Chapter 1

Functions and Limits

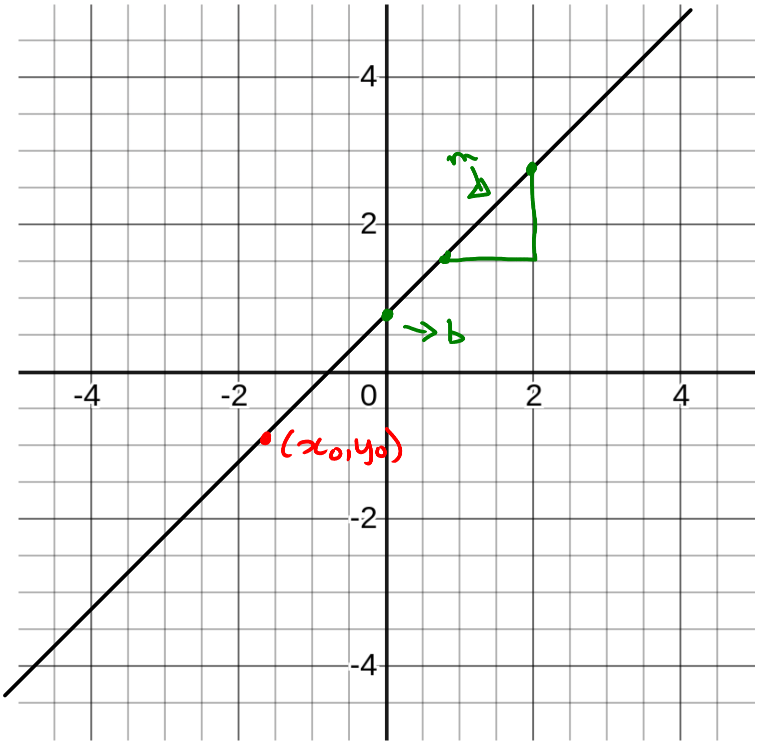
1.2 Mathematical Models: A catalog of Essential Functions

Linear Models.

$$y = f(x) = mx + b$$

.m: the slope

.b: y-intercept



Another formulation (knowing a point):

$$y - y_0 = m(x - x_0)$$

EXAMPLE 1

(a) As dry air moves upward, it expands and cools. If the ground temperature is 20°C and the temperature at a height of 1 km is 10°C, express the temperature T (in °C) as a function of the height h (in kilometers), assuming that a linear model is appropriate.
(b) Draw the graph of the function in part (a). What does the slope represent?
(c) What is the temperature at a height of 2.5 km?

(a) Indep.: h (height)
dep.: T (temperature)

$T = mh + b.$

$\boxed{m}$ $h = 0 \to T = 20$
$h = 1 \to T = 10$

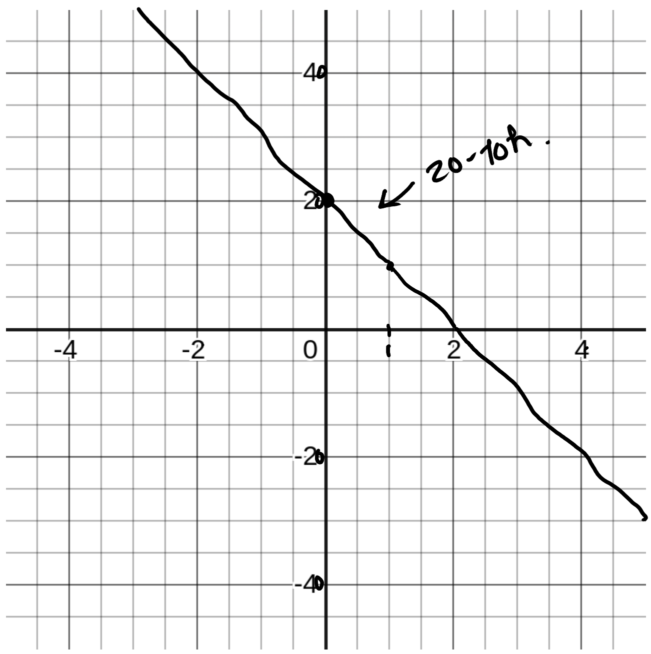
$m = \dfrac{10 - 20}{1 - 0} = -\dfrac{10}{1} = -10$

