MATB42: Assignment #1
Bonus

1. Show that $\int_{-\pi}^{\pi} \sin(kx)\sin(nx)dx = \begin{cases} 0 & , \ k \neq \pm n \\ \pm\pi & , \ k = \pm n \neq 0 \end{cases}$

Assuming $k \neq \pm n$

$$\int_{-\pi}^{\pi} \sin(kx)\sin(nx)dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \cos(kx - nx) - \cos(kx + nx)dx \qquad\qquad \left[\sin(A)\sin(B) = \frac{1}{2}[\cos(A-B) - \cos(A+B)]\right]$$

$$= \frac{1}{2}\left[\int_{-\pi}^{\pi} \cos((k-n)x)dx - \int_{-\pi}^{\pi} \cos((k+n)x)dx\right]$$

$$= \frac{1}{2}\left[\frac{1}{k-n}\Big[\sin((k-n)x)\Big]_{-\pi}^{\pi} - \frac{1}{k+n}\Big[\sin((k+n)x)\Big]_{-\pi}^{\pi}\right] \qquad\qquad \left[\text{Since cos is even}\right]$$

$k, n \in \mathbb{Z} \implies (k \pm n) \in \mathbb{Z}, \text{ but } \forall z \in \mathbb{Z}, \ \sin(z\pi) = 0$

$= 0$

Assuming $k = \pm n$

$$\int_{-\pi}^{\pi} \sin(kx)\sin(nx)dx$$

$$= \int_{-\pi}^{\pi} \sin(kx)\sin(\pm kx)dx$$

$$= \pm\int_{-\pi}^{\pi} \sin^2(kx)dx \qquad\qquad \left[\text{Since sin odd}\right]$$

$$= \pm\int_{-\pi}^{\pi} \frac{1 - \cos(2kx)}{2}dx$$

$$= \pm\frac{1}{2}\left[\int_{-\pi}^{\pi} 1dx - \int_{-\pi}^{\pi} \cos(2kx)dx\right]$$

$$= \pm\frac{1}{2}\Big[2\pi - 0\Big] \qquad\qquad \left[\text{Since cos even}\right]$$

$$= \pm\pi$$

2. For each of the following functions, find the Nth Fourier polynomial, assuming them to be periodic with period 2π. Use symbolic algebra software to graph the first three approximations together with the original function.

(a) $f(x) = x^2$, $-\pi < x \le \pi$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx$$
$$= \frac{2}{3\pi} \left[x^3\right]_0^{\pi} \quad \text{[Since } x^2 \text{ is even]}$$
$$= \frac{2\pi^2}{3}$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \sin(kx) dx$$
$$= 0 \quad \text{[Since } x^2 \text{ is even but sin is odd]}$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos(kx) dx$$
$$= \frac{2}{\pi} \int_0^{\pi} x^2 \cos(kx) dx \quad \text{[Since } x^2 \text{ and cos are even]}$$

Let $u = x^2, du = 2x dx, dv = \cos(kx)dx, v = \dfrac{\sin(kx)}{k}$

$$= \frac{2}{\pi} \left[\left[\frac{1}{k}x^2 \sin(kx)\right]_0^{\pi} - \frac{2}{k} \int_0^{\pi} x \sin(kx) dx \right]$$

Let $u = x, du = 1 dx, dv = \sin(kx)dx, v = -\dfrac{\cos(kx)}{k}$

$$= \frac{2}{\pi} \left[\left[\frac{1}{k}x^2 \sin(kx)\right]_0^{\pi} - \frac{2}{k} \left(\left[-\frac{1}{k}x \cos(kx)\right]_0^{\pi} - \frac{1}{k} \int_0^{\pi} -\cos(kx) dx \right) \right]$$

$$= \frac{2}{\pi} \left[\left[\frac{1}{k}x^2 \sin(kx)\right]_0^{\pi} - \frac{2}{k} \left(\left[-\frac{1}{k}x \cos(kx)\right]_0^{\pi} + \frac{1}{k^2} \left[\sin(kx)\right]_0^{\pi} \right) \right]$$

