Lie Theory
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CHAPTER 1

Lie algebras

We work for now over an arbitrary field but as soon as we start the classification problem we have to confine to $\mathbb{C}$.

Definition 1. Let F be an arbitrary field and $\mathfrak{g}$ a finite-dimensional vector space over F with an operation

$$[,]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$$

called the Lie bracket (or commutator) that is

(1) bilinear;

(2) $[x,x]=0$;

(3) $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$.

Note that if $\operatorname{char} F=2$ the second condition is equivalent to $[x,y]=-[y,x]$. A Lie subalgebra is a subspace of $\mathfrak{g}$ closed under Lie bracket. A homomorphism of Lie algebras is a linear map $f:\mathfrak{g}_1\to\mathfrak{g}_2$ preserving the Lie structure.

Example 0.1. $\mathfrak{gl}_n=M_{n\times n}(F)$ has famous subalgebras: (say $F=\mathbb{C}$)

$$\mathfrak{so}_n=\{x\in\mathfrak{gl}_n : XJ+JX^t=0\}$$

where

$$J=\begin{pmatrix}0 & I_{n/2}\\ I_{n/2} & 0\end{pmatrix}\quad\text{if } n \text{ is even and}\quad J=\begin{pmatrix}1 & & \\ & 0 & I\\ & I & 0\end{pmatrix}\quad\text{if } n \text{ is odd.}$$

More generally J can be taken to be any symmetric bilinear form over our field F. Other examples are $\mathfrak{sp}_n$ and $\mathfrak{sl}_n=\{X:\operatorname{tr}X=0\}$. ⌟

More definitions are to follow: $\mathfrak{g}$ is called abelian if $[\mathfrak{g},\mathfrak{g}]=0$. Note that if $\mathfrak{g}$ is one-dimensional then $\mathfrak{g}$ is abelian. A subalgebra $\mathfrak{h}\subset\mathfrak{g}$ is called an ideal whenever $[x,y]\in\mathfrak{h}$ for all $x\in\mathfrak{g}$ and $y\in\mathfrak{h}$.

1. Solvable, Nilpotent, and Semisimple Lie algebras

DEFINITION 2. With the notation $D_1\mathfrak{g} = [\mathfrak{g},\mathfrak{g}]$ and $D_k\mathfrak{g} = [\mathfrak{g}, D_{k-1}\mathfrak{g}]$ the lower central series for the Lie algebra $\mathfrak{g}$ is

$$\mathfrak{g} \supset [\mathfrak{g},\mathfrak{g}] \supset \cdots.$$

Given $D^1 = D_1$ but $D^k\mathfrak{g} = [D^{k-1}\mathfrak{g}, D^{k-1}\mathfrak{g}]$ we get the derived series

$$\mathfrak{g} \supset [\mathfrak{g},\mathfrak{g}] \supset \cdots.$$

Check that $D_k\mathfrak{g}$ is an ideal of $\mathfrak{g}$.

DEFINITION 3. $\mathfrak{g}$ is nilpotent if $D_k\mathfrak{g} = 0$ for some k and is called solvable if $D^k\mathfrak{g} = 0$ for some k. $\mathfrak{g}$ is called semisimple if it contains no nontrivial solvable ideals and $\mathfrak{g}$ is simple if not one-dimensional ($[\mathfrak{g},\mathfrak{g}] \neq 0$) and it has no nontrivial ideals.

Note that the derived series is contained in the lower central series hence nilpotent implies solvable.

Example 1.1. The subgroup of $\mathfrak{gl}_n$ consisting of strictly upper-triangular matrices is called $\mathfrak{n}_n$ and is nilpotent. The subgroup $\mathfrak{b}_n$ of upper-triangular matrices is solvable. ⌟

LEMMA 1. *The property of being solvable/nilpotent is inherited by the subalgebras and homomorphism images.*

REMARK. Subalgberas of semisimple do not have to be semisimple.

Example 1.2. $\mathfrak{sl}_2$ is simple. Note that the matrices

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \text{and} \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

form a basis (the *standard* basis) of $\mathfrak{sl}_2$ with commutation relations

$$[H,X] = -2X, \quad [H,Y] = -2Y, \text{and} \quad [X,Y] = H.$$

Suppose $0 \neq \mathfrak{h} \subset \mathfrak{sl}_2$ is an ideal. Let $W = aX + bY + cH \in \mathfrak{h}$. If $a \neq 0$ commute with Y twice and get $Y \in \mathfrak{h}$ and likewise for $b \neq 0$ and get $X \in \mathfrak{h}$. If one of X or Y is in $\mathfrak{h}$ then $\mathfrak{h}$ contains H as well and we are done. ⌟

PROPOSITION 1. *There exists a unique maximal solvable ideal (called the* radical *and denoted by* $\mathrm{Rad}(\mathfrak{g})$*) in any Lie algebra* $\mathfrak{g}$.

PROOF. We'll prove that a sum of solvable ideals is a solvable ideal. Then since

$$\mathfrak{a} + \mathfrak{b}/\mathfrak{a} = \mathfrak{b}/\mathfrak{a} \cap \mathfrak{b}$$

then following is an exact sequence

$$0 \to \mathfrak{a} \to \mathfrak{a} + \mathfrak{b} \to \mathfrak{b}/\mathfrak{a} \cap \mathfrak{b} \to 0$$

hence proving the lemma. Same argument with applying Zorn's lemma proves the infinite dimensional case. □

LEMMA 2. *For $\mathfrak{h} \subset \mathfrak{g}$ an ideal, $\mathfrak{g}$ is solvable if and only if $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$ are solvable.*

PROOF. The only important thing to remember is that a Lie algebra $\mathfrak{g}$ is solvable iff there is a series

$$\mathfrak{g}_0 = \mathfrak{g} \supset \mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \cdots \supset \mathfrak{g}_n = 0$$

such that $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ is abelian. And this is easy to show as then $D^k\mathfrak{g} \subset \mathfrak{g}_k$ for all k by an inductive argument. □

So $\mathfrak{g}$ is semisimple if $\mathrm{Rad}(\mathfrak{g}) = \{0\}$.

2. Representations

A *representation* of $\mathfrak{g}$ is as usual a homomorphism of Lie algebras $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ for some vector space V (that we assume to be finite dimensional). A representation is called *faithful* if it is an injective homomorphism. A representation is called irreducible if it has no notrivial subrepresenation.

THEOREM 2.1 (Lie). *If $\mathfrak{g}$ is solvable and $\mathfrak{g} \subset \mathfrak{gl}(V)$ then there is $v \in V$ that is the common eigenvector for all $X \in \mathfrak{g}$.*

If $\mathfrak{g} \subset \mathfrak{gl}(V)$ is a representation, $\mathfrak{g}$ has a one-dimensional subrepresentation if and only if there exists a common eigenvector. But recall that this does not say that every representation of a solvable Lie algebra is a direct sum of 1-dimensional representations.

The mapping

$$\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$

is a homomorphism (by an application of the Jacobi identity) and is hence called the *adjoint representation.* The kernel of ad is

$$Z(\mathfrak{g}) = \{X : [X, Y] = 0, \forall Y\}.$$

Check that if $\mathfrak{g}$ is semisimple then $Z(\mathfrak{g}) = \{0\}$. This proves that every semisimple Lie algebra has a faithful representation.

REMARK. This is true for any Lie algebra by Ado's theorem.

3. Generators, relations and $\dim$ 2

Let $\mathfrak{g}$ be a Lie algebra and $x_1, \cdots, x_n$ a basis satisfying the relations

$$[x_i, x_j] = \sum_{k=1}^{n} a_{ij}^{(k)} x_k.$$

The a_{ij}^k's are called the structure constants. They satisfy by our Lie algebra axioms the following relations

$$a_{ij}^k = -a_{ji}^k \qquad \sum_k a_{ij}^k a_{kl}^m + a_{jl}^k a_{ki}^m + a_{li}^k a_{kj}^m = 0.$$

If $\dim \mathfrak{g} = 2$ an $\mathfrak{g}$ is non-abelian let x, y be any basis. Then $[\mathfrak{g}, \mathfrak{g}] = \langle [x, y] \rangle$ and $\mathfrak{g} \supset [\mathfrak{g}, \mathfrak{g}]$ gives a 1-dimensional subspace. We can pick x to be a generator of $[\mathfrak{g}, \mathfrak{g}]$ and complete it to a basis. Then $[x, y] = cx = 1x = x$ by rescaling of y. So the is a unique non-abeliean Lie algebra in dimension 2. It is solvable but not simple or semi-simple. So there is no semi-simple Lie algebra in dimensions less that 3. $\mathfrak{sl}_2$ is semisimple and 3-dimension.

4. Nilpotency and Engel's theorem

Engel's Theorem (Version 1). If every element is ad-nilpotent then $\mathfrak{g}$ is a nilpotent algebra.

Engel's Theorem (Version 2). If $\mathfrak{g} \subset \mathfrak{gl}(V)$ and consists of nilpotent endomorphisms then there is $v \in V$, such that $X(v) = 0$ for all $X \in \mathfrak{g}$.

Proof of Ver. 1 from Ver. 2. One observation is that if $X \in \mathfrak{gl}(V)$ is a nilpotent element, then the adjoint action

$$\operatorname{ad}(X) : \mathfrak{gl}(V) \to \mathfrak{gl}(V)$$

is nilpotent. In fact X being nilpotent is equivalent to existence of a flag of subspaces

$$0 \subset V_1 \subset \cdots \subset V_{k+1} = V$$

such that $X(V_i) \subset V_{i-1}$. Then for any endomorphism Y of V, and high enough $m >> 0$, $\operatorname{ad}(X)^m(Y)$ kills all of V (in fact for any m, the endomorphism $\operatorname{ad}(X)^m(Y)$ carries V_i into V_{i+k-m}.

Therefore from Version 2, we have that there is a flag

$$\mathfrak{g} = V_0 \supset V_1 \supset \cdots \supset V_k = 0$$

with $[\mathfrak{g}, V_i] \subset V_{i+1}$, from which it follows that $\mathcal{D}_i \mathfrak{g} \subset V_i$. □

Proof. We will run an induction on dimension of $\mathfrak{g}$: Our first step is to prove that there is $\mathfrak{h}$, an ideal of codimension one: Let $\mathfrak{h} \subset \mathfrak{g}$ be a maximal proper subalgebra; we claim that $\mathfrak{h}$ has codimension one and is an ideal. We have that $\operatorname{ad}(\mathfrak{h})$ acts on $\mathfrak{g}$ and preserves $\mathfrak{h}$. So it also acts on $\mathfrak{g}/\mathfrak{h}$. By what we proved above, $\operatorname{ad}(X)$ acts nilpotently on $\mathfrak{gl}(V)$, hence on $\mathfrak{g}$ and consequently on $\mathfrak{g}/\mathfrak{h}$. So by induction hypothesis there is $0 \neq \overline{Y} \in \mathfrak{g}/\mathfrak{h}$ killed by $\mathrm{ad}(X)$, for all $X \in \mathfrak{h}$. In other words there if $Y \in \mathfrak{h} \setminus \mathfrak{h}$, such that $\mathrm{ad}(X)(Y) \in \mathfrak{h}$ for all $X \in \mathfrak{h}$. This implies that, $\mathfrak{h}'$ the span of $\mathfrak{h}$ and Y is a Lie subalgebra of $\mathfrak{g}$, in which $\mathfrak{h}$ sites as an ideal of codimension one. But by maximality of $\mathfrak{h}$, we have $\mathfrak{h}' = \mathfrak{g}$.

So let's assume

$$\mathfrak{g} = \langle \mathfrak{h}, Y \rangle.$$

By induction hypothesis, there is nonzero $v \in V$ such that $X(v) = 0$ for all $X \in \mathfrak{h}$. Let W be the subspace of all such vectors. Then $W \neq \{0\}$. We have to show that there is $w \in W$ such thta $Y(w) = 0$. We show $Y(W) \subset W$ and it therefore suffices to observe that for any $w \in W$, $X(Y(w)) = 0$. But given $w \in W$,

$$X(Y(w)) = \underbrace{Y(X(w))}_{=0} + \underbrace{[X,Y]}_{\in \mathfrak{h}}(w)$$

and we are done. □

Note 4.1. If $X(v) = 0$ for all X, by induction on dimension of V we have that there is a basis in which all elements of $\mathfrak{g}$ are strictly upper triangular.

5. Solvability and Lie's theorem

Lie's Theorem. Let $\mathfrak{g}$ be a solvable Lie algebra and $\mathfrak{g} \subset \mathfrak{gl}(V)$ then there is $v \in V$ that is the common eigenvector for all $X \in \mathfrak{g}$.

Proof. Step 1. We first prove that there is an ideal $\mathfrak{h}$ of codimension one: Consider the abelian quotient algebra $\mathfrak{g}/D\mathfrak{g}$ with quotient map

$$\pi : \mathfrak{g} \to \mathfrak{g}/D\mathfrak{g}.$$

Let $\mathfrak{a}$ be any a codimension one subspace, and consider $\pi^{-1}(\mathfrak{a}) = \mathfrak{h}$.

Step 2. $\mathfrak{g} = \langle \mathfrak{h}, Y \rangle$ and v_0 the common eigenvector for all $X \in \mathfrak{h}$. We have

$$X(v_0) = \lambda(x) v_0, \quad \lambda \in \mathfrak{h}^*$$

Let

$$W = \{v \in V : Xv = \lambda(x)v, \forall X \in \mathfrak{h}\}.$$

So we are done by the next lemma which we state independently. □

Lemma 3. $\mathfrak{h} \subset \mathfrak{g}$ *an ideal.* V *a representation of* $\mathfrak{g}$*, and* $\lambda : \mathfrak{h} \to \mathbb{C}$ *a linear functional.*

$$W = \{v \in V : Xv = \lambda(x)v, \forall X \in \mathfrak{h}\}.$$

Then $Y(W) \subset W$ *for all* $Y \in \mathfrak{g}$.

Proof. For any nonzero $w \in W$ we want $Y(w) \in W$. But

$$X(Y(w)) = \underbrace{Y(X(w))}_{=\lambda(x)Y(w)} + \overbrace{\underbrace{[X,Y]}_{\in\mathfrak{h}}(w)}^{=\lambda([X,Y])w}$$

We want $\lambda([X,Y])w = 0$ for all $X \in \mathfrak{h}$. Let $U = \langle w, Y(w), Y^2(w), \cdots \rangle$. Then by induction $X(U) \subset U$ for all $X \in \mathfrak{h}$: In fact $X(w) = \lambda(X)w$, and $X(Y(w))$ is the linear combination of w and $Y(w)$, and so on. This also shows that in the basis $w, Y(w), \cdots$ the matrices for all $X \in \mathfrak{h}$ are upper-triangular.

