Section 16.7, Problem 6 (5 Pts)

We have

$$\begin{aligned}\vec{r}_u &= \langle \cos v, \sin v, 1\rangle \\ \vec{r}_v &= \langle -u\sin v, u\cos v, 0\rangle .\end{aligned}$$

Therefore

$$\vec{r}_u \times \vec{r}_v = \langle u\cos v, u\sin v, u\rangle \quad \Rightarrow \quad |\vec{r}_u \times \vec{r}_v| = \sqrt{2}u.$$

Hence,

$$\iint_S xyz\, dS = \iint_D u^3 \cos v \sin v\, dA = \int_0^{\pi/2}\int_0^1 \sqrt{2}u^4 \cos v \sin v\, dudv = \frac{1}{5\sqrt{2}}.$$

Section 16.7, Problem 8 (5 Pts)

We have

$$\begin{aligned}\vec{r}_u &= \langle 2v, 2u, 2u\rangle \\ \vec{r}_v &= \langle 2u, -2v, 2v\rangle .\end{aligned}$$

Therefore

$$\vec{r}_u \times \vec{r}_v = \left\langle 8uv, 4(u^2-v^2), -4(u^2+v^2)\right\rangle \quad \Rightarrow \quad |\vec{r}_u \times \vec{r}_v| = 4\sqrt{2}(u^2+v^2),$$

and

$$\begin{aligned}\iint_S (x^2+y^2)\, dS &= \iint_D (4u^2v^2 + (u^2-v^2)^2)4\sqrt{2}(u^2+v^2)\, dA \\ &= \iint_D 4\sqrt{2}(u^2+v^2)^2(u^2+v^2)\, dA \\ &= \iint_D 4\sqrt{2}(u^2+v^2)^3\, dA.\end{aligned}$$

Now,

$$D = \{(u,v)\,:\, u^2+v^2 \le 1\} = \{(r,\theta)\,:\, 0\le r\le 1,\, 0\le\theta\le 2\pi\}$$

and hence

$$\iint_S (x^2+y^2)\, dS = \int_0^{2\pi}\int_0^1 4\sqrt{2}r^7\, drd\theta = \pi\sqrt{2}.$$

Section 16.7, Problem 20 (10 Pts)

We let $S = C\cup T\cup B$, where C is the cylinder, T is the top and B the bottom. Therefore, letting $f(x,y,z) = x^2+y^2+z^2$,

$$\iint_S f(x,y,z)\, dS = \iint_C f(x,y,z)\, dS + \iint_T f(x,y,z)\, dS + \iint_B f(x,y,z)\, dS$$

A parametrization of the cylinder is $\vec{r}(u,v) = \langle 3\cos u, 3\sin u, v\rangle$, with $0 \le u \le 2\pi$ and $0 \le v \le 2$. We then find

$$\begin{aligned}\vec{r}_u &= \langle -3\sin u, 3\cos u, 0\rangle \\ \vec{r}_v &= \langle 0,0,1\rangle.\end{aligned}$$

so that

$$|\vec{r}_u \times \vec{r}_v| = 3.$$

Hence,

$$\iint_C x^2+y^2+z^2\,dS = 3\int_0^{2\pi}\int_0^2 (9\cos^2(u)+9\sin^2(u)+v^2)\,dvdu = 124\pi \approx 389.55.$$

A parametrization of the top is $\vec{r}(u,v) = \langle u\cos v, u\sin v, 2\rangle$ with $0 \le u \le 3$ and $0 \le v \le 2\pi$, so that

$$|\vec{r}_u \times \vec{r}_v| = u$$

and

$$\iint_T (x^2+y^2+z^2)\,dS = \iint_D (u^2+4)u\,dA = \int_0^3\int_0^{2\pi}(u^2+4)u\,dvdu = \frac{153\pi}{2} \approx 240.33.$$

Using $\vec{r}(u,v) = \langle u\cos v, u\sin v, 0\rangle$ as a parametrization of the bottom, we find that

$$|\vec{r}_u \times \vec{r}_v| = u$$

and

$$\iint_B (x^2+y^2+z^2)\,dS = \int_0^3\int_0^{2\pi} u^3\,dvdu = \frac{81\pi}{2} \approx 127.23.$$

Thus, our final answer is

$$\iint_S (x^2+y^2+z^2)\,dS = 124\pi + \frac{153\pi}{2} + \frac{81\pi}{2} = \frac{482\pi}{2} \approx 757.12.$$

Section 16.7, Problem 24 (5 Pts)

