Ringel-Hall algebras and applications to moduli

by

Pooya Ronagh

B.Sc. in Pure Mathematics, Sharif University of Technology, 2009
B.Sc. in Computer Science, Sharif University of Technology, 2009

AN ESSAY SUBMITTED IN PARTIAL FULFILMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

Master of Science

in

THE FACULTY OF GRADUATE STUDIES

(Mathematics)

The University Of British Columbia

(Vancouver)

April 21, 2011

Abstract

In this essay we will survey some of the results of Markus Reineke on geometry of the moduli spaces of stable quiver representations, and Tom Bridgeland on properties of Donaldson-Thomas invariants of Calabi-Yau threefolds. The underlying idea of these results is to assign a suitable Hall algebra to the abelian category of objects of interest in the moduli problem and translate categorical statements about this category into identities in the Hall algebra. An integration on the Hall algebra is defined such that integrating identities in the Hall algebra will then produce generating functions involving invariants that we want to study.

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Acknowledgements

First and foremost I would like to thank my supervisor, Professor Kai Behrend, for his constant support, careful instruction and instructive suggestions ever since my study as a graduate student of the University of British Columbia (UBC). I am indebted to him for his financial support throughout my research that specially made it possible for me to participate in the wonderful School on Moduli Spaces, in January 2011 held at Isaac Newton Institute for Mathematical Sciences, Cambridge (UK).

Specifically, I learnt a lot of central ideas and results surveyed in this essay by attending short courses of Professors Markus Reineke, Daniel Huybrecht, and Brent Doran, and via helpful conversations with them. As a matter of fact, I would like to thank organizers of that workshop, Professors Leticia Brambila-Paz, Peter Newstead, Richard Thomas and Oscar García-Prada for making this possible.

I am grateful to Professor Jim Bryan for edifying conversations with him and for organization of the graduate students reading seminars in algebraic geometry in UBC. I am thankful to the rest of professors and students of the group of algebraic geometry, and the Department of Mathematics of UBC in general, for the amazing research environment in mathematics they have created for us to enjoy.

Pooya Ronagh

The University of British Columbia
May 2011

Introduction

This essay will be an exposition of a technique that first appeared in Reineke's computation of the Betti numbers of the moduli spaces of stable quiver representations [20] and has been used later on in works of Bridgeland [4, 5] to prove properties of Donaldson-Thomas invariants of Calabi-Yau threefolds. In general the idea is to assign a suitable Hall algebra to the category of objects of interest in the moduli problem and translate categorical statements about this category into identities in the Hall algebra. "Integrating" identities in the Hall algebra will then produce generating functions involving invariants that we want to study.

The general setup is that we have an abelian category, $\mathscr{A}$, and we want to study moduli spaces of isomorphism classes of objects in it. We fix a vector d of discrete invariants of the objects of $\mathscr{A}$ that essentially arise from intersection theory of objects of $\mathscr{A}$ (e.g. degree, rank, Chern character). And we choose a notion of stability. We then construct the moduli space $\mathcal{M}_d^{st}(\mathscr{A})$, of stable objects, or variants of it; e.g. moduli of polystable objects or objects together with additional structure such as those that can be written as quotients or come with a framing. The crucial observation is that there is a noncommutative algebra $\mathrm{H}_{\mathscr{A}}$ encoding "quantitative information" about $\mathcal{M}_d^{st}(\mathscr{A})$ such as Betti numbers and Euler characteristics. Our task is to extract this information.

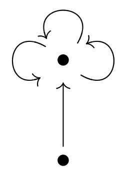
The magic of the theory lies in the construction of a ring homomorphism from the corresponding Hall algebra to a (skew) polynomial ring which is generally named an *integration map*. This homomorphism is in fact an integration map over the moduli stack with respect to a motivic measure. As we will see, the proof of the fact that the integration is a ring homomorphism will heavily rely on

1. $\mathscr{A}$ having cohomological dimension ≤ 1 in the case of Reineke, and
2. $\mathscr{A}$ being a Calabi-Yau abelian category of dimension 3 in case of Bridgeland.

In the former category all $\mathrm{Ext}^{>1}$ groups vanish so that the Euler characteristic depends only on Ext^0 and Ext^1 groups, and in the latter case the $\mathrm{Ext}^{>3}$ groups vanish but Serre duality makes Euler characteristic depend only on Ext^0 and Ext^1 groups again.

To motivate this study, it is worthwhile to comment that both cases are interesting problems to tackle. In the first case the abelian category is $\mathcal{Rep}_k Q$, the category of k-representations of a quiver Q. The huge problem of classifying modules over an algebra is highly related to understanding the case of modules over the path algebra of (probably complicated) quivers. Beside that, there are many hard problems, essentially of linear algebra, that reduce to constructing and studying moduli spaces of representations of a quiver; e.g. understanding normal forms of simultaneous actions of general linear groups. The geometric approach one can have toward these algebra problems is to construct spaces parametrizing these normal forms, and compute algebraic invariants of these moduli spaces relying on the geometry of them.

Such linear algebra problems are of importance to the algebraic geometer as well. Often quiver representations serve as a unified language to encode interesting moduli problems of algebraic geometry. For example, simultaneous action of $\mathrm{Gl}_d(\mathbb{C})$ on $M_{d\times d}(\mathbb{C})^m$ appears in the moduli problem of vector bundles on curves. The action of $\mathrm{Gl}_d \times \mathrm{Gl}_e$ on $M_{d\times e}(\mathbb{C})^m$ via base change shows up when considering vector bundles on $\mathbb{P}^2$. Finally, the construction of Hilbert scheme of points on a smooth Calabi-Yau threefold reduces to studying representations of the quiver in the following figure.



There is a whole theory aside what we will consider in this essay, that aims to associate a quiver to a scheme, such that the category of representations of the quiver is derived equivalent to the category of coherent sheaves on the scheme[3].

The second category is that of coherent sheaves on a projective Calabi-Yau threefold. In the past few decades, string theory has motivated a number of enumerative invariants of Calabi-Yau threefolds. A few interesting ones to the mathematician have been the Gromov-Witten invariants (GW), Donaldson-Thomas invariants (DT), and Pandharipande-Thomas invariant (PT). In [13] and [16] a series of conjectures on the relations of these invariants were proposed. What is mostly known as the MNOP conjecture is the relation of generative functions of GW invariants and DT invariants. However we will focus on that part of the claims in these conjectures that relates the generating functions of the DT and PT invariants. In [4] Bridgeland proved this part of the conjecture using techniques of integration on Hall-algebras that were inspired by the works of Reineke. We will see that beside the technical sophistication of the proof it very much resembles the same ideas that Reineke had used. The DT/PT correspondence is expected to be a major step in proving all claims of the conjectural DT/PT/GW correspondence of MNOP.

The essay is organized in two chapters. In the first chapter we look at works of Reineke. In section 1.1 we review basics of representations of quivers. A good reference for that is a series of lecture notes by William Crawley-Boevey[6]. In section 1.2 we will say a few words on the problem of classification of representations of algebras in general, and path-algebras in particular. Sections 1.3 and 1.4 surveys the construction of the moduli space[11, 19]. We will continue with definition and properties of Hall algebras from section 1.5. The main results of this chapter, i.e. computation of the number of rational points on the moduli of representations of a quiver [18] and the computation of Betti numbers [20] are sketched in section 1.7.

The second chapter will focus on the DT/PT correspondence. We will start in section

2.1 by an overview of the curve counting invariants and state the DT/PT correspondence. We will have to be working in the language of stacks. A useful description of the subcategory of stable pairs will arise from a t-structure on the derived category of coherent sheaves of the underlying Calabi-Yau variety. All of this will be briefly explored in section 2.2. Sections 2.3 and 2.4 will be devoted to definition and some of the properties of the Hall algebra and the integration map. The situation is slightly more complicated than in the case of Reineke since the formal sums that we are eventually interested in are not elements of the Hall algebra, but of an extended version of it. In section 2.5 we will see how Bridgeland overcomes this subtlety by viewing the identities in the extended Hall algebra as limits of identities that hold in the original Hall algebra. The main result of this chapter is the DT/PT correspondence proved in section 2.6.

Chapter 1

Hall algebras and moduli of representations of a quiver

1.1 The category of representations of a quiver

§1.1.1. By a quiver Q, we mean a finite oriented graph; Let I be the set of vertices and E the set of arrows. This structure comes with two maps $h, t : E \to I$ which indicate the vertices at the head and tail of each arrow. But often we will write $\alpha : i \to j$ for an element $\alpha \in E$ for which $t\alpha = i \in I$ and $h\alpha = j \in I$. Let k be an arbitrary field. Then a k-representation,

$$V = ((V_i)_{i \in I}, (V_\alpha : V_{t\alpha} \to V_{h\alpha})_{\alpha \in E})$$

of Q is a collection of finite dimensional k-vector spaces V_i for each $i \in I$, together with k-linear maps

$$V_\alpha : V_{t\alpha} \to V_{h\alpha}$$

along each arrow $\alpha \in E$. A morphism of representations, $f : V \to W$ is a tuple $(f_i : V_i \to W_i)_i$ for each vertex $i \in I$ such that all diagrams

$$\begin{tikzcd} V_i \arrow[r, "V_\alpha"] \arrow[d, "f_i"'] & V_j \arrow[d, "f_j"] \\ W_i \arrow[r, "W_\alpha"] & W_j \end{tikzcd}$$

commute. The composition is defined in the obvious way. Such a map is an isomorphism if each f_i is. A subrepresentation is a tuple $(U_i \subset V_i)_{i \in I}$ such that $V_\alpha(U_i) \subset U_j$ for all α. This defines an abelian category $\mathcal{Rep}_k Q$.

