

1. (a) Find an equation of the tangent plane to the surface  $S$  defined parametrically by  $\Phi(u, v) = (u^2 + v, v, u + v^2)$  at the point  $(9, 0, 3)$ .

$$v = 0$$

$$u + v^2 = 3 \implies u = 3$$

$$\phi_u = (2(3), 0, 1)$$

$$\phi_v = (1, 1, 2(0))$$

$$\phi_u \times \phi_v = (-1, 1, 6)$$

So the tangent plane can be given by

$$0 = ((x - 9, y, z - 3) \cdot (-1, 1, 6))$$

$$0 = (9 - x + y + 6z - 18)$$

$$9 = -x + y + 6z$$

- (b) Use symbolic algebra software to sketch the surface  $S$  and its tangent plane from part (a).



2. Use a surface integral to find the area of the triangle in  $\mathbb{R}^3$  with vertices  $(1,1,0)$ ,  $(1,2,1)$  and  $(3,3,2)$ .

Since the surface is a plane, the tangent plane is just parallel to the plane, so tangent vectors can be given by taking vectors in the plane i.e.  $\phi_u = (3,3,2) - (1,1,0) = (2,2,2)$  and  $\phi_v = (1,2,1) - (1,1,0) = (0,1,1)$ . Now this is a triangle with side lengths of equal magnitude to the tangents, so the integral is given as.

$$\begin{aligned}\|\phi_u \times \phi_v\| &= \|(0, -2, 2)\| = \sqrt{4+4} = 2\sqrt{2} \\ \int_0^1 \int_0^v \|\phi_u \times \phi_v\| \, du \, dv &= \int_0^1 \int_0^v 2\sqrt{2} \, du \, dv \\ &= \int_0^1 2\sqrt{2}v \, dv \\ &= \sqrt{2}\end{aligned}$$

3. Calculate the surface area of the piece of the cone  $x^2 + y^2 - z^2 = 0$  which lies inside the cylinder  $x^2 + y^2 = 4$ .

We can see the radius of the cylinder is 2, so the cone portion that's cut out is the part which has radius less than or equal to 2  $\implies 0 \leq z \leq 2$ . Using polar for the cone,  $0 \leq \theta \leq 2\pi$ .

$$\begin{aligned}
 \Phi(\theta, z) &= (z \cos \theta, z \sin \theta, z) \\
 \phi_\theta &= (-z \sin \theta, z \cos \theta, 0) \\
 \phi_z &= (\cos \theta, \sin \theta, 1) \\
 \phi_\theta \times \phi_z &= (z \cos \theta, z \sin \theta, -z \sin^2 \theta - z \cos^2 \theta) \\
 &= (z \cos \theta, z \sin \theta, -z) \\
 \|\phi_\theta \times \phi_z\| &= z^2 \cos^2 \theta + z^2 \sin^2 \theta + z^2 = 2z^2
 \end{aligned}$$

$$\begin{aligned}
 \int_{\Phi} f \, dS &= \int_0^{2\pi} \int_0^2 2z^2 \, dz \, d\theta \\
 &= \int_0^{2\pi} \left[ \frac{2}{3} z^3 \right]_0^2 d\theta \\
 &= \int_0^{2\pi} \frac{16}{3} d\theta = \frac{32\pi}{3}
 \end{aligned}$$

4. (a) Find the area of the portion of the unit sphere that is cut out by the cone  $z = \sqrt{x^2 + y^2}$ .  
(cf. page 391, #10)

$$\begin{aligned}
 0 &\leq \varphi \leq \pi, \quad 0 \leq \theta \leq 2\pi \\
 \Phi_{\text{sphere}}(\theta, \varphi) &= (\cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi) \\
 \phi_\theta &= (-\sin \theta \sin \varphi, \cos \theta \sin \varphi, 0) \\
 \phi_\varphi &= (\cos \theta \cos \varphi, \sin \theta \cos \varphi, -\sin \varphi) \\
 \phi_\theta \times \phi_\varphi &= (-\cos \theta \sin^2 \varphi, -\sin \theta \sin^2 \varphi, -\sin^2 \theta \sin \varphi \cos \varphi - \cos^2 \theta \sin \varphi \cos \varphi) \\
 &= (-\cos \theta \sin^2 \varphi, -\sin \theta \sin^2 \varphi, -\sin \varphi \cos \varphi) \\
 \|\phi_\theta \times \phi_\varphi\| &= \sqrt{\cos^2 \theta \sin^4 \varphi + \sin^2 \theta \sin^4 \varphi + \sin^2 \varphi \cos^2 \varphi} \\
 &= \sqrt{\sin^2 \varphi} = \sin \varphi
 \end{aligned}$$

