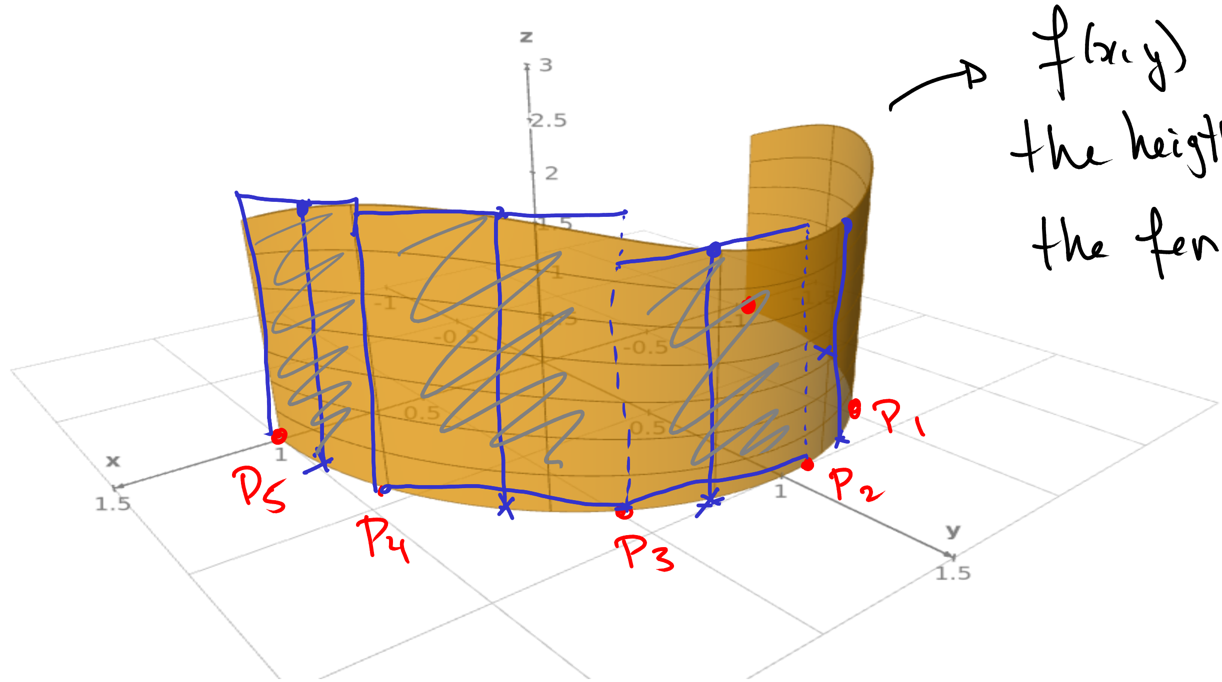
Chapter 16

Vector Calculus

16.2 Line Integrals

Line integrals in 2D.

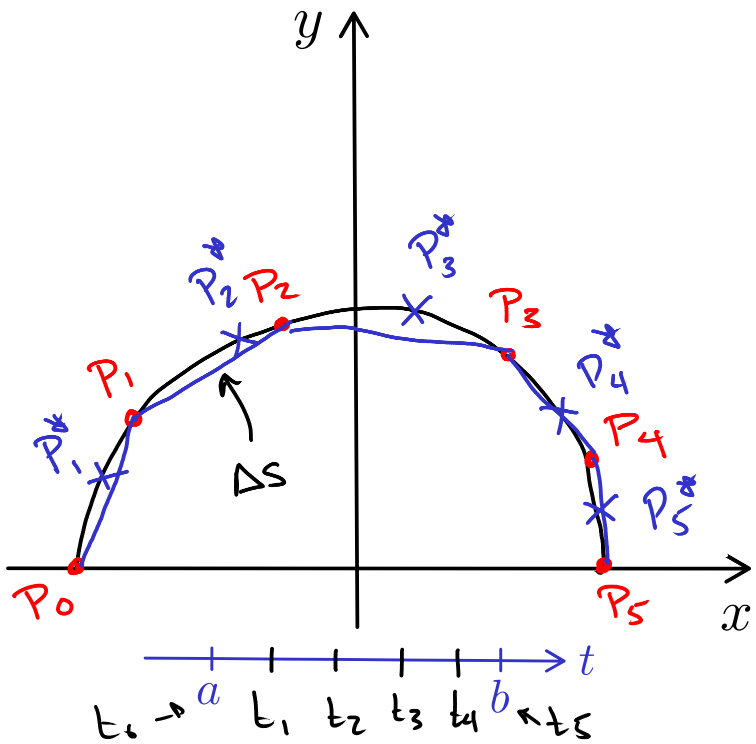
What is the area of the following fancy fence?



