Introduction to Combinatorics: Assigment 3

University of Toronto, Scarborough

Due on March 18 on gradescope by 5pm

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- **Problems 1–4 are worth 23 points each and Problem 5 is worth 8 points**

Problem 1. Provide a **combinatorial** proof of the following identities:

$$\textit{Identity A}: \binom{10}{5} = \binom{8}{3} + \binom{8}{4} + \binom{8}{4} + \binom{8}{5},$$

and

$$\textit{Identity B}: k \cdot \binom{n}{k} = n \cdot \binom{n-1}{k-1}.$$

Proof of identity A (please use the space below only).

Consider a ~~team~~ group of 10 people, where ~~case~~ we are trying to select half (5) of them for a team.

Now take 2 people in the ~~gr~~ group p_1 & p_2

4 cases;

both p_1 & p_2 on team, then out of the 8 other people, 3 more need to join $\Rightarrow \binom{8}{3}$ choices

2X Since either p_1 OR p_2 — only p_1 or p_2 on team; " " " 4 more need to join $\Rightarrow \binom{8}{4}$ choices

both p_1 & p_2 NOT on team " " ", 5 more need to join $\Rightarrow \binom{8}{5}$ choices.

Putting this together with the additive principle (since these cases are mutually exclusive) we get choosing 5 out of 10 $\binom{10}{5} = \binom{8}{3} + 2\binom{8}{4} + \binom{8}{5}$

Proof of identity B (please use the space below only).

As in lecture, LHS is counting how many teams of size k there are $\binom{n}{k}$, where a leader is assigned $(\cdot k)$. If we look ~~at~~ at the RHS, breaking down the multiplication, we have that it is selecting an single person out of n people $(n \cdot)$, then selecting k-1 other people out of the remaining n-1 people in other words, it selects the leader first, then selects the rest of the team, so ~~they~~ ~~both~~ ~~count~~ when combined with the multiplicative principle (since finding them simultaneously) we get both sides counting the same thing (team with leader) hence they are the same,

Problem 2. I. Compute the number of all solutions $(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ of the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 4$$

in the case where

A. $x_i \in \{0, 1\}, i = 1, 2, 3, 4, 5, 6, 7.$

B. $x_i \in \{0, 1, 2, 3, 4\}, i = 1, 2, 3, 4, 5, 6, 7$

II. Compute the number of all solutions $(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ of the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 18$$

in the case where $x_i \in \{2, 3\}, i = 1, 2, 3, 4, 5, 6, 7.$

Solution (please use the space below only).

A. To get a sum of 4, 4 elements need to be $\underline{1}$ & the others need to be 0
$\Rightarrow$ select 4 x_is $\binom{7}{4}$ solutions

B. We can view this as selecting 4 of x_i (with repetition) to assign a "1" to,
~~ea~~ e.g. selecting x_1 twice gives it value 2.
Using the formula for combinations with repetition, we
get $\binom{4+7-1}{4}$ possible solutions, or $\binom{10}{4}$ solutions.

II. We ~~simply~~ can write $18 = 2(x) + 3(7-x)$ as a formula for 18,
where ~~8~~ x is the number of 2s, & $7-x$ # of 3s.
testing $0 \leq x \leq 7$, we get $x = 3$ as the only solution, so
~~x~~ ~~mean~~ there must be 3 twos & 4 threes, if we
just select the 3s, we get $\binom{7}{4}$ possibilities
and these cases simply set 2 to every other variable.

Problem 3. A path on the planar grid $\{(k,l): \ k,l \in \mathbb{Z}\}$ is called **admissible and connected** if it consists of *consecutive* edges joining points of the form

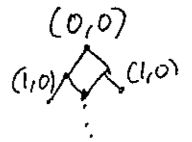
$$(x,y) \to (x+1,y)$$

and

$$(x,y) \to (x,y+1).$$

Compute the number of all possible admissible connected paths from $(0,0)$ to (n,m).

Solution (please use the space below only).



If we consider (0,0) to be the top of the grid, and ~~going~~ going (x+1, y) being left, (x, y+1) being right, ~~the~~ then (n,m) is simply going n times left, down the path, & m times right down the path. Since you make only n+m left/right descisions in total.

This means, we simply need to count how many ~~left/right choices~~ ways we can arrange the choices in n+m total choices. So for n lefts out of n+m, it is

$$\binom{n+m}{n} \; //.$$

Problem 4. Prove (either algebraically or combinatorially) that

$$\sum_{k=0}^{n} k^2 \cdot \binom{n}{k} = n \cdot (n+1) \cdot 2^{n-2}.$$

You may use any identity proved in the notes without proving it here.

Solution (please use the space below only).

Expanding the RHS we get $\underbrace{n^2 2^{n-2}}_{1} + \underbrace{n 2^{n-2}}_{2}$.

First, we can see the LHS represents all possible ways to select a team of any size $\binom{n}{k}$, and select 2 roles on the team, possibly going to the same person (k^2).

For the RHS, suppose the group of (people) elements is $X = \{x_1, x_2, \dots, x_n\}$, then for (1), we can see that we select two possible people x_i, x_j, that will have the two roles and then see if the first $n-2$ elements of $X \setminus \{x_i, x_j\}$ are in our team, (2^{n-2}, two possibilities for all $n-2$ people). If $x_i \neq x_j$ ($i \neq j$) then there are only $n-2$ people left in $X \setminus \{x_i, x_j\}$ and we are done. Otherwise if $x_i = x_j$, then $X \setminus \{x_i, x_j\}$ has $n-1$ elements, so we left out the last element, say x_n. So $n^2 2^{n-2}$ calculated all possibilities where x_n not in the team, so we need to add how many times a team can be made with x_n in it. Fix $x_n \Rightarrow n$ choice for giving both roles & 2^{n-2} (-2 because x_n, and gave one x_i both roles)

So it is (2) $n2^{n-2}$ possibilities. Since each case is defined mutually exclusively, we get that possibilities for selecting 2 roles on any sized team is also $\underline{n^2 2^{n-2} + n2^{n-2}} = \underline{(n+1)n \cdot 2^{n-2}}$

Problem 5. Prove **combinatorially** that

$$\binom{10}{5} = \binom{4}{4} + \binom{5}{4} + \binom{6}{4} + \binom{7}{4} + \binom{8}{4} + \binom{9}{4}.$$

Solution (please use the space below only).

Suppose we are choosing ~~10~~ 5 people from a group of 10

$p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}$.

then immediately, we count the possibilities with $\binom{10}{5}$.

However, it can also be broken into ~~fo~~ the following cases given i. say that p_i in ~~team~~ team, but for p_j where $j > i$ p_j not in the team. I.e. break into cases based on who is the "last" person on the team. Evidently for ~~1<~~ $i \le 10$ this covers all cases.

then given p_i, the last person, we need to find the ~~other~~ $(5-1)$ other people on the team, since ~~the~~ these cases are defined mutually exclusively, we can sum the cases of selecting these 4 people from the $i-1$ other people left.

$\Rightarrow = \sum_{i=1}^{10} \binom{i-1}{4}$, but there is no way to ~~s~~ select 4 people if $i-1 < 4$

$\Rightarrow \sum_{i=5}^{10} \binom{i-1}{4} = \binom{4}{4} + \binom{5}{4} + \binom{6}{4} + \binom{7}{4} + \binom{8}{4} + \binom{9}{4}$

which is equivalent to RHS.