

1. A particle moving on the curve $\gamma(t) = (3t^2, -\sin t, -e^t)$ is released at time $t = \frac{1}{2}$ and flies off on a tangent. What are its coordinates at time $t = 1$.

Calculate the direction of the tangent.

$$\begin{aligned}\gamma'(t) &= (3(2t), -\cos(t), -e^t) = (6t, -\cos(t), -e^t) \\ \gamma'\left(\frac{1}{2}\right) &= \left(3, -\cos\left(\frac{1}{2}\right), -\sqrt{e}\right)\end{aligned}$$

Calculate the position of the particle at $t = \frac{1}{2}$.

$$\gamma\left(\frac{1}{2}\right) = \left(3\left(\frac{1}{4}\right), -\sin\left(\frac{1}{2}\right), -\sqrt{e}\right) = \left(\frac{3}{4}, -\sin\left(\frac{1}{2}\right), -\sqrt{e}\right)$$

This means the particle should be at position $\gamma\left(\frac{1}{2}\right) + \frac{1}{2}\gamma'\left(\frac{1}{2}\right)$ which is equal to:

$$\begin{aligned}&\left(\frac{3}{4}, -\sin\left(\frac{1}{2}\right), -\sqrt{e}\right) + \frac{1}{2}\left(3, -\cos\left(\frac{1}{2}\right), -\sqrt{e}\right) \\ &= \left(\frac{9}{4}, -\frac{2\sin(\frac{1}{2}) + \cos(\frac{1}{2})}{2}, -\frac{3\sqrt{e}}{2}\right)\end{aligned}$$

2. The displacement at time t and horizontal position on a line x of a certain violin string is given by $u = \sin(x - 6t) + \sin(x + 6t)$. Calculate the velocity of the string at $x = 1$ when $t = \frac{1}{3}$.

At time $t = \frac{1}{3}$, the displacement function is given by

$$u = \sin(x - 3) + \sin(x + 3)$$

Taking the rate of change we get

$$\frac{\partial u}{\partial x} = \cos(x - 3) + \cos(x + 3)$$

So the velocity is $\cos(-2) + \cos(4)$

3. Find paths $\gamma(t)$ which represent

(a) the straight line segment in \mathbb{R} from $(1,2,3)$ to $(4,-5,6)$.

$$\gamma(t) = \left(t + 1, 2 - \frac{7t}{3}, t + 3\right), 0 \leq t \leq 3$$

(b) the curve $\{(x, y) \in \mathbb{R}^2 | 3x^2 + 25y^2 = 4\}$

$$\gamma(t) = \left(\frac{2}{\sqrt{3}} \sin(t), \frac{2}{5} \cos(t)\right), 0 \leq t \leq 2\pi$$

(c) the curve of intersection of the cone $z = \sqrt{x^2 + y^2}$ and the sphere $x^2 + y^2 + z^2 = 2x$.

$$\gamma(t) = \left(\frac{\sin(t) + 1}{2}, \frac{\cos(t)}{2}, \frac{\sin(t) + 1}{2}\right), 0 \leq t \leq 2\pi$$

4. Find the arclength of the following curves.

(a) $\gamma(t) = \left(t, \frac{1}{\sqrt{2}}t^2, \frac{1}{3}t^3\right)$ between the origin and $\left(2, 2\sqrt{2}, \frac{8}{3}\right)$.

$$\begin{aligned}\gamma'(t) &= (1, \sqrt{2}t, t^2) \\ \|\gamma'(t)\| &= \sqrt{1 + 2t^2 + t^4} \\ &= \sqrt{(t^2 + 1)^2} \\ &= (t^2 + 1)\end{aligned}\qquad\qquad\begin{aligned}\int_{\gamma} ds &= \int_0^2 \|\gamma'(t)\| dt \\ &= \int_0^2 t^2 dt + \int_0^2 1 dt \\ &= \frac{8}{3} + 2 \\ &= 4 + \frac{2}{3}\end{aligned}$$

