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MATB42: Assignment #6

1. Let $\omega = \dfrac{-y}{x^2+y^2}\ dx + \dfrac{x}{x^2+y^2}\ dy$. Calculate $\displaystyle\int_{\boldsymbol{\gamma}} \omega$ where
   (a) $\boldsymbol{\gamma}$ is the boundary of the triangle with vertices (in order) (0,1), (2,3) and (2,1).
   (b) $\boldsymbol{\gamma}$ is the boundary curve of the region $\left\{(x,y)\in\mathbb{R}^2 \middle| \dfrac{(x-2)^2}{9}+\dfrac{(y+1)^2}{4}\leq 1\right\}$ oriented in a counter clockwise direction
   (c) $\boldsymbol{\gamma}$ is the graph of the polar equation $r = 3+2\sin\theta$ oriented in the clockwise direction.
2. Let $\omega = (y^2+z\ \ln 3)\ dx + (2xy+\sin z)\ dy + (y\cos z + (x+1)\ln 3)\ dz$. Determine if ω is exact. If it is, use the algorithm given in class to find the potential function g.
3. Evaluate the double integral $\displaystyle\int_{-1} 1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 3x^2y^2 dy\ dx$, by first finding an equivalent line integral.
4. Let R be a region in $\mathbb{R}^2$ and let $\boldsymbol{\gamma}$ be a counterclockwise parametrization of ∂R. Let $\boldsymbol{F} = (F_1, F-2)$ be a C^1 vector field defined throughout R and on ∂R and let $\boldsymbol{n}$ be the outward pointing unit normal ector to $\boldsymbol{\gamma}$. Use Green's theorem to give a double integral over R which is equivalent to $\displaystyle\int_{\boldsymbol{\gamma}} \boldsymbol{F}\cdot\boldsymbol{n}\ ds$.
5. Give a parametrization for each of the following surfaces, use a computer algebra sustem to plot the surface and find a unit vector normal to the surface.
   (a) The piece of the cylinder $y^2+z^2=1$ between $x=-1$ and $x=3$.
   (b) The piece of the plane $z = x+y+5$ which lies over the unit disk $x^2+y^2\leq 1$.
   (c) The piece of the sphere $x^2+y^2+z^2=4$ which lies above the plane $z=1$.
   (d) The piece of the plane $x+y+z=1$ which lies above the parallelogram: $0\leq y-x\leq 1, 0\leq y+x2\leq 1$.
6. Let S be the surface given parameterically by $\boldsymbol{\Phi}(u,v) = (u^2, 3v, u^2+v)$ where $(u,v)\in D$, the interior of a triangle with vertices (0,0), (3,0) and (3,3).
   (a) Find the surface area of S.
   (b) Find the equation of the tangent plane to S at the point (4,9,7).
7. Suppose the surface S is the graph of a function $f:\mathbb{R}^2\to\mathbb{R}$. Give a natural parametrization of S (in terms of f) and derive the formula $\|\boldsymbol{\phi}_u\times\boldsymbol{\phi}_v\| = \sqrt{1+\|\text{grad } f\|^2}$
8. A paraboloid of revolution S is parameterized by $\boldsymbol{\Phi}(u,v) = u\cos v, u\sin v, u^2$, $0\leq u\leq 2, 0\leq v\leq 2\pi$.
   (a) Find an equation in x, y and z describing the surface.
   (b) What are the geometric meanings of the parameters u and v?
   (c) Find a unit vector orthogonal to the surface of $\boldsymbol{\Phi}(u,v)$.
   (d) Find the equation for the tangent plane at $\boldsymbol{\Phi}(u_0,v_0) = (1,1,2)$ and express your answer in the following two ways:
      i. parameterized by u and v; and
      ii. in terms of x, y and z.
   (e) Find the area of S.
   (cf. page 424, #16)
9. Let a differentiable function $\boldsymbol{\Phi}:\mathbb{R}^2\to\mathbb{R}^3$ define a parametrized surface.
   (a) Assuming $\boldsymbol{\phi}_u\times\boldsymbol{\phi}_v\neq 0$, show that the range of the linear transformation $D\boldsymbol{\Phi}(u_0,v_0)$ is the plane spanned by $\boldsymbol{\phi}_u$ and $\boldsymbol{\phi}_v$. [Here $\boldsymbol{\phi}_u$ and $\boldsymbol{\phi}_v$ are evaluated at (u_0,v_0).]

(b) Show that $\boldsymbol{w} \perp (\boldsymbol{\phi}_u \times \boldsymbol{\phi}_v)$ if and only if $\boldsymbol{w}$ is in the range of $D\mathbf{\Phi}(u_0, v_0)$.

(c) Show that the tangent plane as defined in terms of $\boldsymbol{\phi}_u \times \boldsymbol{\phi}_v(u_0, v_0)$ is the same as the "parametrized plane"

$$(u, v) \mapsto \mathbf{\Phi}(u_0, v_0) + D\mathbf{\Phi}(u_0, v_0) \begin{bmatrix} u - u_0 \\ v - v_0 \end{bmatrix}$$

(cf. page 383 #20)