$$\begin{aligned}
 z &> 0, \quad 0 \leq \theta \leq 2\pi \\
 \Phi_{\text{cone}}(\theta, z) &= (z \cos \theta, z \sin \theta, z) \\
 \phi_z &= (\cos \theta, \sin \theta, 1) \\
 \phi_\theta &= (-z \sin \theta, z \cos \theta, 0) \\
 \phi_z \times \phi_\theta &= (-z \cos \theta, -z \sin \theta, z) \\
 \|\phi_z \times \phi_\theta\| &= 2z^2
 \end{aligned}$$

For the unit sphere  $x^2 + y^2 + z^2 = 1$ , but the cone is  $x^2 + y^2 = z^2 \implies$  sub  $z$  into sphere gives  $2x^2 + 2y^2 = 1$  So the exact intersection of the surfaces is a circle of radius  $2/\sqrt{2}$  centered at the origin. so the surface cut out is the section of the top of the sphere where  $z \geq 2\sqrt{2} \implies \varphi \leq \frac{\pi}{4}$  from the  $z = \cos \varphi$  portion of the parametrization. So the ranges are  $0 \leq \theta \leq 2\pi, 0 \leq \varphi \leq \frac{\pi}{4}$ . The area is therefore

$$\begin{aligned}
 \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \sin \varphi \, d\varphi \, d\theta &= \int_0^{2\pi} \left[ -\cos \varphi \right]_0^{\frac{\pi}{4}} d\theta \\
 &= 2\pi \left[ -\left(\cos\left(\frac{\pi}{4}\right)\right) - (-\cos(0)) \right] \\
 &= 2\pi \left[ -\frac{\sqrt{2}}{2} + 1 \right] \\
 &= \pi(2 - \sqrt{2})
 \end{aligned}$$

- (b) Find the area of the portion of the cone  $z = \sqrt{x^2 + y^2}$  that is cut out by the unit sphere.  
Plugging in  $x^2 + y^2 = 1/2$  to the cone equation again gives  $z^2 = 1/2 \implies z = \pm \frac{\sqrt{2}}{2}$  but  $z \geq 0$  by the cone definition so  $0 \leq z \leq \frac{\sqrt{2}}{2}$ .

$$\begin{aligned}
 A(\Phi_{\text{cone}}) &= \int_0^{2\pi} \int_0^{\frac{1}{4}} 2z^2 \, dz \, d\theta \\
 &= \int_0^{2\pi} \frac{2}{3} \cdot \frac{1}{4^3} \, d\theta \\
 &= \frac{\pi}{3(16)} \\
 &= \frac{\pi}{48}
 \end{aligned}$$

5. Let  $\Phi : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be a parametrization of a 2-dim surface  $S$  in  $\mathbb{R}^3$ .

(a) Set

$$E = \|\phi_u\|^2, \quad F = \phi_u \cdot \phi_v, \quad G = \|\phi_v\|^2,$$

Show that the surface area of  $S$  is

$$A(S) = \iint_D \sqrt{EG - F^2} dA$$

$$\begin{aligned} \iint_D \sqrt{EG - F^2} dA &= \iint_D \sqrt{\|\phi_u\|^2 \|\phi_v\|^2 - (\phi_u \cdot \phi_v)^2} dA \\ &= \iint_D \sqrt{(\|\phi_u\| \|\phi_v\|)^2 - (\|\phi_u\| \|\phi_v\|)^2 \cos^2 \theta} dA \quad \text{Where } \theta \text{ is the angle between } \phi_u \text{ and } \phi_v. \\ &= \iint_D \sqrt{(\|\phi_u\| \|\phi_v\|)^2 (1 - \cos^2 \theta)} dA \\ &= \iint_D \sqrt{(\|\phi_u\| \|\phi_v\|)^2 (\sin^2 \theta)} dA \\ &= \iint_D \sqrt{\|\phi_u \times \phi_v\|^2} dA \\ &= \iint_D \|\phi_u \times \phi_v\| dA \\ &= \int_{\Phi} 1 dS \end{aligned}$$

