Euclidean Space & Coordinates

Spring 2020

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(Affine) Space

Question

What's a good model for the space we move in?

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Answer

Affine Space.

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What's a good model for the space we move in?

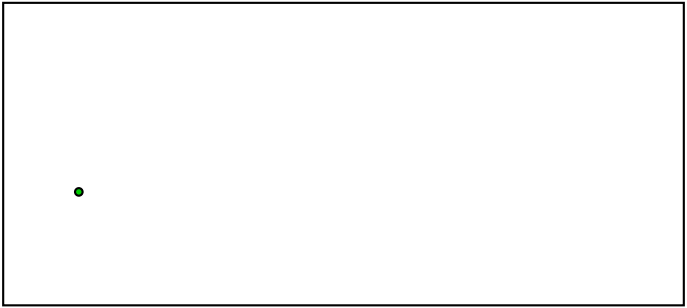
Answer

Affine Space.

Today's discussion covers why.

Ink-Dropping Monks

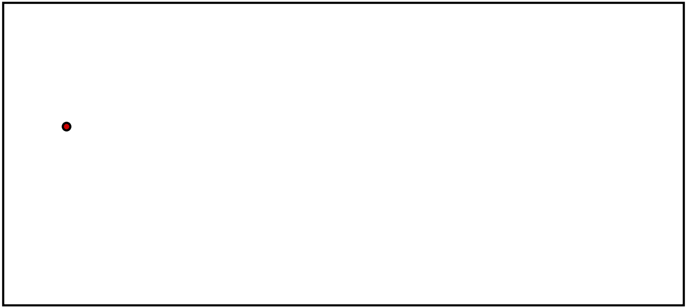
These monks appear, drop some ink on the same place, then disappear:



