ME 599/699 Robot Modeling & Control
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Twists and Wrenches

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- for constant angular velocity $\omega(t) \equiv \omega$, we can solve (1) like a linear system $\dot{x}(t) = Ax(t)$:
$$R(t) = e^{[\omega]t}R(0) \tag{2}$$

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- Let $\hat{\omega} = \omega / \|\omega\|_2$ (unit norm) and $\theta = \dot{\theta} \cdot 1 = \|\omega\|_2$.
- Exponential map $\exp\colon [\hat{\omega}]\theta \in \mathfrak{so}(3) \to R \in \mathrm{SO}(3)$.
  Logarithm map $\log\colon R \in \mathrm{SO}(3) \to [\hat{\omega}]\theta \in \mathfrak{so}(3)$.

Exponential Coordinates

- If we view ω as $\hat{\omega}\theta$, where $\hat{\omega}$ is unit norm and $\theta = \dot{\theta} \cdot 1 = \|\omega\|_2$ then

$$R = e^{[\hat{\omega}]\theta} = I + (\sin\theta)[\hat{\omega}] + (1 - \cos\theta)[\hat{\omega}]^2, \qquad (4)$$

where $e^{(At)} = I + At + A^2t^2/2! + \dots$ is the matrix exponential

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- We may interpret exponential coordinates as coming from a constant angular velocity applied for one second

Angular Velocity Frames

- The equation $\dot{R}(t) = [\omega(t)]R(t)$ involves terms defined in a fixed reference frame, called the *space frame* $\{s\}$ in MR, so really

$$\dot{R}(t) = [\omega_s]R(t) \quad (R = \underbrace{R_b^s}_{\text{RMC}} = \underbrace{R_{sb}}_{\text{MR}})$$

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- The equations may therefore be rewritten as

$$\dot{R}(t) = R(t)[\omega_b]$$

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- The 'angular velocity' corresponding to SE(3) is a *twist*
- Twists for *SE* are not as intuitive as angular velocities for SO(3).

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- If $\omega \in \mathbb{R}^3$ represent velocities for SO(3), twists $\mathcal{V}$ represent velocities for SE(3)
- Consider a homogenous tranformation $T(t) \in \mathrm{SE}(3)$ representing a rigid body pose of $\{b\}$ in $\{s\}$:

$$T(t) = \begin{bmatrix} R(t) & p(t) \\ 0 & 1 \end{bmatrix} \quad (T = \underbrace{T_b^s}_{\text{RMC}} = \underbrace{T_{sb}}_{\text{MR}} = \underbrace{H_b^s}_{\text{HP}}) \tag{5}$$

Body Twist

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- The body twist has simple physical meaning: instantaneous angular velocity of $\{b\}$ as seen in $\{b\}$, and instantaneous velocity of origin of $\{b\}$ as seen in $\{b\}$

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- However, v_s is **not** the velocity of the origin of $\{b\}$ as viewed in $\{s\}$
- v_s a fictitious velocity of the origin of $\{s\}$ as if the space frame $\{s\}$ was rotating about axis ω_s that passes through origin of $\{b\}$ (Fig 3.17 in MR).

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- The remaining notation is about how to transform between $\mathcal{V}_b$ and $\mathcal{V}_s$, or between $[\mathcal{V}_b]$ and $[\mathcal{V}_s]$

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- $\mathcal{V} = \mathcal{S}\dot{\theta}$, just like $\omega = \hat{\omega}\dot{\theta}$
- The result of following a constant spatial twist $\mathcal{V}_s$ for one second can be interpreted as a screw motion : translation along an axis and rotation about that axis.

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- The conversion between expressions is related to the conversion for $[\mathcal{V}_b]$ and $[\mathcal{V}_s]$ (pg. 108 in MR)