Assignment-3

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(MA60102 Probability and Random Processes)

Problem-1 :-

① given $\Omega = \{1, 2, 3 \ldots\}$

and $P(\{\omega\}) = 2^{-\omega}$ for $\omega = 1, 2, 3 \ldots$

$X(\omega) = \omega \quad | \quad Y(\omega) = (-1)^{\omega}$

i) $P(Y=1) = \sum_{y : Y(y)=1} P(Y=y)$

$= \sum_{\omega : (-1)^{\omega}=1} P(\omega)$

$= \sum_{k=1}^{\infty} 2^{-2k} = \frac{2^{-2}}{1-2^{-2}} = \frac{1}{3}$

$\Rightarrow \boxed{P(Y=1) = \frac{1}{3}}$

ii) $P(Y=-1) = \sum_{y : Y(y)=-1} P(Y=y)$

$= \sum_{\omega : (-1)^{\omega}=-1} P(\omega)$

$= \sum_{k=1}^{\infty} 2^{-2k+1} = \frac{2}{3}$

$\boxed{P(Y=-1) = \frac{2}{3}}$

$$\boxed{E[X|Y] = \sum_{x \in X} x \cdot P_{X|Y}(x)}$$

$$E[X|Y=1) = \sum_{\omega=1}^{\infty} \omega \cdot \frac{P(X=\omega \wedge Y=1)}{P(Y=1)}$$

$$= \sum_{\omega=1}^{\infty} \omega \cdot \frac{P(X=\omega \wedge Y=1)}{\frac{1}{3}}$$

$$= \sum_{k=1}^{\infty} 2k \cdot \frac{2^{-2k}}{\frac{1}{3}}$$

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$$

$$\sum_{k=1}^{\infty} k \cdot r^{k-1} = \frac{1}{(1-r)^2}$$

$$\sum_{k=1}^{\infty} k r^k = \frac{r}{(1-r)^2}$$

$$= 6 \sum_{k=1}^{\infty} k \cdot 2^{-2k}$$

$$= 6 \cdot \frac{1/4}{(1-\frac{1}{4})^2} = \frac{8}{3}$$

$$\Rightarrow \boxed{E(X|Y=1) = \frac{8}{3}}$$

$$\Rightarrow E(X|Y=-1) = \sum_{\omega=1}^{\infty} \omega \cdot \frac{P(X=\omega \wedge Y=-1)}{P(Y=-1)}$$

$$= \sum_{\omega=1}^{\infty} \omega \cdot \frac{P(X=\omega \wedge Y=-1)}{\frac{2}{3}}$$

$$= \sum_{k=0}^{\infty} (2k+1) \cdot (1/4)^k \cdot \frac{1}{2} \cdot \frac{3}{2}$$

$$= \left(2 \cdot \sum_{k=0}^{\infty} k \cdot \left(\frac{1}{4}\right)^k + \sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k\right) \cdot \frac{1}{2} \cdot \frac{3}{2}$$

$$= 2 \left(\frac{\frac{1}{4}}{(1-\frac{1}{4})^2} + \frac{1}{1-\frac{1}{4}}\right) \cdot \frac{1}{2} \cdot \frac{3}{2}$$

$$\boxed{= \frac{20}{9} \cdot \frac{3}{2} \cdot \frac{1}{2} = \frac{5}{3}}$$

$$\therefore \boxed{E[X|Y](\omega) = \begin{cases} \frac{8}{3} ; \omega = \text{even} \\ \frac{5}{3} ; \omega = \text{odd} \end{cases}}$$

Problem-2 :

i) $E[X|X]$:

