Chapter 1

Functions and Limits

1.1 Four Ways of Representing a Function

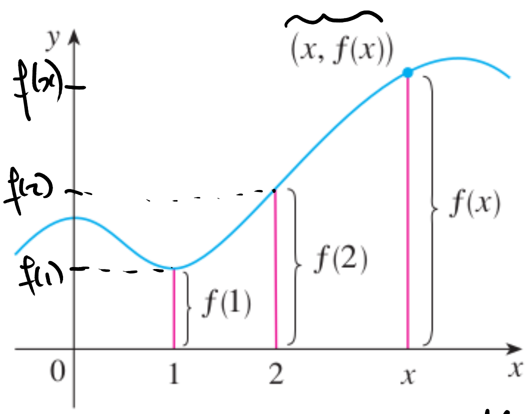
A **function** f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E.

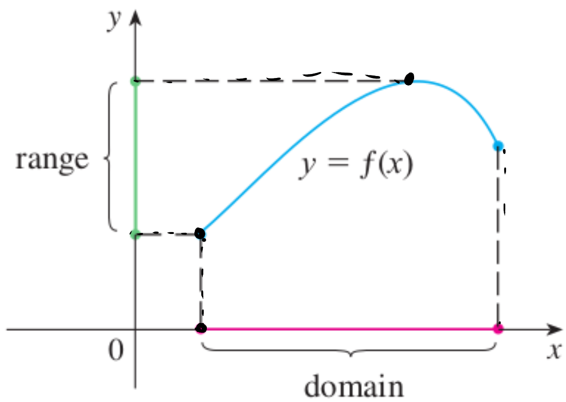
Machine visualization.

Domain: all inputs (x)

Range: all outputs (y)

Graph of a function.





Dependant variable.

usually denoted by y ← $f(x)$

v, s, a

Independant variable.

usually denoted by x

also use t

EXAMPLE 1 The graph of a function f is shown in Figure 6.
(a) Find the values of $f(1)$ and $f(5)$.
(b) What are the domain and range of f?

(a) $f(1) = 3$, $f(5) \approx -0.75$

(b) $\text{Dom}(f) = [0,7]$ $(0 \le x \le 7)$

$\text{Ran}(f) = [-2,4]$.

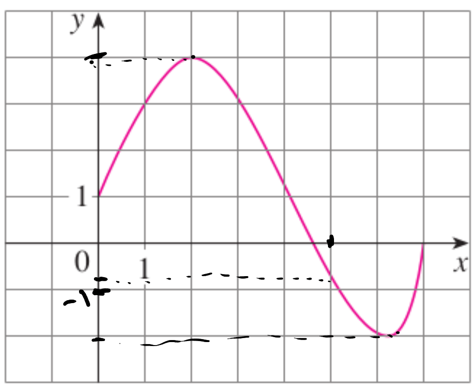
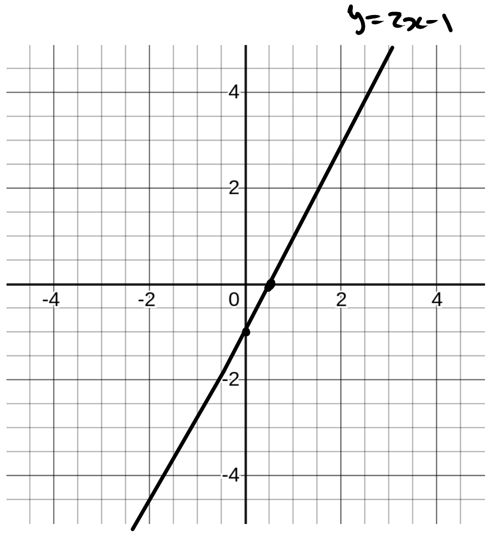


Figure 6

EXAMPLE 2 Sketch the graph and find the domain and range of each function.

