Algebraic Topology
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CHAPTER 1

Homotopy Theory

1. Cofibrations

Last time talked about cofibrations in CG, the space of compactly generated, Hausdorff spaces (see appendix of Hatcher for more information).

Definition 1. $X \in CG$ when this holds: $C \subseteq X$ is closed iff for all K compact, $C \cap K$ is closed.

In particular all CW complexes are in CG.

Proposition 1. *CG is closed under pushouts, coproducts, direct limits (if*

$$A(0) \subseteq A(1) \subseteq A(2) \subseteq \cdots$$

each $A(i) \to A(i+1)$ is a cofibration, then $A(0) \to \varinjlim A(i) = \cup A(i)$ is a cofibration).

Proposition 2. *In any category consider*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \\ D \arrow[r] & E \arrow[r] & F \end{tikzcd}$$

(1) Both squares pushout then rectangle is a pushout.

(2) Left square and rectangle pushout then right square is a pushout.

(3) Right square and rectangle being pushout then it is not true that the left square is a pushout.

Proof. If

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & C \arrow[d, "g"] \\ B \arrow[r] & D \end{tikzcd}$$

is a pushout then

$$\begin{tikzcd} A \times I \cup B \times \{0\} \arrow[r] \arrow[d] & C \times I \cup D \times \{0\} \arrow[d] \\ B \times I \arrow[r] & D \times I \end{tikzcd}$$

is a pushout. Really the top right is the pushout of

$$\begin{tikzcd} C \times \{0\} \arrow[r] \arrow[d] & D \times \{0\} \\ C \times I \end{tikzcd} .$$

If $C \subseteq D$ then it is the subspace $C \times I \cup D \times \{0\} \subseteq D \times I$. Hence if f is a cofibration then g is a cofibration.

In

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & C \times I \arrow[d] \\ B \arrow[r] & D \arrow[r] & C \times I \cup_C D \end{tikzcd}$$

left is a pushout by hypothesis, right is a pushout by defintion. So rectanlge is a pushout.

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A \times I \arrow[r] \arrow[d] & C \times I \arrow[d] \\ B \arrow[r] & A \times I \cup_A B \arrow[r] & C \times I \cup_C D \end{tikzcd}$$

left is a pushout be defintion, rectangle is a pushout by last step. So right is a pushout.

$$\begin{tikzcd} A \times I \arrow[r] \arrow[d] & A \times I \cup B \arrow[r] \arrow[d] & B \times I \arrow[d] \\ C \times I \arrow[r] & C \times I \cup D \arrow[r] & D \times I \end{tikzcd}$$

Left is a pushout by last step. Rectangle is a pushout since in CG. So right is a pushout as desired. □

Next thing to show is that

PROPOSITION 3. *$S^n \to D^{n+1}$ is a cofibration.*

PROOF. $D^{n+1} \times I \subseteq D^{n+1} \times \mathbb{R}$ projecting from $(0, 2)$ gives retraction $D^{n+1} \times I \to S^n \times I \cup D^{n+1} \times \{0\}$.

$$fig(1).$$

□

Example 1.1. For all X, $\varnothing \to X$ is a cofibration. ⌟

2. CW complexes

Definition 2. $f : A \to X$ has the structure of a relative weak CW complex if $X(0) = A$ and we have pushouts

$$\begin{tikzcd} \coprod_{\alpha\in I(i)} S^{n(\alpha)} \arrow[r] \arrow[d] & X(i) \arrow[d] \\ \coprod_{\alpha\in I(i)} D^{n(\alpha)+1} \arrow[r] & X(i+1) \end{tikzcd}$$

with the indexing: $n(\alpha) \in \{-1, 0, \cdots, \}$ and $S^{-1} = \varnothing$ and $D^0 = *$. And

$$X = X(\infty) = \cup X(i)$$

is a (standard) relative CW complex if for every $\alpha \in I(i)$, $n(\alpha) = i - 1$. And if $A = \varnothing$ then X is a (weak) CW complex. In the (standard) CW case then $X_i = X(i)$ is called i-skeleton of X.

Remark. In fact we need to define (weak) relative CW complexes up to an isomorphism of maps: f is a relative CW complex if $f \cong g$ and g can be given the structure of a relative CW complex, where the isomorphism of f and g mean there exists commutative diagrams

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & X' \arrow[r] \arrow[d, "g"] & X \arrow[d, "f"] \\ Y \arrow[r] & Y' \arrow[r] & Y \end{tikzcd}$$

with horizontal arrows isomorphism.

▸ Exercise 1. Show that the (relative) weak CW complexes are the smallest class of maps containing $\varnothing \to *$ and $S^n \to D^{n+1}$ for all $n \geq 0$ and closed under pushouts, coproducts and (countable) direct limits; i.e if

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & B \arrow[d, "g"] \\ C \arrow[r] & D \end{tikzcd}$$

is a pushout then f is relative weak CW complex then so is g.

Proposition 4. *If $f : A \to X$ is a relative CW complex then f is a cofibraiton.*

Proof. Given any diagram below we can complete the diagram:

$$\begin{tikzcd} A \times I \cup X \arrow[r] \arrow[d] & Z \\ X \times I \arrow[ur, dashed, "\varphi"] & \end{tikzcd}$$

Fix $X(0) = A$ and show that $X(i) \to X(i+1)$ is a cofibration. Then $X = \operatorname{colim} X(i)$ so $A \to X$ is a cofibration by previous proposition. For every α, $S^{n(\alpha)} \to D^{n(\alpha)+1}$ is a cofibration. $\coprod_{\alpha\in I(i)} S^{n(\alpha)} \to \coprod_{\alpha\in I(i)} D^{n(\alpha)+1}$ is a cofibration. But then we have a pushout

$$\begin{tikzcd} \coprod_{\alpha\in I(i)} S^{n(\alpha)} \arrow[r] \arrow[d, "i"'] & X(i) \arrow[d, "j"] \\ \coprod_{\alpha\in I(i)} D^{n(\alpha)+1} \arrow[r] & X(i+1) \end{tikzcd}$$

i cofibation $\Rightarrow j$ cofibration. Then direct limit of cofibrations is a cofibration $\Rightarrow A \to X$ is a cofibration. □

For $f : X \to Y$ the mapping cylinder of f is the pushout

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "x\mapsto(x,0)"'] & Y \arrow[d, "i"] \\ X \times I \arrow[r] & M(f) \end{tikzcd}$$

$Y \to M(f)$ is a cofibration. i has strong deformation retraction. And in particular they are homotopy equivalent.

If X is a CW complex then $Y \to M(f)$ is relative (weak) CW complex. If Y is a (weak) CW complex then $X \to M(f)$ via $x \mapsto (x,1)$ is a (weak) relative CW complex. Recall that $D^n \times I$ can be given the structure of a CW complex. There is a pushout

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ X \coprod X \arrow[r] \arrow[d] & Y \coprod X \arrow[d] \\ X \times I \arrow[r] & M(f) \end{tikzcd}$$

is a pushout so if $X \coprod X \to X \times I$ is a relative (weak) CW structure so is $Y \coprod X \to M(f)$. Also $X \to X \coprod Y$ is relative CW if Y is. $Y \to X \coprod Y$ is a relative CW complex if X is. So basically you have to show that $X \coprod X \to X \times I$ is a relative CW complex.

$$\begin{tikzcd} \coprod S^i \times I \cup D^{i+1} \cup D^{i+1} \arrow[r] \arrow[d] & X \coprod X \cup X(i) \times I \arrow[d] \\ \coprod D^{i+1} \times I \arrow[r] & X \coprod X \cup X(i+1) \times I \end{tikzcd}$$

And you can more generally write $\alpha(i)$ for any i.

Definition 3. $f : X \to Y$ is a weak homotopy equivalence if for any $x_0 \in X$ and any n $\pi_n(f) : \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ is an isomorphism.

Theorem 2.1 (Whitehead's). *If X, Y are path-connected (weak) CW complexes and $f : X \to Y$ is a weak homotopy equivalence then f is a homotopy equivalence. And in particular if f is a subcomplex then f has a strong deformation retraction.*

Lemma 1. *Let (X, A) be a relative (weak) CW complex and (Y, B) be just any pair and $B \neq \varnothing$. Suppose for evey n such that $X - A$ has n-cells (have the vanishing of obstruction theory given by) $\pi_n(Y, B, y_0) = 0$ for any $y_0 \in B$. Then for every map $f : (X, A) \to (Y, B)$ of pairs, there exists g such that* $\operatorname{im}(g) \subseteq B$ *and* $f \cong g \operatorname{rel} A$.

Proof. Recall the compression lemma: $\pi_n(Y, B, y_0) = 0$ iff for every $f : (D^n, S^{n-1}, *) \to (Y, B, y_0)$ there exists g such that $f \cong g \operatorname{rel} S^{n-1}$ and $\operatorname{im}(g) \subseteq B$. Suppose

$$A = X(0) \hookrightarrow X(1) \hookrightarrow \cdots \hookrightarrow X(n) \hookrightarrow \cdots X(\infty) = X$$

and we have a sequence of homotopies $H_k : f_k \cong f_{k+1} \operatorname{rel} X(k-1)$ with $f_{k+1}(X(k)) \subseteq B$ for all $k < n$. We will construct the next one: Denote by $\overline{f_n} = f_n|_{X(n)}$ the restriction. We basically want to complete

$$\begin{tikzcd}
X(n-1) \arrow[r, "f|_{X(n-1)}"] \arrow[d] & B \arrow[d] \\
X(n) \arrow[r, "\overline{f_n}"'] \arrow[ur, dashed] & Y
\end{tikzcd}$$

But we also have a pushout

$$\begin{tikzcd}
\coprod S^{n(\alpha)} \arrow[r, "\coprod h(\alpha)"] \arrow[d] & X(n-1) \arrow[r, "f|_{X(n-1)}"] \arrow[d] & B \arrow[d] \\
\coprod D^{n(\alpha)+1} \arrow[r, "\coprod h(\alpha)"'] & X(n) \arrow[r, "\overline{f_n}"'] \arrow[ur, dashed] & Y
\end{tikzcd}$$

So by the compression lemma for

$$\begin{tikzcd}
S^{n(\alpha)} \arrow[r, "f_n h(\alpha)"] \arrow[d] & B \arrow[d] \\
D^{n(\alpha)+1} \arrow[r, "\overline{f_n} h'(\alpha)"'] & Y
\end{tikzcd}$$

there is

$$\begin{tikzcd} S^{n(\alpha)} \times I \arrow[r, "\text{constant}"] \arrow[d] & B \arrow[d] \\ D^{n(\alpha)+1} \times I \arrow[r, "H"'] & Y \end{tikzcd}$$

with $H(-,0) = \overline{f_n} h'(\alpha)$ and $H(-,1)(D^{n(\alpha)+1}) \subseteq B$. So get by taking coproducts $\overline{H_n}$ as the left square below is a pushout:

$$\begin{tikzcd} \coprod S^{n(\alpha)} \times I \arrow[r, "h(\alpha)"] \arrow[d] & X(n-1) \times I \arrow[r, "\text{const. homotopy}"] \arrow[d] & B \arrow[d] \\ \coprod D^{n(\alpha)+1} \times I \arrow[r, "h(\alpha)"'] & X(n) \times I \arrow[r, dashed, "\overline{H_n}"'] & Y \end{tikzcd}$$

$X(n) \hookrightarrow X$ is a cofibration so in

$$\begin{tikzcd} X(n) \times I \cup X \arrow[r, "\overline{H}_n \cup f_n"] \arrow[d] & Y \\ X \times I \arrow[ur, "\exists H_n"'] & \end{tikzcd}$$

call $f_{n+1}(x) \coloneqq H_n(x,1)$ and $H_n(x,0) = f_n(x)$, $H_n : f_n \simeq f_{n+1} \ \text{rel}\ X(n-1)$ and $f_{n+1}(X(n)) \subseteq B$.

Last step is to put all these homotopies together to get

$$H(x,t) = H_k(x, 2^{k+1}t - (2^{k+1} - 1)), t \in [1 - 1/2^k, 1 - 1/2^{k+1}].$$

These glue together to give a homotopy basically since $H_k(x,1) = f_{k+1}(x) = H_{k+1}(x,0)$. H is continuous since $H|_{X(k) \times I}$ is for k. Since H in eventually stationary on each $X(k)$ (just composing finitely many homotopies on each $X(k)$).

∎

Proof of Whitehead's theorem. Suppose $i : X \hookrightarrow Y$ is (weak) relative CW complex. From the long exact sequence on π_k

$$\cdots \to \pi_n X \to \pi_x Y \to \pi_n(Y,X) \xrightarrow{\partial} \pi_{n-1}(X) \to \cdots$$

implies when $\pi_n X \cong \pi_n Y$ that $\pi_n(Y,X) = 0$ for all n. So $\mathrm{id}_Y \simeq g \ \text{rel}\ X$ and $g(Y) \subseteq X$. So g is the homotopy inverse and hence $X \simeq Y$. More generally for any map $f : X \to Y$, where

$Y \cong M(f)$ we have that $i : X \to M(f)$ via $x \mapsto (x, 1)$ making the following

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & M(f) \arrow[d, "p"] \\ & Y \end{tikzcd}$$

commute. Since the down arrows are weak homotopy equivalences (f by hypothesis and p since homotopy equivalence implies weak homotopy equivalence), “2 out of 3” implies that i is a weak homotopy equivalence!

Also i is relative (weak) CW complex hence homotopy equivalence. So $f = i \circ p$ is homotopy equivalence. ☐

3. Cellular and CW approximation

Suppose X, Y are have given CW structures then a map $f : X \to Y$ is called cellular if for every n, $f(X_n) \subseteq Y_n$.

Lemma 2. *Given diagram*

$$\begin{tikzcd} S^n \arrow[r] \arrow[d] & Y \arrow[d] \\ D^{n+1} \arrow[r, "f"] & Y \cup D^k \end{tikzcd}$$

with $k > n+1$ there exists g such that $f \cong g$ rel S^n and $g(D^{n+1}) \subseteq Y$.

Lemma 3. *Given diagram*

$$\begin{tikzcd} S^n \arrow[r] \arrow[d] & Y_{n+1} \arrow[d] \\ D^{n+1} \arrow[r, "f"'] & Y \end{tikzcd}$$

where Y has CW structure, there exists $g : D^{n+1} \to Y$ such that $f \cong g$ rel S^n and $g(D^{n+1}) \subseteq Y_{n+1}$.

Proof. By compactness of D^{n+1}, f factors through f' below; i.e. there exists a commutative diagram

$$\begin{tikzcd} S^n \arrow[r] \arrow[d] & Y_{n+1} \arrow[d] \\ D^{n+1} \arrow[r, "f'"'] \arrow[dr, "f"'] & Y' \arrow[d, hook] \\ & Y \end{tikzcd}$$

where $Y' - Y_{n+1}$ has finitely many cells all of dimension $> n+1$ so apply lemma 2 finitely many times. $\square$

Lemma 4. *Given diagram*

$$\begin{tikzcd} & Y_n \arrow[d] \\ X_n \arrow[ur] \arrow[r] \arrow[d] & Y_{n+1} \arrow[d] \\ X_{n+1} \arrow[r, "f"'] \arrow[ur, dashed, "g"] & Y \end{tikzcd}$$

where X, Y have CW structures, there exists a homotopy $H : X \times I \to Y$ such that

$$H|_{X_{n+1}\times I} : f \cong g \ \text{ rel } X_n \ \textit{and}\ g(X_{n+1}) \cong Y_{n+1}.$$

Proof. As in Whitehead's theorem! First construct $\overline{H} : X_{n+1} \times I \to Y$ then extend to H using the fact that $X_{n+1} \to X_n$ is cofibration. $\square$

Theorem 3.1 (Cellular Approximation). *If X, Y are given CW structures and $f : X \to Y$ is any map then $f \cong g$ with g cellular (relative version is also true).*

Proof. Use lemma 4 and same argument as in Whitehead's theorem (gluing homotopies). $\square$

What remains is

Proof of lemma 2. Given

$$\begin{tikzcd} S^n \arrow[r] \arrow[d] & X \arrow[d] \\ D^{n+1} \arrow[r, "f"'] & X \cup e^k \end{tikzcd}$$

where $k > n+1$. Consider $e^k \cong \mathbb{R}^k$. Since f is a map on a compact set, it is uniformly continuous. Hence there exist ε such that

$$|x - y| < \varepsilon \Rightarrow |f(x) - f(y)| < \frac{1}{2}.$$

Cut $D^{n+1} \cong I^{n+1}$ into cubes with diameters $< \varepsilon$.

$$figure(2)$$

Let $U = f^{-1}B(0, 1/2)$, $V = f^{-1}B(0,1)$ and let K_1 be the union of all cubes intersecting U. Then $K_1 \subseteq V$ and $f(K_1) \subseteq B(0,1)$ and let now K_2 be the union of all cubes which intersect K_1. Note K_1 is closed, ∂K_2 is closed and hence $\partial K_2 \cap K_1 = \varnothing$ and also $f(K_2) \subseteq B(0, 1\frac{1}{2}) \subseteq$

$\mathbb{R}^k$. So there exists some $\psi : K_2 \xrightarrow{0,1}$ and $\psi(\partial K_2) = 0$ and $\psi(K_1) = 1$ by normality of K_2. Triangulate K_2 refining the cubes.

$$fig(3)$$

Then let $g : K_2 \to \mathbb{R}^k \cong e^k$ be the linear map agreeing with f on vertices. Now we just give a homotopy:

$$H(x,t) = (1 - t\psi(x))f(x) + t\psi(x)g(y)$$

Let $h = H(-,1)$. Since H is a homotopy relative to ∂K_2 we can glue with constant homotopy to get

$$\overline{H} : D^{n+1} \to Y \cup D^k : f \cong h' \ \text{ rel } S^n$$

where $h'|_{K_2} = h$. Also $h'_{K_1} = g_{K_1}$. But g_{K_1} is linear on simplices of dimension $n+1$ and they are only finitely many mapping into $\mathbb{R}^k, k > n+1$. So $\text{im}(g)$ is in the union of finitely many hyperplanes in $\mathbb{R}^k$ so is not surjective. Thus $h' : D^{n+1} \to Y \cup D^k$ factors through D^k with a point removed:

$$\begin{tikzcd}
D^{n+1} \arrow[r, "h'"] \arrow[dr] & Y \cup D^k \arrow[d] \\
 & Y \cup D^k - \{u\}
\end{tikzcd}$$

So $h' \cong f$ rel S^n via

$$\begin{tikzcd}
S^n \arrow[r] \arrow[d] & Y \arrow[d] \\
D^{n+1} \arrow[r, "h'"] & Y \cup D^k - \{u\} \arrow[u, bend right, "r"']
\end{tikzcd}$$

the deformation retraction r. □

Remark. From the proof it follows that if f is already cellular on X_n then we can assume $f \cong g$ rel X_n.

Corollary 1. *If $Y = X \cup_{S^n} D^{n+1}$ and $n > 1$, $i : X \to Y$, then $\pi_*(i)$ is isomorphism for $* < n$ and is surjective for $* = n$.*

Proof. For surjectivity let $\alpha = [f] \in \pi_i(Y)$ for any $i \le n$. $f : S^i \to Y$ so $f \cong g$, $g : S^i \to Y$ that factors though X by lemma 2. Suppose $\alpha = [g] \in \pi_k(X)$ for $k < n$ such that $i_*(\alpha) = 0$ then we get diagram

$$\begin{tikzcd}
S^k \arrow[r, "g"] \arrow[d] & X \arrow[d] \\
D^{k+1} \arrow[r, "h"] & Y = X \cup D^{n+1}
\end{tikzcd}$$

since $k + 1 < n + 1$ again by lemma 2, $h \cong f$ rel S^k for $f : D^{k+1} \to X$ hence $[g] = 0$ in $\pi_k(X)$. □

Definition 4. (X, x_0) is n-connected if $\pi_i(X, x_0) = 0, \forall i \le n$.

