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MATB42: Assignment #4
Bonus
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1. (For this question assume that all curves are of class C^k, some $k \geq 3$).

   A curve $\boldsymbol{\gamma} : [a,b] \to \mathbb{R}^n$ is called *regular* if $\boldsymbol{\gamma}'(t) \neq 0$ for any t. For a regular curve $\boldsymbol{\gamma}$, the vector $\boldsymbol{T} = \dfrac{\boldsymbol{\gamma}'(t)}{\|\boldsymbol{\gamma}'(t)\|}$ is called the *unit tangent vector* to the curve.

   (a) If $\boldsymbol{\gamma} : [a,b] \to \mathbb{R}^3$ is a regular curve, show that $\boldsymbol{T}'(t) \cdot \boldsymbol{T}(t) = 0$.
   (see page 235, #16(a))

$$\begin{aligned}
\|\boldsymbol{T}(t)\|^2 &= T_1^2 + T_2^2 + T_3^2 = 1 \\
\frac{d}{dt}\|\boldsymbol{T}(t)\|^2 &= 2T_1T_1' + 2T_2T_2' + T_3T_3' = \frac{d}{dt}1 \\
2(\boldsymbol{T}'(t) \cdot \boldsymbol{T}(t)) &= 2(T_1'T_1 + T_2'T_2 + T_3'T_3) = 0
\end{aligned}$$

   A curve $\boldsymbol{\gamma}(s)$ is said to be *parameterized by arclength* (or have *unit speed*) if $\|\boldsymbol{\gamma}'(s)\| = 1$. The *curvature* κ at a point $\gamma(s)$ of a unit speed curve is defined by $\kappa = \|\boldsymbol{T}'(s)\|$

   (b) (i) If $\boldsymbol{\gamma} : [a,b] \to \mathbb{R}^3$ is a unit speed curve, show that its length is $b - a$.

   The length of $\boldsymbol{\gamma}$ is $\displaystyle\int_{\boldsymbol{\gamma}} d\boldsymbol{s} = \int_a^b \|\gamma'(t)\|\, dt$, but $\|\gamma'(t)\|$ is 1 since $\boldsymbol{\gamma}$ has unit speed. Therefore, the integral is just $b - a$.

   (ii) Show that $\boldsymbol{\sigma}(t) = \frac{1}{\sqrt{2}}(\cos t, \sin t, t)$ is a unit speed curve and compute its curvature.
   (see page 235, #17)

$$\begin{aligned}
\frac{d}{dt}\boldsymbol{\sigma}(t) &= \frac{1}{\sqrt{2}}(\frac{d}{dt}\cos t, \frac{d}{dt}\sin t, \frac{d}{dt}t) \\
&= \frac{1}{\sqrt{2}}(-\sin t, \cos t, 1) \\
\implies \|\frac{d}{dt}\boldsymbol{\sigma}(t)\| &= \frac{\sqrt{\sin^2 t + \cos^2 t + 1}}{\sqrt{2}} \\
&= \frac{\sqrt{2}}{\sqrt{2}} = 1 \quad \text{So } \boldsymbol{\sigma}(t) \text{ is in fact a unit curve.}
\end{aligned}$$

   Since $\boldsymbol{\sigma}(t)$ has unit speed, $\boldsymbol{T}(t)$ is just $\boldsymbol{\sigma}'(t)$, so $\boldsymbol{T}'(t)$ is $\boldsymbol{\sigma}^{(2)}(t)$.

$$\begin{aligned}
\boldsymbol{T}'(t) &= \boldsymbol{\sigma}^{(2)}(t) \\
&= \frac{1}{\sqrt{2}}(\frac{d}{dt} - \sin t, \frac{d}{dt}\cos t, \frac{d}{dt}1) \\
&= \frac{1}{\sqrt{2}}(-\cos t, -\sin t, 0) \\
\implies \|\boldsymbol{T}'(t)\| &= \frac{1}{\sqrt{2}} = \kappa
\end{aligned}$$

   If $\boldsymbol{T}'(t) \neq 0$, $\boldsymbol{N}(t) = \dfrac{\boldsymbol{T}'(t)}{\|\boldsymbol{T}'(t)\|}$ is perpendicular to $\boldsymbol{T}'(t)$ (by part (a)); $\boldsymbol{N}$ is called the *principal normal vector*. The vector $\boldsymbol{B}$, defined by $\boldsymbol{B} = \boldsymbol{T} \times \boldsymbol{N}$, is called the *binormal vector*.

