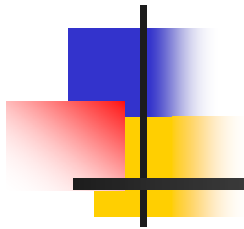


Introduction to Parameter Estimation



**87203 – Multimedial Signal
Processing 1st Module**

Politecnico di Milano –
Polo regionale di Como



Agenda

- Ex1: Mean Estimation
- Ex2: Variance Estimation
- Ex3: Autocorrelation Estimation
- Ex4: Maximum Likelihood Estimation



Exercise 1 (1/6)

- Goal of estimation theory: estimate the value of an unknown parameter (deterministic or random) from a set of observation of a random variable:
 - ES: estimate mean (deterministic) from N samples of a random variable x
 - ES: estimate phase (random) from N samples of a signal:

$$x(n, \vartheta) = \cos(\omega_1 n + \vartheta)$$

Oss: $x(n, \vartheta)$ is a random sequence because is function of the random variable ϑ .



Exercise 1 (2/6)

- HINT: The estimate value $\hat{\vartheta}$ is a random variable because is function of a random variable x (observations).

- BIAS: $B = \vartheta - E[\hat{\vartheta}_N]$

- CONSISTENCE:

- mean $\lim_{N \rightarrow \infty} B = 0$

- variance $\lim_{N \rightarrow \infty} \text{var}\{\hat{\vartheta}_N\} = \lim_{N \rightarrow \infty} \left\{ \left(\hat{\vartheta}_N - E[\hat{\vartheta}_N] \right)^2 \right\} = 0$

- MSE $\lim_{N \rightarrow \infty} E \left\{ \left(\hat{\vartheta}_N - \vartheta \right)^2 \right\} = 0$



Exercise 1 (3/6)

- Goal: Using R realizations of a gaussian noise, estimate the (temporal) mean of the process using two different estimators:

- Sample mean $\mu_s = \frac{1}{N} \sum_{n=1}^N x(n)$

- Average between maximum and minimum value of the realization

$$\mu_a = \frac{\max x(n) + \min x(n)}{2}$$



Exercise 1 (4/6)

- **Hints:**

- "X = sqrt(v)*randn(N,M)+m" builds a N by M matrix containing Gaussian variables of mean "m" and variance "v".
- If x is a row/column vector
 - "y = mean(x)" returns the sample mean of vector x
 - "y = var(x)" returns the variance of vector x
 - "y = max(x)" and "y = min(x)" return, respectively, the maximum and minimum value of vector x.



Exercise 1 (5/6)

- Pseudocode:
 - generate R iid sequence of N samples with gaussian distribution $x \approx N(\eta_x, \sigma_x^2)$
 - $y = \text{sqrt}(v) * \text{randn}(R, N) + m$;
 - for each realization estimate mean with two estimators:
 - for $r = 1:R$
 - $mS(r) = \text{mean}(x(r, :))$;
 - $mM(r) = (\text{max}(x(r, :)) + \text{min}(x(r, :)))/2$;
 - end
 - Compute mean and variance of the two estimators



Exercise 1 (6/6)

- **Question:** try different sequence lengths N :
what is the behavior in terms of the expectation
and variance of the estimators?



Exercise 2 (1/4)

- Goal: Using R realizations of a gaussian noise, estimate the (temporal) variance of the process using the following (biased) estimator:

- Sample variance $\text{var } s = \frac{1}{N} \sum_{i=1}^N (x_i - \hat{\mu})^2$

- Compute the bias of the estimated value



Exercise 2 (2/4)

- Hints:
 - Remember that the bias of an estimator is:

$$B = \theta - E[\hat{\theta}]$$

- The expectation of the sample variance can also be computed analytically

$$E[\hat{\theta}] = \frac{M-1}{M} \vartheta$$



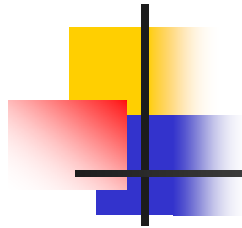
Exercise 2 (3/4)

- Pseudocode:
 - generate R iid sequence of N samples with gaussian distribution $x \approx N(\eta_x, \sigma_x^2)$
 - $y = \text{sqrt}(v) * \text{randn}(R, N) + m$;
 - for each realization estimate the variance:
 - for $r = 1:R$
 - $vS(r) = \text{var}(x(r, :))$;
 - end
 - Compute mean and variance of the estimator



Exercise 2 (4/4)

- **Question:**
 - Which is the trend of the bias increasing the length of sequence?



