ME 599/699 Robot Modeling & Control
Differential Geometry

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Contents

1 Manifolds

Definition 1 (Topological Manifold)**.** A manifold M (manifold) is a Hausdorff topological space that is locally homeomorphic to Euclidean space $\mathbb{R}^m$. The dimension of the manifold becomes m.

Definition 2 (Differentially Manifold)**.** A manifold M (manifold) is a Hausdorff topological space that is locally diffeomorphic to Euclidean space $\mathbb{R}^m$.

Example 1 (Euclidean Space $\mathbb{R}^n$)**.** Every finite-dimensional Euclidean space $\mathbb{R}^n$ is a topological manifold. The identity map is a trivial homeomorphism (and diffeomorphism) mapping this manifold to $\mathbb{R}^n$, that is, $m = n$.

1.1 Coordinates

The notion of the differentiable manifold being locally diffeomorphic to $\mathbb{R}^m$ means that there is a differentiable bijective map $\varphi: U \mapsto \mathbb{R}^m$ where $U \subset M$ is an open subset of M. Since the range space of φ is $\mathbb{R}^m$, the diffeomorphism φ is assigning m-dimensional coordinates to points on the manifold. These m-dimensional coordinates are sometimes referred to as **intrinsic** coordinates. We can perform operations on points in U through operations on their intrinsic coordinates. This indirect operation makes sense precisely because φ is smooth and bijective.

Example 2 (Coordinates for Euclidean Space)**.** Recall that we can describe points in Euclidean space without using coordinates, but to perform computations we assigned coordinates by choosing a frame. That frame can be chosen in many ways, with consistency achieved using homogenous transformations. Note that since we assign coordinates to points in a Euclidean space, a single frame assigns coordinates to all points in Euclidean space.

Manifolds locally look like Euclidean space $\mathbb{R}^m$, but they do not behave like $\mathbb{R}^m$ in a global sense. One consequence of this behavior is that one cannot always assign global coordinates to a manifold, that is, one cannot assign a unique vector $\mathbb{R}^m$ to every point of M and still perform meaningful calculations. The lack of global coordinates is often why we consider embedded manifolds, where we can use the coordinates $\mathbb{R}^n$ together with the constraints h_i to perform calculations.

1.2 Embedded Manifolds

For our purposes, we consider manifolds that are embedded submanifolds. We use l constraints on $\mathbb{R}^n$ to specify M, where $l < n$. Each constraint is represented by a smooth function $h_i: \mathbb{R}^n \to \mathbb{R}$, so that

$$M = \{x \in \mathbb{R}^n: h_i(x) = 0, \forall i \in \{1, \ldots, l\}\} \tag{1}$$

If a point $x \in \mathbb{R}^n$ belongs to the manifold M, then x is said to be the **extrinsic** coordinates of that point on the manifold. The manifold M is said to be embedded in $\mathbb{R}^n$.

Whitney Embedding Theorem An important question is whether there are manifolds for which it is impossible to define extrinsic coordinates, meaning that these manifolds have a geometry so complex that they cannot be embedded into any $\mathbb{R}^n$. The answer is that it is always possible to embed a smooth manifold. Therefore, we can always talk in terms of both extrinsic and intrinsic coordinates for a smooth manifold.

Example 3 (Sphere)**.** We can define the surface $\mathcal{S}^2$ of a sphere of radius 1 as an embedded submanifold of $\mathbb{R}^3$ using the constraint $x^2 + y^2 + z^2 = 1$. That is,

