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MATB42: Assignment #7 1002423727
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1. (a) Find an equation of the tangent plane to the surface S defined parametrically by $\mathbf{\Phi}(u,v) = (u^2+v,\ v,\ u+v^2)$ at the point (9,0,3).

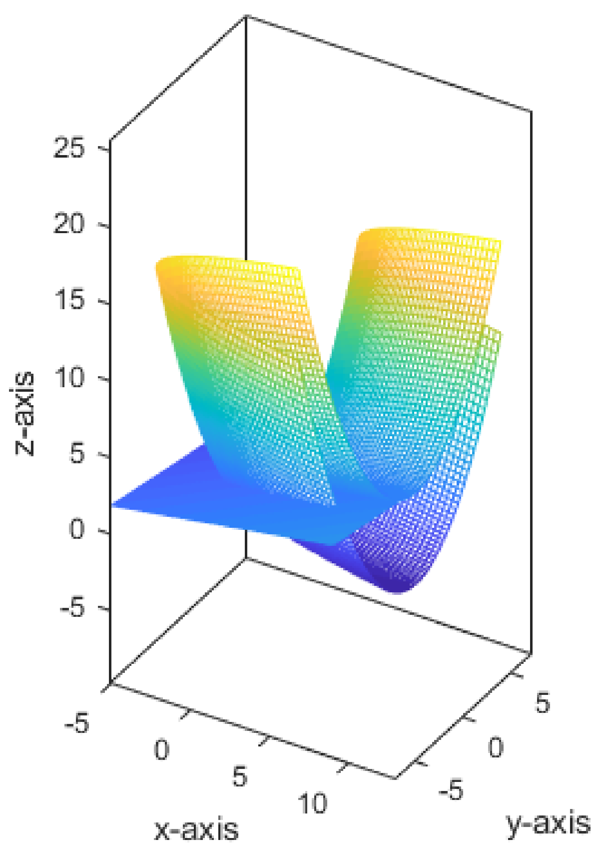
$$v = 0 \qquad\qquad u+v^2 = 3 \implies u = 3$$

$$\begin{aligned}\boldsymbol{\phi}_u &= (2(3),0,1)\\ \boldsymbol{\phi}_v &= (1,1,2(0))\\ \boldsymbol{\phi}_u \times \boldsymbol{\phi}_v &= (-1,1,6)\end{aligned}$$

So the tangent plane can be given by

$$\begin{aligned}0 &= ((x-9,y,z-3)\cdot(-1,1,6))\\ 0 &= (9-x+y+6z-18)\\ 9 &= -x+y+6z\end{aligned}$$

(b) Use symbolic algebra software to sketch the surface S and its tangent plane from part (a).



2. Use a surface integral to find the area of the triangle in $\mathbb{R}^3$ with vertices (1,1,0), (1,2,1) and (3,3,2).

   Since the surface is a plane, the tangent plane is just parallel to the plane, so tangent vectors can be given by taking vectors in the plane i.e. $\boldsymbol{\phi}_u = (3,3,2)-(1,1,0) = (2,2,2)$ and $\boldsymbol{\phi}_v = (1,2,1)-(1,1,0) = (0,1,1)$. Now this is a triangle with side lengths of equal magnitude to the tangents, so the integral is given as.

$$\begin{aligned}
\|\boldsymbol{\phi}_u \times \boldsymbol{\phi}_v\| &= \|(0,-2,2)\| = \sqrt{4+4} = 2\sqrt{2} \\
\int_0^1 \int_0^v \|\boldsymbol{\phi}_u \times \boldsymbol{\phi}_v\| \, du \, dv &= \int_0^1 \int_0^v 2\sqrt{2} \, du \, dv \\
&= \int_0^1 2\sqrt{2} v \, dv \\
&= \sqrt{2}
\end{aligned}$$

3. Calculate the surface area of the piece of the cone $x^2 + y^2 - z^2 = 0$ which lies inside the cylinder $x^2 + y^2 = 4$.

