$$y' + p(x)\,y = 0$$

$$\Rightarrow y' = -p(x)\,y$$

$$\Rightarrow \frac{y'}{y} = -p(x)$$

Math 302

Chapter 2

Section 2.2: Separable Equations

Contents

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A first order differential equation is separable if it can be written as

$$h(y)y' = g(x) \tag{1}$$

where

- the left-hand side is a product of a function h of y with the derivative y'.
- the right-hand side is a function g of the variable x.

EXAMPLE 1. Solve the equation

$$y' = x(1+y^2).$$

Write $y' = \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = x(1+y^2)$

$h(y)$ $\Rightarrow \frac{1}{(1+y^2)} \frac{dy}{dx} = x$ $\to g(x)$

$$\Rightarrow \frac{1}{1+y^2}\,dy = x\,dx$$

$$\Rightarrow \int \frac{1}{1+y^2}\,dy = \int x\,dx + c$$

$$\Rightarrow \arctan(y) = \frac{x^2}{2} + c$$

Apply tan $\Rightarrow \boxed{y = \tan\left(\frac{x^2}{2} + c\right)}$

Trick:

- Write the derivative y' as $\frac{dy}{dx}$.
- Write the ODE in the form $h(y)dy = g(x)dx$.
- Integrate both sides.

EXAMPLE 2.

1. Solve the equation

$$y' = -x/y.$$

2. Solve the initial value problem

$$y' = -x/y, \quad y(1) = 1.$$

1) $y' = \frac{dy}{dx} \Rightarrow y\frac{dy}{dx} = -x$

$$\Rightarrow \int y\,dy = \int -x\,dx$$

$$\Rightarrow \frac{y^2}{2} + c_1 = -\frac{x^2}{2} + c_2$$

$$\Rightarrow \frac{y^2}{2} + \frac{x^2}{2} = c_2 - c_1$$

$$\Rightarrow y^2 + x^2 = 2(c_2 - c_1) \rightarrow c = 2(c_2 - c_1).$$

$$\Rightarrow \boxed{x^2 + y^2 = c}$$

$\hookrightarrow$ implicit solution

Solution (Isolate y)

$$y = \pm\sqrt{c - x^2}$$

2) $y(1) = 1 \Rightarrow 1^2 + 1^2 = c \Rightarrow c = 2$

two poss. $\quad y(x) = \sqrt{2 - x^2} \quad (-2 \le x \le 2)$

or $\quad \cancel{y(x) = -\sqrt{2 - x^2}} \quad (-2 \le x \le 2).$

In the previous examples, we could find an explicit function $y = y(x)$ that is a solution to the ODE. It not always the case though...

Example 3. If possible, find a solution to

$$y' = \frac{2x+1}{5y^4+1}.$$

$y' = \frac{dy}{dx}$ → $\frac{dy}{dx} = \frac{2x+1}{5y^4+1}$

→ $(5y^4+1)\, dy = (2x+1)\, dx$

Integrate → $\boxed{y^5 + y = x^2 + x + c}$

$H(y) = y^5 + y$, $G(x) = x^2 + x$

Rewrite: $\underbrace{\int h(y)}_{H(y)}\, y' = \underbrace{\int g(x)}_{G(x)}$

Terminology: Let the functions $h(y)$ and $g(x)$ be continuous on (c, d) and (a, b) respectively. Suppose

- $H(y)$ is an antiderivative of $h(y)$ on (c, d).
- $G(x)$ is an antiderivative of $h(x)$ on (a, b).
- c is a constant.

Then the implicit equation

$$H(y) = G(x) + c$$

is called an *implicit solution* to (1).

Example 4. Find an implicit solution of

$$y' = \frac{2x+1}{5y^4+1}, \quad y(2) = 1.$$

We already know that y satisfies

$$y^5 + y = x^2 + x + c$$

We have $y(2) = 1$

$$\Rightarrow 1^5 + 1 = 2^2 + 2 + c$$

$$\Rightarrow 2 = 6 + c$$

$$\Rightarrow c = -4$$

Our solution is

$$\boxed{y^5 + y = x^2 + x - 4}$$

Terminology:

Let the functions $h(y)$ and $g(x)$ be continuous on (c, d) and (a, b) respectively. Suppose

- $H(y)$ is an antiderivative of $h(y)$ on (c, d).
- $G(x)$ is an antiderivative of $h(x)$ on (a, b).
- $c = H(y_0) - G(x_0)$.

