CHAPTER 1 CONCEPT CHECK ANSWERS

1. (a) What is a function? What are its domain and range?

   A function f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E. The domain is the set D and the range is the set of all possible values of $f(x)$ as x varies throughout the domain.

   (b) What is the graph of a function?

   The graph of a function f consists of all points (x, y) such that $y = f(x)$ and x is in the domain of f.

   (c) How can you tell whether a given curve is the graph of a function?

   Use the Vertical Line Test: a curve in the xy-plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.

2. Discuss four ways of representing a function. Illustrate your discussion with examples.

   A function can be represented verbally, numerically, visually, or algebraically. An example of each is given below.

   Verbally: An assignment of students to chairs in a classroom (a description in words)

   Numerically: A tax table that assigns an amount of tax to an income (a table of values)

   Visually: A graphical history of the Dow Jones average (a graph)

   Algebraically: A relationship between the area A and side length s of a square: $A = s^2$ (an explicit formula)

3. (a) What is an even function? How can you tell if a function is even by looking at its graph? Give three examples of an even function.

   A function f is even if it satisfies $f(-x) = f(x)$ for every number x in its domain. If the graph of a function is symmetric with respect to the y-axis, then f is even. Examples are $f(x) = x^2$, $f(x) = \cos x$, $f(x) = |x|$.

   (b) What is an odd function? How can you tell if a function is odd by looking at its graph? Give three examples of an odd function.

   A function f is odd if it satisfies $f(-x) = -f(x)$ for every number x in its domain. If the graph of a function is symmetric with respect to the origin, then f is odd. Examples are $f(x) = x^3$, $f(x) = \sin x$, $f(x) = 1/x$.

4. What is an increasing function?

   A function f is increasing on an interval I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I.

5. What is a mathematical model?

   A mathematical model is a mathematical description (often by means of a function or an equation) of a real-world phenomenon. (See the discussion on pages 23–24.)

6. Give an example of each type of function.

   (a) Linear function: $f(x) = 2x + 1$, $f(x) = ax + b$

   (b) Power function: $f(x) = x^2$, $f(x) = x^n$

   (c) Exponential function: $f(x) = 2^x$, $f(x) = b^x$

   (d) Quadratic function: $f(x) = x^2 + x + 1$, $f(x) = ax^2 + bx + c$

   (e) Polynomial of degree 5: $f(x) = x^5 + 2x^4 - 3x^2 + 7$

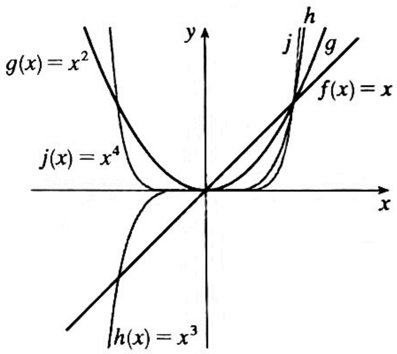
   (f) Rational function: $f(x) = \dfrac{x}{x+2}$, $f(x) = \dfrac{P(x)}{Q(x)}$

   where $P(x)$ and $Q(x)$ are polynomials

7. Sketch by hand, on the same axes, the graphs of the following functions.

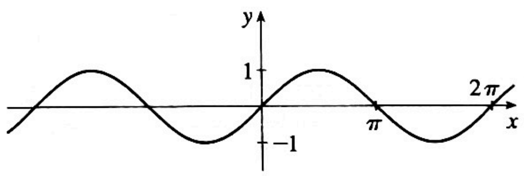
   (a) $f(x) = x$ (b) $g(x) = x^2$

   (c) $h(x) = x^3$ (d) $j(x) = x^4$

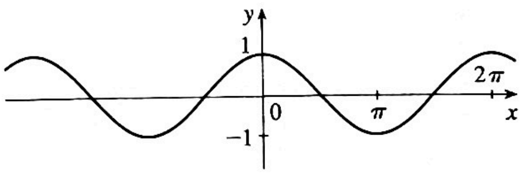


8. Draw, by hand, a rough sketch of the graph of each function.

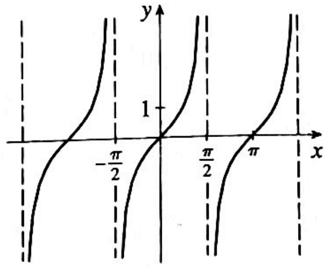
   (a) $y = \sin x$



   (b) $y = \cos x$



   (c) $y = \tan x$



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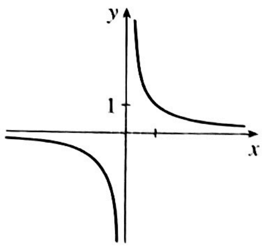
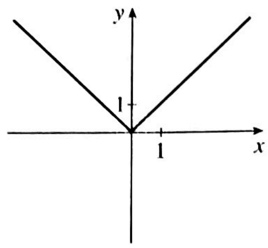
CHAPTER 1 CONCEPT CHECK ANSWERS *(continued)*

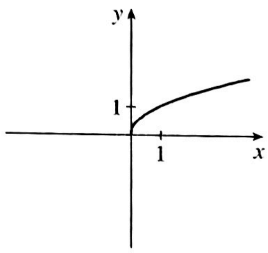
(d) $y = 1/x$



(e) $y = |x|$



(f) $y = \sqrt{x}$



9. Suppose that f has domain A and g has domain B.

(a) What is the domain of $f + g$?