Hence 0-connected is the same as path-connected and 1-connected is the same as simply connected.

Definition 5. Similary (X, A) is n-connected, if $\pi_n(X, A) = 0$ for every $i \le n$.

We are going to assume (as usual) that X and A are path connected. From the long exact sequence of pair (X, A)

$$\cdots \to \pi_i A \to \pi_i(X) \to \pi_i(X, A) \to \pi_{i-1} A \to \pi_{i-1} X \to \cdots$$

we get

Proposition 5. *(X, A) is n-connected if and only if*

$$\pi_i(A) \to \pi_i(X)$$

is an isomorphism for $i < n$ and is surjective if $i = n$.

Example 3.1. $(X \cup_{S^n} D^{n+1}, X)$ is n-connected from corollary 1. ⌟

Proposition 6 (CW approximation). *Say X is path connected. Then there is a (weak) CW-complex Y and weak homotopy equivalence $\eta_X : Y \to X$. (It probably works if X is not path-connected?)*

Remark. $CW(-)$ is a functor and $\eta : CW(-) \to \mathrm{id}$ is a natural transformation. In the above statement $CW(X) = Y$.

Before giving a proof we state for the record the following trivial statement:

Proposition 7. *$(Z, z_0) \mapsto (Z \coprod W, z_0)$ induces an isomorphism on $\pi_i, i \ge 1$ and injection on π_0.*

Proof of proposition 6. For step one consider

$$Y(1) = \coprod_{i \ge 0} \coprod_{f \in \mathrm{Hom}(S^i, X)} S^i.$$

Note that this is functorial. And let $f(1) : Y(1) \to X$ be the obvious map.

<u>Claim</u>: $f(1)$ is surjection on π_*. Precisely speaking, we mean that for any $x_0 \in X$ there is $y_0 \in Y(1)$ such that for any $\alpha \in \pi_*(X, x_0)$ there exists $\beta \in \pi_*(Y(1), y_0)$ satisfying $f(1)_*(\beta) = \alpha$. [1]

In fact let $\alpha = [g : S^i \to X]$ then the composition $S^i_g \to Y(1) \to X$ is g and $\pi_i(g)([\mathrm{id}_{S^i_g}]) = [g]$. Then apply the previous proposition.

[1] A way to get around with the technicality here will be considering the wedge product rather than disjoint union and consider only the pointed maps.

Assuming we have defined $Y(i)$ and $f(i) : Y(i) \to X$ for all $i \leq n$, compatible with the inclusions $k_{i,i+1} : Y(i) \to Y(i+1)$, we will construct $Y(n+1)$ as the pushout

$$\begin{tikzcd}
\coprod_i \coprod_{(f,H)} S^i \arrow[r] \arrow[d] & Y(n) \arrow[d, "k_{n,n+1}"] \arrow[dr, "f(n)"] & \\
\coprod_i \coprod_{(f,H)} D^{i+1} \arrow[r] & Y(n+1) \arrow[r, dashed] & X
\end{tikzcd}$$

where $f : S^i \to Y(n)$ and $H : D^{i+1} \to X$ make

$$\begin{tikzcd}
S^i \arrow[r, "f"] \arrow[d] & Y(n) \arrow[d, "f(n)"] \\
D^{i+1} \arrow[r, "H"] & X
\end{tikzcd}$$

commute. Define f^{n+1} in the obvious way. This is compatible with other maps and is functorial.

Claim: Suppose $\alpha \in \pi_i(Y(n))$ and $f(n)(\alpha) = 0$ as element of $\pi_i(X)$ (slightly different for π_0). Then $k_{n,n+1}(\alpha) = 0$.

Say $[\overline{\alpha}] = \alpha$. $f(n)(\alpha) = 0$ if and only if there is an extension

$$\begin{tikzcd}
S^i \arrow[r, "f(n)(\overline{\alpha})"] \arrow[d] & X \\
D^{i+1} \arrow[ur, dashed] &
\end{tikzcd}$$

which gives a commutative diagram

$$\begin{tikzcd}
S^i \arrow[r, "\overline{\alpha}"] \arrow[d] & Y(n) \arrow[d, "f(n)"] \\
D^{i+1} \arrow[r, "H"] & X
\end{tikzcd}$$

and hence a homotopy of $\overline{\alpha}$ to 0 in $Y(n+1)$.

As the last step of the construction, let

$$Y = \varinjlim Y(n), f = \varinjlim f(n).$$

Note that Y and f are both functorial.

Claim f is a weak homotopy equivalence.

In fact

$$\begin{tikzcd} Y(1) \arrow[r] \arrow[d, "f(1)"'] & Y \arrow[dl, "f"] \\ X & \end{tikzcd}$$

commutes so $f(1)$ is surjective. Consequently f is surjective. For injectivity (π_0 will be different but,) let $\alpha = [g] \in \pi_i(Y)$ and suppose $f_*(\alpha) = 0$. By compactness of S^n

$$\mathrm{Hom}(S^n, \varinjlim Y(i)) = \varinjlim \mathrm{Hom}(S^n, Y(i)).$$

So $g : S^i \to Y$ must factor through $Y(n)$ for some n. So we get a commutative diagram

$$\begin{tikzcd} & Y_n \arrow[dd, "j"] \arrow[dr, "k_{n,n+1}"] & \\ S^i \arrow[ur, "\overline{g}"] \arrow[dr, "g"'] & & Y_{n+1} \arrow[dl, "j'"] \\ & Y & \end{tikzcd} .$$

But $(k_{n,n+1})_*[\overline{g}] = 0$. So $j_*[\overline{g}] = 0$, thus $[g] = 0$. So $f : Y \to X$ is a weak homotopy equivalence. $\square$

Remark. Another construction would be to let adding only the n-cells in the n-th step. Using

$$\begin{tikzcd} \vee S^n \arrow[r] \arrow[d] & Y(n) \arrow[d] \\ \vee D^{n+1} \arrow[r] & Y(n+1) \end{tikzcd}$$

get functorial CW complex (not weak), hence also functorial cellular chains.

4. Hurewicz homomorphism

Consider $[f] \in \pi_n(X, A, x_0) = \pi_n(X, A)$ (or just $\in \pi_n(X)$) where $f : (D^n, \partial D^n) \to (X, A)$ or equivalently $f : S^n \to X$. Get

$$f_* : H_n(D^n, \partial D^n) \to H_n(X, A) \quad (\text{ or } f_* : H_n(S^n) \to H_n(X)).$$

Pick an isomorphism $\mathbb{Z} \cong H_n(D^n, \partial D^n)$ which is really just a choice of ± 1. Get the Hurewicz homomorphism

$$h : \pi_n(X, A) \to H_n(X, A)$$

via $\alpha = [f] \mapsto h(\alpha) := H_*(f)(1)$.

Remark. (1) This is well defined since $f \simeq g$ implies $H_* f = H_* g$.

(2) The isomorphism $H_n(D^n, \partial D^n) \cong \mathbb{Z}$ is needed in the definition so fix that.

Proposition 8. *(a) h is compatible with exact sequences of pairs (for π_* and H_*).*

(b) h is a homomorphism for $n > 1$.

Remark. $n = 1$ is the absolute case is done (abelianization of fundamental group).

Proof. (a) Straightforward, check it basically for the connecting homomorphism. (b) In the absolute case recall that concatenation corresponds to addition of the abelian group π_n. Alternatively $f+g$ is the composition of the pinch map $\psi : S^n \to S^n \vee S^n$ (mod out the equator in the first variable) to $S^n \vee S^n \xrightarrow{f\vee g} X \vee X \xrightarrow{\text{fold}} X$. The folding is explicitly

$$\operatorname{fold}(x_1, x_2) = \begin{cases} x_1 & x_2 = * \\ x_2 & x_1 = * \\ * & x_1 = x_2 = * \end{cases}$$

where the wedge product is represented as $X \vee X = \{(x_1, x_2) : x_1 = * \text{ or } x_2 = *\}$. Now have

$$\begin{aligned} h(f+g) &= (f+g)_*(1) = ((\text{fold})(f\vee g)\psi)_*(1) \\ &\overset{1}{=} \text{fold}_* f_*(1) + g_*(1) \overset{2}{=} f_*(1) + g_*(1) = hf + hg. \end{aligned}$$

Here the equality (1) follows from $\psi(1) = (1,1)$ and the identity $H_n(S^n \vee S^n) \cong H_n(S^n) \oplus H_n(S^n)$. In fact the following

$$\begin{tikzcd} & H_nS^n \oplus H_n(S^n) \arrow[r, "f_*\oplus g_*"] \arrow[d, "\cong"'] & H_n(X) \oplus H_n(X) \arrow[d, "\cong"] \\ H_n(S) \arrow[ur] \arrow[r, "\psi", "1\mapsto(1,1)"'] & H_n(S^n \vee S^n) \arrow[r, "(f\vee g)_*"'] & H_n(X \vee X) \end{tikzcd}$$

commutes since the square commutes on each factor. And (2) follows since $(fold)_* : H_n(X \vee X) \to H_*(X)$ is just $(f,g) \mapsto f+g$. For the relative case it is similar. □

Proposition 9 (Hurewicz's isomorphism). *If (X, A) is $(n-1)$-connected ($n \geq 2$), and X, A are simply connected and $A \neq \varnothing$ then*

$$h : \pi_n(X, A) \to H_n(X, A)$$

(where the first nontrivial groups happen) is isomorphism and $H_i(X, A) = 0$ if $i < n$.

We will prove this later using spectral sequences (or see Hatcher).

Note 4.1. Note that if $n = 1$ and we are in the absolute case ($A = *$) then

$$\pi_1(X) \to H_1(X)$$

is the abelianization. So we actually need $n \geq 2$ condition. ⌟

Example 4.2. In the $n = 2$ case with A not simply connected, say $A = S^1$ and $S^1 \to S^1 \vee S^2 = X$ then from the long exact sequence of pairs

$$0 = \pi_2(S^1) \to \pi_2(S^1 \vee S^2) \to \pi_2(S^1 \vee S^2, S^1) \to \pi_1(S^1)$$

but the second term is isomorphic to $\pi_2(\widetilde{S^1 \vee S^2})$, i.e. π_2 of the covering which is homotopy equivalent to $\bigvee_{i\in\mathbb{Z}} S^2$. Thus this term is infinitely generated but $H_2(S^1 \vee S^2, S^1) \cong \widetilde{H}_2(S^2) \cong \mathbb{Z}$. So we don't get an isomorphism between π_2 and H_2 of $(S^1 \vee S^2, S^1)$. ⌟

Example 4.3.

$$\pi_n(S^n) \cong \mathbb{Z}$$

and

$$\pi_n(S^n \vee S^n) \cong \pi_n(S^n) \oplus \pi_n(S^n) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

⌟

Proposition 10. *If X, Y are simply connected CW complexes and $f : X \to Y$, which is homology isomorphism then it is a weak homotopy equivalence (and consequently homotopy equivalence).*

Proof. We compare long exact sequences: Assume $X \to Y$ is a CW-pair (else consider the mapping cylinder and use $M(f) \simeq Y$). Consider the long exact sequence on π_* and H_*,

$$\begin{tikzcd}
\pi_n X \arrow[r] \arrow[d] & \pi_n Y \arrow[r] \arrow[d] & \pi_n(Y,X) \arrow[r] \arrow[d] & \pi_{n-1}(X) \arrow[r] \arrow[d] & \pi_{n-1}(Y) \arrow[d] \\
H_n(X) \arrow[r] & H_n(Y) \arrow[r] & H_n(Y,X) \arrow[r] & H_{n-1}(X) \arrow[r] & H_{n-1}(Y)
\end{tikzcd}$$

We know that $H_n(Y,X) = 0$ for all n, so by induction on n and from Hurewicz's homomorphism it follows that $\pi_n(Y,X) = 0$ for all n (note that we have assumed simply connectedness). Now finish using Whitehead's theorem. □

5. Moore spaces

Given an abelian group G, and $n \geq 2$, there exists (uniquely up to homotopy equivalence) a CW-complex $M(G,n)$, called a Moore space, such that

$$\widetilde{H}_k M(G,n) = \begin{cases} G & k = n \\ 0 & \text{else} \end{cases} \qquad \text{and} \qquad \pi_k M(G,n) = \begin{cases} G & k = n \\ 0 & i < n \end{cases}.$$

Notation: $P^n(p) = M(\mathbb{Z}/p, n-1)$.

A quick example: $M(\mathbb{Z}, n) = S^n$.

We now show the existence of the Moore space for $G = \mathbb{Z}/\ell$: Either consider the mapping cone of $S^n \xrightarrow{\times\ell} S^n$

$$\begin{tikzcd}
S^n \arrow[r, "\times\ell"] \arrow[dr] & S^n \arrow[r] \arrow[d, "\simeq"] & C(\times\ell) \arrow[d] \\
& M := M(\times\ell) \arrow[r] & M(\times\ell)/S^n
\end{tikzcd}$$

or the pushout

$$\begin{tikzcd}
S^n \arrow[r, "\times\ell"] \arrow[d] & S^n \arrow[d] \\
D^{n+1} \arrow[r] & M(G,n)
\end{tikzcd}$$

as definition. Check that these two definitions are the same. Get long exact sequence for the pair (M, S^n) from the diagram above:

$$H_i(S^n) \to H_i(M) \to H_i(M, S^n) = \widetilde{H}_i(M/S^n) \to H_{i-1}(S^n)$$

For $i \neq 0, n, n+1$ the vanishing of $H_i(M)$ is immediate. For $i = 0$ and $i = 1$ derive $H_i(M, S^n) = 0$ from the long exact sequence again, and for $i = n$

$$0 \to \mathbb{Z} \xrightarrow{\times\ell} \mathbb{Z} \to ? \to 0 \to 0$$

is exact and get $? = \mathbb{Z}/\ell\mathbb{Z}$.

For uniqueness suppose there is another simply connected CW-complex M' with

$$\widetilde{H}_k(M') = \begin{cases} \mathbb{Z}/\ell & k = n \\ 0 & \text{else} \end{cases}$$

By induction and Hurewicz M' is (n-1)-connected. By Hurewicz homomorphism again $\pi_k(M') \cong \mathbb{Z}/\ell$ so there is $S^n \xrightarrow{f} M'$ for which the induced $\pi_n(S^n) \to \pi_n(M')$ via $1 \mapsto [1]$ is surjection on π_n. Hence by Hurewicz it is also sujrective on H_n. On the other hand $0 = \ell[f] \in \pi_k(M')$, meaning we have a homotopy $H : \ell f \simeq 0$ completing the diagram

$$\begin{tikzcd}
S^n \arrow[r, "\times\ell"] \arrow[d] & S^n \arrow[d, "f"] \\
D^{n+1} \arrow[r, "H"'] & M'
\end{tikzcd}.$$

But $M(\mathbb{Z}/\ell, n)$ is a pushout so there is $\varphi : M(\mathbb{Z}, \ell) \to M'$ arising from the pushout. From the triangle

$$\begin{tikzcd}
S^n \arrow[d] \arrow[dr, "f"] & \\
M(\mathbb{Z}/\ell, n) \arrow[r, "\varphi"'] & M'
\end{tikzcd}$$

we get

$$\begin{tikzcd}
\mathbb{Z} \arrow[d, two heads] \arrow[dr, "f_*"] & \\
\mathbb{Z}/\ell \arrow[r, "\varphi_*"'] & \mathbb{Z}/\ell
\end{tikzcd}$$

on the n-th homology level. So φ_* is a surjection and thus an isomorphism. This completes the proof that φ_* is an isomorphism on H_*. Now apply the previous proposition.

6. Eilenberg-Maclane spaces/Postnikov approximations

PROPOSITION 11. *For all X (path-connected as always!) and for all n, there is $Y =: P_n(X)$ with $f : X \to Y$ such that*

$$\pi_i(Y) = \begin{cases} \pi_i(X) & i \leq n \\ 0 & i > n \end{cases}$$

and $\pi_i(f)$ is an isomorphism.

Note again that this construction can be made functorial.

SKETCH OF THE PROOF. Let $Y_n = X$. Assuming Y_k is defined, then Y_{k+1} is the pushout

$$\begin{tikzcd}
\bigvee_{g \in \mathrm{Hom}(S^k, Y_k)} S^k \arrow[r] \arrow[d] & Y_k \arrow[d] \\
\bigvee_{g \in \mathrm{Hom}(S^k, Y_k)} D^{k+1} \arrow[r] & Y_{k+1}
\end{tikzcd}$$

observe that $\pi_i(Y_k) = \pi_i X$ for $i \leq n$ and 0 if $k > i > n$. And let $Y = Y_\infty = \varinjlim Y_k$ and $f : X \to Y$ is the inclusion. □

PROPOSITION 12. *$X \to P_n X$ is relative CW complex and for all $X \to W$ such that $\pi_i(W) = 0$ if $i > n$ then there is a unique (up to homotopy) dashed extension*

$$\begin{tikzcd}
X \arrow[r] \arrow[dr] & P_n X \arrow[d, dashed] \\
& W
\end{tikzcd}$$

so we can think of the functor P_n as a localization.

PROOF. Exercise. □

PROPOSITION 13 (Eilenberg-Mac Lane spaces). *For all $n \geq 1$, and abelian group G there exists a unique (up to homotopy) CW complex $K(G, n)$ such that*

$$\pi_i\ K(G, n) = \begin{cases} G & i = n \\ 0 & else \end{cases}$$

PROOF. $n = 1$ is handled with $K(\pi, 1) = B\pi$ spaces (and G need not be abelian for that matter). So let $n > 1$ the existence is easy: let

$$K(G, n) = P_{n+1}M(G, n)$$

and this is OK since $\pi_n M(G, n) \cong H_n M(G, n) \cong G$. For uniqueness the idea is to use the last proposition: Suppose

$$\pi_i K' = \begin{cases} G & i = n \\ 0 & \text{else} \end{cases}$$

and construct the mapping $M(G, n) \to K'$ as in proof of existence of $M(G, n)$. Then there is $\varphi : P_{n+1}(M(G, n)) \to K'$ through which the former map factors and which is π_* equivalence and thus by Whitehead's theorem a homotopy equivalence. $\square$

Terminology: A product of EM spaces is called a GEM (*generalized Eilenberg- Mac Lane*) space.

Example 6.1. $\pi_i(\prod_i(K(G, n))) = \oplus\pi_i(K(G_i, i)) = G_i$ so we can construct spaces with any homotopy groups we want. ⌟

Example 6.2. $K(\mathbb{Z}, 1) = S^1$ and $K(\mathbb{Z}, 2) = \mathbb{C}P^\infty$. ⌟

CHAPTER 2

Fibrations

1. Fiber bundles and fibrations

DEFINITION 6. Recall $p : E \to B$ has the homotopy lifting property with respect to X if for any solid diagram below there is a dashed extension φ completing the commutative diagram:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & E \arrow[d] \\ X \times I \arrow[r] \arrow[ur, dashed, "\varphi"] & B \end{tikzcd}$$

DEFINITION 7. $p : E \to B$ is a (Hurewicz) fibration if p has HLP with respect to all X.

DEFINITION 8. $p : E \to B$ is called a Serre fibration if p has HLP with respect to all disks $D^n, n \geq 0$.

REMARK. $(D^n \times I, D^n \times \{0\} \cup S^{n-1} \times I)$ is homeomorphic to $(D^n \times I, D^n \times \{0\})$. Therefore p is a Serre fibration if and only if for every solid arrow diagram below there exists a map completing the commutative diagram:

$$\begin{tikzcd} D^n \times \{0\} \cup S^{n-1} \times I \arrow[r] \arrow[d] & E \arrow[d] \\ D^n \times I \arrow[r] \arrow[ur, "\varphi"] & B \end{tikzcd}$$

REMARK. Covering space $\Rightarrow$Hurewicz fibration $\Rightarrow$Serre fibration. Note that covering spaces are Hurewicz fibrations where homotopies lift uniquely.

