ME 599/699 Robot Modeling & Control
Robot Kinematics and Dynamics

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1 Kinematic Chains

Kinematic chains consist of a set of rigid bodies connected to each other through joints.
Start with a base rigid body that's so massive it's fixed. We typically call this the world or inertial frame.
Types of Kinematic Chains:

- Open / Closed
- Serial / Parallel

Simple Joints:

- Prismatic
- Revolute

1.1 Serial Kinematic Chains

We look at serial kinematic chains where all joints are simple.
We number links as 0 for base to n in sequence.
The assumption of single-parameter joints means we can use basic transformations to handle coordinates etc.
These basic transformation are denoted $A_i(q_i)$, where $q_i \in \mathbb{R}$ is the joint variable.
q_i is either an angle θ_i or a distance d_i, depending on the type of simple joint.
Given link i and $i-1$,

$$A_i = \begin{bmatrix} R_i^{i-1} & o_i^{i-1} \\ 0 & 1 \end{bmatrix} \tag{1}$$

Transformations between links i and j is T_j^i, where we are expressing frame j in frame i.

$$T_j^i = \begin{cases} A_{i+1}A_{i+2}\cdots A_{j-1}A_j & i < j \\ I & i = j \\ \left(T_j^i\right)^{-1} & i > j \end{cases} \tag{2}$$

Since we don't want to manually compute R_i^{i-1} for each q_i, we use our previous tricks of composing rotations etc.

1.2 Denavit-Hartenberg Convention

A popular convention that facilitates consistent communication of robot manipulator information. In this convention

- All motion happens along the z axis
- Four numbers instead of 12 numbers (or six)

Based on two restrictions

(DH1) The x_1 axis intersects the z_0 axis.

(DH2) The x_1 axis is orthogonal to the z_0 axis.

This restriction makes the transformation matrix

$$A = \mathrm{Rot}_{z,\theta_i}\mathrm{Trans}_{z,d_i}\mathrm{Trans}_{x,a_i}\mathrm{Rot}_{x,\alpha_i} \tag{3}$$

2 Geometric and Analytic Jacobians

Forward and inverse kinematics are about creating the map $T_n^0(q)$ that provides the end effector pose $\left(o_n^0(q), R_n^0(q)\right)$.
In a similar way, when the link variables q change with time as $\dot{q}$, what is the 'velocity' of the end effector?

2.1 Geometric Jacobian

We represent the end effector velocity as (v_n^0, ω_n^0), where

$$v_n^0 = \dot{o}_n^0 \tag{4a}$$

$$S(\omega_n^0) = \dot{R}_n^0 \left(R_n^0\right)^T \tag{4b}$$

We saw that the desired characterization of the velocity ξ of the end effector is linear in the rate of change of q. That is,

$$\xi \;= \begin{bmatrix} v_n^0 \\ \omega_n^0 \end{bmatrix} = J\dot{q} = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix} \dot{q}, \tag{5}$$

where J is the kinematic Jacobian.
We compute the i^{th} column J_{v_i} of J_v as

$$J_{v_i} = \begin{cases} z_{i-1} & , \text{ if joint } i \text{ is prismatic} \\ z_{i-1} \times (o_n - o_{i-1}) & , \text{ if joint } i \text{ is revolute} \end{cases} \tag{6}$$

We compute the i^{th} column J_{ω_i} of J_ω as

$$J_{\omega_i} = \begin{cases} 0 & , \text{ if joint } i \text{ is prismatic} \\ z_{i-1} & , \text{ if joint } i \text{ is revolute} \end{cases} \tag{7}$$

Note that J is actually a function $J(q)$, since the axes z_i, $i \in \{1, \dots, n-1\}$ depend on q.

