Poon, Keegan
1002423727
MATB42: Assignment #2
Bonus
Jan 30th 2018

1. Let $f(x) = \begin{cases} 0, & -\pi < x < -\frac{\pi}{2} \\ 2, & -\frac{\pi}{2} \le x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} \le x < \pi \end{cases}$

   (a) Find the Fourier series of f.

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\ &= \frac{1}{\pi}\left[\int_{-\pi}^{-\frac{\pi}{2}} 0dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2dx + \int_{\frac{\pi}{2}}^{\pi} 0dx\right] \\ &= \frac{1}{\pi}\left[2\pi\right] \\ &= 2 \end{aligned}$$

$$\begin{aligned} b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx \\ &= \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(kx)dx \\ &= 0 \quad [\text{sin is odd}] \end{aligned}$$

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx \\ &= \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(kx)dx \\ &= \frac{2}{k\pi}\Big[\sin(kx)\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\ &= \frac{2}{k\pi}\left[2\sin\left(\frac{k\pi}{2}\right)\right] \\ &= \frac{4}{k\pi}\sin\left(\frac{k\pi}{2}\right) \end{aligned}$$

This is 0 for even elements, and alternating between 1 and -1 for odd elements.

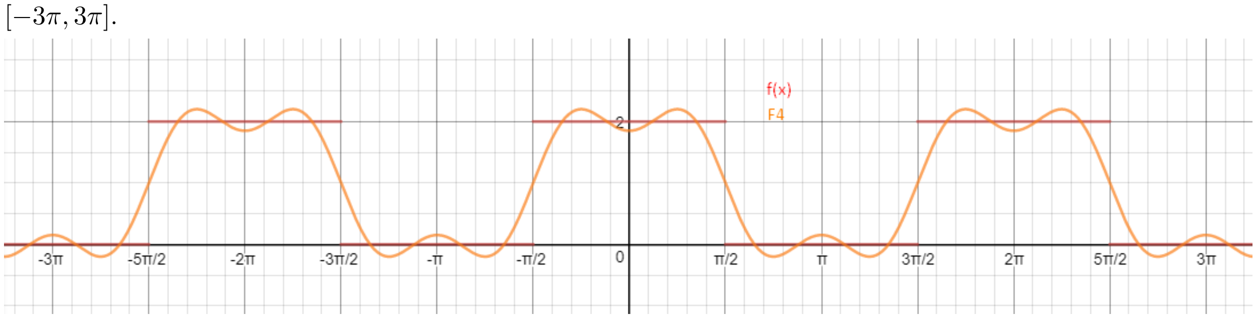
Therefore the Fourier polynomial (for the non-zero terms) is

$$1 + \sum_{l=1}^{\infty}\left[\frac{4(-1)^{l+1}}{(2l-1)\pi}\cos((2l-1)x)\right]$$

   (b) Determine if the Fourier series in part (a) converges. If it does converge, what are the values to which it converges (on $[-\pi, \pi]$).

The function is continuous on its partitions (they are constant functions), so by the theorem the polynomial converges to $f(x)$ on the continuous intervals. On the discontinuities, it converges to 0 at $\frac{\pi}{2}$ and $\frac{-\pi}{2}$ from the Fundemental theorem, and to 0 at π and $-\pi$.

   (c) Use symbolic algeba software to sketch $f(x)$ and its 4^{th} degree Fourier polynomial over the interval $[-3\pi, 3\pi]$.



2. (a) Find the Fourier series of the function $f(x)$ defined by $f(x) = \begin{cases} 0 & , -\pi \le x < 0 \\ x & , 0 \le x < \pi \end{cases}$ and extended from this with period 2π to all of $\mathbb{R}$.

If this Fourier series converges describe the function to which it converges.

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\ &= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)dx + \int_{0}^{\pi} f(x)dx\right] \\ &= \frac{1}{\pi}\left[0 + \int_{0}^{\pi} xdx\right] \\ &= \frac{1}{\pi}\left[\frac{1}{2}\left[x^2\right]_0^{\pi}\right] \\ &= \frac{\pi}{2} \end{aligned}$$

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx \\ &= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0dx + \int_{0}^{\pi} x\cos(kx)dx\right] \end{aligned}$$

