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MATB42: Assignment #8

1. (a) Let $f, g : \mathbb{R}^n \to \mathbb{R}$; $\boldsymbol{F}, \boldsymbol{G} : \mathbb{R}^n \to \mathbb{R}$; and define Δ, the *Laplacian*, by $\Delta f = \sum\limits_{i=1}^{n} \dfrac{\partial^2 f}{\partial x_i^2}$.

   Verify the following identities

   (i) $\operatorname{div}(\boldsymbol{F} + \boldsymbol{G}) = \operatorname{div} \boldsymbol{F} + \operatorname{div} \boldsymbol{G}$.

$$\operatorname{div}(\boldsymbol{F} + \boldsymbol{G}) = \sum_{i=1}^{n} \frac{\partial(\boldsymbol{F} + \boldsymbol{G})}{\partial x_i} = \sum_{i=1}^{n} \frac{\partial \boldsymbol{F}}{\partial x_i} + \frac{\partial \boldsymbol{G}}{\partial x_i} = \sum_{i=1}^{n} \frac{\partial \boldsymbol{F}}{\partial x_i} + \sum_{i=1}^{n} \frac{\partial \boldsymbol{G}}{\partial x_i} = \div \boldsymbol{F} + \div \boldsymbol{G}$$

   (ii) $\operatorname{div}(f\boldsymbol{F}) = f \operatorname{div} \boldsymbol{F} + \boldsymbol{F} \cdot \operatorname{grad} f$.

   (iii) $\Delta(fg) = f\Delta g + g\Delta f + 2(\operatorname{grad} f) \cdot (\operatorname{grad} g)$.

   (b) Let $f, g : D \subset \mathbb{R}^3 \to \mathbb{R}$ be of class C^1. If R is a solid region contained in D then

$$\iiint_R \nabla f \cdot \nabla g \, dV = \iint_{\partial R} f \nabla g \cdot \boldsymbol{n} \, dS - \iiint_R f \nabla^2 g \, dV$$

   $(\nabla^2 g = \operatorname{div}(\nabla g))$.

2. Use the Divergence Theorem to verify your asnwer to question 7 on assignment 8.

3. Let $\boldsymbol{F}(x, y, z) = (x, y^2, e^{yz})$ and let R be a cube centered at the origin with sides of length 2. Evaluate $\int_S \operatorname{div} \boldsymbol{F} \, dV$ directly and by using the Divergence Theorem.

4. Let B be the pyramid with top vertex $(0,0,1)$ and base vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(1,1,0)$. Let S be the 2-dim closed surface bounding B, oriented in the outward direction. Use Gauss' theorem to calculate $\int_S \boldsymbol{F} \cdot d\boldsymbol{S}$, where $\boldsymbol{F}(x, y, z) = (x^2 y, 3y^2 z, 9z^2 x)$.

5. Use the Divergence Theorem to evaluate $\int_S \boldsymbol{F} \cdot d\boldsymbol{S}$, where $\boldsymbol{F}(x, y, z) = \left(z^2 x, \dfrac{y^3}{3} + \tan z, x^2 z + y^2 \right)$ and S is the top half of the unit sphere $x^2 + y^2 + z^2 = 1$, oriented by the unit normalwhich points away from the origin.

6. Let the electric field from a point source at the origin be given by $\boldsymbol{E}(\boldsymbol{x}) = \dfrac{\boldsymbol{x}}{\|\boldsymbol{x}\|^3}$

   (a) What is the outward flux of $\boldsymbol{E}$ across the surface $\dfrac{x^2}{3} + \dfrac{2y^2}{5} + z^2 = 7$.

   (b) Show that the flux of $\boldsymbol{E}$ across that part of the sphere $x^2 + y^2 + z^2 = 25$ with $z \geq 3$ is equal to the flux across that part of the plane $z = 3$ with $x^2 + y^2 \leq 16$.

7. Let $f : \mathbb{R}^n \to \mathbb{R}$ be given by $f(x, y, z) = x^2 yz$ and let η be the 2-form on $\mathbb{R}^3$ given by

$$\eta = (\sin x) \, dx \, dy + (e^y + xyz) \, dx \, dz + (x^2 y^2) \, dy \, dz.$$

   (a) Compute df and $d\eta$.

   (b) Evaluate $df \wedge \eta$.