We parametrize the cone with $\vec{r}(x,y) = \left\langle x, y, \sqrt{x^2+y^2}\right\rangle$, with $1 \le x^2+y^2 \le 9$. Therefore

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \frac{x}{\sqrt{x^2+y^2}} \\ 0 & 1 & \frac{y}{\sqrt{x^2+y^2}} \end{vmatrix} = \left\langle -\frac{x}{\sqrt{x^2+y^2}}, -\frac{y}{\sqrt{x^2+y^2}}, 1\right\rangle$$

However, $\vec{r}_x \times \vec{r}_y$ points upwards. We then choose instead the parametrization $\vec{r}(y,x) = \left\langle x, y, \sqrt{x^2+y^2}\right\rangle$, so that

$$\vec{r}_y \times \vec{r}_x = -\vec{r}_x \times \vec{r}_y = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1\right\rangle.$$

Therefore

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \langle -x, -y, (x^2+y^2)^{3/2}\rangle \cdot \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1 \right\rangle dA$$
$$= -\iint_D \sqrt{x^2+y^2} + (x^2+y^2)^{3/2}\, dA.$$

The region

$$D = \{(x,y) : 1 \le x^2+y^2 \le 9\} = \{(r,\theta) : 1 \le r \le 3,\ 0 \le \theta \le 2\pi\}.$$

Hence,

$$\iint_S \vec{F}\cdot d\vec{S} = -\int_0^{2\pi}\int_1^3 r^2 + r^4\, dr d\theta = -\frac{1712\pi}{15} \approx -358.56.$$

Section 16.7, Problem 26 (5 Pts)

A parametrization of the hemisphere is $\vec{r}(\theta,\phi) = \langle 2\cos\theta\sin\phi, 2\sin\theta\sin\phi, 2\cos\phi\rangle$, with $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi/2$. We then obtain

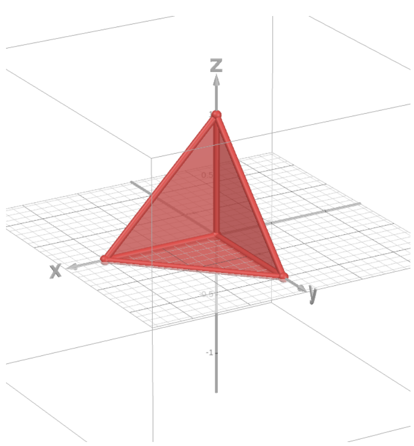
$$\vec{r}_\theta \times \vec{r}_\phi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2\sin\theta\sin\phi & 2\cos\theta\sin\phi & 0 \\ 2\cos\theta\cos\phi & 2\sin\theta\cos\phi & -2\sin\phi \end{vmatrix} = \left\langle -4\cos\theta\sin^2\phi, -4\sin\theta\sin^2\phi, -4\sin\phi\cos\phi \right\rangle$$

It points downwards (towards the origin). Therefore,

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \langle 2\sin\theta\sin\phi, -2\cos\theta\sin\phi, 4\cos(\phi)\rangle \cdot \left\langle -4\cos\theta\sin^2\phi, -4\sin\theta\sin^2\phi, -4\sin\phi\cos\phi \right\rangle dA$$
$$= \iint_D -16\sin\phi\cos^2\phi\, dA$$
$$= -16\int_0^{2\pi}\int_0^{\pi/2} \sin\phi\cos^2\phi\, d\phi d\theta$$
$$= -\frac{32}{3}\pi.$$

Section 16.7, Problem 32 (20 Pts)

Here is a picture of the tetrahedron:



Let's denote the sides in the XZ plane by S_1, in the YZ plane by S_2, in the XY plane by S_3, and the side not in any of the coordinates planes by S_4. Therefore, the tetrahedron denoted by T can be decomposed as

$$T = S_1 \cup S_2 \cup S_3 \cup S_4.$$

From the statement of the problem, T has the positive orientation (so normal vector is pointing outward).
Then, we obtain

$$\iint_T \vec{F} \cdot d\vec{S} = \sum_{j=1}^{4} \iint_{S_j} \vec{F} \cdot d\vec{S}.$$

On S_1. A parametrization of S_1 is $\vec{r}(x,z) = \langle x, 0, z\rangle$, with $0 \leq x \leq 1$ and $0 \leq z \leq 1-x$. Therefore, $\vec{r}_x \cdot \vec{r}_z = \langle 0, -1, 0\rangle$ and hence

$$\iint_{S_1} \vec{F} \cdot d\vec{S} = \iint_D \langle 0, z, x\rangle \cdot \langle 0, -1, 0\rangle \; dA = -\int_0^1 \int_0^{1-x} z \, dzdx = -\frac{1}{6}.$$

