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MATB42: Assignment #2 Jan 30th 2018
Bonus

1. Let $f(x) = \begin{cases} 0, & -\pi < x < -\frac{\pi}{2} \\ 2, & -\frac{\pi}{2} \leq x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} \leq x < \pi \end{cases}$

   (a) Find the Fourier series of f.

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\ &= \frac{1}{\pi}\left[\int_{-\pi}^{-\frac{\pi}{2}} 0dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2dx + \int_{\frac{\pi}{2}}^{\pi} 0dx\right] \\ &= \frac{1}{\pi}\left[2\pi\right] \\ &= 2 \end{aligned}$$

$$\begin{aligned} b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx \\ &= \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(kx)dx \\ &= 0 \quad [\text{sin is odd}] \end{aligned}$$

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx \\ &= \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(kx)dx \\ &= \frac{2}{k\pi}\Big[\sin(kx)\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\ &= \frac{2}{k\pi}\left[2\sin\left(\frac{k\pi}{2}\right)\right] \\ &= \frac{4}{k\pi}\sin\left(\frac{k\pi}{2}\right) \end{aligned}$$

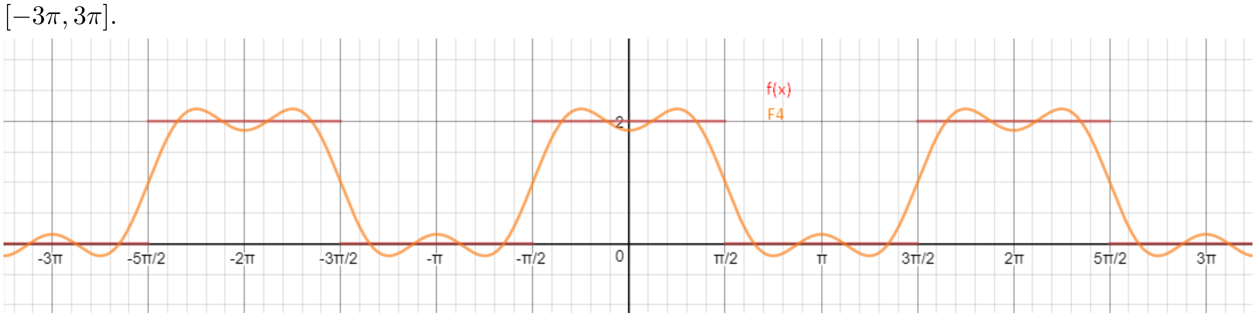
This is 0 for even elements, and alternating between 1 and -1 for odd elements.

Therefore the Fourier polynomial (for the non-zero terms) is

$$1 + \sum_{l=1}^{\infty}\left[\frac{4(-1)^{l+1}}{(2l-1)\pi}\cos((2l-1)x)\right]$$

   (b) Determine if the Fourier series in part (a) converges. If it does converge, what are the values to which it converges (on $[-\pi, \pi]$).
   The function is continuous on its partitions (they are constant functions), so by the fundemendatl theorem the polynomial converges to $f(x)$ on the continuous intervals. On the discontinuities, it converges to 0 at $\frac{\pi}{2}$ and $\frac{-\pi}{2}$ and to 1 at π and $-\pi$ since the theorem states that it converges to the average of the limits around the points. Being continuous functions, it is simply the evaluation of the functions at those points.

   (c) Use symbolic algeba software to sketch $f(x)$ and its 4^{th} degree Fourier polynomial over the interval $[-3\pi, 3\pi]$.



2. (a) Find the Fourier series of the function $f(x)$ defined by $f(x) = \begin{cases} 0 & , -\pi \leq x < 0 \\ x & , 0 \leq x < \pi \end{cases}$ and extended from this with period 2π to all of $\mathbb{R}$.
If this Fourier series converges describe the function to which it converges.

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\ &= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)dx + \int_{0}^{\pi} f(x)dx\right] \\ &= \frac{1}{\pi}\left[0 + \int_{0}^{\pi} xdx\right] \\ &= \frac{1}{\pi}\left[\frac{1}{2}\left[x^2\right]_0^{\pi}\right] \\ &= \frac{\pi}{2} \end{aligned}$$

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx \\ &= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0dx + \int_{0}^{\pi} x\cos(kx)dx\right] \end{aligned}$$

Let $u = x,\ du = dx,$

$dv = \cos(kx),\ v = \dfrac{\sin(kx)}{k}$

$$\begin{aligned} &= \frac{1}{\pi}\left[\frac{1}{k}\left[x\sin(kx)\right]_0^{\pi} - \frac{1}{k}\int_{0}^{\pi} \sin(kx)dx\right] \\ &= -\frac{1}{k\pi}\left[\int_{0}^{\pi} \sin(kx)dx\right] \\ &= \frac{1}{k^2\pi}\left[\cos(kx)\right]_0^{\pi} \\ &= \frac{(-1)^{-k} - 1}{k^2\pi} \end{aligned}$$

$$\begin{aligned} b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx \\ &= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0dx + \int_{0}^{\pi} x\sin(kx)dx\right] \end{aligned}$$

