Math 241

Chapter 4

Section 4.2: Definite Integral

Contents

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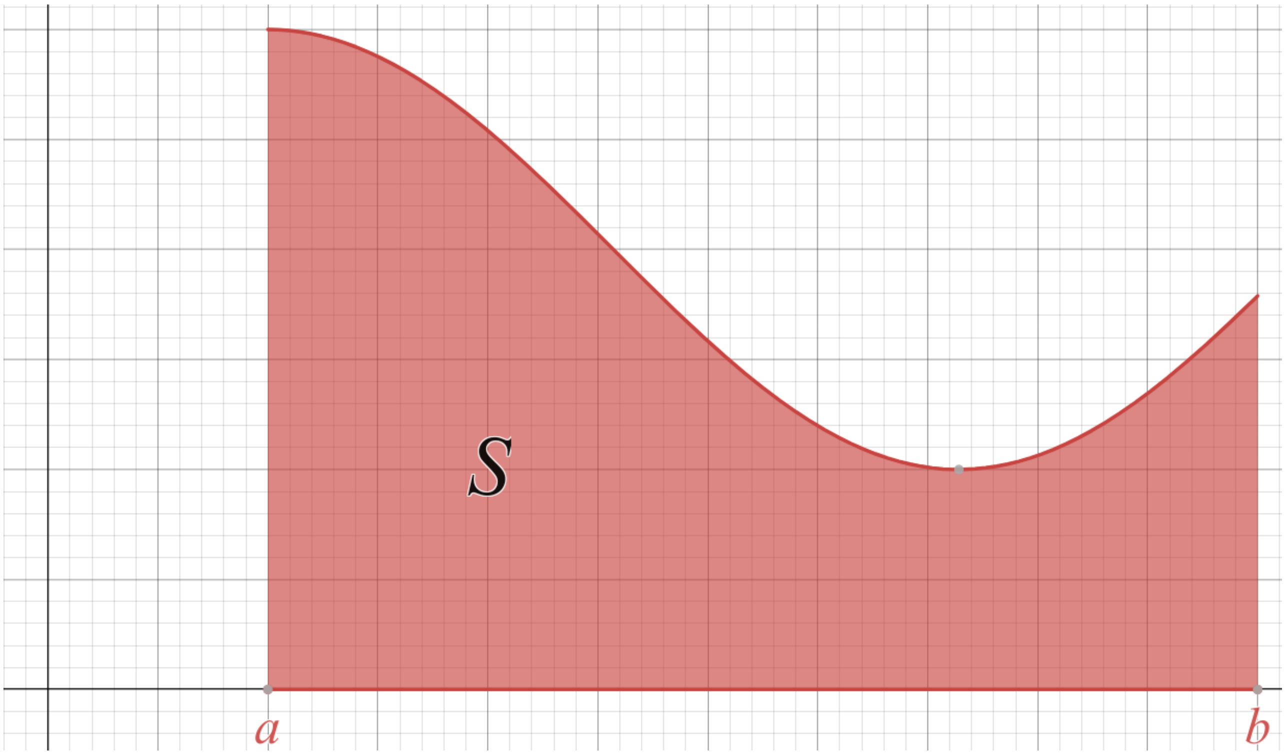
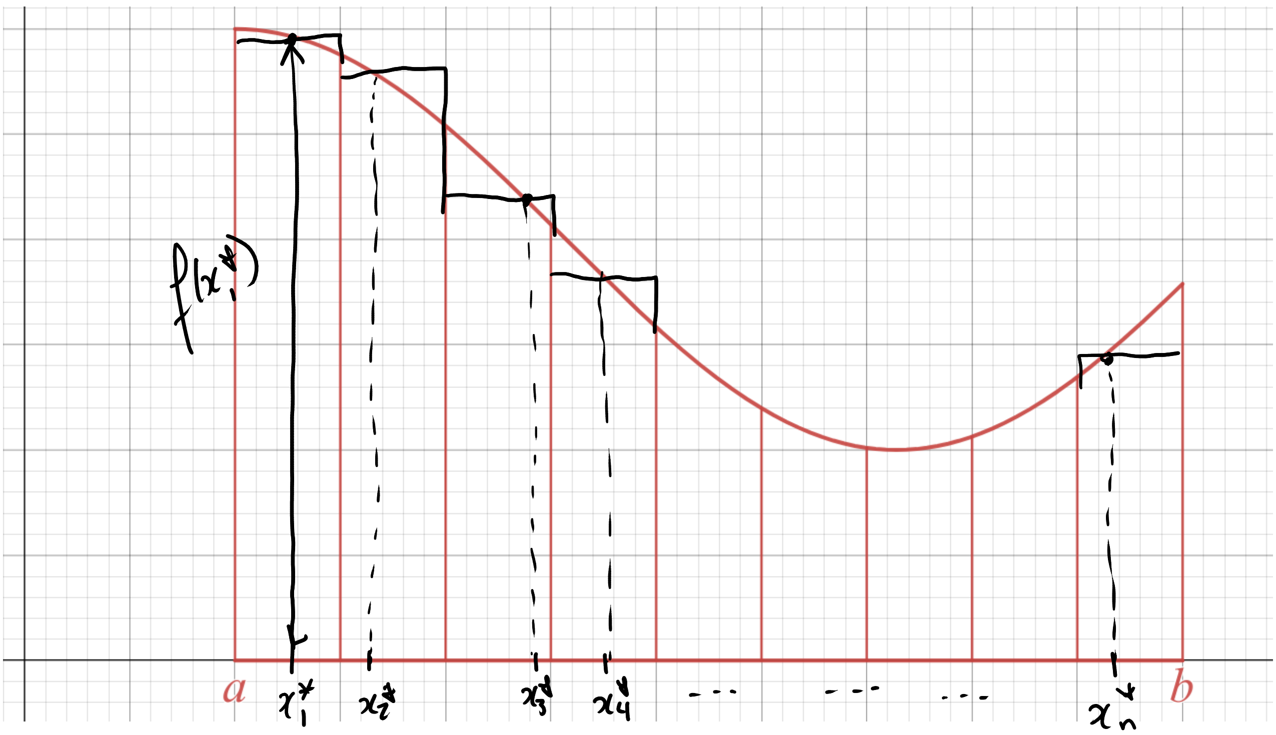
General Definition

