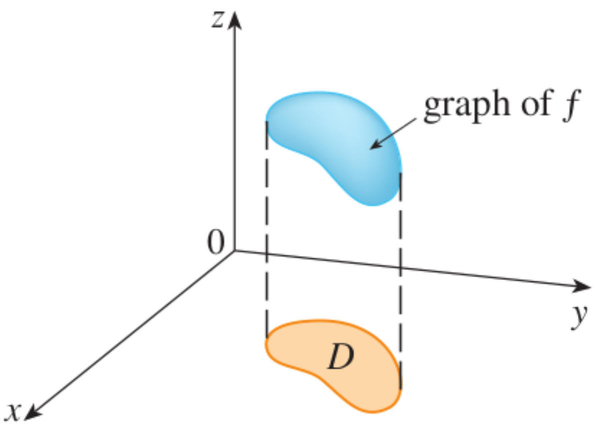
Chapter 15

Multiple Integrals

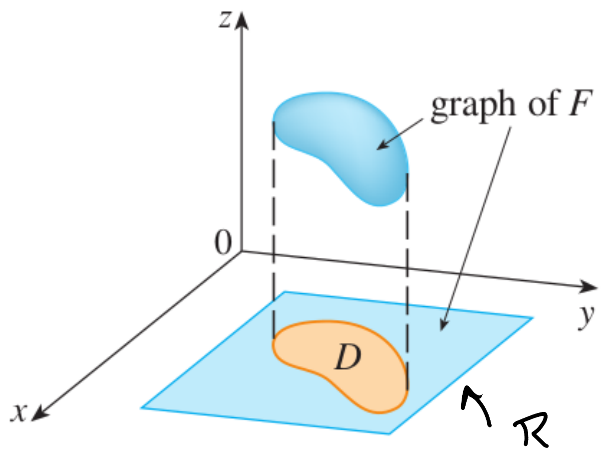
15.2 Double Integrals over genaral regions

Definition.

Given: A function f defined on D



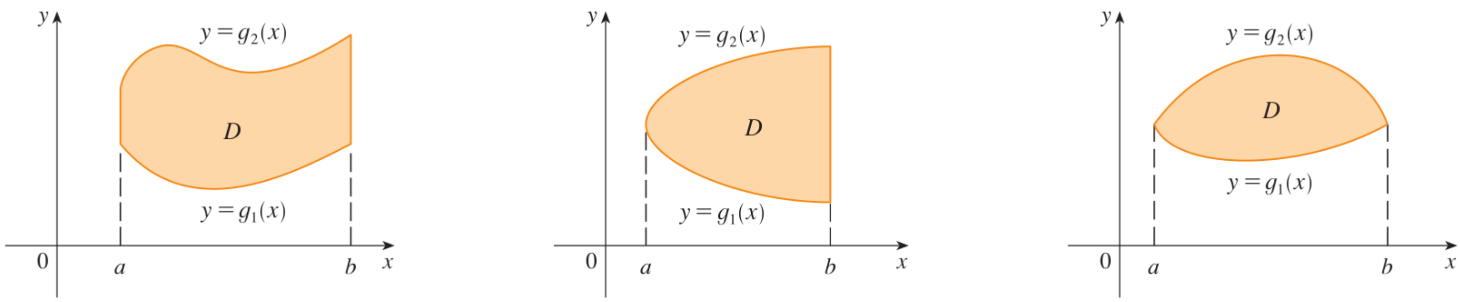
Extend f to a rectangle containing D



$$F(x,y) = \begin{cases} f(x,y), & (x,y) \in D \\ 0, & (x,y) \in R \text{ but } (x,y) \notin D \end{cases}$$

