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MATB42: Assignment #9
Bonus
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1. (a) Let $f, g : \mathbb{R}^n \to \mathbb{R}$; $\boldsymbol{F}$, $\boldsymbol{G} :\ \mathbb{R}^n \to \mathbb{R}^n$; and define Δ, the *Laplacian*, by $\Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$.

Verify the following identities

(i) $\mathrm{div}(\boldsymbol{F}+\boldsymbol{G}) = \mathrm{div}\ \boldsymbol{F} +\ \mathrm{div}\ \boldsymbol{G}$.

$$\mathrm{div}(\boldsymbol{F}+\boldsymbol{G}) = \sum_{i=1}^{n} \frac{\partial(F_i+G_i)}{\partial x_i} = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i} + \frac{\partial G_i}{\partial x_i} = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i} + \sum_{i=1}^{n} \frac{\partial G_i}{\partial x_i} = \mathrm{div}\boldsymbol{F} + \mathrm{div}\boldsymbol{G}$$

(ii) $\mathrm{div}(f\boldsymbol{F}) = f\ \mathrm{div}\ \boldsymbol{F} + \boldsymbol{F}\cdot\ \mathrm{grad} f$.

$$\mathrm{div}(f\boldsymbol{F}) = \sum_{i=1}^{n} \frac{\partial f F_i}{\partial x_i} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} F_i + \frac{\partial F_i}{\partial x_i} f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} F_i + f \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i} = \nabla f \cdot \boldsymbol{F} + f\mathrm{div}\boldsymbol{F}$$

(iii) $\Delta(fg) = f\Delta g + g\Delta f + 2(\mathrm{grad} f)\cdot(\mathrm{grad} g)$.

Proof.

$$\begin{aligned}
\Delta(fg) &= \sum_{i=1}^{n} \frac{\partial^2 fg}{\partial^2 x_i} \\
&= \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left[\frac{\partial f}{\partial x_i} g + \frac{\partial g}{\partial x_i} f\right] \\
&= \sum_{i=1}^{n} \left[\frac{\partial f}{\partial^2 x_i} g + \left(\frac{\partial f}{\partial x_i}\cdot\frac{\partial g}{\partial x_i}\right) + \frac{\partial g}{\partial^2 x_i} f + \left(\frac{\partial f}{\partial x_i}\cdot\frac{\partial g}{\partial x_i}\right)\right] \\
&= g\sum_{i=1}^{n} \frac{\partial f}{\partial^2 x_i} + 2\sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\cdot\frac{\partial g}{\partial x_i}\right) + f\sum_{i=1}^{n}\frac{\partial g}{\partial^2 x_i} \\
&= g\Delta f + 2[\nabla f\cdot\nabla g] + f\Delta g
\end{aligned}$$

□

(b) Let $f, g : D \subset \mathbb{R}^3 \to \mathbb{R}$ be of class C^1. If R is a solid region contained in D then

$$\iiint_R \nabla f\cdot\nabla g\, dV = \iint_{\partial R} f\nabla g\cdot\boldsymbol{n}\, dS - \iiint_R f\nabla^2 g\, dV$$

($\nabla^2 g = \mathrm{div}\ (\nabla g)$).

Proof.

$$\begin{aligned}
&\iiint_R \nabla f\cdot\nabla g\, dV = \iint_{\partial R} f\nabla g\cdot\boldsymbol{n}\, dS - \iiint_R f\nabla^2 g\, dV \\
\iff &\iiint_R \nabla f\cdot\nabla g\, dV + \iiint_R f\nabla^2 g\, dV = \iint_{\partial R} f\nabla g\cdot\boldsymbol{n}\, dS
\end{aligned}$$

$$\iint_{\partial R} f\nabla g\cdot\boldsymbol{n}\, dS \overset{\text{Div Thm}}{=} \iiint_R \mathrm{div}(f\nabla g)\, dV \overset{\text{(ii)}}{=} \iiint_R f(\mathrm{div}\nabla g) + \nabla g\cdot\nabla f\, dV$$
$$= \iiint_R f(\mathrm{div}\nabla g)\, dV + \iiint_R \nabla f\cdot\nabla g\, dV = \iiint_R f\nabla^2 g\, dV + \iiint_R \nabla f\cdot\nabla g\, dV$$

□

2. Use the Divergence Theorem to verify your answer to question 7 on assignment 8.

   Let S be the closed surface that consists of the hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$, and its base $x^2 + y^2 \leq 1$, $z = 0$. let $\boldsymbol{E}$ be the electric field $\boldsymbol{E}(x, y, z) = (2x, 2y, 2z)$. Calculate the electric flux across S using Divergence Theorem.

