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MATB42: Assignment #10 Apr 3rd 2018

1. Let $\boldsymbol{F}$ be a vector field on $\mathbb{R}^3$ given by $\boldsymbol{F} = (F_1, F_2, F_3)$ where F_1, F_2, and F_3 are C^1-functions from $\mathbb{R}^3 \to \mathbb{R}$

   (a) Let η be the 2-form given by

   $$\eta = F_3\, dx\, dy + F_1\, dy\, dz + F_2\, dz\, dx$$

   Show that $d\eta = (\operatorname{div} \boldsymbol{F})\, dx\, dy\, dz$
   (page 489, #6)

$$\begin{aligned}
\eta &= F_3\, dx\, dy + F_1\, dy\, dz + F_2\, dz\, dx \\
d\eta &= d(F_3\, dx\, dy + F_1\, dy\, dz + F_2\, dz\, dx) \\
&= (dF_3)\, dx\, dy + (dF_1)\, dy\, dz + (dF_2)\, dz\, dx \\
&= (\frac{\partial}{\partial x} F_3\, dx + \frac{\partial}{\partial y} F_3\, dy + \frac{\partial}{\partial z} F_3\, dz)\, dx\, dy + (dF_1)\, dy\, dz + (dF_2)\, dz\, dx \\
&= \frac{\partial}{\partial z} F_3\, dz\, dx\, dy + (dF_1)\, dy\, dz + (dF_2)\, dz\, dx \\
&= \frac{\partial}{\partial z} F_3\, dx\, dy\, dz + (\frac{\partial}{\partial x} F_1\, dx + \frac{\partial}{\partial y} F_1\, dy + \frac{\partial}{\partial z} F_1\, dz)\, dy\, dz + (dF_2)\, dz\, dx \\
&= \frac{\partial}{\partial z} F_3\, dx\, dy\, dz + \frac{\partial}{\partial x} F_1\, dx\, dy\, dz + (dF_2)\, dz\, dx \\
&= \frac{\partial}{\partial z} F_3\, dx\, dy\, dz + \frac{\partial}{\partial x} F_1\, dx\, dy\, dz + (\frac{\partial}{\partial x} F_2\, dx + \frac{\partial}{\partial y} F_2\, dy + \frac{\partial}{\partial z} F_2\, dz)\, dz\, dx \\
&= \frac{\partial}{\partial z} F_3\, dx\, dy\, dz + \frac{\partial}{\partial x} F_1\, dx\, dy\, dz + \frac{\partial}{\partial y} F_2\, dy\, dz\, dx \\
&= \frac{\partial}{\partial z} F_3\, dx\, dy\, dz + \frac{\partial}{\partial x} F_1\, dx\, dy\, dz + \frac{\partial}{\partial y} F_2\, dx\, dy\, dz \\
&= \frac{\partial}{\partial x} F_1 + \frac{\partial}{\partial y} F_2 + \frac{\partial}{\partial z} F_3\, dx\, dy\, dz = (\operatorname{div} \boldsymbol{F})\, dx\, dy\, dz
\end{aligned}$$

(b) Show that $dF_1 \wedge dF_2 \wedge dF_3 = (\det D\boldsymbol{F})\, dx\, dy\, dz$

