1. (a)

$$\Omega = \{BB, BG, GB, GG\}, \quad P(\omega) = \frac{1}{4} \ \ \forall\, \omega \in \Omega$$

Let

$$A = \{\text{at least one child is named Lily}\}$$

$$P(\text{Lily} \mid \text{a child}) = \alpha, \quad P(\text{No Lily} \mid \text{a child}) = 1 - \alpha$$

We want:

$$P(A \mid \{GG\}) = P(\text{at least one child named Lily} \mid \{GG\})$$

Method 1

$P(\text{A} \mid \{GG\}) = P(\text{2 children named Lily} \mid \{GG\}) + P(\text{exactly one child named Lily} \mid \{GG\})$

$$= \alpha^2 + \alpha(1-\alpha) + \alpha(1-\alpha) = 2\alpha - \alpha^2$$

Method 2

By Bayes' Theorem:

$$P(A \mid \{GG\}) = \frac{P(A \cap \{GG\})}{P(\{GG\})}$$

$$P(\{GG\}) = \frac{1}{4}, \quad P(A \cap \{GG\}) = \alpha^2 \cdot \frac{1}{4} + 2\alpha(1-\alpha) \cdot \frac{1}{4}$$

$$= \frac{2\alpha - \alpha^2}{4}$$

$$P(A \mid \{GG\}) = \frac{(2\alpha - \alpha^2)/4}{1/4} = 2\alpha - \alpha^2$$

Method 3

$$P(A \mid \{GG\}) = 1 - P(\text{none of children is named Lily})$$

$$= 1 - (1-\alpha)(1-\alpha) = 1 - (1-\alpha)^2$$

$$= 2\alpha - \alpha^2$$

Marking Scheme

- 0.5 mark for any decent attempt or slight mistakes while considering both children's probabilities.
- 1 mark for correct answer.

(b)

Method 1

All possible cases:

$$G_L G,\ G G_L,\ G_L G_L,\ G_L B,\ B G_L$$

By the Total Probability Theorem:

$$P(A) = P(A \mid GG)P(GG) + P(A \mid GB)P(GB) + P(A \mid BG)P(BG) + P(A \mid BB)P(BB)$$

From part (a),

$$P(A \mid G_1G_2) = (2\alpha - \alpha^2), \quad P(G_1G_2) = \frac{1}{4}$$

Hence,

$$P(A) = (2\alpha - \alpha^2) \cdot \frac{1}{2} \cdot \frac{1}{2} + \alpha \cdot \frac{1}{2} \cdot \frac{1}{2} + \alpha \cdot \frac{1}{2} \cdot \frac{1}{2} + 0$$

$$= \frac{4\alpha - \alpha^2}{4}$$

(it's ok if 0 term isn't mentioned)

Marking Scheme

- Correct answer $\rightarrow$ 2 marks
- Cases chosen Total Probability Theorem applied correctly (even if the repetition of necessary cases was missed) $\rightarrow$ 0.5 mark
- If 2 terms out of the expression correct $\rightarrow$ 0.5 mark
- All cases chosen but calculation mistake $\rightarrow$ also given partial credit

Technically, if the answer is wrong, you've been graded out of 1.5 marks based on the approach and to what extent you solved the question, the above is a rough idea. Highly depends on the attempt made.

Method 2

$$P(A) = 1 - P(\text{no Lily})$$

$$= 1 - (1-\alpha)(1-\alpha) \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} - (1-\alpha) \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} - (1-\alpha) \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} - \tfrac{1}{2} \cdot \tfrac{1}{2}$$

$$= \frac{4\alpha - \alpha^2}{4}$$

Again, same marking scheme: marks deducted if any case is forgotten.

Method (3)

$$P(\text{child is a girl named Lily}) = \tfrac{1}{2} \cdot \alpha$$

$$P(\text{child is not a girl named Lily}) = 1 - \tfrac{\alpha}{2}$$

$$P(\text{no Lily at all}) = \left(1 - \tfrac{\alpha}{2}\right)^2$$

$$P(A) = 1 - \left(1 - \tfrac{\alpha}{2}\right)^2$$

$$= \alpha - \frac{\alpha^2}{4}$$

(c)

Using Bayes' Theorem

$$P(\{G_1G_2\} \mid A) = \frac{P(A \mid \{G_1G_2\})\, P(\{G_1G_2\})}{P(A)}$$

From part (a),

$$P(A \mid \{G_1G_2\}) = 2\alpha - \alpha^2, \quad P(\{G_1G_2\}) = \tfrac{1}{4}$$

