MATB42: Assignment #2
Bonus

1. Let $f(x) = \begin{cases} 0, & -\pi < x < -\frac{\pi}{2} \\ 2, & -\frac{\pi}{2} \le 2 < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} \le x < \pi \end{cases}$

   (a) Find the Fourier series of f.

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)dx$$
$$= \frac{1}{\pi} \left[\int_{-\pi}^{-\frac{\pi}{2}} 0\,dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\,dx + \int_{\frac{\pi}{2}}^{\pi} 0\,dx \right]$$
$$= \frac{1}{\pi} \left[2\pi \right]$$
$$= 2$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin(kx)dx$$
$$= \frac{2}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(kx)dx$$
$$= 0 \quad [\sin \text{ is odd}]$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(kx)dx$$
$$= \frac{2}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(kx)dx$$
$$= \frac{2}{k\pi} \left[\sin(kx) \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$
$$= \frac{2}{k\pi} \left[2\sin\left(\frac{k\pi}{2}\right) \right]$$
$$= \frac{4}{k\pi} \sin\left(\frac{k\pi}{2}\right)$$

This is 0 for even elements, and alternating between 1 and -1 for odd elements.

Therefore the Fourier polynomial (for the non-zero terms) is

$$1 + \sum_{l=1}^{\infty} \left[\frac{4(-1)^l}{(2l+1)\pi} \cos((2l+1)x) \right]$$

   (b) Determine if the Fourier series in part (a) converges. If it does converge, what are the values to which it converges (on $[-\pi, \pi]$).

   The function is continuous on its partitions (they are constant functions), so by the theorem the polynomial converges to $f(x)$ on the continuous intervals. On the discontinuities, it converges to 0 at $\frac{\pi}{2}$ and $\frac{-\pi}{2}$ from the Fundemental theorem, and to 0 at π and $-\pi$.

   (c) Use symbolic algeba software to sketch $f(x)$ and its 4th degree Fourier polynomial over the interval $[-3\pi, 3\pi]$.

2. (a) Find the Fourier series of the function $f(x)$ defined by $f(x) = \begin{cases} 0 & , -\pi \le x < 0 \\ x & , 0 \le x < \pi \end{cases}$ and extended from this with period 2π to all of $\mathbb{R}$.

If this Fourier series converges describe the function to which it converges.

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)dx + \int_{0}^{\pi} f(x)dx \right]$$

$$= \frac{1}{\pi}\left[0 + \int_{0}^{\pi} xdx \right]$$

$$= \frac{1}{\pi}\left[\frac{1}{2}\left[x^2 \right]_0^\pi \right]$$

$$= \frac{\pi}{2}$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(kx)dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0dx + \int_{0}^{\pi} x\cos(kx)dx \right]$$

Let $u = x$, $du = dx$,

$$dv = \cos(kx), \quad v = \frac{\sin(kx)}{k}$$

$$= \frac{1}{\pi}\left[\frac{1}{k}\left[x\sin(kx) \right]_0^\pi - \frac{1}{k}\int_0^\pi \sin(kx)dx \right]$$

$$= -\frac{1}{k\pi}\left[\int_0^\pi \sin(kx)dx \right]$$

$$= \frac{1}{k^2\pi}\left[\cos(kx) \right]_0^\pi$$

$$= \frac{(-1)^{-k} - 1}{k^2\pi}$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin(kx)dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0dx + \int_{0}^{\pi} x\sin(kx)dx \right]$$

Let $u = x$, $du = dx$, $dv = \sin(kx)$, $v = -\frac{1}{k}\cos(kx)$

$$= \frac{1}{\pi}\left[-\frac{1}{k}\left[x\cos(kx) \right]_0^\pi + \frac{1}{k}\int_0^\pi \cos(kx)dx \right]$$

$$= \frac{1}{k\pi}\left[-\pi\cos(k\pi) + \frac{1}{k}\left[\sin(kx) \right]_0^\pi \right]$$

$$= \frac{1}{k\pi}\left[-\pi\cos(k\pi) + 0 \right]$$

$$= \frac{(-1)^{k+1}}{k}$$

Therefore the Fourier series of f is

$$F(x) = \frac{\pi}{4} + \sum_{k=1}^{\infty}\left[\frac{(-1)^k - 1}{k^2\pi}\cos(kx) + \frac{(-1)^{k+1}}{k}\sin(kx) \right]$$

Since f is piecewise very smooth (0, x are infinitely differentiable), the series converges to f on $(-\pi, \pi)$ and on both endpoints, it converges to $\frac{\pi}{2}$.

