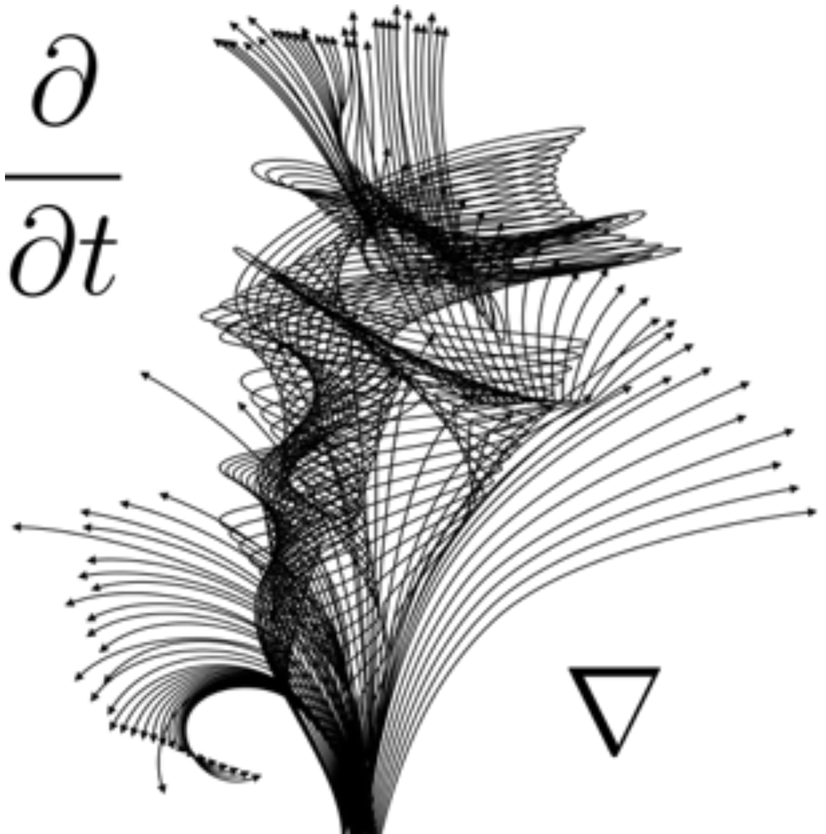
Differential Calculus with Applications to Life Sciences

Math 102:105

Pooya Ronagh

Agenda for today:

- Decays and Growths
- More differential equations

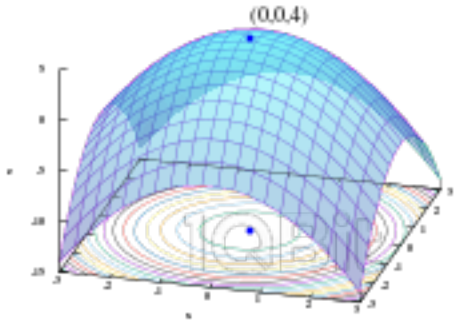


Doubling time

Remember the magic drop with volume 0.05mL.
The bowl of the Millennium Stadium with volume 1.5million metres cubed.

Calculate the exact time at which the entire Stadium is under water.

(A) 44.33 seconds
(B) 44.55 seconds
(C) 44.77 seconds
(D) 44.99 seconds



Doubling time

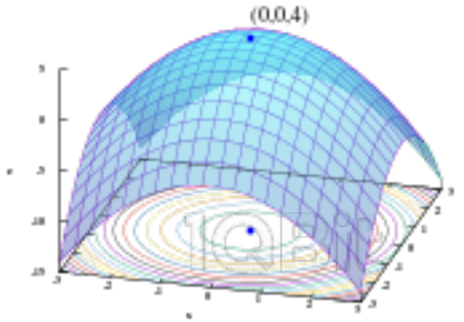
Suppose a phenomena has the trend of an exponential growth given by

$$c(t) = c_0 e^{kt}$$

Growth ⇒ k > 0

Question: How long does it take for c to be double the initial amount?

(A) 2 c0 (B) ln(k) (C) k ln (2) (D) ln(2) / k (E) 2 ln(2)



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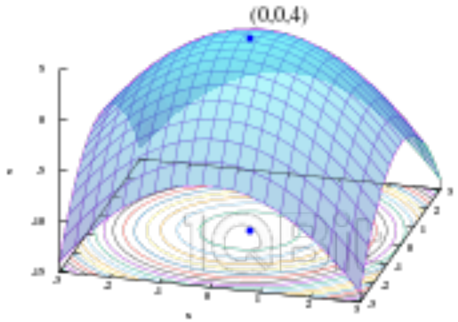


Half-life

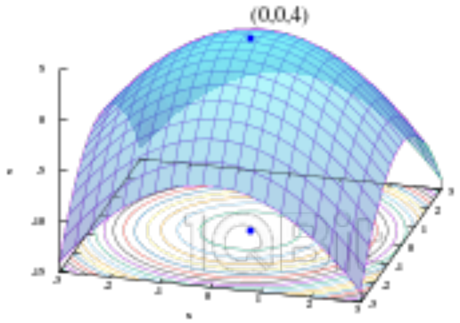
Suppose a phenomena has the trend of an exponential decay given by

$$c(t) = c_0 e^{kt}$$

Decay ⇒ $k < 0$

Question: How long does it take for c to be half the initial amount?

