

## MATB42: Assignment #8

1. (a) Let  $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$ ;  $\mathbf{F}, \mathbf{G} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ; and define  $\Delta$ , the *Laplacian*, by  $\Delta f = \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2}$ .

Verify the following identities

(i)  $\operatorname{div}(\mathbf{F} + \mathbf{G}) = \operatorname{div} \mathbf{F} + \operatorname{div} \mathbf{G}$ .

$$\operatorname{div}(\mathbf{F} + \mathbf{G}) = \sum_{i=1}^n \frac{\partial(F_i + G_i)}{\partial x_i} = \sum_{i=1}^n \frac{\partial F_i}{\partial x_i} + \frac{\partial G_i}{\partial x_i} = \sum_{i=1}^n \frac{\partial F_i}{\partial x_i} + \sum_{i=1}^n \frac{\partial G_i}{\partial x_i} = \operatorname{div} \mathbf{F} + \operatorname{div} \mathbf{G}$$

(ii)  $\operatorname{div}(f\mathbf{F}) = f \operatorname{div} \mathbf{F} + \mathbf{F} \cdot \operatorname{grad} f$ .

$$\operatorname{div}(f\mathbf{F}) = \sum_{i=1}^n \frac{\partial f F_i}{\partial x_i} = \sum_{i=1}^n \frac{\partial f}{\partial x_i} F_i + \frac{\partial F_i}{\partial x_i} f = \sum_{i=1}^n \frac{\partial f}{\partial x_i} F_i + f \sum_{i=1}^n \frac{\partial F_i}{\partial x_i} = \nabla f \cdot \mathbf{F} + f \operatorname{div} \mathbf{F}$$

(iii)  $\Delta(fg) = f\Delta g + g\Delta f + 2(\operatorname{grad} f) \cdot (\operatorname{grad} g)$ .

*Proof.*

$$\begin{aligned} \Delta(fg) &= \sum_{i=1}^n \frac{\partial^2 fg}{\partial x_i^2} \\ &= \sum_{i=1}^n \frac{\partial}{\partial x_i} \left[ \frac{\partial f}{\partial x_i} g + \frac{\partial g}{\partial x_i} f \right] \\ &= \sum_{i=1}^n \left[ \frac{\partial f}{\partial^2 x_i} g + \left( \frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_i} \right) + \frac{\partial g}{\partial^2 x_i} f + \left( \frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_i} \right) \right] \\ &= g \sum_{i=1}^n \frac{\partial f}{\partial^2 x_i} + 2 \sum_{i=1}^n \left( \frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_i} \right) + f \sum_{i=1}^n \frac{\partial g}{\partial^2 x_i} \\ &= g\Delta f + 2[\nabla f \cdot \nabla g] + f\Delta g \end{aligned}$$

□

- (b) Let  $f, g : D \subset \mathbb{R}^3 \rightarrow \mathbb{R}$  be of class  $C^1$ . If  $R$  is a solid region contained in  $D$  then

$$\iiint_R \nabla f \cdot \nabla g \, dV = \iint_{\partial R} f \nabla g \cdot \mathbf{n} \, dS - \iiint_R f \nabla^2 g \, dV$$

$(\nabla^2 g = \operatorname{div}(\nabla g))$ .

*Proof.*

$$\begin{aligned} \iiint_R \nabla f \cdot \nabla g \, dV &= \iint_{\partial R} f \nabla g \cdot \mathbf{n} \, dS - \iiint_R f \nabla^2 g \, dV \\ \iff \iiint_R \nabla f \cdot \nabla g \, dV + \iiint_R f \nabla^2 g \, dV &= \iint_{\partial R} f \nabla g \cdot \mathbf{n} \, dS \end{aligned}$$

$$\begin{aligned} \iint_{\partial R} f \nabla g \cdot \mathbf{n} \, dS &\stackrel{\text{Div Thm}}{=} \iiint_R \operatorname{div}(f \nabla g) \, dV \stackrel{\text{(ii)}}{=} \iiint_R f(\operatorname{div} \nabla g) + \nabla g \cdot \nabla f \, dV \\ &= \iiint_R f(\operatorname{div} \nabla g) \, dV + \iiint_R \nabla f \cdot \nabla g \, dV = \iiint_R f \nabla^2 g \, dV + \iiint_R \nabla f \cdot \nabla g \, dV \end{aligned}$$

□

2. Use the Divergence Theorem to verify your answer to question 7 on assignment 8.

Let  $S$  be the closed surface that consists of the hemisphere  $x^2 + y^2 + z^2 = 1, z \geq 0$ , and its base  $x^2 + y^2 \leq 1, z = 0$ . let  $\mathbf{E}$  be the electric field  $\mathbf{E}(x, y, z) = (2x, 2y, 2z)$ . Calculate the electric flux across  $S$  using Divergence Theorem.