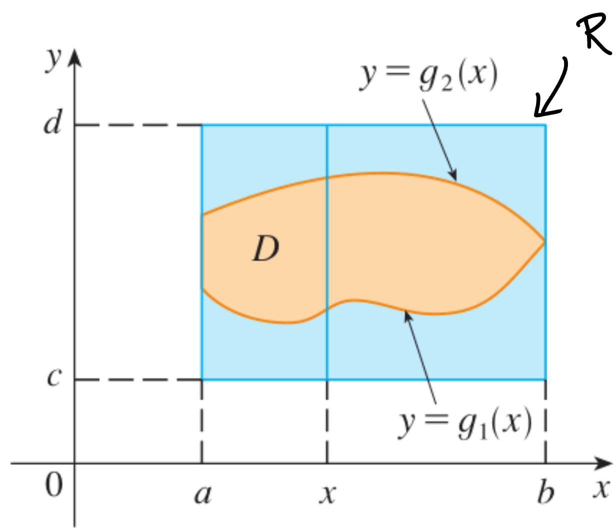
$$\boxed{\iint_D f(x,y)\, dA = \iint_R F(x,y)\, dA}$$

Region of type I.



$$D = \{(x,y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$$

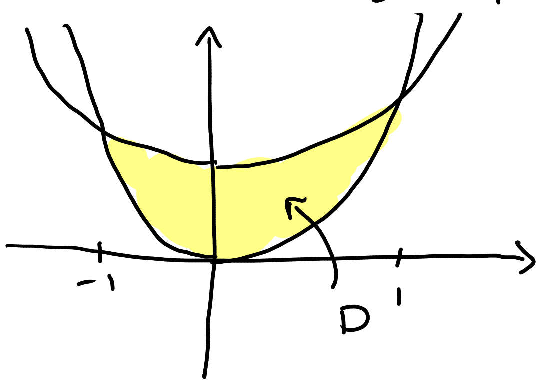
$$\iint_D f(x,y)\, dA = \iint_R F(x,y)\, dA$$
$$= \int_a^b \int_c^d F(x,y)\, dy\, dx$$
$$= \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$$



$$\boxed{\iint_D f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx.}$$

EXAMPLE 1 Evaluate $\iint_D (x + 2y)\, dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

① Picture



$1 + x^2 = 2x^2$

$1 = x^2 \Rightarrow x = \pm 1$

$D = \{ (x, y) : -1 \le x \le 1,\ 2x^2 \le y \le 1 + x^2 \}$.

② Integrate

$g_1(x) = 2x^2$ & $g_2(x) = 1 + x^2$

$$\iint_D x + 2y \, dA = \int_{-1}^{1} \int_{2x^2}^{1+x^2} x + 2y \, dy \, dx$$

$$= \int_{-1}^{1} xy \Big|_{2x^2}^{1+x^2} + y^2 \Big|_{2x^2}^{1+x^2} \, dx$$

$$= \int_{-1}^{1} x\left(1 + x^2 - 2x^2\right) + (1+x^2)^2 - (2x^2)^2 \, dx$$

$$= \int_{-1}^{1} x - x^3 + 1 + 2x^2 + x^4 - 4x^4 \, dx$$

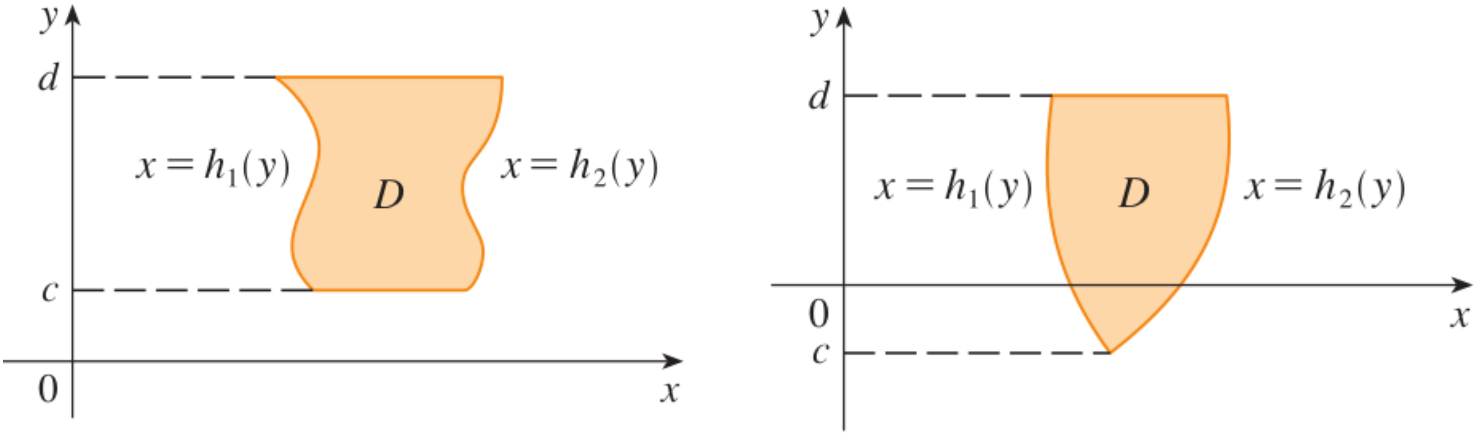
$$= \int_{-1}^{1} 1 + x + 2x^2 - x^3 - 3x^4 \, dx$$

