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MATB42: Assignment #7 Mar 13th 2018

1. (a) Find an equation of the tangent plane to the surface S defined parametrically by $\mathbf{\Phi}(u,v) = (u^2+v,\ v,\ u+v^2)$ at the point (9,0,3).

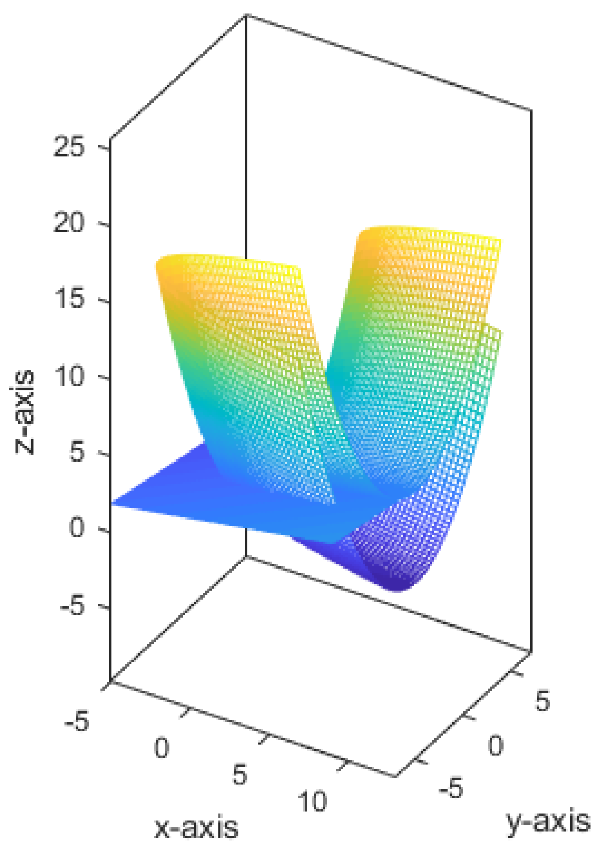
$$v = 0 \qquad\qquad u + v^2 = 3 \implies u = 3$$

$$\begin{aligned} \boldsymbol{\phi}_u &= (2(3), 0, 1) \\ \boldsymbol{\phi}_v &= (1, 1, 2(0)) \\ \boldsymbol{\phi}_u \times \boldsymbol{\phi}_v &= (-1, 1, 6) \end{aligned}$$

So the tangent plane can be given by

$$\begin{aligned} 0 &= ((x-9, y, z-3) \cdot (-1, 1, 6)) \\ 0 &= (9 - x + y + 6z - 18) \\ 9 &= -x + y + 6z \end{aligned}$$

(b) Use symbolic algebra software to sketch the surface S and its tangent plane from part (a).



2. Use a surface integral to find the area of the triangle in $\mathbb{R}^3$ with vertices (1,1,0), (1,2,1) and (3,3,2).

3. Calculate the surface area of the piece of the cone $x^2 + y^2 - z^2 = 0$ which lies inside the cylinder $x^2 + y^2 = 4$.

We can see the radius of the cylinder is 2, so the cone portion that's cut out is the part which has radius less than or equal to 2.

4. (a) Find the area of the portion of the unit sphere that is cut out by the cone $z = \sqrt{x^2 + y^2}$. (cf. page 391, #10)

   (b) Find the area of the portion of the cone $z = \sqrt{x^2 + y^2}$ that is cut out by the unit sphere.

5. Let $\mathbf{\Phi} : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a parametrization of a 2-dim surface S in $\mathbb{R}^3$.

   (a) Set

$$E = \|\boldsymbol{\phi}_u\|^2, \qquad F = \boldsymbol{\phi}_u \cdot \boldsymbol{\phi}_v, \qquad G = \|\boldsymbol{\phi}_v\|^2,$$

   Show that the surface area of S is

$$A(S) = \iint_D \sqrt{EG - F^2}\, dA$$

$$\begin{aligned}
\iint_D \sqrt{EG - F^2}\, dA &= \iint_D \sqrt{\|\boldsymbol{\phi}_u\|^2\|\boldsymbol{\phi}_v\|^2 - (\boldsymbol{\phi}_u \cdot \boldsymbol{\phi}_v)^2}\, dA \\
&= \iint_D \sqrt{(\|\boldsymbol{\phi}_u\|\|\boldsymbol{\phi}_v\|)^2 - (\|\boldsymbol{\phi}_u\|\|\boldsymbol{\phi}_v\|)^2 \cos^2\theta}\, dA \quad \text{Where } \theta \text{ is the angle between } \boldsymbol{\phi}_u \text{ and } \boldsymbol{\phi}_v. \\
&= \iint_D \sqrt{(\|\boldsymbol{\phi}_u\|\|\boldsymbol{\phi}_v\|)^2(1 - \cos^2\theta)}\, dA \\
&= \iint_D \sqrt{(\|\boldsymbol{\phi}_u\|\|\boldsymbol{\phi}_v\|)^2(\sin^2\theta)}\, dA \\
&= \iint_D \sqrt{\|\boldsymbol{\phi}_u \times \boldsymbol{\phi}_v\|^2}\, dA \\
&= \iint_D \|\boldsymbol{\phi}_u \times \boldsymbol{\phi}_v\|\, dA \\
&= \int_{\mathbf{\Phi}} 1\, dS
\end{aligned}$$

