The Z-transform

Multimedial Signal Processing 1st Module

Politecnico di Milano –
Polo regionale di Como

Particulars

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Summary:

- Ex1: The bilateral z-transform and its property
  - Definition
  - ROC
  - Properties:
    - Linearity
    - Frequency shifting
    - Complex conjugation
    - Multiplication

    Sample shifting

    Folding

    Differentiation in the z-domain

    Convolution
- Ex2: Inversion of the z-transform
- Ex3: System representation in the z-domain

Exercise 1 (1/7)

- The z-transform of a sequence x(n) is:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

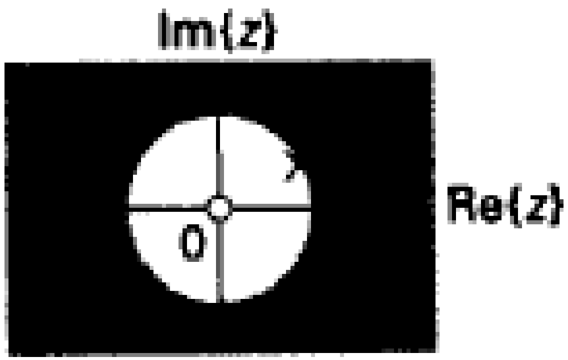
  - Where z is a complex variable $z = |z|e^{j\phi}$

  - The set of values for wich X(z) exist is called the region of convergence (ROC):

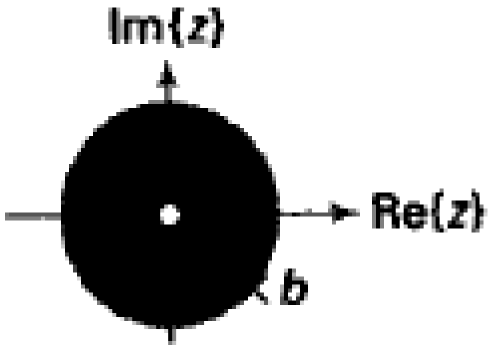
$$R_{x-} \leq |z| \leq R_{x+}$$

Exercise 1 (2/7)

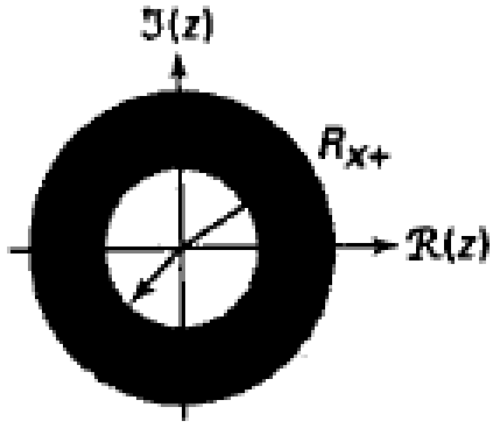
- Since the ROC is defined in terms of magnitude, the shape of the ROC is an open ring:



$a < |z| < \infty$



$0 < |z| < b$



$|a| < |z| < |b| \quad if\ |a| < |b|$

- N.B. The ROC is one contiguous region

Exercise 1 (3/7)

- PROPERTIES:
  - Linearity:

$$Z\left\{\alpha x_1(n)+\beta x_2(n)\right\}=\alpha Z\left\{x_1(n)\right\}+\beta Z\left\{x_2(n)\right\}$$

$$ROC : ROC_{x1} \cap ROC_{x2}$$

  - Sample shifting:

$$Z\left\{x(n-k)\right\}= X\left(z\right)\, z^{-k} \qquad ROC : ROC_{x}$$

Exercise 1 (4/7)

- PROPERTIES:
  - Frequency shifting:

$$Z\left\{x(n)\,a^n\right\} = X\left(\frac{z}{a}\right) \qquad ROC: ROC_x \text{ scaled by } |a|$$

  - Folding:

$$Z\left\{x(-n)\right\} = X\left(\frac{1}{z}\right) \qquad ROC: \text{inverted } ROC_x$$

Exercise 1 (5/7)

- PROPERTIES:
  - Complex conjugation:

$$Z\left\{x^*(n)\right\} = X^*\left(z^*\right) \qquad ROC: ROC_x$$

  - Differentiation in the z-domain:

$$Z\left\{n\,x(n)\right\} = -z\,\frac{dX(z)}{dz} \qquad ROC: ROC_x$$

Exercise 1 (6/7)

- PROPERTIES:
  - Multiplication:

$$Z\{x_1(n)\cdot x_2(n)\} = \frac{1}{2\pi j}\oint_C X_1(v)\,X_2(z/v)\,v^{-1}dv$$

$$ROC : ROC_{x1} \cap invertedROC_{x2}$$

  - Convolution:

$$Z\{x_1(n) * x_2(n)\} = X_1(z)X_2(z) \quad ROC : ROC_{x1} \cap ROC_{x2}$$

Exercise 1: convolution (7/7)

$$Z\{x_1(n) * x_2(n)\} = X_1(z)X_2(z) \quad ROC: ROC_{x1} \cap ROC_{x2}$$

EXAMPLE 4.4 Let $X_1(z) = 2 + 3z^{-1} + 4z^{-2}$ and $X_2(z) = 3 + 4z^{-1} + 5z^{-2} + 6z^{-3}$. Determine $X_3(z) = X_1(z)X_2(z)$.

- x1 = [2 3 4];
- x2 = [3 4 5 6];
- x3 = conv(x1,x2) = 6 17 34 43 38 24
- $X_3(z) = 6 + 17z^{-1} + 34z^{-2} + 43z^{-3} + 38z^{-4} + 24z^{-5}$

  $X_3(z) = \left(2 + 3z^{-1} + 4z^{-2}\right)\left(3 + 4z^{-1} + 5z^{-2} + 6z^{-3}\right)$

Exercise 1: convolution (7/7)

$$Z\{x_1(n) * x_2(n)\} = X_1(z)X_2(z) \quad ROC: ROC_{x1} \cap ROC_{x2}$$

EXAMPLE 4.5 Let $X_1(z) = z + 2 + 3z^{-1}$ and $X_2(z) = 2z^2 + 4z + 3 + 5z^{-1}$. Determine $X_3(z) = X_1(z)X_2(z)$.

- x1 = [1 2 3];   n1 = [-1:1];
- x2 = [3 4 5 6];   n2 = [-2:1];
- [x3, n3] = conv_m(x1,n1,x2,n2)
- x3 =   2   8   17   23   19   15
- n3 =   -3   -2   -1   0   1   2

Exercise 2 (1/18)
Fundamental theorem of algebra

- Every non-zero single-variable polynomial with complex coefficients has exactly as many complex roots as its degree, if each root is counted up to its multiplicity

$$A(z)=\sum_{n=0}^{N} a_n z^{-n} = a_0 \prod_{n=1}^{N}\left(1-z_n z^{-1}\right)$$

- In order to evaluate the output of LTI system it is sufficient to convolve the input transform with each of the N elementar sequences

$$X(z) \longrightarrow \boxed{H(z)} \longrightarrow Y(z)=H(z)\,X(z)$$

Exercise 2 (2/18)

- Fourier Transform:
  - If the ROC of H(z) includes the unit circle, then we can evaluate H(z) on the unit circle:

$$A(z) = 1 - z_0 z^{-1} = \frac{z - z_0}{z} \qquad \left(z = e^{j\phi}\right)$$

Ratio between 2 complex vectors:

