Spectral Estimation

87203 – Multimedial Signal Processing 1st Module

Politecnico di Milano –

Polo regionale di Como

Agenda

NON PARAMETRIC SPECTRAL ESTIMATION

- Ex1: Periodogram
- Ex2: Correlogram
- Ex3: Blackman-Tukey
- Ex4: Bartlett
- Ex5: Welch

PARAMETRIC SPECTRAL ESTIMATION

- Ex6: AutoRegressive model

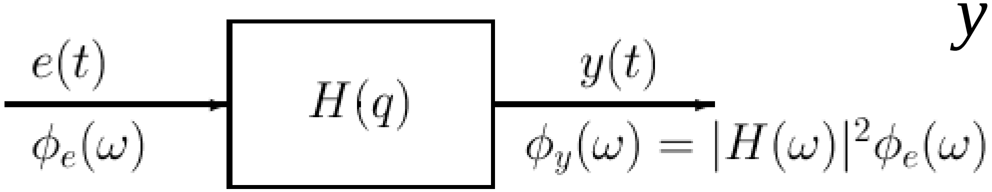
Exercise 1 (1/4)

- Goal of spectral estimation: from one finite record of a stationary data sequence, estimate how the total power is distributed over frequency.
- Power Spectral Density
  - DTFT of the ACS: $\phi(\omega) = \sum_{k=-\infty}^{\infty} r(k)\, e^{-j\omega k}$
  - $\phi(\omega) = \lim_{N\to\infty} E\left\{ \frac{1}{N} \left| \sum_{n=1}^{N} y(n)\, e^{-j\omega n} \right|^2 \right\}$

Exercise 1 (2/4)

- Hint: Remember that AutoCorrelation Sequence is: $$r(k) = E[y(n)\, y^*(n-k)]$$
  - Properties: $$r(k) = r^*(-k)$$ $$r(0) \geq |r(k)|$$
- Hint: Remember that:

$$y(t) = e(t) * h(t)$$



$$r_y(k) = \sum_{p=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} h_p h_m^* r_e(m+k-p)$$

Exercise 1 (3/4)

- Goal: Estimate the power spectral density of the signal “flute2” by means of *periodogram*.

- **Hints on periodogram**: the spectrum estimation using periodogram is given by the following equation:

$$\hat{\phi}_P(\omega) = \frac{1}{N}\left|\sum_{n=1}^{N} y(n)\, e^{-j\omega n}\right|^2$$

Exercise 1 (4/4)

- Pseudocode:
  - load the file flute2.wav
  - consider 50ms of the input signal (y)
  - estimate PSD using parallelogram:
    - N = length(y);
    - M = 2^ceil(log2(N)+1);

      %number of frequency bins

    - phip = (1/N)*abs(fft(y,M)).^2;

Exercise 1b (1/3)

- **Goal**: quantify the bias and variance of the periodogram
- **N.B.:** Periodogram is asymptotically unbiased and has large variance, even for large N.

$$E\left[\hat{\phi}_P(\omega)\right] = \sum_{k=-\infty}^{\infty} w_B(k) r(k)\, e^{-j\omega k} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(\xi) W_B(\omega-\xi)\, d\xi$$

$$\lim_{N\to\infty} \operatorname{var}\left\{\hat{\phi}_P(\omega)\right\} = E\left\{\left[\hat{\phi}_P(\omega_1)-\phi(\omega_1)\right]\left[\hat{\phi}_P(\omega_2)-\phi(\omega_2)\right]\right\}$$

$$= \begin{cases} \phi^2(\omega_1) & \omega_1 = \omega_2 \\ 0 & \omega_1 = \omega_2 \end{cases}$$

Exercise 1b (2/3)

- **Goal**: quantify the bias and variance of the periodogram
- **Pseudocode:**
  - compute R realizations of N samples white noise
    - e = randn(N,R);
  - for each realization:
    - filter white noise by means of a LTI filter

      $$Y(z) = H(z)E(z)$$
    - compute the periodogram spectral estimate

      phip(i,:) = (1/N)*abs(fft(y,N)).^2;
  - end

Exercise 1b (3/3)

- compute the ensemble mean: phip(RxN)
  - phipmean = mean(phip);
- compute the ensemble variance
  - phipvar = var(phip);

