(1) (a) Express e^{-x^2} as a power series.

$$e^{-x^2} = \exp(-x^2) = \sum_{k=0}^{\infty} \frac{(-x^2)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{k!}$$

(b) Express

$$\int_0^x e^{-x^2}\ dt$$

as a power series.

$$\begin{aligned}
\int_0^x e^{-x^2}\ dt &= \int_0^x \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{k!}\ dt &&= \int_0^x \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(-1)^k t^{2k}}{k!}\ dt \\
&= \lim_{n\to\infty} \sum_{k=0}^{n} \int_0^x \frac{(-1)^k t^{2k}}{k!}\ dt &&= \sum_{k=0}^{\infty} \left[\frac{(-1)^k t^{2k+1}}{(2k+1)k!} \right]_0^x \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)k!}
\end{aligned}$$

(2) For $a \in \mathbb{R},\ a \notin \mathbb{N}$, let

$$\binom{a}{k} = \frac{a(a-1)\cdots(a-k+1)}{k!}, \ \binom{a}{0} = 1.$$

(a) Show that

$$f(x) = \sum_{k=0}^{\infty} \binom{a}{k} x^k$$

converges for $|x| < 1$.
Ratio test:

$$\begin{aligned}
\lim_{k\to\infty} \left| \frac{\binom{a}{k+1} x^{k+1}}{\binom{a}{k} x^k} \right| &= \lim_{k\to\infty} \left| \frac{\binom{a}{k+1} x}{\binom{a}{k}} \right| \\
&= \lim_{k\to\infty} \left| \frac{\frac{a(a-1)\cdots(a-(k+1)+2)(a-(k+1)+1)x}{(k+1)!}}{\frac{a(a-1)\cdots(a-k+1)}{k!}} \right| \\
&= \lim_{k\to\infty} \left| \frac{\frac{a(a-1)\cdots(a-k+1)(a-k)x}{(k+1)!}}{\frac{a(a-1)\cdots(a-k+1)}{k!}} \right| \\
&= \lim_{k\to\infty} \left| \frac{(a-k)x}{k+1} \right| \\
&= \lim_{k\to\infty} \left| \frac{ax}{k+1} - \frac{kx}{k+1} + \frac{x}{k+1} - \frac{x}{k+1} \right| \\
&= \lim_{k\to\infty} \left| \frac{(a+1)x}{k+1} - \frac{(k+1)x}{k+1} \right| \\
&= \lim_{k\to\infty} \left| \frac{(a+1)x}{k+1} - x \right| = |x|
\end{aligned}$$

Since the ratio test implies convergence when the result is < 1, this power series converges for $|x| < 1$.

(b) Verify that $f(x)$ is the Taylor series of $(1+x)^a$.
Let $g(x) = (1+x)^a$,

$$\begin{aligned}
g'(x) =& \quad a(1+x)^{a-1} & g'(0) &= a \\
g''(x) =& \quad a(a-1)(1+x)^{a-2} & g''(0) &= a(a-1) \\
&\vdots \\
g^{(n)}(x) =& \quad a(a-1)\cdots(a-n+1)(1+x)^{a-n} & g^{(n)}(0) &= a(a-1)\cdots(a-n+1)
\end{aligned}$$

So the Taylor series of g is given by:

$$\sum_{k=0}^{\infty} \frac{g^{(k)}(0)x^k}{k!} = \sum_{k=0}^{\infty} \frac{a(a-1)\cdots(a-k+1)x^k}{k!} = \sum_{k=0}^{\infty} \binom{a}{k} x^k = f(x)$$

(c) Verify that both $f(x)$ and $(1+x)^a$ satisfy the differential equation

$$(1+x)y' = ay,$$

with initial condition $y(0)$.
Directly solving:

$$\begin{aligned}
(1+x)y' &= ay \\
\frac{y'}{y} &= \frac{a}{1+x} \\
\int \frac{dy}{y} &= \int \frac{a}{1+x}dx \\
\ln y &= a\ln(1+x)+c \\
\ln y &= \ln(1+x)^a + c \\
e^{\ln y} &= e^{\ln(1+x)^a + c} \\
y &= c(1+x)^a
\end{aligned}$$

$$\begin{aligned}
y(0) &= c(1)^a \\
1 &= c \\
\implies y &= (1+x)^a \text{ satifies the IVP}
\end{aligned}$$