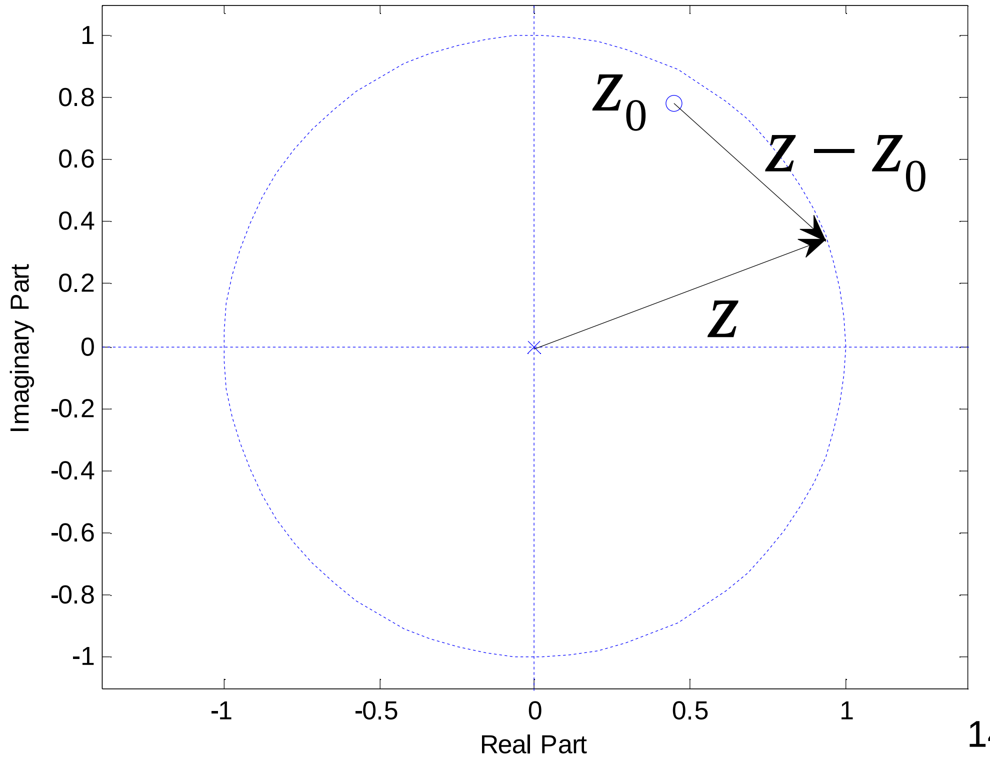
- $\left(z - z_0\right)$ is a vector from the zero to the unit circle
- z is the vector to the unit circle, with fase equal to φ.

$$\left|A(z)\right| = \left|z - z_0\right| \qquad \measuredangle A(z) = \measuredangle\left(z - z_0\right) - \measuredangle(z)$$

Exercise 2 (3/18)

$$A(z)=\frac{z-z_0}{z}=\frac{e^{j\phi}-z_0}{e^{j\phi}}$$

- $(z-z_0)$ is a vector from the zero to the unit circle
- z is the vector to the unit circle, with fase equal to φ.



Exercise 2 (4/18)

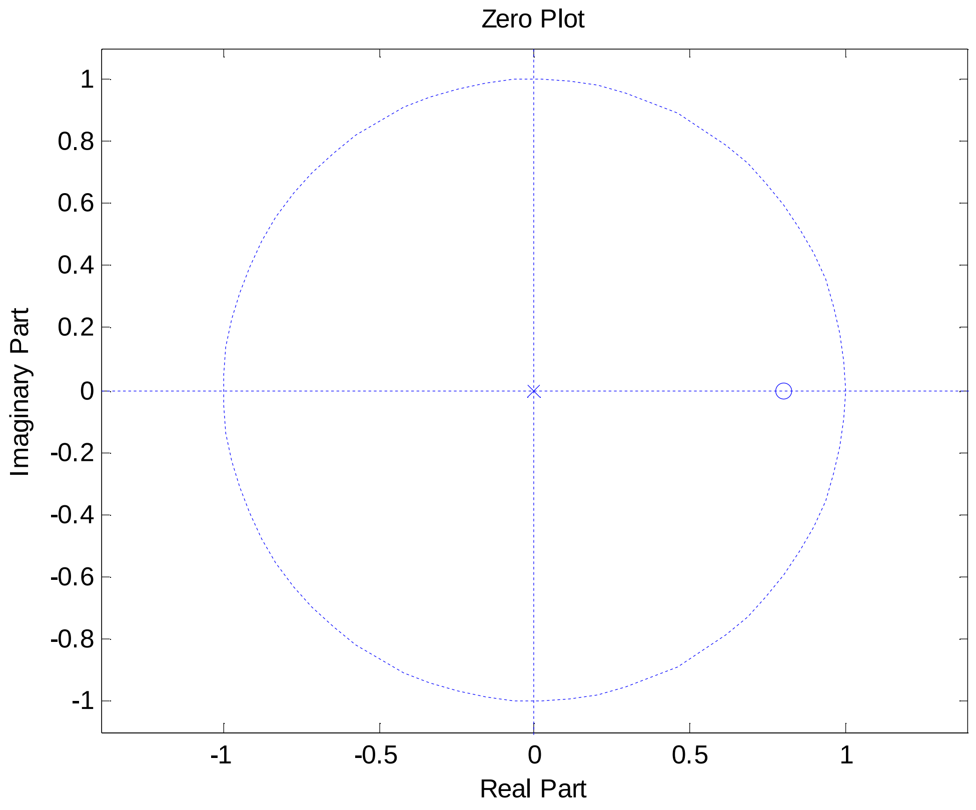
- Matlab provides the functions:
  - "zplane (a,1)" plot in the complex plane the position of the zero (°) of the polynomial A(z) which coefficients are the elements of a.
  - Matlab provides the function "[A w]=freqz(a,1,N)" that returns the N-point complex frequency response "A" and the N-point frequency vector "w" in radians/sample of the filter given numerator coefficients in vectors "a". The frequency response is evaluated at N points equally spaced around the upper half of the unit circle. If N isn't specified, it defaults to 512.

Exercise 2 (5/18)

- Example 1:

$$\{a_n\} = \{1, -0.8\}$$

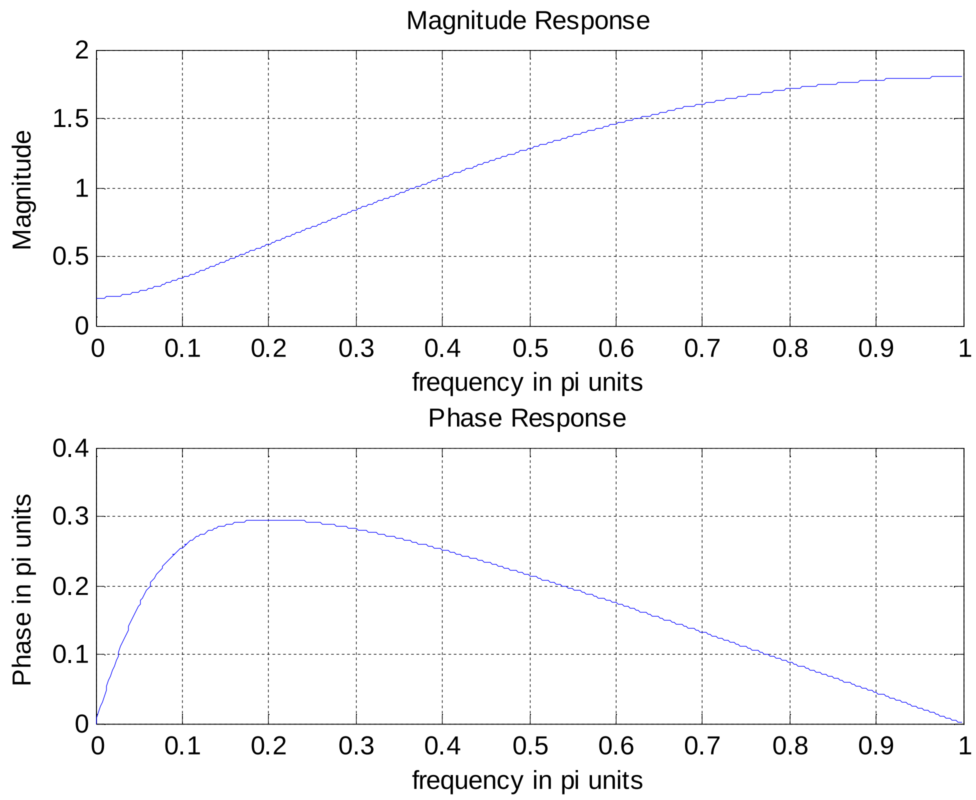
$$A(z) = \frac{z - 0.8}{z}$$



- figure, zplane(a,1), title('Zero Plot');

Exercise 2 (6/18)

