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MATB42: Assignment #1 1002423727
Bonus Jan 23rd 2018

1. Show that $\int_{-\pi}^{\pi} \sin(kx)\sin(nx)dx = \begin{cases} 0 & , \ k \neq \pm n \\ \pm\pi & , \ k = \pm n \neq 0 \end{cases}$

   Assuming $k \neq \pm n$

$$\int_{-\pi}^{\pi} \sin(kx)\sin(nx)dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \cos(kx - nx) - \cos(kx + nx)dx \qquad \left[\sin(A)\sin(B) = \frac{1}{2}[\cos(A-B) - \cos(A+B)]\right]$$

$$= \frac{1}{2}\left[\int_{-\pi}^{\pi} \cos((k-n)x)dx - \int_{-\pi}^{\pi} \cos((k+n)x)dx\right]$$

$$= \frac{1}{2}\left[\frac{1}{k-n}\Big[sin(kx)\Big]_{-\pi}^{\pi} - \frac{1}{k-n}\Big[sin(kx)\Big]_{-\pi}^{\pi}\right] \qquad \text{[Since cos is even]}$$

$k, n \in \mathbb{Z} \implies (k \pm n) \in \mathbb{Z}$, but $\forall z \in \mathbb{Z}, \ \sin(z\pi) = 0$

$= 0$

   Assuming $k = \pm n$

$$\int_{-\pi}^{\pi} \sin(kx)\sin(nx)dx$$

$$= \int_{-\pi}^{\pi} \sin(kx)\sin(\pm kx)dx$$

$$= \pm\int_{-\pi}^{\pi} \sin^2(kx)dx \qquad \text{[Since sin odd]}$$

$$= \pm\int_{-\pi}^{\pi} \frac{1-\cos(2kx)}{2}dx$$

$$= \pm\frac{1}{2}\left[\int_{-\pi}^{\pi} 1dx - \int_{-\pi}^{\pi}\cos(2kx)dx\right]$$

$$= \pm\frac{1}{2}\left[2\pi - 0\right] \qquad \text{[Since cos even]}$$

$$= \pm\pi$$

2. For each of the following functions, find the N^{th} Fourier polynomial, assuming them to be periodic with period 2π. Use symbolic algebra software to graph the first three approximations together with the original function.

   (a) $f(x) = x^2, \ -\pi < x \leq \pi$

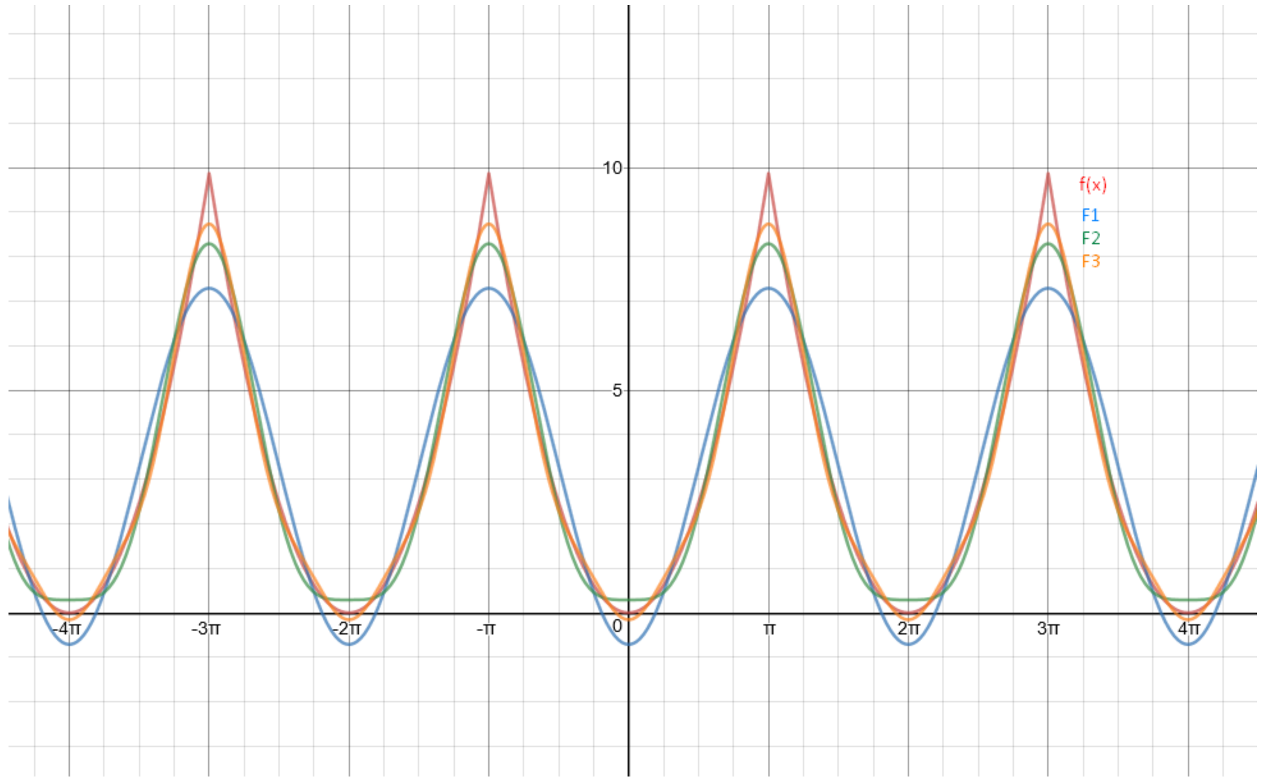
$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx \\ &= \frac{2}{3\pi}\Big[x^3\Big]_0^\pi && \text{[Since } x^2 \text{ is even]} \\ &= \frac{2\pi^2}{3} \end{aligned}$$

