Poon, Keegan
1002423727
MATB42: Assignment #10 Apr 3rd 2018

1. Let $\boldsymbol{F}$ be a vector field on $\mathbb{R}^3$ given by $\boldsymbol{F} = (F_1, F_2, F_3)$ where F_1, F_2, and F_3 are C^1-functions from $\mathbb{R}^3 \to \mathbb{R}$

   (a) Let η be the 2-form given by

$$\eta = F_3\,dx\,dy + F_1\,dy\,dz + F_2\,dz\,dx$$

   Show that $d\eta = (\operatorname{div}\boldsymbol{F})\,dx\,dy\,dz$
   (page 489, #6)

$$\begin{aligned}
\eta &= F_3\,dx\,dy + F_1\,dy\,dz + F_2\,dz\,dx \\
d\eta &= d(F_3\,dx\,dy + F_1\,dy\,dz + F_2\,dz\,dx) \\
&= (dF_3)\,dx\,dy + (dF_1)\,dy\,dz + (dF_2)\,dz\,dx \\
&= (\frac{\partial}{\partial x}F_3\,dx + \frac{\partial}{\partial y}F_3\,dy + \frac{\partial}{\partial z}F_3\,dz)\,dx\,dy + (dF_1)\,dy\,dz + (dF_2)\,dz\,dx \\
&= \frac{\partial}{\partial z}F_3\,dz\,dx\,dy + (dF_1)\,dy\,dz + (dF_2)\,dz\,dx \\
&= \frac{\partial}{\partial z}F_3\,dx\,dy\,dz + (\frac{\partial}{\partial x}F_1\,dx + \frac{\partial}{\partial y}F_1\,dy + \frac{\partial}{\partial z}F_1\,dz)\,dy\,dz + (dF_2)\,dz\,dx \\
&= \frac{\partial}{\partial z}F_3\,dx\,dy\,dz + \frac{\partial}{\partial x}F_1\,dx\,dy\,dz + (dF_2)\,dz\,dx \\
&= \frac{\partial}{\partial z}F_3\,dx\,dy\,dz + \frac{\partial}{\partial x}F_1\,dx\,dy\,dz + (\frac{\partial}{\partial x}F_2\,dx + \frac{\partial}{\partial y}F_2\,dy + \frac{\partial}{\partial z}F_2\,dz)\,dz\,dx \\
&= \frac{\partial}{\partial z}F_3\,dx\,dy\,dz + \frac{\partial}{\partial x}F_1\,dx\,dy\,dz + \frac{\partial}{\partial y}F_2\,dy\,dz\,dx \\
&= \frac{\partial}{\partial z}F_3\,dx\,dy\,dz + \frac{\partial}{\partial x}F_1\,dx\,dy\,dz + \frac{\partial}{\partial y}F_2\,dx\,dy\,dz \\
&= \frac{\partial}{\partial x}F_1 + \frac{\partial}{\partial y}F_2 + \frac{\partial}{\partial z}F_3\,dx\,dy\,dz = (\operatorname{div}\boldsymbol{F})\,dx\,dy\,dz
\end{aligned}$$

   (b) Show that $dF_1 \wedge dF_2 \wedge dF_3 = (\det\, D\boldsymbol{F})\,dx\,dy\,dz$

$$df = \sum_{i=0}^{n} \frac{\partial}{\partial x_i} f\,dx_i$$

$$\begin{aligned}
dF_1 \wedge dF_2 \wedge dF_3 &= (\frac{\partial}{\partial x}F_1\,dx + \frac{\partial}{\partial y}F_1\,dy + \frac{\partial}{\partial z}F_1\,dz) \wedge (\frac{\partial}{\partial x}F_2\,dx + \frac{\partial}{\partial y}F_2\,dy + \frac{\partial}{\partial z}F_2\,dz) \wedge dF_3 \\
&= (\frac{\partial}{\partial x}F_1\,dx \wedge (\frac{\partial}{\partial x}F_2\,dx + \frac{\partial}{\partial y}F_2\,dy + \frac{\partial}{\partial z}F_2\,dz) \\
&\quad + \frac{\partial}{\partial y}F_1\,dy \wedge (\frac{\partial}{\partial x}F_2\,dx + \frac{\partial}{\partial y}F_2\,dy + \frac{\partial}{\partial z}F_2\,dz) \\
&\quad + \frac{\partial}{\partial z}F_1\,dz \wedge (\frac{\partial}{\partial x}F_2\,dx + \frac{\partial}{\partial y}F_2\,dy + \frac{\partial}{\partial z}F_2\,dz)) \wedge dF_3 \\
&= (\frac{\partial}{\partial x}F_1\frac{\partial}{\partial y}F_2\,dx\,dy + \frac{\partial}{\partial x}F_1\frac{\partial}{\partial z}F_2\,dx\,dz) \\
&\quad + \frac{\partial}{\partial y}F_1\,dy \wedge (\frac{\partial}{\partial x}F_2\,dx + \frac{\partial}{\partial y}F_2\,dy + \frac{\partial}{\partial z}F_2\,dz) \\
&\quad + \frac{\partial}{\partial z}F_1\,dz \wedge (\frac{\partial}{\partial x}F_2\,dx + \frac{\partial}{\partial y}F_2\,dy + \frac{\partial}{\partial z}F_2\,dz)) \wedge dF_3
\end{aligned}$$