Let $u = x,\ du = dx,\ dv = \sin(kx),\ v = -\dfrac{1}{k}\cos(kx)$

$$\begin{aligned} &= \frac{1}{\pi}\left[-\frac{1}{k}\left[x\cos(kx)\right]_0^{\pi} + \frac{1}{k}\int_{0}^{\pi} \cos(kx)dx\right] \\ &= \frac{1}{k\pi}\left[-\pi\cos(k\pi) + \frac{1}{k}\left[\sin(kx)\right]_0^{\pi}\right] \\ &= \frac{1}{k\pi}\left[-\pi\cos(k\pi) + 0\right] \\ &= \frac{(-1)^{k+1}}{k} \end{aligned}$$

Therefore the Fourier series of f is

$$F(x) = \frac{\pi}{4} + \sum_{k=1}^{\infty}\left[\frac{(-1)^k - 1}{k^2\pi}\cos(kx) + \frac{(-1)^{k+1}}{k}\sin(kx)\right]$$

Since f is piecewise very smooth (0, x are infinitely differentiable), by the fundemental theorem, the series converges to f on $(-\pi, \pi)$, on both endpoints it converges to $\frac{\pi}{2}$ and at 0, it converges to 0 since the theorem states that it converges to the average of the limits around the points. Being continuous functions, it is simply the evaluation of the functions at those points.

(b) Using the series from part (a) show that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots .$$

$$F(0) = \frac{\pi}{4} + \sum_{k=1}^{\infty}\left[\frac{(-1)^k - 1}{k^2\pi}\right] \qquad\qquad \frac{\pi}{4} = \sum_{k=1}^{\infty}\left[\frac{2}{(2k-1)^2\pi}\right]$$

$$0 = \frac{\pi}{4} + \sum_{k=1}^{\infty}\left[\frac{-2}{(2k-1)^2\pi}\right] \qquad\qquad \frac{\pi^2}{8} = \sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}$$

3. Find the Fourier series for the restriction of the function $f(x) = 3+3x$ to each of the following intervals, $[a, b]$. If the Fourier series converges, to what values will the series converge at the end points?

   (a) $[a, b] = [-\pi, \pi]$

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} 3+3xdx \\ &= \frac{1}{\pi}\left[6\pi + \frac{3}{2}\Big[x^2\Big]_{-\pi}^{\pi}\right] \\ &= \frac{1}{\pi}\Big[6\pi + 0\Big] \\ &= 6 \end{aligned}$$

$$\begin{aligned} b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx \\ &= \frac{3}{\pi}\left[\int_{-\pi}^{\pi}\sin(kx)dx + \int_{-\pi}^{\pi} x\sin(kx)dx\right] \\ &= \frac{6}{\pi}\left[\int_{0}^{\pi} x\sin(kx)dx\right] \quad \text{[Since } x \text{ and sin odd]} \end{aligned}$$

Let $u = x, du = 1dx, dv = \sin(kx)dx, v = -\dfrac{\cos(kx)}{k}$

$$\begin{aligned} &= \frac{6}{\pi}\left[-\frac{1}{k}\Big[x\cos(kx)\Big]_0^{\pi} + \frac{1}{k}\int_0^{\pi}\cos(kx)dx\right] \\ &= \frac{6}{k\pi}\left[\pi(-1)^{k+1} + \frac{1}{k}\Big[\sin(kx)\Big]_0^{\pi}\right] \\ &= \frac{6(-1)^{k+1}}{k} \end{aligned}$$

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx \\ &= \frac{3}{\pi}\left[\int_{-\pi}^{\pi}\cos(kx)dx + \int_{-\pi}^{\pi} x\cos(kx)dx\right] \\ &= \frac{6}{k\pi}\Big[\sin(kx)\Big]_0^{\pi} \quad \text{[Since } x \text{ odd and cos even]} \\ &= 0 \end{aligned}$$

Therefore the Fourier series is defined as

$$F(x) = 3 + \sum_{k=1}^{\infty}\frac{6(-1)^{k+1}}{k}\sin(kx)$$

Linear functions are infinitely differentiable so by the fundemental theorem it will converge to $f(x)$ within the interval, and coverges to 3 at the endpoints since the theorem states that it converges to the average of the limits around the points. Being continuous functions, it is simply the evaluation of the functions at those points.

(b) $[a,b] = [0, 2\pi]$

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\
&= \frac{1}{\pi}\int_{0}^{2\pi} 3+3xdx \\
&= \frac{1}{\pi}\left[6\pi + \frac{3}{2}\Big[x^2\Big]_0^{2\pi}\right] \\
&= \frac{1}{\pi}\left[6\pi + 6\pi^2\right] \\
&= 6(\pi+1)
\end{aligned}$$