$f(x,y)$ is the heigth of the fence.

One way to find it:

- $C : \vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$
- Divide $[a, b]$ in n parts $[t_{i-1}, t_i]$.
- Set $P_i = \vec{r}(t_i)$.
- The P_i's divide C in n segments of length Δs_i.
- Pick a point $P_i^* = (x_i^*, y_i^*)$ between P_{i-1} and P_i.
- Sum each contribution to find:



$$\text{Area} \approx \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta s_i$$

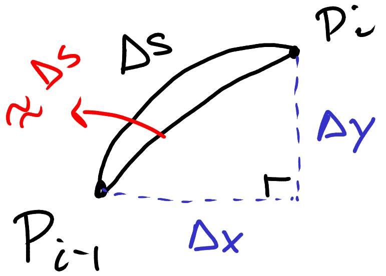
2 Definition If f is defined on a smooth curve C given by Equations 1, then the **line integral of f along C** is

arc length diff.

$$\int_C f(x, y)\, ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta s_i$$

if this limit exists.

Useful Expression



$$\Delta s \approx \sqrt{\Delta x^2 + \Delta y^2}$$

$$\Rightarrow \Delta s \approx \frac{\Delta t}{\Delta t}\sqrt{\Delta x^2 + \Delta y^2}$$

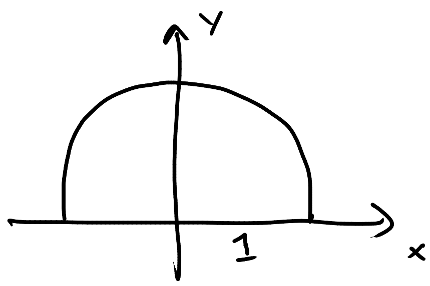
$$\Rightarrow \Delta s \approx \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}\,\Delta t$$

Let $n \to \infty$

$$\boxed{\int_C f(x,y)ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt}$$

EXAMPLE 1 Evaluate $\int_C (2 + x^2y)\,ds$, where C is the upper half of the unit circle $x^2 + y^2 = 1$. $\rightarrow$ $y = \sqrt{1 - x^2}$

① Parametrization



$$\vec{r}(t) = \langle t, \sqrt{1-t^2} \rangle \quad -1 \le t \le 1.$$

$$\vec{r}(\theta) = \langle \cos\theta, \sin\theta \rangle \quad 0 \le \theta \le \pi.$$

$$\frac{dx}{d\theta} = -\sin\theta \qquad \frac{dy}{d\theta} = \cos\theta$$

② Integrate

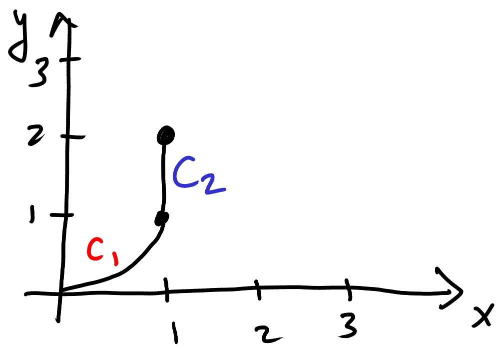
$$\int_C 2 + x^2y\,ds = \int_0^\pi (2 + \cos^2\theta\sin\theta)\left((-\sin\theta)^2 + (\cos\theta)^2\right)^{1/2} d\theta$$

$$= \int_0^\pi (2 + \cos^2\theta\sin\theta)\sqrt{1}\,d\theta$$

$$= \int_0^\pi 2 + \cos^2\theta\sin\theta\,d\theta = \boxed{2\pi + \frac{2}{3}}$$

EXAMPLE 2 Evaluate $\int_C 2x\,ds$, where C consists of the arc C_1 of the parabola $y = x^2$ from (0, 0) to (1, 1) followed by the vertical line segment C_2 from (1, 1) to (1, 2).

