Problem A

Find the inverse Laplace transform of the following transforms.

1) $\dfrac{3}{s^2+4}$.
2) $\dfrac{4}{(s-1)^3}$.
3) $\dfrac{2}{s^2+3s+5}$.
4) $\dfrac{3s}{s^2-s-6}$.
5) $\dfrac{2s+2}{s^2+2s+5}$.
6) $\dfrac{2s-3}{s^2-4}$.
7) $\dfrac{2s+1}{s^2-2s+2}$.
8) $\dfrac{8s^2-4s+12}{s(s^2+4)}$.
9) $\dfrac{2s^2+4s+6}{(s+1)^2(s-1)}$.
10) $\dfrac{s+1}{s(s-1)^2}$.

Problem B

Find the solutions to the following initial value problems.

1) $y''-y'-6y=0$, with $y(0)=1$, $y'(0)=-1$.
2) $y''+3y'+2y=0$, with $y(0)=1$, $y'(0)=0$.
3) $y''+2y'+5y=0$, with $y(0)=2$, $y'(0)=1$.
4) $y''+\omega^2 y=\cos 2t$, with $\omega^2\neq 4$, $y(0)=1$, $y'(0)=0$.
5) $y''+2y'+y=4e^{-t}$, with $y(0)=2$, $y'(0)=-1$.

Complete Solutions

Problem A

1) We can rewrite the expression of the function as followed:

$$\frac{3}{s^2+4} = \frac{3}{2}\left(\frac{2}{s^2+4}\right).$$

Now, we notice that $L(\sin(2t)) = \frac{2}{s^2+4}$. Therefore, the inverse transform is

$$f(t) = L^{-1}\left(\frac{3}{2}\left(\frac{2}{s^2+4}\right)\right) = \frac{3}{2}L^{-1}\left(\frac{2}{s^2+4}\right) = \frac{3}{2}\sin(2t).$$

2) We have to notice that there is a division by a power of $s-1$ in the denominator. We also notice there is a translation of 1 in the expression $s-1$.

Therefore, looking into the table, we see that

$$L^{-1}\left(\frac{2}{s^3}\right) = t^2$$

and since there is a translation, from a result in the lecture notes, we must have that

$$L^{-1}\left(\frac{2}{(s-1)^3}\right) = e^t t^2.$$

Therefore, we obtain

$$f(t) = L^{-1}\left(2\frac{2}{(s-1)^3}\right) = 2L^{-1}\left(\frac{2}{(s-1)^3}\right) = 2t^2e^t.$$

3) We rewrite the expression as

$$\frac{2}{s^2+3s+5} = \frac{2}{s^2+3s+\frac{9}{4}+\frac{11}{4}} = \frac{2}{(s+\frac{3}{2})^2+\frac{11}{4}} = \frac{4}{\sqrt{11}}\left(\frac{\frac{\sqrt{11}}{2}}{(s+\frac{3}{2})^2+\frac{11}{4}}\right).$$

Knowing that $L(\sin(at)) = \frac{a}{s^2+a^2}$ and that $L(e^{at}f(t)) = F(s-a)$, we infer that

$$f(t) = L^{-1}\left(\frac{2}{s^2+3s+5}\right) = \frac{4}{\sqrt{11}}L^{-1}\left(\frac{\frac{\sqrt{11}}{2}}{(s+\frac{3}{2})^2+\frac{11}{4}}\right) = \frac{4}{\sqrt{11}}e^{-\frac{3t}{2}}\sin\left(\frac{\sqrt{11}}{2}t\right).$$

4) We rewrite the expression as followed:

$$\frac{3s}{s^2-s-6} = \frac{3s}{(s-\frac{1}{2})^2-\frac{25}{4}}$$

Now, the numerator can be changed to :

$$3s = 3\left(s-\frac{1}{2}+\frac{1}{2}\right) = 3\left(s-\frac{1}{2}\right)+\frac{3}{2}.$$

