Poon, Keegan

1002423727

1. (a) Let $f, g : \mathbb{R}^n \to \mathbb{R}$; $\boldsymbol{F}, \boldsymbol{G} : \mathbb{R}^n \to \mathbb{R}^n$; and define Δ, the *Laplacian*, by $\Delta f = \displaystyle\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$.

Verify the following identities

(i) $\operatorname{div}(\boldsymbol{F} + \boldsymbol{G}) = \operatorname{div} \boldsymbol{F} + \operatorname{div} \boldsymbol{G}$.

$$\operatorname{div}(\boldsymbol{F} + \boldsymbol{G}) = \sum_{i=1}^{n} \frac{\partial(F_i + G_i)}{\partial x_i} = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i} + \frac{\partial G_i}{\partial x_i} = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i} + \sum_{i=1}^{n} \frac{\partial G_i}{\partial x_i} = \operatorname{div}\boldsymbol{F} + \operatorname{div}\boldsymbol{G}$$

(ii) $\operatorname{div}(f\boldsymbol{F}) = f \operatorname{div} \boldsymbol{F} + \boldsymbol{F} \cdot \operatorname{grad} f$.

$$\operatorname{div}(f\boldsymbol{F}) = \sum_{i=1}^{n} \frac{\partial f F_i}{\partial x_i} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} F_i + \frac{\partial F_i}{\partial x_i} f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} F_i + f \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i} = \nabla f \cdot \boldsymbol{F} + f \operatorname{div}\boldsymbol{F}$$

(iii) $\Delta(fg) = f\Delta g + g\Delta f + 2(\operatorname{grad} f) \cdot (\operatorname{grad} g)$.

Proof.

$$\Delta(fg) = \sum_{i=1}^{n} \frac{\partial^2 fg}{\partial^2 x_i}$$

$$= \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left[\frac{\partial f}{\partial x_i} g + \frac{\partial g}{\partial x_i} f \right]$$

$$= \sum_{i=1}^{n} \left[\frac{\partial f}{\partial^2 x_i} g + \left(\frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_i} \right) + \frac{\partial g}{\partial^2 x_i} f + \left(\frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_i} \right) \right]$$

$$= g \sum_{i=1}^{n} \frac{\partial f}{\partial^2 x_i} + 2 \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_i} \right) + f \sum_{i=1}^{n} \frac{\partial g}{\partial^2 x_i}$$

$$= g\Delta f + 2[\nabla f \cdot \nabla g] + f\Delta g$$

$\square$

(b) Let $f, g : D \subset \mathbb{R}^3 \to \mathbb{R}$ be of class C^1. If R is a solid region contained in D then

$$\iiint_R \nabla f \cdot \nabla g \, dV = \iint_{\partial R} f \nabla g \cdot \boldsymbol{n} \, dS - \iiint_R f \nabla^2 g \, dV$$

$(\nabla^2 g = \operatorname{div}(\nabla g))$.

Proof.

$$\iiint_R \nabla f \cdot \nabla g \, dV = \iint_{\partial R} f \nabla g \cdot \boldsymbol{n} \, dS - \iiint_R f \nabla^2 g \, dV$$

$$\iff \iiint_R \nabla f \cdot \nabla g \, dV + \iiint_R f \nabla^2 g \, dV = \iint_{\partial R} f \nabla g \cdot \boldsymbol{n} \, dS$$

$$\iint_{\partial R} f \nabla g \cdot \boldsymbol{n} \, dS \overset{\text{Div Thm}}{=} \iiint_R \operatorname{div}(f \nabla g) \, dV \overset{\text{(ii)}}{=} \iiint_R f(\operatorname{div}\nabla g) + \nabla g \cdot \nabla f \, dV$$

$$= \iiint_R f(\operatorname{div}\nabla g) \, dV + \iiint_R \nabla f \cdot \nabla g \, dV = \iiint_R f \nabla^2 g \, dV + \iiint_R \nabla f \cdot \nabla g \, dV$$

$\square$

2. Use the Divergence Theorem to verify your answer to question 7 on assignment 8.

3. Let $F(x, y, z) = (x, y^2, e^{yz})$ and let R be a cube centered at the origin with sides of length 2. Evaluate $\int_S \text{div } F \, dV$ directly and by using the Divergence Theorem.

Directly:

$$\int_S \text{div} F \, dV = \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 1 + 2y + ye^{yz} \, dx \, dz \, dy$$

$$= \int_{-1}^1 \int_{-1}^1 2 + 4y + 2ye^{yz} \, dz \, dy$$

$$= \int_{-1}^1 4 + 8y + \left[e^{yz}\right]_{-1}^1 \, dy$$

$$= \int_{-1}^1 4 + 8y + e^y - e^{-y} \, dy$$

$$= 8 + 0 + [e^y]_{-1}^1 + [e^{-y}]_{-1}^1 = 8$$

Divergence:

$$\int_R \text{div} F \, dV = \int_S F \cdot dS$$

where S is the union of 6 planes with $-1 \leq x, y, z \leq 1$ and having normals in the $\pm x, y, z$ directions. The parameterizations can be given by

$$\Phi_{S_1}(x, y) = (x, y, 1) \qquad\qquad \Phi_{S_1'}(x, y) = (x, y, -1)$$
$$\Phi_{S_2}(y, z) = (1, y, z) \qquad\qquad \Phi_{S_2'}(y, z) = (-1, y, z)$$
$$\Phi_{S_3}(x, z) = (x, 1, z) \qquad\qquad \Phi_{S_3'}(x, z) = (x, -1, z)$$

