

1. (For this question assume that all curves are of class  $C^k$ , some  $k \geq 3$ ).

A curve  $\gamma : [a, b] \rightarrow \mathbb{R}^n$  is called *regular* if  $\gamma'(t) \neq 0$  for any  $t$ . For a regular curve  $\gamma$ , the vector  $\mathbf{T} = \frac{\gamma'(t)}{\|\gamma'(t)\|}$  is called the *unit tangent vector* to the curve.

- (a) If  $\gamma : [a, b] \rightarrow \mathbb{R}^3$  is a regular curve, show that  $\mathbf{T}'(t) \cdot \mathbf{T}(t) = 0$ .  
 (see page 235, #16(a))

$$\begin{aligned}\|\mathbf{T}(t)\|^2 &= T_1^2 + T_2^2 + T_3^2 = 1 \\ \frac{d}{dt}\|\mathbf{T}(t)\|^2 &= 2T_1T_1' + 2T_2T_2' + 2T_3T_3' = \frac{d}{dt}1 \\ 2(\mathbf{T}'(t) \cdot \mathbf{T}(t)) &= 2(T_1'T_1 + T_2'T_2 + T_3'T_3) = 0\end{aligned}$$

A curve  $\gamma(s)$  is said to be *parameterized by arclength* (or have *unit speed*) if  $\|\gamma'(s)\| = 1$ . The *curvature*  $\kappa$  at a point  $\gamma(s)$  of a unit speed curve is defined by  $\kappa = \|\mathbf{T}'(s)\|$

- (b) (i) If  $\gamma : [a, b] \rightarrow \mathbb{R}^3$  is a unit speed curve, show that its length is  $b - a$ .

The length of  $\gamma$  is  $\int_a^b \|\gamma'(t)\| dt$ , but  $\|\gamma'(t)\|$  is 1 since  $\gamma$  has unit speed. Therefore, the integral is just  $b - a$ .

- (ii) Show that  $\sigma(t) = \frac{1}{\sqrt{2}}(\cos t, \sin t, t)$  is a unit speed curve and compute its curvature.  
 (see page 235, #17)

$$\begin{aligned}\frac{d}{dt}\sigma(t) &= \frac{1}{\sqrt{2}}\left(\frac{d}{dt}\cos t, \frac{d}{dt}\sin t, \frac{d}{dt}t\right) \\ &= \frac{1}{\sqrt{2}}(-\sin t, \cos t, 1) \\ \implies \left\|\frac{d}{dt}\sigma(t)\right\| &= \frac{\sqrt{\sin^2 t + \cos^2 t + 1}}{\sqrt{2}} \\ &= \frac{\sqrt{2}}{\sqrt{2}} = 1 \quad \text{So } \sigma(t) \text{ is in fact a unit curve.}\end{aligned}$$

Since  $\sigma(t)$  has unit speed,  $\mathbf{T}(t)$  is just  $\sigma'(t)$ , so  $\mathbf{T}'(t)$  is  $\sigma^{(2)}(t)$ .

$$\begin{aligned}\mathbf{T}'(t) &= \sigma^{(2)}(t) \\ &= \frac{1}{\sqrt{2}}\left(\frac{d}{dt}(-\sin t), \frac{d}{dt}(\cos t), \frac{d}{dt}1\right) \\ &= \frac{1}{\sqrt{2}}(-\cos t, -\sin t, 0) \\ \implies \|\mathbf{T}'(t)\| &= \frac{1}{\sqrt{2}} = \kappa\end{aligned}$$

If  $\mathbf{T}'(t) \neq 0$ ,  $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$  is perpendicular to  $\mathbf{T}'(t)$  (by part (a));  $\mathbf{N}$  is called the *principal normal vector*. The vector  $\mathbf{B}$ , defined by  $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ , is called the *binormal vector*.

