ALGEBRAIC STACKS

PROF. KAI BEHREND
LECTURE NOTES OF POOYA RONAGH

Lecture 1

1. Categories Fibered in Groupoids

Definition 1. A groupoid is a category where all morphisms are invertible.

A good example to keep in mind is the following.

Example 1.1. The fundamental groupoid Π of a topological space X: $s, t : \Pi \rightrightarrows X$

$$\Pi = \{(x, [\gamma], y) : x, y \in X, [\gamma] \text{ homotopy class of paths from x to y }\}$$

The objects are points of X and morphisms are given by Π. There is more data: the inverse $i : \Pi \to \Pi$ the identity $e : X \to \Pi$ and the composition $\Pi \times \Pi \to \Pi$.

The next example shows how groupoids appear rather in the theory of stacks.

Example 1.2 (Symmetry groupoids of families of "objects"). Consider triangles, with the notion of equivalence being their similarities in the sense of euclidean geometry. For one equilateral triangle the symmetry group is S_3. For the set of one isosceles triangle and one equilateral triangle the symmetric groupoid is $S_3 \coprod S_2$. Or let's look at a more symmetric family
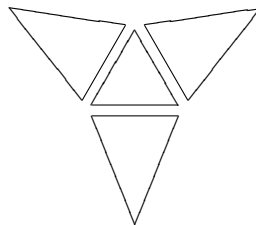
the symmetry groupoid is
$$S_3 \times \{0, 1, 2, 3\} \rightrightarrows \{0, 1, 2, 3\}$$

Date: July 13, 2011.

the former are the morphisms and the latter are the objects! As groupoid (category) this is equivalent to $S_3 \coprod S_2$ from the previous case of the family
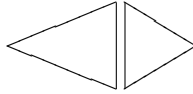
In general a symmetry groupoid is one for which the morphisms set is the set of isomorphisms of one object with another object in the family.

Definition 2. An algebraic (analytic, differentiable, ...) groupoid consists of schemes R, U and morphisms of schemes $s, t : R \to U$, $e : U \to R$, $i : R \to R$ and $m : R_t\times_{U,s} R \to R$ together with a list of axioms which basically guarantee that for every scheme T then $R(T) \rightrightarrows U(T)$ should be a groupoid.

Example 1.3. X scheme, and let $R = U = X$ with all the above maps id_X. The groupoid we get from a scheme T is $\mathrm{Hom}(T, X) \rightrightarrows \mathrm{Hom}(T, X)$ which has as objects the morphisms $T \to X$ and all arrows are identity.

Example 1.4. If $\{U_\alpha\}$ is an open cover of X (Zariski or etale) and $U = \coprod_\alpha U_\alpha$ and $R = \coprod_{\alpha, \beta} U_\alpha \cap U_\beta$.

Example 1.5. Given a morphism $E \to S$ of schemes, the fiber product given an algebraic groupoid, $E \times_S E \rightrightarrows E$.

Example 1.6. Let G be an algebraic group acting on scheme X on the right. The groupoid $\sigma, p : X \times G \rightrightarrows X$ with source map p and target map σ is an algebraic groupoid called the transformation groupoid.

Example 1.7. If the morphism $R \to U \times U$ of schemes is an equivalence relation we can form an algebraic groupoid $R \rightrightarrows U$.

In fact given any algebraic groupoid we can form the map $R \to U \times U$. But this map is an equivalence relation only if it is injective. In particular examples 1.3, 1.4 and 1.5 rise from equivalence relations. From example 1.6 we get an equivalence relation if the action is free.

Now we look at some examples coming from algebraic geometry.

Example 1.8. Elliptic curves up to isomorphism: Look at the Weierstrass family of elliptic curves: $T = \mathbb{C}^2 - \{0\}$ as the parameter space of all pairs of numbers (g_2, g_3)

$$
\begin{array}{ccc}
X & \hookrightarrow & \mathbb{P}^2 \\
 & \searrow & \downarrow \\
 & & T
\end{array}
$$
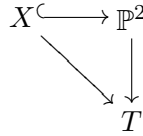
with equations $y^2 = 4x^3 - g_2 x - g_3$. The symmetry groupoid is the transformation groupoid $\mathbb{C}^* \times T \rightrightarrows T$

$$\lambda.(g_2, g_3) = (\lambda^4 g_2, \lambda^6 g_3).$$

This is an action on the parameter space and lifts to the total space of the family via

$$\lambda.(x, y, g_2, g_3) = (\lambda^2 x, \lambda^3 y, \lambda^4 g_2, \lambda^6 g_3).$$

So my claim is that every isomorphism of elliptic curves in the Weierstrass form is given by some λ in the above sense. Prove it! The symmetry groupoid is again $(\sigma, \pi) : \mathbb{C}^* \times T \rightrightarrows T$:

$$X_{(g_2, g_3)} \xrightarrow{\ (\lambda, g_2, g_3)\ } X_{(\lambda^4 g_2, \lambda^6 g_3)}.$$

Note: The base of a symmetry groupoids is always a parameter space!

Example 1.9. I like triangles anyway! Mike Artin himself is reputed to saying that if you study the stack of triangles, you can understand everything about stacks. So we consider triangles up to similarity again. Let N be the differentiable manifold

$$N = \{(a, b, c) \in \mathbb{R}^3 : a + b + c = 2, a, b, c < 1\}$$

this parametrizes a family of triangles. The symmetry groupoid of this family is

$$S_3 \times N \rightrightarrows N.$$

Example 1.10. One should likewise think of vector bundles as "families of vector spaces". Let E be a vector bundle over the scheme T of rank n. The scheme of isomorphisms, $\mathcal{I}som(p_1^* E, p_2^* E)$, is a principal Gl_n-bundle over $T \times T$. Here p_1 and p_2 are the two projections $T \times T \to T$. The symmetry groupoid is then

$$\mathcal{I}som(p_1^* E, p_2^* E) \rightrightarrows T.$$

Now we shall further formalize what we mean by families of objects.

Definition 3. Let $\mathfrak{S}$ be the category of $\mathbb{C}$-schemes (or category of differentiable manifolds, etc.). This is the base category. The objects of $\mathfrak{S}$ serve as parameter spaces of the families. A category fibered in groupoids (CFG) is a morphism of categories $\mathfrak{X} \to \mathfrak{S}$, such that

(1) Pullbacks exists: When $F(x) = S$, we say x is an $\mathfrak{X}$-family parametrized by S. Now given an $\mathfrak{X}$-family x over S and $T \to S$ a morphism in $\mathfrak{S}$ there is a "pullback" $y = x|_T$ fitting in a cartesian square

$$
\begin{array}{ccc}
y & \dashrightarrow & x \\
\downarrow & \square & \downarrow \\
T & \longrightarrow & S
\end{array}
$$

(2) Any diagram as below can be completed as shown by a dotted arrow.

This guarantees that pullbacks are unique up to unique isomorphism.

Remark. Roughly speaking, CFG formalizes the notion of families of mathematical objects which can be pulled back!

The stupidest example is that given any scheme X, have a CFG $\underline{X} \to \mathfrak{S}$. A CFG is called *representable* if it is equivalent over $\mathfrak{S}$ to some $\underline{X}$. If $\mathfrak{X} \to \mathfrak{S}$ is a CFG over $\mathfrak{S}$, check that $\mathfrak{X}(S)$, i.e. the category of objects over S with morphisms those mapping to id_S, is a groupoid for any S. Note that for any $u : S \to T$ we can "choose" by a careful application of the axiom of choice, a pullback

$$u^* : \mathfrak{X}(T) \to \mathfrak{X}(S)$$

via $x \mapsto x|_S$.

