Math 302

Chapter 8

Section 8.2: Laplace Transforms

Contents

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From ODE to Algebra

The Laplace transform is REALLY$^{\text{REALLY}}$ useful to solve ODE.

Example 1. Consider the ODE

$$2y''(t) + 3y'(t) + y(t) = 8e^{-2t}$$

with $y(0) = -4$ and $y'(0) = 2$.

$$Y = \mathcal{L}(y(t)) \Rightarrow \quad \begin{aligned} \mathcal{L}(y') &= sY - y(0) = sY + 4 \\ \mathcal{L}(y'') &= s^2 Y - s f(0) - f'(0) = s^2 + 4s - 2 \\ \mathcal{L}(8e^{-2t}) &= \frac{8}{s+2} \end{aligned}$$

<u>Apply $\mathcal{L}$ on ODE:</u>

$$\mathcal{L}(2y'' + 3y' + y) = \mathcal{L}(8e^{-2t})$$

$$\Rightarrow 2\mathcal{L}(y'') + 3\mathcal{L}(y') + \mathcal{L}(y) = \frac{8}{s+2}$$

$$\Rightarrow 2s^2Y + 8s - 4 + 3sY + 6 + Y = \frac{8}{s+2}$$

$$\Rightarrow (2s^2 + 3s + 1)Y + 8s + 2 = \frac{8}{s+2}$$

$$\Rightarrow (2s+1)(s+1)Y = \frac{8}{s+2} - (8s+2)$$

$$\Rightarrow (2s+1)(s+1)Y = \frac{8 - (8s+2)(s+2)}{s+2}$$

$$\Rightarrow \quad Y(s) = \frac{8 - (8s+2)(s+2)}{(2s+1)(s+1)(s+2)}$$

How do we find $y(t)$??

General Procedure:

1. Apply the Laplace transform to your ODE $ay'' + by' + cy = f(t)$.
2. Apply the properties of the Laplace transform to get
$$a(s^2Y - sf(0) - f'(0)) + b(sY - f(0)) + cY = F.$$
3. Isolate Y:
$$Y = \frac{F + (as+b)f(0) + af'(0)}{as^2 + bs + c}.$$

The last step:

- Take the inverse Laplace transform!

Given a Laplace transform $F(s)$ of an unknown function f, we can go backward to find f.

- We denote the **inverse Laplace transform** by L^{-1}.
- We therefore have $f = L^{-1}(F)$.
- How do we find $L^{-1}(F)$?

Trick: Use the table in the opposite direction!

EXAMPLE 2. Find the inverse Laplace transform of the following functions:

(a) $\dfrac{1}{s^2-1}$.

(b) $\dfrac{s}{s^2+9}$.

(a) From the table : $L(\sinh(t)) = \dfrac{1}{s^2-1}$

$\Rightarrow L^{-1}\left(\dfrac{1}{s^2-1}\right) = \sinh(t)$.

(b) $\dfrac{s}{s^2+9} = \dfrac{s}{s^2+3^2}$

From the table: $L(\cosh(3t)) = \dfrac{s}{s^2+3^2}$

$\Rightarrow L^{-1}\left(\dfrac{s}{s^2+9}\right) = \cosh(3t)$.

Linearity of Inverse Transform

If F and G are Laplace transforms of two unknown functions f and g, then

$$L^{-1}(aF + bG) = aL^{-1}(F) + bL^{-1}(G).$$

Example 3. Find

$$L^{-1}\Big(\overbrace{\frac{8}{s+5} + \frac{7}{s^2+3}}^{H(s)}\Big).$$

Linearity:

$$L^{-1}(H) = 8\,L^{-1}\left(\frac{1}{s+5}\right) + 7\,L^{-1}\left(\frac{1}{s^2+3}\right)$$

table:

$$L^{-1}\left(\frac{1}{s+5}\right) = e^{-5t}$$

&

$$L^{-1}\left(\frac{1}{s^2+3}\right) = L^{-1}\left(\frac{\sqrt{3}/\sqrt{3}}{s^2+(\sqrt{3})^2}\right)$$