Each monk generates a transparency with a single point

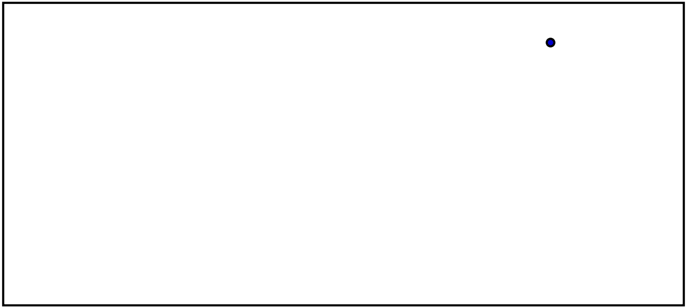
Ink-Dropping Monks

A different monk



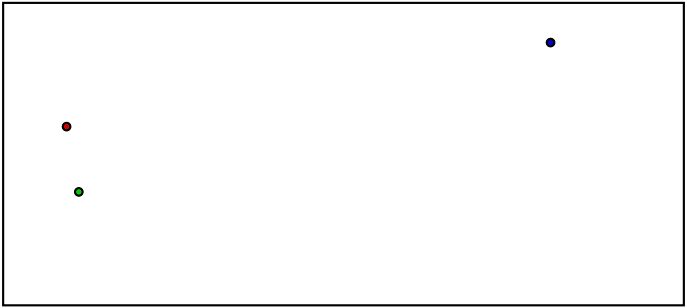
Ink-Dropping Monks

A third monk



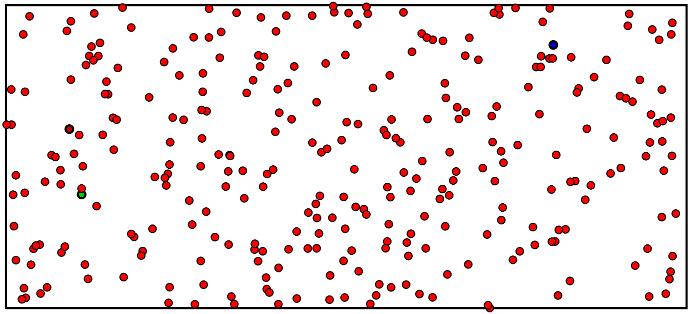
Ink-Dropping Monks

Combine these three transparencies



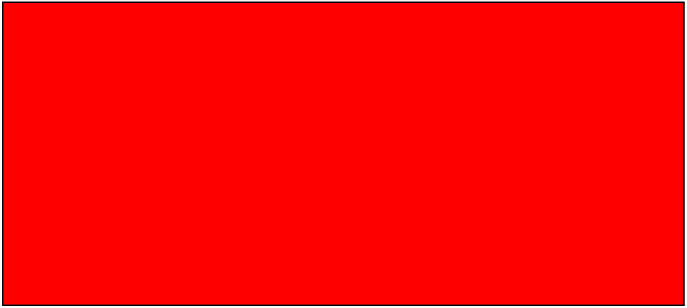
Ink-Dropping Monks

Keep adding points



Ink-Dropping Monks

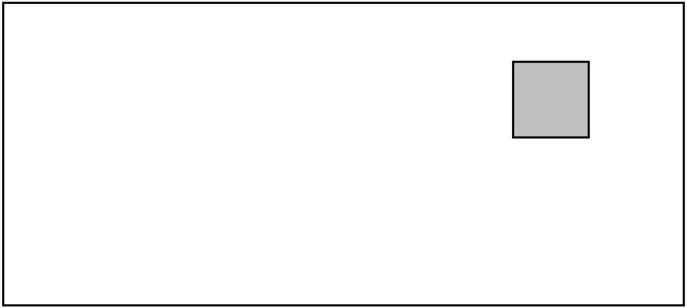
Until we cover the plane



Euclidean space is a collection of points

Describing Space Using Maps

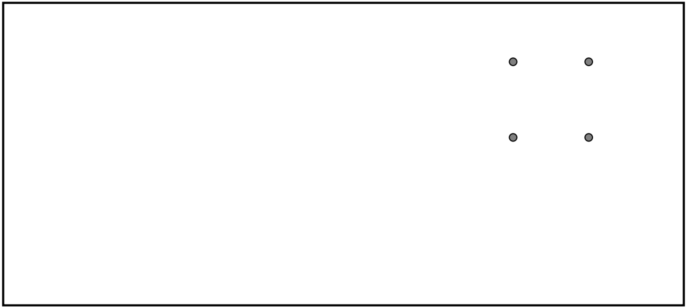
Any map describes space using a subset of these points



This map needs an uncountably infinite set of monks

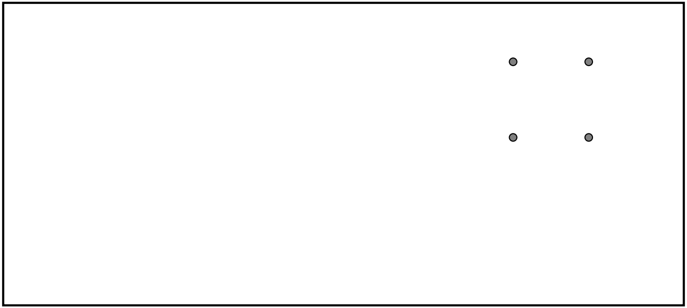
Describing Space Using Maps

Any map describes space using a subset of these points



Here, we need only four monks to generate this map.

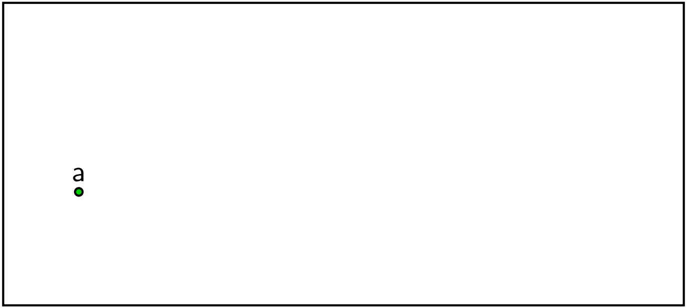
Describing Space Using Maps



To share this map, we need to share these exact four monks, or search among someone else's monks for four monks that produce the same points. This search is slow.

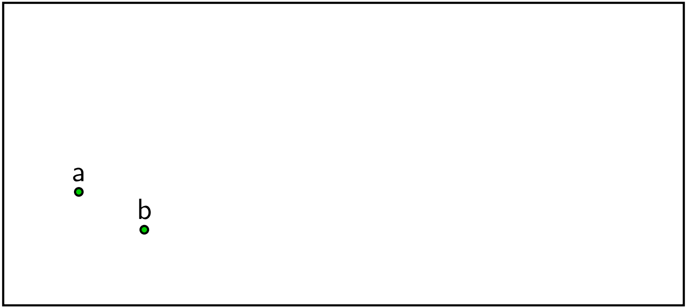
Moving Monks

These monks appear in the same place, drop some ink on the same place, or move their arm, as commanded. Here, they drop once.



