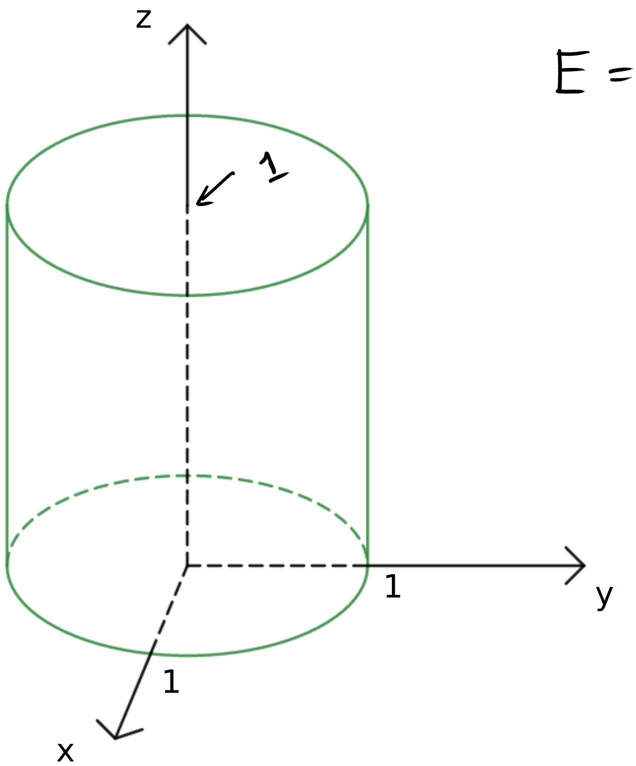
Chapter 15

Multiple Integrals

15.7 Triple integrals in cylindrical coordinates

Cylindrical coordinates

EXAMPLE. Describe the following solid (the interior of a cylinder).



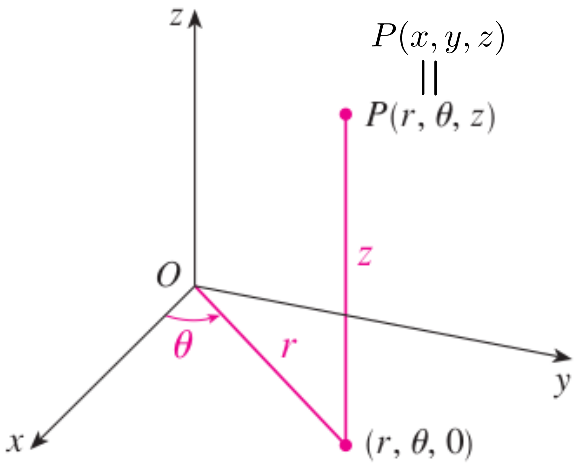
$E = \{ (x, y, z) : \ 0 \le z \le 1, \ 0 \le x^2 + y^2 \le 1 \}$

Because of the circle, it might be difficult to use this description in a triple integral.

Describe the base of the cylinder using polar coordinates:

$E = \{ (r, \theta, z) : \ 0 \le r \le 1, \ 0 \le \theta \le 2\pi, \ 0 \le z \le 1 \}$

Definition (when the main axis is the z-axis)



Cylindrical ⟶ Cartesian

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

Cartesian ⟶ Cylindrical

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan(y/x), \quad z = z$$

EXAMPLE 1

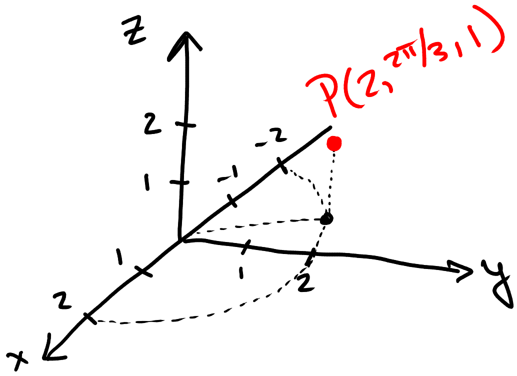
$r \quad \theta \quad z$ $\qquad \frac{\pi}{2} < \frac{2\pi}{3}$

(a) Plot the point with cylindrical coordinates $(2, 2\pi/3, 1)$ and find its rectangular coordinates.

