Solutions MT-03

M244

Q1 Q2 Q3 Q4 Q5 BQ

Question 1

(a) We have $x = 2u\cos v$, $y = 3u\sin v$.

Then

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2\cos v & -2u\sin v \\ 3\sin v & 3u\cos v \end{vmatrix}$$

$$= 6u\cos^2 v + 6u\sin^2 v$$

$$= \boxed{6u}$$

(b) Replacing in the equations of the ellipse:

$$\frac{\cancel{4}u^2\cos^2 v}{\cancel{4}} + \frac{\cancel{9}u^2\sin^2 v}{\cancel{9}} = 1 \rightarrow u^2 = 1$$

$$\rightarrow u = 1.$$

Therefore, the parameters u and v are in a circular region of radius 1. Thus,

$$\text{Area}(D) = \iint_D dA$$

$$= \int_0^{2\pi} \int_0^1 \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv$$

$$= \int_0^{2\pi} \int_0^1 6u \, du\, dv$$

$$= 3 \cdot 2\pi = \boxed{6\pi}$$

Question 2

(a) Along the line $y = x$,

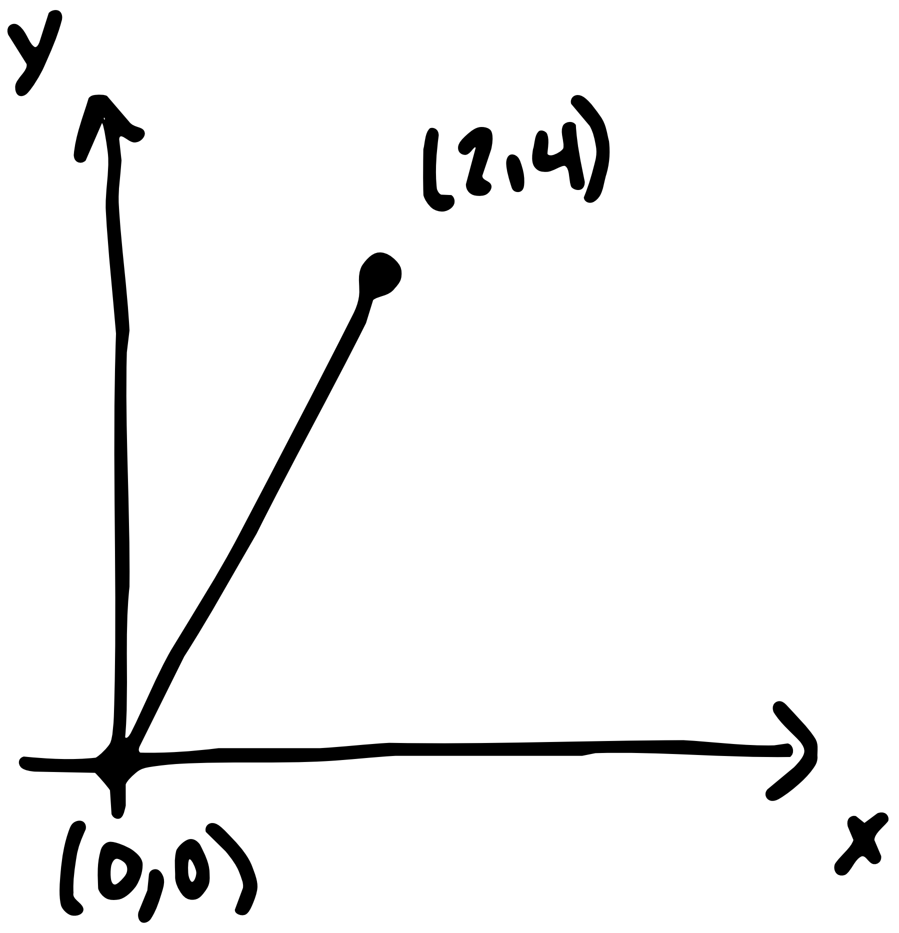
$$\vec{F}(x,y) = \langle 0, x \rangle .$$

So, $\vec{F}$ only points in the direction of the y-axis along the line $y = x$. The $\boxed{\text{Figure I}}$ exhibits this feature.

(b) $Q_x - P_y = 1 - (-1) = 2$, so not conserv.