① Parametrization



$$C = C_1 \sqcup C_2$$

$$C_1 : \vec{r}_1(t) = \langle t, t^2 \rangle, \quad 0 \le t \le 1$$

$$C_2 : \vec{r}_2(t) = \langle 1, 1+t \rangle, \quad 0 \le t \le 1$$

② Integrate.

$$\int_C 2x\,ds = \int_{C_1} 2x\,ds + \int_{C_2} 2x\,ds$$

$$= \int_0^1 2t\sqrt{1+(2t)^2}\,dt + \int_0^1 2\sqrt{0^2+1^2}\,dt$$

$$= \int_0^1 2t\sqrt{1+4t^2}\,dt + \int_0^1 2\,dt$$

$u = 1+4t^2$
$du = 8t\,dt$

$$= \int_1^5 \sqrt{u}\,\frac{du}{4} + 2$$

$$= \frac{\sqrt{125}-1}{6} + 2 = \boxed{\frac{5\sqrt{5}-1}{6} + 2}$$

Remark:

When a curve can be divided into separate different curves, then

$$\int_C f(x,y)\,ds = \int_{C_1} f(x,y)ds + \int_{C_2} f(x,y)\,ds$$

Mass and center of mass. SKIPPED.

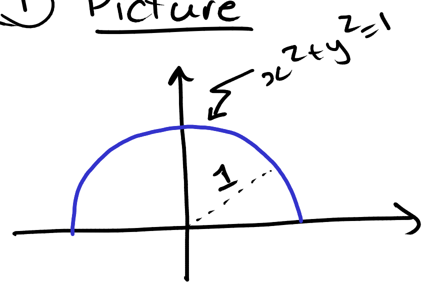
EXAMPLE 3 A wire takes the shape of the semicircle $x^2 + y^2 = 1$, $y \geqslant 0$, and is thicker near its base than near the top. Find the center of mass of the wire if the linear density at any point is proportional to its distance from the line $y = 1$.

center of mass $(\overline{x}, \overline{y})$

$$\overline{x} = \frac{1}{m}\int_C x\rho(x,y)\,ds$$

$$\overline{y} = \frac{1}{m}\int_C y\rho(x,y)\,ds$$

① Picture



$\rho(x,y) = k(1-y)$

$\vec{r}(t) = \langle \cos t, \sin t \rangle$

$0 \le t \le \pi$.

So, $ds = \sqrt{(-\sin t)^2 + (\cos t)^2}\,dt = dt$

② Mass $m = \int_C \rho(x,y)\,ds = \int_0^{\pi} k(1-\sin t)\,dt = k(\pi-2)$

③ Center of mass

$$\overline{x} = \frac{1}{k(\pi-2)}\int_C x\,k(1-y)\,ds = \frac{1}{\pi-2}\int_0^{\pi} \cos t\,(1-\sin t)\,dt$$

$$= 0 \qquad \text{(symmetry)}.$$

$$\overline{y} = \frac{1}{k(\pi-2)}\int_C y\,k(1-y)\,ds = \frac{1}{\pi-2}\int_0^{\pi} \sin t\,(1-\sin t)\,dt$$

$$= \frac{2-\pi/2}{\pi-2} \approx 0.3760$$

Therefore,

$$\boxed{\text{center of mass} = (0, 0.3760)}$$

Line integrals in Space.

What is the average temperature of this wire, if the temperature at each point is given by T = f(x, y, z)? (Here, we assume the wire is normalized, meaning $T = f(x, y, z)/l(C)$ where $l(C)$ is the total length of the curve.)

