Poon, Keegan
1002423727
MATB42: Assignment #10 Apr 3rd 2018

1. Let $\boldsymbol{F}$ be a vector field on $\mathbb{R}^3$ given by $\boldsymbol{F} = (F_1, F_2, F_3)$ where F_1, F_2, and F_3 are C^1-functions from $\mathbb{R}^3 \to \mathbb{R}$

   (a) Let η be the 2-form given by

   $$\eta = F_3\, dx\, dy + F_1\, dy\, dz + F_2\, dz\, dx$$

   Show that $d\eta = (\operatorname{div} \boldsymbol{F})\, dx\, dy\, dz$
   (page 489, #6)

$$\begin{aligned}
\eta &= F_3\, dx\, dy + F_1\, dy\, dz + F_2\, dz\, dx \\
d\eta &= d(F_3\, dx\, dy + F_1\, dy\, dz + F_2\, dz\, dx) \\
&= (dF_3)\, dx\, dy + (dF_1)\, dy\, dz + (dF_2)\, dz\, dx \\
&= (\frac{\partial}{\partial x}F_3\, dx + \frac{\partial}{\partial y}F_3\, dy + \frac{\partial}{\partial z}F_3\, dz)\, dx\, dy + (dF_1)\, dy\, dz + (dF_2)\, dz\, dx \\
&= \frac{\partial}{\partial z}F_3\, dz\, dx\, dy + (dF_1)\, dy\, dz + (dF_2)\, dz\, dx \\
&= \frac{\partial}{\partial z}F_3\, dx\, dy\, dz + (\frac{\partial}{\partial x}F_1\, dx + \frac{\partial}{\partial y}F_1\, dy + \frac{\partial}{\partial z}F_1\, dz)\, dy\, dz + (dF_2)\, dz\, dx \\
&= \frac{\partial}{\partial z}F_3\, dx\, dy\, dz + \frac{\partial}{\partial x}F_1\, dx\, dy\, dz + (dF_2)\, dz\, dx \\
&= \frac{\partial}{\partial z}F_3\, dx\, dy\, dz + \frac{\partial}{\partial x}F_1\, dx\, dy\, dz + (\frac{\partial}{\partial x}F_2\, dx + \frac{\partial}{\partial y}F_2\, dy + \frac{\partial}{\partial z}F_2\, dz)\, dz\, dx \\
&= \frac{\partial}{\partial z}F_3\, dx\, dy\, dz + \frac{\partial}{\partial x}F_1\, dx\, dy\, dz + \frac{\partial}{\partial y}F_2\, dy\, dz\, dx \\
&= \frac{\partial}{\partial z}F_3\, dx\, dy\, dz + \frac{\partial}{\partial x}F_1\, dx\, dy\, dz + \frac{\partial}{\partial y}F_2\, dx\, dy\, dz \\
&= \frac{\partial}{\partial x}F_1 + \frac{\partial}{\partial y}F_2 + \frac{\partial}{\partial z}F_3\, dx\, dy\, dz = (\operatorname{div} \boldsymbol{F})\, dx\, dy\, dz
\end{aligned}$$

   (b) Show that $dF_1 \wedge dF_2 \wedge dF_3 = (detD\boldsymbol{F})\, dx\, dy\, dz$