Also the diagonal entries are all $\lambda(x)$. Then

$$\mathrm{tr}(X|_U) = \dim(U).\lambda(X).$$

Now $[X,Y] \in \mathfrak{h}$, and $0 = \mathrm{tr}([X,Y]|_U) = \dim U.\lambda[X,Y]$ and the claim follows. $\square$

By induction on $\dim V$ we then reach

Corollary 1. *There exists a basis of V such that all elements of $\mathfrak{g}$ are upper triangular. (Reformulation: there exists a flag in V stabilized by $\mathfrak{g}$).*

If $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a represenation, then $\rho(\mathfrak{g})$ is also solvable. We apply this to adjoint representations. By Lie's theorem, there is a flag of subspaces of $\mathfrak{g}$ stable under $\mathrm{ad}(\mathfrak{g})$ (i.e. ideals).

Corollary 2. *For a solvable $\mathfrak{g}$, there is a chain of ideals in $\mathfrak{g}$*

$$\mathfrak{g} = \mathfrak{h}_n \supset \mathfrak{h}_{n-1} \supset \cdots \supset \mathfrak{h}_1 \supset \{0\}$$

such that $\dim \mathfrak{h}_i = i$.

Then every element $X \in [\mathfrak{g},\mathfrak{g}]$ is ad-nilpotent (hence $[\mathfrak{g},\mathfrak{g}]$ is nilpotent). This is because the commutator of an upper-triangular, with anything is strictly upper-triangular.

Summary: We have this short exact sequence

$$0 \to \mathrm{Rad}(\mathfrak{g}) \to \mathfrak{g} \to \underbrace{\mathfrak{g}/\mathrm{Rad}(\mathfrak{g})}_{\mathfrak{g}_{ss}} \to 0$$

then $V = V_0 \otimes L$ where V_0 is an irreducible representation of $\mathfrak{g}_{ss}$ and L is a one-dimensional representation.

6. Cartan's criterion

Our first observation is that for any $x, y, z \in \mathfrak{gl}(V)$ then

$$\operatorname{tr}([x,y],z) = \operatorname{tr}(x.[y,z]).$$

THEOREM 6.1 (Cartan's criterion). *For* $\mathfrak{g} \subset \mathfrak{gl}(V)$ *suppose* $\operatorname{tr}(xy) = 0$ *for any* $x \in [\mathfrak{g},\mathfrak{g}], y \in \mathfrak{g}$. *Then* $\mathfrak{g}$ *is solvable.*

PROOF. Note that $\mathfrak{g}$ is solvable iff $[\mathfrak{g},\mathfrak{g}]$ is nilpotent. The sufficiency is obvious, the necessity follows from Lie's theorem. From Engel's theorem we know that if $x \in [\mathfrak{g},\mathfrak{g}]$ is ad-nilpotent, then $[\mathfrak{g},\mathfrak{g}]$ is nilpotent. So we will show that any $x \in [\mathfrak{g},\mathfrak{g}]$ is nilpotent.

Fix $x \in [\mathfrak{g},\mathfrak{g}]$. If $\operatorname{tr}(xy) = 0$ for any $y \in \mathfrak{gl}(V)$ (suppose over $\mathbb{C}$). Look at Jordan's form of x with eigenvalues $\lambda_1, \cdots, \lambda_n$. Take $y = \operatorname{diag}(\overline{\lambda}_1, \cdots, \overline{\lambda}_n)$. Then $\operatorname{tr}(xy) = 0$ then $\lambda_i = 0$ for any i. The problem with this is that it is not clear why $y \in \mathfrak{g}$. To work around this let

$$M = \{y \in \mathfrak{gl}(V) : [y,[\mathfrak{g},\mathfrak{g}]] \subset \mathfrak{g}\} \supset \mathfrak{g}$$

If $\operatorname{tr}(xy) = 0$ for any $x \in [\mathfrak{g},\mathfrak{g}], y \in \mathfrak{g}$ we have $\operatorname{tr}(xy) = 0$ for any $x \in [\mathfrak{g},\mathfrak{g}], y \in M$. Now we can find our $y \in M$. □

COROLLARY 3. *If in the Lie algebra* $\mathfrak{g}$,

$$\operatorname{tr}(\operatorname{ad} x.\operatorname{ad} y) = 0, \forall x \in [\mathfrak{g},\mathfrak{g}], y \in \mathfrak{g}$$

then $\mathfrak{g}$ *is solvable.*

PROOF. Apply Cartan's criterion to $\operatorname{ad}(\mathfrak{g})$. So $\operatorname{ad}(\mathfrak{g})$ is solvable. Its kernel is $Z(\mathfrak{g})$ which is abelian. So we are done. □

7. Killing form

We define

$$\kappa(x,y) = \operatorname{tr}(\operatorname{ad} x.\operatorname{ad} y)$$

to be the Killing form, and it is a symmetric bilinear form. Notation

$$\kappa_{ij} = (\kappa(x_i, x_j))_{ij}$$

where x_i's form a basis of $\mathfrak{g}$.

The radical of the Killing form is

$$S_k = \{x \in \mathfrak{g} : \kappa(x,y) = 0, \forall y \in \mathfrak{g}\}.$$

A form is called non-degenerate if its radical is zero. This is equivalent to $\det(\kappa_{ij}) \neq 0$. A bilinear form is non-degenerate if and only if it gives an isomorphism between V and V^* via

$$x \mapsto (y \mapsto \kappa(x,y)).$$

THEOREM 7.1. $\mathfrak{g}$ *is semisimple if and only if* κ *is non-degenerate.*

LEMMA 4. *Let* $\mathfrak{h} \subset \mathfrak{g}$ *be an ideal. And*

$$\kappa_{\mathfrak{h}} : \mathfrak{h} \times \mathfrak{h} \to \mathbb{C}$$

its Killing form. Then

$$\kappa_{\mathfrak{h}} = \kappa|_{\mathfrak{h}\times\mathfrak{h}}.$$

PROOF. Let $A : V \to V$ be a linear operator and $\mathrm{im}(A) \subset W \subset V$ where W is a subspace of V. Then

$$\mathrm{tr}(A) = \mathrm{tr}\, A|_W.$$

□

Observation:

$$\kappa([x,y],z) = \kappa(x,[y,z]).$$

COROLLARY 4. *The radical,* S_κ *of* κ *is an ideal in* $\mathfrak{g}$*.*

PROOF OF THE THEOREM. Recall that $\mathfrak{g}$ is semisimple if and only if $\mathfrak{g}$ has no nontrivial abelian ideals.

$S_k \subset \mathrm{Rad}(\mathfrak{g})$: If $x \in S_k$ then

$$\mathrm{tr}(\mathrm{ad}\, x.\, \mathrm{ad}\, y) = 0, \forall y \in \mathfrak{g}$$

(in particular such y is in $[S_k, S_k]$). So by Cartan's criterion S_k is solvable. And since it is an ideal, it is contained in $\mathrm{Rad}(\mathfrak{g})$.

We will show that any abelian ideal is contained in S_k. Let $\mathfrak{h}$ be such an ideal. Pick $x \in \mathfrak{h}, y \in \mathfrak{g}$. We need to show $\kappa(x,y) = 0$. The operator $\mathrm{ad}\, x\, \mathrm{ad}\, y$ maps $\mathfrak{g}$ into $\mathfrak{h}$, and $(\mathrm{ad}\, x\, \mathrm{ad}\, y)^2$ maps $\mathfrak{g}$ into $[\mathfrak{h}, \mathfrak{h}]$, which is zero! Hence $(\mathrm{ad}\, x\, \mathrm{ad}\, y)^2 = 0$ and therefore $\mathrm{tr}(\mathrm{ad}\, x\, \mathrm{ad}\, y) = 0$. □

8. Structure of semisimple algebras

Our next goal is to show that the semisimple algebras are direct sum of simple ones.

DEFINITION 4. $\mathfrak{h}_1, \cdots, \mathfrak{h}_n$ ideals in $\mathfrak{g}$. We say

$$\mathfrak{g} = \mathfrak{h}_1 + \cdots + \mathfrak{h}_n$$

is a direct sum if it is a direct sum as a vector space.

REMARK. $[\mathfrak{h}_i, \mathfrak{h}_j] \subset \mathfrak{h}_i \cap \mathfrak{h}_j = \{0\}$ so then automatically by the above condition we have

$$\mathfrak{g} = \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_n$$

is a direct sum of Lie algebras with $[\mathfrak{h}_i, \mathfrak{h}_j] = 0$.

THEOREM 8.1. *If* $\mathfrak{g}$ *is semisimple then there is a unique decomposition of* $\mathfrak{g}$ *into semisimples.*

$$\mathfrak{g} = \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_n$$

PROOF. let $\mathfrak{h}$ be any ideal in $\mathfrak{g}$. Let

$$\mathfrak{h}^{\perp} = \{y \in \mathfrak{g} : \kappa(x, y) = 0, \forall x \in \mathfrak{h}\}$$

(this is an ideal since $\kappa(x, [y, z]) = \kappa([x, y], z)$). Note that since $\mathfrak{g}$ is semisimple, κ is non-degenerate. Therefore we have $\mathfrak{g} = \mathfrak{h} + \mathfrak{h}^{\perp}$ and all we have to check is that $\mathfrak{h} \cap \mathfrak{h}^{\perp} = \{0\}$. This is the case since

$$\operatorname{tr}(\operatorname{ad} x . \operatorname{ad} y) = 0, \forall x, y \in \mathfrak{h} \cap \mathfrak{h}^{\perp}.$$

Therefore by Cartan's criterion, $\mathfrak{h} \cap \mathfrak{h}^{\perp}$ is solvable. But $\mathfrak{g}$ is semisimple hence the intersection is just the trivial ideal.

Now we do induction on $\dim \mathfrak{g}$. Note that if $\mathfrak{h}_1 \subset \mathfrak{h}$ is an ideal in $\mathfrak{h}$ then it is also an ideal in $\mathfrak{g}$ if $[\mathfrak{h}, \mathfrak{h}^{\perp}] = 0$.

It remains to prove the uniqueness. Let I be an ideal in $\mathfrak{g}$.

$$[I, \mathfrak{g}] = \oplus_i [I, \mathfrak{h}_i]$$

so exactly one of the right hand summands is nonzero because $\mathfrak{h}_i$'s are semisimple. Then $I = \mathfrak{h}_i$. □

COROLLARY 5. *If* $\mathfrak{g}$ *is semisimple then* $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.

COROLLARY 6. *If* $\mathfrak{g}$ *is semisimple, then ideals in* $\mathfrak{g}$ *and homomorphic images of* $\mathfrak{g}$ *are all semisimple. (Note: this is not the case for subalgebras, unlike the solvable or nilpotent ones.)*

CHAPTER 2

Modules and representations

1. Lie algebra modules

Let $\mathfrak{g}$ be a Lie algebra. A $\mathfrak{g}$-module is an action $\mathfrak{g} \times V \to V$, that is compatible with the Lie structures:

(1) $X(\alpha v + \beta w) = \alpha X v + \beta X w$,

(2) $(\alpha X + \beta Y)v = \alpha X v + \beta Y v$

(3) $[X, Y].v = X.Y.v - Y.X.v$

The image of $\mathfrak{g}$ in $\mathrm{End}(V)$: take the (associative!) ring generated by $\rho(\mathfrak{g})$'s. Ultimately we want a 'universal' ring associated with $\mathfrak{g}$ such that representations of $\mathfrak{g}$ are in one-to-one correspondence with modules over this ring.

Schur's lemma still holds. If V, W are $\mathfrak{g}$-modules, then $\mathrm{Hom}_{\mathfrak{g}}(V, W)$ consists of linear maps $f : V \to W$ such that $g.f(v) = f(g.v)$ for $\mathfrak{g} \in W$. $\mathrm{Hom}(V, W) \cong V^* \otimes W$ is a $\mathfrak{g}$-module with action $g.f = g.f(v) - f(g.v)$.

One can generalize the Killing form as follows: suppose $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation. Define

$$\beta(x, y) = \mathrm{tr}(\rho(x), \rho(y)).$$

If ρ is faithful, β is non-degenerate (if $\mathfrak{g}$ is semisimple). Also β is associate:

$$\beta(x, [y, z]) = \beta([x, y], z).$$

The radical, S_β of β is an ideal (solvable).

Let $x_1, \cdots, x_n$ be a basis of $\mathfrak{g}$. Let $y_1, \cdots y_n$ be the dual basis with respect to β:

$$\beta(x_i, y_j) = \delta_{ij}.$$

The Casimir operator

$$C_\rho \coloneqq \sum_i \rho(x_i)\rho(y_i)$$

is a linear operator on V and does not depend on $x_1, \cdots, x_n$.

Lemma 5. *c_ρ commutes with the action of $\mathfrak{g}$.*

PROOF. Fix $X \in \mathfrak{g}$, and let X_i's be the basis of $\mathfrak{g}$ and Y_i's the dual basis:

$$[X, X_i] = \sum a_{ij} X_j, [X, Y_i] = \sum b_{ij} Y_j.$$

Then $b_{ji} = -a_{ij}$ because:

$$-a_{ij} = -\sum_k a_{ik}\beta(X_k, Y_j) = -\beta(\sum_k a_{ik} X_k, Y_j) = \beta([X, X_i], Y_j)$$

$$\beta([X_i, X], Y_j) = \beta(X_i, [X, Y_j]) = \beta(X_i, \sum_k b_{jk} Y_k) = \sum_k b_{jk}\beta(X_i, Y_k) = b_{ji}$$

Finally we do the computation:

$$\begin{aligned}[\rho(X), c_\rho] &= \sum_i [\rho(X), \rho(X_i)\rho(Y_i)] \\ &= \sum_i [\rho(X), \rho(X_i)]\rho(Y_i) + \sum \rho(X_i)[\rho(X), \rho(Y_i)] \\ &= \sum_i \sum_j a_{ij}\rho(X_j)\rho(Y_i) + \sum_i \sum_j \rho(X_i) b_{ij} \rho(Y_j) = 0\end{aligned}$$

□

Example 1.1. $\mathfrak{g} = \mathfrak{sl}_2$ and ρ the identity representation on the 2-dimensional space. Then

$$X^* = Y, Y^* = X, H^* = 1/2 H.$$

So then

$$c_\rho = XY + YX + \frac{1}{2}H^2 = \begin{pmatrix} 3/2 & 0 \\ 0 & 3/2 \end{pmatrix}.$$

⌟

This generalizes by the

LEMMA 6. *Suppose ρ is irreducible. Then c_ρ is the scalar $\frac{\dim \mathfrak{g}}{\dim V}$.*

PROOF. c_ρ is scalar by Schur's lemma. So it remains to compute the trace:

$$\operatorname{tr} c_\rho = \operatorname{tr}(\sum_i \rho(X_i)\rho(Y_i)) = \sum \beta(X_i, Y_i) = \dim \mathfrak{g}.$$

□

Example 1.2. Consider the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}.$$

Then it acts on functions on $\mathbb{H}$ by

$$L_g f(z) = f(g^{-1}.z)$$

for any $g \in \mathrm{SL}_2(\mathbb{R})$. We get an action of $\mathfrak{sl}_2(\mathbb{R})$ on $\mathcal{C}^\infty(\mathbb{H})$, the real smooth functions:

$$X.f = \frac{d}{dt}\Big|_{t=0} f(\exp(-tX).z).$$

Let β be the Casimir operator we found before (either on $\mathfrak{sl}_2$ or its representation in $\mathfrak{gl}_2$).

$$e^{-tX} = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} z = z - t = x - t + iy$$

$$\frac{d}{dt}\Big|_{t=0} f(\exp(-tX).z) = \frac{d}{dt}\Big|_{t=0} f(x(t), y(t)) = -\frac{\partial f}{\partial x}.$$

Continuing similarly we conclude that c_ρ is a second order differential operator and commutes with the action of $\mathfrak{g}$. So this has to be the Laplacian! One in fact can check by hand that it coincides with the Laplacian on $\mathbb{H}$

$$cy^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

⌟

2. Complete reducibility (a special case)

A final goal will be to prove

Weyl's theorem. For the semisimple Lie algebras if V is a representaion, W a subrepresentation, then there is a subrepresenation $U \subset V$ such that $V = U \oplus W$.