$$= \frac{1}{\sqrt{3}}\,L^{-1}\left(\frac{\sqrt{3}}{s^2+(\sqrt{3})^2}\right)$$

$$= \frac{\sin(\sqrt{3}\,t)}{\sqrt{3}}$$

Therefore:

$$L^{-1}(H) = 8e^{-5t} + \frac{7}{\sqrt{3}}\sin(\sqrt{3}\,t).$$

Example 4. Find

$$L^{-1}\Big(\frac{3s+8}{s^2+2s+5}\Big).$$

Notice:

$$s^2 + 2s + 5 = s + 2s + 1 + 4 = (s+1)^2 + 2^2$$

From the table:

set $a = -1$, $\omega = 2$

$$e^{at}\cos(\omega t) \longleftrightarrow \frac{s-a}{(s-a)^2+\omega^2}$$

$$e^{at}\sin(\omega t) \longleftrightarrow \frac{\omega}{(s-a)^2+\omega^2}$$

$$e^{at}\cosh(\omega t) \longleftrightarrow \frac{s-a}{(s-a)^2-\omega^2}$$

$$e^{at}\sinh(\omega t) \longleftrightarrow \frac{\omega}{(s-a)^2-\omega^2}$$

Reshape the numerator:

$$3s + 8 = 3s + 3 - 3 + 8 = 3(s+1) + 5$$

$$\Rightarrow \quad F(s) = \frac{3(s+1)+5}{(s+1)^2 + 2^2}$$

$$= \frac{3(s+1)}{(s+1)^2 + 2^2} + \frac{5}{(s+1)^2 + 2^2}$$

Linearity

$$\Rightarrow \quad L^{-1}(F) = 3L^{-1}\left(\frac{s+1}{(s+1)^3 + 2^2}\right) + \frac{5}{2} L^{-1}\left(\frac{2}{(s+1)^2 + 2^2}\right)$$

$$= 3e^{-t}\cos(2t) + \frac{5}{2}e^{-t}\sin(2t).$$

Inverse Laplace Transform of Rational Functions

Example 5. Find the inverse Laplace transform of

$$F(s) = \frac{3s+2}{s^2 - 3s + 2}.$$

We could write $s^2 - 3s + 2 = \left(s - 3/2\right)^2 + \frac{13}{2}$.

See another way:

<u>Partial Fractions:</u> $s^2 - 3s + 2 = (s-2)(s-1)$

$$F(s) = \frac{A}{s-2} + \frac{B}{s-1} = \frac{3s+2}{s^2-3s+2}$$

$$\Rightarrow \quad \frac{A(s-1) + B(s-2)}{(s-2)(s-1)} = \frac{3s+2}{s^2-3s+2}$$

$$\Leftrightarrow \quad A(s-1) + B(s-2) = 3s+2$$

Setting $s=2 \Rightarrow A = 6+2 = 8$

setting $s=1 \Rightarrow B = -5$

$$\Rightarrow \quad F(s) = \frac{8}{s-2} - \frac{5}{s-1}$$

Applying L^{-1}:

$$L^{-1}(F) = 8\, L^{-1}\left(\frac{1}{s-2}\right) - 5\, L^{-1}\left(\frac{1}{s-1}\right)$$

$$= 8e^{2t} - 5e^{t}.$$

Example 6. Find the inverse transform of

$$F(s) = \frac{6 + (s+1)(s^2 - 5s + 11)}{s(s-1)(s-2)(s+1)}.$$

$$F(s) = \frac{A_1}{s} + \frac{A_2}{s-1} + \frac{A_3}{s-2} + \frac{A_4}{s+1}$$

Heaviside method:

A_1: Forget s in $F(s)$ & set $s = 0$

$$\Rightarrow A_1 = \frac{6 + (0+1)(0^2 - 0 + 11)}{(0-1)(0-2)(0+1)} = \frac{17}{2}$$