The domain of $f + g$ is the intersection of the domain of f and the domain of g; that is, $A \cap B$.

(b) What is the domain of fg?

The domain of fg is also $A \cap B$.

(c) What is the domain of f/g?

The domain of f/g must exclude values of x that make g equal to 0; that is, $\{x \in A \cap B \mid g(x) \neq 0\}$.

10. How is the composite function $f \circ g$ defined? What is its domain?

The composition of f and g is defined by $(f \circ g)(x) = f(g(x))$. The domain is the set of all x in the domain of g such that $g(x)$ is in the domain of f.

11. Suppose the graph of f is given. Write an equation for each of the graphs that are obtained from the graph of f as follows.

(a) Shift 2 units upward: $y = f(x) + 2$

(b) Shift 2 units downward: $y = f(x) - 2$

(c) Shift 2 units to the right: $y = f(x - 2)$

(d) Shift 2 units to the left: $y = f(x + 2)$

(e) Reflect about the x-axis: $y = -f(x)$

(f) Reflect about the y-axis: $y = f(-x)$

(g) Stretch vertically by a factor of 2: $y = 2f(x)$

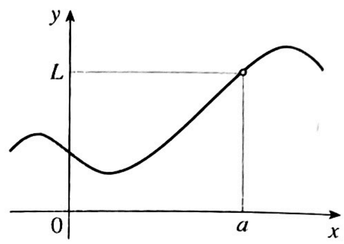
(h) Shrink vertically by a factor of 2: $y = \frac{1}{2}f(x)$

(i) Stretch horizontally by a factor of 2: $y = f(\frac{1}{2}x)$

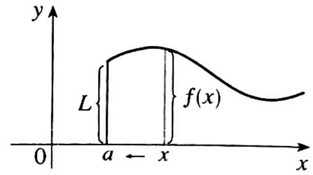
(j) Shrink horizontally by a factor of 2 $y = f(2x)$

12. Explain what each of the following means and illustrate with a sketch.

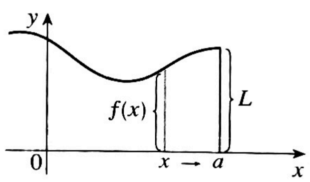
(a) $\lim_{x \to a} f(x) = L$ means that the values of $f(x)$ approach L as the values of x approach a (but $x \neq a$).



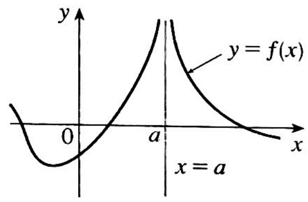
(b) $\lim_{x \to a^+} f(x) = L$ means that the values of $f(x)$ approach L as the values of x approach a through values greater than a.



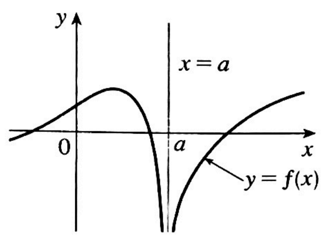
(c) $\lim_{x \to a^-} f(x) = L$ means that the values of $f(x)$ approach L as the values of x approach a through values less than a.



(d) $\lim_{x \to a} f(x) = \infty$ means that the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a (but not equal to a).



(e) $\lim_{x \to a} f(x) = -\infty$ means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a (but not equal to a).



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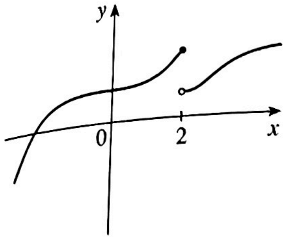
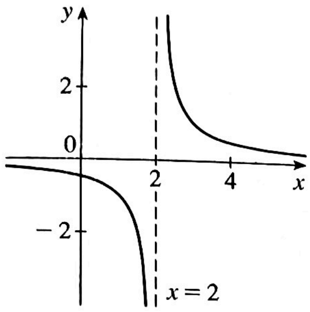
CHAPTER 11 CONCEPT CHECK ANSWERS *(continued)*

13. Describe several ways in which a limit can fail to exist. Illustrate with sketches.

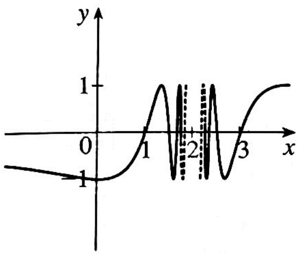
In general, the limit of a function fails to exist when the function values do not approach a fixed number. For each of the following functions, the limit fails to exist at $x = 2$.



