Probability Tutorial: Functions of RVs, Independence, and Conditional Expectation

September 12,2025

Problem 1: Function of Independent Discrete RVs

Let X and Y be the outcomes of two independent rolls of a fair six-sided die. Let a new random variable be defined as $Z = |X - Y|$, the absolute difference of the outcomes.

(a) Find the Probability Mass Function (PMF) of Z.

(b) Calculate the expected value of Z, $E[Z]$.

Hint: *Since the rolls are independent, the probability of any pair (x, y) is $P(X = x, Y = y) = P(X = x)P(Y = y) = 1/36$. To find the PMF of Z, for each possible value z, count how many of the 36 possible outcomes (x, y) result in $|x - y| = z$.*

Problem 2: Sum of Independent Poisson RVs (Convolution)

Let $X \sim \text{Poisson}(\lambda_1)$ and $Y \sim \text{Poisson}(\lambda_2)$ be two independent random variables. Recall that the PMF of a Poisson(λ) variable is $p(k) = \frac{e^{-\lambda}\lambda^k}{k!}$ for $k = 0, 1, 2, \ldots$. Let $Z = X + Y$. Find the PMF of Z.
Hint: *Use the convolution formula for discrete variables. The Binomial Theorem, $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$, will be very useful in simplifying your final sum.*

Problem 3: Joint PMF and Independence

Let the joint PMF of two discrete random variables X and Y be given by the following table:

	Y=1	Y=2	Y=3
X=0	0.1	0.2	0.1
X=1	0.3	0.1	0.2

(a) Find the marginal PMFs of X and Y.

(b) Are X and Y independent? Justify your answer.

(c) Calculate the covariance $Cov(X, Y)$.

Hint: *For independence, the condition $p_{X,Y}(x, y) = p_X(x)p_Y(y)$ must hold for all pairs of (x, y). The formula for covariance is $Cov(X, Y) = E[XY] - E[X]E[Y]$.*

Problem 4: Sum of Independent Binomial RVs

Let $X \sim \text{Binomial}(n_1, p)$ and $Y \sim \text{Binomial}(n_2, p)$ be two independent random variables with the same success probability p. Let $Z = X + Y$.

(a) Find the expected value of Z, $E[Z]$, using linearity of expectation.

(b) Find the Probability Mass Function (PMF) of Z.

Hint: *For (b), use the convolution formula. You will need to use Vandermonde's Identity: $\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}$.*

Problem 5: Sum of a Random Number of RVs (Conditional Expectation)

Let N be a Poisson random variable with parameter λ, representing the number of customers arriving at a store in an hour. Each customer, independently, makes a purchase with probability p. Let X_i be a Bernoulli random variable where $X_i = 1$ if the i-th customer makes a purchase and $X_i = 0$ otherwise. Let $S = \sum_{i=1}^{N} X_i$ be the total number of purchases in that hour. Find the expected number of purchases, $E[S]$.
Hint: *Use the Law of Iterated Expectations:* $E[S] = E[E[S|N]]$. *First, calculate the expectation of* S *given that* N *takes a fixed value* n.

Problem 6: Variance of a Random Sum (Law of Total Variance)

Using the same setup as Problem 5 ($N \sim \text{Poisson}(\lambda)$, $S = \sum_{i=1}^{N} X_i$ with $X_i \sim \text{Bernoulli}(p)$), find the variance of the total number of purchases, $Var(S)$.
Hint: *Use the Law of Total Variance (Eve's Law):* $Var(S) = E[Var(S|N)] + Var(E[S|N])$. *Calculate each of the two terms separately.*