

1. Let $f(x) = \begin{cases} 0, & -\pi < x < -\frac{\pi}{2} \\ 2, & -\frac{\pi}{2} \leq x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} \leq x < \pi \end{cases}$

(a) Find the Fourier series of f .

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx \\ &= \frac{1}{\pi} \left[\int_{-\pi}^{-\frac{\pi}{2}} 0 dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2 dx + \int_{\frac{\pi}{2}}^{\pi} 0 dx \right] \\ &= \frac{1}{\pi} \left[2\pi \right] \\ &= 2 \end{aligned}$$

$$\begin{aligned} b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx \\ &= \frac{2}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(kx) dx \\ &= 0 \quad [\sin \text{ is odd}] \end{aligned}$$

$$\begin{aligned} a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx \\ &= \frac{2}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(kx) dx \\ &= \frac{2}{k\pi} \left[\sin(kx) \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\ &= \frac{2}{k\pi} \left[2 \sin\left(\frac{k\pi}{2}\right) \right] \\ &= \frac{4}{k\pi} \sin\left(\frac{k\pi}{2}\right) \end{aligned}$$

This is 0 for even elements, and alternating between 1 and -1 for odd elements.

Therefore the Fourier polynomial (for the non-zero terms) is

$$1 + \sum_{l=1}^{\infty} \left[\frac{4(-1)^l}{(2l+1)\pi} \cos((2l+1)x) \right]$$

(b) Determine if the Fourier series in part (a) converges. If it does converge, what are the values to which it converges (on $[-\pi, \pi]$).

The function is continuous on its partitions (they are constant functions), so by the theorem the polynomial converges to $f(x)$ on the continuous intervals. On the discontinuities, it converges to 0 at $\frac{\pi}{2}$ and $-\frac{\pi}{2}$ from the Fundamental theorem, and to 0 at π and $-\pi$.

(c) Use symbolic algebra software to sketch $f(x)$ and its 4th degree Fourier polynomial over the interval $[-3\pi, 3\pi]$.



2. (a) Find the Fourier series of the function $f(x)$ defined by $f(x) = \begin{cases} 0 & , -\pi \leq x < 0 \\ x & , 0 \leq x < \pi \end{cases}$ and extended from this with period 2π to all of \mathbb{R} .

If this Fourier series converges describe the function to which it converges.

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx \\ &= \frac{1}{\pi} \left[\int_{-\pi}^0 f(x) dx + \int_0^{\pi} f(x) dx \right] \\ &= \frac{1}{\pi} \left[0 + \int_0^{\pi} x dx \right] \\ &= \frac{1}{\pi} \left[\frac{1}{2} [x^2]_0^{\pi} \right] \\ &= \frac{\pi}{2} \end{aligned}$$

$$\begin{aligned} a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx \\ &= \frac{1}{\pi} \left[\int_{-\pi}^0 0 dx + \int_0^{\pi} x \cos(kx) dx \right] \end{aligned}$$

Let $u = x$, $du = dx$,

$$\begin{aligned} dv &= \cos(kx), v = \frac{\sin(kx)}{k} \\ &= \frac{1}{\pi} \left[\frac{1}{k} [x \sin(kx)]_0^{\pi} - \frac{1}{k} \int_0^{\pi} \sin(kx) dx \right] \\ &= -\frac{1}{k\pi} \left[\int_0^{\pi} \sin(kx) dx \right] \\ &= \frac{1}{k^2\pi} [\cos(kx)]_0^{\pi} \\ &= \frac{(-1)^{-k} - 1}{k^2\pi} \end{aligned}$$

$$\begin{aligned} b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx \\ &= \frac{1}{\pi} \left[\int_{-\pi}^0 0 dx + \int_0^{\pi} x \sin(kx) dx \right] \end{aligned}$$