$$E[X|X=a] = \sum_{a \in X} a \cdot P(X|X=a)$$

$$= \sum_{a \in X} a \cdot P_{(X|X=a)}(a)$$

$$= a \cdot 1$$

$$\boxed{E[X|X=a] = a}$$

$$\boxed{\therefore E[X|X] = X}$$

In (ii) we put $\boxed{X=1}$

$$E[g(Y)|Y] = g(Y) \cdot E[1|Y]$$

$$= g(Y) \cdot 1$$

$$\boxed{\therefore E[g(Y)|Y] = g(Y)} \; *$$

ii) $E[Xg(Y)|Y]$

$$E[Xg(Y)|Y=y] = \sum_{a} a \cdot g(y) \cdot P(X=a|Y=y)$$

$$= g(y) \cdot \sum_{a} a \cdot P(X=a|Y=y)$$

$$= g(y) \cdot E(X|Y=y)$$

ii

$$\boxed{\therefore E[Xg(Y)|Y] = g(Y) \cdot E[X|Y]}$$

(iii) $E[E[X|Y,Z]|Y]$

$$E[E[X|Y,Z]|Y=y] = \sum_z E[X|Y,Z] \, P(Z=z|Y=y)$$

we know

$$E[X|Y,Z] = \sum_a a \cdot P(X=a|Y=y, Z=z)$$

$$\Rightarrow E[E[X|Y,Z]|Y=y] = \sum_z \sum_a a \cdot P(X=a|Y=y,Z=z) \cdot P(Z=z|Y=y)$$

$$= \sum_a a \cdot \sum_z P(X=a|Y=y, Z=z) \cdot P(Z=z|Y=y)$$

we know

$$\sum_z P(X=a|Y=y, Z=z) \cdot P(Z=z|Y=y) = \sum_z P(X=a, Z=z|Y=y)$$

$$= P(X=a|Y=y)$$

$$\Rightarrow E[E[X|Y,Z]|Y=y] = \sum_a a \cdot P(X=a|Y=y)$$

$$= E(X|Y=y)$$

$$\boxed{\therefore E[E[X|Y,Z]|Y] = E[X|Y]}$$

③ Given

$$F_Y(y) = 1 - \frac{2}{(y+1)(y+2)}$$

and $P_{Z|Y}(z|y) = \frac{1}{y^2}$ for $1 \le z \le y^2$

$$\Rightarrow P(Y=y) = F_Y(y) - F_Y(y-1)$$

$$= 1 - \frac{2}{(y+1)(y+2)} - \left(1 - \frac{2}{(y)(y+1)}\right)$$

$$= \frac{2}{y(y+1)} - \frac{2}{(y+1)(y+2)}$$

$$= \frac{2}{y+1}\left(\frac{1}{y} - \frac{1}{y+2}\right) = \frac{4}{y(y+1)(y+2)}$$

$$\boxed{\therefore P(Y=y) = \frac{4}{y(y+1)(y+2)}}$$

$$\cancel{E[X] = \sum_{y \in X} y \cdot P(Y=y)}$$

$$E[Z|Y] = \sum_{z=1}^{y^2} z \cdot P_{Z|Y}(z|y)$$

$$= \sum_{z=1}^{y^2} z \cdot \frac{1}{y^2} = \frac{\cancel{y^2}(y^2+1)}{2\cancel{y^2}}$$

$$\Rightarrow \boxed{E[Z|Y=y] = \frac{y^2+1}{2}}$$

From total expectation theorem.

$$E[Z] = \sum_{y=1}^{\infty} E(Z|Y=y) \cdot P_Y(y)$$

$$\Rightarrow E[Z] = \sum_{y=1}^{\infty} \frac{y^2+1}{2} \cdot \left(\frac{4}{y(y+1)(y+2)}\right)$$

$$= \sum_{y=1}^{\infty} \frac{2(y^2+1)}{y(y+1)(y+2)}$$

$$= \sum_{y=1}^{\infty} \frac{A}{y} + \frac{B}{y+1} + \frac{C}{y+2}$$

$A = 1$
$B = -4$
$C = 5$

$$\Rightarrow \sum_{y=1}^{\infty} \frac{1}{y} - \frac{4}{y+1} + \frac{5}{y+2}$$

$$\Rightarrow \boxed{E[Z] = \sum_{y=1}^{\infty} \frac{1}{y} - \frac{4}{y+1} + \frac{5}{y+2}}$$

$$\sum_{y=1}^{\infty} \frac{1}{y} = \infty$$

$$\boxed{\Rightarrow E[Z] = \infty \qquad E[Z] = \sum_{y=1}^{\infty} \frac{1}{y} - \frac{4}{y+1} + \frac{5}{y+2}}$$

(4) Given for X, Y discrete R.V

mean $= 0$; Variance $= 1$; covariance $= \rho$

$\Rightarrow \sum_{x \in X} x \cdot P_X(x) = 0$ | $\sum_{x} x^2 \cdot P_X(x) = 1$

