

## MATB42: Assignment #10

1. Let  $\mathbf{F}$  be a vector field on  $\mathbb{R}^3$  given by  $\mathbf{F} = (F_1, F_2, F_3)$  where  $F_1$ ,  $F_2$ , and  $F_3$  are  $C^1$ -functions from  $\mathbb{R}^3 \rightarrow \mathbb{R}$

- (a) Let  $\eta$  be the 2-form given by

$$\eta = F_3 dx dy + F_1 dy dz + F_2 dz dx$$

Show that  $d\eta = (\operatorname{div} \mathbf{F}) dx dy dz$

(page 489, #6)

$$\begin{aligned}
 \eta &= F_3 dx dy + F_1 dy dz + F_2 dz dx \\
 d\eta &= d(F_3 dx dy + F_1 dy dz + F_2 dz dx) \\
 &= (dF_3) dx dy + (dF_1) dy dz + (dF_2) dz dx \\
 &= \left( \frac{\partial}{\partial x} F_3 dx + \frac{\partial}{\partial y} F_3 dy + \frac{\partial}{\partial z} F_3 dz \right) dx dy + (dF_1) dy dz + (dF_2) dz dx \\
 &= \frac{\partial}{\partial z} F_3 dz dx dy + (dF_1) dy dz + (dF_2) dz dx \\
 &= \frac{\partial}{\partial z} F_3 dx dy dz + \left( \frac{\partial}{\partial x} F_1 dx + \frac{\partial}{\partial y} F_1 dy + \frac{\partial}{\partial z} F_1 dz \right) dy dz + (dF_2) dz dx \\
 &= \frac{\partial}{\partial z} F_3 dx dy dz + \frac{\partial}{\partial x} F_1 dx dy dz + (dF_2) dz dx \\
 &= \frac{\partial}{\partial z} F_3 dx dy dz + \frac{\partial}{\partial x} F_1 dx dy dz + \left( \frac{\partial}{\partial x} F_2 dx + \frac{\partial}{\partial y} F_2 dy + \frac{\partial}{\partial z} F_2 dz \right) dz dx \\
 &= \frac{\partial}{\partial z} F_3 dx dy dz + \frac{\partial}{\partial x} F_1 dx dy dz + \frac{\partial}{\partial y} F_2 dy dz dx \\
 &= \frac{\partial}{\partial z} F_3 dx dy dz + \frac{\partial}{\partial x} F_1 dx dy dz + \frac{\partial}{\partial y} F_2 dx dy dz \\
 &= \frac{\partial}{\partial x} F_1 + \frac{\partial}{\partial y} F_2 + \frac{\partial}{\partial z} F_3 dx dy dz = (\operatorname{div} \mathbf{F}) dx dy dz
 \end{aligned}$$

