

# ME 599/699 Robot Modeling & Control

## Fall 2021

### Twists and Wrenches

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- ▶ for constant angular velocity  $\omega(t) \equiv \omega$ , we can solve (1) like a linear system  $\dot{x}(t) = Ax(t)$ :

$$R(t) = e^{[\omega]t} R(0) \quad (2)$$

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- ▶ Let  $\hat{\omega} = \omega / \|\omega\|_2$  (unit norm) and  $\theta = \dot{\theta} \cdot 1 = \|\omega\|_2$ .
- ▶ Exponential map  $\exp: [\hat{\omega}]\theta \in \mathfrak{so}(3) \rightarrow R \in \text{SO}(3)$ .  
Logarithm map  $\log: R \in \text{SO}(3) \rightarrow [\hat{\omega}]\theta \in \mathfrak{so}(3)$ .

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- ▶  $\hat{\omega}\theta$  are therefore the exponential coordinates of  $R$
- ▶ We may interpret exponential coordinates as coming from a constant angular velocity applied for one second

## Angular Velocity Frames

- ▶ The equation  $\dot{R}(t) = [\omega(t)]R(t)$  involves terms defined in a fixed reference frame, called the *space frame*  $\{s\}$  in MR, so really

$$\dot{R}(t) = [\omega_s]R(t) \quad (R = \underbrace{R_b^s}_{\text{RMC}} = \underbrace{R_{sb}}_{\text{MR}})$$

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- ▶ The equations may therefore be rewritten as

$$\dot{R}(t) = R(t)[\omega_b]$$

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- ▶ Any Lie group has a similar set of manipulations
- ▶  $\text{SE}(3)$  (homogenous transformations) are also a Lie group
- ▶ The 'angular velocity' corresponding to  $\text{SE}(3)$  is a *twist*
- ▶ Twists for  $\text{SE}$  are not as intuitive as angular velocities for  $\text{SO}(3)$ .

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- ▶ Consider a homogenous transformation  $T(t) \in SE(3)$  representing a rigid body pose of  $\{b\}$  in  $\{s\}$ :

$$T(t) = \begin{bmatrix} R(t) & p(t) \\ 0 & 1 \end{bmatrix} \quad (T = \underbrace{T_b^s}_{\text{RMC}} = \underbrace{T_{sb}}_{\text{MR}} = \underbrace{H_b^s}_{\text{HP}}) \quad (5)$$

# Body Twist

- If the angular velocity in the body frame is  $\omega_b$ , and the velocity of the origin is  $v_b$ , then

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- ▶ The body twist has simple physical meaning:  
instantaneous angular velocity of  $\{b\}$  as seen in  $\{b\}$ , and  
instantaneous velocity of origin of  $\{b\}$  as seen in  $\{b\}$

# Spatial Twist

- ▶ We can convert the body twist  $\mathcal{V}_b = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix}$  into a spatial twist

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- ▶  $v_s$  a fictitious velocity of the origin of  $\{s\}$  as if the space frame  $\{s\}$  was rotating about axis  $\omega_s$  that passes through origin of  $\{b\}$  (Fig 3.17 in MR).

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- ▶ The remaining notation is about how to transform between  $\mathcal{V}_b$  and  $\mathcal{V}_s$ , or between  $[\mathcal{V}_b]$  and  $[\mathcal{V}_s]$

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- ▶  $\mathcal{V} = \mathcal{S}\dot{\theta}$ , just like  $\omega = \hat{\omega}\dot{\theta}$
- ▶ The result of following a constant spatial twist  $\mathcal{V}_s$  for one second can be interpreted as a screw motion : translation along an axis and rotation about that axis.



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- ▶ As usual, a generic force and moment can be expressed in either  $\{b\}$  or  $\{s\}$
- ▶ The conversion between expressions is related to the conversion for  $[\mathcal{V}_b]$  and  $[\mathcal{V}_s]$  (pg. 108 in MR)