ME 599/699 Robot Modeling & Control
Passivity

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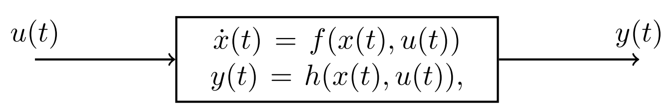
1 Passivity

Consider a system

$$\dot{x}(t) = f(x(t), u(t)) \tag{1}$$
$$y(t) = h(x(t), u(t)), \tag{2}$$

where

- $x \in \mathbb{R}^n$: state
- $y \in \mathbb{R}^m$: output
- $u \in \mathbb{R}^p$: input



This system interacts with the environment through time-varying inputs $u(t)$ and outputs $y(t)$, and this interaction influences the state $x(t)$.

Remark 1. For many systems, the input and output variables match the **effort** and **flow** variables of one-port models used in network-impedance-based modeling.

One way to define this interaction is through two functions known as the **supply rate** $S(y, u)$ and the **storage function** $V(x)$.

Definition 1. Supply Rate. The supply rate is a function $S\colon \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ that quantifies the amount of interaction with the environment.

Definition 2. Storage Function. A storage function is a non-negative function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ of the state.

Example 1. The state of a capacitor is the charge q_c contained in it. A good choice of the storage function is typically the electrical energy stored in the component

$$V(q_c) = \frac{1}{2C} q_c^2,$$

where C is the capacitance of the capacitor.
The input is the current flow i_c and the output is the voltage e_c across the component, however some circuit applications involve switching the two definitions. The supply rate is the power provided/taken from the capacitor:

$$S(e_c, i_c) = e_c i_c.$$

Note that a capacitor satisfies

$$q_c = C e_c, \quad \frac{d}{dt} q_c = i_c.$$

Definition 3 (Passivity.)**.** A system with state x is said to be *passive* with respect to an input-output pair y, u if there exists a supply rate $S(y, u)$ and storage function $V(x)$ such that

$$V(t_1) - V(t_0) \leq \int_{t_0}^{t_1} S\left(y(\tau)u(\tau)\right) d\tau, \quad \forall t_0, t_1 \geq t_0.$$

A passive system is one where the change in its storage function is always less than the integral of the supply rate.

Example 1. Continued Let's compute the time derivative of the storage function:

$$\begin{aligned}
\frac{d}{dt} V(q_c) &= \left(\frac{\partial V}{\partial q_c}\right) \dot{q}_c = \left(\frac{1}{C} q_c\right) \dot{q}_c && (3)\\
&= \frac{1}{C} q_c \dot{q}_c = \frac{1}{C} \left(C e_c\right) \left(i_c\right) && (4)\\
&= e_c i_c && (5)\\
\implies \dot{V}(t) &= S\left(e_c(t), i_c(t)\right) && (6)\\
\implies V(t_1) - V(t_0) &\leq \int_{t_0}^{t_1} S\left(e_c(\tau), i_c(\tau)\right) d\tau && (7)
\end{aligned}$$

Therefore, we conclude that a capacitor is passive with respect to input e_c and output i_c, with storage function given by the energy and supply rate by the electrical power. Passivity here means that the capacitor doesn't create energy, its total energy is always no greater than the energy contained in the total power supplied to the capacitor.

2 Passivity in Robots

For robotic mechanisms, we consider the output to be the joint velocities $\dot{q}$, the input is the non-conservative torques τ applied at the joints by the environment, and the state is $(q, \dot{q})$.
A candidate storage function is the total energy of the system

$$V(q, \dot{q}) = KE(q, \dot{q}) + PE(q),$$

where K is the kinetic energy
$$K = \frac{1}{2}\dot{q}^T M(q)\dot{q},$$
and $PE(q)$ is the potential energy. Note that potential energy may be zero at many configurations, so that **a storage function is usually NOT a Lyapunov function**.

