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MATB42: Assignment #10 Apr 6th 2018

1. Let $\boldsymbol{F}$ be a vector field on $\mathbb{R}^3$ given by $\boldsymbol{F} = (F_1, F_2, F_3)$ where F_1, F_2, and F_3 are C^1-functions from $\mathbb{R}^3 \to \mathbb{R}$

   (a) Let η be the 2-form given by

$$\eta = F_3\, dx\, dy + F_1\, dy\, dz + F_2\, dz\, dx$$

Show that $d\eta = (\operatorname{div} \boldsymbol{F})\, dx\, dy\, dz$
(page 489, #6)

$$\begin{aligned}
\eta &= F_3\, dx\, dy + F_1\, dy\, dz + F_2\, dz\, dx \\
d\eta &= d(F_3\, dx\, dy + F_1\, dy\, dz + F_2\, dz\, dx) \\
&= (dF_3)\, dx\, dy + (dF_1)\, dy\, dz + (dF_2)\, dz\, dx \\
&= (\frac{\partial}{\partial x}F_3\, dx + \frac{\partial}{\partial y}F_3\, dy + \frac{\partial}{\partial z}F_3\, dz)\, dx\, dy + (dF_1)\, dy\, dz + (dF_2)\, dz\, dx \\
&= \frac{\partial}{\partial z}F_3\, dz\, dx\, dy + (dF_1)\, dy\, dz + (dF_2)\, dz\, dx \\
&= \frac{\partial}{\partial z}F_3\, dx\, dy\, dz + (\frac{\partial}{\partial x}F_1\, dx + \frac{\partial}{\partial y}F_1\, dy + \frac{\partial}{\partial z}F_1\, dz)\, dy\, dz + (dF_2)\, dz\, dx \\
&= \frac{\partial}{\partial z}F_3\, dx\, dy\, dz + \frac{\partial}{\partial x}F_1\, dx\, dy\, dz + (dF_2)\, dz\, dx \\
&= \frac{\partial}{\partial z}F_3\, dx\, dy\, dz + \frac{\partial}{\partial x}F_1\, dx\, dy\, dz + (\frac{\partial}{\partial x}F_2\, dx + \frac{\partial}{\partial y}F_2\, dy + \frac{\partial}{\partial z}F_2\, dz)\, dz\, dx \\
&= \frac{\partial}{\partial z}F_3\, dx\, dy\, dz + \frac{\partial}{\partial x}F_1\, dx\, dy\, dz + \frac{\partial}{\partial y}F_2\, dy\, dz\, dx \\
&= \frac{\partial}{\partial z}F_3\, dx\, dy\, dz + \frac{\partial}{\partial x}F_1\, dx\, dy\, dz + \frac{\partial}{\partial y}F_2\, dx\, dy\, dz \\
&= \frac{\partial}{\partial x}F_1 + \frac{\partial}{\partial y}F_2 + \frac{\partial}{\partial z}F_3\, dx\, dy\, dz = (\operatorname{div} \boldsymbol{F})\, dx\, dy\, dz
\end{aligned}$$

(b) Show that $dF_1 \wedge dF_2 \wedge dF_3 = (\det D\boldsymbol{F})\,dx\,dy\,dz$

$$df = \sum_{i=0}^{n} \frac{\partial f}{\partial x_i}\,dx_i$$

$$\begin{aligned}
dF_1 \wedge dF_2 \wedge dF_3 &= (\frac{\partial F_1}{\partial x}\,dx + \frac{\partial F_1}{\partial y}\,dy + \frac{\partial F_1}{\partial z}\,dz) \wedge (\frac{\partial F_2}{\partial x}\,dx + \frac{\partial F_2}{\partial y}\,dy + \frac{\partial F_2}{\partial z}\,dz) \wedge dF_3 \\
&= (\frac{\partial F_1}{\partial x}\,dx \wedge (\frac{\partial F_2}{\partial x}\,dx + \frac{\partial F_2}{\partial y}\,dy + \frac{\partial F_2}{\partial z}\,dz) \\
&\quad + \frac{\partial F_1}{\partial y}\,dy \wedge (\frac{\partial F_2}{\partial x}\,dx + \frac{\partial F_2}{\partial y}\,dy + \frac{\partial F_2}{\partial z}\,dz) \\
&\quad + \frac{\partial F_1}{\partial z}\,dz \wedge (\frac{\partial F_2}{\partial x}\,dx + \frac{\partial F_2}{\partial y}\,dy + \frac{\partial F_2}{\partial z}\,dz)) \wedge dF_3 \\
&= ((\frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial y}\,dx\,dy + \frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial z}\,dx\,dz) \\
&\quad + (\frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial x}\,dy\,dx + \frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial z}\,dy\,dz) \\
&\quad + (\frac{\partial F_1}{\partial z}\frac{\partial F_2}{\partial x}\,dz\,dx + \frac{\partial F_1}{\partial z}\frac{\partial F_2}{\partial y}\,dz\,dy)) \wedge dF_3 \\
&= ((\frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial y} - \frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial x}\,dx\,dy) \\
&\quad + (\frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial z}\frac{\partial F_2}{\partial y}\,dy\,dz) \\
&\quad + (\frac{\partial F_1}{\partial z}\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial z}\,dz\,dx)) \wedge (\frac{\partial F_3}{\partial x}\,dx + \frac{\partial F_3}{\partial y}\,dy + \frac{\partial F_3}{\partial z}\,dz) \\
&= (\frac{\partial F_3}{\partial z}(\frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial y} - \frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial x}\,dx\,dy\,dz) \\
&\quad + \frac{\partial F_3}{\partial x}(\frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial z}\frac{\partial F_2}{\partial y}\,dy\,dz\,dx) \\
&\quad + \frac{\partial F_3}{\partial y}(\frac{\partial F_1}{\partial z}\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial z}\,dz\,dx\,dy)) \\
&= \frac{\partial F_3}{\partial x}(\frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial z}\frac{\partial F_2}{\partial y})\,dx\,dy\,dz \\
&\quad - \frac{\partial F_3}{\partial y}(\frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial z}\frac{\partial F_2}{\partial x})\,dx\,dy\,dz \\
&\quad + \frac{\partial F_3}{\partial z}(\frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial y} - \frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial x})\,dx\,dy\,dz \\
&= \frac{\partial F_3}{\partial x}\begin{vmatrix} \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{vmatrix} - \frac{\partial F_3}{\partial y}\begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial z} \end{vmatrix} + \frac{\partial F_3}{\partial z}\begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{vmatrix}\,dx\,dy\,dz \\
&= \begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \\ \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial z} \end{vmatrix}\,dx\,dy\,dz
\end{aligned}$$

