# ME/AER 676 Robot Modeling & Control
# Spring 2023

## Forward Kinematics & Jacobians

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# Introduction

- We consider robots modeled as links joined in series.
- The degrees of freedom at the joints form the joint variables $q$.
- Task variables $X$ capture quantities describing what the robot must do.
- Traditional robot control focuses on the conversion of joint variables to task variables (forward kinematics) and back (inverse kinematics)

$$X = f(q); \quad q = f^{-1}(X)$$

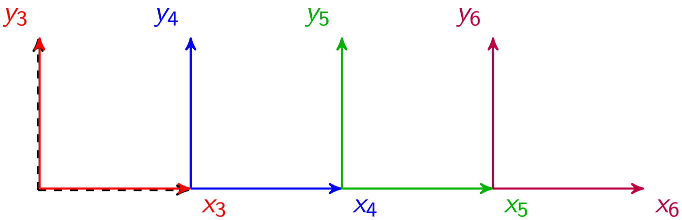
# Forward Kinematics as Homogenous Transformations

- This problem involves composing a number of relative link (homogenous) transformations
- It may be solved numerically, with the specific details depending on how these link transformations are parametrized
- The transformation $(d, R)$ may be represented by
  - origin and Euler angles (URDF)
  - D-H Parameters
  - Twist (Screw Theory)
  - etc. ...

# Serial Kinematic Chains

- We look at serial kinematic chains where all joints are simple.
- We number links as 0 for base to $n$ in sequence.
- The assumption of single-parameter joints means we can use basic transformations to handle coordinate transformations.
- These basic transformation are denoted $A_i(q_i)$, where $q_i \in \mathbb{R}$ is the joint variable.
- $q_i$ is either an angle $\theta_i$ (revolute joints) or a distance $d_i$ (prismatic joints).

# Example: Planar3R



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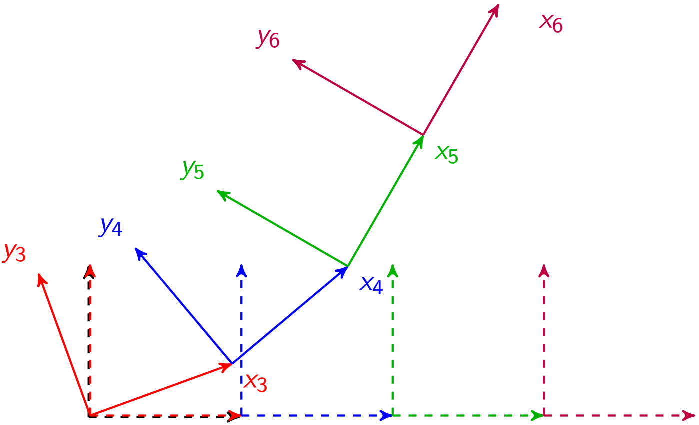

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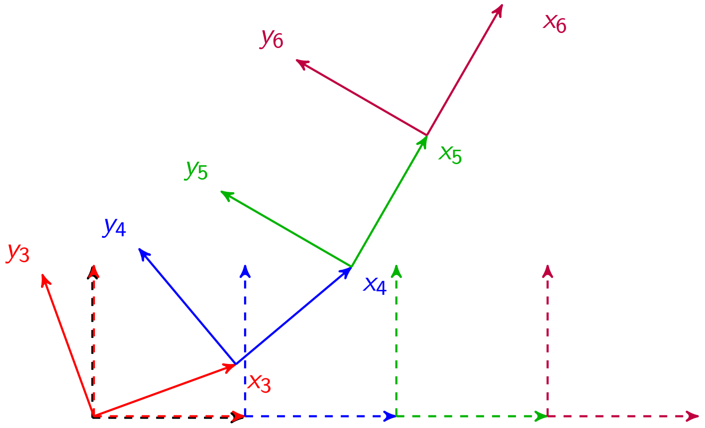


$$o_6^2 = \begin{bmatrix} \mathrm{Rot}_{z,q_1} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{Rot}_{z,q_2} & \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{Rot}_{z,q_3} & \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I_3 & \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

# Forward Kinematics of Serial Chains

Given link $i$ and $i-1$,

$$A_i = \begin{bmatrix} R_i^{i-1} & o_i^{i-1} \\ 0 & 1 \end{bmatrix} \tag{1}$$

Transformations between links $i$ and $j$ is $T_j^i$, where we are expressing frame $j$ in frame $i$.

$$T_j^i = \begin{cases} A_{i+1}A_{i+2}\cdots A_{j-1}A_j & i < j \\ I & i = j \\ \left(T_j^i\right)^{-1} & i > j \end{cases} \tag{2}$$

# Forward Kinematics of Serial Chains

- For an $n$-link serial chain manipulator, the task variables are a combination of
  - Origin of frame $n$ (*end-effector* or *tool* frame)
  - Orientation of frame $n$
- $$T_n^0(q) = \begin{bmatrix} R_n^0(q) & d_n^0(q) \\ 0 & 1 \end{bmatrix}$$
- $X$ is derived from $R_n^0(q)$ and/or $d_n^0(q)$
  i.e. $X = f(q)$

# Modern Robotics

- The book "Modern Robotics" uses exponential coordinates (twists) to represent homogenous transformations.
- It does not follow the D-H convention (next slide).
- The main difference to D-H is that in MR frame $i$ fixed to link $i$ is at joint $i$, not joint $i+1$.
- Videos on FK in this course follow MR's convention of locating frame $i$ at joint $i$.
- Universal Robot Description Formats (URDFs) also follow this approach

# Denavit-Hartenberg Convention

In this convention

- All motion happens along the $z$ axis
- Four numbers are enough to define relative link transformations (instead of 6 or 12).