§1.1.2. There is a well-defined map

$$\begin{aligned} K_0(\mathcal{Rep}_k(Q)) &\to \mathbb{Z}I \\ [V] &\mapsto \underline{\dim} V = (\dim_k V_i)_{i \in I} \end{aligned}$$

since dimensions behave additively along short exact sequences. $\mathbb{Z}I$ is called the *numerical Grothendieck group*. Nevertheless, if Q has no cycles the only irreducible representations are the ones with only a one-dimensional vector space assigned to one vertex and all other vertices and maps are trivial. Hence the above mapping is an isomorphism if Q has no oriented cycles.

§1.1.3. There is an equivalence of categories between $\mathcal{Rep}_k Q$, and $\mathcal{mod}\, kQ$, the category of left-modules over the path algebra kQ. This enables us to do homological algebra on the category $\mathcal{Rep}_k Q$. A path in the quiver Q is a sequence of arrows

$$\omega = \alpha_k \cdots \alpha_1 \tag{1.1.1}$$

such that there are vertices i_s for any $s = 1, \cdots, k+1$, and $\alpha_s : i_s \to i_{s+1}$. We can extend the tail and head functions t, h to the set of all paths: for the path ω above,

$$t(\omega) = i_1, h(\omega) = i_{k+1}.$$

If ω and σ are two paths

$$\omega = \alpha_k \cdots \alpha_1, \sigma = \beta_\ell \cdots \beta_1$$

such that $h(\omega) = t(\sigma)$, the concatenation of the two paths $\omega\sigma$ is defined to be the path

$$\sigma\omega = \beta_\ell \cdots \beta_1 \alpha_k \cdots \alpha_1.$$

Recall that the path-algebra associated to the quiver Q is defined by

Definition 1. The path algebra, kQ of the quiver Q is generated by the set of all paths as in 1.1.1, and elements ε_i for each vertex $i \in I$ modulo the relations

$$\begin{aligned} \varepsilon_i \varepsilon_j &= \delta_{i,j} \varepsilon_i, \\ \varepsilon_{h(\omega)} \omega &= \omega \varepsilon_{t(\omega)} = \omega, \\ \varepsilon_i \omega &= 0 \text{ if } i \neq h(\omega) \\ \omega e_j &= 0 \text{ if } j \neq t(\omega) \end{aligned}$$

The multiplication of paths ω and σ is by concatenation as above and zero otherwise.

It is easy to describe the equivalence of the two categories $\mathcal{Rep}_k Q$ and $\mathcal{mod}\, kQ$; Given a kQ-module V we have projections

$$V_{(i)} = \varepsilon_i V$$

which are k-vector spaces assigned to each vertex $i \in I$. For any arrow $\alpha : i \to j$, let the k-linear map

$$V_\alpha : V_{(i)} \to V_{(j)}$$

be defined via $v \mapsto \alpha.v$. Then the tuple $\big((V_{(i)})_{i\in I}, (V_\alpha)_{\alpha:i\to j}\big)$ forms a k-representation of Q. Conversely from a representation $((V_i)_{i\in I}, (V_\alpha)_{\alpha:i\to j})$ we get a kQ-module V which as a k-vector space is the direct sum

$$V = \bigoplus_{i\in I} V_i$$

with projections $\pi_i : V \to V_i$ and the module structure is prescribed by the definitions

$$\varepsilon_i v = \pi_i v \quad \text{ and } \quad \alpha_m \cdots \alpha_1 v = V_{\alpha_m} \circ \cdots \circ V_{\alpha_1}(\pi_{t(\alpha_1)} v).$$

§1.1.4. Let A be a finite dimensional algebra over the field k. Considered as a left-module over itself, A is projective. So $\mathcal{mod}\, kQ$ has enough projectives. In fact the first step of a projective resolution is then going to be $A \otimes_k M \rightarrow_A M \rightarrow 0$.

Theorem 1.1.1 (Standard resolution)**.** *For every $i \in I$, define*

$$P_{i,j} = \langle \text{ paths } \omega \text{ from } i \text{ to } j \ \rangle$$

to be the vector space generated by paths from i to j and $P_i = \oplus_{j \in I} P_{i,j}$ the vector space of paths starting from i. Then $\oplus_{i \in I} P_i$ is isomorphic to the path algebra kQ viewed as a left module over itself. For any arrow $\alpha : i \longrightarrow j$ we get a morphism $P_\alpha : P_j \rightarrow P_i$ via $\omega \mapsto \omega\alpha$. Then for any left kQ-module V we have a resolution

$$0 \rightarrow \bigoplus_{(\alpha : i \longrightarrow j) \in E} P_j \otimes_k V_{(i)} \xrightarrow{\psi} \bigoplus_{i \in I} P_i \otimes_k V_{(i)} \xrightarrow{\varphi} V \rightarrow 0. \tag{1.1.2}$$

Here φ and ψ are determined by

$$\psi : \omega \otimes v \mapsto \omega\alpha \otimes v - \omega \otimes \alpha.v$$

and

$$\varphi : \omega \otimes v \mapsto \omega.v.$$

Note that for a left kQ-module, V, $P_i \otimes_k V$ is isomorphic as a left kQ-module to $(P_i)^{\oplus \dim V}$ and therefore is a projective left kQ-module, hence the above is a projective resolution.

§1.1.5. One last comment about the structure of modules over the path-algebra kQ is the characterization of the semi-simple and simple ones. Any of the modules P_i we defined above is simple. As noted before, any projective module appearing in the resolutions of the form 1.1.2 can be decomposed into direct sums of P_i's. So in case kQ is finite-dimensional and equivalently when Q has no cycles, these are precisely all the simple representations.

Example 1.1.3. If Q has cycles the above assertion does not hold. Consider the quiver, Q, with one vertex and one loop, α. Since $\alpha\varepsilon = \varepsilon\alpha = \alpha$, the path-algebra of Q is the polynomial ring in variable α. Let us assume that $\operatorname{char} k = 0$, then by Schur's lemma any simple representation of Q is an isomorphism $k \xrightarrow{\times c} k$ for any nonzero scalar $c \in k$. In the category $\mathcal{mod}\, kQ$ this corresponds to the module $k[\alpha]/(\alpha - c)$ over $k[\alpha]$. It follows that these are all the simple modules of the polynomial ring $k[\alpha]$ in one variable. ⌟

§1.1.6. Now we can defined Ext groups in $\mathcal{Rep}_K(Q)$ and an immediate observation is that this is a category of cohomological dimension 1, i.e. $\mathrm{Ext}^{\geq 2} \cong 0$. It is a finite length category (i.e. the objects do not produce infinite filtrations) and the Euler characteristic

$$\dim \operatorname{Hom}(V, W) - \dim \operatorname{Ext}^1(V, W) = \langle \underline{\dim} V, \underline{\dim} W \rangle$$

can be computed using an explicit formula in terms of the dimension vectors. For if $d = (d_i)_{i \in I}$ and $e = (e_i)_{i \in I}$ are respectively the dimension vectors of representations

$V = ((V_i)_{i\in I}, (V_\alpha)_{\alpha\in E})$ and $W = ((W_i)_{i\in I}, (W_\alpha)_{\alpha\in E})$, we can apply $\mathrm{Hom}(-, W)$ to the resolution 1.1.2 and get an exact sequence

$$0 \to \mathrm{Hom}(V,W) \to \mathrm{Hom}(\bigoplus_{i\in I} P_i \otimes_k V_{(i)}, W) \to \mathrm{Hom}(\bigoplus_{\alpha:i\longrightarrow j} P_j \otimes_k V_{(i)}, W) \to \mathrm{Ext}^1(V,W) \to 0$$

We have

$$\begin{aligned}\dim_{kQ} \mathrm{Hom}(P_j \otimes_k V_{(i)}, W) &= \dim_{kQ} \mathrm{Hom}((P_j)^{\oplus \dim V_{(i)}}, W) \\ &= \dim_{kQ} \varepsilon_i V \dim_{kQ} W \varepsilon_j = d_i e_j\end{aligned}$$

and since dimension is additive along exact sequences we conclude that the Euler characteristic is given via the Euler form

$$\langle d, e\rangle := \sum_{i\in I} d_i e_i - \sum_{\alpha:i\longrightarrow j} d_i e_j.$$

1.2 Classification of quiver representations

§1.2.1. Up to isomorphism any quiver representation is just $V = ((k^{d_i}), A_\alpha \in M_{d_j\times d_i}(k))$ and this is isomorphic to $W = ((k^{d_i}), B_\alpha \in M_{d_j\times d_i}(k))$ if and only if $(B_\alpha)_\alpha$ arises from $(A_\alpha)_\alpha$ by simultaneous base change in all k^{d_i}; i.e. automorphisms $(g_i)_{i\in I}$ exist such that for all $\alpha : i \to j$ we have $B_\alpha = g_j A_\alpha g_i^{-1}$.

