Math 302

Chapter 8

Section 8.1: Laplace Transforms

Contents

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From now on,

- the variable t stands for the independent variable (time).

Discrete Process: Power series

Last Chapter : $\sum_{n=0}^{\infty} a_n x^n = A(x)$.

$a_n \longmapsto A(x)$ (transform sequence into a fct.)

Ex.: $1 \longmapsto \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$.

$\frac{1}{n!} \longmapsto \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$

How we extend this for continuous processes?

Continuous Process

Natural Generalization: $n \longrightarrow t \quad (0 \le t < \infty)$

$\Rightarrow \sum_{n=0}^{\infty} a_n x^n = A(x) \longrightarrow \int_0^{\infty} a(t) \underbrace{x^t}_{\text{ugly...}} \, dt = A(x)$

- $0 < x < 1$
- $s = -\ln x$

$\Rightarrow \int_0^{\infty} f(t) \left(e^{\ln x}\right)^t dt = F(s)$

$= \boxed{\int_0^{\infty} f(t) e^{-st} \, dt}$ LAPLACE TRANSFORM.

Remark:

- Recall that, with power series, we were able to solve a differential equation by solving a recurrence relation (so, basically, doing some algebra with a discrete number of data).
- With the Laplace transform, we will also be able to reduce an ODE problem into an algebra one.
- We use the symbol $L(f(t))$ to also denote the Laplace transform $F(s)$.

Example 1. Compute the Laplace transform of the function $f(t) = t$.

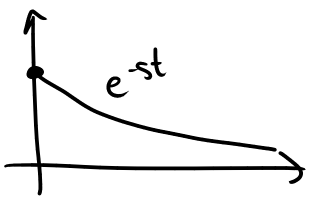
$$F(s) = \int_0^\infty t\, e^{-st}\, dt$$

By parts:

Diff.	Int.	
t	e^{-st}	
1	$e^{-st}/-s$	$+$
0	e^{-st}/s^2	$-$
		$+$

$$\Rightarrow \quad F(s) = -\frac{te^{-st}}{s}\Big|_0^\infty - \frac{e^{-st}}{s^2}\Big|_0^\infty$$

Graph of e^{-st} :



$$\lim_{t\to\infty} e^{-st} = 0$$

$$\lim_{t\to\infty} te^{-st} = 0$$

$$\text{So,} \quad F(s) = -0 + \frac{0\cdot 1}{s} - 0 + \frac{1}{s^2} = \boxed{\frac{1}{s^2}}$$

Here is a sample table of Laplace Transforms.

Function	Transform	Function	Transform
1	$\frac{1}{s}$	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
t	$\frac{1}{s^2}$	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
t^n	$\frac{n!}{s^{n+1}}$	$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$
e^{at}	$\frac{1}{s-a}$	$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$

Table 1: Laplace Transforms (sample)

It is important to check if a function possesses a Laplace transform.

Exponential Order Criterion.

If $f(t)$ is a function satisfying

$$|f(t)| \leq Me^{s_0 t},\ t \geq t_0$$

for some numbers s_0, t_0, and M, then $F(s)$ exists for $s > s_0$.

Remarks:

- Later on, we will see that the Laplace transform exists for discontinuous functions.
- Even more than that, we will apply the Laplace transform on functions taking ∞ as values!

Example 2. The function $f(t) = e^{t^2}$ doesn't have a Laplace transform.

Because, for any s_0

$$\frac{f(t)}{e^{s_0 t}} = e^{t^2 - s_0 t} = e^{t(t-s_0)}$$

Then, if $t > s_0 + 1$

$$\Rightarrow \quad e^{t(t-s_0)} \geq e^{t \cdot 1} = e^t$$

Say $M=2$, for t big enough, say $t \geq t_0 > s_0 + 1$

$$e^t > 2.$$

You can do that for any M!

Desmos illustration:

EXAMPLE 3. Justify that

$$L(\sinh(\omega t)) = \frac{\omega}{s^2 - \omega^2}.$$

Recall: $\sinh(\omega t) = \dfrac{e^{\omega t} - e^{-\omega t}}{2}$.

Therefore,

$$\int_0^\infty \sinh(\omega t)\, e^{-st}\, dt = \int_0^\infty \frac{e^{\omega t} - e^{-\omega t}}{2}\, e^{-st}\, dt$$

$$= \frac{1}{2}\int_0^\infty e^{\omega t} e^{-st} - e^{-\omega t} e^{-st}\, dt$$

$$= \frac{1}{2}\underbrace{\int_0^\infty e^{\omega t} e^{-st}\, dt}_{\mathcal{L}(e^{\omega t})} - \frac{1}{2}\underbrace{\int_0^\infty e^{-\omega t} e^{-st}\, dt}_{\mathcal{L}(e^{-\omega t})}$$

$$\Rightarrow \mathcal{L}(\sinh(\omega t)) = \frac{1}{2}\mathcal{L}(e^{\omega t}) - \frac{1}{2}\mathcal{L}(e^{-\omega t})$$

$$= \frac{1}{2}\frac{1}{s-\omega} - \frac{1}{2}\frac{1}{s+\omega}$$

$$= \frac{\omega}{s^2 - \omega^2}$$

Linearity of Laplace transform:

If f and g are two functions, and a, b are two real numbers, then

$$L(af(t) + bg(t)) = aL(f(t)) + bL(g(t)) = aF(s) + bG(s).$$

You can apply this repeatedly to more than two functions.

