Chapter 1

Functions and Limits

1.1 Four Ways of Representing a Function

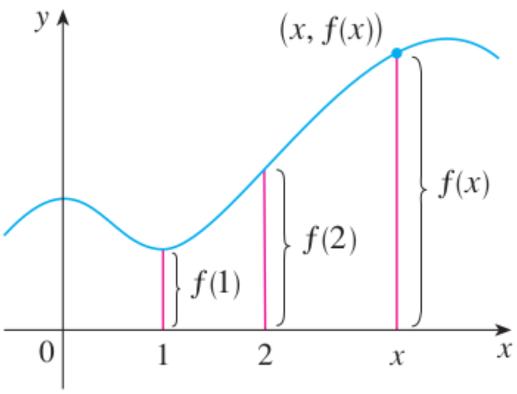
A **function** f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E.

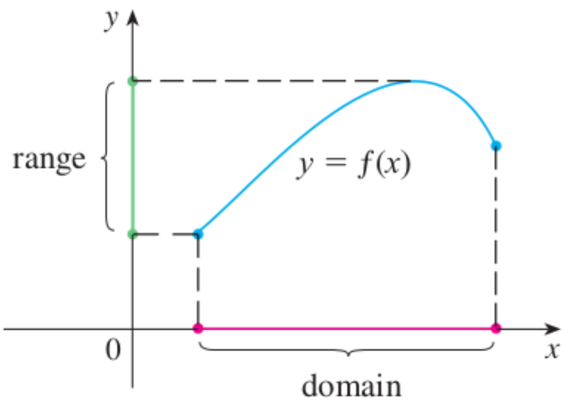
Machine visualization.

Domain: Every admissible inputs

Range: Every possible outputs

Graph of a function.





Dependant variable.

- Usually represents the output.
- Its value depends on the input.

Independant variable.

- Usually represents the input.
- Its value does not depend on anything else.

EXAMPLE 1 The graph of a function f is shown in Figure 6.
(a) Find the values of $f(1)$ and $f(5)$.
(b) What are the domain and range of f?

(a) $f(1) = 3$ $\quad f(5) \approx -0.7$

(b) $\text{Dom} f = [0, 7]$
$\text{Im} f = [-2, 4]$

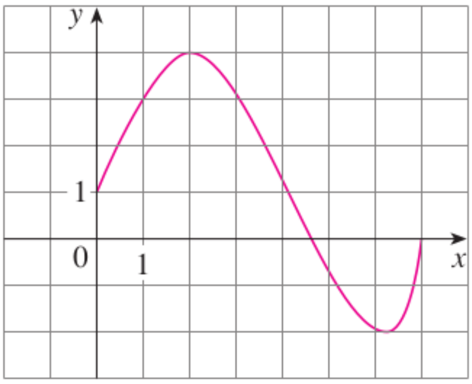


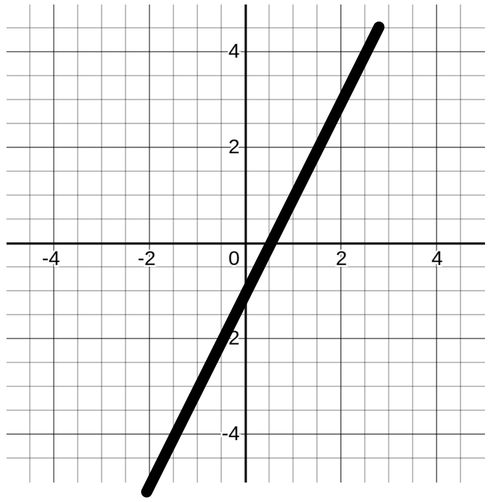
Figure 6

EXAMPLE 2 Sketch the graph and find the domain and range of each function.

(a) $f(x) = 2x - 1$ (b) $g(x) = x^2$

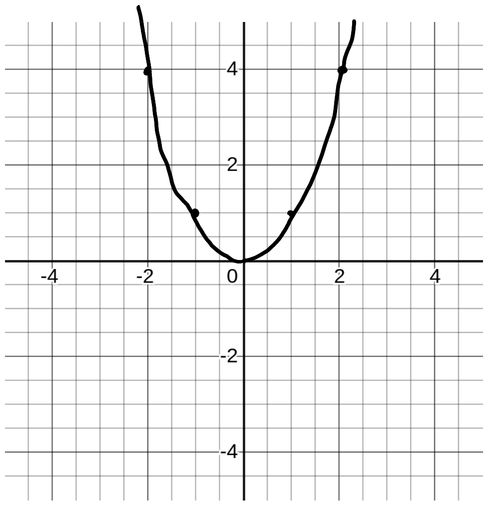
a) $\text{Dom } f = \mathbb{R}$

$\text{Im } f = \mathbb{R}$



b) $\text{Dom } f = \mathbb{R}$

$\text{Im } f = [0, \infty)$



EXAMPLE 3 If $f(x) = 2x^2 - 5x + 1$ and $h \neq 0$, evaluate $\dfrac{f(a+h) - f(a)}{h}$.

$$\frac{2(a+h)^2 - 5(a+h) + 1 - 2a^2 + 5a - 1}{h}$$

$$= \frac{4ah + 2h^2 - 5h}{h}$$

$$= \boxed{4a + 2h - 5}$$

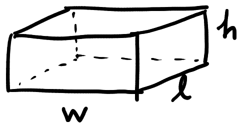
Remark: The fraction $\dfrac{f(a+h) - f(a)}{h}$ is called the DIFFERENCE QUOTIENT.

Representations of functions.

There are four possible ways to represent a function:

- verbally (by a description in words)
- numerically (by a table of values)
- visually (by a graph)
- algebraically (by an explicit formula)

EXAMPLE 5 A rectangular storage container with an open top has a volume of 10 m^3. The length of its base is twice its width. Material for the base costs \$10 per square meter; material for the sides costs \$6 per square meter. Express the cost of materials as a function of the width of the base.



$$10 = wlh = 2w^2h \Rightarrow h = \frac{10}{2w^2} = \frac{5}{w^2}.$$

$$l = 2w$$

$$C = 10 \cdot w \cdot l + 12wh + 12lh$$

$$= 20w^2 + \frac{60}{w} + \frac{60}{w}$$

$$= 20w^2 + \frac{120}{w}$$

Domain of functions given by an explicit formula.

EXAMPLE 6 Find the domain of each function.

