

MATB42: Assignment #10

1. Let \mathbf{F} be a vector field on \mathbb{R}^3 given by $\mathbf{F} = (F_1, F_2, F_3)$ where F_1 , F_2 , and F_3 are C^1 -functions from $\mathbb{R}^3 \rightarrow \mathbb{R}$

- (a) Let η be the 2-form given by

$$\eta = F_3 dx dy + F_1 dy dz + F_2 dz dx$$

Show that $d\eta = (\operatorname{div} \mathbf{F}) dx dy dz$

(page 489, #6)

$$\begin{aligned} \eta &= F_3 dx dy + F_1 dy dz + F_2 dz dx \\ d\eta &= d(F_3 dx dy + F_1 dy dz + F_2 dz dx) \\ &= (dF_3) dx dy + (dF_1) dy dz + (dF_2) dz dx \\ &= \left(\frac{\partial}{\partial x} F_3 dx + \frac{\partial}{\partial y} F_3 dy + \frac{\partial}{\partial z} F_3 dz \right) dx dy + (dF_1) dy dz + (dF_2) dz dx \\ &= \frac{\partial}{\partial z} F_3 dz dx dy + (dF_1) dy dz + (dF_2) dz dx \\ &= \frac{\partial}{\partial z} F_3 dx dy dz + \left(\frac{\partial}{\partial x} F_1 dx + \frac{\partial}{\partial y} F_1 dy + \frac{\partial}{\partial z} F_1 dz \right) dy dz + (dF_2) dz dx \\ &= \frac{\partial}{\partial z} F_3 dx dy dz + \frac{\partial}{\partial x} F_1 dx dy dz + (dF_2) dz dx \\ &= \frac{\partial}{\partial z} F_3 dx dy dz + \frac{\partial}{\partial x} F_1 dx dy dz + \left(\frac{\partial}{\partial x} F_2 dx + \frac{\partial}{\partial y} F_2 dy + \frac{\partial}{\partial z} F_2 dz \right) dz dx \\ &= \frac{\partial}{\partial z} F_3 dx dy dz + \frac{\partial}{\partial x} F_1 dx dy dz + \frac{\partial}{\partial y} F_2 dy dz dx \\ &= \frac{\partial}{\partial z} F_3 dx dy dz + \frac{\partial}{\partial x} F_1 dx dy dz + \frac{\partial}{\partial y} F_2 dx dy dz \\ &= \frac{\partial}{\partial x} F_1 + \frac{\partial}{\partial y} F_2 + \frac{\partial}{\partial z} F_3 dx dy dz = (\operatorname{div} \mathbf{F}) dx dy dz \end{aligned}$$

- (b) Show that $dF_1 \wedge dF_2 \wedge dF_3 = (\det D\mathbf{F}) dx dy dz$