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MATB42: Assignment #10 Apr 3rd 2018

1. Let $\boldsymbol{F}$ be a vector field on $\mathbb{R}^3$ given by $\boldsymbol{F} = (F_1, F_2, F_3)$ where F_1, F_2, and F_3 are C^1-functions from $\mathbb{R}^3 \to \mathbb{R}$

   (a) Let η be the 2-form given by

   $$\eta = F_3\,dx\,dy + F_1\,dy\,dz + F_2\,dz\,dx$$

   Show that $d\eta = (\operatorname{div}\boldsymbol{F})\,dx\,dy\,dz$
   (page 489, #6)

$$\begin{aligned}
\eta &= F_3\,dx\,dy + F_1\,dy\,dz + F_2\,dz\,dx \\
d\eta &= d(F_3\,dx\,dy + F_1\,dy\,dz + F_2\,dz\,dx) \\
&= (dF_3)\,dx\,dy + (dF_1)\,dy\,dz + (dF_2)\,dz\,dx \\
&= (\frac{\partial}{\partial x}F_3\,dx + \frac{\partial}{\partial y}F_3\,dy + \frac{\partial}{\partial z}F_3\,dz)\,dx\,dy + (dF_1)\,dy\,dz + (dF_2)\,dz\,dx \\
&= \frac{\partial}{\partial z}F_3\,dz\,dx\,dy + (dF_1)\,dy\,dz + (dF_2)\,dz\,dx \\
&= \frac{\partial}{\partial z}F_3\,dx\,dy\,dz + (\frac{\partial}{\partial x}F_1\,dx + \frac{\partial}{\partial y}F_1\,dy + \frac{\partial}{\partial z}F_1\,dz)\,dy\,dz + (dF_2)\,dz\,dx \\
&= \frac{\partial}{\partial z}F_3\,dx\,dy\,dz + \frac{\partial}{\partial x}F_1\,dx\,dy\,dz + (dF_2)\,dz\,dx \\
&= \frac{\partial}{\partial z}F_3\,dx\,dy\,dz + \frac{\partial}{\partial x}F_1\,dx\,dy\,dz + (\frac{\partial}{\partial x}F_2\,dx + \frac{\partial}{\partial y}F_2\,dy + \frac{\partial}{\partial z}F_2\,dz)\,dz\,dx \\
&= \frac{\partial}{\partial z}F_3\,dx\,dy\,dz + \frac{\partial}{\partial x}F_1\,dx\,dy\,dz + \frac{\partial}{\partial y}F_2\,dy\,dz\,dx \\
&= \frac{\partial}{\partial z}F_3\,dx\,dy\,dz + \frac{\partial}{\partial x}F_1\,dx\,dy\,dz + \frac{\partial}{\partial y}F_2\,dx\,dy\,dz \\
&= \frac{\partial}{\partial x}F_1 + \frac{\partial}{\partial y}F_2 + \frac{\partial}{\partial z}F_3\,dx\,dy\,dz = (\operatorname{div}\boldsymbol{F})\,dx\,dy\,dz
\end{aligned}$$

(b) Show that $dF_1 \wedge dF_2 \wedge dF_3 = (\det D\boldsymbol{F})\, dx\, dy\, dz$

