System Responses

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1 Introduction

We have seen that transfer functions enable us to predict the response $y(t)$ for inputs $u(t)$ through the s-domain representation:

$$\hat{y}(s) = G(s)\hat{u}(s) - \frac{1}{\alpha(s)}\mathrm{ICT_u} + \frac{1}{\alpha(s)}\mathrm{ICT_y}, \tag{1}$$

where $\hat{y}(s) = L\{y(t)\}$, $\hat{u}(s) = L\{u(t)\}$, and $G(s) = \beta(s)/\alpha(s)$. The first two terms correspond to the **forced response**, and the last term to the **free response**.

We focus on three particular forced responses:

1. Impulse response: $y_i(t)$ when $u(t) = \delta(t)$,
2. Step response: $y_s(t)$ when $u(t) = H(t)$, and
3. Frequency response: $y_f(t)$ when $u(t) = \sin\omega t$ or $\cos\omega t$.

These responses are unique when the initial conditions $y(t_0)$, $\dot{y}(t_0)$, ..., are all zero. When we refer to 'the' impulse response, for example, we are assuming that the initial conditions are zero.

Therefore, we get that the unique responses are

1. Impulse response: $y_i(t) = L^{-1}\left\{G(s)\right\}$ because $\hat{u}(s) = 1$.
2. Step response (including transient response): $y_s(t) = L^{-1}\left\{\frac{G(s)}{s}\right\}$ because $\hat{u}(s) = \frac{1}{s}$.
3. Frequency response: $y_f(t) = L^{-1}\left\{G(s)\frac{\omega}{s^2+\omega^2}\right\}$ or $y(t) = L^{-1}\left\{G(s)\frac{s}{s^2+\omega^2}\right\}$.

We focus on these three responses because they represent many practical types of inputs. A ball bouncing off a box delivers an impulsive force on the box, and we would like to know how that box would react, either in terms of its velocity or its position. A change in the height of the road would represent a step input on the suspension of a vehicle. The vibrations due to an engine would represent sinusoidal inputs acting on the frame of the vehicle containing that engine.

We can combine these three responses to analyze more interesting inputs using the Principle of Superposition: for a linear system, the response of a system to a linear combination of inputs is the linear combination of the responses to each input. For the same initial conditions (possibly all zero), if $y_1(t)$ is response to $u_1(t)$ and $y_2(t)$ is response to $u_2(t)$, then $c_1y_1(t) + c_2y_2(t)$ is the response to $c_1u_1(t) + c_2u_2(t)$.

1.1 Transient Response To Step Inputs

The behavior of a step response $y_s(t)$ for $t \ll \infty$ is known as the **transient response**. Let

$$y_s(\infty) = \lim_{t\to\infty} y_s(t).$$

If the input is $\hat{u}(s) = u_0/s$, we typically want $y_s(\infty) = u_0$, meaning that the output eventually matches the reference input given by u_0. As shown later, for second order systems with positive damping, this condition is always met (this property is not true for second order systems). Given that all step responses for second-order systems will eventually match the reference, the transient response looks at how close the response gets in the time period just after $t = 0$.

We can characterize this transient response using the following quantities:

1. Peak time T_p
2. Rise time T_r
3. Settling time T_s
4. Percent overshoot P_{os}

Figure 1 depicts these quantities for the under-damped step response of a second-order system. More formally, these quantities are defined below. Note that $y_s(t)$ is the step response.

Definition 1 (Rise Time)**.** The rise time T_r of is the time taken for the response to change from 0.1 $y_s(\infty)$ to 0.9 $y_s(\infty)$, assuming a unit step input.

Definition 2 (Peak Time)**.** The peak time T_p is the time taken for the response to reach its maximum value.

Definition 3 (Percent Overshoot)**.** The percent overshoot P_{os} is given by

$$P_{os} = 100\frac{y_s(T_p) - y_s(\infty)}{y_s(\infty)}.$$

Definition 4 (Settling Time)**.** The settling time T_s is the earliest time after which the response remains within 2% of $y_s(\infty)$.

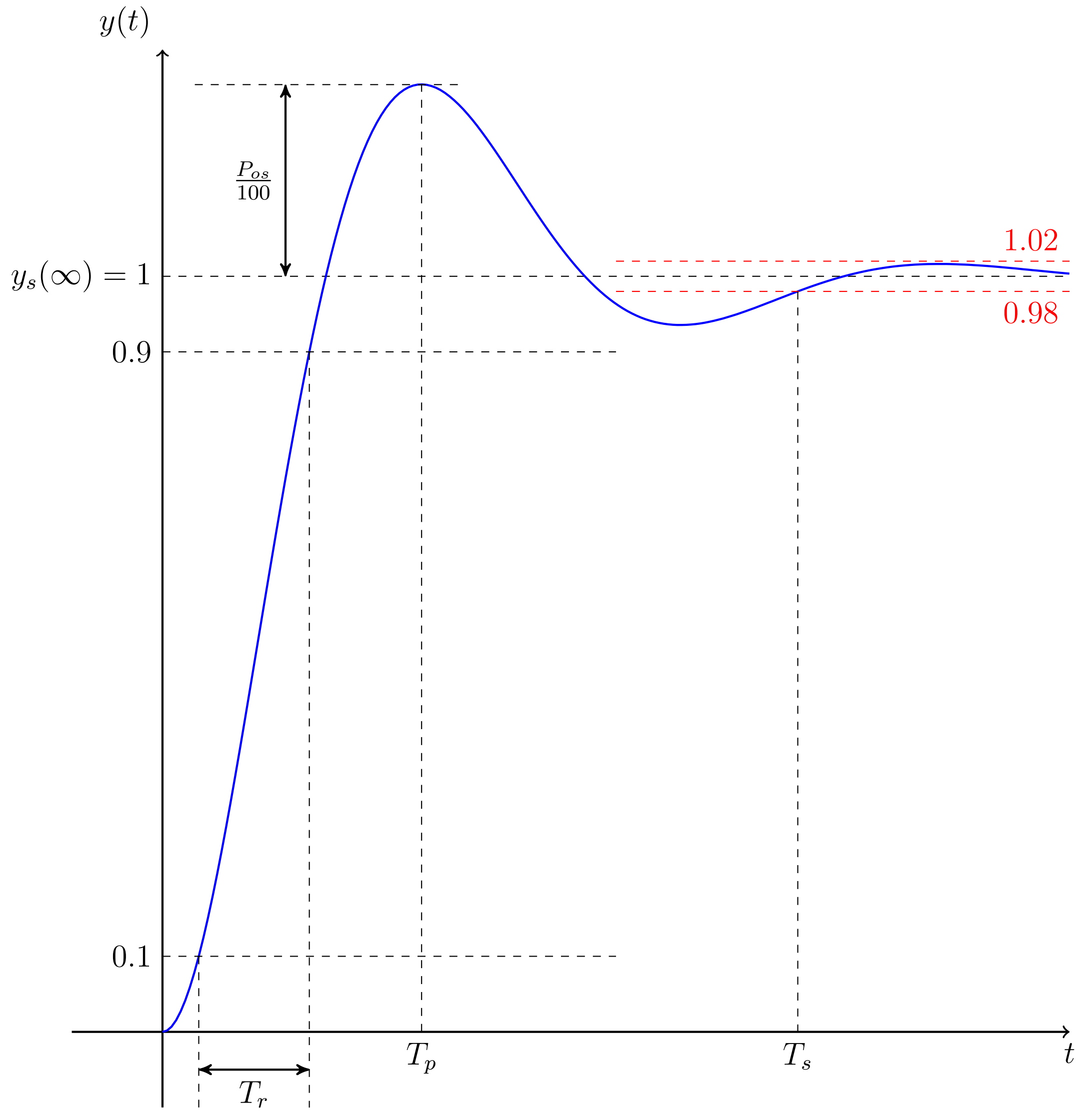


Figure 1: Depiction of transient response characteristics on the step response for an underdamped second-order system.