2. Let ω be a k-form and let η be a ℓ-form. Find $d(d\omega \wedge \eta - \omega \wedge d\eta)$.

$$\begin{aligned} d(d\omega \wedge \eta - \omega \wedge d\eta) &= d(d\omega \wedge \eta) - d(\omega \wedge d\eta) \\ &= (d^2\omega \wedge \eta + (-1)^{k+1}(d\omega \wedge d\eta)) - (d\omega \wedge d\eta + (-1)^k(\omega \wedge d^2\eta)) \\ &= (-1)^{k+1} d\omega \wedge d\eta - d\omega \wedge d\eta \\ &= ((-1)^{k+1} - 1) d\omega \wedge d\eta \end{aligned}$$

3. Determine if $\eta = y\,dx\,dy + xz\,dy\,dz - yz\,dz\,dx$ is exact. If η is exact find a 1-form ω with $d\omega = \eta$. Check if $d\eta = \mathcal{O}$ to see if η closed.

   (compare with page 461, # 22)

$$\begin{aligned} d\eta &= d(y\,dx\,dy + xz\,dy\,dz - yz\,dz\,dx) \\ &= (dy\,dx\,dy + d(xz) \wedge dy\,dz - d(yz) \wedge dz\,dx) \\ &= ((z\,dx + x\,dz) \wedge dy\,dz - (z\,dy + y\,dz) \wedge dz\,dx) \\ &= (z\,dx) \wedge dy\,dz - (z\,dy) \wedge dz\,dx \\ &= z\,dx\,dy\,dz - z\,dx\,dy\,dz = \mathcal{O} \end{aligned}$$

Since the polynomials of x, y and z defined throughout $\mathbb{R}^3$ and η closed, it is exact. By inspection,

$$\omega = xy\,dy + xyz\,dz$$

4. Evaluate $\iint_S \omega$, where $\omega = z\,dx\,dy + x\,dy\,dz + y\,dz\,dx$ and S is the unit sphere, directly and by the Divergence Theorem.

(page 489, #12)

Directly:

Parametrize the sphere S as

$$\mathbf{\Phi}(\varphi, \theta) = (\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi) \text{ with } \theta \in [0, 2\pi],\ \varphi \in [0, \pi]$$

$$\begin{aligned}
\iint_S \omega &= \iint_{\mathbf{\Phi}} z\,dx\,dy + \iint_{\mathbf{\Phi}} x\,dy\,dz + \iint_{\mathbf{\Phi}} y\,dz\,dx \\
&= \int_0^{2\pi}\int_0^{\pi} \cos\varphi \begin{vmatrix} \frac{\partial\cos\theta\sin\varphi}{\partial\varphi} & \frac{\partial\cos\theta\sin\varphi}{\partial\theta} \\ \frac{\partial\sin\theta\sin\varphi}{\partial\varphi} & \frac{\partial\sin\theta\sin\varphi}{\partial\theta} \end{vmatrix} d\varphi\,d\theta + \int_0^{2\pi}\int_0^{\pi} \cos\theta\sin\varphi \begin{vmatrix} \frac{\partial\sin\theta\sin\varphi}{\partial\varphi} & \frac{\partial\sin\theta\sin\varphi}{\partial\theta} \\ \frac{\partial\cos\varphi}{\partial\varphi} & \frac{\partial\cos\varphi}{\partial\theta} \end{vmatrix} d\varphi\,d\theta \\
&\quad + \int_0^{2\pi}\int_0^{\pi} \sin\theta\sin\varphi \begin{vmatrix} \frac{\partial\cos\varphi}{\partial\varphi} & \frac{\partial\cos\varphi}{\partial\theta} \\ \frac{\partial\cos\theta\sin\varphi}{\partial\varphi} & \frac{\partial\cos\theta\sin\varphi}{\partial\theta} \end{vmatrix} d\varphi\,d\theta \\
&= \int_0^{2\pi}\int_0^{\pi} \cos\varphi \begin{vmatrix} \cos\theta\cos\varphi & -\sin\theta\sin\varphi \\ \sin\theta\cos\varphi & \cos\theta\sin\varphi \end{vmatrix} d\varphi\,d\theta + \int_0^{2\pi}\int_0^{\pi} \cos\theta\sin\varphi \begin{vmatrix} \sin\theta\cos\varphi & \cos\theta\sin\varphi \\ -\sin\varphi & 0 \end{vmatrix} d\varphi\,d\theta \\
&\quad + \int_0^{2\pi}\int_0^{\pi} \sin\theta\sin\varphi \begin{vmatrix} -\sin\varphi & 0 \\ \cos\theta\cos\varphi & -\sin\theta\sin\varphi \end{vmatrix} d\varphi\,d\theta \\
&= \int_0^{2\pi}\int_0^{\pi} \sin\varphi\cos^2\varphi\,d\varphi\,d\theta + \int_0^{2\pi}\int_0^{\pi} \cos^2\theta\sin^3\varphi + \sin^2\theta\sin^3\varphi\,d\varphi\,d\theta \\
&= \int_0^{2\pi}\int_0^{\pi} \sin\varphi(\cos^2\varphi + \sin^2\varphi)\,d\varphi\,d\theta \\
&= \int_0^{2\pi}\int_0^{\pi} \sin\varphi\,d\varphi\,d\theta \\
&= 2\pi\Big[-\cos\varphi\Big]_0^{\pi} = 2\pi
\end{aligned}$$

Divergence Theorem:

$$d\omega = dz\,dy\,dx + dx\,dy\,dz + dydzdx = 3\,dx\,dy\,dz$$

$$\begin{aligned}
\iint_S \omega &= \iiint_R d\omega \\
&= 3\int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^2\sin(\varphi)\,d\rho\,d\varphi\,d\theta \\
&= \int_0^{2\pi}\int_0^{\pi} \sin(\varphi)\,d\varphi\,d\theta \\
&= 2\pi\Big[-\cos\varphi\Big]_0^{\pi} = 2\pi
\end{aligned}$$

5. Compute $\int_S \omega$ and use symbolic algebra software to sketch S in each of the following.

(a) $\omega = xz\,dx\,dy + x^2\,dy\,dz + dy\,dz\,dx$

S is the upper hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$ with $\boldsymbol{n}$ pointing upward.

