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CSCD37: Assignment #2

1. Consider the IVODE $y'' = y$ for $t \geq 0$, with initial values $y(0) = 1$ and $y'(0) = 2$.

   (a) Express this second-order ODE as an equivalent system of two first-order ODEs.
   The system of first-order ODEs is given by $y_1 = y$ and $y_2 = y'$. Then the system is $y_1' = y_2$ and $y_2' = y_1$.

   (b) What are the corresponding inital values for the system of ODEs in (a)?
   Since $y = y_1$ and $y' = y_2$ the initial values are just $y_1(0) = 1$ and $y_2(0) = 2$.

   (c) Are solutions of this system stable? Justify your answer.
   We have that $y_1' = f_1(t, y_2)$ and $y_2' = f_2(t, y_1)$, so analyzing the eigenvalues will give the stability of the system. First compute the matrix.

$$\begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

   Now to evaluate the eigenvalues of the matrix.

$$\begin{aligned} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \lambda I &= \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \\ \implies \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} &= (\lambda^2 - 1) = 0 \\ \implies \lambda &= \pm 1 \end{aligned}$$

   There is both a positive and negative eigenvalue so it is unstable in the general case, unless the interval of integration is sufficiently small. In that case, the instability won't start to effect the solution as much.

   (d) Perform one step of forward Euler method for this ODE system using a step size of $h = 0.5$.

$$\begin{aligned} y_k &\approx y(0) \\ y_{k+1} &= y_k + h_k f(t_k, y_k) \\ \implies y_{k+1} &= 1 + (0.5) f(0, 1) \\ \implies y_{k+1} &= 1 + (0.5) y' \end{aligned}$$

   (e) Is the forward Euler method stable for this problem using this step size? Justify your answer.
   The forward Euler method is not stable here since the equation $\|I + hf_y\|_2$ of the amplifcation matrix is > 1.

   (f) Is the backward Euler method stable for this problem using this step size? Justify your answer.
   The backward Euler method is stable for the problem

2. Consider the Trapezoidal Rule

$$y_{k+1} = y_k + \frac{h_k}{2}[f(t_k, y_k) + f(t_{k+1}, y_{k+1})],\ k = 0, 1, 2, \ldots \tag{1}$$

   for integrating the general IVODE $y'(t) = f(t, y(t))$, $y(t_0) = y_0$.

   (a) Show how (1) is derived by combining two appropriate Taylor expansions. What is the truncation error (local error) in (1)?
   From FEM and BEM we have.

$$y(t + h) = y(t) + hy'(t) + \frac{h^2}{2!} y''(t) + \frac{h^3}{3!} y'''(\psi_1)$$

$$y([t+h]-h) = y(t+h) - hy'(t+h) + \frac{(-h)^2}{2!}y''(t+h) + \frac{(-h)^3}{3!}y'''(\psi_2)$$

Adding them together gives

$$2y(t+h) - hy'(t+h) + \frac{(-h)^2}{2!}y''(t+h) + \frac{(-h)^3}{3!}y'''(\psi_2) = 2y(t) + hy'(t) + \frac{h^2}{2!}y''(t) + \frac{h^3}{3!}y'''(\psi_1)$$

$$2y(t+h) = 2y(t) + hy'(t) + hy'(t+h) + \frac{h^2}{2!}y''(t) - \frac{h^2}{2!}y''(t+h) + \frac{h^3}{3!}y'''(\psi_1) + \frac{h^3}{3!}y'''(\psi_2)$$

$$y(t+h) = y(t) + \frac{h}{2}[y'(t) + y'(t+h) + \frac{h}{2!}y''(t) - \frac{h}{2!}y''(t+h) + \frac{h^2}{3!}y'''(\psi_1) + \frac{h^2}{3!}y'''(\psi_2)]$$

$$y(t+h) \approx y(t) + \frac{h}{2}[y'(t) + y'(t+h) + \frac{h^2}{3!}y'''(\psi_1) + \frac{h^2}{3!}y'''(\psi_2)] \text{ Since as } h \to 0,\ t+h \approx t$$

$$\text{Where } \psi_1 \in [t,\, t+h] \,\&\, \psi_2 \in [t+h,\, t]$$

$$y(t+h) \approx y(t) + \frac{h}{2}[y'(t) + y'(t+h)] + \frac{h^3}{3!}y'''(\psi) \text{Where } \psi \in [t,\, t+h]\&$$

