From Nonholonomy to Holonomy: Time-Optimal Velocity Control of Differential Drive Robots

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***Abstract*—There is a large body of literature studying problems such as consensus, flocking, or formation control of multi-agent systems using simple dynamic models, such as first-order integrators. This raises the question of how to implement such results on nonholonomic mobile robot platforms, for example, differential-drive or unicycle-type robots. Since the nonholonomic constraints must be taken into account when commanding a desired velocity change for such robots, we investigate the problem of reaching a desired velocity in minimum time. Using the Pontryagin Maximum Principle, we investigate the time-optimal control for systems with bounded wheel torques as the control input.**

I. Introduction

In this paper, we consider the time-optimal velocity control of planar differential drive robots. Previous work on time-optimal control for the differential drive robot has focused on position control and trajectory planning rather than velocity control [1]–[3]. The contribution of this paper lies in applying the Pontryagin Maximum Principle to the differential drive robot with bounded torque inputs in order to derive time-optimal controls that drive the forward speed, heading angle and angular velocity to desired values.

An important application of this work is in the control of a team of multiple differential drive robots that is operated by a human using some input device, such as a joystick. Typically, the human may command a motion towards a particular direction. From a human factors standpoint, it is important to minimize the delay between the commanded motion and the realization of that commanded motion by the robot formation.

II. Preliminaries

A differential drive robot is shown in Figure 1. The configuration of the robot is $(x, y, \theta)^T \in \mathbb{R}^3$, where (x, y) is the cartesian position of the centroid of the robot and θ is the orientation relative to the world frame.

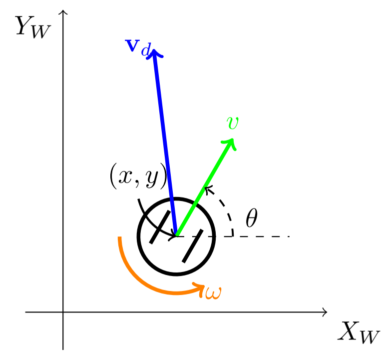

Fig. 1. The differential drive robot with with linear speed v, angular velocity ω and desired velocity $\mathbf{v}_d$.

The kinematic equations of motion are given by

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos(\theta) & 0 \\ \sin(\theta) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix} \tag{1}$$

where v and ω are the linear and angular speed, respectively. The non-holonomic nature of the equations is due to the fact that Equation (1) satisfies the nonintegrable velocity constraint

$$\dot{x}\sin(\theta) - \dot{y}\cos(\theta) = 0 \tag{2}$$

The linear speed v and angular velocity ω are related to the right and left wheel velocities ($\dot{\phi}_R$ and $\dot{\phi}_L$, respectively) as

$$v = r\dot{\phi}_R + r\dot{\phi}_L \tag{3a}$$

$$\omega = \frac{2r}{b}\dot{\phi}_R - \frac{2r}{b}\dot{\phi}_L \tag{3b}$$

where r is the wheel radius and b is the distance between the wheels, respectively.

We take as control inputs the motor torques, in which case we have

$$J\ddot{\phi}_R = u_1, J\ddot{\phi}_L = u_2 \tag{4}$$

where u_1 and u_2 are the net torques at the right and left wheels respectively, and J is the rotational inertia of each wheel about its axis. We assume that these torques are bounded, that is, $|u_1|\leq u_m$ and $|u_2|\leq u_m$ for some $u_m > 0$. Combining the above equations yields

$$\dot{v} = \frac{r}{J}u_1 + \frac{r}{J}u_2 \tag{5}$$

$$\dot{\omega} = \frac{2r}{Jb}u_1 - \frac{2r}{Jb}u_2 \tag{6}$$

where J is the rotational inertia of the robot about the vertical axis through the center of the wheel base. For simplicity, we will normalize the coefficients in Equations (5)-(6) and write

$$\dot{v} = u_1 + u_2 \tag{7}$$

$$\dot{\omega} = u_1 - u_2 \tag{8}$$

Finally, with state vector $q = (v, \theta, \omega)$ and input vector $u = (u_1, u_2)^T$ we can write

$$\begin{aligned} \dot{q} &= \begin{bmatrix} \dot{v} \\ \dot{\theta} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} u_1 + u_2 \\ \omega \\ u_1 - u_2 \end{bmatrix} \\ &= Aq + Bu \end{aligned} \tag{9}$$

where

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \;;\; B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & -1 \end{bmatrix} \tag{10}$$