The dynamics become
$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + \frac{\partial PE(q)}{\partial q} = \tau,$$
where we have so far assumed that the potential energy contains only the gravitational energy, so that $\frac{\partial PE(q)}{\partial q} = G(q)$. Note that technically the gradient $\frac{\partial PE(q)}{\partial q}$ is not a vector, but a dual vector (co-vector), but we don't stress this difference here. Since this term depends on potential energy, it represents conservative forces, of which gravitational force is an example. The term τ contains the non-conservative external forces, where . These external forces usually include:

- Motor torques τ_m
- Damping at joints $-B\dot{q}$
- Contact forces $J^T(q)F$, where $J(q)$ is the Jacobian from q to coordinates of a frame defined at the point of contact.

Other forces are possible, but we focus on these.
The power supplied to the robot is $\dot{q}^T\tau$, which serves as the **supply rate**.
Let's calculate the time-derivative of the storage function:

$$\begin{aligned}
\dot{V} &= \dot{q}^T M(q)\ddot{q} + \frac{1}{2}\dot{q}^T \dot{M}(q)\dot{q} + \left(\frac{\partial PE(q)}{\partial q}\right)^T \dot{q} && (8)\\
&= \dot{q}^T M(q) M^{-1}(q)\left(\tau - C(q,\dot{q})\dot{q} - \frac{\partial PE(q)}{\partial q}\right) + \frac{1}{2}\dot{q}^T \dot{M}(q)\dot{q} + \left(\frac{\partial PE(q)}{\partial q}\right)^T \dot{q} && (9)\\
&= \dot{q}^T\tau + \frac{1}{2}\dot{q}^T\left(\dot{M}(q) - 2C(q,\dot{q})\right)\dot{q} && (10)\\
&= \dot{q}^T\tau && (11)\\
&= S(y,u) && (12)
\end{aligned}$$

Again, we can conclude that a robot is passive from externally applied non-conservative forces/torques to the joint velocity.

2.1 Potential-Shaping Control

Passivity provides an easy way to achieve a certain type of set-point regulation. If the torques τ include a term of the form $-B\dot{q}$, where $B > 0$, either due to motor friction or controlled damping, we can conclude that
$$\dot{V} = -\dot{q}^T B\dot{q} \le 0.$$
Where does $q(t)$ reach? If the potential energy $PE(q)$ has a local minimum $q^\star$, then we can show that $q(t) \to q^\star$ at least locally (when $q(0)$ is close to $q^\star$).

If we want $q^\dagger$ to be the equilibrium, where $q^\dagger \neq q^\star$, we just need to define a new potential energy $PE^\dagger(q)$ which has only one minimum at $q^\dagger$, and then use the motor control

$$\tau_m = -B\dot{q} + \frac{\partial PE(q)}{\partial q} - \frac{\partial PE^\dagger(q)}{\partial q},$$

and the storage function

$$V(q,\dot{q}) = KE(q,\dot{q}) + PE^\dagger(q),$$

to arrive at the conclusion that $q(t) \to q^\dagger$.
Note that this analysis does not account for the presence of any interaction force F.

2.2 Passivity-based Tracking

Suppose we want to track the trajectory $q_d(t)$. Let

$$e = \begin{bmatrix} q(t) - q_d(t) \\ \dot{q}(t) - \dot{q}_d(t) \end{bmatrix} = \begin{bmatrix} \tilde{q} \\ \dot{\tilde{q}} \end{bmatrix}.$$

Assume that the model of the robot is

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + \frac{\partial PE(q)}{\partial q} = \tau_m,$$

We define a controller as

$$\tau_m = M(q)a + C(q,\dot{q})v + \frac{\partial PE(q)}{\partial q} - Kr, \text{ where} \tag{13}$$

$$v = \dot{q}_d - \Lambda\tilde{q} \tag{14}$$
$$a = \dot{v} = \ddot{q}_d - \Lambda\dot{\tilde{q}} \tag{15}$$
$$r = \dot{q} - v = \dot{\tilde{q}} + \Lambda\tilde{q}, \tag{16}$$

with K, Λ diagonal matrices of positive gains.