A_2: Forget $s-1$ in $F(s)$ & set $s=1$

$$\Rightarrow A_2 = \frac{6 + (1+1)(1-5+11)}{1 \cdot (1-2)(1+1)} = -10$$

A_3 Forget $s-2$ in $F(s)$ & set $s=2$

$$\Rightarrow A_3 = \frac{6 + (2+1)(4-10+11)}{2 \cdot (2-1)(2+1)} = \frac{7}{2}$$

A_4: Forget $s+1$ in $F(s)$ & set $s=-1$

$$\Rightarrow A_4 = \frac{6 + (-1+1)(1-5+11)}{(-1)(-2)(-3)} = -1$$

therefore,

$$F(s) = \frac{17/2}{s} - \frac{10}{s-1} + \frac{7/2}{s-2} - \frac{1}{s+1}$$

Apply $\mathcal{L}^{-1}$:

$$\mathcal{L}^{-1}(F) = \frac{17}{2}\mathcal{L}^{-1}\left(\frac{1}{s}\right) - 10\,\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) + \frac{7}{2}\mathcal{L}^{-1}\left(\frac{1}{s-2}\right) - \mathcal{L}^{-1}\left(\frac{1}{s+1}\right)$$

$$= \frac{17}{2} - 10e^{t} + \frac{7}{2}e^{2t} - e^{-t} .$$

General Case:
Suppose your Laplace transform is

$$F(s) = \frac{P(s)}{(s-s_1)(s-s_2)\cdots(s-s_n)}$$

where s_1, s_2, …, s_n are distinct and P is a polynomial of degree less than n. Then

$$F(s) = \frac{A_1}{s-s_1} + \frac{A_2}{s-s_2} + \cdots + \frac{A_n}{s-s_n},$$

- A_1 is computed by letting $s = s_1$ in $G(s) = \dfrac{P(s)}{(s-s_2)\cdots(s-s_n)}$.
- A_2 is computed by letting $s = s_2$ in $G(s) = \dfrac{P(s)}{(s-s_1)(s-s_3)\cdots(s-s_n)}$.
- $\vdots$
- A_n is computed by letting $s = s_n$ in $G(s) = \dfrac{P(s)}{(s-s_1)(s-s_2)\cdots(s-s_{n-1})}$.

Example 7. Write in partial fractions of the following rational function:

$$F(s) = \frac{s^2+1}{s^3-s}.$$

Write: $s^3 - s = s(s^2-1) = s(s-1)(s+1)$.

Seek: $F(s) = \dfrac{A_1}{s} + \dfrac{A_2}{s-1} + \dfrac{A_3}{s+1}$

$\underline{A_1} = \dfrac{0^2+1}{(0-1)(0+1)} = -1$

$\underline{A_2} = \dfrac{1^2+1}{1\cdot(1+1)} = 1$

$\underline{A_3} = \dfrac{(-1)^2+1}{(-1)(-2)} = 1$

$\Rightarrow F(s) = \dfrac{-1}{s} + \dfrac{1}{s-1} + \dfrac{1}{s+1}$.

Rational Functions with Powers in the Denominator

The Heaviside method doesn't work if we encounters powers of monomonials in the denominator. What do we do then?

EXAMPLE 8. Find the partial fraction expansion of

$$F(s) = \frac{8 - (s+2)(4s+10)}{(s+1)(s+2)^2}.$$

Seek: $F(s) = \dfrac{A}{s+1} + \dfrac{B}{s+2} + \dfrac{C}{(s+2)^2}$

Put common denominator but don't simplify!

$$\frac{A(s+2)^2 + B(s+1)(s+2) + C(s+1)}{(s+1)(s+2)^2} = \frac{8 - (s+2)(4s+10)}{(s+1)(s+2)^2}$$

$$\Leftrightarrow A(s+2)^2 + B(s+1)(s+2) + C(s+1) = 8 - (s+2)(4s+10)$$

Set $s=-1$ $\quad A(-1+2)^2 = 8 - (-1+2)(-4+10)$

$\Rightarrow A = 2$

Set $s=-2$ $\quad C(-2+1) = 8$

$\Rightarrow C = -8$

Set $s=0$ $\Rightarrow 2 \cdot 4 + B \cdot 1 \cdot 2 + (-8) = -12$

$\Rightarrow B = -6$

Therefore $F(s) = \dfrac{2}{s+1} - \dfrac{6}{s+2} - \dfrac{8}{(s+2)^2}$

EXAMPLE 9. Using the inverse Laplace transform, complete Example 1.

