FDCL UAV Flight Software

1 Reference Frames

There are six! reference frames formulated. Understanding the definition of each frame, and converting between these frames precisely are critical for successful flight experiments.

1.1 Frames Fixed to UAV

UAV body-fixed frame: b This corresponds to the body-fixed frame defined in [2]. The center of rotor is numbered counterclockwise when observed from the top. The origin of the b-frame is located at the mass center of the UAV, and the first axis points toward the first rotor, and the second axis points toward the second rotor. Thus, the third axis points downward.

IMU-fixed frame: i This corresponds to the frame marked on the surface of the IMU.

VICON object markers frame: m When creating an object from the VICON tracker software, a reference frame fixed to the marker is defined. The orientation of this frame is identical to the above v-frame at the time of object creation. Therefore, it is common that the object is aligned in a specific desired way with respect to the v-frame.

1.2 Framed Fixed to Ground

Local NED frame: n This frame is common in flight dynamics. Its origin is located at the base station. Three axes points toward the due North, the due East, and downward, respectively.

VICON reference frame: v This corresponds to the *world* frame of the VICON system. It is determined when calibrating the VICON sensors with the marker wand. At SEH2200, there is a black elected tape marked on the floor. The first axis points toward the right when sitting at the base station, or the 22nd street; the second axis points front when sitting at the base station, or toward the I street; the third axis points upward.

FDCL SEH2200 frame: f This frame is fixed to SEH2200, and its origin is identical to the origin of the v-frame. The first axis points front when sitting at the base station, or toward the I street; the second axis points toward the right when sitting at the base station, or the 22nd street; the third axis points downward.

1.3 Conversion between Frames

Conversion among UAV-fixed frames Conversion among the three UAV-fixed frames do not change over time, as all of them are fixed to the UAV body.

It will be convenient if the IMU is aligned to the UAV body so that $R_{bi} \in \mathsf{SO}(3)$, i.e., the transform from the i-frame to the b-frame is identical, or a $\frac{\pi}{2}n$ rotation about a b-frame axis. However, in practice, it is challenging to align the IMU in such way. However, the IMU orientation can be precisely determined when designing the layout of the PCB board. As such, the rotation matrix $R_{bi} \in \mathsf{SO}(3)$ will be visually determined.

As discussed above, the orientation of the m-frame relative to the b-frame can be determined when creating the VICON object upon the error in aligning the UAV to the black tapes on the floor.

As both of the IMU and the VICON measures attitudes, the rotation matrix $R_{mi} \in \mathsf{SO}(3)$ from the i-frame to the m-frame can be calibrated as described in Section 2.

Conversion among ground-fixed frames Conversion among the three ground-fixed frames do not change over time, as all of them are fixed to the ground.

The conversion between the v-frame and the f-frame is trivial, as the f-frame is constructed by flipping the v-frame. More explicitly,

$$R_{fv} = [e_2, e_1, -e_3] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad R_{vf} = R_{fv}^T = R_{fv}. \tag{1}$$

The conversion between the n-frame and the v-frame is not straightforward, as the magnetic field in SEH2200 is neither uniform nor stationary. However, as both of the IMU and the VICON measures attitudes, the rotation matrix $R_{nv} \in \mathsf{SO}(3)$ from the v-frame to the n-frame can be calibrated as described in Section 2.

Conversion between a UAV-fixed frame and a ground-fixed frame In contrast to the two types conversion described above, the conversion between a UAV-fixed frame and a ground-fixed frame depends on the actual attitude of the UAV with respect to the ground.

Two types of measurements are available.

- $R_{ni}(t) \in \mathsf{SO}(3)$ is measured by IMU. There are several ways to convert Euler-angles to the corresponding rotation matrix. The following equations should be used for VN100.

```
YPR(0)=data->ypr.yaw*M_PI/180.;
YPR(1)=data->ypr.pitch*M_PI/180.;
YPR(2)=data->ypr.roll*M_PI/180.;

R_ni(0,0)=cos(YPR(0))*cos(YPR(1));
R_ni(0,1)=cos(YPR(0))*sin(YPR(2))*sin(YPR(1)) - cos(YPR(2))*sin(YPR(0));
R_ni(0,2)=sin(YPR(0))*sin(YPR(2)) + cos(YPR(0))*cos(YPR(2))*sin(YPR(1));
R_ni(1,0)=cos(YPR(1))*sin(YPR(0));
R_ni(1,1)=cos(YPR(0))*cos(YPR(2)) + sin(YPR(0))*sin(YPR(2))*sin(YPR(1));
R_ni(1,2)=cos(YPR(2))*sin(YPR(0))*sin(YPR(1)) - cos(YPR(0))*sin(YPR(2));
R_ni(2,0)=-sin(YPR(1));
R_ni(2,1)=cos(YPR(1))*sin(YPR(2));
R_ni(2,2)=cos(YPR(2))*cos(YPR(1));
```