2 First-Order System

Consider the first-order ODE with equations of motion (EoM)

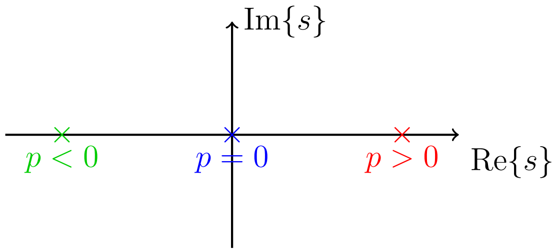
$$\dot{y}(t) + ay(t) = u(t)$$

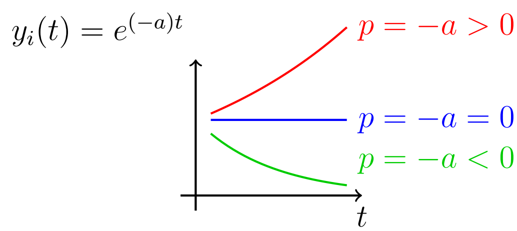
Its transfer function, derived by applying the Laplace transform and setting initial conditions to zero, is

$$G(s) = \frac{1}{s+a}$$

2.1 Impulse Response

$$\hat{y}_i(s) = G(s)\hat{u}(s) = G(s) \cdot 1 = \frac{1}{s+a} \implies y_i(t) = e^{-at}$$





We see that a first order system can show all three possible stability cases (US, LS, and AS) depending on the value of a. Notice that the impulse instantaneously changes the value of y at $t = 0$.

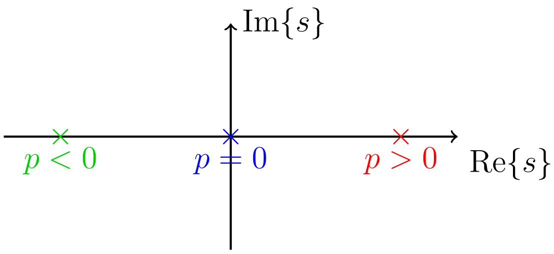
2.2 Step Response

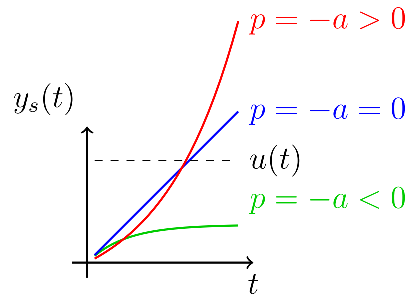
For a first-order system with transfer function $G(s) = \frac{1}{s+a}$,

$$\hat{y}_s(s) = G(s)\hat{u}(s) = G(s) \cdot \frac{1}{s} = \frac{1}{s(s+a)}.$$

Therefore,

$$y_s(t) = \begin{cases} \frac{1}{a} - \frac{e^{-at}}{a}, & \text{if } a \neq 0, \\ t & \text{otherwise.} \end{cases} \tag{2}$$





The impulse response $y_i(t)$ is stable when $a \geq 0$ so that $p \leq 0$, . The step response $y_s(t)$, however, is stable only when $a > 0$, and unstable when $a = 0$.

2.3 Transient Response

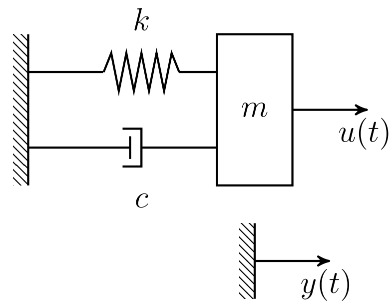
For a first-order system, the transient response is meaningful only if $-a < 0$, so that the quantity $y_i(\infty)$ exists, where $y_i(\infty) = 1/a$. The maximum value of $y_i(s)$ is exactly $1/a$, and so peak time and percent overshoot don't exist for this response. Rise time and settling time does exist, and they are given by the equations

$$T_r = \frac{\log_e 9}{a} \approx \frac{2.2}{a}, \tag{3}$$

$$T_s = \frac{\log_e 50}{a} \approx \frac{3.9}{a} \tag{4}$$

3 Second-Order Systems

Consider the following mechanical system



Its EOM is

$$m\ddot{y}(t) + c\dot{y}(t) + ky(t) = u(t),$$

therefore it has transfer function

$$G(s) = \frac{1}{ms^2 + cs + k}$$

from input $u(t)$ to output $y(t)$.

We can understand this mass-spring-damper system through the general second-order ODE

$$\ddot{y}(t) + 2\xi\omega_n\dot{y}(t) + \omega_n^2 y(t) = b\ u(t),$$

where ω_n is the natural frequency and ξ is the damping ratio. The damping ratio determines the behavior of such second-order systems. We see that

$$G(s) = \frac{b}{s^2 + 2\xi\omega_n s + \omega_n^2},$$

with no zeros, and poles $p_i = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$.

For the MSD,

$$\omega_n = \sqrt{\frac{k}{m}}, \quad \xi = \frac{c}{2\sqrt{km}}.$$

The behavior of such second order system in response to inputs will, as we have seen, depend on the location of the poles. These poles will depend on the damping, and we get four kinds of behaviors corresponding to the following four cases: undamped, under-damped, critically damped, and over-damped.

Undamped: $\xi = 0$ so that the poles are

$$\begin{aligned} p_i &= \pm\omega_n\sqrt{-1} \\ &= \pm\omega_n j, \end{aligned}$$

which are two purely imaginary complex numbers.

Under-damped: $0 < \xi < 1$ so that the poles are

$$
\begin{aligned}
p_i &= -\xi\omega_n \pm \omega_n\sqrt{-1}\sqrt{1-\xi^2} \\
&= -\xi\omega_n + \pm j\omega_n\sqrt{1-\xi^2},
\end{aligned}
$$

which are complex numbers with negative real part and non-zero imaginary part.

Critically damped: $\xi = 1$ so that the poles are

$$p_i = -\omega_n \pm 0,$$

which are two poles equal to the same negative real number.

Over damped: $\xi > 1$ so that the poles are

$$p_i = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1},$$

which are two distinct negative real numbers.

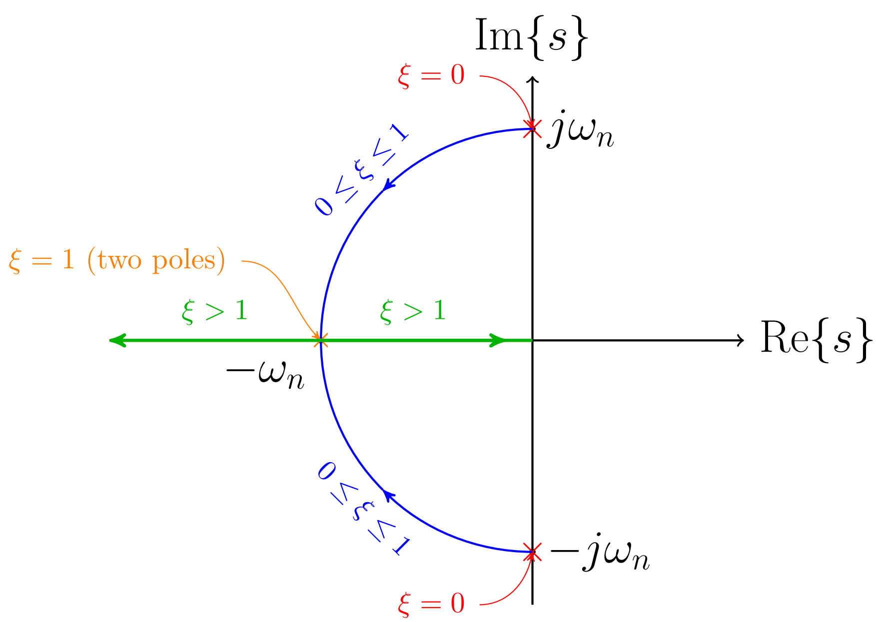


Figure 2: Pole-zero map for second-order systems as ξ goes from 0 to > 1. At $\xi = 0$, the poles are on the imaginary axis (red). As ξ increases from 0 to 1, they travel on a circle of radius ω_n (blue). When $\xi = 1$, the two poles meet at $-\omega_n$(orange). Once $\xi > 1$, the two poles move along the real axis in opposite directions (green).

3.1 Impulse Response

We're now ready to calculate the impulse response corresponding to these different cases.

We have

$$G(s) = \frac{b}{s^2 + 2\xi\omega_n s + \omega_n^2},$$

and $\hat{u}(s) = 1$, so that

$$\hat{y}(s) = G(s)\hat{u}(s) = \frac{b}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

.