(a) $f(x) = 2x - 1$ (b) $g(x) = x^2$

a)



<u>y-intercept.</u> $(x=0)$

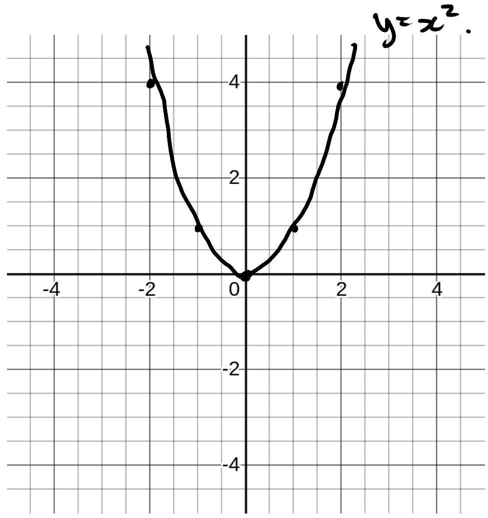
$y = f(0) = -1$

<u>x-intercept</u> $(y=0)$

$0 = f(x) = 2x - 1 \rightarrow 1 = 2x$

$\rightarrow \frac{1}{2} = x$

b)



x	$g(x)$
-2	4
-1	1
0	0
1	1
2	4

EXAMPLE 3 If $f(x) = 2x^2 - 5x + 1$ and $h \neq 0$, evaluate $\dfrac{f(a+h) - f(a)}{h}$.

$f(a+h) = 2(a+h)^2 - 5(a+h) + 1 = 2(a^2 + 2ah + h^2) - 5a - 5h + 1$

$f(a) = 2a^2 - 5a + 1$

$$\frac{f(a+h) - f(a)}{h} = \frac{2(a^2 + 2ah + h^2) - 5a - 5h + 1 - (2a^2 - 5a + 1)}{h}$$

$$= \frac{\cancel{2a^2} + 4ah + 2h^2 \cancel{-5a} - 5h \cancel{+1} \cancel{-2a^2} \cancel{+5a} \cancel{-1}}{h}$$

$$= \frac{4ah + 2h^2 - 5h}{h}$$

$$= \frac{(4a + 2h - 5)\cancel{h}}{\cancel{h}}$$

$$= 4a + 2h - 5$$

Remark: $\dfrac{f(a+h) - f(a)}{h}$ is called the <u>difference quotient</u>

Representations of functions.

There are four possible ways to represent a function:

- verbally (by a description in words)
- numerically (by a table of values)
- visually (by a graph)
- algebraically (by an explicit formula)

EXAMPLE 5 A rectangular storage container with an open top has a volume of 10 m^3. The length of its base is twice its width. Material for the base costs \$10 per square meter; material for the sides costs \$6 per square meter. Express the cost of materials as a function of the width of the base.

$V = 10\,m^3$

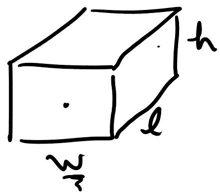
w : width

l : base side

h : height

$V = w \cdot l \cdot h = 10 \to 2w^2 \cdot h = 10$

$l = 2w$ $\quad \to h = \frac{10}{2w^2}$



$b \to 10\,\$/m^2$

$s \to 6\,\$/m^2$

Total cost side: $(2w \cdot h + 2 \cdot l \cdot h) \cdot 6$

+ Total cost base: $w \cdot l \cdot 10$

Total $= 12\,wh + 12\,l \cdot h + 10\,wl = 12\,w \cdot \left(\frac{10}{2w^2}\right) + 12 \cdot (2w)\left(\frac{10}{2w^2}\right) + 10w(2w)$

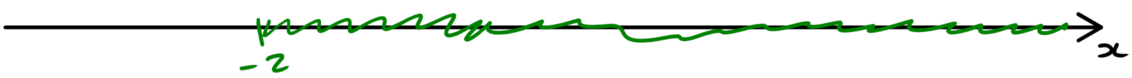
$\to$ Total $= \frac{60}{w} + \frac{120}{w} + 20w^2$.

Domain of functions given by an explicit formula.

EXAMPLE 6 Find the domain of each function.

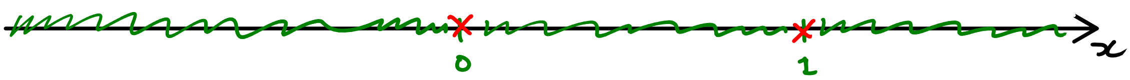
(a) $f(x) = \sqrt{x + 2}$ (b) $g(x) = \dfrac{1}{x^2 - x}$

(a) $\sqrt{\square}$ $\quad \square = x + 2 \geq 0 \to x \geq -2$ $\quad$ Dom$(f) = [-2, \infty)$.