Then the implicit equation

$$H(y) = G(x) + H(y_0) - G(x_0)$$

is called an *implicit solution of the initial value problem.*

Implicit Solutions and Integral Curves

The graph of an implicit solution to

$$h(y)y' = g(x)$$

is an integral curve.

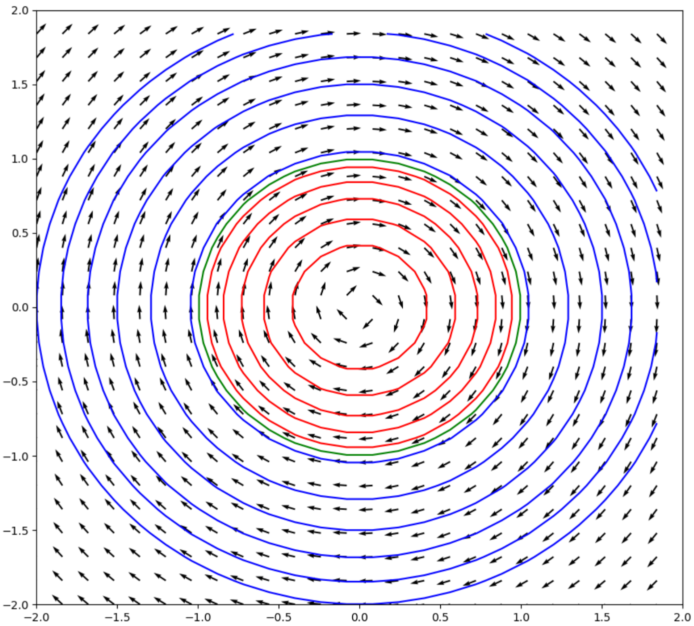

Figure 1: Direction field and implicit solutions of $y' = -\frac{x}{y}$.

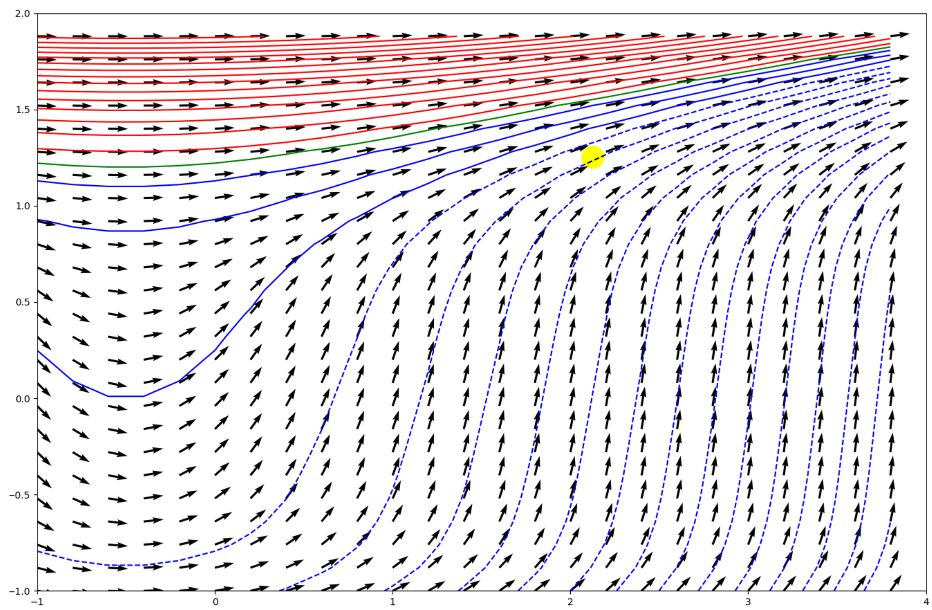

Figure 2: Direction field and implicit solutions of $y' = \frac{2x+1}{5y^4+1}$. In green you can see the implicit solution that satisfies $y(2) = 1$

An equation of the form

$$y' = g(x)p(y)$$

is separable because it can be put in the following forms:

$$h(y) = \frac{1}{p(y)} \qquad \frac{y'}{p(y)} = g(x).$$

Problem:

- The division by $p(y)$ is not possible if $p(y) = 0$.