Idea of fibration: Consider the pullback (B path-connected)

$$\begin{tikzcd} p^{-1}(b) \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ \{b\} \arrow[r, hook] & B. \end{tikzcd}$$

If p is fibration, all fibers (i.e. all $p^{-1}(b)$'s) should be homotopy equivalent and "vary continuously". This comes historically from the notion of fiber bundles:

DEFINITION 9. A fiber bundle structure on E with fiber F is a map $p : E \to B$ such that for all $b \in B$ there is a neighborhood U of b and homeomorphism $h : p^{-1}(U) \to U \times F$ over B. h is called a local trivialization (and there are not unique). B is the base space of the bundle and E is the total space of the bundle and F is the fiber. We will sometimes denote a fiber bundle with these data as $F \to E \xrightarrow{p} B$

Example 1.1. $p_1 : X \times Y \to X$ gives a fiber bundle structure. ⌟

Example 1.2. $E = I \times [-1,1]/ \cong$ with the relation $(0,v) \cong (1,-v)$. The $p : E \to S^1$ is a fiber bundle induced by $I \times [-1,1] \to (I \to S^1)$ and fibers are $[-1,1]$. ⌟

Example 1.3 (Projective spaces)**.**

$$\mathbb{C}^* \to \mathbb{C}^{n+1} - \{0\} \xrightarrow{p} \mathbb{C}^{n+1} - \{0\}/\mathbb{C}^*$$

with the standard trivialization $U_i = (x_i \neq 0)$. One can create analogous spaces using $\mathbb{R}$ or $\mathbb{H}$ (the field of quaternions). The unit vectors versions will respectively be

$$\begin{aligned} S^1 &\to S^{2n+1} \to \mathbb{C}\mathrm{P}(n) \\ S^3 &\to S^{4n+3} \to \mathbb{H}\mathrm{P}(n) \\ S^0 &\to S^n \to \mathbb{R}\mathrm{P}(n). \end{aligned}$$

⌟

THEOREM 1.1. *Suppose $p : E \to B$ is a Serre fibration. Choosing $b_0 \in B, x_0 \in F := p^{-1}(b_0)$ then*

$$p_* : \pi_n(E, F, x_0) \to \pi_n(B, b_0)$$

coming from the map of triples $(E, F, x_0) \xrightarrow{p} (B, b_0, b_0)$ is an isomorphism for all $n \geq 1$ and so there is a long exact sequence

$$\cdots \to \pi_n(F, x_0) \to \pi_n(E, x_0) \xrightarrow{p_*} \pi_n(B, b_0) \xrightarrow{\partial} \pi_{n-1}(F, x_0) \to \cdots$$

Note: ∂ has to be calculated via the isomorphism p_*

$$\begin{tikzcd} \pi_n(B, b_0) \arrow[r] & \pi_{n-1}(F, x_0) . \\ \pi_n(E, F, x_0) \arrow[u, "\cong"', "p_*"] \arrow[ur] & \end{tikzcd}$$

PROOF. We prove the surjectivity only: let $\alpha = [f] \in \pi_n(B)$ for $f : (I^n, \partial I^n) \to (B, b_0)$. Let $J = \partial I^n - \{s_n = 0\}$ as we had denoted before. We want $\widetilde{f} : (I^n, \partial I^n, J) \to (E, F, x_0)$. Define

$$\widetilde{f}|_J : J \to E \quad \text{via } \beta \mapsto x_0.$$

Note that $p\widetilde{f}|_J = f|_J$. E begin a Serre fibration, the following diagram completes

$$\begin{tikzcd} D^{n-1} \times \{1\} \cup S^{n-2} \times I = J \arrow[r, "\widetilde{f}|_J"] \arrow[d] & E \arrow[d, "p"] \\ D^n = I^n \arrow[r, "f"'] \arrow[ur, dashed] & B. \end{tikzcd}$$

So $p\widetilde{f}(\partial I^n) = b_0$ and $\widetilde{f}(\partial I^n) \subseteq p^{-1}(b_0) = F$ and so $[\widetilde{f}] \in \pi_n(E, F, x_0)$ since $\widetilde{f}(J) = x_0$ and also $p_*[\widetilde{f}] = [f]$. □

Proposition 14. *$p : E \to B$ is a Serre fibration if and only if p has the HLP with respect to all weak CW pairs.*

Proof. Sufficiency is obvious. For necessity given a Serre fibration $p : E \to B$ and a relative weak CW complex $X \hookrightarrow Y$ we want to complete

$$\begin{tikzcd} X \times I \cup Y \times \{0\} \arrow[r] \arrow[d] & E \arrow[d] \\ Y \times I \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

to a commutative diagram. Look at the rectangle below with the left square being a pushout

$$\begin{tikzcd} \coprod_{\alpha \in I(i)} S^{n(\alpha)} \times I \cup \coprod D^{n(\alpha)+1} \arrow[r] \arrow[d] & X(i) \times I \cup X(i+1) \times \{0\} \arrow[r] \arrow[d] & E \arrow[d] \\ \coprod D \times I \arrow[r] \arrow[urr, dashed] & X(i+1) \times I \arrow[r] & B \end{tikzcd}$$

So we get lifts $\varphi(i) : X(i+1) \times I \to E$ and then take the direct limit of these lifts. □

Proposition 15. *A fiber bundle $p : E \to B$ is a Serre fibration.*

Remark. If B is paracomplact then p is actually a fibration.

Proof. We want to complete the following

$$\begin{tikzcd} I^n \arrow[r] \arrow[d] & E \arrow[d] \\ I^n \times I \arrow[r, "H"'] \arrow[ur, dashed] & B \end{tikzcd}$$

to a commutative diagram. Break $I^n \times I$ into smaller boxes $I_{\alpha,j} := c_\alpha \times [t_j, t_{j+1}]$ and suppose $\alpha \in J$ where J is some nicely ordered indexing set. Let this partition be fine enough so that $H(I_{\alpha,j}) \subseteq U_{\alpha,j} \subseteq B$ so that $p|_{U_{\alpha,j}}$ is trivial and say $h_{\alpha,j} : p^{-1}(U_{\alpha,j}) \to U_{\alpha,j} \times F$ is trivialization. Inductively assume that the lift has been defined (compatibly) on each c_β for $\beta < \alpha$:

$$\begin{tikzcd}
c_\beta \arrow[r, "\subseteq"] \arrow[d] & I^n \arrow[r] & E \arrow[d] \\
c_\beta \times I \arrow[r, "\subseteq"] \arrow[urr] & I^n \times I \arrow[r, "H"'] & B
\end{tikzcd}$$

To extend to c_α consider the diagram

$$\begin{tikzcd}
D^n \times \{0\} \arrow[r, "\cong"] \arrow[d] \arrow[rrr, bend left, "\rho'"] & (\cup_{\beta<\alpha} c_\beta \cap c_\alpha) \times [t_j, t_{j+1}] \cup c_\alpha \times \{t_j\} \arrow[r] \arrow[d] & p^{-1}(U_{\alpha,j}) \arrow[r, "\cong", "h"'] \arrow[d] & U_{\alpha,j} \times F \arrow[dl, "\pi"'] \\
D^n \times I \arrow[r, "\cong"] \arrow[urr, dashed] \arrow[rr, bend right, "\rho"'] & c_\alpha \times [t_j, t_{j+1}] \arrow[r] & U_{\alpha,j} &
\end{tikzcd}$$

also assume $\varphi_\alpha : c_\alpha \times I \to E$ has been defined for $t \le t_j$ so constructing ψ will define $\varphi|_\alpha(-, t)$ for $t \le t_{j+1}$ putting them together gives φ_α gluing together for all α gives lift $\varphi : I^n \times I \to E$. One can check that

$$\psi(d,t) = h^{-1}(\rho(d,t), \pi_2\rho'(d))$$

works. Note that

$$h\psi((d,0)) = (\rho(d,0), \pi_2\rho'(d)) \overset{☺}{=} (\pi_1\rho'(d), \pi_2\rho'(d))$$

where ☺ equality is because the solid arrow diagram commutes. □

Example 1.4. Compute $\pi_3(S^2) \cong \mathbb{Z}$ using the fibration $S^1 \to S^3 \xrightarrow{\eta} \mathbb{C}P(1) = S^2$. So η is a fiber bundle hence is a Serre fibration. So we look at the long exact sequence on π_*

$$\pi_e(S^1) \to \pi_3(S^3) \to \pi_3(S^2) \to \pi_2(S^1) \to \pi_2(S^3)$$

Since the unviersal cover of S^1 is $\mathbb{R}$, the $\mathbb{R} \to S^1$ induces an isomorphism on π_i for $i \ge 2$ (since it is a convering map) so $\pi_i(S^1) = 0$ if $i \ge 2$. Hence $\pi_3(S^3) \xrightarrow{\pi_3} \pi_3(S^2)$ is isomorphism. By Hurewicz $\pi_3(S^3) \cong \mathbb{Z}$ so $\pi_3(S^2) \cong \mathbb{Z}$. In fact $\pi_i(S^3) \cong \pi_i(S^2)$ for i≥ 3. ⌟

Example 1.5. Similarly consider the fiber bundle $S^3 \to S^7 \to S^4$. Gives exact sequence

$$\pi_n(S^3) \to \pi_n(S^7) \to \pi_n(S^4) \to \pi_{n-1}(S^3)$$

but the first map factors through zero since $S^3 \to S^7 \simeq *$ by cellular approximation so we get short exact sequences

$$0 \to \pi_n(S^7) \to \pi_n(S^4) \to \pi_{n-1}(S^3) \to 0.$$

In fact all of these short exact sequences are split. So $\pi_n(S^4) \cong \pi_n(S67) \oplus \pi_{n-1}(S^3)$. In fact $\Omega S^4 \cong \Omega S^7 \times S^3$ which we will turn back to later! ⌟

Let X be a CW complex. Let $f \in \pi_n X$. Then we can get $\Sigma f \in \pi_n(\Sigma X)$ where $\Sigma X = X \times I/(x,1) \simeq (x,0) \simeq (*,t)$ is the reduced suspension. Σ is a homotopy preserving functor (i.e. $f \simeq g \Rightarrow \Sigma f \simeq \Sigma g$

$$\Sigma : Top_* \to Top_*.$$

So $\alpha = [g] \in \pi_n(S)$ induces $\Sigma\alpha := [\Sigma f] \in \pi_{n+1}(\Sigma X)$.

THEOREM 1.2 (Freudenthal suspension). *Suppose the connectivity of X is at least n,* conn $X \geq n$.*Then $\Sigma : \pi_n(X) \to \pi_{n+1}(\Sigma X)$ is an isomorphism for $i \geq 2n$ and surjective for $i = 2n+1$.*

The functor $\Omega : CG_* \to CG_*$ is the functor: $\Omega X = \mathrm{Map}_*(S^1, X)$ with compact open topology. Then Σ is the left adjoint to Ω:

$$\Theta : \mathrm{Hom}_{CG_*}(\Sigma X, Y) \cong \mathrm{Hom}_{CG_*}(X, \Omega Y)$$

in fact this correspondence is a hemeomorphism. Let us denote Θf by f^a. Then

$$\mathrm{id}^a_{\Sigma X} = \Theta(\mathrm{id}_{\Sigma X}) \in \mathrm{Hom}(X, \Omega\Sigma X)$$

where $[f] \in \pi_n(X) = [\Sigma S^{n-1}, X]$ and $[f^a] \in \pi_{n-1}(\Omega X)$. Theta in fact induces an isomorphism

$$\pi_n X \to \pi_{n-1}(\Omega X)$$

via $[f] \mapsto [\mathrm{id}^a_{\Sigma X}.f] \in \pi_n(\Omega\Sigma X) \cong \pi_{n+1}(\Sigma X)$ which is just Σf.

Recall that $f : X \to Y$ can be turned into a cofibration

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr] & M(f) \arrow[d, two heads] \\ & Y \end{tikzcd}$$

where the injection $X \hookrightarrow M(f)$ is a cofibraiton and $M(f) \twoheadrightarrow Y$ is a fibration. Given $X \xrightarrow{f} Y$. Let

$$E_f = \{(x,p) : f(x) = p(0)\} \subseteq X \times Y^I$$

be the pullback

$$\begin{tikzcd} X \times Y^I \arrow[r] \arrow[d] & Y^I \arrow[d, "ev(0)"] \\ X \arrow[r] & Y \end{tikzcd}$$

Now $p(1) : E_f \to Y$ via $(x,p) \mapsto p(1)$ is a fibration. So we get a commutative diagram

$$\begin{tikzcd} & E_f \arrow[dr, "p(1)"] & \\ X \arrow[ur, "*"] \arrow[rr, "f"'] & & Y \end{tikzcd}$$

where $*$ is a strong deformation retract. Then $p(1)^{-1}(y)$ is a *homotopy fiber* of f. If Y is path connected the homotopy type of $p(1)^{-1}(y)$ is independent of y.

CHAPTER 3

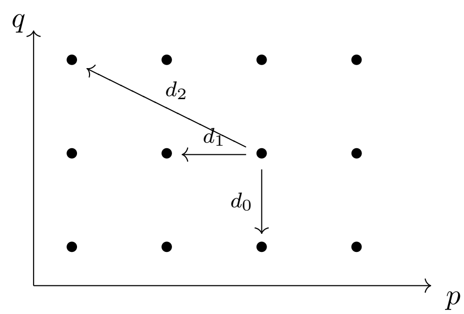
Spectral Sequences

Consider a sequence of bigraded modules (vector spaces) E^r_{pq} $r \geq n \geq 0$ for some starting point n (usually 0, 1 or 2), together with differentials d^r with degree $(-r, r-1)$,

$$d^r : E^r_{pq} \to E^r_{p-r,q+r-1}$$

satisfying $d^r \circ d^r = 0$ and isomorphisms $E^{r+1}_{pq} \cong H_{pq}(E^r, d^r) = \ker d^r_{pq} / \operatorname{im} d^r_{p+r,q-r+1}$. The sequence $E^+ = \{E^r_{pq}, d^r_{pq}\}_r$ is called a spectral sequence. These form a category where arrows are sequences of bigraded maps $E^r_{pq} \to E^{r'}_{pr}$ compatible with the differentials.

E^+ is a first quadrant spectral sequence if $E^r_{pq} \neq 0 \Rightarrow p \geq 0$ and $q \geq 0$.



Total degree of $\alpha \in E^r_{pq}$ is $p+q$. Thus the total degree of d^r is always -1. Suppose E^+ is first quadrant. Then for all p, q, E^r_{pq} stabilizes, i.e. given p, q there is k such that, $r \geq k \Rightarrow E^r_{pq} \cong E^k_{pq}$. We call these terms E^∞_{pq}.

We will consider the first quadrant case from now on. If there exists a finite filtration $0 = F_{-1}H_n \subseteq F_0 H_n \subseteq \cdots \subseteq F_t H_n = H_n$ then

$$E^\infty_{pq} = F_p H_{p+q} / F_{p-1} H_{p+q}$$

is a spectral sequence for which $E^r_{pq} \Rightarrow H_{pq}$.

Example 0.6 (Leray-Serre spectral sequence)**.** Let $F \xrightarrow{i} E \xrightarrow{\pi} B$ be a Serre fibration. We want B to be simply connected. Then there exists a first quadrant spectral sequence [Serre's PhD] with

$$E^2_{pq} = H_p(B, H_q(F)) \Rightarrow H_{p+q}(E)$$

i.e. $\oplus_{p+q=n} E^{\infty}_{pq} = H_n(E)$ up to extensions (with is equality if we work over a field). Also note that

$$H_p(B, H_q(F)) \cong H_pB \otimes H_q(F) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H_{p-1}B, H_qF)$$

where the Tor term is zero if we are over a field.

Consider turning $* \to X$ into a fibration, i.e.

$$PX = * \times_X X^I = \{\varphi : I \to X : \varphi(0) = *\}$$

where $\pi = \mathrm{ev}_1 : PX \to X$ is the evaluation map at 1, $\varphi :\mapsto \varphi(1)$ which is a fibration. And $\pi^{-1}(*) = \Omega X$. So we get the path-loop fibration

$$\Omega X \to PX \to X.$$

Suppose X is path connected in which case PX is contractible, $PX \simeq \{*\}$.

Now we apply the Leray-Serre spectral sequence when $X = S^n$:

$$E^2_{pq} \Rightarrow H_{p+q}(E) = 0 \text{ unless } n = q = 0.$$

Here $E^2_{pq} = H_pB \otimes H_qF$ since H_pB is free and $H_0F \cong \mathbb{Z}$ from the long exact sequence of homotopy. So the second page looks like this:

$$\begin{array}{c|cccc}
q & & & & \\
2 & ? & 0 & 0 & ? \\
1 & ? & 0 & 0 & ? \\
0 & \mathbb{Z} & 0 & 0 & \mathbb{Z} \\
\hline
 & 0 & 1 & n-1 & n \quad p
\end{array}$$

We know that $E^{\infty}_{n0} = 0$ so there is r such that $d^r_{n0}(1) \neq 0$ for some r and is fact is an isomorphism. So

$$E^2_{0,n-1} \cong \mathbb{Z}$$

But that means that $\mathbb{Z} \otimes H_{n-1}F \cong H_0B \otimes H_{n-1}F \cong E^2_{0,n-1} \cong \mathbb{Z}$ so $H_{n-1}F \cong \mathbb{Z}$. Also derive that $E^2_{*,i} = 0$ if $0 < i < n-1$.

By same argument $H_{2n-2}F \cong \mathbb{Z}$ and in general

$$H_iF \cong \begin{cases} \mathbb{Z} & n-1 \mid i, i \geq 0 \\ 0 & \text{else} \end{cases}.$$

Similarly if $\Omega X \simeq S^1$ get path loop fibration $S^1 \to PX \to X$. A spectral sequence argument shows that

$$H_i(X) \cong \begin{cases} \mathbb{Z} & 2 \mid i, i \geq 0 \\ 0 & \text{else} \end{cases}.$$

⌟

1. Construction of spectral sequences

There are generally two ways of getting spectral sequences: filtrations and exzact couples.

1.1. From exact couples. We will talk about exact couples here: An exact couple is a (non-commuting) diagram of modules

$$\begin{tikzcd} \mathcal{E}: D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

which consists of two modules and three maps and is exact at each vertex, e.g. $\ker j = \operatorname{im} i$. Let $d = jk$ and we get $d^2 = 0$. Then we get the derived couple

$$\begin{tikzcd} \mathcal{E}^1: iD \arrow[rr, "i'=i"] & & iD \arrow[dl, "j'=ji^{-1}"] \\ & H(E,d) \arrow[ul, "k=k'"] & \end{tikzcd}$$

that is the maps are determined by, $i' = i|_D, j'(i(d)) = [j(d)], k'[e] = ke$. Check that these are well-defined and $\mathcal{E}^1$ is an exact couple. One can iterate this procedure and get the r-th derive couple

$$\begin{tikzcd} \mathcal{E}^r: D^r = i^r(D) \arrow[rr, "i"] & & D^r \arrow[dl, "j^{(r)}=ji^{-r}"] \\ & E^r = H(E^{r-1}) \arrow[ul, "k"] & \end{tikzcd}$$

i.e $j^{(r)}i^r(d) = [j(d)]$. Suppose now that D, E are bigraded D_{pq}, E_{pq}, such that i has bidegree $(1,-1)$; $i : D_{pq} \to D_{p+1,q-1}$, j has bidegree $(0,0)$ and k has bidegree $(-1,0)$. Then $j^{(r)}$ has bidegree $(-r,r)$, d^r has bidegree $(-r-1,r)$. We have a spectral sequence (E^{r-1}_{pq}, d^r) as above.