$$\mathcal{S}^2 = \{(x, y, z) \in \mathbb{R}^3: x^2 + y^2 + z^2 - 1 = 0\}$$

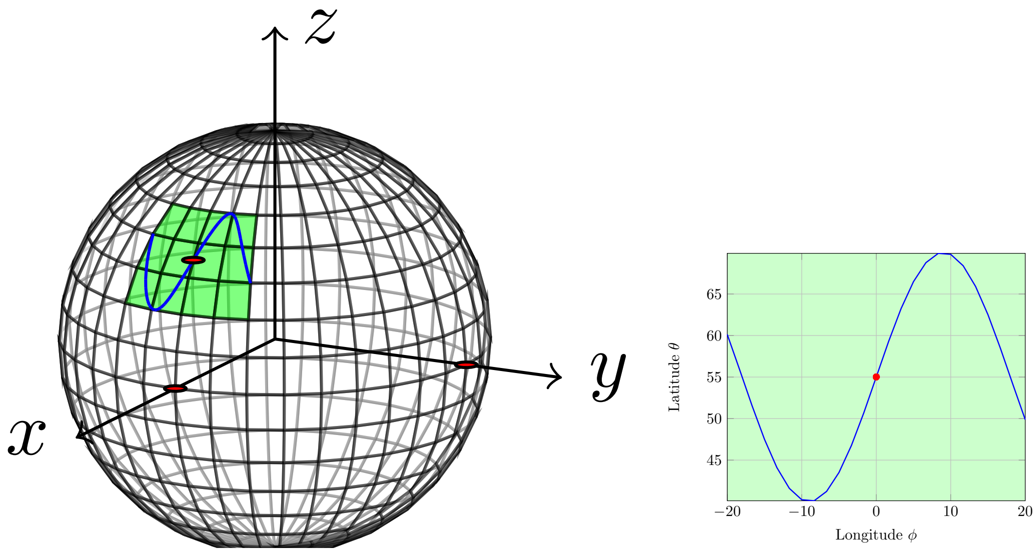

Figure 1: The surface of the sphere (left) can be mapped to $\mathbb{R}^2$ (right) where the x and y axes correspond to longitude and latitude respectively. Since this mapping is a diffeomorphism, the inverse map exists.

Example 4 (Coordinates for $\mathcal{S}^2$)**.** Just as points in Euclidean space can be assigned different coordinates depending on how we define a reference frame, we may be able to assign multiple coordinates to points on a Manifold. The main difference is that these coordinates may only work for a subset of the manifold, unlike Euclidean space.

One example of coordinates for the surface $\mathcal{S}^2$ of a sphere is the longitude ϕ and latitude θ with respect to a point $p^* \in M$ corresponding to $(\phi, \theta) = (0, 0)$ and a great circle ($\theta = 0$) passing through p^*. We can define the map $\varphi \colon \mathcal{S}^2 \mapsto \mathbb{R}^3$ as

$$x = \cos\theta\cos\phi \tag{2}$$

$$y = \cos\theta\sin\phi \tag{3}$$

$$z = \sin\theta \tag{4}$$

As required, φ is a smooth map. In fact, it is smooth for all values of $\mathbb{R}^2$, but is not injective (one-to-one) for all $\mathbb{R}^2$. To make sure that φ is a diffeomorphism, we define its domain as $-\pi/2 < \theta < \pi/2$ and $-\pi < \phi \leq \pi$. One can check that indeed $x^2 + y^2 + z^2 = 1$ for all extrinsic coordinates (x, y, z) corresponding to intrinsic coordinates (θ, ϕ).

2 Tangent Space

We assume that the l differentials dh_i are linearly independent at each point $x \in M$, where M is an embedded manifold with constraint functions h_i. In this case, the dimension of the manifold is $m = n - l$. To such an m-dimensional manifold, we can assign a tangent space T_xM at each $x \in M$ which is an m-dimensional vector space specifying the set of instantaneous velocities possible at x.

Example 5 (Tangent Space to $\mathbb{R}^n$)**.** The tangent space $T_p\mathbb{R}^n$ to a point $p \in \mathbb{R}^n$ is itself another copy of $\mathbb{R}^n$. That is, the possible velocities for a point $p \in \mathbb{R}^n$ creates a space that similar to

$\mathbb{R}^n$. The fact that Euclidean space and its tangent space is the same is the reason that one often confuses point vectors (which are coordinates) with velocity vectors. The fact that the tangent space at each point in a Euclidean space is the same ($\mathbb{R}^n$) is what allows one to treat velocities as free vectors.