(A) 2 c0 (B) ln(-k) (C) -ln (2) /k (D) ln(2)/k (E) - ln(2)



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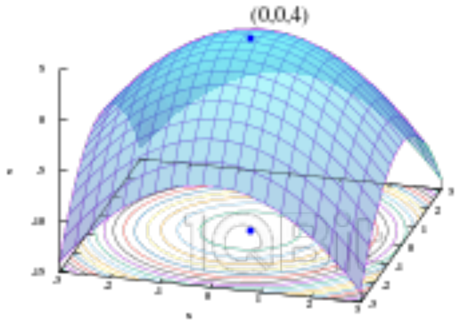


Mean-life

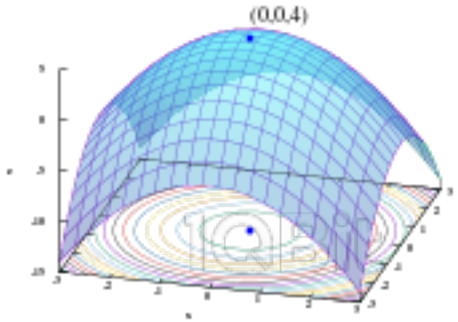
Suppose a phenomena has the trend of an exponential decay given by

$$c(t) = c_0 e^{kt}$$

Decay ⇒ $k < 0$

Question: How long does it take for c to be 1/e its original amount?

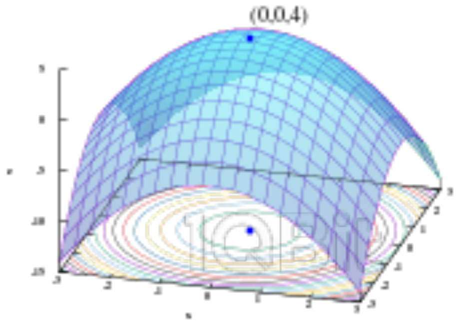
Mean-life = characteristic time = - 1 / k



Differential Equations (2nd encounter)

Last time…

A differential equation is a relationship between the function and its derivative.



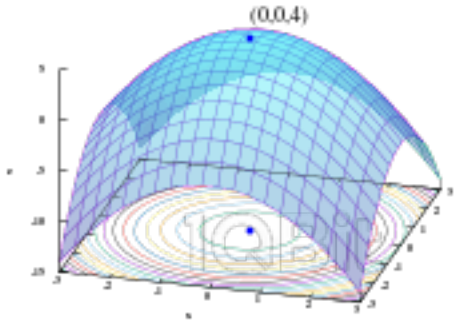
Differential Equations (2nd encounter)

Last time…

A differential equation is a relationship between the function and its derivative.

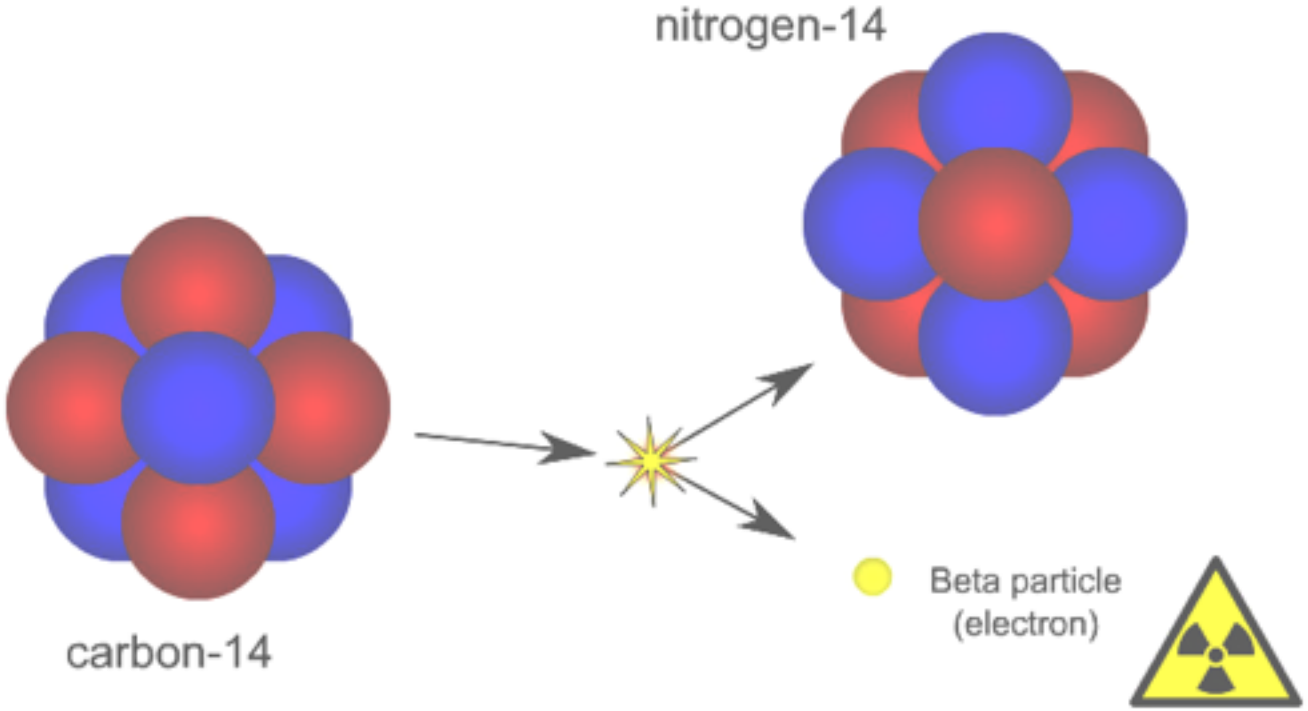
The differential equation $f'(x) = f(x)$, has a solution $f(x) = e^x$