$$df = \sum_{i=0}^{n} \frac{\partial f}{\partial x_i}\, dx_i$$

$$\begin{aligned}
dF_1 \wedge dF_2 \wedge dF_3 &= (\frac{\partial F_1}{\partial x}\, dx + \frac{\partial F_1}{\partial y}\, dy + \frac{\partial F_1}{\partial z}\, dz) \wedge (\frac{\partial F_2}{\partial x}\, dx + \frac{\partial F_2}{\partial y}\, dy + \frac{\partial F_2}{\partial z}\, dz) \wedge dF_3 \\
&= (\frac{\partial F_1}{\partial x}\, dx \wedge (\frac{\partial F_2}{\partial x}\, dx + \frac{\partial F_2}{\partial y}\, dy + \frac{\partial F_2}{\partial z}\, dz) \\
&\quad + \frac{\partial F_1}{\partial y}\, dy \wedge (\frac{\partial F_2}{\partial x}\, dx + \frac{\partial F_2}{\partial y}\, dy + \frac{\partial F_2}{\partial z}\, dz) \\
&\quad + \frac{\partial F_1}{\partial z}\, dz \wedge (\frac{\partial F_2}{\partial x}\, dx + \frac{\partial F_2}{\partial y}\, dy + \frac{\partial F_2}{\partial z}\, dz)) \wedge dF_3 \\
&= ((\frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial y}\, dx\, dy + \frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial z}\, dx\, dz) \\
&\quad + (\frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial x}\, dy\, dx + \frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial z}\, dy\, dz) \\
&\quad + (\frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial x}\, dz\, dx + \frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial y}\, dz\, dy)) \wedge dF_3 \\
&= ((\frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial y} - \frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial x}\, dx\, dy) \\
&\quad + (\frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial y}\, dy\, dz) \\
&\quad + (\frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial z}\, dz\, dx)) \wedge (\frac{\partial F_3}{\partial x}\, dx + \frac{\partial F_3}{\partial y}\, dy + \frac{\partial F_3}{\partial z}\, dz) \\
&= (\frac{\partial F_3}{\partial z}(\frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial y} - \frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial x}\, dx\, dy\, dz) \\
&\quad + \frac{\partial F_3}{\partial x}(\frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial y}\, dy\, dz\, dx) \\
&\quad + \frac{\partial F_3}{\partial y}(\frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial z}\, dz\, dx\, dy)) \\
&= \frac{\partial F_3}{\partial x}(\frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial y})\, dx\, dy\, dz \\
&\quad - \frac{\partial F_3}{\partial y}(\frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial x})\, dx\, dy\, dz \\
&\quad + \frac{\partial F_3}{\partial z}(\frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial y} - \frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial x})\, dx\, dy\, dz \\
&= \frac{\partial F_3}{\partial x} \begin{vmatrix} \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{vmatrix} - \frac{\partial F_3}{\partial y} \begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial z} \end{vmatrix} + \frac{\partial F_3}{\partial z} \begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{vmatrix}\, dx\, dy\, dz \\
&= \begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \\ \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial z} \end{vmatrix}\, dx\, dy\, dz
\end{aligned}$$

2. Let ω be a k-form and let η be a ℓ-form. Find $d(d\omega \wedge \eta - \omega \wedge d\eta)$.

$$\begin{aligned} d(d\omega \wedge \eta - \omega \wedge d\eta) &= d(d\omega \wedge \eta) - d(\omega \wedge d\eta) \\ &= (d^2\omega \wedge \eta + (-1)^{k+1}(d\omega \wedge d\eta)) - (d\omega \wedge d\eta + (-1)^k(\omega \wedge d^2\eta)) \\ &= (-1)^{k+1} d\omega \wedge d\eta - d\omega \wedge d\eta \\ &= ((-1)^{k+1} - 1) d\omega \wedge d\eta \end{aligned}$$

3. Determine if $\eta = y\,dx\,dy + xz\,dy\,dz - yz\,dz\,dx$ is exact. If η is exact find a 1-form ω with $d\omega = \eta$. Check if $d\eta = \mathcal{O}$ to see if η closed.

