MATH 241 COMMON FINAL EXAM, SPRING 2019

You have 120 minutes.

No books, no notes, no electronic devices.

YOU MUST SHOW ALL WORK. NO NEED TO SIMPLIFY ANSWERS.

Name Solutions

Instructor Name ______________________

Section Number ______________________

Grade table (for instructor's use only)

1. (16pts) ________
2. (4pts) ________
3. (6pts) ________
4. (20pts) ________
5. (8pts) ________
6. (6pts) ________
7. (7pts) ________
8. (8pts) ________
9. (10pts) ________
10. (18pts) ________
11. (15pts) ________
12. (8pts) ________
13. (10pts) ________
14. (14pts) ________

Total Score ________ (/150 points)

1. Calculate the following limits. **Do not** use L'Hospital's rule. If the limit is infinite, specify whether it is $+\infty$ or $-\infty$.

   (a) (4pts) $\lim_{x\to 2} \dfrac{x^2-4}{x^2-3x+2}$

$$\frac{x^2-4}{x^2-3x+2} = \frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}(x-1)} = \frac{x+2}{x-1} \quad (x \neq 2)$$

$$\Rightarrow \lim_{x\to 2} \frac{x^2-4}{x^2-3x+2} = \lim_{x\to 2} \frac{x+2}{x-1} = \frac{2+2}{2-1} = \boxed{4}$$

   (b) (4pts) $\lim_{x\to 0} \dfrac{x\cos x}{\sin x}$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \Rightarrow \lim_{x\to 0} \frac{x}{\sin x} = \frac{1}{\lim_{x\to 0} \frac{\sin x}{x}} = 1.$$

$$\text{So, } \lim_{x\to 0} \frac{x\cos x}{\sin x} = \left(\lim_{x\to 0} \frac{x}{\sin x}\right)\left(\lim_{x\to 0} \cos x\right) = 1 \cdot \cos(0) = \boxed{1}$$

   (c) (4pts) $\lim_{x\to 2^-} \dfrac{|2x-4|}{2-x}$

$x \to 2^-$ means x approaches 2 from the left (so, $x<2$).

$$|2x-4| = 2|x-2| = 2(2-x) \quad (x<2)$$

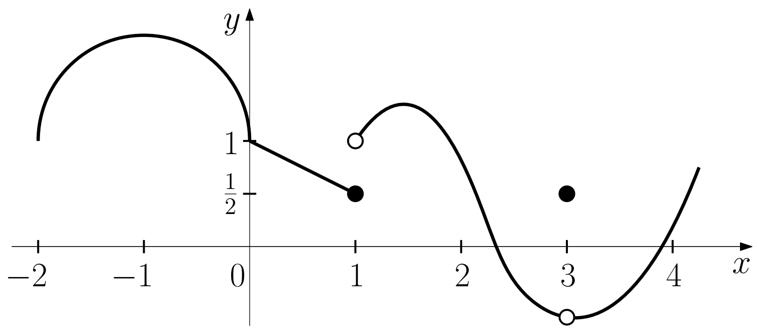
$$\Rightarrow \lim_{x\to 2^-} \frac{|2x-4|}{2-x} = \lim_{x\to 2^-} \frac{2(2-x)}{2-x} = \boxed{2}.$$

   (d) (4pts) $\lim_{x\to\infty} \dfrac{4x^3+\sin x}{2x^3+3}$

$$\frac{4x^3+\sin x}{2x^3+3} = \frac{4+\frac{\sin x}{x^3}}{2+3/x^3} \qquad \lim_{x\to\infty} \frac{\sin x}{x^3} = 0$$

$$\Rightarrow \lim_{x\to\infty} \frac{4x^3+\sin x}{2x^3+3} = \frac{\lim_{x\to\infty} 4+\frac{\sin x}{x^3}}{\lim_{x\to\infty} 2+3/x^3} = \boxed{2}$$

2. Consider the function $f(x)$ whose graph is shown below.



(a) (1pt) Find the following limit: $\lim_{x \to 1^-} f(x)$.

$\boxed{1/2}$

(b) (1pt) Find the values a such that $\lim_{x \to a} f(x)$ does not exist.

