① Types of Error -

Round off Error, Truncation Error, &
Relative Error, & Absolute Error

{ S/W - ADMB. APL, Autochem, Baudline, CC4, etc

In statistics, the term Type I error (also α error, or false positive) and Type II error (β error, or a false negative) are used to describe possible errors made in a statistical decision process.

Type I (α): reject the null hypothesis when the null hypothesis is true, &

Type II (β): Fail to reject the null Hypothesis when the null hypothesis is false.

Null Hypothesis - In statistics hypothesis testing, the null hypothesis (Ho) Formally describes some aspect of the statistical behaviour of a set of data, this description is treated as valid unless the actual behaviour of the data contradicts ~~of the~~ this assumption.

For Example - Imagine flipping a coin three times, for three heads, and then forming the opinion that we have used a two-headed trick coin. clearly this opinion is based on the premise that such a sequence is unlikely to have arisen using

on forget fulness

a normal coin, such sequences occur a quarter of the time on average when using normal unbaised coins. Therefore the opinion that coin is two-headed has little support. Formally the hypothesis to be tested in this Example is " This is a Two-headed coin". one tests it by assessing whether the data contradicts the null hypothesis that " This is a normal Unbias coin".
}

Round off Error -

Round-off errors occur when a fixed number of digits are used to represent exact numbers. Since the number are stored at every stage of computation, round off error is introduced at the end of every arithmetics operation.

Numbers are rounded-off according to the following rule -

(i) Less than half a unit in the n^{th} place, leave the n^{th} digit ~~unaltered~~, unfigured

(ii) Greater than half a unit in the n^{th} place increase the n^{th} digit by unity.

(iii) Exactly half a unit in the n^{th} place, increase the n^{th} digit by unity if it is odd, otherwise leave it unchanged.

The number thus rounded-off is said to be correct to n significant figures.

For Example -

Decimal number	Rounded-off number
30.0567	30.06
0.859378	0.8594
1.6583	1.658
7.9954	8.00

<u>Truncation</u> Error -

Truncation error arise from using an approximation in place of an exact mathematical procedure. Typically, it is the error resulting from the Truncation of the numerical process. We often use some finite number of terms to estimate the sum of an infinite series.

For Example -

$$S = \sum_{i=0}^{\infty} a_i x^i$$

Replace by the finite sum -

$$\sum_{i=0}^{n} a_i x^i$$

The series has been Truncated.

Consider the following infinite series -

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

when we calculate the sine of an angle this series, we cannot use all the terms in the series for computation. We usually terminate the process after a certain term is calculated.

The terms "Truncated" introduce an error which is called truncation error.

For Example

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!}$$

Truncation error when first three terms are added. -

$$\text{Truncation error} = + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!}$$

$$= \frac{(0.2)^3}{6} + \frac{(0.2)^4}{24} + \frac{(0.2)^5}{120} + \frac{(0.2)^6}{720}$$

$$= 0.1402755 \times 10^{-2}$$

Truncation error when First Five terms are added -

$$\text{Truncation error} = 0.27555 \times 10^{-5}$$

Relative Error -

suppose that the true value of a data item is denoted by x_t and its approximate value is denoted by x_a. Then, they are related as follows -

$$\text{True value } (x_t) = \text{Approximation value } (x_a) + \text{Error}$$

OR

$$\text{Error} = \text{True value} - \text{Approximation value}.$$

We Introduce the concept of relative error which is nothing but the "normalized" absolute error. The relative error is defined as follows

$$\text{Relative error} = \frac{|\text{Error}|}{|\text{True value}|}$$

OR

$$RE = \frac{|x_t - x_a|}{|x_t|} = \left| 1 - \frac{x_a}{x_t} \right|$$

For Example - An Approximate value of π is given by $x_a = 22/7 = 3.14285$, and its true value is $x_t = 3.14159$. Find the relative error.

We have - $\text{Error} = x_t - x_a$

$$= 3.14159 - 3.142857$$

$$= -0.0012645$$

$$|\text{Error}| = 0.0012645$$

Relative error $= \dfrac{0.0012645}{3.14159}$

$= 0.000402$.

Absolute Error –

Absolute error is the numerical difference between the true value of a quantity and its approximate value.

Suppose that the ~~data item~~ true value of a data item is denoted by x_t and its approximate value is denoted by x_a. Then the related as follows –

True value = Approximate value – Error

OR

AV – TV

Error = ~~True value – Approximate value~~.

The error may be negative or positive depending on the values of x_t & x_a. In error analysis,

Absolute error (e_a) = | Error |

= | True value – Approximate value |

$$e_a = |x_t - x_a|$$

For Example – Evaluate the sum $S = \sqrt{3} + \sqrt{5} + \sqrt{7}$ to 4 significant digits & find its absolute error.

$x_t = 6.629$

We have

$$\sqrt{3} = 1.732, \quad \sqrt{5} = 2.236 \text{ \&}$$

$$\sqrt{7} = 2.646$$

Hence

$$S = 1.732 + 2.236 + 2.646$$

$$S = 6.614$$

&

$$e_a = 0.0005 + 0.0005 + 0.0005$$

$$\boxed{e_a = 0.0015}$$

—— x ——

Diff. between Roundoff & Truncation error

	Round off.	Truncation
(i)	Roundoff error occurs when a fixed number of digits are used to represent exact numbers	Truncation error occurs when some digits from the number are discarded. i.e. error due to finite representation of an inherently infinite process.
(ii)	Round off error is introduced at the end of every arithmetic opn.	Truncation error is introduced at the start or end of every function
(iii)	One has been added to the second decimal place in the ~~first &~~ Third numbers b/c the third decimal place is greater than five	Truncation error are not contain one adding process.

Bisection method –

The bisection method is one of the simplest and most reliable of iterative methods for the solution of nonlinear equations. This method, also known as ~~binary chopping~~ on half-interval method, relies on the fact that if $f(x)$ is real & continuous in the interval $a \le x \le b$, & $f(a)$, & $f(b)$ are of opposite signs that is

$$f(a)\, f(b) < 0$$

Then there is ~~one~~ at least one real root in the interval b/w a & b (may be more than one root)

Let $x_1 = a$ & $x_2 = b$

Let us also define another point x_0 to be the mid point b/w a and b

i.e. $$x_0 = \frac{x_1 + x_2}{2} = \frac{a+b}{2}$$

Now, there exist the following Three condition

(i) if $f(x_0) = 0$, We have a root at x_0.

(ii) if $f(x_0) \cdot f(x_1) < 0$, There is a root b/w x_0 & x_1.

(iii) if $f(x_0) \cdot f(x_2) < 0$, There is a root b/w x_0 & x_2

It follows that by testing the sign of the function at midpoint.

