Assignment - 1

(MA6.102) Probability and Random Processes

1) Given

Sample space - Ω

σ - Field - $\mathcal{F}$

*A is an atom $A \in \mathcal{F}$

if i) $A \neq \emptyset$

For any $B \in \mathcal{F}$ ii) $B \subseteq A$ then $B = \emptyset$ (or) $B = A$

→ Let X, Y are two distinct atoms $X, Y \in \mathcal{F}$

For any $D, E \in \mathcal{F}$

so (i) $D \subseteq X$ then $D = \emptyset$ (or) $D = X$

(ii) $E \subseteq \cancel{X}$ then $E = \emptyset$ (or) $E = Y$

Since

$X \in \mathcal{F}$ and $Y \in \mathcal{F}$

$\boxed{X \cap Y \in \mathcal{F}}$

~~proof~~ if $A \in \mathcal{F}$ and $B \in \mathcal{F}$

From σ-field.

$A', B' \in \mathcal{F}$

$\Rightarrow A' \cup B' \in \mathcal{F}$

$\Rightarrow (A' \cup B')' \in \mathcal{F}$

$\Rightarrow \boxed{A \cap B \in \mathcal{F}}$ *

so if $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$ (3)

⇒ From (i), (ii),

we know From (i) →

$X \cap Y \subseteq X \Rightarrow \boxed{X \cap Y = \emptyset \text{ (or) } X \cap Y = X}$

and From (ii)

$X \cap Y \subseteq Y \Rightarrow \boxed{X \cap Y = \emptyset \text{ (or) } X \cap Y = Y}$ (4)

From (3), (4)

since $X \neq Y$

$\boxed{X \cap Y = \emptyset}$

∴ Two distinct atoms in a σ-field are necessarily disjoint

② Given

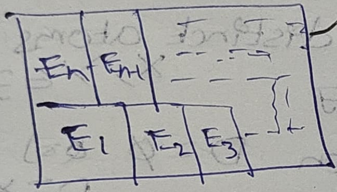
$E_1, E_2 - - - E_n$ are mutually exclusive (and) mutually exhaustive events

i.e (i) $E_i \cap E_j = \phi \quad \forall\, i \neq j$

and (ii) $\bigcup_{i=1}^{n} E_i = \Omega$

So these are

* n-disjoint sets



So smallest σ-field is

$$\boxed{\mathcal{F} = \bigcup_{i \in X} E_i \; ; X \subseteq \{1, 2, 3 - - n\}}$$

Smallest σ-field that contains all events E_i $\forall\, i \in [1, n]$

is

$$\boxed{\mathcal{F} = \left\{ \bigcup_{i \in X} E_i \; ; X \subseteq \{1, 2, 3 - - - n\} \right\}}$$

3) Given

$$P(A) = \frac{3}{4} \; ; \; P(B) = \frac{1}{3}$$

$\Rightarrow \; A \cup B = A \cap (B \setminus A)$

so $P(A \cup B) = P(A) + P(B \setminus A)$ $\quad \because A, B \setminus A$ are disjoint

$$\Rightarrow \boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)} \rightarrow \text{I}$$

Now From (I)

$$P(A \cup B) = \frac{3}{4} + \frac{1}{3} - P(A \cap B) \leq 1$$

$$\Rightarrow \frac{3}{4} + \frac{1}{3} - 1 \leq P(A \cap B)$$

$$\boxed{\frac{1}{12} \leq P(A \cap B)} \rightarrow (1)$$

Now

Let $\boxed{C, D \in \Omega}$

$$C \subseteq D$$

$$\Rightarrow D = C + D \setminus C$$

$$P(D) = P(C) + P(D \setminus C)$$

$$\boxed{\text{so } \boxed{P(D) \geq P(C)} \; \boxed{\text{if } D \supseteq C}}$$

so we know $A \cap B \subseteq B$

$$\therefore P(A \cap B) \subseteq P(B) \Rightarrow \boxed{P(A \cap B) \leq \frac{1}{3}} \rightarrow (2)$$

From (1), (2)

$$\boxed{\therefore \frac{1}{12} \leq P(A \cap B) \leq \frac{1}{3}}$$

④

3-b) For lower bound :

Let $\Omega = \{x_1, x_2, x_3\}$

$A = \{x_1, x_2\}$; $B = \{x_2, x_3\}$

$P(x_1) = \frac{2}{3}$; $P(x_3) = \frac{1}{4}$

$P(x_2) = \frac{1}{12}$

$P(A) = P(x_1) + P(x_2) = \frac{2}{3} + \frac{1}{12} = \frac{3}{4}$

$P(B) = P(x_2) + P(x_3) = \frac{1}{12} + \frac{1}{4} = \frac{1}{3}$

$P(A \cap B) = P(x_2) = \frac{1}{12}$

$\boxed{\therefore P(A \cap B) = \frac{1}{12}}$

For upper bound :

$\Omega = \{x_1, x_2, x_3\}$

$A = \{x_1, x_2\}$; $B = \{x_2\}$

$P(x_1) = \frac{5}{12}$ $\Rightarrow$ $P(A) = P(x_1) + P(x_2) = \frac{3}{4}$

$P(x_2) = \frac{1}{3}$ $\quad P(B) = P(x_2) = \frac{1}{3}$

$P(x_3) = \frac{1}{4}$ $\quad P(A \cap B) = P(x_2) = \frac{1}{3}$

$\boxed{\therefore P(A \cap B) = \frac{1}{3}}$

4)

