Tests of Bayesian Rationality*

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Abstract

What are the testable implications of the hypothesis of Bayesian rationality? This paper argues that the absolute continuity of posteriors with respect to priors constitutes the entirety of the empirical content of this hypothesis. I consider a decision-maker who chooses a sequence of actions and an econometrician who observes the decision-maker's actions but not her signals and is interested in testing the hypothesis that the decision-maker follows Bayes' rule to update her belief. I show that—absent a priori knowledge on the part of the econometrician on the set of models considered by the decision-maker—there are almost no observations that would lead the econometrician to conclude that the decision-maker is not Bayesian. The absolute continuity of posteriors with respect to priors remains the only implication of Bayesian rationality even if the set of actions is sufficiently rich that the decision-maker's actions fully reveal her beliefs and even if the econometrician observes a large number of ex ante identical agents who observe i.i.d. signals and face the same sequence of decision problems.

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1 Introduction

Following the treatise of Savage (1972), the subjective (or Bayesian) theory of probability has become the dominant paradigm in the modeling of decision-making under uncertainty.[1] The dominance of this paradigm in economics is not unwarranted. It allows one to assign probabilities to unique (or rare) events. It has an elegant foundation in the study of rational choice under uncertainty. And it is appealing from a normative point of view—as Epstein and Le Breton (1993) proclaim, "dynamically consistent beliefs must be Bayesian." What is less clear is whether Bayesian rationality is a good positive model of individual behavior. To settle this question requires one to develop formal tests of Bayesian rationality.

A class of tests commonly used in the literature are those based on the martingale property of beliefs. A Bayesian agent's belief sequence is a martingale given the agent's own prior belief. Conversely, any sequence of beliefs that constitutes a martingale given some probability distribution $\mathbb{P}$ can be rationalized as the belief sequence of a Bayesian agent with prior $\mathbb{P}$. This equivalence of Bayesian rationality and the martingale property of beliefs has been known by economists since, at least, Kamenica and Gentzkow (2009).[2]

But although a Bayesian agent's beliefs constitute a martingale with respect to her *subjective belief*, operationalizing the martingale test requires one to sample individuals whose observations depend on an *objective data-generating process* (DGP). The martingale test is thus a valid test of Bayesian rationality only if agents have objectively correct beliefs about the DGP. Said differently, a group of agents could fail a martingale-based test of Bayesian rationality for one of two distinct reasons: (i) the agents may not be Bayesian, or (ii) their belief about the distribution of the signals they observe may not coincide with the true, objective DGP.

This paper's main contribution is to characterize the empirical content of the theory of Bayesian rationality without predicating it on the a priori assumption that individuals' beliefs about the DGP are accurate. I consider an observer (or econometrician) who observes a sequence of decisions made by a decision-maker (DM). The econometrician (he) is wondering whether the observation of any decision sequences would lead him to conclude, with some certainty, that the decision-maker (she) is not Bayesian. The econometrician observes the DM's actions but does not know the full description of her probabilistic model

[1] Savage's theory of subjective probability builds on earlier works by De Finetti (1937), Ramsey (1926), and Von Neumann and Morgenstern (1944).

[2] The necessity of the martingale property for Bayesian rationality is a trivial consequence of the law of iterated expectation. To the best of my knowledge, Kamenica and Gentzkow (2009) are the first in the economics literature to formalize the sufficiency of the martingale property for Bayesian rationality.

of the world—including her belief about the DGP.[3] He attempts to rationalize the sequence of decisions made by the DM by constructing a probabilistic model for the DM and a true DGP, given which the DM's observed choices are consistent with Bayesian rationality. If he is unable to do so, he rejects the DM's Bayesian rationality. The paper's main result is that the econometrician can rationalize almost any sequence of decisions by the DM in the manner just described.

One may object that two obstacles stand in the way of the econometrician's rejecting of the decision-maker's Bayesian rationality. First, the mapping from the DM's beliefs to her actions may not be known to the econometrician or be invertible. So the econometrician may not be able to identify the DM's preferences and her beliefs separately. Second, any seemingly irrational realization of the DM's belief sequence may result from observations by the DM that were unlikely ex ante.

To address these potential objections, I give the econometrician powerful tools to overcome the obstacles outlined in the previous paragraph. First, I assume that the mapping from the DM's beliefs to her actions is invertible and known to the econometrician. This assumption allows the econometrician to identify the DM's beliefs simply by observing her actions. I additionally assume that the econometrician can directly observe the true, objective population distribution of the DM's belief sequence. This assumption represents the limit where the econometrician has access to the belief sequence of a large sample of ex ante identical decision-makers who observe independent signals drawn from a common distribution and who face identical decision problems. The observation of the population distribution allows the econometrician to overcome the second obstacle mentioned above.

I show that few observations would lead the econometrician to conclude that the DM is not Bayesian—despite the unrealistically powerful tools at his disposal. To be more specific, the only testable implication of the theory of Bayesian rationality is the absolute continuity of the posterior with respect to the prior for any prior-posterior pair that is realized with positive probability. The result holds even when the DM's prior belief about the payoff-relevant state agrees with its objective distribution, and it does not rely on the inapplicability of Bayes' rule after contingencies that were assigned zero probability by the DM's prior.

The result characterizes the empirical content of the theory of Bayesian rationality—absent additional a priori assumptions on an individual's probabilistic model of the world. It shows that, in isolation, Bayesian rationality has only weak testable predictions. This

[3] I use the term "probabilistic model of the world" (or simply model) to refer to the subjective probability space $(\Omega, \mathcal{F}, \mu)$ (in the sense of Savage) that rationalizes a rational agent's choices. I use the term "theory" to refer to a hypothesis (such as subjective rationality) being tested by the econometrician.

finding suggests imposing assumptions on what constitutes a reasonable model as a way of obtaining theories with more predictive power. Doing so leads to hybrid theories in which some aspects of individuals' models conform to objective reasonableness requirements while other dimensions of the models are subjective.

I proceed by introducing two such a priori reasonableness assumptions, discussing the contexts in which they are likely to be satisfied, and characterizing their testable implications in conjunction with the assumption of Bayesian rationality. One such reasonableness assumption is that the DM's belief about the DGP conforms to the objective truth. This assumption is likely to be satisfied in lab experiments where the econometrician has complete control over the DGP and can communicate its parameters to the experiment subjects. The martingale property of belief sequences summarizes the empirical content of this theory.

The second reasonableness assumption is that a DM's decisions fully reflect her belief over the entire state space (and not a section of it). This assumption is likely to be satisfied whenever the econometrician can (i) plausibly constrain the set of informative signals observed by the decision-makers and (ii) elicit their beliefs about any event in their model of the world, including their subjective beliefs about the DGP. I show that such a theory imposes sharp restrictions on the prior-posterior pairs consistent with Bayesian rationality. In particular, the decision-maker's prior belief fully pins down her attainable set of Bayesian posteriors.

Related Literature. The paper most closely related to mine is by Shmaya and Yariv (2016). They also study the testable implications of Bayesian rationality and conclude that it is often hard to reject Bayesianism. However, the two papers are different in their assumptions on what is known a priori by the econometrician and the data available to him and their conclusions. While Shmaya and Yariv focus on experimental settings in which the econometrician has a priori knowledge of the decision-makers' set of signals, in this paper's environment, the econometrician cannot rule out the possibility that the decision-makers observe private signals in between the two periods.

The more fundamental difference between the two papers is in what they assume about the econometrician's ability to track individuals over time. In Shmaya and Yariv (2016), the econometrician observes each decision-maker's belief at a single point in time and instead relies on the variation in beliefs between individuals in different treatment groups. The econometrician's inability to exploit the time variation in beliefs leads him to conclude that *any* observation is consistent with Bayesian rationality (absent additional a priori restrictions on the decision-makers' conjectures about the experiment). By contrast, I allow the econometrician to observe how each decision-maker's belief evolves over time. There is then a simple exception to the notion that anything goes: a decision-maker whose subjective belief in an event goes from zero to a non-zero number between subsequent observations could not have followed Bayes' rule. This paper establishes that this is indeed the only such exception.

