

MATB42: Assignment #8

1. A surface  $S$  is obtained by rotation the given figure in the  $xy$ -plane about the  $z$ -axis. (The arc is part of a circle of radius 1 centered at  $(2,0)$ .)



- (a) Paratetrize  $S$  (in pieces) and compute the surface area.

We have that the upper line when rotated, can be parametrized by a restricted cone and similarly for the bottom. The top and bottom respectively can be written as

$$\Phi_1(u, \theta) = ((1-u)3 \cos \theta, (1-u)3 \sin \theta, 3u), \quad 0 \leq u \leq 1, \quad 0 \leq \theta \leq 2\pi$$

$$\Phi_2(u, \theta) = ((1+u)3 \cos \theta, (1+u)3 \sin \theta, -3u), \quad -1 \leq u \leq 0, \quad 0 \leq \theta \leq 2\pi$$

For the circular portion to the left, when rotated around, it will be the inner half of a torus, so the surface is

$$\Phi_3(\theta, \varphi) = ((2 + \cos \varphi) \cos \theta, (2 + \cos \varphi) \sin \theta, \sin \varphi)$$

since it has radius 2 from the origin, and radius 1 from the center of the tube. Also  $\theta \in [0, 2\pi]$ , but  $\varphi$  restricted to  $[\pi/2, 3\pi/2]$  for only the inner half.

$$\mathcal{A}(S) = \int_S dS = \int_{\Phi_1} dS + \int_{\Phi_2} dS + \int_{\Phi_3} dS$$

$$\phi_{1_u} = (-3 \cos \theta, -3 \sin \theta, 3)$$

$$\phi_{1_\theta} = (-(1-u)3 \sin \theta, (1-u)3 \cos \theta, 0)$$

$$\phi_{2_u} = (3 \cos \theta, 3 \sin \theta, -3)$$

$$\phi_{2_\theta} = (-(1+u)3 \sin \theta, (1+u)3 \cos \theta, 0)$$

$$\phi_{3_\theta} = (-(2 + \cos \varphi) \sin \theta, (2 + \cos \varphi) \cos \theta, 0)$$

$$\phi_{3_\varphi} = (-\sin \varphi \cos \theta, -\sin \varphi \sin \theta, \cos \varphi)$$

$$\phi_{1_u} \times \phi_{1_\theta} = (-9(1-u) \cos \theta, -9(1-u) \sin \theta, -9(1-u))$$

$$\Rightarrow \|\phi_{1_u} \times \phi_{1_\theta}\| = \sqrt{2(9^2(1-u)^2)} = \sqrt{2}(9(1-u))$$

$$\phi_{2_u} \times \phi_{2_\theta} = (9(1+u) \cos \theta, 9(1+u) \sin \theta, 9(1+u))$$

$$\Rightarrow \|\phi_{2_u} \times \phi_{2_\theta}\| = \sqrt{2(9^2(1+u)^2)} = \sqrt{2}(9(1+u))$$

$$\phi_{3_\theta} \times \phi_{3_\varphi} = (2 \cos \varphi \cos \theta + \cos \varphi^2 \cos \theta, 2 \cos \varphi \sin \theta + \cos^2 \varphi \sin \theta,$$

$$2 \sin \varphi \sin^2 \theta + \sin \varphi \cos \varphi \sin^2 \theta + 2 \sin \varphi \cos^2 \theta + \sin \varphi \cos \varphi \cos^2 \theta)$$

$$= (\cos \theta(2 \cos \varphi + \cos^2 \varphi), \sin \theta(2 \cos \varphi + \cos^2 \varphi), 2 \sin \varphi + \sin \varphi \cos \varphi)$$

$$\Rightarrow \|\phi_{3_\theta} \times \phi_{3_\varphi}\| = \sqrt{(2 \cos \varphi + \cos^2 \varphi)^2 + (2 \sin \varphi + \sin \varphi \cos \varphi)^2}$$

