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MATB42: Assignment #2
Bonus

1. Let $f(x) = \begin{cases} 0, & -\pi < x < -\frac{\pi}{2} \\ 2, & -\frac{\pi}{2} \leq 2 < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} \leq x < \pi \end{cases}$

   (a) Find the Fourier series of f.

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\ &= \frac{1}{\pi}\left[\int_{-\pi}^{-\frac{\pi}{2}} 0dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2dx + \int_{\frac{\pi}{2}}^{\pi} 0dx\right] \\ &= \frac{1}{\pi}\left[2\pi\right] \\ &= 2 \end{aligned}$$

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx \\ &= \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(kx)dx \\ &= \frac{2}{k\pi}\left[\sin(kx)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\ &= \frac{2}{k\pi}\left[2\sin\left(\frac{k\pi}{2}\right)\right] \\ &= \frac{4}{k\pi}\sin\left(\frac{k\pi}{2}\right) \end{aligned}$$

$$\begin{aligned} b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx \\ &= \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(kx)dx \\ &= 0 \quad [\text{sin is odd}] \end{aligned}$$

This is 0 for even elements, and alternating between 1 and -1 for odd elements.

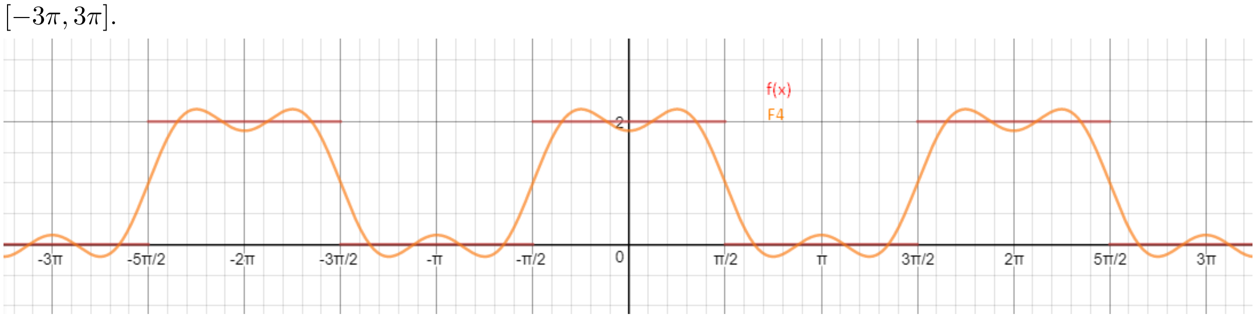
Therefore the Fourier polynomial (for the non-zero terms) is

$$1 + \sum_{l=1}^{\infty}\left[\frac{4(-1)^l}{(2l+1)\pi}\cos((2l+1)x)\right]$$

   (b) Determine if the Fourier series in part (a) converges. If it does converge, what are the values to which it converges (on $[-\pi, \pi]$).

   The function is continuous on its partitions (they are constant functions), so by the theorem the polynomial converges to $f(x)$ on the continuous intervals. On the discontinuities, it converges to 0 at $\frac{\pi}{2}$ and $\frac{-\pi}{2}$ from the Fundemental theorem, and to 0 at π and $-\pi$.

   (c) Use symbolic algeba software to sketch $f(x)$ and its 4^{th} degree Fourier polynomial over the interval $[-3\pi, 3\pi]$.



2. (a) Find the Fourier series of the function $f(x)$ defined by $f(x) = \begin{cases} 0 & , -\pi \leq x < 0 \\ x & , 0 \leq x < \pi \end{cases}$ and extended from this with period 2π to all of $\mathbb{R}$.
If this Fourier series converges describe the function to which it converges.

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\ &= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)dx + \int_{0}^{\pi} f(x)dx\right] \\ &= \frac{1}{\pi}\left[0 + \int_{0}^{\pi} xdx\right] \\ &= \frac{1}{\pi}\left[\frac{1}{2}\left[x^2\right]_0^\pi\right] \\ &= \frac{\pi}{2} \end{aligned}$$

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx \\ &= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0dx + \int_{0}^{\pi} x\cos(kx)dx\right] \end{aligned}$$