Example 1.2.1. For the quiver $\bullet \longrightarrow \bullet$ any representation is isomorphic to the projection $\pi : k^d \to k^e$ on first r coordinates, for some $r \leq e$ by elementary row and column operations. So the three *discrete invariants* $\dim V$, $\dim W$ and $\mathrm{rk}\, f$ classify all representations of this quiver. The problem gets much harder as we consider more complicated quivers. For instance as soon as there is a loop in the quiver the isomorphism classes of representations happen to depend on *continuous parameters*. For the quiver

$$\begin{tikzcd} \bullet \arrow[loop left] \end{tikzcd}$$

the isomorphism classes of representations correspond to conjugacy class of square matrices. Thus over an algebraically closed field, any representation (i.e. an endomorphism of some finite dimensional vector space) is isomorphic to a Jordan canonical form. Therefore we have continuous invariants (eigenvalues) as well as discrete ones (sizes of the Jordan blocks). Another pedagogical example for our purposes is the quiver in the following figure.

$$\begin{tikzcd} & & \bullet & & \\ \bullet \arrow[urr] & \bullet \arrow[ur] & & \bullet \arrow[ul] & \bullet \arrow[ull] \end{tikzcd}$$

For sake of simplicity we fix the dimensions of our vector spaces as being 1 for the vertices of degree 1 and 2 for the vertex of degree -4, "generically" a representation is isomorphic to

$$\begin{tikzcd} & & k^2 & & \\ k \arrow[urr, "\binom{1}{0}"] & k \arrow[ur, "\binom{0}{1}"'] & & k \arrow[ul, "\binom{1}{1}"] & k \arrow[ull, "\binom{\lambda}{\mu}"'] \end{tikzcd}$$

where $\binom{\lambda}{\mu}$ is defined up to scalar multiplication. The geometric approach to solving such classification problems is construction of certain moduli space, which for this example is that of the families of four lines in the 2-space passing through the origin:

$$\mathbb{P}^1 \cong (\mathbb{P}^1)^4_{sst} /\!/ \operatorname{PGL}_2 .$$

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§1.2.2. In the big picture, it is the algebraist's hopeless dream to classify the finite-dimensional representations of a given algebra up to isomorphism. Solving this problem for path-algebras is still a hard specialization of this problem specially in presence of lots of loops or lots of arrows in the underlying quiver. There is a whole theory related to distinguishing such cases. Gabriel's school[9] has famous results on this. For instance one might ask which quivers only have discrete invariants. The necessary and sufficient condition is that the quiver is a disjoint union of the Dynkin diagrams A, D and E where we get a zero dimensional moduli space. This was generalized by Donovan and Freislich [8] to extended Dynkin digrams for which we still get "harmless" continuous invariants. We will not discuss these topics in this essay.

1.3 Stability conditions

§1.3.1. We have an abelian category that is very much like the category of coherent sheaves on a smooth projective curve. We will try to imitate all the necessary notions towards the construct moduli spaces as in the case of vector bundles on curves. In that case we had a canonical choice of stability in terms of rank and degree. Here we do not have such a canonical choice. Rank is analogous to the sum of dimensions of the vector spaces appearing in the representation. But our replacement for degree will be a choice of an additive function $\Theta : \mathbb{Z}I \to \mathbb{Z}$ (so $\Theta(d) = \sum_{i \in I} \Theta_i d_i$). Now we can define the *slope* by

$$\begin{aligned} \mu : \mathbb{Z}^{\geq 0} I - \{0\} &\to \mathbb{Q} \\ d &\mapsto \Theta(d) / \dim d \end{aligned}$$

where $\dim d := \sum_{i \in I} d_i$ is the replacement for the rank. And for any $0 \neq V \in \mathcal{Rep}_K Q$ its slope is defined as the slope of its dimension vector:

$$\mu(V) := \mu(\underline{\dim} V).$$

Definition 2. V is Θ-stable (resp. Θ-semistable) if and only if for any $0 \neq U \subsetneq V$ we have $\mu(U) < \mu(V)$ (resp. $\mu(U) \leq \mu(V)$) and is polystable whenever V is isomorphic to a direct sum of stables of same slope.

§1.3.2. All usual properties of stable and semistable objects hold. The full subcategory $\mathcal{mod}_\mu kQ$ of semistable representations of kQ of slope μ is a full abelian subcategory of $\mathcal{mod}\, kQ$. Obviously the simple objects in $\mathcal{mod}_\mu kQ$ are precisely the stable representations and the polystable representations coincide with the semisimple objects. We can also define Harder-Narasimhan filtrations and prove their uniqueness in the usual way.

Remark. The above is a modification of the θ-(semi-)stability introduced in [11]. Given an additive function

$$\theta : K_0(\mathscr{A}) \to \mathbb{Z}$$

called a *character* on the Grothendieck group of an abelian category $\mathscr{A}$, King defines an object $M \in \mathscr{A}$ to be θ-semistable (resp. θ-stable) if $\theta(M) = 0$ and every subobject $0 \neq M' \subsetneq M$ satisfies $\theta(M') \geq 0$ (resp. $\theta(M') > 0$).

For any choice of character $\Theta = (\Theta_i)_i$ as in §I.1.3.1 assign a second character

$$\theta(d) = \Theta(d) - \underline{\dim}\ d - \Theta_i$$

then θ-(semi)stability in the sense of [11] coincides with Θ-(semi)stability in the sense of §I.1.3.1 by an easy computation.

1.4 Construction of the moduli

§1.4.1. One may expect that the stability condition above coincides with the stability conditions occuring in geometric invariant theory in the content of Hilbert-Mumford numerical criterion in presence of a character that linearizes the action. In this section we will review the construction of the moduli space of representations of a quiver and count some properties of it, more details and proofs can be found in [19].

Let G be a reductive group acting on an affine space R. Let $\chi : G \to \mathbb{G}_m$ be a character of G. We can lift the action of G to the total space of the trivial line bundle $R \times \mathbb{A} \to R$ via $g.(x, z) = (g.x, \chi^{-1}(g)z)$. A *semi-invariant of weight* χ [14, §6] is a regular function $f \in k[R]$ such that for all $g \in G$, $f(g.x) = \chi(g)f(x)$. The ring of semi-invariants is denoted by $k[R]^{G,\chi}$.

A point $x \in R$ is χ-semistable if there is a semi-invariant $f \in k[R_d]^{G,\chi^m}$ with $m \geq 1$ such that $f(x) \neq 0$. Such a point is called stable, if moreover $\dim G.x = \dim G/\Delta$ and the G-action on $\{x \in R : f(x) \neq 0\}$ is closed. Here Δ is the kernel of the representation of G in R induced by the G-action on R.[11]

Powers of χ give the direct sum $\oplus_m k[R]^{G,\chi^m}$ the structure of a graded ring and the projective quotient *in direction* χ of the action of G on R will be

$$R/\!/_\chi G := \mathcal{P}roj \bigoplus_{m \geq 0} k[R]^{G,\chi^m}.$$

We want to apply this construction to our case; We can think of

$$R_d(Q) := \bigoplus_{\alpha : i \longrightarrow j} \mathrm{Hom}(\mathbb{C}^{d_i}, \mathbb{C}^{d_j})$$

as the parameter space of all representations of all types. The reductive group

$$G_d = \prod_{i \in I} \mathrm{GL}(\mathbb{C}^{d_i})$$

acts on it by

$$(g_i)_{i\in I}.(V_\alpha)_{\alpha\in E} = (g_j V_\alpha g_i^{-1})_{\alpha:i\to j}.$$

A first observation is that the action of the diagonally embedded scalar matrices in G_d on $R_d(Q)$ is trivial, so we will not get zero-dimensional stabilizers unless we pass to the action of the projectivization $\mathbb{P}G_d = G_d/k^*$. Secondly, $R_d(Q)/\!\!/\mathbb{P}G_d$ parametrizes merely the semisimple representations. The open locus of isomorphism classes of simple orbits (i.e. the stable objects) consists only of the one dimensional representations P_i defined in §I.1.1.4. Thus we do not always get an interesting moduli space! Before making a second trial to construct moduli spaces we justify the last assertions by the following

Proposition 1. *Let O_V be the orbit of the representation V. O_V is closed if and only if the representation V is semi-simple.*

Proof. For any subrepresentation $W \subset V$ the representation $W \oplus V/W$ is in the closure of O_V. In fact in appropriate basis we have

$$V_\alpha = \begin{pmatrix} W_\alpha & X_\alpha \\ 0 & (V/W)_\alpha \end{pmatrix}$$

for all $\alpha : i \to j$ and some matrices X_α. Then the one-parameter subgroup $\mathbb{G}_m \to R_d$ given by

$$t \mapsto (\begin{pmatrix} W_\alpha & tX_\alpha \\ 0 & (V/W)_\alpha \end{pmatrix})_{\alpha:i\to j}$$

proves this claim. The uniqueness of the closed orbit in any orbit closure is proved as usual [15]. □

Proposition 2. *The quotient variety $R_d(Q)^{st}/\mathbb{P}G_d$ is a moduli space for isomorphism classes of simple representations.*

Proof. For a semi-simple object V to have finite stabilizer $\mathrm{Stab}_{G_d}(V)/k^*$ is equivalent to requiring that

$$\mathrm{Aut}(V) \cong k,$$

(i.e. that the representation is Schurian). These are precisely the simple representations. □

The characters of a general linear group are integer powers of the determinant function. It follows that the characters of $\mathbb{P}G_d$ are of the form

$$\chi_\Theta : \prod_i \mathrm{GL}(\mathbb{C}^{d_i}) \to \mathbb{C}^*$$

$$(g_i)_i \mapsto \prod_{i\in I} \det(g_i)^{\Theta(d)-\dim d.\Theta_i}.$$

Remark. In [11] the character chosen for linearization is χ_θ where $\theta = (\theta_i)_i$. Then we have to put an extra condition

$$\sum d_i\theta_i = 0$$

on the character so that we have a well-defined character of $\mathbb{P}G_d$.

