MATB42: Assignment #7
Bonus

1. (a) Find an equation of the tangent plane to the surface S defined parametrically by $\mathbf{\Phi}(u,v) = (u^2 + v,\ v,\ u + v^2)$ at the point (9,0,3).

$$v = 0 \qquad\qquad u + v^2 = 3 \implies u = 3$$

$$\boldsymbol{\phi}_u = (2(3), 0, 1)$$
$$\boldsymbol{\phi}_v = (1, 1, 2(0))$$
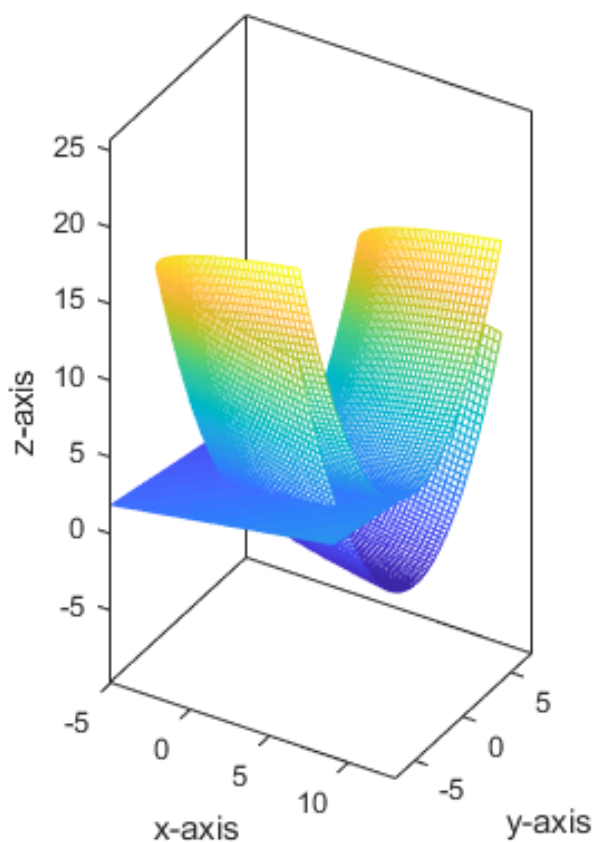$$\boldsymbol{\phi}_u \times \boldsymbol{\phi}_v = (-1, 1, 6)$$

So the tangent plane can be given by

$$0 = ((x - 9, y, z - 3) \cdot (-1, 1, 6))$$
$$0 = (9 - x + y + 6z - 18)$$
$$9 = -x + y + 6z$$

(b) Use symbolic algebra software to sketch the surface S and its tangent plane from part (a).

2. Use a surface integral to find the area of the triangle in $\mathbb{R}^3$ with vertices (1,1,0), (1,2,1) and (3,3,2).

Since the surface is a plane, the tangent plane is just parallel to the plane, so tangent vectors can be given by taking vectors in the plane i.e. $\boldsymbol{\phi}_u = (3,3,2)-(1,1,0) = (2,2,2)$ and $\boldsymbol{\phi}_v = (1,2,1)-(1,1,0) = (0,1,1)$. Now this is a triangle with side lengths of equal magnitude to the tangents, so the integral is given as.

$$\|\boldsymbol{\phi}_u \times \boldsymbol{\phi}_v\| = \|(0,-2,2)\| = \sqrt{4+4} = 2\sqrt{2}$$

$$\int_0^1 \int_0^v \|\boldsymbol{\phi}_u \times \boldsymbol{\phi}_v\| \, du \, dv = \int_0^1 \int_0^v 2\sqrt{2} \, du \, dv$$

$$= \int_0^1 2\sqrt{2}v \, dv$$

$$= \sqrt{2}$$

3. Calculate the surface area of the piece of the cone $x^2 + y^2 - z^2 = 0$ which lies inside the cylinder $x^2 + y^2 = 4$.