Divergence Theorem gives that

$$\iiint_R \operatorname{div} \mathbf{F} dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$$

$S$  is a closed surface, and assuming the default outward normal Divergence Theorem. To integrate over the region, switch to polars restricting  $\varphi \in [0, \pi/2]$  to restrict to the upper half. Together with  $\operatorname{div} \mathbf{E} = 6$  gives:

$$\begin{aligned} \iint_S \mathbf{E} \cdot d\mathbf{S} &= \int_0^1 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} 6\rho^2 \sin \varphi d\varphi d\theta d\rho \\ &= 12\pi \int_0^1 \rho^2 [-\cos \varphi]_0^{\frac{\pi}{2}} d\rho &= 12\pi \left[ \frac{\rho^3}{3} \right]_0^1 \end{aligned}$$

3. Let  $\mathbf{F}(x, y, z) = (x, y^2, e^{yz})$  and let  $R$  be a cube centered at the origin with sides of length 2. Evaluate  $\int_S \operatorname{div} \mathbf{F} dV$  directly and by using the Divergence Theorem.

Directly:

$$\begin{aligned}
 \int_S \operatorname{div} \mathbf{F} dV &= \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 1 + 2y + ye^{yz} dx dz dy \\
 &= \int_{-1}^1 \int_{-1}^1 2 + 4y + 2ye^{yz} dz dy \\
 &= \int_{-1}^1 4 + 8y + [e^{yz}]_{-1}^1 dy \\
 &= \int_{-1}^1 4 + 8y + e^y - e^{-y} dy \\
 &= 8 + 0 + [e^y]_{-1}^1 + [e^{-y}]_{-1}^1 = 8
 \end{aligned}$$

Divergence:

$$\int_R \operatorname{div} \mathbf{F} dV = \int_S \mathbf{F} \cdot d\mathbf{S}$$

where  $S$  is the union of 6 planes with  $-1 \leq x, y, z \leq 1$  and having normals in the  $\pm x, y, z$  directions. The parameterizations can be given by

$$\begin{aligned}
 \Phi_{S_1}(x, y) &= (x, y, 1) & \Phi_{S'_1}(x, y) &= (x, y, -1) \\
 \Phi_{S_2}(y, z) &= (1, y, z) & \Phi_{S'_2}(y, z) &= (-1, y, z) \\
 \Phi_{S_3}(x, z) &= (x, 1, z) & \Phi_{S'_3}(x, z) &= (x, -1, z)
 \end{aligned}$$

When plugging in  $S_3, S'_3$  into  $\mathbf{F}$ , then taking the dot with the normal, we get that the integrands are going to be the same, specifically 1. When doing the same with  $S_2, S'_2$ , due to  $x$  being odd, we get both being negative the other, so integrating over the same range will cancel each other out. Putting this together, the integral over  $S$  is:

$$\begin{aligned}
 \int_S \mathbf{F} \cdot d\mathbf{S} &= 2 \int_{-1}^1 \int_{-1}^1 1 dx dz + \int_{-1}^1 \int_{-1}^1 e^y dx dy + \int_{-1}^1 \int_{-1}^1 -e^{-y} dx dy \\
 &= 8 + 2 \int_{-1}^1 e^y dy - 2 \int_{-1}^1 e^{-y} dy \\
 &= 8 + 2([e^y]_{-1}^1 + [e^{-y}]_{-1}^1) = 8
 \end{aligned}$$

4. Let  $B$  be the pyramid with top vertex  $(0,0,1)$  and base vertices  $(0,0,0)$ ,  $(1,0,0)$ ,  $(0,1,0)$  and  $(1,1,0)$ . Let  $S$  be the 2-dim closed surface bounding  $B$ , oriented in the outward direction. Use Gauss' theorem to calculate  $\int_S \mathbf{F} \cdot d\mathbf{S}$ , where  $\mathbf{F}(x, y, z) = (x^2y, 3y^2z, 9z^2x)$ .

