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1. Let $\boldsymbol{F}$ be a vector field on $\mathbb{R}^3$ given by $\boldsymbol{F} = (F_1, F_2, F_3)$ where F_1, F_2, and F_3 are C^1-functions from $\mathbb{R}^3 \to \mathbb{R}$

   (a) Let η be the 2-form given by

   $$\eta = F_3\,dx\,dy + F_1\,dy\,dz + F_2\,dz\,dx$$

   Show that $d\eta = (\operatorname{div} \boldsymbol{F})\,dx\,dy\,dz$
   (page 489, #6)

$$\begin{aligned}
\eta &= F_3\,dx\,dy + F_1\,dy\,dz + F_2\,dz\,dx \\
d\eta &= d(F_3\,dx\,dy + F_1\,dy\,dz + F_2\,dz\,dx) \\
&= (dF_3)\,dx\,dy + (dF_1)\,dy\,dz + (dF_2)\,dz\,dx \\
&= (\frac{\partial}{\partial x}F_3\,dx + \frac{\partial}{\partial y}F_3\,dy + \frac{\partial}{\partial z}F_3\,dz)\,dx\,dy + (dF_1)\,dy\,dz + (dF_2)\,dz\,dx \\
&= \frac{\partial}{\partial z}F_3\,dz\,dx\,dy + (dF_1)\,dy\,dz + (dF_2)\,dz\,dx \\
&= \frac{\partial}{\partial z}F_3\,dx\,dy\,dz + (\frac{\partial}{\partial x}F_1\,dx + \frac{\partial}{\partial y}F_1\,dy + \frac{\partial}{\partial z}F_1\,dz)\,dy\,dz + (dF_2)\,dz\,dx \\
&= \frac{\partial}{\partial z}F_3\,dx\,dy\,dz + \frac{\partial}{\partial x}F_1\,dx\,dy\,dz + (dF_2)\,dz\,dx \\
&= \frac{\partial}{\partial z}F_3\,dx\,dy\,dz + \frac{\partial}{\partial x}F_1\,dx\,dy\,dz + (\frac{\partial}{\partial x}F_2\,dx + \frac{\partial}{\partial y}F_2\,dy + \frac{\partial}{\partial z}F_2\,dz)\,dz\,dx \\
&= \frac{\partial}{\partial z}F_3\,dx\,dy\,dz + \frac{\partial}{\partial x}F_1\,dx\,dy\,dz + \frac{\partial}{\partial y}F_2\,dy\,dz\,dx \\
&= \frac{\partial}{\partial z}F_3\,dx\,dy\,dz + \frac{\partial}{\partial x}F_1\,dx\,dy\,dz + \frac{\partial}{\partial y}F_2\,dx\,dy\,dz \\
&= \frac{\partial}{\partial x}F_1 + \frac{\partial}{\partial y}F_2 + \frac{\partial}{\partial z}F_3\,dx\,dy\,dz = (\operatorname{div}\boldsymbol{F})\,dx\,dy\,dz
\end{aligned}$$

(b) Show that $dF_1 \wedge dF_2 \wedge dF_3 = (\det\, D\boldsymbol{F})\, dx\, dy\, dz$

