## Section 5.3 — Problem 1 — 10 points

**Find the general solutions to the complementary equation.**
The complementary equation is

$$y'' + 5y' - 6y = 0.$$

The characteristic equation is $r^2 + 5r - 6 = 0$. The roots are $r = 1$ and $r = 5$. So the general solution of the complementary equation is

$$y_h(x) = c_1 e^x + c_2 e^{5x}.$$

**Find a particular solution.**
We have a degree 2 polynomial on the right-hand side of the ODE. We therefore suggest

$$y_{par}(x) = Ax^2 + Bx + C.$$

We have $y' = 2Ax + B$ and $y'' = 2A$. Therefore, after pluging in the ODE:

$$2A + 5(2Ax + B) + Ax^2 + Bx + C = 22 + 18x - 18x^2.$$

Gathering similar terms, we obtain the equation

$$2A + 5B + C + (10A + B)x + Ax^2 = 22 + 18x - 18x^2.$$

We therefore get $A = -18$, $10A + B = 18$, and $2A + 5B + C = -18$. So

$$B = 18 + 180 = 198.$$

Finally, $C = 36 - 990 = -954$. Therefore, the particular solution we were seeking for is

$$y_{par}(x) = -18x^2 + 198x - 954.$$

**General solution.**
Combining $y_h$ and $y_{par}$, we get

$$y(x) = y_h(x) + y_{par}(x) = c_1 e^x + c_2 e^{5x} - 18x^2 + 198x - 954.$$

**Section 5.3 — Problem 3 — 10 points**

**Find the general solution to the complementary equation.**
The complementary equation is

$$y'' + 8y' + 7y = 0.$$

The characteristic equation is $r^2 + 8r + 7 = 0$. The roots are $r = -1$ and $r = -7$. Therefore, the general solution to the complementary equation is

$$y_h(x) = c_1 e^{-x} + c_2 e^{-7x}.$$

**Find a particular solution.**
We have a degree three polynomial on the right-hand side of the polynomial. We therefore suggest

$$y_{par}(x) = Ax^3 + Bx^2 + Cx + D.$$

We have $y'(x) = 3Ax^2 + 2Bx + C$ and $y''(x) = 6Ax + 2B$. After plugging in the ODE, we get

$$6Ax + 2B + 8(3Ax^2 + 2Bx + C) + 7(Ax^3 + Bx^2 + Cx + D) = -8 - x + 24x^2 + 7x^3.$$

Collecting the terms with the same power of $x$, we get

$$2B + 8C + 7D + (6A + 16B + 7C)x + (24A + 7B)x^2 + 24Ax^3 = -8 - x + 24x^2 + 7x^3.$$

Therefore, we see that $24A = 7$, $24A + 7B = 24$, $6A + 16B + 7C = -1$, and $2B + 8C + 7D = -8$. After the dominoes effect, we find that

$$A = 7/24,\ B = 17/7,\ C = -1165/196,\ D = 1700/343.$$

A particular solution is

$$y_{par}(x) = (7/24)x^3 + (17/7)x^2 - (1165/196)x + (1700/343).$$

**General solution.**
The general solution is therefore

$$y(x) = y_h(x) + y_{par}(x) = c_1 e^{-x} + c_2 e^{-7x} + (7/24)x^3 + (17/7)x^2 - (1165/196)x + (1700/343).$$

**Section 5.3 — Problem 7 — 5 points**

Suppose that we could find a particular solution of the form $y_{par}(x) = A + Bx + Cx^2$. Replacing in the ODE $y'$ and $y''$, we find

$$2C + B + 2Cx = 1 + 2x + x^2 \iff (2C + B) + (2C)x + 0x^2 = 1 + 2x + x^2.$$

But, 0 in front of the $x^2$ on the left-hand side can't be equal to the 1 in front of the $x^2$ on the right-hand side. Therefore, the particular solution can't be of the form $A + Bx + Cx^2$.

**Section 5.3 — Problem 15 — 5 points**

Suppose that $y_{par}(x) = Ax^\alpha$, where $A$ is a non-zero constant. To be a solution, the function $y_{par}$ should satisfy the ODE. We have

$$y' = A\alpha x^{\alpha-1} \quad \text{and} \quad y'' = A\alpha(\alpha-1)x^{\alpha-2}.$$

Substituting in the ODE, we get

$$ax^2(A\alpha(\alpha-1)x^{\alpha-2}) + bx(A\alpha x^{\alpha-1}) + cAx^\alpha = Mx^\alpha.$$

After simplifying, we obtain

$$aA\alpha(\alpha-1)x^\alpha + bA\alpha x^\alpha + cAx^\alpha = Mx^\alpha.$$

Dividing through $Ax^\alpha$, we get

$$a\alpha(\alpha-1) + b\alpha + c = M/A.$$

Since $M/A \neq 0$, then $a\alpha(\alpha-1) + b\alpha + c$ can't be zero. This was the claim made.