Close it with the disk of radius 2 on the xy-plane to apply divergence theorem

$$\mathbf{\Phi}(\theta, r) = (r\cos\theta, r\sin\theta, 0),\ r \in [0,2],\ \theta \in [0, 2\pi]$$

$$dx\,dy = \begin{vmatrix} -r\sin\theta & \cos\theta \\ r\cos\theta & \sin\theta \end{vmatrix} = -2r$$

Which is negative, so correct orientation for normal pointing down.

$$dy\,dz = 0 \text{ Since } z \text{ is } 0$$

$$dz\,dx = 0$$

$$\overset{\text{Div Thm}}{\Longrightarrow} \iint_S \omega = \iiint_R d\omega - \iint_{\mathbf{\Phi}} \omega$$

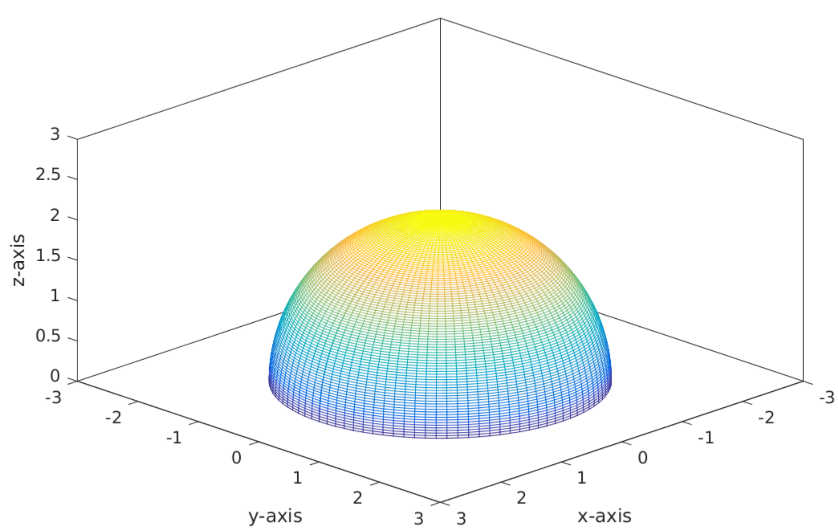
$$\text{But } z = 0 \implies xz\,dx\,dy = 0 \implies \iint_{\mathbf{\Phi}} \omega = 0$$

$$d\omega = x\,dx\,dy\,dz + 2x\,dx\,dy\,dz = 3x\,dx\,dy\,dz$$

$$\iiint_R d\omega = \int_0^2 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} 3(\rho\sin\varphi\cos\theta)\rho^2 \sin\varphi\,d\varphi\,d\theta\,d\rho$$

$$= 0 \text{ Since integrating cos over full period}$$

$$\implies \int_S \omega = 0$$



(b) $\omega = z\,dx\,dy + x\,dy\,dz + y\,dz\,dx$

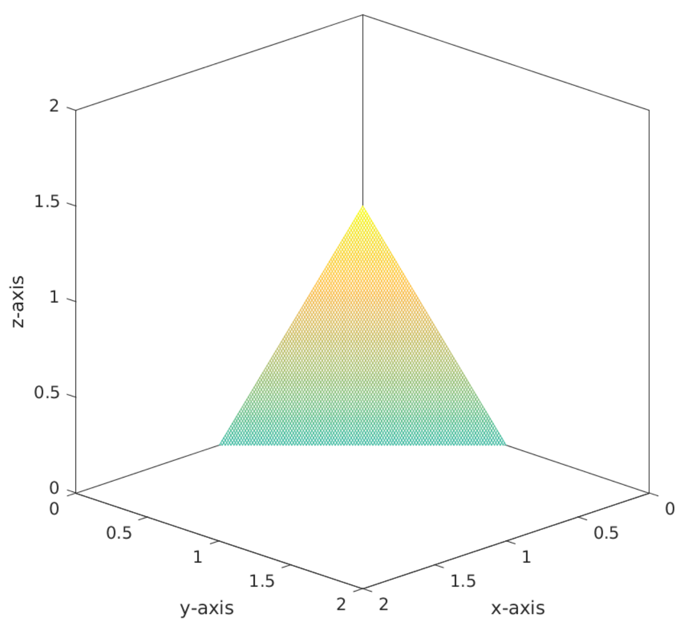
S is the part of the plane $x+y+z=1$ which lies in the first octant oriented by the unit normal which points upward.

