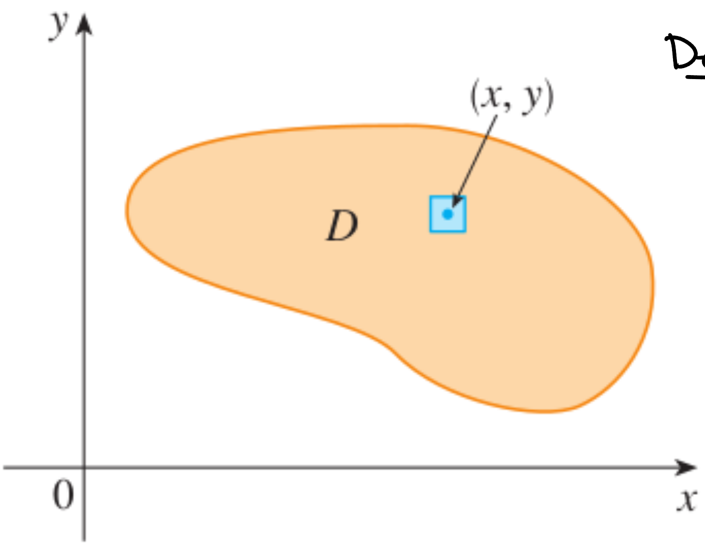
Chapter 15

Multiple Integrals

15.4 Applications of double integrals

Density, mass and charge

given mass density $\rho(x,y)$



Def: $\rho(x,y) = \lim_{\Delta A \to 0} \frac{\Delta m}{\Delta A}$

What is the total mass?

$\Delta m = \rho(x,y)\, \Delta A$

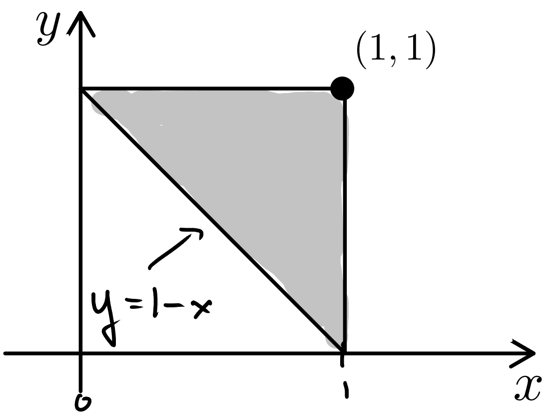
$\xrightarrow{\text{limit}}$ $dm = \rho(x,y)\, dA$

integrate to get

$$m = \iint_D \rho(x,y)\, dA$$

$$Q = \iint_D \sigma(x,y)\, dA$$

EXAMPLE. Mass is distributed over the triangular region D below. The mass density at (x,y) is $\rho(x,y) = xy$, measured in kg/m^2. Find the total mass.



Type I:

$D = \{(x,y) : 0 \le x \le 1,\ 1-x \le y \le 1\}$

$\rho(x,y) = xy$

$$m = \iint_D xy\, dA = \int_0^1 \int_{1-x}^1 xy\, dy\, dx$$

$$= \boxed{\frac{5}{24} \text{ kg}} \approx 0.2083 \text{ kg}.$$

Moments and center of mass.

Moment about the x-axis

$$M_x = \iint_D y\rho(x,y)\,dA$$

Moment about the y-axis

$$M_y = \iint_D x\rho(x,y)\,dA$$

EXAMPLE. Find the moments about the x-axis and y-axis for the lamina from the previous example.

$\rho(x,y) = xy$ and $D = \{(x,y) : 0 \le x \le 1,\ 1-x \le y \le 1\}$.

① $M_x = \iint_D y(xy)\,dA = \int_0^1 \int_{1-x}^1 xy^2\,dy\,dx$

$= \boxed{\frac{3}{20}} = 0.15$

② $M_y = \iint_D x(xy)\,dA = \int_0^1 \int_{1-x}^1 x^2 y\,dy\,dx$

$= \boxed{\frac{3}{20}} = 0.15$

Center of mass $(\overline{x}, \overline{y})$

$$\overline{x} = \frac{M_y}{m} \quad \text{and} \quad \overline{y} = \frac{M_x}{m}$$

EXAMPLE. Find the center of mass for the lamina in the previous examples.

