

1. (a) Let $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$; $\mathbf{F}, \mathbf{G} : \mathbb{R}^n \rightarrow \mathbb{R}^n$; and define Δ , the *Laplacian*, by $\Delta f = \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2}$.

Verify the following identities

(i) $\operatorname{div}(\mathbf{F} + \mathbf{G}) = \operatorname{div} \mathbf{F} + \operatorname{div} \mathbf{G}$.

$$\operatorname{div}(\mathbf{F} + \mathbf{G}) = \sum_{i=1}^n \frac{\partial(F_i + G_i)}{\partial x_i} = \sum_{i=1}^n \frac{\partial F_i}{\partial x_i} + \frac{\partial G_i}{\partial x_i} = \sum_{i=1}^n \frac{\partial F_i}{\partial x_i} + \sum_{i=1}^n \frac{\partial G_i}{\partial x_i} = \operatorname{div} \mathbf{F} + \operatorname{div} \mathbf{G}$$

(ii) $\operatorname{div}(f\mathbf{F}) = f \operatorname{div} \mathbf{F} + \mathbf{F} \cdot \operatorname{grad} f$.

$$\operatorname{div}(f\mathbf{F}) = \sum_{i=1}^n \frac{\partial f F_i}{\partial x_i} = \sum_{i=1}^n \frac{\partial f}{\partial x_i} F_i + \frac{\partial F_i}{\partial x_i} f = \sum_{i=1}^n \frac{\partial f}{\partial x_i} F_i + f \sum_{i=1}^n \frac{\partial F_i}{\partial x_i} = \nabla f \cdot \mathbf{F} + f \operatorname{div} \mathbf{F}$$

(iii) $\Delta(fg) = f\Delta g + g\Delta f + 2(\operatorname{grad} f) \cdot (\operatorname{grad} g)$.

$$\begin{aligned} \Delta(fg) &= \sum_{i=1}^n \frac{\partial^2 fg}{\partial x_i^2} \\ &= \sum_{i=1}^n \frac{\partial}{\partial x_i} \left[\frac{\partial f}{\partial x_i} g + \frac{\partial g}{\partial x_i} f \right] \\ &= \sum_{i=1}^n \left[\frac{\partial f}{\partial^2 x_i} g + \left(\frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_i} \right) + \frac{\partial g}{\partial^2 x_i} f + \left(\frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_i} \right) \right] \\ &= g \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2} + 2 \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_i} \right) + f \sum_{i=1}^n \frac{\partial^2 g}{\partial x_i^2} \\ &= g\Delta f + 2[\nabla f \cdot \nabla g] + f\Delta g \end{aligned}$$

- (b) Let $f, g : D \subset \mathbb{R}^3 \rightarrow \mathbb{R}$ be of class C^1 . If R is a solid region contained in D then

$$\iiint_R \nabla f \cdot \nabla g \, dV = \iint_{\partial R} f \nabla g \cdot \mathbf{n} \, dS - \iiint_R f \nabla^2 g \, dV$$

$(\nabla^2 g = \operatorname{div} (\nabla g))$.

2. Use the Divergence Theorem to verify your answer to question 7 on assignment 8.
3. Let $\mathbf{F}(x, y, z) = (x, y^2, e^{yz})$ and let R be a cube centered at the origin with sides of length 2. Evaluate $\int_S \operatorname{div} \mathbf{F} dV$ directly and by using the Divergence Theorem.

Directly:

$$\begin{aligned}
 \int_S \operatorname{div} \mathbf{F} dV &= \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 1 + 2y + ye^{yz} dx dz dy \\
 &= \int_{-1}^1 \int_{-1}^1 2 + 4y + 2ye^{yz} dz dy \\
 &= \int_{-1}^1 4 + 8y + [e^{yz}]_{-1}^1 dy \\
 &= \int_{-1}^1 4 + 8y + e^y - e^{-y} dy \\
 &= 8 + 0 + [e^y]_{-1}^1 + [e^{-y}]_{-1}^1 = 8
 \end{aligned}$$

Divergence:

$$\int_{-1}^1 \int_{-1}^1$$

4. Let B be the pyramid with top vertex $(0,0,1)$ and base vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(1,1,0)$. Let S be the 2-dim closed surface bounding B , oriented in the outward direction. Use Gauss' theorem to calculate $\int_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = (x^2y, 3y^2z, 9z^2x)$.
5. Use the Divergence Theorem to evaluate $\int_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = \left(z^2x, \frac{y^3}{3} + \tan z, x^2z + y^2 \right)$ and S is the top half of the unit sphere $x^2 + y^2 + z^2 = 1$, oriented by the unit normal which points away from the origin.
6. Let the electric field from a point source at the origin be given by $\mathbf{E}(\mathbf{x}) = \frac{\mathbf{x}}{\|\mathbf{x}\|^3}$
- (a) What is the outward flux of \mathbf{E} across the surface $\frac{x^2}{3} + \frac{2y^2}{5} + z^2 = 7$.
- (b) Show that the flux of \mathbf{E} across that part of the sphere $x^2 + y^2 + z^2 = 25$ with $z \geq 3$ is equal to the flux across that part of the plane $z = 3$ with $x^2 + y^2 \leq 16$.
7. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be given by $f(x, y, z) = x^2yz$ and let η be the 2-form on \mathbb{R}^3 given by

$$\eta = (\sin x) dx dy + (e^y + xyz) dx dz + (x^2y^2) dy dz.$$

- (a) Compute df and $d\eta$.
- (b) Evaluate $df \wedge \eta$.