

MATB42: Assignment #7

1. (a) Find an equation of the tangent plane to the surface  $S$  defined parametrically by  $\Phi(u, v) = (u^2 + v, v, u + v^2)$  at the point  $(9, 0, 3)$ .

$$v = 0 \qquad u + v^2 = 3 \implies u = 3$$

$$\phi_u = (2(3), 0, 1)$$

$$\phi_v = (1, 1, 2(0))$$

$$\phi_u \times \phi_v = (-1, 1, 6)$$

So the tangent plane can be given by

$$0 = ((x - 9, y, z - 3) \cdot (-1, 1, 6))$$

$$0 = (9 - x + y + 6z - 18)$$

$$9 = -x + y + 6z$$

- (b) Use symbolic algebra software to sketch the surface  $S$  and its tangent plane from part (a).



2. Use a surface integral to find the area of the triangle in  $\mathbb{R}^3$  with vertices  $(1, 1, 0)$ ,  $(1, 2, 1)$  and  $(3, 3, 2)$ .
3. Calculate the surface area of the piece of the cone  $x^2 + y^2 - z^2 = 0$  which lies inside the cylinder  $x^2 + y^2 = 4$ .

4. (a) Find the area of the portion of the unit sphere that is cut out by the cone  $z = \sqrt{x^2 + y^2}$ .  
(cf. page 391, #10)
- (b) Find the area of the portion of the cone  $z = \sqrt{x^2 + y^2}$  that is cut out by the unit sphere.
5. Let  $\Phi : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be a parametrization of a 2-dim surface  $S$  in  $\mathbb{R}^3$ .

(a) Set

$$E = \|\phi_u\|^2, \quad F = \phi_u \cdot \phi_v, \quad G = \|\phi_v\|^2,$$

Show that the surface area of  $S$  is

$$A(S) = \iint_D \sqrt{EG - F^2} dA$$

$$\begin{aligned} \iint_D \sqrt{EG - F^2} dA &= \iint_D \sqrt{\|\phi_u\|^2 \|\phi_v\|^2 - (\phi_u \cdot \phi_v)^2} dA \\ &= \iint_D \sqrt{(\|\phi_u\| \|\phi_v\|)^2 - (\|\phi_u\| \|\phi_v\| \cos \theta)^2} dA \quad \text{Where } \theta \text{ is the angle between } \phi_u \text{ and } \phi_v. \\ &= \iint_D \sqrt{(\|\phi_u\| \|\phi_v\|)^2 (1 - \cos^2 \theta)} dA \\ &= \iint_D \sqrt{(\|\phi_u\| \|\phi_v\|)^2 (\sin^2 \theta)} dA \\ &= \iint_D \sqrt{\|\phi_u \times \phi_v\|^2} dA \\ &= \iint_D \|\phi_u \times \phi_v\| dA \\ &= \int_{\Phi} 1 dS \end{aligned}$$

- (b) What does the formula for  $A(S)$  become if the vectors  $\phi_u$  and  $\phi_v$  are orthogonal?  
If the vectors are orthogonal, then the dot product is 0, so the equation reduces to

$$A(S) = \iint_D \|\phi_u\| \|\phi_v\| dA$$

- (c) Use parts (a) and (b) to compute the surface area of a sphere of radius  $a$ .  
(cf. Marsden & Tromba, page 399, # 23.)

$$\begin{aligned} \Phi(\theta, \varphi) &= a(\cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi) \\ \phi_\theta &= a(-\sin \theta \sin \varphi, \cos \theta \sin \varphi, 0) \\ \phi_\varphi &= a(\cos \theta \cos \varphi, \sin \theta \cos \varphi, -\sin \varphi) \\ \|\phi_\theta\| &= a \sin \varphi, \quad \|\phi_\varphi\| = a \\ \implies A(S) &= a^2 \int_0^{2\pi} \int_0^\pi \sin \varphi d\varphi d\theta \\ &= a^2 \int_0^{2\pi} \left[ -\cos \varphi \right]_0^\pi d\theta \\ &= a^2 \int_0^{2\pi} -(-1 - 1) d\theta \\ &= a^2 2 \int_0^{2\pi} 1 d\theta \\ &= 4\pi a^2 \end{aligned}$$

6. For each of the following surfaces  $S$ , sketch  $S$  (using symbolic software) and evaluate the surface integral  $\int_S f dS$ , where  $f(x, y, z) = x$ .

- (a)  $S$  is that part of the surface  $y = 4 - x^2$  between  $z = 0$  and  $z = 1$ , with  $y \geq 0$ .

$$y \geq 0 \implies 4 - x^2 \geq 0 \implies x^2 \leq 4 \implies |x| < 2$$

$$\Phi(x, z) = (x, 4 - x^2, z)$$

$$\phi_x = (1, -2x, 0), \phi_z = (0, 0, 1)$$

$$\phi_x \times \phi_z = (-2x, -1, 0) \implies \|\phi_x \times \phi_z\| = \sqrt{4x^2 + 1}$$

$$\int_S f dS = \int_0^1 \int_{-2}^2 x \sqrt{4x^2 + 1} dx dz$$

The integrand is odd since  $x$  odd and  $\sqrt{4x^2 + 1}$  even, so the integral over  $x$  is 0, making the entire integral 0.

- (b)  $S$  is the upper half of the unit sphere centered at the origin.

Only the upper half so  $0 \leq \theta \leq 2\pi$  and  $0 \leq \varphi \leq \pi/2$ .

$$\Phi(\theta, \varphi) = (\cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi)$$

$$\phi_\theta = (-\sin \theta \sin \varphi, \cos \theta \sin \varphi, 0)$$

$$\phi_\varphi = (\cos \theta \cos \varphi, \sin \theta \cos \varphi, -\sin \varphi)$$

$$\begin{aligned} \phi_\theta \times \phi_\varphi &= (-\cos \theta \sin^2 \varphi, -\sin \theta \sin^2 \varphi, -\sin^2 \theta \sin \varphi \cos \varphi - \cos^2 \theta \sin \varphi \cos \varphi) \\ &= (-\cos \theta \sin^2 \varphi, -\sin \theta \sin^2 \varphi, -\sin \varphi \cos \varphi) \end{aligned}$$

$$\begin{aligned} \|\phi_\theta \times \phi_\varphi\| &= \sqrt{\cos^2 \theta \sin^4 \varphi + \sin^2 \theta \sin^4 \varphi + \sin^2 \varphi \cos^2 \varphi} \\ &= \sqrt{\sin^4 \varphi + \sin^2 \varphi \cos^2 \varphi} \\ &= \sqrt{\sin^2 \varphi} = \sin \varphi \end{aligned}$$

$$\int_{\Phi} f dS = \int_0^{\pi/2} \int_0^{2\pi} \cos \theta \sin^2 \varphi d\theta d\varphi = 0$$

The integral is zero again since integrating  $\cos \theta$  over a whole period is 0.

- (c)  $S$  is that part of the surface  $x = \sin y$  with  $0 \leq y \leq \pi$  and  $0 \leq z \leq 2$ .

$$\Phi(y, z) = (\sin y, y, z)$$

$$\phi_y = (\cos y, 1, 0)$$

$$\phi_z = (0, 0, 1)$$

$$\phi_y \times \phi_z = (1, -\cos y, 0)$$

$$\|\phi_y \times \phi_z\| = \sqrt{1 + \cos^2 y}$$

$$\int_{\Phi} f dS = \int_0^2 \int_0^{\pi} \sin y \sqrt{1 + \cos^2 y} dy dz$$

7. Find the mass of the metallic surface  $S$  given by  $z = 1 - \frac{x^2 + y^2}{2}$  with  $0 \leq x \leq 1$ ,  $0 \leq y \leq 1$ , if the mass density at  $(x, y, z) \in S$  is given by  $m(x, y, z) = xy$ .