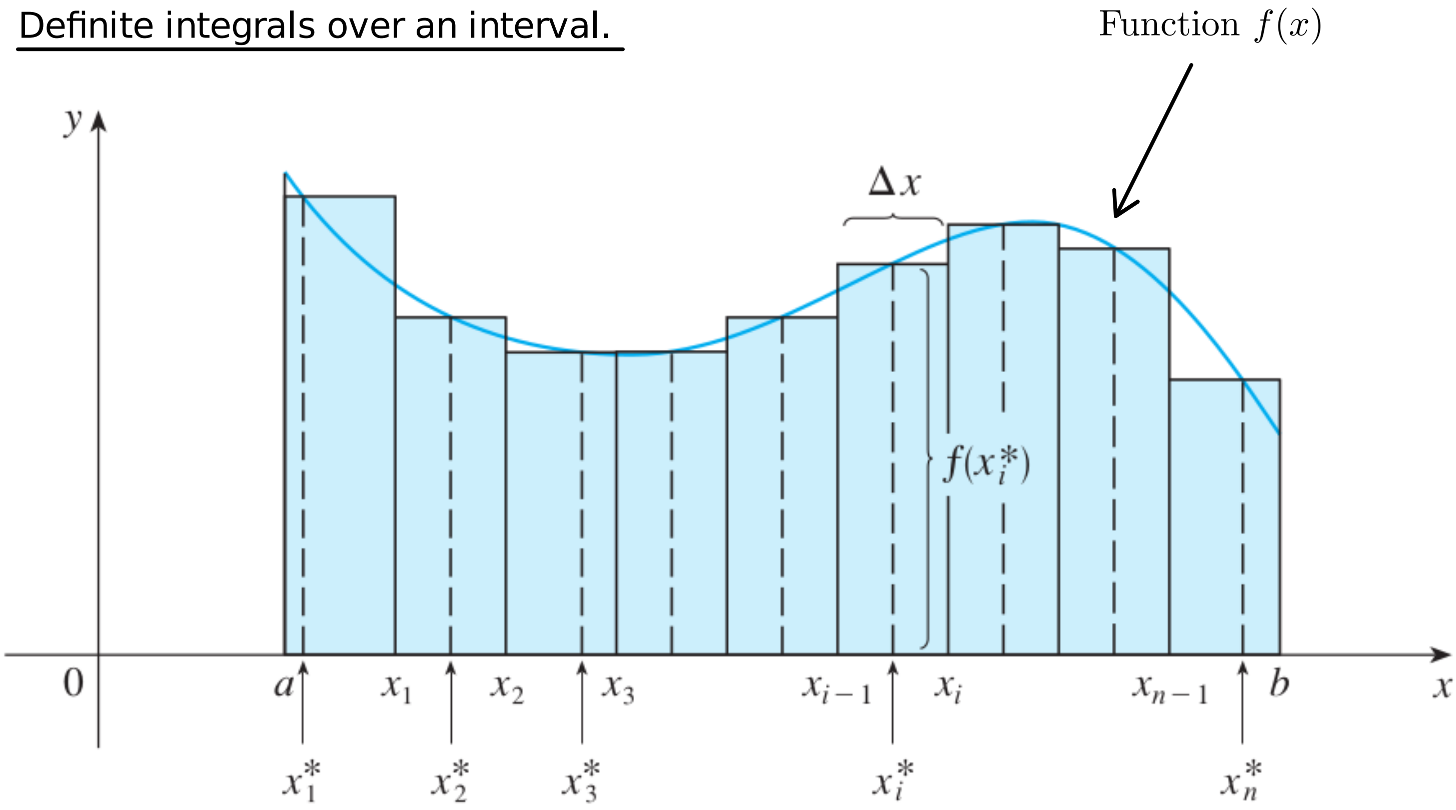
Chapter 15

Multiple Integrals

15.1 Double Integrals over a rectangle

Definite integrals over an interval.