$$\begin{aligned} b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin(kx) dx \\ &= 0 && \text{[Since } x^2 \text{ is even but sin is odd]} \end{aligned}$$

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos(kx) dx \\ &= \frac{2}{\pi}\int_{0}^{\pi} x^2 \cos(kx) dx && \text{[Since } x^2 \text{ and cos are even]} \\ &\text{Let } u = x^2, du = 2xdx, dv = \cos(kx)dx, v = \frac{\sin(kx)}{k} \\ &= \frac{2}{\pi}\left[\Big[\frac{1}{k}x^2\sin(kx)\Big]_0^\pi - \frac{2}{k}\int_0^\pi x\sin(kx)dx\right] \\ &\text{Let } u = x, du = 1dx, dv = \sin(kx)dx, v = -\frac{\cos(kx)}{k} \\ &= \frac{2}{\pi}\left[\Big[\frac{1}{k}x^2\sin(kx)\Big]_0^\pi - \frac{2}{k}\Big(\Big[-\frac{1}{k}x\cos(kx)\Big]_0^\pi - \frac{1}{k}\int_0^\pi -\cos(kx)dx\Big)\right] \\ &= \frac{2}{\pi}\left[\Big[\frac{1}{k}x^2\sin(kx)\Big]_0^\pi - \frac{2}{k}\Big(\Big[-\frac{1}{k}x\cos(kx)\Big]_0^\pi + \frac{1}{k^2}\Big[\sin(kx)\Big]_0^\pi\Big)\right] \\ &= \frac{2}{\pi}\left[0 - \frac{2}{k}\Big(\Big[-\frac{1}{k}x\cos(kx)\Big]_0^\pi + 0\Big)\right] \\ &= -\frac{4}{k\pi}\Big[-\frac{1}{k}x\cos(kx)\Big]_0^\pi \\ &= \frac{4}{k^2}\cos(k\pi) - 0 \\ &= \frac{4}{k^2}(-1)^k \end{aligned}$$

Therefore the N^{th} Fourier polynomial is

$$F_N(x) = \frac{\pi^2}{3} + \sum_{k=1}^{N} \frac{4}{k^2}(-1)^k \cos(kx)$$



10
5
0
-4π
-3π
-2π
-π
π
2π
3π
4π
f(x)
F1
F2
F3

(b) $f(x) = \begin{cases} 0 & , -\pi \leq x < 0 \\ x & , 0 \leq x < \pi \end{cases}$

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx \\ &= \frac{1}{\pi} \left[\int_{-\pi}^{0} f(x) dx + \int_{0}^{\pi} f(x) dx \right] \\ &= \frac{1}{\pi} \left[0 + \int_{0}^{\pi} x dx \right] \\ &= \frac{1}{\pi} \left[\frac{1}{2} \Big[x^2 \Big]_0^{\pi} \right] \\ &= \frac{\pi}{2} \end{aligned}$$

$$\begin{aligned} a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx \\ &= \frac{1}{\pi} \left[\int_{-\pi}^{0} 0 dx + \int_{0}^{\pi} x \cos(kx) dx \right] \end{aligned}$$