$$\begin{aligned}
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx \\
&= \frac{3}{\pi}\left[\int_0^{2\pi}\sin(kx)dx + \int_0^{2\pi} x\sin(kx)dx\right] \\
&\text{Let } u = x, du = 1dx, dv = \sin(kx)dx, v = -\frac{\cos(kx)}{k} \\
&= \frac{3}{k\pi}\left[\Big[\cos(kx)\Big]_0^{2\pi} - \Big[x\cos(kx)\Big]_0^{2\pi} + \int_0^{2\pi}\cos(kx)dx\right] \\
&= \frac{3}{k\pi}\left[-2\pi + \frac{1}{k}\Big[\sin(kx)\Big]_0^{2\pi}\right] \\
&= -\frac{6}{k}
\end{aligned}$$

$$\begin{aligned}
a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx \\
&= \frac{3}{\pi}\left[\int_0^{2\pi}\cos(kx)dx + \int_0^{2\pi} x\cos(kx)dx\right] \\
&\text{Let } u = x,\ du = dx,\ dv = \cos(kx),\ v = \frac{1}{k}\sin(kx) \\
&= \frac{3}{k\pi}\left[\Big[\sin(kx)\Big]_0^{2\pi} + \Big[x\sin(kx)\Big]_0^{2\pi} - \int_0^{2\pi}\sin(kx)dx\right] \\
&= -\frac{3}{k^2\pi}\Big[\cos(kx)\Big]_0^{2\pi} \\
&= 0
\end{aligned}$$

Therefore the Fourier series is defined as

$$F(x) = 3(\pi+1) - \sum_{k=1}^{\infty}\frac{6}{k}\sin(kx)$$

Linear functions are infinitely differentiable so by the fundemental theorem it will converge to $f(x)$ within the interval, and coverges to $3+3\pi$ at the endpoints since the function is continuous, so that is the result of averaging the evaluation of the function at it's endpoints.

4. Find the Fourier series of the function $f(x)$ defined on $[0, 2\pi]$ by $f(x) = x(x-2\pi)$ and extended from this with period 2π to all of $\mathbb{R}$. Use symbolic algebra software to graph the 4^{th} degree Fourier polynomial together with the original function.

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx$$
$$= \frac{1}{\pi}\int_0^{2\pi} x(x-2\pi)dx$$
$$= \frac{1}{\pi}\left[\int_0^{2\pi} x^2\sin(kx)dx - 2\pi\int_0^{2\pi} x\sin(kx)dx\right]$$

Let $u = x,\ du = dx,\ dv = \sin(kx),\ v = -\frac{1}{k}\cos(kx)$

$$= \frac{1}{\pi}\left[\int_0^{2\pi} x^2\sin(kx)dx - 2\pi\Big(-\frac{1}{k}\Big[x\cos(kx)\Big]_0^{2\pi} + \frac{1}{k}\int_0^{2\pi}\cos(kx)dx\Big)\right]$$
$$= \frac{1}{\pi}\left[\int_0^{2\pi} x^2\sin(kx)dx - 2\pi\Big(-\frac{2\pi}{k} + \frac{1}{k^2}\Big[\sin(kx)\Big]_0^{2\pi}\Big)\right]$$
$$= \frac{1}{\pi}\left[\int_0^{2\pi} x^2\sin(kx)dx + \frac{4\pi^2}{k}\right]$$

Let $u = x^2, du = 2xdx,$

$dv = \sin(kx)dx, v = -\frac{\cos(kx)}{k}$

$$= \frac{1}{k\pi}\left[-\Big[x^2\cos(kx)\Big]_0^{2\pi} + \int_0^{2\pi} x\cos(kx)dx + 4\pi^2\right]$$

Let $u = x,\ du = dx,\ dv = \sin(kx),\ v = -\frac{1}{k}\cos(kx)$

$$= \frac{1}{k\pi}\left[\frac{1}{k}\Big[x\sin(kx)\Big]_0^{2\pi} - \frac{1}{k}\int_0^{2\pi}\cos(kx)dx\right]$$
$$= \frac{1}{k\pi}\left[-\frac{1}{k^2}\Big[\sin(kx)\Big]_0^{2\pi}\right]$$
$$= 0$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx$$
$$= \frac{1}{\pi}\left[\int_0^{2\pi} x^2\cos(kx)dx - 2\pi\int_0^{2\pi} x\cos(kx)dx\right]$$

Let $u = x^2, du = 2xdx, dv = \cos(kx)dx, v = \frac{\sin(kx)}{k}$

$$= \frac{1}{\pi}\left[\frac{1}{k}\Big[x^2\sin(kx)\Big]_0^{2\pi} - \frac{2}{k}\int_0^{2\pi} x\sin(kx)dx - 2\pi\int_0^{2\pi} x\cos(kx)dx\right]$$
$$= \frac{1}{\pi}\left[-\frac{2}{k}\int_0^{2\pi} x\sin(kx)dx - 2\pi\int_0^{2\pi} x\cos(kx)dx\right]$$

Let $u = x, du = dx, dv = \sin(kx)dx, v = -\frac{\cos(kx)}{k}$

$$= \frac{1}{\pi}\left[\frac{2}{k^2}\Big[x\cos(kx)\Big]_0^{2\pi} - \frac{1}{k}\int_0^{2\pi}\cos(kx)dx - 2\pi\int_0^{2\pi} x\cos(kx)dx\right]$$
$$= \frac{1}{\pi}\left[\frac{4\pi}{k^2} - \frac{1}{k^2}\Big[\sin(kx)\Big]_0^{2\pi} - 2\pi\int_0^{2\pi} x\cos(kx)dx\right]$$

