

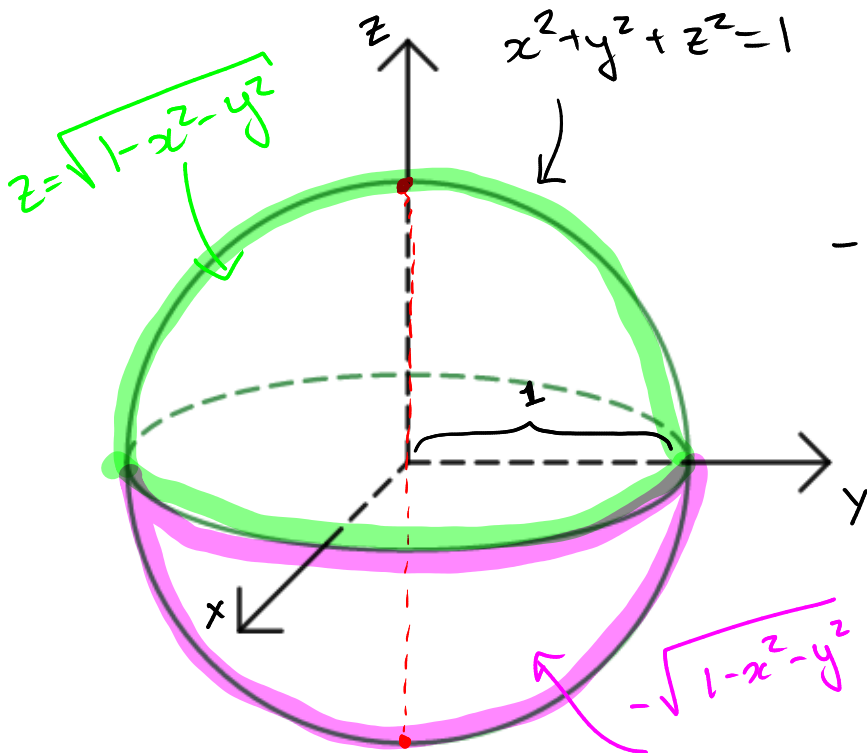
# Chapter 15

## Multiple Integrals

15.8 Triple integrals in spherical coordinates

## Spherical coordinates

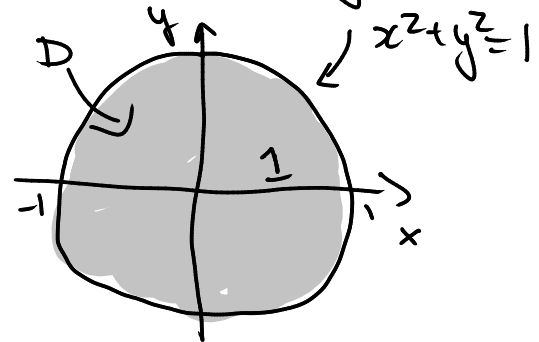
**EXAMPLE.** Describe the solid bounded by the sphere (picture below).



As a type 1:

$$-\sqrt{1-x^2-y^2} \leq z \leq \sqrt{1-x^2-y^2}$$

Shadow on  $xy$ -plane:



$$D = \{(x, y) : -1 \leq x \leq 1, -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}\}$$

$$E = \{(x, y, z) : -1 \leq x \leq 1, -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}, -\sqrt{1-x^2-y^2} \leq z \leq \sqrt{1-x^2-y^2}\}.$$

