

1. (For this question assume that all curves are of class C^k , some $k \geq 3$).

A curve $\gamma : [a, b] \rightarrow \mathbb{R}^n$ is called *regular* if $\gamma'(t) \neq 0$ for any t . For a regular curve γ , the vector $\mathbf{T} = \frac{\gamma'(t)}{\|\gamma'(t)\|}$ is called the *unit tangent vector* to the curve.

- (a) If $\gamma : [a, b] \rightarrow \mathbb{R}^3$ is a regular curve, show that $\mathbf{T}'(t) \cdot \mathbf{T}(t) = 0$.
 (see page 235, #16(a))

$$\begin{aligned}\|\mathbf{T}(t)\|^2 &= T_1^2 + T_2^2 + T_3^2 = 1 \\ \frac{d}{dt}\|\mathbf{T}(t)\|^2 &= 2T_1T_1' + 2T_2T_2' + 2T_3T_3' = \frac{d}{dt}1 \\ 2(\mathbf{T}'(t) \cdot \mathbf{T}(t)) &= 2(T_1'T_1 + T_2'T_2 + T_3'T_3) = 0\end{aligned}$$

A curve $\gamma(s)$ is said to be *parameterized by arclength* (or have *unit speed*) if $\|\gamma'(s)\| = 1$. The *curvature* κ at a point $\gamma(s)$ of a unit speed curve is defined by $\kappa = \|\mathbf{T}'(s)\|$

- (b) (i) If $\gamma : [a, b] \rightarrow \mathbb{R}^3$ is a unit speed curve, show that its length is $b - a$.

The length of γ is $\int_a^b \|\gamma'(t)\| dt$, but $\|\gamma'(t)\|$ is 1 since γ has unit speed. Therefore, the integral is just $b - a$.

- (ii) Show that $\sigma(t) = \frac{1}{\sqrt{2}}(\cos t, \sin t, t)$ is a unit speed curve and compute its curvature.
 (see page 235, #17)

$$\begin{aligned}\frac{d}{dt}\sigma(t) &= \frac{1}{\sqrt{2}}\left(\frac{d}{dt}\cos t, \frac{d}{dt}\sin t, \frac{d}{dt}t\right) \\ &= \frac{1}{\sqrt{2}}(-\sin t, \cos t, 1) \\ \implies \left\|\frac{d}{dt}\sigma(t)\right\| &= \frac{\sqrt{\sin^2 t + \cos^2 t + 1}}{\sqrt{2}} \\ &= \frac{\sqrt{2}}{\sqrt{2}} = 1 \quad \text{So } \sigma(t) \text{ is in fact a unit curve.}\end{aligned}$$

Since $\sigma(t)$ has unit speed, $\mathbf{T}(t)$ is just $\sigma'(t)$, so $\mathbf{T}'(t)$ is $\sigma^{(2)}(t)$.

$$\begin{aligned}\mathbf{T}'(t) &= \sigma^{(2)}(t) \\ &= \frac{1}{\sqrt{2}}\left(\frac{d}{dt}(-\sin t), \frac{d}{dt}(\cos t), \frac{d}{dt}1\right) \\ &= \frac{1}{\sqrt{2}}(-\cos t, -\sin t, 0) \\ \implies \|\mathbf{T}'(t)\| &= \frac{1}{\sqrt{2}} = \kappa\end{aligned}$$

If $\mathbf{T}'(t) \neq 0$, $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$ is perpendicular to $\mathbf{T}'(t)$ (by part (a)); \mathbf{N} is called the *principal normal vector*. The vector \mathbf{B} , defined by $\mathbf{B} = \mathbf{T} \times \mathbf{N}$, is called the *binormal vector*.