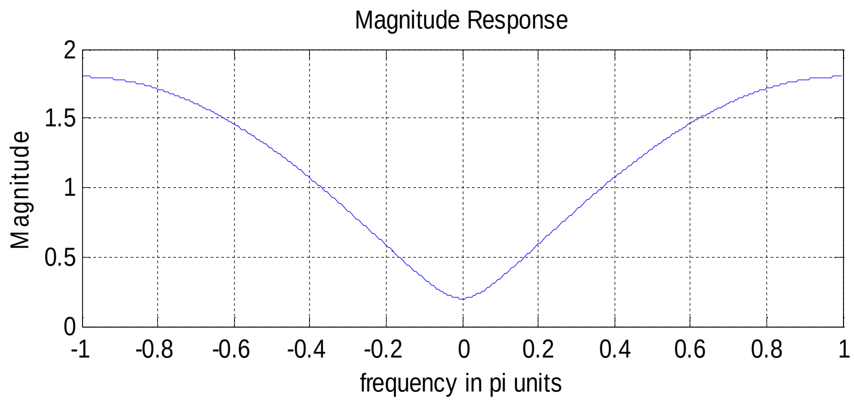
- Example 1:

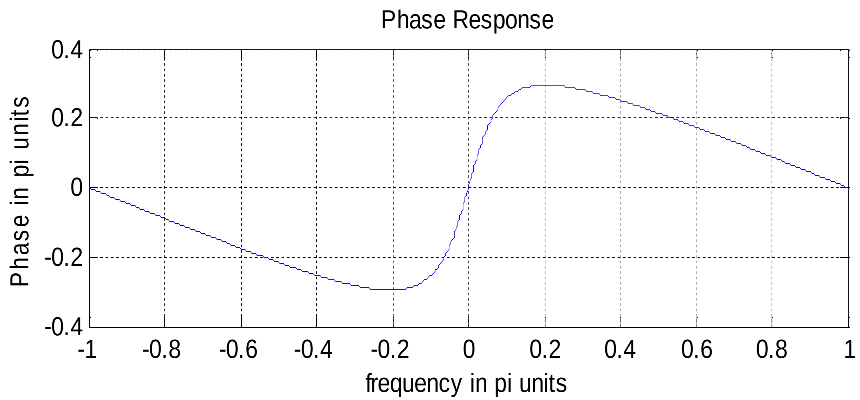


- [A,w] = freqz(a,1);
- magA = abs(A);
- phaA = angle(A);
- figure, subplot(2,1,1);plot(w/pi,magA);grid
- subplot(2,1,2);plot(w/pi,phaA/pi) ;grid

Exercise 2 (7/18)

- Represent also the negative frequencies:
  - w=[-flipud(w(2:end)); w];
  - % flipping the rows
  - A=[conj(flipud(A(2:end))); A];



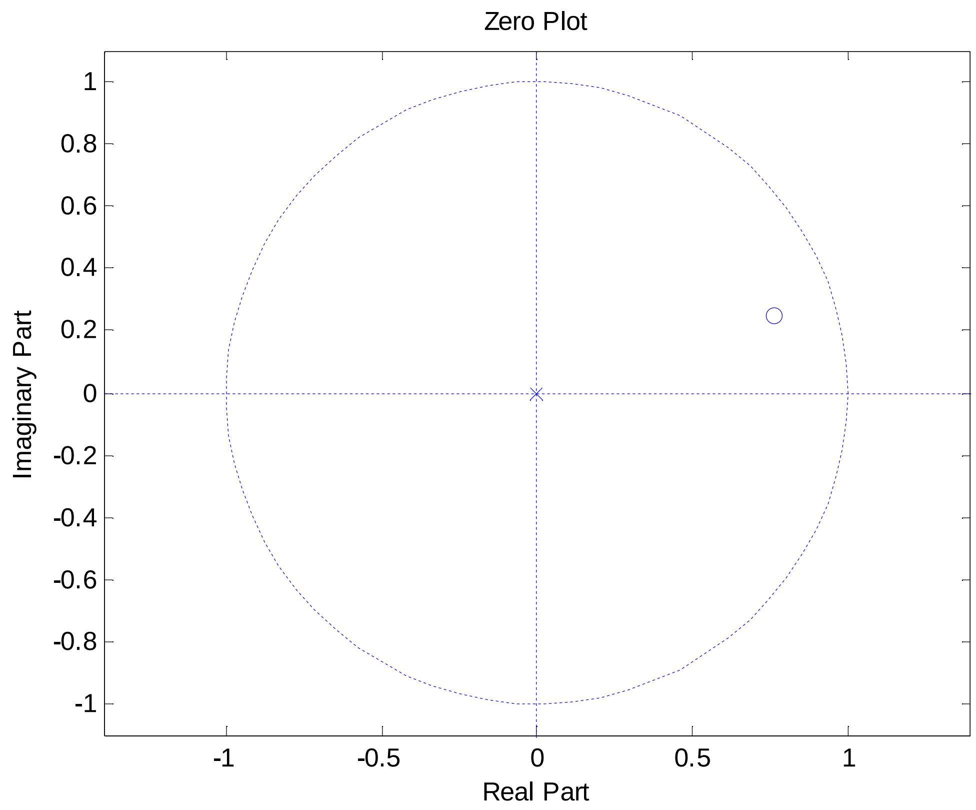


Exercise 2 (8/18)

- Example 2:

$$\{a_n\} = \left\{1, -0.8e^{j\pi/10}\right\}$$

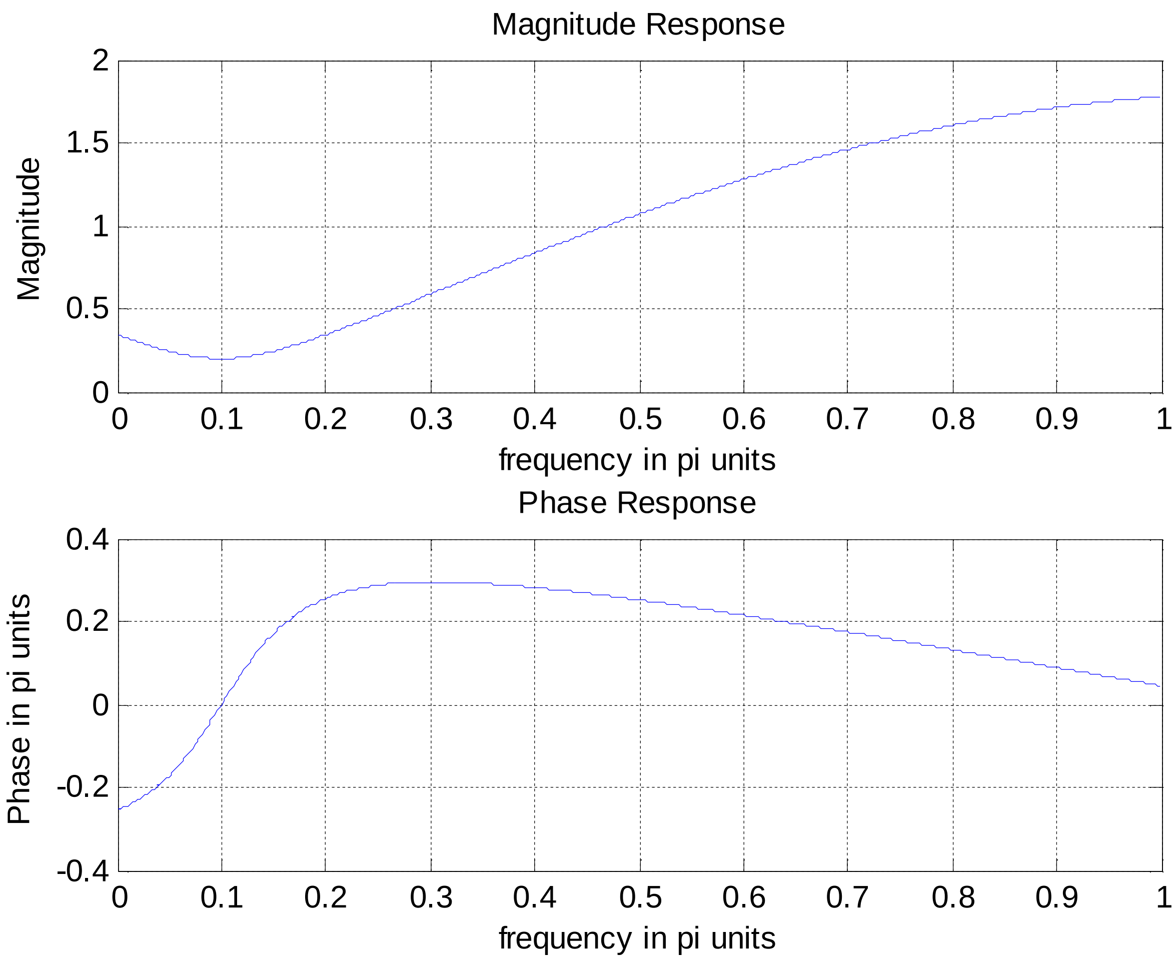
$$A(z) = \frac{z - 0.8e^{j\pi/10}}{z}$$



- figure, zplane(a,1), title('Zero Plot');