(b) $x^2 - x = x(x-1) = 0$ iff $x = 0$ or $x - 1 = 0$

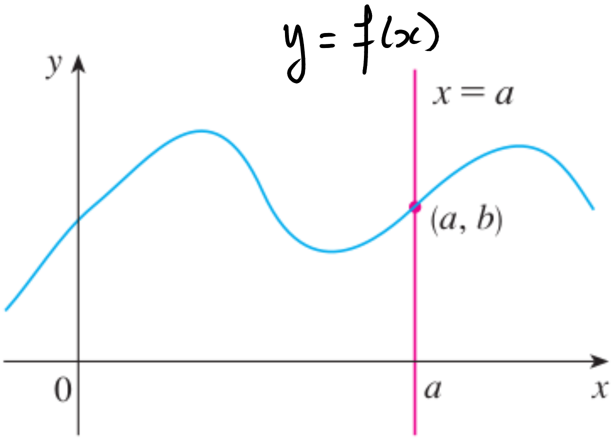
iff $x = 0$ or $x = 1$



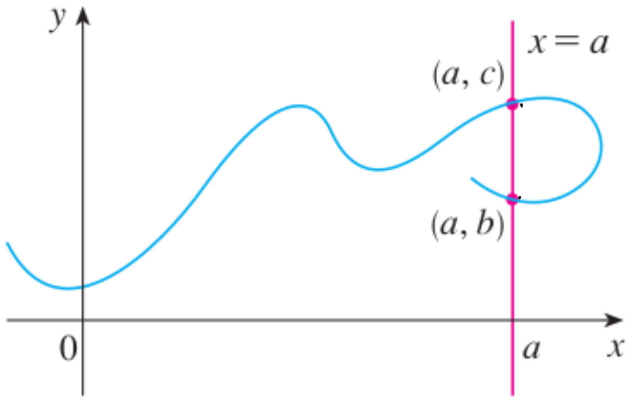
Dom$(g) = (-\infty, 0) \cup (0, 1) \cup (1, \infty)$

Which curves are graphs of functions?

The Vertical Line Test A curve in the xy-plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.



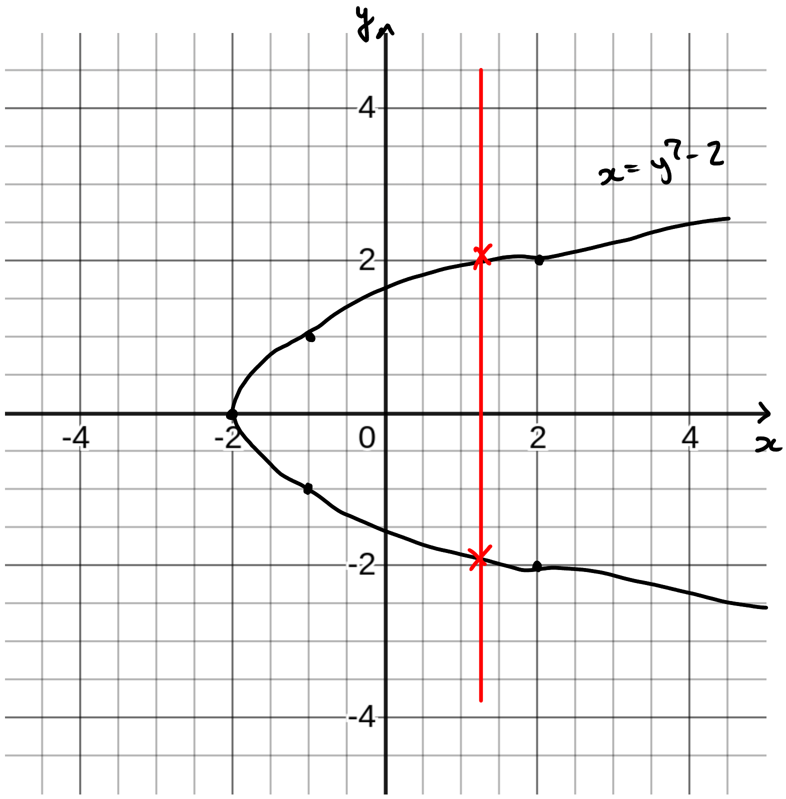
(a) This curve represents a function.



(b) This curve doesn't represent a function.

Example. The parabola $x = y^2 - 2$ is not the graph of a function. Show it using the Vertical Line Test.

x	y
2	-2
-1	-1
-2	0
-1	1
2	2



Piece-wise Functions.

