Chapter 16

Vector Calculus

16.6 Parametric Surfaces

Generic surfaces in 3D.

EXAMPLE. Using Python, draw the surface described by the equation

$$x^2 + y^2 + z^2 = 1.$$

① Cartesian coordinates

Let x, y be such that $x^2+y^2 \le 1$

$$\Rightarrow \quad z = \sqrt{1-x^2-y^2} \longrightarrow \vec{r}(x,y) = \langle x, y, \sqrt{1-x^2-y^2} \rangle$$

$$\text{or} \quad z = -\sqrt{1-x^2-y^2} \longrightarrow \vec{r}(x,y) = \langle x, y, -\sqrt{1-x^2-y^2} \rangle$$

② Spherical coordinates

$$x = \rho\cos\theta\sin\phi \overset{\rho=1}{=} \cos\theta\sin\phi$$

$$y = \rho\sin\theta\sin\phi = \sin\theta\sin\phi$$

$$z = \rho\cos\phi = \cos\phi$$

$$0 \le \theta \le 2\pi$$

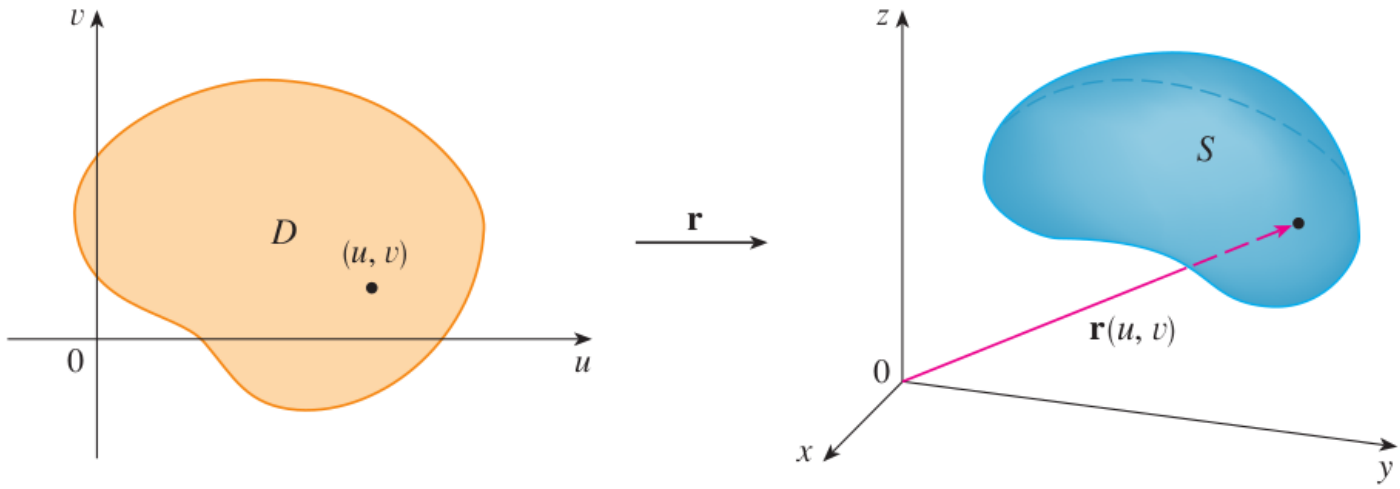
$$0 \le \phi \le \pi$$

$$\vec{r}(\theta,\phi) = \langle \cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi \rangle$$

Polar coordinates

$$\vec{r}(\rho,\theta) = \langle \rho\cos\theta, \rho\sin\theta, \sqrt{1-\rho^2} \rangle$$

$$0 \le \theta \le 2\pi, \quad 0 \le \rho \le 1.$$



$$\boxed{\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle \qquad \Big(\, (u,v) \in D \,\Big)}$$

Vector function or parametrization

3–6 Identify the surface with the given vector equation.

5. $\mathbf{r}(s,t) = \langle \underbrace{s\cos t}_{x}, \underbrace{s\sin t}_{y}, \underbrace{s}_{z} \rangle$ where $0 \le s \le 2$, and $0 \le t \le 2\pi$.