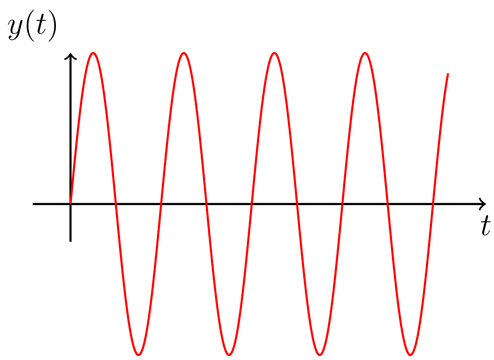
Undamped: $p_i = \pm\omega_n j$

$$\hat{y}(s) = \frac{b}{s^2 + \omega_n^2},$$

therefore

$$y(t) = \frac{b}{\omega_n} \sin \omega_n t.$$

An undamped second order system, such as a mass-spring system, oscillates at the system's natural frequency indefinitely after applying an impulse.



Under-damped: $p_i = -\xi\omega_n + \pm j\omega_n\sqrt{1-\xi^2}$

$$\hat{y}(s) = \frac{b}{s^2 + 2\xi\omega_n s + \omega_n^2} = \frac{b}{s^2 + 2\omega_n s + \xi^2\omega_n^2 + \omega_n^2(1-\xi^2)} = \frac{b}{(s+\xi\omega_n)^2 + \omega_d^2},$$

so that

$$y(t) = \frac{b}{\omega_d} e^{-\xi\omega_n t} \sin(\omega_d t) = \frac{b}{\omega_n\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \sin\left(\omega_n\sqrt{1-\xi^2}t\right)$$

An under-damped mass-spring-damper system will oscillate with decreasing amplitude as time increases, at the system's damped natural frequency.

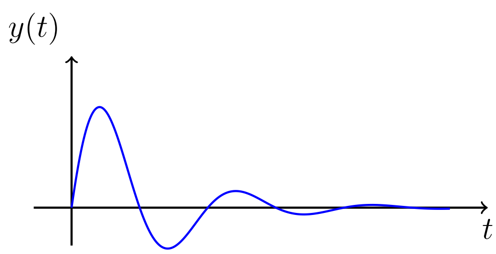
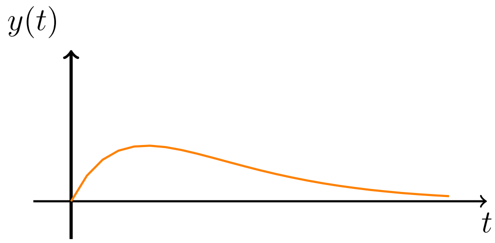


Figure 3: Under-damped second order system: impulse response.

Critically damped: $p_i = -\omega_n \pm 0$



$$\hat{y}(s) = \frac{b}{s^2 + 2\omega_n s + \omega_n^2} = \frac{b}{(s+\omega_n)^2},$$

so that, by s-shift and multiplication-by-time,

$$y(t) = b\ t\ e^{-\omega_n t}.$$

An impulse pushes the output $y(t)$ away from zero, but as $t \to \infty$, $y(t) \to 0$ without oscillations.

Over damped: $p_i = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$

$$\hat{y}(s) = \frac{b}{s^2 + 2\xi\omega_n s + \omega_n^2} \tag{5}$$

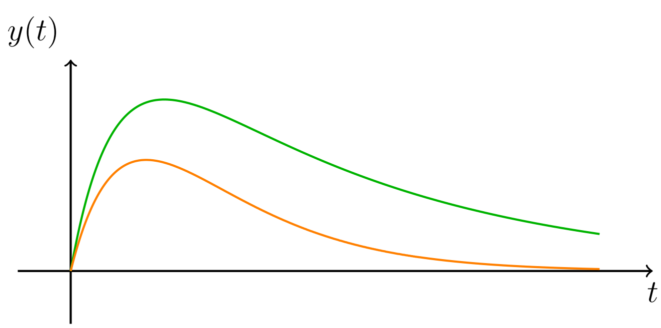
$$= \frac{b}{(s + \xi\omega_n - \omega_n\sqrt{\xi^2-1})(s + \xi\omega_n + \omega_n\sqrt{\xi^2-1})} \tag{6}$$

$$= \frac{b}{2\omega_n\sqrt{\xi^2-1}}\left(\frac{1}{s + \xi\omega_n - \omega_n\sqrt{\xi^2-1}} - \frac{1}{s + \xi\omega_n + \omega_n\sqrt{\xi^2-1}}\right) \tag{7}$$

so that

$$y(t) = \frac{b}{2\omega_n\sqrt{\xi^2-1}}\left(e^{(-\xi\omega_n + \omega_n\sqrt{\xi^2-1})t} - e^{(-\xi\omega_n - \omega_n\sqrt{\xi^2-1})t}\right).$$

An over-damped mass-spring-damper will respond in almost the same way as a critically-damped one, except that the decay to zero will be slower.



Impulse response of second order system: over-damped (green) and critically damped (orange).

Figure 4 shows how the location of the poles of a second order system relate to the impulse response. Question: what happens when $\xi < 0$?

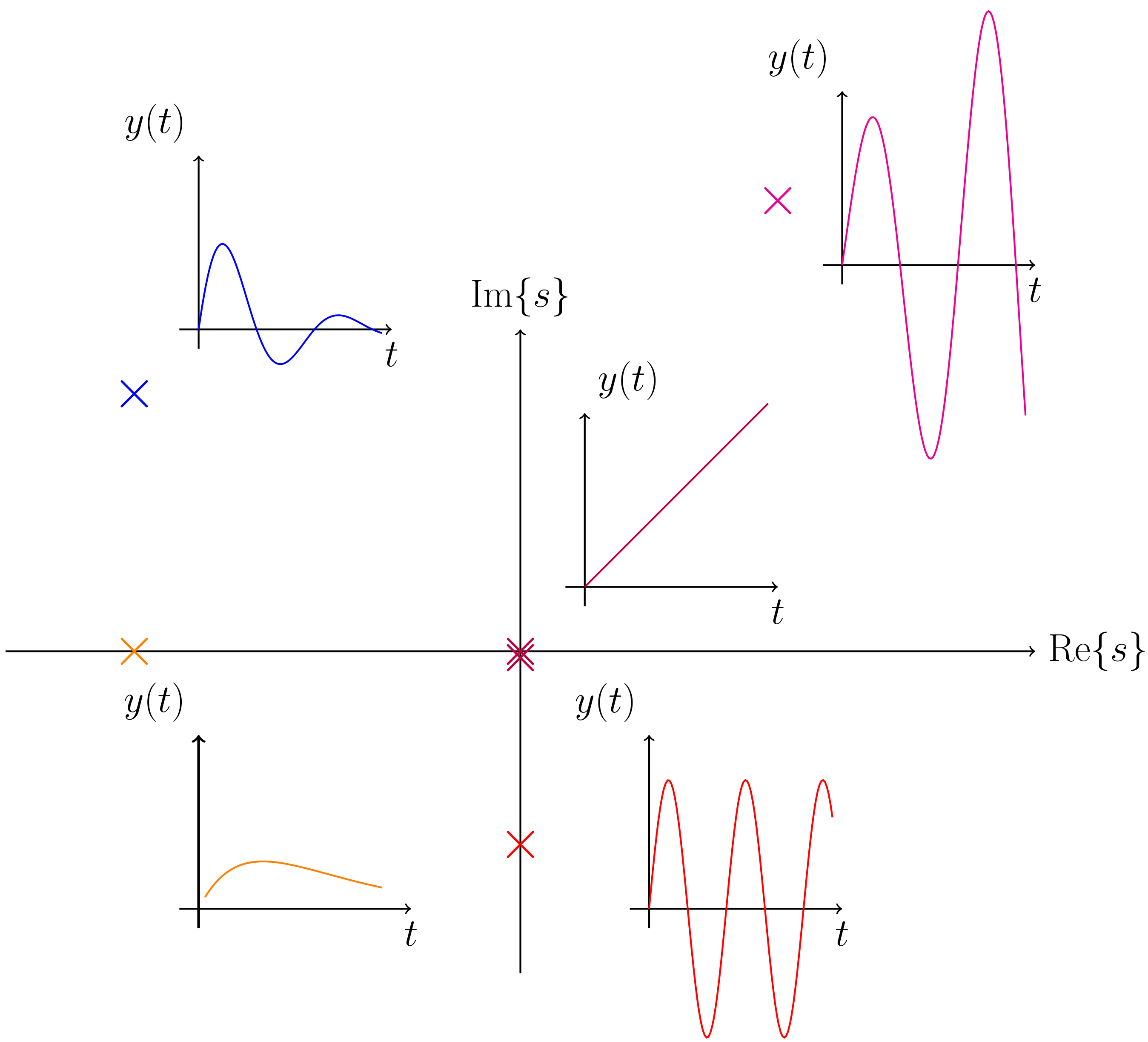


Figure 4: Pole-zero map and impulse response for second-order systems.