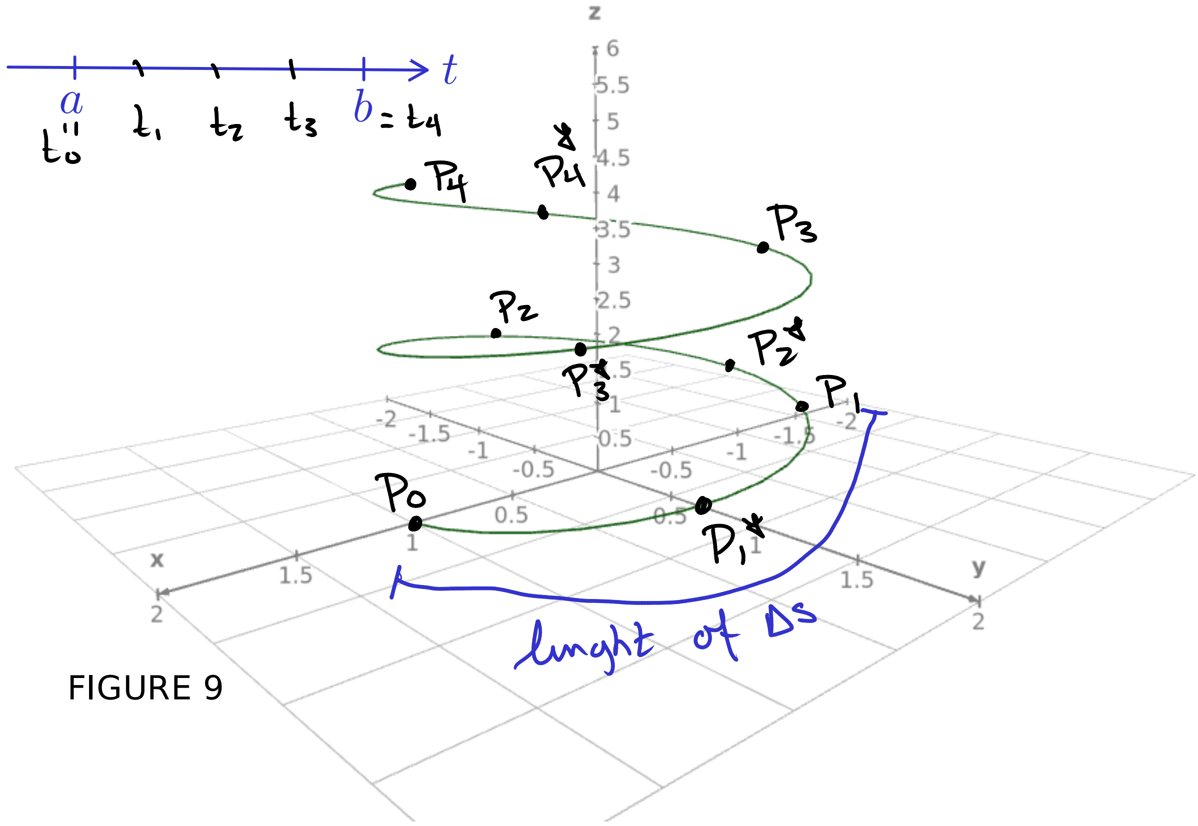


FIGURE 9

Data:

- Divide $[a, b]$ in n equal parts.
- Set $P_i = (x_i, y_i, z_i) = \vec{r}(t_i)$. $= \langle x(t_i), y(t_i), z(t_i) \rangle$
- Select a point $P_i^* = (x_i^*, y_i^*, z_i^*)$ on each segment.

