

MATB42: Assignment #6

1. Let  $\omega = \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy$ . Calculate  $\int_{\gamma} \omega$  where

- (a)  $\gamma$  is the boundary of the triangle with vertices (in order) (0,1), (2,3) and (2,1).
- (b)  $\gamma$  is the boundary curve of the region  $\left\{ (x, y) \in \mathbb{R}^2 \left| \frac{(x-2)^2}{9} + \frac{(y+1)^2}{4} \leq 1 \right. \right\}$  oriented in a counter clockwise direction
- (c)  $\gamma$  is the graph of the polar equation  $r = 3 + 2 \sin \theta$  oriented in the clockwise direction.

2. Let  $\omega = (y^2 + z \ln 3) dx + (2xy + \sin z) dy + (y \cos z + (x + 1) \ln 3) dz$ . Determine if  $\omega$  is exact. If it is, use the algorithm given in class to find the potential function  $g$ .

3. Evaluate the double integral  $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 3x^2 y^2 dy \, dx$ , by first finding an equivalent line integral.

4. Let  $R$  be a region in  $\mathbb{R}^2$  and let  $\gamma$  be a counterclockwise parametrization of  $\partial R$ . Let  $\mathbf{F} = (F_1, F - 2)$  be a  $C^1$  vector field defined throughout  $R$  and on  $\partial R$  and let  $\mathbf{n}$  be the outward pointing unit normal vector to  $\gamma$ . Use Green's theorem to give a double integral over  $R$  which is equivalent to  $\int_{\gamma} \mathbf{F} \cdot \mathbf{n} \, ds$ .

5. Give a parametrization for each of the following surfaces, use a computer algebra system to plot the surface and find a unit vector normal to the surface.
- (a) The piece of the cylinder  $y^2 + z^2 = 1$  between  $x = -1$  and  $x = 3$ .
  - (b) The piece of the plane  $z = x + y + 5$  which lies over the unit disk  $x^2 + y^2 \leq 1$ .
  - (c) The piece of the sphere  $x^2 + y^2 + z^2 = 4$  which lies above the plane  $z = 1$ .
  - (d) The piece of the plane  $x + y + z = 1$  which lies above the parallelogram:  $0 \leq y - x \leq 1, 0 \leq y + x \leq 1$ .
6. Let  $S$  be the surface given parameterically by  $\Phi(u, v) = (u^2, 3v, u^2 + v)$  where  $(u, v) \in D$ , the interior of a triangle with vertices  $(0,0)$ ,  $(3,0)$  and  $(3,3)$ .
- (a) Find the surface area of  $S$ .
  - (b) Find the equation of the tangent plane to  $S$  at the point  $(4,9,7)$ .

7. Suppose the surface  $S$  is the graph of a function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ . Give a natural parametrization of  $S$  (in terms of  $f$ ) and derive the formula  $\|\phi_u \times \phi_v\| = \sqrt{1 + \|\text{grad } f\|^2}$

8. A paraboloid of revolution  $S$  is parameterized by  $\Phi(u, v) = u \cos v, u \sin v, u^2, 0 \leq u \leq 2, 0 \leq v \leq 2\pi$ .
- (a) Find an equation in  $x, y$  and  $z$  describing the surface.
  - (b) What are the geometric meanings of the parameters  $u$  and  $v$ ?
  - (c) Find a unit vector orthogonal to the surface of  $\Phi(u, v)$ .
  - (d) Find the equation for the tangent plane at  $\Phi(u_0, v_0) = (1, 1, 2)$  and express your answer in the following two ways:
    - i. parameterized by  $u$  and  $v$ ; and
    - ii. in terms of  $x, y$  and  $z$ .
  - (e) Find the area of  $S$ .
- (cf. page 424, #16)

9. Let a differentiable function  $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  define a parametrized surface.

- (a) Assuming  $\phi_u \times \phi_v \neq 0$ , show that the range of the linear transformation  $D\Phi(u_0, v_0)$  is the plane spanned by  $\phi_u$  and  $\phi_v$ . [Here  $\phi_u$  and  $\phi_v$  are evaluated at  $(u_0, v_0)$ .]
- (b) Show that  $\mathbf{w} \perp (\phi_u \times \phi_v)$  if and only if  $\mathbf{w}$  is in the range of  $D\Phi(u_0, v_0)$ .
- (c) Show that the tangent plane as defined in terms of  $\phi_u \times \phi_v(u_0, v_0)$  is the same as the "parametrized plane"

$$(u, v) \mapsto \Phi(u_0, v_0) + D\Phi(u_0, v_0) \begin{bmatrix} u - u_0 \\ v - v_0 \end{bmatrix}$$

(cf. page 383 #20)