1. Let $S \subset \mathbb{R}$ be bounded above, and let $a = \sup S$. Prove that for any $\epsilon > 0$, there exists $x \in S$ such that $x > a - \epsilon$.

   Proof. By contradiction: Suppose there is some $\epsilon > 0$ such that $\forall x \in S$, $x \leq a - \epsilon$.

   This means, by definition S is bounded by $a - \epsilon$, but $a > a - \epsilon$, which contradicts that $a = \sup S$. Therefore there must be an element in the range $(a - \epsilon, a)$ in S, $\forall \epsilon$. □

2. Show that $\sup\{r \in \mathbb{Q} | r < a\} = a$, for any $a \in \mathbb{R}$.

   Proof. By definition, S is bounded by a from above ($\forall r \in S = \{r \in \mathbb{Q} | r < a\}, r < a$). Need to show it is the smallest such bound, i.e. $\forall k \in \mathbb{R}, k < a \implies \exists l \in S$ s.t. $k < l < a$. Let $k \in \mathbb{R}, k < a$. Then since $\mathbb{Q}$ dense in $\mathbb{R}$, there is some $q \in \mathbb{Q}$ such that $k < q < a$, and so $q \in S$. So a is the lowest number that can serve as a bound, so $a = \sup S$. □

3. Let $a_1 = 1$ and let $a_{n+1} = (1 - 1/4n^2)a_n$, for any $n \geq 1$.

   (a) Show that $\lim a_n$ exists.
   To show the limit exists, I will use the fact that a bounded monotone sequence converges.

   Proof. Monotone (Strictly Decreasing): Base $a_1 = 1$, $a_2 = 3/4$, $1 > 3/4$.
   Induction : Suppose $a_{n-1} > a_n$, $n \in \mathbb{N}$

$$\Big(1 - \frac{1}{4(n-1)^2}\Big)a_{n-1} > \Big(1 - \frac{1}{4(n-1)^2}\Big)a_n$$
$$\Big(1 - \frac{1}{4(n-1)^2}\Big)a_{n-1} > \Big(1 - \frac{1}{4(n-1)^2}\Big)a_n > \Big(1 - \frac{1}{4(n)^2}\Big)a_n$$
$$\Big(1 - \frac{1}{4(n-1)^2}\Big)a_{n-1} > \Big(1 - \frac{1}{4n^2}\Big)a_n$$
$$a_n > a_{n+1}$$

   Bounded: WTP: $0 < a_k \leq 1$, $\forall k$ Base $a_0 = 1 \in (0, 1]$
   Induction : Suppose $0 < a_n \leq 1, n \in \mathbb{N}$

$$0 < a_n \leq 1$$
$$0 < \Big(1 - \frac{1}{4n^2}\Big)a_n \leq \Big(1 - \frac{1}{4n^2}\Big)$$
$$0 < a_{n+1} \leq \Big(1 - \frac{1}{4n^2}\Big) < 1$$
$$0 < a_{n+1} < 1$$

   Since the sequence is monotone and bounded, it converges.

   □

   (b) What do you think that $\lim a_n$ is?
   ≈ 0.6 from empirical testing.

4. Let $\{s_n\}$ be a sequence such that
$$|s_{n+1} - s_n| < 2^{-N}$$
   for all $n \in \mathbb{N}$. Prove that $\{s_n\}$ is a Cauchy sequence and hence converges.

   Want $\forall \epsilon$, $\exists N$ s.t. $\forall m, n > N \implies |x_m - x_n| < \epsilon$

   Lemma. *Let* $N \in \mathbb{N}$, $\forall m, n \in \mathbb{N}, m, n > N \implies |s_m - s_n| < 2^{-N+1}$

Proof. Given N, choose $N < m < n$ such that the distance $|s_n - s_m|$ is the greatest possible for any choice of m, n. Then:

$$\begin{aligned}
|s_n - s_m| &= |s_n + \sum_{i=m+1}^{n-1}(s_i - s_i) - s_m| \\
&= |s_n - s_{n-1} + \sum_{i=m+1}^{n-2}(s_{i+1} - s_i) + s_{m+1} - s_m| \\
&= |\sum_{i=m}^{n-1}(s_{i+1} - s_i)| \\
&\leq \sum_{i=m}^{n-1}|s_{i+1} - s_i| \\
&< \sum_{i=m}^{n-1} 2^{-i} \\
&= \sum_{i=1}^{n-1} 2^{-i} - \sum_{i=1}^{m-1} 2^{-i} \\
&= \frac{1 - 2^{1-n}}{2(1 - \frac{1}{2})} - \frac{1 - 2^{1-m}}{2(1 - \frac{1}{2})} \\
&= 2^{1-m} - 2^{1-n} \\
&= (2)2^{-m} < 2^{-N+1}
\end{aligned}$$

□

Now to prove the question.

Proof. Given $\epsilon > 0$ choose $N = \log_2(\frac{1}{\epsilon}) + 1$

Now let $m, n > N$, then by the lemma:

$$\begin{aligned}
|s_n - s_m| &< 2^{-N+1} \\
&= 2^{-(\log_2(\frac{1}{\epsilon})+1)+1} \\
&= 2^{\log_2(\epsilon)} \\
&= \epsilon
\end{aligned}$$

Therefore it is Cauchy and converges. □