

## MATB42: Assignment #10

1. Let  $\mathbf{F}$  be a vector field on  $\mathbb{R}^3$  given by  $\mathbf{F} = (F_1, F_2, F_3)$  where  $F_1$ ,  $F_2$ , and  $F_3$  are  $C^1$ -functions from  $\mathbb{R}^3 \rightarrow \mathbb{R}$

- (a) Let  $\eta$  be the 2-form given by

$$\eta = F_3 dx dy + F_1 dy dz + F_2 dz dx$$

Show that  $d\eta = (\operatorname{div} \mathbf{F}) dx dy dz$

(page 489, #6)

$$\begin{aligned}
 \eta &= F_3 dx dy + F_1 dy dz + F_2 dz dx \\
 d\eta &= d(F_3 dx dy + F_1 dy dz + F_2 dz dx) \\
 &= (dF_3) dx dy + (dF_1) dy dz + (dF_2) dz dx \\
 &= \left( \frac{\partial}{\partial x} F_3 dx + \frac{\partial}{\partial y} F_3 dy + \frac{\partial}{\partial z} F_3 dz \right) dx dy + (dF_1) dy dz + (dF_2) dz dx \\
 &= \frac{\partial}{\partial z} F_3 dz dx dy + (dF_1) dy dz + (dF_2) dz dx \\
 &= \frac{\partial}{\partial z} F_3 dx dy dz + \left( \frac{\partial}{\partial x} F_1 dx + \frac{\partial}{\partial y} F_1 dy + \frac{\partial}{\partial z} F_1 dz \right) dy dz + (dF_2) dz dx \\
 &= \frac{\partial}{\partial z} F_3 dx dy dz + \frac{\partial}{\partial x} F_1 dx dy dz + (dF_2) dz dx \\
 &= \frac{\partial}{\partial z} F_3 dx dy dz + \frac{\partial}{\partial x} F_1 dx dy dz + \left( \frac{\partial}{\partial x} F_2 dx + \frac{\partial}{\partial y} F_2 dy + \frac{\partial}{\partial z} F_2 dz \right) dz dx \\
 &= \frac{\partial}{\partial z} F_3 dx dy dz + \frac{\partial}{\partial x} F_1 dx dy dz + \frac{\partial}{\partial y} F_2 dy dz dx \\
 &= \frac{\partial}{\partial z} F_3 dx dy dz + \frac{\partial}{\partial x} F_1 dx dy dz + \frac{\partial}{\partial y} F_2 dx dy dz \\
 &= \frac{\partial}{\partial x} F_1 + \frac{\partial}{\partial y} F_2 + \frac{\partial}{\partial z} F_3 dx dy dz = (\operatorname{div} \mathbf{F}) dx dy dz
 \end{aligned}$$