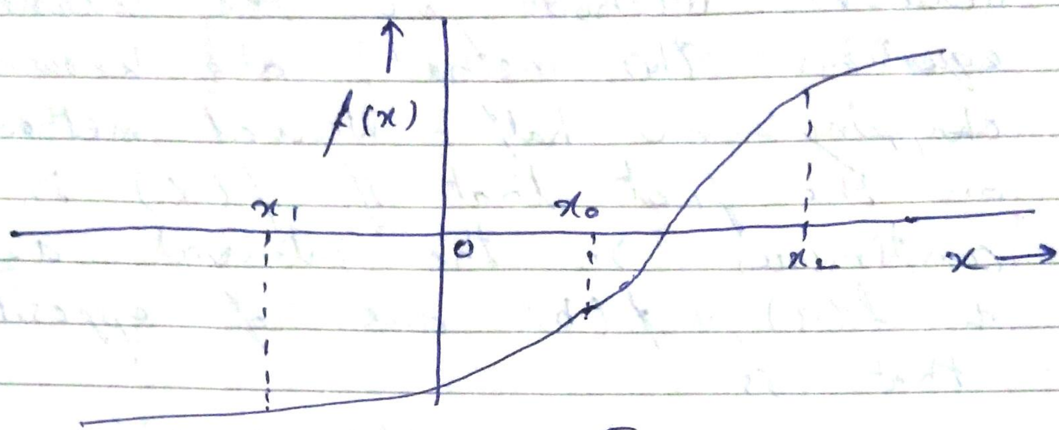

Fig.

Since $f(x_0)$ & $f(x_2)$ are of opposite sign, a root lies b/w x_0 & x_2. We can further divided this ~~x_0 & x_2~~ subinterval into two ~~halves~~ interval to locate a new subinterval containing the root.

This process can be ~~rep~~ repeated until the interval containing the root is as small as we desire.

Theorem :- Prove that bisection method is linearly convergent

Proof^n: In the bisection method, we choose a midpoint x_0 in the interval between x_1 & x_2. Depending on the sign of functions $f(x_0)$, $f(x_1)$ & $f(x_2)$ x_1 or x_2 is set equal to x_0 such that the new interval contains the root.

$$\frac{x_2+x_1}{2^n} = \frac{\Delta x}{2^n}$$

After n iterations, the root must lie within $\pm \Delta x / 2^n$ of our estimate. This means that the error at n^{th} iteration is -

$$E_n = \left| \frac{\Delta x}{2^n} \right|$$

Similarly - $E_{n+1} = \left| \frac{\Delta x}{2^{n+1}} \right| = \frac{E_n}{2}$

That is, the error decreases linearly with each step by a factor of 0.5; The bisection method is, therefore, linearly convergent. and achieve a high degree of accuracy.

Problem 11: Find the root of the equation $x^3 - x - 11 = 0$ which lies between 2 & 3 using bisection method correct to three dicimal places.

Sol^n: Let $f(x) = x^3 - x - 11$

$f(1) = -11 = -Ve.$

$f(2) = -5 = -Ve$

$f(3) = 13 = +Ve$

$\therefore$ a root lies between 2 & 3.

First approximation

$$x_1 = \frac{2+3}{2} = 2.5 \Rightarrow f(x) = x^3 - x - 11 = 0$$

Now $f(2.5) = (2.5)^3 - 2.5 - 11 = 2.125 = +Ve$

$f(x_1) = x_1^3 - x_1 - 11$

Thus the root lies between 2 & 2.5.

$$\therefore \quad x_2 = \frac{2+2.5}{2} = 2.25$$

Now $f(2.25) = (2.25)^3 - 2.25 - 11$

$= -1.859375 = -ve$

Thus, the root lies between 2.5 & 2.25.

$$\therefore \quad x_3 = \frac{2.5+2.25}{2} = 2.375$$

Now $f(2.375) = (2.375)^3 - (2.375) - 11$

$= 0.0214843 = +ve$

Thus, the root lies between 2.25 & 2.375

$$\therefore \quad x_4 = \frac{2.25+2.375}{2} = 2.3125$$

Now $f(2.3125) = (2.3125)^3 - (2.3125) - 11$

$= -0.9460449 = -ve$

Thus, root lies between 2.375 & 2.3125.

$$\therefore \quad x_5 = \frac{2.375+2.3125}{2} = 2.34375$$

Now $f(2.34375) = (2.34375)^3 - (2.34375) - 11$

$= -0.4691467 = -ve$.

Thus, root lies between 2.375 & 2.34375.

$$\therefore \quad x_6 = \frac{2.375+2.34375}{2} = 2.359375$$

Now $f(2.359375) = (2.359375)^3 - (2.359375) - 11$

$= -0.2255592 = -Ve$

Thus, root lies between 2.375 & ~~0.2255592~~ 2.359375

$\therefore \quad x_7 = \frac{2.375 + 2.359375}{2} = 2.3671875$

Now $f(2.3671875) = (2.3671875)^3 - (2.3671875) - 11$

$= -0.1024708 = -Ve$

Thus, root lies between 2.375 & 2.3671875

$\therefore \quad x_8 = \frac{2.375 + 2.3671875}{2} = 2.3710938$

Now $f(2.3710938) = (2.3710938)^3 - (2.3710938) - 11$

$= -0.04060 = -Ve$

Thus, root lies between 2.375 & 2.3710938

$\therefore \quad x_9 = \frac{2.375 + 2.3710938}{2} = 2.3730469$

Now $f(2.3730469) = (2.3730469)^3 - (2.3730469) - 11$

$= -9.585468 \times 10^{-3} = -Ve$

Thus, root lies between 2.375 & 2.3730469.

$\therefore \quad x_{10} = \frac{2.375 + 2.3730469}{2} = 2.3740235$

Now $f(2.3740235) = (2.3740271)^3 - (2.3740235) - 11$

$= 5.943457 \times 10^{-3} = +Ve$

Thus, root lies between 2.3730469 & 2.3740235

$$\therefore x_{11} = \frac{2.3730469 + 2.3740235}{2} = 2.3735352$$

Now $f(2.3735352) = (2.3735352)^3 - (2.3735352) - 11$

$$= -1.822703 \times 10^{-3} = -Ve$$

Thus, root lies between 2.3740235 & 2.3735352

$$\therefore x_{12} = \frac{2.3740235 + 2.3735352}{2} = 2.373779$$

since $x_{11} = x_{12} = 2.373$

Hence, the root is 2.373 correct to 3 decimal places.

$\boxed{\text{Problem: 2}}$: Find a root of the equation $x^3 - 4x - 9 = 0$ using bisection method correct to four decimal places.