Question 3

(a)



$$\vec{r}(t) = \langle 0,0\rangle + t(\langle 2,4\rangle - \langle 0,0\rangle)$$
$$= \langle 2t, 4t\rangle$$

with $0 \le t \le 1$.

Then,
$$\int_C x \, ds = \int_0^1 2t\sqrt{2^2 + 4^2}\, dt$$
$$= 2\sqrt{20}\int_0^1 t\, dt = \sqrt{20} = \boxed{2\sqrt{5}}$$

(b) We have $\vec{r}'(t) = \langle 3t^2, 2t\rangle$. So,

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \langle t^7, -t^6\rangle \cdot \langle 3t^2, 2t\rangle\, dt$$
$$= \int_0^1 3t^9 - 2t^7\, dt$$
$$= \frac{3}{10} - \frac{1}{4} = \frac{12-10}{40} = \boxed{\frac{1}{20}}$$

Question 4

(a) We have $Q_x - P_y = 2x + y^2 - (2x + y^2)$
$= 0$.

So, $\boxed{\text{yes } \vec{F} \text{ is conservative.}}$

Set $\vec{F} = \vec{\nabla} f = \langle f_x, f_y \rangle$. Then

$$\begin{cases} f_x = 2xy + \frac{y^3}{3} & (1) \\ f_y = x^2 + xy^2 & (2) \end{cases}$$

$(1) \Rightarrow f(x,y) = x^2 y + \frac{xy^3}{3} + h(y)$.

$(2) \Rightarrow x^2 + xy^2 + h'(y) = f_y = x^2 + xy^2$

$\Rightarrow h'(y) = 0 \Rightarrow h(y) = C$

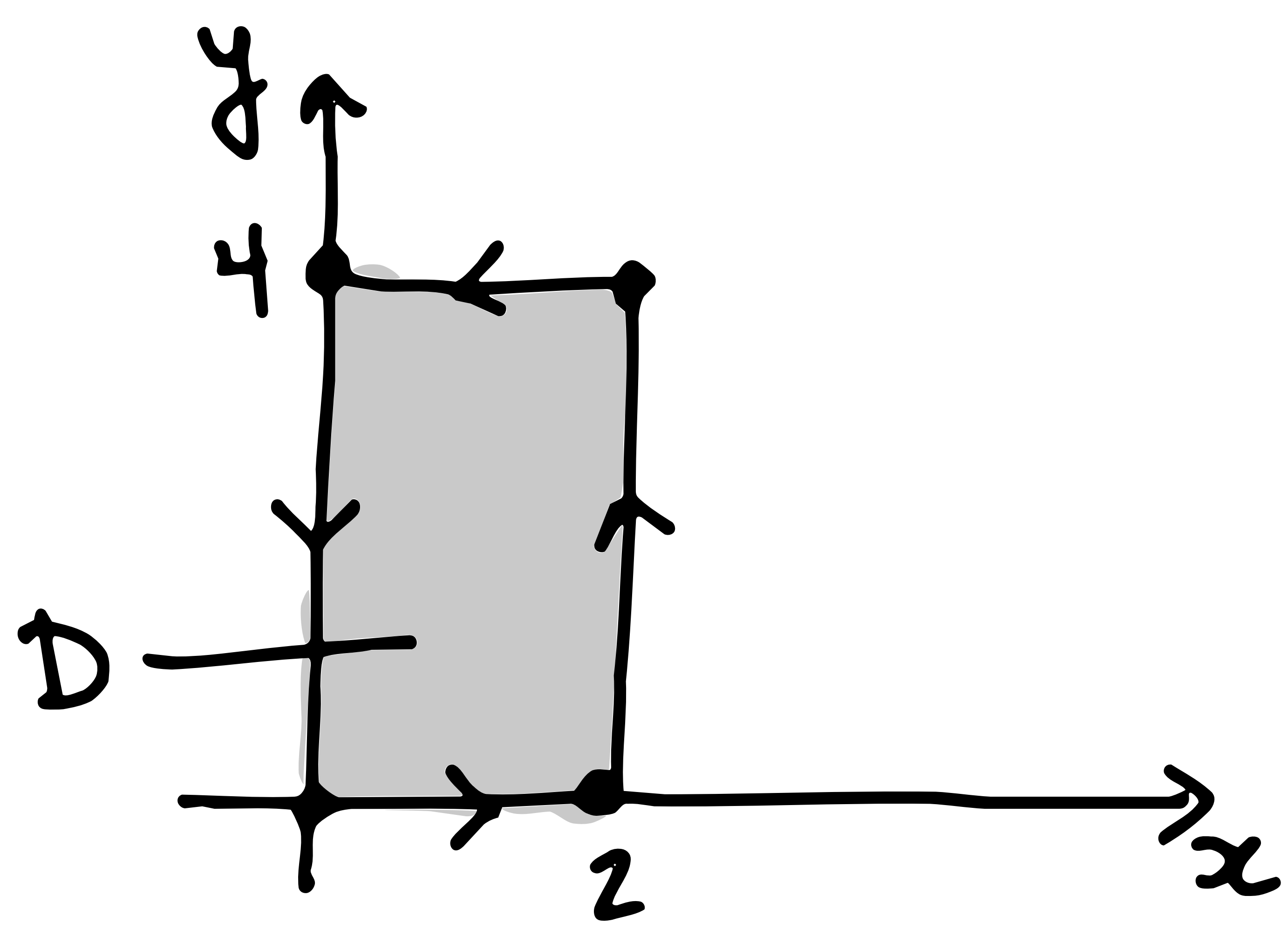
Thus, $\boxed{f(x,y) = x^2 y + \frac{xy^3}{3} + C}$.

