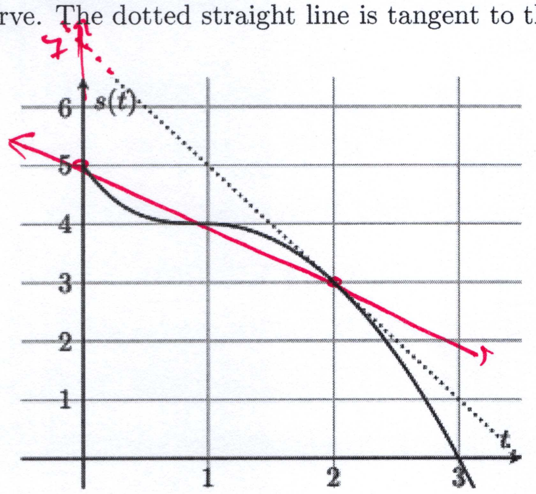
Question 1. (6 points)

A particle moves along a line and its position $s(t)$ in feet after time t seconds is given by the following curve. The dotted straight line is tangent to the curve of $s(t)$ at time $t = 2$ seconds.



(a) (2 points) Find the average velocity of the particle over the time interval $[0, 2]$. Include units with your answer.

$$\frac{3-5}{2-0} = -1 \frac{ft}{sec}$$

+1 correct answer

+1 attempt at finding the slope of the secant line through $(0,5)$ & $(2,3)$

(b) (2 points) Find the instantaneous velocity at time $t = 2$ seconds. Include units with your answer.

$s'(t) = -2$ ft/sec

+2 All or nothing

(c) (2 points) Find the equation of the dotted line.

$\boxed{y = mx+b}$

$y = -2x + 7$

+1 correct slope <u>and</u> y-int.

+1 correct equation (using their slope & y-int)

$\boxed{\text{Point-slope}}$

$y - 3 = -2(x-2)$

+1 correct point on the line <u>and</u> slope.

+1 correct equation (using their point & slope)

Question 2. (9 points)
Compute the following limits, or say if they are $+\infty$ or $-\infty$. Do not use L'Hospital's rule.

(a) (3 points) $\lim_{x\to 2} \frac{x-2}{\sqrt{x}-\sqrt{2}}$

$$\lim_{x\to 2} \frac{x-2}{\sqrt{x}-\sqrt{2}} = \lim_{x\to 2} \frac{(\sqrt{x}-\sqrt{2})(\sqrt{x}+\sqrt{2})}{\sqrt{x}-\sqrt{2}} \qquad (2\text{ pts})$$

$$= \lim_{x\to 2} (\sqrt{x}+\sqrt{2})$$

$$= \sqrt{2}+\sqrt{2} = 2\sqrt{2} \qquad (1\text{ pt})$$

(b) (3 points) $\lim_{x\to 0} \frac{\sin(x^3)}{x}$

$$\lim_{x\to 0} \frac{\sin(x^3)}{x} = \lim_{x\to 0} \frac{\sin(x^3)}{x^3} \cdot \lim_{x\to 0} x^2 \qquad (1\text{ pt})$$

