Algebraic K-theory
Math 600D - Fall 2011

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CHAPTER 1

Zeroeth K-groups

1. K-theory of an abelian category

Recall that if R is a ring, an R-module P is said to be projective if any of the following equivalent conditions are satisfied:

(1) P is a direct summand of free modules,

(2) $0 \to M \to N \to P \to 0$ splits for any sequence of R-modules,

(3) the functor $\mathrm{Hom}_R(P, -)$ is exact,

(4) the following diagram always completes

$$\begin{tikzcd} & P \arrow[d] \arrow[dl, dashed] \\ X \arrow[r, two heads] & Y \end{tikzcd} .$$

The projective dimension, $\mathrm{pd}_R(M)$, of the R-module M is the minimum length of all projective resolutions. In particular $\mathrm{pd}_R M = 0$ whenever M is projective. The global dimension (or homological dimension) of R is the supremum over $\mathrm{pd}_R(M)$ of all R-modules M. A good example to recall is that if R is a regular local ring, then $\mathrm{gl.dim}_R < \infty$ and is equal to its Krull dimension. We will use the notations $R - \mathcal{Mod}$, $R - \mathcal{mod}$ and $\mathcal{P}_R$ respectively for the categories of all, finitely generated, and finitely-generated projective R-modules.

By an abelian category we mean a small preadditive category (i.e. $\mathrm{Hom}(A, B)$ has the structure of an abelian group, compatible with category structure), that has finite (co-)products, has a zero object (i.e. both initial and terminal), is (co-)normal (i.e. every mono(epi-)morphism is the (co-)kernel of some morphism), and finally has epi-mono factorization:

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr] & & N \\ & \mathrm{im}(f) \arrow[ur] & \end{tikzcd}$$

In an abelian category, $\mathcal{A}$, the abelian group $K_0(\mathcal{A})$ is one generated by isomorphism classes of objects of the category with the relations

$$[B] = [A] + [C], \text{ for any short exact sequence } 0 \to A \to B \to C \to 0.$$

This group is universal in the sense that for any abelian group G, and group homomorphism $f : ob(\mathcal{A}) \to G$ respecting the relations $f(B) = f(A) + f(C)$ for any short exact sequence as above, then there is a unique group morphism completing

$$\begin{tikzcd} ob(\mathcal{A}) \arrow[r] \arrow[dr, "f"'] & K_0(A) \arrow[d, dashed, "!"] \\ & G \end{tikzcd} .$$

LEMMA 1. *If $\mathcal{A}$ has countable direct sums then $K_0(\mathcal{A}) = 0$.*

PROOF. Let $B = \coprod_\infty A$ for any object A. Then $A \oplus B = B$ implying $[A] = 0$. □

If $\mathcal{A}$ and $\mathcal{B}$ are two abelian categories and $T : \mathcal{A} \to \mathcal{B}$ is an exact functor we get an induced group homomorphism $K_0(\mathcal{A}) \to K_0(\mathcal{B})$. One therefore concludes that K_0 is a co-variant functor from the category of abelian categories and exact functors between them to the category of abelian groups.

COROLLARY 1. *If $\mathcal{A}$ and $\mathcal{B}$ are equivalent abelian categories, then $K_0(\mathcal{A}) = K_0(\mathcal{B})$.*

It is immediate that $K_0(\mathcal{ab})$, that of the category of abelian groups is isomorphic to $\mathbb{Z}$ and the isomorphism is given by the rank of the abelian group. If $\mathcal{A}$ is however the category of finite abelian groups then $K_0(A)$ is the free abelian group with basis $[\mathbb{Z}/p]$'s. A more general observation is the following

LEMMA 2. *If $\mathcal{A}$ is an abelian category with Jordan-Holder filtrations, i.e.*

$$A = A_n \supseteq A_{n-1} \supseteq \cdots \supseteq A_1 \supseteq A_0 = 0$$

with simple factors, A_i/A_{i-1}, then

$$K_0(\mathcal{A}) = \bigoplus_{\text{simple } S} [S].$$

LEMMA 3. *Two objects A, B in the abelian category $\mathcal{A}$ have the same class in $K_0(\mathcal{A})$ if and only if there is C in $\mathcal{A}$ such that $A \oplus C = B \oplus C$.*

In particular, this motivates the jargon *stably isomorphic* for R-modules: M and N are stably isomorphic R-modules if

$$M \oplus R^k \cong N \oplus R^k$$

for some integer $k > 0$. We say M is *stably free* if

$$M \oplus R^k \cong R^n$$

for some integers $k, n > 0$.

PROOF. One way is obvious. For the other direction, let $[A] = [B]$. Then we can rewrite $[A] - [B] = 0$ as

$$[A] - [B] = \sum_i ([A_i \oplus B_i] - [A_i] - [B_i]) - \sum_j ([A'_j \oplus B'_j] - [A'_j] - [B'_j])$$

So

$$[A] + \sum_i [A'_i \oplus B'_i] + \sum_j ([A_j] + [B_j]) = [B] + \sum_i [A_i \oplus B_i] + \sum_j ([A'_j \oplus B'_j]).$$

Therefore $C = (\bigoplus_i A_i \oplus B_i) \oplus (\bigoplus A'_j \oplus B'_j)$ we prove our claim. $\square$

2. K_0 of local rings

In what follows we do not need R to be necessarily a commutative ring. Our goal is to study the Grothendieck group,

$$K_0(R) := K_0(\mathcal{P}_R),$$

of the category of projective R-modules. It is for instance immediate that if R is any PID, then rank induces $K_0(R) \cong \mathbb{Z}$.

Jacobson radical is the intersection of all maximal ideals as in the commutative case. Then the following version of Nakayama lemma is handy:

THEOREM 2.1 (Nakayama lemma). *let I be an ideal of the ring R containing the Jacobson radical. Suppose M is a finitely-generated R-module. Then $M/IM = 0$ implies $M = 0$.*

THEOREM 2.2 (Kaplansky). *Let R be a local ring and P a projective R-module. Then P is free.*

PROOF. Let $J = J(R)$ be the Jacobson radical of R. Since R is assumed to be a local ring, this is the unique maximal ideal of R. Hence $D = R/J(R)$ is a division ring. Let M be a finitely generated projective R-module. In particular say

$$M \oplus N \cong R^n.$$

Then M/JM and N/JN are D-vector spaces. We now fix $\overline{e}_1, \cdots, \overline{e}_m$ and $\overline{e}'_1, \cdots, \overline{e}'_s$ to form bases of the former vector spaces. By Nakayama's lemma these lift to generators $e_1, \cdots, e_m$ and $e'_1, \cdots, e'_s$ of M and N. It now suffices to show that

$$\{x_1, \cdots, x_n\} = \{e_1, \cdots, e_m, e'_1, \cdots, e'_s\}$$

form a basis of R^n. Let $f_1, \cdots, e_{m+s}$ be the standard basis of $M \oplus N \cong R^n$. So we already have

$$f_i = \sum a_{ij} x_i, x_i = \sum b_{ij} f_j$$

for all $i = 1, \cdots, n$. By the notation $A = (a_{ij})$ and $B = (b_{ij})$ we have $AB = I_n$. Recall now that

$$J(R) = \{x \in R : 1 - ax \text{ is invertible for any } a \in R\}.$$

It follows now that $BA - I \in M_n(J(R))$ proving our claim. $\square$

3. K_0 of some quotient rings

Another interesting variant of K_0 is

$$G_0(R) := K_0(R - \mathcal{mod}).$$

REMARK. If $f : R \to R'$ is a ring homomorphism we get an induced group homomorphism

$$f_* : K_0(R) \to K_0(R')$$

if moreover R' is R-flat then we have an induced homomorphism

$$f_* : G_0(R) \to G_0(R')$$

as well. If R' is a finitely-generated R-module, then the forgetful functor $R'-\mathcal{mod} \to R-\mathcal{mod}$ induced

$$\mathrm{res} : G_0(R') \to G_0(R)$$

and if R' is moreover R-projective, we get an induced

$$\mathrm{res} : K_0(R') \to K_0(R).$$

Our final remark is that the natural inclusion $\mathcal{P}_R \subset R - \mathcal{mod}$ induces the so called Cartan homomorphism

$$c_0 : K_0(R) \to G_0(R).$$

THEOREM 3.1 (Idempotent lifting). *Let R be a ring and J a two-sided ideal of R contained in the Jacobson radical. Let $\overline{R} = R/J$. Then the quotient map induces a functor*

$$\mathcal{P}_R \to \mathcal{P}_{\overline{R}}$$

which is full, additive and satisfies the following: if $f : P \to Q$ is a morphism in $\mathcal{P}_R$ such that $\overline{f} : \overline{P} \to \overline{Q}$ is an isomorphism then f is also an isomorphism.

This is easy to prove. In particular we have

THEOREM 3.2. *If R is J-adicaly complete (i.e. $\varprojlim R/J^n \cong R$), then the above functor $\mathcal{P}_R \to \mathcal{P}_{\overline{R}}$ is bijective.*

PROOF. Given $Q \in \mathcal{P}_{\overline{R}}$ we look for a lift of it $P \in \mathcal{P}_R$. Since Q is projective we may think of it as the image of an idempotent:

$$Q = \mathrm{im}(\overline{\rho}), \overline{\rho} \in M_n\overline{R} = \mathrm{End}_{\overline{R}}\overline{R}^n, \overline{\rho}^2 = \overline{\rho}.$$

All we have to do is to lift $\overline{\rho}$ to idempotent $\rho \in M_n(R)$. Let $A = M_n(R)$, $\overline{A} = M_n(R/J) = M_n(R)/M_n(J)$. Given any $a \in A$, $n > 0$ we have

$$1 = (a + (1-a))^{2n} = \sum_0^{2n} \binom{2n}{j} a^{2n-j}(1-a)^j.$$

But

$$f_n(a) = \sum_0^{n} \binom{2n}{j} a^{2n-j}(1-a)^j \equiv a \mod a^n A.$$

therefore $f_n(a) \equiv a \mod a(1-a)A$. Now choose $a^2 - a = 0$, then $f_n(a)^2 = f_n(a)$. If R is J-adically complete, then A is $M_n(J)$-adically complete. Thus we can form a compatible system of idempotents in R/J^k's, constructing an element $\rho \in \varprojlim A/M_n(J) \cong A$ such that $\rho \mapsto \overline{\rho}$. $\square$

COROLLARY 2. *$K_0(R) = K_0(R/J)$ if R is J-adically complete and $J \subseteq J(R)$.*

4. K-theory of Dedekind domains

Recall that a Dedekind domain R is an intergrally closed noetherian domain of Krull dimension 1.