(a) $f(x) = \sqrt{x+2}$ (b) $g(x) = \dfrac{1}{x^2 - x}$

(a) $[-2, \infty)$

(b) $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$

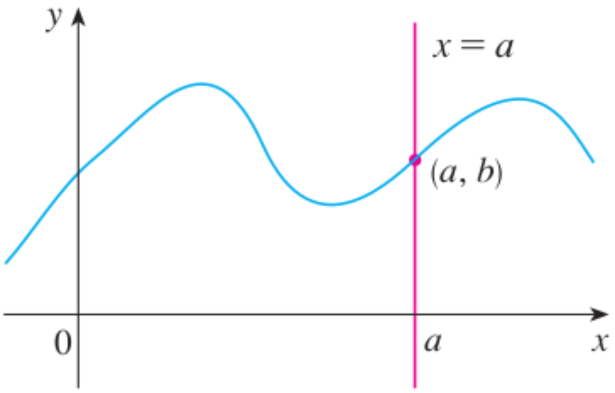
because $x^2 - x \neq 0$

if

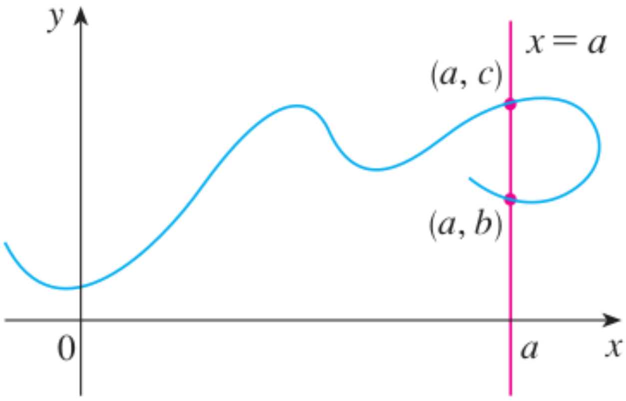
$x \neq 0$ & $x \neq 1$

Which curves are graphs of functions?

The Vertical Line Test A curve in the xy-plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.

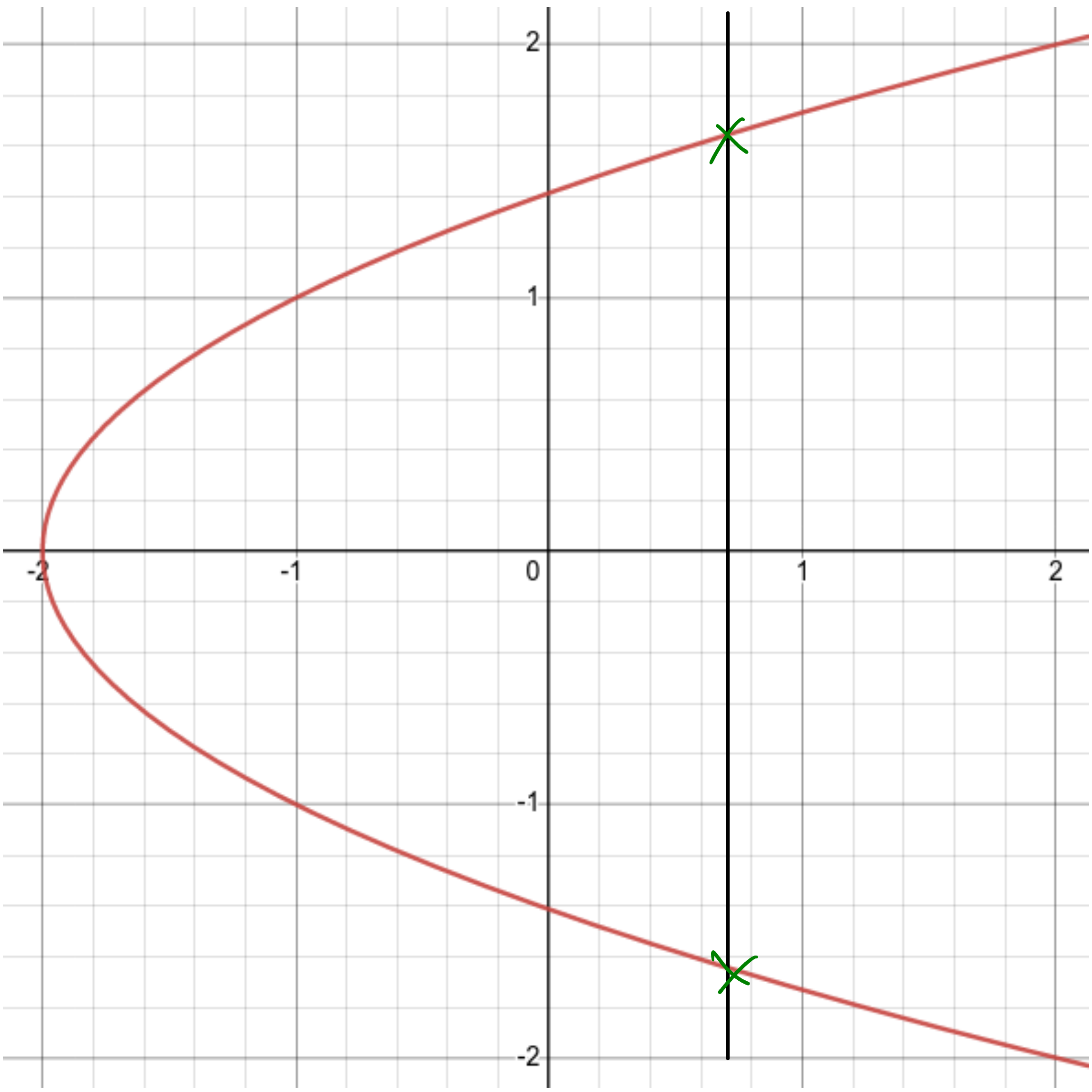


(a) This curve represents a function.



(b) This curve doesn't represent a function.

Example. The parabola $x = y^2 - 2$ is not the graph of a function. Show it using the Vertical Line Test.



Piece-wise Functions.

The functions in the following four examples are defined by different formulas in different parts of their domains. Such functions are called **piecewise defined functions**.