The results of the two papers are thus not comparable, being based on different assumptions about what is known and observed. The econometrician's a priori knowledge of the set of signals in Shmaya and Yariv (2016) makes it easier for him to reject the decision-makers' Bayesian rationality, whereas his inability to track their beliefs over time makes it harder to do so. The two papers nonetheless reach similarly negative conclusions. Shmaya and Yariv (2016) show that anything can be rationalized in their setting; a violation of absolute continuity is the only obstruction to rationalizing an observation in mine.

This paper also contributes to the extensive theoretical literature that considers deviations from rational expectations. This literature can be roughly divided into two strands. In the first strand, such as Esponda and Pouzo (2016, 2021), the deviation from rational expectations is due to the agents' misspecified priors.[4] The second strand of the literature studies the implications of non-Bayesian models of behavior such as representativeness and availability heuristics (Tversky and Kahneman, 1974), confirmation bias (Rabin and Schrag, 1999), and diagnostic expectations (Bordalo, Gennaioli, and Shleifer, 2018).[5] The results of this paper clarify the relationship between these two strands of the literature by showing that almost any non-Bayesian updating rule is observationally equivalent to Bayesian updating given a misspecified prior about the DGP.

Outline. The rest of the paper is organized as follows. Section 2 presents the setup and states the main result of the article. Section 3 illustrates the result in the context of a simple example. Section 4 introduces the mathematical apparatus that is required to formally state the paper's central question and presents additional results that refine and elaborate the main result. Section 5 discusses several generalizations. The proofs are relegated to the appendix.

[4] See also Bohren (2016), Fudenberg, Romanyuk, and Strack (2017), Frick, Iijima, and Ishii (2020), Fudenberg, Lanzani, and Strack (2021), Esponda, Pouzo, and Yamamoto (2021), Hauser and Bohren (2021), and the references therein.

[5] See Epstein, Noor, and Sandroni (2010), Molavi, Tahbaz-Salehi, and Jadbabaie (2018), and Cripps (2018) for other examples of non-Bayesian updating rules.

2 Setup

2.1 The Environment

There are two periods indexed by $t = 0, 1$ and a fixed *payoff-relevant state* s that belongs to the measurable space $(S, \mathcal{S})$. There is a population of ex ante identical decision-makers (DMs), indexed by i, who each choose an action $a^*_{it} \in A$ in period t. The DMs' actions depend on their subjective beliefs about the value of the payoff-relevant state. One can think of the actions as those that maximize the expected utility given the DMs' subjective beliefs and a von Neumann-Morgenstern utility function—but the setup is general enough to accommodate other interpretations. I let $\nu^*_{it} \in \Delta S$ denote i's time-t belief about the value of the payoff-relevant state. The DMs may observe informative signals about the value of the payoff-relevant state between any two periods. This would lead them to revise their beliefs in light of the new information.

The second actor is an econometrician who is interested in testing the hypothesis that the DMs are Bayesian. The econometrician knows what is described in the previous paragraph but does not know anything about the signals observed by the DMs between subsequent periods (if any). He observes the panel data $\{(a^*_{i0}, a^*_{i1})\}_{i=1,\dots,n}$, consisting of the actions of a sample of size n of DMs in periods $t = 0, 1$. He wants to devise a test that would allow him to reject the hypothesis that the DMs use Bayes' rule to update their beliefs between the two periods.

The econometrician's task is challenging for a number of reasons. First, the econometrician's data might be a poor approximation to the true population distribution of actions due to sampling errors or correlated signals. Second, Bayesian rationality is a restriction on the evolution of the DMs' beliefs, but the econometrician only observes their actions, and he may not be able to recover the beliefs by observing the actions.

While important in practice, these issues are not particularly interesting for the purpose of this paper; neither are they unique to the problem of characterizing the empirical content of the theory of Bayesian rationality. That being the case, I instead consider an ideal setting in which the econometrician has exceptionally rich data allowing him to overcome these challenges.

2.2 An Ideal Setting

I make three assumptions to allow the econometrician to overcome the challenges just discussed. First, I assume that the mapping from a DM's belief about the payoff-relevant state, v_{it}^*, to her action, a_{it}^*, is invertible and known by the econometrician. This is equivalent to the assumption that the econometrician directly observes the DMs' beliefs about the payoff-relevant state. It allows me to abstract away from the question of whether preferences and beliefs can be separately identified.

Second, I assume that the signals realized between the two periods are independent and identically distributed across the DMs. This assumption implies that $\{(a_{i0}^*, a_{i1}^*)\}_{i=1,\dots,n}$ is a representative sample drawn from the true population distribution of prior-posterior pairs.

Finally, I assume that n is large enough for the sampling error to be negligible. This assumption, together with the second assumption above, implies that the empirical distribution of prior-posterior pairs well approximates the true population distribution from which the DMs' prior-posterior pairs are drawn.

Note that these assumptions all make it easier for the econometrician to reject the DMs' Bayesian rationality. Therefore, they make the negative results of the paper only stronger: if the econometrician cannot rule out the possibility that the DMs are Bayesian under these unrealistically strong assumptions, then a fortiori, he will not be able to rule out their Bayesian rationality in more realistic settings where these assumptions are violated.

Focusing on this ideal setting simplifies the econometrician's problem significantly. I can use the assumption of large n and i.i.d. signals to replace the DMs with a single representative DM with random beliefs. Since the DMs are ex ante identical, the prior of the representative DM is drawn from a degenerate distribution with unit mass at some $v_0^* \in \Delta S$. On the other hand, the DMs may observe different realizations of the signal between periods 0 and 1, and so, may end up with different posteriors. I let $P_1^* \in \Delta(\Delta S)$ denote the distribution of the representative DM's posterior. The assumption that the mapping from beliefs to actions is invertible and known to the econometrician implies that the econometrician can compute (v_0^*, P_1^*) by observing the DM's actions. Whenever there is no risk of confusion, I refer to the representative DM simply as the DM.

2.3 The Main Result

The econometrician's question is then which pairs (v_0^*, P_1^*), consisting of the representative DM's prior and the probability distribution of her posterior about the payoff-relevant state, are consistent with Bayesian rationality. The main result of the paper establishes that any such pair that satisfies the following condition is consistent with Bayesian rationality:

Condition AC. For any ν_1^* in the support of P_1^*, the probability distribution ν_1^* is absolutely continuous with respect to ν_0^* with an essentially bounded Radon–Nikodym derivative.[6]

This is a weak condition. In particular, it reduces to absolute continuity if S is a finite set. It is easy to see that absolute continuity is often necessary for (ν_0^*, P_1^*) to be consistent with Bayesian rationality: if the prior of a Bayesian agent assigns zero probability to an event, her posterior must also assign zero probability to the event—regardless of the set of signals, the agent's belief about the distribution of signals, and the true DGP.[7] What is more surprising is that Condition AC is sufficient for the observed pair (ν_0^*, P_1^*) to be consistent with Bayesian rationality:

Theorem 1. *Suppose the pair (ν_0^*, P_1^*) of observations, consisting of the representative DM's prior and the distribution of her posterior, satisfies Condition AC. Then it is consistent with Bayesian rationality.*

Condition AC thus encompasses the entire empirical content of the hypothesis of Bayesian rationality. Absent additional a priori restrictions on the what constitutes a reasonable model for the DM or the signals observed by her, any belief sequence that satisfies Condition AC is consistent with the DM's Bayesian rationality. Note that Condition AC is only a restriction on the support of the distribution of posteriors, P_1^*, and not on the probabilities of observing different posteriors in the support of P_1^*. In contrast, rational expectations is a much tighter restriction on P_1^*, requiring posteriors to average out to be equal to the prior, i.e., $\nu_0^* = \int \nu_1 P_1^*(d\nu_1)$.