Solⁿ: Let $f(x) = x^3 - 4x - 9$

$f(1) = -12 = -Ve$

$f(2) = -9 = -Ve$

$f(3) = 6 = +Ve$

Thus the root lies between 2 & 3. Then first approximation to the root is

$$x_1 = \frac{2+3}{2} = 2.5$$

Now, $f(2.5) = -3.375 = -Ve$

Hence, the root lies between 2.5 & 3. Then the second approximation to the root is.

$$x_2 = \frac{2.5 + 3}{2} = 2.75$$

Also, $f(2.75) = 0.7969 = +Ve$.

Hence, the root lies between 2.5 & 2.75

$$\therefore \quad x_3 = \frac{2.5 + 2.75}{2} = 2.625$$

Now, $f(2.625) = -1.4121 = -Ve$.

Hence, the root lies between 2.625 & 2.75

$$\therefore \quad x_4 = \frac{2.625 + 2.75}{2} = 2.6875.$$

Now, $f(2.6875) = 6 \; -0.33911 = -Ve$.

Hence the root lies between 2.6875 & 2.75

$$\therefore \quad x_5 = \frac{2.6875 + 2.75}{2} = 2.71875$$

Now $f(2.71875) = \overset{0.22091}{\cancel{0.06107}} = +Ve$

Hence the root lies between 2.6875 & 2.71875

$$\therefore \quad x_6 = \frac{2.6875 + 2.71875}{2} = 2.703125$$

Now $f(2.703125) = -0.06107 = -Ve$.

Hence the root lies between 2.71875 & 2.703125

$$\therefore \quad x_7 = \frac{2.71875 + 2.703125}{2} = 2.710937$$

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Now, $f(2.710937) = 0.07942 = +ve$

Hence, the root lies between 2.703125 & 2.710937

$$\therefore x_8 = \frac{2.703125 + 2.710937}{2} = 2.707031$$

Now, $f(2.707031) = 9.04925 = +ve.$

Hence the root lies between 2.703125 & 2.707031

$$\therefore x_9 = \frac{2.703125 + 2.707031}{2} = 2.705078$$

Now, $f(2.705078) = -0.02604 = -ve$

Hence the root lies between 2.705078 & 2.707031

$$\therefore x_{10} = \frac{2.705078 + 2.707031}{2} = 2.706013$$

Now $f(2.706054) = -8.50570 = -ve.$

Hence the root lies between 2.706054 & 2.707031

$$\therefore x_{11} = \frac{2.706054 + 2.707031}{2} = 2.706542$$

Now, $f(2.706542) = 2.69739 = +ve.$

Hence, the root lies between 2.706054 & 2.706542

$$\therefore x_{12} = \frac{2.706054 + 2.706542}{2} = 2.706298.$$

Now, $f(2.706298) = -4.11846 = -ve.$

Hence the root lies between 2.706298 & 2.706542

$\therefore \quad x_{13} = \dfrac{2.706298 + 2.706542}{2} = 2.706420$

Now, $f(2.706420) = -1.92448 = -ve.$

Hence the root lies between 2.706420 & 2.706542

$\therefore \quad x_{14} = \dfrac{2.706420 + 2.706542}{2} = 2.706481$

Since $\quad x_{13} = x_{14} = 2.7064$

Hence, the root is 2.7064 correct to 4 decimal places.

Assign. Prob: Find the root of the equation $x^3 - 9x + 1 = 0$ which lies between 2 & 4 using bisection method correct to four decimal places.

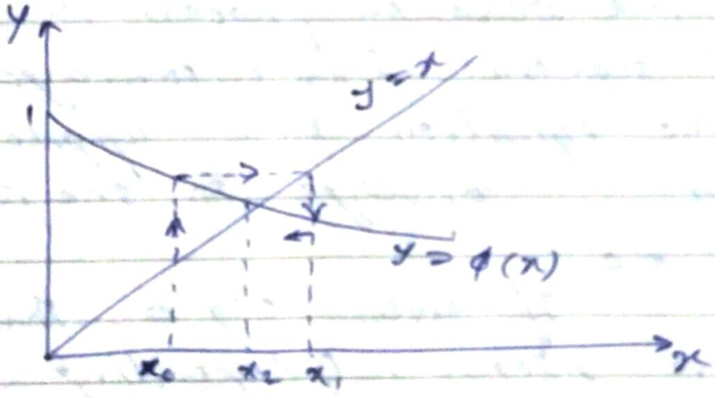
Successive Approximation method OR Iteration method OR Iterative method:

This method can be applied when the equation $f(x) = 0$ can be solved for the unknown x in terms of a function of x. To find the approximate value of the root is that the interval of the unit length is firstly determined ~~and then a point in this interval is taken by guess~~. The iterative method is also called successive approximation method.

Let $f(x) = 0$ be the given equation whose roots are to be determined.

$$x = \phi(x)$$

The roots of $f(x) = 0$ are the same as the points of intersection of the straight line $y = x$ and the curve representating $y = \phi(x)$ in fig.



Let $x = x_0$ be an initial approximation to the actual root.

Then the first approximation is

$$x_1 = \phi(x_0)$$

& the successive approximation are

$$x_2 = \phi(x_1)$$

$$x_3 = \phi(x_2)$$

$$\vdots \qquad \vdots$$

$$x_n = \phi(x_{n-1})$$

or

$$x_{n+1} = \phi(x_n)$$

This relation is known as iterative formula. The method of solving the equation is called iteration method. The function $\phi(x)$ is known as an iteration function.

Convergence of Iteration method :-

Let $x_0, x_1, x_2, \ldots x_i \ldots$ be the successive approximation of a root of the equation $f(x) = 0$.

The iteration process is said to be convergent if sequence $\{x_n\}$ converges to the root of the equation i.e., approximation $x_0, x_1, x_2, \ldots x_i \ldots$ converge to the root of the equation $f(x) = 0$.

If given $\varepsilon > 0$, ~~there~~ there exists positive integer n (depending on ε) such that

$$|x_{i+1} - x_i| < \varepsilon \text{ for } i \geq n$$

Theorem 1:- If a be a root of $f(x) = 0$ which is equivalent to $x = \phi(x)$, I be any interval containing the point $x = a$, $|\phi'(x)| < 1$ for all x in I, then the sequence of approximation $x_0, x_1, x_2 \ldots x_i$ will ~~conv~~ converge to the root a provided the initial approximation x_0 is chosen in I.