Use the natural parametrization for S:

$$\mathbf{\Phi}(x,y) = (x, y, 1-x-y),\ x \in [0,1],\ y \in [0, 1-x]$$

$$dx\,dy = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} > 0\, \forall x, y \implies \text{ Correct orientation}$$

$$\begin{aligned}
\int_S \omega &= \int_0^1 \int_0^{1-x} (1-x-y) + x \begin{vmatrix} 0 & 1 \\ -1 & -1 \end{vmatrix} + y \begin{vmatrix} -1 & -1 \\ 1 & 0 \end{vmatrix} \, dy\, dx \\
&= \int_0^1 \int_0^{1-x} 1\, dy\, dx = \int_0^1 1 - x\, dx \\
&= 1 - \frac{1}{2} = \frac{1}{2}
\end{aligned}$$

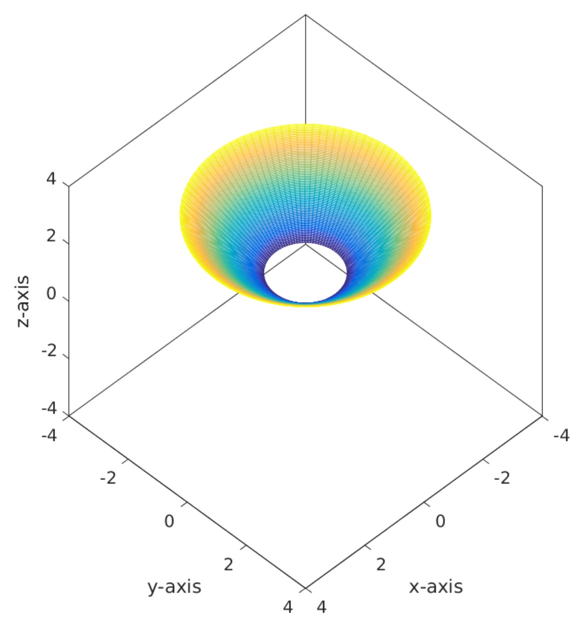


(c) $\omega = xz\,dx\,dy + y\,dx\,dz + z^2\,dy\,dz$

S is the part of the cone $z = \sqrt{x^2+y^2}$ between $z=1$ and $z=3$, oriented by the unit normal with negative z-component.

$$\mathbf{\Phi}(\theta, r) = (r\cos\theta, r\sin\theta, r),\ r \in [1,3],\ \theta \in [0, 2\pi]$$

$$\begin{aligned}
dx\,dy &= \begin{vmatrix} -r\sin\theta & \cos\theta \\ r\cos\theta & \sin\theta \end{vmatrix} = -r < 0 \text{ for } r > 1 \\
dy\,dz &= \begin{vmatrix} r\cos\theta & \sin\theta \\ 0 & 1 \end{vmatrix} = r\cos\theta \\
dz\,dx &= \begin{vmatrix} 0 & 1 \\ -r\sin\theta & \cos\theta \end{vmatrix} = r\sin\theta \\
\implies \omega &= (r\cos\theta)(r)(-r) - (r\sin\theta)(r\sin\theta) + (r)^2(r\cos\theta) \\
&= -r^2\sin^2\theta = -r^2\left(\frac{1}{2} - \frac{\cos(2\theta)}{2}\right) \\
\implies \int_S \omega &= \int_1^3 \int_0^{2\pi} -r^2\left(\frac{1}{2} - \frac{\cos(2\theta)}{2}\right) d\theta\,dr \\
&= \int_1^3 -r^2\pi\,dr = -\pi\left[\frac{r^3}{3}\right]_1^3 = -\frac{26\pi}{3}
\end{aligned}$$



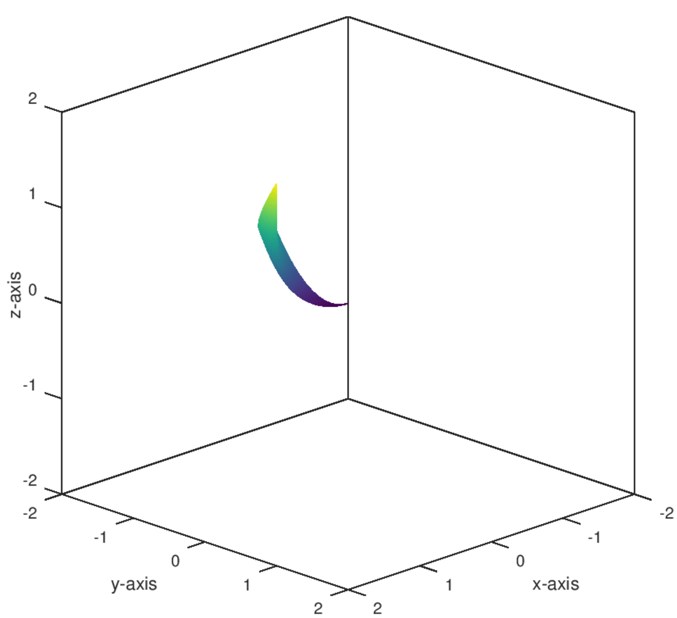
(d) $\omega = z\,dx\,dy + y\,dy\,dz$

S is the oriented surface given by the parametrization

$\Phi(u,v) = (u+v, uv^2, u^2+v^2)$, $0 \le u \le 1$, $0 \le v \le 1$.

$$dx\,dy = \begin{vmatrix} 1 & 1 \\ v^2 & 2uv \end{vmatrix} = 2uv - v^2, \quad dy\,dz = \begin{vmatrix} v^2 & 2uv \\ 2u & 2v \end{vmatrix} = 2v^3 - 4u^2v$$

$$\begin{aligned}
\iint_S \omega &= \int_0^1 \int_0^1 (u^2+v^2)(2uv - v^2) + (uv^2)(2v^3 - 4u^2v)\,du\,dv \\
&= \int_0^1 \int_0^1 (2u^3v - u^2v^2) + (2uv^3 - v^4) + (2uv^5 - 4u^3v^3)\,du\,dv \\
&= \int_0^1 \frac{v}{2} - \frac{v^2}{3} + v^3 - v^4 + v^5 - v^3\,du\,dv \\
&= \int_0^1 \frac{v}{2} - \frac{v^2}{3} - v^4 + v^5\,du\,dv \\
&= \frac{1}{4} - \frac{1}{9} - \frac{1}{5} + \frac{1}{6} = \frac{19}{180}
\end{aligned}$$



6. Verify Stokes' theorem by direct calculation of both sides when the surface S is the piece of the paraboloid $z = x^2 + y^2 - 4$ with $z \leq 0$, oriented by the downward pointing unit normal, and $\omega = (2y - z)\,dx + (x + y^2 - z)\,dy + (4y - 3x)\,dz$.