EXAMPLE 7 A function f is defined by

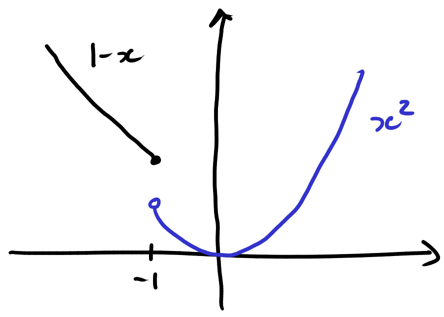
$$f(x) = \begin{cases} 1 - x & \text{if } x \leqslant -1 \\ x^2 & \text{if } x > -1 \end{cases}$$

Evaluate $f(-2)$, $f(-1)$, and $f(0)$ and sketch the graph.

$f(-2) = 1-(-2) = 3$

$f(-1) = 1-(-1) = 2$

$f(0) = 0^2 = 0$



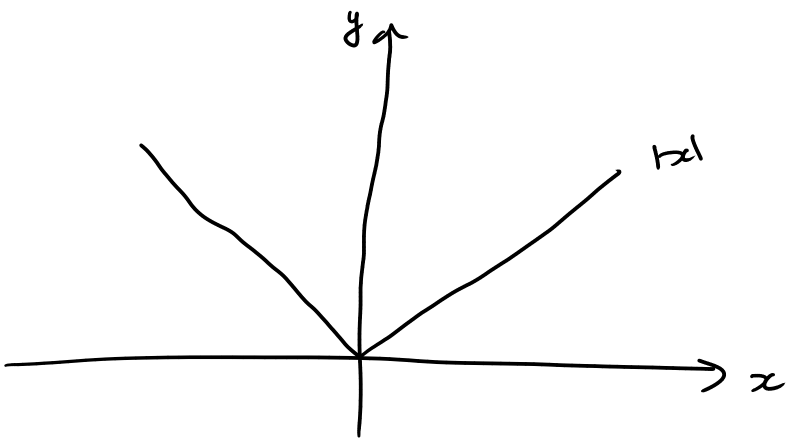
Absolute Value.

$$|a| = a \quad \text{if } a \geqslant 0$$
$$|a| = -a \quad \text{if } a < 0$$

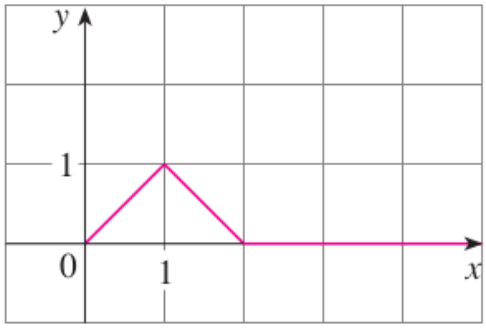
What are the properties of the absolute value:

- $|a| \geq 0$
- $|-a| = |a|$
- $|ab| = |a| \cdot |b|$.

EXAMPLE 8 Sketch the graph of the absolute value function $f(x) = |x|$.



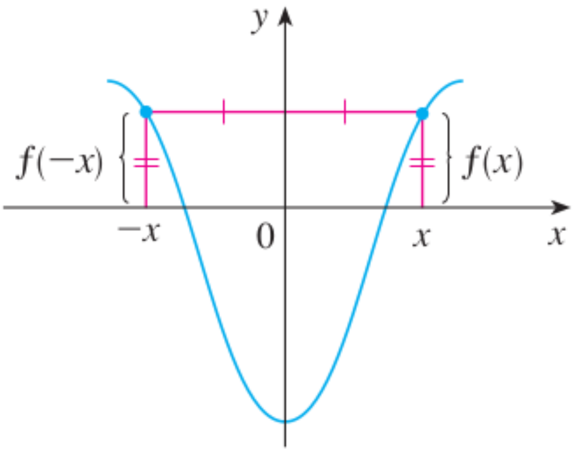
EXAMPLE 9 Find a formula for the function f graphed in Figure 17.



$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2-x, & 1 < x \le 2 \\ 0, & x > 2 \end{cases}.$$

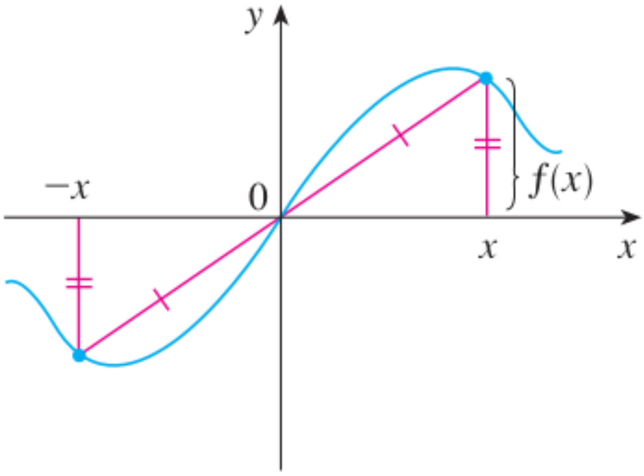
Symmetries.

Even functions



$$f(-x) = f(x)$$

Odd functions.



$$f(-x) = -f(x)$$

EXAMPLE 11 Determine whether each of the following functions is even, odd, or neither even nor odd.

(a) $f(x) = x^5 + x$ (b) $g(x) = 1 - x^4$ (c) $h(x) = 2x - x^2$

(a) $f(-x) = (-x)^5 + (-x) = -x^5 - x = -(x^5 + x) = -f(x)$

$\Rightarrow$ f odd.

(b) $g(-x) = 1 - (-x)^4 = 1 - x^4 = g(x)$

$\Rightarrow$ g even.

(c) $h(-x) = -2x - (-x)^2 = -2x - x^2 \neq$ $h(x)$ & $-h(x)$

neither odd or even.

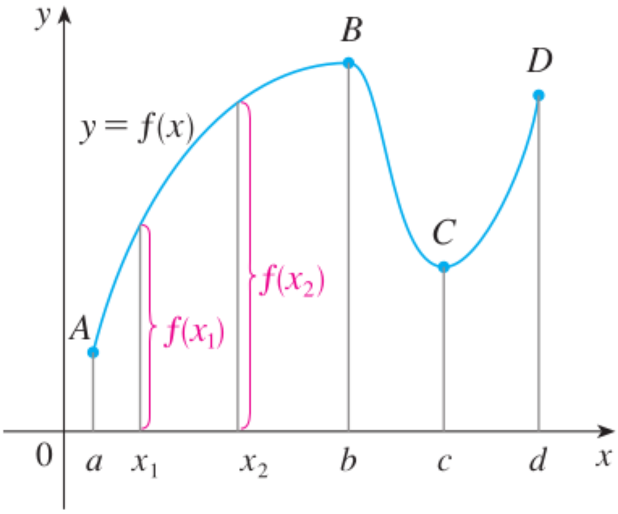
Increasing/Decreasing Functions.