1) Divide the interval in n parts of equal length Δx

2) Name each subinterval $[a, x_1]$, $[x_1, x_2]$, $\ldots$, $[x_{n-1}, b]$

3) Choose some point x_1^* in $[a, x_1]$, x_2^* in $[x_1, x_2]$, $\ldots$, x_n^* in $[x_{n-1}, b]$

$\Rightarrow$ Total Area of rectangles $= f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$

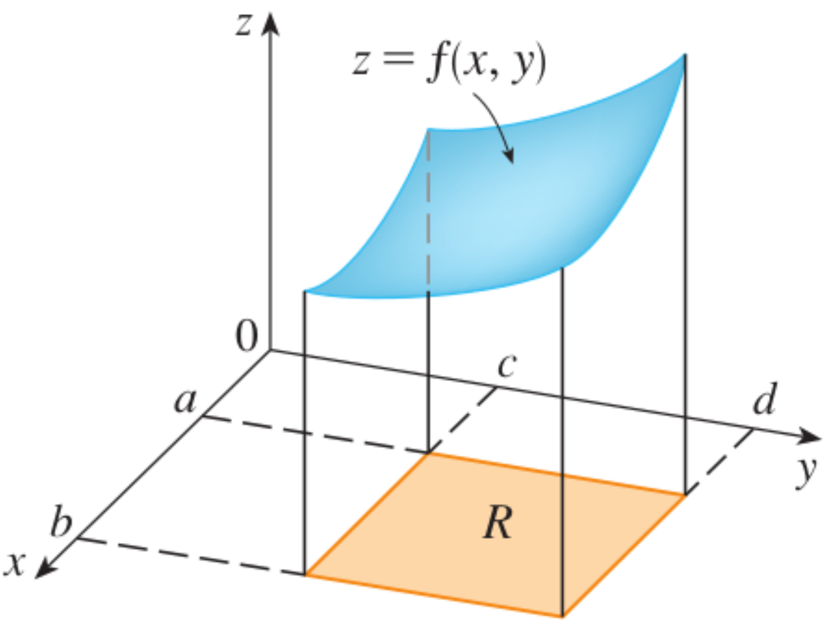
4) Take the limit as $n \to \infty$

$$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x}$$

Useful Fact:

$$\int_a^b f(x)\,dx \cong \sum_{i=1}^{n} f(x_i^*)\,\Delta x \,.$$

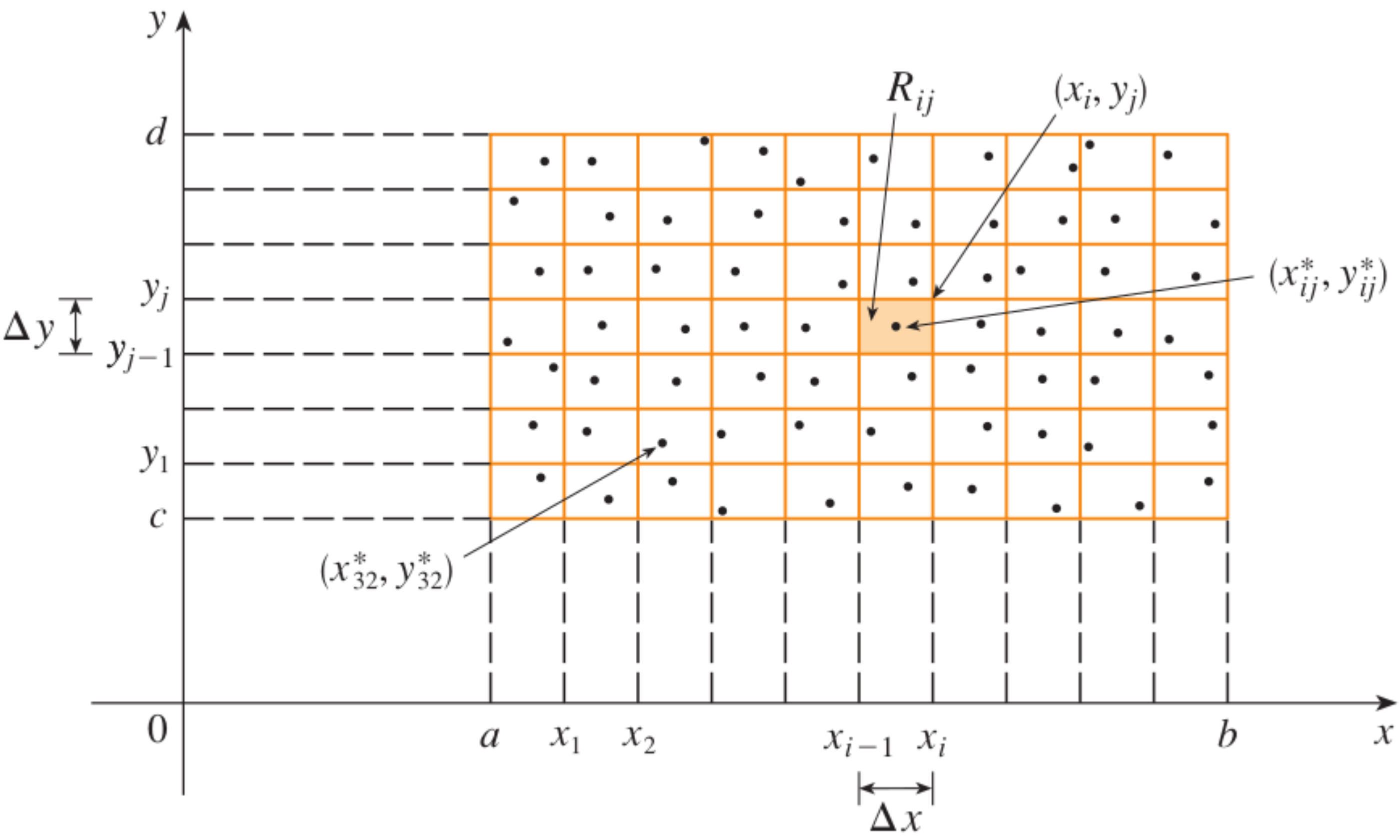
Volumes and Double Integrals.



