Qn 1. Consider a random variable X with the PDF

$$f_X(x) = \begin{cases} p\,\lambda e^{-\lambda x}, & x \geq 0, \\ (1-p)\,\lambda e^{\lambda x}, & x < 0, \end{cases} \qquad \lambda > 0,\ p \in [0,1].$$

$$\int_{-\infty}^{\infty} f_X(x)\,dx = p\lambda \int_0^{\infty} e^{-\lambda x}\,dx + (1-p)\lambda \int_{-\infty}^{0} e^{\lambda x}\,dx$$

$$\int_0^{\infty} e^{-\lambda x}\,dx = \Big[-\tfrac{1}{\lambda}e^{-\lambda x}\Big]_0^{\infty} = \tfrac{1}{\lambda}, \qquad \int_{-\infty}^{0} e^{\lambda x}\,dx = \Big[\tfrac{1}{\lambda}e^{\lambda x}\Big]_{-\infty}^{0} = \tfrac{1}{\lambda},$$

$$\int f_X(x)\,dx = p\lambda \cdot \tfrac{1}{\lambda} + (1-p)\lambda \cdot \tfrac{1}{\lambda} = p + (1-p) = 1$$

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = p\lambda \int_0^{\infty} x e^{-\lambda x}\,dx + (1-p)\lambda \int_{-\infty}^{0} x e^{\lambda x}\,dx$$

$$\int_0^{\infty} x e^{-\lambda x}\,dx = \Big[-\tfrac{x}{\lambda}e^{-\lambda x}\Big]_0^{\infty} + \tfrac{1}{\lambda}\int_0^{\infty} e^{-\lambda x}\,dx = \tfrac{1}{\lambda}\cdot\tfrac{1}{\lambda} = \tfrac{1}{\lambda^2},$$

$$p\lambda \cdot \tfrac{1}{\lambda^2} = \tfrac{p}{\lambda}$$

$$\int_{-\infty}^{0} x e^{\lambda x}\,dx = \Big[\tfrac{x}{\lambda}e^{\lambda x}\Big]_{-\infty}^{0} - \tfrac{1}{\lambda}\int_{-\infty}^{0} e^{\lambda x}\,dx = -\tfrac{1}{\lambda}\cdot\tfrac{1}{\lambda} = -\tfrac{1}{\lambda^2},$$

$$(1-p)\lambda \cdot \big(-\tfrac{1}{\lambda^2}\big) = -\tfrac{1-p}{\lambda}$$

$$E[X] = \tfrac{p-(1-p)}{\lambda} = \tfrac{2p-1}{\lambda}$$

$$E[X^2] = p\lambda \int_0^{\infty} x^2 e^{-\lambda x}\,dx + (1-p)\lambda \int_{-\infty}^{0} x^2 e^{\lambda x}\,dx$$

$$\int_0^{\infty} x^2 e^{-\lambda x} dx = \Big[-\tfrac{x^2}{\lambda}e^{-\lambda x}\Big]_0^{\infty} + \tfrac{2}{\lambda}\int_0^{\infty} x e^{-\lambda x} dx = \tfrac{2}{\lambda}\cdot\tfrac{1}{\lambda^2} = \tfrac{2}{\lambda^3},$$

$$p\lambda \cdot \tfrac{2}{\lambda^3} = \tfrac{2p}{\lambda^2}$$

$$\int_{-\infty}^{0} x^2 e^{\lambda x}\,dx = \int_0^{\infty} t^2 e^{-\lambda t}\,dt = \tfrac{2}{\lambda^3}$$

$$(1-p)\lambda \cdot \tfrac{2}{\lambda^3} = \tfrac{2(1-p)}{\lambda^2}$$

$$E[X^2] = \tfrac{2}{\lambda^2}$$

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2 = \tfrac{2}{\lambda^2} - \tfrac{(2p-1)^2}{\lambda^2} = \tfrac{2-(2p-1)^2}{\lambda^2} = \tfrac{1+4p(1-p)}{\lambda^2}$$

Q2- Proof for Discrete Case

For discrete case:
X and Y will be independent if

$$P(X = a, Y = b) = P(X = a)P(Y = b)$$

Case (i): Suppose X and Y are independent.