(b) $A = \vec{r}(0) = \langle 0, 0 \rangle$, $B = \vec{r}(\pi) = \langle 1, 2 \rangle$.

$\rightarrow \int_C \vec{F} \cdot d\vec{r} = f(B) - f(A) = 2 + \frac{8}{3} = \boxed{\frac{14}{3}}$

Question 5.

1) Picture



$$Q_x - P_y = y + \cancel{\cos x} - \cancel{x \sin x} - (\cancel{\cos x} - \cancel{x \sin x})$$

$$= y.$$

2) Green's Theorem.

$$D = \{ (x,y) : \quad 0 \le x \le 2, \quad 0 \le y \le 4 \}$$

Thus,

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D y \, dA$$

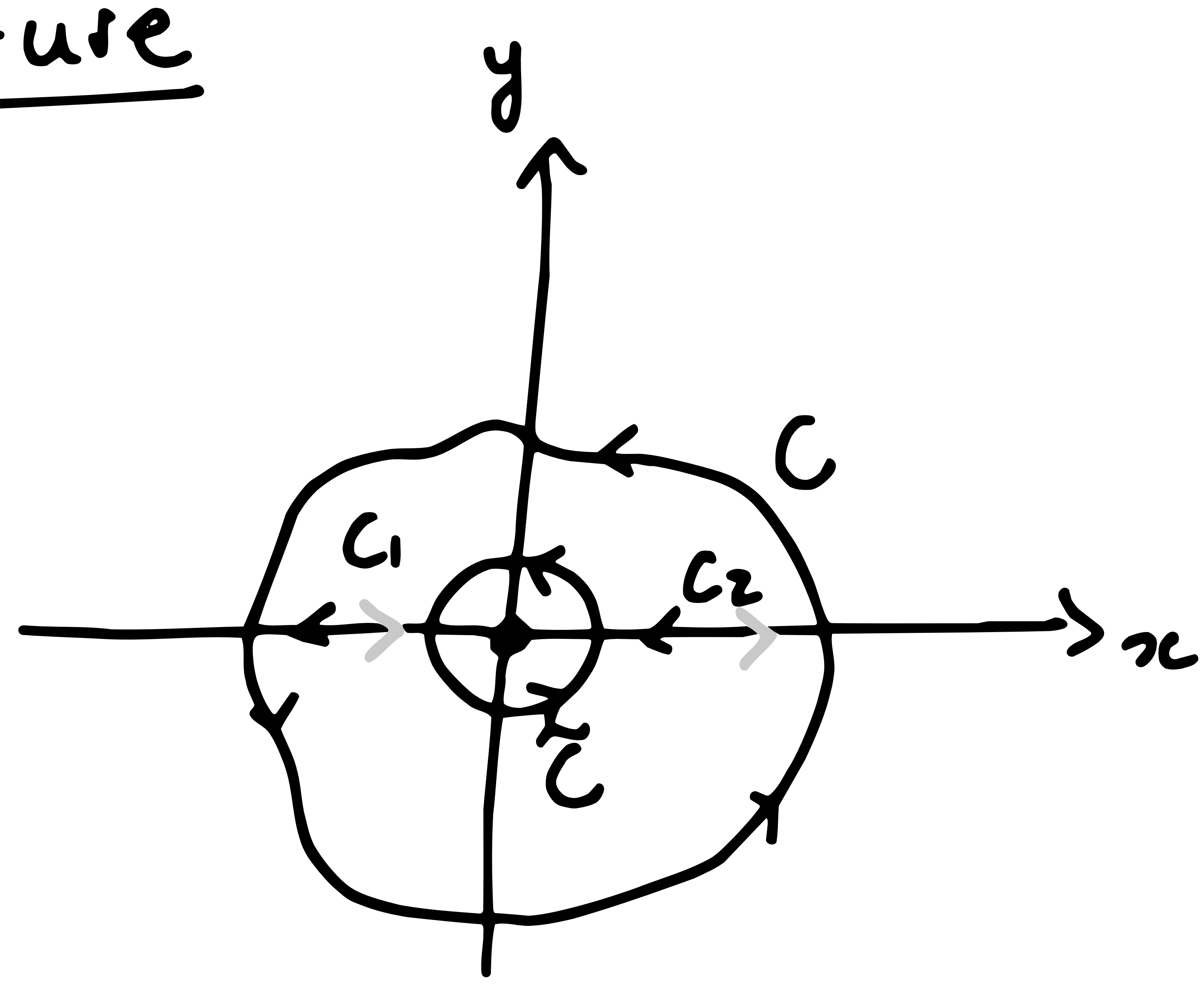
$$= \int_0^2 \int_0^4 y \, dy \, dx$$

$$= 2 \left. \frac{y^2}{2} \right|_0^4$$

$$= 2 \cdot 8 = \boxed{16}$$

BONUS QUESTION

① Picture



$\tilde{C}$: circle of radius ρ inside the curve C.

