**Section 7.1 — Problem A — 5 points**

We have

$$(1-x)^{-1} = \sum_{n=0}^{\infty} x^n.$$

We differentiate, we then get

$$\frac{d}{dx}\Big(\frac{1}{1-x}\Big) = \sum_{n=1}^{\infty} n x^{n-1}$$

and this gives

$$-\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}.$$

## Section 7.1 — Problem B — 20 points

Let $y(x) = \sum_{n=0}^{\infty} a_n x^n$. We have

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

and so the left-hand side becomes

$$\begin{aligned} \sum_{n=1}^{\infty} n a_n x^{n-1} + x &= \sum_{n=1} n x^n \\ \iff a_1 + (2a_2 + 1)x + \sum_{n=2}^{\infty} (n+1) a_{n+1} x^n &= x + \sum_{n=2}^{\infty} n x^n. \end{aligned}$$

Comparing coefficients, we find

$$a_1 = 0, \quad 2a_2 + 1 = 1 \quad \text{and} \quad (n+1)a_{n+1} = n \quad (n \geq 2)$$

and therefore

$$a_1 = 0, \quad a_2 = 0 \quad \text{and} \quad a_{n+1} = \frac{n}{n+1} \quad (n \geq 2).$$

The solution is therefore

$$y(x) = a_0 + \sum_{n=2}^{\infty} \frac{n}{n+1} x^n.$$

We have $y(0) = 0$, then $a_0 = 0$. So, the solution is

$$y(x) = \sum_{n=2}^{\infty} \frac{n}{n+1} x^n.$$

Remark: We can find a close formula for the solution. We have

$$\sum_{n=2}^{\infty} \frac{n}{n+1} x^n = \sum_{n=2}^{\infty} x^n - \sum_{n=2}^{\infty} \frac{x^n}{n+1} = \sum_{n=0}^{\infty} x^n - 1 - x - \frac{1}{x} \sum_{n=2}^{\infty} \frac{x^{n+1}}{n+1}$$

and using the power series representation of $(1-x)^2$ and $-\ln(1-x)$, we then find that

$$\begin{aligned} y(x) &= \frac{1}{1-x} - 1 - x + \frac{1}{x}\left(x + \frac{x^2}{2} + \ln(1-x)\right) \\ &= \frac{1}{1-x} - \frac{x}{2} + \frac{1}{x}\log(1-x). \end{aligned}$$

Section 7.1 — Problem C — 5 points

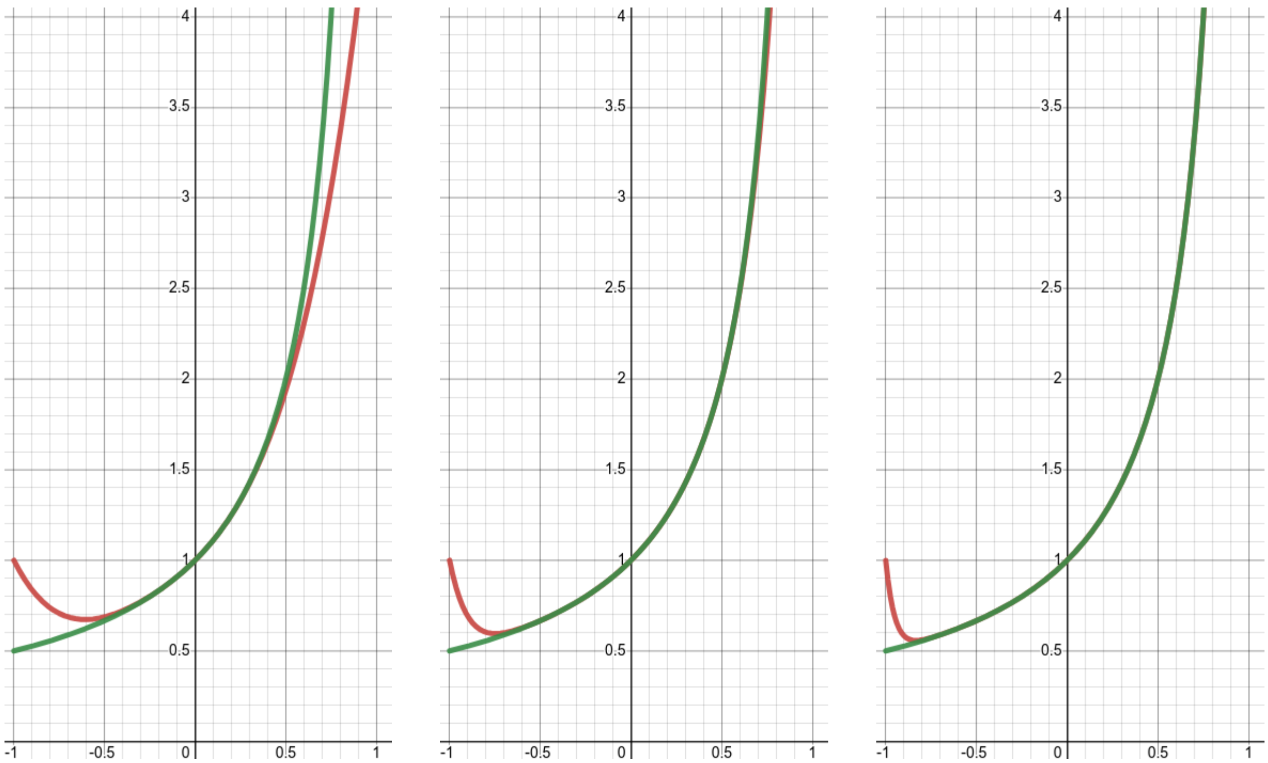
Since the power series of $(1-x)^{-1}$ is

