10

NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

10.1 Introduction

A differential equation arises in a physical situation where we consider the rate of change of one variable with respect to the other. First variable is called as *dependent variable* while second as independent variable.

e.g.
$$\frac{dy}{dx} = f(x, y) \qquad ...(1)$$

We have to find y from (1) for a given value x, when the values are given as

$$x = x_0, \quad y = y_0 \qquad ...(2)$$

This condition is known as *initial condition* or *boundary condition*.

The solution in which y is obtained for given value of x is called *numerical solution* of differential equation.

Here, we record some examples of differential equations which occur in science and engineering.

(*i*) **The law of Motion:** The velocity $v\,(t)$ of a moving body is given by the law

$$m \cdot \frac{dv\,(t)}{dt} = F$$

where m is the mass of the body and F is the force acting on it.

(*ii*) **Kirchhoff's law for an electric circuit:** The voltage across an electric circuit containing an inductance L and a resistance R is given by

$$L \cdot \frac{di}{dt} + iR = V$$

(*iii*) **Radioactive decay:** The radioactive decay of an element is given by

$$\frac{dm}{dt} - km = 0$$

where m is the mass, t is the time and k is the constant rate of decay.

(*iv*) **Law of Cooling:** The Newton's law of cooling states that the rate of loss of heat from a liquid is proportional to the difference of temperature between the liquid and the surroundings. This can be shown mathematically by the differential equation as

$$\frac{dT\,(t)}{dt} = k\,(T_s - T\,(t))$$

where T_s is the temperature of surroundings, $T\,(t)$ is the temperature of the liquid at time t and

k is the constant of proportionality. (V) 2. In Probability theory - Notes 33

Many other examples of physical situations *e.g.* simple harmonic motion, force on a moving boat, heat flow in a rectangular plate are described by differential equations. , Notes - 34

Here, we discuss following methods for numerical solution of ordinary differential equations:

(*i*) Taylor's series method
(*ii*) Picard's method
(*iii*) Euler's method
(*iv*) Euler's modified method
(*v*) Euler's improved method
(*vi*) Runge-Kutta method of 1st, 2nd, 3rd and 4th order
(*vii*) Milne's Predictor and Corrector methods.

10.2 Taylor's Series Method

Let $y = f(x)$ be the solution of the equation

$$\frac{dy}{dx} = f(x, y), \text{ with } y(x_0) = y_0. \quad ...(1)$$

Differentiating (1), we have , w.r.t. x

$$\frac{d^2y}{dx^2} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$$

i.e.,

$$f'' = y'' = f_x + f_y \cdot f' \quad ...(2)$$

Differentiating above equation successively, we can get f''', f^{iv} etc. Putting $x = x_0$, we can get $f''(x_0)$, $f'''(x_0)$, ... etc.

Expanding (1) by Taylor's series about the point x_0, we get

$$f(x) = f(x_0) + \frac{x - x_0}{\lfloor 1} f'(x_0) + \frac{(x - x_0)^2}{\lfloor 2} f''(x_0) + \ldots$$

this may be written as

$$y = f(x) = y_0 + \frac{(x - x_0)}{\lfloor 1} y_0' + \frac{(x - x_0)^2}{\lfloor 2} y_0'' + \ldots$$

putting $x = x_1 = x_0 + h$, we get

$$f(x_1) = y_1 = y_0 + \frac{h}{\lfloor 1} \cdot y_0' + \frac{h^2}{\lfloor 2} \cdot y_0'' + \frac{h^3}{\lfloor 3} \cdot y_0''' + \ldots \quad ...(3)$$

Similarly, we obtain $f(x_2) = y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \ldots$

In general we get -

$$y_{n+1} = y_n + \frac{h}{\lfloor 1} \cdot y_n' + \frac{h^2}{\lfloor 2} \cdot y_n'' + \frac{h^3}{\lfloor 3} \cdot y_n''' + \ldots \quad ...(4)$$

Equation (4) may be written as

$$y_{n+1} = y_n + \frac{h}{\lfloor 1} \cdot y_n' + \frac{h^2}{\lfloor 2} \cdot y_n'' + 0(h^3) \quad ...(5)$$

where $0(h^3)$ represent all the succeeding terms containing the third and higher powers of h. ~~If the terms containing the third and higher powers of h are neglected then the local truncation error in the solution is kh^3 where k is a constant.~~ For a better approximation terms containing higher power

Taylor's series method -

Let $y = f(x)$, be a solution of the equation

$$\frac{dy}{dx} = f(x, y) \quad \text{--- (1)}$$

with $y(x_0) = y_0$

Expanding it by Taylor's series about the point x_0, we get

$$f(x) = f(x_0) + \frac{(x-x_0)}{1!} f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \frac{(x-x_0)^3}{3!} f'''(x_0) + \cdots$$