Let $u = x,\ du = dx,\ dv = \cos(kx),\ v = \frac{1}{k} \sin(kx)$

$$\begin{aligned} &= \frac{1}{\pi} \left[\frac{1}{k} \Big[x \sin(kx) \Big]_0^{\pi} - \frac{1}{k} \int_0^{\pi} \sin(kx) dx \right] \\ &= -\frac{1}{k\pi} \left[\int_0^{\pi} \sin(kx) dx \right] \\ &= \frac{1}{k^2\pi} \Big[\cos(kx) \Big]_0^{\pi} \\ &= \frac{(-1)^{-k} - 1}{k^2\pi} \end{aligned}$$

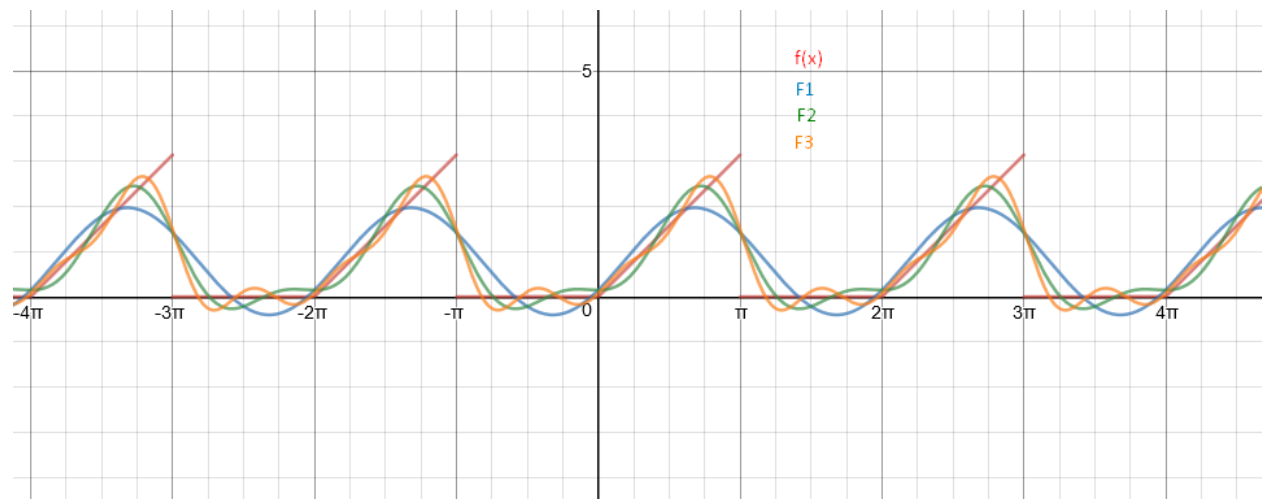
$$\begin{aligned} b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx \\ &= \frac{1}{\pi} \left[\int_{-\pi}^{0} 0 dx + \int_{0}^{\pi} x \sin(kx) dx \right] \end{aligned}$$

Let $u = x,\ du = dx,\ dv = \sin(kx),\ v = -\frac{1}{k} \cos(kx)$

$$\begin{aligned} &= \frac{1}{\pi} \left[-\frac{1}{k} \Big[x \cos(kx) \Big]_0^{\pi} + \frac{1}{k} \int_0^{\pi} \cos(kx) dx \right] \\ &= \frac{1}{k\pi} \left[-\pi \cos(k\pi) + \frac{1}{k} \Big[\sin(kx) \Big]_0^{\pi} \right] \\ &= \frac{1}{k\pi} \Big[-\pi \cos(k\pi) + 0 \Big] \\ &= \frac{(-1)^{k+1}}{k} \end{aligned}$$

Therefore the N^{th} Fourier polynomial is

$$F_N(x) = \frac{\pi}{4} + \sum_{k=1}^{N} \left[\frac{(-1)^k - 1}{k^2\pi} \cos(kx) + \frac{(-1)^{k+1}}{k} \sin(kx) \right]$$


f(x)
F1
F2
F3
5
0
-4π
-3π
-2π
-π
π
2π
3π
4π

3. Find the Fourier series for the function f(x) having period 2π, one period of which is given by

$$f(x) = \begin{cases} 1 & ,0 \le x < \pi \\ x & ,\pi \le x < 2\pi \end{cases}$$

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\ &= \frac{1}{\pi}\left[\int_0^{\pi} f(x)dx + \int_{-\pi}^{0} f(x)dx\right] \\ &= \frac{1}{\pi}\left[\int_0^{\pi} 1dx + \int_{\pi}^{2\pi} xdx\right] \qquad \text{[Period } 2\pi\text{]} \\ &= \frac{1}{\pi}\left[\pi + \left[\frac{x^2}{2}\right]_{\pi}^{2\pi}\right] \\ &= \frac{1}{\pi}\left[\pi + \frac{4\pi^2}{2} - \frac{\pi^2}{2}\right] \\ &= 1 + \frac{3\pi}{2} \end{aligned}$$