3.2 Step Response

The behavior of second-order systems in response to step responses is well-studied. There are two reasons for this situation:

1. Many engineering systems are second-order systems
2. A practical situation for these systems is to track a constant reference signal, which corresponds to a step input signal.

The step input is $\hat{u}_s(s) = 1/s$. The transfer function is

$$G(s) = \frac{b}{s^2 + 2\omega_n \xi s + \omega_n^2}.$$

We want to calculate the step response $y_s(t) = \mathcal{L}^{-1}\{\hat{y}_s(s)\}$.

Stability. The step response $y_s(t)$ corresponds to

$$\hat{y}_s(s) = G(s)\hat{u}_s(s) = \frac{b}{s^2 + 2\xi\omega_n s + \omega_n^2}\frac{1}{s} \tag{8}$$

As long as $\omega_n \neq 0$, this system has only one pole at the origin. If $\xi \geq 0$, then the remaining poles are always in the open left half plane, according to the Routh-Hurwitz criterion. Therefore, the step response is stable for all $\xi \geq 0$.

As observed in Section 3.1, second-order systems have four types based on the damping ratio ξ. We'll now calculate the step response of these four types, separately, below. Figure 5 depicts the nature of these four responses.

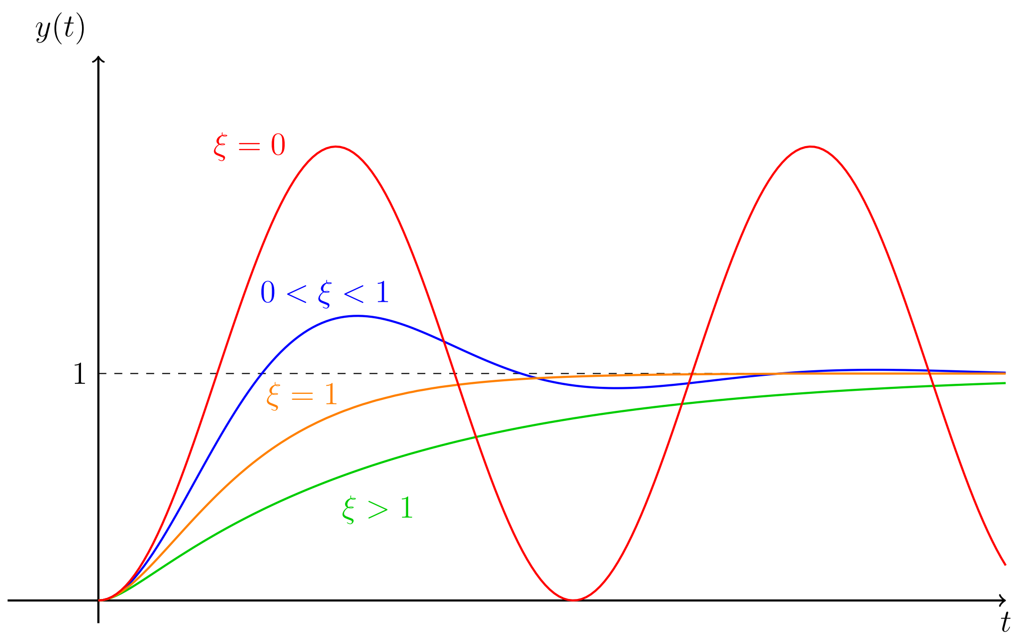


Figure 5: Step response of a second-order system for different damping ratios (same natural frequency ω_n, and $b = \omega_n^2$). Except for the undamped case, all responses satisfy $y_s(t) \to 1$ as $t \to \infty$. All responses are stable.

Under-damped. The poles are complex-valued $p_i = -\xi\omega_n \pm \omega_n\sqrt{1-\xi^2}$

$$\hat{y}_s(s) = G(s)\hat{u}_s(s) = G(s) = \frac{b}{s^2 + 2\omega_n\xi s + \omega_n^2}\frac{1}{s} \tag{9}$$

$$= \frac{k_1}{s} + \frac{k_2}{s^2 + 2\omega_n\xi s + \omega_n^2} \qquad \text{(PFE)} \tag{10}$$

$$= \frac{b/\omega_n^2}{s} - \frac{(b/\omega_n^2)(s + 2\xi\omega_n)}{s^2 + 2\omega_n\xi s + \omega_n^2} \tag{11}$$

$$= \frac{b}{\omega_n^2}\left(\frac{1}{s} - \frac{s + 2\xi\omega_n}{(s+\xi\omega_n)^2 + \omega_n^2(1-\xi^2)}\right) \qquad \text{(simplify denominator)} \tag{12}$$

$$\begin{aligned}
\implies y_s(t) &= \frac{b}{\omega_n^2}\left(1 - \mathcal{L}^{-1}\left\{\frac{s + 2\xi\omega_n}{(s+\xi\omega_n)^2 + \omega_n^2(1-\xi^2)}\right\}\right) \\
&= \frac{b}{\omega_n^2}\left(1 - \mathcal{L}\left\{\frac{s + \xi\omega_n}{(s+\xi\omega_n)^2 + \omega_n^2(1-\xi^2)}\right\} - \mathcal{L}^{-1}\left\{\frac{\xi\omega_n}{(s+\xi\omega_n)^2 + \omega_n^2(1-\xi^2)}\right\}\right) \\
&= \frac{b}{\omega_n^2}\left(1 - e^{-\xi\omega_n t}\mathcal{L}^{-1}\left\{\frac{s}{(s)^2 + \omega_n^2(1-\xi^2)}\right\} - e^{-\xi\omega_n t}\mathcal{L}^{-1}\left\{\frac{\xi\omega_n}{(s)^2 + \omega_n^2(1-\xi^2)}\right\}\right) \\
&= \frac{b}{\omega_n^2}\left(1 - e^{-\xi\omega_n t}\mathcal{L}^{-1}\left\{\frac{s}{(s)^2 + \omega_d^2}\right\} - \frac{\xi e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\mathcal{L}^{-1}\left\{\frac{\omega_d}{(s)^2 + \omega_d^2}\right\}\right) \\
&= \frac{b}{\omega_n^2}\left(1 - e^{-\xi\omega_n t}\cos(\omega_d t) - \frac{\xi}{\sqrt{1-\xi^2}}e^{-\xi\omega_n t}\sin(\omega_d t)\right)
\end{aligned}$$

The step response oscillates about 1 with decreasing amplitude as time increases.

Undamped. We plug in $\xi = 0$ in the above equation to obtain

$$y_s(t) = \frac{b}{\omega_n^2}\left(1 - \cos(\omega_n t)\right). \tag{13}$$

The oscillation of this response about 1 never dies out.

Critically Damped. We **cannot** plug in $\xi = 1$ into the under-damped case to obtain the response (try it to see why). Instead, we begin from the expression (12), and add a simplifying assumption that $b = \omega_n^2$

$$\begin{aligned}
y_s(t) &= \frac{b}{\omega_n^2}\mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s+2\xi\omega_n}{(s+\xi\omega_n)^2 + \omega_n^2(1-\xi^2)}\right\} && \text{(from (12))} \quad (14)\\
&= \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s+2\omega_n}{(s+\omega_n)^2}\right\} && (\text{set } b \to \omega_n^2, \xi \to 1) \quad (15)\\
&= 1 - \mathcal{L}^{-1}\left\{\frac{s+\omega_n}{(s+\omega_n)^2} + \frac{\omega_n}{(s+\omega_n)^2}\right\} && (16)\\
&= 1 - \mathcal{L}^{-1}\left\{\frac{1}{s+\omega_n}\right\} - \mathcal{L}^{-1}\left\{\frac{\omega_n}{(s+\omega_n)^2}\right\} && (17)\\
&= 1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t} && (\text{apply } s\text{-shift}) \quad (18)\\
& && (19)
\end{aligned}$$

This expression indicates that $y_s(t) \to 1$ as $t \to \infty$. Furthermore, the response never goes above 1.

Over-Damped. Note that

$$s^2 + 2\xi\omega_n s + \omega_n^2 = (s-p_1)(s-p_2) = s^2 - (p_1+p_2)s + p_1p_2. \quad (20)$$

Since $\xi > 1$, the poles p_1 and p_2 are distinct and real: $p_i = -\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}$.