We can compute the tangent space as follows: Consider all curves in passing through a point $q \in \mathbb{R}^m$. Each curve γ is a one-dimensional set parametrized by a parameter, say t, belonging to a range, where $\gamma(0) = q$. The derivative $\frac{\partial \gamma}{\partial t}\Big|_{t=0}$ defines a vector tangent to the curve at q, and the collection of all such vectors at q (corresponding to all possible curves through q) forms a vector space at q. When these curves are defined in intrinsic coordinates, the tangent space will turn out to be $\mathbb{R}^m$. When the curves through q are defined in extrinsic coordinates, the tangent space turns out to be a hyperplane tangent to the manifold at p, where $p = \varphi(q)$.
In fact, this process gives an explicit way to assign coordinates to the hyperplane *using the intrinsic coordinates*. Specifically, we can consider a curve through q in $\mathbb{R}^m$ whose tangent is parallel to a coordinate axis of $\mathbb{R}^m$. We can map this curve to its extrinsic coordinates in $\mathbb{R}^n$, and find the derivative of the resulting curve, which gives us a vector in $\mathbb{R}^n$. The derivatives of these two curves (when seen as a curve in extrinsic and intrinsic coordinates) are related through the derivative of the diffeomorphism φ. That is, if $v \in T_x\mathbb{R}^m$ is the velocity at x, the corresponding derivative u of the curve in extrinsic coordinates is

$$u = \frac{\partial \varphi}{\partial q} v. \tag{5}$$

Therefore, given φ, we can use a basis for $\mathbb{R}^m$ (which serves to represent $T_q\mathbb{R}^m$) to generate a basis for T_pM. That is, if $e_1, \ldots, e_m$ is a basis for $\mathbb{R}^m$, then $\frac{\partial \varphi}{\partial q} e_1, \ldots, \frac{\partial \varphi}{\partial x} e_m$ forms a basis for T_pM.

Example 6 (Tangent Space to Sphere)**.** We have the map $\varphi \colon \mathbb{R}^m \mapsto U \subset M$. The partial derivative of φ with respect to (θ, ϕ) at $q = (\theta, \phi)$ is

$$D\varphi = \begin{bmatrix} -\cos\theta \sin\phi & -\sin\theta\cos\phi \\ \cos\theta\cos\phi & -\sin\theta\sin\phi \\ 0 & \cos\theta \end{bmatrix} \tag{6}$$

The tangent space at a point $p \in M \subseteq \mathbb{R}^3$ is simply the span of the columns of $D\varphi$ in (6) with the values $(\phi, \theta) = q$ given by $\varphi^{-1}(p)$.

Remark 1. Note that the tangent space does not change for different diffeomorphisms φ, only the basis for the tangent space will change. In fact, more formal definitions of a smooth manifold insist that the set of diffeomorphisms that map the same point $p \in M$ to different coordinate spaces $\mathbb{R}^m$ be consistent when it comes to assigning coordinates to their tangent spaces, so as to make the tangent space computations independent of the coordinates (diffeomorphism) used.

2.1 Cotangent Space

We can associate any vector space V with a dual space V^* consisting of the space of linear functionals on V. An element of V^* is a real-valued function of V, and is linear with respect to V. Since T_xM is an m-dimensional vector space we can associate a dual space T_x^*M, called the cotangent space, to it.

2.2 Smooth Vector Field

A smooth vector field on a manifold M is a smooth map $f \colon M \to T_xM$. This map is typically represented as a **column** vector of m real-valued functions.

Example 7 (Vector Field on $\mathcal{S}^2$)**.** The first (or second) column of the matrix $D\varphi$ in (6) defines a vector field on $M \subseteq \mathbb{R}^3$.

2.3 Smooth Covector Field

Similarly, a smooth covector field on a manifold M is a smooth map $w\colon M \to T_x^*M$. This map is typically represented as a **row** vector of m real-valued functions.

Example 8 (Covector Field on $\mathcal{S}^2$)**.** The gradient of any smooth scalar function on $\mathcal{S}^2$ defines a covector field on $\mathcal{S}^2$. Taking the function $h(p) = x^2 + y^2 + z^2 - 1$, a covector field on $\mathcal{S}^2$ is $w(p)$ given by

$$w(p) = dh(p) = \begin{bmatrix} 2x & 2y & 2z \end{bmatrix} \tag{7}$$

2.4 Distributions and Codistributions

Let $X_1(x), \ldots, X_k(x)$ be vector fields on M that are linearly independent. A distribution $\Delta(x)$ is the point-wise linear span of these vector fields

$$\Delta = \operatorname{span}\{X_1(x), \ldots, X_k(x)\} \tag{8}$$

This definition ensures that at each $x \in M$, Δ defines a k-dimensional subspace of the tangent space T_xM at x. Similarly, a codistribution $\Omega(x)$ on M is the span of a set of linearly independent covector fields on M.