In Section 4, I introduce the notation needed to formally define what it means for (ν_0^*, P_1^*) to be consistent with Bayesian rationality. In the same section I also discuss some generalizations and limitations of the result. But first, I illustrate the result in the context of a simple example.

3 A Simple Example

This section uses a simple example to illustrate the main result of the paper and the construction that is used in its proof. The payoff-relevant state takes values in the set $S = \{H,$

[6]The Radon–Nikodym derivative $f \equiv d\nu_1^*/d\nu_0^*$ is *essentially bounded* if there exists a constant $c < \infty$ and a set $\widehat{S} \in \mathcal{S}$ with $\nu_0^*(\widehat{S}) = 1$ such that $f(s) \leq c$ for all $s \in \widehat{S}$.

[7]To be precise, the absolute continuity condition is necessary for Bayesian rationality only when the agent's prior assigns positive probability to any signal that is realized with positive probability under the true DGP. In histories where the agent observes a signal that she had assumed to have zero probability, Bayes' rule is inapplicable and Bayesian posteriors are unrestricted. This is why I say that absolute continuity is *often* necessary.

$L\}$. In period $t = 0$, the DM's observed prior about the payoff-relevant state is as follows:

$$\nu_0^* = \begin{array}{|c|}\hline 0.5 \\ \hline 0.5 \\ \hline\end{array},$$

where the number in the top cell is the DM's subjective prior that the payoff-relevant state is H and the number in the bottom cell her is belief that the state is L. The observed distribution of the posterior, P_1^*, is given by

$$P_1^* = \frac{1}{4}\begin{array}{|c|}\hline 0.8 \\ \hline 0.2 \\ \hline\end{array} + \frac{3}{4}\begin{array}{|c|}\hline 1.0 \\ \hline 0.0 \\ \hline\end{array}.$$

That is, with a one-quarter probability the DM's belief that the state is H goes up to 0.8, and with the complementary probability the DM becomes certain that the state is H.

Should the observation of this belief sequence lead the econometrician to conclude that the DM is not Bayesian? The answer may seem to be yes at first. After all, this belief sequence does not constitute a martingale with respect to the DGP that generates the signals observed by the DM: the DM *always* becomes more confident in period $t = 1$ that H is the true state. But the martingale test of Bayesian rationality requires the DM's belief to be a martingale only with respect to her subjective belief about the DGP (and not the true DGP). And the DM's belief may satisfy the martingale property with respect to her subjective prior about the DGP—but not with respect to the true DGP that generates the DM's observations.

The observed pair (ν_0^*, P_1^*) is indeed consistent with the DM's Bayesian rationality. This is implied by Theorem 1 by noting that ν_0^* has full support over S, and so, the pair (ν_0^*, P_1^*) satisfies Condition AC. I illustrate how (ν_0^*, P_1^*) can be rationalized by specifying the set of signals, the DM's belief about the DGP, and the true DGP in such a way that the belief sequence of a Bayesian agent matches the observed prior and distribution of posteriors about the payoff-relevant state.

The following construction rationalizes the observations. Suppose the signal observed between the two periods is drawn from the set $\{0.8^+, 1.0^+, 0.8^-, 1.0^-\}$. Between the two periods, the DM learns which of the four signals is realized. The signals are such that the DM's posterior belief that the state is H conditional on observing signal 0.8^+ is equal to 0.8. Likewise, her posterior that the state is H conditional on observing signal 1.0^+ is equal to 1.0. The signals with the minus superscript are counterfactual signals that are needed to make the DM's belief a martingale with respect to her subjective prior.[8]

[8] One single counterfactual signal suffices for rationalizing the DM's observations. But the construction is cleaner if two counterfactual signals are used.

The uncertainty faced by the DM thus can be represented by the probability space $\Omega = \{H, L\} \times \{0.8^+, 1.0^+, 0.8^-, 1.0^-\}$. Let μ_0 denote the DM's subjective prior over Ω. I want μ_0 to satisfy a number of requirements. First, $\mu_0(H|0.8^+) = 0.8$ and $\mu_0(H|1.0^+) = 1.0$, as previously stated. Second, $\mu_0(H|\{0.8^+, 0.8^-\}) = \nu_0^*(H)$ and $\mu_0(H|\{1.0^+, 1.0^-\}) = \nu_0^*(H)$. This ensures that the DM's belief sequence is a martingale sequence with respect to her own subjective prior. Third, $\mu_0(H) = \nu_0^*(H)$. This ensures that the DM's prior belief about the payoff-relevant state is consistent with the observed prior.

These requirements together with the fact that μ_0 is a probability distribution yield six equations for the eight unknown probabilities $\{\mu_0(\omega) : \omega \in \Omega\}$. Condition AC ensures that these equations have a solution for which $\mu_0(\omega) \in [0, 1]$ for all $\omega \in \Omega$. The following table presents one solution to this system of equations:

	0.8^+	1.0^+	0.8^-	1.0^-
H	0.25	0.25	0	0
L	0.0625	0	0.1875	0.25

The DM's information is represented by the sigma-algebra, which is generated by the partition illustrated in red. According to the DM's prior, the probability that the payoff-relevant state is H is 0.5. This is consistent with the observed prior ν_0^*. The DM's posterior belief about the payoff-relevant state equals $\nu_1^*(H) = 0.8$ conditional on the signal being 0.8^+, equals $\nu_1^*(H) = 1.0$ conditional on the signal being 1.0^+, and equals $\nu_1^*(H) = 0$ otherwise.

It only remains to specify the true DGP. Let η denote the true distribution over the set Ω. The DM's posterior belief that the payoff-relevant state is H equals 0.8 if and only if she observes signal 0.8^+, and such a posterior is observed with probability $1/4$ in the econometrician's data set. Therefore, signal 0.8^+ must have $1/4$ probability given the true DGP, i.e., $\eta(0.8^+) = 1/4$. Likewise, it must be that $\eta(0.8^-) = 3/4$. The true distribution is otherwise unrestricted. One solution to these equations is as follows:

	0.8^+	1.0^+	0.8^-	1.0^-
H	0.125	0.375	0	0
L	0.125	0.375	0	0

Under this distribution, signal 0.8^+ is realized with probability 0.25 and signal 1.0^+ is realized with probability 0.75. Therefore, a quarter of the time the DM will have the posterior belief $\nu_1^*(H) = 0.8$, and the remaining three quarters of the time she will have the posterior belief $\nu_1^*(H) = 1.0$. This is exactly the distribution of posteriors observed by the econometrician.

The DM's belief sequence constitutes a martingale with respect to her subjective prior. The observation of signals 0.8^+ and 1.0^+ raises the DM's belief in the high state—hence the + superscript—while the observation of signals 0.8^- and 1.0^- lowers her belief in the high state. According to the DM's subjective prior, the positive and negative signals are just likely enough to make the DM's belief sequence a martingale. Yet under the true distribution, the negative signals are unlikely.

Note that the objective distribution coincides with the DM's subjective prior about the distribution of the payoff-relevant state. In particular, according to both the DM's subjective prior and the true distribution, the two payoff-relevant states are equally likely ex ante. So, the econometrician can rationalize the observed sequence of beliefs without requiring the DM to hold a prior about the payoff-relevant state that disagrees with the true distribution; the decision-maker only needs to be wrong about the DGP. This is a general feature of the construction used in the proof the main theorem.

Conversely, one can rationalize the observed (ν_0^*, P_1^*) pair *only* if the DM has a misspecified belief about the DGP. If the DM were to hold a correctly specified belief, the subjective distribution μ_0 would have to agree with the objective distribution η on the probabilities of different signals. But then the systems of equations that determine μ_0 and η would have no solution for which μ_0 and η are both proper probability distributions.