   We can see the radius of the cylinder is 2, so the cone portion that's cut out is the part which has radius less than or equal to 2 $\implies 0 \leq z \leq 2$. Using polar for the cone, $0 \leq \theta \leq 2\pi$.

$$\begin{aligned}
\boldsymbol{\Phi}(\theta, z) &= (z\cos\theta, z\sin\theta, z) \\
\boldsymbol{\phi}_\theta &= (-z\sin\theta, z\cos\theta, 0) \\
\boldsymbol{\phi}_z &= (\cos\theta, \sin\theta, 1) \\
\boldsymbol{\phi}_\theta \times \boldsymbol{\phi}_z &= (z\cos\theta, z\sin\theta, -z\sin^2\theta - z\cos^2\theta) \\
&= (z\cos\theta, z\sin\theta, -z) \\
\|\boldsymbol{\phi}_\theta \times \boldsymbol{\phi}_z\| &= z^2\cos^2\theta + z^2\sin^2\theta + z^2 = 2z^2
\end{aligned}$$

$$\begin{aligned}
\int_{\boldsymbol{\Phi}} f \, dS &= \int_0^{2\pi} \int_0^2 2z^2 \, dz \, d\theta \\
&= \int_0^{2\pi} \left[\frac{2}{3} z^3\right]_0^2 d\theta \\
&= \int_0^{2\pi} \frac{16}{3} d\theta = \frac{32\pi}{3}
\end{aligned}$$

4. (a) Find the area of the portion of the unit sphere that is cut out by the cone $z = \sqrt{x^2 + y^2}$. (cf. page 391, #10)

$$
\begin{aligned}
0 \leq \varphi \leq \pi,\ 0 \leq \theta \leq 2\pi \\
\mathbf{\Phi}_{\text{sphere}}(\theta, \varphi) &= (\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi) \\
\boldsymbol{\phi}_\theta &= (-\sin\theta\sin\varphi, \cos\theta\sin\varphi, 0) \\
\boldsymbol{\phi}_\varphi &= (\cos\theta\cos\varphi, \sin\theta\cos\varphi, -\sin\varphi) \\
\boldsymbol{\phi}_\theta \times \boldsymbol{\phi}_\varphi &= (-\cos\theta\sin^2\varphi, -\sin\theta\sin^2\varphi, -\sin^2\theta\sin\varphi\cos\varphi - \cos^2\theta\sin\varphi\cos\varphi) \\
&= (-\cos\theta\sin^2\varphi, -\sin\theta\sin^2\varphi, -\sin\varphi\cos\varphi) \\
\|\boldsymbol{\phi}_\theta \times \boldsymbol{\phi}_\varphi\| &= \sqrt{\cos^2\theta\sin^4\varphi + \sin^2\theta\sin^4\varphi + \sin^2\varphi\cos^2\varphi} \\
&= \sqrt{\sin^2\varphi} = \sin\varphi
\end{aligned}
$$

$$
\begin{aligned}
z > 0,\ 0 \leq \theta \leq 2\pi \\
\Phi_{\text{cone}}(\theta, z) &= (z\cos\theta, z\sin\theta, z) \\
\boldsymbol{\phi}_z &= (\cos\theta, \sin\theta, 1) \\
\boldsymbol{\phi}_\theta &= (-z\sin\theta, z\cos\theta, 0) \\
\boldsymbol{\phi}_z \times \boldsymbol{\phi}_\theta &= (-z\cos\theta, -z\sin\theta, z) \\
\|\boldsymbol{\phi}_z \times \boldsymbol{\phi}_\theta\| &= 2z^2
\end{aligned}
$$

