1. Sol:- $E[X] > 100\,E[Y]$

$$X = \begin{cases} 0 & \text{w.p. } 99/100 \\ 10^5 & \text{w.p. } 1/100 \end{cases}$$

$Y = 1$ (a constant)

$E[Y] = 1$

$E[X] = 10^3$

$$\underbrace{10^3}_{E[X]} > \underbrace{100}_{100\,E[Y]}$$

$$P(Y > X) = \sum_{(x,y):\, x<y} p_X(x)\, p_Y(y)$$

$$= \sum_{x:\, x<1} p_X(x)$$

$$= p_X(0)$$

$$= \frac{99}{100}.$$

So, the given statement is false.

2. sol:- N takes values 1 2 3 - - -

$$P(N=1) = P(X_1 > 2025)$$
$$= 1 - F_X(2025).$$

$$P(N=2) = P(X_1 \le 2025,\ X_2 > 2025)$$
$$= P(X_1 \le 2025)\, P(X_2 > 2025)$$
$$= F_X(2025)\left(1 - F_X(2025)\right).$$

$$P(N=k) = P(X_1 \le 2025 \ - - - - \ X_{k-1} \le 2025,\ X_k > 2025)$$
$$= F_X(2025)^{k-1}\left(1 - F_X(2025)\right).$$

N is Geometric RV with probability of success in each trial $= 1 - F_X(2025)$.

Let $p = 1 - F_X(2025)$.

$$E[N] = \sum_{n=1}^{\infty} n\, p_N(n) = \sum_{n=1}^{\infty} n\,(1-p)^{n-1} p$$
$$= \cancel{p} \cdot {}^{+1}\!/_{p\cancel{2}} = 1/p = \frac{1}{1 - F_X(2025)}.$$

3. Sol':- (a) $\mathbb{1}\{X>Y\}+\mathbb{1}\{Y>Z\}+\mathbb{1}\{Z>X\}$

$$\in \{1, 2\}$$

(b) $\mathbb{1}\{X>Y\}+\mathbb{1}\{Y>Z\}+\mathbb{1}\{Z>X\} \leq 2$

$$\Rightarrow E\left[\mathbb{1}\{X>Y\}+\mathbb{1}\{Y>Z\}+\mathbb{1}\{Z>X\}\right] \leq 2$$

$$\Rightarrow P(X>Y)+P(Y>Z)+P(Z>X) \leq 2.$$

If $a+b+c \leq 2$ then $\min\{a, b, c\} \leq \frac{2}{3}$.

Suppose $\min\{a, b, c\} > \frac{2}{3} \Rightarrow a, b, c > \frac{2}{3}$

$$\Rightarrow a+b+c > 2$$

(a contradiction)

$$\therefore \min\{P(X>Y), P(Y>Z), P(Z>X)\} \leq \frac{2}{3}.$$

4. sol:-

$$M_X(s) = \left(\frac{1}{2}e^{2s} + \frac{1}{2}e^{4s}\right)^7,$$

$$M_Y(s) = e^{8(e^s-1)}$$

$$X = \sum_{i=1}^{7} X_i, \quad X_i\text{'s are i.i.d. with}$$

$$P_{X_i}(2) = \frac{1}{2} = P_{X_i}(4), \quad i = 1 \ldots 7.$$

X takes only even values.

$$\Rightarrow P_X(13) = 0.$$

$$E[X] = \sum_{i=1}^{7} E[X_i] = 3 \times 7 = 21.$$

$$P(X+Y=15) = P(X=14, Y=1)$$

(∵ min value X takes is 14)

$$= P(X=14)\,P(Y=1)$$

$$= \prod_{i=1}^{7} P_{X_i}(2)\;P(Y=1)$$

$$= \left(\frac{1}{2}\right)^7 \cdot e^{-8} \cdot 8.$$

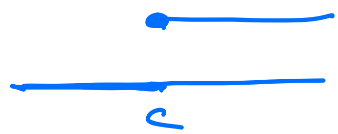
5. sol:- $X_n \xrightarrow{D} c$.

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x) \qquad X = c$$

$$P(|X_n - c| > \varepsilon) = P(X_n - c > \varepsilon \text{ or } X_n - c < -\varepsilon)$$

$$= P(X_n > c+\varepsilon) + P(X_n < c-\varepsilon)$$

$$\leq 1 - F_{X_n}(c+\varepsilon) + P(X_n \leq c-\varepsilon)$$



$$\lim_{n\to\infty} F_{X_n}(c+\varepsilon) = \lim_{n\to\infty} F_X(c+\varepsilon) = 1$$

$$\lim_{n\to\infty} F_{X_n}(c-\varepsilon) = \lim_{n\to\infty} F_X(c-\varepsilon) = 0.$$

so

$$\lim_{n\to\infty} P(|X_n - c| > \varepsilon) \leq 1 - 1 + 0 = 0$$

$$\Rightarrow X_n \xrightarrow{P} c.$$

6. sol:-

Suppose $\lim_{n\to\infty} E\left[\frac{|X_n|}{1+|X_n|}\right] = 0$,

consider, for $\varepsilon > 0$,

$$P(|X_n - 0| > \varepsilon) = P(|X_n| > \varepsilon)$$

$$= P\left(\frac{|X_n|}{1+|X_n|} > \frac{\varepsilon}{1+\varepsilon}\right)$$

$\left(\because f(x) = \frac{x}{1+x} \text{ is increasing in } x\right)$

$$\leq P\left(\frac{|X_n|}{1+|X_n|} \geq \frac{\varepsilon}{1+\varepsilon}\right)$$

