Assignment 5 Solutions

MA6.102 Probability and Random Processes, Monsoon 2025

Question 1

Given:

$$f_{X,Y}(x,y) = c \cdot x(y-x)e^{-y}, \quad 0 \le x \le y < \infty$$

We need to find c such that:

$$\int_0^\infty \int_0^y c \cdot x(y-x)e^{-y}\, dx\, dy = 1$$

First, evaluate the inner integral:

$$\int_0^y x(y-x)\, dx = \int_0^y (xy - x^2)\, dx = y\int_0^y x\, dx - \int_0^y x^2\, dx$$

$$= y\left[\frac{x^2}{2}\right]_0^y - \left[\frac{x^3}{3}\right]_0^y = \frac{y^3}{2} - \frac{y^3}{3} = \frac{y^3}{6}$$

Therefore:

$$\frac{c}{6}\int_0^\infty y^3 e^{-y}\, dy = 1$$

Now evaluate $\int_0^\infty y^3 e^{-y}\, dy$ using integration by parts:

$$\int_0^\infty y^3 e^{-y}\, dy = \left[-y^3 e^{-y}\right]_0^\infty + 3\int_0^\infty y^2 e^{-y}\, dy = 3\int_0^\infty y^2 e^{-y}\, dy$$

$$= 3\left(\left[-y^2 e^{-y}\right]_0^\infty + 2\int_0^\infty y e^{-y}\, dy\right) = 6\int_0^\infty y e^{-y}\, dy$$

$$= 6\left(\left[-y e^{-y}\right]_0^\infty + \int_0^\infty e^{-y}\, dy\right) = 6(0+1) = 6$$

Thus, $c = 1$.

The marginal density of Y is:

$$f_Y(y) = \int_0^y x(y-x)e^{-y}\, dx = e^{-y} \cdot \frac{y^3}{6}, \quad y \ge 0$$

The conditional density is:

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{x(y-x)e^{-y}}{(y^3/6)e^{-y}} = \frac{6x(y-x)}{y^3}, \quad 0 \le x \le y$$

The conditional expectation is:

$$E[X \mid Y = y] = \int_0^y x \cdot f_{X|Y}(x \mid y)\, dx = \frac{6}{y^3} \int_0^y x^2(y - x)\, dx$$

$$= \frac{6}{y^3} \left(y \int_0^y x^2\, dx - \int_0^y x^3\, dx \right) = \frac{6}{y^3} \left(y \cdot \frac{y^3}{3} - \frac{y^4}{4} \right)$$

$$= \frac{6}{y^3} \left(\frac{y^4}{3} - \frac{y^4}{4} \right) = \frac{6}{y^3} \cdot \frac{y^4}{12} = \frac{y}{2}$$

Therefore:

$$E[X \mid Y] = \frac{Y}{2}$$

Question 2

Let X_1 and X_2 be independent and identically distributed random variables with the common PDF $f_X(x) = \frac{1}{x^2}$, for $x > 1$. Define new random variables $Y_1 = X_1 + X_2$ and $Y_2 = \frac{X_2}{X_1+X_2}$. Find the joint PDF, explicitly stating the domain over which it is non-zero.

Solution

We will use the method of Jacobian transformation. For this, we need the joint density of X_1 and X_2, and the Jacobian determinant itself.

Since X_1 and X_2 are independent, their joint PDF is the product of the marginal PDFs:

$$f_{X_1,X_2}(x_1, x_2) = f_{X_1}(x_1) \cdot f_{X_2}(x_2)$$

Given $f_X(x) = \frac{1}{x^2}$ for $x > 1$, we have:

$$f_{X_1,X_2}(x_1, x_2) = \left(\frac{1}{x_1^2}\right)\left(\frac{1}{x_2^2}\right) = \frac{1}{x_1^2 x_2^2}$$