   (c) Show the following about the $\boldsymbol{T}, \boldsymbol{N}$ and $\boldsymbol{B}$ system

   (i) $\dfrac{d\boldsymbol{B}}{dt} \cdot \boldsymbol{B} = 0$ (ii) $\dfrac{d\boldsymbol{B}}{dt} \cdot \boldsymbol{T} = 0$ (iii) $\dfrac{d\boldsymbol{B}}{dt}$ is a scalar multiple of $\boldsymbol{N}$.
   (see page 235, #20)

(i) $\frac{d}{dt}\boldsymbol{B}\cdot\boldsymbol{B} = \frac{d}{dt}\|\boldsymbol{B}\|^2$, but the norm of $\boldsymbol{N}$ and $\boldsymbol{T}$ are 1, so $\|\boldsymbol{B}\|^2 = 1$ which means that $\frac{d}{dt}1 = 0$

(ii) Because $\boldsymbol{B}$ is the cross product of $\boldsymbol{T}$ along with $\boldsymbol{N}$, $\boldsymbol{B}\cdot\boldsymbol{T}$ must be 0, so $\frac{d}{dt}0 = 0$

(iii) From the last two parts, $\frac{d}{dt}\boldsymbol{B}$ is orthogonal to both $\boldsymbol{B}$ and $\boldsymbol{T}$. Since there are only 3 dimensions for $\mathbb{R}^3$, anything orthgonal to both of these must be parallel to each other. We know that $\boldsymbol{N}$ is orthgonal to $\boldsymbol{T}$ by (a), and by definition it is orthgonal to $\boldsymbol{B}$ so both are parallel, i.e. they are scalar multiples.

If $\boldsymbol{\gamma}(s)$ is a unit speed curve we can define the *tortion* τ by $\frac{d\boldsymbol{B}}{ds} = -\tau\boldsymbol{N}$.

(d) Compute the torsion of $\boldsymbol{\sigma}(t) = \frac{1}{\sqrt{2}}(\cos t, \sin t, t)$.

(see page 235, #21(c))

$$\boldsymbol{T}(t) = \frac{1}{\sqrt{2}}(-\sin t, \cos t, 1)\boldsymbol{N}(t) \qquad = (-\cos t, \sin t, 0)$$

2. Sketch the following vector fields including a few flow lines.

(a) $\boldsymbol{F}(x,y) = (1, x^2)$ (b) $\boldsymbol{F}(x,y) = (x^2, x)$ (c) $\boldsymbol{F}(x,y) = (y, -2x)$

(a)

$$\begin{aligned}
\boldsymbol{\gamma}(t) &= (x(t), y(t)) \\
\boldsymbol{\gamma}'(t) &= (x'(t), y'(t)) \\
\implies \frac{\frac{dy(t)}{dt}}{\frac{dx(t)}{dt}} &= \frac{y'(t)}{x'(t)} = \frac{dy}{dx} \\
dy &= x^2,\ dx = 3 \\
\implies \frac{dy}{dx} &= x^2 \\
\implies y &= \frac{x^3}{3} + c
\end{aligned}$$

(b) $\boldsymbol{F}(x,y) = (x^2, x)$

$$\begin{aligned}
dy &= x,\ dx = x^2 \\
\implies \frac{dy}{dx} &= \frac{1}{x} \\
\implies y &= \ln|x| + c \quad x \neq 0
\end{aligned}$$

(c) $\boldsymbol{F}(x,y) = (y, -2x)$

$$\begin{aligned}
dy &= -2x,\ dx = y \\
\implies \frac{dy}{dx} &= \frac{-2x}{y} \\
\implies y\ dy &= -2x\ dx \\
\implies \frac{y^2}{2} + x^2 &= c
\end{aligned}$$

3. Show that the curve $\boldsymbol{c}(t) = (t^2, 2t-1, \sqrt{t}),\ t > 0$ is a flow line of the velocity vector field $\boldsymbol{F}(x, y, z) = (y+1, 2, 1/2z)$

$$\mathbf{c}'(t) = \left(2t, 2, \frac{1}{2\sqrt{t}}\right)$$

$$\boldsymbol{F}(\boldsymbol{c}(t)) = \left(2t - 1 + 1, 2, \frac{1}{2\sqrt{t}}\right) = \left(2t, 2, \frac{1}{2\sqrt{t}}\right) = \mathbf{c}'(t)$$

Therefore, $\boldsymbol{c}$ is a flow line of $\boldsymbol{F}$.