$\boxed{b}$ $20 = -10 \cdot 0 + b \to b = 20$

So, $T = -10h + 20.$

(c) $T(2.5) = -10 \cdot 2.5 + 20 = -5°C.$

(b)



Polynomials.

$$P(x) = a_n x^{\boxed{n}} + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$$

: coefficients (real numbers) : leading coefficient

Domain: $(-\infty, \infty)$ degree is n

Examples.

a) Concrete example.

$$P(x) = \boxed{5}x^5 + 6x^3 + 2x + 1 .$$

$\deg(P) = 5$.

b) Degree 1. $(n=1)$

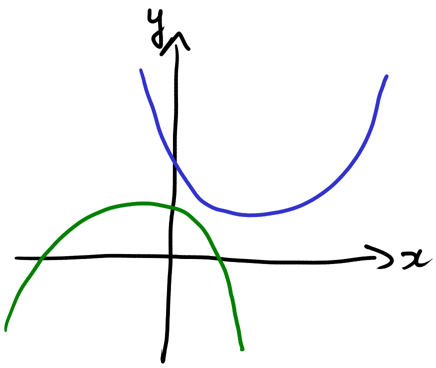
$P(x) = ax + b$

a : slope

b : y-intersect.

c) Degree 2. ($n=2$, parabolas)

$P(x) = ax^2 + bx + c$



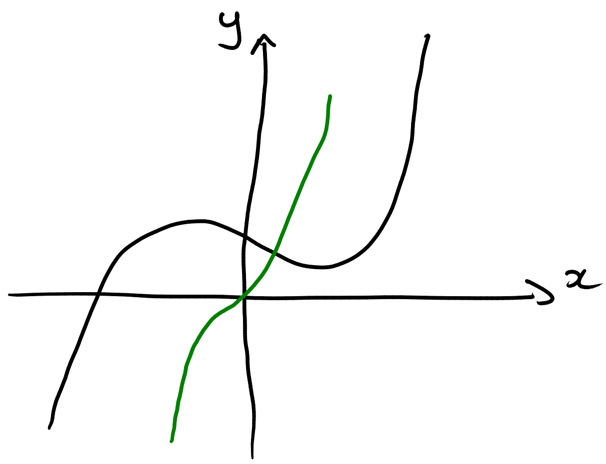
$P(x) = 0$ if

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

provided $b^2 - 4ac \geq 0$

d) Degree 3. (cubics, $n=3$)

