MATB42: Assignment #10

- 1. Let \mathbf{F} be a vector field on \mathbb{R}^3 given by $\mathbf{F} = (F_1, F_2, F_3)$ where F_1, F_2 , and F_3 are C^1 -functions from $\mathbb{R}^3 \to \mathbb{R}$
 - (a) Let η be the 2-form given by

$$\eta = F_3 dx dy + F_1 dy dz + F_2 dz dx$$

Show that $d\eta = (\operatorname{div} \mathbf{F}) dx dy dz$ (page 489, #6)

$$\begin{split} \eta &= F_3 \, dx \, dy + F_1 \, dy \, dz + F_2 \, dz \, dx \\ d\eta &= d(F_3 \, dx \, dy + F_1 \, dy \, dz + F_2 \, dz \, dx) \\ &= (dF_3) \, dx \, dy + (dF_1) \, dy \, dz + (dF_2) \, dz \, dx \\ &= (\frac{\partial}{\partial x} F_3 \, dx + \frac{\partial}{\partial y} F_3 \, dy + \frac{\partial}{\partial z} F_3 \, dz) \, dx \, dy + (dF_1) \, dy \, dz + (dF_2) \, dz \, dx \\ &= \frac{\partial}{\partial z} F_3 \, dz \, dx \, dy + (dF_1) \, dy \, dz + (dF_2) \, dz \, dx \\ &= \frac{\partial}{\partial z} F_3 \, dx \, dy \, dz + (\frac{\partial}{\partial x} F_1 \, dx + \frac{\partial}{\partial y} F_1 \, dy + \frac{\partial}{\partial z} F_1 \, dz) \, dy \, dz + (dF_2) \, dz \, dx \\ &= \frac{\partial}{\partial z} F_3 \, dx \, dy \, dz + \frac{\partial}{\partial x} F_1 \, dx \, dy \, dz + (dF_2) \, dz \, dx \\ &= \frac{\partial}{\partial z} F_3 \, dx \, dy \, dz + \frac{\partial}{\partial x} F_1 \, dx \, dy \, dz + (\partial^2_{z} F_2 \, dx + \frac{\partial}{\partial y} F_2 \, dy + \frac{\partial}{\partial z} F_2 \, dz) \, dz \, dx \\ &= \frac{\partial}{\partial z} F_3 \, dx \, dy \, dz + \frac{\partial}{\partial x} F_1 \, dx \, dy \, dz + (\frac{\partial}{\partial y} F_2 \, dy \, dz \, dx \\ &= \frac{\partial}{\partial z} F_3 \, dx \, dy \, dz + \frac{\partial}{\partial x} F_1 \, dx \, dy \, dz + \frac{\partial}{\partial y} F_2 \, dx \, dy \, dz \\ &= \frac{\partial}{\partial z} F_3 \, dx \, dy \, dz + \frac{\partial}{\partial x} F_1 \, dx \, dy \, dz + \frac{\partial}{\partial y} F_2 \, dx \, dy \, dz \\ &= \frac{\partial}{\partial z} F_3 \, dx \, dy \, dz + \frac{\partial}{\partial x} F_1 \, dx \, dy \, dz + \frac{\partial}{\partial y} F_2 \, dx \, dy \, dz \\ &= \frac{\partial}{\partial z} F_3 \, dx \, dy \, dz + \frac{\partial}{\partial x} F_1 \, dx \, dy \, dz + \frac{\partial}{\partial y} F_2 \, dx \, dy \, dz \\ &= \frac{\partial}{\partial z} F_3 \, dx \, dy \, dz + \frac{\partial}{\partial x} F_1 \, dx \, dy \, dz + \frac{\partial}{\partial y} F_2 \, dx \, dy \, dz \\ &= \frac{\partial}{\partial z} F_3 \, dx \, dy \, dz + \frac{\partial}{\partial x} F_1 \, dx \, dy \, dz + \frac{\partial}{\partial y} F_2 \, dx \, dy \, dz \\ &= \frac{\partial}{\partial z} F_3 \, dx \, dy \, dz + \frac{\partial}{\partial x} F_1 \, dx \, dy \, dz + \frac{\partial}{\partial y} F_2 \, dx \, dy \, dz \\ &= \frac{\partial}{\partial x} F_3 \, dx \, dy \, dz + \frac{\partial}{\partial x} F_3 \, dx \, dy \, dz + \frac{\partial}{\partial x} F_3 \, dx \, dy \, dz + \frac{\partial}{\partial x} F_3 \, dx \, dy \, dz + \frac{\partial}{\partial x} F_3 \, dx \, dy \, dz + \frac{\partial}{\partial x} F_3 \, dx \, dy \, dz + \frac{\partial}{\partial x} F_3 \, dx \, dy \, dz + \frac{\partial}{\partial x} F_3 \, dx \, dy \, dz + \frac{\partial}{\partial x} F_3 \, dx \, dy \, dz + \frac{\partial}{\partial x} F_3 \, dx \, dy \, dz + \frac{\partial}{\partial x} F_3 \, dx \, dy \, dz + \frac{\partial}{\partial x} F_3 \, dx \, dy \, dz + \frac{\partial}{\partial x} F_3 \, dx \, dy \, dz + \frac{\partial}{\partial x} F_3 \, dx \, dy \, dz + \frac{\partial}{\partial x} F_3 \, dx \, dy \, dz + \frac{\partial}{\partial x} F_3 \, dx \, dy \, dz + \frac{\partial}{\partial x} F_3 \, dx \, dy \, dz + \frac{\partial}{\partial x} F_3 \, dx \, dy \, dz + \frac{\partial}$$

