Section 15.8, Problem 2 (5 Pts)

(a) We have

$$x = \rho \cos\theta \sin\phi = 2\cos(\pi/2)\sin(\pi/2) = 0,$$
$$y = \rho \sin\theta \sin\phi = 2\sin(\pi/2)\sin(\pi/2) = 2,$$

and

$$z = \rho\cos(\phi) = 2\cos(\pi/2) = 0$$

.

(b) We have

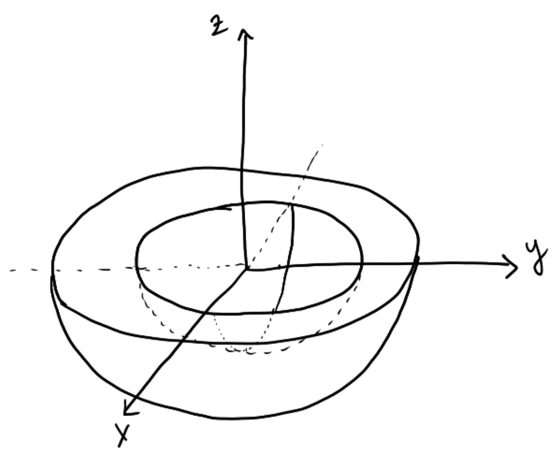
$$x = 4\cos(-\pi/4)\sin(\pi/3) = 4(\sqrt{2}/2)(\sqrt{3}/2) = \sqrt{6},$$
$$y = 4\sin(-\pi/4)\sin(\pi/3) = 4(-\sqrt{2}2)(\sqrt{3}/2) = -\sqrt{6}$$

and

$$z = \rho\cos(\pi/3) = 4(1/2) = 2.$$

Section 15.8, Problem 12 (5 Pts)

There is no restriction on the angle θ. Writing $\rho = \sqrt{x^2+y^2+z^2}$, we see that $1 \leq \sqrt{x^2+y^2+z^2} \leq 2$. Therefore, the solid lies between two spheres centered at the origin of radius 1 and 2 respectively. The latitude ϕ is between $\pi/2$ and π. Here is an illustration of the solid:



It looks like half of the Earth with the center removed (the inner core).

Section 15.8, Problem 22 **(10 Pts)**

In spherical coordinates, we have

$$E = \{(\rho, \theta, \phi) \,:\, 0 \leq \rho \leq 1,\, 0 \leq \theta \leq 2\pi,\, 0 \leq \phi \leq \pi/3\}. \tag{1}$$

Changing to spherical coordinates, we get

$$\begin{aligned}
\iiint_E y^2 z^2 \, dV &= \int_0^{\pi/3} \int_0^{2\pi} \int_0^1 (\rho^2 \sin^2(\theta) \sin^2(\phi))(\rho^2 \cos^2(\phi))\rho^2 \sin\phi \, d\rho d\theta d\phi \\
&= \int_0^{\pi/3} \int_0^{2\pi} \int_0^1 \rho^6 \sin^2(\theta) \sin^3(\phi) \cos^2(\phi) \, d\rho d\phi d\theta \\
&= \Big(\int_0^1 \rho^6 \, d\rho \Big) \Big(\int_0^{2\pi} \sin^2(\theta) \, d\theta \Big) \Big(\int_0^{\pi/3} \sin^2(\phi) \cos^2(\phi) \sin(\phi) \, d\phi \Big) \\
&= \\
&= \int_0^{\pi/3} \int_0^{2\pi} \int_0^1 \rho^5 \sin^2(\theta) \cos^2(\phi) \, d\rho d\theta d\phi \\
&= \Big(\frac{1}{7}\Big) \Big(\Big) \Big(\int_0^{\pi/3} (1 - \cos^2(\phi)) \cos^2(\phi) \sin(\phi) \, d\phi \Big)
\end{aligned}$$

Letting $u = \cos\phi$, we get $du = -\sin(\phi)\, d\phi$ and therefore

$$\begin{aligned}
\int_0^{\pi/3} (1 - \cos^2(\phi)) \cos^2(\phi) \sin(\phi) \, d\phi &= \int_1^{1/2} (1 - u^2) u^2 (-du) \\
&= \int_{1/2}^1 u^2 - u^4 \, du \\
&= \frac{47}{480}.
\end{aligned}$$

Hence, denoting the original integral by I,

$$I = \Big(\frac{1}{7}\Big) \Big(\pi\Big) \Big(\frac{47}{480}\Big) \approx 0.0439. \qquad \triangle$$

Section 15.8, Problem 26 **(10 Pts)**

In spherical coordinates, the cone $z = \sqrt{x^2 + y^2}$ is $\phi = \pi/4$. The equations of the two spheres becomes $\rho = 1$ and $\rho = 2$. Therefore, the solid E can be described as followed:

$$E = \{(\rho, \theta, \phi) \,:\, 1 \leq \rho \leq 2,\, 0 \leq \theta \leq 2\pi,\, 0 \leq \phi \leq \pi/4\}. \tag{2}$$

Therefore,

$$
\begin{aligned}
\iiint_E \sqrt{x^2+y^2+z^2}\,dV &= \int_0^{\pi/4}\int_0^{2\pi}\int_1^2 \rho\,\rho^2\sin(\phi)d\rho d\theta d\phi \\
&= \int_0^{\pi/4}\int_0^{2\pi}\int_1^2 \rho^3\sin(\phi)\,d\rho d\theta d\phi \\
&= \Big(\int_1^2 \rho^3\,d\rho\Big)\Big(\int_0^{2\pi} d\theta\Big)\Big(\int_0^{\pi/4}\sin(\phi)\,d\phi\Big) \\
&= \Big(\frac{16-1}{4}\Big)\Big(2\pi\Big)\Big(\frac{\sqrt{2}-1}{\sqrt{2}}\Big) \\
&= \frac{15(\sqrt{2}-1)\pi}{2\sqrt{2}}. \qquad \triangle
\end{aligned}
$$