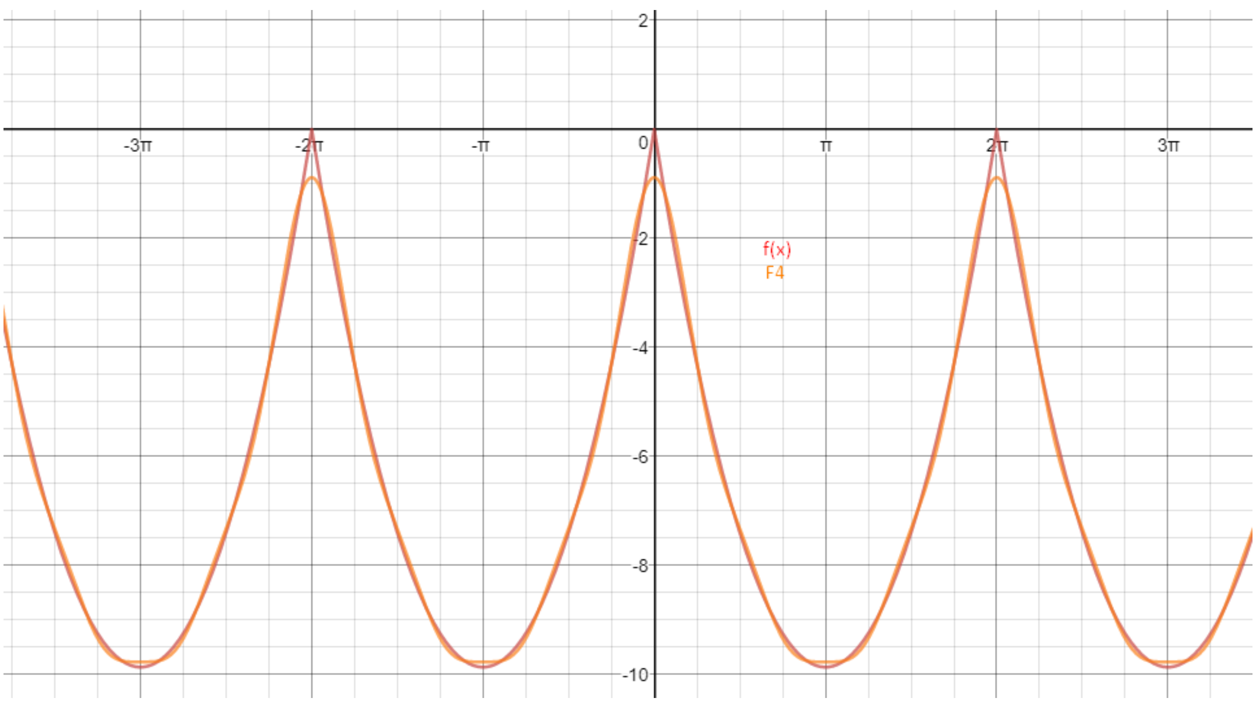
Let $u = x,\ du = dx,\ dv = \cos(kx),\ v = \frac{1}{k}\sin(kx)$

$$= \frac{1}{\pi}\left[\frac{4\pi}{k^2} - 2\pi\Big(\frac{1}{k}\Big[x\sin(kx)\Big]_0^{2\pi} - \frac{1}{k}\int_0^{2\pi}\sin(kx)dx\Big)\right]$$
$$= \frac{1}{\pi}\left[\frac{4\pi}{k^2} + \frac{2\pi}{k}\int_0^{2\pi}\sin(kx)dx\right]$$
$$= \frac{1}{\pi}\left[\frac{4\pi}{k^2} + \frac{2\pi}{k^2}\Big[\cos(kx)\Big]_0^{2\pi}\right]$$
$$= \frac{4}{k^2}$$

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\
&= \frac{1}{\pi}\int_{0}^{2\pi} x(x-2\pi)dx \\
&= \frac{1}{\pi}\left[\int_{0}^{2\pi} x^2 dx - \int_{0}^{2\pi} 2x\pi dx\right] \\
&= \frac{1}{\pi}\left[\frac{1}{3}\Big[x^3\Big]_0^{2\pi} - \pi\Big[x^2\Big]_0^{2\pi}\right] \\
&= \frac{1}{\pi}\left[\frac{8\pi^3}{3} - 4\pi^3\right] \\
&= -\frac{4\pi^2}{3}
\end{aligned}
$$

Therefore the Fourier series of f is

$$
F(x) = -\frac{2\pi^2}{3} + \sum_{k=1}^{\infty} \frac{4}{k^2}\cos(kx)
$$



5. Let $f(x)$ be defined on $[0, 2\pi]$ by $f(x) = x(x - 2\pi)$.

(a) Find the Fourier cosine series of f.

From question 4, we can see that the function is already even, hence the Fourier series of the function itself is a cosine series of f. Namely

$$F(x) = -\frac{2\pi^2}{3} + \sum_{k=1}^{\infty} \frac{4}{k^2}\cos(kx)$$

(b) Find the Fourier sine series of f.