$$df = \sum_{i=0}^{n} \frac{\partial f}{\partial x_i}\, dx_i$$

$$\begin{aligned}
dF_1 \wedge dF_2 \wedge dF_3 &= (\frac{\partial F_1}{\partial x}\, dx + \frac{\partial F_1}{\partial y}\, dy + \frac{\partial F_1}{\partial z}\, dz) \wedge (\frac{\partial F_2}{\partial x}\, dx + \frac{\partial F_2}{\partial y}\, dy + \frac{\partial F_2}{\partial z}\, dz) \wedge dF_3 \\
&= (\frac{\partial F_1}{\partial x}\, dx \wedge (\frac{\partial F_2}{\partial x}\, dx + \frac{\partial F_2}{\partial y}\, dy + \frac{\partial F_2}{\partial z}\, dz) \\
&\quad + \frac{\partial F_1}{\partial y}\, dy \wedge (\frac{\partial F_2}{\partial x}\, dx + \frac{\partial F_2}{\partial y}\, dy + \frac{\partial F_2}{\partial z}\, dz) \\
&\quad + \frac{\partial F_1}{\partial z}\, dz \wedge (\frac{\partial F_2}{\partial x}\, dx + \frac{\partial F_2}{\partial y}\, dy + \frac{\partial F_2}{\partial z}\, dz)) \wedge dF_3 \\
&= ((\frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial y}\, dx\, dy + \frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial z}\, dx\, dz) \\
&\quad + (\frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial x}\, dy\, dx + \frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial z}\, dy\, dz) \\
&\quad + (\frac{\partial F_1}{\partial z}\frac{\partial F_2}{\partial x}\, dz\, dx + \frac{\partial F_1}{\partial z}\frac{\partial F_2}{\partial y}\, dz\, dy)) \wedge dF_3 \\
&= ((\frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial y} - \frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial x}\, dx\, dy) \\
&\quad + (\frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial z}\frac{\partial F_2}{\partial y}\, dy\, dz) \\
&\quad + (\frac{\partial F_1}{\partial z}\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial z}\, dz\, dx)) \wedge (\frac{\partial F_3}{\partial x}\, dx + \frac{\partial F_3}{\partial y}\, dy + \frac{\partial F_3}{\partial z}\, dz) \\
&= (\frac{\partial F_3}{\partial z}(\frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial y} - \frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial x}\, dx\, dy\, dz) \\
&\quad + \frac{\partial F_3}{\partial x}(\frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial z}\frac{\partial F_2}{\partial y}\, dy\, dz\, dx) \\
&\quad + \frac{\partial F_3}{\partial y}(\frac{\partial F_1}{\partial z}\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial z}\, dz\, dx\, dy)) \\
&= \frac{\partial F_3}{\partial x}(\frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial z}\frac{\partial F_2}{\partial y})\, dx\, dy\, dz \\
&\quad - \frac{\partial F_3}{\partial y}(\frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial z}\frac{\partial F_2}{\partial x})\, dx\, dy\, dz \\
&\quad + \frac{\partial F_3}{\partial z}(\frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial y} - \frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial x})\, dx\, dy\, dz \\
&= \frac{\partial F_3}{\partial x}\begin{vmatrix} \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{vmatrix} - \frac{\partial F_3}{\partial y}\begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial z} \end{vmatrix} + \frac{\partial F_3}{\partial z}\begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{vmatrix}\, dx\, dy\, dz \\
&= \begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \\ \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial z} \end{vmatrix}\, dx\, dy\, dz
\end{aligned}$$

2. Let ω be a k-form and let η be a ℓ-form. Find $d(d\omega \wedge \eta - \omega \wedge d\eta)$.

$$\begin{aligned}
d(d\omega \wedge \eta - \omega \wedge d\eta) &= d(d\omega \wedge \eta) - d(\omega \wedge d\eta) \\
&= (d^2\omega \wedge \eta + (-1)^{k+1}(d\omega \wedge d\eta)) - (d\omega \wedge d\eta + (-1)^k(\omega \wedge d^2\eta)) \\
&= (-1)^{k+1} d\omega \wedge d\eta - d\omega \wedge d\eta \\
&= ((-1)^{k+1} - 1) d\omega \wedge d\eta
\end{aligned}$$

3. Determine if $\eta = y\,dx\,dy + xz\,dy\,dz - yz\,dz\,dx$ is exact. If η is exact find a 1-form ω with $d\omega = \eta$. Check if $d\eta = \mathcal{O}$ to see if η closed.

   (compare with page 461, # 22)

$$\begin{aligned}
d\eta &= d(y\,dx\,dy + xz\,dy\,dz - yz\,dz\,dx) \\
&= (dy\,dx\,dy + d(xz) \wedge dy\,dz - d(yz) \wedge dz\,dx) \\
&= ((z\,dx + x\,dz) \wedge dy\,dz - (z\,dy + y\,dz) \wedge dz\,dx) \\
&= (z\,dx) \wedge dy\,dz - (z\,dy) \wedge dz\,dx \\
&= z\,dx\,dy\,dz - z\,dx\,dy\,dz = \mathcal{O}
\end{aligned}$$

Since the polynomials of x, y and z defined throughout $\mathbb{R}^3$ and η closed, it is exact. By inspection,

$$\omega = xy\,dy + xyz\,dz$$