$$= \frac{2}{\pi} \left[0 - \frac{2}{k} \left(\left[-\frac{1}{k}x \cos(kx)\right]_0^{\pi} + 0 \right) \right]$$
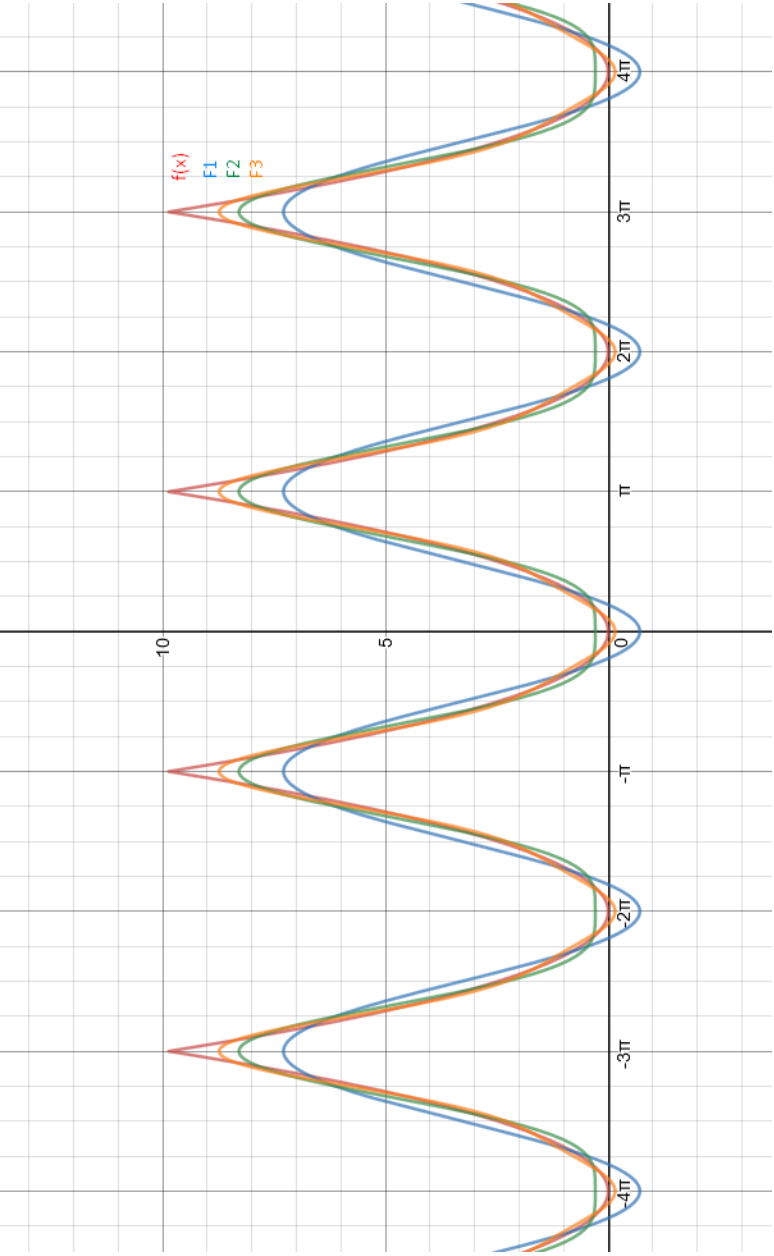
$$= -\frac{4}{k\pi} \left[-\frac{1}{k}x \cos(kx)\right]_0^{\pi}$$

$$= \frac{4}{k^2} \cos(k\pi) - 0$$

$$= \frac{4}{k^2} (-1)^k$$

Therefore the Nth Fourier polynomial is

$$F_N(x) = \frac{\pi^2}{3} + \sum_{k=1}^{N} \frac{4}{k^2}(-1)^k \cos(kx)$$

(b) $f(x) = \begin{cases} 0 & , -\pi \le x < 0 \\ x & , 0 \le x < \pi \end{cases}$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} f(x)dx + \int_{0}^{\pi} f(x)dx \right]$$

$$= \frac{1}{\pi} \left[0 + \int_{0}^{\pi} xdx \right]$$

$$= \frac{1}{\pi} \left[\frac{1}{2} \left[x^2 \right]_{0}^{\pi} \right]$$

$$= \frac{\pi}{2}$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(kx)dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} 0dx + \int_{0}^{\pi} x\cos(kx)dx \right]$$