(b) Using the series from part (a) show that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots .$$

$$F(0) = \frac{\pi}{4} + \sum_{k=1}^{\infty}\left[\frac{(-1)^k - 1}{k^2\pi} \right]$$

$$0 = \frac{\pi}{4} + \sum_{k=1}^{\infty}\left[\frac{-2}{(2k-1)^2\pi} \right]$$

$$\frac{\pi}{4} = \sum_{k=1}^{\infty}\left[\frac{2}{(2k-1)^2\pi} \right]$$

$$\frac{\pi^2}{8} = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$$

3. Find the Fourier series for the restriction of the function $f(x) = 3+3x$ to each of the following intervals, $[a, b]$. If the Fourier series converges, to what values will the series converge at the end points?

(a) $[a, b] = [-\pi, \pi]$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} 3 + 3x dx$$

$$= \frac{1}{\pi} \left[6\pi + \frac{3}{2} [x^2]_{-\pi}^{\pi} \right]$$

$$= \frac{1}{\pi} \left[6\pi + 0 \right]$$

$$= 6$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx)dx$$

$$= \frac{3}{\pi} \left[\int_{-\pi}^{\pi} \sin(kx)dx + \int_{-\pi}^{\pi} x \sin(kx)dx \right]$$

$$= \frac{6}{\pi} \left[\int_{0}^{\pi} x \sin(kx)dx \right] \quad \text{[Since } x \text{ and sin odd]}$$

Let $u = x, du = 1dx, dv = \sin(kx)dx, v = -\frac{\cos(kx)}{k}$

$$= \frac{6}{\pi} \left[-\frac{1}{k} [x \cos(kx)]_{0}^{\pi} + \frac{1}{k} \int_{0}^{\pi} \cos(kx)dx \right]$$

$$= \frac{6}{k\pi} \left[\pi(-1)^{k+1} + \frac{1}{k} [\sin(kx)]_{0}^{\pi} \right]$$

$$= \frac{6(-1)^{k+1}}{k}$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx)dx$$

$$= \frac{3}{\pi} \left[\int_{-\pi}^{\pi} \cos(kx)dx + \int_{-\pi}^{\pi} x \cos(kx)dx \right]$$

$$= \frac{6}{k\pi} [\sin(kx)]_{0}^{\pi} \quad \text{[Since } x \text{ odd and cos even]}$$

$$= 0$$

Therefore the Fourier series is defined as

$$F(x) = 6 + \sum_{k=1}^{\infty} \frac{6(-1)^{k+1}}{k} \sin(kx)$$

Linear functions are infinitely differentiable so it will converge to $f(x)$ within the interval, and coverges to 3 at the endpoints.

(b) $[a, b] = [0, 2\pi]$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)dx$$

$$= \frac{1}{\pi} \int_{0}^{2\pi} 3 + 3xdx$$

$$= \frac{1}{\pi} \left[6\pi + \frac{3}{2} \left[x^2 \right]_{0}^{2\pi} \right]$$

$$= \frac{1}{\pi} \left[6\pi + 6\pi^2 \right]$$

$$= 6(\pi + 1)$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin(kx)dx$$

$$= \frac{3}{\pi} \left[\int_{0}^{2\pi} \sin(kx)dx + \int_{0}^{2\pi} x\sin(kx)dx \right]$$

Let $u = x, du = 1dx, dv = \sin(kx)dx, v = -\frac{\cos(kx)}{k}$

$$= \frac{3}{k\pi} \left[\left[\cos(kx) \right]_{0}^{2\pi} - \left[x\cos(kx) \right]_{0}^{2\pi} + \int_{0}^{2\pi} \cos(kx)dx \right]$$

$$= \frac{3}{k\pi} \left[-2\pi + \frac{1}{k} \left[\sin(kx) \right]_{0}^{2\pi} \right]$$

$$= -\frac{6}{k^2}$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(kx)dx$$

$$= \frac{3}{\pi} \left[\int_{0}^{2\pi} \cos(kx)dx + \int_{0}^{2\pi} x\cos(kx)dx \right]$$

Let $u = x, \ du = dx, \ dv = \cos(kx), \ v = \frac{1}{k}\sin(kx)$

$$= \frac{3}{k\pi} \left[\left[\sin(kx) \right]_{0}^{2\pi} + \left[x\sin(kx) \right]_{0}^{2\pi} - \int_{0}^{2\pi} \sin(kx)dx \right]$$

$$= -\frac{3}{k^2\pi} \left[\cos(kx) \right]_{0}^{2\pi}$$

$$= 0$$

Therefore the Fourier series is defined as

$$F(x) = 6(\pi + 1) + \sum_{k=1}^{\infty} -\frac{6}{k^2} \sin(kx)$$

Linear functions are infinitely differentiable so it will converge to $f(x)$ within the interval, and coverges to $3 + 3\pi$ at the endpoints.

