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MATB42: Assignment #3
Bonus
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1. A particle moving on the curve $\boldsymbol{\gamma}(t) = (3t^2, -\sin t, -e^t)$ is released at time $t = \frac{1}{2}$ and flies off on a tangent. What are its coordinates at time $t = 1$.

   Calculate the direction of the tangent.

$$\boldsymbol{\gamma}'(t) = (3(2t), -\cos(t), -e^t) = (6t, -\cos(t), -e^t)$$
$$\boldsymbol{\gamma}'\left(\frac{1}{2}\right) = \left(3, -\cos\left(\frac{1}{2}\right), -\sqrt{e}\right)$$

   Calculate the position of the particle at $t = \frac{1}{2}$.

$$\boldsymbol{\gamma}\left(\frac{1}{2}\right) = \left(3\left(\frac{1}{4}\right), -\sin\left(\frac{1}{2}\right), -\sqrt{e}\right) = \left(\left(\frac{3}{4}\right), -\sin\left(\frac{1}{2}\right), -\sqrt{e}\right)$$

   This means the particle should be at position $\boldsymbol{\gamma}\left(\frac{1}{2}\right) + \frac{1}{2}\boldsymbol{\gamma}'\left(\frac{1}{2}\right)$ which is equal to:

$$\left(\left(\frac{3}{4}\right), -\sin\left(\frac{1}{2}\right), -\sqrt{e}\right) + \frac{1}{2}\left(3, -\cos\left(\frac{1}{2}\right), -\sqrt{e}\right)$$
$$= \left(\frac{9}{4}, -\frac{2\sin(\frac{1}{2}) + \cos(\frac{1}{2})}{2}, -\frac{3\sqrt{e}}{2}\right)$$

2. The displacement at time $t$ and horizontal position on a line $x$ of a certain violin string is given by $u = \sin(x - 6t) + \sin(x + 6t)$. Calculate the velocity of the string at $x = 1$ when $t = \frac{1}{3}$.

   At time $t = \frac{1}{3}$, the displacement function is given by

$$u = \sin(x - 3) + \sin(x + 3)$$

   Taking the rate of change we get

$$\frac{\partial u}{\partial x} = \cos(x - 3) + \cos(x + 3)$$

   So the velocity is $\cos(-2) + \cos(4)$

3. Find paths $\gamma(t)$ which represent

   (a) the straight line segment in $\mathbb{R}$ from (1,2,3) to (4,-5,6).
   We want $\gamma_1$ to range over 1 to 4, so we can take $\gamma_1(t) = t + 1$ over $0 \leq t \leq 3$
   We want $\gamma_2$ to range over 2 to -5, so we can take $\gamma_1(t) = -\frac{7t}{3} + 2$ over $0 \leq t \leq 3$ to get $-7(3)/3 + 2 = -7 + 2 = -5$.
   We want $\gamma_3$ to range over 3 to 6, so we can take $\gamma_1(t) = t + 3$ over $0 \leq t \leq 3$

$$\gamma(t) = \left(t + 1, 2 - \frac{7t}{3}, t + 3\right),\ 0 \leq t \leq 3$$

   (b) the curve $\{(x, y) \in \mathbb{R}^2 | 3x^2 + 25y^2 = 4\}$
   This is an ellipse, so $x, y$ should be sin cos, exploiting the fact that $\sin^2\alpha + \cos^2\alpha = 1$ take the coefficients of the trig functions so that they cancel the existing ones. $\frac{1}{\sqrt{3}}$ and $\frac{1}{5}$ respectively. But this means the equation equals to 1, to make it 4, multiply each $x, y$ by $\sqrt{4}$ so when it is squared, a factor of four comes out.

$$\gamma(t) = \left(\frac{2}{\sqrt{3}}\sin(t), \frac{2}{5}\cos(t)\right),\ 0 \leq t \leq 2\pi$$

   (c) the curve of intersection of the cone $z = \sqrt{x^2 + y^2}$ and the sphere $x^2 + y^2 + z^2 = 2x$.
   To actually get the intersection, equate the two parameterizations plugging z of the cone into the sphere.

$$\begin{aligned} x^2 + y^2 + x^2 + y^2 &= 2x \\ 2x^2 - 2x + 2y^2 &= 0 \\ 2(x^2 - x + \frac{1}{4}) - \frac{1}{2} + 2y^2 &= 0 \\ 2(x^2 - x + \frac{1}{4}) + 2y^2 &= \frac{1}{2} \\ 2(x - \frac{1}{2})^2 + 2y^2 &= \frac{1}{2} \end{aligned}$$

   From this ellipse equation, looking at the shift of the center of the ellipse and the constant factors, $x, y$ should be set to $\frac{\sin(t)+1}{2}, \frac{\cos(t)}{2}$. Now to figure out $z$, just fit this into one of the earlier equations.

