Math 302

Chapter 8

Section 8.1: Laplace Transforms

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From now on,

- the variable t stands for the independent variable (time).

Discrete Process: Power series

Last Chapter: $\sum_{n=0}^{\infty} a_n x^n = A(x)$.

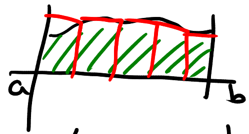
$a_n \longrightarrow A(x)$ (transformed a_n into a function).

Ex.: $1 \longrightarrow \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

$\frac{1}{n!} \longrightarrow \sum_{n=0}^{\infty} \frac{1}{n!} x^n = e^x$

How can we generalize for a continuous spectrum of values.

Continuous Process



Natural: $n \longrightarrow t$ $(0 \le t < \infty)$

$$\sum_{n=0}^{\infty} a_n x^n = A(x) \longrightarrow \int_0^{\infty} a(t)\, x^t \, dt$$

- $0 < x < 1$ $\quad 0 \le s < \infty$
- $s = -\ln x$

↳ $-s = \ln x$

↳ $e^{-s} = x$

$$\Rightarrow \int_0^{\infty} a(t)\,(e^{-s})^t \, dt = \boxed{\int_0^{\infty} a(t)\, e^{-st} \, dt}.$$

$A(s) \longleftarrow$

Remark:

- Recall that, with power series, we were able to solve a differential equation by solving a recurrence relation (so, basically, doing some algebra with a discrete number of data).
- With the Laplace transform, we will also be able to reduce an ODE problem into an algebra one.
- We use the symbol $L(f(t))$ to also denote the Laplace transform $F(s)$.

Example 1. Compute the Laplace transform of the function $f(t) = t$.

$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$
$$= \int_0^\infty t e^{-st}\, dt$$

diff	Int	
t	e^{-st}	
1	$-e^{-st}/s$	$+$
0	e^{-st}/s^2	$-$

$$= -\frac{te^{-st}}{s}\Big|_0^\infty - \frac{e^{-st}}{s^2}\Big|_0^\infty$$
$$= -\frac{0}{s} + 0 - \frac{0}{s^2} + \frac{1}{s^2}$$
$$= \frac{1}{s^2}$$

So, $f(t) = t \xrightarrow{L} F(s) = \frac{1}{s^2}$.

Here is a sample table of Laplace Transforms.

Function	Transform	Function	Transform
1	$\frac{1}{s}$	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
t	$\frac{1}{s^2}$	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
t^n	$\frac{n!}{s^{n+1}}$	$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$
e^{at}	$\frac{1}{s-a}$	$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$

Table 1: Laplace Transforms (sample)

Some Comments on Existence

It is important to check if a function possesses a Laplace transform.

Exponential Order Criterion.

If $f(t)$ is a function satisfying

$$|f(t)| \leq Me^{s_0 t},\ t \geq t_0$$

for some numbers s_0, t_0, and M, then $F(s)$ exists for $s > s_0$.

Remarks:

- Later on, we will see that the Laplace transform exists for discontinuous functions.
- Even more than that, we will apply the Laplace transform on functions taking ∞ as values!

EXAMPLE 2. The function $f(t) = e^{t^2}$ doesn't have a Laplace transform.

$$F(s) = \int_0^\infty e^{t^2} e^{-st}\, dt = \int_0^\infty e^{t^2 - st}\, dt = \int_0^\infty e^{t(t-s)}\, dt$$

If t is big enough $t > s+1 \Rightarrow t-s > 1$

So, $e^{t(t-s)} \geq e^{t \cdot 1} = e^t$

So, $\int_0^\infty e^{t(t-s)}\, dt \geq \int_0^\infty e^t\, dt = \infty$

this is true for any s!

$F(s) = \infty$ for any s!

Example 3. Justify that

$$L(\sinh(\omega t)) = \frac{\omega}{s^2 - \omega^2}.$$

$$\sinh(\omega t) = \frac{e^{\omega t} - e^{-\omega t}}{2}$$

$$\Rightarrow L(\sinh(\omega t)) = \int_0^\infty \sinh(\omega t)\, e^{-st}\, dt$$

$$= \int_0^\infty \left(\frac{e^{\omega t} - e^{-\omega t}}{2}\right) e^{-st}\, dt$$

$$= \int_0^\infty \frac{e^{\omega t} e^{-st}}{2} - \frac{e^{-\omega t} e^{-st}}{2}\, dt$$

$$= \frac{1}{2}\int_0^\infty e^{\omega t} e^{-st}\, dt - \frac{1}{2}\int_0^\infty e^{-\omega t} e^{-st}\, dt$$

$$= \frac{1}{2} L(e^{\omega t}) - \frac{1}{2} L(e^{-\omega t})$$

$$= \frac{1}{2}\left(\frac{1}{s-\omega}\right) - \frac{1}{2}\left(\frac{1}{s+\omega}\right)$$

$$= \frac{s+\omega - (s-\omega)}{2(s^2-\omega^2)} = \boxed{\frac{\omega}{s^2-\omega^2}}$$

Linearity of Laplace transform:

If f and g are two functions, and a, b are two real numbers, then

$$L(af(t) + bg(t)) = aL(f(t)) + bL(g(t)) = aF(s) + bG(s).$$

You can apply this repeatedly to more than two functions. $L(f+g+h) = L(f) + L(g) + L(h)$

Did you notice that

$$L(e^{at}) = \frac{1}{s-a}?$$

- This is $L(1)$, but with a shift $s-a$!!!
- Since $e^{at} = 1 \cdot e^{at}$, we have the following shifting result.

