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1. Let $\omega = \dfrac{-y}{x^2 + y^2}\ dx + \dfrac{x}{x^2 + y^2}\ dy$. Calculate $\displaystyle\int_\gamma \omega$ where

(a) γ is the boundary of the triangle with vertices (in order) $(0,1)$, $(2,3)$ and $(2,1)$.

The triangle does not wrap around the origin, so the winding number is 0. This means the integral should also be 0

(b) γ is the boundary curve of the region $\left\{ (x,y) \in \mathbb{R}^2 \left| \dfrac{(x-2)^2}{9} + \dfrac{(y+1)^2}{4} \le 1 \right. \right\}$ oriented in a counter clockwise direction.

This region is a disk and 0 satifies the equation, so the winding number of its surrounding ellipse is 1 (because the boundary curve is counter clockwise). This means the integral is $1(2\pi) = 2\pi$.

(c) γ is the graph of the polar equation $r = 3 + 2\sin\theta$ oriented in the clockwise direction.

Again, this curve wraps around the origin once, $(r > 0$ since $-2 \le 2\sin\theta \le 2)$, so the integral is -2π since this wraps in the clockwise direction.

2. Let $\omega = (y^2 + z\ \ln 3)\ dx + (2xy + \sin z)\ dy + (y\cos z + (x+1)\ln 3)\ dz$. Determine if ω is exact. If it is, use the algorithm given in class to find the potential function g.

$$\frac{\partial F_1}{\partial y} = 2y \qquad\qquad \frac{\partial F_2}{\partial x} = 2y \qquad\qquad \frac{\partial F_3}{\partial x} = \ln 3$$

$$\frac{\partial F_1}{\partial z} = \ln 3 \qquad\qquad \frac{\partial F_2}{\partial z} = \cos z \qquad\qquad \frac{\partial F_3}{\partial y} = \cos z$$

Since the pairwise differentials are equal, ω is closed. As well, the domain of polynomials and trig functions is $\mathbb{R}$ so the domain of ω is $\mathbb{R}^3$ which is simply connected $\implies \omega$ is exact. Given $F_1 = y^2 + z\ln 3$, $F_2 = 2xy + \sin z$, $F_3 = y\cos z + (x+1)\ln 3$ where $\boldsymbol{F} = (F_1, F_2, F_3)$ then

$$g = \int F_1 dx = xy^2 + xz\ln 3 + f(y,z)$$

$$F_2 = \frac{d}{dy}xy^2 + xz\ln 3 + f(y,z)$$

$$2xy + \sin z = 2xy + f'(y,z) \implies f(y,z) = y\sin z + f(z)$$

$$F_3 = \frac{d}{dz}xy^2 + xz\ln 3 + y\sin z + f(z)$$

$$y\cos z + (x+1)\ln 3 = x\ln 3 + y\cos z + f'(z) \implies f(z) = z\ln 3 + c$$

This means that the potential function g is $xy^2 + z(x+1)\ln 3 + y\sin z + c$

3. Evaluate the double integral $\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 3x^2 y^2 dy\ dx$, by first finding an equivalent line integral.

Note that the region R that is being integrated over is the unit circle in $\mathbb{R}^2$. We can parametrize the boundary of said circle using the path $\boldsymbol{\gamma}(t) : [0, 2\pi] \to \mathbb{R}^2$ by $t \mapsto (\cos t, \sin t)$. Using Green's theorem, choosing $F_2 = x^3 y^2$ and $F_1 = 0$, we have:

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 3x^2 y^2 dy\ dx = \int_{\gamma} F_2 dy = \int_0^{2\pi} \cos^3 t \sin^2 t (\cos t)\ dt$$

$$= \int_0^{2\pi} \cos^4 t \sin^2 t\ dt = \frac{1}{4}\int_0^{2\pi} \cos^2 t (2\sin t \cos t)^2 dt = \frac{1}{8}\int_0^{2\pi}(1+\cos(2t))\sin^2(2t)dt$$

