HW11

6.3Exercise

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21. How many permutations of the letters ABCDEFG contain

- a) the string BCD?
- b) the string CFGA?
- c) the strings BA and GF?
- d) the strings ABC and DE?
- e) the strings ABC and CDE?
- f) the strings CBA and BED?

answer

- a) $5! = 120$
- b) $4! = 24$
- c) $5! = 120$
- d) $4! = 24$
- e) $3! = 6$
- f) 0

31

$*$ 31. How many 4-permutations of the positive integers not exceeding 100 contain three consecutive integers k, $k+1$, $k+2$, in the correct order

- a) where these consecutive integers can perhaps be separated by other integers in the permutation?
- b) where they are in consecutive positions in the permutation?

answer

- a) We need to select 4 numbers, including three consecutive integers k, k+1, k+2 (in the correct order).

First, we need to choose the value of k. Since we're selecting from positive integers not exceeding 100, and we need k, k+1, k+2 to be in range, k can be from 1 to 98, giving us 98 possibilities.

Next, we need to select the fourth number. This number can't be k, k+1, or k+2, so we choose 1 number from the remaining 97 numbers, giving 97 possibilities.

Finally, we need to arrange these 4 numbers so that k, k+1, k+2 appear in the correct order. This means we first determine the positions for k, k+1, k+2, then place the fourth number.

There are $\binom{4}{3} = 4$ ways to choose positions for k, k+1, k+2, with their internal order fixed. The fourth number goes in the remaining position.

Therefore, the total number of permutations is: $98 \times 97 \times 4 = 38024$

- b) In this case, k, k+1, k+2 must appear consecutively in the permutation.

First, we still need to choose the value of k, which ranges from 1 to 98, giving 98 possibilities.

Next, we select the fourth number from the remaining 97 numbers, giving 97 possibilities.

Then, we can treat k, k+1, k+2 as a single unit and arrange it with the fourth number. This is equivalent to arranging 2 elements, which can be done in $2! = 2$ ways.

Therefore, the total number of permutations is: $98 \times 97 \times 2 = 19012$

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38. How many bit strings contain exactly five 0s and 14 1s if every 0 must be immediately followed by two 1s?

answer

So we have five 011 bit strings and four free 1 bit string.
Use the binomial theorem,
there are $\binom{9}{4} = 126$ bit strings.

6.4Exercise

7

7. What is the coefficient of x^9 in $(2-x)^{19}$?

answer

Using the binomial theorem, the expansion of x^9 in $(2-x)^{19}$ is
$(2-x)^{19} = \sum_{k=0}^{19} \binom{19}{k} (2)^{19-k} (-x)^k$
For x^9, $k = 9$.
$\binom{19}{9} = 92378$
$\binom{19}{9} * 2^{10} * (-1)^9 = -94595072$
the coefficient is -94595072

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13. Use the binomial theorem to find the coefficient of $x^a y^b$ in the expansion of $(2x^3 - 4y^2)^7$, where

- a) $a = 9$, $b = 8$.
- b) $a = 8$, $b = 9$.
- c) $a = 0$, $b = 14$.
- d) $a = 12$, $b = 6$.
- e) $a = 18$, $b = 2$.

answer

Using the binomial theorem, the expansion of $(2x^3 - 4y^2)^7$ is:
$(2x^3 - 4y^2)^7 = \sum_{k=0}^{7} \binom{7}{k} (2x^3)^{7-k} (-4y^2)^k$

- a) $a = 9$, $b = 8$:
  For x^9, $k = 4$.
  For y^8, $k = 4$.
  The coefficient is $\binom{7}{4} \cdot 2^3 \cdot (-4)^4 = 35 \cdot 8 \cdot 256 = 71680$

- b) $a = 8, b = 9$:
  No solution exists because if $3(7-k) = 8$, then $7 - k = 8/3$, which is not an integer.
  Therefore, the coefficient is 0.
- c) $a = 0, b = 14$:
  For x^0, $k = 7$.
  For y^{14}, $k = 7$.
  The coefficient is $\binom{7}{7} \cdot 2^0 \cdot (-4)^7 = 1 \cdot 1 \cdot (-4)^7 = -16384$
- d) $a = 12, b = 6$:
  For x^{12}, $k = 3$.
  For y^6, $k = 3$.
  The coefficient is $\binom{7}{3} \cdot 2^4 \cdot (-4)^3 = 35 \cdot 16 \cdot (-64) = -35840$
- e) $a = 18, b = 2$:
  For x^{18}, $k = 1$.
  For y^2, $k = 1$.
  The coefficient is $\binom{7}{1} \cdot 2^6 \cdot (-4)^1 = 7 \cdot 64 \cdot (-4) = -1792$

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27. Show that if n and k are positive integers, then $\binom{n+1}{k} = \frac{(n+1)\binom{n}{k-1}}{k}$. Use this identity to construct an inductive definition of the binomial coefficients.

answer

We need to prove that $\binom{n+1}{k} = \frac{(n+1)\binom{n}{k-1}}{k}$ for positive integers n and k.

