Misspecified Bayesianism

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Abstract

An agent is a misspecified Bayesian if she updates her belief using Bayes' rule given a subjective, possibly misspecified model of her signals. This paper shows that a belief sequence is consistent with misspecified Bayesianism if the prior *contains a grain* of the average posterior, i.e., is a mixture of the average posterior and another distribution. A partition-based variant of the grain condition is both necessary and sufficient. Under correct specification, the grain condition reduces to the usual Bayes plausibility. The condition imposes no restriction on the posterior given a full-support prior over a finite or compact state space. However, it rules out posteriors that have heavier tails than the prior on unbounded state spaces. The results cast doubt on the feasibility of testing Bayesian updating in many environments. They also suggest that many seemingly non-Bayesian updating rules are observationally equivalent to Bayesian updating under misspecified beliefs.

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1 Introduction

Following the treatise of Savage (1972), the Bayesian theory of probability has become the dominant paradigm in the modeling of decision making under uncertainty. This paradigm's dominance in economics is not unwarranted. It allows one to assign probabilities to unique or rare events; it has an elegant foundation in the study of rational choice under uncertainty; and it is appealing from a normative perspective—as Epstein and Le Breton (1993) proclaim, "dynamically consistent beliefs must be Bayesian."

Whether Bayesian updating is an accurate positive model of how individuals actually revise their views is a different matter. A large body of evidence documents violations of Bayesian updating (e.g., Coibion and Gorodnichenko 2015; Bordalo, Gennaioli, Ma, and Shleifer 2020). However, standard tests of Bayes' rule are joint tests of Bayesian updating and the assumption that agents have correctly specified models of the data-generating process. Therefore, any rejection by those tests could be due to non-Bayesian updating, misspecification, or a combination thereof. A natural question is then what restrictions (if any) Bayesian updating imposes on dynamics of beliefs in isolation.

This paper provides an answer. An agent is a *misspecified Bayesian* if she updates her belief using Bayes' rule given an internally consistent but possibly misspecified model of the data she observes. The paper shows that a belief sequence is consistent with misspecified Bayesianism if the prior contains a "grain" of the average posterior. This condition is a relaxation of the Bayes plausibility condition (Kamenica and Gentzkow, 2009) that characterizes the behavior of a "correctly specified Bayesian." While Bayes plausibility requires the prior to equal the average posterior, the grain condition requires the prior to be a mixture of the average posterior with some other probability distribution. The paper also shows that a slightly more permissive, partition-based grain condition is both necessary and sufficient for consistency with misspecified Bayesianism.

The strength of the grain condition is highly dependent on the environment. When the state space is finite, misspecified Bayesianism imposes only a support inclusion requirement: The support of the average posterior must be contained in the support of the prior. If the prior has full support on a finite space, any distribution of posteriors can be rationalized. When the state space is compact and the prior and average posterior admit continuous densities, the same support restriction is again both necessary and sufficient. The picture changes sharply if the state space is unbounded. There the grain condition rules out any set of posteriors whose tails are uniformly heavier than those of the prior. In fact, a single heavier-tailed posterior that is realized with positive probability suffices for rejection.

These findings may appear at odds with the existing results in the literature. Kamenica and Gentzkow (2009) argue that Bayesian updating requires the average posterior to equal the prior. Shmaya and Yariv (2016) argue that any belief sequence in which the prior is in the relative interior of the convex hull of posteriors is consistent with agents' use of Bayes' rule. This paper states two additional theorems to clarify the relationship between these different conditions. The theorems adapt the existing results to general state spaces, thus making them directly comparable to this paper's results. They demonstrate that the earlier results characterize Bayesian updating only under additional assumptions on agents' subjective models. Kamenica and Gentzkow (2009) do so by requiring agents to have correct beliefs about the distribution of signals, whereas Shmaya and Yariv (2016) require the subjective belief to have the same support as the true distribution.

The paper's results have two important implications. First, they suggest that Bayesian updating is essentially unfalsifiable when priors are supported over finite or compact spaces. The only way to test Bayesian updating without invoking extra assumptions about agents' models is by comparing the tails of the prior and posterior over unbounded state spaces. Second, many non-Bayesian updating rules turn out to be observationally equivalent to Bayesian updating with misspecified models. For example, agents with diagnostic expectations (Bordalo, Gennaioli, and Shleifer, 2018) overweight representative states and violate Bayes' rule at face value. Yet they behave *as if* they were Bayesians who believed the signal was less noisy and its noise term was negatively correlated with the state.

The paper thus bridges two strands of work on departures from rational expectations. The first one preserves Bayesian updating but allows misspecification, e.g., Esponda and Pouzo (2016, 2021), Bohren (2016), Frick, Iijima, and Ishii (2020), Fudenberg, Lanzani, and Strack (2021), and Molavi, Tahbaz-Salehi, and Vedolin (2024, 2025). The second strand posits non-Bayesian heuristics, e.g., Tversky and Kahneman (1974), Rabin and Schrag (1999), Epstein, Noor, and Sandroni (2010), and Cripps (2019)—see Ortoleva (2022) for a recent survey. The results suggest that these two types of deviation are empirically hard to disentangle without information such as agents' forecasts of their own future beliefs. In contemporaneous work, Bohren and Hauser (2023) study the question of when non-Bayesian updating rules can be represented as misspecification. They extend Shmaya and Yariv (2016)'s analysis by making agents' forecasts of their future beliefs observable and deriving necessary and sufficient conditions for an updating rule and a forecast to have a misspecified-model representation. In contrast, this paper's focus is characterizing misspecified Bayesianism absent information on agents' beliefs about how they will update their beliefs.

The grain condition is borrowed from the literature on merging of opinions. The merging literature is concerned with the question of whether Bayesian learning can lead agents to forecast the future accurately (i.e., merge to truth) or to play Nash equilibrium strategies. Blackwell and Dubins (1962) show that the absolute continuity of the prior with respect to the true distribution is sufficient to ensure merging. Kalai and Lehrer (1993) introduce the notion of containing a "grain of truth" as a stronger absolute continuity notion that guarantees convergence to an approximate Nash equilibrium in repeated games of incomplete information. While the merging literature studies the long-run convergence of correctly specified Bayesian learners, this paper's focus is on the finite-horizon consistency of belief sequences with Bayesian updating under potentially misspecified models.

The remainder of the paper is organized as follows: Section 2 introduces the paper's conceptual framework and formally defines misspecified Bayesianism. Section 3 presents the main result, examines several special cases and extensions, and discusses the relationship to other related notions in the literature. Section 4 illustrates the theoretical results of the paper in the context of three examples. Section 5 discusses the implications of the results. The proofs are relegated to the appendix.

2 Conceptual Framework

This section introduces the paper's conceptual framework and defines what it means for a belief sequence to be consistent with misspecified Bayesianism.

2.1 Setup

There is a fixed state of the world. The state is denoted by x and belongs to a separable metric space X.[1]

[1] I endow X with the Borel σ-algebra $\mathcal{X} = \mathcal{B}(X)$ and use $\Delta(X)$ to denote the set of probability distributions over $(X, \mathcal{B}(X))$. Since X is a separable metric space, so is $\Delta(X)$. I let $\Delta(\Delta(X))$ denote the set of probability distributions over $(\Delta(X), \mathcal{B}(\Delta(X)))$, where $\mathcal{B}(\Delta(X))$ denotes the Borel σ-algebra of $\Delta(X)$.

The main objects of interest are a prior and a "posterior" about x.[2] The prior is a probability distribution over X denoted by $\mu_0^* \in \Delta(X)$. The posterior is a random variable $\mu_1 \in \Delta(X)$ with distribution $F_1^* \in \Delta(\Delta(X))$. The randomness of the posterior is due to its dependence on a random signal. I denote the signal by s and assume (without loss of generality) that it belongs to the set $S \equiv \Delta(X)$. The *true* distribution of signals given the fixed state of the world is denoted by $\mathbb{P} \in \Delta(S)$.[3]

2.2 Mathematical definitions

The paper's goal is to characterize the conditions under which we can interpret the pair (μ_0^*, F_1^*) as arising from Bayesian updating given some possibly misspecified subjective model of how the signal is generated. This requires that (i) there exists a well-defined subjective distribution over the set of state-signal pairs; (ii) the subjective distribution assigns positive probability to signals that occur with positive probability under the true distribution; and (iii) the distribution is updated following each signal that occurs with positive probability using Bayes' rule. I now formalize each requirement in turn.

The first two requirements can be formalized in a straightforward manner: First, there needs to be a subjective distribution $\mathbb{Q} \in \Delta(X \times S)$ over the set of states and signals. Second, the true distribution of signals $\mathbb{P}$ needs to be absolutely continuous with respect to the S-marginal $\mathbb{Q}_S$. When $\mathbb{P}$ has finite support, this condition reduces to the requirement that $\mathbb{Q}$ does not rule out any signal s such that $\mathbb{P}(\{s\}) > 0$. This requirement ensures that Bayes' rule is applicable following every contingency.[4]

The third requirement is that the subjective distribution is updated using Bayes' rule given the signal. This criterion can be formally expressed through the concept of regular conditional probability. Given the measurable space $(X \times S, \mathcal{X} \times \mathcal{S})$ and probability distribution $\mathbb{Q} \in \Delta(X \times S)$, a *regular conditional probability* of $\mathbb{Q}$ given $\mathcal{S}$ is a mapping $\nu : S \times \mathcal{X} \to [0,1]$ such that (i) $\nu(s, \cdot)$ is a probability distribution on X for every $s \in S$, (ii)

[2] I use the term "posterior" to refer to any probability distribution obtained by updating the prior after observing new information, regardless of whether it is derived from the prior via Bayes' rule—hence the quotation marks.

[3] I endow S with the Borel σ-algebra $\mathcal{S} = \mathcal{B}(S)$ and use $\Delta(S)$ to denote the set of probability distributions over $(S, \mathcal{B}(S))$.

[4] Although this requirement is not necessary for the paper's main finding, having a more demanding notion of Bayesianism strengthens the result by highlighting the fact that the conclusion does not rely on the inapplicability of Bayes' rule after zero-probability events. It also allows me to turn the statement of the main result into an "if and only if" statement.

the mapping $s \mapsto \nu(s, D)$ is measurable for all $D \in \mathcal{X}$, and (iii) the kernel ν satisfies

$$\mathbb{Q}(D \times E) = \int_E \nu(s, D)\mathbb{Q}_S(ds) \tag{1}$$

for all $D \in \mathcal{X}$ and $E \in \mathcal{S}$, where $\mathbb{Q}_S$ is the S-marginal of $\mathbb{Q}$. The regular conditional probability ν defines mapping $\varphi : s \mapsto \nu(s, \cdot)$ from signals to posteriors. Bayesian updating given subjective distribution $\mathbb{Q}$ requires the prior to be updated using this mapping.