Let $u = x$, $du = dx$, $dv = \sin(kx)$, $v = -\frac{1}{k} \cos(kx)$

$$\begin{aligned} &= \frac{1}{\pi} \left[-\frac{1}{k} [x \cos(kx)]_0^{\pi} + \frac{1}{k} \int_0^{\pi} \cos(kx) dx \right] \\ &= \frac{1}{k\pi} \left[-\pi \cos(k\pi) + \frac{1}{k} [\sin(kx)]_0^{\pi} \right] \\ &= \frac{1}{k\pi} \left[-\pi \cos(k\pi) + 0 \right] \\ &= \frac{(-1)^{k+1}}{k} \end{aligned}$$

Therefore the Fourier series of f is

$$F(x) = \frac{\pi}{4} + \sum_{k=1}^{\infty} \left[\frac{(-1)^k - 1}{k^2\pi} \cos(kx) + \frac{(-1)^{k+1}}{k} \sin(kx) \right]$$

Since f is piecewise very smooth ($0, x$ are infinitely differentiable), the series converges to f on $(-\pi, \pi)$ and on both endpoints, it converges to $\frac{\pi}{2}$.

- (b) Using the series from part (a) show that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$$

$$\begin{aligned} F(0) &= \frac{\pi}{4} + \sum_{k=1}^{\infty} \left[\frac{(-1)^k - 1}{k^2\pi} \right] \\ 0 &= \frac{\pi}{4} + \sum_{k=1}^{\infty} \left[\frac{-2}{(2k-1)^2\pi} \right] \end{aligned}$$

$$\begin{aligned} \frac{\pi}{4} &= \sum_{k=1}^{\infty} \left[\frac{2}{(2k-1)^2\pi} \right] \\ \frac{\pi^2}{8} &= \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \end{aligned}$$

3. Find the Fourier series for the restriction of the function $f(x) = 3 + 3x$ to each of the following intervals, $[a, b]$. If the Fourier series converges, to what values will the series converge at the end points?

(a) $[a, b] = [-\pi, \pi]$

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} 3 + 3x dx \\ &= \frac{1}{\pi} \left[6\pi + \frac{3}{2} [x^2]_{-\pi}^{\pi} \right] \\ &= \frac{1}{\pi} [6\pi + 0] \\ &= 6 \end{aligned}$$

$$\begin{aligned} b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx \\ &= \frac{3}{\pi} \left[\int_{-\pi}^{\pi} \sin(kx) dx + \int_{-\pi}^{\pi} x \sin(kx) dx \right] \\ &= \frac{6}{\pi} \left[\int_0^{\pi} x \sin(kx) dx \right] \quad [\text{Since } x \text{ and } \sin \text{ odd}] \end{aligned}$$

Let $u = x, du = 1dx, dv = \sin(kx)dx, v = -\frac{\cos(kx)}{k}$

$$\begin{aligned} &= \frac{6}{\pi} \left[-\frac{1}{k} [x \cos(kx)]_0^{\pi} + \frac{1}{k} \int_0^{\pi} \cos(kx) dx \right] \\ &= \frac{6}{k\pi} \left[\pi(-1)^{k+1} + \frac{1}{k} [\sin(kx)]_0^{\pi} \right] \\ &= \frac{6(-1)^{k+1}}{k} \end{aligned}$$

$$\begin{aligned} a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx \\ &= \frac{3}{\pi} \left[\int_{-\pi}^{\pi} \cos(kx) dx + \int_{-\pi}^{\pi} x \cos(kx) dx \right] \\ &= \frac{6}{k\pi} [\sin(kx)]_0^{\pi} \quad [\text{Since } x \text{ odd and } \cos \text{ even}] \\ &= 0 \end{aligned}$$

Therefore the Fourier series is defined as

$$F(x) = 6 + \sum_{k=1}^{\infty} \frac{6(-1)^{k+1}}{k} \sin(kx)$$

Linear functions are infinitely differentiable so it will converge to $f(x)$ within the interval, and converges to 3 at the endpoints.