The D-H convention is based on two restrictions:

(DH1) The $x_1$ axis intersects the $z_0$ axis.
(DH2) The $x_1$ axis is orthogonal to the $z_0$ axis.

This restriction makes the transformation matrix between link $i$ and $i - 1$ given in (1) reduce to

$$A_i = \mathrm{Rot}_{z,\theta_i}\mathrm{Trans}_{z,d_i}\mathrm{Trans}_{x,a_i}\mathrm{Rot}_{x,\alpha_i} \tag{3}$$

This convention is more common in earlier robotics texts, and is used in many systems.

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- However, the orientation coordinate $(d, R)$ is not a vector! What is $\frac{\mathrm{d}}{\mathrm{dt}}R(t)$?

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$$S = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix},$$
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for any three numbers $\omega_1$, $\omega_2$, $\omega_3$

- Physically, the vector $\omega = \begin{bmatrix} \omega_1 & \omega_2 & \omega_3 \end{bmatrix}^T$ defines the instantaneous angular velocity in base/space frame $\{0\}$

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  - As a vector in 3D indicating the instantaneous axis of rotation in a frame and speed of rotation.
  - As the three derivatives of the three numbers used to parametrize $\mathrm{SO}(3)$ (not a physical vector).

# Jacobians and Forward Velocity Kinematics

$X$ is derived from $R_n^0$ and/or $d_n^0$, where $T_n^0 = \begin{bmatrix} R_n^0 & d_n^0 \\ 0 & 1 \end{bmatrix}$

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$$\text{Forward Velocity Kinematics: } \dot{X} = ? \tag{5}$$

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$$\text{Forward Kinematics: } X = f(q) \tag{4}$$

$$\text{Forward Velocity Kinematics: } \dot{X} = J(q)\dot{q} \tag{5}$$

- $J(q)$: Jacobian matrix
- Size of $J(q)$ depends on joint and task space dimensions
- Derivation of $J(q)$ depends on type of coordinates for joint and task spaces
  - Geometric Jacobians : when representing derivative of task frame orientation as an angular velocity
  - Analytic Jacobians : when representing derivative of task frame orientation as derivative of task orientation variables

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- When orientation is not three numbers, $J(q)$ is geometric, and $\xi \neq \dot{X}(t)$
- Columns of $J(q)$ of geometric Jacobian are derived geometrically

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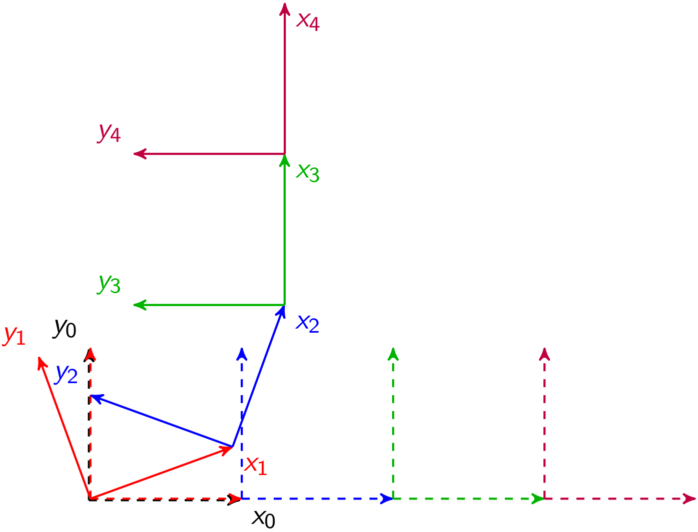

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$$T_2^0 = \begin{bmatrix} \mathrm{Rot}_{z,q_1} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{Rot}_{z,q_2} & \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_2^0 & o_2^0 \\ 0 & 1 \end{bmatrix}$$

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$$T_4^0 = \begin{bmatrix} \mathrm{Rot}_{z,q_1} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{Rot}_{z,q_2} & \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{Rot}_{z,q_3} & \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I_3 & \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix}$$

# Uses of the Jacobian

- Forward Velocity Kinematics: Compute end-effector velocity $\xi$ given joint angle derivatives $\dot{q}$
- Inverse Velocity Kinematics: Compute $\dot{q}$ given $\xi$
- Relates end-effector forces $F$ to joint torques $\tau$ at equilibrium: $\tau = J(q)^T F$
- Defines the manipulability $\mu$ and the manipulability ellipsoid (next slide)

# Manipulability

1. The manipulability $\mu$ is then given by

$$\mu = \Pi_{i=1}^{m} \sigma_i \tag{6}$$

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2. Let $\mathrm{rank}(J) = m$, and $w = U^T \xi$. Then

$$\dot{q} = J^{+} \xi \implies \|\dot{q}\|^2 = \xi^T (J J^T)^{-1} \xi, \text{ where}$$

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3. If $\|\dot{q}\|^2 = 1 = \xi^T (J J^T)^{-1} \xi$ then corresponding $\xi$ form an ellipsoid in space of task velocities $\xi$.

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- This ellipsoid has two physical interpretations:
  - When there's no contact, this ellipsoid describes achievable task velocities given unit-size joint velocities.
  - During static contact, this ellipsoid describes achievable task forces given unit-size joint torques.