frac{y_1 y_2}{y_2+1},\ x_2 = \frac{y_1}{y_2+1}}$$

$$= \lambda^2 e^{-\lambda y_1} \times \left| \frac{x_2^2}{x_1 + x_2} \right|_{x_1 = y_1 y_2,\ x_2 = \frac{y_1}{y_2+1}}$$

$$= \frac{\lambda^2 e^{-\lambda y_1} \cdot y_1 \cancel{z}}{(y_2+1)^2 \cdot \cancel{y_1}}$$

$$\therefore \boxed{f_{Y_1 Y_2}(y_1, y_2) = \frac{\lambda^2 \cdot e^{-\lambda y_1} \cdot y_1}{(1+y_2)^2}}$$

$y_1 \geq 0$
$y_2 > 0$

$$\cancel{f_{Y_1}(y_1) = \int_0^\infty \lambda^2 \cdot e^{-\lambda y}}$$

$\therefore Y_1 = X_1 + X_2$

From convolution property

$$f_{Y_1}(y_1) = \int_0^{y_1} \lambda e^{-\lambda a} \cdot \lambda e^{-\lambda (y_1 - a)} \, da$$

$$= \lambda^2 \int_0^{y_1} e^{-\lambda y_1} \, da$$

$$\boxed{f_{Y_1}(y_1) = \lambda^2 \cdot e^{-\lambda y_1} \cdot y_1}$$

$$F_{Y_2}(y_2) = P\left(\frac{X_1}{X_2} \leq y_2\right) \qquad a_2 > 0$$

$$= P(X_1 \leq X_2 \cdot y_2)$$

$$= \int_{a_2=0}^{\infty} \int_{a_1=0}^{a_2 y_2} \lambda^2 e^{-\lambda(a_1 + a_2)} \, da_1 \, da_2$$

$$= \int_{a_2=0}^{\infty} \lambda^2 \cdot e^{-\lambda a_2} \left[-\frac{e^{-\lambda a_1}}{\lambda}\right]_0^{a_2 y_2} da_2$$

$$= \int_{a_2=0}^{\infty} -\lambda e^{-\lambda a_2} \left(e^{-\lambda a_2 y_2} - 1\right) da_2$$

$$= \left[\frac{-\lambda e^{-\lambda a_2(1+y_2)}}{-\lambda(1+y_2)} - \frac{\lambda e^{-\lambda a_2}}{\lambda}\right]_0^\infty$$

$$\Rightarrow F_{Y_2}(y_2) = \frac{y_2}{1+y_2} = 1 - \frac{1}{1+y_2}$$

$$\Rightarrow f_{Y_2}(y_2) = F'_{Y_2}(y_2)$$

$$= \frac{1}{(1+y_2)^2}$$

$$\boxed{\therefore f_{Y_2}(y_2) = \frac{1}{(1+y_2)^2}}$$

since

$$f_{Y_1 Y_2}(y_1, y_2) = f_{Y_1}(y_1) \cdot f_{Y_2}(y_2)$$

$\boxed{\therefore Y_1 \text{ and } Y_2 \text{ are independent}}$

Problem - 4.7

Given X, Y are jointly continuous

$Z = \max(X, Y)$; $W = \min(X, Y)$

$\Rightarrow$

$$f_{Z,W} = \sum_{i=1}^{2} \frac{f_{XY}(x_i, y_i)}{|J(x_i, y_i)|}$$

$$= \frac{f_{XY}(z, w)}{\begin{vmatrix} \frac{\partial}{\partial x}(x) & \frac{\partial}{\partial y}(x) \\ \frac{\partial}{\partial x}(y) & \frac{\partial}{\partial y}(y) \end{vmatrix}} + \frac{f_{X,Y}(w, z)}{\begin{vmatrix} \frac{\partial}{\partial x}(y) & \frac{\partial}{\partial y}(y) \\ \frac{\partial}{\partial x}(x) & \frac{\partial}{\partial x}(y) \end{vmatrix}}$$

because it $x \geq z$

$\max(x, y) = x$

$\min(x, y) = y$

$$= \frac{f_{X,Y}(z, w)}{1} + \frac{f_{X,Y}(w, z)}{1}$$

~~$\therefore f_{X,Y}\, z, w$~~

$$\boxed{\therefore f_{Z,W}(z, w) = f_{X,Y}(z, w) + f_{X,Y}(w, z)}$$

$\hookrightarrow w \leq z$

$$\therefore f_{(Z,W)}(z, w) = f_{X,Y}(z, w) + f_{X,Y}(w, z) \,;\, w \leq z$$

Problem-5 :