## Definition

Cartesian  $\longrightarrow$  Spherical

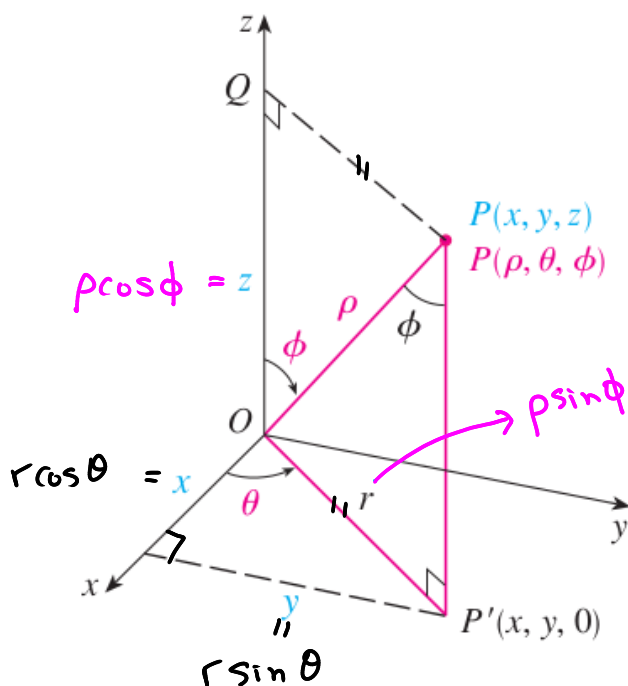
$$x = \rho \sin \phi \cos \theta$$

$$y = \rho \sin \phi \sin \theta$$

$$0 \leq \theta \leq 2\pi$$

$$z = \rho \cos \phi$$

$$0 \leq \phi \leq \pi, \quad \rho \geq 0.$$

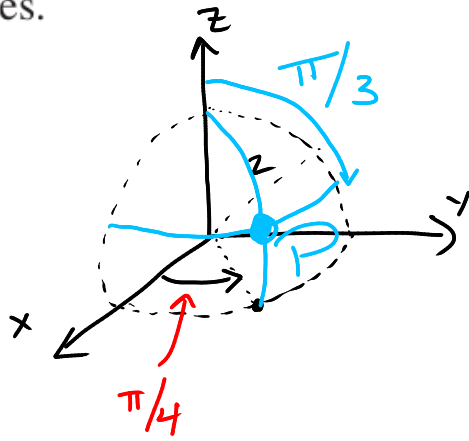


**EXAMPLE 1** The point  $(2, \pi/4, \pi/3)$  is given in spherical coordinates. Plot the point and find its rectangular coordinates.

$$\rho = 2$$

$$\theta = \pi/4$$

$$\phi = \pi/3$$



$$x = \rho \sin \phi \cos \theta = 2 \sin(\pi/3) \cos(\pi/4) = 2 \left( \frac{\sqrt{3}}{2} \right) \left( \frac{\sqrt{2}}{2} \right) = \boxed{\frac{\sqrt{6}}{2}}$$

$$y = \rho \sin \phi \sin \theta = 2 \sin(\pi/3) \sin(\pi/4) = \boxed{\frac{\sqrt{6}}{2}}$$

$$z = \rho \cos \phi = 2 \cos(\pi/3) = 2 \left( \frac{1}{2} \right) = \boxed{1}$$

**EXAMPLE 2** The point  $(0, 2\sqrt{3}, -2)$  is given in rectangular coordinates. Find spherical coordinates for this point.

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{0^2 + (2\sqrt{3})^2 + (-2)^2} = \sqrt{12 + 4} = \boxed{4}$$

We have  $z = \rho \cos \phi \Rightarrow -2 = 4 \cos \phi$

$$\Rightarrow -\frac{1}{2} = \cos \phi$$

$$\Rightarrow \boxed{\phi = \frac{2\pi}{3}} \quad \left( \text{between } 0 \text{ and } \pi \right)$$

$$0 = x = 2 \sin\left(\frac{2\pi}{3}\right) \cos(\theta) = \sqrt{3} \cos(\theta)$$

$$\Rightarrow 0 = \sqrt{3} \cos(\theta) \Rightarrow \cos \theta = 0 \Rightarrow \theta = \pi/2 \text{ or } \theta = 3\pi/2$$

Here  $y = 2\sqrt{3} \geq 0 \Rightarrow \boxed{\theta = \pi/2}$

## Equations of important solids. → surface.

Sphere of radius R.

$$\rho = R$$

$$\sqrt{x^2 + y^2 + z^2} = \sqrt{R^2}$$

$$\Rightarrow \sqrt{x^2 + y^2 + z^2} = R$$

$$\Rightarrow \rho = R$$

$$0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi$$

Half planes.