$$\begin{aligned} \left(\frac{\sin(t) + 1}{2}\right)^2 + \left(\frac{\cos(t)}{2}\right)^2 + z^2 &= \sin(t) + 1 \\ \frac{\sin(t)^2 + 2\sin(t) + 1}{4} + \frac{\cos^2(t)}{4} + z^2 &= \sin(t) + 1 \\ \frac{1}{4} + \frac{1}{4} + \frac{\sin(t)}{2} & \end{aligned}$$

$$\gamma(t) = \left(\frac{\sin(t) + 1}{2}, \frac{\cos(t)}{2}, \frac{\sin(t) + 1}{2}\right),\ 0 \leq t \leq 2\pi$$

4. Find the arclength of the following curves.

(a) $\boldsymbol{\gamma}(t) = \left(t, \frac{1}{\sqrt{2}}t^2, \frac{1}{3}t^3\right)$ between the origin and $\left(2, 2\sqrt{2}, \frac{8}{3}\right)$.

$$\begin{aligned}\boldsymbol{\gamma}'(t) &= \left(1, \sqrt{2}t, t^2\right)\\ \|\boldsymbol{\gamma}'(t)\| &= \sqrt{1+2t^2+t^4}\\ &= \sqrt{(t^2+1)^2}\\ &= (t^2+1)\end{aligned}$$

$$\begin{aligned}\int_{\boldsymbol{\gamma}} ds &= \int_0^2 \|\boldsymbol{\gamma}'(t)\|dt\\ &= \int_0^2 t^2 dt + \int_0^2 1dt\\ &= \frac{8}{3}+2\\ &= 4+\frac{2}{3}\end{aligned}$$

(b) $\boldsymbol{\gamma}(t) = (\cos 3t, \sin 3t, 2t^{\frac{3}{2}})$, $0 \le t \le 2$.

$$\begin{aligned}\boldsymbol{\gamma}'(t) &= (-3\sin 3t, 3\cos 3t, 3t^{\frac{1}{2}})\\ \|\boldsymbol{\gamma}'(t)\| &= 3\sqrt{\sin^2 3t+\cos^2 3t+t}\\ &= 3\sqrt{t+1}\end{aligned}$$

$$\begin{aligned}\int_{\boldsymbol{\gamma}} ds &= \int_0^2 \|\boldsymbol{\gamma}'(t)\|dt\\ &= 3\int_0^2 \sqrt{t+1}dt\\ &\text{Let } u = t+1,\ du = dt\\ &= 3\int_1^3 \sqrt{u}du\end{aligned}$$

$$\begin{aligned}&= 2\left[u^{\frac{3}{2}}\right]_1^3\\ &= 2\sqrt{27}-2\\ &= 6\sqrt{3}-2\end{aligned}$$

(c) $\boldsymbol{\gamma}(t) = (t^2, \sin t - t\cos t, \cos t + t\sin t)$, $0 \le t \le \pi$

$$\begin{aligned}\boldsymbol{\gamma}'(t) &= (2t, \cos t - \cos t + t\sin t, -\sin t + \sin t + t\cos t)\\ &= (2t, t\sin t, t\cos t)\\ \|\boldsymbol{\gamma}'(t)\| &= t\sqrt{4+\sin^2 t+\cos^2 t}\\ &= \sqrt{5}t\end{aligned}$$

$$\begin{aligned}\int_{\boldsymbol{\gamma}} ds &= \int_0^\pi \|\boldsymbol{\gamma}'(t)\|dt\\ &= \sqrt{5}\int_0^\pi tdt\\ &= \frac{\sqrt{5}\pi^2}{2}\end{aligned}$$

5. Evaluate the path integral $\int_\gamma f(x,y,z)ds$, where

(a) $f(x,y,z) = x^2 - y + 3z$ and $\gamma$ is a parameterization of the line segment from the origin to (1,2,1).

$$\begin{aligned}\gamma(t) &= (t, 2t, t) \text{ from } 0 \le t \le 1\\ \gamma'(t) &= (1,2,1)\\ \|\gamma'(t)\| &= \sqrt{6}\end{aligned}$$

$$\begin{aligned}\int_\gamma f(x,y,z)ds &= \int_0^1 f(\gamma(t))\|\gamma'(t)\|dt\\ &= \sqrt{6}\int_0^1 t^2 - 2t + 3tdt = \sqrt{6}\int_0^1 t^2 + tdt\\ &= \sqrt{6}\Big(\frac{1}{3} + \frac{1}{2}\Big)\\ &= \frac{5}{\sqrt{6}}\end{aligned}$$

