End-Semester Examination

(MA6.102) Probability and Random Processes, Monsoon 2025

28 November, 2025

Max. Duration: 3 Hours

Question 1 (3 Marks). Prove or disprove the following statement.
"If X and Y are discrete random variables such that $\mathbb{E}[X] > 100\mathbb{E}[Y]$, then $P(Y > X) < 0.99$."
Hint: Consider a random variable that is almost always zero but takes an extremely large value on rare occasions (similar to a lottery outcome).

Question 2 (2 Marks). Alex is selling his mobile phone and decides to accept the first offer that exceeds Rs. 2025. Assume that the offers are independent random variables with common cumulative distribution function (CDF) F_X. Find the probability mass function (PMF) of the number of offers N received before the phone is sold (including the accepted offer). Also compute $\mathbb{E}[N]$ (A direct answer for $\mathbb{E}[N]$ without derivation will not receive any marks).

Question 3 (3 Marks). Let X, Y, and Z be jointly discrete random variables such that, with probability 1, all three take distinct values.
(a) Find the value of $\mathbb{1}\{X > Y\} + \mathbb{1}\{Y > Z\} + \mathbb{1}\{Z > X\}$, where $\mathbb{1}\{X > Y\}(\omega) = 1$ if $X(\omega) > Y(\omega)$, and 0 otherwise.
(b) Obtain a non-trivial numerical upper bound on $\min\{P(X > Y), P(Y > Z), P(Z > X)\}$.

Question 4 (5 Marks). The moment generating functions (MGFs) associated with two independent discrete random variables X and Y are

$$M_X(s) = \left(\frac{1}{2}\mathrm{e}^{2s} + \frac{1}{2}\mathrm{e}^{4s}\right)^7, \quad M_Y(s) = \mathrm{e}^{8(\mathrm{e}^s - 1)}.$$

Compute $P_X(13)$, $\mathbb{E}[X]$, and $P(X + Y = 15)$.
Hint: If $M_X(s) = M_Y(s)$ for $s \in (-\epsilon, \epsilon)$, then $F_X(t) = F_Y(t)$, for all $t \in \mathbb{R}$. The MGF of Poisson(λ) random variable is given by $\mathrm{e}^{\lambda(\mathrm{e}^s - 1)}$.

Question 5 (3 Marks). Prove or disprove the following statement.
"If a sequence $(X_n)_{n\in\mathbb{N}}$ converges to c (a constant) in distribution, then $(X_n)_{n\in\mathbb{N}}$ also converges to c in probability."

Question 6 (5 Marks). Show that a sequence $(X_n)_{n\in\mathbb{N}}$ converges to 0 in probability if and only if

$$\mathbb{E}\left[\frac{|X_n|}{1 + |X_n|}\right] \to 0 \text{ as } n \to \infty. \tag{1}$$

Hint: Use monotonicity of a specific function, Markov's inequality, total expectation theorem, and the fact that $\frac{|x|}{1+|x|} \leq 1$.

Question 7 (4 Marks). Consider a random process $(X_t; t \in (-\infty, \infty))$ defined by

$$X_t = U \cos t + V \sin t, \tag{2}$$

where U and V are independent random variables, each taking values -2 and 1 with probabilities $\frac{1}{3}$ and $\frac{2}{3}$, respectively. Determine whether X_t is wide-sense stationary (WSS). If it is WSS, is it also strict-sense stationary (SSS)?
Hint: Analyze $\mathbb{E}[X_t^2]$ and $\mathbb{E}[X_t^3]$.

Question 8 (5 Marks). For each of the statements below, write True or False.

*Note: Explanations are **not** required; provide only "True" or "False". Make sure your answer to this question fits on a single page of the answer sheet. Answers that extend beyond one page will NOT be evaluated.*

$P(A \cap C^c) \leq P(B \cap C^c)$

(a) It is possible to have events A, B, and C such that $P(A|C) < P(B|C)$ and $P(A|C^c) < P(B|C^c)$, yet $P(A) > P(B)$.

(b) Let X be a random day of the week, encoded so that Monday = 1, Tuesday = 2, ..., Sunday = 7, with all seven values equally likely. Let Y be the day immediately following X, using the same 1–7 encoding. Then X and Y have the same PMF, and $P(X < Y) = \frac{1}{2}$.

(c) For a non-negative discrete random variable X, we have $\mathbb{E}[X] = \int_0^\infty P(X > x)\, dx$.

(d) It is possible to have two Gaussian random variables (which may not be jointly Gaussian) that are correlated and not independent.

(e) Recall that a sequence of random variables $X_1, X_2, \ldots$ converges in probability to X if

$$\lim_{n\to\infty} P(|X_n - X| > \epsilon) = 0, \text{ for every } \epsilon > 0.$$

If we modify the definition by replacing $P(|X_n - X| > \epsilon)$ with $P(|X_n - X| \geq \epsilon)$, then the resulting definition is not equivalent to the original one.

Uncorr X, Y Gaussian

$P(X = x)$