

CSCD37: Assignment #2

1. Recall the full-Newton algorithm for solving the nonlinear system  $F(x) = 0; F, x \in \mathcal{R}^n$  :

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Generate an initial approximation  $\hat{x}_0$ 
for  $k = 0, 1 \dots$  until convergence
  compute  $-F(\hat{x}_k)$ 
  compute  $\frac{\partial F(\hat{x}_k)}{\partial x}$ 
  solve  $\frac{\partial F(\hat{x}_k)}{\partial x} \Delta_k = -F(\hat{x}_k)$ 
  update  $\hat{x}_{k+1} = \hat{x}_k + \Delta_k$ 
end for

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- (a) This algorithm is computationally expensive. Give a detailed analysis of the cost using standard big-oh notation. You may use results from CSCC37; e.e., the cost (measuring flops) of the LU-factorization of an  $n \times n$  matrix is  $(1/3)n^3 + \mathcal{O}(n^2)$ .

The following analysis makes the assumption that each component of  $F$  can be evaluated in approx 1 flop, and that taking the derivative also takes around 1 flop. For each iteration of the loop, first evaluating the function  $F$  at a point  $x \in \mathcal{R}^n$  will take  $n$  operations. The evaluation of the  $n$  derivatives will also take  $n$ , so evaluating  $n$  derivatives at  $n$  points will take  $n^2$  time. Solving the linear system of the derivative matrix against the original function will take around  $(1/3)n^3 + \mathcal{O}(n^2)$  as was seen in CSCC37. Finally updating a vector with an addition of 2  $n$  length vectors will simply be  $n$  additions. Overall, the entire loop iteration will take  $n^2 + n + (1/3)n^3 + \mathcal{O}(n^2) + n = (1/3)n^3 + \mathcal{O}(n^2) = \mathcal{O}(n^3)$  steps per iteration. This cannot be generalized outside of iterations, as convergence is not always guaranteed and  $k$  may run on infinitely.

- (b) We briefly discussed in lecture the “quasi-Newton algorithm” for solving nonlinear systems. Modify the algorithm above to take advantage of the optimizations we discussed. You do not need to implement the modifications ... pseudo-code will suffice. Be careful to discuss both flop optimizations and convergence issues (“X-test” and “F-test” tolerance, maximum number of iterations, condition of Jacobian, how long to hold Jacobian fixed, etc.).

2. In lecture we derived the divided-difference (Newton) form of the interpolating polynomial for the simple interpolation problem. This question will investigate how the Newton polynomial can be used for osculatory interpolation.

(a) Prove that

$$\lim_{\substack{x_{i+j} \rightarrow x_i \\ 1 \leq j \leq k}} y[x_{i+k}, x_{i+k-1}, \dots, x_i] = \frac{y^{(k)}(x_i)}{k!}$$

Provided  $y \in \mathcal{C}^k$ .

*Proof.* Examine first, the error formula given in lecture,  $E(x) = y(x) - p(x)$  for any polynomial interpolant, then using the Newton polynomial as  $p(x)$ , it produces:

$$E(x) = y(x) - y[x_i] + (x - x_i)y[x_{i+1}, x_i] + \dots + (x - x_i)(x - x_{i+1}) \dots (x - x_{i+k-1})y[x_{i+k}, \dots, x_i]$$

Knowing that  $E(x)$  has  $n+1$  distinct roots (at the interpolation constraints), by Rolle's theorem, gives that the  $k$ th derivative of  $E(x)$  has at least 1 zero  $E^{(k)}(\psi)$  where  $\psi \in \text{span}\{x_i, \dots, x_{i+k}\}$ . Looking at the Newton polynomial, the  $k$ th derivative will only have the leading coefficient times  $k!$  since it is a degree  $k$  polynomial. This coefficient, from the definition can be seen as  $y[x_{i+k}, \dots, x_i]$