(b) What does the formula for  $A(S)$  become if the vectors  $\phi_u$  and  $\phi_v$  are orthogonal?  
If the vectors are orthogonal, then the dot product is 0, so the equation reduces to

$$A(S) = \iint_D \|\phi_u\| \|\phi_v\| dA$$

(c) Use parts (a) and (b) to compute the surface area of a sphere of radius  $a$ .  
(cf. Marsden & Tromba, page 399, # 23.)

$$\begin{aligned} 0 &\leq \varphi \leq \pi, \quad 0 \leq \theta \leq 2\pi \\ \Phi(\theta, \varphi) &= a(\cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi) \\ \phi_\theta &= a(-\sin \theta \sin \varphi, \cos \theta \sin \varphi, 0) \\ \phi_\varphi &= a(\cos \theta \cos \varphi, \sin \theta \cos \varphi, -\sin \varphi) \\ \|\phi_\theta\| &= a \sin \varphi, \quad \|\phi_\varphi\| = a \\ \implies A(S) &= a^2 \int_0^{2\pi} \int_0^\pi \sin \varphi d\varphi d\theta \\ &= a^2 \int_0^{2\pi} \left[ -\cos \varphi \right]_0^\pi d\varphi d\theta \\ &= a^2 \int_0^{2\pi} -(-1 - 1) d\varphi d\theta \\ &= a^2 2 \int_0^{2\pi} 1 d\varphi d\theta \\ &= 4\pi a^2 \end{aligned}$$

6. For each of the following surfaces  $S$ , sketch  $S$  (using symbolic software) and evaluate the surface integral  $\int_S f dS$ , where  $f(x, y, z) = x$ .

(a)  $S$  is that part of the surface  $y = 4 - x^2$  between  $z = 0$  and  $z = 1$ , with  $y \geq 0$ .

$$y \geq 0 \implies 4 - x^2 \geq 0 \implies x^2 \leq 4 \implies |x| < 2$$

$$\Phi(x, z) = (x, 4 - x^2, z)$$

$$\phi_x = (1, -2x, 0), \phi_z = (0, 0, 1)$$

$$\phi_x \times \phi_z = (-2x, -1, 0) \implies \|\phi_x \times \phi_z\| = \sqrt{4x^2 + 1}$$

$$\int_S f dS = \int_0^1 \int_{-2}^2 x \sqrt{4x^2 + 1} dx dz$$

The integrand is odd since  $x$  odd and  $\sqrt{4x^2 + 1}$  even, so the integral over  $x$  is 0, making the entire integral 0.

(b)  $S$  is the upper half of the unit sphere centered at the origin.

Only the upper half so  $0 \leq \theta \leq 2\pi$  and  $0 \leq \varphi \leq \pi/2$ .

$$\Phi(\theta, \varphi) = (\cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi)$$

$$\phi_\theta = (-\sin \theta \sin \varphi, \cos \theta \sin \varphi, 0)$$

$$\phi_\varphi = (\cos \theta \cos \varphi, \sin \theta \cos \varphi, -\sin \varphi)$$

$$\begin{aligned} \phi_\theta \times \phi_\varphi &= (-\cos \theta \sin^2 \varphi, -\sin \theta \sin^2 \varphi, -\sin^2 \theta \sin \varphi \cos \varphi - \cos^2 \theta \sin \varphi \cos \varphi) \\ &= (-\cos \theta \sin^2 \varphi, -\sin \theta \sin^2 \varphi, -\sin \varphi \cos \varphi) \end{aligned}$$

$$\|\phi_\theta \times \phi_\varphi\| = \sqrt{\cos^2 \theta \sin^4 \varphi + \sin^2 \theta \sin^4 \varphi + \sin^2 \varphi \cos^2 \varphi}$$

$$= \sqrt{\sin^4 \varphi + \sin^2 \varphi \cos^2 \varphi}$$

$$= \sqrt{\sin^2 \varphi} = \sin \varphi$$

$$\int_{\Phi} f dS = \int_0^{\pi/2} \int_0^{2\pi} \cos \theta \sin^2 \varphi d\theta d\varphi = 0$$

The integral is zero again since integrating  $\cos \theta$  over a whole period is 0.