$$\theta = c$$

$$0 \leq \rho < \infty.$$

$$0 \leq \phi \leq \pi.$$

$$x = \rho \sin \phi \cos \theta$$

$$y = \rho \sin \phi \sin \theta$$

Replace that in  
 $ax + by = 0$

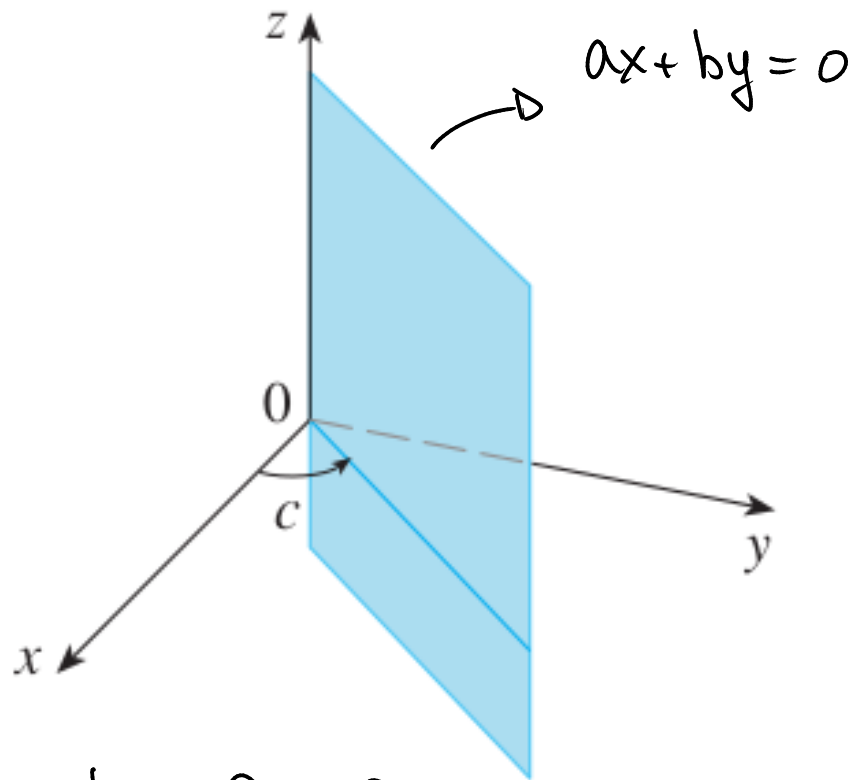
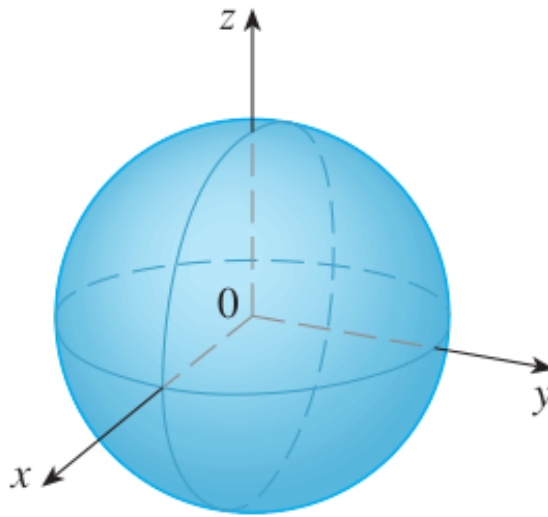
$$\Rightarrow a \rho \sin \phi \cos \theta + b \rho \sin \phi \sin \theta = 0$$

$$\Rightarrow a \cancel{\rho \sin \phi} \cos \theta = -b \cancel{\rho \sin \phi} \sin \theta$$

$$\Rightarrow a \cos \theta = -b \sin \theta$$

$$\Rightarrow -\frac{a}{b} = \frac{\sin \theta}{\cos \theta}$$

$$\Rightarrow -\frac{a}{b} = \tan \theta \Rightarrow \theta = \overbrace{\arctan\left(-\frac{a}{b}\right)}^{=c}$$



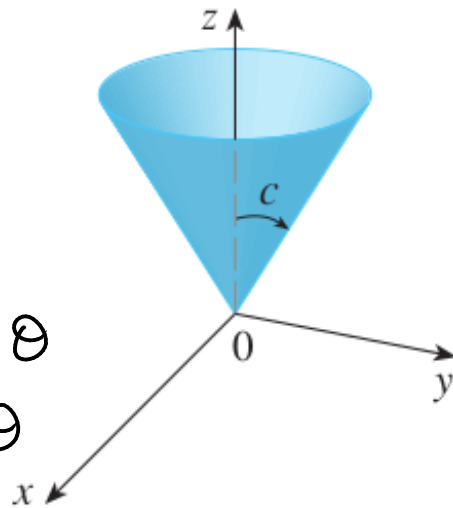
Note:

$$y = \tan(\theta) x$$

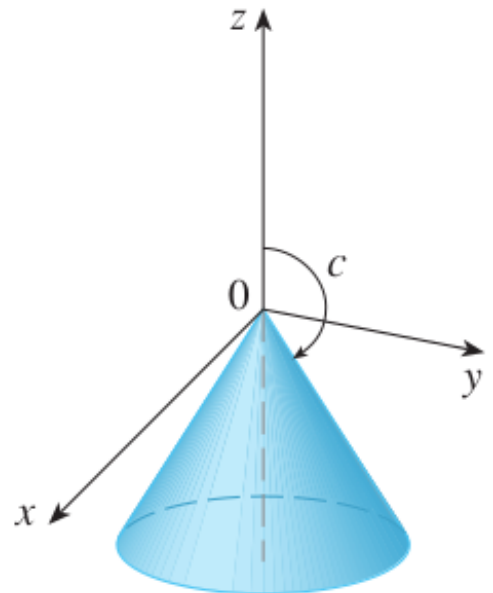
## Cones.

$$\phi = c$$

$$\begin{aligned} x &= \rho \sin \phi \cos \theta \\ y &= \rho \sin \phi \sin \theta \\ z &= \rho \cos \phi \end{aligned}$$



$$0 < c < \pi/2$$

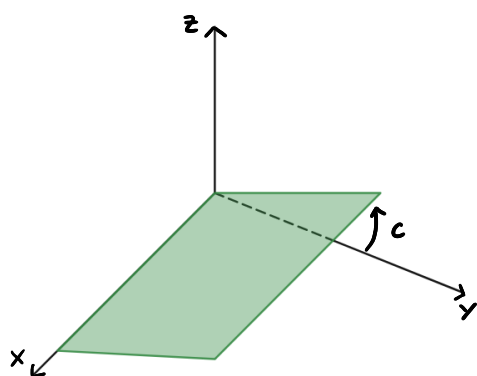


$$\pi/2 < c < \pi$$

$$\begin{aligned} z &= A \sqrt{x^2 + y^2} \Rightarrow \rho \cos \phi = A \sqrt{\rho^2 \sin^2 \phi (\cos^2 \theta + \sin^2 \theta)} \\ &\Rightarrow \rho \cos \phi = A \sqrt{\rho^2 \sin^2 \phi} \end{aligned}$$

- $A > 0 \Rightarrow z \geq 0 \Rightarrow 0 \leq \phi \leq \frac{\pi}{2}$   
 $\Rightarrow \rho \cos \phi = A \rho \sin \phi$   
 $\Rightarrow \frac{1}{A} = \tan \phi \Rightarrow \phi = \arctan\left(\frac{1}{A}\right)$
- $A < 0 \Rightarrow z \leq 0 \Rightarrow \frac{\pi}{2} \leq \phi \leq \pi \Rightarrow \phi = \arctan\left(\frac{1}{A}\right)$

**Question.** Find the equation of the half-plane in the picture below in spherical coordinates. The plane is making an angle of  $c$  with the  $xy$ -plane.