Let $u = \sin(2t),\ du = 2\cos(2t)dt$

$$= \frac{1}{8}\left[\int_0^{2\pi} \sin^2(2t)dt + \frac{1}{2}\int_0^{0} u^2 du\right]$$

$$= \frac{1}{16}\int_0^{2\pi} 1 - \cos(4t)dt$$

$$= \frac{\pi}{8}$$

4. Let R be a region in $\mathbb{R}^2$ and let $\boldsymbol{\gamma}$ be a counterclockwise parametrization of ∂R. Let $\boldsymbol{F} = (F_1, F_2)$ be a C^1 vector field defined throughout R and on ∂R and let $\boldsymbol{n}$ be the outward pointing unit normal vector to $\boldsymbol{\gamma}$. Use Green's theorem to give a double integral over R which is equivalent to $\int_{\gamma} \boldsymbol{F} \cdot \boldsymbol{n}\ ds$.

Knowing that $\boldsymbol{F} \cdot \boldsymbol{n} = \|\boldsymbol{F}\|\|\boldsymbol{n}\| \cos\theta$ shows that no matter the actual direction of $\boldsymbol{F}$ and $\boldsymbol{n}$, as long as they are both angle θ away the dot product doesn't change. With this in mind, rotate $\boldsymbol{F}$ and $\boldsymbol{n}$ by $\frac{\pi}{2}$ so that we can get an expression for $\boldsymbol{n}$. Doing the rotation on n will give a vector on the tangent, as it must be orthogonal, giving the expression $\frac{\gamma'(t)}{\|\gamma'(t)\|}$ as it is unit length for the rotated $\boldsymbol{n}$ hereby referred to be n'. Given the expression for $\boldsymbol{F}$, it can be rotated with

$$F' = \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix} \boldsymbol{F} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = (-F_2, F_1)$$

Now the original integral can be rewritten as

$$\int_{\gamma} F' \cdot n'\ ds = \int_a^b F'(\gamma(t)) \cdot \frac{\gamma'(t)}{\|\gamma'(t)\|}\|\gamma'(t)\|dt$$

$$= \int_a^b F'(\gamma(t)) \cdot \gamma'(t)dt$$

$$= \int_a^b -F_2 dx + F_1 dy$$

$$= \int_{\mathcal{R}} \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} dA \qquad \text{By Green's Theorem (Assuming } \boldsymbol{\gamma} \text{ is smooth)}$$

5. Give a parametrization for each of the following surfaces, use a computer algebra sustem to plot the surface and find a unit vector normal to the surface.

   (a) The piece of the cylinder $y^2 + z^2 = 1$ between $x = -1$ and $x = 3$.

   The cylinder can be parametrized as a circle in the y,z plane i.e. $y, z = (\cos\theta, \sin\theta)$. Restricting x gives
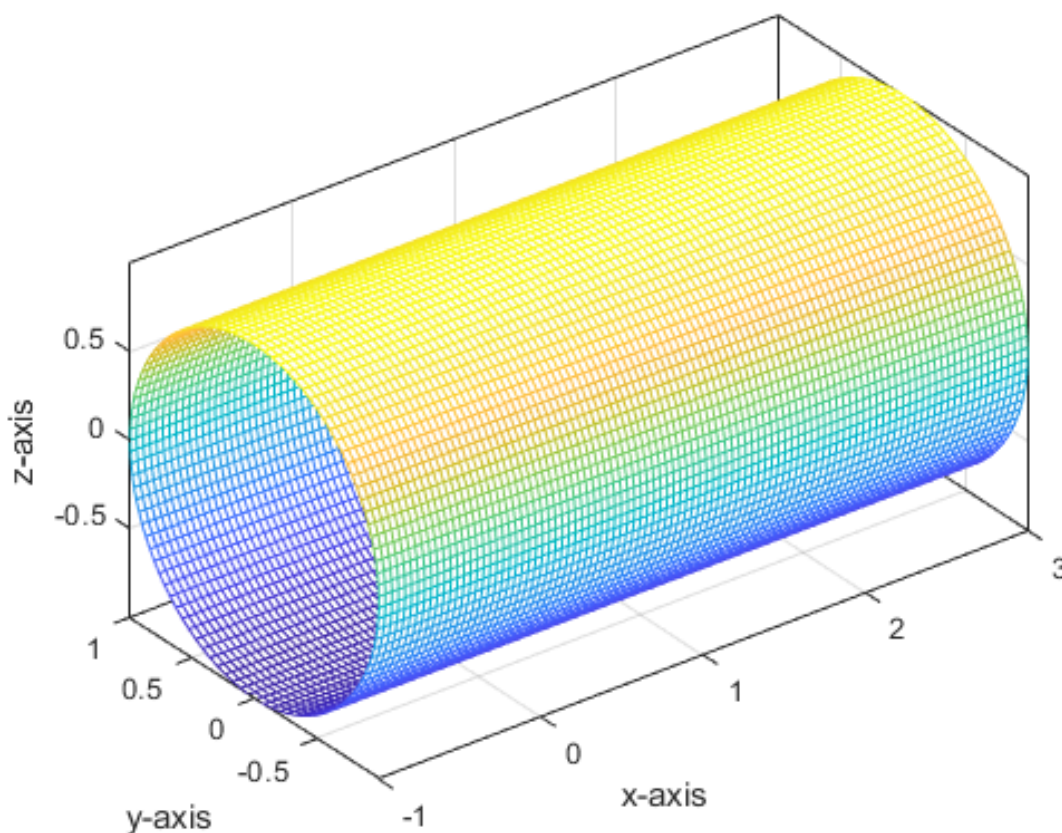
   $$\Phi(x, \theta) = (x, \cos\theta, \sin\theta)$$