$\sum_{y} y \cdot P_Y(y) = 0$ | $\sum_{y} y^2 \cdot P_Y(y) = 1$

$E(X) = E(Y) = 0$ | $E[X^2] = E[Y^2] = 1$

$E[XY] = \rho$

$$\Rightarrow \boxed{\sum_{x,y} xy \cdot P_{XY}(x,y) = \rho}$$

$$\boxed{\max\{x^2, y^2\} = \frac{x^2 + y^2 + |x^2 - y^2|}{2}}$$

$$\Rightarrow E[\max\{x^2, y^2\}] = \frac{E[x^2] + E[y^2] + E[|x^2 - y^2|]}{2}$$

$$= \frac{1 + 1 + E[|x^2 - y^2|]}{2}$$

$$= 1 + \frac{E[|x^2 - y^2|]}{2}$$

By Cauchy-Schwarz inequality

$E[|x^2 - y^2|]$ ~~$\leq \sqrt{}$~~

$$= E[(X-Y)(X+Y)] \leq \sqrt{E[(X-Y)^2]\, E[(X+Y)^2]}$$

$$\leq \sqrt{(1+1-2\rho)(1+1+2\rho)}$$

$$\leq \sqrt{(2-2\rho)(2+2\rho)}$$

$$\leq \sqrt{4 - 4\rho^2}$$

$\Rightarrow E\left[|X^2 - Y^2|\right] \leq \sqrt{4 - 4\rho^2}$

$\Rightarrow E\left[\max\{X^2, Y^2\}\right] \leq 1 + \frac{\sqrt{4 - 4\rho^2}}{2}$

$\leq 1 + \sqrt{1 - \rho^2}$

$$\boxed{\therefore E[\max\{X^2, Y^2\}] \leq 1 + \sqrt{1 - \rho^2}}$$

Problem - 5 :-

Let

$$I_i = \begin{cases} 1 ; \pi(i) = i \\ 0 ; \pi(i) \neq i \end{cases}$$

$$X = \sum_{i=1}^{n} I_i$$

$$\Rightarrow \boxed{P(I_i = 1) = P(\pi(i) = i) = \frac{(n-1)!}{n!} = \frac{1}{n}}$$
$$\Rightarrow E[I_i] = \frac{1}{n}$$

$$E(X) = \sum_{i=1}^{n} E(I_i) = n \cdot \frac{1}{n} = 1$$

$$\Rightarrow \boxed{E[X] = 1}$$

Variance

$$\mathrm{Var}(I_i) = E[I_i^2] - E[I_i]^2$$
$$= \frac{1}{n} - \frac{1}{n^2} = \frac{n-1}{n^2}$$

$$E[I_i I_j] = \frac{(n-2)!}{n!} = \frac{1}{n(n-1)}$$

$$\Rightarrow \operatorname{Cov}(I_i, I_j) = E[I_i I_j] - E[I_i] \cdot E[I_j]$$

$$= \frac{1}{n(n-1)} - \frac{1}{n^2}$$

$$\boxed{\operatorname{Cov}[I_i, I_j] = \frac{1}{n^2(n-1)}}$$

$$\operatorname{Var}(X) = \sum_{i=1}^{n} \operatorname{Var}(I_i) + 2 \sum_{1 \le i < j \le n} \operatorname{Cov}(I_i, I_j)$$

$$= n \cdot \frac{n-1}{n^2} + 2 \cdot \frac{n(n-1)}{2} \cdot \frac{1}{n^2(n-1)}$$

$$= \frac{n-1}{n} + \frac{1}{n}$$

$$= 1$$

$$\boxed{\therefore\ E[X] = 1 \text{ and } \operatorname{Var}(X) = 1}$$

Problem - 6 :-

Given $X_1, X_2, X_3 \dots X_n$ are geometric R.V with parameter P_i

and $X = X_1 + X_2 \dots X_n$

$$\Rightarrow E[X_i] = \frac{1}{P_i}$$

$$\boxed{Var(X_i) = \frac{1-P_i}{P_i^2}}$$

and $\sum_{i=1}^{N} E(X_i) = \mu$ given
$= E[X]$

$$\boxed{\begin{aligned} Var(X) &= \sum_{i=1}^{n} Var(X_i) = \sum_{i=1}^{n} \frac{1-P_i}{P_i^2} \\ &= \sum_{i=1}^{n} \frac{1}{P_i^2} - \frac{1}{P_i} \end{aligned}}$$