The left and right limits are not equal.



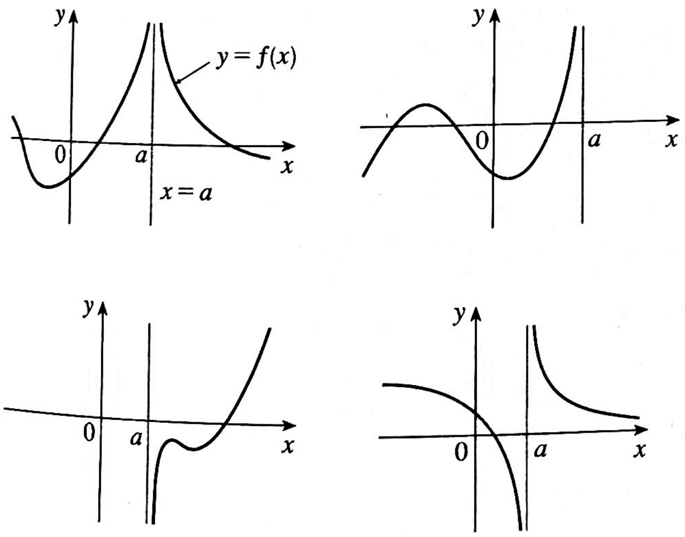
There is an infinite discontinuity.



The function values oscillate between 1 and -1 infinitely often.

14. What does it mean to say that the line $x = a$ is a vertical asymptote of the curve $y = f(x)$? Draw curves to illustrate the various possibilities.

It means that the limit of $f(x)$ as x approaches a from one or both sides is positive or negative infinity.



15. State the following Limit Laws.

(a) Sum Law

The limit of a sum is the sum of the limits:
$\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

(b) Difference Law

The limit of a difference is the difference of the limits:
$\lim_{x\to a}[f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$

(c) Constant Multiple Law

The limit of a constant times a function is the constant times the limit of the function: $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$

(d) Product Law

The limit of a product is the product of the limits:
$\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$

(e) Quotient Law

The limit of a quotient is the quotient of the limits, provided that the limit of the denominator is not 0:

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \quad \text{if } \lim_{x\to a} g(x) \neq 0$$

(f) Power Law

The limit of a power is the power of the limit:
$\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$ (for n a positive integer)

(g) Root Law

The limit of a root is the root of the limit:
$\lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ (for n a positive integer)

16. What does the Squeeze Theorem say?

If $f(x) \leqslant g(x) \leqslant h(x)$ when x is near a (except possibly at a) and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$. In other words, if $g(x)$ is squeezed between $f(x)$ and $h(x)$ near a, and if f and h have the same limit L at a, then g is forced to have the same limit L at a.

17. (a) What does it mean for f to be continuous at a?

A function f is continuous at a number a if the value of the function at $x = a$ is the same as the limit when x approaches a; that is, $\lim_{x\to a} f(x) = f(a)$.

(b) What does it mean for f to be continuous on the interval $(-\infty, \infty)$? What can you say about the graph of such a function?

A function f is continuous on the interval $(-\infty, \infty)$ if it is continuous at every real number a.

The graph of such a function has no hole or break in it.

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CONCEPT CHECK ANSWERS *(continued)*

18. **(a)** Give examples of functions that are continuous on $[-1, 1]$.

$f(x) = x^3 - x$, $g(x) = \sqrt{x + 2}$, $y = \sin x$, $y = \tan x$, $y = 1/(x - 3)$, and $h(x) = |x|$ are all continuous on $[-1, 1]$.

(b) Give an example of a function that is not continuous on $[0, 1]$.

$$f(x) = \frac{1}{x - \frac{1}{2}} \qquad \left[f(x) \text{ is not defined at } x = \tfrac{1}{2}\right]$$

19. What does the Intermediate Value Theorem say?

If f is continuous on $[a, b]$ and N is any number between $f(a)$ and $f(b)$ $[f(a) \neq f(b)]$, Then there exists a number c in (a, b) such that $f(c) = N$. In other words, a continuous function takes on every intermediate value between the function values $f(a)$ and $f(b)$.

CHAPTER 2 CONCEPT CHECK ANSWERS

1. Write an expression for the slope of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$.

   The slope of the tangent line is given by

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad \text{or} \quad \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

2. Suppose an object moves along a straight line with position $f(t)$ at time t. Write an expression for the instantaneous velocity of the object at time $t = a$. How can you interpret this velocity in terms of the graph of f?

