Linear State Space Systems

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Linear State-Variable Equations

Suppose that we are given the state-variable equations

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

where $x(t)$: state; $y(t)$: output, $u(t)$: input, and t: time.

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Suppose further that we know the initial condition (IC) $x(t_0)$, and input $u(t)$ for $t \in [t_0, t_f]$. We want to understand how the output $y(t)$ will behave over the time interval $[t_0, t_f]$.

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To do so, we may either

- Explicitly solve for $x(t)$, because $y(t) = Cx(t) + Du(t)$
- Use A, B, C, and D to predict the behavior of solutions $x(t)$ given ICs and input.

Explicit Solution

The explicit solution for $y(t)$ is

$$
\begin{aligned}
y(t) &= Ce^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t) \\
&= \text{free response} + \text{forced response}
\end{aligned}
$$

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The term e^{At} is a matrix derived from A and t using the matrix exponential:

$$e^{At} = I + \frac{1}{1}At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \frac{1}{4!}A^4t^4 + \cdots \tag{1}$$

Note: I is the matrix identity

Matrix Exponential

$$e^{At} = I + \frac{1}{1}At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \frac{1}{4!}A^4t^4 + \cdots \tag{2}$$

Properties:

- $e^{A(t_1+t_2)} = e^{At_1}e^{At_2}$
- $\frac{d}{dt}e^{At} = Ae^{At}$

Consequence:

The solution to the differential equation $\dot{x}(t) = Ax(t)$ with initial condition $x(t_0)$ is

$$x(t) = e^{A(t-t_0)}x(t_0).$$

Proof

Suppose we define $x(t) = e^{At}v$.
Then,

$$\begin{aligned}\dot{x}(t) &= \frac{d}{dt}\left(e^{At}v\right) = \frac{d}{dt}\left(e^{At}\right)v && (v \text{ is a constant}) \\ &= \left(Ae^{At}\right)v = A\left(e^{At}v\right) && (\text{Property of } e^{At}) \\ &= Ax(t)\end{aligned}$$

At the initial time t_0, $x(t_0) = e^{At_0}v$, so that $v = e^{-At_0}x(t_0)$, and

$$x(t) = e^{At}e^{-At_0}x(t_0) = e^{A(t-t_0)}x(t_0).$$

Note: free response of output is $Ce^{A(t-t_0)}x(t_0)$

Example

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}, B = 0, D = 0 \tag{3}$$

Let

$$x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Find the free response.

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Find the free response.

Solution:

Full Explicit Solution

Using similar algebra, we will find that solution $y(t)$ system

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

with initial condition (IC) $x(t_0)$, and input $u(t)$ for $t \in [t_0, t_f]$ is

$$y(t) = Ce^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

Laplace Transform

$$\dot{x}(t) = Ax(t) + Bu(t); \qquad y(t) = Cx(t) + Du(t)$$

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Laplace Transform of first equation:

$$s\hat{x}(s) - x(t_0) = A\hat{x} + B\hat{u}(s)$$

$$\implies (sI - A)\hat{x}(s) = x(t_0) + B\hat{u}(s)$$

$$\hat{x}(s) = (sI - A)^{-1}x(t_0) + (sI - A)^{-1}B\hat{u}(s)$$

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Laplace Transform of first equation:

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Laplace Transform of second equation:

$$\begin{aligned} \hat{y}(s) &= C\hat{x}(s) + D\hat{u}(s) \\ \implies \hat{y}(s) &= C(sI - A)^{-1}x(t_0) + C(sI - A)^{-1}B\hat{u}(s) + D\hat{u}(s) \end{aligned}$$

To find the Laplace transform, set $x(t_0) = 0$ to obtain

$$\begin{aligned} \hat{y}(s) &= C(sI - A)^{-1}B\hat{u}(s) + D\hat{u}(s) \\ \implies \hat{y}(s) &= \left(C(sI - A)^{-1}B + D\right)\hat{u}(s) = G(s)\hat{u}(s) \\ \implies G(s) &= C(sI - A)^{-1}B + D \end{aligned}$$

Transfer Function

$$G(s) = C(sI - A)^{-1}B + D$$

$G(s)$ is a matrix function of the complex variable s.

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$G(s)$ is a matrix function of the complex variable s.

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The number of columns equals the number of inputs.

The $(i, j)^{\mathrm{th}}$ element of $G(s)$ is the transfer function from the j^{th} input to the i^{th} output.

