MATB42: Assignment #7

1. (a) Find an equation of the tangent plane to the surface S defined parametrically by $\Phi(u,v) = (u^2 + v, v, u + v^2)$ at the point (9,0,3).

$$v = 0 \qquad \qquad u + v^2 = 3 \implies u = 3$$

$$\phi_u = (2(3), 0, 1)$$

$$\phi_v = (1, 1, 2(0))$$

$$\phi_u \times \phi_v = (-1, 1, 6)$$

So the tangent plane can be given by

$$0 = ((x - 9, y, z - 3) \cdot (-1, 1, 6))$$

$$0 = (9 - x + y + 6z - 18)$$

$$9 = -x + y + 6z$$

(b) Use symbolic algebra software to sketch the surface S and its tangent plane from part (a).



- 2. Use a surface integral to find the area of the triangle in \mathbb{R}^3 with vertices (1,1,0), (1,2,1) and (3,3,2).
- 3. Calculate the surface area of the piece of the cone $x^2 + y^2 z^2 = 0$ which lies inside the cylinder $x^2 + y^2 = 4$.

We can see the radius of the cylinder is 2, so the cone portion that's cut out is the part which has radius less than or equal to $2 \implies 0 \le z \le 2$. Using polar for the cone, $0 \le \theta \le 2\pi$.

$$\begin{aligned} \Phi(\theta, z) &= (z \cos \theta, z \sin \theta, z) \\ \phi_{\theta} &= (-z \sin \theta, z \cos \theta, 0) \end{aligned} \qquad \int_{\Phi} f \, dS = \int_{0}^{2\pi} \int_{0}^{2} 2z^{2} \, dz \, d\theta \\ \phi_{z} &= (\cos \theta, \sin \theta, 1) \end{aligned} \qquad = \int_{0}^{2\pi} \left[\frac{2}{3} z^{3} \right]_{0}^{2} d\theta \\ &= (z \cos \theta, z \sin \theta, -z \sin^{2} \theta - z \cos^{2} \theta) \end{aligned} \qquad = \int_{0}^{2\pi} \frac{16}{3} d\theta = \frac{32\pi}{3} \end{aligned}$$

4. (a) Find the area of the portion of the unit sphere that is cut out by the cone $z = \sqrt{x^2 + y^2}$. (cf. page 391, #10)

$$\Phi_{\text{sphere}}(\theta, \varphi) = (\cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi)$$
$$\Phi_{\text{cone}}(\theta, z) = (z \cos \theta, z \sin \theta, z)$$

For the unit sphere $x^2 + y^2 + z^2 = 1$, but the cone is $x^2 + y^2 = z^2 \implies$ sub z into sphere gives $2x^2 + 2y^2 = 1$

- (b) Find the area of the portion of the cone $z=\sqrt{x^2+y^2}$ that is cut out by the unit sphere. So the exact intersection of the surfaces is a circle of radius 1/4 centered at the origin. Plugging in $x^2+y^2=1/2$ to the equation again gives $z^2=1/2 \implies z=\frac{1}{4}$
- 5. Let $\Phi: D \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a parametrization of a 2-dim surface S in \mathbb{R}^3 .
 - (a) Set

$$E = \|\phi_u\|^2,$$
 $F = \phi_u \cdot \phi_v,$ $G = \|\phi_v\|^2,$

Show that the surface area of S is

$$A(S) = \iint_D \sqrt{EG - F^2} \, dA$$

$$\begin{split} \iint_D \sqrt{EG - F^2} \, dA &= \iint_D \sqrt{\|\phi_u\|^2 \|\phi_v\|^2 - (\phi_u \cdot \phi_v)^2} \, dA \\ &= \iint_D \sqrt{(\|\phi_u\| \|\phi_v\|)^2 - (\|\phi_u\| \|\phi_v\|)^2 \cos^2\theta} \, dA \quad \text{Where θ is the angle between ϕ_u and ϕ_v.} \\ &= \iint_D \sqrt{(\|\phi_u\| \|\phi_v\|)^2 (1 - \cos^2\theta)} \, dA \\ &= \iint_D \sqrt{(\|\phi_u\| \|\phi_v\|)^2 (\sin^2\theta)} \, dA \\ &= \iint_D \sqrt{\|\phi_u \times \phi_v\|^2} \, dA \\ &= \iint_D \|\phi_u \times \phi_v\| \, dA \end{split}$$

(b) What does the formula for A(S) become if the vectors ϕ_u and ϕ_v are orthogonal? If the vectors are orthogonal, then the dot product is 0, so the equation reduces to

 $=\int_{\mathbf{\Phi}} 1 dS$

$$A(S) = \iint_D \|\phi_u\| \|\phi_v\| dA$$

(c) Use parts (a) and (b) to compute the surface area of a sphere of radius a. (cf. Marsden & Tromba, page 399, # 23.)

