

MATB42: Assignment #10

1. Let \mathbf{F} be a vector field on \mathbb{R}^3 given by $\mathbf{F} = (F_1, F_2, F_3)$ where F_1 , F_2 , and F_3 are C^1 -functions from $\mathbb{R}^3 \rightarrow \mathbb{R}$

- (a) Let η be the 2-form given by

$$\eta = F_3 dx dy + F_1 dy dz + F_2 dz dx$$

Show that $d\eta = (\operatorname{div} \mathbf{F}) dx dy dz$

(page 489, #6)

$$\begin{aligned}
 \eta &= F_3 dx dy + F_1 dy dz + F_2 dz dx \\
 d\eta &= d(F_3 dx dy + F_1 dy dz + F_2 dz dx) \\
 &= (dF_3) dx dy + (dF_1) dy dz + (dF_2) dz dx \\
 &= \left(\frac{\partial}{\partial x} F_3 dx + \frac{\partial}{\partial y} F_3 dy + \frac{\partial}{\partial z} F_3 dz \right) dx dy + (dF_1) dy dz + (dF_2) dz dx \\
 &= \frac{\partial}{\partial z} F_3 dz dx dy + (dF_1) dy dz + (dF_2) dz dx \\
 &= \frac{\partial}{\partial z} F_3 dx dy dz + \left(\frac{\partial}{\partial x} F_1 dx + \frac{\partial}{\partial y} F_1 dy + \frac{\partial}{\partial z} F_1 dz \right) dy dz + (dF_2) dz dx \\
 &= \frac{\partial}{\partial z} F_3 dx dy dz + \frac{\partial}{\partial x} F_1 dx dy dz + (dF_2) dz dx \\
 &= \frac{\partial}{\partial z} F_3 dx dy dz + \frac{\partial}{\partial x} F_1 dx dy dz + \left(\frac{\partial}{\partial x} F_2 dx + \frac{\partial}{\partial y} F_2 dy + \frac{\partial}{\partial z} F_2 dz \right) dz dx \\
 &= \frac{\partial}{\partial z} F_3 dx dy dz + \frac{\partial}{\partial x} F_1 dx dy dz + \frac{\partial}{\partial y} F_2 dy dz dx \\
 &= \frac{\partial}{\partial z} F_3 dx dy dz + \frac{\partial}{\partial x} F_1 dx dy dz + \frac{\partial}{\partial y} F_2 dx dy dz \\
 &= \frac{\partial}{\partial x} F_1 + \frac{\partial}{\partial y} F_2 + \frac{\partial}{\partial z} F_3 dx dy dz = (\operatorname{div} \mathbf{F}) dx dy dz
 \end{aligned}$$

(b) Show that $dF_1 \wedge dF_2 \wedge dF_3 = (\det D\mathbf{F}) dx dy dz$

$$df = \sum_{i=0}^n \frac{\partial f}{\partial x_i} dx_i$$

$$\begin{aligned}
dF_1 \wedge dF_2 \wedge dF_3 &= \left(\frac{\partial F_1}{\partial x} dx + \frac{\partial F_1}{\partial y} dy + \frac{\partial F_1}{\partial z} dz \right) \wedge \left(\frac{\partial F_2}{\partial x} dx + \frac{\partial F_2}{\partial y} dy + \frac{\partial F_2}{\partial z} dz \right) \wedge dF_3 \\
&= \left(\frac{\partial F_1}{\partial x} dx \wedge \left(\frac{\partial F_2}{\partial x} dx + \frac{\partial F_2}{\partial y} dy + \frac{\partial F_2}{\partial z} dz \right) \right. \\
&\quad + \frac{\partial F_1}{\partial y} dy \wedge \left(\frac{\partial F_2}{\partial x} dx + \frac{\partial F_2}{\partial y} dy + \frac{\partial F_2}{\partial z} dz \right) \\
&\quad \left. + \frac{\partial F_1}{\partial z} dz \wedge \left(\frac{\partial F_2}{\partial x} dx + \frac{\partial F_2}{\partial y} dy + \frac{\partial F_2}{\partial z} dz \right) \right) \wedge dF_3 \\
&= \left(\left(\frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial y} dx dy + \frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial z} dx dz \right) \right. \\
&\quad + \left(\frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial x} dy dx + \frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial z} dy dz \right) \\
&\quad \left. + \left(\frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial x} dz dx + \frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial y} dz dy \right) \right) \wedge dF_3 \\
&= \left(\left(\frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial y} - \frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial x} \right) dx dy \right. \\
&\quad + \left(\frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial y} \right) dy dz \\
&\quad \left. + \left(\frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial z} \right) dz dx \right) \wedge \left(\frac{\partial F_3}{\partial x} dx + \frac{\partial F_3}{\partial y} dy + \frac{\partial F_3}{\partial z} dz \right) \\
&= \left(\frac{\partial F_3}{\partial z} \left(\frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial y} - \frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial x} \right) dx dy \right. \\
&\quad + \frac{\partial F_3}{\partial x} \left(\frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial y} \right) dy dz \\
&\quad \left. + \frac{\partial F_3}{\partial y} \left(\frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial z} \right) dz dx \right) \\
&= \frac{\partial F_3}{\partial x} \left(\frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial y} \right) dx dy dz \\
&\quad - \frac{\partial F_3}{\partial y} \left(\frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial x} \right) dx dy dz \\
&\quad + \frac{\partial F_3}{\partial z} \left(\frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial y} - \frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial x} \right) dx dy dz \\
&= \frac{\partial F_3}{\partial x} \begin{vmatrix} \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{vmatrix} - \frac{\partial F_3}{\partial y} \begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial z} \end{vmatrix} + \frac{\partial F_3}{\partial z} \begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{vmatrix} dx dy dz \\
&= \begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \\ \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial z} \end{vmatrix} dx dy dz
\end{aligned}$$

2. Let ω be a k -form and let η be a ℓ -form. Find $d(d\omega \wedge \eta - \omega \wedge d\eta)$.

$$\begin{aligned} d(d\omega \wedge \eta - \omega \wedge d\eta) &= d(d\omega \wedge \eta) - d(\omega \wedge d\eta) \\ &= (d^2\omega \wedge \eta + (-1)^{k+1}(d\omega \wedge d\eta)) - (d\omega \wedge d\eta + (-1)^k(\omega \wedge d^2\eta)) \\ &= (-1)^{k+1}d\omega \wedge d\eta - d\omega \wedge d\eta \\ &= ((-1)^{k+1} - 1)d\omega \wedge d\eta \end{aligned}$$

3. Determine if $\eta = y \, dx \, dy + xz \, dy \, dz - yz \, dz \, dx$ is exact. If η is exact find a 1-form ω with $d\omega = \eta$. Check if $d\eta = \mathcal{O}$ to see if η closed.
(compare with page 461, # 22)

$$\begin{aligned}
 d\eta &= d(y \, dx \, dy + xz \, dy \, dz - yz \, dz \, dx) \\
 &= (dy \, dx \, dy + d(xz) \wedge dy \, dz - d(yz) \wedge dz \, dx) \\
 &= ((z \, dx + x \, dz) \wedge dy \, dz - (z \, dy + y \, dz) \wedge dz \, dx) \\
 &= (z \, dx) \wedge dy \, dz - (z \, dy) \wedge dz \, dx \\
 &= z \, dx \, dy \, dz - z \, dx \, dy \, dz = \mathcal{O}
 \end{aligned}$$

Since the polynomials of x , y and z defined throughout \mathbb{R}^3 and η closed, it is exact.