In addition, the control input $u \in U \subset \mathbb{R}^2$, where $U = [-u_m, u_m] \times [-u_m, u_m]$. The problem we consider is then, given an initial velocity $q(0) = q_0 = (v_0, \theta_0, \omega_0)$ and desired final velocity $q(t_f) = q_d = (v_d, \theta_d, \omega_d$, find a control u with $u(t) \in U$ for all $t \geq 0$, that transfers the state from q_0 to q_d in minimum time, i.e. that minimizes

$$J = \int_0^{t_f} dt \tag{11}$$

III. The Maximum Principle

Consider a dynamical system

$$\dot{q} = f(q, u) \tag{12}$$

with state $q \in \mathbb{R}^n$, input $u \in U \subset \mathbb{R}^p$ and vector field $f : \mathbb{R}^n \times U \to \mathbb{R}^n$. Let q_0 and q_d represent initial and target states in $\mathbb{R}^n$, respectively. Assume that there exists some control $u(t)$ defined on $[0, t_f]$ such that the corresponding trajectory $q(t)$ defined on $[0, t_f]$ has the property that $q(0) = q_0$ and $q(t_f) = q_d$. The pair $(q(t), u(t))$, where t is defined on $[0, t_f]$, is called a controlled trajectory. Out of all controlled trajectories that achieve the desired change of state, the time-optimal control problem consists of finding one for which the final time t_f is minimum.

The *Pontryagin Maximum Principle* [4]–[7] provides necessary conditions that such trajectories must satisfy. Any controlled trajectory meeting these necessary conditions is called an extremal. The time-optimal trajectories are a subset of the extremals.

We introduce the *adjoint state* $\psi \in \mathbb{R}^n$, and Hamiltonian H given by

$$H(q, \mu, \psi, u) = -\mu + \psi^T f(q, u) \tag{13}$$

where $\mu \in \{0, 1\}$, and state the following:

Theorem III.1 (Maximum Principle for the time-optimal control problem)**.** *Consider system* (12) *with U a compact subset of $\mathbb{R}^p$. Let there exist an adjoint state $\psi \in \mathbb{R}^n$, a Hamiltonian function H given by* (13)*, an extremal denoted by the triple $(q^*(t), \psi^*(t), u^*(t))$ and the extremal Hamiltonian $H^*(t) = H(q^*(t), \mu, \psi^*(t), u^*(t))$ defined on $t \in I = [0, t_f]$. Then the following are true*

N1: For all $t \in I$, $(\mu, \psi^*(t)) \neq 0$.

N2: For almost all $t \in I$, the adjoint state satisfies

$$\dot{\psi} = -\frac{\partial H}{\partial q}(q^*(t), \mu, \psi^*(t), u^*(t)) \tag{14}$$

N3: For almost all $t \in I$, $u^*(t)$ satisfies

$$H^*(t) = \max_{u \in U} H(q^*(t), \mu, \psi^*(t), u) \tag{15}$$

N4: For almost all $t \in I$, $H^*(t) = 0$.

IV. Time Optimal Control

We begin by constructing the extremals through application of the Pontryagin Maximum Principle. Since the system (9) is a controllable linear system, it can be shown that extremals exist over some compact time interval $I = [0, t_f]$. Given an extremal $(q^*(t), \psi^*(t), u^*(t))$, we refer to $q^*(t)$ as the extremal trajectory and $u^*(t)$ as the extremal control.