Let $u = x,\ du = dx,$

$dv = \cos(kx),\ v = \dfrac{\sin(kx)}{k}$

$$\begin{aligned} &= \frac{1}{\pi}\left[\frac{1}{k}\left[x\sin(kx)\right]_0^{\pi} - \frac{1}{k}\int_0^{\pi}\sin(kx)dx\right] \\ &= -\frac{1}{k\pi}\left[\int_0^{\pi}\sin(kx)dx\right] \\ &= \frac{1}{k^2\pi}\left[\cos(kx)\right]_0^{\pi} \\ &= \frac{(-1)^{-k}-1}{k^2\pi} \end{aligned}$$

$$\begin{aligned} b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx \\ &= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0dx + \int_{0}^{\pi} x\sin(kx)dx\right] \end{aligned}$$

Let $u = x,\ du = dx,\ dv = \sin(kx),\ v = -\frac{1}{k}\cos(kx)$

$$\begin{aligned} &= \frac{1}{\pi}\left[-\frac{1}{k}\left[x\cos(kx)\right]_0^{\pi} + \frac{1}{k}\int_0^{\pi}\cos(kx)dx\right] \\ &= \frac{1}{k\pi}\left[-\pi\cos(k\pi) + \frac{1}{k}\left[\sin(kx)\right]_0^{\pi}\right] \\ &= \frac{1}{k\pi}\left[-\pi\cos(k\pi) + 0\right] \\ &= \frac{(-1)^{k+1}}{k} \end{aligned}$$

Therefore the Fourier series of f is

$$F(x) = \frac{\pi}{4} + \sum_{k=1}^{\infty}\left[\frac{(-1)^k - 1}{k^2\pi}\cos(kx) + \frac{(-1)^{k+1}}{k}\sin(kx)\right]$$

Since f is piecewise very smooth (0, x are infinitely differentiable), the series converges to f on $(-\pi,\pi)$ and on both endpoints, it converges to $\frac{\pi}{2}$.

(b) Using the series from part (a) show that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots.$$

$$\begin{aligned} F(0) &= \frac{\pi}{4} + \sum_{k=1}^{\infty}\left[\frac{(-1)^k - 1}{k^2\pi}\right] \\ 0 &= \frac{\pi}{4} + \sum_{k=1}^{\infty}\left[\frac{-2}{(2k-1)^2\pi}\right] \end{aligned}$$

$$\begin{aligned} \frac{\pi}{4} &= \sum_{k=1}^{\infty}\left[\frac{2}{(2k-1)^2\pi}\right] \\ \frac{\pi^2}{8} &= \sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} \end{aligned}$$

3. Find the Fourier series for the restriction of the function $f(x) = 3+3x$ to each of the following intervals, $[a, b]$. If the Fourier series converges, to what values will the series converge at the end points?

   (a) $[a, b] = [-\pi, \pi]$

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} 3+3xdx \\ &= \frac{1}{\pi}\left[6\pi + \frac{3}{2}\Big[x^2\Big]_{-\pi}^{\pi}\right] \\ &= \frac{1}{\pi}\left[6\pi + 0\right] \\ &= 6 \end{aligned}$$

$$\begin{aligned} b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx \\ &= \frac{3}{\pi}\left[\int_{-\pi}^{\pi}\sin(kx)dx + \int_{-\pi}^{\pi} x\sin(kx)dx\right] \\ &= \frac{6}{\pi}\left[\int_{0}^{\pi} x\sin(kx)dx\right] \quad \text{[Since } x \text{ and sin odd]} \end{aligned}$$

Let $u = x, du = 1dx, dv = \sin(kx)dx, v = -\frac{\cos(kx)}{k}$

$$\begin{aligned} &= \frac{6}{\pi}\left[-\frac{1}{k}\Big[x\cos(kx)\Big]_0^{\pi} + \frac{1}{k}\int_0^{\pi}\cos(kx)dx\right] \\ &= \frac{6}{k\pi}\left[\pi(-1)^{k+1} + \frac{1}{k}\Big[\sin(kx)\Big]_0^{\pi}\right] \\ &= \frac{6(-1)^{k+1}}{k} \end{aligned}$$

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx \\ &= \frac{3}{\pi}\left[\int_{-\pi}^{\pi}\cos(kx)dx + \int_{-\pi}^{\pi} x\cos(kx)dx\right] \\ &= \frac{6}{k\pi}\Big[\sin(kx)\Big]_0^{\pi} \quad \text{[Since } x \text{ odd and cos even]} \\ &= 0 \end{aligned}$$

Therefore the Fourier series is defined as

$$F(x) = 3 + \sum_{k=1}^{\infty}\frac{6(-1)^{k+1}}{k}\sin(kx)$$

Linear functions are infinitely differentiable so it will converge to $f(x)$ within the interval, and coverges to 3 at the endpoints.

