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MATB42: Assignment #8 Mar 20th 2018

1. (a) Let $f, g : \mathbb{R}^n \to \mathbb{R}$; $\boldsymbol{F}, \boldsymbol{G} : \mathbb{R}^n \to \mathbb{R}^n$; and define Δ, the *Laplacian*, by $\Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$.

Verify the following identities

(i) $\text{div}(\boldsymbol{F} + \boldsymbol{G}) = \text{div } \boldsymbol{F} + \text{ div } \boldsymbol{G}$.

$$\text{div}(\boldsymbol{F} + \boldsymbol{G}) = \sum_{i=1}^{n} \frac{\partial(F_i + G_i)}{\partial x_i} = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i} + \frac{\partial G_i}{\partial x_i} = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i} + \sum_{i=1}^{n} \frac{\partial G_i}{\partial x_i} = \text{div}\boldsymbol{F} + \text{div}\boldsymbol{G}$$

(ii) $\text{div}(f\boldsymbol{F}) = f \text{ div } \boldsymbol{F} + \boldsymbol{F} \cdot \text{ grad} f$.

$$\text{div}(f\boldsymbol{F}) = \sum_{i=1}^{n} \frac{\partial f F_i}{\partial x_i} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} F_i + \frac{\partial F_i}{\partial x_i} f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} F_i + f \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i} = \nabla f \cdot \boldsymbol{F} + f \text{div} \boldsymbol{F}$$

(iii) $\Delta(fg) = f\Delta g + g\Delta f + 2(\text{grad} f) \cdot (\text{grad} g)$.

Proof.

$$\begin{aligned}
\Delta(fg) &= \sum_{i=1}^{n} \frac{\partial^2 fg}{\partial^2 x_i} \\
&= \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left[\frac{\partial f}{\partial x_i} g + \frac{\partial g}{\partial x_i} f \right] \\
&= \sum_{i=1}^{n} \left[\frac{\partial f}{\partial^2 x_i} g + \left(\frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_i} \right) + \frac{\partial g}{\partial^2 x_i} f + \left(\frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_i} \right) \right] \\
&= g \sum_{i=1}^{n} \frac{\partial f}{\partial^2 x_i} + 2 \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_i} \right) + f \sum_{i=1}^{n} \frac{\partial g}{\partial^2 x_i} \\
&= g\Delta f + 2[\nabla f \cdot \nabla g] + f \Delta g
\end{aligned}$$

□

(b) Let $f, g : D \subset \mathbb{R}^3 \to \mathbb{R}$ be of class C^1. If R is a solid region contained in D then

$$\iiint_R \nabla f \cdot \nabla g \, dV = \iint_{\partial R} f \nabla g \cdot \boldsymbol{n} \, dS - \iiint_R f \nabla^2 g \, dV$$

($\nabla^2 g = \text{div } (\nabla g)$).

Proof.

$$\begin{aligned}
&\iiint_R \nabla f \cdot \nabla g \, dV = \iint_{\partial R} f \nabla g \cdot \boldsymbol{n} \, dS - \iiint_R f \nabla^2 g \, dV \\
\iff &\iiint_R \nabla f \cdot \nabla g \, dV + \iiint_R f \nabla^2 g \, dV = \iint_{\partial R} f \nabla g \cdot \boldsymbol{n} \, dS
\end{aligned}$$

$$\begin{aligned}
&\iint_{\partial R} f \nabla g \cdot \boldsymbol{n} \, dS \overset{\text{Div Thm}}{=} \iiint_R \text{div}(f \nabla g) \, dV \overset{\text{(ii)}}{=} \iiint_R f(\text{div} \nabla g) + \nabla g \cdot \nabla f \, dV \\
= &\iiint_R f(\text{div} \nabla g) \, dV + \iiint_R \nabla f \cdot \nabla g \, dV = \iiint_R f \nabla^2 g \, dV + \iiint_R \nabla f \cdot \nabla g \, dV
\end{aligned}$$

□