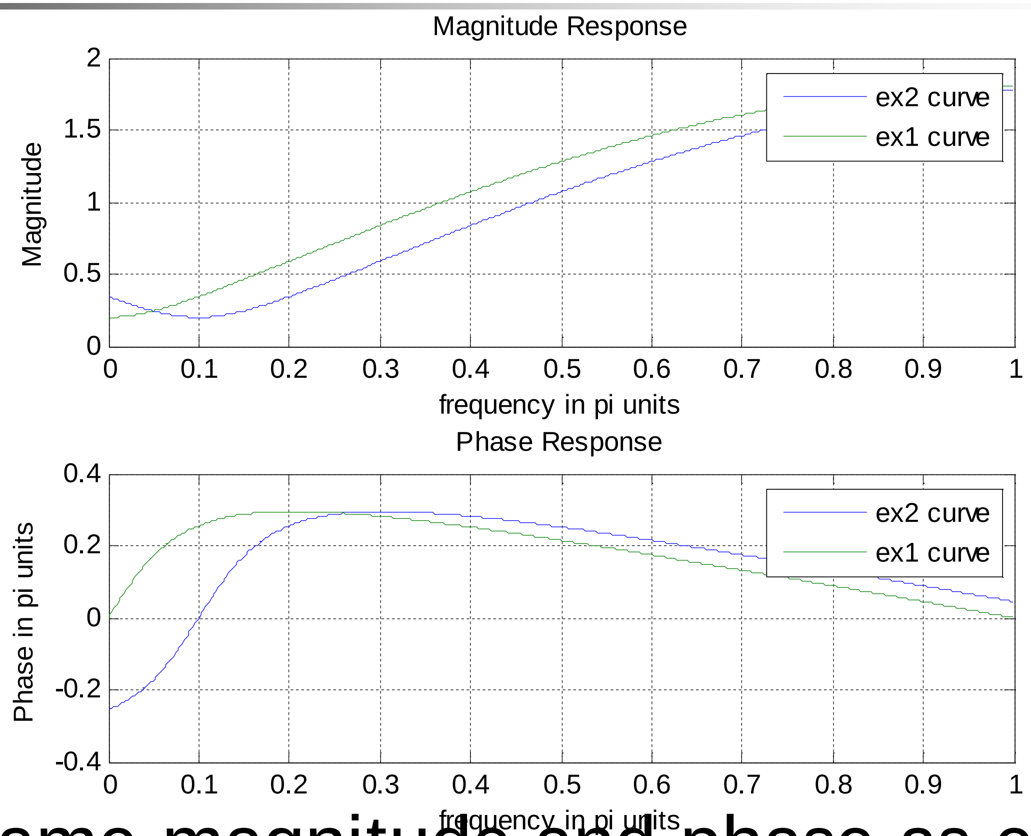
Exercise 2 (9/18)

- Example 2:



Exercise 2 (10/18)

- Example 2:



- It has the same magnitude and phase as ex 1, but translated on the right of an angle equal to φ_0 .

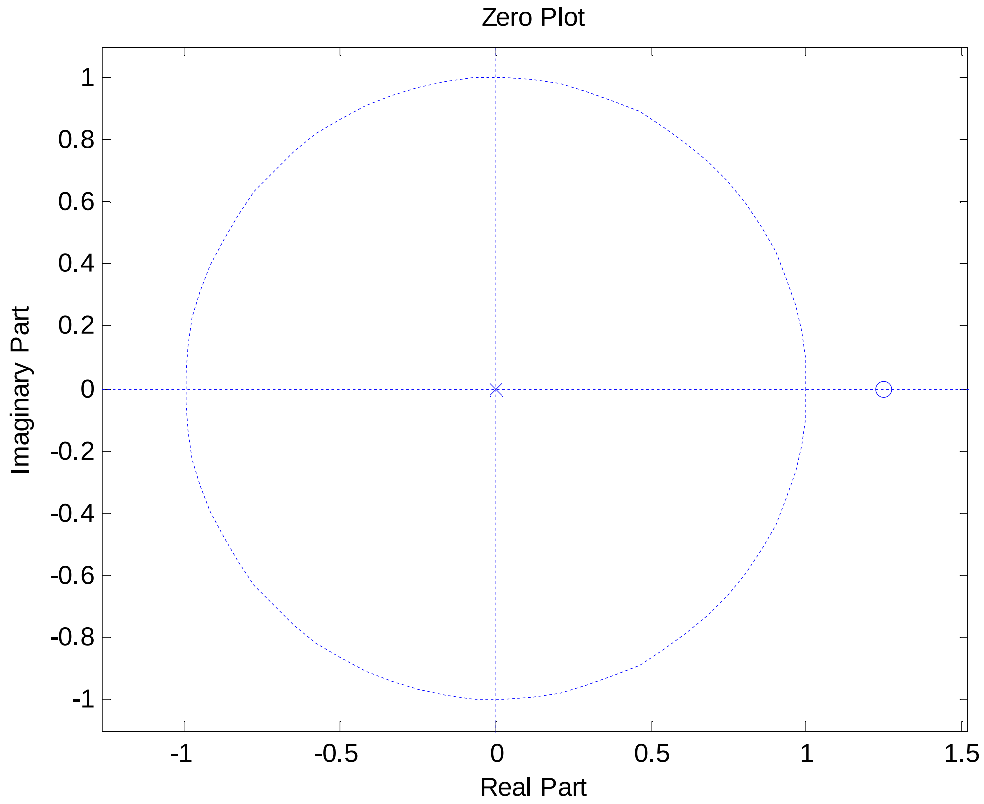
Exercise 2 (11/18)

- Example 3:

$$\{a_n\} = \{1, -1.25\}$$

$$A(z) = \frac{z - 1.25}{z}$$

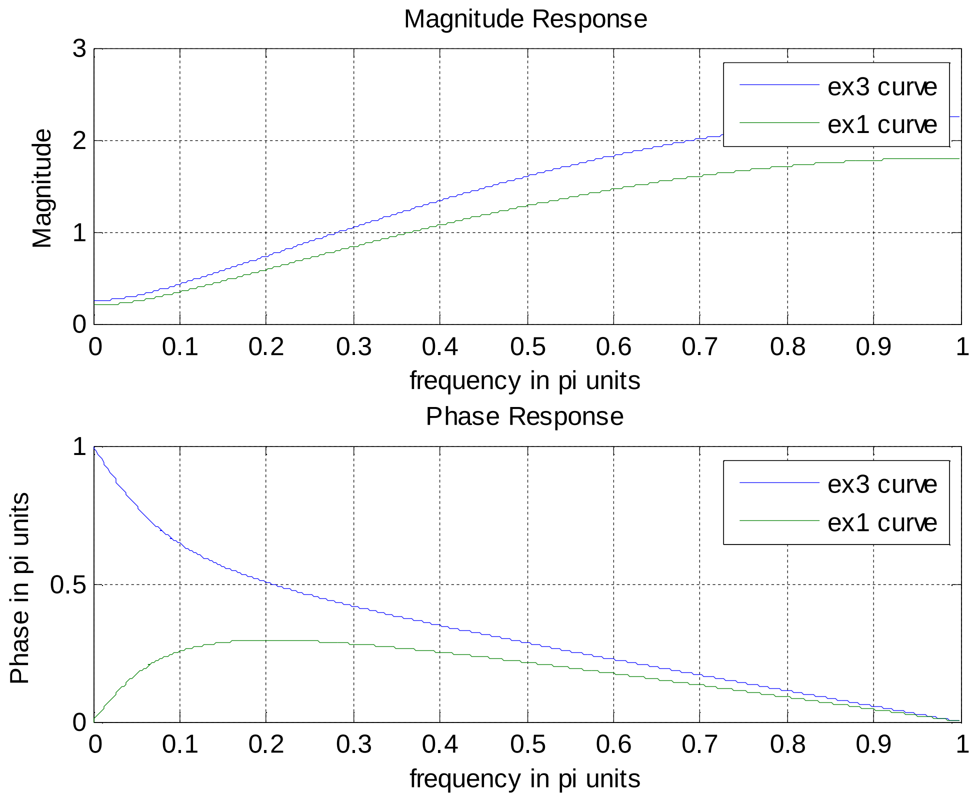
$$1.25 = \frac{1}{0.8}$$



- figure, zplane(a,1), title('Zero Plot');