(b) $[a,b] = [0, 2\pi]$

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\
&= \frac{1}{\pi}\int_0^{2\pi} 3 + 3xdx \\
&= \frac{1}{\pi}\left[6\pi + \frac{3}{2}\Big[x^2\Big]_0^{2\pi}\right] \\
&= \frac{1}{\pi}\left[6\pi + 6\pi^2\right] \\
&= 6(\pi + 1)
\end{aligned}$$

$$\begin{aligned}
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx \\
&= \frac{3}{\pi}\left[\int_0^{2\pi} \sin(kx)dx + \int_0^{2\pi} x\sin(kx)dx\right] \\
&\text{Let } u = x, du = 1dx, dv = \sin(kx)dx, v = -\frac{\cos(kx)}{k} \\
&= \frac{3}{k\pi}\left[\Big[\cos(kx)\Big]_0^{2\pi} - \Big[x\cos(kx)\Big]_0^{2\pi} + \int_0^{2\pi}\cos(kx)dx\right] \\
&= \frac{3}{k\pi}\left[-2\pi + \frac{1}{k}\Big[\sin(kx)\Big]_0^{2\pi}\right] \\
&= -\frac{6}{k}
\end{aligned}$$

$$\begin{aligned}
a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx \\
&= \frac{3}{\pi}\left[\int_0^{2\pi}\cos(kx)dx + \int_0^{2\pi} x\cos(kx)dx\right] \\
&\text{Let } u = x,\ du = dx,\ dv = \cos(kx),\ v = \frac{1}{k}\sin(kx) \\
&= \frac{3}{k\pi}\left[\Big[\sin(kx)\Big]_0^{2\pi} + \Big[x\sin(kx)\Big]_0^{2\pi} - \int_0^{2\pi}\sin(kx)dx\right] \\
&= -\frac{3}{k^2\pi}\Big[\cos(kx)\Big]_0^{2\pi} \\
&= 0
\end{aligned}$$

Therefore the Fourier series is defined as

$$F(x) = 3(\pi + 1) - \sum_{k=1}^{\infty}\frac{6}{k}\sin(kx)$$

Linear functions are infinitely differentiable so it will converge to $f(x)$ within the interval, and coverges to $3 + 3\pi$ at the endpoints.

4. Find the Fourier series of the function $f(x)$ defined on $[0, 2\pi]$ by $f(x) = x(x - 2\pi)$ and extended from this with period 2π to all of $\mathbb{R}$. Use symbolic algebra software to graph the 4^{th} degree Fourier polynomial together with the original function.

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x(x-2\pi)dx$$

$$= \frac{1}{\pi}\left[\int_0^{2\pi} x^2\sin(kx)dx - 2\pi\int_0^{2\pi} x\sin(kx)dx\right]$$

Let $u = x,\ du = dx,\ dv = \sin(kx),\ v = -\frac{1}{k}\cos(kx)$

$$= \frac{1}{\pi}\left[\int_0^{2\pi} x^2\sin(kx)dx - 2\pi\Big(-\frac{1}{k}\Big[x\cos(kx)\Big]_0^{2\pi} + \frac{1}{k}\int_0^{2\pi}\cos(kx)dx\Big)\right]$$

$$= \frac{1}{\pi}\left[\int_0^{2\pi} x^2\sin(kx)dx - 2\pi\Big(-\frac{2\pi}{k} + \frac{1}{k^2}\Big[\sin(kx)\Big]_0^{2\pi}\Big)\right]$$

$$= \frac{1}{\pi}\left[\int_0^{2\pi} x^2\sin(kx)dx + \frac{4\pi^2}{k}\right]$$

Let $u = x^2, du = 2xdx,$

$dv = \sin(kx)dx, v = -\frac{\cos(kx)}{k}$

$$= \frac{1}{k\pi}\left[-\Big[x^2\cos(kx)\Big]_0^{2\pi} + \int_0^{2\pi} x\cos(kx)dx + 4\pi^2\right]$$

Let $u = x,\ du = dx,\ dv = \sin(kx),\ v = -\frac{1}{k}\cos(kx)$

$$= \frac{1}{k\pi}\left[\frac{1}{k}\Big[x\sin(kx)\Big]_0^{2\pi} - \frac{1}{k}\int_0^{2\pi}\cos(kx)dx\right]$$

$$= \frac{1}{k\pi}\left[-\frac{1}{k^2}\Big[\sin(kx)\Big]_0^{2\pi}\right]$$

$$= 0$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx$$

$$= \frac{1}{\pi}\left[\int_0^{2\pi} x^2\cos(kx)dx - 2\pi\int_0^{2\pi} x\cos(kx)dx\right]$$

