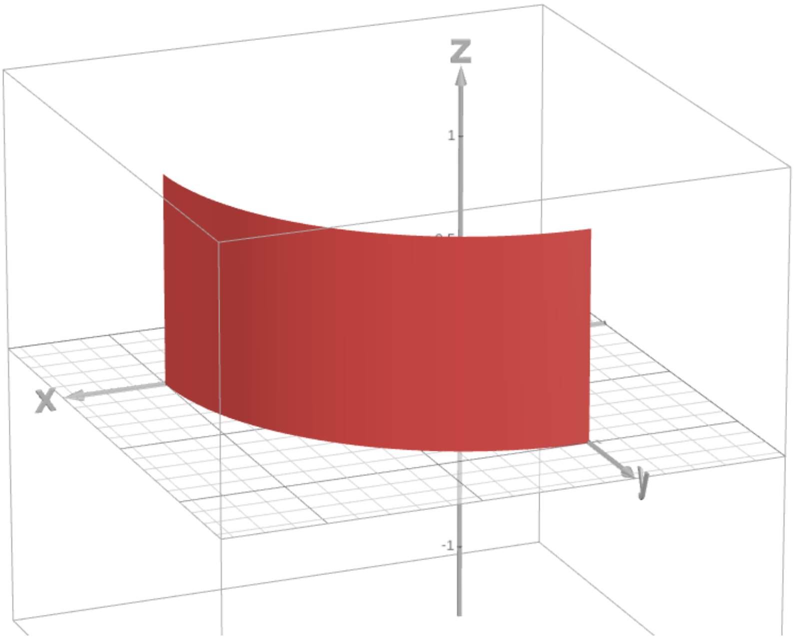
Chapter 16

Vector Calculus

16.7 Surface Integrals

Surface Differential

EXAMPLE. Find the area of the following parametric surface S:



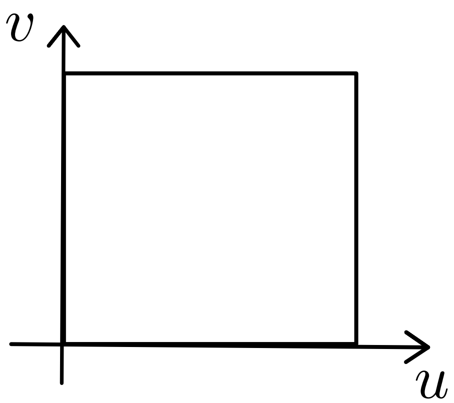
https://www.desmos.com/3d/728faf627a

Parametric Equations

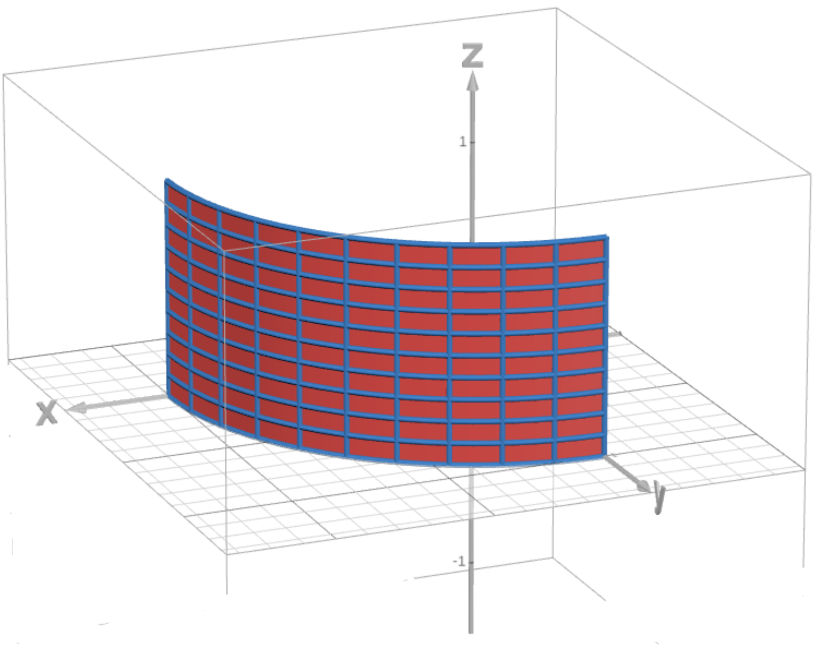
$$x = \cos\left((\pi/2)u\right)$$
$$y = \sin\left((\pi/2)u\right)$$
$$z = v$$

