MATB42: Assignment #8

1. (a) Let  $f, g : \mathbb{R}^n \to \mathbb{R}$ ;  $\mathbf{F}, \mathbf{G} : \mathbb{R}^n \to \mathbb{R}^n$ ; and define  $\Delta$ , the Laplacian, by  $\Delta f = \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2}$ .

Verify the following identities

(i)  $\operatorname{div}(\mathbf{F} + \mathbf{G}) = \operatorname{div} \mathbf{F} + \operatorname{div} \mathbf{G}$ .

$$\operatorname{div}(\boldsymbol{F} + \boldsymbol{G}) = \sum_{i=1}^{n} \frac{\partial (F_i + G_i)}{\partial x_i} = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i} + \frac{\partial G_i}{\partial x_i} = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i} + \sum_{i=1}^{n} \frac{\partial G_i}{\partial x_i} = \operatorname{div} \boldsymbol{F} + \operatorname{div} \boldsymbol{G}$$

(ii)  $\operatorname{div}(f\mathbf{F}) = f \operatorname{div} \mathbf{F} + \mathbf{F} \cdot \operatorname{grad} f$ .

$$\operatorname{div}(f\mathbf{F}) = \sum_{i=1}^{n} \frac{\partial f F_{i}}{\partial x_{i}} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_{i}} F_{i} + \frac{\partial F_{i}}{\partial x_{i}} f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_{i}} F_{i} + f \sum_{i=1}^{n} \frac{\partial F_{i}}{\partial x_{i}} = \nabla f \cdot \mathbf{F} + f \operatorname{div} \mathbf{F}$$

(iii)  $\Delta(fg) = f\Delta g + g\Delta f + 2(\operatorname{grad} f) \cdot (\operatorname{grad} g)$ .

$$\begin{split} \Delta(fg) &= \sum_{i=1}^n \frac{\partial^2 fg}{\partial^2 x_i} \\ &= \sum_{i=1}^n \frac{\partial}{\partial x_i} \bigg[ \frac{\partial f}{\partial x_i} g + \frac{\partial g}{\partial x_i} f \bigg] \end{split}$$

(b) Let  $f, g: D \subset \mathbb{R}^3 \to \mathbb{R}$  be of class  $C^1$ . If R is a solid region contained in D then

$$\iiint_{R} \nabla f \cdot \nabla g \, dV = \iint_{\partial R} f \nabla g \cdot \boldsymbol{n} \, dS - \iiint_{R} f \nabla^{2} g \, dV$$

- 2. Use the Divergence Theorem to verify your asswer to question 7 on assignment 8.
- 3. Let  $F(x, y, z) = (x, y^2, e^{yz})$  and let R be a cube centered at the origin with sides of length 2. Evaluate  $\int_S \operatorname{div} F dV$  directly and by using the Divergence Theorem.
- 4. Let B be the pyramid with top vertex (0,0,1) and base vertices (0,0,0), (1,0,0), (0,1,0) and (1,1,0). Let S be the 2-dim closed surface bounding B, oriented in the outward direction. Use Gauss' theorem to calculate  $\int_{S} \mathbf{F} \cdot d\mathbf{S}$ , where  $\mathbf{F}(x,y,z) = (x^2y, 3y^2z, 9z^2x)$ .
- 5. Use the Divergence Theorem to evaluate  $\int_S \mathbf{F} \cdot d\mathbf{S}$ , where  $\mathbf{F}(x,y,z) = \left(z^2 x, \frac{y^3}{3} + \tan z, x^2 z + y^2\right)$  and S is the top half of the unit sphere  $x^2 + y^2 + z^2 = 1$ , oriented by the unit normalwhich points away from the origin.
- 6. Let the electric field from a point source at the origin be given by  $E(x) = \frac{x}{\|x\|^3}$ 
  - (a) What is the outward flux of E across the surface  $\frac{x^2}{3} + \frac{2y^2}{5} + z^2 = 7$ .
  - (b) Show that the flux of  $\boldsymbol{E}$  across that part of the sphere  $x^2+y^2+z^2=25$  with  $z\geq 3$  is equal to the flux across that part of the plane z=3 with  $x^2+y^2\leq 16$ .
- 7. Let  $f:\mathbb{R}^n\to\mathbb{R}$  be given by  $f(x,y,z)=x^2yz$  and let  $\eta$  be the 2-form on  $\mathbb{R}^3$  given by

$$\eta = (\sin x) dx dy + (e^y + xyz) dx dz + (x^2y^2) dy dz$$

(a) Compute df and  $d\eta$ .

 $(\nabla^2 q = \operatorname{div}(\nabla q)).$ 

(b) Evaluate  $df \wedge \eta$ .