Math 302

Chapter 2

Section 2.1: Linear First Order Differential Equation

Contents

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A first order ODE is said to be **linear** (abbreviated LFODE) if it can be written as

$$y' + p(x)y = f(x). \quad \rightarrow \quad y' = \underbrace{f(x) - p(x)y} \tag{1}$$

- Example: $y' + 3y/x^2 = 1.$ $p(x) = 3/x^2, \; f(x) = 1$
- Example: $xy' - 8x^2y = \sin x.$ $\rightarrow \div x \rightarrow \; y' - 8xy = \frac{\sin x}{x} \; (x \neq 0)$

More Terminology

- A first order ODE that is not of the form (1), then the ODE is said to be **nonlinear**.
  - Example: $xy' + 3y^2 = 2x.$
  - Example: $\underline{yy'} + \underline{e^y} = \underline{\tan(xy)}.$
- When $f(x) = 0$ for any x, then $y' + p(x)y = 0$ is said to be **homogeneous**.
  - Example: $y' + 3y/x^2 = 0.$
  - Example: $xy' - 8x^2y = 0.$
- When $f(x)$ is not zero, then the LODE is said to be **nonhomogeneous**.

Example 1. Find all the solutions to

$$y' = \frac{1}{x^2}$$

$p(x) = 0$
$f(x) = 1/x^2$

Integrate $\Rightarrow$ $y(x) = -\frac{1}{x} + c$ valid on $(-\infty, 0) \cup (0, \infty)$

$\hookrightarrow (*)$

$(*)$ is called a one-parameter family of functions.

$y(x, c)$

General Solution

We say that a function $y = y(x, c)$ is a **general solution** to (1) if

- For each fixed parameter c, the resulting function $y = y(x, c)$ is a solution to (1) on an an open interval (a, b).
- If $y_1 = y_1(x)$ is a solution to (1) on (a, b), then y_1 can be obtained from the formula $y = y(x, c)$ by choosing c appropriately.

$y_1(x, 2) = -\frac{1}{x} + 2$ $\rightarrow$ come from $-\frac{1}{x^2} + c$

with $c = 2$.

We now find the general solution to

$$y' + p(x)y = 0 \tag{2}$$

where p is continuous on an interval (a, b).

EXAMPLE 2. Let a be a constant (fixed).

1. Find the general solution of $y' - ay = 0$.
2. Solve the initial value problem

$$y' - ay = 0, \quad y(x_0) = y_0.$$

1) Rewrite $y' - ay = 0$ as $y' = ay \to y(x) = Ce^{ax}$

Verify: $y' = Cae^{ax} = ay$ ✓

Another approach

$y \neq 0. \to \frac{y'}{y} = a$

Here, $\frac{y'}{y} = (\ln|y|)' = \frac{d}{dx}(\ln|y|)$

$\Rightarrow \frac{d}{dx}(\ln|y|) = a$

$\underset{\Rightarrow}{\text{integrate}} \ln|y| = ax + k$

$\to e^{\ln|y|} = e^{ax+k} \Rightarrow |y| = e^{ax}\underline{e^{k}}$

The exponential function is always positive:

$\Rightarrow y$ is either always positive or always negative.

Set
$$c = \begin{cases} e^{k} & \text{if } y>0 \\ -e^{k} & \text{if } y<0. \end{cases}$$

Therefore
$$y(x) = ce^{ax} = c\, e^{-\int (-a)\,dx}$$
$p(x) = -a$

(b) We know the general solution:
$$y(x) = ce^{ax}.$$

$$y(x_0) = y_0 \Rightarrow ce^{ax_0} = y_0 \Rightarrow c = \frac{y_0}{e^{ax_0}}$$

Therefore,
$$y(x) = \frac{y_0}{e^{ax_0}} e^{ax} = y_0\, e^{a(x-x_0)}$$

Remark: $y(x) = y_0\, e^{-\int_{x_0}^{x} (-a)\,dx}$ $p(x)$.

Example 3.

$$\to \quad y' + \frac{y}{x} = 0 \qquad p(x) = \frac{1}{x}$$

1. Find the general solution of $xy' + y = 0$.
2. Solve the initial value problem

$$xy' + y = 0, \quad y(1) = 3.$$

1) Write: $y' = \frac{dy}{dx}$

$$\Rightarrow \quad x\frac{dy}{dx} + y = 0$$

$$\Rightarrow \quad x\frac{dy}{dx} = -y$$

$$\Rightarrow \quad \frac{dy}{y} = -\frac{1}{x}\,dx$$

$$\Rightarrow \quad \int \frac{dy}{y} = \int -\frac{1}{x}\,dx + k$$

$$\Rightarrow \quad \ln|y| = -\ln|x| + k$$

$$\Rightarrow \quad |y| = e^{-\ln|x| + k}$$

$$\Rightarrow \quad |y| = \cancel{e}^{\cancel{\ln} |x|^{-1}}\; e^{k}$$

$$\Rightarrow \quad |y| = \frac{1}{|x|} \cdot e^{k} \quad \Rightarrow \quad |y| = \frac{e^{k}}{|x|}$$

Let $c = \begin{cases} e^{k}, & \text{if } y > 0 \\ -e^{k}, & \text{if } y < 0 \end{cases}$

$$\Rightarrow \quad y(x) = \frac{c}{x}.$$

2) $y(x) = \dfrac{c}{x}$ is the general solution.