- (c) Show the following about the  $\mathbf{T}, \mathbf{N}$  and  $\mathbf{B}$  system

$$(i) \frac{d\mathbf{B}}{dt} \cdot \mathbf{B} = 0 \quad (ii) \frac{d\mathbf{B}}{dt} \cdot \mathbf{T} = 0 \quad (iii) \frac{d\mathbf{B}}{dt} \text{ is a scalar multiple of } \mathbf{N}.$$

(see page 235, #20)

- (i)  $\frac{d}{dt} \mathbf{B} \cdot \mathbf{B} = \frac{d}{dt} \|\mathbf{B}\|^2$ , but the norm of  $\mathbf{N}$  and  $\mathbf{T}$  are 1, so  $\|\mathbf{B}\|^2 = 1$  which means that  $\frac{d}{dt} 1 = 0$
- (ii) Because  $\mathbf{B}$  is the cross product of  $\mathbf{T}$  along with  $\mathbf{N}$ ,  $\mathbf{B} \cdot \mathbf{T}$  must be 0, so  $\frac{d}{dt} 0 = 0$
- (iii) From the last two parts,  $\frac{d}{dt} \mathbf{B}$  is orthogonal to both  $\mathbf{B}$  and  $\mathbf{T}$ . Since there are only 3 dimensions for  $\mathbb{R}^3$ , anything orthogonal to both of these must be parallel to each other. We know that  $\mathbf{N}$  is orthogonal to  $\mathbf{T}$  by (a), and by definition it is orthogonal to  $\mathbf{B}$  so both are parallel, i.e. they are scalar multiples.

If  $\gamma(s)$  is a unit speed curve we can define the *torsion*  $\tau$  by  $\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$ .

- (d) Compute the torsion of  $\sigma(t) = \frac{1}{\sqrt{2}}(\cos t, \sin t, t)$ .  
(see page 235, #21(c))

$$\begin{aligned}\mathbf{T}(t) &= \frac{1}{\sqrt{2}}(-\sin t, \cos t, 1) \\ \mathbf{N}(t) &= (-\cos t, -\sin t, 0) \\ \mathbf{B}(t) &= (0 + \sin t, -\cos t - 0, \sin^2 t + \cos^2 t) \\ \mathbf{B}(t) &= (\sin t, -\cos t, 1) \\ \frac{d}{dt} \mathbf{B}(t) &= (\cos t, \sin t, 0) = -\mathbf{N} \implies \tau = 1\end{aligned}$$

2. Sketch the following vector fields including a few flow lines.

$$(a) \mathbf{F}(x, y) = (1, x^2) \quad (b) \mathbf{F}(x, y) = (x^2, x) \quad (c) \mathbf{F}(x, y) = (y, -2x)$$

(a)

$$\begin{aligned} \gamma(t) &= (x(t), y(t)) \\ \gamma'(t) &= (x'(t), y'(t)) \\ \implies \frac{\frac{dy(t)}{dt}}{\frac{dx(t)}{dt}} &= \frac{y'(t)}{x'(t)} = \frac{dy}{dx} \\ dy &= x^2, \quad dx = 3 \\ \implies \frac{dy}{dx} &= x^2 \\ \implies y &= \frac{x^3}{3} + c \end{aligned}$$

(b)  $\mathbf{F}(x, y) = (x^2, x)$

$$\begin{aligned} dy &= x, \quad dx = x^2 \\ \implies \frac{dy}{dx} &= \frac{1}{x} \\ \implies y &= \ln|x| + c \quad x \neq 0 \end{aligned}$$

(c)  $\mathbf{F}(x, y) = (y, -2x)$

$$\begin{aligned} dy &= -2x, \quad dx = y \\ \implies \frac{dy}{dx} &= \frac{-2x}{y} \\ \implies y \, dy &= -2x \, dx \\ \implies \frac{y^2}{2} + x^2 &= c \end{aligned}$$

3. Show that the curve  $\mathbf{c}(t) = (t^2, 2t - 1, \sqrt{t})$ ,  $t > 0$  is a flow line of the velocity vector field  $\mathbf{F}(x, y, z) = (y + 1, 2, 1/2z)$

$$\mathbf{c}'(t) = \left(2t, 2, \frac{1}{2\sqrt{t}}\right)$$

$$\mathbf{F}(\mathbf{c}(t)) = \left(2t - 1 + 1, 2, \frac{1}{2\sqrt{t}}\right) = \left(2t, 2, \frac{1}{2\sqrt{t}}\right) = \mathbf{c}'(t)$$

Therefore,  $\mathbf{c}$  is a flow line of  $\mathbf{F}$ .

