IonQ-SKKU Challenge
Your name here
Fall 2023

Problem Set 1 – Solution

1. *(Spin chains 101.)* Show that the spin interaction terms acting on a common pair of sites along different axes commute. Concretely, let

$$[A, B] = AB - BA$$

denote the *commutator* of A and B, let $S_j^\alpha = \sigma_j^\alpha \sigma_{j+1}^\alpha$ denote the spin-spin interaction term at the $(j, j+1)$-st sites along the α direction, and show that

$$[S_j^\alpha, S_j^\beta] = 0.$$

Now show that, conversely, spin interaction terms at adjacent pairs of sites do **not** commute; that is, prove that

$$[S_j^\alpha, S_{j+1}^\beta] \neq 0.$$

Solution

1. $[S_j^\alpha, S_j^\beta] = 0$

$\sigma_j^\alpha, \sigma_j^\beta = 0$, they are anti commute which means

$\sigma_j^\alpha \sigma_j^\beta + \sigma_j^\beta \sigma_j^\alpha = 0$

$\sigma_j^\alpha \sigma_j^\beta = -\sigma_j^\beta \sigma_j^\alpha$

So when we change the sequence, -1 is multiplied to the value. and beacuse that sequnce is twice, $-1 \times -1 = 1$ that means,

$\sigma_j^\alpha \sigma_{j+1}^\alpha \sigma_j^\beta \sigma_{j+1}^\beta = \sigma_j^\beta \sigma_{j+1}^\beta \sigma_j^\alpha \sigma_{j+1}^\alpha$

Solution

2. $[S_j^\alpha, S_{j+1}^\beta] \neq 0$

In this case, interaction is only occur in the j+1 term. So the result is anti commute which means,

$\sigma_j^\alpha \sigma_{j+1}^\alpha \sigma_{j+1}^\beta \sigma_{j+2}^\beta = -\sigma_{j+1}^\beta \sigma_{j+2}^\beta \sigma_j^\alpha \sigma_{j+1}^\alpha$

2. *(Schrodinger dynamics.)* Suppose A is a complex $m \times m$ matrix. In addition, suppose that for each $t \geq 0$, $z(t)$ is a complex m-vector. Use the defining series for the matrix exponential to show that $z(t) = e^{tA} z(0)$ is a solution to

$$z'(t) = Az(t).$$

Solution

$$z(t) = e^{At} z(0) = \sum_{k=1}^{\text{inf}} \frac{(At)^k}{k!} z(0)$$

$$z'(t) = \sum_{k=1}^{\text{inf}} k \frac{A^k t^{k-1}}{k!} z(0)$$

$$z'(t) = A \sum_{k=1}^{\text{inf}} \frac{A^{k-1} t^{k-1}}{(k-1)!} z(0) = Ae^{tA} z(0) = Az(t)$$

3. *(Foundations: $N = 2$ case.)*

   (a) Use the binomial theorem to show that if A and B are commuting $m \times m$ matrices then

   $$e^A e^B = e^{A+B}.$$

   In addition, show that this generally fails if A and B do **not** commute; that is, find two matrices A, B such that $[A, B] \neq 0$ and

   $$e^A e^B \neq e^{A+B}.$$

   Solution

   1. $A^n B = A^{n-1} B A = A^{n-2} B A^2 ... = B A^n$

   Right equation $= \sum_{k=0}^{\text{inf}} \sum_{l=0}^{k} \frac{B^l A^{k-l}}{l!(k-l)!}$

   for given number l : $\sum_{k=l}^{\text{inf}} \frac{B^l}{l!} \frac{A^{k-l}}{(k-l)!} = \frac{B^l}{l!} \sum_{k=0}^{\text{inf}} \frac{A^k}{k!}$

   So $l = 0$ to inf , $= \sum_{l=0}^{\text{inf}} \frac{B^l}{l!} \sum_{k=0}^{\text{inf}} \frac{A^k}{k!}$

   2. $A = i\pi/2X, B = i\pi/2Z$ (X Z is Pauli gate, Not consider glober phase)
   $e^{i\pi/2X} = -iX$, $e^{i\pi/2Z} = -iZ$ $XZ = Y = e^{i\pi/2Y} \neq e^{i\pi/2(X+Z)}$

(b) Write $\mathcal{H}_S = \mathcal{H}_{\text{even}} + \mathcal{H}_{\text{odd}}$ as a sum of even- and odd-indexed spin interaction terms.

Briefly explain why the summands in H_{even} (H_{odd}) pairwise commute; in other words, justify the following equalities:

$$e^{-it\mathcal{H}_{\text{even}}} = \prod_j B_{2j}(\theta) \quad \text{and similarly} \quad e^{-itH_{\text{odd}}} = \prod_j B_{2j-1}(\theta),$$

with B_j acting as the block operator on the $(j, j+1)$-st sites.

Solution

Because $(2j, 2j+1)$ and $(2j+2, 2j+3)$ is not interact in Hamiltonian(they only interact with adjacent part), the interaction between odd(even) part is independent. So because of this reason, if their sequence reversed, this can preserve the result.

(c) What is the depth of the circuit illustrated above? Compare this to the depth of the circuit that directly Trotterizes

$$U(t) \approx \left(\prod_j B_j(\theta/n)\right)^n.$$

Solution

(N: number of node)

Each step have two interaction that is occur between odd side and even side. So each step, depth is 2. But in direct Trotterizes, interaction occur in all of N blocks. This makes depth more bigger that is N each step.

Result is $2n$(odd,even) and Nn(direct)

4. *(Staggered magnetization.)* Suppose $\mathcal{O}$ is a Hermitian observable whose matrix with respect to the computational basis is diagonal. Let λ_x denote the eigenvalue of $\mathcal{O}$ corresponding to the computational basis state $|x\rangle$. Let $|\psi\rangle = \sum_x \alpha_x |x\rangle$ denote any quantum state, written as a superposition over the computational basis, and let $p_x = |\alpha_x|^2$ denote the probability of observing $|\psi\rangle$ in the state $|x\rangle$.

Show that the expectation value

$$\langle\psi|\, \mathcal{O}\, |\psi\rangle = \sum_x p_x \lambda_x$$

is simply a weighted average of the eigenvalues of $\mathcal{O}$.

Solution

$\mathcal{O}\,|\psi\rangle = \sum\limits_x \mathcal{O}\,\alpha_x|x\rangle = \sum\limits_x \lambda_x\alpha_x|x\rangle$

So, $\langle\psi|\,\mathcal{O}\,|\psi\rangle = \sum\limits_x \langle x|\alpha_x^*\lambda_x\alpha_x|x\rangle = \sum\limits_x p_x\lambda_x$

5. *(YBE-powered compression.)*

   (a) Comment on your results and include your plots showing four magnetization curves on the same set of axes for each N.

   Solution

   Fill this in!

   (b) Use Matsumoto's Monoid Lemma or an explicit basis of the (finite-dimensional) 0-Hecke algebra to prove our compression scheme existence theorem.

   Solution

   Fill this in!