OR

The iteration method $x_{i+1} = \phi(x_i)$ is convergent, if $|\phi'(x)| < 1$

Proof:- Given that a is a root of $x = \phi(x)$ we have

$$a = \phi(a)$$

Successive approximation of the root of the equation $f(x) = 0$ are obtained by.

$$x_{i+1} = \phi(x_i) \qquad \text{put } n = 0, 1, 2, 3, \ldots n,$$

$$x_1 = \phi(x_0)$$

$$x_2 = \phi(x_1)$$

$$\cdots\cdots$$

$$\cdots\cdots$$

$$\cdots\cdots$$

$$\left.\begin{aligned} x_n &= \phi(x_{n-1}) \\ x_{n+1} &= \phi(x_n) \end{aligned}\right\} \quad \text{——} \; (1)$$

By mean value theorem

$$\frac{\phi(x_n) - \phi(x_{n-1})}{x_n - x_{n-1}} = \phi'(c_n) \quad \text{——} \; (2)$$

where $x_{n-1} < c_n < x_n$

from equation (1)

$$\phi(x_n) = x_{n+1}$$

$$\phi(x_{n-1}) = x_n$$

$\therefore$ Putting the values of $\phi(x_n)$ & $\phi(x_{n-1})$ in equation (ii) ; we have

$$\frac{x_{n+1} - x_n}{x_n - x_{n-1}} = \phi'(c_n)$$

$$\therefore \; |x_{n+1} - x_n| = |\phi'(c_n)| \, |x_n - x_{n-1}|$$

Replacing n by $n-1, n-2, \ldots 1$

$$|x_n - x_{n-1}| = |\phi'(c_{n-1})| \, |x_{n-1} - x_{n-2}|$$

. . . .

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$$|x_3 - x_2| = |\phi'(c_2)| \, |x_2 - x_1|$$

$$|x_2 - x_1| = |\phi'(c_1)| \, |x_1 - x_0|$$

Multiplying we get.

$$|x_{n+1} - x_n| \, |x_n - x_{n-1}| \cdots\cdots |x_3 - x_2| \, |x_2 - x_1| =$$

$$= |\phi'(c_n)| \, |\phi'(c_{n-1})| \cdots\cdots |\phi'(c_1)|$$

$$|x_n - x_{n-1}| \, |x_{n-1} - x_{n-2}| \cdots\cdots |x_2 - x_1|$$

$$|x_1 - x_0|$$

Cancelling out common factors, we get.

$$|x_{n+1} - x_n| = |\phi'(c_n)| \, |\phi'(c_{n-1})| \cdots\cdots |\phi'(c_1)| \, |x_1 - x_0|$$

If $|\phi'(x)| \leq k$ for all values of x lying in the interval (a,b) such that.

$$a < x_0 < x_1 < \ldots\ldots < x_n < \ldots < b$$

$|\phi'(c_n)||\phi'(c_{n-1})|\ldots|\phi'(c_2)||\phi'(c_1)|$ all are less than or equal to k.

$$\therefore |x_{n+1} - x_n| \leq k^n |x_1 - x_0|$$

Iteration method converges if

$$\lim_{n\to\infty} |x_{n+1} - x_n| = 0$$

OR if $k^n|x_1 - x_0| \to 0$ as $n \to \infty$

i.e. if $k < 1$ OR $|\phi'(x)| < 1$

Proved

$\boxed{\text{Problem : 2}}$:- Find a root (near 1), of the following equatⁿ using iterative method (upto six digits) where initial value 1.5,?

$$x^3 + 2x^2 + 10x - 20 = 0.$$

Solⁿ :- Given $x^3 + 2x^2 + 10x - 20 = 0$

Let, $f(x) = x^3 + 2x^2 + 10x - 20$ $\quad \because f(x) = 0$

$\therefore x = 1, 2, \ldots$

$f(0) = -20 = -Ve$

$f(1) = (1)^3 + 2(1)^2 + 10(1) - 20$

$= 1 + 2 + 10 - 20$

$f(1) = -7 = -Ve$

$f(2) = (2)^3 + 2(2)^2 + 10(2) - 20$

$= 8 + 2 \times 4 + 20 - 20$

$f(2) = 16 = +ve.$

$\therefore$ root lies b/w 1 & 2

Rewriting the given equation as

$10x = 20 - 2x^2 - x^3$

$x = \frac{1}{10}(20 - 2x^2 - x^3) = \phi(x)$, say

Taking initially $x_0 = 1.5$ & using iteration method the successive approximation are.

The First approximation:

$$x_1 = \phi(x_0) = \frac{20 - 2x_0^2 - x_0^3}{10}$$

$$x_1 = \frac{20 - 2(1.5)^2 - (1.5)^3}{10} = 1.21250$$

The Second approximation:

$$x_2 = \phi(x_1) = \frac{20 - 2x_1^2 - x_1^3}{10}$$

$$= \frac{20 - 2(1.2125)^2 - (1.2125)^3}{10}$$

$$x_2 = 1.527123$$

The Third approximation:

$$x_3 = \phi(x_2) = \frac{20 - 2x_2^2 - x_2^3}{10}$$

$$= \frac{20 - 2(1.527123)^2 - (1.527123)^3}{10}$$

$$x_3 = \phi(x_2) = 1.1766655$$

The Fourth approximation :

$$x_4 = \phi(x_3) = \frac{20 - 2x_3^2 - x_3^3}{10}$$

$$= \frac{20 - 2(1.1766655)^2 - (1.1766655)^3}{10}$$

$$x_4 = 1.5601774$$

The Fifth approximation :

$$x_5 = \phi(x_4) = \frac{20 - 2x_4^2 - x_4^3}{10}$$

$$= \frac{20 - 2(1.5601774)^2 - (1.5601774)^3}{10}$$

$$x_5 = 1.1333981$$

The Sixth approximation :

$$x_6 = \phi(x_5) = \frac{20 - 2x_5^2 - x_5^3}{10}$$

$$x_6 = \frac{20 - 2(1.1333981)^2 - (1.1333981)^3}{10}$$

$$x_6 = 1.5974863$$

$$\phi(x_6) = x_7 = \frac{20 - 2x_6^2 - x_6^3}{10}$$

The seventh approximation

$$x_7 = \phi(x_6) = \frac{20 - 2(1.5974863)^2 - (1.5974863)^3}{10}$$

$$x_7 = 1.0819348$$

The Eighth approximation :

$$x_8 = \phi(x_7) = \frac{20 - 2x_7^2 - x_7^3}{10}$$

$$= \frac{20 - 2(1.0819348)^2 - (1.0819348)^3}{10}$$

$$x_8 = \phi(x_7) = 1.6392339$$

The Nineth approximation :

$$x_9 = \phi(x_8) = \frac{20 - 2x_8^2 - x_8^3}{10}$$

$$= \frac{20 - 2(1.6392339)^2 - (1.6392339)^3}{10}$$

$$x_9 = 1.02210588$$

The Tenth approximation :

$$x_{10} = \phi(x_9) = \frac{20 - 2x_9^2 - x_9^3}{10}$$

$$= \frac{20 - 2(1.02210588)^2 - (1.02210588)^3}{10}$$

$$x_{10} = \phi(x_9) = 1.6842804$$

The Eleventh approximation :

$$x_{11} = \phi(x_{10}) = \frac{20 - 2x_{10}^2 - x_{10}^3}{10}$$

$$= \frac{20 - 2(1.6842804)^2 - (1.6842804)^3}{10}$$

$$x_{11} = \phi(x_{10}) = 0.954843$$

Hence the root is 0.954843.