The following example shows that regular conditional probability reduces to the usual conditional probability for positive-probability signals:

Example 1. Suppose $\mathbb{Q}(\{s\}) > 0$ for some $s \in S$. Then equation (1) implies that

$$\mathbb{Q}(D \times \{s\}) = \int_{\{s\}} \nu(s, D)\mathbb{Q}_S(ds) = \nu(s, D)\mathbb{Q}(\{s\}).$$

Therefore,

$$\nu(s, D) = \frac{\mathbb{Q}(D \times \{s\})}{\mathbb{Q}(\{s\})}.$$

Regular conditional probability ν determines the posterior as a function of the realized signal. However, it does not specify the distribution of posteriors. In particular, for any event $D \in \mathcal{X}$, $\nu(s, D)$ is a random variable whose distribution depends on the distribution of the signal.

To determine the probability with which each posterior is realized, one needs to use the true distribution of signals. Given a regular conditional probability ν and the true distribution of signals $\mathbb{P}$, the posterior about the state x is distributed according to the probability distribution $F_{\nu,\mathbb{P}} \in \Delta(\Delta(X))$, defined as

$$F_{\nu,\mathbb{P}}(E) \equiv \mathbb{P}\left(\{s \in S : \nu(s, \cdot) \in E\}\right) \tag{2}$$

for all $E \in \mathcal{S}$. This is the observed distribution of posteriors when agents who hold subjective distribution $\mathbb{Q}$ (with the corresponding ν) observe signals distributed according to $\mathbb{P}$ and update their priors using Bayes' rule.

2.3 Misspecified Bayesianism

I can now define what it means for a belief sequence to be consistent with misspecified Bayesianism.

Definition 1. Given the true distribution of signals $\mathbb{P} \in \Delta(S)$, a pair (μ_0^*, F_1^*), consisting of a prior and a distribution for posteriors, is *consistent with misspecified Bayesianism* if there exists a subjective distribution $\mathbb{Q} \in \Delta(X \times S)$ that satisfies the following conditions:

(a) $\mathbb{Q}_X = \mu_0^*$,

(b) $\mathbb{P}$ is absolutely continuous with respect to $\mathbb{Q}_S$,

(c) $\mathbb{Q}$ has a regular conditional probability ν such that $F_{\nu,\mathbb{P}} = F_1^*$,

where $\mathbb{Q}_X$ and $\mathbb{Q}_S$ are the X- and S-marginals of the subjective distribution $\mathbb{Q}$, respectively, and $F_{\nu,\mathbb{P}}$ is the distribution of posteriors defined in (2).

This definition formalizes the intuitive notion of misspecified Bayesianism laid out earlier. Rationalizing a belief sequence requires finding a subjective distribution $\mathbb{Q}$ that explains the observed changes in beliefs. Such a conjectured $\mathbb{Q}$ is a joint distribution for the state and signal that must satisfy three conditions: Condition (a) of Definition 1 simply requires the conjectured distribution to be consistent with the prior. Condition (b) is the requirement that the subjective distribution assigns positive probability to signals that are realized with positive probability. Condition (c) requires that the distribution of posteriors F_1^* matches the distribution obtained when starting with the conjectured distribution $\mathbb{Q}$, observing signals as per $\mathbb{P}$, and updating the subjective distribution using Bayes' rule.

A minimal requirement for (μ_0^*, F_1^*) to be consistent with Bayesianism given $\mathbb{P}$ is that $F_1^* = \mathbb{P} \circ \varphi^{-1}$ for some measurable mapping $\varphi : S \to \Delta(X)$. If no such φ existed, no updating rule—Bayesian or non-Bayesian—could generate a distribution F_1^* of posteriors based on a signal distributed according to $\mathbb{P}$. To rule out such uninteresting cases, in the remainder of the paper, I assume that F_1^* and $\mathbb{P}$ can be linked via $F_1^* = \mathbb{P} \circ \varphi^{-1}$ for *some* (possibly unknown) mapping φ. The question then is whether there exists an updating rule φ that corresponds to Bayes' rule given a subjective distribution $\mathbb{Q}$.

3 Results

The paper's main result establishes a necessary and sufficient condition for a (μ_0^*, F_1^*) pair to be consistent with misspecified Bayesianism. This section presents this characterization result, examines several special cases and extensions of the result, and discusses the relationship to the notion of Bayes plausibility.

3.1 Preliminaries

Before presenting the result, I introduce two definitions that are used in its statement. The first is the following:

Definition 2. Given a measurable subset E of the set of posteriors $\Delta(X)$ with $F_1^*(E) > 0$, *average posterior over* E is the probability distribution over X defined as

$$\overline{\mu}_{1|E}^*(D) \equiv \frac{\int_E \mu(D) F_1^*(d\mu)}{F_1^*(E)}$$

for any measurable set D. I refer to $\overline{\mu}_1^* \equiv \overline{\mu}_{1|\Delta(X)}^*$ as the *average posterior*.

That $\overline{\mu}_{1|E}^*$ is a probability distribution over X for any set E with $F_1^*(E) > 0$ is easy to see: $\overline{\mu}_{1|E}^*(D) \in [0,1]$ for any D, $\overline{\mu}_{1|E}^*(\emptyset) = 0$ for the empty set $\emptyset$, $\overline{\mu}_{1|E}^*(X) = 1$ for the entire space X, and $\overline{\mu}_{1|E}^*$ is countably additive.

The paper's main result establishes that a (μ_0^*, F_1^*) pair is consistent with misspecified Bayesianism if and only if the support of F_1^* can be partitioned into a collection of measurable cells such that the average posterior over a cell of positive F_1^* measure satisfies an absolute continuity condition with respect to the prior. The appropriate absolute continuity notion is the following:

Definition 3 (Kalai and Lehrer, 1993)**.** For probability distributions P and Q defined over the same measurable space, P *contains a grain of* Q if $P = \epsilon Q + (1-\epsilon)Q'$ for some $\epsilon \in (0,1]$ and some probability measure Q'.

The following proposition states two alternative definitions, which are equivalent to Definition 3:

Proposition 1. *For probability distributions P and Q defined over the same measurable space, the following are equivalent:*

(i) P contains a grain of Q.

(ii) The Radon–Nikodym derivative $f \equiv \frac{dQ}{dP}$ exists and is bounded P-almost surely.

(iii) There exists a constant $c \geq 1$ such that $Q(E) \leq cP(E)$ for any measurable set E.

The proposition illustrates that the grain condition is stronger than absolute continuity. Q is absolutely continuous with respect to P if $Q(E) > 0$ for any event E for which $P(E) > 0$, whereas P contains a grain of Q when the ratio $Q(E)/P(E)$ is bounded uniformly in E. The condition in Definition 3 can thus be seen as a form of "uniform absolute continuity." The following example illustrates the wedge between absolute continuity and the grain condition:

Example 2. Consider two continuous distributions P and Q over the reals with densities f_P and f_Q. If $\operatorname{supp} Q \subseteq \operatorname{supp} P$, then Q is absolutely continuous with respect to P. However, P contains a grain of Q only if f_Q/f_P is bounded. In particular, P does *not* contain a grain of any Q that has heavier tails than P. For example, a normal distribution never contains a grain of a Laplace or exponential distribution. See Subsection 3.4 for more on tail restrictions implied by the grain condition.

3.2 Characterization of misspecified Bayesianism

With Definitions 1–3 in hand, I can state the paper's main result.

Theorem 1. *The pair (μ_0^*, F_1^*) is consistent with misspecified Bayesianism* if and only if *there exists a measurable partition of the set of posteriors $\Delta(X)$ into sets $\{E_k\}_k$ such that, for every E_k with $F_1^*(E_k) > 0$, the prior μ_0^* contains a grain of the average posterior $\overline{\mu}^*_{1|E_k}$ over E_k.*

Misspecified Bayesianism is thus fully characterized by the condition that the prior contains a grain of a set of average posteriors. Absent additional a priori restrictions on what constitutes a reasonable subjective distribution, any belief sequence that satisfies this condition is consistent with Bayesian updating. It is easy to see that the absolute continuity of realized posteriors with respect to the prior is necessary for Bayesianism: If the prior of a Bayesian agent assigns zero probability to an event, her posterior must also assign zero probability to the event—regardless of the agent's subjective belief and the true distribution of signals. What is more surprising is that consistency with misspecified Bayesianism requires the stronger condition that the prior contains a grain of average posteriors. Furthermore, this condition is both necessary and sufficient, so it cannot be weakened.

When the true signal distribution is supported on a finite set, misspecified Bayesianism is characterized by two easier-to-check conditions. The next result considers that case:

Proposition 2. *Consider a pair (μ_0^*, F_1^*), with F_1^* supported on a finite set. The following are equivalent:*

(i) The pair (μ_0^*, F_1^*) is consistent with misspecified Bayesianism.

(ii) The prior μ_0^* contains a grain of any posterior μ_1 in the support of F_1^*.

(iii) The prior μ_0^* contains a grain of the average posterior $\overline{\mu}_1^* \equiv \int \mu F_1^*(d\mu)$.

When the support of F_1^* is finite, the partition in Theorem 1 can be taken to be the trivial partition (with one cell containing the entire space) or the singleton partition (with each cell a singleton). That makes it easier to check whether a pair (μ_0^*, F_1^*) is consistent with misspecified Bayesianism. Condition (iii) is particularly useful since it only requires the knowledge of the average posterior.

Theorem 1 characterizes misspecified Bayesianism for arbitrary state spaces. I now turn to several special cases and show that the grain condition manifests itself very differently across them. When the state space is bounded—finite or compact—the grain condition adds no substantive restrictions beyond a basic support restriction. In contrast, for unbounded state spaces, it puts a real constraint on the tail behavior of the posterior.

3.3 Finite or compact state spaces

I start with the special case where the state space X is finite, a common situation in applications. The following result establishes an easy-to-check condition that characterizes consistency with misspecified Bayesianism in that case:

Proposition 3. *Suppose the state space X is finite. Then the pair (μ_0^*, F_1^*) is consistent with misspecified Bayesianism* if and only if

$$\operatorname{supp} \overline{\mu}_1^* \subseteq \operatorname{supp} \mu_0^*,$$

where $\overline{\mu}_1^* \equiv \int \mu F_1^*(d\mu)$ denotes the average posterior.

This result characterizes misspecified Bayesianism in discrete settings. The average posterior cannot assign positive probability to states that have zero probability according to the prior. The necessity of this property is apparent given Bayes' rule; the proposition goes a step further by establishing its sufficiency. A corollary of Proposition 3 is the following:

Corollary 1. *Suppose the state space X is finite and μ_0^* has full support over X. Then the pair (μ_0^*, F_1^*) is consistent with misspecified Bayesianism for every distribution F_1^* of posteriors.*

Misspecified Bayesianism imposes *no* restrictions on posteriors when the state space is finite and the prior assigns positive probability to every state. The result casts doubt on the possibility of deciding whether decision makers are Bayesian in many common scenarios. I discuss this implication further in Section 5.