   The instantaneous velocity at time $t = a$ is

$$v(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

   It is equal to the slope of the tangent line to the graph of f at the point $P(a, f(a))$.

3. If $y = f(x)$ and x changes from x_1 to x_2, write expressions for the following.

   (a) The average rate of change of y with respect to x over the interval $[x_1, x_2]$:

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

   (b) The instantaneous rate of change of y with respect to x at $x = x_1$:

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

4. Define the derivative $f'(a)$. Discuss two ways of interpreting this number.

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

   or, equivalently,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

   The derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ (with respect to x) when $x = a$ and also represents the slope of the tangent line to the graph of f at the point $P(a, f(a))$.

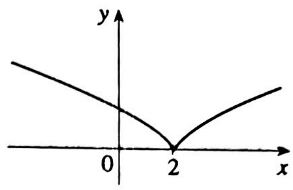
5. (a) What does it mean for f to be differentiable at a?

   f is differentiable at a if the derivative $f'(a)$ exists.

   (b) What is the relation between the differentiability and continuity of a function?

   If f is differentiable at a, then f is continuous at a.

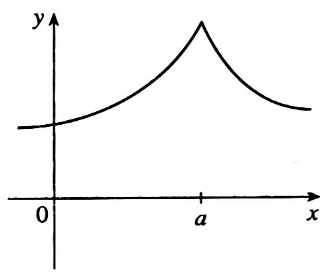
   (c) Sketch the graph of a function that is continuous but not differentiable at $a = 2$.



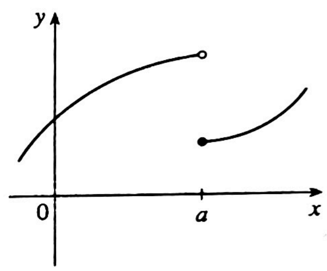
   The graph of f changes direction abruptly at $x = 2$, so f has no tangent line there.

6. Describe several ways in which a function can fail to be differentiable. Illustrate with sketches.

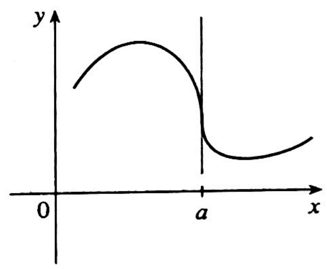
   A function is not differentiable at any value where the graph has a "corner," where the graph has a discontinuity, or where it has a vertical tangent line.



A corner



A discontinuity



A vertical tangent

7. What are the second and third derivatives of a function f? If f is the position function of an object, how can you interpret f'' and f'''?

   The second derivative f'' is the derivative of f', and the third derivative f''' is the derivative of f''.

   If f is the postition function of an object, then f' is the velocity function of the object, f'' is the acceleration function, and f''' is the jerk function (the rate of change of acceleration).

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8. State each differentiation rule both in symbols and in words.

(a) The Power Rule

If n is any real number, then $\frac{d}{dx}(x^n) = nx^{n-1}$.

To find the derivative of a variable raised to a constant power, we multiply the expression by the exponent and then subtract one from the exponent.

(b) The Constant Multiple Rule

If c is a constant and f is a differentiable function, then

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$$

The derivative of a constant times a function is the constant times the derivative of the function.

(c) The Sum Rule

If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

The derivative of a sum of functions is the sum of the derivatives.

(d) The Difference Rule

If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

The derivative of a difference of functions is the difference of the derivatives.

(e) The Product Rule

If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

The derivative of a product of two functions is the first function times the derivative of the second function plus the second function times the derivative of the first function.

(f) The Quotient Rule

If f and g are both differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

The derivative of a quotient of functions is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

(g) The Chain Rule

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

$$F'(x) = f'(g(x))g'(x)$$

The derivative of a composite function is the derivative of the outer function evaluated at the inner function times the derivative of the inner function.

9. State the derivative of each function.

(a) $y = x^n$: $y' = nx^{n-1}$

(b) $y = \sin x$: $y' = \cos x$

(c) $y = \cos x$: $y' = -\sin x$

(d) $y = \tan x$: $y' = \sec^2 x$

(e) $y = \csc x$: $y' = -\csc x \cot x$

(f) $y = \sec x$: $y' = \sec x \tan x$

(g) $y = \cot x$: $y' = -\csc^2 x$

10. Explain how implicit differentiation works.

Implicit differentiation consists of differentiating both sides of an equation with respect to x, treating y as a function of x. Then we solve the resulting equation for y'.

11. Give several examples of how the derivative can be interpreted as a rate of change in physics, chemistry, biology, economics, or other sciences.