From part (b),

$$P(A) = \frac{4\alpha - \alpha^2}{4}$$

Thus,

$$P(\{G_1G_2\} \mid A) = \frac{(2\alpha - \alpha^2) \cdot \frac{1}{4}}{(4\alpha - \alpha^2)/4} = \frac{2\alpha - \alpha^2}{4\alpha - \alpha^2} = \frac{2 - \alpha}{4 - \alpha}$$

Marking Scheme

- 1 mark for correct Bayes' theorem application
- 0.5 for numerator correct
- 0.5 for denominator correct
- It's ok if you don't simplify the final answer, no marks deducted for that in any part of the question.

Q2. Solution

(a) - 2 marks

The pre-image sets for the random variable X, $\{X \leq x\}$, are defined as a piecewise function of x.

$$\{X \leq x\} = \begin{cases} \emptyset & \text{if } x < 0 \\ \{0\} & \text{if } 0 \leq x < 1 \\ \{-1, 0, 1\} & \text{if } 1 \leq x < 2 \\ \{-2, -1, 0, 1, 2\} & \text{if } 2 \leq x < 3 \\ \Omega \text{ or } \{\text{-3,-2,-1,0,1,2,3}\} & \text{if } x \geq 3 \end{cases}$$

(b) and (c) - 3 marks

For X to be a random variable, the smallest σ-field, $\mathcal{F}$ (which is $\sigma(X)$), must contain all the pre-image sets calculated in (a).

This σ-field is the same as the one generated by the mutually exclusive and exhaustive "atoms" (partition events) derived from the sets in (a):

- $E_1 = \{\mathbf{0}\}$
- $E_2 = \{-1, 0, 1\} \setminus \{0\} = \{\mathbf{1}, -\mathbf{1}\}$
- $E_3 = \{-2, -1, 0, 1, 2\} \setminus \{-1, 0, 1\} = \{\mathbf{2}, -\mathbf{2}\}$
- $E_4 = \Omega \setminus \{-2, -1, 0, 1, 2\} = \{\mathbf{3}, -\mathbf{3}\}$

The smallest σ-field $\mathcal{F}$ is the set of all possible unions of these 4 atoms, i.e.

$$\mathcal{F} = \left\{ \bigcup_{i \in I} E_i ; I \subseteq \{1, 2, 3, 4\} \right\}$$

Since there are 4 atoms in the generating partition, the cardinality of $\mathcal{F}$ is $|\mathcal{F}| = 2^4 = 16$.

Q3. Solution

We first define the probability space $(\Omega, \mathcal{F}, P)$:

- $\Omega = \mathbb{R}$ [1 mark]
- $\mathcal{F}$ = Borel σ-algebra on $\mathbb{R}$ [1 Mark]
- $P((a, b]) = F(b) - F(a)$ [1 mark]
  (This extends to any Borel set as any set can be written as a countable union of sets of the form $(a, b]$ and their complements & the probability law above can be used on it)

This construction satisfies all the axioms of probability. [Proving these axioms fetches 0.5 marks]

Define the random variable $\mathbf{X}$ based on the sample space:

$$\mathbf{X}(\omega) = \omega, \quad \text{for } \omega \in \mathbb{R}$$

Now, we find the cumulative distribution function (CDF) of $\mathbf{X}$:

$$\begin{aligned}
P(\mathbf{X} \leq x) &= P(\{\omega \in \Omega : \mathbf{X}(\omega) \leq x\}) \\
&= P(\{\omega \in \mathbb{R} : \omega \leq x\}) \quad (\text{Since } \mathbf{X}(\omega) = \omega) \\
&= P((-\infty, x]) \\
&= P\left(\bigcup_{n=1}^{\infty} (-n, x]\right) \quad (\text{Writing the interval as an increasing union}) \\
&= \lim_{n \to \infty} P((-n, x]) \quad [\text{By continuity of probability}] \\
&= \lim_{n \to \infty} [F(x) - F(-n)] \quad (\text{Applying the definition of P}) \\
&= F(x) - \lim_{n \to \infty} F(-n) \\
&= F(x) - 0 \quad (\text{Property of F given in the question, } \lim_{y \to -\infty} F(y) = 0) \\
&= F(x). \quad [\text{1.5 marks for proving that the constructed RV } \mathbf{X} \text{ has F(x) as the CDF}]
\end{aligned}$$