Problem : 1 : Find the root of the polynomial $x^3 + x^2 - 1 = 0$ correct up to three decimal places by iteratⁿ method, where initial value is 0.5 ?

Solⁿ : Given $x^3 + x^2 - 1 = 0$ $\because f(x) = 0$

Let $f(x) = x^3 + x^2 - 1$

$f(0) = -1 = -ve$

$f(1) = 1 + 1 - 1 = 2 - 1$

$f(1) = 1 = +ve.$

i.e., the root lies b/w 0 & 1.

Rewriting the given equation as,

$$x = \frac{1}{\sqrt{x+1}} = \phi(x), \text{ say.}$$

Taking initially $x_0 = 0.5$ & using iteration method the successive approximation are.

$$x_1 = \phi(x_0) = \frac{1}{\sqrt{(0.5+1)}} = 0.816$$

$$x_2 = \phi(x_1) = \frac{1}{\sqrt{(0.816+1)}} = 0.742$$

$$x_3 = \phi(x_2) = \frac{1}{\sqrt{(0.742+1)}} = 0.758$$

$$x_4 = \phi(x_3) = \frac{1}{\sqrt{(0.758+1)}} = 0.754$$

$$x_5 = \phi(x_4) = \frac{1}{\sqrt{(0.754+1)}} = 0.755$$

$$x_6 = \phi(x_5) = \frac{1}{\sqrt{(0.755+1)}} = 0.755$$

Hence x_5 & x_6 being almost the same, the root is 0.755 correct to 3 decimal place.

Problem 3: Find the real root of $x = (5-x)^{1/3}$ correct three decimal places by using iteration method, where initial value is 1.5 ?.

Sol^n :

$$x = (5-x)^{1/3}$$

$$x^3 = 5 - x$$

Let $f(x) = x^3 + x - 5$

$f(0) = -5 = -Ve$

$f(1) = -3 = -Ve$

$f(2) = 5 = +Ve$

From the given equation, we get.

$$x = (5-x)^{1/3} = \phi(x)$$

Taking initially $x_0 = 1.5$ using iteration method the successive approximation are.

$$x_1 = \phi(x_0) = (5 - 1.5)^{1/3} = 1.518$$

$$x_2 = \phi(x_1) = (5 - 1.518)^{1/3} = 1.516$$

$$x_3 = \phi(x_2) = (5 - 1.516)^{1/3} = 1.516$$

since $x_2 = x_3$

Hence the root is $\underline{1.516}$

Newton - Raphson Formula or Method -

This method is also known as Newton - Iterative method & a particular form of the iterative method. When an approximation value of a root of an equation is given, a better & closer approximation to the root can be found using this method.

Suppose x_0 is an approximation value of a root of the equation $f(x)=0$, & suppose x_0+h is the exact value of the corresponding root, where h is very small quantity. Then.

$$f(x_0+h) = 0 \quad \text{——— (1)}$$

Since x_0+h is the root of the equation $f(x)=0$.
Expanding equation (i) by Taylor's theorem, we get.

$$f(x_0+h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0) + \cdots = 0$$

Since h is very small, neglecting second & higher order terms & taking the first approximat.

$$f(x_0) + hf'(x_0) = 0 \Rightarrow h = \frac{-f(x_0)}{f'(x_0)},$$ where $f'(x_0) \neq 0$

$$\therefore \quad x_1 = x_0 + h = x_0 - \frac{f(x_0)}{f'(x_0)} \quad \text{——— (2)}$$

Equation (2) gives the improved value of the root

over the previous one. we get a Second app.
Now substituting x_1 in place of x_0 & x_2 for x_1 in equation ②, we get.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \quad \text{———} \quad ③$$

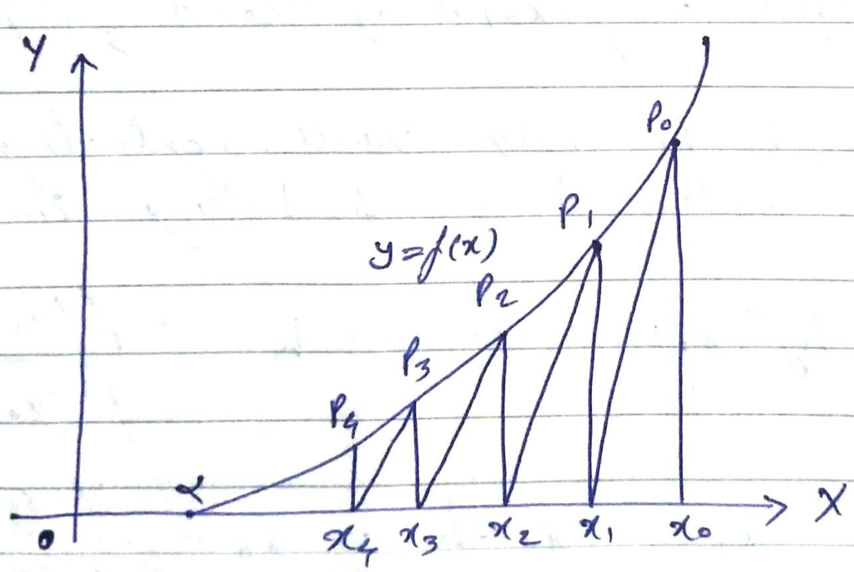
In general, we can get an approximation or the Newton iteration formula.

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

OR

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{———} \quad ④$$

From this formula, we can calculate successive better values of the root.
Equation ④ known as Newton-Raphson method.



Let the curve $f(x)=0$ meet the x-axis at $x=\alpha$. It means that α is the original root of $f(x)=0$.

EXAMPLE :

$\boxed{\text{Problem :-}}$ Apply Newton-Raphson formula to calculate at least one root of the equation (upto Correct four decimal places ~~six digit~~)

$$x^3 + 2x^2 + 10x - 20 = 0$$

where I. Value is 1.2

Soln :

Roots

Let $f(x) = 0$

So, $f(x) = x^3 + 2x^2 + 10x - 20$

$\therefore f'(x) = 3x^2 + 4x + 10$ $\because$ F.O. Diff.

