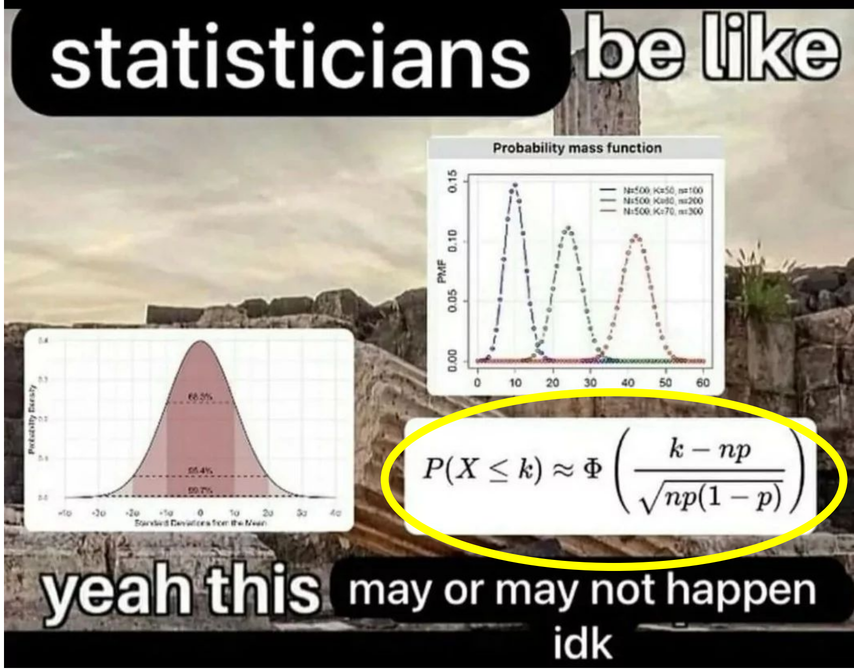
Probability and Random Processes

Tutorial-7

7-11-2025

Agenda

- Quiz-2 Solutions and Queries
- Convergence
- WLLN, SLLN, CLT
- Problems



Quiz 2

(MA6.102) Probability and Random Processes, Monsoon 2025

28 October, 2025

Max. Duration: 45 Minutes

Question 1 (5 marks). Consider two jointly continuous random variables X and Y with joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 2 & \text{if } x > 0, y > 0, \text{ and } x + y \leq 1 \\ 0 & \text{otherwise} \end{cases}.$$

(a) Compute the probability $P(X < Y)$.
(b) Determine the conditional CDF $F_{X|\{X<Y\}}$, and then obtain the conditional PDF $f_{X|\{X<Y\}}$.

Question 2 (5 marks). Let X and Y be jointly continuous random variables with joint PDF

$$f_{X,Y}(x,y) = x + y, \ (x,y) \in [0,1]^2.$$

Define new random variables $Z = X^2$ and $W = X(1+Y)$.
(a) Determine the range (support) of the pair (Z, W), i.e., the subset of $\mathbb{R}^2$ where (Z, W) can take values.
(b) Find the joint PDF $f_{Z,W}$.

Question 3 (5 Marks). Let $M_X(s) = \mathbb{E}[\mathrm{e}^{sX}]$ be finite for $s \in (-c, c)$, for some $c > 0$. Show that

$$\lim_{n\to\infty} \left(M_X\left(\frac{s}{n}\right)\right)^n = \mathrm{e}^{s\mathbb{E}[X]}.$$

Hint: $n \log M_X(\frac{s}{n}) = \frac{\log M_X(\frac{s}{n})}{\frac{1}{n}}$.

Convergence in Distribution

We say a sequence of RVs $X_1, X_2, \ldots$ converges to X in distribution if

$$\lim_{n \to \infty} F_{X_n}(x) = F_X(x),$$

for all points x at which $F_X(x) = P(X \le x)$ is continuous.