Therefore, using the linearity of the inverse Laplace transform, we have

$$\begin{aligned} f(t) = L^{-1}\left(\frac{3s}{s^2-s-6}\right) &= L^{-1}\left[3\left(\frac{s-\frac{1}{2}}{(s-\frac{1}{2})^2-\frac{25}{4}}\right)+\frac{3}{2}\left(\frac{1}{(s-\frac{1}{2})^2-\frac{25}{4}}\right)\right] \\ &= 3L^{-1}\left(\frac{s-\frac{1}{2}}{(s-\frac{1}{2})^2-\frac{25}{4}}\right)+\frac{3}{2}L^{-1}\left(\frac{1}{(s-\frac{1}{2})^2-\frac{25}{4}}\right). \end{aligned}$$

We know, however, that $L(e^{at}f(t)) = F(s-a)$ where $F = L(f)$ and also that

$$L(\cosh(at)) = \frac{s}{s^2-a^2} \quad \text{et} \quad L(\sinh(at)) = \frac{a}{s^2-a^2}.$$

Therefore, we can conclude that

$$f(t) = 3e^{\frac{t}{2}}\cosh(\tfrac{5}{2}t)+\tfrac{3}{5}e^{\frac{t}{2}}\sinh(\tfrac{5}{2}t).$$

Another approach is to notice that

$$\frac{s}{s^2-s-6} = \frac{s}{(s-2)(s+3)} = \frac{2}{5(s+2)}+\frac{3}{5(s-3)}.$$

Therefore, we obtain

$$f(t) = \frac{6}{5}e^{-2t}+\frac{9}{5}e^{3t}.$$

We can check that the two solutions are equivalent:

$$\begin{aligned} 3e^{t/2}\cosh(5t/2)+\tfrac{3}{5}e^{t/2}\sinh(5t/2) &= \tfrac{3}{2}(e^{3t}+e^{-2t})+\frac{3}{10}(e^{3t}-e^{-2t}) \\ &= \frac{15+3}{10}e^{3t}+\frac{15-3}{10}e^{-2t} \\ &= \frac{9}{5}e^{3t}+\frac{6}{5}e^{-2t}. \end{aligned}$$

5) We can rewrite the expression in the following way:

$$\frac{2s+2}{s^2+2s+5} = 2\frac{s+1}{(s+1)^2+4}.$$

Therefore, we see that

$$f(t) = L^{-1}\left(\frac{2s+2}{s^2+2s+5}\right) = 2L^{-1}\left(\frac{s+1}{(s+1)^2+4}\right).$$

We have $L^{-1}(e^{at}f(t)) = F(s-a)$ where $F = L(f)$ and $L(\cos(at)) = \frac{s}{s^2+a^2}$. We can then see that

$$f(t) = 2e^{-t}\cos(2t).$$

6) We rewrite the expression as

$$\frac{2s-3}{s^2-4} = 2\left(\frac{s}{s^2-4}\right) - \frac{3}{2}\left(\frac{2}{s^2-4}\right).$$

Therefore, fro the linearity of the inverse Laplace transform, we obtain

$$\begin{aligned} f(t) = L^{-1}\left(\frac{2s-3}{s^2-4}\right) &= 2L^{-1}\left(\frac{s}{s^2-4}\right) - \frac{3}{2}L^{-1}\left(\frac{2}{s^2-4}\right) \\ &= 2\cosh(2t) - \tfrac{3}{2}\sinh(2t). \end{aligned}$$

Another way to do the same problem is to find the partial fractions decomposition of the expression:

$$\frac{2s-3}{(s^2-4)} = \frac{7}{4(s+2)} + \frac{1}{4(s-2)}.$$

Therefore, we find that

$$f(t) = \tfrac{7}{4}e^{-2t} + \tfrac{1}{4}e^{2t}.$$

We can check that the two solutions are equivalent:

$$2\cosh(2t) - \tfrac{3}{2}\sinh(2t) = e^{2t} + e^{-2t} - \tfrac{3}{4}(e^{2t} - e^{-2t}) = \tfrac{1}{4}e^{2t} + \tfrac{7}{4}e^{-2t}.$$

7) We rewrite the expression as

$$\frac{2s+1}{s^2-2s+2} = 2\left(\frac{s-1}{(s-1)^2+1}\right) + 3\left(\frac{1}{(s-1)^2+1}\right)$$