For the unit sphere $x^2 + y^2 + z^2 = 1$, but the cone is $x^2 + y^2 = z^2 \implies$ sub z into sphere gives $2x^2 + 2y^2 = 1$ So the exact intersection of the surfaces is a circle of radius $2/\sqrt{2}$ centered at the origin. so the surface cut out is the section of the top of the sphere where $z \geq 2\sqrt{2} \implies \varphi \leq \frac{\pi}{4}$ from the $z = \cos\varphi$ portion of the parametrization. So the ranges are $0 \leq \theta \leq 2\pi$, $0 \leq \varphi \leq \frac{\pi}{4}$. The area is therefore

$$
\begin{aligned}
\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \sin\varphi \, d\varphi \, d\theta &= \int_0^{2\pi} \Big[-\cos\varphi\Big]_0^{\frac{\pi}{4}} d\theta \\
&= 2\pi\Big[-(\cos\left(\frac{\pi}{4}\right)) - (-\cos(0))\Big] \\
&= 2\pi\Big[-\frac{\sqrt{2}}{2} + 1\Big] \\
&= \pi(2 - \sqrt{2})
\end{aligned}
$$

(b) Find the area of the portion of the cone $z = \sqrt{x^2 + y^2}$ that is cut out by the unit sphere. Plugging in $x^2 + y^2 = 1/2$ to the cone equation again gives $z^2 = 1/2 \implies z = \pm\frac{\sqrt{2}}{2}$ but $z \geq 0$ by the cone definition so $0 \leq z \leq \frac{\sqrt{2}}{2}$.

$$
\begin{aligned}
A(\mathbf{\Phi}_{\text{cone}}) &= \int_0^{2\pi} \int_0^{\frac{1}{4}} 2z^2 \, dz \, d\theta \\
&= \int_0^{2\pi} \frac{2}{3} \cdot \frac{1}{4^3} \, d\theta \\
&= \frac{\pi}{3(16)} \\
&= \frac{\pi}{48}
\end{aligned}
$$

5. Let $\mathbf{\Phi} : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a parametrization of a 2-dim surface S in $\mathbb{R}^3$.

(a) Set

$$E = \|\boldsymbol{\phi}_u\|^2, \qquad F = \boldsymbol{\phi}_u \cdot \boldsymbol{\phi}_v, \qquad G = \|\boldsymbol{\phi}_v\|^2,$$

Show that the surface area of S is

$$A(S) = \iint_D \sqrt{EG - F^2}\, dA$$

$$\begin{aligned}
\iint_D \sqrt{EG - F^2}\, dA &= \iint_D \sqrt{\|\boldsymbol{\phi}_u\|^2 \|\boldsymbol{\phi}_v\|^2 - (\boldsymbol{\phi}_u \cdot \boldsymbol{\phi}_v)^2}\, dA \\
&= \iint_D \sqrt{(\|\boldsymbol{\phi}_u\| \|\boldsymbol{\phi}_v\|)^2 - (\|\boldsymbol{\phi}_u\| \|\boldsymbol{\phi}_v\|)^2 \cos^2\theta}\, dA \quad \text{Where } \theta \text{ is the angle between } \boldsymbol{\phi}_u \text{ and } \boldsymbol{\phi}_v. \\
&= \iint_D \sqrt{(\|\boldsymbol{\phi}_u\| \|\boldsymbol{\phi}_v\|)^2 (1 - \cos^2\theta)}\, dA \\
&= \iint_D \sqrt{(\|\boldsymbol{\phi}_u\| \|\boldsymbol{\phi}_v\|)^2 (\sin^2\theta)}\, dA \\
&= \iint_D \sqrt{\|\boldsymbol{\phi}_u \times \boldsymbol{\phi}_v\|^2}\, dA \\
&= \iint_D \|\boldsymbol{\phi}_u \times \boldsymbol{\phi}_v\|\, dA \\
&= \int_{\mathbf{\Phi}} 1\, dS
\end{aligned}$$

(b) What does the formula for $A(S)$ become if the vectors $\boldsymbol{\phi}_u$ and $\boldsymbol{\phi}_v$ are orthogonal?
If the vectors are orthogonal, then the dot product is 0, so the equation reduces to

$$A(S) = \iint_D \|\boldsymbol{\phi}_u\| \|\boldsymbol{\phi}_v\|\, dA$$

(c) Use parts (a) and (b) to compute the surface area of a sphere of radius a.
(cf. Marsden & Tromba, page 399, # 23.)