- $R_{vm}(t) \in \mathsf{SO}(3)$ is measured by VICON. There are several ways to convert a quaternion to the corresponding rotation matrix. The following equations should be used for VICON VRPN.

```
R_vm(0,0) = 1-(2*(tdata.quat[1])*(tdata.quat[1]))-(2*(tdata.quat[2])*(tdata.quat[2]));
R_vm(0,1) = (2*tdata.quat[0]*tdata.quat[1])-(2*tdata.quat[3]*tdata.quat[2]);
R_vm(0,2) = (2*tdata.quat[0]*tdata.quat[2])+(2*tdata.quat[3]*tdata.quat[1]);
R_vm(1,0) = (2*tdata.quat[0]*tdata.quat[1])+(2*tdata.quat[3]*tdata.quat[2]);
R_vm(1,1) = 1-(2*(tdata.quat[0])*(tdata.quat[0]))-(2*(tdata.quat[2])*(tdata.quat[2]));
R_vm(1,2) = (2*(tdata.quat[1])*(tdata.quat[2]))-(2*(tdata.quat[3])*(tdata.quat[0]));
R_vm(2,0) = (2*tdata.quat[0]*tdata.quat[2])-(2*tdata.quat[3]*tdata.quat[1]);
```

```
R_vm(2,1) = (2*tdata.quat[0]*tdata.quat[3])+(2*tdata.quat[2]*tdata.quat[1]);
R_vm(2,2) = 1-(2*(tdata.quat[0])*(tdata.quat[0]))-(2*(tdata.quat[1])*(tdata.quat[1]));
```

1.4 REFERENCE FRAME SELECTION: IMPORTANT

In indoor flight experiments at SEH2200, we use the following two reference frames:

- The body-fixed frame, or the b-frame, as the UAV-fixed frame.
- The FDCL SEH2200 frame, or the f-frame, as the ground-fixed frame.

In the implementation of the controller or the generation of tracking commands,

- $x \in \mathbb{R}^3$ is the position of the origin of the b-frame, or the center of gravity of the UAV, resolved in the f-frame.
- $v = \dot{x} \in \mathbb{R}^3$.
- $R \triangleq R_{fb} \in \mathsf{SO}(3)$ is the rotation matrix from the b-frame to the f-frame.
- $\Omega \in \mathbb{R}^3$ is the angular velocity resolved in the b-frame.

IMU The output of the IMU corresponds to $(R_{\mathrm{IMU}}, \Omega_{\mathrm{IMU}}, a_{\mathrm{IMU}}) = (R_{ni}, \Omega_i, a_{\mathrm{IMU}})$, where a_{IMU} is the relative acceleration with respect to the gravitational acceleration resolved in the i-frame. They can be converted into (R, Ω, a) as

$$R(t) = R_{fb}(t) = R_{fv}R_{vn}R_{ni}(t)R_{ib}, \tag{2}$$

$$\dot{R}(t) = R_{fv}R_{vn}R_{ni}(t)\hat{\Omega}_i(t)R_{ib} = R_{fv}R_{vn}R_{ni}(t)R_{ib}\widehat{R_{ib}^T\Omega_i(t)} = R(t)\widehat{R_{ib}^T\Omega_i(t)} = R(t)\hat{\Omega}(t) \tag{3}$$

$$\Omega(t) = \Omega_b(t) = R_{bi}\Omega_i(t), \tag{4}$$

$$a(t) = \ddot{x}(t) = R_{fi}(t)a_{\mathrm{IMU}} + ge_3 = R_{fv}R_{vn}R_{ni}(t)a_{\mathrm{IMU}} + ge_3 = R_{fb}(t)R_{bi}a_{\mathrm{IMU}} + ge_3. \tag{5}$$

VICON The output of the VICON measurements corresponds to (x_v, R_{vm}). They can be converted into (x, R) as

$$x(t) = x_f(t) = R_{fv}x_v(t), \tag{6}$$

$$R(t) = R_{fb}(t) = R_{fv}R_{vm}(t)R_{mb}. \tag{7}$$

2 VICON and IMU Calibration

This section describes computational algorithms to identify two fixed rotation matrices $R_{nv}, R_{mi} \in \mathsf{SO}(3)$.

