Today

- $\ln(x)$ as inverse function for e^x.
- Derivative of $\ln(x)$ aka $\log(x)$.
- Derivative of a^x.
- Converting between a^x and e^{kx}.

Derivative of $f(x)=a^x$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{a^{x+h} - a^x}{h}$$

$$= a^x \lim_{h\to 0} \frac{a^h - 1}{h} = C_a \, a^x$$

$$\lim_{h\to 0} \frac{a^h - 1}{h} \approx ?? \qquad = C_a$$

Find a special value of a.

- When is $C_a = \lim_{h \to 0} \frac{a^h - 1}{h} = 1$?

Want $\frac{a^h - 1}{h} \approx 1$ (for h small)

$$a^h - 1 \approx h$$

$$a^h \approx h + 1$$

$$a \approx (h+1)^{\frac{1}{h}}$$

With h=0.00001, a≈2.71826823719.

What is this special a value? a=e!

We just found a function that is its own derivative! $f(x)=e^x$.

$$f'(x) = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h}$$

1

$$= e^x \lim_{h\to 0} \frac{e^h - 1}{h}$$

$$= e^x$$

This is precisely how e is defined - the number whose exponential function is its own derivative.

Differential equations

- What real number is the same as its own square?
  - Equivalent to asking “what x satisfies the equation $x=x^2$?”
  - Call this an algebraic equation.
- What function is equal to its own derivative?
  - Equivalent to asking “what f(x) satisfies $f'(x)=f(x)$?”
  - Call this a “differential equation”. (DE)

DE example: Which of the following satisfies $f'(x)=f(x)$?

(A) $f(x) = 2^x$

(B) $f(x) = e^x$

(C) $f(x) = x^{-1}$

(D) $f(x) = -x^{-1}$

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A note about units

- e is a “pure” number without units, called dimensionless.
- This means e^a for any a is also dimensionless.
- Furthermore, the exponent, a, must also be dimensionless.
- If $y(t) = y_0e^{-kt}$, and t is time in seconds, what must be the units of k?

What is the definition of the inverse function of $f(x)$?

(A) The function $g(x)$ for which $g(f(x))=x$.

(B) The mirror image of graph of $f(x)$ in the line $y=x$.

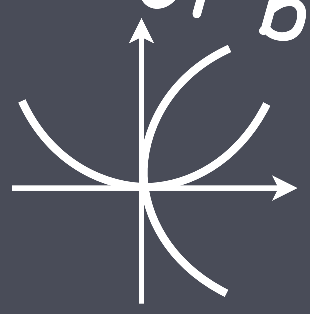
(C) $1/f(x)$

(D) $-f(x)$

What is the definition of the inverse function of f(x)?

(A) The function g(x) for which g(f(x))=x. $g(x) = f^{-1}(x)$

(B) The mirror image of graph of f(x) in the line y=x. <---might not be a function.



(C) 1/f(x)

(D) -f(x)

$f^{-1}(x)$ is the function that goes backwards through f(x). If you plug the output of f(x) into $f^{-1}(x)$, you will get back to x.

Let $f(x)=e^x$. Define $\ln(x)$ to be $f^{-1}(x)$.

Which of the following is false?

(A) If $a=e^b$ and $c=e^d$ then $\ln(a/c) = b-d$.

(B) If $a=e^b$ and $c=e^d$ then $\ln(a-c) = b/d$.

(C) If $c=a^d$ then $\ln(c) = d \ln(a)$.

(D) If $a=e^b$ and $c=a^d$ then $\ln(c) = bd$.

(E) If $a=e^b$ and $c=e^d$ then $\ln(ac) = b+d$.

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$\ln(x)$ = natural logarithm of x = $\log_e(x)$)

Derivative of ln(x)

- If $y = \ln(x)$ then $e^y = e^{\ln(x)} =$

(A) 1

(B) x

(C) 1/x

(D) e

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(A) 1

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(D) e

Derivative of $\ln(x)$

- If $y = \ln(x)$ then $e^y = e^{\ln(x)} = f(f^{-1}(x)) = x$.
- Implicit differentiation:

(A) $e^{y'} = 1$

(B) $e^y y' = 1$

(C) $e^y = x'$

(D) $ye^{y-1} = 1$

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$g(x) = \ln(x)$

--> $g'(x) = 1/x$

- Solve for y': $y' = e^{-y} = 1/x$

$f(x)=a^x$. $f'(x)=C_a a^x$. C_a=??

- Recall that we got stuck on this derivative.
- Time to get unstuck...

$$f(x) = e^{\ln(2)x}.$$

(A) $f'(x) = e^{\ln(2)x}$.

(B) $f'(x) = \ln(2)e^{\ln(2)x}$.

(C) $f'(x) = \ln(2)\cdot 1/2\cdot e^{\ln(2)x}$.

(D) $f'(x) = \ln(2)xe^{\ln(2)x-1}$.

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$$f(x) = e^{\ln(2)x}.$$

(A) $f(x) = 2x.$

(B) $f(x) = (e^{\ln(2)})^x = 2^x.$

(C) $f(x) = e^{\ln(2)}\, e^x = 2e^x.$

(D) $f(x) = e^{\ln(x^2)} = x^2.$

$$f(x) = e^{\ln(2)x}.$$

(A) $f(x) = 2x.$

(B) $f(x) = (e^{\ln(2)})^x = 2^x.$

(C) $f(x) = e^{\ln(2)}\, e^x = 2e^x.$

(D) $f(x) = e^{\ln(x^2)} = x^2.$

From the last two clicker Qs…

- $f(x) = e^{\ln(2)x}$ --> $f'(x) = \ln(2)e^{\ln(2)x}$.
- $f(x) = e^{\ln(2)x}$ --> $f(x) = 2^x$.
- So $f(x) = 2^x$ --> $f'(x) = 2^x \ln(2)$.
- In general, $f(x) = a^x$ --> $f'(x) = a^x \ln(a)$.

What value of k makes $a^x = e^{kx}$?

(A) $k=e^a$

(B) $k=e^{-a}$

(C) $k=\ln(a)$

(D) $k=-\ln(a)$

(E) $k=\ln(-a)$

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$$a^x = (e^k)^x$$
$$a = e^k$$
$$\ln(a) = \ln(e^k)$$
$$\ln(a) = k\ \ln(e)$$
$$\ln(a) = k$$

$f(x) = a^x = e^{\ln(a)x}$

--> $f'(x) = a^x \ln(a)$.

Which of following is the graph of ln(x)?

Which of following is the graph of ln(x)?

(A)

(B)

(C)

(D)