(b) $[a, b] = [0, 2\pi]$

$$\begin{aligned}
 a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx \\
 &= \frac{1}{\pi} \int_0^{2\pi} 3 + 3x dx \\
 &= \frac{1}{\pi} \left[6\pi + \frac{3}{2} [x^2]_0^{2\pi} \right] \\
 &= \frac{1}{\pi} \left[6\pi + 6\pi^2 \right] \\
 &= 6(\pi + 1)
 \end{aligned}$$

$$\begin{aligned}
 a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx \\
 &= \frac{3}{\pi} \left[\int_0^{2\pi} \cos(kx) dx + \int_0^{2\pi} x \cos(kx) dx \right]
 \end{aligned}$$

Let $u = x$, $du = dx$, $dv = \cos(kx)$, $v = \frac{1}{k} \sin(kx)$

$$\begin{aligned}
 &= \frac{3}{k\pi} \left[\left[\sin(kx) \right]_0^{2\pi} + \left[x \sin(kx) \right]_0^{2\pi} - \int_0^{2\pi} \sin(kx) dx \right] \\
 &= -\frac{3}{k^2\pi} \left[\cos(kx) \right]_0^{2\pi} \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx \\
 &= \frac{3}{\pi} \left[\int_0^{2\pi} \sin(kx) dx + \int_0^{2\pi} x \sin(kx) dx \right]
 \end{aligned}$$

Let $u = x$, $du = 1dx$, $dv = \sin(kx)dx$, $v = -\frac{\cos(kx)}{k}$

$$\begin{aligned}
 &= \frac{3}{k\pi} \left[\left[\cos(kx) \right]_0^{2\pi} - \left[x \cos(kx) \right]_0^{2\pi} + \int_0^{2\pi} \cos(kx) dx \right] \\
 &= \frac{3}{k\pi} \left[-2\pi + \frac{1}{k} \left[\sin(kx) \right]_0^{2\pi} \right] \\
 &= -\frac{6}{k^2}
 \end{aligned}$$

Therefore the Fourier series is defined as

$$F(x) = 6(\pi + 1) + \sum_{k=1}^{\infty} -\frac{6}{k^2} \sin(kx)$$

Linear functions are infinitely differentiable so it will converge to $f(x)$ within the interval, and covers to $3 + 3\pi$ at the endpoints.

4. Find the Fourier series of the function $f(x)$ defined on $[0, 2\pi]$ by $f(x) = x(x - 2\pi)$ and extended from this with period 2π to all of \mathbb{R} . Use symbolic algebra software to graph the 4th degree Fourier polynomial together with the original function.

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} x(x - 2\pi) dx$$

$$= \frac{1}{\pi} \left[\int_0^{2\pi} x^2 \sin(kx) dx - 2\pi \int_0^{2\pi} x \sin(kx) dx \right]$$

$$\text{Let } u = x, du = dx, dv = \sin(kx), v = -\frac{1}{k} \cos(kx)$$

$$= \frac{1}{\pi} \left[\int_0^{2\pi} x^2 \sin(kx) dx - 2\pi \left(-\frac{1}{k} \left[x \cos(kx) \right]_0^{2\pi} \right. \right.$$

$$\left. + \frac{1}{k} \int_0^{2\pi} \cos(kx) dx \right]$$

$$= \frac{1}{\pi} \left[\int_0^{2\pi} x^2 \sin(kx) dx - 2\pi \left(-\frac{2\pi}{k} \right. \right.$$

$$\left. + \frac{1}{k^2} \left[\sin(kx) \right]_0^{2\pi} \right]$$

$$= \frac{1}{\pi} \left[\int_0^{2\pi} x^2 \sin(kx) dx + \frac{4\pi^2}{k} \right]$$