Let $u = x$, $du = dx$, $dv = \cos(kx)$, $v = \frac{1}{k}\sin(kx)$

$$= \frac{1}{\pi} \left[\frac{1}{k} \left[x\sin(kx) \right]_{0}^{\pi} - \frac{1}{k} \int_{0}^{\pi} \sin(kx)dx \right]$$

$$= -\frac{1}{k\pi} \left[\int_{0}^{\pi} \sin(kx)dx \right]$$

$$= \frac{1}{k^2\pi} \left[\cos(kx) \right]_{0}^{\pi}$$

$$= \frac{(-1)^{-k} - 1}{k^2\pi}$$
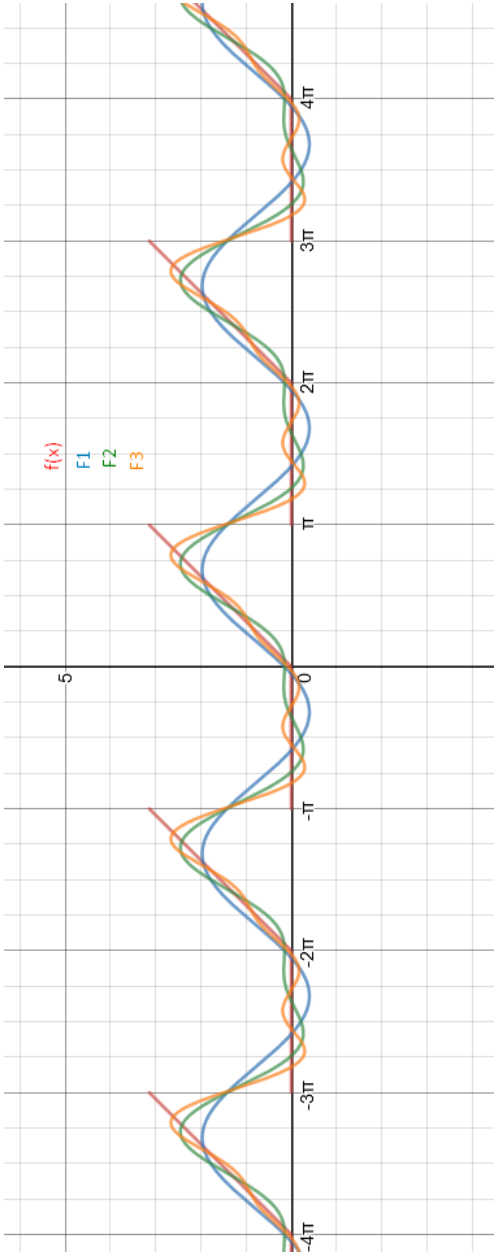
$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin(kx)dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} 0dx + \int_{0}^{\pi} x\sin(kx)dx \right]$$

Let $u = x$, $du = dx$, $dv = \sin(kx)$, $v = -\frac{1}{k}\cos(kx)$

$$= \frac{1}{\pi} \left[-\frac{1}{k} \left[x\cos(kx) \right]_{0}^{\pi} + \frac{1}{k} \int_{0}^{\pi} \cos(kx)dx \right]$$

$$= \frac{1}{k\pi} \left[-\pi\cos(k\pi) + \frac{1}{k} \left[\sin(kx) \right]_{0}^{\pi} \right]$$

$$= \frac{1}{k\pi} \left[-\pi\cos(k\pi) + 0 \right]$$

$$= \frac{(-1)^{k+1}}{k}$$

Therefore the Nth Fourier polynomial is

$$F_N(x) = \frac{\pi}{4} + \sum_{k=1}^{N} \left[\frac{(-1)^k - 1}{k^2\pi} \cos(kx) + \frac{(-1)^{k+1}}{k} \sin(kx) \right]$$

