Math 302

Chapter 2

Section 2.2: Separable Equations

Contents

Created by: Pierre-Olivier Parisé

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What Is a Separable First Order ODE

A first order differential equation is separable if it can be written as

$$h(y)y' = g(x) \tag{1}$$

where

- the left-hand side is a product of a function h of y with the derivative y'.
- the right-hand side is a function g of the variable x.

Example 1. Solve the equation

$$y' = x(1+y^2).$$

Write $y' = \frac{dy}{dx}$ $\Rightarrow$ $\frac{dy}{dx} = x(1+y^2)$

$\Rightarrow$ $\frac{dy}{1+y^2} = x\,dx$ (separation of varibles)

Have the for $h(y)\,dy = g(x)\,dx$

$$\Rightarrow \int \frac{dy}{1+y^2} = \int x\,dx + c$$

$$\Rightarrow \arctan(y) = \frac{x^2}{2} + c$$

$$\overset{\tan}{\Rightarrow} \tan(\arctan(y)) = \tan\left(\frac{x^2}{2} + c\right)$$

$$\Rightarrow \boxed{y(x) = \tan\left(\frac{x^2}{2} + c\right)}$$

Trick:

- Write the derivative y' as $\frac{dy}{dx}$.
- Write the ODE in the form $h(y)dy = g(x)dx$.
- Integrate both sides.

Example 2.

1. Solve the equation
$$y' = -x/y.$$
2. Solve the initial value problem
$$y' = -x/y, \quad y(1) = 1.$$

1) Write $y' = \frac{dy}{dx}$ $\Rightarrow$ $\frac{dy}{dx} = \frac{-x}{y}$

$\Rightarrow$ $y\,dy = -x\,dx$

$\Rightarrow$ $\frac{y^2}{2} = \frac{-x^2}{2} + k$

$\Rightarrow$ $y^2 = -x^2 + 2k$

Write $c^2 = 2k$, $c > 0$ $\Rightarrow$ $x^2 + y^2 = c^2$. (Integral curve)

Solutions: $y(x) = \sqrt{c^2 - x^2}$ or $y(x) = -\sqrt{c^2 - x^2}$, $-c \leq x \leq c$.

2) $y(1) = 1$ & 1 is positive $\Rightarrow$ $y(x) = \sqrt{c^2 - x^2}$ (y is positive).

Therefore $1 = \sqrt{c^2 - 1}$ $\Rightarrow$ $1 = c^2 - 1$ $\Rightarrow$ $c^2 = 2$

$\Rightarrow$ $c = \pm\sqrt{2}$

$\Rightarrow$ $c = \sqrt{2}$ (bec. $c > 0$)

$$\Rightarrow \boxed{y(x) = \sqrt{2 - x^2}, \quad -\sqrt{2} \leq x \leq \sqrt{2}.}$$

In the previous examples, we could find an explicit function $y = y(x)$ that is a solution to the ODE. It not always the case though...

EXAMPLE 3. If possible, find a solution to

$$y' = \frac{2x+1}{5y^4+1}.$$

Write $\frac{dy}{dx} = y'$ $\Rightarrow$ $\frac{dy}{dx} = \frac{2x+1}{5y^4+1}$

$\Rightarrow$ $(5y^4+1)\,dy = (2x+1)\,dx$

$\Rightarrow$ $y^5 + y = x^2 + x + c$

To find y, we have to find the roots of a fifth degree polynomial in y... Extremely difficult!

So, we leave it as

$y^5 + y = x^2 + x + c$ $\leftarrow$ Implicit solutions.

Terminology: Let the functions $h(y)$ and $g(x)$ be continuous on (c, d) and (a, b) respectively. Suppose

- $H(y)$ is an antiderivative of $h(y)$ on (c, d).
- $G(x)$ is an antiderivative of $h(x)$ on (a, b).
- c is a constant.

Then the implicit equation

$$H(y) = G(x) + c$$

is called an *implicit solution* to (1).

Example 4. Find an implicit solution of

$$y' = \frac{2x+1}{5y^4+1}, \quad y(2) = 1.$$

An implicit solution to the ODE is

$$y^5 + y = x^2 + x + c.$$

From the hypothesis, $x = 2 \Rightarrow y = 1$

$$\Rightarrow \quad 1^5 + 1 = 2^2 + 2 + c$$

$$\Rightarrow \quad 2 = 6 + c$$

$$\Rightarrow \quad c = -4.$$

Therefore, the implicit solution is

$$\boxed{y^5 + y = x^2 + x - 4}$$

implicit solution of the IVP.

