ODE Examples

Spring 2020

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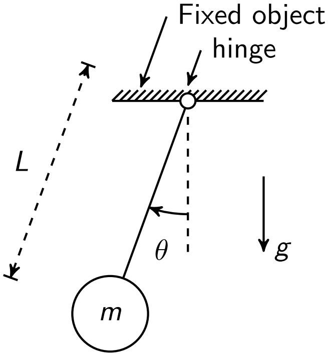
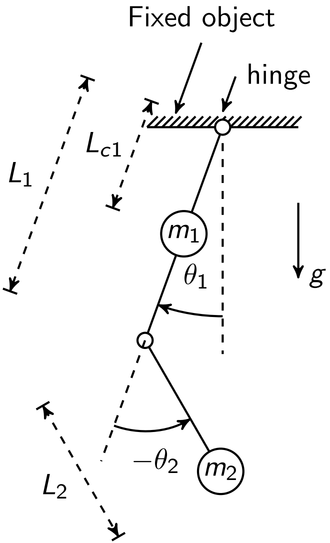
Simple Pendulum

- Mass suspended by rigid massless string
- Downward position is $\theta = 0$
- Force due to gravity



$$mL^2\ddot{\theta} + mgL\sin\theta = 0$$

Double Pendulum



Double Pendulum

Equations of Motion:

$$\left(m_1 L_{c1}^2 + m_2 L_1^2\right)\ddot{\theta}_1 + d(\theta_2)\ddot{\theta}_2 + 2h(\theta_2)\dot{\theta}_1\dot{\theta}_2 + h(\theta_2)\dot{\theta}_2^2 \\ + m_2 L_2 g\cos(\theta_1+\theta_2) + (m_1 L_{c1} + m_2 L_1)\, g\cos\theta_1 = 0$$

$$d(\theta_2)\ddot{\theta}_1 + m_2 L_2^2\ddot{\theta}_2 - h(\theta_2)\dot{\theta}_1^2 + m_2 L_2 g\cos(\theta_1+\theta_2) = 0$$

where $d(\theta_2) = \left(m_2 L_2^2 + m_2 L_1 L_2 \cos\theta_2\right)$ and
$h(\theta_2) = -m_2 L_1 L_2 \sin\theta_2$

Double Pendulum

Equations of Motion:

$$\left(m_1 L_{c1}^2 + m_2 L_1^2\right)\ddot{\theta}_1 + d(\theta_2)\ddot{\theta}_2 + 2h(\theta_2)\dot{\theta}_1\dot{\theta}_2 + h(\theta_2)\dot{\theta}_2^2 + m_2 L_2 g\cos(\theta_1+\theta_2) + \left(m_1 L_{c1} + m_2 L_1\right) g\cos\theta_1 = 0$$

$$d(\theta_2)\ddot{\theta}_1 + m_2 L_2^2\ddot{\theta}_2 - h(\theta_2)\dot{\theta}_1^2 + m_2 L_2 g\cos(\theta_1+\theta_2) = 0$$

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More terms if links have rotational inertia in addition to mass.

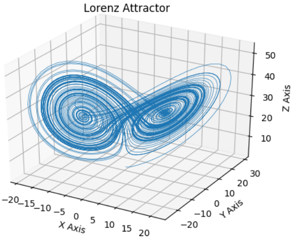
Lorenz Attractor

Simplified model of convection in the atmosphere.
First simulation model to exhibit chaos.

$$\dot{x}_1 = 10(x_2 - x_1) \tag{1}$$

$$\dot{x}_2 = (28 - x_3)x_1 - x_2 \tag{2}$$

$$\dot{x}_3 = x_1 x_2 - \frac{8}{3}x_3 \tag{3}$$



Second Order Linear Dynamical Systems

The ODE for a second order linear system is

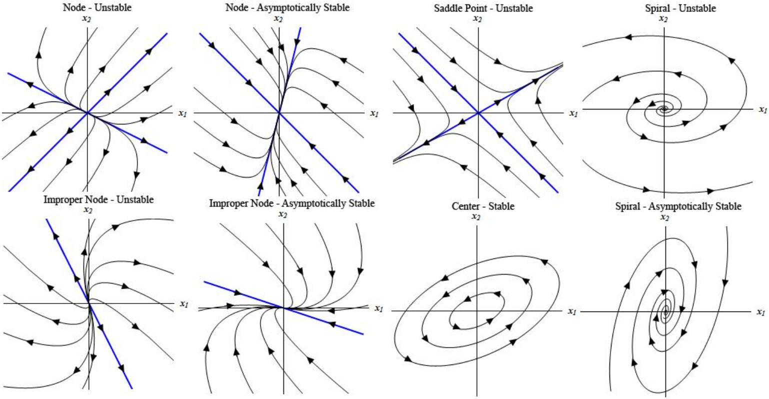
$$\ddot{q}(t) + a_1\dot{q}(t) + a_0 q(t) = 0$$

In state-space form, with $x = \begin{bmatrix} q & \dot{q} \end{bmatrix}^T$, we have

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} x = Ax.$$

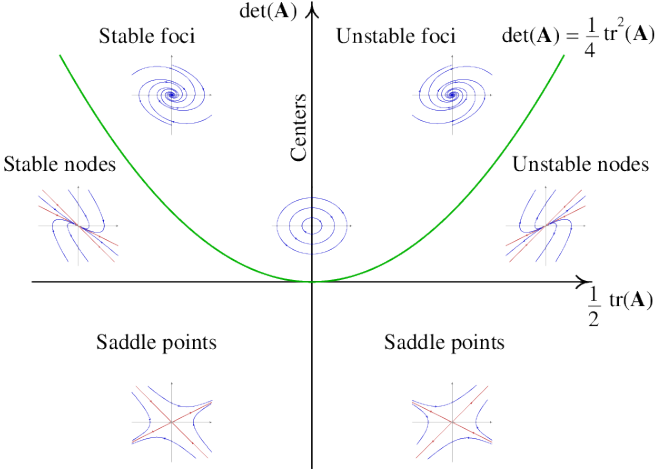
The possible behaviors of the solutions of this system depend on the matrix A.

Second Order Linear Dynamical Systems



Depending on A, solutions from multiple initial conditions will fill a plot of $\dot{q}$ vs q in ways qualitatively similar to these diagrams

Second Order Linear Dynamical Systems



If the eigenvalues of A are λ_1 and λ_2, then

$$\det(A) = \lambda_1 \lambda_2; \quad \mathrm{tr}(A) = \lambda_1 + \lambda_2.$$