Math 302

Chapter 6

Section 6.1 and 6.2: Spring-mass System

Contents

Created by: Pierre-Olivier Parisé
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Setup

Consider an object with a mass m suspended from a spring of negligible mass.

- When the object is at rest and the forces sum to zero, we say that the spring-mass system is in **equilibrium**.
- When the system is at equilibrium, we call the position of the object the **equilibrium position**.
- We let $y(t)$ be the position of the object with respect to the equilibrium position.

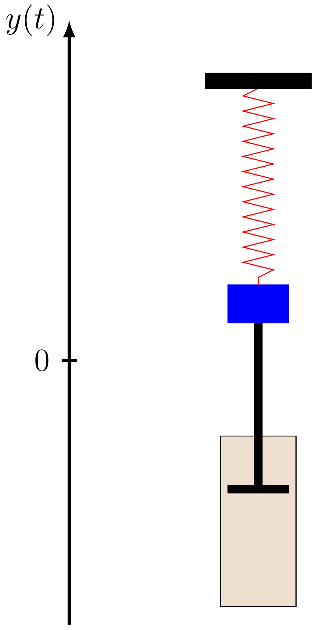

Figure: Illustration of the spring-mass system with damping.

From Newton's second Law, we have

$$my'' =$$

The force F_s is related to y by

$$F_s = mg - ky.$$

Substituting this expression of F_s in the expression of my'', we obtain the following ODE called the **equation of motion**:

$$my'' + cy' + ky = F.$$

We'll consider no exterior force and no damping. Therefore, the ODE becomes

$$my'' + ky = 0 \iff y'' + (k/m)y = 0.$$

The solution is therefore given by

$$y(t) = c_1 \cos\left(t\sqrt{\frac{k}{m}}\right) + c_2 \sin\left(t\sqrt{\frac{k}{m}}\right).$$

Remark:

- An important quantity is Δl, the difference between the natural length of a spring and it's length at the equilibrium. We have

$$mg = k\Delta l \iff \frac{k}{m} = \frac{g}{\Delta l}.$$

Using some trigonometry identities, we can rewrite the solution as

$$y(t) = R\cos\left(t\sqrt{\frac{k}{m}} - \phi\right)$$

where R and ϕ are constants related to c_1 and c_2 in the following way:

$$c_1 = R\cos\phi \quad \text{and} \quad c_2 = R\sin\phi.$$

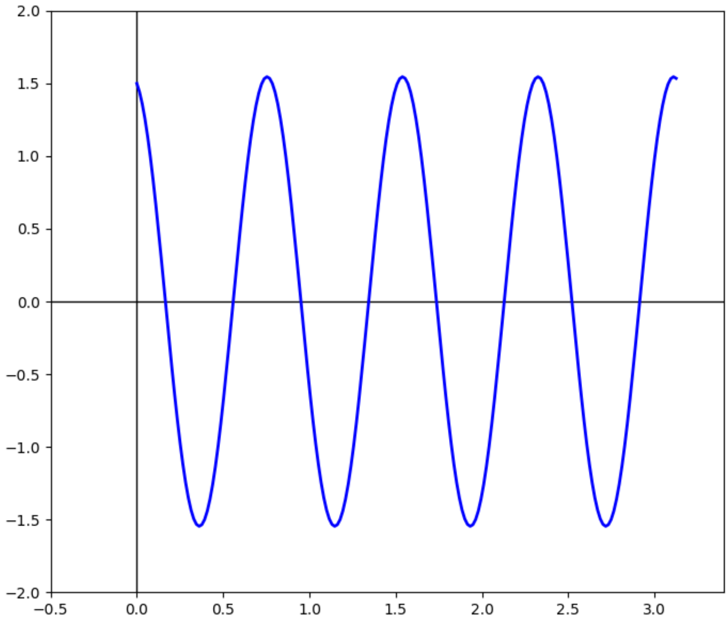

Figure 1: Example of a harmonic motion: $y(t) = (3/2)\cos(8t) - (3/8)\sin(8t)$ or $y(t) = (3\sqrt{17}/8)\cos(8t + 0.245)$. See Example 6.1.1, page 269 and Example 6.1.2, page 272.

We now consider an external force F of the form

$$F(t) = F_0 \cos(\omega t).$$

So, we want to solve

$$y'' + (k/m)y = \frac{F_0}{m}\cos(\omega t).$$

Two cases:

- If $\omega \neq \sqrt{k/m}$, then the solution is periodic and do not blow up!
- If $\omega = \sqrt{k/m}$, then the solution blows up!

