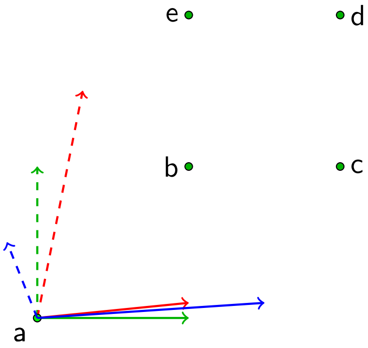
Cartesian Coordinates and Rigid Transformations

Spring 2020

Hasan Poonawala

Same Vector Space, Different Bases



A basis and an origin together form a **coordinate frame** or **reference frame**.

Change Of Basis

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When we use a different basis, the coordinates assigned to a point must change, in order to correctly regenerate that point using the new basis.

Let A, B, C, ... be different coordinate frames.

A point p then has coordinates p^A, p^B, p^C ... corresponding to each basis.

Change Of Vector Space Basis

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Let the coordinates of p in frame A be $p^A = (\alpha_1^A, \alpha_2^A, \ldots, \alpha_n^A)$, so that the point p can be expressed as

$$p = \sum_j^n \alpha_j^A e_A^j$$

Note that p is an abstract point equivalent to the coordinate-given combination of the basis $\{e_A^1, e_A^2, \ldots, e_A^n\}$.

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Note that p is an abstract point equivalent to the coordinate-given combination of the basis $\{e_A^1, e_A^2, \ldots, e_A^n\}$.

Similarly, if $p^B = (\beta_1^B, \beta_2^B, \ldots, \beta_n^B)$, then

$$p = \sum_i^n \beta_i^B e_B^i$$

Change Of Vector Space Basis

So, we can write

$$e_B^i = \sum_i^n T_{ji} e_A^i; \quad p = \sum_i^n \beta_i^B e_B^i; \quad p = \sum_j^n \alpha_j^A e_A^j \tag{1}$$

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Combining the first and second equation in (1), we get

$$\begin{aligned} p = \sum_i^n \beta_i^B e_B^i &= \sum_i^n \beta_i^B \left(\sum_j^n T_{ji} e_A^j \right) \\ &= \sum_j^n \left(\sum_i^n \left(\beta_i^B T_{ji} \right) \right) e_A^j \end{aligned} \tag{2}$$

Comparing (2) to the third equation in (1), we get

$$\alpha_j^A = \sum_i^n \left(\beta_i^B T_{ji} \right).$$

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To see this, we abuse some notation and write:

$$p = \begin{bmatrix} e_A^1 & e_A^2 & \cdots & e_A^n \end{bmatrix} \begin{bmatrix} \alpha_1^A \\ \alpha_2^A \\ \vdots \\ \alpha_n^A \end{bmatrix} = \begin{bmatrix} e_B^1 & e_B^2 & \cdots & e_B^n \end{bmatrix} \begin{bmatrix} \beta_1^B \\ \beta_2^B \\ \vdots \\ \beta_n^B \end{bmatrix}$$

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The coordinates of e_B^i in frame A give:

$$e_B^1 = \begin{bmatrix} e_A^1 & \cdots & e_A^n \end{bmatrix} \begin{bmatrix} T_{11} \\ \vdots \\ T_{n1} \end{bmatrix}, e_B^2 = \begin{bmatrix} e_A^1 & \cdots & e_A^n \end{bmatrix} \begin{bmatrix} T_{12} \\ \vdots \\ T_{n2} \end{bmatrix}, \ldots$$

Change Of Vector Space Basis

We can collect these expressions for point e_B^i as

$$\begin{bmatrix} e_B^1 & e_B^2 & \cdots & e_B^n \end{bmatrix} = \begin{bmatrix} e_A^1 & e_A^2 & \cdots & e_A^n \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{bmatrix},$$

So that

$$\begin{bmatrix} e_B^1 & e_B^2 & \cdots & e_B^n \end{bmatrix} \begin{bmatrix} \beta_1^B \\ \beta_2^B \\ \vdots \\ \beta_n^B \end{bmatrix} = \begin{bmatrix} e_A^1 & e_A^2 & \cdots & e_A^n \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{bmatrix} \begin{bmatrix} \beta_1^B \\ \beta_2^B \\ \vdots \\ \beta_n^B \end{bmatrix}$$

