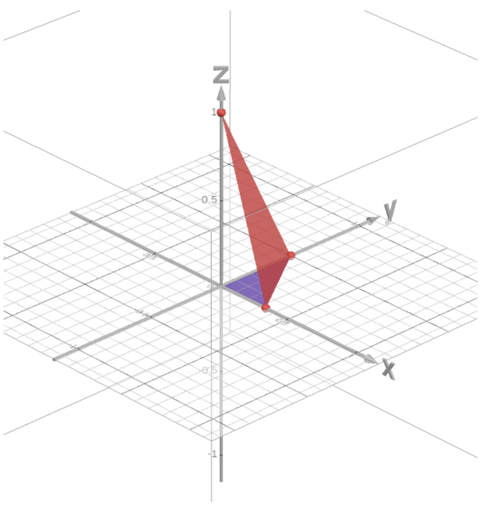
Section 16.8, Problem 2 (10 Pts)

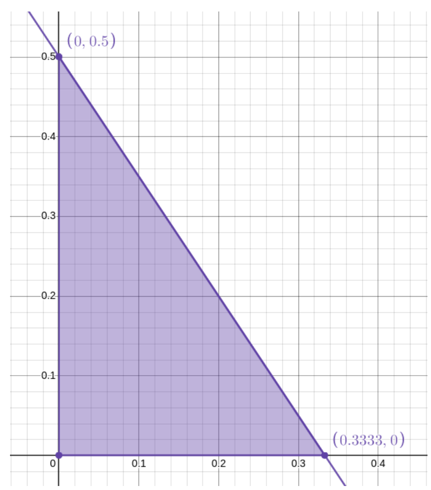
The paraboloid crosses the XY-plane in a circle $x^2 + y^2 = 1$. Since the surface S is oriented upwards, it induced the counterclockwise orientation on the circle. Parametrize the circle by $\vec{r}(t) = \langle \cos t, \sin t, 0 \rangle$ for $0 \le t \le 2\pi$ and using Stokes' Theorem

$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot \vec{r}\,' \, dt = \int_0^{2\pi} \langle 0, \sin^2(t), \cos t \sin t \rangle \cdot \langle -\sin t, \cos t, 0 \rangle \; dt$$
$$= \int_0^{2\pi} \cos(t) \sin^2(t) \, dt = \boxed{0}.$$

Section 16.8, Problem 8 (15 Pts)

The surface containing the curve C is the inside S of the triangle and this triangle lies in the plane of equation $3x + 2y + z = 1$.





Since the curve is oriented counterclockwise as seen from above, using the "Rule of Thumb", the surface S is oriented positively. Let $\vec{r}(x, y) = \langle x, y, 1 - 3x - 2y \rangle$, so that $\vec{r}_x \times \vec{r}_y = \langle 3, 2, 1 \rangle$, with

$$D = \{(x, y) \,:\, 0 \le x \le 1/3, \, 0 \le y \le 1/2 - 3x/2\}.$$

Therefore, from Stokes' Theorem,

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}.$$

We have

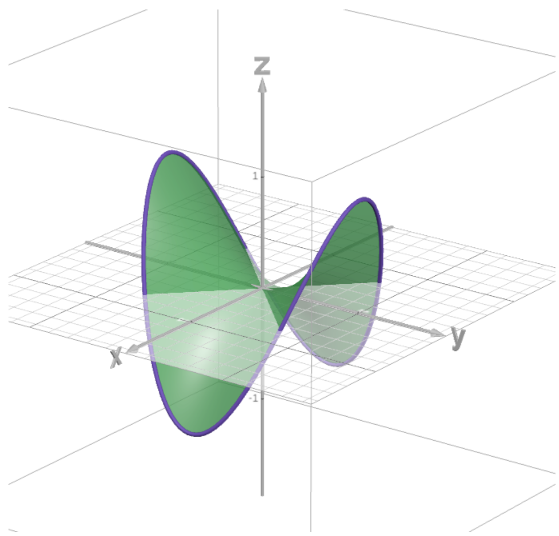
$$\operatorname{curl} \vec{F} = \langle x - y, -y, 1 \rangle$$

and therefore

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{r} &= \int_0^{1/3}\int_0^{1/2-3x/2} \langle x-y,-y,1\rangle\cdot\langle 3,2,1\rangle\ dydx\\ &= \int_0^{1/3}\int_0^{1/2-3x/2} 3x-5y+1\,dydx = \boxed{\frac{1}{24}}.\end{aligned}$$

Section 16.8, Problem 12a & 12b (15 Pts)

a) We will use the surface S represented in the figure below. Also, the curve C is pictured in purple.