Suppose we have a region S under the graph of a function $y = f(x)$ from $x = a$ to $x = b$.



• Divide the interval $[a, b]$ in n subintervals of equal length $\Delta x = (b - a)/n$.



• Select some number x_i^* in each $[x_{i-1}, x_i]$ (can be any number within the subinterval).

• Form the sum: $S_n = \sum_{i=1}^{n} f(x_i^*)\Delta x = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x.$ ← Riemann Sum.

Definite Integral: For a continuous function f, the definite integral of f is defined by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(\sum_{i=1}^{n} f(x_i^*)\Delta x \right).$$

Important Remarks:

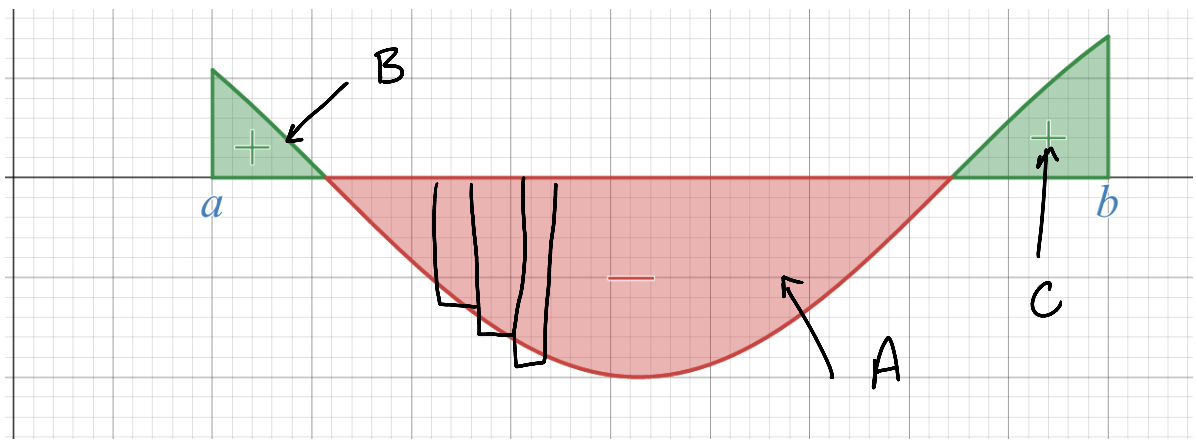
- Description of the terminology:
  - Symbol $\int$: integral symbol ("continuous" sum).
  - a: lower bound.
  - b: upper bound.
  - $f(x)$: integrand (expression to integrate).
  - dx: var. of integration
- The definite integral is a **number!** It does not depend on x! This means that