This may be written as -

$$y = f(x) = y_0 + \frac{(x-x_0)}{1!} y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \frac{(x-x_0)^3}{3!} y_0''' + \cdots$$

Putting $x = x_1 = x_0 + h$, we get -

$$f(x_1) = y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots \quad \text{--- (2)}$$

similarly we obtain

$$\Rightarrow y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \cdots \quad \text{--- (3)}$$

In General we obtain

$$y_{n+1} = y_n + \frac{h}{1!} y_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' + \cdots \quad (4)$$

Equation (4) may be written as -

$$y_{n+1} = y_n + \frac{h}{1!} y_n' + \frac{h^2}{2!} y_n'' + o(h^3) \quad \text{--- (5)}$$

where $o(h^3)$ means that all the succeeding terms containing the third & higher powers of h. If the terms containing the third & higher powers of h are neglected then the local truncation error in the solution is kh^3 where k is a constant. For a better approximation terms containing higher powers of h are considered.

Note :- Taylor's series method is applicable only when the various derivatives of $f(x,y)$ exist & the value of $(x-x_0)$ in the expansion of $y=f(x)$ near x_0 must be very small so that the series converge.

Ex 1 :- Solve $\frac{dy}{dx} = x+y$, Initially $y(1)=0$, numerically upto $x=1.2$ with $h=0.1$, by using taylor's Series method?

Sol^n :- We have $x_0 = 1,\ y_0 = 0$ &

$$y' = \frac{dy}{dx} = x+y \Rightarrow y_0' = 1+0 = 1$$

$$y'' = \frac{d^2y}{dx^2} = 1+y' \Rightarrow y_0'' = 1+1 = 2$$

$$y''' = \frac{d^3y}{dx^3} = y'' \Rightarrow y_0''' = 2$$

$$y^{iv} = \frac{d^4y}{dx^4} = y''' \Rightarrow y_0^{iv} = 2$$

$$y^{v} = \frac{d^5y}{dx^5} = y^{iv} \Rightarrow y_0^{v} = 2,$$

. . . .

Substituting the above values in taylor's Series method -

$$y_1 = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{iv} + \frac{h^5}{5!}y_0^{v} + \cdots$$

We get -

$$y_1 = 0 + (0.1) + \frac{(0.1)^2}{2}.2 + \frac{(0.1)^3}{6}.2 + \frac{(0.1)^4}{24}.2 + \frac{(0.1)^5}{120}.2 + \cdots$$

$\Rightarrow y_1 = 0.11033847 \quad \therefore y_1 = y(0.1) \approx \underline{0.110}.$

Now

$x_1 = x_0 + h = 1 + 0.1 = \underline{1.1}$.

We have

$y_1' = x_1 + y_1 = 1.1 + 0.110 = 1.21$

$y_1'' = 1 + y_1' = 1 + 1.21 = 2.21$

$y_1''' = y_1'' = 2.21$

$y_1^{iv} = 2.21$

$y_1^{v} = 2.21$

$\ldots\ldots\ldots$

$\ldots\ldots\ldots$

$y_3 = 0.232 + 0.1432 + 0.01216 +$

$+ 0.000405 + 0.00001$

$y_3 = y(1.2) = 0.3878$

substituting the above values in Taylor's Series method, we get -

$$y_2 = 0.110 + (0.1)(1.21) + \frac{(0.1)^2}{2}(2.21) + \frac{(0.1)^3}{6}(2.21) + \frac{(0.1)^4}{24}(2.21) + \frac{(0.1)^5}{120}\cdot(2.21)$$

$\Rightarrow y_2 = 0.232$ (approximately)

$\therefore y(0.2) = 0.232$.

Now $x_2 = x_0 + 2 \cdot h$
$= 1 + 2 \times 0.1$
$x_2 = 1 + 0.2 = 1.2$

$y_2' = x_2 + y_2 = 1.2 + 0.232$
$y_2' = 1.432$

$y_2'' = 1 + y_2' = 1 + 1.432$
$= 2.432$

$y_2''' = y_2'' = 2.432$

$y_2^{iv} = y_2''' = 2.432$

$y_2^{v} = y_2^{iv} = 2.432$

eq. ①
$$y_3 = y_2 + \frac{h}{1!}y_2' + \frac{h^2}{2!}y_2'' + \frac{h^3}{3!}y_2''' + \frac{h^4}{4!}y_2^{iv} + \frac{h^5}{5!}y_2^{v} + \cdots$$

Ex. 2:- Given $\frac{dy}{dx} = 1 + xy$ with the initial condition that $y = 1$, when $x = 0$ compute $y(0.1)$ correct to four places of decimal by using Taylor's series method.