Using polar coordinates to parametrize the region D inside the cylinder, we get

$$\vec{r}(u,v) = \left\langle u\cos v, u\sin v, u^2\sin^2(v) - u^2\cos^2(v)\right\rangle$$

with $D = \{(u,v)\,:\,0\le u\le 1, 0\le v\le 2\pi\}$. Using Stokes' Theorem,

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}.$$

Now, we have $2u^2\sin(v)\cos^2(v) + 2u^2\sin^3(v)$

$$\vec{r}_u\times\vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k}\\ \cos v & \sin v & 2u\sin^2(v) - 2u\cos^2(v))\\ -u\sin v & u\cos v & 4u^2\sin(v)\cos(v)\end{vmatrix} = \left\langle 2u^2\cos(v), -2u^2\sin(v), u\right\rangle$$

We see that the second component $-2u^2\sin(v)$ of $\vec{r}_u\times\vec{r}_v$ can be rewritten as $-2uy$, where $y = u\sin(v)$ is the second component of the position vector $\vec{r}$ of S. Since $2u$ is always positive, the vector $\vec{r}_u\times\vec{r}_v$ will always point in opposite direction of the y axis, given us the upward orentation.

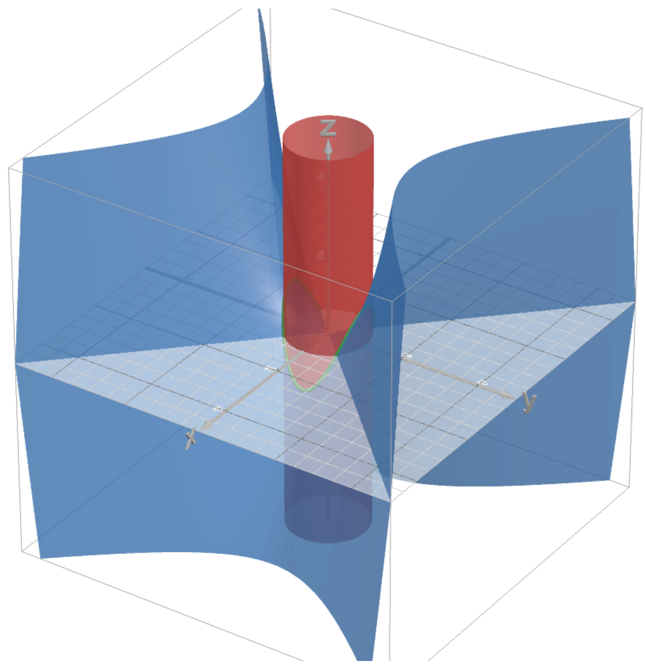
Also, we compute

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & (1/3)x^3 & xy \end{vmatrix} = \langle x, -y, 0\rangle$$

Hence,

$$\begin{aligned} \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} &= \iint_D \langle u\cos v, -u\sin v, 0\rangle \cdot \left\langle 2u^2\cos(v), -2u^2\sin(v), u\right\rangle\, dA \\ &= \int_0^{2\pi}\int_0^1 2u^3\cos^2(v) + 2u^3\sin^2(v)\, dudv \\ &= \int_0^{2\pi}\int_0^1 2u^3\, dudv \\ &= (2\pi)(1/2) = \boxed{\pi}. \end{aligned}$$

b) See the picture in part (a) or the picture below:



Section 16.8, Problem 18 (10 Pts)

Let S be the part of the surface $z = 2xy$ that is inside the curve C. We can parametrize S in the following way:

$$\vec{r}(x,y) = \langle x, y, 2xy\rangle$$

with $D = \{(x,y) \,:\, x^2+y^2 \le 1\}$. Let

$$\vec{F}(x,y,z) = \langle P, Q, R\rangle = \left\langle y+\sin x, z^2+\cos y, x^3\right\rangle.$$

Using Stokes' Theorem, we can write

$$\int_C Pdx + Qdy + Rdz = \int_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}.$$

We compute

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2y \\ 0 & 1 & 2x \end{vmatrix} = \langle -2y, -2x, 1 \rangle$$

and

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y + \sin x & z^2 + \cos y & x^3 \end{vmatrix} = \left\langle -2z, -3x^2, 1 \right\rangle .$$

Since the z component is positive, the vector $\vec{r}_x \times \vec{r}_y$ points upwards. However, the curve is oriented negatively and we should use $-\vec{r}_x \times \vec{r}_y$. Therefore,

$$\begin{aligned} \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} &= \iint_D \left\langle -4xy, -3x^2, -1 \right\rangle \cdot \langle 2y, 2x, -1 \rangle \; dA \\ &= \iint_D -8xy^2 + -6x^3 + 1 \, dA \end{aligned}$$

Notice that if $f(x, y) = 8xy^2$, then $f(-x, y) = -8xy^2 = -f(x, y)$ and $f(x, -y) = f(x, y)$. Since D is symmetric about the x-axis and the y-axis, we obtain

$$\iint_D 8xy^2 = 0.$$

Using a similar argument with $g(x, y) = x^3$, we obtain

$$\iint_D 6x^3 = 0.$$

Therefore,

$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \iint_D 1 \, dA = \operatorname{Area}(D) = \pi.$$