$$F_{XY}(m, y) = P(X \leq m, Y \leq y)$$

$$= \sum_{a \leq m} \sum_{b \leq y} P(X = a, Y = b)$$

$$= \sum_{a \leq m} \sum_{b \leq y} P(X = a)P(Y = b)$$

$$= \left(\sum_{a \leq m} P(X = a) \right) \left(\sum_{b \leq y} P(Y = b) \right)$$

$$= F_X(m) \cdot F_Y(y)$$

Case (ii): Suppose $F_{XY}(m, y) = F_X(m)F_Y(y)$.

$$P(X = a, Y = b) = P(X \leq a, Y \leq b) - P(X < a, Y \leq b) - P(X \leq a, Y < b) + P(X < a, Y < b)$$

$$= F_{XY}(a, b) - F_{XY}(a^-, b) - F_{XY}(a, b^-) + F_{XY}(a^-, b^-)$$

[Split all the CDFs]

$$= [F_X(a) - F_X(a^-)][F_Y(b) - F_Y(b^-)]$$

$$= P(X = a) \cdot P(Y = b)$$

Hence proved.

Proof for Continuous Case

Case (A): If X and Y are independent,

$$P(X \leq x, Y \leq y) = P(X \leq x)P(Y \leq y)$$

X and Y are independent if for all Borel sets $A, B \subset \mathbb{R}$,

$$P(X \in A, Y \in B) = P(X \in A) \cdot P(Y \in B)$$

Now consider the events

$$\{X \leq x\} = \{X \in (-\infty, x]\}, \quad \{Y \leq y\} = \{Y \in (-\infty, y]\}$$

Since X and Y are independent, we can apply

$$A = (-\infty, x], \quad B = (-\infty, y]$$

$$\Rightarrow P(X \leq x, Y \leq y) = P(X \leq x) \cdot P(Y \leq y)$$

$$\Rightarrow F_{XY}(x, y) = F_X(x)F_Y(y)$$

Case (B): Given $F_{XY}(x, y) = F_X(x)F_Y(y)$

$$\text{Now, } f_{XY}(x, y) = \frac{\partial^2}{\partial x \, \partial y} F_{XY}(x, y)$$

$$\frac{\partial F_{XY}(x, y)}{\partial x} = F_Y(y) \frac{dF_X(x)}{dx} = F_Y(y) f_X(x)$$

$$\frac{\partial^2 F_{XY}(x, y)}{\partial x \, \partial y} = F_X(x) \frac{dF_Y(y)}{dy} = F_X(x) f_Y(y)$$

$$\therefore \quad f_{XY}(x, y) = f_X(x) f_Y(y)$$

Hence, X and Y are independent.

$\boxed{\text{Hence proved.}}$

Question 3

Let X and Y be the two random variables representing the points chosen from the interval $[a, b]$. Since they are chosen from a uniform distribution, their probability density function (PDF) is:

$$f(z) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq z \leq b \\ 0 & \text{otherwise} \end{cases}$$

Since X and Y are independent, their joint PDF is the product of their individual PDFs:

$$f(x, y) = f_X(x) f_Y(y) = \frac{1}{(b-a)^2} \quad \text{for } x, y \in [a, b]$$

We want to find the expected distance between the two points, which is $E[|X - Y|]$. Using the definition of expected value for continuous variables, we have:

$$E[|X - Y|] = \int_a^b \int_a^b |x - y| f(x, y)\, dx\, dy = \frac{1}{(b-a)^2} \int_a^b \int_a^b |x - y|\, dx\, dy$$

The domain of integration is a square in the xy-plane. To handle the absolute value $|x - y|$, we can split the integral into two regions: one where $x \geq y$ and one where $y > x$. Due to the symmetry of the problem, the integrals over these two regions are equal. We can therefore calculate the integral over one region (e.g., where $x \geq y$) and multiply the result by 2.

$$\int_a^b \int_a^b |x - y|\, dx\, dy = 2 \int_a^b \int_a^x (x - y)\, dy\, dx$$

First, we evaluate the inner integral with respect to y:

$$\begin{aligned} \int_a^x (x - y)\, dy &= \left[xy - \frac{y^2}{2} \right]_{y=a}^x \\ &= \left(x^2 - \frac{x^2}{2} \right) - \left(ax - \frac{a^2}{2} \right) \\ &= \frac{x^2}{2} - ax + \frac{a^2}{2} \\ &= \frac{1}{2}(x - a)^2 \end{aligned}$$