Did you notice that

$$L(e^{at}) = \frac{1}{s-a}?$$

- This is $L(1)$, but with a shift $s-a$!!!
- Since $e^{at} = 1 \cdot e^{at}$, we have the following shifting result.

Shifting Theorem:

If $f(t)$ is a function with a Laplace transform $F(s)$, then

$$L(e^{at}f(t)) = F(s-a).$$

Example 4. Find the Laplace transform of

(a) $f(t) = e^{at}\sin(\omega t)$.

(b) $f(t) = e^{at}\cos(\omega t)$.

(a) $\mathcal{L}(\sin \omega t) = \dfrac{\omega}{s^2+\omega^2} \longrightarrow \mathcal{L}(e^{at}\sin \omega t) = \dfrac{\omega}{(s-a)^2+\omega^2}$

(b) $\mathcal{L}(\cos(\omega t)) = \dfrac{s}{s^2+\omega^2} \longrightarrow \mathcal{L}(e^{at}\cos(\omega t)) = \dfrac{s-a}{(s-a)^2+\omega^2}.$

Powers and Derivatives

Did you notice that

$$L(t) = \frac{1}{s^2} = -\frac{d}{ds}\left(\frac{1}{s}\right)?$$

- This is the derivative of $L(1)$, but with a different sign.
- Since $t = 1 \cdot t$, we have the following result.

Powers Transformed in Derivatives.

If f has a Laplace transform and n is a positive integer, then

$$L(t^n f(t)) = (-1)^n F^{(n)}(s).$$

Example 5. Find the Laplace transform of

(a) $f(t) = t\cos(\omega t)$.

(b) $f(t) = t\sinh(\omega t)$.

(c) $f(t) = te^{at}$.

(d) $f(t) = t\sin(2t) + t^2\cos(t)\sin(t)$.

(a) $\mathcal{L}(\sinh(\omega t)) = \dfrac{\omega}{s^2 - \omega^2}$.

$$\Rightarrow \mathcal{L}(t\sinh(\omega t)) = -\frac{d}{ds}\left(\frac{\omega}{s^2-\omega^2}\right) = \frac{2s\omega}{(s^2-\omega^2)^2}$$

(b) $F(s) = \mathcal{L}(t\sin 2t) + \mathcal{L}(t^2\cos t\sin t)$

- $\mathcal{L}(t\sin 2t) = -\dfrac{d}{ds}\left(\dfrac{2}{s^2+4}\right) = \dfrac{4s}{(s^2+4)^2}$
- $\cos t\sin t = \dfrac{\sin 2t}{2} \Rightarrow \mathcal{L}(t^2\sin 2t) = \dfrac{d^2}{ds^2}\left(\dfrac{2}{s^2+4}\right) = \dfrac{4(3s^2-4)}{(s^2+4)^3}$

$$\Rightarrow F(s) = \frac{4s}{(s^2+4)^2} + \frac{2(3s^2-4)}{(s^2+4)^3}.$$

Did you notice that

$$L(\cos(t)) = \frac{s}{s^2+1} = s \cdot \frac{1}{s^2+1} = s\mathcal{L}(\sin t) + \sin(0) \quad . \qquad ?$$

Derivatives Transformed in Powers:

If $f, f', \ldots, f^{(n)}$ have a Laplace transform for $n \geq 1$, then

$$L(f^{(n)}(t)) = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s f^{(n-2)}(0) - f^{(n-1)}(0).$$

Most relevant formulas:

- $n = 1$: $L(f'(t)) = sF(s) - f(0)$.
- $n = 2$: $L(f''(t)) = s^2 F(s) - sf(0) - f'(0)$.
- $n = 3$: $L(f^{(3)}(t)) = s^3 F(s) - s^2 f(0) - sf'(0) - f''(0)$.

EXAMPLE 6. Find the Laplace transform of

(a) $f(t) = \cos^2(t)$.

(b) $g(t) = \sin^2(t)$.

(a) Notice that $f'(t) = -2\cos(t)\sin(t) = -\sin(2t)$

So, $\mathcal{L}(f'(t)) = -\frac{2}{s^2+4}$.

But, $\mathcal{L}(f'(t)) = s\mathcal{L}(f(t)) - f(0)$

$$\Rightarrow \quad -\frac{2}{s^2+4} = sF(s) - 1$$

$$\Rightarrow \quad \frac{1}{s} - \frac{2}{s(s^2+4)} = F(s) \Rightarrow \boxed{F(s) = \frac{s^2+2}{s(s^2+4)}}$$

(b) A little trick: $\cos^2 t + \sin^2 t = 1$

Laplace $\Rightarrow \frac{s^2+2}{s(s^2+4)} + F(s) = \frac{1}{s} \Rightarrow \boxed{F(s) = \frac{1}{s} - \frac{s^2+2}{s(s^2+4)}}$