Problem Formulation The IMU measures $R_{ni} \in \mathsf{SO}(3)$ which represents the linear transformation of a representation of a vector from the i-frame to the n-frame, and the VICON measures $R_{vm} \in \mathsf{SO}(3)$ that is the rotation matrix from the m-frame to the v-frame. They are related as

$$R_{ni}(t) = R_{nv}R_{vm}(t)R_{mi}, \tag{8}$$

where $R_{mi} \in \mathsf{SO}(3)$ is the fixed rotation matrix from the i-frame to the m-frame, and $R_{nv} \in \mathsf{SO}(3)$ is the fixed rotation matrix from the v-frame to the n-frame.

The objective of this section is to compute R_{mi} and R_{nv} for several measurements of $\{(R_{ni}(t), R_{vm}(t)) \in \mathsf{SO}(3) \times \mathsf{SO}(3) \,|\, t \in \{t_1, t_2, \ldots t_N\}\}$ for a positive integer $N \geq 2$.

Wahba's Problem For notational convenience, let

$$R = R_{ni}, \quad Q = R_{vm}, \quad R_{nv} = X, \quad R_{mi} = Y.$$

Using these, (8) can be rewritten for two sets of measurements as

$$R_i = XQ_iY, \quad R_j = XQ_jY,$$

for $i, j \in \{1, \ldots, N\}$. We have $Y = Q_j^T X^T R_j$ from the second equation. Substituting it into the first one,

$$R_i = XQ_iQ_j^T X^T R_j,$$

which is rearranged

$$R_iR_j^T = XQ_iQ_j^T X^T, \tag{9}$$

which follows the form of $A = XBX^T$, or equivalently $AX = XB$, that is an equation well-known in sensor calibrations.

We solve it using the matrix exponential. Define $r_{ij}, q_{ij} \in \mathbb{R}^3$ such that

$$R_iR_j^T = \exp \hat{r}_{ij}, \quad Q_iQ_j^T = \exp \hat{q}_{ij}. \tag{10}$$

Using this, (9) is rewritten as $\exp \hat{r}_{ij} = X \exp \hat{q}_{ij} X^T = \exp(\widehat{Xq_{ij}})$, which is equivalent to

$$r_{ij} = Xq_{ij}. \tag{11}$$

The problem of finding $X \in \mathsf{SO}(3)$ for given $\{(r_{ij}, q_{ij}) \in \mathbb{R}^3 \times \mathbb{R}^3 \,|\, i, j \in \{1, \ldots, N\}\}$ has been investigated in the field of attitude determination. In particular, the least-square determination approach is recognized as Wahba's problem.

One of the most popular solution to Wahba's problem is based on the singular value decomposition. Define the matrix $Z \in \mathbb{R}^{3\times 3}$ as

$$Z = \sum_{i,j=1}^{N} w_{ij} r_{ij} q_{ij}^T, \tag{12}$$

where $w_{ij} \in \mathbb{R}$ denotes a positive weighting parameters. Let the singular value decomposition of Z be $Z = USV^T$. The least-square solution of (11) minimizing $\mathcal{J} = \sum_{i,j=1}^{N} w_{ij} \|r_{ij} - Xq_{ij}\|^2$ is given by

$$X = U \,\mathrm{diag}[1, 1, \det[U]\det[V]]\, V^T. \tag{13}$$

Compute R_{nv} In summary, the rotation matrix from the v-frame to the n-frame, namely $X = R_{nv} \in \mathsf{SO}(3)$ is determined by

1. Collect $\{(R_{ni}(t), R_{vm}(t)) \in \mathsf{SO(3)} \times \mathsf{SO(3)} \,|\, t \in \{t_1, t_2, \ldots t_N\}\}$.

2. Define $\{(r_{ij}, q_{ij}) \in \mathbb{R}^3 \times \mathbb{R}^3 \,|\, i, j \in \{1, \ldots, N\}\}$ from (10).

3. Define Z as (12), and perform singular value decomposition to obtain $Z = USV^T$.

4. We have $R_{nv} = U \,\mathrm{diag}[1, 1, \det[U]\det[V]]\, V^T$.

Compute R_{mi} The rotation matrix $Y = R_{mi}$ can be constructed directly by using the above solution via $Y = Q_j^T X^T R_j$ for any $j \in \{1, \ldots, N\}$. In order to compute R_{mi} in the lease-square sense, the above procedure can be repeated as follows.