Example 4.1. Examples are $\mathbb{Z}$, $k[x]$, ring of integers in an algebraic number field. Let K be a finite extension of $\mathbb{Q}$ and

$$\mathcal{O}_K = \{\theta \in K : f(\theta) = 0 \text{ for some monic polynomial } f(x) \in \mathbb{Z}[x]\}$$

be the integral closure of $\mathbb{Z}$ in K

$$\begin{tikzcd} \mathcal{O}_K \arrow[r, no head] \arrow[d, no head] & K \arrow[d, no head] \\ \mathbb{Z} \arrow[r, phantom, "\subseteq"] & \mathbb{Q} \end{tikzcd}$$

Then $\mathcal{O}_K$ is a Dedekind domain. If R is a noetherian, integrally closed ring and $\mathfrak{p}$ is a prime ideal of height one, then $R_{\mathfrak{p}}$ is also a Dedekind domain. ⌟

So let R be a Dedekind domaion and K its function field.

DEFINITION 1. A fractional ideal of K is a nonzero R-submodule I of F such that there is an element $a \in I$ with $aI \subseteq R$.

Fractional ideals form an abelian monoid under multiplication with (1) the identity element. If I is a fractional ideal , there exists a fractional ideal

$$I^{-1} = \{a \in K : aI \subseteq R\}$$

and that $II^{-1} = R$. So the fractional ideals of R form a group and the principal fractional ideals form a subgroup.

DEFINITION 2. The class group $C(R)$ for a Dedekind domaion R is defined to be

$$C(R) = \frac{\text{the group of fractional ideals}}{\text{subgroup of principal fractional ideals}}$$

THEOREM 4.1. *If R is a Dedekind domain, then every fractional ideal is finitely generated and projective.*

PROOF. Let I be a fractional ideal, since $I^{-1}I = R$ there are elements $x_1, \cdots, x_n \in I^{-1}$ and $y_1, \cdots y_n \in I$ such that

$$\sum_{i=1}^{n} x_i y_i = 1$$

If $b \in I$, then $b = \sum(bx_i)y_i$ with $bx_i \in II^{-1} = R$. Hence $y_!, \cdots, y_n$ generate I.

Consider the homomorphism $R^n \to I$ via $e_i \mapsto y_i$. This has a splitting with right inverse

$$b \mapsto (bx_1, \cdots, bx_n)$$

Hence I is a direct summand of a free modli and therefore projective. □

COROLLARY 3. *If R is a Dedekind domain, then every finitely generated projective R-modulie is isomorphic to a direct sum of ideals. In particular the isomorphism classes of ideals generate $K_0(R)$.*

PROOF. Let $M \in \mathcal{P}_R$. Then $M \subseteq R^n$. Proof is by induction on n. Let $\pi : R^n \to R$ be the projection to the last factor. Then $\pi(M) \subseteq R$, hence $\pi(M)$ is an ideal in R. If $\pi(M) = 0$ we are done otherwise, we may assume that

$$0 \neq \pi(M) = I \subseteq R.$$

As I is projective we get $M \cong \ker \pi|_M \oplus I$. But $\ker \pi \subseteq R^{n-1}$ and we are done by induction hypothesis. □

It is now easy to prove

THEOREM 4.2. *If R is a Dedekind domain, then $K_0(R) \cong \mathbb{Z} \oplus C(R)$.*

PROOF IDEA. Any finitely-generated projective module, P, of rank$n$ can be expressed as $R^{n-1} \oplus I$ for a fractional ideal I. The mapping $K_0(R) \to \mathbb{Z}$ is just the rank map. □

5. K-theory of rings

Recall that $K_0(R) = K_0(\mathcal{P}_R)$ and $G_0(R) = K_0(R - \mathcal{mod})$.

THEOREM 5.1 (Devisage, Hellor). *Let $\mathcal{A}$ be an abelian category, $\mathcal{C}$ and $\mathcal{B}$ full subcategories with $\mathcal{C} \subseteq \mathcal{B}$ and such that*

1. *$\mathcal{C}$ is closed in $\mathcal{A}$ with respect to subobjects and quotient objects.*
2. *Eery object of $\mathcal{B}$ has afinite filtration with all quotients in $\mathcal{C}$.*

Then the canonical map $K_0(\mathcal{C}) \to K_0(\mathcal{B})$ is an isomorphism.

PROOF. The inverse to the natural map $K_0(\mathcal{C}) \to K_0(\mathcal{B}$ is defined by $f : K_0(\mathcal{B}) \to K_0(\mathcal{C})$ as follows. Let $B \in \mathcal{B}$ and

$$B = B_0 \supseteq B_1 \supset \cdots \supset B_n = 0$$

such that all factors are in $\mathcal{C}$. Check that $[B] \mapsto \sum[B_i/B_{i+1}]$ works. □

COROLLARY 4. *Let $\mathcal{A}$ be an abelian category in which each object has finite length (with respect to simple objects). Then $K_0(\mathcal{A})$ is the free abelian group on $[S]$ where S varies over a set of representatives of the simple objects of $\mathcal{A}$.*

THEOREM 5.2. *Let $\mathcal{A}$ be an abelian category and $\mathcal{P}$ be the full subcategory of projective objects (i.e. those for which* $\mathrm{Hom}(P,-)$ *is exact). If* $\mathrm{pd}A$ *(via resolution by projectives) is finite for every object $A \in \mathcal{A}$, the the natural map*

$$I : K_0(\mathcal{P}) \to K_0(\mathcal{A})$$

is an isomorphism.

PROOF. Check that $A \mapsto \sum(-1)^i P_i$ works. For this we need Schanuel's lemma that we recall here.

LEMMA 4 (Schanuel's). *Let $\mathcal{A}$ be an abelian category and*

$$0 \to B \to P \to A \to 0 \text{ and } 0 \to B' \to P' \to A' \to 0$$

are exact sequences with $A \cong A'$ and P, P' projective. Then $B \oplus P' \cong B' \oplus P$.

This proves independence of the projective resolution once we pass to long exact sequences from this. In fact let $P^\bullet \to A$ and $P'^\bullet \to A$ are two resolutions. By Schanuel's lemma $P_n \oplus P'_{n-1} \oplus \cdots$ is isomorphic to $P'_n \oplus P_{n-1} \oplus \cdots$. Thus

$$\sum_{odd} P_i \oplus \sum_{even} P'_k \cong \sum_{even} P_i \oplus \sum_{odd} P'_k.$$

Next step is to show that this map is independant of the choice of representative for the class of A. □

COROLLARY 5. *If R is regular (i.e. every finitely generated R-module has a finite projective resolution) then $K_0(R) \to G_0(R)$ is an isomorphism.*

COROLLARY 6. *If R is a regular commutative ring, then $K_0(R) \to G_0(R)$ is an isomorphism.*

When working with commutative rings the notion of regularity is given equivalently by

REMARK. A local ring $(A, \mathfrak{m})$ is regular whenever $\mathrm{krull}\dim A = \dim_k \mathfrak{m}/\mathfrak{m}^2$. A ring is regular if $A_\mathfrak{p}$ is regular for all prime ideals in A (equivalently if so for maximal ideals).

In general if R is a ring (domain), we can define $\mathrm{Pic}(R)$ to be the group generated by the set of all invertible ideals of R. Then we always have a surjection

$$K_0(R) \twoheadrightarrow \mathrm{Pic}(R).$$

It is not always the case that $K_0(R) = G_0(R)$. For instance let $R = k[t^2, t^3] \subset k[t]$. Then $\mathrm{Pic}(A) \cong k_+$, the additive group of k.

Remark. The above remarks motivate a general definition of Krull dimension for an R-module. And this is given as

$$\dim_R M := \dim(R/\mathrm{Ann}_R M).$$

Remark. $G_0(A) = K_0(A - \mathit{mod})$ has two broad classes: that of the free modules (a $\mathbb{Z}$-contribution) and those with finite length (giving a free group on simple modules).

6. K-theory of rings of polynomials

The question we are going to consider is conditions under which $K_0(R[t]) \cong K_0(R)$. This is not always the case but, we will see that this holds when R has finite homological dimension. We will need the following tools:

Theorem 6.1 (Hilbert's Syzygy). *If R is a ring such that $\mathrm{gl.dim}\, R \leq n$, then $R[t]$ has $\mathrm{gl.dim}\, R[t] \leq n+1$.*

Proposition 1. *If $0 \to M_1 \to M_2 \to M_3 \to 0$ is an exact sequence of R-modules, then*

$$\mathrm{pd}\, M_2 \leq \max\{\mathrm{pd}\, M_1, \mathrm{pd}\, M_2\}$$

and equality holds if and only if $\mathrm{pd}\, M_3 \neq \mathrm{pd}\, M_1 + 1$.

Proposition 2. $\mathrm{pd}_k M \geq n$ *then* $\mathrm{Tor}^i_R(M, N) = 0$ *for all* $i \geq n+1$ *and for all modules* N.

This proves

Lemma 5. *Let x be a (central) nonzero divisor in a ring S. If M is a nonzero S/x-module with $\mathrm{pd}_{S/x} M < \infty$ then $\mathrm{pd}_X M = 1 + \mathrm{pd}_{S/x} M$.*

Theorem 6.2. *If R is a ring with $\mathrm{gl\,dim}_R \geq n$ then $R[t]$ has $\mathrm{gl\,dim} \leq n+1$.*

Proof. Step 1. Let $S = R[x]$. As S is R-flat, hence $\mathrm{pd}_S\, M[x] \leq \mathrm{pd}_R\, M$. For projective resolution of R-modules

$$P^\bullet \twoheadrightarrow M$$

we tensor by $R[x]$ and get

$$R[x] \otimes_R P^\bullet \twoheadrightarrow R[x] \otimes_R M$$

which is a projective resolution of $R[x] \otimes_R M \cong M[x]$. Check that the previous lemma implies that

$$\mathrm{gl\,dim}\, R[x] \geq 1 + \mathrm{gl\,dim}\, R.$$

Step 2. Let M be an S-module and consider M_R. Consider the S-modules

$$0 \to S \otimes_R M_r \xrightarrow{\beta} S \otimes_R M_R \xrightarrow{\mu} M \to 0$$

where $\mu : s \otimes m \to sm$ and $\beta : s \otimes m \mapsto s(x \otimes m - 1 \otimes xm)$. (Check that this is exact).

Step 3. Use comparison statement for projective dimension on exact sequence to conclude that $\mathrm{pd}_S M \leq 1 + \mathrm{pd}_R M$. So

$$\mathrm{gl}\dim S \leq 1 + \mathrm{gl}\dim R$$

and we are done. □

THEOREM 6.3. *For noetherian R with finite global dimension, we have*

$$K_0(R[t]) \cong K_0(R) \cong K_0(R[t, t^{-1}]).$$

PROOF. We have split short exact sequences

$$0 \to tR[t] \to R[t] \to R \to 0$$

and

$$0 \to (t-1)R[t, t^{-1}] \to R[t, t^{-1}] \to R \to 0$$

of R -modules. Hence $K_0(R[t])$ contains $K_0(R)$ as a direct summand.

