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MATB42: Assignment #8 Mar 20th 2018

1. (a) Let $f, g : \mathbb{R}^n \to \mathbb{R}$; $\boldsymbol{F}$, $\boldsymbol{G}$: $\mathbb{R}^n \to \mathbb{R}^n$; and define Δ, the *Laplacian*, by $\Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$.

Verify the following identities

(i) $\text{div}(\boldsymbol{F} + \boldsymbol{G}) = \text{div } \boldsymbol{F} + \text{ div } \boldsymbol{G}$.

$$\text{div}(\boldsymbol{F} + \boldsymbol{G}) = \sum_{i=1}^{n} \frac{\partial(F_i + G_i)}{\partial x_i} = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i} + \frac{\partial G_i}{\partial x_i} = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i} + \sum_{i=1}^{n} \frac{\partial G_i}{\partial x_i} = \text{div}\boldsymbol{F} + \text{div}\boldsymbol{G}$$

(ii) $\text{div}(f\boldsymbol{F}) = f \text{ div } \boldsymbol{F} + \boldsymbol{F} \cdot \text{ grad} f$.

$$\text{div}(f\boldsymbol{F}) = \sum_{i=1}^{n} \frac{\partial f F_i}{\partial x_i} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} F_i + \frac{\partial F_i}{\partial x_i} f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} F_i + f \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i} = \nabla f \cdot \boldsymbol{F} + f \text{div} \boldsymbol{F}$$

(iii) $\Delta(fg) = f\Delta g + g\Delta f + 2(\text{grad} f) \cdot (\text{grad} g)$.

Proof.

$$\begin{aligned}
\Delta(fg) &= \sum_{i=1}^{n} \frac{\partial^2 fg}{\partial^2 x_i} \\
&= \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left[\frac{\partial f}{\partial x_i} g + \frac{\partial g}{\partial x_i} f \right] \\
&= \sum_{i=1}^{n} \left[\frac{\partial f}{\partial^2 x_i} g + \left(\frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_i} \right) + \frac{\partial g}{\partial^2 x_i} f + \left(\frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_i} \right) \right] \\
&= g \sum_{i=1}^{n} \frac{\partial f}{\partial^2 x_i} + 2 \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_i} \right) + f \sum_{i=1}^{n} \frac{\partial g}{\partial^2 x_i} \\
&= g\Delta f + 2[\nabla f \cdot \nabla g] + f\Delta g
\end{aligned}$$

□

(b) Let $f, g : D \subset \mathbb{R}^3 \to \mathbb{R}$ be of class C^1. If R is a solid region contained in D then

$$\iiint_R \nabla f \cdot \nabla g \, dV = \iint_{\partial R} f \nabla g \cdot \boldsymbol{n} \, dS - \iiint_R f \nabla^2 g \, dV$$

($\nabla^2 g = \text{div } (\nabla g)$).

Proof.

$$\begin{aligned}
&\iiint_R \nabla f \cdot \nabla g \, dV = \iint_{\partial R} f \nabla g \cdot \boldsymbol{n} \, dS - \iiint_R f \nabla^2 g \, dV \\
\iff &\iiint_R \nabla f \cdot \nabla g \, dV + \iiint_R f \nabla^2 g \, dV = \iint_{\partial R} f \nabla g \cdot \boldsymbol{n} \, dS
\end{aligned}$$

$$\begin{aligned}
&\iint_{\partial R} f \nabla g \cdot \boldsymbol{n} \, dS \stackrel{\text{Div Thm}}{=} \iiint_R \text{div}(f \nabla g) \, dV \stackrel{\text{(ii)}}{=} \iiint_R f(\text{div} \nabla g) + \nabla g \cdot \nabla f \, dV \\
&= \iiint_R f(\text{div} \nabla g) \, dV + \iiint_R \nabla f \cdot \nabla g \, dV = \iiint_R f \nabla^2 g \, dV + \iiint_R \nabla f \cdot \nabla g \, dV
\end{aligned}$$

□

2. Use the Divergence Theorem to verify your answer to question 7 on assignment 8.

3. Let $\boldsymbol{F}(x, y, z) = (x, y^2, e^{yz})$ and let R be a cube centered at the origin with sides of length 2. Evaluate $\displaystyle\int_S$ div $\boldsymbol{F}\, dV$ directly and by using the Divergence Theorem.

Directly:

$$\begin{aligned}
\int_S \text{div}\boldsymbol{F}\, dV &= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} 1 + 2y + ye^{yz}\, dx\, dz\, dy \\
&= \int_{-1}^{1}\int_{-1}^{1} 2 + 4y + 2ye^{yz}\, dz\, dy \\
&= \int_{-1}^{1} 4 + 8y + \left[e^{yz}\right]_{-1}^{1} dy \\
&= \int_{-1}^{1} 4 + 8y + e^{y} - e^{-y}\, dy \\
&= 8 + 0 + [e^y]_{-1}^{1} + [e^{-y}]_{-1}^{1} = 8
\end{aligned}$$

Divergence:

$$\int_R \text{div}\boldsymbol{F}\, dV = \int_S \boldsymbol{F} \cdot d\boldsymbol{S}$$

where S is the union of 6 planes with $-1 \leq x, y, z \leq 1$ and having normals in the $\pm x, y, z$ directions. The parameterizations can be given by

$$\begin{aligned}
\boldsymbol{\Phi}_{S_1}(x, y) &= (x, y, 1) & \boldsymbol{\Phi}_{S_1'}(x, y) &= (x, y, -1) \\
\boldsymbol{\Phi}_{S_2}(y, z) &= (1, y, z) & \boldsymbol{\Phi}_{S_2'}(y, z) &= (-1, y, z) \\
\boldsymbol{\Phi}_{S_3}(x, z) &= (x, 1, z) & \boldsymbol{\Phi}_{S_3'}(x, z) &= (x, -1, z)
\end{aligned}$$