§1.4.2. As we hoped for to be able to use geometric invariant theory, χ_Θ-(semi)stability of §I.1.4.1 coincides with the notion of Θ-(semi)stability in §I.1.3.1 by [11, proposition 3.1]. So we can interpret the GIT-quotient as the moduli space of semistable objects. The closed orbits in $R_d(Q)^{sst}$ correspond precisely to polystable objects, so the GIT-quotient

$$\pi : R_d \dashrightarrow M_d^{\Theta-pst}(Q) = R_d(Q)^{sst}/\!/G_d := \mathcal{Proj}(\oplus_{n\geq 0}\mathbb{C}[R_d(Q)]^{G_d,\chi_\Theta^n})$$

parametrizes the Θ-polystable representations of dimension vector d. There is a projective morphism from the above space to the affine GIT-quotient

$$\mathcal{M}_d^{0-pst} = R_d(Q)/\!/G_d := \mathcal{Spec}(\mathbb{C}[R_d(Q)]^{G_d}).$$

Observe that the latter is a point if Q has no oriented cycles. There is an open locus

$$\mathcal{M}_d^{\Theta-st}(Q) = R_d^{\Theta-st}(Q)/G_d := \pi(R_d^{\Theta-st}(Q)) \subseteq M_d^{\Theta-pst}(Q)$$

which is smooth and irreducible since we are taking geometric quotient of a smooth irreducible affine universal bundle. We can use the Euler form to comment on the dimension of it also:

$$\dim \mathcal{M}_d^{\Theta-st}(Q) = 1 - \langle d, d\rangle$$

if nonempty.

Remark. This Euler form was of homological meaning but now we see that it also appears in the moduli, so the moduli has some homological information in it as well.

§1.4.3. The choice of Θ is an important issue. We may not get any interesting collection of (semi)stable objects with some choices of it. For a given choice of Θ we say

Definition 3. The dimension vector d is coprime if $\dim d$ and $\Theta(d)$ are coprime.

If d is coprime then semi-stability and stability coincide so in this case we have that

$$\mathcal{M}_d^{pst} = \mathcal{M}_d^{st}$$

is a smooth projective scheme (over the affine quotient).

1.5 Hall algebras

§1.5.1. Let k be a finite field $k = \mathbb{F}_q$. We will associate a completed version of the Hall algebras to the category $\mathcal{Rep}_k(Q)$, given as a $\mathbb{Q}$-vector space by

$$\mathrm{H}_k(Q) := \prod_{d\in\mathbb{N}I} \mathbb{Q}^{G_d(k)}(R_d(Q)(k)),$$

the set of arbitrary $G_d(k)$-invariant functions $f : R_d(Q)(k) \to \mathbb{Q}$. The product structure is defined by

$$(f * g)(V) = \sum_{U\subseteq V} f(U)g(V/U)$$

where $f : R_d(Q)(k) \to \mathbb{Q}$ and $g : R_e(Q)(k) \to \mathbb{Q}$. Here V ranges over elements of $R_{d+e}(Q)(k)$ and the summation is over all subrepresentations $U \subseteq V$ with dimension vector d. V/U makes sense since f and g are invariant functions, so if we make a choice of basis for V/U we get an element of $R_e(Q)(k)$. We conclude that the above sum is finite and well-defined.

$\mathrm{H}_k(Q)$ is an associative $\mathbb{N}I$-graded $\mathbb{Q}$-algebra with unit $1 : R_0(Q)(k) \to \mathbb{Q}$ that assigns 1 to the single point of $R_0(Q)(k)$.

§1.5.2. Let $\mathbf{1} : V \mapsto 1$ be the constant function and for any $\mu \in \mathbb{Q}$ define

$$\mathbf{1}_\mu(V) = \begin{cases} 1 & V \text{ is semistable of slope } \mu \text{ or if } V \text{ is the zero representation} \\ 0 & \text{otherwise.} \end{cases}$$

The above elements of $H_k(Q)$ are related as follows.

$$\begin{aligned} \overleftarrow{\prod_{\mu\in\mathbb{Q}}} \mathbf{1}_\mu(V) &:= \sum_{\mu_1 > \cdots > \mu_s} (\mathbf{1}_{\mu_1} * \cdots * \mathbf{1}_{\mu_s})(V) \\ &= \sum_{\mu_1 > \cdots > \mu_s} \sum_{0=V_0 \subset \cdots \subset V_s = V} (\mathbf{1}_{\mu_1}(V_1/V_0) * \cdots * \mathbf{1}_{\mu_s}(V_s/V_{s-1})) \end{aligned}$$

The latter summation is the cardinality of the set

$\{0 = V_0 \subset \cdots \subset V_s = V :$ all factors are semistable and their slopes are strictly decreasing$\}$

which by definition is the number of Harder-Narasimhan filtrations of V. But the Harder-Narasihman filtration is unique hence we have proved the following lemma which implies that we can extract information about stability of representations from the Hall algebra.

Lemma 1 (Harder-Narasimhan recursion)**.** *We have an identity* $\mathbf{1} = \overleftarrow{\prod}_{\mu\in\mathbb{Q}} \mathbf{1}_\mu$.

1.6 Integration on Hall algebras

§1.6.1. Let $\mathbb{Q}_q[\![I]\!] = \mathbb{Q}[\![t_i : i \in I]\!]$ be the skew ring of power series in variables t_i, with the product structure

$$t^d.t^e = q^{-\langle e,d\rangle} t^{d+e}$$

where we have used the notation $t^d = \prod t_i^{d_i}$ for convenience and q is the size of the finite field as in the previous section.

Definition 4. The *integration* map on the Hall algebra $\mathrm{H}_k(Q)$ is defined via

$$\begin{aligned} \int : \mathrm{H}_k(Q) &\to \mathbb{Q}_q[\![I]\!] \\ f &\mapsto \sum_{[V]} \frac{f(V)}{|\operatorname{Aut}(V)|} t^{\underline{\dim} V} \end{aligned}$$

here $[V]$ ranges over all isomorphism classes.

Our key observation is the following

Proposition 3. $\int : \mathrm{H}_k(Q) \to \mathbb{Q}_q [\![I]\!]$ *is a* $\mathbb{Q}$*-algebra homomorphism.*

Proof. Note that the characteristic functions 1_V of orbits (i.e. isomorphism classes in $\mathcal{Rep}_k(Q)$) form a basis for the algebra. So it suffices to prove $\int(\mathbf{1}_V * \mathbf{1}_W) = \int \mathbf{1}_V * \int \mathbf{1}_W$. This is equivalent to proving

$$\sum_X \#\{U \subseteq X : U \cong V, X/U \cong W\} \frac{1}{|\operatorname{Aut}(X)|} t^{\underline{\dim} X} = \frac{q^{-\langle \dim W, \dim V\rangle}}{|\operatorname{Aut} V||\operatorname{Aut} W|} t^{\dim V + \dim W}. \tag{1.6.1}$$

Consider the set of all short exact sequences $0 \to V \to X \to W \to 0$ of representations and call it $\mathcal{S}$. Two short exact sequences are equivalent if they fit into a commutative diagram

$$\begin{tikzcd} V \arrow[r] \arrow[d, "\mathrm{id}"'] & X \arrow[r] \arrow[d, "\cong"] & W \arrow[d, "\mathrm{id}"] \\ V \arrow[r] & Y \arrow[r] & W \end{tikzcd}$$

commutes. Then as a set $\operatorname{Ext}^1(W,V)_X$ is the set of extensions $(V \to X \to W)$ where the middle term is isomorphic to X, up to the above equivalence of extensions.

On the set $\mathcal{S}$ the groups $\operatorname{Aut} V$, $\operatorname{Aut} W$ and $\operatorname{Aut} X$ act. It is clear that $\operatorname{Aut} V \times \operatorname{Aut} W$ acts freely but simple diagram chasing shows that $\operatorname{Aut} X$ acts with stabilizer $1 + \alpha \operatorname{Hom}(W,V)\beta$ where α and β are the maps fitting in the short exact sequence of the trivial extension

$$0 \to V \xrightarrow{\alpha} V \oplus W \xrightarrow{\beta} W \to 0.$$

If we mod out $\mathcal{S}$ by $\operatorname{Aut} X$ then we get the orbit space $\operatorname{Ext}^1(W,V)_X$ and if we mod out by $\operatorname{Aut} V \times \operatorname{Aut} W$ we get our set of interest

$$\{U \subseteq X : U \cong V, X/U \cong W\}.$$

So the left hand side of equation 1.6.1 is equal to

$$\begin{aligned} &\sum_X \frac{|\operatorname{Ext}^1(W,V)_X||\operatorname{Aut} X|}{|\operatorname{Aut} V||\operatorname{Aut} W||\operatorname{Hom}(W,V)|} \frac{1}{|\operatorname{Aut} X|} t^{\underline{\dim} V + \underline{\dim} W} \\ &= \frac{t^{\underline{\dim} V + \underline{\dim} W}}{|\operatorname{Aut} V||\operatorname{Aut} X||\operatorname{Hom}(W,V)|} \sum_X |\operatorname{Ext}^1(W,V)_X| \\ &= \frac{t^{\underline{\dim} V + \underline{\dim} W}}{|\operatorname{Aut} V||\operatorname{Aut} X|} \frac{|\operatorname{Ext}^1(W,V)|}{|\operatorname{Hom}(W,V)|} \\ &= \frac{t^{\underline{\dim} V + \underline{\dim} W}}{|\operatorname{Aut} V||\operatorname{Aut} X|} q^{\dim \operatorname{Ext}^1(W,V) - \dim \operatorname{Hom}(W,V)}. \end{aligned}$$

This completes the proof. ∎

1.7 Cohomology and arithmetic of $\mathcal{M}_d^{\Theta-pst}(Q)$

§1.7.1. We want to exploit the identity $\int \mathbf{1} = \int \prod_\mu \mathbf{1}_\mu$. On the left hand side we have

$$\int \mathbf{1} = \sum_{[V]} \frac{1}{|\operatorname{Aut} V|} t^{\underline{\dim} V} = \sum_{d \in \mathbb{N}I} \frac{|R_d(Q)(k)|}{|G_d(k)|}.t^d = \sum_{d \in \mathbb{N}I} \frac{q^{-\langle d,d \rangle}}{\prod_{i \in I} \prod_{j=1}^{d_i}(1-q^{-j})} t^d.$$

and on the other side

$$\int \mathbf{1}_\mu = 1 + \sum_{d:\mu(d)=\mu} \frac{|R_d^{sst}(Q)(k)|}{|G_d(k)|} t^d.$$

Since integration is a homomorphism we get

$$\sum_{d \in \mathbb{N}I} \frac{q^{-\langle d,d \rangle}}{\prod_{i \in I} \prod_{j=1}^{d_i}(1-q^{-j})} t^d = \overleftarrow{\prod_{\mu \in \mathbb{Q}}} (1 + \sum_{d:\mu(d)=\mu} \frac{|R_d^{sst}(Q)(k)|}{|G_d(k)|} t^d).$$

Comparing coefficients gives a recursive expression for $\frac{|R_d^{sst}(Q)(k)|}{|G_d(k)|}$ in terms of coefficients on the left hand side. This counts the number of rational points over finite fields:
Corollary 1. *We have*

$$\frac{|R_d^{sst}(Q)(k)|}{|G_d(k)|} = \sum (-1)^{s-1} q^{-\sum_{i<j} \langle d^j, d^i \rangle} \prod_{i=1}^{s} \frac{|R_{d^i}^{sst}(Q)|}{|G_{d^i}|}$$

where the sum is over all partitions $d_1 + \cdots + d_s = d$ of d into non-zero dimension vectors such that $\mu(\sum_{j=1}^{i} d^j) > \mu(d)$ for all $i < s$.

