MATB42: Assignment #4
Bonus

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1. (For this question assume that all curves are of class C^k, some $k \geq 3$).
   A curve $\boldsymbol{\gamma} : [a,b] \to \mathbb{R}^n$ is called *regular* if $\boldsymbol{\gamma}'(t) \neq 0$ for any t. For a regular curve $\boldsymbol{\gamma}$, the vector $\boldsymbol{T} = \dfrac{\boldsymbol{\gamma}'(t)}{\|\boldsymbol{\gamma}'(t)\|}$ is called the *unit tangent vector* to the curve.

   (a) If $\boldsymbol{\gamma} : [a,b] \to \mathbb{R}^3$ is a regular curve, show that $\boldsymbol{T}'(t) \cdot \boldsymbol{T}(t) = 0$.
   (see page 235, #16(a))

$$\begin{aligned}
\|\boldsymbol{T}(t)\|^2 &= T_1^2 + T_2^2 + T_3^2 \\
\frac{d}{dt}\|\boldsymbol{T}(t)\|^2 &= 2T_1T_1' + 2T_2T_2' + T_3T_3' \\
2(\boldsymbol{T}'(t) \cdot \boldsymbol{T}(t)) = 2(T_1'T_1 + T_2'T_2 + T_3'T_3) &= \frac{d}{dt}1 = 0
\end{aligned}$$

   A curve $\boldsymbol{\gamma}(s)$ is said to be *parameterized by arclength* (or have *unit speed*) if $\|\boldsymbol{\gamma}'(s)\| = 1$. The *curvature* κ at a point $\gamma(s)$ of a unit speed curve is defined by $\kappa = \|\boldsymbol{T}'(s)\|$

   (b) (i) If $\boldsymbol{\gamma} : [a,b] \to \mathbb{R}^3$ is a unit speed curve, show that its length is $b - a$.
   The length of $\boldsymbol{\gamma}$ is $\displaystyle\int_{\boldsymbol{\gamma}} d\boldsymbol{s} = \int_a^b \|\boldsymbol{\gamma}'(t)\|\,dt$, but $\|\boldsymbol{\gamma}'(t)\|$ is 1 since $\boldsymbol{\gamma}$ has unit speed. Therefore, the integral is just $b - a$.

   (ii) Show that $\boldsymbol{\sigma}(t) = \frac{1}{\sqrt{2}}(\cos t, \sin t, t)$ is a unit speed curve and compute its curvature.
   (see page 235, #17)

$$\begin{aligned}
\frac{d}{dt}\boldsymbol{\sigma}(t) &= \frac{1}{\sqrt{2}}\left(\frac{d}{dt}\cos t, \frac{d}{dt}\sin t, \frac{d}{dt}t\right) \\
&= \frac{1}{\sqrt{2}}(-\sin t, \cos t, 1) \\
\implies \left\|\frac{d}{dt}\boldsymbol{\sigma}(t)\right\| &= \frac{\sqrt{\sin^2 t + \cos^2 t + 1}}{\sqrt{2}} \\
&= \frac{\sqrt{2}}{\sqrt{2}} = 1 \quad \text{So } \boldsymbol{\sigma}(t) \text{ is in fact a unit curve.}
\end{aligned}$$

   Since $\boldsymbol{\sigma}(t)$ has unit speed, $\boldsymbol{T}(t)$ is just $\boldsymbol{\sigma}'(t)$, so $\boldsymbol{T}'(t)$ is $\boldsymbol{\sigma}^{(2)}(t)$.

$$\begin{aligned}
\boldsymbol{T}'(t) &= \boldsymbol{\sigma}^{(2)}(t) \\
&= \frac{1}{\sqrt{2}}\left(\frac{d}{dt} - \sin t, \frac{d}{dt}\cos t, \frac{d}{dt}1\right) \\
&= \frac{1}{\sqrt{2}}(-\cos t, -\sin t, 0) \\
\implies \|\boldsymbol{T}'(t)\| &= \frac{1}{\sqrt{2}} = \kappa
\end{aligned}$$

   If $\boldsymbol{T}'(t) \neq 0$, $\boldsymbol{N}(t) = \dfrac{\boldsymbol{T}'(t)}{\|\boldsymbol{T}'(t)\|}$ is perpendicular to $\boldsymbol{T}'(t)$ (by part (a)); $\boldsymbol{N}$ is called the *principal normal vector*. The vector $\boldsymbol{B}$, defined by $\boldsymbol{B} = \boldsymbol{T} \times \boldsymbol{N}$, is called the *binormal vector*.