$\boxed{a = 1}$

(c) (1pt) Find the values a such that $f(x)$ is discontinuous at $x = a$.

$\boxed{a=1}$ & $\boxed{a = 3}$

(d) (1pt) Find the values a such that $f'(a)$ does not exist.

$\boxed{a = 0}$, $\boxed{a=1}$ & $\boxed{a=3}$

3. (6pts) Use the definition of the derivative to compute $f'(1)$ for $f(x) = \dfrac{4}{x+1}$.
(Warning: you will not get credit if you use the rules of differentiation.)

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{4}{2+h} - 2}{h}$$

$$= \lim_{h \to 0} \frac{-2\cancel{h}}{\cancel{h}(2+h)}$$

$$= \lim_{h \to 0} \frac{-2}{2+h} = \boxed{-1}$$

4. Differentiate the following functions. You do not need to simplify your answers.

(a) (5pts) $f(x) = \dfrac{x-1}{x+1}$

$$f'(x) = \frac{(x-1)'(x+1) - (x-1)(x+1)'}{(x+1)^2}$$

$$= \frac{x+1 - (x-1)}{(x+1)^2}$$

$$= \boxed{\frac{2}{(x+1)^2}}$$

(b) (5pts) $f(x) = (x-1)\cos(1+x^2)$

$$f'(x) = (x-1)'\cos(1+x^2) + (x-1)\left(\cos(1+x^2)\right)'$$

$$= \cos(1+x^2) + (x-1)(-\sin(1+x^2)\cdot 2x)$$

$$= \boxed{\cos(1+x^2) - (x-1)x\sin(1+x^2)}$$

Problem continues on the next page...

(c) (5pts) $f(x) = \left(x^2 + \frac{1}{\sqrt{x+1}} + 3^3\right)^{\frac{3}{2}}$

$$f'(x) = \frac{3}{2}\left(x^2 + \frac{1}{\sqrt{x+1}} + 81\right)^{1/2}\left(2x - \frac{3}{2(x+1)^{3/2}}\right)$$

(we used the Chain Rule).

(d) (5pts) $f(x) = \int_0^{x^2} \frac{dt}{1+\sin^2 t}$

Let $g(x) = \int_0^x \frac{dt}{1+\sin^2 t}$. Then

$f(x) = g(x^2)$.

Chain Rule $\Rightarrow$ $f'(x) = g'(x^2) \cdot 2x$

FTC Part 1 $\Rightarrow$ $g'(x) = \frac{1}{1+\sin^2 x}$

So,

$$f'(x) = \boxed{\frac{2x}{1+\sin^2(x^2)}}.$$

5. Consider the equation $x^5 + 2x - 1 = 0$.

(a) (6pts) Use the Intermediate Value Theorem to show that the given equation has a solution in the interval $[0, 1]$.

Let $f(x) = x^5 + 2x - 1$

f is continuous (polynomial).

$f(0) = -1$.

$f(1) = 2$.

Taking $L = 0 \in (-1, 2)$ $\Rightarrow$ IVT implies there is a $0 < x < 1$ s.t. $f(x) = L = 0$

$\Rightarrow$ $x^5 + 2x - 1 = 0$.

(b) (2pts) Use Rolle's theorem or the Mean Value Theorem to show that the given equation cannot have more than one solution in the interval $[0, 1]$.

We have $f'(x) = 5x^4 + 2$.

Suppose there is another number, say y, s.t. $f(y) = 0$.

Then, $f(y) = f(x)$. By Rolle's theorem, there is a number c between x & y s.t. $f'(c) = 0$.

But $f'(x) = 5x^4 + 2 \geq 2$ for every $0 \leq x \leq 1$.

this is a contradiction. There is only one root.

6. (6pts) Use linear approximation and the fact that $27^{-\frac{1}{3}} = \dfrac{1}{3}$ to estimate $28^{-\frac{1}{3}}$.

7. (7pts) Find an equation of the tangent line to the curve $x^2y^2 - 2x = 9 - y$ at the point $(-2, 1)$.

