5.3 Exercise

15

In Exercises 12–19 f_n is the nth Fibonacci number.
$*$ 15. Show that $f_0 f_1 + f_1 f_2 + \cdots + f_{2n-1} f_{2n} = f_{2n}^2$ when n is a positive integer.

answer

Pf: by induction

Invariant:
$P(n)$ is $f_0 f_1 + f_1 f_2 + \cdots + f_{2n-1} f_{2n} = f_{2n}^2$.
Base case:
$f_0 = 0$, $f_1 = 1$, $f_2 = 1$, $P(1)$ is true.
Inductive Step:
we assume that $P(n)$ is true.
$S(n) = f_0 f_1 + f_1 f_2 + \cdots + f_{2n-1} f_{2n}$,
$R(n) = f_{2n}^2$,
$S(n) = R(n)$
$S(n+1) - S(n) = f_{2n} f_{2n+1} + f_{2n+1} f_{2n+2} = 2 f_{2n} f_{2n+1} + f_{2n+1}^2$
$R(n+1) - R(n) = f_{2n+2}^2 - f_{2n}^2 = (f_{2n+2} - f_{2n})(f_{2n+2} + f_{2n}) = f_{2n+1}(2 f_{2n} + f_{2n+1})$
So $S(n+1) = R(n+1)$
$P(n+1)$ is true.
Q.E.D

28

28. Let S be the subset of the set of ordered pairs of integers defined recursively by
    Basis step: $(0,0) \in S$.
    Recursive step: If $(a,b) \in S$, then $(a+2, b+3) \in S$ and $(a+3, b+2) \in S$.

- a) List the elements of S produced by the first five applications of the recursive definition.
- b) Use strong induction on the number of applications of the recursive step of the definition to show that $5 \mid a+b$ when $(a,b) \in S$.
- c) Use structural induction to show that $5 \mid a+b$ when $(a,b) \in S$.

answer

- a) $(0,0),(2,3),(3,2),(4,6),(5,5)$
- b) Pf: by strong induction
- Invariant $P(n)$ is n_{th} Point $(a_n, b_n) \in S, 5|a_n + b_n$
  Base case: $5 \mid 0+0$
  Inductive Step: If $P(0) \wedge P(1)... \wedge P(n)$ is true,
  then $P(n+1)$'s father is i_{th} point (a_i, b_i),
  then $(a_{n+1}, b_{n+1}) = (a_i + 2, b_i + 3)$ or $(a_i + 3, b_i + 2)$
  because $5|a_i + b_i$, then $5|a_i + 3 + b_i + 2$ and $5|a_i + 2 + b_i + 3$
  then $P(0) \wedge P(1)... \wedge P(n) \wedge P(n+1)$ is true.
  Q.E.D
- c)Pf:

note

结构归纳总结如下:
(c) 用结构归纳（structural induction）证明

对所有 $(a,b) \in S$，都有 $5 \mid (a+b)$。

递归定义回顾

- 基例：$(0,0) \in S$。
- 递归步：若 $(a,b) \in S$，则
  $(a+2,\ b+3) \in S, \qquad (a+3,\ b+2) \in S.$

结构归纳要求我们检查基例与每一种生成规则，证明目标性质在"父结点"真时，其所有"子结点"也真。

1. 基例

$$a+b = 0+0 = 0, \qquad 0 \equiv 0 \pmod 5.$$

因此 $5 \mid (a+b)$ 对基例成立。

2. 归纳（递归）步

设已知某一父结点 $(a,b)\in S$ 满足

$$5 \mid (a+b) \quad\Longrightarrow\quad a+b=5k \ (\text{某整数 } k).$$

我们要证明按照递归规则生成的子结点也满足性质。

- 子结点 1：$(a+2,\ b+3)$
  $(a+2)+(b+3)=a+b+5=5k+5=5(k+1),$
  显然仍被 5 整除。
- 子结点 2：$(a+3,\ b+2)$
  $(a+3)+(b+2)=a+b+5=5k+5=5(k+1),$
  亦被 5 整除。

因此，只要父结点满足性质，两个子结点必然满足。

3. 结论

性质

$P(a,b):\ 5 \mid (a+b)$

对基例成立；并且若对某元素成立则对它的所有递归后继也成立。依据结构归纳原理，可得

$$\boxed{\forall\,(a,b)\in S,\ 5 \mid (a+b)}$$

32

32. Prove that in a bit string, the string 01 occurs at most one more time than the string 10.

answer

Pf: by induction

Invariant:

$P(n)$: for a n length bit string end with 1,the string 01 occurs **at most one more time** than the string 10.