$y(1) = 3$ & $y(1) = c$

$\Rightarrow 3 = c \Rightarrow y(x) = \dfrac{3}{x}$

Remark:

$\dfrac{c}{x} = c e^{-\int \frac{1}{x}\, dx}$ $\leftarrow = p(x)$

General facts:

- The general solution to (2) is given by

$$y = ce^{-P(x)}$$

where $P(x) = \int p(x)\, dx$ is any antiderivative of $p(x)$.

- The solution to the IVP

$$y' + p(x)y = 0, \quad y(x_0) = y_0$$

is given by

$$y(x) = y_0 e^{-\int_{x_0}^{x} p(x)\, dx}.$$

We now want to find the general solution to

$$y' + p(x)y = f(x)$$

where the functions $p(x)$ and $f(x)$ are continuous on an open interval (a, b).

Remark:

- The homogeneous part $y' + p(x)y = 0$ is called the **complementary equation**.

EXAMPLE 4. Find the general solution of

$$y' + 2y = x^3e^{-2x}.$$

1) Solve the complementary eq.

$y' + 2y = 0$ complementary equation.

$\rightarrow y' = -2y \rightarrow y(x) = ce^{-2x}$

or

$y(x) = ce^{-\int 2\,dx} = ce^{-2x}$

2) Make c a function of x! (Variation of parameter)

Let $y(x) = u(x)e^{-2x}$

$\Rightarrow y'(x) = u'e^{-2x} - 2ue^{-2x}$

Replace y' & y in the ODE

$\Rightarrow y' + 2y = x^3e^{-2x} \Rightarrow u'e^{-2x} \cancel{-2ue^{-2x}} \cancel{+2ue^{-2x}} = x^3e^{-2x}$

$\Rightarrow u'\cancel{e^{-2x}} = x^3\cancel{e^{-2x}}$

$\Rightarrow u' = x^3$

integrate $\Rightarrow$ $$u(x) = \frac{x^4}{4} + c$$

So

$$y(x) = u(x)e^{-2x}$$
$$= \boxed{\left(\frac{x^4}{4} + c\right)e^{-2x}}$$

Summary of The Method

- Find a function y_1 such that $y_1' + p(x)y_1 = 0$
- Write $y = uy_1$ where u is an unknown function.
- Solve $u'y_1 = f(x)$.
- Substitute u in y.

EXAMPLE 5.

1. Find the general solution

$$y' + (\cot x)y = x \csc x.$$

2. Solve the initial value problem

$$y' + (\cot x)y = x \csc x, \quad y(\pi/2) = 1.$$

1) Complementary equation

$$y' + (\cot x)\, y = 0$$

Write $y' = \frac{dy}{dx}$ $\Rightarrow$ $\frac{dy}{dx} + (\cot x)\, y = 0$

$$\Rightarrow \frac{dy}{dx} = (-\cot x)\, y$$

$$\Rightarrow \frac{dy}{y} = -\cot x \, dx$$

$$\Rightarrow \int \frac{dy}{y} = \int -\frac{\cos x}{\sin x}\, dx + k$$

$u = \sin x$

$du = \cos x \, dx$

$$\Rightarrow \ln|y| = -\int \frac{1}{u}\, du + k$$

$$\Rightarrow \ln|y| = -\ln|\sin x| + k$$

$$\Rightarrow |y| = \frac{e^k}{|\sin x|}$$

$$\Rightarrow \quad y(x) = \frac{c}{\sin x}$$

<u>Variation of parameter:</u>

$$y(x) = \frac{u}{\sin x} \quad \to \quad y' = \frac{u'}{\sin x} - \frac{u\cos x}{\sin^2 x}$$

Replace y & y' in the ODE:

$$\frac{u'}{\sin x} - \frac{u\cos x}{\sin^2 x} + \frac{(\cot x)\, u}{\sin x} = x \csc x$$

$$\Rightarrow \frac{u'}{\sin x} - \cancel{\frac{u\cos x}{\sin^2 x}} + \cancel{\frac{\cos x\, u}{\sin^2 x}} = x \csc x$$

$$\Rightarrow \frac{u'}{\sin x} = \frac{x}{\sin x} \Rightarrow u' = x \Rightarrow u = \frac{x^2}{2} + c$$

$$\text{so,} \quad y(x) = \frac{u}{\sin x} = \boxed{\frac{\left(\frac{x^2}{2} + c\right)}{\sin x}}$$

General Theorem

Suppose

- $p(x)$ and $f(x)$ are continuous on an interval (a, b)
- y_1 is a solution to the complementary equation.

Then the general solution to $y' + p(x)y = f(x)$ is

$$y(x) = y_1(x)\left(c + \int \frac{f(x)}{y_1(x)}\,dx\right)$$

for each x in (a, b).

Existence Theorem

Suppose

- $p(x)$ and $f(x)$ are continuous on an interval (a, b).
- y_1 is a solution to the complementary equation.
- x_0 is an arbitrary number in (a, b) and y_0 is an arbitrary number.

Then the boundary value problem

$$y' + p(x)y + f(x), \quad y(x_0) = y_0$$

has a unique solution which is of the form

$$y(x) = y_1(x)\left(\frac{y_0}{y_1(x_0)} + \int_{x_0}^{x} \frac{f(t)}{y_1(t)}\,dt\right)$$

for each x in (a, b).