Misspecified Bayesianism continues to impose only weak restrictions when the state space is infinite but compact and priors and posteriors have well-behaved densities. The following proposition considers that case:

Proposition 4. *Let the state space X be a compact subset of $\mathbb{R}^n$. Suppose μ_0^* and $\overline{\mu}_1^*$ admit continuous densities m_0^* and $\overline{m}_1^*$ with respect to the Lebesgue measure such that $m_0^*(x) > 0$ for every $x \in \operatorname{supp} \mu_0^*$. Then the pair (μ_0^*, F_1^*) is consistent with misspecified Bayesianism* if and only if

$$\operatorname{supp} \overline{\mu}_1^* \subseteq \operatorname{supp} \mu_0^*,$$

where $\overline{\mu}_1^* \equiv \int \mu F_1^*(d\mu)$ denotes the average posterior.

Intuitively, the uniformity in the grain condition has no bite once the state space is bounded. On a compact set and with a continuous Radon–Nikodym derivative, simple absolute continuity already implies uniform absolute continuity. Moreover, in this setting absolute continuity of $\overline{\mu}_1^*$ with respect to μ_0^* reduces to the support condition $\operatorname{supp} \overline{\mu}_1^* \subseteq \operatorname{supp} \mu_0^*$. Hence, when the state space is compact, misspecified Bayesianism imposes no restrictions beyond ruling out an expansion of the prior's support.

3.4 Unbounded state spaces and tail behavior

Propositions 3 and 4 suggest that misspecified Bayesianism does not impose any meaningful restrictions on belief sequences when the state space is bounded. The conclusion changes dramatically when the state space is unbounded, as this subsection shows.

If the prior is supported on a non-compact set, then misspecified Bayesianism restricts the heaviness of the posterior's tails. For concreteness, I focus on the case where the state space is $X = \mathbb{R}^n$ and use the following (partial) tail order:

Definition 4. Let P, Q be probability distributions that have full support on $\mathbb{R}^n$. If

$$\lim_{r\to\infty} \frac{Q(\|x\| > r)}{P(\|x\| > r)} = \infty,$$

then Q has *heavier tails* than P.[5]

The following uniform version of this partial order will also be useful:

[5]The definition is independent of the choice of norm because of the equivalence of norms on $\mathbb{R}^n$.

Definition 5. Let Q be a set of probability distributions and P be a probability distribution with full support on $\mathbb{R}^n$. If

$$\lim_{r\to\infty} \inf_{Q\in\mathcal{Q}} \frac{Q(\|x\| > r)}{P(\|x\| > r)} = \infty,$$

then distributions in Q have *uniformly heavier tails* than P.

The next result shows that Bayesian updating *cannot* lead to tails that are uniformly heavier:

Proposition 5. *Suppose the state space is given by $X = \mathbb{R}^n$. If there exists an F_1^*-positive measure set of posteriors whose tails are uniformly heavier than the prior μ_0^*, then (μ_0^*, F_1^*) is* not *consistent with misspecified Bayesianism.*

If the support of the distribution of posteriors is finite, we can dispense with uniformity and get the following sharper result:

Corollary 2. *Suppose the state space is given by $X = \mathbb{R}^n$, and consider a pair (μ_0^*, F_1^*), with F_1^* supported on a finite set. If there exists a posterior $\mu_1 \in \operatorname{supp} F_1^*$ whose tails are heavier than those of μ_0^*, then (μ_0^*, F_1^*) is* not *consistent with misspecified Bayesianism.*

Bayesian updating can redistribute the prior's probability mass. However, shifting mass into the tail regions requires signals that were themselves extremely unlikely under the prior. Allowing for misspecified beliefs about signal probabilities enlarges the set of attainable posteriors, because signals that are rare in reality may be deemed likely by the subjective model. Yet even under such misspecification there is a hard limit: Any posterior that is heavier-tailed than the prior cannot be produced by Bayesian updating—no matter how misspecified the subjective model. Observing such a heavy-tailed posterior is then a telltale sign of violations of Bayes' rule.

3.5 Extensions

I next discuss two straightforward extensions of this paper's framework and results.

More than two periods. Consider a sequence of beliefs $(\mu_t^*)_t$, where each μ_t^* is now a random variable. Let $F_{t+1}^*(\cdot|\mu_t^*)$ denote the distribution of μ_{t+1}^* conditional on μ_t^*. A straightforward generalization of Theorem 1 characterizes misspecified Bayesianism in this case: The sequence $(\mu_t^*)_t$ is consistent with misspecified Bayesianism if (and only if) for almost any μ_t^*, the pair $(\mu_t^*, F_{t+1}^*(\cdot|\mu_t^*))$ satisfies the grain condition of Theorem 1.

Time-varying states and filtering. Suppose there is a time-varying state x_t, which belongs to a separable metric space. Theorem 1 already covers this case by defining $x \equiv (x_t)_t$ and taking X to be the separable metric space of infinite sequences.

3.6 Bayes plausibility and related notions

Definition 1 puts no restrictions on what constitutes a reasonable subjective distribution $\mathbb{Q}$. The assumption that any well-defined subjective distribution is permissible is made in keeping with Savage (1972)'s idea of purely subjective probability. I assume that any subjective distribution $\mathbb{Q}$ that can rationalize (μ_0^*, F_1^*) is a valid subjective distribution. In other words, rationality of beliefs is not judged by what those beliefs are but by how they are updated. I next discuss two alternatives to this assumption proposed in the literature and how they change the conclusion of Theorem 1.

The first alternative I consider imposes a correctly specified belief about the distribution of signals. This assumption leads to Bayes plausibility (or the martingale property of Bayesian beliefs): The average posterior must equal the prior. Aumann and Maschler (1995) and Kamenica and Gentzkow (2009) show that this is indeed the only restriction Bayesian updating imposes on beliefs. The following theorem adapts this result to general metric spaces. More importantly, however, it highlights the fact that Bayes plausibility characterizes Bayesianism *only* under the assumption of correct beliefs about the distribution of signals.

Theorem 2 (Kamenica and Gentzkow, 2009)**.** *The pair (μ_0^*, F_1^*) is consistent with Bayesianism given a subjective distribution $\mathbb{Q}$ with the S-marginal satisfying $\mathbb{Q}_S = \mathbb{P}$* if and only if *$\mu_0^* = \overline{\mu}_1^* \equiv \int \mu F_1^*(d\mu)$.*

A more permissive notion of Bayesianism is proposed by Shmaya and Yariv (2016). They allow for incorrect beliefs about the distribution of signals—as long as the supports of those beliefs coincide with the support of the true distribution. The following theorem generalizes Shmaya and Yariv (2016)'s Lemma 1 to general metric state spaces and arbitrary true signal distributions. It reduces to their result when both X and $\operatorname{supp}\mathbb{P}$ are finite sets. However, its main significance is to clarify that Shmaya and Yariv (2016)'s conclusion relies on an a priori restriction on what constitutes a reasonable subjective distribution.

Theorem 3 (Shmaya and Yariv, 2016)**.** *The following statements are equivalent:*

(i) The pair (μ_0^*, F_1^*) is consistent with Bayesianism given a subjective distribution $\mathbb{Q}$ with an S-marginal $\mathbb{Q}_S$ that is absolutely continuous with respect to $\mathbb{P}$.

(ii) There exists a probability measure $\lambda \in \Delta(S)$ such that λ and F_1^* are mutually absolutely continuous and $\mu_0^* = \int \mu\lambda(d\mu)$.

Theorems 2 and 3 show that misspecified Bayesianism is a less restrictive notion than either of the notions considered by Kamenica and Gentzkow (2009) and Shmaya and Yariv (2016). The following table summarizes the relationship between different notions of Bayesianism and the conditions that characterize them:

Bayes plausibility (KG, 2009)		Shmaya and Yariv (2016)		misspecified Bayesianism
$\mu_0^* = \int \mu F_1^*(d\mu)$	$\Longrightarrow$	$\mu_0^* = \int \mu\lambda(d\mu), \quad \lambda \sim F_1^*$	$\Longrightarrow$	$\mu_0^* = \epsilon \int \mu F_1^*(d\mu) + (1-\epsilon)\mu_1'$

4 Illustrative Examples

I next use several examples to illustrate the theoretical results of the paper.

4.1 An example with a finite state space

The first example illustrates the proof of the "if" direction of Theorem 1 in a discrete-state setting. The state takes values in the set $X = \{H, L\}$. The prior is the uniform distribution over X. The distribution of posteriors F_1^* is as follows: with a one-quarter probability, the belief that the state is H goes up to 0.8; with the remaining three-quarters probability, the belief that the state is H goes up to 1.0. Is this belief sequence consistent with misspecified Bayesianism? The answer is yes. This conclusion follows Corollary 1 by noting that μ_0^* has full support over X. In what follows, I illustrate how (μ_0^*, F_1^*) can be rationalized.

F_1^* imposes some restrictions on the true distribution of signals $\mathbb{P}$ and the mapping φ used to form beliefs. Since the posterior takes on two values, there are at least two signals that are realized with positive probability. The observation of one set of signals moves the belief that the state is H to 0.8. Since with a one-quarter probability, the posterior is $\mu_1(H) = 0.8$, the signals that lead to this posterior must have probability $\mathbb{P}(\{s : \varphi(s) = (\mu_1(H) = 0.8)\}) = 0.25$. Likewise, there is a set of signals that has true probability $\mathbb{P}(\{s : \varphi(s) = (\mu_1(H) = 1)\}) = 0.75$ and leads to the posterior that the state is

H with certainty. With slight abuse of notation, I refer to the $\{s : \varphi(s) = (\mu_1(H) = 0.8)\}$ and $\{s : \varphi(s) = (\mu_1(H) = 1)\}$ events simply as the $s = 0.8$ and $s = 1$ signals, respectively.[6]

I illustrate how (μ_0^*, F_1^*) can be rationalized by finding a subjective distribution $\mathbb{Q}$ such that the belief sequence of a Bayesian agent with subjective distribution $\mathbb{Q}$ matches the prior and the distribution of posteriors. The distribution $\mathbb{Q}$ needs to satisfy three requirements for it to rationalize the prior μ_0^* and posterior distribution F_1^*. First, $\mathbb{Q}$ must be consistent with μ_0^*; i.e., $\mathbb{Q}(H) = \mu_0^*(H) = 0.5$. Second, it must assign positive probability to the $s = 0.8$ and $s = 1$ signals for Bayes' rule to be applicable after the observation of those signals. Third, the posterior conditional on the $s = 0.8$ and $s = 1$ signals must be consistent with the corresponding posteriors; i.e., $\mathbb{Q}(H|s = 0.8) = 0.8$ and $\mathbb{Q}(H|s = 1.0) = 1.0$.