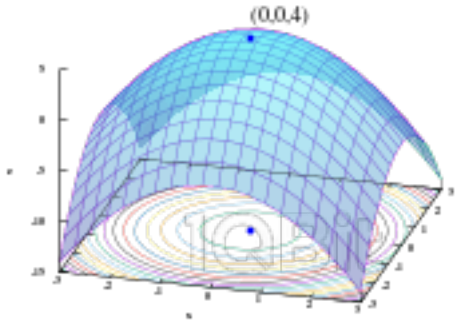
Question: What are other solutions to this DE?



Example (Carbon dating)

The amount of Carbon-14 in a sample decreases at a rate proportional to how much C-14 is present. Write a DE for the amount of C-14 at time t.





Example (Carbon dating)

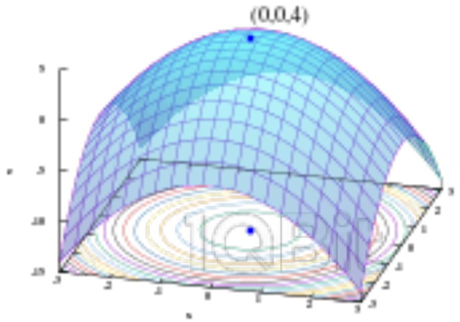
The amount of Carbon-14 in a sample decreases at a rate proportional to how much C-14 is present. Write a DE for the amount of C-14 at time t.

$$(A) \quad C'(t) = k\,C(t) \quad \text{where} \quad k > 0$$

$$(B) \quad C'(t) = k\,C(t) \quad \text{where} \quad k < 0$$

$$(C) \quad C(t) = C_0\,e^{kt}$$

$$(D) \quad C'(t) = C_0\,e^{-kt}$$



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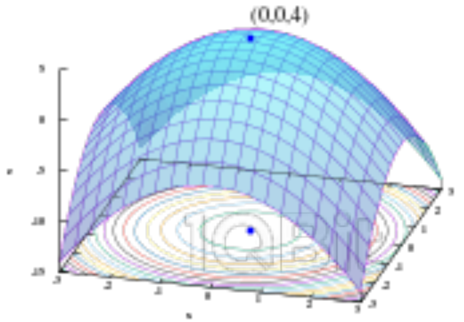
This one is a **solution** to the DE, not the DE itself!

$$(D) \quad C'(t) = C_0\,e^{-kt}$$



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$$(B) \quad C'(t) = -k\,C(t) \quad \text{where} \quad k > 0$$

also written as

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This one is a **solution** to the DE, not the DE itself!

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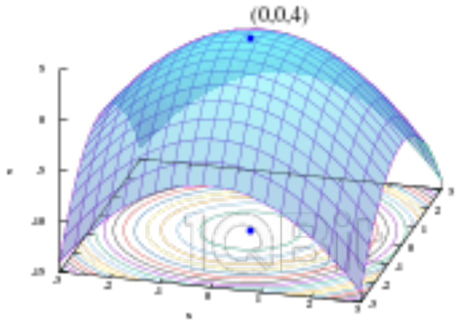
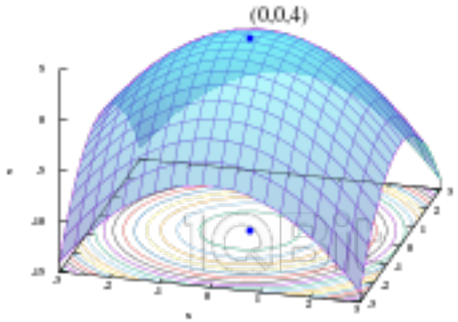


Example (Carbon dating)

The amount of Carbon-14 in a sample decreases at a rate proportional to how much C-14 is present. Write a DE for the amount of C-14 at time t.

Question: What is a solution to this DE if the original amount of Carbon-14 was 17 units?



Initial Value Problems

The amount of Carbon-14 in a sample decreases at a rate proportional to how much C-14 is present. Write a DE for the amount of C-14 at time t.