   (c) Show the following about the $\boldsymbol{T}, \boldsymbol{N}$ and $\boldsymbol{B}$ system
   (i) $\dfrac{d\boldsymbol{B}}{dt} \cdot \boldsymbol{B} = 0$ (ii) $\dfrac{d\boldsymbol{B}}{dt} \cdot \boldsymbol{T} = 0$ (iii) $\dfrac{d\boldsymbol{B}}{dt}$ is a scalar multiple of $\boldsymbol{N}$.
   (see page 235, #20)

If $\boldsymbol{\gamma}(s)$ is a unit speed curve we can define the *tortion* τ by $\frac{d\boldsymbol{B}}{ds} = -\tau \boldsymbol{N}$.

(d) Compute the torsion of $\boldsymbol{\sigma}(t) = \frac{1}{\sqrt{2}}(\cos t, \sin t)$.
(see page 235, #21(c))

2. Sketch the following vector fields including a few flow lines.

(a) $\boldsymbol{F}(x, y) = (1, x^2)$ (b) $\boldsymbol{F}(x, y) = (x^2, x)$ (c) $\boldsymbol{F}(x, y) = (y, -2x)$

(a)

$$
\begin{aligned}
\boldsymbol{\gamma}(t) &= (x(t), y(t)) \\
\boldsymbol{\gamma}'(t) &= (x'(t), y'(t)) \\
\implies \frac{\frac{dy(t)}{dt}}{\frac{dx(t)}{dt}} &= \frac{y'(t)}{x'(t)} = \frac{dy}{dx} \\
dy &= x^2,\ dx = 3 \\
\implies \frac{dy}{dx} &= x^2 \\
\implies y &= \frac{x^3}{3} + c
\end{aligned}
$$

(b) $\boldsymbol{F}(x, y) = (x^2, x)$

(c) $\boldsymbol{F}(x, y) = (y, -2x)$

3. Show that the curve $\boldsymbol{c}(t) = (t^2, 2t-1, \sqrt{t}),\ t > 0$ is a flow line of the velocity vector field $\boldsymbol{F}(x,y,z) = (y+1, 2, 1/2z)$

$$\mathbf{c}'(t) = \left(2t, 2, \frac{1}{2\sqrt{t}}\right)$$

$$\boldsymbol{F}(\boldsymbol{c}(t)) = \left(2t - 1 + 1, 2, \frac{1}{2\sqrt{t}}\right) = \left(2t, 2, \frac{1}{2\sqrt{t}}\right) = \mathbf{c}'(t)$$

Therefore, $\boldsymbol{c}$ is a flow line of $\boldsymbol{F}$.

4. Find the work done by the force field $\boldsymbol{F}(x, y, z) = (xy, yz, zx)$ in moving a particle along the twisted cubic, $\boldsymbol{\gamma}(t) = (t, t^2, t^3)$, from $t = 0$ to $t = 1$.

$$\begin{aligned}
\int_{\gamma} \boldsymbol{F} \cdot ds &= \int_0^1 \boldsymbol{F}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t)\ dt \\
&= \int_0^1 (t)(t^2)(1) + (t^2)(t^3)(2t) + (t^3)(t)(3t^2) dt \\
&= \int_0^1 t^3 + 2t^6 + 3t^6 dt \\
&= \int_0^1 t^3 + 5t^6 \\
&= \frac{1}{4}\left[t^4\right]_0^1 + \frac{5}{7}\left[t^7\right]_0^1 \\
&= \frac{1}{4} + \frac{5}{7} = \frac{27}{28}
\end{aligned}$$

5. Evaluate each of the following integrals:

(a) $\displaystyle\int_\gamma xy\ dx + y^2 dy, \quad \gamma(t) = (\cos t, \sin t), 0 \leq t \leq \frac{\pi}{2}.$

$$\begin{aligned}\int_\gamma \omega \cdot ds &= \int_0^{\frac{\pi}{2}} \sin t \cos t(-\sin t) + \sin^2 t \cos t\ dt \\ &= 0\end{aligned}$$