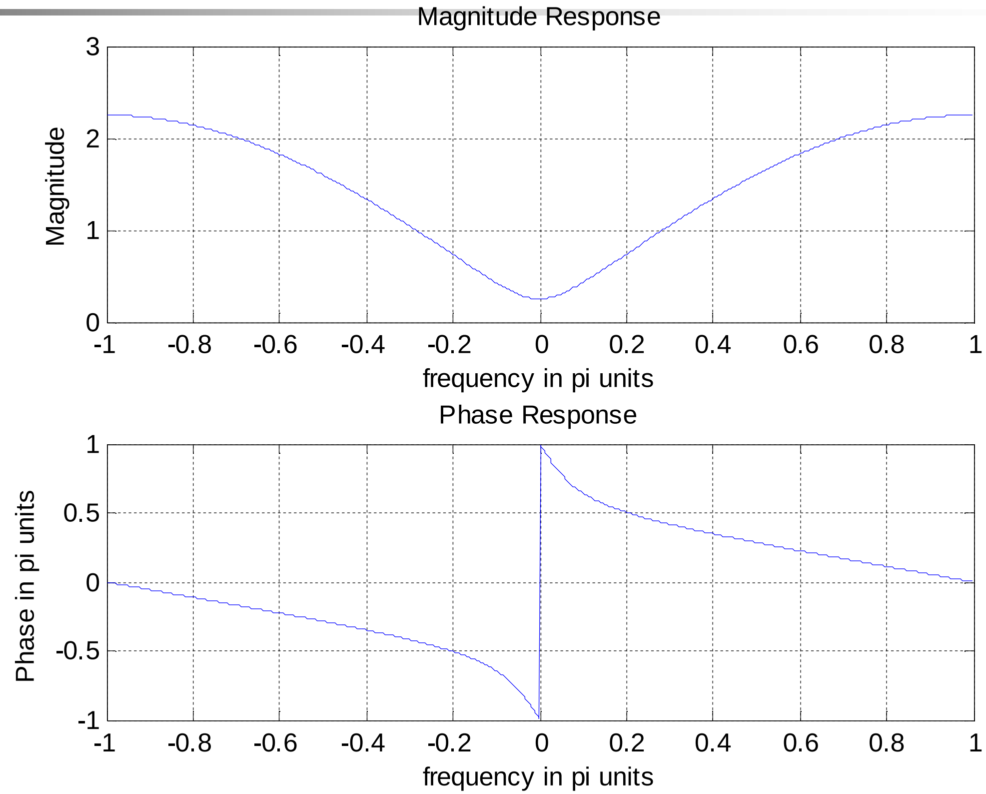
Exercise 2 (12/18)

- Example 3:



Exercise 2 (13/18)

- Example 3:



- It has the same magnitude as ex 1 (a part for a scale factor), but the phase shows an abrupt discontinuity of 2π in the origin of axes.

Exercise 2 (14/18)

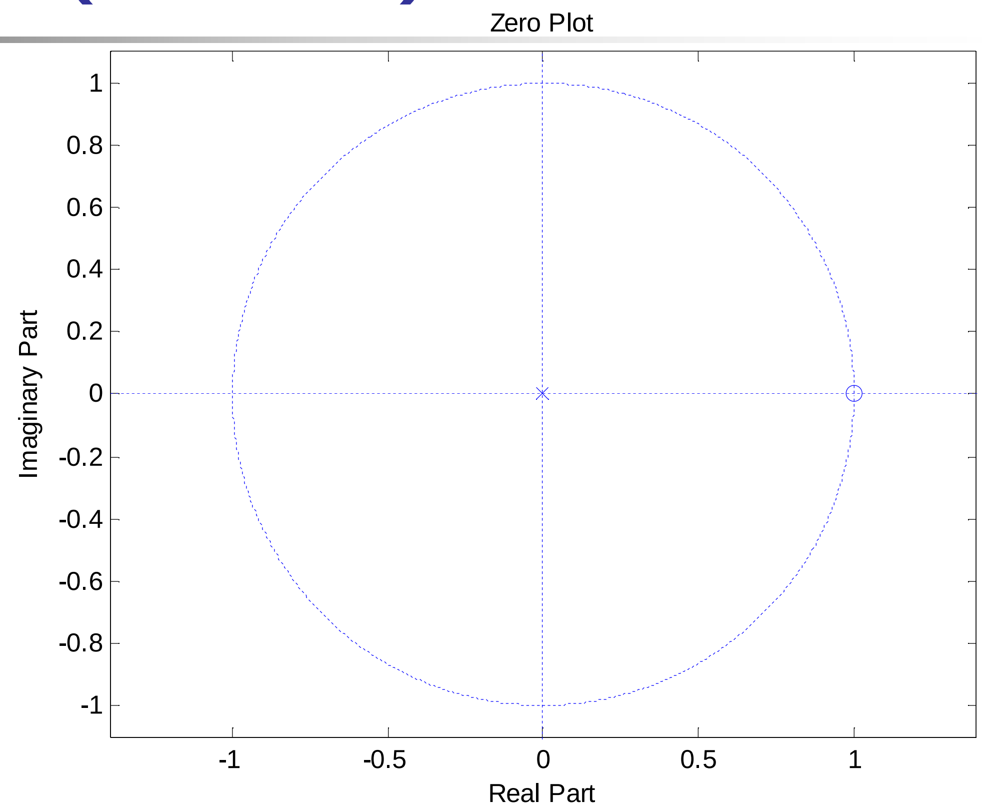
- HINT:
- MINIMUM PHASE ZERO (inside the unit circle): when the vector z makes a complete phase circle, also the vector $\left(z - z_0\right)$ does (we return to the initial phase value)

- MAXIMUM PHASE ZERO (outside the unit circle): when the vector z makes a complete phase circle, the vector $\left(z - z_0\right)$ does not (we return to the initial phase value unless a jump equal to 2π)

Exercise 2 (15/18)

- Example 4:

$$\{a_n\} = \{1, -1\}$$



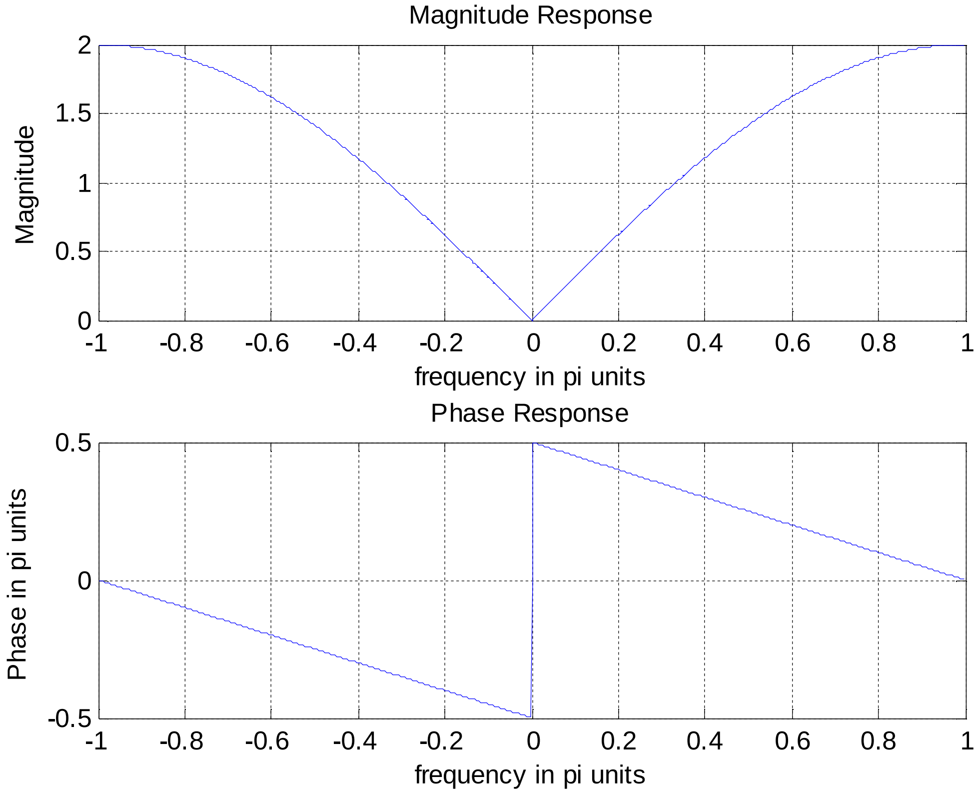
$$A(z) = 1 - 1e^{-j\varphi} = e^{-j\varphi/2}\left(e^{j\varphi/2} - e^{-j\varphi/2}\right) = 2j\sin\left(\varphi/2\right)e^{-j\varphi/2}$$