$$\text{Let } u = x^2, du = 2x dx,$$

$$dv = \sin(kx) dx, v = -\frac{\cos(kx)}{k}$$

$$= \frac{1}{k\pi} \left[-\left[x^2 \cos(kx) \right]_0^{2\pi} + \int_0^{2\pi} x \cos(kx) dx + 4\pi^2 \right]$$

$$\text{Let } u = x, du = dx, dv = \sin(kx), v = -\frac{1}{k} \cos(kx)$$

$$= \frac{1}{k\pi} \left[\frac{1}{k} \left[x \sin(kx) \right]_0^{2\pi} - \frac{1}{k} \int_0^{2\pi} \cos(kx) dx \right]$$

$$= \frac{1}{k\pi} \left[-\frac{1}{k^2} \left[\sin(kx) \right]_0^{2\pi} \right]$$

$$= 0$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx$$

$$= \frac{1}{\pi} \left[\int_0^{2\pi} x^2 \cos(kx) dx - 2\pi \int_0^{2\pi} x \cos(kx) dx \right]$$

$$\text{Let } u = x^2, du = 2x dx, dv = \cos(kx) dx, v = \frac{\sin(kx)}{k}$$

$$= \frac{1}{\pi} \left[\frac{1}{k} \left[x^2 \sin(kx) \right]_0^{2\pi} - \frac{2}{k} \int_0^{2\pi} x \sin(kx) dx \right.$$

$$\left. - 2\pi \int_0^{2\pi} x \cos(kx) dx \right]$$

$$= \frac{1}{\pi} \left[-\frac{2}{k} \int_0^{2\pi} x \sin(kx) dx - 2\pi \int_0^{2\pi} x \cos(kx) dx \right]$$

$$\text{Let } u = x, du = dx, dv = \sin(kx) dx, v = -\frac{\cos(kx)}{k}$$

$$= \frac{1}{\pi} \left[\frac{2}{k^2} \left[x \cos(kx) \right]_0^{2\pi} - \frac{1}{k} \int_0^{2\pi} \cos(kx) dx \right.$$

$$\left. - 2\pi \int_0^{2\pi} x \cos(kx) dx \right]$$

$$= \frac{1}{\pi} \left[\frac{4\pi}{k^2} - \frac{1}{k^2} \left[\sin(kx) \right]_0^{2\pi} - 2\pi \int_0^{2\pi} x \cos(kx) dx \right]$$

$$\text{Let } u = x, du = dx, dv = \cos(kx), v = \frac{1}{k} \sin(kx)$$

$$= \frac{1}{\pi} \left[\frac{4\pi}{k^2} - 2\pi \left(\frac{1}{k} \left[x \sin(kx) \right]_0^{2\pi} - \frac{1}{k} \int_0^{2\pi} \sin(kx) dx \right) \right]$$

$$= \frac{1}{\pi} \left[\frac{4\pi}{k^2} + \frac{2\pi}{k} \int_0^{2\pi} \sin(kx) dx \right]$$

$$= \frac{1}{\pi} \left[\frac{4\pi}{k^2} + \frac{2\pi}{k^2} \left[\cos(kx) \right]_0^{2\pi} \right]$$

$$= \frac{4}{k^2}$$

$$\begin{aligned}
 a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx \\
 &= \frac{1}{\pi} \int_0^{2\pi} x(x - 2\pi) dx \\
 &= \frac{1}{\pi} \left[\int_0^{2\pi} x^2 dx - \int_0^{2\pi} 2x\pi dx \right] \\
 &= \frac{1}{\pi} \left[\frac{1}{3} [x^3]_0^{2\pi} - \pi [x^2]_0^{2\pi} \right] \\
 &= \frac{1}{\pi} \left[\frac{8\pi^3}{3} - 4\pi^3 \right] \\
 &= -\frac{4\pi^2}{3}
 \end{aligned}$$

Therefore the Fourier series of f is

$$F(x) = -\frac{2\pi^2}{3} + \sum_{k=1}^{\infty} \frac{4}{k^2} \cos(kx)$$



5. Let $f(x)$ be defined on $[0, 2\pi]$ by $f(x) = x(x - 2\pi)$.

(a) Find the Fourier cosine series of f .