P(opposite face is tails | heads occured)

$$= \frac{P(\text{opposite face is tails} \cap \text{heads occured})}{P(\text{heads occured})}$$

$$\cancel{\frac{1}{3} \times 1 +}$$

$$\cancel{\frac{1}{3} \times \frac{1}{2} \times 0 + \frac{1}{3} \times 0 \times 1}$$

selecting coin, getting heads, opp face is tails

$$\Rightarrow P = \frac{\frac{1}{3} \times 1 \times 0 + \frac{1}{3} \times 0 \times 1 + \frac{1}{3} \times \frac{1}{2} \times 1}{\frac{1}{3} \times 1 + \frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times 0}$$

selecting coin, getting heads

$$\frac{\frac{1}{6}}{\frac{1}{3} + \frac{1}{6}} = \frac{\frac{1}{2}}{1 + \frac{1}{2}} = \boxed{\frac{1}{3}}$$ *

$\therefore$ P(opposite face is tails | heads occured) $= \frac{1}{3} = 0.333\ldots$

5) Given $C = \{A_1, A_2 --- A_n\}$ (4)

are mutully independent events

i.e $P\left(\bigcap_{i \in B} A_i\right) = \prod_{i \in B} P(A_i) \; \Big| \; B \subseteq \{1, 2, 3 -- n\}$

Now

$$\boxed{B = \{A_1, A_2, A_3 -- A_{i-1}, A_i^c, A_{i+1}, --- A_n\}}$$

Now to check B be a collection of mutually independent

(i) $\Rightarrow$ $\boxed{P\left(\bigcap_{i \in D - \{i\}} B_i\right) = \prod_{i \in D-\{i\}} P(B_i)}$ $\boxed{D \subseteq \{1, 2 --- n\}}$

$\because$ we know A_1, A_2 -- ~~are~~ mutually independent

(ii) with A_i^c

$\Rightarrow$

$$P\left(A_i^c \cap \bigcap_{i \in D - \{i\}} A_i\right) = P\left(\bigcap_{i \in D-\{i\}} A_i\right) - P\left(\bigcap_{i \in D-\{i\}} A_i \cap A_i\right) \rightarrow (3)$$

$\boxed{\because P(A \cap B) = P(A) - P(A \cap B^c)}$

Now we know

$$P \bigcap_{i \in D-\{i\}} A_i = \prod_{i \in D-\{i\}} P(A_i) \rightarrow (1)$$

and $P\left(A_i \bigcap_{i \in D-\{i\}} A_i\right) = P(A_i) \cdot P\left(\bigcap_{i \in D-\{i\}} A_i\right) \rightarrow (2)$

$\Rightarrow$ From ①, ②, ③

$$\Rightarrow P\left(A_i^c \cap \bigcap_{j \in D-\{i\}} A_j\right) = \prod_{j \in D-\{i\}} P(A_j) - P(A_i) \cdot \prod_{j \in D-\{i\}} P(A_j)$$

$$= \prod_{j \in (D-\{i\})} P(A_j)\,(1 - P(A_i))$$

$$P\left(A_i^c \cap \bigcap_{j \in D-\{i\}} A_j\right) = \prod_{j \in (D-\{i\})} P(A_j) \cdot P(A_i^c)$$

$A_j = B_j \;;\; B_i = A_i^c$

$$\Rightarrow P\left(B_i \cap \bigcap_{j \in D-\{i\}} B_j\right) = \prod_{j \in D-\{i\}} P(B_j) \cdot P(B_i)$$

$$\Rightarrow \boxed{P\left(\bigcap_{j \in D} B_j\right) = \prod_{j \in D} P(B_j)} \quad D \subseteq \{1, 2, \ldots, n\}$$

$\therefore$ $B = (C \setminus A_i) \cup (A_i^c)$ for any $i \in [1:n]$ is also a collection of mutually independent events.

6) Given $P(E) = \frac{1}{1000}$

Let C = Alice and Bob declare that E occurred

D = E actually occurred

$$P(D|C) = \frac{P(C|D) \cdot P(D)}{P(C|D) \cdot P(D) + P(C|D^c) \cdot P(D^c)}$$

For E occurring

$$= \frac{\frac{1}{1000} \times \frac{9}{10} \times \frac{9}{10}}{\frac{1}{1000} \times \frac{9}{10} \times \frac{9}{10} + \frac{999}{1000} \times \frac{1}{10} \times \frac{1}{10}}$$

For Alice and Bob to declare E occurred

For E not to occure

$$= \frac{81}{81+999} = \frac{81}{1080} = \boxed{0.075}$$

Probability of E actually occurred given that Alice and Bob declare E occurred is 0.075

7)

Let

$P(i)$ is probability of ultimately bankrupted if he starting from i

For getting tails — For getting heads

$$\Rightarrow P(i) = \frac{1}{2} \cdot P(i-1) + \frac{1}{2} \cdot P(i+1)$$

~~For~~ lost one rupee — got one rupee

$$\Rightarrow 2P(i) = P(i-1) + P(i+1)$$

$$\boxed{P(i) - P(i-1) = P(i+1) - P(i)}$$

difference is constant

So Let

$$P(i) = c_1 + c_2 i$$

we know

$$P(0) = 1 \; ; \; P(N) = 0$$

$N = 2000000$

$$\Rightarrow P(0) = c_1 = 1$$

$$P(N) = c_1 + c_2 N = 0 \Rightarrow c_2 = \frac{-1}{N}$$

So

$$\boxed{P(i) = 1 - \frac{i}{N}}$$

$\therefore$ Probability of ultimately bankrupted starting from i rupees

is $P(i) = 1 - \frac{i}{N}$

so

$$P(200) = 1 - \frac{200}{2000000}$$

$$= 0.9999$$

$\therefore$ Probability of ultimately bankrupted

$= 0.9999$