$$\sum_{n=0}^{\infty} x^n,$$

we have

$$T_N(x) = 1 + x + x^2 + \cdots + x^{N-1} + x^N.$$

Below, we used Desmos to plot the graph of $T_4$, $T_1$0, $T_{20}$. The graph of $(1-x)^{-1}$ is in green.




(a) $T_4$ (b) $T_{10}$ (c) $T_{20}$

**Section 7.1 — Problem D — 10 points**

Set $y(x) = \sum_{n=0}^{\infty} a_n x^n$. We have

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Therefore, we get

$$\begin{aligned} x^2 y'' &= \sum_{n=2}^{\infty} n(n-1) a_n x^n \\ 2xy' &= \sum_{n=1}^{\infty} 2n a_n x^n \\ 3xy &= \sum_{n=0}^{\infty} 3a_n x^{n+1}. \end{aligned}$$

We then get

$$\begin{aligned} x^2y'' + 2xy' - 3xy &= \sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=1}^{\infty} 2n a_n x^n - \sum_{n=0}^{\infty} 3a_n x^{n+1} \\ &= \sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=1}^{\infty} 2n a_n x^n - \sum_{n=1}^{\infty} 3a_{n-1} x^n \\ &= (2a_1 - 3a_0)x + \sum_{n=2}^{\infty} \Big( \big( n(n-1) + 2n \big) a_n - 2a_{n-1} \Big) x^n \\ &= (2a_1 - 3a_0)x + \sum_{n=2}^{\infty} \Big( n(n+1) a_n - 2a_{n-1} \Big) x^n. \end{aligned}$$

Therefore, the expression can be rewritten as a power series $\sum_{n=0}^{\infty} c_n x^n$, where

$$c_0 = 0, \quad c_1 = 2a_1 - 3a_0 \quad \text{and} \quad c_n = n(n+1)a_n - 2a_{n-1}.$$

**Section 7.1 — Problem E — 10 points**

Let $y(x) = \sum_{n=0}^{\infty} a_n x^n$. Then, we have

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

We therefore get

$$\begin{aligned} xy'' &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-1} \\ (4+2x)y' &= \sum_{n=0}^{\infty} 4a_n x^n + \sum_{n=1}^{\infty} 2n a_n x^n \\ (2+x)y &= \sum_{n=0}^{\infty} 2a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+1}. \end{aligned}$$

We can therefore get

$$\begin{aligned} xy'' + (4+2x)y' + (2+x)y &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-1} + \sum_{n=0}^{\infty} 4a_n x^n + \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} 2a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+1} \\ &= \sum_{n=1}^{\infty} (n+1) n a_n x^n + \sum_{n=0}^{\infty} (6+2n) a_n x^n + \sum_{n=1}^{\infty} a_{n-1} x^n \\ &= 6a_0 + \sum_{n=1}^{\infty} \Big( (n^2 + n + 2n + 6) a_n + a_{n-1} \Big) x^n \\ &= 6a_0 + \sum_{n=1}^{\infty} \Big( (n^2 + 3n + 6) a_n + a_{n-1} \Big) x^n. \end{aligned}$$

We can then rewrite the expression as a power series $\sum_{n=0}^{\infty} c_n x^n$ with

$$c_0 = 6a_0 \quad \text{and} \quad c_n = (n^2 + 3n + 6) a_n + a_{n-1}.$$

TOTAL (POINTS): 50.