10. $y(0.6) = 0.21$[illegible]; $y(0.8) = 0.4153$, $y(1) = 0.7027$.

11. 3.4394.

12. $y(1.2) = 1.0228$; $y(1.4) = 1.8847$.

13. $y(0.2) = 1.9227$.

14. $y(0.1) = 0.095$
$y(0.2) = 0.18098$.

15. $y(0.1) = 1.1105$
$y(0.2) = 1.25026$.

16. $y(0.2) = 1.0095$.

17. $y(0.25) = 1.0625$.

18. $y(2) = 5.0516$.

19. $y(0.2) = 0.1802$
$y(0.4) = 0.3228$
$y(0.6) = 0.4324$
$y(0.8) = 0.5146$
$y(1.0) = 0.5748$.

20. $y(0.2) = 0.10833$
$y(0.4) = 0.162083$
$y(0.6) = 0.345520$
$y(0.8) = 0.621380$
$y(1.0) = 0.980345$.

10.7 Runge-Kutta Method

The R-K method are actually a family of met

This method is named after two German mathematicians Carl Runge (1856-1927) and Wilhelm Kutta (1867-1944). It was developed to avoid the computation of higher order derivations which the Taylor's method may involve. Runge-Kutta formulae possess the advantage of requiring only the function values at some selected points. These methods agree with Taylor's series solutions upto the term h^r where r differs from method to method and is called the order of that method.

(*i*) **First order Runge-Kutta Method:** Consider the first order differential equation

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0 \qquad \text{...(1)}$$

By Euler's method we know that

$$y_1 = y_0 + hf(x_0, y_0)$$

Expanding by Taylor's series, we get

$$y_1 = y(x_0 + h) = y_0 + h\,y_0' + \frac{h^2}{\underline{|2}}\,y_0'' + \ldots \qquad \text{...(2)}$$

It follows that the Euler's method agrees with the Taylor's series solution upto the term in h.

Hence, the Euler's method is the Runge-Kutta method of first order.

(*ii*) **Second order Runge-Kutta Method:** We know that modified Euler's method

$$y_1 = y_0 + \frac{h}{2}\,[f(x_0, y_0) + f(x_0 + h, y_1)] \qquad \text{...(3)}$$

putting $y_1 = y_0 + hf(x_0, y_0)$ on the right hand side of equation (3), we get,

$$y_1 = y_0 + \frac{h}{2}\,[f_0 + f(x_0 + h, y_0 + hf_0)] \qquad \text{...(4)}$$

[where $f_0 = f(x_0, y_0)$].

Expanding left hand side by Taylor's series, we get

$$y_1 = y(x_0 + h) = y_0 + hy_0' + \frac{h^2}{\underline{|2}}\,y_0'' + \frac{h^3}{\underline{|3}}\,y_0''' + \ldots \qquad \text{...(5)}$$

Expanding $f(x_0 + h, x_0 + hf_0)$ by Taylor's series for a function of two variables, equation (4) gives

$$y_1 = y_0 + \frac{h}{2}\left[f_0 + \left\{f(x_0, y_0) + h\left(\frac{\partial f}{\partial x}\right)_0 + hf_0\left(\frac{\partial f}{\partial y}\right)_0 + 0\,(h^2) + \right\}\right]$$

$$= y_0 + \frac{1}{2}\left[hf_0 + hf_0 + h^2\left\{\left(\frac{\partial f}{\partial x}\right)_0 + \left(\frac{\partial f}{\partial y}\right)_0\right\} + 0\,(h^3)\right]$$

$$= y_0 + hf_0 + \frac{h^2}{2} f_0' + 0\,(h^3) \qquad \left[\because \frac{df\,(x, y)}{dx} = \frac{\partial f}{\partial x} = f\frac{\partial f}{\partial y}\right]$$

$$= y_0 + hy_0' + \frac{h^2}{\underline{|2}} y_0'' + 0\,(h^3) \qquad \text{...(6)}$$

Equating (5) and (6), it follows that the modifies Euler's method agrees with Taylor's series solution upto the term in h^2.

Hence, the modifies Euler's method is the Runge-Kutta method of second order.

Therefore, the second order Runge-Kutta formula is

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2), \text{ where } k_1 = hf\,(x_0, y_0)$$

and $$k_2 = hf\,(x_0 + h, y_0 + k_1)$$

(*iii*) **Third Order Runge-Kutta Method:** On the lines of above methods, one can see that Runge-Kutta method agree with the Taylor's series solution upto the terms in h^3. *i.e.*,

$$y_1 = y_0 + \frac{h}{6}\left[f\,(x_0, y_0) + 4f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f\,(x_0, y_0)\right) + f\,(x_0 + h, y_0 + hf\,(x_0 + h, y_0 + hf\,(x_0, y_0)))\right] \qquad \text{...(7)}$$

The third order Runge-Kutta formula is

$$y_1 = y_0 + \frac{1}{6}(k_1 + 4k_2 + k_3)$$

where $$k_1 = hf\,(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\,(x_0 + h,\ y_0 + k')$$

where $$k' = hf\,(x_0 + h, y_0 + k_1).$$ This formula is known as R-K method T.S.

(*iv*) **Fourth Order Runge-Kutta Method:** The Runge-Kutta fourth order method considers with the Taylor's Series solution upto terms of h^4, we know that the Taylor's series solution can be expressed in terms of $f\,(x, y)$ and its partial derivatives of various orders.

