Problem 10

Let $f(x) = \frac{x^2+5x}{25-x^2}$.

A. We have $25 - x^2 = 0$ when $x^2 = 25$. So the denominator is 0 when $x = \pm 5$. The domain is then $(-\infty, -5) \cup (-5, 5) \cup (5, \infty)$.

B. For $x = 0$, we find $f(0) = 0$. Also, we have $f(x) = 0$ when $x^2 + 5x = 0$. So the x-intercept is $x = 0$.

C. No symmetry, unfortunately.

D. We first find the HAs and then the VAs.

(I) We first rewrite the function as followed:

$$f(x) = \frac{x^2(1 + 5/x)}{x^2(25/x^2 - 1)} = \frac{1 + 5/x}{25/x - 1}.$$

We can now see that

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1 + 5/x}{25/x - 1} = \frac{1}{-1} = -1$$

and

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{1 + 5/x}{25/x - 1} = \frac{1}{-1} = -1.$$

Therefore, $y = -1$ is a HA for $x \to \infty$ and $y = -1$ is a HA for $x \to -\infty$.

(II) We have $25 - x^2 = (5 - x)(5 + x)$ and $x^2 + 5x = x(x + 5)$. Therefore, the expression of the function becomes

$$f(x) = \frac{x(x + 5)}{(5 - x)(5 + x)}.$$

Recall that we might have some problems at $x = -5$ and $x = 5$ because of the division by zero.

We first have

$$\lim_{x\to-5} f(x) = \lim_{x\to-5} \frac{x}{5 - x} = \frac{-5}{5 - (-5)} = -\frac{1}{2}$$

and therefore there is no VA at $x = -5$.

Let's now examine the other possible problem at $x = 5$. We have

$$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^-} \frac{x}{5 - x} = \frac{5}{0^+} = \infty$$

and

$$\lim_{x \to 5^+} f(x) = \lim_{x \to 5^+} \frac{x}{5 - x} = \frac{5}{0^-} = -\infty.$$

Therefore, we have a VA at $x = 5$.

E. The derivative of the function is

$$f'(x) = \frac{5}{(x - 5)^2}.$$

There is one critical number, which is $x = 5$ because the derivative does not exist there.

The second derivative of the function is

$$f''(x) = -\frac{10}{(x - 5)^3}.$$

There is one possible inflection point which is $x = 5$ because the second derivative does not exist there.

F. We will now construct the table

(a) Recall that $x = 5$ is a critical number. The sign of the derivative does not change because $(x - 5)^2 \geq 0$. Therefore, $f'(x) > 0$ when $x \neq 5$ and the function is increasing there.

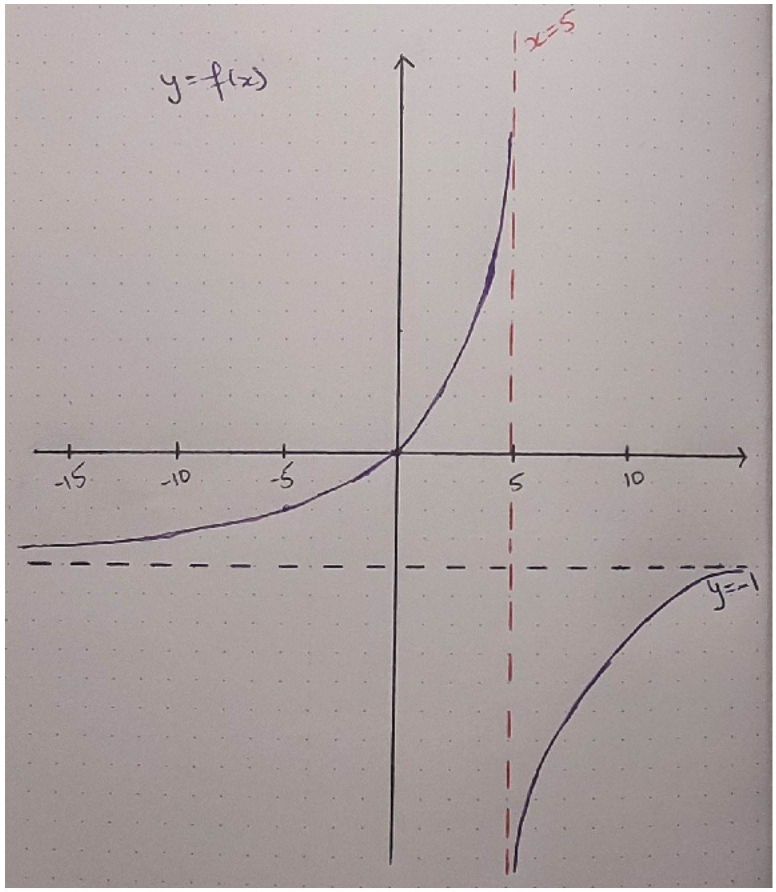
(b) We have only one possible inflection point, at $x = 5$. When $x < 5$, then $x - 5 < 0$, so that $(x - 5)^3 < 0$. Therefore, because of the multiplication by -10, we obtain that $f''(x) > 0$. When $x > 5$, then $x - 5 > 0$ and $(x - 5)^3 > 0$. Therefore, we get that $f''(x) < 0$.