$$= 1 \cdot \lim_{x\to 0} x^2 \qquad (1\text{ pt})$$

$$= 1 \cdot 0^2 = 0 \qquad (1\text{ pt})$$

(c) (3 points) $\lim_{x\to -\infty} \frac{x}{\sqrt{1+x^2}}$

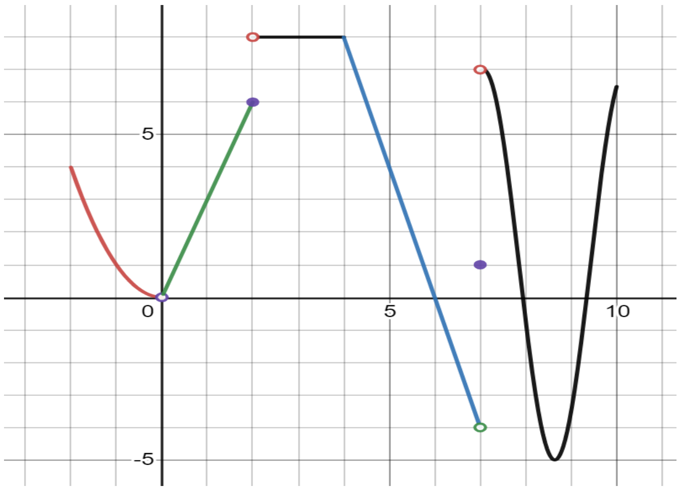
$$\lim_{x\to -\infty} \frac{x}{\sqrt{1+x^2}} = \lim_{x\to -\infty} \frac{1}{\frac{\sqrt{1+x^2}}{x}} \qquad (1\text{ pt})$$

$$= \lim_{x\to -\infty} \frac{1}{-\sqrt{\frac{1}{x^2}+1}} \qquad (1\text{ pt})$$

$$= \frac{1}{-\sqrt{0+1}} = -1 \qquad (1\text{ pt})$$

Question 3. (4 points)

Consider the graph of the function $y = f(x)$ below on the interval $[-2, 10]$ where $f(0)$ is undefined. You do not need to justify your answers.



(a) (1 points) What is $\lim_{x\to 0} f(x)$? ('Does not exist' is a possible answer)

$$\lim_{x \to 0} f(x) = 0$$

(b) (1 points) What is $\lim_{x\to 4} f'(x)$? ('Does not exist' is a possible answer)

Does not exist

(c) (1 points) Where (if anywhere) is the function not continuous?

$x = 0, \quad x = 2, \quad x = 7$

(d) (1 points) Where (if anywhere) is the function not differentiable?

$x = 0, \quad x = 2, \quad x = 4, \quad x = 7$

Question 4. (6 points)

Use the limit definition of the derivative to compute $f'(x)$ when $f(x) = 2 + \frac{1}{x}$.

You will get no credit for computing the derivative without using the definition.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{2} + \frac{1}{x+h} - (\cancel{2} + \frac{1}{x})}{h}$$

2 points

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{x+h} - \frac{1}{x}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{\cancel{x} - (\cancel{x}+h)}{x(x+h)}\right)$$

$$= \lim_{h\to 0} \frac{1}{\cancel{h}} \cdot \frac{-\cancel{h}}{x(x+h)}$$

$$= \lim_{h\to 0} -\frac{1}{x(x+h)}$$

2 points.

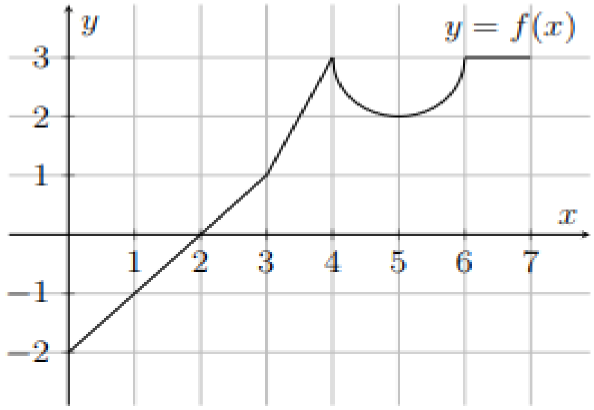
$$= -\frac{1}{x(x+0)}$$

$$= -\frac{1}{x^2}$$

$$\therefore f'(x) = -\frac{1}{x^2}$$

2 points.

Question 5. (8 points)
Consider the graph of the function $y = f(x)$ below. For t in the interval $[0, 7]$ we define $g(t) = \int_0^t f(x)dx$.



(a) (3 points) Find $g(0)$ and $g(3)$.

$g(0) - 1\text{pt.}$
$g(3) - 2\text{pt.}$

$g(0) = \int_0^0 f(x)dx = 0$

$g(3) = \int_0^3 f(x)\,dx = \frac{1}{2}(2)(-2) + \frac{1}{2}(1)(1)$
$= -2 + \frac{1}{2}$
$= -\frac{3}{2}$

(b) (2 points) Find all critical points of the function g on the interval $(0, 7)$.