3. Find the Fourier series for the function f(x) having period 2π, one period of which is given by

$$f(x) = \begin{cases} 1 & ,0 \le x < \pi \\ x & ,\pi \le x < 2\pi \end{cases}$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)dx$$

$$= \frac{1}{\pi}\left[\int_0^\pi f(x)dx + \int_{-\pi}^0 f(x)dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^\pi 1dx + \int_\pi^{2\pi} xdx\right] \qquad \text{[Period } 2\pi\text{]}$$

$$= \frac{1}{\pi}\left[\pi + \left[\frac{x^2}{2}\right]_\pi^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\pi + \frac{4\pi^2}{2} - \frac{\pi^2}{2}\right]$$

$$= 1 + \frac{3\pi}{2}$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(kx)dx$$

$$= \frac{1}{\pi}\left[\int_0^\pi \cos(kx)dx + \int_\pi^{2\pi} x\cos(kx)dx\right] \quad \text{[Period } 2\pi\text{]}$$

Let $u = x$, $du = dx$, $dv = \cos(kx)$, $v = \frac{1}{k}\sin(kx)$

$$= \frac{1}{\pi}\left[\frac{1}{k}\left[\sin(kx)dx\right]_0^\pi + \frac{1}{k}\left[x\sin(kx)\right]_\pi^{2\pi} - \frac{1}{k}\int_\pi^{2\pi} \sin(kx)dx\right]$$

$$= \frac{-1}{k\pi}\left[\int_\pi^{2\pi} \sin(kx)dx\right]$$

$$= \frac{1}{k^2\pi}\left[\cos(kx)dx\right]_\pi^{2\pi}$$

$$= \frac{1 - (-1)^k}{k^2\pi}$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin(kx)dx$$

$$= \frac{1}{\pi}\left[\int_0^\pi \sin(kx)dx + \int_\pi^{2\pi} x\sin(kx)dx\right] \quad \text{[Period } 2\pi\text{]}$$

Let $u = x$, $du = dx$, $dv = \sin(kx)$, $v = -\frac{1}{k}\cos(kx)$

$$= \frac{1}{\pi}\left[\frac{-1}{k}\left[\cos(kx)\right]_0^\pi + \frac{-1}{k}\left[x\cos(kx)\right]_\pi^{2\pi} + \frac{1}{k}\int_\pi^{2\pi} \cos(kx)dx\right]$$

$$= \frac{1}{k\pi}\left[-\cos(k\pi) + 1 + -2\pi + \pi\cos(k\pi) + \int_\pi^{2\pi} \cos(kx)dx\right]$$

$$= \frac{1}{k\pi}\left[(\pi - 1)(-1)^k + 1 - 2\pi + \left[\sin(kx)\right]_\pi^{2\pi}\right]$$

$$= \frac{1}{k\pi}\left[(\pi - 1)(-1)^k + 1 - 2\pi\right]$$

$$= \frac{(\pi - 1)(-1)^k + 1 - 2\pi}{k\pi}$$

Therefore the Fourier series for $f(x)$ is

$$F(x) = \frac{2 + 3\pi}{4} + \sum_{k=1}^{\infty}\left[\frac{1 - (-1)^k}{k^2\pi}\cos(kx) + \frac{(\pi - 1)(-1)^k + 1 - 2\pi}{k\pi}\sin(kx)\right]$$

4. Let $f(x) = \begin{cases} 0 & ,-\pi \leq x < -1 \\ \frac{1}{2} & ,-1 \leq x < 1 \\ 0 & ,1 \leq x < \pi \end{cases}$

(a) What fraction of the energy of f is contained in the constant term of its Fourier series?

$$E(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2 dx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$= \frac{1}{\pi}\left[0 + \int_{-1}^{1}\frac{1}{2}dx + 0\right]$$

$$= \frac{1}{\pi}\left[0 + \int_{-1}^{1}\frac{1}{4}dx + 0\right]$$

$$= \frac{1}{2\pi}$$

$$= \frac{1}{\pi}$$

$$E(a_0) = \frac{1}{2}a_0^2$$

$$\frac{E(a_0)}{E(f)} = \frac{\frac{1}{2\pi^2}}{\frac{1}{2\pi}}$$

$$= \frac{1}{\pi}\left[0 + \int_{-1}^{1}\frac{1}{4}dx + 0\right]$$

$$= \frac{1}{\pi}$$

$$= \frac{1}{2\pi^2}$$

$$\approx 32\%$$

(b) Find a formula for the energy of the kth harmonic of f.