We can see the radius of the cylinder is 2, so the cone portion that's cut out is the part which has radius less than or equal to 2 $\implies 0 \le z \le 2$. Using polar for the cone, $0 \le \theta \le 2\pi$.

$$\mathbf{\Phi}(\theta, z) = (z\cos\theta, z\sin\theta, z)$$
$$\boldsymbol{\phi}_\theta = (-z\sin\theta, z\cos\theta, 0)$$
$$\boldsymbol{\phi}_z = (\cos\theta, \sin\theta, 1)$$
$$\boldsymbol{\phi}_\theta \times \boldsymbol{\phi}_z = (z\cos\theta, z\sin\theta, -z\sin^2\theta - z\cos^2\theta)$$
$$= (z\cos\theta, z\sin\theta, -z)$$
$$\|\boldsymbol{\phi}_\theta \times \boldsymbol{\phi}_z\| = z^2\cos^2\theta + z^2\sin^2\theta + z^2 = 2z^2$$

$$\int_{\mathbf{\Phi}} f \, dS = \int_0^{2\pi} \int_0^2 2z^2 \, dz \, d\theta$$
$$= \int_0^{2\pi} \left[\frac{2}{3}z^3\right]_0^2 d\theta$$
$$= \int_0^{2\pi} \frac{16}{3} d\theta = \frac{32\pi}{3}$$

4. (a) Find the area of the portion of the unit sphere that is cut out by the cone $z = \sqrt{x^2 + y^2}$. (cf. page 391, #10)

$$0 \leq \varphi \leq \pi, \ 0 \leq \theta \leq 2\pi$$
$$\mathbf{\Phi}_{\text{sphere}}(\theta, \varphi) = (\cos\theta \sin\varphi, \sin\theta \sin\varphi, \cos\varphi)$$
$$\boldsymbol{\phi}_\theta = (-\sin\theta \sin\varphi, \cos\theta \sin\varphi, 0)$$
$$\boldsymbol{\phi}_\varphi = (\cos\theta \cos\varphi, \sin\theta \cos\varphi, -\sin\varphi)$$
$$\boldsymbol{\phi}_\theta \times \boldsymbol{\phi}_\varphi = (-\cos\theta \sin^2\varphi, -\sin\theta \sin^2\varphi, -\sin^2\theta \sin\varphi \cos\varphi - \cos^2\theta \sin\varphi \cos\varphi)$$
$$= (-\cos\theta \sin^2\varphi, -\sin\theta \sin^2\varphi, -\sin\varphi \cos\varphi)$$
$$\|\boldsymbol{\phi}_\theta \times \boldsymbol{\phi}_\varphi\| = \sqrt{\cos^2\theta \sin^4\varphi + \sin^2\theta \sin^4\varphi + \sin^2\varphi \cos^2\varphi}$$
$$= \sqrt{\sin^2\varphi} = \sin\varphi$$

$$z > 0, \ 0 \leq \theta \leq 2\pi$$
$$\mathbf{\Phi}_{\text{cone}}(\theta, z) = (z\cos\theta, z\sin\theta, z)$$
$$\boldsymbol{\phi}_z = (\cos\theta, \sin\theta, 1)$$
$$\boldsymbol{\phi}_\theta = (-z\sin\theta, z\cos\theta, 0)$$
$$\boldsymbol{\phi}_z \times \boldsymbol{\phi}_\theta = (-z\cos\theta, -z\sin\theta, z)$$
$$\|\boldsymbol{\phi}_z \times \boldsymbol{\phi}_\theta\| = 2z^2$$

For the unit sphere $x^2 + y^2 + z^2 = 1$, but the cone is $x^2 + y^2 = z^2 \implies$ sub z into sphere gives $2x^2 + 2y^2 = 1$ So the exact intersection of the surfaces is a circle of radius $2/\sqrt{2}$ centered at the origin. so the surface cut out is the section of the top of the sphere where $z \geq 2\sqrt{2} \implies \varphi \leq \frac{\pi}{4}$ from the $z = \cos\varphi$ portion of the parametrization. So the ranges are $0 \leq \theta \leq 2\pi, \ 0 \leq \varphi \leq \frac{\pi}{4}$. The area is therefore

$$\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \sin\varphi \, d\varphi \, d\theta = \int_0^{2\pi} \left[-\cos\varphi\right]_0^{\frac{\pi}{4}} d\theta$$
$$= 2\pi\left[-\left(\cos\left(\frac{\pi}{4}\right)\right) - (-\cos(0))\right]$$
$$= 2\pi\left[-\frac{\sqrt{2}}{2} + 1\right]$$
$$= \pi(2 - \sqrt{2})$$