$g'(t) = \frac{d}{dt}\int_0^t f(x)dx = f(t)$

$f(t) = 0 \iff t = 2$

FTC I - 1pt.
$t = 2$ - 1pt.

(c) (3 points) Find the absolute maximum value of g on the interval $[0, 3]$.

$g'(t)$: $-$ before 2, $+$ after 2 $\Rightarrow g(2)$ is a local minimum

$g(0) = 0$, $g(3) = -\frac{3}{2} \Rightarrow$ absolute maximum is $g(0) = 0$

$g(2)$ is local min. - 2pt.
$g(0)$ is abs. max - 1pt.

(Could just do it geometrically.)

Question 6. (8 points)

For this question, let g be a differentiable function such that g and g' take the following values:

$$g(\pi/3) = 1, \quad g'(\pi/3) = 0, \quad g(2) = 2, \quad g'(2) = 4,$$

(a) (4 points) Compute $f'(\pi/3)$ if $f(x) = \dfrac{g(x)}{\sin(x)}$.

By quotient rule,

$$f'(x) = \frac{g'(x)\sin(x) - g(x)\cos(x)}{\sin^2(x)} \longrightarrow 2 \text{ pts}$$

$$\Rightarrow f'(\pi/3) = \frac{g'(\pi/3)\sin(\pi/3) - g(\pi/3)\cos(\pi/3)}{\sin^2(\pi/3)} \longrightarrow 1 \text{ pt}$$

$$= \frac{0\cdot\sin(\pi/3) - 1\cdot\cos(\pi/3)}{\sin^2(\pi/3)}$$

$$= -\frac{\cos(\pi/3)}{\sin^2(\pi/3)} \longrightarrow 1 \text{ pt}$$

$$= -2/3 \longrightarrow 0 \text{ pt, not necessary}$$

(b) (4 points) Compute $h'(2)$ if $h(x) = \sqrt{1 + 2\,g(x)}$. $= (1+2g(x))^{1/2}$

By chain rule,

$$h'(x) = \frac{1}{2}(1+2g(x))^{-1/2}(2g'(x)) \longrightarrow 2 \text{ pts}$$

$$\Rightarrow h'(2) = (1+2g(2))^{-1/2}\, g'(2) \longrightarrow 1 \text{ pt}$$

$$= (1+2(2))^{-1/2}\cdot 4$$

$$= 4(5)^{-1/2} \longrightarrow 1 \text{ pt}$$

$$= \frac{4}{\sqrt{5}} = \frac{4\sqrt{5}}{5} \longrightarrow \text{Not necessary}$$

Question 7. (9 points)

Let y be defined implicitly as $4\sqrt{3x-y} = xy + 9$.

(a) (5 points) Find $\frac{dy}{dx}$ in terms of x and y.

Derivative $4\frac{3-dy/dx}{2\sqrt{3x-y}} = y + x\frac{dy}{dx}$ 2 Points for implicit differentiation

Isolate. $\frac{6}{\sqrt{3x-y}} - y = x\frac{dy}{dx} + \frac{2\,dy/dx}{\sqrt{3x-y}}$

$\Rightarrow \frac{6 - y\sqrt{3x-y}}{\sqrt{3x-y}} = \frac{x\sqrt{3x-y}+2}{\sqrt{3x-y}}\left(\frac{dy}{dx}\right)$

2 pts for moving the $\frac{dy}{dx}$ terms to one side and factoring

$\Rightarrow \boxed{dy/dx = \frac{6-y\sqrt{3x-y}}{2+x\sqrt{3x-y}}}$ 1 pt for simplifying

(b) (4 points) Find the equation of the tangent line to the curve at the point $(1, -1)$.

