Chapter 15

Multiple Integrals

15.9 Change of variables in multiple integrals

Change of variable from Calculus I

If $x = g(u)$, then

$$\int_a^b f(x)\,dx = \int_c^d f(g(u))g'(u)\,du$$

where $a = g(c)$ and $b = g(d)$.

Change of Variable in polar coordinate.

If $x = r\cos\theta$ and $y = r\sin\theta$, then

$$\iint_D f(x,y)\,dA = \iint_S f(r\cos\theta, r\sin\theta)r\,drd\theta$$

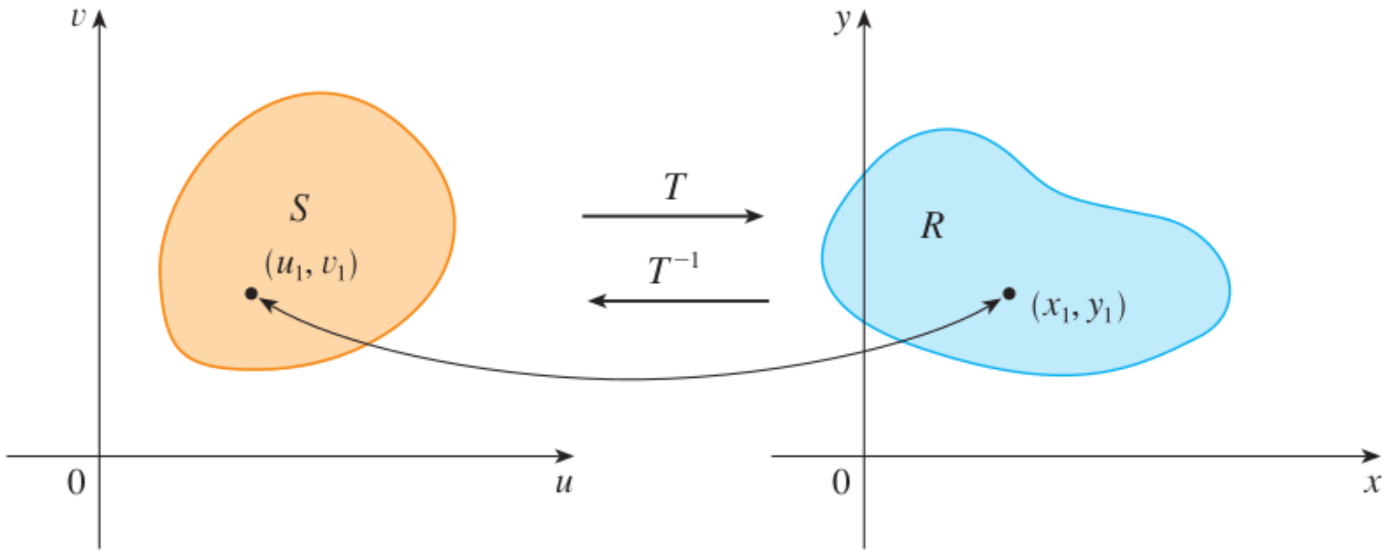
where R is a region in the xy-plane and
S is a region in the $r\theta$-plane.

General transformation in 2D.

EXAMPLE 1 A transformation is defined by the equations

$$x = u^2 - v^2 \qquad y = 2uv$$

Find the image of the square $S = \{(u, v) \mid 0 \leqslant u \leqslant 1,\ 0 \leqslant v \leqslant 1\}$.



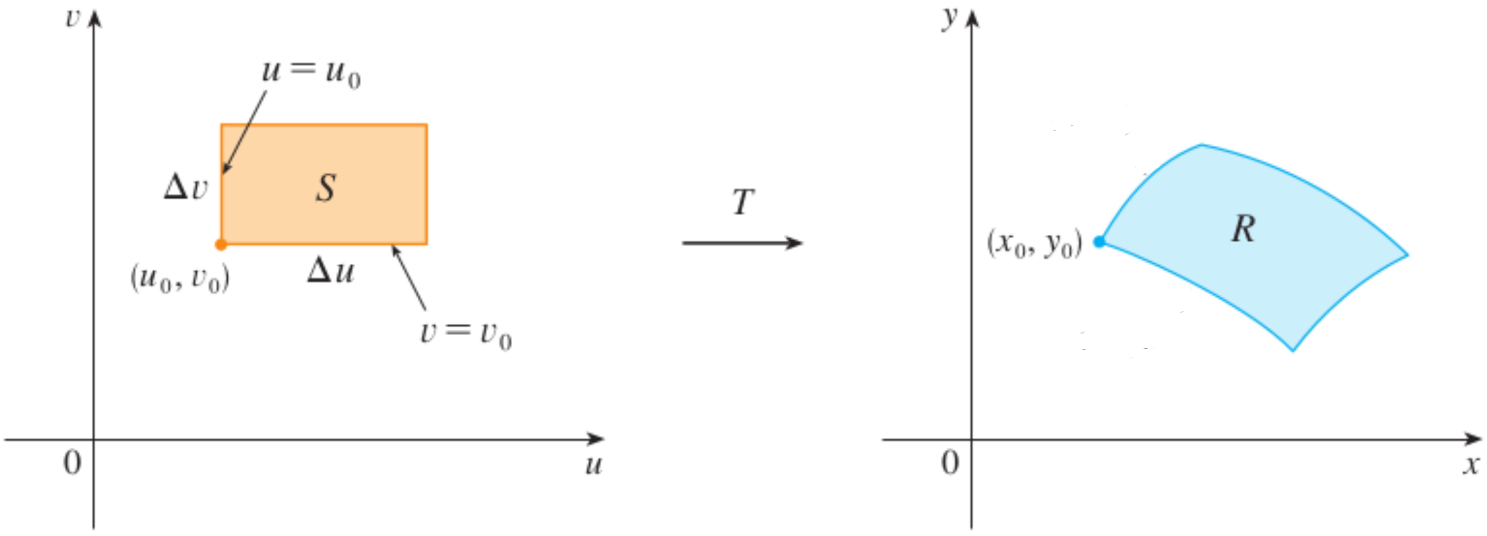
Two equations for x and y:

$$(x, y) = T(u, v) \quad \Longleftrightarrow \quad x = x(u, v) \text{ and } y = y(u, v)$$

Image: The region R is the set of possible outputs.

Domain: The region S is the set of all possible inputs.

Effect of a change of variables in double integral.



Goal: Find how dA is transformed after the transformation.

$$\boxed{dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dvdu}$$

type I

or

$$\boxed{dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv}$$

type II

Remarks:

EXAMPLE 2 Use the change of variables $x = u^2 - v^2$, $y = 2uv$ to evaluate the integral $\iint_R y\, dA$, where R is the region bounded by the x-axis and the parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$, $y \geqslant 0$.

EXAMPLE 3 Evaluate the integral $\iint_R e^{(x+y)/(x-y)}\, dA$, where R is the trapezoidal region with vertices $(1, 0)$, $(2, 0)$, $(0, -2)$, and $(0, -1)$.

Effect of change of variable in Triple integrals.

Spherical coordinates.

$$(x, y, z) = T(\rho, \theta, \phi) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)$$

This implies that

$$dV = \underbrace{\rho^2 \sin\phi}_{\text{Jacobien of the transformation.}}\, d\rho\, d\theta\, d\phi$$

Transformation in 3D:

- A function T from a region S in the uvw-space into a region R in the xyz-space.
- So $$(x, y, z) = T(u, v, w)$$ $$\Updownarrow$$ $$x = x(u, v, w),\ y = y(u, v, w) \text{ and } z = z(u, v, w)$$

Jacobian in 3D:

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

$$\boxed{\iiint\limits_R f(x, y, z)\, dV = \iiint\limits_S f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du\, dv\, dw}$$

Important fact: If $T^{-1} : R \to S$ exists, then $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} = \dfrac{1}{\frac{\partial(x,y,z)}{\partial(u,v,w)}}$

56. Use the transformation $x = u^2,\ y = v^2,\ z = w^2$ to find the volume of the region bounded by the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$ and the coordinate planes.