$$= \sqrt{(4 \cos^2 \varphi + 4 \cos^3 \varphi + \cos^4 \varphi) + (4 \sin^2 \varphi + 4 \sin \varphi^2 \cos \varphi + \sin^2 \varphi \cos^2 \varphi)}$$

$$= \sqrt{4 + 4 \cos^3 \varphi + \cos^4 \varphi + 4 \sin \varphi^2 \cos \varphi + \sin^2 \varphi \cos^2 \varphi}$$

$$= \sqrt{4 + 4 \cos^3 \varphi + \cos^2 \varphi + 4 \sin \varphi^2 \cos \varphi}$$

$$= \sqrt{4 + 4 \cos \varphi + \cos^2 \varphi}$$

$$= \sqrt{(\cos \varphi + 2)^2}$$

$$= \cos \varphi + 2$$

$$\begin{aligned}
\int_{\Phi_1} dS &= \int_0^1 \int_0^{2\pi} 9\sqrt{2} - 9\sqrt{2}u \, d\theta \, du = 9(2\pi)\sqrt{2} - 9\pi\sqrt{2} = 9\pi\sqrt{2} \\
\int_{\Phi_2} dS &= \int_{-1}^0 \int_0^{2\pi} 9\sqrt{2} + 9\sqrt{2}u \, d\theta \, du = 9\pi\sqrt{8} - 9\pi\sqrt{2} = 9\pi\sqrt{2} \\
\int_{\Phi_3} dS &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_0^{2\pi} \cos \varphi + 2 \, d\theta \, d\varphi = -8\pi + 4\pi^2 \\
\implies \mathcal{A}(S) &= 2(9\pi\sqrt{2}) - 8\pi + 4\pi^2
\end{aligned}$$

(b) Use a computer algebra system to sketch  $S$ .



2. Let  $S$  be the cone with vertex  $(2,3,3)$  and base the circle  $x^2 + y^2 = 1$  in the  $xy$ -plane.

(a) Parametrize  $S$

Starting with a base of a circle, we get  $(\cos \theta, \sin \theta, 1)$  with  $0 \leq \theta \leq 2\pi$ . To change into a cone multiply  $x$  and  $y$  by  $(1 - u)$  with  $0 \leq u \leq 1$  and finally to shift the vertex, add  $(2u, 3u, 2u)$  where  $z = 2u$  since the base equation already has a 1, so  $1 + ku \leq 3 \implies k \leq 2$ .

$$\implies \Phi(u, \theta) = ((1 - u) \cos \theta + 2u, (1 - u) \sin \theta + 3u, 1 + 2u)$$

(b) Use a computer algebra system to sketch  $S$ .



(c) Write down the integral that would give the surface area of  $S$ . (You are not expected to evaluate the integral.)

$$\begin{aligned} \phi_\theta &= (-(1 - u) \sin \theta, (1 - u) \cos \theta, 0) \\ \phi_u &= (-\cos \theta + 2, -\sin \theta + 3, 2) \\ \phi_\theta \times \phi_u &= ((2(1 - u) \cos \theta), (2(1 - u) \sin \theta), \\ &\quad (-(1 - u) \sin \theta)(-\sin \theta + 3) - ((1 - u) \cos \theta)(-\cos \theta + 2)) \\ &= ((2 - 2u) \cos \theta, (2 - 2u) \sin \theta, (1 - u) \sin^2 \theta - (3 - 3u) \sin \theta + (1 - u) \cos^2 \theta - (2 - 2u) \cos \theta) \\ &= ((2 - 2u) \cos \theta, (2 - 2u) \sin \theta, (1 - u) - (3 - 3u) \sin \theta - (2 - 2u) \cos \theta) \\ \|\phi_\theta \times \phi_u\| &= \sqrt{(2 - 2u)^2 \cos^2 \theta + (2 - 2u)^2 \sin^2 \theta + ((1 - u) - (3 - 3u) \sin \theta - (2 - 2u) \cos \theta)^2} \\ &= \sqrt{(2 - 2u)^2 + ((1 - u) - (3 - 3u) \sin \theta - (2 - 2u) \cos \theta)^2} \\ \implies \mathcal{A}(S) &= \int_0^1 \int_0^{2\pi} \sqrt{(2 - 2u)^2 + ((1 - u) - (3 - 3u) \sin \theta - (2 - 2u) \cos \theta)^2} d\theta du \end{aligned}$$