For a small segment of length Δs, the approximate temperature is

$$f(x_i^*, y_i^*, z_i^*)\Delta s \quad /l(C)$$

Summing the contribution of all segments:

$$\text{Average Temperature} \approx \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\underline{\Delta s} \quad /l(C)$$

Letting $n \to \infty$:

$$T = \boxed{\int_C f(x, y, z)\, \underline{ds} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\Delta s} \quad /l(C)$$

Useful Formula for the path integral

$$\boxed{\int_C f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt}$$

EXAMPLE 5 Evaluate $\int_C y \sin z \, ds$, where C is the circular helix given by the equations $x = \cos t$, $y = \sin t$, $z = t$, $0 \leqslant t \leqslant 2\pi$. (See Figure 9.)

Here $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$, $0 \le t \le 2\pi$.

So $\frac{dx}{dt} = -\sin t$, $\frac{dy}{dt} = \cos t$ $\frac{dz}{dt} = 1$

$$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2} = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2}$$

$$= \sqrt{2}$$

So,

$$\int_C y \sin z \, ds = \int_0^{2\pi} \sin t \, \sin t \, \sqrt{2} \, dt$$

$$= \int_0^{2\pi} \sin^2 t \, \sqrt{2} \, dt \quad \left(\text{use } \sin^2 t = \frac{1-\cos 2t}{2} \right)$$

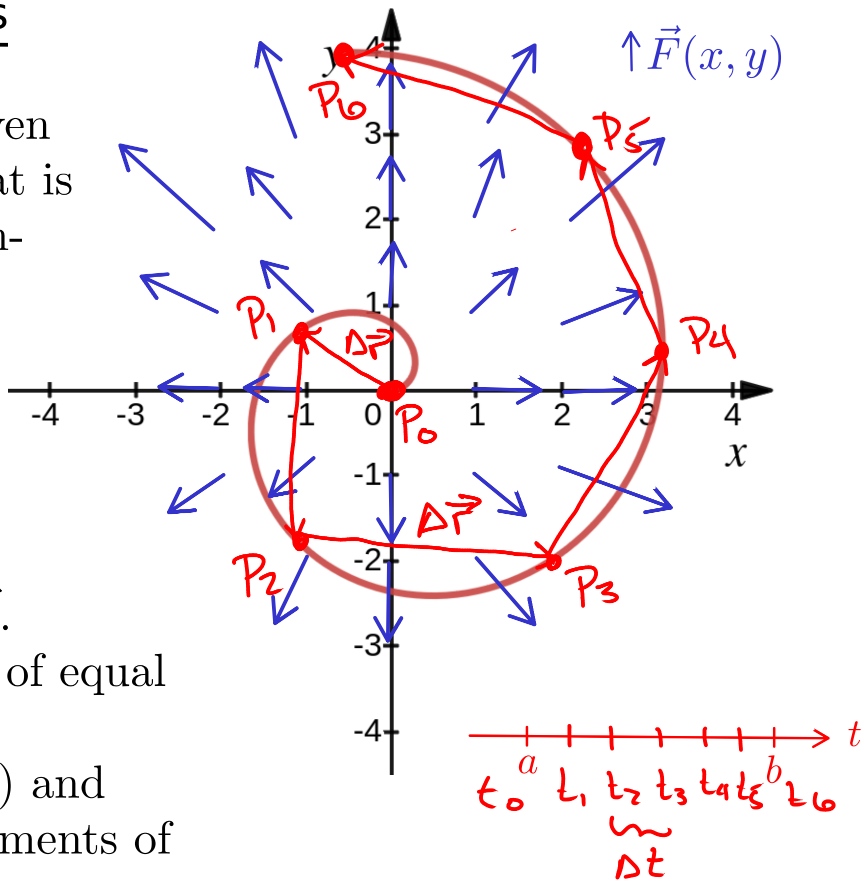
$$= \boxed{\sqrt{2}\,\pi}$$

When a curve C is the combination of multiple curves:

$$\int_C f(x,y,z)\,ds = \int_{C_1} f(x,y,z)\,ds + \int_{C_2} f(x,y,z)\,ds$$

Line integrals of Vector Fields

If the current of a river is given by a vector field $\vec{F}(x, y)$, what is the work done by a dog swimming on a spiral path?



Data:

- $C : \vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$.
- Divide $[a, b]$ in n parts of equal size Δt.
- Let $P_i = \vec{r}(t_i) = (x_i, y_i)$ and C is divided into n segments of direction $\Delta\vec{r}$.

Approximation of the work (W):

- The work W_i done on each segment is: $\approx \vec{F}(x_i,y_i) \bullet \Delta\vec{r}$

  $[\vec{r}(t_1) - \vec{r}(t_2)]/\Delta t \approx \vec{r}\,'(t_1)$
- But, $\Delta\vec{r} \approx \vec{r}\,'(t_i)\Delta t$. Therefore,

  $w_i \approx \vec{F}(x_i,y_i) \bullet \vec{r}\,'(t_i)\,\Delta t$
- Summing all the contribution:

  $$W \approx \sum_{i=1}^{n} w_i \approx \sum_{i=1}^{n} [\vec{F}(x_i,y_i) \bullet \vec{r}\,'(t_i)]\,\Delta t \,.$$
- Letting $n \to \infty$:

  $$\int_C \vec{F} \bullet d\vec{r}$$

$$W = \boxed{\lim_{n\to\infty} \sum_{i=1}^{n} \vec{F}(\vec{r}(t_i)) \cdot \vec{r}\,'(t_i)\Delta t = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt}$$

Remark: Exactly the same formula for 3D vector fields.

EXAMPLE. Find the work done by the force field $\vec{F}(x,y) = x\vec{i} + y\vec{j}$ along the path C given by

$$\vec{r}(t) = t\cos(2t)\vec{i} + t\sin(2t)\vec{j} \quad (0 \le t \le 4).$$

We have $\vec{r}\,'(t) = \langle \cos 2t - 2t\sin(2t), \sin 2t + 2t\cos 2t \rangle$.

So,

$$W = \int_C \vec{F} \cdot d\vec{r}$$

$$= \int_0^4 \langle t\cos 2t, t\sin 2t \rangle \cdot \langle \cos 2t - 2t\sin(2t), \sin 2t + 2t\cos 2t \rangle \, dt$$

$$= \int_0^4 t\cos^2 2t - 2t^2 \cancel{\cos 2t \sin 2t} + t\sin^2 2t + 2t^2 \cancel{\cos 2t \sin 2t} \, dt$$

$$= \int_0^4 t \, dt = \boxed{8}$$

Remark:

- Another notation used for the integral of a vector field along a path C is: Writing $d\vec{r} = \vec{r}\,'(t)\,dt$

$$\Rightarrow \int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F} \cdot \vec{r}\,'(t) \, dt$$

- Two other path integrals: If $\vec{r}\,'(t) = \langle x'(t), y'(t) \rangle$ and $\vec{F} = \langle P, Q \rangle$

  - w.r.t x: $\int_C P \, dx = \int_a^b P(x(t), y(t))\, x'(t) \, dt$

  - w.r.t. y: $\int_C Q \, dy = \int_a^b Q(x(t), y(t))\, y'(t) \, dt$.

So,

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P\,dx + Q\,dy \quad \left(= \int_C P\,dx + \int_C Q\,dy \right)$$

EXAMPLE. Find the work done by the force field $\vec{F}(x,y,z) = xy\vec{i} + yz\vec{j} + zx\vec{k}$ along the path C given by

$$x = t, \quad y = t^2, \quad z = t^3 \qquad (\underset{a}{0} \le t \le \underset{b}{1}).$$

$$\vec{r}\,'(t) = \langle 1, 2t, 3t^2 \rangle$$

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_0^1 \langle t^3, t^5, t^4 \rangle \cdot \langle 1, 2t, 3t^2 \rangle \, dt$$

$$= \int_0^1 t^3 + 2t^6 + 3t^6 \; dt$$

$$= \int_0^1 t^3 + 5t^6 \; dt$$

$$= \boxed{\frac{27}{21}} \approx 1.27\ldots$$

Remark:

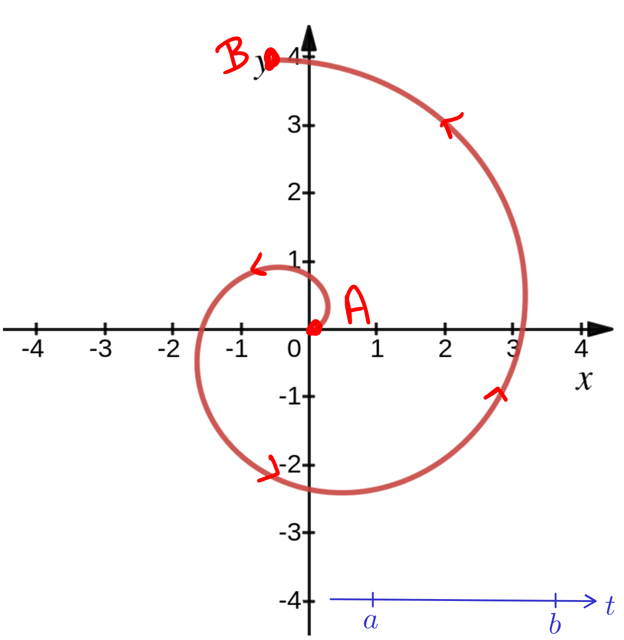
- Three other path integrals: $\vec{F} = \langle P, Q, R \rangle$, $\vec{r}\,'(t) = \langle x', y', z' \rangle$

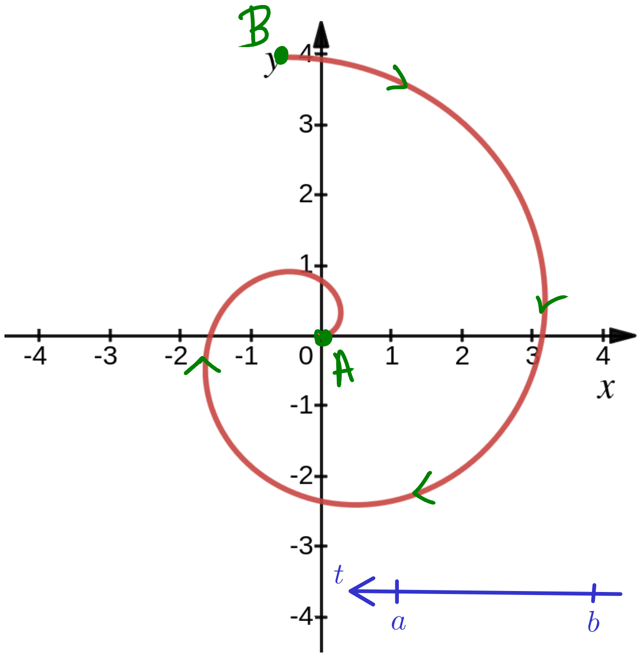
w.r.t. x : $\displaystyle\int_C P\,dx = \int_a^b P(x(t), y(t), z(t))\, x'(t)\, dt$

w.r.t. y : $\displaystyle\int_C Q\,dy = \int_a^b Q(x(t), y(t), z(t))\, y'(t)\, dt$

w.r.t. z : $\displaystyle\int_C R\,dz = \int_a^b R(x(t), y(t), z(t))\, z'(t)\, dt.$

Orientation





$C: \vec{r}(t) = \langle x(t), y(t) \rangle$

$\vec{r}(a) = A$, $\vec{r}(b) = B$

$-C: \vec{r}(-u)$ $\quad a < b$

where $-b \leq u \leq -a$

$$\boxed{\int_{-C} \vec{F} \cdot d\vec{r} = -\int_{C} \vec{F} \cdot d\vec{r}.}$$

What about the other type of path integrals?

- Line integral of scalar valued functions:

$$\boxed{\int_{-C} f\, ds = \int_{C} f\, ds.}$$

- Line integrals coming from vector valued functions:

$$\int_{-C} f\, dx = -\int_{C} f\, dx \qquad \int_{-C} f\, dy = -\int_{C} f\, dy$$

$$\int_{-C} f\, dz = -\int_{C} f\, dz$$