Math 302

Chapter 7

Section 7.1: Review of Power Series

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Fall 2022

Why Power Series?

Most of the differential equation of order 2 we have encountered are constant coefficients ODE. In most real-life application, the coefficients will be **variable coefficients** such as

- **Bessel's equation** of order n:

$$x^2y'' + xy' + (x^2 - n^2)y = 0.$$

- **Legendre's equation** of order n:

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0.$$

The methods we used in chapter 5won't be of use in those situations. This is why we need power series and the **power series method**.

Basic Definitions

- A **Power series** centered at a number a is an expression involving an infinite sum of powers of $(x - a)$:

$$\sum_{n=0}^{\infty} a_n(x-a)^n. = a_0 + a_1(x-a) + a_2(x-a)^2 + \dots$$

- If $a = 0$, we simply write

$$\boxed{\sum_{n=0}^{\infty} a_n x^n.}$$

We will confine ourselves to power series centered at $a = 0$.

- A power series **converges** on an interval I provided that for any x in this interval I, the following limit exists

$$\lim_{N\to\infty} \sum_{n=0}^{N} a_n x^n.$$

$\to S_0, S_1, S_2, \ldots$
$N=0 \quad N=1 \quad N=2 \quad \ldots$

$N=0,\ s_0 = a_0$
$N=1,\ s_1 = a_0 + a_1 x$
$N=2,\ s_2 = a_0 + a_1 x + a_2 x$
$\vdots$

- If a function f is expressed as a power series on I, we then write

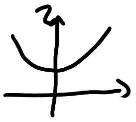
$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

and called this a **power series representation** of f.

Some examples of power series representations of some famous[1] function

- $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad -\infty < x < \infty$
- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad -\infty < x < \infty$
- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad -\infty < x < \infty$
- $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}, \quad -\infty < x < \infty$
- $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}, \quad -\infty < x < \infty.$
- $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}, \quad -1 < x < 1$
- $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{n=0}^{\infty} x^n, \quad -1 < x < 1$
- $(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} x^3 + \cdots, \quad -1 < x < 1$

$\frac{e^x + e^{-x}}{2}$



$\frac{e^x - e^{-x}}{2}$

Remark:

- The **Taylor series** of f is

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

- The **Maclaurin series** of f is

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n.$$

[1]They made the coverpage of New York Times magazine several times for their influence on the world.

Differentiation

If $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

and in general

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) a_n x^{n-k}.$$

Example 1. Differentiate the power series representation of $\sin x$.

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\Rightarrow \frac{d}{dx}(\sin x) = \sum_{n=0}^{\infty} \frac{d}{dx}\left(\frac{(-1)^n}{(2n+1)!} x^{2n+1} \right)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (2n+1) x^{2n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{\cancel{(2n+1)}(2n)!} \cdot \cancel{(2n+1)} x^{2n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$$= \cos(x).$$

Identity Principle or Uniqueness of Power series

If $f(x)=\sum_{n=0}^{\infty} a_n x^n$ and $g(x)=\sum_{n=0}^{\infty} b_n x^n$, then

$$f(x)=g(x) \quad \Longleftrightarrow \quad a_n=b_n, \text{ for all } n\geq 0.$$

Consequence: We have

$$\sum_{n=0} a_n x^n = 0 = \sum_{n=0}^{\infty} 0\cdot x^n$$

if, and only if, $a_n=0$ for all $n\geq 0$.

EXAMPLE 2. Find $y(x)$ if

$$y'=\sum_{n=1}^{\infty} x^{n-1} \quad \text{and} \quad y(0)=0.$$

$$y(x)=\sum_{n=0}^{\infty} a_n x^n \longrightarrow y'(x)=\sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$\Rightarrow \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} x^{n-1}$$

$$\Rightarrow 1\cdot a_1 + 2a_2 x + 3a_3 x^2 + \ldots$$

$$= 1 + x + x^2 + \ldots$$

$$\Rightarrow a_1 = 1, \quad 2a_2 = 1, \quad 3a_3 = 1, \ldots, n a_n = 1$$

$$\Rightarrow a_n = \frac{1}{n}, \text{ for any } n \geq 1.$$

So, $$y(x) = a_0 + \sum_{n=1}^{\infty} \frac{1}{n} x^n.$$

We have $y(0)=0 \Rightarrow 0 = a_0 + \cancel{\sum_{n=1}^{\infty} \frac{1}{n}\cdot 0^n} \to 0$

$\Rightarrow a_0 = 0$

Answer $$y(x) = \sum_{n=1}^{\infty} \frac{1}{n} x^n = -\log(1-x).$$

Sum, Difference and Multiplication by A Constant

If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ are two power series, then

- $f(x) + g(x) = \sum_{n=0}^{\infty} (a_n + b_n) x^n$.
- $f(x) - g(x) = \sum_{n=0}^{\infty} (a_n - b_n) x^n$.
- $cf(x) = \sum_{n=0}^{\infty} (c a_n) x^n$.