By N. R. Formulae :

we have $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$$x_{n+1} = x_n - \left[\frac{x_n^3 + 2x_n^2 + 10x_n - 20}{3x_n^2 + 4x_n + 10}\right]$$

$$= \frac{2(x_n^3 + x_n^2 + 10)}{3x_n^2 + 4x_n + 10} \quad \text{—— (1)}$$

Now we can see that $f(1) = -7 < 0$ &
$f(2) = 16 > 0$

Therefore the root lies between 1 & 2.

Suppose $x_0 = 1.2$ is the initial approximation.

$\therefore f(1.2) < 0$

substituting $n = 0$ in equation (i)
First approximation x_1 is given by,

$$x_1 = \frac{2(x_0^3 + x_0^2 + 10)}{3x_0^2 + 4x_0 + 10} = \frac{2[(1.2)^3 + (1.2)^2 + 10]}{[3(1.2)^2 + 4(1.2) + 10]}$$

$$x_1 = \frac{26.336}{19.12} = 1.3774059.$$

The second approximation x_2 is.

$$x_2 = \frac{2(x_1^3 + x_1^2 + 10)}{3x_1^2 + 4x_1 + 10}$$

$$= \frac{2[(1.3774059)^3 + (1.3774059)^2 + 10]}{[3(1.3774059)^2 + 4(1.3774059) + 10]}$$

$$x_2 = \frac{29.021052}{21.201364} = 1.3688295$$

The Third approximation x_3 is -

$$x_3 = \frac{2(x_2^3 + x_2^2 + 10)}{3x_2^2 + 4x_2 + 10}$$

$$= \frac{2[(1.3688295)^3 + (1.3688295)^2 + 10]}{[3(1.3688295)^2 + 4(1.3688295) + 10]}$$

$$x_3 = \frac{28.876924}{21.0964} = 1.3688081.$$

The Fourth approximation x_4 is -

$$x_4 = \frac{2(x_3^3 + x_3^2 + 10)}{3x_3^2 + 4x_3 + 10}$$

$$x_4 = \frac{2[(1.3688081)^3 + (1.3688081)^2 + 10}{3(1.3688081)^2 + 4(1.3688081) + 10}$$

$$x_4 = \frac{28.876567}{21.09614} = 1.3688081$$

$\therefore x_3 = x_4$

Hence the root is 1.3688081.

Illustrative Examples

I.V 2.5

Example 2.10 Use Newton-Raphson method to find a root of the equation $x^3 - 3x - 5 = 0$.

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Solution: Given $f(x) = x^3 - 3x - 5 = 0$

$\therefore \quad f(2) = 8 - 6 - 5 = -3$

and $\quad f(3) = 27 - 9 - 5 = 13.$

Therefore, a real root lies between 2 and 3.

Now, $\quad f'(x) = 3x^2 - 3$

Let the initial approximation be $x_0 = 2.$

$\therefore \quad f'(2) = 12 - 3 = 9.$

Using Newton-Raphson formula, we get

Iteration 1. $$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 2 - \frac{(-3)}{9} = 2.333333.$$

Iteration 2. $$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.333333 - \frac{f(2.333333)}{f'(2.333333)}$$

$$= 2.333333 - \frac{0.703704}{13.333333}$$

$$= 2.280555.$$

Iteration 3. $$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.280555 - \frac{f(2.280555)}{f'(2.280555)}$$

$$= 2.279020.$$

Iteration 4. $$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 2.279020 - \frac{f(2.279020)}{f'(2.279020)}$$

$$= 2.279019$$

Hence, the correct root is **2.2790**.

Example 2.11 Find the real root of the equation $x \sin x + \cos x = 0$, using Newtons Raphson method.

Solution. We have $$f(x) = x \sin x + \cos x$$

Thus, $$f'(x) = x \cos x.$$

Thus, the iteration formula is $x_{n+1} = x_n - \dfrac{x_n \sin x_n + \cos x_n}{x_n \cos x_n}$.

Now, on taking $x_0 = \pi$, we get the successive iterates as

$$x_1 = x_0 - \frac{x_0 \sin x_0 + \cos x_0}{x_0 \cos x_0}$$

$$= \pi - \frac{\pi . \sin \pi + \cos \pi}{\pi . \cos \pi} = \pi - \frac{(-1)}{\pi . (-1)} = 3.14159 - 0.31831$$

$$= 2.82328$$

$$f(x_1) = 2.82328 . \sin 2.82328 + \cos 2.82328 = -0.06618$$

Now, $$x_2 = x_1 - \frac{x_1 \sin x_1 + \cos x_1}{x_1 \cos x_1} = 2.82328 - \frac{2.82328 \sin 2.82328 + \cos 2.82328}{2.82328 . \cos 2.82328}$$

$$= 2.79860.$$

$$f(2.7986) = 2.7986 . \sin 2.7986 + \cos 2.7986$$

$$= -0.00056$$

$\therefore$ $$x_3 = 2.7986 - \frac{2.7986 \sin 2.7986 + \cos 2.7986}{2.7986 . \cos 2.7986}$$

$$= 2.7986 - \frac{-0.00056}{-2.63559} = 2.79839$$

False Position OR Regula - Falsi Method :

OR

Regula - Falsi Method of Roots -

This is the oldest method for finding the real root of an equation & closely remembler
+ Secant mth) the bisection method. In this method, we choose two points x_0 & x_1 such that $f(x_0)$ & $f(x_1)$ are of opposite signs. Since the graph of $y = f(x)$ crosses the x-axis between these two points, a root must lie in between these points.

Now the equation of the ~~chord joining the~~ joining the two points $[x_0, f(x_0)]$ & $[x_1, f(x_1)]$ is :

$$\frac{\cancel{y(x)} - f(x_0)}{x - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \quad \text{———} \textcircled{1}$$

The point of Intersection in the present case is given by putting, $\cancel{f(x)} = 0$ in equation ①.
Thus,

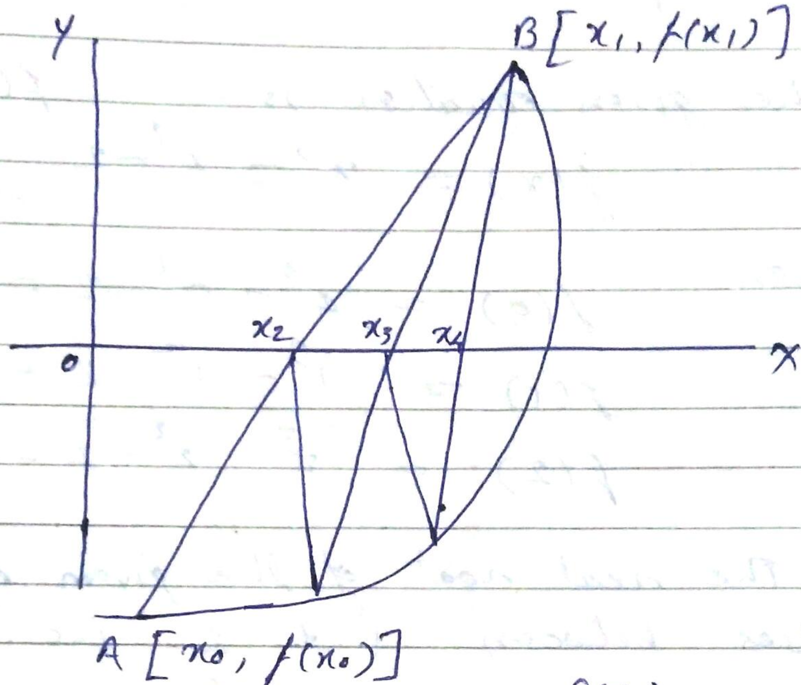
$$\cancel{x =} \frac{-f(x_0)}{x - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$-\frac{f(x_0)}{f(x_1) - f(x_0)}(x_1 - x_0) = x - x_0$$

