MATB42: Assignment #8

1. (a) Let $f, g : \mathbb{R}^n \to \mathbb{R}$; $\mathbf{F}, \mathbf{G} : \mathbb{R}^n \to \mathbb{R}^n$; and define Δ , the Laplacian, by $\Delta f = \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2}$.

Verify the following identities

(i) $\operatorname{div}(\mathbf{F} + \mathbf{G}) = \operatorname{div} \mathbf{F} + \operatorname{div} \mathbf{G}$.

$$\operatorname{div}(\boldsymbol{F} + \boldsymbol{G}) = \sum_{i=1}^{n} \frac{\partial (F_i + G_i)}{\partial x_i} = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i} + \frac{\partial G_i}{\partial x_i} = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i} + \sum_{i=1}^{n} \frac{\partial G_i}{\partial x_i} = \operatorname{div} \boldsymbol{F} + \operatorname{div} \boldsymbol{G}$$

(ii) $\operatorname{div}(f\mathbf{F}) = f \operatorname{div} \mathbf{F} + \mathbf{F} \cdot \operatorname{grad} f$.

$$\operatorname{div}(f\mathbf{F}) = \sum_{i=1}^{n} \frac{\partial f F_{i}}{\partial x_{i}} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_{i}} F_{i} + \frac{\partial F_{i}}{\partial x_{i}} f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_{i}} F_{i} + f \sum_{i=1}^{n} \frac{\partial F_{i}}{\partial x_{i}} = \nabla f \cdot \mathbf{F} + f \operatorname{div} \mathbf{F}$$

(iii) $\Delta(fg) = f\Delta g + g\Delta f + 2(\text{grad}f) \cdot (\text{grad}g).$ Proof.

$$\begin{split} \Delta(fg) &= \sum_{i=1}^n \frac{\partial^2 fg}{\partial^2 x_i} \\ &= \sum_{i=1}^n \frac{\partial}{\partial x_i} \left[\frac{\partial f}{\partial x_i} g + \frac{\partial g}{\partial x_i} f \right] \\ &= \sum_{i=1}^n \left[\frac{\partial f}{\partial^2 x_i} g + \left(\frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_i} \right) + \frac{\partial g}{\partial^2 x_i} f + \left(\frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_i} \right) \right] \\ &= g \sum_{i=1}^n \frac{\partial f}{\partial^2 x_i} + 2 \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_i} \right) + f \sum_{i=1}^n \frac{\partial g}{\partial^2 x_i} \\ &= g \Delta f + 2 [\nabla f \cdot \nabla g] + f \Delta g \end{split}$$

(b) Let $f, g: D \subset \mathbb{R}^3 \to \mathbb{R}$ be of class C^1 . If R is a solid region contained in D then

$$\iiint_{R} \nabla f \cdot \nabla g \, dV = \iint_{\partial R} f \nabla g \cdot \boldsymbol{n} \, dS - \iiint_{R} f \nabla^{2} g \, dV$$

 $(\nabla^2 g = \operatorname{div}(\nabla g)).$

Proof.

$$\iiint_{R} \nabla f \cdot \nabla g \, dV = \iint_{\partial R} f \nabla g \cdot \boldsymbol{n} \, dS - \iiint_{R} f \nabla^{2} g \, dV$$

$$\iff \iiint_{R} \nabla f \cdot \nabla g \, dV + \iiint_{R} f \nabla^{2} g \, dV = \iint_{\partial R} f \nabla g \cdot \boldsymbol{n} \, dS$$

$$\iint_{\partial R} f \nabla g \cdot \boldsymbol{n} \, dS \stackrel{\text{Div} Thm}{=} \iiint_{R} \operatorname{div}(f \nabla g) \, dV \stackrel{\text{(ii)}}{=} \iiint_{R} f(\operatorname{div} \nabla g) + \nabla g \cdot \nabla f \, dV$$
$$= \iiint_{R} f(\operatorname{div} \nabla g) \, dV + \iiint_{R} \nabla f \cdot \nabla g \, dV = \iiint_{R} f \nabla^{2} g \, dV + \iiint_{R} \nabla f \cdot \nabla g \, dV$$

2. Use the Divergence Theorem to verify your answer to question 7 on assignment 8.	

3. Let $\mathbf{F}(x,y,z) = (x,y^2,e^{yz})$ and let R be a cube centered at the origin with sides of length 2. Evaluate $\int_S \operatorname{div} \mathbf{F} \, dV$ directly and by using the Divergence Theorem. Directly:

$$\int_{S} \operatorname{div} \mathbf{F} \, dV = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} 1 + 2y + y e^{yz} \, dx \, dz \, dy$$

$$= \int_{-1}^{1} \int_{-1}^{1} 2 + 4y + 2y e^{yz} \, dz \, dy$$

$$= \int_{-1}^{1} 4 + 8y + \left[e^{yz} \right]_{-1}^{1} \, dy$$

$$= \int_{-1}^{1} 4 + 8y + e^{y} - e^{-y} \, dy$$

$$= 8 + 0 + \left[e^{y} \right]_{-1}^{1} + \left[e^{-y} \right]_{-1}^{1} = 8$$

Divergence:

$$\int_{R} \operatorname{div} \boldsymbol{F} \, dV = \int_{S} \boldsymbol{F} \cdot d\boldsymbol{S}$$

where S is the union of 6 planes with $-1 \le x, y, z \le 1$ and having normals in the $\pm x, y, z$ directions. The parameterizations can be given by

$$\begin{split} & \Phi_{S_1}(x,y) = (x,y,1) \\ & \Phi_{S_2}(y,z) = (1,y,z) \\ & \Phi_{S_3}(x,z) = (x,1,z) \end{split} \qquad \begin{split} & \Phi_{S_1'}(x,y) = (x,y,-1) \\ & \Phi_{S_2'}(y,z) = (-1,y,z) \\ & \Phi_{S_3'}(x,z) = (x,-1,z) \end{split}$$

