Assignment - 6

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(MA6.102) Probability and Random Processes

Problem-1:

Given $h : R \longrightarrow [0, M]$ is non-negative function taking values bounded by some number M

we know from Markov's Inequality

$$P(h(x) \geq a) \leq \frac{E[h(x)]}{a} \; ; \; a > 0, \; h(x) \rightarrow \text{non-ve R.V}$$

$\Rightarrow P(h(x) \geq a)$ $\cancel{a \leq 0}$ $\boxed{\cancel{a \leq M}}$

$$= P(-h(x) \leq -a) = P(M - h(x) \leq M - a)$$

$$\Rightarrow P(M - h(x) \leq M - a) = 1 - P(M - h(x) \geq M - a)$$

From Markov's Inequality $\boxed{M - a > 0 ; a \geq 0}$

$$* \; P((M - h(x)) \geq (M - a)) \leq \frac{E[M - h(x)]}{M - a}$$

$\Rightarrow$

$$1 - P((M - h(x)) \geq (M - a)) \geq 1 - \frac{E[M - h(x)]}{M - a}$$

$$P(h(x) \geq a) \geq 1 - \frac{E[M - h(x)]}{M - a}$$

$$P(h(x) \geq a) \geq \frac{M - a - E[M - h(x)]}{M - a}$$

$$P(h(x) \geq a) \geq \frac{\cancel{M} - a - \cancel{M} + E[h(x)]}{M - a}$$

$$\boxed{\therefore P(h(x) \geq a) \geq \frac{E[h(x)] - a}{M - a}}$$

whenever $0 \leq a < M$

Problem-2 :- Given $X_n(\omega) = \frac{n}{n+1}\omega + (1-\omega)^n$ & $\Omega = [0,1]$

$\Rightarrow \lim_{n\to\infty} X_n(\omega)$

$\Rightarrow \lim_{n\to\infty} \left(\frac{n}{n+1}\omega + (1-\omega)^n\right)$

$\Rightarrow \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}}\omega + (1-\omega)^n\right)$

$= \omega + \lim_{n\to\infty} \left((1-\omega)^n\right)$.

$\Rightarrow \forall\, \omega \in (0,1] \Rightarrow \lim_{n\to\infty} (1-\omega)^n = 0$

$$\boxed{\therefore \lim_{n\to\infty} X_n(\omega) = \omega} \quad \forall\, \omega \in (0,1]$$

Given $X(\omega) = \omega \quad \forall\, \Omega$

$$\Rightarrow \boxed{\therefore \lim_{n\to\infty} X_n(\omega) = X(\omega)} \quad \forall\, \omega \in (0,1]$$

$P\{0\} \Rightarrow$ by probability law $P([a,b]) = b-a$

$P\{0\} = P([0,0]) = 0-0 = 0$

$(0,1] = \Omega / \{0\} \Rightarrow P(0,1] = P(\Omega) - P\{0\}$

$= 1-0$

$= 1$

$$\boxed{\therefore P(0,1] = 1}$$

$$\boxed{\therefore P\left(\left\{\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\right\}\right) = 1}$$

$\therefore$ X_n converges to X almost surely.

Problem - 3 :-

Given Alvin selects n people at random and estimator M_n is obtained by dividing S_n (the no. of smokers in his sample by n)

→ S_n is no. of smokers in his sample of n

↳ [Same as no. of heads in n Tosses]

$\therefore S_n \simeq \text{binomial}(n, f)$

$\Rightarrow E[S_n] = n \cdot f$ | $f \to$ true fraction of smokers

$Var[S_n] = nf(1-f)$ | $\therefore E[\text{binomial}] = n \cdot p$

$Var[\text{binomial}] = np(1-p)$

Now $M_n = \frac{S_n}{n}$

$\therefore E[M_n] = \frac{E[S_n]}{n} = \frac{nf}{n} = f$

$Var[M_n] = \frac{Var(S_n)}{n^2} = \frac{nf(1-f)}{n^2} = \frac{f(1-f)}{n}$

$$\boxed{\therefore E[M_n] = f \ ; \ Var[M_n] = \frac{f \cdot (1-f)}{n}}$$

Now by chebyshev's inequality

$$P(|M_n - f| \geq \epsilon) \leq \frac{Var(M_n)}{\epsilon^2} = \frac{f(1-f)}{n\epsilon^2}$$