A function f is called **increasing** on an interval I if

$$f(x_1) < f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in } I$$

It is called **decreasing** on I if

$$f(x_1) > f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in } I$$



- From A to B:

  Increasing

- From B to C:

  decreasing

- From C to D:

  increasing

Example. Where is the function $f(x) = x^2$ increasing? Where is it decreasing?

increasing on $(0, \infty)$

decreasing on $(-\infty, 0)$.

Chapter 1

Functions and Limits

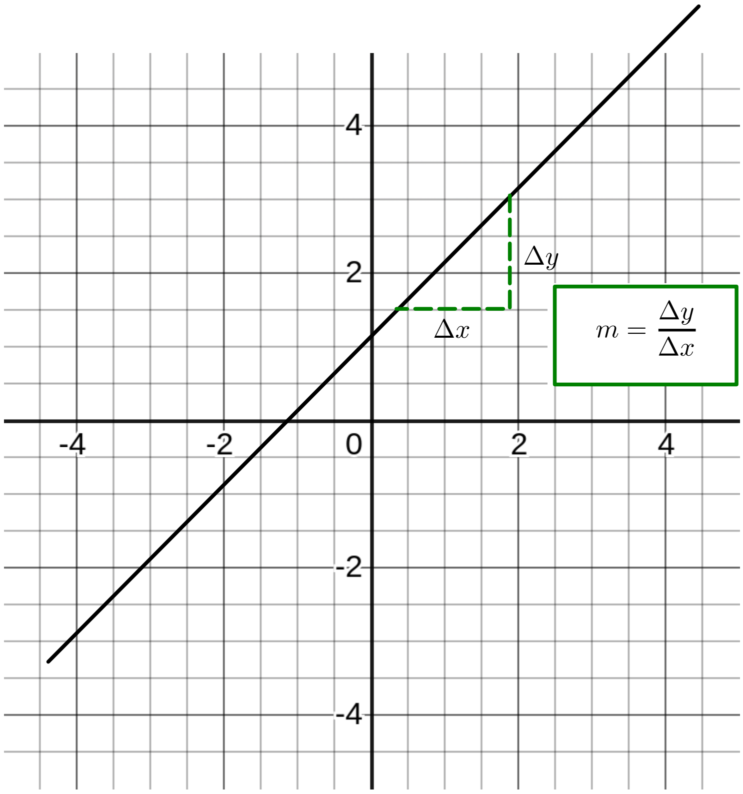
1.2 Mathematical Models: A catalog of Essential Functions

Linear Models.

$$\boxed{y = f(x) = mx + b}$$

.m: the slope

.b: y-intercept



Another formulation (point-slope):

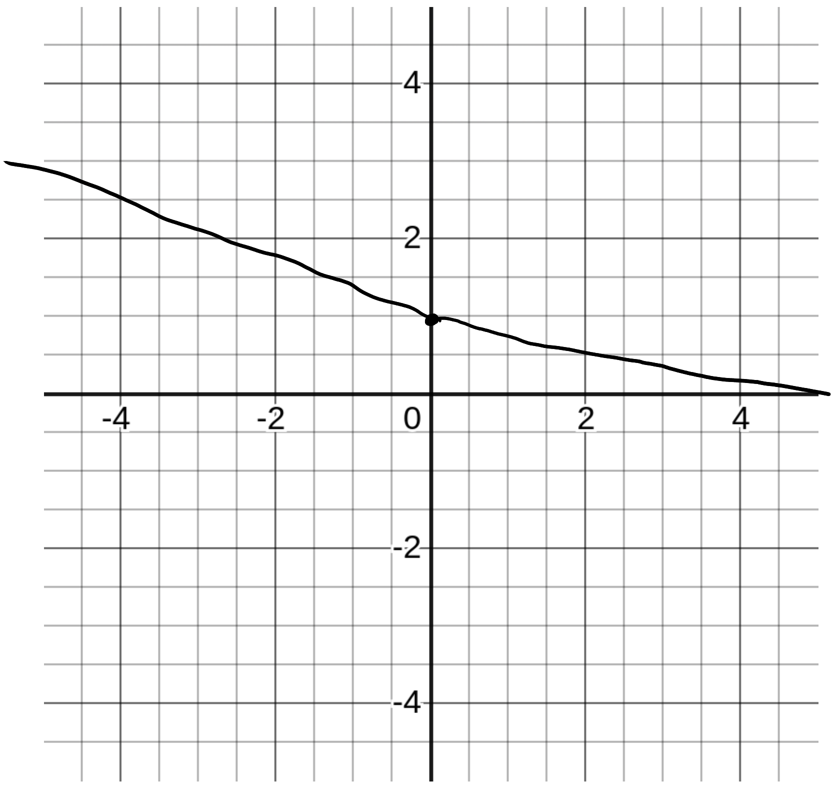
$$\boxed{y - y_0 = m(x - x_0)}$$

Example. A line passes through the points (0, 1) and (3, 1/2). Find the equation of the line and sketch its graph.

$$m = \frac{1/2 - 1}{3 - 0} = \frac{-1}{6}$$

$$\Rightarrow \quad y - 1 = -\frac{1}{6}(x - 0)$$

$$\Rightarrow \quad y = -\frac{x}{6} + 1$$



Polynomials.

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$$

: Coefficients

: Leading coefficient

: Degree of polynomial

Domain: All the numbers (real numbers).

Examples.

a) Concrete example.

$2x$

$x^{10} + x^5 + 2$

$x^2 + 2x + 9$

b) Degree 1.