Terminology:

Let the functions $h(y)$ and $g(x)$ be continuous on (c, d) and (a, b) respectively. Suppose

- $H(y)$ is an antiderivative of $h(y)$ on (c, d).
- $G(x)$ is an antiderivative of $h(x)$ on (a, b).
- $c = H(y_0) - G(x_0)$.

Then the implicit equation

$$H(y) = G(x) + H(y_0) - G(x_0)$$

is called an *implicit solution of the initial value problem.*

Implicit Solutions and Integral Curves

The graph of an implicit solution to

$$h(y)y' = g(x)$$

is an integral curve.

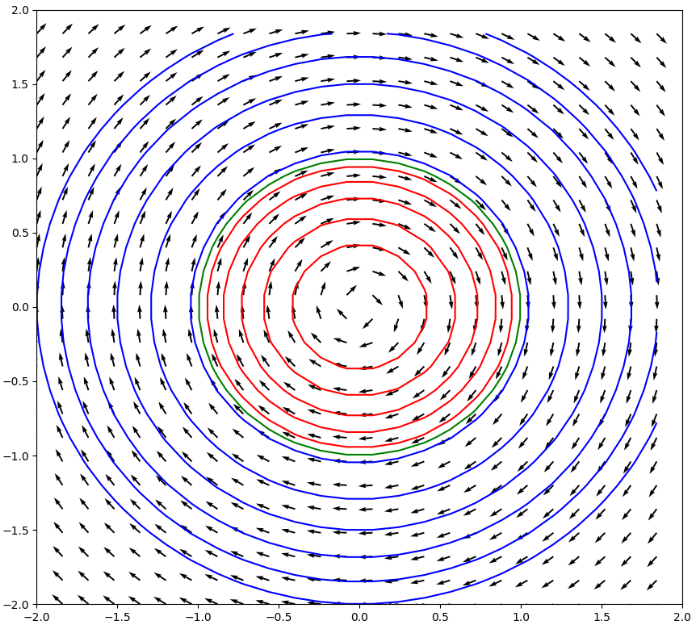

Figure 1: Direction field and implicit solutions of $y' = -\frac{x}{y}$.

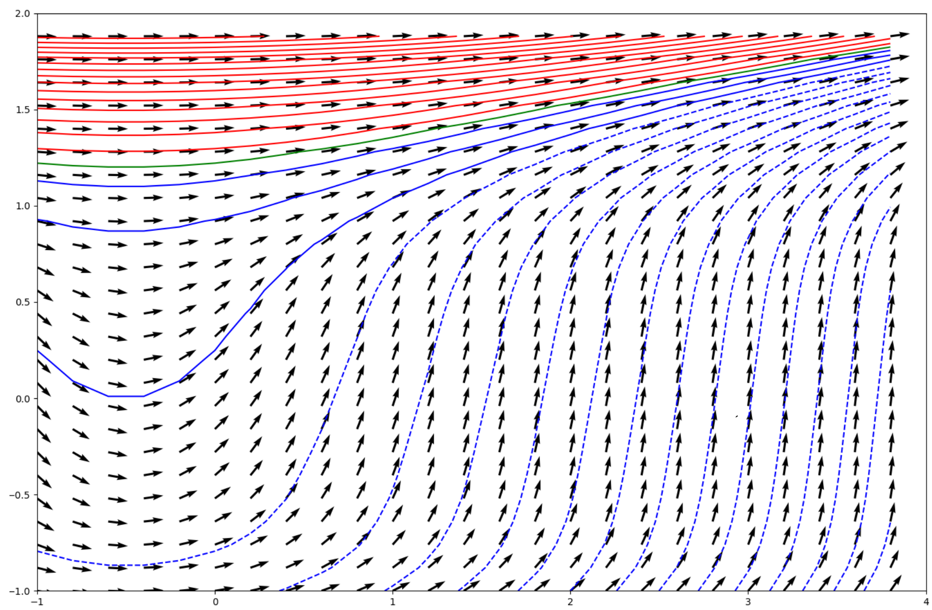

Figure 2: Direction field and implicit solutions of $y' = \frac{2x+1}{5y^4+1}$. In green you can see the implicit solution that satisfies $y(2) = 1$

An equation of the form

$$y' = g(x)p(y)$$

is separable because it can be put in the following forms:

$$h(y) = \frac{1}{p(y)} \quad \frac{y'}{p(y)} = g(x).$$

Problem:

- The division by $p(y)$ is not possible if $p(y) = 0$.

