MATB42: Assignment #4
Bonus

1. (For this question assume that all curves are of class C^k, some $k \geq 3$).

A curve $\boldsymbol{\gamma} : [a, b] \to \mathbb{R}^n$ is called *regular* if $\boldsymbol{\gamma}'(t) \neq 0$ for any t. For a regular curve $\boldsymbol{\gamma}$, the vector $\boldsymbol{T} = \dfrac{\boldsymbol{\gamma}'(t)}{\|\boldsymbol{\gamma}'(t)\|}$ is called the *unit tangent vector* to the curve.

(a) If $\boldsymbol{\gamma} : [a, b] \to \mathbb{R}^3$ is a regular curve, show that $\boldsymbol{T}'(t) \cdot \boldsymbol{T}(t) = 0$.
(see page 235, #16(a))

A curve $\boldsymbol{\gamma}(s)$ is said to be *parameterized by arclength* (or have *unit speed*) if $\|\boldsymbol{\gamma}'(s)\| = 1$. The *curvature* κ at a point $\gamma(s)$ of a unit speed curve is defined by $\kappa = \|\boldsymbol{T}'(s)\|$

(b) (i) If $\boldsymbol{\gamma} : [a, b] \to \mathbb{R}^3$ is a unit speed curve, show that its length is $b - a$.

The length of $\boldsymbol{\gamma}$ is $\displaystyle\int_{\gamma} d\boldsymbol{s} = \int_a^b \|\boldsymbol{\gamma}'(t)\| \, dt$, but $\|\boldsymbol{\gamma}'(t)\|$ is 1 since $\boldsymbol{\gamma}$ has unit speed. Therefore, the integral is just $b - a$.

(ii) Show that $\boldsymbol{\sigma}(t) = \frac{1}{\sqrt{2}}(\cos t, \sin t, t)$ is a unit speed curve and compute its curvature.
(see page 235, #17)

$$\frac{d}{dt}\boldsymbol{\sigma}(t) = \frac{1}{\sqrt{2}}(\frac{d}{dt}\cos t, \frac{d}{dt}\sin t, \frac{d}{dt}t)$$
$$= \frac{1}{\sqrt{2}}(-\sin t, \cos t, 1)$$
$$\implies \|\frac{d}{dt}\boldsymbol{\sigma}(t)\| = \frac{\sqrt{\sin^2 t + \cos^2 t + 1}}{\sqrt{2}}$$
$$= \frac{\sqrt{2}}{\sqrt{2}} = 1 \quad \text{So } \boldsymbol{\sigma}(t) \text{ is in fact a unit curve.}$$

Since $\boldsymbol{\sigma}(t)$ has unit speed, $\boldsymbol{T}(t)$ is just $\boldsymbol{\sigma}'(t)$, so $\boldsymbol{T}'(t)$ is $\boldsymbol{\sigma}^{(2)}(t)$.

$$\boldsymbol{T}'(t) = \boldsymbol{\sigma}^{(2)}(t)$$
$$= \frac{1}{\sqrt{2}}(\frac{d}{dt}-\sin t, \frac{d}{dt}\cos t, \frac{d}{dt}1)$$
$$= \frac{1}{\sqrt{2}}(-\cos t, -\sin t, 0)$$
$$\implies \|\boldsymbol{T}'(t)\| = \frac{1}{\sqrt{2}} = \kappa$$

If $\boldsymbol{T}'(t) \neq 0$, $\boldsymbol{N}(t) = \dfrac{\boldsymbol{T}'(t)}{\|\boldsymbol{T}'(t)\|}$ is perpendicular to $\boldsymbol{T}'(t)$ (by part (a)); $\boldsymbol{N}$ is called the *principal normal vector*. The vector $\boldsymbol{B}$, defined by $\boldsymbol{B} = \boldsymbol{T} \times \boldsymbol{N}$, is called the *binormal vector*.

(c) Show the following about the $\boldsymbol{T}, \boldsymbol{N}$ and $\boldsymbol{B}$ system

(i) $\dfrac{d\boldsymbol{B}}{dt} \cdot \boldsymbol{B} = 0$ (ii) $\dfrac{d\boldsymbol{B}}{dt} \cdot \boldsymbol{T} = 0$ (iii) $\dfrac{d\boldsymbol{B}}{dt}$ is a scalar multiple of $\boldsymbol{N}$.
(see page 235, #20)

If $\boldsymbol{\gamma}(s)$ is a unit speed curve we can define the *tortion* τ by $\frac{d\boldsymbol{B}}{ds} = -\tau\boldsymbol{N}$.

