Poon, Keegan
1002423727
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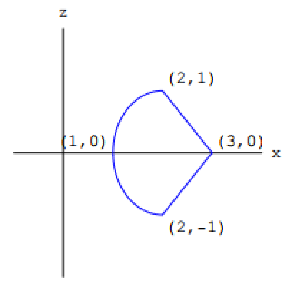
MATB42: Assignment #8

1. A surface S is obtained by rotation the given figure in the xy-plane about the z-axis. (The arc is part of a circle of radius 1 centered at (2,0).)



   (a) Paratemetrize S (in pieces) and compute the surface area.

   We have that the upper line when rotated, can be parametrized by a restricted cone and similarly for the bottom. The top and bottom respectively can be written as

$$\mathbf{\Phi}_1(u,\theta) = ((1-u)3\cos\theta, (1-u)3\sin\theta, 3u),\ 0 \le u \le 1,\ 0 \le \theta \le 2\pi$$
$$\mathbf{\Phi}_2(u,\theta) = ((1+u)3\cos\theta, (1+u)3\sin\theta, -3u),\ -1 \le u \le 0,\ 0 \le \theta \le 2\pi$$

For the circular portion to the left, when rotated around, it will be the inner half of a torus, so the surface is

$$\mathbf{\Phi}_3(\theta,\varphi) = ((2+\cos\varphi)\cos\theta, (2+\cos\varphi)\sin\theta, \sin\phi)$$

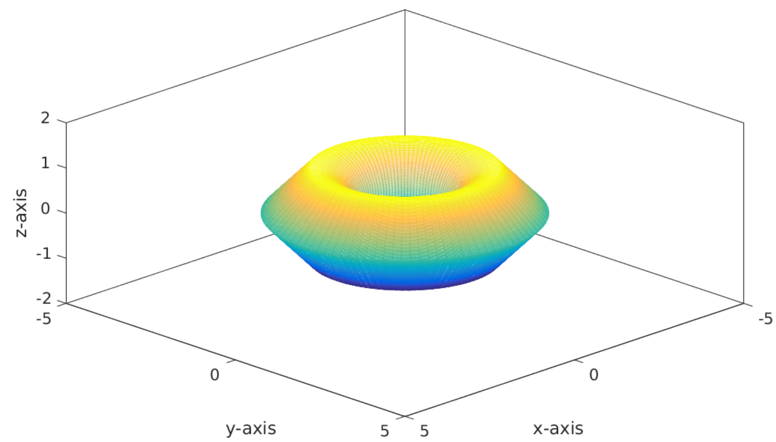
since it has radius 2 from the origin, and radius 1 from the center of the tube. Also $\theta \in [0, 2\pi]$, but φ restricted to $[\pi/2, 3\pi/2]$ for only the inner half.

$$\mathcal{A}(S) = \int_S dS = \int_{\mathbf{\Phi}_1} dS + \int_{\mathbf{\Phi}_2} dS + \int_{\mathbf{\Phi}_3} dS$$

$$\boldsymbol{\phi}_{1_u} = (-3\cos\theta, -3\sin\theta, 3) \qquad \boldsymbol{\phi}_{1_\theta} = (-(1-u)3\sin\theta, (1-u)3\cos\theta, 0)$$
$$\boldsymbol{\phi}_{2_u} = (3\cos\theta, 3\sin\theta, -3) \qquad \boldsymbol{\phi}_{2_\theta} = (-(1+u)3\sin\theta, (1+u)3\cos\theta, 0)$$

$$\begin{aligned} \boldsymbol{\phi}_{1_u} \times \boldsymbol{\phi}_{1_\theta} &= (-9(1-u)\cos\theta, -9(1-u)\sin\theta, -9(1-u)) \\ \implies \|\boldsymbol{\phi}_{1_u} \times \boldsymbol{\phi}_{1_\theta}\| &= \sqrt{2(9^2(1-u)^2)} = \sqrt{2}(9(1-u)) \\ \boldsymbol{\phi}_{2_u} \times \boldsymbol{\phi}_{2_\theta} &= (9(1+u)\cos\theta, 9(1+u)\sin\theta, 9(1+u)) \\ \implies \|\boldsymbol{\phi}_{2_u} \times \boldsymbol{\phi}_{2_\theta}\| &= \sqrt{2(9^2(1-u)^2)} = \sqrt{2}(9(1-u)) \end{aligned}$$

   (b) Use a computer algebra system to sketch S.


