Introduction to Combinatorics: Assigment 3

University of Toronto, Scarborough

Due on March 18 on gradescope by 5pm

- Name:
- Student ID:
- **Problems 1–4 are worth 23 points each and Problem 5 is worth 8 points**

Problem 1. Provide a **combinatorial** proof of the following identities:

$$\textit{Identity A}: \binom{10}{5} = \binom{8}{3} + \binom{8}{4} + \binom{8}{4} + \binom{8}{5},$$

and

$$\textit{Identity B}: \ k \cdot \binom{n}{k} = n \cdot \binom{n-1}{k-1}.$$

Proof of identity A (please use the space below only).

Proof of identity B (please use the space below only).

Problem 2. I. Compute the number of all solutions $(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ of the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 4$$

in the case where

A. $x_i \in \{0, 1\}, i = 1, 2, 3, 4, 5, 6, 7.$

B. $x_i \in \{0, 1, 2, 3, 4\}, i = 1, 2, 3, 4, 5, 6, 7$

II. Compute the number of all solutions $(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ of the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 18$$

in the case where $x_i \in \{2, 3\}, i = 1, 2, 3, 4, 5, 6, 7.$

Solution (please use the space below only).

Problem 3. A path on the planar grid $\{(k,l): \ k,l \in \mathbb{Z}\}$ is called **admissible and connected** if it consists of *consecutive* edges joining points of the form

$$(x,y) \to (x+1,y)$$

and

$$(x,y) \to (x,y+1).$$

Compute the number of all possible admissible connected paths from $(0,0)$ to (n,m).

Solution (please use the space below only).

Problem 4. Prove (either algebraically or combinatorially) that

$$\sum_{k=0}^{n} k^2 \cdot \binom{n}{k} = n \cdot (n+1) \cdot 2^{n-2}.$$

You may use any identity proved in the notes without proving it here.

Solution (please use the space below only).

Problem 5. Prove **combinatorially** that

$$\binom{10}{5} = \binom{4}{4} + \binom{5}{4} + \binom{6}{4} + \binom{7}{4} + \binom{8}{4} + \binom{9}{4}.$$

Solution (please use the space below only).