To extend this as an odd function, define the f on the range $[-2\pi, 0]$ as $f(x) = -((x+2\pi)((x+2\pi)-2\pi)) = -x(x+2\pi)$. Note that this definition of f now has a period of 4π.

$$\begin{aligned}
b_k &= \frac{1}{2\pi}\int_{-2\pi}^{2\pi} f(x)\sin\left(\frac{kx}{2}\right)dx \\
&= -\frac{1}{\pi}\left[\int_{-2\pi}^{0} x(x+2\pi)\sin\left(\frac{kx}{2}\right)dx\right] \quad [f \text{ and } \sin \text{ are both odd so the integrand is even}] \\
&= -\frac{1}{\pi}\left[\int_{-2\pi}^{0} x^2\sin\left(\frac{kx}{2}\right)dx + 2\pi\int_{-2\pi}^{0} x\sin\left(\frac{kx}{2}\right)dx\right]
\end{aligned}$$

Let $u = x^2,\ du = 2xdx,\ dv = \sin\left(\frac{kx}{2}\right)dx,\ v = -\frac{2\cos(\frac{kx}{2})}{k}$

$$\begin{aligned}
&= -\frac{1}{\pi}\left[-\frac{2}{k}\left[x^2\cos\left(\frac{kx}{2}\right)\right]_{-2\pi}^{0} + \frac{4}{k}\int_{-2\pi}^{0} x\cos\left(\frac{kx}{2}\right)dx + 2\pi\int_{-2\pi}^{0} x\sin\left(\frac{kx}{2}\right)dx\right] \\
&= -\frac{1}{\pi}\left[\frac{8\pi^2(-1)^k}{k} + \frac{4}{k}\int_{-2\pi}^{0} x\cos\left(\frac{kx}{2}\right)dx + 2\pi\int_{-2\pi}^{0} x\sin\left(\frac{kx}{2}\right)dx\right]
\end{aligned}$$

Let $u = x,\ du = dx,\ dv = \cos\left(\frac{kx}{2}\right),\ v = \frac{2}{k}\sin\left(\frac{kx}{2}\right)$

$$\begin{aligned}
&= -\frac{1}{\pi}\left[\frac{8\pi^2}{k}(-1)^k + \frac{4}{k^2}\left[x\sin\left(\frac{kx}{2}\right)\right]_{-2\pi}^{0} - \frac{8}{k^2}\int_{-2\pi}^{0} \sin\left(\frac{kx}{2}\right)dx + 2\pi\int_{-2\pi}^{0} x\sin\left(\frac{kx}{2}\right)dx\right] \\
&= -\frac{1}{\pi}\left[\frac{8\pi^2}{k}(-1)^k + \frac{16}{k^3}\left[\cos\left(\frac{kx}{2}\right)\right]_{-2\pi}^{0} + 2\pi\int_{-2\pi}^{0} x\sin\left(\frac{kx}{2}\right)dx\right]
\end{aligned}$$

Let $u = x, du = dx, dv = \sin\left(\frac{kx}{2}\right)dx, v = -\frac{2\cos(\frac{kx}{2})}{k}$

$$\begin{aligned}
&= -\frac{1}{\pi}\left[\frac{8\pi^2}{k}(-1)^k + \frac{16}{k^3}(1-(-1)^k) + \frac{4\pi}{k}\left(-\left[x\cos\left(\frac{kx}{2}\right)\right]_{-2\pi}^{0} + \int_{-2\pi}^{0}\cos\left(\frac{kx}{2}\right)dx\right)\right] \\
&= -\frac{1}{\pi}\left[\frac{8\pi^2}{k}(-1)^k + \frac{16}{k^3}(1-(-1)^k) + \frac{4\pi}{k}\left(2\pi(-1)^{k+1} + \frac{2}{k}\left[\sin\left(\frac{kx}{2}\right)\right]_{-2\pi}^{0}\right)\right] \\
&= -\frac{1}{\pi}\left[\frac{8\pi^2}{k}(-1)^k + \frac{16}{k^3}(1-(-1)^k) + \frac{8\pi^2}{k}(-1)^{k+1}\right] \\
&= \frac{16}{k^3\pi}((-1)^k - 1)
\end{aligned}$$

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-2\pi}^{2\pi} f(x)dx \\
&= 0 \quad [\text{Since } f \text{ is defined odd}]
\end{aligned}$$