2
1
0
-1
-2
z-axis
-5
0
5
y-axis
5
0
-5
x-axis

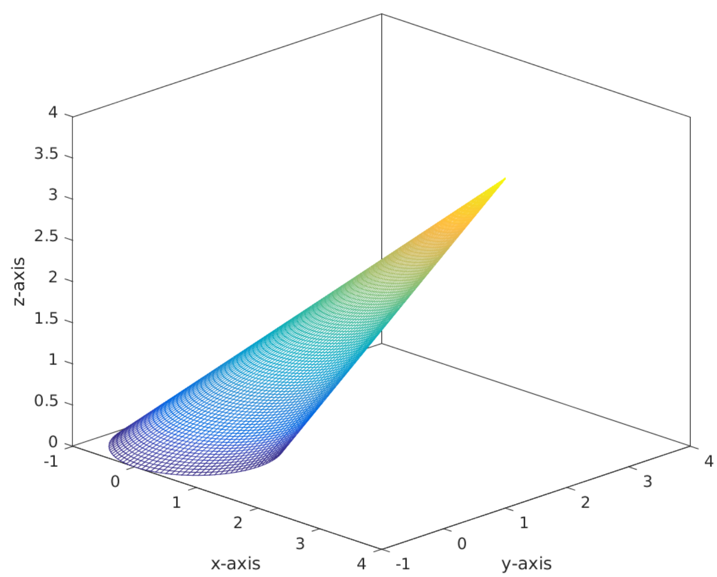
2. Let S be the cone with vertex (2,3,3) and base the circle $x^2 + y^2 = 1$ in the xy-plane.

   (a) Paratemetrize S

   Starting with a base of a circle, we get $(\cos\theta, \sin\theta, 1)$ with $0 \le \theta \le 2\pi$. To change into a cone multiply x and y by $(1-u)$ with $0 \le u \le 1$ and finally to shift the vertex, add $(2u, 3u, 2u)$ where $z = 2u$ since the base equation already has a 1, so $1 + ku <= 3 \implies k \le 2$.

$$\implies \mathbf{\Phi}(u,\theta) = ((1-u)\cos\theta + 2u, (1-u)\sin\theta + 3u, 1 + 2u)$$

   (b) Use a computer algebra system to sketch S.



   (c) Write down the integral that would give the surface area of S. (You are not expected to evaluate the integral.)

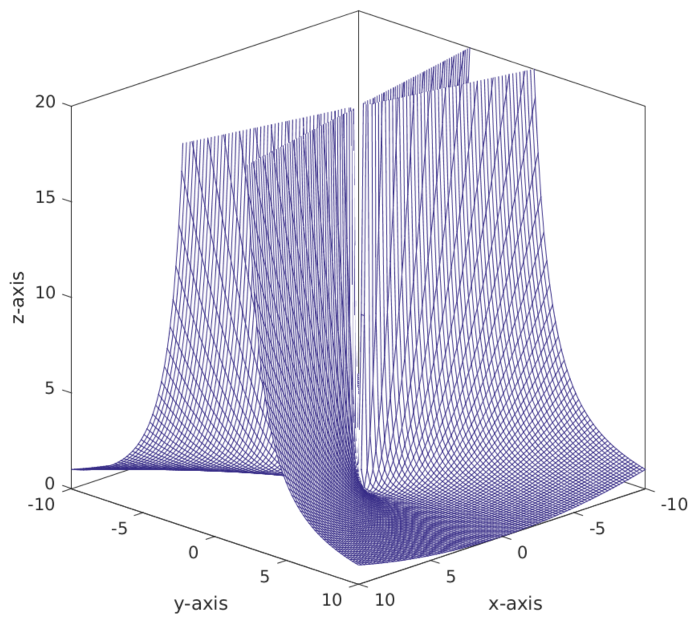
$$\begin{aligned}
\boldsymbol{\phi}_\theta &= (-(1-u)\sin\theta,\ (1-u)\cos\theta,\ 0)\\
\boldsymbol{\phi}_u &= (-\cos\theta + 2,\ -\sin\theta + 3,\ 2)\\
\boldsymbol{\phi}_\theta \times \boldsymbol{\phi}_u &= ((2(1-u)\cos\theta), (2(1-u)\sin\theta),\\
&\quad (-(1-u)\sin\theta)(-\sin\theta + 3) - ((1-u)\cos\theta)(-\cos\theta + 2))\\
&= ((2-2u)\cos\theta, (2-2u)\sin\theta, (1-u)\sin^2\theta - (3-3u)\sin\theta + (1-u)\cos^2\theta - (2-2u)\cos\theta)\\
&= ((2-2u)\cos\theta, (2-2u)\sin\theta, (1-u) - (3-3u)\sin\theta - (2-2u)\cos\theta)\\
\|\boldsymbol{\phi}_\theta \times \boldsymbol{\phi}_u\| &= \sqrt{(2-2u)^2\cos^2\theta + (2-2u)^2\sin^2\theta + ((1-u) - (3-3u)\sin\theta - (2-2u)\cos\theta)^2}\\
&= \sqrt{(2-2u) + ((1-u) - (3-3u)\sin\theta - (2-2u)\cos\theta)^2}\\
\implies \mathcal{A}(S) &= \int_0^1 \int_0^{2\pi} \sqrt{(2-2u) + ((1-u) - (3-3u)\sin\theta - (2-2u)\cos\theta)^2}\, d\theta\, du
\end{aligned}$$

