

## MATB42: Assignment #10

1. Let  $\mathbf{F}$  be a vector field on  $\mathbb{R}^3$  given by  $\mathbf{F} = (F_1, F_2, F_3)$  where  $F_1$ ,  $F_2$ , and  $F_3$  are  $C^1$ -functions from  $\mathbb{R}^3 \rightarrow \mathbb{R}$

- (a) Let  $\eta$  be the 2-form given by

$$\eta = F_3 dx dy + F_1 dy dz + F_2 dz dx$$

Show that  $d\eta = (\operatorname{div} \mathbf{F}) dx dy dz$

(page 489, #6)

$$\begin{aligned}
 \eta &= F_3 dx dy + F_1 dy dz + F_2 dz dx \\
 d\eta &= d(F_3 dx dy + F_1 dy dz + F_2 dz dx) \\
 &= (dF_3) dx dy + (dF_1) dy dz + (dF_2) dz dx \\
 &= \left( \frac{\partial}{\partial x} F_3 dx + \frac{\partial}{\partial y} F_3 dy + \frac{\partial}{\partial z} F_3 dz \right) dx dy + (dF_1) dy dz + (dF_2) dz dx \\
 &= \frac{\partial}{\partial z} F_3 dz dx dy + (dF_1) dy dz + (dF_2) dz dx \\
 &= \frac{\partial}{\partial z} F_3 dx dy dz + \left( \frac{\partial}{\partial x} F_1 dx + \frac{\partial}{\partial y} F_1 dy + \frac{\partial}{\partial z} F_1 dz \right) dy dz + (dF_2) dz dx \\
 &= \frac{\partial}{\partial z} F_3 dx dy dz + \frac{\partial}{\partial x} F_1 dx dy dz + (dF_2) dz dx \\
 &= \frac{\partial}{\partial z} F_3 dx dy dz + \frac{\partial}{\partial x} F_1 dx dy dz + \left( \frac{\partial}{\partial x} F_2 dx + \frac{\partial}{\partial y} F_2 dy + \frac{\partial}{\partial z} F_2 dz \right) dz dx \\
 &= \frac{\partial}{\partial z} F_3 dx dy dz + \frac{\partial}{\partial x} F_1 dx dy dz + \frac{\partial}{\partial y} F_2 dy dz dx \\
 &= \frac{\partial}{\partial z} F_3 dx dy dz + \frac{\partial}{\partial x} F_1 dx dy dz + \frac{\partial}{\partial y} F_2 dx dy dz \\
 &= \frac{\partial}{\partial x} F_1 + \frac{\partial}{\partial y} F_2 + \frac{\partial}{\partial z} F_3 dx dy dz = (\operatorname{div} \mathbf{F}) dx dy dz
 \end{aligned}$$

(b) Show that  $dF_1 \wedge dF_2 \wedge dF_3 = (\det D\mathbf{F}) dx dy dz$

$$df = \sum_{i=0}^n \frac{\partial f}{\partial x_i} dx_i$$

$$\begin{aligned}
dF_1 \wedge dF_2 \wedge dF_3 &= \left( \frac{\partial F_1}{\partial x} dx + \frac{\partial F_1}{\partial y} dy + \frac{\partial F_1}{\partial z} dz \right) \wedge \left( \frac{\partial F_2}{\partial x} dx + \frac{\partial F_2}{\partial y} dy + \frac{\partial F_2}{\partial z} dz \right) \wedge dF_3 \\
&= \left( \frac{\partial F_1}{\partial x} dx \wedge \left( \frac{\partial F_2}{\partial x} dx + \frac{\partial F_2}{\partial y} dy + \frac{\partial F_2}{\partial z} dz \right) \right. \\
&\quad + \frac{\partial F_1}{\partial y} dy \wedge \left( \frac{\partial F_2}{\partial x} dx + \frac{\partial F_2}{\partial y} dy + \frac{\partial F_2}{\partial z} dz \right) \\
&\quad \left. + \frac{\partial F_1}{\partial z} dz \wedge \left( \frac{\partial F_2}{\partial x} dx + \frac{\partial F_2}{\partial y} dy + \frac{\partial F_2}{\partial z} dz \right) \right) \wedge dF_3 \\
&= \left( \left( \frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial y} dx dy + \frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial z} dx dz \right) \right. \\
&\quad + \left( \frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial x} dy dx + \frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial z} dy dz \right) \\
&\quad \left. + \left( \frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial x} dz dx + \frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial y} dz dy \right) \right) \wedge dF_3 \\
&= \left( \left( \frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial y} - \frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial x} \right) dx dy \right. \\
&\quad + \left( \frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial y} \right) dy dz \\
&\quad \left. + \left( \frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial z} \right) dz dx \right) \wedge \left( \frac{\partial F_3}{\partial x} dx + \frac{\partial F_3}{\partial y} dy + \frac{\partial F_3}{\partial z} dz \right) \\
&= \left( \frac{\partial F_3}{\partial z} \left( \frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial y} - \frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial x} \right) dx dy \right. \\
&\quad + \frac{\partial F_3}{\partial x} \left( \frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial y} \right) dy dz \\
&\quad \left. + \frac{\partial F_3}{\partial y} \left( \frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial z} \right) dz dx \right) \\
&= \frac{\partial F_3}{\partial x} \left( \frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial y} \right) dx dy dz \\
&\quad - \frac{\partial F_3}{\partial y} \left( \frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial x} \right) dx dy dz \\
&\quad + \frac{\partial F_3}{\partial z} \left( \frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial y} - \frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial x} \right) dx dy dz \\
&= \frac{\partial F_3}{\partial x} \begin{vmatrix} \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{vmatrix} - \frac{\partial F_3}{\partial y} \begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial z} \end{vmatrix} + \frac{\partial F_3}{\partial z} \begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{vmatrix} dx dy dz \\
&= \begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \\ \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial z} \end{vmatrix} dx dy dz
\end{aligned}$$