Example 9 (Tangent space as a Codistribution)**.** If we view each column of the matrix $D\varphi$ in (6) as a vector field, say $X_1(p)$ and $X_2(p)$, then at a given $p \in M$, the tangent space T_pM is precisely the distribution $\Delta = \operatorname{span}\{X_1(p), X_2(p)\}$, that is, $T_pM = \Delta$.

3 Lie Groups

Definition 3 (Lie Group)**.** A Lie group is a finite dimensional smooth manifold G together with a group structure on G, such that the multiplication $G \times G \to G$ and the attaching of an inverse $g \mapsto g^{-1}\colon G \to G$ are smooth maps.

Example 10 ($SO(3)$)**.** The space of rotation matrices forms a Lie group under matrix multiplication. The dimension of the manifold of rotation matrices is 3. $SO(3)$ is also called a Matrix Lie Group.

3.1 Lie Algebra and the Tangent Space of Lie Groups

To every Lie group G we can associate a Lie algebra whose underlying vector space $\mathfrak{g}$ is the tangent space of the Lie group at the identity element.

Definition 4 (Lie Algebra)**.** A Lie algebra is a vector space $\mathfrak{g}$, together with a non-associative operation called the Lie bracket, an alternating bilinear map

$$\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \; (x, y) \mapsto [x, y],$$

satisfying the Jacobi identity:

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] \;=\; 0, \forall\; x, y, z \in \mathfrak{g}.$$

Remark 2. Alternating: $[x, x] = 0$; Non-associative: $[[x, y], z] \neq [x, [y, z]]$

Example 11 (Lie Algebra of $SO(3)$)**.** The Lie algebra of $SO(3)$ consists of a vector space $\mathfrak{so}(3)$ and a Lie bracket given by the usual matrix commutator. $\mathfrak{so}(3)$ is the set of 3×3 real skew-symmetric matrices, and the Lie bracket is

$$[R_1, R_2] = R_1 R_2 - R_2 R_1.$$

Properties of $\mathfrak{so}(3)$:

- Linear
- Interpretation as cross product
- $x^T S x = 0$ for any x.
- $S(Ra) = RS(a)R^T$

3.2 Exponential Map

Definition 5. The exponential of $X \in \mathfrak{g}$ is given by $\exp(X) = \gamma(1)$ where $\gamma\colon \mathbb{R} \to G$ is the unique one-parameter subgroup of G whose tangent vector at the identity is equal to X.

Example 12. For a Matrix Lie Group, the exponential of X is

$$\exp(X) = \sum_{k=0}^{\infty} \frac{X^k}{k!} = I + X + \frac{1}{2}X^2 + \frac{1}{6}X^3 + \cdots.$$

Example 13. The unit circle centered at 0 in the complex plane is a Lie group (called the circle group) whose tangent space at 1 can be identified with the imaginary line in the complex plane, $\{it : t \in \mathbb{R}\}$. The exponential map for this Lie group is given by

$$it \mapsto \exp(it) = e^{it} = \cos(t) + i\sin(t),$$

that is, the same formula as the ordinary complex exponential.

4 Dynamical Systems, Tangents, Vector Fields

Vector fields are used to define differential equations, since they pick elements from the tangent space at each point of a space.
Consider an autonomous nonlinear differential equation on a state space $X \subseteq \mathbb{R}^n$ given by

$$\dot{x} = f(x). \tag{9}$$

The function $f(x)$ is precisely a vector field on X, and $f(x) \in T_x X$.
Consider a nonlinear system with inputs of the form

$$\begin{aligned} \dot{x} &= f(x) + g_1(x)u_1 + \cdots g_m(x)u_m \\ &= f(x) + G(x)u \end{aligned} \tag{10}$$

where $f(x)$, $g_1(x)$, $\cdots$, $g_m(x)$ are smooth vector fields on M, $u \in \mathbb{R}^m$, and the i^{th} column of $G(x)$ is $g_i(x)$. We assume $M = \mathbb{R}^n$ for simplicity. System (10) is known as an affine input system, since the dynamics (vector field) are affine in the input u. Note that the vector field can still be nonlinear in the state x.