The Hamiltonian function in this case is

$$\begin{aligned} H &= -\mu + \psi^T(Aq + Bu) \\ &= -\mu + \psi_1(u_1 + u_2) + \psi_2\omega + \psi_3(u_1 - u_2) \\ &= (\psi_1 + \psi_3)u_1 + (\psi_1 - \psi_3)u_2 + \psi_2\omega \end{aligned} \tag{16}$$

The adjoint state dynamics are

$$\dot{\psi} = -\frac{\partial H}{\partial q} = -\frac{\partial}{\partial q}\left(-\mu + \psi^T(Aq + Bu)\right) \tag{17}$$

$$= -A^T\psi \tag{18}$$

which yields

$$\begin{bmatrix} \dot{\psi}_1 \\ \dot{\psi}_2 \\ \dot{\psi}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\psi_2 \end{bmatrix} \tag{19}$$

The solution of this system, given initial condition $\psi(0) = (\psi_1(0), \psi_2(0), \psi_3(0))$, is simply

$$\psi_1(t) = \psi_1(0), \psi_2(t) = \psi_2(0), \psi_3(t) = \psi_2(0)t + \psi_3(0) \tag{20}$$

A. Classification of extremals

We are now in a position to determine extremals $(q^*(t), \psi^*(t), u^*(t))$. For any extremal, the function H^* is maximized at each time instant t (necessary condition $N3$), implying that

$$u_1(t) = u_m \ \text{sign}(\psi_1(t) + \psi_3(t)) \tag{21a}$$
$$u_2(t) = u_m \ \text{sign}(\psi_1(t) - \psi_3(t)) \tag{21b}$$

The initial condition $\psi(0)$ determines $\psi^*(t)$ according to (20), which in turn determines $u^*(t)$ according to (21) and hence $q^*(t)$, given $q(0)$.

The possible extremals are thus determined by the initial conditions $\psi(0)$. Clearly, three possibilities exist:

1) $\psi(t) \equiv 0$
2) $\psi(t) \equiv \psi(0) \neq 0$
3) $\psi(t) \neq \psi(0) \ \forall t > 0$

For convenience, we define the following *switching functions*:

$$\sigma_1(t) = (\psi_1(t) + \psi_3(t)) \tag{22}$$
$$\sigma_2(t) = (\psi_1(t) - \psi_3(t)) \tag{23}$$

Case 1: If $\psi(t) \equiv 0$ then $H = -\mu + \psi_2 \omega$ and any control $u^*(t) \in U, \ \forall t \in I$ would satisfy $N3$. Such a case is known as a doubly-singular control. However, such a control cannot be an extremal control due to the fact that $N1$ and $N4$ cannot simultaneously hold, and thus $\psi(t) \equiv 0$ cannot be part of a valid extremal.

Case 2: If $\psi(t) \equiv \psi(0) \neq 0$, then $\psi_2(0) = 0$. Consider either of the two mutually exclusive possibilities:

$S1$: $\sigma_1(t) = \sigma_1(0) = 0$
$S2$: $\sigma_2(t) = \sigma_2(0) = 0$

When $S1$ ($S2$) holds, the coefficient of u_1 (u_2) in (16) is zero, while the coefficient of u_1 (u_2) is a non-zero constant. This implies that extremal controls are of the form where one motor torque is $\pm u_m$ over the interval of definition of the trajectory, while the other control is arbitrary. These are singular extremals.

If $\psi(t) \equiv \psi(0) \neq 0$ and neither $S1$ nor $S2$ hold then, according to (21) the motor torques are constant with maximum possible magnitude and no switching occurs.

Case 3: Finally, if $\psi_2(0) \neq 0$, implying that $\psi(t) \neq \psi(0) \ \forall t > 0$, then $\psi_1(t)$ is constant and $\psi_3(t)$ is linear in time t, $\sigma_1(t)$ (or $\sigma_2(t)$) either monotonically increases of monotonically decreases, with exactly one time instant where its value is zero. Since $u_1 = u_m \text{sign}(\sigma_1)$ and $u_2 = u_m \text{sign}(\sigma_2)$, this implies that the motor torques are piecewise constant (with value $\pm u_m$) with no more than one switch.

To summarize, the application of the Pontryagin Maximum Principle results in the conclusion that all extremal controls consist of only two possible cases:

$C1$ At least one motor torque is constant with value u_m or $-u_m$ over I. The other motor torque is arbitrary and may have multiple switches or no switches.