   Divergence Theorem gives that

$$\iiint_R \text{div}\boldsymbol{F}\, dV = \iint_S \boldsymbol{F} \cdot d\boldsymbol{S}$$

   S is a closed surface, and assuming the default outward normal Divergence Theorem. To integrate over the region, switch to polars restricting $\varphi \in [0, \pi/2]$ to restrict to the upper half. Together with $\text{div}\boldsymbol{E} = 6$ gives:

$$\iint_S \boldsymbol{E} \cdot d\boldsymbol{S} = \int_0^1 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} 6\rho^2 \sin\varphi\, d\varphi\, d\theta\, d\rho = 12\pi \int_0^1 \rho^2 \big[-\cos\varphi\big]_0^{\frac{\pi}{2}}\, d\rho = 12\pi \left[\frac{\rho^3}{3}\right]_0^1 = 4\pi$$

3. Let $\boldsymbol{F}(x, y, z) = (x, y^2, e^{yz})$ and let R be a cube centered at the origin with sides of length 2. Evaluate $\displaystyle\int_S \text{div } \boldsymbol{F}\, dV$ directly and by using the Divergence Theorem.

   Directly:

$$\begin{aligned}
\int_S \text{div}\boldsymbol{F}\, dV &= \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 1 + 2y + ye^{yz}\, dx\, dz\, dy \\
&= \int_{-1}^1 \int_{-1}^1 2 + 4y + 2ye^{yz}\, dz\, dy
\end{aligned}
\qquad
\begin{aligned}
&= \int_{-1}^1 4 + 8y + \big[e^{yz}\big]_{-1}^1\, dy \\
&= \int_{-1}^1 4 + 8y + e^y - e^{-y}\, dy \\
&= 8 + 0 + [e^y]_{-1}^1 + [e^{-y}]_{-1}^1 = 8
\end{aligned}$$

   Divergence:

$$\int_R \text{div}\boldsymbol{F}\, dV = \int_S \boldsymbol{F} \cdot d\boldsymbol{S}$$

   where S is the union of 6 planes with $-1 \leq x, y, z \leq 1$ and having normals in the $\pm x, y, z$ directions. The parameterizations can be given by

$$\begin{aligned}
\boldsymbol{\Phi}_{S_1}(x, y) &= (x, y, 1) \\
\boldsymbol{\Phi}_{S_2}(y, z) &= (1, y, z) \\
\boldsymbol{\Phi}_{S_3}(x, z) &= (x, 1, z)
\end{aligned}
\qquad
\begin{aligned}
\boldsymbol{\Phi}_{S_1'}(x, y) &= (x, y, -1) \\
\boldsymbol{\Phi}_{S_2'}(y, z) &= (-1, y, z) \\
\boldsymbol{\Phi}_{S_3'}(x, z) &= (x, -1, z)
\end{aligned}$$

   When plugging in S_3, S_3' into $\boldsymbol{F}$, then taking the dot with the normal, we get that the integrands are going to be the same, specifically 1. When doing the same with S_2, S_2, due to x being odd, we get both being negative the other, so integrating over the same range will cancel each other out. Putting this together, the integral over S is:

$$\begin{aligned}
\int_S \boldsymbol{F} \cdot d\boldsymbol{S} &= 2 \int_{-1}^1 \int_{-1}^1 1\, dx\, dz + \int_{-1}^1 \int_{-1}^1 e^y\, dx\, dy + \int_{-1}^1 \int_{-1}^1 -e^{-y}\, dx\, dy \\
&= 8 + 2 \int_{-1}^1 e^y\, dy - 2 \int_{-1}^1 e^{-y}\, dy \\
&= 8 + 2([e^y]_{-1}^1 + [e^{-y}]_{-1}^1) = 8
\end{aligned}$$

4. Let B be the pyramid with top vertex (0,0,1) and base vertices (0,0,0), (1,0,0), (0,1,0) and (1,1,0). Let S be the 2-dim closed surface bounding B, oriented in the outward direction. Use Gauss' theorem to calculate $\int_S \boldsymbol{F} \cdot d\boldsymbol{S}$, where $\boldsymbol{F}(x,y,z) = (x^2y, 3y^2z, 9z^2x)$.

