ME 599/699 Robot Modeling & Control
Nonholonomy

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Nonholonomic Systems

So far we have dealt with holonomic constraints when deriving the Euler Lagrangian equations, and environment constraints (force control). In fact, Euler Lagrangian equations are derived for systems with holonomic constraints using the principle of virtual work to account for the constraint forces. In this lecture, we look at another type of constraint known as **nonholonomic constraints**.
The main idea is that a nonholonomic constraint is a constraint on the velocity of the system that does not become a constraint on the position of the system. By contrast, a holonomic constraint can be a velocity constraint, but that velocity constraint is equivalent to a constraint on position. For holonomic constraints, differentiation and integration from one form (velocity or position) constraints makes sense. For nonholonomic constraints, the integration does not. In fact, nonholonomic constraints are precisely non-integrable.
Consider a configuration space $\mathcal{Q}$ and let $q \in \mathbb{R}^n$ denote the vector of generalzied coordinates defining the system configuration.

Definition 1. A set of $k < n$ constraints

$$h_i(q) = 0, \quad i \in \{1, \dots, k\} \tag{1}$$

is called holonomic, where each $h_i \colon Q \mapsto \mathbb{R}$ is smooth.

We can differentiate the constraints to obtain

$$\langle dh_i, \dot{q} \rangle = 0, \quad i \in \{1, \dots, k\} \tag{2}$$

If a point q for which $h_i(q) = 0$ for all $i \in \{1, \dots, k\}$ moves with a velocity $\dot{q}(t)$ satisfying the constraints above, then $q(t)$ will always satisfy the holonomic constraints.
More generally, we can express a constraint as

$$\langle w_i, \dot{q} \rangle = 0, \quad i \in \{1, \dots, k\} \tag{3}$$

where $w_i(q)$ are covectors. The question is when can we express $w_i(q)$ as the differential of a set of smooth maps $h_i(q)$?

Definition 2. Constraints of the form

$$\langle w_i, \dot{q} \rangle = 0, \quad i \in \{1, \dots, k\} \tag{4}$$

are **holonomic** if there exist smooth functions $h_1, \dots, h_k$ such that

$$w_i(q) = dh_i(q), \quad i \in \{1, \dots, k\}. \tag{5}$$

and **nonholonomic** if no such functions $h_1, \dots, h_k$ exist.

Involutivity and Holonomy The quantities $w_1(q), \ldots, w_k(q)$ together form a codistribution Ω over $\mathcal{Q}$. This codistribution defines subspaces that do not contain any component of the tangent vectors corresponding to feasible velocities. Using Ω, we can characterize the space of feasible velocities through a distribution Δ, that is, a finite set of vector fields. Let $\Delta = \{g_1, \ldots, g_{n-k}\}$, where

$$\langle w_i, g_j \rangle = 0, \forall \ i, j \tag{6}$$

We refer to distirbution Δ so derived from Ω as the annihalator of Ω, denoted as $\Omega^{\perp}$.

Lemma 1. *A set of constraints given by codistribution $\Omega = \{w_1, \ldots, w_k\}$ is holonomic if and only if the annihalator distribution $\Omega^{\perp}$ is involutive.*

If we view each w_i as the derivative of some unknown function h_i, then we have that

$$\langle dh_i, g_j \rangle = 0, \forall \ i, j \tag{7}$$

or

$$L_{g_j} h_i = 0, \forall \ i, j \tag{8}$$

Essentially, if the annihalator distribution is involutive, then by the Frobenius Theorem we can solve the partial differential equations $L_{g_j} h_i$ to obtain the function $h_i(q)$. In other words, the constraint $\langle w_i \dot{q} \rangle = 0$ is integrable.

Driftless Control Systems

For a system with constraints given by a codistribution $\Omega = \{w_1, \ldots, w_k\}$, the vector fields $\Omega^{\perp} = \{g_1, \ldots, g_{n-k}\}$ define the possible velocities at each point. We can model the dynamics of the system as

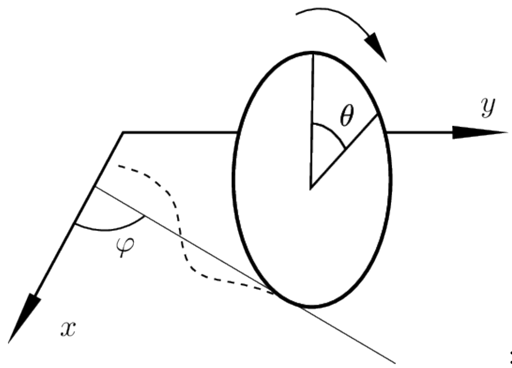
$$\dot{q} = g_1(q)u_1 + \cdots + g_m(q)u_m \tag{9}$$

for suitable coefficients u_i, where $m = n - k$. One interpretation of the coefficients of u_i are as control inputs. If these coefficients are truly under our control, then (9) is called a **driftless** system because when $u = 0$ then $\dot{q} = 0$. If any of the coefficients are fixed at a non-zero value, those fixed coefficients define a drift term. We focus on driftless systems.