Let $\mathfrak{g}$ be a semisimple Lie algebra, $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ a faithful representation. Let's consider a special case: that of when $W \subset V$ is an invariant subspace of codimension one and itself irreducible as a representation of $\mathfrak{g}$. We show that W has an invariant complement, i.e.

$$0 \to W \to V \to V/W \to 0$$

is a split sequence of representations.

Take $c_\rho \in \mathrm{End}(V)$. Then $c_\rho(V) \subset W$ because it is a linear combination of product of elements of $\mathfrak{g}$.

Look at $c_\rho|_W$. Since W is irreducible, $c_\rho|_W$ has to be a nonzero scalar because of the trace. $\ker(c_\rho) \cap W = \{0\}$ then $\ker(c_\rho)$ is the dimension one complement.

When W is not irreducible but still codimension one, we do an easy induction on $\dim W$. If W is not codimension one, consider $\mathrm{Hom}(V, W)$ as a $\mathfrak{g}$-module. This contains a submodule $\mathfrak{V}$ corresponding to the scalar action on W, and in its codimension one, there is $\mathfrak{W}$ with zero action on W. Let f be the generator of the complement of $\mathfrak{W}$ in $\mathfrak{V}$. Then $\ker(f)$ is the complement of W.

3. Jordan decomposition

We know from linear algebra that the Jordan decomposition gives a basis independent decomposition

$$X = X_s + X_n$$

where X_n is nilpotent and X_s is diagonalizable (the minimal polynomial has distinct roots). Moreover X_s and X_n are polynomials in X. To have an analogue of this decomposition in an arbitrary Lie algebra, we fix a faithful representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ and then we have

$$\rho(X) = \rho(X)_s + \rho(X)_n$$

but then the question is that do the two right hand terms have to be in $\rho(\mathfrak{g})$. Do they depend on the representation.

The answer is that all is fine if $\mathfrak{g}$ is semisimple.

Example 3.1. For $\mathfrak{g} = \mathfrak{gl}(1)$,

$$\rho_1 : \mathfrak{g} \to \mathfrak{gl}_2 \text{ via } t \mapsto \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}$$

has semi-simple image,

$$\rho_2 : \mathfrak{g} \to \mathfrak{gl}_2 \text{ via } t \mapsto \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}$$

has nilpotent image and $t \mapsto \begin{pmatrix} t & t \\ 0 & 0 \end{pmatrix}$ has neither type image. ⌟

We will show that if $\mathfrak{g}$ is semisimple, then

(1) X_s, X_n are elements of $\mathfrak{g}$,

(2) If $\rho : \mathfrak{g} \to \mathfrak{gl}(V')$ is a homomorphism, then $\rho(X) = \rho(X_s) + \rho(X_n)$ is the Jordan decomposition in $\mathfrak{gl}(V')$.

Remark. If $\mathfrak{g}$ is semisimple, we know that $\operatorname{ad}\mathfrak{g} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is injective. So

$$\operatorname{ad}(X) = \operatorname{ad}(X_s) + \operatorname{ad}(X_n)$$

is called the *abstract* or *absolute* or even the *universal* Jordan decomposition of $\mathfrak{g}$.

Example 3.2. If $\mathfrak{g} = \mathfrak{so}(\mathbb{C})$, then the diagonal and the nilpotent parts of any orthogonal matrix are orthogonal. ⌟

Proof of the first assertion. We can see that $\mathfrak{g} \subset \mathfrak{sl}(V)$ as $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. Also X_s and X_n are in $\mathfrak{sl}(V)$ by considering their trace. We shall think of V as a $\mathfrak{g}$-module. Then for every submodule, W, we define

$$S_W = \{Y \in \mathfrak{gl}(V) : Y(W) \subset W \text{ and } \operatorname{tr} Y|_W = 0\}.$$

Thus $\mathfrak{g}$ is in all S_W's and so are X_s and X_n as they are polynomials in X. We also define

$$\mathfrak{M} = \{A \in \mathfrak{gl}(V) : [A, \mathfrak{g}] \subset \mathfrak{g}\}.$$

Note that if $A \in \mathfrak{M}$ then $A_s, A_n \in \mathfrak{M}$. Our claim is that

$$\mathfrak{g} = \underbrace{\mathfrak{M} \cap \Big(\bigcap_{W \text{ irred.}} S_W \Big)}_{\mathfrak{g}'}.$$

Note that $\mathfrak{g}'$ is already a subalgebra of $\mathfrak{gl}(V)$ and $\mathfrak{g}$ is an ideal of $\mathfrak{g}'$. By the adjoint-action of $\mathfrak{g}$ on $\mathfrak{g}'$, we have that $\mathfrak{g}'$ is a $\mathfrak{g}$-module and $\mathfrak{g}$ is a submodule hence

$$\mathfrak{g}' = \mathfrak{g} \oplus U$$

such that $[\mathfrak{g}, U] = 0$. We want to show that $U = \{0\}$. Take $Y \in U$, we will show that for every irreducible $\mathfrak{g}$-submodule, W, $Y|_W = 0$. Y commutes with $\mathfrak{g}$ on W, therefore by Schur's lemma $Y|_W$ is a scalar, but at the same time traceless, hence proving the claim. □

Proof of the second assertion. If $\rho : \mathfrak{g} \to \mathfrak{g}'$ is a representation of $\mathfrak{g} \subset \mathfrak{gl}(V)$ in $\mathfrak{gl}(V') \supset \mathfrak{g}'$, since $\mathfrak{g}$ can be written as a direct sum of simple ideals, $\operatorname{im} \mathfrak{g} = \mathfrak{g}/(\oplus \mathfrak{g}_i)$: $\mathfrak{g}$ is a direct sum of eigenspaces, then $\mathfrak{g}'$ is a direct sum of eigenspaces of $\rho(X_s)$. So $\rho(X_s)$ is semisimple in $\mathfrak{gl}(V')$. $\rho(X_n)$ is obviously nilpotent. Now we use the uniqueness or Jordan decomposition in $\mathfrak{gl}(V')$. □

4. Representations of $\mathfrak{sl}_2(\mathbb{C})$

Recall our notation for the standard basis:

$$\mathfrak{sl}_2(\mathbb{C}) = \langle X, Y, H \rangle.$$

Say V is a $\mathfrak{g}$-module under $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$. H is semisimple in the standard representation so $\rho(H)$ is also semisimple on V and therefore

$$V = \oplus_\lambda V_\lambda.$$

Definition 5. λ is called a weight of V if $V_\lambda \neq \{0\}$.

Lemma 7. *If λ is a weight and $v \in V_\lambda$, then*

$$Xv \in V_{\lambda+2}, Yv \in V_{\lambda-2}.$$

Proof. This is just an easy computation, for instance:

$$H.(X.v) = [H, X]v + X.Hv = 2Xv + X(\lambda v) = (\lambda + 2)Xv.$$

□

Now if V is finite-dimensional, there exists a weight λ such that $V_{\lambda+2} = \{0\}$. Therefore any $v_0 \in V_\lambda$ is killed by X. v_0 is the maximal vector of weight λ. We use a more general notation

$$v_{-1} = 0, \quad v_0 \text{ as defined }, \quad v_\ell = \frac{1}{\ell!} Y^\ell v_0.$$

Then we have

$$Hv_\ell = (\lambda - 2\ell)v_\ell, \tag{4.1}$$

$$Yv_\ell = (\ell+1)v_{\ell+1}, \tag{4.2}$$

$$Xv_\ell = (\lambda - \ell + 1)v_{\ell-1}. \tag{4.3}$$

In fact $v_\ell \in V_{\lambda-2\ell}$ so at some point $v_{m+1} = 0$. So $\{v_0, \cdots, v_m\}$ is a set of linearly independent vectors spanning a submodule of dimension $m+1$ and by irreducibility of V this is going to be all of V. When $\ell = m+1$, we have

$$Xv_{m+1} = (\lambda - m)v_m = 0$$

from 4.3 so $\lambda = m$. Thus $v_0 \in V_m$ and the weights are

$$\{m, m-2, \cdots, -m\}$$

and all V_ℓ are one-dimensional for $\ell \in \{m, m-2, \cdots, -m\}$. The integer m is called the highest weight as it is the weight of the maximal vector, v_0 (i.e. the one with the highest weight). Let

$$V^{(m)} := \langle v_0, \cdots, v_m \rangle$$

with the action of $\mathfrak{sl}_2(\mathbb{C})$ determined by 4.1 to 4.2. We have proved that

Proposition 2. *Every irreducible $\mathfrak{sl}_2$-module is isomorphic to $V^{(m)}$.*

Example 4.4. The standard representation is a $V^{(1)}$ and the adjoint representation is a $V^{(2)}$.

⌟

Remark. As a remark observe that every finite dimensional $\mathfrak{sl}_2$-module is now just a direct sum of irreducible ones and the number of summands is

$$\dim V_0 + \dim V_1$$

where the first term is the number of 0-eigenspaces of H and the second term is the number of 1-eigenspaces of it.

CHAPTER 3

Root space decomposition

If $\mathfrak{g}$ is a semisimple Lie-algebra, then $\mathfrak{g}$ is a $\mathfrak{g}$-module via adjoint action. If every element of $\mathfrak{g}$ was ad-nilpotent then we already know that $\mathfrak{g}$ would have been nilpotent. Otherwise, there is some element $X \in \mathfrak{g}$ such that $X_s \neq 0$ and $\operatorname{ad} X = \operatorname{ad} X_s$. So we can talk about subalgebras consisting of semisimple elements then.

1. Toral algebras

DEFINITION 6. A toral subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ is a subalgebra that consists of semisimple elements only.

Example 1.1. In $\mathfrak{sl}_2$ the diagonal elements form a maximal toral subalgebra. The subalgebra of elements $\begin{pmatrix} 0 & x \\ 0 & x \end{pmatrix}$ is just conjugate to the former one. ⌟

PROPOSITION 3. *$\mathfrak{h}$ has to be abelian.*

PROOF. For any $X \in \mathfrak{h}$, we will show that $\operatorname{ad}_{\mathfrak{h}} X$ has only eigenvalue 0 on $\mathfrak{h}$. Let $Y \in \mathfrak{h}$ be an eigenvector:

$$[X, Y] = aY.$$

We diagonalize Y. Then $\operatorname{ad} Y$ is also diagonal. Thus $[Y, X]$ has to be a linear combination of eigenvectors for $\operatorname{ad} Y$ corresponding to nonzero eigenvalues. But from above we have $[Y, X] = -aY$, which is contradictory. □

2. The decomposition

Thus when working with a semisimple $\mathfrak{g}$, we can fix a maximal toral subalgebra $\mathfrak{h}$ and simultaneously diagonalize $\mathfrak{h}$. Let

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [\mathfrak{h}, X] = \alpha(\mathfrak{h})X\}$$

be an eigenspace where $\alpha \in \mathfrak{h}^*$ is a linear functional on $\mathfrak{h}$. $\mathfrak{g}_0$ coincides with $C_{\mathfrak{g}}(\mathfrak{h})$ and we have a decomposition

$$\mathfrak{g} = C_{\mathfrak{g}}(\mathfrak{h}) \oplus \left(\bigoplus_{\alpha \neq 0} \mathfrak{g}_\alpha \right).$$

$$\varphi = \{\alpha : \mathfrak{g}_\alpha \neq \{0\}\}$$

is called the set of roots of $\mathfrak{g}$, and each $\mathfrak{g}_\alpha$ is called a root space.

Example 2.1. For $\mathfrak{sl}_3 \subset \mathfrak{gl}_3$ there is a basis consisting of

$$h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and e_{ij}'s for all $i \neq j$. The maximal torus is generated by h_1 and h_2.

α	$\mathfrak{g}_\alpha$	$\alpha(h_1)$	$\alpha(h_2)$
α_1	$\langle e_{12} \rangle$	2	-1
α_2	$\langle e_{13} \rangle$	1	1
α_3	$\langle e_{21} \rangle$	-2	1
α_4	$\langle e_{23} \rangle$	-1	2
α_5	$\langle e_{31} \rangle$	-1	-1
α_6	$\langle e_{32} \rangle$	1	-2

where as we see $\alpha_5 = -\alpha_2$ and $\alpha_1 = -\alpha_3$ and finally $\alpha_1 + \alpha_4 = \alpha_2$. ⌟

LEMMA 8. *If $\alpha, \beta \in \mathfrak{h}^*$*

(1) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$.

(2) If $X \in \mathfrak{g}_\alpha$ and $\alpha \neq 0$ then $\operatorname{ad} X$ is nilpotent.

(3) If $\alpha + \beta \neq 0$ then $\kappa(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$.

PROOF. First assertion follows from an easy computation:

$$[h, [X, Y]] = [[h, X], Y] + [X, [h, Y]] = \alpha(h)[X, Y] + \beta(h)[X, Y].$$

Let Y be a linear combination of elements of $\mathfrak{g}_\beta$ for the second part. Here $\beta \in \Phi \cup \{0\}$. Then $[X, Y]$ is an element of $\mathfrak{g}_{\beta+\alpha}$. By dimensional considerations $\operatorname{ad} X$ has to be nilpotent.

Finally for the last claim, let $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_\beta$, and pick $h \in \mathfrak{h}$ such that $(\alpha + \beta)(h) \neq 0$. Then

$$\alpha(h)\kappa(X, Y) = \kappa([h, X], Y) = -\kappa([X, h], Y) = -\kappa(X, [h, Y]) = -\beta(h)\kappa(X, Y).$$

Therefore $\kappa(X, Y) = 0$. □

Example 2.2. $\mathfrak{g} = \mathfrak{sl}_2 = \mathfrak{h} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ where $\mathfrak{h}$ is spanned by H and is the maximal torus. The root $\alpha : \langle H \rangle \to \mathbb{C}$ maps $H \mapsto 2$. ⌟

PROPOSITION 4. $C_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$.

PROOF. Let $C := C_{\mathfrak{g}}\mathfrak{h}$.

