Section 15.4, Problem 6 (10 Pts)

The description of the domain is

$$D = \{(x, y) \,:\, 0 \leq y \leq 2/5,\, y/2 \leq x \leq 1 - 2y\}.$$

The mass is given by

$$m = \iint_D x \, dA = \int_0^{2/5} \int_{y/2}^{1-2y} x \, dx dy = \frac{2}{25} = 0.08.$$

The center of mass is $(\overline{x}, \overline{y})$, where $\overline{x} = M_y/m$ and $\overline{y} = M_x/m$. We have

$$M_y = \iint_D x(x) \, dA = \int_0^{2/5} \int_{y/2}^{1-2y} x^2 \, dx dy = \frac{3}{750} = 0.041333$$

and

$$M_x = \iint_D y(x) \, dA = \int_0^{2/5} \int_{y/2}^{1-2y} xy \, dx dy = \frac{7}{750} \approx 0.005417.$$

Therefore,

$$\overline{x} = \frac{3/750}{2/25} = \frac{31}{60} \approx 0.5167 \quad \text{and} \quad \overline{y} = \frac{7/750}{2/25} = \frac{7}{60} \approx 0.1167. \qquad \triangle$$

Section 15.4, Problem 14 (10 Pts)

In polar coordinates,

$$D = \{(r, \theta) \,:\, 1 \leq r \leq 2, 0 \leq \theta \leq \pi.\}$$

The mass density is $\rho(x, y) = 1/\sqrt{x^2 + y^2}$. Using polar coordinates,

$$m = \iint_D \frac{1}{\sqrt{x^2 + y^2}} \, dA = \int_0^{\pi} \int_1^2 \frac{1}{r} r \, dr d\theta = \int_0^{\pi} \int_1^2 \, dr d\theta = \pi.$$

Also, we have

$$M_y = \int_0^{\pi} \int_1^2 \frac{r \cos\theta}{r} r \, dr d\theta = \int_0^{\pi} \int_1^2 r \cos\theta \, dr d\theta = 0$$

and

$$M_x = \int_0^{\pi} \int_1^2 \frac{r \sin\theta}{r} r \, dr d\theta = \int_0^{\pi} \int_1^2 r \sin\theta \, dr d\theta = 3.$$

Therefore,

$$\overline{x} = \frac{M_y}{m} = 0 \quad \text{and} \quad \overline{y} = \frac{M_x}{m} = \frac{3}{\pi}.$$

Note: It is possible to deduce from the symmetry of the region D that $M_y = 0$ and the following fact: the mapping $x \mapsto x/\sqrt{x^2 + y^2}$ is an odd function.

Section 15.6, Problem 6 **(10 Pts)**

Denote by I the value of the integral. So

$$\begin{aligned} I = \int_0^1 \int_0^1 \frac{z}{y+1} \Big(x\Big)\Big|_0^{\sqrt{1-z^2}} \, dzdy &= \int_0^1 \int_0^1 \frac{z}{y+1} (\sqrt{1-z^2}) \, dzdy \\ &= \int_0^1 \int_1^0 \frac{-(1/2)\sqrt{u}}{y+1} \, dudy \\ &= (1/2) \int_0^1 \int_0^1 \frac{\sqrt{u}}{y+1} \, dudy \\ &= (1/2) \Big(\int_0^1 \frac{1}{y+1} \, dy \Big) \Big(\int_0^1 u^{1/2} \, du \Big) \\ &= (1/2) \Big(\ln(y+1) \Big)\Big|_0^1 \Big(\frac{2u^{3/2}}{3} \Big)\Big|_0^1 \\ &= (1/3) \ln 2. \end{aligned}$$

△

Section 15.6, Problem 14 **(10 Pts)**

The description of E is

$$E = \{(x, y, z) \,:\, 0 \le y \le 2, -1 \le x \le 1, x^2 - 1 \le z \le 1 - x^2\}.$$

Therefore, we obtain

$$\begin{aligned} \iiint_E (x-y) \, dV = \int_0^2 \int_{-1}^1 \int_{x^2-1}^{1-x^2} (x-y) \, dzdxdy &= \int_0^2 \int_{-1}^1 (x-y) \Big(z\Big)\Big|_{x^2-1}^{1-x^2} \, dxdy \\ &= \int_0^2 \int_{-1}^1 (x-y)(2-2x^2) \, dxdy \\ &= 2 \int_0^2 \int_{-1}^1 (x - x^3 - y + x^2 y) \, dxdy \\ &= 2 \int_0^2 (x^2/2 - x^4/4 - xy + x^3 y/3)|_{-1}^1 \, dy \\ &= 2 \int_0^2 -2y + 2y/3 \, dy \\ &= 2(-y^2 + y^2/3)|_0^2 \\ &= -16/3 \approx -5.3333. \end{aligned}$$

△

Section 15.6, Problem 22 **(10 Pts)**

The y value is bounded by $y = -1$ and $y = 4 - z$. Isolating z from the equation of the cylinder, we get $z = -\sqrt{4 - x^2}$ as a lower bound and $z = \sqrt{4 - x^2}$ as an upper bound, with $-1 \le x \le 1$. Therefore,

$$E = \{(x, y, z) \,:\, -1 \le x \le 1, -1 \le y \le 4 - z, -\sqrt{4 - x^2} \le z \le \sqrt{4 - x^2}\}$$

Therefore,

$$
\begin{aligned}
Vol(E) = \iiint_E dV &= \int_{-1}^{1}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{-1}^{4-z} dydzdx \\
&= \int_{-1}^{1}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (5-z)\, dzdx.
\end{aligned}
$$

The region in the xz-plane is a disk (inside a circle of radius 2). We can therefore use the polar coordinates with $x = r\cos\theta$ and $z = r\sin\theta$ and

$$
\begin{aligned}
Vol(E) &= \int_0^{2\pi}\int_0^2 (5 - r\sin\theta) r\, drd\theta \\
&= \int_0^{2\pi} (5r^2/2 - (r^3/3)\sin\theta)|_0^2\, d\theta \\
&= \int_0^{2\pi} 10 - (8/3)\sin\theta\, d\theta \\
&= \Big(10\theta + (1/3)\cos\theta\Big)\Big|_0^{2\pi} \\
&= 20\pi.
\end{aligned}
$$

$\triangle$