In physics, interpretations of the derivative include velocity, linear density, electrical current, power (the rate of change of work), and the rate of radioactive decay. Chemists can use derivatives to measure reaction rates and the compressibility of a substance under pressure. In biology the derivative measures rates of population growth and blood flow. In economics, the derivative measures marginal cost (the rate of change of cost as more items are produced) and marginal profit. Other examples include the rate of heat flow in geology, the rate of performance improvement in psychology, and the rate at which a rumor spreads in sociology.

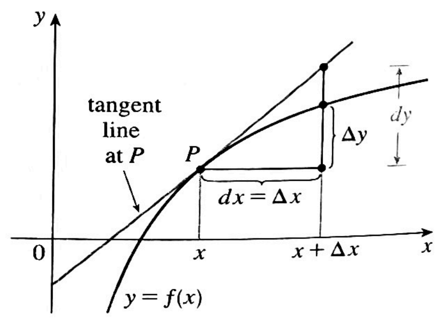
12. (a) Write an expression for the linearization of f at a.

$$L(x) = f(a) + f'(a)(x - a)$$

(b) If $y = f(x)$, write an expression for the differential dy.

$$dy = f'(x)\,dx$$

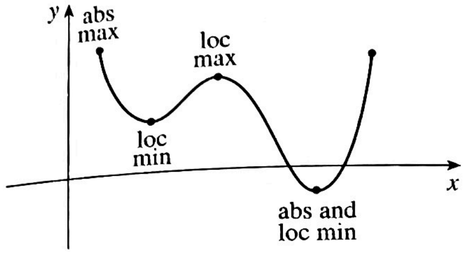
(c) If $dx = \Delta x$, draw a picture showing the geometric meanings of Δy and dy.



CHAPTER 3 CONCEPT CHECK ANSWERS

1. Explain the difference between an absolute maximum and a local maximum. Illustrate with a sketch.

   The function value $f(c)$ is the absolute maximum value of f if $f(c)$ is the largest function value on the entire domain of f, whereas $f(c)$ is a local maximum value if it is the largest function value when x is near c.



2. What does the Extreme Value Theorem say?

   If f is a continuous function on a closed interval $[a, b]$, then it always attains an absolute maximum and an absolute minimum value on that interval.

3. (a) State Fermat's Theorem.

   If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

   (b) Define a critical number of f.

   A critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

4. Explain how the Closed Interval Method works.

   To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$, we follow these three steps:

   - Find the critical numbers of f in the interval (a, b) and compute the values of f at these numbers.
   - Find the values of f at the endpoints of the interval.
   - The largest of the values from the previous two steps is the absolute maximum value; the smallest of these values is the absolute minimum value.

5. (a) State Rolle's Theorem.

   Let f be a function that satisfies the following three hypotheses:

   - f is continuous on the closed interval $[a, b]$.
   - f is differentiable on the open interval (a, b).
   - $f(a) = f(b)$

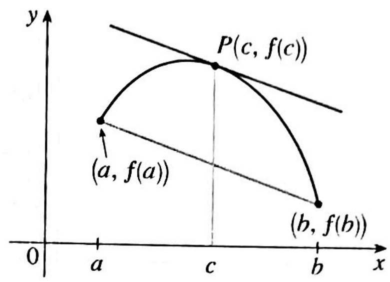
   Then there is a number c in (a, b) such that $f'(c) = 0$.

   (b) State the Mean Value Theorem and give a geometric interpretation.

   If f is continuous on the interval $[a, b]$ and differentiable on (a, b), then there exists a number c between a and b such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

   Geometrically, the theorem says that there is a point $P(c, f(c))$, where $a < c < b$, on the graph of f where the tangent line is parallel to the secant line that connects $(a, f(a))$ and $(b, f(b))$.



6. (a) State the Increasing/Decreasing Test.

   If $f'(x) > 0$ on an interval, then f is increasing on that interval.

   If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

   (b) What does it mean to say that f is concave upward on an interval I?

   f is concave upward on an interval if the graph of f lies above all of its tangents on that interval.

   (c) State the Concavity Test.

   If $f''(x) > 0$ on an interval, then the graph of f is concave upward on that interval.

   If $f''(x) < 0$ on an interval, then the graph of f is concave downward on that interval.

   (d) What are inflection points? How do you find them?

   Inflection points on the graph of a continuous function f are points where the curve changes from concave upward to concave downward or from concave downward to concave upward. They can be found by determining the values at which the second derivative changes sign.

7. (a) State the First Derivative Test.

   Suppose that c is a critical number of a continuous function f.

   - If f' changes from positive to negative at c, then f has a local maximum at c.
   - If f' changes from negative to positive at c, then f has a local minimum at c.
   - If f' is positive to the left and right of c, or negative to the left and right of c, then f has no local maximum or minimum at c.

