Introduction to Parameter Estimation

87203 – Multimedial Signal Processing 1st Module

Politecnico di Milano –
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Agenda

- Ex1: Mean Estimation
- Ex2: Variance Estimation
- Ex3: Autocorrelation Estimation
- Ex4: Maximum Likelihood Estimation

Exercise 1 (1/6)

- Goal of estimation theory: estimate the value of an unknown parameter (deterministic or random) from a set of observation of a random variable:
  - ES: estimate mean (deterministic) from N samples of a random variable x
  - ES: estimate phase (random) from N samples of a signal:

$$x(n,\vartheta) = \cos(\omega_1 n + \vartheta)$$

Oss: $x(n,\vartheta)$ is a random sequence because is function of the random variable ϑ.

Exercise 1 (2/6)

- HINT: The estimate value $\hat{\vartheta}$ is a random variable because is function of a random variable x (observations).
  - BIAS: $B = \vartheta - E[\hat{\vartheta}_N]$
  - CONSISTENCE:
    - mean $\lim_{N \to \infty} B = 0$
    - variance $\lim_{N \to \infty} \mathrm{var}\{\hat{\vartheta}_N\} = \lim_{N \to \infty} \left\{ \left(\hat{\vartheta}_N - E[\hat{\vartheta}_N] \right)^2 \right\} = 0$
    - MSE $\lim_{N \to \infty} E\left\{ \left(\hat{\vartheta}_N - \vartheta \right)^2 \right\} = 0$

Exercise 1 (3/6)

- Goal: Using R realizations of a gaussian noise, estimate the (temporal) mean of the process using two different estimators:
  - Sample mean $$\mu s = \frac{1}{N}\sum_{n=1}^{N} x(n)$$
  - Average between maximum and minimum value of the realization $$\mu a = \frac{\max x(n) + \min x(n)}{2}$$

Exercise 1 (4/6)

- **Hints**:
  - "X = sqrt(v)*randn(N,M)+m" builds a N by M matrix containing Gaussian variables of mean "m" and variance "v".
  - If x is a row/column vector
    - "y = mean(x)" returns the sample mean of vector x
    - "y = var(x)" returns the variance of vector x
    - "y = max(x)" and "y = min(x)" return, respectively, the maximum and minimum value of vector x.

Exercise 1 (5/6)

- Pseudocode:
  - generate R iid sequence of N samples with gaussian distribution $x \approx N(\eta_x, \sigma_x^2)$
    - y = sqrt(v)*randn(R,N)+m ;
  - for each realization estimate mean with two estimators:
    - for r = 1:R

      mS(r) = mean(x(r,:));

      mM(r) = (max(x(r,:)) + min(x(r,:)))/2;
    - end
  - Compute mean and variance of the two estimators

Exercise 1 (6/6)

- **Question**: try different sequence lengths N: what is the behavior in terms of the expectation and variance of the estimators?

Exercise 2 (1/4)

- Goal: Using R realizations of a gaussian noise, estimate the (temporal) variance of the process using the following (biased) estimator:

  - Sample variance $$\operatorname{var} s = \frac{1}{N}\sum_{i=1}^{N}(x_i - \hat{\mu})^2$$

- Compute the bias of the estimated value

Exercise 2 (2/4)

- Hints:
  - Remember that the bias of an estimator is:

$$B = \theta - E[\hat{\theta}]$$

  - The expectation of the sample variance can also be computed analytically

$$E[\hat{\theta}] = \frac{M-1}{M}\vartheta$$

Exercise 2 (3/4)

- Pseudocode:
  - generate R iid sequence of N samples with gaussian distribution $x \approx N(\eta_x, \sigma_x^2)$
    - y = sqrt(v)*randn(R,N)+m ;
  - for each realization estimate the variance:
    - for r = 1:R

      vS(r) = var(x(r,:));
    - end
  - Compute mean and variance of the estimator

Exercise 2 (4/4)

- **Question**:
  - Which is the trend of the bias increasing the length of sequence?

Exercise 3 (1/8)

- Goal: Using R realizations of a gaussian process, estimate the autocorrelation of the process with two different estimators:

  - **Method 1**: estimation of the autocorrelation assuming ergodic processes

  - **Method 2**: estimation of the autocorrelation using set average

Exercise 3 (2/8)

- **Hints on method 1**:
  - Given a realization of a random process, compute its autocorrelation using the following equation (biased):

$$\hat{r}(k) = \frac{1}{N} \sum_{t=k+1}^{N} y(t) y^*(t-k), \quad k \geq 0$$

Exercise 3 (3/8)

- Pseudocode:
  - generate R iid sequence of N samples with gaussian distribution $x \approx N(\eta_x, \sigma_x^2)$
    - y = sqrt(v)*randn(R,N)+m ;
  - for each realization estimate the autocorrelation:
    - For each lag of the autocorrelation k
      - For each sample y(t) of the sequence:
        - Compute the product y(t)*y(t-k)
        - Sum the product in the buffer

Exercise 3 (4/8)

- **Hints on method 2**:
  - Given a set of realization of a random process, compute the autocorrelation according to the following equation:

$$r(k) = E[x(t)x(t-k)]$$

  - The expectation is taken over all the realizations

Exercise 3 (5/8)

- Pseudocode:
  - generate R iid sequence of N samples with gaussian distribution $x \approx N\,(\eta_x, \sigma_x^2)$
    - y = sqrt(v)*randn(R,N)+mean ;
  - For each lag of the autocorrelation m
    - For each sample k
      - For each realization i
        - vec(i)=vec(i)+x(k,i)*x(k-m+1,i);
      - Compute the mean of vec
    - r(lag) = r(lag) + mean(vec)

Exercise 3 (6/8)

- Hint: Matlab provides the function

  “[r lag]=xcorr(x)”

  that produces an estimate of the autocorrelation (2N-1 samples) of the stationary sequence “x”.

  “lag” is the vector of lag indices [-N+1:1:N-1].

Exercise 3 (7/8)

- Goal: Define a AR(1) random process with the difference equation:

$$x(n) = \rho x(n-1) + z(n) \qquad z(n) \approx N\,(0,1)$$

- Generate R realizations of the process, each having length N.
- For each realization, provide an estimate of the autocorrelation r(k).

Exercise 3 (8/8)

- The true autocorrelation function is given by

$$r_x(k) = \sigma_x^2 \rho^{-|k|} = \frac{\sigma_z^2}{1-\rho^2} \rho^{-|k|}$$

- Compute $E[\hat{r}(k)]$ and compare it with r(k) to verify the bias.
- Compute $E\left[\left(\hat{r}(k) - E[\hat{r}(k)]\right)^2\right]$ and plot is as a function of k.

What is the effect of N on the variance?

Exercise 4 (1/4)

- Goal of Maximum Likelihood estimation: estimate the value of an unknown parameter (deterministic) from a N-point dataset:

$$x(n) = s(\vartheta) + w \qquad \hat{\vartheta} = g(\{x(n)\})$$

- ML: $\hat{\vartheta} = \arg\max_{\vartheta}\left\{f(\vec{x} \mid \vec{\vartheta})\right\} = \arg\max_{\vartheta}\left\{L(\vec{\vartheta} \mid \vec{x})\right\}$

Exercise 4 (2/4)

- Goal: Estimate the mean of a Gaussian process given the observed data x and the variance σ_x^2
- Hint: $L(\vec{\vartheta} \mid \vec{x}) = \left(\frac{1}{\sqrt{2\pi\sigma^2}} \right)^N \exp\left\{ -\frac{1}{2\sigma^2} \sum_{n=1}^{N} (x_n - \vartheta)^2 \right\}$

Exercise 4 (3/4)

- Pseudocode:
  - generate N samples of the gaussian process
    - x = sqrt(varx)*randn(N,1)+theta ;
  - compute the likelihood function:
    - thetas =[-6:0.01:6]';
    - L = ones(length(thetas),1);
    - for n=1:N % for each sample
      - L = L.*(1/(sqrt(2*pi*varx))).*
        exp((-1/(2*varx))*(x(n)-thetas).^2);
      - % is the product of the marginal function
    - end

Exercise 4 (4/4)

- Pseudocode (continued):
  - find maximum value of likelihood function and its index:
    - [Lmax ind]=max(L);
    - thetast=thetas(ind)

- OSS: L is a function of theta --> it assume a different value for each theta

$$L(\vec{\vartheta} \mid \vec{x}) = \left\lfloor L(\vartheta_1 \mid \vec{x}) \quad L((\vartheta_2 \mid \vec{x}) \quad \ldots \quad L((\vartheta_{l_\vartheta} \mid \vec{x}) \right\rfloor$$