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Since

$$p = \begin{bmatrix} e_A^1 & e_A^2 & \cdots & e_A^n \end{bmatrix} \begin{bmatrix} \alpha_1^A \\ \alpha_2^A \\ \vdots \\ \alpha_n^A \end{bmatrix},$$

we find that transforming coordinates is a linear operation represented by matrix operations:

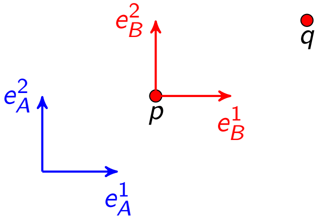
$$\begin{bmatrix} \alpha_1^A \\ \alpha_2^A \\ \vdots \\ \alpha_n^A \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{bmatrix} \begin{bmatrix} \beta_1^B \\ \beta_2^B \\ \vdots \\ \beta_n^B \end{bmatrix}$$

More compactly: $p^B = \left(T_B^A\right)^{-1} p^A$, where to example

$$T_B^A = \left[\left(e_B^1\right)^A \quad \left(e_B^2\right)^A \quad \cdots \quad \left(e_B^n\right)^A \right].$$

Change Of Origin

Suppose points p, q have coordinates p^A, q^A in a frame A. Consider frame B whose origin is at p, with the same basis elements for its vector space as the frame A. What is q^B?



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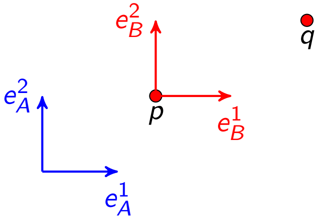

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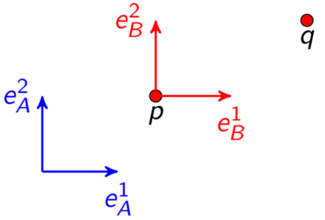

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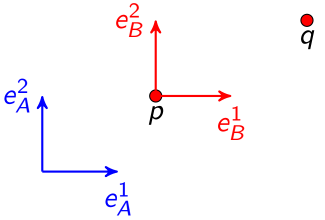

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Precisely because vectors are free, the coordinates of v in frame B will be the same as that in frame A. So, $q^B = q^A - p^A$.

In general, $q^B = q^A -$ (coordinates of origin of B in A)

Change Of Frames

Combining previous discussions, we get that:

To map coordinates from one frame to another, we express the coordinates of the basis vectors (through, say, matrix T_B^A) and the origin of one frame in another (through, say coordinate vector o_B^A), and use the map

$$p^B = \left(T_B^A\right)^{-1}(p^A - o_B^A)$$

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If all bases for the plane give us two numbers, what's special about a basis where the two elements are at 90 degrees , and have the same 'length'?

Norms and Distances

Let's reconsider our hard won example:

$$T_B^A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}; \quad q^A = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \implies q^B = \left(T_B^A\right)^{-1} q^A = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

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Note that $\|q^B\|_B = \|\left(T_B^A\right)^{-1} q^A\|_A$.

Q: What kinds of matrices preserve the norms of the vectors they act upon?

Special Orthogonal Group in Three Dimensions

if $T_B^A \in SO(3)$, then we'd have norm-preservation.

Definition ($SO(3)$)

The Special Orthogonal Group $SO(3)$ is the set of matrices $R \in \mathbb{R}^{3\times 3}$ such that

$$R^T R = Id, \text{and } \det R = 1$$

.

$SO(3)$ is known as the orientation group **and** the rotation group.

