Problem 6

By the Divergence Theorem, we have

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} \, dV.$$

The surface is a rectangular box S. The inside of the box is

$$E = \{(x, y, z) \,:\, 0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq c\}.$$

We have

$$\operatorname{div} \vec{F} = 2xyz + 2xyz + 2xyz = 6xyz.$$

So, we obtain

$$\iint_S \vec{F} \cdot d\vec{S} = \int_0^c \int_0^b \int_0^a 6xyz \, dxdydz = 3a^2b^2c^2/4.$$

Problem 18

Let S_1 be the bottom of the paraboloid. This is the disk

$$S_1 = \{(x, y, z) \,:\, x^2 + y^2 \leq 1, \; z = 1\}.$$

Let $\tilde{S} := S \cup S_1$, with the outward orientation. By the Divergence Theorem, we have

$$\iint_{\tilde{S}} \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} \, dV.$$

The solid E bounded by $\tilde{S}$ is

$$E = \{(x, y, z) \,:\, x^2 + y^2 \leq 1, 1 \leq z \leq 2 - x^2 - y^2\}.$$

The divergence of $\vec{F}$ is

$$\operatorname{div} \vec{F} = 0 + 0 + 1 = 1.$$

So, we obtain

$$\iint_{\tilde{S}} \vec{F} \cdot d\vec{S} = \iint_D \Big(\int_1^{2-x^2-y^2} 1 \, dz \Big) dA = \iint_D 1 - x^2 - y^2 \, dA,$$

where $D := \{(x, y) : x^2 + y^2 \leq 1\}$. We can compute this integral by passing to polar coordinates. We then obtain

$$\int_0^{2\pi} \int_0^1 (1 - r^2) r \, dr d\theta = \pi/2.$$

The flux of $\vec{F}$ through $\tilde{S}$ is then $\pi/2$.

This is not exactly the flux of $\vec{F}$ through S. To obtain the flux through S, we have to write

$$\pi/2 = \iint_{\tilde{S}} \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot d\vec{S} + \iint_{S_1} \vec{F} \cdot d\vec{S}$$

and so

$$\iint_S \vec{F} \cdot d\vec{S} = \pi/2 - \iint_{S_1} \vec{F} \cdot d\vec{S}.$$

Recall that $\tilde{S}$ had the outward orientation, so to be consistent with that choice, S_1 has the downward orientation. A normal vector to S_1 pointing downward is $\vec{n} = \langle 0, 0, -1 \rangle$ and so

$$\iint_{S_1} \vec{F} \cdot d\vec{S} = \iint_{S_1} \vec{F} \cdot \langle 0, 0, -1 \rangle \; dS = \iint_{S_1} -z \, dS = -\iint_{S_1} z \, dS$$

But, when on S_1, we have $z = 1$, and so the double integral represents the area of the disk. The disk has radius 1 and we then obtain

$$\iint_S \vec{F} \cdot d\vec{S} = \pi/2 + \pi = 3\pi/2.$$