(d) Compute the torsion of $\boldsymbol{\sigma}(t) = \frac{1}{\sqrt{2}}(\cos t, \sin t)$.
(see page 235, #21(c))

2. Sketch the following vector fields including a few flow lines.

   (a) $\boldsymbol{F}(x, y) = (1, x^2)$ (b) $\boldsymbol{F}(x, y) = (x^2, x)$ (c) $\boldsymbol{F}(x, y) = (y, -2x)$

3. Show that the curve $c(t) = (t^2, 2t - 1, \sqrt{t})$, $t > 0$ is a flow line of the velocity vector field $F(x, y, z) = (y + 1, 2, 1/2z)$

$$c'(t) = \left(2t, 2, \frac{1}{2\sqrt{t}}\right)$$

$$F(c(t)) = \left(2t - 1 + 1, 2, \frac{1}{2\sqrt{t}}\right) = \left(2t, 2, \frac{1}{2\sqrt{t}}\right) = c'(t)$$

Therefore, c is a flow line of F.

4. Find the work done by the force field $\boldsymbol{F}(x, y, z) = (xy, yz, zx)$ in moving a particle along the twisted cubic, $\boldsymbol{\gamma}(t) = (t, t^2, t^3)$, from $t = 0$ to $t = 1$.

$$\int_\gamma \boldsymbol{F} \cdot ds = \int_0^1 \boldsymbol{F}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t) \ dt$$

$$= \int_0^1 (t)(t^2)(1) + (t^2)(t^3)(2t) + (t^3)(t)(3t^2) dt$$

$$= \int_0^1 t^3 + 2t^6 + 3t^6 dt$$

$$= \int_0^1 t^3 + 5t^6$$

$$= \frac{1}{4}\left[t^4\right]_0^1 + \frac{5}{7}\left[t^7\right]_0^1$$

$$= \frac{1}{4} + \frac{5}{7} = \frac{27}{28}$$

5. Evaluate each of the following integrals:

(a) $\int_\gamma xy\ dx + y^2 dy, \quad \gamma(t) = (\cos t, \sin t), 0 \le t \le \frac{\pi}{2}.$

$$\int_\gamma \omega \cdot ds = \int_0^{\frac{\pi}{2}} \sin t \cos t(-\sin t) + \sin^2 t \cos t\ dt$$
$$= 0$$

(b) $\int_\gamma F \cdot ds, \quad F(x,y,z) = (y,z,x), \quad \gamma(t) = \left(t, -2t^2, \frac{1}{3}t^3\right), \quad 0 \le t \le 1.$

$$\int_\gamma F \cdot ds = \int_0^1 F(\gamma(t)) \cdot \gamma'(t)\ dt$$
$$= \int_0^1 (-2t^2)(1) + (\frac{1}{3}t^3)(-4t) + (t)(t^2)dt$$
$$= \int_0^1 -2t^2 - \frac{4}{3}t^4 + t^3 dt$$
$$= -\frac{2}{3}\left[t^3\right]_0^1 - \frac{4}{15}\left[t^5\right]_0^1 + \frac{1}{4}\left[t^4\right]_0^1$$
$$= -\frac{10}{15} - \frac{4}{15} + \frac{1}{4} = -\frac{56}{60} + \frac{15}{60} = \frac{41}{60}$$

(c) $\int_\gamma z\ dx - xyz\ dy + 2x^2\ dz, \quad \gamma$ is the parabola $z = x^2, y = 0$, from (-1,0,1) to (1,0,1).

Can parameterize γ by $\gamma(t) = (t, 0, t^2)$, $-1 \le t \le 1$, as on the parabola y is constant 0, x goes from $-1 \to 1$ and z goes from $1 \to 0 \to 1$.

$$\int_\gamma \omega \cdot ds = \int_{-1}^1 (t^2)(1) - (t)(0)(t^2)(0) + 2(t)^2(2t)\ dt$$
$$= \int_{-1}^1 t^2 + 4t^3\ dt$$
$$= \frac{2}{3}\left[t^3\right]_0^1 \quad \text{Exploiting even/odd}$$
$$= \frac{2}{3}$$

(d) $\int_\gamma F \cdot ds, \quad F(x,y,z) = (2xy, x^2 + e^z, ye^z), \quad \gamma$ consists of straight line segments joining, in order, the points (1,1,0), (2,0,5) and (0,3,0).

