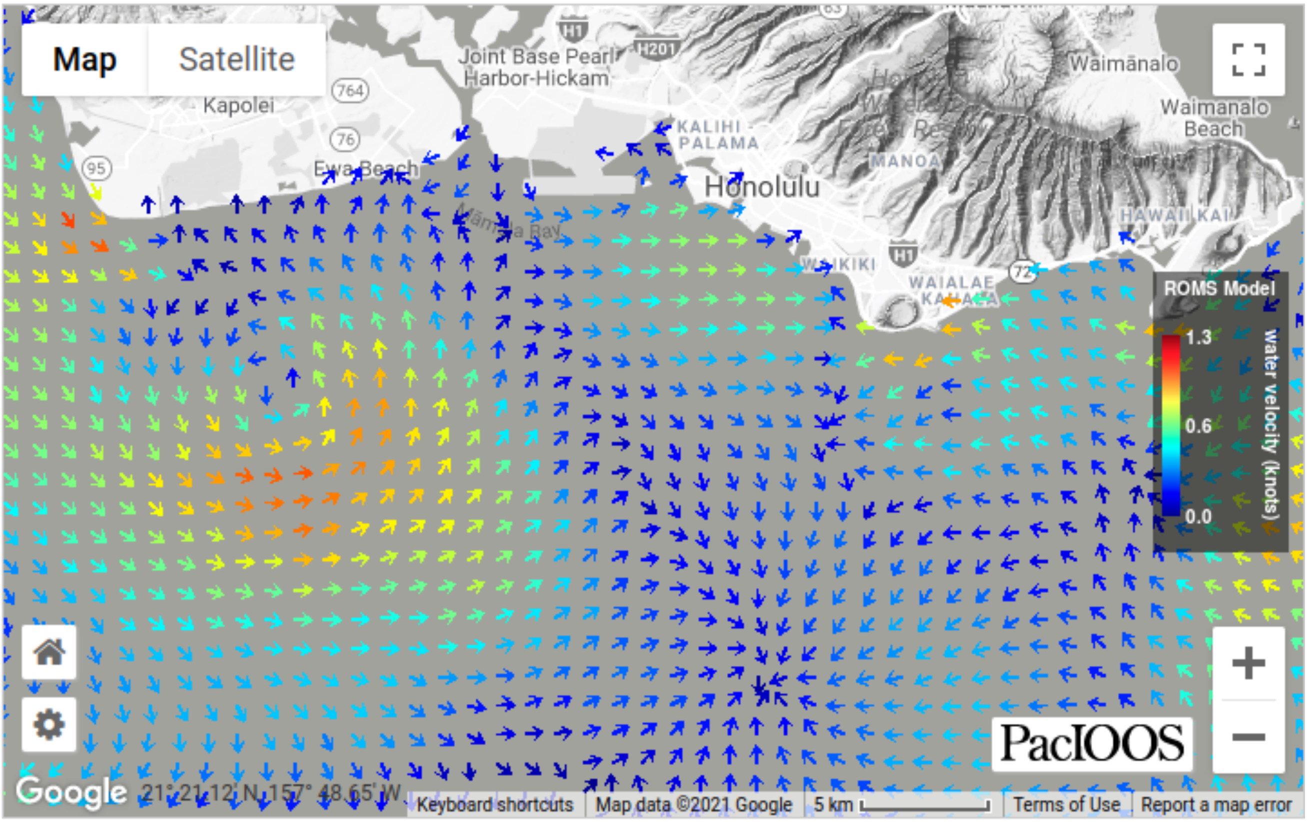
Chapter 16

Vector Calculus

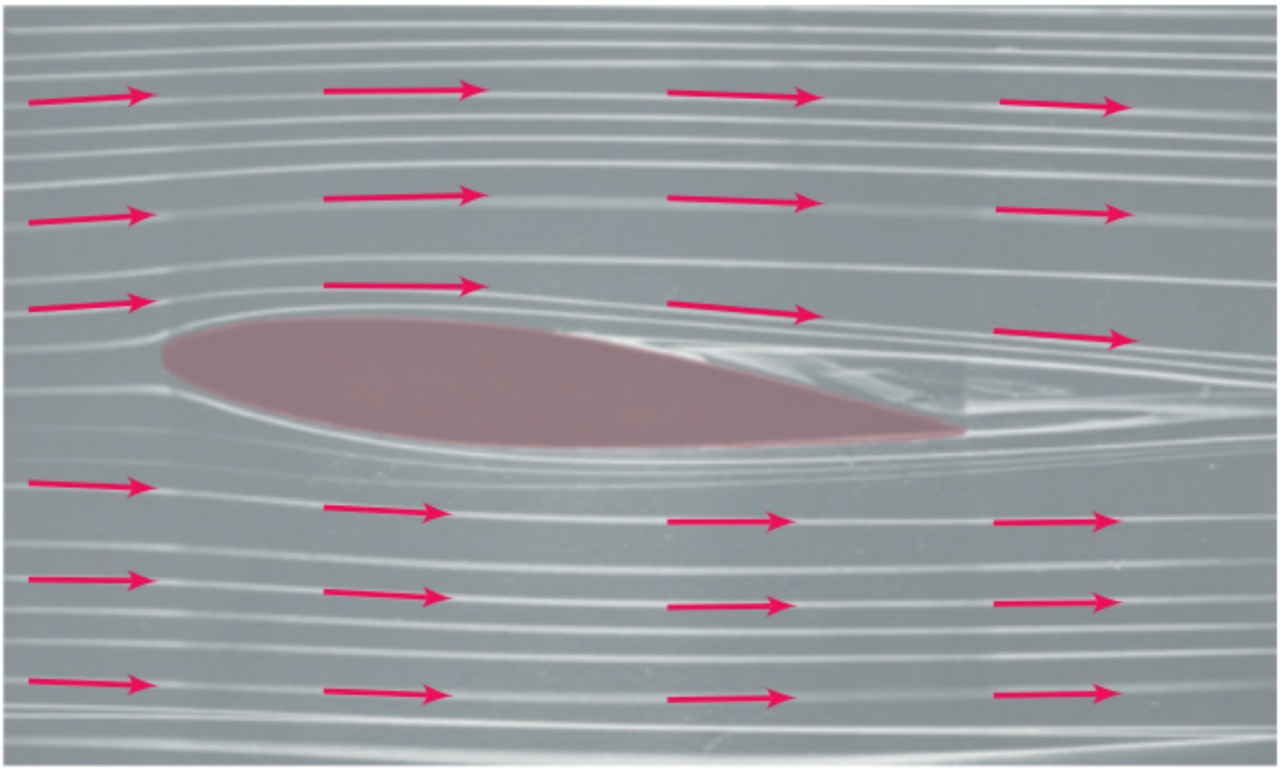
16.1 Vector Fields

Examples.



Map retrieved from