# Chapter 16

# Vector Calculus

## 16.5 Curl and Divergence

## Curl.

For a vector field $\vec{F} = \langle P, Q, R \rangle$,

$$\boxed{\operatorname{curl} \vec{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle .}$$

Another way to write the curl:

$$\vec{\nabla} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \implies \operatorname{curl} \vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

Note:

---

**EXAMPLE 1** If $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$, find curl $\mathbf{F}$.

THEOREM.

Let $\vec{F} = \langle P, Q, R \rangle$ . Assume that

- The functions $P, Q, R$ have continuous partial derivatives.
- $\text{curl}\vec{F} = \vec{0}$.

Then, $\vec{F}$ is conservative.

Note: This generalizes the condition for convervative vector fields in 2D.

---

**EXAMPLE 3**

(a) Show that

$$\mathbf{F}(x, y, z) = y^2z^3\,\mathbf{i} + 2xyz^3\,\mathbf{j} + 3xy^2z^2\,\mathbf{k}$$

is a conservative vector field.

(b) Find a function $f$ such that $\mathbf{F} = \nabla f$.

# Divergence.

For a vector field $\vec{F} = \langle P, Q, R \rangle$ ,

$$\boxed{\operatorname{div} \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.}$$

Another way to write the divergence:

$$\vec{\nabla} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \implies \operatorname{div} \vec{F} = \vec{\nabla} \cdot \vec{F}.$$

---

**EXAMPLE 4** If $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$, find div $\mathbf{F}$.

---

**THEOREM.**

Let $\vec{F} = \langle P, Q, R \rangle$ . Assume that

- The functions $P, Q, R$ have continuous partial derivatives.

Then, $\operatorname{div}(\operatorname{curl} \vec{F}) = 0$.

Intuition behind this result:

**EXAMPLE 5** Show that the vector field $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$ can't be written as the curl of another vector field, that is, $\mathbf{F} \neq \operatorname{curl} \mathbf{G}$.

---

## Laplace's Equation.

$$\Delta f = \operatorname{div} \vec{\nabla} f = f_{xx} + f_{yy} + f_{zz} \quad \Longrightarrow \quad \boxed{\Delta f = 0}$$

Functions satisfying Laplace's equation are called HARMONIC functions.

## Vector Form of Green's Theorem.

$$\boxed{\int_C \vec{F} \cdot d\vec{r} = \iint_D (\operatorname{curl} \vec{F}) \cdot \vec{k}\, dA}$$