$$\begin{aligned} \implies \frac{d^k}{dx^k} E(x) &= y^{(k)}(x) - y[x_{i+k}, \dots, x_i]k! \\ \frac{d^k}{dx^k} E(\psi) &= y^{(k)}(\psi) - y[x_{i+k}, \dots, x_i]k! \\ 0 &= y^{(k)}(\psi) - y[x_{i+k}, \dots, x_i]k! \\ y[x_{i+k}, \dots, x_i]k! &= y^{(k)}(\psi) \\ y[x_{i+k}, \dots, x_i] &= \frac{y^{(k)}(\psi)}{k!} \\ \implies \lim_{\substack{x_{i+j} \rightarrow x_i \\ 1 \leq j \leq k}} y[x_{i+k}, \dots, x_i] &= \lim_{\substack{x_{i+j} \rightarrow x_i \\ 1 \leq j \leq k}} \frac{y^{(k)}(\psi)}{k!} \\ \implies \lim_{\substack{x_{i+j} \rightarrow x_i \\ 1 \leq j \leq k}} y[x_{i+k}, \dots, x_i] &= \frac{y^{(k)}(x_i)}{k!} \end{aligned}$$

The last line is since if  $\forall x_{i+j}, |x_{i+j} - x_i| < \varepsilon \implies \text{span}\{x_i, \dots, x_{i+k}\} \subset (x_i - \varepsilon, x_i + \varepsilon) \implies \psi \rightarrow x_i$ .

□

- (b) The result in (a) tells us that divided differences can be replaced with derivatives as data points coincide. Using this result, construct a divided difference table to find the coefficients of the Newton polynomial of degree 6 or less that satisfies the following interpolation conditions:

$$\begin{array}{llll} p(-1) = 4 & p(0) = 7 & p(1) = 28 & p(2) = 247 \\ & p'(0) = 6 & p'(1) = 56 & \\ & & p''(1) = 140 & \end{array}$$



Editor - C:\Users\Keegan\schoo\cscd37\ a2q2c.m

a7q1b.m x approx.m x a2q2c.m x +

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1 - size = 7;
2 - vandermonde = zeros(size);
3 - coeff1 = 0:(size-1);
4 - coeff2 = [0,0,2,3*2,4*3,5*4,6*5];
5 - powers0 = 0:(size-1);
6 - powers1 = [0,0:(size-2)];
7 - powers2 = [0,0,0:(size-3)];
8 - x = [-1,0,1,2];
9 - vandermonde(1,:) = x(1).^powers0;
10 - vandermonde(2,:) = x(2).^powers0;
11 - vandermonde(3,:) = (x(2).^(powers1)).*coeff1;
12 - vandermonde(4,:) = x(3).^powers0;
13 - vandermonde(5,:) = (x(3).^(powers1)).*coeff1;
14 - vandermonde(6,:) = (x(3).^(powers2)).*coeff2;
15 - vandermonde(7,:) = x(4).^powers0;
16 - y = [4 7 6 28 56 140 247];
17 - soln = vandermonde\y';
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Command Window

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Trial>> soln

soln =

    7.0000
    6.0000
    5.0000
    4.0000
    3.0000
    2.0000
    1.0000

fx Trial>> |
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3. Consider the function  $y \in \mathcal{C}^{n+1}$  and the polynomial  $p \in \mathcal{P}_n$  which satisfies

$$p^{(j)}(x_i) = y^{(j)}(x_i); \quad j = 0, \dots, j_i; \quad i = 0, \dots, k; \quad \sum_{i=0}^k (j_i + 1) = n + 1;$$

with all of the  $x_i$  distinct. The error in this polynomial interpolant is given by

$$E(x) = y(x) - p(x) = \frac{y^{(n+1)}(\xi)}{(n+1)!} (x - x_0)^{j_0+1} (x - x_1)^{j_1+1} \dots (x - x_k)^{j_k+1} \quad (1)$$

where  $\xi \in \text{span}\{x_0, \dots, x_k, x\} = [\min\{x_0, \dots, x_k, x\}, \max\{x_0, \dots, x_k, x\}]$  provided  $y \in \mathcal{C}^{n+1}$  on  $\text{span}\{x_0, \dots, x_k, x\}$ .

This is a fundamental formula in Numerical Approximation. The following is an outline of a possible derivation of (1). In this question you will expand this outline by proving certain key statements.

If  $x = x_i$ ,  $i = 0, \dots, k$ , then  $y(x) - p(x) = 0$  since  $p$  interpolates  $y$  at these  $k+1$  points. Also,  $E(x_i) = 0$  in (1) since  $(x_i - x_i)^{j_i+1} = 0$ . Therefore, (1) holds when  $x = x_i$ .