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx$$

$$= 0 \text{ [Since sin is odd and } f \text{ is even]}$$

$$= \frac{1}{2\pi}\int_{-1}^{1}\cos(kx)dx$$

$$= \frac{1}{\pi}\int_{0}^{1}\cos(kx)dx \qquad \text{[cos is even]}$$

$$= \frac{1}{k\pi}\sin(k)$$

$$E(k^{th} \text{ harmonic}) = A_k^2 = \sqrt{a_k^2 + b_k^2}^2 = a_k^2$$

$$= \frac{\sin^2(k)}{k^2\pi^2}$$

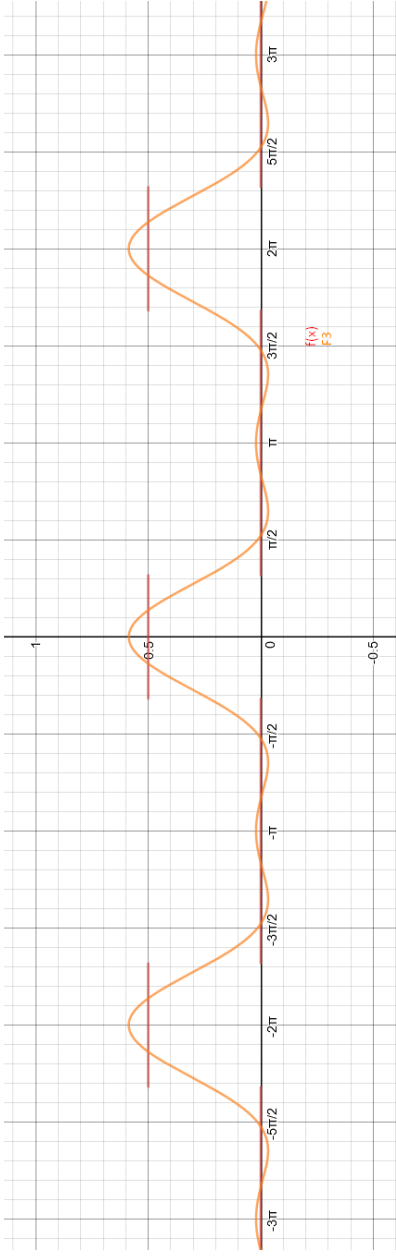(c) How many terms of the Fourier series of f are needed to capture 80% of the energy of f?

$$\frac{\sin^2(k)}{k^2\pi^2}\Big/\frac{1}{2\pi} = \frac{2\sin^2(1)}{1^2\pi} \approx 45\%$$

$$\frac{2\sin^2(2)}{2^2\pi} \approx 13\%$$

$$32\% + 45\% + 13\% = 90\% \geq 80\%$$

Including the constant term, three terms are required to capture 80% of the energy of f.

(d) Find F_N, the Nth Fourier polynomial of f, and use symbolic algebra software to graph f and F_3 on the interval $[-3\pi, 3\pi]$.

$$F_N(x) = \frac{1}{2\pi} + \sum_{k=1}^{N}\left[\frac{\sin(k)}{k\pi}\cos(kx)\right]$$

5. Find the Fourier series for the function $f(x)$ (of period 4) which corresponds to $y = x^2 - 4$ on the interval $[-2, 2]$.

$$a_k = \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{2\pi k x}{4}\right)dx$$

$$= \int_{0}^{2}(x^2 - 4)\cos\left(\frac{\pi k x}{2}\right)dx \quad [f(x) \text{ and } \cos \text{ are even}]$$

$$a_0 = \frac{2}{4}\int_{-2}^{2} f(x)dx$$

$$= \int_{0}^{2} x^2 \cos\left(\frac{\pi k x}{2}\right)dx - 4\int_{0}^{2}\cos\left(\frac{\pi k x}{2}\right)dx$$

$$= \int_{0}^{2} x^2 - 4dx \quad [f(x) \text{ is even}]$$

Let $u = x^2, du = 2xdx, dv = \cos\left(\frac{\pi k x}{2}\right)dx, v = \frac{2\sin\left(\frac{\pi k x}{2}\right)}{k\pi}$

$$= \left[\frac{1}{3}\left[x^3\right]_0^2 - 4\left[x\right]_0^2\right]$$

$$= \frac{2}{k\pi}\left[x^2\sin\left(\frac{\pi k x}{2}\right)\right]_0^2 - \frac{4}{k\pi}\int_{0}^{2} x\sin\left(\frac{\pi k x}{2}\right)dx$$