Example 5. Find all solutions to

$$y' = 2xy^2.$$

Write $y' = \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = 2x y^2$

- $\underline{y = 0}$ this is a solution.
- $\underline{y \neq 0}$. $\frac{dy}{y^2} = 2x\,dx \Rightarrow -\frac{1}{y} = x^2 + c$

$$\Rightarrow -\frac{1}{x^2+c} = y$$

Therefore, $\boxed{y(x) = 1 \;\&\; y(x) = -\frac{1}{x^2+c}}$

Example 6. Find all solutions of

$$y' = \frac{1}{2}x(1-y^2).$$

Zeros of $1-y^2$ are $y=\pm 1$

- $\underline{y=1}$ $\quad y'=0$ & $1-y^2=0 \Rightarrow y=1$ solution.
- $\underline{y=-1}$ $\quad y'=0$ & $1-y^2=0 \Rightarrow y=-1$ solution.
- $\underline{y\neq 1 \text{ \& } y\neq -1}$

$$\frac{y'}{1-y^2} = \frac{1}{2}x$$

$$\Rightarrow \cancel{\tfrac{1}{2}}\left(\frac{1}{1-y}+\frac{1}{1+y}\right)y' = \cancel{\tfrac{1}{2}}x$$

$$\Rightarrow \left(\frac{1}{1-y}+\frac{1}{1+y}\right)y' = x$$

$$\overset{\text{integrate}}{\Rightarrow} -\ln|1-y| + \ln|1+y| = \frac{x^2}{2}+k$$

$$\Rightarrow \ln\left(\frac{|1+y|}{|1-y|}\right) = \frac{x^2}{2}+k$$

$$\overset{\text{exp.}}{\Rightarrow} \frac{|1+y|}{|1-y|} = e^{\frac{x^2}{2}}e^{k}$$

$$\Rightarrow \frac{1+y}{1-y} = ce^{x^2/2} \qquad \left(c = \begin{cases} e^{k}, & +\text{sign} \\ -e^{k}, & -\text{sign} \end{cases} \text{ \& } c\neq 0 \right)$$

$$\Rightarrow 1+y = ce^{x^2/2} - ce^{x^2/2}y$$

$$\Rightarrow y(x) = -\frac{1-ce^{x^2/2}}{1+ce^{x^2/2}} \qquad (*)$$

Remarks: • if $c=0 \Rightarrow y=-1$. So the solution $y=-1$ is included in $(*)$ if we add $c=0$

• $y=1$ is not included in our solution $(*)$. But if $c \to \infty$, then $y \to 1$.

Answer:

$$y(x) = -\frac{1-ce^{x^2/2}}{1+ce^{x^2/2}} \quad c \text{ any real number}$$

&

$$y(x) = 1.$$

Remark: In the textbook, they obtain

$$y(x) = \frac{1+ce^{-x^2/2}}{1-ce^{-x^2/2}}$$

This is almost the same solution:

$$
\begin{aligned}
-\frac{1-ce^{x^2/2}}{1+ce^{x^2/2}} &= -\frac{1+(-c)e^{x^2/2}}{1-(-c)e^{x^2/2}} \\
&\overset{a=-c}{=} -\frac{1+ae^{x^2/2}}{1-ae^{x^2/2}} \\
&= -\frac{\cancel{ae^{x^2/2}}}{\cancel{ae^{x^2/2}}}\left(\frac{\frac{1}{a}e^{-x^2/2}+1}{\frac{1}{a}e^{-x^2/2}-1}\right) \\
&= \frac{1+\frac{1}{a}e^{-x^2/2}}{1-\frac{1}{a}e^{-x^2/2}} \quad \rightarrow \quad \text{same form } \frac{1+ce^{-x^2/2}}{1-ce^{-x^2/2}}.
\end{aligned}
$$