§1.7.2. We make the assumption that d is coprime and Θ is generic, in which setting

$$R_d(Q)^{st} \to \mathcal{M}_d^{\Theta-poly}(Q) = \mathcal{M}_d^{\Theta-st}(Q)$$

is a $\mathbb{P}G_d$-principal bundle over smooth base.

By Seshari's variant of geometric invariant theory over rings[22] there is a scheme X over $\mathcal{S}pec\,\mathbb{Z}$ such that $\mathcal{S}pec\,\mathbb{C} \times_{\mathbb{Z}} X = \mathcal{M}_d^{\Theta-st}(Q)$. Hence over the k-points the above fact implies

$$|\mathcal{M}_d^{\Theta-st}(Q)(k)| = \frac{|R_d(Q)^{st}(k)|}{|\mathbb{P}G_d(k)|} = (q-1)\frac{|R_d(Q)^{sst}(k)|}{|G_d(k)|}. \tag{1.7.1}$$

By corollary 1 we conclude that $|\mathcal{M}_d^{\Theta-st}(Q)(\mathbb{F}_q)|$ is a rational polynomial in q.

§1.7.3. Recall that Deligne's theorem (Weil's conjecture)[7] shows that if the number of rational points of a scheme it is counted by a rational polynomial in the size of the field then it is actually a polynomial with integer coefficients. In our case $\mathcal{M}_d^{\Theta-st}(Q)$ is a smooth projective and the same result of Deligne shows that this polynomial is the Poincare polynomial

$$|\mathcal{M}_d^{\Theta-st}(Q)(\mathbb{F}_q)| = \sum_{i \geq 0} \dim H^i(\mathcal{M}_d^{\Theta-st}(Q), \mathbb{Q}) q^{i/2}.$$

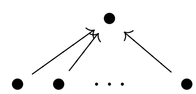
In particular the coefficients are non-negative integers.

One can explicitly work out the recursive formula obtained from corollary 1 and equation 1.7.1 to get

$$\sum_i \dim H^i(\mathcal{M}_d^{st}(Q),\mathbb{Q})q^{i/2} = (q-1)\sum(-1)^{s-1}q^{-\sum_{k\leq\ell}\langle d^\ell,d^k\rangle}\prod_{k=1}^{s}\prod_{i\in I}\prod_{j=1}^{d_i^k}(1-q^{-j})^{-1}. \tag{1.7.2}$$

where the sum is over all partitions $d^1+\cdots+d^s = d$ of d into non-zero dimension vectors such that $\mu(\sum_{j=1}^i d^j) > \mu(d)$ for all $i < s$. This gives an explicit computation of all the Betti numbers of our moduli space. Note that $q = 1$ is a singularity of the right hand series so this result fails to give us the computation of the Euler characteristic.

Example 1.7.3. It is possible to make formula 1.7.2 explicit for several classes of quivers. One such class of examples is that of quivers of the form



with vertices $i_0, i_1, \cdots, i_n$ and arrows $i_k \to i_0$ for $k = 1, \cdots, n$. We restrict ourselves to the dimension vectors of the form $(m, 1, \cdots, 1)$. Making right hand side of formula 1.7.2 is relatively easy when the stability condition corresponds to $\Theta(d) = -d_0$. However the counting problem of all possible partitions get involved in the cases that this choice of Θ is not coprime and we have to find an alternative one.

For instance it is straightforward to work out the case of dimension vector $d = (2, 1, 1, 1)$ when the quiver has four vertices. In this case the above choice of Θ is coprime. We need to consider all partitions of d such that the partial sums $\sum_{j=1}^i d^j = (b, a_1, a_2, a_3)$ satisfy

$$a_1 + a_2 + a_3 > \frac{3b}{2}.$$

We may rewrite the formula 1.7.2 as

$$\sum_i \dim H^i(\mathcal{M}_d^{st}(Q),\mathbb{Q})q^{i/2} = 1 + \frac{q^5}{(q+1)(q-1)^4}S_1 + \frac{q^5}{(q-1)^4}S_2 + \frac{q^4}{(q-1)^4}S_3$$

where S_1 is the number of nontrivial partitions ending with $d_0^s = 2$, S_2 is the number of partitions ending with

$$d^s = (1,0,0,0), d^{s-1} = (1,1,1,1)$$

and S_3 is the number of partitions ending with $d_0^s = 1$ and one of d^{s-1} or d^{s-2} is from the set

$$\{(1,1,0,1),(1,0,1,1),(0,1,1,1)\}.$$

Counting shows that $S_1 = S_2 = S_3 = 0$ which agrees with the fact that the moduli space of three points on $\mathbb{P}^1$, i.e. $(\mathbb{P}^1)^3_{ss}/\!\!/\,\mathbb{P}\mathrm{GL}_2$ is a point. ⌟

Chapter 2

Hall algebras and DT/PT correspondence

2.1 A review of curve counting invariants

§2.1.1. In this chapter, we will be working over the base field $\mathbb{C}$. Throughout, M will denote a fixed smooth complex projective Calabi-Yau threefold and by this we mean that the canonical bundle K_M is trivial and $H^1(M, \mathcal{O}_M) = 0$. Let $\mathcal{L}$ be a fixed very ample line bundle on M as well. For a class $\beta \in H_2(M, \mathbb{Z})$ and an integer $n \in \mathbb{Z}$ there is a scheme $\mathrm{Hilb}_M(\beta, n)$ that parametrizes the projections

$$\mathcal{O}_M \twoheadrightarrow E$$

where E is a coherent sheaf of Hilbert polynomial

$$\chi(E(k)) = k \int_\beta c_1(\mathcal{L}) + n.$$

In particular the Chern character of E is

$$(\mathrm{rk}(E), ch_1(E), ch_2(E), ch_3(E)) = (1, 0, -\beta, -n). \tag{2.1.1}$$

The Donaldson-Thomas invariant of M corresponding to the pair (β, n) is now defined to be

$$\mathrm{DT}_M(\beta, n) = \sum_{n \in \mathbb{Z}} n \chi(\nu^{-1}_{\mathrm{Hilb}_M}(n)).$$

Here ν is the Behrend function [1] of this scheme, which is a constructible function

$$\nu_{\mathrm{Hilb}_M} : \mathrm{Hilb}_M(\beta, n) \to \mathbb{Z}.$$

A key property of the Behrend function that we would like to recall is that if $f : X \to Y$ is a smooth morphism of relative dimension n then the Behrend functions of the two schemes are related via

$$\nu_X = (-1)^n f^*(\nu_Y). \tag{2.1.2}$$

This can be used to extend Behrend functions to any algebraic stack locally of finite type over $\mathbb{C}$.

§2.1.2. There is another invariant associated to M and (β, n) defined by Pandharipande and Thomas [17]. They first construct another moduli space

$$\mathrm{Hilb}^{\#}_{M}(\beta, n)$$

parametrizing objects

$$\mathcal{O}_M \xrightarrow{s} E$$

where E still has the same Hilbert polynomial as above (hence of rank one) but

1. is a pure sheaf (i.e. has no subsheaf supported in dimension zero), and
2. $\mathrm{coker}(s)$ is supported in dimension zero.

Such a map $\mathcal{O}_X \xrightarrow{s} E$ is called a *stable pair.* To construct this scheme as a moduli space Le Potier [12] considers the GIT problem of pairs (E, s), consisting of a sheaf E with support of dimension at most 1 and a section $\mathcal{O}_M \xrightarrow{s} E$. There is a certain slope-stability condition on these objects, for which stable objects coincide with the stable pairs defined above and the semistable objects coincide with maps $\mathcal{O}_X \xrightarrow{s} E$ which are again pure but instead of condition (2) satisfy

(2') s is a nonzero section.

Consequently the scheme $\mathrm{Hilb}^{\#}_{M}(\beta, n)$ can be constructed using geometric invariant theory as the moduli space of semistable pairs. Now the *Pandharipande-Thomas invariant* of M for vector (β, n) is the same weighted Euler characteristic computed for this scheme

$$\mathrm{PT}_M(\beta, n) = \sum_{n \in \mathbb{Z}} n\chi(\nu^{-1}_{\mathrm{Hilb}^{\#}_{M}}(n)).$$

§2.1.3. It was conjectured (together with many other things) that the above two invariants are related as follows [17, Conjecture 3.3.]:

1. The Laurent series
$$\mathrm{DT}(\beta) = \sum_{n \in \mathbb{Z}} \mathrm{DT}(\beta, n) t^n$$
and
$$\mathrm{PT}(\beta) = \sum_{n \in \mathbb{Z}} \mathrm{PT}(\beta, n) t^n$$
are related by the identity
$$\mathrm{DT}(\beta) = \mathrm{PT}(\beta).\mathrm{DT}(0). \tag{2.1.3}$$
2. $\mathrm{PT}(\beta)$ is the Laurent expansion of a rational function of t which is invariant under $t \mapsto t^{-1}$.