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx \\ &= \frac{1}{\pi}\left[\int_0^{\pi} \cos(kx)dx + \int_{\pi}^{2\pi} x\cos(kx)dx\right] \text{[Period } 2\pi\text{]} \\ &\text{Let } u = x,\ du = dx,\ dv = \cos(kx),\ v = \frac{1}{k}\sin(kx) \\ &= \frac{1}{\pi}\left[\frac{1}{k}\Big[\sin(kx)dx\Big]_0^{\pi} + \frac{1}{k}\Big[x\sin(kx)\Big]_{\pi}^{2\pi} - \frac{1}{k}\int_{\pi}^{2\pi} \sin(kx)dx\right] \\ &= \frac{-1}{k\pi}\left[\int_{\pi}^{2\pi} \sin(kx)dx\right] \\ &= \frac{1}{k^2\pi}\Big[\cos(kx)dx\Big]_{\pi}^{2\pi} \\ &= \frac{1-(-1)^k}{k^2\pi} \end{aligned}$$

$$\begin{aligned} b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx \\ &= \frac{1}{\pi}\left[\int_0^{\pi} \sin(kx)dx + \int_{\pi}^{2\pi} x\sin(kx)dx\right] \quad \text{[Period } 2\pi\text{]} \\ &\text{Let } u = x,\ du = dx,\ dv = \sin(kx),\ v = -\frac{1}{k}\cos(kx) \\ &= \frac{1}{\pi}\left[\frac{-1}{k}\Big[\cos(kx)\Big]_0^{\pi} + \frac{-1}{k}\Big[x\cos(kx)\Big]_{\pi}^{2\pi} + \frac{1}{k}\int_{\pi}^{2\pi} \cos(kx)dx\right] \\ &= \frac{1}{k\pi}\left[-\cos(k\pi) + 1 + -2\pi + \pi\cos(k\pi) + \int_{\pi}^{2\pi} \cos(kx)dx\right] \\ &= \frac{1}{k\pi}\left[(\pi-1)(-1)^k + 1 - 2\pi + \Big[\sin(kx)\Big]_{\pi}^{2\pi}\right] \\ &= \frac{1}{k\pi}\left[(\pi-1)(-1)^k + 1 - 2\pi\right] \\ &= \frac{(\pi-1)(-1)^k + 1 - 2\pi}{k\pi} \end{aligned}$$

Therefore the Fourier series for $f(x)$ is

$$F(x) = \frac{2+3\pi}{4} + \sum_{k=1}^{\infty}\left[\frac{1-(-1)^k}{k^2\pi}\cos(kx) + \frac{(\pi-1)(-1)^k + 1 - 2\pi}{k\pi}\sin(kx)\right]$$

4. Let $f(x) = \begin{cases} 0 & , -\pi \leq x < -1 \\ \frac{1}{2} & , -1 \leq x < 1 \\ 0 & , 1 \leq x < \pi \end{cases}$

(a) What fraction of the energy of f is contained in the constant term of its Fourier series?

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\ &= \frac{1}{\pi}\left[0 + \int_{-1}^{1} \frac{1}{2}dx + 0\right] \\ &= \frac{1}{\pi} \end{aligned}$$

$$\begin{aligned} E(f) &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2 dx \\ &= \frac{1}{\pi}\left[0 + \int_{-1}^{1} \frac{1}{4}dx + 0\right] \\ &= \frac{1}{2\pi} \end{aligned}$$

$$\begin{aligned} E(a_0) &= \frac{1}{2}a_0^2 \\ &= \frac{1}{\pi}\left[0 + \int_{-1}^{1} \frac{1}{4}dx + 0\right] \\ &= \frac{1}{2\pi^2} \end{aligned}$$

$$\begin{aligned} \frac{E(a_0)}{E(f)} &= \frac{\frac{1}{2\pi^2}}{\frac{1}{2\pi}} \\ &= \frac{1}{\pi} \\ &\approx 32\% \end{aligned}$$

(b) Find a formula for the energy of the k^{th} harmonic of f.