Let $u = x^2, du = 2xdx, dv = \cos(kx)dx, v = \frac{\sin(kx)}{k}$

$$= \frac{1}{\pi}\left[\frac{1}{k}\Big[x^2\sin(kx)\Big]_0^{2\pi} - \frac{2}{k}\int_0^{2\pi} x\sin(kx)dx - 2\pi\int_0^{2\pi} x\cos(kx)dx\right]$$

$$= \frac{1}{\pi}\left[-\frac{2}{k}\int_0^{2\pi} x\sin(kx)dx - 2\pi\int_0^{2\pi} x\cos(kx)dx\right]$$

Let $u = x, du = dx, dv = \sin(kx)dx, v = -\frac{\cos(kx)}{k}$

$$= \frac{1}{\pi}\left[\frac{2}{k^2}\Big[x\cos(kx)\Big]_0^{2\pi} - \frac{1}{k}\int_0^{2\pi}\cos(kx)dx - 2\pi\int_0^{2\pi} x\cos(kx)dx\right]$$

$$= \frac{1}{\pi}\left[\frac{4\pi}{k^2} - \frac{1}{k^2}\Big[\sin(kx)\Big]_0^{2\pi} - 2\pi\int_0^{2\pi} x\cos(kx)dx\right]$$

Let $u = x,\ du = dx,\ dv = \cos(kx),\ v = \frac{1}{k}\sin(kx)$

$$= \frac{1}{\pi}\left[\frac{4\pi}{k^2} - 2\pi\Big(\frac{1}{k}\Big[x\sin(kx)\Big]_0^{2\pi} - \frac{1}{k}\int_0^{2\pi}\sin(kx)dx\Big)\right]$$

$$= \frac{1}{\pi}\left[\frac{4\pi}{k^2} + \frac{2\pi}{k}\int_0^{2\pi}\sin(kx)dx\right]$$

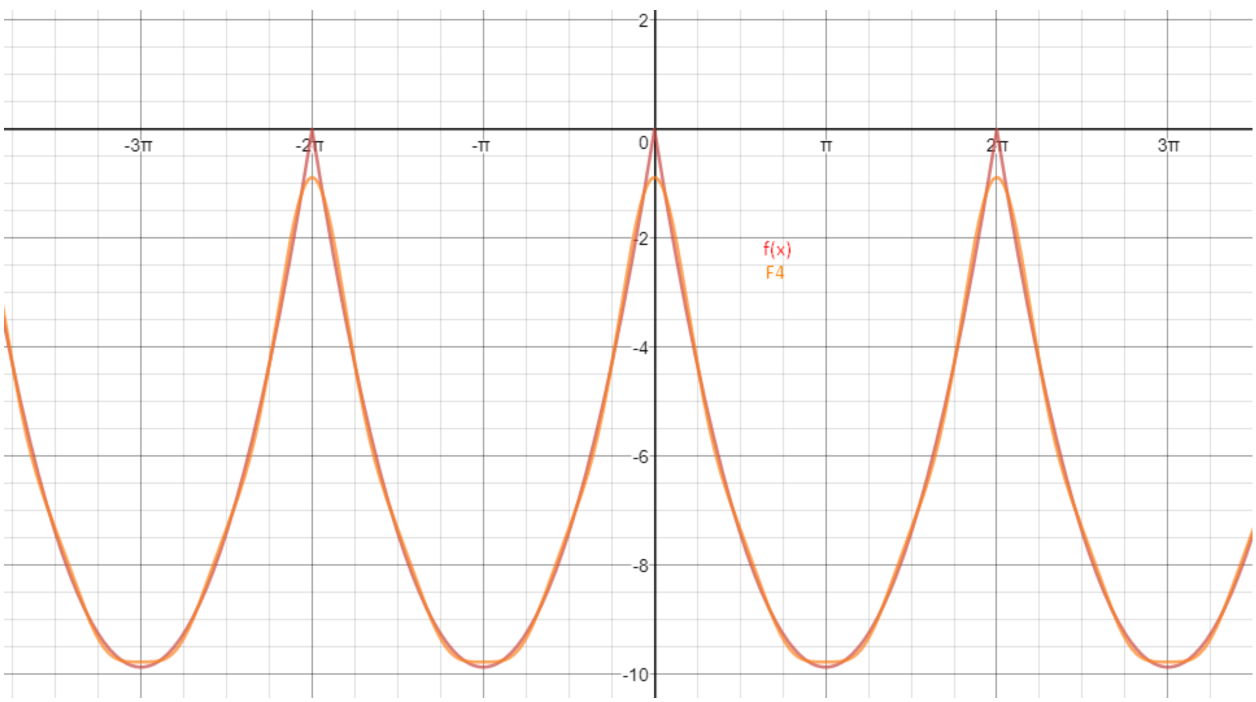
$$= \frac{1}{\pi}\left[\frac{4\pi}{k^2} + \frac{2\pi}{k^2}\Big[\cos(kx)\Big]_0^{2\pi}\right]$$