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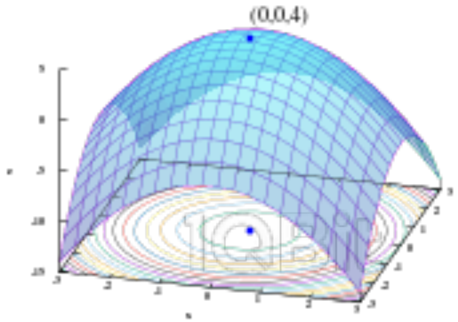
$$C(t) = 17e^{-kt}$$



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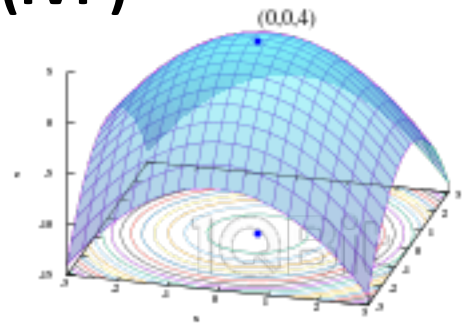
Differential equation(DE) + Initial Condition(IC) = Initial Value Problem (IVP)



You should be able to:

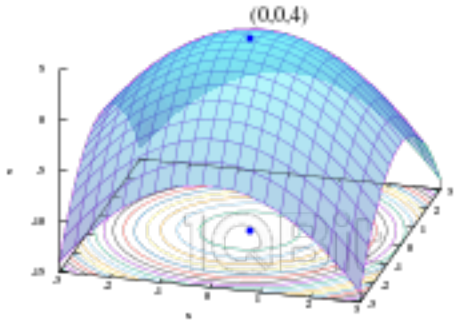
(1) Take a word problem of the form "Quantity blah changes at a rate proportional to how much blah there is" and write down the DE:

$$Q'(t) = kQ(t)$$

(2) Write down the solution to this equation:

$$Q(t) = Q_0 e^{kt}$$

(3) Determine k and Q_0 from given values or percentage of Q at two different times (i.e. data).
(4) Determine half-life/doubling time from data or k.

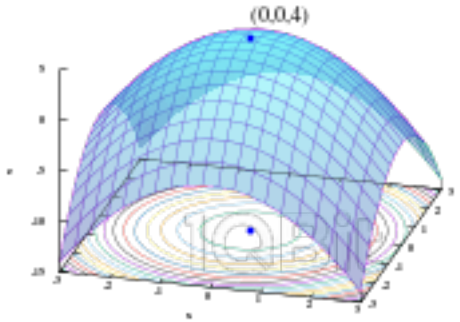


More on DEs

The differential equation

$$Q'(t) = kQ(t)$$

is called a **linear differential equation**.

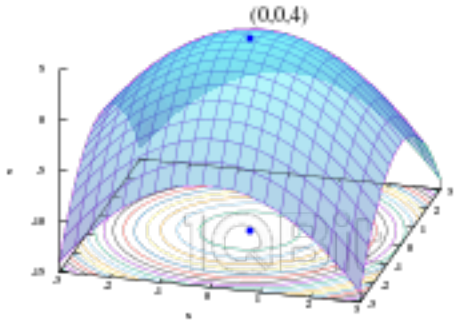


More on DEs

The differential equation

$$y' = ky$$

is called a **linear differential equation**.



More on DEs

The differential equation

$$y' = ky + a$$

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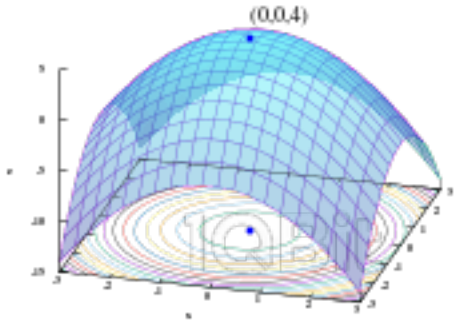

More on DEs

The differential equation

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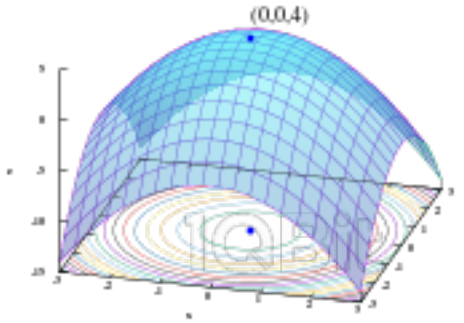
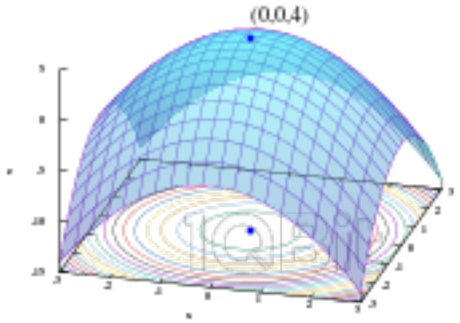
Some nonlinear differential equations:

$$y' = g - y^2 \qquad y' = -sin(y) \qquad (y')^2 = c$$



The logistic equation

$$\frac{dN}{dt} = bN - cN = (b-c)N$$



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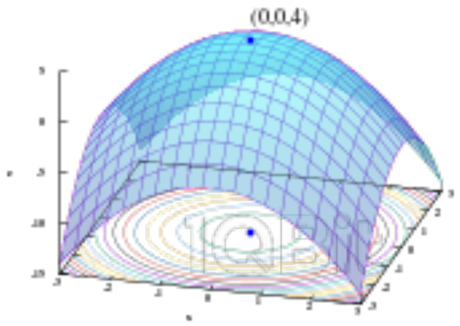
Constant birth rate