2.2 Analytic Jacobian

The geometric Jacobian $J(q)$ is not the partial derivative of any map from q to $\left(o_n^0(q), R_n^0(q)\right)$. In particular, the angular velocity ω_n^0 is usually not the derivative of the coordinates representing the configuration.
Suppose we represent the position and orientation of the end effector using vectors $d(q) \in \mathbb{R}^3$ and $\alpha(q) \in \mathbb{R}^3$, so that

$$X = \begin{bmatrix} d(q) \\ \alpha(q) \end{bmatrix}, \tag{8}$$

and

$$\dot{X} = \begin{bmatrix} \dot{d} \\ \dot{\alpha} \end{bmatrix} = J_a(q)\dot{q}, \tag{9}$$

where $J_a(q)$ is known as the analytic Jacobian. It maps the rates of change of the link angles, $\dot{q}$ to the rates of change of the chosen configuration X of the end-effector.
To derive it, we use the geometric Jacobian. To do so, note that we can define the angular velocity ω in terms of the rates of change of a parametrization such as Euler angles. For example, let α be $Z - Y - Z$ Euler angles (ϕ, θ, ψ) such that $R = R_{z,\psi} R_{y,\theta} R_{z,\phi}$ and $\dot{R} = S(\omega)R$, we can obtain a map $\omega = B(\alpha)\dot{\alpha}$. Then,

$$J(q)\dot{q} = \begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} \dot{d} \\ B(\alpha)\dot{\alpha} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & B(\alpha) \end{bmatrix} J_a(q)\dot{q}, \tag{10}$$

When $\det B(\alpha) \neq 0$, we can derive

$$J_a(q) = \begin{bmatrix} I & 0 \\ 0 & B^{-1}(\alpha) \end{bmatrix} J(q). \tag{11}$$

2.3 Singularities

Once we have an expression for the J, we can use it to find $\dot{q}$ given some desired values for v_0, ω_0. To do so, we must solve

$$\xi = J\dot{q} \tag{12}$$

If J is an invertible 6×6 matrix, we are done. Often, J is not invertible, because it is not square of because it does not have full rank when square.

Singularities Suppose $J \in \mathbb{R}^{6\times n}$, where n is the number of simple joints in a kinematic chain. The rank $r(J)$ of $J(q)$ is $\min(6, n)$, and the set of reachable velocities is a subspace with dimension $r(J)$. Ideally, we want $r(J) = 6$ so that any arbitrary velocity can be achieved at any configuration.
This situation fails when J has rank less than the dimension of ξ, which for an end-effector is always 6. The matrix J depends on q, and at some configurations, J may lose rank. These configurations are known as singularities or singular configurations.
It's not just that the singular point prevents calculation of $\dot{q}$, but that J is ill-conditioned near it.

- Some singular points become unreachable under perturbations of the system mechanical parameters.

2.4 Decoupling Singularities

For a manipulator comprising a 3-DOF arm and a 3-DOF wrist, the Jacobian $J(q)$ can be made to exhibit a block diagonal structure that makes studying its singular configurations easier. The main step that achieves this is to ensure that o_6 coincides with the already coincident origins o_3, o_4, o_5. Then,

$$J = \begin{bmatrix} J_{11} & 0 \\ J_{12} & J_{22} \end{bmatrix}, \tag{13}$$

and so

$$\det J = \det J_{11} \det J_{22}. \tag{14}$$

For the three-link articulated manipulator, we can derive that

$$\det J_{11} = a_2 a_3 \sin\theta_3 \left(a_2 \cos\theta_2 + a_3 \cos(\theta_2 + \theta_3)\right). \tag{15}$$

2.5 Inverse Velocity

When $n > 6$, and $r(J) = 6$, we can reach any velocity ξ, however the matrix J is not invertible. To compute $\dot{q}$ from ξ, we use a pseudo-inverse J^+. One example for J^+ is the right inverse $J^T(JJ^T)^{-1}$. That is, $\dot{q} = J^+\xi + (I - J^+J)b$, where $b \in \mathbb{R}^n$ is an arbitrary vector that does not affect ξ. If we want to minimize the norm of $\dot{q}$, we choose $b = 0$.