we know

$$\sum_{i=1}^{n} \frac{1}{P_i} = \mu$$

$$\Rightarrow Var(X) = \sum_{i=1}^{n} \frac{1}{P_i^2} - \mu$$

By Cauchy-Schwarz inequality

$$\sqrt{\frac{a_1^2 + a_2^2 \dots a_n^2}{n}} \geq \frac{a_1 + a_2 \dots a_n}{n}$$

$$\Rightarrow \frac{1}{n} \sum_{i=1}^{n} \frac{1}{P_i^2} \geq \left(\frac{\sum_{i=1}^{n} \frac{1}{P_i}}{n} \right)^2$$

$$\Rightarrow \boxed{\sum_{i=1}^{n} \frac{1}{P_i^2} \geq \frac{\mu^2}{n}}$$

$$\Rightarrow \sum_{i=1}^{n}\left(\frac{1}{p_i}\right)^2 \geq \frac{\mu^2}{n}$$

↳ equality holds iff $\frac{1}{p_1}=\frac{1}{p_2}\cdots\frac{1}{p_n}=\frac{\mu}{n}$

$$\therefore \operatorname{Var}(X)_{min}=\frac{\mu^2}{n}-\mu$$

$$\boxed{\operatorname{Var}(X)_{min}=\mu\left(\frac{\mu}{n}-1\right)}$$

For $\frac{1}{p_i}=\frac{\mu}{n} \Rightarrow \boxed{p_i=\frac{n}{\mu}}$

Var(X) is minimized when -

$$\boxed{\begin{array}{l}\therefore \text{For } p_1=p_2=p_3\cdots p_n=\frac{n}{\mu}\\ \operatorname{Var}(X)_{min}=\mu\left(\frac{\mu}{n}-1\right)\end{array}}$$

Problem-7 :-

given X_1, X_2, X_3 be independent R.V

and $P_{X_i}(k)=(1-p_i)\,p_i^{k-1}$

$k=1,2\cdots\cdots$

(i)

$P(X_1<X_2<X_3)$

$$=\sum_{i=1}^{\infty}P(X_1=i)\cdot\sum_{j=i+1}^{\infty}P(X_2=j)\cdot\sum_{k=j+1}^{\infty}P(X_3=k)$$

$$=\sum_{i=1}^{\infty}P(X_1=i)\cdot\sum_{j=i+1}^{\infty}P(X_2=j)\cdot\sum P(X_3>j)$$

$$\Rightarrow P(X_3 > j) = \sum_{q=j+1}^{\infty} P(X_3 = q)$$

$$= \sum_{q=j+1}^{\infty} (1-P_3)\, P_3^{q-1}$$

$$= (1-P_3) \sum_{q=j+1}^{\infty} P_3^{q-1}$$

$$= (1-P_3) \sum_{m=0}^{\infty} P_3^{m} \cdot P_3^{j}$$

$$\cancel{(1-P_3)}\, P_3^{j} \cdot \frac{1}{\cancel{1-P_3}} = P_3^{j}$$

$$\boxed{\therefore P(X_3 > j) = P_3^{j}}$$

$$\Rightarrow P(X_1 < X_2 < X_3) = \sum_{q=1}^{\infty} P(X_1 = q) \cdot \sum_{j=q+1}^{\infty} P(X_2 = j) \cdot P_3^{j}$$

$$= \sum_{q=1}^{\infty} P(X_1 = q) \sum_{j=q+1}^{\infty} (1-P_2)\, P_2^{j-1} \cdot P_3^{j}$$

$$= \sum_{q=1}^{\infty} P(X_1 = q)(1-P_2)\, P_3 \sum_{j=q+1}^{\infty} (P_2 P_3)^{j-1}$$

$$= \sum_{q=1}^{\infty} P(X_1 = q)(1-P_2)\, P_3 \cdot \frac{(P_2 P_3)^{q}}{1-P_2P_3}$$

