

1. Show that $\int_{-\pi}^{\pi} \sin(kx) \sin(nx) dx = \begin{cases} 0 & , k \neq \pm n \\ \pm\pi & , k = \pm n \neq 0 \end{cases}$

Assuming $k \neq \pm n$

$$\begin{aligned} & \int_{-\pi}^{\pi} \sin(kx) \sin(nx) dx \\ &= \frac{1}{2} \int_{-\pi}^{\pi} \cos(kx - nx) - \cos(kx + nx) dx && [\sin(A) \sin(B) = \frac{1}{2} [\cos(A - B) - \cos(A + B)]] \\ &= \frac{1}{2} \left[\int_{-\pi}^{\pi} \cos((k - n)x) dx - \int_{-\pi}^{\pi} \cos((k + n)x) dx \right] \\ &= \frac{1}{2} \left[\frac{1}{k - n} [\sin(kx)]_{-\pi}^{\pi} - \frac{1}{k + n} [\sin(kx)]_{-\pi}^{\pi} \right] && [\text{Since cos is even}] \\ &k, n \in \mathbb{Z} \implies (k \pm n) \in \mathbb{Z}, \text{ but } \forall z \in \mathbb{Z}, \sin(z\pi) = 0 \\ &= 0 \end{aligned}$$

Assuming $k = \pm n$

$$\begin{aligned} & \int_{-\pi}^{\pi} \sin(kx) \sin(nx) dx \\ &= \int_{-\pi}^{\pi} \sin(kx) \sin(\pm kx) dx \\ &= \pm \int_{-\pi}^{\pi} \sin^2(kx) dx && [\text{Since sin odd}] \\ &= \pm \int_{-\pi}^{\pi} \frac{1 - \cos(2kx)}{2} dx \\ &= \pm \frac{1}{2} \left[\int_{-\pi}^{\pi} 1 dx - \int_{-\pi}^{\pi} \cos(2kx) dx \right] \\ &= \pm \frac{1}{2} [2\pi - 0] && [\text{Since cos even}] \\ &= \pm\pi \end{aligned}$$

2. For each of the following functions, find the N^{th} Fourier polynomial, assuming them to be periodic with period 2π . Use symbolic algebra software to graph the first three approximations together with the original function.

(a) $f(x) = x^2, -\pi < x \leq \pi$

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx \\ &= \frac{2}{3\pi} \left[x^3 \right]_0^{\pi} \quad [\text{Since } x^2 \text{ is even}] \\ &= \frac{2\pi^2}{3} \end{aligned} \qquad \begin{aligned} b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \sin(kx) dx \\ &= 0 \quad [\text{Since } x^2 \text{ is even but } \sin \text{ is odd}] \end{aligned}$$

$$\begin{aligned} a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos(kx) dx \\ &= \frac{2}{\pi} \int_0^{\pi} x^2 \cos(kx) dx \quad [\text{Since } x^2 \text{ and } \cos \text{ are even}] \end{aligned}$$

Let $u = x^2, du = 2x dx, dv = \cos(kx) dx, v = \frac{\sin(kx)}{k}$

$$= \frac{2}{\pi} \left[\left[\frac{1}{k} x^2 \sin(kx) \right]_0^{\pi} - \frac{2}{k} \int_0^{\pi} x \sin(kx) dx \right]$$

Let $u = x, du = 1 dx, dv = \sin(kx) dx, v = -\frac{\cos(kx)}{k}$

$$= \frac{2}{\pi} \left[\left[\frac{1}{k} x^2 \sin(kx) \right]_0^{\pi} - \frac{2}{k} \left(\left[-\frac{1}{k} x \cos(kx) \right]_0^{\pi} - \frac{1}{k} \int_0^{\pi} -\cos(kx) dx \right) \right]$$

$$= \frac{2}{\pi} \left[\left[\frac{1}{k} x^2 \sin(kx) \right]_0^{\pi} - \frac{2}{k} \left(\left[-\frac{1}{k} x \cos(kx) \right]_0^{\pi} + \frac{1}{k^2} \left[\sin(kx) \right]_0^{\pi} \right) \right]$$

$$= \frac{2}{\pi} \left[0 - \frac{2}{k} \left(\left[-\frac{1}{k} x \cos(kx) \right]_0^{\pi} + 0 \right) \right]$$