   As part of your solution, provide a sketch showing the appropriate orientations. (For this question you may draw the sketch by hand or use symbolic algebra software.)

   Stokess' Theorem states that:

$$\int_{\partial_S} \omega = \int_S d\omega$$

   Calculation of $\displaystyle\int_{\partial_S} \omega$

   The boundary curve of the plane is the circle at $z = 0$ with radius 2. Since the normal vector is downward pointing, the curve is parametrized clockwise. So parametrize the curve as $\boldsymbol{\gamma}(\theta) = (-2\cos\theta, 2\sin\theta, 0)$

$$\begin{aligned}
&\int_0^{2\pi} (2(2\sin\theta) - 0)(2\sin\theta) + ((-2\cos\theta) + (2\sin\theta)^2 - 0)(2\cos\theta)\,d\theta \\
&= \int_0^{2\pi} 8\sin^2\theta - 4\cos^2\theta + 8\sin^2\theta\cos\theta\,d\theta \\
&= \int_0^{2\pi} 4(1 - \cos(2\theta)) - 2(1 + \cos(2\theta)) + 8\sin^2\theta\cos\theta\,d\theta \\
&= \int_0^{2\pi} 2 + 8\sin^2\theta\cos\theta\,d\theta \\
&= 4\pi + \left[\frac{8\sin^3\theta}{3}\right]_0^{2\pi} = 4\pi
\end{aligned}$$

   Calculation of $\displaystyle\int_S d\omega$

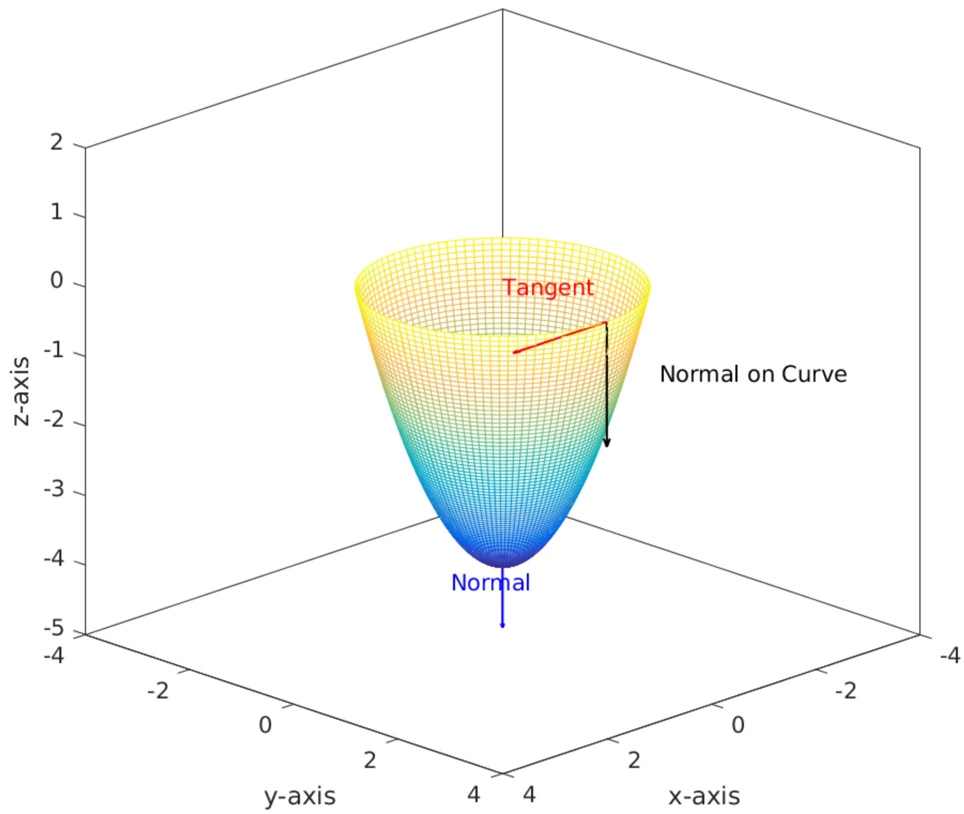
   (Using the parametrization $\boldsymbol{\Phi}(\theta, r) = (r\cos\theta, r\sin\theta, r^2 - 4),\ r \in [0, 2],\ \theta \in [0, 2\pi]$)

$$dx\,dy = \begin{vmatrix} -r\sin\theta & \cos\theta \\ r\cos\theta & \sin\theta \end{vmatrix} = -r$$

   So the orientation is in the correct direction.

$$\begin{aligned}
d\omega &= d(2y - z)\,dx + d(x + y^2 - z)\,dy + d(4y - 3x)\,dz \\
&= (2\,dy - \,dz) \wedge dx + (dx - dz) \wedge dy + (4\,dy - 3\,dx) \wedge dz \\
&= -2\,dx\,dy - \,dz\,dx + dx\,dy + dy\,dz + 4\,dy\,dz + 3\,dz\,dx \\
&= -\,dx\,dy + 5\,dy\,dz + 2\,dz\,dx
\end{aligned}$$

$$\begin{aligned}
\implies \int_S d\omega &= \int_0^{2\pi}\int_0^2 -\begin{vmatrix} -r\sin\theta & \cos\theta \\ r\cos\theta & \sin\theta \end{vmatrix} + 5\begin{vmatrix} r\cos\theta & \sin\theta \\ 0 & 2r \end{vmatrix} + 2\begin{vmatrix} 0 & 2r \\ -r\sin\theta & \cos\theta \end{vmatrix}\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^2 r + 5(2r^2\cos\theta) + 2(2r^2\sin\theta)\,dr\,d\theta \\
&= 2\pi\int_0^2 r\,dr = 4\pi
\end{aligned}$$



z-axis
2
1
0
-1
-2
-3
-4
-5
Tangent
Normal on Curve
Normal
-4
-2
0
2
4
y-axis
4
2
0
-2
-4
x-axis

7. Let $\omega = yz\,dx - xz\,dy + xy\,dz$ and let $\boldsymbol{\gamma}(t) = (2\cos t, 2\sin t, 4)$, $0 \le t \le 2\pi$.

$$\begin{aligned}
d\omega &= d(yz) \wedge dx - d(xy) \wedge dy + d(xy) \wedge dz \\
&= (z\,dy + y\,dz) \wedge dx - (z\,dx + x\,dz) \wedge dy + (y\,dx + x\,dy) \wedge dz \\
&= z\,dy\,dx + y\,dz\,dx - z\,dx\,dy - x\,dz\,dy + y\,dx\,dz + x\,dy\,dz \\
&= -2z\,dx\,dy + 2x\,dy\,dz
\end{aligned}$$

(a) Let S be the piece of the surface $z = x^2 + y^2$ with $z \le 4$. Use Stokes' theorem to give an integral over S which is equivalent to $\int_{\boldsymbol{\gamma}} \omega$. Verify by directly computing both integrals.