Example 1 (Elbow Manipulator)**.** For example, let the task X be the position of the end effector of a planar elbow manipulator. The configuration is $q_1, q_2 = \theta_1, \theta_2$ (both angles). We can compute

$$\dot{X} = J \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}. \tag{16}$$

We find that $\mu = |\det J| = a_1 a_2 |s_2|$ ($|\det J|$ for non-redundant manipulators). When θ_2 is small, we have the least ability to move in all directions.

2.6 Manipulability

Suppose that $\xi \in \mathbb{R}^m$, so that $J(q) \in \mathbb{R}^{m\times n}$. We use the minimum norm solution $\dot{q} = J^+\xi$ to obtain link variable velocities from end-effector velocities.
We can derive

$$\|\dot{q}\|^2 = \xi^T (JJ^T)^{-1} \xi \tag{17}$$

If $r(J) = m$, so that J is full rank, then we can define a manipulability ellipsoid in $\mathbb{R}^m$ as follows. Let $J = U\Sigma V$, the singular value decomposition.
Then

$$\xi^T (JJ^T)^{-1} \xi = (U^T\xi)^T \Sigma_m^{-2} (U^T\xi), \tag{18}$$

in which

$$\Sigma_m^{-2} = \begin{bmatrix} \sigma_1^{-2} & & & \\ & \sigma_2^{-2} & & \\ & & \ddots & \\ & & & \sigma_m^{-2} \end{bmatrix} \tag{19}$$

Substituting $w = U^T\xi$, we finally get that

$$\|\dot{q}\|^2 = w^T\sigma_1^{-2}w = \sum_{i=1}^{m}\frac{w_i^2}{\sigma_i^2} \tag{20}$$

If we look at unit norm velocities in the joint space, these vectors form an ellipsoid defined by σ_i^2 in the space w which is a rotated version of ξ.
The manipulability μ is then given by

$$\mu = \Pi_{i=1}^{m}\sigma_i \tag{21}$$

Example 2 (Planar Elbow Manipulator)**.** For example, let the task X be the position of the end effector of a planar elbow manipulator. The configuration is $q_1, q_2 = \theta_1, \theta_2$ (both angles). We can compute

$$\dot{X} = J\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{bmatrix} = \begin{bmatrix}-a_1s_1 - a_2s_{12} & -a_2s_{12}\\ a_1c_1 + a_2c_{12} & a_2c_{12}\end{bmatrix}\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{bmatrix}. \tag{22}$$

We find that $\mu = |\det J| = a_1a_2|s_2|$ ($|\det J|$ for non-redundant manipulators). When θ_2 is small, we have the least ability to move in all directions.

3 Static Force/Torque Relationships

Let $F = \begin{bmatrix}F_x & F_y & F_x & n_x & n_y & n_z\end{bmatrix}$ be the vector of forces and moments at the end effector. Let τ be the corresponding vector of joint torques. Then F and τ are related by

$$\tau = J^T(q)F \tag{23}$$

3.1 Derivation

One way to derive this relationship is the principle of virtual work. The idea is to imagine infinitesmal displacements δX and δq which satisfy the system constraints on motion. These displacements are called virtual displacements,. The total work done by F and τ when achieving these virtual displacements is

$$\delta w = F^T\delta X - \tau^T\delta q. \tag{24}$$

Since $\delta X = J(q)\delta q$, which is one constraint on the displacements due to the system, we obtain that

$$\delta w = (F^TJ - \tau^T)\delta q. \tag{25}$$

The principle of virtual work says for a system in equilibrium, the total work done under any virtual displacement satisfying the constraints must be zero. Thus, (23) holds.

3.2 Manipulability

The map from the force to the torque is characterized by J. Manipulability determines our ability to apply force (or accelerate) in certain directions.
In this case, it is sometimes good to have near singular configurations, since we get more force in some directions for the same input energy (norm of torque).