We had

$$Y(s) = \frac{8 - (8s+2)(s+2)}{(2s+1)(s+1)(s+2)} = \frac{\left(\frac{8-(8s+2)(s+2)}{2}\right)}{(s+\frac{1}{2})(s+1)(s+2)}$$

Partial decomposition:

$$Y = \frac{A_1}{s+\frac{1}{2}} + \frac{A_2}{s+1} + \frac{A_3}{s+2}$$

$\underline{s=-1/2}$ $\quad A_1 = \frac{22}{3}$ $\qquad \underline{s=-1}$ $\quad A_2 = -14$ $\qquad \underline{s=-2}$ $\quad A_3 = \frac{8}{3}$

$$\Rightarrow \quad Y(s) = \frac{\frac{22}{3}}{s+1/2} - \frac{14}{s+1} + \frac{8/3}{s+2}$$

Applying the inverse L^{-1} :

$$y(t) = L^{-1}(Y) = \frac{22}{3} L^{-1}\left(\frac{1}{s+1/2}\right) - 14 L^{-1}\left(\frac{1}{s+1}\right) + \frac{8}{3} L^{-1}\left(\frac{1}{s+2}\right)$$

$$= \frac{22}{3} e^{-t/2} - 14 e^{-t} + \frac{8}{3} e^{-2t}$$

Answer:

$$y(t) = \frac{22}{3} e^{-t/2} - 14 e^{-t} + \frac{8}{3} e^{-2t}.$$

Example 10. Use the Laplace transform to solve the initial value problem:

$$y'' - 6y' + 5y = 3e^{2t}, \quad y(0) = 2, \; y'(0) = 3.$$

① Apply $\mathcal{L}$ $\qquad Y = \mathcal{L}(y)$

$$\mathcal{L}(y'') - 6\mathcal{L}(y') + 5\mathcal{L}(y) = 3\mathcal{L}(e^{2t})$$

$$\Rightarrow s^2 Y - f(0)s - f'(0) - 6(sY - f(0)) + 5Y = \frac{3}{s-2}$$

$$\Rightarrow s^2 Y - 2s - 3 - 6(sY - 2) + 5Y = \frac{3}{s-2}$$

$$\Rightarrow (s^2 - 6s + 5)Y - 2s + 9 = \frac{3}{s-2}$$

$$\Rightarrow (s-5)(s-1)Y = 2s - 9 + \frac{3}{s-2}$$

$$\Rightarrow (s-5)(s-1)Y = \frac{(2s-9)(s-2) + 3}{s-2}$$

$$\Rightarrow Y(s) = \frac{(9-2s)(s-2) + 3}{(s-5)(s-1)(s-2)}.$$

② Simplify Expression of Y

Partial fractions:

$$Y(s) = \frac{A_1}{s-5} + \frac{A_2}{s-1} + \frac{A_3}{s-2}$$

$s=5$: $A_1 = \frac{1}{2}$ $\qquad$ $s=1$: $A_2 = \frac{5}{2}$ $\qquad$ $s=2$: $A_3 = -1$

Therefore

$$Y(s) = \frac{1/2}{s-5} + \frac{5/2}{s-1} - \frac{1}{s-2}$$

③ Inverse Laplace Transform

$$y(t) = \frac{1}{2} L^{-1}\left(\frac{1}{s-5}\right) + \frac{5}{2} L^{-1}\left(\frac{1}{s-1}\right) - L^{-1}\left(\frac{1}{s-2}\right)$$

$$= \boxed{\frac{1}{2} e^{5t} + \frac{5}{2} e^{t} - e^{2t}}.$$