   (b) What does the formula for $A(S)$ become if the vectors $\boldsymbol{\phi}_u$ and $\boldsymbol{\phi}_v$ are orthogonal?
   If the vectors are orthogonal, then the dot product is 0, so the equation reduces to

$$A(S) = \iint_D \|\boldsymbol{\phi}_u\|\|\boldsymbol{\phi}_v\|\, dA$$

   (c) Use parts (a) and (b) to compute the surface area of a sphere of radius a.
   (cf. Marsden & Tromba, page 399, # 23.)

$$\begin{aligned}
\mathbf{\Phi}(\theta, \varphi) &= a(\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi) \\
\boldsymbol{\phi}_\theta &= a(-\sin\theta\sin\varphi, \cos\theta\sin\varphi, 0) \\
\boldsymbol{\phi}_\varphi &= a(\cos\theta\cos\varphi, \sin\theta\cos\varphi, -\sin\varphi) \\
\|\boldsymbol{\phi}_\theta\| &= a\sin\varphi, \quad \|\boldsymbol{\phi}_\varphi\| = a \\
\implies A(S) &= a^2 \int_0^{2\pi} \int_0^{\pi} \sin\varphi\, d\varphi\, d\theta \\
&= a^2 \int_0^{2\pi} \Big[-\cos\varphi\Big]_0^{\pi}\, d\varphi\, d\theta \\
&= a^2 \int_0^{2\pi} -(-1-1)\, d\varphi\, d\theta \\
&= a^2 2 \int_0^{2\pi} 1\, d\varphi\, d\theta \\
&= 4\pi a^2
\end{aligned}$$

6. For each of the following surfaces S, sketch S (using symbolic software) and evaluate the surface integral $\int_S f\,dS$, where $f(x, y, z) = x$.

   (a) S is that part of the surface $y = 4 - x^2$ between $z = 0$ and $z = 1$, with $y \geq 0$.

$$y \geq 0 \implies 4 - x^2 \geq 0 \implies x^2 \leq 4 \implies |x| < 2$$

$$\begin{aligned} \mathbf{\Phi}(x, z) &= (x, 4 - x^2, z) \\ \boldsymbol{\phi}_x &= (1, -2x, 0),\ \boldsymbol{\phi}_z = (0, 0, 1) \\ \boldsymbol{\phi}_x \times \boldsymbol{\phi}_z &= (-2x, -1, 0) \implies \|\boldsymbol{\phi}_x \times \boldsymbol{\phi}_z\| = \sqrt{4x^2 + 1} \\ \int_S f dS &= \int_0^1 \int_{-2}^2 x\sqrt{4x^2 + 1}\,dx\,dz \end{aligned}$$

   The integrand is odd since x odd and $\sqrt{4x^2 + 1}$ even, so the integral over x is 0, making the entire integral 0.

   (b) S is the upper half of the unit sphere centered at the origin.
   Only the upper half so $0 \leq \theta \leq 2\pi$ and $0 \leq \varphi \leq \pi/2$.

$$\begin{aligned} \mathbf{\Phi}(\theta, \varphi) &= (\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi) \\ \boldsymbol{\phi}_\theta &= (-\sin\theta\sin\varphi, \cos\theta\sin\varphi, 0) \\ \boldsymbol{\phi}_\varphi &= (\cos\theta\cos\varphi, \sin\theta\cos\varphi, -\sin\varphi) \\ \boldsymbol{\phi}_\theta \times \boldsymbol{\phi}_\varphi &= (-\cos\theta\sin^2\varphi, -\sin\theta\sin^2\varphi, -\sin^2\theta\sin\varphi\cos\varphi - \cos^2\theta\sin\varphi\cos\varphi) \\ &= (-\cos\theta\sin^2\varphi, -\sin\theta\sin^2\varphi, -\sin\varphi\cos\varphi) \\ \|\boldsymbol{\phi}_\theta \times \boldsymbol{\phi}_\varphi\| &= \sqrt{\cos^2\theta\sin^4\varphi + \sin^2\theta\sin^4\varphi + \sin^2\varphi\cos^2\varphi} \\ &= \sqrt{\sin^4\varphi + \sin^2\varphi\cos^2\varphi} \\ &= \sqrt{\sin^2\varphi} = \sin\varphi \\ \int_{\mathbf{\Phi}} f\,dS &= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \cos\theta\sin^2\varphi\,d\theta\,d\varphi = 0 \end{aligned}$$

   The integral is zero again since integrating $\cos\theta$ over a whole period is 0.

   (c) S is that part of the surface $x = \sin y$ with $0 \leq y \leq \pi$ and $0 \leq z \leq 2$.

$$\begin{aligned} \mathbf{\Phi}(y, z) &= (\sin y, y, z) \\ \boldsymbol{\phi}_y &= (\cos y, 1, 0) \\ \boldsymbol{\phi}_z &= (0, 0, 1) \\ \boldsymbol{\phi}_y \times \boldsymbol{\phi}_z &= (1, -\cos y, 0) \\ \|\boldsymbol{\phi}_y \times \boldsymbol{\phi}_z\| &= \sqrt{1 + \cos^2 y} \\ \int_{\mathbf{\Phi}} f\,dS &= \int_0^2 \int_0^\pi \sin y\sqrt{1 + \cos^2 y}\,dy\,dz \end{aligned}$$

7. Find the mass of the metallic surface S given by $z = 1 - \dfrac{x^2 + y^2}{2}$ with $0 \leq x \leq 1$, $0 \leq y \leq 1$, if the mass density at $(x, y, z) \in S$ is given by $m(x, y, z) = xy$.