Assignment-2

2025122010
M. Saipoojith

MA6.102 Probability and Random Processes

Problem 1:

Given $\Omega = \{-2, -1, 0, 1, 2\}$

(a) Smallest sigma field with $X(\omega) = \omega^2$

$X(-2) = (-2)^2 = 4$

$X(-1) = (-1)^2 = 1$

$X(0) = 0$

$X(1) = 1$

$X(2) = 4$

$\Rightarrow$ $X^{-1}(4) = \{-2, 2\} = A_1$

$X^{-1}(1) = \{-1, 1\} = A_2$

$X^{-1}(0) = \{0\} = A_3$

Disjoint sets

$\therefore$ Smallest σ-field is

$$\boxed{\bigcup_{i \in B} A_i \quad \text{where } B \subseteq \{1, 3\}}$$

$\hookrightarrow$ Union of all subsets.

$\Rightarrow$ Smallest σ-field is

$\therefore \{\emptyset, \Omega, \{0\}, \{-2, 2\}, \{-1, 1\}, \{0, -1, 1\}, \{0, -2, 2\}, \{-1, 1, -2, 2\}\}$

Smallest σ-field for $X(\omega) = \omega^2$

$\hookrightarrow 2^3 = 8$ elements.

(b) $X(\omega) = \omega + 1$

$X(-2) = -1$
$X(-1) = 0$
$X(0) = 1$
$X(1) = 2$
$X(2) = 3$

$\Rightarrow$ $X^{-1}(-1) = \{-2\} = A_1$
$X^{-1}(0) = \{-1\} = A_2$
$X^{-1}(1) = \{0\} = A_3$
$X^{-1}(2) = \{1\} = A_4$
$X^{-1}(3) = \{2\} = A_5$

$\Rightarrow \Omega = \{A_1, A_2, A_3, A_4, A_5\}$

→ all A_i are disjoint sets

so smallest sigma field is ~~entire power set of Ω~~

$\Rightarrow \boxed{\bigcup_{i \in B} A_i \text{ where } B \subseteq \{1,2,3,4,5\}}$

→ every subset of $(A_1, A_2, A_3, A_4, A_5)$ is σ-field. [union of all subsets]

$\boxed{\begin{array}{l} \therefore \text{ smallest } \sigma\text{-field is} \\ \bigcup_{i \in B} A_i \text{ where } B \subseteq (1,2,3,4,5) \\ 2^5 = 32 \text{ subsets of } \Omega \end{array}}$

Problem - 2 (a)

Given $\Omega = [0,1] \times [0,1]$ be the unit square

$P(A) = \text{area of } A$ for any $A \subseteq \Omega$

$X(\omega) =$ distance from ω to nearest edge

$\omega = (a, y) \in [0,1]^2$

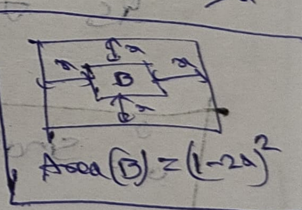
$$\Rightarrow X(a,y) = \min(a-0,\, y-0,\, 1-a,\, 1-y)$$

$$\Rightarrow \{X \le a\} = \begin{cases} \emptyset & a < 0 \\ \{(a,y) \in [0,1]^2 : \min(a, y, 1-a, 1-y) \le a\} & 0 \le a < \frac{1}{2} \\ [0,1]^2 & a \ge \frac{1}{2} \end{cases}$$

when $\boxed{0 \le a < \frac{1}{2}}$

$$\Rightarrow F_X(a) = P(X \le a)$$

$$= 1 - P(X > a)$$ Area which has min side length $> a$



$$= 1 - (1-2a)^2$$

$$= 1 - (1 + 4a^2 - 4a)$$

$$= 4a(1-a)$$

$$\boxed{F_X(a) = 4a(1-a)}$$

$\therefore$ CDF F_X is

$$F_X(a) = \begin{cases} 0 & a < 0 \\ 4a(1-a) & 0 \le a < \frac{1}{2} \\ 1 & a \ge \frac{1}{2} \end{cases}$$