We have $X = R_j Y^T Q_j^T$, which yield $R_i = R_j Y^T Q_j^T Q_i Y$ that is equivalent to

$$R_j^T R_i = Y^T Q_j^T Q_i Y,$$

which is comparable to (9). As such, the procedure to compute X can be repeated by using the following definition of r_{ij}, q_{ij} instead of (10),

$$R_j^T R_i = \exp \hat{r}_{ij}, \quad Q_j^T Q_i = \exp \hat{q}_{ij}. \tag{14}$$

In summary, the rotation matrix from the i-frame to the m-frame, namely $Y = R_{mi} \in \mathsf{SO(3)}$ is determined by

1. Collect $\{(R_{ni}(t), R_{vm}(t)) \in \mathsf{SO(3)} \times \mathsf{SO(3)} \,|\, t \in \{t_1, t_2, \ldots t_N\}\}$.

2. Define $\{(r_{ij}, q_{ij}) \in \mathbb{R}^3 \times \mathbb{R}^3 \,|\, i, j \in \{1, \ldots, N\}\}$ from (14).

3. Define Z as (12), and perform singular value decomposition to obtain $Z = USV^T$.

4. We have $R_{mi} = V \,\mathrm{diag}[1, 1, \det[U]\det[V]]\, U^T$.

3 State Estimation

3.1 Equations of Motion

The equations of motion for (x, v, R) are given by

$$\begin{align}
\dot{x} &= v, \tag{15}\\
\dot{v} &= a = RR_{bi}(a_{\mathrm{IMU}} + b_a + w_a) + g e_3, \tag{16}\\
\dot{R} &= R\hat{\Omega} = R(R_{bi}(\Omega_{\mathrm{IMU}} + b_\Omega + w_\Omega))^\wedge, \tag{17}\\
\dot{b}_a &= w_{b_a}, \tag{18}\\
\dot{b}_\Omega &= w_{b_\Omega}, \tag{19}
\end{align}$$

where the acceleration measurement and the angular velocity measurements, namely $(a_{\mathrm{IMU}}, \Omega_{\mathrm{IMU}})$ are treated as an exogenous time-varying signal. As such, the measurement noise for the acceleration and the angular velocity, namely $w_a, w_\Omega \in \mathbb{R}^3$ are considered as the process noise, instead of the measurement noise in the formulation of the estimator. The above equation also includes the bias $b_a, b_\Omega \in \mathbb{R}^3$ in the angular velocity measurement and the acceleration measurement. More specifically

$$a = RR_{bi}(a_{\mathrm{IMU}} + b_a + w_a) + ge_3, \tag{20}$$

$$\Omega = R_{bi}(\Omega_{\mathrm{IMU}} + b_\Omega + w_\Omega). \tag{21}$$

Let $(\delta x, \delta v, \delta R = R\hat{\eta}, \delta b_a, \delta b_\Omega)$ be the perturbation of the state for $\eta \in \mathbb{R}^3$, from the ideal case of the absence of noise. After ignoring the higher order terms of perturbations and noise, the linearized equations of motion for $\delta x, \delta v, \delta b_a, \delta b_\Omega$ are given by

$$\begin{aligned}
\delta\dot{x} &= \delta v,\\
\delta\dot{v} &= R\hat{\eta}R_{bi}(a_{\mathrm{IMU}} + b_a) + RR_{bi}(\delta b_a + w_a) = -R(R_{bi}(a_{\mathrm{IMU}} + b_a))^\wedge \eta + RR_{bi}\delta b_a + RR_{bi}w_a,\\
\delta\dot{b}_a &= w_{b_a},\\
\delta\dot{b}_\Omega &= w_{b_\Omega}.
\end{aligned}$$

Also the perturbation of (17) is written as

$$\begin{aligned}
\frac{d}{dt}(\delta R) &= \frac{d}{dt}(R\hat{\eta}) = R\hat{\Omega}\hat{\eta} + R\dot{\hat{\eta}} = R(R_{bi}(\Omega_{\mathrm{IMU}} + b_\Omega))^\wedge\hat{\eta} + R\dot{\hat{\eta}}\\
&= \delta(\dot{R}) = R\hat{\eta}(R_{bi}(\Omega_{\mathrm{IMU}} + b_\Omega))^\wedge + R(R_{bi}\delta b_\Omega)^\wedge + R(R_{bi}w_\Omega)^\wedge,
\end{aligned}$$

which yields

$$\dot{\eta} = -(R_{bi}(\Omega_{\mathrm{IMU}} + b_\Omega))^\wedge\eta + R_{bi}\delta b_\Omega + R_{bi}w_\Omega,$$

after ignoring the higher-order terms of the process noise and the perturbation.