$x \quad y \quad z$

(b) Find cylindrical coordinates of the point with rectangular coordinates $(3, -3, -7)$.

(a)



$\frac{2\pi}{3} - \frac{\pi}{2} = \frac{\pi}{6}$

$x = 2\cos(2\pi/3) = 2\cdot\left(-\frac{1}{2}\right) = -1$

$y = 2\sin(2\pi/3) = 2\frac{\sqrt{3}}{2} = \sqrt{3}$

$z = 1$

(b) $r = \sqrt{x^2+y^2} = \sqrt{3^2+(-3)^2} = \sqrt{18} = \underline{3\sqrt{2}}$

$\theta = \text{Arctan}\left(\frac{y}{x}\right) = \text{Arctan}\left(\frac{-3}{3}\right) = \underline{\frac{-\pi}{4} \text{ or } \frac{7\pi}{4}}$

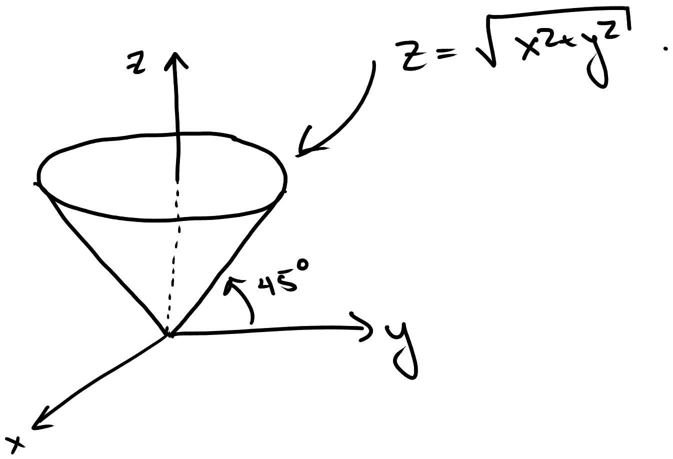
$z = \underline{-7}$

EXAMPLE 2 Describe the surface whose equation in cylindrical coordinates is $z = r$.

$r = \sqrt{x^2+y^2} \longrightarrow z = \sqrt{x^2+y^2}$

$\longrightarrow z^2 = x^2+y^2, \quad z \geq 0.$

<u>Cone</u>:



Note: Principle axis (the z-axis) can be any other axis (x-axis or y-axis) in some applications.

EXAMPLE. Write the equation in cylindrical coordinates and identify the surface.

$$z = x^2 - y^2$$

$$x = r\cos\theta \qquad\qquad z = r^2\cos^2\theta - r^2\sin^2\theta$$

$$y = r\sin\theta \quad \longrightarrow \quad = r^2(\cos^2\theta - \sin^2\theta)$$

$$z = z \qquad\qquad = r^2\cos(2\theta)$$

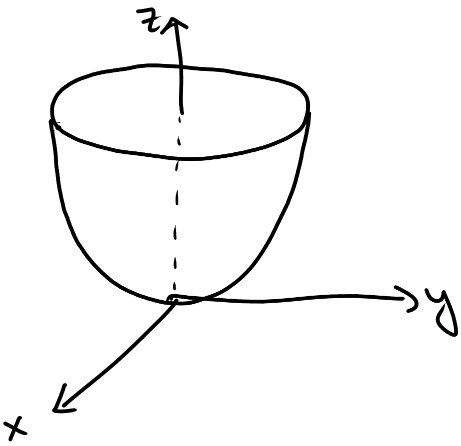
So, $z = r^2\cos(2\theta)$, $r \geq 0$, $0 \leq \theta \leq 2\pi$

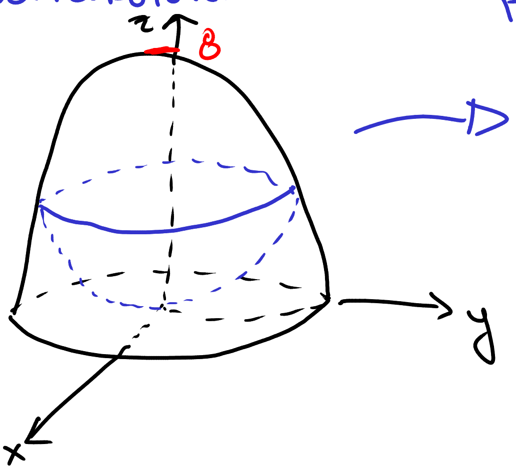
Surface's name: hyperbolic paraboloid.

