ME 599/699 Robot Modeling & Control
Cartesian Frames & Rigid Body Pose

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Contents

1 Preview

1. Points in three dimensional space (or the two dimensional plane) do not have intrinsic coordinates. These points form a real affine space.
2. Every cartesian coordinate frame assigns its own unique coordinate to a point in n-dimensional space. These coordinates form a real coordinate space $\mathbb{R}^n$ that possesses an inner product, a norm, and a metric.
3. The same point in space can have multiple coordinates, each corresponding to a different frame.
4. We can relate descriptions of the same point in space in different coordinate frames via rigid coordinate transformations.
5. We can describe the motion of multiple points on a moving rigid body occurs by describing the motion of a body-fixed coordinate frame.

Definition 1 (Group)**.** A group G is a set together with a binary operation $\cdot$ that satisfies the following properties for all $a, b, c \in G$:

(i) Closure: $a \cdot b \in G$;

(ii) Associativity: $a \cdot (b \cdot c) = (a \cdot) b \cdot c$;

(iii) Existence of identity element $e \in G$ such that $a \cdot e = e \cdot a = a$;

(iv) Existence of inverse element $d \in G$ such that $d \cdot a = a \cdot d = e$.

2 Cartesian Coordinates

We've seen the an n-dimensional Euclidean space consists of a collection of points, together with the notion of translation as implied by a inner product space $\mathbb{R}^n$.

This inner product helps identify whether two translations are collinear or not, in effect defining parallel lines in Euclidean space.

Definition 2 (Cartesian Coordinates)**.** Identifying a point in Euclidean space with the zero vector of $\mathbb{R}^n$, and defining an orthogonal basis for $\mathbb{R}^n$ equips Euclidean space with Cartesian coordinates.

2.1 Body-Fixed Frames

We may define a coordinate frame that moves with a rigid body in $\mathbb{R}^n$ by choosing $n+1$ non-trivial points on the rigid body. One point becomes the origin, the remaining points define n independent basis vectors. For 3D, we need four points. Every point on the rigid body can then be assigned a unique coordinate relative to this frame which is constant for all time.

This frame is known as a body-fixed frame. Unless specified, we assume that the n independent basis vectors are orthogonal and normal, so that the frame is a cartesian frame.

3 Homogenous Transformations

Example 1 (Robot And Camera)**.** A robot needs to pick something up, and a camera tells it where it is. If the robot and camera are using different reference frames, how do you convert the position from the camera into a position that makes sense for the robot?

Let p^A and p^B be the coordinates of a point p in frames A and B respectively. We want a to find a map $g: \mathbb{R}^n \mapsto \mathbb{R}^n$ such that $p^A = g(p^B)$ for any point p in Euclidean space. The following theorem says that such a map must necessarily be affine.

Theorem 1 (Ulam-Mazur)**.** *Let U, V be normed spaces over $\mathbb{R}$. If mapping $g: U \mapsto V$ is a bijective isometry, then g is affine.*

Corollary 2. *Any coordinate transformation g between a pair of three dimensional cartesian coordinate spaces X and X' with the same orientation is parametrized by a pair (d, R) where $d \in \mathbb{R}^3$ and $R \in SO(3)$. Thus, $g(p) = Rp + d$.*

Proof. Assignment. □

Problem 1 (HW 2)**.** Prove Corollary 2
Hint: Given two coordinates p^A and q^A, and a map g that maps them to coordinates p^B and q^B respectively, what properties of coordinates p^B and q^B hold independent of the map g?

Problem 2 (HW 2)**.** Show that for two given cartesian coordinate frames, the parameters (d, R) of the coordinate transformation are unique.

Problem 3 (HW 2)**.** Let the affine transformation from frame A to frame B be parametrized by (d, R). Express the affine transformation that maps coordinates in frame B to coordinates in frame A in terms of R and d?
[Aside: Why is the derivation of this expression valid?]

The unique transformation that maps a point's coordinates in one frame to its coordinates in another frame is an affine map. We can convert this affine map between two Euclidean spaces of dimension 3 into a linear map between two subsets of $\mathbb{R}^4$.