Given X, Y, Z be independent uniformly distributed on $(0,1]$

$$P(XY < Z^2)$$

$$= P\left(X < \frac{Z^2}{Y}\right)$$

$z \in (0,1)$
$x \in \left(0, \frac{z^2}{y}\right)$
$y \in (z^2, 1)$

$z \in (0,1)$
$x \in (0,1)$
$y \in (0, z^2)$

$$\Rightarrow P\left(X < \frac{Z^2}{Y}\right) = \int_{x=0}^{z^2/y} \int_{y=z^2}^{1} \int_{z=0}^{1} dx\, dy\, dz \cdot 1$$

uniform

$$+ \int_{x=0}^{1} \int_{y=0}^{z^2} \int_{z=0}^{1} dx\, dy\, dz \cdot 1$$

$$\Rightarrow P\left(X < \frac{Z^2}{Y}\right)$$

$$= \int_{z=0}^{1} \int_{y=z^2}^{1} \frac{z^2}{y}\, dy\, dz + \int_{z=0}^{1} \int_{y=0}^{z^2} dy\, dz$$

$$\cancel{\int_0^1 \frac{z^2}{y} \Big|_{1-z^2}}$$

$$= \int_{z=0}^{1} -z^2 \ln(z^2)\, dz + \left[\frac{z^3}{3}\right]_{z=0}^{1}$$

$$= \frac{2}{(2+1)^2} + \frac{1}{3} - 0 = \frac{2}{9} + \frac{1}{3}$$

$$= \frac{5}{9}$$

$$\boxed{\therefore P(XY < Z^2) = \frac{5}{9}}$$

$$\therefore P(XY < Z^2) = \frac{5}{9}$$

Problem 67

given, F_X F_Y be strictly increasing

since F_X and F_Y are strictly increasing

they are invertible.

Let

$$g(x) = F_Y^{-1}(F_X(x))$$

$$Z = g(X)$$

$$F_Z(y) = P(g(X) \le y)$$

$$= P(F_Y^{-1}(F_X(X)) \le y)$$

$$F_Z(y) = \cancel{F_Y^{-1}} P(F_X(X) \le F_Y(y))$$

$\because F_Y^{-1}(u) \le y \iff u \le F_Y(y)$

$$F_Z(y) = P(F_X(X) \le F_Y(y))$$

$$= P(X \le F_X^{-1}(F_Y(y)))$$

$$F_Z(y) = F_X(F_X^{-1}(F_Y(y)))$$

$$= F_Y(y)$$

$$\therefore \boxed{F_Z(y) = F_Y(y)} \quad \forall y \in \mathbb{R}$$

$\therefore$ if $g(x) = F_Y^{-1}(F_X(x))$ then $F_Z(y) = F_Y(y)$ $\forall y \in \mathbb{R}$

Problem-7 :

(a) $\phi_X(t) = E\left[e^{itX}\right]$

$$= \int_{-\infty}^{\infty} e^{itx} \cdot E(x)$$

$$|\phi_X(t)| = \left|\int_{-\infty}^{\infty} e^{itx} \cdot E(x)\right| \leq \int_{-\infty}^{\infty} |e^{itx}| |E(x)|$$

$$|\phi_X(t)| \leq \int_{-\infty}^{\infty} |E(x)|$$

$$\boxed{\phi_X(t) \leq 1}$$

if $\phi_X(2\pi) = 1$

$\hookrightarrow$ $e^{i2\pi X} = 1$

$\Rightarrow$ $\cos(2\pi x) = 1$; $\sin(2\pi x) = 0$

$\Rightarrow$ x should be integer

$$\boxed{\therefore \phi_X(2\pi) = 1 \Rightarrow P(X \in Z) = 1}$$

(b) $$\frac{1}{2\pi}\int_0^{2\pi} e^{-itn} \phi_X(t)\, dt = \frac{1}{2\pi}\int_0^{2\pi} e^{-itn} \sum_{k \in Z} e^{itk} P_k\, dt$$

$$\Rightarrow \frac{1}{2\pi}\sum_{k \in Z} P_k \cdot \int_0^{2\pi} e^{it(k-n)}\, dt \quad \left|\ \int_0^{2\pi} e^{it(k-n)}\, dt = \begin{cases} 2\pi & k=n \\ 0 \end{cases}\right.$$

$$\Rightarrow \frac{1}{2\pi} \cdot P_n \cdot 2\pi$$

$$\boxed{\therefore P(n) = \frac{1}{2\pi}\int_0^{2\pi} e^{-itn} \phi_X(t)\, dt} \quad n \in Z$$