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MATB42: Assignment #6

1. Let $\omega = \dfrac{-y}{x^2+y^2}\,dx + \dfrac{x}{x^2+y^2}\,dy$. Calculate $\displaystyle\int_\gamma \omega$ where

   (a) γ is the boundary of the triangle with vertices (in order) (0,1), (2,3) and (2,1).

   (b) γ is the boundary curve of the region $\left\{ (x,y) \in \mathbb{R}^2 \,\middle|\, \dfrac{(x-2)^2}{9} + \dfrac{(y+1)^2}{4} \leq 1 \right\}$ oriented in a counter clockwise direction

   (c) γ is the graph of the polar equation $r = 3 + 2\sin\theta$ oriented in the clockwise direction.

2. Let $\omega = (y^2 + z \ \ln 3)\ dx + (2xy + \sin z)\ dy + (y \cos z + (x+1)\ln 3)\ dz$. Determine if ω is exact. If it is, use the algorithm given in class to find the potential function g.

3. Evaluate the double integral $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 3x^2y^2 dy\, dx$, by first finding an equivalent line integral.

4. Let R be a region in $\mathbb{R}^2$ and let $\boldsymbol{\gamma}$ be a counterclockwise parametrization of ∂R. Let $\boldsymbol{F} = (F_1, F - 2)$ be a C^1 vector field defined throughout R and on ∂R and let $\boldsymbol{n}$ be the outward pointing unit normal ector to $\boldsymbol{\gamma}$. Use Green's theorem to give a double integral over R which is equivalent to $\displaystyle\int_{\boldsymbol{\gamma}} \boldsymbol{F} \cdot \boldsymbol{n}\, ds$.

5. Give a parametrization for each of the following surfaces, use a computer algebra sustem to plot the surface and find a unit vector normal to the surface.

   (a) The piece of the cylinder $y^2 + z^2 = 1$ between $x = -1$ and $x = 3$.

   (b) The piece of the plane $z = x + y + 5$ which lies over the unit disk $x^2 + y^2 \leq 1$.

   (c) The piece of the sphere $x^2 + y^2 + z^2 = 4$ which lies above the plane $z = 1$.

   (d) The piece of the plane $x + y + z = 1$ which lies above the parallelogram: $0 \leq y - x \leq 1, 0 \leq y + x2 \leq 1$.

6. Let S be the surface given parameterically by $\boldsymbol{\Phi}(u, v) = (u^2, 3v, u^2 + v)$ where $(u, v) \in D$, the interior of a triangle with vertices (0,0), (3,0) and (3,3).

   (a) Find the surface area of S.

   (b) Find the equation of the tangent plane to S at the point (4,9,7).

7. Suppose the surface S is the graph of a function $f : \mathbb{R}^2 \to \mathbb{R}$. Give a natural parametrization of S (in terms of f) and derive the formula $\|\boldsymbol{\phi}_u \times \boldsymbol{\phi}_v\| = \sqrt{1 + \|\text{grad } f\|^2}$

8. A paraboloid of revolution S is parameterized by $\boldsymbol{\Phi}(u, v) = u \cos v, u \sin v, u^2$, $0 \leq u \leq 2, 0 \leq v \leq 2\pi$.

   (a) Find an equation in x, y and z describing the surface.

   (b) What are the geometric meanings of the parameters u and v?

   (c) Find a unit vector orthogonal to the surface of $\boldsymbol{\Phi}(u, v)$.

   (d) Find the equation for the tangent plane at $\boldsymbol{\Phi}(u_0, v_0) = (1, 1, 2)$ and express your answer in the following two ways:

      i. parameterized by u and v; and

      ii. in terms of x, y and z.

   (e) Find the area of S.
   (cf. page 424, #16)

9. Let a differentiable function $\boldsymbol{\Phi} : \mathbb{R}^2 \to \mathbb{R}^3$ define a parametrized surface.

   (a) Assuming $\boldsymbol{\phi}_u \times \boldsymbol{\phi}_v \neq 0$, show that the range of the linear transformation $D\boldsymbol{\Phi}(u_0, v_0)$ is the plane spanned by $\boldsymbol{\phi}_u$ and $\boldsymbol{\phi}_v$. [Here $\boldsymbol{\phi}_u$ and $\boldsymbol{\phi}_v$ are evaluated at (u_0, v_0).]

   (b) Show that $\boldsymbol{w} \perp (\boldsymbol{\phi}_u \times \boldsymbol{\phi}_v)$ if and only if $\boldsymbol{w}$ is in the range of $D\boldsymbol{\Phi}(u_0, v_0)$.

   (c) Show that the tangent plane as defined in terms of $\boldsymbol{\phi}_u \times \boldsymbol{\phi}_v(u_0, v_0)$ is the same as the "parametrized plane"

$$(u, v) \mapsto \boldsymbol{\Phi}(u_0, v_0) + D\boldsymbol{\Phi}(u_0, v_0) \begin{bmatrix} u - u_0 \\ v - v_0 \end{bmatrix}$$

   (cf. page 383 #20)