   (compare with page 461, # 22)

$$\begin{aligned} d\eta &= d(y\,dx\,dy + xz\,dy\,dz - yz\,dz\,dx) \\ &= (dy\,dx\,dy + d(xz) \wedge dy\,dz - d(yz) \wedge dz\,dx) \\ &= ((z\,dx + x\,dz) \wedge dy\,dz - (z\,dy + y\,dz) \wedge dz\,dx) \\ &= (z\,dx) \wedge dy\,dz - (z\,dy) \wedge dz\,dx \\ &= z\,dx\,dy\,dz - z\,dx\,dy\,dz = \mathcal{O} \end{aligned}$$

Since the polynomials of x, y and z defined throughout $\mathbb{R}^3$ and η closed, it is exact. By inspection,

$$\omega = xy\,dy + xyz\,dz$$

4. Evaluate $\iint_S \omega$, where $\omega = z\,dx\,dy + x\,dy\,dz + y\,dz\,dx$ and S is the unit sphere, directly and by the Divergence Theorem.

(page 489, #12)

Directly:

Parametrize the sphere S as

$$\mathbf{\Phi}(\varphi,\theta) = (\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi) \text{ with } \theta\in[0,2\pi],\ \varphi\in[0,\pi]$$

$$\begin{aligned}
\iint_S \omega &= \iint_{\mathbf{\Phi}} z\,dx\,dy + \iint_{\mathbf{\Phi}} x\,dy\,dz + \iint_{\mathbf{\Phi}} y\,dz\,dx \\
&= \int_0^{2\pi}\int_0^{\pi} \cos\varphi \begin{vmatrix} \frac{\partial\cos\theta\sin\varphi}{\partial\varphi} & \frac{\partial\cos\theta\sin\varphi}{\partial\theta} \\ \frac{\partial\sin\theta\sin\varphi}{\partial\varphi} & \frac{\partial\sin\theta\sin\varphi}{\partial\theta} \end{vmatrix} d\varphi\,d\theta + \int_0^{2\pi}\int_0^{\pi} \cos\theta\sin\varphi \begin{vmatrix} \frac{\partial\sin\theta\sin\varphi}{\partial\varphi} & \frac{\partial\sin\theta\sin\varphi}{\partial\theta} \\ \frac{\partial\cos\varphi}{\partial\varphi} & \frac{\partial\cos\varphi}{\partial\theta} \end{vmatrix} d\varphi\,d\theta \\
&\quad + \int_0^{2\pi}\int_0^{\pi} \sin\theta\sin\varphi \begin{vmatrix} \frac{\partial\cos\varphi}{\partial\varphi} & \frac{\partial\cos\varphi}{\partial\theta} \\ \frac{\partial\cos\theta\sin\varphi}{\partial\varphi} & \frac{\partial\cos\theta\sin\varphi}{\partial\theta} \end{vmatrix} d\varphi\,d\theta \\
&= \int_0^{2\pi}\int_0^{\pi} \cos\varphi \begin{vmatrix} \cos\theta\cos\varphi & -\sin\theta\sin\varphi \\ \sin\theta\cos\varphi & \cos\theta\sin\varphi \end{vmatrix} d\varphi\,d\theta + \int_0^{2\pi}\int_0^{\pi} \cos\theta\sin\varphi \begin{vmatrix} \sin\theta\cos\varphi & \cos\theta\sin\varphi \\ -\sin\varphi & 0 \end{vmatrix} d\varphi\,d\theta \\
&\quad + \int_0^{2\pi}\int_0^{\pi} \sin\theta\sin\varphi \begin{vmatrix} -\sin\varphi & 0 \\ \cos\theta\cos\varphi & -\sin\theta\sin\varphi \end{vmatrix} d\varphi\,d\theta \\
&= \int_0^{2\pi}\int_0^{\pi} \sin\varphi\cos^2\varphi\,d\varphi\,d\theta + \int_0^{2\pi}\int_0^{\pi} \cos^2\theta\sin^3\varphi + \sin^2\theta\sin^3\varphi\,d\varphi\,d\theta \\
&= \int_0^{2\pi}\int_0^{\pi} \sin\varphi(\cos^2\varphi + \sin^2\varphi)\,d\varphi\,d\theta \\
&= \int_0^{2\pi}\int_0^{\pi} \sin\varphi\,d\varphi\,d\theta \\
&= 2\pi\Big[-\cos\varphi\Big]_0^{\pi} = 2\pi
\end{aligned}$$