$$\int_a^b f(x)\,dx = \int_a^b f(r)\,dr = \int_a^b f(t)\,dt = \dots$$

- The expression S_n are called **Riemann Sums**.
- When $f(x) \geq 0$, then $\int_a^b f(x)\,dx$ is the area of the region S:

$$\text{Area}(S) = \int_a^b f(x)\,dx.$$

- If $f(x)$ is negative somewhere, then $\int_a^b f(x)\,dx$ is the **net area** between the graph of $y = f(x)$ and the horizontal line $y = 0$ (the x-axis).

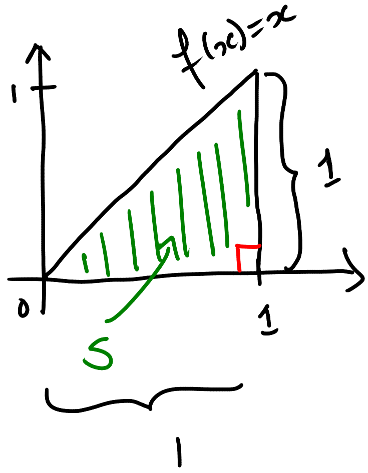


$$\int_a^b f(x)\,dx = B + C - A$$

Example 1. Find the value of the following integrals.

(a) $\int_0^1 x\,dx$. (b) $\int_{-1}^1 x\,dx$. (c) $\int_0^2 |x-1|\,dx$.

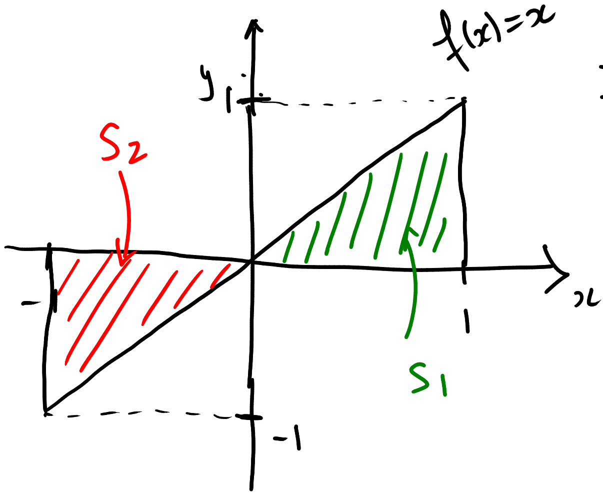
(a)



$f(x) = x$, $a = 0$, $b = 1$

$$\int_0^1 x\,dx = \text{Area}(S) = \frac{1\cdot 1}{2} = \boxed{\frac{1}{2}}.$$

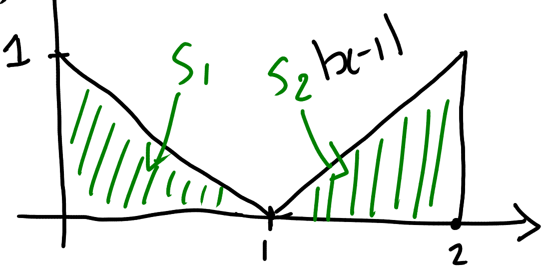
(b)



$f(x) = x$, $a = -1$, $b = 1$

$$\int_{-1}^1 x\,dx = \text{Net area} = \text{Area}(S_1) - \text{Area}(S_2) = \frac{1}{2} - \frac{1}{2} = \boxed{0}$$

(c)



$f(x) = |x-1|$, $a = 0$, $b = 2$

$$\int_0^2 |x-1|\,dx = \text{Area}(S_1) + \text{Area}(S_2) = \frac{1}{2} + \frac{1}{2} = \boxed{1}.$$

Useful Trick: Try to intepret the integral geometrically!