4 Tests of Bayesian Rationality

In this section, I generalize the insights of the example discussed in the previous section by characterizing the testable implications of Bayesian rationality. The main challenge is to formally express what it means for a decision-maker to be subjectively rational when there is a true underlying data-generating process that determines the signals observed by the DM. I start by introducing a framework that combines elements of subjective and objective probabilities: the econometrician is interested in testing the hypothesis that the DM has an internally consistent *subjective* probability systems but can only use samples drawn from an *objective* distribution. I then formally state the question of whether a belief sequences is consistent with Bayesian rationality.

4.1 Technical Assumptions

I maintain the following standard technical assumptions throughout the remainder of the paper. Every set X is assumed to be a complete separable metric space that is endowed with its corresponding Borel sigma-algebra $\mathcal{X}$. The set of probability distributions over

$(X, \mathcal{X})$ is denoted by ΔX and is endowed with the topology of weak convergence and the corresponding Borel sigma-algebra, which I denote by $\mathcal{B}(\Delta X)$. Finally, I assume that the DM's observed prior, ν_0^*, is a probability distribution over $(S, \mathcal{S})$ and the observed distribution of the DM's posterior, P_1^*, is a probability distribution over $(\Delta S, \mathcal{B}(\Delta S))$.

4.2 Formalism

The underlying uncertainty and the DM's information can be fully captured by a measure space $(\Omega, \mathcal{F})$ and sigma-algebras $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}$. The pair $(\Omega, \mathcal{F})$ is an abstract measurable space that captures all the uncertainty faced by the DM. Each $\omega \in \Omega$ is a complete description of all the variables that affect the DM's decisions. It includes, at a minimum, the value of the payoff-relevant state s, but it also includes the description of any signals that may be observed by the DM between the two periods.

I refer to ω as the *state of the world* to contrast it with s, the payoff-relevant state.[9] I let $\mathbf{S}(\omega) \in S$ denote the value of the payoff-relevant state when the state of the world is given by ω, with $\mathbf{S} : \Omega \to S$ a measurable mapping. The sigma-algebra $\mathcal{F}_t$ captures the DM's information in period t—the DM's actions in period t are measurable with respect to $\mathcal{F}_t$. I assume without loss of generality that $\mathcal{F}_0$ is the trivial sigma-algebra. Therefore, $\mathcal{F}_1$ represents the information content of the signal observed by the DM between periods 0 and 1.

A Bayesian DM's subjective prior about the state of the world can be represented by a probability distribution $\mu_0 : \mathcal{F} \to [0,1]$. The probability distribution μ_0 captures the DM's belief both about the payoff-relevant state and about the distribution of the signal. The DM's induced subjective prior about the payoff-relevant state is given by the probability distribution $\nu_0 : \mathcal{S} \to [0,1]$, defined as

$$\nu_0(B) \equiv \mu_0\left(\mathbf{S}^{-1}(B)\right) \tag{1}$$

for any arbitrary event $B \in \mathcal{S}$.[10]

The DM is Bayesian if her posterior about the state of the world is obtained from her prior by conditioning on the sigma-algebra $\mathcal{F}_1$. This requirement can be stated formally using the concept of regular conditional probability. A mapping $\mu_1 : \Omega \times \mathcal{F} \to [0,1]$ is a *subfield regular conditional probability* for $(\Omega, \mathcal{F}, \mu_0)$ given $\mathcal{F}_1$ if (i) the mapping $\omega \mapsto \mu_1(\omega, B)$ is $\mathcal{F}_1$-measurable for all $B \in \mathcal{F}$, (ii) $\mu_1(\omega, \cdot)$ is a probability distribution over $(\Omega, \mathcal{F})$ for every $\omega \in \Omega$,

[9]While the set Ω is chosen by the econometrician to rationalize his observations, the set of payoff-relevant states is fixed and known to him. The set S thus can be thought of as the largest slice of the state space capturing the underlying uncertainty over which the DM's beliefs can be elicited by the econometrician (either directly or indirectly).

[10]The probability measure ν_0 is known as the *pushforward measure*.

and (iii)

$$\mu_0(B \cap E) = \int_E \mu_1(\omega, B)\mu_0(d\omega) \tag{2}$$

for any $B \in \mathcal{F}$ and $E \in \mathcal{F}_1$.[11] The notion of regular conditional probability is the natural generalization of the elementary notion of conditional probability to probability distributions with uncountable supports. Condition (i) is the measurability requirement: the posteriors need to be the same conditional on any two states that are indistinguishable given $\mathcal{F}_1$. Condition (ii) is the requirement that posteriors are well-defined probability distributions. And condition (iii) is the appropriate statement of Bayes' rule for distributions with uncountable supports. It is the internal consistency requirement that underpins any test of Bayesian updating.

The consistency condition (2) is not directly verifiable by the econometrician. The econometrician only observes the DM's prior and posterior *over* S, but Bayesian rationality requires the DM's prior and posterior to satisfy the consistency requirement (2) *over* Ω—and S is in general only a slice of Ω.[12] Yet equation (2) induces a consistency requirement for the DM's belief over S. More specifically, the regular conditional probability μ_1 and the random variable $\mathbf{S} : \Omega \to S$ define a regular conditional probability $\nu_1 : \Omega \times \mathcal{S} \to [0, 1]$ as follows: for any $\omega \in \Omega$ and $B \in \mathcal{S}$,

$$\nu_1(\omega, B) \equiv \mu_1\left(\omega, \mathbf{S}^{-1}(B)\right). \tag{3}$$

Intuitively, $\nu_1(\omega, B)$ is the DM's posterior belief that the payoff-relevant state belongs to set B conditional on the event that the realized state of the world is ω.

While the DM's prior is a *subjective* probability distribution, the distribution of her posterior depends on the *objective* distribution of the signals she observes. Given the set of states of the world $(\Omega, \mathcal{F})$, the DM's information structure $\mathcal{F}_1$, the mapping $\mathbf{S}$ from the state of the world to the payoff-relevant state, and the DM's prior μ_0, equation (3) defines the DM's posterior belief about the value of the payoff-relevant state. The DM's posterior is a random variable whose realization depends on the signal observed by the DM. Therefore, the distribution of the DM's posterior depends on the objective distribution from which her signal is drawn. I let $\mathbb{P} \in \Delta\Omega$ denote the objective probability distribution that determines the distribution of the signal observed by the DM. Given $((\Omega, \mathcal{F}), \mathbf{S}, \mu_0, \mathcal{F}_1, \mathbb{P})$, the DM's posterior about the payoff-relevant state is distributed according to the probability distribution $P_1 \in \Delta(\Delta S)$, defined as

$$P_1(B_1) = \mathbb{P}\left(\{\omega \in \Omega : \nu_1(\omega, \cdot) \in B_1\}\right) \tag{4}$$

[11] For a proof of the existence of a regular conditional probability when the underlying space is Polish, see Faden (1985).

[12] Refer to footnote 9 for a discussion of the conceptual difference between S and Ω.

for any $B_1 \in \mathcal{B}(\Delta S)$, where ν_1 is the regular conditional probability defined in (3).

4.3 Tests of Bayesian Rationality

I can now use the notation introduced in the previous subsection to formally state what it means for the DM's belief sequence to be consistent with Bayesian rationality.

Definition 1. A pair $\left(\nu_0^*, P_1^*\right)$ of observations consisting of the DM's prior and the distribution of her posterior is *consistent with Bayesian rationality given* $(\Omega, \mathcal{F}, \mathcal{F}_1, \mathbf{S}, \mu_0, \mathbb{P})$ if $\nu_0 = \nu_0^*$ and $P = P^*$, where ν_0 is defined in (1) and P_1 is defined in (4).