3. Let S be the self-intersecting rectangle in $\mathbb{R}^3$ given by the implicit equation $x^2 - y^2 z = 0$.

   (a) Give a parametrization of S and use a computer algebra system to provide a sketch.

$$x^2 - y^2 z = 0 \implies y^2 z = x^2 \implies z = \left(\frac{x}{y}\right)^2$$

$$\mathbf{\Phi}(x, y) = \left(x, y, \left(\frac{x}{y}\right)^2\right)$$



   (b) Is your parametrization one-to-one? Explain.
   Yes, if $\mathbf{\Phi}(x_0, y_0) = \mathbf{\Phi}(x_1, y_1)$ then $\Phi_1(x_0, y_0) = \Phi_2(x_1, y_1) \implies x_0 = x_1$, and $\Phi_2(x_0, y_0) = \Phi_2(x_1, y_1) \implies y_0 = y_1$. This means that $\mathbf{\Phi}(x_0, y_0) = \mathbf{\Phi}(x_1, y_1) \implies (x_0, y_0) = (x_1, y_1)$ so it is one to one.

   (c) Find the equation of the tangent plane to S at $\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)$.

$$\boldsymbol{\phi}_x = \left(1, 0, \frac{2x}{y^2}\right), \ \boldsymbol{\phi}_y = \left(0, 1, \frac{-2x^2}{y^3}\right)$$

$$\boldsymbol{\phi}_x \times \phi_y = \left(\frac{-2x}{y^2}, \frac{2x^2}{y^3}, 1\right)$$

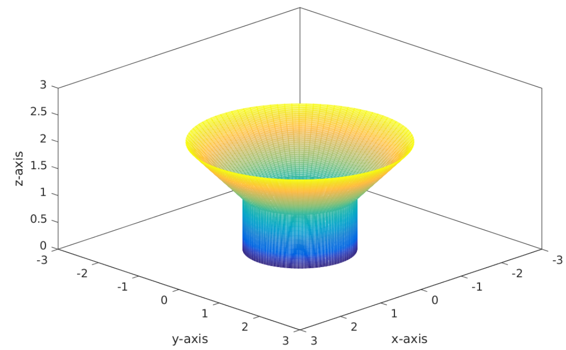
$$(\boldsymbol{\phi}_x \times \phi_y)\left(\frac{1}{4}, \frac{1}{2}\right) = \left(\frac{-1/2}{1/4}, \frac{1/8}{1/8}, 1\right) = (-2, 1, 1)$$

So the tangent plane is defined by the equation

$$(-2)(x - 1/4) + (y - 1/2) + (z - 1/4) = 0 \Leftrightarrow -2x + y + z = 1/4$$

4. Let S be the surface defined by $x^2 + y^2 = 1$ for $0 \le z \le 1$ and by $x^2 + y^2 = z^2$ for $1 \le z \le 2$.

(a) Use symbolic algebra software to sketch S.



(b) Evaluate $\int_S \boldsymbol{F} \cdot d\boldsymbol{S}$ where $\boldsymbol{F}(x, y, z) = (-y, x, z)$ and S is oriented by outward pointing normals.
S can be parametrized piecewise by $\boldsymbol{\Phi}_1(u, \theta) = (\cos\theta, \sin\theta, u)$ for $0 \le u \le 1$ and $\boldsymbol{\Phi}_2(u, \theta) = (u\cos\theta, u\sin\theta, u)$ for $1 \le u \le 2$
The derivatives of each are

$$\boldsymbol{\phi}_{1_u} = (0, 0, 1) \qquad \boldsymbol{\phi}_{1_\theta} = (-\sin\theta, \cos\theta, 0)$$
$$\boldsymbol{\phi}_{2_u} = (\cos\theta, \sin\theta, 1) \qquad \boldsymbol{\phi}_{2_\theta} = (-u\sin\theta, u\cos\theta, 0)$$