$$\begin{aligned}
&0 \le \varphi \le \pi,\ 0 \le \theta \le 2\pi \\
\mathbf{\Phi}(\theta, \varphi) &= a(\cos\theta \sin\varphi, \sin\theta \sin\varphi, \cos\varphi) \\
\boldsymbol{\phi}_\theta &= a(-\sin\theta \sin\varphi, \cos\theta \sin\varphi, 0) \\
\boldsymbol{\phi}_\varphi &= a(\cos\theta \cos\varphi, \sin\theta \cos\varphi, -\sin\varphi) \\
\|\boldsymbol{\phi}_\theta\| &= a \sin\varphi, \quad \|\boldsymbol{\phi}_\varphi\| = a \\
\implies A(S) &= a^2 \int_0^{2\pi} \int_0^{\pi} \sin\varphi\, d\varphi\, d\theta \\
&= a^2 \int_0^{2\pi} \Big[-\cos\varphi \Big]_0^{\pi} d\varphi\, d\theta \\
&= a^2 \int_0^{2\pi} -(-1 - 1)\, d\varphi\, d\theta \\
&= a^2 2 \int_0^{2\pi} 1\, d\varphi\, d\theta \\
&= 4\pi a^2
\end{aligned}$$

6. For each of the following surfaces S, sketch S (using symbolic software) and evaluate the surface integral $\int_S f\, dS$, where $f(x, y, z) = x$.

   (a) S is that part of the surface $y = 4 - x^2$ between $z = 0$ and $z = 1$, with $y \geq 0$.

$$y \geq 0 \implies 4 - x^2 \geq 0 \implies x^2 \leq 4 \implies |x| < 2$$

$$\begin{aligned}
\mathbf{\Phi}(x, z) &= (x, 4 - x^2, z) \\
\boldsymbol{\phi}_x &= (1, -2x, 0),\ \boldsymbol{\phi}_z = (0, 0, 1) \\
\boldsymbol{\phi}_x \times \boldsymbol{\phi}_z &= (-2x, -1, 0) \implies \|\boldsymbol{\phi}_x \times \boldsymbol{\phi}_z\| = \sqrt{4x^2 + 1} \\
\int_S f dS &= \int_0^1 \int_{-2}^2 x\sqrt{4x^2 + 1}\, dx\, dz
\end{aligned}$$

   The integrand is odd since x odd and $\sqrt{4x^2 + 1}$ even, so the integral over x is 0, making the entire integral 0.

   (b) S is the upper half of the unit sphere centered at the origin.
   Only the upper half so $0 \leq \theta \leq 2\pi$ and $0 \leq \varphi \leq \pi/2$.

$$\begin{aligned}
\mathbf{\Phi}(\theta, \varphi) &= (\cos\theta \sin\varphi, \sin\theta \sin\varphi, \cos\varphi) \\
\boldsymbol{\phi}_\theta &= (-\sin\theta \sin\varphi, \cos\theta \sin\varphi, 0) \\
\boldsymbol{\phi}_\varphi &= (\cos\theta \cos\varphi, \sin\theta \cos\varphi, -\sin\varphi) \\
\boldsymbol{\phi}_\theta \times \boldsymbol{\phi}_\varphi &= (-\cos\theta \sin^2\varphi, -\sin\theta \sin^2\varphi, -\sin^2\theta \sin\varphi \cos\varphi - \cos^2\theta \sin\varphi \cos\varphi) \\
&= (-\cos\theta \sin^2\varphi, -\sin\theta \sin^2\varphi, -\sin\varphi \cos\varphi) \\
\|\boldsymbol{\phi}_\theta \times \boldsymbol{\phi}_\varphi\| &= \sqrt{\cos^2\theta \sin^4\varphi + \sin^2\theta \sin^4\varphi + \sin^2\varphi \cos^2\varphi} \\
&= \sqrt{\sin^4\varphi + \sin^2\varphi \cos^2\varphi} \\
&= \sqrt{\sin^2\varphi} = \sin\varphi \\
\int_{\mathbf{\Phi}} f\, dS &= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \cos\theta \sin^2\varphi\, d\theta\, d\varphi = 0
\end{aligned}$$

   The integral is zero again since integrating $\cos\theta$ over a whole period is 0.