(b) Find the area of the portion of the cone $z = \sqrt{x^2 + y^2}$ that is cut out by the unit sphere. Plugging in $x^2 + y^2 = 1/2$ to the cone equation again gives $z^2 = 1/2 \implies z = \pm\frac{\sqrt{2}}{2}$ but $z \geq 0$ by the cone definition so $0 \leq z \leq \frac{\sqrt{2}}{2}$.

$$A(\mathbf{\Phi}_{\text{cone}}) = \int_0^{2\pi} \int_0^{\frac{1}{4}} 2z^2 \, dz \, d\theta$$
$$= \int_0^{2\pi} \frac{2}{3} \cdot \frac{1}{4^3} \, d\theta$$
$$= \frac{\pi}{3(16)}$$
$$= \frac{\pi}{48}$$

5. Let $\boldsymbol{\Phi} : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a parametrization of a 2-dim surface S in $\mathbb{R}^3$.

(a) Set

$$E = \|\boldsymbol{\phi}_u\|^2, \qquad\qquad F = \boldsymbol{\phi}_u \cdot \boldsymbol{\phi}_v, \qquad\qquad G = \|\boldsymbol{\phi}_v\|^2,$$

Show that the surface area of S is

$$A(S) = \iint_D \sqrt{EG - F^2}\, dA$$

$$
\begin{aligned}
\iint_D \sqrt{EG - F^2}\, dA &= \iint_D \sqrt{\|\boldsymbol{\phi}_u\|^2 \|\boldsymbol{\phi}_v\|^2 - (\boldsymbol{\phi}_u \cdot \boldsymbol{\phi}_v)^2}\, dA \\
&= \iint_D \sqrt{(\|\boldsymbol{\phi}_u\|\|\boldsymbol{\phi}_v\|)^2 - (\|\boldsymbol{\phi}_u\|\|\boldsymbol{\phi}_v\|)^2 \cos^2\theta}\, dA \quad \text{Where } \theta \text{ is the angle between } \boldsymbol{\phi}_u \text{ and } \boldsymbol{\phi}_v. \\
&= \iint_D \sqrt{(\|\boldsymbol{\phi}_u\|\|\boldsymbol{\phi}_v\|)^2(1 - \cos^2\theta)}\, dA \\
&= \iint_D \sqrt{(\|\boldsymbol{\phi}_u\|\|\boldsymbol{\phi}_v\|)^2(\sin^2\theta)}\, dA \\
&= \iint_D \sqrt{\|\boldsymbol{\phi}_u \times \boldsymbol{\phi}_v\|^2}\, dA \\
&= \iint_D \|\boldsymbol{\phi}_u \times \boldsymbol{\phi}_v\|\, dA \\
&= \int_{\boldsymbol{\Phi}} 1\, dS
\end{aligned}
$$

(b) What does the formula for $A(S)$ become if the vectors $\boldsymbol{\phi}_u$ and $\boldsymbol{\phi}_v$ are orthogonal?
If the vectors are orthogonal, then the dot product is 0, so the equation reduces to

$$A(S) = \iint_D \|\boldsymbol{\phi}_u\|\|\boldsymbol{\phi}_v\|\, dA$$

(c) Use parts (a) and (b) to compute the surface area of a sphere of radius a.
(cf. Marsden & Tromba, page 399, # 23.)