Let $u = x$, $du = dx$,

$dv = \cos(kx)$, $v = \dfrac{\sin(kx)}{k}$

$$\begin{aligned} &= \frac{1}{\pi}\left[\frac{1}{k}\left[x\sin(kx)\right]_0^\pi - \frac{1}{k}\int_0^\pi \sin(kx)dx\right] \\ &= -\frac{1}{k\pi}\left[\int_0^\pi \sin(kx)dx\right] \\ &= \frac{1}{k^2\pi}\left[\cos(kx)\right]_0^\pi \\ &= \frac{(-1)^{-k}-1}{k^2\pi} \end{aligned}$$

$$\begin{aligned} b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx \\ &= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0dx + \int_{0}^{\pi} x\sin(kx)dx\right] \end{aligned}$$

Let $u = x$, $du = dx$, $dv = \sin(kx)$, $v = -\frac{1}{k}\cos(kx)$

$$\begin{aligned} &= \frac{1}{\pi}\left[-\frac{1}{k}\left[x\cos(kx)\right]_0^\pi + \frac{1}{k}\int_0^\pi \cos(kx)dx\right] \\ &= \frac{1}{k\pi}\left[-\pi\cos(k\pi) + \frac{1}{k}\left[\sin(kx)\right]_0^\pi\right] \\ &= \frac{1}{k\pi}\left[-\pi\cos(k\pi) + 0\right] \\ &= \frac{(-1)^{k+1}}{k} \end{aligned}$$

Therefore the Fourier series of f is

$$F(x) = \frac{\pi}{4} + \sum_{k=1}^{\infty}\left[\frac{(-1)^k - 1}{k^2\pi}\cos(kx) + \frac{(-1)^{k+1}}{k}\sin(kx)\right]$$

Since f is piecewise very smooth (0, x are infinitely differentiable), the series converges to f on $(-\pi, \pi)$ and on both endpoints, it converges to $\frac{\pi}{2}$.

(b) Using the series from part (a) show that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots.$$

$$\begin{aligned} F(0) &= \frac{\pi}{4} + \sum_{k=1}^{\infty}\left[\frac{(-1)^k - 1}{k^2\pi}\right] \\ 0 &= \frac{\pi}{4} + \sum_{k=1}^{\infty}\left[\frac{-2}{(2k-1)^2\pi}\right] \end{aligned}$$

$$\begin{aligned} \frac{\pi}{4} &= \sum_{k=1}^{\infty}\left[\frac{2}{(2k-1)^2\pi}\right] \\ \frac{\pi^2}{8} &= \sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} \end{aligned}$$

3. Find the Fourier series for the restriction of the function $f(x) = 3+3x$ to each of the following intervals, $[a, b]$. If the Fourier series converges, to what values will the series converge at the end points?

(a) $[a, b] = [-\pi, \pi]$

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} 3+3xdx \\ &= \frac{1}{\pi}\left[6\pi + \frac{3}{2}\Big[x^2\Big]_{-\pi}^{\pi}\right] \\ &= \frac{1}{\pi}\Big[6\pi + 0\Big] \\ &= 6 \end{aligned}$$

$$\begin{aligned} b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx \\ &= \frac{3}{\pi}\left[\int_{-\pi}^{\pi} \sin(kx)dx + \int_{-\pi}^{\pi} x\sin(kx)dx\right] \\ &= \frac{6}{\pi}\left[\int_{0}^{\pi} x\sin(kx)dx\right] \quad \text{[Since } x \text{ and sin odd]} \end{aligned}$$

Let $u = x, du = 1dx, dv = \sin(kx)dx, v = -\frac{\cos(kx)}{k}$

$$\begin{aligned} &= \frac{6}{\pi}\left[-\frac{1}{k}\Big[x\cos(kx)\Big]_0^{\pi} + \frac{1}{k}\int_0^{\pi}\cos(kx)dx\right] \\ &= \frac{6}{k\pi}\left[\pi(-1)^{k+1} + \frac{1}{k}\Big[\sin(kx)\Big]_0^{\pi}\right] \\ &= \frac{6(-1)^{k+1}}{k} \end{aligned}$$

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx \\ &= \frac{3}{\pi}\left[\int_{-\pi}^{\pi} \cos(kx)dx + \int_{-\pi}^{\pi} x\cos(kx)dx\right] \\ &= \frac{6}{k\pi}\Big[\sin(kx)\Big]_0^{\pi} \quad \text{[Since } x \text{ odd and cos even]} \\ &= 0 \end{aligned}$$

Therefore the Fourier series is defined as

$$F(x) = 6 + \sum_{k=1}^{\infty}\frac{6(-1)^{k+1}}{k}\sin(kx)$$

Linear functions are infinitely differentiable so it will converge to $f(x)$ within the interval, and coverges to 3 at the endpoints.