(c)  $S$  is that part of the surface  $x = \sin y$  with  $0 \leq y \leq \pi$  and  $0 \leq z \leq 2$ .

$$\begin{aligned}
\Phi(y, z) &= (\sin y, y, z) \\
\phi_y &= (\cos y, 1, 0) \\
\phi_z &= (0, 0, 1) \\
\phi_y \times \phi_z &= (1, -\cos y, 0) \\
\|\phi_y \times \phi_z\| &= \sqrt{1 + \cos^2 y} \\
\int_{\Phi} f \, dS &= \int_0^2 \int_0^\pi \sin y \sqrt{1 + \cos^2 y} \, dy \, dz
\end{aligned}$$

$$\text{Let } u = \cos y, \, du = -\sin y \, dy$$

$$= 2 \int_{-1}^1 \sqrt{1 + u^2} \, du$$

$$\text{Let } u = \tan \theta, \, du = \sec^2(\theta) \, d\theta$$

$$= 2 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{\frac{1}{\cos^2 \theta}} \sec^2 \theta \, du$$

$$= 2 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^3 \theta \, d\theta$$

$$\text{Let } u = \sec \theta, \, du = \sec \theta \tan \theta \, d\theta, \, dv = \sec^2 \theta \, d\theta, \, v = \tan \theta$$

$$= 2 \left( \left[ \sec \theta \tan \theta \right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} - \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^2 \theta \sec \theta \, d\theta \right)$$

$$= 2 \left( \left[ \sec \theta \tan \theta \right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} - \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\sec^2 \theta - 1) \sec \theta \, d\theta \right)$$

$$2 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^3 \theta \, d\theta = 2 \left( \left[ \sec \theta \tan \theta \right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} - \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^3 \theta \, d\theta + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec \theta \, d\theta \right)$$

$$4 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^3 \theta \, d\theta = 2 \left( \left[ \sec \theta \tan \theta \right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec \theta \, d\theta \right)$$

$$2 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^3 \theta \, d\theta = \left( \left[ \sec \theta \tan \theta \right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec \theta \, d\theta \right)$$

$$= \left( \left[ \sec \theta \tan \theta \right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec \theta \left( \frac{\sec \theta + \tan \theta}{\sec \theta + \tan \theta} \right) d\theta \right)$$

$$= \left( \left[ \sec \theta \tan \theta \right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} + \left[ \ln |\sec \theta + \tan \theta| \right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \right)$$

$$= 2\sqrt{2} + \ln |1 + \sqrt{2}| - \ln |\sqrt{2} - 1|$$

7. Find the mass of the metallic surface  $S$  given by  $z = 1 - \frac{x^2 + y^2}{2}$  with  $0 \leq x \leq 1$ ,  $0 \leq y \leq 1$ , if the mass density at  $(x, y, z) \in S$  is given by  $m(x, y, z) = xy$ .

$$\Phi(x, y) = (x, y, 1 - \frac{x^2 + y^2}{2})$$

$$\phi_x = (1, 0, -x)$$

$$\phi_y = (0, 1, -y)$$

$$\|\phi_x \times \phi_y\| = \sqrt{x^2 + y^2 + 1}$$

$$\int_{\Phi} f dS = \int_0^1 \int_0^1 xy \sqrt{x^2 + y^2 + 1} dx dy$$

$$\text{Let } u = x^2 + y^2 + 1, du = 2x dx$$

$$= \frac{1}{2} \int_0^1 \int_{y^2+1}^{y^2+2} y \sqrt{u} du dy$$

$$= \frac{1}{3} \int_0^1 y \left[ u^{\frac{3}{2}} \right]_{y^2+1}^{y^2+2} dy$$

$$= \frac{1}{3} \int_0^1 y \left[ (y^2 + 2)^{\frac{3}{2}} - (y^2 + 1)^{\frac{3}{2}} \right] dy$$