3. Let  $S$  be the self-intersecting rectangle in  $\mathbb{R}^3$  given by the implicit equation  $x^2 - y^2z = 0$ .

(a) Give a parametrization of  $S$  and use a computer algebra system to provide a sketch.

$$x^2 - y^2z = 0 \implies y^2z = x^2 \implies z = \left(\frac{x}{y}\right)^2$$

$$\Phi(x, y) = \left(x, y, \left(\frac{x}{y}\right)^2\right)$$



(b) Is your parametrization one-to-one? Explain.

Yes, if  $\Phi(x_0, y_0) = \Phi(x_1, y_1)$  then  $\Phi_1(x_0, y_0) = \Phi_2(x_1, y_1) \implies x_0 = x_1$ , and  $\Phi_2(x_0, y_0) = \Phi_2(x_1, y_1) \implies y_0 = y_1$ . This means that  $\Phi(x_0, y_0) = \Phi(x_1, y_1) \implies (x_0, y_0) = (x_1, y_1)$  so it is one to one.

(c) Find the equation of the tangent plane to  $S$  at  $\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)$ .

$$\begin{aligned}\phi_x &= \left(1, 0, \frac{2x}{y^2}\right), \quad \phi_y = \left(0, 1, \frac{-2x^2}{y^3}\right) \\ \phi_x \times \phi_y &= \left(\frac{-2x}{y^2}, \frac{2x^2}{y^3}, 1\right) \\ (\phi_x \times \phi_y)\left(\frac{1}{4}, \frac{1}{2}\right) &= \left(\frac{-1/2}{1/4}, \frac{1/8}{1/8}, 1\right) \\ &= (-2, 1, 1)\end{aligned}$$

So the tangent plane is defined by the equation

$$(-2)(x - 1/4) + (y - 1/2) + (z - 1/4) = 0 \Leftrightarrow -2x + y + z = 1/4$$

4. Let  $S$  be the surface defined by  $x^2 + y^2 = 1$  for  $0 \leq z \leq 1$  and by  $x^2 + y^2 = z^2$  for  $1 \leq z \leq 2$ .

(a) Use symbolic algebra software to sketch  $S$ .



(b) Evaluate  $\int_S \mathbf{F} \cdot d\mathbf{S}$  where  $\mathbf{F}(x, y, z) = (-y, x, z)$  and  $S$  is oriented by outward pointing normals.

$S$  can be parametrized piecewise by  $\Phi_1(u, \theta) = (\cos \theta, \sin \theta, u)$  for  $0 \leq u \leq 1$  and  $\Phi_2(u, \theta) = (u \cos \theta, u \sin \theta, u)$  for  $1 \leq u \leq 2$

The derivatives of each are

$$\phi_{1_u} = (0, 0, 1)$$

$$\phi_{1_\theta} = (-\sin \theta, \cos \theta, 0)$$

$$\phi_{2_u} = (\cos \theta, \sin \theta, 1)$$

$$\phi_{2_\theta} = (-u \sin \theta, u \cos \theta, 0)$$

So their respective normals are

$$\phi_{1_u} \times \phi_{1_\theta} = (-\cos \theta, -\sin \theta, 0)$$

$$\phi_{2_u} \times \phi_{2_\theta} = (-u \cos \theta, -u \sin \theta, u)$$

Examining the rightmost point (If projected into the  $xy$ -plane), where  $\theta = 0$ , both vectors will point towards the left as  $-\cos(0) = -1$  (since  $u > 0$ ). These normals are orientation reversing, so their integrals will need to be of the opposite sign.