(b) Show that  $dF_1 \wedge dF_2 \wedge dF_3 = (\det D\mathbf{F}) dx dy dz$

$$df = \sum_{i=0}^n \frac{\partial f}{\partial x_i} dx_i$$

$$\begin{aligned}
dF_1 \wedge dF_2 \wedge dF_3 &= \left( \frac{\partial F_1}{\partial x} dx + \frac{\partial F_1}{\partial y} dy + \frac{\partial F_1}{\partial z} dz \right) \wedge \left( \frac{\partial F_2}{\partial x} dx + \frac{\partial F_2}{\partial y} dy + \frac{\partial F_2}{\partial z} dz \right) \wedge dF_3 \\
&= \left( \frac{\partial F_1}{\partial x} dx \wedge \left( \frac{\partial F_2}{\partial x} dx + \frac{\partial F_2}{\partial y} dy + \frac{\partial F_2}{\partial z} dz \right) \right. \\
&\quad + \frac{\partial F_1}{\partial y} dy \wedge \left( \frac{\partial F_2}{\partial x} dx + \frac{\partial F_2}{\partial y} dy + \frac{\partial F_2}{\partial z} dz \right) \\
&\quad \left. + \frac{\partial F_1}{\partial z} dz \wedge \left( \frac{\partial F_2}{\partial x} dx + \frac{\partial F_2}{\partial y} dy + \frac{\partial F_2}{\partial z} dz \right) \right) \wedge dF_3 \\
&= \left( \left( \frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial y} dx dy + \frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial z} dx dz \right) \right. \\
&\quad + \left( \frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial x} dy dx + \frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial z} dy dz \right) \\
&\quad \left. + \left( \frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial x} dz dx + \frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial y} dz dy \right) \right) \wedge dF_3 \\
&= \left( \left( \frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial y} - \frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial x} \right) dx dy \right. \\
&\quad + \left( \frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial y} \right) dy dz \\
&\quad \left. + \left( \frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial z} \right) dz dx \right) \wedge \left( \frac{\partial F_3}{\partial x} dx + \frac{\partial F_3}{\partial y} dy + \frac{\partial F_3}{\partial z} dz \right) \\
&= \left( \frac{\partial F_3}{\partial z} \left( \frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial y} - \frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial x} \right) dx dy \right. \\
&\quad + \frac{\partial F_3}{\partial x} \left( \frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial y} \right) dy dz \\
&\quad \left. + \frac{\partial F_3}{\partial y} \left( \frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial z} \right) dz dx \right) \\
&= \frac{\partial F_3}{\partial x} \left( \frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial y} \right) dx dy dz \\
&\quad - \frac{\partial F_3}{\partial y} \left( \frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial x} \right) dx dy dz \\
&\quad + \frac{\partial F_3}{\partial z} \left( \frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial y} - \frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial x} \right) dx dy dz \\
&= \frac{\partial F_3}{\partial x} \begin{vmatrix} \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{vmatrix} - \frac{\partial F_3}{\partial y} \begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial z} \end{vmatrix} + \frac{\partial F_3}{\partial z} \begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{vmatrix} dx dy dz \\
&= \begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \\ \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial z} \end{vmatrix} dx dy dz
\end{aligned}$$

2. Let  $\omega$  be a  $k$ -form and let  $\eta$  be a  $\ell$ -form. Find  $d(d\omega \wedge \eta - \omega \wedge d\eta)$ .

$$\begin{aligned}
 d(d\omega \wedge \eta - \omega \wedge d\eta) &= d(d\omega \wedge \eta) - d(\omega \wedge d\eta) \\
 &= (d^2\omega \wedge \eta + (-1)^{k+1}(d\omega \wedge d\eta)) - (d\omega \wedge d\eta + (-1)^k(\omega \wedge d^2\eta)) \\
 &= (-1)^{k+1}d\omega \wedge d\eta - d\omega \wedge d\eta \\
 &= ((-1)^{k+1} - 1)d\omega \wedge d\eta
 \end{aligned}$$

3. Determine if  $\eta = y \, dx \, dy + xz \, dy \, dz - yz \, dz \, dx$  is exact. If  $\eta$  is exact find a 1-form  $\omega$  with  $d\omega = \eta$ . Check if  $d\eta = \mathcal{O}$  to see if  $\eta$  closed.

(compare with page 461, # 22)

$$\begin{aligned}
 d\eta &= d(y \, dx \, dy + xz \, dy \, dz - yz \, dz \, dx) \\
 &= (dy \, dx \, dy + d(xz) \wedge dy \, dz - d(yz) \wedge dz \, dx) \\
 &= ((z \, dx + x \, dz) \wedge dy \, dz - (z \, dy + y \, dz) \wedge dz \, dx) \\
 &= (z \, dx) \wedge dy \, dz - (z \, dy) \wedge dz \, dx \\
 &= z \, dx \, dy \, dz - z \, dx \, dy \, dz = \mathcal{O}
 \end{aligned}$$

Since the polynomials of  $x$ ,  $y$  and  $z$  defined throughout  $\mathbb{R}^3$  and  $\eta$  closed, it is exact. By inspection,

$$\omega = xy \, dy + xyz \, dz$$

4. Evaluate  $\iint_S \omega$ , where  $\omega = z dx dy + x dy dz + y dz dx$  and  $S$  is the unit sphere, directly and by the Divergence Theorem.

