MATH 302

CHAPTER 7

SECTION 7.1: REVIEW OF POWER SERIES

CONTENTS

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Why Power Series?

Most of the differential equation of order 2 we have encountered are constant coefficients ODE. In most real-life application, the coefficients will be **variable coefficients** such as

- **Bessel's equation** of order n:
$$x^2y'' + xy' + (x^2 - n^2)y = 0.$$
- **Legendre's equation** of order n:
$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0.$$

The methods we used in chapter 5won't be of use in those situations. This is why we need power series and the **power series method**.

Basic Definitions

- A **Power series** centered at a number a is an expression involving an infinite sum of powers of $(x - a)$:
$$\sum_{n=0}^{\infty} a_n(x-a)^n.$$
- If $a = 0$, we simply write
$$\sum_{n=0}^{\infty} a_n x^n.$$

We will confine ourselves to power series centered at $a = 0$.

- A power series **converges** on an interval I provided that for any x in this interval I, the following limit exists

$$\lim_{N\to\infty} \sum_{n=0}^{N} a_n x^n.$$

- If a function f is expressed as a power series on I, we then write

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

  and called this a **power series representation** of f.

Some examples of power series representations of some famous[1] function

- $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad -\infty < x < \infty$
- $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad -\infty < x < \infty$
- $\sin x = x - \frac{x^3}{6} + \frac{x^5}{60} - \frac{x^7}{5040} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \quad -\infty < x < \infty$
- $\cosh x = 1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} + \ldots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}, \quad -\infty < x < \infty$
- $\sinh x = x + \frac{x^3}{6} + \frac{x^5}{60} + \frac{x^7}{5040} + \ldots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}, \quad -\infty < x < \infty$
- $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \quad -1 < x < 1$
- $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n, \quad -1 < x < 1.$
- $(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} x^3 + \ldots, \quad -1 < x < 1.$

Remark:

- The **Taylor series** of f is

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

- The **Maclaurin series** of f is

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n.$$

[1]They made the coverpage of New York Times magazine several times for their influence on the world.

Differentiation

If $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

and in general

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) a_n x^{n-k}.$$

EXAMPLE 1. Differentiate the power series representation of $\sin x$.

$$\frac{d}{dx}(\sin x) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}\right)$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{d}{dx}\left(\frac{x^{2n+1}}{(2n+1)!}\right)$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (2n+1) x^{2n}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{\cancel{(2n+1)}(2n)!} \cancel{(2n+1)} x^{2n}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$
$$= \cos(x) \ ! \ :)$$

Identity Principle or Uniqueness of Power series

If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$, then

$$f(x) = g(x) \quad \iff \quad a_n = b_n, \text{ for all } n \geq 0.$$

Consequence: We have

$$\sum_{n=0} a_n x^n = 0$$

if, and only if, $a_n = 0$ for all $n \geq 0$.

Example 2. Find $y(x)$ if

$$y' = \sum_{n=1}^{\infty} x^n \quad \text{and} \quad y(0) = 0.$$

Write $y(x) = \sum_{n=0}^{\infty} a_n x^n$.

Then, $y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$.

So,

$$y'(x) = \sum_{n=1}^{\infty} x^{n-1}$$

$$\iff \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} x^{n-1}$$

$$\iff n a_n = 1, \quad n \geq 1$$

$$\iff a_n = \frac{1}{n}, \quad n \geq 1$$

We therefore see that

$$y(x) = a_0 + \sum_{n=1}^{\infty} a_n x^n$$

$$= a_0 + \sum_{n=1}^{\infty} \frac{x^n}{n}$$

Since $y(0) = 0 \Rightarrow 0 = a_0 + \sum 0^n/n \Rightarrow a_0 = 0$.

So, $y(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = -\log(1-x)$ $\overset{!!}{\text{o}}$.

Sum, Difference and Multiplication by A Constant

If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ are two power series, then

- $f(x) + g(x) = \displaystyle\sum_{n=0}^{\infty} (a_n + b_n) x^n$.
- $f(x) - g(x) = \displaystyle\sum_{n=0}^{\infty} (a_n - b_n) x^n$.
- $cf(x) = \displaystyle\sum_{n=0}^{\infty} (ca_n) x^n$.

EXAMPLE 3. Use the definition of $\cosh(x)$ and $\sinh(x)$ to find its power series representation.