Given:

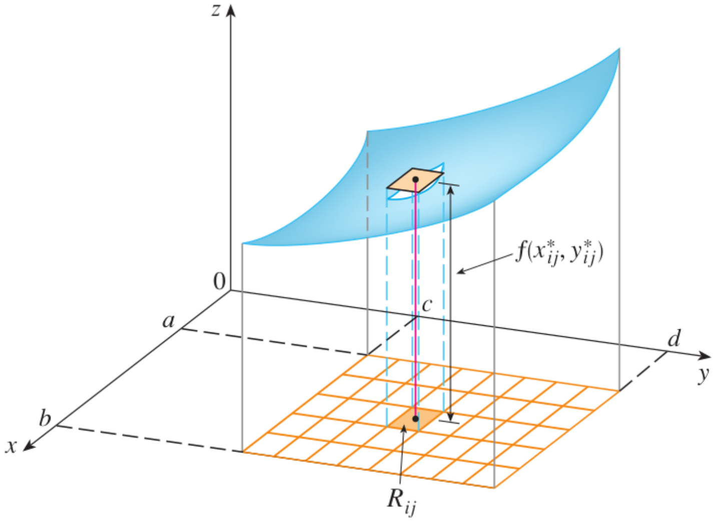
- A function $z = f(x, y)$
- The domain $R = [a, b] \times [c, d]$

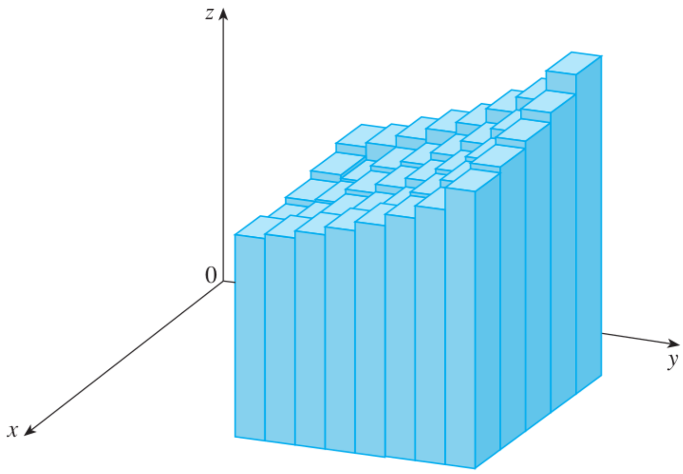
1st Step: Divide the domain to create a grid.



1) Divide [a, b] in m equal parts Δx
2) Divide [c, d] in n equal parts Δy
3) Create the rectangle $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$
4) Select a point (x_{ij}^*, y_{ij}^*) in R_{ij}

2nd Step: Approximate the volume by "buildings"





1) Volume of a building: $\Delta A \cdot f(x_{ij}^*, y_{ij}^*)$
2) Total volume:

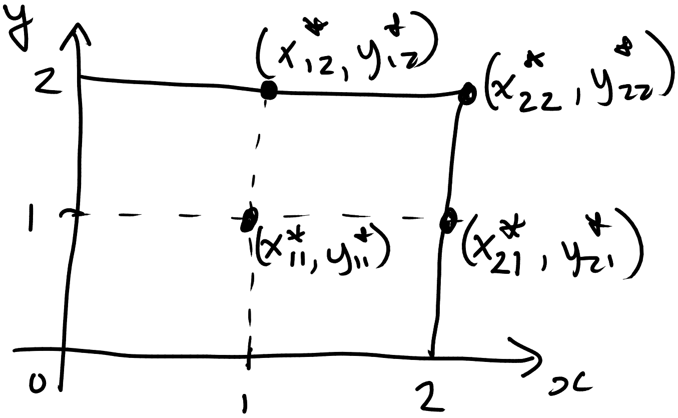
$$V = \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \cdot \Delta A$$