The linearized equations of motion are rearranged into a matrix form with $\mathbf{x} = [\delta x; \delta v; \eta; \delta b_a; \delta b_\Omega] \in \mathbb{R}^{15}$ and $\mathbf{w} = [w_a; w_\Omega; w_{b_a}; w_{b_\Omega}] \in \mathbb{R}^{12}$ as

$$\begin{aligned}
\dot{\mathbf{x}} &= \begin{bmatrix} 0 & I_{3\times3} & 0 & 0 & 0\\ 0 & 0 & -R(R_{bi}(a_{\mathrm{IMU}} + b_a))^\wedge & RR_{bi} & 0\\ 0 & 0 & -(R_{bi}(\Omega_{\mathrm{IMU}} + b_\Omega))^\wedge & 0 & R_{bi}\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 & 0 & 0 & 0\\ RR_{bi} & 0 & 0 & 0\\ 0 & R_{bi} & 0 & 0\\ 0 & 0 & I_{3\times3} & 0\\ 0 & 0 & 0 & I_{3\times3} \end{bmatrix}\mathbf{w}\\
&\triangleq A(t)\mathbf{x} + F(t)\mathbf{w}.
\end{aligned} \tag{22}$$

3.2 Prediction

Mean Let $h_k = t_k - t_{k-1}$ be the discrete time step. The mean values $(\bar{x}, \bar{v}, \bar{R})$ are updated by discretizing (15)–(17) in the absence of the process noise via the following second order explicit method:

$$\bar{R}_k = \bar{R}_{k-1} \exp\left\{\frac{h_k}{2}(\bar{\Omega}_{k-1} + \bar{\Omega}_k)^\wedge\right\}, \tag{23}$$

$$\bar{x}_k = \bar{x}_{k-1} + h_k \bar{v}_{k-1} + \frac{h_k^2}{2}\bar{a}_{k-1}, \tag{24}$$

$$\bar{v}_k = \bar{v}_{k-1} + \frac{h_k}{2}(\bar{a}_{k-1} + \bar{a}_k), \tag{25}$$

$$\bar{b}_{a_k} = \bar{b}_{a_{k-1}}, \tag{26}$$

$$\bar{b}_{\Omega_k} = \bar{b}_{\Omega_{k-1}}, \tag{27}$$

with

$$\bar{\Omega}_k = R_{bi}(\Omega_{\mathrm{IMU}_k} + b_{\Omega_k}), \tag{28}$$

$$\bar{a}_k = \bar{R}_k R_{bi}(a_{\mathrm{IMU}_k} + b_k) + g e_3. \tag{29}$$

Covariance The linearized equation is discretized according to [1, p 330] as

$$\mathbf{x}_k = A_{k-1}\mathbf{x}_{k-1} + F_{k-1}\mathbf{w}_{k-1}, \tag{30}$$

where

$$A_{k-1} = I_{15\times 15} + h_k A(t_{k-1})\Psi, \tag{31}$$

$$F_{k-1} = h_k \Psi F(t_{k-1}), \tag{32}$$

$$\Psi = I_{15\times 15} + \frac{h_k}{2}A(t_{k-1})\left(I + \frac{h_k}{3}A(t_{k-1})\left(I + \cdots\left(I + \frac{h_k}{N}A(t_{k-1})\right)\right)\right). \tag{33}$$

Let the covariance of $\mathbf{x}_k$ be $P_k \in \mathbb{R}^{15\times 15}$ and let the covariance of $\mathbf{w}_k$ be $Q_k \in \mathbb{R}^{12\times 12}$. It is updated as

$$P_k = A_{k-1}P_{k-1}A_{k-1}^T + F_{k-1}Q_{k-1}F_{k-1}^T. \tag{34}$$

3.3 Correction by IMU

In addition to $(\Omega_{\mathrm{IMU}}, a_{\mathrm{IMU}})$, the IMU measures R_{ni}. The measurement equation is given by

$$R_{\mathrm{IMU}} = R_{nv}R_{vf}RR_{bi}\exp\hat{\zeta}.$$

where $\zeta \in \mathbb{R}^3$ denotes the measurement noise for R_{IMU}. The estimation of the IMU measurement is given by

$$\bar{R}_{\mathrm{IMU}} = R_{nv}R_{vf}\bar{R}R_{bi}.$$

The difference between R_{IMU} and $\bar{R}_{\mathrm{IMU}}$ is referred to as the residual error, which is formulated by $\eta, \zeta \in \mathbb{R}^3$ as

$$\begin{aligned}R_{\mathrm{IMU}} &= R_{nv}R_{vf}\bar{R}\exp\hat{\eta}R_{bi}\exp\hat{\zeta}\\ &= R_{nv}R_{vf}\bar{R}R_{bi} + R_{nv}R_{vf}\bar{R}\hat{\eta}R_{bi} + R_{nv}R_{vf}\bar{R}R_{bi}\hat{\zeta},\end{aligned}$$

for small η and ζ, which is rearranged into

$$\bar{R}^T R_{vf}^T R_{nv}^T R_{\mathrm{IMU}} R_{bi}^T = I_{3\times 3} + \hat{\eta} + \widehat{R_{bi}\zeta},$$

which yields

$$(\bar{R}^T R_{vf}^T R_{nv}^T R_{\mathrm{IMU}} R_{bi}^T - I_{3\times 3})^\vee = \eta + R_{bi}\zeta.$$