$$= \frac{4}{k^2}$$

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\
&= \frac{1}{\pi}\int_{0}^{2\pi} x(x-2\pi)dx \\
&= \frac{1}{\pi}\left[\int_{0}^{2\pi} x^2dx - \int_{0}^{2\pi} 2x\pi dx\right] \\
&= \frac{1}{\pi}\left[\frac{1}{3}\left[x^3\right]_0^{2\pi} - \pi\left[x^2\right]_0^{2\pi}\right] \\
&= \frac{1}{\pi}\left[\frac{8\pi^3}{3} - 4\pi^3\right] \\
&= -\frac{4\pi^2}{3}
\end{aligned}
$$

Therefore the Fourier series of f is

$$
F(x) = -\frac{2\pi^2}{3} + \sum_{k=1}^{\infty}\frac{4}{k^2}\cos(kx)
$$



5. Let $f(x)$ be defined on $[0, 2\pi]$ by $f(x) = x(x - 2\pi)$.

(a) Find the Fourier cosine series of f.

From question 4, we can see that the function is already even, hence the Fourier series of the function itself is a cosine series of f. Namely

$$F(x) = -\frac{2\pi^2}{3} + \sum_{k=1}^{\infty} \frac{4}{k^2}\cos(kx)$$

(b) Find the Fourier sine series of f.

To extend this as an odd function, define the f on the range $[-2\pi, 0]$ as $f(x) = -((x+2\pi)((x+2\pi)-2\pi)) = -x(x+2\pi)$. Note that this definition of f now has a period of 4π.

$$\begin{aligned}
b_k &= \frac{1}{2\pi}\int_{-2\pi}^{2\pi} f(x)\sin\left(\frac{kx}{2}\right)dx \\
&= -\frac{1}{\pi}\left[\int_{-2\pi}^{0} x(x+2\pi)\sin\left(\frac{kx}{2}\right)dx\right] \quad [f \text{ and } \sin \text{ are both odd so the integrand is even}] \\
&= -\frac{1}{\pi}\left[\int_{-2\pi}^{0} x^2\sin\left(\frac{kx}{2}\right)dx + 2\pi\int_{-2\pi}^{0} x\sin\left(\frac{kx}{2}\right)dx\right]
\end{aligned}$$

Let $u = x^2,\ du = 2xdx,\ dv = \sin\left(\frac{kx}{2}\right)dx,\ v = -\frac{2\cos(\frac{kx}{2})}{k}$

$$\begin{aligned}
&= -\frac{1}{\pi}\left[-\frac{2}{k}\left[x^2\cos\left(\frac{kx}{2}\right)\right]_{-2\pi}^{0} + \frac{4}{k}\int_{-2\pi}^{0} x\cos\left(\frac{kx}{2}\right)dx + 2\pi\int_{-2\pi}^{0} x\sin\left(\frac{kx}{2}\right)dx\right] \\
&= -\frac{1}{\pi}\left[\frac{8\pi^2(-1)^k}{k} + \frac{4}{k}\int_{-2\pi}^{0} x\cos\left(\frac{kx}{2}\right)dx + 2\pi\int_{-2\pi}^{0} x\sin\left(\frac{kx}{2}\right)dx\right]
\end{aligned}$$

Let $u = x,\ du = dx,\ dv = \cos\left(\frac{kx}{2}\right),\ v = \frac{2}{k}\sin\left(\frac{kx}{2}\right)$

$$\begin{aligned}
&= -\frac{1}{\pi}\left[\frac{8\pi^2}{k}(-1)^k + \frac{4}{k^2}\left[x\sin\left(\frac{kx}{2}\right)\right]_{-2\pi}^{0} - \frac{8}{k^2}\int_{-2\pi}^{0}\sin\left(\frac{kx}{2}\right)dx + 2\pi\int_{-2\pi}^{0} x\sin\left(\frac{kx}{2}\right)dx\right] \\
&= -\frac{1}{\pi}\left[\frac{8\pi^2}{k}(-1)^k + \frac{16}{k^3}\left[\cos\left(\frac{kx}{2}\right)\right]_{-2\pi}^{0} + 2\pi\int_{-2\pi}^{0} x\sin\left(\frac{kx}{2}\right)dx\right]
\end{aligned}$$