Step 1. C contains all semisimple and nilpotent parts of its elements: given $X \in C$, it commutes with $\mathfrak{h}$ therefore $\operatorname{ad}_{\mathfrak{g}} X$ maps $\mathfrak{h}$ to 0. Then $(\operatorname{ad}_{\mathfrak{g}} X)_s$ and $(\operatorname{ad}_{\mathfrak{g}} X)_n$ also map $\mathfrak{h}$ to 0 (as they are polynomials in $\operatorname{ad}_{\mathfrak{g}} X$ without constant term).

Step 2. All semisimple elements in C have to lie in $\mathfrak{h}$: This is because the sum of commuting semisimple elements are semisimple and $\mathfrak{h}$ is maximal.

Step 3. $\kappa|_C$ is non-degenerate. For $z \in C$, already $\kappa(z, \mathfrak{g}_\alpha) = 0$ for any α by previous proposition. If $\kappa(z, C) = 0$ then $z \in S_\kappa$ which is a contradiction as S_κ is empty.

Step 4. $\kappa|_{\mathfrak{h}}$ is non-degenerate. Assume $\kappa(z, h) = 0$ for all $z \in \mathfrak{h}$, so z would belong to the radical of $\kappa|_C$ If y is nilpotent in C then

$$\operatorname{tr}(\operatorname{ad} z \operatorname{ad} y) = 0.$$

By step 1 then $\kappa(z, C) = 0$ which contradicts step 3.

Step 5. C is nilpotent. For this it suffices to show that for any $X \in C$, $\operatorname{ad}_C X$ is nilpotent. If $X = X_s + X_n$ then $X_s \in \mathfrak{h}$ so $\operatorname{ad}_C(X_s) = 0$. X_n is nilpotent so $\operatorname{ad}_C X_n$ is nilpotent, completing the proof.

Step 6. $\mathfrak{h} \cap [C, C] = \{0\}$. This is clear since

$$\kappa(\mathfrak{h}, [C, C]) = \kappa([\mathfrak{h}, C], C) = 0$$

but we know that κ is non-degenerate on h.

Step 7. C is abelian. We need a small

LEMMA 9. *If C is a nilpotent Lie algebra, and I is a nonzero ideal then $I \cap Z(C) \neq \{0\}$.*

PROOF. I is a C-module under the adjoint action so there is a common vector killed by all of C. $\square$

From this we conclude that $[C, C] \cap Z(C) \neq \{0\}$. Let $z \neq 0$ be an element of this intersection. This can not be semisimple as it would be in $\mathfrak{h}$ then, and we have shown $\mathfrak{h} \cap [C, C] = \{0\}$. This $z = s + n$ with the nilpotent part nonzero: $n \neq 0$. So n is an element of $[C, C] \cap Z(C)$. So $\kappa(n, C) = 0$ which is a contradiction as $\kappa|_C$ is non-degenerate.

Step 8. We finally show that $C = \mathfrak{h}$. If $C \neq \mathfrak{h}$, there is $z = s + n \in C$ which is not semisimple. So $0 \neq n \in C$. But then $\kappa(n, C) = 0$ implying that $n = 0$. $\square$

COROLLARY 7. *$\kappa|_{\mathfrak{h}}$ is non-degenerate.*

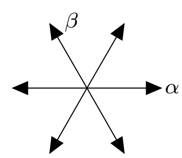
This is going to be a very useful fact in the next chapter. We will be using this to identify $\mathfrak{h}$ and $\mathfrak{h}^*$: for any $\alpha \in \Phi$, we denote by t_α the element of $\mathfrak{h}$ satisfying

$$\kappa(t_\alpha, h) = \alpha(h), \quad \forall h \in \mathfrak{h}.$$

3. Properties of the root spaces

Our goal is to find a *real* euclidean vector space where the roots live and get information about length and angles of them with respect to each other in that space.

Example 3.1. In the example of $\mathfrak{sl}_3 \subset \mathfrak{gl}_3$ above let $\alpha = \alpha_1$ and $\beta = \alpha_4$. Then we have a configuration



generating all α_i's. In fact with respect to the induced inner product $(.,.)$ on $\mathfrak{h}^*$ defined by

$$(\lambda, \mu) = \kappa(t_\lambda, t_\mu)$$

we have $(\alpha, \beta) = \kappa(t_\alpha, t_\beta) = \beta(t_\alpha) = \alpha(t_\beta)$. So let us find $t_\beta = ah_1 + ah_2$ where h_1 and h_2 are the elements of the basis of $\mathfrak{h} \subset \mathfrak{sl}_3$ as before. In this concrete example we may find a, b from computation of $\kappa(t_\beta, h_1) = \beta(h_1)$ and $\kappa(t_\beta, h_2) = \beta(h_1)$. It turns out that

$$t_\alpha = \frac{1}{6}h_1, t_\beta = \frac{1}{6}h_2, (\alpha, \beta) = -\frac{1}{6}, (\alpha, \alpha) = (\beta, \beta) = \frac{1}{3}.$$

This justifies the picture above; we think of α_i's as vectors in the $\mathbb{R}$-span of them with angles given by $(.,.)$. ⌟

Proposition 5.

(1) Φ *spans* $\mathfrak{h}^*$.

(2) $\alpha \in \Phi$ *then* $-\alpha \in \Phi$.

(3) For $x \in \mathfrak{g}_\alpha$ *and* $y \in \mathfrak{g}_{-\alpha}$, *we have* $[x, y] = \kappa(x, y)t_\alpha$.

(4) If $\alpha \in \Phi$, *then* $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ *is one-dimensional, with basis* t_α.

(5) $\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) \neq 0$ *for any* $\alpha \in \Phi$.

(6) For any $\alpha \in \Phi$, *and nonzero* $x \in \mathfrak{g}_\alpha$, *there is* $y \in \mathfrak{g}_{-\alpha}$, *such that* $\langle x, y, [x, y]\rangle$ *is a subalgebra isomorphic to* $\mathfrak{sl}(2, F)$. *The element* $h = [x, y] \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \in \mathfrak{h}$ *is given via*

$$h_\alpha = \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}$$

and therefore $h_\alpha = -h_{-\alpha}$.

Proof.

(1) If $\alpha(h) = 0$ for all $\alpha \in \Phi$, then $h \in Z(\mathfrak{g})$ so $h = 0$.

(2) If $-\alpha \notin \Phi$, then $\mathfrak{g}_{-\alpha} = 0$ and hence $\kappa(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ for all $\beta \in \mathfrak{h}^*$. Thus $\kappa(\mathfrak{g}_\alpha, \mathfrak{g}) = 0$ contradicting the nondegeneracy of κ.

(3) For any $h \in \mathfrak{g}$ we have

$$\kappa(h, [x, y]) = \kappa([h, x], y) = \alpha(h)\kappa(x, y) = \kappa(t_\alpha, h)\kappa(x, y) = \kappa(h, \kappa(x, y)t_\alpha)$$

hence by nondegeneracy we get the result.

(4) Given $\alpha \neq 0$, if $\kappa(\alpha, \mathfrak{g}_{-\alpha}) = 0$ then κ is degenerate on the whole space. Therefore $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is nonzero.

(5) By last part t_α, x and y form a three dimensional solvable algebra S. Thus $\mathrm{ad}_L\, s$ is nilpotent for all $s \in [S, S]$. So $\mathrm{ad}_L\, t_\alpha$ is both semisimple and nilpotent and thus $t_\alpha \in Z(L) = 0$ contrary to the choice of t_α.

(6) This follows by computing commutators of the three elements easily.

□

We have seen that $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ is an $\mathfrak{sl}_2$-module via the adjoint representation for each embedding

$$\mathfrak{sl}_2 \hookrightarrow \mathfrak{g}$$

via $X \mapsto x_\alpha, Y \mapsto y_\alpha$ and $H \mapsto h_\alpha$. There are submodules

$$M = \mathfrak{h} \oplus \bigoplus_{c\alpha\in\Phi} \mathfrak{g}_{c\alpha}$$

on which the X_α acts by $X\alpha.\mathfrak{g}_{c\alpha} \subseteq g_{(c+1)\alpha}$, etc. So $\mathfrak{g}_{c\alpha}$'s are weight spaces of M with weight $c\alpha(h_\alpha) = 2c$ so they are one-dimensional. So the only possible submodules of this type appear for $c = \pm 1$ in which cases

$$M = \langle h_\alpha, x_\alpha, y_\alpha \rangle \oplus \text{ rest of } \mathfrak{h}.$$

In particular the only multiples of α that can be roots are $\pm\alpha$.

Now let's fix α and let $\beta \neq -\alpha$. Then

$$K = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_{\beta+i\alpha}$$

is a sum over weight spaces of our $\mathfrak{sl}_2$-module with weights $\beta(h_\alpha) + 2i \in \mathbb{Z}$. Then K is an irreducible submodule and say has highest weight $\beta(h_\alpha) + 2q$ and lowest weight $\beta(h_\alpha) - 2r$. The corresponding roots in $\mathfrak{g}$ are

$$\beta + q\alpha, \cdots, \beta - r\alpha.$$

Therefore $\beta(h_\alpha) = r - q$ and the above is called the *α-string through β*.

Also this provides an update on lemma 8:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \mathfrak{g}_\alpha}$$

where the second-type summands are all one-dimensional; the only occurrences of multiples of α are $\pm\alpha$; and that for $\beta \neq -\alpha$ we have

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \begin{cases} \mathfrak{g}_{\alpha+\beta} & \text{if } \alpha+\beta \text{ is a root} \\ 0 & \text{otherwise.} \end{cases}$$

Now we can as well prove that $(\alpha, \beta) \in \mathbb{Q}$ for all $\alpha, \beta \in \Phi$. Let $\alpha_1, \cdots, \alpha_\ell$ be a basis for $\mathfrak{h}^*$. Then for any $\beta \in \Phi$, we set $\beta = \sum_{i=1}^{\ell} c_i \alpha_i$ so

$$\frac{2}{(\alpha_j, \alpha_j)}(\beta, \alpha_j) = \sum_i c_i (\alpha_i, \alpha_j) \frac{2}{(\alpha_j, \alpha_j)}$$

and hence we have a system of linear equations with $\mathbb{Z}$-coefficients

$$\beta(h_{\alpha_j}) = \sum c_i \alpha_i (h_{\alpha_j}) \quad \forall j$$

completing the proof of our claim: In the $\mathbb{Q}$-space spanned by the roots, $E_{\mathbb{Q}}$, $(.,.)$ is a $\mathbb{Q}$-valued non-degenerate bilinear form.

Also for any $\lambda, \mu \in \mathfrak{h}^*$ we have

$$(\lambda, \mu) = \operatorname{tr}(\operatorname{ad} t_\lambda, \operatorname{ad} t_\mu) = \sum_{\alpha \in \Phi} \alpha(t_\lambda)\alpha(t_\mu) = \sum_{\alpha \in \Phi} (\lambda, \alpha)(\mu, \alpha).$$

In particular $(\lambda, \lambda) = \sum_{\alpha \in \Phi} (\lambda, \alpha)^2$. Thus $(.,.)$ is a positive definite bilinear pairing on $E_{\mathbb{Q}}$.

The fractions $\frac{2(\alpha,\beta)}{(\beta,\beta)}$ turn out to play important role in the upcoming chapter. One reminder about them is that they are always integer-valued.

CHAPTER 4

Root systems

1. Basic definitions

Pick a basis for $\mathfrak{h}^*$ from Φ, say $\alpha_1, \cdots, \alpha_\ell$. Then we make a $\mathbb{Q}$-span of $\langle \alpha_1, \cdots, \alpha_\ell \rangle$ and tensor this with $\mathbb{R}$ to get an honest real-valued vector space of the same dimension as $\mathfrak{h}$ with an inner product $(,)$ on it.

Recall that we had $h_\alpha \subset [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \langle t_\alpha \rangle$. We have

$$\beta(h_\alpha) = \frac{2(\beta, \alpha)}{(\alpha, \alpha)}$$

from $\mathfrak{sl}_2$-module $\oplus_{i \in \mathbb{Z}} \mathfrak{g}_{\beta + i\alpha}$. Thus $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$. Thus $(\beta, \alpha) \in \mathbb{Z}$ for any $\alpha, \beta \in \Phi$. Then also $(,)$ is positive-definite:

$$(\lambda, \lambda) = \sum \underbrace{(\lambda, \alpha)^2}_{\in \mathbb{Q}} > 0.$$

So Φ corresponds to a finite collection of vectors in a real Euclidean vector space (i.e. a real vector space with an inner product on it).

Our next observation is that if $\beta - r\alpha, \cdots, \beta + q\alpha$ are the roots corresponding to weights $\beta(h_\alpha) + 2i$ in $\oplus_i \mathfrak{g}_{\beta + i\alpha}$ then $\beta - \beta(h_\alpha).\alpha$ is also in Φ.

Definition 7. Let E be a real Euclidean vector space. A finite collection Φ of vectors in E is called a *(reduced) root system* if

- R1 Φ is finite, spans E and $0 \notin \Phi$.
- R2 If $\alpha \in \Phi$, then Φ contains $\pm\alpha$ and no other multiples of α.
- R3 For any $\alpha, \beta \in \Phi$, then the reflection $\sigma_\alpha(\beta) \in \Phi$.
- R4 For any $\alpha, \beta \in \Phi$, then $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$.

The reflection σ_α is the reflection about the hyperplane orthogonal to α. So it fixes P_α and maps $\alpha \mapsto -\alpha$:

$$x \mapsto x - \frac{2(x, \alpha)}{(\alpha, \alpha)} \alpha.$$

Notation: $\langle \beta, \alpha \rangle := \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$.

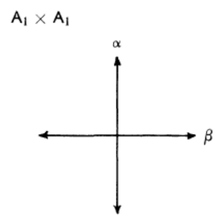
Remark. The axioms (R1), (R3) and (R4) imply that the only multiples of $\alpha \in \Phi$ are $\pm 2\alpha$ and $\pm 1/2\alpha$ [HW #3]. In fact in some sources the axioms of a root system are confined to (R1), (R3) and (R4). If the root system has (R2) then it is called a *reduced* root system. Non-reduced root systems arise over $\mathbb{R}$. Over $\mathbb{C}$ non-reduced systems cannot arise from Lie algebras so we will stick to our nomenclature.

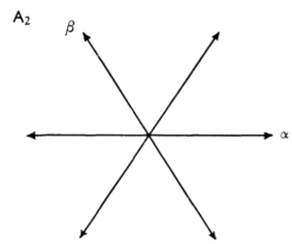
So by results of the previous chapter, the root space of a semisimple Lie algebra over $\mathbb{C}$ is a root system.

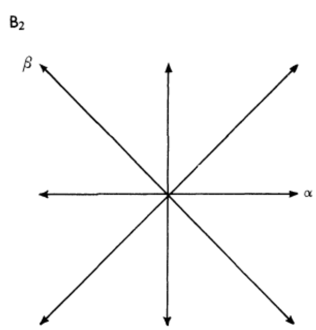
Example 1.1. Let the rank of the Euclidean space be $\ell = \dim E = \dim \mathfrak{h}$.[1]

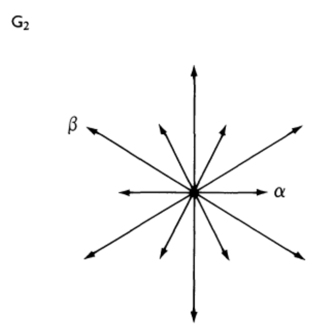
(1) $\ell = 1$

(2) $\ell = 2$









⌟

[1]Pictures are copied from Humphreys' book.

