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MATB42: Assignment #7 Mar 13th 2018

1. (a) Find an equation of the tangent plane to the surface S defined parametrically by $\mathbf{\Phi}(u,v) = (u^2+v,\ v,\ u+v^2)$ at the point (9,0,3).

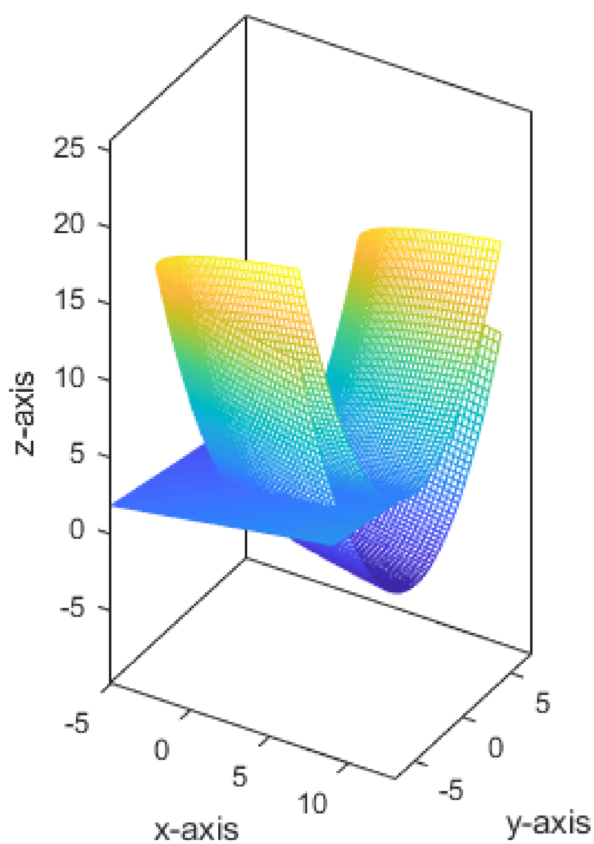
$$v = 0 \qquad\qquad u+v^2 = 3 \implies u = 3$$

$$\begin{aligned} \boldsymbol{\phi}_u &= (2(3), 0, 1) \\ \boldsymbol{\phi}_v &= (1, 1, 2(0)) \\ \boldsymbol{\phi}_u \times \boldsymbol{\phi}_v &= (-1, 1, 6) \end{aligned}$$

So the tangent plane can be given by

$$\begin{aligned} 0 &= ((x-9, y, z-3)\cdot(-1,1,6)) \\ 0 &= (9 - x + y + 6z - 18) \\ 9 &= -x + y + 6z \end{aligned}$$

(b) Use symbolic algebra software to sketch the surface S and its tangent plane from part (a).



2. Use a surface integral to find the area of the triangle in $\mathbb{R}^3$ with vertices (1,1,0), (1,2,1) and (3,3,2).

3. Calculate the surface area of the piece of the cone $x^2+y^2-z^2=0$ which lies inside the cylinder $x^2+y^2=4$.

We can see the radius of the cylinder is 2, so the cone portion that's cut out is the part which has radius less than or equal to 2 $\implies 0 \leq z \leq 2$.

$$\begin{aligned}
\mathbf{\Phi}(\theta, z) &= (z\cos\theta, z\sin\theta, z) \\
\boldsymbol{\phi}_\theta &= (-z\sin\theta, z\cos\theta, 0) \\
\boldsymbol{\phi}_z &= (\cos\theta, \sin\theta, 1) \\
\boldsymbol{\phi}_\theta \times \boldsymbol{\phi}_z &= (z\cos\theta, z\sin\theta, -z\sin^2\theta - z\cos^2\theta) \\
&= (z\cos\theta, z\sin\theta, -z) \\
\|\boldsymbol{\phi}_\theta \times \boldsymbol{\phi}_z\| &= z^2\cos^2\theta + z^2\sin^2\theta + z^2 = 2z^2
\end{aligned}$$

4. (a) Find the area of the portion of the unit sphere that is cut out by the cone $z = \sqrt{x^2+y^2}$. (cf. page 391, #10)

   (b) Find the area of the portion of the cone $z = \sqrt{x^2+y^2}$ that is cut out by the unit sphere.

