Practice Problem Set 2

(MA6.102) Probability and Random Processes, Monsoon 2025

Problem 1. Suppose X and Y have the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} c(1-x-y), & \text{if } x \geq 0, y \geq 0, x+y \leq 1 \\ 0, & \text{otherwise} \end{cases}.$$

Find c, f_X, and $f_{Y|X}$.

Problem 2. Let X_1, X_2, and X_3 be independent and identically distributed Bernoulli(p) random variables, and

$$Y_1 = \max\{X_2, X_3\},\ Y_2 = \max\{X_1, X_3\},\ Y_3 = \max\{X_1, X_2\},$$
$$Y = Y_1 + Y_2 + Y_3.$$

Find $\mathbb{E}[Y]$ and Var(Y).

Problem 3. Let X be a random variable such that

$$M_X(s) = a + b\mathrm{e}^{2s} + c\mathrm{e}^{4s},\ \mathbb{E}[X] = 3,\ \mathrm{Var}(X) = 2.$$

Find a, b, and c, and the PMF P_X.

Problem 4. Let X and Y be two independent and identically distributed discrete random variables with common PMF $P(x)$. Assume that $P(x)$ is symmetric around zero, i.e., $P(-x) = P(x)$, for all x. Show that the PMF of $X+Y$ is also symmetric around zero and is largest at zero.

Problem 5. Let $X_1, Y_1, X_2, Y_2, \ldots$ are independent random variables and uniformly distributed over the interval $[0,1]$, and let

$$W = \frac{\sum_{i=1}^{16} X_i - \sum_{i=1}^{16} Y_i}{16}.$$

Find an approximate value to the quantity $P(|W - \mathbb{E}[W]| < 0.001)$ in terms of the CDF of standard Gaussian random variable $\mathcal{N}(0,1)$.

Problem 6. Suppose that a sequence of random variables X_n converges in distribution to a constant c. Show that X_n converges in probability to c.

Problem 7. Suppose $X_0 \geq 0$, and $X_n = \sqrt{X_{n-1}} + 6$, for $n \geq 1$. Does X_n converge almost surely? If so, identify the limit.
Hint. Use $|6 + \sqrt{x} - 9| \leq |6 + \frac{x}{3} - 9|$, *for* $x \geq 0$.

Problem 8. Let $(N_t, t \in [0,\infty))$ be a Poisson process with rate λ. Find the probability that there are two arrivals in $(0,2]$ and three arrivals in $(1,4]$.

Problem 9. Let $(X_t, t \in \mathbb{R})$ be a continuous-time WSS process with autocorrelation function

$$R_X(\tau) = \begin{cases} 3 - |\tau|, & -2 \leq \tau \leq 2, \\ 1, & \text{otherwise.} \end{cases}$$

Find $\mathbb{E}[(X_{2023} + X_{2024} + X_{2025})^2]$.

Problem 10. Let $g : \mathbb{R} \to \mathbb{R}$ be a periodic function with period T, i.e., $g(t+T) = g(t)$, for all $t \in \mathbb{R}$. Consider the random process

$$X(t) = g(t+U), \text{ for all } t \in \mathbb{R},$$

where U is a random variable uniformly distributed over the interval $[0, T]$. Is $X(t)$ a wide-sense stationary (WSS) process?

All the best for end-semester examinations