(b) $f(x,y,z) = xyz$ and $\gamma(t) = (-\sin 2t, \sqrt{2}\cos 2t, \sin 2t)$, $0 \le t \le \frac{\pi}{4}$

$$\begin{aligned}\gamma'(t) &= (-2\cos 2t, -2\sqrt{2}\sin 2t, 2\cos 2t)\\ \|\gamma'(t)\| &= \sqrt{4\cos^2 2t + 8\sin^2 2t + 4\cos^2 2t}\\ &= \sqrt{8} = 2\sqrt{2}\end{aligned}$$

$$\begin{aligned}\int_\gamma f(x,y,z)ds &= \int_0^{\frac{\pi}{4}} f(\gamma(t))\|\gamma'(t)\|dt\\ &= -2\sqrt{2}\sqrt{2}\int_0^{\frac{\pi}{4}} \sin^2 2t\cos 2tdt =\\ &\text{Let } u = \sin(2t),\ du = 2\cos(2t)dt\\ &= -2\int_0^{\sin(\frac{\pi}{4})} u^2dt\end{aligned}$$

$$\begin{aligned}&= -\frac{2}{3}\Big[\sin(2t)^3\Big]_0^{\frac{\pi}{4}}\\ &= -\frac{2}{3}\sin\Big(\frac{\pi}{2}\Big)^3\\ &= -\frac{2}{3}\end{aligned}$$

(c) $f(x,y,z) = xyz$ and $\gamma(t) = (t, 2t, 3t)$, $0 \le t \le 2$.

$$\begin{aligned}\gamma'(t) &= (1,2,3)\\ \|\gamma'(t)\| &= \sqrt{1+4+9}\\ &= \sqrt{14}\end{aligned}$$

$$\begin{aligned}\int_\gamma f(x,y,z)ds &= \int_0^2 f(\gamma(t))\|\gamma'(t)\|dt\\ &= 6\sqrt{14}\int_0^2 t^3dt\\ &= \frac{6}{4}\sqrt{14}(2^4)\\ &= 24\sqrt{14}\end{aligned}$$

(d) $f(x,y,z) = 3x + xy + z^3$ and $\gamma(t) = (\cos 4t, \sin 4t, 3t)$, $0 \le t \le 2\pi$.

$$\begin{aligned}\gamma'(t) &= (-4\sin 4t, 4\cos 4t, 3)\\ \|\gamma'(t)\| &= \sqrt{16(\sin^2 4t + \cos^2 4t) + 9}\\ &= \sqrt{25}\\ &= 5\end{aligned}$$

$$\begin{aligned}
\int_{\boldsymbol{\gamma}} f(x,y,z)ds &= \int_0^{2\pi} f(\boldsymbol{\gamma}(t))\|\boldsymbol{\gamma}'(t)\|dt \\
&= 5\int_0^{2\pi} 3\cos 4t + \sin 4t\cos 4t + 27t^3 dt \\
&= 5\int_0^{2\pi} (3+\sin 4t)\cos 4t dt \\
&+ 5\int_0^{2\pi} +27t^3 dt
\end{aligned}$$

Let $u = 3+\sin(4t),\ du = 4\cos(4t)dt$

$$\begin{aligned}
&= \frac{5}{4}\int_3^{3+\sin(8\pi)} (3+u)du + \frac{5(27)}{4}\Big[t^4\Big]_0^{2\pi} \\
&= \frac{5}{4}\int_3^{3} (3+u)du + \frac{5(27)}{4}\Big[t^4\Big]_0^{2\pi} \\
&= 20(27)\pi^4 \\
&= 540\pi^4
\end{aligned}$$

(e) $f(x,y,z) = z\cos y$ and $\boldsymbol{\gamma}(t) = (-2t, 3t, t),\ 0 \le t \le 2$.

$$\begin{aligned}
\boldsymbol{\gamma}'(t) &= (-2,3,1) \\
\|\boldsymbol{\gamma}'(t)\| &= \sqrt{4+9+1} \\
&= \sqrt{14}
\end{aligned}$$

$$\begin{aligned}
\int_{\boldsymbol{\gamma}} f(x,y,z)ds &= \int_0^{2} f(\boldsymbol{\gamma}(t))\|\boldsymbol{\gamma}'(t)\|dt \\
&= \sqrt{14}\int_0^2 t\cos 3t dt
\end{aligned}$$

Let $u = t,\ du = dt,\ dv = \cos 3t,\ v = \dfrac{\sin 3t}{3}$

$$\begin{aligned}
&= \sqrt{14}\Big(\Big[\frac{t\sin 3t}{3}\Big]_0^2 - \int_0^2 \frac{\sin 3t}{3}dt\Big) \\
&= \frac{\sqrt{14}}{3}\Big(2\sin 6 + \frac{1}{3}\Big[\cos 3t\Big]_0^2\Big) \\
&= \frac{\sqrt{14}}{3}\Big(2\sin 6 + \frac{1}{3}\Big[\cos 6 - 1\Big]\Big) \\
&= \frac{2\sqrt{14}}{3}\sin 6 + \frac{\sqrt{14}}{9}\cos 6 - \frac{\sqrt{14}}{9}
\end{aligned}$$