① Implicit diff.

$$2xy^2 + x^2\,2yy' - 2 = -y'$$

② Isolate y'

$$2x^2yy' + y' = 2 - 2xy^2$$

$$\Rightarrow (2x^2y+1)\,y' = 2 - 2xy^2$$

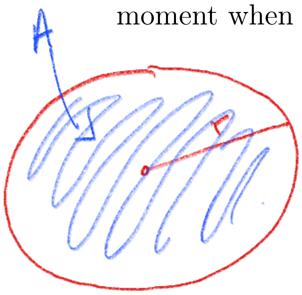
$$\Rightarrow y' = \frac{2(1-xy^2)}{2x^2y+1}$$

③ Tangent line

$$m = y'(-2) = \frac{2(1+2)}{8+1} = \frac{2}{3}$$

$$\Rightarrow \boxed{y - 1 = \frac{2}{3}(x+2)}$$

8. (8pts) A stone dropped in a pond sends out a circular ripple whose radius increases at a constant rate of 3 ft/sec. How rapidly is the area enclosed by the ripple increasing at the moment when the radius is equal to 30 ft?



r: radius

A: area enclosed by the ripple

$\frac{dr}{dt} = 3$

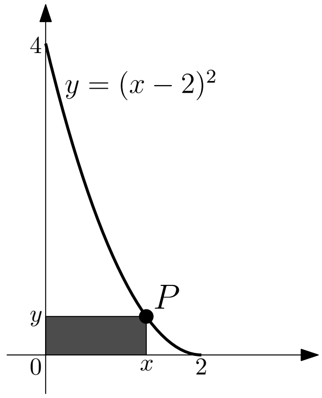
Goal: find $\left.\frac{dA}{dt}\right|_{r=30}$.

We have $A = \pi r^2$.

Then $\frac{dA}{dt} = \pi 2r \frac{dr}{dt}$

$\Rightarrow \frac{dA}{dt} = 2\pi(30) \cdot 3 = \boxed{180\pi \text{ ft}^2/\text{sec}}$

9. (10pts) Consider the parabola $y = (x-2)^2$. Find the coordinates (x, y) of the point P lying on this parabola between $x = 0$ and $x = 2$ such that the *perimeter* of the rectangle shown below is the smallest.



x: width
y: height.
P: perimeter

Goal. Find smallest P.

① Find function.

$P = 2x + 2y$. But $y = (x-2)^2$

$\Rightarrow \quad P = 2x + 2(x-2)^2$

② optimize.

$\frac{dP}{dx} = 2 + 4(x-2) = 4x - 2.$

$\frac{dP}{dx} = 0 \iff x = \frac{1}{2}.$

Also $\frac{d^2P}{dx^2} = 4 \Rightarrow$ by the second derivative test, $x = \frac{1}{2}$ gives the absolute minimum.

③ Answer.

perimeter $= 2(1/2) + 2(\frac{1}{2} - 2)^2 = 1 + \frac{9}{2} = \boxed{\frac{11}{2}}$

$x = \boxed{\frac{1}{2}}$

$y = (\frac{1}{2} - 2)^2 = \boxed{\frac{9}{4}}$

10. Let $f(x) = \dfrac{x+1}{(x-1)^2}$. Then $f'(x) = -\dfrac{x+3}{(x-1)^3}$, $f''(x) = \dfrac{2(x+5)}{(x-1)^4}$.

(a) (4pts) Find vertical and horizontal asymptotes of the graph of $f(x)$, if there are any.

$$\lim_{x\to+\infty} \frac{x+1}{(x-1)^2} = \lim_{x\to\infty} \frac{1+1/x}{x(1-1/x)^2} = 0$$

So HA at $y=0$.

$$\lim_{x\to-\infty} \frac{x+1}{(x-1)^2} = \lim_{x\to-\infty} \frac{1+1/x}{x(1-1/x)^2} = 0.$$

So HA at $y=0$.

$$\lim_{x\to 1} \frac{x+1}{(x-1)^2} = \frac{2}{0^2} = +\infty \rightarrow \text{VA at } x=1.$$

(b) (4pts) Find intervals on which $f(x)$ is increasing and intervals on which it is decreasing. Find critical numbers of $f(x)$, if there are any.

