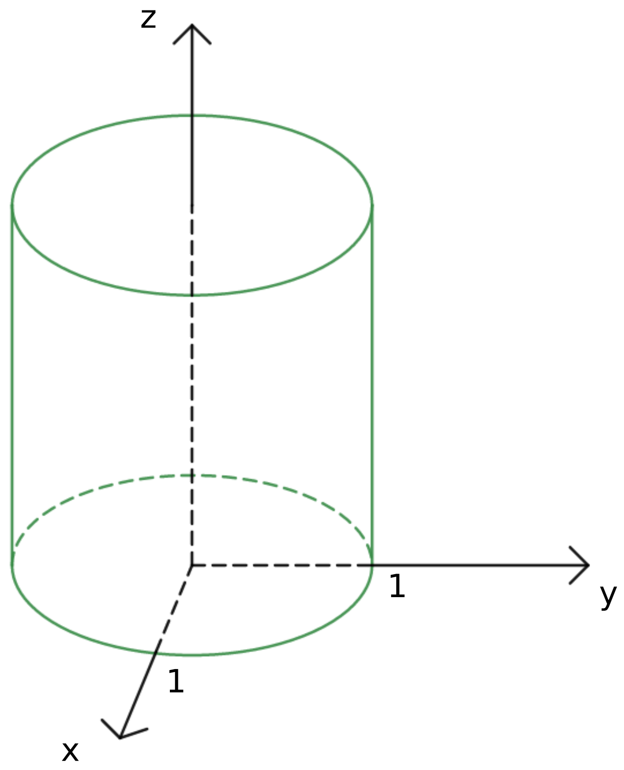
Chapter 15

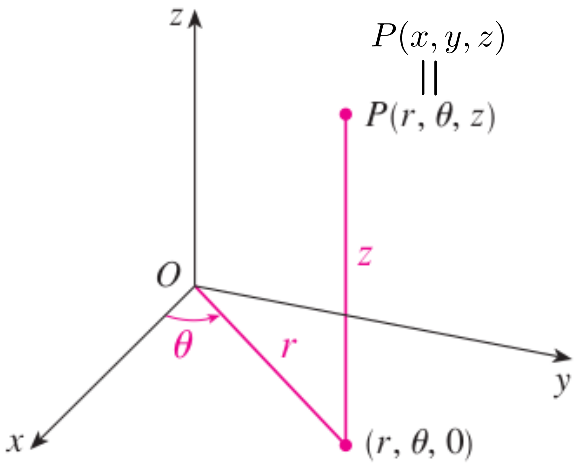
Multiple Integrals

15.7 Triple integrals in cylindrical coordinates

EXAMPLE. Describe the following solid (the interior of a cylinder).



Definition (when the main axis is the z-axis)



Cylindrical ⟶ Cartesian

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

Cartesian ⟶ Cylindrical

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan(y/x), \quad z = z$$

EXAMPLE 1

(a) Plot the point with cylindrical coordinates $(2, 2\pi/3, 1)$ and find its rectangular coordinates.

(b) Find cylindrical coordinates of the point with rectangular coordinates $(3, -3, -7)$.

EXAMPLE 2 Describe the surface whose equation in cylindrical coordinates is $z = r$.

Note: Principle axis (the z-axis) can be any other axis (x-axis or y-axis) in some applications.

EXAMPLE. Write the equation in cylindrical coordinates and identify the surface.

$$z = x^2 - y^2$$

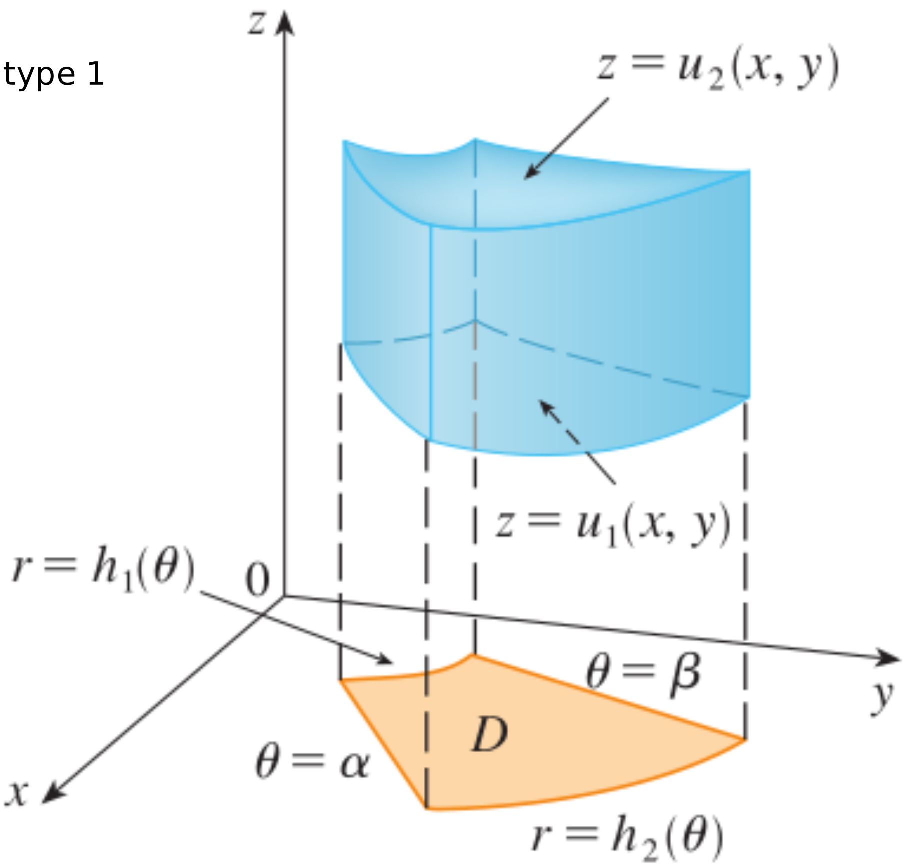
EXAMPLE. Sketch the solid described by the given inequalities:

$$r^2 \leq z \leq 8 - r^2$$

Question. What is the equation of a plane in cylindrical coordinates?

Evaluating triple integrals in cylindrical coordinates.

Solid of type 1



- $E = \{(x, y, z) \,:\, (x, y) \in D \text{ and } u_1(x, y) \leq z \leq u_2(x, y)\}$

- Describe D in polar coordinates.

$$\iiint_E f(x, y, z)\, dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \left[\int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, dz \right] r dr d\theta$$

Note: Can be adapted to type 2 and type 3 solids.

EXAMPLE. A solid E lies within the cylinder $x^2 + y^2 = 1$, below the plane $z = 4$, and above the paraboloid $z = 1 - x^2 - y^2$. Find the value of the integral

$$\iiint_E x^2 + y^2 \, dV$$

EXAMPLE 4 Evaluate $\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} (x^2 + y^2)\, dz\, dy\, dx.$