Constant death rate

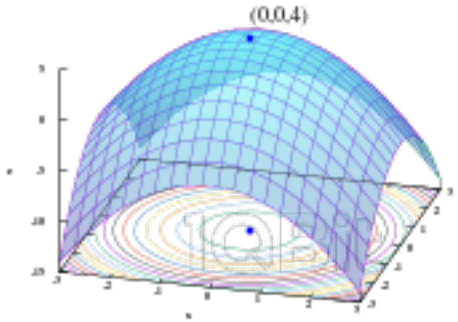


The logistic equation

$$\frac{dN}{dt} = bN - (cN)N$$

Constant birth rate

Death rate growing
as the population grows



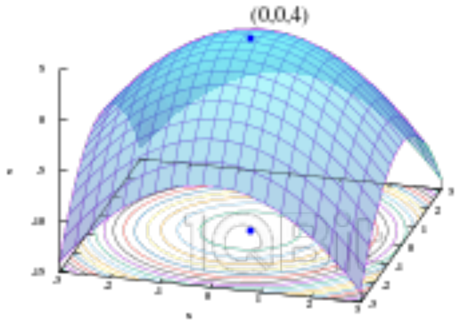
The logistic equation

$$\frac{dN}{dt} = bN - cN^2$$

Often re-written in the following form:

$$\frac{dN}{dt} = rN(1 - \frac{N}{K})$$

Think of this as a model for the population of species in a bounded environment.

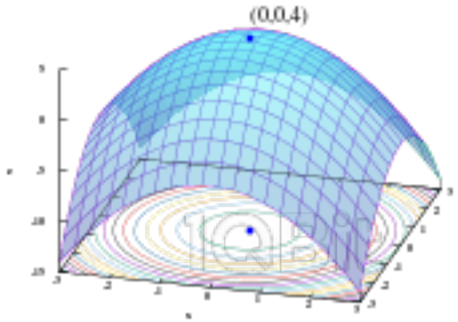


Qualitative analysis

- Finding a formula for a solution to a DE is great but not always an easy task
  - With f' = f we got lucky!

- Qualitative analysis = extract information about the general solution without solving
  - general solution = when we don't restrict ourselves to an IV

- Useful tools:
  - Slope fields
  - Plotting y' versus y (state space/phase line)

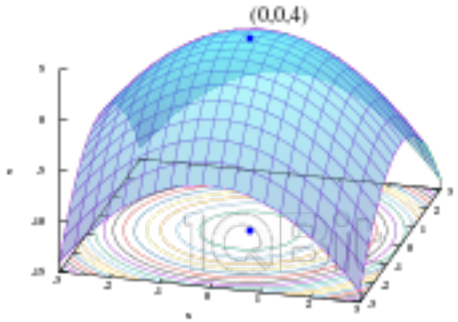


Steady states

Consider the DE

$$x' = x(1 - x)$$

expressing a law of motion for a particle (so x = position, x’ = velocity)



Steady states

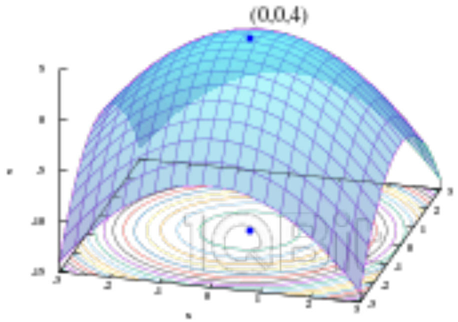
Consider the DE

$$x' = x(1 - x)$$

expressing a law of motion for a particle (so x = position, x' = velocity)

In what position will the particle stay still?

(A) x = 0 (B) x = -1 (C) x = 1 (D) x = 1/2



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Steady state

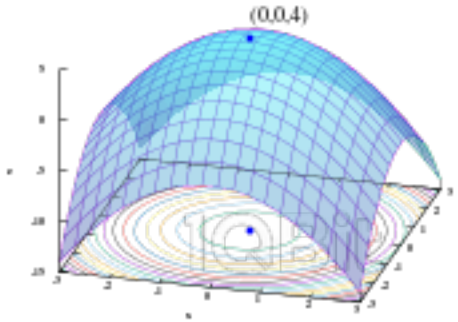
(A) x = 0

(B) x = -1

Steady state

(C) x = 1

(D) x = 1/2



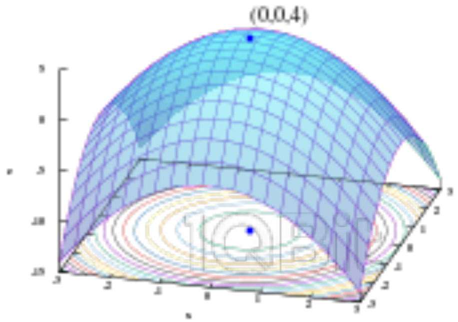
Steady states

Similarly: what are the steady states of:

$$x' = -x(x-1)(x+1)$$

Similarly: at what populations does the population become steady?