(b) $[a,b] = [0, 2\pi]$

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\
&= \frac{1}{\pi}\int_{0}^{2\pi} 3+3xdx \\
&= \frac{1}{\pi}\left[6\pi + \frac{3}{2}\left[x^2\right]_0^{2\pi}\right] \\
&= \frac{1}{\pi}\left[6\pi + 6\pi^2\right] \\
&= 6(\pi+1)
\end{aligned}$$

$$\begin{aligned}
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx \\
&= \frac{3}{\pi}\left[\int_0^{2\pi} \sin(kx)dx + \int_0^{2\pi} x\sin(kx)dx\right]
\end{aligned}$$

Let $u = x, du = 1dx, dv = \sin(kx)dx, v = -\dfrac{\cos(kx)}{k}$

$$\begin{aligned}
&= \frac{3}{k\pi}\left[\left[\cos(kx)\right]_0^{2\pi} - \left[x\cos(kx)\right]_0^{2\pi} + \int_0^{2\pi}\cos(kx)dx\right] \\
&= \frac{3}{k\pi}\left[-2\pi + \frac{1}{k}\left[\sin(kx)\right]_0^{2\pi}\right] \\
&= -\frac{6}{k^2}
\end{aligned}$$

$$\begin{aligned}
a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx \\
&= \frac{3}{\pi}\left[\int_0^{2\pi} \cos(kx)dx + \int_0^{2\pi} x\cos(kx)dx\right]
\end{aligned}$$

Let $u = x,\ du = dx,\ dv = \cos(kx),\ v = \dfrac{1}{k}\sin(kx)$

$$\begin{aligned}
&= \frac{3}{k\pi}\left[\left[\sin(kx)\right]_0^{2\pi} + \left[x\sin(kx)\right]_0^{2\pi} - \int_0^{2\pi}\sin(kx)dx\right] \\
&= -\frac{3}{k^2\pi}\left[\cos(kx)\right]_0^{2\pi} \\
&= 0
\end{aligned}$$

Therefore the Fourier series is defined as

$$F(x) = 6(\pi+1) + \sum_{k=1}^{\infty} -\frac{6}{k^2}\sin(kx)$$

Linear functions are infinitely differentiable so it will converge to $f(x)$ within the interval, and coverges to $3+3\pi$ at the endpoints.

4. Find the Fourier series of the function $f(x)$ defined on $[0, 2\pi]$ by $f(x) = x(x - 2\pi)$ and extended from this with period 2π to all of $\mathbb{R}$. Use symbolic algebra software to graph the 4^{th} degree Fourier polynomial together with the original function.

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx$$
$$= \frac{1}{\pi}\int_0^{2\pi} x(x-2\pi)dx$$
$$= \frac{1}{\pi}\left[\int_0^{2\pi} x^2\sin(kx)dx - 2\pi\int_0^{2\pi} x\sin(kx)dx\right]$$

Let $u = x,\ du = dx,\ dv = \sin(kx),\ v = -\frac{1}{k}\cos(kx)$

$$= \frac{1}{\pi}\left[\int_0^{2\pi} x^2\sin(kx)dx - 2\pi\Big(-\frac{1}{k}\Big[x\cos(kx)\Big]_0^{2\pi} + \frac{1}{k}\int_0^{2\pi}\cos(kx)dx\Big)\right]$$
$$= \frac{1}{\pi}\left[\int_0^{2\pi} x^2\sin(kx)dx - 2\pi\Big(-\frac{2\pi}{k} + \frac{1}{k^2}\Big[\sin(kx)\Big]_0^{2\pi}\Big)\right]$$
$$= \frac{1}{\pi}\left[\int_0^{2\pi} x^2\sin(kx)dx + \frac{4\pi^2}{k}\right]$$

Let $u = x^2, du = 2xdx,$

$$dv = \sin(kx)dx, v = -\frac{\cos(kx)}{k}$$
$$= \frac{1}{k\pi}\left[-\Big[x^2\cos(kx)\Big]_0^{2\pi} + \int_0^{2\pi} x\cos(kx)dx + 4\pi^2\right]$$

Let $u = x,\ du = dx,\ dv = \sin(kx),\ v = -\frac{1}{k}\cos(kx)$

$$= \frac{1}{k\pi}\left[\frac{1}{k}\Big[x\sin(kx)\Big]_0^{2\pi} - \frac{1}{k}\int_0^{2\pi}\cos(kx)dx\right]$$
$$= \frac{1}{k\pi}\left[-\frac{1}{k^2}\Big[\sin(kx)\Big]_0^{2\pi}\right]$$
$$= 0$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx$$
$$= \frac{1}{\pi}\left[\int_0^{2\pi} x^2\cos(kx)dx - 2\pi\int_0^{2\pi} x\cos(kx)dx\right]$$