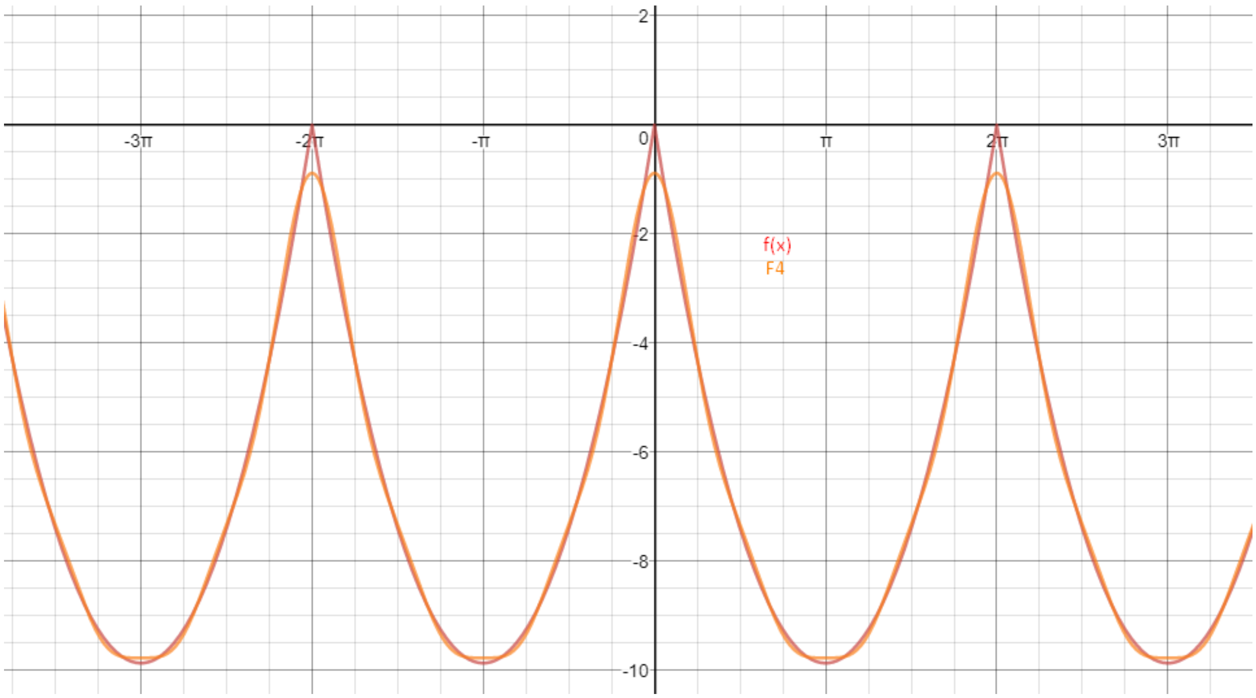
$$\begin{aligned}
a_k &= \frac{1}{2\pi}\int_{-2\pi}^{2\pi} f(x)\cos\left(\frac{kx}{2}\right)dx \\
&= 0 \quad [\text{Since } f \text{ is defined odd}]
\end{aligned}$$

The Fourier sine series is thusly

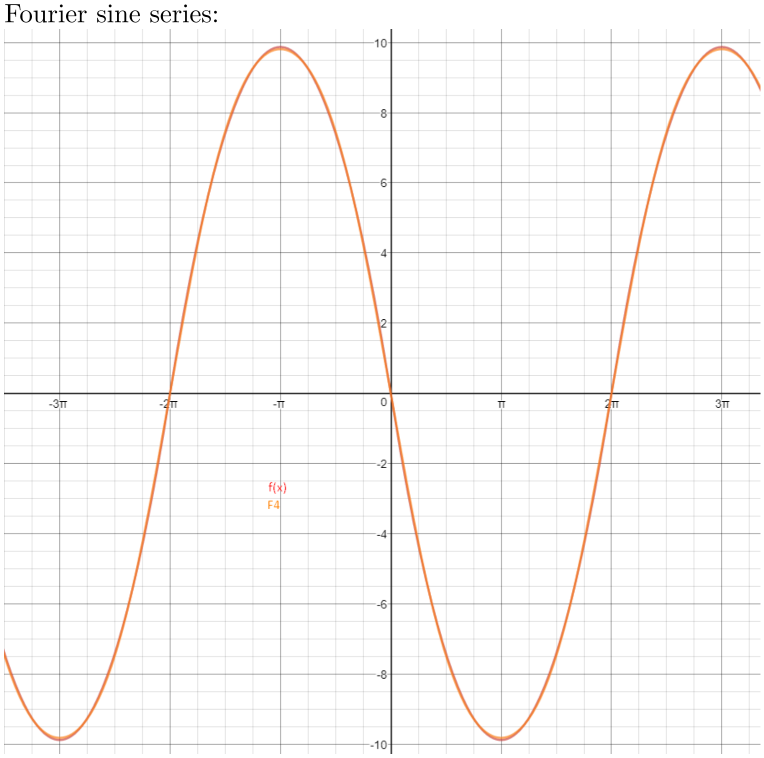
$$F(x) = \sum_{k=1}^{\infty} \frac{16}{k^3\pi}((-1)^k - 1)\sin\left(\frac{kx}{2}\right)$$

(c) Use symbolic algebra software to graph the 4^{th} degree Fourier polynomials from parts (a) and (b) together with the original function.