To simplify the notation, we use the following notation:

$$\omega_0 := \sqrt{k}m.$$

First case: $\omega_0 \neq \omega$

The solution is

$$y(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega^2)}\sin(\omega t).$$

If we suppose that $y(0) = 0$ and $y'(0) = 0$, then from some clever tricks and trig identities, we can simplify the expression of $y(t)$ to

$$\begin{aligned} y(t) &= \frac{2F_0}{m(\omega_0^2 - \omega^2)}\sin\left(\frac{[\omega_0 - \omega]t}{2}\right)\sin\left(\frac{[\omega_0 + \omega]t}{2}\right) \\ &= R(t)\sin\left(\frac{[\omega_0 + \omega]t}{2}\right). \end{aligned}$$

<u>Remark:</u>

- The function $R(t)$ is called a **beat**.

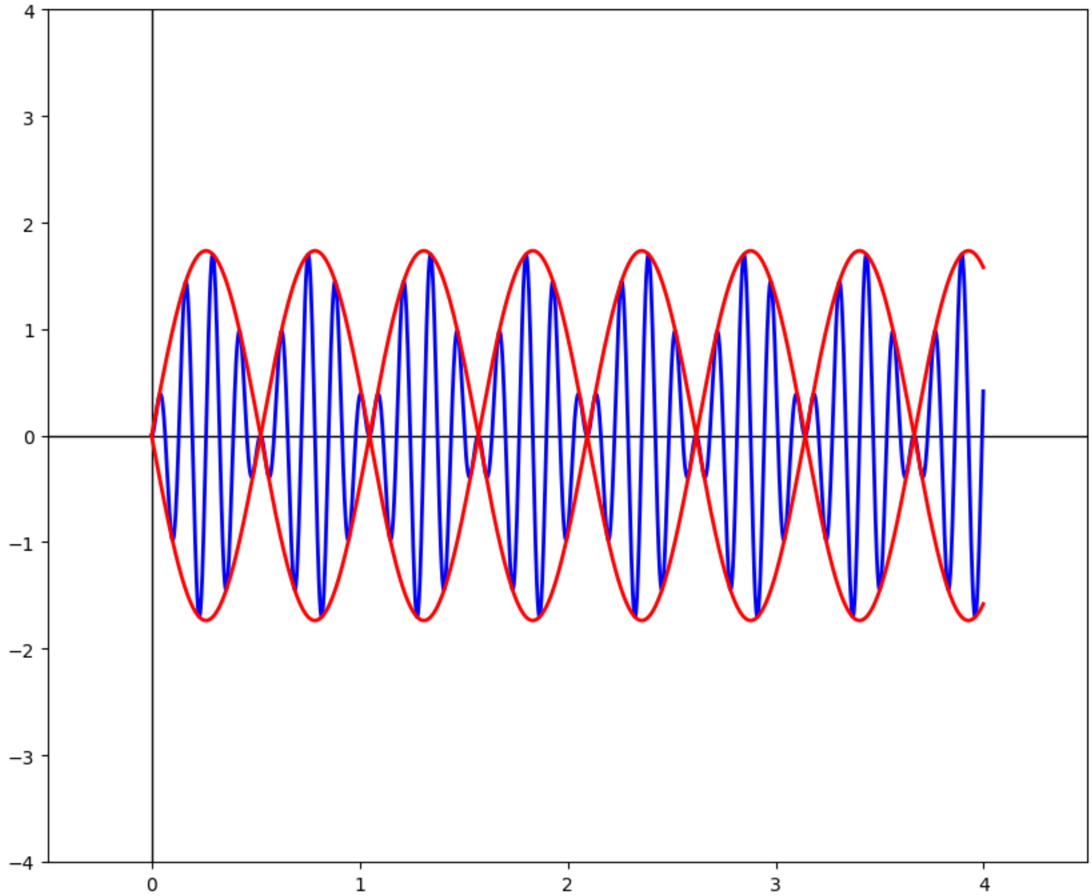

Figure 2: In blue: Solution with $F_0 = 4000$, $m = 4$, $\omega_0 = 54$ and $w = 42$. In red: the *beat*.

