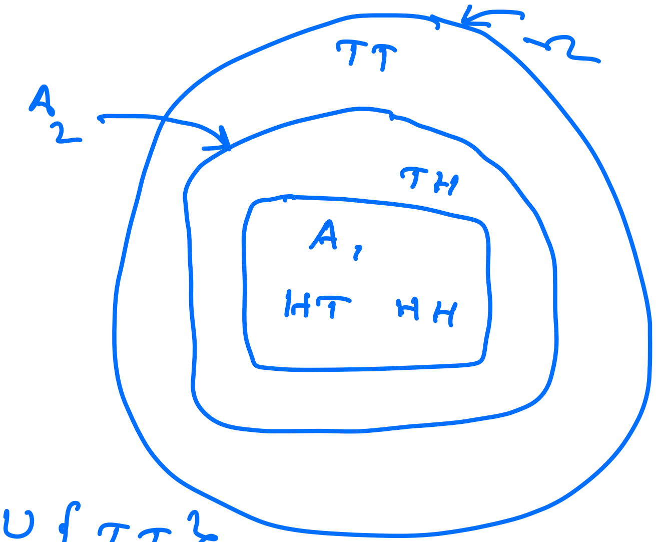
Mid-Sem Solutions

1. sol:- $A_i = \{\omega = (\omega_1, \omega_2) : \omega_j = H \text{ for some } j \in [1:i]\}$

$A_1 = \{HT, HH\}$

$A_2 = \{HT, HH, TH\}$



$\Omega = \{HT, HH\} \cup \{TH\} \cup \{TT\}$

The smallest σ-field that contains A_1 and A_2 is equal to the smallest σ-field that contains the above mutually exclusive and exhaustive events.

$$\mathcal{F} = \left\{ \bigcup_{i \in \mathcal{I}} E_i : \mathcal{I} \subseteq [1:n] \right\} \text{ where}$$

$E_1, E_2, \ldots, E_n$ are mutually excl. & exh. events.

$$\mathcal{F} = \{\Omega, \phi, \{HT, HH\}, \{TH\}, \{TT\}, \{HT, HH, TH\}, \{HT, HH, TT\}, \{TH, TT\}\}$$

$\{HH, TH\} \notin \mathcal{F}$.

2. Sol:-

$$P\left(\bigcap_{i=1}^{n} A_i\right) = 1 - P\left(\left(\bigcap_{i=1}^{n} A_i\right)^c\right)$$

$$= 1 - P\left(\bigcup_{i=1}^{n} A_i^c\right)$$

$$\geq 1 - \sum_{i=1}^{n} P(A_i^c)$$

$$= 1 - n + \sum_{i=1}^{n} P(A_i)$$

$$\Longrightarrow P\left(\bigcap_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} P(A_i) - (n-1).$$

3. sol:- $Z = X + Y$

$$P_Z(z) = \sum_{k=0}^{\infty} P_{X,Y}(k, z-k)$$

$$= \sum_{k=0}^{\infty} P_X(k)\, P_Y(z-k)$$

$$= \sum_{k=0}^{z} \frac{e^{-\lambda} \cdot \lambda^k}{k!} \cdot \frac{e^{-\mu} \cdot \mu^{z-k}}{(z-k)!}$$

$$= \frac{e^{-(\lambda+\mu)}}{z!} \sum_{k=0}^{z} \binom{z}{k} \lambda^k \mu^{z-k}$$

$$= \frac{e^{-(\lambda+\mu)}}{z!} (\lambda+\mu)^z$$

$\Rightarrow Z \sim \text{Poisson}(\lambda+\mu)$.