   (b) State the Second Derivative Test.

   Suppose f'' is continuous near c.

   - If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
   - If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

(continued)

(c) What are the relative advantages and disadvantages of these tests?

The Second Derivative Test is sometimes easier to use, but it is inconclusive when $f''(c) = 0$ and fails if $f''(c)$ does not exist. In either case the First Derivative Test must be used.

8. Explain the meaning of each of the following statements.

(a) $\lim_{x\to\infty} f(x) = L$ means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large.

(b) $\lim_{x\to-\infty} f(x) = L$ means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large negative.

(c) $\lim_{x\to\infty} f(x) = \infty$ means that the values of $f(x)$ can be made arbitrarily large by requiring x to be sufficiently large.

(d) The curve $y = f(x)$ has the horizontal asymptote $y = L$.

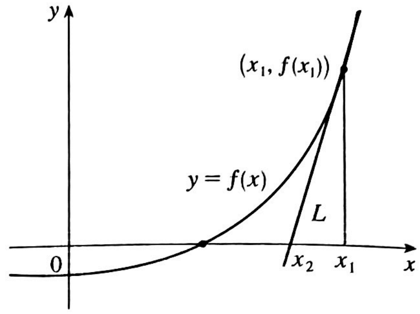
The line $y = L$ is called a horizontal asymptote of the curve $y = f(x)$ if either $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$.

9. If you have a graphing calculator or computer, why do you need calculus to graph a function?

Calculus reveals all the important aspects of a graph, such as local extreme values and inflection points, that can be missed when relying solely on technology. In many cases we can find exact locations of these key points rather than approximations. Using derivatives to identify the behavior of the graph also helps us choose an appropriate viewing window and alerts us to where we may wish to zoom in on a graph.

10. (a) Given an initial approximation x_1 to a root of the equation $f(x) = 0$, explain geometrically, with a diagram, how the second approximation x_2 in Newton's method is obtained.

We find the tangent line L to the graph of $y = f(x)$ at the point $(x_1, f(x_1))$. Then x_2 is the x-intercept of L.



(b) Write an expression for x_2 in terms of x_1, $f(x_1)$, and $f'(x_1)$.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

(c) Write an expression for x_{n+1} in terms of x_n, $f(x_n)$, and $f'(x_n)$.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

(d) Under what circumstances is Newton's method likely to fail or to work very slowly?

Newton's method is likely to fail or to work very slowly when $f'(x_1)$ is close to 0. It also fails when $f'(x_i)$ is undefined.

11. (a) What is an antiderivative of a function f?

A function F is an antiderivative of f if $F'(x) = f(x)$.

(b) Suppose F_1 and F_2 are both antiderivatives of f on an interval I. How are F_1 and F_2 related?

They are identical or they differ by a constant.

CHAPTER 4 CONCEPT CHECK ANSWERS

1. (a) Write an expression for a Riemann sum of a function f on an interval $[a, b]$. Explain the meaning of the notation that you use.

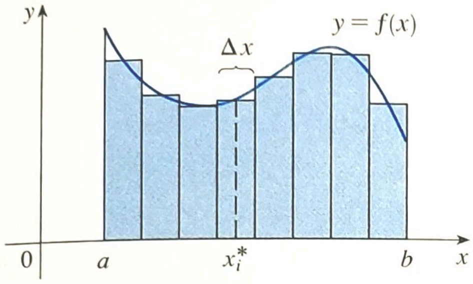
If f is defined for $a \leqslant x \leqslant b$ and we divide the interval $[a, b]$ into n subintervals of equal width Δx, then a Riemann sum of f is

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

where x_i^* is a point in the ith subinterval.

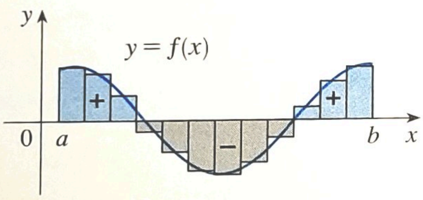
(b) If $f(x) \geqslant 0$, what is the geometric interpretation of a Riemann sum? Illustrate with a diagram.

If f is positive, then a Riemann sum can be interpreted as the sum of areas of approximating rectangles, as shown in the figure.



(c) If $f(x)$ takes on both positive and negative values, what is the geometric interpretation of a Riemann sum? Illustrate with a diagram.

If f takes on both positive and negative values then the Riemann sum is the sum of the areas of the rectangles that lie above the x-axis and the negatives of the areas of the rectangles that lie below the x-axis (the areas of the blue rectangles minus the areas of the gray rectangles).



2. (a) Write the definition of the definite integral of a continuous function from a to b.