Let $u = x, du = dx, dv = \sin\left(\frac{kx}{2}\right)dx, v = -\frac{2\cos(\frac{kx}{2})}{k}$

$$\begin{aligned}
&= -\frac{1}{\pi}\left[\frac{8\pi^2}{k}(-1)^k + \frac{16}{k^3}(1-(-1)^k) + \frac{4\pi}{k}\left(-\left[x\cos\left(\frac{kx}{2}\right)\right]_{-2\pi}^{0} + \int_{-2\pi}^{0}\cos\left(\frac{kx}{2}\right)dx\right)\right] \\
&= -\frac{1}{\pi}\left[\frac{8\pi^2}{k}(-1)^k + \frac{16}{k^3}(1-(-1)^k) + \frac{4\pi}{k}\left(2\pi(-1)^{k+1} + \frac{2}{k}\left[\sin\left(\frac{kx}{2}\right)\right]_{-2\pi}^{0}\right)\right] \\
&= -\frac{1}{\pi}\left[\frac{8\pi^2}{k}(-1)^k + \frac{16}{k^3}(1-(-1)^k) + \frac{8\pi^2}{k}(-1)^{k+1}\right] \\
&= \frac{16}{k^3\pi}((-1)^k - 1)
\end{aligned}$$

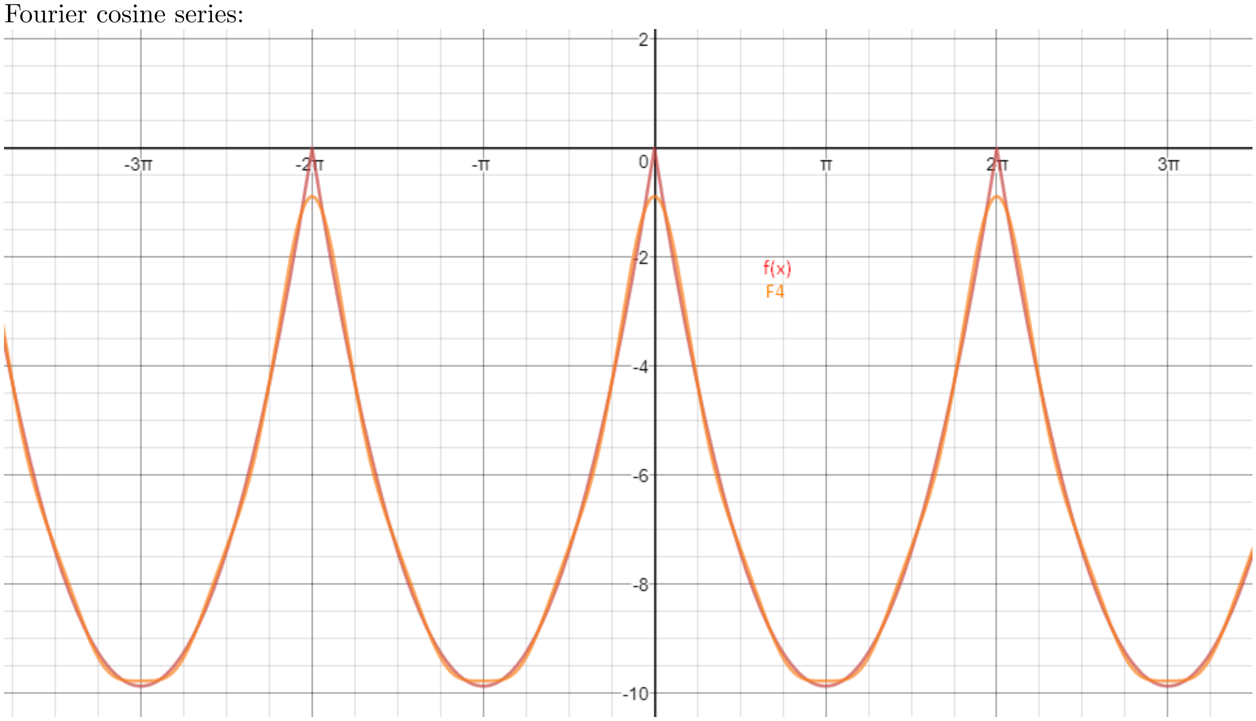
$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-2\pi}^{2\pi} f(x)dx \\
&= 0 \quad [\text{Since } f \text{ is defined odd}]
\end{aligned}
\qquad
\begin{aligned}
a_k &= \frac{1}{2\pi}\int_{-2\pi}^{2\pi} f(x)\cos\left(\frac{kx}{2}\right)dx \\
&= 0 \quad [\text{Since } f \text{ is defined odd}]
\end{aligned}$$

The Fourier sine series is thusly

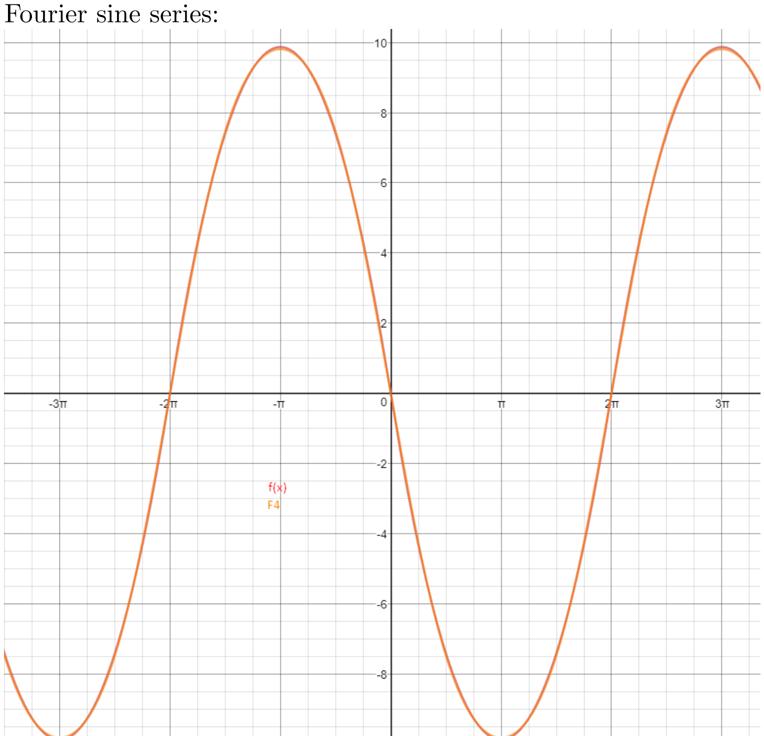
$$F(x) = \sum_{k=1}^{\infty} \frac{16}{k^3 \pi}((-1)^k - 1)\sin\left(\frac{kx}{2}\right)$$

(c) Use symbolic algebra software to graph the 4^{th} degree Fourier polynomials from parts (a) and (b) together with the original function.

Fourier cosine series:



Fourier sine series:



6. Find the Fourier series for the following functions:

(a) $f(x) = \sin^2 x + \sin^3 x$

$$\begin{aligned}
\sin^2 x + \sin^3 x &= (1/2i)^2(e^{ix} - e^{-ix})^2 + (1/2i)^3(e^{ix} - e^{-ix})^3 \\
\text{[Binomial Theorem]} &= (-1/4)(e^{2ix} - 2(e^{(ix-ix)}) + e^{-2ix}) + (-1/8i)(e^{3ix} - 3(e^{(2ix-ix)}) + 3(e^{(ix-2ix)}) - e^{-3ix}) \\
&= (-1/4)(e^{2ix} + e^{-2ix} - 2) + (-1/8i)(e^{3ix} - e^{-3ix} - 3(e^{(ix)}) + 3(e^{(-ix)})) \\
&= (-1/2)(\cos(2x) - 2) + (-1/4)(\sin(3x) - 3\sin(x)) \\
&= 1 + \frac{3}{4}\sin(x) - \frac{1}{2}\cos(2x) - \frac{1}{4}\sin(3x)
\end{aligned}$$

(b) $f(x) = \sin^4 x$

$$\begin{aligned}
\sin^4 x &= (1/2i)^4(e^{ix} - e^{-ix})^4 \\
\text{[Binomial Theorem]} &= (1/16)(e^{4ix} - 4e^{3ix-ix} + 6e^{2ix-2ix} - 4e^{ix-3ix} + e^{4ix}) \\
&= (1/16)(6 - 4e^{2ix} - 4e^{-2ix} + e^{4ix} + e^{4ix}) \\
&= (1/8)(3 - 4\cos(2x) + \cos(4x)) \\
&= \frac{3}{8} - \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)
\end{aligned}$$