We have

$$\begin{aligned}
\hat{y}_s(s) &= \frac{1}{s}\frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} && (b \to \omega_n^2) \quad (21)\\
\hat{y}_s(s) &= \frac{1}{s}\frac{p_1p_2}{s^2 - (p_1+p_2)s + p_1p_2} && ((20)) \quad (22)\\
&= \frac{k_1}{s} + \frac{k_2}{s-p_1} + \frac{k_3}{s-p_2} && \text{(PFE)} \quad (23)\\
&= \frac{1}{s} + \frac{p_2}{p_1-p_2}\frac{1}{(s-p_1)} + \frac{p_1}{p_2-p_1}\frac{1}{(s-p_2)} && \text{(Apply PFE coefficient rules)} \quad (24)\\
\implies y_s(t) &= 1 + \frac{p_2}{p_1-p_2}e^{p_1 t} + \frac{p_1}{p_2-p_1}e^{p_2 t} && (25)
\end{aligned}$$

Like the critically damped case, $y_s(t) \to 1$ as $t \to \infty$. Furthermore, the response never goes above 1. The over-damped behavior comes from the fact that one of the poles satisfies $-\omega_n < p_i < 0$, which decays to zero more slowly than the critically damped case where both poles are at $-\omega_n$.

3.3 Transient Response

For a second-order under-damped system,

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$$

.

For a second-order under-damped system,

$$P_{os} = 100e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}}$$

.

The settling time is difficult to compute exactly. A good estimate is

$$T_s \approx \frac{4}{\xi\omega_n}.$$

For a second-order system, we do not have closed-form expressions for the rise time.

4 Frequency Response

For a transfer function $G(s)$, the response to sinusoidal inputs of the form $\sin \omega t$ or $\cos \omega t$ is the frequency response $y_f(t)$ of $G(s)$. We focus on these two particular inputs almost any periodic signal can be represented as a linear combination of these two terms, possibly with differing amplitudes and frequencies.

4.1 Full Frequency Response

Let's focus on the input $u(t) = \cos \omega t$. Then,

$$y_f(t) = \mathcal{L}^{-1}\left\{\hat{y}(s)\right\} = \mathcal{L}^{-1}\left\{G(s)\hat{u}(s)\right\} = \mathcal{L}^{-1}\left\{G(s)\frac{s}{s^2+\omega^2}\right\}.$$

In general, we would compute $y_f(t)$ using the PFE of $G(s)\frac{s}{s^2+\omega^2}$.

Example 1. Let $G(s) = \frac{s}{s+a}$. Then

$$\begin{aligned} y_f(t) &= \mathcal{L}^{-1}\left\{G(s)\frac{s}{s^2+\omega^2}\right\} && (26)\\ &= \mathcal{L}^{-1}\left\{\frac{s}{(s+a)}\frac{s}{(s^2+\omega^2)}\right\} && (27)\\ &= \mathcal{L}^{-1}\left\{\frac{k_1}{s+a} + \frac{k_2}{s+j\omega} + \frac{k_3}{s-j\omega}\right\} && (28) \end{aligned}$$

We apply the distinct roots rule:

$$k_1 = \frac{s^2}{(s+a)(s^2+\omega^2)}(s+a)\bigg|_{s=-a} = \frac{a^2}{a^2+\omega^2} \tag{29}$$

$$\begin{aligned} k_2 &= \frac{s^2}{(s+a)(s^2+\omega^2)}(s+j\omega)\bigg|_{s=-j\omega} = \frac{s^2}{(s+a)(s-j\omega)}\bigg|_{s=-j\omega} = \frac{-\omega^2}{(-j\omega+a)(-2j\omega)} && (30)\\ &= \frac{\omega}{2}\frac{1}{(-j^2\omega+ja)} = \frac{\omega}{2}\frac{1}{(\omega+ja)} && (31)\\ &= \frac{\omega(\omega-ja)}{2(a^2+\omega^2)} && (32) \end{aligned}$$

$$\begin{aligned} k_3 &= \frac{s^2}{(s+a)(s^2+\omega^2)}(s-j\omega)\bigg|_{s=j\omega} = \frac{s^2}{(s+a)(s+j\omega)}\bigg|_{s=j\omega} = \frac{-\omega^2}{(j\omega+a)(2j\omega)} && (33)\\ &= \frac{-\omega}{2}\frac{1}{(j^2\omega+ja)} = \frac{\omega}{2}\frac{1}{(\omega-ja)} && (34)\\ &= \frac{\omega(\omega+ja)}{2(a^2+\omega^2)} && (35) \end{aligned}$$

So, we get

$$y(t) = \mathcal{L}^{-1}\left\{\frac{a^2}{(a^2+\omega^2)}\frac{1}{(s+a)} + \frac{\omega}{2(a^2+\omega^2)}\left(\frac{\omega - ja}{s+j\omega} + \frac{\omega + ja}{s - j\omega}\right)\right\} \tag{36}$$

Let's use variables to simplify computations. Let $z_1 = \omega - ja$ and $z_2 = j\omega$. The last term, removing the constant, is then

$$\begin{aligned} \text{last term} &= \frac{z_1}{s+z_2} + \frac{\bar{z}_1}{s+\bar{z}_2} && (37)\\ &= \frac{(z_1+\bar{z}_1)s + z_1\bar{z}_2 + \bar{z}_1 z_2}{s^2 + (z_2+\bar{z}_2)s + z_2\bar{z}_2} && (38)\\ &= \frac{2\omega s - 2a\omega}{s^2+\omega^2} && (39)\\ &= \frac{2\omega(s-a)}{s^2+\omega^2} && (40) \end{aligned}$$

Finally

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{a^2}{(a^2+\omega^2)}\frac{1}{(s+a)} + \frac{\omega^2}{(a^2+\omega^2)}\frac{s-a}{(s^2+\omega^2)}\right\} && (41)\\ &= \mathcal{L}^{-1}\left\{\frac{a^2}{(a^2+\omega^2)}\frac{1}{(s+a)}\right\} + \mathcal{L}^{-1}\left\{\frac{\omega^2}{(a^2+\omega^2)}\frac{s-a}{(s^2+\omega^2)}\right\} && (42)\\ &= \frac{a^2e^{-at}}{(a^2+\omega^2)} + \frac{\omega^2}{(a^2+\omega^2)}\cos\omega t - \frac{a\omega}{(a^2+\omega^2)}\sin\omega t && (43) \end{aligned}$$

□

That's a lot of work. What if

$$G(s) = \frac{s(s+1)(s+2)}{(s^2+s+4)(s^2+4s+9)}?$$

How long would it take to solve *that* PFE?

Note that for our example, as $t \to \infty$, the first term disappears, and $y_f(t) \to$ sinusoidal terms. The computations involving the PFE due to the input $\hat{u}(s)$ seems familiar. These two ideas allow us to approximate the frequency response that remains after some time has passes.

4.2 Steady-State Frequency Response: Asymptotically Stable Systems

Consider a system with transfer function $G(s)$, where $G(s)$ is asymptotically stable. All its poles are in the OLHP; let there be n of them. For simplicity, let's assume the roots are distinct. This system's frequency response, for $u(t)\cos\omega t$ is

$$
\begin{aligned}
y_f(t) &= \mathcal{L}^{-1}\left\{G(s)\frac{s}{s^2+\omega^2}\right\} && (44)\\
&= \mathcal{L}^{-1}\left\{\frac{k_1}{(s-p_1)}+\frac{k_2}{(s-p_2)}+\cdots\frac{k_n}{(s-p_n)}+\frac{c_1}{(s-j\omega)}+\frac{c_2}{(s+j\omega)}\right\} && (45)\\
&= \mathcal{L}^{-1}\left\{\frac{k_1}{(s-p_1)}+\frac{k_2}{(s-p_2)}+\cdots\frac{k_n}{(s-p_n)}\right\}+\mathcal{L}^{-1}\left\{\frac{c_1}{(s-j\omega)}+\frac{c_2}{(s+j\omega)}\right\} && (46)\\
&= y_t(t)+y_{ss}(t) && (47)
\end{aligned}
$$

The solution $y_t(t)$ is the transient forced response, and since all the poles are in the OLHP,

$$\lim_{t\to\infty} y_t(t) = 0.$$

In other words, $y_t(t)$ is a transient signal, which decays exponentially fast to zero. What remains is the steady-state forced response $y_{ss}(t)$. Note that it's more specifically the steady-state forced frequency response to an input with a single frequency, but we don't use the full phrase. So,

$$\lim_{t\to\infty} y_f(t) = y_{ss}(t).$$

So, we just need to calculate

$$y_{ss}(t) = \mathcal{L}^{-1}\left\{\frac{c_1}{(s-j\omega)}+\frac{c_2}{(s+j\omega)}\right\} \tag{48}$$