$\rho(x,y) = xy$ and $D = \{(x,y) : 0 \le x \le 1,\ 1-x \le y \le 1\}$.

$m = \frac{5}{24}$, $M_x = M_y = \frac{3}{20}$

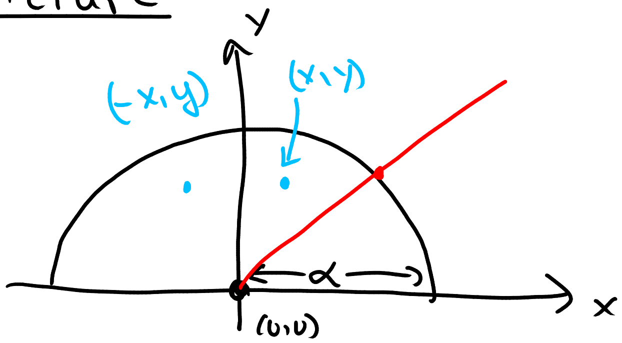
① $\overline{x} = \frac{M_y}{m} = \frac{3/20}{5/24} = \frac{3 \cdot \cancel{4} \cdot 6}{5 \cdot \cancel{4} \cdot 5} = \frac{18}{25}$

② $\overline{y} = \frac{M_x}{m} = \frac{3/20}{5/24} = \frac{18}{25}$.

Center of mass: $\boxed{(\overline{x}, \overline{y}) = \left(\frac{18}{25}, \frac{18}{25}\right)}$.

EXAMPLE 3 The density at any point on a semicircular lamina is proportional to the distance from the center of the circle. Find the center of mass of the lamina.

① Picture



$$D = \left\{ (x,y) : \sqrt{x^2+y^2} \le a,\ y \ge 0 \right\}$$

$$\rho(x,y) = k\sqrt{x^2+y^2}$$

$$x\sqrt{x^2+y^2} \ @ (x,y)$$
$$-x\sqrt{x^2+y^2} \ @ (-x,y)$$

② Mass.

$$m = \iint_D \rho(x,y)\,dA$$
$$= \iint_D k\sqrt{x^2+y^2}\,dA$$
$$= \int_0^{\pi}\int_0^{\alpha} k\,r\ r\,dr\,d\theta$$
$$= \int_0^{\pi}\int_0^{\alpha} k r^2\,dr\,d\theta = \frac{k\alpha^3\pi}{3}$$

③ Moments

$$M_y = \iint_D x\left(k\sqrt{x^2+y^2}\right)dA = 0 \quad \text{(by symmetry).}$$

$$M_x = \iint_D y\left(k\sqrt{x^2+y^2}\right)dA$$
$$= \int_0^{\pi}\int_0^{\alpha} r\sin\theta\,(k\,r)\ r\,dr\,d\theta$$
$$= \int_0^{\pi}\int_0^{\alpha} k r^3 \sin\theta\,dr\,d\theta = k\left(\int_0^{\pi}\sin\theta\,d\theta\right)\left(\int_0^{\alpha} r^3\,dr\right)$$
$$= k\cdot 2\cdot\frac{\alpha^4}{4} = k\frac{\alpha^4}{2}.$$

So, $$(\bar{x},\bar{y}) = \left(\frac{0}{\frac{k\alpha^3\pi}{3}},\ \frac{\cancel{k}\alpha^{\cancel{4}}/2}{\cancel{k}\cancel{\alpha^3}\pi/3}\right) = \boxed{\left(0, \frac{3\alpha}{2\pi}\right)}$$

Moment of Inertia.

Inertia about the x-axis

$$\boxed{I_x = \iint_D y^2 \rho(x,y)\, dA}$$

Inertia about the y-axis

$$\boxed{I_y = \iint_D x^2 \rho(x,y)\, dA}$$

Inertia about the origin

$$\boxed{I_0 = \iint_D (x^2 + y^2) \rho(x,y)\, dA}$$