(b) $\displaystyle\int_\gamma \boldsymbol{F} \cdot d\boldsymbol{s}, \quad \boldsymbol{F}(x,y,z) = (y,z,x), \quad \boldsymbol{\gamma}(t) = \left(t, -2t^2, \frac{1}{3}t^3\right), \quad 0 \leq t \leq 1.$

$$\begin{aligned}\int_\gamma \boldsymbol{F} \cdot ds &= \int_0^1 \boldsymbol{F}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t)\ dt \\ &= \int_0^1 (-2t^2)(1) + (\frac{1}{3}t^3)(-4t) + (t)(t^2)dt \\ &= \int_0^1 -2t^2 - \frac{4}{3}t^4 + t^3 dt \\ &= -\frac{2}{3}\left[t^3\right]_0^1 - \frac{4}{15}\left[t^5\right]_0^1 + \frac{1}{4}\left[t^4\right]_0^1 \\ &= -\frac{10}{15} - \frac{4}{15} + \frac{1}{4} = -\frac{56}{60} + \frac{15}{60} = \frac{41}{60}\end{aligned}$$

(c) $\displaystyle\int_\gamma z\ dx - xyz\ dy + 2x^2\ dz, \quad \boldsymbol{\gamma}$ is the parabola $z = x^2, y = 0$, from (-1,0,1) to (1,0,1).

Can parameterize $\boldsymbol{\gamma}$ by $\boldsymbol{\gamma}(t) = (t, 0, t^2)$, $-1 \leq t \leq 1$, as on the parabola y is constant 0, x goes from $-1 \to 1$ and z goes from $1 \to 0 \to 1$.

$$\begin{aligned}\int_\gamma \omega \cdot ds &= \int_{-1}^1 (t^2)(1) - (t)(0)(t^2)(0) + 2(t)^2(2t)\ dt \\ &= \int_{-1}^1 t^2 + 4t^3\ dt \\ &= \frac{2}{3}\left[t^3\right]_0^1 \quad \text{Exploiting even/odd} \\ &= \frac{2}{3}\end{aligned}$$

(d) $\displaystyle\int_\gamma \boldsymbol{F} \cdot d\boldsymbol{s}, \quad \boldsymbol{F}(x,y,z) = (2xy, x^2 + e^z, ye^z), \quad \boldsymbol{\gamma}$ consists of straight line segments joining, in order, the points (1,1,0), (2,0,5) and (0,3,0).

Note: By inspection $g = x^2y + ye^z$ is a potential function for $\boldsymbol{F}$. Also, straight line segments, being linear functions are smooth. Furthermore, $F(x,y,z)$ is smooth since polynomials and exponential functions are each smooth. Therefore, GFTC applies, and $\displaystyle\int_\gamma \boldsymbol{F} \cdot d\boldsymbol{s} = g(1,1,0) - g(0,3,0) = ((1)^2(1) + (1)e^{(0)}) - ((0)^2(3) + (3)e^{(0)}) = 2 - 3 = -1.$

6. (a) Let $\boldsymbol{F}(x,y) = (y,-x)$. Find $\displaystyle\int_\gamma \boldsymbol{F}\cdot d\boldsymbol{s}$ from (1,0) to (0,-1) along

   (i) the straight line segment joining these points
   Parameterize the path as $t \mapsto (1-t,-t)$ where $0 \le t \le 1$.

$$\begin{aligned}\int_\gamma \boldsymbol{F}\cdot ds &= \int_0^1 \boldsymbol{F}(\boldsymbol{\gamma}(t))\cdot\boldsymbol{\gamma}'(t)\ dt\\ &= \int_0^1 (-t)(-1)+(t)(-1)\ dt\\ &= \int_0^1 t-t\ dt = 0\end{aligned}$$

