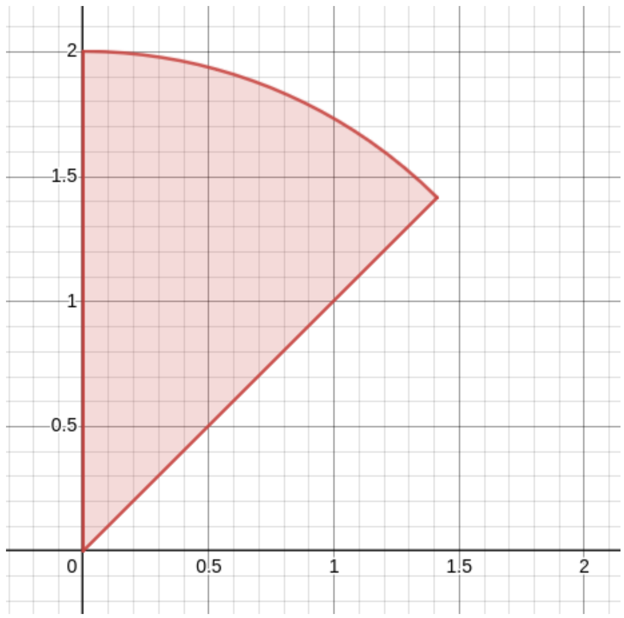
Section 15.3, Problem 8 **(10 Pts)**

A picture of the region is presented below.



Using polar coordinates, the equation of the circle $x^2+y^2=4$ becomes $r=2$. Therefore, $0\leq r\leq 2$. The equations $y=x$ and $x=0$ will be used to find the bounds on the angle. When $y=x$, we then have $r\sin\theta = r\cos\theta$, so that $\tan\theta = 1$. Therefore, $\theta_1=\pi/4$. When $x=0$, we then have $r\cos\theta=0$, so that $\cos\theta=0$. Therefore, $\theta_2=\frac{\pi}{2}$. The region, in polar coordinates, is therefore

$$R=\{(r,\theta)\,:\,0\leq r\leq 2 \text{ and } \frac{\pi}{4}\leq\theta\leq\frac{\pi}{2}\}.$$

The integral in polar coordinates turns out to be

$$\begin{aligned}\iint_R (2x-y)\,dA &= \int_{\pi/4}^{\pi/2}\int_0^2 (2r\cos\theta - r\sin\theta)r\,drd\theta \\ &= \int_{\pi/4}^{\pi/2}\int_0^2 2r^2\cos\theta - r^2\sin\theta\,drd\theta.\end{aligned}$$

We can now evaluate the iterated integrals:

$$\begin{aligned}\iint_R (2x-y)\,dA &= \int_{\pi/4}^{\pi/2}\left(2\frac{r^3}{3}\cos\theta - \frac{r^3}{3}\sin\theta\right)\Big|_0^2\,d\theta \\ &= \int_{\pi/4}^{\pi/2}\left(\frac{16}{3}\cos\theta - \frac{8}{3}\sin\theta\right)d\theta \\ &= \frac{8}{3}\,(2\sin\theta+\cos\theta)|_{\pi/4}^{\pi/2} \\ &= \frac{16}{3}-4\sqrt{2}.\end{aligned}$$

△

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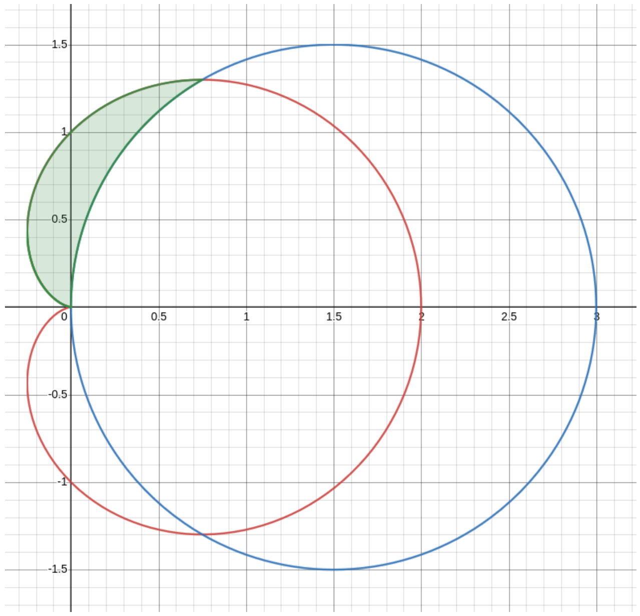
Section 15.3, Problem 14 **(10 Pts)**

In fact, the problem asked to find the area inside the cardioid and outside the circle. By symmetry, we can restrict ourselves to the part of the region above the x-axis (see the picture below).