$\left(\because \{X > x\} \Rightarrow \{X \geq x\}\right)$

$$\leq E\left[\frac{|X_n|}{1+|X_n|}\right] \cdot \frac{1+\varepsilon}{\varepsilon}$$

$$\Rightarrow \lim_{n\to\infty} P(|X_n| > \varepsilon) \leq \lim_{n\to\infty} E\left[\frac{|X_n|}{1+|X_n|}\right] \cdot \frac{1+\varepsilon}{\varepsilon}$$

$$= 0.$$

$$\therefore X_n \xrightarrow{P} 0.$$

Suppose $X_n \xrightarrow{P} 0$, i.e., $\lim_{n\to\infty} P(|X_n|>\varepsilon)=0$, for every $\varepsilon>0$.

$$E\left[\frac{|X_n|}{1+|X_n|}\right] =$$

$$E\left[\frac{|X_n|}{1+|X_n|} \,\Big|\, |X_n|>\varepsilon\right] P(|X_n|>\varepsilon) +$$

$$E\left[\frac{|X_n|}{1+|X_n|} \,\Big|\, |X_n|\le\varepsilon\right] P(|X_n|\le\varepsilon)$$

$$\le 1 \cdot P(|X_n|>\varepsilon) + \frac{\varepsilon}{1+\varepsilon}\cdot 1$$

$$\left(\because \frac{|x_n|}{1+|x_n|} \le 1, \ \& \ |X_n|\le\varepsilon \Rightarrow \frac{|x_n|}{1+|x_n|} \le \frac{\varepsilon}{1+\varepsilon}\right)$$

$$\lim_{n\to\infty} E\left[\frac{|X_n|}{1+|X_n|}\right] \le \lim_{n\to\infty} P(|X_n|>\varepsilon) + \frac{\varepsilon}{1+\varepsilon}$$

$$= \frac{\varepsilon}{1+\varepsilon}$$

Since this is true for every $\varepsilon>0$ we have

$$\Rightarrow \lim_{n\to\infty} E\left[\frac{|x_n|}{1+|x_n|}\right] = 0$$

7. Sol:- $X_t = U\cos t + V\sin t \quad t \in (-\infty, \infty)$

$E[U] = -2 \cdot \frac{1}{3} + 1 \cdot \frac{2}{3} = 0$

$E[V] = -2 \cdot \frac{1}{3} + 1 \cdot \frac{2}{3} = 0$

$E[UV] = E[U]E[V] = 0.$

$E[U^2] = E[V^2]$

$= 4 \cdot \frac{1}{3} + 1 \cdot \frac{2}{3}$

$= 2.$

$\mu_x(t) = E[X_t] = E[U\cos t + V\sin t]$

$= E[U]\cos t + E[V]\sin t$

$= 0.$

$R_x(t_1, t_2) = E[X_{t_1} X_{t_2}]$

$= E[(U\cos t_1 + V\sin t_1)(U\cos t_2 + V\sin t_2)]$

$= E[U^2]\cos t_1 \cos t_2 + E[V^2]\sin t_1 \sin t_2$
$+ E[UV](\cos t_1 \sin t_2 + \sin t_1 \cos t_2)$

$= 2\cos(t_1 - t_2) + 0 \longrightarrow$ fn of $(t_1 - t_2)$.

$\therefore X_t$ is WSS

$E[U^3] = E[V^3]$

$= \frac{1}{3}(-8) + \frac{2}{3} \cdot 1 = -2.$

$E[X_t^3] = E[(U\cos t + V\sin t)^3]$

$= E[U^3\cos^3 t + V^3\sin^3 t + 3U^2V\cos^2 t\sin t$

$+ 3UV^2\cos t\sin^2 t]$

$= -2(\cos^3 t + \sin^3 t) + 3\underbrace{E[U^2V]}_{=0}\cos^2 t\sin t$

$+ 3\underbrace{E[UV^2]}_{=0}\cos t\sin^2 t$

$= -2(\cos^3 t + \sin^3 t).$

$\Rightarrow E[X_t^3]$ depends on t.

If X_t would have been SSS then $E[X_t^3]$ should not depend on t because F_{X_t} should be the same for every $t \in \mathbb{R}$.

$\Rightarrow X_t$ is not SSS.

8. Sol:-

(a) FALSE

(b) FALSE

(c) TRUE

(d) TRUE

(e) FALSE