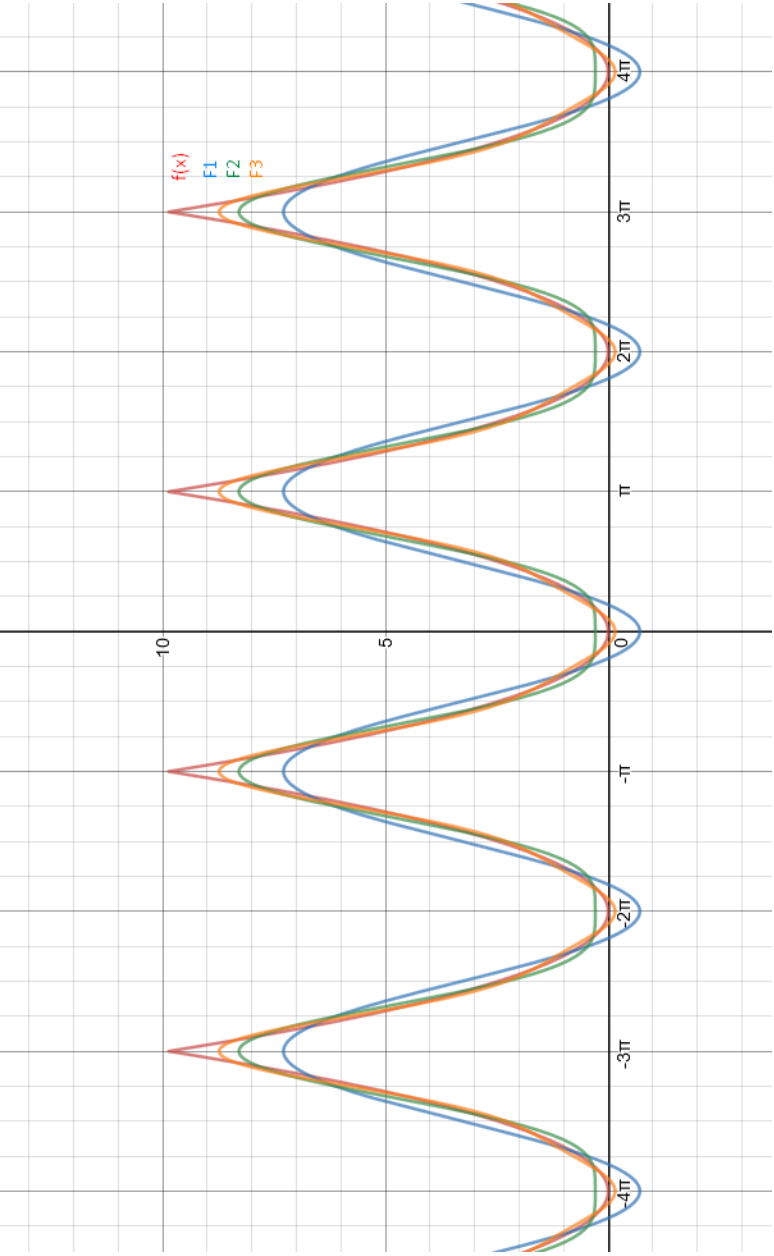
$$= -\frac{4}{k\pi} \left[-\frac{1}{k} x \cos(kx) \right]_0^{\pi}$$

$$= \frac{4}{k^2} \cos(k\pi) - 0$$

$$= \frac{4}{k^2} (-1)^k$$

Therefore the N^{th} Fourier polynomial is

$$F_N(x) = \frac{\pi^2}{3} + \sum_{k=1}^N \frac{4}{k^2} (-1)^k \cos(kx)$$



$$(b) \quad f(x) = \begin{cases} 0 & , -\pi \leq x < 0 \\ x & , 0 \leq x < \pi \end{cases}$$

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx \\ &= \frac{1}{\pi} \left[\int_{-\pi}^0 f(x) dx + \int_0^{\pi} f(x) dx \right] \\ &= \frac{1}{\pi} \left[0 + \int_0^{\pi} x dx \right] \\ &= \frac{1}{\pi} \left[\frac{1}{2} [x^2]_0^{\pi} \right] \\ &= \frac{\pi}{2} \end{aligned}$$

$$\begin{aligned} a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx \\ &= \frac{1}{\pi} \left[\int_{-\pi}^0 0 dx + \int_0^{\pi} x \cos(kx) dx \right] \end{aligned}$$

$$\text{Let } u = x, \quad du = dx, \quad dv = \cos(kx), \quad v = \frac{1}{k} \sin(kx)$$

$$\begin{aligned} &= \frac{1}{\pi} \left[\frac{1}{k} [x \sin(kx)]_0^{\pi} - \frac{1}{k} \int_0^{\pi} \sin(kx) dx \right] \\ &= -\frac{1}{k\pi} \left[\int_0^{\pi} \sin(kx) dx \right] \\ &= \frac{1}{k^2\pi} [\cos(kx)]_0^{\pi} \\ &= \frac{(-1)^{-k} - 1}{k^2\pi} \end{aligned}$$