$$\int_S \mathbf{F} d\mathbf{S} = \int_{\Phi_1} \mathbf{F}(\Phi_1) \cdot d\mathbf{S} + \int_{\Phi_2} \mathbf{F}(\Phi_2) \cdot d\mathbf{S}$$

$$\int_{\Phi_1} \mathbf{F}(\Phi_1) \cdot d\mathbf{S} = - \int_0^1 \int_0^{2\pi} -(\sin \theta)(-\cos \theta) + (\cos \theta)(-\sin \theta) + (u)(0) d\theta du = 0$$

$$\begin{aligned} \int_{\Phi_2} \mathbf{F}(\Phi_2) \cdot d\mathbf{S} &= - \int_0^1 \int_0^{2\pi} -(u \sin \theta)(-u \cos \theta) + (u \cos \theta)(-u \sin \theta) + (u)(u) d\theta du \\ &= - \int_0^1 \int_0^{2\pi} u^2 d\theta du = -\frac{2\pi}{3} \end{aligned}$$

5. Evaluate the (vector) surface integral  $\int_S \mathbf{F} \cdot d\mathbf{S}$  in each of the following cases.

- (a)  $\mathbf{F}(x, y, z) = (1, x, z)$ ,  $S$  is the upper hemisphere  $x^2 + y^2 + z^2 = 1$ ,  $z \geq 0$ , with  $\mathbf{n}$  pointing upward.  
A parametrization for  $S$  is given by  $\Phi(\theta, \varphi) = (\cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi)$ , where  $0 \leq \theta \leq 2\pi$  and  $0 \leq \varphi \leq \frac{\pi}{2}$

$$\begin{aligned}\phi_\theta &= (-\sin \theta \sin \varphi, \cos \theta \sin \varphi, 0) \\ \phi_\varphi &= (\cos \theta \cos \varphi, \sin \theta \cos \varphi, -\sin \varphi) \\ \phi_\theta \times \phi_\varphi &= (-\cos \theta \sin^2 \varphi, -\sin \theta \sin^2 \varphi, -\sin^2 \theta \sin \varphi \cos \varphi - \cos^2 \theta \sin \varphi \cos \varphi) \\ &= (-\cos \theta \sin^2 \varphi, -\sin \theta \sin^2 \varphi, -\sin \varphi \cos \varphi)\end{aligned}$$

Since  $\sin \varphi, \cos \varphi \geq 0$  for  $\varphi \in [0, \pi/2]$  this normal is orientation reversing, the sign needs to be flipped.

$$\begin{aligned}\int_S \mathbf{F} \cdot d\mathbf{S} &= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} (1, \cos \theta \sin \varphi, \cos \varphi) \cdot (\cos \theta \sin^2 \varphi, \sin \theta \sin^2 \varphi, \sin \varphi \cos \varphi) d\varphi d\theta \\ &= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \cos \theta \sin^2 \varphi + (\cos \theta \sin \varphi)(\sin \theta \sin^2 \varphi) + (\cos \varphi)(\sin \varphi \cos \varphi) d\varphi d\theta \\ &= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \cos \theta \sin^2 \varphi + \cos \theta \sin \theta \sin^3 \varphi + \sin \varphi \cos^2 \varphi d\varphi d\theta \\ &= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \cos \theta \sin^2 \varphi + \frac{1}{2} \sin(2\theta) \sin^3 \varphi d\varphi d\theta + \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \sin \varphi \cos^2 \varphi d\varphi d\theta \\ &= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \sin \varphi \cos^2 \varphi d\varphi d\theta \text{ [Integrate over full period of } \theta] \\ \text{Let } u &= \cos \varphi, du = -\sin \varphi \\ &= \int_0^{2\pi} -\int_1^0 u^2 d\varphi d\theta \\ &= 2\pi \int_0^1 u^2 d\varphi = \frac{2\pi}{3}\end{aligned}$$

- (b)  $\mathbf{F}(x, y, z) = (2, x, z + y)$ ,  $S$  is that part of the plane  $x + y + z = 1$  which lies in the first octant and  $\mathbf{n}$  points upward.