The definition considers one extreme case where the econometrician has a priori knowledge about every aspect of the environment: the underlying state space (specifying, among other things, the set of signals), the DM's prior on the entire state space, and the true DGP. Given this knowledge, the only distribution of posteriors that is consistent with the DM's Bayesian rationality and her subjective prior is the one defined in (4). I next consider the other extreme where the econometrician does not have any a priori knowledge about the environment.

Definition 2. A pair $\left(\nu_0^*, P_1^*\right)$ of observations consisting of the DM's prior and the distribution of her posterior is *consistent with Bayesian rationality* if it is consistent with Bayesian rationality given some $(\Omega, \mathcal{F}, \mathcal{F}_1, \mathbf{S}, \mu_0, \mathbb{P})$.

The definition does not confound Bayesian rationality with other restrictions on what constitutes an objectively reasonable model of the world (such as having a correct prior about the DGP). The econometrician rejects the DM's Bayesian rationality in the sense of Definition 2 only if there is no internally consistent subjective probability that rationalizes the observed prior and posterior distribution. It is this definition that is used to state the main question of the paper:

Question 1. Which pairs $\left(\nu_0^*, P_1^*\right)$ of priors and posterior distributions are consistent with the DM's Bayesian rationality?

The question formalizes an intuitive scenario. The econometrician observes the prior ν_0^* and distribution P_1^* of posteriors but has no a priori knowledge of other aspects of the environment. He chooses the tuple $(\Omega, \mathcal{F}, \mathcal{F}_1, \mathbf{S}, \mu_0, \mathbb{P})$ in an attempt to explain his observation as resulting from Bayesian updating by a decision-maker with prior μ_0 in a world described by $(\Omega, \mathcal{F}, \mathcal{F}_1, \mathbf{S}, \mathbb{P})$. If he is unable to find a tuple $(\Omega, \mathcal{F}, \mathcal{F}_1, \mathbf{S}, \mu_0, \mathbb{P})$ that explains his observation, he concludes that the DM is not Bayesian. Theorem 1 establishes that he can rationalize any observation that satisfies Condition AC.

The proof of the theorem is constructive. The construction generalizes the one in the example of Section 3. The econometrician constructs a large enough state space Ω, an objective distribution $\mathbb{P}$, and a subjective belief μ_0 for the DM under which the DM's prior about the probability of the payoff-relevant state s coincides with v_0^* and the distribution of her Bayesian posterior about s coincides with P_1^*. The construction requires the econometrician to postulate an objective probability distribution that is in general different from the subjective prior held by the agent.

But the econometrician observes the DM's prior belief about the payoff-relevant state. Can requiring the objective distribution to respect the observed prior of the DM restrict the set of observations that are consistent with Bayesian rationality? The next theorem shows that the answer to this question is negative. The econometrician can rationalize almost any distribution of posteriors using an objective distribution that agrees with the DM's prior about the distribution of the payoff-relevant state.

Before stating the theorem, I formally define what it means for an objective distribution $\mathbb{P}$ to agree with the DM's prior about the payoff-relevant state. Recall that $(\Omega, \mathcal{F})$ denotes the underlying state space and $\mathbf{S}$ is the random variable that determines the value of the payoff-relevant state as a function of the underlying state. Given $(\Omega, \mathcal{F})$ and $\mathbf{S}$, the distribution of the payoff-relevant state implied by the objective distribution $\mathbb{P}$ is given by

$$\eta_0(B) \equiv \mathbb{P}\left(\mathbf{S}^{-1}(B)\right) \tag{5}$$

for any arbitrary event $B \in \mathcal{S}$.

Definition 3. Given the underlying probability space $(\Omega, \mathcal{F})$ and the random variable $\mathbf{S}$, the objective probability $\mathbb{P}$ *agrees with the subjective prior* v_0^* about the distribution of the payoff-relevant state if $\eta_0 = v_0^*$, where η_0 is defined in (5).

The next theorem is a generalization of Theorem 1. It establishes that requiring agreement with the subjective prior does not put any restrictions, above and beyond Condition AC, on the set of observations that are consistent with Bayesian rationality.

Theorem 2. *Suppose the pair (v_0^*, P_1^*) of observations, consisting of the DM's prior and the distribution of her posterior, satisfies Condition AC. Then it is consistent with Bayesian rationality given an objective probability $\mathbb{P}$ that agrees with the subjective prior v_0^* on the distribution of the payoff-relevant state.*

The theorem has a striking consequence. Bayesian rationality does not impose any meaningful restriction on the distribution of posteriors *even if* the DM's observed prior agrees with the objective distribution of the payoff-relevant state. Even if the econometrician observes the DM's prior belief over the set S and even under the assumption that the DM has a correct prior over S, there is no restriction on the DM's Bayesian posterior other than the absolute continuity Condition AC. Note that the set S can be arbitrarily large—the econometrician may elicit the DM's beliefs about the probabilities of an arbitrarily large set of events. And yet, there is almost no distribution of posteriors that cannot be made a martingale with respect to the DM's prior over S by choosing a sufficiently large Ω.

Intuitively, if the econometrician only observes the dynamics of an DM's belief about the variables that belong to S, then there is always a richer state space Ω that encodes more complex models of the world over which variables in S are defined such that the observed belief dynamic about S is rational given some subjective prior over Ω. The next result further refines Theorem 2 by establishing that the probability space Ω can be chosen independently of the observation $\left(\nu_0^*, P_1^*\right)$ that the econometrician is attempting to rationalize.

Theorem 3. *Given any space of payoff-relevant states* $(S, \mathcal{S})$*, there exists a probability space* $(\Omega, \mathcal{F})$*, a random variable* $\mathbf{S}$*, and a sub-sigma-algebra* $\mathcal{F}_1 \subset \mathcal{F}$ *such that any pair* $\left(\nu_0^*, P_1^*\right)$ *satisfying Condition AC is consistent with Bayesian rationality given* $(\Omega, \mathcal{F}, \mathcal{F}_1, \mathbf{S}, \mu_0, \mathbb{P})$ *for some* μ_0 *and* $\mathbb{P}$.

The results presented so far cast doubts on the possibility of deciding whether decision-makers are Bayesian in non-experimental settings. In such settings, an econometrician may be able to impute the DM's beliefs over some set S. But there is no guarantee that any such set S captures *all* the uncertainty that the DM believes to be relevant to what she thinks about $s \in S$. In particular, given any set S such that the DM's belief on S can be elicited by the econometrician, it may be the case that the relevant uncertainty (from the point of view of the DM) is captured by a larger set Ω. The theorems then show that, for any such set S (capturing all that the econometrician can learn about the DM's beliefs), he can postulate a larger set Ω (capturing all the uncertainty that is relevant to the DM) such that almost any belief sequence over S can be rationalized by fine-tuning the DM's beliefs over Ω.

In experimental settings where the econometrician controls the DM's observations, meanwhile, he can elicit her belief over a set S that can be plausibly assumed to be large enough to capture all the uncertainty that is relevant to the DM's decisions. In other words, in experimental settings the set Ω can be taken to be fixed and known to the econometrician. Moreover, the econometrician can elicit the beliefs over the entire set Ω (so that S coincides with Ω). Then there are relatively tight restrictions on the belief sequences that are consistent with Bayesian rationality. These restrictions are spelled out in the following proposition for the case where S is finite and ν_0^* has full support.[13]

Proposition 1. *Suppose* $(\Omega, \mathcal{F}) = (S, \mathcal{S})$ *and* $\mathbf{S} = id_\Omega$, *where* S *is a finite set. Given a prior* ν_0^* *with full support and a distribution of posteriors* P_1^*, *there exists some* $\mathcal{F}_1$, μ_0, *and* $\mathbb{P}$ *such that* (ν_0^*, P_1^*) *is consistent with Bayesian rationality given* $(\Omega, \mathcal{F}, \mathcal{F}_1, \mathbf{S}, \mu_0, \mathbb{P})$ *if and only if*

(i) $\operatorname{supp} \nu \cap \operatorname{supp} \widehat{\nu} = \emptyset$ *for any distinct* $\nu, \widehat{\nu} \in \operatorname{supp} P_1^*$;

(ii) $\nu(\widehat{S}) = \nu_0^*(\widehat{S} | \operatorname{supp} \nu)$ *for any* $\widehat{S} \in \mathcal{S}$ *and any* $\nu \in \operatorname{supp} P_1^*$.