Derivatives	$x <$	-5	$< x <$	5	$< x$
$f'(x)$	$+$		$+$	$\nexists$	$+$
$f''(x)$	$+$		$+$	$\nexists$	$-$
$f(x)$		DNE		VA	

(c) We now see from the table that

- There is no maximum at $x = 5$.
- There is an inflection point at $x = 5$.

G. We can know sketch the graph of the function (see next page).


y = f(x)
x=5
-15
-10
-5
5
10
y=-1

Problem 20

Let $f(x) = \frac{x^3}{x-2}$.

A. We see that $x - 2 = 0$ when $x = 2$. Therefore the domain is $(-\infty, 2) \cup (2, \infty)$.

B. The y-intercept is $f(0) = 0$. The x-intercept is the values of x giving $f(x) = 0$. The only x-intercept is then $x = 0$.

C. There is no symmetry.

D. We will first find the HAs and then the VAs.

(I) We first see that

$$f(x) = \frac{xx^2}{x(1-2/x)} = \frac{x^2}{1-2/x}.$$

Since $\lim_{x\to\infty} \frac{2}{x} = 0$ and $\lim_{x\to\infty} x^2 = \infty$, we have

$$\lim_{x\to\infty} f(x) = \frac{\lim_{x\to\infty} x^2}{\lim_{x\to\infty} 1 - 2/x} = \frac{\infty}{1} = \infty.$$

Similarly, we have $\lim_{x\to-\infty} f(x) = \infty$. There is no HA.

(II) We have a problem when $x = 2$. Let's examine more closely this problem. We have

$$\lim_{x\to 2^-} \frac{x^3}{x-2} = \frac{(2^-)^3}{0^-} = \frac{8}{0^-} = -\infty.$$

We also have

$$\lim_{x\to 2^+} \frac{x^3}{x-2} = \frac{8}{0^+} = \infty.$$

There is a VA at $x = 2$.

E. The derivative of $f(x)$ is

$$f'(x) = \frac{3x^2(x-2) - x^3}{(x-2)^2} = \frac{2x^2(x-3)}{(x-2)^2}$$

We find the critical numbers. The derivative does not exist when $x - 2 = 0$, so when $x = 2$. The derivative is 0 if

$$\frac{2x^2(x-3)}{(x-2)^2} = 0 \iff 2x^2(x-3) = 0 \iff x = 0 \text{ or } x = 3.$$

The second derivative of $f(x)$ is

$$f''(x) = \frac{2x(x^2 - 6x + 12)}{(x-2)^3}.$$

It is zero when $x = 0$ or $x^2 - 6x + 12 = 0$. But the polynomial $x^2 - 6x + 12$ is never zero because its discrimant is

$$b^2 - 4ac = 36 - 48 = -12 < 0.$$

The second derivative does not exist when $x = 2$.

F. We now construct the table.

(I) The critical numbers are $x = 0$ and $x = 3$. Since $(x-2)^2 \geq 0$ and $x^2 \geq 0$, the sign of the derivative is determined by the sign of the factor $(x-3)$. So, when $x < 3$, we have $x - 3 < 0$ and therefore $f'(x) < 0$. When $x > 3$, we have $x - 3 > 0$ and therefore $f'(x) > 0$. We input this information in the table.

(II) The possible inflection points are $x = 0$ and $x = 2$. Since $x^2 - 6x + 12 \geq 0$, the sign of the second derivative is determined by the sign of x and of $(x-2)$. If $x < 0$, then $x - 2 < 0$ and therefore the overall sign is $f''(x) > 0$. When $x > 0$ but $x < 2$, then $x - 2 < 0$ and the overall sign if $f''(x) < 0$. When $x > 2$, then $x > 0$ and $x - 2 > 0$ and therefore the overall sign is still $f'(x) > 0$. We input this in the table.

Derivatives	$x <$	0	$< x <$	2	$< x <$	3	$< x$
$f'(x)$	$-$	0	$-$	$\nexists$	$-$	0	$+$
$f''(x)$	$+$	0	$-$	$\nexists$	$+$		$+$
$f(x)$	↘		↘	VA	↘		↗

(III) We now see from the table that

- There is no maximum at $x = 0$ and $x = 2$ from the First derivative test.
- There is a local minimum at $x = 3$ from the First derivative test. We have $f(3) = 27$.
- There is an inflection point at $x = 0$ and $x = 2$.

G. We are now ready to sketch the graph of the function.