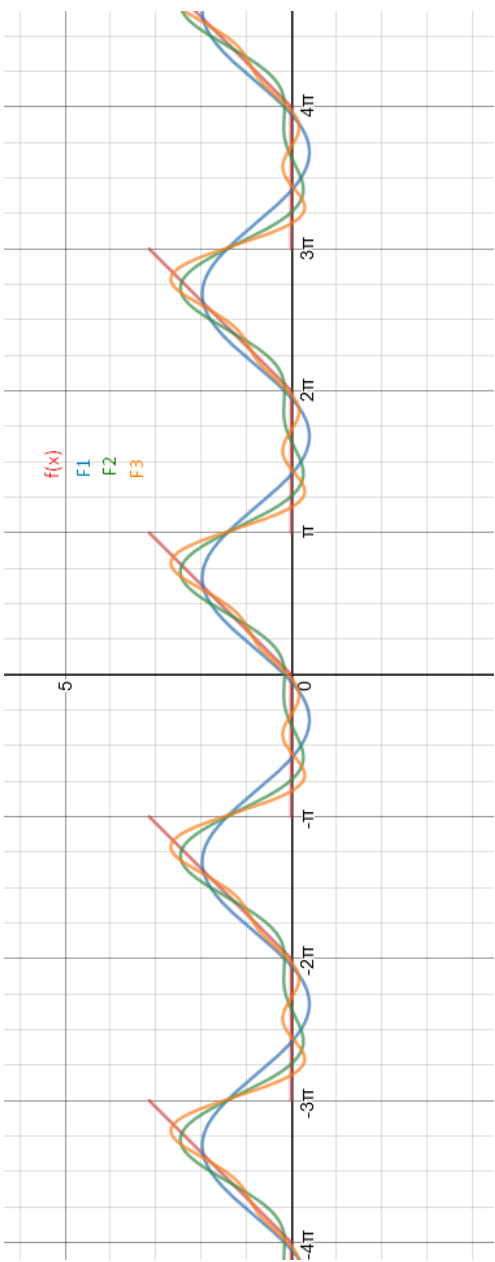
$$\begin{aligned} b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx \\ &= \frac{1}{\pi} \left[\int_{-\pi}^0 0 dx + \int_0^{\pi} x \sin(kx) dx \right] \end{aligned}$$

$$\text{Let } u = x, \quad du = dx, \quad dv = \sin(kx), \quad v = -\frac{1}{k} \cos(kx)$$

$$\begin{aligned} &= \frac{1}{\pi} \left[-\frac{1}{k} [x \cos(kx)]_0^{\pi} + \frac{1}{k} \int_0^{\pi} \cos(kx) dx \right] \\ &= \frac{1}{k\pi} \left[-\pi \cos(k\pi) + \frac{1}{k} [\sin(kx)]_0^{\pi} \right] \\ &= \frac{1}{k\pi} \left[-\pi \cos(k\pi) + 0 \right] \\ &= \frac{(-1)^{k+1}}{k} \end{aligned}$$

Therefore the N^{th} Fourier polynomial is

$$F_N(x) = \frac{\pi}{4} + \sum_{k=1}^N \left[\frac{(-1)^k - 1}{k^2\pi} \cos(kx) + \frac{(-1)^{k+1}}{k} \sin(kx) \right]$$



3. Find the Fourier series for the function $f(x)$ having period 2π , one period of which is given by

$$f(x) = \begin{cases} 1 & , 0 \leq x < \pi \\ x & , \pi \leq x < 2\pi \end{cases}$$

