Math 302

Chapter 5

Section 5.3: Nonhomogeneous Linear Equations

Contents

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Our goal is to find the solutions to

$$y'' + p(x)y' + q(x)y = f(x). \tag{1}$$

Nomenclature:

- the equation $y'' + p(x)y' + q(x)y = 0$ is the **complementary equation** for (1).
- a **particular solution** is a solution y_{par} of (1).

EXAMPLE 1. Find a particular solution to the following ODE:

$$y'' - 2y' + y = 4x.$$

Trick: Guess!

What we know: $y(x) = A \Rightarrow y' = 0$ & $y'' = 0$
$y(x) = Ax+B \Rightarrow y' = A$ & $y'' = 0$.

Suggest $y(x) = Ax+B$.

$\Rightarrow y' = A$ & $y'' = 0$

$\Rightarrow 0 - 2A + Ax+B = 4x$

$\Rightarrow Ax + B - 2A = 4x + 0$

$\Rightarrow A = 4$ & $B - 2A = 0$

$\Rightarrow A = 4$ & $B - 8 = 0$

$\Rightarrow A = 4$ & $B = 8$

So,

$$\boxed{y_{par}(x) = 4x + 8}$$

Assumptions:

1) Suppose $\{y_1, y_2\}$ is a fundamental set of solutions to

$$y'' + p(x)y' + q(x)y = 0.$$

2) Suppose y_{par} is a particular solution to

$$y'' + p(x)y' + q(x)y = f(x).$$

Conclusion:

- Then the $y = y_{par} + c_1 y_1 + c_2 y_2$ is the general solution of

$$y'' + p(x)y' + q(x)y = f(x).$$

EXAMPLE 2.

a) Find the general solution of

$$y'' - 2y' + y = -3 - x + x^2.$$

b) Solve the following IVP:

$$y'' - 2y' + y = -3 - x + x^2, \quad y(0) = -2,\ y'(0) = 1.$$

a) 1) General sol. to complementary Eq.

$y'' - 2y' + y = 0 \rightarrow r^2 - 2r + 1 = 0$

$\rightarrow (r-1)(r-1) = 0$

$\rightarrow r = 1$

$\Rightarrow y(x) = (c_1 x + c_2) e^x . \rightarrow c_1 \underbrace{xe^x}_{y_1} + c_2 \underbrace{e^x}_{y_2}$

2) Find a particular solution.

Suggest: $y(x) = Ax^2 + Bx + C$.

$y'(x) = 2Ax + B$

$y''(x) = 2A$

$$\Rightarrow \quad 2A - 2(2Ax + B) + Ax^2 + Bx + C = -3 - x + x^2$$

$$\Rightarrow \quad 2A - 2B + C - 4Ax + Bx + Ax^2 = -3 - x + x^2$$

$$\Rightarrow \quad 2A - 2B + C + (B - 4A)x + Ax^2 = -3 - x + x^2$$

$$\Rightarrow \begin{cases} 2A - 2B + C = -3 & * \quad \rightarrow \quad 2 - 6 + C = -3 \rightarrow C = 1 \\ B - 4A = -1 & ** \quad \rightarrow \quad B - 4 = -1 \rightarrow B = 3 \\ A = 1 & *** \end{cases}$$

So, $\quad y_{par}(x) = x^2 + 3x + 1$

3) General solution:

$$\boxed{y(x) = \underbrace{c_1 x e^x + c_2 e^x}_{\text{homogeneous solution denoted by } y_h} + x^2 + 3x + 1}$$

EXAMPLE 3. Suppose that we know that $y_1(x) = x^4/15$ is a particular solution to

$$x^2y'' + 4xy' + 2y = 2x^4$$

and that $y_2(x) = x^2/3$ is a particular solution to

$$x^2y'' + 4xy' + 2y = 4x^2.$$

Find a particular solution to

$$x^2y'' + 4xy' + 2y = 2x^4 + 4x^2.$$

Say $y(x) = \frac{x^4}{15} + \frac{x^2}{3} = y_1(x) + y_2(x)$.

$\Rightarrow \quad y' = y_1' + y_2'$

& $\quad y'' = y_1'' + y_2''$

Then,

$$x^2(y_1'' + y_2'') + 4x(y_1' + y_2') + 2(y_1 + y_2)$$

$$= x^2y_1'' + x^2y_2'' + 4xy_1' + 4xy_2' + 2y_1 + 2y_2$$

$$= x^2y_1'' + 4xy_1' + 2y_1 + x^2y_2'' + 4xy_2' + 2y_2$$

$$= 2x^4 + 4x^2$$

$y' = \frac{4x^3}{15} + \frac{2x}{3}$ & $y'' = \frac{12}{15}x^2 + \frac{2}{3}$

$$x^2\left(\frac{12}{15}x^2 + \frac{2}{3}\right) + 4x\left(\frac{4x^3}{15} + \frac{2x}{3}\right) + \frac{2x^4}{15} + \frac{2x^2}{3}$$

$$= \frac{12}{15}x^4 + \frac{2x^2}{3} + \frac{16}{15}x^4 + \frac{8x^2}{3} + \frac{2x^4}{15} + \frac{2x^2}{3} = \frac{30}{15}x^4 + \frac{12}{3}x^2$$

$$= 2x^4 + 4x^2$$

General Fact: If y_1 is a particular solution to

$$y'' + p(x)y' + q(x)y = f_1(x)$$

and y_2 is a particular solution to

$$y'' + p(x)y' + q(x)y = f_2(x)$$

then $y_{par} = y_1 + y_2$ is a particular solution to

$$y'' + p(x)y' + q(x)y = f_1(x) + f_2(x).$$