If f is a continuous function on the interval $[a, b]$, then we divide $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0\ (= a), x_1, x_2, \ldots, x_n\ (= b)$ be the endpoints of these subintervals. Then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

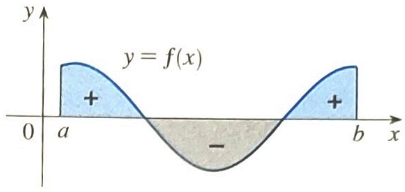
where x_i^* is any sample point in the ith subinterval $[x_{i-1}, x_i]$.

(b) What is the geometric interpretation of $\int_a^b f(x)\,dx$ if $f(x) \geqslant 0$?

If f is positive, then $\int_a^b f(x)\,dx$ can be interpreted as the area under the graph of $y = f(x)$ and above the x-axis for $a \leqslant x \leqslant b$.

(c) What is the geometric interpretation of $\int_a^b f(x)\,dx$ if $f(x)$ takes on both positive and negative values? Illustrate with a diagram.

In this case $\int_a^b f(x)\,dx$ can be interpreted as a "net area," that is, the area of the region above the x-axis and below the graph of f (labeled "+" in the figure) minus the area of the region below the x-axis and above the graph of f (labeled "–").



3. State the Midpoint Rule.

If f is a continuous function on the interval $[a, b]$ and we divide $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$, then

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\bar{x}_i)\,\Delta x$$

where $\bar{x}_i$ = midpoint of $[x_{i-1}, x_i] = \frac{1}{2}(x_{i-1} + x_i)$.

4. State both parts of the Fundamental Theorem of Calculus.

Suppose f is continuous on $[a, b]$.

Part 1. If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$.

Part 2. $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f, that is, $F' = f$.

5. (a) State the Net Change Theorem.

The integral of a rate of change is the net change:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

(b) If $r(t)$ is the rate at which water flows into a reservoir, what does $\int_{t_1}^{t_2} r(t)\,dt$ represent?

$\int_{t_1}^{t_2} r(t)\,dt$ represents the change in the amount of water in the reservoir between time t_1 and time t_2.

(continued)

6. Suppose a particle moves back and forth along a straight line with velocity $v(t)$, measured in feet per second, and acceleration $a(t)$.

(a) What is the meaning of $\int_{60}^{120} v(t)\,dt$?

$\int_{60}^{120} v(t)\,dt$ represents the net change in position (the displacement) of the particle from $t = 60$ s to $t = 120$ s, in other words, in the second minute.

(b) What is the meaning of $\int_{60}^{120} |v(t)|\,dt$?

$\int_{60}^{120} |v(t)|\,dt$ represents the total distance traveled by the particle in the second minute.

(c) What is the meaning of $\int_{60}^{120} a(t)\,dt$?

$\int_{60}^{120} a(t)\,dt$ represents the change in velocity of the particle in the second minute.

7. (a) Explain the meaning of the indefinite integral $\int f(x)\,dx$.

The indefinite integral $\int f(x)\,dx$ is another name for an antiderivative of f, so $\int f(x)\,dx = F(x)$ means that $F'(x) = f(x)$.

(b) What is the connection between the definite integral $\int_a^b f(x)\,dx$ and the indefinite integral $\int f(x)\,dx$?

The connection is given by Part 2 of the Fundamental Theorem:

$$\int_a^b f(x)\,dx = \int f(x)\,dx\Big]_a^b$$

if f is continuous on $[a, b]$.

8. Explain exactly what is meant by the statement that "differentiation and integration are inverse processes."

Part 1 of the Fundamental Theorem of Calculus can be rewritten as

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

which says that if f is integrated and then the result is differentiated, we arrive back at the original function f.

Since $F'(x) = f(x)$, Part 2 of the theorem (or, equivalently, the Net Change Theorem) states that

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

This says that if we take a function F, first differentiate it, and then integrate the result, we arrive back at the original function, but in the form $F(b) - F(a)$.

Also, the indefinite integral $\int f(x)\,dx$ represents an antiderivative of f, so

$$\frac{d}{dx}\int f(x)\,dx = f(x)$$

9. State the Substitution Rule. In practice, how do you use it?

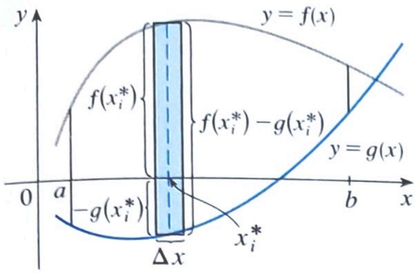
If $u = g(x)$ is a differentiable function and f is continuous on the range of g, then

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$$

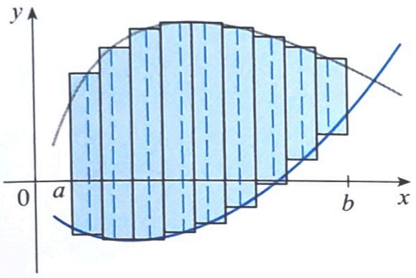
In practice, we make the substitutions $u = g(x)$ and $du = g'(x)\,dx$ in the integrand in order to make the integral simpler to evaluate.