From question 4, we can see that the function is already even, hence the Fourier series of the function itself is a cosine series of f . Namely

$$F(x) = -\frac{2\pi^2}{3} + \sum_{k=1}^{\infty} \frac{4}{k^2} \cos(kx)$$

(b) Find the Fourier sine series of f .

To extend this as an odd function, define the f on the range $[-2\pi, 0]$ as $f(x) = -((x + 2\pi)((x + 2\pi) - 2\pi)) = -x(x + 2\pi)$. Note that this definition of f now has a period of 4π .

$$\begin{aligned} b_k &= \frac{1}{2\pi} \int_{-2\pi}^{2\pi} f(x) \sin\left(\frac{kx}{2}\right) dx \\ &= -\frac{1}{\pi} \left[\int_{-2\pi}^0 x(x + 2\pi) \sin\left(\frac{kx}{2}\right) dx \right] \quad [f \text{ and } \sin \text{ are both odd so the integrand is even}] \\ &= -\frac{1}{\pi} \left[\int_{-2\pi}^0 x^2 \sin\left(\frac{kx}{2}\right) dx + 2\pi \int_{-2\pi}^0 x \sin\left(\frac{kx}{2}\right) dx \right] \end{aligned}$$

$$\begin{aligned} \text{Let } u &= x^2, du = 2x dx, dv = \sin\left(\frac{kx}{2}\right) dx, v = -\frac{2 \cos\left(\frac{kx}{2}\right)}{k} \\ &= -\frac{1}{\pi} \left[-\frac{2}{k} \left[x^2 \cos\left(\frac{kx}{2}\right) \right]_{-2\pi}^0 + \frac{4}{k} \int_{-2\pi}^0 x \cos\left(\frac{kx}{2}\right) dx + 2\pi \int_{-2\pi}^0 x \sin\left(\frac{kx}{2}\right) dx \right] \\ &= -\frac{1}{\pi} \left[\frac{8\pi^2(-1)^k}{k} + \frac{4}{k} \int_{-2\pi}^0 x \cos\left(\frac{kx}{2}\right) dx + 2\pi \int_{-2\pi}^0 x \sin\left(\frac{kx}{2}\right) dx \right] \end{aligned}$$

$$\begin{aligned} \text{Let } u &= x, du = dx, dv = \cos\left(\frac{kx}{2}\right), v = \frac{2}{k} \sin\left(\frac{kx}{2}\right) \\ &= -\frac{1}{\pi} \left[\frac{8\pi^2}{k}(-1)^k + \frac{4}{k^2} \left[x \sin\left(\frac{kx}{2}\right) \right]_{-2\pi}^0 - \frac{8}{k^2} \int_{-2\pi}^0 \sin\left(\frac{kx}{2}\right) dx + 2\pi \int_{-2\pi}^0 x \sin\left(\frac{kx}{2}\right) dx \right] \\ &= -\frac{1}{\pi} \left[\frac{8\pi^2}{k}(-1)^k + \frac{16}{k^3} \left[\cos\left(\frac{kx}{2}\right) \right]_{-2\pi}^0 + 2\pi \int_{-2\pi}^0 x \sin\left(\frac{kx}{2}\right) dx \right] \end{aligned}$$