Our goal is now to calculate what c_1 and c_2 are. Since these two coefficients correspond to poles of $\hat{y}_{ss}(s)$ with multiplicity 1 (no matter what they are, all other poles are off the imaginary axis),

$$
\begin{aligned}
c_1 &= G(s)\frac{s}{s^2+\omega^2}(s-j\omega)\bigg|_{s=j\omega} && (49)\\
&= G(s)\frac{s}{s+j\omega}\bigg|_{s=j\omega} && (50)\\
&= G(j\omega)\frac{j\omega}{2j\omega} && (51)\\
&= \frac{G(j\omega)}{2} && (52)
\end{aligned}
$$

and,

$$
\begin{aligned}
c_2 &= G(s)\frac{s}{s^2+\omega^2}(s+j\omega)\bigg|_{s=-j\omega} && (53)\\
&= G(s)\frac{s}{s-j\omega}\bigg|_{s=-j\omega} && (54)\\
&= G(-j\omega)\frac{-j\omega}{-2j\omega} && (55)\\
&= \frac{G(-j\omega)}{2} && (56)
\end{aligned}
$$

Now, let $z_1 = G(j\omega)$. When $G(s)$ contains polynomials with only real coefficients, which holds for the physical systems we study, then $G(-j\omega) = \bar{z}_1$. Let $z_2 = -j\omega$ So, we obtain

$$
y_{ss}(t) = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{z_1}{(s+z_2)} + \frac{\bar{z}_1}{(s+\bar{z}_2)}\right\}, \quad (57)
$$

and we've seen how to simplify that.

$$
y_{ss}(t) = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{(z_1+\bar{z}_1)s + z_1\bar{z}_2 + \bar{z}_1 z_2}{(s^2+\omega^2)}\right\} \quad (58)
$$

We use the fact that z_2 is purely imaginary to note that $z_2 = -\bar{z}_2$, so that

$$
\begin{aligned}
y_{ss}(t) &= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{(z_1+\bar{z}_1)s + (z_1-\bar{z}_1)j\omega}{(s^2+\omega^2)}\right\} && (59)\\
&= \mathcal{L}^{-1}\left\{\frac{\mathrm{Re}\{z_1\}s - \mathrm{Im}\{z_1\}\omega}{(s^2+\omega^2)}\right\} && (60)\\
&= \mathrm{Re}\{G(j\omega)\}\cos\omega t - \mathrm{Im}\{G(j\omega)\}\sin\omega t && (61)\\
&= |G(j\omega)|\cos(\omega t + \angle G(j\omega)) && (62)
\end{aligned}
$$

Fact: Consider the system $G(s)$ where $G(s)$ is AS. Assume that $u(t) = A\cos\omega t$, then the steady-state forced response is

$$
y_{ss}(t) = A|G(j\omega)|\cos(\omega t + \angle G(j\omega)).
$$

Notes

1. If $u(t) = B\sin\omega t$, then $y_{ss}(t) = B|G(j\omega)|\sin(\omega t + \angle G(j\omega))$.

2. If $u(t) = A_1 \cos\omega_1 t + A_2 \cos\omega_2 t + \cdots + A_n \cos\omega_n t$, then

$$\begin{aligned} y_{ss}(t) = & A_1|G(j\omega_1)| \cos\left(\omega_1 t + \angle G(j\omega_1)\right) + A_2|G(j\omega_2)| \cos\left(\omega_2 t + \angle G(j\omega_2)\right) \\ & + \cdots + A_n|G(j\omega_n)| \cos\left(\omega_n t + \angle G(j\omega_n)\right). \end{aligned} \tag{63}$$

3. If $G(s)$ is unstable, then $y_{ss}(t)$ does not exist.

4.3 Steady State Frequency Response: Lyapunov Stable Systems

We've seen that we can solve for the steady state response of AS systems. Why couldn't we do this for Lyapunov stable systems?

The answer boils down to the possibility of $\hat{y}(s)$ having multiple poles on the IA when $G(s)$ is LS. Is there any LS system that would have a steady-state forced response? Yes, consider $G_1(s) = \frac{1}{s}$, so that when $\hat{u}(s) = \frac{\omega}{s^2+\omega^2}$,

$$\hat{y}_1(s) = \frac{\omega}{s(s^2+\omega^2)} = \frac{1}{\omega}\left(\frac{1}{s} - \frac{s}{s^2+\omega^2}\right),$$

so that $y_1(t) = \omega^{-1}(1 - \cos(\omega t))$

Any other? No, not for all values of ω.

Example 2. Let $G_2(s) = \omega_n^2/(s^2+\omega_n^2)$ which is LS. Then, when $\hat{u}(s) = \omega/(s^2+\omega^2)$,

$$\hat{y}_2(s) = \frac{\omega\omega_n^2}{(s^2+\omega_n^2)(s^2+\omega^2)} = \frac{\omega_n}{\omega^2-\omega_n^2}\left(\frac{\omega\omega_n}{(s^2+\omega_n^2)} - \frac{\omega\omega_n}{(s^2+\omega^2)}\right).$$

Then,

$$y_2(t) = \frac{\omega_n}{\omega^2-\omega_n^2}(\omega\sin(\omega_n t) - \omega_n \sin(\omega t)).$$

If $\omega \neq \omega_n$, then $y_2(t)$ exists and is the steady state forced response of $G_2(s)$.

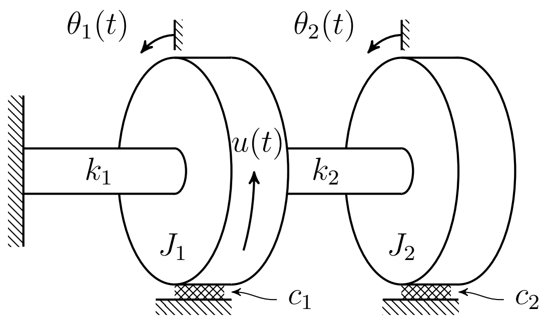
However, for most t,

$$\lim_{\omega\to\omega_n} y_2(t) = \infty.$$

This phenomenon is known as resonance. This one pathological case is why we avoid including LS systems in the formula for $y_{ss}(t)$, even though it would apply in many cases of a LS system responding to a sinusoidal input. Even with damping, we still get this behavior of having large output magnitude for small input oscillations, as $\omega \to \omega_n$. □

5 Problems

Problem 1. Consider the following system:



Let $J_1 = J_2 = 1$ kg m^2, $c_1 = 3$ Ns/m, $c_2 = 1$ Ns /m, $k_1 = 2$ N/m, and $k_2 = 1$ N/m.

All initial conditions are zero.

Let $y = \theta_2$.

a) Find the impulse response
b) Find the final value of the step response. How would you compute the step response?
c) Is there any (continuous) bounded input that would make the response unstable?
d) Find the steady-state forced response to input $u(t) = \sin 2t + \cos t$

Solution: (Later steps motivate earlier steps)

Step 1. Free body diagrams

Step 2. Equations of Motion

Step 3. Laplace Transform from $\hat{u}(s)$ to $\hat{y}(s) = \hat{\theta}_2(s)$

Step 4. Poles of $G(s)$

Step 5. Partial Fraction Expansion of $G(s)$

Step 6. inverse Laplace transforms for impulse

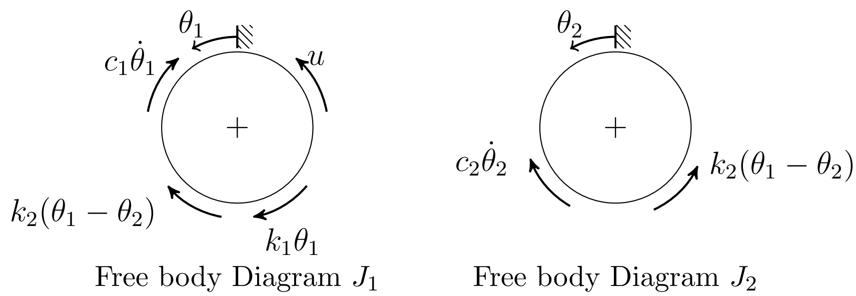
Step 7. Characterize System Stability

Step 8. Final Value Theorem

Step 9. Characterize bounded inputs

Step 10. Use either inverse Laplace transforms or Formulae for steady-state forced frequency response

Free-body Diagrams:



The equations of motion are

$$J_1\ddot{\theta}_1(t)+c_1\dot{\theta}_1(t)+(k_1+k_2)\theta_1(t)-k_2\theta_2(t)=u(t)$$
$$J_2\ddot{\theta}_2(t)+c_2\dot{\theta}_2(t)+k_2\theta_2(t)-k_2\theta_1(t)=0$$

We take the Laplace transform, and assume initial conditions are zero:

$$(J_1s^2+c_1s+k_1+k_2)\hat{\theta}_1(s)-k_2\hat{\theta}_2(s)=\hat{u}(s) \tag{64}$$
$$(J_2s^2+c_2s+k_2)\hat{\theta}_2(s)-k_2\hat{\theta}_1(s)=0 \tag{65}$$

To derive the relationship between $\hat{y}(s)=\hat{\theta}_2(s)$ and $\hat{u}(s)$, we must eliminate $\hat{\theta}_1(s)$, since $\theta_1(t)$ and $\theta_2(t)$ depend on each other, unlike the input $u(t)$ or the initial conditions. The input and initial conditions do not depend on either $\theta_1(t)$ or $\theta_2(t)$.

$$(J_1s^2+c_1s+k_1+k_2)\hat{\theta}_1(s)-k_2\hat{\theta}_2(s)=\hat{u}(s) \tag{66}$$
$$\implies (J_1s^2+c_1s+k_1+k_2)\frac{(J_2s^2+c_2s+k_2)}{k_2}-k_2\hat{\theta}_2(s)=\hat{u}(s) \tag{67}$$
$$\implies \left[(J_1s^2+c_1s+k_1+k_2)(J_2s^2+c_2s+k_2)-k_2^2\right]\hat{\theta}_2(s)=k_2\hat{u}(s) \tag{68}$$
$$\implies \hat{\theta}_2(s)=\left(\frac{k_2}{(J_1s^2+c_1s+k_1+k_2)(J_2s^2+c_2s+k_2)-k_2^2}\right)\hat{u}(s) \tag{69}$$
$$\implies G(s)=\frac{k_2}{(J_1s^2+c_1s+k_1+k_2)(J_2s^2+c_2s+k_2)-k_2^2} \tag{70}$$

So, we get

$$G(s) = \frac{k_2}{(J_1J_2)s^4 + (J_1c_2 + J_2c_1)s^3 + (J_1k_2 + J_2k_1 + J_2k_2 + c_1c_2)s^2 + (c_1k_2 + c_2k_1 + c_2k_2)s + k_1k_2} \tag{71}$$

$$= \frac{1}{s^2 + 4s^3 + 7s^2 + 6s + 2} \tag{72}$$

The poles of $G(s)$ are $-1, -1, -1 \pm j$. Therefore, $G(s)$ is **asymptotically stable (AS)**, and we can rewrite it as

$$G(s) = \frac{1}{(s+1)^2(s+1-j)(s+1+j)} \tag{73}$$

a) Impulse Response We need to calcule $y_i(t) = \mathcal{L}^{-1}\{G(s)\}$:

This step calls for a Partial Fraction Expansion

$$\hat{y}_i(s) = k_0 + \frac{k_1}{(s+1)^2} + \frac{k_2}{(s+1)} + \frac{k_3}{(s+1-j)} + \frac{k_4}{(s+1+j)}$$

$k_0 = 0$ since $n = 4 > m = 0$

According to the rules laid out in the notes on Laplace transforms, we can compute three coefficients that corresponds to multiplicity of corresponding poles. These coefficients are k_1, k_3 and k_4.

$$\hat{y}(s) = \frac{1}{(s+1)^2(s+1-j)(s+1+j)} = \frac{k_1}{(s+1)^2} + \frac{k_2}{(s+1)} + \frac{k_3}{(s+1-j)} + \frac{k_4}{(s+1+j)} \tag{74}$$

Multiplying (74) by $(s-p)^{\text{multiplicity of } p}$ and then substituting $s = p$ always gets rid of everything on the right hand side except the coefficient of $1/(s-p)^{\text{multiplicity of } p}$.

$$k_1 = \left.\frac{(s+1)^2}{(s+1)^2(s+1-j)(s+1+j)}\right|_{s=-1} = \left.\frac{1}{(s+1-j)(s+1+j)}\right|_{s=-1}$$

$$k_3 = \left.\frac{(s+1-j)}{(s+1)^2(s+1-j)(s+1+j)}\right|_{s=-1+j} = \left.\frac{1}{(s+1)^2(s+1+j)}\right|_{s=-1+j}$$

$$k_4 = \left.\frac{(s+1+j)}{(s+1)^2(s+1-j)(s+1+j)}\right|_{s=-1-j} = \left.\frac{}{(s+1)^2(s+1-xj)}\right|_{s=-1-j}$$

After some complex number algebra, $k_1 = 1$, $k_3 = j0.5$, $k_4 = -j0.5$.

To get k_2, start by multiplying (74) by $(s-p)^{\text{ multiplicity of } p}$, where here $p=-1$:

$$\frac{(s+1)^2}{(s+1)^2(s+1-j)(s+1+j)} = \frac{k_1}{(s+1)^2}(s+1)^2 + \frac{k_2}{(s+1)}(s+1)^2 + \frac{k_3}{(s+1-j)}(s+1)^2 + \frac{k_4}{(s+1+j)}(s+1)^2 \tag{75}$$

$$\implies \frac{1}{(s+1-j)(s+1+j)} = k_1 + k_2(s+1) + \frac{k_3}{(s+1-j)}(s+1)^2 + \frac{k_4}{(s+1+j)}(s+1)^2 \tag{76}$$

$$\implies \frac{1}{(s+1)^2+1} = k_1 + k_2(s+1) + \frac{k_3}{(s+1-j)}(s+1)^2 + \frac{k_4}{(s+1+j)}(s+1)^2 \tag{77}$$

Differentiate both sides:

$$\frac{d}{ds}\left(\frac{1}{(s+1)^2+1}\right) = \frac{d}{ds}\left(k_1 + k_2(s+1) + \frac{k_3}{(s+1-j)}(s+1)^2 + \frac{k_4}{(s+1+j)}(s+1)^2\right) \tag{78}$$

$$\implies \frac{-2(s+1)}{((s+1)^2+1)^2} = 0 + k_2 + \frac{d}{ds}\left(\frac{k_3}{(s+1-j)}(s+1)^2\right) + \frac{d}{ds}\left(\frac{k_4}{(s+1+j)}(s+1)^2\right) \tag{79}$$

When we substitute $s=-1$, we know that the terms containing k_3 and k_4 must still contain $(s+1)$ in the numerator. So, we know that they will be zero.

$$\left.\frac{-2(s+1)}{((s+1)^2+1)^2}\right|_{s=-1} = 0 + k_2|_{s=-1} + \left.\frac{d}{ds}\left(\frac{k_3}{(s+1-j)}(s+1)^2\right)\right|_{s=-1} + \left.\frac{d}{ds}\left(\frac{k_4}{(s+1+j)}(s+1)^2\right)\right|_{s=-1} \tag{80}$$

$$\implies 0 = k_2 + 0 + 0 \tag{81}$$

So, $k_2 = 0$, and

$$\hat{y}_i(s) = \frac{1}{(s+1)^2} + \frac{j0.5}{(s+1-j)} + \frac{-j0.5}{(s+1+j)}$$

So,

$$y_i(t) = \mathcal{L}^{-1}\left\{\hat{y}_i(s)\right\} = te^{-t} + \mathcal{L}^{-1}\left\{\frac{j0.5}{(s+1-j)} + \frac{-j0.5}{(s+1+j)}\right\}$$

There are two approaches to dealing with the second inverse.

1.
$$\mathcal{L}^{-1}\left\{\frac{j0.5}{(s+1-j)} + \frac{-j0.5}{(s+1+j)}\right\} = \mathcal{L}^{-1}\left\{\frac{-1}{(s+1)^2+1}\right\} = -e^{-t}\sin t \tag{82}$$

2. An incorrect statement made in class was that $\mathcal{L}^{-1}\left\{1/(s+a)\right\} = e^{-at}$ does not work when a is complex. It actually does work, however e^{-at} is not a real-valued function. The presence of the complex conjugate $-\bar{a}$ as a pole takes care of this issue.