$$= \left[\frac{8}{3} - 8\right]$$

$$- \frac{8}{k\pi}\left[\sin\left(\frac{\pi k x}{2}\right)\right]_0^2$$

$$= -\frac{16}{3}$$

$$= \frac{8}{k\pi}\sin\left(\frac{\pi k x}{2}\right) - \frac{4}{k\pi}\int_{0}^{2} x\sin\left(\frac{\pi k x}{2}\right)dx - \frac{8}{k\pi}\sin\left(\frac{\pi k x}{2}\right)$$

Let $u = x, du = 1dx, dv = \sin\left(\frac{\pi k x}{2}\right)dx, v = -\frac{2\cos\left(\frac{\pi k x}{2}\right)}{k\pi}$

$$= -\frac{4}{k\pi}\left[-\frac{2}{k\pi}\left[x\cos\left(\frac{\pi k x}{2}\right)\right]_0^2 + \frac{2}{k\pi}\int_{0}^{2}\cos\left(\frac{\pi k x}{2}\right)dx\right]$$

$$= -\frac{4}{k\pi}\left[-\frac{4\cos(\pi k)}{k\pi} + \frac{2}{k^2\pi^2}\left[\sin\left(\frac{\pi k x}{2}\right)\right]_0^2\right]$$

$$= -\frac{4}{k\pi}\left[-\frac{4(-1)^k}{k\pi} + \frac{\sin(\pi k)}{k^2\pi^2}\right]$$

$$b_k = \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{2\pi k x}{4}\right)dx$$

$$= \frac{16(-1)^k}{k^2\pi^2}$$

$$= 0 \text{ [Since } f \text{ is even, but sin is odd]}$$

Therefore the Fourier series for $f(x)$ is

$$F(x) = -\frac{8}{3} + \sum_{k=1}^{\infty}\frac{16(-1)^k}{k^2\pi^2}\cos\left(\frac{k\pi x}{2}\right)$$

6. (a) Suppeose $f(x)$ has a continuous derivative $f'(x)$ on $[0, 2\pi]$. Let a_k and b_k the kth Fourier coefficients of f and let a'_k and b'_k be those of f'. Show that

$$a'_k = kb_k + \frac{f(2\pi) - f(0)}{\pi}$$

$$b'_k = -ka_k.$$

By definition,

$$a'_k = \frac{1}{\pi} \int_0^{2\pi} f'(x)\cos(kx)dx$$

Let $u = \cos(kx)$, $du = -k\sin(kx)dx$, $dv = f'(x)dx$, $v = f(x)$

$$= \frac{1}{\pi}\left[\left[f(x)\cos(kx)\right]_0^{2\pi} + k\int_0^{2\pi}\sin(kx)f(x)dx\right]$$

$$= \frac{1}{\pi}\left[\left[f(x)\cos(kx)\right]_0^{2\pi} + k\pi\left[\frac{1}{\pi}\int_0^{2\pi}\sin(kx)f(x)dx\right]\right]$$

$$= \frac{1}{\pi}\left[f(2\pi)\cos(2k\pi) - f(0)\cos(0) + k\pi b_k\right] \quad \text{[Def of } b_k\text{]}$$

$$= \frac{1}{\pi}\left[f(2\pi) - f(0) + k\pi b_k\right]$$

$$= kb_k + \frac{f(2\pi) - f(0)}{\pi} \quad \text{As wanted}$$

By definition,

$$b'_k = \frac{1}{\pi}\int_0^{2\pi} f'(x)\sin(kx)dx$$

Let $u = \sin(kx)$, $du = k\cos(kx)dx$, $dv = f'(x)dx$, $v = f(x)$

$$= \frac{1}{\pi}\left[\left[f(x)\sin(kx)\right]_0^{2\pi} - k\int_0^{2\pi}\cos(kx)f(x)dx\right]$$

$$= \frac{1}{\pi}\left[\left[f(x)\sin(kx)\right]_0^{2\pi} - k\pi\left[\frac{1}{\pi}\int_0^{2\pi}\cos(kx)f(x)dx\right]\right]$$

$$= \frac{1}{\pi}\left[-f(2\pi)\sin(2k\pi) + f(0)\sin(0) - k\pi a_k\right] \quad \text{[Def of } a_k\text{]}$$

$$= -ka_k \quad \text{As wanted}$$

(b) Use part (a) to find all the Fourier coefficients of the restriction of $f(x) = e^{\lambda x}$ to the interval $[0, 2\pi]$ in terms of the constant λ.