$$\Rightarrow x = x_0 - \frac{f(x_0)}{f(x_1) - f(x_0)}(x_1 - x_0)$$

Thus the second approximation to the root of of $f(x) = 0$ is given by

$$x_2 = x_0 - \frac{f(x_0)}{f(x_1)-f(x_0)}(x_1-x_0) \quad \text{--- (ii)}$$



If now $f(x_2)$ & $f(x_0)$ are of opposite sign, then the root lies between x_0 & x_2 and we replace x_1 by x_2 in equation (ii) & obtain the next approximation. Otherwise, we replace x_0 by x_2 & generate the next approximation. The procedure is repeated till the root is obtained to the desired accuracy.

Probl : Find by Regula-falsi method, the real root of the equation $x^3 - x^2 - 2 = 0$, correct to three decimal places.

Solⁿ: The given equation is - $f(x) = 0$

$$f(x) = x^3 - x^2 - 2 \quad \text{--- (1)}$$

Then

$$f(0) = 0^3 - 0^2 - 2 = -2 = -Ve$$

$$f(1) = 1^3 - 1^2 - 2 = -2 = -Ve$$

$$f(2) = 2^3 - 2^2 - 2 = 2 = +Ve$$

∴ The real root of the given equation $f(x) = 0$ lies between 1 & 2 i.e., in the interval (1,2)

Also we know that for Regula-falsi method,

For the ~~First~~ second Approximation x_2:

X_1 OR
$$x_2 = x_0 - \frac{f(x_0)(x_1 - x_0)}{f(x_1) - f(x_0)} \quad \text{--- (2)}$$

Here $x_0 = 1$, $x_1 = 2$

$$f(x_0) = f(1) = 1^3 - 1^2 - 2 = -2 = -Ve$$

$$f(x_1) = f(2) = 2^3 - 2^2 - 2 = 2 = +Ve$$

∴ from Relation (2)

$$\cancel{X_1} = x_2 = 1 - \frac{(2-1)(-2)}{2-(-2)} = 1 - \frac{1 \times -2}{2+2} = 1 + \frac{2}{4}$$

$$= 1.5 \quad \text{--- (3)}$$

For the ~~second~~ Third Approximation x_3 -

X_2 OR
$$x_3 = x_1 - \frac{f(x_1)(x_2 - x_1)}{f(x_2) - f(x_1)} \quad \text{--- (4)}$$

Here $x_1 = 1.5$ & $x_2 = 2$.

$$f(x_1) = f(1.5) = (1.5)^3 - (1.5)^2 - 2$$
$$= 3.375 - 2.25 - 2$$
$$= -0.875$$

& $f(x_2) = f(2) = \cancel{(1.5)^3} \; 2$

$\therefore$ from equation (4)

$$X_2 \text{ or } x_3 = (1.5) - \frac{(-0.875)(2-1.5)}{2-(-0.875)}$$
$$= 1.5 + \frac{(0.875)(0.5)}{2.875}$$
$$= 1.5 + 0.152 = \cancel{1.652}\ 1.652 \quad \text{——— (5)}$$

For the ~~third~~ Fourth approximation x_4 =

$$X_3 \text{ or } x_4 = x_2 - \frac{f(x_2)(x_3 - x_2)}{f(x_3) - f(x_2)} \quad \text{——— (6)}$$

Here, ~~x_1~~ $x_2 = 1.652$, $x_3 = 2$ given

$$f(x_2) = f(1.652) = (1.652)^3 - (1.652)^2 - 2$$
$$= 4.50847 - 2.7291 - 2$$
$$= -0.2207$$

& $f(x_3) = f(2) = 2$

$\therefore$ from equation (6)

$$X_3 \text{ or } x_4 = 1.652 - \frac{(-0.2207)(2-1.652)}{2-(-0.2207)}$$

$$x_4 = 1.652 + \frac{(0.348)(0.2207)}{2.2207}$$

$$= 1.652 + 0.0345 = 1.6865 \quad \text{—— (7)}$$

For the ~~fourth~~ fifth approximation x_5 -

$$x_5 = x_3 - \frac{f(x_3)(x_4 - x_3)}{f(x_4) - f(x_3)} \quad \text{—— (8)}$$

Here $x_3 = 1.6865$, $x_4 = 2$ given.

$$f(x_3) = f(1.6865) = (1.6865)^3 - (1.6865)^2 - 2$$

$$= 4.7969 - 2.8443 - 2$$

$$f(x_3) = \quad = -0.0474$$

& $f(x_4) = f(2) = 2.$

$\therefore$ from equation (8)

$$x_5 = 1.6865 - \frac{(2 - 1.6865)(-0.0474)}{2 - (-0.0474)}$$

$$= 1.6865 + \frac{(0.3135)(0.0474)}{2.0474}$$

$$= 1.6865 + 0.0073 = 1.6938 \quad \text{—— (9)}$$

For the ~~fifth~~ Sixth approximation x_6 -

$$x_6 = x_4 - \frac{f(x_4)(x_5 - x_4)}{f(x_5) - f(x_4)} \quad \text{—— (10)}$$

Here $x_4 = 1.6938$, $x_5 = 2$ given.