(b) $\gamma(t) = (\cos 3t, \sin 3t, 2t^{\frac{3}{2}})$, $0 \leq t \leq 2$.

$$\begin{aligned}\gamma'(t) &= (-3\sin 3t, 3\cos 3t, 3t^{\frac{1}{2}}) \\ \|\gamma'(t)\| &= 3\sqrt{\sin^2 3t + \cos^2 3t + t} \\ &= 3\sqrt{t+1}\end{aligned}$$

$$\begin{aligned}\int_{\gamma} ds &= \int_0^2 \|\gamma'(t)\| dt &= 2\left[u^{\frac{3}{2}}\right]_1^3 \\ &= 3\int_0^2 \sqrt{t+1} dt &= 2\sqrt{27} - 2 \\ &\text{Let } u = t + 1, \, du = dt &= 6\sqrt{3} - 2 \\ &= 3\int_1^3 \sqrt{u} du\end{aligned}$$

(c) $\gamma(t) = (t^2, \sin t - t \cos t, \cos t + t \sin t)$, $0 \leq t \leq \pi$

$$\begin{aligned}\gamma'(t) &= (2t, \cos t - \cos t + t \sin t, -\sin t + \sin t + t \cos t) \\ &= (2t, t \sin t, t \cos t) \\ \|\gamma'(t)\| &= t\sqrt{4 + \sin^2 t + \cos^2 t} \\ &= \sqrt{5}t\end{aligned}\qquad\qquad\begin{aligned}\int_{\gamma} ds &= \int_0^{\pi} \|\gamma'(t)\| dt \\ &= \sqrt{5} \int_0^{\pi} t dt \\ &= \frac{\sqrt{5}\pi^2}{2}\end{aligned}$$

5. Evaluate the path integral $\int_{\gamma} f(x, y, z) ds$, where

(a) $f(x, y, z) = x^2 - y + 3z$ and γ is a parameterization of the line segment from the origin to $(1, 2, 1)$.

$$\begin{aligned}\gamma(t) &= (t, 2t, t) \text{ from } 0 \leq t \leq 1 \\ \gamma'(t) &= (1, 2, 1) \\ \|\gamma'(t)\| &= \sqrt{6}\end{aligned}\quad \begin{aligned}\int_{\gamma} f(x, y, z) ds &= \int_0^1 f(\gamma(t)) \|\gamma'(t)\| dt \\ &= \sqrt{6} \int_0^1 t^2 - 2t + 3t dt = \sqrt{6} \int_0^1 t^2 + t dt \\ &= \sqrt{6} \left(\frac{1}{3} + \frac{1}{2} \right) \\ &= \frac{5}{\sqrt{6}}\end{aligned}$$

(b) $f(x, y, z) = xyz$ and $\gamma(t) = (-\sin 2t, \sqrt{2} \cos 2t, \sin 2t)$, $0 \leq t \leq \frac{\pi}{4}$

$$\begin{aligned}\gamma'(t) &= (-2 \cos 2t, -2\sqrt{2} \sin 2t, 2 \cos 2t) \\ \|\gamma'(t)\| &= \sqrt{4 \cos^2 2t + 8 \sin^2 2t + 4 \cos^2 2t} \\ &= \sqrt{8} = 2\sqrt{2}\end{aligned}$$

$$\begin{aligned}\int_{\gamma} f(x, y, z) ds &= \int_0^{\frac{\pi}{4}} f(\gamma(t)) \|\gamma'(t)\| dt &= -\frac{2}{3} \left[\sin(2t)^3 \right]_0^{\frac{\pi}{4}} \\ &= -2\sqrt{2} \int_0^{\frac{\pi}{4}} \sin^2 2t \cos 2t dt = &= -\frac{2}{3} \sin\left(\frac{\pi}{2}\right)^3 \\ \text{Let } u = \sin(2t), \, du = 2 \cos(2t) dt & &= -\frac{2}{3} \\ &= -2 \int_0^{\sin(\frac{\pi}{4})} u^2 dt\end{aligned}$$