Since $\frac{d}{dx}e^{\lambda x} = \lambda e^{\lambda x}$, the coefficients of every term of the Fourier polynomial of the derivative function will simply be that of the original, multiplied by λ.

$$\lambda a_k = kb_k + \frac{e^{2\pi\lambda} - 1}{\pi}$$

$$\lambda a_k = -k\left(\frac{ka_k}{\lambda}\right) + \frac{e^{2\pi\lambda} - 1}{\pi}$$

$$\lambda a_k + \frac{k^2 a_k}{\lambda} = \frac{e^{2\pi\lambda} - 1}{\pi}$$

$$a_k\left(\lambda + \frac{k^2}{\lambda}\right) = \frac{e^{2\pi\lambda} - 1}{\pi}$$

$$a_k = \frac{e^{2\pi\lambda} - 1}{\pi\left(\lambda + \frac{k^2}{\lambda}\right)}$$

$$a_k = \frac{e^{2\pi\lambda} - 1}{\frac{\pi(\lambda^2 + k^2)}{\lambda}}$$

$$a_k = \frac{\lambda(e^{2\pi\lambda} - 1)}{\pi(\lambda^2 + k^2)}$$

$$\lambda b_k = -ka_k$$

$$b_k = -\frac{ka_k}{\lambda}$$

$$b_k = -\frac{k}{\lambda}\left(\frac{\lambda(e^{2\pi\lambda} - 1)}{\pi(\lambda^2 + k^2)}\right)$$

$$b_k = -\frac{k(e^{2\pi\lambda} - 1)}{\pi(\lambda^2 + k^2)}$$

7. Find two Fourier expansions for the restriction of the function $f(x) = \sin x$ to the interval $[0, \pi]$. In one expansion all the sine terms should have zero coefficient, in the other all cosine terms should have coefficient.

Even expansion (sine terms have zero coefficient)

$$f(-x) = \begin{cases} -\sin(x), & \text{if } -\pi \le x < 0 \\ \sin(x), & \text{if } 0 \le x < \pi \end{cases}$$

This works since sin odd, so $-\sin(-x) = \sin(x)$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \sin(x) \cos(kx) dx$$

$$= \frac{1}{\pi} \left[\int_0^{\pi} \sin((k+1)x) dx - \int_0^{\pi} \sin((k-1)x) dx \right]$$

$$= \frac{1}{\pi} \left[-\frac{1}{k+1} \Big[\cos((k+1)x) \Big]_0^{\pi} + \frac{1}{k-1} \Big[\cos((k-1)x) \Big]_0^{\pi} \right]$$

$$= \frac{1}{\pi} \left[-\frac{(-1)^{(k+1)} - 1}{k+1} + \frac{(-1)^{(k-1)} - 1}{k-1} \right]$$

$$= \frac{(-1)^{(k+1)} - 1}{\pi} \left[\frac{1}{k-1} - \frac{1}{k+1} \right]$$

$$= \frac{(-1)^{(k+1)} - 1}{\pi} \left[\frac{k+1-k+1}{k^2 - 1^2} \right]$$

$$= 2 \frac{(-1)^{(k+1)} - 1}{\pi(k^2 - 1)} \text{ [Valid for } k > 1]$$

For k = 1

$$= \frac{1}{\pi} \left[\int_0^{\pi} \sin(2x) dx - \int_0^{\pi} \sin((1-1)x) dx \right]$$

$$= \frac{1}{\pi} \int_0^{\pi} \sin(2x) dx$$

$$= -\frac{1}{2\pi} \Big[\cos(2x) \Big]_0^{\pi}$$

$$= -\frac{1}{2\pi} \Big[1 - 1 \Big]$$

$$= 0$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} f(x) dx \text{ [Defined to be even]}$$

$$= \frac{2}{\pi} \int_0^{\pi} \sin(x) dx$$

$$= \frac{2}{\pi} \left[-\Big[\cos(x) \Big]_0^{\pi} \right]$$

$$= \frac{2}{\pi} \Big[1 + 1 \Big]$$

$$= \frac{4}{\pi}$$

$$b_k = 0 \qquad \text{[Since } f \text{ defined even]}$$

Therefore the even Fourier expansion for $f(x)$ is

$$F_N = \frac{2}{\pi} + \sum_{k=2}^{\infty} 2 \frac{(-1)^{(k+1)} - 1}{\pi(k^2 - 1)} \cos(kx)$$