4 Dynamics

So far, our control was based of an independent joint model that isolated each link and used robustness to account for this huge assumption. Gravity and dynamic coupling will limit the success of this approach in highly complex or high-energy motions. Therefore, we need to start considering a coupled dynamics model. We use the Euler-Lagrange framework to obtain the classic robot dynamics model given by

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau + J^TF_{ext} - B\dot{q}, \tag{26}$$

where q is the configuration, $\dot{q}$ the joint rates of change, F are the external forces (excluding gravity and viscous damping), and τ are the motor torques. The terms D, C, B, J, and g are the mass or inertia matrix, Coriolis matrix, damping coefficient matrix, geometric Jacobian, and conservative force vector (usually gravity) respectively.
q is often referred to as a generalized coordinate in dynamics. This coordinate represents degrees of freedoms after accounting for all holonomic constraints in the system.
We can derive these equations by defining the Lagrangian $\mathcal{L}$ of the system, which is the difference between the kinetic and potential energies of the system. This derivation leads to all terms on the left hand side of (26). The terms on the right are essentially non-conservative external forces acting on the system.

4.1 Kinetic Energy

The kinetic energy of the robot is the sum of the kinetic energies of its link. Since each link is a rigid body, we know how to calculate its kinetic energy. Each link has a mass m and principal inertia I about its center-of-mass q that has velocity v and angular velocity ω in the world frame. The kinetic energy is given by

$$\mathcal{K} = \frac{1}{2}mv^T v + \frac{1}{2}\omega^T \mathcal{I}\omega, \tag{27}$$

where $\mathcal{I}$ is the inertia of the link with respect to the world frame given by RIR^T where R is orientation of principal axes in world frame. The equation for I is

$$I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}, \tag{28}$$

where

$$I_{xx} = \int\int\int (y^2 + z^2)\rho(x, y, z)dxdydz \tag{29}$$

$$I_{yy} = \int\int\int (x^2 + z^2)\rho(x, y, z)dxdydz \tag{30}$$

$$I_{zz} = \int\int\int (x^2 + y^2)\rho(x, y, z)dxdydz \tag{31}$$

and

$$I_{xy} = I_{yx} = -\int\int\int xy \;\rho(x, y, z)dxdydz \tag{32}$$

$$I_{xz} = I_{zx} = -\int\int\int xz \;\rho(x, y, z)dxdydz \tag{33}$$

$$I_{yz} = I_{zy} = -\int\int\int yz \;\rho(x, y, z)dxdydz \tag{34}$$

When we combine these kinetic energies, we get

$$\begin{aligned} K &= \sum_i \frac{1}{2}m_i v_i^T v_i + \frac{1}{2}\omega_i \mathcal{I}\omega_i \\ &= \sum_i \frac{1}{2}m_i \dot{q}^T J_{v_i}^T(q) J_{v_i}(q)\dot{q} + \frac{1}{2}\dot{q}^T J_{\omega_i}^T(q) R_i(q) I_i R_i^T(q) J_{\omega_i}\dot{q} \\ &= \frac{1}{2}\dot{q}^T D(q)\dot{q} \\ &= \sum_{i,j} d_{ij}(q)\dot{q}_i\dot{q}_j \end{aligned} \tag{35}$$

4.2 Potential Energy

Potential energy P due to gravity is

$$P = \sum_{i=1}^{n} m_i g \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} c_i^0(q) \tag{36}$$

where $c_i^0(q)$ is the location of the center of mass of link i in the world frame (and not the origin of the i^{th} frame).

5 Euler Lagrange Equations

See the following online resource
Given a Lagrangian $K - P$, we can derive an equation of motion for each generalized coordinate q_k as

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} - \frac{\partial \mathcal{L}}{\partial q_k} = \tau_j \tag{37}$$

Given the expressions for K and P above, we end up with

$$\sum_{j=1}^{n} d_{kj}(q)\ddot{q}_j + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ijk}(q)\dot{q}_i\dot{q}_j + g_k(q) = \tau_k, \quad k \in \{1, \ldots, n\} \tag{38}$$

where

$$c_{ijk} = \frac{1}{2}\left\{\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right\} \tag{39}$$

$$g_k = \frac{\partial P}{\partial q_k} \tag{40}$$

The compact representation of these equations is

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = \tau, \tag{41}$$

where $c_{kj} = \sum_{i=1}^{n} c_{ijk}(q)\dot{q}_i$. Adding non-conservative forces such as viscous friction and externally applied forces yields (26).