This joint PDF is non-zero on the domain $D_X = \{(x_1, x_2) : x_1 > 1, x_2 > 1\}$.
We are given the transformations:

$$Y_1 = X_1 + X_2$$
$$Y_2 = \frac{X_2}{X_1 + X_2}$$

We now solve for x_1 and x_2 in terms of y_1 and y_2. From the definition of Y_2:

$$Y_2 = \frac{X_2}{Y_1} \quad \Rightarrow \quad X_2 = Y_1 Y_2$$

From the definition of Y_1:

$$X_1 = Y_1 - X_2 = Y_1 - Y_1 Y_2 = Y_1(1 - Y_2)$$

The inverse transformations are:

$$x_1 = y_1(1 - y_2)$$
$$x_2 = y_1 y_2$$

Finding the Jacobian J:

We compute the Jacobian with respect to x_1 and x_2: $J = \det \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{vmatrix}$

From the transformations $y_1 = x_1 + x_2$ and $y_2 = \frac{x_2}{x_1+x_2}$:

$$\frac{\partial y_1}{\partial x_1} = 1, \quad \frac{\partial y_1}{\partial x_2} = 1$$
$$\frac{\partial y_2}{\partial x_1} = -\frac{x_2}{(x_1 + x_2)^2}, \quad \frac{\partial y_2}{\partial x_2} = \frac{x_1}{(x_1 + x_2)^2}$$

The determinant is: $J = (1)\left(\frac{x_1}{(x_1+x_2)^2}\right) - (1)\left(-\frac{x_2}{(x_1+x_2)^2}\right) = \frac{x_1+x_2}{(x_1+x_2)^2} = \frac{1}{x_1+x_2}$

Since $x_1 + x_2 = y_1$, we have $|J| = \frac{1}{y_1}$.

The formula for the joint PDF using this Jacobian is: $f_{Y_1,Y_2}(y_1, y_2) = \frac{f_{X_1,X_2}(x_1(y_1,y_2),x_2(y_1,y_2))}{|J|}$

Alternative Method: We can also compute the Jacobian with respect to y_1 and y_2:

$$J' = \det \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}$$

Computing the partial derivatives:

$$\frac{\partial x_1}{\partial y_1} = 1 - y_2, \quad \frac{\partial x_1}{\partial y_2} = -y_1$$
$$\frac{\partial x_2}{\partial y_1} = y_2, \quad \frac{\partial x_2}{\partial y_2} = y_1$$

The determinant is: $J' = (1 - y_2)(y_1) - (-y_1)(y_2) = y_1 - y_1y_2 + y_1y_2 = y_1$

Note that $J' = \frac{1}{J}$, so $|J'| = y_1$, and the transformation formula becomes: $f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(x_1(y_1, y_2), x_2(y_1, y_2)) \cdot |J'|$

Both methods give the same result.

We map the original domain D_X to the new domain D_Y using the inverse transformation. The original domain D_X is defined by two conditions: $x_1 > 1$ and $x_2 > 1$.

Substituting the inverse functions into these inequalities:
1. $y_1(1 - y_2) > 1$
2. $y_1y_2 > 1$

These two inequalities define the domain D_Y where the joint PDF of Y is non-zero. (We can also deduce that $y_1 > 2$, $0 < y_2 < 1$, and $y_1 > \max\left(\frac{1}{y_2}, \frac{1}{1-y_2}\right)$).

The formula for the joint PDF of the new variables is: $f_{Y_1,Y_2}(y_1, y_2) = \frac{f_{X_1,X_2}(x_1(y_1,y_2),x_2(y_1,y_2))}{|J|}$

Substituting $x_1 = y_1(1 - y_2)$ and $x_2 = y_1y_2$: $f_{Y_1,Y_2}(y_1, y_2) = \frac{1}{[y_1(1-y_2)]^2[y_1y_2]^2} \cdot y_1 = \frac{y_1}{y_1^4y_2^2(1-y_2)^2}$

$$f_{Y_1,Y_2}(y_1, y_2) = \frac{1}{y_1^3y_2^2(1 - y_2)^2}$$

This PDF is non-zero on the domain D_Y:

$$D_Y = \{(y_1, y_2) : y_1y_2 > 1 \text{ and } y_1(1 - y_2) > 1\}$$

Final Answer:

$$\boxed{f_{Y_1,Y_2}(y_1, y_2) = \frac{1}{y_1^3y_2^2(1 - y_2)^2}}$$

for the domain:

$$\boxed{y_1y_2 > 1 \text{ and } y_1(1 - y_2) > 1}$$

and $f_{Y_1,Y_2}(y_1, y_2) = 0$ elsewhere.