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx \\ &= \frac{1}{2\pi}\int_{-1}^{1} \cos(kx)dx \\ &= \frac{1}{\pi}\int_{0}^{1} \cos(kx)dx \qquad \text{[cos is even]} \\ &= \frac{1}{k\pi}\sin(k) \end{aligned}$$

$$\begin{aligned} b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx \\ &= 0 \text{ [Since sin is odd and } f \text{ is even]} \end{aligned}$$

$$\begin{aligned} E(k^{th}\text{ harmonic}) = A_k^2 &= \sqrt{a_k^2 + b_k^2}^2 = a_k^2 \\ &= \frac{\sin^2(k)}{k^2\pi^2} \end{aligned}$$

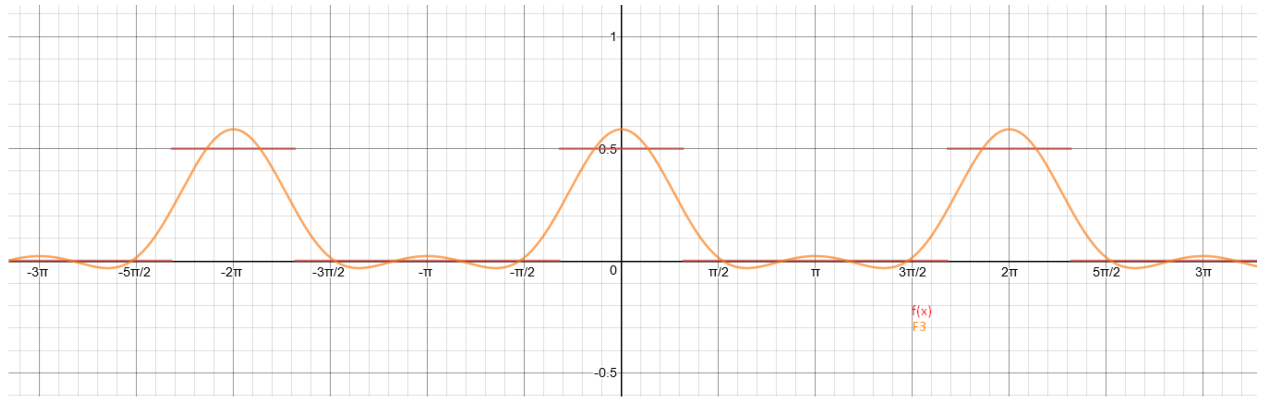
(c) How many terms of the Fourier series of f are needed to capture 80% of the energy of f?

$$\begin{aligned} \frac{\sin^2(k)}{k^2\pi^2}\Big/\frac{1}{2\pi} = \frac{2\sin^2(1)}{1^2\pi} &\approx 45\% \\ \frac{2\sin^2(2)}{2^2\pi} &\approx 13\% \\ 32\% + 45\% + 13\% = 90\% &\geq 80\% \end{aligned}$$

Including the constant term, three terms are required to capture 80% of the energy of f.

(d) Find F_N, the N^{th} Fourier polynomial of f, and use symbolic algebra software to graph f and F_3 on the interval $[-3\pi, 3\pi]$.

$$F_N(x) = \frac{1}{2\pi} + \sum_{k=1}^{N}\left[\frac{\sin(k)}{k\pi}\cos(kx)\right]$$



1
0.5
0
-0.5
-3π
-5π/2
-2π
-3π/2
-π
-π/2
π/2
π
3π/2
2π
5π/2
3π
f(x)
F3

5. Find the Fourier series for the function $f(x)$ (of period 4) which corresponds to $y = x^2 - 4$ on the interval $[-2, 2]$.

$$\begin{aligned} a_0 &= \frac{2}{4}\int_{-2}^{2} f(x)dx \\ &= \int_0^2 x^2 - 4dx \quad [f(x) \text{ is even}] \\ &= \left[\frac{1}{3}\Big[x^3\Big]_0^2 - 4\Big[x\Big]_0^2\right] \\ &= \left[\frac{8}{3} - 8\right] \\ &= -\frac{16}{3} \end{aligned}$$

$$\begin{aligned} a_k &= \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{2\pi kx}{4}\right)dx \\ &= \int_0^2 (x^2-4)\cos\left(\frac{\pi kx}{2}\right)dx \quad [f(x) \text{ and } \cos \text{ are even}] \\ &= \int_0^2 x^2\cos\left(\frac{\pi kx}{2}\right)dx - 4\int_0^2 \cos\left(\frac{\pi kx}{2}\right)dx \end{aligned}$$