- Plot the true spectrum Φ(ω)
- Plot the average of R spectral estimates and compare the averaged spectrum and the true PSD
- Plot $E\left[\hat{\phi}_P(\omega)\right] \pm \sqrt{\operatorname{var}\left[\hat{\phi}_P(\omega)\right]}$ and compare with the true PSD

Exercise 1c (1/4)

- **Goal**: quantify the bias and variance of the periodogram
- **Procedure**: filter a white noise by means of the following LTI filter

$$Y(z) = [H_1(z) + H_2(z)]E(Z)$$

- With the following poles and zeros

$$p_1 = \rho_P e^{j\pm\pi\omega_1} \quad p_2 = \rho_P e^{j\pm\pi\omega_2}$$

$$z_1 = \rho_Z e^{j\pm\pi\omega_1} \quad z_2 = \rho_Z e^{j\pm\pi\omega_2}$$

$$\rho_Z = 0.95$$

$$\rho_P = 0.99$$

$$\omega_1 = 0.6\pi$$

$$\omega_2 = 2(0.3+\alpha)\pi$$

$$\alpha = 0.05$$

Exercise 1c (2/4)

- **Hint**

$$H_i(z) = \frac{B_i(z)}{A_i(z)} = \frac{(1-\rho_Z e^{+j\omega_i} z^{-1})(1-\rho_Z e^{-j\omega_i} z^{-1})}{(1-\rho_P e^{+j\omega_i} z^{-1})(1-\rho_P e^{-j\omega_i} z^{-1})} = \frac{1-2\rho_Z \cos(\omega_1) z^{-1} + \rho_Z^2 z^{-2}}{1-2\rho_P \cos(\omega_1) z^{-1} + \rho_P^2 z^{-2}}$$

$$H(z) = H_1(z) + H_2(z) = \frac{B_1(z)}{A_1(z)} + \frac{B_2(z)}{A_2(z)} = \frac{B_1(z)A_2(z) + B_2(z)A_1(z)}{A_1(z)A_2(z)}$$

$$C(z) = A(z)B(z) \Rightarrow \mathbf{c} = conv(\mathbf{a}, \mathbf{b})$$

$$C(z) = A(z) + B(z) \Rightarrow \mathbf{c} = \mathbf{a} + \mathbf{b}$$

Exercise 1c (3/4)

- **Goal**: quantify the bias and variance of the periodogram
- **Pseudocode:**
  - compute R realizations of N samples white noise
    - e = randn(N,R);
  - for each realization:
    - filter white noise by means of a LTI filter

$$Y(z) = H(z)E(z)$$

    - compute the periodogram spectral estimate

      phip(i,:) = (1/N)*abs(fft(y,N)).^2;
  - end

Exercise 1c (4/4)

- compute the ensemble mean: phip(RxN)
  - phipmean = mean(phip);
- compute the ensemble variance
  - phipvar = var(phip);

- Plot the true spectrum Φ(ω)
- Plot the average of R spectral estimates and compare the averaged spectrum and the true PSD
- Plot $E\left[\hat{\phi}_P(\omega)\right] \pm \sqrt{\mathrm{var}\left[\hat{\phi}_P(\omega)\right]}$ and compare with the true PSD

Exercise 2 (1/3)

- Goal: Estimate the power spectral density of the signal “flute2” by means of *correlogram*.

- **Hints on correlogram**: the spectrum estimation using correlogram is given by the following equation:

$$\hat{\phi}_C(\omega) = \sum_{k=-(N-1)}^{N-1} \hat{r}(k)\, e^{-j\omega k}$$

$$\hat{r}(k) = \frac{1}{N} \sum_{n=k+1}^{N} y(n)\, y^*(n-k) \quad k \geq 0$$

Exercise 2 (2/3)

- Pseudocode:
  - load the file flute2.wav
  - consider 50ms of the input signal (y)
  - estimate ACS
    - [r lags] = xcorr(y, 'biased');
    - r = circshift(r,N);
  - estimate PSD using correlogram:
    - N = length(y);
    - M = 2^ceil(log2(2*N-1)+1);

      %number of frequency bins
    - phic = fft(r,M);

Exercise 2 (3/3)

- Hint: Matlab provides the functions:
  - “[r lag]=xcorr(x,’biased’)” that produces a biased estimate of the autocorrelation (2N-1 samples) of the stationary sequence “x”. “lag” is the vector of lag indices [-N+1:1:N-1].
  - “r = circshift(r,N)” that circularly shifts the values in the array r by N elements. If N is positive, the values of r are shifted down (or to the right). If it is negative, the values of r are shifted up (or to the left).