Divergence Theorem:

$$d\omega = dz\,dy\,dx + dx\,dy\,dz + dydzdx = 3\,dx\,dy\,dz$$

$$\begin{aligned}
\iint_S \omega &= \iiint_R d\omega \\
&= 3\int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho^2\sin(\varphi)\,d\rho\,d\varphi\,d\theta \\
&= \int_0^{2\pi}\int_0^{\pi} \sin(\varphi)\,d\varphi\,d\theta \\
&= 2\pi\Big[-\cos\varphi\Big]_0^{\pi} = 2\pi
\end{aligned}$$

5. Compute $\int_S \omega$ and use symbolic algebra software to sketch S in each of the following.

(a) $\omega = xz\,dx\,dy + x^2\,dy\,dz + dy\,dz\,dx$

S is the upper hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$ with $\boldsymbol{n}$ pointing upward.

Close it with the disk of radius 2 on the xy-plane to apply divergence theorem

$$\boldsymbol{\Phi}(\theta, r) = (r\cos\theta, r\sin\theta, 0),\ r \in [0,2],\ \theta \in [0, 2\pi]$$

$$dx\,dy = \begin{vmatrix} -r\sin\theta & \cos\theta \\ r\cos\theta & \sin\theta \end{vmatrix} = -2r$$

Which is negative, so correct orientation for normal pointing down.

$dy\,dz = 0$ Since z is 0

$dz\,dx = 0$

$$\xRightarrow{\text{Div Thm}} \iint_S \omega = \iiint_R d\omega - \iint_{\boldsymbol{\Phi}} \omega$$

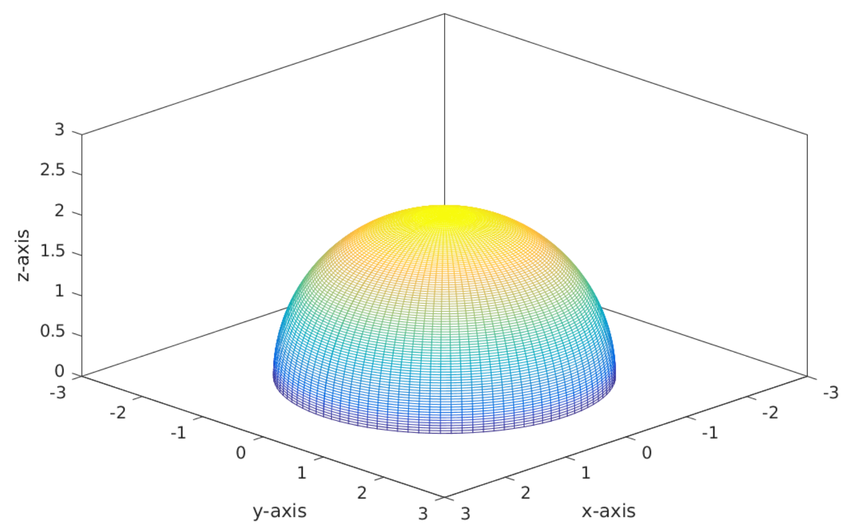
But $z = 0 \implies xz\,dx\,dy = 0 \implies \iint_{\boldsymbol{\Phi}} \omega = 0$

$$d\omega = x\,dx\,dy\,dz + 2x\,dx\,dy\,dz = 3x\,dx\,dy\,dz$$

$$\iiint_R d\omega = \int_0^2 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} 3(\rho\sin\varphi\cos\theta)\rho^2 \sin\varphi\,d\varphi\,d\theta\,d\rho$$