CHAPTER 5 CONCEPT CHECK ANSWERS

1. (a) Draw two typical curves $y = f(x)$ and $y = g(x)$, where $f(x) \geqslant g(x)$ for $a \leqslant x \leqslant b$. Show how to approximate the area between these curves by a Riemann sum and sketch the corresponding approximating rectangles. Then write an expression for the exact area.



A Riemann sum that approximates the area between these curves is $\sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\,\Delta x$. A sketch of the corresponding approximating rectangles:

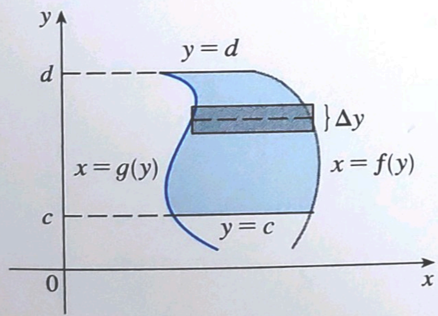


An expression for the exact area is

$$\lim_{n \to \infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\,\Delta x = \int_a^b [f(x) - g(x)]\,dx$$

(b) Explain how the situation changes if the curves have equations $x = f(y)$ and $x = g(y)$, where $f(y) \geqslant g(y)$ for $c \leqslant y \leqslant d$.

Instead of using "top minus bottom" and integrating from left to right, we use "right minus left" and integrate from bottom to top: $A = \int_c^d [f(y) - g(y)]\,dy$



2. Suppose that Sue runs faster than Kathy throughout a 1500-meter race. What is the physical meaning of the area between their velocity curves for the first minute of the race?

It represents the number of meters by which Sue is ahead of Kathy after 1 minute.

3. (a) Suppose S is a solid with known cross-sectional areas. Explain how to approximate the volume of S by a Riemann sum. Then write an expression for the exact volume.

We slice S into n "slabs" of equal width Δx. The volume of the ith slab is approximately $A(x_i^*)\,\Delta x$, where x_i^* is a sample point in the ith slab and $A(x_i^*)$ is the cross-sectional area of S at x_i^*. Then the volume of S is approximately $\sum_{i=1}^{n} A(x_i^*)\,\Delta x$ and the exact volume is

$$V = \lim_{n \to \infty} \sum_{i=1}^{n} A(x_i^*)\,\Delta x = \int_a^b A(x)\,dx$$

(b) If S is a solid of revolution, how do you find the cross-sectional areas?

If the cross-section is a disk, find the radius in terms of x or y and use $A = \pi(\text{radius})^2$. If the cross-section is a washer, find the inner radius r_{in} and outer radius r_{out} and use $A = \pi(r_{\text{out}}^2) - \pi(r_{\text{in}}^2)$.

4. (a) What is the volume of a cylindrical shell?

$$V = 2\pi rh\,\Delta r = (\text{circumference})(\text{height})(\text{thickness})$$

(b) Explain how to use cylindrical shells to find the volume of a solid of revolution.

We approximate the region to be revolved by rectangles, oriented so that revolution forms cylindrical shells rather than disks or washers. For a typical shell, find the circumference and height in terms of x or y and calculate

$$V = \int_a^b (\text{circumference})(\text{height})(dx \text{ or } dy)$$

(c) Why might you want to use the shell method instead of slicing?

Sometimes slicing produces washers or disks whose radii are difficult (or impossible) to find explicitly. On other occasions, the cylindrical shell method leads to an easier integral than slicing does.

5. Suppose that you push a book across a 6-meter-long table by exerting a force $f(x)$ at each point from $x = 0$ to $x = 6$. What does $\int_0^6 f(x)\,dx$ represent? If $f(x)$ is measured in newtons, what are the units for the integral?

$\int_0^6 f(x)\,dx$ represents the amount of work done. Its units are newton-meters, or joules.

6. (a) What is the average value of a function f on an interval $[a, b]$?

$$f_{\text{ave}} = \frac{1}{b - a} \int_a^b f(x)\,dx$$

(b) What does the Mean Value Theorem for Integrals say? What is its geometric interpretation?

If f is continuous on $[a, b]$, then there is a number c in $[a, b]$ at which the value of f is exactly equal to the average value of the function, that is, $f(c) = f_{\text{ave}}$. This means that for positive functions f, there is a number c such that the rectangle with base $[a, b]$ and height $f(c)$ has the same area as the region under the graph of f from a to b.