Starting with the definition of binomial coefficients:
$\binom{n+1}{k} = \frac{(n+1)!}{k!(n+1-k)!}$

We can rewrite this as:
$\binom{n+1}{k} = \frac{(n+1)!}{k!(n+1-k)!} = \frac{(n+1) \cdot n!}{k \cdot (k-1)! \cdot (n+1-k)!}$

Now, we can recognize that $\binom{n}{k-1} = \frac{n!}{(k-1)!(n-(k-1))!} = \frac{n!}{(k-1)!(n-k+1)!}$

Substituting this into our expression:

$\binom{n+1}{k} = \frac{(n+1) \cdot n!}{k \cdot (k-1)! \cdot (n+1-k)!} = \frac{(n+1)}{k} \cdot \frac{n!}{(k-1)!(n-k+1)!} = \frac{(n+1)\binom{n}{k-1}}{k}$

Therefore, $\binom{n+1}{k} = \frac{(n+1)\binom{n}{k-1}}{k}$ is proven.

Using this identity, we can construct an inductive definition of binomial coefficients:

Base cases:

- $\binom{n}{0} = 1$ for all $n \geq 0$
- $\binom{n}{n} = 1$ for all $n \geq 0$

Inductive step:

- For $1 \leq k \leq n$, $\binom{n+1}{k} = \frac{(n+1)\binom{n}{k-1}}{k}$

31

$*$ 31. Prove the hockeystick identity $\sum_{k=0}^{r} \binom{n+k}{k} = \binom{n+r+1}{r}$ whenever n and r are positive integers,

- a) using a combinatorial argument.
- b) using Pascal's identity.

answer

- a) Combinatorial Proof:
  Pf:
  Consider selecting r people from a group of n+r+1 people. By definition, there are $\binom{n+r+1}{r}$ ways to do this.
  Alternatively, we can classify the selection based on the position of the last person selected:
  - If the last person selected is in position n+0, then we need to select r-0 people from the first n-1+0 people, giving $\binom{n-1}{r}$ ways
  - If the last person selected is in position n+1, then we need to select r-1 people from the first n-1+1 people, giving $\binom{n}{r-1}$ ways
  - If the last person selected is in position n+2, then we need to select r-1 people from the first n-1+2 people, giving $\binom{n+1}{r-1}$ ways
  - ...
  - If the last person selected is in position n+r, then we need to select r-1 people from the first n-1+r people, giving $\binom{n+r-1}{r-1}$ ways

  By reindexing, these cases can be represented as $\sum_{k=0}^{r} \binom{n+k-1}{r-1}$. Using the combinatorial identity $\binom{n+k-1}{r-1} = \binom{n+k}{k}$, we get $\sum_{k=0}^{r} \binom{n+k}{k} = \binom{n+r+1}{r}$.

Q.E.D

- b) Proof using Pascal's identity:

  Pf: By induction

  Base case:

  For $r = 1$:

  $\sum_{k=0}^{1} \binom{n+k}{k} = \binom{n}{0} + \binom{n+1}{1} = 1 + (n+1) = n+2 = \binom{n+1+1}{1}$, which is true.

  Inductive Step:

  Assume that for some r ≥ 1, $\sum_{k=0}^{r} \binom{n+k}{k} = \binom{n+r+1}{r}$ holds.

  For r+1, we have:

  $\sum_{k=0}^{r+1} \binom{n+k}{k} = \sum_{k=0}^{r} \binom{n+k}{k} + \binom{n+r+1}{r+1}$

  By our induction hypothesis, $\sum_{k=0}^{r} \binom{n+k}{k} = \binom{n+r+1}{r}$

  So $\sum_{k=0}^{r+1} \binom{n+k}{k} = \binom{n+r+1}{r} + \binom{n+r+1}{r+1}$

  Using Pascal's identity $\binom{n+r+1}{r} + \binom{n+r+1}{r+1} = \binom{n+r+2}{r+1}$

  Therefore $\sum_{k=0}^{r+1} \binom{n+k}{k} = \binom{n+r+2}{r+1}$, which matches our formula.

  Q.E.D

6.5Exercise

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15. How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 21$, where x_i, $i = 1, 2, 3, 4, 5$, is a nonnegative integer such that

- a) $x_1 \geq 1$?
- b) $x_i \geq 2$ for $i = 1, 2, 3, 4, 5$?
- c) $0 \leq x_1 \leq 10$?
- d) $0 \leq x_1 \leq 3$, $1 \leq x_2 < 4$, and $x_3 \geq 15$?

answer

Just consider to tuck 4 columns into 22 slots, collision is allowed. And a_i is the quantity of gap from the previous column.