$G(s)$ From EOM in 2 Ways

Input-Output Differential Equations:

$$a_n y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_2 \ddot{y}(t) + a_1 \dot{y}(t) + a_0 y(t)$$
$$= b_m u^{(m)}(t) + b_{m-1} u^{(m-1)}(t) + \cdots + b_2 \ddot{u}(t) + b_1 \dot{u}(t) + b_0 u(t).$$

$$\text{Define} \quad \alpha(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_2 s^2 + a_1 s + a_0$$
$$\beta(s) = b_m s^m + b_{m-1} s^{m-1} + \cdots + b_2 s^2 + b_1 s + b_0.$$

Then,

$$G(s) = \frac{\beta(s)}{\alpha(s)}$$

Linear State-Variable Equations:

$$\dot{x}(t) = Ax(t) + Bu(t) \quad ; \quad y(t) = Cx(t) + Du(t)$$
$$\Longrightarrow \; G(s) = C(sI - A)^{-1}B + D$$

Poles and Eigenvalues

$$G(s) = \frac{\beta(s)}{\alpha(s)}$$

$$G(s) = C(sI - A)^{-1}B + D$$

How are they related?

$$\alpha(s) = \det(sI - A)$$

The poles of $G(s)$ = roots of $\alpha(s)$ = eigenvalues of A.

$\chi_A(s) = \det(sI - A)$ is known as the characteristic polynomial of A.

det is the determinant of a matrix.

Matrix Computations

Given a linear state-variable equation with matrices A, B, C, D, the matrix (function) $sI - A$ is important.

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Given a linear state-variable equation with matrices A, B, C, D, the matrix (function) $sI - A$ is important.

We perform two operations on this matrix:

- Determinant $\det(sI - A)$
- Matrix Inverse $(sI - A)^{-1}$

Matrix Computations

Determinant

The determinant of a $n \times n$ square matrix M is given by

$$\det M = \sum_{\sigma \in S_n} (-1)^{N_\sigma} \Pi_i^n M_{i,\sigma(i)},$$

where S_n be the set permutations of $(1, 2, 3, \ldots, n)$, N_σ is the number of pairwise exchanges of elements of σ required to convert σ into $(1, 2, \ldots, n)$.

Inverse

The inverse of a $n \times n$ square matrix M, denoted M^{-1}, is a matrix whose $(i, j)^{\text{th}}$ element $M_{i,j}^{-1}$ is given by

$$M_{i,j}^{-1} = (-1)^{(i+j)} \frac{\det M_{[i,j]}}{\det M},$$

where $M_{[i,j]}$ is an $(n-1) \times (n-1)$ matrix obtained by deleting the i^{th} **column** and j^{th} **row** of M.

2×2 Matrix

Example

Let

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

.

Calculate M^{-1}.

Solution:

$$M^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

2×2 Matrix

$$\det M = ad - bc.$$

By deleting the i^{th} **column** and j^{th} **row** of M, we get

$$M_{[1,1]} = d \qquad M_{[1,2]} = b$$
$$M_{[2,1]} = c \qquad M_{[2,2]} = a$$

The $(i,j)^{\text{th}}$ entry of M^{-1} is then

$$M^{-1}_{1,1} = (-1)^{(1+1)} \frac{\det M_{[1,1]}}{\det M} = \frac{d}{ad - bc}$$
$$M^{-1}_{1,2} = (-1)^{(1+2)} \frac{\det M_{[1,2]}}{\det M} = \frac{-b}{ad - bc}$$
$$M^{-1}_{2,1} = (-1)^{(2+1)} \frac{\det M_{[2,1]}}{\det M} = \frac{-c}{ad - bc}$$
$$M^{-1}_{2,2} = (-1)^{(2+2)} \frac{\det M_{[2,2]}}{\det M} = \frac{a}{ad - bc}$$

2×2 Matrix

We just derived:

$$M^{-1}_{1,1} = \frac{d}{ad - bc} \qquad M^{-1}_{1,2} = \frac{-b}{ad - bc}$$
$$M^{-1}_{2,1} = \frac{-c}{ad - bc} \qquad M^{-1}_{2,2} = \frac{a}{ad - bc}$$

Therefore,

$$M^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Example: Revisited

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}, B = 0, D = 0 \tag{4}$$

Let

$$x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Previously: Found $y_{free}(t) = e^{-t} - e^{-2t}$ through matrix exponential formula.
Now: Find $y_{free}(t)$ using Laplace transforms.