Problem 3

Let X_1 and X_2 be independent exponential random variables with parameter λ. Find the joint PDF of

$$Y_1 = X_1 + X_2, \qquad Y_2 = \frac{X_1}{X_2},$$

and check whether Y_1 and Y_2 are independent.

$$f_{X_1,X_2}(x_1,x_2) = \begin{cases} \lambda^2 e^{-\lambda(x_1+x_2)}, & x_1 > 0, x_2 > 0, \\ 0, & \text{otherwise.} \end{cases}$$

$$Y_1 = X_1 + X_2, \quad Y_2 = \frac{X_1}{X_2}$$

$$X_1 = Y_1 \frac{Y_2}{1+Y_2}, \quad X_2 = \frac{Y_1}{1+Y_2}$$

Support: $y_1 > 0, y_2 > 0$.

Method 1

$$J_1 = \det\left(\frac{\partial(x_1,x_2)}{\partial(y_1,y_2)}\right) = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} \dfrac{y_2}{1+y_2} & \dfrac{y_1}{(1+y_2)^2} \\ \dfrac{1}{1+y_2} & -\dfrac{y_1}{(1+y_2)^2} \end{vmatrix}$$

$$= \left(\frac{y_2}{1+y_2}\right)\left(-\frac{y_1}{(1+y_2)^2}\right) - \left(\frac{y_1}{(1+y_2)^2}\right)\left(\frac{1}{1+y_2}\right)$$

$$= -\frac{y_1(y_2+1)}{(1+y_2)^3} = -\frac{y_1}{(1+y_2)^2}$$

$$|J_1| = \frac{y_1}{(1+y_2)^2}$$

$$f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}(x_1,x_2)\,|J_1| = \lambda^2 e^{-\lambda(x_1+x_2)} \frac{y_1}{(1+y_2)^2}$$
$$= \lambda^2 e^{-\lambda y_1} \frac{y_1}{(1+y_2)^2} = \lambda^2 y_1 e^{-\lambda y_1}(1+y_2)^{-2}$$

Method 2

$$J_2 = \det\left(\frac{\partial(y_1,y_2)}{\partial(x_1,x_2)}\right) = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ \frac{1}{x_2} & -\frac{x_1}{x_2^2} \end{vmatrix}$$

$$= (1)\left(-\frac{x_1}{x_2^2}\right) - (1)\left(\frac{1}{x_2}\right) = -\frac{x_1+x_2}{x_2^2}$$

$$|J_2| = \frac{x_1+x_2}{x_2^2}$$

$$f_{Y_1,Y_2}(y_1,y_2) = \frac{f_{X_1,X_2}(x_1,x_2)}{|J_2|} = \frac{\lambda^2 e^{-\lambda(x_1+x_2)}}{\left(\dfrac{x_1+x_2}{x_2^2}\right)} = \lambda^2 e^{-\lambda(x_1+x_2)}\left(\frac{x_2^2}{x_1+x_2}\right)$$

$$= \lambda^2 e^{-\lambda y_1}\left(\frac{(\frac{y_1}{1+y_2})^2}{y_1}\right) = \lambda^2 e^{-\lambda y_1}\left(\frac{y_1^2/(1+y_2)^2}{y_1}\right)$$
$$= \lambda^2 e^{-\lambda y_1}\left(\frac{y_1}{(1+y_2)^2}\right) = \lambda^2 y_1 e^{-\lambda y_1}(1+y_2)^{-2}$$

$$f_{Y_1,Y_2}(y_1,y_2) = \begin{cases} \lambda^2 y_1 e^{-\lambda y_1}(1+y_2)^{-2}, & y_1>0, y_2>0, \\ 0, & \text{otherwise.} \end{cases}$$

