HW4🚀

Exercise 2.1

7

7. Determine whether each of these pairs of sets are equal.

- a) $1, 3, 3, 3, 5, 5, 5, 5, 5 = 5, 3, 1$
- b) $1 = 1, 1$
- c) $\emptyset = \emptyset$

answer

- a) T
- b) F
- c) T

12

12. Determine whether these statements are true or false.

- a) $\emptyset \in \{\emptyset\}$
- b) $\emptyset \in \{\emptyset, \{\emptyset\}\}$
- c) $\{\emptyset\} \in \{\emptyset\}$
- d) $\{\emptyset\} \in \{\{\emptyset\}\}$
- e) $\{\emptyset\} \subset \{\emptyset, \{\emptyset\}\}$
- f) $\{\{\emptyset\}\} \subset \{\emptyset, \{\emptyset\}\}$
- g) $\{\{\emptyset\}\} \subset \{\{\emptyset\}, \{\emptyset\}\}$

answer

- a) T
- b) T
- c) T
- d) T
- e) T
- f) T
- g) T

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27. Prove that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ if and only if $A \subseteq B$.

answer

Pf:
$(\implies)$:(by contradiction)
we assume that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ but $A \nsubseteq B$.
then there must exist a element e that $e \in A$ but $e \notin B$.
By the definition of power set, then $e \in \mathcal{P}(A)$ but $e \notin \mathcal{P}(B)$, which is contradictory to the assumption.
S0 $(\mathcal{P}(A) \subseteq \mathcal{P}(B)) \implies (A \subseteq B)$

$(\impliedby)$:
For any subset $S \subseteq A$, because $A \subseteq B$, so $S \subseteq B$.
By the definition of power set, if $S \in \mathcal{P}(A)$, and $S \in \mathcal{P}(B)$.
By the definition of subset, $\mathcal{P}(A) \subseteq \mathcal{P}(B)$

So $\mathcal{P}(A) \subseteq \mathcal{P}(B \iff A \subseteq B$.
Q.E.D

35

35. Find A^2 if

- a) $A = \{0, 1, 3\}$.
- b) $A = \{1, 2, a, b\}$.

answer

- a) $A^2 = \{(0,0),(0,1),(0,3),(1,0),(1,1),(1,3),(3,0),(3,1),(3,3)\}$
- b) $A^2 = \{(1,1),(1,2),(1,a),(1,b),(2,1),(2,2),(2,a),(2,b),(a,1),(a,2),(a,a),(a,b),(b,1),(b,2),(b,a),(b,b)\}$

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39. How many different elements does A^n have when A has m elements and n is a positive integer?

answer

m^n

42

42. Explain why $(A \times B) \times (C \times D)$ and $A \times (B \times C) \times D$ are not the same.

answer

we assume that:

- $A : \{1\}$
- $B : \{2\}$
- $C : \{3\}$
- $D : \{4\}$

$(A \times B) \times (C \times D) = \{(1, 2)\} \times \{(3, 4)\} = \{((1, 2), (3, 4))\}$

$A \times (B \times C) \times D = \{1\} \times \{(2, 3)\} \times \{4\} = \{(1, (2, 3), 4)\}$

Therefore, $(A \times B) \times (C \times D)$ and $A \times (B \times C) \times D$ are not the same because their element structures are different. The former is a set containing ordered pairs where each element is itself an ordered pair; while the latter is a set containing ordered triples.

Q.E.D

Exercise 2.2

20. Let A, B, and C be sets. Show that

- a) $(A \cup B) \subseteq (A \cup B \cup C)$.
- b) $(A \cap B \cap C) \subseteq (A \cap B)$.
- c) $(A \setminus B) \setminus C \subseteq A \setminus C$.
- d) $(A \setminus C) \cap (C \setminus B) = \emptyset$.
- e) $(B \setminus A) \cup (C \setminus A) = (B \cup C) \setminus A$.

answer

- a) assume that $x \in A \cup B$.
  then $x \in A$ or $x \in B$.
  If $x \in A$, because $A \subset A \cup B \cup C$, so $x \in A \cup B \cup C$.
  If $x \in B$, because $B \subset A \cup B \cup C$, so $x \in A \cup B \cup C$.
  So by definition, $(A \cup B) \subseteq (A \cup B \cup C)$.
- b)
  If $x \in A \cap B \cap C$, then $x \in A$ and $x \in B$ and $x \in C$.
  then $x \in A$ and $x \in B$, then $x \in A \cap B$.
  So by definition, $(A \cap B \cap C) \subseteq (A \cap B)$.
- c)
  If $x \in (A \setminus B) \setminus C$, then $x \in A$ and $x \notin B$ and $x \notin C$. then $x \in A$ and $x \notin C$, then $x \in A \setminus C$.

So by definition, $(A \setminus B) \setminus C \subseteq A \setminus C$

- d)
  If $x \in A \setminus C$, then $x \in A$ and $x \notin C$, then $x \notin C$, then $x \in \overline{C}$.
  If $x \in C \setminus B$, then $x \in C$ and $x \notin B$, then $x \in C$.
  So $x \in \overline{C} \cap C$, so $x \in \emptyset$.
  then $(A \setminus B) \setminus C \subseteq A \setminus C$.
- e)
  If $x \in (B \setminus A)$, then $x \in B$ and $x \notin A$.
  If $x \in (C \setminus A)$, then $x \in C$ and $x \notin A$.
  So if $x \in (B \setminus A) \cup (C \setminus A)$, $x \in B$ or $x \in C$ and $x \notin A$.
  So $x \in (B \cup C) \setminus A$.
  So by definition, $(B \setminus A) \cup (C \setminus A) = (B \cup C) \setminus A$.