Let $u = x^2, du = 2xdx, dv = \cos\left(\frac{\pi kx}{2}\right)dx, v = \frac{2\sin(\frac{\pi kx}{2})}{k\pi}$

$$\begin{aligned} &= \frac{2}{k\pi}\left[x^2\sin\left(\frac{\pi kx}{2}\right)\right]_0^2 - \frac{4}{k\pi}\int_0^2 x\sin\left(\frac{\pi kx}{2}\right)dx \\ &\quad - \frac{8}{k\pi}\left[\sin\left(\frac{\pi kx}{2}\right)\right]_0^2 \\ &= \frac{8}{k\pi}\sin\left(\frac{\pi kx}{2}\right) - \frac{4}{k\pi}\int_0^2 x\sin\left(\frac{\pi kx}{2}\right)dx - \frac{8}{k\pi}\sin\left(\frac{\pi kx}{2}\right) \end{aligned}$$

Let $u = x, du = 1dx, dv = \sin\left(\frac{\pi kx}{2}\right)dx, v = -\frac{2\cos(\frac{\pi kx}{2})}{k\pi}$

$$\begin{aligned} &= -\frac{4}{k\pi}\left[-\frac{2}{k\pi}\left[x\cos\left(\frac{\pi kx}{2}\right)\right]_0^2 + \frac{2}{k\pi}\int_0^2 \cos\left(\frac{\pi kx}{2}\right)dx\right] \\ &= -\frac{4}{k\pi}\left[-\frac{4\cos(\pi k)}{k\pi} + \frac{2}{k^2\pi^2}\left[\sin\left(\frac{\pi kx}{2}\right)\right]_0^2\right] \\ &= -\frac{4}{k\pi}\left[-\frac{4(-1)^k}{k\pi} + \frac{\sin(\pi k)}{k^2\pi^2}\right] \\ &= \frac{16(-1)^k}{k^2\pi^2} \end{aligned}$$

$$\begin{aligned} b_k &= \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{2\pi kx}{4}\right)dx \\ &= 0 \text{ [Since } f \text{ is even, but sin is odd]} \end{aligned}$$

Therefore the Fourier series for $f(x)$ is

$$F(x) = -\frac{8}{3} + \sum_{k=1}^{\infty} \frac{16(-1)^k}{k^2\pi^2}\cos(kx)$$

6. (a) Suppeose $f(x)$ has a continuous derivative $f'(x)$ on $[0, 2\pi]$. Let a_k and b_k the k^{th} Fourier coefficients of f and let a'_k and b'_k be those of f'. Show that

$$\begin{aligned} a'_k &= kb_k + \frac{f(2\pi) - f(0)}{\pi} \\ b'_k &= -ka_k. \end{aligned}$$

By definition,

$$a'_k = \frac{1}{\pi}\int_0^{2\pi} f'(x)\cos(kx)dx$$

Let $u = \cos(kx), du = -k\sin(kx)dx, dv = f'(x)dx, v = f(x)$

$$= \frac{1}{\pi}\left[\Big[f(x)\cos(kx)\Big]_0^{2\pi} + k\int_0^{2\pi}\sin(kx)f(x)dx\right]$$

$$= \frac{1}{\pi}\left[\Big[f(x)\cos(kx)\Big]_0^{2\pi} + k\pi\Big[\frac{1}{\pi}\int_0^{2\pi}\sin(kx)f(x)dx\Big]\right]$$

$$= \frac{1}{\pi}\left[f(2\pi)\cos(2k\pi) - f(0)\cos(0) + k\pi b_k\right] \text{ [Def of } b_k]$$

$$= \frac{1}{\pi}\left[f(2\pi) - f(0) + k\pi b_k\right]$$

$$= kb_k + \frac{f(2\pi) - f(0)}{\pi} \text{ As wanted}$$

By definition,

$$b'_k = \frac{1}{\pi}\int_0^{2\pi} f'(x)\sin(kx)dx$$

Let $u = \sin(kx), du = k\cos(kx)dx, dv = f'(x)dx, v = f(x)$

$$= \frac{1}{\pi}\left[\Big[f(x)\sin(kx)\Big]_0^{2\pi} - k\int_0^{2\pi}\cos(kx)f(x)dx\right]$$

$$= \frac{1}{\pi}\left[\Big[f(x)\sin(kx)\Big]_0^{2\pi} - k\pi\Big[\frac{1}{\pi}\int_0^{2\pi}\cos(kx)f(x)dx\Big]\right]$$

$$= \frac{1}{\pi}\left[-f(2\pi)\sin(2k\pi) + f(0)\sin(0) - k\pi a_k\right] \text{ [Def of } a_k]$$

$$= -ka_k \text{ As wanted}$$

(b) Use part (a) to find all the Fourier coefficients of the restriction of $f(x) = e^{\lambda x}$ to the interval $[0, 2\pi]$ in terms of the constant λ.