Notice that

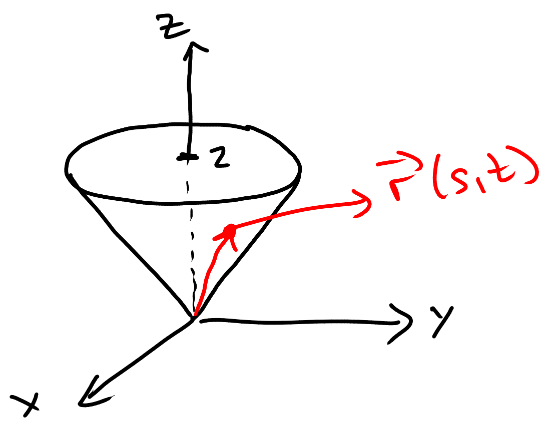
$$x^2 + y^2 = s^2\cos^2 t + s^2 \sin^2 t$$
$$= s^2 = z^2$$

So, $z^2 = x^2 + y^2 \xrightarrow[z \ge 0]{} z = \sqrt{x^2+y^2}$.

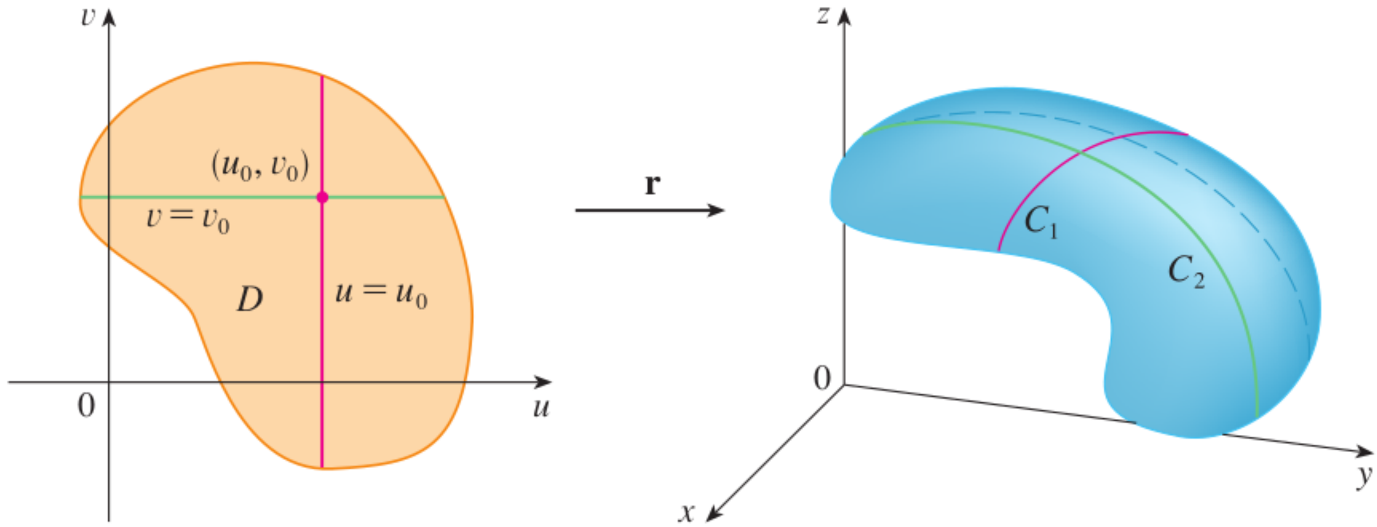
cone

(cône)

That's a cone



Grid curves.



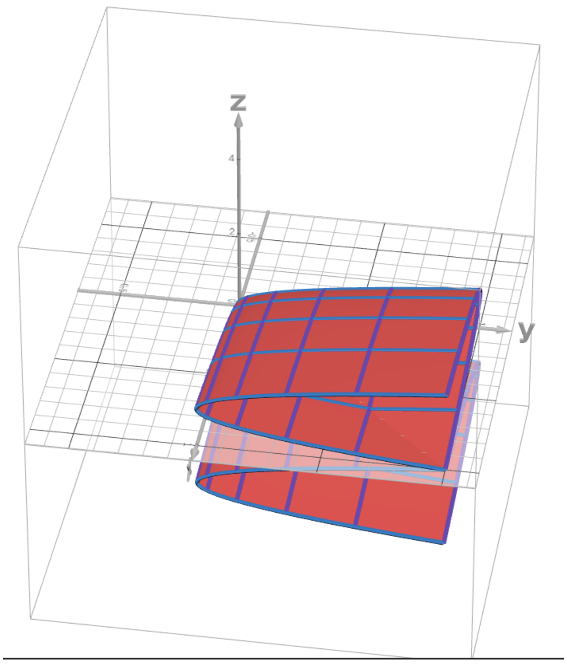
- $C_1 : \vec{r}(v) = \vec{r}(u_0, v)$
- $C_2 : \vec{r}(u) = \vec{r}(u, v_0)$

7–12 Use a computer to graph the parametric surface. Get a printout and indicate on it which grid curves have u constant and which have v constant.