(c) $f(x) = \cos^7 x$

$$\begin{aligned}
\cos^7 x &= (1/2)^7(e^{ix} + e^{-ix})^7 \\
\text{[Binomial Theorem]} &= (1/128)(e^{7ix} + 7e^{6ix-ix} + 21e^{5ix-2ix} + 35e^{4ix-3ix} \\
&\quad + 35e^{3ix-4ix} + 21e^{2ix-5ix} + 7e^{ix-6ix} + e^{-7ix}) \\
&= (1/128)(35e^{ix} + 35e^{-ix} + 21e^{3ix} + 21e^{-3ix} + 7e^{5ix} + 7e^{-5ix} + e^{7ix} + e^{-7ix}) \\
&= (1/64)(35\cos(x) + 21\cos(3x) + 7\cos(5x) + \cos(7x)) \\
&= \frac{35}{64}\cos(x) + \frac{21}{64}\cos(3x) + \frac{7}{64}\cos(5x) + \frac{1}{64}\cos(7x))
\end{aligned}$$

(*Hint:* Recall that $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$ and $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$)

The next question is for those among you who have previously seen complex numbers. It gives another approach to Fourier series.

7. Suppose

i. $f(x)$ is a real values function of x,

ii. $f(x) = \displaystyle\sum_{n=-\infty}^{\infty} C_n e^{inx}$ on $[-\pi, \pi]$, where the C_n are complex constants, and

iii. that the term by term theorem holds true in this case

(a) Express the C_n as integrals involving f.

Multiplying by e^{-ikx} on both sides (where $k \in \mathbb{Z}$) gives the expression:

$$\begin{aligned}
e^{-ikx} f(x) &= \sum_{n=-\infty}^{\infty} C_n e^{inx} e^{-ikx} \\
\implies \int_{-\pi}^{\pi} e^{-ikx} f(x) dx &= \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} C_n e^{inx-ikx} dx \\
\int_{-\pi}^{\pi} e^{-ikx} f(x) dx &= \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} C_n e^{i(n-k)x} dx \quad \text{[Due to the term by term theorem]}
\end{aligned}$$

Now there are two cases to consider as $n \in (-\infty, \infty)$

When $n \neq k$

$$\begin{aligned}
\int_{-\pi}^{\pi} C_n e^{i(n-k)x} dx &= C_n \int_{-\pi}^{\pi} e^{i(n-k)x} dx \\
&= C_n \frac{1}{i(n-k)} \left[e^{i(n-k)x} \right]_{-\pi}^{\pi} \\
&= C_n \frac{2}{(n-k)} \frac{1}{2i} \left[e^{i(n-k)\pi} - e^{-i(n-k)\pi} \right] \\
&= C_n \frac{2}{(n-k)} \sin((n-k)\pi) = 0
\end{aligned}$$

When $n = k$

$$\begin{aligned}
\int_{-\pi}^{\pi} C_n e^{i(n-k)x} dx &= C_n \int_{-\pi}^{\pi} e^{0} dx \\
&= C_n (2\pi) dx
\end{aligned}$$

So that means

$$\begin{aligned}
\int_{-\pi}^{\pi} e^{-ikx} f(x) dx &= \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} C_n e^{i(n-k)x} dx \\
&= 2\pi C_k \\
\implies \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikx} f(x) dx &= C_k
\end{aligned}$$

(b) Find the Fourier coefficients of f in terms of the C_n.

$$\begin{aligned}
a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx \\
&= \frac{1}{\pi} \sum_{n=-\infty}^{\infty} C_n \int_{-\pi}^{\pi} e^{inx} dx \\
&= 2C_0 \quad \text{[From (a), if } n = 0\text{, integral is } 2\pi\text{, o/w 0]}
\end{aligned}$$

$$\begin{aligned}
a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx \\
&= \frac{1}{\pi} \sum_{n=-\infty}^{\infty} C_n \frac{1}{2i} \int_{-\pi}^{\pi} e^{inx} (e^{ikx} - e^{-ikx}) dx \\
&= \frac{1}{\pi} \sum_{n=-\infty}^{\infty} C_n \frac{1}{2i} \int_{-\pi}^{\pi} e^{i(n+k)x} dx - \frac{1}{2i} \int_{-\pi}^{\pi} e^{-ikx} dx
\end{aligned}$$

(c) Find the C_n in terms of the Fourier coefficients of f.