Fourier cosine series:



Fourier sine series:



6. Find the Fourier series for the following functions:

   (a) $f(x) = \sin^2 x + \sin^3 x$

$$\begin{aligned}
\sin^2 x + \sin^3 x &= (1/2i)^2(e^{ix} - e^{-ix})^2 + (1/2i)^3(e^{ix} - e^{-ix})^3 \\
\text{[Binomial Theorem]} &= (-1/4)(e^{2ix} - 2(e^{(ix-ix)}) + e^{-2ix}) + (-1/8i)(e^{3ix} - 3(e^{(2ix-ix)}) + 3(e^{(ix-2ix)}) - e^{-3ix}) \\
&= (-1/4)(e^{2ix} + e^{-2ix} - 2) + (-1/8i)(e^{3ix} - e^{-3ix} - 3(e^{(ix)}) + 3(e^{(-ix)})) \\
&= (-1/2)(\cos(2x) - 2) + (-1/4)(\sin(3x) - 3\sin(x)) \\
&= 1 + \frac{3}{4}\sin(x) - \frac{1}{2}\cos(2x) - \frac{1}{4}\sin(3x)
\end{aligned}$$

   (b) $f(x) = \sin^4 x$

$$\begin{aligned}
\sin^4 x &= (1/2i)^4(e^{ix} - e^{-ix})^4 \\
\text{[Binomial Theorem]} &= (1/16)(e^{4ix} - 4e^{3ix-ix} + 6e^{2ix-2ix} - 4e^{ix-3ix} + e^{4ix}) \\
&= (1/16)(6 - 4e^{2ix} - 4e^{-2ix} + e^{4ix} + e^{4ix}) \\
&= (1/8)(3 - 4\cos(2x) + \cos(4x)) \\
&= \frac{3}{8} - \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)
\end{aligned}$$

   (c) $f(x) = \cos^7 x$

$$\begin{aligned}
\cos^7 x &= (1/2)^7(e^{ix} + e^{-ix})^7 \\
\text{[Binomial Theorem]} &= (1/128)(e^{7ix} + 7e^{6ix-ix} + 21e^{5ix-2ix} + 35e^{4ix-3ix} \\
&\quad + 35e^{3ix-4ix} + 21e^{2ix-5ix} + 7e^{ix-6ix} + e^{-7ix}) \\
&= (1/128)(35e^{ix} + 35e^{-ix} + 21e^{3ix} + 21e^{-3ix} + 7e^{5ix} + 7e^{-5ix} + e^{7ix} + e^{-7ix}) \\
&= (1/64)(35\cos(x) + 21\cos(3x) + 7\cos(5x) + \cos(7x)) \\
&= \frac{35}{64}\cos(x) + \frac{21}{64}\cos(3x) + \frac{7}{64}\cos(5x) + \frac{1}{64}\cos(7x))
\end{aligned}$$

   (*Hint:* Recall that $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$ and $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$)

The next question is for those among you who have previously seen complex numbers. It gives another approach to Fourier series.

7. Suppose

   i. $f(x)$ is a real values function of x,

   ii. $f(x) = \displaystyle\sum_{n=-\infty}^{\infty} C_n e^{inx}$ on $[-\pi, \pi]$, where the C_n are complex constants, and

   iii. that the term by term theorem holds true in this case

   (a) Express the C_n as integrals involving f.

Multiplying by e^{-ikx} on both sides (where $k \in \mathbb{Z}$) gives the expression:

$$
\begin{aligned}
e^{-ikx} f(x) &= \sum_{n=-\infty}^{\infty} C_n e^{inx} e^{-ikx} \\
\implies \int_{-\pi}^{\pi} e^{-ikx} f(x) dx &= \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} C_n e^{inx-ikx} dx \\
\int_{-\pi}^{\pi} e^{-ikx} f(x) dx &= \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} C_n e^{i(n-k)x} dx \quad \text{[Due to the term by term theorem]}
\end{aligned}
$$

Now there are two cases to consider as $n \in (-\infty, \infty)$

When $n \neq k$

$$
\begin{aligned}
\int_{-\pi}^{\pi} C_n e^{i(n-k)x} dx &= C_n \int_{-\pi}^{\pi} e^{i(n-k)x} dx \\
&= C_n \frac{1}{i(n-k)} \left[e^{i(n-k)x} \right]_{-\pi}^{\pi} \\
&= C_n \frac{2}{(n-k)} \frac{1}{2i} \left[e^{i(n-k)\pi} - e^{-i(n-k)\pi} \right] \\
&= C_n \frac{2}{(n-k)} \sin((n-k)\pi) = 0
\end{aligned}
$$

When $n = k$

$$
\begin{aligned}
\int_{-\pi}^{\pi} C_n e^{i(n-k)x} dx &= C_n \int_{-\pi}^{\pi} e^0 dx \\
&= C_n (2\pi) dx
\end{aligned}
$$

So that means

$$
\begin{aligned}
\int_{-\pi}^{\pi} e^{-ikx} f(x) dx &= \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} C_n e^{i(n-k)x} dx \\
&= 2\pi C_k \\
\implies \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikx} f(x) dx &= C_k
\end{aligned}
$$

(b) Find the Fourier coefficients of f in terms of the C_n.