$$df = \sum_{i=0}^{n} \frac{\partial f}{\partial x_i}\, dx_i$$

$$\begin{aligned}
dF_1 \wedge dF_2 \wedge dF_3 &= (\frac{\partial F_1}{\partial x}\, dx + \frac{\partial F_1}{\partial y}\, dy + \frac{\partial F_1}{\partial z}\, dz) \wedge (\frac{\partial F_2}{\partial x}\, dx + \frac{\partial F_2}{\partial y}\, dy + \frac{\partial F_2}{\partial z}\, dz) \wedge dF_3 \\
&= (\frac{\partial F_1}{\partial x}\, dx \wedge (\frac{\partial F_2}{\partial x}\, dx + \frac{\partial F_2}{\partial y}\, dy + \frac{\partial F_2}{\partial z}\, dz) \\
&\quad + \frac{\partial F_1}{\partial y}\, dy \wedge (\frac{\partial F_2}{\partial x}\, dx + \frac{\partial F_2}{\partial y}\, dy + \frac{\partial F_2}{\partial z}\, dz) \\
&\quad + \frac{\partial F_1}{\partial z}\, dz \wedge (\frac{\partial F_2}{\partial x}\, dx + \frac{\partial F_2}{\partial y}\, dy + \frac{\partial F_2}{\partial z}\, dz)) \wedge dF_3 \\
&= ((\frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial y}\, dx\, dy + \frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial z}\, dx\, dz) \\
&\quad + (\frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial x}\, dy\, dx + \frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial z}\, dy\, dz) \\
&\quad + (\frac{\partial F_1}{\partial z}\frac{\partial F_2}{\partial x}\, dz\, dx + \frac{\partial F_1}{\partial z}\frac{\partial F_2}{\partial y}\, dz\, dy)) \wedge dF_3 \\
&= ((\frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial y} - \frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial x}\, dx\, dy) \\
&\quad + (\frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial z}\frac{\partial F_2}{\partial y}\, dy\, dz) \\
&\quad + (\frac{\partial F_1}{\partial z}\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial z}\, dz\, dx)) \wedge (\frac{\partial F_3}{\partial x}\, dx + \frac{\partial F_3}{\partial y}\, dy + \frac{\partial F_3}{\partial z}\, dz) \\
&= (\frac{\partial F_3}{\partial z}(\frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial y} - \frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial x}\, dx\, dy\, dz) \\
&\quad + \frac{\partial F_3}{\partial x}(\frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial z}\frac{\partial F_2}{\partial y}\, dy\, dz\, dx) \\
&\quad + \frac{\partial F_3}{\partial y}(\frac{\partial F_1}{\partial z}\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial z}\, dz\, dx\, dy)) \\
&= \frac{\partial F_3}{\partial x}(\frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial z}\frac{\partial F_2}{\partial y})\, dx\, dy\, dz \\
&\quad - \frac{\partial F_3}{\partial y}(\frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial z}\frac{\partial F_2}{\partial x})\, dx\, dy\, dz \\
&\quad + \frac{\partial F_3}{\partial z}(\frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial y} - \frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial x})\, dx\, dy\, dz \\
&= \frac{\partial F_3}{\partial x}\begin{vmatrix} \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{vmatrix} - \frac{\partial F_3}{\partial y}\begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial z} \end{vmatrix} + \frac{\partial F_3}{\partial z}\begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{vmatrix}\, dx\, dy\, dz \\
&= \begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \\ \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial z} \end{vmatrix}\, dx\, dy\, dz
\end{aligned}$$

2. Let ω be a k-form and let η be a ℓ-form. Find $d(d\omega \wedge \eta - \omega \wedge d\eta)$.

$$\begin{aligned}
d(d\omega \wedge \eta - \omega \wedge d\eta) &= d(d\omega \wedge \eta) - d(\omega \wedge d\eta) \\
&= (d^2\omega \wedge \eta + (-1)^{k+1}(d\omega \wedge d\eta)) - (d\omega \wedge d\eta + (-1)^k(\omega \wedge d^2\eta)) \\
&= (-1)^{k+1} d\omega \wedge d\eta - d\omega \wedge d\eta \\
&= ((-1)^{k+1} - 1) d\omega \wedge d\eta
\end{aligned}$$

3. Determine if $\eta = y\,dx\,dy + xz\,dy\,dz - yz\,dz\,dx$ is exact. If η is exact find a 1-form ω with $d\omega = \eta$. Check if $d\eta = \mathcal{O}$ to see if η closed.

   (compare with page 461, # 22)

$$\begin{aligned}
d\eta &= d(y\,dx\,dy + xz\,dy\,dz - yz\,dz\,dx) \\
&= (dy\,dx\,dy + d(xz) \wedge dy\,dz - d(yz) \wedge dz\,dx) \\
&= ((z\,dx + x\,dz) \wedge dy\,dz - (z\,dy + y\,dz) \wedge dz\,dx) \\
&= (z\,dx) \wedge dy\,dz - (z\,dy) \wedge dz\,dx \\
&= z\,dx\,dy\,dz - z\,dx\,dy\,dz = \mathcal{O}
\end{aligned}$$

Since the polynomials of x, y and z defined throughout $\mathbb{R}^3$ and η closed, it is exact. By inspection,

$$\omega = xy\,dy + xyz\,dz$$

4. Evaluate $\iint_S \omega$, where $\omega = z\,dx\,dy + x\,dy\,dz + y\,dz\,dx$ and S is the unit sphere, directly and by the Divergence Theorem.

(page 489, #12)

Directly:

Parametrize the sphere S as

$$\mathbf{\Phi}(\varphi,\theta) = (\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi) \text{ with } \theta\in[0,2\pi],\ \varphi\in[0,\pi]$$

