Problem 40

(From Section 16.6) We have

$$\vec{r}_u = \langle 1, -3, 1 \rangle$$
$$\vec{r}_v = \langle 1, 0, -1 \rangle .$$

So, $\vec{r}_u \times \vec{r}_v = \langle 3, 2, -3 \rangle$. Thus,

$$\iint_S dS = \iint_D |\vec{r}_u \times \vec{r}_v| \, dA = \int_{-1}^{1} \int_0^2 \sqrt{22} \, dudv = 4\sqrt{22}.$$

Notice here that $\vec{r}_u$ and $\vec{r}_v$ are perpendicular and so the angle form between them is $\pi/2$. Hence,

$$|\vec{r}_u \times \vec{r}_v| = |\vec{r}_u||\vec{r}_v| \sin(\pi/2) = |\vec{r}_u||\vec{r}_v|.$$

Beware, this is not true all the time! We need the vectors $\vec{r}_u$ and $\vec{r}_v$ to be orthogonal, which is not always the case.

Problem 44

(From Section 16.6) The domain D of the parameters x, y are inside the triangle. More precisely, we have

$$D = \{(x, y) \,:\, 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1 - x\}.$$

By the formula for surfaces obtained from an equation $z = f(x, y)$, we can use te following formula

$$A(S) = \iint_D \sqrt{1 + z_x^2 + z_y^2} \, dA.$$

We have $z_x = -4x$ and $z_y = 1$. Thus, we obtain

$$A(S) = \int_0^1 \int_0^{1-x} \sqrt{2 + 16x^2} \, dydx = \sqrt{2} \int_0^1 (1 - x)\sqrt{1 + 8x^2} \, dx \approx 0.72828$$

Problem 10

Intersection the plane with the xy-plane, we get our domain:

$$D = \{(x, y) \,:\, 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1 - x\}.$$

Now, we have $z = 4 - 2x - 2y$, so that

$$dS = \sqrt{1 + z_x^2 + z_y^2} \, dA = \sqrt{1 + 4 + 4} \, dA = 3 \, dA.$$

Thus we get

$$\iint_S xz\,dS = \int_0^1\int_0^{1-x} 3x(4-2x-2y)\,dydx = 3\int_0^1 x(1-x) - 2x^2(1-x) - x(1-x)^2\,dx = -1/12.$$

Problem 16

In spherical coordinates, a cone is $\phi = \pi/4$. Thus, we have

$$S = \{(\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)\ :\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi/4\}.$$

We have $dS = |\vec{r}_\theta \times \vec{r}_\phi|\,dA$. So

$$\begin{aligned}\vec{r}_\theta &= \langle -\sin\theta\sin\phi, \cos\theta\sin\phi, 0\rangle\\ \vec{r}_\phi &= \langle \cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi\rangle\,.\end{aligned}$$

So, we these tangent vectors, we see that

$$|\vec{r}_\theta \times \vec{r}_\phi| = \sin\phi.$$

Thus, $dS = \sin\phi\,dA$ and

$$\begin{aligned}\iint_S y^2\,dS &= \int_0^{\pi/4}\int_0^{2\pi} \sin^2\theta\sin^2\phi\sin\phi\,d\phi d\theta\\ &= \int_0^{\pi/4} \sin^2\theta\,d\theta(4/3)\\ &= (1/8)(\pi-2)(4/3)\\ &= \frac{\pi-2}{6}.\end{aligned}$$

Problem 44

A parametrization of the hemisphere is

$$\vec{r}(\phi,\theta) = \langle 3\cos\theta\sin\phi, 3\sin\theta\sin\phi, 3\cos\phi\rangle$$

where $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi/2$. The cross product of the tangent vectors is

$$\vec{r}_\phi \times \vec{r}_\theta = \left\langle 9\sin^2\phi\cos\theta, 9\sin^2\phi\sin\theta, 9\sin\phi\cos\phi\right\rangle = 9\sin\phi\,\langle\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\rangle\,.$$

The vector $\vec{r}_\phi \times \vec{r}_\theta$ is pointing outward from the sphere (because it's pointing in the same direction as the position vector). Thus we get

$$\begin{aligned}\rho\vec{v}\cdot(\vec{r}_\phi \times \rho\vec{r}_\theta) &= 9\rho\sin\phi\,\langle 3\sin\theta\sin\phi, 3\cos\theta\sin\phi, 0\rangle\cdot\langle\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\rangle\\ &= 9\rho\sin\phi(3\sin\theta\cos\theta\sin^2\phi + 3\sin\theta\cos\theta\sin^2\phi)\\ &= 54\rho\sin\theta\cos\theta\sin^3\phi.\end{aligned}$$

Thus, we have

$$\begin{aligned}
\iint_S \rho\vec{v}\cdot d\vec{S} &= \iint_D \rho\vec{v}\cdot(\vec{r}_\phi\times\vec{r}_\theta)\,dA \\
&= 54\rho\int_0^{2\pi}\int_0^{\pi/2}\sin\theta\cos\theta\sin^3\phi\,d\phi d\theta \\
&= 54\rho\Big(\int_0^{2\pi}\sin\theta\cos\theta\,d\theta\Big)\Big(\int_0^{\pi/2}\sin^3\phi\,d\phi\Big) \\
&= 0.
\end{aligned}$$

Thus, the net flow is 0 kg/s.