Section 15.8, Problem 30 **(10 Pts)**

The equation of the xy-plane in spherical coordinates is $\phi=\pi/2$. The equation of the sphere is $\rho=2$ and the equation of the cone is $\phi=\pi/4$. Therefore,

$$E=\{(\rho,\theta,\phi)\,:\,0\le\rho\le 2,\,0\le\theta\le 2\pi,\,\pi/4\le\phi\le\pi/2\}. \tag{3}$$

The volume is given by

$$
\begin{aligned}
\mathrm{Vol}(E)=\iiint_E dV &= \int_{\pi/4}^{\pi/2}\int_0^{2\pi}\int_0^2 \rho^2\sin\phi\,d\rho d\theta d\phi \\
&= \Big(\int_0^2 \rho^2\,d\rho\Big)\Big(\int_0^{2\pi} d\theta\Big)\Big(\int_{\pi/4}^{\pi/2}\sin\phi\,d\phi\Big) \\
&= \Big(\frac{8}{3}\Big)(2\pi)\Big(\frac{\sqrt{2}}{2}\Big) \\
&= \frac{8\pi\sqrt{2}}{3}. \qquad \triangle
\end{aligned}
$$

Section 15.9, Problem 8 **(10 Pts)**

The map is $(x,y)=T(u,v)=(v,u(1+v^2))$. We will analyse how the transformation T acts on each side of the square S.

① $u=0$, v varies from 0 to 1. In this case, we have $(x,y)=(v,0)$, for $0\le v\le 1$. This is a horizontal segment on the x-axis, starting at $(0,0)$ and ending at $(1,0)$.

② $u=1$, v varies from 0 to 1. In this case, we have $(x,y)=(v,1+v^2)$, for $0\le v\le 1$. Therefore, $x=v$ and $y=1+v^2$. Replacing x in the expression of y, we get $y=1+x^2$, for $0\le x\le 1$. This is a segment of a parabola, starting at $(0,1)$ and ending at $(1,2)$.

③ u varies from 0 to 1, $v=0$. In this case, we have $(x,y)=(0,u)$, for $0\le u\le 1$. This is a vertical segment on the y-axis, starting at $(0,0)$ and ending at $(0,1)$.

④ u varies from 0 to 1, $v=1$. In this case, we have $(x,y)=(1,2u)$, for $0\le u\le 1$. Therefore, $x=1$ and $y=2u$. This is a vertical segment parallel to the y-axis, starting at $(1,0)$ and ending at $(1,2)$.

A representation of the square S in the uv-plane and its image in the xy-plane is illustrated in the picture below.

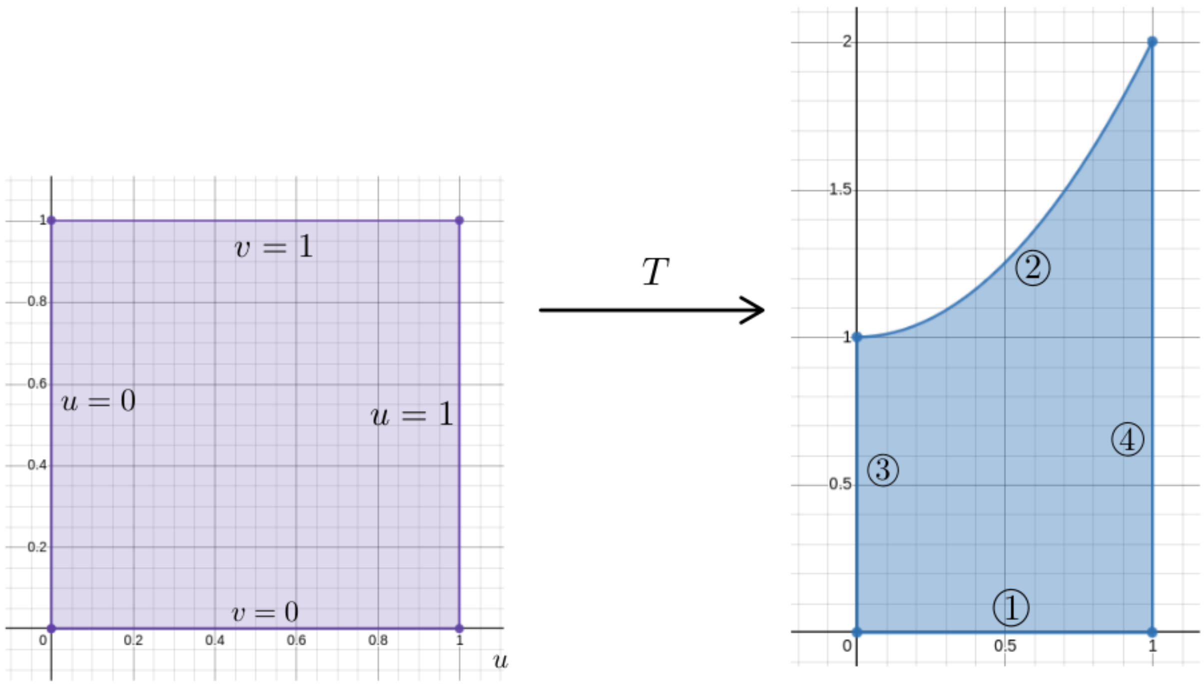


Figure (a) Square S

Figure (b) Image $T(S)$ of the square S

It's like if S was stretched from one of its corners. △