   (c) S is that part of the surface $x = \sin y$ with $0 \leq y \leq \pi$ and $0 \leq z \leq 2$.

$$
\begin{aligned}
\mathbf{\Phi}(y,z) &= (\sin y, y, z)\\
\boldsymbol{\phi}_y &= (\cos y, 1, 0)\\
\boldsymbol{\phi}_z &= (0,0,1)\\
\boldsymbol{\phi}_y \times \boldsymbol{\phi}_z &= (1, -\cos y, 0)\\
\|\boldsymbol{\phi}_y \times \boldsymbol{\phi}_z\| &= \sqrt{1+\cos^2 y}\\
\int_{\mathbf{\Phi}} f\,dS &= \int_0^2 \int_0^\pi \sin y\sqrt{1+\cos^2 y}\,dy\,dz\\
&\text{Let } u=\cos y,\ du=-\sin y\,dy\\
&= 2\int_{-1}^{1} \sqrt{1+u^2}\,du\\
&\text{Let } u=\tan\theta,\ du=sec^2(\theta)\,d\theta\\
&= 2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{\frac{1}{\cos^2\theta}}\sec^2\theta\,du\\
&= 2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^3\theta\,d\theta\\
&\text{Let } u=\sec\theta,\ du=\sec\theta\tan\theta\,d\theta,\ dv=\sec^2\theta\,d\theta,\ v=\tan\theta\\
&= 2\Big(\Big[\sec\theta\tan\theta\Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} - \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^2\theta\sec\theta\,d\theta\Big)\\
&= 2\Big(\Big[\sec\theta\tan\theta\Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} - \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\sec^2\theta-1)\sec\theta\,d\theta\Big)\\
2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^3\theta\,d\theta &= 2\Big(\Big[\sec\theta\tan\theta\Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} - \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^3\theta\,d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{4}} \sec\theta\,d\theta\Big)\\
4\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^3\theta\,d\theta &= 2\Big(\Big[\sec\theta\tan\theta\Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec\theta\,d\theta\Big)\\
2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^3\theta\,d\theta &= \Big(\Big[\sec\theta\tan\theta\Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec\theta\,d\theta\Big)\\
&= \Big(\Big[\sec\theta\tan\theta\Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec\theta\Big(\frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}\Big)\,d\theta\Big)\\
&= \Big(\Big[\sec\theta\tan\theta\Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} + \Big[\ln|\sec\theta+\tan\theta|\Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\Big)\\
&= 2\sqrt{2} + \ln|1+\sqrt{2}| - \ln|\sqrt{2}-1|
\end{aligned}
$$

7. Find the mass of the metallic surface S given by $z = 1 - \dfrac{x^2 + y^2}{2}$ with $0 \leq x \leq 1$, $0 \leq y \leq 1$, if the mass density at $(x, y, z) \in S$ is given by $m(x, y, z) = xy$.

$$
\begin{aligned}
\mathbf{\Phi}(x, y) &= (x, y, 1 - \frac{x^2 + y^2}{2}) \\
\boldsymbol{\phi}_x &= (1, 0, -x) \\
\boldsymbol{\phi}_y &= (0, 1, -y) \\
\|\boldsymbol{\phi}_x \times \boldsymbol{\phi}_y\| &= \sqrt{x^2 + y^2 + 1} \\
\int_{\mathbf{\Phi}} f \, dS &= \int_0^1 \int_0^1 xy\sqrt{x^2 + y^2 + 1} \, dx \, dy \\
&\text{Let } u = x^2 + y^2 + 1, \; du = 2x \, dx \\
&= \frac{1}{2} \int_0^1 \int_{y^2+1}^{y^2+2} y\sqrt{u} \, du \, dy \\
&= \frac{1}{3} \int_0^1 y \left[u^{\frac{3}{2}} \right]_{y^2+2}^{y^2+1} dy \\
&= \frac{1}{3} \int_0^1 y \left[(y^2 + 2)^{\frac{3}{2}} - (y^2 + 1)^{\frac{3}{2}} \right] dy \\
&\text{Let } u = y^2 + 1, \; du = 2y \, dy \\
&= \frac{1}{6} \int_1^2 (u + 1)^{\frac{3}{2}} - (u)^{\frac{3}{2}} \, du \\
&= \frac{1}{15} \left[(u + 1)^{\frac{5}{2}} - (u)^{\frac{5}{2}} \right]_1^2 \\
&= \frac{1}{15} \left[(3^{\frac{5}{2}} - 2^{\frac{5}{2}}) - (2^{\frac{5}{2}} - 1) \right] \\
&= \frac{1}{15} \left[3^{\frac{5}{2}} - 2^{\frac{7}{2}} - 1 \right] \\
&= \frac{1}{15} \left[9\sqrt{3} - 8\sqrt{2} + 1 \right]
\end{aligned}
$$