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx \\ &= \frac{1}{\pi} \left[\int_0^{\pi} f(x) dx + \int_{-\pi}^0 f(x) dx \right] \\ &= \frac{1}{\pi} \left[\int_0^{\pi} 1 dx + \int_{\pi}^{2\pi} x dx \right] \quad [\text{Period } 2\pi] \\ &= \frac{1}{\pi} \left[\pi + \left[\frac{x^2}{2} \right]_{\pi}^{2\pi} \right] \\ &= \frac{1}{\pi} \left[\pi + \frac{4\pi^2}{2} - \frac{\pi^2}{2} \right] \\ &= 1 + \frac{3\pi}{2} \end{aligned}$$

$$\begin{aligned} a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx \\ &= \frac{1}{\pi} \left[\int_0^{\pi} \cos(kx) dx + \int_{\pi}^{2\pi} x \cos(kx) dx \right] \quad [\text{Period } 2\pi] \\ \text{Let } u &= x, \quad du = dx, \quad dv = \cos(kx), \quad v = \frac{1}{k} \sin(kx) \\ &= \frac{1}{\pi} \left[\frac{1}{k} \left[\sin(kx) dx \right]_0^{\pi} + \frac{1}{k} \left[x \sin(kx) \right]_{\pi}^{2\pi} - \frac{1}{k} \int_{\pi}^{2\pi} \sin(kx) dx \right] \\ &= \frac{-1}{k\pi} \left[\int_{\pi}^{2\pi} \sin(kx) dx \right] \\ &= \frac{1}{k^2\pi} \left[\cos(kx) dx \right]_{\pi}^{2\pi} \\ &= \frac{1 - (-1)^k}{k^2\pi} \end{aligned}$$

$$\begin{aligned} b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx \\ &= \frac{1}{\pi} \left[\int_0^{\pi} \sin(kx) dx + \int_{\pi}^{2\pi} x \sin(kx) dx \right] \quad [\text{Period } 2\pi] \\ \text{Let } u &= x, \quad du = dx, \quad dv = \sin(kx), \quad v = -\frac{1}{k} \cos(kx) \\ &= \frac{1}{\pi} \left[\frac{-1}{k} \left[\cos(kx) \right]_0^{\pi} + \frac{-1}{k} \left[x \cos(kx) \right]_{\pi}^{2\pi} + \frac{1}{k} \int_{\pi}^{2\pi} \cos(kx) dx \right] \\ &= \frac{1}{k\pi} \left[-\cos(k\pi) + 1 + -2\pi + \pi \cos(k\pi) + \int_{\pi}^{2\pi} \cos(kx) dx \right] \\ &= \frac{1}{k\pi} \left[(\pi - 1)(-1)^k + 1 - 2\pi + \left[\sin(kx) \right]_{\pi}^{2\pi} \right] \\ &= \frac{1}{k\pi} \left[(\pi - 1)(-1)^k + 1 - 2\pi \right] \\ &= \frac{(\pi - 1)(-1)^k + 1 - 2\pi}{k\pi} \end{aligned}$$

Therefore the Fourier series for $f(x)$ is

$$F(x) = \frac{2 + 3\pi}{4} + \sum_{k=1}^{\infty} \left[\frac{1 - (-1)^k}{k^2\pi} \cos(kx) + \frac{(\pi - 1)(-1)^k + 1 - 2\pi}{k\pi} \sin(kx) \right]$$

4. Let $f(x) = \begin{cases} 0 & , -\pi \leq x < -1 \\ \frac{1}{2} & , -1 \leq x < 1 \\ 0 & , 1 \leq x < \pi \end{cases}$

(a) What fraction of the energy of f is contained in the constant term of its Fourier series?

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx \\ &= \frac{1}{\pi} \left[0 + \int_{-1}^1 \frac{1}{2} dx + 0 \right] \\ &= \frac{1}{\pi} \end{aligned} \qquad \begin{aligned} E(f) &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 dx \\ &= \frac{1}{\pi} \left[0 + \int_{-1}^1 \frac{1}{4} dx + 0 \right] \\ &= \frac{1}{2\pi} \end{aligned}$$

$$\begin{aligned} E(a_0) &= \frac{1}{2} a_0^2 \\ &= \frac{1}{\pi} \left[0 + \int_{-1}^1 \frac{1}{4} dx + 0 \right] \\ &= \frac{1}{2\pi^2} \end{aligned} \qquad \begin{aligned} \frac{E(a_0)}{E(f)} &= \frac{\frac{1}{2\pi^2}}{\frac{1}{2\pi}} \\ &= \frac{1}{\pi} \\ &\approx 32\% \end{aligned}$$

(b) Find a formula for the energy of the k^{th} harmonic of f .