$$\begin{aligned} f_{Y_1}(y_1) &= \int_0^\infty \lambda^2 y_1 e^{-\lambda y_1}(1+y_2)^{-2}\,dy_2 \\ &= \lambda^2 y_1 e^{-\lambda y_1}\int_0^\infty (1+y_2)^{-2}\,dy_2 \\ &= \lambda^2 y_1 e^{-\lambda y_1}\left[-(1+y_2)^{-1}\right]_0^\infty = \lambda^2 y_1 e^{-\lambda y_1}, \quad y_1>0. \end{aligned}$$

$$\begin{aligned} f_{Y_2}(y_2) &= \int_0^\infty \lambda^2 y_1 e^{-\lambda y_1}(1+y_2)^{-2}\,dy_1 \\ &= (1+y_2)^{-2}\int_0^\infty \lambda^2 y_1 e^{-\lambda y_1}\,dy_1 \\ &= (1+y_2)^{-2}\cdot 1 = (1+y_2)^{-2}, \quad y_2>0. \end{aligned}$$

$$f_{Y_1}(y_1)\,f_{Y_2}(y_2) = \left(\lambda^2 y_1 e^{-\lambda y_1}\right)\cdot\left((1+y_2)^{-2}\right) = f_{Y_1,Y_2}(y_1,y_2).$$

Therefore, Y_1 and Y_2 are independent.

Problem 5.

$$X, Y, Z \overset{\text{iid}}{\sim} \mathrm{Unif}(0,1), \qquad W = XY.$$

$$\begin{aligned}
F_W(w) = P(XY \le w) &= \int_0^1 P\left(X \le \frac{w}{y}\right) dy \\
&= \int_0^1 \min\left(1, \frac{w}{y}\right) dy = \int_0^w 1\, dy + \int_w^1 \frac{w}{y}\, dy \\
&= w + w \ln\left(\frac{1}{w}\right) = w - w \ln w, \qquad 0 < w \le 1,
\end{aligned}$$

$$f_W(w) = F_W'(w) = -\ln w, \qquad 0 < w < 1.$$

$$\begin{aligned}
P(XY < Z^2) &= \iint_{[0,1]^2} \mathbf{1}_{\{w < z^2\}}\, f_W(w)\, f_Z(z)\, dw\, dz \\
&= \int_0^1 f_W(w)\, P(Z > \sqrt{w})\, dw \\
&= \int_0^1 (-\ln w)\,(1 - \sqrt{w})\, dw \\
&= \int_0^1 (-\ln w)\, dw - \int_0^1 (-\ln w)\sqrt{w}\, dw.
\end{aligned}$$

$$\int_0^1 (-\ln w)\, dw = 1, \qquad \int_0^1 (-\ln w)\sqrt{w}\, dw = -\int_0^1 w^{1/2} \ln w\, dw = \frac{4}{9}.$$

$$P(XY < Z^2) = 1 - \frac{4}{9} = \frac{5}{9}.$$

$$\boxed{P(XY < Z^2) = \tfrac{5}{9}.}$$

Question 6

Let F_X and F_Y be strictly increasing CDFs of random variables X and Y. Since they are strictly increasing, the ordinary inverses F_X^{-1} and F_Y^{-1} exist on $(0,1)$. Define

$$g(x) = F_Y^{-1}\big(F_X(x)\big), \qquad Z = g(X).$$

We prove $F_Z(y) = F_Y(y)$ by starting from $F_Y(y)$:

$$\begin{aligned}
F_Y(y) &= F_X\big(F_X^{-1}(F_Y(y))\big) && \text{(inverse identity)} \\
&= \mathbb{P}\big\{X \le F_X^{-1}(F_Y(y))\big\} && \text{(definition of } F_X\text{)} \\
&= \mathbb{P}\{F_X(X) \le F_Y(y)\} && \text{(apply increasing } F_X \text{ to both sides)} \\
&= \mathbb{P}\big\{F_Y^{-1}(F_X(X)) \le y\big\} && \text{(apply increasing } F_Y^{-1} \text{ to both sides)} \\
&= \mathbb{P}\{Z \le y\} && \text{(since } Z = F_Y^{-1}(F_X(X))\text{)} \\
&= F_Z(y).
\end{aligned}$$

Thus $Z = g(X)$ has the same CDF as Y.

Question 7

(a)

Yes, the statement is **true**.