$$\frac{dN}{dt} = bN - cN^2$$



Stable and unstable

$$x' = x(1 - x)$$

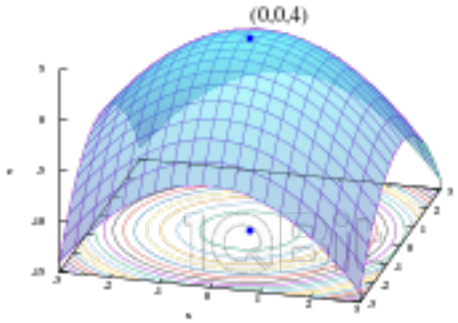
Steady state

x = 0

Steady state

x = 1

Steady states can be **stable** or **unstable**.



Stable and unstable

$$x' = x(1-x)$$

Steady state

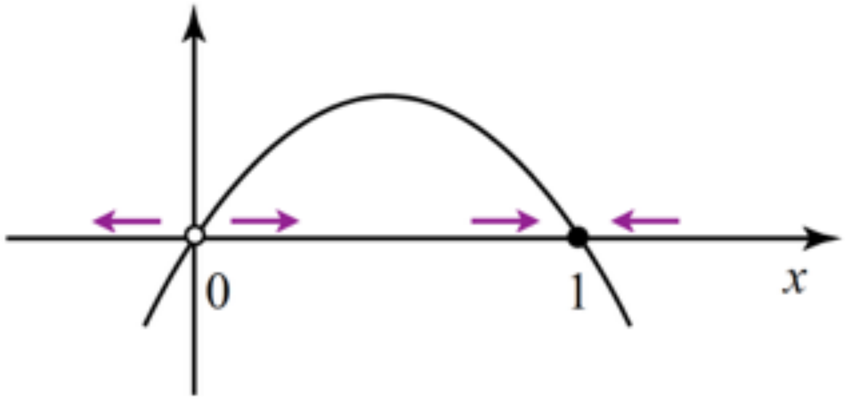
x = 0