So their respective normals are

$$\boldsymbol{\phi}_{1_u} \times \boldsymbol{\phi}_{1_\theta} = (-\cos\theta, -\sin\theta, 0) \qquad \boldsymbol{\phi}_{2_u} \times \boldsymbol{\phi}_{2_\theta} = (-u\cos\theta, -u\sin\theta, u)$$

Examining the rightmost point (If projected into the xy-plane), where $\theta = 0$, both vectors will point towards the left as $-\cos(0) = -1$ (since $u > 0$). These normals are orientation reversing, so their integrals will need to be of the opposite sign.

$$\int_S \boldsymbol{F}\, d\boldsymbol{S} = \int_{\boldsymbol{\Phi}_1} \boldsymbol{F}(\boldsymbol{\Phi}_1) \cdot d\boldsymbol{S} + \int_{\boldsymbol{\Phi}_2} \boldsymbol{F}(\boldsymbol{\Phi}_2) \cdot d\boldsymbol{S}$$

$$\int_{\boldsymbol{\Phi}_1} \boldsymbol{F}(\boldsymbol{\Phi}_1) \cdot d\boldsymbol{S} = -\int_0^1 \int_0^{2\pi} -(\sin\theta)(-\cos\theta) + (\cos\theta)(-\sin\theta) + (u)(0)\, d\theta\, du = 0$$
$$\int_{\boldsymbol{\Phi}_2} \boldsymbol{F}(\boldsymbol{\Phi}_2) \cdot d\boldsymbol{S} = -\int_0^1 \int_0^{2\pi} -(u\sin\theta)(-u\cos\theta) + (u\cos\theta)(-u\sin\theta) + (u)(u)\, d\theta\, du$$
$$= -\int_0^1 \int_0^{2\pi} u^2\, d\theta\, du = -\frac{2\pi}{3}$$

5. Evaluate the (vector) surface integral $\int_S \boldsymbol{F} \cdot d\boldsymbol{S}$ in each of the following cases.

(a) $\boldsymbol{F}(x, y, z) = (1, x, z)$, S is the upper hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$, with $\boldsymbol{n}$ pointing upward. A parametrization for S is given by $\boldsymbol{\Phi}(\theta, \varphi) = (\cos\theta \sin\varphi, \sin\theta \sin\varphi, \cos\varphi)$, where $0 \leq \theta \leq 2\pi$ and $0 \leq \varphi \leq \frac{\pi}{2}$

$$\begin{aligned}
\boldsymbol{\phi}_\theta &= (-\sin\theta\sin\varphi, \cos\theta\sin\varphi, 0) \\
\boldsymbol{\phi}_\varphi &= (\cos\theta\cos\varphi, \sin\theta\cos\varphi, -\sin\varphi) \\
\boldsymbol{\phi}_\theta \times \boldsymbol{\phi}_\varphi &= (-\cos\theta\sin^2\varphi, -\sin\theta\sin^2\varphi, -\sin^2\theta\sin\varphi\cos\varphi - \cos^2\theta\sin\varphi\cos\varphi) \\
&= (-\cos\theta\sin^2\varphi, -\sin\theta\sin^2\varphi, -\sin\varphi\cos\varphi)
\end{aligned}$$

Since $\sin\varphi, \cos\varphi \geq 0$ for $\varphi \in [0, \pi/2]$ this normal is orientation reversing, the sign needs to be flipped.

$$\begin{aligned}
\int_S \boldsymbol{F} \cdot d\boldsymbol{S} &= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} (1, \cos\theta\sin\varphi, \cos\varphi) \cdot (\cos\theta\sin^2\varphi, \sin\theta\sin^2\varphi, \sin\varphi\cos\varphi)\, d\varphi\, d\theta \\
&= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \cos\theta\sin^2\varphi + (\cos\theta\sin\varphi)(\sin\theta\sin^2\varphi) + (\cos\varphi)(\sin\varphi\cos\varphi))\, d\varphi\, d\theta \\
&= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \cos\theta\sin^2\varphi + \cos\theta\sin\theta\sin^3\varphi + \sin\varphi\cos^2\varphi\, d\varphi\, d\theta \\
&= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \cos\theta\sin^2\varphi + \frac{1}{2}\sin(2\theta)\sin^3\varphi\, d\varphi\, d\theta + \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \sin\varphi\cos^2\varphi\, d\varphi\, d\theta \\
&= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \sin\varphi\cos^2\varphi\, d\varphi\, d\theta \text{ [Integrate over full period of } \theta] \\
&\text{Let } u = \cos\varphi,\ du = -\sin\varphi \\
&= \int_0^{2\pi} - \int_1^0 u^2\, d\varphi\, d\theta \\
&= 2\pi \int_0^1 u^2\, d\varphi = \frac{2\pi}{3}
\end{aligned}$$