Playing with Lower and Upper Bounds

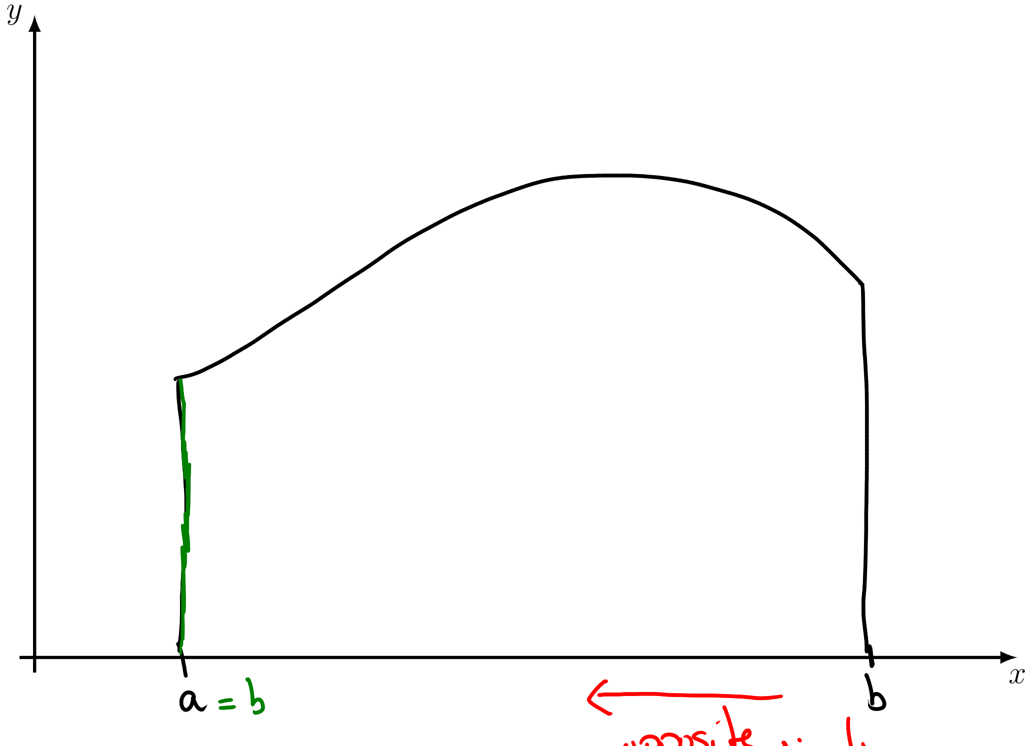
- If we change the order of the lower and upper bounds, then

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

- If the lower and upper bounds are equal, the definite integral is zero, that is

$$\int_a^a f(x)\,dx = 0.$$

Illustration:



Algebraic operations

For two continuous functions $f(x)$ and $g(x)$ on the interval $[a, b]$,

- Addition: $\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$
- Substraction: $\int_a^b (f(x) - g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx.$
- Multiplication by constant: $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx.$

$$x^n \longrightarrow \frac{x^{n+1}}{n+1}$$

Useful Formulas

Go to Desmos: https://www.desmos.com/calculator/mr9ba23hpz.

- $\int_a^b 1\,dx =$ $b-a$
- $\int_a^b x\,dx =$ $\frac{b^2}{2} - \frac{a^2}{2} = \frac{b^2-a^2}{2}$
- In general,

$$\int_a^b x^n\,dx = \frac{b^{n+1} - a^{n+1}}{n+1}.$$

EXAMPLE 2. Using the properties of the integral and the formulas, find the value of the following integrals.

(a) $\int_0^1 2x^2 - x^4\,dx$.

(b) $\int_{-2}^2 4x^4 - 3x^2\,dx$.