$$\text{Let } u = y^2 + 1, du = 2y dy$$

$$= \frac{1}{6} \int_1^2 (u + 1)^{\frac{3}{2}} - (u)^{\frac{3}{2}} du$$

$$= \frac{1}{15} \left[ (u + 1)^{\frac{5}{2}} - (u)^{\frac{5}{2}} \right]_1^2$$

$$= \frac{1}{15} \left[ (3^{\frac{5}{2}} - 2^{\frac{5}{2}}) - (2^{\frac{5}{2}} - 1) \right]$$

$$= \frac{1}{15} \left[ 3^{\frac{5}{2}} - 2^{\frac{7}{2}} - 1 \right]$$

$$= \frac{1}{15} \left[ 9\sqrt{3} - 8\sqrt{2} + 1 \right]$$



## Bonus

- (a) Calculate the surface area of  $S$ , the piece of the cone  $x^2 + y^2 = z^2$  lying over the disk  $x^2 + y^2 \leq 4$ ,  $z = 0$

Then calculate  $\int_S x^2 z dS$

Parametrize the surface as

$$\begin{aligned}\Phi(\theta, r) &= (r \cos \theta, r \sin \theta, \sqrt{r^2 \cos^2 \theta + r^2 \sin^2 \theta}) \\ &= (r \cos \theta, r \sin \theta, r)\end{aligned}$$

Where:  $0 \leq \theta \leq 2\pi$ ,  $0 \leq r \leq 2$

$$\phi_\theta = (-r \sin \theta, r \cos \theta, 0)$$

$$\phi_r = (\cos \theta, \sin \theta, 1)$$

$$\begin{aligned}\phi_\theta \times \phi_r &= (r \cos \theta, r \sin \theta, -r \sin^2 \theta - r \cos^2 \theta) \\ &= (r \cos \theta, r \sin \theta, -r)\end{aligned}$$

$$\begin{aligned}\|\phi_\theta \times \phi_r\| &= \sqrt{r^2 \cos^2 \theta + r^2 \sin^2 \theta + r^2} \\ &= \sqrt{2r^2} = \sqrt{2}r\end{aligned}$$

$$\begin{aligned}\int_0^{2\pi} \int_0^2 \sqrt{2}r dr d\theta &= \int_0^{2\pi} 2\sqrt{2}d\theta = 4\sqrt{2}\pi \\ \int_S x^2 z dS &= \int_0^{2\pi} \int_0^2 r^2 \cos^2 \theta r \sqrt{2}r dr d\theta = \int_0^{2\pi} \int_0^2 r^4 \cos^2 \theta \sqrt{2} dr d\theta \\ &= \sqrt{2} \int_0^{2\pi} \left[ \frac{r^5}{5} \right]_0^2 \cos^2 \theta d\theta = \frac{32}{5} \sqrt{2} \int_0^{2\pi} \cos^2 \theta d\theta \\ &= \frac{32}{5} \sqrt{2} \int_0^{2\pi} 1/2 + (1/2) \cos 2\theta d\theta = \frac{32\pi}{5} \sqrt{2}\end{aligned}$$

- (b) Parametrize  $S$  which is part of the plane  $z = 2x + 4y$  that is part of the first octant & below  $z = 6$ .

$$\Phi(x, y) = (x, y, 2x + 4y)$$

Now this is in the first octant, so  $x, y > 0 \implies 2x + 4y > 0$ . It must also satisfy  $z < 6 \implies 2x + 4y < 6$ . Ignoring  $y$ ,  $x < 3$ , now fixing  $x$ ,  $y < \frac{6-2x}{4}$ . so the bounds are:

$$0 \leq x < 3, 0 \leq y < \frac{6-2x}{4}$$

- (c)  $S$  is that part of the cone  $x = 2\sqrt{x^2 + y^2}$  with  $x \leq 4$  in the first octant.