Exercise 3 (1/2)

- Goal: Estimate the power spectral density of the signal “flute2” by means of *Blackman-Tukey method*.

- **Hints on B-T method:** the spectrum estimation using BT method is given by the following equation:

$$\hat{\phi}_{BT}(\omega) = \sum_{k=-(M-1)}^{M-1} w(k)\hat{r}(k)\, e^{-j\omega k} \qquad W(\omega) \geq 0$$

Exercise 3 (2/2)

- Pseudocode:
  - load the file flute2.wav
  - consider 50ms of the input signal -->N = length(y);
  - estimate ACS
    - [r lags] = xcorr(y, 'biased');
  - window with a bartlett window of the same length
    - rw = r.*bartlett(2*N-1);
    - r = circshift(r,N);
  - estimate PSD using BT:
    - Nfft = 2^ceil(log2(2*N-1)+1);
    - phiBT = real(fft(r,Nfft));

Exercise 3b (1/3)

- **Goal**: quantify the bias and variance of the BT method
- **Pseudocode:**
  - compute R realizations of N samples white noise
    - e = randn(N,R);
  - for each realization:
    - filter white noise by means of a LTI filter

$$Y(z) = H(z)E(z)$$

    - compute the BT spectral estimate
  - end

Exercise 3b (2/3)

- compute the ensemble mean: phip(RxN)
  - phipmean = mean(phip);
- compute the ensemble variance
  - phipvar = var(phip);

- Plot the true spectrum $\Phi(\omega)$
- Plot the average of R spectral estimates and compare the averaged spectrum and the true PSD
- Plot $E\left[\hat{\phi}_P(\omega)\right] \pm \sqrt{\operatorname{var}\left[\hat{\phi}_P(\omega)\right]}$ and compare with the true PSD

Exercise 3b (3/3)

- QUESTION: Can you see any difference between periodogram and BT method?

Exercise 4 (1/3)

- Goal: Estimate the power spectral density of the signal “flute2” by means of *Bartlett method*.

- **Basic idea:** split up the available sample of N observations into L = N/M subsamples of M observations each, then average the periodograms obtained from the subsamples for each value of ω.

Exercise 4 (2/3)

- **Matematically:** the spectrum estimation using Bartlett method is given by the following equations:

$$y_i(n) = y((i-1)M + n) \quad n = 1,\dots,M$$

$$i = 1,\dots,L \equiv \frac{N}{M}$$

$$\hat{\phi}_i(\omega) = \frac{1}{M}\left|\sum_{n=1}^{M} y_i(n)\, e^{-j\omega n}\right|^2 \qquad \hat{\phi}_B(\omega) = \frac{1}{L}\sum_{i=1}^{L}\hat{\phi}_i(\omega)$$

Exercise 4 (3/3)

- Pseudocode:
  - load the file flute2.wav
  - consider 50ms of the input signal -->N = length(y);
  - define the number of subsequences L and the number of samples for each of them M=ceil(N/L)
  - for each subsequence:
    - consider the right samples: yl = y(1+l*M : M+l*M);
    - estimate periodogram: (1/M)*abs(fft(yl)).^2
  - mean periodograms of the subsequences:
    - phil = phil + (1/M)*abs(fft(yl)).^2;
    - phiB=phil/L;

Exercise 5 (1/4)

- Goal: Estimate the power spectral density of the signal “flute2” by means of ***Welch method***.

- **Basic idea:** similar to Bartlett method but:
- allow overlap of subsequences
- use data window for each periodogram

Exercise 5 (2/4)

- **Matematically:** the spectrum estimation using Welch method is given by the following equations:

$$y_i(n) = y((i-1)K + n) \quad n = 1,\dots,M$$
$$i = 1,\dots,S$$

$$\hat{\phi}_i(\omega) = \frac{1}{MP}\left|\sum_{n=1}^{M} v(n) y_i(n)\, e^{-j\omega n}\right|^2$$

$$P = \frac{1}{M}\sum_{n=1}^{M} |v(n)|^2$$

$$\hat{\phi}_W(\omega) = \frac{1}{S}\sum_{i=1}^{S} \hat{\phi}_i(\omega)$$

Exercise 5 (3/4)