4. Find the Fourier series of the function $f(x)$ defined on $[0, 2\pi]$ by $f(x) = x(x - 2\pi)$ and extended from this with period 2π to all of $\mathbb{R}$. Use symbolic algebra software to graph the 4th degree Fourier polynomial together with the original function.

$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx$

$= \frac{1}{\pi}\int_0^{2\pi} x(x-2\pi)dx$

$= \frac{1}{\pi}\left[\int_0^{2\pi} x^2\sin(kx)dx - 2\pi\int_0^{2\pi} x\sin(kx)dx\right]$

Let $u = x$, $du = dx$, $dv = \sin(kx)$, $v = -\frac{1}{k}\cos(kx)$

$= \frac{1}{\pi}\left[\int_0^{2\pi} x^2\sin(kx)dx - 2\pi\left(-\frac{1}{k}\left[x\cos(kx)\right]_0^{2\pi}\right.\right.$

$\left.\left. + \frac{1}{k}\int_0^{2\pi}\cos(kx)dx\right)\right]$

$= \frac{1}{\pi}\left[\int_0^{2\pi} x^2\sin(kx)dx - 2\pi\left(-\frac{2\pi}{k}\right.\right.$

$\left.\left. + \frac{1}{k^2}\left[\sin(kx)\right]_0^{2\pi}\right)\right]$

$= \frac{1}{\pi}\left[\int_0^{2\pi} x^2\sin(kx)dx + \frac{4\pi^2}{k}\right]$

Let $u = x^2$, $du = 2xdx$,

$dv = \sin(kx)dx$, $v = -\frac{\cos(kx)}{k}$

$= \frac{1}{k\pi}\left[-\left[x^2\cos(kx)\right]_0^{2\pi} + \int_0^{2\pi} x\cos(kx)dx + 4\pi^2\right]$

Let $u = x$, $du = dx$, $dv = \sin(kx)$, $v = -\frac{1}{k}\cos(kx)$

$= \frac{1}{k\pi}\left[\frac{1}{k}\left[x\sin(kx)\right]_0^{2\pi} - \frac{1}{k}\int_0^{2\pi}\cos(kx)dx\right]$

$= \frac{1}{k\pi}\left[-\frac{1}{k^2}\left[\sin(kx)\right]_0^{2\pi}\right]$

$= 0$

$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx$

$= \frac{1}{\pi}\left[\int_0^{2\pi} x^2\cos(kx)dx - 2\pi\int_0^{2\pi} x\cos(kx)dx\right]$

Let $u = x^2$, $du = 2xdx$, $dv = \cos(kx)dx$, $v = \frac{\sin(kx)}{k}$

$= \frac{2}{\pi}\left[\frac{1}{k}\left[x^2\sin(kx)\right]_0^{2\pi} - \frac{2}{k}\int_0^{2\pi} x\sin(kx)dx\right.$

$\left. - \pi\int_0^{2\pi} x\cos(kx)dx\right]$

$= \frac{2}{\pi}\left[-\frac{2}{k}\int_0^{2\pi} x\sin(kx)dx - \pi\int_0^{2\pi} x\cos(kx)dx\right]$

Let $u = x$, $du = dx$, $dv = \sin(kx)dx$, $v = -\frac{\cos(kx)}{k}$

$= \frac{2}{\pi}\left[\frac{2}{k^2}\left[x\cos(kx)\right]_0^{2\pi} - \frac{1}{k}\int_0^{2\pi}\cos(kx)dx\right.$

$\left. - \pi\int_0^{2\pi} x\cos(kx)dx\right]$

$= \frac{2}{\pi}\left[\frac{4\pi}{k^2} - \frac{1}{k^2}\left[\sin(kx)\right]_0^{2\pi} - \pi\int_0^{2\pi} x\cos(kx)dx\right]$