The rest of this essay will explain a proof of part (1) of the above conjecture. Before that we will fix some notations. We will need to work with moduli stacks rather than the above moduli spaces. We start with the abelian category of coherent sheaves on M. $\mathscr{A}$ will always denote this category and $\mathcal{M}$ the moduli stack of objects of $\mathscr{A}$. This means that the objects of $\mathcal{M}$ over a scheme S are coherent sheaves on $S \times M$ flat over S. We will use the same notation for a scheme and the stack it represents. In particular from now on Hilb_M and $\mathrm{Hilb}_M^{\#}$ will rather denote corresponding stacks over $\mathbb{C}$. For a full subcategory $\mathscr{B}$ of $\mathscr{A}$, that yields an algebraic stack, the corresponding open substack of $\mathcal{M}$ will be denoted by

$$\mathcal{M}_{\mathscr{B}} \subset \mathcal{M}_{\mathscr{A}}.$$

2.2 The moduli stacks and some of its useful variants

§2.2.1. Let $K(\mathscr{A})$ be the Grothendieck group of the category $\mathscr{A}$. The bilinear form

$$\chi(E,F) = \sum_i (-1)^i \dim_{\mathbb{C}} \mathrm{Ext}^i(E,F)$$

is called the Euler form on $K(\mathscr{A})$. By Serre duality, the sets of left and right orthogonal objects to $\mathscr{A}$ with respect to the Euler form, i.e.

$\{E : \chi(E,F) = 0$ for all objects F in $\mathscr{A}\ \}$ and $\{F : \chi(E,F) = 0$ for all objects E in $\mathscr{A}\ \}$,

are the same subgroup $K(\mathscr{A})^{\perp}$ and therefore the quotient

$$N(M) = K(\mathscr{A})/K(\mathscr{A})^{\perp}$$

called the *numerical Grothendieck group* carries a well-defined bilinear form. There is a monoid $\Gamma \subset N(M)$ consisting of classes of sheaves.

§2.2.2. The two full subcategories

$$\mathscr{T} = \{E \in \mathscr{A} : \dim \mathrm{supp}(E) = 0\}$$

and

$$\mathscr{F} = \{E \in \mathscr{A} : \mathrm{Hom}_{\mathscr{A}}(T,E) = 0, \forall T \in \mathscr{T}\}$$

form a torsion pair. By tilting the standard t-structure of $D^b(\mathscr{A})$ with respect to this torsion pair, we get another t-structure on $D^b(\mathscr{A})$ with heart

$$\mathscr{A}^{\#} = \{E \in D : H^0(E) \in \mathscr{F}, H^1(E) \in \mathscr{T}, \text{ and } H^i(E) = 0 \text{ otherwise}\}.$$

In particular $\mathcal{O}_M$ is in the heart. The nice thing about this t-structure is that a stable pair as defined above is precisely an epimorphism in $\mathscr{A}^{\#}$ whose image is supported in dimension at most one. In fact take any short exact sequence

$$0 \to \ker f \to \mathcal{O}_M \xrightarrow{f} E \to 0$$

in $\mathscr{A}^{\#}$. Applying cohomology to it, the nontrivial part of the long exact sequence is

$$0 \to H^0(\ker f) \to \mathcal{O}_M \to H^0(E) \to H^1(\ker) \to 0.$$

So E has no higher cohomologies and therefore is a sheaf $H^0(E) = E \in \mathscr{A}$. But $\mathscr{A} \cap \mathscr{A}^\# = \mathscr{F}$ and thus $E \in \mathscr{F}$. Conversely a pair $\mathcal{O}_M \xrightarrow{f} E$ fits in a triangle

$$\mathcal{I} \to \mathcal{O}_M \to E \to \mathcal{I}[1]$$

in $\mathscr{A}$ by rotating the triangle $\mathcal{O}_M \xrightarrow{f} E \to Cf \to \mathcal{O}_M[1]$. We get the long exact sequence

$$0 \to H^0(\mathcal{I}) \to \mathcal{O}_M \to H^0(E) \to H^1(\mathcal{I}) \to 0.$$

So $\mathcal{I} = H^0(\mathcal{I})$ is a subsheaf of $\mathcal{O}_M$. But $\mathcal{O}_M$ is an object of $\mathscr{F}$ and therefore $\mathcal{I} \in \mathscr{F}$. Since $\mathscr{F}$ is a subcategory of $\mathscr{A}^\#$ we also get $\operatorname{coker} f = H^1(\mathcal{I}) \in \mathscr{T}$. We conclude that $\mathcal{I} \in \mathscr{A}^\#$ and that the triangle is a short exact sequence in $\mathscr{A}^\#$.

§2.2.3. We shall restrict to the category of coherent sheaves supported in dimension at most one. These form a full subcategory $\mathscr{A}_{\leq 1} \subset \mathscr{A}$. The objects of it form an open and closed substack

$$\mathcal{M}_{\leq 1} \subset \mathcal{M}.$$

They also generate a subgroup $N_{\leq 1}(M) \subset N(M)$ and the monoid Γ intersects it in

$$\Gamma_{\leq 1} = N_{\leq 1}(M) \cap \Gamma.$$

In view of the characterization 2.1.1 this *effective cone* can be identified [4, Lemma 2.2] with

$$\Gamma_{\leq 1} = \{(\beta, n) \in A_1(M) \oplus \mathbb{Z} : \beta > 0, \text{ or } \beta = 0 \text{ and } n \geq 0\}.$$

Also observe that for any $\gamma \in \Gamma$ there is an open and closed substack $\mathcal{M}_\gamma \subset \mathcal{M}$ which results in the decomposition

$$\mathcal{M}_{\leq 1} = \coprod_{\gamma \in \Gamma_{\leq 1}} \mathcal{M}_\gamma. \tag{2.2.1}$$

We will make a small abuse of notation and denote the full subcategory

$$\mathscr{F} \cap \mathscr{A}_{\leq 1} \subset \mathscr{A}_{\leq 1}$$

by the same notation $\mathscr{F}$. This should arise no confusions since in what follows all equations involving elements of Hall algebras will be that of categories restricted to sheaves supported in dimensions ≤ 1.

§2.2.4. The moduli spaces constructed in section 2.1 can be viewed as open substacks of the *stack of framed sheaves* denote by $\mathcal{M}(\mathcal{O})$. The stack $\mathcal{M}(\mathcal{O})$ is an algebraic stack with objects lying over a scheme S being pairs (E, γ) of an S-flat coherent sheaf E on $S \times M$ and a section

$$\gamma : \mathcal{O}_{S \times M} \to E.$$

There is an obvious morphism

$$q : \mathcal{M}(\mathcal{O}) \to \mathcal{M} \tag{2.2.2}$$

which is representable and of finite type. Then

$$\mathrm{Hilb}_M \subset \mathcal{M}(\mathcal{O}), \text{ and } \mathrm{Hilb}_M^\# \subset \mathcal{M}(\mathcal{O})$$

are the open substacks whose $\mathbb{C}$-valued points are morphisms $\mathcal{O}_M \to E$ that are epimorphisms in the category $\mathscr{A}$ and $\mathscr{A}^\#$ respectively.

2.3 Motivic Hall algebras

§2.3.1. Let $\mathfrak{S}$ be an algebraic stack, locally of finite type over $\mathbb{C}$. We say that $\mathfrak{S}$ has affine stabilizers if for any $\mathbb{C}$-valued point $x \in \mathfrak{S}(\mathbb{C})$, the algebraic group $\mathcal{I}som_{\mathbb{C}}(x, x)$ is affine. By [4, Proposition 3.4] the condition of having affine stabilizers is equivalent to existence of a variety R with an action of a general linear group G, and a representable morphism of stacks

$$f : [R/G] \to \mathfrak{S}$$

such that the induced mapping $[R/G](\mathbb{C}) \to \mathfrak{S}(\mathbb{C})$ on $\mathbb{C}$-valued points is a bijection.

§2.3.2. Let $\mathfrak{S}$ be an algebraic stack, locally of finite type over $\mathbb{C}$ with affine stabilizers. There is a 2-category of algebraic stacks over $\mathfrak{S}$. Let $\mathrm{St}/\mathfrak{S}$ be the full subcategory of objects $X \xrightarrow{f} \mathfrak{S}$ for which X if of finite type over $\mathbb{C}$. We say the morphism $X \to \mathfrak{S}$ has affine stabilizers if X does. The *relative Grothendieck group* $K(\mathrm{St}/\mathfrak{S})$ is the complex vector space spanned by equivalence classes of objects

$$[X \xrightarrow{f} \mathfrak{S}]$$

with affine stabilizers modulo the relations

- for every pair of stacks X and Y

$$[X_1 \xrightarrow{f_1} \mathfrak{S}] + [X_2 \xrightarrow{f_2} \mathfrak{S}] = [X_1 \coprod X_2 \xrightarrow{f_1 \coprod f_2} \mathfrak{S}],$$

- if $g : X_1 \to X_2$ is a geometric bijection over $\mathfrak{S}$, then

$$[X_1 \xrightarrow{f_1} \mathfrak{S}] = [X_2 \xrightarrow{f_2} \mathfrak{S}],$$

- for every pair of Zariski fibrations (i.e. presentable morphisms of stacks with all their pullbacks to schemes being Zariski trivial fibrations of schemes)

$$h_1 : X_1 \to Y, \text{ and } h_2 : X_2 \to Y$$

  with same fibers and for every morphism $g : Y \to \mathfrak{S}$, we require

$$[X_1 \xrightarrow{g \circ h_1} \mathfrak{S}] = [X_2 \xrightarrow{g \circ h_2} \mathfrak{S}].$$

Every scheme is a representable stack so we have a homomorphism of commutative rings

$$K(\mathrm{Var}/\mathbb{C}) \to K(\mathrm{St}/\mathbb{C}).$$

The reason we impose the condition of having affine stabilizers is to obtain an isomorphism

$$K(\mathrm{Var}/\mathbb{C})[\mathbb{L}^{-1}][(\mathbb{L}^i - 1)^{-1} : i \geq 1] \cong K(\mathrm{Var}/\mathbb{C})[\mathrm{GL}(d)^{-1} : d \geq 1] \xrightarrow{\cong} K(\mathrm{St}/\mathbb{C}).$$

Here $\mathbb{L} = [\mathbb{A}^1] \in K(\mathrm{Var}/\mathbb{C})$ is the class of affine line.