Here we take

$$\left.\begin{aligned} k_1 &= hf\,(x, y) \\ k_2 &= hf\,(x + mh, y + mk_1) \\ k_3 &= hf\,(x + nh, y + nk_2) \\ k_4 &= hf\,(x + ph, y + pk_3) \end{aligned}\right\} \qquad \text{...(8)}$$

to avoid the calculations of derivatives.

Now, $$y\,(x + h) \approx y\,(x) + ak_1 + bk_2 + ck_3 + dk_4 \qquad \text{...(9)}$$

We know that by Taylor's series

$$y\,(x + h) = y\,(x) + hy' + \frac{h^2}{\underline{|2}} y'' + \frac{h^3}{\underline{|3}} y''' + \frac{h^4}{\underline{|4}} y^{iv} + 0\,(h^5) \qquad \text{...(10)}$$

RUNGE - KUTTA METHODS —

These methods were devised by C. Runge and extended by W. Kutta a few years later. The Runge-Kutta methods are actually a family of methods, of which the second order and fourth order methods are widely used. As we have seen in the preceding section, we can improve the accuracy of those methods by taking smaller step sizes. Much greater accuracy can be obtained by using Runge-Kutta methods. These are equivalent of approximating the ~~ac~~ exact solution by matching the first n terms of the Taylor's series method. We will discuss only the second and fourth order methods. In these methods, first the slope at some of the intermediate points is computed and then weighted average of slopes is used to extrapolate the next solution point.

RUNGE- KUTTA SECOND ORDER METHODS —

The Runge-Kutta second order methods ~~are actually a family of methods~~, ~~each of~~ which matches the Taylor series ~~method~~ solution up to the second degree terms in h, where h is the step size.

In these method the interval $[x_1, x_f]$ is divided into subintervals and a weighted average of derivatives (slopes) at these intervals is used to determine the value of dependent variable. One

Picard's
advantage of these methods is that they, like Euler's method one step methods i.e, in order to evaluate y_{i+1}, we need information only at the preceding point (x_i, y_i)

* → Fig.

Consider the following differential equation

$$\frac{dy}{dx} = f(x, y)$$

with an initial condition :

$$y = y_1 \text{ at } x = x_1$$

At the starting point, compute the slope of the curve as $f(x_1, y_1)$. Let it be S_1. Now compute the slope of the curve at point (x_2, y_1+S_1h) as (x_2, y_1+S_1h), where $x_2 = x_1 + h$. Let this new slope be S_2. Find the average of these two slopes, and then compute the value of the dependent variable y from the following equation

$$y_2 = y_1 + hs$$

where

$$s = \frac{s_1 + s_2}{2}$$

$$s_1 = f(x_1, y_1)$$

$$s_2 = (x_1 + h, y_1 + s_1 h)$$

Hence starting from point $f(x_1, y_1)$ we obtained the second point (x_2, y_2). Similarly starting from second point, we can obtain the third point. And this process is repeated till we find the solution in the desired interval.

In general, the value of y for the $(i+1)^{th}$ point on the solution curve is obtained from the i^{th} solution point using the formulae.

$$y_{i+1} = y_i + hs \quad \text{——— ①}$$

where

$$s = \frac{s_i + s_{i+1}}{2}, \quad s_i = f(x_i, y_i) \text{ and}$$

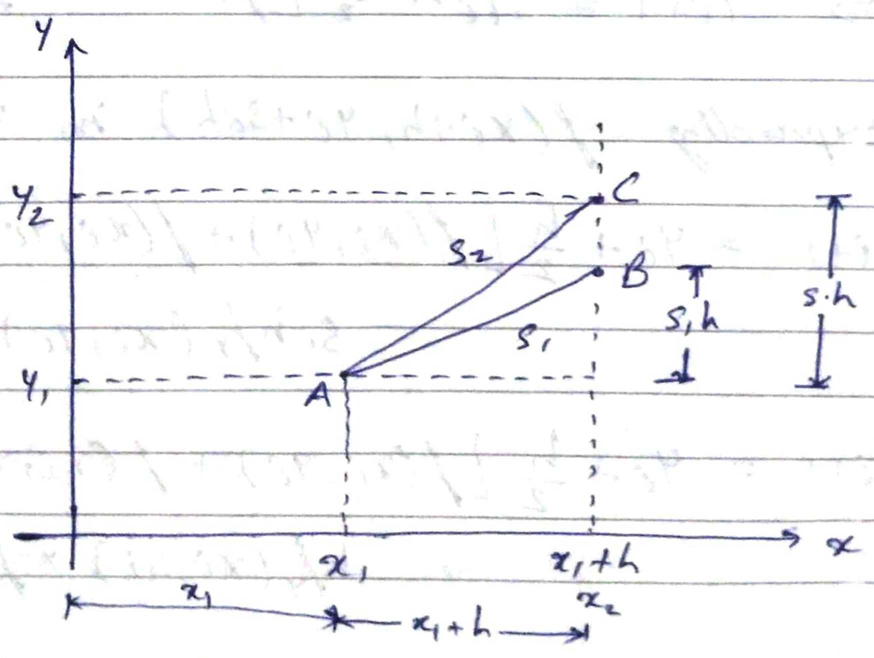
$$s_{i+1} = (x_i + h, y_i + s_i h)$$

This formulae for the Runge-Kutta second order method is also known as Heun's method.

Geometric Interpretation –

From the starting point (x_1, y_1) labelled as A, draw a line with slope $s_1 \, (= f(x_1, y_1))$.