Example: Revisited

Let's first construct $M = (sI - A)$:

$$M = (sI - A) = s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} (s+1) & -1 \\ 0 & (s+2) \end{bmatrix}$$

Example: Revisited

Recall: $$P = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \implies P^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Therefore

$$\begin{bmatrix} (s+1) & -1 \\ 0 & (s+2) \end{bmatrix}^{-1} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} (s+2) & 1 \\ 0 & (s+1) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{s+1} & \frac{1}{(s+1)(s+2)} \\ 0 & \frac{1}{s+2} \end{bmatrix}$$

Example: Revisited

$$\begin{aligned}
\hat{y}_{free}(s) &= C(sI - A)^{-1}x(t_0) \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{s+1} & \frac{1}{(s+1)(s+2)} \\ 0 & \frac{1}{s+2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \frac{1}{(s+1)(s+2)} \\
&= \frac{1}{s+1} - \frac{1}{s+2} \\
\implies y_{free}(t) &= L^{-1}\left\{\frac{1}{s+1}\right\} - L^{-1}\left\{\frac{1}{s+2}\right\} \\
&= e^{-t} - e^{-2t}
\end{aligned}$$

Example

Consider the input-output differential equation

$$\ddot{y}(t) + 11\dot{y}(t) + 30y(t) = u(t).$$

Let all initial conditions be zero.

1. Convert this IO DE into a state-variable equation
2. Use this state-variable equation to compute $G(s)$
3. Find the eigenvalues of A and the poles of $G(s)$

Example: State-Variable Equation

$$\ddot{y}(t) + 11\dot{y}(t) + 30y(t) = u(t).$$

Target: $\dot{x} = Ax + Bu$, $y = Cx + Du$.
Define $x_1(t) = y(t)$, $x_2(t) = \dot{x}_1(t) = \dot{y}(t)$.

Example: State-Variable Equation

$$\ddot{y}(t) + 11\dot{y}(t) + 30y(t) = u(t).$$

Target: $\dot{x} = Ax + Bu$, $y = Cx + Du$.
Define $x_1(t) = y(t)$, $x_2(t) = \dot{x}_1(t) = \dot{y}(t)$.
This step already gives us the output equation: $y = x_1$

To get the state derivative:

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} x_2 \\ u - 11\dot{y} - 30y \end{bmatrix} = \begin{bmatrix} x_2 \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} x_2 \\ u - 11x_2 - 30x_1 \end{bmatrix}$$

So,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -30x_1 - 11x_2 + u \\ y &= x_1 \end{aligned}$$

Example: Linear State-Variable Equation

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -30x_1 - 11x_2 + u \\ y &= x_1 \end{aligned}$$

$$\begin{aligned} \implies \dot{x}_1 &= (0)x_1 + (1)x_2 + (0)u \\ \dot{x}_2 &= (-30)x_1 + (-11)x_2 + (1)u \\ y &= (1)x_1 + (0)x_2 + (0)u \end{aligned}$$

$$\implies \dot{x} = \begin{bmatrix} 0 & 1 \\ -30 & -11 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{5}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \end{bmatrix} u \tag{6}$$

Example: Transfer Function From SV

We derived:

$$A = \begin{bmatrix} 0 & 1 \\ -30 & -11 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}, D = 0$$

$$\begin{aligned} G(s) &= C(sI - A)^{-1}B + D \\ &= C \begin{bmatrix} s & -1 \\ 30 & s+11 \end{bmatrix}^{-1} B + 0 \\ &= C \frac{1}{s(s+11) - (-30)} \begin{bmatrix} s+11 & 1 \\ -30 & s \end{bmatrix} B \end{aligned}$$

Example: Transfer Function From SV

$$
\begin{aligned}
G(s) &= C \frac{1}{s(s+11) - (-30)} \begin{bmatrix} s+11 & 1 \\ -30 & s \end{bmatrix} B \\
&= \frac{1}{s^2 + 11s + 30} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s+11 & 1 \\ -30 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \frac{1}{s^2 + 11s + 30} \begin{bmatrix} 1 \end{bmatrix} = \frac{1}{s^2 + 11s + 30}
\end{aligned}
$$

Example: Poles & Eigenvalues

$$G(s) = \frac{1}{s^2 + 11s + 30}$$

Since $\xi_A(s) = \det(sI - A)^{-1} = s^2 + 11s + 30$, the eigenvalues of A are: ______

The poles of $G(s)$ are ______

Observe:

$$\begin{aligned}
&\ddot{y}(t) + 11\dot{y}(t) + 30y(t) = u(t) \\
\Longrightarrow\ & s^2\hat{y}(s) + 11s\hat{y}(s) + 30\hat{y}(s) = \hat{u}(s) \quad (\text{Initial conditions} = 0) \\
\Longrightarrow\ & (s^2 + 11s + 30)\hat{y}(s) = \hat{u}(s) \\
\Longrightarrow\ & \hat{y}(s) = \frac{1}{s^2 + 11s + 30}\hat{u}(s)
\end{aligned}$$