One also needs to specify the subjective probability of observing signals other than 0.8 and 1.0. I start by assuming that, according to $\mathbb{Q}$, the signal can only take values $s = 0.8$ and $s = 1.0$. This assumption constrains $\mathbb{Q}$ to satisfy $\mathbb{Q}(\{(x, s) : s \in \{0.8, 1.0\}\}) = 1$. This constraint, together with the requirements previously discussed and the requirement that $\mathbb{Q}(x, s) \geq 0$ for any (x, s), yields a mixed system of equalities and inequalities for the four unknown probabilities $\mathbb{Q}(H, 0.8)$, $\mathbb{Q}(L, 0.8)$, $\mathbb{Q}(H, 1.0)$, and $\mathbb{Q}(L, 1.0)$:

$$\mathbb{Q}(H, 0.8) + \mathbb{Q}(L, 0.8) > 0, \tag{3}$$

$$\mathbb{Q}(H, 1.0) + \mathbb{Q}(L, 1.0) > 0, \tag{4}$$

$$\frac{\mathbb{Q}(H, 0.8)}{\mathbb{Q}(H, 0.8) + \mathbb{Q}(L, 0.8)} = 0.8, \tag{5}$$

$$\frac{\mathbb{Q}(H, 1.0)}{\mathbb{Q}(H, 1.0) + \mathbb{Q}(L, 1.0)} = 1.0, \tag{6}$$

$$\mathbb{Q}(H, 0.8), \mathbb{Q}(L, 0.8), \mathbb{Q}(H, 1.0), \mathbb{Q}(L, 1.0) \geq 0, \tag{7}$$

$$\mathbb{Q}(H, 0.8) + \mathbb{Q}(H, 1.0) = 0.5, \tag{8}$$

$$\mathbb{Q}(L, 0.8) + \mathbb{Q}(L, 1.0) = 0.5. \tag{9}$$

It is easy to verify that this system does not have a solution.

Thus, for the belief sequence to be consistent with misspecified Bayesianism, the subjective distribution must entertain the possibility that the signal takes values outside

[6]This is not an abuse of notation under the assumption that φ is the identity mapping. Note that the assumption that φ is the identity mapping is innocuous in this example since F_1^* only identifies $\mathbb{P} \circ \varphi^{-1} = F_1^*$—but not $\mathbb{P}$ or φ. Nonetheless, the construction in the example can be easily modified to allow for the possibility that F_1^*, $\mathbb{P}$, and φ are separately identified. I do not pursue this extension here since it would lead to additional notational complexity without offering any new insights. See the proof of Theorem 1 for the general construction.

the set $\{0.8, 1.0\}$. With slight abuse of notation, I let $s = \ominus$ denote the event that the signal takes a value outside the set $\{0.8, 1.0\}$. Constraints (8) and (9) must now be modified as follows:

$$\mathbb{Q}(H, 0.8) + \mathbb{Q}(H, 1.0) + \mathbb{Q}(H, \ominus) = 0.5, \tag{10}$$

$$\mathbb{Q}(L, 0.8) + \mathbb{Q}(L, 1.0) + \mathbb{Q}(L, \ominus) = 0.5. \tag{11}$$

The remaining requirements, expressed in equations (3)–(7), remain intact. However, $\mathbb{Q}$ must now additionally satisfy the two non-negativity requirements:

$$\mathbb{Q}(H, \ominus), \mathbb{Q}(L, \ominus) \geq 0. \tag{12}$$

Equations (3)–(7) and (10)–(12) constitute a mixed system of equalities and inequalities for the six unknown probabilities $\mathbb{Q}(H, 0.8)$, $\mathbb{Q}(L, 0.8)$, $\mathbb{Q}(H, 1.0)$, $\mathbb{Q}(L, 1.0)$, $\mathbb{Q}(H, \ominus)$, and $\mathbb{Q}(L, \ominus)$. The fact that the average posterior has the same support as the prior is sufficient to ensure that this system has a solution. One such solution—and the one corresponding to the proof of Theorem 1—is as follows:

	0.8	1.0	$\ominus$
H	0.25	0.25	0
L	0.0625	0	0.4375

Note that the pair (μ_0^*, F_1^*) can be rationalized only if we allow for a misspecified belief about the distribution of signals. If the subjective distribution $\mathbb{Q}$ were to agree with the true distribution $\mathbb{P}$ on the probabilities of different signals, the system of equalities and inequalities that determines $\mathbb{Q}$ would have no solution.

4.2 Fattened tails and inconsistency with misspecified Bayesianism

The previous example may suggest that misspecified Bayesianism imposes no restrictions on the posterior when the prior has full support. The next example illustrates that this is not true when the state space is unbounded. I consider a pair (μ_0^*, F_1^*) that is not consistent with misspecified Bayesianism even though every posterior in the support of F_1^* is absolutely continuous with respect to the prior.

The state space is the real line: $X = \mathbb{R}$. The prior is that the state is normally distributed with mean zero and unit variance. With one-half probability, the posterior is the exponential distribution with mean one; with the remaining one-half probability, the posterior is the (mirrored) exponential distribution supported over $(-\infty, 0]$ with mean minus one. Therefore, the prior μ_0^* is the standard normal distribution, and the average posterior $\overline{\mu}_1^*$ is the Laplace distribution with mean zero and the scale parameter equal to one.

The prior and both realizations of the posterior are non-atomic probability measures over $X = \mathbb{R}$. Moreover, the support of the posteriors is a subset of the support of the prior. Therefore, the two realized posteriors are absolutely continuous with respect to the prior. However, the prior does *not* contain a grain of the average posterior (or either of the two realizations of the posterior), because the two posteriors in the support of F_1^* both have heavier tails than the prior. Therefore, Corollary 2 implies that (μ_0^*, F_1^*) is inconsistent with misspecified Bayesianism.

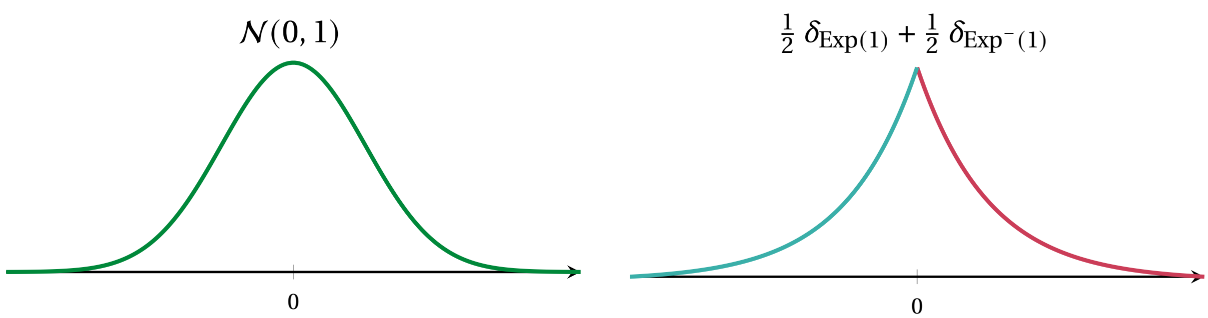

Figure 1. Subsection 4.2's example. The prior (left) and posterior (right). This prior-posterior pair is *not* consistent with misspecified Bayesianism.

4.3 The necessity of using a partition in Theorem 1

When the support of F_1^* is finite, Proposition 2 states that the partition in Theorem 1 can be chosen to be either the trivial partition or the singleton partition. The next example shows that that is not the case when the support of F_1^* is infinite. Then a (μ_0^*, F_1^*) pair may be consistent with Bayesianism even if it violates conditions (ii) and (iii) of Proposition 2.

I start with a subjective distribution $\mathbb{Q}$ and the assumption that the posterior is generated from the prior using Bayes' rule—hence the induced prior-posterior pair will be consistent with misspecified Bayesianism by construction. I then argue that the prior does not contain a grain of any realization of the posterior or the average posterior. However, there exists a measurable partition of the support of the distribution of posteriors such that the prior contains a grain of the average posterior over every cell of the partition.

The state belongs to the real line: $X = \mathbb{R}$. According to the subjective distribution $\mathbb{Q}$, the prior distribution of the state is the standard normal distribution and the signal equals the state with probability one. Hence, the posterior always equals the realization of the signal. The true signal distribution $\mathbb{P}$ is as follows: The support of $\mathbb{P}$ coincides with the support of the subjective distribution. However, the true signal distribution is Laplace$(0, 1)$, i.e., the Laplace distribution with location parameter $\mu = 0$ and scale parameter $b = 1$. Therefore, the posterior is always a point mass at some $Z \in \mathbb{R}$, with Z distributed according to the Laplace$(0, 1)$ distribution.[7]

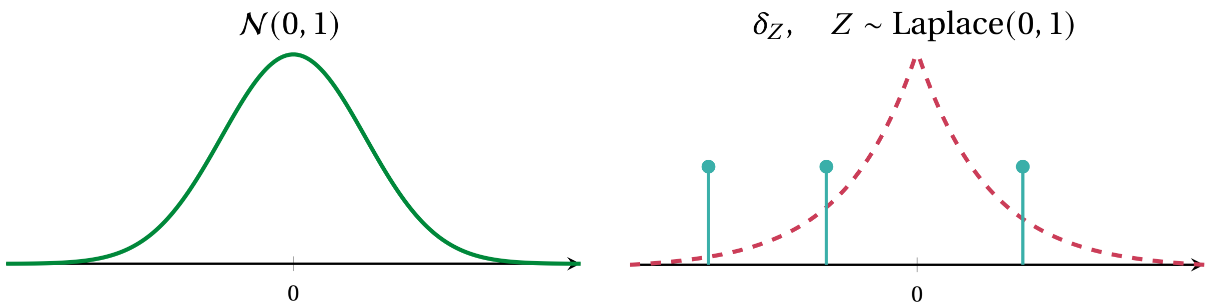

Figure 2. Subsection 4.3's example. The prior (left) and posterior (right). Each point mass in the right panel is a realization of the posterior; the dashed line shows the distribution of the location of those point masses.

The Radon–Nikodym derivative of the Laplace$(0, 1)$ distribution with respect to the normal distribution is unbounded. Therefore, the prior does *not* contain a grain of the average posterior, and statement (iii) in Proposition 2 does not hold. Similarly, the prior does not contain a grain of any of the posteriors in the support of the distribution of posteriors, so statement (ii) in Proposition 2 does not hold either. Yet, by construction, the induced (μ_0^*, F_1^*) pair is consistent with Bayesianism, and must therefore, satisfy the grain condition in Theorem 1.

The distribution of posteriors indeed satisfies the partition version of the grain condition. To see this, consider a partition of the reals into a countable union of non-empty intervals D_k of finite length, and consider the countable partition of the support of the

[7]The example can be formalized as follows: $X = \mathbb{R}$, and $\operatorname{supp}\mathbb{P} = \operatorname{supp}\mathbb{Q}_S = \{\delta_x : x \in \mathbb{R}\}$, where δ_x denotes the point mass at x. For any measurable set $D \subset \mathbb{R}$, $\mathbb{P}(\{\delta_x : x \in D\}) = F_{\text{Laplace}(0,1)}(D)$ and $\mathbb{Q}_S(\{\delta_x : x \in D\}) = F_{\mathcal{N}(0,1)}(D)$. Finally, $\mathbb{Q}$ is defined via equation (1), where $\nu(\delta_x, D) = \mathbb{1}\{x \in D\}$ and $\nu(s, D)$ is arbitrary when s is not a point mass at some $x \in \mathbb{R}$.

distribution of posteriors into sets $E_k \equiv \{\delta_x : x \in D_k\}$. The average posterior over set E_k is equal to the truncated distribution obtained from restricting the Laplace$(0, 1)$ distribution to D_k. Since the Radon–Nikodym derivative of the truncated Laplace$(0, 1)$ distribution with respect to the standard normal distribution is bounded, the prior contains a grain of the average posterior over E_k.