Shifting Theorem:

If $f(t)$ is a function with a Laplace transform $F(s)$, then

$$L(e^{at} f(t)) = F(s-a).$$

EXAMPLE 4. Find the Laplace transform of

(a) $f(t) = e^{at}\sin(\omega t)$.

(b) $f(t) = e^{at}\cos(\omega t)$.

(a) $L(e^{at} \sin \omega t)$??

$$L(\sin \omega t) = \frac{\omega}{s^2+\omega^2}$$

$$\Rightarrow L(e^{at} \sin \omega t) = \boxed{\frac{\omega}{(s-a)^2+\omega^2}}$$

(b) $$L(\cos(\omega t)) = \frac{s}{s^2+\omega^2}$$

$$\Rightarrow L(e^{at} \cos \omega t) = \boxed{\frac{s-a}{(s-a)^2+\omega^2}}$$

Did you notice that

$$L(t) = \frac{1}{s^2} = -\frac{d}{ds}\left(\frac{1}{s}\right)?$$

- This is the derivative of $L(1)$, but with a different sign.
- Since $t = 1 \cdot t$, we have the following result.

Powers Transformed in Derivatives.

If f has a Laplace transform and n is a positive integer, then

$$L(t^n f(t)) = (-1)^n F^{(n)}(s).$$

Example 5. Find the Laplace transform of

(a) $f(t) = t\cos(\omega t)$.

$\rightarrow$ (b) $f(t) = t\sinh(\omega t)$.

$\rightarrow$ (c) $f(t) = te^{at}$.

$\rightarrow$ (d) $f(t) = t\sin(2t) + t^2\cos(t)\sin(t)$.

(a) mult. by $t \rightarrow -\frac{d}{ds} L(\cos(\omega t))$ $\quad L(t\cos\omega t)$

$$L(\cos\omega t) = \frac{s}{s^2+\omega^2} \Rightarrow -\frac{d}{ds}\left(\frac{s}{s^2+\omega^2}\right)$$

$$= -\frac{s^2+\omega^2-2s^2}{(s^2+\omega^2)^2} = \boxed{\frac{s^2-\omega^2}{(s^2+\omega^2)^2}}$$

(d) $L(f(t)) = L(t\sin 2t) + L(t^2 \cos t \sin t)$

trig. identity.

$$= -\frac{d}{ds}\left(\frac{2}{s^2+4}\right) + L\left(t^2\,\frac{\sin 2t}{2}\right)$$

$$= \frac{-2s}{(s^2+4)^2} + \frac{1}{2}L(t^2 \sin 2t)$$

$$= \frac{-2s}{(s^2+4)^2} + \frac{1}{2}\frac{d^2}{ds^2}\left(\frac{2}{s^2+4}\right)$$

$$= \boxed{-\frac{2s}{(s^2+4)^2} + \frac{4(3s^2-4)}{(s^2+4)^3}}$$

Did you notice that

$$\frac{d}{dt}(\sin t) \leftarrow L(\cos(t)) = \frac{s}{s^2+1} = s \cdot \frac{1}{s^2+1} = s L(\sin t) - \sin(0) \quad ?$$

Derivatives Transformed in Powers:

If $f, f', \ldots, f^{(n)}$ have a Laplace transform for $n \geq 1$, then

$$L(f^{(n)}(t)) = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0).$$

Most relevant formulas:

- $n = 1$: $L(f'(t)) = sF(s) - f(0)$.
- $n = 2$: $L(f''(t)) = s^2 F(s) - s f(0) - f'(0)$.
- $n = 3$: $L(f^{(3)}(t)) = s^3 F(s) - s^2 f(0) - s f'(0) - f''(0)$.

EXAMPLE 6. Find the Laplace transform of

(a) $f(t) = \cos^2(t)$.

(b) $g(t) = \sin^2(t)$.

trig ident.

(a) $f'(t) = -2\cos(t)\sin(t) = -\sin(2t)$

$L(f'(t)) = s F(s) - f(0)$ $= \cos^2(0) = 1$

$\Rightarrow L(-\sin(2t)) = sF(s) - 1$

$\Rightarrow -\frac{2}{s^2+4} = sF(s) - 1$

$L(\cos^2 t)$.

$\Rightarrow -\frac{2}{s^2+4} + 1 = sF(s) \Rightarrow \boxed{F(s) = \frac{s^2+2}{s(s^2+4)}}$

(b) $\cos^2 t + \sin^2 t = 1$

$\Rightarrow L(\cos^2 t) + L(\sin^2 t) = L(1)$

$\Rightarrow \frac{s^2+2}{s(s^2+4)} + L(\sin^2 t) = \frac{1}{s}$

$\Rightarrow L(\sin^2 t) = \boxed{\frac{1}{s} - \frac{s^2+2}{s(s^2+4)}}$