Math 302

Chapter 5

Section 5.6: Reduction Of Order

Contents

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What Is Reduction Of Order

We study the ODE

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x).$$

where $P_0(x)$, $P_1(x)$, $P_2(x)$, $F(x)$ are continuous functions in the variable x.

Goal: Find the general solutions to the ODE above.

Trick:

- Have a solution to the complementary equation.
- Use variation of parameter.

Example 1. Find the general solution of

$$xy'' - (2x+1)y' + (x+1)y = x^2$$

given that $y_1(x) = e^x$ is a solution to the complementary equation.

1) Use Var. of param.

$$y(x) = u(x)\, y_1(x) = u(x)\, e^x$$

So
$$y' = u'e^x + ue^x$$
$$y'' = u''e^x + 2u'e^x + ue^x$$

Replace in the ODE

$$\Rightarrow x\left(u''e^x + 2u'e^x + ue^x\right) - (2x+1)\left(u'e^x + ue^x\right) + (x+1)ue^x = x^2$$

$$\Rightarrow xe^x u'' + \cancel{2xe^x u'} + \cancel{xe^x u} - \cancel{2xe^x u'} - \cancel{2xe^x u} - u'e^x - \cancel{ue^x} + \cancel{xue^x} + \cancel{e^x u} = x^2$$

$$\Rightarrow xe^x u'' - e^x u' = x^2$$

2) Change the order!

Let $z = u' \longrightarrow z' = u''$

So,
$$xe^x z' - e^x z = x^2$$

(I) Compl. Eq.

$$xe^x z' - e^x z = 0 \Rightarrow \frac{z'}{z} = \frac{1}{x}$$

$$\Rightarrow \ln|z| = \ln|x| + k$$

$$\Rightarrow z = Kx \qquad (K = \pm e^k)$$

(II) Var. of Param.

$$z(x) = v(x)\, x \rightarrow z' = v'x + v$$

$$\Rightarrow xe^x v' x + \cancel{xe^x v} - \cancel{e^x v x} = x^2$$

$$\Rightarrow v' = e^{-x}$$

$$\Rightarrow v(x) = -e^{-x} + c_1$$

So, $$z(x) = -xe^{-x} + c_1 x$$

3) Integrate

$$z = u' \Rightarrow u' = -xe^{-x} + c_1 x$$

$$\Rightarrow u(x) = (x-1)e^{-x} + c_1\frac{x^2}{2} + c_2$$

4) Replace in y

$$\boxed{y = ue^x = (x-1) + \frac{c_1 e^x x^2}{2} + c_2 e^x}$$

Example 2. Find the general solution of

$$x^2y'' + xy' - y = x^2 + 1$$

given that $y_1(x) = x$ is a solution to the complementary equation.

1) Var. of Param.

$$y(x) = u(x) \cdot x \Rightarrow \begin{array}{l} y' = u'x + u \\ y'' = u''x + 2u' \end{array}$$

Replace in the ODE:

$$x^2(u''x + 2u') + x(u'x + u) - ux = x^2 + 1$$

$$\Rightarrow x^3 u'' + 2x^2u' + x^2u' + \cancel{xu} - \cancel{ux} = x^2+1$$

$$\Rightarrow x^3 u'' + 3x^2 u' = x^2 + 1$$

2) Reduction of order:

Let $z = u' \Rightarrow x^3 z' + 3x^2 z = x^2 + 1$

I. Compl. Eq.

$$x^3 z' + 3x^2 z = 0$$

$$\Rightarrow \frac{z'}{z} = -\frac{3}{x}$$

$$\Rightarrow \ln|z| = -3\ln|x|$$

$$\Rightarrow z = \frac{K}{x^3} \rightarrow z_1 = \frac{1}{x^3}$$

II. Var. Param.

$$z = \frac{v}{x^3} \Rightarrow z' = \frac{v'}{x^3} - \frac{3v}{x^4}$$

$$\Rightarrow \quad x^3\left(\frac{v'}{x^3} - \frac{3v}{x^4}\right) + \frac{3x^2 v}{x^3} = x^2+1$$

$$\Rightarrow \quad v' - \cancel{\frac{3v}{x}} + \cancel{\frac{3v}{x}} = x^2+1$$

$$\Rightarrow \quad v' = x^2+1 \quad \Rightarrow \quad v(x) = \frac{x^3}{3} + x + c_1$$

So, $\quad z = \dfrac{1}{3} + \dfrac{1}{x^2} + \dfrac{c_1}{x^3}$

3) Integrate

$$z = u' \quad \Rightarrow \quad u = \frac{x}{3} - \frac{1}{x} - \frac{c_1}{2x^2} + c_2$$

4) Replace in y

$$\boxed{y(x) = u \cdot x = \frac{x^2}{3} - 1 - \frac{c_1}{2x} + c_2 x}$$