On S_2. A parametrization of S_2 is $\vec{r}(z,y) = \langle 0, y, z\rangle$, for $0 \leq z \leq 1$ and $0 \leq y \leq 1-z$. Therefore, $\vec{r}_z \times \vec{r}_y = \langle -1, 0, 0\rangle$ and hence

$$\iint_{S_2} \vec{F} \cdot d\vec{S} = \iint_D \langle y, z-y, 0\rangle \cdot \langle -1, 0, 0\rangle \; dA = -\int_0^1 \int_0^{1-z} y \, dydz = -\frac{1}{6}.$$

On S_3. A parametrization of S_3 is $\vec{r}(y,x) = \langle x, y, 0\rangle$, for $0 \leq y \leq 1$ and $0 \leq x \leq 1-y$. Therefore, $\vec{r}_y \times \vec{r}_x = \langle 0, 0, -1\rangle$ and hence

$$\iint_{S_3} \vec{F} \cdot d\vec{S} = \iint_D \langle y, -y, x\rangle \cdot \langle 0, 0, -1\rangle \; dA = -\int_0^1 \int_0^{1-y} x \, dxdy = -\frac{1}{6}.$$

On S_4. A parametrization of S_4 is $\vec{r}(x,y) = \langle x, y, 1-x-y\rangle$, for $0 \leq x \leq 1$ and $0 \leq y \leq 1-x$. We then have

$$\vec{r}_x = \langle 1, 0, -1\rangle \quad \text{and} \quad \vec{r}_y = \langle 0, 1, -1\rangle$$

so that

$$\vec{r}_x \times \vec{r}_y = \langle 1, 1, 1\rangle .$$

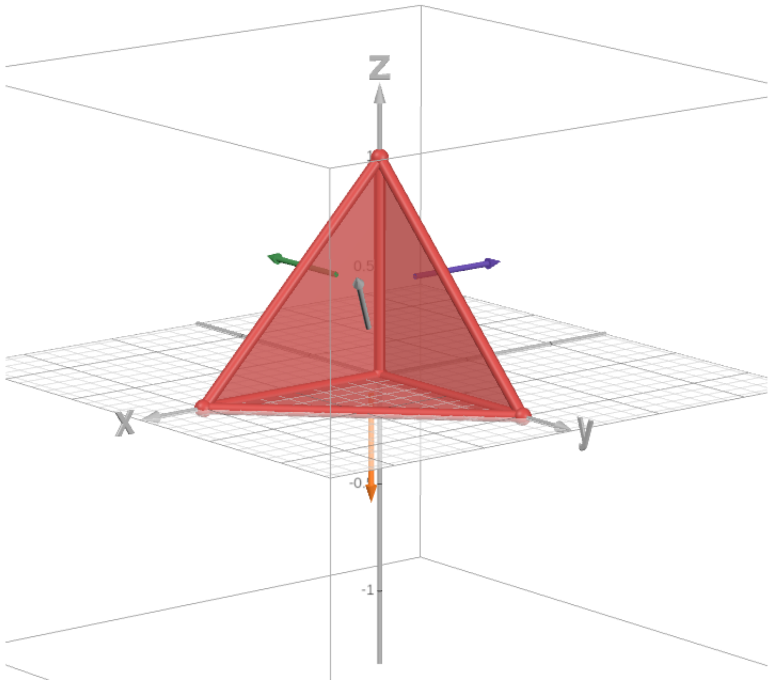
Hence,

$$\iint_{S_4} \vec{F} \cdot d\vec{S} = \iint_D \langle y, 1-x-2y, x\rangle \cdot \langle 1, 1, 1\rangle \; dA = \int_0^1 \int_0^{1-x} (1-y) \, dydx = \frac{1}{3}$$

Final Answer. Summing up, we obtain

$$\iint_T \vec{F} \cdot d\vec{S} = -\frac{3}{6} + \frac{1}{3} = \boxed{-\frac{1}{6}} .$$

On the next page, there is a picture of the vectors $\vec{r}_u \times \vec{r}_v$ for each faces of the tetrahedron. Copy-paste `https://www.desmos.com/3d/fc7631c3a2` in an internet browser to access the Desmos app and see the tetrahedron live).


z
x
y
0.5
-1