$$= \boxed{\frac{32}{15}} \approx 1.3167.$$

$$\left(x + \frac{x^2}{2} + \frac{2x^3}{3} - \frac{x^4}{4} - \frac{3x^5}{5}\right)\Big|_{-1}^{1}$$

Region of Type II.

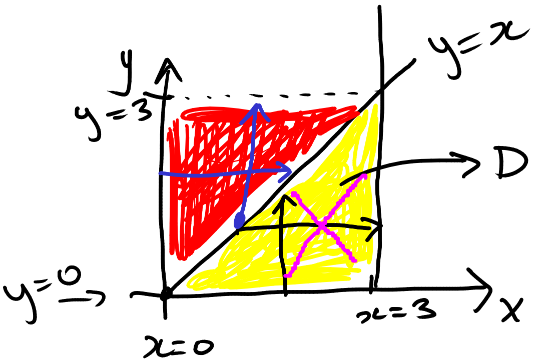
$$D = \{(x, y) \mid c \leqslant y \leqslant d,\ h_1(y) \leqslant x \leqslant h_2(y)\}$$



$$\boxed{\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy}$$

EXAMPLE. Evaluate $\iint_D e^{-y^2}\, dA$, where D is the region bounded by the lines $x = 0, y = 3$ and $x = y$.

① Picture.



$D = \{(x,y) : 0 \le x \le 3,\ x \le y \le 3\}$ TYPE I

$= \{(x,y) : 0 \le x \le y,\ 0 \le y \le 3\}$ TYPE II.

② Integrate

$c = 0,\ d = 3,\ h_1(y) = 0,\ h_2(y) = y.$

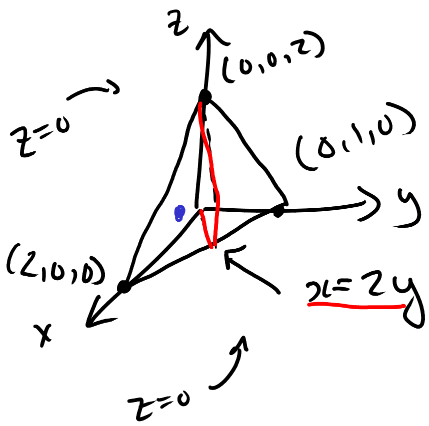
$$\iint_D e^{-y^2}\, dA = \int_0^3 \int_0^y e^{-y^2}\, dx\, dy$$ u-sub $(u = y^2)$

$$= \int_0^3 \left(y e^{-y^2} \right) dy = \frac{1}{2} \int_0^9 e^{-u}\, du$$

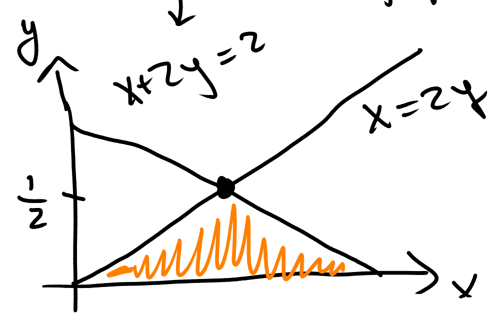
$$= \boxed{\frac{1}{2}\left(1 - e^{-9}\right)}$$

EXAMPLE. Find the volume of the tetrahedron bounded by the planes $x + 2y + z = 2$, $x = 2y$, $y = 0$, and $z = 0$.