http://www.pacioos.hawaii.edu/currents/model-oahu/

ONERA photograph, Werle, 1974

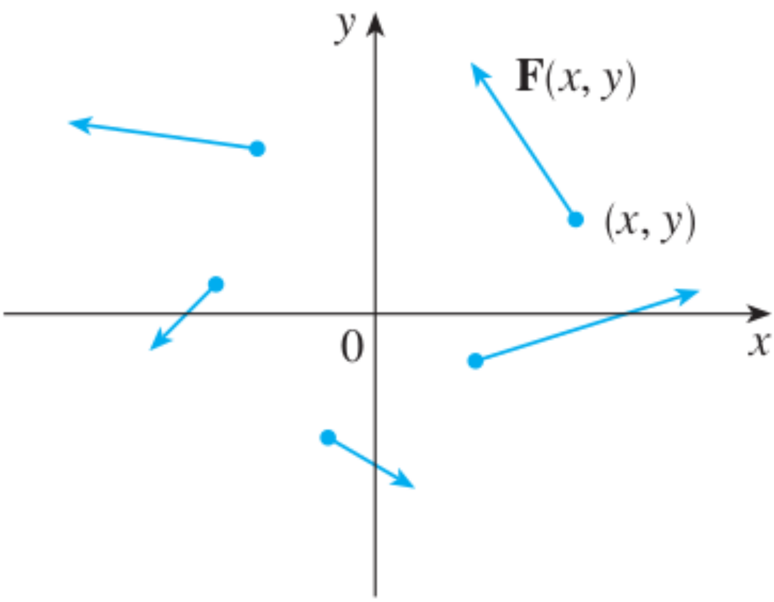


(b) Airflow past an inclined airfoil

Vector Fields in 2D.