$$\iiint_R \operatorname{div}\boldsymbol{F}\, dV = \iint_S \boldsymbol{F} \cdot d\boldsymbol{S}$$

$$\begin{aligned}
\operatorname{div}\boldsymbol{F} &= 2xy + 6yz + 18zx \\
\iiint_R \operatorname{div}\boldsymbol{F}\, dV &= \int_0^1 \int_0^{1-z} \int_0^{1-z} 2xy + 6yz + 18zx \, dx\, dy\, dz \\
&= \int_0^1 \int_0^{1-z} (1-z)^2 y + 6yz(1-z) + 9z(1-z)^2 \, dy\, dz \\
&= \int_0^1 (1/2)(1-z)^4 + 3z(1-z)^3 + 9z(1-z)^3 \, dz \\
&= \int_0^1 -\frac{23z^4}{2} + 34z^3 - 33z^2 + 10z + \frac{1}{2} \, dz \\
&= -\frac{23}{10} + \frac{34}{4} - 11 + 5 + \frac{1}{2} \\
&= \frac{7}{10}
\end{aligned}$$

5. Use the Divergence Theorem to evaluate $\int_S \boldsymbol{F} \cdot d\boldsymbol{S}$, where $\boldsymbol{F}(x,y,z) = \left(z^2x, \frac{y^3}{3} + \tan z, x^2z + y^2\right)$ and S is the top half of the unit sphere $x^2 + y^2 + z^2 = 1$, oriented by the unit normal which points away from the origin.

For the Divergence Theorem to apply, the integral must be over a closed, oriented outwards. To use Divergence Theorem the surface needs to be closed, so add the disk S' on the xy-plane to the surface. Then

$$\iiint_R \mathrm{div}\boldsymbol{F}\, dV - \iint_{S'} \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_S \boldsymbol{F} \cdot d\boldsymbol{S}$$

The disk can be parametrized as $\boldsymbol{\Phi}(r,\theta) = (r\cos\theta, r\sin\theta, 0)$ with $r \in [0,1], \theta \in [0,2\pi]$ The normal, pointing outward needs to be downward.

$$\begin{aligned}
\iint_{S'} \boldsymbol{F} \cdot d\boldsymbol{S} &= \int_0^1 \int_0^{2\pi} (*, *, (r\sin\theta)^2) \cdot (0,0,-1)\, d\theta\, dr \\
&= -\int_0^1 \int_0^{2\pi} (r\sin\theta)^2\, d\theta\, dr \\
&= -\int_0^1 \int_0^{2\pi} r^2(1/2 - (\cos\theta)/2)\, d\theta\, dr \\
&= -\int_0^1 \pi r^2\, dr = -\frac{\pi}{3}
\end{aligned}$$

Meanwhile, the region enclosed by this, is the top half of the unit sphere, so switch to polars, restricting $\varphi \in [0, \pi/2]$ to remain above the xy-plane.

$$\mathrm{div}\boldsymbol{F} = z^2 + y^2 + x^2 = \rho$$

$$\begin{aligned}
\iiint_R \mathrm{div}\boldsymbol{F}\, dV &= \int_0^1 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \rho^3 \sin\varphi\, d\varphi\, d\theta\, d\rho \\
&= 2\pi \int_0^1 \rho^3 \big[-\cos\varphi\big]_0^{\frac{\pi}{2}}\, d\rho = \frac{\pi}{2} \\
\implies \iint_S \boldsymbol{F} \cdot d\boldsymbol{S} &= \frac{\pi}{2} + \frac{\pi}{3} = \frac{5\pi}{6}
\end{aligned}$$

6. Let the electric field from a point source at the origin be given by $\boldsymbol{E}(\boldsymbol{x}) = \dfrac{\boldsymbol{x}}{\|\boldsymbol{x}\|^3}$

(a) What is the outward flux of $\boldsymbol{E}$ across the surface $\dfrac{x^2}{3} + \dfrac{2y^2}{5} + z^2 = 7$.

The surface is an ellipsoid centered around the origin. Since $\boldsymbol{E}$ follows the formula for Gauss' Law and the ellipsoid is closed and bounded, the flux of the surface is just 4π.

(b) Show that the flux of $\boldsymbol{E}$ across that part of the sphere $x^2 + y^2 + z^2 = 25$ with $z \geq 3$ is equal to the flux across that part of the plane $z = 3$ with $x^2 + y^2 \leq 16$.