In practice, $\bar{R}^T R_{vf}^T R_{nv}^T R_{\mathrm{IMU}} R_{bi}^T - I_{3\times 3}$ may not be skew-symmetric. After projecting it to the space of skew-symmetric matrices,

$$\frac{1}{2}(\bar{R}^T R_{vf}^T R_{nv}^T R_{\mathrm{IMU}} R_{bi}^T - R_{bi}R_{\mathrm{IMU}}^T R_{nv}R_{vf}\bar{R})^\vee = \delta z = \eta + R_{bi}\zeta = H\mathbf{x} + R_{bi}\zeta, \tag{35}$$

where $H = [0_{3\times 3}, 0_{3\times 3}, I_{3\times 3}, 0_{3\times 3}, 0_{3\times 3}] \in \mathbb{R}^{3\times 15}$.

Observability For given discrete approximation, we have

$$\begin{aligned}H\Phi(t_1,t_0) &= HA_0 = H(I_{15\times 15} + hA(t_0))\\ &= [0, 0, I + hA_{33}(t_0), 0, hR_b i],\\ H\Phi(t_2,t_0) &= HA_1A_0 = [0, 0, I + hA_{33}(t_1), 0, hR_b i]A_0,\\ &= [0, 0, (I + hA_{33}(t_1))(I + hA_{33}(t_0)), 0, (I + hA_{33}(t_1))hR_{bi}],\\ &\vdots\end{aligned}$$

As such it is clear that $(\delta x, \delta v, \delta b_a)$ are not observable from the IMU measurement R_{ni}.

Define $T \in \mathbb{R}^{15\times 15}$ as

$$T = \begin{bmatrix} 0_{3\times 3} & 0_{3\times 3} & I_{3\times 3} & 0_{3\times 3} & 0_{3\times 3}\\ 0_{3\times 3} & 0_{3\times 3} & 0_{3\times 3} & 0_{3\times 3} & I_{3\times 3}\\ I_{3\times 3} & 0_{3\times 3} & 0_{3\times 3} & 0_{3\times 3} & 0_{3\times 3}\\ 0_{3\times 3} & I_{3\times 3} & 0_{3\times 3} & 0_{3\times 3} & 0_{3\times 3}\\ 0_{3\times 3} & 0_{3\times 3} & 0_{3\times 3} & I_{3\times 3} & 0_{3\times 3}\end{bmatrix}.$$

with $T^{-1} = T$. Let $\chi = Tx \in \mathbb{R}^{15}$. We have the similarity transform, $\bar{A} = TAT^{-1}$, $\bar{F} = TF$, and

$\bar{H} = HT^{-1}$,

$$
\begin{aligned}
\dot{\chi} &= \begin{bmatrix} -(R_{bi}(\Omega_{\mathrm{IMU}} + b_\Omega))^\wedge & R_{bi} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{3\times 3} & 0 \\ -R(R_{bi}(a_{\mathrm{IMU}} + b_a))^\wedge & 0 & 0 & 0 & RR_{bi} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \chi + \begin{bmatrix} 0 & R_{bi} & 0 & 0 \\ 0 & 0 & 0 & I_{3\times 3} \\ 0 & 0 & 0 & 0 \\ RR_{bi} & 0 & 0 & 0 \\ 0 & 0 & I_{3\times 3} & 0 \end{bmatrix} \mathbf{w} \\
&\triangleq \bar{A}(t)\chi + \bar{F}(t)\mathbf{w}, \qquad (36) \\
\delta z &= \begin{bmatrix} I_{3\times 3} & 0 & 0 & 0 & 0 \end{bmatrix} \chi \qquad (37) \\
&\triangleq \bar{H}\chi, \qquad (38)
\end{aligned}
$$

The first six elements of χ, namely $\chi_o = T_o[\eta, \delta\Omega] \in \mathbb{R}^6$ corresponds to the observable subspace, where $T_0 \in \mathbb{R}^{6\times 15}$ is

$$
T_o = \begin{bmatrix} I_{3\times 3} & 0 & 0 & 0 & 0 \\ 0 & I_{3\times 3} & 0 & 0 & 0 \end{bmatrix}.
$$