$$= \sum_{q=1}^{\infty} P(X_1 = q) \cdot \frac{(1-P_2) \cdot P_2^{q} \cdot P_3^{q+1}}{1-P_2P_3}$$

$$= \sum_{q=1}^{\infty} (1-P_1)\, P_1^{q-1} \cdot \frac{(1-P_2)\, P_2^{q} \cdot P_3^{q+1}}{1-P_2P_3}$$

$$= \frac{(1-P_1)(1-P_2)\, P_3}{1-P_2P_3} \sum_{q=1}^{\infty} P_1^{q-1} \cdot (P_2 P_3)^{q}$$

$$\Rightarrow P(X_1 < X_2 < X_3) = \frac{(1-P_1)(1-P_2)P_3}{1-P_2P_3} \cdot \sum_{i=1}^{\infty} P_1^{i-1} (P_2P_3)^i$$

$$= \frac{(1-P_1)(1-P_2)P_3}{1-P_2P_3} \cdot \frac{P_2P_3}{1-P_1P_2P_3}$$

$$= \frac{(1-P_1)(1-P_2)P_2 \cdot P_3^2}{(1-P_2P_3)(1-P_1P_2P_3)}$$

$$\boxed{\therefore P(X_1 < X_2 < X_3) = \frac{(1-P_1)(1-P_2)P_2 \cdot P_3^2}{(1-P_2P_3)(1-P_1P_2P_3)}}$$

ii)

$$P(X_1 \le X_2 \le X_3) = \sum_{i=1}^{\infty} P(X_1=i) \cdot \sum_{j=i}^{\infty} P(X_2=j) \cdot \sum_{k=j}^{\infty} P(X_3=k)$$

$$= \sum_{i=1}^{\infty} P(X_1=i) \cdot \sum_{j=i}^{\infty} P(X_2=j) \cdot P_3(X_3 \ge j)$$

we know

$$P_3(X_3 > j) = P_3^j$$

$$P_3(X_3 \ge j) = P_3(X_3 > j-1) = P_3^{j-1}$$

$$\Rightarrow P(X_1 \le X_2 \le X_3) = \sum_{i=1}^{\infty} P(X_1=i) \cdot \sum_{j=i}^{\infty} P(X_2=j) \cdot P_3^{j-1}$$

$$= \sum_{i=1}^{\infty} P(X_1=i) \cdot \sum_{j=i}^{\infty} (1-P_2) \cdot P_2^{j-1} \cdot P_3^{j-1}$$

$$= \sum_{i=1}^{\infty} P(X_1=i) \cdot (1-P_2) \sum_{j=i}^{\infty} (P_2P_3)^{j-1}$$

$$=$$

$$\Rightarrow P(X_1 \le X_2 \le X_3) = \sum_{q=1}^{\infty} P(X_1 = q)(1-P_2) \sum_{j=q}^{\infty} (P_2 P_3)^{j-1}$$

$$= \sum_{q=1}^{\infty} P(X_1 = q)(1-P_2) \cdot \frac{(P_2 P_3)^{q-1}}{1 - P_2 P_3}$$

$$= \frac{(1-P_2)}{1-P_2P_3} \sum_{q=1}^{\infty} P(X_1 = q)(P_2 P_3)^{q-1}$$

$$= \frac{(1-P_2)}{1-P_2P_3} \sum_{q=1}^{\infty} (1-P_1) P_1^{q-1} \cdot (P_2 P_3)^{q-1}$$

$$= \frac{(1-P_1)(1-P_2)}{1-P_2P_3} \sum_{q=1}^{\infty} (P_1 P_2 P_3)^{q-1}$$

$$= \frac{(1-P_1)(1-P_2)}{1-P_2P_3} \times \frac{1}{1-P_1P_2P_3}$$

$$\Rightarrow \boxed{\therefore P(X_1 \le X_2 \le X_3) = \frac{(1-P_1)(1-P_2)}{(1-P_2P_3)(1-P_1P_2P_3)}}$$

and

$$\boxed{P(X_1 < X_2 < X_3) = \frac{(1-P_1)(1-P_2)P_2P_3^2}{(1-P_2P_3)(1-P_1P_2P_3)}}$$