Note: By inspection $g = x^2 y + ye^z$ is a potential function for F. Also, straight line segments, being linear functions are smooth. Furthermore, $F(x,y,z)$ is smooth since polynomials and exponential functions are each smooth. Therefore, GFTC applies, and $\int_\gamma F \cdot ds = g(1,1,0) - g(0,3,0) =$

$((1)^2(1) + (1)e^{(0)}) - ((0)^2(3) + (3)e^{(0)})) = 2 - 3 = -1.$

6. (a) Let $\boldsymbol{F}(x, y) = (y, -x)$. Find $\displaystyle\int_\gamma \boldsymbol{F} \cdot d\boldsymbol{s}$ from (1,0) to (0,-1) along

   (i) the straight line segment joining these points
   Parameterize the path as $t \mapsto (1 - t, -t)$ where $0 \le t \le 1$.

$$\int_\gamma \boldsymbol{F} \cdot d\boldsymbol{s} = \int_0^1 \boldsymbol{F}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t) \ dt$$

$$= \int_0^1 (-t)(-1) + (t)(-1) \ dt$$

$$= \int_0^1 t - t \ dt = 0$$

   (ii) three-quarters of the unit circle centered at the origin traced in the counter-clockwise direction.
   Parameterize the path as $t \mapsto (\sin -t, \cos -t) = (-\sin t, \cos t)$ where $0 \le t \le \frac{3\pi}{2}$. Using $-t$ since it is counter-clockwise

$$\int_\gamma \boldsymbol{F} \cdot d\boldsymbol{s} = \int_0^{\frac{3\pi}{2}} \boldsymbol{F}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t) \ dt$$

$$= \int_0^{\frac{3\pi}{2}} (\cos t)(-\cos t) - (-\sin t)(-\sin t) \ dt$$

$$= \int_0^{\frac{3\pi}{2}} -1 \ dt = -\frac{3\pi}{2}$$

   (b) Can your answers for part (a) help you determine if the 1-form $\omega = y \ dx - x \ dy$ is exact? Explain.
   Yes, we can determine that it is not exact. If ω were to be exact then $\boldsymbol{F}$ would be conservative implying that the line integral would be independant of path. Since the integrals are different, this is evidently not the case.

7. Let c be the curve obtained by intersecting the cylinder $y^2 + z^2 = 4$ and the surface $x = yz$ in $\mathbb{R}^3$.

(a) Give a parametrization, $\gamma(t)$, of the curve c.

The cylinder simply describes a circle of radius 2 in 2 dimensions, so y and z can be parameterized as $t \mapsto (2\sin t, 2\cos t)$. To add the additional constraint of the surface, just check what x is, given the y and z. $x = (2\sin t)(2\cos t) = 4\sin t \cos t$.

Given these conditions, $\gamma(t)$ is given by $(2\sin t, 2\cos t, 4\sin t \cos t)$, $0 \le t \le 2\pi$.

(b) Evaluate $\displaystyle\int_c \boldsymbol{F} \cdot d\boldsymbol{s}$, where $\boldsymbol{F}(x, y, z) = (2xy, 4y, x^2)$.

$$\int_\gamma \boldsymbol{F} \cdot d\boldsymbol{s} = \int_0^{2\pi} \boldsymbol{F}(\gamma(t)) \cdot \gamma'(t)\ dt$$

$$= \int_0^{2\pi} 2(2\sin t)(2\cos t)(2\cos t) + 4(2\cos t)(-2\sin t) + (2\sin t)^2(4(\cos^2 t - \sin^2 t))\ dt$$

$$= \int_0^{2\pi} 16\sin t \cos^2 t - 16\cos t \sin t + 16\sin^2 t \cos^2 t - 16\sin^4 t\ dt$$

$$= \int_0^{2\pi} 16(\cos^2 t - \cos t)\sin t\ dt + \int_0^{2\pi} 16\sin^2 t \cos^2 t - 16\sin^4 t\ dt$$

Let $u = \cos t,\ du = -\sin t$

$$= -\int_{\cos 0}^{\cos 2\pi} 16(u^2 - u)\ dt + \int_0^{2\pi} 16\sin^2 t \cos^2 t - 16\sin^4 t\ dt$$

$$= -\int_1^1 16(u^2 - u)\ dt + \int_0^{2\pi} 16\sin^2 t \cos^2 t - 16\sin^4 t\ dt$$

$$= \int_0^{2\pi} 16\sin^2 t \cos^2 t - 16\sin^4 t\ dt$$

$$= \int_0^{2\pi} 16\left(\frac{(1 - \cos 2t)(1 + \cos 2t)}{4}\right) - 16\left(\frac{(1 - \cos 2t)^2}{4}\right) dt$$

$$= \int_0^{2\pi} 4\left(1 - \cos^2 2t\right) - 4\left(1 - 2\cos 2t + \cos^2 2t\right) dt$$

$$= \int_0^{2\pi} -8\cos^2 2t + 8\cos 2t\ dt$$

$$= -8\int_0^{2\pi} \cos^2 2t\ dt + 8\int_0^{2\pi} \cos 2t\ dt$$

$$= -8\int_0^{2\pi} \cos^2 2t\ dt + 4\left[\sin 2t\right]_0^{2\pi}$$

$$= -8\int_0^{2\pi} \cos^2 2t\ dt$$

$$= -4\int_0^{2\pi} 1 + \cos 4t\ dt$$

$$= -8\pi - \int_0^{2\pi} \cos 4t\ dt$$

$$= -8\pi$$