Exercise 2 (16/18)

$$A(z) = 1 - 1e^{-j\varphi} = e^{-j\varphi/2}\left(e^{j\varphi/2} - e^{-j\varphi/2}\right) = 2j\sin\left(\varphi/2\right)e^{-j\varphi/2}$$

$$\left|A(z)\right| = 2\sin\left(\varphi/2\right)$$

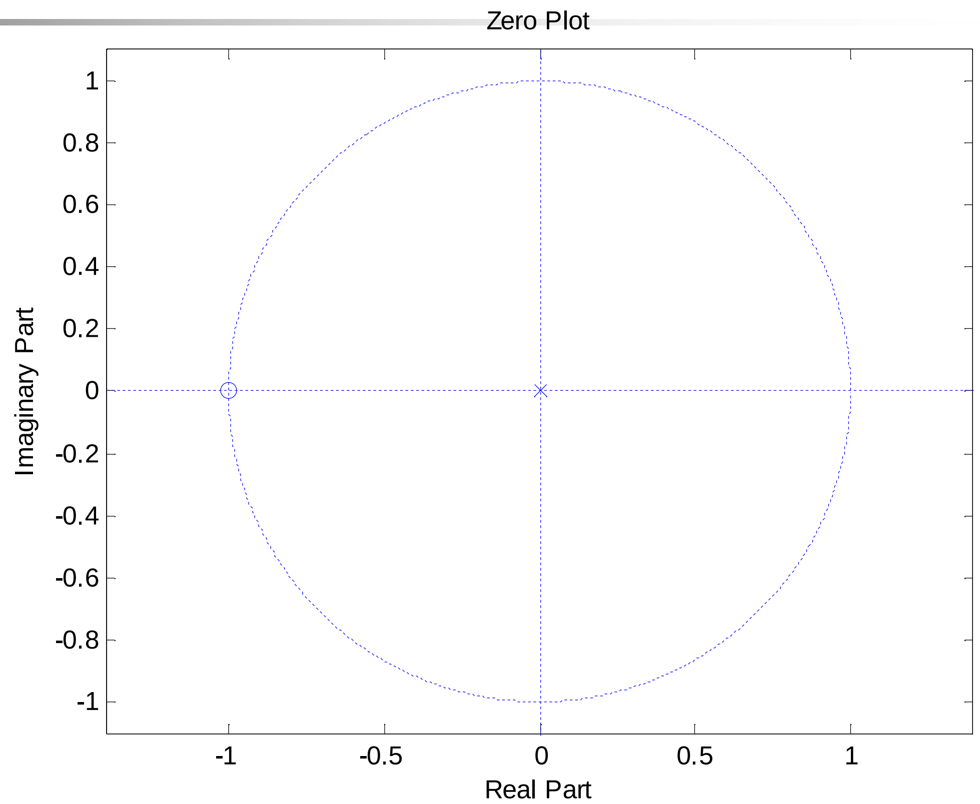
$$\measuredangle\left\{A(z)\right\} = -\varphi/2 + \pi/2$$



Exercise 2 (17/18)

- Example 5:

$$\{a_n\} = \{1,1\}$$



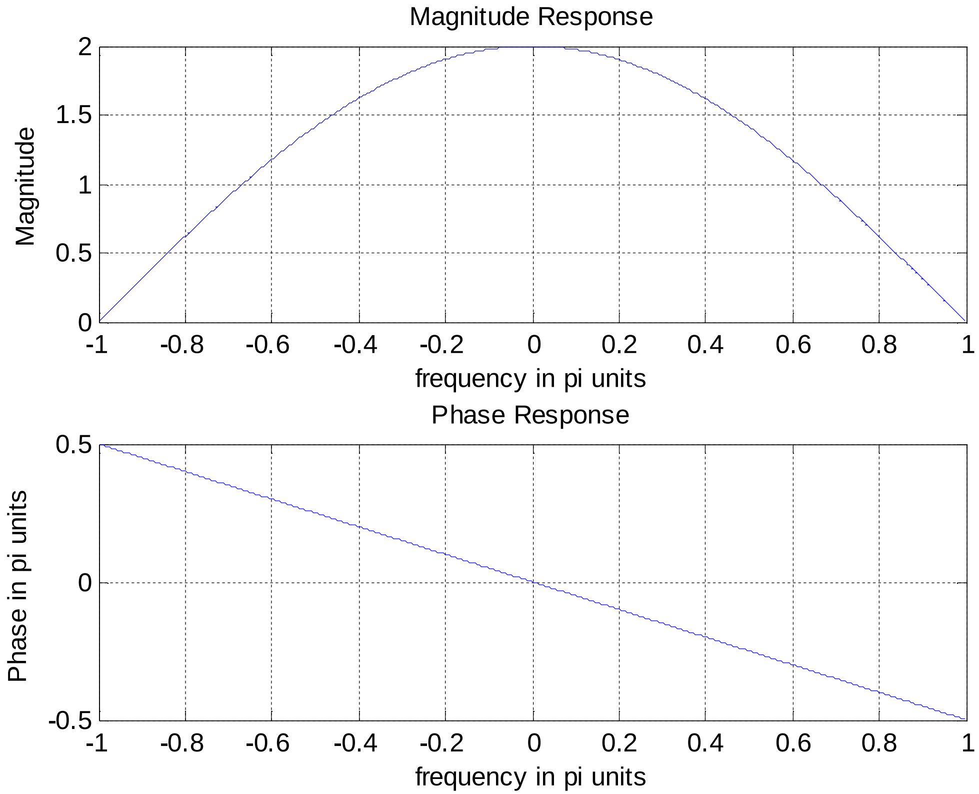
$$A(z) = 1 + 1e^{-j\varphi} = e^{-j\varphi/2}\left(e^{j\varphi/2} + e^{-j\varphi/2}\right) = 2\cos\left(\varphi/2\right)e^{-j\varphi/2}$$

Exercise 2 (18/18)

$$A(z) = 1 + 1e^{-j\varphi} = e^{-j\varphi/2}\left(e^{j\varphi/2} + e^{-j\varphi/2}\right) = 2\cos\left(\varphi/2\right)e^{-j\varphi/2}$$

$$\left|A(z)\right| = 2\cos\left(\varphi/2\right)$$

$$\measuredangle\left\{A(z)\right\} = -\varphi/2$$



Exercise 3 (1/6)

- The inverse z-tranform of a complex function X(z) is:

$$x(n) = \frac{1}{2\pi} \oint_C X(z)\, z^{n-1} dz \qquad z = |z| e^{j\omega}$$

  - Where C is a counterclockwise contour encircling the origin and lying on the ROC

Exercise 3 (2/6)

- Let be x(n) a sequence with a rational transform:

$$X(z) = \frac{A(z)}{B(z)}$$

  - Where B(z) and A(z) are polynomials in z^{-1}.
  - We can numerically compute the inverse z-ransform using the "filter" function as you can see in the next example

Exercise 3 (3/6)

- Example 4.6: Given

$$X(z) = \frac{0.25z^{-3} - 0.5z^{-4} + 0.0625z^{-5}}{1 - z^{-1} + 0.75z^{-2} - 0.25z^{-3} + 0.0625z^{-4}} \quad |z| > 0.5$$

  - Check that it is the z-transform of

$$x(n) = (n-2)(0.5)^{n-2}\cos\left[\frac{\pi}{3}(n-2)\right]u(n-2)$$

Exercise 3 (4/6)