As $S = R[t]$ is R-flat, we have $f_* : G_0(R) \to G_0(S)$. Also $\mathrm{hd}_{R[t]} R = 1$. Now take a short exact sequence

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

of $R[t]$-modules and we derive with tensoring with R as an $R[t]$-module. Because of projective dimension of R higher Tors vanish and our ong exact sequence reduces to

$$\begin{aligned} 0 \to \mathrm{Tor}_1^{R[t]}(R, M_1) &\to \mathrm{Tor}_1^{R[t]}(R, M_2) \to \mathrm{Tor}_1^{R[t]}(R, M_3) \\ &\to R \otimes_{R[t]} M_1 \to R \otimes_{R[t]} M_2 \to R \otimes_{[R[t]} M_3 \to 0 \end{aligned}$$

We will show that f_* has a right inverse, φ and then prove that f_* is also injective: $\varphi : G_0(R[t]) \to G_0(R)$ is define by

$$[M] \mapsto [\underbrace{R \otimes_{R[t]} M}_{M/tM}] - [\mathrm{Tor}_1^{R[t]}(R, M)]$$

and is well-defined by our long exact sequence. Now

$$M \xrightarrow{f_*} \underbrace{[R[t] \otimes_R M]}_{[M[t]]} \xrightarrow{\varphi_*} \underbrace{[R \otimes_{R[t]} M[t]]}_{[M]} - \underbrace{[\mathrm{Tor}_1^{R[t]}(R, R[t] \otimes_R M)]}_{0}$$

so f_* is a split surjection. It remains to show it is injective.

For this part we use the following observation of Grothendieck: When R is increasing-filtered $R = \operatorname{inj\,lim} R_n$, we get a grading on R by factors of the filtration.

$$\begin{tikzcd} G_0(R) \otimes_{\mathbb{Z}} \mathbb{Z}[t] \arrow[r, phantom, "\cong"] \arrow[d, phantom, "\cong"] & G_0^{gr}(R) \\ K_0(grR-mod) & \end{tikzcd}$$

amd we jave

$$\begin{tikzcd} G_0^{gr}(R) \arrow[r, "\theta"] \arrow[d, "\text{forget}"] & G_0^{gr}(R[t,s]) \arrow[d, "\psi_*"] \\ G_0(R) \arrow[r, hook] & G_0(R[t]) \end{tikzcd}$$

So the completion of the proof is by showing that this is a commutative diagram, ψ_* is sujrective and therefore that θ is an isomorphism. $\square$

Corollary 7. *If* $\operatorname{gl\,dim} R < \infty$*, so is* $\operatorname{gl\,dim} R[x_1, \cdots, x_n]$.

Corollary 8. *For a field k, $K_0(k[x_1, \cdots, x_n]) \cong G_0(k[x_1, \cdots, x_n]) \cong \mathbb{Z}$.*

So this shows that every projective $k[x_1, \cdots, x_n]$-module P is stably free, i.e.

$$P \oplus R^k \cong R^{k+t}.$$

Can we do a cancellation then? The ideas to come are

Quillen/Suslin: Stably free $k[x_1, \cdots, x_n]$-projective moduels are free.

Bass cancellation: Let R bea commutative noetherian domain of Krull dimension d. The every stably free R-module of rank $> d$ is a free module over R.

7. K-theory of topological spaces

Let X be a compact, Hausdorff topological space and $\mathrm{Vect}_F(X)$ be the commutative monoid of isomorphism classes of F-vector bundles over X, under the Whitney sum operation and X being the unit. Here F is the field $\mathbb{R}$ or $\mathbb{C}$. The topological K-theory of X is defined by

$$K_F^0(X) = K_0(\mathrm{Vect}_F(X)).$$

Then $X \mapsto \mathrm{Vect}_F(X)$ is a contravariant functor from the category of compact Hausdorff spaces to the category of abelian groups.

Theorem 7.1 (Swan). *Let $R = C^F(X)$ be the ring of continuous F-valued functions on X nad $\Gamma(X, E)$ be the R-module of sections of E, then*

(1) $\Gamma(X, E)$ is finitely generated and projective over R.

(2) Conversely, every finitely-generated projective R-module arises this way.

(3) *The map* $E \mapsto \Gamma(X, E)$ *induces an isomorphism of categories betwenn* $\mathrm{Vect}_F(X)$ *and* $\mathcal{P}_R$*, hence an isomorphism*

$$K^0_F(X) \to K_0(R).$$

8. K-theory of schemes

Let $\mathrm{Vect}(X)$ be the category of algebraic vector bundles on the ringed space $(X, \mathcal{O}_X)$ (for us, locally free $\mathcal{O}_X$-module with finite rank at every point). In the case $(X, \mathcal{O}_X)$ is the affine scheme $\mathcal{S}pec\, R$, with R a commutative noetherian ring, then

$$\mathrm{Vect}(X) = \mathcal{P}_R$$

since free $\mathcal{O}_X$-modules are precisely the projective ones.

PROPOSITION 3. *For every scheme* X *and* $\mathcal{O}_X$*-module* $\mathcal{F}$*, the following are equivalent:*

(1) $\mathcal{F}$ *is a vector bundle on* X*.*

(2) $\mathcal{F}$ *is a coherent sheaf and the stalks* $\mathcal{F}_x$ *are free* $\mathcal{O}_{X,x}$*-modules of finite rank.*

(3) *For every affine open* $U = \mathcal{S}pec\, R$ *in* X*,* $\mathcal{F}|_U$ *is a finitely-generated projective* R*-module.*

So we have a dictionary

sheaf	R-module
coherent sheaf	finitely-generated R-modules
locally free of finite rank (vector bundles)	finitely-generated projective modules

So we have analogously $K^0(\mathrm{Vect}(X) = K_0(R)$ at least in the affine case. In general we have a map

$$K^0(\mathrm{Vect}(X) \to \mathrm{Pic}(X)$$

by highest exterior powers.

CHAPTER 2

K_1 of a ring

Let R be a noetherian ring, not necessarily commutative. We already saw that matrices over R play a role in $K_0(R)$. Let $\mathcal{A}$ be the full subcategory of an abelian category which contains 0, is closed under $\oplus$, and is equivalent to a small category.

Notation: $\mathcal{A}$ as above, $\mathcal{A}[x, x^{-1}]$ is defined to be the following category: objects are pairs (A, f) where A is an object of $\mathcal{A}$ and $f : A \to A$ is an isomorphism. Morphisms are

$$\begin{array}{ccc} A & \xrightarrow{f} & A \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ B & \xrightarrow{g} & B \end{array} .$$

A sequence $0 \to (A', f') \to (A, f) \to (A'', f'') \to 0$ is then exact if and only $0 \to A' \to A \to A'' \to 0$ is exact.

DEFINITION 3. $K_1(\mathcal{A})$ is the abelian group whose generators are objects $[(A, f)] \in ob(\mathcal{A}[x, x^{-1}])$. with relations

(a) If $0 \to \; [(A', f')] \to \; [(A, f)] \to \; [(A'', f'')] \to 0$ is exact then
$$[(A, f)] = [(A', f')] + [(A'', f'')].$$

(b) $[(A, fg)] = [(A, f)] + [(A, g)]$.

Clearly if $\mathcal{B}$ is a category like $\mathcal{A}$ and $T : \mathcal{A} \to \mathcal{B}$ is a functor that preserves exact sequence then we have an induced homomorphism $K_1(\mathcal{A}) \to K_1(\mathcal{B})$.

REMARK. For C a compact Hausdorff topological space $K^1(X) = K^0(SX)$.

DEFINITION 4. If R is a ring, then $K_1(R) := K_1(\mathcal{P}_R)$.

If $\varphi : R \to R'$ is a ring homomorphism, then the functor $T(M) = R' \otimes_R M$ preserves exact sequence of projective modules hence we get a functor from the induce morphisms
$$\varphi_* : K_1(R) \to K_1(R').$$

PROPOSITION 4. *Let R be a ring and let $\mathcal{A} = \mathcal{P}_R$. Let $\mathcal{F}$ be the category of finitely-generated free R-modules. The inclusion $\mathcal{F} \subset \mathcal{P}_R$ induces an isomorphism $K_1(\mathcal{F}) \to K_1(R)$.*

PROOF. Clearly by relation (b) we have $[(A, 1_A)] = 0$ for any $A \in \mathcal{A}$. Now let $[(P, f)] \in K_1(R)$, then there is a projective module Q such that $P \oplus Q \cong R^n =: F \in \mathcal{F}$. Let $g : F \to F$ be defined by $g = (f \oplus 1_Q)$. Define $j : K_1(R) \to K_1(\mathcal{F})$ via $[(P, f)] \mapsto [(P \oplus Q, f \oplus 1_Q)]$. Then

$$ij[(P, f)] = [(F, g)] = [(P, f)] + [(Q, 1|_Q)] = [(P, f)].$$

Note that j is well-defined since if $P \oplus Q' = F', g' = (f \oplus 1|_{Q'})$. To show that $[(F, g)] = [(F', g')]$ consider

$$[(P \oplus Q \oplus P \oplus Q', f \oplus 1 \oplus 1 \oplus 1)] \in K_1(\mathcal{F})$$

we have

$$[(P \oplus Q \oplus P \oplus Q', f \oplus 1 \oplus 1 \oplus 1)] = [(P \oplus Q \oplus P \oplus Q', 1 \oplus 1 \oplus f \oplus 1)]$$

So $[(P \oplus Q, f \oplus q)] = [(P \oplus Q', f \oplus 1)]$. □

LaTeX2e

Let $\mathrm{GL}(n, R)$ be the group of $n \times n$ matices over R that are invertible. The embeddings

$$\mathrm{GL}(n, R) \subset \mathrm{GL}(n + 1, R), A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

give the definition of the infinite general linear group

$$\mathrm{GL}(R) = \varinjlim_n \mathrm{GL}(n, R).$$

If $f \in \mathrm{Gl}(n, R)$, consider $[(R^n, f)] \in K_1(\mathcal{F})$. We get a homomorphism

$$\mathrm{GL}(n, R) \to K_1(\mathcal{F})$$

and also $\mathrm{Gl}(R) \to K_1(\mathcal{F})$. Now let the matrices $E_{ij}(a) = \mathrm{id} + ae_{ij}$ be the *elementary* matrices $(i \neq j)$ and $E(n, R) \subset \mathrm{GL}(n, R)$ be the subgroup generated by elementary matrices

$$E(R) = \varinjlim E(n, R).$$

THEOREM 0.1 (Whitehead). *Let R be any ring. Then $E(R) = [\mathrm{GL}(R), \mathrm{GL}(R)]$.*

PROOF. Observe that $E_{ij}(a)^{-1} = (I - ae_{ij})$. To show the converse inclusion for $A \in \mathrm{GL}(n, R)$,

$$\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \in E(2n, R).$$

One can go from the above matrix to I_{2n} by a series of steps sonsiting of multiplication by elements of $E(2n, R)$. Then observe that

$$\begin{pmatrix} [A, B] & 0 \\ 0 & I \end{pmatrix} \in E(2n, R).$$

□

By the next result, we have an equivalent definition for $K_1(R)$:

$$K_1(R) \cong \frac{\mathrm{GL}(R)}{E(R)}.$$

Note that as $E(R) = [\mathrm{GL}(R), \mathrm{GL}(R)]$, we immediately have that $K_1(R)$ is the abelianization of $\mathrm{GL}(R)$. Note that universal property

$$\begin{tikzcd} \mathrm{GL}(R) \arrow[r, two heads] \arrow[dr] & K_1(R) \arrow[d, dashed] \\ & G \end{tikzcd}$$

for any abelian group G.

