

## MATB42: Assignment #10

1. Let  $\mathbf{F}$  be a vector field on  $\mathbb{R}^3$  given by  $\mathbf{F} = (F_1, F_2, F_3)$  where  $F_1$ ,  $F_2$ , and  $F_3$  are  $C^1$ -functions from  $\mathbb{R}^3 \rightarrow \mathbb{R}$

- (a) Let  $\eta$  be the 2-form given by

$$\eta = F_3 dx dy + F_1 dy dz + F_2 dz dx$$

Show that  $d\eta = (\operatorname{div} \mathbf{F}) dx dy dz$

(page 489, #6)

$$\begin{aligned} \eta &= F_3 dx dy + F_1 dy dz + F_2 dz dx \\ d\eta &= d(F_3 dx dy + F_1 dy dz + F_2 dz dx) \\ &= (dF_3) dx dy + (dF_1) dy dz + (dF_2) dz dx \\ &= \left( \frac{\partial}{\partial x} F_3 dx + \frac{\partial}{\partial y} F_3 dy + \frac{\partial}{\partial z} F_3 dz \right) dx dy + (dF_1) dy dz + (dF_2) dz dx \\ &= \frac{\partial}{\partial z} F_3 dz dx dy + (dF_1) dy dz + (dF_2) dz dx \\ &= \frac{\partial}{\partial z} F_3 dx dy dz + \left( \frac{\partial}{\partial x} F_1 dx + \frac{\partial}{\partial y} F_1 dy + \frac{\partial}{\partial z} F_1 dz \right) dy dz + (dF_2) dz dx \\ &= \frac{\partial}{\partial z} F_3 dx dy dz + \frac{\partial}{\partial x} F_1 dx dy dz + (dF_2) dz dx \\ &= \frac{\partial}{\partial z} F_3 dx dy dz + \frac{\partial}{\partial x} F_1 dx dy dz + \left( \frac{\partial}{\partial x} F_2 dx + \frac{\partial}{\partial y} F_2 dy + \frac{\partial}{\partial z} F_2 dz \right) dz dx \\ &= \frac{\partial}{\partial z} F_3 dx dy dz + \frac{\partial}{\partial x} F_1 dx dy dz + \frac{\partial}{\partial y} F_2 dy dz dx \\ &= \frac{\partial}{\partial z} F_3 dx dy dz + \frac{\partial}{\partial x} F_1 dx dy dz + \frac{\partial}{\partial y} F_2 dx dy dz \\ &= \frac{\partial}{\partial x} F_1 + \frac{\partial}{\partial y} F_2 + \frac{\partial}{\partial z} F_3 dx dy dz = (\operatorname{div} \mathbf{F}) dx dy dz \end{aligned}$$

- (b) Show that  $dF_1 \wedge dF_2 \wedge dF_3 = (\det D\mathbf{F}) dx dy dz$

2. Let  $\omega$  be a  $k$ -form and let  $\eta$  be a  $\ell$ -form. Find  $d(d\omega \wedge \eta - \omega \wedge d\eta)$ .

3. Determine if  $\eta = y \, dx \, dy + dz \, dy \, dz - yz \, dz \, dx$  is exact. If  $\eta$  is exact find a 1-form  $\omega$  with  $d\omega = \eta$ .  
(compare with page 461, # 22)