$$
\begin{aligned}
0 &\le \varphi \le \pi, \; 0 \le \theta \le 2\pi \\
\boldsymbol{\Phi}(\theta, \varphi) &= a(\cos\theta \sin\varphi, \sin\theta \sin\varphi, \cos\varphi) \\
\boldsymbol{\phi}_\theta &= a(-\sin\theta \sin\varphi, \cos\theta \sin\varphi, 0) \\
\boldsymbol{\phi}_\varphi &= a(\cos\theta \cos\varphi, \sin\theta \cos\varphi, -\sin\varphi) \\
\|\boldsymbol{\phi}_\theta\| &= a\sin\varphi, \quad \|\boldsymbol{\phi}_\varphi\| = a \\
\implies A(S) &= a^2 \int_0^{2\pi} \int_0^{\pi} \sin\varphi \, d\varphi \, d\theta \\
&= a^2 \int_0^{2\pi} \Big[-\cos\varphi\Big]_0^{\pi} d\varphi \, d\theta \\
&= a^2 \int_0^{2\pi} -(-1 - 1)\, d\varphi \, d\theta \\
&= a^2 2 \int_0^{2\pi} 1 \, d\varphi \, d\theta \\
&= 4\pi a^2
\end{aligned}
$$

6. For each of the following surfaces S, sketch S (using symbolic software) and evaluate the surface integral $\int_S f \, dS$, where $f(x, y, z) = x$.

(a) S is that part of the surface $y = 4 - x^2$ between $z = 0$ and $z = 1$, with $y \geq 0$.

$$y \geq 0 \implies 4 - x^2 \geq 0 \implies x^2 \leq 4 \implies |x| < 2$$

$$\Phi(x, z) = (x, 4 - x^2, z)$$
$$\phi_x = (1, -2x, 0), \quad \phi_z = (0, 0, 1)$$
$$\phi_x \times \phi_z = (-2x, -1, 0) \implies \|\phi_x \times \phi_z\| = \sqrt{4x^2 + 1}$$
$$\int_S f \, dS = \int_0^1 \int_{-2}^2 x \sqrt{4x^2 + 1} \, dx \, dz$$

The integrand is odd since x odd and $\sqrt{4x^2 + 1}$ even, so the integral over x is 0, making the entire integral 0.

(b) S is the upper half of the unit sphere centered at the origin.

Only the upper half so $0 \leq \theta \leq 2\pi$ and $0 \leq \varphi \leq \pi/2$.

$$\Phi(\theta, \varphi) = (\cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi)$$
$$\phi_\theta = (-\sin \theta \sin \varphi, \cos \theta \sin \varphi, 0)$$
$$\phi_\varphi = (\cos \theta \cos \varphi, \sin \theta \cos \varphi, -\sin \varphi)$$
$$\phi_\theta \times \phi_\varphi = (-\cos \theta \sin^2 \varphi, -\sin \theta \sin^2 \varphi, -\sin^2 \theta \sin \varphi \cos \varphi - \cos^2 \theta \sin \varphi \cos \varphi)$$
$$= (-\cos \theta \sin^2 \varphi, -\sin \theta \sin^2 \varphi, -\sin \varphi \cos \varphi)$$
$$\|\phi_\theta \times \phi_\varphi\| = \sqrt{\cos^2 \theta \sin^4 \varphi + \sin^2 \theta \sin^4 \varphi + \sin^2 \varphi \cos^2 \varphi}$$
$$= \sqrt{\sin^4 \varphi + \sin^2 \varphi \cos^2 \varphi}$$
$$= \sqrt{\sin^2 \varphi} = \sin \varphi$$
$$\int_\Phi f \, dS = \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \cos \theta \sin^2 \varphi \, d\theta \, d\varphi = 0$$

The integral is zero again since integrating $\cos \theta$ over a whole period is 0.