$0 \le u \le 1,\ 0 \le v \le 1.$

1. Divide the uv-region in small rectangles.



2. Approximate the area of each small piece.



3. Sum up.

4. Compute the Area.

Surface Area Differential:

$$\boxed{dS = |\vec{r}_u \times \vec{r}_v| dA}$$

Integral of scalar-valued functions.

Data:

- A surface S.
- A parametrization $\vec{r}(u, v)$ of the surface with domain D.
- A scalar-valued function $f(x, y, z)$.

$$\boxed{\iint_S f(x, y, z)\, dS = \iint_D f(\vec{r}(u, v))|\vec{r}_u \times \vec{r}_v|\, dA}$$

5–20 Evaluate the surface integral.

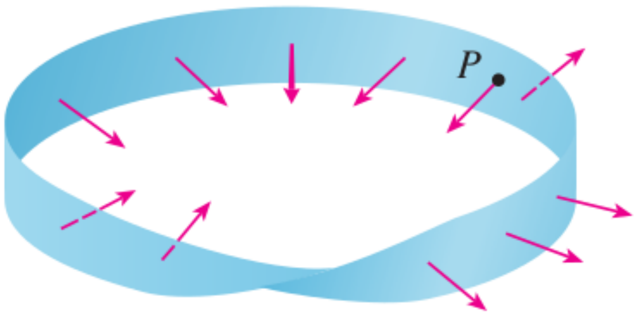
5. $\iint_S (x + y + z)\, dS$,
S is the parallelogram with parametric equations $x = u + v$, $y = u - v$, $z = 1 + 2u + v$, $0 \leqslant u \leqslant 2$, $0 \leqslant v \leqslant 1$

Example.

Evaluate $\displaystyle\iint_S z\,dS$, where S is the surface whose sides are given by the cylinder $x^2 + y^2 = 1$ from $z = 0$ to $z = 2$ and whose bottom is the disk $x^2 + y^2 \leq 1$ in the plane $z = 0$.

Surface integral of Vector Fields.

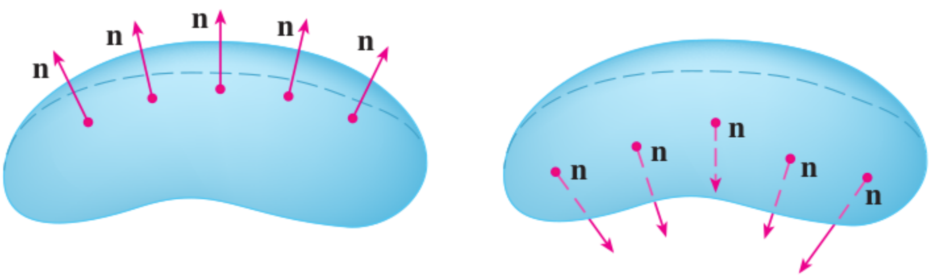
- Non-orientable surfaces.



https://www.desmos.com/3d/45663aa8e7

- Orientable surface.

https://www.desmos.com/3d/b9f507b01b



- We assume that the surface is orientable, so that there is a well-defined normal vector.

- For a parametric surface with parametrization $\vec{r}(u, v)$:

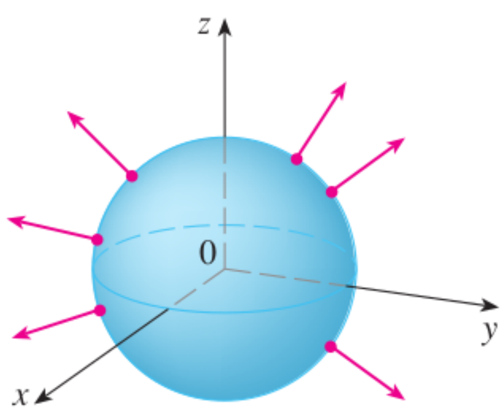
$$\boxed{\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}}$$

Example.