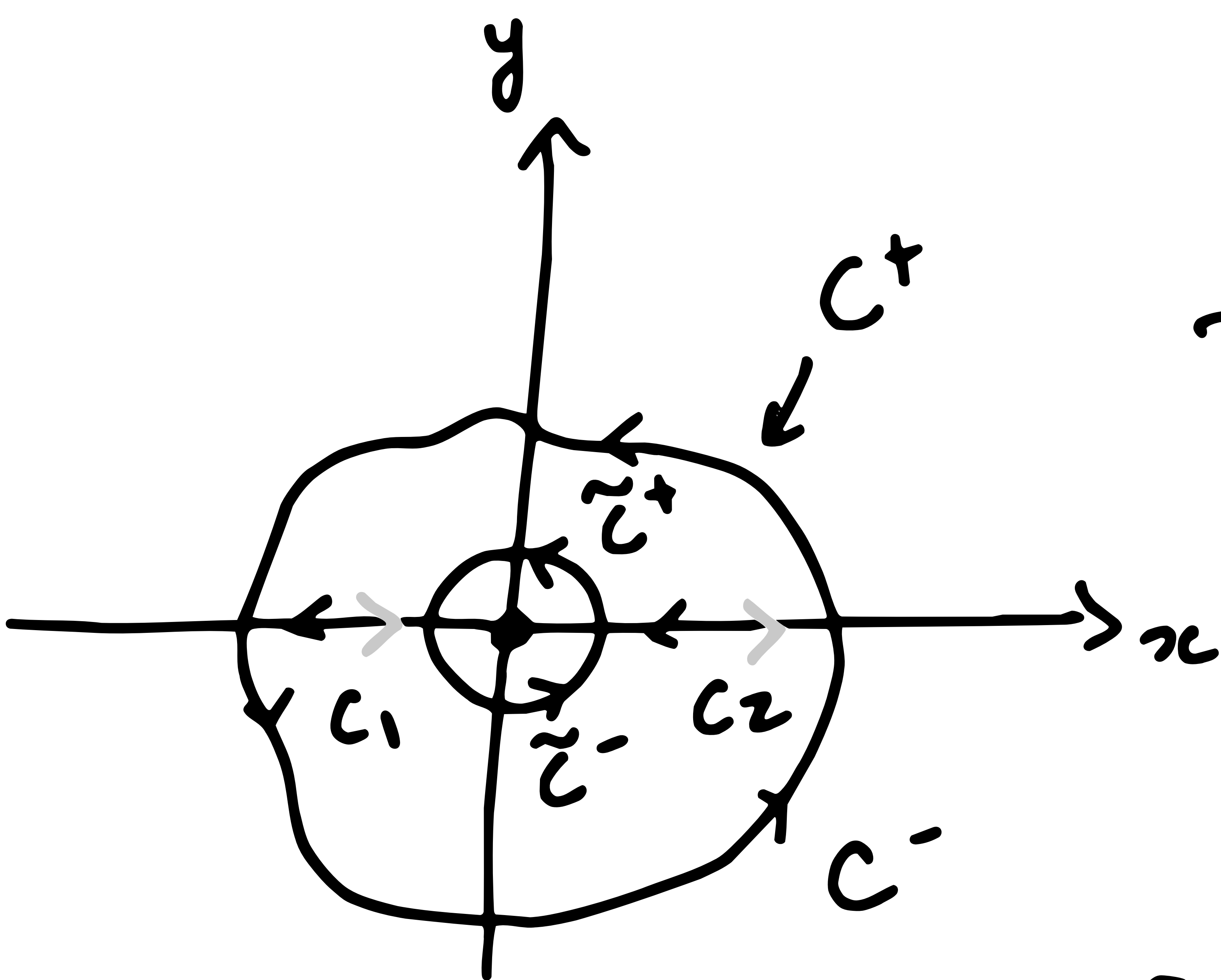
② $\vec{F}$ is conservative

$$Q_x = \frac{(x^2+y^2) - 2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$P_y = \frac{-(x^2+y^2) + 2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

Thus, $\vec{F}$ is conservative.

② Reduction to a circle



Since the path

$D = C^+ \cup (-C_1) \cup (-\tilde{C}^+) \cup (-C_2)$

is a closed path and bounds a simply connected region, by Green's theorem:

$$\int_D \vec{F} \cdot dr = 0$$

$$\Rightarrow \int_{C^+} \vec{F} \cdot d\vec{r} - \int_{C_1} \vec{F} \cdot d\vec{r}$$

$$= \int_{\tilde{C}^+} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}. \quad (*)$$

Similarly:

$$\int_{C^-} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$$

$$= \int_{\tilde{C}^-} \vec{F} \cdot d\vec{r} - \int_{C_1} \vec{F} \cdot d\vec{r} \quad (**)$$

Summing (*) and (**) together

$$\Rightarrow \int_{C^+} \vec{F} \cdot d\vec{r} + \int_{C^-} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} - \int_{C_1} \vec{F} \cdot d\vec{r}$$