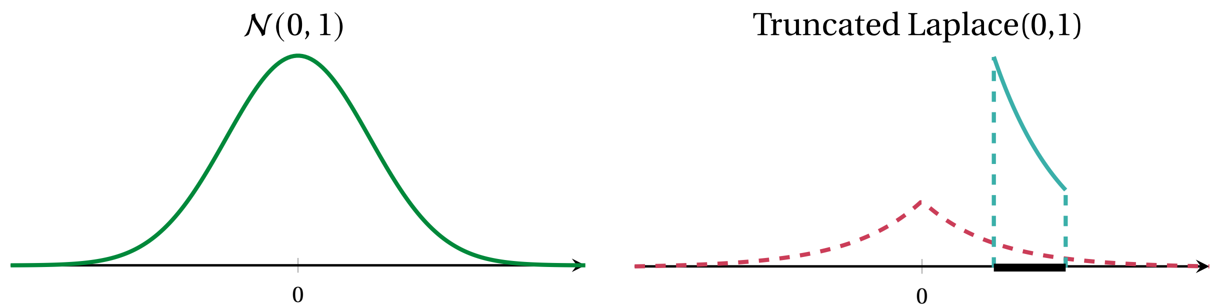

Figure 3. Subsection 4.3's example. The prior (left) and the average posterior over a cell of the partition of supp F_1^* (right).

Although the prior and the average posterior are identical in the last two examples, only the current example is consistent with misspecified Bayesianism. This shows that knowledge of the average posterior is, in general, insufficient to determine consistency. Only when the posterior distribution has finite support can one decide consistency simply by examining the prior and the average posterior.

5 Applications and Implications

5.1 Tests of Bayesian updating

Consider an econometrician who wants to decide whether a group of agents updates their beliefs about a state of the world $x \in X$ using Bayes' rule. There are two periods $t = 0, 1$, and a large number of agents indexed by $i \in I$. Agent i's time-t belief about the value of x is a probability distribution, denoted by $\mu_{it} \in \Delta(X)$. Agents' beliefs may evolve between the periods due to new information. Let s_i denote the signal observed by agent i between the two periods.

The econometrician does not know the realization of signals agents use to update their beliefs. However, he can elicit what each agent believes about the state of the world in each of the two periods and perfectly observe μ_{it} for all i and $t = 0, 1$. The econometrician's question is then whether the agents' observed belief paths $\{(\mu_{i0}, \mu_{i1})\}_{i\in I}$ are consistent with Bayesian updating given some subjective belief about the distribution of signals.

I make several assumptions. First, agents are ex ante identical. In particular, $\mu_{i0} = \mu_0^*$ for some $\mu_0^* \in \Delta(X)$ and all $i \in I$. Second, agents' signals are independent and identically distributed, with $\mathbb{P} \in \Delta(S)$ denoting the true distribution of signals given the fixed state of the world. Third, agents all use the same mapping $\varphi = \varphi_i : s_i \mapsto \mu_{i1}$ to form their beliefs as a function of their signals. Fourth, the number of agents is large enough that the empirical distribution of observed posteriors $\{\mu_{i1}\}_{i\in I}$ provides an arbitrarily good approximation to the corresponding population distribution (by the law of large numbers). Specifically, I assume that the econometrician can perfectly observe the population distribution of agents' posterior beliefs, denoted by $F_1^* \in \Delta(\Delta(X))$. Fifth, the econometrician is assumed to know everything described so far—except for the mapping φ used by agents to update their beliefs. Finally and for simplicity, I assume that $\mathbb{P}$ and F_1^* have finite support. This assumption serves no purpose other than to allow me to rely on Proposition 2 instead of the more complicated characterization in Theorem 1.

The assumptions that agents have identical priors, observe i.i.d. signals, and use identical mappings to update their beliefs all help with the identification of agents' subjective beliefs. The econometrician observes each agent only after the realization of a single signal. Without the homogeneity assumptions, how an agent behaves after a signal would not be informative of how other agents would have behaved if they had observed that same signal. In contrast, the assumption that $\mathbb{P}$, μ_0^*, and F_1^* are perfectly observed by the econometrician limits what he can freely choose in order to rationalize his observations. However, the assumptions ultimately all help the econometrician conclude that agents must *not* be Bayesian.

Yet the econometrician can only reject agents' misspecified Bayesianism if he observes a violation of the grain condition. Proposition 2 implies that the econometrician's observations are consistent with Bayesian updating by agents if and only if μ_0^* contains a grain of the average posterior $\overline{\mu}_1^* \equiv \int \mu F_1^*(d\mu)$. If the state space X is finite and agents' prior has full support over X, then *any* distribution of posteriors is consistent with agents being Bayesian. On the other hand, if the prior is supported on an unbounded subset of the Euclidean space, misspecified Bayesianism disallows the average posterior from having heavier tails than the prior's tails.

This finding has two important implications. First, it suggests that essentially any test of Bayesian updating proposed in the literature is a joint test of Bayesian updating and the assumption that agents have a correctly specified model of the data-generating process. Therefore, any rejection by those tests could be due to non-Bayesian updating, misspecification, or a combination thereof. Second, it suggests that one can indeed test Bayesian updating, in isolation, by looking at the change in belief tails. In particular, the posterior having heavier tails than the prior provides strong evidence against Bayesian updating.

5.2 Equivalence of Bayesian and non-Bayesian updating rules

The paper's results suggest that many non-Bayesian updating rules are observationally equivalent to Bayesian updating under a misspecified subjective model. Consider an agent who observes signals generated according to some stochastic process $\mathbb{P}$ and updates her belief using some non-Bayesian updating rule φ. As long as φ does not lead to violations of the grain condition, it is *as if* the agent used Bayes' rule to update her belief—but under the assumption that her data is described by a subjective process $\mathbb{Q} \neq \mathbb{P}$. The following example makes this point concrete in the context of diagnostic expectations (Bordalo, Gennaioli, and Shleifer, 2018; Bordalo, Gennaioli, Ma, and Shleifer, 2020).

I focus on the updating step of diagnostic expectations since the deviation from the benchmark framework appears in that step. There is a state $x \in \mathbb{R}$ and a signal $s \in \mathbb{R}$ about the state. The true distribution of the signal is given by

$$s = x + \epsilon, \qquad \epsilon \sim \mathcal{N}(0, \sigma_\epsilon^2). \tag{$\mathbb{P}$}$$

All agents—whether correctly specified Bayesian, diagnostic, or misspecified Bayesian—share the prior that x is normally distributed with some mean $\bar{x}$ and some variance σ^2.[8] Let μ_0^* denote that prior.

First consider a Bayesian agent who has a correctly specified model of the data. The agent's posterior belief is that x is normally distributed with mean $\bar{x}+K(s-\bar{x})$ and variance $(1-K)\sigma^2$, where $K \equiv \sigma^2/(\sigma^2+\sigma_\epsilon^2)$ is the Kalman gain. Let μ_1 denote the posterior of the agent.

Next consider a diagnostic agent who starts with the same prior as the correctly specified Bayesian but updates her belief by overweighting representative states. The agent's posterior probability of state x is given by $\mu_1^\theta(x) = \mu_1(x)R(x)^\theta \frac{1}{Z}$, where $R(x)$ is a measure

[8] In Bordalo et al. (2018, 2020), the state is time-varying and the prior itself is obtained from Kalman filtering. However, the dynamics of the prior are orthogonal to the distortion from diagnostic expectations.

of the representativeness of state x given the signal, θ is the diagnosticity parameter, and Z is a normalization factor. Proposition 1 in Bordalo et al. (2020) shows that μ_1^θ is normal with variance $(1-K)\sigma^2$ and mean $\bar{x}+(1+\theta)K(s-\bar{x})$. That is, the posterior uncertainty of the diagnostic agent matches that of the correctly specified Bayesian, whereas the mean of her posterior is more sensitive to the signal.

The belief sequence of the diagnostic agent satisfies the grain condition. The prior belief is normal, while the posterior belief is normal with a fixed variance and a mean that is itself normally distributed. Therefore, the agent's prior always contains a grain of the average diagnostic posterior. Theorem 1 thus implies that the diagnostic agent's belief sequence is consistent with Bayesianism given some subjective distribution $\mathbb{Q}$ for the signal.

The following parsimonious subjective model rationalizes diagnostic expectations. Consider an agent who starts with prior μ_0^* but believes that the distribution of the signal is as follows:

$$s = x + \epsilon, \qquad \epsilon \sim \mathcal{N}\left(\frac{-\theta}{1+\theta}x, \frac{\sigma_\epsilon^2}{(1+\theta)^2}\right). \tag{$\mathbb{Q}$}$$

It is straightforward to verify that the distribution of the Bayesian posterior of the agent coincides with that of a diagnostic agent with parameter θ.[9]

This rationalization offers an alternative interpretation of diagnostic agents' behavioral bias. Diagnostic agents behave *as if* they believe that the signal is less noisy conditional on the state and corrupted by a noise term that is negatively correlated with the state. This belief increases the Kalman gain without altering the agents' posterior uncertainty.

[9]In fact, a stronger observational equivalence holds here: the Bayesian posterior of the misspecified agent coincides with that of the diagnostic agent *signal-by-signal*.

Proofs

Proof of Proposition 1

The proof involves showing (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (i).

Proof of (i) $\implies$ (ii). If $P = \epsilon Q + (1-\epsilon)Q'$, then $Q(E) \leq \frac{1}{\epsilon}P(E)$ for any measurable set E. Therefore, Q is absolutely continuous with respect to P, and so, by the Radon–Nikodym theorem, there exists a derivative $f \equiv \frac{dQ}{dP}$. I finish the proof by showing that f is bounded P-almost surely. Toward a contradiction, suppose that for any positive constant C there exists a measurable set E with $P(E) > 0$ such that $f > C$ on E. Then,

$$Q(E) = \int_E dQ = \int_E f\,dP > C\int_E dP = CP(E).$$

Since C is arbitrary, there exists no constant $\epsilon > 0$ such that $Q(E) \leq \frac{1}{\epsilon}P(E)$ for all E, a contradiction.