In the other direction, if $a\alpha(\alpha-1) + b\alpha + c \neq 0$, then there is some constant $M \neq 0$ such that

$$a\alpha(\alpha-1) + b\alpha + c = M.$$

Multiplying by $x^\alpha$, we obtain

$$a\alpha(\alpha-1)x^\alpha + b\alpha x^\alpha + cx^\alpha = Mx^\alpha$$

which can be rewritten as

$$ax^2\alpha(\alpha-1)x^{\alpha-2} + bx\alpha x^\alpha + cx^\alpha = Mx^\alpha.$$

Letting $y(x) = x^\alpha$, we therefore see that

$$ax^2y'' + bxx' + cy = Mx^\alpha.$$

Therefore, $y = x^\alpha$ is a particular solution (here, with $A = 1$).

**Section 5.4 — Problem 15 — 10 points**

**Find the general solution to the complementary equation.**

The complementary equation is

$$y'' - 3y' + 2y.$$

The characteristic equation is $r^2 - 3r + 2 = 0$. Therefore, the roots are $r = 1$ and $r = 2$. So, the general solution to the complementary equation is

$$y_h(x) = c_1 e^x + c_2 e^{2x}.$$

**Find a particular solution.**

The right-hand side if of the form exponential times a polynomial. Also, one of the root does not appear in the exponential. We therefore suggest

$$y_{par}(x) = Ae^{3x} + Bxe^{3x}.$$

We have

$$y' = 3Ae^{3x} + Be^{3x} + 3Bxe^{3x} \quad \text{and} \quad y'' = 9Ae^{3x} + 3Be^{3x} + 9Bxe^{3x} + 3Be^{3x}.$$

Replacing this in the ODE, we find

$$9Ae^{3x} + 3Be^{3x} + 9Bxe^{3x} + 3Be^{3x} - 3(3Ae^{3x} + Be^{3x} + 3Bxe^{3x}) + 2(Ae^{3x} + Bxe^{3x}) = e^{3x} + xe^{3x}.$$

Collecting similar terms together, we get

$$(2A + 3B)e^{3x} + (2B)xe^{3x} = e^{3x} + xe^{3x}$$

We should have the same number of $e^{3x}$ and $xe^{3x}$ on both sides. Therefore, we find that

$$2B = 1 \quad \text{and } 2A + 3B = 1.$$

We find that $B = 1/2$ and $A = -1/4$. The particular solution is therefore

$$y_{par}(x) = -\frac{e^{3x}}{4} + \frac{xe^{3x}}{2}.$$

**General solution.**

The general solution is therefore

$$y(x) = y_h(x) + y_{par}(x) = c_1 e^x + c_2 e^{2x} - \frac{e^{3x}}{4} + \frac{xe^{3x}}{2}.$$

**Section 5.4 — Problem 19 — 10 points**

**Find the general solution to the complementary equation.**
The characteristic equation is $r^2 - 2r + 1 = 0$. There is only one root, $r = 1$. Therefore, the solution is

$$y_h(x) = c_1 e^x + c_2 x e^x.$$

**Find a particular solution.**
We have that both $e^x$ and $xe^x$ are solutions to the complementary equation. Therefore, based on the lecture notes, we suggest

$$y_{par}(x) = x^2 e^x (Ax + B) = Ax^3 e^x + Bx^2 e^x.$$

We have

$$\begin{aligned} y' &= 3Ax^2 e^x + Ax^3 e^x + 2Bxe^x + Bx^2 e^x \\ y'' &= Ax^3 e^x + (6A + B)x^2 e^x + (6A + 4B)xe^x + 2Be^x \end{aligned}$$

We plug this in the ODE:

$$\begin{aligned} Ax^3 e^x + (6A + B)x^2 e^x + (6A + 4B)xe^x + 2Be^x - 2(3Ax^2 e^x + Ax^3 e^x + 2Bxe^x + Bx^2 e^x) \\ + Ax^3 e^x + Bx^2 e^x = 2e^x - 12xe^x. \end{aligned}$$

Collecting similar terms, we obtain

$$6Axe^x + 2Be^x = 2e^x - 12xe^x.$$

We must have $6A = 2$ and $2B = -12$. Therefore, we conclude that $A = 1/3$ and $B = -6$. A particular solution to the ODE is

$$y_{par}(x) = \frac{1}{3} x^3 e^x - 6x^2 e^x.$$

**General solution.**
The general solution is

$$y(x) = y_h(x) + y_{par}(x) = c_1 e^x + c_2 x e^x + \frac{x^3 e^x}{3} - 6x^2 e^x.$$

Total (Points): 50.