Let $u = x^2, du = 2xdx, dv = \cos(kx)dx, v = \frac{\sin(kx)}{k}$

$$= \frac{2}{\pi}\left[\frac{1}{k}\Big[x^2\sin(kx)\Big]_0^{2\pi} - \frac{2}{k}\int_0^{2\pi} x\sin(kx)dx - \pi\int_0^{2\pi} x\cos(kx)dx\right]$$
$$= \frac{2}{\pi}\left[-\frac{2}{k}\int_0^{2\pi} x\sin(kx)dx - \pi\int_0^{2\pi} x\cos(kx)dx\right]$$

Let $u = x, du = dx, dv = \sin(kx)dx, v = -\frac{\cos(kx)}{k}$

$$= \frac{2}{\pi}\left[\frac{2}{k^2}\Big[x\cos(kx)\Big]_0^{2\pi} - \frac{1}{k}\int_0^{2\pi}\cos(kx)dx - \pi\int_0^{2\pi} x\cos(kx)dx\right]$$
$$= \frac{2}{\pi}\left[\frac{4\pi}{k^2} - \frac{1}{k^2}\Big[\sin(kx)\Big]_0^{2\pi} - \pi\int_0^{2\pi} x\cos(kx)dx\right]$$

Let $u = x,\ du = dx,\ dv = \cos(kx),\ v = \frac{1}{k}\sin(kx)$

$$= \frac{2}{\pi}\left[\frac{4\pi}{k^2} - \pi\Big(\frac{1}{k}\Big[x\sin(kx)\Big]_0^{2\pi} - \frac{1}{k}\int_0^{2\pi}\sin(kx)dx\Big)\right]$$
$$= \frac{2}{\pi}\left[\frac{4\pi}{k^2} + \frac{\pi}{k}\int_0^{2\pi}\sin(kx)dx\right]$$
$$= \frac{2}{\pi}\left[\frac{4\pi}{k^2} + \frac{\pi}{k^2}\Big[\cos(kx)\Big]_0^{2\pi}\right]$$
$$= \frac{8}{k^2}$$

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\
&= \frac{1}{\pi}\int_{0}^{2\pi} x(x-2\pi)dx \\
&= \frac{1}{\pi}\left[\int_{0}^{2\pi} x^2dx - \int_{0}^{2\pi} 2x\pi dx\right] \\
&= \frac{1}{\pi}\left[\frac{1}{3}\left[x^3\right]_0^{2\pi} - \pi\left[x^2\right]_0^{2\pi}\right] \\
&= \frac{1}{\pi}\left[\frac{8\pi^3}{3} - 4\pi^3\right] \\
&= -\frac{4\pi^2}{3}
\end{aligned}
$$

Therefore the Fourier series of f is

$$-\frac{2\pi^2}{3} + \sum_{k=1}^{\infty} \frac{8}{k^2}\cos(kx)$$

5. Let $f(x)$ be defined on $[0, 2\pi]$ by $f(x) = x(x - 2\pi)$.
   (a) Find the Fourier cosine series of f.
   From question 4, we can see that the function is already even, hence the fourier series of the function itself is a cosine series of f. Namely

   $$-\frac{2\pi^2}{3} + \sum_{k=1}^{\infty} \frac{8}{k^2}\cos(kx)$$

   (b) Find the Fourier sine series of f.
   (c) Use symbolic algebra software to graph the 4^{th} degree Fourier polynomials from parts (a) and (b) together with the original function.

6. Find the Fourier series for the following functions:
   (a) $f(x) = \sin^2 x + \sin^3 x$
   (b) $f(x) = \sin^4 x$
   (c) $f(x) = \cos^7 x$
   (*Hint:* Recall that $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$ and $\sin\theta = \frac{e^{i\theta}+e^{-i\theta}}{2i}$)

The next question is for those among you who have previously seen complex numbers. It gives another approach to Fourier series.

7. Suppose
   i. $f(x)$ is a real values function of x,
   ii. $f(x) = \displaystyle\sum_{n=-\infty}^{\infty} C_n einx$ on $[-\pi, \pi]$, where the C_n are complex constants, and
   iii. that the term by term theorem holds true in this case
   (a) Express the C_n as integrals involving f.
   (b) Find the Fourier coefficients of f in terms of the C_n.
   (c) Find the C_n in terms of the Fourier coefficients of f.