$$
\begin{aligned}
a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx \\
&= \frac{1}{\pi} \sum_{n=-\infty}^{\infty} C_n \int_{-\pi}^{\pi} e^{inx} dx \\
&= 2C_0 \quad \text{[From (a), if } n = 0\text{, integral is } 2\pi\text{, o/w 0]}
\end{aligned}
$$

$$
\begin{aligned}
a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx \\
&= \frac{1}{\pi}\sum_{n=-\infty}^{\infty} C_n \frac{1}{2}\int_{-\pi}^{\pi} e^{inx}(e^{ikx} + e^{-ikx})dx \\
&= \frac{1}{\pi}\sum_{n=-\infty}^{\infty} C_n \frac{1}{2}\Big[\int_{-\pi}^{\pi} e^{i(n+k)x}dx + \int_{-\pi}^{\pi} e^{-i(k-n)x}dx\Big]
\end{aligned}
$$

Again the integrals are 2π respectively when $n = \pm k$ and 0 otherwise

$$
\begin{aligned}
&= \frac{1}{2\pi}\left[2\pi C_{-k} + 2\pi C_k\right] \\
&= \left[C_{-k} + C_k\right]
\end{aligned}
$$

$$
\begin{aligned}
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx \\
&= \frac{1}{\pi}\sum_{n=-\infty}^{\infty} C_n \frac{1}{2i}\int_{-\pi}^{\pi} e^{inx}(e^{ikx} - e^{-ikx})dx \\
&= \frac{1}{\pi}\sum_{n=-\infty}^{\infty} C_n \frac{1}{2i}\Big[\int_{-\pi}^{\pi} e^{i(n+k)x}dx - \int_{-\pi}^{\pi} e^{-i(k-n)x}dx\Big]
\end{aligned}
$$

Again the integrals are 2π respectively when $n = \pm k$ and 0 otherwise

$$
\begin{aligned}
&= -\frac{i}{2\pi}\left[2\pi C_{-k} - 2\pi C_k\right] \\
&= -i\left[C_{-k} - C_k\right]
\end{aligned}
$$

(c) Find the C_n in terms of the Fourier coefficients of f.

$$
\begin{aligned}
\frac{a_k - ib_k}{2} &= \frac{-\left[C_{-k} - C_k\right] + \left[C_{-k} + C_k\right]}{2} \\
&= \frac{2C_k}{2} \\
&= C_k
\end{aligned}
$$

Bonus

Let $f(x)$ be the function (of period 2π) which corresponds to $y = x^2$ on the interval $[-\pi, \pi]$.

(a) Find the Fourier series of the function $f(x)$. If this Fourier series converges describe the function to which it converges.

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx \\ &= \frac{1}{3\pi}\Big[x^3\Big]_{-\pi}^{\pi} \\ &= \frac{2\pi^2}{3} \end{aligned}$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx$$

Notice that f even, so integral is 0

$= 0$

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos(kx)dx \end{aligned}$$

Let $u = x^2, du = 2xdx, dv = \cos(kx)dx, v = \dfrac{\sin(kx)}{k}$

$$= \frac{1}{\pi}\Bigg[-\frac{2}{k}\int_{-\pi}^{\pi} x\sin(kx)dx\Bigg]$$

Let $u = x, du = dx, dv = \sin(kx)dx, v = -\dfrac{\cos(kx)}{k}$

$$\begin{aligned} &= \frac{1}{\pi}\Bigg[-\frac{2}{k}\Big[-\frac{1}{k}\Big[x\cos(kx)\Big]_{-\pi}^{\pi} + \frac{1}{k}\int_{-\pi}^{\pi}\cos(kx)dx\Big]\Bigg] \\ &= \frac{1}{\pi}\Bigg[-\frac{2}{k}\Big[-\frac{1}{k}\Big[2\pi(-1)^k\Big] + \frac{1}{k^2}\Big[\sin(kx)\Big]_{-\pi}^{\pi}\Big]\Bigg] \\ &= \frac{1}{\pi}\Bigg[-\frac{2}{k}\Big[-\frac{1}{k}\Big[2\pi(-1)^k\Big]\Big]\Bigg] \\ &= \frac{4(-1)^k}{k^2} \end{aligned}$$

Hence, the Fourier series is

$$F(x) = \frac{\pi^2}{3} + \sum_{k=1}^{\infty}\frac{4(-1)^k}{k^2}\cos(kx)$$

By the fundemental theorem, the series converges to $f(x)$ on $(-\pi, \pi)$ since x^2 is continuous, and on the endpoints it converges to pi^2 since the limits are both equal to that at the endpoints.

(b) Find the value of $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$ *(Hint: Let $x = 0$)*

$$\begin{aligned} F(0) &= \frac{\pi^2}{3} + \sum_{k=1}^{\infty}\frac{4(-1)^k}{k^2} \\ 0 &= \frac{\pi^2}{3} + \sum_{k=1}^{\infty}\frac{4(-1)^k}{k^2} \\ \frac{\pi^2}{3} &= \sum_{k=1}^{\infty}\frac{4(-1)^{k+1}}{k^2} \\ \frac{\pi^2}{12} &= \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2} \end{aligned}$$

The sum converges to $\dfrac{\pi^2}{12}$.