MATH 302

CHAPTER 5

SECTION 5.6: REDUCTION OF ORDER

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CREATED BY: PIERRE-OLIVIER PARISÉ
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What Is Reduction Of Order

We study the ODE

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x).$$

where $P_0(x)$, $P_1(x)$, $P_2(x)$, $F(x)$ are continuous functions in the variable x.

Goal: Find the general solutions to the ODE above.

Trick:

- Have a solution to the complementary equation.
- Use variation of parameter.

Example 1. Find the general solution of

$$xy'' - (2x+1)y' + (x+1)y = x^2$$

given that $y_1(x) = e^x$ is a solution to the complementary equation.

1) Var. of Param.

$$y(x) = u(x)\, y_1(x) = ue^x$$

$$\Rightarrow \quad y' = u'e^x + ue^x$$

$$y'' = u''e^x + 2u'e^x + ue^x$$

Replace in ODE:

$$x(u''e^x + 2u'e^x + ue^x) - (2x+1)(u'e^x + ue^x) + (x+1)ue^x = x^2$$

$$\Rightarrow \quad xu''e^x + \cancel{xue^x} - \cancel{2xue^x} - u'e^x - \cancel{ue^x} + \cancel{xue^x} + \cancel{ue^x} = x^2$$

$$\Rightarrow \quad xu''e^x - u'e^x = x^2$$

② Change of Variable

$z = u' \Rightarrow z' = u''$

So, $x z' e^{x} - z e^{x} = x^{2}$

① Compl. Eq.

$x z' e^{x} - z e^{x} = 0$

$\Rightarrow \frac{z'}{z} = \frac{1}{x} \Rightarrow \ln|z| = \ln|x| + k$

$\Rightarrow z(x) = Kx$

$(K = \pm e^{k})$

② Var. of Param.

$z(x) = v(x) \cdot x \Rightarrow z' = v'x + v$

Replace in the ODE $\Rightarrow v' = e^{-x}$

$\Rightarrow v(x) = -e^{-x} + c_1$

So, $z(x) = (-e^{-x} + c_1)x = c_1 x - x e^{-x}$

④ Integrate.

$z = u' \Rightarrow u(x) = \int c_1 x - x e^{-x}\, dx + c_2$

$= \frac{c_1 x^2}{2} + (x-1)e^{-x} + c_2$

⑤ Answer:

$y(x) = u(x) \cdot e^{x} = \boxed{\frac{c_1 x^2 e^{x}}{2} + c_2 e^{x} + (x-1)}.$

Example 2. Find the general solution of

$$x^2y'' + xy' - y = x^2 + 1$$

given that $y_1(x) = x$ is a solution to the complementary equation.

(1) $y(x) = u(x) \cdot x \quad \Rightarrow \quad y' = u'x + u$
$y'' = u''x + 2u'$

Replace in the ODE:

$$\Rightarrow \quad x^2(u''x + 2u') + x(u'x + u) - ux = x^2 + 1$$

$$\Rightarrow \quad x^3u'' + 2x^2u' + x^2u' + \cancel{xu} - \cancel{ux} = x^2 + 1$$

$$\Rightarrow \quad x^3u'' + 3x^2u' = x^2 + 1$$

(2) $z = u' \quad \Rightarrow \quad z' = u''$

$$\Rightarrow x^3z' + 3x^2z = x^2 + 1$$

$$\Rightarrow \frac{d}{dx}(x^3z) = x^2 + 1$$

$$\Rightarrow \int \frac{d}{dx}(x^3z)\, dx = \int x^2 + 1\, dx + c_1$$

$$\Rightarrow \quad x^3z = \frac{x^3}{3} + x + c_1$$

$$\Rightarrow \quad z = \frac{1}{3} + \frac{1}{x^2} + \frac{c_1}{x^3}$$

(3) Integrate

$$z = u' \quad \Rightarrow \quad u(x) = \frac{x}{3} - \frac{1}{x} - \frac{c_1}{2x^2} + c_2$$

4 $y(x) = ux = \boxed{\frac{x^2}{3} - 1 - \frac{c_1}{2x} + c_2 x}$