$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

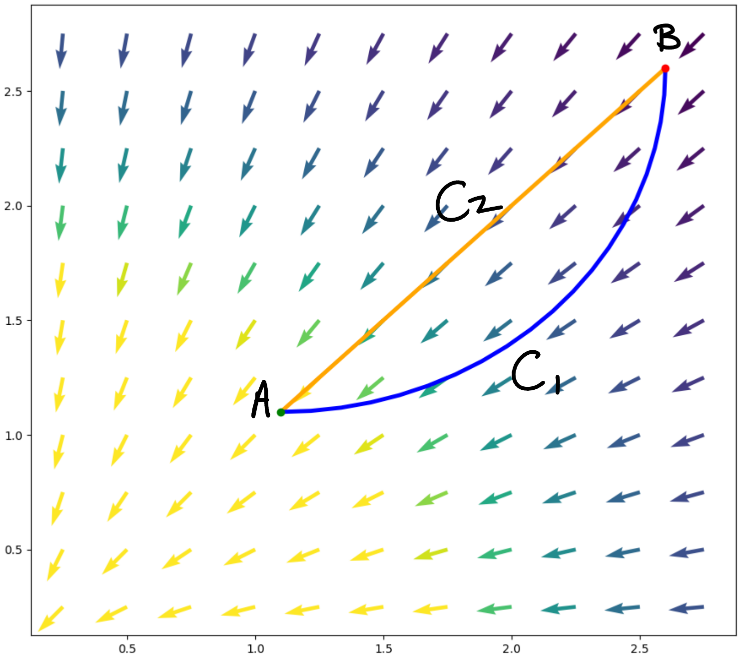
Chapter 16

Vector Calculus

16.3 The Fundamental Theorem for Line Integrals

The Theorem

We want to travel from A(green) to B(red). in space. Assume there is only one massive object creating the gravitational field as in the picture. There are two possible paths from A to B illustrated in orange and blue.



Which path will make us work less?

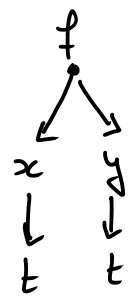
Recall that $\vec{F}$ is conserv.

$$\Rightarrow \vec{\nabla} f = \vec{F}.$$

Work along C_1

$$W_1 = \int_{C_1} \vec{F} \cdot d\vec{r} = \int_a^b \vec{\nabla} f \cdot \vec{r}\,'(t)\, dt$$

$$= \int_a^b \left(f_x \frac{dx}{dt} + f_y \frac{dy}{dt} \right) dt$$



$$= \int_a^b \frac{d}{dt}\left(f(\vec{r}(t)) \right) dt$$

$$= f(\vec{r}(b)) - f(\vec{r}(a)) = f(B) - f(A).$$

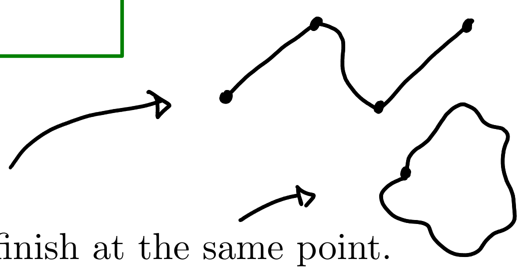
Work on C_2 $\quad W_2 = \int_{C_2} \vec{F} \cdot d\vec{r} = f(B) - f(A).$

Hence $W_1 = W_2 \Rightarrow$ Same work!

Let C be a smooth curve parametrized by $\vec{r}(t)$, $a \le t \le b$. Then

$$\boxed{\int_C \vec{\nabla} f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)),}$$

for any function f with a continuous $\vec{\nabla} f$.



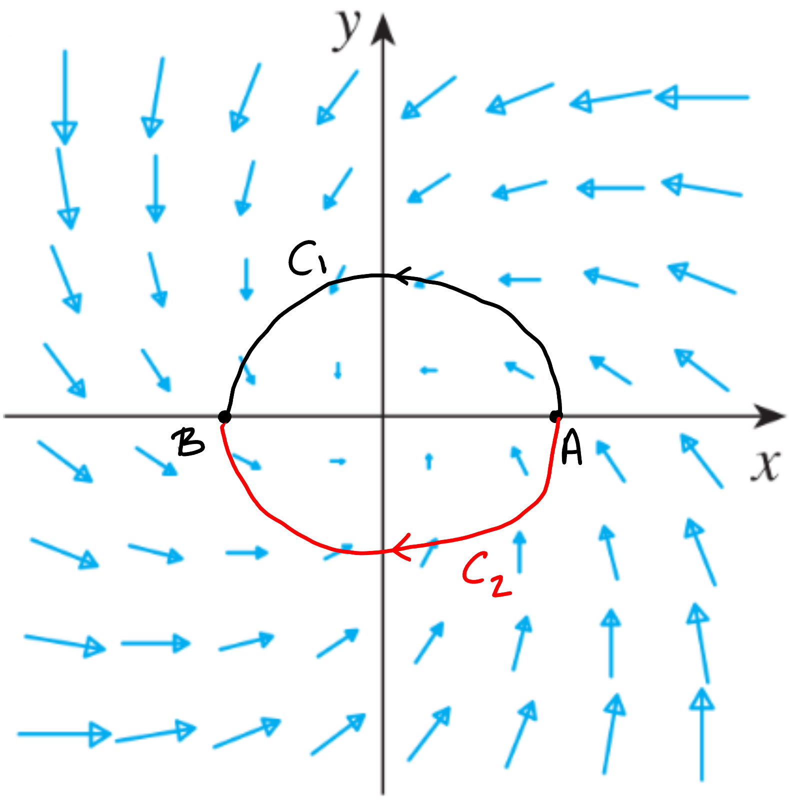
- By a path, we mean a piecewise smooth curve C.
- By a closed path, we mean a path that start and finish at the same point.