§2.3.3. There is an algebraic stack, $\mathcal{M}^{(2)}$, of short exact sequence of objects of $\mathscr{A}$ with morphisms $a_1, a_2, b : \mathcal{M}^{(2)} \to \mathcal{M}$ sending a short exact sequence

$$0 \to A_1 \to B \to A_2 \to 0$$

to the sheaves A_1, A_2 and B respectively. The motivic Hall algebra [4, Section 4] is the vector space

$$\mathrm{H}(\mathscr{A}) = K(\mathrm{St}/\mathcal{M}).$$

We define the so called *convolution product* on elements of the Hall algebra via

$$[X_1 \xrightarrow{f_1} \mathcal{M}] * [X_2 \xrightarrow{f_2} \mathcal{M}] = [Z \xrightarrow{b \circ h} \mathcal{M}]$$

where h is the morphism filling the cartesian square

$$\begin{tikzcd} Z \arrow[r, "h"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{M}^{(2)} \arrow[r, "b"] \arrow[d, "(a_1,a_2)"] & \mathcal{M} \\ X_1 \times X_2 \arrow[r] & \mathcal{M} \times \mathcal{M}. & \end{tikzcd}$$

With this product structure $H(\mathscr{A})$ is a non-commutative associative unital algebra over $K(\mathrm{St}/\mathbb{C})$ with unit element $[\mathcal{S}pec(\mathbb{C}) \hookrightarrow \mathcal{M}]$.

§2.3.4. Some of the objects of our interest live in an "infinite-type" version of the Hall-algebra. It is conceptually worthwhile to define this algebra since when working with formal sums over this extended algebra we will not have to worry about the possible convergence issues. Given an algebraic stack $\mathfrak{S}$ we first define an infinite-type Grothendieck group $L(\mathrm{St}_\infty/\mathfrak{S})$ by considering symbols $[f : X \to \mathfrak{S}]$ as in §II.2.3.2 but with the condition that X is merely locally of finite type over $\mathbb{C}$ modulo the same relations as before except the first one.

- If $g : X_1 \to X_2$ is a geometric bijection over $\mathfrak{S}$, then

$$[X_1 \xrightarrow{f_1} \mathfrak{S}] = [X_2 \xrightarrow{f_2} \mathfrak{S}],$$

- for every pair of Zariski fibrations

$$h_1 : X_1 \to Y, \text{ and } h_2 : X_2 \to Y$$

  with same fibers and for every morphism $g : Y \to \mathfrak{S}$, we require

$$[X_1 \xrightarrow{g \circ h_1} \mathfrak{S}] = [X_2 \xrightarrow{g \circ h_2} \mathfrak{S}].$$

Note that otherwise the three conditions together give the zero vector space. The infinite-type Hall algebra associated to $\mathscr{A}$ is the vector space

$$\mathrm{H}_\infty(\mathscr{A}) = L(\mathrm{St}_\infty/\mathscr{A})$$

with analogous product structure as in §II.2.3.3.

§2.3.5. Now we list some of the interesting elements of the infinite-type Hall algebra. For any open substack $\mathcal{N} \subseteq \mathcal{M}$ we will use the notation

$$1_{\mathcal{N}} = [\mathcal{N} \hookrightarrow \mathcal{M}] \in \mathrm{H}_\infty(\mathscr{A}).$$

By pulling back $\mathcal{N} \hookrightarrow \mathcal{M}$ along q defined in equation 2.2.2 we get

$$1^{\mathcal{O}}_{\mathcal{N}} = [\mathcal{N}(\mathcal{O}) \xrightarrow{q} \mathcal{M}] \in \mathrm{H}_\infty(\mathscr{A}).$$

The restriction of q to the Hilbert scheme and moduli stack of stable pairs are the other element of interest to us

$$\mathcal{H} = [\mathrm{Hilb}_M \xrightarrow{q} \mathcal{M}] \in \mathrm{H}_\infty(\mathscr{A}), \quad \mathcal{H}^{\#} = [\mathrm{Hilb}^{\#}_M \xrightarrow{q} \mathcal{M}] \in \mathrm{H}_\infty(\mathscr{A}).$$

2.4 Integration on Hall algebras

§2.4.1. Let $\mathbf{H} \subset \mathrm{H}(\mathscr{A})$ be the subalgebra spanned by symbols

$$[X \xrightarrow{f} \mathcal{M}]$$

that factor through the inclusion $\mathcal{M}_{\leq 1} \to \mathcal{M}$. Recall that $\mathrm{H}(\mathscr{A})$ is an algebra over the ring $K(\mathrm{St}/\mathbb{C})$. In view of the homomorphisms of commutative rings

$$K(\mathrm{Var}/\mathbb{C}) \to K(\mathrm{Var}/\mathbb{C})(\mathbb{L}^{-1}) \to K(\mathrm{St}/\mathbb{C})$$

the $K(\mathrm{Var}/\mathbb{C})(\mathbb{L}^{-1})$-submodule of $\mathrm{H}(\mathscr{A})$ spanned by regular classes (i.e. that of maps $[f : X \to \mathcal{M}]$ with X a variety) is closed under the convolution product and is therefore a $K(\mathrm{Var}/\mathbb{C})[\mathbb{L}^{-1}]$-subalgebra

$$\mathbf{H}_{reg} \subset \mathbf{H}.$$

A next observation [4, Theorem 5.1] is that the quotient

$$\mathbf{H}_{sc} = \mathbf{H}_{reg}/(\mathbb{L} - 1)\mathbf{H}_{reg}$$

known as the *semi-classical Hall algebra* is a commutative $K(\mathrm{Var}/\mathbb{C})$-algebra and is equipped with an extra structure, that of a Poisson bracket defined by

$$\{f, g\} = \frac{f * g - g * f}{\mathbb{L} - 1} \mod (\mathbb{L} - 1).$$

By equation 2.2.1 these are all $\Gamma_{\leq 1}$-graded algebras.

§2.4.2. Now let $\mathbb{C}(\Gamma_{\leq 1}) = \bigoplus_{\gamma \in \Gamma_{\leq 1}} \mathbb{C}.x^\gamma$ be the monoid-algebra of $\Gamma_{\leq 1}$ over field $\mathbb{C}$. It is a Poisson algebra with commutative product

$$x^\gamma * x^\delta = x^{\gamma+\delta}$$

and trivial Poisson bracket. We reserve the notation x^β for $x^{(\beta,0)}$ and q for $x^{(0,1)}$.

Our integration map is then a homomorphism of $\Gamma_{\leq 1}$-graded Poisson algebras

$$I : \mathbf{H}_{sc} \to \mathbb{C}[\Gamma_{\leq 1}]$$

which is uniquely determined [4, Theorem 5.2] by the property that

$$I([X \xrightarrow{f} \mathcal{M}_\gamma]) = \chi(X, f^*(\nu_{\mathcal{M}_{\leq 1}})).x^\gamma.$$

Here $\nu : \mathcal{M}_{\leq 1} \to \mathbb{Z}$ is the Behrend function of the stack $\mathcal{M}_{\leq 1}$ and the weight Euler characteristic χ is defined as usual:

$$\chi(X, f^*(\nu_{\mathcal{M}_{\leq 1}})) = \sum_{n \in \mathbb{Z}} n\chi((\nu_{\mathcal{M}_{\leq 1}} \circ f)^{-1}(n)).$$

§2.4.3. One final modification of the Hall algebra is to extend it to the group of formal sums of elements of $\mathbf{H}$ of the form

$$a = \sum_{\gamma \in \Delta \subset \Gamma_{\leq 1}} a_\gamma$$

that satisfy some boundedness conditions from below. These conditions are not very complicated [5, Section 5.2] but for sake of simplicity of notations we will neglect them and use the same symbol $\mathbf{H}$ for the group of formal sums created thus. Let $\pi_\gamma : \mathbf{H} \to \mathbf{H}_\gamma$ be the projection on the γ-graded piece of $\mathbf{H}$. We extend the $K(\mathrm{St}/\mathbb{C})$-algebra structure provided by the convolution product via

$$\pi_\sigma(a * b) = \sum_{\gamma + \delta = \sigma} \pi_\gamma(a) * \pi_\delta(b).$$

This also extends $\mathbf{H}_{reg}, \mathbf{H}_{sc}$ and the (graded) homomorphism I of Poisson algebras.