Example 1.1 (Bockstein Spectral Sequence (BSS), due to Browder)**.** Consider the short exact sequence $0 \to \mathbb{Z} \xrightarrow{\times \ell} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/\ell \to 0$ where ℓ is a prime. Tensor this with C_*X (or any free chain complex) and get short exact seqeunce of complexes

$$0 \to C_*(X,\mathbb{Z}) \to C_*(X,\mathbb{Z}) \to C_*(X,\mathbb{Z}/\ell) \to 0.$$

The long exact sequence on H_* gives an exact couple

$$\begin{tikzcd}
H_*(X,\mathbb{Z}) \arrow[rr, "\times\ell"] & & H_*(X,\mathbb{Z}) \arrow[dl, "\pi_*"] \\
& H_*(X,\mathbb{Z}/\ell) \arrow[ul, "j"] &
\end{tikzcd} .$$

The associated spectral sequence is the BSS. ⌟

▸ Exercise 2. Calculate BSS replacing C_*X with the chain complex

$$\cdots \to 0 \to M_1 = \mathbb{Z} \xrightarrow{\ell^k} M_0 = \mathbb{Z} \to 0 \to \cdots$$

given any integer $k \geq 1$. We do the first cases here for illustration: when $k = 1$ the exact couple is

$$\begin{tikzcd}
H_*C \arrow[rr, "i"] & & H_*C \arrow[dl, "j"] \\
& H_*(C \otimes \mathbb{Z}/\ell) \arrow[ul, "k"] &
\end{tikzcd}$$

where k is the connecting homomorphism. In degree 0 we get

$$\begin{tikzcd}
\mathbb{Z}/\ell \arrow[rr, "0"] & & \mathbb{Z}/\ell \arrow[dl, "j", "\cong"'] \\
& \mathbb{Z}/\ell \arrow[ul, "k"] &
\end{tikzcd}$$

and in degree 1

$$\begin{tikzcd}
0 \arrow[rr] & & 0 \arrow[dl] \\
& \mathbb{Z}/\ell \arrow[ul] &
\end{tikzcd} .$$

Thus the BSS is given by

$$E_0^1 = \mathbb{Z}/\ell$$
$$E_1^1 = \mathbb{Z}/\ell, d = kj : E_1^1 \cong \mathbb{Z}/\ell \xrightarrow{\cong} E_0^1 \cong \mathbb{Z}/\ell$$
$$E_*^2 = 0.$$

In $k = 2$ case,

degree 1 degree 0

$$\begin{tikzcd}
0 \arrow[rr] & & 0 \arrow[dl] & \mathbb{Z}/\ell^2 \arrow[rr, "\times\ell"] & & \mathbb{Z}/\ell^2 \arrow[dl, "\pi"] \\
& \mathbb{Z}/\ell \arrow[ul] \arrow[urr, dashed, "\delta"] & & & \mathbb{Z}/\ell \arrow[ul, "\delta"] &
\end{tikzcd}$$

The differential is

$$\begin{aligned} d^1 = jk = \pi\delta : E_1^1 \cong \mathbb{Z}/\ell &\to \mathbb{Z}/\ell \cong E_0^1 \\ [1] &\mapsto \pi\delta[1] = \pi[\ell] = 0. \end{aligned}$$

Now passing to the derived couple is given by

$$\begin{tikzcd}
0 \arrow[rr] & & 0 \arrow[dl] & \mathbb{Z}/\ell \arrow[rr, "\times\ell"] & & \mathbb{Z}/\ell \arrow[dl, "ji^{-1}"] . \\
& \mathbb{Z}/\ell \arrow[ul] \arrow[urr, dashed, "{[\delta]}"'] & & & \mathbb{Z}/\ell \arrow[ul] &
\end{tikzcd}$$

with differential of the spectral sequence being

$$\begin{aligned} d^2 = j'k' = (\pi\frac{1}{\ell})[\delta] : E_1^2 \cong \mathbb{Z}/\ell &\to \mathbb{Z}/\ell \cong E_0^2 \\ [1] &\mapsto \pi\frac{1}{\ell}[\ell] = \pi[1] = [1] \end{aligned}$$

which is an isomorphism and therefore $E_*^3 = E_*^\infty = 0$.

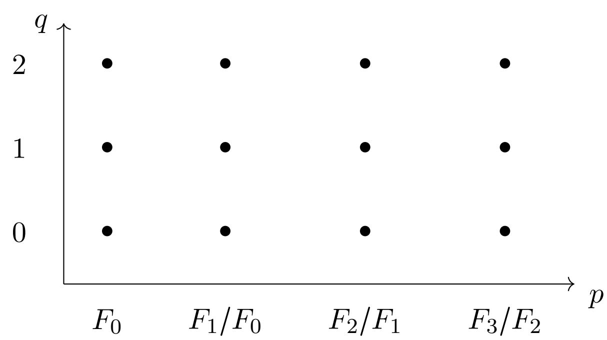
1.2. From filtrations on chain complexes. Let C be a chain complex. A filtration on C is a sequence of sub-chain-complexes $F_i = F_i(C)$

$$0 = F_{-1} \subseteq F_0 \subseteq \cdots \subseteq F_p \subseteq \cdots \subseteq C.$$

The filtration is *exhaustive* (or *co-omplete*) if $C = \cup F_pC$ (*complete* corresponds to the limit being 0). Also assume C is non-negatively graded so that we get a first quadrant spectral sequence. The zero page is

$$E^0_{pq} = (F_pC)_{p+q}/(F_{p-1}C)_{p+q}.$$

and d^0 is induced by the differentials on quotient.



So in first page $E^1_{p,q} = H_{p+q}(E^0_{p,*}, d^0)$. Now let

$$A_p^r = \{c \in F_pC : d(c) \in F_{p-r}C\}$$

i.e. $c \in F_p$ is a cycle mod $F_{p-r}C$; you may call is an *approximate cycle*. As a piece of notation also define

$$\eta_p : F_pC \to F_pC/F_{p-1}C = E^0_p$$

(and we are dropping q's from our notations). Then

$$\begin{aligned} Z^r_p &= \eta_p(A^r_p) \subseteq E^0_p \\ B^{r+1}_{p-r} &= \eta_{p-r}d(A^r_p) \subseteq E^0_{p-r} \\ Z^\infty_p &= \cap^\infty_{r=1} Z^r_p \\ B^\infty_p &= \cup^\infty_{r=1} B^r_p \end{aligned}$$

Observe that we have filtration

$$\begin{aligned} 0 \subseteq B^0_p \subseteq B^1_p \subseteq \cdots \subseteq B^\infty_p \\ \subseteq Z^\infty_p \subseteq \cdots \subseteq Z^0_p = E^0_p \end{aligned}$$

The r-th page would be Z^r_p/B^r_p:

$$\begin{aligned} A^r_p \cap F_{p-1}C &= A^{r-1}_{p-1} \\ Z^r_p &\cong A^r_p/A^{r-1}_{p-1} \\ E^r_p = Z^r_p/B^r_p &\cong \frac{A^r_p + F_{p-1}C}{dA^{r-1}_{p+r-1} + F_{p-1}(C)} \cong \frac{A^r_p}{d(A^{r-1}_{p+1}) + A^{r-1}_{p-1}} \end{aligned}$$

and the most important part defining the differential:

$$\begin{tikzcd} d^r_p : E^r_p \arrow[r, dashed] & E^r_{p-r} & \\ Z^r_p \arrow[u, two heads] \arrow[r, dashed, "d"] & Z^r_{p-r} \arrow[u, two heads] & \text{defined since } d^2 = 0 \\ A^r_p \arrow[u, two heads] \arrow[r, "d"'] & A^r_{p-r} \arrow[u, two heads] & d \text{ lands in here since } d^2 = 0 \end{tikzcd}$$

d^r_p is unique since the vertical maps are surjections. To see that d^r_p exists (making the diagram commute) it suffices that $d'(B^r_p) = 0$. But since $d\,\mathrm{im}(dA^{r-1}_{p+r-1}) = 0$ this is the case. Clearly $(d^r)^2 = 0$. Finally check that $E^{r+1} \cong H_*(E^r)$ to complete the construction of our spectral sequence.

Example 1.2. Let our chain complex be concentrated only in degrees 3 and 4, where

$$C_4 \cong \mathbb{Z}e_{31} \oplus \mathbb{Z}e_{22}, \quad C_3 = \mathbb{Z}e_{21} \oplus \mathbb{Z}e_{12}$$

and we have chosen the indices this way since if the differential $d : C_4 \to C_3$ is given via $(a,b) \mapsto (a+kb, b)$ then

$$F_1C = \mathbb{Z}e_{21}, F_2C = \mathbb{Z}e_{12} \oplus \mathbb{Z}e_{22} \oplus \mathbb{Z}e_{21}, F_3C = C$$

$$E^0: \begin{array}{c|cccc} q & & & & \\ & 0 & \mathbb{Z} & \mathbb{Z} & 0 \\ & 0 & 0 & \mathbb{Z} & \mathbb{Z} \\ & 0 & 0 & 0 & 0 \\ \hline & & & & p \end{array} \qquad E^1: \begin{array}{c|cccc} q & & & & \\ & 0 & \mathbb{Z} & 0 & 0 \\ & 0 & 0 & \mathbb{Z}/k & \mathbb{Z} \\ & 0 & 0 & 0 & 0 \\ \hline & & & & p \end{array}$$

$$E^2: \begin{array}{c|cccc} q & & & & \\ & 0 & \mathbb{Z} & 0 & 0 \\ & 0 & 0 & \mathbb{Z}/k & \mathbb{Z} \\ & 0 & 0 & 0 & 0 \\ \hline & & & & p \end{array} \qquad E^3: \begin{tikzcd} 0 & \mathbb{Z} & 0 & 0 \\ 0 & 0 & 0 & k\mathbb{Z} \arrow[ull] \\ 0 & 0 & 0 & 0 \end{tikzcd}$$

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▸ EXERCISE 3. Compute the differentials in the previous example.

In a filtration of positively graded chain complex C, that is co-complete we have the convergence $E^r \Rightarrow F_pH_{p+q}/F_{p+1}H_{p+q}$ and one can check that this is the same as the spectral sequence associated to the exact couple

$$\begin{tikzcd} H_*F_i \arrow[rr, "\text{shift}"] & & \oplus_i H_*F_{i+1} \arrow[dl] \\ & H_*(F_{i+1}/F_i) \arrow[ul] & \end{tikzcd} .$$

Let $\pi : E \to B$ be a fibration and assume that B has a CW structure. Let $f : B' \to B$ be a weak homotopy equivalence. The by long exact sequence on π_*, $f' : \pi^{-1}B' \to B'$ is also a weak homotopy equivalence. Let B_n be the n-skeleton of B. Consider the filtration

$$F_n = \operatorname{im} C_*(\pi^{-1}B_n)$$

where $\pi^{-1}B_n \to E$ is the pullback. Then $F_{-1} = 0$ and filtration is co-complete. Since

$$\begin{tikzcd} D^r \arrow[r] \arrow[dr, dashed] & E \\ & \pi^{-1}(B_n) \arrow[u, hook] \end{tikzcd}$$

factors for some n through $\pi^{-1}B_n$ since D^n is compact. So the associated spectral sequence is a Serre spectral sequence. As always

$$E^0_p = F_p/F_{p-1}$$
$$E^1_p = H_*(F_p/F_{p-1}).$$

Note 1.3 (Recall Cellular homology). Let $C^{\text{cell}}_n = \oplus_{\alpha\in I(n)}\mathbb{Z}_\alpha$ be the freely generated $\mathbb{Z}$-module generated by n-cells in a (not weak) CW structure. To define a map $C^{\text{cell}}_n X \to C^{\text{cell}}_{n-1}$. We have a mapping that is the composition

$$S^{n-1}_\alpha \xrightarrow{f(\alpha)} X_{n-1} \xrightarrow{q} X_{n-1}/X_{n-2} \cong \bigvee_{\beta\in I(n-1)} S^{n-1}_\beta \xrightarrow{p_\beta} S^{n01}_\beta.$$

Let $p_\beta \oplus \mathbb{Z} \to \mathbb{Z}$ be the projection. And consider $\deg(p_\beta q f(\alpha) 1_\beta)$. (Notice S^{n-1} is compact so this will be a finite sum.) This induces a differential $d_n : C^{\text{cell}}_n \to C^{\text{cell}}_{n-1}$. ⌟

The following squares are pullbacks:

$$\begin{tikzcd}
\pi^{-1}(B_{n-1}) \arrow[r] \arrow[d] & \pi^{-1}B_n \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\
B_{n-1} \arrow[r, hook] & B_n \arrow[r] & B
\end{tikzcd}$$

and we have short exact sequence $0 \to F_{p-1} \to F_p \to F_p/F_{p-1} \to 0$ in the form

$$0 \to C_*(\pi^{-1}B_{n-1}) \to C_*(\pi^{-1}B_n) \to C_*(\pi^{-1}B_m, \pi^{-1}B_{n-1}) \to 0.$$

So $H_*(F_p/F_{p-1}) \cong H_*(\pi^{-1}B_n, \pi^{-1}B_{n-1})$. But by excision

$$H_*(B_n, B_{n-1}) \cong H_n(\coprod_{\alpha\in I(n)} D^n, \coprod_{\alpha\in I(n)} S^{n-1})$$
$$H_*(\pi^{-1}B_n, \pi^{-1}B_{n-1}) \cong H_n(\coprod_{\alpha\in I(n)} \pi^{-1}D^n, \coprod_{\alpha\in I(n)} \pi^{-1}S^{n-1}).$$

Consider

$$\begin{tikzcd}
i_*D^n : \arrow[r, equal] & \pi^{-1}D^n \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\
& D^n \arrow[r, "i"] & B
\end{tikzcd}$$

$i \simeq * =: 0$ so by proposition 4.62 of Hatcher, i_*D^n is fiber homotopy equivalent to 0_*D^n, i.e. there are maps f, g making the following diagram commute.

$$\begin{tikzcd}
i_*D^n \arrow[rr, bend right, "f"'] \arrow[dr, "\pi"'] & & 0_*D^n \arrow[ll, bend right, "g"'] \arrow[dl, "\pi"] \\
& D^n &
\end{tikzcd}$$

where by homotopies over D^n, $gf \overset{H}{\simeq} \mathrm{id}_{i_* D^n}$ (for some homotopy H) and $fg \simeq \mathrm{id}_{0_* D^n}$. Thus this induces an equivalence of pairs

$$(i_* D^n, i_* S^{n-1}) \cong (0_* D^n, 0_* S^{n-1}) \cong (D^n \times F, S^{n-1} \times F)$$

where $F = \pi^{-1}(*)$.

Now

$$\begin{aligned} H_*(F_n, F_{n-1}) &\cong H_*(\coprod \pi^{-1} D^n, \prod \pi^{-1} S^{n-1}) \\ &\cong \oplus_{\alpha \in I(n)} H_*(\pi^{-1} D^n, \pi^{-1} S^{n-1}) \\ &\cong \oplus H_*(D^n \times F, S^{n-1} \times F) \\ &\cong C_n^{\mathrm{cell}}(B) \otimes H_* F. \end{aligned}$$

The last equality above holds because,

$$\begin{tikzcd} & & & & \mathbb{Z}_\alpha \otimes H_*(F) \arrow[d, equal, "\text{in degree n}"] & \\ & \cdots \arrow[r] & H_*(S^{n-1} \times F) \arrow[r] \arrow[d, equal, "\text{Kunneth}"] & H_*(D^n \times F) \arrow[r, "0"] \arrow[d, equal] & H_*(D^n \times F, S^{n-1} \times F) \arrow[r] & \cdots \\ 0 \arrow[r] & \widetilde{H}_*(S^{n-1}) \otimes H_*(F) \arrow[r] & H_*(S^{n-1}) \otimes H_*(F) \arrow[r] & H_*(F) \arrow[r] & 0 & \end{tikzcd}$$

Note 1.4. If the base is not simply connected the above argument does not work. (Then say at $n = 0$, in $S^{n-1} \times F$ we can have disjoint copies of F and the kernel is not the above one??)

So $E^1_{pq} = C_p^{\mathrm{cell}}(B) \otimes H_q(F)$ and $d^1 = d^{\mathrm{cell}} \otimes \mathrm{id}$ so

$$E^2 \cong H_*^{\mathrm{cell}} B \otimes H_* F \oplus \mathrm{Tor}(H_* B, H_* F)$$

ignoring the degrees.

▸ Exercise 4. Work out the "fibration" $S^1 \vee S^2 \to S^1$: You may turn this into a fibration $F \to \pi^{-1}(*) \to PS^1, \pi : PS^1 \to S^1$. Here F is homotopy equivalent to the covering space of $S^1 \vee S^2$.

CHAPTER 4

H-spaces and algebras

1. H-spaces

Definition 10. An H-space (H, φ) is a topological space H with base point e and a product map

$$\begin{aligned} \varphi : H \times H &\to H \\ (x, y) &\mapsto x.y \end{aligned}$$

making

$$\begin{tikzcd} H \vee H \arrow[d] \arrow[dr, "\text{fold}"] & \\ H \times H \arrow[r, "\varphi"'] & H \end{tikzcd}$$

commute relative to $\{e\}$ and only up to homotopy.

Definition 11. H is homotopy associative if

$$\begin{tikzcd} H \times H \times H \arrow[r, "\varphi \times \mathrm{id}_H"] \arrow[d, "\mathrm{id}_H \times \varphi"'] & H \times H \arrow[d, "\varphi"] \\ H \times H \arrow[r, "\varphi"'] & H \end{tikzcd}$$

commutes relative to $\{(e, e, e)\}$ and up to homotopy.

Definition 12. H has homotopy inverse $\widehat{\cdot} : H \to H$ if $\widehat{e} = e$ and

$$\begin{tikzcd} & H \times H \arrow[r, "\mathrm{id} \times \widehat{\cdot}"] & H \times H \arrow[dr, "\varphi"] & \\ H \arrow[ur, "\Delta"] \arrow[rrr, bend right, "\{e\}"'] & & & H \end{tikzcd}$$

commutes relative to $\{e\}$ and up to homotopy.

Example 1.1. Topological groups are homotopy associative H spaces with homotopy inverse (take all homotopies to be trivial). So are loop spaces ΩX: $\Omega X = \mathrm{Map}_*(S^1, X)$ with compact open topology. The *box product* is

$$\begin{aligned} \varphi : \Omega X \times \Omega X &\to \Omega X \\ (\alpha, \beta) &\mapsto \alpha \end{aligned}$$

traversing the first and then the second path with twice speed and

$$\begin{aligned} \overset{\frown}{\cdot} : \Omega X &\to \Omega X \\ \alpha &\mapsto \overline{\alpha} : t \mapsto \alpha(t-1). \end{aligned}$$

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Definition 13. A homotopy associative H-space with homotopy inverse is called an H-group.

One can dualize all the above notions:

Definition 14. A co-H-space is a pair (C, ψ) with C a topological space with base point $\{e\}$, and coproduct $\psi : C \to C \vee C$ making

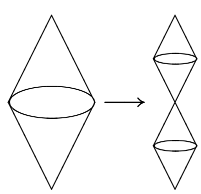
$$\begin{tikzcd} & C \vee C \arrow[d] \\ C \arrow[ur, "\psi"] \arrow[r, "\Delta"'] & C \times C \end{tikzcd}$$

commute up to homotopy.

Homotopy co-associativity and homotopy co-inverses are defined likewise. A co-H-group is a homotopy co-associative co-H-space with homotopy co-inverse.

Example 1.2. ΣX (the reduced suspension) is a co-H-group, where the coproduct is the pinching morphism along the equator:

$\psi :$



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Proposition 16. *(1) If Y is an H-group and X is any space, then $[X, Y]_*$ is a group.*

(2) If X is a co-H-group and Y is any space, then $[X, Y]_*$ is a group.

(3) If Y is an H-group and X is a co-H-group then $[X, Y]_*$ is abelian (and both multiplications in the above cases give the same result).