Example 1.11. Let G be an algebraic groupoid. Define BG to be the CFG of G-bundles, $BG \to \mathfrak{S}$. The objects are principal G-bundles, $E \to S$. We may define a principal G-bundle as a morphism $E \to S$ of schemes making the following diagram cartesian:

$$
\begin{array}{ccc}
E \times G & \xrightarrow{\ \sigma\ } & E \\
{\scriptstyle p}\downarrow & \square & \downarrow {\scriptstyle \text{smooth surjective}} \\
E & \longrightarrow & S
\end{array}
$$

Morphisms are commutative G-equivariant diagrams of G-bundles (they are automatically cartesian).

Let $\mathfrak{X} \to \mathfrak{S}$ be a CFG and x over S be an $\mathfrak{X}$-family. Then we can define the symmetry groupoid of x over S via

$$
\begin{array}{ccc}
R & \longrightarrow & \underline{S} \\
\downarrow & \square & \downarrow x \\
\underline{S} & \xrightarrow{\;x\;} & \mathfrak{X}
\end{array}
$$

If R is representable then $R \rightrightarrows S$ is the symmetry groupoid of x/S and this will be the general notion of groupoid from now on for us! We are going to throw in the condition that this symmetry groupoid is representable and the condition of being locally finite type often.

Definition 4. The family x/S is a *versal family* if

    (1) The symmetry groupoid is representable; and,

    (2) $s, t : R \rightrightarrows S$, s, t are smooth.

Moreover x/S is said to be *complete* if every object of $\mathfrak{X}(\mathbb{C})$ occurs as a pullback along $\mathcal{S}pec\,\mathbb{C} \to S$:

$$
\begin{array}{ccc}
x|_s & \longrightarrow & x \\
\downarrow & & \downarrow \\
\mathcal{S}pec\,\mathbb{C} & \xrightarrow{\;s\;} & S
\end{array}
$$

If $R \rightrightarrows U$ is an algebraic groupoid then is a corresponding CFG $[R \rightrightarrows U]$ in general. For instance $[G \rightrightarrows *]$ for algebraic group G gives the CFG BG. In fact the general construction can be thought of a generalization of the way we pass from G to BG.

If x/S is versal and complete then we can also construct a morphism

$$\mathfrak{S} \to [R \rightrightarrows S]$$

via $y \mapsto \mathcal{I}som(y, x)$ where $[R \rightrightarrows S]$ is the symmetry groupoid of x/S.

Example 1.12. If all objects of $\mathfrak{X}(\mathbb{C})$ are isomorphic and $G = \mathrm{Aut}(obj)$ (for example in case of vector bundles) get $\mathfrak{X} \to BG$ via $y/S \mapsto (\mathcal{I}som(y, x_S)/S)$ where the latter is a principal G-bundle over S. And $G \rightrightarrows \mathcal{S}pec\,\mathbb{C}$ is a symmetric groupoid of x.

Definition 5. $\mathfrak{X}$ is an *algebraic stack* if there is versal and complete family such that $\mathfrak{X} \to [R \rightrightarrows U]$ is an equivalence of CFGs.

2. STACKS

Lecture 2

We start by an example:

Example 2.1. Let $Vect_n$ be the CFG of vector bundles of rank n. The objects are vector bundles $E \to S$ over scheme S and morphisms are morphisms of vector bundles that are cartesian:

$$
\begin{array}{ccc}
F & \longrightarrow & E \\
\downarrow & \square & \downarrow \\
T & \longrightarrow & S
\end{array}
$$

It is a CFG, i.e. we can pullback vector bundles and pullback is unique up to unique isomorphism. Choose pullbacks! The more interesting feature is that we can glue vector bundles. This precisely means that the following two conditions are satisfied:

(1) We can glue isomorphisms of vector bundles: If $\{S_i \to S\}$ is an etale cover of a scheme S and E and F are two vector bundles over S and we have isomorphisms

$$
E|_{S_i} \overset{\varphi_i}{\cong} F|_{S_i},
$$

such that for all i,j we have $\varphi_i|_{S_{ij}} = \varphi_j|_{S_{ij}}$, then there is a unique $\varphi : E \to F$ such that $\varphi_{S_i} = \varphi_i$.

(2) We can glue vector bundles themselves: If $\{S_i \to S\}$ is an etale covering and E_i over S_i is vector bundle and

$$
E_i|_{S_{ij}} \overset{\varphi_{ij}}{\to} E_j|_{S_{ij}}
$$

such that for all i,j,k we have $\varphi_{jk}\varphi_{ij} = \varphi_{ij}$ over S_{ijk}, then there is a vector bundle E over S and $\varphi_i : E|_{S_i} \to E_i$ such that $\varphi_{ij}\varphi_i = \varphi_j$ (and note that all pullbacks are isomorphisms).

Remark. In general you need to use descent theory and use the fact that $\coprod S_i \to S$ is faithfully flat.

Remark. These 2 axioms make sense for every CFG over schemes or more generally for any CFG over a base Grothendieck site (interesting other examples are: analytics spaces, differentiable manifolds).

Definition 6. If $\mathfrak{X}$ satisfies (1) and (2) above it is a *stack*. If it only satisfies (1) it is called a *prestack*.

Example 2.2. $\overline{\mathcal{M}}_{1,1}$ is the CFG of flat projective families, $C \to T$ of curves of arithmetic genus $g = 1$, with a section $p : T \to C$. This is a stack by descent theory.

We now refine some of the notions we covered last time.

Example 2.3. Let $R \rightrightarrows U$ be an algebraic groupoid and assume that

$$
(s,t) : R \to U \times U
$$

is quasi-affine. Define the CFG $[U/R] = [R \rightrightarrows U]$ of $(R \rightrightarrows U)$-torsors as follows. The objects are pairs of a morphism of schemes with etale local sections, and banal groupoid $(P \to T, P \times_T P \rightrightarrows P)$ with a square morphism of groupoids:

$$
\begin{array}{ccc}
P \times_T P & \longrightarrow & R \\
\big\updownarrow & \square & \big\updownarrow \\
P & \longrightarrow & U
\end{array}
$$

To prove the stack axioms we need quasi-affineness in the assumptions otherwise we will have to work with algebraic spaces instead of schemes.   ⌡

Example 2.4. Let $G \times X \rightrightarrows X$ be the transformation groupoid of G, where $G \times X \to X \times X$ is affine. If you work out the definition above you see that the data of a torsor over T is equivalently a pair $(P/T, P \to X)$ where P is a G-torsor (i.e. a principal homogeneous G-bundle) and $P \to X$ is a G-equivariant morphism. These data is given equivalently by the following diagram

$$
\begin{array}{ccc}
P \times_T P \cong G \times P & \longrightarrow & G \times X \\
\big\updownarrow & \square & \big\updownarrow \\
P & \longrightarrow & X \\
\big\downarrow & & \\
T & &
\end{array}
$$

⌡

Remark. Recall the notation of fiber product of CFGs. Recall further that if we have a diagram of CFGs

$$
\begin{array}{ccc}
\mathfrak{W} & \longrightarrow & \mathfrak{Y} \\
\big\downarrow & \nearrow & \big\downarrow \\
\mathfrak{X} & \longrightarrow & \mathfrak{Z}
\end{array}
$$

we get an induced morphism of CFG's over $\mathfrak{S}$, $\mathfrak{W} \to \mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$. If this is an quivalence of categories then the diagram is called 2-cartesian.