5. Let $\mathbf{\Phi} : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a parametrization of a 2-dim surface S in $\mathbb{R}^3$.

   (a) Set

$$E = \|\boldsymbol{\phi}_u\|^2, \qquad F = \boldsymbol{\phi}_u \cdot \boldsymbol{\phi}_v, \qquad G = \|\boldsymbol{\phi}_v\|^2,$$

   Show that the surface area of S is

$$A(S) = \iint_D \sqrt{EG - F^2}\, dA$$

$$\begin{aligned}
\iint_D \sqrt{EG - F^2}\, dA &= \iint_D \sqrt{\|\boldsymbol{\phi}_u\|^2\|\boldsymbol{\phi}_v\|^2 - (\boldsymbol{\phi}_u \cdot \boldsymbol{\phi}_v)^2}\, dA \\
&= \iint_D \sqrt{(\|\boldsymbol{\phi}_u\|\|\boldsymbol{\phi}_v\|)^2 - (\|\boldsymbol{\phi}_u\|\|\boldsymbol{\phi}_v\|)^2\cos^2\theta}\, dA \quad \text{Where } \theta \text{ is the angle between } \boldsymbol{\phi}_u \text{ and } \boldsymbol{\phi}_v. \\
&= \iint_D \sqrt{(\|\boldsymbol{\phi}_u\|\|\boldsymbol{\phi}_v\|)^2(1 - \cos^2\theta)}\, dA \\
&= \iint_D \sqrt{(\|\boldsymbol{\phi}_u\|\|\boldsymbol{\phi}_v\|)^2(\sin^2\theta)}\, dA \\
&= \iint_D \sqrt{\|\boldsymbol{\phi}_u \times \boldsymbol{\phi}_v\|^2}\, dA \\
&= \iint_D \|\boldsymbol{\phi}_u \times \boldsymbol{\phi}_v\|\, dA \\
&= \int_{\mathbf{\Phi}} 1\, dS
\end{aligned}$$

   (b) What does the formula for $A(S)$ become if the vectors $\boldsymbol{\phi}_u$ and $\boldsymbol{\phi}_v$ are orthogonal?
   If the vectors are orthogonal, then the dot product is 0, so the equation reduces to

$$A(S) = \iint_D \|\boldsymbol{\phi}_u\|\|\boldsymbol{\phi}_v\|\, dA$$

   (c) Use parts (a) and (b) to compute the surface area of a sphere of radius a.

(cf. Marsden & Tromba, page 399, # 23.)

$$
\begin{aligned}
\mathbf{\Phi}(\theta,\varphi) &= a(\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi)\\
\boldsymbol{\phi}_\theta &= a(-\sin\theta\sin\varphi, \cos\theta\sin\varphi, 0)\\
\boldsymbol{\phi}_\varphi &= a(\cos\theta\cos\varphi, \sin\theta\cos\varphi, -\sin\varphi)\\
\|\boldsymbol{\phi}_\theta\| &= a\sin\varphi, \quad \|\boldsymbol{\phi}_\varphi\| = a\\
\implies A(S) &= a^2\int_0^{2\pi}\int_0^{\pi}\sin\varphi\,d\varphi\,d\theta\\
&= a^2\int_0^{2\pi}\Big[-\cos\varphi\Big]_0^{\pi}\,d\varphi\,d\theta\\
&= a^2\int_0^{2\pi}-(-1-1)\,d\varphi\,d\theta\\
&= a^2 2\int_0^{2\pi}1\,d\varphi\,d\theta\\
&= 4\pi a^2
\end{aligned}
$$

6. For each of the following surfaces S, sketch S (using symbolic software) and evaluate the surface integral $\int_S f\,dS$, where $f(x,y,z)=x$.