Example

Problem: Find p^B if $p^A = (1, 1)$.
Solution: From the diagram,

$$e_B^1 = e_A^1$$
$$e_B^2 = e_A^1 + e_A^2$$
$$\implies T_B^A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Apply the formula:

$$p^B = \left(T_B^A\right)^{-1} p^A$$
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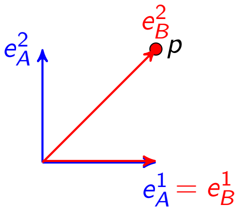

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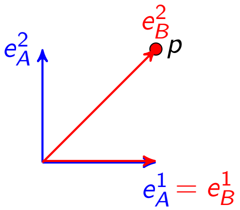

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$$\left(T_A^B\right)^T T_A^B = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

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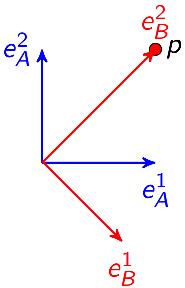

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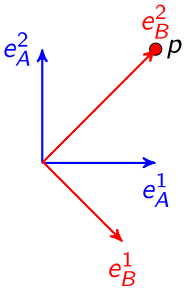

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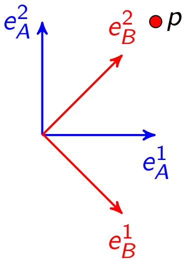

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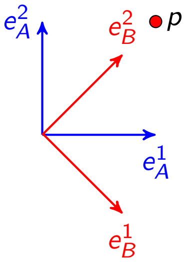

norm-preserving!

$$\left(T_A^B\right)^T T_A^B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Orthonormal Vectors

We have seen that

$$T_B^A = \begin{bmatrix} \left(e_B^1\right)^A & \left(e_B^2\right)^A & \cdots & \left(e_B^n\right)^A \end{bmatrix}.$$

Therefore,

$$\left(T_B^A\right)^T T_B^A = \begin{bmatrix} \left(\left(e_B^1\right)^A\right)^T \\ \left(\left(e_B^2\right)^A\right)^T \\ \vdots \\ \left(\left(e_B^1\right)^A\right)^T \end{bmatrix} \begin{bmatrix} \left(e_B^1\right)^A & \left(e_B^2\right)^A & \cdots & \left(e_B^n\right)^A \end{bmatrix}$$

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$$= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Effectively, the coordinates of basis vectors of B in frame A are unit length and perpendicular to each other.

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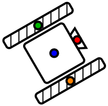
- To preserve distance, the linear part of the affine map must be in SO(3)
- $T_B^A \in SO(3)$ when basis vectors are all unit length, mutually perpendicular.
- The coordinate transformation is then $p^B = \left(R_B^A\right)^{-1}\left(p^A - o_B^A\right)$ ▸ mobile robot

Coordinate Transformation Vs Rigid Motion



Consider a robot with a center, a camera in ‘front’, and two wheels to the side.

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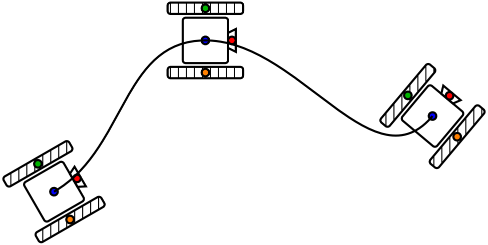

Whenever we move the robot, the distances between these points don't change.

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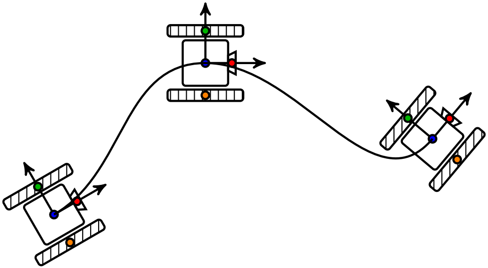
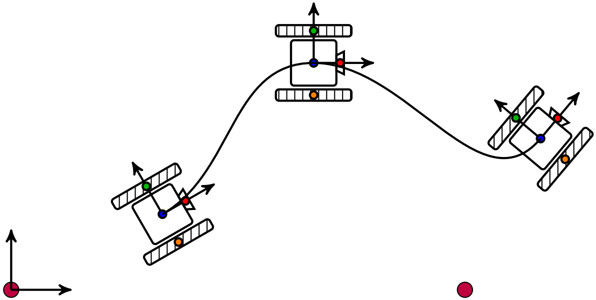


As the robot moves, we can take a snapshot of these points, and they each define a coordinate frame for Euclidean space.