$f'(x)=0$ or $f'(x) \nexists \rightarrow x=-3$ & $x=1$ are CN.

x		-3		1	
$f'(x)$	$-$	0	$+$	$\nexists$	$-$
$f(x)$	$\searrow$		$\nearrow$	$\nexists$	$\searrow$

- f increasing on $(-3,1)$
- f decreasing on $(-\infty,-3)$ & $(1,\infty)$.

(c) (2pts) Find x-coordinates of local minima and local maxima of $f(x)$, if any exist.

By the first derivative test, there is a local minimum at $x=-3$.

There are no local maximum.

Problem continues on the next page...

(d) (4pts) Find intervals on which $f(x)$ is concave up and intervals on which it is concave down. Find x-coordinates of inflection points of the graph of $f(x)$, if there are any.

$f''(x) = 0$ or $f''(x)$ $\nexists$ $\to$ $x = -5$ & $x = 1$.

x		-5		1	
$f''(x)$	$-$	0	$+$	$\nexists$	$+$
f	$\frown$		$\smile$		$\smile$

- f is concave down on $(-\infty, -5)$ & $(1, \infty)$
- f is concave up on $(-5, 1)$.

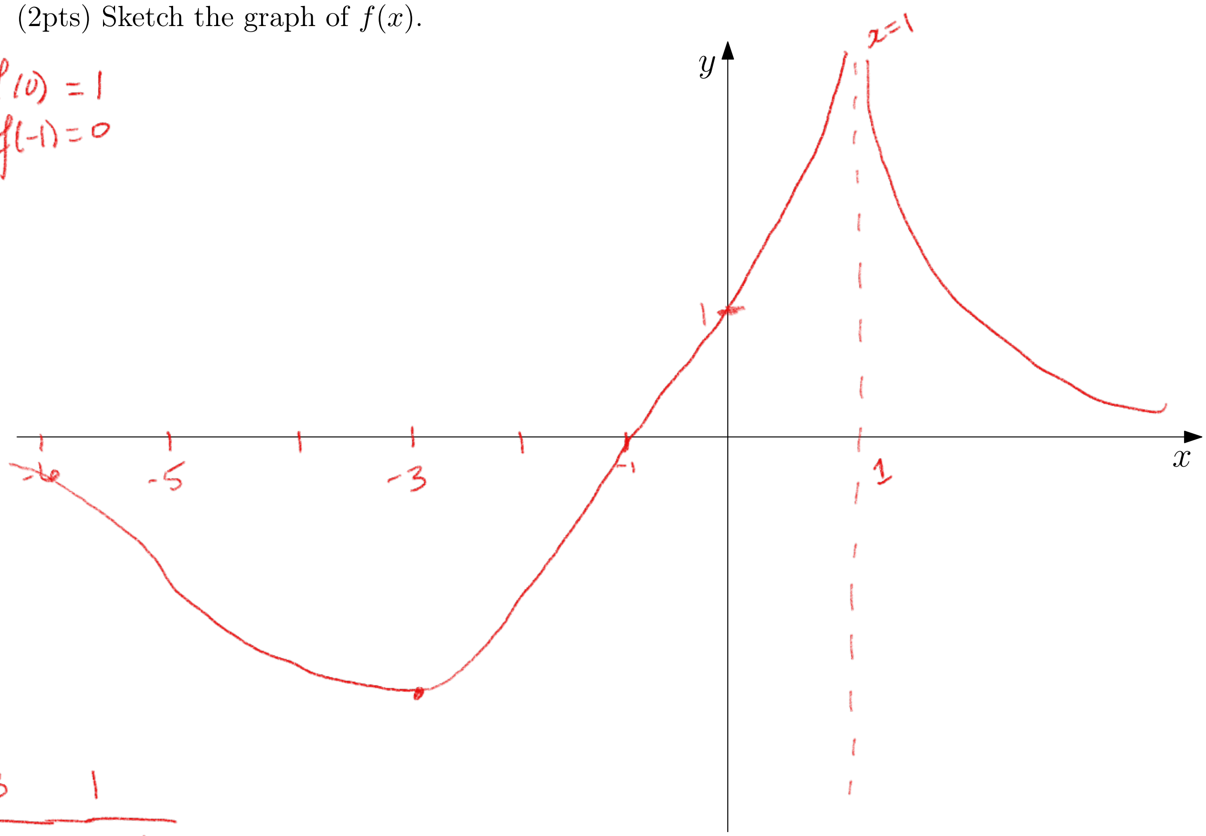
(e) (2pts) Find the minimum value of $f(x)$ on the interval $[2, 5]$.