(b) Show that  $dF_1 \wedge dF_2 \wedge dF_3 = (\det D\mathbf{F}) dx dy dz$

$$df = \sum_{i=0}^n \frac{\partial f}{\partial x_i} dx_i$$

$$\begin{aligned}
dF_1 \wedge dF_2 \wedge dF_3 &= \left( \frac{\partial F_1}{\partial x} dx + \frac{\partial F_1}{\partial y} dy + \frac{\partial F_1}{\partial z} dz \right) \wedge \left( \frac{\partial F_2}{\partial x} dx + \frac{\partial F_2}{\partial y} dy + \frac{\partial F_2}{\partial z} dz \right) \wedge dF_3 \\
&= \left( \frac{\partial F_1}{\partial x} dx \wedge \left( \frac{\partial F_2}{\partial x} dx + \frac{\partial F_2}{\partial y} dy + \frac{\partial F_2}{\partial z} dz \right) \right. \\
&\quad + \frac{\partial F_1}{\partial y} dy \wedge \left( \frac{\partial F_2}{\partial x} dx + \frac{\partial F_2}{\partial y} dy + \frac{\partial F_2}{\partial z} dz \right) \\
&\quad \left. + \frac{\partial F_1}{\partial z} dz \wedge \left( \frac{\partial F_2}{\partial x} dx + \frac{\partial F_2}{\partial y} dy + \frac{\partial F_2}{\partial z} dz \right) \right) \wedge dF_3 \\
&= \left( \left( \frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial y} dx dy + \frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial z} dx dz \right) \right. \\
&\quad + \left( \frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial x} dy dx + \frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial z} dy dz \right) \\
&\quad \left. + \left( \frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial x} dz dx + \frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial y} dz dy \right) \right) \wedge dF_3 \\
&= \left( \left( \frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial y} - \frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial x} \right) dx dy \right. \\
&\quad + \left( \frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial y} \right) dy dz \\
&\quad \left. + \left( \frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial z} \right) dz dx \right) \wedge \left( \frac{\partial F_3}{\partial x} dx + \frac{\partial F_3}{\partial y} dy + \frac{\partial F_3}{\partial z} dz \right) \\
&= \left( \frac{\partial F_3}{\partial z} \left( \frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial y} - \frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial x} \right) dx dy \right. \\
&\quad + \frac{\partial F_3}{\partial x} \left( \frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial y} \right) dy dz \\
&\quad \left. + \frac{\partial F_3}{\partial y} \left( \frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial z} \right) dz dx \right) \\
&= \frac{\partial F_3}{\partial x} \left( \frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial y} \right) dx dy dz \\
&\quad - \frac{\partial F_3}{\partial y} \left( \frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial x} \right) dx dy dz \\
&\quad + \frac{\partial F_3}{\partial z} \left( \frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial y} - \frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial x} \right) dx dy dz \\
&= \frac{\partial F_3}{\partial x} \begin{vmatrix} \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{vmatrix} - \frac{\partial F_3}{\partial y} \begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial z} \end{vmatrix} + \frac{\partial F_3}{\partial z} \begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{vmatrix} dx dy dz \\
&= \begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \\ \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial z} \end{vmatrix} dx dy dz
\end{aligned}$$

2. Let  $\omega$  be a  $k$ -form and let  $\eta$  be a  $\ell$ -form. Find  $d(d\omega \wedge \eta - \omega \wedge d\eta)$ .

$$\begin{aligned} d(d\omega \wedge \eta - \omega \wedge d\eta) &= d(d\omega \wedge \eta) - d(\omega \wedge d\eta) \\ &= (d^2\omega \wedge \eta + (-1)^{k+1}(d\omega \wedge d\eta)) - (d\omega \wedge d\eta + (-1)^k(\omega \wedge d^2\eta)) \\ &= (-1)^{k+1}d\omega \wedge d\eta - d\omega \wedge d\eta \\ &= ((-1)^{k+1} - 1)d\omega \wedge d\eta \end{aligned}$$

3. Determine if  $\eta = y \, dx \, dy + xz \, dy \, dz - yz \, dz \, dx$  is exact. If  $\eta$  is exact find a 1-form  $\omega$  with  $d\omega = \eta$ . Check if  $d\eta = \mathcal{O}$  to see if  $\eta$  closed.

(compare with page 461, # 22)

$$\begin{aligned} d\eta &= d(y \, dx \, dy + xz \, dy \, dz - yz \, dz \, dx) \\ &= (dy \, dx \, dy + d(xz) \wedge dy \, dz - d(yz) \wedge dz \, dx) \\ &= ((z \, dx + x \, dz) \wedge dy \, dz - (z \, dy + y \, dz) \wedge dz \, dx) \\ &= (z \, dx) \wedge dy \, dz - (z \, dy) \wedge dz \, dx \\ &= z \, dx \, dy \, dz - z \, dx \, dy \, dz = \mathcal{O} \end{aligned}$$

Since the polynomials of  $x$ ,  $y$  and  $z$  defined throughout  $\mathbb{R}^3$  and  $\eta$  closed, it is exact.