Define a homogenization $h: \mathbb{R}^3 \mapsto \mathbb{R}^4$

$$h\left(p^A\right) = \begin{bmatrix} p^A \\ 1 \end{bmatrix}. \tag{1}$$

We refer to the vector $h\left(p^A\right)$ as the homogenous representation of coordinate p^A. The transformation between homogenous representations of coordinates in different frames is linear. Mathematically, if $p^A = Rp^B + d$, then

$$h\left(p^A\right) = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix} h\left(p^B\right). \tag{2}$$

The matrix $\begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix}$ represents a homogenous transformation, and forms a group.

Problem 4 (HW 1)**.** Show that homogenous transformations form a group.

Definition 3 (Special Euclidean Group)**.** A rigid motion is a pair (d, R) where $d \in \mathbb{R}^3$ and $R \in SO(3)$. The group of all rigid motions is known as the **Special Euclidean Group** and is denoted by $SE(3)$. We see that $SE(3) = \mathbb{R}^3 \times SO(3)$.

3.1 Notation

The superscript of a coordinate denotes the frame it is defined in. The coordinate transformation that takes points in frame B to frame A is denoted as (d_B^A, R_B^A), so that

$$p^A = R_B^A p^B + d_B^A.$$

4 Rigid Body Pose

We have shown that given two frames A and B, there's a unique affine transformation (d, R) that maps coordinates of a point in frame B to its coordinates in frame A, where $R \in SO(3)$ and $d \in \mathbb{R}^3$. Given any other frame C, the transformation that maps coordinates of frame C into coordinates in frame A is given by a transformation (d', R') where $(d', R') \neq (d, R)$. Since the affine transformation is unique for any frame, the pair (d, R) serves as a configuration of frame B in frame A.

Since we can associate a coordinate frame to a rigid body, we can associate the configuration of that frame to the rigid body. Therefore, the configuration of any rigid body in some coordinate frame, also known as its pose, is described by a pair (d, R) where $R \in SO(3)$ and $d \in \mathbb{R}^3$. This pair is associated with the rigid body, and it comes from the body-fixed coordinate frame.

4.1 Rotations

The matrix R is an element of $SO(3)$, which is a subset of the more general linear group $GL(\mathbb{R}^3)$. The group $GL(\mathbb{R}^3)$ is the space of linear bijective transformations between $\mathbb{R}^3 \mapsto \mathbb{R}^3$ with functional composition as the group operation. The matrix R, which is part of the pose of a rigid body, is known as the orientation matrix, and exactly gives the orientation of one rigid body with respect to another.

We have seen one interpretation of R as a map from one frame to another frame rotated with respect to the first. Since the map is unique, R also serves to represent the orientation of the second frame with respect to the first. A rotation matrix can also represent a rotation within the same (or **current**) frame.

Most important property: $R^T = R^{-1}$

The rotation matrix is effectively defining a basis for $\mathbb{R}^3$.

Definition 4 (Basis)**.** A basis B of a vector space V over a field $\mathbb{F}$ is a linearly independent subset of V that spans V.

An orthonormal bases has unit elements and mutually perpendicular vectors.

The relationships for change of bases from linear transformation has direct interpretations in terms of rotation operations.

Similarity Transform

A linear map defined in a coordinate frame has a matrix representation in that frame. A similarity transform maps the representation of that linear transformation into another coordinate frame.

Suppose T^A represents a linear map defined in frame A. Let the orientation of frame B with respect to frame A be R_B^A. Then the same linear map in frame B is given by matrix T^B

$$\begin{aligned}
T^B &= \left(R_B^A\right)^{-1} \; T^A \; R_B^A \\
\implies T^A &= R_B^A \; T^B \; \left(R_B^A\right)^{-1} \\
\implies T^B &= R_A^B \; T^A \; \left(R_A^B\right)^{-1} \\
\implies T^A &= \left(R_A^B\right)^{-1} \; T^B \; R_A^B
\end{aligned}$$

4.2 Basic Rotations

Consider three frames rotated about each one of the world frame axes by an angle θ.

Each rotation is given by

$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$R_{y,\theta} = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

$$R_{z,\theta} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4.3 Composition of Rotations

We can consider a rotation matrix to represent a rotation relative to a frame. Consider a rigid body in frame A, where its frame is coincident with that of A. We perform a rotation corresponding to rotation matrix R_1. The rigid body's frame is now different from frame A, and we call it frame B. We perform a second rotation corresponding to a matrix R_2, say $R_2 = R_{x,\pi/3}$. However, this rotation can be applied in two ways to our rigid body, depending on which frame R_2 is relative to: the original frame A, or the frame B that is coincident with the body. We get two different final poses for the rigid body depending on which one we choose.