The proposition clarifies the scope and logic of Theorems 1–3. The theorems rely on the assumption that the econometrician does not have any a priori knowledge about what constitutes a complete description of the uncertainty relevant to the DM's decisions, and so, is unable to observe the DM's beliefs over the underlying state space. He is thus free to reverse-engineer the DM's beliefs over the underlying state space in order to rationalize his observations. Proposition 1 shows how observing what the decision-maker believes about the DGP ties the econometrician's hand.

5 Concluding Remarks

The main result of the paper is that the only testable prediction of Bayesian rationality is the absolute continuity of the posterior with respect to the prior for any prior-posterior pair that is realized with positive probability. As long as this condition is satisfied, any sequence of beliefs and actions can be rationalized as those of a Bayesian agent. The paper also shows that this negative conclusion can be overturned if the beliefs about the DGP itself can be elicited.

I conclude the paper by discussing two modifications of the setup studied in the paper, which do *not* overturn the negative results. Suppose first that the DM's actions or beliefs are observed over a horizon that is longer than two periods. Theorems 1–3 trivially generalize to such a setting: as long as the econometrician does not have any a priori knowledge of the true DGP and does not observe the DM's subjective beliefs about the DGP, the only observations that are not consistent with Bayesian rationality are those that violate absolute continuity.

[13] The extensions to the cases where S is not finite or ν_0^* does not have full support are straightforward. But such extensions are awkward to state since they require care when dealing with zero probability events. I do not pursue those extensions here for the sake of exposition.

A perhaps more interesting modification is to assume that the econometrician observes some—but not all—of the signals observed by the DM and knows the conditional distributions of the observed signals. It remains the case that absolute continuity is the only testable prediction of Bayesian rationality. As long as the econometrician cannot rule out the possibility that the DM observes private signals unbeknown to him, he can reject her Bayesian rationality only if he observes a violation of absolute continuity.

A Proofs

Proof of Theorems 1–3

Since Theorems 1 and 2 are corollaries of Theorem 3, I only prove Theorem 3. Let $\Omega = S \times \Delta S \times \{+,-\}$, and let $\mathcal{F}$ denote the product sigma-algebra. A generic element of Ω is denoted by $(s, \nu^{\diamond})$, where s is an element of S, ν is a probability distribution over S, and $\diamond \in \{+,-\}$. Let $\mathbf{S} : \Omega \to S$ be the canonical projection onto S, that is, the mapping that maps $(s, \nu^{\diamond})$ to s. Let $\mathcal{F}_1$ be the smallest sub-sigma-algebra of $\mathcal{F}$ that makes all sets of the form $S \times \{\nu\} \times \{\diamond\}$ for $\nu \in \Delta S$ and $\diamond \in \{+,-\}$ measurable. That is, given the sigma-algebra $\mathcal{F}_1$, the DM learns the realized value of $\nu^{\diamond}$ between periods zero and one but learns nothing else about ω.

In the remainder of the proof, I fix an observed tuple (ν_0^*, P_1^*) that satisfies Condition AC and show how it can be rationalized by the appropriate choice of the subjective prior μ_0 and the objective distribution $\mathbb{P}$ (together with the tuple $(\Omega, \mathcal{F}, \mathbf{S}, \mathcal{F}_1)$ chosen above). Note that while Ω, $\mathcal{F}$, $\mathbf{S}$, and $\mathcal{F}_1$ are independent of (ν_0^*, P_1^*), the probability distributions μ_0 and $\mathbb{P}$ do depend on it. I specify μ_0 such that the DM's posterior belief over S conditional on observing signal ν^+ is given by ν. I then choose $\mathbb{P}$ such that the probability of signal ν^+ is consistent with the distribution of ν in the observed distribution of posteriors P_1^*.

I start by constructing the regular conditional probability (rcp) $\nu_1^* : \Omega \times \mathcal{S} \to [0,1]$ that represents the DM's posterior belief about the payoff relevant state $s \in S$ conditional on $\mathcal{F}_1$. First, I fix some $\nu \in \operatorname{supp} P_1^*$ and specify $\nu_1^*(\omega, \cdot)$ for all ω of the form $\omega = (s, \nu^{\diamond})$ with $\diamond \in \{+,-\}$ and $s \in S$ arbitrary. Since ν is absolutely continuous with respect to ν_0^* by Condition AC, there exists a Radon–Nikodym derivative $f_\nu \equiv d\nu/d\nu_0^* : S \to \mathbb{R}_+$ such that

$$\nu(\widehat{S}) = \int_{\widehat{S}} f_\nu(s)\nu_0^*(ds)$$

for any $\widehat{S} \in \mathcal{S}$. Let $\epsilon_\nu = 1/\operatorname{ess\,sup}_{s\in S} f_\nu(s)$. Since ν_0^* and ν are both probability measures, $\epsilon_\nu \le 1$. Moreover, by Condition AC, $\epsilon_\nu > 0$. With the definition of ϵ_ν in hand, I can specify $\nu_1^*\left((s, \nu^{\diamond}), \cdot\right)$ for $\diamond \in \{+,-\}$ and $s \in S$. For any $\widehat{S} \in \mathcal{S}$ and $s \in S$, let $\nu_1^*\big((s,\nu^+), \widehat{S}\big) = \nu(\widehat{S})$. If $\epsilon_\nu = 1$, pick $\nu_1^*\big((s,\nu^-), \cdot\big)$ to be an arbitrary probability measure over S; otherwise, let

$$\nu_1^*\big((s,\nu^-), \widehat{S}\big) = \frac{1}{1-\epsilon_\nu}\nu_0^*(\widehat{S}) - \frac{\epsilon_\nu}{1-\epsilon_\nu}\nu(\widehat{S}). \tag{6}$$

I still have to verify that $\nu_1^*\big((s,\nu^-), \cdot\big)$ as defined above is indeed a probability measure over S. In order to see this, first note that $\nu_1^*\big((s,\nu^-), \cdot\big)$ is countably additive since both ν_0^* and ν are probability measures and thus countably additive. Moreover,

$$\nu_1^*\big((s,\nu^-), S\big) = \frac{1}{1-\epsilon_\nu}\nu_0^*(S) - \frac{\epsilon_\nu}{1-\epsilon_\nu}\nu(S) = \frac{1}{1-\epsilon_\nu} - \frac{\epsilon_\nu}{1-\epsilon_\nu} = 1,$$

and

$$\nu_1^*\big((s,\nu^-),\emptyset\big) = \frac{1}{1-\epsilon_\nu}\nu_0^*(\emptyset) - \frac{\epsilon_\nu}{1-\epsilon_\nu}\nu(\emptyset) = 0,$$

where both equalities are due to the fact that ν_0^* and ν are probability measures over S. Finally, for any set $\widehat{S} \in \mathcal{S}$,

$$\begin{aligned}\nu_1^*\big((s,\nu^-),\hat{S}\big) &= \frac{1}{1-\epsilon_\nu}\left(\int_{\widehat{S}} \nu_0^*(ds) - \epsilon_\nu \int_{\widehat{S}} f(s)\nu_0^*(ds)\right)\\ &\geq \frac{1}{1-\epsilon_\nu}\left(\int_{\widehat{S}} \nu_0^*(ds) - \int_{\widehat{S}} \nu_0^*(ds)\right) = 0.\end{aligned}$$

This proves that $\nu_1^*\big((s,\nu^-),\cdot\big)$ is a probability distribution over S. It remains to specify $\nu_1^*\big((s,\nu^\diamond),\cdot\big)$ for $\nu \notin \operatorname{supp} P_1^*$, $\diamond \in \{+,-\}$, and $s \in S$. I simply set $\nu_1^*\big((s,\nu^\diamond),\cdot\big) = \nu_0^*(\cdot)$, indicating that the DM's posterior equals her prior when she observes a signal $\nu^\diamond$ with $\nu \notin \operatorname{supp} P_1^*$. This completes the description of ν_1^*. Note that, by construction, the mapping $\omega \mapsto \nu_1^*(\omega, B)$ is $\mathcal{F}_1$-measurable for any $B \in \mathcal{S}$, as is required for ν_1^* to be a rcp.

I can now construct the probability distribution $\mu_0 : \mathcal{F} \to [0,1]$ that represents the DM's prior. Let λ denote an arbitrary probability distribution over ΔS, and for any measurable sets $\widehat{S} \subseteq S$ and $\widehat{\Delta S} \subseteq \Delta S$, let

$$\begin{aligned}\mu_0(\widehat{S} \times \widehat{\Delta S} \times \{+\}) &\equiv \int_{\widehat{\Delta S}} \nu_1^*(\nu^+, \widehat{S})\epsilon_\nu \lambda(d\nu),\\ \mu_0(\widehat{S} \times \widehat{\Delta S} \times \{-\}) &\equiv \int_{\widehat{\Delta S}} \nu_1^*(\nu^-, \widehat{S})(1-\epsilon_\nu)\lambda(d\nu),\end{aligned}$$

where ν_1^* is as in the previous paragraph and I am using the shorthand notation $\nu_1^*(\nu^+,\widehat{S}) \equiv \nu_1^*\big((s,\nu^+),\widehat{S}\big)$ and $\nu_1^*(\nu^-,\widehat{S}) \equiv \nu_1^*\big((s,\nu^-),\widehat{S}\big)$.

I complete the construction by specifying the true probability distribution $\mathbb{P}$ over Ω. Let $\mathbb{P}$ be the probability measure supported on $S \times \operatorname{supp} P_1^* \times \{+\}$ defined as $\mathbb{P}(\widehat{S} \times \widehat{\Delta S} \times \{+\}) = \nu_0^*(\widehat{S})P_1^*(\widehat{\Delta S})$ for any measurable sets $\widehat{S} \subseteq S$ and $\widehat{\Delta S} \subseteq \Delta S$. That is, according to $\mathbb{P}$, only states in $S \times \operatorname{supp} P_1^* \times \{+\}$ have positive probability, the probability that the DM observes signal ν^+ is given by the frequency of posterior ν in the target distribution P_1^*, and the objective probability distribution $\mathbb{P}$ agrees with the DM's observed prior ν_1^* over the set of payoff-relevant variables S. This completes the construction.

To complete the proof, I need to show that $\nu_0 = \nu_0^*$ and that ν_1 is distributed according to P_1^* given the objective prior $\mathbb{P}$, where ν_0 and ν_1 are defined in equations (1) and (3), respectively.

By the law of total probability, for any set $\widehat{S} \in \mathcal{S}$,

$$\begin{aligned}
\nu_0(\widehat{S}) &= \mu_0(\mathbf{S}^{-1}(\widehat{S})) \\
&= \mu_0(\widehat{S} \times \Delta S \times \{+\}) + \mu_0(\widehat{S} \times \Delta S \times \{-\}) \\
&= \int_{\Delta S} \Big(\nu_1^*(\nu^+, \widehat{S})\epsilon_\nu + \nu_1^*(\nu^-, \widehat{S})(1-\epsilon_\nu) \Big) \lambda(d\nu) \\
&= \int_{\Delta S} \nu_0^*(\widehat{S})\lambda(d\nu) = \nu_0^*(\widehat{S}),
\end{aligned}$$

where in the fourth equality I am using (6) and the fact that $\nu_1^*(\nu^+, \widehat{S}) = \nu(\widehat{S}) = \nu_0^*(\widehat{S})$ for any set $\widehat{S} \in \mathcal{S}$ whenever $\epsilon_\nu = 1$.

I next characterize the subfield rcp, μ_1, for $(\Omega, \mathcal{F}, \mu_0)$ given $\mathcal{F}_1$, defined in (2). I do so by guessing some μ_1 and verifying that my guess is indeed the subfield rcp. For any $\omega = (s, \nu^\diamond) \in \Omega$ and measurable sets $\widehat{S} \subseteq S$ and $\widehat{\Delta S} \subseteq \Delta S$, let

$$\begin{aligned}
\mu_1\big((s, \nu^\diamond), \widehat{S} \times \widehat{\Delta S} \times \{+\}\big) &\equiv \nu_1^*(\nu^\diamond, \widehat{S})\mathbb{1}\{\nu \in \widehat{\Delta S}\}\mathbb{1}\{\diamond = +\}, \\
\mu_1\big((s, \nu^\diamond), \widehat{S} \times \widehat{\Delta S} \times \{-\}\big) &\equiv \nu_1^*(\nu^\diamond, \widehat{S})\mathbb{1}\{\nu \in \widehat{\Delta S}\}\mathbb{1}\{\diamond = -\}.
\end{aligned}$$

Since the sigma algebra $\mathcal{F}$ is generated by the sets of the form $\widehat{S} \times \widehat{\Delta S} \times \{\diamond\}$ for $\diamond \in \{+, -\}$, the above equations uniquely determine the probability measure $\mu_1(\omega, \cdot)$ over $(\Omega, \mathcal{F})$. Moreover, since the mapping $\omega \mapsto \nu_1^*(\omega, B)$ is $\mathcal{F}_1$-measurable for any $B \in \mathcal{S}$, the mapping $\omega \mapsto \mu_1(\omega, B)$ is also $\mathcal{F}_1$-measurable for any $B \in \mathcal{F}$. Thus, to show that μ_1 is the subfield rcp for $(\Omega, \mathcal{F}, \mu_0)$ given $\mathcal{F}_1$, I only need to show that it satisfies (2). Since $\mathcal{F}$ is generated by sets of the form $\widehat{S} \times \widehat{\Delta S} \times \{\diamond\}$ for $\diamond \in \{+, -\}$ and $\mathcal{F}_1$ is generated by sets of the form $S \times \widehat{\Delta S} \times \{\diamond\}$ for $\diamond \in \{+, -\}$, I only need to check (2) for sets B of the form $B = \widetilde{S} \times \widetilde{\Delta S} \times \{\widetilde{\diamond}\}$ and E of the form $E = S \times \widehat{\Delta S} \times \{\widehat{\diamond}\}$. If $\widehat{\diamond} \neq \widetilde{\diamond}$, then both sides of (2) are trivially zero. So I assume that $\widehat{\diamond} = \widetilde{\diamond}$. I only present the argument for the case $\widehat{\diamond} = \widetilde{\diamond} = +$; the case $\widehat{\diamond} = \widetilde{\diamond} = -$ is similar. By construction,

$$\begin{aligned}
\int_E \mu_1(\omega, B)\mu_0(d\omega) &= \int_{S \times \widehat{\Delta S} \times \{+\}} \nu_1^*(\nu^\diamond, \widetilde{S})\mathbb{1}\{\nu \in \widetilde{\Delta S}\}\mathbb{1}\{\diamond = +\}\mu_0(ds \times d\nu \times d\diamond) \\
&= \int_{S \times (\widehat{\Delta S} \cap \widetilde{\Delta S}) \times \{+\}} \nu_1^*(\nu^+, \widetilde{S})\mu_0(ds \times d\nu \times d\diamond) \\
&= \int_{\widehat{\Delta S} \cap \widetilde{\Delta S}} \nu_1^*(\nu^+, \widetilde{S})\nu_1^*(\nu^+, S)\epsilon_\nu \lambda(d\nu) \\
&= \int_{\widehat{\Delta S} \cap \widetilde{\Delta S}} \nu_1^*(\nu^+, \widetilde{S})\epsilon_\nu \lambda(d\nu) \\
&= \mu_0\big(\widetilde{S} \times (\widehat{\Delta S} \cap \widetilde{\Delta S}) \times \{+\}\big) = \mu_0(B \cap E),
\end{aligned}$$

where the first equality is by the definition of μ_1, the third and fifth equalities are by the definition of μ_0, and the fourth inequality is using the fact that $\nu_1^*(\omega, S) = 1$ for all $\omega \in \Omega$.

