Math 244

Chapter 16

Section 16.9: Divergence Theorem

Contents

Created by: Pierre-Olivier Parisé
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Divergence in 3D

Definition 1. If $\vec{F} = \langle P, Q, R \rangle$ is a vector field in 3D, then

$$\operatorname{div} \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Another way to write curl $\vec{F}$ is as followed. Define

$$\vec{\nabla} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \quad \Longrightarrow \quad \operatorname{div} \vec{F} = \vec{\nabla} \cdot \vec{F}.$$

Example 1. Find the divergence of $\vec{F} = \langle xz, xyz, -y^2 \rangle$.

Solution.

$$\begin{aligned}
\operatorname{div} \vec{F} &= \vec{\nabla} \cdot \vec{F} \\
&= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle xz, xyz, -y^2 \rangle \\
&= \frac{\partial (xz)}{\partial x} + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(-y^2) \\
&= z + xz + 0 \\
&= \boxed{z + xz}
\end{aligned}$$

THEOREM 1. Let $\vec{F} = \langle P, Q, R\rangle$ and assume P, Q, R have continuous second partial derivatives. Then

$$\operatorname{div}(\operatorname{curl}\vec{F}) = 0. \qquad \left(\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{F}\right)\right)$$

EXAMPLE 2. Show that $\vec{F}(x, y, z) = \langle xz, xyz, -y^2\rangle$ can't be written as the curl of some other vector field.

SOLUTION.

By contradiction, assume that there is a vector field $\vec{G}$ such that

$$\vec{F} = \operatorname{curl}\vec{G}.$$

By thm. 1,

$$\operatorname{div}\vec{F} = \operatorname{div}(\operatorname{curl}\vec{G}) = 0.$$

But,

$$\operatorname{div}\vec{F} = z + xz \neq 0.$$

This is a contradiction!

So, $\vec{F} \neq \operatorname{curl}\vec{G}$, for any $\vec{G}$. □

Theorem 2. Assume

- S be a closed surface with positive orientation (outward orientation).
- $\vec{F} = \langle P, Q, R\rangle$ with P, Q, R having continuous partial derivatives.

Then,

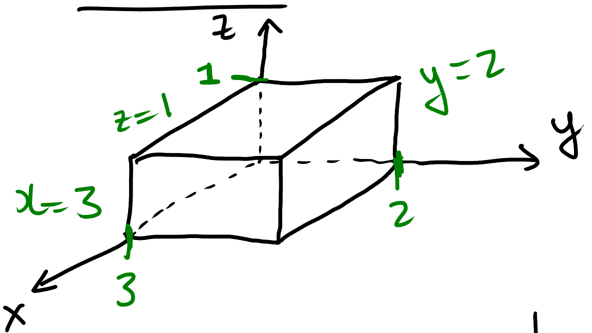
$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\,dV,$$

where E is the solid bounded by S.

Example 3. Let $\vec{F}(x,y,z) = \langle xye^z, xy^2z^3, -ye^z\rangle$ and S is the surface of the box bounded by the coordinates planes and the planes $x = 3$, $y = 2$, and $z = 1$. Compute the flux of $\vec{F}$ across S.

Solution.

Picture



$$E = \{(x,y,z) : 0 \le x \le 3,\ 0 \le y \le 2,\ 0 \le z \le 1\}$$

$$\operatorname{div}\vec{F} = 2xyz^3$$

Divergence Theorem

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} \, dV$$

$$= \int_0^1 \int_0^2 \int_0^3 2xyz^3 \, dx \, dy \, dz$$

$$= \left(\int_0^3 2x \, dx \right) \left(\int_0^2 y \, dy \right) \left(\int_0^1 z^3 \, dz \right)$$

$$= 9 \cdot 2 \cdot \frac{1}{4}$$

$$= \boxed{\frac{9}{2}}$$