Definition 7. From any x/S get a morphism $\underline{S} \xrightarrow{s} \mathfrak{X}$ a morphism. If in the diagram

$$
\begin{array}{ccc}
\mathcal{I}som(x,x) & \longrightarrow & \underline{S} \\
\big\downarrow & \square & \big\downarrow{\scriptstyle x} \\
\underline{S} & \xrightarrow{x} & \mathfrak{X}
\end{array}
$$

have $\mathcal{I}som(x,x) \cong \underline{R}$ for some R then $R \rightrightarrows S$ is an algebraic groupoid and is called the symmetry groupoid of x/S.

Assume $R \rightrightarrows S$ is the symmetry groupoid of x/S. Then if $R \rightrightarrows S$ is smooth x is called *versal*. If every $\mathfrak{X}$-family y/T is locally induced from x/S, i.e. there is a cover $T' \to T$ and $T' \to S$ such that $y|_{T'} = x|_{T'}$, then x is called *complete*.

Theorem 2.1. *Suppose $\mathfrak{X}$ over $\mathfrak{S}$ is a stack. Suppose $R \rightrightarrows U$ with $R \to U \times U$ quasi-affine, is the symmetry groupoid of a complete vesal (may not need versal here!) family x/U. Then*

    (1) For any object y/T, $Isom(y|_{T\times U}, x|_{T\times U})/T$ is an $(R \rightrightarrows U)$-torsor.

    (2) Then $\mathfrak{X} \to [U/R]$ via $y/T \mapsto \mathcal{I}som(y|_{T\times U}, x|_{T\times U})/T$ is an isomorphism of CFG's over $\mathfrak{S}$.

Rather than explaining the proof we will do an example.

Example 2.5. Let $\mathfrak{X} = Vect_n$ over $\mathfrak{S}$. Then $\mathbb{C}^n/\ast$ is a complete versal family with symmetry groupoid $\mathrm{Gl}_n \rightrightarrows \ast$. And the above equivalence is $Vect_n \to BGL_n$ via

$$E/T \mapsto \mathcal{I}som(E, \mathbb{C}_T^n) = \text{ frame bundle of } E \ .$$

Definition 8. The stack $\mathfrak{X}$ over $\mathfrak{S}$ is an algebraic stack if there is a complete versal family.

Remark. We have had to assume the condition of quasi-affine diagonals. A more general definition of algebraic stack can be given if one starts over assuming algebraic spaces. But this class of algebraic stacks with quasi-affine diagonals already covers lots of algebraic stacks; e.g. all Deligne-Mumford stacks and lots of Artin stacks.

Example 2.6. $\overline{\mathcal{M}}_{1,1}$ is an algebraic stack because the Weierstrass family is versal and complete. Now the theorem applies and we have

$$\overline{\mathcal{M}}_{1,1} \cong [\mathbb{C}^2 \smallsetminus \{0\}/\mathbb{C}^\ast] = \mathbb{P}(4,6).$$

Note also that our theorem implies that any family of elliptic curves $E \to T$ is actually a $\mathbb{C}^2 \smallsetminus \{0\}/\mathbb{C}^\ast$-torsor.

Other versal families for $\mathcal{M}_{1,1}$ are for instance $\mathbb{H}$ upper half plane parametrizing a family of elliptic curves with $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ for $\tau \in \mathbb{H}$ but in the analytic category. A symmetry groupoid for this family is in fact a transformation groupoid $\mathrm{SL}_2\mathbb{Z} \times \mathbb{H} \rightrightarrows \mathbb{H}$. Another versal family is the Legendre family $y^2 = x(x-1)(x-\lambda)$ for $\lambda \in \mathbb{P}^1 - \{0, 1, \infty\}$. The symmetry groupoid will be left as an exercise.

In general can often find vesal families by rigidifying: for instance if (E, P, ω) is an elliptic curve with base point P and differential ω are parametrized by $W = \mathbb{C}^2 -$ discriminant. And then I can see that this choice of additional data for this case, i.e. the differential, is unique up to $\mathbb{C}^\ast$. That is why the symmetry groupoid is given by the $\mathbb{C}^\ast$-action.

3. General words on motivations for working with stacks

3.1. **Analogy with schemes.** Say X is a scheme and $\{U_\alpha\}$ and affine open (or etale) cover. Then $X = \coprod U_\alpha / \coprod U_{\alpha\beta}$ as a quotient of $\coprod U_\alpha$ by the equivalence relation given by the overlaps.

$$
\begin{array}{ccc}
\coprod U_{\alpha\beta} & \longrightarrow & \coprod U_\alpha \\
\downarrow & & \downarrow \\
\coprod U_\alpha & \longrightarrow & X
\end{array}
$$

If $\mathfrak{X}$ algebraic stack, and x/U complete versal family, then $\mathfrak{X} = [U/R]$ is a quotient of the algebraic groupoid suggested this time by another diagram

$$
\begin{array}{ccc}
R & \xrightarrow{\text{smooth}} & \underline{U} \\
{\scriptstyle\text{smooth}}\downarrow & \square & \downarrow{\scriptstyle\text{smooth epi}} \\
\underline{U} & \xrightarrow[\text{smooth epi}]{} & \mathfrak{X}
\end{array}
$$

And we want $\underline{U} \to \mathfrak{X}$ to reflect the geometry of $\mathfrak{X}$, that is why we need it to be a nice map. And smooth epimorphism is just one. If you can get away with etale morphism then your stack is moreover Deligne-Mumford. Nevertheless it at least has to be flat to have any descent theory working for it. But there is a theorem of Artin that if a flat atlast exists then a smooth one also exists.

Lecture 3

3.2. **Analogy with coarse moduli spaces.** Given a CFG (stack) $\mathfrak{X} \to \mathfrak{S}$, you should think of this as a moduli problem! Classically one searches for a scheme $X \in \mathfrak{S}$ such that $\mathfrak{X} \cong \underline{X}$. If such object exists then X is know as a fine moduli space. But as soon as there are objects with automorphisms there is no such object, since $\underline{X}(S)$ is a set and $\mathfrak{X}(S)$ is not equivalent to a set.

Definition 9. A scheme X is a coarse moduli space for $\mathfrak{X}$ if given $\pi : \mathfrak{X} \to \underline{X}$ such that for all $\mathfrak{X} \to \underline{Y}$ there exists a unique factorization:

$$
\begin{array}{ccc}
\mathfrak{X} & \xrightarrow{\pi} & \underline{X} \\
& \searrow & \vdots \\
& & \underline{Y}
\end{array}
$$

There are general theorems on when coarse moduli spaces exists, that we do not go through. (One can always pass from $\mathfrak{X}$ to its equivalence classes and one get a presheaf, then we sheafify and get a sheaf but this sheaf might not be representable by a scheme.) If $\underline{X}$ does not exist or has poor properties then we rather work with the stack $\mathfrak{X}$, the moduli problem itself.

Example 3.1. If there exists a complete versal family x/U whose symmetry groupoid is the transformation groupoid $G \times U \rightrightarrows U$. Then we have seen that the stack is algebraic and $\mathfrak{X} = [U/G]$. A categorical quotient U/G is then a coarse moduli space:

$$
\begin{array}{ccc}
G \times U & \longrightarrow & U \\
\downarrow & \square & \downarrow \\
U & \longrightarrow & \mathfrak{X} = [U/G] \\
& & \searrow \\
& & U/G
\end{array}
$$

Often we use GIT to construct the coarse moduli spaces.