Slope $y' = \frac{6+1\sqrt{3\cdot 1+1}}{2+1\sqrt{3\cdot 1+1}} = \frac{6+2}{2+2} = 2$ 2 pts for correct slope

y-intercept. $1 = 2\cdot 1 + b \Rightarrow b = -1$ 1 pt for correct y int

So, $\boxed{y = 2x - 3}$ 1 pt for plugging slope and yint into eq.

3/4 pts if their answer is correct but is in point-slope form.

Question 8. (10 points)

The volume of a spherical oil balloon expands at a rate of $100\text{cm}^3/s$ when oil is pumped into it. How fast the radius is growing when the radius is 25 cm? Do not simplify your answer.

Hint: The volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.

2 pts. for naming variables* and/or writing down what we know

V = volume* r = radius

$\frac{dV}{dt} = 100\text{cm}^3/s$

want $\frac{dr}{dt}$ when $r = 25$ cm

1 pt. for identifying what we're trying to find

1 pt for relationship between V and r

$V = \frac{4}{3}\pi r^3$

2 pts for differentiating both sides

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

1 pt for chain rule

$100 = 4\pi (25)^2 \frac{dr}{dt}$

1 pt for filling in what's known

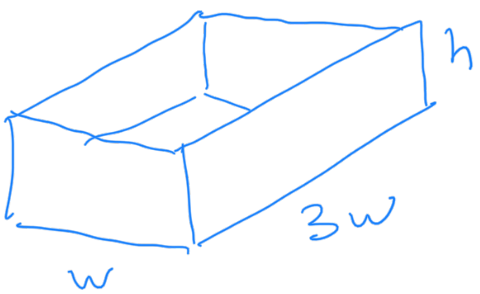
1 pt for rearranging (can happen before filling things in)

$\frac{dr}{dt} = \frac{100}{2500\pi} = \frac{1}{25\pi}$ cm/s

1 pt for correct units

* V and r are pretty standard, so ok to omit if rest is correct

Question 9. (10 points)
Alice is making for a large rectangular base aquarium tank without a lid of volume 18ft^3. The length of the base of the tank is three times its width. In order to minimize the cost she needs to minimize the amount of material used. Find the dimensions of the tank that minimize the amount of material used.



$$V = 3w^2 h = 18$$

$$\Rightarrow \quad h = \frac{6}{w^2} \quad (1)$$ 2 pts

$$A = 3w^2 + 2wh + 2(3w)h$$
$$= 3w^2 + 8wh$$

use (1) :

$$A = 3w^2 + 8w\left(\frac{6}{w^2}\right) = 3w^2 + \frac{48}{w}$$ 2 pts

min A means $\frac{dA}{dw} = 0$ 2 pts $= 6w - \frac{48}{w^2}$

$$\Rightarrow \quad w = \frac{8}{w^2} \qquad w^3 = 8$$

$$w = 2 \text{ ft}$$ 4 pts

$$\Rightarrow \quad h = \frac{6}{4} = \frac{3}{2} \text{ ft}$$

We can verify that it is a min by using the 2rd derivative test :

$$\frac{d^2A}{dw^2} = 6 + \frac{96}{w^3} > 0$$

Question 10. (10 points)
Consider the function F defined on the interval $[-1, 6]$ by

$$F(s) = s^3 + 6s^2 - 1.$$

(a) (3 points) On what subintervals (if any) of $[-1, 6]$ is F increasing and on which is it decreasing?

Derivatives: $F'(s) = 3s^2 + 12s$ (1pt)

Zeros: $3s^2 + 12s = 0$ if $s = 0$ or $s = -4$ (1pt)
$s = -4$ to reject because outside of $[-1, 6]$.

Table: Write $F'(s) = 3s(s+4)$. (1pt)

s	-1		0		6
$3s$		$-$	0	$+$	
$s+4$		$+$	$+$	$+$	
F'		$-$	0	$+$	

Answer:
- $F' < 0$ on $(-1, 0)$ $\to$ F decreasing on $(-1, 0)$
- $F' > 0$ on $(0, 6)$ $\to$ F increasing on $(0, 6)$

(b) (3 points) Where (if anywhere) does F have a local minimum or a local maximum?