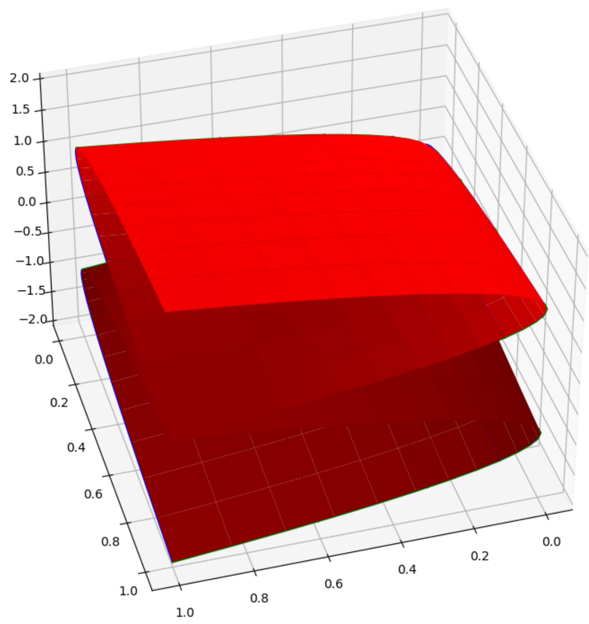
7. $\mathbf{r}(u, v) = \langle u^2, v^2, u + v \rangle$, $-1 \leqslant u \leqslant 1$, $-1 \leqslant v \leqslant 1$ https://www.desmos.com/3d/fb472f71c5

If $u = u_0$, then
$C_1 : \vec{r}(v) = \langle u_0^2, v^2, u_0 + v \rangle$

If $v = v_0$, then
$C_2 : \vec{r}(u) = \langle u^2, v_0^2, u + v_0 \rangle$

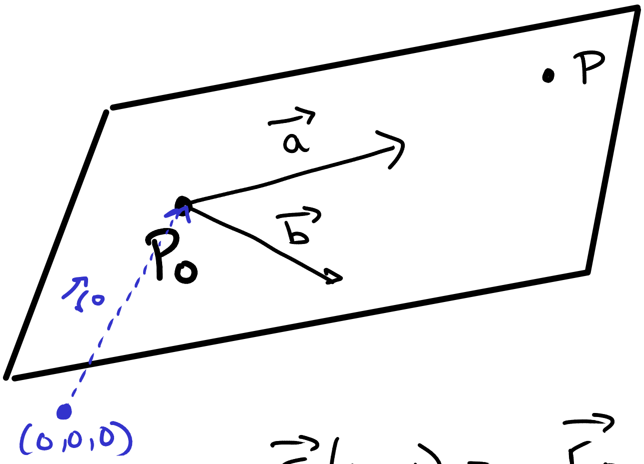


With Desmos



With Python

EXAMPLE 3 Find a vector function that represents the plane that passes through the point P_0 with position vector $\mathbf{r}_0$ and that contains two nonparallel vectors $\mathbf{a}$ and $\mathbf{b}$.



$\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$

$\vec{a} = \langle a_0, a_1, a_2 \rangle$

$\vec{b} = \langle b_0, b_1, b_2 \rangle$

$$\vec{r}(u,v) = \vec{r}_0 + u\vec{a} + v\vec{b}$$

where $-\infty < u < \infty$ and $-\infty < v < \infty$.

Also,

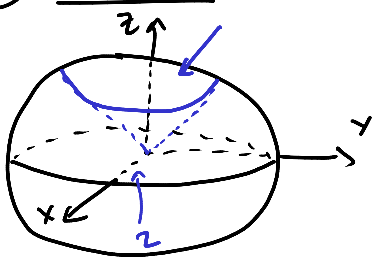
$$\vec{r}(u,v) = \langle x_0 + ua_0 + vb_0,\ y_0 + ua_1 + vb_1,\ z_0 + ua_2 + vb_2 \rangle$$

$-\infty < u < \infty$, $-\infty < v < \infty$.

19–26 Find a parametric representation for the surface.