(page 489, #12)

Directly:

Parametrize the sphere  $S$  as

$$\Phi(\varphi, \theta) = (\cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi) \text{ with } \theta \in [0, 2\pi], \varphi \in [0, \pi]$$

$$\begin{aligned} \iint_S \omega &= \iint_{\Phi} z dx dy + \iint_{\Phi} x dy dz + \iint_{\Phi} y dz dx \\ &= \int_0^{2\pi} \int_0^{\pi} \cos \varphi \begin{vmatrix} \frac{\partial \cos \theta \sin \varphi}{\partial \varphi} & \frac{\partial \cos \theta \sin \varphi}{\partial \theta} \\ \frac{\partial \sin \theta \sin \varphi}{\partial \varphi} & \frac{\partial \sin \theta \sin \varphi}{\partial \theta} \end{vmatrix} d\varphi d\theta + \int_0^{2\pi} \int_0^{\pi} \cos \theta \sin \varphi \begin{vmatrix} \frac{\partial \sin \theta \sin \varphi}{\partial \varphi} & \frac{\partial \sin \theta \sin \varphi}{\partial \theta} \\ \frac{\partial \cos \varphi}{\partial \varphi} & \frac{\partial \cos \varphi}{\partial \theta} \end{vmatrix} d\varphi d\theta \\ &\quad + \int_0^{2\pi} \int_0^{\pi} \sin \theta \sin \varphi \begin{vmatrix} \frac{\partial \cos \varphi}{\partial \varphi} & \frac{\partial \cos \varphi}{\partial \theta} \\ \frac{\partial \cos \theta \sin \varphi}{\partial \varphi} & \frac{\partial \cos \theta \sin \varphi}{\partial \theta} \end{vmatrix} d\varphi d\theta \\ &= \int_0^{2\pi} \int_0^{\pi} \cos \varphi \begin{vmatrix} \cos \theta \cos \varphi & -\sin \theta \sin \varphi \\ \sin \theta \cos \varphi & \cos \theta \sin \varphi \end{vmatrix} d\varphi d\theta + \int_0^{2\pi} \int_0^{\pi} \cos \theta \sin \varphi \begin{vmatrix} \sin \theta \cos \varphi & \cos \theta \sin \varphi \\ -\sin \varphi & 0 \end{vmatrix} d\varphi d\theta \\ &\quad + \int_0^{2\pi} \int_0^{\pi} \sin \theta \sin \varphi \begin{vmatrix} -\sin \varphi & 0 \\ \cos \theta \cos \varphi & -\sin \theta \sin \varphi \end{vmatrix} d\varphi d\theta \\ &= \int_0^{2\pi} \int_0^{\pi} \sin \varphi \cos^2 \varphi d\varphi d\theta + \int_0^{2\pi} \int_0^{\pi} \cos^2 \theta \sin^3 \varphi + \sin^2 \theta \sin^3 \varphi d\varphi d\theta \\ &= \int_0^{2\pi} \int_0^{\pi} \sin \varphi (\cos^2 \varphi + \sin^2 \varphi) d\varphi d\theta \\ &= \int_0^{2\pi} \int_0^{\pi} \sin \varphi d\varphi d\theta \\ &= 2\pi \left[ -\cos \varphi \right]_0^{\pi} = 2\pi \end{aligned}$$

Divergence Theorem:

$$d\omega = dz dy dx + dx dy dz + dy dz dx = 3 dx dy dz$$

$$\begin{aligned} \iint_S \omega &= \iiint_R d\omega \\ &= 3 \int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^2 \sin(\varphi) d\rho d\varphi d\theta \\ &= \int_0^{2\pi} \int_0^{\pi} \sin(\varphi) d\varphi d\theta \\ &= 2\pi \left[ -\cos \varphi \right]_0^{\pi} = 2\pi \end{aligned}$$

5. Compute  $\int_S \omega$  and use symbolic algebra software to sketch  $S$  in each of the following.

(a)  $\omega = xz \, dx \, dy + x^2 \, dy \, dz + dy \, dz \, dx$

$S$  is the upper hemisphere  $x^2 + y^2 + z^2 = 4, z \geq 0$  with  $\mathbf{n}$  pointing upward.

Close it with the disk of radius 2 on the  $xy$ -plane to apply divergence theorem

$$\Phi(\theta, r) = (r \cos \theta, r \sin \theta, 0), \quad r \in [0, 2], \quad \theta \in [0, 2\pi]$$

$$dx \, dy = \begin{vmatrix} -r \sin \theta & \cos \theta \\ r \cos \theta & \sin \theta \end{vmatrix} = -2r$$

Which is negative, so correct orientation for normal pointing down.

$$dy \, dz = 0 \quad \text{Since } z \text{ is } 0$$

$$dz \, dx = 0$$

$$\stackrel{\text{Div Thm}}{\implies} \iint_S \omega = \iiint_R d\omega - \iint_{\Phi} \omega$$

$$\text{But } z = 0 \implies xz \, dx \, dy = 0 \implies \iint_{\Phi} \omega = 0$$

$$d\omega = x \, dx \, dy \, dz + 2x \, dx \, dy \, dz = 3x \, dx \, dy \, dz$$

$$\iiint_R d\omega = \int_0^2 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} 3(\rho \sin \varphi \cos \theta) \rho^2 \sin \varphi \, d\varphi \, d\theta \, d\rho$$

$$= 0 \quad \text{Since integrating } \cos \text{ over full period}$$

$$\implies \int_S \omega = 0$$



(b)  $\omega = z \, dx \, dy + x \, dy \, dz + y \, dz \, dx$

$S$  is the part of the plane  $x + y + z = 1$  which lies in the first octant oriented by the unit normal which points upward.

Use the natural parametrization for  $S$ :

$$\Phi(x, y) = (x, y, 1 - x - y), \quad x \in [0, 1], \quad y \in [0, 1 - x]$$

$$dx \, dy = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} > 0 \, \forall x, y \implies \text{Correct orientation}$$

$$\begin{aligned} \int_S \omega &= \int_0^1 \int_0^{1-x} (1 - x - y) + x \begin{vmatrix} 0 & 1 \\ -1 & -1 \end{vmatrix} + y \begin{vmatrix} -1 & -1 \\ 1 & 0 \end{vmatrix} dy \, dx \\ &= \int_0^1 \int_0^{1-x} 1 \, dy \, dx = \int_0^1 1 - x \, dx \\ &= 1 - \frac{1}{2} = \frac{1}{2} \end{aligned}$$



(c)  $\omega = xz \, dx \, dy + y \, dx \, dz + z^2 \, dy \, dz$

$S$  is the part of the cone  $z = \sqrt{x^2 + y^2}$  between  $z = 1$  and  $z = 3$ , oriented by the unit normal with negative  $z$ -component.

$$\Phi(\theta, r) = (r \cos \theta, r \sin \theta, r), \quad r \in [1, 3], \quad \theta \in [0, 2\pi]$$

$$dx \, dy = \begin{vmatrix} -r \sin \theta & \cos \theta \\ r \cos \theta & \sin \theta \end{vmatrix} = -r < 0 \text{ for } r > 1$$

$$dy \, dz = \begin{vmatrix} r \cos \theta & \sin \theta \\ 0 & 1 \end{vmatrix} = r \cos \theta$$

$$dz \, dx = \begin{vmatrix} 0 & 1 \\ -r \sin \theta & \cos \theta \end{vmatrix} = r \sin \theta$$

$$\Rightarrow \omega = (r \cos \theta)(r)(-r) - (r \sin \theta)(r \sin \theta) + (r)^2(r \cos \theta)$$

$$= -r^2 \sin^2 \theta = -r^2 \left( \frac{1}{2} - \frac{\cos(2\theta)}{2} \right)$$

$$\Rightarrow \int_S \omega = \int_1^3 \int_0^{2\pi} -r^2 \left( \frac{1}{2} - \frac{\cos(2\theta)}{2} \right) d\theta \, dr$$

$$= \int_1^3 -r^2 \pi \, dr = -\pi \left[ \frac{r^3}{3} \right]_1^3 = -\frac{26\pi}{3}$$