$$\Phi(\theta, \varphi) = a(\cos\theta \sin\varphi, \sin\theta \sin\varphi, \cos\varphi)$$

$$\phi_{\theta} = a(-\sin\theta \sin\varphi, \cos\theta \sin\varphi, 0)$$

$$\phi_{\varphi} = a(\cos\theta \cos\varphi, \sin\theta \cos\varphi, -\sin\varphi)$$

$$\|\phi_{\theta}\| = a\sin\varphi, \quad \|\phi_{\varphi}\| = a$$

$$\Rightarrow A(S) = a^{2} \int_{0}^{2\pi} \int_{0}^{\pi} \sin\varphi \, d\varphi \, d\theta$$

$$= a^{2} \int_{0}^{2\pi} \left[-\cos\varphi \right]_{0}^{\pi} d\varphi \, d\theta$$

$$= a^{2} \int_{0}^{2\pi} -(-1-1) \, d\varphi \, d\theta$$

$$= a^{2} 2 \int_{0}^{2\pi} 1 \, d\varphi \, d\theta$$

$$= 4\pi a^{2}$$

- 6. For each of the following surfaces S, sketch S (using symbolic software) and evaluate the surface integral $\int_S f \, dS$, where f(x, y, z) = x.
 - (a) S is that part of the surface $y = 4 x^2$ between z = 0 and z = 1, with $y \ge 0$.

$$\mathbf{\Phi}(x,z) = (x,4-x^2,z)$$

 $y > 0 \implies 4 - x^2 > 0 \implies x^2 < 4 \implies |x| < 2$

$$\begin{aligned} \phi_x &= (1, -2x, 0), \ \phi_z = (0, 0, 1) \\ \phi_x &\times \phi_z = (-2x, -1, 0) \implies \|\phi_x \times \phi_z\| = \sqrt{4x^2 + 1} \\ \int_S f dS &= \int_0^1 \int_0^2 x \sqrt{4x^2 + 1} \, dx \, dz \end{aligned}$$

The integrand is odd since x odd and $\sqrt{4x^2 + 1}$ even, so the integral over x is 0, making the entire integral 0.

(b) S is the upper half of the unit sphere centered at the origin.

Only the upper half so $0 \le \theta \le 2\pi$ and $0 \le \varphi \le \pi/2$.

$$\begin{split} & \Phi(\theta,\varphi) = (\cos\theta\sin\varphi,\sin\theta\sin\varphi,\cos\varphi) \\ & \phi_{\theta} = (-\sin\theta\sin\varphi,\cos\theta\sin\varphi,0) \\ & \phi_{\varphi} = (\cos\theta\cos\varphi,\sin\theta\cos\varphi,-\sin\varphi) \\ & \phi_{\theta} \times \phi_{\varphi} = (-\cos\theta\sin^{2}\varphi,-\sin\theta\sin^{2}\varphi,-\sin^{2}\theta\sin\varphi\cos\varphi-\cos^{2}\theta\sin\varphi\cos\varphi) \\ & = (-\cos\theta\sin^{2}\varphi,-\sin\theta\sin^{2}\varphi,-\sin\varphi\cos\varphi) \\ & \|\phi_{\theta} \times \phi_{\varphi}\| = \sqrt{\cos^{2}\theta\sin^{4}\varphi+\sin^{2}\theta\sin^{4}\varphi+\sin^{2}\varphi\cos^{2}\varphi} \\ & = \sqrt{\sin^{4}\varphi+\sin^{2}\varphi\cos^{2}\varphi} \\ & = \sqrt{\sin^{2}\varphi} = \sin\varphi \\ & \int_{\Phi} f \, dS = \int_{0}^{\frac{\pi}{2}} \int_{0}^{2\pi} \cos\theta\sin^{2}\varphi \, d\theta \, d\varphi = 0 \end{split}$$

The integral is zero again since integrating $\cos \theta$ over a whole period is 0.

(c) S is that part of the surface $x = \sin y$ with $0 \le y \le \pi$ and $0 \le z \le 2$.

$$\begin{split} & \Phi(y,z) = (\sin y, y, z) \\ & \phi_y = (\cos y, 1, 0) \\ & \phi_z = (0, 0, 1) \\ & \phi_y \times \phi_z = (1, -\cos y, 0) \\ & \|\phi_y \times \phi_z\| = \sqrt{1 + \cos^2 y} \\ & \int_{\Phi} f \, dS = \int_0^2 \int_0^\pi \sin y \sqrt{1 + \cos^2 y} \, dy \, dz \end{split}$$

7. Find the mass of the metallic surface S given by $z=1-\frac{x^2+y^2}{2}$ with $0 \le x \le 1, \ 0 \le y \le 1$, if the mass density at $(x,y,z) \in S$ is given by m(x,y,z)=xy.