3) Take the limit as $m, n \to \infty$:

$$\boxed{\iint_R f(x,y)\,dA = \lim_{n,m\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A}$$

Useful Fact: $\iint_R f(x,y)\,dA \cong \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A$

EXAMPLE 1 Estimate the volume of the solid that lies above the square $R = [0, 2] \times [0, 2]$ and below the elliptic paraboloid $z = 16 - x^2 - y^2$. Divide R into four equal squares and choose the sample point to be the upper right corner of each square R_{ij}. Sketch the solid and the approximating rectangular boxes.



$(x_{11}^*, y_{11}^*) = (1, 1)$ $(x_{21}^*, y_{21}^*) = (2, 1)$

$(x_{12}^*, y_{12}^*) = (1, 2)$ $(x_{22}^*, y_{22}^*) = (2, 2)$

$f(x, y) = 16 - x^2 - y^2$.

$\Delta A = 1$

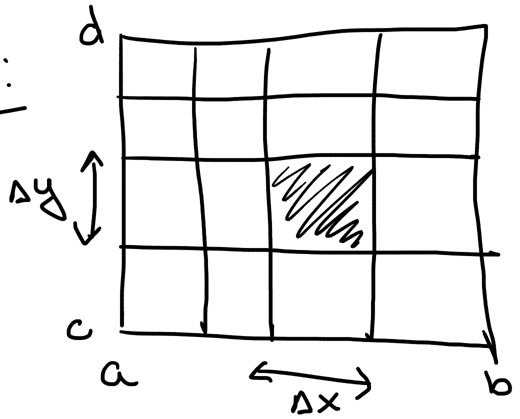
$$\iint_R 16 - x^2 - y^2 \, dA \cong \sum_{i=1}^{2} \sum_{j=1}^{2} \left[16 - (x_{ij}^*)^2 - (y_{ij}^*)^2\right] \Delta A$$

$$= (16 - 1^2 - 1^2) \cdot 1 + (16 - 1^2 - 2^2) \cdot 1$$

$$+ (16 - 2^2 - 1^2) \cdot 1 + (16 - 2^2 - 2^2) \cdot 1$$

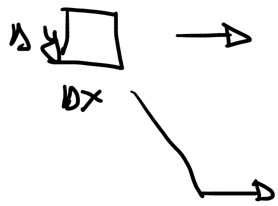
$$= \boxed{44}$$

Grid:



$[a,b]$ in m parts Δx.

$[c,d]$ in n parts Δy



$$\Delta A = \Delta x\, \Delta y \xrightarrow{m,n\to\infty} dA = dx\,dy \quad (1)$$

$$\Delta A = \Delta y\, \Delta x \xrightarrow{m,n\to\infty} dA = dy\,dx \quad (2)$$

$$\frac{d}{dx}(2) = 0 \qquad \frac{d}{dx}(2x) = 2$$

$$\frac{\partial}{\partial x}(x^2 + \cancel{y^2}^{\to 0}) = 2x$$

$$\frac{\partial}{\partial y}(x y^2) = x\frac{\partial}{\partial y}(y^2) = 2xy.$$

Fubini's Theorem:

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)dxdy = \int_a^b \int_c^d f(x,y)\,dydx$$

Called "iterated integrals"

EXAMPLE 4 Evaluate the iterated integrals.

(a) $\int_0^3 \left[\int_1^2 x^2 y\, dy\right] dx = I_1$ (b) $\int_1^2 \left[\int_0^3 x^2 y\, dx\right] dy = I_2$

(a) $I_1 =$ first $\int_1^2 x^2 y\, dy$

second $\int_0^3 (\ldots)\, dx$

$$\int_1^2 x^2 y\, dy = x^2 \int_1^2 y\, dy = x^2 \frac{y^2}{2}\Big|_1^2 = x^2\left(2 - \frac{1}{2}\right)$$

$$= \frac{3x^2}{2}$$

$$I_1 = \underbrace{\int_0^3 \int_1^2 x^2 y\, dy}_{=\frac{3x^2}{2}}\, dx = \int_0^3 \frac{3x^2}{2}\, dx = \frac{3}{2} \frac{x^3}{3}\Big|_0^3 = \boxed{\frac{27}{2}}$$

(b) $\int_0^3 x^2 y\, dx = y \int_0^3 x^2\, dx = y\, \frac{x^3}{3}\Big|_0^3 = 9y$

$$I_2 = \int_1^2 \underbrace{\int_0^3 x^2 y\, dx}_{9y}\, dy = \int_1^2 9y\, dy = \frac{9y^2}{2}\Big|_1^2$$

$$= 18 - \frac{9}{2}$$

$$= \boxed{\frac{27}{2}}$$

Notice: $I_1 = I_2$. The order doesn't matter.