1 Definition Let D be a set in $\mathbb{R}^2$ (a plane region). A **vector field on** $\mathbb{R}^2$ is a function $\mathbf{F}$ that assigns to each point (x, y) in D a two-dimensional vector $\mathbf{F}(x, y)$.

Representation.



Component Functions

$$\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$$

- P : x-component of $\vec{F}$
- Q : y-component of $\vec{F}$

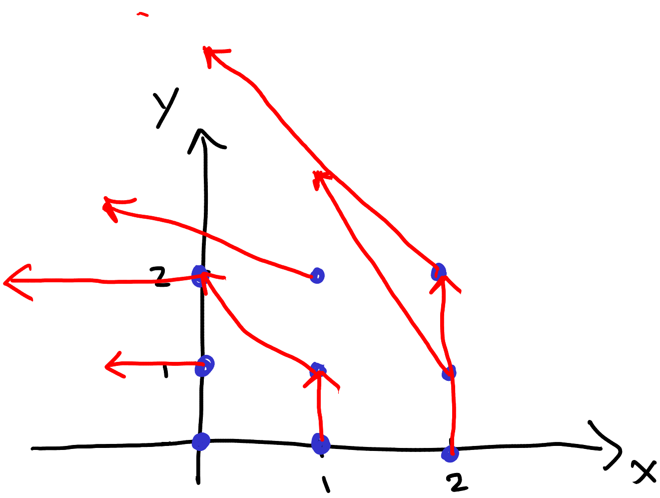
Remark:

- P and Q are real-valued functions.
- $\vec{F}$ is continuous if P and Q are continuous.

EXAMPLE 1 A vector field on $\mathbb{R}^2$ is defined by $\mathbf{F}(x, y) = -y\,\mathbf{i} + x\,\mathbf{j}$. Describe $\mathbf{F}$ by sketching some of the vectors $\mathbf{F}(x, y)$ ~~as in Figure 3.~~

$\vec{i} = \langle 1, 0 \rangle$

$\vec{j} = \langle 0, 1 \rangle$



y/x	0	1	2
2	$\langle -2, 0 \rangle$	$\langle -2, 1 \rangle$	$\langle -2, 2 \rangle$
1	$\langle -1, 0 \rangle$	$\langle -1, 1 \rangle$	$\langle -1, 2 \rangle$
0	$\langle 0, 0 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 2 \rangle$

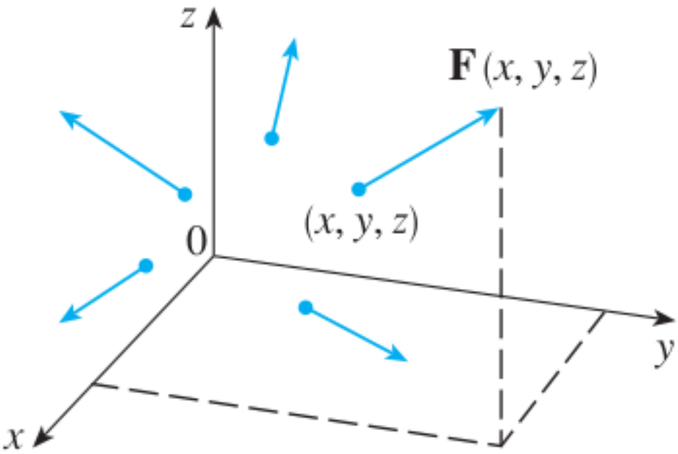
$\vec{F}(0,0) = \langle -y, x \rangle = \langle 0, 0 \rangle$.

$\vec{F}(1,0) = \langle 0, 1 \rangle$

Vector Fields in 3D.