(c) $f(x, y, z) = xyz$ and $\gamma(t) = (t, 2t, 3t)$, $0 \leq t \leq 2$.

$$\begin{aligned}\gamma'(t) &= (1, 2, 3) \\ \|\gamma'(t)\| &= \sqrt{1 + 4 + 9} \\ &= \sqrt{14}\end{aligned}\quad \begin{aligned}\int_{\gamma} f(x, y, z) ds &= \int_0^2 f(\gamma(t)) \|\gamma'(t)\| dt \\ &= 6\sqrt{14} \int_0^2 t^3 dt \\ &= \frac{6}{4} \sqrt{14} (2^4) \\ &= 24\sqrt{14}\end{aligned}$$

(d) $f(x, y, z) = 3x + xy + z^3$ and $\gamma(t) = (\cos 4t, \sin 4t, 3t)$, $0 \leq t \leq 2\pi$.

$$\begin{aligned}\gamma'(t) &= (-4 \sin 4t, 4 \cos 4t, 3) \\ \|\gamma'(t)\| &= \sqrt{16(\sin^2 4t + \cos^2 4t) + 9} \\ &= \sqrt{25} \\ &= 5\end{aligned}$$

$$\begin{aligned}
\int_{\gamma} f(x, y, z) ds &= \int_0^{2\pi} f(\gamma(t)) \|\gamma'(t)\| dt \\
&= 5 \int_0^{2\pi} 3 \cos 4t + \sin 4t \cos 4t + 27t^3 dt \\
&= 5 \int_0^{2\pi} (3 + \sin 4t) \cos 4t dt \\
&\quad + 5 \int_0^{2\pi} 27t^3 dt
\end{aligned}$$

$$\begin{aligned}
&\text{Let } u = 3 + \sin(4t), \quad du = 4 \cos(4t) dt \\
&= \frac{5}{4} \int_3^{3+\sin(8\pi)} (3+u) du + \frac{5(27)}{4} \left[t^4 \right]_0^{2\pi} \\
&= \frac{5}{4} \int_3^3 (3+u) du + \frac{5(27)}{4} \left[t^4 \right]_0^{2\pi} \\
&= 20(27)\pi^4 \\
&= 540\pi^4
\end{aligned}$$

(e) $f(x, y, z) = z \cos y$ and $\gamma(t) = (-2t, 3t, t)$, $0 \leq t \leq 2$.

$$\begin{aligned}
\gamma'(t) &= (-2, 3, 1) \\
\|\gamma'(t)\| &= \sqrt{4 + 9 + 1} \\
&= \sqrt{14}
\end{aligned}$$

$$\begin{aligned}
\int_{\gamma} f(x, y, z) ds &= \int_0^2 f(\gamma(t)) \|\gamma'(t)\| dt \\
&= \sqrt{14} \int_0^2 t \cos 3t dt
\end{aligned}$$

$$\begin{aligned}
&\text{Let } u = t, \quad du = dt, \quad dv = \cos 3t, \quad v = \frac{\sin 3t}{3} \\
&= \sqrt{14} \left(\left[\frac{t \sin 3t}{3} \right]_0^2 - \int_0^2 \frac{\sin 3t}{3} dt \right) \\
&= \frac{\sqrt{14}}{3} \left(2 \sin 6 + \frac{1}{3} \left[\cos 3t \right]_0^2 \right) \\
&= \frac{\sqrt{14}}{3} \left(2 \sin 6 + \frac{1}{3} [\cos 6 - 1] \right) \\
&= \frac{2\sqrt{14}}{3} \sin 6 + \frac{\sqrt{14}}{9} \cos 6 - \frac{\sqrt{14}}{9}
\end{aligned}$$

(f) $f(x, y, z) = x + z^2$ and $\gamma(t) = (t, \ln t, 2\sqrt{2t})$, $1 \leq t \leq 2$.