Parametrize  $S$  as  $\Phi(x, y) = (x, y, 1 - x - y)$  where  $x \in [0, 1]$ ,  $y \in [0, 1 - x]$

$$\phi_x = (1, 0, -1), \phi_y = (0, 1, -1), \phi_x \times \phi_y = (1, 1, 1)$$

$z$  is positive for  $\mathbf{n}$ , so it is orientation preserving.

$$\begin{aligned}\int_S \mathbf{F} \cdot d\mathbf{S} &= \int_0^1 \int_0^{1-x} (2, x, (1 - x - y) + y) \cdot (1, 1, 1) dy dx \\ &= \int_0^1 \int_0^{1-x} 2 + x + 1 - x dy dx = \int_0^1 \int_0^{1-x} 3 dy dx \\ &= \int_0^1 3(1 - x) dx = \int_0^1 3 - 3x dx \\ &= 3 - \frac{3}{2} = \frac{3}{2}\end{aligned}$$

(c) Marsden & Tromba, page 425, #22.

Let  $S$  be the part of the cone  $z^2 = x^2 + y^2$  with  $z$  between 1 and 2 oriented by the normal pointing out of the cone. Compute  $\iint_S \mathbf{F} \cdot d\mathbf{S}$ , where  $\mathbf{F}(x, y, z) = (x^2, y^2, z^2)$ .

The parametrization for  $S$  is given by  $\Phi(z, \theta) = (z \cos \theta, z \sin \theta, z)$ ,  $z \in [1, 2]$ ,  $\theta \in [0, 2\pi]$

$$\phi_z = (\cos \theta, \sin \theta, 1), \quad \phi_\theta = (-z \sin \theta, z \cos \theta, 0), \quad \phi_z \times \phi_\theta = (-z \cos \theta, -z \sin \theta, z)$$

Since  $x, y$  are negative, the normal vector points inwards, this is an orientation reversing normal, so the sign needs to be flipped.

$$\begin{aligned} \int_S \mathbf{F} \cdot d\mathbf{S} &= - \int_0^{2\pi} \int_1^2 (z^2 \cos^2 \theta, z^2 \sin^2 \theta, z^2) \cdot (-z \cos \theta, -z \sin \theta, z) dz d\theta \\ &= \int_0^{2\pi} \int_1^2 z^3 \cos^3 \theta + z^3 \sin^3 \theta - z^3 dz d\theta \\ &= (16/4 - 1/4) \int_0^{2\pi} (\cos^3 \theta + \sin^3 \theta - 1) d\theta \\ &= \frac{15}{4} \left[ \int_0^{2\pi} \cos \theta (1 - \sin^2 \theta) d\theta + \int_0^{2\pi} \sin \theta (1 - \cos^2 \theta) d\theta + \int_0^{2\pi} -1 d\theta \right] \\ &= \frac{15}{4} \left[ \int_0^0 (1 - u^2) du - \int_1^1 (1 - u^2) du + \int_0^{2\pi} -1 d\theta \right] \\ &= -\frac{15\pi}{2} \end{aligned}$$

6. Let  $S$  be the portion of the plane  $x - 2y + z = 1$  that is cut off by the coordinate planes and the plane  $x + y = 1$ . Let  $\mathbf{V}$  be the velocity field  $\mathbf{V}(x, y, z) = (y, z, x^2)$ . Find the flow across  $S$  when  $\mathbf{n}$  points upward. Explain your answer.



7. Let  $S$  be the closed surface that consists of the hemisphere  $x^2 + y^2 + z^2 = 1, z \geq 0$ , and its base  $x^2 + y^2 \leq 1, z = 0$ . let  $\mathbf{E}$  be the electric field  $\mathbf{E}(x, y, z) = (2x, 2y, 2z)$ . Directly calculate the electric flux across  $S$ .