Given $P(|M_n - f| \geq \epsilon) \leq \delta$

$$\Rightarrow \frac{f(1-f)}{n\epsilon^2} \le \delta$$

$$\boxed{\therefore \frac{f(1-f)}{\delta\epsilon^2} \le n}$$

Given that

Alvin chooses the n to be smallest possible for which chebyshevs true.

$$\Rightarrow \boxed{n = \frac{f\cdot(1-f)}{\delta\epsilon^2}}$$

(a) Now if $\epsilon' = \frac{2}{3}\epsilon$

$$n' = \frac{f(1-f)}{\delta\epsilon'^2} = \underbrace{\frac{f(1-f)}{\delta\epsilon^2}}_{n} \cdot \frac{9}{4}$$

$$\boxed{\therefore n' = \frac{9n}{4}}$$

(b) Now if $\delta' = \frac{3}{5}\cdot\delta$

$$n' = \frac{f(1-f)}{\delta'\cdot\epsilon^2} = \frac{f(1-f)}{\frac{3}{5}\delta\cdot\epsilon^2} = \underbrace{\frac{f(1-f)}{\delta\epsilon^2}}_{n} \cdot \frac{5}{3}$$

$$\boxed{\therefore n' = \frac{5}{3}n}$$

$$\boxed{\begin{array}{l} \therefore \text{ if } \epsilon' = \frac{2}{3}\epsilon \Rightarrow n' = \frac{9n}{4} \\ \text{if } \delta' = \frac{3}{5}\cdot\delta \Rightarrow n' = \frac{5n}{3} \end{array}}$$

Problem - ~~1~~4

9) Given

$$X_n \xrightarrow{as} X$$

and

$$Y_n \xrightarrow{as} Y$$

$$\Rightarrow P\left(\{\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\}\right) = 1$$

and $$P\left(\{\omega : \lim_{n\to\infty} Y_n(\omega) = Y(\omega)\}\right) = 1$$

Now let

$$A = \{\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\}$$

and $$B = \{\omega : \lim_{n\to\infty} Y_n(\omega) = Y(\omega)\}$$

Let $C = A \cap B \Rightarrow P(C) = 1 - P(A^c) P(B^c)$

$$\boxed{P(C) = 1}$$

Now

$$\lim_{n\to\infty} X_n(\omega) = X(\omega) \quad \forall\, \omega \in C$$

$$\therefore C \subseteq A$$

and $$\lim_{n\to\infty} X_n(\omega) = X(\omega) \quad \forall\, \omega \in C$$

$$\therefore C \subseteq B$$

Now $$\lim_{n\to\infty} (X_n(\omega) + Y_n(\omega))$$

$$= \lim_{n\to\infty} X_n(\omega) + \lim_{n\to\infty} Y_n(\omega)$$

$$= X(\omega) + Y(\omega) \quad \forall\, \omega \in C$$

$\therefore \lim_{n\to\infty} (X_n(\omega) + Y_n(\omega)) = X(\omega) + Y(\omega)$

$\forall\ \boxed{\omega \in C}$

and $P(C) = 0$

$$\boxed{\therefore P(\{\omega : \lim_{n\to\infty} (X_n(\omega) + Y_n(\omega)) = X(\omega) + Y(\omega)\}) = 1}$$

$$\boxed{\therefore X_{n(\omega)} + Y_{n(\omega)} \xrightarrow{as} X_{(\omega)} + Y_{(\omega)}}$$