4.3.1 Current Frame

If the rotation R_2 is relative to the current frame of the rigid body (frame B), then the combined effect of the two successive rotations in frame A is a post-multiplication of the sequence of rotations: R_1 then R_2. That is, $R' = R_1 R_2$.

4.3.2 Fixed Frame

If the rotation R_2 is relative to the fixed frame A, then the combined effect of the two successive rotations in frame A is a pre-multiplication of the sequence of rotations: R_1 then R_2. That is, $R'' = R_2 R_1$.

How to derive: We have a rotation R_2 in frame A. We can express it in frame B via a similarity transform

$$R_3 = R_1^{-1} R_2 R_1$$

. We've converted the rotation in frame A to its representation in frame B, so that the sequence of rotations are with respect to the current frame.

$$\begin{aligned} R'' &= R_1 R_3 && (3) \\ &= R_1 \left(R_1^{-1} R_2 R_1 \right) && (4) \\ &= R_2 R_1 && (5) \end{aligned}$$

4.3.3 Non-commutation

Since matrix multiplication is non-commutative, in general $R' \neq R''$.

4.4 Parametrizations of SO(3)

Although the representation R has nine elements, the space $SO(3)$ is three dimensional. One way to see this is to note that $R^T R = I$, which introduces six constraints on the elements of R. We now look at two popular ways to parametrize R as a three-dimensional vector.

4.4.1 Euler Angle Representation

Euler angles consist of three angles corresponding to three consecutive basic rotations. These rotations either use two axes (proper Euler) or three axes (Tait-Bryan). Furthermore, we may designate the rotations to be with respect to a world frame (extrinsic) or the body frame (intrinsic).

Proper Euler: There are six proper Euler conventions:

1. X-Y-X (Rotate about X, then Y, then X again)
2. X-Z-X
3. Y-X-Y
4. Y-Z-Y
5. Z-X-Z
6. Z-Y-Z (Common in astrophysics)

These are doubled when considering intrinsic (body-frame) and extrinsic (world-frame) rotations.

Tait-Bryan:

1. X-Y-Z
2. X-Z-Y
3. Y-X-Z
4. Y-Z-X
5. Z-X-Y
6. Z-Y-X

Also double if you allow both intrinsic and extrinsic rotations. This representation includes the yaw-roll-pitch method common in aerospace literature. Instead, if someone says roll-pitch-yaw, then we have different numerical values.

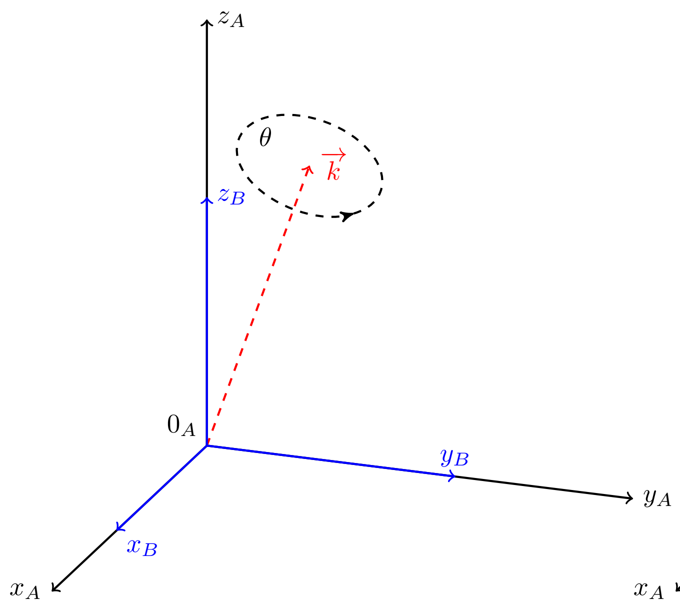
The main drawback of Euler-angles: non-uniqueness of values of three angles at singular points.