$Q(n)$: for a n length bit string end with 0,the string 01 occurs at most **the same time** as the string

10.

Base case:

$P(2)$: for length 2 bit string end with 1, 01, 11, all satify the assumption.

$Q(2)$: for length 2 bit string end with 0, 00, 10, all satify the assumption.

Inductive Step:

we assume that $P(n), Q(n)$ is true.

then for all $n+1$ length bit string,

- case 1:if the fisrt n length string end with 0, then in fisrt n length string, the string 01 occurs at most **the same time** as the string 10.
  - the last bit is 0, then there is at most one more time 01 appears than 10.
  - the last bit is 1, then there is at most the same time 01 appears as 10.
- case 2:if the fisrt n length string end with 1, then the string 01 occurs **at most one more time** than the string 10.
  - the last bit is 0, then there is at most the same time 01 appears as 10.
  - the last bit is 1, then there is at most one more time 01 appears than 10.

So $P(n+1), Q(n+1)$ is true.

So in a bit string, the string 01 occurs at most one more time than the string 10.

Q.E.D

47

47. Use generalized induction as was done in Example 13 to show that if $a_{m,n}$ is defined recursively by $a_{0,0} = 0$ and

$$a_{m,n} = \begin{cases} a_{m-1,n} + 1 & \text{if } n = 0 \text{ and } m > 0 \\ a_{m,n-1} + 1 & \text{if } n > 0 \end{cases}$$

then $a_{m,n} = m + n$ for all $(m, n) \in \mathbb{N} \times \mathbb{N}$.

answer

Pf: by generalized induction

Invariant:

$P(m,n)$ is for $(m_0, n_0) \in \mathbb{N} \times \mathbb{N}$ $m_0 \le m, n_0 \le n$, $a_{m_0,n_0} = m_0 + n_0$

Base case:

$P(0,0)$: $a_{0,0} = 0 + 0 = 0$

Inductive Step:

If $P(m, n)$ is true,

for $(m, n+1)$, $a_{m,n+1} = a_{m,n} + 1 = m + n + 1$

for $(m+1, n)$

- if $n = 0$, $a_{m+1,n} = a_{m,n} + 1 = m + n + 1$
- if $n > 0$, $a_{m+1,n} = a_{m+1,n-1} + 1 = m + 1 + n - 1 + 1 = m + n + 1$

So $P(m+1, n)$ is true and $P(m, n+1)$ is true.

Q.E.D

56

Consider the following inductive definition of a version of Ackermann's function. This function was named after Wilhelm Ackermann, a German mathematician who was a student of the great mathematician David Hilbert. Ackermann's function plays an important role in the theory of recursive functions and in the study of the complexity of certain algorithms involving set unions. (There are several different variants of this function. All are called Ackermann's function and have similar properties even though their values do not always agree.)

$$A(m, n) = \begin{cases} 2n & \text{if } m = 0 \\ 0 & \text{if } m \geq 1 \text{ and } n = 0 \\ 2 & \text{if } m \geq 1 \text{ and } n = 1 \\ A(m-1, A(m, n-1)) & \text{if } m \geq 1 \text{ and } n \geq 2 \end{cases}$$

Exercises 50–57 involve this version of Ackermann's function.

$*$ 56. Prove that $A(m+1, n) \geq A(m, n)$ whenever m and n are nonnegative integers.

answer

Pf:

- if $n = 0$, $A(m+1, n) = A(m, n) = 0$
- if $n = 1$, $A(m+1, n) = A(m, n) = 2$
- if $m = 0,\ n \geq 2$, $A(m+1, n) = A(1, n) = A(0, A(0, n-1)) = 2(A(0, n-1)) = 4n - 4$, $A(m, n) = A(0, n) = 2n$, for $n \geq 2$ and $n \in \mathbb{N}$, $4n - 4 \geq 2n$
- if $m \geq 1, n \geq 2$

note

这题很难

要证

$$\forall\, m,n \in \mathbb{Z}_{\ge 0} \qquad A(m+1,n)\ \ge\ A(m,n).$$

下面分两步完成证明：

1. 先证明引理：在固定 m 时，$A(m,n)$ 随 n 单调不减；
2. 再利用该引理对 m 做归纳证明主命题。

1. 引理：$A(m,n)$ 对第二个参数 n 单调不减

引理 1 对所有非负整数 m，若 $n_1 \ge n_0$ 则
$A(m,n_1)\ \ge\ A(m,n_0).$

证明（双重归纳）

外层对 m 归纳。

- **基例** $m=0$
  $A(0,n)=2n$，显然当 n 增大时数值不减。
- **归纳假设**
  设命题对 $m=k$ 成立，即 $A(k,n)$ 对 n 单调不减。
- **要证** $m=k+1$
  对该固定 m，再对 n 归纳。
  - **基例** $n=0\to 1$
    $A(k+1,0)=0\le A(k+1,1)=2$。