Let $u = x$, $du = dx$, $dv = \cos(kx)$, $v = \frac{1}{k}\sin(kx)$

$= \frac{2}{\pi}\left[\frac{4\pi}{k^2} - \pi\left(\frac{1}{k}\left[x\sin(kx)\right]_0^{2\pi} - \frac{1}{k}\int_0^{2\pi}\sin(kx)dx\right)\right]$

$= \frac{2}{\pi}\left[\frac{4\pi}{k^2} + \frac{\pi}{k}\int_0^{2\pi}\sin(kx)dx\right]$

$= \frac{2}{\pi}\left[\frac{4\pi}{k^2} + \frac{\pi}{k^2}\left[\cos(kx)\right]_0^{2\pi}\right]$

$= \frac{8}{k^2}$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)dx$$

$$= \frac{1}{\pi} \int_{0}^{2\pi} x(x - 2\pi)dx$$

$$= \frac{1}{\pi} \left[\int_{0}^{2\pi} x^2 dx - \int_{0}^{2\pi} 2x\pi dx \right]$$

$$= \frac{1}{\pi} \left[\frac{1}{3} \left[x^3 \right]_{0}^{2\pi} - \pi \left[x^2 \right]_{0}^{2\pi} \right]$$

$$= \frac{1}{\pi} \left[\frac{8\pi^3}{3} - 4\pi^3 \right]$$

$$= -\frac{4\pi^2}{3}$$

Therefore the Fourier series of f is

$$F(x) = -\frac{2\pi^2}{3} + \sum_{k=1}^{\infty} \frac{8}{k^2} \cos(kx)$$

5. Let $f(x)$ be defined on $[0, 2\pi]$ by $f(x) = x(x - 2\pi)$.

   (a) Find the Fourier cosine series of f.

   From question 4, we can see that the function is already even, hence the fourier series of the function itself is a cosine series of f. Namely

   $$F(x) - \frac{2\pi^2}{3} + \sum_{k=1}^{\infty} \frac{8}{k^2} \cos(kx)$$

   (b) Find the Fourier sine series of f.

   To extend this as an odd function, define the f on the range $[-2\pi, 0]$ as $f(x) = -((x + 2\pi)((x + 2\pi) - 2\pi)) = -x(x + 2\pi)$. Note that this definition of f now has a period of 4π.

$$a_0 = \frac{1}{2\pi} \int_{-2\pi}^{2\pi} f(x)dx$$

$$= \frac{1}{2\pi} \int_{-2\pi}^{2\pi} f(x)dx$$

$$= \frac{1}{2\pi} \left[-\int_{-2\pi}^{0} x(x + 2\pi)dx + \int_{0}^{2\pi} x(x - 2\pi)dx \right]$$

$$= \frac{1}{2\pi} \left[-\int_{-2\pi}^{0} x^2 + 2\pi x dx - \frac{4\pi^3}{3} \right]$$

$$= -\frac{1}{2\pi} \left[\int_{-2\pi}^{0} x^2 dx + 2\pi \int_{-2\pi}^{0} x dx + \frac{4\pi^3}{3} \right]$$

$$= -\frac{1}{2\pi} \left[\frac{1}{3} \left[x^3 \right]_{-2\pi}^{0} + \pi \left[x^2 \right]_{-2\pi}^{0} + \frac{4\pi^3}{3} \right]$$

$$= -\frac{1}{2\pi} \left[\frac{8\pi^3}{3} - 4\pi^3 + \frac{4\pi^3}{3} \right]$$

(c) Use symbolic algebra software to graph the 4th degree Fourier polynomials from parts (a) and (b) together with the original function.

6. Find the Fourier series for the following functions:

   (a) $f(x) = \sin^2 x + \sin^3 x$

   (b) $f(x) = \sin^4 x$

   (c) $f(x) = \cos^7 x$

   (*Hint:* Recall that $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$ and $\sin\theta = \frac{e^{i\theta}+e^{-i\theta}}{2i}$)

   The next question is for those among you who have previously seen complex numbers. It gives another approach to Fourier series.

7. Suppose

   i. $f(x)$ is a real values function of x,

   ii. $f(x) = \sum_{n=-\infty}^{\infty} C_n einx$ on $[-\pi, \pi]$, where the C_n are complex constants, and

   iii. that the term by term theorem holds true in this case

   (a) Express the C_n as integrals involving f.

   (b) Find the Fourier coefficients of f in terms of the C_n.

   (c) Find the C_n in terms of the Fourier coefficients of f.