Problem-35

For a function to be valid CDF

it should satisfy

1) $\lim_{x\to\infty} F_X(x) = 1$; $\lim_{x\to-\infty} F_X(x) = 0$

2) if $x \le y \Rightarrow F_X(x) \le F_X(y)$

3) $\lim_{\varepsilon\to 0^+} F_X(x+\varepsilon) = F_X(x)$

For $\boxed{(a)\quad 1-(1-F_X(x))^{\gamma}}$

(i) $\lim_{x\to\infty} 1-(1-F_X(x))^{\gamma} = 1-(1-\lim_{x\to\infty} F_X(x))^{\gamma}$

$= 1-(1-1)^{\gamma} = 1$

and $\lim_{x\to-\infty} 1-(1-F_X(x))^{\gamma} = 1-(1-\lim_{x\to-\infty} F_X(x))^{\gamma}$

$= 1-(1-0)^{\gamma} = 0$

(ii) if $x < y$

$F_X(x) \le F_X(y)$

$\Rightarrow (1-F_X(x))^{\gamma} \ge (1-F_X(y))^{\gamma}$

$\Rightarrow \boxed{1-(1-F_X(x))^{\gamma} \le 1-(1-F_X(y))^{\gamma}}$

so it is inc

(iii)

$\lim_{\varepsilon\to 0^+} 1-(1-F_X(x+\varepsilon))^{\gamma}$

$= 1-(1-\lim_{\varepsilon\to 0^+} F_X(x+\varepsilon))^{\gamma}$

$= 1-(1-F_X(x))^{\gamma}$,

right continuous

$\boxed{\therefore\ 1-(1-F_X(x))^{\gamma} \text{ is CDF}}$

For (b) $F_X(x) + (1-F_X(x)) \log(1-F_X(x))$

Let $g(x) = F_X(x) + (1-F_X(x)) \log(1-F_X(x))$

i) $\lim_{x\to\infty} g(x) = \lim_{x\to\infty} F_X(x) + (1 - \lim_{x\to\infty} F_X(x)) \log(1 - \lim_{x\to\infty} F_X(x))$

$= 1 + (1-1) \log(1 - \lim_{x\to\infty} F_X(x))$

$\boxed{\lim_{x\to\infty} g(x) = 1}$

$\lim_{x\to-\infty} (g(x)) = \lim_{x\to-\infty} F_X(x) + (1 - \lim_{x\to-\infty} F_X(x)) \cdot \log(1 - \lim_{x\to-\infty} F_X(x))$

$= 0 + (1-0) \cdot \log(1-0)$

$= \log(1) = 0$

$\Rightarrow \boxed{\lim_{x\to-\infty} g(x) = 0}$

ii) if $x < y$

$F_X(x) \le F_Y(y)$

$\Rightarrow 1 - F_X(x) \ge 1 - F_Y(y)$

$\because \log(1-F_X(x)) < 0$

$\Rightarrow$

$\log(1-F_X(x)) < \log(1-F_X(y))$
(-ve) (-ve)

$\boxed{\log(1-F_X(x)) \cdot (1-F_X(x)) \le (1-F_Y(y)) \cdot \log(1-F_Y(y))}$

$\Rightarrow \boxed{F_X(x) + \log(1-F_X(x)) \cdot (1-F_X(x)) \le F_X(y) + \log(1-F_X(y)) (1-F_X(y))}$

inc (non-dec function)

iii) $\lim_{\varepsilon \to 0^+} g(a+\varepsilon)$

$$= \lim_{\varepsilon \to 0^+} \left(F_X(a+\varepsilon) + (1 - F_X(a+\varepsilon)) \log(1 - F_X(a+\varepsilon))\right)$$

$$= \lim_{\varepsilon \to 0^+} \left(F_X(a+\varepsilon)\right) + \left(1 - \lim_{\varepsilon \to 0^+} F_X(a+\varepsilon)\right) \log\left(1 - \lim_{\varepsilon \to 0^+} F_X(a+\varepsilon)\right)$$

$$= F_X(a) + (1 - F_X(a)) \log(1 - F_X(a))$$

$$= g(a)$$

$$\boxed{\therefore \lim_{\varepsilon \to 0^+} g(a+\varepsilon) = g(a)}$$

so $g(a)$ is valid CDF

$$\Rightarrow \boxed{\therefore F_X(a) + (1 - F_X(a)) \log(1 - F_X(a)) \text{ is a valid CDF}}$$