Remark. If $\mathfrak{X}$ is an algebraic stack then it has a universal family $\mathfrak{X} = [U/R]$ for algebraic groupoid $R \rightrightarrows U$. Indeed we can rewrite

$$
\begin{array}{ccc}
R & \longrightarrow & U \\
\downarrow & \square & \downarrow \\
U & \longrightarrow & \mathfrak{X}
\end{array}
$$

as

$$
\begin{array}{ccc}
P \times_T P & \longrightarrow & R \\
\downdownarrows & \square & \downdownarrows \\
P & \longrightarrow & U \\
\downarrow & \square & \downarrow \\
T & \longrightarrow & \mathfrak{X}
\end{array}
$$

and $R \rightrightarrows U \to \mathfrak{X}$ is a universal $(R \rightrightarrows U)$-torsor over $\mathfrak{X}$ and the $(R \rightrightarrows U)$-torsor on the left is the pullback of the universal one. For instance if $\mathfrak{X} = BG$ and E/T is arbitrary G-torsor then $* \to BG$ is the universal G-torsor and E is the pullback

$$
\begin{array}{ccc}
E & \longrightarrow & * \\
\downarrow & \square & \downarrow \\
T & \longrightarrow & BG
\end{array}
$$

4. Some basic properties of algebraic stacks and their morphisms

4.1. **Representable morphisms.** Let $\mathfrak{X} \xrightarrow{f} \mathfrak{Y}$ be a morphism of algebraic stacks (morphism of CFGs). Then f is representable if for any morphism $\underline{Y} \to \mathfrak{Y}$ where Y is a scheme,

then X below is a scheme:

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & \square & \downarrow \\ \mathfrak{X} & \longrightarrow & \mathfrak{Y} \end{array}$$

If you get algebraic space for plugging in an algberaic space Y then you get a different notion, but let us stick to this one.

For an algebraic space $\mathfrak{X}$, the diagonal $\mathfrak{X} \overset{\Delta}{\to} \mathfrak{X} \times \mathfrak{X}$ is always representable. Assume that $\mathfrak{X}$ has an atlas

$$\mathfrak{X} = [R \rightrightarrows U].$$

The claim is that S is a scheme over T (quasi-affine).

We can do things locally for the etale topology in T so without loss of generality there exists a lift $T \to U \times Y$ and hence S is a scheme. (This is a general way of showing representability.)

Definition 10. Let $\wp$ be one of the properties of being etale, smooth, unramified, affine, finite , proper, etc. If $\mathfrak{X} \overset{f}{\to} \mathfrak{Y}$ is representable and for all $Y \to \mathfrak{Y}$ the induces morphism g from the cartesian diagram

$$\begin{array}{ccc} X & \overset{g}{\longrightarrow} & Y \\ \downarrow & \square & \downarrow \\ \mathfrak{X} & \overset{f}{\longrightarrow} & \mathfrak{Y} \end{array}$$

has property $\wp$ then f is said to have property $\wp$.

Remark. For any property $\wp$ of morphism of schemes, if g' in the diagram

$$\begin{array}{ccc} X' & \overset{g'}{\longrightarrow} & Y' \\ \downarrow & \square & \downarrow {\scriptstyle \text{smooth surjection}} \\ X & \overset{g}{\longrightarrow} & Y \end{array}$$

has property $\wp$ one can conclude that g also has the property $\wp$ and is preserved under arbitrary base change.

Definition 11. An algebraic stack is Deligne-Mumford (DM), if $\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is unramified. For us an Artin stack is one that is not Deligne-Mumford.

Theorem 4.1. *If $\mathfrak{X}$ is DM then there exists a complete versal family with etale symmetry groupoid $R \rightrightarrows U$ (i.e. both source and target are etale).*

Example 4.1. $\overline{\mathcal{M}}_{1,1}$ is DM. In fact we have

$$\mathbb{C} \coprod \mathbb{C} \to \mathbb{C}^2 - \{0\} \to \overline{\mathcal{M}}_{1,1}$$

for the two copies of $\mathbb{C}$ corresponding to $g_2 = 1$ and $g_3 = 1$ respectively.   ⌟

The converse of the above theorem is also the case:

Proposition 1. *If a stack comes from an etale groupoid then it is Deligne-Mumford.*

If $\wp$ is a property local in source and target, then f'' having $\wp$ implies that f has $\wp$.

$$
\begin{array}{ccc}
X'' \xrightarrow{\text{smooth surj}} X' & \longrightarrow & X \\
\end{array}
$$

with arrows f'', f', f and bottom $Y' \xrightarrow{\text{smooth surj}} Y$.

Some examples of local properties are being smooth, etale, locally of finite type, locally of finite presentation, etc.

Definition 12. If $\mathfrak{X} \to \mathfrak{Y}$ is a morphism of algebraic stacks such that there exists a diagram

$$
\begin{array}{ccc}
X & \xrightarrow{g} & Y \\
\text{smooth surj} \downarrow & \nearrow & \downarrow \text{smooth surj} \\
\mathfrak{X} & \xrightarrow{f} & \mathfrak{Y}
\end{array}
$$

with g having property $\wp$, we say f has property $\wp$.

Remark. The algebraic stack $\mathfrak{X}$ is smooth if $\mathfrak{X} \to \underline{S} = \mathfrak{S}$ is smooth.

Example 4.2. Let $[R \rightrightarrows U] = \mathfrak{X}$ be an algebraic stack for the groupoid $s, t : R \rightrightarrows U$ such that s, t are smooth. If U is smooth then $\mathfrak{X}$ is smooth.   ⌟

We define the dimension of an algebraic stack $\mathfrak{X} = [R \rightrightarrows U]$ by,

$$\dim \mathfrak{X} := 2 \dim U - \dim R.$$

For instance $\dim BG = -\dim G$ and $\dim \overline{\mathcal{M}}_{1,1} = 4 - 3 = 1$.

Example 4.3. The stack of euclidean triangles up to similarity is a smooth differentiable stack of dimension 1 (DM).   ⌟

Essentially you can think of DM as a stack with all objects having discrete automorphism groups. Let us see an example of an Artin stack now.

Example 4.4. Say C is a smooth projective curve over k and have $\deg O(1) = 1$. Let $X = Bun_{(r,d)}$ be the stack of vector bundles of rank r and degree d. By this we mean that the objects over T are families of vector bundles over C parametrized by T, $E \to X \times T$, such that for any $t \in T$

$$\begin{array}{ccc} E_t & \longrightarrow & E \\ \downarrow & \square & \downarrow \\ C & \xrightarrow{\mathrm{id}\times t} & C \times T \end{array}$$

E_t has degree d and rank r. Then morphisms are cartesian squares

$$\begin{array}{ccc} F & \longrightarrow & E \\ \downarrow & \square & \downarrow \\ X \times S & \xrightarrow{\mathrm{id}_X \times f} & X \times T. \end{array}$$

Now let $Bun_{(r,d)}^{\leq n}(C)$ be the stack of vector bundles E such that $E(n)$ is regular; i.e.

$$\dim \Gamma(C, E(n)) = d + r(n + 1 - g) = n = h.$$

This is an open substack of $Bun_{(r,d)}(C)$; i.e. for any diagram

$$\begin{array}{ccc} T^{\leq n} & \longrightarrow & T \\ \downarrow & \square & \downarrow \\ Bun^{\leq n} & \xrightarrow{u} & Bun \end{array}$$

the upper morphism is an open immersion. This is implied by a theorem about vector bundles that the fibered product above is represented by the open subscheme of all $t \in T$ such that $E_t(n)$ is regular.

$Bun^{\leq n}$ is a quotient stack. In fact

$$Bun^{\leq n} = [Quot(\mathcal{O}^h)^{reg}/\mathrm{Gl}_h]$$

where $Quot$ is the moduli scheme of objects of the form $(E, s_1, \cdots, s_h)$ where E is a regular vector bundle and $s_1, \cdots, s_h$ is a basis of global sections. Now we can write Bun as the colimit

$$Bun := \bigcup_n Bun^{\leq n},$$

of open substacks which are quotient stacks.