Soln:- Given $\frac{dy}{dx} = 1 + xy$ & $y(0) = 1$

$\therefore y_1(0) = 1 + 0 \times 1 = 1$

Differentiating the given equation w.r.t. x, we get -

$$\frac{d^2y}{dx^2} = y + x\frac{dy}{dx}$$

$$y_0'' = 1 + 0 \times 1 = 1$$

similarly -

$$\frac{d^3y}{dx^3} = x\frac{d^2y}{dx^2} + 2\frac{dy}{dx}$$

$$\Rightarrow \quad y_0''' = 2.$$

&

$$\frac{d^4y}{dx^4} = x.\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2}$$

$$\Rightarrow \quad y_0^{iv} = 3,$$

from Taylor's series method, we have -

$$y_1 = 1 + h\,y_0' + \frac{h^2}{2}y_0'' + \frac{h^3}{6}y_0''' + \frac{h^4}{24}y_0^{iv} +$$

$$\therefore \; y(0.1) = 1 + (0.1)(1) + \frac{(0.1)^2}{2}(1) + \frac{(0.1)^3}{6}(2) + \frac{(0.1)^4}{24}(3)$$

$$y(0.1) = 1.1053425$$

$$\therefore \; y(0.1) = 1.1053$$

Correct to four decimal places.

Ex:-3:- Apply the Taylor's series method to find the value of $y(1.1)$ & $y(1.2)$ correct to three decimal places given that $\frac{dy}{dx} = xy^3$, $y(1) = 1$, taking the first three terms of the Taylor's series expansion.

Sol^n:- Given $\frac{dy}{dx} = xy^3$, $y_0 = 1$, $x_0 = 1$, $h = 0.1$

$$y_0' = x_0 y_0^3 = (1.1)^3 = 1$$

EULER'S METHOD –

Consider the first order & first degree differential eq.

$$\frac{dy}{dx} = f(x, y) \quad \text{—— (1)}$$

with the condition that $y(x_0) = y_0$. Suppose we want to find the approximation value of y say y_n when $x = x_n$. We divide the interval $[x_0, x_n]$ into n- subintervals of equal length say h, with the division points $x_0, x_1, \ldots, x_n$, where $x_r = x_0 + rh$, $(r = 1, 2, \ldots, n)$

Let us assume that

$$f(x, y) \approx f(x_{r-1}, y_{r-1})$$

in $[x_{r-1}, x_r]$. Integrating eq. (1) in $[x_{r-1}, x_r]$, we get.

$$\int_{x_{r-1}}^{x_r} dy = \int_{x_{r-1}}^{x_r} f(x, y)\, dx$$

$$\Rightarrow [y_r - y_{r-1}] = \int_{x_{r-1}}^{x_r} f(x, y)\, dx$$

$$\Rightarrow y_r \approx y_{r-1} + f(x_{r-1}, y_{r-1}) \int_{x_{r-1}}^{x_r} dx$$

$$\Rightarrow y_r \approx y_{r-1} + f(x_{r-1}, y_{r-1})(x_r - x_{r-1})$$

$$\therefore \quad y_r \approx y_{r-1} + h f(x_{r-1}, y_{r-1}) \quad \text{—— (2)}$$

Eq. (2) is called Euler's iteration formula.

Taking $r = 1, 2, \ldots, n$ in eq. (2), we get the successive approximatively of y as follows

$$y_1 = y(x_1) = y_0 + h f(x_0, y_0)$$

$$y_2 = y(x_2) = y_1 + h f(x_1, y_1)$$

. . . .

. . . .

$$y_n = y(x_n) = y_{n-1} + h f(x_{n-1}, y_{n-1}) \quad \text{OR}$$

$$y_{n+1} = y(x_{n+1}) = y_n + h f(x_n, y_n)$$

Note :- Euler's method has limited usage because of the large error that is accumulated as the process proceeds. The process is very slow & to obtain reasonable accuracy with euler's method we have to take a smaller value of h. Further the method should not be used for a larger range of x as the values found by this method go on becoming farther & farther away from the true values. To avoid this difficulty one can choose Euler's modified method to solve the eq. ①.

MODIFIED EULER'S METHOD :-

From Euler's iteration formulae we have known that

$$y_r \approx y_{r-1} + h f(x_{r-1}, y_{r-1}) \quad \text{—— ③}$$

Let $y(x_r) = y_r$ denote the initial value using ③ an approximate value of y_r^0 can be calculated as -

$$y_r^{(0)} = y_{r-1} + \int_{x_{r-1}}^{x_r} f(x, y)\, dx$$

$$\Rightarrow y_r^{(0)} \approx y_{r-1} + h f(x_{r-1}, y_{r-1}) \quad \text{—— ④}$$

Replacing $f(x, y)$ by $f(x_{r-1}, y_{r-1})$ in $x_{r-1} \leq x < x_r$, using Trapezoidal rule in $[x_{r-1}, x_r]$, we can write

$$y_r^{(0)} = y_{r-1} + \frac{h}{2}\left[f(x_{r-1}, y_{r-1}) + f(x_r, y_r)\right]$$

Replacing $f(x_r, y_r)$ by its approximate value $f(x_r, y_r^{(0)})$, at the end point of the interval $[x_{r-1}, x_r]$, we get

$$y_r^{(1)} = y_{r-1} + \frac{h}{2}\left[f(x_{r-1}, y_{r-1}) + f(x_r, y_r^{(0)})\right]$$

where $y_r^{(1)}$ is the first approximation to $y_r = y(x_r)$

proceeding as above we get the iteration formulae -

$$y_r^{(n)} = y_{r-1} + \frac{h}{2}\left[f(x_{r-1}, y_{r-1}) + f(x_r, y_r^{(n-1)})\right]$$

where y_r^{n} denoted the nth approximation to y_r. —— (5)

$\therefore$ we have

$$y_r \approx y_r^{(n)} = y_{r-1} + \frac{h}{2}\left[f(x_{r-1}, y_{r-1}) + f(x_r, y_r^{(n-1)})\right].$$

Ex:- 1:- Solve the equation $\frac{dy}{dx} = 1-y$, with the initial condition $x=0$, $y=0$, using Euler's algorithm & tabulate the solutions at $x = 0.1, 0.2, 0.3$.

Soln:- Given $\frac{dy}{dx} = 1-y$, with the initial condition $x=0$, $y=0$.

$$\therefore f(x,y) = 1-y.$$

we have

$$h = 0.1$$

$$\therefore x_0 = 0, \; y_0 = 0$$

$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

$$x_2 = 0.2, \; x_3 = 0.3$$

Taking $n=0$ in

$$y_{n+1} = y_n + h \cdot f(x_n, y_n)$$

we get $y_1 = y_0 + h f(x_0, y_0)$

$$= 0 + (0.1)(1-0) = 0.1$$

$\therefore \quad y_1 = 0.1$ i.e. $y(0.1) = 0.1$.

$$y_2 = y_1 + h f(x_1, y_1)$$

$$y_2 = 0.1 + (0.1)(1-y_1)$$

$$= 0.1 + (0.1)(1-0.1)$$

$$\therefore \quad y_2 = 0.19$$

$$\therefore \quad y_2 = y(0.2) = 0.19$$

$$y_3 = y_2 + h \cdot f(x_2, y_2)$$

$$\therefore \quad y_3 = 0.19 + (0.1)(1-y_2)$$

$$= 0.19 + (0.1)(1-0.19)$$

$$= 0.19 + (0.1)(0.81)$$

$$= 0.271$$

$$\therefore \quad y_3 = y(0.3) = 0.271$$

x	solution by Euler's method
0	0
0.1	0.1
0.2	0.19
0.3	0.271

Ex:- 2 :- Given $\frac{dy}{dx} = x^3+y$, $y(0) = 1$, Compute $y(0.2)$ by Euler's method taking $h = 0.01$.

Sol^n :- Given $\frac{dy}{dx} = x^3+y$.

with the initial condition $y(0) = 1$.

$\therefore$ we have $f(x,y) = x^3+y$

$$x_0 = 0 \;, y_0 = 1, \; h = 0.01$$

$$x_1 = x_0+h = 0+0.01 = 0.01$$

$$x_2 = x_0+2.h = 0+2(0.01) = 0.02$$

Applying Euler's formulae we get.

$$y_1 = y_0 + h\,f(x_0,y_0)$$

$$\therefore \quad y_1 = 1+(0.01)(x_0^3+y_0)$$

$$= 1+(0.01)(0^3+1)$$

$$= 1.01$$

$$\therefore \; y_1 = y(0.01) = 1.01$$

$$y_2 = y_1 + hf(x_1,y_1)$$

$$= 1.01+(0.01)[x_1^3+y_1]$$

$$= 1.01+(0.01)[(0.01)^3+1.01] = 1.0201$$

$$\therefore \; y_2 = y(0.02) = 1.0201$$

Ex:- 3 :- Solve by Euler's method the following differential equation $x = 0.1$ correct to four decimal places $\frac{dy}{dx} = \frac{y-x}{y+x}$ with the initial condition $y(0) = 1$.

Sol^n :- Here

$$\frac{dy}{dx} = \frac{y-x}{y+x}$$

$$\Rightarrow f(x,y) = \frac{y-x}{y+x},$$

the initial condition is $y(0) = 1$.