Now, we substitute this result into the outer integral and integrate with respect to x:

$$\begin{aligned} 2 \int_a^b \frac{1}{2}(x - a)^2\, dx &= \int_a^b (x - a)^2\, dx \\ &= \left[\frac{(x - a)^3}{3} \right]_a^b \\ &= \frac{(b - a)^3}{3} - \frac{(a - a)^3}{3} \\ &= \frac{(b - a)^3}{3} \end{aligned}$$

Finally, we substitute this result back into the expression for the expected value:

$$E[|X - Y|] = \frac{1}{(b-a)^2} \cdot \frac{(b-a)^3}{3} = \frac{b-a}{3}$$

Thus, the expected distance between the two points is $\mathbf{\frac{b-a}{3}}$.

Q4-

(a)

$$\begin{aligned} E[X|A] &= \sum_x x \cdot P(X = x|A) \\ &= \sum_x x \cdot \frac{P(\{X = x\} \cap A)}{P(A)} \\ &= \frac{1}{P(A)} \sum_x x \cdot P(\{X = x\} \cap A) \\ &= \frac{1}{P(A)} E[1_A X] \end{aligned}$$

Here, 1_A is the indicator random variable for the event A. The expectation of the product $1_A X$ is:

$$E[1_A X] = \sum_x x \cdot P(1_A X = x) = \sum_x x \cdot P(\{X = x\} \cap A)$$

(b)

Let D_1 and D_2 be the outcomes of the first and second dice rolls, respectively. $X = D_1 + D_2$. $A_i = \{D_1 = i\}$.

$$\begin{aligned} E[X|A_i] &= E[D_1 + D_2|A_i] \\ &= E[D_1|A_i] + E[D_2|A_i] \end{aligned}$$

Given A_i, D_1 is fixed at i.

$$E[D_1|A_i] = i$$

Since D_1 and D_2 are independent,

$$E[D_2|A_i] = E[D_2] = \frac{1+2+3+4+5+6}{6} = 3.5$$

Therefore,

$$E[X|A_i] = i + 3.5$$

Q5-

We are given the CDF of a random variable X:

$$F_X(x) = \begin{cases} 0, & x < a, \\ 1 - \dfrac{a^3}{x^3}, & x \geq a, \end{cases} \qquad a > 0$$

PDF

The probability density function is obtained by differentiating the CDF:

$$f_X(x) = \frac{d}{dx} F_X(x) = \begin{cases} 0, & x < a, \\ \dfrac{3a^3}{x^4}, & x \geq a. \end{cases}$$

Mean

$$\mathbb{E}[X] = \int_a^\infty x f_X(x)\, dx = 3a^3 \int_a^\infty x^{-3}\, dx = 3a^3 \left[-\tfrac{1}{2}x^{-2}\right]_a^\infty = \tfrac{3a}{2}.$$

Second Moment

$$\mathbb{E}[X^2] = \int_a^\infty x^2 f_X(x)\, dx = 3a^3 \int_a^\infty x^{-2}\, dx = 3a^3 \left[-x^{-1}\right]_a^\infty = 3a^2.$$

Variance

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \big(\mathbb{E}[X]\big)^2 = 3a^2 - \left(\tfrac{3a}{2}\right)^2 = \tfrac{3}{4}a^2.$$

Q6

To find the value of the constant c, we use the property that the integral of a PDF over its entire domain must be equal to 1.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy = 1$$

Given the limits $0 < y < \infty$ and $-y \le x \le y$, the equation becomes:

$$\int_{0}^{\infty}\int_{-y}^{y} c(y^2 - x^2)e^{-y}dxdy = 1 \tag{1}$$

First, we solve the inner integral with respect to x:

$$\begin{aligned}\int_{-y}^{y} c(y^2 - x^2)e^{-y}dx &= ce^{-y}\int_{-y}^{y}(y^2 - x^2)dx\\ &= ce^{-y}\left[y^2x - \frac{x^3}{3}\right]_{-y}^{y}\\ &= ce^{-y}\left[\left(y^3 - \frac{y^3}{3}\right) - \left(-y^3 - \frac{(-y)^3}{3}\right)\right]\\ &= ce^{-y}\left[\frac{2y^3}{3} - \left(-\frac{2y^3}{3}\right)\right]\\ &= c\frac{4}{3}y^3e^{-y}\end{aligned}$$