(f) $f(x,y,z) = x + z^2$ and $\boldsymbol{\gamma}(t) = (t, \ln t, 2\sqrt{2t}),\ 1 \le t \le 2$.

$$\begin{aligned}
\boldsymbol{\gamma}'(t) &= \Big(1, \frac{1}{t}, \sqrt{\frac{2}{t}}\Big) \\
\|\boldsymbol{\gamma}'(t)\| &= \sqrt{1 + \frac{2}{t} + \frac{1}{t^2}} \\
\int_{\boldsymbol{\gamma}} f(x,y,z)ds &= \int_1^{2} f(\boldsymbol{\gamma}(t))\|\boldsymbol{\gamma}'(t)\|dt \\
&= \int_1^2 (t+8t)\sqrt{1+\frac{2}{t}+\frac{1}{t^2}}dt \\
&= \int_1^2 9t\sqrt{1+\frac{2}{t}+\frac{1}{t^2}}dt \\
&= 9\int_1^2 \sqrt{t^2+2t+1}dt
\end{aligned}$$

$$\begin{aligned}
&= 9\int_1^2 (t+1)dt \\
&= 9\Big(\Big[\frac{t^2}{2}\Big]_1^2 + \Big[1\Big]_1^2\Big) \\
&= 9\Big(\frac{3}{2}+1\Big) \\
&= 9\Big(\frac{5}{2}\Big) \\
&= \frac{45}{2}
\end{aligned}$$

6. Find the average value of the following functions over the helix $\boldsymbol{\gamma}(t) = (\sin t, \cos t, t)$, $0 \le t \le 2\pi$.

$$\begin{aligned}\boldsymbol{\gamma}'(t) &= (\cos t, -\sin t, 1)\\ \|\boldsymbol{\gamma}'(t)\| &= \sqrt{\sin^2 t + \cos^2 t + 1}\\ &= \sqrt{2}\end{aligned}$$

The length of the curve $\boldsymbol{\gamma}$ is:

$$\begin{aligned}\int_{\boldsymbol{\gamma}} ds &= \int_0^{2\pi} \|\boldsymbol{\gamma}'(t)\| dt\\ &= \sqrt{2}\int_0^{2\pi} 1 dt\\ &= 2\sqrt{2}\pi\end{aligned}$$

So the average value of the function should just be: $\dfrac{\text{path integral}}{\text{length of } \boldsymbol{\gamma}}$

(a) $f(x,y,z) = y \sin z$

$$\begin{aligned}\frac{1}{\int_0^{2\pi} \|\boldsymbol{\gamma}'(t)\| dt}\int_{\boldsymbol{\gamma}} f(x,y,z) ds &= \frac{1}{2\sqrt{2}\pi}\int_0^{2\pi} f(\boldsymbol{\gamma}(t))\|\boldsymbol{\gamma}'(t)\| dt\\ &= \frac{1}{2\sqrt{2}\pi}\int_0^{2\pi} \cos t \sin t dt\\ &\text{Let } u = \sin(t),\ du = \cos(t)dt\\ &= \frac{1}{2\sqrt{2}\pi}\int_0^{0} u du\\ &= 0\end{aligned}$$

(b) $f(x,y,z) = x^3 + \cos^2 z$

$$\begin{aligned}\frac{1}{\int_0^{2\pi} \|\boldsymbol{\gamma}'(t)\| dt}\int_{\boldsymbol{\gamma}} f(x,y,z) ds &= \frac{1}{2\sqrt{2}\pi}\int_0^{2\pi} f(\boldsymbol{\gamma}(t))\|\boldsymbol{\gamma}'(t)\| dt\\ &= \frac{1}{2\sqrt{2}\pi}\int_0^{2\pi} \sin^3 t + \cos^2 t dt\\ &= \frac{1}{2\sqrt{2}\pi}\Big(\int_0^{2\pi} (\sin t)\frac{1-\cos^2 t}{2} dt + \int_0^{2\pi} \frac{1+\cos 2t}{2} dt\Big)\\ &\text{Let } u = \cos(t),\ du = -\sin(t)dt\\ &= \frac{1}{2\sqrt{2}\pi}\Big(-\int_1^{1} \frac{1-u^2}{2} du + \int_0^{2\pi} \frac{1+\cos 2t}{2} dt\Big)\\ &= \frac{1}{4\sqrt{2}\pi}\Big(\int_0^{2\pi} 1 + \cos 2t dt\Big)\\ &= \frac{1}{4\sqrt{2}\pi}\Big(2\pi + \Big[\frac{\sin 2t}{2}\Big]_0^{2\pi}\Big)\\ &= \frac{1}{2\sqrt{2}}\end{aligned}$$