Table: (1pt.)

s	-1		0		6
$3s$		$-$	0	$+$	
$s+4$		$+$	$+$	$+$	
F'		$-$	0	$+$	

1st Test: (2pt)
$F'(s) < 0$ for $s < 0$ &
$F'(s) > 0$ for $s > 0$.

$\boxed{\text{OR}}$

2nd Test: $F''(s) = 6s + 12$ (1pt)
$F'(0) = 0$ & $F''(0) > 0$ (1pt)

Answer: local min at $s = 0$

(c) (4 points) On what subintervals (if any) of $[-1, 6]$ is F concave up or concave down, and where (if anywhere) does F have an inflection point?

Second Derivative: $F''(s) = 6s + 12$. (1pt)

Zero(s): $F''(s) = 0 \iff s = -2$. (1pt)

Table: Write $F''(s) = 6(s+2)$.

s	-2	-1		6
6			$+$	
$s+2$			$+$	
F''			$+$	

(2pt)

Answer:
- $F'' > 0$ on $(-1, 6)$ $\to$ F $\circlearrowleft$ there.
- No inflection point.

General comment: If derivative miscalculated, but everything else okay, remove only the pt. for that part.

Question 11. (8 points)

For this question, consider the equation $x^3 + 2x - 2 = 0$

(a) (4 points) Show that the given equation has a solution in the interval $[-1, 1]$. [*Hint:* Use Intermediate Value Theorem]

Using a proof:

Let $f(x) = x^3 + 2x - 2$, then

1pt $f(-1) = -5 < 0$

1pt $f(1) = 1 > 0$

1pt $f(x)$ is continuous

1pt (Acceptable) logical statement

↳ Intermediate value theorem

↳ or $f(x)$ cannot jump over 0

↳ or cannot lift the pen (draw)

Using a sketch for a) and b) [Sec. 3.6]

1pt $f(-1) = -5$

1pt $f(0) = -2$

1pt $f(1) = 1$

1pt $f'(x) > 0 \quad \forall x$

1pt $f''(x) < 0 \quad x \in (-1, 0)$

1pt $f''(x) > 0 \quad x \in (0, 1)$

1pt $f''(x) = 0 \quad x = 0$

1pt Sketch

No points if no work

No points for "magical" roots

No points for "should look like this"

(b) (4 points) Show that the given equation cannot have more than one solution in the interval $[-1, 1]$. [*Hint:* You may consider using Rolle's theorem or Mean Value Theorem]

The student can sketch a graph. See

Proof by contradiction:

Let $a \neq b$ and $f(a) = f(b) = 0$

1pt $f'(x) = 3x^2 + 2$

1pt $f'(x) > 0$

2pts (Acceptable) logical statement

↳ Contradicts Rolle's (or MVT) $f'(x) = 0$

So there is at most one solution of $f(x) = 0$.

Conceptual proof:

1pt $f'(x) = 3x^2 + 2$

1pt $f(x)$ is always increasing

$f(x)$ is continuous (May be in a))

2pts (Acceptable) logical statement

E.g. Since $f(x)$ is continuous and always increasing on $[-1,1]$, then $f(x)$ cannot cross 0 more than once. Hence, $f(x) = 0$ cannot have more than one solution in $\mathbb{R}$

Do not be too picky on words

No logical statement, no points

Question 12. (10 points)