PROOF. In (1) the group operation is the composition

$$f.g : X \xrightarrow{\Delta} X \times X \xrightarrow{f\times g} Y \times Y \xrightarrow{\varphi} Y$$

and in (2)

$$f.g : X \xrightarrow{\psi} X \vee X \xrightarrow{f\vee g} Y \vee Y \xrightarrow{\text{fold}} Y.$$

For (3) look at [Bredon, p. 442]. □

REMARK. The question of studying pairs of spaces X and Y for which $\Omega X \simeq \Omega Y$ (delooping!) is related to the above ideas. In fact, recall the fibration $F = \Omega X \to PX \xrightarrow{ev} X$ and observe that $PX \simeq *$. The long exact sequence on π_* looks in part like

$$\cdots \to \pi_n(\Omega X) \to \pi_n(PX) = 0 \to \pi_n(X) \xrightarrow{\cong} \pi_n(\Omega X) \to \cdots$$

thus $\pi_n X \cong \pi_{n-1}(\Omega X)$. So it is hopeless to try to recover X and Y from the loop spaces. One need more structure on the spaces to tackle the problem.

2. (co)-Algebras

Fix R (or k), your favorite underlying commutative ring or field. An algebra is a triple (A, φ, μ) with <u>multiplication</u> $\varphi : A \otimes A \to A$ and <u>unit</u> $\mu : R \to A$.

$$\begin{tikzcd} R \otimes A \arrow[r, "\mu\otimes \mathrm{id}"] \arrow[dr, equal, "\sim"] & A \otimes A \arrow[d, "\varphi"] & A \otimes R \arrow[l, "\mathrm{id}\otimes\mu"'] \arrow[dl, equal, "\sim"'] \\ & A & \end{tikzcd}$$

We may want to have a grading on A and want φ to respect the grading, $\varphi(A^k \otimes A^s) \subseteq A^{k+s}$, $\mathrm{im}(\mu) \subseteq A^0$. φ is said to be associative:

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\varphi\otimes \mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\varphi"'] \arrow[dr, phantom, "\circlearrowleft"] & A \otimes A \arrow[d, "\varphi"] \\ A \otimes A \arrow[r, "\varphi"'] & A \end{tikzcd}$$

(without the position of parentheses matter) and is <u>(graded-)commutative</u> if

$$\begin{tikzcd} A \otimes A \arrow[rr, "T"] \arrow[dr] & & A \otimes A \arrow[dl, "\varphi"] \\ & A & \end{tikzcd}$$

commutes such that $T(\alpha \otimes \beta) = (-1)^{|\alpha||\beta|}\beta \otimes \alpha$, where $|.|$ denotes the degree.

A coalgebra is the dual object: a triple (C, Δ, η) where C is an R-module, $\Delta : C \to C \otimes C$ and $\eta : C \to R$ and everything is defined dually likewise! A (co)algebra is connected if $A^0 = R$ and it is non-negatively graded.

REMARK. When the coalgebra C is connected then for $\alpha \in C^n$

$$\Delta(\alpha) = \sum_{i+j=n} \alpha_i' \otimes \alpha_j'' \in (C \otimes C)_n = \oplus_{i+j=n} C_i \otimes C_j$$

$$\mathrm{id}_C \otimes \eta(\Delta(\alpha)) = \sum_{i+j=n} \alpha_i' \otimes \underbrace{\eta(\alpha_j'')}_{=0 \text{if } j\neq 0} = \alpha_n' \otimes \alpha_0'' = \alpha \otimes 1$$

So we can change α_n', α_0'' so that $\alpha_n' = \alpha$ and $\alpha_0'' = 1$. Similarly for the other factor. So we can write

$$\Delta(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha + \sum_{0<i<n} \alpha_i' \otimes \alpha_{n-i}''$$

and therefore we have a *reduced diagonal* $\overline{\Delta}$ given via

$$\overline{\Delta}(\alpha) = \Delta(\alpha) - (\alpha \otimes 1 + 1 \otimes \alpha).$$

If $\overline{\Delta}(\alpha) = 0$ then α is called *primitive*, and the R-module of such elements is denoted by $P(C)$.

Example 2.1. $H^*(X, R)$ is an associative commutative algebra and $H_*(X, R)$ is a co-associative co-commutative coalgebra and are connected if X is path-connected. ⌟

3. Hopf algebras

DEFINITION 15 (Hopf algebra). A Hopf algebra is a quintuple $(A, \varphi, \Delta, \mu, \eta)$ where (A, φ, μ) is an algebra and (A, Δ, η) is a coalgebra and the structures are compatible: φ is a coalgebra map (or equivalently Δ is an algebra map), i.e.

$$\begin{tikzcd}
A \otimes A \arrow[r, "\Delta\otimes\Delta"] \arrow[d, "\varphi"] \arrow[dr, "\Delta(A\otimes A)"] & A \otimes A \otimes A \otimes A \arrow[d, "\mathrm{id}\otimes T\otimes \mathrm{id}"] \arrow[dd, bend left=60, "\varphi_{A\otimes A}"] \\
A \arrow[dr] & A \otimes A \otimes A \otimes A \arrow[d, "\varphi\otimes\varphi"] \\
 & A \otimes A
\end{tikzcd}$$

commutes (but be aware that T introduces signs).

REMARK. In a Hopft algebra A, $P(A)$ is a Lie algebra with product

$$[\alpha, \beta] = \alpha\beta - (-1)^{|\alpha||\beta|}\beta\alpha$$

where the products on the right hand side are multiplication in A. For if $\alpha, \beta \in P(A)$

$$\begin{aligned}
\Delta[\alpha,\beta] &= \Delta(\alpha)\Delta(\beta) - (-1)\Delta\beta\delta\alpha \qquad \text{since } \Delta \text{ is linear and an algebra map} \\
&= (\alpha \otimes 1 + 1 \otimes \alpha)(\beta \otimes 1 + 1 \otimes \beta) \\
&\quad - (-1)^{|\alpha||\beta|}(\beta \otimes 1 + 1 \otimes \beta)(\alpha \otimes 1 + 1 \otimes \alpha) \\
&= \left(\alpha\beta - (-1)^{|\alpha||\beta|}\beta\alpha\right) \otimes 1 + 1 \otimes \left(\alpha\beta - (-1)^{|\alpha||\beta|}\beta\alpha\right) \\
&= [\alpha,\beta] \otimes 1 + 1 \otimes [\alpha,\beta].
\end{aligned}$$

Example 3.1. To be safe (?) let k be an underlying field and (X, φ) be a path-connected H-space. So we know that $H^0(X,k) \cong k$. Then $H^*(X,k)$ is a connected, commutative, Hopf algebra. If $X = \Omega Y$ or is a topological group then $H^*(X,k)$ will be co-associative: $\Delta : X \to X \times X$ induces the isomorphism fitting into the following diagram:

$$\begin{tikzcd}
H^*(X) \otimes H^*(X) \arrow[r, "\cong"] \arrow[dr, "\varphi"'] & H^8(X \times X) \arrow[d, "H^*(\Delta)"] \\
& H^*X
\end{tikzcd}$$

Since X is an H-space we also have $X \times X \xrightarrow{\varphi} X$ which induces

$$\begin{tikzcd}
H^*(X) \arrow[r, "H^*(\varphi)"] \arrow[rr, bend right, "\Delta"'] & H^*(X \times X) \arrow[r, "\cong"] & H^*(X) \otimes H^*(X) \arrow[l, bend right]
\end{tikzcd}$$

completing the construction of our Hopf algebra structure. Similarly $H_*(X,k)$ is a Hopf algebra. If $X = \Omega Y$ then this is associative, but typically not commutative. However if $X = \Omega^2 Y$ then $H_*(X,k)$ will be commutative. Here $\Omega^2 Y := \Omega\Omega Y$ fitting in the adjunction

$$[\Sigma^2 X, Y] \cong [\Sigma X, \Omega Y] \cong [X, \Omega^2 Y].$$

⌟

Note 3.2. Note that $\mathrm{Hom}(A \times B, Y) \cong \mathrm{Hom}(B, \mathrm{Hom}(A,Y))$ true as sets or as mapping spaces if spaces satify basic topological properties! Similarly

$$\mathrm{Map}_*(S^1 \wedge X, Y) \cong \mathrm{Map}_*(X, \mathrm{Map}_*(S^1, Y))$$

where $X \wedge Y = X \times Y / X \vee Y$ is the smash product. Since $\Sigma X \cong S^1 \wedge X$ we have

$$\mathrm{Map}_*(\Sigma X, Y) \cong \mathrm{Map}_*(X, \Omega Y)$$

which is compatible with homotopies and therefore

$$[\Sigma X, Y] \cong [X, \Omega Y]$$

i.e. Σ and Ω are adjoint functors (actually $\pi_0 \mathrm{Map}_*(\Sigma X, Y) = [\Sigma X, Y]$). ⌟

Let $R[\alpha]$ be the polynomial algebra over R on one generator α with positive even degree $|\alpha|$. Let α be primitive so that $\Delta\alpha = 1\otimes\alpha + \alpha\otimes 1$ and in particular

$$\begin{aligned}\Delta(\alpha^n) &= (\Delta\alpha)^n = (\alpha\otimes 1 + 1\otimes\alpha)^n \\ &= \sum_{i=0}^{n}\binom{n}{i}\alpha^i\otimes\alpha^{n-i}\end{aligned}$$

Note that the coproduct on generators always determine the coproduct. And here since this algebra is free the above formula gives a well-defined coproduct.

Let $E(\alpha) = \wedge_R[\alpha]$ be the exterior algebra on generator α with the degree of α being odd. And assume α is primitive. Then $E(\alpha)$ is a free graded commutative algebra on α. This will imply that the formula for Δ is well-defined.

Example 3.3. Say $\mathrm{char}(k) \neq 2$ and $|\alpha| = 2$ contrary to the above assumption. We want to have $\Delta(\alpha) = 1\otimes\alpha + \alpha\otimes 1$. Then

$$\begin{aligned}0 = \Delta(\alpha^2) &= (\Delta\alpha)(\Delta\alpha) \\ &= (1\otimes\alpha + \alpha\otimes 1)(1\otimes\alpha + \alpha\otimes 1) \\ &= \underbrace{1\otimes\alpha^2}_{=0} + \alpha\otimes\alpha + (-1)^{|\alpha||\alpha|}\alpha\otimes\alpha + \underbrace{\alpha^2\otimes 1}_{=0} \\ &= 2\alpha\otimes\alpha \neq 0.\end{aligned}$$

Therefore we do not get a Hopf algebra unless in characteristic 2. More generally $k[\alpha]/(\alpha^{p^k})$ with α primitive, determines a Hopf algebra if and only if $\mathrm{char}(k) = p$. ⌟

If A and B are Hopf algebras then so is $A\otimes B$ in the obvious way.

THEOREM 3.1. *If* $\mathrm{char}(k) = 0$*, A is a (graded-)commutative associative connected Hopf algebra. Then as algebras (and not Hopf algebras) we have an isomorphism*

$$A \cong E(a_1, \cdots, a_n, \cdots)\otimes k[b_1, \cdots, b_\ell, \cdots]$$

with $|a_i|$'s odd and $|b_j|$'s even.

Notation: We denote the above by $\bigwedge(a_1, \cdots, a_n, \cdots, b_1, \cdots, b_\ell, \cdots)$ the free graded commutative algebra on the a_i and b_j.

Example 3.4. We have seen using Serre's spectral sequence that $H_i(\Omega S^n, \mathbb{Z}) \cong \begin{cases}\mathbb{Z} & n-1|i \\ 0 & \text{otherwise}\end{cases}$.
Thus

$$H^i(\Omega S^n, \mathbb{Q}) \cong \begin{cases}\mathbb{Q} & r-1|i \\ 0 & \text{otherwise.}\end{cases}$$

The latter is a Hopf algebra so if n is odd we have $H^i(\Omega S^n, \mathbb{Q}) \cong \mathbb{Q}[a]$ with $|a| = n-1$.

Working over$R = \mathbb{Z}$ let $|a|$ be even and $|c_n| = n|a|$ then the *divided powers algebra*

$$\Gamma(a) = \oplus_{n\in\mathbb{Z}^{\geq 0}} c_n\mathbb{Z}$$

is defined via $c_0 = 1$, $c_1 = a$ and $c_ic_j = \binom{n}{j}c_n$. Then $\Delta(c_n) = \sum_{i+j=n} c_i \otimes c_j$. And note that $\Gamma(a) \cong k[a]$ if $\operatorname{char} k = 0$ (in fact a Hopf algebra isomorphism if the isomorphism is defined in a convenient way). Finally

$$\Gamma(a)^* = \operatorname{Hom}_{\mathbb{Z}}(\Gamma(a), \mathbb{Z}) \cong R[a]$$

as Hopf algebras. ⌟

We work out the proof only when A^n is finite dimensional for all n, i.e. A is of *finite type*. A CW-complex X is called finite type if H_*X is finite type so this is a concrete case where the following prove goes through.

Proof. Say A has a generating set $a_1, \cdots, a_n, \cdots$ with $|a_i| \leq |a_{i+1}|$. Set $A(0) = k$, and $A(n)$ the subalgebra (not sub-Hopf algebra) generated by $a_1, \cdots, a_n$. It turns out that $A(n)$ is also a sub-Hopf algebra:

$$\Delta(a_i) = a_i \otimes 1 + 1 \otimes a_i + \text{ something in } \bigoplus_{p+q=|a_i|, p>0} A^p \otimes A^q$$

so the q's above have to be $|q| < |a_i|$ therefore the extra terms are in

$$\oplus A^p \otimes A^q \subseteq A(n-1) \otimes A(n-1).$$

We conclude that $\Delta A(n) \subseteq A(n) \otimes A(n)$. Assume $A(n-1) \cong \gamma(a_1, \cdots, a_{n-1})$ if $|a_n|$ even. There is a surjection $A(n-1) \otimes k[a_n] \xrightarrow{f} A(n)$ (uses commutativity). It is injective as well: say $f(\alpha) = \sum_{0\leq i\leq t} \alpha_i a_n^i, \alpha_i \in A(n-1)$.

$$\begin{aligned}\Delta f\alpha &= \sum_{0\leq i\leq t} \Delta(\alpha_i a_n^i) \\ &= \sum_{0\leq i\leq t} (\Delta\alpha_i)(\Delta a_n)^i \quad \text{since } \textit{Delta} \text{ is an algebra map.}\end{aligned}$$

But observe that

$$(\Delta\alpha_i)(\Delta a_n)^i = (\alpha_i \otimes 1 + A(n-1) \otimes A(n-1) + 1 \otimes \alpha_i)(a_n \otimes 1 + A(n-1) \otimes A(n-1) + 1 \otimes a_n)^i.$$

so $\Delta f(\alpha) = \sum_{0\leq i\leq t} (\alpha_i \otimes 1)(a_n \otimes 1 + 1 \otimes a_n)^i$.

Let $I \subseteq A(n)$ be the ideal generated by $A(n-1)$ and a_n^2. Consider the composition

$$A(n) \xrightarrow{\Delta} A(n) \otimes A(n) \to A(n) \otimes A(n)/I$$

The image of $f\alpha$ under composition is therefore

$$\sum_{0\leq i\leq t} (\alpha_i \otimes 1)(i a_n^{i-1} \otimes a_n) + \sum_{0\leq i\leq t} (\alpha \otimes 1)(a_n^i \otimes 1)$$

so if $f(\alpha) = 0$ then $\Delta f(\alpha) = 0$ thus we should have

$$A(n) \in \sum_{1 \leq i \leq t} i\alpha_i a_n^{i-1} = 0.$$

If we take $\alpha \in \ker f$ of lowest non-zero degree the we have found an element of lower degree in the kernel which is a contradiction. So α has to be of degree zero. But f is injective on degree zero and we are done with proving the injectivity.

We conclude that f is an isomorphism. The case of $|a_n|$ being odd is similar, so taking unions (direct limits) we get that $A \cong \gamma(a_1, c \ldots, a_n)$. $\square$

Corollary 2. $H_i^*(\Omega X; \mathbb{Q}) \cong \wedge(a_1, \cdots, a_n, \cdots)$.

Remark. The quesiton of realizability of an algebra as the cohomology of a space or more specifically the loop space of some topological space in rational coefficient will be done later in this course.

$$\exists X, H^*(\Omega X, \mathbb{Q}) \cong \bigwedge(a_1, \cdots, a_n, \cdots).$$

The question is difficult integrally though. It is known however that any polynomial algebra $\mathbb{Z}[x_1, \cdots, x_k]$ can be realized with a space $H^*(X, \mathbb{Z}) \cong \mathbb{Z}[x_1, \cdots, x_k]$.

CHAPTER 5

Localization (rationalization)

There are two approaches: Serre classes ($\mathcal{C}$ classes) of 50's and localization of 60's. We will study the latter in what follows. All spaces will be simply connected, CW complexes.

DEFINITION 16. For a topological space X, suppose there is a space $X_{(0)}$ and a map $\eta : X \to X_{(0)}$ such that η induces an isomorphism on $H_*(-, \mathbb{Q})$, $\widetilde{H}_*(X_{(0)}, \mathbb{Z})$ is a vector space over $\mathbb{Q}$ and for all $f : X \to Y$ such that $\widetilde{H}_*(Y, \mathbb{Z})$ is a $\mathbb{Q}$-vector space, there is a unique **?**
map $\varphi : X_{(0)} \to Y$ (up to homotopy), making

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[dr, "f"'] & X_{(0)} \arrow[d, dashed, "\varphi"] \\ & Y \end{tikzcd}$$

commute up to homotopy. Then $X \to X_{(0)}$ is called the *rationalization* of X.

Aside: There's a construction to show that $\widetilde{H}_*(S^n, \mathbb{Z})$ is a $\mathbb{Q}$-vector space. This is a construction by Sullivan to make a *rational sphere* by a telescope construction from maps $S^n \xrightarrow{\times p} S^n$ for all primes appearing infinitely many times.

PROPOSITION 17. *There is a functorial localization $\eta_X : X \to X_{(0)}$ on the category of simply connected CW complexes.*

Denote $\eta_X(f)$ by $f_{(0)}$, then $f_{(0)}$ is homotopy equivalence if and only if $H_*(f) \otimes \mathbb{Q}$ is an isomorphism if and only if (using simply connectedness) $\pi_*(f) \otimes \mathbb{Q}$ is an isomorphism (for certain tensor product). Localizations preserves fibrations (everything is simply connected again).

Recall $K(\Pi, n)$ for abelian group Π:

$$\pi_i K(\Pi, n) = \begin{cases} \Pi & i = n \\ 0 & \text{otherwise.} \end{cases}$$

By this we mean a more general construction than what we presented above which covers additive abelian groups of infinite generation; In gist, take a presentation of any abelian group G, $0 \to \oplus Z \to \oplus Z \to G \to 0$. Consider a map $\vee S^n \to \vee S^n$ such that passing to

H_* gives the injective homomorphism of the sequence. Construct a space $K(G,n)$ by the gluing map.

If we assume $K(\Pi,n)$ is a CW complex then it is determined up to homotopy. We want to calculate the localizations of these spaces. Note that

$$\Omega K(\Pi,n) \sim K(\Pi,n-1). \tag{0.5}$$

In fact this is a weak homotopy equivalence from what we know. But a result of Milnor implies that loop spaces are CW-complexes and therefore by Whitehead's theorem, the computation of π_* as follows, results the existence of the homotopy equivalence 0.5 of CW-complexes. Using the long exact sequence on π_* for the fibration

$$\Omega K(\Pi,n) \to PK(\Pi,n) \to K(\Pi,n)$$

so $\pi_{*-1}\Omega K(\Pi,n) \cong \pi_* K(\Pi,n)$.[1]

Example 0.6 (Localization of $\mathbb{C}P^\infty$)**.** The fibrations $S^1 \to S^{2n+1} \to \mathbb{C}P^n$ form commutative diagrams with compatible actions:

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d, equal] & S^{2n+1} \arrow[r] \arrow[d] & \mathbb{C}P^n \arrow[d] \\ S^1 \arrow[r] & S^{2n+3} \arrow[r] & \mathbb{C}P^{n+1}. \end{tikzcd}$$

Taking the direct limit we get a fibration

$$S^1 \to S^\infty \to \mathbb{C}P^\infty.$$

We know that $\pi_*(\varinjlim(X_i)) \cong \varinjlim \pi_*(X_i)$ since S^n is compact, and since $\pi_*(S^n) \to \pi_*(S^{n+1})$ is the zero map we get that $\pi_*(S^\infty) = 0$ and therefore $S^\infty \sim *$ since S^∞ has the a natural CW-structure (or directly see that $S^\infty \sim *$). $S^1 \sim K(\mathbb{Z},1)$ as we know.[2] We conclude that $\Omega\mathbb{C}P^\infty \sim S^1$ and

$$\pi_i(\mathbb{C}P^\infty) = \begin{cases} \mathbb{Z} & i = 2 \\ 0 & \text{otherwise} \end{cases}$$

thus $\mathbb{C}P^\infty = K(\mathbb{Z},2)$.