The characteristic function is defined as $\phi_X(t) = \mathbb{E}[e^{\mathrm{i}tX}]$. We are given the condition $\phi_X(2\pi) = 1$. By definition, this means $\mathbb{E}[e^{\mathrm{i}2\pi X}] = 1$. Using Euler's formula, $e^{\mathrm{i}\theta} = \cos(\theta) + \mathrm{i}\sin(\theta)$, we can expand the expectation:

$$\mathbb{E}[\cos(2\pi X) + \mathrm{i}\sin(2\pi X)] = 1$$

By the linearity of expectation, this separates into real and imaginary parts:

$$\mathbb{E}[\cos(2\pi X)] + \mathrm{i}\mathbb{E}[\sin(2\pi X)] = 1 + 0\mathrm{i}$$

Equating the real and imaginary parts, we get two conditions:

$$\mathbb{E}[\cos(2\pi X)] = 1$$
$$\mathbb{E}[\sin(2\pi X)] = 0$$

Let's focus on the first condition. The cosine function has a maximum value of 1, so $\cos(2\pi X) \leq 1$ for all outcomes of X. If the expected value of a random variable is equal to its maximum possible value, the random variable must equal that maximum value with probability 1.

Therefore, we must have $\mathbb{P}(\cos(2\pi X) = 1) = 1$. The equation $\cos(\theta) = 1$ holds if and only if $\theta = 2\pi k$ for some integer $k \in \mathbb{Z}$. Applying this to our case:

$$2\pi X = 2\pi k \implies X = k \quad \text{for some } k \in \mathbb{Z}$$

This means the random variable X must take an integer value with probability 1. Thus, we have shown that $\mathbb{P}(X \in \mathbb{Z}) = 1$. Note that the second expectation condition is also satisfied.

(b)

We want to show that $P_X(n) = \frac{1}{2\pi}\int_0^{2\pi} e^{-\mathrm{i}tn}\phi_X(t)\,dt$.

For an integer-valued random variable X, the characteristic function is given by the sum over all possible integer outcomes k:

$$\phi_X(t) = \mathbb{E}[e^{\mathrm{i}tX}] = \sum_{k\in\mathbb{Z}} e^{\mathrm{i}tk} P_X(k)$$

We start with the right-hand side (RHS) of the equation to be proved and substitute this definition.

$$\begin{aligned}
\text{RHS} &= \frac{1}{2\pi}\int_0^{2\pi} e^{-\mathrm{i}tn}\phi_X(t)\,dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{-\mathrm{i}tn}\left(\sum_{k\in\mathbb{Z}} e^{\mathrm{i}tk} P_X(k)\right) dt
\end{aligned}$$

We interchange the order of integration and summation:

$$\begin{aligned}
\text{RHS} &= \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} P_X(k)\int_0^{2\pi} e^{-\mathrm{i}tn}e^{\mathrm{i}tk}\,dt \\
&= \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} P_X(k)\int_0^{2\pi} e^{\mathrm{i}t(k-n)}\,dt
\end{aligned}$$

Now, we evaluate the integral $\int_0^{2\pi} e^{\mathrm{i}t(k-n)}\,dt$ based on two cases for the integers k and n.

- **Case 1: $k = n$.** The exponent is 0, so the integrand is $e^0 = 1$.

$$\int_0^{2\pi} 1\,dt = [t]_0^{2\pi} = 2\pi$$

- **Case 2: $k \neq n$.** The integral evaluates to:

$$\int_0^{2\pi} e^{\mathrm{i}t(k-n)}\, dt = \left[\frac{e^{\mathrm{i}t(k-n)}}{\mathrm{i}(k-n)} \right]_0^{2\pi} = \frac{e^{\mathrm{i}2\pi(k-n)} - e^0}{\mathrm{i}(k-n)}$$

  Since $k - n$ is a non-zero integer, $e^{\mathrm{i}2\pi(k-n)} = 1$. Thus, the integral is $\frac{1-1}{\mathrm{i}(k-n)} = 0$.