- Pseudocode:
  - b = [0 0 0 0.25 -0.5 0.0625]; % numerator
  - a = [1 -1 0.75 -0.25 0.0625]; % denominator
  - [delta, n] = impseq(0,0,7) % impulse
  - x = filter(b,a,delta) % check sequence
  - xv = [(n-2).*((1/2).^(n-2)).*(cos(pi*(n-2)/3))].*stepseq(2,0,7)
    % original sequence

Exercise 3 (5/6)

- When X(z) is a rational function,

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{1 + a_1 z^{-1} + \ldots + a_N z^{-N}}$$

- it can be expressed as a sum of first order factors using the partial fraction expansion:

$$X(z) = \sum_{k=1}^{N_p} \sum_{l=1}^{r_k} \frac{R_{k+l-1}}{\left(1 - p_k z^{-1}\right)^l} + \underbrace{\sum_{k=0}^{M-N} C_k z^{-k}}_{M \geq N}$$

Exercise 3 (6/6)

- Where p_k is the k-th pole of X(z) and it has multiplicity r_k. R_k is the reduez at $p_{k,l}$:

$$R_{k+l-1} = \left. \frac{X(z)}{\sum_{k=0}^{M-N} C_k z^{-k}} \left(1 - p_k z^{-1}\right) \right|_{z=p_k}$$

- Matlab provides the function:
  - "[r p k]=residuez(b,a)" that computes residues, poles and direct terms of the partial-fraction expansion of the transfer function B(z)/A(z).

Exercise 4 (1/14)

- The system function is:

$$H(z) = \sum_{n=-\infty}^{\infty} h(n)\, z^{-n} \qquad\qquad R_{h-} \le |z| \le R_{h+}$$

- The output of the system can be computed as:

$$X(z) \longrightarrow \boxed{H(z)} \longrightarrow Y(z) = H(z)\,X(z)$$

$$ROC_y : ROC_x \cap ROC_h$$

Exercise 4 (2/14)

$$X(z) \longrightarrow \boxed{H(z)} \longrightarrow Y(z) = H(z)\,X(z)$$

- When LTI systems are described by difference equation:

$$y(n) + \sum_{k=1}^{N} a_k y(n-k) = \sum_{l=0}^{M} b_l x(n-l)$$

$$Y(z) + \sum_{k=1}^{N} a_k z^{-k} Y(z) = \sum_{l=0}^{M} b_l z^{-l} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{1 + a_1 z^{-1} + \ldots + a_N z^{-N}} = \frac{B(z)}{A(z)}$$

Exercise 4 (3/14)

$$X(z) \longrightarrow \boxed{H(z)} \longrightarrow Y(z) = H(z)\,X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{B(z)}{A(z)} = \frac{\sum_{l=0}^{M} b_l z^{-l}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

- factorizating

$$H(z) = \frac{Y(z)}{X(z)} = b_0 z^{N-M} \frac{\prod_{l=1}^{M} (z - z_l)}{\prod_{k=1}^{N} (z - p_k)}$$

Exercise 4 (4/14)

- Matlab provides the functions:
  - “p=roots(a)” that computes the roots “p” of the polynomial defined from the coefficients “a”.
  - “a=poly(p)” that computes the coefficients “a” of a polynomial whose roots are the elements of “p”.
  - “zplane (z,p)” or equivalently “zplane(b,a)” plot in the complex plane the position of the zeros (°) and poles (x) of the transfer function of the filter.

Exercise 4 (5/14)

- Given the equation:

$$1 \cdot y(n) = -2\rho\cos(\theta)\, y(n-1) - \rho^2 y(n-2) + x(n) + 2\,x(n-1) + x(n-2)$$

- Compute and plot its zeros and poles
  - rho = 0.9;   theta = pi/8;
  - a = [1 2*rho*cos(theta) rho^2]
  - b = [1 2 1]
  - p = roots(a);   z = roots(b);
  - figure, zplane(b,a)

Exercise 4 (6/14)

- Given the equation:

$$1\cdot y(n) = -2\rho\cos(\theta)\,y(n-1) - \rho^2 y(n-2) + x(n) + 2\,x(n-1) + x(n-2)$$

- Compute and plot its impulse response
  - [delta, n] = impseq(0,0,100)
  - h = filter(b,a,delta);
  - figure, plot(n,h), title('Impulse Responce‘)

Exercise 4 (7/14)

- Transfer function:
  - If the ROC of H(z) includes the unit circle, then we can evaluate H(z) on the unit circle:

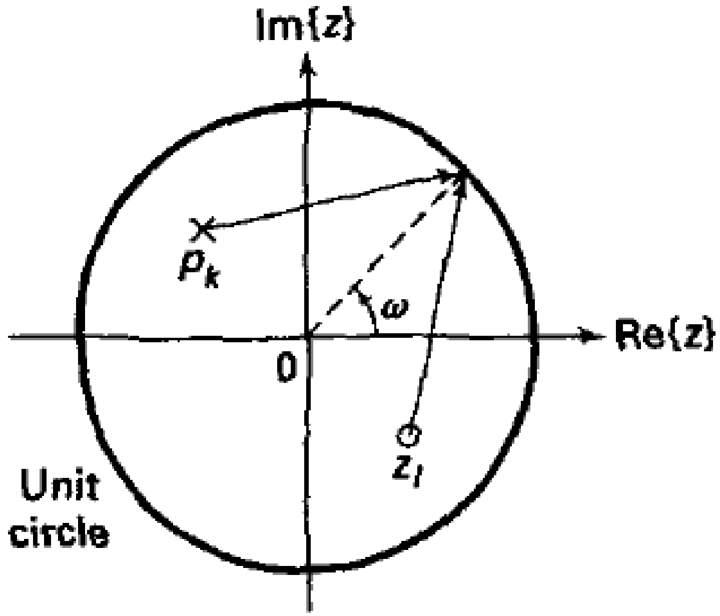
$$H(e^{j\omega}) = b_0 e^{j(N-M)\omega} \frac{\prod_{l=1}^{M} \left(e^{j\omega} - z_l\right)}{\prod_{k=1}^{N} \left(e^{j\omega} - p_k\right)}$$

  - $\left(e^{j\omega} - z_l\right)$ is a vector from the zero to the unit circle
  - $\left(e^{j\omega} - p_k\right)$ is a vector from the pole to the unit circle

Exercise 4 (8/14)

- Modulus:

$$\left|H(e^{j\omega})\right| = |b_0| \frac{\prod_{l=1}^{M} \left|e^{j\omega} - z_l\right|}{\prod_{k=1}^{N} \left|e^{j\omega} - p_k\right|}$$



- Phase:

$$\angle H(e^{j\omega}) = [0 \; or \; \pi] + [(N-M)\omega] + \sum_{l=1}^{M} \angle\left(e^{j\omega} - z_l\right) - \sum_{k=1}^{N} \angle\left(e^{j\omega} - p_k\right)$$

Exercise 4 (9/14)

- GOAL: compute the transfer function of a IIR filter:

$$H(\omega) = \frac{B(\omega)}{A(\omega)} = \frac{b_0 + b_1 e^{-j\omega} + \ldots + b_M e^{-j\omega M}}{1 + a_1 e^{-j\omega} + \ldots + a_N e^{-j\omega N}}$$

- Matlab provides the function "[H w]=freqz(b,a,N)" that returns the N-point complex frequency response "H" and the N-point frequency vector "w" in radians/sample of the filter given numerator and denominator coefficients in vectors "b" and "a". The frequency response is evaluated at N points equally spaced around the upper half of the unit circle. If N isn't specified, it defaults to 512.