$$\begin{aligned} \text{Let } u &= x, du = dx, dv = \sin\left(\frac{kx}{2}\right) dx, v = -\frac{2 \cos\left(\frac{kx}{2}\right)}{k} \\ &= -\frac{1}{\pi} \left[\frac{8\pi^2}{k}(-1)^k + \frac{16}{k^3} (1 - (-1)^k) + \frac{4\pi}{k} \left(- \left[x \cos\left(\frac{kx}{2}\right) \right]_{-2\pi}^0 + \int_{-2\pi}^0 \cos\left(\frac{kx}{2}\right) dx \right) \right] \\ &= -\frac{1}{\pi} \left[\frac{8\pi^2}{k}(-1)^k + \frac{16}{k^3} (1 - (-1)^k) + \frac{4\pi}{k} \left(2\pi(-1)^{k+1} + \frac{2}{k} \left[\sin\left(\frac{kx}{2}\right) \right]_{-2\pi}^0 \right) \right] \\ &= -\frac{1}{\pi} \left[\frac{8\pi^2}{k}(-1)^k + \frac{16}{k^3} (1 - (-1)^k) + \frac{8\pi^2}{k}(-1)^{k+1} \right] \\ &= \frac{16}{k^3\pi} ((-1)^k - 1) \end{aligned}$$

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-2\pi}^{2\pi} f(x) dx \\ &= 0 \quad [\text{Since } f \text{ is defined odd}] \end{aligned}$$

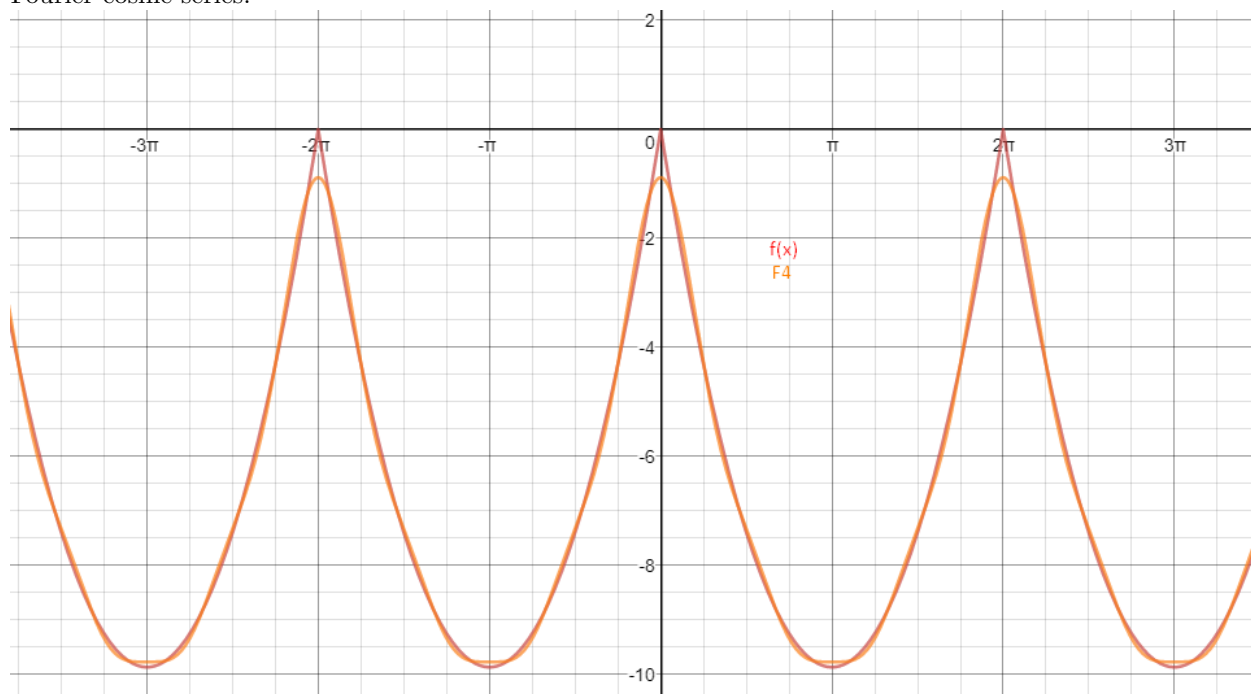
$$\begin{aligned} a_k &= \frac{1}{2\pi} \int_{-2\pi}^{2\pi} f(x) \cos\left(\frac{kx}{2}\right) dx \\ &= 0 \quad [\text{Since } f \text{ is defined odd}] \end{aligned}$$