(Φ, E) is *isomorphic* to (Φ', E') if and only if there is a linear isomorphism $A : E \to E'$ mapping Φ to Φ' such that

$$\langle A(\beta), A(\alpha)\rangle = \langle \beta, \alpha\rangle, \quad \forall \alpha, \beta \in \Phi.$$

The *dual* of Φ is the collection of the co-roots

$$\alpha^\vee := \frac{2\alpha}{(\alpha, \alpha)}$$

and is denote by $\Phi^\vee$. One can see that $\Phi^\vee$ is a root system.

Note that

$$\langle \beta, \alpha\rangle\langle \alpha, \beta\rangle = 4\cos^2(\theta)$$

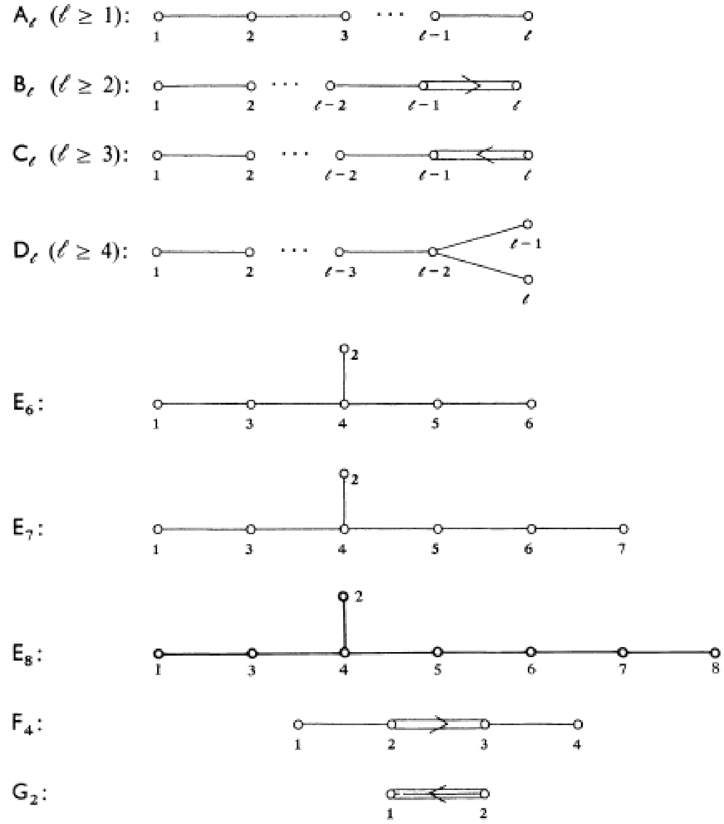
where θ is the angle between α and β, which is in $[0, 1)$ provided that $\alpha \neq \pm\beta$. Also we know that $\langle \beta, \alpha\rangle$ and $\langle \alpha, \beta\rangle$ are integers of the same sign, so there are not too many possiblities:

$\langle \alpha, \beta\rangle$	$\langle \beta, \alpha\rangle$	θ	$\frac{\Vert\beta\Vert^2}{\Vert\alpha\Vert^2}$	label
0	0	$\pi/2$	any	$A_1 \times A_1$
1	1	$\pi/3$	1	A_2
-1	-1	$2\pi/3$	1	A_2
1	2	$\pi/4$	2	B_2, C_2
-1	-2	$3\pi/4$	2	B_2, C_2
1	3	$\pi/6$	3	G_2
-1	-3	$5\pi/6$	3	G_2

2. Coxeter graphs and Dynkin diagrams

In what follows we will have to find a good way of choosing α, β and more generally a 'good' bases $\{\alpha_1, \cdots, \alpha_\ell\}$ for E. But once this task if done with the outcome $\Delta \subset \Phi$ we will call the elements of this simple roots and use it to draw more intrinsic graphs called the Coxeter graphs by putting a vertix for every simple root, and connecting any α and β by the number of edges $\langle \alpha, \beta\rangle\langle \beta, \alpha\rangle$. So the possible number of edges varies form zero to three. We will see that it is 'almost' possible to reconstruct Φ from the corresponding Coxeter graph, but only from the Euclidean geometry of E we see that there are only a few possible Coxeter graphs labeled by A_ℓ, B_ℓ and C_ℓ at the same time, D_ℓ for any $\ell \geq 4$ and then the exceptional cases of E_6, E_7, E_8, F_4 and finally G_2.

In a root system at most two lengths of roots can occur. If there are two different lengths then we have a multiple edge and we will put an arrow pointing to the shorter root. The result if called a *Dynkin diagrams*. This will distinguish B's and C's:



These will classify simple algebras. The connected components of the Dynkin diagrams will correspond to simple ideals in $\mathfrak{g}$.

The Classification Theorem. Dynkin diagrams are in one-to-one correspondence with simple Lie algebras. Modulo the definition of Δ we can construct a unique Dynkin diagrtam for a simple Lie algebra (so the diagram is independent of $\mathfrak{h}$ and Δ and in other words any two simple Lie algebras with same Dynking diagrams are isomorphic).

2.1. Prerequisites to classification of Dynkin diagrams. First we state a useful

Lemma 10. *If $\alpha, \beta \in \Phi$, are non-proportional roots, if $(\alpha, \beta) > 0$ then $\alpha - \beta$ is a root and if $(\alpha, \beta) < 0$ then $\alpha + \beta$ is a root.*

Proof. For instance $(\alpha, \beta) > 0$, is equivalent to $\langle \alpha, \beta \rangle = 0$ in that case $\langle \beta, \alpha \rangle \langle \alpha, \beta \rangle$ is either 1, 2 or 3 therefore one of the two terms is 1. Suppose $\langle \beta, \alpha \rangle = 1$, the

$$\sigma_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha = \beta - \alpha \in \Phi.$$

□

Corollary 8. *If $\{\beta + i\alpha\} \subset \Phi$ is an α-string through β, then*

(1) σ_α reverses the string.

(2) Every vector $\beta + i\alpha$ for $-r \le i \le q$ is a root.

(3) $q - r = \langle \beta, \alpha \rangle$ and in particular no root strings of length greater that 4 may exist.

Definition 8. $\Delta \subset \Phi$ is called a base if it spans E as a vector space, and every element $\gamma \in \Phi$ can be written as

$$\gamma = \sum_{\alpha \in \Delta} k_\alpha \alpha$$

with all k_α either non-negative integers or non-positive ones.

The labels we choice without an explanation in the above diagrams form bases. If $(\alpha, \beta) > 0$ then $\beta - \alpha \in \Phi$. This proves the following

Lemma 11. *If $\Delta \subset \Phi$ is a base, then $(\alpha, \beta) \le 0$ for all $\alpha, \beta \in \Delta$.*

Suppose we choose a base $\Delta \subset \Phi$ and put a partial ordering on E by

$\gamma \succcurlyeq 0$ if it is a combination of elements of Δ with non-negative coefficients

and $\mu - \nu \succcurlyeq 0$ if $\mu - \nu \succcurlyeq 0$ To show that a bases exists, pick $\gamma \notin P_\alpha$ for any α, and pick a half space defined by P_γ, then call a root δ in that half-space *decomposable* if there are α, β in the same half-space such that $\delta = \alpha + \beta$ and otherwise *indecomposable*. Then one can see that the set of indecomposable roots in P_γ form a base.

Definition 9. The Cartan matrix of a root system Φ is defined to be

$$(\langle \alpha_i, \alpha_j \rangle_{ij})$$

where $\{\alpha_1, \cdots, \alpha_\ell\}$ is a base for Φ.

There are some facts about independence of choices. In particular all bases are obtained by the method we used. $E \setminus \bigcup_{\alpha \in \Phi} P_\alpha$ is a union of connected components called Weyl chambers. On each Weyl chamber the sign of (x, α) (for x in the chamber) does not depend on x for any α. So there is a one-to-one correspondence

$$\{ \text{ bases } \} \leftrightarrow \{ \text{ Weyl chambers } \}.$$

2.2. The Weyl group.

Definition 10. The Weyl group of Φ is the group of all $\sigma_\alpha (\alpha \in \Phi)$.

Every element of W gives a permutation of Φ (so W is a finite subgroup of $S_{|\Phi|}$. As an aside each Weyl group is a Coxeter group i.e. a group of the form

$$\langle r_i : (r_i r_j)^{m_{ij}} = e, m_{ij} \in \mathbb{Z}, m_{ii} = 1, m_{ij} \geq 2\rangle.$$

The group $\mathrm{Aut}(\Phi)$ is the group of all linear operators $A : E \to E$ perserving Φ such that

$$\langle A(\beta), A(\alpha)\rangle = \langle \beta, \alpha\rangle.$$

Note that under any linear operator as such $\langle , \rangle$ is automatically preserves since

$$A\sigma_\alpha A^{-1} = \sigma_{A(\alpha)}.$$

Then $W \subset \mathrm{Aut}(\Phi)$. A next observation is that W is normal in $\mathrm{Aut}(\Phi)$.

THEOREM 2.1. *Let Δ be a base of Φ. Then*

(1) If $\gamma \in E$, and $\gamma \notin P_\alpha, \alpha \in \Phi$, then there is $w \in W$ such that

$$(w(\gamma), \alpha) > 0, \forall \alpha \in \Phi.$$

(i.e. W acts transitively on the Weyl chambers).

(2) Let Δ' be another base. Then there is $w \in W$ such that $w(\Delta') = \Delta$.

(3) For $\alpha \in \Phi$, there is $w \in W$ such that $w(\alpha) \in \Delta$.

(4) $W = \langle \sigma_\alpha : \alpha \in \Delta\rangle$.

(5) If $w(\Delta) = \Delta$ then $w = 1$.

PROOF.

(1) Take $\delta = \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha$. Here

$$\Phi^+ = \{\alpha \in \Phi : \alpha > 0\}$$

For any $\gamma \in E$, and $w \in W$ such that $(w(\gamma), \delta)$ is maximal over $w \in W$, we have

$$(\sigma_\alpha w\gamma, \delta) \leq (w\gamma, \delta).$$

But $(\sigma_\alpha w\gamma, \delta) = (w\gamma, \delta) - (w\gamma, \alpha)$ proving the claim since $\gamma \notin P_\alpha$.

(2) follows from (1).

(3) For $\alpha \in \Phi$, use $w \in W$ to send Δ' to Δ.

(4) $W' = \langle \sigma_\alpha : \alpha \in \Delta\rangle$ acts transitively on Weyl chambers by part (1). So $\beta \in \Phi$ there is $w \in W'$, such that $\alpha := w\beta \in \Delta$. Then $\sigma_\beta = w^{-1}\sigma_\alpha w$.

(5) Define the length $\ell(w)$ as the minimum of the length of the expression

$$w = \sigma_{\alpha_1} \cdots \sigma_{\alpha_t}$$

using elements of Δ. Then the first claim is that

LEMMA 12. *$\ell(W)$ coincides with the number of positive roots that w maps to negative roots.*

The key fact is that if $w = \sigma_{\alpha_1} \cdots \sigma_{\alpha_t}$ and suppose

$$(\sigma_1 \cdots \sigma_{t-1})(\alpha_t) \preccurlyeq 0$$

then the expression can be shortened to

$$w = \sigma_1 \cdots \widehat{\sigma}_s \cdots \sigma_{t-1}.$$

So the above lemma follows by induction on $\ell(w)$. If $\ell(w) = 0$ then $w = 1$ then $n(w) = 0$. If $w = \sigma_{\alpha_1} \cdots \sigma_{\alpha_t}$ is the reduced expression then $w(\alpha_t) \preccurlyeq 0$. So

$$w\sigma_{\alpha_t} = \sigma_{\alpha_1} \cdots \sigma_{\alpha_{t-1}}(\alpha) \succcurlyeq 0$$

but all other positive roots are permuted by σ_{α_t} so $n(w\sigma_{\alpha_t}) = n(w) - 1$.

□

From part (1) and (5) above an a little work on the boundary we have

COROLLARY 9. *The closure of a Weyl chamber is a fundamental domain for the aciton of W on E.*

W preserves Φ, so it preserves $\langle , \rangle$ and respects the lengths in such a way that we have

COROLLARY 10. *The Dynking diagram does not depend on Δ.*

The Weyl group contains the data of the action of $\mathfrak{g}$ on itself via conjugation.

$$1 \to W \to \mathrm{Aut}(\Phi) \to \Gamma \to 1.$$

Then

$$\mathrm{Aut}(\Phi) = W \rtimes \Gamma$$

is an extension through the elements of $\mathrm{Aut}(\mathfrak{g})$ that are not conjugations.

2.3. Classification of Dynkin diagrams. We proceed by showing that the above are the only possible Dynkin diagrams. We will be using some weight c_i for each vertex i in the following various steps. Let $u_i = \alpha_i / \|\alpha_i\|$ for each simple root α_i.

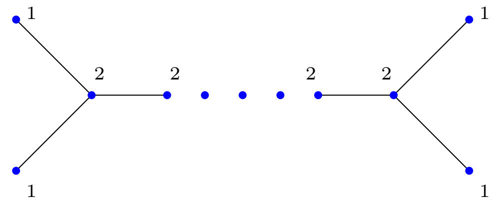
Step 1. There is no loops: Let $c_i = 1$ for any vertex in a simple loop and 0 otherwise. Then for $\overline{u} = \sum c_i u_i$ we have

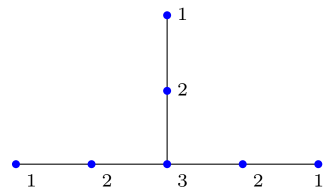
$$(\overline{u}, \overline{u}) = \sum c_i^2 (u_i, u_i) + 2 \sum_{i<j} c_i c_j (u_i, u_j) = \ell + 2 \sum_{i<j, i \leftrightarrow j} c_i c_j \geq \ell + 2\ell(-\frac{1}{2}) \leq 0.$$

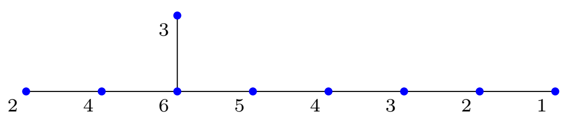
Then it has to be zero which implies $\overline{u} = 0$ but u_i's are chosen from simple roots.

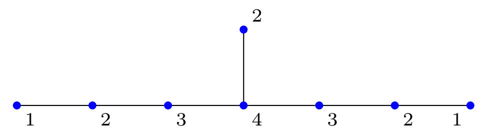
Now we rule out specific branchings in our tree. Just as above in all cases we let c be half the sum of the weights of the neighbors.