   where $-1 \leq x \leq 3$ and $0 \leq \theta \leq 2\pi$.

   To find a normal vector, first need to figure out the tangent vectors. $\phi_x = (1, 0, 0), \phi_\theta = (0, -\sin\theta, \cos\theta)$, so a normal vector is

   $$\phi_x \times \phi_\theta = (0, -\cos\theta, -\sin\theta)$$

   The magnitude of said vector is already one, so it is a unit normal vector to the surface.

   (b) The piece of the plane $z = x + y + 5$ which lies over the unit disk $x^2 + y^2 \leq 1$.

   Since working with a disk, switch to polar coordinates to describe the domain where $x, y = (r\cos\theta, r\sin\theta)$ having $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$.
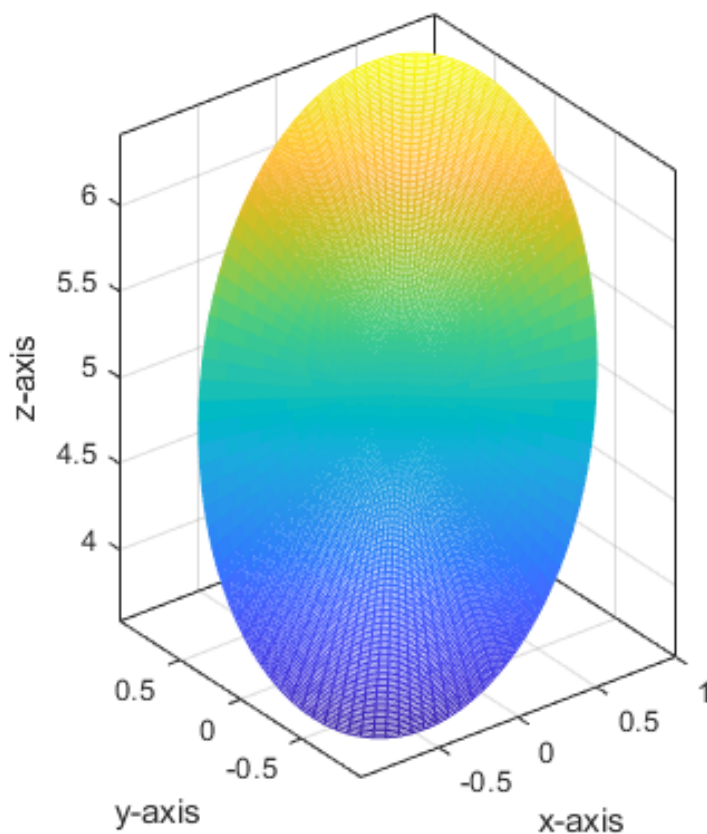
   Now the surface can be parameterized by the given expression for z:

   $$\Phi(r, \theta) = (r\cos\theta, r\sin\theta, r(\cos\theta + \sin\theta) + 5)$$

   The tangent vectors of this plane are $\phi_r = (\cos\theta, \sin\theta, \cos\theta + \sin\theta)$ and $\phi_\theta = (-r\sin\theta, r\cos\theta, r(-\sin\theta + \cos\theta))$. The normal vector is therefore $\phi_r \times \phi_\theta = (\sin\theta r(\cos\theta - \sin\theta) - (\cos\theta + \sin\theta)r\cos\theta, (\cos\theta +$

$\sin\theta)(-r\sin\theta)-(\cos\theta)r(\cos\theta-\sin\theta),\cos\theta(r\cos\theta)-\sin\theta(-r\sin\theta))=(-r,-r,r)$. Dividing by the magnitude of the vector $\sqrt{3r^2}=\sqrt{3}r$ gives the unit vector

$$\frac{1}{\sqrt{3}}(-1,-1,1)$$

(c) The piece of the sphere $x^2+y^2+z^2=4$ which lies above the plane $z=1$.

The sphere can be parametrized in a standard way as $\boldsymbol{\Phi}(u,v)\mapsto(2\cos u\sin v,2\sin u\sin v,2\cos v)$ To restrict this surface to only the portion that lies above $z=1$, take the z of the parametrization and restrict it. This gives $2\cos v>1\implies\cos v>1/2\implies v<\arccos 1/2=\pi/3$ since $\arccos$ strictly decreasing so $0<v<\pi/3$. The other angle being restricted to $0\le u\le 2\pi$. The tangent

vectors to the plane are

$$\phi_u = (-2\sin u \sin v, 2\cos u \sin v, 0)$$
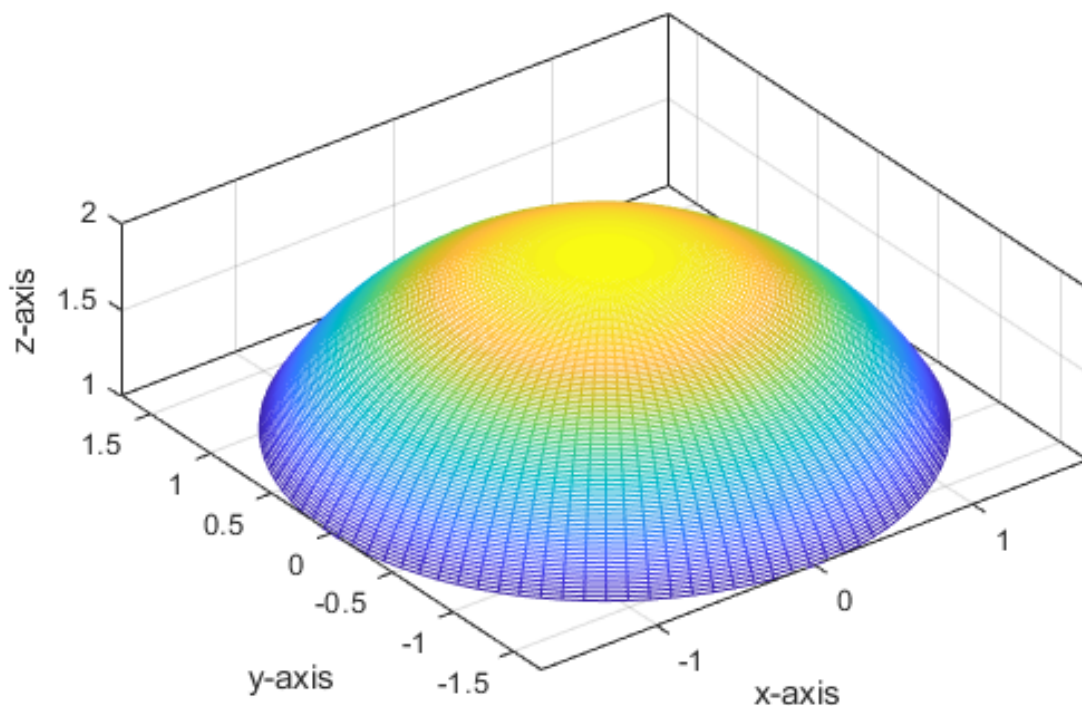$$\phi_v = (2\cos u \cos v, 2\sin u \cos v, -2\sin v)$$

Now the normal vector is

$$\phi_u \times \phi_v = (-4\cos u \sin^2 v, -4\sin u \sin^2 v, -4\sin^2 u \cos v \sin v - 4\cos^2 u \cos v \sin v)$$
$$= -4(\cos u \sin^2 v, \sin u \sin^2 v, \cos v \sin v)$$

The magnitude of which is

$$\|\phi_u \times \phi_v\| = 4\sqrt{\cos^2 u \sin^4 v + \sin^2 u \sin^4 v + \cos^2 v \sin^2 v}$$
$$= 4\sqrt{(\cos^2 u + \sin^2 u)\sin^4 v + \cos^2 v \sin^2 v}$$
$$= 4\sqrt{(\sin^2 v + \cos^2 v)\sin^2 v}$$
$$= 4|\sin v|$$

So the unit normal vector is $= -\dfrac{1}{|\sin v|}(\cos u \sin^2 v, \sin u \sin^2 v, \cos v \sin v)$

(d) The piece of the plane $x + y + z = 1$ which lies above the parallelogram: $0 \le y - x \le 1, 0 \le y + 2x \le 1$.

Want to parametrize easier, so choose variables to exploit this, $u = y - x, v = y + 2x$

$$2u + v = 3y \qquad \Longrightarrow \qquad y = \frac{2u + v}{3}$$

$$v - u = 3x \qquad \Longrightarrow \qquad x = \frac{v - u}{3}$$

$$z = 1 - x - y \qquad \Longrightarrow \qquad z = \frac{1}{3}(3 - v + u - 2u - v) = \frac{3 - 2v - u}{3}$$

So $\boldsymbol{\Phi}(u, v) \mapsto \frac{1}{3}(v - u, 2u + v, 3 - 2v - u)$ where $0 \le u, v \le 1$. Then the tangents are

$$\boldsymbol{\phi}_u = \frac{1}{3}(-1, 2, -1)$$

$$\boldsymbol{\phi}_v = \frac{1}{3}(1, 1, -2)$$

$$\boldsymbol{\phi}_u \times \boldsymbol{\phi}_v = \frac{1}{3}((2)(-2) - (-1)(1), (-1)(1) - (-1)(-2), (-1)(1) - (2)(1))$$

$$= \frac{1}{3}(-4 + 1, -1 - 2, -1 - 2) = -(1, 1, 1)$$

So the unit normal is $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$
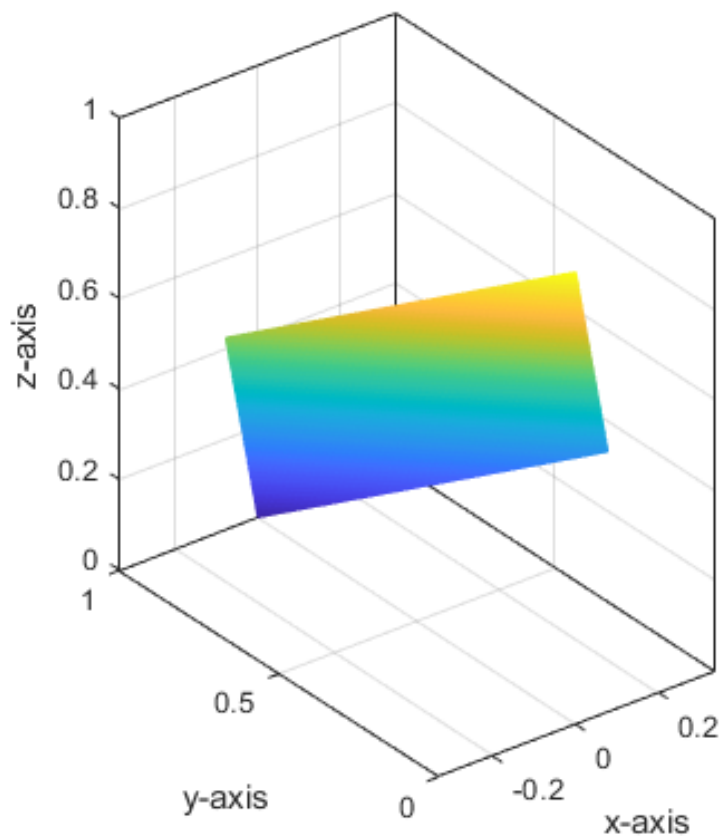
6. Let S be the surface given parameterically by $\mathbf{\Phi}(u, v) = (u^2, 3v, u^2 + v)$ where $(u, v) \in D$, the interior of a triangle with vertices (0,0), (3,0) and (3,3).

  (a) Find the surface area of S.

$$\begin{aligned}
\boldsymbol{\phi}_u &= (2u, 0, 2u)\\
\boldsymbol{\phi}_v &= (0, 3, 1)\\
\boldsymbol{\phi}_u \times \boldsymbol{\phi}_v &= (0 - 6u, 0 - 2u, 6u - 0) = (-6u, -2u, 6u)\\
\|\boldsymbol{\phi}_u \times \boldsymbol{\phi}_v\| &= \sqrt{36u^2 + 4u^2 + 36u^2} = \sqrt{76}u \text{ u positive in the triangle}\\
\int_D \|\boldsymbol{\phi}_u \times \boldsymbol{\phi}_v\| dA &= \int_0^3 \int_0^u \sqrt{76}u\, dv\, du = \int_0^3 \sqrt{76}u^2\, du = \frac{\sqrt{76}}{3}3^3 = 9\sqrt{76}
\end{aligned}$$

  (b) Find the equation of the tangent plane to S at the point (4,9,7).

  The equation for the tangent plane is given by $(\boldsymbol{\phi}_u \times \boldsymbol{\phi}_v \cdot (4 - x, 9 - y, 7 - z)) = 0$. We know that $\mathbf{\Phi}(u_0, v_0) = (4, 9, 7)$ so $u_0 = 2$, $v_0 = 3$. This means the normal vector is $(-12, -4, 12) \implies$ the tangent plane is $(-12)(4 - x) + (-4)(9 - y) + (12)(7 - z) = 0$.

$$\implies 12x + 4y - 12z = -36$$

7. Suppose the surface S is the graph of a function $f : \mathbb{R}^2 \to \mathbb{R}$. Give a natural parametrization of S (in terms of f) and derive the formula $\|\boldsymbol{\phi}_u \times \boldsymbol{\phi}_v\| = \sqrt{1 + \|\text{grad } f\|^2}$

The surface S can be parametrized naturally as $\mathbf{\Phi}(u, v) = (u, v, f(u, v))$

This means the tangent vectors can be written as

$$\begin{aligned}
\boldsymbol{\phi}_u &= (1, 0, \frac{\partial}{\partial u} f(u, v))\\
\boldsymbol{\phi}_v &= (0, 1, \frac{\partial}{\partial v} f(u, v))\\
\boldsymbol{\phi}_u \times \boldsymbol{\phi}_v &= (-\frac{\partial}{\partial u} f(u, v), -\frac{\partial}{\partial v} f(u, v), 1)\\
\|\boldsymbol{\phi}_u \times \boldsymbol{\phi}_v\| &= \sqrt{(\frac{\partial}{\partial u} f(u, v))^2 + (\frac{\partial}{\partial v} f(u, v))^2 + 1}\\
&= \sqrt{(\frac{\partial}{\partial u} f(u, v), \frac{\partial}{\partial v} f(u, v)) \cdot (\frac{\partial}{\partial u} f(u, v), \frac{\partial}{\partial v} f(u, v)) + 1}\\
&= \sqrt{\nabla f \cdot \nabla f + 1}\\
&= \sqrt{\|\nabla f\|^2 + 1}
\end{aligned}$$

8. A paraboloid of revolution S is parameterized by $\mathbf{\Phi}(u, v) = (u \cos v, u \sin v, u^2)$, $0 \le u \le 2, 0 \le v \le 2\pi$.

   (a) Find an equation in x, y and z describing the surface.

      Looking at the first two elements of the surface, $u \cos v$ and $u \sin v$, $x^2 + y^2 = u^2 \cos^2 v + u^2 \sin^2 v = u^2 = z^2$. So the equation is just $x^2 + y^2 = z^2$.

   (b) What are the geometric meanings of the parameters u and v?

      The v-curves are the rings of the paraboloid, at a given height level, while u-curves are the parabolas that stretch down the paraboloid vertically, so v is the angle on the paraboloid, and u is the height.

   (c) Find a unit vector orthogonal to the surface of $\mathbf{\Phi}(u, v)$.

   $$\boldsymbol{\phi}_u = (\cos v, \sin v, 2u)$$
   $$\boldsymbol{\phi}_v = (-u \sin v, u \cos v, 0)$$
   $$\boldsymbol{\phi}_u \times \boldsymbol{\phi}_v = (-2u^2 \cos v, -2u^2 \sin v, u \cos^2 v + u \sin^2 v)$$
   $$= u(-2u \cos v, -2u \sin v, 1)$$
   $$\|\boldsymbol{\phi}_u \times \boldsymbol{\phi}_v\| = u\sqrt{4u^2 \cos^2 v + 4u^2 \sin^2 v + 1}$$
   $$= u\sqrt{4u^2 + 1}$$
   $$\text{Normal unit vector } \boldsymbol{n}(u, v) = \frac{1}{\sqrt{4u^2 + 1}}(-2u \cos v, -2u \sin v, 1)$$

   (d) Find the equation for the tangent plane at $\mathbf{\Phi}(u_0, v_0) = (1, 1, 2)$ and express your answer in the following two ways:

   $$u_0^2 = 2 \implies u_0 = \sqrt{2}$$
   $$\sqrt{2} \cos(v_0) = \sqrt{2} \sin(v_0) = 1$$
   $$\implies \sin(v_0) = \cos(v_0) = \frac{1}{\sqrt{2}} \implies v_0 = \frac{\pi}{2}$$

   i. parameterized by u and v; and

   $$\boldsymbol{\phi}_u(\sqrt{2}, \frac{\pi}{2}) = (0, 1, 2\sqrt{2})$$
   $$\boldsymbol{\phi}_v(\sqrt{2}, \frac{\pi}{2}) = (-\sqrt{2}, 0, 0)$$
   $$\mathbf{\Phi}_{\text{tangent}}(u, v) = \boldsymbol{\phi}_u(\sqrt{2}, \frac{\pi}{2})u + \boldsymbol{\phi}_v(\sqrt{2}, \frac{\pi}{2})v + (1, 1, 2)$$
   $$= (0, u, 2\sqrt{2}u) + (-\sqrt{2}, 0, 0)v + (1, 1, 2)$$
   $$= (1 - \sqrt{2}v, 1 + u, 2(1 + \sqrt{2}u))$$

   ii. in terms of x, y and z.

   $$\boldsymbol{n}(u_0, v_0) = \frac{1}{3}(0, -2\sqrt{2}, 1)$$
   $$\boldsymbol{n} \cdot (x - 1, y - 1, z - 2) = 0$$
   $$(0 + \frac{-2\sqrt{2}}{3}(y - 1) + \frac{1}{3}(z - 2)) = 0$$
   $$\frac{-2\sqrt{2}}{3}y + \frac{2\sqrt{2}}{3} + \frac{z}{3} - \frac{2}{3} = 0$$
   $$\frac{-2\sqrt{2}}{3}y + \frac{z}{3} = \frac{1 - 2\sqrt{2}}{3}$$

(e) Find the area of S.

