Math 302

$M\,dx + N\,dy = 0$

$M_y = N_x$

F s.t. $F_x = M$

$F_y = N$

Chapter 2

Section 2.6: Integrating Factors

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Example 1. Verify if

$$\underbrace{(\cancel{3x} + 2y^3)}_{M}dx + \underbrace{2xy}_{N}dy = 0$$

is exact.

We have $M_y = 6y^2$ & $N_x = 2y \Rightarrow M_y \neq N_x$
$\Rightarrow$ not exact!.

Consider a fct. $\mu = \mu(x,y)$ & multiply the DE by μ:

$$\Rightarrow \underbrace{\mu(3x+2y^3)}_{\text{new } M}dx + \underbrace{\mu(2xy)}_{\text{new } N}dy = 0$$

Idea: can we find μ s.t.

$$\frac{\partial}{\partial y}\left(\mu(3x+2y^3)\right) = \frac{\partial}{\partial x}\left(\mu\, 2xy\right) ?$$

$\mu(x,y) = x$ for this ODE.

$$\Rightarrow x(3x+2y^3)\,dx + 2x^2y\,dy = 0 \quad \text{(New ODE)}$$

A function $\mu = \mu(x, y)$ is an **integrating factor** for

$$M(x, y)dx + N(x, y)dy = 0$$

if the equation

$$\mu(x, y)M(x, y)dx + \mu(x, y)N(x, y)dy = 0$$

is exact.

Let's start with the equation

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0. \tag{1}$$

Trick: By the exactness condition

$$\Rightarrow \quad (\mu M)_y = (\mu N)_x$$

$$\Rightarrow \quad \mu(M_y - N_x) = \mu_x N - \mu_y M.$$

$$\underset{P}{\overset{2}{x}} \quad \underset{Q}{\overset{3}{y}}$$

Suppose that $\mu(x,y) = P(x)\,Q(y)$.

$$\Rightarrow \quad PQ(M_y - N_x) = P'QN - PQ'M$$

$\div PQ$

$$\Rightarrow \quad M_y - N_x = \frac{P'}{P} N - \frac{Q'}{Q} M$$

1st case: $\div N$

$$\Rightarrow \frac{M_y - N_x}{N} = \frac{P'}{P} - \cancel{\frac{Q'}{Q}\frac{M}{N}}^{=0}$$

if $\dfrac{M_y - M_x}{N}$ is ind. of y

fct in x →

$$\Rightarrow \frac{M_y - N_x}{N} = \frac{P'}{P}$$

← fct in x.

2nd case $\div(-M)$

$$\Rightarrow \frac{N_x - M_y}{M} = \frac{Q'}{Q}$$

→ Find Q by integrating.

General Facts: Let M, N, M_y, N_x be continuous on an open rectangle R.

- if $(M_y - N_x)/N$ is independent of y, then
$$\mu(x,y) = \pm e^{\int p(x)\,dx}$$
is an integrating factor for (1) where $p(x) = (M_y - N_x)/N$.
- if $(N_x - M_y)/M$ is independent of x, then
$$\mu(x,y) = \pm e^{\int q(y)\,dy}$$
is an integrating factor for (1) where $q(y) = (N_x - M_y)/M$.

EXAMPLE 2. Find an integrating factor for the equation

$$\underbrace{(2xy^3 - 2x^3y^3 - 4xy^2 + 2x)}_{M}dx + \underbrace{(3x^2y^2 + 4y)}_{N}dy = 0.$$

1) $$\frac{M_y - N_x}{N} = \frac{\cancel{6xy^2} - 6x^3y^2 - 8xy - \cancel{6xy^2}}{3x^2y^2 + 4y}$$

$$= \frac{-6x^3y^2 - 8xy}{3x^2y^2+4y} = \frac{-2x\cancel{(3x^2y^2+4y)}}{\cancel{3x^2y^2+4y}}$$

$$= -2x \quad (\text{ind. of } y)$$

$$\Rightarrow \mu(x,y) = e^{\int -2x\,dx} = \boxed{e^{-x^2}}$$

Example 3. Find an integrating factor for the equation

$$\underline{2xy^3}dx + (\underline{3x^2y^2 + x^2y^3 + 1})dy = 0$$

and solve the equation.

1) $M_y = 6xy^2$, $N_x = 6xy^2 + 2xy^3$ $\to$ $M_y \neq N_x$ $\to$ Not Exact!

2) $\frac{M_y - N_x}{N} = \frac{\cancel{6xy^2} - \cancel{6xy^2} - 2xy^3}{3x^2y^2 + x^2y^3 + 1} = \frac{-2xy^3}{3x^2y^2 + x^2y^3 + 1}$

$\frac{N_x - M_y}{M} = \frac{\cancel{6xy^2} + 2xy^3 - \cancel{6xy^2}}{2xy^3} = \frac{2xy^3}{2xy^3} = 1$

$\Rightarrow \quad \mu = e^{\int 1 dy} = e^y$

Multiply $\Rightarrow$ $\underbrace{2xy^3e^y}_{\text{new } M} dx + \underbrace{(\dots)e^y}_{\text{new } N} dy = 0$

4) Solve: Find F s.t.

$F_x = 2xy^3e^y$ & $F_y = (3x^2y^2 + x^2y^3 + 1)e^y$

(A) Integrate: $F(x,y) = x^2y^3e^y + c(y)$ $\to$ y^2

(B) Diff $F_y = \cancel{3x^2y^2e^y} + \cancel{x^2y^3e^y} + c'(y)$

$= (\cancel{3x^2y^2} + \cancel{x^2y^3} + 1)e^y$ From

$\Rightarrow \quad c'(y) = e^y$

$\Rightarrow \quad c(y) = e^y + K$ (after integrating)

Now, $F(x,y) = x^2y^3e^y + e^y \qquad (K=0)$

Solution $\boxed{x^2y^3e^y + e^y = c}$