(b) Show that $dF_1 \wedge dF_2 \wedge dF_3 = (\det D\mathbf{F}) dx dy dz$

$$df = \sum_{i=0}^{n} \frac{\partial f}{\partial x_i} \, dx_i$$

$$\begin{split} dF_1 \wedge dF_2 \wedge dF_3 &= (\frac{\partial F_1}{\partial x} \, dx + \frac{\partial F_1}{\partial y} \, dy + \frac{\partial F_1}{\partial z} \, dz) \wedge (\frac{\partial F_2}{\partial x} \, dx + \frac{\partial F_2}{\partial y} \, dy + \frac{\partial F_2}{\partial z} \, dz) \\ &= (\frac{\partial F_1}{\partial x} \, dx \wedge (\frac{\partial F_2}{\partial x} \, dx + \frac{\partial F_2}{\partial y} \, dy + \frac{\partial F_2}{\partial z} \, dz) \\ &+ \frac{\partial F_1}{\partial y} \, dy \wedge (\frac{\partial F_2}{\partial x} \, dx + \frac{\partial F_2}{\partial y} \, dy + \frac{\partial F_2}{\partial z} \, dz) \\ &+ \frac{\partial F_1}{\partial z} \, dz \wedge (\frac{\partial F_2}{\partial x} \, dx + \frac{\partial F_2}{\partial y} \, dy + \frac{\partial F_2}{\partial z} \, dz) \wedge dF_3 \\ &= ((\frac{\partial F_1}{\partial x} \, \frac{\partial F_2}{\partial y} \, dx \, dy + \frac{\partial F_1}{\partial x} \, \frac{\partial F_2}{\partial z} \, dx \, dz) \\ &+ (\frac{\partial F_1}{\partial y} \, \frac{\partial F_2}{\partial x} \, dx \, dy + \frac{\partial F_1}{\partial y} \, \frac{\partial F_2}{\partial z} \, dy \, dz) \\ &+ (\frac{\partial F_1}{\partial z} \, \frac{\partial F_2}{\partial x} \, dz \, dx + \frac{\partial F_1}{\partial z} \, \frac{\partial F_2}{\partial y} \, dy \, dy) \wedge dF_3 \\ &= ((\frac{\partial F_1}{\partial x} \, \frac{\partial F_2}{\partial y} - \frac{\partial F_1}{\partial y} \, \frac{\partial F_2}{\partial x} \, dx \, dy) \\ &+ (\frac{\partial F_1}{\partial z} \, \frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial z} \, \frac{\partial F_2}{\partial y} \, dx \, dy) \\ &+ (\frac{\partial F_1}{\partial z} \, \frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial z} \, \frac{\partial F_2}{\partial z} \, dz \, dx)) \wedge (\frac{\partial F_3}{\partial x} \, dx + \frac{\partial F_3}{\partial y} \, dy + \frac{\partial F_3}{\partial z} \, dz) \\ &= (\frac{\partial F_3}{\partial z} (\frac{\partial F_1}{\partial x} \, \frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial z} \, \frac{\partial F_2}{\partial z} \, dx \, dy) \, dz) \\ &+ \frac{\partial F_3}{\partial z} (\frac{\partial F_1}{\partial z} \, \frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial z} \, \frac{\partial F_2}{\partial z} \, dx \, dy \, dz) \\ &+ \frac{\partial F_3}{\partial x} (\frac{\partial F_1}{\partial y} \, \frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial z} \, \frac{\partial F_2}{\partial z} \, dx \, dy \, dz) \\ &+ \frac{\partial F_3}{\partial x} (\frac{\partial F_1}{\partial y} \, \frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial z} \, \frac{\partial F_2}{\partial z} \, dx \, dy \, dz) \\ &= \frac{\partial F_3}{\partial x} (\frac{\partial F_1}{\partial y} \, \frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial z} \, \frac{\partial F_2}{\partial z} \, dx \, dy \, dz \\ &+ \frac{\partial F_3}{\partial x} (\frac{\partial F_1}{\partial y} \, \frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial z} \, \frac{\partial F_2}{\partial y} \, dx \, dy \, dz \\ &= \frac{\partial F_3}{\partial x} (\frac{\partial F_1}{\partial y} \, \frac{\partial F_2}{\partial z} \, \frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial z} \, \frac{\partial F_2}{\partial z} \, \frac{\partial F_2}{\partial z} \, dx \, dy \, dz \\ &= \frac{\partial F_3}{\partial x} \left(\frac{\partial F_1}{\partial y} \, \frac{\partial F_2}{\partial z} \, \frac{\partial F_2}{\partial z} \right) \, dx \, dy \, dz \\ &= \frac{\partial F_3}{\partial x} \left(\frac{\partial F_1}{\partial y} \, \frac{\partial F_2}{\partial z} \, \frac{\partial F_2}{\partial z} \right) \, \frac{\partial F_2}{\partial x} \, \frac{\partial F_2}{\partial z} \, dx \, dy \, dz \\ &= \frac{\partial F_3}{\partial x} \left(\frac{\partial F_1}{\partial y} \, \frac{\partial F_2}{\partial z} \right$$

2. Let ω be a k-form and let η be a ℓ -form. Find $d(d\omega \wedge \eta - \omega \wedge d\eta)$.

$$\begin{split} d(d\omega \wedge \eta - \omega \wedge d\eta) &= d(d\omega \wedge \eta) - d(\omega \wedge d\eta) \\ &= (d^2\omega \wedge \eta + (-1)^{k+1}(d\omega \wedge d\eta)) - (d\omega \wedge d\eta + (-1)^k(\omega \wedge d^2\eta)) \\ &= (-1)^{k+1}d\omega \wedge d\eta - d\omega \wedge d\eta \\ &= ((-1)^{k+1} - 1)d\omega \wedge d\eta \end{split}$$

3. Determine if $\eta = y \, dx \, dy + xz \, dy \, dz - yz \, dz \, dx$ is exact. If η is exact find a 1-form ω with $d\omega = \eta$. Check if $d\eta = \mathcal{O}$ to see if η closed.

(compare with page 461, # 22)

$$\begin{split} d\eta &= d(y \, dx \, dy + xz \, dy \, dz - yz \, dz \, dx) \\ &= (dy \, dx \, dy + d(xz) \wedge dy \, dz - d(yz) \wedge dz \, dx) \\ &= ((z \, dx + x \, dz) \wedge dy \, dz - (z \, dy + y \, dz) \wedge dz \, dx) \\ &= (z \, dx) \wedge dy \, dz - (z \, dy) \wedge dz \, dx \\ &= z \, dx \, dy \, dz - z \, dx \, dy \, dz = \mathcal{O} \end{split}$$

Since the polynomials of x, y and z defined throughout \mathbb{R}^3 and η closed, it is exact.