- Pseudocode:
  - load the file flute2.wav
  - consider 50ms of the input signal -->N = length(y);
  - define:
    - the number of samples for each subsequence: M
    - the number of new samples for each subsequence: K=M/4
    - the number of subsequences: S= N/K - (M-K)/K;
    - the window: v = hamming(M) ;
    - P = (1/M)*sum(v.^2);

Exercise 5 (4/4)

- Pseudocode (continued):
  - for each subsequence:
    - consider the right samples:
      - xs = x(1+s*K : M+s*K) ;
    - window the subsequence:
      - v.*xs
    - estimate periodogram:
      - (1/(M*P))*abs(fft(v.*xs)).^2
  - mean periodograms of the subsequences:
    - phis = phis+ (1/(M*P))*abs(fft(v.*xs)).^2 ;
    - phiW = phis/S;

Agenda

NON PARAMETRIC SPECTRAL ESTIMATION

- Ex1: Periodogram
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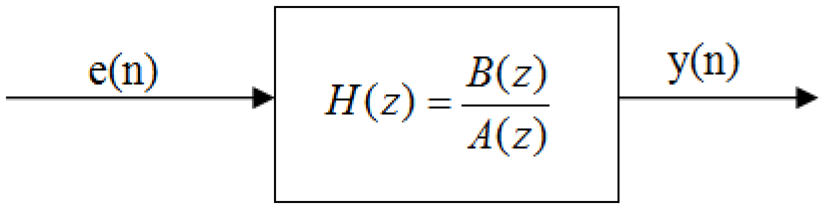
- Ex4: Bartlett
- Ex5: Welch

PARAMETRIC SPECTRAL ESTIMATION

- Ex6: AutoRegressive model

Exercise 6 (1/11)

- The parametric of model-based methods of spectral estimation assume that the signal satisfies a generating model with known functional form, and then proceed in estimating the parameters in the assumed model Power Spectral Density



$$\phi(\omega) = \left| \frac{B(\omega)}{A(\omega)} \right|^2 \sigma_e^2$$

Exercise 6 (2/11)

- Hint: We have y(n).

From the estimated $\{a_n\}$ and $\{b_m\}$, we can compute the correspondent $\hat{\phi}(\omega)$:

- Depending on the values assumed by m and n we can have the following cases:
  - if $m = 0$ and $n \neq 0$, autoregressive model (AR), $A(z)Y(z) = E(z)$
  - if $m \neq 0$ and $n = 0$, moving average model (MA), $Y(z) = B(z)E(z)$
  - if $m \neq 0$ and $n \neq 0$, ARMA model (autoregressive, moving average), $A(z)Y(z) = B(z)E(z)$

Exercise 6 (3/11)

- For AR class we have:

$$\begin{cases} r(0)+\sum_{i=1}^{n} a_i r(-i) = \sigma_e^2 \sum_{j=1}^{m} b_j h_j^* = \sigma_e^2 \\ r(k)+\sum_{i=1}^{n} a_i r(k-i) = 0 \qquad\qquad k>0 \end{cases}$$

Exercise 6 (4/11)

- For AR class we have:

$$\begin{bmatrix} r(0) & r(-1) & \dots & r(-n) \\ r(1) & r(0) & \dots & . \\ . & . & . & . \\ r(n) & . & \dots & r(0) \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ . \\ a_n \end{bmatrix} = \begin{bmatrix} \sigma^2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Exercise 6 (5/11)

- For AR class we have:

$$\begin{bmatrix} r(1) \\ . \\ r(n) \end{bmatrix} + \begin{bmatrix} r(0) & . & r(-n+1) \\ . & . & . \\ r(n-1) & . & r(0) \end{bmatrix} \begin{bmatrix} a_1 \\ . \\ a_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$r_n + R_n\theta = 0 \qquad\qquad \theta = [a_1, a_2, ..., a_n]$$

Exercise 6 (6/11)

- Solution:

$$\theta = -R_n^{-1} r_n \qquad \theta = [a_1, a_2, \ldots, a_n]$$

$$\sigma^2 = r(0) + \sum_{i=1}^{n} a_i r(-i)$$

  - Hint: if the true ACS is unknown, we can replace r(k) with $\hat{r}(k)$
  - The PSD estimate is: $\hat{\phi}(\omega) = \dfrac{\hat{\sigma}^2}{\left|\hat{A}(\omega)\right|^2}$

Exercise 6 (7/11)

- Goal: Estimate the power spectral density of the signal "y" by means of AR model.