Step 2. For single bonds, the only possibilities are (1) trees, or (2) only one branching in a vertex. And if a branching happens it is either of the form D_n, E_6, E_7 or E_8. Here are the weights that lead to contradiction:

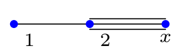








Step 3. A triple bond cannot have neighbor vertices:

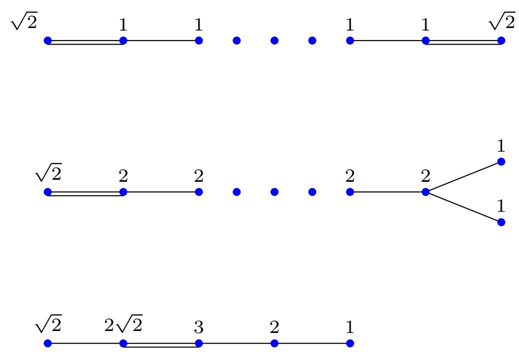


Then if we let $\overline{u} = u_1 + 2u_2 + xu_3$ we get

$$(\overline{u}, \overline{u}) = 1 + 4 + x^2 - 2(1 + \sqrt{3}x)$$

so if $x = \sqrt{3}$ then $(\overline{u}, \overline{u})$ must be zero. So the only possible case is that of G_2.

Step 4. For double bonds we have:



So that the only possible case is F_4.

We have already seen that A, B and C families can be realized by Lie algebras. We will see that the other remaining cases which we introduced previously can actually occure as well.

CHAPTER 5

Reconstructing the Lie algebras from root systems

1. From Dynkin diagrams to root systems

The first fact is that the root system is uniquely determined by the Dynkin diagram.

PROPOSITION 6. *If* (φ, E) *and* (φ', E') *are root systems with corresponding bases* Δ *and* Δ' *and* $\pi : \Delta \to \Delta'$ *is such that*

$$\langle \alpha_i, \alpha_j \rangle = \langle \pi(\alpha_i), \pi(\alpha_j) \rangle$$

then π *extends (uniquely) to an isomorphism* $\Phi \to \Phi'$.

This is easily seen from the Weyl group W. From π we have a homomorphism of the Weyl groups and then $W \cong W'$ implying that $\Phi = W.\Delta = W'.\Delta' = \Phi'$. The constructive way to find the root system is as follows. We define a *height* for any

$$\beta = \sum_{\alpha \in \Delta} m_\alpha . \alpha, m_\alpha \geq 0, \text{ or } m_\alpha \leq 0.$$

Then $ht(\beta) = \sum m_\alpha$. We can actually always write $\beta = \alpha_1 + \cdots + \alpha_s$ which $\alpha_i \in \Delta$ such that every partial sum is a root. The algorithm is to build Φ by height, using strings: we start by the roots of heigh one. Then for any pair $\alpha_i \neq \alpha_j$ the integer r for the α_j-string through α_i is 0 so the integer q equals $-\langle \alpha_i, \alpha_j \rangle$. We repeat this procedure long enough.

DEFINITION 11. φ decomposable if $\Phi = \Phi_1 \cup \Phi_2$ such that $\alpha \perp \beta$ for all $\alpha \in \Phi_1$ and $\beta \in \Phi_2$.

If $\Delta = \Delta_1 \sqcup \Delta_2$ with each element of Δ_1 perpendicular to any of Δ_2 we have

$$\Phi = W.(\Delta_1 \sqcup \Delta_2) = W_1 \Delta_1 \cup W_2 \Delta_2$$

Then $W = W_1 \times W_2$. This shows that

LEMMA 13. Φ *is indecomposable if and only if the Dynkin diagram is connected.*

COROLLARY 11. $\mathfrak{g}$ *is simple if and only if the Dynkin diagram is connected.*

PROOF. Let Δ_1 be a connected component of Δ. Let M be the span of all h_α for $\alpha \in \Delta_1$ together with elements of $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$. So $\Phi = \Phi_1 \cup \Phi_2$ with $\Phi_1 \perp \Phi_2$. Then if $\beta \notin \Phi_1$ we have $\beta \in \Phi_2$ and $\alpha + \beta$ is not a root for $\alpha \in \Delta_1$. Thus $\operatorname{ad} x_\beta$ (where x_β is a generator of $\mathfrak{g}_\beta$) kills x_α. Therefore $\operatorname{ad} x_\beta(y_\alpha) \in \mathfrak{g}_{\beta-\alpha} = 0$. Since $\beta - \alpha$ is also not a root. We conclude that M is a nontrivial ideal. $\square$

It is easy to see that if $\mathfrak{g} = \oplus \mathfrak{g}_i$ and $\mathfrak{h} \subset \mathfrak{g}$ is the maximal toral subalgebra of $\mathfrak{g}$ then $\mathfrak{h}_i := \mathfrak{h} \cap \mathfrak{g}_i$ is the maximal toral subalgebra of $\mathfrak{g}_i$. Also note the terminology for h_α's; co-roots!

So we know already that $\mathfrak{g}$ is generated by $\mathfrak{h}$ and $\mathfrak{g}_\alpha$ for all $\alpha \in \Phi$. We also know that

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \begin{cases} \mathfrak{g}_{\alpha+\beta} & \alpha + \beta \in \Phi \\ 0 & \text{otherwise} \end{cases} .$$

The next proposition gives us a *standard* set of generators for $\mathfrak{g}$.

PROPOSITION 7. *$\mathfrak{g}$ is generated by x_α and y_α for all $\alpha \in \Delta$.*

PROOF. $\mathfrak{h}$ is contained in M. So we need to prove that for all $\beta \in \Phi$, $\mathfrak{g}_\alpha \subset \langle x_\alpha, y_\alpha \rangle$. ☐

Suppose we have an isomorphism $\pi : (\Phi, E) \to (\Phi', E')$ of root systems. π initially is preserving $\langle , \rangle$. Rescale Φ so that π becomes an isometry. So we may assume that it is preserving $(,)$ as well. Suppose Φ is a root system of $\mathfrak{g}$ and Φ' be that of $\mathfrak{g}'$. Let $\mathfrak{h}$ and $\mathfrak{h}'$ be the respective maximal toral subalgebras. Then $\{h_\alpha\}$ and $\{h'_\alpha\}$ are basis for $\mathfrak{h}$ and $\mathfrak{h}'$. By isometry we know that $t_\alpha \mapsto t'_{\pi(\alpha)}$. So we define the map $\mathfrak{h} \to \mathfrak{h}'$ via $h_\alpha \mapsto h'_{\pi(\alpha)}$.

THE ISOMORPHISM THEOREM. Let $\{x_\alpha\}$ be an arbitrary set of nonzero vectors in $\mathfrak{g}_\alpha$'s for all $\alpha \in \Delta$. let $y_\alpha \in \mathfrak{g}_{-\alpha}$ be such that $[x_\alpha, y_\alpha] = h_\alpha$. Similarly choose $x'_{\pi(\alpha)}$ is $\mathfrak{g}'$ and get $y'_{\pi(\alpha)}$'s likewise. Then there exists a unique isomorphism of Lie algebras $\widetilde{\pi} : \mathfrak{g} \to \mathfrak{g}'$ such that $\widetilde{\pi}(x_\alpha) = x'_{\pi(\alpha)}$, $\widetilde{\pi}(y_\alpha) = y'_{\pi(\alpha)}$ (and of course $\widetilde{(}h_\alpha) = h'_{\pi(\alpha)}$).

PROOF. Uniqueness follows from our previous proposition. We may assume that $\mathfrak{g}$ and $\mathfrak{g}'$ are simple. For existence the idea is to construct a subalgebra $D \subset \mathfrak{g} \oplus \mathfrak{g}'$ such that the projections $D \to \mathfrak{g}$ and $D \to \mathfrak{g}'$ are isomorphisms. Let $\overline{x}_\alpha = (x_\alpha, x'_{\pi(\alpha)}) \in \mathfrak{g} \oplus \mathfrak{g}'$ and so for $\overline{t}_\alpha$'s for $\alpha \in \Delta$. Let D be the Lie algebra generated by $\overline{x}_\alpha$ and $\overline{y}_\alpha$ for $\alpha \in \Delta$. We know that $\ker \pi_1 = (0, \mathfrak{g}')$ and $\ker \pi_2 = (\mathfrak{g}, 0)$. We need to prove that $\ker \pi_i \cap D = \{0\}$ and for this it suffices to show that

$$D \neq \mathfrak{g} \oplus \mathfrak{g}'.$$

let β be the maximal root in $\mathfrak{g}$ with respect to the partial order defined by Δ. Consider $\mathfrak{g}_\alpha \oplus \mathfrak{g}'_{\pi(\beta)}$. Note that $\pi(\beta)$ is the maximal root of $\mathfrak{g}'$. Let $\overline{x} = (x_\beta, x'_{\pi(\beta)})$ and M be the subspace of $\mathfrak{g} \oplus \mathfrak{g}'$ spanned by all

$$(*) \qquad \operatorname{ad} \overline{y}_{\alpha_1} \cdots \operatorname{ad} \overline{y}_{\alpha_n} (\overline{x})$$

where α's do not have to be distinct. We will show that M is stable under D. Any expression like * lives in $\mathfrak{g}_{\beta - \sum \alpha_i} \oplus \mathfrak{g}'_{\beta - \sum \pi(\alpha_i)}$. Then if we append $\operatorname{ad} \overline{y}_\alpha$ to * we stay in M be definition. By an inductive argument $\operatorname{ad} \overline{h}_\alpha$'s stabilize M, and finally from $\operatorname{ad} x_\alpha(\overline{x}) = 0$ we see that $\operatorname{ad} x_\alpha$'s stabilize M completing the proof. ☐

REMARK. At this point the classification of Lie algebras is complete:

(1) We started with complete reducibility to get reduced to understanding the simple ideals.

(2) We started by Engel's theorem, Killing form we get the decomposition
$$\mathfrak{g} = C(\mathfrak{h}) \oplus (\oplus_{\alpha\in\mathfrak{h}^*}\mathfrak{g}_\alpha) = \mathfrak{h} \oplus (\oplus_{\alpha\in\mathfrak{h}^*}\mathfrak{g}_\alpha).$$

(3) From the classification of $\mathfrak{s}l_2$-modules we got that $\{\alpha\}$'s for a root system.

(4) We pick a set of simple roots, Δ. We got the classification of Dynkin diagrams. This classifies the root systems.

(5) The Dynkin diagram is independent of the choice of Δ thanks to the Weyl group action.

(6) The Isomorphism Theorem implies that isomorphism root systems give isomorphism Lie algebras.

(7) We will show that Φ does not depend on the choice of $\mathfrak{h}$ by conjugacy properties of $\mathfrak{h}$'s.

(8) One last fact that we will not prove is that if we start with a Dynkin diagram there is a root system Φ corresponding to it. We will see that there exists a Lie algebra $\mathfrak{g}$ with root system Φ. So this will imply existence of Lie algebras from Dynkin diagrams as well [Humphreys, Chapter 12].

2. Automorphisms and conjugacy

Let k be a field and A an algebra over k (note: in particular the Lie bracket gives the structure of an algebra to a Lie algebra) then a derivative
$$D : A \to A$$
is a linear mapping such that $D(ab) = D(a)b + aD(b)$.

Example 2.1. Let $A = k[x_1, \cdots, x_n]$ and let $D = \frac{\partial}{\partial x_i}$. Or let $\mathfrak{g}$ be a Lie algebra and then $\operatorname{ad} x$ is a derivative for any $x \in \mathfrak{g}$. ⌟

We have a k-vector space from the set of all derivatives: Der A. This is actually a Lie sub-algebra of $\mathfrak{gl}(A)$ via
$$[D_1, D_2] = D_1D_2 - D_2D_1.$$
This gives a lot of new interesting ways of constructing Lie algebras but the point is that all the new interesting ones are going to be highly infinite dimensional.

Suppose $D : A \to A$ is locally nilpotent (given any x, $D^k x = 0$ for high k). Then
$$\exp(D) = \sum_{k=0}^{n-1} \frac{1}{k!} D^k \in \mathrm{GL}(A).$$

and $\exp(D) : A \to A$ turns out to be an automorphism of A with inverse $\exp(-D)$.

For us the main example is when $\mathfrak{g}$ is a Lie algebra and $x \in \mathfrak{G}$ then $\exp(\operatorname{ad} x) \in \operatorname{Aut}(\mathfrak{g})$ is defined when x is ad-nilpotent and the subgroup $\operatorname{Int}(\mathfrak{g})$ of all automorphisms of $\mathfrak{g}$ generated by $\exp(\operatorname{ad} x)$'s for all ad-nilpotent x is called the group of inner automorphisms of $\mathfrak{g}$.

Remark. $\operatorname{Int}(\mathfrak{g})$ is a Lie group with tangent algebra being $\operatorname{ad}\mathfrak{g}$. If $\mathfrak{g}$ is semisimple then $\operatorname{ad}\mathfrak{g} \cong \mathfrak{g}$. This is the group with the trivial center (called the adjoint group corresponding to $\mathfrak{g}$). In fact $\operatorname{Int}\mathfrak{g}$ is the identity component of the Lie group $\operatorname{Aut}(\mathfrak{g})$.

Reconciling exp(ad) with $\mathfrak{g} \subset \mathfrak{gl}(V)$. Take $x \in \mathfrak{g}$ nilpotent (hence ad-nilpotent) and then for any $y \in \mathfrak{g}$ we have

$$(\exp x) y (\exp x)^{-1} = (\exp(\operatorname{ad} x))(y).$$

To verify this use

$$\lambda_x : \operatorname{End}(V) \to \operatorname{End}(V) \quad \text{via} \quad A \mapsto XA$$

and also

$$\rho_{-x} : \operatorname{End}(V) \to \operatorname{End}(V) \quad \text{via} \quad A \mapsto A(-X)$$

and then notice that

$$\operatorname{ad} x = \lambda_x + \rho_{-x}$$

and then exponentiate the right hand side in $\operatorname{End}(\operatorname{End}(V))$.

Now we come back to our simple roots $x_\alpha \in \mathfrak{g}_\alpha$ with $\alpha \in \Delta$. Suppose $\sigma : \Phi \to \Phi$ is an automorphism of Φ. We know that it lifts to an automorphism of $\mathfrak{g}$. Then we let $\widetilde{\sigma}$:.

Example 2.2. Take $-\mathrm{id} : \Phi \to \Phi$ via $\alpha \mapsto -\alpha$. Then this induces the map $h \mapsto -h$ on $\mathfrak{h}$ so

$$\widetilde{\sigma}(h_\alpha) = -h_\alpha = h_{-\alpha}.$$

We can define where the x_α go for $\alpha \in \Delta$. We want $x_\alpha \mapsto -y_\alpha$ where $(x_\alpha, y_\alpha) = h_\alpha$ and we know that this has to be in $\mathfrak{g}_{-\alpha}$. Then we will have $y_\alpha \mapsto -x_\alpha$ because of the relations $[x_\alpha, y_\alpha] = h_\alpha$. We get (by the isomorphism theorem) a $\widetilde{\sigma} : \mathfrak{g} \to \mathfrak{g}$ such that $\widetilde{\sigma} = \mathrm{id}$. ⌟

Example 2.3. In $\mathfrak{sl}_2$ we have σ, the conjugation by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ which coincides with $\widetilde{\sigma}$ from this construction. ⌟

We expect to see a relation between W and $\operatorname{Int}(\mathfrak{g})$. We want to associate a σ to an element $w \in W$ and then associate $\widetilde{\sigma} \in \operatorname{Aut}(\mathfrak{g})$ to that but the problem is that $\sigma \mapsto \widetilde{\sigma}$ is not a group homomorphism.

