

MATB42: Assignment #8

1. A surface  $S$  is obtained by rotation the given figure in the  $xy$ -plane about the  $z$ -axis. (The arc is part of a circle of radius 1 centered at  $(2,0)$ .)

- (a) Paratetrize  $S$  (in pieces) and compute the surface area.

We have that the upper line when rotated, can be parametrized by a restricted cone and similarly for the bottom. The top and bottom respectively can be written as



$$\Phi(u, \theta) = ((1 - u)3 \cos \theta, (1 - u)3 \sin \theta, 3u), \quad 0 \leq u \leq 1, \quad 0 \leq \theta \leq 2\pi$$

$$\Phi(u, \theta) = ((1 + u)3 \cos \theta, (1 + u)3 \sin \theta, -3u), \quad -1 \leq u \leq 0, \quad 0 \leq \theta \leq 2\pi$$

For the circular portion to the left, when rotated around, it will be the inner half of a torus, so the equation will be

- (b) Use a computer algebra system to sketch  $S$ .

2. Let  $S$  be the cone with vertex  $(2,3,3)$  and base the circle  $x^2 + y^2 = 1$  in the  $xy$ -plane.

- (a) Parametrize  $S$  Starting with a base of a circle, we get  $(\cos \theta, \sin \theta, 1)$ . To change into a cone multiply  $x$  and  $y$  by  $(1 - u)$  and finally to shift the vertex, add  $(2u, 3u, 2u)$  where  $z = 2u$  since the base equation already has a 1, so it needs to be one less.

$$\implies \Phi(u, \theta) = ((1 - u) \cos \theta + 2u, (1 - u) \sin \theta + 3u, 1 + 2u)$$

- (b) Use a computer algebra system to sketch  $S$ .
- (c) Write down the integral that would give the surface area of  $S$ . (You are not expected to evaluate the integral.)

$$\phi_\theta = (-(1 - u) \sin \theta, (1 - u) \cos \theta, 0)$$

$$\phi_u = (-\cos \theta + 2, -\sin \theta + 3, 2)$$

$$\phi_\theta \times \phi_u = ((2(1 - u) \cos \theta), (2(1 - u) \sin \theta),$$

$$(-(1 - u) \sin \theta)(-\sin \theta + 3) - ((1 - u) \cos \theta)(-\cos \theta + 2))$$

$$= ((2 - 2u) \cos \theta, (2 - 2u) \sin \theta, (1 - u) \sin^2 \theta - (3 - 3u) \sin \theta + (1 - u) \cos^2 \theta - (2 - 2u) \cos \theta)$$

$$= ((2 - 2u) \cos \theta, (2 - 2u) \sin \theta, (1 - u) - (3 - 3u) \sin \theta - (2 - 2u) \cos \theta)$$

$$\|\phi_\theta \times \phi_u\| = \sqrt{(2 - 2u) \cos \theta, (2 - 2u) \sin \theta, (1 - u) - (3 - 3u) \sin \theta - (2 - 2u) \cos \theta}$$

3. Let  $S$  be the self-intersecting rectangle in  $\mathbb{R}^3$  given by the implicit equation  $x^2 - y^2z = 0$ .
  - (a) Give a parametrization of  $S$  and use a computer algebra system to provide a sketch.
  - (b) Is your parametrization one-to-one? Explain.
  - (c) Find the equation of the tangent plane to  $S$  at  $\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)$ .
4. Let  $S$  be the surface defined by  $x^2 + y^2 = 1$  for  $0 \leq z \leq 1$  and by  $x^2 + y^2 = z^2$  for  $1 \leq z \leq 2$ .
  - (a) Use symbolic algebra software to sketch  $S$ .
  - (b) Evaluate  $\int_S \mathbf{F} \cdot d\mathbf{S}$  where  $\mathbf{F}(x, y, z) = (-y, x, z)$  and  $S$  is oriented by outward pointing normals.
- (a) Evaluate the (vector) surface integral  $\int_S \mathbf{F} \cdot d\mathbf{S}$  in each of the following cases.
  - i.  $\mathbf{F}(x, y, z) = (1, x, z)$ ,  $S$  is the upper hemisphere  $x^2 + y^2 + z^2 = 1$ ,  $z \geq 0$ , with  $\mathbf{n}$  pointing upward.
  - ii.  $\mathbf{F}(x, y, z) = (2, x, z + y)$ ,  $S$  is that part of the plane  $x + y + z = 1$  which lies in the first octant and  $\mathbf{n}$  points upward.
  - iii. Marsden & Tromba, page 425, #22.
5. Let  $S$  be the portion of the plane  $x - 2y + z = 1$  that is cut off by the coordinate planes and the plane  $x + y = 1$ . Let  $\mathbf{V}$  be the velocity field  $\mathbf{V}(x, y, z) = (y, z, x^2)$ . Find the flow across  $S$  when  $\mathbf{n}$  points upward. Explain your answer.
6. Let  $S$  be the closed surface that consists of the hemisphere  $x^2 + y^2 + z^2 = 1$ ,  $z \geq 0$ , and its base  $x^2 + y^2 \leq 1$ ,  $z = 0$ . Let  $\mathbf{E}$  be the electric field  $\mathbf{E}(x, y, z) = (2x, 2y, 2z)$ . Directly calculate the electric flux across  $S$ .