Math 302

Chapter 4

Section 4.2: Cooling and Mixing

Contents

Created by: Pierre-Olivier Parisé
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Recall that Newton's law of cooling is given by

constant

$$T' = -k(T - T_m) \tag{1}$$

where $k > 0$ is a constant, T is the temperature of the object, and T_m is the temperature of the medium (surrounding).

EXAMPLE 1. Find the solution to (1) subject to the additional condition $T_0 = T(0)$.

T_m is constant: (1) is a seperable ODE.

So, $\dfrac{T'}{T-T_m} = -k$ $\quad \left(T' = \dfrac{dT}{dt}\right)$

$\xRightarrow{\text{integrate}}$ $\ln|T-T_m| = -kt + C$

$\Rightarrow$ $|T-T_m| = e^{-kt} e^{C}$

$\Rightarrow$ $T - T_m = ce^{-kt}$ $\quad (c = \pm e^{C})$

$\Rightarrow$ $T(t) = T_m + ce^{-kt}$.

We have $T_0 = T(0)$

$\Rightarrow$ $T_0 = T_m + c$ $\Rightarrow$ $c = T_0 - T_m$

The solution to

$T' = -k(T-T_m)$ & $T(0) = T_0$

is $\boxed{T(t) = T_m + (T_0 - T_m)e^{-kt}}$

EXAMPLE 2. A ceramic insulator is baked at 400°C and cooled in a room in which the temperature is 25°C. After 4 minutes the temperature of the insulator is 200°C. What is its temperature after 8 minutes?

From the previous page:

$$T(t) = T_m + (T_0 - T_m)e^{-kt}$$

$T_m = 25°C$

$T_0 = 400°C$

$$\Rightarrow \quad T(t) = 25 + (375)e^{-kt}.$$

Find k.

$$T(4) = 200 \quad \Rightarrow \quad 200 = 25 + 375e^{-4k}$$

$$\Rightarrow \quad 175 = 375e^{-4k}$$

$$\Rightarrow \quad \frac{175}{375} = e^{-4k}$$

$$\Rightarrow \ln\left(\frac{175}{375}\right) = -4k \Rightarrow k = \frac{-1}{4}\ln\left(\frac{7}{15}\right)$$

So $T(t) = 25 + 375e^{[\ln(7/15)]t/4}$

$$= 25 + 375e^{\ln(7/15)^{t/4}}$$

$a\ln(4) = \ln(4^a)$

$\frac{t}{4}\ln(7/15) = \ln(7/15)^{t/4}.$

$$\Rightarrow \quad T(t) = 25 + 375\,(7/15)^{t/4}$$

Find T(8)

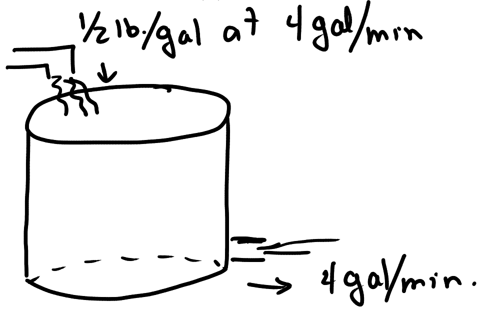
$$T(8) = 25 + 375\,(7/15)^{8/4}$$

$$= 25 + 375\,(7/15)^2 \approx \boxed{107°C}$$

Example 3. A tank initially contains 40 pounds of salt dissolved in 600 gallons of water. Starting at $t_0 = 0$, water that contains $1/2$ pound of salt per gallon is poured into the tank at the rate of $4\text{gal}/\text{min}$ and the mixture is drained from the tank at the same rate. We assume that the mixture is stirred instantly so that the salt is always uniformly distributed throughout the mixture.

1. Find a differential equation for the quantity $Q(t)$ of salt in the tank at time $t > 0$, and solve the equation to determine $Q(t)$.
2. Find $\lim_{t\to\infty} Q(t)$.

1.



Since rate in = rate out

$\Downarrow$

Volume of solution is constant.

$Q(t)$: Quantity of salt (lb)

$Q'(t)$: rate of change of Q.

t: time in minutes.

$$Q' = \text{rate in} - \text{rate out}$$

Rate in: 4 gal/min at ½ lb/gal

$$\Rightarrow 4 \cdot \cancel{\text{gal}}/\text{min} \cdot (1/2)\ \text{lb}/\cancel{\text{gal}} = 2\ \text{lb}/\text{min}.$$

$$\Rightarrow \text{rate in} = 2\ \text{lb}/\text{min}$$

Rate out: (Concentration of salt in the tank)

$\times$ (rate that is drained)

$$= \frac{Q(t)}{\text{Vol. tank}}\ \text{lb}/\cancel{\text{gal}} \times 4\ \cancel{\text{gal}}/\text{min}$$

$$= \frac{Q(t)}{600} \cdot 4\ \text{lb}/\text{min} = \frac{Q}{150}\ \text{lb}/\text{min}.$$

So, $Q' = 2 - \dfrac{Q}{150} = \dfrac{300 - Q}{150}$, $Q(0) = 40\text{lb}$.

Solve ODE

$$Q' = \frac{300-Q}{150}$$

$$\Leftrightarrow \quad \frac{dQ}{dt} = \frac{300-Q}{150} \Leftrightarrow dQ = \frac{300-Q}{150}\,dt$$

$$\boxed{\begin{array}{l} u = 300-Q \\ du = -dQ \end{array}}$$

$$\Rightarrow \quad \int \frac{dQ}{300-Q} = \int \frac{dt}{150}$$

$$\Rightarrow \quad -\ln|300-Q| = \frac{t}{150} + K$$

$$\Rightarrow \quad \frac{1}{|300-Q|} = e^{t/150} \cdot e^{K}$$

$$\Rightarrow \quad 300 - Q = ce^{-t/150} \qquad (c = \pm e^{-K})$$

$$\Rightarrow \quad Q(t) = 300 - ce^{-t/150}$$

Find c

$$40 = Q(0) = 300 - c \Rightarrow c = 260$$

Therefore, $\boxed{Q(t) = 300 - 260e^{-t/150}}$

2) $$\lim_{t\to\infty} Q(t) = \lim_{t\to\infty} 300 - 260 \lim_{t\to\infty} e^{-t/150}$$

$$= 300 - 260 \cdot 0$$

$$= \boxed{300}$$