Second case: $\omega_0 = \omega$

The solution is

$$y(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{F_0}{2m\omega_0} t \sin(\omega_0 t).$$

We can show that the values of $y(t)$ oscillate between the lines

$$y = \frac{F_0}{2m\omega_0} t \quad \text{and} \quad y = -\frac{F_0}{2m\omega_0} t.$$

Remark:

- As $t \to \infty$, we see that the oscillations of $y(t)$ become bigger and bigger.
- Such a phenomena is called **resonance**.
- A resonance may be dangerous for, amongs other examples, suspended bridges.

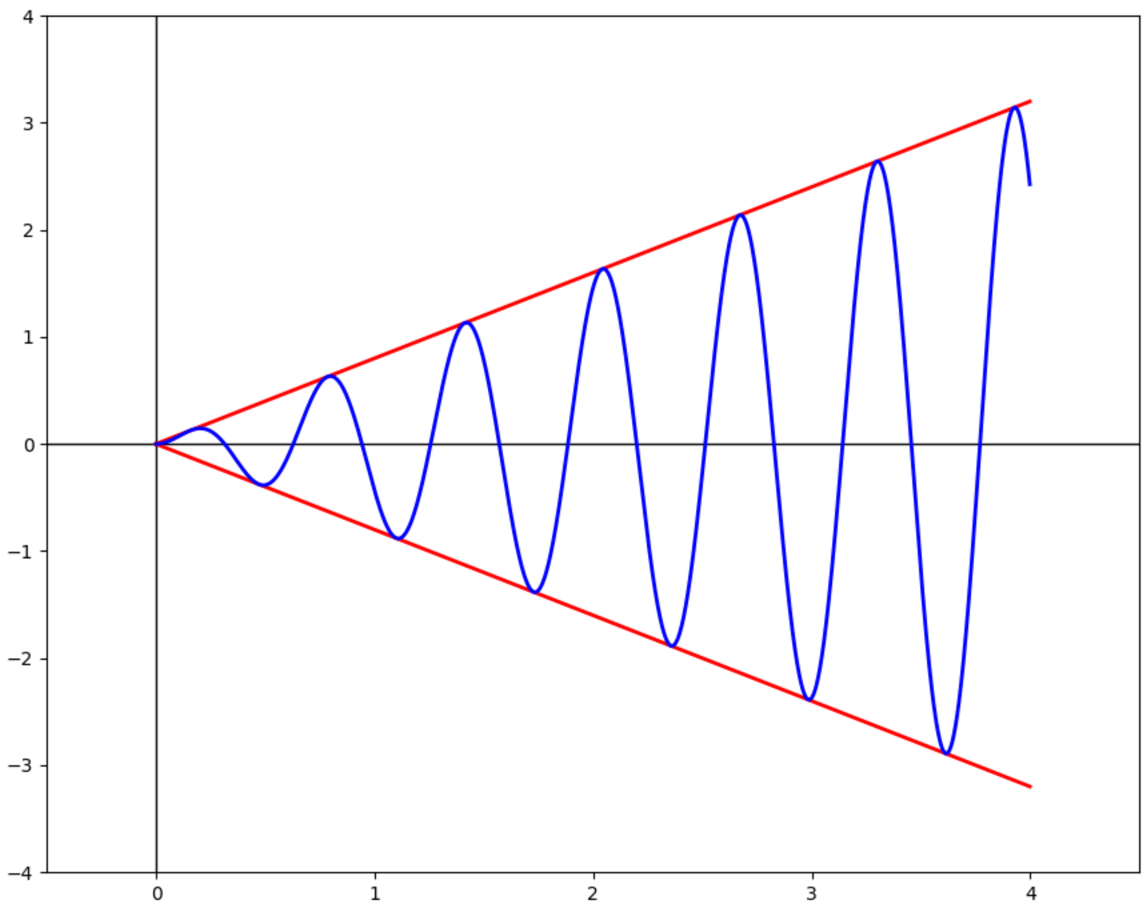

Figure 3: In blue: Graph of the solution with $c_1 = 0$ and $c_2 = 0$ corresponding to $y(0) = y'(0) = 0$. In red: The shell containing the solution (maximum and minimum possible values).

