

MATB42: Assignment #8

1. (a) Let $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$; $\mathbf{F}, \mathbf{G} : \mathbb{R}^n \rightarrow \mathbb{R}$; and define Δ , the *Laplacian*, by $\Delta f = \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2}$.

Verify the following identities

- (i) $\operatorname{div}(\mathbf{F} + \mathbf{G}) = \operatorname{div} \mathbf{F} + \operatorname{div} \mathbf{G}$.
- (ii) $\operatorname{div}(f\mathbf{F}) = f \operatorname{div} \mathbf{F} + \mathbf{F} \cdot \operatorname{grad} f$.
- (iii) $\Delta(fg) = f\Delta g + g\Delta f + 2(\operatorname{grad} f) \cdot (\operatorname{grad} g)$.

- (b) Let $f, g : D \subset \mathbb{R}^3 \rightarrow \mathbb{R}$ be of class C^1 . If R is a solid region contained in D then

$$\iiint_R \nabla f \cdot \nabla g \, dV = \iint_{\partial R} f \nabla g \cdot \mathbf{n} \, dS - \iiint_R f \nabla^2 g \, dV$$

$$(\nabla^2 g = \operatorname{div} (\nabla g)).$$

2. Use the Divergence Theorem to verify your answer to question 7 on assignment 8.
3. Let $\mathbf{F}(x, y, z) = (x, y^2, e^{yz})$ and let R be a cube centered at the origin with sides of length 2. Evaluate $\int_S \operatorname{div} \mathbf{F} \, dV$ directly and by using the Divergence Theorem.
4. Let B be the pyramid with top vertex $(0,0,1)$ and base vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(1,1,0)$. Let S be the 2-dim closed surface bounding B , oriented in the outward direction. Use Gauss' theorem to calculate $\int_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = (x^2y, 3y^2z, 9z^2x)$.
5. Use the Divergence Theorem to evaluate $\int_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = \left(z^2x, \frac{y^3}{3} + \tan z, x^2z + y^2 \right)$ and S is the top half of the unit sphere $x^2 + y^2 + z^2 = 1$, oriented by the unit normal which points away from the origin.
6. Let the electric field from a point source at the origin be given by $\mathbf{E}(\mathbf{x}) = \frac{\mathbf{x}}{\|\mathbf{x}\|^3}$