TABLE I
NOTATION FOR FOUR TORQUE MODES

	$u_1 = u_m$	$u_1 = -u_m$
$u_2 = u_m$	$+\beta$	$-\alpha$
$u_2 = -u_m$	α	$-\beta$

$C2$ Both motor torques are piecewise constant, with values in $\{-u_m, +u_m\}$, and with exactly one switch for each motor at time instants t_1 and t_2 such that $t_1, t_2 \in (0, t_f)$.

B. Switching controls

We have characterized all possible pairs $(\psi^*(t), u^*(t))$ that satisfy the conditions of the Pontryagin Maximum Principle. Given an initial condition q_0, each such control $u^*(t)$ generates a trajectory $q^*(t)$, such that $(q^*(t), \psi^*(t), u^*(t))$ is an extremal. We must search among these extremals and find those such that $(q^*(t), \psi^*(t), u^*(t))$ is defined on $t \in I = [0, t_f]$ and $q(t_f) = q_d$, and then select the one where the corresponding final time t_f is the least.

For a $C2$ control, since each motor will switch exactly once, any $C2$ extremal consists of at most two instants of switching, and therefore at most three time intervals of time in which one of the four controls in Table I is used. We call each time interval a phase of the extremal. We will refer to the controls used during any such interval using Table I. If the control $u_1 = u_m$, $u_2 = -u_m$ is used during a phase, for example, we refer to that phase as a $+\alpha$ phase. The possible sequences of control phases that are valid $C2$ extremal controls sequences are

1) $\pm\alpha \rightarrow \pm\beta \rightarrow \mp\alpha$
2) $\pm\alpha \rightarrow \mp\alpha$
3) $\pm\beta \rightarrow \pm\alpha \rightarrow \mp\beta$
4) $\pm\beta \rightarrow \mp\beta$

where the arrow denotes a transition between one control phase (on the left of the arrow) to another control (on the right) at some time instant. Since both motors must switch exactly once, the control in the last phase is always the reversal of the control in the first phase.

For any $C1$ extremal control, one motor never switches, the other motor may switch. Extremal controls of the form $C1$ include singular controls, where the switching motor may switch arbitrarily. We want to identify a special subset of $C1$ controls where the switching motor switches no more than twice. We will say that such controls are of type $C1_{ns}$. The possible $C1_{ns}$ extremal control sequences are

1) $\pm\beta \rightarrow \pm\alpha \rightarrow \pm\beta$
2) $\pm\beta \rightarrow \pm\alpha$
3) $\pm\beta$
4) $\pm\alpha \rightarrow \pm\beta$
5) $\pm\alpha$

The next subsection deals with the selection of the time-optimal control given q_0 and q_d, under the assumption that the desired angular velocity is zero.

C. Synthesis for goal states with zero angular velocity

In this subsection, we will find the time optimal control for initial states of the form (v, θ, ω) and goal state at $(v_d, \theta_d, 0)$. Due to the fact that $f(q, u) = f(q + [v, \theta, 0]^T, u)$, we can change coordinates such that the target state as $(0, 0, 0)$. Thus, if the initial forward velocity is $v(0)$ and initial orientation is $\theta(0)$, then $v = v(0) - v_d$ and $\theta = \theta(0) - \theta_d$. In other words, solving the time-optimal control from any initial state to any other goal state (with zero angular velocity) is equivalent to solving the time-optimal control problem from any initial state to the origin.

Let the initial condition be $q_0 = (v, \theta, \omega)$. Let the initial time be $t = 0$. The motor switches occur at $\bar{t}_1 = t_1$ and $\bar{t}_2 = t_1 + t_2$. The total duration is $\bar{t}_3 = t_1 + t_2 + t_3$. Given the control inputs $u_1(t)$ and $u_2(t)$, we can obtain the state at $t = \bar{t}_3$ through straightforward integration of equation (9). If the control is of the form $C2$, then the state at $t = \bar{t}_3$ is given by

$$\theta(\bar{t}_3) = s_1 u_m t_1^2 + 2 s_1 u_m t_1 t_2 + 2 s_1 u_m t_1 t_3 - s_1 u_m t_3^2 + \theta \tag{24a}$$

$$\omega(\bar{t}_3) = \omega + 2 s_1 u_m t_1 - 2 s_1 u_m t_3 \tag{24b}$$

$$v(\bar{t}_3) = v + 2 s_2 u_m t_2 \tag{24c}$$

where $s_1 = 1$ if the control in the first phase is $+\alpha$ and $s_1 = -1$ if the control in the first phase is $-\alpha$. Similarily, $s_2 = 1$ if the control in the first phase is $+\beta$ and $s_2 = -1$ if the control in the first phase is $-\beta$.