We consider the spring-mass system with $c \neq 0$. We therefore have

$$my'' + cy' + ky = 0.$$

The solutions will now depend on the roots of the characteristic polynomial:

$$r_1 = \frac{-c - \sqrt{c^2 - 4mk}}{2m} \quad \text{and} \quad r_2 = \frac{-c + \sqrt{c^2 - 4mk}}{2m}.$$

Underdamped Motion

- The motion of the mass is said to be **underdamped** if $c < \sqrt{4mk}$.
- If $\omega_1 = \frac{\sqrt{4mk - c^2}}{2m}$, then the expression of the roots become

$$r_1 = -\frac{c}{2m} - i\omega_1 \quad \text{and} \quad r_2 = -\frac{c}{2m} + i\omega_1$$

- The general solution is therefore

$$y(t) = e^{-ct/2m}\Big(c_1 \cos(\omega_1 t) + c_2 \sin(\omega_1 t)\Big).$$

- The expression of the solution can be simplified to

$$y(t) = Re^{-ct/2m} \cos\Big(\omega_1 t - \phi\Big).$$

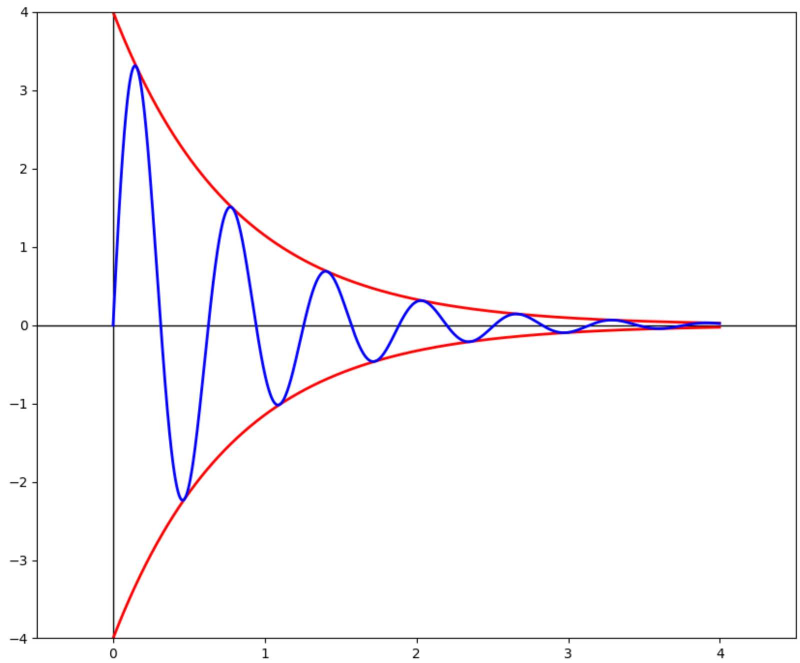

Figure 4: In blue: Graph of the solution when $c < \sqrt{4mk}$. In red: Graph of $\pm Re^{-ct/2m}$.

Overdamped Motion

- The system is said to be **overdamped** if $c > \sqrt{4mk}$.
- The roots are

$$r_1 = \frac{-c - \sqrt{c^2 - 4mk}}{2m} \quad \text{and} \quad r_2 = \frac{-c + \sqrt{c^2 - 4mk}}{2m}.$$

  We have $r_1 < r_2 < 0$.
- The general solution is

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

- We remark that $\lim_{t\to\infty} y(t) = 0$.