The Fourier sine series is thusly

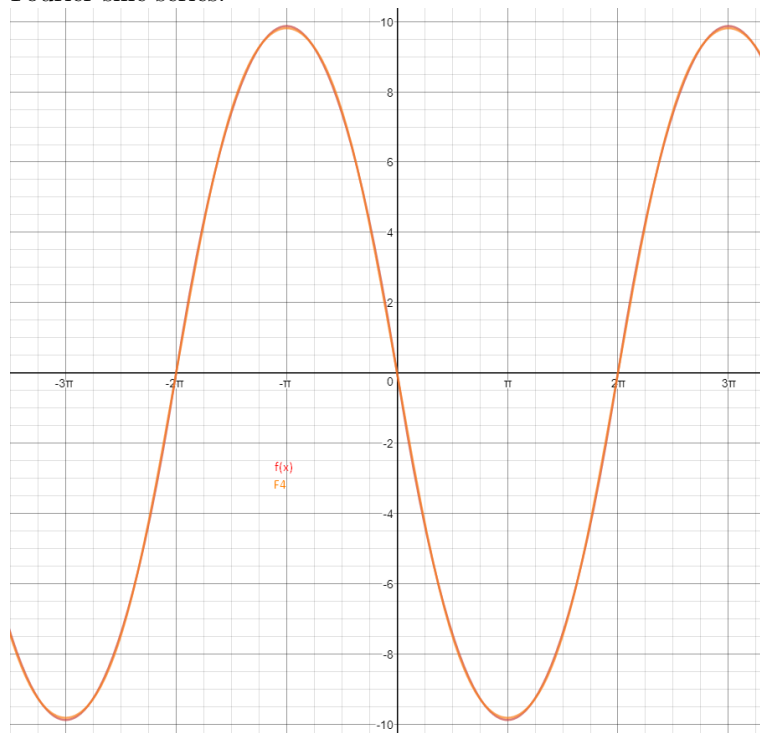
$$F(x) = \sum_{k=1}^{\infty} \frac{16}{k^3 \pi} ((-1)^k - 1) \sin\left(\frac{kx}{2}\right)$$

- (c) Use symbolic algebra software to graph the 4th degree Fourier polynomials from parts (a) and (b) together with the original function.

Fourier cosine series:



Fourier sine series:



6. Find the Fourier series for the following functions:

(a) $f(x) = \sin^2 x + \sin^3 x$

$$\begin{aligned}\sin^2 x + \sin^3 x &= (1/2i)^2(e^{ix} - e^{-ix}) + (1/2i)^3(e^{ix} - e^{-ix}) \\ &= (-1/4)(e^{2ix} - 2(e^{ix-i x}) + e^{-2ix}) + (-1/8i)(e^{3ix} - 3(e^{2ix-ix}) + 3(e^{ix-2ix}) - e^{-3ix}) \\ &= (-1/4)(e^{2ix} + e^{-2ix} - 2) + (-1/8i)(e^{3ix} - e^{-3ix} - 3(e^{ix}) + 3(e^{-ix})) \\ &= (-1/2)(\cos(2x) - 2) + (-1/4)(\sin(3x) - 3\sin(x))\end{aligned}$$

(b) $f(x) = \sin^4 x$

(c) $f(x) = \cos^7 x$

(Hint: Recall that $\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$ and $\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$)

The next question is for those among you who have previously seen complex numbers. It gives another approach to Fourier series.

7. Suppose

i. $f(x)$ is a real valued function of x ,

ii. $f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}$ on $[-\pi, \pi]$, where the C_n are complex constants, and

iii. that the term by term theorem holds true in this case

(a) Express the C_n as integrals involving f .

(b) Find the Fourier coefficients of f in terms of the C_n .

(c) Find the C_n in terms of the Fourier coefficients of f .