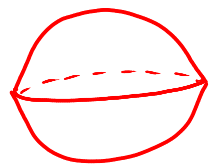
EXAMPLE. Sketch the solid described by the given inequalities:

$$r^2 \leq z \leq 8 - r^2$$

$$r = \sqrt{x^2+y^2} \longrightarrow \underbrace{x^2+y^2}_{\text{paraboloid}} \leq z \leq \underbrace{8-x^2-y^2}_{\text{paraboloid(ish)}}$$



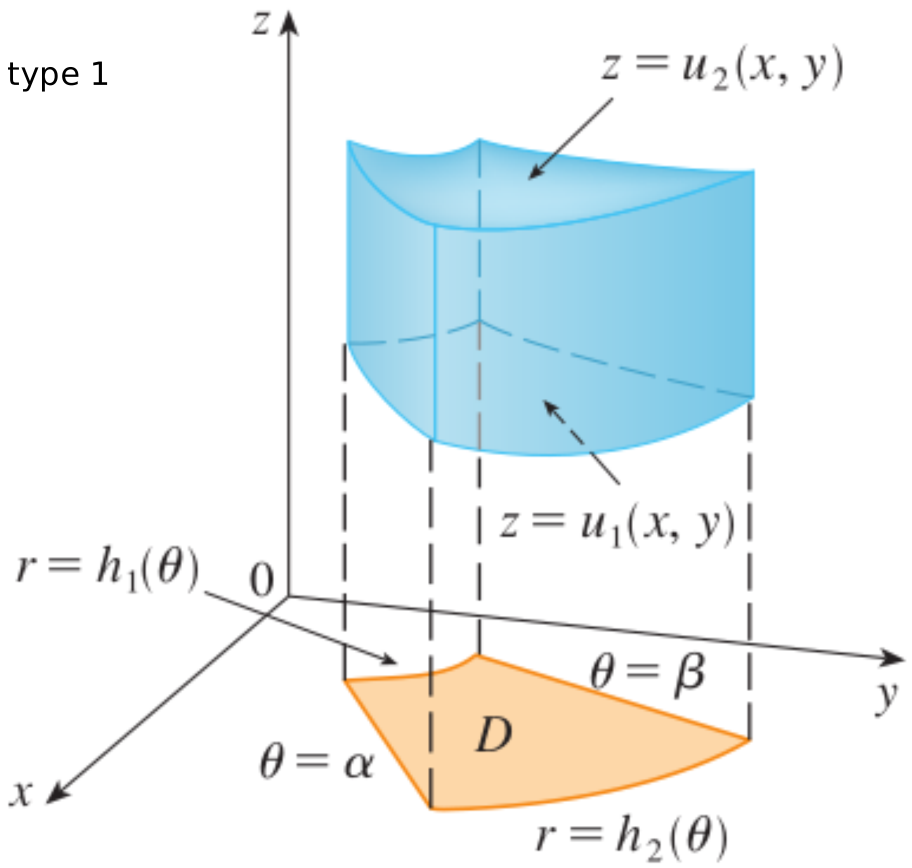




Question. What is the equation of a plane in cylindrical coordinates?

Evaluating triple integrals in cylindrical coordinates.

