Problem A

Find an expression of the following functions involving only unit step functions.

1) $f(t) = \begin{cases} 0 & t < 2 \\ (t-2)^2 & t > 2. \end{cases}$

2) $f(t) = \begin{cases} 0 & t < \pi \\ t - \pi & t \in [\pi, 2\pi] \\ 0 & t > 2\pi. \end{cases}$

Problem B

Find the Laplace transform of the following functions.

1) $f(t) := tu(t-1)$.
2) $f(t) := te^{2t}u(t-1)$.
3) $f(t) := (t^2-1)u(t-1)$.
4) $f(t) := t\sin tu(t-2\pi)$.
5) $f(t) := \cos tu(t-a)$.

Problem C

Find the inverse Laplace transform of the following functions.

1) $F(s) = \frac{3!}{(s-2)^4}$.
2) $F(s) = \frac{2(s-1)e^{-2s}}{s^2-2s+2}$.
3) $F(s) = \frac{(s-2)e^{-s}}{s^2-4s+3}$.
4) $F(s) = \frac{e^{-2s}}{s^2+s-2}$.
5) $F(s) = \frac{2e^{-2s}}{s^2-4}$.
6) $F(s) = \frac{e^{-s}+e^{-2s}-e^{-3s}-e^{-4s}}{s}$.

Complete Solutions

Problem A

1) The expression of f in terms of unit step functions is

$$(t-2)^2 u_2(t).$$

2) First of all, the expression of the function is $t-\pi$, when $t \geq \pi$. Therefore, the function should take the following form:

$$(t-\pi)u_\pi(t).$$

Second of all, the expression $t-\pi$ vanish when $t > 2\pi$. Therefore, the function should also have the following part

$$-(t-\pi)u_{2\pi}(t).$$

Therefore, we obtain

$$f(t) = (t-\pi)\left(u_\pi(t) - u_{2\pi}(t)\right).$$

Problem B

1) We rewrite the expression of the function as

$$tu(t-1) = (t-1)u(t-1) + u(t-1).$$

In this form, we can apply the Laplace transform directly. We find that

$$F(s) = e^{-s}L(t) + e^{-s}L(1) = \frac{e^{-s}}{s^2} + \frac{e^{-s}}{s}.$$

2) First of all, we have

$$te^{2t} = (t-1)e^{2t} + e^{2t} = e^2(t-1)e^{2(t-1)} + e^2 e^{2(t-1)}.$$

Therefore, the expression of the function becomes

$$f(t) = e^2(t-1)e^{2(t-1)}u(t-1) + e^2 e^{2(t-1)}u(t-1).$$

Also, since $L(te^{2t}) = \frac{1}{(s-2)^2}$ and $L(e^{2t}) = \frac{1}{s-2}$, applying the Laplace transform implies that

$$F(s) = \frac{e^2 e^{-s}}{(s-2)^2} + \frac{e^2 e^{-s}}{s-2} = \frac{e^{-(s-2)}}{(s-2)^2} + \frac{e^{-(s-2)}}{s-2}.$$

Finally, after simplifying the answer, we get

$$F(s) = e^{-(s-2)}\left(\frac{1}{(s-2)^2} + \frac{1}{s-2}\right).$$

3) The expression of the function is not in the form to apply the Laplace transform correctly. We therefore rewrite the expression in the following way:

$$\begin{aligned} f(t) = (t^2 + 2t - 2t + 1 - 1 - 1)u(t-1) &= (t^2 - 2t + 1)u(t-1) + (2t-2)u(t-1) \\ &= (t-1)^2 u(t-1) + 2(t-1)u(t-1). \end{aligned}$$

We can now apply the Laplace transform and then obtain

$$F(s) = \frac{2e^{-s}}{s^3} + \frac{2e^{-s}}{s^2}.$$

The simplified answer is

$$F(s) = 2e^{-s}\left(\frac{1}{s^3} + \frac{1}{s^2}\right).$$

4) We have to put the expression of the function in an appropriate form to apply the Laplace transform. The expression can be rewritten as followed:

$$f(t) = (t - 2\pi)\sin t u(t-2\pi) + 2\pi \sin t u(t - 2\pi).$$

From a trigonometry identity (or the periodicity of the sin function), we find that