$ax+b \quad , \quad a \neq 0$

c) Degree 2.

ax^2+bx+c

quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

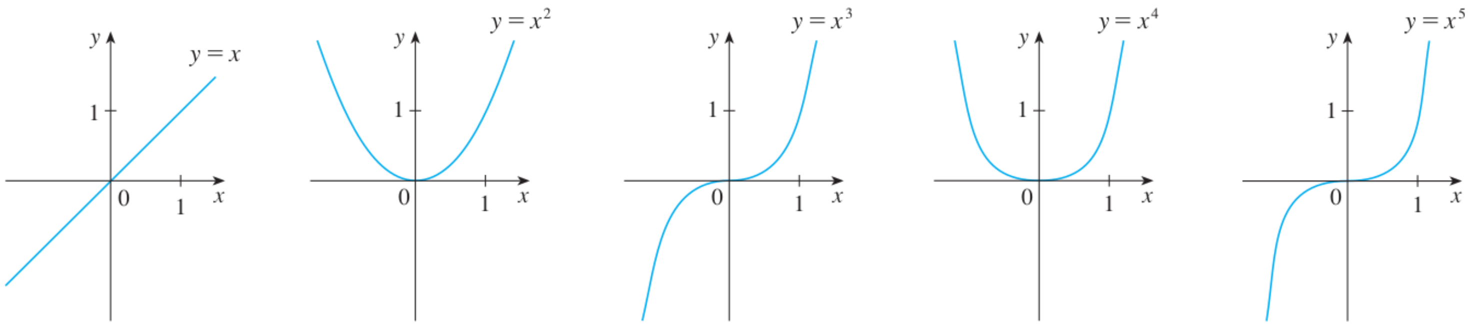
d) Degree 3.

ax^3+bx^2+cx+d

Power Functions.

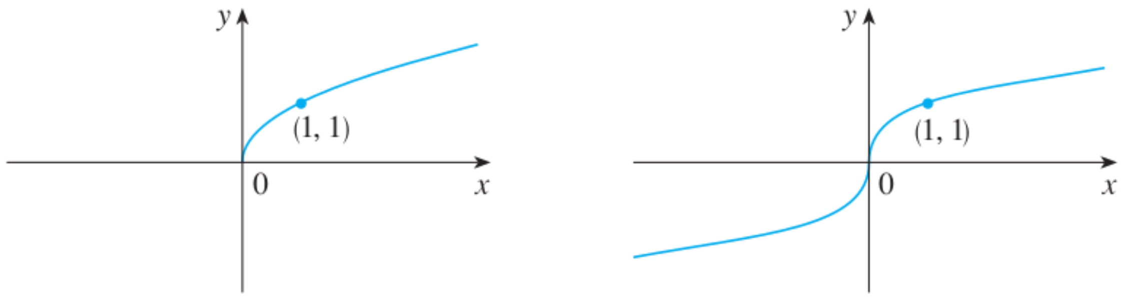
$$\boxed{f(x) = x^a}$$

i) a is a positive integer or is zero.



Domain: All the numbers (real numbers).

ii) a is the reciprocal of a positive integer.

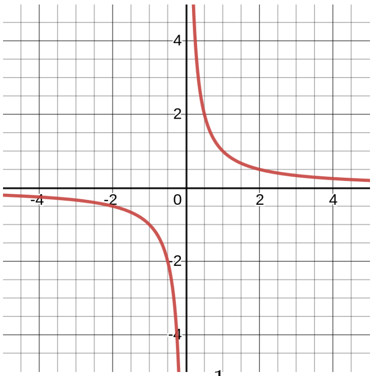


(a) $f(x) = \sqrt{x}$ (b) $f(x) = \sqrt[3]{x}$

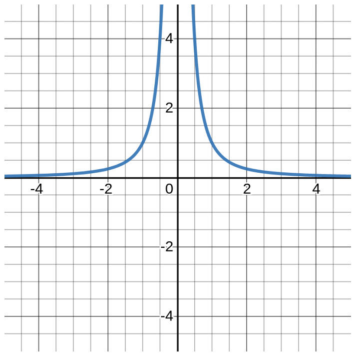
Domain: For odd integer ---> all the numbers (Real numbers).

For even integer ----> Positive numbers or zero.

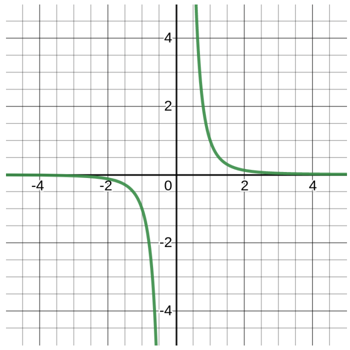
iii) When a is a negative integer.



$f(x) = \frac{1}{x}$



$f(x) = \frac{1}{x^2}$



$f(x) = \frac{1}{x^3}$

Domain: All the numbers except zero.

Rational Functions.

$$\boxed{f(x) = \frac{P(x)}{Q(x)}}$$

P: polynomial

Q: polynomial

Domain: all the numbers except the number x such that Q(x) = 0.

Example. Find the domain of the function $f(x) = \dfrac{2x^4 - x^2 + 1}{x^2 - 4}$.

$x^2 - 4 \neq 0$ iff $x \neq \pm 2$

$\Rightarrow$ $\text{Dom} f = (-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

Algebraic Functions.

An algebraic function f is a function that can be expressed only in term of the basic operations :

- summation ;
- substraction ;
- multiplication ;
- division ;
- extracting roots (i.e. taking $\sqrt[n]{\cdot}$).

Domain: Depends on the components of the function.

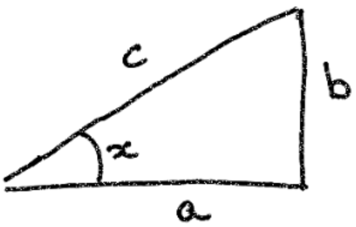
Examples. Find the domain of the following function $g(x) = \dfrac{x^4 - 16x^2}{x + \sqrt{x}} + (x-2)\sqrt[3]{x+1}$.

$x + \sqrt{x} \neq 0 \Rightarrow x \neq 0$.

$\sqrt{x} \rightarrow x \geq 0$.

So, $\text{Dom} f = (0, \infty)$.

Trigonometric Functions.

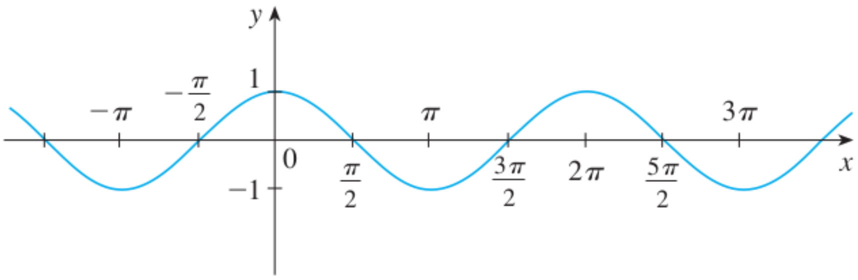


$\cos x = \dfrac{a}{c}$ $\quad$ $\sec x = \dfrac{1}{\cos x}$

$\sin x = \dfrac{b}{c}$ $\quad$ $\operatorname{cosec} x = \dfrac{1}{\sin x}$

$\tan x = \dfrac{\sin x}{\cos x}$ $\quad$ $\cot x = \dfrac{1}{\tan x}$.

i) Cosine function.