Exercise 4 (10/14)

EXAMPLE 4.11 Given a causal system

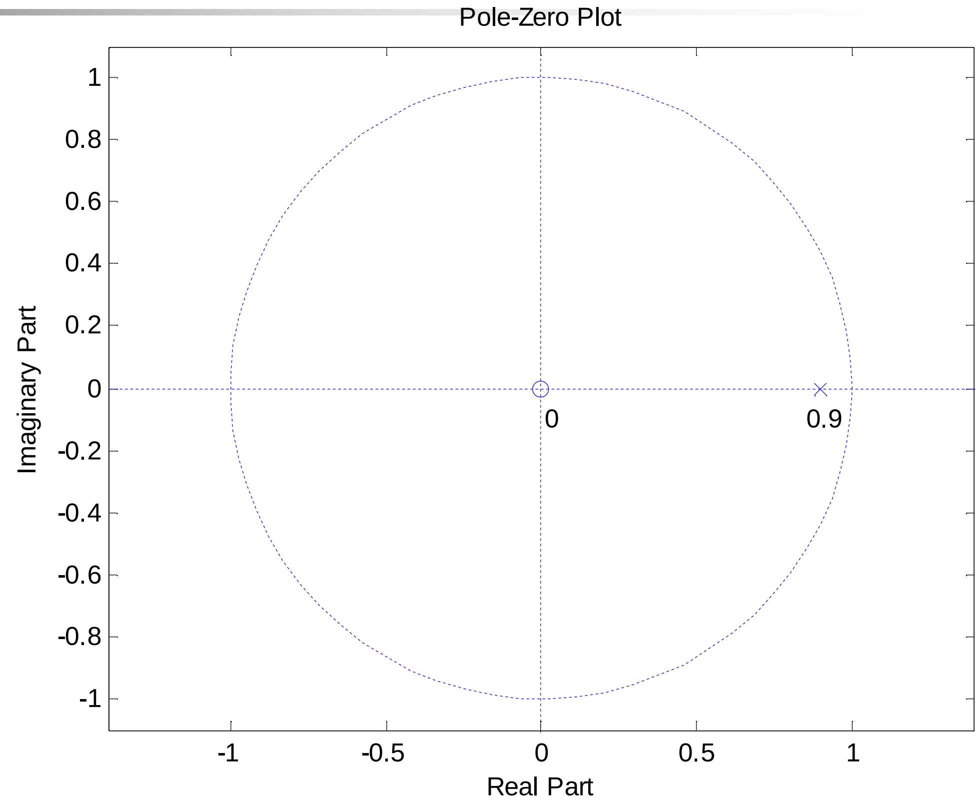
$$y(n) = 0.9y(n-1) + x(n)$$

a. Find $H(z)$ and sketch its pole-zero plot.
b. Plot $\left|H(e^{j\omega})\right|$ and $\angle H(e^{j\omega})$.
c. Determine the impulse response $h(n)$.

$$H(z) = \frac{1}{1-0.9z^{-1}} \quad |z| > 0.9 \qquad h(n) = (0.9)^n\, u(n)$$

Exercise 4 (11/14)

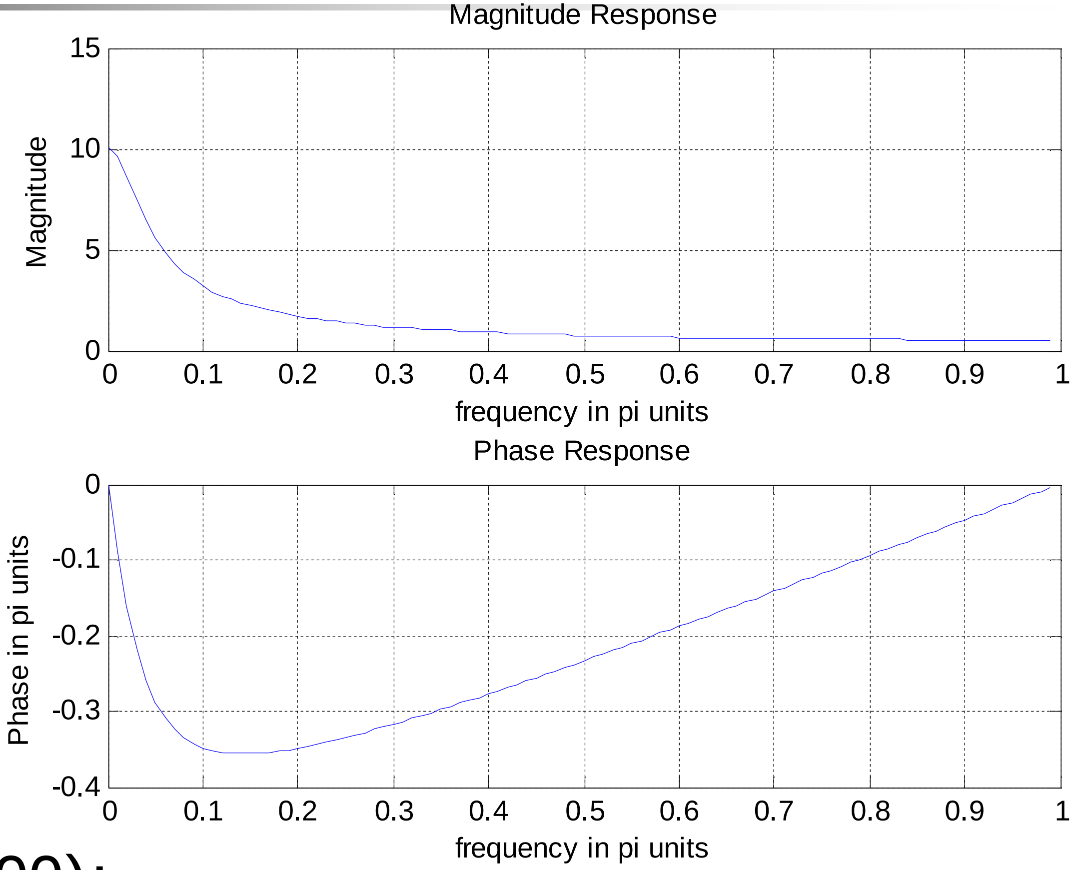
- Example 4.11 a:



- p = roots(a) = 0.9
- z = roots(b) = 0
- figure, zplane(b,a), title('Pole-Zero Plot');

Exercise 4 (12/14)

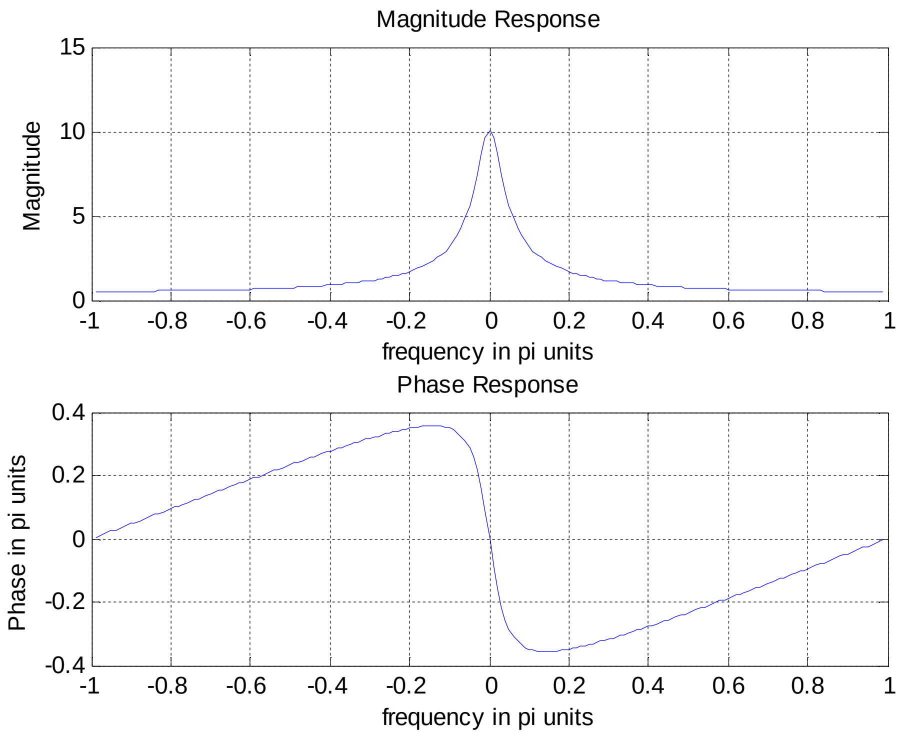
- Example 4.11 b:



- [H,w] = freqz(b,a,100);
- magH = abs(H); phaH = angle(H);

Exercise 4 (13/14)

- Represent also the negative frequencies:
  - w=[-flipud(w(2:end)); w]; % flipping the rows
  - H=[conj(flipud(H(2:end))); H];



Exercise 4 (14/14)

- Example 4.11 c:
- [delta, n] = impseq(0,0,100);
- h = filter(b,a,delta);
- h_r = (0.9).^n.*stepseq(0,0,100);