MATB42: Assignment #10

1. Let $\boldsymbol{F}$ be a vector field on $\mathbb{R}^3$ given by $\boldsymbol{F} = (F_1, F_2, F_3)$ where F_1, F_2, and F_3 are C^1-functions from $\mathbb{R}^3 \to \mathbb{R}$

   (a) Let η be the 2-form given by

   $$\eta = F_3 \, dx \, dy + F_1 \, dy \, dz + F_2 \, dz \, dx$$

   Show that $d\eta = (\operatorname{div} \boldsymbol{F}) \, dx \, dy \, dz$
   (page 489, #6)

   $$\eta = F_3 \, dx \, dy + F_1 \, dy \, dz + F_2 \, dz \, dx$$
   $$d\eta = d(F_3 \, dx \, dy + F_1 \, dy \, dz + F_2 \, dz \, dx)$$
   $$= (dF_3) \, dx \, dy + (dF_1) \, dy \, dz + (dF_2) \, dz \, dx$$
   $$= (\frac{\partial}{\partial x} F_3 \, dx + \frac{\partial}{\partial y} F_3 \, dy + \frac{\partial}{\partial z} F_3 \, dz) \, dx \, dy + (dF_1) \, dy \, dz + (dF_2) \, dz \, dx$$
   $$= \frac{\partial}{\partial z} F_3 \, dz \, dx \, dy + (dF_1) \, dy \, dz + (dF_2) \, dz \, dx$$
   $$= \frac{\partial}{\partial z} F_3 \, dx \, dy \, dz + (\frac{\partial}{\partial x} F_1 \, dx + \frac{\partial}{\partial y} F_1 \, dy + \frac{\partial}{\partial z} F_1 \, dz) \, dy \, dz + (dF_2) \, dz \, dx$$
   $$= \frac{\partial}{\partial z} F_3 \, dx \, dy \, dz + \frac{\partial}{\partial x} F_1 \, dx \, dy \, dz + (dF_2) \, dz \, dx$$
   $$= \frac{\partial}{\partial z} F_3 \, dx \, dy \, dz + \frac{\partial}{\partial x} F_1 \, dx \, dy \, dz + (\frac{\partial}{\partial x} F_2 \, dx + \frac{\partial}{\partial y} F_2 \, dy + \frac{\partial}{\partial z} F_2 \, dz) \, dz \, dx$$
   $$= \frac{\partial}{\partial z} F_3 \, dx \, dy \, dz + \frac{\partial}{\partial x} F_1 \, dx \, dy \, dz + \frac{\partial}{\partial y} F_2 \, dy \, dz \, dx$$
   $$= \frac{\partial}{\partial z} F_3 \, dx \, dy \, dz + \frac{\partial}{\partial x} F_1 \, dx \, dy \, dz + \frac{\partial}{\partial y} F_2 \, dx \, dy \, dz$$
   $$= \frac{\partial}{\partial x} F_1 + \frac{\partial}{\partial y} F_2 + \frac{\partial}{\partial z} F_3 \, dx \, dy \, dz = (\operatorname{div} \boldsymbol{F}) \, dx \, dy \, dz$$

(b) Show that $dF_1 \wedge dF_2 \wedge dF_3 = (\det \, D\boldsymbol{F}) \, dx \, dy \, dz$

$$df = \sum_{i=0}^{n} \frac{\partial f}{\partial x_i} \, dx_i$$

$$
\begin{aligned}
dF_1 \wedge dF_2 \wedge dF_3 &= (\frac{\partial F_1}{\partial x} \, dx + \frac{\partial F_1}{\partial y} \, dy + \frac{\partial F_1}{\partial z} \, dz) \wedge (\frac{\partial F_2}{\partial x} \, dx + \frac{\partial F_2}{\partial y} \, dy + \frac{\partial F_2}{\partial z} \, dz) \wedge dF_3 \\
&= (\frac{\partial F_1}{\partial x} \, dx \wedge (\frac{\partial F_2}{\partial x} \, dx + \frac{\partial F_2}{\partial y} \, dy + \frac{\partial F_2}{\partial z} \, dz) \\
&\quad + \frac{\partial F_1}{\partial y} \, dy \wedge (\frac{\partial F_2}{\partial x} \, dx + \frac{\partial F_2}{\partial y} \, dy + \frac{\partial F_2}{\partial z} \, dz) \\
&\quad + \frac{\partial F_1}{\partial z} \, dz \wedge (\frac{\partial F_2}{\partial x} \, dx + \frac{\partial F_2}{\partial y} \, dy + \frac{\partial F_2}{\partial z} \, dz)) \wedge dF_3 \\
&= ((\frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial y} \, dx \, dy + \frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial z} \, dx \, dz) \\
&\quad + (\frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial x} \, dy \, dx + \frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial z} \, dy \, dz) \\
&\quad + (\frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial x} \, dz \, dx + \frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial y} \, dz \, dy)) \wedge dF_3 \\
&= ((\frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial y} - \frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial x} \, dx \, dy) \\
&\quad + (\frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial y} \, dy \, dz) \\
&\quad + (\frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial z} \, dz \, dx)) \wedge (\frac{\partial F_3}{\partial x} \, dx + \frac{\partial F_3}{\partial y} \, dy + \frac{\partial F_3}{\partial z} \, dz) \\
&= (\frac{\partial F_3}{\partial z} (\frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial y} - \frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial x} \, dx \, dy \, dz) \\
&\quad + \frac{\partial F_3}{\partial x} (\frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial y} \, dy \, dz \, dx) \\
&\quad + \frac{\partial F_3}{\partial y} (\frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial z} \, dz \, dx \, dy)) \\
&= \frac{\partial F_3}{\partial x} (\frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial y}) \, dx \, dy \, dz \\
&\quad - \frac{\partial F_3}{\partial y} (\frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial x}) \, dx \, dy \, dz \\
&\quad + \frac{\partial F_3}{\partial z} (\frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial y} - \frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial x}) \, dx \, dy \, dz \\
&= \frac{\partial F_3}{\partial x} \begin{vmatrix} \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{vmatrix} - \frac{\partial F_3}{\partial y} \begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial z} \end{vmatrix} + \frac{\partial F_3}{\partial z} \begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{vmatrix} \, dx \, dy \, dz \\
&= \begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \\ \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial z} \end{vmatrix} \, dx \, dy \, dz
\end{aligned}
$$