Now assume  $x \neq x_i$  for any  $i = 0, \dots, k$  and consider  $x$  fixed. Let  $F(t) = y(t) - p(t) - CW(t)$  where  $C = [y(x) - p(x)]/W(x)$  is a constant and

$$W(t) = (t - x_0)^{j_0+1} (t - x_1)^{j_1+1} \dots (t - x_k)^{j_k+1} \quad (2)$$

is a polynomial of degree  $n+1$ . Clearly  $F(x) = 0$ , and also

$$F^{(j)}(x_i) = 0; \quad j = 0, \dots, j_i; \quad i = 0, \dots, k. \quad (3)$$

Therefore, counting multiplicities,  $F(t)$  has at least  $n+2$  zeros in  $\text{span}\{x_0, \dots, x_k, x\}$ , which implies  $F^{(n+1)}(t)$  has at least 1 zero in  $\text{span}\{x_0, \dots, x_k, x\}$ , or, in other words,

$$F^{(n+1)}(\xi) = 0, \quad \xi \in \text{span}\{x_0, \dots, x_k, x\}. \quad (4)$$

But

$$F^{(n+1)}(t) = \frac{d^{n+1}}{dt^{n+1}} [y(t) - p(t) - CW(t)] = y^{(n+1)}(t) - (n+1)!C. \quad (5)$$

Therefore,

$$F^{(n+1)}(\xi) = 0 \implies y^{(n+1)}(\xi) - (n+1)!C = 0 \implies C = \frac{y^{(n+1)}(\xi)}{(n+1)!}.$$

But

$$C = \frac{y(x) - p(x)}{W(x)} \implies y(x) - p(x) = \frac{y^{(n+1)}(\xi)}{(n+1)!} W(x).$$

Now for the statements you must prove:

- (a) Prove that  $W(t)$  in (2) is a polynomial of degree  $n+1$ .

*Proof.* Polynomials are closed under exponentiation and basic operations, so it is sufficient to show the highest power to be  $n+1$ . For each term  $(t - x_i)^{j_i+1}$ , the highest power of  $t$  is  $j_i + 1$  respectively using the Binomial Theorem. This means that the final product will highest power of all of these terms together, so the power will be  $\sum_{i=0}^k j_i + 1 = n+1$  by definition.  $\square$

- (b) Prove (3)

*Proof.* We know that  $p$  must approximate  $y$  exactly at the interpolation points, so evidently for a point on the  $j$ th derivative,  $y^{(j)}(x_i) - p^{(j)}(x_i)$  will be 0. For the other term  $CW(t)$ , taking the derivative only  $j$  times will result in the term  $(x - x_i)$  to show up in every term generated by the product rule since it has power of  $(j_i + 1)$ , hence  $W(t)$  will also be 0 making  $F^{(j)}(x_i) = 0$  at all interpolation points.  $\square$

- (c) Explain how (4) follows from  $F(t)$  having at least  $n + 2$  zeros in  $\text{span}\{x_0, \dots, x_k\}$ .

**Lemma.** *Counting multiplicities, if  $F(t)$  is  $\mathcal{C}^1$  over  $\text{span}\{x_0, \dots, x_k\}$  has at least  $k$  zeros in  $\text{span}\{x_0, \dots, x_k\}$  then  $F'(t)$  has at least  $k - 1$  zeros in  $\text{span}\{x_0, \dots, x_k\}$ .*

*Proof.* Suppose the distinct roots of  $F(t)$  are ordered as  $a_1 < a_2 < \dots < a_s$  with multiplicities  $m(a_i)$  such that  $\sum_{i=1}^s m(a_i) = n$ . From Rolle's Theorem, we have that there is some  $\rho \in (a_i, a_{i+1})$ ,  $1 \leq i < s$  where  $F'(\rho) = 0$  immediately giving us  $s - 1$  roots in the derivative. Now counting the multiplicities of the previous roots, we get that there are  $\sum_{i=1}^s (m(a_i) - 1)$  more roots, which summing together gives  $\sum_{i=1}^s m(a_i) - s + (s - 1) = n - 1$  roots, which are all in the span.  $\square$

From this, it follows immediately that taking  $n + 1$  derivatives of a function with  $n + 2$  roots will have one root in the same span.