2. Let ω be a k-form and let η be a ℓ-form. Find $d(d\omega \wedge \eta - \omega \wedge d\eta)$.

3. Determine if $\eta = y\,dx\,dy + dz\,dy\,dz - yz\,dz\,dx$ is exact. If η is exact find a 1-form ω with $d\omega = \eta$. (compare with page 461, # 22)

4. Evaluate $\iint_S \omega$, where $\omega = z\,dx\,dy + x\,dy\,dz + y\,dz\,dx$ and S is the unit sphere, directly and by the Divergence Theorem.

(page 489, #12)

5. Compute $\displaystyle\int_S \omega$ and use symbolic algebra software to sketch S in each of the following.

(a) $\omega = xz\,dx\,dy + x^2\,dy\,dz + dy\,dz\,dx$
S is the upper hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$ with $\boldsymbol{n}$ pointing upward.

(b) $\omega = z\,dx\,dy + x\,dy\,dz + y\,dz\,dx$
S is the part of the plane $x + y + z = 1$ which lies in the first octant oriented by the unit normal which points upward.

(c) $\omega = xz\,dx\,dy + y\,dx\,dz + z^2\,dy\,dz$
S is the part of the cone $z = \sqrt{x^2 + y^2}$ between $z = 1$ and $z = 3$, oriented by the unit normal with negative z-component.

(d) $\omega = z\,dx\,dy + y\,dy\,dz$
S is the oriented surface given by the parametrization
$\Phi(u, v) = (u + v, uv^2, u^2 + v^2)$, $0 \leq u \leq 1$, $0 \leq v \leq 1$.

6. Verify Stokes' theorem by direct calculation of both sides when the surface S is the piece of the paraboloid $z = x^2 + y^2 - 4$ with $z \leq 0$, oriented by the downward pointing unit normal, and $\omega = (2y - z)\, dx + (x + y^2 - z)\, dy + (4y - 3x)\, dz$.

   As part of your solution, provide a sketch showing the appropriate orientations. (For this question you may draw the skwtch by hand or use symbolic algebra software.)

7. Let $\omega = yz\,dx - xz\,dy + xy\,dz$ and let $\gamma(t) = (2\cos t, 2\sin t, 4)$, $0 \le t \le 2\pi$.

(a) Let S be the piece of the surface $z = x^2 + y^2$ with $z \le 4$. Use Stokes' theorem to give an integral over S which is equivalent to $\displaystyle\int_\gamma \omega$. Verify by directly computing both integrals.

(b) Let S' be the part of the plane $z = 4$ with $x^2 + y^2 \le 4$. Use Stokes' theorem to give an integral over S' which is equivalent to $\displaystyle\int_\gamma \omega$. Verify by direct computation.

(c) Can you give another explanation as to why the integrals you get over S and S' should have the same value?

8. Let $\boldsymbol{F}(x, y, z) = (e^{z^2}, 4z - y, 8x \sin y)$. Find $\displaystyle\int_S (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{S}$ where S is the unit sphere oriented with the outward normal.

9. (a) Marsden & Tromba, page 451, # 13.
   (b) Marsden & Tromba, page 451, # 15.
   (c) Use symbolic algebra software to sketch the surfaces in parts (a) and (b).

10. (a) Let $\boldsymbol{F}$ and $\boldsymbol{G}$ be vector fields on $\mathbb{R}^3$ and let $f : \mathbb{R}^3 \to \mathbb{R}$. Verify the following identities.

(i) $\operatorname{div}(\boldsymbol{F} \times \boldsymbol{G}) = \boldsymbol{G} \cdot \operatorname{curl} \boldsymbol{F} - \boldsymbol{F} \cdot \operatorname{curl} \boldsymbol{G}$.

(ii) $\operatorname{curl}(f\boldsymbol{F}) = f\operatorname{curl}\boldsymbol{F} + (\operatorname{grad} f) \times \boldsymbol{F}$.

(b) Let R be a closed region in $\mathbb{R}^3$ with boundary ∂R. Prove the identity

$$\int_{\partial R} (\boldsymbol{F} \times \operatorname{curl}\boldsymbol{G}) \cdot d\boldsymbol{S} = \int_R (\operatorname{curl}\boldsymbol{F}) \cdot (\operatorname{curl}\boldsymbol{G})\, dV - \int_R \boldsymbol{F} \cdot \operatorname{curl}(\operatorname{curl}\boldsymbol{G})\, dV$$

(page 490, #2)