① Picture



squeeze



TYPE II

$$D = \{(x,y) : 0 \le y \le \tfrac{1}{2},\ 2y \le x \le 2-2y\}.$$

② Integrate.

height = z
$= 2 - x - 2y$

$$\mathrm{Vol}(S) = \iint_D \text{height}\, dA$$

$$= \int_0^{1/2} \int_{2y}^{2-2y} 2 - x - 2y \, dx\, dy$$

$$= \int_0^{1/2} \left(2x - \frac{x^2}{2} - 2yx\right)\Big|_{2y}^{2-2y} dy$$

$$= \int_0^{1/2} 2(2-2y) - \frac{(2-2y)^2}{2} - 2y(2-2y)$$
$$- \left(4y - 2y^2 - 4y^2\right) \, dy$$

$$= \boxed{\frac{1}{3}} \approx 0.33\ldots$$

EXAMPLE 5 Evaluate the iterated integral $\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx$.

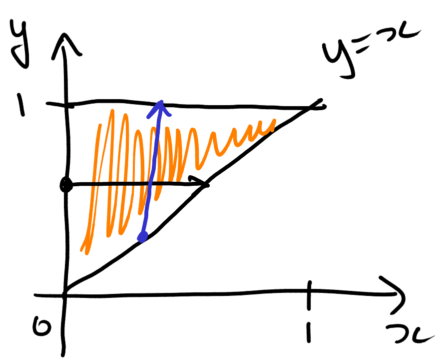
① Picture

TYPE I.

$D = \{(x,y) : 0 \le x \le 1,\ x \le y \le 1\}.$

$= \{(x,y) : 0 \le x \le y,\ 0 \le y \le 1\}$

TYPE II.



$$\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx = \iint_D \sin(y^2)\,dA.$$

② Integrate

$$\iint_D \sin(y^2)\,dA = \int_0^1 \int_0^y \sin(y^2)\,dx\,dy$$

$$= \int_0^1 \sin(y^2)\ y\ dy$$

u-sub

$u = y^2 \rightarrow du = 2y\,dy$

$$= \int_0^1 \sin(u)\ \frac{du}{2}$$

$$= \frac{1}{2}\ (-\cos u)\Big|_0^1$$

$$= \boxed{\frac{1-\cos(1)}{2}}$$

$\int_0^1 x + \cos x \, dx = \int_0^1 x \, dx + \int_0^1 \cos x \, dx$

$\boxed{6}$ $$\iint_D \big(f(x,y)+g(x,y)\big)\,dA = \iint_D f(x,y)\,dA + \iint_D g(x,y)\,dA$$

$\boxed{7}$ $$\iint_D c f(x,y)\,dA = c\iint_D f(x,y)\,dA$$

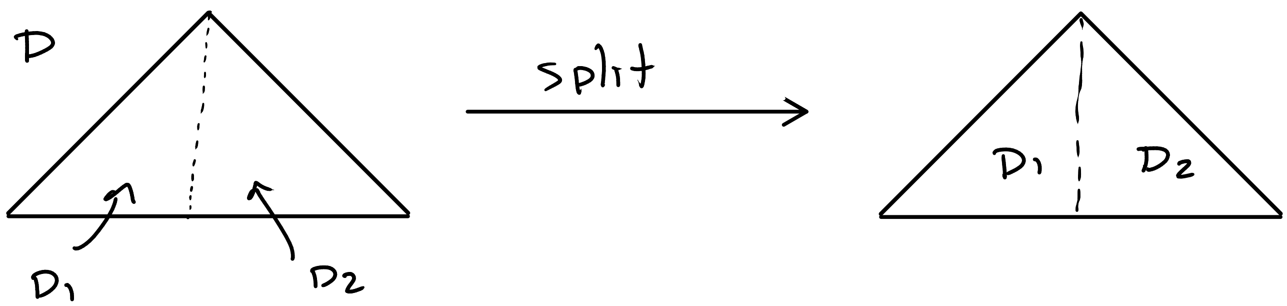
$\iint_D 1+x^2+y^2 \, dA \geq \iint_D x^2+y^2 \, dA$.

