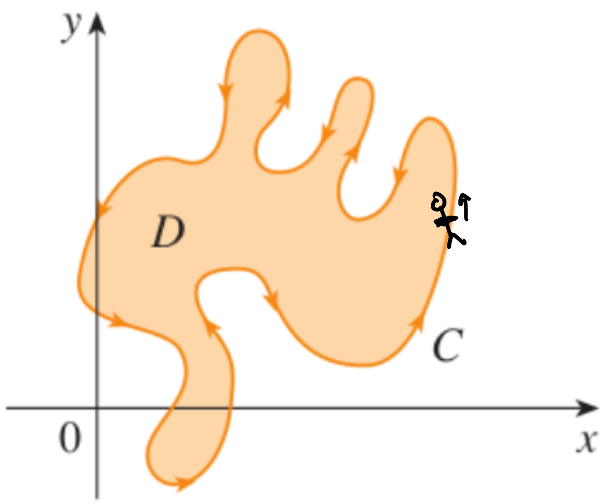
Chapter 16

Vector Calculus

16.4 Green's Theorem

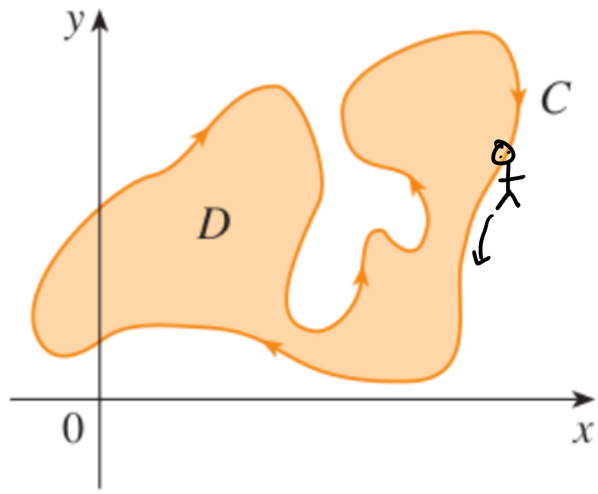
Orientation of closed curves



(a) Positive orientation

Domain D is always on the left.

Syn: counter clockwise



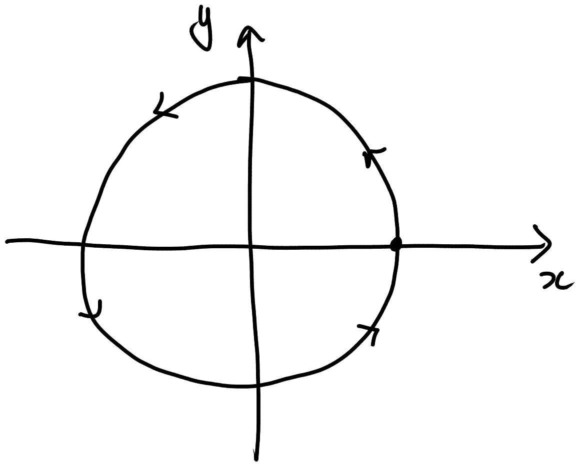
(b) Negative orientation

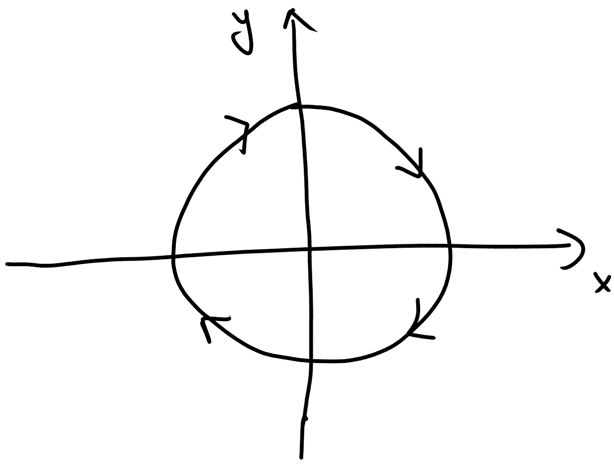
Domain D is always on the right.

Syn.: clockwise.

EXAMPLE.

Give a parametrization of the positively oriented circle of radius 1 centered at the origin. Find a parametrization giving the negative orientation?





Positive orientation:

$\vec{r}(t) = \langle \cos t, \sin t \rangle$

$0 \le t \le 2\pi$

Negative Orientation:

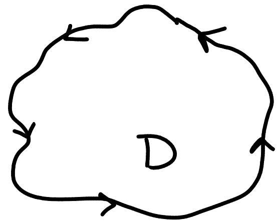
$\vec{r}(t) = \langle \cos(-t), \sin(-t) \rangle$

$= \langle \cos t, -\sin t \rangle$.

$-2\pi \le t \le 0$

Green's Theorem.

C : closed path with positive orientation.
D : region bounded by C.
If $\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$, with P, Q continuously differentiable, then



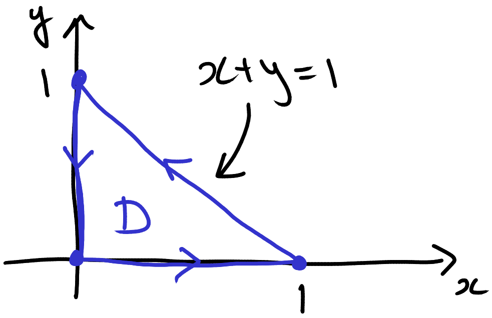
$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

Remarks:

- The symbol $\oint_C$ means the path has a positive orientation.
- The left-hand side measures how $\vec{F}$ follows the direction of C.
- The right-hand side measures the tendency of $\vec{F}$ to rotate in the direction of C in the region enclosed by it.

EXAMPLE 1 Evaluate $\oint_C x^4\,dx + xy\,dy$, where C is the triangular curve consisting of the line segments from $(0, 0)$ to $(1, 0)$, from $(1, 0)$ to $(0, 1)$, and from $(0, 1)$ to $(0, 0)$.

① Picture



② Green's Theorem

$D = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 1-x\}$

So

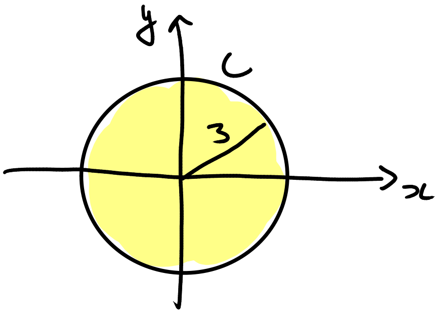
$$\oint_C \vec{F}\cdot d\vec{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA.$$

Here, $P = x^4$ $Q = xy$. So

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^1\int_0^{1-x} y - 0\ dy\,dx = \boxed{\frac{1}{2}}.$$

EXAMPLE 2 Evaluate $\oint_C \overset{P}{(3y - e^{\sin x})}\, dx + \overset{Q}{\left(7x + \sqrt{y^4 + 1}\right)}\, dy$, where C is the circle $x^2 + y^2 = 9$.

① Picture



$$D = \left\{ (r,\theta) : \begin{array}{l} 0 \le r \le 3 \\ 0 \le \theta \le 2\pi \end{array} \right\}$$

② Green's Theorem.

$$Q_x - P_y = 7 - 3 = 4$$

$$\oint_C \vec{F} \bullet d\vec{r} = \iint_D 4 \, dA$$

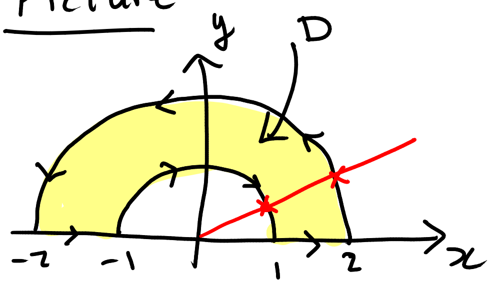
$$= 4 \, \text{Area}(D)$$

$$= 4 \, \pi \, 3^2$$

$$= \boxed{36\pi}$$

EXAMPLE 4 Evaluate $\oint_C y^2\, dx + 3xy\, dy$, where C is the boundary of the semiannular region D in the upper half-plane between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

① Picture



$$Q_x - P_y = y$$

$$D = \{ (r,\theta) : 1 \le r \le 2, \; 0 \le \theta \le \pi \}.$$

② Green's Theorem. $\vec{F} = \langle P, Q \rangle$.

$$\oint_C \vec{F} \bullet d\vec{r} = \iint_D y \, dA$$

$$= \int_0^{\pi} \int_1^2 r \sin\theta \; r\, dr\, d\theta$$

$$= \left(\int_1^2 r^2 \, dr \right) \left(\int_0^{\pi} \sin\theta \, d\theta \right)$$

$$= \frac{7}{3} \cdot 2 = \boxed{\frac{14}{3}}$$

Computing Areas with Green's Theorem

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D Q_x - P_y \, dA$$

Recall: $A(D) = \iint_D 1 \, dA$

① $A(D) = \oint_C x \, dy.$

Set $Q = x$ and $P = 0$ in Green's Theorem.

② $A(D) = -\oint_C y \, dx.$

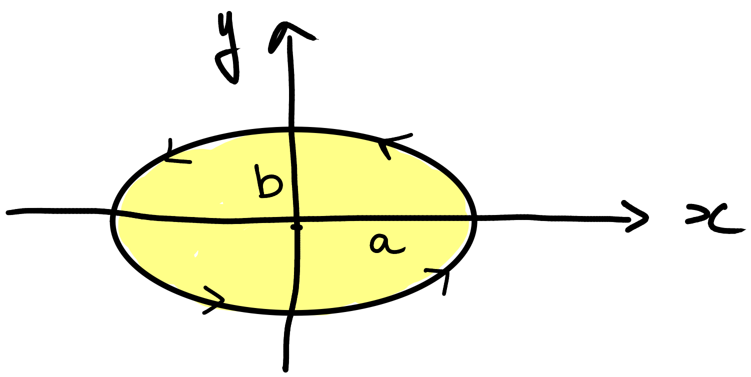
Set $Q = 0$ and $P = -y$ in Green's Theorem.

③ $A(D) = \frac{1}{2}\left(\oint_C x \, dy - y \, dx\right).$

Set $Q = \frac{x}{2}$ and $P = \frac{-y}{2}$ in Green's Theorem.

EXAMPLE 3 Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

① Picture



$\vec{r}(t) = \langle a\cos t, b\sin t \rangle$

$0 \le t \le 2\pi$

$\rightarrow \vec{r}\,'(t) = \langle \underbrace{-a\sin t}_{x'(t)}, \underbrace{b\cos t}_{y'(t)} \rangle$

② Area By formula ③

$$\text{Area}(D) = \frac{1}{2} \oint_C x\,dy - y\,dx$$

$$= \frac{1}{2} \int_0^{2\pi} a\cos t \, b\cos t - b\sin t(-a\sin t) \, dt$$

$$= \frac{1}{2} \int_0^{2\pi} ab \cos^2 t + ab \sin^2 t \, dt$$

$$= \frac{ab}{2} \int_0^{2\pi} \underbrace{\cos^2 t + \sin^2 t}_{=1} \, dt = \boxed{ab\pi}$$