Exercise 3 (1/8)

- Goal: Using R realizations of a gaussian process, estimate the autocorrelation of the process with two different estimators:
 - **Method 1**: estimation of the autocorrelation assuming ergodic processes
 - **Method 2**: estimation of the autocorrelation using set average



Exercise 3 (2/8)

- **Hints on method 1:**
 - Given a realization of a random process, compute its autocorrelation using the following equation (biased):

$$\hat{r}(k) = \frac{1}{N} \sum_{t=k+1}^N y(t)y^*(t-k), \quad k \geq 0$$



Exercise 3 (3/8)

- Pseudocode:
 - generate R iid sequence of N samples with gaussian distribution $x \approx N(\eta_x, \sigma_x^2)$
 - $y = \text{sqrt}(v) * \text{randn}(R, N) + m$;
 - for each realization estimate the autocorrelation:
 - For each lag of the autocorrelation k
 - For each sample $y(t)$ of the sequence:
 - Compute the product $y(t) * y(t-k)$
 - Sum the product in the buffer



Exercise 3 (4/8)

- **Hints on method 2:**
 - Given a set of realization of a random process, compute the autocorrelation according to the following equation:

$$r(k) = E[x(t)x(t - k)]$$

- The expectation is taken over all the realizations



Exercise 3 (5/8)

- Pseudocode:

- generate R iid sequence of N samples with gaussian distribution

$$x \approx N(\eta_x, \sigma_x^2)$$

- $y = \text{sqrt}(v) * \text{randn}(R, N) + \text{mean};$

- For each lag of the autocorrelation m

- For each sample k

- For each realization i

- $\text{vec}(i) = \text{vec}(i) + x(k, i) * x(k - m + 1, i);$

- Compute the mean of vec

- $r(\text{lag}) = r(\text{lag}) + \text{mean}(\text{vec})$



Exercise 3 (6/8)

- Hint: Matlab provides the function
"`[r lag]=xcorr(x)`"
that produces an estimate of the autocorrelation
($2N-1$ samples) of the stationary sequence " x ".

" lag " is the vector of lag indices $[-N+1:1:N-1]$.



Exercise 3 (7/8)

- Goal: Define a AR(1) random process with the difference equation:

$$x(n) = \rho x(n-1) + z(n) \qquad z(n) \approx N(0,1)$$

- Generate R realizations of the process, each having length N.
- For each realization, provide an estimate of the autocorrelation $r(k)$.



Exercise 3 (8/8)

- The true autocorrelation function is given by

$$r_x(k) = \sigma_x^2 \rho^{-|k|} = \frac{\sigma_z^2}{1 - \rho^2} \rho^{-|k|}$$

- Compute $E[\hat{r}(k)]$ and compare it with $r(k)$ to verify the bias.
- Compute $E[(\hat{r}(k) - E[\hat{r}(k)])^2]$ and plot it as a function of k .

What is the effect of N on the variance?



Exercise 4 (1/4)

- Goal of Maximum Likelihood estimation: estimate the value of an unknown parameter (deterministic) from a N-point dataset:

$$x(n) = s(\mathcal{G}) + w \qquad \hat{\mathcal{G}} = g(\{x(n)\})$$

- ML: $\hat{\mathcal{G}} = \arg \max_{\mathcal{G}} \left\{ f(\vec{x} | \vec{\mathcal{G}}) \right\} = \arg \max_{\mathcal{G}} \left\{ L(\vec{\mathcal{G}} | \vec{x}) \right\}$



Exercise 4 (2/4)

- Goal: Estimate the mean of a Gaussian process given the observed data x and the variance σ_x^2

- Hint:
$$L(\vec{\mathcal{G}} | \vec{x}) = \left(\frac{1}{\sqrt{2\pi\sigma^2}} \right)^N \exp \left\{ -\frac{1}{2\sigma^2} \sum_{n=1}^N (x_n - \mathcal{G})^2 \right\}$$



Exercise 4 (3/4)

- Pseudocode:
 - generate N samples of the gaussian process
 - $x = \text{sqrt}(\text{varx}) * \text{randn}(N,1) + \text{theta}$;
 - compute the likelihood function:
 - $\text{thetas} = [-6:0.01:6]'$;
 - $L = \text{ones}(\text{length}(\text{thetas}),1)$;
 - for $n=1:N$ % for each sample
 - $L = L .* (1/(\text{sqrt}(2 * \pi * \text{varx}))) .* \exp((-1/(2 * \text{varx})) * (x(n) - \text{thetas}).^2)$;
 - % is the product of the marginal function
 - end



Exercise 4 (4/4)

- Pseudocode (continued):
 - find maximum value of likelihood function and its index:
 - [Lmax ind]=max(L);
 - thetast=thetas(ind)
 - OSS: L is a function of theta --> it assume a different value for each theta

$$L(\vec{\mathcal{G}} | \vec{x}) = \begin{bmatrix} L(\mathcal{G}_1 | \vec{x}) & L(\mathcal{G}_2 | \vec{x}) & \dots & L(\mathcal{G}_{l_g} | \vec{x}) \end{bmatrix}$$