$$\begin{aligned} a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx \\ &= \frac{1}{2\pi} \int_{-1}^1 \cos(kx) dx \\ &= \frac{1}{\pi} \int_0^1 \cos(kx) dx \quad [\cos \text{ is even}] \\ &= \frac{1}{k\pi} \sin(k) \end{aligned} \qquad \begin{aligned} b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx \\ &= 0 \quad [\text{Since } \sin \text{ is odd and } f \text{ is even}] \end{aligned}$$

$$\begin{aligned} E(k^{th} \text{ harmonic}) &= A_k^2 = \sqrt{a_k^2 + b_k^2}^2 = a_k^2 \\ &= \frac{\sin^2(k)}{k^2\pi^2} \end{aligned}$$

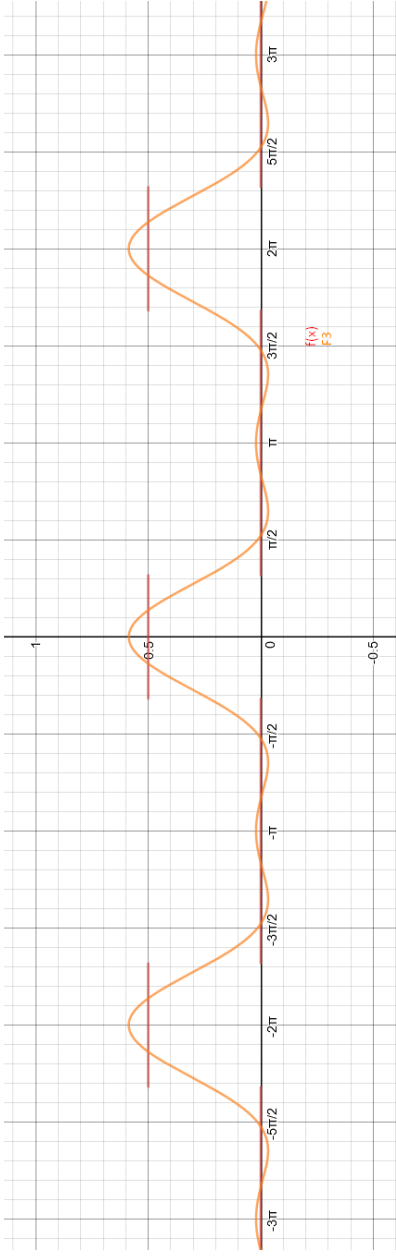
(c) How many terms of the Fourier series of f are needed to capture 80% of the energy of f ?

$$\begin{aligned} \frac{\sin^2(k)}{k^2\pi^2} / \frac{1}{2\pi} &= \frac{2\sin^2(1)}{1^2\pi} \approx 45\% \\ \frac{2\sin^2(2)}{2^2\pi} &\approx 13\% \\ 32\% + 45\% + 13\% &= 90\% \geq 80\% \end{aligned}$$

Including the constant term, three terms are required to capture 80% of the energy of f .

(d) Find F_N , the N^{th} Fourier polynomial of f , and use symbolic algebra software to graph f and F_3 on the interval $[-3\pi, 3\pi]$.

$$F_N(x) = \frac{1}{2\pi} + \sum_{k=1}^N \left[\frac{\sin(k)}{k\pi} \cos(kx) \right]$$



5. Find the Fourier series for the function $f(x)$ (of period 4) which corresponds to $y = x^2 - 4$ on the interval $[-2, 2]$.

$$\begin{aligned}
 a_0 &= \frac{1}{4} \int_{-2}^2 f(x) dx \\
 &= \frac{1}{4} \int_{-2}^2 (x^2 - 4) dx \quad [f(x) \text{ is even}] \\
 &= \left[\frac{1}{12} x^3 - 4x \right]_{-2}^2 \\
 &= \left[\frac{8}{3} - 8 \right] \\
 &= -\frac{16}{3}
 \end{aligned}$$

$$\begin{aligned}
 a_k &= \frac{1}{2} \int_{-2}^2 f(x) \cos\left(\frac{2\pi kx}{4}\right) dx \\
 &= \int_0^2 (x^2 - 4) \cos\left(\frac{\pi kx}{2}\right) dx \quad [f(x) \text{ and } \cos \text{ are even}] \\
 &= \int_0^2 x^2 \cos\left(\frac{\pi kx}{2}\right) dx - 4 \int_0^2 \cos\left(\frac{\pi kx}{2}\right) dx
 \end{aligned}$$