Therefore, using the linearity of the inverse Laplace transform, we find that

$$\begin{aligned} f(t) = L^{-1}\left(\frac{2s+1}{s^2-2s+1}\right) &= 2L^{-1}\left(\frac{s-1}{(s-1)^2+1}\right) + 3L^{-1}\left(\frac{1}{(s-1)^2+1}\right) \\ &= 2e^t\cos t + 3e^t\sin t. \end{aligned}$$

8) First of all, we have

$$\frac{8s^2-4s+12}{s(s^2+4)} = \frac{8s}{s^2+4} - \frac{4}{s^2+4} + \frac{12}{s(s^2+4)} = 8\frac{s}{s^2+4} - 2\frac{2}{s^2+4} + 6\left(\frac{1}{s}\right)\left(\frac{2}{s^2+4}\right).$$

Therefore, after applying the inverse Laplace transform and using the linearity, we obtain

$$f(t) = 8L^{-1}\left(\frac{s}{s^2+4}\right) - 2L^{-1}\left(\frac{2}{s^2+4}\right) + 6L^{-1}\left[\left(\frac{1}{s}\right)\left(\frac{2}{s^2+4}\right)\right].$$

We know that $L^{-1}\left(\frac{s}{s^2+4}\right) = \cos(2t)$ and $L^{-1}\left(\frac{2}{s^2+4}\right) = \sin(2t)$. From a result in the lecture notes (in section 8.4), when a Laplace transform is divided by s, the original function comes from an integral:

$$L^{-1}\left(\frac{F}{s}\right) = \int_0^t f(\tau)\,d\tau,$$

where $F = L(f)$. In our case, this gives us

$$L^{-1}\left(\frac{\frac{2}{s^2+4}}{s}\right) = \int_0^t \sin 2\tau \, d\tau = \frac{-\cos(2\tau)}{2}\bigg|_0^t = \frac{1-\cos(2t)}{2}.$$

Therefore, we obtain

$$f(t) = 8\cos(2t) - 2\sin(2t) + 3 - 3\cos(2t) = 5\cos(2t) - 2\sin(2t) + 3.$$

This last calculations using the integral is in fact a shortcut from section 8.4. Therefore, I give you another approach using only the notions from sections 8.1 and 8.2. We had

$$\frac{8s^2-4s+12}{s(s^2+4)} = \frac{8s}{s^2+4} - \frac{4}{s^2+4} + \frac{12}{s(s^2+4)}.$$

We rewrite the last term using partial fractions:

$$\frac{12}{s(s^2+4)} = \frac{3}{s} - \frac{3s}{s^2+4}.$$

Since $L(1) = \frac{1}{s}$ and $L(\cos(2t)) = \frac{s}{s^2+4}$, we get, by the linearity of the Laplace transform:

$$L^{-1}\left(\frac{12}{s(s^2+4)}\right) = 3L^{-1}\left(\frac{1}{s}\right) - 3L\left(\frac{s}{s^2+4}\right) = 3 - 3\cos(2t).$$

Collecting all the previous results for the other expressions of the function, we get

$$\begin{aligned} f(t) = L^{-1}\left(\frac{8s^2-4s+12}{s(s^2+4)}\right) &= 8L^{-1}\left(\frac{s}{s^2+4}\right) - 2L^{-1}\left(\frac{2}{s^2+4}\right) + L^{-1}\left(\frac{12}{s(s^2+4)}\right) \\ &= 8\cos(2t) - 2\sin(2t) + 3 - 3\cos(2t) \\ &= 5\cos(2t) - 2\sin(2t) + 3. \end{aligned}$$

9) The numerator of the fraction is written as:

$$2s^2+4s+6 = 2(s^2+2s+3) = 2((s+1)^2+2) = 2(s+1)^2+4.$$

This implies that we can rewrite the expression as followed:

$$\frac{2s^2+4s+6}{(s+1)^2(s-1)} = \frac{2}{s-1} + \frac{4}{(s+1)^2(s-1)}.$$

Applying the inverse Laplace transform, we find that

$$f(t) = 2L^{-1}\left(\frac{1}{s-1}\right) + 4L^{-1}\left(\frac{1}{(s+1)^2(s-1)}\right).$$