Note that the boundary of S at $z = 4$ is $\boldsymbol{\gamma}$. This means that, if the orientation is compatible, Stokes' theorem will apply. Since the orientation of the curve is counter clockwise, the normal vector should be pointing up for the surface.

To parametrize S, use a similar parametrization to the one in question 6.

$$\boldsymbol{\Phi}(r, \theta) = (r\cos\theta, r\sin\theta, r^2), \; r \in [0, 2], \; \theta \in [0, 2\pi]$$

The normal vector is also oriented correctly.

$$dx\,dy = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r \ge 0$$

$$\overset{\text{Stokes' Thm}}{\Longrightarrow} \int_{\partial S = \boldsymbol{\gamma}} \omega = \int_S d\omega$$

$$\begin{aligned}
\int_{\boldsymbol{\gamma}} \omega &= \int_0^{2\pi} (2\sin t)(4)(-2\sin t) - (2\cos t)4(2\cos t)\,dt \\
&= \int_0^{2\pi} -16(\sin^2 t + \cos^2 t)\,dt \\
&= \int_0^{2\pi} -16\,dt = -32\pi \\
\int_S d\omega &= \int_0^2 \int_0^{2\pi} -2(r^2)(r) + 2(r\cos\theta) \begin{vmatrix} \sin\theta & r\cos\theta \\ 2r & 0 \end{vmatrix} \,d\theta\,dr \\
&= \int_0^2 \int_0^{2\pi} -2r^3 - 4(r^3\cos\theta)\,d\theta\,dr \\
&= \int_0^2 \int_0^{2\pi} -2r^3 - 2(r^3(1 + \cos(2\theta)))\,d\theta\,dr \\
&= \int_0^2 \int_0^{2\pi} -4r^3 - 2r^3\cos(2\theta)\,d\theta\,dr \\
&= \int_0^2 -8\pi r^3\,d\theta\,dr \\
&= -(2^3)\pi \left[\frac{r^4}{2^2}\right]_0^2 = -32\pi
\end{aligned}$$

(b) Let S' be the part of the plane $z = 4$ with $x^2 + y^2 \leq 4$. Use Stokes' theorem to give an integral over S' which is equivalent to $\int_\gamma \omega$. Verify by direct computation.

Note that the disk S' can be parametrized as $\mathbf{\Phi}(r, \theta) = (r\cos\theta, r\sin\theta, 4)$, $r \in [0, 2]$, $\theta \in [0, 2\pi]$ as we have that

$$dx\,dy = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r \geq 0$$

Also, the same conditions hold for the normal vector as the curve at the boundary has not changed. (S' also intersects with the $z = 4$ plane at the radius 2 disk.

$$\begin{aligned}
\int_{\boldsymbol{\gamma}} \omega &= \int_1^{2\pi} (2\sin t)(4)(-2\sin t) - (2\cos t)4(2\cos t)\,dt \\
&= \int_0^{2\pi} -16(\sin^2 t + \cos^2 t)\,dt \\
&= \int_0^{2\pi} -16\,dt = -32\pi \\
\int_{S'} d\omega &= \int_0^2 \int_0^{2\pi} -2(4)(r) + 0\,d\theta\,dr \\
&= \int_0^2 -16\pi r\,d\theta\,dr \\
&= -16\pi \left[\frac{r^2}{2}\right]_0^2 = -32\pi
\end{aligned}$$

(c) Can you give another explanation as to why the integrals you get over S and S' should have the same value?

If we take the vector field over S and S' as the flow of a substance through the surface, then intuitively, the rate of flow is restricted by boundary $\boldsymbol{\gamma}$. This means that, if the boundary remains the same, the flow cannot change independantly of the shape of the surface.

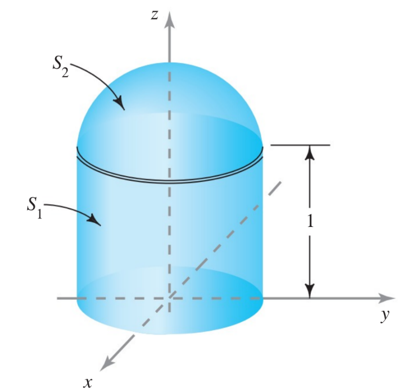
8. Let $\boldsymbol{F}(x, y, z) = (e^{z^2}, 4z - y, 8x \sin y)$. Find $\int_S (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{S}$ where S is the unit sphere oriented with the outward normal.

The unit sphere is a closed bounded surface. Trig, poly and exponential functions are both defined everywhere in $\mathbb{R}$ as well as inifinitely differentiable, so $\boldsymbol{F}$ is defined everywhere within the unit sphere, which is the region that S bounds. For the region $R =$ unit sphere, $\partial R = S$ so we have that

$$\int_{S=\partial R} (\operatorname{curl} \boldsymbol{F}) \cdot \, d\boldsymbol{S} = 0$$

9. (a) Marsden & Tromba, page 451, # 13.