Since $\frac{d}{dx}e^{\lambda x} = \lambda e^{\lambda x}$, the coefficients of every term of the Fourier polynomial of the derivative function will simply be that of the original, multiplied by λ.

$$\begin{aligned}
\lambda a_k &= kb_k + \frac{e^{2\pi\lambda} - 1}{\pi} \\
\lambda a_k &= -k\Big(\frac{ka_k}{\lambda}\Big) + \frac{e^{2\pi\lambda} - 1}{\pi} \\
\lambda a_k + \frac{k^2 a_k}{\lambda} &= \frac{e^{2\pi\lambda} - 1}{\pi} \\
a_k\Big(\lambda + \frac{k^2}{\lambda}\Big) &= \frac{e^{2\pi\lambda} - 1}{\pi} \\
a_k &= \frac{e^{2\pi\lambda} - 1}{\pi(\lambda + \frac{k^2}{\lambda})} \\
a_k &= \frac{e^{2\pi\lambda} - 1}{\frac{\pi(\lambda^2 + k^2)}{\lambda}} \\
a_k &= \frac{\lambda(e^{2\pi\lambda} - 1)}{\pi(\lambda^2 + k^2)}
\end{aligned}$$

$$\begin{aligned}
\lambda b_k &= -ka_k \\
b_k &= -\frac{ka_k}{\lambda} \\
b_k &= -\frac{k}{\lambda}\Big(\frac{\lambda(e^{2\pi\lambda} - 1)}{\pi(\lambda^2 + k^2)}\Big) \\
b_k &= -\frac{k(e^{2\pi\lambda} - 1)}{\pi(\lambda^2 + k^2)}
\end{aligned}$$

7. Find two Fourier expansions for the restriction of the function $f(x) = \sin x$ to the interval $[0, \pi]$. In one expansion all the sine terms should have zero coefficient, in the other all cosine terms should have coefficient.

Even expansion (sine terms have zero coefficient)

$$f(-x) = \begin{cases} -\sin(x), & \text{if } -\pi \leq x < 0 \\ \sin(x), & \text{if } 0 \leq x < \pi \end{cases}$$

This works since sin odd, so $-\sin(-x) = \sin(x)$

$$\begin{aligned}
a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx \\
&= \frac{2}{\pi}\int_{0}^{\pi} \sin(x)\cos(kx)dx \\
&= \frac{1}{\pi}\left[\int_0^{\pi} \sin((k+1)x)dx - \int_0^{\pi} \sin((k-1)x)dx\right] \\
&= \frac{1}{\pi}\left[-\frac{1}{k+1}\Big[\cos((k+1)x)\Big]_0^{\pi} + \frac{1}{k-1}\Big[\cos((k-1)x)\Big]_0^{\pi}\right] \\
&= \frac{1}{\pi}\left[-\frac{(-1)^{(k+1)}-1}{k+1} + \frac{(-1)^{(k-1)}-1}{k-1}\right] \\
&= \frac{(-1)^{(k+1)}-1}{\pi}\left[\frac{1}{k-1} - \frac{1}{k+1}\right] \\
&= \frac{(-1)^{(k+1)}-1}{\pi}\left[\frac{k+1-k+1}{k^2-1^2}\right] \\
&= 2\frac{(-1)^{(k+1)}-1}{\pi(k^2-1)} [\text{Valid for } k > 1]
\end{aligned}$$

For k = 1

$$\begin{aligned}
&= \frac{1}{\pi}\left[\int_0^{\pi} \sin(2x)dx - \int_0^{\pi} \sin((1-1)x)dx\right] \\
&= \frac{1}{\pi}\int_0^{\pi} \sin(2x)dx \\
&= -\frac{1}{2\pi}\Big[\cos(2x)\Big]_0^{\pi} \\
&= -\frac{1}{2\pi}\Big[1-1\Big] \\
&= 0
\end{aligned}$$

