Poon, Keegan
1002423727
MATB42: Assignment #6 Mar 6th 2018

1. Let $\omega = \frac{-y}{x^2+y^2}\ dx + \frac{x}{x^2+y^2}\ dy$. Calculate $\int_{\boldsymbol{\gamma}} \omega$ where
   (a) $\boldsymbol{\gamma}$ is the boundary of the triangle with vertices (in order) (0,1), (2,3) and (2,1).
   (b) $\boldsymbol{\gamma}$ is the boundary curve of the region $\left\{(x,y) \in \mathbb{R}^2 \middle| \frac{(x-2)^2}{9} + \frac{(y+1)^2}{4} \le 1\right\}$ oriented in a counter clockwise direction
   (c) $\boldsymbol{\gamma}$ is the graph of the polar equation $r = 3 + 2\sin\theta$ oriented in the clockwise direction.
2. Let $\omega = (y^2 + z\ \ln 3)\ dx + (2xy + \sin z)\ dy + (y\cos z + (x+1)\ln 3)\ dz$. Determine if ω is exact. If it is, use the algorithm given in class to find the potential function g.
3. Evaluate the double integral $\int_{-1} 1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 3x^2y^2 dy\ dx$, by first finding an equivalent line integral.
4. Let R be a region in $\mathbb{R}^2$ and let $\boldsymbol{\gamma}$ be a counterclockwise parametrization of ∂R. Let $\boldsymbol{F} = (F_1, F-2)$ be a C^1 vector field defined throughout R and on ∂R and let $\boldsymbol{n}$ be the outward pointing unit normal ector to $\boldsymbol{\gamma}$. Use Green's theorem to give a double integral over R which is equivalent to $\int_{\boldsymbol{\gamma}} \boldsymbol{F}\cdot\boldsymbol{n}\ ds$.
5. Give a parametrization for each of the following surfaces, use a computer algebra sustem to plot the surface and find a unit vector normal to the surface.
   (a) The piece of the cylinder $y^2 + z^2 = 1$ between $x = -1$ and $x = 3$.
   (b) The piece of the plane $z = x + y + 5$ which lies over the unit disk $x^2 + y^2 \le 1$.
   (c) The piece of the sphere $x^2 + y^2 + z^2 = 4$ which lies above the plane $z = 1$.
   (d) The piece of the plane $x + y + z = 1$ which lies above the parallelogram: $0 \le y - x \le 1, 0 \le y + x2 \le 1$.
6. Let S be the surface given parameterically by $\boldsymbol{\Phi}(u,v) = (u^2, 3v, u^2+v)$ where $(u,v) \in D$, the interior of a triangle with vertices (0,0), (3,0) and (3,3).
   (a) Find the surface area of S.
   (b) Find the equation of the tangent plane to S at the point (4,9,7).
7. Suppose the surface S is the graph of a function $f : \mathbb{R}^2 \to \mathbb{R}$. Give a natural parametrization of S (in terms of f) and derive the formula $\|\boldsymbol{\phi}_u \times \boldsymbol{\phi}_v\| = \sqrt{1 + \|\text{grad } f\|^2}$
8. A paraboloid of revolution S is parameterized by $\boldsymbol{\Phi}(u,v) = u\cos v, u\sin v, u^2$, $0 \le u \le 2, 0 \le v \le 2\pi$.
   (a) Find an equation in x, y and z describing the surface.
   (b) What are the geometric meanings of the parameters u and v?
   (c) Find a unit vector orthogonal to the surface of $\boldsymbol{\Phi}(u,v)$.
   (d) Find the equation for the tangent plane at $\boldsymbol{\Phi}(u_0,v_0) = (1,1,2)$ and express your answer in the following two ways:
      i. parameterized by u and v; and
      ii. in terms of x, y and z.
   (e) Find the area of S.
   (cf. page 424, #16)
9. Let a differentiable function $\boldsymbol{\Phi} : \mathbb{R}^2 \to \mathbb{R}^3$ define a parametrized surface.
   (a) Assuming $\boldsymbol{\phi}_u \times \boldsymbol{\phi}_v \ne 0$, show that the range of the linear transformation $D\boldsymbol{\Phi}(u_0,v_0)$ is the plane spanned by $\boldsymbol{\phi}_u$ and $\boldsymbol{\phi}_v$. [Here $\boldsymbol{\phi}_u$ and $\boldsymbol{\phi}_v$ are evaluated at (u_0,v_0).]

(b) Show that $\boldsymbol{w} \perp (\boldsymbol{\phi}_u \times \boldsymbol{\phi}_v)$ if and only if $\boldsymbol{w}$ is in the range of $D\mathbf{\Phi}(u_0, v_0)$.

(c) Show that the tangent plane as defined in terms of $\boldsymbol{\phi}_u \times \boldsymbol{\phi}_v(u_0, v_0)$ is the same as the "parametrized plane"

$$(u, v) \mapsto \mathbf{\Phi}(u_0, v_0) + D\mathbf{\Phi}(u_0, v_0) \begin{bmatrix} a \\ b \end{bmatrix}$$