4. Find the work done by the force field $\boldsymbol{F}(x, y, z) = (xy, yz, zx)$ in moving a particle along the twisted cubic, $\boldsymbol{\gamma}(t) = (t, t^2, t^3)$, from $t = 0$ to $t = 1$.

$$\begin{aligned}
\int_{\gamma} \boldsymbol{F} \cdot ds &= \int_0^1 \boldsymbol{F}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t)\ dt \\
&= \int_0^1 (t)(t^2)(1) + (t^2)(t^3)(2t) + (t^3)(t)(3t^2) dt \\
&= \int_0^1 t^3 + 2t^6 + 3t^6 dt \\
&= \int_0^1 t^3 + 5t^6 \\
&= \frac{1}{4}\left[t^4\right]_0^1 + \frac{5}{7}\left[t^7\right]_0^1 \\
&= \frac{1}{4} + \frac{5}{7} = \frac{27}{28}
\end{aligned}$$

5. Evaluate each of the following integrals:

(a) $\displaystyle\int_\gamma xy\ dx + y^2 dy, \quad \gamma(t) = (\cos t, \sin t), 0 \le t \le \frac{\pi}{2}.$

$$\begin{aligned}\int_\gamma \omega \cdot ds &= \int_0^{\frac{\pi}{2}} \sin t \cos t(-\sin t) + \sin^2 t \cos t\ dt \\ &= 0\end{aligned}$$

(b) $\displaystyle\int_\gamma \boldsymbol{F} \cdot d\boldsymbol{s}, \quad \boldsymbol{F}(x,y,z) = (y,z,x), \quad \boldsymbol{\gamma}(t) = \left(t, -2t^2, \frac{1}{3}t^3\right), \quad 0 \le t \le 1.$

$$\begin{aligned}\int_\gamma \boldsymbol{F} \cdot ds &= \int_0^1 \boldsymbol{F}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t)\ dt \\ &= \int_0^1 (-2t^2)(1) + (\frac{1}{3}t^3)(-4t) + (t)(t^2) dt \\ &= \int_0^1 -2t^2 - \frac{4}{3}t^4 + t^3 dt \\ &= -\frac{2}{3}\left[t^3\right]_0^1 - \frac{4}{15}\left[t^5\right]_0^1 + \frac{1}{4}\left[t^4\right]_0^1 \\ &= -\frac{10}{15} - \frac{4}{15} + \frac{1}{4} = -\frac{56}{60} + \frac{15}{60} = -\frac{41}{60}\end{aligned}$$

(c) $\displaystyle\int_\gamma z\ dx - xyz\ dy + 2x^2\ dz, \quad \boldsymbol{\gamma}$ is the parabola $z = x^2, y = 0$, from (-1,0,1) to (1,0,1).

Can parameterize $\boldsymbol{\gamma}$ by $\boldsymbol{\gamma}(t) = (t, 0, t^2)$, $-1 \le t \le 1$, as on the parabola y is constant 0, x goes from $-1 \to 1$ and z goes from $1 \to 0 \to 1$.

$$\begin{aligned}\int_\gamma \omega \cdot ds &= \int_{-1}^1 (t^2)(1) - (t)(0)(t^2)(0) + 2(t)^2(2t)\ dt \\ &= \int_{-1}^1 t^2 + 4t^3\ dt \\ &= \frac{2}{3}\left[t^3\right]_0^1 \quad \text{Exploiting even/odd} \\ &= \frac{2}{3}\end{aligned}$$

(d) $\displaystyle\int_\gamma \boldsymbol{F} \cdot d\boldsymbol{s}, \quad \boldsymbol{F}(x,y,z) = (2xy, x^2 + e^z, ye^z), \quad \boldsymbol{\gamma}$ consists of straight line segments joining, in order, the points (1,1,0), (2,0,5) and (0,3,0).