Coordinate Transformation Vs Rigid Motion



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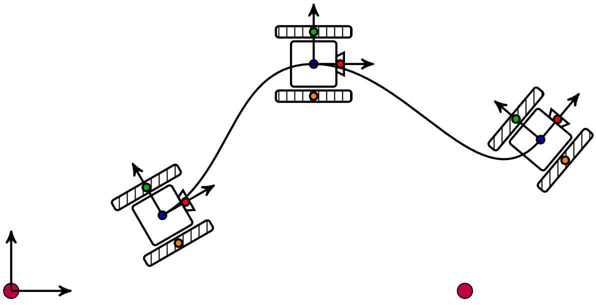

Q1: How would the robot compare observations of either purple point over time? A1: Coordinate transformations

Q2: How do we keep track of all the points on the robots?

A2: Coordinate transformations, but reinterpreted. ▸ rigid motion

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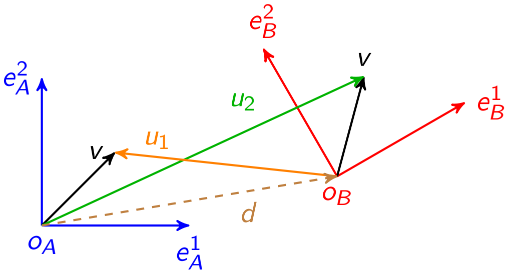

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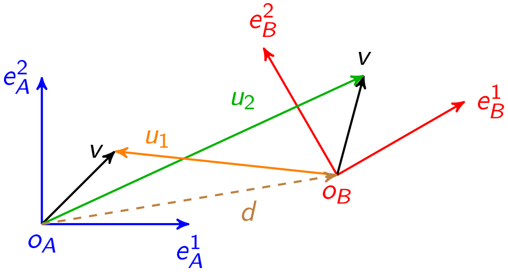
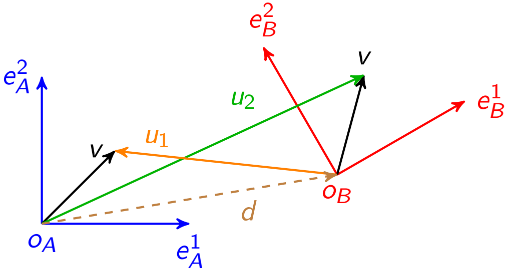


- If we view u_1 as coordinates in frame B, we've changed coordinates of v from world to body frame.
- If we view u_2 as coordinates in frame A, we've moved the point $o_A \oplus v$ relative to frame A.

Rigid Body Pose



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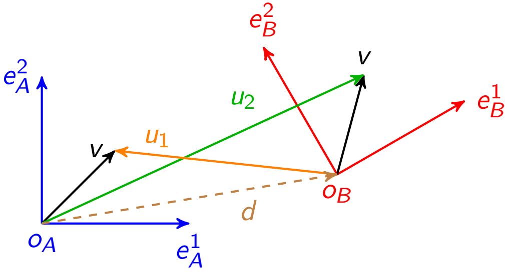

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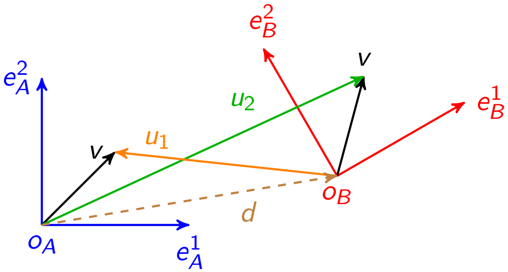