Since $f(x)$ is decreasing, the minimum is at $x = 5$.

So, $f(5) = \boxed{\frac{5}{16}}$

(f) (2pts) Sketch the graph of $f(x)$.

$f(0) = 1$
$f(-1) = 0$



x		-5		-3		1	
f'	$\searrow$		$\searrow$		$\nearrow$		$\searrow$
f''	$\frown$		$\smile$		$\smile$		$\smile$
f	$\searrow$		$\searrow$		$\nearrow$		

11. Evaluate the following integrals.

(a) (5pts) $\int_0^{\frac{\pi}{2}} 4\sin^2 x \cos x dx$

$u = \sin x$

$du = \cos x dx$

$\frac{\pi}{2} \mapsto 1$

$0 \mapsto 0$

$$\to \int_0^{\pi/2} 4\sin^2 x \cos x \, dx$$

$$= \int_0^1 4u^2 du$$

$$= \frac{4}{3} u^3 \Big|_0^1 = \boxed{\frac{4}{3}}$$

(b) (5pts) $\int \frac{6x + 6x^2}{\sqrt{3x^2 + 2x^3}} dx = I$

$u = 3x^2 + 2x^3$

$du = (6x + 6x^2) dx$

$$\to I = \int \frac{du}{\sqrt{u}}$$

$$= 2u^{1/2} + C$$

$$= 2(3x^2 + 2x^3)^{1/2} + C$$

(c) (5pts) $\int \left(5x^{\frac{2}{3}} + 4x(1 - x^2) - \frac{2}{\sqrt[3]{x}}\right) dx$

$$\int 5x^{2/3} + 4x - 4x^3 - x^{-1/3} \, dx$$

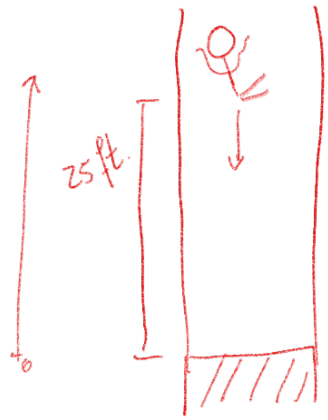
$$= 5\int x^{2/3} dx + 4\int x \, dx - 4\int x^3 dx - \int x^{-1/3} dx$$

$$= 5 \frac{x^{5/3}}{5/3} + 4\frac{x^2}{2} - 4\frac{x^4}{4} - \frac{x^{2/3}}{2/3} + C$$

$$= 3x^{5/3} + 2x^2 - x^4 - \frac{3}{2}x^{2/3} + C$$

12. In Lewis Carroll's book Alice's Adventures in Wonderland, there is a part where Alice falls down a well, but fortunately she slows down as she is falling. Suppose that our reference time point is such that at $t = 0$ Alice is 25 ft above the bottom of the well. Assume also that her speed at that moment is 10 ft/sec, and is decreasing at the rate of 2 ft/sec^2 from that moment on.

(a) (4pts) Find the function that describes Alice's position at time t for $t \geq 0$.



$t=0 \to 25\,ft = s_0$

$-10\,ft/sec = v_0$

$2\,ft/sec^2 = a$

Acceleration.

$a(t) = 2$

Velocity.

$v(t) = C + 2t$

$\Rightarrow -10 = C$

$\Rightarrow v(t) = -10 + 2t$

Position.

$s(t) = -10t + t^2 + C$

Now, $s(0) = 25 \Rightarrow C = 25$.

Thus, $s(t) = t^2 - 10t + 25$

(b) (2pts) After how many seconds will Alice reach the bottom of the well?

$s(t) = 0 \iff t^2 - 10t + 25 = 0$

$\iff (t-5)^2 = 0$

$\iff t = 5$.

Alice with reach the bottom after 5 sec.

(c) (2pts) What will Alice's speed be when she reaches the bottom?