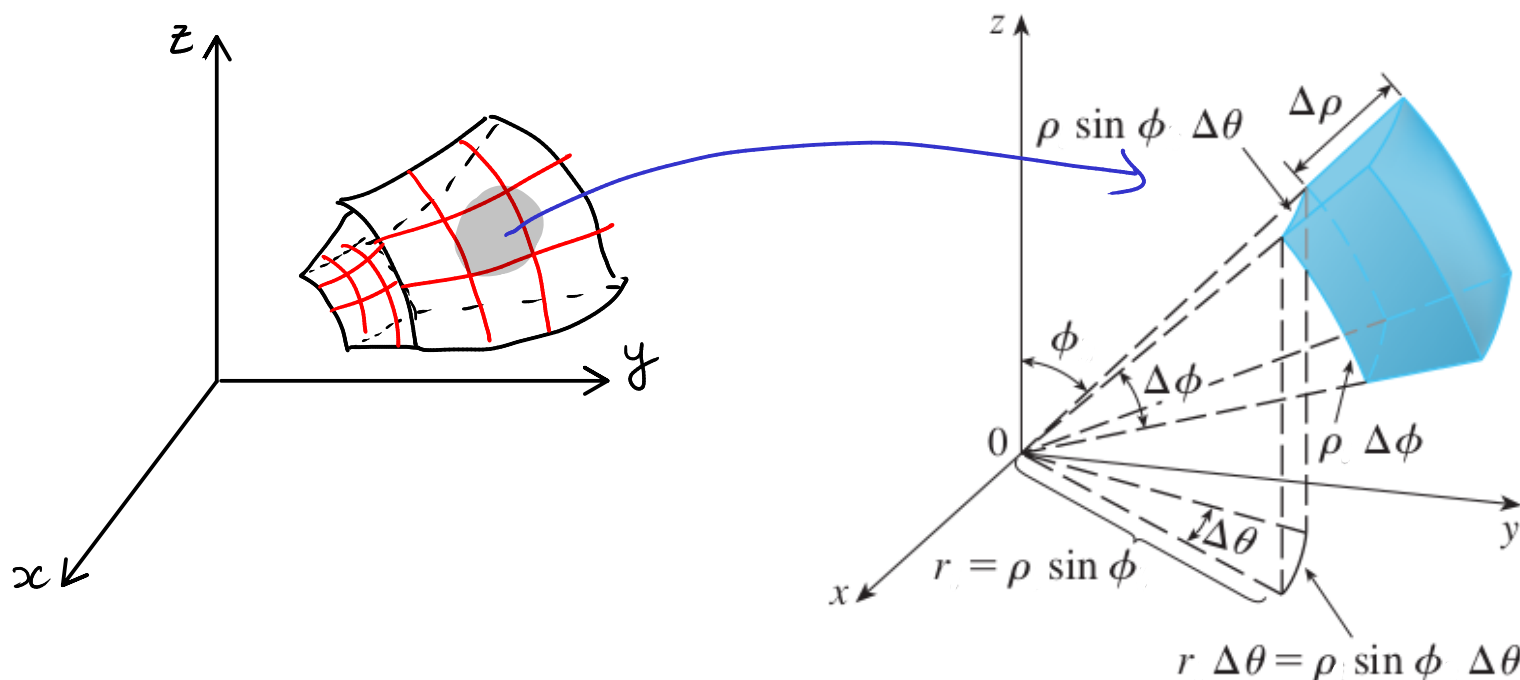
$$\begin{aligned} ay + bz &= 0 \Rightarrow a \rho \sin \phi \sin \theta + b \rho \cos \phi = 0 \\ &\Rightarrow a \rho \sin \phi \sin \theta = -b \rho \cos \phi \\ &\Rightarrow \sin \theta = -\frac{b}{a} \cot \phi \end{aligned}$$

Not as nice as the plane  
 $ax + by = 0 \dots$

# Evaluating integrals in spherical coordinates.

## Spherical Wedge

$$E = \{(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) : a \leq \rho \leq b, \alpha \leq \theta \leq \beta, c \leq \phi \leq d\}$$



We can show that

$$\Delta V = \bar{\rho}^2 \sin \bar{\phi} \Delta \rho \Delta \theta \Delta \phi$$

$\bar{\rho}$  : mid point  
subdivisions  
 $\bar{\phi}$  : midpoint  
subdivisions

As the number of subdivisions goes to infinity, we obtain

$$dV = \rho^2 \sin \phi d\rho d\theta d\phi$$

Formula for the change of variable (in spherical coordinates).

$$\iiint_E f(x, y, z) dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin(\phi) \cos(\theta), \rho \sin(\phi) \sin(\theta), \rho \cos(\phi)) \rho^2 \sin(\phi) d\rho d\theta d\phi$$

$$x = \rho \sin \phi \cos \theta$$

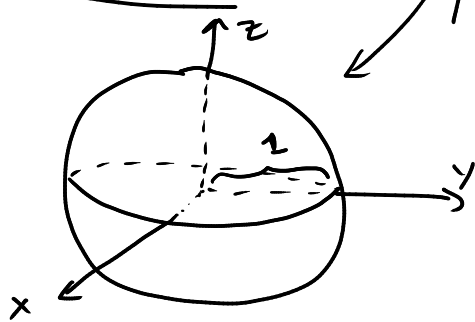
$$z = \rho \cos \phi$$

$$y = \rho \sin \phi \sin \theta$$

**EXAMPLE 3** Evaluate  $\iiint_B e^{(x^2+y^2+z^2)^{3/2}} dV$ , where  $B$  is the unit ball:

$$B = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$$

① Picture



Radius  $0 \leq \rho \leq 1$

Angles.