$$\sin t = \sin(t - 2\pi)$$

and then we see that

$$f(t) = (t-2\pi)\sin(t-2\pi)u(t-2\pi) + 2\pi\sin(t-2\pi)u(t-2\pi).$$

We can now apply the Laplace transform to the last expression. Starting with the fact that $L(tg) = -G'$, we get $L(t\sin t) = \frac{2s}{(s^2+1)^2}$ et therefore

$$F(s) = \frac{2se^{-2\pi s}}{(s^2+1)^2} + \frac{2\pi e^{-2\pi s}}{s^2+1}.$$

The simplified answer is

$$F(s) = 2e^{-2\pi s}\left(\frac{s}{(s^2+1)^2} + \frac{\pi}{s^2+1}\right).$$

5) From a trigonometry identity, we have

$$\cos(t) = \cos(t - a + a) = \cos(t-a)\cos(a) - \sin(t-a)\sin(a).$$

Therefore, the expression of the function can be rewritten as

$$f(t) = \cos(a)\cos(t-a)u(t-a) - \sin(a)\sin(t-a)u(t-a).$$

We can now apply the Laplace transform and get

$$F(s) = \cos(a)\frac{se^{-sa}}{s^2+1} - \sin(a)\frac{e^{-sa}}{s^2+1}.$$

Therefore, we get

$$F(s) = \frac{e^{-sa}}{s^2+1}\left(s\cos a - \sin a\right).$$

Problem C

In this problem, to simplify the notation, we use the following shortcuts:

$$f_a(t) = f(t-a) \quad \text{and} \quad u_a(t) = u(t-a).$$

Therefore, when you see f_a and/or u_a in the text, these refer to the above.

1) From the table, we immediately have

$$f(t) = t^3 e^{2t}.$$

2) The denominator can be rewritten as $(s-1)^2+1$. Then, from the table, we get

$$L^{-1}\left(\frac{s-1}{(s-1)^2+1}\right) = e^t \cos t.$$

Since the expression of F contained an e^{-2s}, we find that

$$f(t) = 2e^{t-2}\cos(t-2)u(t-2)$$

because $L^{-1}(Fe^{-as}) = f_a u_a$.

3) The denominator can be factored:

$$s^2 - 4s + 3 = (s-2)^2 - 1.$$

Then from the table, we get

$$L^{-1}\left(\frac{s-2}{(s-2)^2-1}\right) = e^{2t}\cosh(t).$$

Finaly, since the expression of F contained an e^{-s}, we find that

$$f(t) = e^{2(t-1)}\cosh(t-1)u(t-1)$$

because $L^{-1}(Fe^{-as}) = f_a u_a$.

4) The denominator can be rewritten as

$$s^2 + s - 2 = (s+2)(s-1)$$

and then the expression of F becomes

$$F(s) = \frac{e^{-2s}}{3(s-1)} - \frac{e^{-2s}}{3(s+2)}$$

after using a partial fractions decomposition. Therefore, applying the inverse Laplace transform and keeping in mind that $L^{-1}(Fe^{-as}) = f_a u_a$, we find that

$$f(t) = \tfrac{1}{3}e^{t-2}u(t-2) - \tfrac{1}{3}e^{-2(t-2)}u(t-2).$$

5) Using a partial fractions decomposition, we can rewrite F as followed:

$$F(s) = \left(\frac{1}{4(s-2)} - \frac{1}{4(s+2)}\right) 2e^{-2s}.$$

Applying the inverse Laplace transform and keeping in mind that $L^{-1}(Fe^{-as}) = f_a u_a$, we find that

$$f(t) = \left(\frac{1}{2}e^{2(t-2)} - \frac{1}{2}e^{-2(t-2)}\right) u(t-2).$$

This solution is also equivalent to:

$$f(t) = \sinh(2(t-2))u(t-2)$$

because $\sinh(x) = \frac{e^x - e^{-x}}{2}$.

Notice that we could use the table directly. Indeed, we know that $L(\sinh(2t)) = \frac{2}{s^2-4}$. Therefore, since $L(g(t-a)u(t-a)) = e^{-as}G(s)$, we find that

$$f(t) = \sinh(2(t-2))u(t-2).$$

6) Applying directly the fact that $L^{-1}\left(\frac{e^{-as}}{s}\right) = U_a$, we find that

$$f(t) = u(t-1) + u(t-2) - u(t-3) - u(t-4).$$