$= 0$ Since integrating cos over full period

$$\implies \int_S \omega = 0$$



(b) $\omega = z\,dx\,dy + x\,dy\,dz + y\,dz\,dx$

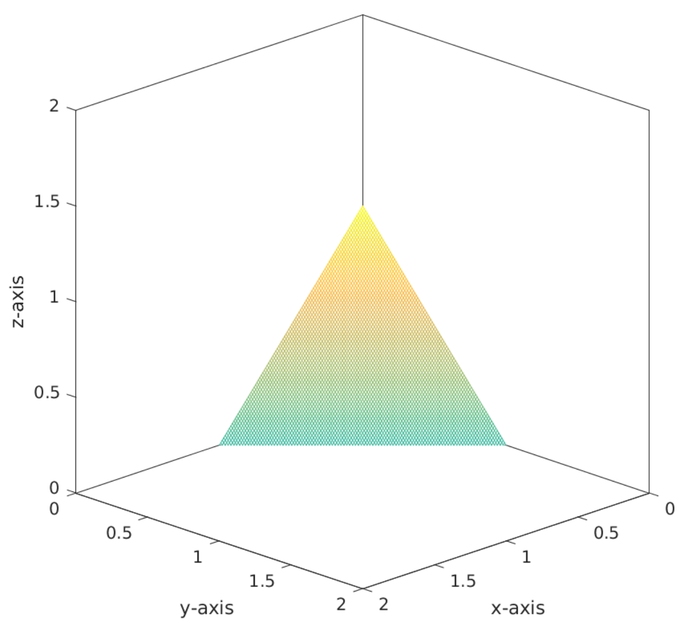
S is the part of the plane $x + y + z = 1$ which lies in the first octant oriented by the unit normal which points upward.

Use the natural parametrization for S:

$$\mathbf{\Phi}(x, y) = (x, y, 1 - x - y),\ x \in [0, 1],\ y \in [0, 1 - x]$$

$$dx\,dy = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} > 0\,\forall x, y \implies \text{Correct orientation}$$

$$\begin{aligned}\int_S \omega &= \int_0^1 \int_0^{1-x} (1 - x - y) + x \begin{vmatrix} 0 & 1 \\ -1 & -1 \end{vmatrix} + y \begin{vmatrix} -1 & -1 \\ 1 & 0 \end{vmatrix} dy\,dx \\ &= \int_0^1 \int_0^{1-x} 1\,dy\,dx = \int_0^1 1 - x\,dx \\ &= 1 - \frac{1}{2} = \frac{1}{2}\end{aligned}$$

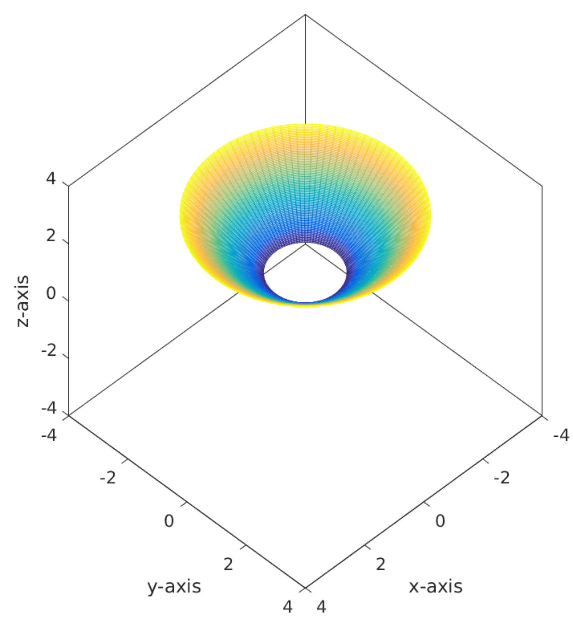


(c) $\omega = xz\,dx\,dy + y\,dx\,dz + z^2\,dy\,dz$

S is the part of the cone $z = \sqrt{x^2+y^2}$ between $z = 1$ and $z = 3$, oriented by the unit normal with negative z-component.