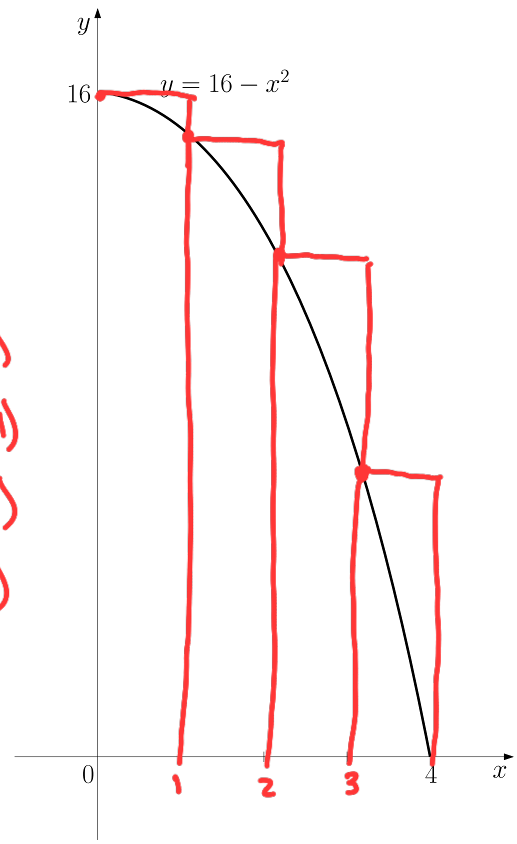
We have $v(t) = s'(t) = 2t - 10$

$\Rightarrow v(5) = 10 - 10 = 0$.

Alice's speed when she reaches the bottom is 0 ft/sec.

13. Consider the function $f(x) = 16 - x^2$ pictured below.

(a) (6pts) Estimate $\int_0^4 f(x)dx$ with a Riemann sum using four subintervals of equal width and left endpoints.



$n = 4$

$\Delta x = \frac{4-0}{4} = 1$

$x_1 = 0$, $x_2 = 1$

$x_3 = 2$, $x_4 = 3$.

$$\int_0^4 f(x)dx \approx \Delta x f(x_1) + \Delta x f(x_2) + \Delta x f(x_3) + \Delta x f(x_4)$$

$$= 1\cdot(16-0^2) + 1\cdot(16-1^2) + 1\cdot(16-4) + 1\cdot(16-9)$$

$$= 16 + 15 + 12 + 5$$

$$= \boxed{48}$$

(b) (2pts) Sketch the rectangles that you used in part (a) on the provided graph.

(c) (2pts) Find the exact value of $\int_0^4 f(x)dx$.

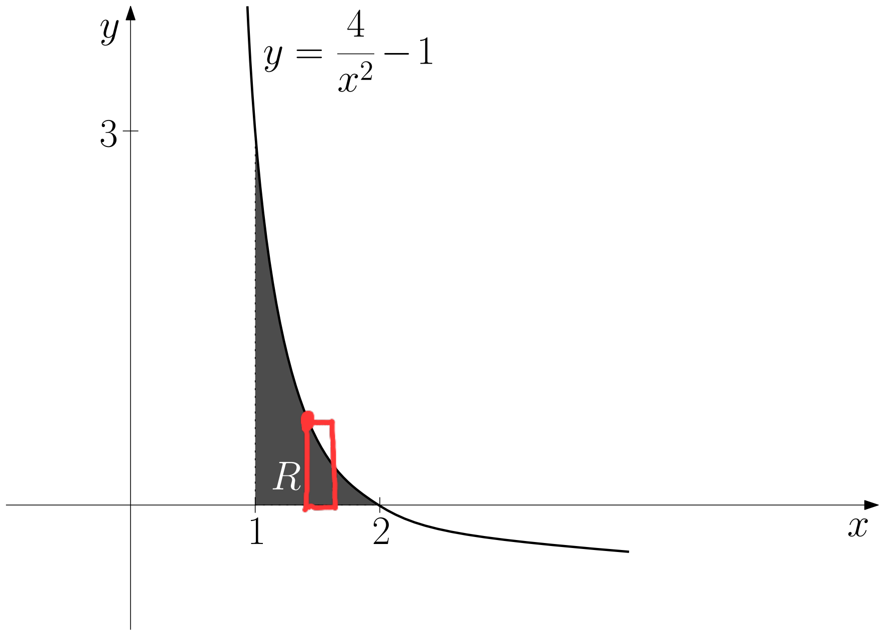
$$\int_0^4 f(x)dx = \int_0^4 16 - x^2 \, dx = 16x - \frac{x^3}{3}\Big|_0^4$$