$$\begin{aligned}
 \text{Let } u &= x^2, du = 2x dx, dv = \cos\left(\frac{\pi kx}{2}\right) dx, v = \frac{2 \sin\left(\frac{\pi kx}{2}\right)}{k\pi} \\
 &= \frac{2}{k\pi} \left[x^2 \sin\left(\frac{\pi kx}{2}\right) \right]_0^2 - \frac{4}{k\pi} \int_0^2 x \sin\left(\frac{\pi kx}{2}\right) dx \\
 &\quad - \frac{8}{k\pi} \left[\sin\left(\frac{\pi kx}{2}\right) \right]_0^2 \\
 &= \frac{8}{k\pi} \sin\left(\frac{\pi kx}{2}\right) - \frac{4}{k\pi} \int_0^2 x \sin\left(\frac{\pi kx}{2}\right) dx - \frac{8}{k\pi} \sin\left(\frac{\pi kx}{2}\right)
 \end{aligned}$$

$$\begin{aligned}
 \text{Let } u &= x, du = 1 dx, dv = \sin\left(\frac{\pi kx}{2}\right) dx, v = -\frac{2 \cos\left(\frac{\pi kx}{2}\right)}{k\pi} \\
 &= -\frac{4}{k\pi} \left[-\frac{2}{k\pi} \left[x \cos\left(\frac{\pi kx}{2}\right) \right]_0^2 + \frac{2}{k\pi} \int_0^2 \cos\left(\frac{\pi kx}{2}\right) dx \right] \\
 &= -\frac{4}{k\pi} \left[-\frac{4 \cos(\pi k)}{k\pi} + \frac{2}{k^2 \pi^2} \left[\sin\left(\frac{\pi kx}{2}\right) \right]_0^2 \right] \\
 &= -\frac{4}{k\pi} \left[-\frac{4(-1)^k}{k\pi} + \frac{\sin(\pi k)}{k^2 \pi^2} \right] \\
 &= \frac{16(-1)^k}{k^2 \pi^2}
 \end{aligned}$$

$$\begin{aligned}
 b_k &= \frac{1}{2} \int_{-2}^2 f(x) \sin\left(\frac{2\pi kx}{4}\right) dx \\
 &= 0 \quad [\text{Since } f \text{ is even, but } \sin \text{ is odd}]
 \end{aligned}$$

Therefore the Fourier series for $f(x)$ is

$$F(x) = -\frac{8}{3} + \sum_{k=1}^{\infty} \frac{16(-1)^k}{k^2 \pi^2} \cos(kx)$$

6. (a) Suppose $f(x)$ has a continuous derivative $f'(x)$ on $[0, 2\pi]$. Let a_k and b_k the k^{th} Fourier coefficients of f and let a'_k and b'_k be those of f' . Show that

$$\begin{aligned}a'_k &= kb_k + \frac{f(2\pi) - f(0)}{\pi} \\b'_k &= -ka_k.\end{aligned}$$

By definition,

$$a'_k = \frac{1}{\pi} \int_0^{2\pi} f'(x) \cos(kx) dx$$

Let $u = \cos(kx)$, $du = -k \sin(kx) dx$, $dv = f'(x) dx$, $v = f(x)$

$$\begin{aligned}&= \frac{1}{\pi} \left[\left[f(x) \cos(kx) \right]_0^{2\pi} + k \int_0^{2\pi} \sin(kx) f(x) dx \right] \\&= \frac{1}{\pi} \left[\left[f(x) \cos(kx) \right]_0^{2\pi} + k\pi \left[\frac{1}{\pi} \int_0^{2\pi} \sin(kx) f(x) dx \right] \right] \\&= \frac{1}{\pi} \left[f(2\pi) \cos(2k\pi) - f(0) \cos(0) + k\pi b_k \right] \text{ [Def of } b_k] \\&= \frac{1}{\pi} \left[f(2\pi) - f(0) + k\pi b_k \right] \\&= kb_k + \frac{f(2\pi) - f(0)}{\pi} \text{ As wanted}\end{aligned}$$