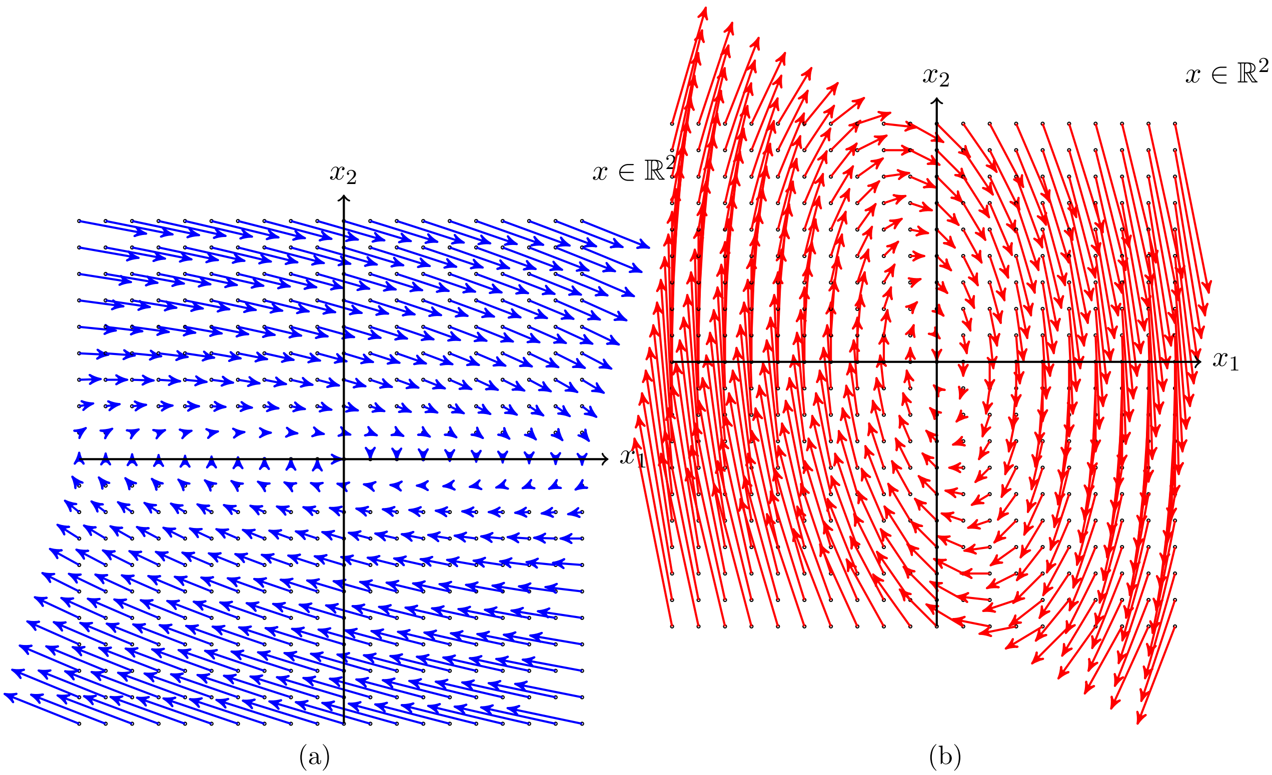

Figure 2: Two vector fields on $\mathbb{R}^2$

Example 14 (Linear Systems)**.** Compare (10) to the linear system $\dot{x} = Ax + Bu$.

Example 15 (Angular Velocity)**.** Suppose we have a rigid body at orientation given by angle $\theta(t)$ about some axis $\overrightarrow{k}$ at time t. We typically refer to its angular velocity as $\dot{\theta}$, assuming a frame conveniently aligned with $\overrightarrow{k}$. Its angular velocity ω is $\omega = \dot{\theta}k$ in the same frame that defines k. Now, suppose we have a point $p(t)$ on this body. We have learned that its velocity is $\omega \times p(t)$. What happens when the orientation is given by a rotation matrix?
Or, how do we compute the velocity of a point $q(t)$ when

$$q(t) = R(t)q + d(t)?$$

Derive $\dot{R}(t) = SR(t)$ when $RR^T = I$. Explain that this version works in world frame.