Convergence in Mean-Square sense

We say a sequence of RVs $X_1, X_2, \ldots$ converges in mean-square sense to X if

$$\lim_{n \to \infty} E[(X_n - X)^2] = 0.$$

Convergence in Probability

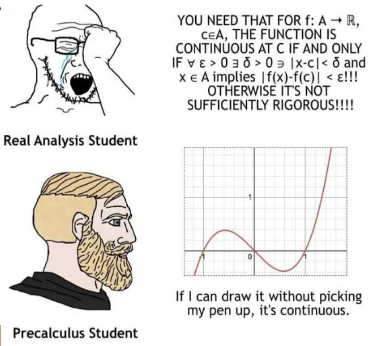
Let $X_1, X_2, \ldots$ be a sequence of RVs on some probability space $(\Omega, \mathcal{F}, P)$. We say $(X_n)_{n \in \mathbb{N}}$ converges to another RV X in probability if

$$\lim_{n \to \infty} P(|X_n - X| > \varepsilon) = 0 \quad \text{for all } \varepsilon > 0.$$

Almost sure Convergence

A sequence of random variables $X_1, X_2, \ldots$ is said to converge almost surely if

$$P(\{\omega : \lim_{n \to \infty} X_n(\omega) = X(\omega)\}) = 1$$



Convergence in Distribution but not in Probability

- In a fair coin toss, the sample space is {H, T} with P(H) = P(T) = 1/2. Consider a sequence of RVs in which all the RVs are the same and map H to 0 and T to 1.
- Consider a 'limit' RV which maps H to 1 and T to 0.
- It is clear that they converge in distribution (They have the exact same CDF)
- However, $P(\{\omega : |X_n(\omega) - X(\omega)| > \varepsilon\}) = 1$ for all $n \in N$ (even for the limit as $n \to \infty$)
- Thus, this does not converge in probability.

The Weak Law of Large Numbers (WLLN)

Let $X_1, X_2, \dots$ be independent identically distributed random variables with mean μ. For every $\epsilon > 0$, we have

$$\mathbf{P}\big(|M_n - \mu| \geq \epsilon\big) = \mathbf{P}\left(\left|\frac{X_1 + \cdots + X_n}{n} - \mu\right| \geq \epsilon\right) \to 0, \qquad \text{as } n \to \infty.$$

The Strong Law of Large Numbers (SLLN)

Let $X_1, X_2, \dots$ be a sequence of independent identically distributed random variables with mean μ. Then, the sequence of sample means $M_n = (X_1 + \cdots + X_n)/n$ converges to μ, *with probability 1*, in the sense that

$$\mathbf{P}\left(\lim_{n\to\infty} \frac{X_1 + \cdots + X_n}{n} = \mu\right) = 1.$$

The difference between the weak and the strong law is subtle and deserves close scrutiny. The weak law states that the probability $\mathbf{P}\big(|M_n - \mu| \geq \epsilon\big)$ of a significant deviation of M_n from μ goes to zero as $n \to \infty$. Still, for any finite n, this probability can be positive and it is conceivable that once in a while, even if infrequently, M_n deviates significantly from μ. The weak law provides no conclusive information on the number of such deviations, but the strong law does. According to the strong law, and with probability 1, M_n converges to μ. This implies that for any given $\epsilon > 0$, the difference $|M_n - \mu|$ will exceed ϵ only a finite number of times.

Source : Intro to Probability by Bertsekas and Tsitsiklis

The Central Limit Theorem

Let $X_1, X_2, \dots$ be a sequence of independent identically distributed random variables with common mean μ and variance σ^2, and define

$$Z_n = \frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}.$$

Then, the CDF of Z_n converges to the standard normal CDF

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2}\, dx,$$

in the sense that

$$\lim_{n\to\infty} \mathbf{P}(Z_n \leq z) = \Phi(z), \qquad \text{for every } z.$$

Normal Approximation Based on the Central Limit Theorem

Let $S_n = X_1 + \cdots + X_n$, where the X_i are independent identically distributed random variables with mean μ and variance σ^2. If n is large, the probability $\mathbf{P}(S_n \leq c)$ can be approximated by treating S_n as if it were normal, according to the following procedure.