THEOREM 0.2. *Let R be any ring. Then the map $\mathrm{GL}(R)/E(R) \to K_1(R) = K_1(\mathcal{F})$ is an isomorphism.*

PROOF. We already know that $\mathrm{GL}(R) \twoheadrightarrow K_1(R)$. This gives surjectivity. We construct $j : K_1(\mathcal{F}) \to \mathrm{GL}(R)/E(R)$ which is an inverse to the map. Let $[(F, f)] \in K_1(\mathcal{F}$ where F is a rank n free module. Take $(e_1, \cdots, e_n)$ to be a basis for F. Let A be the corresponding matrix for f. Then $j[(F, f)] = [A]$.

If A' is another matrix obtained as above, there is an isomorphism

$$\varphi : (F \oplus F, f \oplus 1) \to (F \oplus F, 1 \oplus f)$$

in $\mathcal{F}(x, x^{-1})$ making

$$\begin{tikzcd} F \oplus F \arrow[r] \arrow[d] & F \oplus F \arrow[d] \\ F \oplus F \arrow[r] & F \oplus F \end{tikzcd}$$

commute. Therefore $\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$ and $\begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix}$ give the same element in $\mathrm{GL}(R)/E(R)$. This shows that j is well-defined.

It also preserves relations of K_1; for the relation

$$0 \to (F', f') \to (F, f) \to (F'', f'') \to 0$$

pick a basis $e'_1, \cdots, e'_n$ for F', extend it to a basis $e'_1, \cdots, e'n, e''_1, \cdots, e''_m$ for F so that the images of e''_i form a basis for F''. If A', A'' correspond respectively to f' and f'', we get

$$A = \begin{pmatrix} A' & B \\ 0 & A'' \end{pmatrix}$$

for the matrix corresponding to f. But

$$A = \begin{pmatrix} A' & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & A'' \end{pmatrix} \begin{pmatrix} I & B \\ 0 & I \end{pmatrix}$$

and therefore $j[(F,f)] = j[(F',f')] + j[(F'',f'')]$. $\square$

Corollary 9. *If F is a field, then $K_1(F) \cong F^\times$ via the determinant map.*

1. Stable range results

If R is commutative then
$$\det : \mathrm{GL}(R) \to R^*$$
is split. Let $SK_1(R) = \ker(\det)$ so $K_1(R) \cong R^* \oplus SK_1(R)$. Let
$$\mathrm{SL}(n,R) = \ker(\mathrm{GL}(n,R) \to R^*), \mathrm{SL}(R) = \varinjlim \mathrm{SL}(n,R).$$
So we have
$$E(R) \subset \mathrm{SL}(R) \subset \mathrm{GL}(R).$$
For a general ring R, define
$$SK_1(R) := \mathrm{Sl}(R)/E(R).$$
IF R is a field, then $\mathrm{SL}(n,R) = E(n,R)$, hence $SK_1(R) = 0$ and $K_1(R) \cong R^*$.

Now the questions are:

(1) When are the following maps surjective/isomorphisms?
$$\mathrm{GL}(n,R)/E(n,R) \to K_1(R).$$

(2) Is $\mathrm{GL}(n,R)/E(n,R)$ a group? an abelian group?

Definition 5. The integer n defines a stable range for $\mathrm{GL}(R)$ if whenever $r > n$, and $(a_1, \cdots, a_r)$ is a unimodular row, then there exist $b_1, \cdots, b_r \in R$ such that
$$(a_1 + a_r b_1, \cdots, a_{r-1} a_r b_{r-1})$$
is a unimodular row.

Lemma 6. *If n defines a stable range for $\mathrm{Gl}(R)$, and if $r > n$ and $(a_1, \cdots, a_r)$ is a unimodular row, then there is $A \in E(r,R)$ such that*
$$(a_1, \cdots, a_r)A = (1, 0, \cdots, 0).$$

Theorem 1.1. *IF n defines a stable range for R, then*

(1) $\mathrm{Gl}(m,R)/E(m,R) \to \mathrm{GL}(R)/E(R)$ is surjective for all $m \geq n+1$.

(2) $E(m,R)$ is a normal subgroup of $\mathrm{Gl}(m,R)$, if $m \geq n+1$.

(3) $\mathrm{Gl}(r,R)/E(r,R)$ is an abelian group if $r \geq 2n$.

Remark. Bass, Milnor and Serre, proved that if R is commutative, the stable range for R is 1. If R is the ring of integers of an algebraic number field, then $st.r(R) = \{0\}$.

PROPOSITION 5. *If D is a division ring, then the inclusion $D^* \hookrightarrow \mathrm{GL}(1,D) \hookrightarrow \mathrm{GL}(D)$ induces a surjection*

$$D^*/[D^*,D^*] \twoheadrightarrow K_1(D).$$

REMARK. IF $(R,\mathfrak{m})$ is a local ring, then the inclusion $R^* \hookrightarrow \mathrm{GL}(1,R) \to \mathrm{Gl}(R)$ induces a surjection

$$R^*_{ab} \twoheadrightarrow K_1(R).$$

One can in fact define a non-commutative determinant for the local rings to get an isomorphism

$$E^{\times}_{ab} \xrightarrow{\cong} K_1(R).$$

For semi-local rings, again there is a surjection

$$R^{\times}_{ab} \twoheadrightarrow K_1(R).$$

The idea in defining a non-commutative determinant for local rings is

(1) In any $n \times n$ matrix in $\mathrm{Gl}(n,R)$, any given row has at least on element which is not in $\mathfrak{m}$ and is therefore a unit.
(2) Use induction to define the usual determinant along with the last step to define determinant.

2. K_1 and topology

These are mainly works of Milnor and C. T. C. Wall. Suppose (K,L) is a pair, consisting of a finite connected CW complex K and a subcomplex L which is a deformation retract of K. Then

$$G := \pi_1(K) = \pi_1(L)$$

as we know. Milnor defined an element $\tau(K,L)$ called the Whitehead torsion, in the Whitehead group

$$Wh(G) := K_1(\mathbb{Z}[G])/\langle\{\pm g\}\rangle.$$

Whitehead torsion has applications in surgery theory, Poincare conjecture, etc. Milnor showed that the Whitehead torsion of a homotopy equivalence between finite CW-complexes can be used to distinguish between homotopy and simple homotopy types. Some of the computations done where for $G = \mathbb{Z}/5$ when $Wh(G)$ is infinite and if $G \cong \mathbb{Z}/2$ the $Wh(G) = \{1\}$. For $R = k[x,y]/(x^2+y^2-1)$ we get

$$SK_1(R) \neq 0, SK_1(R) \cong \mathbb{Z}/2\mathbb{Z}.$$

Then Wall constructs the Wall obstruction: for a topological space X, dominated by a finite complex, he defines a generalized Euler characteristic $\chi(X) \in K_1(\mathbb{Z}[G])$ when $G = \pi_1(X)$. He then shows that X has the homotopy type of a finite complex if and only if $\chi(X) \in \mathbb{Z}$.

3. Description of K_1 using group cohomology

Recall the definition of a G-module for an arbitrary group G.

Example 3.1. If K is a field and L/K is a Galois extension, then the Galois group $G = Gal(L/K)$ acts on $L^{\times}$. ⌟

If G is any group. $\mathbb{Z}$ is the trivial G-module. We have definitions

$$H^0(G, M) = M^G = \{m \in M : g.m = m, \forall g \in G\}$$

and the dual notion is

$$H_0(G, M) = M_G = M/I_G M.$$

Here I_G is the augmentation ideal defined to be the kernel of the augmentation map

$$\mathbb{Z}[G] \xrightarrow{e} \mathbb{Z}, \text{ via } \sum a_g g \mapsto \sum ag.$$

As an example $H_0(G, \mathbb{Z}) = \mathbb{Z}$.

Define

$$Z(G, M) = \{f : G \to M : f(g_1 g_2) = g_1 f(g_2) + g_2 f(g_1)\}$$

to be the set of Crossed homomorphisms. For $m \in M$, $f_m : G \to M$ via $g \mapsto gm - m$ is a crossed homomorphism. Then let

$$B^1(G, M)$$

be the gorup of boundaries. Then the group cohomology of M is defined as

$$H^1(G, m) = Z(G, M)/B^1(G, M).$$

In particular

$$H^1(G, \mathbb{Z}) = \mathrm{Hom}(G, \mathbb{Z}) = \mathrm{Hom}(G^{ab}, \mathbb{Z}).$$

Once defined systematically, one gets

$$H_1(G, \mathbb{Z}) = H_1(G, \mathbb{Z})^{\vee} = G^{ab}.$$

A consequence of this is a description

$$K_1(R) = H_1(\mathrm{GL}(R), \mathbb{Z}).$$

4. Baby K-theory long exact sequence

Theorem 4.1 (Resolution theorem). *Suppose $\mathcal{M}$ and $\mathcal{P}$ are categories with exact sequences both contained in an abelian category $\mathcal{A}$ and $\mathcal{P} \subset \mathcal{M}$ is a full subcategory. Assume the following*

(1) For each object M of $\mathcal{M}$, there is an epimorphism $P \twoheadrightarrow M$ in $\mathcal{A}$ with $P \in ob(\mathcal{P})$ such that every endomorphism of M lifts to an endomorphism of P.

(2) If $\cdots \to P_n \xrightarrow{d_n} P_{n-1} \to \cdots \to P_0 \to M \to 0$ is exact in $\mathcal{M}$ with $P_i \in ob(\mathcal{P})$ then $\ker d_n \in ob(\mathcal{P})$ for $n >> 0$.

(3) If $0 \to M_1 \to M_2 \to M_3 \to 0$ is a short exact sequence in $\mathcal{A}$ with $M_2, M_3 \in ob(\mathcal{M})$ then $M_1 \in ob(\mathcal{M})$.

Then the inclusion $\mathcal{P} \subset \mathcal{M}$ induces an isomorphism $K_1(\mathcal{P}) \xrightarrow{\cong} K_1(\mathcal{M})$.

PROOF. Let $[M, \alpha]$ be an object in $K_1(\mathcal{M})$, $\alpha : M \to M$ an automorphism of M. Consider $\alpha \oplus \alpha^{-1} \in \mathrm{Aut}(M \oplus M)$. Then $\alpha \oplus \alpha^{-1}$ is a product of matrices

$$\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} = \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -A^{-1} & 1 \end{pmatrix} \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

so it can be written as a product of elementary automorphisms of the form

$$\begin{pmatrix} 1_M & \beta \\ 0 & M \end{pmatrix}, \begin{pmatrix} 1_M & 0 \\ \gamma & 1_M \end{pmatrix}, \quad \beta, \gamma \in \mathrm{End}(M).$$

Lift β, γ to elements in $\mathrm{End}(P)$ by (1).

$$\begin{tikzcd} P \arrow[r, two heads] \arrow[d, "\widetilde{\beta},\widetilde{\gamma}"'] & M \arrow[d, "\beta,\gamma"] \\ P \arrow[r, two heads] & M \end{tikzcd}$$

Hence we can lift $\alpha \oplus \alpha^{-1}$ to an automorphism of $P \oplus P$. Since the kernel is in M by (3) we keep repeating this process and take the corresponding alternating sums of the resolution obtained in this manner. $\square$

We have similar corollaries from this theorem as in case of K_0:

(1) $K_1(\mathcal{P}_R) \cong K_1(R-\mathit{mod}) := G_1(R)$ if the global dimension of R is finite.

(2) $K_1(R[t]) = K_1(R)$ if $\mathrm{gl.dim}\, R < \infty$.

(3) $K_1(R[t, t^{-1}]) = K_1(R) \oplus K_0(R)$.

Let $f : R \to R'$ be a homomorphism of rings. This gives a homomorphism $K_1(R) \to K_1(R')$. We define a category $\mathcal{P}_f$ with objects (A, g, B) where A, B are projective left R-modules and $g : R' \otimes_R A \to R' \otimes_R B$ is an isomorphism. A morphism between (A, g, B) and (A', g', B') is a pair (α, β) where $\alpha : A \to A'$, $\beta : B \to B'$ such that

$$\begin{tikzcd} R' \otimes_R A \arrow[r, "1\otimes\alpha"] \arrow[d, "g"'] & R' \otimes_R A' \arrow[d, "g'"] \\ R' \otimes_R B \arrow[r, "1\otimes\beta"] & R' \otimes_R B' \end{tikzcd}$$

is commutative and

$$0 \to (A', g', B') \to (A, g, B) \to (A'', g'', B'') \to 0$$

is exact (meaning $0 \to A' \to A \to A'' \to 0$ and $0 \to B' \to B \to B'' \to 0$ are exact).

Definition 6. Let $K_0(R,f)$ be an abelian group defined similar to K_0 and K_1 with generators $(A,g,B) \in ob(\mathcal{P}_f)$ with the scissor relations and an extra relation

$$(A,gh,B) = (A,h,C) + (C,g,B).$$

One can check that if we associate to $\theta \in K_1(R')$ the triple (R^n, θ, R^n) (viewing θ as an $n \times n$ matrix in $\mathrm{GL}(n,R')$) we get a well-defined mapping

$$K_1(R') \to K_0(R,f).$$

It is straightforward to check that

Theorem 4.2. *Let $f : R \to R'$ be a homomorphism of rings. The the sequence*

$$K_1(R) \to K_0(R'') \to K_0(R,f) \to K_0(R) \to K_0(R')$$

is exact.

4.1. Localization sequence. Let R be commutative and $S \subset R$ be a multiplicative closed set and suppose $\mathrm{gl.dim}\, R < \infty$. Let $S - \mathcal{tor}$ be the subcategory of $R - \mathcal{mod}$ consisting of objects $M \in R - \mathcal{mod}$ such that $S^{-1}M = 0$. $S - \mathcal{tor}$ is closed with respect to exact sequence: If

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is an exact sequence of R-modules, then if $S^{-1}M_i = 0$ for any two for the i's, $S^{-1}M_i = 0$ for the other one. One gets an exact sequence

$$K_0(S - \mathcal{tor}) \to K_0(R - \mathcal{mod}) \to K_0(S^{-1}R - \mathcal{mod}) \to 0$$

called the localization sequence.

Remark. If fact $K_0(S - \mathcal{tor}) \cong K_0(R,f)$ for the inclusion mapping $f : R \to S^{-1}R$.

CHAPTER 3

K_2 groups

1. Milnor's K_2

Let R be a ring and $E(R)$ be the set of elementary matrices. $E(R)$ is generated by $E_{ij}(\lambda), \lambda \in R$ with relations

(1)
$$E_{ij}(r).E_{ij}(s) = E_{ij}(r+s)$$

(2)
$$[E_{ij}(s), E_{k\ell}(t)] = \begin{cases} E_{i\ell} & j = k, i \neq \ell \\ \mathrm{id} & j \neq k, i \neq \ell \\ E_{ikj}(-ts) & j \neq k, i = \ell \end{cases} .$$

The Steinberg group, $St(R)$, is on the other hand defined as a non-abelian group with generators $x_{ij}(t), t \in R$ where $i \neq j$ ranges over positive integers with the following relations:

(1)
$$x_{ij}(s)x_{ij}(t) = x_{ij}(s+t).$$

(2)
$$[x_r(s), x_{k\ell}(t)] = \begin{cases} x_{i\ell}(st) & j = k, i \neq \ell \\ 1 & i \neq \ell, r \neq k \\ x_{kj}(-ts) & j \neq k, i = \ell \end{cases} .$$

We can view St as a functor form the category of rings to the category of groups.

Remark. Since $[a,b]^{-1} = [b,a]$ the relation
$$[x_{ij}(s_{)}, x_{ki}(t)] = x_{kj}(-ts)$$
holds. We clearly have a group homomorphism $f : St(R) \twoheadrightarrow E(R)$.

Generalizing the above notion with define $St_n(R)$ as the quotient of the free group on symbols $x_{ij}^{(n)}(a)$ for $1 \leq i, j \leq n, i \neq j$ and $a \in R$. The relations will be as before. Then we take the direct limit
$$St(R) := \varinjlim St_n(R).$$

DEFINITION 7. $K_2(R)$ is defined to be the kernel of the surjection:

$$K_2(R) := \ker(f : St(R) \twoheadrightarrow E(R)).$$

LEMMA 7. *$K_2(R)$ is the center of $St(R)$.*

PROOF. Let $x \in Z(St(R))$. Then $f(x) \in Z(E(R))$ as f is surjective. So it suffices to show that $Z(E(R))$ is trivial. This is clear since if $A \in \mathrm{GL}(n, R)$ is in the center, $A.E_{ij}(1) = E_{ij}(1).A$ hence A is a diagonal matrix with all entries on the diagonal equal. Therefore $A = \mathrm{id}$.

For the other direction let $\alpha \in K_2(R) \cap \mathrm{im}(St_{n-1}(R) \to St(R))$. So we can write α as a word in $x_{ij}(s)$ with $i, j < n$. Let A_n be the subgroup of $St(R)$:

$$A_n = \langle \{x_{in}(t) : 1 \leq i \leq n-1, t \in R\} \rangle.$$

By the relation

$$[x_{ij}(s), x_{k\ell}(r)] = 1, \text{ if } i \neq \ell, j \neq k, r, s \in R$$

we see that A_n is commutative. From the relation $x_{ij}(s)x_{ij}(t) = x_{ij}(s+t)$ we have $x_{ij}(0) = 1$. Hence each element of A_n has a unique expression of the form

$$x_{1n}(a_1)\cdots x_{n-1,n}(a_{n-1}).$$

Thus $f|_{A_n} : A_n \to E(R)$ is an isomorphism onto the group of matrices in $\mathrm{SL}_n(R)$ of the form

$$\begin{pmatrix} 1 & 0 & \cdots & a_1 \\ 0 & 1 & \cdots & a_2 \\ & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Further if $i, j < n$ then

$$x_{ij}(a)x_{kn}(b)x_{ij}(-a) = \begin{cases} x_{kn}(b) & j \neq k \\ x_{in}(ab)x_{kn}(b) & j = k \end{cases}.$$

Therefore α normalizes A_n. But $f|_{A_n}$ is injective and $f(\alpha) = 1$ as $\alpha \in K_2(R) = \ker(St(R) \to E(R))$. Hence α centralizes A_n:

$$[\alpha, x_{in}(a)] = 1, \quad \forall 1 \leq i \leq n-1, a \in R.$$

Similarly $[\alpha, x_{ij}(a)] = 1$ for any $j \in \{1, \cdots, n-1\}$ and $a \in R$. Thus α commtues with $x_{ij}(ab)$ for all $a, b \in R$ and $1 \leq j \neq i \leq n-1$. This holds for all large n completing the proof. □

2. K_2 as an obstruction element

If G acts on M, the second group cohomology $H^2(G, M)$ coincides with the (group of) central extensions:

$$0 \to M \to E \to G \to 0, M \subset Z(E).$$

So we suspect a relation between $K_2(R)$ and central extensions. We will explore this relation in what follows.

DEFINITION 8. An extension $1 \to A \to E \to G \to 1$ is called a universal central extension if

(1) it is central,

(2) given any other central extension $1 \to A' \to E' \to G \to 1$ then there is a unique surjective $\varphi : E \to G$ such that

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r] \arrow[d] & E \arrow[r, "\varphi"] \arrow[d] & G \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & A' \arrow[r] & E' \arrow[r] & G \arrow[r] & 1 \end{tikzcd}$$

commutes.

THEOREM 2.1. *A group G has a universal central extension if and only if G is perfect (i.e. $G = [G,G]$). In this case, a central extension (E,φ) is universal if and only if the following two conditions hold:*

(i) E is perfect,

(ii) all central extensions of E are trivial.

PROOF. For necessity suppose G has a nontrivial abelian quotient A. Let $\psi : G \to A$ be the quotient map. Then if

$$1 \to A' \to E \xrightarrow{\varphi} G \to 1$$

is a central extension of G we have two distinct morphisms from (E,φ) to $(G \times A \xrightarrow{p_1} G_1)$:

$$\begin{tikzcd} E \arrow[r, "\varphi"] \arrow[d, "(\varphi{,}1)"'] & G \arrow[d, equal] \\ G \times A \arrow[r, "p_1"'] & G \end{tikzcd}$$

and

$$\begin{tikzcd} E \arrow[r, "\varphi"] \arrow[d, "(\varphi{,}\psi\circ\varphi)"'] & G \arrow[d, equal] \\ G \times A \arrow[r, "p_1"'] & G \end{tikzcd}$$

so (E,φ) cannot be universal.

For sufficiency, consider a presentation of G with a free group F and a subgroup R of relations:

$$1 \to R \to F \to G \to 1.$$

Step 1. We have a surjection

$$(*) \qquad \frac{[F,F]}{[F,R]} \twoheadrightarrow \frac{[F,F]}{R} = [F/R, F/R] = [G,G] = G.$$

We will show that this is in fact a central extension of G.

Step 2. Show that any central extension of G satisfying (i) and (ii) is universal. Let (E, φ) be a central extension of G satisfying (i) and (ii). Let (E', φ') be any other central extension of G. If

$$\begin{tikzcd} E \arrow[r, "\varphi"] \arrow[d, "\psi{,}\psi'"'] & G \arrow[r] \arrow[d, equal] & 1 \\ E' \arrow[r, "\varphi'"'] & G \arrow[r] & 1 \end{tikzcd}$$

are two morphisms, for $x \in E$ we have $\varphi' \circ \psi(x) = \varphi' \circ \psi'(x) = \varphi(x)$. So $\psi(x) = c_x \psi'(x)$ where $c_x \in A' = \ker \varphi'$. If $y \in E$, then $\psi(y) = c_y \psi'(y)$ for $c_y \in A' = \ker \varphi'$ and c_x, c_y are central. Hence

$$\psi([x, y]) = [\psi(x), \psi(y) = [\psi'(x), \psi'(y)].$$

Thus $\psi = \psi'$ on $[E, E]$ but $E = [E, E]$ showing uniqueness.

We still need to construct a morphism $(E, \varphi) \to (E', \varphi')$ to show existence. Let $E'' = E \times_G E'$. Since φ, φ' are surjective, $\pi : E'' \to E$ is surjective. Note that $\ker(\pi_1) \cong A' := \ker \varphi'$. Therefore $\pi : E'' \to E$ is a central extension of E. Check the remaining details.

Step 3. Show that (*) satisfies (i) and (ii). Let $E = \frac{[F,F]}{[F,R]}$, $\varphi : E \twoheadrightarrow G$. We have a diagram

$$\begin{tikzcd} E \arrow[d, hook] \arrow[dr, "\varphi"] & \\ E_1 = F/[F,R] \arrow[r, two heads, "\varphi"] \arrow[d, two heads] & F/R = G \arrow[dl, equal] \\ F/R & \end{tikzcd}$$

So $\ker \varphi_1 \subset R/[F, R]$ and hence $[\ker \varphi_1, E_1] \subset [R/[F, R], F/[F, R]]$, So $\ker \varphi_1$ is central in E_1 and therefore φ, φ_1 are central extensions of G. We check (i) and (ii) for E.

(i) Clearly E_1 has the property that $[E_1, E_1] = E$. Let $e_1 \in E_1$ then there is $e \in E$ such that $\varphi(e) = \varphi_1(e_1)$ therefore $\varphi_1(ee_1^{-1}) = 1$. Therefore $ee_1^{-1} \in \ker \varphi_1$. Hence

$$E = [E_1, E_1] = [E \ker \varphi_1, E \ker \varphi_1] = [E, E]$$

as $\ker \varphi_1$ is central. We conclude that E is perfect.

(ii) Let $1 \to A \to E_1 \xrightarrow{\psi} E \to 1$ be any central extension. Define $E_3 = E_1 \times_G E_2$. Consider $\pi_1 : E_3 \to E_1$. The claim is that (E_3, π_1) is a central extension.

$$\ker(\pi_1) \cong \ker(\varphi \circ \psi : E_2 \to G).$$

But $E = [E, E]$ hence $\psi([E_2, E_2]) = [\psi(E_2), \psi(E_2)] = [E, E]$. Hence $E_2 = [E_2, E_2]A$. Also

$$\psi([E_2, \ker \varphi \circ \psi]) \subset [E, \ker \varphi] = 1$$

as $\ker\varphi$ is central. We conclude from all this that if $x \in \ker\varphi\circ\psi$ then x commutes with $[E_2, E_2]$ and A (as A is central). Therefore x commutes with all of E_2. Hence E_3 is central. Consider

$$\begin{tikzcd} & F \arrow[dl, dashed, "\eta"'] \arrow[d, two heads] \\ E_3 \arrow[r, two heads, "\pi_1"'] & E_1 = F/[F,R] \end{tikzcd}$$

Use η to get a homomorphism $\theta : F \to E_2$ such that if $x \in F$, then $\varphi\circ\psi(\theta(x))$ is the image of x in $G = F/R$. Therefore θ descends to

$$\overline{\theta} : F/[F,R] = E_1 \to E_2.$$

and together with the identity map on E_1 this gives a splitting $E_1 \to E_1 \times E_2 = E_3$. Then we restrict this to E gives the desired trivialization proving (ii). □

COROLLARY 10. *$K_2(R)$ is the kernel of a universal central extension of $E(R)$.*

3. Constructing elements of K_2

Let R be any ring and consider two matrices $A, B \in E(R)$ which commute. Choose representatives $a, b \in St(R)$ under $f : St(R) \to E(R)$. Then we define

$$A * B := aba^{-1}b^{-1} \in K_2(R)$$

which is independent of the liftings as K_2R is central.

LEMMA 8. *The following statements hold:*

1. *Star is skew-symmetric:* $(A * B)^{-1} = B * A$.
2. *Star is bimultiplicative:* $A_1A_2 * B = (A_1 * B)(A_2 * B)$.
3. $(PAP^{-1}) * (PBP^{-1}) = A * B$.

At least when R is commutative, given $u, v \in R^*$ then

$$D_u = \begin{pmatrix} u & 0 & 0 \\ 0 & u^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D'_v = \begin{pmatrix} v & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & v^{-1} \end{pmatrix}$$

then D_u and D'_v commute giving $\{u, v\} = [u, v] \in K_2(R)$.

REMARK. We haven't shown that the elements constructed this way are nontrivial. We do not even know yet that $K_2(R)$ is nontrivial. In case of fields for instance, a highly nontrivial fact is that $\{-1, -1\} \neq 0$, and more generally $\{a, 1 - a\} \neq 0$.

A central simple algebra is a finite dimensional algebra over its center with no proper nontrivial two sided ideals. Examples are $M_n(F)$, the quaternion algebra $\mathbb{H}$, $M_n(D)$ if D is a division algebra.

THEOREM 3.1. *The Steinberg group $St(R)$ is the universal central extension of $K_2(R)$.*

PROOF. It is easy to see that $St(R)$ is perfect. All we have to do is to construct a section for a central extension

$$1 \to C \to Y \xrightarrow{\varphi} St(R) \to 1.$$

For $x_{ij}(a) \in St(R)$, choose an index k distinct from i and j and let

$$y = \varphi^{-1}(x_{ik}(\ell)), y' = \varphi^{-1}(x_{kj}(a)).$$

Then we let $s_{ij}(a) = [y, y']$. Note that $\varphi(s_{ij}(a)) = x_{ij}(a)$. One shall show now that $s_{ij}(a)$ is independent of the choice of index k, and satisfies Steinberg relations. For this we consider any central extension

$$1 \to C \to Y \to St(n, R) \to 1 (n \geq 0)$$

then for $x, x' \in St(n, R)$ the symbol $[\varphi^{-1}(x), \varphi^{-1}(x')] \in Y$ will denote the commutator $[y, y']$ where $y \in \varphi^{-1}(x)$ and $y' \in \varphi^{-1}(x')$. Using the Steinberg relations one makes the following

Observation: If $j \neq k$ and $i \neq \ell$ and $a, b \in R$ then $[\varphi^{-1}(x_{ij}(a)), \varphi^{-1}(x_{k\ell}(b))] = 1$ in Y.

So if we choose four distinct indices h, i, j, k and $u \in \varphi^{-1}(x_{ki}(1)), v \in \varphi^{-1}(x_{ij}(a))$ and $w \in \varphi^{-1}(x_{jk}(b)$ then $[u, v] = 1$. Let G be the subgroup of Y generated by u, v, w. Then $G' = [G, G]$ is generated by elements in $\varphi^{-1}(x_{hj}(a))$, $\varphi^{-1}(x_{ik}(ab))$ and $\varphi^{-1}(x_{hk}(ab))$. This implies that $G'' = 1$ by Steinberg relations. Therefore

$$[\varphi^{-1}x_{hj}(a), \varphi^{-1}x_{jk}(b)] = [\varphi^{-1}x_{hi}(a), \varphi^{-1}x_{ik}(ab)].$$

We set $a = 1$ and conclude

$$s_{hk}(b) = [\varphi^{-1}x_{hj}(1), \varphi^{-1}x_{jk}(b)]$$

i.e. that s_{ij} is independent of the chosen index h. Also we have shown that s_{ij} satisfies the first Steinberg relation. For the second relation apply $[u, v][v, w] = [u, vw][v, [w, u]]$ to $u \in \varphi^{-1}(x_{ij}(1)), v \in \varphi^{-1}(x_{hi}(a)), w \in \varphi^{-1}(x_{jk}(b))$.

□

DEFINITION 9. A monomial matrix is one in $\mathrm{GL}(n, R)$ that can be expressed as a product PD where P is a permuation matrix and D is a diagonal matrix.

Let $W \subset St(n, R)$ be the subgroup generated by all the w_{ij}s. Then and important fact is that ff R is commutative, then $\varphi(W) \subset \mathrm{GL}(n, R)$ via $\varphi : St(R) \to E(R) \subset \mathrm{Gl}(R)$. In fact $\varphi(W)$ is athe set of all monomial matrices of determinant 1.

Our goal is to prove

THEOREM 3.2. *If R is a commutative ring, the Steinberg symbols map $R^\times \times R^\times \xrightarrow{\{,\}} K_2(R)$ satisfying $\{u, 1-u\} = 1$ for $u \in R^\times$ and $1-u \in R^\times$.*

We define two new symbols:

$$w_{ij}(u) = x_{ij}(u)x_{ji}(-u^{-1})x_{ij}(u)$$
$$h_{ij}(u) = w_{ij}(u)w_{ij}(-1).$$

We see that there are relations such as $[h_{12}(u), h_{13}(v)] = h_{13}(uv)h_{13}(u)^{-1}h_{13}(v)^{-1}$. The following facts are just tedious computations using definitions:

LEMMA 9. *We have $w_{ij}(u)^{-1} = w_{ij}(-u)$, $h_{ij}(1) = 1$ and $w_{ij}(u) = w_{ji}(-u^{-1})$. In addition if $u, v \in R^\times$ and $i \neq j$ and $k \neq \ell$ then*

$$w_{k\ell}(u)w_{ij}(v)w_{k\ell}(u)^{-1} = \begin{cases} w_{ij}(-v) & \text{if } j, k, \ell \text{ are distinct} \\ w_{ij}(-u^{-1}v) & \text{if } k = i \text{ and } j, i, \ell \text{ are distinct} \\ w_{i\ell}(-v) & \text{if } k = j \text{ and } j, i, \ell \text{ are distinct} \\ w_{ji}(-u^{-1}vu^{-1}) & \text{if } k = i \text{ and } j = \ell. \end{cases}$$

Take $u = v$ and we get

$$w_{k\ell}(u)w_{ij}(u)w_{k\ell}(k)^{-1} = w_{ji}(-u^{-1}).$$

If R is commutative, $\{u, v\} = \{v, u\}^{-1}$, $\{u, u_2, v\} = \{u, v\}\{u_2, v\}$ and also

LEMMA 10. *If R is a commutative ring and $u, v \in R^\times$ then $h_{12}(uv) = h_{12}(u)h_{12}(v)\{u, v\}^{-1}$.*

PROOF OF THE THEOREM. For $\{u, -u\} = 1$ we have to show that

$$h_{12}(u)h_{12}(-u) = h_{12}(-u^2).$$

Which is the case since the left hand side can be written as

$$w_{12}(u)w_{12}(-1)w_{12}(-u)w_{12}(-1) = w_{21}(u^{-2})w_{12}(-1) = w_{12}(-u^2)w_{12}(-1) = h_{12}(-u^2).$$

For $\{u, 1-u\} = 1$ we will be showing that

$$h_{12}(u - u^2) = h_{12}(u)h_{12}(1-u)$$

and this can be shown starting from the right hand sind and rewriting everything in terms of w's and x's. □

Finally we have some nonzero elements

PROPOSITION 6.

(1) $\{a, 1\} = 1 = \{1, a\}$.

(2) $\{a^{-1}, b\} = \{a, b^{-1}\} = \{a, b\}^{-1}$.

(3) $\{a, b\} = \{b, a\}^{-1}$.

4. Case of fields

Let F be a field, and let $T(F^\times) = \oplus_n T_n(F^\times) = \oplus_n (F^\times)^{\oplus n}$ be the torus. Then $K_*^M(F) = T(F^\times)/I$ where I is the 2-sided ideal generated by all elements of the form $a \otimes (1-a)$ for all $1 \neq a \in F^\times$.

THEOREM 4.1 (Matsumoto). *If F is a field, then $K_2(F)$ has a presentation as the free abelian group on the symbols $\{a,b\}$ with $a,b \in F^\times$ and subject to the relations*

(1) $(a_1a_2, b) = (a_1, b)(a_2, b)$,

(2) $(a,b) = (b,a)^{-1}$,

(3) $(a, 1-a) = 1$.

Example 4.1 (Symbols on a field)**.** A symbol on F with values in an abelian group G is a map $(,) : F^\times \times F^\times \to G$ such that it satisfies the above three relations. From Matsumoto's theorem there is a bijection between symbols on F with values on G and homomorphism from $K_2(F)$ to G. One type of tame symbols are the *tame* symbols for a field (F,ν) with a discrete valuation $\nu : F^\times \to \mathbb{Z}$ on it. Let $\mathcal{O}_\nu$ be the valuation ring $\{x \in F^\times : \nu(x) \geq 0\}$ which is a local ring. Then form the natural $\psi : \mathbb{P}_\nu^\times \to k(\nu)^\times$ we have

$$T_\nu : F^\times \times F^\times \to k(\nu)^\times$$

via $(a,b) = \psi((-1)^{\nu(a)+\nu(b)} \frac{a^{\nu(b)}}{b^{\nu(a)}}$.

The differential symbols are useful in the context of algebraic geometry. Let F be a field again and $\Omega_F^1 = \Omega_{F/\mathbb{Z}}^1$ be the F-vector space of Kahler differentials of F. As an F-vector space Ω_F^1 is spanned by the symbols da for all $a \in F^\times$ given via

$$d(ab) = adb + bda.$$

Then $\Omega_F^2 = \wedge^2 \Omega_F^1$. The differential symbol $F^\times \times F^\times \to \Omega_F^2$ is

$$(a,b) \mapsto \frac{da}{a} \wedge \frac{db}{b}.$$

⌟

Let L/F be a field extension. Then $F^\times \hookrightarrow L^\times$ induces $L_2(F) \to K_2(L)$. If L/F is finite, then there exists homomorphisms $\mathrm{tr} : K_i(L) \to K_i(F)$ induced by the norm. For instance in the case of $i = 2$ if $(\lambda, \theta) \in K_2(l)$ such that $\lambda \in F, \theta \in L$ then

$$\mathrm{tr}(\lambda, \theta) = (\lambda, N_{L/F}\theta).$$

CHAPTER 4

Galois Cohomology

The setup is this: G is a group. M is a G-module, i.e. M is a module over $\mathbb{Z}[G]$. And of course this need not be commutative, so whenever we neglect saying, we assume a left action. We say M is a *trivial* module if $g.m = m$ for all $g \in G$. For G-modules, M and N, $\mathrm{Hom}(M, N)$ as groups, has a G-module structure via

$$(g.f)(m) = gf(g^{-1}m).$$

On $M \otimes N$ we put also the G-module structure $g(m \otimes n) = gm \otimes gn$. If M and N are G-modules then

$$\mathrm{Hom}_G(M, N) := \mathrm{Hom}_{\mathbb{Z}[[G]]}(M, N) = \{f : M \to N, f(gm) = g.f(m)\}.$$

Then we defined

$$M^G = H^0(G, M) := \{m \in M : gm = m, \forall g \in G\} \cong \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, M).$$

where we consider $\mathbb{Z}$ as a trivial G-module. The last isomorphism if given by $f \mapsto f(1)$ in the reverse direction. The cohomology groups of G with coefficients in M are a sequence fo abelina groups

$$H^q(G, A) \quad q = 0, 1, 2, \cdots$$

such that

(1) $H^0(G, M) = M^G$.

(2) For $q \geq 0$ the assigment $M \mapsto H^q(G, M)$ is a covariant functor.

(3) Given an exact sequence $0 \to M' \to M \to M'' \to 0$ of G-module there exists a long exact sequence constructed from connecting homomorphisms

$$\delta_q : H^q(G, M'') \to H^{q+1}(G, M')$$

that fits into a long exact sequence

$$0 \to H^0(G, M') \to H^0(G, M) \to H^0(G, M'') \to H^1(G, M') \to \cdots$$

and that furthermore δ is functorial .

(4) Let $H \subset G$ and define $N = \mathrm{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], M)$ (such modules are called co-induced modules from H to G). Then $H^q(G, N) = H^q(H, M)$ for all $q \geq 1$.

Theorem 0.2. *For any group G and any G-module M, the cohomology groups $H^q(G, M)$ exists for $q \geq 0$ and are unique up to functorial isomorphisms.*

The idea is to left-derive $M \to M^G$.

PROOF. Suppose we have two cohomology theories H and $\widetilde{H}$. From exact sequence $0 \to M \hookrightarrow M^* \to M' \to 0$ where $M^* = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], M)$, we have

$$\begin{tikzcd}
0 \arrow[r] & M^G \arrow[r] \arrow[d, equal] & (M^*)^G \arrow[r] \arrow[d, equal] & (M')^G \arrow[r] \arrow[d, equal] & H^1(G,M) \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & M^G \arrow[r] & (M^*)^G \arrow[r] & (M')^G \arrow[r] & \widetilde{H}^1(G,M) \arrow[r] & 0
\end{tikzcd}$$

we get $f_1 : H^1(G,M) \to \widetilde{H}^1(G,M)$ such that $f_1\delta_0 = \widetilde{\delta}_0$. As δ_0 and $\widetilde{\delta}_0$ are functorial in M we get that f_1 is also functorial. The general proof follows by induction.

For existence we define

$$H^q(G,M) = \mathrm{Ext}^q_{\mathbb{Z}[G]}(\mathbb{Z}, M).$$

To compute $\mathrm{Ext}^q_{\mathbb{Z}[G]}(\mathbb{Z}, M)$ we find a $\mathbb{Z}[G]$-projective resolution of $\mathbb{Z}$. Let P_i be the free $\mathbb{Z}[G]$-module $G^{\oplus i+1}$ given the module structure

$$g.(g_0, \cdots, g_i) = (gg_0, \cdots, gg_i).$$

P_i is a free $\mathbb{Z}[G]$-module with basis $\{(1, g_1, \cdots, g_o) : g_k \in G\}$. The differentials $d_i : P_i \to P_{i-1}$ are defined via

$$(g_0, \cdots, g_i) \mapsto \sum (-1)^i (g_0, \cdots, \widehat{g}_i, \cdots, g_i).$$

This gives an exact sequence of $\mathbb{Z}[G]$-modules $P^\bullet \to \mathbb{Z}$.

For connecting homomorphisms the maps $\delta_i : \mathrm{Hom}_{\mathbb{Z}[G]}(P_i, M) \to \mathrm{Hom}_{\mathbb{Z}[G]}(P_{i+1}, M)$ work:

$$\begin{aligned}
\delta_i(\theta)(x_1, \cdots, x_{i+1}) = x_1\delta_1(x_2, \cdots, x_{i+1}) \\
+ \cdots + \sum_{1 \le j \le i} (-1)^j \theta(x_1, \cdots, x_j x_{j+1}, \cdots, x_{i+1}) \\
+ (-1)^{i+1}\theta(x_1, \cdots, x_n).
\end{aligned}$$

□

1. Lower cohomology groups

So we have expressions

$$H^1(G,M) = B^1(G,M) = \{f_m : m \in M, f_m : G \to M \text{ such that } f_m(g) = gm - m\}.$$

For the second cohomology we have $H^2(G,M) = Z^2(G,M)/B^2(G,M)$ where

$$Z^2(G,M) = \{f : G \times G \to M : x_1 f(x_2, x_3) - f(x_1, x_2, x_3) + f(x_1, x_2, x_3) - f(x_1, x_2) = 0\}$$

$$B^2(G,M) = \{\delta h : G_1 \times G_1 \to M : h : G \to M \text{ and } \delta h(x_1, x_2) = x_1 h(x_2) + h(x_1) - h(x_1 x_2)\}.$$

If f is a co-cycle, for $(x, 1, 1) \in G^3$ we have $xf(1,1) = f(x,1)$. The map $f^* : G^2 \to M$ given by $f^*(x_1, x_2) = f(x_1, x_2) - f(x, 1)$ is verified to be a 2-cocyle. In fact $f^* = f - \delta h$ where

$h : G \to M$ is given via $h(x) = f(1,1)$. Hence f and f^* are cohomologous. So every 2-cocyle is cohomologous to a normalized 2-cocyle (i.e. a 2-cocyle $\widetilde{f}$ such that $\widetilde{f}(x,1) = \widetilde{f}(1,x) = 0$ for any $x \in G$).

2. Computations

Let L/K be a Galois extension and $G = \mathrm{Gal}(L,K)$. Then we have the famous

THEOREM 2.1 (Hilbert's theorem 90). $H^1(G, L^*) = (e)$.

PROOF. Let $f \in Z^1(G, L^*)$. By Dedekind's theorem elements of G are linearly independent over L. Hence there are elements $a, b \in L^*$ such that

$$\sum_{\tau \in G} f(\tau)\tau(b) = a.$$

By cocyle condition we have $f(\sigma\tau) = \sigma f(\tau) f(\sigma)$ so for any $\sigma \in G$

$$\sigma(a) = \sum_{\tau \in G} \sigma f(\tau)\sigma\tau(b) = \sum_{\tau \in G} f(\sigma\tau) f(\sigma)^{-1}\sigma\tau(b) = a f(\sigma)^{-1}.$$

Therefor e$f(\sigma) = \sigma(a^{-1}).a$ and hence f is a coboundary. □

COROLLARY 11. *Let L/K be a finite cyclic extension and σ be a generator of G. Let $a \in L^\times$ then $N_{L/K}(a) = 1$ if and only if there is $b \in L^\times$ such that $a = \sigma b/b$.*

PROOF. If $a = \sigma b/b$ then

$$N_{L/K}(a) = a.\sigma a.\cdots.\sigma^{n-1}a = 1$$

trivially. Conversely suppose $a \in L^\times$ with norm 1. Check that the map $\sigma \mapsto a$ can be extended to a 1-cocyle on G. But by Hilbert's theorem 90 then this is a coboundary. Hence there is $b \in L^*$ such that $f(\sigma) = \sigma b/b$. Therefore $a = (\sigma b)b^{-1}$. □

PROPOSITION 7. *If L/K is Galois and $G = \mathrm{Gal}(L/K)$ then for all $n \geq 1$ we have $H^n(G, L) = 0$.*

PROOF. There is a normal basis for L/K, i.e. there is $a \in L$ such that $\{a, \sigma a, \cdots\}$ forms a basis. Any $b \in L$ can hence be written as

$$b = \sum b_\sigma \sigma(a).$$

Then the map $L \to \mathrm{Hom}_Z(\mathbb{Z}[G], K)$ via $b \mapsto (\sigma \mapsto b_{\sigma^{-1}})$ is an isomorphism of G-modules (check this). Hence L is coinduced. Therefore $H^n(G, L) = 0$ for all $n \geq 1$. □

3. Maps in the level of cohomology

3.1. Inflation and restriction. Let G, G' be groups, M a G-module and M' is a G'-module. Let $f : G' \to G$ be a group homomorphism. Suppose $\varphi : M \to M'$ is a G'-homomorphism. Then (M, M') is said to be (f, φ)-compatible and we get a homomorphism $H^n(G, M) \to H^n(G', M')$ for all $n \geq 0$ from

$$(G')^n \xrightarrow{f^n} G^n \to M \xrightarrow{\varphi} M'.$$

Example 3.1. For $H \hookrightarrow G$ this is called the restriction homomorphism $H^n(G, M) \to H^n(H, M)$. $\lrcorner$

Example 3.2. If $H \subset G$ is a normal subgroup we have the quotient mapping $G \twoheadrightarrow G/H$. The action of G on M induces an action of G/H on M^H and we get homomorphisms

$$H^n(G/M, M^H) \to H^n(G, M).$$

There are called the inflation maps denoted by $\inf^n$. $\lrcorner$

The inflation and restriction maps are functorial and commute with the connecting homomorphism.

3.2. Corestriction. Let M be a G-module and let N be co-induced as

$$N = \mathrm{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], M).$$

Then $H^n(f) : H^n(G, M) \to H^n(G, N) \cong H^n(H, M)$ is the restriction map.

If $H \subset G$ is a subgroup of finite index, let $\{x_i : i \in I\}$ be a set of right coset representatives of H in G. We have a G-linear map

$$\mathrm{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], M) \to M$$

via $f \mapsto \sum_{i \in I} x_i^{-1} f(x_i)$. One can observe that this homomorphism is independent of the choice of representatives. It is moreover functorial and we get induced homomorphims

$$\begin{tikzcd}
H^n(G, \mathrm{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], M)) \arrow[r] \arrow[d, equal] & H^n(G, M) \\
H^n(H, M) \arrow[ur, "\text{cores}"'] &
\end{tikzcd}$$

called the corestriction map.

The corestriction commutes with the connecting homomorphisms. For $n = 0$ this is just the averaging map

$$\begin{aligned} M^H &\to M^G \\ m &\mapsto \sum x_i m. \end{aligned}$$

PROPOSITION 8. *Let G be a group and $H \subset G$ a subgroup. Let M be a G-module. The composite*

$$H^n(G,M) \xrightarrow{\text{res}} H^n(H,M) \xrightarrow{\text{cores}} H^n(H,M)$$

is multiplication by $k = [G : H]$.

Example 3.3 (Kummer theory)**.** Let L/F be a Galois field extension with Galois group $\mathrm{Gal}(L/F) = G$. Take $L = \overline{F}$ in characteristic zero and let $G = \mathrm{Gal}(\overline{F}/F)$. Let $\mu_n \subset \overline{F}$ be the Galois module consisting of n-th roots of 1. We have

$$0 \to \mu_n \to \overline{F}^\times \to \overline{F}^\times \to 1$$

where the sujrection is $x \mapsto x^n$. So

$$0 \to H^0(G,\mu_n) \to H^0(G,\overline{F}^\times) \to H^0(G,\overline{F}^\times) \to H^1(G,\mu_n) \to H^1(F,\overline{F}^\times) = 0$$

is the result of taking the long exact sequence. So we get

$$0 \to \mu_n(F) \to F^\times \xrightarrow{n} F^\times \to F^\times/(F^\times)^n.$$

Taking $n = 2$ we have

$$H^1(G_F,\mu_2) = F^\times/(F^\times)^2.$$

⌟

3.3. Cup product. Suppose G is a group and M,N are two G-modules. The $M \otimes_Z N$ is again a G-module. Then for all $p,q \geq 0$ there exists unique homomorphisms

$$\cup : H^p(G,M) \otimes H^q(G,N) \to H^{p+q}(G,M \otimes N).$$

such that

(1) are functorial in the sense that

$$\begin{tikzcd} H^p(G,M) \otimes H^q(G,N) \arrow[r, "\cup"] \arrow[d, "f\otimes g"'] & H^{p+q}(G,M\otimes N) \arrow[d, "f\otimes g"] \\ H^p(G,M') \otimes H^q(G,N') \arrow[r, "\cup"'] & H^{p+q}(G,M'\otimes N') \end{tikzcd}$$

is commutative.

(2) commutes with boundary, restriction, corestriction and inflation maps.

In fact explicitly the cup product can be expressed as follows: if $a : G^p \to M$ and $b : G^q \to N$ are elements are $H^p(G,M)$ and $H^q(G,N)$ respectively then

$$(a \cup b)(g_1,\cdots,g_{p+q}) = a(g_1,\cdots,g_p) \otimes b(g_{p+1},\cdots,g_{p+q}).$$

Example 3.4. Recall that $H^1(G_F, \mu_n) = F^\times/(F^\times)^n$. The special elements in $H^2(G_F, \mu_2)$ are of the form

$$\{\lambda_1, \lambda_2\}, \quad \lambda_i \in F^\times/(F^\times)^2.$$

These are called symbols in Galois cohomology. We have

$$K_1^M(F) = F^\times = H^0(G_F, \overline{F}^\times).$$

Thus we get a map $K_1^M(F) \to H^1(G_F, \mu_2)$ which is a homomorphism: $\lambda \mapsto [\lambda] \in F^\times/(F^\times)^2$. It follows that since $K_n^M(F)$ is generated by symbols $\{\lambda_1, \cdots, \lambda_n\}$ then $H^n(G_F, \mu_2)$ is generated by

$$[\lambda_1] \cup \cdots \cup [\lambda_n].$$

There is a theorem of Merkurjev showing that there is an isomorphism $K_2^M(F) \to H^2(G, \mu_2)$ induced as such and called the norm residue homomorphism. ⌟

PROPOSITION 9. *Let $H \subset G$ be a normal subgroup and A a G-module. If $H^i(H, A) = 0$ for $1 \le i \le n-1$ (no condition if $n = 1$) then the sequence*

$$0 \to H^n(G/H, A^H) \xrightarrow{\text{inf}} H^n(G, A) \xrightarrow{\text{res}} H^n(H, A)$$

is exact.

PROOF. By induction on n. Let $n = 1$ and let $[f] \in H^1(G/H, A^H)$ such that $\inf[f] = 0$. Then the cocyle $\inf f$ becaomes a coboundary, i.e. there is $a \in A$ such taht $f(\sigma) = \sigma a - a$ for all $\sigma \in G$. Since $f|_H = 0$ then $a \in A^h$ then $f \in B^1(G/H, A^H)$ and therefore the inflation is injective.

The composition $\text{res} \circ \inf = 0$. since $H \to G \to G/H$ is the zero map. Let $f \in Z^1(G, A)$ be such that $\text{res}(f) = 0$. Then $f(\sigma) = \sigma a - a$ for all $\sigma \in H$ and $a \in A$. Let $f_a : G \to A$ be $f_a(g) = ga - a$. Thus $[f]$ and $[f']$ are equal in $H^1(G, A)$ where $f' = f - f_a$. But $f'|_H = 0$ and hence f' induces a 1-cocyle: $f'' : G/H \to A^H$. Then

$$\inf[f''] = [f]$$

proving the exactness at $H^1(G, A)$. Suppose $n > 1$ and assume the proposition is true for $n - 1$. Consider the exact sequence

$$0 \to A \to A^\times \to A' \to 0$$

where $A^\times = \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)$. Taking the long exact sequence we get the short exact sequence

$$0 \to A^H \to (A^\times)^H \to (A')^H \to 0$$

of G/H-modules.

Now $A^\times$ is coinduced also as an A-module and $(A^\times)^H$ is coinduced as a G/H-module. Taking the long exact sequence associated to the above short exact sequence and comparing it to that of the quotient $0 \to G \to H \to G/H \to 0$ the isomorphism for $n \ge 2$ are all proved. $\square$

COROLLARY 12. *Let $A = L^\times$ and $G = \mathrm{Gal}(L/K)$ be finite. Let $H \subseteq G$ be a normal subgroup. We get*

$$0 \to H^2(G/H, F^\times) \xrightarrow{\text{inf}} H^2(G, L^\times) \xrightarrow{\text{res}} H^2(H, L^\times)$$

is exact.

REMARK. Just as in more general setups we have a Hochschild-Serre spectral sequence with page two

$$E_2^{p,q} = H^p(G/H, H^q(H, A))$$

which converges to $H^n(G, A)$. The construction is the same as any Grothendieck spectral sequence.

4. Profinite groups

Let K be a field and L a Galois extension, not necessarily finite. Let $G = \mathrm{Gal}(L/K)$. In this case G comes with a topology called the pro-finite topology. The opens are the normal subgroups corresponding to subfields of L that are finite and Galois over K. The fact that

$$L = \cup K_\alpha = \varinjlim K_\alpha$$

where K_α ranges over all intermediate field extensions is equivalence to

$$G = \varprojlim G/G_\alpha.$$

Also note that G is a profinite group in the sense that it is the inverse limit of an inverse system of finite groups. Then

$$G \subset \prod_\alpha G/G_\alpha$$

is a compact, totally disconnected, Hausdorff topological group if G_α's have discrete topology.

DEFINITION 10. Let G be a profinite gorup and A a G-module. Then A is a discrete G-module if $G \times A \to A$ is continuous when A is given the discrete topology.

This is equivalently the case when $A = \cup A^{G'}$ for G' ranging over open subgroups of G.

$$H_c^n(G, A) = H_c^n(\varprojlim G/H_\alpha, A) = \varinjlim H^n(G/H_\alpha, A) = \varinjlim H^n(G/H_\alpha, A^{H_\alpha}).$$