- (c) Show the following about the \mathbf{T}, \mathbf{N} and \mathbf{B} system

$$(i) \frac{d\mathbf{B}}{dt} \cdot \mathbf{B} = 0 \quad (ii) \frac{d\mathbf{B}}{dt} \cdot \mathbf{T} = 0 \quad (iii) \frac{d\mathbf{B}}{dt} \text{ is a scalar multiple of } \mathbf{N}.$$

(see page 235, #20)

- (i) $\frac{d}{dt} \mathbf{B} \cdot \mathbf{B} = \frac{d}{dt} \|\mathbf{B}\|^2$, but the norm of \mathbf{N} and \mathbf{T} are 1, so $\|\mathbf{B}\|^2 = 1$ which means that $\frac{d}{dt} 1 = 0$
- (ii) Because \mathbf{B} is the cross product of \mathbf{T} along with \mathbf{N} , $\mathbf{B} \cdot \mathbf{T}$ must be 0, so $\frac{d}{dt} 0 = 0$
- (iii) From the last two parts, $\frac{d}{dt} \mathbf{B}$ is orthogonal to both \mathbf{B} and \mathbf{T} . Since there are only 3 dimensions for \mathbb{R}^3 , anything orthogonal to both of these must be parallel to each other. We know that \mathbf{N} is orthogonal to \mathbf{T} by (a), and by definition it is orthogonal to \mathbf{B} so both are parallel, i.e. they are scalar multiples.

If $\gamma(s)$ is a unit speed curve we can define the *torsion* τ by $\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$.

- (d) Compute the torsion of $\sigma(t) = \frac{1}{\sqrt{2}}(\cos t, \sin t, t)$.
(see page 235, #21(c))

$$\mathbf{T}(t) = \frac{1}{\sqrt{2}}(-\sin t, \cos t, 1)$$

$$\mathbf{N}(t) = (-\cos t, -\sin t, 0)$$

$$\mathbf{B}(t) = (0 + \sin t, -\cos t - 0, \sin^2 t + \cos^2 t)$$

$$\mathbf{B}(t) = (\sin t, -\cos t, 1)$$

2. Sketch the following vector fields including a few flow lines.

$$(a) \mathbf{F}(x, y) = (1, x^2) \quad (b) \mathbf{F}(x, y) = (x^2, x) \quad (c) \mathbf{F}(x, y) = (y, -2x)$$

(a)

$$\begin{aligned} \gamma(t) &= (x(t), y(t)) \\ \gamma'(t) &= (x'(t), y'(t)) \\ \implies \frac{\frac{dy(t)}{dt}}{\frac{dx(t)}{dt}} &= \frac{y'(t)}{x'(t)} = \frac{dy}{dx} \\ dy &= x^2, \quad dx = 3 \\ \implies \frac{dy}{dx} &= x^2 \\ \implies y &= \frac{x^3}{3} + c \end{aligned}$$

(b) $\mathbf{F}(x, y) = (x^2, x)$

$$\begin{aligned} dy &= x, \quad dx = x^2 \\ \implies \frac{dy}{dx} &= \frac{1}{x} \\ \implies y &= \ln|x| + c \quad x \neq 0 \end{aligned}$$

(c) $\mathbf{F}(x, y) = (y, -2x)$

$$\begin{aligned} dy &= -2x, \quad dx = y \\ \implies \frac{dy}{dx} &= \frac{-2x}{y} \\ \implies y \, dy &= -2x \, dx \\ \implies \frac{y^2}{2} + x^2 &= c \end{aligned}$$

3. Show that the curve $\mathbf{c}(t) = (t^2, 2t - 1, \sqrt{t})$, $t > 0$ is a flow line of the velocity vector field $\mathbf{F}(x, y, z) = (y + 1, 2, 1/2z)$

$$\mathbf{c}'(t) = \left(2t, 2, \frac{1}{2\sqrt{t}} \right)$$

$$\mathbf{F}(\mathbf{c}(t)) = \left(2t - 1 + 1, 2, \frac{1}{2\sqrt{t}} \right) = \left(2t, 2, \frac{1}{2\sqrt{t}} \right) = \mathbf{c}'(t)$$

Therefore, \mathbf{c} is a flow line of \mathbf{F} .

