Math 302

Chapter 2

Section 2.6: Integrating Factors

Contents

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Fall 2022

Example 1. Verify if

$$\underbrace{(3x + 2y^3)}_{=M}dx + \underbrace{2xy}_{=N}dy = 0 \qquad (*)$$

is exact.

$M_y = 6y^2$ & $N_x = 2y$ $\Rightarrow$ $M_y \neq N_x$

$\Rightarrow$ Not exact.

There is something we can do!

Multiply by $\mu(x,y)$ $(*)$

$$\Rightarrow \underbrace{\mu(x,y)(3x+2y^3)}_{\text{New } M}dx + \underbrace{\mu(x,y)2xy}_{\text{New } N}dy = 0$$

Would like: $(\mu M)_y = (\mu N)_x \Leftrightarrow \mu_y(3x+2y^3) + \mu 6y^2 = \mu_x(2xy) + \mu(2y)$

Such a function is $\mu(x,y) = x$

&

$$x(3x+2y^3)dx + x(2xy)dy = 0$$

$$\Updownarrow$$

$$(3x^2+2xy^3)dx + 2x^2y\,dy = 0$$

is now exact!! :)

A function $\mu = \mu(x, y)$ is an **integrating factor** for

$$M(x, y)dx + N(x, y)dy = 0$$

if the equation

$$\mu(x, y)M(x, y)dx + \mu(x, y)N(x, y)dy = 0$$

is exact.

Let's start with the equation

$$\overbrace{\mu(x,y)M(x,y)}^{\text{New } M}dx + \overbrace{\mu(x,y)N(x,y)}^{\text{new } N}dy = 0. \tag{1}$$

Trick: To be exact, we must satisfy:

$$(\mu M)_y = (\mu N)_x$$

$$\Leftrightarrow \quad \mu_y M + \mu M_y = \mu_x N + \mu N_x$$

$$\Leftrightarrow \quad \mu(M_y - N_x) = \mu_x N - \mu_y M.$$

Suppose that $\mu(x,y) = P(x)\,Q(y) \rightarrow \begin{array}{l}\mu_x = P'(x)\,Q(y)\\ \mu_y = P(x)\,Q'(y)\end{array}$

$$\Rightarrow \quad PQ\,(M_y - N_x) = P'\,QN - PQ'\,M$$

$\div PQ$

$$\Rightarrow \quad M_y - N_x = \frac{P'(x)}{P(x)}N - \frac{Q'(y)}{Q(y)}M$$

- 1st case: $\div N$

$$\frac{M_y - N_x}{N} = \frac{P'(x)}{P(x)} - \frac{Q'(y)}{Q(y)}\frac{M}{N}$$

- 2nd case: $\div M$

$$\frac{M_y - N_x}{M} = \frac{P'(x)}{P(x)}\frac{N}{M} - \frac{Q'(y)}{Q(y)}$$

General Facts: Let M, N, M_y, N_x be continuous on an open rectangle R.

- if $(M_y - N_x)/N$ is independent of y, then

$$\mu(x,y) = \pm e^{\int p(x)\,dx}$$

is an integrating factor for (1) where $p(x) = (M_y - N_x)/N$.

- if $(N_x - M_y)/M$ is independent of x, then

$$\mu(x,y) = \pm e^{\int q(y)\,dy}$$

is an integrating factor for (1) where $q(y) = (N_x - M_y)/M$.

EXAMPLE 2. Find an integrating factor for the equation

$$\underbrace{(2xy^3 - 2x^3y^3 - 4xy^2 + 2x)}_{M}dx + \underbrace{(3x^2y^2 + 4y)}_{N}dy = 0.$$

$$M_y = 6xy^2 - 6x^3y^2 - 8xy \quad \rightarrow \quad M_y \neq N_x \rightarrow \text{not exact.}$$

$$N_x = 6xy^2$$

So,

$$p(x) = \frac{M_y - N_x}{N} = \frac{-6x^3y^2 - 8xy}{3x^2y^2 + 4y}$$

$$= -2x\left(\frac{\cancel{3x^2y^2 + 4y}}{\cancel{3x^2y^2 + 4y}}\right)$$

$$= -2x \quad \text{ind. of } x\,!$$

So, $$\mu(x,y) = \pm e^{\int -2x\,dx} = \pm e^{-x^2}$$

choose $+$

$$\Rightarrow \boxed{\mu(x,y) = e^{-x^2}} \quad \text{integrating factor.}$$

Example 3. Find an integrating factor for the equation

$$\underbrace{2xy^3}_{M}dx + (\underbrace{3x^2y^2 + x^2y^3 + 1}_{N})dy = 0$$

and solve the equation.

$M_y = 6xy^2$ & $N_x = 6xy^2 + 2xy^3 \Rightarrow$ Not exact!

- $\dfrac{M_y - N_x}{N} = \dfrac{6xy^2 - 6xy^2 - 2xy^3}{3x^2y^2 + x^2y^3 + 1} = \dfrac{-2xy^3}{3x^2y^2 + x^2y^3 + 1}$ (depends on x & y) $\times$

- $\dfrac{N_x - M_y}{M} = \dfrac{\cancel{6xy^2} + 2xy^3 - \cancel{6xy^2}}{2xy^3} = 1$ (ind. of x)

So $\mu(x,y) = \pm e^{\int 1 dy} = \pm e^y$

$\to \mu(x,y) = e^y$.

Solving ODE.

1) Mult. by μ: $e^y 2xy^3 dx + e^y(3x^2y^2 + x^2y^3 + 1)dy = 0$

$\hookrightarrow$ exact.

2) Find F:

$F_x = 2xy^3e^y$ & $F_y = (3x^2y^2 + x^2y^3 + 1)e^y$

(A) $F = x^2y^3e^y + g(y)$

(B) $F_y = \cancel{3x^2y^2e^y} + \cancel{x^2y^3e^y} + g'(y)$

$= (\cancel{3x^2y^2} + \cancel{x^2y^3} + 1)e^y$

$\Rightarrow g'(y) = e^y$

$\Rightarrow g(y) = e^y$ (don't need the const. c).

So, $F(x,y) = x^2 y^3 e^y + e^y = (x^2y^3 + 1)e^y$

Solution: $\boxed{(x^2y^3+1)e^y = c}$