Note: By inspection $g = x^2y + ye^z$ is a potential function for $\boldsymbol{F}$. Also, straight line segments, being linear functions are smooth. Furthermore, $F(x,y,z)$ is smooth since polynomials and exponential functions are each smooth. Therefore, GFTC applies, and $\displaystyle\int_\gamma \boldsymbol{F} \cdot d\boldsymbol{s} = g(0,3,0) - g(1,1,0) = ((0)^2(3) + (3)e^{(0)}) - ((1)^2(1) + (1)e^{(0)}) = 3 - 2 = 1.$

6. (a) Let $\boldsymbol{F}(x, y) = (y, -x)$. Find $\displaystyle\int_\gamma \boldsymbol{F} \cdot d\boldsymbol{s}$ from (1,0) to (0,-1) along

(i) the straight line segment joining these points
Parameterize the path as $t \mapsto (1-t, -t)$ where $0 \le t \le 1$.

$$\begin{aligned}
\int_\gamma \boldsymbol{F} \cdot ds &= \int_0^1 \boldsymbol{F}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t)\ dt \\
&= \int_0^1 -(-t)(-1) + (1-t)(-1)\ dt \\
&= \int_0^1 -1\ dt = -1
\end{aligned}$$

(ii) three-quarters of the unit circle centered at the origin traced in the counter-clockwise direction.
Parameterize the path as $t \mapsto (\sin -t, \cos -t) = (-\sin t, \cos t)$ where $0 \le t \le \frac{3\pi}{2}$. Using $-t$ since it is counter-clockwise

$$\begin{aligned}
\int_\gamma \boldsymbol{F} \cdot ds &= \int_0^{\frac{3\pi}{2}} \boldsymbol{F}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t)\ dt \\
&= \int_0^{\frac{3\pi}{2}} (\cos t)(-\cos t) - (-\sin t)(-\sin t)\ dt \\
&= \int_0^{\frac{3\pi}{2}} -1\ dt = -\frac{3\pi}{2}
\end{aligned}$$

(b) Can your answers for part (a) help you determine if the 1-form $\omega = y\ dx - x\ dy$ is exact? Explain.
Yes, we can determine that it is not exact. If ω were to be exact then $\boldsymbol{F}$ would be conservative implying that the line integral would be independant of path. Since the integrals are different, this is evidently not the case.

7. Let $\boldsymbol{c}$ be the curve obtained by intersecting the cylinder $y^2 + z^2 = 4$ and the surface $x = yz$ in $\mathbb{R}^3$.

(a) Give a parametrization, $\boldsymbol{\gamma}(t)$, of the curve $\boldsymbol{c}$.

The cylinder simply describes a circle of radius 2 in 2 dimensions, so y and z can be parameterized as $t \mapsto (2\sin t, 2\cos t)$. To add the additional constraint of the surface, just check what x is, given the y and z. $x = (2\sin t)(2\cos t) = 4\sin t\cos t$.

Given these conditions, $\boldsymbol{\gamma}(t)$ is given by $(4\sin t\cos t, 2\sin t, 2\cos t)$, $0 \le t \le 2\pi$.

(b) Evaluate $\displaystyle\int_{\boldsymbol{c}} \boldsymbol{F} \cdot d\boldsymbol{s}$, where $\boldsymbol{F}(x, y, z) = (2xy, 4y, x^2)$.

$$
\begin{aligned}
\int_{\gamma} \boldsymbol{F} \cdot ds &= \int_0^{2\pi} \boldsymbol{F}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t)\ dt \\
&= \int_0^{2\pi} 2(4\sin t\cos t)(2\sin t)(4(\cos^2 t - \sin^2 t)) + 4(2\sin t)(2\cos t) + (4\cos t\sin t)^2(-2\sin t)\ dt \\
&= \int_0^{2\pi} 16\sin^2 t\cos t(\cos^2 t - \sin^2 t) + 8\sin t\cos t - 32\cos^2 t\sin^3 t\ dt \\
&= \int_0^{2\pi} 16\sin^2 t\cos^3 t - 16\sin^4 t\cos t + 8\sin t\cos t - 32\cos^2 t\sin^3 t\ dt \\
&= \int_0^{2\pi} 16\sin^2 t(1 - \sin^2 t)\cos t - 16\sin^4 t\cos t + 8\sin t\cos t\ dt - 32\int_0^{2\pi} \cos^2 t(1 - \cos^2 t)\sin t\ dt \\
&\text{Let } u = \sin t,\ du = \cos t \\
&= \int_0^{0} 16u^2(1 - u^2) - 16u^4 + 8u\ du - 32\int_0^{2\pi} \cos^2 t(1 - \cos^2 t)\sin t\ dt \\
&= -32\int_0^{2\pi} \cos^2 t(1 - \cos^2 t)\sin t\ dt \\
&\text{Let } u = \cos t,\ du = -\sin t \\
&= 32\int_1^{1} u^2(1 - u^2)\ du \\
&= 0
\end{aligned}
$$