2 Definition Let E be a subset of $\mathbb{R}^3$. A **vector field on** $\mathbb{R}^3$ is a function $\mathbf{F}$ that assigns to each point (x, y, z) in E a three-dimensional vector $\mathbf{F}(x, y, z)$.

Representation.



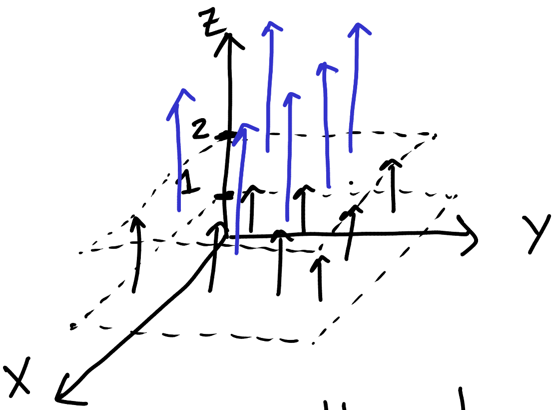
Component Functions. $\langle P, Q, R \rangle$

$$P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$$

- P : x-component of $\vec{F}$
- Q : y-component of $\vec{F}$
- R : z-component of $\vec{F}$

$\vec{F}$ is continuous if P, Q, R are continuous.

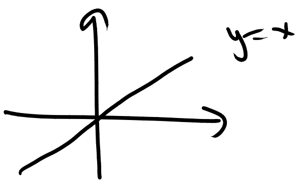
EXAMPLE 2 Sketch the vector field on $\mathbb{R}^3$ given by $\mathbf{F}(x, y, z) = z\,\mathbf{k}$. $= \langle 0, 0, z \rangle$



on the plane $z = k$

$\hookrightarrow$ $F(x, y, k) = \langle 0, 0, k \rangle$

(x,y,z)	$\vec{F}(x,y,z)$
$(0,0,1)$	$\langle 0,0,1 \rangle$
$(0,1,0)$	$\langle 0,0,0 \rangle$
$(1,1,1)$	$\langle 0,0,1 \rangle$
$(1,0,1)$	$\langle 0,0,1 \rangle$
$(1,0,2)$	$\langle 0,0,2 \rangle$
$\vdots$	$\vdots$



Remark:

A vector field is continuous if each of its component function (that is P, Q, R) are continuous.

EXAMPLE 4 Newton's Law of Gravitation tells you that the magnitude of the force of attraction between two objects of mass m and M is

$$F = \frac{mMG}{r^2}$$

where G is the gravitational constant, and r is the distance between the two objects. Find the vector field describing the gravitational field.

Assume M is located at the origin.

Let $\vec{x} = \langle x, y, z \rangle$ be the position vector. Then m will be attracted towards M in the direction $-\dfrac{\vec{x}}{\|\vec{x}\|}$.

Therefore, the magnitude of the grav. field is F and

$$\vec{F}(\vec{x}) = F\left(\frac{-\vec{x}}{\|\vec{x}\|}\right) = \frac{-mMG}{r^2\|\vec{x}\|}\vec{x} = -\frac{mMG}{\|\vec{x}\|^3}\vec{x}$$

<u>More Examples:</u>

- Force field around an electric charge Q :

$$\vec{F}(\vec{x}) = \frac{\varepsilon_0 qQ}{\|\vec{x}\|^3}\vec{x}$$

- Electric Field around the charge Q :

$$\vec{E}(\vec{x}) = \frac{\vec{F}(\vec{x})}{q} = \frac{\varepsilon_0 Q}{\|\vec{x}\|^3}\vec{x}$$

Gradient Fields.