Examples of Nonholonomic Systems

There are two common physical situations where nonholonomic constraints arise:

1. Rolling without slipping , for example
   - Wheeled mobile robots
   - Unicycles
   - Manipulation of rigid objects
2. When angular momentum is conserved, for example
   - Satellites
   - Terrestrial robots that lose contact with the ground



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Example 1 (Unicycle)**.** The configuration of the Unicycle is (x, y, ϕ, θ). The condition that no slipping occurs at the contact point boils down to the constraints

$$\dot{x} - r\dot{\theta}\cos\phi = 0 \tag{10}$$

$$\dot{y} - r\dot{\theta}\sin\phi = 0 \tag{11}$$

so that

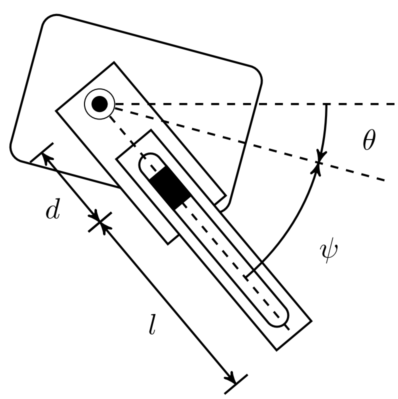
$$w_1 = \begin{bmatrix}1 & 0 & 0 & -r\cos\phi\end{bmatrix}, \text{ and} \tag{12}$$

$$w_2 = \begin{bmatrix}1 & 0 & 0 & -r\sin\phi\end{bmatrix}. \tag{13}$$

One possible annihalator distribution $\Omega^\perp$ is

$$g_1 = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \quad g_2 = \begin{bmatrix}r\sin\phi\\r\cos\phi\\0\\1\end{bmatrix}. \tag{14}$$

One can check that $\Omega^\perp$ is not involutive.



Example 2 (Hopping Robot)**.** THe configuraton of the hopping robot when in the air is $q = [\psi\ l\ \theta]^T$. The angular momentum conservation law is given by

$$I\dot{\theta} + m(l+d)^2(\dot{\theta} + \dot{\psi}) = 0, \tag{15}$$

corresponding to the covector

$$w = \begin{bmatrix} m(l+d)^2 & 0 & I + m(l+d)^2 \end{bmatrix} \tag{16}$$

One possible annihalator distribution is

$$g_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad g_2 = \begin{bmatrix} 1 \\ 0 \\ \frac{-2Im(l+d)}{[I+m(l+d)^2]^2} \end{bmatrix}. \tag{17}$$

One can check that $\Omega^{\perp}$ is not involutive.

Controllability

Given a driftless system, we can represent it in the form (9). When $m < n$, we cannot instantaneously move in all directions (assuming $M = \mathbb{R}^n$). This statement implies that some constraints are active on the system.
Since we cannot move instantaneously in all directions, we may wonder if we can move from one configuration to any other configuration. If we can, the system is said to be controllable. Chow's theorem provides a check for when a driftless system is controllable.
The condition involves the involutive closure $\bar{\Delta}(q)$ of a distribution Δ. The involutive closure is the smallest distribution $\bar{\Delta}$ such that $\bar{\Delta}$ is involutive and $\Delta \subset \bar{\Delta}$.

Theorem 1 (Chow's Theorem)**.** *The driftless system*

$$\dot{q} = g_1(q)u_1 + \cdots + g_m(q)u_m \tag{18}$$

is controllable if and only if the $\text{rank}\bar{\Delta}(q) = n$ *at each* $q \in \mathbb{R}^n$*, where* $\bar{\Delta}(q)$ *is the involutive closure of* $\Delta(q)$*.*

Summary:

1. A constrained system is modeled as a driftless system
2. Check whether it's controllable
3. If so, Design Control!
4. Roger Brockett showed that nonholonomic systems don't have smooth stabilizing control. (Compare to $u = -Kx$ for controllable linear systems)
5. This limitation makes planning more popular than pure feedback control.

Control of Nonholonomic Systems

Once we know a system is controllable, we need to choose the control. For a linear system $\dot{x} = Ax + Bu$, we know that once a system is controllable, we can choose a control $u = -Kx$ that stabilizes the system to $x = 0$. Note that the control is a continuous function of the state, and therefore is a continuous function of time along any solution of the closed-loop system.
For nonholonomic systems, a famous result shows that one cannot design such a continuous stabilizing feedback system as was possible for controllable linear systems. The implication of this result is that we have to define piece-wise continuous control laws to control nonholonomic systems.

Geometric Phase

One of the main ideas in nonholonomic control is **geometric phase** where the available vector fields are combined in a way that creates a motion unachievable by any individual vector fields. This control approach essentially uses the idea that while the annihalator of the nonholonomic constraints is not involutive, its closure is involutive, enabling us to move in any direction.