- **归纳步骤** 设 $n \ge 1$，且已知

$$A(k+1,n) \;\ge\; A(k+1,n-1). \tag{IH_n}$$

  由于 $n \ge 1$，有 $n+1 \ge 2$，根据递归定义
  $A(k+1,n+1) = A\big(k,\ A(k+1,n)\big)$，
  $A(k+1,n) = A\big(k,\ A(k+1,n-1)\big)$。
  - 由 IH_n 得 $A(k+1,n) \ge A(k+1,n-1)$。
  - 由外层归纳假设知当 $m=k$ 时函数对第二参单调不减，于是
    $A\big(k,\ A(k+1,n)\big) \;\ge\; A\big(k,\ A(k+1,n-1)\big)$。
    即 $A(k+1,n+1) \ge A(k+1,n)$。

综上 $A(k+1,n)$ 对所有 n 单调不减，引理得证。 $\square$

2. 主命题：$A(m+1,n) \ge A(m,n)$

定理 对所有 m,n 有
$A(m+1,n) \;\ge\; A(m,n).$

证明（对 m 归纳，利用引理 1）

- **基例** $m=0$
  - $n=0$: $A(1,0)=0=A(0,0)$.
  - $n=1$: $A(1,1)=2=A(0,1)$.
  - $n \ge 2$: 由定义
    $A(1,n) = A\big(0, A(1,n-1)\big) = 2\,A(1,n-1).$
    而引理 1 给出 $A(1,n-1) \ge n$，故
    $A(1,n) = 2\,A(1,n-1) \;\ge\; 2n = A(0,n).$
    因此 $m=0$ 情形成立。
- **归纳假设**
  设对某 $m=k$ 成立：

$$\forall n,\ A(k+1,n) \ge A(k,n). \tag{IH_k}$$

- **要证** $m=k+1$
  需示 $\forall n,\ A(k+2,n) \ge A(k+1,n)$。
  - $n=0,1$ 与基例完全相同，显然成立。

- 令 $n \ge 2$。根据递归式
  $A(k+2,n) = A\big(k+1,\ A(k+2,n-1)\big)$,

  $A(k+1,n) = A\big(k,\ A(k+1,n-1)\big)$.

  i. 由 IH_k（把 n 换成 $n-1$）得

  $$A(k+2,n-1)\ \ge\ A(k+1,n-1). \tag{①}$$

  ii. 引理 1 告知 $\forall x$，$A(k+1,x)$ 随 x 单调不减，于是由 (①)

  $$A\big(k+1,\ A(k+2,n-1)\big)\ \ge\ A\big(k+1,\ A(k+1,n-1)\big). \tag{②}$$

  iii. 再次使用 IH_k（这回把 n 换成 $A(k+1,n-1)$）可得

  $$A\big(k+1,\ A(k+1,n-1)\big)\ \ge\ A\big(k,\ A(k+1,n-1)\big). \tag{③}$$

  iv. 联结 (②)(③) 即
  $A(k+2,n) = A\big(k+1,\ A(k+2,n-1)\big)\ \ge\ A\big(k,\ A(k+1,n-1)\big) = A(k+1,n)$.

综上，对所有 n 均有 $A(k+2,n) \ge A(k+1,n)$，归纳完成。

因此定理得证。□

59

59. Use strong induction to prove that a function F defined by specifying $F(0)$ and a rule for obtaining $F(n+1)$ from the values $F(k)$ for $k = 0, 1, 2, \ldots, n$ is well defined.

answer

Pf: by strong induction

Invariant:

$P(n)$: for n, $F(n)$ is unique.

Base case:

$F(0)$ is unique.

So $P(0)$ is true.

Inductive Step:

we assume $P(0) \wedge P(1) \wedge \ldots \wedge P(n)$ is true.

then $F(0), F(1) \ldots F(n)$ is unique.