   (ii) three-quarters of the unit circle centered at the origin traced in the counter-clockwise direction.
   Parameterize the path as $t \mapsto (\sin -t, \cos -t) = (-\sin t, \cos t)$ where $0 \le t \le \frac{3\pi}{2}$. Using $-t$ since it is counter-clockwise

$$\begin{aligned}\int_\gamma \boldsymbol{F}\cdot ds &= \int_0^{\frac{3\pi}{2}} \boldsymbol{F}(\boldsymbol{\gamma}(t))\cdot\boldsymbol{\gamma}'(t)\ dt\\ &= \int_0^{\frac{3\pi}{2}} (\cos t)(-\cos t)-(-\sin t)(-\sin t)\ dt\\ &= \int_0^{\frac{3\pi}{2}} -1\ dt = -\frac{3\pi}{2}\end{aligned}$$

(b) Can your answers for part (a) help you determine if the 1-form $\omega = y\ dx - x\ dy$ is exact? Explain.
Yes, we can determine that it is not exact. If ω were to be exact then $\boldsymbol{F}$ would be conservative implying that the line integral would be independant of path. Since the integrals are different, this is evidently not the case.

7. Let $\boldsymbol{c}$ be the curve obtained by intersecting the cylinder $y^2+z^2=4$ and the surface $x=yz$ in $\mathbb{R}^3$.

(a) Give a parametrization, $\boldsymbol{\gamma}(t)$, of the curve $\boldsymbol{c}$.

The cylinder simply describes a circle of radius 2 in 2 dimensions, so y and z can be parameterized as $t \mapsto (2\sin t, 2\cos t)$. To add the additional constraint of the surface, just check what x is, given the y and z. $x=(2\sin t)(2\cos t)=4\sin t\cos t$.

Given these conditions, $\boldsymbol{\gamma}(t)$ is given by $(2\sin t, 2\cos t, 4\sin t\cos t)$, $0\le t\le 2\pi$.

(b) Evaluate $\displaystyle\int_{\boldsymbol{c}} \boldsymbol{F}\cdot d\boldsymbol{s}$, where $\boldsymbol{F}(x,y,z)=(2xy,4y,x^2)$.

$$
\begin{aligned}
\int_{\boldsymbol{\gamma}} \boldsymbol{F}\cdot ds &= \int_0^{2\pi} \boldsymbol{F}(\boldsymbol{\gamma}(t))\cdot\boldsymbol{\gamma}'(t)\ dt \\
&= \int_0^{2\pi} 2(2\sin t)(2\cos t)(2\cos t)+4(2\cos t)(-2\sin t)+(2\sin t)^2(4(\cos^2 t-\sin^2 t))\ dt \\
&= \int_0^{2\pi} 16\sin t\cos^2 t-16\cos t\sin t+16\sin^2 t\cos^2 t-16\sin^4 t\ dt \\
&= \int_0^{2\pi} 16(\cos^2 t-\cos t)\sin t\ dt+\int_0^{2\pi} 16\sin^2 t\cos^2 t-16\sin^4 t\ dt \\
&\text{Let } u=\cos t,\ du=-\sin t \\
&= -\int_{\cos 0}^{\cos 2\pi} 16(u^2-u)\ dt+\int_0^{2\pi} 16\sin^2 t\cos^2 t-16\sin^4 t\ dt \\
&= -\int_1^1 16(u^2-u)\ dt+\int_0^{2\pi} 16\sin^2 t\cos^2 t-16\sin^4 t\ dt \\
&= \int_0^{2\pi} 16\sin^2 t\cos^2 t-16\sin^4 t\ dt \\
&= \int_0^{2\pi} 16\Big(\frac{(1-\cos 2t)(1+\cos 2t)}{4}\Big)-16\Big(\frac{(1-\cos 2t)^2}{4}\Big)\ dt \\
&= \int_0^{2\pi} 4\Big(1-\cos^2 2t\Big)-4\Big(1-2\cos 2t+\cos^2 2t\Big)\ dt \\
&= \int_0^{2\pi} -8\cos^2 2t+8\cos 2t\ dt \\
&= -8\int_0^{2\pi}\cos^2 2t\ dt+8\int_0^{2\pi}\cos 2t\ dt \\
&= -8\int_0^{2\pi}\cos^2 2t\ dt+4\Big[\sin 2t\Big]_0^{2\pi} \\
&= -8\int_0^{2\pi}\cos^2 2t\ dt \\
&= -4\int_0^{2\pi} 1+\cos 4t\ dt \\
&= -8\pi-\int_0^{2\pi}\cos 4t\ dt \\
&= -8\pi
\end{aligned}
$$