Note also that $\pi_*(X_{(0)}) \cong \pi_* X \otimes \mathbb{Q}$ so $\mathbb{C}P^\infty_{(0)} = K(\mathbb{Q},2)$. So **?**

$$\mathbb{C}P^\infty_{(0)} \sim K(\mathbb{Q},2) \sim \Omega K(\mathbb{Q},3)$$

and in particular $\mathbb{C}P^\infty_{(0)}$ is a loop space. ⌟

Before doing another example we state a general fact that will be lectured on later:

[1] It is more generally the case that $\pi_{*-1}(\Omega X) \cong \pi_*(X)$ if X is simply connected.

[2] And in fact $S^1_{(0)} \sim K(\mathbb{Q},1)$.

THEOREM 0.2. *$K(\Pi, n)$ represents $H^*(-,\Pi)$: One can think of $[-, K(\Pi, n)]$ as a functor from the category of pointed homotopy classes of maps to groups. If Π is cyclic*

$$[-, K(\Pi, n)] \cong H^*(-,\Pi) \text{ via } f \mapsto f^*(\iota_n),$$

where $\iota_n \in H^n K(\Pi, n) \cong \pi_n K(\Pi, n) \cong \Pi$ is the generator.

Example 0.7 (Localization of ΩS^3)**.** We start with the observation that

$$H^*(\mathbb{C}P^\infty, \mathbb{Z}) \cong \mathbb{Z}[a], |a| = 2.$$

In fact $H^*(\mathbb{C}P^n, \mathbb{Z}) \cong \mathbb{Z}[a]/(a^{n+1})$ and $H^*(\mathbb{C}P^n, \mathbb{Z}) \to H^*(\mathbb{C}P^{n-1}, \mathbb{Z})$ maps a to a under the isomorphism above.

We want to find $H^*(\mathbb{C}P^\infty, \mathbb{Q}) \cong \bigwedge(a_1, \cdots, a_n, \cdots)$. Observe that

$$H^0(\mathbb{C}P^\infty, \mathbb{Q}) = \mathbb{Q}, \quad H^1(\mathbb{C}P^\infty, \mathbb{Q}) = 0, \quad H^2(\mathbb{C}P^\infty, \mathbb{Q}) \cong \mathbb{Q}.a, \quad \cdots.$$

Therefore there is an injection[3]

$$\mathbb{Q}[a] \to H^* \mathbb{C}P^\infty.$$

Checking dimensions implies that the map is also surjective:

$$\dim(\mathbb{Q}[a])_k = \begin{cases} 1 & 2|k, k \geq 0 \\ 0 & \text{otherwise} \end{cases}, \dim(H^* \mathbb{C}P^\infty)_k = \begin{cases} 1 & 2|k, k \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

So any injective (graded) map $\mathbb{Q}[a] \to H^* \mathbb{C}P^\infty$ is sujrective. On the other hand

$$H^*(\Omega S^3; \mathbb{Q}) \cong \mathbb{Q}[a'], |a'| = 2.$$

(Compute $H^*(\Omega S^3, \mathbb{Q})$ as a ring with the Serre spectral sequence; Then take the map of algebras $\mathbb{Q}[a'] \to H^*(\Omega S^3, \mathbb{Q}), |a'| = 2$ via $a' \mapsto \iota_2$. This is injective, and by dimension count of the $\mathbb{Q}$-vector spaces in all degrees it is surjective.)

By 0.2 there is a map $f : \Omega S^3 \to \mathbb{C}P^\infty$, with $f(a) = a'$. Since f^* is an algebra map it must be an isomorphism, after tensoring with $\mathbb{Q}$. Thus $\Omega S^3_{(0)} \sim K(\mathbb{Q}, 2)$. Really $H^*(f, \mathbb{Q})$ is an isomorphism, hence $H^*(f_{(0)}, \mathbb{Z})$ is isomorphism: This follows from the commutative diagram

$$\begin{tikzcd}
\widetilde{H}^*(X, \mathbb{Q}) \arrow[r, "\cong"] \arrow[d, "\cong"'] & \widetilde{H}^*(X_{(0)}, \mathbb{Z}) \cong \mathbb{Q} \arrow[d] \\
\widetilde{H}^*(Y, \mathbb{Q}) \arrow[r, "\cong"'] & \widetilde{H}^*(Y_{(0)}, \mathbb{Z}) \cong \mathbb{Q}
\end{tikzcd}$$

$f_{(0)}$ is a homotopy equivalence $\Omega S^3_{(0)} \xrightarrow{f_{(0)}} K(\mathbb{Q}, 2)$. ⌟

We will end this section with a generalization of the above example, that is the following

[3]When we omit the coefficient in cohomologies we are assuming rational coefficients unless stated otherwise.

THEOREM 0.3. *$\Omega S^{2n+1}_{(0)} \sim K(\mathbb{Q}, 2n)$ and $S^{2n+1}_{(0)} \sim K(\mathbb{Q}, 2n+1)$ for all $n \geq 1$.*

In fact, the computation in previous example motivates

$$H^*(K(\mathbb{Q}, 2n)) \cong \mathbb{Q}[a], |a| = 2n, H^*(K(\mathbb{Q}, 2n+1)) = E(a), |a| = 2n+1,$$

where $E(a)$ is the exterior algebra in one generator. We will be proving these in what follows.

LEMMA 5. *There exists $g : S^{2n+1}_{(0)} \to K(\mathbb{Q}, 2n+1)$ that is an isomorphism on π_{2n+1} and hence on H_{2n+1}.*

PROOF. $\pi_{2n+1}(K(\mathbb{Z}, 2n+1)) \cong \mathbb{Z}$ with generator ι_{2n+1}. Let $g' : S^{2n+1} \to K(\mathbb{Z}, 2n+1)$ be representing ι. Thus $\pi_{2n+1}(g')$ is an isomorphism. By $\pi_*(f_{(0)}) = \pi_* f \otimes \mathbb{Q}$, $g = g'_{(0)}$ is an isomorphism on $\pi_{2n+1} \cong H_{2n+1}$. □

PROPOSITION 18. *Assume $H^*(K(\mathbb{Q}, 2n)) = \mathbb{Q}[a], |a| = 2n$. Then*

(1) $f = \Omega g$ is a homotopy equivalence.

(2) g is a homotopy equivalence.

PROOF. We have a map between fibration sequences

$$\begin{tikzcd}
(\Omega S^{2n+1})_{(0)} \arrow[r] \arrow[d, "\cong"] & (PS^{2n+1})_{(0)} \arrow[r] \arrow[d, "\cong"] & S^{2n+1}_{(0)} \arrow[d, equal] \\
\Omega(S^{2n+1}_{(0)}) \arrow[r] \arrow[d] & PS^{2n+1}_{(0)} \arrow[r] \arrow[d] & S^{2n+1}_{(0)} \arrow[d] \\
\underset{\cong K(\mathbb{Q},2n)}{\Omega K(\mathbb{Q}, 2n+1)} \arrow[r] & PK(\mathbb{Q}, 2n+1) \arrow[r] & K(\mathbb{Q}, 2n+1)
\end{tikzcd}$$

Deriving the long exact sequence of π_* on the fibration sequence we get that f is an isomorphism on π_{2n} and consequently on H_{2n} and also on H^{2n}. $H^*(\Omega S^{2n+1}) \cong \mathbb{Q}[a'], |a'| = 2n$, hence $H^*(f)(a') = \alpha a$ for some $\alpha \neq 0$. Hence $H^*(f)$ is an isomorphism. Thus $H_*(f)$ (which is an algebra morphism) is an isomorphism so f is a homotopy equivalence by Whitehead's theorem.

For g use the same long exact sequence on π_* and note that $\pi_*(f)$ is an isomorphism in all dimensions. □

PROPOSITION 19. *Assuming $H^*(K(\mathbb{Q}, 2n+1)) \cong E(a), |a| = 2n+1$, we have*

$$H^*(K(\mathbb{Q}, 2n+2)) \cong \mathbb{Q}[b], |b| = 2n+2.$$

Proof. By our assumption

$$H_i(K(\mathbb{Q},2n+1)) = \begin{cases} \mathbb{Q} & i = 0, 2n+1 \\ 0 & \text{otherwise} \end{cases}.$$

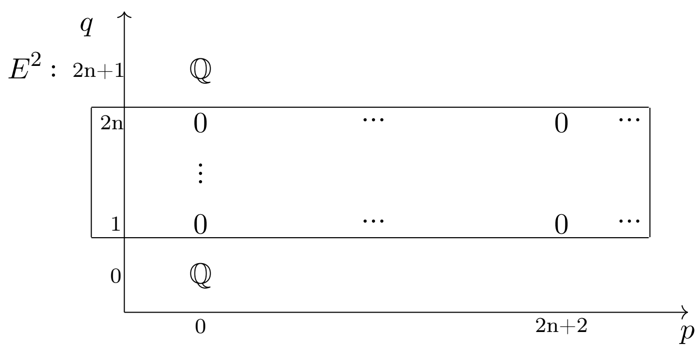
Consider Serre spectral sequence for homotopy fibration sequence

$$\underset{\sim \Omega K(\mathbb{Q},2n+2)}{K(\mathbb{Q},2n+1)} \longrightarrow \underset{\sim PK(\mathbb{Q},2n+2)}{*} \longrightarrow K(\mathbb{Q},2n+2)$$

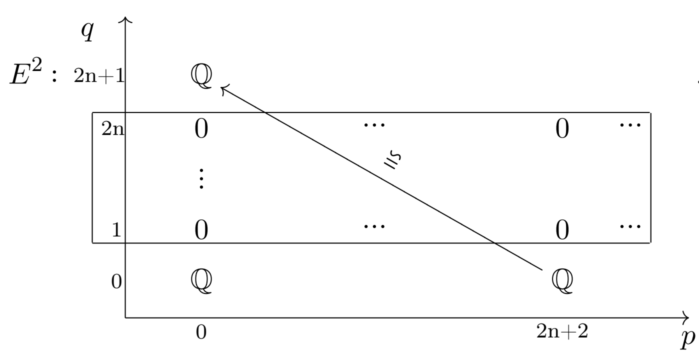
in which

$$E^2_{pq} \cong H_p(K(\mathbb{Q},2n+2)) \otimes H_q(K(\mathbb{Q},2n+1)) \Rightarrow H_*(*) \cong \mathbb{Q}.$$

So we know that the spectral sequence looks like



The only page at which $E_{0,2n+1}$ may degenerate at 0 is $r = 2n+1$. We also want $E_{2n+2,0}$ to degenerate at 0 on the same page (otherwise it will never stabilize at 0). Therefore the d^r-differential has to be the isomorphism shown below



.

Likewise analysis shows that all $E^2_{k,0}, 1 \le k \le 2n+1$ should already be zero in the second page.

Now from $E^2_{2n+2,0} = \mathbb{Q}$ we conclude that $E^2_{2n+2,2n+1} \cong \mathbb{Q}$ also. We can now use the same argument on the terms $E^2k, 0, 2n+3 \leq k \leq 4n+2$. Inductively we conclude that

$$H^i(K(\mathbb{Q}, 2n+2)) = \begin{cases} \mathbb{Q} & 2n+2|i, i \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Hence as a commutative, connected, Hopf-algebra over a field of characteristic zero

$$H^i(K(\mathbb{Q}, 2n+2)) \cong \mathbb{Q}[b], |b| = 2n+2$$

as desired. □

PROOF OF THEOREM 0.3. This is the result of the induction starting with

$$H^i(\mathbb{C}P^\infty_{(0)}) \cong \begin{cases} \mathbb{Q} & i \text{ even }, i \geq 0 \\ 0 & i \text{ odd} \end{cases}$$

and induction steps of propositions 18 and 19. □

So we have computed $\pi_*(S^{2n+1}) \otimes \mathbb{Q}$. What about the even dimensional spheres?

$$\pi_*(S^{2n}) \otimes \mathbb{Q} = ?$$

Consider $f : S^{2n}_{(0)} \to K(\mathbb{Q}, 2n)$ such that $\pi_{2n}(f)$ is an isomorphism. This map exists since $K(\mathbb{Q}, 2n)$ represents H^{2n}. Form the Serre spectral sequence associated to the fibration sequence

$$F \to S^{2n}_{(0)} \to K(\mathbb{Q}, 2n).$$

1. Cohomology Serre spectral sequence

Given a fibration sequence $F \to E \to B$ where B is simply connected and all spaces are path-connected, there exists a spectral sequence

$$E_2^{pq} \Rightarrow H^*(E)$$

with the E_2-terms

$$E_2^{pq} \cong H^p(B) \otimes H^q(F).$$

This is also a spectral sequence of algebras. In particular the differentials satisfy the Leibnitz rule

$$d^r(ab) = (d^r a)b + (-1)^{|a|} a d^r b, \tag{1.1}$$

where the product structure is induced by the product structure on the tensor product. Note that since d^r satisfies the Leibnitz rule **??**, E^{r+1} has an induced algebra structure.

1.1. Edge homomorphism. In the Serre spectral sequence of the fibration

$$F \xrightarrow{i} E \xrightarrow{p} B$$

as above (B simply-connected and F connected) we have the mappings

$$\begin{tikzcd} H_n(E) \arrow[r] \arrow[rrr, bend left, "H_n(p)"] & E^{\infty}_{n,0} \arrow[r] & E^2_{n,0} \arrow[r] & H_n(B) \\ \cap & \cong & & \\ F_n H & F_n H / F_{n-1} H. & & \end{tikzcd}$$

The composition $E^{\infty}_{n,0} \to E^2_{n,0} \to H_n(B)$ is called the *edge homomorphism.*

We wanted to find $\pi_i(S^{2n}) \otimes \mathbb{Q}$. Consider a fibration sequence

$$F \to S^{2n}_{(0)} \xrightarrow{f} K(\mathbb{Q}, 2n)$$

so that f is $\pi_{>n}$-isomorphism. Take the associated cohomology Serre spectral sequence:

$$E_2^{pq} \cong H^p K(\mathbb{Q}, 2n) \otimes H^q F.$$

We know that $H^0 F \cong \mathbb{Q}$ so

$$E_2^{p,0} \cong H^p K(\mathbb{Q}, 2n) \cong \mathbb{Q}[a], |a| = 2n.$$

Also the edge homomorphism

$$H^p B \xrightarrow{\cong} E_2^{p,0} \twoheadrightarrow E_{\infty}^{p,0} \to H^p(S^{2n}_{(0)})$$

tells us

$$E_{\infty}^{pq} \cong H^p S^{2n}_{(0)} = \begin{cases} \mathbb{Q} & p = 0, 2n \\ 0 & \text{otherwise} \end{cases}.$$

So in the second page of the spectral sequence, since $\operatorname{im} d^r \cap (E_r^{2n,0}) = 0$

$$H^q F = 0, 0 < q < 4n - 1.$$

So there is $\alpha \in E_{4n}^{0,4n-1}$ such that $d^{4n}\alpha \neq 0$ and hence $E_2^{0,4n-1} \cong \mathbb{Q}$. Therefore

$$E_2^{p,4n} \cong \begin{cases} \mathbb{Q} & 2n|p, p \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

So since this is a spectral sequence of algebras we can rewrite this computation as

$$E_2^{*,4n-1} \cong \mathbb{Q}[a] \otimes \mathbb{Q}\langle b \rangle, |a| = 2n, |b| = 4n - 1.$$

So since $d^r(a) = 0 = d^r(b)$ then $d^r(E_r^{*,4n-1}) = 0$ if $r < 4n$. Also since $d^{4r}(1 \otimes b) = a^2 \otimes 1$

$$d^{4n}(a^k \otimes b) = (d^{4r} a^k).b + (-1)^{|a^k|} a^k.d^{4r} b = a^{k+2} \otimes 1.$$

So we get that

$$E_\infty^{pq} \cong \begin{cases} \mathbb{Q} & p = 0, 2n, q \leq 4n - 1 \\ 0 & \text{otherwise.} \end{cases}$$

Consequently $E_2^{0,q} = 0, q \geq 4n$ so

$$H^i F = \begin{cases} \mathbb{Q} & i = 0, 4n - 1 \\ 0 & \text{otherwise.} \end{cases}$$

If $\alpha \in H^q F$, the $d^r(\alpha) = 0$ for all r: The only interesting cases are d^{2n} and d^{4n}. If $d^{2n}\alpha = c(a \otimes b), c \neq 0$ then we should have

$$d^{4n}(a \otimes b) = 0$$

but this is a contradiction by the computation we did: $d^{4n}(a \otimes b) = a^3 \otimes 1$. So we should have $d^{2n}(\alpha) = 0$. For the $4n$-page just use the fact that d^{4n} is a differential, so $d^{4n}(\alpha) = 0$. For all other r

$$d^r(\alpha) \in E_r^{r,q-r} = 0$$

but $E_\infty^{0,q} = 0$ if $q \neq 0$ (by edge homomorphism).

We conclude that

$$H_i(F) \cong H^i(F) \cong \begin{cases} \mathbb{Q} & i = 0, 4n - 1 \\ 0 & \text{otherwise.} \end{cases}$$

Claim $F \sim F_{(0)}$: Because in the long exact sequence on π_* for $F \to S^{2n}_{(0)} \to B_{(0)}$ we get that

$$\pi_*(F) \cong \pi_* F \otimes \mathbb{Q}.$$

Then

$$H^i(F, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Q} & i = 4n - 1 \\ 0 & \text{otherwise.} \end{cases}$$

So $F \sim M(\mathbb{Q}, 4n - 1) = S^{4n-1}_{(0)} = K(\mathbb{Q}, 4n - 1)$.

Now use long exact sequence on π_* for $F \to S^{2n}_{(0)} \to K(\mathbb{Q}, 2n)$ so

$$\pi_i(S^{2n}_{(0)}) = \begin{cases} \mathbb{Q} & i = 2n, 4n - 1 \\ 0 & \text{otherwise.} \end{cases}$$

This completes our comuptation.

CHAPTER 6

Loop suspensions

Recall we have adjoint functors Σ and Ω and unit of adjunction $X \to \Omega\Sigma X$. Let X be any pointed space (with base point $*$). Let JX be the free monoid on X with identity $*$, i.e.

$$JX = \{x_1 \cdots x_k : k \in \mathbb{N}, x_i \in X\} / \simeq$$

where the equivalence relation is $x_1 \cdots x_i \cdots x_n = x_1 \cdots \widehat{x_i} \cdots x_n$ if $x_i = *$. In fact let $J_n X \subset JX$ be the set of all k-tuples for all $k \leq n$ of JX:

$$J_n X = X^n / \simeq$$

with the quotient topology. Then

$$JX = \varinjlim J_n X,$$

and it is a *topological monoid* by concatenation. This is called *James' construction.* It is an associative H-space, with a *strict* identity; i.e. $\varphi(e, x) = x$ (not just upto homotopy).

There is an obvious inclusion $\eta : X \to JX$ via $x \mapsto x$. JX has the universal property that given $f : X \to H$ there is $\overline{f} : JX \to H$ making

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "f"'] & JX \arrow[d, dashed, "\overline{f}"] \\ & H \end{tikzcd}$$

commutes exactly. $\overline{f}$ is well-defined since e is strict (we define it by $(x_1 \cdots x_n) \mapsto (x_1(x_2(\cdots(x_n)\cdots)))$ or any other choice of parenthesis).