$$= 64 - \frac{64}{3} - 0$$

$$= \frac{192 - 64}{3}$$

$$= \boxed{\frac{128}{3}}$$

14. Consider the region R bounded by the x-axis, the curve $y = \frac{4}{x^2} - 1$, and the line $x = 1$.



(a) (6pts) Find the area of the region R.

$f(x) = \frac{4}{x^2} - 1$

$g(x) = 0.$

$$A(R) = \int_1^2 \left(\frac{4}{x^2} - 1\right) - 0 \, dx$$

$$= \int_1^2 \frac{4}{x^2} - 1 \, dx$$

$$= -4x^{-1}\Big|_1^2 - x\Big|_1^2$$

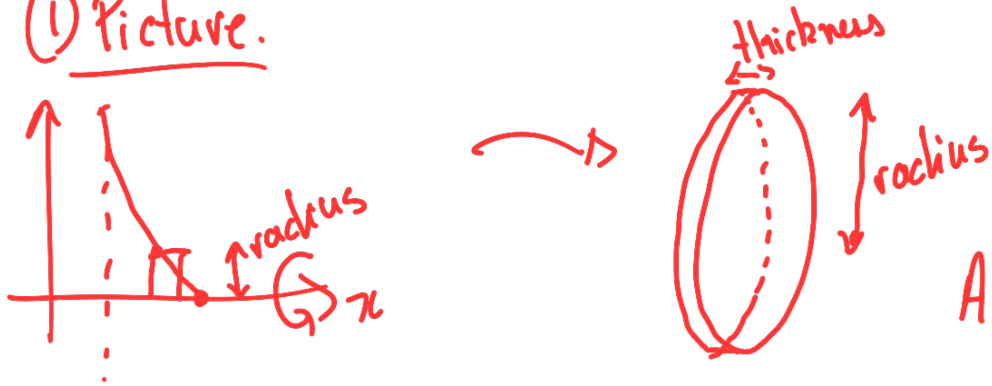
$$= -\frac{4}{2} + \frac{4}{1} - 2 + 1$$

$$= \boxed{1}$$

Problem continues on the next page...

(b) (4pts) Set up **but do not evaluate** the integral that gives the volume of the solid obtained by revolving the region R about the x-axis.

① Picture.



radius = y

thickness = dx

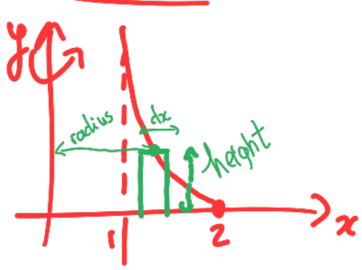
$A(x) = \pi y^2 = \pi\left(\frac{4}{x^2} - 1\right)^2$

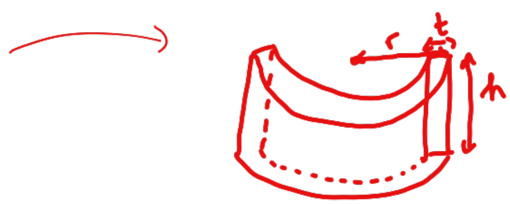
② Volume.

$$V = \int_1^2 A(x)\,dx = \int_1^2 \pi\left(\frac{4}{x^2} - 1\right)^2 dx$$

(c) (4pts) Set up **but do not evaluate** the integral that gives the volume of the solid obtained by revolving the region R about the y-axis.

① Picture.





$r = x$

$h = y$

$t = dx$

② Volume.

$$V = \int_1^2 2\pi\,(\text{radius})(\text{height})\ \text{thickness}$$

$$= \int_1^2 2\pi\, x\left(\frac{4}{x^2} - 1\right) dx$$