Discrete Mathematics
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Homework 1

1. Prove that there is no positive integer n such that $n^2 + n^3 = 100$. .

   Given n belongs to Z^+,

   $$n^2 + n^3 = 100$$
   $$n^2(n+1) = 100$$

   Let's presume the assumption that $\exists$ n which satisfies the condition which means n^2 is a factor of 100

   The factors of 100 which are perfect squares are

   $$n^2 = 1, 4, 25, 100$$
   $$\Rightarrow n = 1, 2, 5, 10$$

   By using induction, our assumption is wrong.

   This proves that there is no positive integer n such that $n^2 + n^3 = 100$

2. Prove that $n^2 + 1 \geq 2^n$ when n is a positive integer with $1 \leq n \leq 4$

   Given to prove $n^2 + 1 \geq 2^n$ when n is a positive integer with $1 \leq n \leq 4$

   $$n^2 + 1 \geq 2^n$$
   $$\Rightarrow n^2 - 2^n \geq -1$$

   Let's see how the function $f(n) = n^2 - 2^n$ acts in our domain $n \in [1, 4]$.

   $$df/dn = 2n - 2^n ln2$$
   $$\text{For } n = 1, f(n) = -1$$

   Differential is grater than zero for $1 \leq n \leq 3$

   $$\Rightarrow \text{f(n) is } \textit{increasing} \text{ in this interval}$$

   But it starts decreasing from somewhere between 3 and 4 as differential f(4) is negative.

   $f(4) = 0$

   So, the function $f(n)$ increases from -1 and decreases to 0 in the interval $[1, 4]$.

$$\Rightarrow n^2 - 2^n \geq -1 \text{ for the interval } n \in [1, 4]$$

Hence $n^2 + 1 \geq 2^n$ for interval $n \in [1, 4]$

3. Find a compound proposition involving the propositional variables p, q, r, and s that is true when exactly three of these propositional variables are true and is false otherwise

   The compound proposition given below gives true when any 3 out of 4 variables are true.

   $$(\neg p \cap q \cap r \cap s) \cup (p \cap \neg q \cap r \cap s) \cup (p \cap q \cap \neg r \cap s) \cup (p \cap q \cap r \cap \neg s)$$

4. Let P(x) and Q(x) be propositional functions. Show that ∃x(P(x) → Q(x)) and ∀x P(x) → ∃x Q(x) always have the same truth value

   The first proposition ∃x(P(x) → Q(x)) is same as ∃x(¬ (P ∩ ¬ Q))

   As we know that ¬(¬(P)) = P and ¬ (P → Q) is equivalent to P ∩ ¬ Q .

   The second proposition ∀x P(x) → ∃x Q(x) is true except when[∀x P(x) is true and ∃x Q(x) is false]

   Now, the first proposition is only false when [P is true and Q is false] for ∃x which is same as ∃x Q(x) is false and ∀x P(x) is true.

   Hence the both propositions have same truth table.

5. Suppose that A and B are sets such that the power set of A is a subset of the power set of B. Does it follow that A is a subset of B.

   Given power set of A is a sub set of power set of B.

   Let's denote P(A) to be the power set of A: it's elements are subsets of A.

   $$\Rightarrow P(A) \subseteq A$$

   As we know that any set S is subset of it's own set S

   $$\Rightarrow SetA \in P(A)$$

   From the conditions,

   $$\text{P(A)} \subset \text{P(B) and P(B)} \subseteq \text{B}$$
   $$\Rightarrow P(A) \subset P(B) \subseteq B$$
   $$\Rightarrow P(A) \subset B \text{ [Set A} \in \text{P(A)]}$$
   $$\Rightarrow A \subset B$$

   Hence, we conclude that A is subset of B.

6. Let A and B be sets. Show that A $\subseteq$ B if and only if A $\cap$ B = A

   To show that A $\subseteq$ B $\Leftrightarrow$ A $\cap$ B = A

   $A \subseteq B \Rightarrow$ Every element of set A is in set B. [Subset definition]

   Formal definition of intersection of 2 sets :

   Given two sets, say (A) and (B), the **intersection** of these sets consists of all elements that are **common to both** (A) and (B).

   In our case of A and B,

   $$\text{A} \subseteq \text{B [Every element of A is in B.]}$$
   $$\Rightarrow \text{Common elements of A and B is/are in set A.}$$
   $$\Rightarrow A \cap B = A$$

   Hence, we showed that $A \subseteq B \Leftrightarrow A \cap B = A$