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37. Prove or disprove that for all sets A, B, and C, we have

- a) $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$.
- b) $\overline{A} \times \overline{(B \cup C)} = \overline{A \times (B \cup C)}$.

answer

- a)
  Pf:(by definition)
  we assume that

$$
\begin{aligned}
&A = \{a_1, a_2, ..., a_{n_a}\} \\
&B = \{b_1, b_2, ..., b_{n_b}, e_1, e_2, ..., e_n\} \\
&C = \{c_1, c_2, ..., c_{n_c}, e_1, e_2, ..., e_n\} \\
&(n_a, n_b, n_c, n = 0, 1, 2...) \\
&(b_i \neq c_j \quad i = 1, 2, ...n_b; j = 1, 2, ...n_c)
\end{aligned}
$$

For the left part:

$$
\begin{aligned}
B \setminus C &= b_1, b_2, ...b_{n_b} \\
A \times (B \setminus C) &= \{(a_i, b_j)\} \quad (i = 1, 2, ...n_a; j = 1, 2, ...n_b)
\end{aligned}
$$

For the right part:

$$
\begin{aligned}
A \times B &= \{(a_i, b_j), (a_i, e_k)\} \quad (i = 1, 2, ...n_a; j = 1, 2, ...n_b; k = 1, 2, ...n) \\
A \times C &= \{(a_i, c_j), (a_i, e_k)\} \quad (i = 1, 2, ...n_a; j = 1, 2, ...n_c; k = 1, 2, ...n) \\
(A \times B) \setminus (A \times C) &= \{(a_i, b_j)\} \quad (i = 1, 2, ...n_a; j = 1, 2, ...n_b)
\end{aligned}
$$

So

$$A \times (B \setminus C) = (A \times B) \setminus (A \times C)$$

- b)
  Dispf:
  we assume $A = \{1\}, B = \{2\}, C = \{3\}$ and $U = \{1, 2, 3, (1, 1), (1, 2), (1, 3)\}$
  Then we have:

$$\overline{A} \times \overline{(B \cup C)} = \{2, 3\} \times \{1\} = \{(2, 1), (3, 1)\}$$
$$\overline{A \times (B \cup C)} = \overline{\{(1, 2), (1, 3)\}} = \{1, 2, 3, (1, 1), (2, 2), (2, 3), (3, 2), (3, 3)\}$$

Therefore, $\overline{A} \times \overline{(B \cup C)} \neq \overline{A \times (B \cup C)}$, which disproves the statement.

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$*$ 47. Suppose that A, B, and C are sets such that $A \oplus C = B \oplus C$. Must it be the case that $A = B$?

answer

Pf:(by contradiction)
If there exist a x that $x \in A$ but $x \notin B$,

- case1: $x \in C$, then $x \notin A \oplus C$, but $x \in B \oplus C$, which is contradictory to the premise $A \oplus C = B \oplus C$
- case2: $x \notin C$, then $x \in A \oplus C$, but $x \notin B \oplus C$, which is contradictory to the premise $A \oplus C = B \oplus C$

So there is no x that that $x \in A$ but $x \notin B$, so $A \subseteq B$.
In the same reason, there is no x that $x \in B$ but $x \notin A$, so $B \subseteq A$.
So $A = B$

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56. Find $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$ if for every positive integer i,

- a) $A_i = i, i + 1, i + 2, \ldots$.
- b) $A_i = 0, i$.
- c) $A_i = (0, i)$, that is, the set of real numbers x with $0 < x < i$.
- d) $A_i = (i, \infty)$, that is, the set of real numbers x with $x > i$.

answer

- a) $\bigcup_{i=1}^{\infty} A_i = \{1, 2, \ldots\infty\}$, $\bigcap_{i=1}^{\infty} A_i = \emptyset$
- b) $\bigcup_{i=1}^{\infty} A_i = \{0, 1, 2, \ldots\infty\}$, $\bigcap_{i=1}^{\infty} A_i = \{0\}$
- c) $\bigcup_{i=1}^{\infty} A_i = (0, \infty)$, $\bigcap_{i=1}^{\infty} A_i = (0, 1)$
- d) $\bigcup_{i=1}^{\infty} A_i = (1, \infty)$, $\bigcap_{i=1}^{\infty} A_i = \emptyset$

note

之前的错误答案：

- c) $\bigcup_{i=1}^{\infty} A_i = \{(0, \infty)\}$, $\bigcap_{i=1}^{\infty} A_i = \{(0, 1)\}$
- d) $\bigcup_{i=1}^{\infty} A_i = \{(1, \infty)\}$, $\bigcap_{i=1}^{\infty} A_i = \{\emptyset\}$
  题目中已经说明，$A_i = (0, i)$, that is, **the set of real numbers** x with $0 < x < i$.
  所以$(0, i)$本身就是一个集合。没必要在外面再套一个。

71

The Jaccard similarity $J(A, B)$ of the finite sets A and B is $J(A, B) = \frac{|A \cap B|}{|A \cup B|}$, with $J(\emptyset, \emptyset) = 1$. The Jaccard distance $d_J(A, B)$ between A and B equals $d_J(A, B) = 1 - J(A, B)$.

71. Find $J(A, B)$ and $d_J(A, B)$ for these pairs of sets.

- a) $A = 1, 3, 5$, $B = 2, 4, 6$
- b) $A = 1, 2, 3, 4$, $B = 3, 4, 5, 6$
- c) $A = 1, 2, 3, 4, 5, 6$, $B = 1, 2, 3, 4, 5, 6$
- d) $A = 1$, $B = 1, 2, 3, 4, 5, 6$

answer

- a) 0,1
- b) $\frac{1}{3}$,$\frac{2}{3}$
- c) 1,0
- d) $\frac{1}{6}$,$\frac{5}{6}$