Cutting this equation off at the first two terms, gives the approxmiation

$$y_{k+1} = y_k + \frac{h}{2}[f(t_k, y_k) + f(t_{k+1}, y_{k+1})]$$

and this has the evident truncation error of

$$\frac{h^3}{3!}y'''(\psi)$$

(b) Derive the global error propagation formula for (1), showing how the global error at t_{k+1} is related to the global error at t_k and the local error t_k. Recall that this is a *general* IVP with $y, f : \mathbb{R}^n \to \mathbb{R}^n$.

$$y(t_{k+1}) - y_{k+1} = y'(t_k) + \frac{h}{2}[f(t_k, y(t_k)) + f(t_{k+1}, y(t_{k+1}))] - y_k - \frac{h}{2}[f(t_k, k) + f(t_{k+1}, y_{k+1})] + \frac{h^3}{3!}y'''(\psi)$$

$$\text{Using MVT to simplify}$$

$$y(t_{k+1}) - y_{k+1} = y(t_k) - y_k + \frac{h}{2}[f_y(t_k, \eta_k)(y(t_k) - y_k) + f_y(t_{k+1}, \eta_{k+1})(y(t_{k+1}) - y_{k+1})] + \frac{h^3}{3!}y'''(\psi)$$

$$[I + f_y(t_{k+1}, \eta_{k+1})](y(t_{k+1}) - y_{k+1}) = \frac{h}{2}[I + f_y(t_k, \eta_k)](y(t_k) - y_k) + \frac{h^3}{3!}y'''(\psi)$$

$$y(t_{k+1}) - y_{k+1} = \frac{h}{2}[I + f_y(t_{k+1}, \eta_{k+1})]^{-1}[I + f_y(t_k, \eta_k)](y(t_k) - y_k) + [I + f_y(t_{k+1}, \eta_{k+1})]^{-1} + \frac{h^3}{3!}y'''(\psi)$$

(c) When applied to the model problem $y'(t) = \lambda y(t), y : \mathbb{R} \to \mathbb{R}, \lambda \in \mathbb{R}$, (1) simplifies. Write out the simplified form.

$$y_{k+1} = y_k + \frac{h}{2}[f(t_k, y_k) + f(t_{k+1}, y_{k+1})]$$

$$y_{k+1} = y_k + \frac{h}{2}[\lambda y_k + \lambda y_{k+1}]$$

$$y_{k+1} - h\lambda y_{k+1} = y_k\left(1 + \frac{h}{2}\lambda\right)$$

$$y_{k+1} = y_k\left(\frac{2 + h\lambda}{2 - h\lambda}\right)$$

$$\vdots$$

$$y_k = y_0\left(\frac{2 + h\lambda}{2 - h\lambda}\right)^k$$

(d) Derive the growth factor for the simplified form.

$$y_{k+1} = y_k \left(\frac{2 + h\lambda}{2 - h\lambda} \right)$$

So the growth factor is

$$\left(\frac{2 + h\lambda}{2 - h\lambda} \right)$$

(e) Using the growth factor for the simplified form derived in (d), define and sketch the region of absolute stability for (1). *Show all of your work.*

$$\left| \frac{2 + h\lambda}{2 - h\lambda} \right| \leq 1$$
$$|2 + h\lambda| \leq |2 - h\lambda|$$

This holds for any $h\lambda \leq 0$, so the region of absolute stability is just the entirety of $\mathbb{C}^-$.

(f) Is (1) A-stable? Is it L-stable? *Justify your answer.*

From above, the region of stability is $\mathbb{C}^-$ so it must include it, meaning it is A-stable. However, it is when $h\lambda \mapsto -\infty$ the fact that there is only a sum with a constant, means the limit approaches 1, and is thus not L-stable.

(g) (1) clearly is more expensive per step than both the forward Euler method (FEM) and backward Euler method (BEM) discussed in lecture. What advantage(s), if any, does (1) have over FEM and BEM?

This iteration has a much smaller local error, with a cubed factor of the stepsize so it is much more accurate then either of the other methods, having a 2^{nd} derivative multiplied by h^2. Not only this, but it also maintains the A-stability that BEM has, so it also will work very well on stiff ODEs.