A parametrization for the hemisphere is given by  $\Phi(\theta, \varphi) = (\cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi)$ , where  $0 \leq \theta \leq 2\pi$  and  $0 \leq \varphi \leq \frac{\pi}{2}$

$$\begin{aligned}\phi_\theta &= (-\sin \theta \sin \varphi, \cos \theta \sin \varphi, 0) \\ \phi_\varphi &= (\cos \theta \cos \varphi, \sin \theta \cos \varphi, -\sin \varphi) \\ \phi_\theta \times \phi_\varphi &= (-\cos \theta \sin^2 \varphi, -\sin \theta \sin^2 \varphi, -\sin^2 \theta \sin \varphi \cos \varphi - \cos^2 \theta \sin \varphi \cos \varphi) \\ &= (-\cos \theta \sin^2 \varphi, -\sin \theta \sin^2 \varphi, -\sin \varphi \cos \varphi)\end{aligned}$$

But this normal is pointed inward, so needs sign to be flipped.

$$\begin{aligned}\int_0^{2\pi} \int_0^{\frac{\pi}{2}} \mathbf{E}(\Phi) \cdot (\phi_\theta \times \phi_\varphi) d\varphi d\theta &= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} 2 \cos^2 \theta \sin^3 \varphi + 2 \sin^2 \theta \sin^3 \varphi + 2 \sin \varphi \cos^2 \varphi d\varphi d\theta \\ &= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} 2 \sin^3 \varphi + 2 \sin^3 \varphi + 2 \sin \varphi \cos^2 \varphi d\varphi d\theta \\ &= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} 2 \sin^3 \varphi + 2 \sin \varphi \cos^2 \varphi d\varphi d\theta \\ &= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} 2 \sin \varphi d\varphi d\theta = \int_0^{2\pi} 2 d\theta = 4\pi\end{aligned}$$

While the circle is parametrized as  $\Phi(\theta, r) = (r \cos \theta, r \sin \theta, 0)$ ,  $r \in [0, 1]$ ,  $\theta \in [0, 2\pi]$

$$\begin{aligned}\phi_\theta &= (-r \sin \theta, r \cos \theta, 0), \quad \phi_r = (\cos \theta, \sin \theta, 0) \\ \phi_\theta \times \phi_r &= (0, 0, -r)\end{aligned}$$

This points away from the upper hemisphere, so it is the correct orientation.

$$\int_0^{2\pi} \int_0^1 \mathbf{E}(\Phi) \cdot (\phi_\theta \times \phi_r) dr d\theta = \int_0^{2\pi} \int_0^1 0 dr d\theta = 0$$

So the integral is the sum which is  $4\pi$ .

## Bonus

Evaluate  $\int_S \mathbf{F} \cdot d\mathbf{S}$  where  $\mathbf{F}(x, y, z) = (x, y, z)$  and  $S$  is the part of the sphere  $x^2 + y^2 + z^2 = 1$  where  $\frac{1}{2} \leq z \leq \frac{\sqrt{3}}{2}$  and  $\vec{n}$  points inward.

A parametrization for  $S$  is given by  $\Phi(\theta, \varphi) = (\cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi)$ , where  $0 \leq \theta \leq 2\pi$  and  $\frac{\pi}{6} \leq \varphi \leq \frac{\pi}{3}$

$$\begin{aligned}\phi_\theta &= (-\sin \theta \sin \varphi, \cos \theta \sin \varphi, 0) \\ \phi_\varphi &= (\cos \theta \cos \varphi, \sin \theta \cos \varphi, -\sin \varphi) \\ \phi_\theta \times \phi_\varphi &= (-\cos \theta \sin^2 \varphi, -\sin \theta \sin^2 \varphi, -\sin^2 \theta \sin \varphi \cos \varphi - \cos^2 \theta \sin \varphi \cos \varphi) \\ &= (-\cos \theta \sin^2 \varphi, -\sin \theta \sin^2 \varphi, -\sin \varphi \cos \varphi)\end{aligned}$$

Given the parameterized ranges for  $\varphi$ ,  $\sin \varphi \cos \varphi$  is always positive, so negative that is always pointing inwards. Therefore it is orientation preserving.

$$\mathbf{F}(\Phi) \cdot (\phi_\theta \theta \phi_\varphi) = [-\sin^3 \varphi \cos^2 \theta - \sin^3 \varphi \sin^2 \theta - \sin \varphi \cos^2 \varphi]$$