- (d) Prove (5).

*Proof.* Since  $p$  is a polynomial of degree  $n$ , taking  $n + 1$  derivatives will immediately cause it to vanish. On the otherhand, for  $W(t)$ , we know it is a polynomial of degree  $n + 1$  from (a), so taking the  $(n + 1)$ th derivative gives us the leading coefficient times  $(n + 1)!$ . Looking back at how  $W(t)$  is setup though, since none of the  $t$  in each binomial term have any coefficient the resulting expanded polynomials leading coefficient will be 1. This means the  $(n + 1)$ th derivative of  $CW(t)$  is just  $C(n + 1)!$ .

$$\begin{aligned} \implies F^{(n+1)}(t) &= \frac{d^{n+1}}{dt^{n+1}}[y(t) - p(t) - CW(t)] \\ &= \frac{d^{n+1}}{dt^{n+1}}y(t) - \frac{d^{n+1}}{dt^{n+1}}p(t) - \frac{d^{n+1}}{dt^{n+1}}CW(t) \\ &= y^{(n+1)}(t) - (n + 1)!C. \end{aligned}$$

$\square$

4. In lecture we proved that the roots of the Chebyshev polynomial

$$T_k(x) = \cos(k \cos^{-1}(x)), k = 0, 1, \dots \quad (6)$$

are the optimal interpolation points on  $[-1, 1]$ .

(a) Prove that (6) is a polynomial of degree  $k$  for all  $k \geq 0$ .

*Proof.* For the case of  $k = 0, 1$ , we have that

$$\begin{aligned} \cos(k \arccos(x))|_{k=0} &= \cos(0) = 1 \\ \cos(k \arccos(x))|_{k=1} &= \cos(\arccos(x)) = x \end{aligned}$$

So these cases hold. Using induction, assume that  $T_n(x)$  is a polynomial for  $n < k$ , then there are two inductive cases:

First  $k = 2t$  is even

$$\begin{aligned} \cos(k \arccos(x)) &= \cos(2(t \arccos(x))) \\ &= 2(\cos(t \arccos(x)))^2 - 1 \text{ [Double angle identity]} \\ &= 2(T_t(x))^2 - 1 \end{aligned}$$

This is again a polynomial since  $T_t$  is one by IH, so this case holds.

Second  $k = 2t + 1$  is odd

$$\begin{aligned} \cos(k \arccos(x)) &= \cos(2(t \arccos(x)) + \arccos(x)) \\ &\stackrel{\text{angle sum}}{=} \cos(2t \arccos(x))(\cos(\arccos(x))) - \sin(2t \arccos(x))(\sin(\arccos(x))) \\ &= xT_{2t}(x) - \sin(2t \arccos(x))(\sin(\arccos(x))) \\ &\stackrel{\text{double angle}}{=} xT_{2t}(x) - 2\cos(t \arccos(x))\sin(t \arccos(x))(\sin(\arccos(x))) \\ &= xT_{2t}(x) - 2T_t(x)\sin(t \arccos(x))(\sin(\arccos(x))) \\ &\stackrel{\text{angle prod}}{=} xT_{2t}(x) - T_t(x)(\cos((t-1)\arccos(x)) - \cos((t+1)\arccos(x))) \\ &= xT_{2t}(x) - T_t(x)(T_{t-1} - T_{t+1}) \end{aligned}$$

Which is again a polynomial all the lower degrees of  $T$  are polynomials by IH, so both cases hold.  $\square$

(b) Derive the leading coefficient of (6) (i.e., the coefficient of  $x^k$ ).

The coefficient is  $2^{k-1}$  for  $k \geq 1$  and 1 for  $k = 0$ .

*Proof.* Again, using induction, the case of 1 and 0 are trivial, as  $T_0(x) = 1$  and  $T_1(x) = x$ . For the inductive case, consider again even and odd, and assume that it holds for  $n < k$ .