$$\mathbf{\Phi}(\theta, r) = (r\cos\theta, r\sin\theta, r),\ r \in [1,3],\ \theta \in [0, 2\pi]$$

$$\begin{aligned}
dx\,dy &= \begin{vmatrix} -r\sin\theta & \cos\theta \\ r\cos\theta & \sin\theta \end{vmatrix} = -r < 0 \text{ for } r > 1 \\
dy\,dz &= \begin{vmatrix} r\cos\theta & \sin\theta \\ 0 & 1 \end{vmatrix} = r\cos\theta \\
dz\,dx &= \begin{vmatrix} 0 & 1 \\ -r\sin\theta & \cos\theta \end{vmatrix} = r\sin\theta \\
\implies \omega &= (r\cos\theta)(r)(-r) - (r\sin\theta)(r\sin\theta) + (r)^2(r\cos\theta) \\
&= -r^2\sin^2\theta = -r^2\left(\frac{1}{2} - \frac{\cos(2\theta)}{2}\right) \\
\implies \int_S \omega &= \int_1^3 \int_0^{2\pi} -r^2\left(\frac{1}{2} - \frac{\cos(2\theta)}{2}\right) d\theta\,dr \\
&= \int_1^3 -r^2\pi\,dr = -\pi\left[\frac{r^3}{3}\right]_1^3 = -\frac{26\pi}{3}
\end{aligned}$$



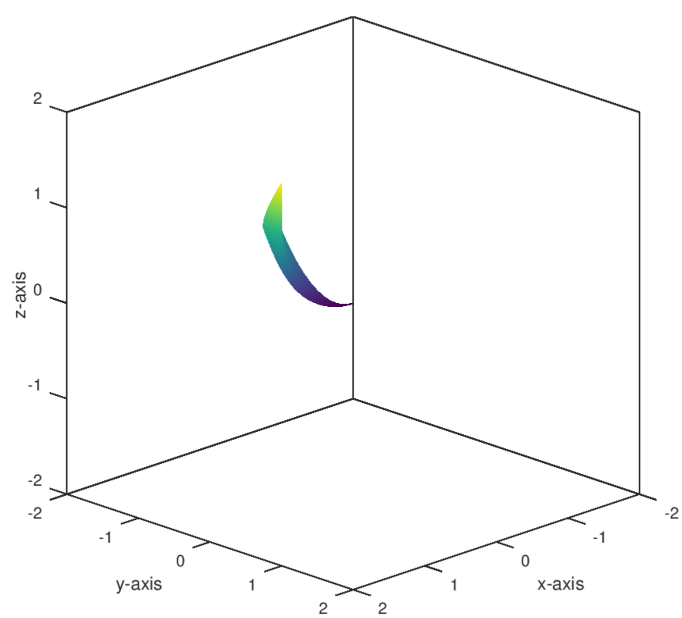
(d) $\omega = z\,dx\,dy + y\,dy\,dz$

S is the oriented surface given by the parametrization

$\Phi(u,v) = (u+v, uv^2, u^2+v^2)$, $0 \le u \le 1$, $0 \le v \le 1$.

$$dx\,dy = \begin{vmatrix} 1 & 1 \\ v^2 & 2uv \end{vmatrix} = 2uv - v^2, \quad dy\,dz = \begin{vmatrix} v^2 & 2uv \\ 2u & 2v \end{vmatrix} = 2v^3 - 4u^2v$$

$$\begin{aligned}
\iint_S \omega &= \int_0^1 \int_0^1 (u^2+v^2)(2uv - v^2) + (uv^2)(2v^3 - 4u^2v)\,du\,dv \\
&= \int_0^1 \int_0^1 (2u^3v - u^2v^2) + (2uv^3 - v^4) + (2uv^5 - 4u^3v^3)\,du\,dv \\
&= \int_0^1 \frac{v}{2} - \frac{v^2}{3} + v^3 - v^4 + v^5 - v^3\,du\,dv \\
&= \int_0^1 \frac{v}{2} - \frac{v^2}{3} - v^4 + v^5\,du\,dv \\
&= \frac{1}{4} - \frac{1}{9} - \frac{1}{5} + \frac{1}{6} = \frac{19}{180}
\end{aligned}$$



6. Verify Stokes' theorem by direct calculation of both sides when the surface S is the piece of the paraboloid $z = x^2 + y^2 - 4$ with $z \leq 0$, oriented by the downward pointing unit normal, and $\omega = (2y - z)\,dx + (x + y^2 - z)\,dy + (4y - 3x)\,dz$.