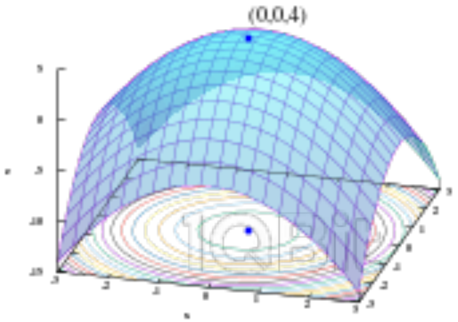
Unstable

Steady state

x = 1

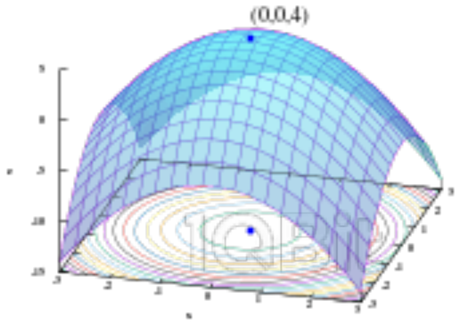
Stable



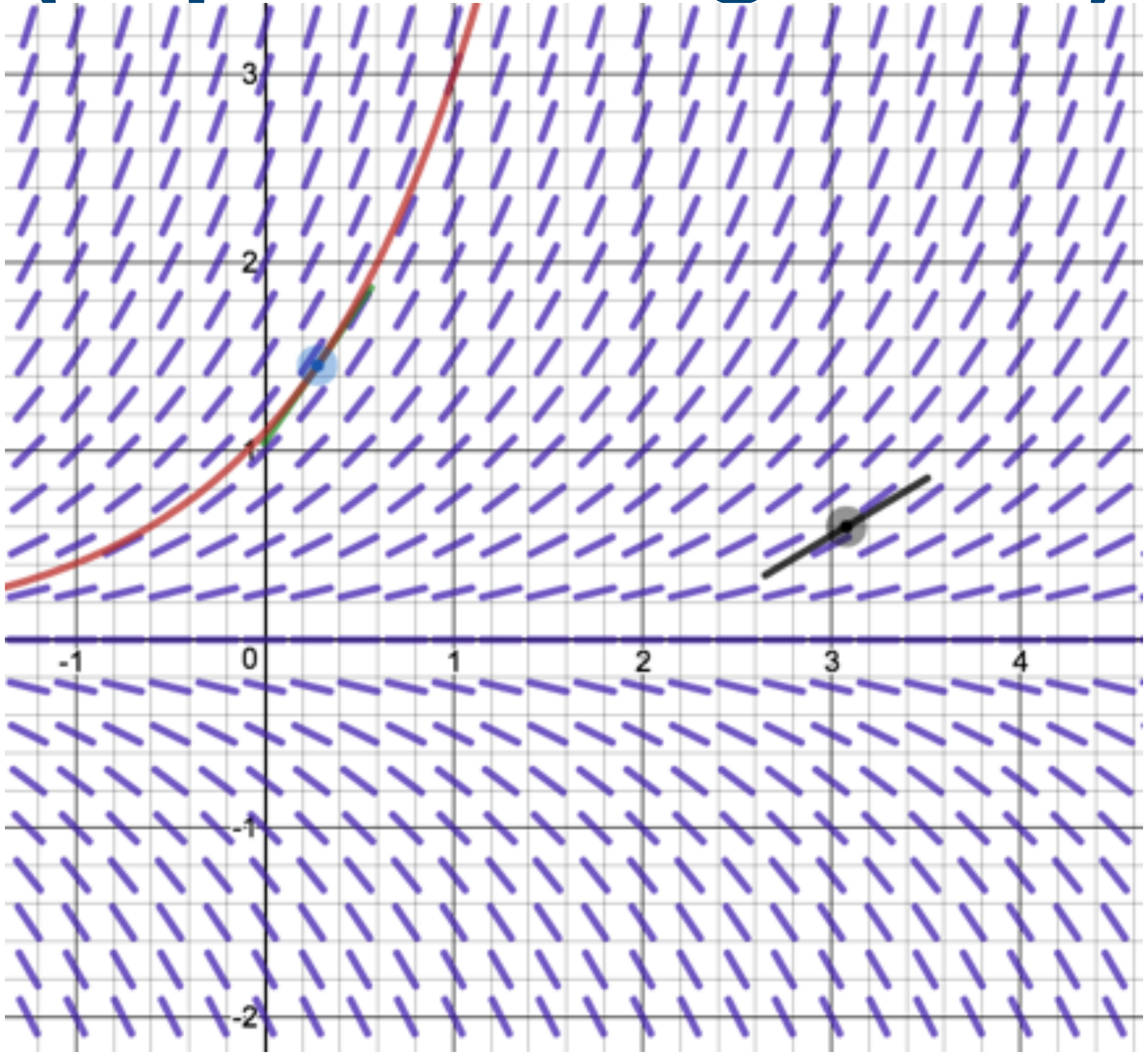


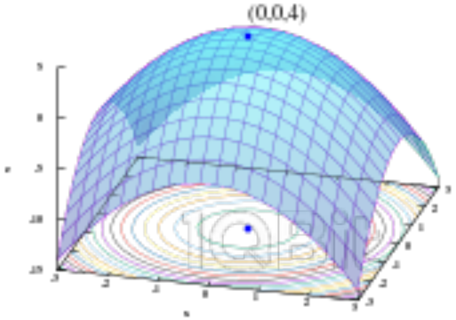
Slope field

Let's look at our simple DE again: $f' = f$

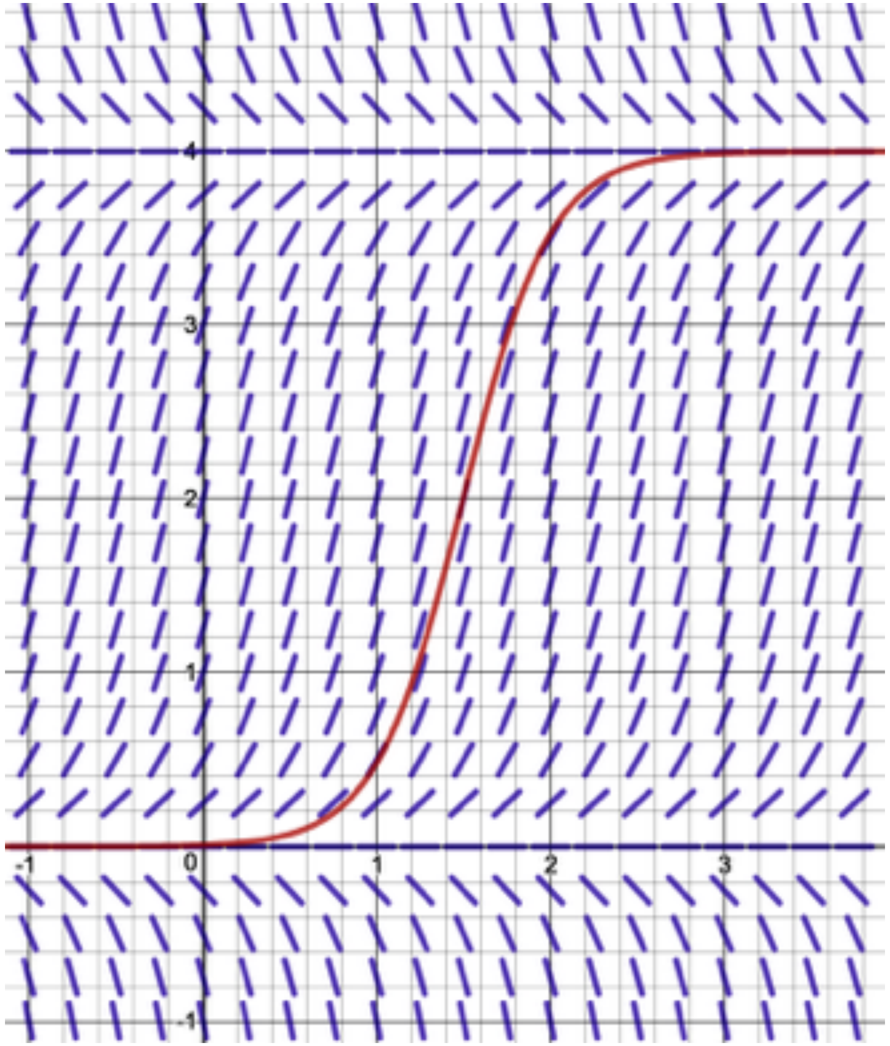