Consider the function $f(x) = 9 - x^2$ on the interval $[0, 3]$

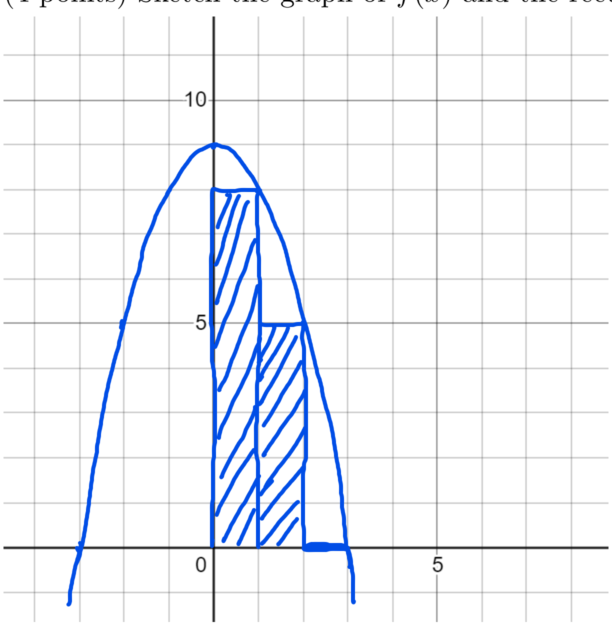
(a) (4 points) Estimate $\int_0^3 f(x)dx$ with a Riemann sum using $n = 3$ subintervals of equal width and right endpoints.

$\sum_{i=1}^{3} h \cdot f(x_i^*) = 1 \cdot f(1) + 1 \cdot f(2) + 1 \cdot f(3) = 8 + 5 + 0 = 13$

(−2 pts): used some other points instead of right endpoints (e.g. left endpoints)

(−1 pt): small calculation mistake

(b) (4 points) Sketch the graph of $f(x)$ and the rectangles that you used in part (a) on the same axis.



(−2 pts): used some other points instead of right endpoints

(−1 pt): f(x) was graphed "slightly" incorrectly

(c) (2 points) Is this estimate over or under the actual answer? Or is it impossible to tell? Explain your answer.

(1 pt): correct answer

(+1 pt): any reasonable explanation

(1 pt): if student did everything correctly but for other points instead of right endpoints

Question 13. (5 points)

A function $f(x)$ is defined by $f(x) = \int_2^{x^2} \sqrt{1+t^3}\,dt$. Find $f'(x)$ by using the Fundamental Theorem of Calculus.

Make substitution $u = x^2$. → 1pt

Note that $\frac{du}{dx} = 2x$. → 1pt

$$f'(x) = \frac{d}{dx}\int_2^{x^2}\sqrt{1+t^3}\,dt$$

$$= \frac{d}{du}\left[\int_2^{u}\sqrt{1+t^3}\,dt\right]\frac{du}{dx}$$ Chain Rule → 1pt

$$= \sqrt{1+u^3}\,\frac{du}{dx}$$ FTC → 1pt

$$= \sqrt{1+(x^2)^3}\,(2x)$$ Substitute $u = x^2$ and $du/dx = 2x$ → 1pt

Antiderivative : $-\dfrac{(x^2+x+1)^{-2}}{2} + C$ (3pts)

$f(1) = 1 \Rightarrow C = \dfrac{19}{18}$ (2pts)

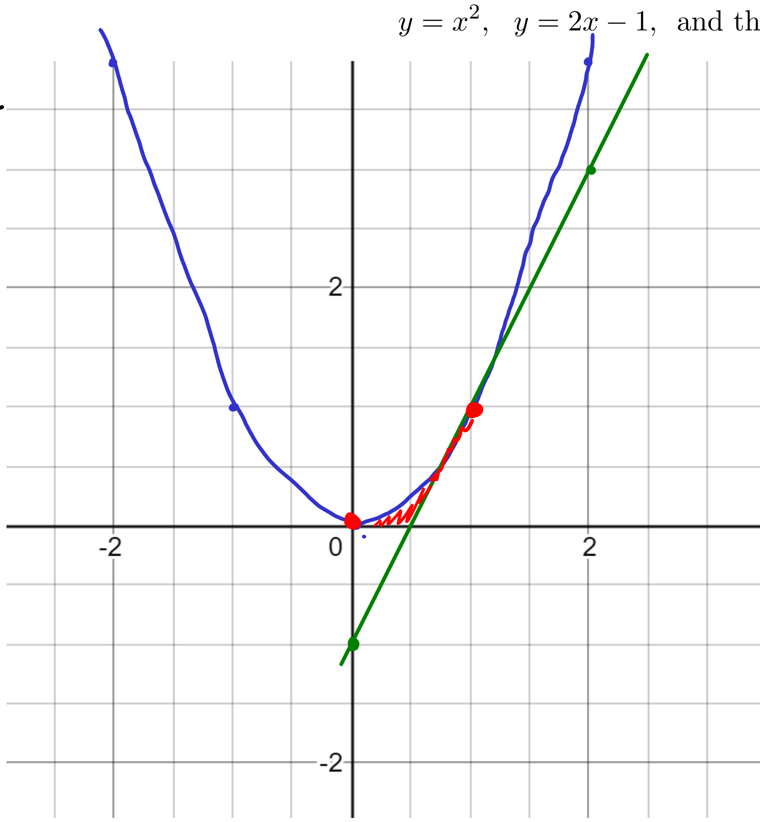
$$f(x) = -\frac{(x^2+x+1)^{-2}}{2} + \frac{19}{18}$$

Question 15. (7 points)

(a) (4 points) Graph and shade the area bounded by the following. Label points of intersection points of the graph.

$$y = x^2, \quad y = 2x - 1, \quad \text{and the x-axis.}$$

① Graph



② Intercepts.

$x^2 = 2x - 1$

$\Leftrightarrow (x-1)^2 = 0 \Leftrightarrow x = 1.$

(b) (3 points) Set up the integral(s) that gives the area of the shaded region. **DO NOT EVALUATE** the integral.

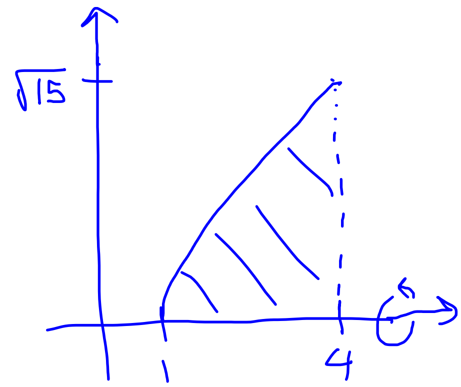
From the intercepts, $0 \leq x \leq 1$.

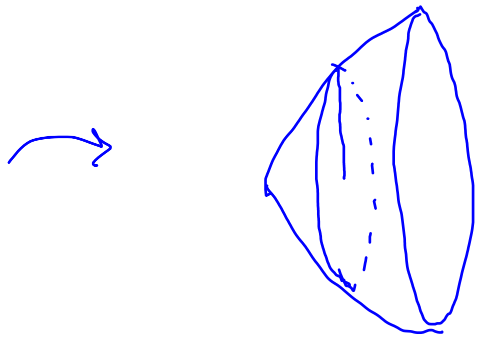
the function $x \mapsto x^2$ is above the function $x \mapsto 2x-1$ on the interval $[0,1]$.

So,

$$A = \int_0^1 x^2 - (2x-1)\, dx = \boxed{\int_0^1 x^2 - 2x + 1\, dx}$$

Question 16. (5 points)
Find the volume of the solid generated using the area bounded by $y = 0,\ y = \sqrt{x^2 - 1},\ 1 \leq x \leq 4$ rotated about the $x-$axis.





$$A(x) = \pi\left(\sqrt{x^2-1}\right)^2 = \pi(x^2-1)$$

$$V = \int_1^4 A(x)\,dx = \int_1^4 \pi(x^2-1)\,dx$$

$$= \pi\left[\frac{x^3}{3} - x\right]_1^4$$

$$= \pi\left(\left(\frac{64}{3} - 4\right) - \left(\frac{1}{3} - 1\right)\right) \qquad \left(\frac{63}{3} = 21\right)$$

$$= 18\pi.$$

Rubric

2 pts - correct cross-sectional area
2 pts - correct integral
1 pt - correct answer.