Example 3. Use the definition of $\cosh(x)$ and $\sinh(x)$ to find its power series representation.

1) $\cosh(x) = \dfrac{e^x + e^{-x}}{2} = \dfrac{e^x}{2} + \dfrac{e^{-x}}{2}$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \frac{x^n}{n!} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \frac{x^n}{n!} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2n!} + \frac{(-1)^n}{2n!} \right) x^n$$

$$= \left(\frac{1}{2} + \frac{1}{2} \right) x^0 + \left(\frac{1}{2} - \frac{1}{2} \right) x$$
$$+ \left(\frac{1}{2(2!)} + \frac{1}{2(2!)} \right) x^2 + \cdots$$

$$= x^0 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

Product with Polynomials

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$.

- $g(x) = cx$, then

$$f(x)g(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right) cx$$
$$= \sum_{n=0}^{\infty} a_n c x^{n+1} = \sum_{n=1}^{\infty} a_{n-1} c x^n$$

- $g(x) = cx^2$, then

$$f(x)g(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right) cx^2$$
$$= \sum_{n=0}^{\infty} a_n c x^{n+2} = \sum_{n=2}^{\infty} a_{n-2} c x^n .$$

- $g(x) = cx^3$, then

$$f(x)g(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right) cx^3$$
$$= \sum_{n=0}^{\infty} c a_n x^{n+3} = \sum_{n=3}^{\infty} a_{n-3} c x^n .$$

EXAMPLE 4. If $f(x) = \sum_{n=0}^{\infty} a_n x^n$, find the expression of

(a) xf'.

(b) $(2-x)f''$.

(a) $$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$\Rightarrow \quad x f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1+1} = \sum_{n=1}^{\infty} n a_n x^n .$$

(b) $$f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$(2-x) f''(x) = 2f''(x) - x f''(x)$$
$$= 2\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$
$$= \sum_{n=2}^{\infty} 2n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^{n-1}$$

$$\stackrel{?}{=} \sum_{n=2}^{\infty} \left(2n(n-1)a_n - n(n-1)a_n \right) x$$

How can we shift the index?

$\leq$

Shifting

For any integer k, if

$$y(x) = \sum_{n=n_0}^{\infty} a_n x^{n-k}.$$

$m = n-k$

$n = n_0 \longrightarrow m_0 = n_0 - k$

$m + k = n$

then

$$y(x) = \sum_{n=n_0-k}^{\infty} a_{n+k} x^n.$$

EXAMPLE 5. Complete Example 4.

$$(2-x)f'' = \sum_{n=2}^{\infty} 2n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} [-n(n-1)a_n] x^{n-1}$$

$$= \sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2} x^n + \sum_{n=1}^{\infty} [-(n+1)n\, a_{n+1}] x^n$$

$$= 2(2)(1)a_2 + \sum_{n=1}^{\infty} 2(n+2)(n+1)a_{n+2} x^n + \sum_{n=1}^{\infty} (-(n+1)n\, a_{n+1}) x^n$$

$$= 4a_2 + \sum_{n=1}^{\infty} [2(n+2)(n+1)a_{n+2} - (n+1)n\, a_{n+1}] x^n = \sum_{n=0}^{\infty} b_n x^n$$

$$\Rightarrow b_0 = 4a_2 \quad \& \quad b_n = 2(n+2)(n+1)a_{n+2} - (n+1)n\, a_{n+1}.$$

Example 6. Express $2y - xy''$ as a power series $\sum_{n=0}^{\infty} c_n x^n$.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

So,

$$2y = \sum_{n=0}^{\infty} 2a_n x^n$$

$$xy'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-1}$$

$$2y - xy'' = \sum_{n=0}^{\infty} 2a_n x^n - \sum_{n=2}^{\infty} n(n-1) a_n x^{n-1}$$

$$= \sum_{n=0}^{\infty} 2a_n x^n - \sum_{n=1}^{\infty} (n+1)n\, a_{n+1} x^n$$

$$= 2a_0 + \sum_{n=1}^{\infty} 2a_n x^n - \sum_{n=1}^{\infty} (n+1)n\, a_{n+1} x^n$$

$$= 2a_0 + \sum_{n=1}^{\infty} \left(2a_n - (n+1)n\, a_{n+1}\right) x^n$$

$$= \sum_{n=0}^{\infty} c_n x^n$$

with $c_0 = 2a_0$, $c_n = 2a_n - (n+1)n\, a_{n+1}$.