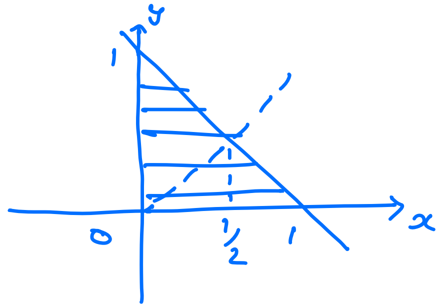
Quiz 2 Solutions

1. sol:- $f_{XY}(x,y) = \begin{cases} 2 & \text{if } x > 0,\ y > 0,\ x+y \le 1 \\ 0 & \text{o.w.} \end{cases}$



(a) $P(X<Y)$

$$= \int_{x=0}^{1/2} \int_{y=x}^{1-x} 2\, dy\, dx$$

$$= \int_{x=0}^{1/2} 2\,(1-2x)\, dx$$

$$= 2\left[x - x^2\right]_0^{1/2} = \tfrac{1}{2}.$$ [1 Mark]

(b) $F_{X|\{X<Y\}}(x) = P(X \le x \mid X<Y)$

$$= \frac{P(X \le x,\ X<Y)}{P(X<Y)}$$ (1 Mark)

For $x<0$, $F_{X|\{X<Y\}}(x) = 0$. [0.5 Marks]

for $x \ge 1/2$, $F_{X|\{X<Y\}}(x) = 1$ because

$\{X<Y\} \Longrightarrow \{X \le x\}$ for $x \ge 1/2$. [0.5 Marks]

For $0 \le x < \frac{1}{2}$,

$$F_{X \mid \{X<Y\}}(x) = \frac{P(X \le x, X < Y)}{P(X<Y)}$$

$$= \frac{P((X,Y) \in B_x)}{P(X<Y)}, \quad B_x = \{(u,v) \in \mathcal{X} \times \mathcal{Y} : u \le x, u < v\}$$

$$= \int_{u=0}^{x} \int_{v=u}^{1-u} 2 \, du \, dv \cdot 2 \quad \text{[0.5 marks]}$$

$$= 4 \int_{u=0}^{x} (1-2u) \, du$$

$$= 4 \left[u - u^2 \right]_0^x$$

$$= 4(x - x^2). \quad \text{[0.5 marks]}$$

$$\therefore F_{X \mid \{X<Y\}}(x) = \begin{cases} 0, & x < 0 \\ 4(x-x^2), & 0 \le x < \frac{1}{2} \\ 1, & x \ge \frac{1}{2} \end{cases}.$$

$$f_{X \mid \{X < Y\}}(x) = \frac{d}{dx} F_{X \mid \{X < Y\}}(x)$$

$$= \begin{cases} \underline{4(1-2x)} & 0 \leq x < \frac{1}{2} \\ \underline{0} & \text{o.w.} \end{cases}$$

[1 Mark]

Solution

Given:

$$f_{X,Y}(x,y) = x + y, \quad (x,y) \in [0,1]^2$$

and the transformation:

$$z = x^2, \quad w = x(1+y)$$

(a)

$$z = x^2 \Rightarrow x = \sqrt{z}, \quad z \in [0,1]$$
$$w = x(1+y) = \sqrt{z}(1+y) \implies \sqrt{z} \le w \le 2\sqrt{z}$$

Thus,

$$(z,w) \in \{(z,w) : 0 \le z \le 1,\ \sqrt{z} \le w \le 2\sqrt{z}\}$$

(1 mark for right answer)

Partial marks only for $w \in [0,2]$ or if one term of the w-interval is correct. No partial marks for writing z interval only.

(b)

$$z = g_1(x,y) = x^2, \quad w = g_2(x,y) = x(1+y)$$

Then,

$$x = \sqrt{z}, \quad 1 + y = \frac{w}{\sqrt{z}} \Rightarrow y = \frac{w}{\sqrt{z}} - 1$$