*



Let this line intersects the vertical line at x_1+h at a point labelled B, whose co-ordinates are (x_1+h, y_1+s_1h). Now compute the slope of the curve at point B. Let this slope be $s_2\ (= f(x_1+h, y_1+s_1h))$. Compute the average of these two slopes i.e. s_1 and s_2. Let this average slope be s. Now draw a line from point A with slope s. Let this new line intersects the vertical line at x_1+h at a point labelled C. The point C, whose co-ordinates are (x_1+h, y_1+sh), is taken as the second solution point. Then in the similar manner we proceed from the second solution point to obtain the third solution point, and so on.

We will now show that Runge-Kutta second order methods are identical to Taylor series method. Since

$$y_{i+1} = y_i + \frac{h}{2}(s_i + s_{i+1})$$

$$\Rightarrow \quad y_{i+1} = y_i + \frac{h}{2}[f(x_i, y_i) + f(x_i+h, y_i+s_ih)]$$

Expanding $f(x_i+h, y_i+s_ih)$ in Taylor series,

$$y_{i+1} = y_i + \frac{h}{2}[f(x_i, y_i) + f(x_i, y_i) + hf_x(x_i, y_i) + s_ihf_y(x_i, y_i) + \cdots]$$

$$y_{i+1} = y_i + \frac{h}{2}[f(x_i, y_i) + f(x_i, y_i) + hf_x(x_i, y_i) + hf_y(x_i, y_i) \times f(x_i, y_i) + \cdots]$$

$$\Rightarrow y_{i+1} = y_i + hf(x_i, y_i) + \frac{h^2}{2}\left[f_x(x_i, y_i) + f_y(x_i, y_i) \times f(x_i, y_i) + \cdots\right]$$

which is ~~identifi~~ identical to the Taylor series method, and hence the proof.

$\boxed{\text{EXMP:7}}$:- Given $\frac{dy}{dx} = xy$ with $y(1) = 5$

Find the solution ~~correct to three decimal position~~ in the interval $[1, 1.5]$ using step size $h = 0.1$, by using R-K Second-order method?

Soln :- The formulae for second order Runge-kutta method (Heun's method) is

$$y_{i+1} = y_i + hs$$

where $s = \frac{s_1 + s_2}{2}$, $s_1 = f(x_i, y_i)$, $s_2 = f(x_i + h, y_i + hs_1)$

$$f(x, y) = xy$$

Here $x_1 = 1$, $y_1 = 5$, $h = 0.1$

The Runge-kutta second order formulae can be written as

$$y_{i+1} = y_i + 0.1 \times s$$

For $i = 1$

$$y_2 = y_1 + 0.1 \times s$$

$$s_1 = f(x_1, y_1) = f(1, 5)$$

$S_1 = 1 \times 5 = 5$

$S_2 = f(x_1 + 0.1, y_1 + 0.1 \times S_1)$

$= f(1.1, 5 + 0.1 \times 5)$

$= f(1.1, 5.5)$

$S_2 = 1.1 \times 5.5 = 6.05$

$$S = \frac{S_1 + S_2}{2} = \frac{5 + 6.05}{2} = \frac{11.05}{2}$$

$S = 5.525$

Thus $y_2 = 5 + 0.1 \times 5.525$

$y_2 = 5.553$.

Thus, the second solution point is obtained as

$(x_2, y_2) = (1.1, 5.553)$

For $i = 2$

$y_3 = y_2 + 0.1 \times S$

$S_1 = f(x_2, y_2) = f(1.1, 5.553)$

$= 1.1 \times 5.553$

$S_1 = 6.108$

$S_2 = f(x_2 + 0.1, y_2 + 0.1 \times S_1)$

$= f(1.2, 5.553 + 0.1 \times 6.108)$

$= f(1.2, 6.164) = 1.2 \times 6.164$

$S_2 = 7.397$

$s = (6.108 + 7.397)/2$

$s = 6.752$

Therefore

$y_3 = 5.553 + 0.1 \times 6.752$

$y_3 = 6.228$

Thus, the third solution point is obtained as $(x_3, y_3) = (1.2, 6.228)$

For $i = 3$

$y_4 = y_3 + 0.1 \times s$

$s_1 = f(x_3, y_3)$

$= f(1.2, 6.228) = 1.2 \times 6.228$

$s_1 = 7.474$

$s_2 = f(x_3 + 0.1, y_3 + 0.1 \times s_1)$

$= f(1.3, 6.228 + 0.1 \times 7.475)$

$= f(1.3, 6.975) = 1.3 \times 6.975$

$s_2 = 9.068$

$$s = \frac{7.475 + 9.068}{2}$$

$s = 8.271$

Therefore

$y_4 = 6.228 + 0.1 \times 8.271$

$= 7.055$

Thus, the fourth solution point is obtained as $(x_4, y_4) = (1.3, 7.055)$

For $i = 4$,

$y_5 = y_4 + 0.1 \times s$

$s_1 = f(x_4, y_4)$

$= f(1.3, 7.055) = 1.3 \times 7.055$

$s_1 = 9.172$

$s_2 = f(x_4 + 0.1, y_4 + 0.1 \times s_1)$

$= f(1.4, 7.055 + 0.1 \times 9.172)$

$= f(1.4, 7.972) = 1.4 \times 7.972$

$s_2 = 11.161$

$$s = \frac{(9.172 + 11.161)}{2}$$

$s = 10.166$

There fore

$y_5 = 7.055 + 0.1 \times 10.166$

$= 8.072$

Thus, the fifth solution point is obtained as

$(x_5, y_5) = (1.4, 8.072)$.