$P(x) = ax^3 + bx^2 + cx + d$

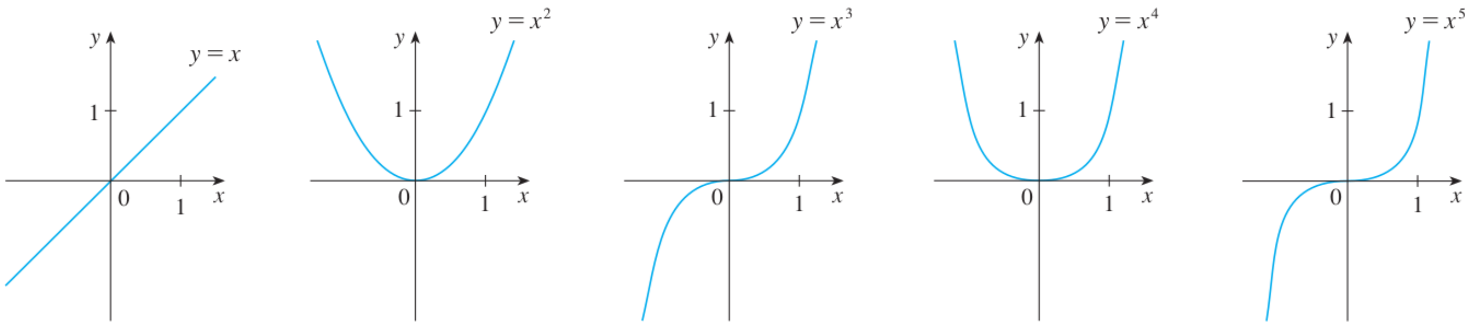


Power Functions.

$$\boxed{f(x) = x^a}$$

i) a is a positive integer or is zero. $(a=1,\ a=2,\ a=3, \ldots)$

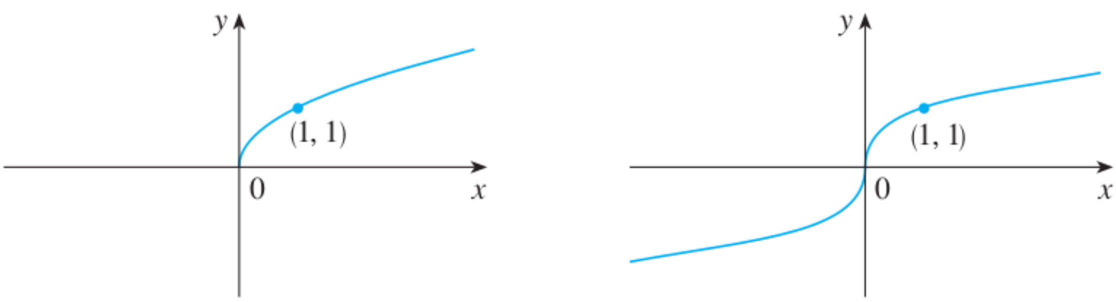
<u>monomials.</u>



Domain: $(-\infty, \infty)$

ii) a is the reciprocal of a positive integer. $(a = 1/2,\ a = 1/3,\ a = 1/4, \ldots)$

$f(x) = x^{1/n}$ or $f(x) = \sqrt[n]{x}$.



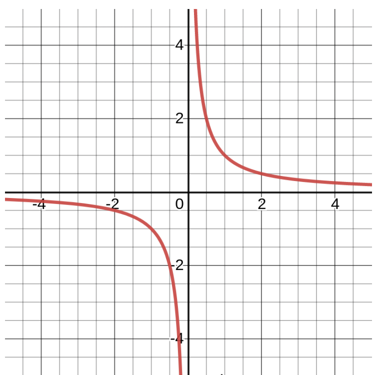
(a) $f(x) = \sqrt{x}$ (b) $f(x) = \sqrt[3]{x}$

Domain: n is even: $\mathrm{dom}(f) = [0, \infty)$

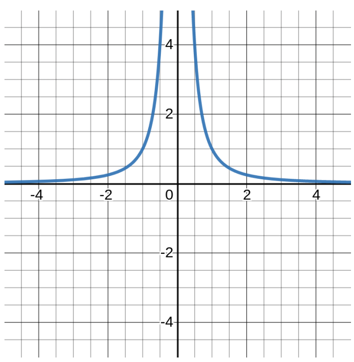
n is odd: $\mathrm{dom}(f) = (-\infty, \infty)$.

iii) When a is a negative integer. $(a=-1,\ a=-2, \ldots)$

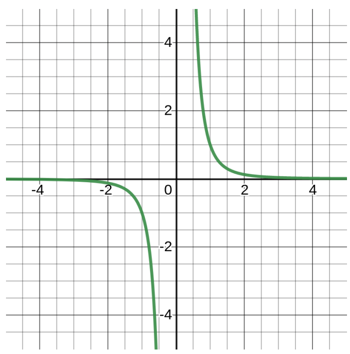
$$f(x) = x^{-a} = \frac{1}{x^a}.$$



$f(x) = \dfrac{1}{x}$



$f(x) = \dfrac{1}{x^2}$



$f(x) = \dfrac{1}{x^3}$

Domain: $(-\infty, 0) \cup (0, \infty)$.

