Problem 2

By Stokes' Theorem, we have

$$\iint_S \operatorname{curl} \vec{F}\, d\vec{S} = \int_C \vec{F} \cdot d\vec{r}.$$

The surface S is the part of the paraboloïd oriented upward. The boundary of S is the circle $x^2 + y^2 = 1$ (when we let $z = 0$ in the equation of the paraboloïd). We parametrize the circle with

$$\vec{r}(\theta) = \langle \cos\theta, \sin\theta, 0\rangle \quad (0 \leq \theta \leq 2\pi).$$

The surface induces the counterclockwise orientation on C. Thus, we get

$$\begin{aligned}\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} &= \int_0^{2\pi} \Big\langle \cos^2\theta \sin(0), \sin^2\theta, \cos\theta\sin\theta \Big\rangle \cdot \langle -\sin\theta, \cos\theta, 0\rangle \; dt \\ &= \int_0^{2\pi} \sin^2\theta\cos\theta \, dt = 0.\end{aligned}$$

Problem 8

By Stokes' Theorem, we have

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}$$

The curve C is the boundary of the surface S and a parametrization for the surface S is

$$\vec{r}(x, y) = \langle x, y, 1 - 3x - 2y\rangle \quad (0 \leq x \leq 1/3,\, 0 \leq y \leq (1-3x)/2\ ).$$

Since C is oriented counterclockwise, S must be positively oriented. So, a normal vector to S would be

$$\vec{r}_x \times \vec{r}_y = \langle 3, 3, 1\rangle .$$

The curl $\vec{F}$ is

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 1 & (x + yz) & xy - \sqrt{z} \end{vmatrix} = \langle x - y, -y, 1\rangle .$$

So, we obtain

$$\operatorname{curl} \vec{F} \cdot (\vec{r}_x \times \vec{r}_y) = 3x - 3y - 3y + 1 = 3x - 6y + 1.$$

Thus, we finally obtain

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{1/3} \int_0^{(1-3x)/2} 3x - 6y + 1 \, dydx = 1/36.$$