The functions in the following four examples are defined by different formulas in different parts of their domains. Such functions are called **piecewise defined functions**.

EXAMPLE 7 A function f is defined by

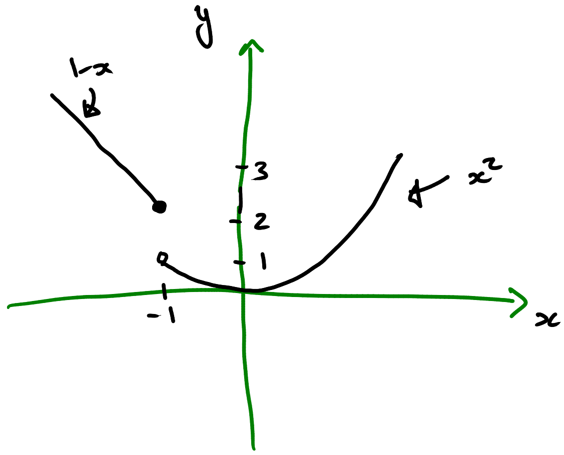
$$f(x) = \begin{cases} 1 - x & \text{if } x \leqslant -1 \\ x^2 & \text{if } x > -1 \end{cases}$$

Evaluate $f(-2)$, $f(-1)$, and $f(0)$ and sketch the graph.

$f(-2) = 1-(-2) = 3$

$f(-1) = 1-(-1) = 2$

$f(0) = 0^2 = 0$



Absolute Value.

$$|a| = a \quad \text{if } a \geqslant 0$$
$$|a| = -a \quad \text{if } a < 0$$

$a=2 \to |2|=2$

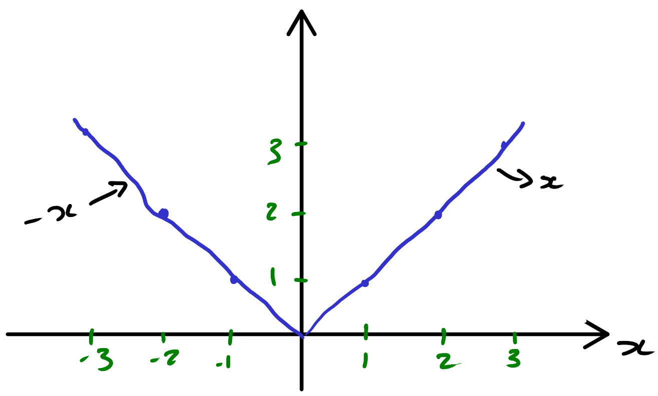
$a=-2 \to |-2|=2$

Properties: $|-a| = |a|$, $|ab| = |a| \cdot |b|$

$|-2| = |2|$, $|2 \cdot 3| = |2| \cdot |3|$

EXAMPLE 8 Sketch the graph of the absolute value function $f(x) = |x|$.

x	$f(x)$
-2	2
-1	1
0	0
1	1
2	2



$$f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0. \end{cases}$$

EXAMPLE 9 Find a formula for the function f graphed in Figure 17.

$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2-x, & 1 < x \le 2 \\ 0, & 2 < x \le 5 \end{cases}$$

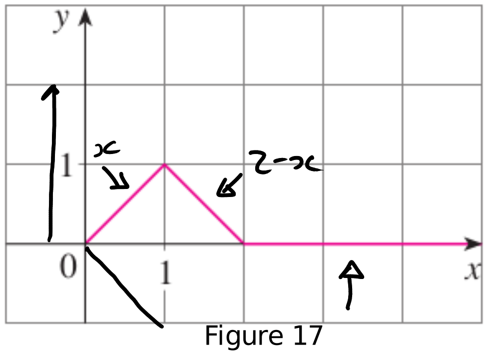
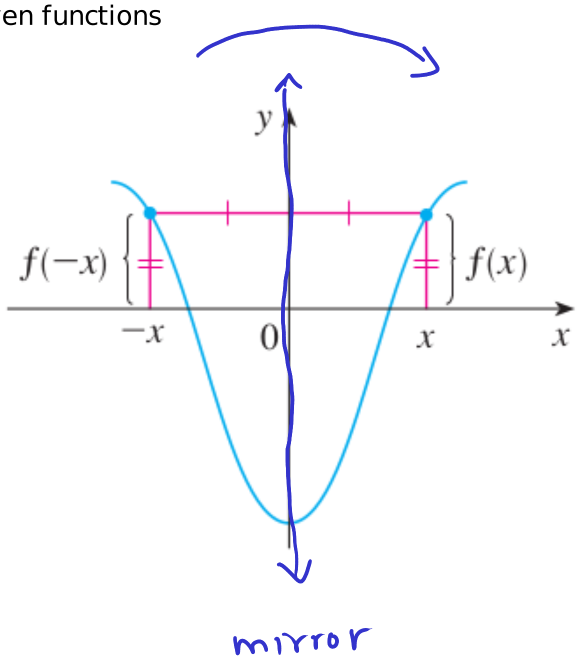


