Section 16.3, Problem 2 **(5 Pts)**

At $t = 0$, we have $(x, y) = (1, 0)$ and at $t = 1$, we have $(x, y) = (2, 2)$. Therefore, by the Fundamental Theorem of line integral, we get

$$\int_C \vec{\nabla} f \cdot d\vec{r} = f(2, 2) - f(1, 0).$$

From the table, $f(2, 2) = 9$ adn $f(1, 0) = 3$. Hence,

$$\int_C \vec{\nabla} f \cdot d\vec{r} = 9 - 3 = 6.$$

Section 16.3, Problem 6 **(5 Pts)**

We have

$$Q_x - P_y = e^x - e^x = 0.$$

Therefore, $\vec{F}$ is conservative. We want to find f such that $\vec{\nabla} f = \vec{F}$. We write $\vec{\nabla} f = \langle f_x, f_y \rangle$, so that

$$f_x = ye^x \quad \text{and} \quad f_y = e^x + e^y.$$

Integrating the first equation with respect to x, we get

$$f(x, y) = ye^x + c(y),$$

where $c(y)$ is a function of y. We now take the derivative with respect to y and use the second equation:

$$f_y = e^x + c'(y) = e^x + e^y.$$

Simplifying, we get $c'(y) = e^y$, so that $c(y) = e^y + c$, where c is a constant. Therefore,

$$f(x, y) = ye^x + e^y + c.$$

Section 16.3, Problem 12 **(10 Pts)**

We first notice that

$$Q_x - P_y = 4xy - 4xy = 0.$$

Therefore $\vec{F}$ is conservative. We now find a potential function for $\vec{F}$. We set $\vec{\nabla} f = \vec{F}$, so that

$$f_x = 3 + 2xy^2 \quad \text{and} \quad f_y = 2x^2 y.$$

Integrating the first equation with respect to x, we get

$$f(x, y) = 3x + x^2 y^2 + c(y).$$

Then, differentiating with respect to y and using the second equation, we get

$$f_y = 2x^2y + c'(y) = 2x^2y.$$

This implies that $c'(y) = 0$, so that $c(y) = c$, a constant. Hence,

$$f(x, y) = 3x + x^2y^2 + c.$$

We can apply the fundamental theorem for line integrals. We have

$$\int_C \vec{F} \cdot d\vec{r} = f(4, 1/4) - f(1, 1) = 9.$$

Section 16.3, Problem 26 **(5 Pts)**

The vector field is conservative because it does not seem to have a tendency to rotate about a point.

Section 16.4, Problem 6 **(5 Pts)**

The curve C bounds a region D which can be described as followed:

$$D = \{(x, y) : 0 \leq x \leq 2, x/2 \leq y \leq 1\}.$$

Using Green's Theorem, we have

$$\int_C \vec{F} \cdot d\vec{r} = \iint_D Q_x - P_y \, dA.$$

We have

$$Q_x - P_y = 2x - 2y = 2(x - y)$$

and hence

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^2 \int_{x/2}^1 2(x - y) \, dydx = 0.$$

Section 16.4, Problem 8 **(5 Pts)**

We have $Q_x - P_y = 2y^3 - 4y^3 = -2y^3$. Using Green's Theorem, we have

$$\int_C \vec{F} \cdot d\vec{r} = -\iint_D 2y^3 \, dA.$$

where D is the interior of the ellipse. We can rewrite the ellipse as

$$\left(\frac{x}{\sqrt{2}}\right)^2 + y^2 = 1.$$

We therefore use the transformation $T(r, \theta) = (\sqrt{2}r\cos\theta, r\sin\theta)$, for $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$. The Jacobian of this transformation is

$$\begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \sqrt{2}\cos\theta & -\sqrt{2}r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = \sqrt{2}r.$$

Therefore, we get

$$-\iint_D 2y^3\,dA = -2\sqrt{2}\int_0^{2\pi}\int_0^1 r^4\sin^3(\theta)\,drd\theta = 0.$$