Then because $F(n+1)$ is obtained from the values $F(k)$ for $k = 0, 1, 2, \ldots, n$,
So $F(n+1)$ is unique.
So $P(0) \wedge P(1) \wedge ... \wedge P(n) \wedge P(n+1)$ is true.
So F is well defined.
Q.E.D

5.4 Exercise

14

14. Give a recursive algorithm for finding a **mode** of a list of integers. (A **mode** is an element in the list that occurs at least as often as every other element.)

answer

```
def find_mode(lst):
    if lst is empty:
        return None
    if lst has only one element:
        return that element

    first = lst[0]
    rest = lst[1:]

    mode_rest = find_mode(rest)

    if count(first, lst) >= count(mode_rest, lst):
        return first
    else:
        return mode_rest

def count(x, lst):
    if lst is empty:
        return 0
    if lst[0] == x:
        return 1 + count(x, lst[1:])
    else:
        return count(x, lst[1:])
```

17

17. Describe a recursive algorithm for multiplying two nonnegative integers x and y based on the fact that $xy = 2(x \cdot (y/2))$ when y is even and $xy = 2(x \cdot \lfloor y/2 \rfloor) + x$ when y is odd, together with the initial condition $xy = 0$ when $y = 0$.

answer

```
def calc(x, y):
    if y == 0: return 0;
    if y % 2 == 0: return 2 * calc(x * (y // 2))
    if y % 2 == 1: return 2 * calc(x * (y - 1)/2) + x
```

23

23. Devise a recursive algorithm for computing n^2 where n is a nonnegative integer, using the fact that $(n+1)^2 = n^2 + 2n + 1$. Then prove that this algorithm is correct.

answer

```
def square(x):
    if x == 0: return 0;
    return square(x-1) + 2*(x-1) + 1
```

35

35. Give iterative and recursive algorithms for finding the n_{th} term of the sequence defined by $a_0 = 1$, $a_1 = 3$, $a_2 = 5$, and $a_n = a_{n-1} \cdot a_{n-2}^2 \cdot a_{n-3}^3$. Which is more efficient?

answer

- Iterative:

```
a[0]=1
a[1]=3
a[2]=5
for(int i=3; i<=n; i++){
    a[n] = a[n-1] * pow(a[n-2],2) * pow(a[n-3],3);
}
return a[n]
```

- Recursive:

```
int a(int i){
    switch{
        case 0:{
            return 1;
        }
        case 1:{
            return 3;
        }
        case 2:{
            return 5;
        }
        return a(i-1) * pow(a[i-2],2) * pow(a[i-3],3);
    }
}
```

The iterative algorithm is more efficient.

The quick sort is an efficient algorithm. To sort $a_1, a_2, \ldots, a_n$, this algorithm begins by taking the first element a_1 and forming two sublists, the first containing those elements that are less than a_1, in the order they arise, and the second containing those elements greater than a_1, in the order they arise. Then a_1 is put at the end of the first sublist. This procedure is repeated recursively for each sublist, until all sublists contain one item. The ordered list of n items is obtained by combining the sublists of one item in the order they occur.

53

53. What is the largest number of comparisons needed to order a list of four elements using the quick sort algorithm?

answer

Worst case: Always choose the first element as the boundary.
$3 + 2 + 1 = 6$

54

54. What is the least number of comparisons needed to order a list of four elements using the quick sort algorithm?

answer

Best case: Always choose the middle element as the boundary.
$3 + 1 = 4$

55

55. Determine the worst-case complexity of the quick sort algorithm in terms of the number of comparisons used.

answer

For worst case, the times of comparison is:
$n - 1 + n - 2 + n - 3 ... + 1 = \frac{(n-1)n}{2}$
The time complexity is:
$O(n^2)$

6.1 Exercise

24

24. How many positive integers between 1000 and 9999 inclusive

- a) are divisible by 9?

- b) are even?
- c) have distinct digits?
- d) are not divisible by 3?
- e) are divisible by 5 or 7?
- f) are not divisible by either 5 or 7?
- g) are divisible by 5 but not by 7?
- h) are divisible by 5 and 7?

answer

$9999 - 1000 + 1 = 9000$

- a) $\lfloor 9999/9 \rfloor - \lceil 1000/9 \rceil + 1 = 1000$
- b) $9000/2 = 4500$
- c) $9 * 9 * 8 * 7 = 4536$
- d) $\lfloor 9999/3 \rfloor - \lceil 1000/3 \rceil + 1 = 3000$, $9000 - 3000 = 6000$
- e) divisible by 5: $\lfloor 9999/5 \rfloor - \lceil 1000/5 \rceil + 1 = 1800$
  divisible by 7: $\lfloor 9999/7 \rfloor - \lceil 1000/7 \rceil + 1 = 1286$
  divisible by 35: $\lfloor 9999/35 \rfloor - \lceil 1000/35 \rceil + 1 = 257$
  $1800 + 1286 - 257 = 2829$
- f) $9000 - 2829 = 6171$
- g) $1800 - 257 = 1543$
- h) 257