(a) $\int_0^1 2x^2 - x^4\,dx = \int_0^1 2x^2\,dx - \int_0^1 x^4\,dx$

$= 2\int_0^1 x^2\,dx - \int_0^1 x^4\,dx$

$= 2\left(\frac{1^3-0^3}{3}\right) - \frac{1^5-0^5}{5}$

$= 2\left(\frac{1}{3}\right) - \frac{1}{5}$

$= \boxed{\frac{14}{15}}$

(b) $\int_{-2}^2 4x^4 - 3x^2\,dx = \int_{-2}^2 4x^4\,dx - \int_{-2}^2 3x^2\,dx$

$= 4\int_{-2}^2 x^4\,dx - 3\int_{-2}^2 x^2\,dx$

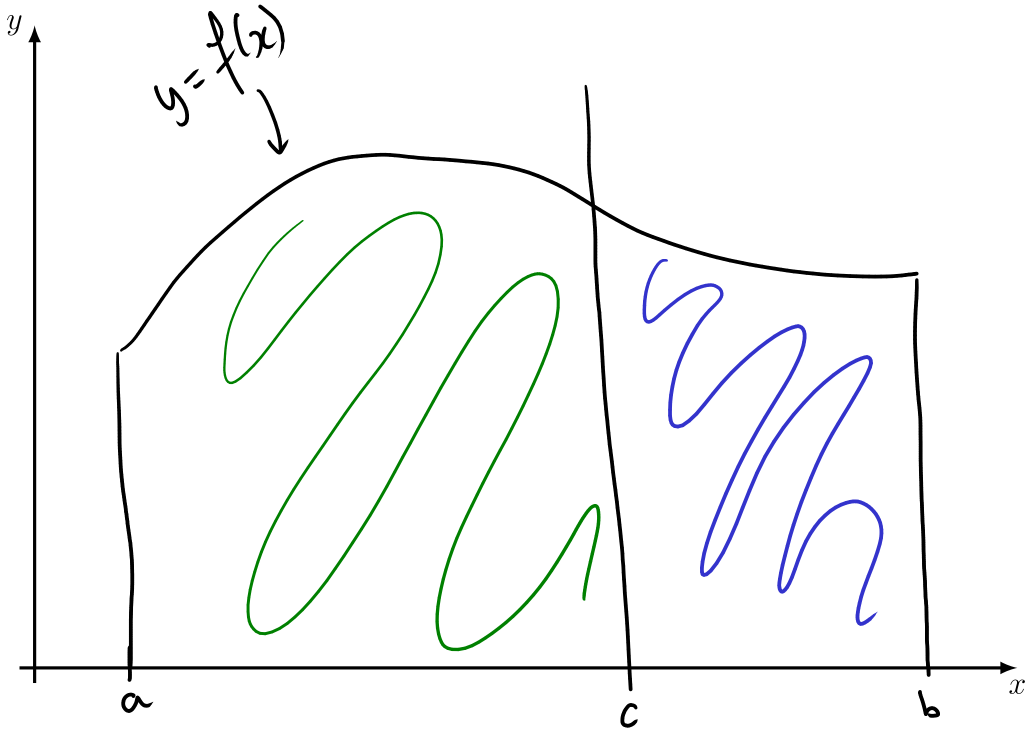
$= 4\left(\frac{2^5-(-2)^5}{5}\right) - 3\left(\frac{2^3-(-2)^3}{3}\right) = \boxed{\frac{176}{5}}$

Cutting the domain

Let $a < c < b$ and $f(x)$ be a continuous function on $[a, b]$. Then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Illustration:



EXAMPLE 3. If it is known that $\int_0^{10} f(x)\,dx = 17$ and $\int_0^8 f(x)\,dx = 12$, then find $\int_8^{10} f(x)\,dx$.

$$\int_0^{10} f(x)\,dx = \int_0^8 f(x)\,dx + \int_8^{10} f(x)\,dx$$

$$\Rightarrow \quad 17 = 12 + \int_8^{10} f(x)\,dx$$

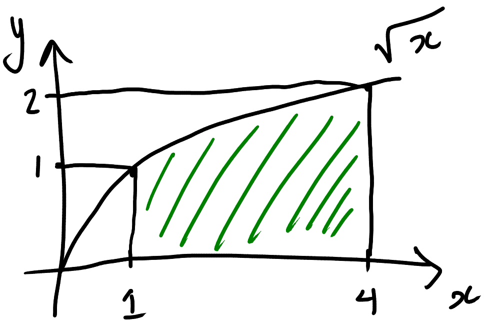
$$\Rightarrow \quad \int_8^{10} f(x)\,dx = 17 - 12 = \boxed{5}$$

Comparison Properties

- If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq 0$.
- If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.
- If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

EXAMPLE 4. Use the last comparison property to estimate $\int_1^4 \sqrt{x}\,dx$.



$1 \leq x \leq 4$

$\Rightarrow \sqrt{1} \leq \sqrt{x} \leq \sqrt{4}$

$\Rightarrow 1 \leq \sqrt{x} \leq 2$

min. = m   max. = M

So,

$$1(4-1) \leq \int_1^4 \sqrt{x}\,dx \leq 2(4-1)$$

$$\Rightarrow 3 \leq \int_1^4 \sqrt{x}\,dx \leq 6$$

$$\int_1^4 \sqrt{x}\,dx \approx \frac{6+3}{2} = 4.5$$

(Formula: $\int_1^4 \sqrt{x}\,dx = \frac{4^{3/2} - 1^{3/2}}{3/2} = \frac{7 \cdot 2}{3} = \frac{14}{3}$