Odd expansion (cosine terms have zero coefficient)

$$f(-x) = \sin(x), \text{ on } [-\pi, \pi]$$

This works, since sin is already an odd function

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)dx$$
$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(x)dx$$
$$= 0 \text{ [sin is odd]}$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin(kx)dx$$
$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(x)\sin(kx)dx$$
$$= 1 \text{ if } k = 1, \ 0 \text{ otherwise [Refer to q1]}$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(kx)dx$$
$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(x)\cos(kx)dx$$
$$= 0 \text{ [sin is odd, cos even]}$$

So the Fourier expansion is simply $\sin(x)$

Bonus

Find the Fourier series for the function $y = f(x)$ of period 2π , if one period is given by

$$f(x) = \begin{cases} -\frac{1}{2} - \frac{x}{2\pi} & , -\pi \le x < 0 \\ \frac{1}{2} - \frac{x}{2\pi} & , 0 \le x < \pi \end{cases}$$

This function is odd over the period since $(-\frac{1}{2} - \frac{-x}{2\pi}) = (-\frac{1}{2} + \frac{x}{2\pi}) = -(\frac{1}{2} - \frac{x}{2\pi})$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)dx$$

$$= 0 \quad [\text{Since } f \text{ odd}]$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(kx)dx$$

$$= 0 \quad [\text{Since } f \text{ odd and cos even}]$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin(kx)dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} \left[-\frac{1}{2} - \frac{x}{2\pi} \right] \sin(kx)dx + \int_{0}^{\pi} \left[\frac{1}{2} - \frac{x}{2\pi} \right] \sin(kx)dx \right]$$

$$= \frac{1}{\pi} \left[-\int_{-\pi}^{0} \left[\frac{(\pi + x)\sin(kx)}{2\pi} \right] dx + \int_{0}^{\pi} \left[\frac{(\pi - x)\sin(kx)}{2\pi} \right] dx \right]$$

Let $u = (\pi + x), du = 1dx, dv = \sin(kx)dx, v = -\dfrac{\cos(kx)}{k}$

$$= \frac{1}{2\pi^2} \left[\int_{0}^{\pi} \left[(\pi - x)\sin(kx) \right] dx + \left[(\pi + x)\frac{\cos(kx)}{k} \right]_{-\pi}^{0} \right.$$

$$\left. - \int_{-\pi}^{0} \frac{\cos(kx)}{k} dx \right]$$

$$= \frac{1}{2\pi^2} \left[\int_{0}^{\pi} \left[(\pi - x)\sin(kx) \right] dx + \frac{\pi}{k} - \frac{1}{k^2} \left[\sin(kx) \right]_{-\pi}^{0} \right]$$

$$= \frac{1}{2\pi^2} \left[\int_{0}^{\pi} \left[(\pi - x)\sin(kx) \right] dx + \frac{\pi}{k} \right]$$

Let $u = (\pi - x), du = -1dx, dv = \sin(kx)dx, v = -\dfrac{\cos(kx)}{k}$

$$= \frac{1}{2\pi^2} \left[-\left[\frac{(\pi - x)\cos(kx)}{k} \right]_{0}^{\pi} - \int_{0}^{\pi} \frac{\cos(kx)}{k} dx + \frac{\pi}{k} \right]$$

$$= \frac{1}{2\pi^2} \left[\frac{\pi}{k} - \left[\frac{\sin(kx)}{k^2} \right]_{0}^{\pi} + \frac{\pi}{k} \right]$$

$$= \frac{1}{k\pi}$$

Therefore the Fourier series for $f(x)$ is

$$F(x) = \sum_{k=1}^{\infty} \frac{1}{k\pi} \sin(kx)$$