ii) Given

$X_n \xrightarrow{P} X$

$Y_n \xrightarrow{P} Y$

$\Rightarrow \lim_{n\to\infty} P(|X_n - X| > \varepsilon) = 0$

$\lim_{n\to\infty} P(|Y_n - Y| > \varepsilon) = 0$ $\forall\ \varepsilon > 0$

(or) $\lim_{n\to\infty} P(|X_n - X| > \frac{\varepsilon}{2}) = 0$

and $\lim_{n\to\infty} P(|Y_n - Y| > \frac{\varepsilon}{2}) = 0$

$\Rightarrow |(X_n - X) + (Y_n - Y)| \le |X_n - X| + |Y_n - Y|$

$\Rightarrow P(|(X_n - X) + (Y_n - Y)| > \varepsilon) \le P(|X_n - X| > \frac{\varepsilon}{2}) + P(|Y_n - Y| > \frac{\varepsilon}{2})$

$\therefore$ as $n \to \infty$ R.H.S is 0

$$\boxed{\therefore \lim_{n\to\infty} P(|(X_n + Y_n) - (X + Y)| > \varepsilon) = 0}$$

$$\boxed{\therefore X_n + Y_n \xrightarrow{P} X + Y}$$

(iii) Given $X_n \xrightarrow{D} X$

$Y_n \xrightarrow{D} Y$

$\Rightarrow \lim_{n\to\infty} F_{X_n}(x) = F_X(x)$

and $\lim_{n\to\infty} F_{Y_n}(y) = F_Y(y)$

Now $\lim_{n\to\infty} F_{(X_n+Y_n)}(a) = ?$ cannot tell

$\therefore$ If $X_n \xrightarrow{D} X$ and $Y_n \xrightarrow{D} Y$

$\boxed{X_n + Y_n \text{ may or may not converge in distribution to } X+Y}$

Problem 5: Given $N_t, t \geq 0$ be a poisson process with rate $\lambda > 0$

no. of arrivals in time t

for $t_1 < t_2$

$\boxed{b > a}$

$P(N_{t_1} = a, N_{t_2} = b) = P(N_{t_1} = a, N_{t_2} - N_{t_1} = b-a)$

we know $N_{t_1}, N_{t_2} - N_{t_1}$ is independent

$= P(N_{t_1} = a) \cdot P(N_{t_2} - N_{t_1} = b-a)$

$= \frac{e^{-\lambda t_1} \cdot (\lambda t_1)^a}{a!} \cdot \frac{e^{-\lambda(t_2 - t_1)} \cdot (\lambda(t_2 - t_1))^{b-a}}{(b-a)!}$

because we know $P(N_t = k) = \frac{e^{-\lambda t} \cdot (\lambda t)^k}{k!}$

for $b \geq a \geq 0$ For $t_2 \geq t_1$

$\therefore P(N_{t_1} = a, N_{t_2} = b) = \frac{e^{-\lambda t_1} \cdot (\lambda t_1)^a}{a!} \cdot \frac{e^{-\lambda(t_2 - t_1)} (\lambda(t_2 - t_1))^{b-a}}{(b-a)!}$,

if $b < a$ (o.w) $\Rightarrow P(N_{t_1} = a, N_{t_2} = b) = 0$

Problem-6:

Given $X_t = A\cos(\omega_c t + \phi)$

→ ω_c is non-zero const

→ A, ϕ are independent R.V $P(A \geq 0) = 1$ and $E[A^2] < \infty$

and ϕ is uniformly distributed $[0, 2\pi]$

$$f_\phi(x) = \begin{cases} \frac{1}{2\pi} & \forall x \in [0, 2\pi] \\ 0 & \text{o.w.} \end{cases}$$

$$E[X_t] = E[A\cos(\omega_c t + \phi)]$$

$$= A \cdot E[\cos(\omega_c t + \phi)]$$

$\because A, \phi$ are independent

$$\Rightarrow E[X_t] = A \cdot \int_0^{2\pi} \frac{1}{2\pi} \cdot \cos(\omega_c t + \phi)\, d\phi$$