3. Borel and Cartan subalgebras

What happens for different $\mathfrak{h}$? The main point is that in a semisimple $\mathfrak{g}$, all $\mathfrak{h}$'s are conjugate. There is an element of $\mathrm{Int}(\mathfrak{g})$ taking $\mathfrak{h}$ to $\mathfrak{h}'$.

DEFINITION 12. A Cartan subalgebra in $\mathfrak{g}$ is one that is nilpotent and its own normalizer.

Example 3.1. If $\mathfrak{g}$ is semisimple, $\mathfrak{h}$ is a maximal toral subalgebra then $\mathfrak{h}$ is a Cartan simple algerba. (Try to check this!) ⌟

In fact in Chapter 15 of [Humphreys] we see that more is true:

THEOREM 3.1. *If $\mathfrak{g}$ is semisimple, then $\mathfrak{h}$ is a Cartan subalgebra if and only if $\mathfrak{h}$ is a maximal toral subalgebra.*

So to prove our main point we have to show that the Cartan subalgebra's are conjugate.

THEOREM 3.2. *If $\mathfrak{g}$ is any Lie algebra (not necessarily semisimple) then all Cartan subalgebras are conjugate. (Here we certainly use algebraic closedness of the underlying field).*

DEFINITION 13. A Borel subalgebra is a maximal solvable subalgebra in $\mathfrak{g}$.

For example the set of all upper-triangular matrices is a Borel subalgebra. For the proof we need to pass to the world of Lie groups. One way to do this is to take the corresponding group for $\mathfrak{g}$ to be $\mathrm{Int}(\mathfrak{g})$: we take x_α, y_α's and consider the exponentiation $\exp \operatorname{ad} x_\alpha$ and $\exp \operatorname{ad} y_\alpha$'s.

PROOF STEPS FOR THE THEOREM. Here is the program of proving this:

(1) If $\mathfrak{g}$ is solvable, all Cartan subalgebras are conjugate.

(2) Embed any Cartan subalgebra $\mathfrak{h}$ in a Borel subalgebra. This is possible since $\mathfrak{h}$ is nilpotent and a Borel subalgebra is a maximal solvable one.

(3) We associate groups G and B to $\mathfrak{g}$ and $\mathfrak{b}$.

(4) We prove that any two Borel subalgebras in semisimple $\mathfrak{g}$ are conjuagte: Given Borel subgroups B and B', G/B is a projective variety (say from the Bruhat decomposition). If B' is another Borel subalgebra then B' acts on G/B. We know that if a solvable algebraic gorup acts on a projective variety then it has a fixed point. So let $x \in G/B$ be such that for any $b \in B'$ we have
$$b(xB) = xB.$$
Therefore $x^{-1}bx \in B$ for any $b \in B'$. This shows conjugacy.

(5) We conclude that the corresponding Borel subalgebras $\mathfrak{b}$ and $\mathfrak{b}'$ are conjugate. This is the result of our association $\mathfrak{g} \mapsto G = \mathrm{Int}(\mathfrak{g})$.

(6) Under conjugation $\mathfrak{h}' \subset \mathfrak{b}'$ maps to $\mathfrak{h}'' \subset B$. But any two Cartan subalgebras of B are conjugate by Step 1.

□

REMARK. If G is any group with Lie algebra $\mathfrak{g}$ then $\mathrm{Int}(\mathfrak{g})$ is the group

$$G^{sc}/Z(G^{sc})$$

were G^{sc} is the simply connected component of G containing the identity element of G.

4. From root systems to Borel subalgebras

Let $\mathfrak{g}$ be semisimple. We start by fixing $\mathfrak{h}$ a Cartan subalgebra. Then the borel sublagebras containing $\mathfrak{h}$ are in one-to-one correspondence with bases for Φ relative to $\mathfrak{h}$. This gives a way for going from Δ to $\mathfrak{b}$:

$$\mathfrak{b} = \mathfrak{h} \oplus (\oplus_{\alpha>0} \mathfrak{g}_\alpha).$$

The the claim is that any Borel subalgebra containing $\mathfrak{h}$ is of this form. Why is $\mathfrak{b}$ solvable? Look at $\mathrm{ad}\, x_\alpha$ acting on $\mathfrak{b}$. Then we know that $\alpha \in \Delta$ pushes the root vectors up (increases their height). Why is it maximal? ... All 's are conjugate and all $\mathfrak{h}$'s are conjugate. This shows independence from the choice of $\mathfrak{h}$.

Example 4.1. In the $\mathfrak{s}l_3$ picture in the Weyl chamber giving the simple roots $\{\alpha, \beta\}$ we get the Borel subalgebra of all matrices of the form

$$\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$$

and if we take α and $-\alpha - \beta$ we get the Borel subalgebra of all matrices of the form

$$\begin{pmatrix} * & * & 0 \\ 0 & * & 0 \\ * & * & * \end{pmatrix}.$$

⌟

REMARK. So we have a one-to-one correspondence between

$$\text{Weyl chambers} \leftrightarrow \text{ simple roots } \leftrightarrow \text{ Borel subalgebras }.$$

5. The automorphism group of a Lie algebra

Let $\mathfrak{g}$ be a semisimple Lie algebra. Fix $\mathfrak{h}$ and Δ. Let $=(\Delta)$ be the Borel subalgebra. Let τ be an automoprhism of $\mathfrak{h}$. Then $\tau()$ is a Borel subalgebra. Then there is $\sigma_1 \in \mathrm{Int}(\mathfrak{g})$ such that

$$\sigma_1 \tau(\mathfrak{b}) = \mathfrak{b}.$$

Now $\sigma_1\tau$ takes $\mathfrak{h}$ to some other Cartan subalgebra in $\mathfrak{b}$. Now let σ be in $\mathrm{Int}(\mathfrak{b})$ chosen such that

$$\sigma_2\sigma_1\tau(\mathfrak{h}) = \mathfrak{h}.$$

Then $\sigma_2\sigma_1\tau$ given an automorphism of Φ which preserves Δ since it is taking $\mathfrak{b}$ to $\mathfrak{b}$. We know that

$$\mathrm{Aut}(\Phi) = W \rtimes \Gamma.$$

Given $\gamma \in \Gamma$ lift is to $\widetilde{\gamma} \in \mathrm{Aut}(\Phi)$. Then the claim is that $\gamma, \widetilde{\gamma}$ exists such taht $\widetilde{\gamma}\sigma_2\sigma_1\tau$ takes h_α to h_α and x_α to $c_\alpha x_\alpha$ and y_α to $c^{-1}{}_\alpha x_\alpha$ for any $\alpha \in \Delta$. Therefore $\widetilde{\gamma}\sigma_2\sigma_1\tau$ has to take $\mathfrak{g}_\alpha$ to itself for any $\alpha \in \Delta$.

One can see that there is an element of $\mathrm{Int}(\mathfrak{g})$ that unfolds the rescaling of the x_α's and y_α's. We conclude that

$$\mathrm{Aut}(\mathfrak{g}) = \mathrm{Int}(\mathfrak{g}) \rtimes \Gamma.$$

CHAPTER 6

Universal enveloping algebras

The motivation is that if G a finite group there is a one-to-one correspondence between representations of G and $\mathbb{C}[G]$-modules (which are unital associative algebras by the way). For locally compact G be may associate $\mathcal{H}(G)$ with same purposes (without a unit in general). What we want to do is to make such a ring for $\mathfrak{g}$.

Another motivational comment is that we have seen the representations $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ have products and we used this to defined Jordan decompositions and Casimir elements. We use this to prove complete reducibility. We have also seen that the Jordan decomposition is independent of the representation. This suggests that there should be a construction $\mathcal{U}(\mathfrak{g})$, an algebra such that product in it captures the properties of $\rho(X)\rho(Y)$ which are independent of ρ. By the fact that $[X,Y] \mapsto \rho(X)\rho(Y) - \rho(Y)\rho(X)$ it is clear how we want the product structure of $\mathcal{U}(\mathfrak{g})$ be related to the Lie brackets.

For the construction, thinking of $\mathfrak{g}$ as a vector space we start by

$$T(\mathfrak{g}) = F \oplus \mathfrak{g} \oplus \mathfrak{g}^{\otimes 2} \oplus \cdots$$

There is a natural product structure

$$(v_1 \otimes \cdots \otimes v_m)(u_1 \otimes \cdots \otimes u_n) = v_1 \otimes \cdots \otimes v_m \otimes u_1 \otimes \cdots \otimes u_n.$$

We define $\mathcal{U}(\mathfrak{g}) = T(\mathfrak{g})/\mathcal{I}$ where $\mathcal{I}$ is the 2-sided ideal in $T(\mathfrak{g})$ generated by all

$$(x \otimes y - y \otimes x) - [x,y].$$

1. Universal property

Recall that for a vector space V over field F, the space

$$T(V) = F \oplus \bigoplus_{n>0} V^{\otimes n}$$

is an associative algebra with unit $1 \in F$. Recall that this space has the universal property that for any linear mapping $V \to A$ for a unital associative algebra V over F, there is a unique lift $\widetilde{\varphi} : T(V) \to A$ which preserves the units $1_T \mapsto 1_A$ and makes the following diagram commute

$$\begin{tikzcd} V \arrow[r] \arrow[dr, "\varphi"'] & T(V) \arrow[d, dashed, "\widetilde{\varphi}"] \\ & A \end{tikzcd}$$

Note that there is a canonical mapping $i : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ (which need not be injective at this moment). The universal property for $\mathcal{U}(\mathfrak{g})$ reads: for any unital associative algebra A and a linear mapping $\varphi : \mathfrak{g} \to A$ satisfying

$$\varphi([x,y]) = \varphi(x)\varphi(y) - \varphi(y)\varphi(x)$$

there is a unique lift $\widetilde{\varphi} : \mathcal{U}(\mathfrak{g}) \to A$. In fact by the universal property of $T(\mathfrak{g})$, there exists a map $\widehat{\varphi} : T(\mathfrak{g}) \to A$ which contains $\mathcal{I}$ in its kernel. Note that by the same universal property an associative algebra with the mentioned property is unique.

Proposition 8. *There is a one-to-one correspondence between the representations of $\mathfrak{g}$ and the $\mathcal{U}(\mathfrak{g})$-modules.*

Proof. Given a $\mathcal{U}(\mathfrak{g})$-module the map $i : \mathfrak{g} \hookrightarrow \mathcal{U}(\mathfrak{g})$ induces a representation of $\mathfrak{g}$ and conversely if $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation then there is a unique homomorphism of algebras, $\widetilde{\rho} : \mathcal{U}(\mathfrak{g}) \to \mathfrak{gl}(V)$, turning V into a $\mathcal{U}(\mathfrak{g})$-module. □

Remark. If $\mathfrak{g}$ has a faithful representation, then $i : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ has to be injective for trivial reasons.

Theorem 1.1 (Poincare-Birkhoff-Witt). *Suppose $\mathfrak{g}$ has a countable basis $\{x_j\}$. The the set*

$$\{1\} \cup \{x_{i(1)} \cdots x_{i(m)} : i(1) \leq \cdots \leq u(m), m \in \mathbb{Z}\}$$

form a basis of $\mathcal{U}(\mathfrak{g})$. Here $x_{i(1)} \cdots x_{i(m)} = \pi(x_{i(1)} \otimes \cdots \otimes x_{i(m)})$ where π is the quotient $T(\mathfrak{g}) \to T(\mathfrak{g})/\mathcal{I}$.

We will present an alternative version of PBW and show that it implies the above theorem. Let us fix a notation

$$T^m V = V^{\otimes m}, \text{ and } T_m V = \oplus_{i=0}^m T^i.$$

Then $\pi : T(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ maps T_m to $\pi(T_m) =: \mathcal{U}_m$. Finally with the convention that $\mathcal{U}_{-1} = 0$ we set

$$\mathcal{G}^m := \mathcal{U}_m / \mathcal{U}_{m-1} (\forall m \geq 0), \text{ and } \mathcal{G} := \bigoplus_{m=0}^{\infty} G^m.$$

Obviously there is a well-defined induced mapping

$$\mathcal{U}_m/\mathcal{U}_{m-1} \times \mathcal{U}_p/\mathcal{U}_{p-1} \to \mathcal{U}_{m+p}/\mathcal{U}_{m+p-1}$$

given by the product structure of $\mathcal{U}$, hence $\mathcal{G}$ is a graded algebra.

Lemma 14. *The natural mapping $\varphi : T(\mathfrak{g}) \to \mathcal{G}$ factors through the symmetric tensor algebra*

$$S(\mathfrak{g}) = T(\mathfrak{g}) / \langle \{x \otimes y - y \otimes x\} \rangle.$$

PROOF. This is obvious as $\varphi(x\otimes y - y\otimes x) = [x,y] \in \mathcal{U}_1$ whereas $x\otimes y - y\otimes x \in \mathcal{U}_2$. $\square$

So we have a surjection $S(\mathfrak{g}) \twoheadrightarrow \mathcal{G}$.

THEOREM 1.2 (Poincare-Birkhoff-Witt (2)). *This map is an isomorphism of algebras.*

COROLLARY 12. *Let $W \subseteq T^m(\mathfrak{g})$ be a subspace such that W maps isomorphically onto $S^m(\mathfrak{g})$. Then $\pi(W)$ is a complement of $\mathcal{U}_{m-1}$ in $\mathcal{U}_m$.*

PROOF. It following by starring at the following commutative diagram!

$$\begin{tikzcd}
 & \mathcal{U}_m \arrow[dr, "\text{mod } U_{m-1}"] & \\
W \subseteq T^m \arrow[ur, "\pi"] \arrow[dr, "\text{symm}"'] & & G^m \\
 & S^m \arrow[ur, "\text{isom } \varphi"'] &
\end{tikzcd}$$

$\square$

COROLLARY 13. *$i : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ is injective.*

PROOF. This is the case of $m = 1$ in the previous corollary. $\square$

COROLLARY 14 (Poincare-Birkhoff-Witt theorem).

PROOF. The subspace of T^m spanned by all $x_{i(1)} \otimes \cdots \otimes x_{i(m)}$ map isomorphically onto S^m. Therefore $\pi(W)$ is a complement of $\mathcal{U}_{m-1}$ in $\mathcal{U}_m$. $\square$

COROLLARY 15. *If $\mathfrak{h} \hookrightarrow \mathfrak{g}$ then $\mathcal{U}(\mathfrak{h}) \hookrightarrow \mathcal{U}(\mathfrak{g})$.*

2. Generators, relations, and the existence theorem

Let X be any set. The free Lie algebra L_X on the set X is defined by the universal property depicted in the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[dr, "\varphi"'] & L_X \arrow[d, dashed, "!"] \\
 & A.
\end{tikzcd}$$

Here $i : X \to L_X$ is canonically chosen. In case of existence, such a Lie algebra is unique for trivial reasons. For the existence, we start with a vector space V in basis X. Then let L_X be the Lie subalgebra of the tensor algebra $T(V)$, generated by X. Another construction is suggested in discussion problems.