- Pseudocode:
  - Consider the signal y defined by the differential equation:

    $y(n)=a_1\, y(n-1) + a_2\, y(n-2) + a_3\, y(n-3) + z(n)$
  - Estimate $\{a_p\}$ and σ_z with an AR model (order p)
  - Plot estimated PSD and compare with the true PSD

Exercise 6 (8/11)

- Hint: Matlab provides the functions:
  - "[r lag]=xcorr(x,'biased')" that produces a biased estimate of the autocorrelation (2N-1 samples) of the stationary sequence "x". "lag" is the vector of lag indices [-N+1:1:N-1].
  - "R=toeplitz(C,R)" that produces a non-symmetric Toeplitz matrix having C as its first column and R as its first row.

    "R=toeplitz(R)" is a symmetric (or Hermitian) Toeplitz matrix.

Exercise 6 (9/11)

- **Pseudocode:**
  - Consider the signal y defined by the differential equation:

    $y(n)=a_1\ y(n-1) + a_2\ y(n-2) + a_3\ y(n-3) + z(n)$

    - sigmae = 10;
    - a = poly([0.99 0.99*exp(j*pi/4) 0.99*exp(-j*pi/4)])
    - b = 1 ;
    - z = sigmae*randn(N,1);
    - y = filter(b, a, z);

Exercise 6 (10/11)

- **Pseudocode** (continued):
  - Estimate $\{a_p\}$ and σ_z with an AR model (order n)
    - n=3;
    - r = xcorr(y , 'biased');
    - Rx = toeplitz(r(N:N+n-1), r (N:-1:N-n+1));
    - rz = r(N+1:N+n) ;
    - theta = -Rx^(-1)*rz;
    - varz = r(N) +sum(theta.*r(N-1:-1:N-n));

Exercise 6 (11/11)

- **Pseudocode** (continued):
  - Plot estimated PSD and compare with the true PSD
    - plot(w, 10*log10(sigmae^2*abs(H).^2))
    - hold on
    - [He, w] = freqz(1 ,[1; theta], Nfft);
    - plot(w, 10*log10(varz*abs(He).^2), 'r')
    - legend ('true', 'estimated');
    - hold off

Exercise 6b (1/1)

- An all-pole LTI filter is defined by the following poles:

  $[0.99, 0.9e^{+j0.2}, 0.9e^{-j0.2}, 0.95e^{+j0.5}, 0.95e^{-j0.5}]$

  - Feed a white noise process e(n) ($r_e(n) = \delta(n)$) into the system to produce one realization of N = 1000 samples.
  - Plot the true PSD
  - Estimate the PSD using the Bartlett method.
  - Estimate the PSD using a parametric method. Verify the effect of the number of parameters.

Exercise 6c (1/3)

- **Goal**: Estimate the PSD, using AR model, of the signal obtained filtering a white noise by means of the following LTI filter

$$Y(z) = [H_1(z) + H_2(z)]E(Z)$$

- With the following poles and zeros

$$\rho_Z = 0.95$$
$$\rho_P = 0.99$$
$$\omega_1 = 0.6\pi$$
$$\omega_2 = 2(0.3+\alpha)\pi$$
$$\alpha = 0.05$$

$$p_1 = \rho_P e^{j\pm\pi\omega_1} \quad p_2 = \rho_P e^{j\pm\pi\omega_2}$$

$$z_1 = \rho_Z e^{j\pm\pi\omega_1} \quad z_2 = \rho_Z e^{j\pm\pi\omega_2}$$

$$H(z) = H_1(z) + H_2(z) = \frac{B_1(z)}{A_1(z)} + \frac{B_2(z)}{A_2(z)} = \frac{B_1(z)A_2(z) + B_2(z)A_1(z)}{A_1(z)A_2(z)}$$

Exercise 6c (2/3)

- **Pseudocode:**
  - compute R realizations of N samples white noise
    - e = randn(N,R);
  - for each realization:
    - filter white noise by means of a LTI filter

$$Y(z) = H(z)E(z)$$

    - compute the AR spectral estimate
    - [P, w] = freqz(1, [1 theta'], w);
    - Shat(r,:)=var_s*abs(P).^2;
  - end

Exercise 6c (3/3)

- compute the ensemble mean: Shat(RxN)
  - ExpShat=mean(Shat,1);

- Plot the true spectrum $\Phi(\omega)$
- Plot the average of R spectral estimates and compare the averaged spectrum and the true PSD

- Observe what happens varying the order p