And if H is strictly associative then $\overline{f}$ is strictly multiplicative.

$$\begin{tikzcd} JX \times JX \arrow[r] \arrow[d, "\overline{f} \times \overline{f}"'] & JX \arrow[d, "\overline{f}"] \\ H \times H \arrow[r, "\varphi"] & H. \end{tikzcd}$$

In fact if H is strictly associative the choice of parenthesis does not matter since H is strictly associative.

We will show that JX is a model for $\Omega\Sigma X$ (i.e. it is homotopy equivalent to $\Omega\Sigma X$; in fact more it true, it is multiplicative homotopy equivalent to this space).

Definition 17 (Moore loops). Let $(X,*)$ be a based space. The More loops of X is
$$\Omega^m X = \{(\eta, s) : \eta : \mathbb{R}^{\geq 0} \to X, s \in \mathbb{R}^{\geq 0}, \eta(t) = * \text{ if } t \geq s \text{ or } t = 0\}$$
with subspace topology of $\mathrm{Map}(\mathbb{R}^{\geq 0}, X) \times \mathbb{R}^{\geq 0}$.

$\Omega^M X$ is associative H-space with strict identity. The structure is given as follows:
$$\begin{aligned} \Omega^M X \times \Omega^M X &\to \Omega^M X \\ (\eta, s) \times (\eta', s') &\mapsto (\eta \,\square\, \eta', s + s') \end{aligned}$$
where
$$\eta \,\square\, \eta'(t) = \begin{cases} \eta(t) & 0 \leq t \leq s \\ \eta'(t-s) & s \leq t \end{cases}$$
and $e = (e_*, 0)$, i.e. $e_*(t) = *$ is the identity map.

Consider the continuous mapping
$$\begin{aligned} f : \Omega X &\to \Omega^M X \\ \eta &\mapsto (\eta, 1) \end{aligned}$$
and the rescaling map
$$\begin{aligned} g : \Omega^M X &\to \Omega X \\ (\eta, s) &\mapsto \eta' : \eta'(t) = \eta(st). \end{aligned}$$

Proposition 20. *g and f are homotopy multiplicative inverse homotopy equivalences.*

Proof. $gf = \mathrm{id}_{\Omega X}$ and $fg \simeq \mathrm{id}_{\Omega^M X}$ by scaling to length 1. Rescaling also gives a homotopy
$$\begin{tikzcd} \Omega X \times \Omega X \arrow[r, "f \times f"] \arrow[d] & \Omega^M X \times \Omega^M X \arrow[d] \\ \Omega X \arrow[r, "f"'] & \Omega^M X \end{tikzcd}$$
by the top right composition one is mapping a path to some $(*, 2)$ and by the bottom left composition to $(*, 1)$. For the other direction let r be the rescaling $(\eta, s) \mapsto (\overline{\eta}, 1/2)$. Then the outer pentagon commutes exactly in
$$\begin{tikzcd} & \Omega^M X \times \Omega^M X \arrow[dr, "\mathrm{id}"] & \\ \Omega^M X \times \Omega^M X \arrow[ur, "r \times r"] \arrow[rr] \arrow[d, "g \times g"'] & & \Omega^M X \times \Omega^M X \arrow[d, "g"] \\ \Omega X \times \Omega X \arrow[rr] & & \Omega X. \end{tikzcd}$$
$\square$

Recall the map

$$\begin{aligned} &X \to \Omega\Sigma X \\ &x \mapsto \varphi, \varphi(t) = (x,t) \in \Sigma X = X \times I/\cong . \end{aligned}$$

This factors through $X \to \Omega^M \Sigma X$ via $x \mapsto (\varphi, 1)$. We need an extra hypothesis $* \to X$ is a cofibration (one says X is *well-pointed*).

Proposition 21. *Suppose $(X,*)$ is well-pointed and $f : X \to \Omega^M\Sigma X$ is the map above. Then $f \simeq g$ such that $g(*) = e$.*

Proof. Consider

$$\begin{tikzcd} * \times I \cup X \times \{0\} \arrow[r, "H\cup f"] \arrow[d] & \Omega^M \Sigma X \\ X \times I. \arrow[ur, dashed, "\widetilde{H}"'] & \end{tikzcd}$$

where H is a path in $\Omega^M\Sigma X$ from $f(*)$ to e: say $f(*) = (\varphi, 1)$ where $\varphi(t) = (*, t)$. Then

$$H(s) = (\varphi, 1 - s).$$

Now $\widetilde{H}$ making the diagram commute is our homotopy. □

Notation: Let $g : X \to \Omega^M\Sigma X$ be the mapping

$$g(x) = \widetilde{H}(x, 1)$$

and $\theta : JX \to \Omega^M\Sigma X$ its multiplicative extension.

We will show that $H_*(\theta)$ is an isomorphism.

Theorem 0.1. *Let k be any field. Then $H_*(\Omega\Sigma X; k) \cong T\widetilde{H}_*(X; k)$ as algebras.*

Recall that if V is a (graded) vector space, then $TV = \oplus_{0\le n<\infty} V^{\otimes n}$, called the *tensor algebra* and has the obvious multiplicative structure.

In fact in case of $X = S^n$ and $\operatorname{char} k = 0$, we know that

$$H^*(\Omega S^{n+1}, k) = k[a]$$

where a is primitive for degree reasons. So we know $H^*(\Omega S^{n+1}, l)$ as Hopf algebra and

$$(H^*(\Omega S^{n+1}, k))^* \cong H_*(\Omega S^{n+1}, k)$$

as Hopf algebras (universal coefficients theorem).

Recall that $X \wedge X = X \times X / X \vee X$ and from that

$$\widetilde{H}_*(X \wedge X) \cong \widetilde{H}_*(X) \otimes \widetilde{H}_*(X).$$

Proposition 22. *If X is connected, $\Sigma(X \times X) \cong \Sigma(X \vee X) \vee \Sigma(X \wedge X)$.*

PROOF. Consider the cofibration sequence

$$X \vee X \to X \times X \to X \wedge X. \quad (*)$$

By this we mean a sequence $A \hookrightarrow B \to C$ where C is a quotient of a cofibration $A \hookrightarrow B$. We have maps

$$\begin{aligned} p_i : X \times X &\to X \\ (x_1, x_2) &\mapsto x_i. \end{aligned}$$

Then $(*)$ gives a sequence

$$\begin{tikzcd} \Sigma(X \vee X) \arrow[rr] \arrow[dd, "\mathrm{id}"'] & & \Sigma(X \times X) \arrow[rr] \arrow[d, "\text{pinch twice}"] & & \Sigma(X \wedge X) \arrow[dd, "\mathrm{id}"] \\ & & \Sigma(X \times X) \vee \Sigma(X \times X) \vee \Sigma(X \times X) \arrow[d, "\Sigma p_1 \vee \Sigma p_2 \vee \Sigma \pi"'] & & \\ \Sigma X \vee \Sigma X \arrow[rr] & & \Sigma X \vee \Sigma X \vee \Sigma(X \wedge X) \arrow[rr] & & \Sigma(X \wedge X) \end{tikzcd}$$

The diagram commutes up to homotopy and the bottom row is cofibration sequence. So using long exact sequence on H_* of bottom and top cofibration sequences and the five lemma we get that

$$\Sigma(X \times X) \to \Sigma X \vee \Sigma X \vee \Sigma(X \wedge X)$$

is homology isomorphism and hence homotopy isomorphism since the suspension of connected spaces are simply connected. Note that we also using the fact that the (weak) product of CW complexes are CW complexes and Σ of CW complexes are CW complexes. Now we apply Whitehead's theorem. $\square$

REMARK. The above is a space level version of the homology decomposition

$$H_*(X \times X) \cong H_*(X) \otimes H_*(X)$$

the last one being only linearly equivalent to

$$H_*(X) \oplus \widetilde{H}_*(X) \oplus \widetilde{H}_*(X) \oplus \widetilde{H}_*(X \wedge X).$$

The idea is to start from a decomposition of algebraic invariants and prove a decomposition of objects with more structure.

Recall

$$T^n X = \{(x_1, \cdots, x_n) \in X^n : x_i = * \text{ for some } i \}$$

is the *flat wedge* of X.

PROPOSITION 23. *(1) Let X be a CW-complex, then the diagram*

$$\begin{tikzcd} T^n X \arrow[r] \arrow[d] & J_{n-1} X \arrow[d] \\ X^n \arrow[r] & J_n X \end{tikzcd}$$

is a pushout.

(2) $X^n/T^nX \cong J_nX/J_{n-1}X$.

(3) *Also* $T^nX \to X^n$ *is a cofibration so* $J_{n-1}X \to J_nX$ *is a cofibration.*

Proof. (1) is obvious. (2) follows since if

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & C \end{tikzcd}$$

is a pushout, then

$$\begin{tikzcd} C/A \arrow[r, "f"] & D/B \arrow[l, bend left, "\varphi"] \end{tikzcd}$$

is a homeomorphism. If φ exists such that the diagram commutes, φ induced by map $D \to C/A$ determind by projection $C \to C/A$ and

$$\begin{tikzcd} & A/A \arrow[dr] & \\ B \arrow[ur] \arrow[rr] & & C/A \end{tikzcd}$$

clearly agree on A so we get a map $D/B \to B/A$. Check f and φ are inverses. We skip the proof of (3). □

Proposition 24. $T^nX \to X^n \to X^{\wedge n}$ *is a cofibration sequence i.e.* $X^n/T^nX \cong X^{\wedge n}$.

Proof. Look at $X \vee Y \to X \times Y \to X \wedge Y = X \times Y/X \vee Y$. So

$$X^{\wedge n} = X^n/\cong$$

where the equivalence is generated by $(x_1, \cdots, x_n) \cong *$ if some $x_i = *$. This proves the assertion. □

Corollary 3. $J_{n-1}X \to J_nX \to X^{\wedge n}$ *is a cofibration sequence.*

Proof. Follows from two previous propositions. □

Since $J_1X = X$ we get

$$\begin{tikzcd} & H_*X \arrow[r] \arrow[dl] \arrow[d] & H_*JX \\ \widetilde{H}_*X \arrow[r] & T\widetilde{H}_*X. \arrow[ur, dashed, "\varphi"'] & \end{tikzcd}$$

So take φ to be the unique algebra extension.

Proposition 25. *If $H_*X = H_*(X,R)$ is a free R-module then the map φ above is an isomorphism.*

Proof. Let

$$T_n(\widetilde{H}_*X) = \oplus_{0\leq i\leq n}(\widetilde{H}_*X)^{\otimes i}$$

be the tensors of lengths $\leq n$.

Claims:

(1) $\varphi(T_n\widetilde{H}_*X) \subseteq H_*J_nX$.

(2) $\widetilde{H}_*(X)^{\otimes n} \hookrightarrow T_n\widetilde{H}_*(X) \xrightarrow{\varphi} H_*J_nX \to H_*(X^{\wedge n})$ is an isomorphism.

Considering how φ is defined we have

$$\begin{tikzcd}
(\widetilde{H}_*X)^{\otimes n} \arrow[r] & (H_*X)^{\otimes n} \arrow[r] \arrow[dr] \arrow[ddr] & H_*(J_1X)^{\otimes n} \arrow[r] \arrow[d] & H_*(JX)^{\otimes n} \arrow[d, "\text{multiply}"] \\
& & H_*(X)^{\otimes n} \arrow[ur] \arrow[d] \arrow[r, phantom, "\circ"] & H_*(JX) \\
& & H_*(X^n) \arrow[r] \arrow[ur] & H_*(J_nX) \arrow[u]
\end{tikzcd}$$

which proves claim (1).

For claim (2) look at

$$\begin{tikzcd}
\widetilde{H}_*(X)^{\otimes n} \arrow[r] \arrow[dr] \arrow[rr, bend left, "\dagger"] & H_*(X^n) \arrow[r] \arrow[d] & \widetilde{H}_*(X^{\wedge n}) \arrow[d, "\ddagger"] \\
& H_*(J_n(X) \arrow[r] & \widetilde{H}_*(X^{\wedge n})
\end{tikzcd}$$

$\dagger$ is an isomorphism by Kunneth formula and induction. $\ddagger$ is also an isomorphism since cofibers are heomoemorphic.

So we get maps between short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & T_{n-1}\widetilde{H}_*(X) \arrow[r] \arrow[d] & T_n\widetilde{H}_*X \arrow[r] \arrow[d] & \widetilde{H}_*(X)^{\otimes n} \arrow[r] \arrow[d, "\cong\text{by claim (1)}"] & 0 \\
0 \arrow[r] & H_*(J_{n-1}X) \arrow[r] & H_*(J_nX) \arrow[r] & \widetilde{H}_*(X^{\wedge n}) \arrow[r] & 0
\end{tikzcd}$$

where the vertical maps are induced by φ. So by induction on n we get that $T_n\widetilde{H}_*X \to H_*(J_nX)$ is an isomorphism.

So we have a sequence of isomorphisms

$$\begin{aligned} T\widetilde{H}_*X &\cong \varinjlim T_n\widetilde{H}_*X \\ &\cong \varinjlim H_*(J_nX) \\ &\cong H_* \varinjlim J_nX && \text{since } \varinjlim \text{ commutes with } H_* \\ &\cong H_*(JX) && \text{since } JX \coloneqq \varinjlim J_nX \end{aligned}$$

Therefore φ is an isomorphism. ☐

Proposition 26. *The map $JX \to \Omega\Sigma X$ from the last class is a homotopy equivalence.*

Proof. We are assuming everything is of finite type. We have a homotopy commuting diagram

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & JX \arrow[d, "\Theta"] \\ & \Omega\Sigma X \end{tikzcd}$$

where Θ is homotopy multiplcative. So taking H_* we get multiplcative map $H_*(\Theta)$. We have a commutative diagram

$$\begin{tikzcd} & T(\widetilde{H}_*(X)) \arrow[d, "\cong"] \\ H_*X \arrow[ur] \arrow[r] \arrow[dr] \arrow[ddr] & H_*(JX) \arrow[d, "H_*\Theta"] \\ & H_*(\Omega\Sigma X) \\ & T(\widetilde{H}_*(X)) \arrow[u, "\cong \text{ by Bott-Samuelson theorem}"'] \end{tikzcd}$$

so $H_*(Theta, k)$ is an isomorphism for any field k. So $H_*(\Theta, \mathbb{Z})$ is an isomorphism. So Θ is a homotopy equivalentce. Since X is a CW complex TX is a CW complex and $\Omega\Sigma X$ is homotopy equivalent to a CW-complex. ☐

Proposition 27. $\Sigma JX \cong \Sigma(\bigvee_{1\le i<\infty} X^{\wedge i})$.

Proof. Consider the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & T_nX \arrow[r] \arrow[d] & X^n \arrow[r] \arrow[d] & X^{\wedge n} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & J_{n-1}X \arrow[r] & J_nX \arrow[r] & X^{\wedge n} \arrow[r] & 0. \end{tikzcd}$$

The top row splits after suspension. Therefore so does the bottom row. So we have

$$\Sigma J_n X \simeq \Sigma J_{n-1} X \cup \Sigma X^n.$$

Now induction and passing to $\varinjlim$ gives the proposition. □

THEOREM 0.2 (Freudenthal suspension theorem). *Suppose the connectivity of X is $\geq n-1$. Then*

$$\pi_i X \to \pi_i(\Omega\Sigma X) \cong \pi_{i+1}(\Sigma X)$$

is an isomorphism for $i \geq 2n-1$ and is a surjection for $i \leq 2n-1$.

PROOF. Look at Serre spectral sequence for fibration

$$F \to X \to \Omega\Sigma X.$$

Then $E^2_{p,0} \cong T(\widetilde{H}_*(X))$. Lowest degree homology is at least $H_n(X)$ so the lowest degree homology in the cokernel will at least be

$$\widetilde{H}_n X \otimes \widetilde{H}_n X.$$

So connectivity of the fiber, F, is at least $2n-2$. Then we use the long exact sequence onπ_* to complete the proof. □

1. Another homotopy computation

Recall that for the fibration

$$F \xrightarrow{g} \Omega S^3 \xrightarrow{f} \mathbb{C}P^\infty = K(\mathbb{Z}, 2)$$

By Serre's method, g induces an isomorphism on π_i for $i > 2$ and $\pi_i(F) = 0$ for $i \leq 2$, and f induces isomorphism on π_2 and is zero on π_i for $i > 2$. As a result of applying Hurewitcz's theorem we had

$$\pi_3 F = H_3(F, \mathbb{Z}) = \pi_3(\Omega S^3) \cong \pi_4 S^3.$$

We now want to compute $H_3(F, \mathbb{Z})$. As an algebra

$$H_*(\Omega S^3, \mathbb{Z}) \cong \mathbb{Z}[a], \quad |a| = 2.$$

This is a consequence of Bott-Samuelson theorem: we have $\Omega S^3 = \Omega\Sigma S^2$ and hence

$$H_*(\Omega S^3) = T(\widetilde{H}_*(S^2, R)) = T(R.a) \cong R[a]$$

since $\widetilde{H}_*(S^2)$ is a free R-module.

Moreover as Hopf algebras we have an isomorphism

$$H^*(\mathbb{C}P^\infty, \mathbb{Z}) \cong \mathbb{Z}[a']$$

with a' primitive (just work out the diagonal $\Delta(a')$ to see this). Since

$$\Delta(a')^n = \sum \binom{i+j}{i} (a')^i \otimes (a')^j$$

the dual algebra is the *divided powers algebra*,

$$\Gamma(\alpha) := \oplus_{i\geq 0}\mathbb{Z}\alpha_i$$

where $\alpha_i = (a^i)^*$ and $\alpha_i\alpha_j = \binom{i+j}{u}\alpha_{i+j}$:

$$H_*(\mathbb{C}P^\infty,\mathbb{Z}) \cong (H_*(\mathbb{C}P^\infty,\mathbb{Z}))^* \cong \Gamma(\alpha) \quad \text{both as Hopf algebras.}$$

In $\mathbb{Z}/p$ coefficients we have $H_*(f,\mathbb{Z}/p)$ is an isomorphism for $* < 2p$ and is zero if $* = 2p$.

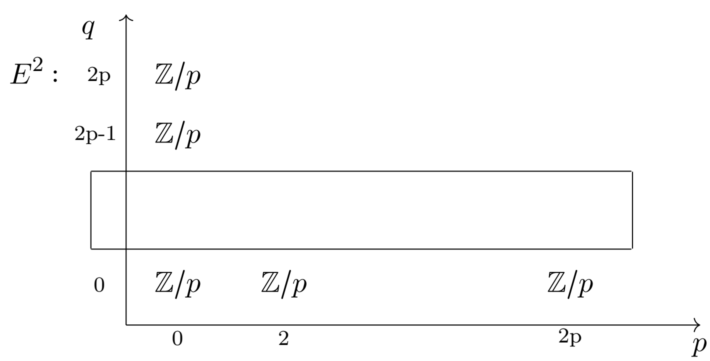
$$H_*(\Omega S^3,\mathbb{Z}/p) \cong \mathbb{Z}/p[a] \to \Gamma(\alpha)\otimes\mathbb{Z}/p \cong H*(\mathbb{C}P^\infty,\mathbb{Z}/p)$$
$$a^p \mapsto (\alpha_1)^p = p!(\alpha_p) = 0$$

Note 1.1. In cases of both coefficients $H_*(f)$ is algebra map, since $f = \Omega g$. So f is a Hopf algebra map. ⌟

In the next step we look at Serre's spectral sequence with coefficients in $\mathbb{Z}/p$: in the second page we have

$$E_{\ell,0} \cong H_\ell B \cong H_\ell\mathbb{C}P^\infty.$$

so



The edge homomorphism is

$$H_\ell(\Omega S^3) \to E^\infty_{\ell,0} \cong E^2_{\ell,0} \to H_2(\mathbb{C}P^\infty)$$

which is an isomorphism. Hence

$$E^\infty_{2p,0} = 0.$$

So on second page we need to have

$$E^2_{0,2p-1} \cong \mathbb{Z}/p.$$

Also $E^2_{0,2p} \cong \mathbb{Z}/p$ since $\oplus_{i+j=2p}E^\infty_{ij} = \mathbb{Z}/p \cong H_{2p}(\Omega S^3)$ but we don't need this.