If the control is of the form $C1$, then by integration of (9) we can compute the state at $t = \bar{t}_3$ as

$$v(\bar{t}_3) = v + 2 s_3 u_m t_1 + 2 s_5 u_m t_3 \tag{25a}$$

$$\theta(\bar{t}_3) = \theta + \omega(t_1 + t_2) + s_4 u_m t_2^2 + (\omega + 2 s_4 u_m t_2) t_3 \tag{25b}$$

$$\omega(\bar{t}_3) = \omega + 2 s_4 u_m t_2 \tag{25c}$$

where $s_3 = 1$ ($s_5 = 1$) if the control in the first (third) phase is $+\beta$ and $s_3 = -1$ ($s_3 = -1$) if the control in the first (third) phase is $-\beta$. Similarly, $s_4 = 1$ if the control in the second phase is $+\alpha$ and $s_4 = -1$ if the control in the second phase is $-\alpha$.

Since the goal state is the origin,

$$v(\bar{t}_3) = \theta(\bar{t}_3) = \omega(\bar{t}_3) = 0 \tag{26}$$

We can solve for t_1, t_2 and t_3 by substituting (26) and all possible values of s_i, $i \in \{1, 2, 3, 4, 5\}$ in Equations (24) and (25). We accept only those solutions where $t_1, t_2, t_3 \geq 0$, and from among such solutions we pick the one such that $t_1 + t_2 + t_3$ is minimum.

The formulation above suggests that we must compute 2^5 or 64 solutions of Equations (24) and (25). In fact, there are just 6 possible solutions, since we require $t_2 \geq 0$. Note that (24c) reduces to $0 = 2 s_2 u_m t_2 + v$, which immediately implies that $s_2 = -\text{sign}(v)$. Again, from (25c), $0 = \omega + 2 s_4 u_m t_2$, which yields $s_4 = -\text{sign}(\omega)$. Hence, only s_1, s_3 and s_5 are unknown.

Thus, the solution of (24) and (26) is

$$t_1 = \frac{1}{4 s_1 u_m} \left(-(2\omega + s_1 |v|) + \sqrt{\Delta} \right) \tag{27a}$$

$$t_2 = \frac{|v|}{2} \tag{27b}$$

$$t_3 = \frac{1}{4 s_1 u_m} \left(-(s_1 |v|) + \sqrt{\Delta} \right) \tag{27c}$$

$$\Delta = 2\omega^2 + |v|^2 - 8 s_1 u_m \theta \tag{27d}$$

and the solution of (25) and (26) is

$$t_1 = -\frac{|\omega|}{4 u_m} - \frac{\theta}{\omega} \tag{28a}$$

$$t_2 = \frac{|\omega|}{2 u_m} \tag{28b}$$

$$t_3 = \left(-\frac{s_3}{s_5} \right) \left(\frac{v}{2 s_3 u_m} + \frac{|\omega|}{4 u_m} - \frac{\theta}{\omega} \right) \tag{28c}$$

As mentioned above, we must compute six solutions, and find the solutions with minimum total time $t_1 + t_2 + t_3$, provided t_1, t_2 and t_3 are non-negative. Given the variables s_i, t_1 and t_2 for the minimum-time solution, we can determine the the initial torques $u_1(0)$ and $u_2(0)$ and the times at which they switch, if at all. In this manner, we have obtained the time-optimal control from the initial state to a goal state with zero angular velocity.