23. The part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the cone $z = \sqrt{x^2 + y^2}$

① Picture



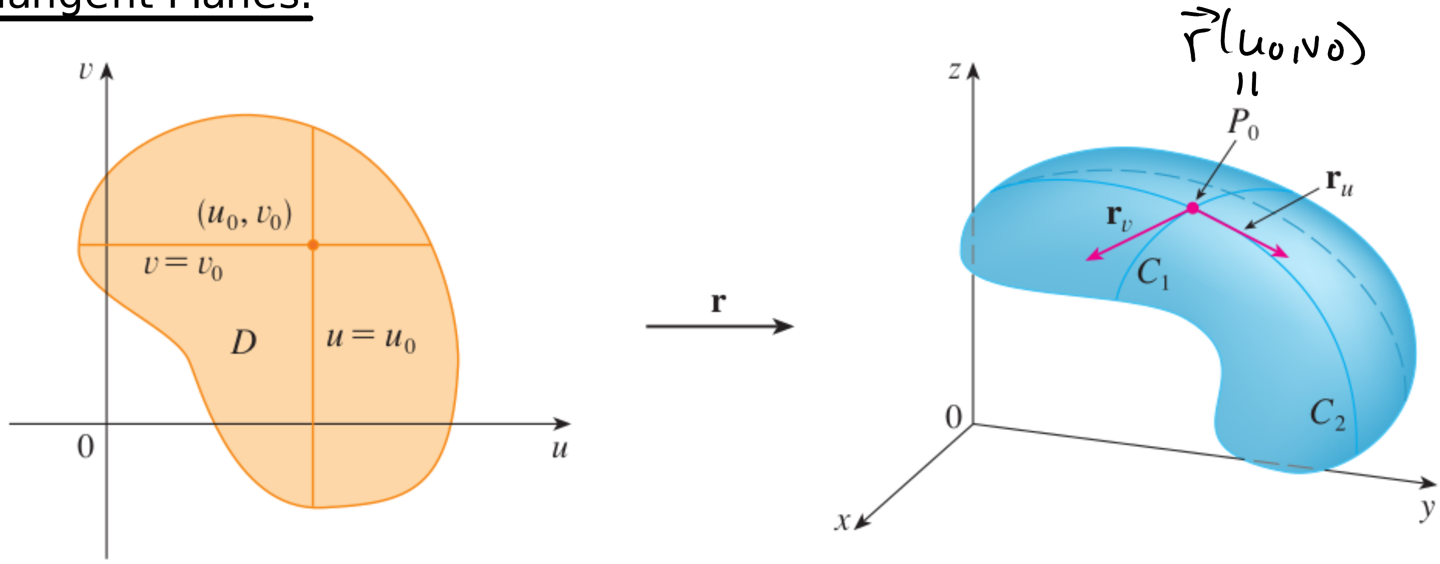
② Parametrization

A cone : $\phi = \pi/4$.

$$\vec{r}(u,v) = \langle 2\cos u \sin v,\ 2\sin u \sin v,\ 2\cos v \rangle$$

$0 \le \underset{(=\theta)}{u} \le 2\pi \qquad 0 \le \underset{(=\phi)}{v} \le \pi/4$.

Tangent Planes.



- $u = u_0$ (constant). $\vec{r}(v) = \vec{r}(u_0, v)$ represents the curve C_1. Hence, $\vec{r}_v$ is the tangent vector to C_1 at P_0.
- $v = v_0$ (constant). $\vec{r}(u) = \vec{r}(u, v_0)$ represents the curve C_2. Hence, $\vec{r}_u$ is the tangent vector to C_2 at P_0.

Equation of the tangent plane at P_0 :

$$\boxed{\vec{r}(u,v) = \langle x_0, y_0, z_0 \rangle + u\vec{r}_u(u_0, v_0) + v\vec{r}_v(u_0, v_0)}$$

where $-\infty < u < \infty, -\infty < v < \infty$.

37–38 Find an equation of the tangent plane to the given parametric surface at the specified point. Graph the surface and the tangent plane. https://www.desmos.com/3d/dfc50f1356

37. $\mathbf{r}(u, v) = u^2\,\mathbf{i} + 2u\sin v\,\mathbf{j} + u\cos v\,\mathbf{k}$; $u = 1,\ v = 0$

1 Partials

$$\vec{r}_u = \langle 2u, 2\sin v, \cos v \rangle \longrightarrow \vec{r}_u = \langle 2, 0, 1 \rangle$$

$$\vec{r}_v = \langle 0, 2u\cos v, -u\sin v \rangle \longrightarrow \vec{r}_v = \langle 0, 2, 0 \rangle$$

for $P_0 \to \vec{r}(1,0)$.

2 Tangent plane $\quad \vec{r}_0 = \langle x_0, y_0, z_0 \rangle = \langle 1, 0, 1 \rangle$

$$\Rightarrow \vec{r}(u,v) = \langle 1,0,1 \rangle + u\langle 2,0,1 \rangle + v\langle 0,2,0 \rangle$$
$$= \langle 1+2u, 2v, 1+u \rangle, \quad -\infty < u < \infty, \ -\infty < v < \infty$$