

1. (a) Let  $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$ ;  $\mathbf{F}, \mathbf{G} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ; and define  $\Delta$ , the *Laplacian*, by  $\Delta f = \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2}$ .

Verify the following identities

(i)  $\operatorname{div}(\mathbf{F} + \mathbf{G}) = \operatorname{div} \mathbf{F} + \operatorname{div} \mathbf{G}$ .

$$\operatorname{div}(\mathbf{F} + \mathbf{G}) = \sum_{i=1}^n \frac{\partial(F_i + G_i)}{\partial x_i} = \sum_{i=1}^n \frac{\partial F_i}{\partial x_i} + \frac{\partial G_i}{\partial x_i} = \sum_{i=1}^n \frac{\partial F_i}{\partial x_i} + \sum_{i=1}^n \frac{\partial G_i}{\partial x_i} = \operatorname{div} \mathbf{F} + \operatorname{div} \mathbf{G}$$

(ii)  $\operatorname{div}(f\mathbf{F}) = f \operatorname{div} \mathbf{F} + \mathbf{F} \cdot \operatorname{grad} f$ .

$$\operatorname{div}(f\mathbf{F}) = \sum_{i=1}^n \frac{\partial f F_i}{\partial x_i} = \sum_{i=1}^n \frac{\partial f}{\partial x_i} F_i + \frac{\partial F_i}{\partial x_i} f = \sum_{i=1}^n \frac{\partial f}{\partial x_i} F_i + f \sum_{i=1}^n \frac{\partial F_i}{\partial x_i} = \nabla f \cdot \mathbf{F} + f \operatorname{div} \mathbf{F}$$

(iii)  $\Delta(fg) = f\Delta g + g\Delta f + 2(\operatorname{grad} f) \cdot (\operatorname{grad} g)$ .

$$\begin{aligned} \Delta(fg) &= \sum_{i=1}^n \frac{\partial^2 fg}{\partial x_i^2} \\ &= \sum_{i=1}^n \frac{\partial}{\partial x_i} \left[ \frac{\partial f}{\partial x_i} g + \frac{\partial g}{\partial x_i} f \right] \\ &= \sum_{i=1}^n \left[ \frac{\partial f}{\partial^2 x_i} g + \left( \frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_i} \right) + \frac{\partial g}{\partial^2 x_i} f + \left( \frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_i} \right) \right] \\ &= g \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2} + 2 \sum_{i=1}^n \left( \frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_i} \right) + f \sum_{i=1}^n \frac{\partial^2 g}{\partial x_i^2} \\ &= g\Delta f + 2[\nabla f \cdot \nabla g] + f\Delta g \end{aligned}$$

- (b) Let  $f, g : D \subset \mathbb{R}^3 \rightarrow \mathbb{R}$  be of class  $C^1$ . If  $R$  is a solid region contained in  $D$  then

$$\iiint_R \nabla f \cdot \nabla g \, dV = \iint_{\partial R} f \nabla g \cdot \mathbf{n} \, dS - \iiint_R f \nabla^2 g \, dV$$

$(\nabla^2 g = \operatorname{div} (\nabla g))$ .

2. Use the Divergence Theorem to verify your answer to question 7 on assignment 8.
3. Let  $\mathbf{F}(x, y, z) = (x, y^2, e^{yz})$  and let  $R$  be a cube centered at the origin with sides of length 2. Evaluate  $\int_S \operatorname{div} \mathbf{F} dV$  directly and by using the Divergence Theorem.

Directly:

$$\begin{aligned} \int_S \operatorname{div} \mathbf{F} dV &= \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 (1 + 2y + ye^{yz}) dx dz dy \\ &= \int_{-1}^1 \int_{-1}^1 (2 + 4y + 2ye^{yz}) dz dy \\ &= \int_{-1}^1 (4 + 8y + [e^{yz}]_{-1}^1) dy \\ &= \int_{-1}^1 (4 + 8y + e^y - e^{-y}) dy \\ &= 8 + 0 + [e^y]_{-1}^1 + [e^{-y}]_{-1}^1 \end{aligned}$$

4. Let  $B$  be the pyramid with top vertex  $(0,0,1)$  and base vertices  $(0,0,0)$ ,  $(1,0,0)$ ,  $(0,1,0)$  and  $(1,1,0)$ . Let  $S$  be the 2-dim closed surface bounding  $B$ , oriented in the outward direction. Use Gauss' theorem to calculate  $\int_S \mathbf{F} \cdot d\mathbf{S}$ , where  $\mathbf{F}(x, y, z) = (x^2y, 3y^2z, 9z^2x)$ .
5. Use the Divergence Theorem to evaluate  $\int_S \mathbf{F} \cdot d\mathbf{S}$ , where  $\mathbf{F}(x, y, z) = \left( z^2x, \frac{y^3}{3} + \tan z, x^2z + y^2 \right)$  and  $S$  is the top half of the unit sphere  $x^2 + y^2 + z^2 = 1$ , oriented by the unit normal which points away from the origin.
6. Let the electric field from a point source at the origin be given by  $\mathbf{E}(\mathbf{x}) = \frac{\mathbf{x}}{\|\mathbf{x}\|^3}$
- What is the outward flux of  $\mathbf{E}$  across the surface  $\frac{x^2}{3} + \frac{2y^2}{5} + z^2 = 7$ .
  - Show that the flux of  $\mathbf{E}$  across that part of the sphere  $x^2 + y^2 + z^2 = 25$  with  $z \geq 3$  is equal to the flux across that part of the plane  $z = 3$  with  $x^2 + y^2 \leq 16$ .
7. Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be given by  $f(x, y, z) = x^2yz$  and let  $\eta$  be the 2-form on  $\mathbb{R}^3$  given by

$$\eta = (\sin x) dx dy + (e^y + xyz) dx dz + (x^2y^2) dy dz.$$

- Compute  $df$  and  $d\eta$ .
- Evaluate  $df \wedge \eta$ .