When plugging in S_3 , S'_3 into \mathbf{F} , then taking the dot with the normal, we get that the integrands are going to be the same, specifically 1. When doing the same with S_2 , S_2 , due to x being odd, we get both being negative the other, so integrating over the same range will cancel each other out. Putting this together, the integral over S is:

$$\int_{S} \mathbf{F} \cdot d\mathbf{S} = 2 \int_{-1}^{1} \int_{-1}^{1} 1 \, dx \, dz + \int_{-1}^{1} \int_{-1}^{1} e^{y} \, dx \, dy + \int_{-1}^{1} \int_{-1}^{1} -e^{-y} \, dx \, dy$$
$$= 8 + 2 \int_{-1}^{1} e^{y} \, dy - 2 \int_{-1}^{1} e^{-y} \, dy$$
$$= 8 + 2 ([e^{y}]_{-1}^{1} + [e^{-y}]_{-1}^{1}) = 8$$

4. Let B be the pyramid with top vertex (0,0,1) and base vertices (0,0,0), (1,0,0), (0,1,0) and (1,1,0). Let S be the 2-dim closed surface bounding B, oriented in the outward direction. Use Gauss' theorem to calculate $\int_{S} \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x,y,z) = (x^2y, 3y^2z, 9z^2x)$.

$$\int_{R} \operatorname{div} \boldsymbol{F} \, dV = \int_{S} \boldsymbol{F} \cdot d\boldsymbol{S}$$

$$\operatorname{div} \mathbf{F} = 2xy + 6yz + 18zx$$

$$\int_{R} \operatorname{div} \mathbf{F} = \int_{0}^{1} \int_{0}^{1-z} \int_{0}^{1-z} 2xy + 6yz + 18zx \, dx \, dy \, dz$$

$$= \int_{0}^{1} \int_{0}^{1-z} (1-z)^{2}y + 6yz(1-z) + 9z(1-z)^{2} \, dy \, dz$$

$$= \int_{0}^{1} (1/2)(1-z)^{4} + 3z(1-z)^{3} + 9z(1-z)^{3} \, dz$$

$$= \int_{0}^{1} -\frac{23z^{4}}{2} + 34z^{3} - 33z^{2} + 10z + \frac{1}{2} \, dz$$

$$= -\frac{23}{10} + \frac{34}{4} - 11 + 5 + \frac{1}{2}$$

$$= \frac{7}{10}$$

5. Use the Divergence Theorem to evaluate $\int_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x,y,z) = \left(z^2x, \frac{y^3}{3} + \tan z, x^2z + y^2\right)$ and S is the top half of the unit sphere $x^2 + y^2 + z^2 = 1$, oriented by the unit normal which points away from the origin.

For the Divergence Theorem to apply, the integral must be over a closed, oriented outwards. To use Divergence Theorem the surface needs to be closed, so add the disk S' on the xy-plane to the surface. Then

$$\iiint_R \operatorname{div} \boldsymbol{F} \, dV - \iint_{S'} \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_S \boldsymbol{F} \cdot d\boldsymbol{S}$$

The disk can be parametrized as $\Phi(r,\theta) = (r\cos\theta, r\sin\theta, 0)$ with $r \in [0,1], \theta \in [0,2\pi]$ The normal, pointing outward needs to be downward.

$$\iint_{S'} \mathbf{F} \cdot d\mathbf{S} = \int_0^1 \int_0^{2\pi} (*, *, (r \sin \theta)^2) \cdot (0, 0, -1) \, d\theta \, dr$$

$$= -\int_0^1 \int_0^{2\pi} (r \sin \theta)^2 \, d\theta \, dr$$

$$= -\int_0^1 \int_0^{2\pi} r^2 (1/2 - (\cos \theta)/2) \, d\theta \, dr$$

$$= -\int_0^1 \pi r^2 \, dr = -\frac{\pi}{3}$$

Meanwhile, the region enclosed by this, is the top half of the unit sphere, so switch to polars, restricting $\varphi \in [0, \pi/2]$ to remain above the xy-plane.

$$\operatorname{div} \boldsymbol{F} = z^2 + y^2 + x^2 = \rho$$

$$\iiint_R \operatorname{div} \boldsymbol{F} \, dV = \int_0^1 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \rho^3 \sin \varphi \, d\varphi \, d\theta \, d\rho \qquad \qquad = 2\pi \int_0^1 \rho^3 \left[-\cos \varphi \right]_0^{\frac{\pi}{2}} d\rho$$

- 6. Let the electric field from a point source at the origin be given by $E(x) = \frac{x}{\|x\|^3}$
 - (a) What is the outward flux of \boldsymbol{E} across the surface $\frac{x^2}{3} + \frac{2y^2}{5} + z^2 = 7$.
 - (b) Show that the flux of E across that part of the sphere $x^2 + y^2 + z^2 = 25$ with $z \ge 3$ is equal to the flux across that part of the plane z = 3 with $x^2 + y^2 \le 16$.
- 7. Let $f:\mathbb{R}^n\to\mathbb{R}$ be given by $f(x,y,z)=x^2yz$ and let η be the 2-form on \mathbb{R}^3 given by

$$\eta = (\sin x) \, dx \, dy + (e^y + xyz) \, dx \, dz + (x^2y^2) \, dy \, dz.$$

- (a) Compute df and $d\eta$.
- (b) Evaluate $df \wedge \eta$.