PROPOSITION 9. *Let $\mathfrak{g}$ be a semisimple finite-dimensional Lie algebra with a given maximal toral subalgebra $\mathfrak{h} \subset \mathfrak{g}$, a root system Φ and a base of simple roots $\Delta = \{\alpha_1, \cdots, \alpha_\ell\}$. Then for each i pick*

(1) *$h_i \in \mathfrak{h}$ such that $\alpha_i(h_{\alpha_i}) = 2$;*

(2) *x_i an arbitrary generator of $\mathfrak{g}_{\alpha_i}$;*

(3) *y_i such that $[x_i, y_i] = h_i$.*

Then $\mathfrak{g}$ is generated by $\{x_i, y_i, h_i : i\}$ with the following relations

S1 $[h_i, h_j] = 0$ *for all i, j;*

S2 $[x_i, y_i] = h_i$ *and* $[x_i, y_j] = 0$ *if $i \neq j$;*

S3 $[h_i, x_j] = \langle \alpha_j, \alpha_i \rangle x_j$ *and* $[h_i, y_j = -\langle \alpha_j, \alpha_i \rangle y_j$;

S_{ij}^+ *assuming $i \neq j$,* $(\operatorname{ad} x_i) - \langle \alpha_j, \alpha_i \rangle + 1(x_j) = 0$;

S_{ij}^- *assuming $i \neq j$,* $(\operatorname{ad} y_i) - \langle \alpha_j, \alpha_i \rangle + 1(y_j) = 0$.

PROOF. $S1 - S3$ are easy to see. For $S_{ij}^\pm$ note that $\alpha_j - \alpha_i$ is not a root. Hence the α_i-string through α_j is $\alpha_j, \alpha_j + \alpha_i, \cdots, \alpha_j + q\alpha_i$ where $-q = \langle \alpha_j, \alpha_i \rangle$. □

THEOREM 2.1 (Serre's existence). *Let Δ be a Dynkin diagram. The lie algebra $\mathfrak{g}$ on generators $\{x_i, y_i, h_i : i\}$ with relations above is a semisimple finite-dimensional Lie algebra which has Δ as its Dynkin diagram.*

REMARK. The Lie algebra $\mathfrak{g}_0$ on same set of generators as in the above theorem, but only satisfying relations $S1 - S3$ is a Lie algebra with finite dimensional Cartan subalgebra $\mathfrak{h}$. The relations $S_{ij}^\pm$ insures that $\mathfrak{g}$ is finite-dimensional.

CHAPTER 7

Representation Theory

1. Weight lattice

Recall that E is a real vector space spanned by roots in Φ. We shall fix $\Delta \subset \Phi$ a set of simple roots. Recall that this space has an inner product $(.,.)$ on it. The weight lattice is the lattice of elements

$$\{\lambda \in E : \langle \lambda, \alpha \rangle \in \mathbb{Z}, \forall \alpha \in \Phi\}.$$

Another interesting lattice is the root lattice, which is the $\mathbb{Z}$-module generated by Δ.

For example consider the corresponding lattice for $\mathfrak{sl}_2$, $Q \subset P$. Then $\#P/Q = 2$. In fact P/Q has the information about how many different groups can be associated with $\mathfrak{g}$; if G is a group with Lie algebra $\mathfrak{g}$ one can construct a *character lattice* X as an intermediate lattice

$$Q \subseteq X \subseteq P.$$

1.1. Basis for P.

We know that $\Delta = \{\alpha_1, \cdots, \alpha_\ell\}$ is a basis for E. Thus the vectors $\frac{2\alpha_i}{(\alpha_i,\alpha_i)}$ still form a basis for E. Take $\{\lambda_i\}$ a dual basis for the latter:

$$P = \langle \lambda_i, \alpha_j \rangle = (\lambda_i, \frac{2\alpha_i}{(\alpha_i, \alpha_i)}) = \delta_{ij}.$$

Then for any $\lambda = \sum m_i \lambda_i$ we have $\langle \lambda, \alpha_i \rangle = m_i$ so P is the $\mathbb{Z}$-span of $\{\lambda_i\}$. These are called the *fundamental weights*.

Note that since $\langle \alpha_i, \alpha_j \rangle \in \mathbb{Z}$ for all i and j, we have $Q \subseteq P$. Moreover the elements $\langle \alpha_i, \alpha_j \rangle$ are the entries of the Cartan matrix, C, and satisfy

$$\alpha_i = \sum m_{ij} \lambda_i, \quad m_{ij} = \langle \alpha_i, \alpha_j \rangle.$$

Hence the transition matrix from basis of P to basis of Q is the C and therefore

$$\#(P/Q) = |\det C|.$$

1.2. Dominant weights.

Definition 14. A weight is called *dominant* if it lies in the closure of the positive Weyl chamber:

$$\langle \lambda, \alpha \rangle \geq 0, \forall \alpha \in \Delta.$$

It is called *strongly dominant* if the above inequalities are all strict.

Note that $\sigma_{\alpha_i}\lambda_j = \lambda_j - \delta_{ij}\alpha_i \in P$ hence P is preserved by the Weyl group, W. Our work until now shows that

LEMMA 15. *For any $\lambda \in P$ there is a unique dominant weight λ' such that $\lambda' \in W.\lambda$.*

Recall the partial order on E: $\lambda \succcurlyeq \mu$ iff $\lambda - \mu$ is a sum of non-negative roots. Then it is obvious that

LEMMA 16. *If λ is dominant then $\sigma\lambda \preccurlyeq \lambda$ for any $\sigma \in W$. If λ is strongly dominant then the inequalities are strict unless $\sigma = \mathrm{id}$.*

However note that we can have dominant μ and $\lambda \succcurlyeq \mu$ such that λ is not dominant. We will use the notation P^+ for the set of dominant weights.

LEMMA 17. *Fix $\lambda \in P^+$. Then the set of dominant weights μ such that $\mu \preccurlyeq \lambda$ is finite.*

PROOF. All such weights μ, satisfy $(\mu,\mu) \leq (\lambda,\lambda)$ so they are in the intersection of a discrete lattice and a bounded set in E. □

DEFINITION 15. A set Π of weights is called *saturated* if for any $\lambda \in \Pi$ and $\alpha \in \Phi$, given any integer i such that

$$0 \leq i \leq \langle \lambda, \alpha \rangle$$

we have $\lambda - i\alpha \in \Pi$.

Obviously such Π is stable under W since we may take $i = \langle \lambda, \alpha \rangle$.

Example 1.1. The root system Φ is saturated. Also take $\lambda \in P^+$ and consider its W-orbit. Then the intersection of P and the closure of the convex hull of $W.\lambda$ is saturated. ⌟

If Π is saturated we say it has an element λ of *highest weight* if $\lambda \in \Pi$ and for all $\mu \in \Pi$ we have $\mu \preccurlyeq \lambda$.

LEMMA 18. *Let Π be a saturated set of weights with highest weight λ. Then if μ is dominant and $\mu \preccurlyeq \lambda$ we have $\mu \in \Pi$.*

PROOF. Suppose for some set of integers $k_\alpha \in \mathbb{Z}_{\geq 0}$ we have $\mu' = \mu + \sum_{\alpha\in\Delta} k_\alpha \alpha \in \Pi$. We will show that that if we reduce at least one of the k_α's by one unit we still stay in Π. If $\mu' \neq \mu$ then $k_\alpha > 0$ for at least one α. From $(\sum k_\alpha\alpha, \sum k_\alpha\alpha) > 0$ we have

$$\sum_{\beta\in\Delta} k_\beta(\sum_{\alpha\in\Delta} k_\alpha\alpha, \beta) > 0.$$

So for at least one $\beta \in \Delta$ we have $(\sum k_\alpha\alpha, \beta) > 0$ and $k_\alpha > 0$. Since μ is dominant $\langle \mu, \beta \rangle \geq 0$ hence $\langle \mu', \beta \rangle > 0$. Since Π is saturated we may subtract β form μ', as many as $\langle \mu', \beta \rangle$ times and stay in Π. □

Also we deduce that

COROLLARY 16. *For any $\lambda \in P^+$ there is a unique saturated set with highest weight λ.*

2. Classification of representations

Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra, Φ a root system and Δ a set of simple roots. We know that we may view any $\mathfrak{g}$-module V as a $\mathcal{U}(\mathfrak{g})$-module.

If V is finite-dimensional we can diagonalize $\mathfrak{h}$. Then we get a decomposition

$$V = \oplus_{\lambda \in \mathfrak{h}^*} V_\lambda$$

into eigenspaces $V_\lambda = \{v : h.v = \lambda(h)v\}$. We say λ is a weight if $V_\lambda \neq 0$.

Example 2.1. In case of the adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ the weights of ad are just precisely the roots. Note also that $\mathfrak{h} = V_0$. ⌟

Example 2.2. For $\mathfrak{g} = \mathfrak{sl}_2$ every $m \in \mathbb{Z}$ gives an irreducible representation V_m of dimension m. One should think of $m \in \mathbb{Z}$ as an element of the weight lattice in $\mathfrak{h}^*$. ⌟

Remark. The summation $\sum_\lambda V_\lambda$ is always a direct sum as a result of linear independence of eigenvectors for distinct eigenvalues.

Remark. In general (not assuming finite-dimensionality) $V' = \oplus_\lambda V_\lambda$ is not necessarily equal to V but has the structure of a $\mathfrak{g}$-submodule $V' \subseteq V$. In fact we know that $\mathfrak{g}$ is generated by $\langle X_\alpha, Y_\alpha, h_\alpha \rangle$ for all $\alpha \in \Delta$. V' is obviously stable under h_α. And since

$$h.X_\alpha v = X_\alpha(h(v)) + [h, X_\alpha]v = (\lambda + \alpha(h))X_\alpha(v)$$

we observe that X_α maps V_λ to $V_{\lambda+\alpha}$ and Y_α maps it to $V_{\lambda-\alpha}$.

2.1. Standard cyclic modules.

Definition 16. Let V be a $\mathfrak{g}$-module. $v^+ \in V$ is called a *maximal vector* if $\mathfrak{g}_\alpha . v^+ = 0$ for all $\alpha > 0$. V is a standard cyclic module (of highest weight λ) if there exists $v^+ \in V_\lambda$ which is a maximal vector.

Note that if V is a finite dimensional it has to be standard cyclic of some weight λ. Consider $\mathfrak{b} = \mathfrak{h} \oplus_{\alpha>0} \mathfrak{g}_\alpha$. By Lie's theorem $\mathfrak{b}$ must have a common eigenvector, v^+. Then v^+ has to be killed by $\mathfrak{g}_\alpha$ for any $\alpha > 0$.

Now suppose V is a standard cyclic $\mathfrak{g}$-module and v^+ is a maximal vector. Consider $V' = \mathcal{U}(\mathfrak{g}).v^+$ and let $\Phi^+ = \{\beta_1, \cdots, \beta_m\}$ be a set of positive roots. V' has interesting properties we count below:

(1) V' is spanned by $y_{\beta_1}^{n_1} \cdots y_{\beta_m}^{n_m} . v^+$ where y_β is a generator of $\mathfrak{g}_\beta$.

In fact we may write $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{b}$ where $\mathfrak{n}^-$ is the nilpotent part generated by y_β's ($\beta < 0$). Then

$$\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{n}^-) \oplus \mathcal{U}(\mathfrak{b}).$$

Now notice that $\mathcal{U}(\mathfrak{n}^-)$ has basis $y_{\beta_1}^{n_1} \cdots y_{\beta_m}^{n_m}$ by PBW, and $\mathcal{U}(\mathfrak{b})$ stabilizes $\langle v^+ \rangle$.

(2) Weights of V' are of the form

$$\mu = \lambda - \sum_{\alpha \in \Delta} k_\alpha \alpha, \quad k_\alpha \in \mathbb{Z}^+.$$

In fact y_β maps V_λ to $V_{\lambda-\beta}$ so $\dim V_\mu$ is equal to the number of ways of writing μ as $\mu = \lambda - \sum_{\alpha \in \Phi^+} k_\alpha \alpha$ for $k_\alpha \geq 0$.

(3) $\dim V_\lambda = 1$.

(4) Each submodule of $\mathcal{U}(\mathfrak{g}).v^+$ is a direct sum of its weight spaces.

(5) $\mathcal{U}(\mathfrak{g}).v^+$ is indecomposable; has a unique maximal proper submodule, namely $\oplus_{\mu<\lambda} V_\mu$, and has a unique irreducible quotient.

(6) Every homomorphic image of $\mathcal{U}(\mathfrak{g})v^+$ is a standard cyclic module of the highest weight λ.

Theorem 2.1. *For any $\lambda \in \mathfrak{h}^+$ there is a unique (up to isomorphism) irreducible standard cyclic module of highest weight λ (it can be infinite-dimensional).*

Proof of existence. Take highest weight λ. Let D_λ be the one-dimensional $\mathfrak{b}$-module $D_\lambda := \langle v^+ \rangle$. So D_λ has the structure of a $\mathcal{U}(\mathfrak{b})$-module. Let $Z(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} D_\lambda$ (with maximal vector $1 \otimes v^+$). Let $Y(\lambda)$ be the maximal proper submodule of $Z(\lambda)$. Then $V(\lambda) = Z(\lambda)/Y(\lambda)$. □

Example 2.3. For $\mathfrak{sl}(\mathbb{C})$ with $\lambda \in \mathbb{Z}_+$ we have $V(\lambda) = Z(\lambda)/Z(-1)$. ⌟

So every irreducible finite dimensional module has to be of the form $V(\lambda)$. Let $\alpha \in \Delta$ be a simple root and let S_α be the copy of $\mathfrak{sl}_2$ given by Jacobson-Marozov theorem. Then $V(\lambda) = \oplus V_\mu$ is a finite dimensional S_α-module where μ ranges over integral weights (in the sense of last class; i.e. $\mu \in P$) and $\mu(h_\alpha) = \langle \mu, \alpha \rangle \in \mathbb{Z}$. Since $\lambda(h_\alpha)$ is the highest weight of V as an $\mathfrak{sl}_2$-module we have $\langle \lambda, \alpha \rangle \geq 0$ for any α So λ is a dominant integral weight.

The main result is that when V is a finite-dimensional module and $\Pi(V)$ is the set of weights of V, then $\Pi(V)$ is saturated (so for any $\alpha \in \Phi$, $\mu \in \Pi(V)$ the α-string through μ is in $\Pi(V)$) and we have a one-to-one correspondence

$$\left\{ \begin{array}{c} \text{finite-dimensional} \\ \text{irreducible representations} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{saturated sets of highest weight } \lambda \\ \text{with } \lambda \text{ a dominant integral weight} \end{array} \right\}.$$