$$= \int_{\tilde{C}^+} \vec{F} \cdot d\vec{r} + \int_{\tilde{C}^-} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} - \int_{C_1} \vec{F} \cdot d\vec{r}$$

$$\Rightarrow \int_{C} \vec{F} \cdot d\vec{r} = \int_{\tilde{C}} \vec{F} \cdot d\vec{r}.$$

4. <u>Compute path integral on $\tilde{C}$.</u>

Now, $\vec{r}(t) = \langle \rho\cos t, \rho\sin t \rangle$ so that

$$\int_{C} \vec{F} \cdot d\vec{r} = \int_{\tilde{C}} \vec{F} \cdot d\vec{r}$$

$$= \int_0^{2\pi} \left\langle \frac{-\rho\sin t}{\rho^2}, \frac{\rho\cos t}{\rho^2} \right\rangle \cdot \langle -\rho\sin t, \rho\cos t \rangle \, dt$$

$$= \int_0^{2\pi} \frac{\rho^2\sin^2 t + \rho^2\cos^2 t}{\rho^2} \, dt$$

$$= \boxed{2\pi}.$$

(5) Green's Theorem.

There is no contradiction in our version of Green's Theorem because the vector field is not defined at $\langle 0,0\rangle$. In the version of the lecture notes, we required the vector field to be defined at every point of the plane!