Bun is not of finite type only locally of finite type. If Y_n is the closed complement of $Bun^{\leq n}$, then

$$\lim_{n \to \infty} \mathrm{codim}(Y_n, Bun) = \infty.$$

However Bun is connected and smooth (since $Quot$ is smooth) of dimension

$$\dim H^1(C, \operatorname{End} E) - \dim H^0(\operatorname{End} E) = -\chi(\operatorname{End} E) = r^2(g-1).$$

Example 4.5. If $C = \mathbb{P}^1$ and $(r,d) = (2,0)$ then $E \cong \mathcal{O}(n) \oplus \mathcal{O}(-n)$. The bundles $\mathcal{O} \oplus \mathcal{O}$ and $\mathcal{O}(n) \oplus \mathcal{O}(-n)$ are both memebrs of a connected family. The open substacks suggested by the above example are given by

$$Bun = B\operatorname{Aut}(\mathcal{O} \oplus \mathcal{O}) \cup B\operatorname{Aut}(\mathcal{O}(1) \oplus \mathcal{O}(-1)) \cup \cdots.$$

The first substack is a dense open substack of BGL_2 of dimension -4. Bun itself is a smooth Artin stack of dimension -4. The next term is a dense open substack of

$$B\begin{pmatrix} \mathbb{C}^* & \Gamma(\mathcal{O}(2)) \\ 0 & \mathbb{C}^* \end{pmatrix}$$

of dimension -5. The next term is of dimension -7, and so on.

Remark. Over a finite field $k = \mathbb{F}_q$, $\mathfrak{X}(\mathbb{F}_q)$ is the fiber of $\mathfrak{X}$ over $\mathcal{S}pec\,\mathbb{F}_q$ and

$$\#\mathfrak{X}(\mathfrak{F}_q) = \sum_{\text{isom. classes}} \frac{1}{\#\operatorname{Aut}(X)}.$$

If $\mathfrak{X}$ is of finite type $\#\mathfrak{X}(\mathbb{F}_q)$ is finite. $\#Bun(\mathbb{F}_q)$ is also finite even though Bun is not of finite type.

▸ **Exercise 1.** Calculate this for the case of $\mathbb{P}^1_k$ when $(r,d) = (2,0)$.

Finally, there are two useful notions that we will not be touching are that of gerbes and orbifolds:

Definition 13. $\mathfrak{X} \to X$ is a gerbe over X if

(1) $\mathfrak{X} \to X$ is an epimorphism (i.e has local sections in the etale topology),

(2) $\mathfrak{X} \to \mathfrak{X} \times_{\mathfrak{X}} \mathfrak{X}$ is an epimorphism.

Definition 14. A smooth connected Deligne-Mumford stack which is generically a scheme is called an orbifold.

5. Cohomology of algebraic stacks

Lecture 4

Our setup will be $\pi : \mathfrak{X} \to \mathfrak{S}$ a stack over site $\mathfrak{S}$. For us $\mathfrak{S}$ will be $\mathcal{S}ch/S$ with etale topology or say $(C^\infty - mfld)$ with analytic topology. Then $\mathfrak{X}$ has an induced Grothendieck topology: $(X_i \to X)$ is a covering family if $\pi(X_i \to X)$ is one in $\mathfrak{S}$. So $\mathfrak{X}$ is a site and now we want to make sense of sheaves and cohomology on a site.

Definition 15 (Sheaves, or rather a *big sheaves*, on $\mathfrak{X}$)**.** CFG's $F \to \mathfrak{X}$ with unique pullbacks are called presheaves. For such CFG's the stack axioms are reduced to sheaf axioms. In case they hold F is called a sheaf on $\mathfrak{X}$.

An *abelian sheaf* will just mean a sheaf $F \to \mathfrak{X}$ such that $F(x)$ is an abelian group for any x and that for any map $y \to x$ in $\mathfrak{X}$ the pullback maps $F(x) \to F(y)$ are additive. These form an abelian category with enough injectives. If F is such a sheaf then define

$$H^i(\mathfrak{X}, F) = \mathrm{Ext}^i(\mathbb{Z}_\mathfrak{X}, F)$$

where $\mathbb{Z}_\mathfrak{X}$ is the constant sheaf.

Remark. For any scheme S in $\mathfrak{S}$ there is a *small (etale) site* over S, which is the category of schemes over S with etale structure map. A sheaf over this site is differed from a big sheaf and is called a *small sheaf*. Note that a big sheaf $F \to \mathfrak{X}$ defines for any x/S a small sheaf on S denoted by F_S or F_x.

Example 5.1. A representable etale morphism $\mathfrak{Y} \to \mathfrak{X}$ of stacks, is a sheaf on $\mathfrak{X}$ (called a *small sheaf* on $\mathfrak{X}$); e.g. $\mathbb{Z}_\mathfrak{X}$ or $\mathbb{C}_\mathfrak{X}$ ($\mathbb{C}$ with discrete topology). ⌟

Example 5.2. The structure sheaf of $\mathfrak{X}$ is given by $\mathcal{O}_\mathfrak{X}(x) = \mathcal{O}_S(S)$ for all x/S. This is not small but representable by $\mathbb{A}^1 \times \mathfrak{X} \to \mathfrak{X}$. ⌟

Remark. If the base field is $\mathbb{C}$, do not confuse $\mathbb{A}^1 \times \mathfrak{X}$ and $\mathbb{C}_\mathfrak{X}$. The former is not etale over $\mathfrak{X}$, but the latter is infinitely many copies of $\mathfrak{X}$ indexed by the complex numbers lying etale over $\mathfrak{X}$.

Definition 16. A quasi-coherent sheaf is a big sheaf F such that F_S is a quasi-coherent sheaf for all x/S and satisfying for all

$$
\begin{array}{ccc}
y & \longrightarrow & x \\
\downarrow & & \downarrow \\
\downarrow & \xrightarrow{\ f\ } & \downarrow \\
T & \longrightarrow & S
\end{array}
$$

that $f^{-1}F_S \cong F_T$.

Example 5.3. Say $\mathfrak{S} = (C^\infty - \mathit{mfld})$ with analytic topology. Then $\Omega^q_\mathfrak{X}$ is defined by

$$\Omega^q(x) = \Omega^q(S)$$

for all x/S. This is a big sheaf. It satisfies the first condition for being quasi-coherent but fails on the second condition. Therefore is not quasi-coherent. ⌟

Example 5.4 (of cohomology)**.** Let $F = \mathcal{O}^* = \mathbb{G}_m \times \mathfrak{X}/\mathfrak{X}$.

$$H^1(\mathfrak{X}, \mathcal{O}^*) = \mathrm{Ext}^1(\mathbb{Z}_\mathfrak{X}, \mathcal{O}^*)$$

which is by general principles the isomorphism classes of extensions

$$1 \to \mathcal{O}_\mathfrak{X}^* \to E \xrightarrow{\pi} \mathbb{Z}_\mathfrak{X} \to 0$$

or just isomorphism classes of $\mathbb{G}_m$-torsors (if you pull-back π with $\mathfrak{X} \xrightarrow{1} \mathbb{Z}_\mathfrak{X}$, we get the stack $\pi^{-1}(1)$ which is a $\mathbb{G}_m$-torsor). This is identical also to the family of isomorphism classes of line bundles on $\mathfrak{X}$, which is defined to be the Picard group $Pic(\mathfrak{X})$. So if $\mathfrak{X} = [X/G]$ then this is the family of G-equivarient line bundles on X and if more generally $\mathfrak{X} = [U/R]$ we get the class of descent data for line bundles over $R \rightrightarrows U$. These latter statements make it possible to calculate the Picard groups in concrete examples.

