ME 599/699 Robot Modeling & Control
Mathematical Descriptions of Physical Space

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Contents

1 Introduction

This document lays the mathematical foundations for describing physical two and three dimensional space. The central message is that there are **infinite** ways to mathematically describe physical space. For example, two separate LIDAR sensors on a robot may not describe locations of the same object in space using the same coordinates. When we perform mathematical computations for robot motion, we need to be careful that we account for such differences.

Section 2 describes vector spaces and inner product spaces. Section 3 connects inner product spaces with a mathematical characterization of physical space as a Euclidean space. Section 4 describes graphs as an alternative description of space, which often serves as a useful abstraction for certain problems.

2 Vector Spaces

Definition 1 (Group)**.** A group G is a set together with a binary operation $\cdot$ that satisfies the following properties for all $a, b, c \in G$:

(i) Closure: $a \cdot b \in G$;

(ii) Associativity: $a \cdot (b \cdot c) = (a \cdot) b \cdot c$;

(iii) Existence of identity element $e \in G$ such that $a \cdot e = e \cdot a = a$;

(iv) Existence of inverse element $d \in G$ such that $d \cdot a = a \cdot d = e$.

Example 1. Real numbers form a group under addition.

Example 2. Real numbers without 0 form a group under multiplication.

Definition 2 (Field)**.** A field $\mathbb{F}$ is a set together with two operations – addition $+\colon \mathbb{F} \times \mathbb{F} \mapsto \mathbb{F}$ and multiplication $\cdot\colon \mathbb{F} \times \mathbb{F} \mapsto \mathbb{F}$ – that satisfy the eight axioms listed below.

(i) Addition and multiplication are associative

(ii) Addition and multiplication are commutative

(iii) Existence of additive and multiplicative identity elements

(iv) Existence of inverse element for addition for each $v \in V$

(v) Existence of inverse element for multiplication for each $v \in V$ except for the additive identity

(vi) Distributivity of multiplication with respect to addition

Example 3. Real numbers are a field under usual addition and multiplication.

Definition 3 (Vector space)**.** A vector space over a field $\mathbb{F}$ is a set V together with two operations – vector addition $+\colon V \times V \mapsto V$ and scalar multiplication $\cdot\colon \mathbb{F} \times V \mapsto V$ – that satisfy the eight axioms listed below, for all $u, v, w \in V$ and $a, b \in \mathbb{F}$.

(i) Addition is associative: $u + (v + w) = (u + v) + w$;

(ii) Addition is commutative: $u + v = v + u$;

(iii) Existence of identity element $0 \in V$ such that $v + 0 = v$;

(iv) Existence of inverse element $x \in V$ such that $v + x = 0$;

(v) Compatibility of scalar multiplication with respect to field multiplication: $a \cdot (bv) = (a \cdot b)b$;

(vi) Existence of identity element $e \in \mathbb{F}$ under scalar multiplication such that $ev = v$;

(vii) Distributivity of scalar multiplication with respect to vector addition: $a \cdot (u + v) = a \cdot u + b \cdot v$;

(viii) Distributivity of scalar multiplication with respect to field addition: $(a + b) \cdot u = a \cdot u + b \cdot u$.

Example 4. The set of n-tuples of real numbers, denoted $\mathbb{R}^n$, over the field of real numbers form a vector space when addition and scalar multiplication of these n-tuples are taken to be element-wise addition and scalar multiplication. The 0 vector is the vector with all elements 0, and the inverse of $v \in \mathbb{R}^n$ is $-v = (-1) \cdot v$.

Definition 4 (Vector Space Basis)**.** A basis B of a vector space V is a set of vectors in V such that all other vectors can be written as a finite linear combination of the elements of B.

Remark 1 (Basis for $\mathbb{R}^n$)**.** A basis for vector space $\mathbb{R}^n$ contains exactly n linearly independent vectors.

Remark 2 (Coordinates for $\mathbb{R}^n$)**.** A basis for $\mathbb{R}^n$ equips each point $x \in \mathbb{R}^n$ with a coordinate given by the n coefficients of the basis vectors in the linear combination that yields x.

Definition 5 (Inner Product Space)**.** An inner product on a vector space V defined over a field $\mathbb{F}$ is a function $\langle \cdot, \cdot \rangle\colon V \times V \mapsto \mathbb{F}$ with the following properties

(i) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for all $x, y \in V$;

(ii) $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$, for all $x, y, z \in V$ and $a, b \in \mathbb{F}$;

(iii) $\langle x, x \rangle \geq 0$, for all $x \in V$, and $\langle x, x \rangle = 0 \iff x = 0$.

An inner product space is a vector space equipped with a suitable inner product.