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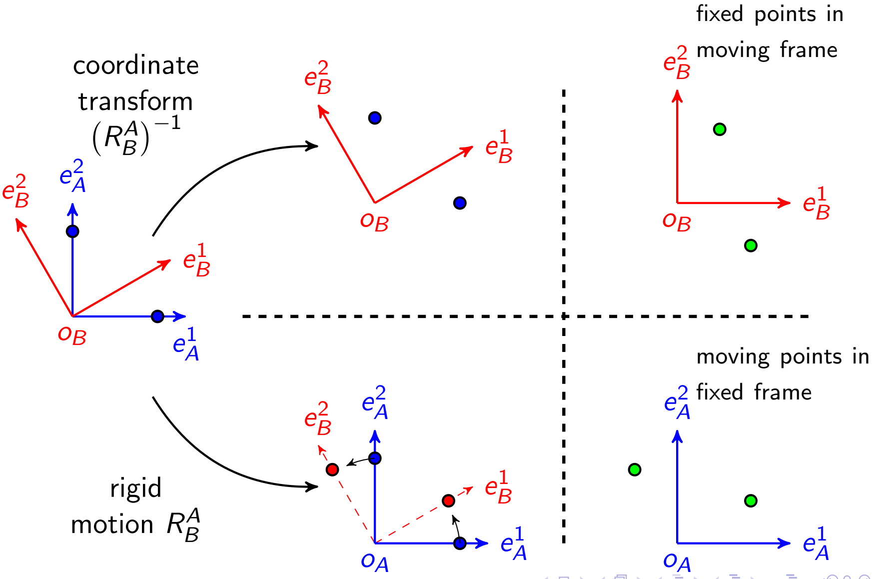
Move in frame A = reorient by R and then move by d : $Rv + d$

Example



coordinate
transform
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e_B^1
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e_A^1
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fixed points in
moving frame
e_B^2
o_B
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motion R_B^A
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Special Euclidean Group $\mathrm{SE}(3)$

Coordinates of points in 3D Euclidean space $= p^A \in \mathbb{R}^3$
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Instead, we define an identity element (it's a group): the reference coordinate frame.

Homogenous Transformations

We can convert the affine map between two Euclidean spaces of dimension 3 into a linear map between two subsets of $\mathbb{R}^4$.

Define a homogenization $h\colon \mathbb{R}^3 \mapsto \mathbb{R}^4$ as $h\left(p^A\right) = \begin{bmatrix} p^A \\ 1 \end{bmatrix}$.

If $p^A = Rp^B + d$, then

$$h\left(p^A\right) = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix} h\left(p^B\right). \tag{6}$$

The matrix $\begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix}$ represents a homogenous transformation, and forms a group.
You show the group structure in HW2.

An Example

On Board: Three robot poses

Checkpoint

- The coordinate transformation is $p^B = \left(R_B^A\right)^{-1}\left(p^A - o_B^A\right)$
- Norm-preserving coordinate transformation $=$ rigid motion of points within the same coordinate frame.
- Set of rigid body poses/rigid motions forms a group: $\mathrm{SE}(3)$
- After choosing a reference frame, we assign coordinates – aka rigid body pose – (d, R) to frame (Torsor structure)

Back to SO(3)

- Rigid bodies correspond to cartesian frames

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- The G-Torsor nature is why SO(3) is called both the rotation group and the orientation group.
- Assigning coordinates to an orientation is the same as defining the rotation that generates that frame.

Basic Rotations

Consider three frames rotated about each one of the world frame axes by an angle θ.

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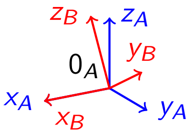

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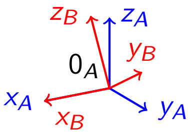

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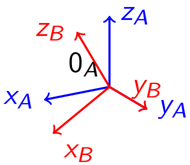

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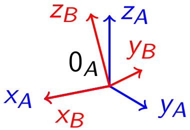

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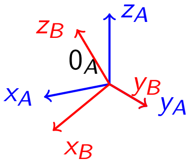

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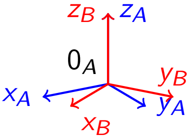

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For rotations, they do. In general, $R_1 R_2 \neq R_2 R_1$.

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How would you pick the right transformation? Why did we not consider R_A^B?

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Change-of-Identity For Rotations

- The rotation R^A is relative to frame A.
- A general orientation P has coordinates R_P^A in frame A
- Rotating this point results in an orientation $R^A R_P^A$ in frame A:

$$R_P^A \mapsto R^A R_P^A$$

- But note that $R_P^A = R_C^A R_P^C$
- Therefore :

$$R_C^A R_P^C \mapsto R^A R_C^A R_P^C, \text{or}$$

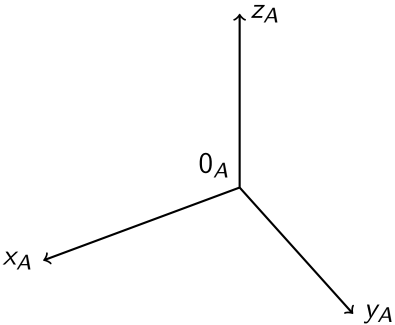
$$R_P^C \mapsto \left(R_C^A\right)^{-1} R^A R_C^A R_P^C, \text{or}$$

- Therefore, a rotation R^A in frame A becomes a rotation

$$R^B = \left(R_C^A\right)^{-1} R^A R_C^A$$

in frame C.