▸ **Exercise 2.** Show that

$$H^1(\mathcal{M}_{1,1}, \mathcal{O}^*) = \mathbb{Z}/12\mathbb{Z},$$
$$and \quad H^1(\overline{\mathcal{M}}_{1,1}, \mathcal{O}^*) = \mathbb{Z}.$$

5.1. **Cech Cohomology.** One other good way to calculate cohomology is Cech cohomology. Suppose $R \rightrightarrows U$ is a groupoid presentation for the algebraic stack $\mathfrak{X}$ (this is the jargon for saying that $R \rightrightarrows U$ is the symmetry groupoid of a complete versal object). We construct a simplicial scheme by

$$X_0 = U,$$
$$X_1 = R,$$
$$X_p = \underbrace{U \times_\mathfrak{X} \cdots \times_\mathfrak{X} U}_{p+1 \text{ times}}.$$

This is often written as

$$\cdots X_2 \substack{\longrightarrow\\[-0.6em]\longrightarrow\\[-0.6em]\longrightarrow} X_1 \rightrightarrows X_0 \to \mathfrak{X}.$$

Theorem 5.1. *If F is a big abelian sheaf on $\mathfrak{X}$ then there exists an E_1-spectral sequence*

$$H^q(X_p, F_{X_p}) \Rightarrow H^{p+q}(\mathfrak{X}, F)$$

the former is the usual cohomology of small sheaves on etale sites of the schemes X_p.

Example 5.5. Let $\mathfrak{X} \xrightarrow{\Delta} \mathfrak{X} \times \mathfrak{X}$ be an affine diagonal. Then without loss of generality we can assume that U is affine by replacing it with an affine open cover U' of U as in

$$\begin{array}{ccc}
R' & \longrightarrow & U' \\
& \searrow & \\
& R \longrightarrow U & \\
\downarrow & \square & \downarrow \\
U' \longrightarrow & U \longrightarrow & \mathfrak{X}.
\end{array}$$

Then we still have

$$\mathfrak{X} = [R \rightrightarrows U] = [R' \rightrightarrows U'].$$

The cartesian square

$$
\begin{array}{ccc}
R & \longrightarrow & U \times U \\
\downarrow & \square & \downarrow \\
\mathfrak{X} & \longrightarrow & \mathfrak{X} \times \mathfrak{X}
\end{array}
$$

then shows that R is affine. Likewise one observes that all X_p's are affine. Then all higher cohomologies vanish if F is quasi-coherent and we get

$$H^p(\mathfrak{X}, F) = \text{ p-th Cech coholmology of the complex } \Gamma(X_\bullet, F_{X_\bullet}).$$

That's all I wanted to say about the quasi-coherent sheaves.

5.2. **DeRham cohomology.** Now let me talk about cohomology with constant coefficients. The easiest thing to do is maybe to pass to the associated analytic stack assuming that the stack is smooth. If the algebraic stack is smooth the associated analytic stack is $\mathcal{C}^\infty$. Then the cohomology with constant coefficients can be computed using deRham cohomology.

Once again let $\mathfrak{S}$ be the analytic site of $\mathcal{C}^\infty$ manifolds. And then $\mathfrak{X} = [R \rightrightarrows U]$ will be a Lie groupoid (i.e. R, U are $\mathcal{C}^\infty$-manifolds and s, t are $\mathcal{C}^\infty$-submersions). Since we are taking fiber products of smooth submersions, $X_\bullet$ exists in the category, and is the simplicial manifold. And then we can work out the Cech-deRham double complex following Bott-Tu.

$$
\begin{array}{ccc}
\vdots & & \vdots \\
\uparrow\scriptstyle d & & \uparrow\scriptstyle d \\
\Omega^1(X_0) & \xrightarrow{\partial} \Omega^1(X_1) & \xrightarrow{\partial} \cdots \\
\uparrow\scriptstyle d & & \uparrow\scriptstyle d \\
\Omega^0(X_0) & \xrightarrow{\partial} \Omega^0(X_1) & \xrightarrow{\partial} \cdots
\end{array}
$$

Thus as usual the cohomology of the total complex will be identical to

$$H^p_{dR}(\mathfrak{X}) = H^p(\mathfrak{X}, \mathbb{R}).$$

So in particular it does not depend on the choice of the groupoid presentation. You can show this as an exercise in homological algebra.

We define the differential forms on $\mathfrak{X}$ as kernels of ∂ in the above complex:

$$
\begin{array}{ccccccccc}
& & \vdots & & \vdots & & \vdots & & \\
& & \big\uparrow & & d\big\uparrow & & d\big\uparrow & & \\
0 & \longrightarrow & \Omega^1(\mathfrak{X}) & \longrightarrow & \Omega^1(X_0) & \xrightarrow{\;\partial\;} & \Omega^1(X_1) & \xrightarrow{\;\partial\;} & \cdots \\
& & \big\uparrow & & d\big\uparrow & & d\big\uparrow & & \\
0 & \longrightarrow & \Omega^0(\mathfrak{X}) & \longrightarrow & \Omega^0(X_0) & \xrightarrow{\;\partial\;} & \Omega^0(X_1) & \xrightarrow{\;\partial\;} & \cdots
\end{array}
$$

Example 5.6. If $\mathfrak{X}$ is the quotient stack of a Lie group acting on a manifold, $\mathfrak{X} = [X/G]$, the simplicial scheme is given by $X_q = G^p \times X$. Then $H_{dR}^\bullet(\mathfrak{X}) = H_G^\bullet(X)$ equivariant cohomology.

Reference: Everything I am saying today can be found in Trieste lecture notes on cohomology of stacks.

Example 5.7. If $\mathfrak{X} = BG$ is the classifying stack of the compact Lie group G. Then
$$
H_{dR}^p(BG) = (S^{2p}\mathfrak{g}^\vee)^G.
$$
So for example for the simplest nontrivial Lie group S^1 we get
$$
H_{dR}^*(BS^1) = \mathbb{R}[c].
$$

Remark. There exists a cup product structure $\cup$ on $H^*(\mathfrak{X}) = H_{dR}^*(\mathfrak{X})$.

5.3. **Cohomology with compact supports.** Let $\mathfrak{X}$ be oriented, meaning that R and U are oriented and the source and target maps of the groupoid presentation $s, t : R \rightrightarrows U$ are both compatibly oriented. Let $n = \dim \mathfrak{X}$ and r be the relative dimension of X_0 over $\mathfrak{X}$ (the relative dimension of the source and target maps). Then the cohomology with compact support $H_c^*(\mathfrak{X})$ is the total cohomology of the third quadrant double complex of the following figure:

$$
\begin{array}{ccccc}
\cdots \longrightarrow & \Omega^{n+2r}(X_1) & \xrightarrow{\;\partial_!\;} & \Omega^{n+r}(X_0) & \\
& d\big\uparrow & & d\big\uparrow & \nwarrow \\
\cdots \longrightarrow & \Omega^{n+2r-1}(X_1) & \xrightarrow{\;\partial_!\;} & \Omega^{n+r-1}(X_0) & \text{degree } n \\
& \big\uparrow & & \big\uparrow & \nwarrow \\
& \vdots & & \vdots & \text{degree } n\text{-}1
\end{array}
$$

Here $\partial_!$ is the alternating sum of integration along fibers which is well-defined since we are considering compactly supported cochains. The cokernels of the row maps are defined to be the compactly supported forms on the stack.