4. Find the work done by the force field $\mathbf{F}(x, y, z) = (xy, yz, zx)$ in moving a particle along the twisted cubic, $\boldsymbol{\gamma}(t) = (t, t^2, t^3)$, from $t = 0$ to $t = 1$.

$$\begin{aligned}\int_{\gamma} \mathbf{F} \cdot ds &= \int_0^1 \mathbf{F}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t) \, dt \\&= \int_0^1 (t)(t^2)(1) + (t^2)(t^3)(2t) + (t^3)(t)(3t^2) dt \\&= \int_0^1 t^3 + 2t^6 + 3t^6 dt \\&= \int_0^1 t^3 + 5t^6 \\&= \frac{1}{4} [t^4]_0^1 + \frac{5}{7} [t^7]_0^1 \\&= \frac{1}{4} + \frac{5}{7} = \frac{27}{28}\end{aligned}$$

5. Evaluate each of the following integrals:

(a) $\int_{\gamma} xy \, dx + y^2 dy, \quad \gamma(t) = (\cos t, \sin t), 0 \leq t \leq \frac{\pi}{2}.$

$$\begin{aligned} \int_{\gamma} \omega \cdot ds &= \int_0^{\frac{\pi}{2}} \sin t \cos t (-\sin t) + \sin^2 t \cos t \, dt \\ &= 0 \end{aligned}$$

(b) $\int_{\gamma} \mathbf{F} \cdot d\mathbf{s}, \quad \mathbf{F}(x, y, z) = (y, z, x), \quad \gamma(t) = \left(t, -2t^2, \frac{1}{3}t^3\right), \quad 0 \leq t \leq 1.$

$$\begin{aligned} \int_{\gamma} \mathbf{F} \cdot d\mathbf{s} &= \int_0^1 \mathbf{F}(\gamma(t)) \cdot \gamma'(t) \, dt \\ &= \int_0^1 (-2t^2)(1) + \left(\frac{1}{3}t^3\right)(-4t) + (t)(t^2) \, dt \\ &= \int_0^1 -2t^2 - \frac{4}{3}t^4 + t^3 \, dt \\ &= -\frac{2}{3} \left[t^3\right]_0^1 - \frac{4}{15} \left[t^5\right]_0^1 + \frac{1}{4} \left[t^4\right]_0^1 \\ &= -\frac{10}{15} - \frac{4}{15} + \frac{1}{4} = -\frac{56}{60} + \frac{15}{60} = -\frac{41}{60} \end{aligned}$$

(c) $\int_{\gamma} z \, dx - xyz \, dy + 2x^2 \, dz, \quad \gamma$ is the parabola $z = x^2, y = 0$, from $(-1, 0, 1)$ to $(1, 0, 1)$.

Can parameterize γ by $\gamma(t) = (t, 0, t^2)$, $-1 \leq t \leq 1$, as on the parabola y is constant 0, x goes from $-1 \rightarrow 1$ and z goes from $1 \rightarrow 0 \rightarrow 1$.

$$\begin{aligned} \int_{\gamma} \omega \cdot ds &= \int_{-1}^1 (t^2)(1) - (t)(0)(t^2)(0) + 2(t)^2(2t) \, dt \\ &= \int_{-1}^1 t^2 + 4t^3 \, dt \\ &= \frac{2}{3} \left[t^3\right]_0^1 \quad \text{Exploiting even/odd} \\ &= \frac{2}{3} \end{aligned}$$

(d) $\int_{\gamma} \mathbf{F} \cdot d\mathbf{s}, \quad \mathbf{F}(x, y, z) = (2xy, x^2 + e^z, ye^z), \quad \gamma$ consists of straight line segments joining, in order, the points $(1, 1, 0)$, $(2, 0, 5)$ and $(0, 3, 0)$.

Note: By inspection $g = x^2y + ye^z$ is a potential function for \mathbf{F} . Also, straight line segments, being linear functions are smooth. Furthermore, $F(x, y, z)$ is smooth since polynomials and exponential functions are each smooth. Therefore, GFTC applies, and $\int_{\gamma} \mathbf{F} \cdot d\mathbf{s} = g(0, 3, 0) - g(1, 1, 0) = ((0)^2(3) + (3)e^{(0)}) - ((1)^2(1) + (1)e^{(0)}) = 3 - 2 = 1.$

6. (a) Let $\mathbf{F}(x, y) = (y, -x)$. Find $\int_{\gamma} \mathbf{F} \cdot d\mathbf{s}$ from $(1,0)$ to $(0,-1)$ along

(i) the straight line segment joining these points

Parameterize the path as $t \mapsto (1-t, -t)$ where $0 \leq t \leq 1$.

$$\begin{aligned} \int_{\gamma} \mathbf{F} \cdot d\mathbf{s} &= \int_0^1 \mathbf{F}(\gamma(t)) \cdot \gamma'(t) \, dt \\ &= \int_0^1 -(-t)(-1) + (1-t)(-1) \, dt \\ &= \int_0^1 -1 \, dt = -1 \end{aligned}$$

(ii) three-quarters of the unit circle centered at the origin traced in the counter-clockwise direction.