Finally, I show that $\nu_1 = \nu_1^*$ and $P_1 = P_1^*$, where ν_1 is defined as in equation (3) and P_1 is defined as in equation (4). That $\nu_1 = \nu_1^*$ trivially follows the definition of μ_1. On the other hand, under the objective prior $\mathbb{P}$, slices $S \times \{\nu^+\}$ are distributed according to $P_1^*(\nu)$ while $S \times \Delta S \times \{-\}$ has zero probability. Therefore, P_1 as defined in (4) is equal to the observed distribution of posteriors P_1^*. The proof is complete once I argue that $\mathbb{P}(\mathbf{S}^{-1}(\widehat{S})) = \nu_0^*(\widehat{S})$ for all sets $\widehat{S} \in \mathcal{S}$. But this is trivially true by construction.

$\square$

Proof of Proposition 1

The "if" direction. Let $S(\nu) \in \mathcal{S} = \mathcal{F}$ denote the support of $\nu \in \operatorname{supp} P_1^*$, and let $S^c(\nu)$ denote its complement. Define

$$\mathcal{P} = \left\{S(\nu) : \nu \in \operatorname{supp} P_1^*\right\} \cup \left\{\bigcap_{\nu \in \operatorname{supp} P_1^*} S^c(\nu)\right\}.$$

Since $\operatorname{supp} \nu \cap \operatorname{supp} \widehat{\nu} = \emptyset$ for any distinct $\nu, \widehat{\nu} \in \operatorname{supp} P_1^*$ by condition (i) of the proposition, $\mathcal{P}$ is a partition of Ω. Let $\mathcal{F}_1 \subseteq \mathcal{F}$ denote the sigma-algebra over $\Omega = S$ generated by $\mathcal{P}$, let $\mu_0 = \nu_0^* \in \Delta S = \Delta\Omega$, and let $\mathbb{P}$ be any probability measure over $\Omega = S$ that satisfies $\mathbb{P}(S(\nu)) = P_1^*(\nu)$ for all $\nu \in \operatorname{supp} P_1^*$. By construction, $\nu_0 = \mu_0 \circ \mathbf{S}^{-1} = \mu_0 = \nu_0^*$.

I next show that $P_1 = P_1^*$, where P_1 is defined in equation (4). Note that $\mu_1(\omega, \cdot)$ is a probability distribution over $(\Omega, \mathcal{F})$ for any $\omega \in \Omega$; it takes value $\mu_0(\cdot|\widehat{\Omega})$ whenever the state of the world ω belongs to cell $\widehat{\Omega}$ of partition $\mathcal{P}$. Therefore, since $\mathbf{S}$ is the identity mapping, ν_1 as defined in equation (3) takes value $\mu_0(\cdot|\widehat{\Omega}) = \nu_0(\cdot|\widehat{\Omega}) = \nu_0^*(\cdot|\widehat{\Omega})$ whenever ω belongs to $\widehat{\Omega}$. Recall that $S(\nu) = \operatorname{supp} \nu$ and $\nu_0^*(\cdot|\operatorname{supp} \nu) = \nu(\cdot)$ by condition (ii) of the proposition. Hence, ν_1 takes value ν whenever $\omega \in S(\nu)$. Finally, note that by construction $S(\nu)$ has probability $P_1^*(\nu)$ according to the objective probability measure $\mathbb{P}$. Therefore, ν_1 is equal to ν with probability $P_1^*(\nu)$ and so $P_1 = P_1^*$.

The "only if" direction. I first show (ii). Since $\Omega = S$ is a finite set and $\mu_0 = \mu_0 \circ \mathbf{S}^{-1} = \nu_0 = \nu_0^*$ has full support over Ω, for any $\mathcal{F}_1$ there exists a partition $\mathcal{P} = \{\Omega_k : k \in K\}$ of Ω such that $\mu_1(\omega, \cdot)$ takes value $\mu_0(\cdot|\widehat{\Omega}) \in \Delta S = \Delta\Omega$ whenever the state of the world ω belongs to cell $\widehat{\Omega}$ of $\mathcal{P}$. Therefore, $\nu_1(\omega, \cdot) = \mu_1(\omega, \mathbf{S}^{-1}(\cdot)) = \mu_1(\omega, \cdot)$ only takes values in the set $\{\mu_0(\cdot|\Omega_k) : k \in K\}$. Moreover, using the assumption that μ_0 has full support over Ω and Bayes' rule, I can conclude that $\operatorname{supp} \mu_0(\cdot|\Omega_k) = \Omega_k$ for all $k \in K$. On the other hand, $\mu_0 = \mu_0 \circ \mathbf{S}^{-1} = \nu_0 = \nu_0^*$. Therefore, for any realization ν of ν_1, as defined in (3), and any set $\widehat{S} \in \mathcal{S}$, we have that $\nu(\widehat{S}) = \nu_0^*(\widehat{S}|\operatorname{supp} \nu)$. Noting that, when $\mathbb{P}$ is the objective prior, ν_1 is distributed according to P_1, defined in (4), and that $P_1^* = P_1$ completes the proof of (ii).

I show (i) by contradiction. Toward a contradiction, suppose that there exist distinct ν, $\widehat{\nu} \in \operatorname{supp} P_1^*$ such that $\operatorname{supp} \nu \cap \operatorname{supp} \widehat{\nu} = S_0 \neq \emptyset$. Note that the supports of ν and $\widehat{\nu}$ cannot be equal because otherwise by (ii) ν and $\widehat{\nu}$ are not distinct. Let $S(\nu) = \operatorname{supp} \nu \setminus S_0$ and let $S(\widehat{\nu}) = \operatorname{supp} \widehat{\nu} \setminus S_0$. Since $\operatorname{supp} \nu \neq \operatorname{supp} \widehat{\nu}$, at least one of $S(\nu)$ and $S(\widehat{\nu})$ is non-empty. Without loss of generality, assume that $S(\nu) \neq \emptyset$. Since $P_1^* = P_1$, where P_1 is defined in (4), ν and $\widehat{\nu}$ are also realizations of $\mu_0(\mathbf{S}^{-1}(\cdot)|\mathcal{F}_1) = \mu_0(\cdot|\mathcal{F}_1)$. Therefore, $\operatorname{supp} \nu$ and $\operatorname{supp} \widehat{\nu}$ are measurable with respect to $\mathcal{F}_1$, and so are S_0, $S(\nu)$, and $S(\widehat{\nu})$ since they are intersections of measurable sets. Therefore,

$$\mu_0(\mathbf{S}^{-1}(S_0)|\mathcal{F}_1) = \mu_0(S_0|\mathcal{F}_1) = \mathbb{1}\{\omega \in S_0\} \in \{0,1\}. \tag{7}$$

In other words, the agent learns whether the state belongs to set S_0 given the information revealed by $\mathcal{F}_1$ in between periods 0 and 1. On the other hand, since (i) S is a finite set, (ii) neither of S_0 and $S(\nu)$ is empty, and (iii) $\operatorname{supp} \nu = S_0 \cup S(\nu)$, we have that $\nu(S_0) \in (0,1)$. Thus, equation (7) implies that ν cannot be a realization of P_1, contradicting the assumption that $\nu \in \operatorname{supp} P_1^* = P_1$. □

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