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & \Omega^{n+2r}(X_1) & \xrightarrow{\ \partial_!\ } & \Omega^{n+r}(X_0) & \dashrightarrow & \Omega_c^n(\mathfrak{X}) & \longrightarrow & 0 \\
& & \big\uparrow{\scriptstyle d} & & \big\uparrow{\scriptstyle d} & & \big\vert & & \\
\cdots & \longrightarrow & \Omega^{n+2r-1}(X_1) & \xrightarrow{\ \partial_!\ } & \Omega^{n+r-1}(X_0) & \dashrightarrow & \Omega_c^{n-1}(\mathfrak{X}) & \longrightarrow & 0 \\
& & \big\uparrow & & \big\uparrow & & \big\vert & & \\
& & \vdots & & \vdots & & \vert & &
\end{array}
$$

There are a few more remarks on the structure of this cohomology groups with compact support. Firstly, $H_c^*(\mathfrak{X})$ is a module over $H^*(\mathfrak{X})$ via

$$H^i(\mathfrak{X}) \otimes H_c^j(\mathfrak{X}) \to H_c^{i+j}(\mathfrak{X}),$$

$$\tau \otimes \gamma \mapsto \tau \cap \gamma.$$

If we pullback an n-form and integrate it on X_0, we get a well-defined integral

$$\int_{\mathfrak{X}} : H_c^n(\mathfrak{X}) \to \mathbb{R}.$$

This induces the Poincare pairing

$$H^*(\mathfrak{X}) \otimes H_c^*(\mathfrak{X}) \to \mathbb{R}$$

$$\omega \otimes \gamma \mapsto \int_{\mathfrak{X}} \omega \cap \gamma.$$

Assume $\mathfrak{X}$ is of finite type and R and U admit finite good covers then we get a Poincare duality, i.e.

$$H^p(\mathfrak{X}) \otimes H_c^{n-p}(\mathfrak{X}) \to \mathbb{R}$$

is a perfect pairing of finite dimensional vector spaces.

Example 5.8. $\begin{cases} H_c^p(BS^1) = \mathbb{R}, & \text{if } p \text{ is odd and negative} \\ 0 & \text{otherwise.} \end{cases}$

5.4. **Case of Deligne-Mumford stacks.** This phenomena happens only for Artin stacks. Now lets talk about Deligne-Mumford stacks: We do this on an etale site $s, t : R \rightrightarrows U$ etale. In particular all X_p have the same dimension n which is also the dimension of the stack. We throw in another assumption that $R \to U \times U$ is proper with finite fibers. (These are the Deligne-Mumford stacks in the differentiable world.) Then one can prove that

$$H_{dR}^k(\mathfrak{X}) = \ k\text{-th cohomology of } (\Omega^\bullet(\mathfrak{X}), d)$$

and

$$H_c^k(\mathfrak{X}) = \ k\text{-th cohomology of } (\omega_C^\bullet(\mathfrak{X}), d).$$

In particular $H^k(\mathfrak{X})$ and $H^k_c(\mathfrak{X})$ all vanish except for $k \in [0, n]$. This is one distinction of Artin and DM stacks. We see that DM stacks hence behave a lot more like spaces.

5.5. **Case of proper DM stacks.** We want to take integrals so we pass to proper DM stacks. In addition to our assumptions above we will hence assume that the coarse moduli space (as a topological space given via $U/\overline{R}$ where $\overline{R} = \mathrm{im}(R \to U \times U)$) is compact. Then we have in fact that

$$H^k(\mathfrak{X}) = H^k_c(\mathfrak{X})$$

and the isomorphism is induced by

$$
\begin{array}{ccc}
\Omega^k(\mathfrak{X}) & - - \to & \Omega^k(X_0) \\
\downarrow & & \uparrow{\scriptstyle r=0} \\
\Omega^k(X_0) & \xrightarrow{\omega \mapsto \rho\omega} & \Omega^k_c(\mathfrak{X})
\end{array}
$$

where ρ is a *partition of unity* for $R \rightrightarrows U$ with compact support. A partition of unity as such means that $\rho : U = X_0 \to \mathbb{R}$ is a $\mathcal{C}^\infty$ function, where $t * s^* \rho \cong 1$. The push forward is by integration along the fibers.

Remark. Not all groupoids have partition of unity so one might need to change the goupoid.

In particular there exists

$$\int_{\mathfrak{X}} : H^n(\mathfrak{X}) \to \mathbb{R}$$

where for $\omega \in \Omega^n(\mathfrak{X})$ we have $\int_{\mathfrak{X}} \omega = \int_{X_0} \rho.\omega|_{X_0}$. Finally this gives Poincare duality this time on the cohomology of $\mathfrak{X}$ (no need to compact support anymore). So

$$H^k(\mathfrak{X}) \otimes H^{n-k}(\mathfrak{X}) \to \mathbb{R}$$

is a perfect pairing.

Example 5.9. Here is one of the simplest examples. When $\dim \mathfrak{X} = 0$ then $\mathfrak{X} = \coprod BG_i$ for finite groups G_i. So without loss of generality we may consider the case of $\mathfrak{X} = BG$ for finite group G. A groupoid presenting this stack is the zero-dimensional Lie group $G \rightrightarrows *$. The only choice for ρ is then $\rho = 1/|G|$. $1 \in H^0(\mathfrak{X})$ is a canonical element for which

$$\int_{\mathfrak{X}} 1 = \int_{BG} 1 = \frac{1}{|G|}.$$

For the general case of $\mathfrak{X} = \coprod BG_i$

$$\int_{\mathfrak{X}} 1 = \#\mathfrak{X} = \sum_{\sigma \text{ an isom class}} \frac{1}{\# \mathrm{Aut}(\sigma)}.$$

6. Intersection theory

Let $\mathfrak{Y} \xrightarrow{f} \mathfrak{X}$ be a proper representable morphism of $\mathcal{C}^\infty$ stacks. Being proper implies that if we pull-back a form with compact support we get one with compact support as well. This induces

$$f^* : H_c^i(\mathfrak{X}) \to H_c^i(\mathfrak{Y})$$

and dually via Poincare duality we get

$$f_! : H^{\dim \mathfrak{Y} - i}(\mathfrak{Y}) \to H^{\dim \mathfrak{X} - i}(\mathfrak{X}).$$

If $i = \dim \mathfrak{Y}$ and we look at the effect of $f_!$ on the element 1 we get a definition for the class of $\mathfrak{Y}$:

$$cl(\mathfrak{Y}) = f_! 1 \in H^{\mathrm{codim}(\mathfrak{Y}, \mathfrak{X})}(\mathfrak{X}).$$

So in particular we get a class for any substack.

Example 6.1. Let $E \to \mathfrak{X}$ be a vector bundle of rank r with zero section $0 : \mathfrak{X} \to E$. We define the Euler class by

$$e(E) = 0^*(cl(\mathfrak{X}) \text{ in } E) \in H^r(\mathfrak{X}).$$

Definition 17. If $\mathfrak{X}$ is proper and DM then

$$e(\mathfrak{X}) = \int_{\mathfrak{X}} e(T_{\mathfrak{X}})$$

is the euler number of $\mathfrak{X}$. Note that if you have an etale groupoid presentation $R \rightrightarrows U$ for the stack then the tangent bundle of U and tangent bundle of R give the gluing data of a vector bundle on $\mathfrak{X}$ which is defined to be the tangent bundle of it.

Remark. There is a type of Gauss-Bonnet theorem for proper smooth DM stacks:

$$e(\mathfrak{X}) = \text{ orbifold Euler characteristic of } \mathfrak{X} = \chi_{orb}(\mathfrak{X}).$$

The latter is defined for any (not necessarily proper) DM-stack by

(1) $\chi_{orb}(\mathfrak{X}) = \chi_{top}(\mathfrak{X})$ if $\mathfrak{X}$ is a variety.