5.1 Example

Planar Elbow Manipulator Use the DH joint variables as generalized coordinates. Compute

$$v_{c1} = J_{v_{c1}}\dot{q} = \begin{bmatrix} -l_{c1}\sin q_1 & 0 \\ l_{c1}\cos q_1 & 0 \\ 0 & 0 \end{bmatrix}\dot{q} \tag{42}$$

$$v_{c1} = J_{v_{c2}}\dot{q} = \begin{bmatrix} -l_1\sin q_1 - l_{c2}\sin(q_1+q_2) & -l_{c2}\sin(q_1+q_2) \\ l_{c1}\cos q_1 + l_{c2}\cos(q_1+q_2) & l_{c2}\cos(q_1+q_2) \\ 0 & 0 \end{bmatrix}\dot{q} \tag{43}$$

Also,

$$J_{\omega 1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad J_{\omega 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}, \quad R_{\omega 1} = R_{z,q_1}, \qquad R_{\omega 2} = R_{z,q_1+q_2}. \tag{44}$$

Therefore, $J_{\omega 1}^T R_{z,q_1} = J_{\omega 1}$ and $J_{\omega 2}^T R_{z,q_1+q_2} = J_{\omega 2}$. In turn,

$$J_{\omega 1}^T R_{z,q_1} I_1 R_{z,q_1}^T J_{\omega 1} = \begin{bmatrix} I_{zz,1} & 0 \\ 0 & 0 \end{bmatrix}, \quad J_{\omega 2}^T R_{z,q_1+q_2} I_2 R_{z,q_1+q_2}^T J_{\omega 2} = \begin{bmatrix} I_{zz,2} & I_{zz,2} \\ I_{zz,2} & I_{zz,2} \end{bmatrix}. \tag{45}$$

Therefore

$$\begin{aligned} D(q) &= m_1 J_{v_{c1}}^T J_{v_{c1}} + m_2 J_{v_{c2}}^T J_{v_{c2}} + \begin{bmatrix} I_{zz,1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} I_{zz,2} & I_{zz,2} \\ I_{zz,2} & I_{zz,2} \end{bmatrix} \\ &= \begin{bmatrix} m_1 l_{c1}^2 + m_2(l_1^2 + l_{c2}^2 + 2l_1 l_{c2}\cos q_2) + I_{zz,1} + I_{zz,2} & m_2(l_{c2}^2 + 2l_1 l_{c2}\cos q_2) + I_{zz,2} \\ m_2(l_{c2}^2 + 2l_1 l_{c2}\cos q_2) + I_{zz,2} & m_2 l_{c2}^2 + I_{zz,2} \end{bmatrix} \end{aligned} \tag{46}$$

Let $h = -m_2 l_1 l_{c2} \sin q_2$. Then,

$$c_{111} = c_{222} = c_{122} = 0, \quad c_{121} = c_{211} = c_{221} = h, \quad c_{112} = -h. \tag{47}$$

$$C(q, \dot{q}) = \begin{bmatrix} h\dot{q}_2 & h\dot{q}_2 + h\dot{q}_1 \\ -h\dot{q}_1 & 0 \end{bmatrix} \tag{48}$$

We have that

$$P = m_1 g l_{c1} \sin q_1 + m_2 g(l_1 \sin q_1 + l_{c2}\sin(q_1 + q_2)) \tag{49}$$

