Math 302

Chapter 8

Section 8.3: Unit Step Function

Contents

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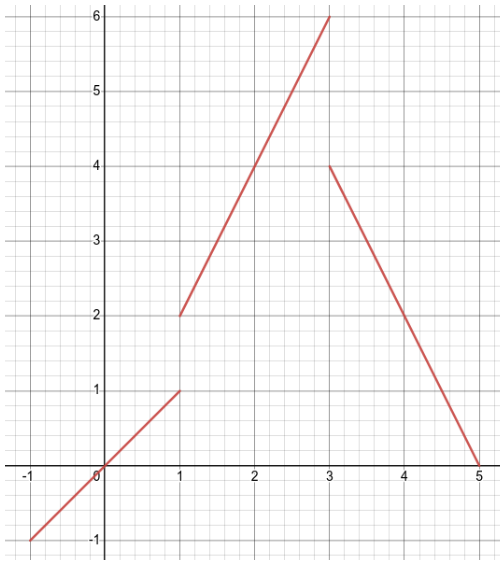
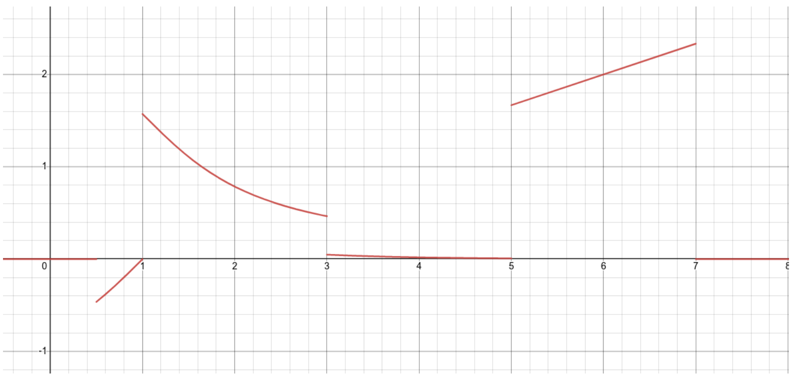
Piecewise Continuous Functions

A piecewise continuous function f is

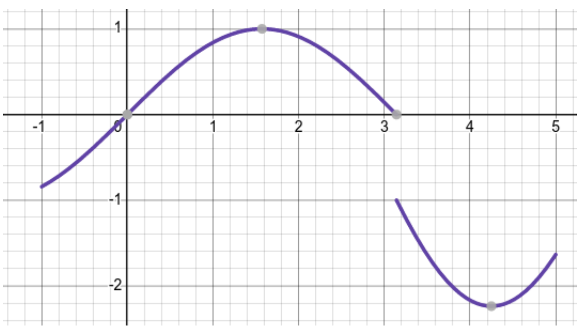
- a function defined on a finite number of intervals $[t_0, t_1]$, $[t_1, t_2]$, ..., $[t_{n-1}, t_n]$;
- such that it is continuous on each interval (t_0, t_1), (t_1, t_2), ..., (t_{n-1}, t_n).



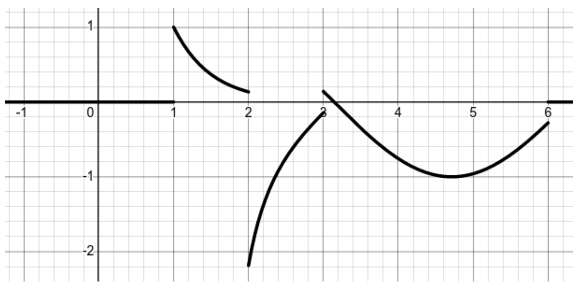
(a) A function $f(x)$



(b) A function $k(x)$



(c) A function $g(x)$



(d) A function $h(x)$

Example 1. Find the Laplace transform of

$$f(t) = \begin{cases} t & 0 < t \le 1 \\ 2t & 1 < t \le 3 \\ 10 - 3t & 3 < t \le 5 \\ 0 & 5 < t. \end{cases}$$

$$F(s) = \int_0^\infty f(t) e^{-st} \, dt$$

$$= \int_0^1 t e^{-st} \, dt + \int_1^3 2t e^{-st} \, dt$$

$$+ \int_3^5 (10-3t) e^{-st} \, dt + \cancel{\int_5^\infty 0 e^{-st} \, dt} \to 0 .$$

$$= \frac{1 - e^{-s}}{s} + \frac{1 - e^{-s}}{s^2} + 2\left(\frac{e^{-s} - 3e^{-3s}}{s} + \frac{e^{-s} - e^{-3s}}{s^2} \right)$$

$$+ \frac{e^{-3s} + 5e^{-5s}}{s} + 3 \frac{e^{-5s} - e^{-3s}}{s^2}$$

To make the work easier with piecewise continuous function, we introduce the **unit step function**:

$$u(t) := \begin{cases} 0 & t < 0 \\ 1 & t \geq 0. \end{cases}$$

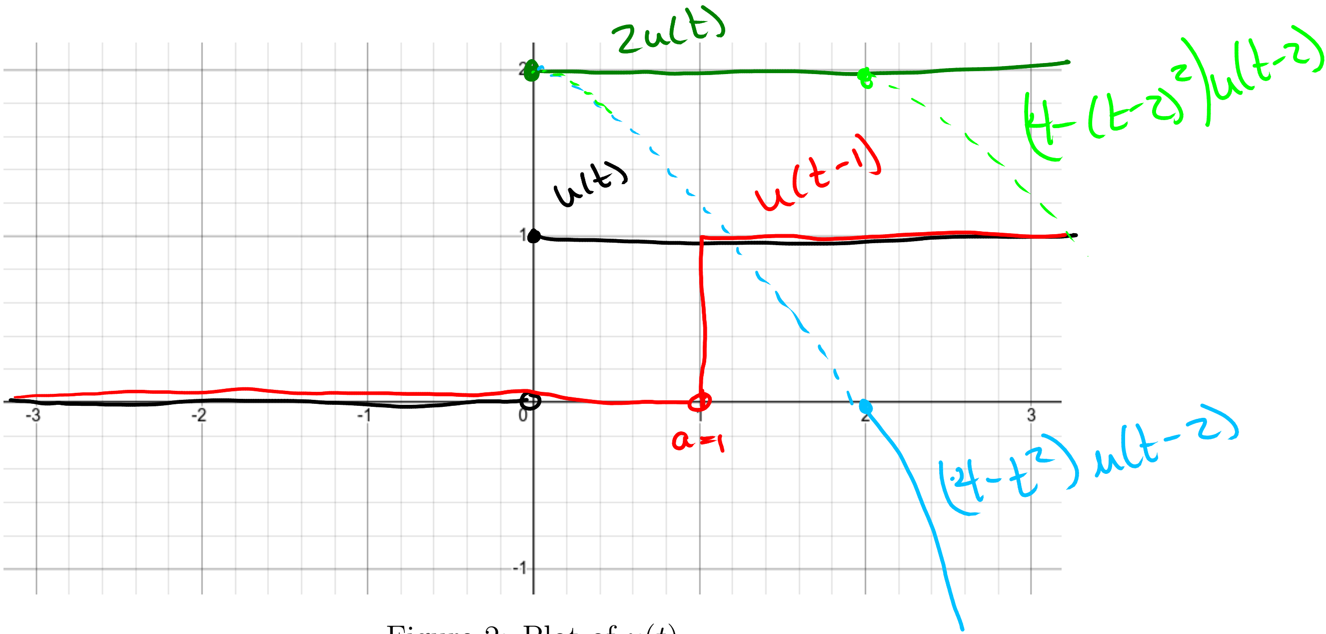


Figure 2: Plot of $u(t)$

Basic Operations

- Translation by a units:

$$u(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a. \end{cases}$$

- Multiplication by c:

$$cu(t) = \begin{cases} 0 & t < 0 \\ c & t \geq 0. \end{cases}$$

- Activation of a function $f(t)$ at time a:

$$f(t)u(t-a) = \begin{cases} 0 & t < a \\ f(t) & t \geq a. \end{cases}$$

- Destruction of a function $f(t)$ at time b and activation of a function $g(t)$ at time b:

$$f(t)u(t-a) + (g(t) - f(t))u(t-b) = \begin{cases} 0 & t < a \\ f(t) & a \leq t < b \\ g(t) & b \leq t. \end{cases}$$

EXAMPLE 2. Rewrite the function $f(t)$ in Example 1 using the unit step function.

$$f(t) = \begin{cases} t, & 0 \leq t \leq 1 \\ 2t, & 1 < t \leq 3 \\ 10-3t, & 3 < t \leq 5 \\ 0, & t > 5 . \end{cases}$$

$$t\,u(t) - t\,u(t-1)$$

$$2t\,u(t-1) - 2t\,u(t-3)$$

$$(10-3t)u(t-3) - (10-3t)\,u(t-5)$$

$$\begin{aligned} f(t) &= t\,u(t) - t\,u(t-1) + 2t\,u(t-1) \\ &\quad - 2t\,u(t-3) + (10-3t)\,u(t-3) \\ &\quad - (10-3t)\,u(t-5) \\ &= t\,u(t) + t\,u(t-1) + (10-5t)\,u(t-3) \\ &\quad - (10-3t)\,u(t-5) . \end{aligned}$$

Let $a \geq 0$ be a real number and f be a function with a Laplace transform $F(s)$.