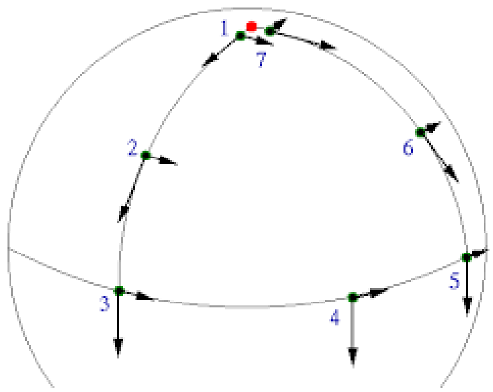


Figure 1: Parallel motion over a loop still causes a rotation.

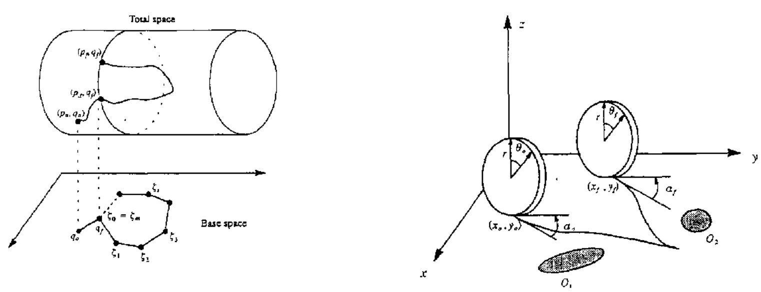


Figure 2: A loop over θ, ϕ causes a displacement in the direction of nonholonomic constraint.

Example 3. Differential Drive Robot The unicycle's nonholonomic constraints Ω result in a annihalator distribution $\Delta = \text{span}\{g_1, g_2\}$ where

$$g_1 = \begin{bmatrix} r\sin\phi \\ r\cos\phi \\ 0 \\ 1 \end{bmatrix}, \quad g_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}. \tag{19}$$

Therefore, the dynamics can be seen as

$$\dot{q} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\phi} \\ \dot{\theta} \end{bmatrix} = g_1(q)u_1 + g_2(q)u_2, \tag{20}$$

where we can interpret u_1 as the instantaneous forward velocity and u_2 as the instantaneous angular motion.
This model is often used for differential drive robots.

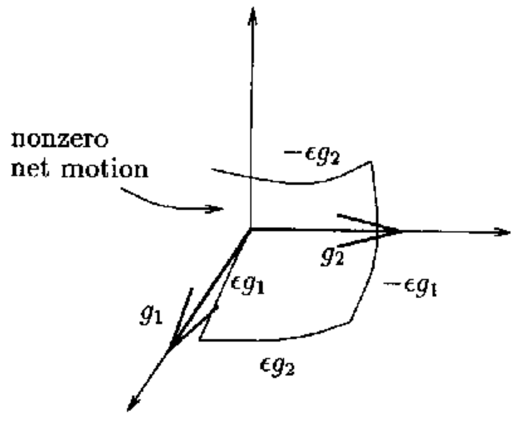


Figure 3: A loop defined by two vector fields causes a non-zero motion in a direction outside the plane of the two vector fields.

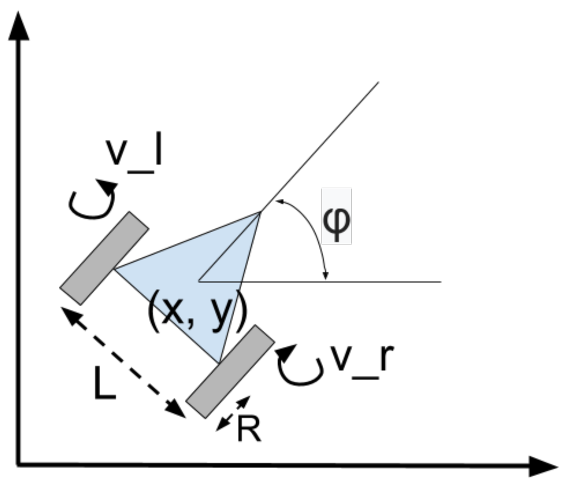


Figure 4: A differential drive robot.

We simplify the model by ignoring the wheel variable θ

$$\dot{q} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \cos\phi u_1 \\ \sin\phi u_1 \\ u_2 \end{bmatrix} \tag{21}$$

We transform this system by redefining state and control

$$x_1 = x \qquad v_1 = \cos\phi u_1 \tag{22}$$

$$x_2 = \phi \qquad v_2 = u_2 \tag{23}$$

$$x_3 = y \tag{24}$$

So that

$$\dot{x}_1 = v_1 \quad (= \dot{x}) \tag{25}$$

$$\dot{x}_2 = v_2 \quad (= \dot{\phi}) \tag{26}$$

$$\dot{x}_3 = x_2 v_1 \tag{27}$$

We can choose v_1 and v_2 to get to any x and ϕ we want. However, y will also change. To fix y, we use geometric phase. The idea is to create a loop trajectory in x and ϕ. Trajectory starts at x^* and ϕ^* and ends where it starts. Due to geometric phase, at the end of the loop, $x_3 = y$ has changed. For example, the trajectory that results from using control

$$v_1 = a\sin(\omega t) \tag{28}$$

$$v_2 = b\cos(\omega t) \tag{29}$$