

MATB42: Assignment #7

- (a) Find an equation of the tangent plane to the surface S defined parametrically by $\Phi(u, v) = (u^2 + v, v, u + v^2)$ at the point $(9, 0, 3)$.

$$v = 0 \qquad u + v^2 = 3 \implies u = 3$$

$$\phi_u = (2(3), 0, 1)$$

$$\phi_v = (1, 1, 2(0))$$

$$\phi_u \times \phi_v = (-1, 1, 6)$$

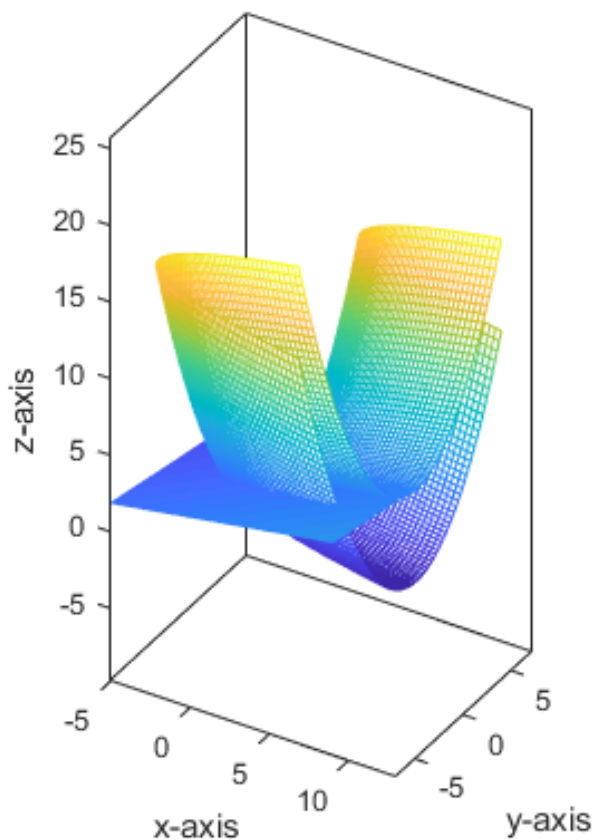
So the tangent plane can be given by

$$0 = ((x - 9, y, z - 3) \cdot (-1, 1, 6))$$

$$0 = (9 - x + y + 6z - 18)$$

$$9 = -x + y + 6z$$

- (b) Use symbolic algebra software to sketch the surface S and its tangent plane from part (a).



- Use a surface integral to find the area of the triangle in \mathbb{R}^3 with vertices $(1, 1, 0)$, $(1, 2, 1)$ and $(3, 3, 2)$.
- Calculate the surface area of the piece of the cone $x^2 + y^2 - z^2 = 0$ which lies inside the cylinder $x^2 + y^2 = 4$.

We can see the radius of the cylinder is 2, so the cone portion that's cut out is the part which has radius less than or equal to 2 $\implies 0 \leq z \leq 2$. Using polar for the cone, $0 \leq \theta \leq 2\pi$.

$$\begin{aligned}
\Phi(\theta, z) &= (z \cos \theta, z \sin \theta, z) \\
\phi_\theta &= (-z \sin \theta, z \cos \theta, 0) \\
\phi_z &= (\cos \theta, \sin \theta, 1) \\
\phi_\theta \times \phi_z &= (z \cos \theta, z \sin \theta, -z \sin^2 \theta - z \cos^2 \theta) \\
&= (z \cos \theta, z \sin \theta, -z) \\
\|\phi_\theta \times \phi_z\| &= z^2 \cos^2 \theta + z^2 \sin^2 \theta + z^2 = 2z^2
\end{aligned}
\qquad
\begin{aligned}
\int_{\Phi} f \, dS &= \int_0^{2\pi} \int_0^2 2z^2 \, dz \, d\theta \\
&= \int_0^{2\pi} \left[\frac{2}{3} z^3 \right]_0^2 d\theta \\
&= \int_0^{2\pi} \frac{16}{3} d\theta = \frac{32\pi}{3}
\end{aligned}$$