(b) $\boldsymbol{F}(x, y, z) = (2, x, z + y)$, S is that part of the plane $x + y + z = 1$ which lies in the first octant and $\boldsymbol{n}$ points upward.
Parametrize S as $\boldsymbol{\Phi}(x, y) = (x, y, 1 - x - y)$ where $x \in [0, 1]$, $y \in [0, 1 - x]$

$$\boldsymbol{\phi}_x = (1, 0, -1),\ \boldsymbol{\phi}_y = (0, 1, -1),\ \boldsymbol{\phi}_x \times \boldsymbol{\phi}_y = (1, 1, 1)$$

z is positive for $\boldsymbol{n}$, so it is orientation preserving.

$$\begin{aligned}
\int_S \boldsymbol{F} \cdot d\boldsymbol{S} &= \int_0^1 \int_0^{1-x} (2, x, (1 - x - y) + y) \cdot (1, 1, 1)\, dy\, dx \\
&= \int_0^1 \int_0^{1-x} 2 + x + 1 - x\, dy\, dx = \int_0^1 \int_0^{1-x} 3\, dy\, dx \\
&= \int_0^1 3(1 - x)\, dx = \int_0^1 3 - 3x\, dx \\
&= 3 - \frac{3}{2} = \frac{3}{2}
\end{aligned}$$

(c) Marsden & Tromba, page 425, #22.

Let S be the part of the cone $z^2 = x^2 + y^2$ with z between 1 and 2 oriented by the normal pointing out of the cone. Compute $\iint_S \boldsymbol{F} \cdot d\boldsymbol{S}$, where $\boldsymbol{F}(x, y, z) = (x^2, y^2, z^2)$.

The parametrization for S is given by $\boldsymbol{\Phi}(z, \theta) = (z\cos\theta, z\sin\theta, z)$, $z \in [1, 2]$, $\theta \in [0, 2\pi]$

$$\boldsymbol{\phi}_z = (\cos\theta, \sin\theta, 1),\ \boldsymbol{\phi}_\theta = (-z\sin\theta, z\cos\theta, 0),\ \boldsymbol{\phi}_z \times \boldsymbol{\phi}_\theta = (-z\cos\theta, -z\sin\theta, z)$$

Since x, y are negative, the normal vector points inwards, this is an orientation reversing normal, so the sign needs to be flipped.

$$\begin{aligned}
\int_S \boldsymbol{F} \cdot d\boldsymbol{S} &= -\int_0^{2\pi}\int_1^2 (z^2\cos^2\theta, z^2\sin^2\theta, z^2) \cdot (-z\cos\theta, -z\sin\theta, z)\, dz\, d\theta \\
&= \int_0^{2\pi}\int_1^2 z^3\cos^3\theta + z^3\sin^3\theta - z^3\, dz\, d\theta \\
&= (16/4 - 1/4)\int_0^{2\pi} (\cos^3\theta + \sin^3\theta - 1)\, d\theta \\
&= \frac{15}{4}\left[\int_0^{2\pi} \cos\theta(1 - \sin^2\theta)\, d\theta + \int_0^{2\pi} \sin\theta(1 - \cos^2\theta)\, d\theta + \int_0^{2\pi} -1\, d\theta\right] \\
&= \frac{15}{4}\left[\int_0^0 (1 - u^2)\, du - \int_1^1 (1 - u^2)\, du + \int_0^{2\pi} -1\, d\theta\right] \\
&= -\frac{15\pi}{2}
\end{aligned}$$

6. Let S be the portion of the plane $x - 2y + z = 1$ that is cut off by the coordinate planes and the plane $x + y = 1$. Let $\boldsymbol{V}$ be the velocity field $\boldsymbol{V}(x, y, z) = (y, z, x^2)$. Find the flow across S when $\boldsymbol{n}$ points upward. Explain your answer.

7. Let S be the closed surface that consists of the hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$, and its base $x^2 + y^2 \leq 1$, $z = 0$. let $\boldsymbol{E}$ be the electric field $\boldsymbol{E}(x, y, z) = (2x, 2y, 2z)$. Directly calculate the electric flux across S.