(2) If $\mathfrak{X} \to \mathfrak{Y}$ representable finite etale cover of degree d then $\chi_{orb}(\mathfrak{X}) = d\chi_{orb}(\mathfrak{Y})$.

Example 6.2. $\overline{\mathcal{M}}_{1,1}$ is a Deligne-Mumford stack with two of its stacky points p and q having respectively $\mathbb{Z}_4$ or $\mathbb{Z}_6$ as automorphism group.
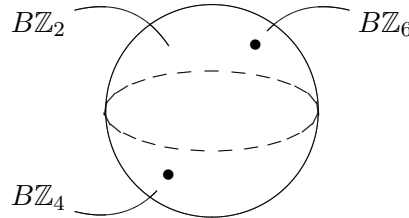
The rest of the points have $\mathbb{Z}_2$ automorphism group. Thus $\overline{\mathcal{M}}_{1,1} \smallsetminus \{p,q\}$ is a degree $1/2$ cover of the coarse moduli space which is $\mathbb{C}^*$ which is a variety. Hence the contribution of it to the Euler characteristic is 0. We conclude that

$$e(\overline{\mathcal{M}}_{1,1}) = 1/4 + 1/6 + 0 = 5/12.$$

For $\mathcal{M}_{1,1}$ we should remove one of the generic points so

$$\chi_{top}(\mathcal{M}_{1,1}) = 5/12 - 1/2 = -1/12.$$

I want to talk about intersection numbers now. All what I am talking about here is not hard to prove. In fact the proofs always reduce to the case of manifolds by taking covers.

Proposition 2. *Suppose the following is a pullback diagram.*

$$
\begin{array}{ccc}
\mathfrak{W} & \longrightarrow & \mathfrak{Z} \\
\Big\downarrow{\scriptstyle u} & \searrow^{f} & \Big\downarrow \\
\mathfrak{Y} & \longrightarrow & \mathfrak{X}
\end{array}
$$

If $\mathfrak{W}$ is smooth, and all maps are representable, proper and injective on tangent spaces (i.e. submersions) then the cup product of the classes of $\mathfrak{Y}$ and $\mathfrak{Z}$ in $\mathfrak{X}$ is given by

$$cl(\mathfrak{Y}) \cup cl(\mathfrak{Z}) = f_! e(u^* N_{\mathfrak{Y}/\mathfrak{X}}/N_{\mathfrak{W}/\mathfrak{Z}})$$

This is essentially a local statement and therefore by the above comment easy to proof. Now if $\mathfrak{X}$ is furthermore proper we can define the intersection numbers as in

Definition 18. If $\mathfrak{X}$ proper, we define the intersection number of $\mathfrak{Y}$ and $\mathfrak{Z}$ by the real number $\int_{\mathfrak{X}} cl(\mathfrak{Y}) \cup cl(\mathfrak{Z})$.

Proposition 2 has the following two corollaries in special cases.

Corollary 1. *If $\mathfrak{Y}$ and $\mathfrak{Z}$ intersect transversally (this has the usual meaning and can be checked etale locally on the groupoid presentation) and if $\dim Y + \dim \mathfrak{Z} = \dim \mathfrak{X}$ where*

$$\int_{\mathfrak{X}} cl(\mathfrak{Y}) \cap cl(\mathfrak{Z}) = \#\mathfrak{W} \in \mathbb{Q}$$

where $\#\mathfrak{W}$ is the number of points of $\mathfrak{W}$ in the stacky sense we talked about earlier.

Corollary 2. *If $\mathfrak{Y} = \mathfrak{Z}$ then*

$$\int_{\mathfrak{X}} cl(\mathfrak{Y})^2 = \int_{\mathfrak{W}} e(u^* N_{\mathfrak{Y}/\mathfrak{X}}).$$

I will finish by an example of this self-intersection number.

Example 6.3. Say $\mathfrak{X} = \overline{\mathcal{M}}_{1,2}$. We can think of this as the universal elliptic curve over $\overline{\mathcal{M}}_{1,1}$: over infinity of $\overline{\mathcal{M}}_{1,1}$ there is a rational fiber and the other fibers are elliptic curves. Of course all elliptic curves have a distinguished point giving a universal section $\mathfrak{Y} = \overline{\mathcal{M}}_{1,1} \hookrightarrow \mathfrak{X}$. In this case $\mathfrak{Y} \to \mathfrak{X}$ is an embedding so the pullback in proposition 2 is $\mathfrak{W} = \mathfrak{Y}$. By the last corollary

$$\int_{\mathfrak{X}} cl(\mathfrak{Y})^2 = \int_{\overline{\mathcal{M}}_{1,1}} e(N)$$

where N is the normal bundle of $\mathfrak{Y}$ in $\mathfrak{X}$. To compute this we choose a morphism $f : \mathbb{P}^1 \to \overline{\mathcal{M}}_{1,1}$. And we let E be the pullback as in

$$
\begin{array}{ccc}
E & \longrightarrow & \overline{\mathcal{M}}_{1,2} \\
\downarrow & \square & \downarrow \\
\mathbb{P}^1 & \xrightarrow{\ f\ } & \overline{\mathcal{M}}_{1,1}
\end{array}
$$

E is some kind of an elliptic fibration on $\mathbb{P}^1$ with a section $\mathbb{P}^1 \hookrightarrow E$. We have

$$\int_{\mathbb{P}^1} e(N_{\mathbb{P}^1/E}) = \int_{\mathbb{P}^1} f^* e(N) = \int_{\overline{\mathcal{M}}_{1,1}} f_! f^* e(N) = \deg(f) \int_{\overline{\mathcal{M}}_{1,1}} e(N).$$

The degree of f is twice the number of rational fibers in $E/\mathbb{P}^1$, for we have a copy of $B\mathbb{Z}_2$ sitting in $\overline{\mathcal{M}}_{1,1}$ as the classifying stack of the singular fiber in $\overline{\mathcal{M}}_{1,2}$. The pullback, D, of this via f will be the support of all the rational fibers in $E/\mathbb{P}^1$.

$$
\begin{array}{ccc}
D & \longrightarrow & B\mathbb{Z}_2 \\
k \downarrow & \square & \uparrow j \\
\mathbb{P}^1 & \xrightarrow{\ f\ } & \overline{\mathcal{M}}_{1,1}
\end{array}
$$

Now $\deg f = \deg k$. $B\mathbb{Z}_2 \to *$ has degree $^1\!/_2$ so $^1\!/_2 \deg f$ is the number of points in D.

It remains to study

$$\int_{\mathbb{P}^1} e(N_{\mathbb{P}^1/E}) = \deg(N_{\mathbb{P}^1/E}) =: \deg(E/\mathbb{P}^1)$$

then we may write

$$\int_{\overline{\mathcal{M}}_{1,1}} e(N) = \frac{\deg(E/\mathbb{P}^1)}{2 \times \text{ number of rational fibers}} = \frac{\deg(E/\mathbb{P}^1)}{2\chi_{top}(E)}$$

for any choice of $E/\mathbb{P}^1$. So pick $E/\mathbb{P}^1$ to be the pencil of cubics with 8 generic points in $\mathbb{P}^2$. Then E is the blow-up of $\mathbb{P}^2$ at 9 points. Thus $\chi(E) = 12$ and $\deg(E/\mathbb{P}^1) = -1$. The number we wanted to compute is therefore $-1/24$.

Department of mathematics, University of British Columbia
Room 121 - 1984 Mathematics Road, BC, Canada V6T 1Z2

E-mail address: `pooya@math.ubc.ca`