(cf. page 424, #16)

$$\int_D dS = \int_0^2 \int_0^{2\pi} u\sqrt{4u^2 + 1}\, dv\, du$$

$$= 2\pi \int_0^2 u\sqrt{4u^2 + 1}\, du$$

Let $x = 4u^2 + 1, dx = 8u\ du$

$$= \frac{\pi}{4} \int_1^{17} \sqrt{x}\, dx$$

$$= \frac{\pi}{6} \left[x^{\frac{3}{2}} \right]_1^{17}$$

$$= \frac{\pi}{6} (17^{\frac{3}{2}} - 1)$$

9. Let a differentiable function $\mathbf{\Phi} : \mathbb{R}^2 \to \mathbb{R}^3$ define a parametrized surface.

(a) Assuming $\boldsymbol{\phi}_u \times \boldsymbol{\phi}_v \neq 0$, show that the range of the linear transformation $D\mathbf{\Phi}(u_0, v_0)$ is the plane spanned by $\boldsymbol{\phi}_u$ and $\boldsymbol{\phi}_v$. [Here $\boldsymbol{\phi}_u$ and $\boldsymbol{\phi}_v$ are evaluated at (u_0, v_0).]

If $\mathbf{\Phi}(u, v) = (\phi_1(u, v), \phi_2(u, v), \phi_3(u, v))$. Then:

$$D\mathbf{\Phi}(u, v) = \begin{bmatrix} \phi_{1u}(u, v) & \phi_{1v}(u, v) \\ \phi_{2u}(u, v) & \phi_{2v}(u, v) \\ \phi_{3u}(u, v) & \phi_{3v}(u, v) \end{bmatrix} = \begin{bmatrix} \boldsymbol{\phi}_u(u, v) & \boldsymbol{\phi}_v(u, v) \end{bmatrix}$$

This means that for any vector $\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2$, we get that

$$D\mathbf{\Phi}(u_0, v_0)\boldsymbol{x} = x_1\boldsymbol{\phi}_u(u_0, v_0) + x_2\boldsymbol{\phi}_v(u_0, v_0)$$

Which is just a linear combination of both tangent vectors, i.e. the range is the span of the tangent vectors.

(b) Show that $\boldsymbol{w} \perp (\boldsymbol{\phi}_u \times \boldsymbol{\phi}_v)$ if and only if $\boldsymbol{w}$ is in the range of $D\mathbf{\Phi}(u_0, v_0)$.

Since the cross of the tangent vectors gives a normal to the plane, the normal to the normal of the plane is simply any and every vector in the plane. $\boldsymbol{w}$ must be on the tangent plane generated by the vectors. However, we showed earlier that range($D\mathbf{\Phi}(u_0, v_0)$) is the plane spanned by the 2 tangent vectors, so $\boldsymbol{w}$ must be in the range.

(c) Show that the tangent plane as defined in terms of $\boldsymbol{\phi}_u \times \boldsymbol{\phi}_v(u_0, v_0)$ is the same as the "parametrized plane"

$$(u, v) \mapsto \mathbf{\Phi}(u_0, v_0) + D\mathbf{\Phi}(u_0, v_0) \begin{bmatrix} u - u_0 \\ v - v_0 \end{bmatrix}$$

(cf. page 383 #20)

Given any point (x, y, z) on the plane we have:

$$(x, y, z) = \mathbf{\Phi}(u_0, v_0) + D\mathbf{\Phi}(u_0, v_0) \begin{bmatrix} u - u_0 \\ v - v_0 \end{bmatrix}$$

$$(x, y, z) - \mathbf{\Phi}(u_0, v_0) = D\mathbf{\Phi}(u_0, v_0) \begin{bmatrix} u - u_0 \\ v - v_0 \end{bmatrix}$$

$$((x, y, z) - \mathbf{\Phi}(u_0, v_0)) \cdot \boldsymbol{\phi}_u \times \boldsymbol{\phi}_v = (D\mathbf{\Phi}(u_0, v_0) \begin{bmatrix} u - u_0 \\ v - v_0 \end{bmatrix}) \cdot \boldsymbol{\phi}_u \times \boldsymbol{\phi}_v$$

But we know from (b) that any possible point in the range of $D\mathbf{\Phi}(u_0, v_0)$ is in the plane, so when you dot that with the normal to the plane the RHS is 0. So the equaion result is

$$((x, y, z) - \mathbf{\Phi}(u_0, v_0)) \cdot \boldsymbol{\phi}_u \times \boldsymbol{\phi}_v = 0$$

Which is the other equation for a plane using $\boldsymbol{\phi}_u \times \boldsymbol{\phi}_v$.