and $\boxed{1 - (1 - F_X(a))^{\gamma} \text{ is also a valid CDF}}$

Problem 4r

we have to show

$$E[N] = \sum_{i=1}^{\infty} P(N \geq i)$$

$\Rightarrow$ ~~$E[N]$~~ $= \frac{RHS}{L}$

$$\sum_{i=1}^{\infty} P(N \geq i) = \sum_{i=1}^{\infty} \sum_{n=i}^{\infty} P(N = n)$$

$$= \sum_{n=1}^{\infty} \sum_{i=1}^{n} P(N = n)$$

$$\sum_{i=1}^{\infty} P(N \geq i) = \sum_{n=1}^{\infty} P(N = n) \cdot \sum_{i=1}^{n} 1$$

$$\sum_{i=1}^{\infty} P(N \geq i) = \sum_{n=1}^{\infty} P(N = n) \cdot n$$

$$\sum_{i=1}^{\infty} P(N \geq i) = \sum_{n=1}^{\infty} n \cdot P(N = n)$$

$$\sum_{i=1}^{\infty} P(N \geq i) = E[N]$$

$$\boxed{\therefore E[N] = \sum_{i=1}^{\infty} P(N \geq i)}$$

Problem-5:

we have to give non-constant R.V

Sol: $E\left[\frac{1}{X}\right] = \frac{1}{E[X]}$

we can take

$E\left[\frac{1}{X}\right] = E[X] = 1$

and $X = \{-1, 3, \frac{1}{3}\}$ (assume)

Let $P\{X = -1\} = P_1$

and $P\{X = 3\} = P\{X = \frac{1}{3}\} = P_2$

$\Rightarrow$ $P\{X = -1\} + P\{X = 3\} + P\{X = \frac{1}{3}\} = 1$

$$\boxed{P_1 + 2P_2 = 1} \rightarrow (1)$$

and

For $E[X] = 1$

$\hookrightarrow \sum_a a \cdot P(X=a) = 1$

$$\Rightarrow \boxed{-P_1 + 3P_2 + \frac{1}{3}P_2 = 1}$$

For $E\left[\frac{1}{X}\right] = 1$

$\hookrightarrow \sum_a \frac{1}{a} P(X=a) = 1$

$$\boxed{-P_1 + \frac{1}{3}P_2 + 3P_2 = 1} \rightarrow (2)$$

From ①, ②

$$-1 + 2P_2 + \frac{1}{3}P_2 + 3P_2 = 1$$

$$\left(5 + \frac{1}{3}\right)P_2 = 2$$

$$\frac{16 P_2}{3} = 2$$

$$P_2 = \frac{3}{8} \quad ; \quad P_1 = \frac{2}{8}$$

$\therefore$ we can take

$\Omega = \{A_1, A_2, A_3\}$ and R.V $\to X$ as

$$\boxed{\begin{array}{lll} X(A_1) = -1 & & P\{X=-1\} = \frac{2}{8} \\ X(A_2) = 3 & \text{and} & P\{X=3\} = \frac{3}{8} \\ X(A_3) = \frac{1}{3} & & P\{X=-3\} = \frac{3}{8} \end{array}}$$

$X = \{-1, 3, \frac{1}{3}\}$

↳ X will satisfy

$$E\left[\frac{1}{X}\right] = \frac{1}{E[X]}$$

$\boxed{X \to \text{non constant R.V}}$

Problem 6 r

(a) binomial R.v

$$\boxed{P_X(k) = {}^nC_k \cdot p^k \cdot (1-p)^{n-k}} \quad k = 0, 1, \ldots n$$

$$\frac{P_X(k+1)}{P_X(k)} = \frac{{}^nC_{k+1} \cdot p^{k+1} \cdot \cancel{(1-p)}^{n-k-1}}{{}^nC_k \cdot \cancel{p^k} \cdot (1-p)^{n-k}}$$

$$= \boxed{\frac{{}^nC_{k+1}}{{}^nC_k} \cdot \frac{p}{1-p}} = \frac{\dfrac{n!}{(k+1)!\,(n-k-1)!}}{\dfrac{n!}{k! \times (n-k)!}} \times \frac{p}{1-p}$$

$$\frac{P_X(k+1)}{P_X(k)} = \frac{n-k}{k+1} \cdot \frac{p}{1-p}$$

Similarly $\quad \dfrac{P_X(k)}{P_X(k-1)} = \dfrac{n-k+1}{k} \cdot \dfrac{p}{1-p}$