Let S be the capped cylindrical surface shown in Figure 1. S is the union of two surfaces, S_1 and S_2, where S_1 is the set of (x, y, z) with $x^2 + y^2 = 1$, $0 \leq z \leq 1$, and S_2 is the set of (x, y, z) with $x^2 + y^2 + (z-1)^2 = 1$, $z \geq 1$. Set $\boldsymbol{F}(x, y, z) = (zx + z^2y + x)\boldsymbol{i} + (z^3yx + y)\boldsymbol{j} + z^4x^2\boldsymbol{k}$. Compute $\iint_S (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{S}$. (HINT: Stokes' theorem holds for this surface.)



Incomplete

(b) Marsden & Tromba, page 451, # 15.

Evaluate the integral $\iint_S (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{S}$, where S is the portion of the surface of a sphere defined by $x^2 + y^2 + z^2 = 1$ and $x + y + z =\geq 1$, and where $\boldsymbol{F} = \boldsymbol{r} \times (\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$, $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$. Assuming a default outward orientation for the sphere, Stokes' theorem applies where the boundary of S is the circle on the plane $x + y + z = 1$.

$$
\begin{aligned}
\boldsymbol{F} &= \boldsymbol{r} \times (1,1,1) = (x,y,z) \times (1,1,1) \\
&= (y - z, z - x, x - y) \\
\nabla \times \boldsymbol{F} &= \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y - z & z - x & x - y \end{vmatrix} \\
&= (-1 - 1, -1 - 1, -1 - 1) \qquad = -2(1,1,1) \\
\implies -2\, d\omega &= -2(1,1,1) \\
\overset{\text{Inspection}}{\implies} -2\omega &= -2(x + y + z)
\end{aligned}
$$

Now to parametrize the curve of intersection of the sphere and plane.

$$
\begin{aligned}
1 &= x^2 + y^2 + (1 - x - y)^2 \\
1 &= x^2 + y^2 + 1 - 2x + 2xy - 2y + x^2 + y^2 \\
0 &= x^2 + y^2 + xy - x - y \\
0 &= x^2 + x(y-1) + y^2 - y \\
0 &= x^2 + x(y-1) + \frac{(y-1)^2}{4} - \frac{(y-1)^2}{4} + y^2 - y \\
0 &= \left(x + \frac{y-1}{2}\right)^2 - \frac{(y-1)^2}{4} + y^2 - y \\
0 &= \left(x + \frac{y-1}{2}\right)^2 - \frac{y^2}{4} + \frac{y}{2} - \frac{1}{4} + y^2 - y
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{4} &= \left(x + \frac{y-1}{2}\right)^2 + \frac{3y^2}{4} - \frac{y}{2} \\
\frac{1}{4} &= \left(x + \frac{y-1}{2}\right)^2 + \frac{3}{4}\left(y^2 - \frac{4y}{3(2)}\right) \\
\frac{1}{4} &= \left(x + \frac{y-1}{2}\right)^2 + \frac{3}{4}\left(y^2 - \frac{2y}{3} + \frac{1}{9}\right) - \frac{3}{4(9)} \\
\frac{1}{4} &= \left(x + \frac{y-1}{2}\right)^2 + \frac{3}{4}\left(y - \frac{1}{3}\right)^2 - \frac{1}{12} \\
\frac{1}{4} + \frac{1}{12} &= \left(x + \frac{y-1}{2}\right)^2 + \frac{3}{4}\left(y - \frac{1}{3}\right)^2 \\
\frac{1}{3} &= \left(x + \frac{y-1}{2}\right)^2 + \frac{3}{4}\left(y - \frac{1}{3}\right)^2
\end{aligned}
$$

The equation is that of a shifted ellipse. Solve both terms in terms of cos and sin.

$$\begin{aligned}
\frac{3}{4}\left(y-\frac{1}{3}\right)^2 &= \frac{1}{3}\cos^2\theta \\
\frac{1}{4}\left(y-\frac{1}{3}\right)^2 &= \frac{1}{9}\cos^2\theta \\
\left(y-\frac{1}{3}\right)^2 &= \frac{4}{9}\cos^2\theta \\
y-\frac{1}{3} &= \frac{2}{3}\cos\theta \\
y &= \frac{2}{3}\cos\theta+\frac{1}{3}
\end{aligned}$$

$$\begin{aligned}
\left(x+\frac{y-1}{2}\right)^2 &= \frac{1}{3}\sin^2\theta \\
x+\frac{y-1}{2} &= \frac{1}{\sqrt{3}}\sin\theta \\
x &= \frac{1}{\sqrt{3}}\sin\theta-\frac{y-1}{2} \\
x &= \frac{1}{\sqrt{3}}\sin\theta+\frac{1}{2}\left(\frac{2}{3}\cos\theta-\frac{1}{3}-1\right) \\
x &= \frac{1}{\sqrt{3}}\sin\theta-\frac{1}{3}\cos\theta+\frac{1}{3}
\end{aligned}$$

Plugging that back into the plane equation gives

$$\begin{aligned}
z &= 1-x-y \\
z &= 1-\left(\frac{1}{\sqrt{3}}\sin\theta-\frac{1}{3}\cos\theta+\frac{1}{3}\right)-\left(\frac{2}{3}\cos\theta+\frac{1}{3}\right) \\
z &= 1-\frac{1}{\sqrt{3}}\sin\theta+\frac{1}{3}\cos\theta-\frac{1}{3}-\frac{2}{3}\cos\theta-\frac{1}{3} \\
z &= \frac{1}{3}-\frac{1}{\sqrt{3}}\sin\theta-\frac{1}{3}\cos\theta
\end{aligned}$$

So the curve is given as

$$\begin{aligned}
\gamma(\theta) &= \left(\frac{1}{\sqrt{3}}\sin\theta-\frac{1}{3}\cos\theta+\frac{1}{3},\ \frac{2}{3}\cos\theta+\frac{1}{3},\ \frac{1}{3}-\frac{1}{\sqrt{3}}\sin\theta-\frac{1}{3}\cos\theta\right) \\
\gamma'(\theta) &= \left(\frac{1}{\sqrt{3}}\cos\theta+\frac{1}{3}\sin\theta,\ -\frac{2}{3}\sin\theta,\ \frac{-1}{\sqrt{3}}\cos\theta+\frac{1}{3}\sin\theta\right)
\end{aligned}$$