2D

$$\boxed{\vec{\nabla} f(x,y) = f_x(x,y)\vec{i} + f_y(x,y)\vec{j}} = \langle f_x, f_y \rangle$$

f_x : derivative w.r.t. x $\quad \partial f/\partial x$

f_y : derivative w.r.t. y. $\quad \partial f/\partial y$

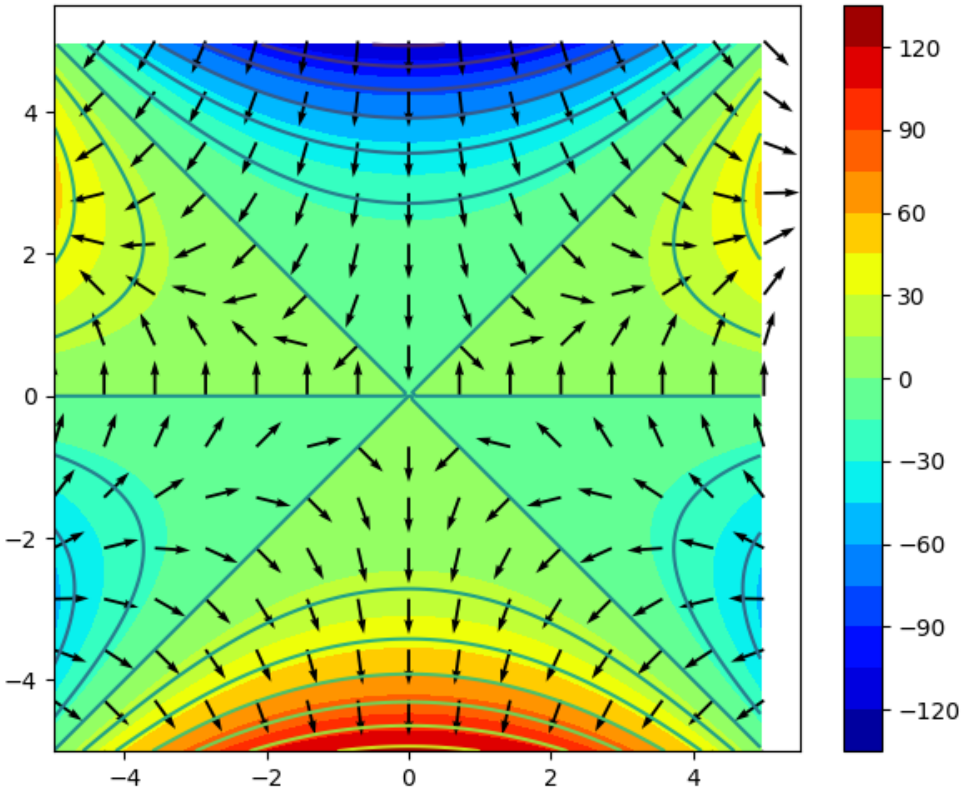
3D

$$\boxed{\vec{\nabla} f(x,y,z) = f_x(x,y,z)\vec{i} + f_y(x,y,z)\vec{j} + f_z(x,y,z)\vec{k}}$$

f_z : derivative w.r.t. z $\quad \frac{\partial f}{\partial z}$.

EXAMPLE 6 Find the gradient vector field of $f(x, y) = x^2y - y^3$. Plot the gradient vector field together with a contour map of f. How are they related?

$$\vec{\nabla} f = \langle 2xy, \; x^2 - 3y^2 \rangle.$$



Level curves:

$f(x,y) = c$

Conservative Vector Fields.

- A vector field $\vec{F}$ is conservative if there is a scalar-valued function f such that
$$\vec{F} = \vec{\nabla} f$$

- The function f is called the potential function of $\vec{F}$.

EXAMPLE. Show that the Gravitational field is conservative.

Goal: find an f such that $\vec{\nabla} f = \frac{-mMG}{\|\vec{x}\|^3} \vec{x}$.

f must satisfies:

$$f_x = \frac{-mMG}{\|\vec{x}\|^3} x = \frac{-mMG\,x}{\left(\sqrt{x^2+y^2+z^2}\right)^3}$$

$$f_y = \frac{-mMG\,y}{\left(\sqrt{x^2+y^2+z^2}\right)^3}$$

$$f_z = \frac{-mMG\,z}{\left(\sqrt{x^2+y^2+z^2}\right)^3}$$

The function f is

$$\boxed{f(x,y,z) = \frac{mMG}{\sqrt{x^2+y^2+z^2}}}$$