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MATB42: Assignment #10 Apr 3rd 2018

1. Let $\boldsymbol{F}$ be a vector field on $\mathbb{R}^3$ given by $\boldsymbol{F} = (F_1, F_2, F_3)$ where F_1, F_2, and F_3 are C^1-functions from $\mathbb{R}^3 \to \mathbb{R}$

   (a) Let η be the 2-form given by

$$\eta = F_3\,dx\,dy + F_1\,dy\,dz + F_2\,dz\,dx$$

   Show that $d\eta = (\operatorname{div}\boldsymbol{F})\,dx\,dy\,dz$
   (page 489, #6)

$$\begin{aligned}
\eta &= F_3\,dx\,dy + F_1\,dy\,dz + F_2\,dz\,dx \\
d\eta &= d(F_3\,dx\,dy + F_1\,dy\,dz + F_2\,dz\,dx) \\
&= (dF_3)\,dx\,dy + (dF_1)\,dy\,dz + (dF_2)\,dz\,dx \\
&= (\frac{\partial}{\partial x}F_3\,dx + \frac{\partial}{\partial y}F_3\,dy + \frac{\partial}{\partial z}F_3\,dz)\,dx\,dy + (dF_1)\,dy\,dz + (dF_2)\,dz\,dx \\
&= \frac{\partial}{\partial z}F_3\,dz\,dx\,dy + (dF_1)\,dy\,dz + (dF_2)\,dz\,dx \\
&= \frac{\partial}{\partial z}F_3\,dx\,dy\,dz + (\frac{\partial}{\partial x}F_1\,dx + \frac{\partial}{\partial y}F_1\,dy + \frac{\partial}{\partial z}F_1\,dz)\,dy\,dz + (dF_2)\,dz\,dx \\
&= \frac{\partial}{\partial z}F_3\,dx\,dy\,dz + \frac{\partial}{\partial x}F_1\,dx\,dy\,dz + (dF_2)\,dz\,dx \\
&= \frac{\partial}{\partial z}F_3\,dx\,dy\,dz + \frac{\partial}{\partial x}F_1\,dx\,dy\,dz + (\frac{\partial}{\partial x}F_2\,dx + \frac{\partial}{\partial y}F_2\,dy + \frac{\partial}{\partial z}F_2\,dz)\,dz\,dx \\
&= \frac{\partial}{\partial z}F_3\,dx\,dy\,dz + \frac{\partial}{\partial x}F_1\,dx\,dy\,dz + \frac{\partial}{\partial y}F_2\,dy\,dz\,dx \\
&= \frac{\partial}{\partial z}F_3\,dx\,dy\,dz + \frac{\partial}{\partial x}F_1\,dx\,dy\,dz + \frac{\partial}{\partial y}F_2\,dx\,dy\,dz \\
&= \frac{\partial}{\partial x}F_1 + \frac{\partial}{\partial y}F_2 + \frac{\partial}{\partial z}F_3\,dx\,dy\,dz = (\operatorname{div}\boldsymbol{F})\,dx\,dy\,dz
\end{aligned}$$

   (b) Show that $dF_1 \wedge dF_2 \wedge dF_3 = (\det\, D\boldsymbol{F})\,dx\,dy\,dz$

2. Let ω be a k-form and let η be a ℓ-form. Find $d(d\omega \wedge \eta - \omega \wedge d\eta)$.

3. Determine if $\eta = y\,dx\,dy + dz\,dy\,dz - yz\,dz\,dx$ is exact. If η is exact find a 1-form ω with $d\omega = \eta$. (compare with page 461, # 22)

4. Evaluate $\iint_S \omega$, where $\omega = z\,dx\,dy + x\,dy\,dz + y\,dz\,dx$ and S is the unit sphere, directly and by the Divergence Theorem.

(page 489, #12)

5. Compute $\displaystyle\int_S \omega$ and use symbolic algebra software to sketch S in each of the following.

(a) $\omega = xz\,dx\,dy + x^2\,dy\,dz + dy\,dz\,dx$
S is the upper hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$ with $\boldsymbol{n}$ pointing upward.

(b) $\omega = z\,dx\,dy + x\,dy\,dz + y\,dz\,dx$
S is the part of the plane $x + y + z = 1$ which lies in the first octant oriented by the unit normal which points upward.

(c) $\omega = xz\,dx\,dy + y\,dx\,dz + z^2\,dy\,dz$
S is the part of the cone $z = \sqrt{x^2 + y^2}$ between $z = 1$ and $z = 3$, oriented by the unit normal with negative z-component.

(d) $\omega = z\,dx\,dy + y\,dy\,dz$
S is the oriented surface given by the parametrization
$\Phi(u, v) = (u + v, uv^2, u^2 + v^2)$, $0 \leq u \leq 1$, $0 \leq v \leq 1$.

6. Verify Stokes' theorem by direct calculation of both sides when the surface S is the piece of the paraboloid $z = x^2 + y^2 - 4$ with $z \leq 0$, oriented by the downward pointing unit normal, and $\omega = (2y - z)\,dx + (x + y^2 - z)\,dy + (4y - 3x)\,dz$.