Now, we write $\frac{1}{(s+1)^2(s-1)}$ using partial fractions. The decomposition in partial fractions should be

$$\frac{1}{(s+1)^2(s-1)} = \left(-\frac{1}{4}\right)\left(\frac{1}{s+1}\right) - \frac{1}{2}\left(\frac{1}{(s+1)^2}\right) + \frac{1}{4}\left(\frac{1}{s-1}\right).$$

Therefore, we find, after applying the inverse Laplace transform, that

$$L^{-1}\left(\frac{1}{(s+1)^2(s-1)}\right) = -\frac{1}{4}L^{-1}\left(\frac{1}{s+1}\right) - \frac{1}{2}L^{-1}\left(\frac{1}{(s+1)^2}\right) + \frac{1}{4}L^{-1}\left(\frac{1}{s-1}\right).$$

We then get

$$\begin{aligned} f(t) &= 2L^{-1}\left(\frac{1}{s-1}\right) - L^{-1}\left(\frac{1}{s+1}\right) - 2L^{-1}\left(\frac{1}{(s+1)^2}\right) + L^{-1}\left(\frac{1}{s-1}\right) \\ &= 3L^{-1}\left(\frac{1}{s-1}\right) - L^{-1}\left(\frac{1}{s+1}\right) - 2L^{-1}\left(\frac{1}{(s+1)^2}\right) \\ &= 3e^t - e^{-t} - 2te^{-t}. \end{aligned}$$

10) We write the fraction $\frac{s+1}{s(s-1)^2}$ in its partial fraction decomposition. We have

$$\frac{s+1}{s(s-1)^2} = \frac{1}{s} - \frac{1}{s-1} + \frac{2}{(s-1)^2}.$$

Therefore, after applying the inverse Laplace transform, we obtain

$$\begin{aligned} f(t) &= L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{1}{s-1}\right) + 2L^{-1}\left(\frac{1}{(s-1)^2}\right) \\ &= 1 - e^t + 2te^t. \end{aligned}$$

Problem B

In the solutions below, the capital letters refer to the Laplace transform of a given function (always denoted by lower-case letters). I encourage you to verify your solutions using the technics from Chapter 5 (using the characteristic polynomial).

1) First of all, we apply the Laplace transform to the ODE:

$$s^2Y - sy(0) - y'(0) - sY + y(0) - 6Y = 0.$$

Using the initial conditions, we find out that

$$(s^2 - s - 6)Y - s + 1 + 1 = 0.$$

Therefore, the last expression can be rewritten as

$$(s-3)(s+2)Y = s - 2$$

and, after isolating Y, we find out that

$$Y = \frac{s-2}{(s-3)(s+2)}.$$

Expanding the last expression in partial fractions, we obtain:

$$Y = \frac{4}{5(s+2)} + \frac{1}{5(s-3)}.$$

Taking the inverse transform, we therefore get

$$y(t) = \tfrac{4}{5}e^{-2t} + \tfrac{1}{5}e^{3t}.$$

2) We first apply the Laplace transform to the ODE to obtain

$$s^2Y - sy(0) - y'(0) + 3sY - 3y(0) + 2Y = 0.$$

Using the initial condition, the last expression becomes

$$(s^2 + 3s + 2)Y - s - 3 = 0.$$

After isolating Y, we find out that

$$Y = \frac{s+3}{s^2+3s+2}.$$

We have that $s^2 + 3s + 2 = (s+2)(s+1)$. Therefore, the partial fraction expansion of the last expression is:

$$Y = \frac{2}{s+1} - \frac{1}{s+2}.$$

Taking the inverse transforms and using the table, we obtain

$$y(t) = 2e^{-t} - e^{-2t}.$$

3) Apply the Laplace transform to get

$$s^2Y - sy(0) - y'(0) + 2sY - 2y(0) + 5Y = 0$$

and after substituting the initial conditions, we get

$$(s^2 + 2s + 5)Y - 2s - 1 - 4 = 0.$$

After isolating Y, we obtain

$$Y = \frac{2s+5}{s^2+2s+5}.$$

We can rewrite the denominator in the last expression as $s^2+2s+5 = (s+1)^2+4$. Therefore, the last expression takes the following form:

$$Y = \frac{2(s+1)}{(s+1)^2+4} + \frac{3}{(s+1)^2+4} = \frac{2(s+1)}{(s+1)^2+4} + \frac{3}{2}\left(\frac{2}{(s+1)^2+4}\right).$$