Examining the tangent at $\theta=0$ shows that the direction is the wrong way (clockwise), so when taking the intergral, the orientation needs to be reversed.

$$\begin{aligned}
\int_\gamma \omega &= \int_0^{2\pi} 2(x(\theta)+y(\theta)+z(\theta))\|\gamma'(\theta)\|\,d\theta \\
&= 2\int_0^{2\pi}\sqrt{\left(\frac{1}{\sqrt{3}}\cos\theta\right)^2+\left(-\frac{2}{3}\sin\theta\right)^2+\left(\frac{-1}{\sqrt{3}}\cos\theta+\frac{1}{3}\sin\theta\right)^2}\,d\theta \\
&= 2\int_0^{2\pi}\frac{1}{3}\cos^2\theta+\frac{4}{9}\sin^2\theta+\frac{1}{3}\cos^2\theta-\frac{2}{3\sqrt{3}}\sin\theta\cos\theta+\frac{1}{9}\sin^2\theta\,d\theta \\
&= 2\int_0^{2\pi}\frac{1}{9}\cos^2\theta+\frac{5}{9}-\frac{2}{3\sqrt{3}}\sin\theta\cos\theta+\frac{1}{9}\sin^2\theta\,d\theta
\end{aligned}$$

(c) Use symbolic algebra software to sketch the surfaces in parts (a) and (b).

10. (a) Let $\boldsymbol{F}$ and $\boldsymbol{G}$ be vector fields on $\mathbb{R}^3$ and let $f : \mathbb{R}^3 \to \mathbb{R}$. Verify the following identities.

(i) $\operatorname{div}(\boldsymbol{F} \times \boldsymbol{G}) = \boldsymbol{G} \cdot \operatorname{curl} \boldsymbol{F} - \boldsymbol{F} \cdot \operatorname{curl} \boldsymbol{G}$.

Proof. Say $\boldsymbol{F} = (F_1, F_2, F_3)$ and $\boldsymbol{G} = (G_1, G_2, G_3)$

$$\begin{aligned}
\boldsymbol{F} \times \boldsymbol{G} &= (F_2G_3 - F_3G_2,\ F_3G_1 - F_1G_3,\ F_1G_2 - F_2G_1) \\
\operatorname{div}(\boldsymbol{F} \times \boldsymbol{G}) &= \frac{\partial}{\partial x}(F_2G_3 - F_3G_2) + \frac{\partial}{\partial y}(F_3G_1 - F_1G_3) + \frac{\partial}{\partial z}(F_1G_2 - F_2G_1) \\
&= (F_{2x}G_3 + F_2G_{3x} - F_{3x}G_2 - F_3G_{2x}) + (F_{3y}G_1 + F_3G_{1y} - F_{1y}G_3 - F_1G_{3y}) \\
&\quad + F_{1z}G_2 + F_1G_{2z} - F_{2z}G_1 - F_2G_{1z} \\
&= G_1(F_{3y} - F_{2z}) + G_2(F_{1z} - F_{3x}) + G_3(F_{2x} - F_{1y}) \\
&\quad + F_1(G_{2z} - G_{3y}) + F_2(G_{3x} - G_{1z}) + F_3(G_{1y} - G_{2x}) \\
&= \boldsymbol{G} \cdot (\nabla \times \boldsymbol{F}) - \boldsymbol{F} \cdot (\nabla \times \boldsymbol{G})
\end{aligned}$$

□

(ii) $\operatorname{curl}(f\boldsymbol{F}) = f \operatorname{curl} \boldsymbol{F} + (\operatorname{grad} f) \times \boldsymbol{F}$.

Proof.

$$f\boldsymbol{F} = (fF_1,\ fF_2,\ fF_3)$$

$$\begin{aligned}
\operatorname{curl}(f\boldsymbol{F}) &= \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ fF_1 & fF_2 & fF_3 \end{vmatrix} \\
&= (f_yF_3 + fF_{3y} - f_zF_2 - fF_{2z},\ +f_zF_1 + fF_{1z} - f_xF_3 + fF_{3x}, \\
&\quad + f_xF_2 + fF_{2_x} - f_yF_1 - fF_{1y}) \\
&= (fF_{3y} - fF_{2z},\ +fF_{1z} + fF_{3x},\ +fF_{2_x} - fF_{1y}) \\
&\quad + (f_yF_3 - f_zF_2,\ f_zF_1 - f_xF_3,\ f_xF_2 - f_yF_1) \\
&= f(F_{3y} - F_{2z},\ +F_{1z} + F_{3x},\ +F_{2_x} - F_{1y}) \\
&\quad + (f_x,\ f_y,\ f_z) \times (F_1,\ F_2,\ F_3) \\
&= f \operatorname{curl} \boldsymbol{F} + \nabla f \times \boldsymbol{F}
\end{aligned}$$

□

(b) Let R be a closed region in $\mathbb{R}^3$ with boundary ∂R. Prove the identity

$$\int_{\partial R} (\boldsymbol{F} \times \operatorname{curl}\boldsymbol{G}) \cdot d\boldsymbol{S} = \int_R (\operatorname{curl}\boldsymbol{F}) \cdot (\operatorname{curl}\boldsymbol{G})\, dV - \int_R \boldsymbol{F} \cdot \operatorname{curl}(\operatorname{curl}\boldsymbol{G})\, dV$$

(page 490, #2)

Proof.

$$\begin{aligned}
\int_{\partial R} (\boldsymbol{F} \times \operatorname{curl}\boldsymbol{G}) \cdot d\boldsymbol{S} &\overset{\text{Div Thm}}{=} \int_R \operatorname{div}(\boldsymbol{F} \times \operatorname{curl}\boldsymbol{G}) \cdot dV \\
&\overset{\text{(i)}}{=} \int_R (\operatorname{curl}\boldsymbol{F}) \cdot (\operatorname{curl}\boldsymbol{G}) \cdot dV - \int_R \boldsymbol{F} \cdot \operatorname{curl}(\operatorname{curl}\boldsymbol{G}) \cdot dV
\end{aligned}$$

□