A line integral of a vector field $\vec{F}$ is said to be independent of path in a region if

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r},$$

for any path C_1 and C_2 having the same starting point and ending point.

EXAMPLE. Determine if the line integral of the following vector field $\vec{F}$ is independent of path.



• On C_1 :

Because the vectors in $\vec{F}$ seems to point in a similar direction as $\vec{r}\,'(t)$

$\Rightarrow \vec{F} \cdot \vec{r}\,'(t) > 0$.

$\Rightarrow \int_{C_1} \vec{F} \cdot d\vec{r} > 0$.

• On C_2 $\vec{F}$ point in opposite direction $\vec{r}\,'(t)$.

$\Rightarrow \int_{C_2} \vec{F} \cdot d\vec{r} < 0$.

It is not indepent of path.

Equivalent way:

$\vec{F}$ is independent of path is equivalent to satisfying the following condition:

$$\int_C \vec{F} \cdot d\vec{r} = 0$$

for every closed path C.

Link between conservative fields and independence of path

Theorem

Suppose a vector field $\vec{F}$ is defined on the whole of $\mathbb{R}^2$ (resp. $\mathbb{R}^3$) If the line integral $\int_C \vec{F} \cdot d\vec{r}$ is independent of path, then $\vec{F}$ is a conservative vector field.

Proof: See the end of page 1129 in the textbook.

An easier way in 2D

Theorem

Assume $\vec{F}(x, y) = P(x, y)\vec{\imath} + Q(x, y)\vec{\jmath}$ is defined for every point (x, y). Assume that P and Q have continuous partial derivatives. Then the following are equivalent:

a) $\vec{F}$ is conservative.
b) $Q_x - P_y = 0$.

- The quantity $Q_x - P_y$ expresses the tendency of a vector field to rotate about a point.

EXAMPLE 2 Determine whether or not the vector field

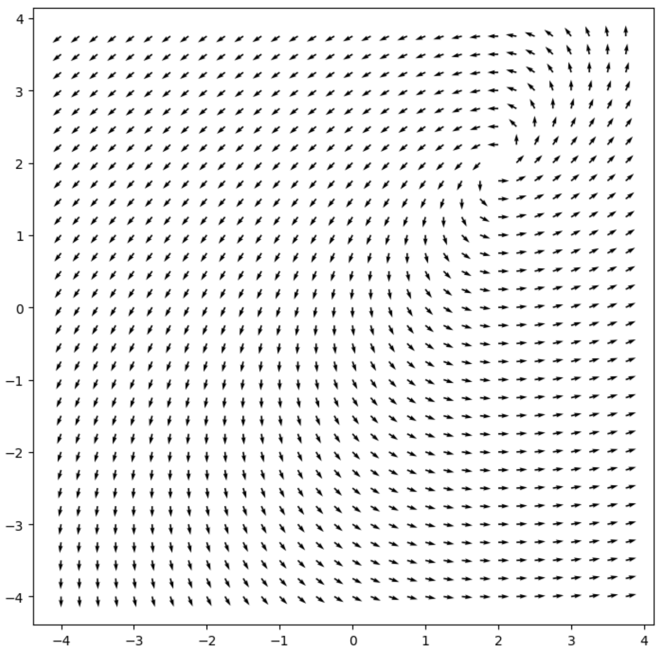
$$\mathbf{F}(x, y) = (x - y)\,\mathbf{i} + (x - 2)\,\mathbf{j}$$

is conservative.

$P = x - y$

$Q = x - 2$

$Q_x - P_y = 1 - (-1) = 2 \neq 0.$



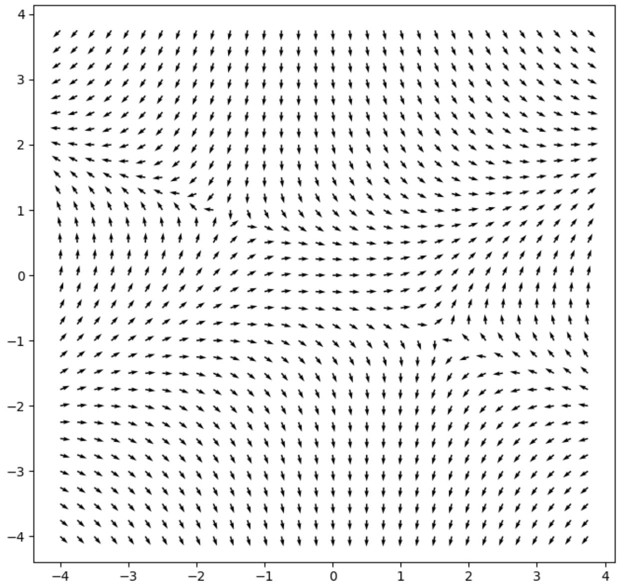
EXAMPLE 3 Determine whether or not the vector field

$$\mathbf{F}(x, y) = (3 + 2xy)\,\mathbf{i} + (x^2 - 3y^2)\,\mathbf{j}$$

is conservative.

$P = 3 + 2xy \longrightarrow Q_x - P_y = 2x - 2x = 0.$

$Q = x^2 - 3y^2$



EXAMPLE 4

(a) If $\mathbf{F}(x, y) = (3 + 2xy)\,\mathbf{i} + (x^2 - 3y^2)\,\mathbf{j}$, find a function f such that $\mathbf{F} = \nabla f$.

(b) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve given by

$$\mathbf{r}(t) = e^t \sin t\,\mathbf{i} + e^t \cos t\,\mathbf{j} \qquad 0 \leqslant t \leqslant \pi$$