Solid of type 1



- $E = \{(x, y, z) : (x, y) \in D \text{ and } u_1(x, y) \leq z \leq u_2(x, y)\}$

$$\iiint_E f(x,y,z)\,dV = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right) dA$$

- Describe D in polar coordinates.

$$D = \{(r, \theta) : h_1(\theta) \leq r \leq h_2(\theta),\ \alpha \leq \theta \leq \beta\}$$

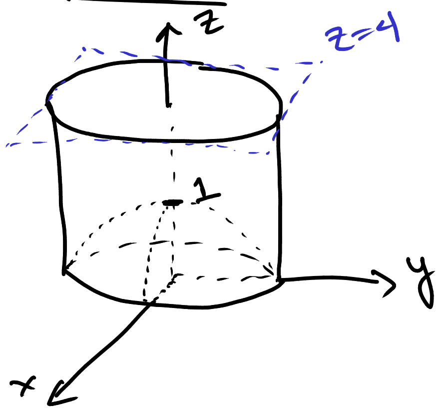
$$\boxed{\iiint_E f(x,y,z)\,dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \left[\int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\,dz\right] \underbrace{r\,dr\,d\theta}_{dA}}$$

Note: Can be adapted to type 2 and type 3 solids.

EXAMPLE. A solid E lies within the cylinder $x^2 + y^2 = 1$, below the plane $z = 4$, and above the paraboloid $z = 1 - x^2 - y^2$. Find the value of the integral

$$\iiint_E x^2 + y^2 \, dV$$

① Picture



set $z=0 \Rightarrow 0 = 1-x^2-y^2$

$\Rightarrow x^2+y^2=1$

Z-values $1-x^2-y^2 \le z \le 4$

Shadow: $D = \{(x,y) : 0 \le x^2+y^2 \le 1\}$

② Integrate $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

$\Rightarrow D = \{(r,\theta) : 0 \le r \le 1, \ 0 \le \theta \le 2\pi\}$

$$\iiint_E x^2+y^2 \, dV = \iint_D \left(\int_{1-x^2-y^2}^{4} x^2+y^2 \, dz \right) dA$$

$$= \int_0^{2\pi} \int_0^1 \int_{1-r^2}^{4} r^2 \, dz \; r \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^1 r^2 (4 - (1-r^2)) \, r \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^1 r^3 (3 + r^2) \, dr \, d\theta$$

$$= \left(\int_0^{2\pi} d\theta \right) \left(\int_0^1 3r^3 + r^5 \, dr \right) = 2\pi \left(\frac{3}{4} r^4 + \frac{r^6}{6} \right) \Bigg|_0^1$$

$$= \boxed{\frac{11\pi}{6}}$$

$$= \iiint_E x^2+y^2\,dV$$

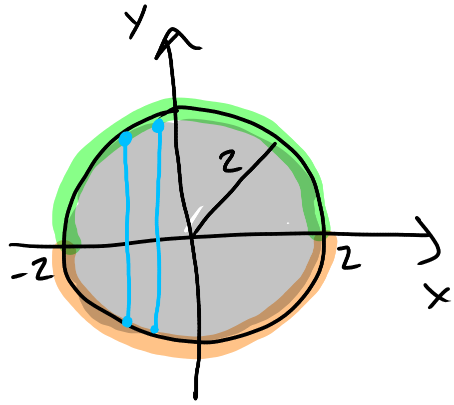
EXAMPLE 4 Evaluate $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dz\,dy\,dx.$

$$E = \{(x,y,z) : -2 \le x \le 2,\ -\sqrt{4-x^2} \le y \le \sqrt{4-x^2},\ \sqrt{x^2+y^2} \le z \le 2\}$$

Describe E in cylindrical coordinates

$y = -\sqrt{4-x^2}$ $\xrightarrow{\text{square}}$ $x^2+y^2=4$

$y = \sqrt{4-x^2}$ $\xrightarrow{\text{square}}$ $x^2+y^2=4$



$$D = \{(r,\theta) : 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}.$$

$$\iiint_E x^2+y^2\,dV = \int_0^{2\pi}\int_0^2\int_r^2 r^2\,dz\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 r^3(2-r)\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 2r^3 - r^4\,dr\,d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^2 2r^3-r^4\,dr\right)$$

$$= 2\pi\left(\frac{r^4}{2} - \frac{r^5}{5}\right)\Big|_0^2$$

$$= 2\pi\left(8 - \frac{32}{5}\right) = \boxed{\frac{16\pi}{5}}$$