And it is governed by

$$
\begin{aligned}
\dot{\chi}_o &= (T_o \bar{A}(t) T_o^T) T_o \chi + T_o \bar{F}(t)\mathbf{w}, \\
&= \begin{bmatrix} -(R_{bi}(\Omega_{\mathrm{IMU}} + b_\Omega))^\wedge & R_{bi} \\ 0 & 0 \end{bmatrix} \chi + \begin{bmatrix} 0 & R_{bi} & 0 & 0 \\ 0 & 0 & 0 & I_{3\times 3} \end{bmatrix} \mathbf{w} \\
&\triangleq \bar{A}_o(t)\chi_o + \bar{F}_o(t)\mathbf{w}, \qquad (39) \\
\delta z &= (\bar{H} T_o^T) T_o \chi = \begin{bmatrix} I_{3\times 3} & 0 \end{bmatrix} \chi_o \\
&\triangleq \bar{H}_o \chi_o. \qquad (40)
\end{aligned}
$$

Also

$$
P_o = \mathrm{E}[(\chi_o - \bar{\chi}_o)(\chi_o - \bar{\chi}_o)^T] = \mathrm{E}[T_o(\chi - \bar{\chi})(\chi - \bar{\chi})T_o^T] = \mathrm{E}[T_o T(\mathbf{x} - \bar{\mathbf{x}})(\mathbf{x} - \bar{\mathbf{x}})T^T T_o^T] = T_o T P T^T T_o^T.
$$

Correction The above equation is considered as the linearized measurement equation for EKF with the residual or the innovation term δz. Let $V_\zeta \in \mathbb{R}^{3\times 3}$ be the covariance of the measurement error ζ. The residual (or innovation) covariance $S \in \mathbb{R}^{3\times 3}$ is

$$
\begin{aligned}
S &= \bar{H}_o P_o \bar{H}_o^T + R_{bi} V_\zeta R_{bi}^T = (H T^T T_o^T)(T_o T P T^T T_o^T) T_o T H^T + R_{bi} V_\zeta R_{bi}^T \\
&= H P H^T + R_{bi} V_\zeta R_{bi}^T. \qquad (41)
\end{aligned}
$$

The Kalman gain $\bar{K}_o \in \mathbb{R}^{6\times 3}$ is given by

$$
\bar{K}_o = P_o \bar{H}_o^T S^{-1} = (T_o T P T^T T_o^T)(T_o T H^T), \qquad (42)
$$

and the a posteriori value of the perturbed state is given by

$$\chi^+ = \bar{K}_o \delta z. \tag{43}$$

Or equivalently,

$$\mathbf{x}^+ = T^T T_o^T \bar{K}_o \delta z \triangleq K \delta z. \tag{44}$$

The a posteriori state is

$$\bar{R}^+ = \bar{R}^- \exp(\hat{\eta}^+), \quad b_\Omega^+ = b_\Omega^- + \delta b_\Omega^+, \tag{45}$$

and the a posteriori covariance is

$$P^+ = (I_{15\times 15} - KH)P^- = (I_{15\times 15} - KH)P^-(I_{15\times 15} - KH)^T + KR_{bi}V_\zeta R_{bi}^T K^T, \tag{46}$$

where the latter is known as the Joseph form, which is implemented to preserve the symmetry and the positive-definiteness of P in numerical computation.

3.4 Correction via VICON

The VICON measures the location in the v-frame and the attitude from the m-frame to the v-frame, namely $(x_{\mathrm{VICON}}, R_{\mathrm{VICON}})$.

$$\begin{aligned} x_{\mathrm{VICON}} &= R_{vf}x + \zeta_x, \\ R_{\mathrm{VICON}} &= R_{vm} \exp \hat{\zeta}_R = R_{vf} R R_{bi} R_{im} \exp \hat{\zeta}_R, \end{aligned}$$

where ζ_x, ζ_R denote the measurement noise for x_{VICON} and R_{VICON}, respectively.

Similar to above, the estimate of R_{VICON} is given by $\bar{R}_{\mathrm{VICON}} = R_{vf} R R_{bm}$, and the difference between R_{VICON} and $\bar{R}_V$ is formulated as

$$\begin{aligned} R_{\mathrm{VICON}} &= R_{vf} \bar{R} \exp \hat{\eta} R_{bi} R_{im} \exp \hat{\zeta}_R \\ &= R_{vf} \bar{R} R_{bi} R_{im} + R_{vf} \bar{R} \hat{\eta} R_{bi} R_{im} + R_{vf} \bar{R} R_{bi} R_{im} \hat{\zeta}_R, \end{aligned}$$

after ignoring the higher order terms of ζ_R and η. This is rearranged into

$$\bar{R}^T R_{vf}^T R_{\mathrm{VICON}} R_{im}^T R_{bi}^T = I_{3\times 3} + \hat{\eta} + \widehat{R_{bm}\zeta_R},$$

or equivalently

$$(\bar{R}^T R_{vf}^T R_{\mathrm{VICON}} R_{im}^T R_{bi}^T - I_{3\times 3})^\vee = \eta + R_{bm}\zeta_R.$$