So,

$$(x,y) = \left(\sqrt{z}, \frac{w}{\sqrt{z}} - 1\right)$$

is the unique solution of the system of equations:

$$z = x^2, \quad w = x(1+y)$$

(1 mark)

$$J(x,y) = \begin{vmatrix} \dfrac{\partial g_1(x,y)}{\partial x} & \dfrac{\partial g_1(x,y)}{\partial y} \\ \dfrac{\partial g_2(x,y)}{\partial x} & \dfrac{\partial g_2(x,y)}{\partial y} \end{vmatrix} \quad \text{(0.5 mark)}$$

$$= \begin{vmatrix} 2x & 0 \\ 1+y & x \end{vmatrix} = 2x^2 \quad \text{(1 mark)}$$

$$f_{Z,W}(z,w) = \frac{f_{X,Y}(x,y)}{|J(x,y)|} \quad \text{where} \quad x = \sqrt{z},\ y = \frac{w}{\sqrt{z}} - 1 \quad [1\text{ mark}]$$

$$= \frac{\sqrt{z} + \dfrac{w}{\sqrt{z}} - 1}{2z} \quad [0.5\text{ marks}]$$

$$f_{Z,W}(z,w) = \begin{cases} \dfrac{z + w - \sqrt{z}}{2z\sqrt{z}}, & 0 \le z \le 1,\ \sqrt{z} \le w \le 2\sqrt{z} \\ 0, & \text{otherwise} \end{cases}$$

Same scheme applies if the inverse Jacobian is used.

other method

$$F_{Z,W}(z,w) = P(Z \le z, W \le w)$$
$$= P(X^2 \le z, X(1+Y) \le w) \quad [1 \text{ mark}]$$
$$\Rightarrow X \le \sqrt{z}, \quad 0 \le X \le \sqrt{z}$$
$$X(1+Y) \le w \Rightarrow 1 + Y \le \frac{w}{X} \Rightarrow Y \le \frac{w}{X} - 1, \quad Y \ge 0$$
$$\Rightarrow 0 \le Y \le \min\{1, \tfrac{w}{X} - 1\}, \quad \text{and } w > 0,\ 0 < X \le \sqrt{z}$$

Finding limits and getting CDF – 2 marks (check below):

$$F_{Z,W}(z,w) = \int_{x=0}^{w/2} \int_{y=0}^{1} (x+y)\,dy\,dx + \int_{x=w/2}^{\sqrt{z}} \int_{y=0}^{\frac{w}{x}-1} (x+y)\,dy\,dx$$

On solving,

$$= \left(w + \frac{1}{2}\right)(z - \frac{w^2}{2}) - \frac{1}{2}\left[x^2 + \frac{y^2}{2}\right]_{w/2}^{\sqrt{z}} - w\,[2x \ln x]_{\frac{w}{2}}^{\sqrt{z}}$$

Next,

$$f_{Z,W}(z,w) = \frac{\partial^2 F}{\partial z\,\partial w}$$

Only some terms depend on $w \Rightarrow \frac{\partial z_1}{\partial z\,\partial w} = 0$, so solve the other terms.

$$\frac{\partial F}{\partial z} = \frac{1}{2\sqrt{z}}\left[w - \sqrt{z} + \frac{w^2}{2z} - \frac{w}{\sqrt{z}} + \frac{1}{2}\right]$$
$$f_{Z,W}(z,w) = \frac{\partial^2 F}{\partial z\,\partial w} = \frac{1}{2\sqrt{z}}\left\{1 + \frac{w}{z} - \frac{1}{\sqrt{z}}\right\} \quad [1 \text{ mark}]$$

3. Sol:- Let $f(s) = M_x\left(\frac{s}{n}\right)^n$

$$\Rightarrow \log f(s) = n \log M_x(s/n)$$ [1 Mark]

$$\lim_{n\to\infty} \log f(s) = \lim_{n\to\infty} n \log M_x(s/n)$$

$$= \lim_{n\to\infty} \frac{\log M_x(s/n)}{1/n}$$

$$= \lim_{t\to 0} \frac{\log M_x(st)}{t}$$

[1 Mark]

$$= \frac{0}{0} \text{ form as } M_x(0) = 1.$$

By L'Hospital's rule

$$\lim_{t\to 0} \frac{\log M_x(st)}{t}$$

$$= \lim_{t\to 0} \frac{\frac{1}{M_x(st)} \cdot s M_x'(st)}{1}.$$ [2 marks]

$$= \frac{1}{M_x(0)} \quad s M_x'(0)$$

$$= \frac{1}{1} \cdot s E[x]$$ [1 mark]

(as $M_x(0) = 1$ & $M_x'(0) = E[x]$)

$$= s E[x]$$

We have

$$\lim_{n\to\infty} \log f(s) = s E[x]$$

$$\Rightarrow \log \lim_{n\to\infty} f(s) = s E[x]$$

(because log is a continuous function)

$$\Rightarrow \lim_{n\to\infty} f(s) = e^{sE[x]}.$$

$$\therefore \lim_{n\to\infty} \left(M_x(s/n)\right)^n = e^{sE[x]}.$$