$1+x^2+y^2 \geq x^2+y^2$

$\boxed{8}$ If $f(x,y) \geq g(x,y)$ on D, then $\displaystyle\iint_D f(x,y)\,dA \geq \iint_D g(x,y)\,dA$

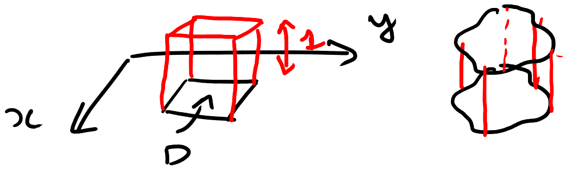
$\boxed{9}$ If $D = D_1 \cup D_2$, with $D_1 \cap D_2 = \emptyset$, then

$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$$



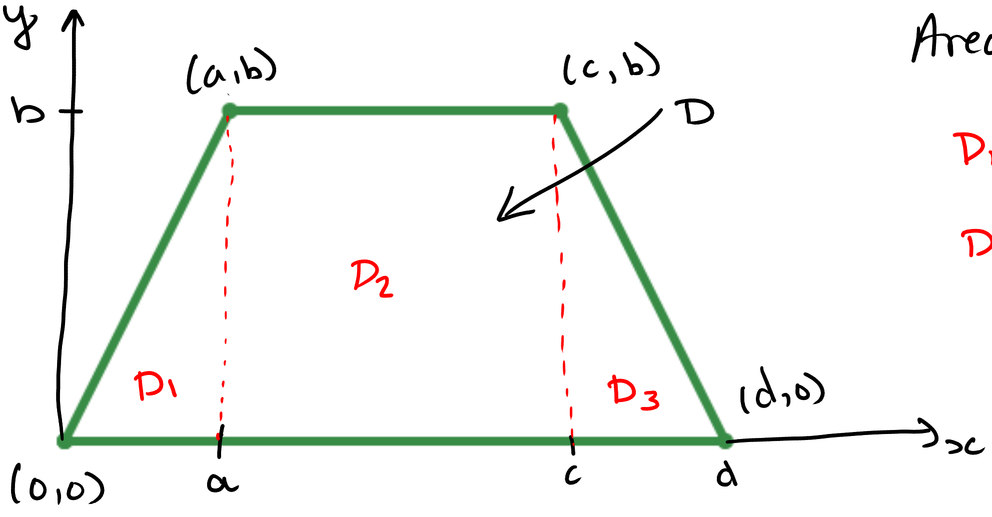
$f(x,y)=1$

$\boxed{10}$ $Area(D) = \displaystyle\iint_D 1\,dA$



$\boxed{11}$ If $m \leq f(x,y) \leq M$, then $m \cdot Area(D) \leq \displaystyle\iint_D f(x,y)\,dA \leq M \cdot Area(D)$

Example. Find the area of the trapezoid below:



$\text{Area}(D) = \iint_D 1\, dA$.

D_1 : triangle (& D_3).

D_2: rectangle.

$$\begin{aligned}
\text{Area}(D) &= \iint_D 1\, dA \\
&= \iint_{D_1} 1\, dA + \iint_{D_2} 1\, dA + \iint_{D_3} 1\, dA \quad \text{[Prop 9]} \\
&= \text{Area}(D_1) + \text{Area}(D_2) + \text{Area}(D_3) \\
&= \text{Area}(\triangle_{a,b}) + \text{Area}(\square_{b,\ c-a}) + \text{Area}(\triangle_{b,\ d-c}) \\
&= \frac{b \cdot a}{2} + b(c-a) + \frac{b(d-c)}{2} \\
&= \frac{b \cdot a}{2} + \frac{2bc - 2ba}{2} + \frac{bd - bc}{2} \\
&= \frac{bc - ab + bd}{2} \\
&= \boxed{\frac{b}{2}(c - a + d)} \qquad \left(\frac{h(b+B)}{2}\right).
\end{aligned}$$

Challenge. Find the area of the hexagone below using properties 9 and 10: