Math 302

Chapter 8

Section 8.3: Unit Step Function

Contents

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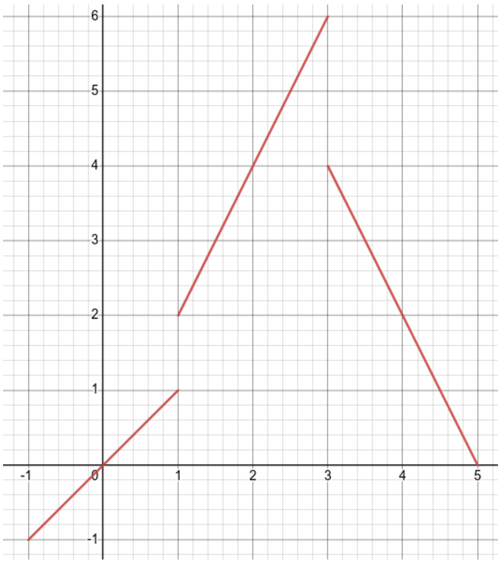
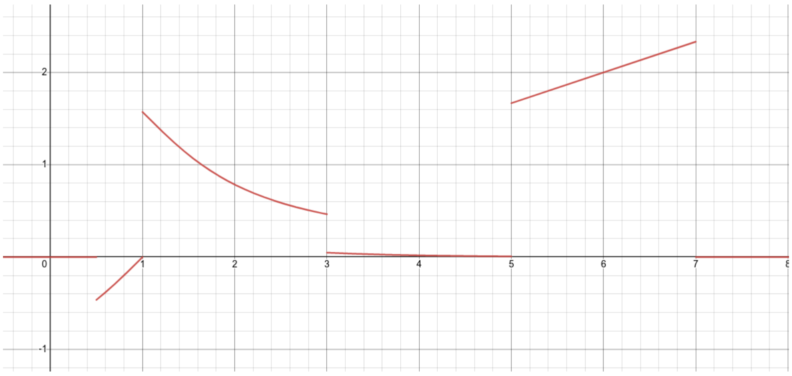
Piecewise Continuous Functions

A piecewise continuous function f is

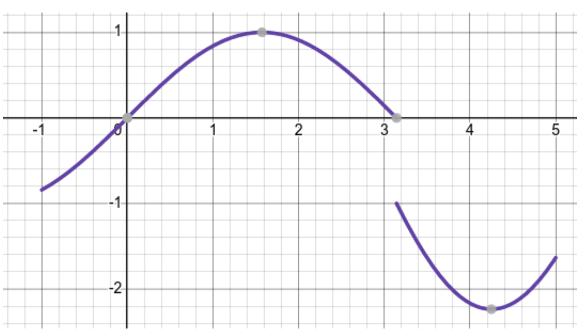
- a function defined on a finite number of intervals $[t_0, t_1]$, $[t_1, t_2]$, $\ldots$, $[t_{n-1}, t_n]$;
- such that it is continuous on each interval (t_0, t_1), (t_1, t_2), $\ldots$, (t_{n-1}, t_n).



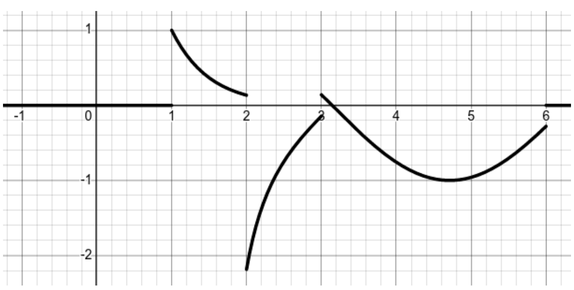
(a) A function $f(x)$



(b) A function $k(x)$



(c) A function $g(x)$



(d) A function $h(x)$

EXAMPLE 1. Find the Laplace transform of

$$f(t) = \begin{cases} \cancel{t} & 0 \leq t \leq 1 \\ 2t & 1 < t \leq 3 \\ 10 - 3t & 3 < t \leq 5 \\ 0 & 5 < t. \end{cases}$$

$0 \leq x \leq 1$
$1 < x \leq 3$
$3 < x \leq 5$
$5 < x$

By definition,

$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$
$$= \int_0^1 t e^{-st}\, dt + \int_1^3 2t\, e^{-st}\, dt + \int_3^5 (10-3t)\, e^{-st}\, dt$$

By parts :

t	e^{-st}	
1	$e^{-st}/-s$	$+$
0	e^{-st}/s^2	$-$
		$+$

$10-3t$	e^{-st}	
-3	$e^{-st}/-s$	$+$
0	e^{-st}/s^2	$-$
		$+$

$$\Rightarrow \int_0^1 t e^{-st}\, dt = -\frac{te^{-st}}{s}\Big|_0^1 - \frac{e^{-st}}{s^2}\Big|_0^1 = \frac{1-e^{-s}}{s} + \frac{1-e^{-s}}{s^2}$$

$$\Rightarrow \int_1^3 t e^{-st}\, dt = -\frac{te^{-st}}{s}\Big|_1^3 - \frac{e^{-st}}{s^2}\Big|_1^3 = \frac{e^{-s} - 3e^{-3s}}{s} + \frac{e^{-s} - e^{-3s}}{s^2}$$

$$\Rightarrow \int_3^5 (10-3t)\, e^{-st} = -\frac{(10-3t)\,e^{-st}}{s}\Big|_3^5 + 3\,\frac{e^{-st}}{s^2}\Big|_3^5$$

$$= \frac{e^{-3s} + 5e^{-5s}}{s} + 3\,\frac{e^{-5s} - e^{-3s}}{s^2}$$

So,

$$F(s) = \frac{1-e^{-s}}{s} + \frac{1-e^{-s}}{s^2} + 2\left(\frac{e^{-s} - 3e^{-3s}}{s} + \frac{e^{-s} - e^{-3s}}{s^2}\right)$$
$$+ \frac{e^{-3s} + 5e^{-5s}}{s} + 3\,\frac{e^{-5s} - e^{-3s}}{s^2} \quad \ldots$$

To make the work easier with piecewise continuous function, we introduce the **unit step function**:

$$u(t) := \begin{cases} 0 & t < 0 \\ 1 & t \geq 0. \end{cases}$$

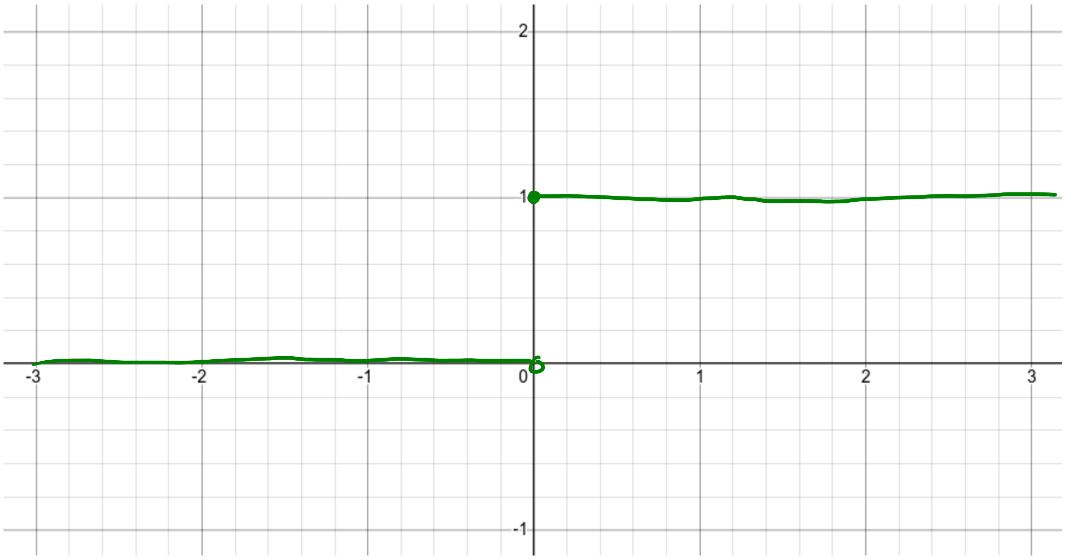


Figure 2: Plot of $u(t)$

Basic Operations

- Translation by a units:

$$u(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a. \end{cases}$$

- Multiplication by c:

$$cu(t) = \begin{cases} 0 & t < 0 \\ c & t \geq 0. \end{cases}$$

- Activation of a function $f(t)$ at time a:

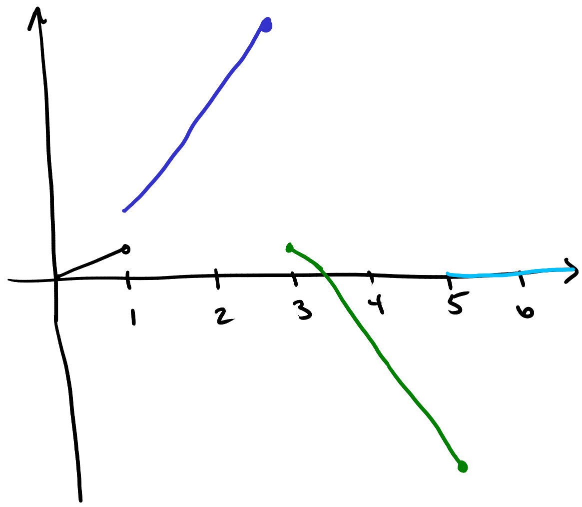
$$f(t)u(t-a) = \begin{cases} 0 & t < a \\ f(t) & t \geq a. \end{cases}$$

- Destruction of a function $f(t)$ at time b and activation of a function $g(t)$ at time b:

$$f(t)u(t-a) + (g(t) - f(t))u(t-b) = \begin{cases} 0 & t < a \\ f(t) & a \leq t < b \\ g(t) & b \leq t. \end{cases}$$

Example 2. Rewrite the function $f(t)$ in Example 1 using the unit step function.

$$f(x) = \begin{cases} x & 0 \le x \le 1 \\ 2x & 1 < x \le 3 \\ 10-3x & 3 < x \le 5 \\ 0 & 5 < x \end{cases}$$



$\underline{0 \le t \le 1}$

$$\underbrace{t\,u(t)}_{\text{switch on}} - \underbrace{t\,u(t-1)}_{\text{switch off.}}$$

$\underline{2 \le t \le 3}$

$$\underbrace{2t\ u(t-1)}_{\text{switch on}} - \underbrace{2t\ u(t-3)}_{\text{switch off}}$$

$\underline{3 \le t \le 5}$

$$\underbrace{(10-3t)u(t-3)}_{\text{switch on}} - \underbrace{(10-3t)\,u(t-5)}_{\text{switch off}}$$

<u>Therefore:</u>

$$\begin{aligned} f(t) &= t u(t) - t u(t-1) + 2t\,u(t-1) - 2t\,u(t-3) \\ &\quad + (10-3t)\,u(t-3) - (10-3t)\,u(t-5) \\ &= t\ u(t) + t\ u(t-1) + (10-5t)\,u(t-3) \\ &\quad + (10-3t)\ u(t-5). \end{aligned}$$

EXAMPLE 3. A farmer has a field of potatoes of 1 kilometer long. An automated watering system starts at 5:00AM and stops at 8:00AM. The spite of water is 1000 liters per hour. Give an expression of the function $W(t)$ of water used during the day using the unit step function.

t: time in hour, from 00:00.

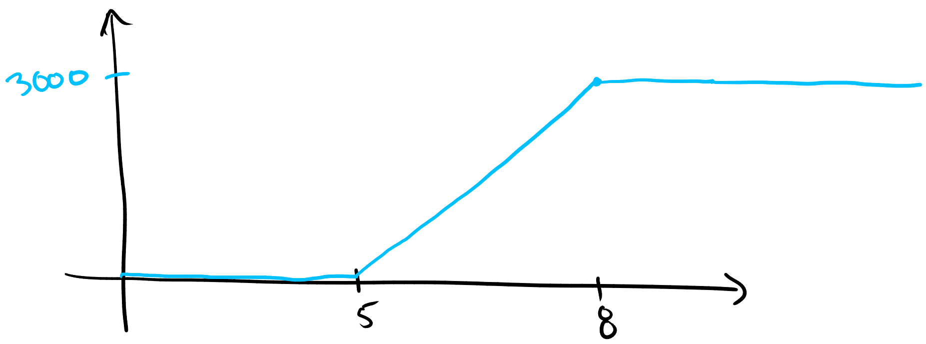
Water used : $1000 \cdot \text{TIME}$.

So,

$$\underbrace{1000(t-5)\,u(t-5)}_{\text{switch on}} - \underbrace{1000(t-5)\,u(t-8)}_{\text{switch off.}}$$

Total water used : $3 \cdot 1000 = 3000$

$$\Rightarrow \boxed{W(t) = 1000(t-5)\left[u(t-5) - u(t-8)\right] + 3000\,u(t-8)}$$



Let $a \geq 0$ be a real number and f be a function with a Laplace transform $F(s)$.

- $L(u(t-a)) = \dfrac{e^{-sa}}{s}$.
- $L(u(t-a)f(t)) = e^{-sa}L(f(t+a))$.
- $L(u(t-a)f(t-a)) = e^{-sa}F(s)$.

EXAMPLE 4. Find the Laplace transform of

$$f(t) = \begin{cases} \sin(t) & , 0 \leq t < \pi/2 \\ \cos(t) - 3\sin(t) & , \pi/2 \leq t < \pi \\ 3\cos(t) & , t \geq \pi. \end{cases}$$

Write

$$f(t) = \sin t \left[u(t) - u(t - \pi/2)\right]$$
$$+ \cos(t) - 3\sin(t) \left[u\left(t - \tfrac{\pi}{2}\right) - u(t-\pi)\right]$$
$$+ 3\cos(t)\, u(t-\pi)$$

$$\Rightarrow f(t) = \sin t\, u(t)^{①} + [\cos t - 4\sin t]^{②} u(t - \pi/2)$$
$$+ [3\sin t + 2\cos t]^{③} u(t-\pi)$$

Use the first formula: $\mathcal{L}(u(t-a)f(t)) = e^{-sa}L(f(t+a))$

① $L(\sin t\, u(t)) = e^{-s(0)} L(\sin(t+0))$
$$= 1 \cdot L(\sin t)$$
$$= \frac{1}{s^2+1}$$

② $L((\cos t - 4\sin t)u(t - \pi/2)) = e^{-s\frac{\pi}{2}} L\left(\cos\left(t + \tfrac{\pi}{2}\right) - 4\sin\left(t + \tfrac{\pi}{2}\right)\right)$

Now, $\cos(t+\frac{\pi}{2}) = -\sin(t)$

& $\sin(t+\frac{\pi}{2}) = \cos(t)$

$$\Rightarrow \quad L(\cos(t+\tfrac{\pi}{2})) = L(-\sin t) = -\frac{1}{s^2+1}$$

$$\& \quad L(\sin(t+\pi/2)) = L(\cos t) = \frac{s}{s^2+1}$$

So, $$② = -\frac{e^{-\frac{s\pi}{2}}}{s^2+1} + \frac{se^{-s\pi/2}}{s^2+1}$$

$$③ \quad L([3\sin t + 2\cos t]\, u(t-\pi)) = e^{-\pi s} L(3\sin(t+\pi) + 2\cos(t+\pi))$$

We have

$$\sin(t+\pi) = -\sin t$$

$$\& \quad \cos(t+\pi) = -\cos t$$

$$\Rightarrow \quad L(\sin(t+\pi)) = -L(\sin t) = \frac{-1}{s^2+1}$$

$$\& \quad L(\cos(t+\pi)) = -L(\cos t) = -\frac{s}{s^2+1}$$

So, $$③ = -\frac{3e^{-\pi s}}{s^2+1} - \frac{2se^{-\pi s}}{s^2+1}$$

Therefore:

$$F(s) = \frac{1}{s^2+1} - \frac{e^{-\frac{s\pi}{2}}}{s^2+1} + \frac{se^{-s\pi/2}}{s^2+1} - \frac{3e^{-\pi s}}{s^2+1} - \frac{2se^{-\pi s}}{s^2+1}.$$

EXAMPLE 5. Find

$$S = L^{-1}\Big(\frac{1}{s^2} - e^{-s}\Big(\frac{1}{s^2} + \frac{2}{s}\Big) + e^{-4s}\Big(\frac{4}{s^3} + \frac{1}{s}\Big)\Big)$$

$$S = L^{-1}\left(\frac{1}{s^2}\right) - L^{-1}\left(e^{-s}/s^2\right) - 2L^{-1}\left(e^{-s}/s\right) + 4L^{-1}\left(\frac{e^{-4s}}{s^3}\right) + L^{-1}\left(\frac{e^{-4s}}{s}\right)$$

- $L^{-1}\left(\frac{1}{s^2}\right) = t$
- $L^{-1}\left(\frac{e^{-s}}{s^2}\right) = u(t-1)\,(t-1)$
- $L^{-1}\left(\frac{e^{-s}}{s}\right) = u(t-1)$
- $L^{-1}\left(\frac{e^{-4s}}{s^3}\right) = \frac{1}{2}u(t-4)\,(t-4)^2$
- $L^{-1}\left(\frac{e^{-4s}}{s}\right) = u(t-4)$.

$\mathcal{L}(1) = \frac{1}{s}$

$\mathcal{L}(t) = \frac{1}{s^2}$

$L(t^2) = \frac{2}{s^3}$

Therefore,

$$f(t) = t - (t-1)\,u(t-1) - 2u(t-1) + 2(t-4)^2 u(t-4) + u(t-4)$$

We can now allow the forcing function to be a discontinuous function (piecewise continuous).