We are interested in morphisms of stacks $f : X \to \mathcal{M}_{\leq 1}$ that can be written as a formal sum

$$\sum_{\gamma \in S} [X_\gamma \xrightarrow{f} \mathcal{M}_{\leq 1}]$$

in the above algebra (this requires $X_\gamma = f^{-1}(\mathcal{M}_\gamma)$ to be of finite type for all γ and S is an index set satisfying boundedness conditions). It turns out that the open and closed subschemes

$$\mathrm{Hilb}_{M,\leq 1} = \mathcal{M}_{\leq 1} \cap \mathrm{Hilb}_M, \mathrm{Hilb}^\#_{M,\leq 1} = \mathcal{M}_{\leq 1} \cap \mathrm{Hilb}^\#_M$$

give morphisms

$$q : \mathrm{Hilb}_{M,\leq 1} \to \mathcal{M}_{\leq 1}, \quad q : \mathrm{Hilb}^\#_{M,\leq 1} \to \mathcal{M}_{\leq 1}$$

of the mentioned type. We let $\mathcal{H}$ and $\mathcal{H}^\#$ denote their corresponding elements in $\mathbf{H}$. So we can integrate them and we get

$$I(\mathcal{H}) = \sum_{(\beta,n) \in \Gamma_{\leq 1}} (-1)^n \mathrm{DT}(\beta, n) x^\beta q^n = \mathrm{DT}_\beta(-q).x^\beta \tag{2.4.1}$$

$$I(\mathcal{H}^\#) = \sum_{(\beta,n) \in \Gamma_{\leq 1}} (-1)^n \mathrm{PT}(\beta, n) x^\beta q^n = \mathrm{PT}_\beta(-q).x^\beta. \tag{2.4.2}$$

The component of the Hilbert scheme parametrizing zero-dimension subschemes of M also gives a morphism $q : \mathrm{Hilb}_{M,0} \to \mathcal{M}_{\leq 1}$ of the desire type and the integral of the corresponding element $\mathcal{H}_0$ of the algebra is

$$I(\mathcal{H}_0) = \sum_{(\beta,n)\in\Gamma_{\leq 1}} \mathrm{DT}(0,n)q^n = \mathrm{DT}_0(-q). \tag{2.4.3}$$

The reason equations 2.4.1, 2.4.2 and 2.4.3 hold is basically that Behrend functions of $\mathcal{M}_{\leq 1}$ and our moduli schemes are related with the following Behrend function identity proved by Pandharipande, Thomas and Bridgeland [4, Theorem 3.1]

$$\nu_{\mathcal{M}(\mathcal{O})} = (-1)^{\chi\circ q}.\,(\nu_{\mathcal{M}} \circ q)$$

which is analogous to the identity of equation 2.1.2.

2.5 Truncated moduli stacks and identities in limit

§2.5.1. As in the case of Hall algebras associated to representations of quivers the idea is to prove some identities in a certain Hall algebra and integrate them in a sense similar to that of Chapter 1. The subtlety is that some of the elements of our interest will not be in the Hall algebra we shall be working in but in an extended one (see §II.2.3.4). In order to overcome this technical difficulty we will introduce some truncated versions of the above moduli spaces in terms of the stability conditions and define a notion of *identity in limit* as will come up. The stability condition that Bridgeland works with is the slope stability

$$\mu(\gamma) = \frac{c_3(\gamma)}{c_2(\gamma)[\mathcal{L}]}.$$

Since $c_2(\gamma)$ can be zero, the slopes are rational numbers ranging in $(-\infty, +\infty]$. Stability, semistability, Harder-Narasimhan filtrations are all defined in the usual way. The important observation is that for any interval $I \subset (-\infty, +\infty]$, there is a full subcategory

$$\mathscr{S}_I \subset \mathscr{A}_{\leq 1}$$

consisting of zero objects and sheaves whose Harder-Narasimhan factors all have slopes in I. In particular

$$\mathscr{T} = \mathscr{S}_{\{+\infty\}}, \quad \text{and} \quad \mathscr{F} = \mathscr{S}_{(-\infty,+\infty)}.$$

§2.5.2. In $H_\infty(\mathscr{A})$ we have identities

$$1_{\mathcal{M}} = 1_{\mathcal{M}_{\mathscr{T}}} * 1_{\mathcal{M}_{\mathscr{F}}} \tag{2.5.1}$$

$$1^{\mathcal{O}}_{\mathcal{M}} = 1^{\mathcal{O}}_{\mathcal{M}_{\mathscr{T}}} * 1^{\mathcal{O}}_{\mathcal{M}_{\mathscr{F}}}. \tag{2.5.2}$$

The proofs [4, Lemma 4.1 and 4.2] are nice exercises in 2-categories. For instance to prove 2.5.1 it suffices to prove that the right hand side product

$$[Z \to \mathcal{M}]$$

given by the diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & \mathcal{M}^{(2)} \arrow[r] \arrow[d] & \mathcal{M} \\ \mathcal{M}_{\mathscr{T}} \times \mathcal{M}_{\mathscr{F}} \arrow[r] & \mathcal{M} \times \mathcal{M} & \end{tikzcd}$$

is an equivalence on $\mathbb{C}$-valued points as in §II.2.3.2. But the groupoid of S-valued points of Z is identical to short exact sequences

$$0 \to P \to E \to Q \to 0$$

of S-flat sheaves, with P in $\mathscr{T}$ and Q in $\mathscr{F}$. $(\mathscr{T}, \mathscr{F})$ is a torsion pair so we are done since any object E fits in a unique short exact sequence as above. This is the type of argument that will be used in proving the other Hall algebra identities that will show up as well. Hence we will skip the proofs and refer the reader to [4] for details.

§2.5.3. To over come the convergence issue in the above identities when working in $\mathbf{H}$, we exchange them by truncated ones

$$1_{\mathcal{M}_{\mathscr{S}_{[\mu,\infty]}}} = 1_{\mathscr{T}} * 1_{\mathcal{M}_{\mathscr{S}_{[\mu,\infty)}}}, \quad \text{and} \quad 1^{\mathcal{O}}_{\mathcal{M}_{\mathscr{S}_{[\mu,\infty]}}} = 1^{\mathcal{O}}_{\mathscr{T}} * 1^{\mathcal{O}}_{\mathcal{M}_{\mathscr{S}_{[\mu,\infty)}}}$$

and observe that they hold as $\mu \to -\infty$. Such an 'approximation' for the identities in $\mathrm{H}_\infty(\mathscr{A})$ will be written shortly as

$$1^{\mathcal{O}}_{\mathcal{M}} \approx 1^{\mathcal{O}}_{\mathcal{M}_{\mathscr{T}}} * 1^{\mathcal{O}}_{\mathcal{M}_{\mathscr{F}}} \tag{2.5.3}$$

$$1_{\mathcal{M}} \approx 1_{\mathcal{M}_{\mathscr{T}}} * 1_{\mathcal{M}_{\mathscr{F}}}. \tag{2.5.4}$$

2.6 DT/PT correspondence

§2.6.1. We want to prove

$$I(\mathcal{H}) = I(\mathcal{H}_0) I(\mathcal{H}^{\#}). \tag{2.6.1}$$

There are three identities

$$1^{\mathcal{O}}_{\mathcal{M}} \approx \mathcal{H} * 1_{\mathcal{M}} \tag{2.6.2}$$

$$1^{\mathcal{O}}_{\mathcal{M}_{\mathscr{T}}} = \mathcal{H}_0 * 1_{\mathcal{M}_{\mathscr{T}}} \tag{2.6.3}$$

$$1^{\mathcal{O}}_{\mathcal{M}_{\mathscr{F}}} \approx \mathcal{H}^{\#} * 1_{\mathcal{M}_{\mathscr{F}}} \tag{2.6.4}$$

where only the middle one contains terms of finite type over $\mathbb{C}$ and holds strictly. The other two hold in limits as $\mu \to -\infty$.

There is a whole machinery of Joyce that can be used to show that the elements

$$1_{\mathcal{M}}, 1_{\mathcal{M}_{\mathscr{T}}}, \text{ and } 1_{\mathcal{M}_{\mathscr{F}}}$$

are invertible [4, Theorem 6.3]. So we have

$$\begin{aligned}
\mathcal{H} * 1_{\mathcal{M}_{\mathscr{T}}} &\approx 1^{\mathcal{O}}_{\mathcal{M}} * 1^{-1}_{\mathcal{M}} * 1_{\mathcal{M}_{\mathscr{T}}} \\
&\approx 1^{\mathcal{O}}_{\mathcal{M}_{\mathscr{T}}} * 1^{\mathcal{O}}_{\mathcal{M}_{\mathscr{F}}} * 1^{-1}_{\mathcal{M}} * 1_{\mathcal{M}_{\mathscr{T}}} && \text{by 2.5.1} \\
&\approx 1^{\mathcal{O}}_{\mathcal{M}_{\mathscr{T}}} * 1^{\mathcal{O}}_{\mathcal{M}_{\mathscr{F}}} * 1^{-1}_{\mathcal{M}} * 1_{\mathcal{M}_{\mathscr{T}}} && \text{by 2.5.1} \\
&\approx 1^{\mathcal{O}}_{\mathcal{M}_{\mathscr{T}}} * 1^{\mathcal{O}}_{\mathcal{M}_{\mathscr{F}}} * 1^{-1}_{\mathcal{M}_{\mathscr{F}}} * 1^{-1}_{\mathcal{M}_{\mathscr{T}}} * 1_{\mathcal{M}_{\mathscr{T}}} && \text{by 2.5.2} \\
&\approx \mathcal{H}_0 * 1_{\mathcal{M}_{\mathscr{T}}} * \mathcal{H}^{\#}.
\end{aligned}$$

Now $\mathcal{H}^{\#}$ and $1_{\mathcal{M}_{\mathscr{T}}}$ commute up to a coefficient arising from the Poisson structure which vanishes after integration (see details in [4, Corollary 6.4]). This proves identity 2.6.1.

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