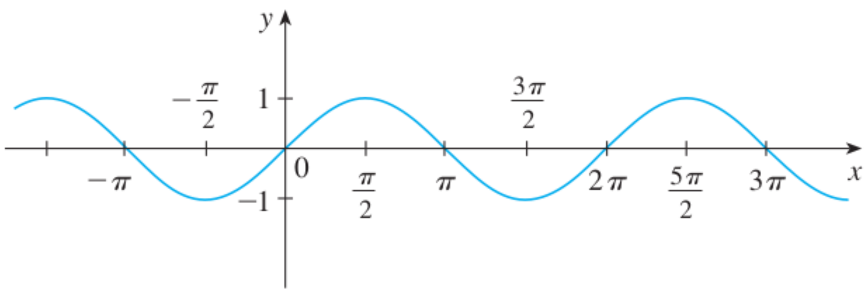
Domain: All of the numbers

Range: the interval [-1, 1]

Zeros: $x = \dfrac{(2k+1)\pi}{2}$, $k = \ldots, -2, -1, 0, 1, 2, \ldots$

Other: $\cos(-x) = \cos(x)$

ii) Sine Function.



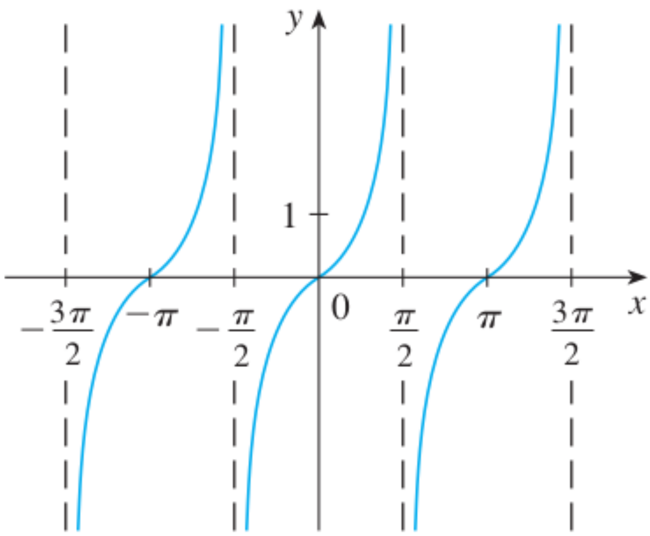
Domain: All the numbers

Range: [-1, 1]

Zeros: $x = k\pi$, $k = \ldots, -2, -1, 0, 1, 2, \ldots$

Other:
- $\sin(-x) = -\sin(x)$
- $\sin^2(x) + \cos^2(x) = 1$
- See trigonometric sheet

iii) Tangent Function.



Domain: $(-\infty, \infty) - \{\ldots, -3\pi/2, -\pi/2, \pi/2, 3\pi/2\}$

Range: all numbers

Zeros: same as the cos (x).

Other:

EXAMPLE 5 What is the domain of the function $f(x) = \dfrac{1}{1 - 2\cos x}$?

$1 - 2\cos x = 0$ if $\cos x = \frac{1}{2}$ if $x = \frac{\pi}{3} + 2k\pi$

or $x = \frac{5\pi}{3} + 2k\pi$

So, $\text{Dom} f = \mathbb{R} \setminus \left\{ \frac{\pi}{3} + 2k\pi, \frac{5\pi}{3} + 2k\pi : k = \ldots, -1, 0, 1, \ldots \right\}$.

Chapter 1

Functions and Limits

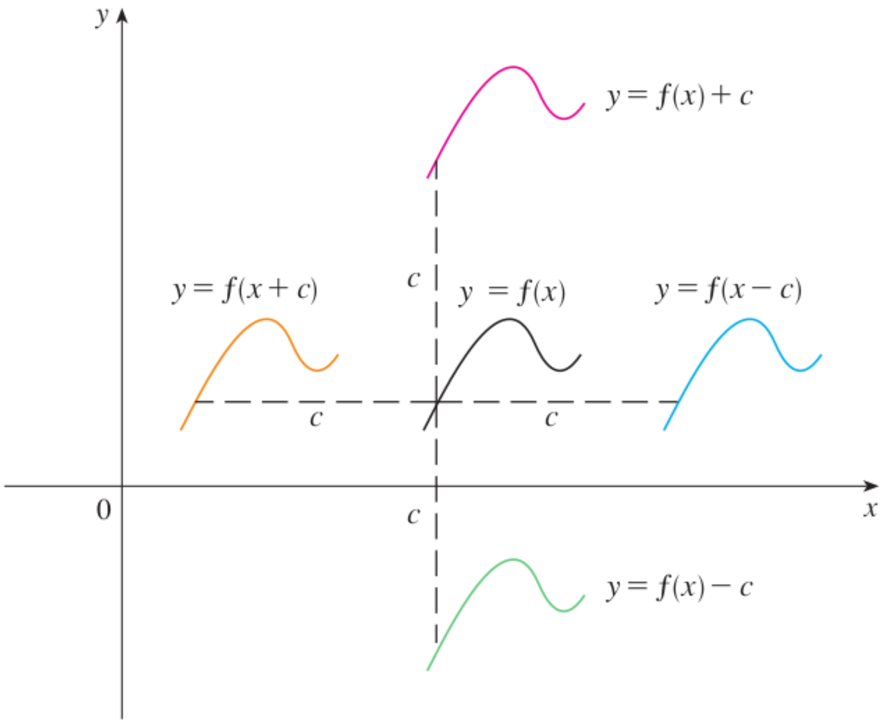
1.3 New Functions from Old Functions

<u>Transformations of Functions.</u>

Translation.