Parameterize the path as $t \mapsto (\sin -t, \cos -t) = (-\sin t, \cos t)$ where $0 \leq t \leq \frac{3\pi}{2}$. Using $-t$ since it is counter-clockwise

$$\begin{aligned} \int_{\gamma} \mathbf{F} \cdot d\mathbf{s} &= \int_0^{\frac{3\pi}{2}} \mathbf{F}(\gamma(t)) \cdot \gamma'(t) \, dt \\ &= \int_0^{\frac{3\pi}{2}} (\cos t)(-\cos t) - (-\sin t)(-\sin t) \, dt \\ &= \int_0^{\frac{3\pi}{2}} -1 \, dt = -\frac{3\pi}{2} \end{aligned}$$

(b) Can your answers for part (a) help you determine if the 1-form $\omega = y \, dx - x \, dy$ is exact? Explain.

Yes, we can determine that it is not exact. If ω were to be exact then \mathbf{F} would be conservative implying that the line integral would be independent of path. Since the integrals are different, this is evidently not the case.

7. Let \mathbf{c} be the curve obtained by intersecting the cylinder $y^2 + z^2 = 4$ and the surface $x = yz$ in \mathbb{R}^3 .

(a) Give a parametrization, $\gamma(t)$, of the curve \mathbf{c} .

The cylinder simply describes a circle of radius 2 in 2 dimensions, so y and z can be parameterized as $t \mapsto (2 \sin t, 2 \cos t)$. To add the additional constraint of the surface, just check what x is, given the y and z . $x = (2 \sin t)(2 \cos t) = 4 \sin t \cos t$.

Given these conditions, $\gamma(t)$ is given by $(4 \sin t \cos t, 2 \sin t, 2 \cos t)$, $0 \leq t \leq 2\pi$.

(b) Evaluate $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$, where $\mathbf{F}(x, y, z) = (2xy, 4y, x^2)$.

$$\begin{aligned}
 \int_{\gamma} \mathbf{F} \cdot d\mathbf{s} &= \int_0^{2\pi} \mathbf{F}(\gamma(t)) \cdot \gamma'(t) \, dt \\
 &= \int_0^{2\pi} 2(4 \sin t \cos t)(2 \sin t)(4(\cos^2 t - \sin^2 t)) + 4(2 \sin t)(2 \cos t) + (4 \cos t \sin t)^2(-2 \sin t) \, dt \\
 &= \int_0^{2\pi} 16 \sin^2 t \cos t (\cos^2 t - \sin^2 t) + 8 \sin t \cos t - 32 \cos^2 t \sin^3 t \, dt \\
 &= \int_0^{2\pi} 16 \sin^2 t \cos^3 t - 16 \sin^4 t \cos t + 8 \sin t \cos t - 32 \cos^2 t \sin^3 t \, dt \\
 &= \int_0^{2\pi} 16 \sin^2 t (1 - \sin^2 t) \cos t - 16 \sin^4 t \cos t + 8 \sin t \cos t \, dt - 32 \int_0^{2\pi} \cos^2 t (1 - \cos^2 t) \sin t \, dt \\
 &\text{Let } u = \sin t, \, du = \cos t \\
 &= \int_0^0 16u^2(1 - u^2) - 16u^4 + 8u \, du - 32 \int_0^{2\pi} \cos^2 t (1 - \cos^2 t) \sin t \, dt \\
 &= -32 \int_0^{2\pi} \cos^2 t (1 - \cos^2 t) \sin t \, dt \\
 &\text{Let } u = \cos t, \, du = -\sin t \\
 &= 32 \int_1^{-1} u^2(1 - u^2) \, du \\
 &= 0
 \end{aligned}$$