1)
$$\cosh(x) = \frac{e^{x}}{2} + \frac{e^{-x}}{2}$$
$$= \frac{1}{2} \sum_{n=0}^{\infty} \frac{x^n}{n!} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$$
$$= \sum_{n=0}^{\infty} \frac{1}{2n!} \left(1 + (-1)^n\right) x^n$$
$$= \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}.$$

2)
$$\sinh(x) = \frac{e^{x}}{2} - \frac{e^{-x}}{2}$$
$$= \frac{1}{2} \sum_{n=0}^{\infty} \frac{x^n}{n!} - \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$$
$$= \sum_{n=0}^{\infty} \frac{1}{2n!} \left(1 - (-1)^n\right) x^n$$
$$= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}.$$

Product with Polynomials

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$.

- $g(x) = cx$, then

$$f(x)g(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right) cx$$
$$= \sum_{n=0}^{\infty} (ca_n) x^{n+1} = \sum_{n=1}^{\infty} (ca_{n-1}) x^n$$

- $g(x) = cx^2$, then

$$f(x)g(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right) cx^2$$
$$= \sum_{n=0}^{\infty} (ca_n) x^{n+2} = \sum_{n=2}^{\infty} (ca_{n-2}) x^n$$

- $g(x) = cx^3$, then

$$f(x)g(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right) cx^3$$
$$= \sum_{n=0}^{\infty} (ca_n) x^{n+3} = \sum_{n=3}^{\infty} (ca_{n-3}) x^n .$$

Example 4. If $f(x) = \sum_{n=0}^{\infty} a_n x^n$, find the expression of

(a) xf'.

(b) $(2-x)f''$.

(a) $$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + \cdots$$
$$= \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$\Rightarrow \quad x f'(x) = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^{n+1}$$
$$= \sum_{n=1}^{\infty} n a_n x^n .$$

(b) $$(2-x) f''(x) = 2f(x) - x f''(x) .$$

- $$2f''(x) = \sum_{n=2}^{\infty} 2n(n-1) a_n x^{n-2}$$
- $$-x f''(x) = -x \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$
$$= \sum_{n=2}^{\infty} -n(n-1) a_n x^{n-1}$$

Therefore, we get

$$(2-x)f'' = \sum_{n=2}^{\infty} 2n(n-1)\, a_n x^{n-2} + \sum_{n=2}^{\infty} -n(n-1)\, a_n x^{n-1}$$

For now, we can't add the series together.

But we can do a shifting !

Shifting

For any integer k, if

$$y(x) = \sum_{n=n_0}^{\infty} a_n x^{n-k}$$

then

$$y(x) = \sum_{n=n_0-k}^{\infty} a_{n+k} x^n$$

EXAMPLE 5. Complete Example 4.

$$(2-x)f'' = \sum_{n=2}^{\infty} 2n(n-1)\, a_n\, x^{n-2}$$

$$+ \sum_{n=2}^{\infty} (-n(n-1)) a_n x^{n-1}$$

$$= \sum_{n=0}^{\infty} 2(n+2)(n+1)\, a_{n+2}\, x^n$$

$$+ \sum_{n=1}^{\infty} -(n+1)n\, a_{n+1}\, x^n$$

$$= 2(2)(1)\, a_2 + \sum_{n=1}^{\infty} 2(n+2)(n+1) a_{n+2} x^n$$

$$+ \sum_{n=1}^{\infty} -(n+1)n\, a_{n+1}\, x^n$$

$$= 4a_2 + \sum_{n=1}^{\infty} \big(2(n+2)(n+1) a_{n+2} - (n+1)n\, a_{n+1}\big) x^n$$

$$= \sum_{n=0}^{\infty} c_n x^n$$

where $c_0 = 4a_2$ & $c_n = 2(n+2)(n+1) a_{n+2} - (n+1)n\, a_{n+1}$.

EXAMPLE 6. Express $2y - xy''$ as a power series $\sum_{n=0}^{\infty} c_n x^n$.

Let $y(x) = \sum_{n=0}^{\infty} a_n x^n$.

Then $y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

$$\Rightarrow \quad x y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-1}$$

$$= \sum_{n=1}^{\infty} (n+1) n \, a_{n+1} x^n$$

So,

$$2y - xy'' = \sum_{n=0}^{\infty} 2a_n x^n - \sum_{n=1}^{\infty} (n+1) n \, a_{n+1} x^n$$

$$= 2a_0 + \sum_{n=1}^{\infty} 2a_n x^n - \sum_{n=1}^{\infty} (n+1) n \, a_{n+1} x^n$$

$$= 2a_0 + \sum_{n=1}^{\infty} \left(2a_n - (n+1) n \, a_{n+1}\right) x^n$$

$$= \sum_{n=0}^{\infty} c_n x^n$$

where $c_0 = 2a_0$

&

$c_n = 2a_n - (n+1) n \, a_{n+1}$.