Vertical and Horizontal Shifts Suppose $c > 0$. To obtain the graph of

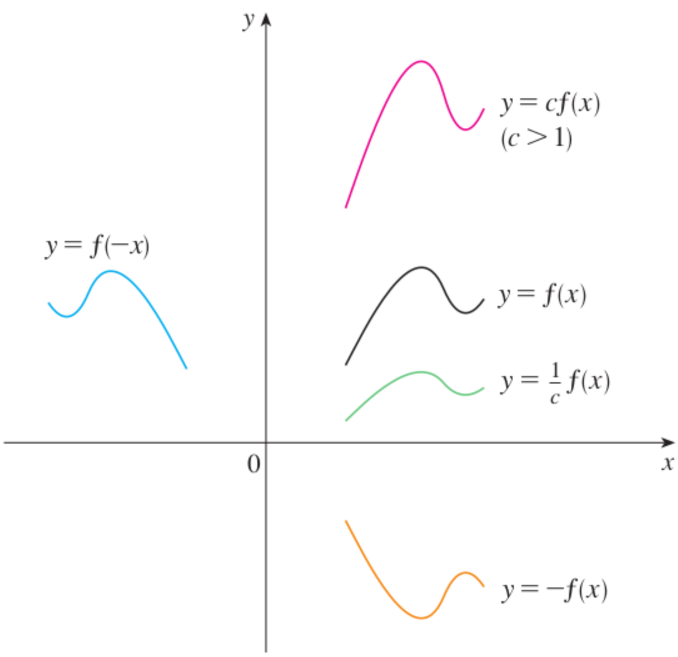
$y = f(x) + c$, shift the graph of $y = f(x)$ a distance c units upward
$y = f(x) - c$, shift the graph of $y = f(x)$ a distance c units downward
$y = f(x - c)$, shift the graph of $y = f(x)$ a distance c units to the right
$y = f(x + c)$, shift the graph of $y = f(x)$ a distance c units to the left



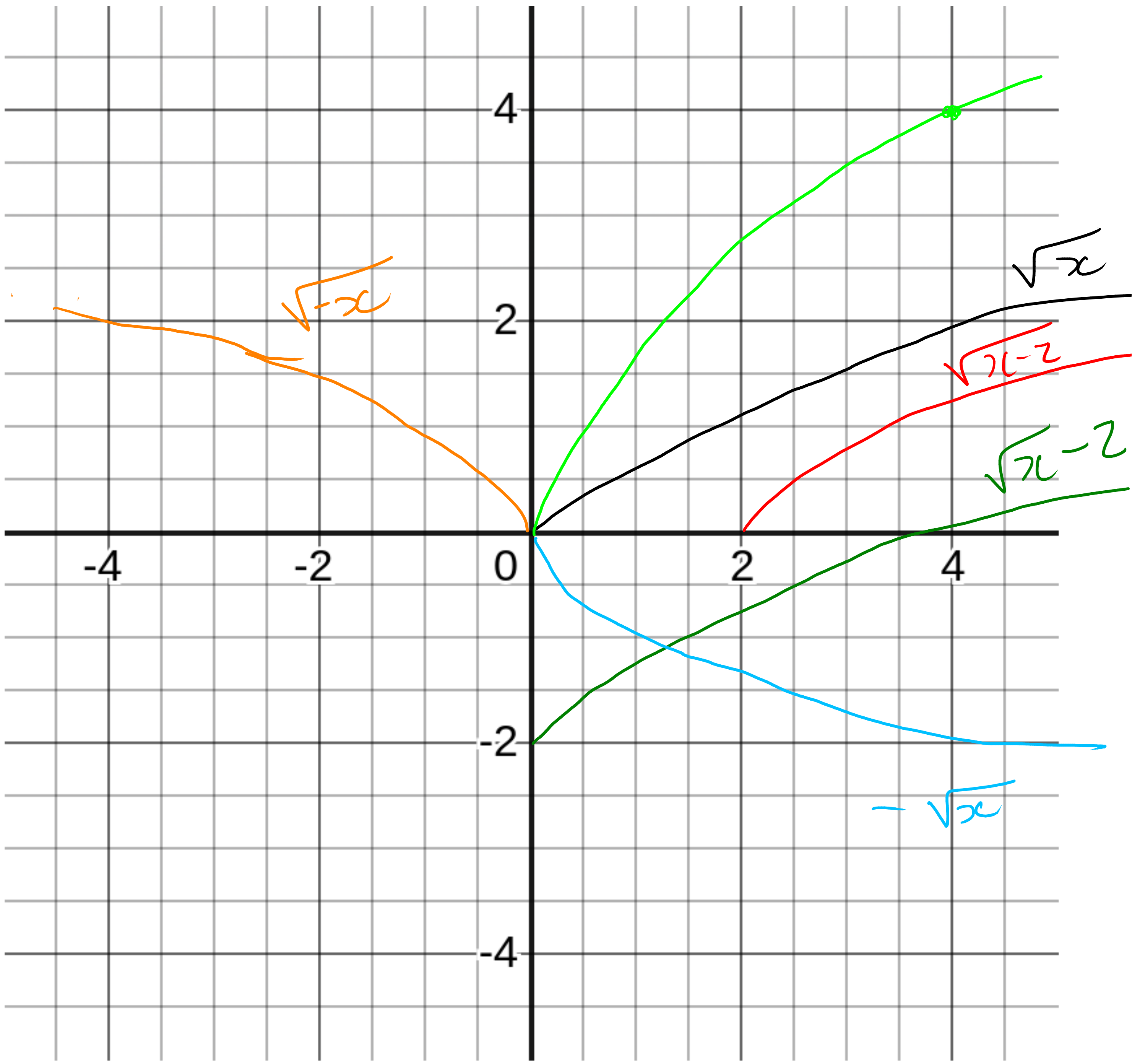
Stretching and reflecting.