4. (a) Find the area of the portion of the unit sphere that is cut out by the cone $z = \sqrt{x^2 + y^2}$.
(cf. page 391, #10)

$$\begin{aligned}
\Phi_{\text{sphere}}(\theta, \varphi) &= (\cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi) \\
\phi_\theta &= (-\sin \theta \sin \varphi, \cos \theta \sin \varphi, 0) \\
\phi_\varphi &= (\cos \theta \cos \varphi, \sin \theta \cos \varphi, -\sin \varphi) \\
\phi_\theta \times \phi_\varphi &= (-\cos \theta \sin^2 \varphi, -\sin \theta \sin^2 \varphi, -\sin^2 \theta \sin \varphi \cos \varphi - \cos^2 \theta \sin \varphi \cos \varphi) \\
&= (-\cos \theta \sin^2 \varphi, -\sin \theta \sin^2 \varphi, -\sin \varphi \cos \varphi) \\
\|\phi_\theta \times \phi_\varphi\| &= \sqrt{\cos^2 \theta \sin^4 \varphi + \sin^2 \theta \sin^4 \varphi + \sin^2 \varphi \cos^2 \varphi} \\
&= \sqrt{\sin^2 \varphi} = \sin \varphi
\end{aligned}$$

$$\begin{aligned}
\Phi_{\text{cone}}(\theta, z) &= (z \cos \theta, z \sin \theta, z) \\
\phi_z &= (\cos \theta, \sin \theta, 1) \\
\phi_\theta &= (-z \sin \theta, z \cos \theta, 0) \\
\phi_z \times \phi_\theta &= (-z \cos \theta, -z \sin \theta, z) \\
\|\phi_z \times \phi_\theta\| &= 2z^2
\end{aligned}$$

For the unit sphere $x^2 + y^2 + z^2 = 1$, but the cone is $x^2 + y^2 = z^2 \implies$ sub z into sphere gives $2x^2 + 2y^2 = 1$ So the exact intersection of the surfaces is a circle of radius $2/\sqrt{2}$ centered at the origin. so the surface cut out is the section of the top of the sphere where $z \geq 2\sqrt{2} \implies \varphi \leq \frac{\pi}{4}$ from the $z = \cos \varphi$ portion of the parametrization. So the ranges are $0 \leq \theta \leq 2\pi$, $0 \leq \varphi \leq \frac{\pi}{4}$. The area is therefore

$$\begin{aligned}
\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \sin \varphi \, d\varphi \, d\theta &= \int_0^{2\pi} \left[-\cos \varphi \right]_0^{\frac{\pi}{4}} d\theta \\
&= 2\pi \left[-\left(\cos\left(\frac{\pi}{4}\right)\right) - (-\cos(0)) \right] \\
&= 2\pi \left[-\frac{\sqrt{2}}{2} + 1 \right] \\
&= \pi(2 - \sqrt{2})
\end{aligned}$$

- (b) Find the area of the portion of the cone $z = \sqrt{x^2 + y^2}$ that is cut out by the unit sphere.

Plugging in $x^2 + y^2 = 1/2$ to the cone equation again gives $z^2 = 1/2 \implies z = \pm \frac{\sqrt{2}}{2}$ but $z \geq 0$ by the cone definition so $0 \leq z \leq \frac{\sqrt{2}}{2}$.

$$\begin{aligned} A(\Phi_{\text{cone}}) &= \int_0^{2\pi} \int_0^{\frac{1}{4}} 2z^2 dz d\theta \\ &= \int_0^{2\pi} \frac{2}{3} \cdot \frac{1}{4^3} d\theta \end{aligned}$$

5. Let $\Phi : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be a parametrization of a 2-dim surface S in \mathbb{R}^3 .

(a) Set

$$E = \|\phi_u\|^2, \quad F = \phi_u \cdot \phi_v, \quad G = \|\phi_v\|^2,$$

Show that the surface area of S is

$$A(S) = \iint_D \sqrt{EG - F^2} dA$$

$$\begin{aligned} \iint_D \sqrt{EG - F^2} dA &= \iint_D \sqrt{\|\phi_u\|^2 \|\phi_v\|^2 - (\phi_u \cdot \phi_v)^2} dA \\ &= \iint_D \sqrt{(\|\phi_u\| \|\phi_v\|)^2 - (\|\phi_u\| \|\phi_v\|)^2 \cos^2 \theta} dA \quad \text{Where } \theta \text{ is the angle between } \phi_u \text{ and } \phi_v. \\ &= \iint_D \sqrt{(\|\phi_u\| \|\phi_v\|)^2 (1 - \cos^2 \theta)} dA \\ &= \iint_D \sqrt{(\|\phi_u\| \|\phi_v\|)^2 (\sin^2 \theta)} dA \\ &= \iint_D \sqrt{\|\phi_u \times \phi_v\|^2} dA \\ &= \iint_D \|\phi_u \times \phi_v\| dA \\ &= \int_{\Phi} 1 dS \end{aligned}$$

(b) What does the formula for $A(S)$ become if the vectors ϕ_u and ϕ_v are orthogonal?
If the vectors are orthogonal, then the dot product is 0, so the equation reduces to

$$A(S) = \iint_D \|\phi_u\| \|\phi_v\| dA$$

(c) Use parts (a) and (b) to compute the surface area of a sphere of radius a .