The closed loop becomes

$$M(q)\dot{r} + C(q,\dot{q})r + Kr = 0 \tag{17}$$

Choose a storage function $V(q,\dot{q})$, which in this case is also a Lyapunov function:

$$V = \frac{1}{2}r^T M(q) r + \tilde{q}^T \Lambda K \tilde{q} \tag{18}$$

$$\begin{aligned} \dot{V} &= -r^T K r + 2\tilde{q}^T \Lambda K \dot{\tilde{q}} + \frac{1}{2}r^T \left(\dot{M}(q) - 2C\right) r \\ &= -\tilde{q}^T \Lambda^T K \Lambda \tilde{q} - \dot{\tilde{q}}^T K \dot{\tilde{q}} \\ &= -e^T Q e \end{aligned} \tag{19}$$

If $M(q)$ is bounded then we can conclude that $e = 0 \iff V = 0$ so that the origin is GAS.
Again, our analysis assumes that $F = 0$.

2.3 Passivity-based Interaction

When $F \neq 0$, we cannot ensure tracking using the previous arguments, because

$$\begin{aligned} \dot{V} &= \dot{q}^T \tau_m + \dot{q}^T J^T(q) F && (20) \\ &= \dot{q}^T \tau_m + \xi^T F, && (21) \\ & && (22) \end{aligned}$$

where ξ is the velocity of the contact frame, in which the contact force is F.

Like impedance control, if we know how the environment behaves (the force it generates at the contact in response to motion of the contact), we may be able to choose τ_m intelligently to achieve force tracking or position tracking.

Passivity control allows us a different type guarantee: if the environment is passive, then the robot-environment can be made stable by choosing τ_m to make the robot passive, **without knowing anything else about the environment, like its impedance**.
Let the robot and environment storage functions be V_r and V_e respectively. If the only interaction is at the robot-environment contact, we get the the supply rate to the robot is

$$S_r = \dot{q}^T \tau_m + \xi^T F.$$

The important idea is that the supply rate of the environment is

$$S_e = -\xi^T F.$$

Therefore, we can define a new storage function

$$V_{re} = V_r + V_e,$$

and see that

$$\dot{V}_{re} = \dot{q}^T \tau_m.$$

Assuming that the storage functions are bounded when the state is bounded, we can make the system stable by choosing $\tau_m = 0$.
Saying more than that requires us to know the storage function and supply rate of the environment. If we know that, for example, that $F = K\xi$, which would destabilize the system, we may choose $\tau_m = -\rho(\xi) K \dot{q}$ where $\rho(\xi) \geq \|\xi\|^2$.

3 Applications

3.1 Bipedal Robots

Potential-energy shaping is frequently used to dictate the behavior of a robot by modifying the potential energy of that robot. Gravity compensated PD control is an example of potential-energy shaping. A non-trivial example is low-power walking of bipedal robots on flat ground and uphill [3]. The idea was that some mechanisms walk steadily down slopes with no energy inputs. The energy lost at foot-strike balanced out the energy gained from potential energy. By comping up with a suitable potential energy function, those same motions could be achieved on flat ground, and do not require much additional energy inputs.

3.2 Teleoperation

Remote physical interaction used to perform poorly due to the issue of time-delay in the medium transferring real-time physical signal information between the master device and the remote device. Passivity theory suggested that the instability was due to a fake added energy resulting from the delay in signals. Loosely speaking, the power-content of out-of-phase signals is different from the real power present in synched versions. This power-mismatch built up energy, causing instability. In effect, the time-delay in the medium made it become a fake source of power that flowed into the master and remote devices. The solution was to make the medium itself passive, so that the three systems together: master, medium, and remote, formed a interconnected system that was passive by construction [2].

3.3 Power System Control

Electrical systems are also networks of components, and passivity naturally applies to the analysis and control of such systems [1].

References

[1] V. H. Garcia-Rodriguez, J. R. Garcia-Sanchez, R. Silva-Ortigoza, E. Hernandez-Marquez, H. Taud, M. Ponce-Silva, and G. Saldana-Gonzalez. Passivity based control for the boost converter-inverter-dc motor system. In *2017 International Conference on Mechatronics, Electronics and Automotive Engineering (ICMEAE)*, pages 77–81, 2017.

[2] Emmanuel Nuno, Luis Basanez, and Romeo Ortega. Passivity-based control for bilateral teleoperation: A tutorial. *Automatica*, 47(3):485 – 495, 2011.

[3] Mark W. Spong. Passivity based control of the compass gait biped. *IFAC Proceedings Volumes*, 32(2):506 – 510, 1999. 14th IFAC World Congress 1999, Beijing, Chia, 5-9 July.