4.4.2 Axis/Angle Representation

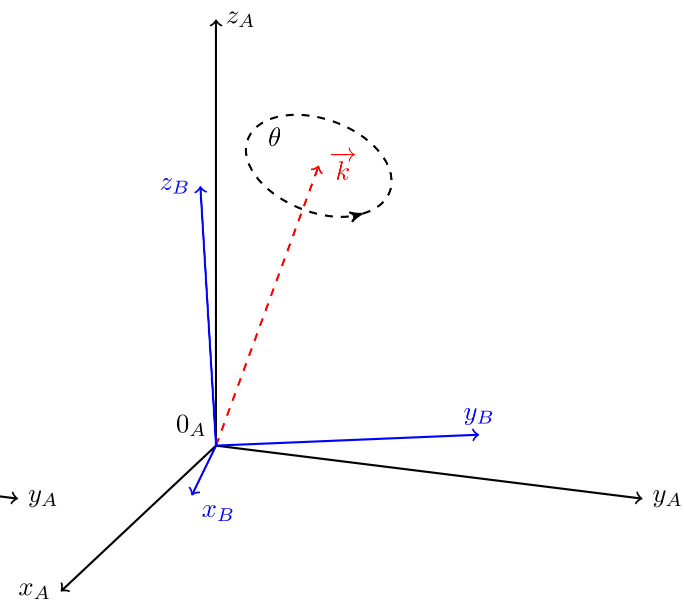
This representation is related to quaternions. The idea is that any orientation in a frame can be reached by rotating a coordinate frame by some angle $\theta \in [0, 2\pi)$ around some vector $\overrightarrow{k} \in \mathbb{R}^3$ in that frame. How do we represent that orientation?

Consider a frame C identical to the world frame A. Define β, rotation of C about world y, then α, rotation of C about world z that aligns world z_C with $\overrightarrow{k}$. In effect, β and α parametrize the unit-norm 3-dimensional vector $\overrightarrow{k}$. Then $R_C^A = R_{z,\alpha} R_{y,\beta}$. We want to find the rotation matrix $R_{k,\theta}$ in frame A corresponding to a rotation about $\overrightarrow{k}$, given that it represents a rotation about z_C in frame C by θ.

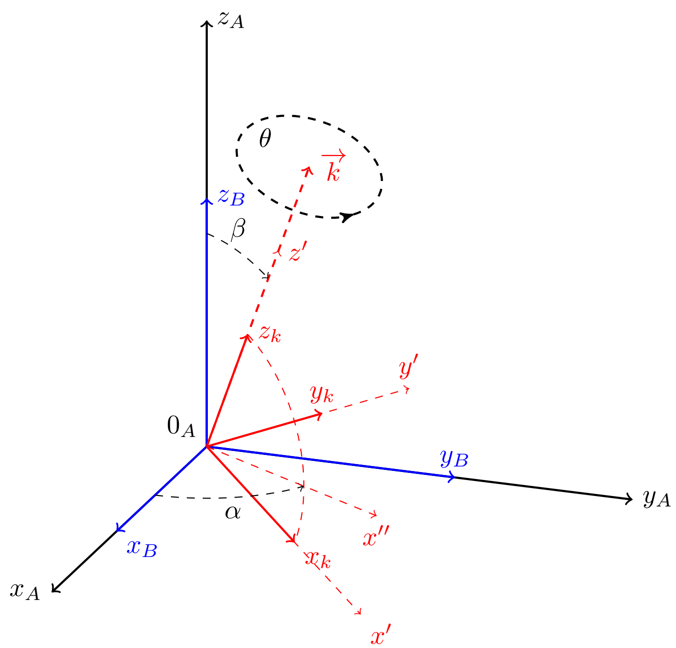
$$
\begin{aligned}
R_C^A &= R_{z,\alpha} R_{y,\beta}. \\
R_{k,\theta} &= R_C^A R_{z,\theta} \left(R_C^A\right)^{-1} && \left(\text{using } T^A = R_B^A \; T^B \; \left(R_B^A\right)^{-1}\right) \\
R_{k,\theta} &= R_{z,\alpha} R_{y,\beta} R_{z,\theta} R_{y,-\beta} R_{z,-\alpha}.
\end{aligned}
$$



(a) Axis-Angle Rotation: Before



(b) Axis-Angle Rotation: After



(c) Axis-Angle Rotation: β and α. Rotate a copy of the world frame about world y by β, then rotate the result about world x by α. The final result is the $x' - y' - k$ frame. The vector x'' lies in both the world $x - y$ plane and the plane formed by vectors k and x'.