2. Let ω be a k-form and let η be a ℓ-form. Find $d(d\omega \wedge \eta - \omega \wedge d\eta)$.

$$d(d\omega \wedge \eta - \omega \wedge d\eta) = d(d\omega \wedge \eta) - d(\omega \wedge d\eta)$$
$$= (d^2\omega \wedge \eta + (-1)^{k+1}(\omega \wedge d\eta)) - (d\omega \wedge d\eta + (-1)^k(\omega \wedge d^2\eta)) \quad = (-1)^{k+1}(\omega \wedge d\eta) - d\omega \wedge d\eta$$

3. Determine if $\eta = y\,dx\,dy + dz\,dy\,dz - yz\,dz\,dx$ is exact. If η is exact find a 1-form ω with $d\omega = \eta$.
   (compare with page 461, # 22)

4. Evaluate $\iint_S \omega$, where $\omega = z\,dx\,dy + x\,dy\,dz + y\,dz\,dx$ and S is the unit sphere, directly and by the Divergence Theorem.

(page 489, #12)

5. Compute $\int_S \omega$ and use symbolic algebra software to sketch S in each of the following.

(a) $\omega = xz\, dx\, dy + x^2\, dy\, dz + dy\, dz\, dx$

S is the upper hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$ with $\boldsymbol{n}$ pointing upward.

(b) $\omega = z\, dx\, dy + x\, dy\, dz + y\, dz\, dx$

S is the part of the plane $x + y + z = 1$ which lies in the first octant oriented by the unit normal which points upward.

(c) $\omega = xz\, dx\, dy + y\, dx\, dz + z^2\, dy\, dz$

S is the part of the cone $z = \sqrt{x^2 + y^2}$ between $z = 1$ and $z = 3$, oriented by the unit normal with negative z-component.

(d) $\omega = z\, dx\, dy + y\, dy\, dz$

S is the oriented surface given by the parametrization
$\Phi(u, v) = (u + v, uv^2, u^2 + v^2)$, $0 \leq u \leq 1$, $0 \leq v \leq 1$.

6. Verify Stokes' theorem by direct calculation of both sides when the surface S is the piece of the paraboloid $z = x^2 + y^2 - 4$ with $z \leq 0$, oriented by the downward pointing unit normal, and $\omega = (2y - z)\,dx + (x + y^2 - z)\,dy + (4y - 3x)\,dz$.

As part of your solution, provide a sketch showing the appropriate orientations. (For this question you may draw the sketch by hand or use symbolic algebra software.)

7. Let $\omega = yz\,dx - xz\,dy + xy\,dz$ and let $\gamma(t) = (2\cos t, 2\sin t, 4)$, $0 \leq t \leq 2\pi$.

(a) Let S be the piece of the surface $z = x^2 + y^2$ with $z \leq 4$. Use Stokes' theorem to give an integral over S which is equivalent to $\int_\gamma \omega$. Verify by directly computing both integrals.

(b) Let S' be the part of the plane $z = 4$ with $x^2 + y^2 \leq 4$. Use Stokes' theorem to give an integral over S' which is equivalent to $\int_\gamma \omega$. Verify by direct computation.

(c) Can you give another explanation as to why the integrals you get over S and S' should have the same value?

8. Let $\boldsymbol{F}(x, y, z) = (e^{z^2}, 4z - y, 8x \sin y)$. Find $\displaystyle\int_S (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{S}$ where S is the unit sphere oriented with the outward normal.

9. (a) Marsden & Tromba, page 451, # 13.

   (b) Marsden & Tromba, page 451, # 15.

   (c) Use symbolic algebra software to sketch the surfaces in parts (a) and (b).

10. (a) Let $\boldsymbol{F}$ and $\boldsymbol{G}$ be vector fields on $\mathbb{R}^3$ and let $f : \mathbb{R}^3 \to \mathbb{R}$. Verify the following identities.

    (i) $\operatorname{div}(\boldsymbol{F} \times \boldsymbol{G}) = \boldsymbol{G} \cdot \operatorname{curl}\boldsymbol{F} - \boldsymbol{F} \cdot \operatorname{curl}\boldsymbol{G}$.

    (ii) $\operatorname{curl}(f\boldsymbol{F}) = f\operatorname{curl}\boldsymbol{F} + (\operatorname{grad} f) \times \boldsymbol{F}$.

(b) Let R be a closed region in $\mathbb{R}^3$ with boundary ∂R. Prove the identity

$$\int_{\partial R} (\boldsymbol{F} \times \operatorname{curl}\boldsymbol{G}) \cdot d\boldsymbol{S} = \int_{R} (\operatorname{curl}\boldsymbol{F}) \cdot (\operatorname{curl}\boldsymbol{G}) \, dV - \int_{R} \boldsymbol{F} \cdot \operatorname{curl}(\operatorname{curl}\boldsymbol{G}) \, dV$$

(page 490, #2)