Rational Functions.

$$f(x) = \frac{P(x)}{Q(x)}$$

P: Polynomial.

Q: Polynomial.

Domain: all real numbers except $\underbrace{x \text{ s.t. } Q(x)=0}_{\text{zeros of } Q}$.

Example. Find the domain of the function $f(x) = \dfrac{2x^4 - x^2 + 1}{x^2 - 4}$ $\to$ $x^2 - 4 = 0$ if $x = \pm 2$

$\text{Dom}(f) = (-\infty, \infty) \setminus \{-2, 2\}$

$= (-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

Algebraic Functions.

An algebraic function f is a function that can be expressed only in term of the basic operations :

- summation ; $\to$ powers x^n, $x^{1/n}$
- substraction ; $\to$ power x^n, $x^{1/n}$
- multiplication ;
- division ;
- extracting roots (i.e. taking $\sqrt[n]{\cdot}$).

Domain: depends on the function.

look for $\sqrt{\square}$, $\frac{1}{\square}$ (neg. numbers $\sqrt{\ }$ or division by zero).

Examples. Find the domain of the following function $g(x) = \dfrac{x^4 - 16x^2}{\underline{x + \sqrt{x}}} + (x-2)\sqrt[3]{x+1}$.

$x = 0$

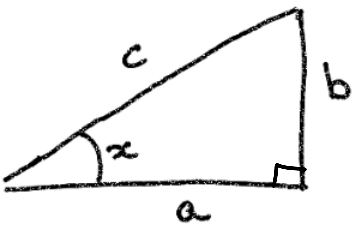
$\cancel{x = -1}$

$\sqrt{x} \to x \geq 0$

~~$\times$ $\text{Dom}(g) = (-\infty, \infty) \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$~~ .

$\text{Dom}(g) = (0, \infty)$

Trigonometric Functions.

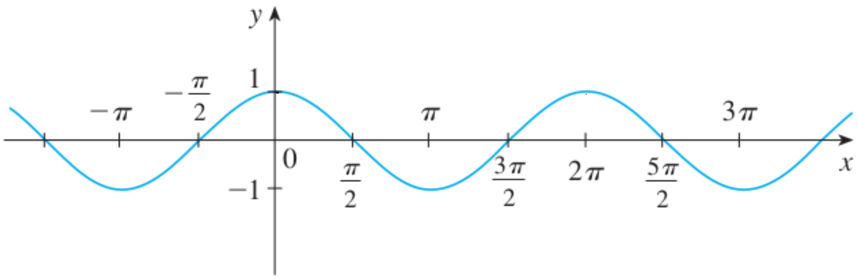


$$\cos x = \frac{a}{c} \qquad \sec x = \frac{1}{\cos x}$$

$$\sin x = \frac{b}{c} \qquad \operatorname{cosec} x = \frac{1}{\sin x}$$

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{1}{\tan x}.$$

i) Cosine function.



Domain: $(-\infty, \infty)$

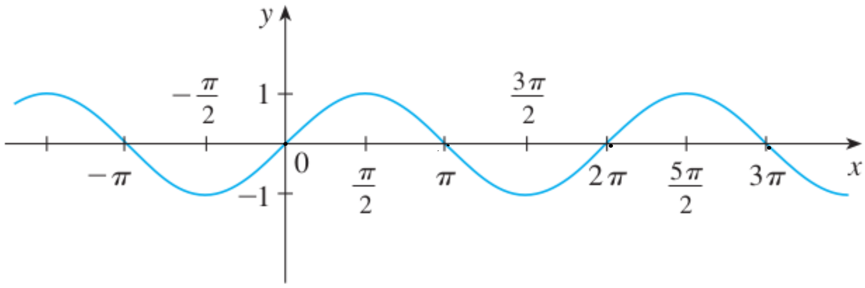
Range: $-1 \leq \cos x \leq 1$

Zeros: $x = \frac{\pi}{2} + \underset{\text{integer}}{k}\pi = \frac{(2k+1)}{2}\pi$

Other: $\cos x = 1 \iff x = 2k\pi$, k int.