Verify power series:

$$f'(x) = \sum_{k=0}^{\infty} k\binom{a}{k}x^{k-1}$$

$$\begin{aligned}
(1+x)f'(x) &= \sum_{k=0}^{\infty} k\binom{a}{k}x^{k-1} + \sum_{k=0}^{\infty} k\binom{a}{k}x^{k} \\
&= \sum_{k=-1}^{\infty} (k+1)\binom{a}{k+1}x^{k} + \sum_{k=0}^{\infty} k\binom{a}{k}x^{k} \\
&= \sum_{k=0}^{\infty} (k+1)\binom{a}{k+1}x^{k} + \sum_{k=0}^{\infty} k\binom{a}{k}x^{k} && \text{Since } k+1|_{k=-1} = 0 \\
&= \sum_{k=0}^{\infty} \left[(k+1)\binom{a}{k+1} + k\binom{a}{k}\right]x^{k} \\
&= \sum_{k=0}^{\infty} \left[(k+1)\frac{a(a-1)\cdots(a-k+1)}{(k+1)!}(a-k) + k\binom{a}{k}\right]x^{k} \\
&= \sum_{k=0}^{\infty} \left[\frac{a(a-1)\cdots(a-k+1)}{k!}(a-k) + k\binom{a}{k}\right]x^{k} \\
&= \sum_{k=0}^{\infty} \left[\binom{a}{k}(a-k) + k\binom{a}{k}\right]x^{k} \\
&= \sum_{k=0}^{\infty} a\binom{a}{k}x^{k} = a\sum_{k=0}^{\infty} \binom{a}{k}x^{k} = af(x) \\
f(0) &= \sum_{k=0}^{\infty} \binom{a}{k}0^{k} = 1 \implies \text{The power series also satisfies the IVP}
\end{aligned}$$

(3) (a) Show that

$$\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2} \ , \text{ for } |x| < 1.$$

From lecture we know that $\dfrac{1}{1-x} = \displaystyle\sum_{k=1}^{\infty} x^k$ for $|x| < 1$.

$$\begin{aligned}
x\frac{d}{dx}\frac{1}{1-x} &= x\frac{d}{dx}\sum_{k=1}^{\infty} x^k \\
x\frac{1}{(1-x)^2} &= x\frac{d}{dx}\lim_{n\to\infty}\sum_{k=1}^{n} x^k \\
\frac{x}{(1-x)^2} &= x\lim_{n\to\infty}\sum_{k=1}^{n}\frac{d}{dx}x^k \\
&= x\sum_{k=1}^{\infty} kx^{k-1} \\
&= \sum_{k=1}^{\infty} kx^k
\end{aligned}$$

Since we know that integration and differentiation of power series holds the radius of convergence constant, this also holds for $|x| < 1$, multiplying by a factor of x will not change it either.

(b) Find an explicit formula for

$$\sum_{n=1}^{\infty} n^2 x^n.$$

$$\begin{aligned}
x\frac{d}{dx}\frac{x}{(1-x)^2} &= x\frac{d}{dx}\sum_{k=1}^{\infty} kx^k \\
x\frac{(1-x)^2 + 2x(1-x)}{(1-x)^4} &= x\frac{d}{dx}\lim_{n\to\infty}\sum_{k=1}^{n} kx^k \\
\frac{x(1-x)((1-x)+2x)}{(1-x)^4} &= x\lim_{n\to\infty}\sum_{k=1}^{n}\frac{d}{dx}kx^k \\
\frac{x(1-x)(1+x)}{(1-x)^4} &= x\sum_{k=1}^{\infty} k^2x^{k-1} \\
\frac{x+x^2}{(1-x)^3} &= \sum_{k=1}^{\infty} k^2x^k
\end{aligned}$$

(4) Let $f_k(x) = (1/k)\sin kx$. So f_k is differentiable on $\mathbb{R}$. Let $f(x) = 0$, for all $x \in \mathbb{R}$.

(a) Show that $f_k \to f$ uniformly on $\mathbb{R}$.

Since $\sin x$ is bounded by $-1 \leq \sin x \leq 1,\ \forall x \in \mathbb{R} \implies 0 \leq |\sin(x)| \leq 1$ so Squeeze Theorem applies using the bounds of $0 \leq |\frac{\sin kx}{k}| \leq \frac{1}{k}$. Since $\frac{1}{k}$ converges to 0, so does f, independently of x hence f converges uniformly.

(b) Show that $\lim_k f_k'(x)$ is not defined for all $x \in \mathbb{R}$.

First, the sequence is defined $f_k'(x) = \cos(kx)$. Assume it did converge, then by definition we can say

$$\forall \varepsilon\ \exists N \text{ s.t. } \forall n > N,\ |f_n(x) - L| < \varepsilon \text{ where } L \text{ is the limit.}$$

So choose $\varepsilon < 0.5,\ x = \frac{\pi}{2}$. It can be seen that $|f_n(\frac{\pi}{2})|$ cycles between 1 and 0 as n runs through the integers, so we can fix $n > N$ to get

$$f_{n+1}\left(\frac{\pi}{2}\right) - f_n\left(\frac{\pi}{2}\right) = 1$$

So in this case $|f_{n+1}(x) - L| = |1 + f_n(\frac{\pi}{2}) - L| \implies |1 - \varepsilon| < |1 + f_n(\frac{\pi}{2}) - L| < |1 + \varepsilon| \implies |f_{n+1} - L| > |1 - \varepsilon| > \varepsilon$ since $\varepsilon < 0.5$ which is a contradiction since it should hold for all $n > N$.