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MATB42: Assignment #8 Mar 20th 2018

1. (a) Let $f, g : \mathbb{R}^n \to \mathbb{R}$; $\boldsymbol{F}, \boldsymbol{G} : \mathbb{R}^n \to \mathbb{R}^n$; and define Δ, the *Laplacian*, by $\Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$.

   Verify the following identities

   (i) $\text{div}(\boldsymbol{F}+\boldsymbol{G}) = \text{div } \boldsymbol{F} + \text{ div } \boldsymbol{G}$.

$$\text{div}(\boldsymbol{F}+\boldsymbol{G}) = \sum_{i=1}^{n} \frac{\partial(F_i+G_i)}{\partial x_i} = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i} + \frac{\partial G_i}{\partial x_i} = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i} + \sum_{i=1}^{n} \frac{\partial G_i}{\partial x_i} = \text{div}\boldsymbol{F} + \text{div}\boldsymbol{G}$$

   (ii) $\text{div}(f\boldsymbol{F}) = f \text{ div } \boldsymbol{F} + \boldsymbol{F} \cdot \text{ grad} f$.

$$\text{div}(f\boldsymbol{F}) = \sum_{i=1}^{n} \frac{\partial f F_i}{\partial x_i} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} F_i + \frac{\partial F_i}{\partial x_i} f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} F_i + f \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i} = \nabla f \cdot \boldsymbol{F} + f \text{div} \boldsymbol{F}$$

   (iii) $\Delta(fg) = f\Delta g + g\Delta f + 2(\text{grad} f) \cdot (\text{grad} g)$.

$$\begin{aligned}
\Delta(fg) &= \sum_{i=1}^{n} \frac{\partial^2 fg}{\partial^2 x_i} \\
&= \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left[\frac{\partial f}{\partial x_i} g + \frac{\partial g}{\partial x_i} f \right] \\
&= \sum_{i=1}^{n} \left[\frac{\partial f}{\partial^2 x_i} g + \left(\frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_i} \right) + \frac{\partial g}{\partial^2 x_i} f + \left(\frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_i} \right) \right] \\
&= \sum_{i=1}^{n} \frac{\partial f}{\partial^2 x_i} g + 2 \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_i} \right) + \sum_{i=1}^{n} \frac{\partial g}{\partial^2 x_i} f
\end{aligned}$$

   (b) Let $f, g : D \subset \mathbb{R}^3 \to \mathbb{R}$ be of class C^1. If R is a solid region contained in D then

$$\iiint_R \nabla f \cdot \nabla g \, dV = \iint_{\partial R} f \nabla g \cdot \boldsymbol{n} \, dS - \iiint_R f \nabla^2 g \, dV$$

   ($\nabla^2 g = \text{div } (\nabla g)$).

2. Use the Divergence Theorem to verify your asnwer to question 7 on assignment 8.

3. Let $\boldsymbol{F}(x, y, z) = (x, y^2, e^{yz})$ and let R be a cube centered at the origin with sides of length 2. Evaluate $\int_S \text{div } \boldsymbol{F} \, dV$ directly and by using the Divergence Theorem.

4. Let B be the pyramid with top vertex (0,0,1) and base vertices (0,0,0), (1,0,0), (0,1,0) and (1,1,0). Let S be the 2-dim closed surface bounding B, oriented in the outward direction. Use Gauss' theorem to calculate $\int_S \boldsymbol{F} \cdot d\boldsymbol{S}$, where $\boldsymbol{F}(x, y, z) = (x^2 y, 3y^2 z, 9z^2 x)$.

5. Use the Divergence Theorem to evaluate $\int_S \boldsymbol{F} \cdot d\boldsymbol{S}$, where $\boldsymbol{F}(x, y, z) = \left(z^2 x, \frac{y^3}{3} + \tan z, x^2 z + y^2 \right)$ and S is the top half of the unit sphere $x^2 + y^2 + z^2 = 1$, oriented by the unit normalwhich points away from the origin.

6. Let the electric field from a point source at the origin be given by $\boldsymbol{E}(\boldsymbol{x}) = \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|^3}$

   (a) What is the outward flux of $\boldsymbol{E}$ across the surface $\frac{x^2}{3} + \frac{2y^2}{5} + z^2 = 7$.

(b) Show that the flux of $\boldsymbol{E}$ across that part of the sphere $x^2 + y^2 + z^2 = 25$ with $z \geq 3$ is equal to the flux across that part of the plane $z = 3$ with $x^2 + y^2 \leq 16$.

7. Let $f : \mathbb{R}^n \to \mathbb{R}$ be given by $f(x, y, z) = x^2yz$ and let η be the 2-form on $\mathbb{R}^3$ given by

$$\eta = (\sin x)\, dx\, dy + (e^y + xyz)\, dx\, dz + (x^2y^2)\, dy\, dz.$$

(a) Compute df and $d\eta$.

(b) Evaluate $df \wedge \eta$.