

MATB42: Assignment #6

1. Let $\omega = \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy$. Calculate $\int_{\gamma} \omega$ where

- (a) γ is the boundary of the triangle with vertices (in order) (0,1), (2,3) and (2,1).
- (b) γ is the boundary curve of the region $\left\{ (x, y) \in \mathbb{R}^2 \left| \frac{(x-2)^2}{9} + \frac{(y+1)^2}{4} \leq 1 \right. \right\}$ oriented in a counter clockwise direction
- (c) γ is the graph of the polar equation $r = 3 + 2 \sin \theta$ oriented in the clockwise direction.

2. Let $\omega = (y^2 + z \ln 3) dx + (2xy + \sin z) dy + (y \cos z + (x + 1) \ln 3) dz$. Determine if ω is exact. If it is, use the algorithm given in class to find the potential function g .

3. Evaluate the double integral $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 3x^2 y^2 dy \, dx$, by first finding an equivalent line integral.

4. Let R be a region in \mathbb{R}^2 and let γ be a counterclockwise parametrization of ∂R . Let $\mathbf{F} = (F_1, F - 2)$ be a C^1 vector field defined throughout R and on ∂R and let \mathbf{n} be the outward pointing unit normal vector to γ . Use Green's theorem to give a double integral over R which is equivalent to $\int_{\gamma} \mathbf{F} \cdot \mathbf{n} \, ds$.

5. Give a parametrization for each of the following surfaces, use a computer algebra system to plot the surface and find a unit vector normal to the surface.
- (a) The piece of the cylinder $y^2 + z^2 = 1$ between $x = -1$ and $x = 3$.
 - (b) The piece of the plane $z = x + y + 5$ which lies over the unit disk $x^2 + y^2 \leq 1$.
 - (c) The piece of the sphere $x^2 + y^2 + z^2 = 4$ which lies above the plane $z = 1$.
 - (d) The piece of the plane $x + y + z = 1$ which lies above the parallelogram: $0 \leq y - x \leq 1, 0 \leq y + x \leq 1$.

6. Let S be the surface given parameterically by $\Phi(u, v) = (u^2, 3v, u^2 + v)$ where $(u, v) \in D$, the interior of a triangle with vertices $(0,0)$, $(3,0)$ and $(3,3)$.

(a) Find the surface area of S .

(b) Find the equation of the tangent plane to S at the point $(4,9,7)$.

7. Suppose the surface S is the graph of a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$. Give a natural parametrization of S (in terms of f) and derive the formula $\|\phi_u \times \phi_v\| = \sqrt{1 + \|\text{grad } f\|^2}$

8. A paraboloid of revolution S is parameterized by $\Phi(u, v) = u \cos v, u \sin v, u^2, 0 \leq u \leq 2, 0 \leq v \leq 2\pi$.
- (a) Find an equation in x, y and z describing the surface.
 - (b) What are the geometric meanings of the parameters u and v ?
 - (c) Find a unit vector orthogonal to the surface of $\Phi(u, v)$.
 - (d) Find the equation for the tangent plane at $\Phi(u_0, v_0) = (1, 1, 2)$ and express your answer in the following two ways:
 - i. parameterized by u and v ; and
 - ii. in terms of x, y and z .
 - (e) Find the area of S .
- (cf. page 424, #16)

9. Let a differentiable function $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ define a parametrized surface.

- (a) Assuming $\phi_u \times \phi_v \neq 0$, show that the range of the linear transformation $D\Phi(u_0, v_0)$ is the plane spanned by ϕ_u and ϕ_v . [Here ϕ_u and ϕ_v are evaluated at (u_0, v_0) .]
- (b) Show that $\mathbf{w} \perp (\phi_u \times \phi_v)$ if and only if \mathbf{w} is in the range of $D\Phi(u_0, v_0)$.
- (c) Show that the tangent plane as defined in terms of $\phi_u \times \phi_v(u_0, v_0)$ is the same as the "parametrized plane"

$$(u, v) \mapsto \Phi(u_0, v_0) + D\Phi(u_0, v_0) \begin{bmatrix} u - u_0 \\ v - v_0 \end{bmatrix}$$

(cf. page 383 #20)