(d)  $\omega = z \, dx \, dy + y \, dy \, dz$

$S$  is the oriented surface given by the parametrization

$$\Phi(u, v) = (u + v, uv^2, u^2 + v^2), \quad 0 \leq u \leq 1, \quad 0 \leq v \leq 1.$$

$$dx \, dy = \begin{vmatrix} 1 & 1 \\ v^2 & 2uv \end{vmatrix} = 2uv - v^2, \quad dy \, dz = \begin{vmatrix} v^2 & 2uv \\ 2u & 2v \end{vmatrix} = 2v^3 - 4u^2v$$

$$\begin{aligned} \iint_S \omega &= \int_0^1 \int_0^1 (u^2 + v^2)(2uv - v^2) + (uv^2)(2v^3 - 4u^2v) \, du \, dv \\ &= \int_0^1 \int_0^1 (2u^3v - u^2v^2) + (2uv^3 - v^4) + (2uv^5 - 4u^3v^3) \, du \, dv \\ &= \int_0^1 \left( \frac{v}{2} - \frac{v^2}{3} + v^3 - v^4 + v^5 - v^3 \right) \, dv \\ &= \int_0^1 \left( \frac{v}{2} - \frac{v^2}{3} - v^4 + v^5 \right) \, dv \\ &= \frac{1}{4} - \frac{1}{9} - \frac{1}{5} + \frac{1}{6} = \frac{19}{180} \end{aligned}$$





6. Verify Stokes' theorem by direct calculation of both sides when the surface  $S$  is the piece of the paraboloid  $z = x^2 + y^2 - 4$  with  $z \leq 0$ , oriented by the downward pointing unit normal, and  $\omega = (2y - z) dx + (x + y^2 - z) dy + (4y - 3x) dz$ .

As part of your solution, provide a sketch showing the appropriate orientations. (For this question you may draw the sketch by hand or use symbolic algebra software.)

Stokes' Theorem states that:

$$\int_{\partial S} \omega = \int_S d\omega$$

Calculation of  $\int_{\partial S} \omega$

The boundary curve of the plane is the circle at  $z = 0$  with radius 2. Since the normal vector is downward pointing, the curve is parametrized clockwise. So parametrize the curve as  $\gamma(\theta) = (-2 \cos \theta, 2 \sin \theta, 0)$

$$\begin{aligned} & \int_0^{2\pi} (2(2 \sin \theta) - 0)(2 \sin \theta) + ((-2 \cos \theta) + (2 \sin \theta)^2 - 0)(2 \cos \theta) d\theta \\ &= \int_0^{2\pi} 8 \sin^2 \theta - 4 \cos^2 \theta + 8 \sin^2 \theta \cos \theta d\theta \\ &= \int_0^{2\pi} 4(1 - \cos(2\theta)) - 2(1 + \cos(2\theta)) + 8 \sin^2 \theta \cos \theta d\theta \\ &= \int_0^{2\pi} 2 + 8 \sin^2 \theta \cos \theta d\theta \\ &= 4\pi + \left[ \frac{8 \sin^3 \theta}{3} \right]_0^{2\pi} = 4\pi \end{aligned}$$

Calculation of  $\int_S d\omega$

(Using the parametrization  $\Phi(\theta, r) = (r \cos \theta, r \sin \theta, r^2 - 4)$ ,  $r \in [0, 2]$ ,  $\theta \in [0, 2\pi]$ )

$$dx dy = \begin{vmatrix} -r \sin \theta & \cos \theta \\ r \cos \theta & \sin \theta \end{vmatrix} = -r$$

So the orientation is in the correct direction.

$$\begin{aligned} d\omega &= d(2y - z) dx + d(x + y^2 - z) dy + d(4y - 3x) dz \\ &= (2 dy - dz) \wedge dx + (dx - dz) \wedge dy + (4 dy - 3 dx) \wedge dz \\ &= -2 dx dy - dz dx + dx dy + dy dz + 4 dy dz + 3 dz dx \\ &= -dx dy + 5 dy dz + 2 dz dx \\ \Rightarrow \int_S d\omega &= \int_0^{2\pi} \int_0^2 \begin{vmatrix} -r \sin \theta & \cos \theta \\ r \cos \theta & \sin \theta \end{vmatrix} + 5 \begin{vmatrix} r \cos \theta & \sin \theta \\ 0 & 2r \end{vmatrix} + 2 \begin{vmatrix} 0 & 2r \\ -r \sin \theta & \cos \theta \end{vmatrix} dr d\theta \\ &= \int_0^{2\pi} \int_0^2 r + 5(2r^2 \cos \theta) + 2(2r^2 \sin \theta) dr d\theta \\ &= 2\pi \int_0^2 r dr = 4\pi \end{aligned}$$