4. Find the work done by the force field  $\mathbf{F}(x, y, z) = (xy, yz, zx)$  in moving a particle along the twisted cubic,  $\boldsymbol{\gamma}(t) = (t, t^2, t^3)$ , from  $t = 0$  to  $t = 1$ .

$$\begin{aligned}\int_{\gamma} \mathbf{F} \cdot ds &= \int_0^1 \mathbf{F}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t) \, dt \\&= \int_0^1 (t)(t^2)(1) + (t^2)(t^3)(2t) + (t^3)(t)(3t^2) dt \\&= \int_0^1 t^3 + 2t^6 + 3t^6 dt \\&= \int_0^1 t^3 + 5t^6 \\&= \frac{1}{4} [t^4]_0^1 + \frac{5}{7} [t^7]_0^1 \\&= \frac{1}{4} + \frac{5}{7} = \frac{27}{28}\end{aligned}$$

5. Evaluate each of the following integrals:

(a)  $\int_{\gamma} xy \, dx + y^2 dy, \quad \gamma(t) = (\cos t, \sin t), 0 \leq t \leq \frac{\pi}{2}.$

$$\begin{aligned} \int_{\gamma} \omega \cdot ds &= \int_0^{\frac{\pi}{2}} \sin t \cos t (-\sin t) + \sin^2 t \cos t \, dt \\ &= 0 \end{aligned}$$

(b)  $\int_{\gamma} \mathbf{F} \cdot d\mathbf{s}, \quad \mathbf{F}(x, y, z) = (y, z, x), \quad \gamma(t) = \left(t, -2t^2, \frac{1}{3}t^3\right), \quad 0 \leq t \leq 1.$

$$\begin{aligned} \int_{\gamma} \mathbf{F} \cdot d\mathbf{s} &= \int_0^1 \mathbf{F}(\gamma(t)) \cdot \gamma'(t) \, dt \\ &= \int_0^1 (-2t^2)(1) + \left(\frac{1}{3}t^3\right)(-4t) + (t)(t^2) \, dt \\ &= \int_0^1 -2t^2 - \frac{4}{3}t^4 + t^3 \, dt \\ &= -\frac{2}{3} \left[t^3\right]_0^1 - \frac{4}{15} \left[t^5\right]_0^1 + \frac{1}{4} \left[t^4\right]_0^1 \\ &= -\frac{10}{15} - \frac{4}{15} + \frac{1}{4} = -\frac{56}{60} + \frac{15}{60} = -\frac{41}{60} \end{aligned}$$

(c)  $\int_{\gamma} z \, dx - xyz \, dy + 2x^2 \, dz, \quad \gamma$  is the parabola  $z = x^2, y = 0$ , from  $(-1, 0, 1)$  to  $(1, 0, 1)$ .

Can parameterize  $\gamma$  by  $\gamma(t) = (t, 0, t^2)$ ,  $-1 \leq t \leq 1$ , as on the parabola  $y$  is constant 0,  $x$  goes from  $-1 \rightarrow 1$  and  $z$  goes from  $1 \rightarrow 0 \rightarrow 1$ .

$$\begin{aligned} \int_{\gamma} \omega \cdot ds &= \int_{-1}^1 (t^2)(1) - (t)(0)(t^2)(0) + 2(t)^2(2t) \, dt \\ &= \int_{-1}^1 t^2 + 4t^3 \, dt \\ &= \frac{2}{3} \left[t^3\right]_0^1 \quad \text{Exploiting even/odd} \\ &= \frac{2}{3} \end{aligned}$$

(d)  $\int_{\gamma} \mathbf{F} \cdot d\mathbf{s}, \quad \mathbf{F}(x, y, z) = (2xy, x^2 + e^z, ye^z), \quad \gamma$  consists of straight line segments joining, in order, the points  $(1, 1, 0)$ ,  $(2, 0, 5)$  and  $(0, 3, 0)$ .

Note: By inspection  $g = x^2y + ye^z$  is a potential function for  $\mathbf{F}$ . Also, straight line segments, being linear functions are smooth. Furthermore,  $F(x, y, z)$  is smooth since polynomials and exponential functions are each smooth. Therefore, GFTC applies, and  $\int_{\gamma} \mathbf{F} \cdot d\mathbf{s} = g(0, 3, 0) - g(1, 1, 0) = ((0)^2(3) + (3)e^{(0)}) - ((1)^2(1) + (1)e^{(0)}) = 3 - 2 = 1.$

6. (a) Let  $\mathbf{F}(x, y) = (y, -x)$ . Find  $\int_{\gamma} \mathbf{F} \cdot d\mathbf{s}$  from  $(1, 0)$  to  $(0, -1)$  along

(i) the straight line segment joining these points

Parameterize the path as  $t \mapsto (1 - t, -t)$  where  $0 \leq t \leq 1$ .

$$\begin{aligned} \int_{\gamma} \mathbf{F} \cdot d\mathbf{s} &= \int_0^1 \mathbf{F}(\gamma(t)) \cdot \gamma'(t) \, dt \\ &= \int_0^1 -(-t)(-1) + (1-t)(-1) \, dt \\ &= \int_0^1 -1 \, dt = -1 \end{aligned}$$

(ii) three-quarters of the unit circle centered at the origin traced in the counter-clockwise direction.