We have computed the switching times for all possible $C2$ extremal controls, however we have only looked at a subset of all possible $C1$ extremal controls, namely, those that are of the form $C1_{ns}$. We now establish that for any state (v, θ, ω) such that a singular $C1$ extremal control exists that results in a transition to the origin, then the corresponding $C1_{ns}$ control has the same duration.

Lemma IV.1. *Let $q_0 = (v_0, \theta_0, \omega_0) \in \mathbb{R}^3$. Let there exist an extremal $(q^*(t), \psi^*(t), u^*(t))$ defined on $I = [0, t_f]$ such that $q^*(0) = q_0$ and $q^*(t_f) = (0, 0, 0)$, such that $u^*(t)$ is a singular control. Then there exists a $C1_{ns}$ control $\bar{u}^*(t)$ and corresponding extremal $(\bar{q}^*(t), \bar{\psi}^*(t), \bar{u}^*(t))$ defined on $[0, \bar{t}_f]$ such that $\bar{q}^*(0) = q_0$, $\bar{q}^*(\bar{t}_f) = (0, 0, 0)$ and $\bar{t}_f = t_f$.*

Proof: We can compute the initial right and left wheel speeds ($\dot{\phi}_R(0)$ and $\dot{\phi}_L(0)$ respectively) from v_0 and ω_0 using (3). Since one motor never switches, the duration of any $C1$ extremal control is exactly the time taken to bring the larger (in magnitude) of the initial wheel speeds to zero. Thus,

$$\bar{t}_3 = \frac{\max\left(|\dot{\phi}_R(0)|, |\dot{\phi}_L(0)| \right)}{u_m} \tag{29}$$

Any $C1$ extremal control with initial condition q_0 must have this same duration. Let $|\dot{\phi}_R(0)| > |\dot{\phi}_L(0)|$. The right motor torque $u_1^*(t)$ is given by

$$u_1^*(t) = -s u_m$$

where $s = \text{sign}(\dot{\phi}_R(0))$.

Consider the functional

$$h(u_2(t)) = \int_0^{\bar{t}_3} \int_0^t u_2(\tau) d\tau dt \tag{30}$$

and the functional

$$\gamma(u_2(t)) \ = \dot{\phi}_L(\bar{t}_3) = \dot{\phi}_L(0) + \int_0^{\bar{t}_3} u_2(\tau)d\tau$$

The left motor torque $u_2(t)$ can be arbitrary. However, we must have $\dot{\phi}_L(\bar{t}_3) = 0$, so that $\gamma(u_2(t)) = 0$ for any valid $C1$ extremal. Additionally, we must have $\theta(\bar{t}_3) = 0$. This implies

$$\theta(\bar{t}_3) = \theta_0 + \omega_0 \bar{t}_3 - \frac{su_m \bar{t}_3^2}{2} - \int_0^{\bar{t}_3} \int_0^t u_2(\tau)d\tau dt = 0$$

or

$$h(u_2(t)) = k(q_0)$$

where $k(q_0) = (\theta_0 + \omega_0 \bar{t}_3) - \frac{su_m \bar{t}_3^2}{2}$ is a constant which depends on the initial condition. Clearly, the given singular control $u_2^*(t)$ is such that $\gamma(u_2^*(t)) = 0$ and $h(u_2^*(t)) = k(q_0)$. We claim that a control that only switches twice also satisfies these conditions. Consider the control $u_2^{ns}(t)$ defined as

$$u_2^{ns}(t) = \begin{cases} -su_m & \text{if } t < t_1 \\ su_m & \text{if } t_1 < t < t_1 + t_2 \\ -su_m & \text{if } t_1 + t_2 < t < \bar{t}_3 \end{cases} \tag{31}$$

From the condition that $\gamma(u_2^{ns}(t)) = 0$ we obtain

$$t_2 = \frac{|\omega_0|}{2u_m} \tag{32}$$

Consider the function

$$\begin{aligned} g(t_1) &= h(u_2^{ns}(t)) \\ &= \frac{su_m}{2}(t_1^2 + 2t_1t_2 - t_2^2) + \frac{su_m}{2}(\bar{t}_3 - (t_1 + t_2))^2 \\ &\quad + su_m(t_1 - t_2)(\bar{t}_3 - (t_1 + t_2)) \end{aligned} \tag{33}$$

where $t_1 \in [0, (\bar{t}_3 - t_2)]$ is the only unknown. In order for $u_2^{ns}(t)$ to be a valid control, we must have $g(t^{ns}) = h(u_2^{ns}(t)) = k(q_0) = h(u_2^*(t))$ for some $t^{ns} \in [0, (\bar{t}_3 - t_2)]$.