By definition,

$$b'_k = \frac{1}{\pi} \int_0^{2\pi} f'(x) \sin(kx) dx$$

Let $u = \sin(kx)$, $du = k \cos(kx) dx$, $dv = f'(x) dx$, $v = f(x)$

$$\begin{aligned}&= \frac{1}{\pi} \left[\left[f(x) \sin(kx) \right]_0^{2\pi} - k \int_0^{2\pi} \cos(kx) f(x) dx \right] \\&= \frac{1}{\pi} \left[\left[f(x) \sin(kx) \right]_0^{2\pi} - k\pi \left[\frac{1}{\pi} \int_0^{2\pi} \cos(kx) f(x) dx \right] \right] \\&= \frac{1}{\pi} \left[-f(2\pi) \sin(2k\pi) + f(0) \sin(0) - k\pi a_k \right] \text{ [Def of } a_k] \\&= -ka_k \text{ As wanted}\end{aligned}$$

- (b) Use part (a) to find all the Fourier coefficients of the restriction of $f(x) = e^{\lambda x}$ to the interval $[0, 2\pi]$ in terms of the constant λ .

Since $\frac{d}{dx}e^{\lambda x} = \lambda e^{\lambda x}$, the coefficients of every term of the Fourier polynomial of the derivative function will simply be that of the original, multiplied by λ .

$$\begin{aligned}
 \lambda a_k &= k b_k + \frac{e^{2\pi\lambda} - 1}{\pi} & \lambda b_k &= -k a_k \\
 \lambda a_k &= -k \left(\frac{k a_k}{\lambda} \right) + \frac{e^{2\pi\lambda} - 1}{\pi} & b_k &= -\frac{k a_k}{\lambda} \\
 \lambda a_k + \frac{k^2 a_k}{\lambda} &= \frac{e^{2\pi\lambda} - 1}{\pi} & b_k &= -\frac{k}{\lambda} \left(\frac{\lambda(e^{2\pi\lambda} - 1)}{\pi(\lambda^2 + k^2)} \right) \\
 a_k \left(\lambda + \frac{k^2}{\lambda} \right) &= \frac{e^{2\pi\lambda} - 1}{\pi} & b_k &= -\frac{k(e^{2\pi\lambda} - 1)}{\pi(\lambda^2 + k^2)} \\
 a_k &= \frac{e^{2\pi\lambda} - 1}{\pi(\lambda + \frac{k^2}{\lambda})} \\
 a_k &= \frac{e^{2\pi\lambda} - 1}{\frac{\pi(\lambda^2 + k^2)}{\lambda}} \\
 a_k &= \frac{\lambda(e^{2\pi\lambda} - 1)}{\pi(\lambda^2 + k^2)}
 \end{aligned}$$

7. Find two Fourier expansions for the restriction of the function $f(x) = \sin x$ to the interval $[0, \pi]$. In one expansion all the sine terms should have zero coefficient, in the other all cosine terms should have coefficient.

Even expansion (sine terms have zero coefficient)

$$f(-x) = \begin{cases} -\sin(x), & \text{if } -\pi \leq x < 0 \\ \sin(x), & \text{if } 0 \leq x < \pi \end{cases}$$

This works since \sin odd, so $-\sin(-x) = \sin(x)$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \sin(x) \cos(kx) dx$$

$$= \frac{1}{\pi} \left[\int_0^{\pi} \sin((k+1)x) dx - \int_0^{\pi} \sin((k-1)x) dx \right]$$

$$= \frac{1}{\pi} \left[-\frac{1}{k+1} \left[\cos((k+1)x) \right]_0^{\pi} + \frac{1}{k-1} \left[\cos((k-1)x) \right]_0^{\pi} \right] = \frac{2}{\pi} \left[-\left[\cos(x) \right]_0^{\pi} \right]$$

$$= \frac{1}{\pi} \left[-\frac{(-1)^{(k+1)} - 1}{k+1} + \frac{(-1)^{(k-1)} - 1}{k-1} \right]$$

$$= \frac{(-1)^{(k+1)} - 1}{\pi} \left[\frac{1}{k-1} - \frac{1}{k+1} \right]$$

$$= \frac{(-1)^{(k+1)} - 1}{\pi} \left[\frac{k+1 - k+1}{k^2 - 1^2} \right]$$

$$= 2 \frac{(-1)^{(k+1)} - 1}{\pi(k^2 - 1)} \text{ [Valid for } k > 1]$$

For $k = 1$

$$= \frac{1}{\pi} \left[\int_0^{\pi} \sin(2x) dx - \int_0^{\pi} \sin((1-1)x) dx \right]$$

$$= \frac{1}{\pi} \int_0^{\pi} \sin(2x) dx$$

$$= -\frac{1}{2\pi} \left[\cos(2x) \right]_0^{\pi}$$

$$= -\frac{1}{2\pi} [1 - 1]$$

$$= 0$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} f(x) dx \text{ [Defined to be even]}$$