Moving Monks

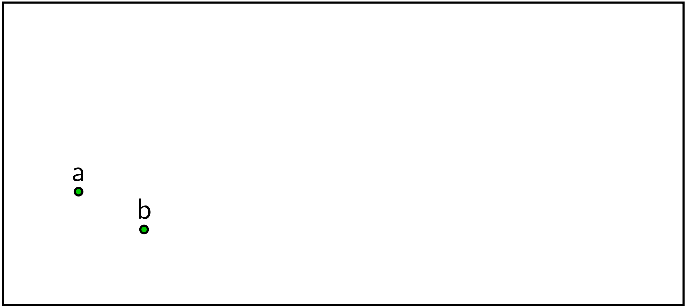
Drop, move, drop



Each 'move' takes a point and generates a new point.

Moving Monks

Drop, move, drop

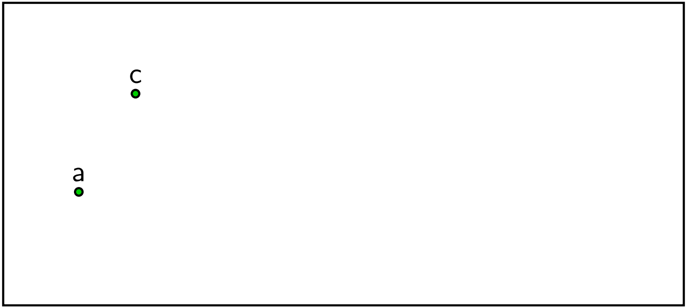


One ‘moving monk’ can generate what two ink-dropping monks generate.

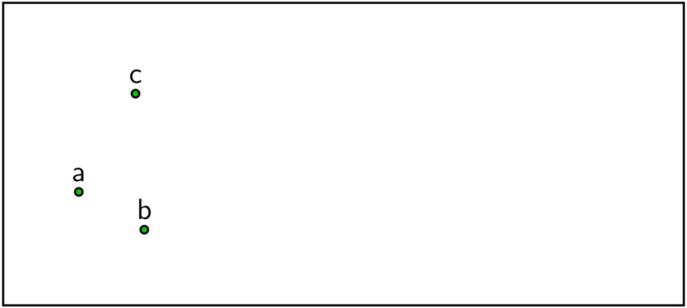
$$\text{Move} = \text{relationship between points}$$

Moving Monks

Drop, different move, drop

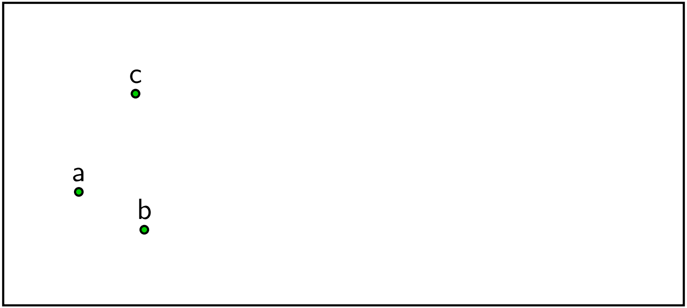


Moving Monks



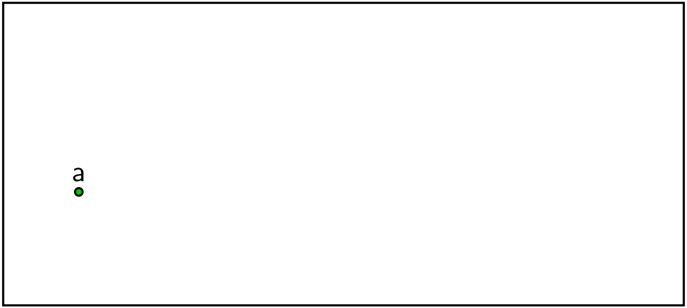
What moves are available? Are they available at all points?
Given two points, is there always a move that relates them?

Moving Monks



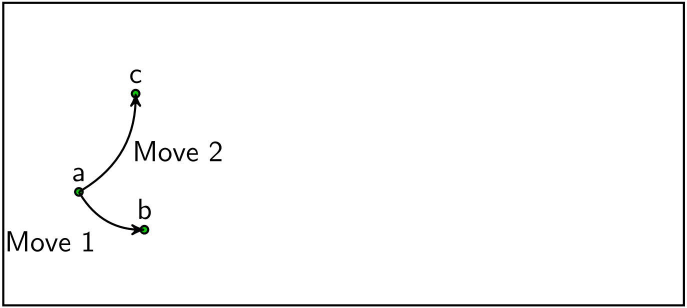
The possible moves form an inner product space

Moves $\neq$ Points



Moves $\neq$ Points

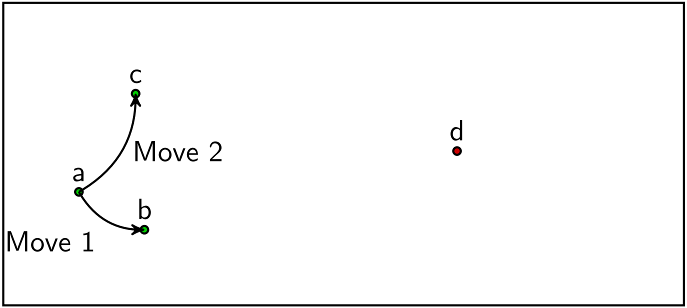
Moves create points.