Example 6. Solve the initial value problem

$$y'' - y = f(t), \quad y(0) = -1, \, y'(0) = 2,$$

where

$$f(t) = \begin{cases} t & 0 \leq t < 1 \\ 1 & t \geq 1. \end{cases}$$

Write

$$f(t) = t\,[u(t) - u(t-1)] + u(t-1)$$
$$= t\,u(t) + (1-t)\,u(t-1).$$

Take Laplace transform:

$$L(y'') = s^2 Y - s y(0) - y'(0) = s^2 Y + s - 2$$
$$L(y') = sY - y(0) = sY + 1$$
$$L(y) = Y.$$
$$L(f) = \frac{-1}{s^2} + \frac{e^{-s}}{s^2}$$

The transformed ODE is therefore:

$$s^2 Y + s - 2 - Y = \frac{-1}{s^2} + \frac{e^{-s}}{s^2}$$

$$\Rightarrow \quad (s^2 - 1) Y = 2 - s + \frac{e^{-s} - 1}{s^2}$$

$$\Rightarrow \quad Y = \frac{2-s}{s^2-1} + \frac{e^{-s}-1}{s^2(s^2-1)}$$

Now, $\quad s^2-1 = (s-1)(s+1)$

$$\Rightarrow \quad Y = \frac{2-s}{(s-1)(s+1)} + \frac{e^{-s}-1}{s^2(s-1)(s+1)}$$

$$= \frac{1}{(s-1)(s+1)} - \frac{1}{s+1} + \frac{e^{-s}-1}{s^2(s-1)(s+1)}$$

1) $\displaystyle \frac{1}{(s-1)(s+1)} = \frac{1/2}{s-1} + \frac{1/2}{s+1}$

2) $\displaystyle \frac{1}{s^2(s-1)(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{s+1}$

$$\Leftrightarrow \quad 1 = As(s-1)(s+1) + B(s-1)(s+1) + Cs^2(s+1) + Ds^2(s-1)$$

$\underline{s=0}$ $\quad 1 = B(-1)(1) \Rightarrow \underline{B=-1}$

$\underline{s=1}$ $\quad 1 = C(1)^2(2) \Rightarrow \underline{C=1/2}$

$\underline{s=-1}$ $\quad 1 = D(-1)^2(-2) \Rightarrow \underline{D=-1/2}$

$\underline{s=3}$ $\quad 1 = A(3)(2)(4) + (-1)(2)(4) + (1/2)(9)(4) + (-1/2)(9)(2)$

$\Leftrightarrow 1 = 24A - 8 + 18 - 9$

$\Leftrightarrow 1 = 24A + 1 \Leftrightarrow \underline{A=0}$

So, $\displaystyle \frac{1}{s^2(s^2-1)} = \frac{-1}{s^2} + \frac{1/2}{s-1} - \frac{1/2}{s+1}$

Therefore,

$$Y(s) = \cancel{\frac{1/2}{s-1}} + \cancel{\frac{1/2}{s+1}} - \cancel{\frac{1}{s+1}} - \frac{e^{-s}}{s^2} + \frac{1}{2}\frac{e^{-s}}{s-1} - \frac{1}{2}\frac{e^{-s}}{s+1}$$
$$+ \frac{1}{s^2} - \cancel{\frac{1}{2}\cdot\frac{1}{s-1}} + \cancel{\frac{1}{2}\frac{1}{s+1}}$$

$$\Rightarrow \quad Y(s) = \frac{1}{s^2} - \frac{e^{-s}}{s^2} + \frac{1}{2}\frac{e^{-s}}{s-1} - \frac{1}{2}\frac{e^{-s}}{s+1}$$

- $L(t) = -\frac{1}{s^2}$ $\qquad$ • $L(u(t-1)(t-1)) = -\frac{e^{-s}}{s^2}$

- $L\left(u(t-1)\,e^{t-1}\right) = \frac{e^{-(s-1)}}{s-1} = e^{1}\cdot\frac{e^{-s}}{s-1}$

- $L\left(u(t-1)\,e^{t+1}\right) = \frac{e^{-(s+1)}}{s+1} = e^{-1}\,\frac{e^{-s}}{s+1}$

Therefore,

$$\boxed{y(t) = -t + (t-1)\,u(t-1) + \frac{1}{2e}\,e^{t-1}\,u(t-1) - \frac{e}{2}\,e^{t+1}\,u(t-1)}$$