$$= \frac{A}{2\pi} \cdot \int_0^{2\pi} \cos(\omega_c t + \phi)\, d\phi$$

$$= \frac{A}{2\pi} \cdot 0$$

$$\boxed{E[X_t] = 0}$$

$\boxed{\therefore \mu_t \text{ is independent on } t}$

Now $R_X(t_1, t_2) = E[X_{t_1}, X_{t_2}]$

$$= E\left[A^2 \cos(\omega_c t_1 + \phi) \cdot \cos(\omega_c t_2 + \phi)\right]$$

$$R_X(t_1, t_2) = E[A^2] \cdot E\left[\cos(\omega_c t_1 + \phi) \cdot \cos(\omega_c t_2 + \phi)\right]$$

$$\Rightarrow R_X(t_1, t_2) = E[A^2] \cdot E\left[\frac{1}{2}\cos(\omega_c(t_1 + t_2) + 2\phi) + \cos\omega_c(t_1 - t_2)\right]$$

$$\Rightarrow R_X(t_1, t_2) = E[A^2] \cdot \frac{1}{2}\left[E[\cos(\omega_c(t_1 + t_2) + 2\phi)] + E[\cos\omega_c(t_1 - t_2)]\right]$$

$$E[\cos\omega_c(t_1 + t_2) + 2\phi] = \int_0^{2\pi} \frac{1}{2\pi} \cdot \cos(\omega_c(t_1 + t_2) + 2\phi)\, d\phi$$

$$= 0$$

and $E[\cos(\omega_c(t_1 - t_2))] = \cos(\omega_c(t_1 - t_2))$

$$\boxed{\therefore R_X(t_1, t_2) = E[A^2] \cdot \frac{1}{2} \cdot \cos(\omega_c(t_1 - t_2))}$$

Since $R_X(t_1, t_2)$ is function of $t_1 - t_2$ and ~~it~~ μ it is independent of t

$$\boxed{\therefore X_t \text{ is wide-sense stationary (WSS)}}$$

For strict-sense stationary

$$X_{t+n} = A\cos(\omega_c(t+n) + \phi)$$

$$= A\cos(\omega_c t + \omega_c n + \phi)$$

$$= A\cos(\omega_c t + B) \qquad B \to \omega_c n + \phi$$

we know ϕ is Uniform $(0, 2\pi)$ so B is also Uniform $(0, 2\pi)$

B is uniform $(0, 2\pi)$

because cos is periodic with period 2π

so $X_{(\omega t + n)} = A \cos(\omega_c t + B)$ B or uniform $(0, 2\pi)$

$X(\omega) = A \cos(\omega_c t + \phi)$ $\phi = \text{uniform}(0, 2\pi)$

$\therefore$ It will have same n^{th} order distribution

$$\boxed{F_{X_1, X_2, X_3, X_4, \ldots x_n}(\omega) = F_{X_{1+n} X_{2+n} \ldots}(\omega)}$$

$\boxed{\therefore X_t \text{ is strict-sense stationary also}}$

Problem - 7 :-

Given X_t is WSS process

with $R_X(\tau) = e^{-a\tau}$; $a > 0$ $\forall$ $\tau \in \mathbb{R}$

* Power spectral density is Fourier Transform of $R_X(t)$

$$\Rightarrow S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) \cdot e^{-j2\pi f\tau} d\tau$$

we know that $R_X(\tau)$ is real and even and $S_X(f)$ is real and even

$$\Rightarrow S_X(f) = 2\int_0^{\infty} R_X(\tau) \cdot \cos(2\pi f\tau) \, d\tau$$

$$= 2\int_0^{\infty} e^{-a|\tau|} \cdot \cos(2\pi f\tau) \, d\tau$$

$$\boxed{\therefore \int_0^{\infty} e^{-at} \cos(bt) \, dt = \frac{a}{a^2+b^2}}$$

$$\hookrightarrow S_X(f) = 2\int_0^{\infty} e^{-a\tau} \cdot \cos(2\pi f\tau) \, d\tau$$

$$\boxed{S_X(f) = \frac{2a}{a^2 + (2\pi f)^2}}$$

$\therefore$ Power spectral density of X_t is

$$S_X(f) = \frac{2a}{a^2 + (2\pi f)^2}$$