Taking $h = 0.02$, we get

$$x_1 = 0.02, \quad x_2 = 0.04, \quad x_3 = 0.06, \quad x_4 = 0.08$$

$$x_5 = 0.1.$$

Using Euler's formula we get.

$$y_1 = y(0.02) = y_0 + hf(x_0, y_0)$$

$$= y_0 + h\left(\frac{y_0 - x_0}{y_0 + x_0}\right)$$

$$= 1 + (0.02)\left(\frac{1-0}{1+0}\right)$$

$$= 1.0200$$

$$\therefore y(0.02) = 1.0200$$

$$y_2 = y(0.04) = y_1 + hf(x_1, y_1)$$

$$= y_1 + h\left(\frac{y_1 - x_1}{y_1 + x_1}\right)$$

$$= 1.0200 + (0.02)\left(\frac{1.02 - 0.02}{1.02 + 0.02}\right)$$

$$= 1.0392$$

$$y_2 = y(0.04) = 1.0392,$$

$$y_3 = y(0.06) = y_2 + h\left(\frac{y_2 - x_2}{y_2 + x_2}\right)$$

$$= 1.0392 + (0.02)\left[\frac{1.0392 - 0.04}{1.0392 + 0.04}\right]$$

$$\therefore\ y_3 = y(0.06) = 1.0577.$$

$$y_4 = y(0.08) = y_3 + hf(x_3, y_3)$$

$$= y_3 + h\left[\frac{y_3 - x_3}{y_3 + x_3}\right]$$

$$= 1.0577 + (0.02)\left[\frac{1.0577 - 0.06}{1.0577 + 0.06}\right]$$

$$= 1.0756$$

$$\therefore\ y_4 = y(0.08) = 1.0756$$

$$y_5 = y(0.1) = y_4 + hf(x_4, y_4)$$

$$= y_4 + h\left(\frac{y_4 - x_4}{y_4 + x_4}\right)$$

$$= 1.0756 + (0.02)\left[\frac{1.0756 - 0.08}{1.0756 - 0.08}\right]$$

$$= 1.0928$$

$$\therefore\ y(0.1) = 1.0928.$$

Ex:-4:- Solve the Euler's modified method the following differential equation for $x = 0.02$ by taking $h = 0.01$

$$\frac{dy}{dx} = x^2 + y,\ y = 1,\ \text{when } x = 0.$$

Solⁿ:- we have $f(x, y) = x^2 + y$

$h = 0.01,\ x_0 = 0,\ y_0 = y(0) = 1,\ x_1 = 0.01,\ x_2 = 0.02$

8. Find $y(0.1)$, give that $\frac{dy}{dx} = x + z$, $\frac{dz}{dx} = x - y^2$ and $y(0) = 2$, $z(0) = 1$.
9. Find $y(0.2)$ and $y(0.4)$, given that $y'' + xy = 0$ and $y(0) = 1$, $y'(0) = 0.5$.
10. Find $y(0.1)$, given $y' = x^2 y - 1$, $y(0) = 1$.

ANSWERS

1. $y(0.1) = 1.1053$.
2. $y = x + \frac{x^3}{6} + \frac{x^5}{120} + \ldots$
3. 0.4158.
4. 1.22788.
5. $y(0.1) = 0.995$, $y(0.2) = 0.9801$.
6. $y(0.1) = 1.00501252$.
7. $y(1.1) = 1.225$, $y(1.2) = 1.512$; $y(1.3) = 1.874$.
8. $y(0.1) = 2.0845$.
10. $y(0.1) = 0.9003$.

10.3 Picard's Method of Successive Approximations

Consider the first order differential equation

$$\frac{dy}{dx} = f(x, y) \quad \Rightarrow \quad dy = f(x,y)\,dx \qquad \ldots(1)$$

with initial condition $y(x_0) = y_0$.

Integrating the differential equation (1), we obtain,

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y)\,dx \quad \Rightarrow \quad y - y_0 = \int_{x_0}^{x} f(x,y)\,dx$$

or

$$y = y_0 + \int_{x_0}^{x} f(x, y)\,dx. \qquad \ldots(2)$$

In equation (2), the unknown function y appears under the integral sign, is called an *integral equation*. This equation can be solved by the method of successive approximations. The first approximation to y is obtained by putting y_0 for y on right side of (2), and can be written as

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx.$$

For a second approximation $y^{(2)}$, we put $y = y^{(1)}$ in $f(x, y)$ and integrate (2) as

$$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})\,dx.$$

Proceeding in this way, we get $y^{(3)}, y^{(4)}, \ldots, y^{(n-1)}$ and $y^{(n)}$, where

$$\left.\begin{aligned} y^{(n)} &= y_0 + \int_{x_0}^{x} f(x, y^{(n-1)})\,dx \\ \text{with } y^{(0)} &= y_0 \end{aligned}\right\} \qquad \ldots(3)$$

eq. (3) is known as P. M.

Hence, this method gives a sequence of approximations $y^{(1)}, y^{(2)}, \ldots, y^{(n)}$ and it can be proved that if the function $f(x, y)$ is bounded in some region about the point (x_0, y_0) and if $f(x, y)$ satisfies the Lipschitz condition, viz.

$$|f(x, y) - f(x, \bar{y})| \le k\,|y - \bar{y}|,\ k \text{ being a constant} \qquad ...(4)$$

then the sequence $y^{(1)}, y^{(2)}, \ldots$ converges to the solution of equation (1).

upto fourth approximation

Example 10.6 Solve the differential equation using Picard's method $\frac{dy}{dx} = x + y$, with initial condition $y = 1$ when $x = 0$. Approximate y when $x = 0.1$ and 0.2.

Solution: Given $f(x, y) = x + y$

Integrating the given differential equation between the limits, we have

$$\int_1^y dy = \int_0^x (x + y)\, dx$$

or
$$y = 1 + \int_0^x (x + y)\, dx.$$

First approximation y_1 is obtained by replacing y by 1.

i.e.,
$$y_1 = 1 + \int_0^x (x + 1)\, dx$$

$$y_1 = 1 + \frac{x^2}{2} + x = 1 + x + \frac{x^2}{2}.$$

Second approximation y_2 is obtained by replacing y by y_1 in $(y + x)$ *i.e.,*

$$y_2 = 1 + \int_0^x (x + y_1)\, dx = 1 + \int_0^x \left(x + 1 + x + \frac{x^2}{2}\right) dx$$

$$= 1 + x + x^2 + \frac{x^3}{6}.$$

Third approximation is obtained by

$$y_3 = 1 + \int_0^x (x + y_2)\, dx = 1 + \int_0^x \left(x + 1 + x + x^2 + \frac{x^3}{6}\right) dx$$

$\Rightarrow$
$$y_3 = 1 + \int_0^x \left(1 + 2x + x^2 + \frac{x^3}{6}\right) dx$$

or
$$y_3 = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}.$$

The fourth approximation is obtained by

$$y_4 = 1 + \int_0^x (x + y_3)\, dx$$

$$= 1 + \int_0^x \left(x + 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}\right) dx$$

$$= 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}.$$

Hence, the solution upto fourth approximation is given by above equation.

Now, let us consider,

$$y = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}$$

when $x = 0.1$

$$y = 1 + 0.1 + (0.1)^2 + \frac{(0.1)^3}{3} + \frac{(0.1)^4}{12} + \frac{(0.1)^5}{120}$$

$$= 1 + 0.1 + 0.01 + \frac{0.001}{3} + \frac{0.0001}{12} + \frac{0.00001}{120}$$

$$= \mathbf{1.110342}.$$

when $x = 0.2$

$$y = 1 + 0.2 + (0.2)^2 + \frac{(0.2)^3}{3} + \frac{(0.2)^4}{12} + \frac{(0.2)^5}{120}$$

$$= 1 + 0.2 + 0.04 + \frac{0.008}{3} + \frac{0.0016}{12} + \frac{0.00032}{120}$$

$$= \mathbf{1.24280}.$$

Example 10.7 By using Picard's method, solve the differential equation $\frac{dy}{dx} = xe^y$, with the condition $y = 0$ when $x = 0$. Also estimate y at $x = 0.1, 0.2, 1.0$.

Solution: We have $\frac{dy}{dx} = f(x, y) = xe^y$ and $y_0 = 0, \quad x_0 = 0$

First approximation is given by

$$y_1 = y_0 + \int_0^x f(x, y)\, dx$$

$$= 0 + \int_0^x xe^0\, dx = \frac{x^2}{2}.$$

Second approximation y_2 is obtained by

$$y_2 = y_0 + \int_0^x xe^{y_1}\, dx = 0 + \int_0^x xe^{\frac{x^2}{2}}\, dx$$

$$y_2 = e^{\frac{x^2}{2}} - 1.$$

Now, we see that it is difficult to find the next approximations, as it is difficult to integrate further. Therefore, we assume here

$$y = y_2 = e^{x^2/2} - 1$$

When $x = 0.1$

$$y = e^{\frac{(0.1)^2}{2}} - 1 = 0.00501252$$

When $x = 0.2$

$$y = e^{\frac{(0.2)^2}{2}} - 1 = 0.02020134$$

and when $x = 1$

$$y = e^{\frac{(1)^2}{2}} - 1 = e^{0.5} - 1 = 0.64872127$$

Now, let us consider,

$$y = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}$$

when $x = 0.1$

$$y = 1 + 0.1 + (0.1)^2 + \frac{(0.1)^3}{3} + \frac{(0.1)^4}{12} + \frac{(0.1)^5}{120}$$

$$= 1 + 0.1 + 0.01 + \frac{0.001}{3} + \frac{0.0001}{12} + \frac{0.00001}{120}$$

$$= \mathbf{1.110342}.$$

when $x = 0.2$

$$y = 1 + 0.2 + (0.2)^2 + \frac{(0.2)^3}{3} + \frac{(0.2)^4}{12} + \frac{(0.2)^5}{120}$$

$$= 1 + 0.2 + 0.04 + \frac{0.008}{3} + \frac{0.0016}{12} + \frac{0.00032}{120}$$

$$= \mathbf{1.24280}.$$

Example 10.7 By using Picard's method, solve the differential equation $\frac{dy}{dx} = xe^y$, with the condition $y = 0$ when $x = 0$. Also estimate y at $x = 0.1, 0.2, 1.0$.

Solution: We have $\frac{dy}{dx} = f(x, y) = xe^y$ and $y_0 = 0, \quad x_0 = 0$

First approximation is given by

$$y_1 = y_0 + \int_0^x f(x, y)\, dx$$

$$= 0 + \int_0^x xe^0\, dx = \frac{x^2}{2}.$$

Second approximation y_2 is obtained by

$$y_2 = y_0 + \int_0^x xe^{y_1}\, dx = 0 + \int_0^x xe^{\frac{x^2}{2}}\, dx$$

$$y_2 = e^{\frac{x^2}{2}} - 1.$$

Now, we see that it is difficult to find the next approximations, as it is difficult to integrate further. Therefore, we assume here

$$y = y_2 = e^{x^2/2} - 1$$

When $x = 0.1$

$$y = e^{\frac{(0.1)^2}{2}} - 1 = 0.00501252$$

When $x = 0.2$

$$y = e^{\frac{(0.2)^2}{2}} - 1 = 0.02020134$$

and when $x = 1$

$$y = e^{\frac{(1)^2}{2}} - 1 = e^{0.5} - 1 = 0.64872127$$

If we compare with the exact solution of $\frac{dy}{dx} = xe^y$,

i.e., $$y = -\ell n\left(1 - \frac{x^2}{2}\right)$$

at $x = 0.1$, 0.2 and 1, we get

$$y = 0.0050125,\ y = 0.0202027,\ y = 0.6933147 \text{ respectively.}$$

Example 10.8 Find the solution of $\frac{dy}{dx} = 1 + xy$ which passes through (0, 1) in the interval (0, 0.5) such that the value of y is ~~correct to three decimal places,~~ (use the whole interval as one interval only). Take $h = 0.1$. evaluate upto Third approx.

Solution: We have,

$$f(x, y) = \frac{dy}{dx} = 1 + xy; \text{ with } y_0 = 1 \text{ and } x_0 = 0.$$

Now, first approximation,

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx = 1 + \int_0^x \{1 + (xy_0)\}\, dx$$

$$= 1 + \int_0^x (1 + x)\, dx = 1 + x + \frac{x^2}{2}. \qquad ...(1)$$

Second approximation

$$y^{(2)} = y_0 + \int_0^x f(x, y^{(1)})\, dx = y_0 + \int_0^x \{1 + (xy^{(1)})\}\, dx$$

$$= 1 + \int_0^x \left(1 + x + x^2 + \frac{x^3}{2}\right) dx = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}. \qquad ...(2)$$

Third approximation

$$y^{(3)} = 1 + \int_0^x \left\{1 + x\left(1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}\right)\right\} 6x$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \frac{x^6}{48}. \qquad ...(3)$$

When $x = 0$, $y = 1$ *i.e.*, $y_0 = y^{(0)} = 1$.

When $x = 0.1$, $y^{(1)} = 1.105$, $y^{(2)} = 1.10533$.

But we need the accuracy up to 3 decimal places.

Therefore, when $x = 0.1$, $y = 1.105$.

When $x = 0.2$, $y^{(1)} = 1.202$, $y^{(2)} = 1.2027$.

Therefore, when $x = 0.2$, $y = 1.203$.

When $x = 0.3$, $y^{(1)} = 1.345$, $y^{(2)} = 1.35501$, $y^{(3)} = 1.3551$.

Therefore, when $x = 0.3$, $y = 1.355$.

When $x = 0.4$; $y^{(1)} = 1.48$, $y^{(2)} = 1.50453$, $y^{(3)} = 1.5052$.

Therefore, when $x = 0.4$, $y = 1.505$.

When $x = 0.5$, $y^{(1)} = 1.625$, $y^{(2)} = 1.6744725$ and $y^{(3)} = 1.6765$.

Therefore, we have $y = 1.677$ when $x = 0.5$.