Projecting the left hand side onto the skew-symmetric matrices,

$$\frac{1}{2}(\bar{R}^T R_{vf}^T R_{\mathrm{VICON}} R_{im}^T R_{bi}^T - R_{bi} R_{im} R_{\mathrm{VICON}}^T R_{vf} \bar{R})^\vee = \eta + R_{bm}\zeta_R.$$

Similarly,

$$x_{\mathrm{VICON}} = R_{vf}\bar{x} + R_{vf}\delta x + \zeta_x.$$

The linearized measurement equation is written as

$$\delta z = \begin{bmatrix} \frac{1}{2}(\bar{R}^T R_{vf}^T R_{\text{VICON}} R_{im}^T R_{bi}^T - R_{bi} R_{im} R_{\text{VICON}}^T R_{vf} \bar{R})^\vee \\ x_V - R_{vf}\bar{x} \end{bmatrix} = \begin{bmatrix} \eta \\ R_{vf}\delta x \end{bmatrix} + \begin{bmatrix} R_{bi} R_{im} \zeta_R \\ \zeta_x \end{bmatrix} = H\mathbf{x} + G\mathbf{v},$$

where $\mathbf{v} = [\zeta_R; \zeta_x] \in \mathbb{R}^6$, and the matrices $H \in \mathbb{R}^{6\times 15}$ and $G \in \mathbb{R}^{6\times 6}$ are given by

$$H = \begin{bmatrix} 0 & 0 & I_{3\times 3} & 0 & 0 \\ R_{vf} & 0 & 0 & 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} R_{bi} R_{im} & 0 \\ 0 & I_{3\times 3} \end{bmatrix}.$$

Let $V \in \mathbb{R}^{6\times 6}$ be the covariance of $\mathbf{v}$. The residual covariance $S \in \mathbb{R}^{6\times 6}$ is given by

$$S = HPH^T + GVG^T.$$

The Kalman gain $K \in \mathbb{R}^{15\times 6}$ is given by

$$K = PH^T S^{-1}, \tag{47}$$

and the a posteriori value of the perturbed state is given by

$$\mathbf{x}^+ = K\delta z. \tag{48}$$

The a posteriori state is

$$x^+ = x^- + \delta x^+, \quad v^+ = v^- + \delta v^+, \quad \bar{R}^+ = \bar{R}^- \exp(\hat{\eta}^+), \quad b_a^+ = b_a^- + \delta b_a^+, \quad b_\Omega^+ = b_\Omega^- + \delta b_\Omega^+. \tag{49}$$

The a posteriori covariance is

$$P^+ = (I_{15\times 15} - KH)P^- = (I_{15\times 15} - KH)P^-(I_{15\times 15} - KH)^T + KGVG^T K^T, \tag{50}$$

where the latter is known as the Joseph form, which is implemented to preserve the symmetry and the positive-definiteness of P in numerical computation.

3.5 Low-Pass Filter for Ω

In the above formulation of Kalman filter, the angular velocity measurement is considered as a known time-varying signal. Even after correcting it with the gyro bias, $\bar{\Omega}_k$ obtained by (28) is subject to high-frequency noise, which causes undesired irregular vibrations in the corresponding control input.

A simple first order low-pass filter is represented by

$$G(s) = \frac{1}{\tau s + 1},$$

where τ represents the time-constant, i.e., the time which the initial condition becomes reduced by the factor of $\frac{1}{e} = 0.368$. For the sinusoidal inputs,

$$|G(j\omega)| = \frac{1}{\sqrt{1+\omega^2\tau^2}}$$

Therefore, the bandwidth, the frequency where $|G(j\omega)|^2 = 0.5$ becomes $\omega = \frac{1}{\tau}$. For example, when the bandwidth is set for $f = 100\,\mathrm{Hz}$, or equivalently $\omega = 2\pi f$, then the time constant is chosen as

$$\tau = \frac{1}{\omega} = \frac{1}{2\pi f} = 0.0016\,\mathrm{sec}.$$

In the continuous time, the filter equation is written as

$$\tau\dot{y} + y = u,$$

which can be approximated by

$$\tau(y_k - y_{k-1}) + hy_k = hu_k,$$

or equivalently

$$y_k = \frac{1}{h+\tau}(\tau y_{k-1} + hu_k) = (1-\alpha)y_{k-1} + \alpha u_k,$$

where $\alpha = \frac{h}{\tau+h}$.

References

[1] W. Brogan. *Modern Control Theory*. Prentice Hall, 1991.

[2] T. Lee, M. Leok, and N. McClamroch. Geometric tracking control of a quadrotor UAV on SE(3). In *Proceedings of the IEEE Conference on Decision and Control*, pages 5420–5425, Atlanta, GA, Dec. 2010.