$\cos x = -1 \iff x = (2k+1)\pi$, k int.

ii) Sine Function.



Domain: $(-\infty, \infty)$

Range: $-1 \leq \sin x \leq 1$

Zeros: $x = k\pi$, k integer.

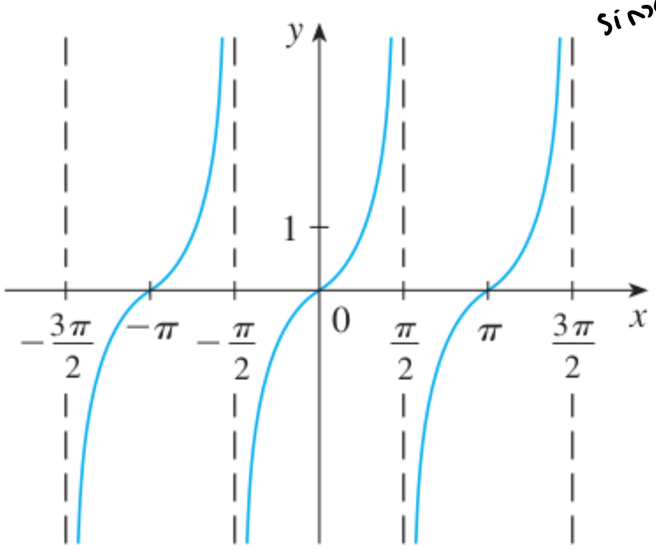
Other:

$\sin x = 1 \iff x = \frac{\pi}{2} + k2\pi$, k int.

$\sin x = -1 \iff x = \frac{3\pi}{2} + k2\pi$, k int.

iii) Tangent Function.

$\tan x = \frac{\boxed{\sin x}}{\cos x} = 0$ if $\sin x = 0$



Domain: $(-\infty, \infty) \setminus \{ \frac{\pi}{2} + k\pi : k \text{ int.} \}$.

Range: $(-\infty, \infty)$

Zeros: $x = k\pi$, k int.

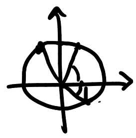
Other:

$\tan(x + \pi) = \tan(x)$

(π-periodic).

EXAMPLE 5 What is the domain of the function $f(x) = \dfrac{1}{\underbrace{1 - 2\cos x}}$?

$1 - 2\cos x = 0$ if $\frac{1}{2} = \cos x$



if $x = \frac{\pi}{3} + 2k\pi$ $\quad x = -\frac{\pi}{3} + 2k\pi$

$$\mathrm{Dom}(f) = (-\infty, \infty) \setminus \left(\left\{ \frac{\pi}{3} + 2k\pi : k \text{ int.} \right\} \cup \left\{ -\frac{\pi}{3} + 2k\pi : k \text{ int.} \right\} \right)$$

$$= (-\infty, \infty) \text{ except } \frac{\pi}{3} + 2k\pi \text{ \& } -\frac{\pi}{3} + 2k\pi \text{ (} k \text{ int.)}$$

EXAMPLE 6 Classify the following functions as one of the types of functions that we have discussed.

~~(a) $f(x) = 5^x$~~

(b) $g(x) = x^5$

(c) $h(x) = \dfrac{1 + x}{1 - \underline{\sqrt{x}}}$

(d) $u(t) = \underline{1} - \underline{t} + \underline{5t}^4$

(b) power fct with $a = 5$ (monomial).

(c) algebraic function

(d) Polynomial (deg = 4).