Example 16 (Rigid Body Dynamics)**.** Let $x(t) \in \mathbb{R}^3$ be the location of the origin of a frame attached to a rigid body at time t, relative to an inertial frame. Let $R(t)$ be its orientation relative to that same inertial frame. Let I_0 anad $\omega_0(t)$ be the rotational inertia and angular velocity of the rigid body in the inertial frame. The dynamics of the rigid body pose in the inertial frame are given by

$$\dot{x}(t) = v(t) \tag{11}$$
$$m\dot{v}(t) = f(t) \tag{12}$$
$$\dot{R}(t) = S(\omega_0)R(t) \tag{13}$$
$$\frac{d}{dt}(I_0\omega_0) = \tau_0 \tag{14}$$

The orientation dynamics is easier to express in a body-fixed frame

$$I\dot{\omega} + \omega \times I\omega = \tau \tag{15}$$

Remark 3 (Quadrotor Dynamics)**.** A quadrotor is often treated as a rigid body on which acts three torques

4.1 Lie Algebra

Choosing a feedback control $u = k(x)$ for the system (10) is like choosing a vector field out of the distribution implied by $f(x)$ and $G(x)$. Our ability to dictate the evolution of x with time therefore depends on this distribution. We now introduce some algebraic operations that help analyze the possible behaviors allowed by a distribution consisting of a finite set of linearly independent vector fields.

4.2 Lie Bracket

Let f and g be differentiable vector fields on $\mathbb{R}^n$. The Lie bracket of f and g, denoted $[f, g]$, is a vector field on $\mathbb{R}^n$ given by

$$[f, g](x) = \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x) \tag{16}$$

where $\frac{\partial g}{\partial x}$ is the Jacobian of $g(x)$.
Note that the Lie Bracket maps two vector fields on $\mathbb{R}^n$ into another vector field on $\mathbb{R}^n$.

Example 17 (Lie bracket as Commutation)**.**

We also denote $[f, g]$ as $\mathrm{ad}_f(g)$, so that we can define repeated Lie brackets with respect to f through the recursion $\mathrm{ad}_f^k(g) = [f, \mathrm{ad}_f^{k-1}(g)]$, where $\mathrm{ad}_f^0(g) = g$.

4.3 Lie Derivative

Let $f \colon \mathbb{R}^n \to \mathbb{R}^n$ be a vector field on $\mathbb{R}^n$ and $h \colon \mathbb{R}^n \to \mathbb{R}$ be a scalar function. The Lie derivative of h with respect to f, denoted $L_f h$, is given by

$$L_f h = \frac{\partial h}{\partial x} f(x) \tag{17}$$

The Lie derivative yields another scalar function, implying that we can define repeated Lie brackets as $L_f^k h = L_f(L_f^{k-1} h)$, where $L_f^0 h = h$.
The Lie Brackets and Derivatives satisfy the following identity:

$$L_{[f,g]} h = L_f L_g h - L_g L_f h. \tag{18}$$

Combined with the definition of repeated Lie brackets/derivatives, this identity gets used in showing many results.

4.4 Involutivity

A distribution $\Delta = \mathrm{span}\{X_1, \ldots, X_k\}$ is involutive if and only if there exist scalar functions $\alpha_{ijk} \colon \mathbb{R}^n \to \mathbb{R}$ such that

$$[X_i, X_j] = \sum_{k=1}^{m} \alpha_{ijk} X_k, \;\; \forall \;\, i, j, k \tag{19}$$

In other words, a distribution is involutive if it is closed with respect to the Lie bracket operation.

5 Feedback Linearization

The motivation for feedback linearization is to create a systematic procedure for designing controllers for nonlinear systems of the form (10) using well-known linear system design principles, when possible. We use ideas from differential geometry to characterize when a system can be feedback linearized.

Example 18. Feedback Linearization Consider the system

$$\dot{x}_1 = a \sin x_2 \tag{20a}$$
$$\dot{x}_2 = -x_1^2 + u \tag{20b}$$

It isn't clear how to choose u so as to influence x_1. If we change the variables, locally, through the transformation

$$y_1 = x_1 \tag{21a}$$
$$y_2 = a \sin x_2 = \dot{x}_1, \tag{21b}$$

The dynamics become

$$\dot{y}_1 = y_2 \tag{22a}$$
$$\dot{y}_2 = a \cos(x_2)(-x_1^2 + u) \tag{22b}$$

Choosing

$$u = \frac{1}{a \cos x_2} v + x_1^2 \tag{23}$$

yields

$$\dot{y}_1 = y_2 \tag{24a}$$
$$\dot{y}_2 = v \tag{24b}$$

which we know how to design for and analyze.
Suppose we get a closed-loop response $y(t)$ by using some control $v = -Ky$. The response in the original coordinates is