$$= \frac{2}{\pi} \int_0^{\pi} \sin(x) dx$$

$$= \frac{2}{\pi} \left[-\left[\cos(x) \right]_0^{\pi} \right]$$

$$= \frac{2}{\pi} [1 + 1]$$

$$= \frac{4}{\pi}$$

$$b_k = 0 \quad \text{[Since } f \text{ defined even]}$$

Therefore the even Fourier expansion for $f(x)$ is

$$F_N = \frac{2}{\pi} + \sum_{k=2}^{\infty} 2 \frac{(-1)^{(k+1)} - 1}{\pi(k^2 - 1)}$$

Odd expansion (cosine terms have zero coefficient)

$$f(-x) = \sin(x), \text{ on } [-\pi, \pi]$$

This works, since \sin is already an odd function

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(x) dx \\ &= 0 \text{ [sin is odd]} \end{aligned}$$

$$\begin{aligned} b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(x) \sin(kx) dx \\ &= 1 \text{ if } k = 1, 0 \text{ otherwise [Refer to q1]} \end{aligned}$$

$$\begin{aligned} a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(x) \cos(kx) dx \\ &= 0 \text{ [sin is odd, cos even]} \end{aligned}$$

So the Fourier expansion is simply $\sin(x)$

Bonus

Find the Fourier series for the function $y = f(x)$ of period 2π , if one period is given by

$$f(x) = \begin{cases} -\frac{1}{2} - \frac{x}{2\pi} & , -\pi \leq x < 0 \\ \frac{1}{2} - \frac{x}{2\pi} & , 0 \leq x < \pi \end{cases}$$

This function is odd over the period since $(-\frac{1}{2} - \frac{-x}{2\pi}) = (-\frac{1}{2} + \frac{x}{2\pi}) = -(\frac{1}{2} - \frac{x}{2\pi})$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$= 0 \quad [\text{Since } f \text{ odd}]$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx$$

$$= 0 \quad [\text{Since } f \text{ odd and } \cos \text{ even}]$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^0 \left[-\frac{1}{2} - \frac{x}{2\pi} \right] \sin(kx) dx + \int_0^{\pi} \left[\frac{1}{2} - \frac{x}{2\pi} \right] \sin(kx) dx \right]$$

$$= \frac{1}{\pi} \left[- \int_{-\pi}^0 \left[\frac{(\pi+x) \sin(kx)}{2\pi} \right] dx + \int_0^{\pi} \left[\frac{(\pi-x) \sin(kx)}{2\pi} \right] dx \right]$$

$$\text{Let } u = (\pi+x), du = 1dx, dv = \sin(kx)dx, v = -\frac{\cos(kx)}{k}$$

$$= \frac{1}{2\pi^2} \left[\int_0^{\pi} \left[(\pi-x) \sin(kx) \right] dx + \left[(\pi+x) \frac{\cos(kx)}{k} \right]_{-\pi}^0 \right]$$

$$- \int_{-\pi}^0 \frac{\cos(kx)}{k} dx \Big]$$

$$= \frac{1}{2\pi^2} \left[\int_0^{\pi} \left[(\pi-x) \sin(kx) \right] dx + \frac{\pi}{k} - \frac{1}{k^2} \left[\sin(kx) \right]_{-\pi}^0 \right]$$

$$= \frac{1}{2\pi^2} \left[\int_0^{\pi} \left[(\pi-x) \sin(kx) \right] dx + \frac{\pi}{k} \right]$$

$$\text{Let } u = (\pi-x), du = -1dx, dv = \sin(kx)dx, v = -\frac{\cos(kx)}{k}$$

$$= \frac{1}{2\pi^2} \left[- \left[\frac{(\pi-x) \cos(kx)}{k} \right]_0^{\pi} - \int_0^{\pi} \frac{\cos(kx)}{k} dx + \frac{\pi}{k} \right]$$

$$= \frac{1}{2\pi^2} \left[\frac{\pi}{k} - \left[\frac{\sin(kx)}{k^2} \right]_0^{\pi} + \frac{\pi}{k} \right]$$

$$= \frac{1}{k\pi}$$

Therefore the Fourier series for $f(x)$ is

$$F(x) = \sum_{k=1}^{\infty} \frac{1}{k\pi} \sin(kx)$$