Figure 17

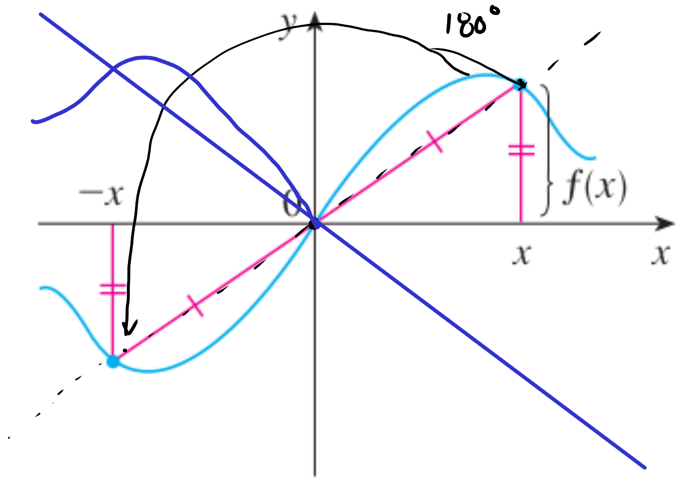
Symmetries.

Even functions



$f(-x) = f(x)$

Odd functions.



$f(-x) = -f(x)$

EXAMPLE 11 Determine whether each of the following functions is even, odd, or neither even nor odd.

(a) $f(x) = x^5 + x$ (b) $g(x) = 1 - x^4$ (c) $h(x) = 2x - x^2$

(a) $f(-x) = (-x)^5 + (-x) = (-1)^5 (x)^5 - x$

$= -x^5 - x = -(x^5 + x) = -f(x)$

So, f is odd.

(b) $g(-x) = 1 - (-x)^4 = 1 - (-1)^4 (x)^4 = 1 - x^4 = g(x)$

So, g is even.

(c) $h(-1) = -2 - 1 = -3$

$h(1) = 2 - 1 = 1$ $\rightarrow$ $-3 \neq 1$, $-3 \neq -1$ $\rightarrow$ h is nor even, nor odd.

Increasing/Decreasing Functions.

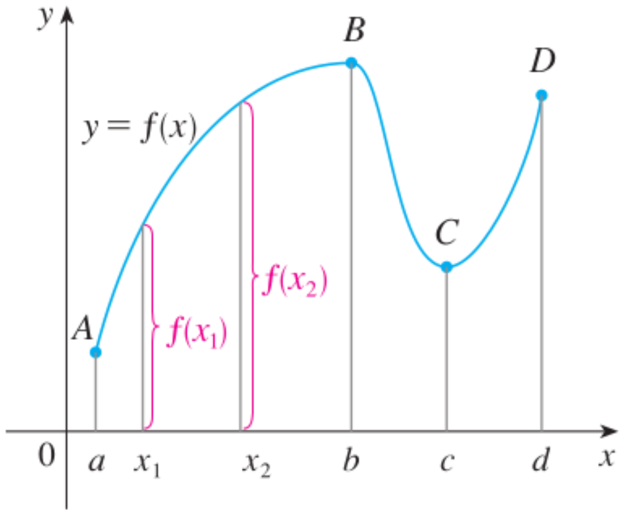
$[0,1] \to 0 \le x \le 1$
$(0,1) \to 0 < x < 1$

A function f is called **increasing** on an interval I if

$$y_1 = f(x_1) < f(x_2) = y_2 \qquad \text{whenever } x_1 < x_2 \text{ in } I$$

It is called **decreasing** on I if

$$f(x_1) > f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in } I$$



- From A to B: Increasing.
- From B to C: Decreasing.
- From C to D: Increasing.

Example. Where the function $f(x) = x^2$ is increasing? Where is it decreasing?

Increasing: $x > 0$.

Decreasing: $x < 0$.