$$\dot{x}_1 = y_1 \tag{25a}$$
$$\dot{x}_2 = \sin^{-1} \frac{y_2}{a} \tag{25b}$$

5.1 Single Input Systems

A system $\dot{x} = f(x) + g(x)u$ is feedback linearizable if there exists a diffeomorphism $T\colon U \to \mathbb{R}^n$ defined on an open region $U \subseteq \mathbb{R}^n$ containing the origin, and nonlinear feedback $u = \alpha(x) + \beta(x)v$, with $\beta(x) \neq 0$ on U, such that the transformed state $y = T(x)$ satisfies the system of linear equations $\dot{y} = Ay + bu$ where A and b represent as a chain of integrators.
Since $y = T(x)$, and T is a diffeomorphism, we can derive

$$\dot{y} = \frac{\partial T}{\partial x} \dot{x} \tag{26}$$
$$\implies Ay + bv = \frac{\partial T}{\partial x}(f(x) + g(x)u) \tag{27}$$
$$\implies AT(x) + bv = \frac{\partial T}{\partial x}(f(x) + g(x)u) \tag{28}$$

Going by each component of the n equations, we get

$$T_2 = L_f T_1 + L_g T_1 u \tag{29}$$
$$T_3 = L_f T_2 + L_g T_2 u \tag{30}$$
$$\vdots$$
$$v = L_f T_n + L_g T_n u \tag{31}$$

Since $T(x)$ is independent of u, but v depends on u, we get

$$L_g T_1 = L_g T_2 = \cdots = L_g T_{n-1} = 0 \tag{32}$$
$$L_g T_n \neq 0 \tag{33}$$

thereby reducing the n components to

$$T_{i+1} = L_f T_i, \quad i \in \{1, \ldots, n-1\} \tag{34}$$
$$v = L_f T_n + L_g T_n u \tag{35}$$

We now work to eliminate T_i for $i \geq 2$. We do this by using the relationship between Lie brackets and Lie derivatives in (18). This relationship implies that (34) and (35) become

$$L_{\mathrm{ad}_f^k(g)} T_1 = 0, \quad k \in \{0, 1, \ldots, n-2\} \tag{36}$$
$$L_{\mathrm{ad}_f^{n-1}(g)} T_1 \neq 0 \tag{37}$$

If we can find T_1 satisfying the conditions above, we can find $T_2, \ldots T_n$ inductively, and then find u.
First of all, we need $\mathrm{ad}_f^k(g)$ for $k \in \{0, \ldots, n-1\}$ to be independent so that (37) is satisfiable. For (36) to have a solution, we know that $\mathrm{ad}_f^k(g)$ for $k \in \{0, \ldots, n-2\}$ must lead to an involutive distribution, by the Frobenius Theorem (see below).

Theorem 1. *A system $\dot{x} = f(x) + g(x)u$ is feedback linearizable if and only if there exists an open region $U \subseteq \mathbb{R}^n$ containing the origin in which*

1. *$\mathrm{ad}_f^k(g)$ for $k \in \{0, \ldots, n-1\}$ are linearly independent in U.*
2. *$\Delta = \mathrm{span}\{g, \mathrm{ad}_f(g), \ldots, \mathrm{ad}_f^{n-2}(g)\}$ is involutive in U.*

5.2 Frobenius Theorem

This theorem is concerned with the existence of a solution to a system of partial differential equations in terms of a distribution corresponding to those PDEs.

Definition 6. A distribution $\Delta = \mathrm{span}\,\{X_1, \ldots, X_m\}$ on $\mathbb{R}^n$ is said to be completely integrable if and only if there are $n - m$ linearly independent functions $h_1, \ldots, h_{n-m}$ satisfying the system of partial differential equations

$$L_{X_i} h_j = 0, \text{ for } 1 \leq i \leq m, 1 \leq j \leq n - m \tag{38}$$

Theorem 2 (Frobenius Theorem)**.** *A distribution Δ is completely integrable if and only if it is involutive.*