$$\begin{aligned}
\iint_S \omega &= \iint_{\mathbf{\Phi}} z\,dx\,dy + \iint_{\mathbf{\Phi}} x\,dy\,dz + \iint_{\mathbf{\Phi}} y\,dz\,dx \\
&= \int_0^{2\pi}\int_0^{\pi} \cos\varphi \begin{vmatrix} \frac{\partial\cos\theta\sin\varphi}{\partial\varphi} & \frac{\partial\cos\theta\sin\varphi}{\partial\theta} \\ \frac{\partial\sin\theta\sin\varphi}{\partial\varphi} & \frac{\partial\sin\theta\sin\varphi}{\partial\theta} \end{vmatrix} d\varphi\,d\theta + \int_0^{2\pi}\int_0^{\pi} \cos\theta\sin\varphi \begin{vmatrix} \frac{\partial\sin\theta\sin\varphi}{\partial\varphi} & \frac{\partial\sin\theta\sin\varphi}{\partial\theta} \\ \frac{\partial\cos\varphi}{\partial\varphi} & \frac{\partial\cos\varphi}{\partial\theta} \end{vmatrix} d\varphi\,d\theta \\
&\quad + \int_0^{2\pi}\int_0^{\pi} \sin\theta\sin\varphi \begin{vmatrix} \frac{\partial\cos\varphi}{\partial\varphi} & \frac{\partial\cos\varphi}{\partial\theta} \\ \frac{\partial\cos\theta\sin\varphi}{\partial\varphi} & \frac{\partial\cos\theta\sin\varphi}{\partial\theta} \end{vmatrix} d\varphi\,d\theta \\
&= \int_0^{2\pi}\int_0^{\pi} \cos\varphi \begin{vmatrix} \cos\theta\cos\varphi & -\sin\theta\sin\varphi \\ \sin\theta\cos\varphi & \cos\theta\sin\varphi \end{vmatrix} d\varphi\,d\theta + \int_0^{2\pi}\int_0^{\pi} \cos\theta\sin\varphi \begin{vmatrix} \sin\theta\cos\varphi & \cos\theta\sin\varphi \\ -\sin\varphi & 0 \end{vmatrix} d\varphi\,d\theta \\
&\quad + \int_0^{2\pi}\int_0^{\pi} \sin\theta\sin\varphi \begin{vmatrix} -\sin\varphi & 0 \\ \cos\theta\cos\varphi & -\sin\theta\sin\varphi \end{vmatrix} d\varphi\,d\theta \\
&= \int_0^{2\pi}\int_0^{\pi} \sin\varphi\cos^2\varphi\,d\varphi\,d\theta + \int_0^{2\pi}\int_0^{\pi} \cos^2\theta\sin^3\varphi + \sin^2\theta\sin^3\varphi\,d\varphi\,d\theta \\
&= \int_0^{2\pi}\int_0^{\pi} \sin\varphi(\cos^2\varphi + \sin^2\varphi)\,d\varphi\,d\theta \\
&= \int_0^{2\pi}\int_0^{\pi} \sin\varphi\,d\varphi\,d\theta \\
&= 2\pi\Big[-\cos\varphi\Big]_0^{\pi} = 2\pi
\end{aligned}$$

Divergence Theorem:

$$d\omega = dz\,dy\,dx + dx\,dy\,dz + dydzdx = 3\,dx\,dy\,dz$$

$$\begin{aligned}
\iint_S \omega &= \iiint_R d\omega \\
&= 3\int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho^2\sin(\varphi)\,d\rho\,d\varphi\,d\theta \\
&= \int_0^{2\pi}\int_0^{\pi} \sin(\varphi)\,d\varphi\,d\theta \\
&= 2\pi\Big[-\cos\varphi\Big]_0^{\pi} = 2\pi
\end{aligned}$$

5. Compute $\int_S \omega$ and use symbolic algebra software to sketch S in each of the following.

(a) $\omega = xz\,dx\,dy + x^2\,dy\,dz + dy\,dz\,dx$
S is the upper hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$ with $\boldsymbol{n}$ pointing upward.
Close it with the disk of radius 2 on the xy-plane to apply divergence theorem

$$\mathbf{\Phi}(\theta, r) = (r\cos\theta, r\sin\theta, 0),\ r \in [0,2],\ \theta \in [0, 2\pi]$$

$$dx\,dy = \begin{vmatrix} -r\sin\theta & \cos\theta \\ r\cos\theta & \sin\theta \end{vmatrix} = -2r$$

Which is negative, so correct orientation for normal pointing down.

$dy\,dz = 0$ Since z is 0

$dz\,dx = 0$

$$\overset{\text{Div Thm}}{\Longrightarrow} \iint_S \omega = \iiint_R d\omega - \iint_{\mathbf{\Phi}} \omega$$

$$\text{But } z = 0 \implies xz\,dx\,dy = 0 \implies \iint_{\mathbf{\Phi}} \omega = 0$$

$$d\omega = x\,dx\,dy\,dz + 2x\,dx\,dy\,dz = 3x\,dx\,dy\,dz$$

$$\iiint_R d\omega = \int_0^2 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} 3(\rho\sin\varphi\cos\theta)\rho^2 \sin\varphi\,d\varphi\,d\theta\,d\rho$$

$$= 0 \text{ Since integrating cos over full period}$$

$$\implies \int_S \omega = 0$$