Proof of (ii) $\implies$ (iii). Suppose the Radon–Nikodym derivative $f \equiv \frac{dQ}{dP}$ satisfies $f \leq c$ for some $c \geq 1$ and up to sets of P-measure zero. For any measurable set E,

$$Q(E) = \int_E dQ = \int_E f\,dP \leq c\int_E dP = cP(E).$$

Proof of (iii) $\implies$ (i). Let $\epsilon \equiv 1/c \leq 1$. When $\epsilon = 1$, then Q' can be chosen arbitrarily. This is because $Q(E) \leq P(E)$ implies $Q(E^c) \geq P(E^c)$, where E^c denotes the complement of E. But $Q(E^c) \leq P(E^c)$ by assumption. Therefore, $Q(E) = P(E)$. Since E is an arbitrary measurable set, $Q = P$. When $\epsilon < 1$, set $Q' = \frac{1}{1-\epsilon}P - \frac{\epsilon}{1-\epsilon}Q$. To finish the proof, I need to argue that such a Q' is a probability measure. Note that $Q'(E) = \frac{1}{1-\epsilon}P(E) - \frac{\epsilon}{1-\epsilon}Q(E) \geq \frac{1}{1-\epsilon}P(E) - \frac{1}{1-\epsilon}P(E) = 0$. Moreover, since P and Q are probability measures, Q' returns zero for the empty set, returns one for the entire space, and is countably additive. Therefore, Q' is a probability measure. $\square$

Proof of Theorem 1

Proof of the "if" direction. The proof of this direction is constructive. Given the measurable space $(X, \mathcal{X})$ and the true signal distribution $\mathbb{P}$, I construct the subjective distribution $\mathbb{Q}$ that rationalizes an observed pair (μ_0^*, F_1^*) satisfying the assumption of the theorem. By assumption, $F_1^* = \mathbb{P} \circ \varphi^{-1}$ for some φ, and there exists a measurable partition $\{E_k\}_k$ of the set of posteriors $\Delta(X)$ into sets such that, for every E_k with $F_1^*(E_k) > 0$, μ_0^* contains a grain of the average posterior $\overline{\mu}^*_{1|E_k}$ over E_k. Let K denote the indices of the cells E_k for which $F_1^*(E_k) > 0$. Since at most countably many of E_k have positive measure, K is a countable set. For any $k \in K$, since μ_0^* contains a grain of $\overline{\mu}^*_{1|E_k}$, there exist some $\epsilon_k \in (0,1]$ and some probability measure $\mu'_k \in \Delta(X)$ such that $\mu_0^* = \epsilon_k \overline{\mu}^*_{1|E_k} + (1-\epsilon_k)\mu'_k$.

I start by constructing the regular conditional probability $\nu : S \times \mathcal{X} \to [0,1]$ that represents agents' posterior about the state $x \in X$ conditional on the signal s. Let $\ominus \in S$ denote a signal such that $\mathbb{P}(\ominus) = 0$. Such a signal always exists since $S = \Delta(X)$ is uncountable, but there are at most countably many signals $s \in S$ such that $\mathbb{P}(s) > 0$. For any $s \in S$ such that $\varphi(s) \in \operatorname{supp} F_1^*$ and $s \neq \ominus$, set $\nu(s, D) = \varphi(s)(D)$ for all $D \in \mathcal{X}$. Set $\nu(\ominus, \cdot) = \mu'$, where μ' is an arbitrary probability measure over X if $1 - \sum_{k\in K} \epsilon_k F_1^*(E_k) = 0$ and is given by

$$\mu'(D) \equiv \sum_{k\in K} \frac{(1-\epsilon_k)F_1^*(E_k)}{1 - \sum_{k\in K} \epsilon_k F_1^*(E_k)} \mu'_k(D)$$

for all $D \in \mathcal{X}$ if $1 - \sum_{k\in K} \epsilon_k F_1^*(E_k) > 0$. Note that μ' is always a probability measure over X since $\epsilon_k \leq 1$ for all $k \in K$ and $\sum_{k\in K} \frac{(1-\epsilon_k)F_1^*(E_k)}{1-\sum_{k\in K} \epsilon_k F_1^*(E_k)} = 1$. Finally, set $\nu(s, D) = \mu_0^*(D)$ for any $s \in S$ such that $\varphi(s) \notin \operatorname{supp} F_1^* \cup \{\ominus\}$ and all $D \in \mathcal{X}$, indicating that agents' posterior equals their prior conditional on any signal realized with zero probability. Note that, by construction, the mapping $s \mapsto \nu(s, D)$ is measurable for any $D \in \mathcal{X}$, and $\nu(s, \cdot)$ is a probability distribution on $(X, \mathcal{X})$ for all $s \in S$.

I can now define the subjective distribution $\mathbb{Q}$, starting with its S-marginal distribution $\mathbb{Q}_S$. Let

$$\mathbb{Q}_S(E) \equiv \sum_{k\in K} \epsilon_k \mathbb{P}\left(E \cap \varphi^{-1}(E_k)\right) + \left(1 - \sum_{k\in K} \epsilon_k F_1^*(E_k)\right) \mathbb{1}\{\ominus \in E\},$$

for all $E \in \mathcal{S}$. The fact that $\epsilon_k \leq 1$ for all $k \in K$ implies that $\sum_{k\in K} \epsilon_k F_1^*(E_k) \leq 1$. Therefore, $\mathbb{Q}_S(E)$ is a probability distribution over S. Moreover, since $\mathbb{Q}_S(E) \geq \sum_{k\in K} \epsilon_k \mathbb{P}(E \cap \varphi^{-1}(E_k))$ for all $E \in \mathcal{S}$, the true distribution $\mathbb{P}$ is absolutely continuous with respect to $\mathbb{Q}_S$. Next, let

$$\mathbb{Q}(D \times E) \equiv \int_E \nu(s, D)\mathbb{Q}_S(ds) \tag{13}$$

for all $D \in \mathcal{X}$ and $E \in \mathcal{S}$. Since the sigma-algebra $(\mathcal{X} \times \mathcal{S})$ over $(X \times S)$ is generated by sets of the form $D \times E$ with $D \in \mathcal{X}$ and $E \in \mathcal{S}$, the above expression fully specifies the probability distribution $\mathbb{Q}$. Furthermore, comparing equations (1) and (13) shows that ν is indeed a regular conditional probability of $\mathbb{Q}$ given $\mathcal{S}$.

It remains to show that $\mathbb{Q}_X = \mu_0^*$ and that the distribution of posteriors $F_{\nu,\mathbb{P}}$, defined in equation (2), coincides with the observed posterior distribution F_1^*. Note that, by definition, $F_1^* = \mathbb{P} \circ \varphi^{-1}$. Let

$$\hat{S} \equiv \operatorname{supp} \mathbb{P} = \{s \in S : \varphi(s) \in \operatorname{supp} F_1^*\}.$$

For any $D \in \mathcal{X}$,

$$\begin{aligned}
\mathbb{Q}_X(D) &= \int_S \nu(s, D) \mathbb{Q}_S(ds) \\
&= \int_{\hat{S} \setminus \{\ominus\}} \varphi(s)(D) \mathbb{Q}_S(ds) + \mathbb{Q}_S(\{\ominus\}) \nu(\ominus, D) \\
&= \sum_{k \in K} \epsilon_k \int_{\hat{S} \cap \varphi^{-1}(E_k)} \varphi(s)(D) \mathbb{P}(ds) + \left(1 - \sum_{k \in K} \epsilon_k F_1^*(E_k)\right) \mu'(D) \\
&= \sum_{k \in K} \epsilon_k \int_{\varphi(\hat{S}) \cap E_k} \mu(D) \mathbb{P} \circ \varphi^{-1}(d\mu) + \left(1 - \sum_{k \in K} \epsilon_k F_1^*(E_k)\right) \mu'(D) \\
&= \sum_{k \in K} \epsilon_k \int_{E_k} \mu(D) F_1^*(d\mu) + \left(1 - \sum_{k \in K} \epsilon_k F_1^*(E_k)\right) \mu'(D) \\
&= \sum_{k \in K} F_1^*(E_k) \epsilon_k \overline{\mu}^*_{1|E_k}(D) + \left(1 - \sum_{k \in K} \epsilon_k F_1^*(E_k)\right) \mu'(D) \\
&= \sum_{k \in K} F_1^*(E_k) \left(\mu_0^*(D) - (1 - \epsilon_k) \mu_k'(D)\right) + \left(1 - \sum_{k \in K} \epsilon_k F_1^*(E_k)\right) \mu'(D) \\
&= \mu_0^*(D) - \sum_{k \in K} (1 - \epsilon_k) F_1^*(E_k) \mu_k'(D) + \left(1 - \sum_{k \in K} \epsilon_k F_1^*(E_k)\right) \mu'(D).
\end{aligned}$$

If $\epsilon_k = 1$ for all $k \in K$, then the last two terms in the above display are both zero, and so, $\mathbb{Q}_X(D) = \mu_0^*(D)$. If, on the other hand, $\epsilon_k < 1$ for some $k \in K$, then $1 - \sum_{k \in K} \epsilon_k F_1^*(E_k) > 0$, and the last two terms cancel out given the definition of μ', again resulting in $\mathbb{Q}_X(D) = \mu_0^*(D)$.

Lastly, I show that $F_{\nu,\mathbb{P}} = F_1^*$. Note that

$$\mathbb{P}\left(\left\{s \in S : \varphi(s) \notin \operatorname{supp} F_1^*\right\}\right) = \mathbb{P}\left(\{s \in S : s \notin \operatorname{supp} \mathbb{P}\}\right) = 0.$$

On the other hand, by construction, $\mathbb{P}(\ominus) = 0$. Therefore, for any $E \in \mathcal{S}$,

$$\begin{aligned} F_{\nu,\mathbb{P}}(E) &= \mathbb{P}\left(\{s \in S : \nu(s,\cdot) \in E\}\right) \\ &= \mathbb{P}\left(\{s \in S : \nu(s,\cdot) \in E, \varphi(s) \in \operatorname{supp} F_1^*, s \neq \ominus\}\right) \\ &= \mathbb{P}\left(\{s \in S : \varphi(s) \in E\}\right) = F_1^*(E). \end{aligned}$$

This completes the proof of the first direction.

Proof of the "only if" direction. Let $\mathbb{Q}$ denote agents' subjective distribution on $X \times S$, and let ν denote the regular conditional probability of $\mathbb{Q}$ given $\mathcal{S}$. The existence of ν follows the assumption that $\mathbb{Q}$ satisfies condition (c) of Definition 1. Since the signal labels have no inherent meaning, I can label any signal by the posterior belief it induces. More specifically, I can assume, without loss of generality, that $\nu(\mu, D) = \mu(D)$ for any $\mu \in S = \Delta(X)$.[10] Given those signal labels, $F_1^* = F_{\nu,\mathbb{P}} = \mathbb{P}$.