Taking the inverse transform, we get

$$y(t) = 2e^{-t}\cos(2t) + \tfrac{3}{2}e^{-t}\sin(2t).$$

4) Apply the Laplace transform to the ODE to get

$$s^2Y - sy(0) - y'(0) + \omega^2 Y = \frac{s}{s^2+4}$$

and using the initial conditions, we get

$$s^2Y - s + \omega^2 Y = \frac{s}{s^2+4}.$$

Isolating Y, we obtain

$$Y = \frac{s}{s^2+\omega^2} + \frac{s}{(s^2+4)(s^2+\omega^2)}.$$

We now want to rewrite the last expression in its partial fraction decomposition. We let

$$\frac{s}{(s^2+4)(s^2+\omega^2)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+\omega^2}$$

and we have to find the values of the constants A, B, C and D such that

$$\frac{s}{(s^2+4)(s^2+\omega^2)} = \frac{(A+C)s^3 + (B+D)s^2 + (A\omega^2+4C)s + B\omega^2 + 4D}{(s^2+4)(s^2+\omega^2)}$$

Therefore, we must have that $A+C=0$, $B+D=0$, $A\omega^2+4C=1$ and $B\omega^2+4D=0$. Knowing that $\omega^2-4 \neq 0$, we find that $D=-B$ et then

$$(\omega^2-4)B = 0 \Rightarrow B = 0.$$

Furthermore, since $A+C=0$ and $A\omega^2+4C=1$, we find that

$$(\omega^2-4)A = 1 \Rightarrow A = \frac{1}{\omega^2-4}.$$

Therefore, we have $B=D=0$ and $A=\frac{1}{\omega^2-4}$ and also that $C=-\frac{1}{\omega^2-4}$. This gives the following partial fraction decomposition:

$$\frac{s}{(s^2+4)(s^2+\omega)} = \frac{\left(\frac{1}{\omega^2-4}\right)s}{s^2+4} - \frac{\left(\frac{1}{\omega^2-4}\right)s}{s^2+\omega^2}.$$

By what we just found, we can rewrite Y as followed:

$$Y = \frac{s}{s^2+\omega^2} + \frac{\left(\frac{1}{\omega^2-4}\right)s}{s^2+4} - \frac{\left(\frac{1}{\omega^2-4}\right)s}{s^2+\omega^2} = \frac{\left(\frac{\omega^2-5}{\omega^2-4}\right)s}{s^2+\omega^2} + \frac{\left(\frac{1}{\omega^2-4}\right)s}{s^2+4}.$$

Taking the inverse trnasform, we find that

$$y(t) = \left(\frac{\omega^2-5}{\omega^2-4}\right)\cos(\omega t) + \left(\frac{1}{\omega^2-4}\right)\cos(2t).$$

5) Taking the Laplace transform of the EDO, we find that

$$s^2Y - sy(0) - y'(0) + 2sY - 2y(0) + Y = \frac{4}{s+1}$$

and using the initial conditions, we obtain

$$s^2Y - 2s + 1 + 2sY - 4 + Y = \frac{4}{s+1}.$$

Collecting the terms with a Y in them, we obtain the following expression:

$$(s^2 + 2s + 1)Y = 2s + 3 + \frac{4}{s+1}.$$

Now, isolating Y and taking into account that $s^2 + 2s + 1 = (s+1)^2$, we see that

$$Y = \frac{2s+3}{(s+1)^2} + \frac{4}{(s+1)^3} = \frac{2}{s+1} + \frac{1}{(s+1)^2} + \frac{4}{(s+1)^3}.$$

Taking the inverse transform, we see that

$$y(t) = 2e^{-t} + te^{-t} + 2t^2e^{-t} = (2 + t + 2t^2)e^{-t}.$$