Therefore,

$$g_1(q) = m_1 g l_{c1} \cos q_1 + m_2 g l_1 \cos q_1 + m_2 g l_{c2} \cos(q_1 + q_2) \tag{50}$$

$$g_2(q) = m_2 g l_{c2} \cos(q_1 + q_2) \tag{51}$$

5.2 Properties of the Euler Lagrange Equations

5.2.1 Skew Symmetry and Passivity

Proposition 1. *The matrix $\dot{D}(Q) - 2C$ is skew symmetric.*

Proof. The $(k, j)^{\text{th}}$ element of the matrix $N = \dot{D}(Q) - 2C$ is

$$\begin{aligned} n_{kj} &= \dot{d}_{kj} - 2c_{kj} \\ &= \sum_{i=1}^{n} \frac{\partial d_{kj}}{\partial q_i}\dot{q}_i - 2\sum_{i=1}^{n} c_{ijk}\dot{q}_i \\ &= \sum_{i=1}^{n} \left[\frac{\partial d_{kj}}{\partial q_i} - c_{ijk}\right]\dot{q}_i \\ &= \sum_{i=1}^{n} \left[\frac{\partial d_{kj}}{\partial q_i} - 2\frac{1}{2}\left\{\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right\}\right]\dot{q}_i \\ &= \sum_{i=1}^{n} \left[\frac{\partial d_{ij}}{\partial q_k} - \frac{\partial d_{ki}}{\partial q_j}\right]\dot{q}_i \end{aligned} \tag{52}$$

The expression for n_{jk} will be

$$n_{jk} = \left[\frac{\partial d_{ik}}{\partial q_j} - \frac{\partial d_{ji}}{\partial q_k}\right]\dot{q}_i \tag{53}$$

Since $d_{ij} = d_{ji}$ and $d_{ik} = d_{ki}$, we see that $n_{jk} = -n_{kj}$. Therefore, N is skew symmetric

□

This skew symmetry property is related to a concept known as passivity. Passivity is a property that arises in several types of systems. In the case of dynamical systems, one interpretation is that a passive system (the system exhibits passivity) doesn't produce energy of its own. Electrical circuits made with passive components behave this way, which is where the term passivity comes from.
A system $\dot{x} = f(x, u)$, $y = h(x, u)$ with input u and output y is passive if there exists a storage function $H(x)$ which is positive semi-definite on the state space such that

$$H(x(t)) - H(x(0)) \leq \int_0^T y(s)^T u(s) ds.$$

Passive systems have some well-understood behaviors such as

1. Stability
2. L_2 gain stability
3. Behavior of feedback interconnections

For the conservative robot equations, the storage function is

$$H = \frac{1}{2}\dot{q}^T D(q)\dot{q} + P(q) \tag{54}$$

We have that

$$\begin{aligned}
\dot{H} &= \dot{q}^T D(q)\ddot{q} + \frac{1}{2}\dot{q}^T \dot{D}(q)\dot{q} + \dot{q} + \frac{\partial P(q)}{\partial q} \\
&= \dot{q}^T \left(\tau - C(q,\dot{q})\dot{q} - \frac{\partial P(q)}{\partial q}\right) + \dot{q}^T \dot{D}(q)\dot{q} + \dot{q} + \frac{\partial P(q)}{\partial q} \\
&= \dot{q}^T \tau + \frac{1}{2}\dot{q}^T \left(\dot{D}(q) - 2C(q,\dot{q})\right)\dot{q} \\
&= \dot{q}^T \tau
\end{aligned} \tag{55}$$

5.2.2 Bounds on Inertia Matrix

For a system with revolute joints, there exist λ_m and λ_M such that

$$\lambda_m I_{n\times n} \leq D(q) \leq \lambda_M I_{n\times n} < \infty \tag{56}$$

If the joints are not revolute, then the upper bound by ∞ goes away. The remaining inequalities are still valid, since the matrix $D(q)$ is always real, symmetric, and positive definite.

5.3 Linearity in Parameters

Unsurprisingly, we can derive a function $Y(q,\dot{q},\ddot{q})$ and parameter set θ such that

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = Y(q,\dot{q},\ddot{q})\theta \tag{57}$$

The key idea is that for the same robot (same mechanism), Y is unchanging, and the equations are linear in the parameters θ.