Summary Final

Power Series & Laplace transform

Laplace Transform

see table !

Another notation: $L(f(t))$.

$$F(s)=\int_0^\infty f(t)e^{-st}\,dt$$

Properties

- Linear: $L(af+bg)=aL(f)+bL(g)$
- Shift: $L(e^{at}f(t)) = F(s-a)$
- Powers $\to$ Derivatives: $L(t^n f(t))=(-1)^n F^{(n)}(s)$.
- Derivatives $\to$ Powers:
  - * $L(f')=sF-f(0)$
  - * $L(f'')=s^2F-sf(0)-f'(0)$.

Solve ODEs

Strategy:

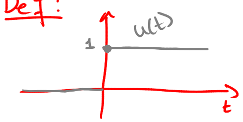
1) Write $Y=L(y)$.
2) Apply Laplace transform
3) Isolate $Y(s)$.
4) Inverse Laplace transform

trick: use table opposite direction !

1) Partial Fractions

Unit Step

Def:



translation: $u(t-a)=\begin{cases}1, & t\geq a\\ 0, & t<a\end{cases}$

Activation: $f(t)u(t-a)=\begin{cases}f(t), & t\geq a\\ 0, & t<a\end{cases}$

Desactivation: $f(t)u(t-a)-f(t)u(t-b)=\begin{cases}f(t), & a\leq t<b\\ 0, & \text{elsewhere}\end{cases}$

Laplace transforms:

- $L(u(t-a))=e^{-sa}$
- $L(u(t-a)f(t))=e^{-sa}L(f(t+a))$
- $L(u(t-a)f(t-a))=e^{-sa}F(s)$

Convolution

Def:

$$f*g(t)=\int_0^t f(x)g(t-x)\,dx$$

$\to$ Laplace: $L(f*g)=L(f)L(g)$

consequence: $L\left(\int_0^t f(x)\,dx\right)=\dfrac{F(s)}{s}$.

$\to$ Solve integro-Diff:

1) Apply Laplace.
2) Spot a convolution
3) Isolate Y
4) Find inverse

Power series

$$\sum_{n=0}^{\infty} a_n(x-a)^n \quad\text{or}\quad \sum_{n=0}^{\infty} a_n x^n$$

Taylor Polynomials: $\sum_{n=0}^{N} a_n x^n = T_N(x)$.

$\to$ $\left(\sum_{n=0}^{\infty} a_n x^n\right)' = \sum_{n=1}^{\infty} n a_n x^{n-1}$

$\left(\sum_{n=0}^{\infty} a_n x^n\right)'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$

$\to$ $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$ $\to$ $a_n=b_n$ for any n.

Algebra of operations:

- $\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty}(a_n+b_n)x^n$
- $c\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} c a_n x^n$
- $cx^k\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+k}$
- $\sum_{n=n_0}^{\infty} a_n x^{n-k} \leftrightarrow \sum_{n=n_0-k}^{\infty} a_{n+k}x^n$

Power Series Solution

$\to$ ODE (to solve).

$A(x)y''+B(x)y'+C(x)y=0$

A, B, C: polynomials.

Ordinary & Singular

- x_0 ord. point if $A(x_0)\neq 0$.
- x_0 sing. point if $A(x_0)=0$.

$\to$ Radius convergence of a solution:

R = distance from a to the nearest singular point.

$\left(y(x)=\sum_{n=0}^{\infty} a_n(x-a)^n\right)$.

$\to$ Strategy:

1) Write $y(x)=\sum_{n=0}^{\infty} a_n x^n$.
2) Write expression y' & y''.
3) Plug y, y' & y'' in the ODE.
4) Simplify in one power series
5) Find recurrence relation.
6) Find $a_2, a_3, a_4, a_5, a_6 \ldots$
7) With IC, find a_0 & a_1.