An inner product defines the notion of angle between two vectors, specifically defining when two vectors are orthogonal (perpendicular) to each other.

Example 5. Vector space $\mathbb{R}^n$ equipped with the usual dot product forms an inner product space. Two vectors in $\mathbb{R}^n$ are orthogonal when the angle between them is $90°$.

Definition 6 (Norm)**.** A norm on a vector space V defined over field $\mathbb{F}$ (which is a subfield of the complex numbers $\mathbb{C}$) is a function $p: V \mapsto \mathbb{R}$ with the following properties:

For all $a \in \mathbb{F}$ and $x, y \in V$,

(i) $p(x + y) \leq p(x) + p(y)$;

(ii) $p(ax) = |a|p(x)$;

(iii) If $p(x) = 0$ then $x = 0$.

A **norm** defines a notion of **size** of vectors.

Example 6. An inner product space V with field $\mathbb{R}$ may be equipped with a norm p as follows:

$$p(u) = \sqrt{\langle u, u\rangle}.$$

Remark 3. For real vector spaces defined over $\mathbb{R}$, the symbol $\|\cdot\|$ is often used to denote the norm, instead of $p(\cdot)$.

Definition 7 (Metric)**.** A metric on a space X is a function $d: X \times X \mapsto \mathbb{R}$ with the following properties

(i) $d(x, y) \geq 0$ for all $x, y \in X$, and $d(x, y) = 0 \iff x = y$;

(ii) $d(x, y) = d(y, x)$, for all $x, y \in X$;

(iii) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

A **metric** defines a notion of **distance** on a space.

Example 7. An inner product space V may be equipped with a norm $\|\cdot\|$, which then defines a metric $d: V \times V \to \mathbb{R}$ as

$$d(u, v) = \|u - v\|.$$

3 Euclidean Space

Euclidean Space is a model for physical space. Mathematically, this model turns out be that of an affine space. An affine space consists of elements called points. The main idea is that these **points are not vectors**, however differences between points become vectors. What this means is that we can't assign numbers to a point without using another (reference) point.

Definition 8 (Affine Space)**.** An affine space is a set A together with a vector space $\overrightarrow{A}$, and a transitive and free action of the additive group of $\overrightarrow{A}$ on the set A. The elements of the affine space A are called points, and the elements of the associated vector space $\overrightarrow{A}$ are called vectors, translations, or sometimes free vectors. Explicitly, the definition above means that the action is a mapping, generally denoted as an addition

$$A \times \overrightarrow{A} \to A \tag{1}$$

$$(a, v) \mapsto a + v \tag{2}$$

Free implies that the only the 0 element of a vector space maps a point back to itself. Transitive means any two points define a unique element of the vector space.

Definition 9 (Euclidean Space)**.** A Euclidean space is an affine space with the vector space given by an inner product space.

To reiterate:

Remark 4. An point in n-dimensional Euclidean space is not a vector.

When we describe points as numbers, what we are doing is implicitly using a reference point (origin) and a vector space with a basis to describe points. This process is natural to us because an n-dimensional Euclidean space is *isomorphic* to $\mathbb{R}^n$. That is, we can always create a one-to-one correspondence between a Euclidean space and the inner product space $\mathbb{R}^n$. Concretely, after choosing a point in Euclidean space to be the origin, Euclidean space is indistinguishable from $\mathbb{R}^n$. The **problem** is that we can create **infinite** such correspondences.

Definition 10 (Cartesian Coordinates)**.** Identifying a point in Euclidean space with the zero vector of $\mathbb{R}^n$, and defining an orthogonal basis for $\mathbb{R}^n$ equips Euclidean space with Cartesian coordinates.

The fact that Euclidean space may be numerically handled through a Euclidean vector space $\mathbb{R}^n$ gives us something else: a notion of distance. This distance is known as **Euclidean distance**, and is the usual distance derived from the dot product (see Section 2). Once we have a notion of distance, we are able to define a topology (see Appendix A) on Euclidean space, which leads to **mathematical descriptions of motion** in Euclidean space through the tools of calculus.

4 Graphs

A graph G is a pair (V, E) where V is as set with often a finite, but possible countable, number of members, and $E \subset V \times V$ is a set of ordered or unordered pairs that defines a kind of topology on V.

When E is ordered, the graph G is directed. When E is unordered, the graph is undirected.

A Topology

A set X is a collection of distinct objects. When we define a concept of closeness between elements of a set, we equip the set with a topology. The importance of an abstract concept like topology in practice is that it allows us to predict the effect of imprecision and approximation; of being close but not quite exact.

A.1 Neighborhoods

More concretely, we define neighborhoods that are subsets of X with specific properties.