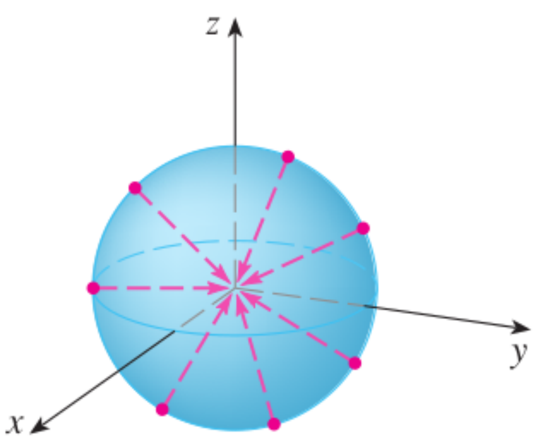
Find a normal vector at every point of a sphere of equation

$$x^2 + y^2 + z^2 = 1$$

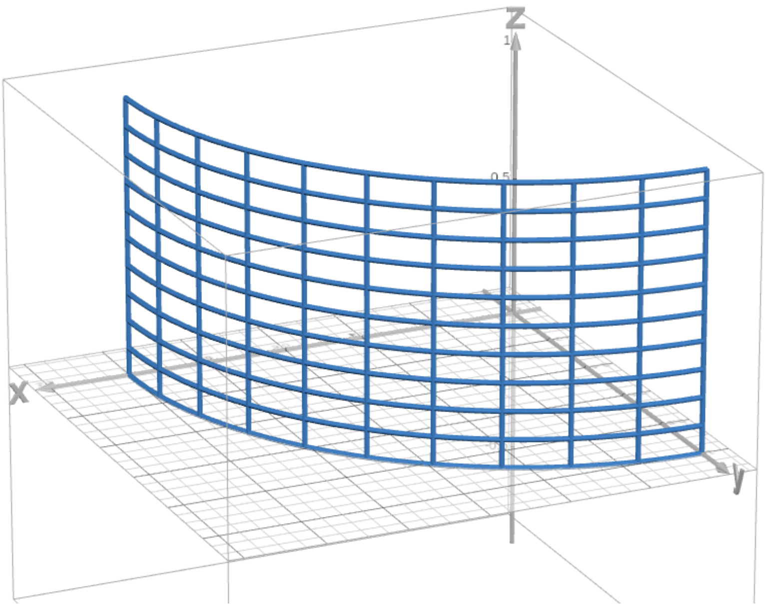
Positive orientation of a closed surface.



Negative orientation of a closed surface.



Flux integral (or Surface integral).



https://www.desmos.com/3d/d51cd6d708

Data:

- An orientable surface S.
- A parametrization $\vec{r}(u,v)$ of the surface.
- A vector field $\vec{F}(x,y,z)$.

$$\boxed{\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\,dS = \iint_D \vec{F}\cdot(\vec{r}_u\times\vec{r}_v)\,dA}$$

EXAMPLE.

Find the flux integral of $\vec{F}(x,y,z) = \langle xy, yz, zx\rangle$ through the part of the paraboloid $z = 4 - x^2 - y^2$ lying above the square $[0,1]\times[0,1]$ and with upward orientation.

EXAMPLE.

Find the flux integral of $\vec{F}(x, y, z) = \langle x, 2y, 3z \rangle$ if S is a cube with diagonal $(0, 0, 0)$ to $(1, 1, 1)$ and S has the positive orientation.

Gauss' Law

The net charge enclosed by a closed surface S is

$$Q = \varepsilon_0 \iint_S \vec{E} \cdot d\vec{S}$$

where $\vec{E}$ is the electric field and ε_0 is the permittivity of free space.