Example. Evaluate the following integral: $R=[0,1]\times[0,1]$

$$I=\int_0^1\left[\int_0^1 v(u^2+v^2)^4\,du\right]dv$$

$$\int_0^1 v(u^2+v^2)^4\,du = v\int_0^1 (u^2+v^2)^4\,du$$

Change the order! (to use a u-sub method).

$$I=\int_0^1\int_0^1 v(u^2+v^2)^4\,dv\,du$$

$$w=u^2+v^2 \longrightarrow dw=2v\,dv \longrightarrow \frac{dw}{2}=v\,dv$$

$$I=\int_0^1\int_{u^2}^{u^2+1} w^4\,\frac{dw}{2}\,du$$

$$=\int_0^1 \frac{1}{2}\,\frac{w^5}{5}\Big|_{u^2}^{u^2+1}\,du$$

$$=\frac{1}{10}\int_0^1 (u^2+1)^5-u^{10}\,du$$

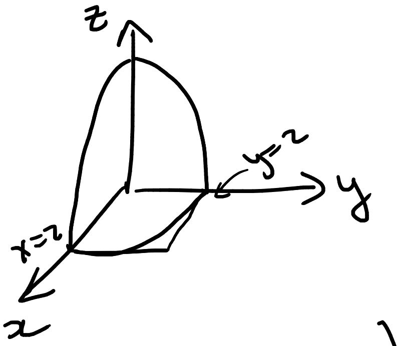
$$=\frac{1}{10}\int_0^1 \cancel{u^{10}}+5u^8+10u^6+10u^4+5u^2+1-\cancel{u^{10}}\,du$$

$$=\frac{1}{10}\int_0^1 5u^8+10u^6+10u^4+5u^2+1\,du$$

$$=\boxed{\frac{419}{630}}$$

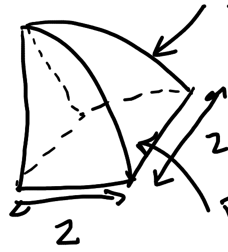
EXAMPLE 7 Find the volume of the solid S that is bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$, and the three coordinate planes.

$\hookrightarrow z = 16 - x^2 - 2y^2$



solid $\longrightarrow$

S $\rightarrow$



$z = 16 - x^2 - 2y^2$

$R = [0,2] \times [0,2]$

$$\text{Vol}(S) = \iint_R 16 - x^2 - 2y^2 \, dA .$$

So,

$$\text{Vol}(S) = \int_0^2 \int_0^2 16 - x^2 - 2y^2 \, dx\, dy$$

$$= \int_0^2 16x - \frac{x^3}{3} - 2y^2 x \Big|_0^2 \, dy$$

$$= \int_0^2 32 - \frac{8}{3} - 4y^2 \, dy$$

$$= 32y - \frac{8}{3}y - \frac{4y^3}{3} \Big|_0^2$$

$$= \boxed{64 - \frac{16}{3} - \frac{32}{3}}$$

$$= \frac{196 - 16 - 32}{3} = \frac{148}{3} \approx 49.33$$

EXAMPLE 8 If $R = \underbrace{[0, \pi/2]}_{x} \times \underbrace{[0, \pi/2]}_{y}$, then compute $\iint_R \sin x \cos y \, dA$.

$$\iint_R \sin x \cos y \, dA = \int_0^{\pi/2} \int_0^{\pi/2} \sin x \cos y \, dx \, dy$$

$$= \int_0^{\pi/2} \cos y \underbrace{\left(\int_0^{\pi/2} \sin x \, dx \right)}_{\text{number.}} dy$$

$$= \left(\int_0^{\pi/2} \sin x \, dx \right) \left(\int_0^{\pi/2} \cos y \, dy \right)$$

$$= -\cos x \Big|_0^{\pi/2} \quad \sin y \Big|_0^{\pi/2}$$

$$= 1 \cdot 1 = \boxed{1}$$

=

Useful Fact:

$f(x,y)$

$$\iint_R g(x)h(y) \, dA = \left(\int_a^b g(x) \, dx \right) \left(\int_c^d h(y) \, dy \right)$$

where $R = [a, b] \times [c, d]$