For $i = 5$

$y_6 = y_5 + 0.1 \times s$

$s_1 = f(x_5, y_5) = f(1.4, 8.072) = 1.4 \times 8.072$

$s_1 = 11.301$

$$S_2 = f(x_5 + 0.1,\ y_5 + 0.1 \times S_1)$$

$$= f(1.5,\ 8.072 + 0.1 \times 11.301)$$

$$= f(1.5,\ 9.202) = 1.5 \times 9.202$$

$$S_2 = 13.803$$

$$S = \frac{11.301 + 13.803}{2}$$

$$S = 12.552$$

Therefore

$$y_6 = 8.072 + 0.1 \times 12.552$$

$$= 9.327$$

Thus, the sixth and final solution point is obtained as $(x_6, y_6) = (1.5, 9.327)$

The complete solution of the given differential equations is given as.

i	0	1	2	3	4	5
x_i	1.0	1.1	1.2	1.3	1.4	1.5
y_i	5.0	5.525	6.228	7.055	8.072	9.327.

RUNGE - KUTTA FOURTH ORDER METHODS —

The error in the second order Runge-kutta method is $O(h^3)$ per step. However, if more precision is required, then we can use the fourth order Runge-kutta method in which the error is $O(h^5)$ per step.

In Runge-kutta fourth order methods, the slope at four points including the starting point is computed and then the weighted average of these slope is computed as —

$$S = \frac{1}{6}(S_1 + 2S_2 + 2S_3 + S_4)$$

where

$$S_1 = f(x_1, y_1)$$

$$S_2 = f\left(x_1 + \frac{h}{2}, y_1 + \frac{h}{2} S_1\right)$$

$$S_3 = f\left(x_1 + \frac{h}{2}, y_1 + \frac{h}{2} S_2\right)$$

$$S_4 = f(x_1 + h, y_1 + hS_3)$$

The value of the dependent variable y is computed as

$$y_2 = y_1 + hs$$

In the similar manner, starting from the second solution point we compute the third point. This process is repeated till we find the solution in the desired interval.

In general, the $(i+1)^{th}$ point of the solution curve is obtained from the i^{th} point using the following equation.

$$y_{i+1} = y_i + hs \qquad (1)$$

where

$$s = \frac{1}{6}(s_1 + 2s_2 + 2s_3 + s_4)$$

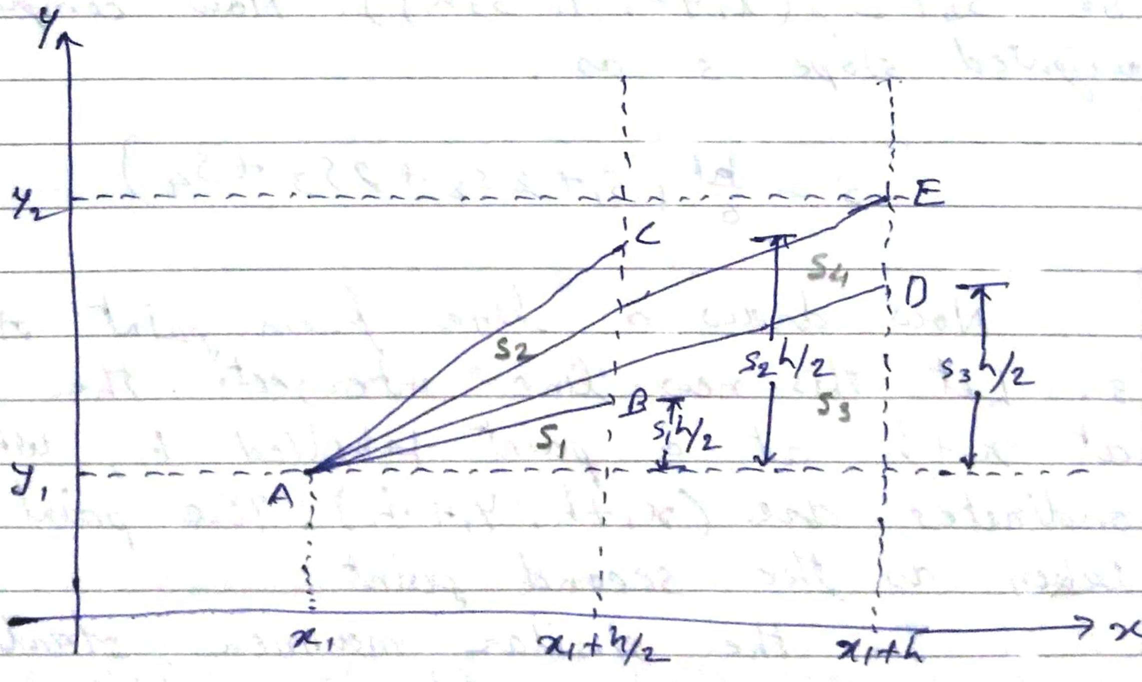
$$s_1 = f(x_i, y_i)$$

$$s_2 = f\left(x_i + \frac{h}{2},\ y_i + \frac{h}{2}s_1\right)$$

$$s_3 = f\left(x_i + \frac{h}{2},\ y_i + \frac{h}{2}s_2\right)$$

$$s_4 = f(x_i + h,\ y_i + hs_3)$$

GEOMETRIC INTERPRETATION -



From the starting point (x_1, y_1) labelled as A, draw a line with slope s_1 $(= f(x_1, y_1))$. Let this line intersects the vertical line at $x_1 + h/2$ at a point labelled B, whose co-ordinates are $(x_1 + h/2, y_1 + s_1 h/2)$. Now compute the slope of the curve at point B.

Let this slope be s_2 $(= f(x_1 + h/2, y_1 + s_1 h/2))$ Now draw a line from point A with slope s_2. Let this new line intersects the vertical line at $x_1 + h/2$ at a point labelled C. whose co-ordinates are $(x_1 + h/2, y_1 + s_2 h/2)$. Now compute the slope of the curve at point C. Let this slope be s_3 $(= f(x_1 + h/2, y_1 + s_2 h/2))$. Now draw a line from point A with slope s_3. Let this new line intersects the vertical line at $x_1 + h$ at a point labelled D, whose co-ordinates are $(x_1 + h, y_1 + s_3 h)$. Now compute the slope of the curve at point D. Let this slope be s_4 $(= f(x_1 + h, y_1 + s_3 h))$. Now compute the weighted slope s as.

$$s = \frac{h}{6}(s_1 + 2s_2 + 2s_3 + s_4)$$

Now draw a line from point A with slope s. Let this new line intersect the vertical line at $x_1 + h$ at a point labelled E, whose co-ordinates are $(x_1 + h, y_1 + sh)$. The point E is taken as the second point.

In the similar manner, starting from the second solution point, we obtain the third solution point and so on. This process goes on till we obtain the solution in the desired interval.

The formulae for the fourth order Runge-Kutta method is identical to the Taylor series truncated after the term containing h^4. The proof is left as an exercise for the students.

Exmp :- Given $\frac{dy}{dx} = xy$ with $y(1) = 5$

Find the solution correct to three decimal position in the interval $[1, 1.5]$ using step size $h = 0.1$ by using fourth-order R-K method ?

Sol^n :- The formulae for fourth order Runge-Kutta method is

$$y_{i+1} = y_i + hs$$

where $$s = \frac{1}{6}(s_1 + 2s_2 + 2s_3 + s_4)$$

$$s_1 = f(x_i, y_i)$$

$$s_2 = f(x_i + \frac{h}{2}, y_i + \frac{h}{2} s_1)$$

$$s_3 = f(x_i + \frac{h}{2}, y_i + \frac{h}{2} s_2)$$

$$s_4 = f(x_i + h, y_i + h s_3)$$

$$\because f(x, y) = xy$$

$$x_1 = 1, \quad y_1 = 5, \quad h = 0.1$$

The Runge-Kutta fourth order formulae can be written as

$$y_{i+1} = y_i + 0.1 * s$$

For $i=1$

$y_2 = y_1 + 0.1 \times S$

$s_1 = f(x_1, y_1)$

$= f(1, 5) = 1 \times 5$

$S_1 = 5$

$s_2 = f(x_1 + 0.05, y_1 + 0.05 \times s_1)$

$= f(1.05, 5 + 0.05 \times 5)$

$= f(1.05, 5.25) = 1.05 \times 5.25$

$s_2 = 5.513$

$S_3 = f(x_1 + 0.05, y_1 + 0.05 \times S_2)$

$= f(1.05, 5 + 0.05 \times 5.513)$

$= f(1.05, 5.276)$

$= 1.05 \times 5.276$

$S_3 = 5.540$

$S_4 = f(x_1 + 0.1, y_1 + 0.1 \times S_3)$

$= f(1.1, 5 + 0.1 \times 5.540)$

$= f(1.1, 5.554) = 1.1 \times 5.554$

$S_4 = 6.109$

$S = \frac{1}{6}[5 + 2 \times 5.513 + 2 \times 5.540 + 6.109]$

$= 5.536.$

Therefore

$y_2 = 5 + 0.1 \times 5.536$

$y_2 = 5.554$

Thus, the second solution point is obtained as

$(x_2, y_2) = (1.1, 5.554)$

For $i = 2$

$y_3 = y_2 + 0.1 \times S$

$S_1 = f(x_2, y_2)$

$= f(1.1, 5.554) = 1.1 \times 5.554$

$S_1 = 6.109$

$S_2 = f(x_2 + 0.05, y_2 + 0.05 \times S_1)$

$= f(1.15, 5.554 + 0.05 \times 6.109)$

$= f(1.15, 5.859) = 1.15 \times 5.859$

$S_2 = 6.738$

$S_3 = f(x_2 + 0.05, y_2 + 0.05 \times S_2)$

$= f(1.15, 5.554 + 0.05 \times 6.738)$

$= f(1.15, 5.891) = 1.15 \times 5.891$

$S_3 = 6.775$

$S_4 = f(x_2 + 0.1, y_2 + 0.1 \times S_3)$

$= f(1.2, 5.554 + 0.1 \times 6.775)$

$$= f(1.2,\ 6.232) = 1.2 \times 6.232$$

$$S_4 = 7.478$$

$$S = \frac{1}{6}\left[6.109 + 2 \times 6.738 + 2 \times 6.775 + 7.478\right]$$

$$= 6.769$$

Therefore $y_3 = 5.554 + 0.1 \times 6.769$

$$y_3 = 6.231$$

Thus, the third solution point is obtained as

$(x_3, y_3) = (1.2,\ 6.231)$

For $i = 3$

$$y_4 = y_3 + 0.1 \times S$$

$$S_1 = f(x_3, y_3)$$

$$= f(1.2,\ 6.231) = 1.2 \times 6.231$$

$$S_1 = 7.477$$

$$S_2 = f(x_3 + 0.05,\ y_3 + 0.05 \times S_1)$$

$$= f(1.25,\ 6.231 + 0.05 \times 7.477)$$

$$= f(1.25,\ 6.604)$$

$$S_2 = 1.25 \times 6.604 = 8.256$$

$$S_3 = f(x_3 + 0.05,\ y_3 + 0.05 \times S_2)$$

$$= f(1.25,\ 6.231 + 0.05 \times 8.256)$$

$= f(1.25, 6.644) = 1.25 \times 6.644$

$S_3 = 8.305$

$S_4 = f(x_3 + 0.01, y_3 + 0.1 \times S_3)$

$= f(1.3, 6.231 + 0.1 \times 8.305)$

$= f(1.3, 7.062) = 1.3 \times 7.062$

$S_4 = 9.181$

$S = \frac{1}{6}[7.477 + 2 \times 8.256 + 2 \times 8.305 + 9.181]$

$= 8.297$

Therefore $y_4 = 6.231 + 0.1 \times 8.297$

$y_4 = 7.061$

Thus, the fourth solution point is obtained as $(x_4, y_4) = (1.3, 7.061)$

For $i = 4$.

$y_5 = y_4 + 0.1 \times S$

$S_1 = f(x_4, y_4)$

$= f(1.3, 7.061) = 1.3 \times 7.061$

$S_1 = 9.179$

$S_2 = f(x_4 + 0.05, y_4 + 0.05 \times S_1)$

$= f(1.35, 7.061 + 0.05 \times 9.179)$

$S_2 = f(1.35, 7.520) = 1.35 \times 7.520$

$s_2 = 10.152$

$s_3 = f(x_4 + 0.05, y_4 + 0.05 \times s_2)$

$= f(1.35, 7.061 + 0.05 \times 10.152)$

$= f(1.35, 7.569) = 1.35 \times 7.569$

$s_3 = 10.218$

$s_4 = f(x_4 + 0.1, y_4 + 0.1 \times s_3)$

$= f(1.4, 7.061 + 0.1 \times 10.218)$

$= f(1.4, 9.402) = 1.4 \times 9.402$

$s_4 = 1.4 \times 9.402 = 13.163$

$s = \frac{1}{6}[9.179 + 2 \times 10.152 + 2 \times 10.218 + 14.305]$

$s = 10.514$

Therefore

$y_5 = 7.061 + 0.1 \times 10.514$

$y_5 = 8.112$

Thus, the fifth solution point is obtained as $(x_5, y_5) = (1.4, 8.112)$

For $i = 5$.

$y_6 = y_5 + 0.1 \times s$.

$S_1 = f(x_5, y_5)$

$= f(1.4, 8.112) = 1.4 \times 8.112$

$S_1 = 11.357$

$S_2 = f(x_5 + 0.05, y_5 + 0.05 \times S_1)$

$= f(1.45, 8.112 + 0.05 \times 11.357)$

$= f(1.45, 8.680) = 1.45 \times 8.680$

$S_2 = 12.586$

$S_3 = f(x_5 + 0.05, y_5 + 0.05 \times S_2)$

$= f(1.45, 8.112 + 0.05 \times 12.586)$

$= f(1.45, 8.741) = 1.45 \times 8.741$

$S_3 = 12.675$

$S_4 = f(x_5 + 0.1, y_5 + 0.1 \times S_3)$

$= f(1.5, 8.112 + 0.1 \times 12.675)$

$= f(1.5, 9.380) = 1.5 \times 9.380$

$S_4 = 14.070$

$S = \frac{1}{6}[11.357 + 2 \times 12.586 + 2 \times 12.675 + 14.070]$

$S = 12.658$

Therefore

$$y_6 = 8.112 + 0.1 \times 12.658$$

$$y_6 = 9.378$$

Thus the sixth solution and final solution point is obtained as $(x_6, y_6) = (1.5,\ 9.378)$

The complete solution of the given differential equations is given as:

i	0 ~~1~~	1 ~~2~~	2 ~~3~~	3 ~~4~~	4 ~~5~~	5 ~~6~~
x_i	1.0	1.1	1.2	1.3	1.4	1.5
y_i	5.0	5.554	6.231	7.061	8.112	9.378

Exmp: Using Runge-kutta method solve the following 4th & 2nd order differential equation, step size $h = 0.1$

$$\frac{dy}{dx} = 3x + y^2, \quad y(1) = 1.2 \quad ; \text{ Find } y(1.2)$$

Sol^n:- Given that

$$F(x, y) = 3x + y^2$$

$$x_0 = 1, \quad y_0 = 1.2$$

$$x_1 = 1.1, \quad h = 0.1$$

$$S_1 = 0.1 (3x_0 + y_0^2)$$

$$S_1 = 0.1 (3 + (1.2)^2)$$

$$S_1 = 0.444$$

$$S_2 = h.f (x_0 + h/2, y_0 + \frac{S_1}{2})$$

$$S_2 = 0.1 \left[3 + \frac{0.1}{2}, (1.2 + \frac{0.444}{2})^2 \right]$$

$$S_2 = 0.1 \left[3.05 + (1.422)^2 \right]$$

$$S_2 = 0.517$$

$$S_3 = h.f \left[x_0 + h/2, (y_0 + \frac{S_2}{2})^2 \right]$$

$$S_3 = 0.1 \left[3(1 + \frac{0.1}{2}), (y_0 + \frac{0.517}{2})^2 \right]$$

$$S_3 = 3.05 + 1.4585$$

$$S_3 = 0.528$$

$s_4 = h \cdot f(x_0 + h, (y_0 + s_3)^2]$

$s_4 = h[3(x_0 + h) \cdot (y_0 + s_3)^2]$

$= 0.1[3(1 + 0.1) \cdot (1.2 + 0.528)^2]$

$s_4 = 0.629$

$$s = \frac{s_1 + 2s_2 + 2s_3 + s_4}{6}$$

$$s = \frac{0.444 + 2 \times 0.517 + 2 \times 0.528 + 0.629}{6}$$

$s = 0.527$

$x_1 = x_0 + h \qquad y_1 = y_0 + k$

$x_1 = 1 + 0.1 \qquad y_1 = 1.2 + 0.527$

$x_1 = 1.1 \qquad y_1 = 1.727$

$y_2 = 2.4910$

Exmp:- Solve following differential equation for Runge-Kutta fourth order method. step size $h = 0.5$

$\frac{dy}{dx} = \frac{1}{x+y}$; $y(0) = 1$ Find $y(0.5)$

Solⁿ :- $x_0 = 0$, $y_0 = 1$, $x_1 = 0.5$, $h = x_1 - x_0 = 0.5 - 0$
$h = 0.5$

* $s_1 = h \cdot f(x_0\ y_0)$

$s_1 = \frac{h}{x_0 + y_0} = \frac{0.5}{0+1} = 0.5$

$$S_2 = h \cdot f\left(x_0 + h/2,\ y_0 + \frac{S_1}{2}\right]$$

$$S_2 = \frac{h}{x_0 + \frac{h}{2} + y_0 + \frac{S_1}{2}} = \frac{0.5}{0 + \frac{0.5}{2} + 1 + \frac{0.5}{2}}$$

$$S_2 = \frac{0.5}{1.5} = 0.3333$$

$$S_3 = h \cdot f.\left(x_0 + h/2,\ y_0 + \frac{S_2}{2}\right)$$

$$S_3 = \frac{h}{x_0 + h/2 + y_0 + \frac{S_2}{2}} = \frac{0.5}{0 + \frac{0.5}{2} + 1 + \frac{0.3333}{2}}$$

$$S_3 = 0.353$$

$$S_4 = h \cdot f.(x_0 + h,\ y_0 + S_3) = \frac{h}{x_0 + h + y_0 + S_3} =$$

$$S_4 = \frac{0.5}{0 + 0.5 + 1 + 0.3333} = 0.270$$

$$S = \frac{S_1 + 2S_2 + 2S_3 + S_4}{6} = \frac{0.5 + 2 \times 0.3333 + 2 \times 0.353 + 0.270}{6}$$

$$S = 0.357$$

If x change from 0 to 0.5 then value of y change by 0.357. $\therefore$ value of y at 0.5 will by $y_1 = y_0 + S = y_1 = 1 + 0.357$

$$y_1 = \underline{1.357}$$

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Ex :- Solve following differential equation using Runge kutta method - step size $h = 0.1$

$$\frac{dy}{dx} = 3x + \frac{y}{2}, \quad y(0) = 1, \quad y(0.1) = ?$$

Solⁿ :- Given that

$$\frac{dy}{dx} = f(x,y) = 3x + \frac{y}{2}$$

$$x_0 = 0, \quad y_0 = 1, \quad x_1 = 0.1$$

$$h = x_1 - x_0 = 0.1 - 0 = 0.1$$

$$S_1 = h \cdot f(x_0, y_0) = 0.1\left[3x_0 + \frac{y_0}{2}\right]$$

$$S_1 = 0.1\left[3 \times 0 + \frac{1}{2}\right] = 0.05$$

$$S_2 = h \cdot f\left(x_0 + h/2, y_0 + \frac{S_1}{2}\right) = 0.1\left[3 \times 0 + \frac{0.1}{2} + 1 + \frac{0.05}{2}\right]$$

$$S_2 = 0.06625$$

$$S_3 = h \cdot f\left(x_0 + h/2, y_0 + \frac{S_2}{2}\right) = 0.1\left[3 \times 0 + \frac{0.1}{2} + \frac{\left(1 + \frac{0.6625}{2}\right)}{2}\right]$$

$$S_3 = 0.666562$$

$$S_4 = h \cdot f(x_0 + h, y_0 + S_3) = 0.1\left[3(x_0 + h) + \frac{y_0 + S_3}{2}\right]$$

$$S_4 = 0.83333$$

$$S = \frac{S_1 + 2S_2 + 2S_3 + S_4}{6} = \frac{0.05 + 2 \times 0.6625 + 2 \times 0.6665 + 0.8333}{6}$$

$$S = 0.66524$$

If the value of x changes from 0 to 0.1 then increment in y now be $y_0 + k$.

y at 0.1 will be, $y_1 = y_0 + S \Rightarrow y_1 = 1.66524$.

Ex:- Solve the following Differential Equation using the Runge-Kutta fourth order method.

$\frac{dy}{dx} = y - x$, $y(0) = 2$, find $y(0.1)$ & $y(0.2) = ?$

$$= 0.1\,[3\,(1 + 0.1) + (1.2 + 0.52775)^2]$$
$$= 0.6285$$

$$\therefore \quad y(1.1) = y_0 + \frac{1}{6}\,(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= 1.2 + \frac{1}{6}\,(0.444 + 2 \times 0.5172 + 2 \times 0.52775 + 0.6285)$$
$$= \mathbf{1.72707.} \quad \textbf{Ans.}$$

PROBLEM SET 10.4

Solve the following differential equations using Runge-Kutta method:

1. Given $\frac{dy}{dx} = 3x + \frac{1}{2}\,y$, with $y\,(0) = 1$. Find $y\,(0.1)$.
2. Given $\frac{dy}{dx} = xy$ with $x = 1$, $y\,(1) = 2$, find $y\,(1.4)$, taking $h = 0.2$.
3. Given $\frac{dy}{dx} = xy + y^2$, $y\,(0) = 1$. Evaluate y for $x = 0.1, 0.2, 0.3$.
4. Solve $\frac{dy}{dx} = \frac{1}{x + y}$ for $x = 0.5$, to $x = 2$, $(h = 0.5)$ with $x_0 = 0$, $y_0 = 1$.
5. Evaluate $y\,(1.4)$ given $\frac{dy}{dx} = x + y$, $y\,(1.2) = 2$.
6. Compute $y\,(0.3)$, given $\frac{dy}{dx} + y + xy^2 = 0$, $y\,(0) = 1$, taking $h = 0.1$.
7. Obtain the value of y at $x = 0.2$, if y satisfies $\frac{dy}{dx} - x^2\,y = x$; $y\,(0) = 1$, taking $h = 0.1$.
8. Solve $y' = \frac{y - x}{y + x}$, given $y\,(0) = 1$, to obtain $y\,(0.2)$.
9. Given $\frac{dy}{dx} = x^3 + \frac{1}{2}\,y$, $y\,(1) = 2$. Find $y\,(1.1)$, $y\,(1.2)$.
10. Given $\frac{dy}{dx} = \frac{\sin h\,(0.5y + x)}{1.5} + 0.5y$, $y\,(0) = 0$. Find y for $0 \le x \le 0.2$ with $h = 0.1$.
11. Given $\frac{dy}{dx} = \sqrt{x + y}$, $y\,(0.4) = 0.41$, $h = 0.4$. Find $y\,(0.8)$.
12. Solve $\frac{dy}{dx} = \mathrm{x\,y} + 1$ as $\mathrm{x} = 0.2, 0.4, 0.6$ given $\mathrm{y}(0) = 2$, taking $\mathrm{h} = 0.2$.
13. Given $\frac{dy}{dx} = 1 + y\sin x - x^2$, $y\,(0) = 0$, $h = 0.1$, find $y\,(0.1)$.
14. Evaluate y in the interval $0 \le x \le 1$, given $\frac{dy}{dx} = -2xy^2$, $y\,(0) = 1$, $h = 0.2$.
15. Find $y\,(20.6)$, given $\frac{dy}{dx} = \log\frac{x}{y}$, $y\,(20) = 5$.
16. Solve the differential equation $\frac{dy}{dx} = 1 + xz$, $\frac{dz}{dx} = -xy$ for $x = 0.3$, using fourth order Runge-Kutta method. The initial values are $x = 0$, $y = 0$, $z = 1$.

17. Solve the system $\frac{dy}{dx} = xz + 1$, $\frac{dz}{dx} = -xy$ for $x = 0.3\ (0.3),\ (0.9)$ taking $x = 0,\ y = 0,\ z = 1$.

18. Solve $\frac{dy}{dx} = -xz$, $\frac{dz}{dx} = y^2$, given $y(0) = 1$, $z(0) = 1$, for $x = 0\ (0.2)\ (0.4)$.

ANSWERS

1. 1.066524.
2. 3.232107
3. $y(0.1) = 1.1169$, $y(0.3) = 1.504$, $y(0.2) = 1.2773$.
4. $y(0.5) = 1.3571$, $y(1) = 1.5837$, $y(1.5) = 1.7565$, $y(2.0) = 1.8950$.
5. 2.7299.
6. $y(0.1) = 0.9006$, $y(0.2) = 0.8046$, $y(0.3) = 0.7144$.
7. 1.0224
8. 1.165.
9. $y(1.1) = 2.2213$, $y(1.2) = 2.4914$.
10. $y(0.1) = 0.003432$, $y(0.2) = 0.014155$.
11. $y(0.4) = 0.848$.
12. 3.072
14. 0.9615328, 0.8620525, 0.7352784, 0.6097519, 0.5000073.
15. 5.35599.
16. $y = 0.3448$, $z = 0.99$,
17. $y(0.3) = 0.3448$, $z(0.3) = 0.99$, $y(0.6) = 0.7738$, $z(0.6) = 0.9121$, $y(0.9) = 1.255$, $z(0.9) = 0.6806$.

10.8 Predictor—Corrector Methods

The methods which we discuss above are Taylor's Series, Euler's, Runge-Kutta and Picards. These all methods are called *single-step methods*, because they use only the information from one previous point to compute the successive point. The Predictor—Corrector methods are methods which require function values at $x_n, x_{n-1}, x_{n-2}, \ldots$ for the computation of the function value at x_{n+1}. A predictor formula is used to predict the value of y at x_{n+1} and then the corrector formula is used to improve the value of y_{n+1}. Here, we discuss two predictor-corrector formulae viz. Milne's method and Adams-Bashforth-Moulton method.

10.8.1 Milne's Method

Let us consider the differential equation

$$\frac{dy}{dx} = f(x, y),\quad y(x_0) = y_0. \qquad \ldots(1)$$

We have to estimate successively $y_1 = y(x_0 + h) = y(x_1)$, $y_2 = y(x_0 + 2h) = y(x_2)$, $y_3 = y(x_0 + 3h) = y(x_3)$ where h is a suitable accepted spacing, which is very small.

The Newton's forward difference formula is

$$f(x, y) = f_0 + u\,\Delta f_0 + \frac{u(u-1)}{\underline{|2}}\Delta^2 f_0 + \frac{u(u-1)(u-2)}{\underline{|3}}\Delta^3 f_0 + \ldots$$

where
$$u = \frac{x - x_0}{h}. \qquad \ldots(2)$$

On putting (2) in the relation

$$y_4 = y_0 + \int_{x_0}^{x_4} f(x, y)\, dx \qquad \ldots(3)$$

We get
$$y_4 = y_0 + \int_{x_0}^{x_4} \left[f_0 + u\Delta f_0 + \frac{u(u-1)}{\underline{|2}}\Delta^2 f_0 + \frac{u(u-1)(u-2)}{\underline{|3}}\Delta^3 f_0 + \ldots \right] dx$$