   (a) S is that part of the surface $y=4-x^2$ between $z=0$ and $z=1$, with $y\geq 0$.

$$
y\geq 0 \implies 4-x^2\geq 0 \implies x^2\leq 4 \implies |x|<2
$$

$$
\begin{aligned}
\mathbf{\Phi}(x,z) &= (x, 4-x^2, z)\\
\boldsymbol{\phi}_x &= (1,-2x,0),\ \boldsymbol{\phi}_z = (0,0,1)\\
\boldsymbol{\phi}_x\times\boldsymbol{\phi}_z &= (-2x,-1,0) \implies \|\boldsymbol{\phi}_x\times\boldsymbol{\phi}_z\| = \sqrt{4x^2+1}\\
\int_S f dS &= \int_0^1\int_{-2}^{2} x\sqrt{4x^2+1}\,dx\,dz
\end{aligned}
$$

   The integrand is odd since x odd and $\sqrt{4x^2+1}$ even, so the integral over x is 0, making the entire integral 0.

   (b) S is the upper half of the unit sphere centered at the origin.
   Only the upper half so $0\leq\theta\leq 2\pi$ and $0\leq\varphi\leq\pi/2$.

$$
\begin{aligned}
\mathbf{\Phi}(\theta,\varphi) &= (\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi)\\
\boldsymbol{\phi}_\theta &= (-\sin\theta\sin\varphi, \cos\theta\sin\varphi, 0)\\
\boldsymbol{\phi}_\varphi &= (\cos\theta\cos\varphi, \sin\theta\cos\varphi, -\sin\varphi)\\
\boldsymbol{\phi}_\theta\times\boldsymbol{\phi}_\varphi &= (-\cos\theta\sin^2\varphi, -\sin\theta\sin^2\varphi, -\sin^2\theta\sin\varphi\cos\varphi - \cos^2\theta\sin\varphi\cos\varphi)\\
&= (-\cos\theta\sin^2\varphi, -\sin\theta\sin^2\varphi, -\sin\varphi\cos\varphi)\\
\|\boldsymbol{\phi}_\theta\times\boldsymbol{\phi}_\varphi\| &= \sqrt{\cos^2\theta\sin^4\varphi+\sin^2\theta\sin^4\varphi+\sin^2\varphi\cos^2\varphi}\\
&= \sqrt{\sin^4\varphi+\sin^2\varphi\cos^2\varphi}\\
&= \sqrt{\sin^2\varphi} = \sin\varphi\\
\int_{\mathbf{\Phi}} f\,dS &= \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\cos\theta\sin^2\varphi\,d\theta\,d\varphi = 0
\end{aligned}
$$

   The integral is zero again since integrating $\cos\theta$ over a whole period is 0.

(c) S is that part of the surface $x = \sin y$ with $0 \le y \le \pi$ and $0 \le z \le 2$.

$$\begin{aligned}
\mathbf{\Phi}(y, z) &= (\sin y, y, z)\\
\boldsymbol{\phi}_y &= (\cos y, 1, 0)\\
\boldsymbol{\phi}_z &= (0, 0, 1)\\
\boldsymbol{\phi}_y \times \boldsymbol{\phi}_z &= (1, -\cos y, 0)\\
\|\boldsymbol{\phi}_y \times \boldsymbol{\phi}_z\| &= \sqrt{1 + \cos^2 y}\\
\int_{\mathbf{\Phi}} f \, dS &= \int_0^2 \int_0^\pi \sin y \sqrt{1 + \cos^2 y} \, dy \, dz
\end{aligned}$$

7. Find the mass of the metallic surface S given by $z = 1 - \dfrac{x^2 + y^2}{2}$ with $0 \le x \le 1$, $0 \le y \le 1$, if the mass density at $(x, y, z) \in S$ is given by $m(x, y, z) = xy$.