(cf. Marsden & Tromba, page 399, # 23.)

$$\begin{aligned}
\Phi(\theta, \varphi) &= a(\cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi) \\
\phi_\theta &= a(-\sin \theta \sin \varphi, \cos \theta \sin \varphi, 0) \\
\phi_\varphi &= a(\cos \theta \cos \varphi, \sin \theta \cos \varphi, -\sin \varphi) \\
\|\phi_\theta\| &= a \sin \varphi, \quad \|\phi_\varphi\| = a \\
\Rightarrow A(S) &= a^2 \int_0^{2\pi} \int_0^\pi \sin \varphi \, d\varphi \, d\theta \\
&= a^2 \int_0^{2\pi} \left[-\cos \varphi \right]_0^\pi d\theta \\
&= a^2 \int_0^{2\pi} -(-1 - 1) \, d\theta \\
&= a^2 2 \int_0^{2\pi} 1 \, d\theta \\
&= 4\pi a^2
\end{aligned}$$

6. For each of the following surfaces S , sketch S (using symbolic software) and evaluate the surface integral $\int_S f \, dS$, where $f(x, y, z) = x$.

- (a) S is that part of the surface $y = 4 - x^2$ between $z = 0$ and $z = 1$, with $y \geq 0$.

$$y \geq 0 \implies 4 - x^2 \geq 0 \implies x^2 \leq 4 \implies |x| < 2$$

$$\begin{aligned}
\Phi(x, z) &= (x, 4 - x^2, z) \\
\phi_x &= (1, -2x, 0), \quad \phi_z = (0, 0, 1) \\
\phi_x \times \phi_z &= (-2x, -1, 0) \implies \|\phi_x \times \phi_z\| = \sqrt{4x^2 + 1} \\
\int_S f \, dS &= \int_0^1 \int_{-2}^2 x \sqrt{4x^2 + 1} \, dx \, dz
\end{aligned}$$

The integrand is odd since x odd and $\sqrt{4x^2 + 1}$ even, so the integral over x is 0, making the entire integral 0.

- (b) S is the upper half of the unit sphere centered at the origin.

Only the upper half so $0 \leq \theta \leq 2\pi$ and $0 \leq \varphi \leq \pi/2$.

$$\begin{aligned}
\Phi(\theta, \varphi) &= (\cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi) \\
\phi_\theta &= (-\sin \theta \sin \varphi, \cos \theta \sin \varphi, 0) \\
\phi_\varphi &= (\cos \theta \cos \varphi, \sin \theta \cos \varphi, -\sin \varphi) \\
\phi_\theta \times \phi_\varphi &= (-\cos \theta \sin^2 \varphi, -\sin \theta \sin^2 \varphi, -\sin^2 \theta \sin \varphi \cos \varphi - \cos^2 \theta \sin \varphi \cos \varphi) \\
&= (-\cos \theta \sin^2 \varphi, -\sin \theta \sin^2 \varphi, -\sin \varphi \cos \varphi) \\
\|\phi_\theta \times \phi_\varphi\| &= \sqrt{\cos^2 \theta \sin^4 \varphi + \sin^2 \theta \sin^4 \varphi + \sin^2 \varphi \cos^2 \varphi} \\
&= \sqrt{\sin^4 \varphi + \sin^2 \varphi \cos^2 \varphi} \\
&= \sqrt{\sin^2 \varphi} = \sin \varphi \\
\int_\Phi f \, dS &= \int_0^{\pi/2} \int_0^{2\pi} \cos \theta \sin^2 \varphi \, d\theta \, d\varphi = 0
\end{aligned}$$

The integral is zero again since integrating $\cos \theta$ over a whole period is 0.

(c) S is that part of the surface $x = \sin y$ with $0 \leq y \leq \pi$ and $0 \leq z \leq 2$.

$$\begin{aligned}\Phi(y, z) &= (\sin y, y, z) \\ \phi_y &= (\cos y, 1, 0) \\ \phi_z &= (0, 0, 1) \\ \phi_y \times \phi_z &= (1, -\cos y, 0) \\ \|\phi_y \times \phi_z\| &= \sqrt{1 + \cos^2 y} \\ \int_{\Phi} f \, dS &= \int_0^2 \int_0^\pi \sin y \sqrt{1 + \cos^2 y} \, dy \, dz\end{aligned}$$

7. Find the mass of the metallic surface S given by $z = 1 - \frac{x^2 + y^2}{2}$ with $0 \leq x \leq 1$, $0 \leq y \leq 1$, if the mass density at $(x, y, z) \in S$ is given by $m(x, y, z) = xy$.