For  $k = 2t$  even, the leading coefficient  $\mathcal{LC}(T_k)$ :

$$\begin{aligned} \mathcal{LC}(T_k) &= 2\mathcal{LC}(T_t)^2 \text{ From the recurrence in (a)} \\ &= 2(2^{t-1})^2 = 2(2^{2t-2}) = 2^{k-1} \text{ By IH} \end{aligned}$$

For  $k = 2t + 1$  odd,

$$\begin{aligned} \mathcal{LC}(T_k) &= \mathcal{LC}(T_{k-1}) + \mathcal{LC}(T_t)\mathcal{LC}(T_{t+1}) \text{ From the other recurrence in (a)} \\ &= 2^{k-2} + (2^{t-1})(2^t) \text{ From IH} \\ &= 2^{k-2} + 2^{2t-1} \\ &= 2^{k-2} + 2^{k-2} = 2^{k-1} \end{aligned}$$

□

- (c) Derive the roots of (6) (i.e., the so-called Chebyshev points).

The roots for a polynomial of degree  $k$  are  $x_i = \cos\left(\frac{(2i+1)\pi}{2k}\right)$  where  $i = 0, \dots, k-1$ .

*Proof.* Given the definition for  $T_k$ , plugging in  $x_i$  gives

$$\begin{aligned}\cos(k \arccos(x_i)) &= \cos(k \arccos(\cos\left(\frac{(2i+1)\pi}{2k}\right))) \\ &= \cos\left(k \frac{(2i+1)\pi}{2k}\right) \\ &= \cos\left(\frac{(2i+1)\pi}{2}\right) \\ &= 0 \quad \text{When } i \in \mathbb{N}\end{aligned}$$

□

- (d) Complete the proof showing why the Chebyshev points are optimal.

Since the Chebyshev polynomial is in fact a polynomial, with coefficient  $2^{k-1}$ , we can write it out as  $2^{k-1}(x - x_1)(x - x_2) \cdots (x - x_k)$ , where  $x_i, 1 \leq i \leq k$  are the roots of  $T_k$ . It can also be converted to a monic polynomial by dividing by  $2^{k-1}$ . Note that it is also bound between -1 and 1 because of the cos definition of  $T_k$ . To show that they are optimal, we prove the following:

**Theorem.** If  $W(x)$  is a monic polynomial of degree  $\leq k$ , then  $\max_x |W(x)| \geq \max_x \left| \frac{T_k(x)}{2^{k-1}} \right| = \frac{1}{2^{k-1}}$

*Proof.* Suppose  $\max_x |W(x)| < \max_x \left| \frac{T_k(x)}{2^{k-1}} \right|$ , then in particular  $|W(y_i)| < \left| \frac{T_k(y_i)}{2^{k-1}} \right| = \frac{1}{2^{k-1}}$  where  $y_i = \cos\left(\frac{i\pi}{k}\right)$  since looking at the definition of  $T_k$  these are the absolute maximums, specifically

$$T_k(y_i) = \begin{cases} 1 & i \text{ even} \\ -1 & i \text{ odd} \end{cases}$$

□

Now consider the polynomial  $q(x) = \frac{T_k(x)}{2^{k-1}} - W(x)$  which is in  $\mathcal{P}_{k-1}$  since both polynomials are monic, hence the highest power will cancel. Now look at what happens in the case of plugging in  $y_i$ . So

$$q(y_i) = \frac{T_k(y_i)}{2^{k-1}} - W(y_i) = \begin{cases} > 0 & i \text{ even} \\ < 0 & i \text{ odd} \end{cases}$$

This is the case, since at even points  $T_k$  is positive, and the absolute value of  $W$  is less than  $T_k/2^{k-1}$  so it could not possibly subtract more than 1. Likewise for the  $i$  odd case it cannot add enough since the absolute value is less than  $T_k/2^{k-1}$ . This means that  $q(x)$  has a root in each interval  $[y_i, y_{i+1}]$  for  $i = 0, \dots, k-1$  so  $q$  has  $k$  roots, but  $q \in \mathcal{P}_{k-1}$  so  $q = 0$ , so  $\frac{T_k(x)}{2^{k-1}} - W(x) = 0$ , which is a contradiction to our assumption. Hence,  $\frac{T_k(x)}{2^{k-1}}$  is the smallest possible polynomial that  $W$  could be, so using the Chebyshev points will make  $W$  optimal.