$$\int_R \operatorname{div} \mathbf{F} dV = \int_S \mathbf{F} \cdot d\mathbf{S}$$

$$\operatorname{div} \mathbf{F} = 2xy + 6yz + 18zx$$

$$\begin{aligned} \int_R \operatorname{div} \mathbf{F} &= \int_0^1 \int_0^{1-z} \int_0^{1-z} 2xy + 6yz + 18zx \, dx \, dy \, dz \\ &= \int_0^1 \int_0^{1-z} (1-z)^2 y + 6yz(1-z) + 9z(1-z)^2 \, dy \, dz \\ &= \int_0^1 (1/2)(1-z)^4 + 3z(1-z)^3 + 9z(1-z)^3 \, dz \\ &= \int_0^1 -\frac{23z^4}{2} + 34z^3 - 33z^2 + 10z + \frac{1}{2} \, dz \\ &= -\frac{23}{10} + \frac{34}{4} - 11 + 5 + \frac{1}{2} \\ &= \frac{7}{10} \end{aligned}$$

5. Use the Divergence Theorem to evaluate  $\int_S \mathbf{F} \cdot d\mathbf{S}$ , where  $\mathbf{F}(x, y, z) = \left( z^2 x, \frac{y^3}{3} + \tan z, x^2 z + y^2 \right)$  and  $S$  is the top half of the unit sphere  $x^2 + y^2 + z^2 = 1$ , oriented by the unit normal which points away from the origin.

For the Divergence Theorem to apply, the integral must be over a closed, oriented outwards. To use Divergence Theorem the surface needs to be closed, so add the disk  $S'$  on the  $xy$ -plane to the surface. Then

$$\iiint_R \operatorname{div} \mathbf{F} dV - \iint_{S'} \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot d\mathbf{S}$$

The disk can be parametrized as  $\Phi(r, \theta) = (r \cos \theta, r \sin \theta, 0)$  with  $r \in [0, 1], \theta \in [0, 2\pi]$  The normal, pointing outward needs to be downward.

$$\begin{aligned} \iint_{S'} \mathbf{F} \cdot d\mathbf{S} &= \int_0^1 \int_0^{2\pi} (*, *, (r \sin \theta)^2) \cdot (0, 0, -1) d\theta dr \\ &= - \int_0^1 \int_0^{2\pi} (r \sin \theta)^2 d\theta dr \\ &= - \int_0^1 \int_0^{2\pi} r^2 (1/2 - (\cos \theta)/2) d\theta dr \\ &= - \int_0^1 \pi r^2 dr = -\frac{\pi}{3} \end{aligned}$$

Meanwhile, the region enclosed by this, is the top half of the unit sphere, so switch to polars, restricting  $\varphi \in [0, \pi/2]$  to remain above the  $xy$ -plane.

$$\operatorname{div} \mathbf{F} = z^2 + y^2 + x^2 = \rho$$

$$\begin{aligned} \iiint_R \operatorname{div} \mathbf{F} dV &= \int_0^1 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \rho^3 \sin \varphi d\varphi d\theta d\rho \\ &= 2\pi \int_0^1 \rho^3 [-\cos \varphi]_0^{\frac{\pi}{2}} d\rho = \frac{\pi}{2} \\ \implies \iint_S \mathbf{F} \cdot d\mathbf{S} &= \frac{\pi}{2} + \frac{\pi}{3} = \frac{5\pi}{6} \end{aligned}$$

6. Let the electric field from a point source at the origin be given by  $\mathbf{E}(\mathbf{x}) = \frac{\mathbf{x}}{\|\mathbf{x}\|^3}$

- (a) What is the outward flux of  $\mathbf{E}$  across the surface  $\frac{x^2}{3} + \frac{2y^2}{5} + z^2 = 7$ .
- (b) Show that the flux of  $\mathbf{E}$  across that part of the sphere  $x^2 + y^2 + z^2 = 25$  with  $z \geq 3$  is equal to the flux across that part of the plane  $z = 3$  with  $x^2 + y^2 \leq 16$ .

7. Let  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  be given by  $f(x, y, z) = x^2 yz$  and let  $\eta$  be the 2-form on  $\mathbb{R}^3$  given by

$$\eta = (\sin x) \, dx \, dy + (e^y + xyz) \, dx \, dz + (x^2 y^2) \, dy \, dz.$$

- (a) Compute  $df$  and  $d\eta$ .
- (b) Evaluate  $df \wedge \eta$ .