So the spectral sequence yields

$$H_{2p-1}(F,\mathbb{Z}/p) = \mathbb{Z}/p, H_i(F,\mathbb{Z}_{(p)}) = 0 \text{ if } i < 2p-1.$$

From the short exact sequence

$$0 \to \mathbb{Z}_{(p)} \to \mathbb{Z}_{(p)} \to \mathbb{Z}/p \to 0$$

we get

$$H_{2p-1}(F, \mathbb{Z}_{(p)}) = \mathbb{Z}_{(p)} \text{ or } \mathbb{Z}_{p^n}$$

but we can exclude Z/p^n from knowing $H_*(f) \otimes \mathbb{Q}$ is isomorphism so $H_*(F) \otimes \mathbb{Q} = 0$. On the other hand assuming finite type, and localization techniques we have

$$H_i(F, \mathbb{Z}_{(p)}) = 0, i < 2p - 1$$

(so $F_{(p)}$ is $2p-2$ connected). By Hurewicz's theorem

$$\pi_{2p-1}(F) \otimes \mathbb{Z}_{(p)} \neq 0.$$

Again from $H_*(f) \otimes \mathbb{Q}$ being an isomorphism, we have

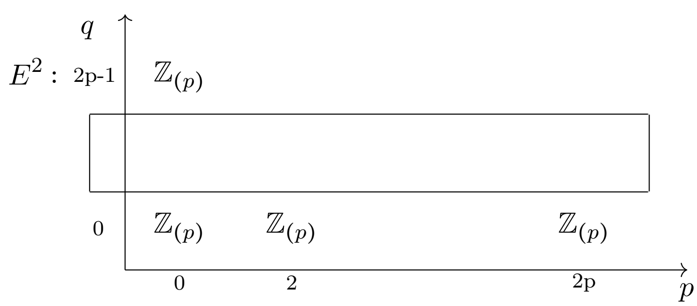
$$p^k H_{2p-1}(F, \mathbb{Z}_{(p)}) \cong \pi_{2p-1}(F) \otimes \mathbb{Z}_{(p)} = 0$$

so $\pi_{2p-1}(F) \otimes \mathbb{Z}_{(p)} \cong Z/p^n$.

Now we look at the Serre spectral sequence of

$$F_{(p)} \to \Omega S^3_{(p)} \to \mathbb{C}P^\infty_{(p)}$$

with $\mathbb{Z}$-coefficients.



By the edge homomorphism

$$H_{2p}(\Omega S^3) \to E^\infty_{2p,0} \to E^2_{2p,0} \cong H_{2p}(\mathbb{C}P^\infty)$$

is multiplication by p, $\mathbb{Z}_{(p)} \to \mathbb{Z}_{(p)}$. So $\alpha_p \notin Z^\infty_{2p}$ and $p\alpha_p \in Z^\infty_{2p}$ aso there is $\beta \in E^2_{0,2p-1}$ and $p\beta = 0$. And β mut generate $E^2_{0,2p-1}$. So

$$E^2_{0,2p-1} = \mathbb{Z}/p \cong H_{2p-1}(F_{(p)}, \mathbb{Z}_{(p)}) \cong \pi_{2p-1}(F) \otimes \mathbb{Z}_{(p)}$$

Also $\widetilde{H}_\ell(F_{(p)}, \mathbb{Z}) = 0, \ell < 2p - 1$. Since $\pi_i(F) \cong \pi_i(\Omega S^3) \cong \pi_{i+1}(S^3)$ for $i > 2$

$$\pi_i(S^3) \otimes \mathbb{Z}_{(p)} = 0, 3 < i < 2p$$

and get

$$\pi_{2p}(S^3) \oplus \mathbb{Z}_{(p)} = \mathbb{Z}/p.$$

CHAPTER 7

Classifying spaces

1. Category of G-bundles

Let G be a topological group, a space with a continuous right G-action is called a right G-space. We will usually be working with right actions and shortening the above term by G-space. Morphisms of G-spaces are G-equivariant maps. We know that if X is a G-space then the is an open map $\pi : X \to X/G$ to the quotient space with quotient topology.

Definition 18. A map $\xi : X \xrightarrow{p} B$ is a G-bundle if it is isomorphic to $X' \to X'/G$ for some G-space X.

Example 1.1. The action of $\mathrm{GL}(n, \mathbb{R})$ on $\mathbb{R}^n$ gives a quotient space $\mathbb{R}^n/\mathrm{GL}(n, \mathbb{R})$ consisting of two points with one of them being open and the other one not open. It is obvious from this example that G-bundles may not be fibrations. ⌟

Definition 19. a G-space X is free is $xs = x$ implies $s = 1$.

If X is free then we make a notation

$$X^* := \{(x, xs) : x \in X, s \in G\} \subset X \times X.$$

The map $\tau : X^* \to G$ via $(x, xs) \mapsto s$ is well-defined since G is free and is called the *translation function.*

Definition 20. A G-space X is principal if X is free and $\tau : X^* \to G$ is continuous. A G-bundle $X \xrightarrow{p} B$ is principal if X is a principal G-space.

Proposition 28. *A principal G-bundle $X \xrightarrow{p} B$ has fiber G (i.e. for all $b \in B$, $p^{-1}(b)$ is homeomorphic to G).*

Proof. Let $b \in B$ pick $x \in p^{-1}(b)$ define $u : G \to p^{-1}(b)$ via $x \mapsto u(s) := xs$. The inverse is $y \mapsto \tau(x, y)$. □

Example 1.2. Let $G = \mathbb{Z}/2$ act on S^n. Then $S^n \to S^n/G$ is principal. Clearly the action is free and observe that τ from the space

$$\{(x, \pm x) : x \in S\} \subset S^n \times S^n$$

to $\{1, -1\}$ is continuous. ⌟

If X is a G-space with associated G-bundle $X \xrightarrow{p} B$ then any map $B' \xrightarrow{f} B$ can pullback p to a G-bundle on B':

$$\begin{tikzcd} X' \arrow[r, "\overline{f}"] \arrow[d, "p'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "p"] \\ B' \arrow[r, "f"'] & B. \end{tikzcd}$$

The G-action on X' is given via

$$(x, b').s = (xs, b').$$

It is clear that $\overline{f}$ is a map of G-spaces. Also if X is principal then X' is one as well.

Definition 21. For any map $f : Y \to Y'$ of G-spaces we get a commutative diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & Y' \arrow[d] \\ Y/G \arrow[r] & Y'/G \end{tikzcd}$$

a map of G-bundles is something isomorphic to this (here meaning only consistent homeomorphisms).

Let $\mathcal{Bun}[G]$ be the category of G-bundles and $G - \mathcal{Bun}_B$ is the groupoid of G-bundles over B. Any map $f : B' \to B$ gives a functor

$$f^* : \mathcal{Bun}_B[G] \to \mathcal{Bun}_{B'}[G].$$

The fact that this is a groupoid will be left unproved,

Proposition 29. *If $f \in \mathrm{Hom}_{\mathcal{Bun}_B[G]}(X', X)$ then f is an isomorphism.*

Proof. Let f be the equivariant map over B in the diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & X' \arrow[dl, "p'"] \\ & B & \end{tikzcd}$$

Injectivity: If $f(x_1) = f(x_2)$ then $p(x_1) = p'(f(x)) = p'(f(x_2)) = p(x_2)$. So $x_1 s = x_2$ for some $s \in G$. Thus

$$f(x_1)s = f(x_1 s) = f(x_2) = f(x_1).$$

Therefore $x = 1$ since ξ' is free. So $x_1 = x_2$.

Sujrectivity: Given $x' \in X'$, let $x \in X$ be such that $p(x) = p'(x')$ (by surjectivity of p). So $p'(x') = p(x) = p'(f(x))$. Thus $x' = f(x)s$ for some $x \in G$. Therefore $x' = f(xs)$ so f is surjective. Now check that f^{-1} is continuous. $\square$

2. Fiber bundles

Let $\xi : (X \xrightarrow{p} B)$ be a principal G-bundle. Let also F be a left G-space and we have a diagonal right action of G on $X \times F$ via

$$(x, f).s = (xs, s^{-1}f)$$

and the orbit space will be denoted by

$$X_F := X \times F/G = X \times_G F.$$

The map $p_F : X_F \to B$ is the unique factorization of

$$X \times F \to X \to B$$

through $X \times_G F$. Then $\xi_F : (X_F \xrightarrow{p_F} B)$ is called a fiber bundle over B with fiber F (as a G-space).

One Session missing here

DEFINITION 22. A covering $\{U_i\}$ of X is a numerable overing if there is a locally finite partition of unity $\{U_i\}_{i \in S}$. In particular

$$\overline{U_i((0,1])} \subseteq U_i, \forall i \in S.$$

For Hausdorff spaces and paracompact spaces all coverings are numerable. A G-bundle ξ is numerable if it is trivial over a numerable covering.

PROPOSITION 30. *For a map of bases $f : B' \to B$, numerable G-bundle ξ pulls back to numerable $f^*(\xi)$.*

PROOF. Use $u_i' = u_i \circ f$ and $U_i' = f^{-1}(U_i)$. □

For convenience we write $E(\xi)$ for the total space of a G-bundle ξ.

PROPOSITION 31. *Given $r : B \times I \to B \times I$ via $(b, t) \mapsto (b, 1)$ and ξ a numerable G-bundle over $B \times I$ then there is a G-morphism $(g, r : \xi \to \xi$ such that there is a commutative diagram*

$$\begin{tikzcd}
E(\xi) \arrow[r, "g"] \arrow[d] & E(\xi) \arrow[d] \\
B \times I \arrow[r, "r"'] & B \times I.
\end{tikzcd}$$

COROLLARY 4. *ξ and $r^*\xi$ are isomorphic over $B \times I$.*

PROOF. There is a unique map making the diagram

$$\begin{tikzcd}
E(\xi) \arrow[dr, dashed] \arrow[drr, bend left, "g"] \arrow[ddr, bend right] & & \\
& E(r^*\xi) \arrow[r] \arrow[d] & E(\xi) \arrow[d] \\
& B \times I \arrow[r] & B \times I
\end{tikzcd}$$

commute. So check that it is a G-equivariant map and by last session it must be an isomorphism. ☐

THEOREM 2.1. *Given $f, g : B' \to B$ and ξ a bundle over B, if $f \cong g$ that $f * (\xi) \cong g^*(\xi)$.*

PROOF. Let H be a homotopy from f to g. Let $E_0 : B' \to B; \times I$ be $b \mapsto (b, 0)$. We have $HE_0 = f$, and $HrE_0 = g$. Now we have

$$f^*(\xi) = (HE_0)^*(\xi) = E_0^* H^*(\xi) \cong E_0^* r^*(H^*(\xi)) \cong (HrE_0)^*(\xi) = g^*(\xi).$$

☐

Notation: $k_G(B)$ is the set of isomorphism classes of numerable principal G-bundles over B. And by isomorphism we mean isomorphisms over the identity map.

COROLLARY 5. *For $f \in [X, Y]$ we get $k_G(f) : k_G(Y) \to k_G(X)$. And $k_G(f)$ is a bijection if f is a homotopy equivalence.*

DEFINITION 23. A principal G-bundle $\omega : E_0 \xrightarrow{p_0} B_0$ is a universal (numerable) G-bundle if ω is numerable and

$$[-, B_0] \to k_G(-)$$

is an isomorphism of contravariant functors, i.e. it's a natural transformation such that for all X,

$$[X, B_0] \to k_G(X)$$

is a bijection.

Note that $[-, B_0]$ is clearly a functor on the homotopy category $\mathcal{T}op_{\cong}$ so is $k_G(0)$ morphisms given by pullback.

PROPOSITION 32. *A numerable principal G-bundle ω is universal if and only if*

(1) $\forall \xi$ principal numerable G-bundle over X, there is $f : X \to B_0$, $\xi \cong f^*(\omega)$ over X.

(2) For any $f, g : X \to B_0$. $f^*(\omega) \cong g^*(\omega)$ implies $f \simeq g$.

3. Milnor's construction

DEFINITION 24 (The join). $A * B$ is a topological space $A \times B \times I/\cong$ with the relations

$$(a, b, 1) = (a, b', 1), (a, b, 0) = (a', b, 0).$$

We can view it as having points $(t_0 a, t_1 b)$ with $t_0 + t_1 = 1$ with relations

$$(0a, t_1 b) = (0a', t_1 b).$$

$A * B$ fits into a pushout

$$\text{(I)} \quad \begin{tikzcd} A * B \arrow[r] \arrow[d] & A \times CB \arrow[d] \\ CA \times B \arrow[r] & A * B \end{tikzcd}$$

where CX is the notation for cone on X. Consider the homotopy pushout

$$\text{(II)} \quad \begin{tikzcd} A \vee B \arrow[r] \arrow[d] & A \vee CB \arrow[d] \\ CA \vee B \arrow[r] & CA \vee CB \end{tikzcd}$$

the inclusion $\mathrm{II} \to \mathrm{I}$ is a cofibration on each space in part.

$$\operatorname{colim}(\mathrm{I}/\mathrm{II}) \cong \operatorname{colim} \mathrm{I}/\operatorname{colim} \mathrm{II} \cong A * B$$

since $\operatorname{colim} \mathrm{II} \simeq *$. But

$$\mathrm{I}/\mathrm{II}: \quad \begin{tikzcd} A \wedge B \arrow[r] \arrow[d] & A \wedge CB (\simeq *) \\ (* \simeq) CA \wedge B & \end{tikzcd}$$

and this diagram is "equivalent" to

$$\begin{tikzcd} A \wedge B \arrow[r] \arrow[d] & C(A \wedge B) \arrow[d] \\ * \arrow[r] & \Sigma(A \wedge B) \end{tikzcd}$$

and we conclude that $\operatorname{colim} \mathrm{I}/\mathrm{II} \simeq \Sigma(A \wedge B)$. And finally that

$$A * B \simeq \Sigma A \wedge B.$$

We make the notation $E_nG = *_nG = (*_{n-1}G) * G$. And as a set we should view it as
$$\{(t_0x_0, t_1x_1, \cdots, t_{n-1}x_{n-1}) : \sum t_i = 1\}/\cong$$
with the relations that if $t_i = 0$ then
$$(\cdots, t_2x_i, \cdots) \cong (\cdots, t_ix_i', \cdots).$$
Clearly $E_nG \to E_{n+1}G$ via
$$(t_0x_0, \cdots, t_{n-1}x_{n-1}) \mapsto (t_0x_0, \cdots, t_{n-1}x_{n-1}, 0).$$
As an aside note that one can show that $E_nG \to E_{n+1}G$ is null-homotopic.

We now make the final definition
$$EG := *_\infty G := \operatorname{colim}_n(E_nG).$$

Proposition 33. *$EG \simeq *$.*

Proof. $E_nG \simeq E_{n-1}G * G \simeq \Sigma(E_{n-1}G \wedge G)$. So it's connected to all degrees. □

E_nG with diagonal action is a free G-space. In fact, since for any $(t_0g_0, \cdots, t_{n-1}g_{n-1})$ we have $\sum t_i = 1$, some $t_i \neq 0$ so the action is free. The action is continuous as well (check it!). So we can define
$$B_nG := E_nG/G, \quad BG := EG/G.$$

Proposition 34. *$E_nG \xrightarrow{p_n} B_nG$ and $\omega_G : EG \xrightarrow{p} BG$ are locally trivial, numerable, principal G-bundles.*

Proof. We will only work out the proofs of local triviality and numerability for the EG case. Let $t_i : EG \to I$ be given via
$$(t_0x_0, \cdots, t_ix_i, \cdots) \mapsto t_i.$$
There is a unique $u_i : BG \xrightarrow{0,1}$ such that $u_ip = t_i$. Let
$$v_i = u_i^{-1}((0,1]) = p(t_i^{-1}((0,1])) \subseteq B.$$
To show that ωG is trivial over v_i, we show that there is a section over v_i,
$$s_i : v_i \to t_i^{-1}(0,1] \subseteq E_G.$$
So define $s_i' : t_i^{-1}(0,1] \to t_i^{-1}(0,1]$ that is constant on G-orbits
$$s_i' : t_i^{-1}(0,1] \to t_i^{-1}(0,1]$$
to be constant on G-orbits, via
$$s_i' : t_i^{-1}(0,1] \to t_i^{-1}(0,1]$$
$$(t_0x_0, \cdots, t_ix_i, \cdots) \mapsto a(x_i)^{-1}.$$
s_i' is well-defined since $t_i \neq 0$. And $s_i(ag) = s_i'(a)$ for all $g \in G$, hence s_i' induces
$$s_i : t_i^{-1}(0,1]/G \to t_i^{-1}(0,1].$$

It is easy to check that $ps_i = \mathrm{id}_{v_i}$. So ω_G is locally trivial, also numerable since the u_i's give a partition of unity. $\square$

COROLLARY 6. *$EG \xrightarrow{p} BF$ is a fibration with fiber ΩBG and so B is an inverse to Ω on the homotopy category.*

PROOF. $EG \simeq *$ and $EG \xrightarrow{p} BF$ a fibration, since ω_G is locally trivial G-bundle over a CW complex. Therefore

$$p^{-1}(*) \simeq \Omega BG.$$

The constructions we have for E_nG, B_nG, EG and BG are clearly functorial on group homomorphisms. So B is a functor

$$\mathcal{T}op\ group \to \mathcal{T}op.$$

(probably we need some topological restrictions on the category of Topological groups we are considering.) Recall that Ω can be considered as a functor

$$\mathcal{T}op \to \mathcal{T}op\ group.$$

This is due to Milnor. The first thing one has to show is an equivalence of categories between simplicial sets and topological spaces. Then he works on simplicial sets to create a fibration $PX \to X$. PX turns out to be contractible so the fiber is the loop space ΩX. The way the fibration is constructed ΩX has a natural group structure on it as well. $\square$

From now on we work with CW complexes. So all coverings are numerable since CW complexes are paracompact.

THEOREM 3.1. *For a principal G-bundle, ξ, over B, there is $f : B \to BG$ such that*

$$\xi \cong f^*(\omega_G).$$

PROOF. Since ξ is numerable there is $\{u_n\}$ over B and $U_n = u_n^{-1}(0,1]$, trivializing ξ for all $n \geq 0$. (Quesiton: Why can we assume $\{U_n\}$ is a countable collection?) There is a trivialization

$$h_n : U_n \times G \cong E(\xi|_{U_n}) \subseteq E_\xi).$$

Let $g_n : U_n \times G \to G$ be the projection and $p : E(\xi) \to B$. Define $g : E(\xi) \to EG$

$$z \mapsto (u_0(p(z)g_0h_0^{-1}(z), \cdots, u_n(p(z))g_nh_n^{-1}(z), \cdots).$$

If h_n is not defined, then $u_np(z) = 0$. We conclude that g is well-defined and that $g(zs) = g(z)s$ for all $s \in G$ since h_n is G-equivariant. So g induces $f : B \to BG$ and $(g, f) : \xi \to \omega_G$ is a bundle map, i.e.

$$\begin{tikzcd} E(\xi) \arrow[r, "g"] \arrow[d] & EG \arrow[d] \\ B \arrow[r, "f"'] & BG \end{tikzcd}$$

commutes. Hence $\xi \cong f^*(\omega_G)$. $\square$

THEOREM 3.2. *For f and g are maps $B \to BG$, and $f^*(\omega_G) \cong g^*(\omega_G)$ then $f \simeq g$.*

This completes the proof of $[B, BG] \to k_G(B)$ being a bijection.