Proof. First, examining the sphere, closing it with the disk at $z = 3$ allows the Divergence Theorem to be applied, but that is exactly the other surface given. Label the union of the two surfaces S. Since both the hemisphere and plane are above the xy-plane, $\mathbf{0} \notin S$, and since $\boldsymbol{E}$ satifies inverse square we have that the flux of the surface S is 0 by Gauss'. Now applying divergence, since S is closed and bounded, we have that the sum of the flux of the hemisphere S_1 and the plane S_2 is 0. Looking symbolically, we have

$$\int_{S_1} \boldsymbol{E} \cdot \boldsymbol{n}_1 \, dS_1 = -\int_{S_2} \boldsymbol{E} \cdot \boldsymbol{n}_2 \, dS_2$$

But now they are oriented in opposite direction since the outward normal of S has the normal of S_1 pointing away from the origin, and vice versa for S_2. If we reorient S_2 in the same direction, (with $-\boldsymbol{n}_2$) we get:

$$\int_{S_1} \boldsymbol{E} \cdot \boldsymbol{n}_1 \, dS_1 = \int_{S_2} \boldsymbol{E} \cdot \boldsymbol{n}_2 \, dS_2$$

So the flux of both surfaces is the same. $\square$

7. Let $f : \mathbb{R}^n \to \mathbb{R}$ be given by $f(x, y, z) = x^2yz$ and let η be the 2-form on $\mathbb{R}^3$ given by

$$\eta = (\sin x)\, dx\, dy + (e^y + xyz)\, dx\, dz + (x^2y^2)\, dy\, dz.$$

(a) Compute df and $d\eta$.

f is a 0 form so $df = \displaystyle\sum_{i=0}^{n} \frac{\partial}{\partial x_i} f\, dx_i \implies df = 2xyz\, dx + x^2z\, dy + x^2y\, dz$

$$\begin{aligned}
d\eta &= d(\sin x) \wedge dx \wedge dy + d(e^y + xyz) \wedge dx \wedge dz + d(x^2y^2) \wedge dy \wedge dz \\
&= (\cos x\, dx + 0\, dy + 0\, dz) \wedge dx \wedge dy + (yz\, dx + e^y + xz\, dy + xy\, dz) \wedge dx \wedge dz + d(x^2y^2) \wedge dy \wedge dz \\
&= (yz\, dx + e^y + xz\, dy + xy\, dz) \wedge dx \wedge dz + d(x^2y^2) \wedge dy \wedge dz \\
&= (yz\, dx + e^y + xz\, dy + xy\, dz) \wedge dx \wedge dz + (2xy^2\, dx + 2x^2y\, dy) \wedge dy \wedge dz \\
&= (e^y + xz)\, dy \wedge dx \wedge dz + (2xy^2)\, dx \wedge dy \wedge dz \\
&= -(e^y + xz)\, dx \wedge dy \wedge dz + (2xy^2)\, dx \wedge dy \wedge dz \\
&= (2xy^2 - e^y + xz)\, dx \wedge dy \wedge dz
\end{aligned}$$

(b) Evaluate $df \wedge \eta$.

$$\begin{aligned}
&(2xyz\, dx + x^2z\, dy + x^2y\, dz) \wedge ((\sin x)\, dx\, dy + (e^y + xyz)\, dx\, dz) \\
&= (2xyz)\, dx \wedge ((\sin x)\, dx\, dy + (e^y + xyz)\, dx\, dz) \\
&\quad + (x^2z)\, dy \wedge ((\sin x)\, dx\, dy + (e^y + xyz)\, dx\, dz) \\
&\quad + (x^2y)\, dz \wedge ((\sin x)\, dx\, dy + (e^y + xyz)\, dx\, dz) \\
&= (x^2z)\, dy \wedge ((e^y + xyz)\, dx\, dz) + (x^2y)\, dz \wedge ((\sin x)\, dx\, dy) \\
&= ((x^2z)(e^y + xyz)\, dy\, dx\, dz) + ((x^2y)(\sin x)\, dz\, dx\, dy) \\
&= -((x^2z)(e^y + xyz)\, dx\, dy\, dz) + ((x^2y)(\sin x)\, dx\, dy\, dz) \\
&= ((x^2y)(\sin x) - (x^2z)(e^y + xyz))\, dx\, dy\, dz \\
&= x^2(y \sin x - ze^y - xyz^2)\, dx\, dy\, dz
\end{aligned}$$

Bonus

Use Gauss' Divergence Theorem to evaluate $\boldsymbol{F} = (x^3, 0, z^3)$ over the surface S, where S is the upper hemisphere of $x^2 + y^2 + z^2 = 4$.