Thus, the tabulated values are as follows:

x :	0	0.1	0.2	0.3	0.4	0.5
y :	1.000	1.105	1.203	1.355	1.505	1.677

Example 10.9 Use Picards method to approximate y when $x = 0.1$ given that $y = 1$ when $x = 0$ and $\frac{dy}{dx} = \frac{y-x}{y+x}$.

Solution: We have

$$f(x, y) = \frac{y-x}{y+x}, \quad x_0 = 0, \quad y_0 = 1.$$

Now, first approximation,

$$y^{(1)} = y_0 + \int_0^x f(x, y_0)\, dx = 1 + \int_0^x \frac{1-x}{1+x}\, dx$$

$$= 1 + [2 \log(1+x) - x]_0^x = 1 - x + 2 \log(1+x).$$

Second approximation,

$$y^{(2)} = y_0 + \int_{x_0}^x f(x, y^{(1)})\, dx$$

$$= 1 + \int_0^x \frac{1-2x+2\log(1+x)}{1+2\log(1+x)}\, dx$$

$$= 1 + x - 2\int_0^x \frac{x}{1+2\log(1+x)}\, dx$$

$$= 1 + x - 2\int_0^t \frac{e^{2t}}{1+2t}\, dt + 2\int_0^t \frac{e^t}{1+2t}\, dt$$

where $t = \log(1+x)$.

Which is difficult to integrate.

When $x = 0$, $y = 1$.

When $x = 0.1$, $y^{(1)} = 1 - 0.1 + 2\log(1.1) = 0.9828$.

$y^{(2)}$ = can not be obtained

Hence, in this example only first approximation which is 0.9828 can be obtained.

Example 10.10 Approximate y and z by using Picard's method for the particular solution of $\frac{dy}{dx} = x + z$, $\frac{dz}{dx} = x - y^2$, with initial condition $y = 2$, $z = 1$ when $x = 0$.

Solution: Let $f(x, y, z) = x + z$ and $\phi(x, y, z) = x - y^2$ $x_0 = 0$, $y_0 = 2$, $z_0 = 1$.

$$\frac{dy}{dx} = f(x, y, z), \quad y = y_0 + \int_{x_0}^x f(x, y, z)\, dx \qquad ...(1)$$

Again we have $$\frac{dz}{dx} = \phi(x, y, z), \quad z = z_0 + \int_{x_0}^x \phi(x, y, z)\, dx \qquad ...(2)$$

Now first approximation of y is

$$y^{(1)} = y_0 + \int_0^x f(x, y_0, z_0)\, dx = 2 + \int_0^x (x + 1)\, dx$$

$$= 2 + x + \frac{x^2}{2}.$$

Also first approximation of z is

$$z^{(1)} = z_0 + \int_0^x \phi\ (x, y_0, z_0)\, dx = 1 + \int_0^x (x - 4)\, dx$$

$$= 1 - 4x + \frac{x^2}{2}.$$

Hence, the second approximation $y^{(2)}$ and $z^{(2)}$ are

$$y^{(2)} = y_0 + \int_0^x f(x, y^{(1)}, z^{(1)})\, dx$$

$$= 2 + \int_0^x (x + z^{(1)})\, dx = 2 + \int_0^x \left(x + 1 - 4x + \frac{x^2}{2}\right) dx$$

$$= 2 + x - \frac{3}{2} x^2 + \frac{1}{6} x^3$$

and

$$z^{(2)} = z_0 + \int_0^x \phi\ (x, y^{(1)}, z^{(1)})\, dx = 1 + \int_0^x [x - \{y^{(1)}\}^2]\, dx$$

$$= 1 + \int_0^x \left[x - \left(2 + x + \frac{x^2}{2}\right)^2\right] dx$$

$$= 1 - 4x - \frac{3x^2}{2} - x^3 - \frac{x^4}{4} - \frac{x^5}{20}$$

PROBLEM SET 10.2

1. Solve $y' = y - x^2$, $y(0) = 1$, by Picard's method upto the third approximation. Hence, find the value of $y(0.1)$, $y(0.2)$.
2. Solve the differential equation $\frac{dy}{dx} = x \sin \pi y$ by Picard's method. Given $y(0) = \frac{1}{2}$.
3. Solve $\frac{dy}{dx} = x^2 + y^2$, $y(0) = 1$, Picard's method.
4. Solve the differential equation $\frac{dy}{dx} = e^x - y$, $y(0) = 0$ by Picard's method.
5. Using Picard's method, solve the differential equation $\frac{dy}{dx} = x + y^2 + 1$, with $y(0) = 0$.
6. Solve $y' = 1 + y^2$, given $y(0) = 0$ using Picard's method. Evaluate $y(0.2)$ and $y(0.4)$.
7. Using Picard's method solve the differential equation $y' = 2x - y$ with $y(1) = 3$. Also find $y(1.1)$.