- a) $\binom{20+5-1}{5-1} = \binom{24}{4}$
- b) $\binom{15}{4} = 1365$
- c) $\sum_{k=0}^{10} \binom{24-k}{3}$
- d) $x_3' = x_3 - 15$, $x_1 + x_2 + x_3' + x_4 + x_5 = 6$

-

x_1	x_2	$6 - x_1 - x_2$	$\binom{\text{remained}+3-1}{3-1}$
0	1	5	$\binom{7}{2} = 21$
0	2	4	$\binom{6}{2} = 15$
0	3	3	$\binom{5}{2} = 10$
1	1	4	$\binom{6}{2} = 15$
1	2	3	$\binom{5}{2} = 10$
1	3	2	$\binom{4}{2} = 6$
2	1	3	$\binom{5}{2} = 10$
2	2	2	$\binom{4}{2} = 6$
2	3	1	$\binom{3}{2} = 3$
3	1	2	$\binom{4}{2} = 6$
3	2	1	$\binom{3}{2} = 3$
3	3	0	$\binom{2}{2} = 1$

so the answer is 106

21

21. A Swedish tour guide has devised a clever way for his clients to recognize him. He owns 13 pairs of shoes of the same style, customized so that each pair has a unique color. How many ways are there for him to choose a left shoe and a right shoe from these 13 pairs

- a) without restrictions and which color is on which foot matters?
- b) so that the colors of the left and right shoe are different and which color is on which foot matters?
- c) so that the colors of the left and right shoe are different but which color is on which foot does not matter?
- d) without restrictions, but which color is on which foot does not matter?

answer

- a) $13 * 13 = 169$
- b) $13 * 12 = 156$
- c) $\binom{13}{2} = 78$
- d) $13 + 78 = 91$

42

42. How many ways are there to travel in $xyzw$ space from the origin $(0,0,0,0)$ to the point $(4,3,5,4)$ by taking steps one unit in the positive x, positive y, positive z, or positive w direction?

answer

Consider there are 4 x steps, 3 y steps, 5 z steps and 4 w steps, and we need to compose these steps.
So there is $\frac{16!}{3!4!4!5!} = 50450400$

8.1Exercise

7

7.

- a) Find a recurrence relation for the number of bit strings of length n that contain a pair of consecutive 0s.
- b) What are the initial conditions?
- c) How many bit strings of length seven contain two consecutive 0s?

answer

- a) If the n length bit string
  - start with 1: then there is a_{n-1} possibilities.
  - start with 00: then there is 2^{n-2} possibilities.
  - start with 01: then there is a_{n-2} possibilities.

so the answer is $a_{n-1} + a_{n-2} + 2^{n-2}$

- b) $a_0 = 0$, $a_1 = 0$, $a_2 = 1$
- c) 94

17

$*$ 17.

- a) Find a recurrence relation for the number of ternary strings of length n that do not contain consecutive symbols that are the same.
- b) What are the initial conditions?
- c) How many ternary strings of length six do not contain consecutive symbols that are the same?

answer

- a) Consider a n length string, by symmetry, we know that the quantity of the target string start with 0 is equal to the quantity start with 1 and 2. So for the $n+1$ length, we just need to make the last number be different with the previous.
  So $a_{n+1} = 2 * a_n$
- b) $a_0 = 1, \quad a_1 = 3$
- c) 96

26

26.

- a) Find a recurrence relation for the number of ways to completely cover a 2 × n checkerboard with 1 × 2 dominoes. [Hint: Consider separately the coverings where the position in the top right corner of the checkerboard is covered by a domino positioned horizontally and where it is covered by a domino positioned vertically.]
- b) What are the initial conditions for the recurrence relation in part (a)?
- c) How many ways are there to completely cover a 2 × 17 checkerboard with 1 × 2 dominoes?

answer

- a) Consider the right up corner. that domino is:

- **horizon**: then there is $a_n - 2$ possibility.
- **vertical**:then there is $a_n - 1$ possibility.

so $a_n = a_{n-1} + a_{n-2}$

- b) $a_1 = 1 \quad a_2 = 2$
- c) 2584

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Let $\{a_n\}$ be a sequence of real numbers. The backward differences of this sequence are defined recursively as shown next. The first difference ∇a_n is $\nabla a_n = a_n - a_{n-1}$. The $(k+1)$st difference $\nabla^{k+1} a_n$ is obtained from $\nabla^k a_n$ by $\nabla^{k+1} a_n = \nabla^k a_n - \nabla^k a_{n-1}$.

48. Show that $a_{n-1} = a_n - \nabla a_n$.

answer

we know ∇a_n is $\nabla a_n = a_n - a_{n-1}$.
So $a_{n-1} = a_n - \nabla a_n$