Vertical and Horizontal Stretching and Reflecting Suppose $c > 1$. To obtain the graph of

$y = cf(x)$, stretch the graph of $y = f(x)$ vertically by a factor of c
$y = (1/c)f(x)$, shrink the graph of $y = f(x)$ vertically by a factor of c
$y = f(cx)$, shrink the graph of $y = f(x)$ horizontally by a factor of c
$y = f(x/c)$, stretch the graph of $y = f(x)$ horizontally by a factor of c
$y = -f(x)$, reflect the graph of $y = f(x)$ about the x-axis
$y = f(-x)$, reflect the graph of $y = f(x)$ about the y-axis



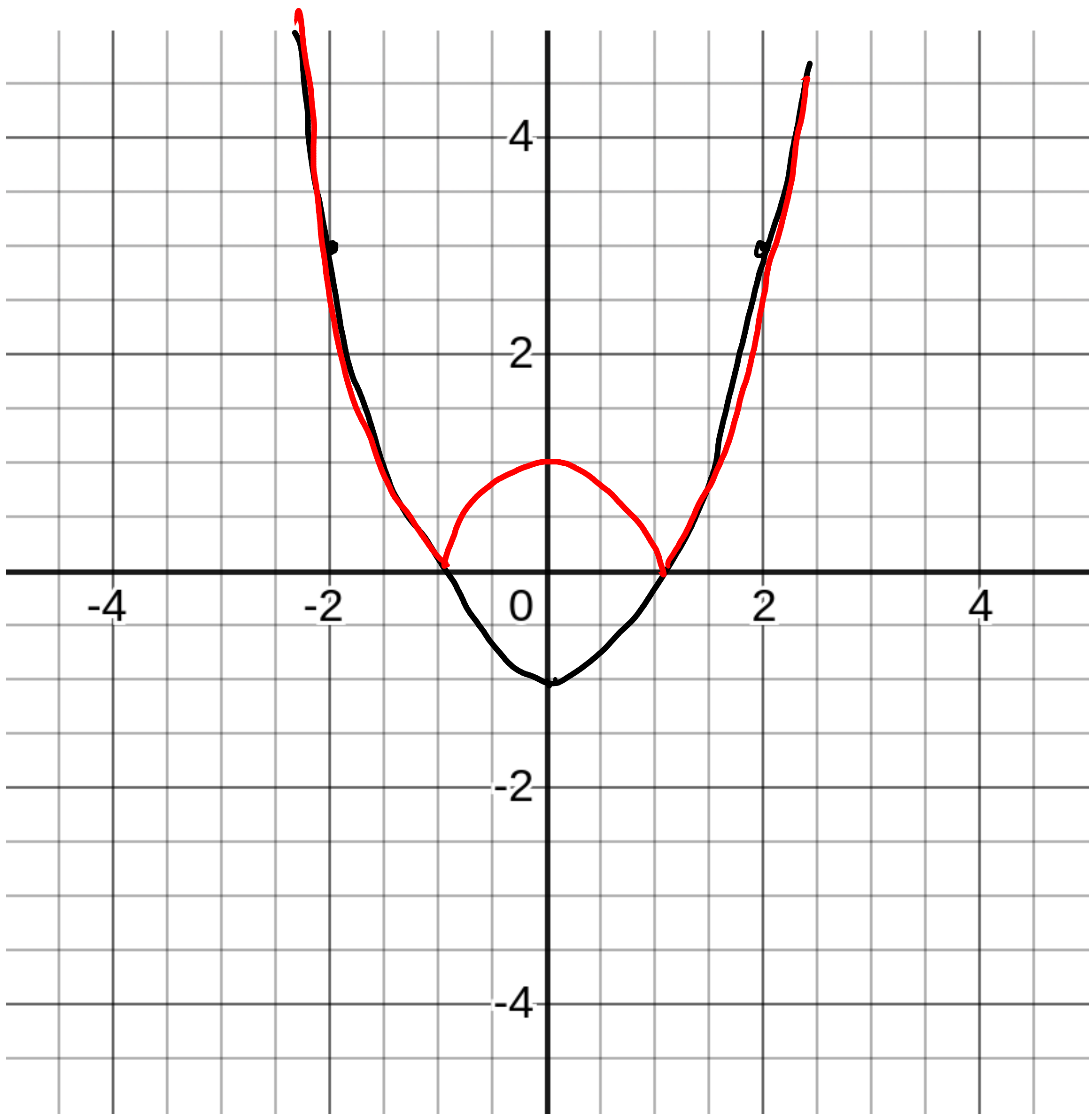
EXAMPLE 1 Given the graph of $y = \sqrt{x}$, use transformations to graph $y = \sqrt{x} - 2$, $y = \sqrt{x-2}$, $y = -\sqrt{x}$, $y = 2\sqrt{x}$, and $y = \sqrt{-x}$.



EXAMPLE 5 Sketch the graph of the function $y = |x^2 - 1|$.

$x^2 - 1$

$|x^2 - 1|$



Combinaisons of Functions.

Adding.

$$(f+g)(x) = f(x) + g(x)$$

Domain = $\mathrm{Dom}(f) \cap \mathrm{Dom}(g)$

Substracting.

$$(f-g)(x) = f(x) - g(x)$$

Domain = $\mathrm{Dom}(f) \cap \mathrm{Dom}(g)$

Multiplying.

$$(fg)(x) = f(x)g(x)$$

Domain = $\mathrm{Dom}(f) \cap \mathrm{Dom}(g)$

Dividing.

$$(f/g)(x) = f(x)/g(x)$$

Domain = every x in $\mathrm{Dom}(f) \cap \mathrm{Dom}(g)$ for which $g(x) \neq 0$.

Example. Find the domain of the function

$h(x) = \sqrt{x} + \sqrt{2-x}$.

$[0,2]$

because

$\sqrt{x} \to [0,\infty)$

$\sqrt{2-x} \to (-\infty, 2]$.

Example Find the domain of the function $h(x) = \dfrac{x^2}{x-1}$.

$x-1 \neq 0$ if $x \neq 1$

$\mathrm{Dom}\, f = (-\infty, 1) \cup (1, \infty)$

Composite of two functions (Composition).

Definition Given two functions f and g, the **composite function** $f \circ g$ (also called the **composition** of f and g) is defined by

$$(f \circ g)(x) = f(g(x))$$

Domain = every x in the $\mathrm{Dom}(g)$ such that $g(x)$ is in $\mathrm{Dom}(f)$.

EXAMPLE 6 If $f(x) = x^2$ and $g(x) = x - 3$, find the composite functions $f \circ g$ and $g \circ f$.

$$f(g(x)) = (x-3)^2 = x^2 - 6x + 9$$

$$g(f(x)) = x^2 - 3$$

EXAMPLE 9 Given $F(x) = \cos^2(x + 9)$, find functions f, g, and h such that $F = f \circ g \circ h$.

$h(x) = x + 9$

$g(x) = \cos x$

$f(x) = x^2$.

Example. Find the domain of the function $h(x) = \sqrt{x+2}$.

$x \geq -2$ so $x+2 \geq 0$

$\Rightarrow$ Dom $h = [-2, \infty)$.