So, the integral is non-zero only when $k = n$. This causes the infinite sum to collapse to a single term.

$$\begin{aligned}
\text{RHS} &= \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} P_X(k) \cdot \begin{cases} 2\pi & \text{if } k = n \\ 0 & \text{if } k \neq n \end{cases} \\
&= \frac{1}{2\pi} (P_X(n) \cdot 2\pi) \\
&= P_X(n)
\end{aligned}$$

The right-hand side equals the left-hand side, which completes the proof.

Problem 4

Given Formula

Let (X, Y) be jointly continuous with joint pdf $f_{X,Y}(x, y)$. Let

$$Z = g_1(X, Y), \qquad W = g_2(X, Y),$$

$$g_1(x, y) = z, \qquad g_2(x, y) = w$$

(x_i, y_i), $i = 1, \ldots, n$. Define

$$J(x_i, y_i) = \left| \det \begin{pmatrix} \frac{\partial g_1(x,y)}{\partial x} & \frac{\partial g_1(x,y)}{\partial y} \\ \frac{\partial g_2(x,y)}{\partial x} & \frac{\partial g_2(x,y)}{\partial y} \end{pmatrix} \right|_{(x,y)=(x_i,y_i)}.$$

Then the joint pdf of (Z, W) is

$$\boxed{f_{Z,W}(z, w) = \sum_{i=1}^{n} \frac{f_{X,Y}(x_i, y_i)}{J(x_i, y_i)}}.$$

Apply to $Z = \min\{X, Y\},\ W = \max\{X, Y\}$

We take

$$g_1(x, y) = \min\{x, y\}, \qquad g_2(x, y) = \max\{x, y\}.$$

Solve $g_1(x, y) = z,\ g_2(x, y) = w$

Fix (z, w) with $z < w$. The equations

$$\min\{x, y\} = z, \qquad \max\{x, y\} = w$$

have exactly two solutions:

$$(x_1, y_1) = (z, w), \qquad (x_2, y_2) = (w, z).$$

Compute J at each region

- $(x_1, y_1) = (z, w)$ with $z < w$: Therefore

$$J(x_1, y_1) = \left|\det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right| = 1.$$

Reason

$$\begin{aligned}
\frac{\partial z}{\partial x}(z, w) &= \lim_{h \to 0} \frac{z(z+h, w) - z(z, w)}{h} = \lim_{h \to 0} \frac{\min(z+h, w) - \min(z, w)}{h} \\
&= \lim_{h \to 0} \frac{(z+h) - z}{h} = 1, \\
\frac{\partial z}{\partial y}(z, w) &= \lim_{h \to 0} \frac{z(z, w+h) - z(z, w)}{h} = \lim_{h \to 0} \frac{\min(z, w+h) - \min(z, w)}{h} \\
&= \lim_{h \to 0} \frac{z - z}{h} = 0, \\
\frac{\partial w}{\partial x}(z, w) &= \lim_{h \to 0} \frac{w(z+h, w) - w(z, w)}{h} = \lim_{h \to 0} \frac{\max(z+h, w) - \max(z, w)}{h} \\
&= \lim_{h \to 0} \frac{w - w}{h} = 0, \\
\frac{\partial w}{\partial y}(z, w) &= \lim_{h \to 0} \frac{w(z, w+h) - w(z, w)}{h} = \lim_{h \to 0} \frac{\max(z, w+h) - \max(z, w)}{h} \\
&= \lim_{h \to 0} \frac{(w+h) - w}{h} = 1.
\end{aligned}$$

$$J(z, w) = \begin{pmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \left|\det J(z, w)\right| = 1.$$

- $(x_2, y_2) = (w, z)$ with $w > z$: Therefore

$$J(x_2, y_2) = \left|\det \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right| = 1.$$

Plugging into formula

$$f_{Z,W}(z, w) = \frac{f_{X,Y}(x_1, y_1)}{J(x_1, y_1)} + \frac{f_{X,Y}(x_2, y_2)}{J(x_2, y_2)} = f_{X,Y}(z, w) + f_{X,Y}(w, z), \qquad (z < w).$$

Since $Z = \min\{X, Y\} \leq W = \max\{X, Y\}$ by definition,

$$f_{Z,W}(z, w) = \begin{cases} f_{X,Y}(z, w) + f_{X,Y}(w, z), & z \leq w, \end{cases}$$