Math 302

Chapter 8

Section 8.4: Convolution

Contents

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The Story of The Matches

- Suppose we have a number of matches we need to light.
- At each second, so at $t = 0$, $t = 1$, $t = 2$, $t = 3$, ..., $t = n$, we light a certain number of matches. Denote by $f(t)$ the number of matches lit at time t.
- Each matches give off smoke. Denote by $g(t)$ the smoke produced by a match after t seconds.

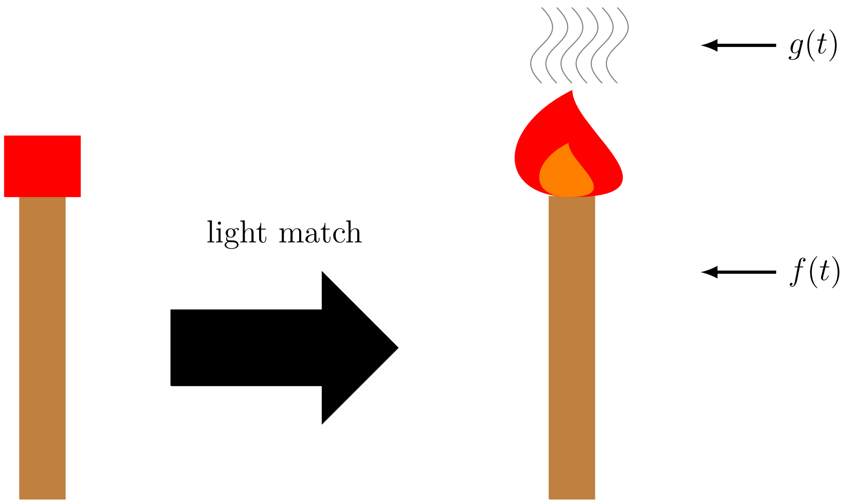

Figure 1: The Matches Problem

Question: What is the total quantity of smoke in the air after a certain time t?

Times (t)	$Q(t)$

The total contribution of the matches after n seconds:

$$Q(t) =$$

What if we have a continuous phenomena?

Definition

The convolution of a function $f(t)$ with another function $g(t)$ is the new function $(f*g)(t)$ defined by

$$(f*g)(t) = \int_0^t f(x)g(t-x)\,dx.$$

Example 1. Let

$$f(t) = u(t) - u(t-1) \quad \text{and} \quad g(t) = u(t) - u(t-1).$$

Compute $f*g$.

Desmos: https://www.desmos.com/calculator/h50sct4xeq

Laplace Transform

The nice properties of the convolution is a direct connection with the Laplace transform.

Example 2. Let $f(t) = e^t$ and $g(t) = e^{-t}$.

(a) Compute $f * g$.

(b) Find $L(f * g)$.

(c) Compare with $L(f)L(g)$.

Tranform of Convolution: If

- $f(t)$ is a function with Laplace transform $F(s)$;
- $g(t)$ is a function with Laplace transform $G(s)$;

then

$$L(f * g) = L(f)L(g) = F(s)G(s).$$

Example 3. Find the inverse Laplace transform of the following function:

$$\frac{1}{s^2(s^2+4)}.$$

Laplace Transforms of Integrals

As a special case of the Laplace transform of a convolution, we can take the Laplace transform of an integral.

Example 4. Suppose f has a Laplace transform given by $F(s)$. Find the Laplace transform of

$$h(t) = \int_0^t f(x)\, dx.$$

Other related results:

- For $g(t) = \int_0^t \int_0^x f(u)\, du\, dx$, we have $G(s) = F(s)/s^2$.
- For a function $g(t)$ given as three integrals, then $G(s) = F(s)/s^3$.
- For a function $g(t)$ given as n integrals, then $G(s) = F(s)/s^n$.

We can solve more than just an ODE!

EXAMPLE 5. Find the solution to the following integro-differential equation

$$\int_0^t y(u)\,du + y'(t) = t,$$

where $y(0) = 0$.

Example 6. Find the general solution to the following integral equation

$$y(t) = \sin(t) - 2\int_0^t y(u)\cos(t-u)\,du.$$