Extrinsic vs Intrinsic Rotations



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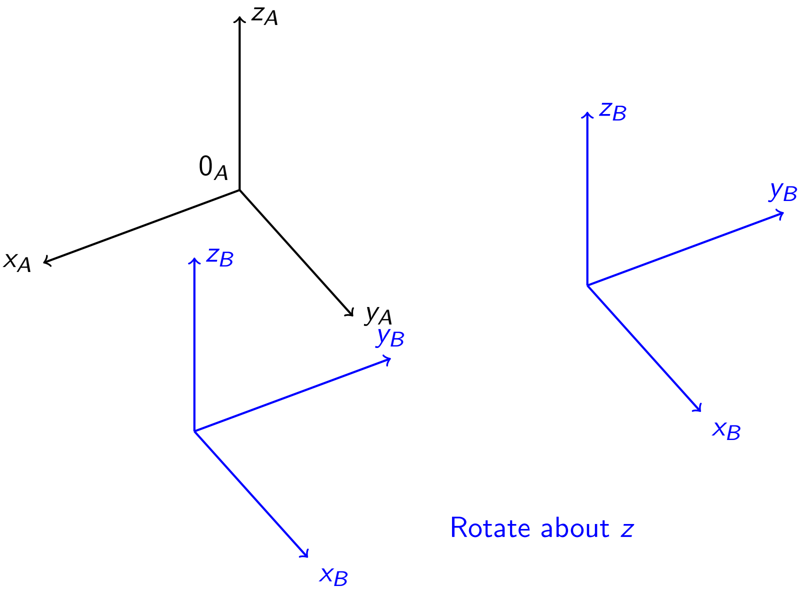
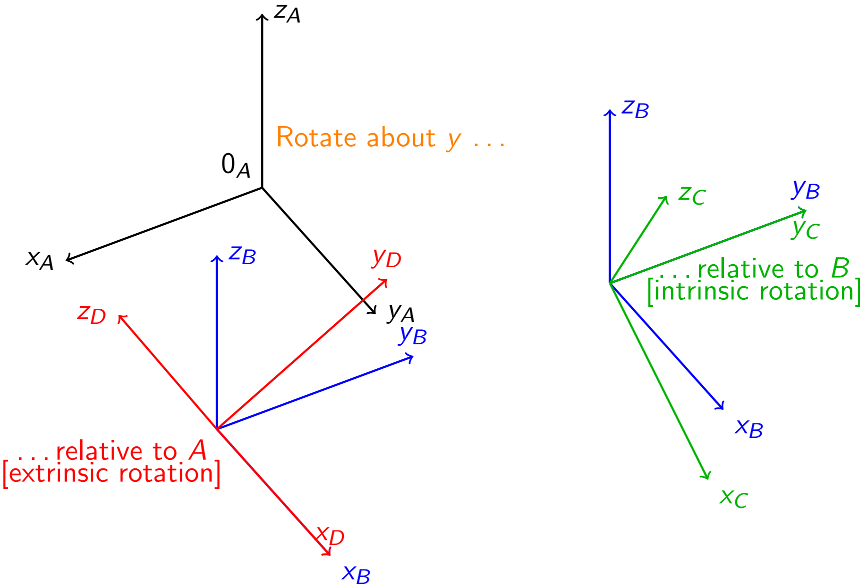


Rotate about z

Extrinsic vs Intrinsic Rotations



Extrinsic vs Intrinsic Rotations

- A first rigid motion corresponding to rotation R_1 relative to a frame A produces frame B
- A second rigid motion rotation R_2 can be applied relative to either A or B.
- When applied relative to B, the second rotation is an intrinsic rotation. $R = R_1 R_2$.
- When applied relative to A, the second rotation is an extrinsic rotation. $R = R_2 R_1$.