There will be two regions, a region R_1 from $\theta = \pi/3$ to $\theta = \pi/2$ and another region R_2 from $\theta = \pi/2$ to $\theta = \pi$. We explicitly have

$$R_1 = \{(r, \theta) \,:\, 3\cos\theta \leq r \leq 1 + \cos\theta,\, \pi/3 \leq \theta \leq \pi/2\}$$

and

$$R_2 = \{(r, \theta) \,:\, 0 \leq r \leq 1 + \cos\theta,\, \pi/2 \leq \theta \leq \pi\}.$$

Therefore, we obtain

$$\begin{aligned} Area(R) &= 2\Big(\iint_{R_1} dA + \iint_{R_2} dA\Big) \\ &= 2\Big(\int_{\pi/3}^{\pi/2}\int_{3\cos\theta}^{1+\cos\theta} r dr d\theta + \int_{\pi/2}^{\pi}\int_{0}^{1+\cos\theta} r dr d\theta\Big) \\ &= 2\Big(1 - \frac{\pi}{4} + \frac{3\pi}{8} - 1\Big) = \frac{\pi}{4}. \end{aligned}$$

△

Section 15.3, Problem 20 **(10 Pts)**

The function is $f(x, y) = \sqrt{x^2 + y^2}$ and therefore, the volume of the solid is given by

$$\text{Vol}(S) = \iint_D \sqrt{x^2 + y^2}\, dA,$$

where D is the region over which is the integration.

The region D is between two circles (an annulus):

$$D = \{(x, y) \,:\, 1 \leq x^2 + y^2 \leq 4\}$$

which can be described in polar coordinates, with $r = \sqrt{x^2 + y^2}$ as

$$D = \{(r, \theta)\,;\, 1 \leq r \leq 2,\, 0 \leq \theta \leq 2\pi\}.$$

So

$$\begin{aligned}\mathrm{Vol}(S) &= \int_0^{2\pi}\int_1^2 r^2 \, dr d\theta \\ &= \Big(\int_0^{2\pi} d\theta\Big)\Big(\int_1^2 r^2\, dr\Big) = \frac{14}{3}\pi. \end{aligned} \qquad \triangle$$

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Section 15.3, Problem 32 **(10 Pts)**

From the bounds in the integrals, we see that

$$D = \{(x, y) \,:\, 0 \leq x \leq 2,\, 0 \leq y \leq \sqrt{2x - x^2}\}.$$

The upper bound $y = \sqrt{2x - x^2}$ can be rewritten as

$$y^2 = 2x - x^2 \iff x^2 - 2x + y^2 = 0 \iff (x-1)^2 + y^2 = 1.$$

So this is a circle centered at $(1, 0)$ of radius 1. Therefore, the region D is the upper half region enclosed by this circle.
Letting $x = r\cos\theta$ and $y = r\sin\theta$, we see that

$$y^2 = 2x - x^2 \Rightarrow r^2\sin^2(\theta) = 2r\cos\theta - r^2\cos^2(\theta) \Rightarrow r^2 = 2r\cos\theta \Rightarrow r = 2\cos\theta.$$

This circle intersects the x axis at $x = 0$ and $x = 2$. When $x = 0$, we obtain $\theta = \pi/2$ and when $x = 2$, we obtain $\theta = 0$. Therefore,

$$\begin{aligned}\int_0^2\int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dydx &= \iint_D \sqrt{x^2+y^2}\, dA \\ &= \int_0^{\pi/2}\int_0^{2\cos\theta} r^2\, drd\theta \\ &= \int_0^{\pi/2} \frac{8\cos^3(\theta)}{3}\, d\theta \\ &= \int_0^{\pi/2} \frac{8}{3}\Big(\cos\theta - \sin^2(\theta)\cos\theta\Big)\, d\theta \\ &= \frac{8}{3}\Big(\sin\theta\Big)\Big|_0^{\pi/2} - \frac{8}{3}\int_0^1 u^2\, du \\ &= \frac{8}{3} - \frac{8}{9} = \frac{16}{9}.\end{aligned}$$