Since $\mathbb{Q}$ satisfies condition (a) of Definition 1 and ν is a regular conditional probability of $\mathbb{Q}$ given $\mathcal{S}$,

$$\mu_0^*(D) = \mathbb{Q}_X(D) = \int_S \nu(s, D)\mathbb{Q}_S(ds) \tag{14}$$

for all $D \in \mathcal{X}$. On the other hand, by condition (b) of Definition 1, $\mathbb{P}$ is absolutely continuous with respect to $\mathbb{Q}_S$. Hence, by the Radon–Nikodym theorem, there exists a Radon–Nikodym derivative $f \equiv \frac{d\mathbb{P}}{d\mathbb{Q}_S}$. For $k \in \mathbb{N}$, define

$$E_k \equiv \left\{s \in S : f(s) \in [k-1, k)\right\}.$$

Since f is a measurable function, E_k is a measurable subset of S for any $k \in \mathbb{N}$. Furthermore, the sets $\{E_k\}_{k\in\mathbb{N}}$ partition the set of posteriors $S = \Delta(X)$. For any k such that $F_1^*(E_k) > 0$ and any $D \in \mathcal{X}$,

$$\overline{\mu}^*_{1|E_k}(D) = \frac{1}{F_1^*(E_k)} \int_{E_k} \mu(D) F_1^*(d\mu) = \frac{1}{F_1^*(E_k)} \int_{E_k} \mu(D) F_{\nu,\mathbb{P}}(d\mu) = \frac{1}{F_1^*(E_k)} \int_{E_k} \nu(s, D)\mathbb{P}(ds). \tag{15}$$

[10] Given a subjective distribution $\mathbb{Q}$ on $X \times S$ with regular conditional probability ν and a true distribution for signals $\mathbb{P} \in \Delta(S)$, define $\tilde{\nu}(\mu, D) \equiv \mu(D)$, $\tilde{\mathbb{Q}}_S(E) \equiv \mathbb{Q}(\{s \in S : \nu(s,\cdot) \in E\})$, $\tilde{Q}(D \times E) = \int_E \tilde{\nu}(s, D)\tilde{\mathbb{Q}}_S(ds)$, $\tilde{\mathbb{P}}(E) \equiv \mathbb{P}(\{s \in S : \nu(s,\cdot) \in E\})$ for any $\mu \in S = \Delta(X)$ and any measurable sets $D \subseteq X$ and $E \subseteq S$. Then $\tilde{\mathbb{Q}}_X = \mathbb{Q}_X$ and $F_{\tilde{\nu},\tilde{\mathbb{P}}} = F_{\nu,\mathbb{P}}$; that is, the prior and distribution of posteriors induced by $\tilde{\mathbb{Q}}$ and $\tilde{\mathbb{P}}$ coincide with those induced by $\mathbb{Q}$ and $\mathbb{P}$.

Since f is the Radon–Nikodym derivative of $\mathbb{P}$ with respect to $\mathbb{Q}_S$,

$$\int_{E_k} \nu(s,D)\mathbb{P}(ds) = \int_{E_k} \nu(s,D)f(s)\mathbb{Q}_S(ds) \le k\int_{E_k} \nu(s,D)\mathbb{Q}_S(ds) \le k\int_S \nu(s,D)\mathbb{Q}_S(ds), \tag{16}$$

where the first inequality is by the definition of set E_k, and the second inequality is due to the fact that $\int_{S\setminus E_k} \nu(s,D)\mathbb{Q}_S(ds) \ge 0$. Equations (14)–(16) imply

$$\overline{\mu}^*_{1|E_k}(D) = \frac{1}{F_1^*(E_k)}\int_{E_k} \nu(s,D)\mathbb{P}(ds) \le \frac{k}{F_1^*(E_k)}\int_S \nu(s,D)\mathbb{Q}_S(ds) = \frac{k}{F_1^*(E_k)}\mu_0^*(D).$$

The $k/F_1^*(E_k)$ constant in the above inequality is independent of D. Therefore, by Proposition 1, μ_0^* contains a grain of $\overline{\mu}^*_{1|E_k}$. □

Proof of Proposition 2

Theorem 1 establishes (iii) $\implies$ (i) by choosing the trivial partition, so I only need to show that (i) $\implies$ (ii) $\implies$ (iii).

Proof of (i) $\implies$ (ii). Fix a posterior μ in the support of F_1^*. I show that μ_0^* contains a grain of μ. Let $\mathbb{Q}$ denote agents' subjective distribution on $X \times S$, and let ν denote the regular conditional probability of $\mathbb{Q}$ given $\mathcal{S}$. Let $S_\mu = \{s \in S : \nu(s,\cdot) = \mu\}$ denote the set of signals that engender μ as the posterior. The assumptions that the support of F_1^* is finite and μ is in the support of F_1^* imply that $F_1^*(\mu) > 0$, and so, $\mathbb{P}(S_\mu) > 0$. Since $\mathbb{P}$ is absolutely continuous with respect to $\mathbb{Q}_S$, it must also be that $\mathbb{Q}_S(S_\mu) > 0$. Since ν is the regular conditional probability of $\mathbb{Q}$ given $\mathcal{S}$,

$$\mathbb{Q}(D \times S_\mu) = \int_{S_\mu} \nu(s,D)\mathbb{Q}_S(ds) = \mu(D)\mathbb{Q}_S(S_\mu)$$

for any $D \in \mathcal{X}$. On the other hand,

$$\mu_0^*(D) = \mathbb{Q}_X(D) = \mathbb{Q}(D \times S) \ge \mathbb{Q}(D \times S_\mu).$$

Therefore,

$$\mu(D) = \frac{\mathbb{Q}(D \times S_\mu)}{\mathbb{Q}_S(S_\mu)} \le \frac{1}{\mathbb{Q}_S(S_\mu)}\mu_0^*(D),$$

and by Proposition 1, μ_0^* contains a grain of μ.

Proof of (ii) $\implies$ (iii). By assumption, for any $\mu \in \operatorname{supp} F_1^*$, there exist some $\epsilon_\mu \in (0,1]$ and some probability measure $\mu'_\mu \in \Delta(X)$ such that $\mu_0^* = \epsilon_\mu \mu + (1-\epsilon_\mu)\mu'_\mu$. Therefore,

$$\overline{\mu}_1^* = \sum_{\mu \in \operatorname{supp} F_1^*} F_1^*(\mu)\mu = \sum_{\mu \in \operatorname{supp} F_1^*} F_1^*(\mu)\left(\frac{\mu_0^*}{\epsilon_\mu} - \frac{1-\epsilon_\mu}{\epsilon_\mu}\mu'_\mu\right),$$

and so,

$$\mu_0^* = \frac{1}{\sum_{\mu \in \operatorname{supp} F_1^*} \frac{F_1^*(\mu)}{\epsilon_\mu}}\left(\overline{\mu}_1^* + \sum_{\mu \in \operatorname{supp} F_1^*} \frac{(1-\epsilon_\mu)F_1^*(\mu)}{\epsilon_\mu}\mu'_\mu\right). \tag{17}$$

Let $\epsilon \equiv \left(\sum_{\mu \in \operatorname{supp} F_1^*} \frac{F_1^*(\mu)}{\epsilon_\mu}\right)^{-1}$ denote the weighted harmonic mean of $\{\epsilon_\mu\}_{\mu \in \operatorname{supp} F_1^*}$. Since all ϵ_μ are in the $(0,1]$ interval, so is ϵ. Let

$$\mu' \equiv \frac{\epsilon}{1-\epsilon} \sum_{\mu \in \operatorname{supp} F_1^*} \frac{(1-\epsilon_\mu)F_1^*(\mu)}{\epsilon_\mu}\mu'_\mu$$

denote the weighted average of probability measures μ'_μ. Since the weights are positive and add up to one, μ' is a probability distribution over X. Equation (17) can thus be written as $\mu_0^* = \epsilon\overline{\mu}_1^* + (1-\epsilon)\mu'$, establishing that μ_0^* contains a grain of $\overline{\mu}_1^*$. □

Proof of Proposition 3

Proof of the "if" direction. If $\operatorname{supp} \overline{\mu}_1^* \subseteq \operatorname{supp} \mu_0^*$, then

$$\mu_0^*(\{x\}) = 0 \implies \overline{\mu}_1^*(\{x\}) = 0. \tag{18}$$

Define

$$c \equiv \max_{\{x \in X : \mu_0^*(\{x\}) > 0\}} \frac{\overline{\mu}_1^*(\{x\})}{\mu_0^*(\{x\})}.$$

Since X is a finite set, c is well-defined. Furthermore, since μ_0^* and $\overline{\mu}_1^*$ are probability distributions over X, which also satisfy (18), $c \geq 1$. Therefore, for all $x \in X$,

$$\overline{\mu}_1^*(\{x\}) \leq c\mu_0^*(\{x\}),$$

where the inequality follows the definition of c for any x for which $\mu_0^*(\{x\}) > 0$ and follows equation (18) for other x. The above display establishes that μ_0^* contains a grain of $\overline{\mu}_1^*$. The result then follows Theorem 1 by choosing the trivial partition of $\Delta(X)$.

Proof of the "only if" direction. Let $\mathbb{Q}$ denote agents' subjective distribution on $X \times S$, and let ν denote the regular conditional probability of $\mathbb{Q}$ given $\mathcal{S}$. By equations (14) and (15) in the proof of Theorem 1,

$$\mu_0^*(D) = \int_S \nu(s, D)\mathbb{Q}_S(ds), \tag{19}$$

$$\overline{\mu}_1^*(D) \equiv \overline{\mu}_{1|S}^*(D) = \int_S \nu(s, D)\mathbb{P}(ds). \tag{20}$$

for any $D \in \mathcal{X}$. Fix some $x \in X$ *not* in the support of μ_0^*, and let $\hat{S} \equiv \{s \in S : \nu(s, \{x\}) > 0\}$. Since $\mu_0^*(\{x\}) = 0$, by equation (19), $\mathbb{Q}_S(\hat{S}) = 0$. Therefore, since $\mathbb{P}$ is absolutely continuous with respect to $\mathbb{Q}_S$, it must be that $\mathbb{P}(\hat{S}) = 0$. Equation (20) then implies that $\overline{\mu}_1^*(\{x\}) = 0$; that is, x is not in the support of $\overline{\mu}_1^*$. □

Proof of Proposition 4

Proof of the "if" direction. By assumption, m_0^* and $\overline{m}_1^*$ are continuous functions and $m_0^*(x) > 0$ for all $x \in \operatorname{supp} \mu_0^*$. Therefore, $\overline{m}_1^*(x)/m_0^*(x)$ is a continuous function over the compact support of μ_0^*, and so, it is bounded. But since μ_0^* and $\overline{\mu}_1^*$ have densities, the Radon–Nikodym derivative $d\overline{\mu}_1^*/d\mu_0^*$ is equal to the ratio of densities $\overline{m}_1^*(x)/m_0^*(x)$ on the support of μ_0^*. (Since $\operatorname{supp} \overline{\mu}_1^* \subseteq \operatorname{supp} \mu_0^*$, the Radon–Nikodym derivative is arbitrary and irrelevant off the support of μ_0^*.) Thus, Proposition 1 implies that μ_0^* contains a grain of $\overline{\mu}_1^*$. The result then follows Theorem 1 by choosing the trivial partition.