- $L(u(t-a)) = \dfrac{e^{-sa}}{s}$. $\to$ $a=0$ $L(u(t)) = \dfrac{1}{s}$
- $L(u(t-a)f(t)) = e^{-sa}L(f(t+a))$.
- $L(\underline{u(t-a)f(t-a)}) = e^{-sa}\underline{F(s)}$.

Example 3. Find the Laplace transform of

$$f(t) = \begin{cases} \sin(t) & , 0 \leq t < \pi/2 \\ \cos(t) - 3\sin(t) & , \pi/2 \leq t < \pi \\ 3\cos(t) & , t \geq \pi. \end{cases}$$

$$f(t) = u(t)\sin(t) - u(t-\pi/2)\sin(t)$$
$$+ u(t-\pi/2)[\cos t - 3\sin t]$$
$$- u(t-\pi)[\cos t - 3\sin t]$$
$$+ u(t-\pi)3\cos t$$

$$\Rightarrow f(t) = \overset{①}{u(t)\sin t} + \overset{②}{u(t-\pi/2)[\cos t - 4\sin t]}$$
$$+ u(t-\pi)[4\cos t - 3\sin t].③$$

$$① \to L(u(t)\sin(t)) = e^{0}L(\underbrace{f(t)}_{\sin t}) = \frac{1}{s^2+1}$$

$$② \to L(u(t-\pi/2)f(t)) = e^{-s\pi/2}L(f(t+\tfrac{\pi}{2}))$$

$$f(t+\pi/2) = \cos(t+\pi/2) - 4\sin(t+\pi/2)$$

$\cos(t+\pi/2) = -\sin(t)$
$\sin(t+\pi/2) = \cos(t)$

$$\Rightarrow \quad L(\cos(t+\pi/2)) = -L(\sin t) = \frac{-1}{s^2+1}$$

$$L(\sin(t+\pi/2)) = L(\cos t) = \frac{s}{s^2+1}$$

$$\Rightarrow (2) = -\frac{e^{-s\pi/2}}{s^2+1} - 4\frac{se^{-s\pi/2}}{s^2+1}.$$

$$\underline{(3)} \quad L(u(t-\pi)f(t)) = e^{-\pi s} L(f(t+\pi))$$

$$f(t+\pi) = 4\cos(t+\pi) - 3\sin(t+\pi)$$

$\cos(t+\pi) = -\cos t$
$\sin(t+\pi) = -\sin t$

$$\Rightarrow \quad L(\cos(t+\pi)) = -L(\cos t) = -\frac{s}{s^2+1}$$

&

$$L(\sin(t+\pi)) = -L(\sin t) = -\frac{1}{s^2+1}$$

$$\Rightarrow (3) = -\frac{4se^{-\pi s}}{s^2+1} + 3\frac{e^{-\pi s}}{s^2+1}$$

So, $F(s) = \dfrac{1}{s^2+1} - \dfrac{e^{-s\pi/2}}{s^2+1} - \dfrac{4se^{-s\pi/2}}{s^2+1} - \dfrac{4se^{-\pi s}}{s^2+1} + 3\dfrac{e^{-\pi s}}{s^2+1}$

Example 4. Find

$$L^{-1}\Big(\overbrace{\frac{1}{s^2} - e^{-s}\Big(\frac{1}{s^2} + \frac{2}{s}\Big) + e^{-4s}\Big(\frac{4}{s^3} + \frac{1}{s}\Big)}^{H(s)}\Big)$$

$$L^{-1}(H) = L^{-1}\Big(\frac{1}{s^2}\Big) - L^{-1}\big(e^{-s}/s^2\big) - 2L^{-1}\big(e^{-s}/s\big)$$
$$+4L^{-1}\Big(\frac{e^{-4s}}{s^3}\Big) + L\Big(\frac{e^{-4s}}{s}\Big)$$

$$\frac{1}{s^2} \longrightarrow t$$

$$e^{-s}\cdot\frac{1}{s^2} \xrightarrow[f(t)=t]{a=1} u(t-1)f(t-1) = u(t-1)\cdot(t-1)$$

$$e^{-s}\cdot\frac{1}{s} \xrightarrow[f(t)=1]{a=1} u(t-1)f(t-1) = u(t-1)$$

$$e^{-4s}\cdot\underbrace{\frac{1}{s^3}}_{F(s)} \xrightarrow[f(t)=\frac{t^2}{2}]{a=4} u(t-4)f(t-4) = u(t-4)\frac{(t-4)^2}{2}$$

$$e^{-4s}\cdot\frac{1}{s} \xrightarrow[f(t)=1]{a=4} u(t-4)$$

$$\Rightarrow f(t) = t - u(t-1)(t-1) - 2u(t-1) + u(t-4)(t-4)^2$$
$$+ u(t-4)$$