4. Evaluate  $\iint_S \omega$ , where  $\omega = z dx dy + x dy dz + y dz dx$  and  $S$  is the unit sphere, directly and by the Divergence Theorem.

(page 489, #12)

Directly:

Parametrize the sphere  $S$  as

$$\Phi(\varphi, \theta) = (\cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi) \text{ with } \theta \in [0, 2\pi], \varphi \in [0, \pi]$$

$$\begin{aligned} \iint_S \omega &= \iint_{\Phi} z dx dy + \iint_{\Phi} x dy dz + \iint_{\Phi} y dz dx \\ &= \int_0^{2\pi} \int_0^{\pi} \cos \varphi \begin{vmatrix} \frac{\partial \cos \theta \sin \varphi}{\partial \varphi} & \frac{\partial \cos \theta \sin \varphi}{\partial \theta} \\ \frac{\partial \sin \theta \sin \varphi}{\partial \varphi} & \frac{\partial \sin \theta \sin \varphi}{\partial \theta} \end{vmatrix} d\varphi d\theta + \int_0^{2\pi} \int_0^{\pi} \cos \theta \sin \varphi \begin{vmatrix} \frac{\partial \sin \theta \sin \varphi}{\partial \varphi} & \frac{\partial \sin \theta \sin \varphi}{\partial \theta} \\ \frac{\partial \cos \varphi}{\partial \varphi} & \frac{\partial \cos \varphi}{\partial \theta} \end{vmatrix} d\varphi d\theta \\ &\quad + \int_0^{2\pi} \int_0^{\pi} \sin \theta \sin \varphi \begin{vmatrix} \frac{\partial \cos \varphi}{\partial \varphi} & \frac{\partial \cos \varphi}{\partial \theta} \\ \frac{\partial \cos \theta \sin \varphi}{\partial \varphi} & \frac{\partial \cos \theta \sin \varphi}{\partial \theta} \end{vmatrix} d\varphi d\theta \\ &= \int_0^{2\pi} \int_0^{\pi} \cos \varphi \begin{vmatrix} \cos \theta \cos \varphi & -\sin \theta \sin \varphi \\ \sin \theta \cos \varphi & \cos \theta \sin \varphi \end{vmatrix} d\varphi d\theta + \int_0^{2\pi} \int_0^{\pi} \cos \theta \sin \varphi \begin{vmatrix} \sin \theta \cos \varphi & \cos \theta \sin \varphi \\ -\sin \varphi & 0 \end{vmatrix} d\varphi d\theta \\ &\quad + \int_0^{2\pi} \int_0^{\pi} \sin \theta \sin \varphi \begin{vmatrix} -\sin \varphi & 0 \\ \cos \theta \cos \varphi & -\sin \theta \sin \varphi \end{vmatrix} d\varphi d\theta \\ &= \int_0^{2\pi} \int_0^{\pi} \sin \varphi \cos^2 \varphi d\varphi d\theta + \int_0^{2\pi} \int_0^{\pi} \cos^2 \theta \sin^3 \varphi + \sin^2 \theta \sin^3 \varphi d\varphi d\theta \\ &= \int_0^{2\pi} \int_0^{\pi} \sin \varphi (\cos^2 \varphi + \sin^2 \varphi) d\varphi d\theta \\ &= \int_0^{2\pi} \int_0^{\pi} \sin \varphi d\varphi d\theta \\ &= 2\pi \left[ -\cos \varphi \right]_0^{\pi} = 2\pi \end{aligned}$$

Divergence Theorem:

$$d\omega = dz dy dx + dx dy dz + dy dz dx = 3 dx dy dz$$

$$\begin{aligned} \iint_S \omega &= \iiint_R d\omega \\ &= 3 \int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^2 \sin(\varphi) d\rho d\varphi d\theta \\ &= \int_0^{2\pi} \int_0^{\pi} \sin(\varphi) d\varphi d\theta \\ &= 2\pi \left[ -\cos \varphi \right]_0^{\pi} = 2\pi \end{aligned}$$