$$\begin{aligned}
\gamma'(t) &= \left(1, \frac{1}{t}, \sqrt{\frac{2}{t}} \right) \\
\|\gamma'(t)\| &= \sqrt{1 + \frac{2}{t} + \frac{1}{t^2}} \\
\int_{\gamma} f(x, y, z) ds &= \int_1^2 f(\gamma(t)) \|\gamma'(t)\| dt \\
&= \int_1^2 (t + 8t) \sqrt{1 + \frac{2}{t} + \frac{1}{t^2}} dt \\
&= \int_1^2 9t \sqrt{1 + \frac{2}{t} + \frac{1}{t^2}} dt \\
&= 9 \int_1^2 \sqrt{t^2 + 2t + 1} dt
\end{aligned}$$

$$\begin{aligned}
&= 9 \int_1^2 (t+1) dt \\
&= 9 \left(\left[\frac{t^2}{2} \right]_1^2 + \left[t \right]_1^2 \right) \\
&= 9 \left(\frac{3}{2} + 1 \right) \\
&= 9 \left(\frac{5}{2} \right) \\
&= \frac{45}{2}
\end{aligned}$$

6. Find the average value of the following functions over the helix $\gamma(t) = (\sin t, \cos t, t)$, $0 \leq t \leq 2\pi$.

$$\begin{aligned}\gamma'(t) &= (\cos t, -\sin t, 1) \\ \|\gamma'(t)\| &= \sqrt{\sin^2 t + \cos^2 t + 1} \\ &= \sqrt{2}\end{aligned}$$

The length of the curve γ is:

$$\begin{aligned}\int_{\gamma} ds &= \int_0^{2\pi} \|\gamma'(t)\| dt \\ &= \sqrt{2} \int_0^{2\pi} 1 dt \\ &= 2\sqrt{2}\pi\end{aligned}$$

So the average value of the function should just be: $\frac{\text{path integral}}{\text{length of } \gamma}$

(a) $f(x, y, z) = y \sin z$

$$\begin{aligned}\frac{1}{\int_0^{2\pi} \|\gamma'(t)\| dt} \int_{\gamma} f(x, y, z) ds &= \frac{1}{2\sqrt{2}\pi} \int_0^{2\pi} f(\gamma(t)) \|\gamma'(t)\| dt \\ &= \frac{1}{2\sqrt{2}\pi} \int_0^{2\pi} \cos t \sin t dt \\ \text{Let } u &= \sin(t), \quad du = \cos(t) dt \\ &= \frac{1}{2\sqrt{2}\pi} \int_0^0 u du \\ &= 0\end{aligned}$$

(b) $f(x, y, z) = x^3 + \cos^2 z$

$$\begin{aligned}\frac{1}{\int_0^{2\pi} \|\gamma'(t)\| dt} \int_{\gamma} f(x, y, z) ds &= \frac{1}{2\sqrt{2}\pi} \int_0^{2\pi} f(\gamma(t)) \|\gamma'(t)\| dt \\ &= \frac{1}{2\sqrt{2}\pi} \int_0^{2\pi} \sin^3 t + \cos^2 t dt \\ &= \frac{1}{2\sqrt{2}\pi} \left(\int_0^{2\pi} (\sin t) \frac{1 - \cos^2 t}{2} dt + \int_0^{2\pi} \frac{1 + \cos 2t}{2} dt \right) \\ \text{Let } u &= \cos(t), \quad du = -\sin(t) dt \\ &= \frac{1}{2\sqrt{2}\pi} \left(- \int_1^1 \frac{1 - u^2}{2} du + \int_0^{2\pi} \frac{1 + \cos 2t}{2} dt \right) \\ &= \frac{1}{4\sqrt{2}\pi} \left(\int_0^{2\pi} 1 + \cos 2t dt \right) \\ &= \frac{1}{4\sqrt{2}\pi} \left(2\pi + \left[\frac{\sin 2t}{2} \right]_0^{2\pi} \right) \\ &= \frac{1}{2\sqrt{2}}\end{aligned}$$