For the Divergence Theorem to apply, the surface needs to be closed, so add the disk at $z = 0$ to enclose the upper hemisphere. Now the theorem applies for $S' = S \cup S_D$ (where S_D is the disk) and gives that

$$\iiint_R \operatorname{div}\boldsymbol{F}\, dV = \iint_S \boldsymbol{F} \cdot d\boldsymbol{S} + \iint_{S_D} \boldsymbol{F} \cdot d\boldsymbol{S}$$

$$\iff \iiint_R \operatorname{div}\boldsymbol{F}\, dV - \iint_{S_D} \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_S \boldsymbol{F} \cdot d\boldsymbol{S}$$

Evaluate $\displaystyle\iiint_R \operatorname{div}\boldsymbol{F}\, dV$ using spherical polars, restricting $\varphi \in [0, \pi/2]$ to restrict to upper hemisphere, and the other variables the default limits for a sphere.

$$\begin{aligned}
\iiint_R \operatorname{div}\boldsymbol{F}\, dV &= \int_0^2 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} (3(\rho\cos\theta\sin\varphi)^2 + 3(\rho\cos\varphi)^2)\rho^2 \sin\varphi \, d\varphi\, d\theta\, d\rho \\
&= \int_0^2 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} 3\rho^2(\cos^2\theta\sin^2\varphi + \cos^2\varphi)\rho^2 \sin\varphi \, d\varphi\, d\theta\, d\rho \\
&= \int_0^2 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} 3\rho^4((1-\sin^2\theta)\sin^2\varphi + \cos^2\varphi) \sin\varphi \, d\varphi\, d\theta\, d\rho \\
&= \int_0^2 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} 3\rho^4(\sin^2\varphi + \cos^2\varphi - \sin^2\theta\sin^2\varphi) \sin\varphi \, d\varphi\, d\theta\, d\rho \\
&= \int_0^2 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} 3\rho^4(1 - \sin^2\theta(1-\cos^2\varphi)) \sin\varphi \, d\varphi\, d\theta\, d\rho
\end{aligned}$$

Let $u = \cos\varphi$, $du = -\sin\varphi$

$$\begin{aligned}
&= \int_0^2 \int_0^{2\pi} \int_1^0 -3\rho^4(1 - \sin^2\theta(1-u))\, du\, d\theta\, d\rho \\
&= \int_0^2 \int_0^{2\pi} \int_1^0 -3\rho^4(1 - \sin^2\theta + u^2\sin^2\theta)\, du\, d\theta\, d\rho \\
&= \int_0^2 \int_0^{2\pi} -3\rho^4 \left[u - u\sin^2\theta + \frac{1}{3}u^3\sin^2\theta\right]_1^0 d\theta\, d\rho \\
&= \int_0^2 \int_0^{2\pi} -3\rho^4(-1 + \sin^2\theta - \frac{1}{3}\sin^2\theta)\, d\theta\, d\rho
\end{aligned}$$

$$\begin{aligned}
&= \int_0^2 \int_0^{2\pi} 3\rho^4 \left(1 - \frac{1-\cos(2\theta)}{3}\right) d\theta\, d\rho \\
&= \int_0^2 \int_0^{2\pi} 3\rho^4 \left(\frac{2}{3} + \frac{\cos(2\theta)}{3}\right) d\theta\, d\rho \\
&= \int_0^2 \left[3\rho^4\left(\frac{2}{3}\right)\theta\right]_0^{2\pi} d\rho \\
&= \int_0^2 \rho^4(4\pi)\, d\rho \\
&= \left[\frac{4\rho^5\pi}{5}\right]_0^2 = \frac{128\pi}{5}
\end{aligned}$$

Evaluate $\displaystyle\iint_{S_D} \boldsymbol{F} \cdot d\boldsymbol{S}$ using parametrization of $\boldsymbol{\Phi}(\theta, r) = (r\cos\theta, r\sin\theta, 0)$, $r \in [0,2], \theta \in [0, 2\pi]$, with downward normal to point outward from upper hemispher.

$$\iint_{S_D} \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_{S_D} (*, *, 0) \cdot (0, 0, -1)\, dS = 0$$

$$\implies \iint_S \boldsymbol{F} \cdot d\boldsymbol{S} = \frac{128\pi}{5}$$