There's a one-to-one correspondence betweens moves and points, **given a fixed starting point**.

Moves $\neq$ Points

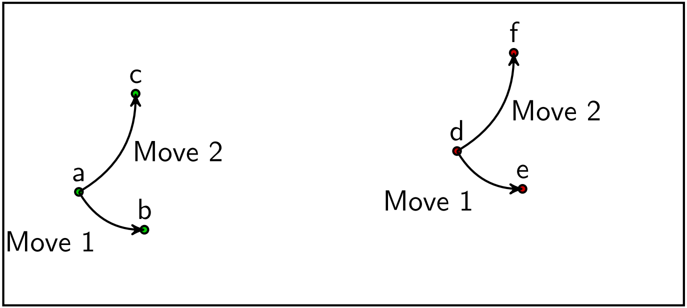
Summon a monk who drops a point at d



(We're changing the starting point.)

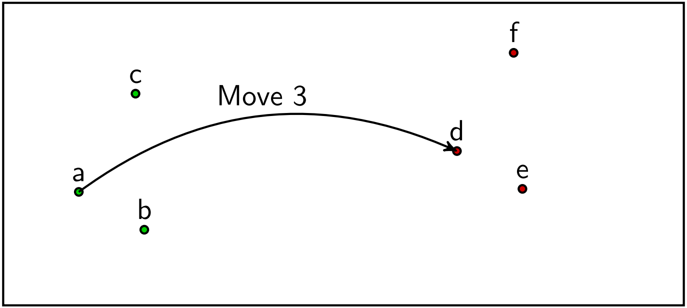
Moves $\neq$ Points

He moves the same way as the previous monk.



The same moves applied to different points produces different points.

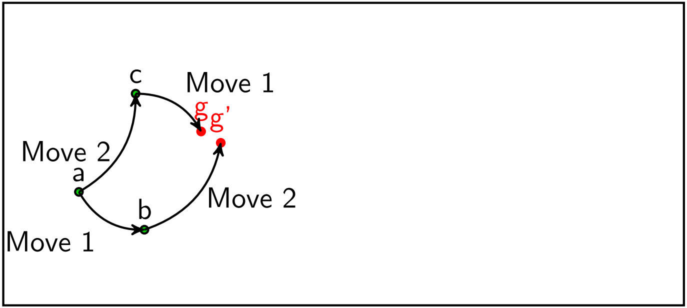
Moves $\neq$ Points



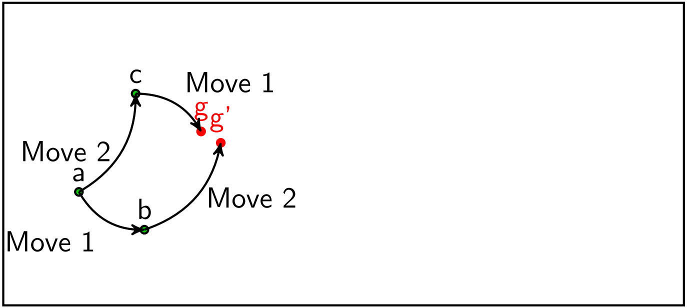
We can always find a move that relates two points.

Moves $\neq$ Points

Does the switched sequence of moves result in the same point?



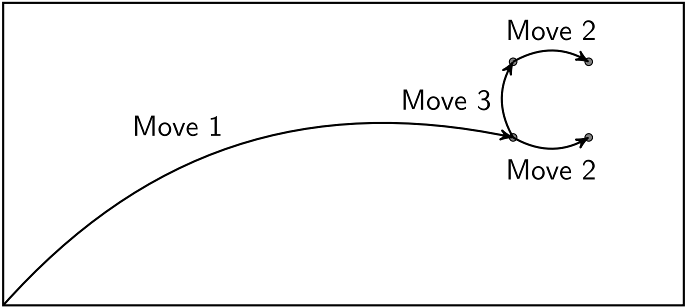
Moves $\neq$ Points



In Affine Spaces, including Euclidean space, $g = g'$.