Definition 11 (Topological Space via Neighborhoods)**.** Let $\mathbf{N}(x)$ be the neighborhood function that provides the neighborhoods of $x \in X$. X with $\mathbf{N}(x)$ defines a topological space if it satisfies the following axioms:

i) If N is a neighborhood of x ($N \in \mathbf{N}(x)$), then $x \in N$

ii) If $N \subset X$ contains an element of $\mathbf{N}(x)$, then $N \in \mathbf{N}(x)$

iii) The intersection of two neighborhoods of x is also a neighborhood of x

iv) Any neighborhood N of x includes a neighborhood M of x such that N is a neighborhood of each point of M.

Example 8. The real number line $\mathbb{R}$, with N being a neighborhood of x if it contains an open interval containing x, forms a topological space. An open interval of the real line is a connected line segment that does not contain the end points.

The set of neighborhoods of x, which is $\mathbf{N}(x)$, is mind-bogglingly large. For example, we could create a countably infinite set of disjoint intervals of the real line where only one of these intervals contains x, and this infinitely large set would still be called a neighborhood of x.

To see that this set $X = \mathbb{R}$ with $\mathbf{N}(x)$ as given satisfies the requirements of being a topological space, we mainly need to apply the definition and keep in mind the properties of open intervals.

i) By definition, if $N \in \mathbf{N}(x)$, then $x \in N$

ii) If a set M contains a subset N that belongs to $\mathbf{N}(x)$, then there's an open interval U such that $x \in U \subset N$. Since $N \subset M$, in turn we may say that $x \in U \subset M$, which implies that $M \in \mathbf{N}(x)$

iii) If N_1 and N_2 are two neighborhoods, they contain open intervals U_1 and U_2 such that $x \in U_1$, $x \in U_2$. Clearly $U_3 = U_1 \cap U_2 \neq \emptyset$, since it contains x. Moreover, U_3 is an open interval since it is a non-empty intersection of two open intervals. Since $x \in U_3 \subset N_1 \cap N_2$, and U_3 is an open interval, we conclude that $N_1 \cap N_2$ is a neighborhood of x, so $N_1 \cap N_2 \in \mathbf{N}(x)$.

iv) Any neighborhood N of x contains at least one open interval U that contains x. For each y in U, we may treat N as a neighborhood of y, since $y \in U \subset N$, and U is an open interval. Thus, any neighborhood N of x contains a set U such that N is a neighborhood of each point in U.

Remark 5. Several mathematical proofs are as tedious but straightforward as this. Apply the recent definitions, using a few arguments involving mathematical properties/facts/definitions that are considered to be widely known (properties of open intervals of the real line, in this case).

A.2 Open Sets

An alternate characterization of topological spaces can be given in terms of a collection τ of subsets of X that satisfy specific properties:

Definition 12 (Topological Space via Open Sets)**.** A topological space if an ordered space (X, τ), where X is a set and τ is a collection of subsets of X satisfying

1. The empty set and X belong to τ
2. Any arbitrary union of members of τ still belongs to τ
3. The intersection of **finite** number of members of τ still belongs to τ.

The elements of τ that satisfy the conditions above are called the **open sets** and the collection τ is called a **topology** on X.

To make things confusing, one can use neighborhoods to define open sets:

Definition 13 (Open Sets via Neighborhoods)**.** Given a set of neighborhoods, a subset $U \subset X$ is *open* if U is a neighborhood for all points in U.

Example 9. Returning to Example 8, the closed interval $I = [0, 1]$ is a neighborhood of any point in I **except** for 0 and 1, since you can't fit an open interval containing 0 (or 1) into I. By contrast, the open interval $I' = (0, 1)$ is a neighborhood for all points in I'. Therefore, it is an open set. In general, unions of open intervals are precisely the open sets in the real line.

What is the corresponding version in $\mathbb{R}^n$ of an open interval in $\mathbb{R}$? Euclidean distance allows us to come up with one answer. An open set is a ball $B_r(x)$, where $x \in \mathbb{R}^n$, $r > 0$ is a radius, and

$$B_r(x) = \{y \in \mathbb{R}^n \colon \|y - x\| < r\}.$$

It is important to use $< r$ rather than $\leq r$ to ensure that $B_r(x)$ is open, much like $[0, 1]$ is not open in $\mathbb{R}$ but $(0, 1)$ is.

Example 10 (Topology Of $\mathbb{R}^n$)**.** The Euclidean vector space $\mathbb{R}^n$ equipped with a metric forms a topology where the open sets are balls centered at any point and with any radius.

B Analysis

A Banach space is a normed vector space that is complete.

A Hilbert space is a normed vector space, induced by an inner product space, that is complete. A Banach space need not be a Hilbert space, since a norm may not come from an inner product.