$$\Rightarrow \frac{P_X(k+1)}{P_X(k)} \cdot \frac{P_X(k-1)}{P_X(k)} = \frac{n-k}{k+1} \cdot \frac{\cancel{p}}{\cancel{1-p}} \cdot \frac{k}{(n-k+1)} \frac{\cancel{1-p}}{\cancel{p}}$$

$$= \frac{k\,(n-k)}{(k+1)\,(n-k+1)} \leq 1 \qquad (\leq 1,\ \leq 1)$$

$$\boxed{\therefore P_X(k-1)\, P_X(k+1) \leq P_X(k)^2}$$

For binomial R.v

99) For Poisson R.V

$\hookrightarrow P_X(k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$

$$\frac{P_X(k+1)}{P_X(k)} = \frac{\cancel{e^{-\lambda}} \cdot \frac{\lambda^{k+1}}{(k+1)!}}{\cancel{e^{-\lambda}} \cdot \frac{\lambda^k}{k!}} = \frac{\lambda}{k+1}$$

$$\Rightarrow \boxed{\frac{P_X(k+1)}{P_X(k)} = \frac{\lambda}{k+1}}$$

Simillarly

$$\frac{P_X(k)}{P_X(k-1)} = \frac{\lambda}{k}$$

$$\Rightarrow \frac{P_X(k+1)}{P_X(k)} \cdot \frac{P_X(k-1)}{P_X(k)} = \frac{\cancel{\lambda}}{k+1} \cdot \frac{k}{\cancel{\lambda}}$$

$$= \frac{k}{k+1} \leq 1$$

$$\Rightarrow \frac{P_X(k+1)\, P_X(k-1)}{P_X^2(k)} \leq 1$$

$$\boxed{\therefore P_X(k+1) \cdot P_X(k-1) \leq P_X(k)^2}$$

For Poisson R.V

Eg 2 We can take geometric R.V

$$\boxed{P_X(k) = (1-p)^{k-1} \cdot p} \quad k = 1, 2 \ldots$$

$$P_X(k-1) \cdot P_X(k+1) = (1-p)^{k-2} \cdot p \cdot (1-p)^{k} \cdot p$$

$$= (1-p)^{2k-2} \cdot p^2$$

$$= (P_X(k))^2$$

$\therefore$ For geometric R.V

$\hookrightarrow P_X(k) = (1-p)^{k-1} \cdot p$

$$\boxed{P_X(k-1) \cdot P_X(k+1) = (P_X(k))^2}$$

Problem-7

Poisson R.V

$\hookrightarrow P_X(k) = \dfrac{e^{-\lambda} \cdot \lambda^k}{k!}$

Let $\boxed{g(X) = t^X}$ $\quad k = 0, 1, 2 \ldots \infty$

By LOTUS

$$E[g(X)] = \sum_{a} t^{a} \cdot P(X=a)$$

$$= \sum_{a} t^{a} \cdot \frac{e^{-\lambda} \cdot \lambda^{a}}{a!}$$

$$= e^{-\lambda} \sum_{a=0}^{\infty} \frac{(\lambda t)^{a}}{a!}$$

$$= e^{-\lambda} \cdot e^{\lambda t}$$

$$\boxed{E(g(X)) = e^{\lambda(t-1)} \quad \text{for } g(X) = t^X}$$

$$\boxed{\because \sum_{k=0}^{\infty} \frac{a^k}{k!} = e^{a}}$$

For $g(x) = t^x$

$$\boxed{E[g(x)] = e^{\lambda(t-1)}} \rightarrow (1)$$

For

(a) $g(x) = e^{-3x}$

From (1)

$$E[g(x)] = e^{\lambda(e^{-3}-1)}$$

$$\neq \theta = e^{-3\lambda}$$

So $\boxed{g(x) = e^{-3x} \text{ is biased}}$

(b) $g(x) = (-2)^x$

From (1)

$$E[g(x)] = e^{\lambda(-2-1)}$$

$$= e^{-3\lambda}$$

$$= \theta$$

$\therefore$ $g(x) = (-2)^x$ is unbiased

$\therefore$ $g(x) = e^{-3x}$ is biased

$g(x) = (-2)^x$ is unbiased