MATH 302

CHAPTER 8

SECTION 8.1: LAPLACE TRANSFORMS

CONTENTS

CREATED BY: PIERRE-OLIVIER PARISÉ
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From now on,

- the variable t stands for the independent variable (time).

Discrete Process: Power series

Continuous Process

Remark:

- Recall that, with power series, we were able to solve a differential equation by solving a recurrence relation (so, basically, doing some algebra with a discrete number of data).
- With the Laplace transform, we will also be able to reduce an ODE problem into an algebra one.
- We use the symbol $L(f(t))$ to also denote the Laplace transform $F(s)$.

Example 1. Compute the Laplace transform of the function $f(t) = t$.

Here is a sample table of Laplace Transforms.

Function	**Transform**	**Function**	**Transform**
1	$\frac{1}{s}$	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
t	$\frac{1}{s^2}$	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
t^n	$\frac{n!}{s^{n+1}}$	$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$
e^{at}	$\frac{1}{s-a}$	$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$

Table 1: Laplace Transforms (sample)

Some Comments on Existence

It is important to check if a function possesses a Laplace transform.

Exponential Order Criterion.

If $f(t)$ is a function satisfying

$$|f(t)| \leq Me^{s_0 t},\, t \geq t_0$$

for some numbers s_0, t_0, and M, then $F(s)$ exists for $s > s_0$.

Remarks:

- Later on, we will see that the Laplace transform exists for discontinuous functions.
- Even more than that, we will apply the Laplace transform on functions taking ∞ as values!

Example 2. The function $f(t) = e^{t^2}$ doesn't have a Laplace transform.

Example 3. Justify that

$$L(\sinh(\omega t)) = \frac{\omega}{s^2 - \omega^2}.$$

Linearity of Laplace transform:

If f and g are two functions, and a, b are two real numbers, then

$$L(af(t) + bg(t)) = aL(f(t)) + bL(g(t)) = aF(s) + bG(s).$$

You can apply this repeatedly to more than two functions.

First Shift Theorem

Did you notice that

$$L(e^{at}) = \frac{1}{s-a}?$$

- This is $L(1)$, but with a shift $s-a$!!!
- Since $e^{at} = 1 \cdot e^{at}$, we have the following shifting result.

Shifting Theorem:

If $f(t)$ is a function with a Laplace transform $F(s)$, then

$$L(e^{at} f(t)) = F(s-a).$$

Example 4. Find the Laplace transform of

(a) $f(t) = e^{at}\sin(\omega t)$.

(b) $f(t) = e^{at}\cos(\omega t)$.

Powers and Derivatives

Did you notice that

$$L(t) = \frac{1}{s^2} = -\frac{d}{ds}\left(\frac{1}{s}\right)?$$

- This is the derivative of $L(1)$, but with a different sign.
- Since $t = 1 \cdot t$, we have the following result.

Powers Transformed in Derivatives.

If f has a Laplace transform and n is a positive integer, then

$$L(t^n f(t)) = (-1)^n F^{(n)}(s).$$

Example 5. Find the Laplace transform of

(a) $f(t) = t\cos(\omega t)$.

(b) $f(t) = t\sinh(\omega t)$.

(c) $f(t) = te^{at}$.

(d) $f(t) = t\sin(2t) + t^2\cos(t)\sin(t)$.

Did you notice that

$$L(\cos(t)) = \frac{s}{s^2+1} = \qquad\qquad\qquad\qquad ?$$

Derivatives Transformed in Powers:

If $f, f', \ldots, f^{(n)}$ have a Laplace transform for $n \geq 1$, then

$$L(f^{(n)}(t)) = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s f^{(n-2)}(0) - f^{(n-1)}(0).$$

Most relevant formulas:

- $n = 1$: $L(f'(t)) = sF(s) - f(0)$.
- $n = 2$: $L(f''(t)) = s^2 F(s) - s f(0) - f'(0)$.
- $n = 3$: $L(f^{(3)}(t)) = s^3 F(s) - s^2 f(0) - s f'(0) - f''(0)$.

Example 6. Find the Laplace transform of

(a) $f(t) = \cos^2(t)$.

(b) $g(t) = \sin^2(t)$.