When plugging in S_3, S_3' into F, then taking the dot with the normal, we get that the integrands are going to be the same, specifically 1. When doing the same with S_2, S_2, due to x being odd, we get both being negative the other, so integrating over the same range will cancel each other out. Putting this together, the integral over S is:

$$\int_S F \cdot dS = 2 \int_{-1}^1 \int_{-1}^1 1 \, dx \, dz + \int_{-1}^1 \int_{-1}^1 e^y \, dx \, dy + \int_{-1}^1 \int_{-1}^1 -e^{-y} \, dx \, dy$$

$$= 8 + 2 \int_{-1}^1 e^y \, dy - 2 \int_{-1}^1 e^{-y} \, dy$$

$$= 8 + 2([e^y]_{-1}^1 + [e^{-y}]_{-1}^1) = 8$$

4. Let B be the pyramid with top vertex $(0,0,1)$ and base vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(1,1,0)$. Let S be the 2-dim closed surface bounding B, oriented in the outward direction. Use Gauss' theorem to calculate $\int_S \boldsymbol{F} \cdot d\boldsymbol{S}$, where $\boldsymbol{F}(x,y,z) = (x^2y, 3y^2z, 9z^2x)$.

$$\int_R \operatorname{div}\boldsymbol{F}\, dV = \int_S \boldsymbol{F} \cdot d\boldsymbol{S}$$

$$\operatorname{div}\boldsymbol{F} = 2xy + 6yz + 18zx$$

$$\int_R \operatorname{div}\boldsymbol{F} = \int_0^1 \int_0^{1-z} \int_0^{1-z} 2xy + 6yz + 18zx\, dx\, dy\, dz$$

$$= \int_0^1 \int_0^{1-z} (1-z)^2 y + 6yz(1-z) + 9z(1-z)^2\, dy\, dz$$

$$= \int_0^1 (1/2)(1-z)^4 + 3z(1-z)^3 + 9z(1-z)^3\, dz$$

$$= \int_0^1 -\frac{23z^4}{2} + 34z^3 - 33z^2 + 10z + \frac{1}{2}\, dz$$

$$= -\frac{23}{10} + \frac{34}{4} - 11 + 5 + \frac{1}{2}$$

$$= \frac{7}{10}$$

5. Use the Divergence Theorem to evaluate $\int_S \boldsymbol{F} \cdot d\boldsymbol{S}$, where $\boldsymbol{F}(x, y, z) = \left(z^2 x, \dfrac{y^3}{3} + \tan z, x^2 z + y^2\right)$

and S is the top half of the unit sphere $x^2 + y^2 + z^2 = 1$, oriented by the unit normal which points away from the origin.

For the Divergence Theorem to apply, the integral must be over a closed, oriented outwards. To use Divergence Theorem the surface needs to be closed, so add the disk S' on the xy-plane to the surface. Then

$$\iiint_R \operatorname{div} \boldsymbol{F} \, dV - \iint_{S'} \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_S \boldsymbol{F} \cdot d\boldsymbol{S}$$

The disk can be parametrized as $\boldsymbol{\Phi}(r, \theta) = (r\cos\theta, r\sin\theta, 0)$ with $r \in [0, 1], \theta \in [0, 2\pi]$ The normal, pointing outward needs to be downward.

$$\iint_{S'} \boldsymbol{F} \cdot d\boldsymbol{S} = \int_0^1 \int_0^{2\pi} (*, *, (r\sin\theta)^2) \cdot (0, 0, -1) \, d\theta \, dr$$

$$= -\int_0^1 \int_0^{2\pi} (r\sin\theta)^2 \, d\theta \, dr$$

$$= -\int_0^1 \int_0^{2\pi} r^2 (1/2 - (\cos\theta)/2) \, d\theta \, dr$$

$$= -\int_0^1 \pi r^2 \, dr = -\frac{\pi}{3}$$

Meanwhile, the region enclosed by this, is the top half of the unit sphere, so switch to polars, restricting $\varphi \in [0, \pi/2]$ to remain above the xy-plane.

$$\operatorname{div} \boldsymbol{F} = z^2 + y^2 + x^2 = \rho$$

$$\iiint_R \operatorname{div} \boldsymbol{F} \, dV = \int_0^1 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \rho^3 \sin\varphi \, d\varphi \, d\theta \, d\rho$$

$$= 2\pi \int_0^1 \rho^3 \left[-\cos\varphi \right]_0^{\frac{\pi}{2}} \, d\rho = \frac{\pi}{4}$$

6. Let the electric field from a point source at the origin be given by $\boldsymbol{E}(\boldsymbol{x}) = \dfrac{\boldsymbol{x}}{\|\boldsymbol{x}\|^3}$

   (a) What is the outward flux of $\boldsymbol{E}$ across the surface $\dfrac{x^2}{3} + \dfrac{2y^2}{5} + z^2 = 7$.

   (b) Show that the flux of $\boldsymbol{E}$ across that part of the sphere $x^2 + y^2 + z^2 = 25$ with $z \geq 3$ is equal to the flux across that part of the plane $z = 3$ with $x^2 + y^2 \leq 16$.

7. Let $f : \mathbb{R}^n \to \mathbb{R}$ be given by $f(x, y, z) = x^2 yz$ and let η be the 2-form on $\mathbb{R}^3$ given by

$$\eta = (\sin x) \, dx \, dy + (e^y + xyz) \, dx \, dz + (x^2 y^2) \, dy \, dz.$$

   (a) Compute df and $d\eta$.
   (b) Evaluate $df \wedge \eta$.