Now, we substitute this back into the outer integral:

$$\int_{0}^{\infty} c\frac{4}{3}y^3e^{-y}dy = \frac{4c}{3}\int_{0}^{\infty} y^3e^{-y}dy = 1$$

The integral

$$\int_{0}^{\infty} y^3e^{-y}dy$$

is a standard integral and evalutaes to 6

$$\implies 8c = 1$$

Thus, the value of the constant is:

$$c = \frac{1}{8}$$

The marginal PDF for Y is found by integrating the joint PDF with respect to x:

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx = \int_{-y}^{y}\frac{1}{8}(y^2 - x^2)e^{-y}dx$$

From our calculation for c, we already evaluated this integral:

$$\int_{-y}^{y}(y^2 - x^2)e^{-y}dx = \frac{4}{3}y^3e^{-y}$$

Multiplying by $c = \frac{1}{8}$, we get:

$$f_Y(y) = \frac{1}{8}\cdot\frac{4}{3}y^3e^{-y} = \frac{1}{6}y^3e^{-y}$$

The marginal PDF for Y is:

$$f_Y(y) = \frac{1}{6}y^3e^{-y}, \quad \text{for } 0 < y < \infty$$

The marginal PDF for X is found by integrating the joint PDF with respect to y:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$$

The domain is defined by $-y \leq x \leq y$ and $y > 0$. This implies $y \geq x$ and $y \geq -x$, which simplifies to $y \geq |x|$. The limits for y are from $|x|$ to ∞.

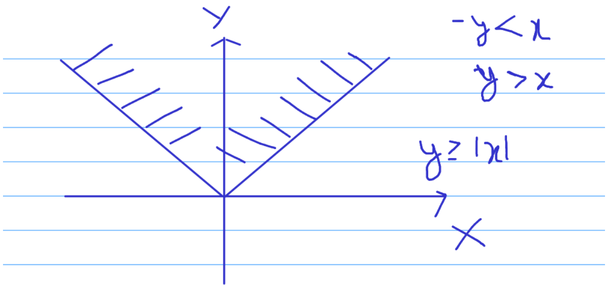

Figure 1: Region of integration

$$f_X(x) = \int_{|x|}^{\infty} \frac{1}{8}(y^2 - x^2)e^{-y}dy$$

Evaluating this integral (e.g., via integration by parts) yields:

$$\begin{aligned}
\int (y^2 - x^2)e^{-y}dy &= (x^2 - y^2 - 2y - 2)e^{-y} \\
\implies f_X(x) &= \frac{1}{8}\left[(x^2 - y^2 - 2y - 2)e^{-y}\right]_{|x|}^{\infty} \\
&= \frac{1}{8}\left(\lim_{y\to\infty}\left[(x^2 - y^2 - 2y - 2)e^{-y}\right] - \left[(x^2 - |x|^2 - 2|x| - 2)e^{-|x|}\right]\right) \\
&= \frac{1}{8}\left[0 - (x^2 - x^2 - 2|x| - 2)e^{-|x|}\right] \quad (\text{since } x^2 = |x|^2) \\
&= \frac{1}{8}\left[-(-2|x| - 2)e^{-|x|}\right] \\
&= \frac{1}{8}(2|x| + 2)e^{-|x|} = \frac{1}{4}(1 + |x|)e^{-|x|}
\end{aligned}$$

The marginal PDF for X is:

$$f_X(x) = \frac{1}{4}(1 + |x|)e^{-|x|}, \quad \text{for } -\infty < x < \infty$$

Problem 7

Let X_1, X_2, X_3 be independent continuous random variables with common PDF f_X. Express $P(X_1 < X_2 < X_3)$ as an integral involving f_X and evaluate its value.

Solution:

Given X_1, X_2, X_3 be i.i.d. continuous random variables with a common PDF $f_X(x)$, let's assume the CDF to be $F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$.