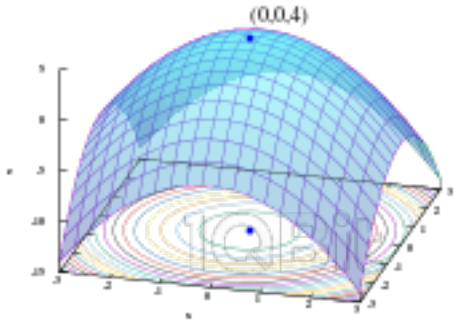
Slope field (exponential growth)





Slope field (logistic equation)





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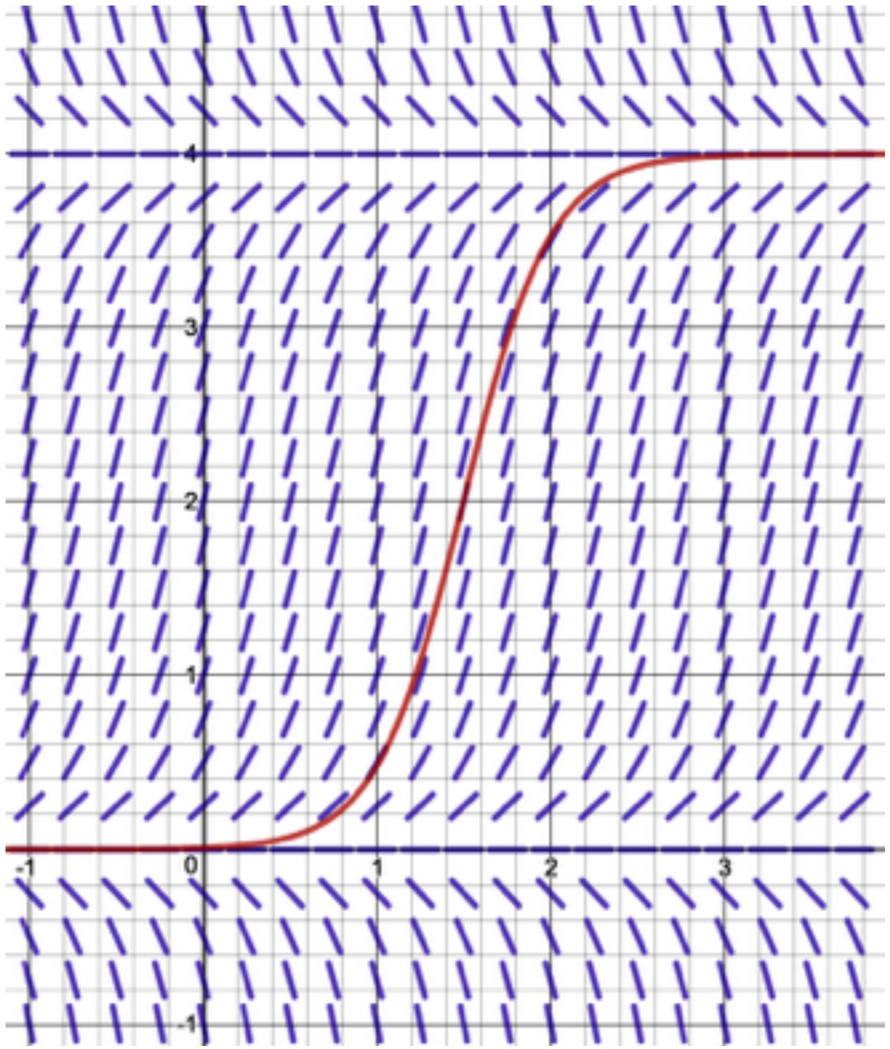

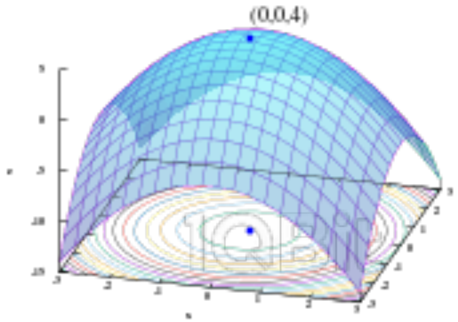

See you next week!

Nov 3	WW 8
Nov 7	PL10.1
Nov 9	PL10.2