EXAMPLE 5. Find all solutions to

$$y' = 2xy^2.$$

Here $\Rightarrow \quad \frac{y'}{y^2} = 2x \quad (\text{if } y \neq 0)$

Two cases:

1) $y = 0 \rightarrow y' = 0 \rightarrow \boxed{y(x) = c}$ (constant). $\boxed{y=0}$

2) $y \neq 0 \rightarrow \frac{y'}{y^2} = 2x \rightarrow \frac{dy}{y^2} = 2x\,dx$

$$\rightarrow -\frac{1}{y} = x^2 + c$$

$$\rightarrow \boxed{\frac{-1}{x^2+c} = y(x)}$$

Example 6. Find all solutions of

$$y' = \frac{1}{2}x(1-y^2).$$

divide by $1-y^2$ $\to$ $1-y^2=0$ when $y = \pm 1$

<u>3 cases.</u>

<u>$y=1$</u> (Is that a solution).

$y'=0$ & $\frac{1}{2}x(1-y^2) = 0$

$\Rightarrow$ $y' = \frac{1}{2}x(1-y^2)$

$\Rightarrow$ $\boxed{y=1}$ is a solution

<u>$y=-1$</u>

$y' = 0$ & $\frac{1}{2}x(1-y^2) = 0$

$\Rightarrow$ $y' = \frac{1}{2}x(1-y^2)$

$\Rightarrow$ $\boxed{y=-1 \text{ is a solution}}$

<u>$y \neq 1$ & $y \neq -1$</u>

$$\frac{y'}{1-y^2} = \frac{1}{2}x$$

$y' = \frac{dy}{dx}$ $\Rightarrow$ $\frac{dy}{1-y^2} = \frac{1}{2}x\,dx$

$$\Rightarrow \underbrace{\int \frac{dy}{1-y^2}}_{A} = \int \frac{1}{2}x\,dx + c$$

(A) Partial Fractions.

$$\frac{1}{1-y^2} = \frac{1}{(1-y)(1+y)} = \frac{A}{1-y} + \frac{B}{1+y}$$

$$= \frac{A(1+y) + B(1-y)}{(1-y)(1+y)}$$

$$= \frac{A+B + (A-B)y}{1-y^2}$$

$\Rightarrow$ $A+B=1$ & $A-B=0$

$\Rightarrow$ $A=B=\frac{1}{2}$

So, $\dfrac{1}{1-y^2} = \dfrac{1/2}{1-y} + \dfrac{1/2}{1+y}$.

$$\int \frac{1}{1-y^2}\,dy = \int \frac{1/2}{1-y} + \frac{1/2}{1+y}\,dy = -\frac{1}{2}\ln|1-y| + \frac{1}{2}\ln|1+y|$$

So,

$$-\frac{1}{2}\ln|1-y| + \frac{1}{2}\ln|1+y| = \frac{x^2}{4} + k$$

$$\Rightarrow \quad -\ln|1-y| + \ln|1+y| = \frac{x^2}{2} + k$$

$$\Rightarrow \quad \ln|1-y|^{-1} + \ln|1+y| = \frac{x^2}{2} + k$$

$$\Rightarrow \quad \ln\left(\frac{|1+y|}{|1-y|}\right) = \frac{x^2}{2} + k$$

$$\Rightarrow \quad \frac{|1+y|}{|1-y|} = e^{x^2/2 + k} = e^{x^2/2}e^{k}$$

Let $c = \pm e^{k}$

$$\Rightarrow \quad \frac{1+y}{1-y} = ce^{x^2/2}$$

$$\Rightarrow \quad 1+y = (1-y)\,ce^{x^2/2}$$

$$\Rightarrow \quad y\left(1+ce^{x^2/2}\right) = ce^{x^2/2} - 1$$

$$\Rightarrow \quad \boxed{y = \frac{ce^{x^2/2}-1}{1+ce^{x^2/2}}}$$