Since the random variables are independent, we know their joint PDF is the product of their individual PDFs:

$$f_{X_1,X_2,X_3}(x_1, x_2, x_3) = f_X(x_1)f_X(x_2)f_X(x_3)$$

We need to evaluate:

$$P(X_1 < X_2 < X_3) = \iiint_{x_1<x_2<x_3} f_X(x_1)f_X(x_2)f_X(x_3)\,dx_1\,dx_2\,dx_3$$

This is equivalent to the integral (as seen above):

$$P(X_1 < X_2 < X_3) = \int_{-\infty}^{\infty} f_X(x_3)\left(\int_{-\infty}^{x_3} f_X(x_2)\left(\int_{-\infty}^{x_2} f_X(x_1)\,dx_1\right)dx_2\right)dx_3$$

We first evaluate the innermost integral with respect to x_1. By the definition of the CDF, this is:

$$\int_{-\infty}^{x_2} f_X(x_1)\,dx_1 = F_X(x_2)$$

Substituting this back, we get:

$$P(X_1 < X_2 < X_3) = \int_{-\infty}^{\infty} f_X(x_3)\left(\int_{-\infty}^{x_3} F_X(x_2)f_X(x_2)\,dx_2\right)dx_3$$

Solving the integral with respect to x_2:

$$\int_{-\infty}^{x_3} F_X(x_2)f_X(x_2)\,dx_2$$

Now, if $u = F_X(x_2)$, $du = f_X(x_2)\,dx_2$. The limits of integration become $F_X(-\infty) = 0$ and $F_X(x_3)$. We thus have:

$$\int_{0}^{F_X(x_3)} u\,du = \left[\frac{u^2}{2}\right]_0^{F_X(x_3)} = \frac{[F_X(x_3)]^2}{2}$$

Substituting this result back:

$$P(X_1 < X_2 < X_3) = \int_{-\infty}^{\infty} \frac{[F_X(x_3)]^2}{2} f_X(x_3)\,dx_3$$

Now, again, if $v = F_X(x_3)$, $dv = f_X(x_3)\,dx_3$. The limits become $F_X(-\infty) = 0$ and $F_X(\infty) = 1$.

$$\begin{aligned} P(X_1 < X_2 < X_3) &= \int_0^1 \frac{v^2}{2}\,dv \\ &= \frac{1}{2}\left[\frac{v^3}{3}\right]_0^1 \\ &= \frac{1}{6} \end{aligned}$$

$$\boxed{P(X_1 < X_2 < X_3) = \frac{1}{6}}$$

Intuition

Since X_1, X_2, X_3 are independent with common PDF f_X, their joint PDF is:

$$f_{X_1,X_2,X_3}(x_1, x_2, x_3) = f_X(x_1)f_X(x_2)f_X(x_3)$$

The probability $P(X_1 < X_2 < X_3)$ can be expressed as:

$$P(X_1 < X_2 < X_3) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{I}_{x_1<x_2<x_3} f_X(x_1)f_X(x_2)f_X(x_3)\,dx_1\,dx_2\,dx_3$$

where $\mathbf{I}_{x_1<x_2<x_3}$ is the indicator function that is 1 when $x_1 < x_2 < x_3$, and 0 otherwise.

We can introduce the indicator function definition through the integral bounds. For this, we must consider one of the RVs to be free (ideally X_1 or X_3), and then constrain the other 2. Say X_1 can be freely picked, thus:

- $X_1 = x_1$ where, $x_1 \in (-\infty, \infty)$
- $X_2 = x_2$ where, $x_2 \in (x_1, \infty)$ since $x_1 < x_2$
- $X_3 = x_3$ where, $x_3 \in (x_2, \infty)$ since $x_2 < x_3$

$$P(X_1 < X_2 < X_3) = \int_{x_1=-\infty}^{\infty}\int_{x_2=x_1}^{\infty}\int_{x_3=x_2}^{\infty} f_X(x_1)f_X(x_2)f_X(x_3)\,dx_3\,dx_2\,dx_1$$

Since X_1, X_2, X_3 are given to be i.i.d., all orderings of these three variables are equally likely by symmetry. There is no inherent reason to believe that any one of them is more likely to be larger or smaller than the others! We have $3! = 6$ possible orderings of the three distinct values, viz.

$$\begin{gathered} X_1 < X_2 < X_3 \\ X_1 < X_3 < X_2 \\ X_2 < X_1 < X_3 \\ X_2 < X_3 < X_1 \\ X_3 < X_1 < X_2 \\ X_3 < X_2 < X_1 \end{gathered}$$

By symmetry, each ordering has the same probability. That is,

$$\sum_{\text{all orderings}} P(\text{ordering}) = 1$$

Therefore:

$$\boxed{P(X_1 < X_2 < X_3) = \frac{1}{6}}$$