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\
&= \frac{2}{\pi}\int_0^{\pi} f(x)dx \text{ [Defined to be even]} \\
&= \frac{2}{\pi}\int_0^{\pi} \sin(x)dx \\
&= \frac{2}{\pi}\left[-\Big[\cos(x)\Big]_0^{\pi}\right] \\
&= \frac{2}{\pi}\Big[1+1\Big] \\
&= \frac{4}{\pi}
\end{aligned}$$

$b_k = 0$ [Since f defined even]

Therefore the even Fourier expansion for $f(x)$ is

$$F_N = \frac{2}{\pi} + \sum_{k=2}^{\infty} 2\frac{(-1)^{(k+1)}-1}{\pi(k^2-1)}$$

Odd expansion (cosine terms have zero coefficient)

$$f(-x) = \sin(x), \text{ on } [-\pi, \pi]$$

This works, since sin is already an odd function

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(x)dx \\ &= 0 \text{ [sin is odd]} \end{aligned}$$

$$\begin{aligned} b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(x)\sin(kx)dx \\ &= 1 \text{ if } k = 1,\ 0 \text{ otherwise [Refer to q1]} \end{aligned}$$

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(x)\cos(kx)dx \\ &= 0 \text{ [sin is odd, cos even]} \end{aligned}$$

So the Fourier expansion is simply $\sin(x)$

Bonus

Find the Fourier series for the function $y = f(x)$ of period 2π , if one period is given by

$$f(x) = \begin{cases} -\frac{1}{2} - \frac{x}{2\pi} & , -\pi \le x < 0 \\ \frac{1}{2} - \frac{x}{2\pi} & , 0 \le x < \pi \end{cases}$$

This function is odd over the period since $(-\frac{1}{2} - \frac{-x}{2\pi}) = (-\frac{1}{2} + \frac{x}{2\pi}) = -(\frac{1}{2} - \frac{x}{2\pi})$

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\ &= 0 \ \ \text{[Since } f \text{ odd]} \\ a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx \\ &= 0 \ \ \text{[Since } f \text{ odd and cos even]} \end{aligned}$$

$$\begin{aligned} b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx \\ &= \frac{1}{\pi}\left[\int_{-\pi}^{0}\left[-\frac{1}{2} - \frac{x}{2\pi}\right]\sin(kx)dx + \int_{0}^{\pi}\left[\frac{1}{2} - \frac{x}{2\pi}\right]\sin(kx)dx\right] \\ &= \frac{1}{\pi}\left[-\int_{-\pi}^{0}\left[\frac{(\pi+x)\sin(kx)}{2\pi}\right]dx + \int_{0}^{\pi}\left[\frac{(\pi-x)\sin(kx)}{2\pi}\right]dx\right] \end{aligned}$$

Let $u = (\pi + x), du = 1dx, dv = \sin(kx)dx, v = -\dfrac{\cos(kx)}{k}$

$$\begin{aligned} &= \frac{1}{2\pi^2}\left[\int_{0}^{\pi}\Big[(\pi - x)\sin(kx)\Big]dx + \left[(\pi + x)\frac{\cos(kx)}{k}\right]_{-\pi}^{0} - \int_{-\pi}^{0}\frac{\cos(kx)}{k}dx\right] \\ &= \frac{1}{2\pi^2}\left[\int_{0}^{\pi}\Big[(\pi - x)\sin(kx)\Big]dx + \frac{\pi}{k} - \frac{1}{k^2}\Big[\sin(kx)\Big]_{-\pi}^{0}\right] \\ &= \frac{1}{2\pi^2}\left[\int_{0}^{\pi}\Big[(\pi - x)\sin(kx)\Big]dx + \frac{\pi}{k}\right] \end{aligned}$$

Let $u = (\pi - x), du = -1dx, dv = \sin(kx)dx, v = -\dfrac{\cos(kx)}{k}$

$$\begin{aligned} &= \frac{1}{2\pi^2}\left[-\left[\frac{(\pi - x)\cos(kx)}{k}\right]_{0}^{\pi} - \int_{0}^{\pi}\frac{\cos(kx)}{k}dx + \frac{\pi}{k}\right] \\ &= \frac{1}{2\pi^2}\left[\frac{\pi}{k} - \left[\frac{\sin(kx)}{k^2}\right]_{0}^{\pi} + \frac{\pi}{k}\right] \\ &= \frac{1}{k\pi} \end{aligned}$$

Therefore the Fourier series for $f(x)$ is

$$F(x) = \sum_{k=1}^{\infty}\frac{1}{k\pi}\sin(kx)$$