$f(x_4) = f(1.6938) = (1.6938)^3 - (1.6938)^2 - 2$

$= 4.85944 - 2.86895 - 2$

$= -0.00951$

$f(x_5) = f(2) = 2.$

$\therefore$ from equation 9.

$$x_6 = 1.6938 - \frac{(2-1.6938)(-0.00951)}{2-(-0.00951)}$$

$$= 1.6938 + \frac{(0.3062)(0.00951)}{2.00951}$$

$= 1.6938 + 0.001449$

$= 1.69525$

$x_6 = 1.6953$ —— (10)

For the ~~sixth~~ seventh approximation x_7 -

$$x_7 = x_5 - \frac{f(x_5)(x_6 - x_5)}{f(x_6) - f(x_5)} \quad \text{—— (11)}$$

Here

$x_5 = 1.6953, \quad x_6 = 2$

$f(x_5) = f(1.6953) = (1.6953)^3 - (1.6953)^2 - 2$

$= 4.87236 - 2.87404 - 2$

$= -0.00168$

$f(x_6) = f(2) = 2$

$\therefore$ from equation (11)

$$x_7 = 1.6953 - \frac{(2-1.6953)(-0.00168)}{2-(-0.00168)}$$

$$= 1.6953 + \frac{(0.3047)(0.00168)}{2.00168}$$

$$= 1.6953 + \frac{0.0005119}{2.00168}$$

$$= 1.6953 + 0.000256$$

$$= 1.695556 = 1.6956 \quad \text{—— (12)}$$

From equation (11) & (12) we get the required root, correct to three decimal place 1.695

Probl : : Solve $x^3 - 9x + 1 = 0$, for the root lying between 2 & 4 by regula-falsi method.

Ans - 2.942
$x_5 = x_6$

Sol^n :-

$\because f(x) = 0$

$f(x) = x^3 - 9x + 1$

$f(0) = 1 = +Ve$

$f(1) = (1)^3 - 9(1) + 1 = 1 - 9 + 1 = -7 = -Ve$

$f(2) = (2)^3 - 9(2) + 1 = 8 - 18 + 1 = -9 = -Ve$

$f(3) = 27 - 27 + 1 = 1 = +Ve$

$f(4) = 64 - 36 + 1 = 19 = +Ve$

Equation (5) is the generalized Regula-False Formula for ...

Some Illustrative Examples

Example 2.7 Find a real root of the equation $f(x) = x^3 - x - 1 = 0$ by using Regula-Falsi Method. correct upto 2 decimal places.

Solution: Here the function is

$$f(x) = x^3 - x - 1 = 0$$

$$f(1) = 1^3 - 1 - 1 = -1 < 0$$

$$\therefore \quad f(2) = 8 - 2 - 1 = 5 > 0$$

i.e., $\quad f(1) \cdot f(2) < 0.$

Hence, there exist a root between 1 and 2. So, we take $x_0 = 1$ and $x_1 = 2$. Therefore, by using Regula-false formula, we get

$$x_2 = x_1 - \frac{f(x_1)}{f(x_1) - f(x_0)} \cdot (x_1 - x_0)$$

i.e.,

$$x_2 = 2 - \frac{5}{5-(-1)} \cdot (2-1)$$

$$= 2 - \frac{5}{6}$$

$$= 1.166667$$

$$f(x_2) = f(1.166667) = (1.166667)^3 - 1.166667 - 1 = -0.578704$$

$\therefore \quad f(1.166667) \cdot f(2) < 0.$; $x_2 = 1.166667$, $x_1 = 2$

Hence,

$$x_3 = x_2 - \frac{f(x_2)}{f(x_2) - f(x_1)} \cdot (x_2 - x_1)$$

$$= 1.166667 - \frac{(-0.578704)}{-0.578704 - 5} (1.166667 - 2)$$

$$= 1.253112$$

$$f(x_3) = f(1.253112) = (1.253112)^3 - 1.253112 - 1 = -0.285363$$

$\therefore \quad f(x_3) f(x_1) < 0.$; $x_3 = 1.253112$, $x_1 = 2$

Hence,

$$x_4 = x_3 - \frac{f(x_3)}{f(x_3) - f(x_1)} (x_3 - x_1)$$

$$= 1.253112 - \frac{-0.285363}{-0.285363 - 5} (1.253112 - 2)$$

$$= 1.293437$$

$$f(x_4) = f(1.293437) = (1.293437)^3 - 1.293437 - 1 = -0.129544$$

$\therefore \quad f(x_4) \cdot f(x_1) < 0.$; $x_4 = 1.293437$, $x_1 = 2$

Hence,

$$x_5 = x_4 - \frac{f(x_4)}{f(x_4) - f(x_1)} (x_4 - x_1)$$

$$= 1.293437 - \frac{-0.129544}{-0.129544 - 5} (1.293437 - 2)$$

$$= 1.311281$$

$$f(x_5) = f(1.311281) = (1.311281)^3 - 1.311281 - 1 = -0.056589$$

$\therefore$ $f(x_5) \cdot f(x_1) < 0.$; $x_5 = 1.311281,\ x_1 = 2$

So that $$x_6 = x_5 - \frac{f(x_5)}{f(x_5) - f(x_1)} \cdot (x_5 - x_1)$$

$$= 1.311281 - \frac{-0.56589}{-0.56589 - 5}(1.311281 - 2)$$

$$= 1.318989.$$

Hence, the root of the equation is 1.31, correct to 2-decimal places.

Example 2.8 Solve the equation $x \tan x = -1$ by Regula false method starting with $x_0 = 2.5$ and $x_1 = 3$ correct to 3 decimal places.

Solution: Let $f(x) = x \tan x + 1$

$$f(2.5) = 2.5 \tan (2.5) + 1 = -0.867556$$

$$f(3) = 3 \tan (3) + 1 = 0.572360$$

$\therefore$ The root will lies between 2.5 and 3.

On taking $x_0 = 2.5$ and $x_1 = 3$, we have

$$x_2 = x_1 - \frac{f(x_1)}{f(x_1) - f(x_0)}(x_1 - x_0)$$

$$= 3 - \frac{0.572360}{0.572360 - (-0.867556)}(3 - 2.5)$$

$$= 2.801252.$$

Now, $$f(2.801252) = 2.801252 \tan (0.2801252) + 1$$

$$= 0.008021 = +\text{ve}$$

$\therefore$ The root will lies between 2.5 and 2.801252.

Now, $$x_3 = x_2 - \frac{f(x_2)}{f(x_2) - f(x_1)} \cdot (x_2 - x_1),$$ where $x_2 = 2.801252$ and $x_1 = 2.5$

$$= 2.801252 - \frac{0.008021}{0.008021 - (-0.867556)}(2.801252 - 2.5)$$

$$= 2.798492.$$

Now, $$f(2.798492) = 2.798492 \tan (2.798492) + 1$$

$$= 0.000297\ (+\text{ve}).$$

$\therefore$ Root will lies between 2.5 and 2.798492.

Hence $$x_4 = x_3 - \frac{f(x_3)}{f(x_3) - f(x_2)}(x_3 - x_2),$$ where $x_3 = 2\ 798492$ and $x_2 = 2.5$

$\therefore$ $$x_4 = 2.798492 - \frac{0.000297}{0.000297 - (-0.867556)}(2.798492 - 2.5)$$

$$= 2.798390.$$

Hence, the root correct to three places is 2.798.