The right hand side of Equation (30) is clearly maximized when $u_2(\tau) = u_m$, however such a control violates the condition $\gamma(u_2(t)) = 0$. The control that maximizes $h(u_2(t))$ and satisfies $\gamma(u_2(t)) = 0$ is given by

$$u_2^{max}(t) = \begin{cases} u_m & \text{if } t < t_{sw} \\ -u_m & \text{if } t_{sw} < t \end{cases}$$

where t_{sw} is such that $u_2^{max}(t)$ satisfies $\gamma(u_2^{max}(t)) = 0$. Similarily,

$$u_2^{min}(t) = \begin{cases} -u_m & \text{if } t < t_{sw} \\ u_m & \text{if } t_{sw} < t \end{cases}$$

will minimize $h(u_2(t))$ where $u_2^{min}(t)$ satisfies $\gamma(u_2^{min}(t)) = 0$ for suitable t_{sw}. One can check that $h(u_2^{min}(t)) = g(0)$ and $h(u_2^{max}(t)) = g(\bar{t}_3 - t_2)$. Thus, $g(0) < h(u_2^*(t)) = k(q_0) < g(\bar{t}_3 - t_2)$. Since $g(t_1)$ is a continuous function of t_1, we can find $t^{ns} \in [0, \bar{t}_3 - t_2]$ such that $g(t^{ns}) = k(q_0) = h(u_2^*(t))$, proving the result. ∎

Remark 1. Lemma IV.1 implies that if we consider all possible $C1_{ns}$ trajectories from an initial state q_0, then we do not need to consider singular controls even if they exist, since one of the $C1_{ns}$ controls will result in the trajectory reaching the goal state in the same time.

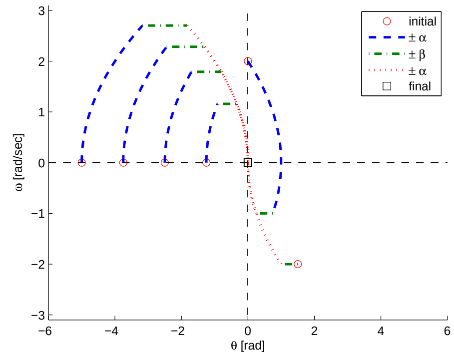

Fig. 2. Time-optimal trajectories for different initial conditions plotted in the $\theta - \omega$ plane. For all simulations, $v(0) = 1m/s$. The initial and final values of (θ, ω) are marked by circles and squares respectively. Trajectories corresponding to control phases $\pm\beta$ are represented by dashed-dotted lines and those for $\pm\alpha$ are represented by dashed or dotted lines.

For any initial condition, we can compute the time-optimal control using the method above, and simulate the open-loop implementation of this control. The results for four initial conditions are plotted in the plane $v = 0$ in Figure 2. For all plotted trajectories, $v(0) = 1m/s$. The circle indicates the initial values of θ and ω for each trajectory. The time-optimal control for the for the initial condition $(1m/s, 1.5rad, -2rad/sec)$ is a $C1_{ns}$ control. The time-optimal controls for the remaining initial conditions are $C2$ controls. The open-loop controls result in all trajectories reaching the origin, as can be seen in the Figure. Note that the time-based switch from α to $\pm\beta$ control phase for appears to occur along a specific line in the plot.

V. Conclusion

We have derived time-optimal controls that enable a torque-controlled differential drive wheeled mobile robot to reach a desired constant velocity in the plane in minimum time, for any initial velocity. The controls have been obtained as functions of time, and are designed to be implemented in open loop. Plots of the trajectories due to these open-loop control suggest that a switching feedback control law may be obtained to achieve the minimum-time transition to the origin, and this has been investigated in [8].

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