$\therefore P(X+Y=n) = P(Z=n) = \dfrac{e^{-(\lambda+\mu)} \cdot (\lambda+\mu)^n}{n!}$.

$$P_{X|\{X+Y=n\}}(k) = \frac{P(X=k,\ X+Y=n)}{P(X+Y=n)}$$

$$= \frac{P(X=k,\ Y=n-k)}{P(X+Y=n)}$$

$$= \frac{\dfrac{\cancel{e^{-\lambda}} \cdot \lambda^{k}}{k!} \cdot \dfrac{\cancel{e^{-\mu}} \cdot \mu^{n-k}}{(n-k)!}}{\dfrac{\cancel{e^{-(\lambda+\mu)}} (\lambda+\mu)^{n}}{n!}}$$

$$= \binom{n}{k} \left(\frac{\lambda}{\lambda+\mu}\right)^{k} \left(\frac{\mu}{\lambda+\mu}\right)^{n-k}, \quad k \in [0:n].$$

This is binomial distribution, Binomial $\left(n, p = \frac{\lambda}{\lambda+\mu}\right)$.

4. sol:- $E[ZW] = E[(X+Y)|X-Y|]$

$= E[X|X-Y|] + E[Y|X-Y|]$

$$E[X|X-Y|] = \sum_{x,y\in\{0,1\}} x|x-y| \frac{1}{4}$$

$$= \frac{1}{4}(1) = 1/4$$

Similarly $E[Y|X-Y|] = 1/4$

$\Rightarrow E[ZW] = 1/2$

$E[Z] = E[X+Y] = 1.$

$$E[W] = E[|X-Y|] = \sum_{x,y\in\{0,1\}} |x-y| \frac{1}{4} = 1/2.$$

$\therefore E[ZW] = E[Z]E[W].$

$\Rightarrow$ Z and W are uncorrelated.

$P_{Z,W}(0,0) = P(Z=0, W=0) = P(X+Y=0, |X-Y|=0)$

$= P(X=0, Y=0) = 1/4$

$P(Z=0) = P(X+Y=0) = 1/4$, $P(W=0) = P(|X-Y|=0) = 1/2.$

$\Rightarrow$ Z and W are not independent.

5. Sol:— (a) $Y \in \{0, 1, 2, ---\}$

$$P_Y(y) = P(Y=y)$$

$$= P(\lfloor X \rfloor = y)$$

$$= P(y \le X < y+1)$$

$$= \int_y^{y+1} f_X(x)\,dx$$

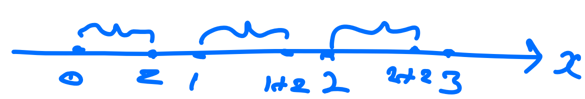
$$= \int_y^{y+1} \lambda e^{-\lambda x}\,dx$$

$$= \left[-e^{-\lambda x}\right]_y^{y+1} = -e^{-\lambda(y+1)} + e^{-\lambda y}$$

$$= e^{-\lambda y}(1-e^{-\lambda}),$$

$y = 0, 1, 2, ---.$

(b) $F_Z(z) = P(Z \le z)$



$$= P(X - \lfloor X \rfloor \le z)$$

$$= P\left(X \in \bigcup_{y=0}^{\infty} [y, y+z]\right)$$

$$= \sum_{y=0}^{\infty} P(X \in [y, y+z]$$

Consider $P(X \in [y, y+z])$

$$= \int_y^{y+z} \lambda e^{-\lambda x} dx = e^{-\lambda y}(1-e^{-\lambda z}).$$

$$\Rightarrow F_Z(z) = \sum_{y=0}^{\infty} e^{-\lambda y}(1-e^{-\lambda z})$$

$$= \frac{1-e^{-\lambda z}}{1-e^{-\lambda}}, \quad 0 \le z < 1.$$

$$\Rightarrow f_Z(z) = \frac{d}{dz} F_Z(z)$$

$$= \frac{1}{1-e^{-\lambda}} e^{-\lambda z}(+\lambda), \quad 0 \le z < 1.$$

$$\therefore f_Z(z) = \begin{cases} \dfrac{\lambda e^{-\lambda z}}{1-e^{-\lambda}} & 0 \le z < 1 \\ 0 & \text{o.w.} \end{cases}$$