Bonus

(a) Calculate the surface area of S, the piece of the cone $x^2+y^2=z^2$ lying over the disk $x^2+y^2 \leq 4,\ z=0$

Then calculate $\displaystyle\int_S x^2 z dS$

Parametrize the surface as

$$\begin{aligned}
\mathbf{\Phi}(\theta, r) &= (r\cos\theta, r\sin\theta, \sqrt{r^2\cos^2\theta + r^2\sin^2\theta}) \\
&= (r\cos\theta, r\sin\theta, r) \\
&\text{Where: } 0 \leq \theta \leq 2\pi,\ 0 \leq r \leq 2 \\
\boldsymbol{\phi}_\theta &= (-r\sin\theta, r\cos\theta, 0) \\
\boldsymbol{\phi}_r &= (\cos\theta, \sin\theta, 1) \\
\boldsymbol{\phi}_\theta \times \boldsymbol{\phi}_r &= (r\cos\theta, r\sin\theta, -r\sin^2\theta - r\cos^2\theta) \\
&= (r\cos\theta, r\sin\theta, -r) \\
\|\boldsymbol{\phi}_\theta \times \boldsymbol{\phi}_r\| &= \sqrt{r^2\cos^2\theta + r^2\sin\theta + r^2} \\
&= \sqrt{2r^2} = \sqrt{2}r \\
\int_0^{2\pi}\int_0^2 \sqrt{2}r\,dr\,d\theta &= \int_0^{2\pi} 2\sqrt{2}d\theta = 4\sqrt{2}\pi \\
\int_S x^2 z\,dS &= \int_0^{2\pi}\int_0^2 r^2\cos^2\theta r\sqrt{2}r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 r^4\cos^2\theta\sqrt{2}\,dr\,d\theta \\
&= \sqrt{2}\int_0^{2\pi}\left[\frac{r^5}{5}\right]_0^2 \cos^2\theta\,d\theta = \frac{32}{5}\sqrt{2}\int_0^{2\pi}\cos^2\theta\,d\theta \\
&= \frac{32}{5}\sqrt{2}\int_0^{2\pi} 1/2 + (1/2)\cos 2\theta\,d\theta = \frac{32\pi}{5}\sqrt{2}
\end{aligned}$$

(b) Parametrize S which is part of the plane $z = 2x+4y$ that is part of the first octant & below $z=6$.

$$\mathbf{\Phi}(x,\ y) = (x, y, 2x+3y)$$

Now this is in the first octant, so $x, y > 0 \implies 2x+3y > 0$. It must also satisfy $z < 6 \implies 2x+3y < 6$. Ignoring y, $x < 3$, now fixing x, $y < \frac{6-2x}{3}$. so the bounds are:

$$0 \leq x < 3,\ 0 \leq y < \frac{6-2x}{3}$$

(c) S is that part of the cone $x = 2\sqrt{x^2+y^2}$ with $x \leq 4$ in the first octant.

$$\mathbf{\Phi}(\theta, r) = (2\sqrt{r^2\cos^2\theta + r^2\cos^2\theta}, r\cos\theta, r\sin\theta)$$

$$= (2r, r\cos\theta, r\sin\theta)$$

For the bounds, $x \leq 4 \implies r \leq 2$, and first quadrant $\implies r > 0,\ 0 \leq \theta \leq \pi/2$