Parameterize the path as  $t \mapsto (\sin -t, \cos -t) = (-\sin t, \cos t)$  where  $0 \leq t \leq \frac{3\pi}{2}$ . Using  $-t$  since it is counter-clockwise

$$\begin{aligned} \int_{\gamma} \mathbf{F} \cdot d\mathbf{s} &= \int_0^{\frac{3\pi}{2}} \mathbf{F}(\gamma(t)) \cdot \gamma'(t) \, dt \\ &= \int_0^{\frac{3\pi}{2}} (\cos t)(-\cos t) - (-\sin t)(-\sin t) \, dt \\ &= \int_0^{\frac{3\pi}{2}} -1 \, dt = -\frac{3\pi}{2} \end{aligned}$$

(b) Can your answers for part (a) help you determine if the 1-form  $\omega = y \, dx - x \, dy$  is exact? Explain.

Yes, we can determine that it is not exact. If  $\omega$  were to be exact then  $\mathbf{F}$  would be conservative implying that the line integral would be independent of path. Since the integrals are different, this is evidently not the case.

7. Let  $\mathbf{c}$  be the curve obtained by intersecting the cylinder  $y^2 + z^2 = 4$  and the surface  $x = yz$  in  $\mathbb{R}^3$ .

(a) Give a parametrization,  $\gamma(t)$ , of the curve  $\mathbf{c}$ .

The cylinder simply describes a circle of radius 2 in 2 dimensions, so  $y$  and  $z$  can be parameterized as  $t \mapsto (2 \sin t, 2 \cos t)$ . To add the additional constraint of the surface, just check what  $x$  is, given the  $y$  and  $z$ .  $x = (2 \sin t)(2 \cos t) = 4 \sin t \cos t$ .

Given these conditions,  $\gamma(t)$  is given by  $(4 \sin t \cos t, 2 \sin t, 2 \cos t)$ ,  $0 \leq t \leq 2\pi$ .

(b) Evaluate  $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$ , where  $\mathbf{F}(x, y, z) = (2xy, 4y, x^2)$ .

$$\begin{aligned}
 \int_{\gamma} \mathbf{F} \cdot d\mathbf{s} &= \int_0^{2\pi} \mathbf{F}(\gamma(t)) \cdot \gamma'(t) dt \\
 &= \int_0^{2\pi} 2(4 \sin t \cos t)(2 \sin t)(4(\cos^2 t - \sin^2 t)) + 4(2 \sin t)(2 \cos t) + (4 \cos t \sin t)^2(-2 \sin t) dt \\
 &= \int_0^{2\pi} 16 \sin^2 t \cos t (\cos^2 t - \sin^2 t) + 8 \sin t \cos t - 32 \cos^2 t \sin^3 t dt \\
 &= \int_0^{2\pi} 16 \sin^2 t \cos^3 t - 16 \sin^4 t \cos t + 8 \sin t \cos t - 32 \cos^2 t \sin^3 t dt \\
 &= \int_0^{2\pi} 16 \sin^2 t (1 - \sin^2 t) \cos t - 16 \sin^4 t \cos t + 8 \sin t \cos t dt - 32 \int_0^{2\pi} \cos^2 t (1 - \cos^2 t) \sin t dt \\
 &\text{Let } u = \sin t, \quad du = \cos t \\
 &= \int_0^0 16u^2(1 - u^2) - 16u^4 + 8u du - 32 \int_0^{2\pi} \cos^2 t (1 - \cos^2 t) \sin t dt \\
 &= -32 \int_0^{2\pi} \cos^2 t (1 - \cos^2 t) \sin t dt \\
 &\text{Let } u = \cos t, \quad du = -\sin t \\
 &= 32 \int_1^{-1} u^2(1 - u^2) du \\
 &= 0
 \end{aligned}$$