(c) S is that part of the surface $x = \sin y$ with $0 \leq y \leq \pi$ and $0 \leq z \leq 2$.

$$\boldsymbol{\Phi}(y, z) = (\sin y, y, z)$$
$$\boldsymbol{\phi}_y = (\cos y, 1, 0)$$
$$\boldsymbol{\phi}_z = (0, 0, 1)$$
$$\boldsymbol{\phi}_y \times \boldsymbol{\phi}_z = (1, -\cos y, 0)$$
$$\|\boldsymbol{\phi}_y \times \boldsymbol{\phi}_z\| = \sqrt{1 + \cos^2 y}$$
$$\int_{\boldsymbol{\Phi}} f \, dS = \int_0^2 \int_0^\pi \sin y \sqrt{1 + \cos^2 y} \, dy \, dz$$

Let $u = \cos y, \; du = -\sin y \, dy$

$$= 2 \int_{-1}^1 \sqrt{1 + u^2} \, du$$

Let $u = \tan\theta, \; du = sec^2(\theta) \, d\theta$

$$= 2 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{\frac{1}{\cos^2\theta}} \sec^2\theta \, du$$

$$= 2 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^3\theta \, d\theta$$

Let $u = \sec\theta, \; du = \sec\theta\tan\theta \, d\theta, \; dv = \sec^2\theta \, d\theta, \; v = \tan\theta$

$$= 2\left(\Big[\sec\theta\tan\theta \Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} - \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^2\theta \sec\theta \, d\theta \right)$$

$$= 2\left(\Big[\sec\theta\tan\theta \Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} - \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\sec^2\theta - 1) \sec\theta \, d\theta \right)$$

$$2 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^3\theta \, d\theta = 2\left(\Big[\sec\theta\tan\theta \Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} - \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^3\theta \, d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{4}} \sec\theta \, d\theta \right)$$

$$4 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^3\theta \, d\theta = 2\left(\Big[\sec\theta\tan\theta \Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec\theta \, d\theta \right)$$

$$2 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^3\theta \, d\theta = \left(\Big[\sec\theta\tan\theta \Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec\theta \, d\theta \right)$$

$$= \left(\Big[\sec\theta\tan\theta \Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec\theta \left(\frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta} \right) d\theta \right)$$

$$= \left(\Big[\sec\theta\tan\theta \Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} + \Big[\ln|\sec\theta + \tan\theta| \Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \right)$$

$$= 2\sqrt{2} + \ln|1 + \sqrt{2}| - \ln|\sqrt{2} - 1|$$

7. Find the mass of the metallic surface S given by $z = 1 - \dfrac{x^2 + y^2}{2}$ with $0 \le x \le 1$, $0 \le y \le 1$, if the mass density at $(x, y, z) \in S$ is given by $m(x, y, z) = xy$.

$$\Phi(x, y) = (x, y, 1 - \frac{x^2 + y^2}{2})$$

$$\phi_x = (1, 0, -x)$$

$$\phi_y = (0, 1, -y)$$

$$\|\phi_x \times \phi_y\| = \sqrt{x^2 + y^2 + 1}$$

$$\int_\Phi f \, dS = \int_0^1 \int_0^1 xy\sqrt{x^2 + y^2 + 1} \, dx \, dy$$

Let $u = x^2 + y^2 + 1$, $du = 2x \, dx$

$$= \frac{1}{2} \int_0^1 \int_{y^2+1}^{y^2+2} y\sqrt{u} \, du \, dy$$

$$= \frac{1}{3} \int_0^1 y \left[u^{\frac{3}{2}} \right]_{y^2+2}^{y^2+1} dy$$

$$= \frac{1}{3} \int_0^1 y \left[(y^2 + 2)^{\frac{3}{2}} - (y^2 + 1)^{\frac{3}{2}} \right] dy$$

Let $u = y^2 + 1$, $du = 2y \, dy$

$$= \frac{1}{6} \int_1^2 (u + 1)^{\frac{3}{2}} - (u)^{\frac{3}{2}} \, du$$

$$= \frac{1}{15} \left[(u + 1)^{\frac{5}{2}} - (u)^{\frac{5}{2}} \right]_1^2$$

$$= \frac{1}{15} \left[(3^{\frac{5}{2}} - 2^{\frac{5}{2}}) - (2^{\frac{5}{2}} - 1) \right]$$

$$= \frac{1}{15} \left[3^{\frac{5}{2}} - 2^{\frac{7}{2}} - 1 \right]$$

$$= \frac{1}{15} \left[9\sqrt{3} - 8\sqrt{2} + 1 \right]$$