1. Calculate the mean $n\mu$ and the variance $n\sigma^2$ of S_n.
2. Calculate the normalized value $z = (c - n\mu)/\sigma\sqrt{n}$.
3. Use the approximation

$$\mathbf{P}(S_n \leq c) \approx \Phi(z),$$

where $\Phi(z)$ is available from standard normal CDF tables.

Problem 1

- An elevator has a maximum weight limit of 10,000 Kg. 100 identical boxes are being loaded into the elevator. The weight of each box is a random variable. We do not know the distribution of the weight of the boxes but we know the mean to be 98 Kg and standard deviation to be 10 Kg of that distribution.
    - What is the expected total weight of the boxes?
    - What is the standard deviation of the total weight?
    - What is the probability that the lift will be overloaded?

Problem 2

Let $X_1, X_2, X_3, \cdots$ be a sequence of i.i.d. $Uniform(0,1)$ random variables. Define the sequence Y_n as

$$Y_n = \min(X_1, X_2, \cdots, X_n).$$

Prove the following convergence results independently (i.e, do not conclude the weaker convergence modes from the stronger ones).

a. $Y_n \xrightarrow{d} 0$.
b. $Y_n \xrightarrow{p} 0$.
c. $Y_n \xrightarrow{L^r} 0$, for all $r \geq 1$.
d. $Y_n \xrightarrow{a.s} 0$.

Borel-Cantelli Lemma

Theorem 7.5
Consider the sequence $X_1, X_2, X_3, \cdots$. If for all $\epsilon > 0$, we have

$$\sum_{n=1}^{\infty} P(|X_n - X| > \epsilon) < \infty,$$

then $X_n \xrightarrow{a.s.} X$.

Problem 3

Problem 1 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space $\Omega = [0, 1]$ with the uniform (Lebesgue) measure $\mathbb{P}$. Let the limit random variable be $X(\omega) = 0$ for all ω.*

We construct a sequence X_k using indicator functions. We first define a sequence of intervals I_k in "rounds".

- ***Round** n $(n = 1, 2, \ldots)$ consists of n disjoint intervals $J_{n,j} = \left[\frac{j-1}{n}, \frac{j}{n}\right]$ for $j = 1, \ldots, n$.*

The full sequence of intervals I_k is formed by listing all intervals from Round 1, then Round 2, then Round 3, and so on. (i.e., $I_1 = J_{1,1}$, $I_2 = J_{2,1}$, $I_3 = J_{2,2}$, $I_4 = J_{3,1}$, $I_5 = J_{3,2}$, $I_6 = J_{3,3}$, $I_7 = J_{4,1}, \ldots$)

For each k, we define the random variable $X_k(\omega) = \mathbb{1}_{I_k}(\omega)$.

Part 1: Convergence in Probability

Show that X_k **converges in probability** to X (i.e., $X_k \xrightarrow{p} X$).

Show that for any $\epsilon > 0$, $\lim_{k\to\infty} \mathbb{P}(|X_k(\omega) - X(\omega)| > \epsilon) = 0$.

1. Use $\epsilon = 0.5$. The 'bad set' is $B_k = \{\omega \mid |X_k(\omega)| > 0.5\} = I_k$.
2. The probability is $\mathbb{P}(B_k) = \mathbb{P}(I_k)$, which is the length of the interval I_k.
3. Show that as $k \to \infty$, the round number $n \to \infty$, and thus length$(I_k) \to 0$.

Part 2: Almost Sure Convergence

Show that X_k **does not converge almost surely** to X.

Show that the set of ω for which $\lim_{k\to\infty} X_k(\omega) = 0$ has probability 0.

1. Pick any $\omega_0 \in [0, 1]$.
2. In any given Round n, ω_0 must belong to exactly one interval $J_{n,j}$.
3. This means $X_k(\omega_0) = 1$ for at least one k in every single round.
4. Therefore, $X_k(\omega_0) = 1$ infinitely often.
5. Conclude that the sequence of numbers $X_k(\omega_0)$ does not converge to 0. Since ω_0 was arbitrary, this holds for all $\omega \in [0, 1]$.

THANK YOU