

ME/AER 676 Robot Modeling & Control

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Forward Kinematics & Jacobians

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Introduction

- ▶ We consider robots modeled as joined links.
- ▶ The degrees of freedom at the joints for the joint variables q .
- ▶ Task variables x capture quantities describing what the robot must do.
- ▶ Traditional robot control focuses on the conversion of joint variables to task variables (forward kinematics) and back (inverse kinematics)

$$x = f(q); \quad q = f^{-1}(x)$$

Forward Kinematics as Homogenous Transformations

- ▶ This problem involves composing a number of relative link (homogenous) transformations
- ▶ It may be solved numerically, with the specific details depending on how these link transformations are parametrized
- ▶ The transformation (d, R) may be represented by
 - ▶ D-H Parameters
 - ▶ Twist (Screw Theory)
 - ▶ origin and Euler angles
 - ▶ etc. . . .

Serial Kinematic Chains

- ▶ We look at serial kinematic chains where all joints are simple.
- ▶ We number links as 0 for base to n in sequence.
- ▶ The assumption of single-parameter joints means we can use basic transformations to handle coordinate transformations.
- ▶ These basic transformation are denoted $A_i(q_i)$, where $q_i \in \mathbb{R}$ is the joint variable.
- ▶ q_i is either an angle θ_i or a distance d_i , depending on the type of simple joint.

Forward Kinematics of Serial Chains

Given link i and $i - 1$,

$$A_i = \begin{bmatrix} R_i^{i-1} & o_i^{i-1} \\ 0 & 1 \end{bmatrix} \quad (1)$$

Transformations between links i and j is T_j^i , where we are expressing frame j in frame i .

$$T_j^i = \begin{cases} A_{i+1}A_{i+2} \cdots A_{j-1}A_j & i < j \\ I & i = j \\ (T_j^i)^{-1} & i > j \end{cases} \quad (2)$$

Forward Kinematics of Serial Chains

- ▶ For an n -link serial chain manipulator, the task variables are a combination of
 - ▶ Origin of frame n (*end-effector* or *tool* frame)
 - ▶ Orientation of frame n



$$T_n^0 = \begin{bmatrix} R_n^0 & d_n^0 \\ 0 & 1 \end{bmatrix}$$

- ▶ x is derived from R_n^0 and/or d_n^0

Modern Robotics

- ▶ The book “Modern Robotics” uses exponential coordinates (twists) to represent homogenous transformations.
- ▶ It does not follow the D-H convention (next slide).
- ▶ The main difference to D-H is that frame i fixed to link i is at joint i , not joint $i + 1$.
- ▶ Videos on FK in this course follow MR's convention of locating frame i at joint i .
- ▶ Universal Robot Description Formats (URDFs) also follow this approach

Denavit-Hartenberg Convention

In this convention

- ▶ All motion happens along the z axis
- ▶ Four numbers are enough to define relative link transformations (instead of 6 or 12).

The D-H convention is based on two restrictions:

(DH1) The x_1 axis intersects the z_0 axis.

(DH2) The x_1 axis is orthogonal to the z_0 axis.

This restriction makes the transformation matrix between link i and $i - 1$ given in (1) reduce to

$$A_i = \text{Rot}_{z,\theta_i} \text{Trans}_{z,d_i} \text{Trans}_{x,a_i} \text{Rot}_{x,\alpha_i} \quad (3)$$

This convention is more common in earlier robotics texts, and is used in many systems.

Jacobians and Forward Velocity Kinematics

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$$\text{Forward Kinematics: } x = f(q) \quad (4)$$

$$\text{Forward Velocity Kinematics: } \dot{x} = ? \quad (5)$$

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$$\text{Forward Kinematics: } x = f(q) \quad (4)$$

$$\text{Forward Velocity Kinematics: } \dot{x} = J(q)\dot{q} \quad (5)$$

- ▶ $J(q)$: Jacobian matrix
- ▶ Size of $J(q)$ depends on that of joint and task spaces
- ▶ Derivation of $J(q)$ depends on type of coordinates for joint and task spaces
 - ▶ Geometric Jacobians : when representing derivative of task frame orientation as an angular velocity
 - ▶ Analytic Jacobians : when representing derivative of task frame orientation as derivative of task orientation variables

Uses of the Jacobian

- ▶ Forward Velocity Kinematics: Compute end-effector velocity ξ given joint angle derivatives \dot{q}
- ▶ Inverse Velocity Kinematics: Compute \dot{q} given ξ
- ▶ Relates end-effector forces F to joint torques τ at equilibrium:
$$\tau = J(q)^T F$$
- ▶ Defines the manipulability μ and the manipulability ellipsoid (next slide)

Manipulability

1. The manipulability μ is then given by

$$\mu = \prod_{i=1}^m \sigma_i \quad (6)$$

where σ_i are the singular values of $J \in \mathbb{R}^{m \times n}$; $J = U\Sigma V$.

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2. Let $\text{rank}(J) = m$, and $w = U^T \xi$. Then

$$\dot{q} = J^+ \xi \implies \|\dot{q}\|^2 = \xi^T (JJ^T)^{-1} \xi, \text{ where}$$

$$\xi^T (JJ^T)^{-1} \xi = (U^T \xi)^T \Sigma_m^{-2} (U^T \xi) = w^T \Sigma_m^{-2} w = \sum_{i=1}^m \frac{w_i^2}{\sigma_{m_i}^2}$$

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3. If $\|\dot{q}\|^2 = 1 = \xi^T (JJ^T)^{-1} \xi$ then corresponding ξ form an ellipsoid in space of task velocities ξ .

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- ▶ This ellipsoid has two physical interpretations:
 - ▶ When there's no contact, this ellipsoid describes achievable task velocities given unit-size joint velocities.
 - ▶ During static contact, this ellipsoid describes achievable task forces given unit-size joint torques.