Describing Space Using Maps

Instead of sharing monks, we share moves



Main Message

Points are qualitative but not quantitative. To work with numbers, we

- fix a point (the origin)
- Use a set of moves to relate other points the origin
- use the inner product and basis elements of the set of moves to compute coordinates

The structure of Euclidean space allows the vector space operations to be consistent with respect to the set of points

Affine Space

Definition (Affine Space)

An affine space is a set A together with a vector space $\overrightarrow{A}$, and a transitive and free action of the additive group of $\overrightarrow{A}$ on the set A.

$$A \times \overrightarrow{A} \to A \tag{1}$$

$$(a, v) \mapsto a \oplus v \tag{2}$$

Free: $a \oplus v = a$ only if $v = 0$. Transitive: for any $a, b \in A$, $\exists v \in \overrightarrow{A}$ such that $b = a \oplus v$.

If $+$ is vector addition, then

$$a \oplus (v + w) = (a \oplus v) \oplus w = (a \oplus w) \oplus v$$

But the following distribution attempt makes no sense:

$$a \oplus (v + w) \neq (a \oplus v) + (a \oplus w)$$

because $(a \oplus v) + (a \oplus w)$ is not really a thing

Euclidean Space

Euclidean Space: An affine space where $\overrightarrow{A}$ is a vector space over the reals.

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The vector space gives us some nice properties:

- Its dimension defines the dimension of Euclidean space
- Its basis allows us to assign coordinates to each point relative to an origin point.

When the vector space is 'over the reals' we get additional benefits:

- We get a natural inner product, norm, and distance
- The coordinates are n ordered real numbers

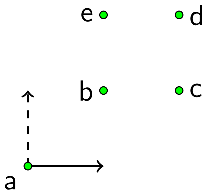
These natural benefits are why Euclidean space is called $\mathbb{R}^n$, hiding all the assumptions underneath.

Computations

Let's do some calculations:

Our basis is the solid and dashed arrows.

Coordinate of e^1 (solid) is :
Coordinate of e^2 (dashed) is :
Coordinate of a is :
Coordinate of b is :
Coordinate of c is :
Coordinate of d is :
Coordinate of e is :

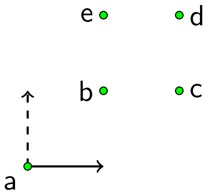


Computations

Let's do some calculations:

Our basis is the solid and dashed arrows.

Coordinate of e^1 (solid) is : (1,0)
Coordinate of e^2 (dashed) is : (0,1)
Coordinate of a is : (0,0)
Coordinate of b is : (1,1)
Coordinate of c is : (2,1)
Coordinate of d is : (2,2)
Coordinate of e is : (1,2)



Computations

Reconstruct the points:

Our basis is now the red solid and dashed arrows.

Coordinate of e^1 (solid) is : (1,0)
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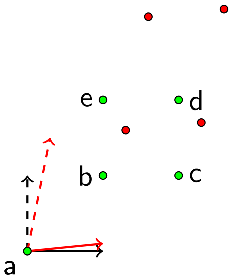

Computations

Reconstruct the points:

Our basis is now the red solid and dashed arrows. Instead, consider the blue arrows

Coordinate of e^1 (solid) is : (1,0)
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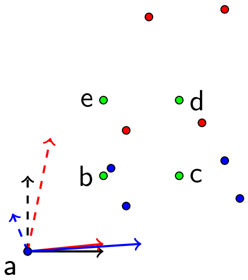
Coordinate of a is : (0,0)
Coordinate of b is : (1,1)
Coordinate of c is : (2,1)
Coordinate of d is : (2,2)
Coordinate of e is : (1,2)



Coordinates

Given n coordinates, you may proceed to build a map .

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Are you sure your origin and basis are the same as those that generated the coordinates you have?

What's the price of getting it wrong?

Calculating Distances

The inner product on the vector space gives us a notion of size.

$$\|v\| = \sqrt{\langle v, v\rangle}$$

The distance between points x and y is $d(x, y) = \|v\|$ where

$$x = y \oplus v \quad \text{,or} \quad y = x \oplus v$$

.

The actual numbers we assign to v depend on the basis we define.

If you and I define unit-norm vectors (or bases) differently, we won't agree on how far things are from one another.

Our descriptions of where things are will be incompatible.

Check Point

We now understand what it means to assign coordinates to points in a Euclidean space.

We can quantitatively describe points in space.

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Interpreting descriptions can go wrong.

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Interpreting descriptions can go wrong.

Managing multiple quantitative descriptions of space is central to problems in robotics.