Proof of the "only if" direction. Theorem 1 implies that there exists a measurable partition of the set of posteriors $\Delta(X)$ into sets $\{E_k\}_k$ such that, for every E_k with $F_1^*(E_k) > 0$, the prior μ_0^* contains a grain of the average posterior $\overline{\mu}_{1|E_k}^*$ over E_k. Therefore, for every E_k with $F_1^*(E_k) > 0$, $\overline{\mu}_{1|E_k}^*$ is absolutely continuous with respect to μ_0^*, and consequently, $\operatorname{supp} \overline{\mu}_{1|E_k}^* \subseteq \operatorname{supp} \mu_0^*$. Let K denote the indices of the cells E_k for which $F_1^*(E_k) > 0$. Since at most countably many of E_k have positive measure, K is a countable set. Therefore,

$$\overline{\mu}_1^* = \sum_{k \in K} F_1^*(E_k)\overline{\mu}_{1|E_k}^*.$$

Because the support of each $\overline{\mu}_{1|E_k}^*$ is contained in the support of μ_0^* and $\overline{\mu}_1^*$ is their convex combination, the support of $\overline{\mu}_1^*$ is contained in $\bigcup_{k \in K} \operatorname{supp} \overline{\mu}_{1|E_k}^* \subseteq \operatorname{supp} \mu_0^*$. □

Proof of Proposition 5

By assumption, there exists an F_1^*-positive measure set of posteriors H such that posteriors in H have uniformly heavier tails than μ_0^*. Consider an arbitrary measurable partition of the set of posteriors $\Delta(X)$ into sets $\{E_k\}_k$. Since $F_1^*(H) > 0$, there exists a cell E_{k^*} of the partition such that $F_1^*(H \cap E_{k^*}) > 0$. Since the distributions in H have uniformly heavier tails than μ_0^*, for every M, there exists some R such that, for all $r > R$,

$$\frac{\mu_1(\|x\| > r)}{\mu_0^*(\|x\| > r)} > M,$$

for all $\mu_1 \in H$. Therefore, for every M, there exists some R such that

$$\begin{aligned}\frac{\overline{\mu}^*_{1|E_{k^*}}(\|x\| > r)}{\mu_0^*(\|x\| > r)} &= \frac{1}{F_1^*(E_{k^*})}\int_{E_{k^*}} \frac{\mu_1(\|x\| > r)}{\mu_0^*(\|x\| > r)} F_1^*(d\mu_1) \\ &\geq \frac{1}{F_1^*(E_{k^*})}\int_{H\cap E_{k^*}} \frac{\mu_1(\|x\| > r)}{\mu_0^*(\|x\| > r)} F_1^*(d\mu_1) \\ &> \frac{MF_1^*(H\cap E_{k^*})}{F_1^*(E_{k^*})},\end{aligned}$$

for any $r > R$. Hence,

$$\lim_{r\to\infty} \frac{\overline{\mu}^*_{1|E_{k^*}}(\|x\| > r)}{\mu_0^*(\|x\| > r)} = \infty.$$

That is, $\overline{\mu}^*_{1|E_{k^*}}$ has heavier tails than μ_0^*. Towards a contradiction, suppose μ_0^* contains a grain of $\overline{\mu}^*_{1|E_{k^*}}$. Proposition 1 then implies that there exists a constant $c \geq 1$ such that $\overline{\mu}^*_{1|E_{k^*}}(E) \leq c\mu_0^*(E)$ for any measurable set E, a contradiction to the fact that $\overline{\mu}^*_{1|E_{k^*}}(\|x\| > r)/\mu_0^*(\|x\| > r)$ grows without bound as r goes to infinity. Thus, μ_0^* does not contain a grain of $\overline{\mu}^*_{1|E_{k^*}}$. Since the partition was arbitrary and $F_1^*(E_{k^*}) > 0$, by Theorem 1, the pair (μ_0^*, F_1^*) is not consistent with misspecified Bayesianism. □

Proof of Theorem 2

Proof of the "if" direction. The proof of this direction is constructive. I construct the regular conditional probability $\nu : S \times \mathcal{X} \to [0,1]$ by setting $\nu(s, D) = \varphi(s)(D)$ for all $s \in S$ such that $\varphi(s) \in \operatorname{supp} F_1^*$ and all $D \in \mathcal{X}$ and setting $\nu(s, D) = \mu_0^*(D)$ for all $s \in S$ such that $\varphi(s) \notin \operatorname{supp} F_1^*$ and all $D \in \mathcal{X}$. By construction, $\nu(s, \cdot)$ is a probability distribution on $(X, \mathcal{X})$, and the mapping $s \mapsto \nu(s, D)$ is measurable for all $D \in \mathcal{X}$. I set the S-marginal $\mathbb{Q}_S$ of the subjective distribution equal to the true distribution $\mathbb{P}$ of signals and define $\mathbb{Q}$ as in

(13). By construction, ν is a regular conditional probability of $\mathbb{Q}$ given $\mathcal{S}$. Next, note that, for any $D \in \mathcal{X}$,

$$\mathbb{Q}_X(D) = \int_S \nu(s,D)\mathbb{P}(ds) = \int_{\operatorname{supp} F_1^*} \mu(D) F_1^*(d\mu) = \overline{\mu}_1^*(D) = \mu_0^*(D),$$

where the last equality follows the assumption that $\overline{\mu}_1 = \mu_0^*$. Moreover, by an argument similar to the one in the proof of Theorem 1,

$$F_{\nu,\mathbb{P}}(E) = \mathbb{P}\left(\{s \in S : \nu(s,\cdot) \in E\}\right) = \mathbb{P}\left(\{s \in S : \varphi(s) \in E\}\right) = F_1^*(E)$$

for all $E \in \mathcal{S}$. This shows that the subjective distribution $\mathbb{Q}$ constructed above rationalizes the observed pair (μ_0^*, F_1^*).

Proof of the "only if" direction. By equations (14) and (15) in the proof of Theorem 1, $\mu_0^*(D) = \int_S \nu(s,D)\mathbb{Q}_S(ds)$ and $\overline{\mu}_1^*(D) = \int_S \nu(s,D)\mathbb{P}(ds)$ for any $D \in \mathcal{X}$. The assumption that $\mathbb{Q}_S = \mathbb{P}$ completes the proof. □

Proof of Theorem 3

Proof of (i) $\implies$ (ii). Suppose (μ_0^*, F_1^*) is consistent with Bayesianism given a subjective distribution $\mathbb{Q}$ with an S-marginal $\mathbb{Q}_S$ that is absolutely continuous with respect to $\mathbb{P}$, and let ν denote the regular conditional probability of $\mathbb{Q}$ given $\mathcal{S}$. The existence of ν follows the assumption that $\mathbb{Q}$ satisfies condition (b) of Definition 1. I define $\lambda \in \Delta(S)$ as follows:

$$\lambda(E) \equiv \mathbb{Q}_S(\{s \in S : \nu(s,\cdot) \in E\})$$

for all $E \in \mathcal{S}$. I next show that λ and F_1^* are mutually absolutely continuous and $\mu_0^* = \int \mu\lambda(d\mu)$. Since $\mathbb{Q}$ satisfies condition (a) and ν is a regular conditional probability of $\mathbb{Q}$ given $\mathcal{S}$,

$$\mu_0^* = \mathbb{Q}_X = \int_S \nu(s,\cdot)\mathbb{Q}_S(ds) = \int_S \mu\lambda(d\mu),$$

where the last equality is by definition. On the other hand, for all $E \in \mathcal{S}$,

$$\lambda(E) = \mathbb{Q}_S(\{s \in S : \nu(s,\cdot) \in E\})$$

and

$$F_1^*(E) = F_{\nu,\mathbb{P}}(E) = \mathbb{P}(\{s \in S : \nu(s,\cdot) \in E\}),$$

where the first equality above is by condition (c) of Definition 1. Therefore, since $\mathbb{Q}_S$ and $\mathbb{P}$ are mutually absolutely continuous by assumption, so are λ and F_1^*.

Proof of (ii) $\implies$ (i). Suppose there exists a probability measure $\lambda \in \Delta(S)$ such that λ and F_1^* are mutually absolutely continuous and $\mu_0^* = \int \mu\lambda(d\mu)$. By the Radon–Nikodym theorem, there are derivatives $f \equiv \frac{d\lambda}{dF_1^*}$ and $\frac{1}{f} \equiv \frac{dF_1^*}{d\lambda}$. Set $\nu(s,D) = \varphi(s)(D)$ for all $s \in S$ and $D \in \mathcal{X}$, and set $\mathbb{Q}_S(ds) = f(\varphi(s))\mathbb{P}(ds)$. I need to show that $\mathbb{Q}_S$, as defined above, is indeed a probability distribution on $(S, \mathcal{S})$. By construction, $\mathbb{Q}_S(E) \geq 0$ for all $E \in \mathcal{S}$, and $\mathbb{Q}_S(\emptyset) = 0$. Next, note that

$$\int_S \mathbb{Q}_S(ds) = \int_S f(\varphi(s))\mathbb{P}(ds) = \int_S f(\mu)\mathbb{P}\circ\varphi^{-1}(d\mu) = \int_S f(\mu)F_1^*(d\mu) = \int_S \lambda(d\mu) = 1,$$

where the first equality is by definition, the second one uses the change-of-variables formula for pushforward measures, the third equality is due to the fact that $F_1^* = \mathbb{P}\circ\varphi^{-1}$, the fourth one uses the definition of f, and the last equality is because λ is a probability measure on S. Finally, $\mathbb{Q}_S$ is countably additive since $\mathbb{P}$ is countably additive. Therefore, $\mathbb{Q}_S$ is a well-defined probability distribution. I finish the construction by defining $\mathbb{Q}$ as in equation (1). Note that, by construction, ν is a conditional probability of $\mathbb{Q}$ given $\mathcal{S}$. Furthermore, by an argument similar to the one in the above display,

$$\mathbb{Q}_X = \int_S \nu(s,\cdot)\mathbb{Q}_S(ds) = \int_S \varphi(s)f(\varphi(s))\mathbb{P}(ds) = \int_S \mu f(\mu)F_1^*(d\mu) = \int_S \mu\lambda(d\mu) = \mu_0^*,$$

where the last equality is by assumption. Therefore, condition (a) of Definition 1 is satisfied. Furthermore, since $\mathbb{Q}_S(ds) = f(\varphi(s))\mathbb{P}(ds)$ and $\mathbb{P}(ds) = \frac{1}{f(\varphi(s))}\mathbb{Q}_S(ds)$, probability distributions $\mathbb{Q}_S$ and $\mathbb{P}$ are mutually absolutely continuous. That is, condition (b) of Definition 1 is satisfied, and $\mathbb{Q}_S$ is absolutely continuous with respect to $\mathbb{P}$. On the other hand,

$$F_{\nu,\mathbb{P}}(E) = \mathbb{P}(\{s \in S : \nu(s,\cdot) \in E\}) = \mathbb{P}(\{s \in S : \varphi(s) \in E\}) = \mathbb{P}\circ\varphi^{-1}(E) = F_1^*(E)$$

for all $E \in \mathcal{S}$, implying that condition (c) is also satisfied. □

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