Figure 1: Axis/Angle Rotations

4.5 A Concept Chart

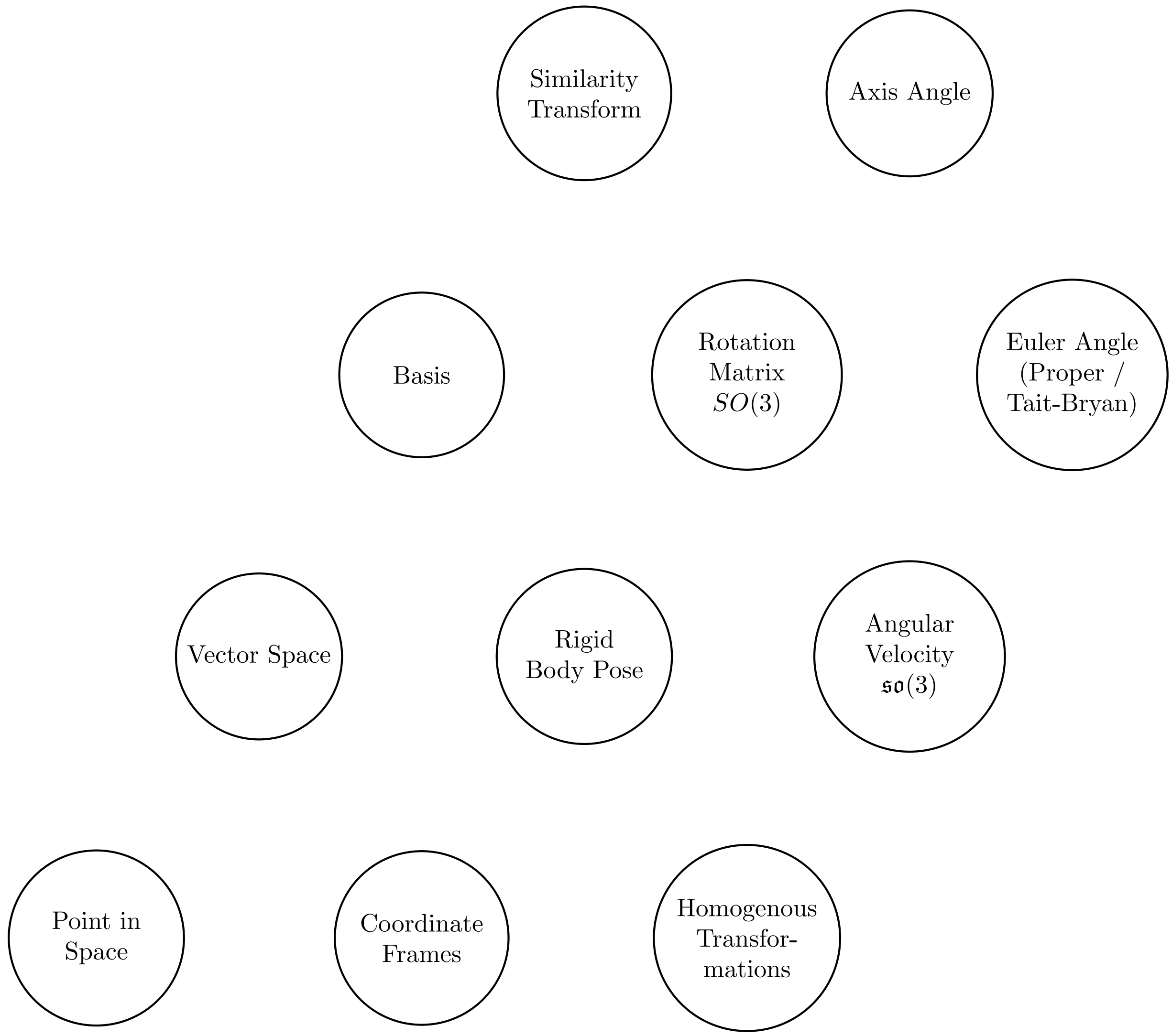


Figure 2: Connect The Dots