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{j0.5}{(s+1-j)} + \frac{-j0.5}{(s+1+j)}\right\} &= \mathcal{L}^{-1}\left\{\frac{j0.5}{(s+1-j)}\right\} - \mathcal{L}^{-1}\left\{\frac{j0.5}{(s+1-j)}\right\} && (83)\\
&= j0.5e^{(-1+j)t} - j0.5e^{(-1-j)t} && (84)\\
&= j0.5e^{-t}e^{jt} - j0.5e^{-t}e^{-jt} && (85)\\
&= \frac{je^{-t}}{2}\left(e^{jt} - e^{-jt}\right) && (86)
\end{aligned}$$

Since $e^{jx} = \cos x + j\sin x$, and $e^{-jx} = \cos x - j\sin x$, we get

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{j0.5}{(s+1-j)} + \frac{-j0.5}{(s+1+j)}\right\} &= \frac{je^{-t}}{2}\left(e^{jt} - e^{-jt}\right) && (87)\\
&= \frac{je^{-t}}{2}\left(2j\sin t\right) && (88)\\
&= -e^{-t}\sin t && (89)
\end{aligned}$$

So, $y_i(t) = te^{-t} - e^{-t}\sin t$

b) Final value of step response

$$y_{step}(t) = \mathcal{L}^{-1}\left\{G(s)\frac{1}{s}\right\}$$

Since G(s) is AS, we can apply the Final Value Theorem.

$$\lim_{t\to\infty} y_{step}(t) = \lim_{s\to 0} s\hat{y}_{step}(s) = \lim_{s\to 0} s\ G(s)\frac{1}{s} = \lim_{s\to 0} G(s) = \frac{1}{2}. \quad \text{(using (72))}$$

Step response:

$$y_{step}(t) = \mathcal{L}^{-1}\left\{G(s)\frac{1}{s}\right\} = \frac{1}{s(s^4+4s^3+7s^2+6s+2)} = \frac{1}{s(s+1)^2(s+1-j)(s+1+j)}$$

One option: PFE and the usual. Alternate: use the rule for how the Laplace transform changes under integration of the time-domain function.

c) Bounded inputs If $u_{bounded}(t)$ is bounded, and if $\hat{u}_{bounded}(s)$ is its transfer function, then its poles are in the open left half plane (OLHP) or lie on the imaginary axis (IA) with multiplcity 1. $G(s)$ is asymptotically stable, and has all poles in the OLHP.

So, $\hat{y}_{bounded}(s) = G(s)\hat{u}_{bounded}(s)$ must have all its poles either in the OLHP or on the IA with multiplicity 1. Therefore, $y_{bounded}(t) = \mathcal{L}^{-1}\left\{\hat{y}_{bounded}(s)\right\}$ is LS, and is bounded.

Ans: There is no bounded input that destabilizes $G(s)$.

d) Frequency Response The input is $u(t) = \sin 2t + \cos t$. Since $G(s)$ is AS, we just need to find $G(j2)$ and $G(j)$ to calculate the steady-state forced response to this input, which exists.

$$\begin{aligned}
G(j2) &= \frac{1}{(j2)^4 + 4(j2)^3 + 7(j2)^2 + 6(j2) + 2} && (90)\\
&= \frac{1}{16 \cdot j^4 + 4 \cdot 8 \cdot j^3 + 7 \cdot 4 \cdot j^2 + j12 + 2} && (91)\\
&= \frac{1}{16 - j32 - 28 + j12 + 2} && (92)\\
&= \frac{1}{-10 - j20} = \frac{-1}{10}\frac{1}{(1 + j2)} && (93)\\
&= \frac{-1 + j2}{50} && (94)
\end{aligned}$$

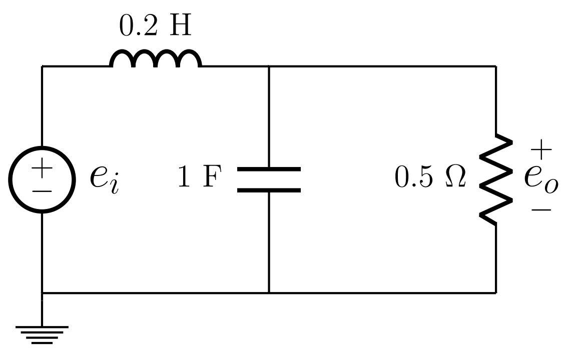
$$\begin{aligned}
G(j) &= \frac{1}{(j)^4 + 4(j)^3 + 7(j)^2 + 6(j) + 2} && (95)\\
&= \frac{1}{j^4 + 4 \cdot j^3 + 7 \cdot j^2 + j6 + 2} && (96)\\
&= \frac{1}{1 - j4 - 7 + j6 + 2} && (97)\\
&= \frac{1}{-4 + j2} = \frac{1}{2}\frac{1}{(-2 + j)} && (98)\\
&= \frac{-2 - j}{10} && (99)
\end{aligned}$$

$$|G(j2)| = \frac{\sqrt{5}}{50}, \quad \angle G(j2) = \pi + \tan^{-1} -2, \quad |G(j)| = \frac{\sqrt{5}}{10}, \quad \angle G(j2) = \pi + \tan^{-1} 0.5$$

Therefore, using all the results from steady-state frequency response notes, and the linearity of the system (principle of superposition)

$$y_{ss}(t) = \frac{\sqrt{5}}{50}\sin(2t + \pi + \tan^{-1} 2) + \frac{\sqrt{5}}{10}\cos(t + \pi + \tan^{-1} 0.5)$$

Problem 2. Consider the circuit below:

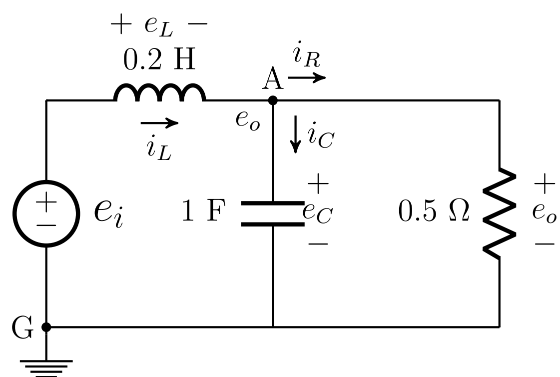


The initial conditions are $e_o(0) = 0$ V and $\dot{e}_o(0) = 0$ V/s. The input $e_i(t) = 6$ V.

Find the free response, forced response, and total response e_o.

Solution:

We add nodes and assign voltages and currents as shown on the right. Note that $e_o = e_C$.



KVL Around outer loop yields

$$e_i = e_L + e_o.$$

To find an expression for e_L, KCL at A will be useful:

$$i_L = i_R + i_C \tag{100}$$

$$\implies i_L = \frac{e_o}{0.5} + 1 \cdot \dot{e}_o \tag{101}$$

$$\implies \frac{d}{dt} i_L = 2\dot{e}_o + \ddot{e}_o \tag{102}$$

$$\implies \frac{e_i - e_o}{0.2} = 2\dot{e}_o + \ddot{e}_o \tag{103}$$

$$\implies \ddot{e}_o + 2\dot{e}_o + 5e_o = 5e_i \tag{104}$$

The initial conditions are zero, which simplifies the procedure of taking the Laplace transform:

$$(s^2 + 2s + 5)\hat{e}_o(s) = 5\hat{e}_i(s) \implies \hat{e}_o(s) = \frac{5}{s^2 + 2s + 5}\hat{e}_i(s) \tag{105}$$

In general, this is a very good point to stop and double check your work, since any error here carries over to the computations involving finding poles and the response.

Since the initial conditions are zero, the free response is zero.

$$e_{o,free}(t) = 0$$

For the forced response, $\hat{e}_i(s) = 6/s$

$$\begin{aligned}\hat{e}_o(s) &= \frac{5}{s^2+2s+5}\hat{e}_i(s) &(106)\\ &= 6\frac{5}{s(s^2+2s+5)} &(107)\end{aligned}$$

This type of term is easy to expand:

$$\frac{c}{s(as^2+bs+c)} = \frac{1}{s} - \frac{as+b}{as^2+bs+c} \tag{108}$$

Therefore

$$\begin{aligned}\hat{e}_o(s) &= \frac{6}{s} - 6\frac{s+2}{s^2+2s+5} &(109)\\ &= \frac{6}{s} - 6\frac{s+2}{(s+1)^2+2^2} &(110)\\ &= \frac{6}{s} - 6\frac{s+1+1}{(s+1)^2+2^2} &(111)\\ &= \frac{6}{s} - 6\frac{s+1}{(s+1)^2+2^2} - 6\frac{1}{(s+1)^2+2^2} &(112)\\ \implies e_{o,forced}(t) &= 6 - 6e^{-t}\cos 2t - 6e^{-t}\sin 2t &(113)\end{aligned}$$