When plugging in S_3, S_3' into $\boldsymbol{F}$, then taking the dot with the normal, we get that the integrands are going to be the same, specifically 1. When doing the same with S_2, S_2, due to x being odd, we get both being negative the other, so integrating over the same range will cancel each other out. Putting this together, the integral over S is:

$$\begin{aligned}
\int_S \boldsymbol{F} \cdot d\boldsymbol{S} &= 2\int_{-1}^{1}\int_{-1}^{1} 1\, dx\, dz + \int_{-1}^{1}\int_{-1}^{1} e^{y}\, dx\, dy + \int_{-1}^{1}\int_{-1}^{1} -e^{-y}\, dx\, dy \\
&= 8 + 2\int_{-1}^{1} e^{y}\, dy - 2\int_{-1}^{1} e^{-y}\, dy \\
&= 8 + 2([e^{y}]_{-1}^{1} + [e^{-y}]_{-1}^{1}) = 8
\end{aligned}$$

4. Let B be the pyramid with top vertex (0,0,1) and base vertices (0,0,0), (1,0,0), (0,1,0) and (1,1,0). Let S be the 2-dim closed surface bounding B, oriented in the outward direction. Use Gauss' theorem to calculate $\int_S \boldsymbol{F} \cdot d\boldsymbol{S}$, where $\boldsymbol{F}(x, y, z) = (x^2y, 3y^2z, 9z^2x)$.

$$\int_R \mathrm{div}\boldsymbol{F}\, dV = \int_S \boldsymbol{F} \cdot d\boldsymbol{S}$$

$$\begin{aligned}
\mathrm{div}\boldsymbol{F} &= 2xy + 6yz + 18zx \\
\int_R \mathrm{div}\boldsymbol{F} &= \int_0^1 \int_0^{1-z} \int_0^{1-z} 2xy + 6yz + 18zx\, dx\, dy\, dz \\
&= \int_0^1 \int_0^{1-z} (1-z)^2 y + 6yz(1-z) + 9z(1-z)^2\, dy\, dz \\
&= \int_0^1 (1/2)(1-z)^4 + 3z(1-z)^3 + 9z(1-z)^3\, dz \\
&= \int_0^1 -\frac{23z^4}{2} + 34z^3 - 33z^2 + 10z + \frac{1}{2}\, dz \\
&= -\frac{23}{10} + \frac{34}{4} - 11 + 5 + \frac{1}{2} \\
&= \frac{7}{10}
\end{aligned}$$

5. Use the Divergence Theorem to evaluate $\int_S \boldsymbol{F} \cdot d\boldsymbol{S}$, where $\boldsymbol{F}(x, y, z) = \left(z^2 x, \frac{y^3}{3} + \tan z, x^2 z + y^2\right)$ and S is the top half of the unit sphere $x^2 + y^2 + z^2 = 1$, oriented by the unit normal which points away from the origin.

For the Divergence Theorem to apply, the integral must be over a closed, oriented outwards. To use Divergence Theorem the surface needs to be closed, so add the disk S' on the xy-plane to the surface. Then

$$\iiint_R \operatorname{div}\boldsymbol{F}\, dV - \iint_{S'} \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_S \boldsymbol{F} \cdot d\boldsymbol{S}$$

The disk can be parametrized as $\boldsymbol{\Phi}(r, \theta) = (r\cos\theta, r\sin\theta, 0)$ with $r \in [0, 1], \theta \in [0, 2\pi]$ The normal, pointing outward needs to be downward.

$$\begin{aligned}
\iint_{S'} \boldsymbol{F} \cdot d\boldsymbol{S} &= \int_0^1 \int_0^{2\pi} (*, *, (r\sin\theta)^2) \cdot (0, 0, -1)\, d\theta\, dr \\
&= -\int_0^1 \int_0^{2\pi} (r\sin\theta)^2\, d\theta\, dr \\
&= -\int_0^1 \int_0^{2\pi} r^2(1/2 - (\cos\theta)/2)\, d\theta\, dr \\
&= -\int_0^1 \pi r^2\, dr = -\frac{\pi}{3}
\end{aligned}$$

Meanwhile, the region enclosed by this, is the top half of the unit sphere, so switch to polars, restricting $\varphi \in [0, \pi/2]$ to remain above the xy-plane.

$$\operatorname{div}\boldsymbol{F} = z^2 + y^2 + x^2 = \rho$$

$$\iiint_R \boldsymbol{F}\, dV = \rho^3 \sin\theta$$

6. Let the electric field from a point source at the origin be given by $\boldsymbol{E}(\boldsymbol{x}) = \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|^3}$

   (a) What is the outward flux of $\boldsymbol{E}$ across the surface $\frac{x^2}{3} + \frac{2y^2}{5} + z^2 = 7$.

   (b) Show that the flux of $\boldsymbol{E}$ across that part of the sphere $x^2 + y^2 + z^2 = 25$ with $z \geq 3$ is equal to the flux across that part of the plane $z = 3$ with $x^2 + y^2 \leq 16$.

7. Let $f : \mathbb{R}^n \to \mathbb{R}$ be given by $f(x, y, z) = x^2 yz$ and let η be the 2-form on $\mathbb{R}^3$ given by

$$\eta = (\sin x)\, dx\, dy + (e^y + xyz)\, dx\, dz + (x^2 y^2)\, dy\, dz.$$

   (a) Compute df and $d\eta$.

   (b) Evaluate $df \wedge \eta$.