$$0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi$$

$$E = \{(\rho, \theta, \phi) : 0 \leq \rho \leq 1, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi\}.$$

② Integrate

$$\iiint_E e^{(x^2+y^2+z^2)^{3/2}} dV \quad \longrightarrow \quad \rho^2 = x^2 + y^2 + z^2 \rightarrow \text{shortcut}.$$

$$= \int_0^\pi \int_0^{2\pi} \int_0^1 \left( e^{(\rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta + \rho^2 \cos^2 \phi)^{3/2}} \right) \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$$

$$\rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$$

$$= \int_0^\pi \int_0^{2\pi} \int_0^1 e^{(\rho^2 \sin^2 \phi + \rho^2 \cos^2 \phi)^{3/2}} \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$$

$$= \int_0^\pi \int_0^{2\pi} \int_0^1 e^{(\rho^2)^{3/2}} \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$$

$$= \int_0^\pi \int_0^{2\pi} \int_0^1 e^{\rho^3} \cdot \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$$

$$= \left( \int_0^1 e^{\rho^3} \rho^2 \, d\rho \right) \left( \int_0^{2\pi} d\theta \right) \left( \int_0^\pi \sin \phi \, d\phi \right) = \boxed{4\pi \left( \frac{e-1}{3} \right)}.$$

$$u = \rho^3 \rightarrow du = 3\rho^2 \, d\rho$$

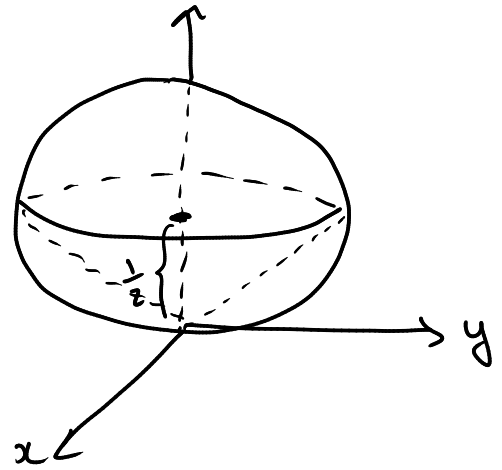
**EXAMPLE 4** Use spherical coordinates to find the volume of the solid that lies above the cone  $z = \sqrt{x^2 + y^2}$  and below the sphere  $x^2 + y^2 + z^2 = z$ .

### ① Picture

$$x^2 + y^2 + z^2 = z$$

$$x^2 + y^2 + z^2 - z + \frac{1}{4} = \frac{1}{4}$$

$$x^2 + y^2 + \left(z - \frac{1}{2}\right)^2 = \frac{1}{4} = \left(\frac{1}{2}\right)^2$$



Replace eq. cone in eq. sphere:

$$x^2 + y^2 + x^2 + y^2 = \sqrt{x^2 + y^2}$$

$$\Rightarrow 2 \underbrace{(x^2 + y^2)}_{(\sqrt{x^2 + y^2})^2} = \sqrt{x^2 + y^2} \Rightarrow 2 \sqrt{x^2 + y^2} = 1$$

$$\Rightarrow x^2 + y^2 = \frac{1}{4}$$

Cone

$$\phi = \pi/4$$

Sphere

$$x^2 + y^2 + z^2 = z$$

$$\Rightarrow \rho^2 = \rho \cos \phi$$

$$\Rightarrow \rho = \cos \phi$$

$$E = \{(\rho, \theta, \phi) : 0 \leq \rho \leq \cos \phi, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \frac{\pi}{4}\}$$

### ② Volume

$$\begin{aligned} \text{Vol}(E) &= \iiint_E 1 \, dV = \int_0^{\pi/4} \int_0^{2\pi} \int_0^{\cos \phi} \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi \\ &= \int_0^{\pi/4} \int_0^{2\pi} \sin \phi \frac{\cos^3 \phi}{3} \, d\theta \, d\phi \quad \left( \begin{array}{l} u = \cos \phi \\ du = -\sin \phi \, d\phi \end{array} \right) \\ &= \boxed{\frac{\pi}{24}} \end{aligned}$$