Section 16.4, Problem 14 **(5 Pts)**

We have $P = \sqrt{x^2+1}$ and $Q = \tan^{-1}(x)$, so that

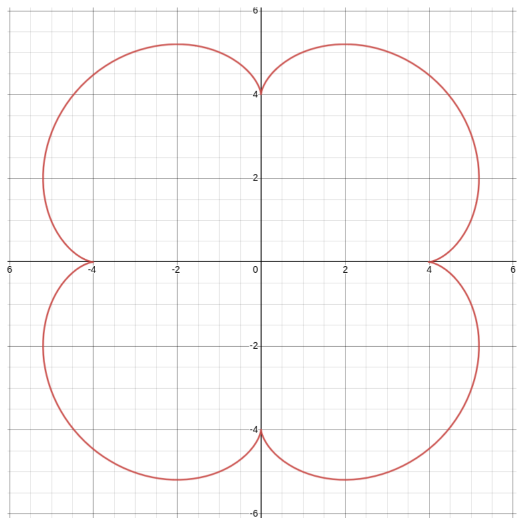
$$Q_x - P_y = \frac{1}{1+x^2} - 0 = \frac{1}{1+x^2}.$$

Therefore, by Green's Theorem,

$$\int_C \vec{F}\cdot d\vec{r} = \iint_D \frac{1}{1+x^2}\,dA = \int_0^1\int_x^1 \frac{1}{1+x^2}\,dydx = \frac{\pi}{4} - \frac{\ln(2)}{2}.$$

Section 16.4, Problem 20 **(5 Pts)**

Here is a picture of the epicycloid, obtained using Desmos:



Using the formulas for the area, we find that

$$\text{Area(D)} = \int_C xdy$$

We have

$$dy = y'(t)\,dt = (5\cos(t) - 5\cos(5t))\,dt.$$

Hence,

$$\begin{aligned}\text{Area}(D) &= \int_0^{2\pi}(5\cos(t)-\cos(5t))(5\cos(t)-5\cos(5t))\,dt\\ &= 5\int_0^{2\pi} 5\cos^2(t) - 6\cos(t)\cos(5t) + \cos^2(5t)\,dt\\ &= 30\pi.\end{aligned}$$

Section 16.4, Problem 21 **(5 Pts)**

a) Parametrize the line segment as $\vec{r}(t) = \langle x_1 + (x_2 - x_1)t, y_1 + (y_2 - y_1)t \rangle$. Then, we have

$$\begin{aligned}
\int_C x dy - y dx &= \int_0^1 \langle -(y_1 + (y_2 - y_1)t, x_1 + (x_2 - x_1)t \rangle \cdot \langle (x_2 - x_1), (y_2 - y_1) \rangle \, dt \\
&= \int_0^1 -y_1(x_2 - x_1) - (y_2 - y_1)(x_2 - x_1)t + x_1(y_2 - y_1) + (x_2 - x_1)(y_2 - y_1)t \, dt \\
&= \int_0^1 -y_1(x_2 - x_1) + x_1(y_2 - y_1) \, dt \\
&= -y_1 x_2 + y_1 x_1 + x_1 y_2 - x_1 y_1 \\
&= x_1 y_2 - x_2 y_1.
\end{aligned}$$

b) Let C_1, C_2, ..., C_{n-1}, C_n represent the sides of the polygon. For example, C_1 is the segment joining (x_1, y_1) to (x_2, y_2) and C_n is the segment joining (x_n, y_n) to (x_1, y_1). Let $C = C_1 \cup C_2 \cup \cdots \cup C_n$ be the boundary of the polygon.

A formula for the area of the polygon D is

$$\text{Area}(D) = \frac{1}{2} \int_C x dy - y dx = \frac{1}{2} \sum_{j=1}^{n} \int_{C_j} x dy - y dx.$$

On each segment C_j, with $1 \leq j < n$, we have

$$\int_{C_j} x dy - y dx = x_j y_{j+1} - x_{j+1} y_j$$

and

$$\int_{C_n} x dy - y dx = x_n y_1 - x_1 y_n.$$

Plugging that in the equation for the area of the polygon, we get

$$\text{Area}(D) = \frac{1}{2} \Big((x_1 y_2 - x_2 y_1) + (x_2 y_3 - x_3 y_2) + \ldots + (x_{n-1} y_n - x_n y_{n-1}) + (x_n y_1 - x_1 y_n) \Big).$$

TOTAL: 50 Pts.