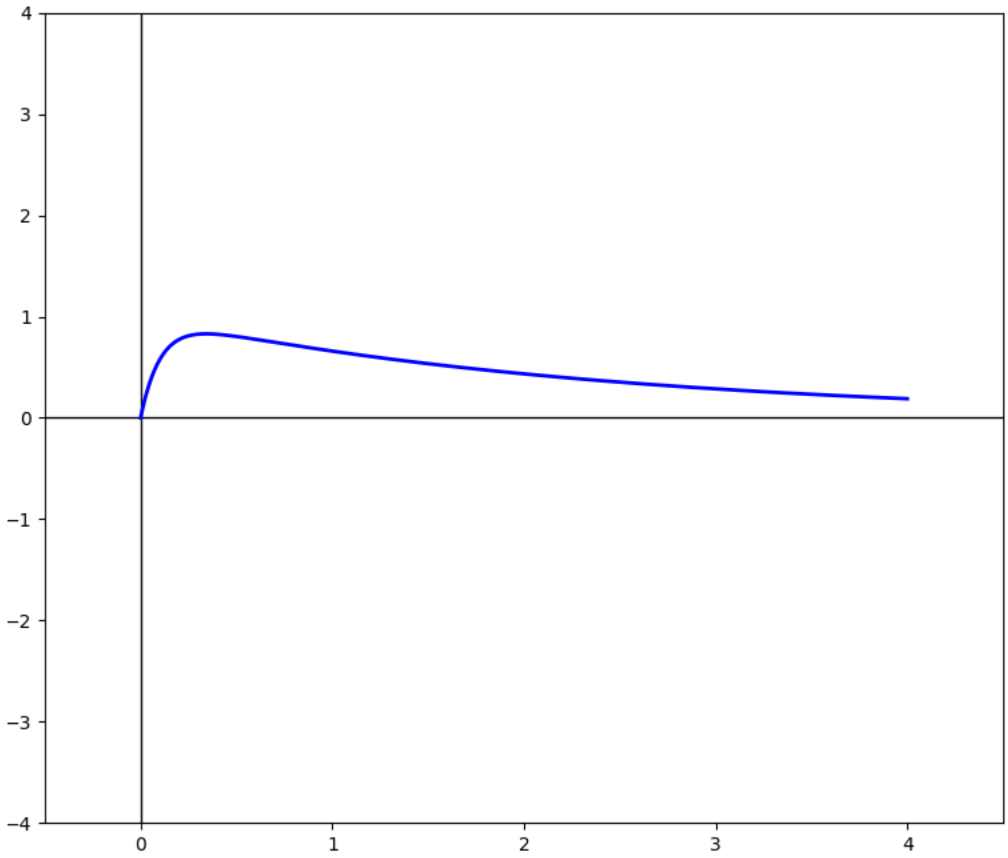

Figure 5: Overdamped System

Critically Damped Motion

- The system is said to be **critically damped** if $c = \sqrt{4mk}$.
- The roots are

$$r_1 = r_2 = -\frac{c}{2m}.$$

- The general solution is therefore

$$y(t) = e^{-ct/2m}(c_1 + c_2 t).$$

- We see that $\lim_{t\to\infty} y(t) = 0$.

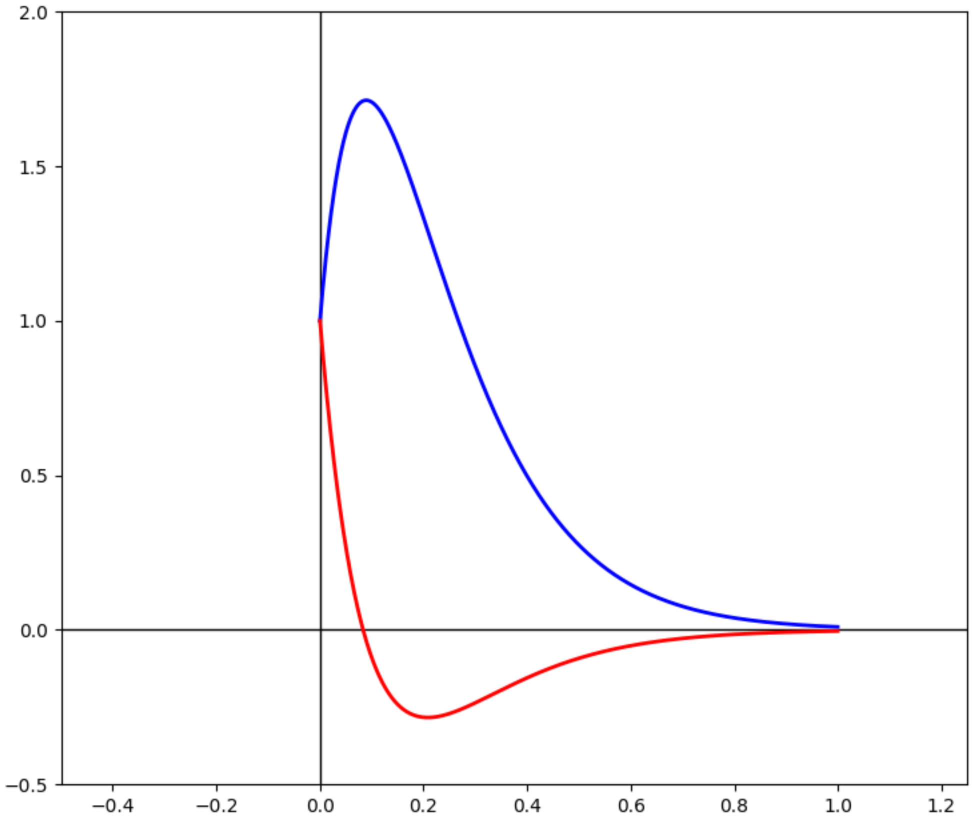

Figure 6: In blue: $y(t) = e^{-8t}(1 + 28t)$. In red: $y(t) = e^{-8t}(1 - 12t)$.