Assignment 6

(MA6.102) Probability and Random Processes, Monsoon 2025

Release date: 12 November 2025, Due date: 20 November 2025

INSTRUCTIONS

- Discussions with other students are not discouraged. However, all write-ups must be done individually with your own solutions.
- Any plagiarism when caught will be heavily penalised.
- Be clear and precise in your writing.

Problem 1. If $h : \mathbb{R} \to [0, M]$ is a non-negative function taking values bounded by some number M, then

$$P(h(X) \geq a) \geq \frac{\mathbb{E}[h(X)] - a}{M - a}, \text{whenever } 0 \leq a < M. \tag{1}$$

Problem 2. Consider the sample space $\Omega = [0, 1]$ with uniform probability law, i.e., $P([a, b]) = b - a$, for all $0 \leq a \leq b \leq 1$. Define a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ by $X_n(\omega) = \frac{n}{n+1}\omega + (1 - \omega)^n$. Also, define another random variable X on this sample space as $X(\omega) = \omega$, for all $\omega \in \Omega$. Show that X_n converges to X almost surely.

Problem 3. In order to estimate f, the true fraction of smokers in a large population, Alvin selects n people at random. His estimator M_n is obtained by dividing S_n, the number of smokers in his sample, by n, i.e., $M_n = \frac{S_n}{n}$. Alvin chooses the sample size n to be the smallest possible number for which the Chebyshev's inequality yields a guarantee that

$$P(|M_n - f| \geq \epsilon) \leq \delta,$$

where ϵ and δ are some prespecified tolerances. Determine how the value of n recommended by the Chebyshev's inequality changes in the following cases.

(a) The value of ϵ is reduced to $\frac{2}{3}$ of its original value.

(b) The probability δ is reduced to $\frac{3}{5}$ of its original value.

Problem 4. Suppose that X_n converges almost surely to X and Y_n converges almost surely to Y. Show that $X_n + Y_n$ converges almost surely to $X + Y$. Does the corresponding result also hold for convergence in probability and convergence in distribution?

Problem 5. Let $\{N_t,\, t \geq 0\}$ be a Poisson process with rate $\lambda > 0$. Find the joint probability mass function of (N_{t_1}, N_{t_2}) for $0 \leq t_1 \leq t_2$.

Problem 6. Let $X_t = A\cos(\omega_c t + \Theta)$, where ω_c is a non-zero constant, A and Θ are independent random variables with $P(A > 0) = 1$ and $\mathbb{E}[A^2] < \infty$. If Θ is uniformly distributed over $[0, 2\pi]$, show that X_t is wide-sense stationary (WSS). Is X_t strict-sense stationary also?

Problem 7. Consider a WSS process X_t with autocorrelation $R_X(\tau) = \mathrm{e}^{-a|\tau|}$, where $a > 0$, for all $\tau \in \mathbb{R}$. Find the power spectral density of X_t.