27

27. A committee is formed consisting of one representative from each of the 50 states in the United States, where the representative from a state is either the governor or one of the two senators from that state. How many ways are there to form this committee?

answer

3^{50}

45

45. At a large university, 434 freshmen, 883 sophomores, and 43 juniors are enrolled in an introductory algorithms course. How many sections of this course need to be scheduled to accommodate all these students if each section contains 34 students?

answer

By the generlized pigeonhole priciple,
If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.
There is $434 + 883 + 43 = 1360$ students(pigeons).
We want one box have 34 pigeons.
so $\lceil 1360/k \rceil = 34$
k is 40.

67

67. How many ways are there to arrange the letters a, b, c and d such that a is not followed immediately by b?

answer

all arrange: $4! = 24$
a followed by b: $3! = 6$
so the answer is $24 - 6 = 18$.

6.2 Exercise

9

9. Let n be a positive integer. Show that in any set of n consecutive integers there is exactly one divisible by n.

answer

We note that the set of n consecutive integers is $\{a_1, a_2, ..., a_n\}$.
Then $m_i \equiv a_i \mod n, 0 \leq m_i < n$
So the question is equal to, there is exactly one 0 in $\{m_1, m_2, ..., m_n\}$
Exist at least one 0:
If $m_1 = 0$, Done.
If $m_1 \neq 0$, then the m_{n-m_1+1} is zero.
At most one 0:
If there exist two 0, note the smallest one as a_m, then the next one is at least $a_m + n$, then there is at least $n + 1$ element in the set.
Q.E.D

17

17. How many numbers must be selected from the set $\{1, 2, 3, 4, 5, 6\}$ to guarantee that at least one pair of these numbers add up to 7?

answer

there is 3 possible pair:
$(1, 6)$
$(2, 5)$
$(3, 4)$
Firstly, pick one number from each pair.
Then pick any number.
So the answer is 4.

35

In the 17th century, there were more than 800,000 inhabitants of Paris. At the time, it was believed that no one had more than 200,000 hairs on their head. Assuming these numbers are correct and that everyone has at least one hair on their head (that is, no one is completely bald), use the pigeonhole principle to show, as the French writer Pierre Nicole did, that there had to be two Parisians with the same number of hairs on their heads. Then use the generalized pigeonhole principle to show that there had to be at least five Parisians at that time with the same number of hairs on their heads.

answer

The number of possible hair counts ranges from 1 to 200,000, so there are 200,000 possible "pigeonholes" (distinct hair counts).
There were more than 800,000 inhabitants of Paris (the "pigeons").
Since the number of pigeons (800,000+) exceeds the number of pigeonholes (200,000), by the pigeonhole principle, at least two pigeons must occupy the same pigeonhole. Therefore, at least two Parisians must have had exactly the same number of hairs on their heads.

The generalized pigeonhole principle states that if n objects are placed into k boxes, then at least one box contains at least $\lceil n/k \rceil$ objects.

In this case:

- $n = 800,000+$ (Parisians)
- $k = 200,000$ (possible hair counts)

Therefore, at least one hair count must be shared by at least $\lceil 800,000/200,000 \rceil = \lceil 4 \rceil = 4$ Parisians.

However, since there are actually more than 800,000 Parisians, we can be more precise:
$\lceil (800,001)/200,000 \rceil = \lceil 4.00005 \rceil = 5$

Thus, by the generalized pigeonhole principle, at least five Parisians must have had exactly the same number of hairs on their heads.

46

46. There are 51 houses on a street. Each house has an address between 1000 and 1099, inclusive. Show that at least two houses have addresses that are consecutive integers.

answer

Pf: by contradiction
We asssume that all address is not consecutive.
Then sort all address ascendingly, noted as $a_1, a_2, ...a_51$
because all address are not consecutive.
$a_2 \geq a_1 + 2$
$a_3 \geq a_2 + 2$

...

$a_{51} \geq a_{50} + 2$

so $a_{51} \geq a_1 + 100$

but because all addres are between 1000 and 1099.

So $a_{51} \leq a_1 + 99$

which is contradictory to the assumption.

So at least two houses have addresses that are consecutive integers.

Q.E.D