   As part of your solution, provide a sketch showing the appropriate orientations. (For this question you may draw the sketch by hand or use symbolic algebra software.)

$$\int_{\partial S} \omega = \int_S d\omega$$

The boundary curve of the plane is the circle at $z = 0$ with radius 2. Since the normal vector is downward pointing, the curve is parametrized counter clockwise. So parametrize the curve as $\boldsymbol{\gamma}(\theta) = (2\cos\theta, 2\sin\theta, 0)$

$$\int_0^{2\pi} (2(2\sin\theta) - 0)(-2\sin\theta) + ((2\cos\theta) + (2\sin\theta)^2 - 0)(2\cos\theta)\,d\theta$$

$$= \int_0^{2\pi} -8\sin^2\theta + 4\cos^2\theta + 8\sin^2\theta\cos\theta\,d\theta$$

$$= \int_0^{2\pi} -4(1 - \cos(2\theta)) + 2(1 + \cos(2\theta)) + 8\sin^2\theta\cos\theta\,d\theta \qquad = \int_0^{2\pi} -2 + 8\sin^2\theta\cos\theta\,d\theta$$

7. Let $\omega = yz\,dx - xz\,dy + xy\,dz$ and let $\gamma(t) = (2\cos t, 2\sin t, 4)$, $0 \le t \le 2\pi$.

(a) Let S be the piece of the surface $z = x^2 + y^2$ with $z \le 4$. Use Stokes' theorem to give an integral over S which is equivalent to $\displaystyle\int_\gamma \omega$. Verify by directly computing both integrals.

(b) Let S' be the part of the plane $z = 4$ with $x^2 + y^2 \le 4$. Use Stokes' theorem to give an integral over S' which is equivalent to $\displaystyle\int_\gamma \omega$. Verify by direct computation.

(c) Can you give another explanation as to why the integrals you get over S and S' should have the same value?

8. Let $\boldsymbol{F}(x, y, z) = (e^{z^2}, 4z - y, 8x \sin y)$. Find $\int_S (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{S}$ where S is the unit sphere oriented with the outward normal.

9. (a) Marsden & Tromba, page 451, # 13.
   (b) Marsden & Tromba, page 451, # 15.
   (c) Use symbolic algebra software to sketch the surfaces in parts (a) and (b).

10. (a) Let $\boldsymbol{F}$ and $\boldsymbol{G}$ be vector fields on $\mathbb{R}^3$ and let $f : \mathbb{R}^3 \to \mathbb{R}$. Verify the following identities.

   (i) $\operatorname{div}(\boldsymbol{F} \times \boldsymbol{G}) = \boldsymbol{G} \cdot \operatorname{curl} \boldsymbol{F} - \boldsymbol{F} \cdot \operatorname{curl} \boldsymbol{G}$.

   (ii) $\operatorname{curl}(f\boldsymbol{F}) = f\operatorname{curl}\boldsymbol{F} + (\operatorname{grad} f) \times \boldsymbol{F}$.

   (b) Let R be a closed region in $\mathbb{R}^3$ with boundary ∂R. Prove the identity

$$\int_{\partial R} (\boldsymbol{F} \times \operatorname{curl}\boldsymbol{G}) \cdot d\boldsymbol{S} = \int_R (\operatorname{curl}\boldsymbol{F}) \cdot (\operatorname{curl}\boldsymbol{G})\, dV - \int_R \boldsymbol{F} \cdot \operatorname{curl}(\operatorname{curl}\boldsymbol{G})\, dV$$

   (page 490, #2)