Math 302

Chapter 5

Section 5.5: The Method of Undetermined Coefficient II

Contents

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When The Force Function Is A Trig. Function

We consider the following first basic case:

$$ay'' + by' + cy = F\cos\omega x + G\sin\omega x$$

where F, G and α are fixed real numbers.

Case I

When $\cos\omega x$ and $\sin\omega x$ are not solution to the complementary equation $ay'' + by' + cy = 0$.

Example 1. Find the general solution to

$$y'' - 2y' + y = 5\cos 2x + 10\sin 2x.$$

1) Comp. Eq.

$y'' - 2y' + y = 0 \to r^2 - 2r + 1 = (r-1)^2$

$\to$ root is 1 (repeated)

$\to y_1 = e^x$ & $y_2 = xe^x$

2) Guess y_{par}.

$y_{par}(x) = A\cos 2x + B\sin 2x$

$\Rightarrow y' = -2A\sin 2x + 2B\cos 2x$

& $y'' = -4A\cos 2x - 4B\sin 2x$

Replace in ODE:

$$-4A\cancel{\cos}2x - 4B\cancel{\sin}2x + 4A\cancel{\sin}2x - 4B\cancel{\cos}2x + A\cancel{\cos}2x + B\cancel{\sin}2x = 5\cos 2x + 10\sin 2x$$

$\Rightarrow (-3A - 4B)\cos 2x + (-3B + 4A)\sin 2x = 5\cos 2x + 10\sin 2x$

$\Rightarrow \underset{(I)}{-3A - 4B = 5}$ & $\underset{(II)}{-3B + 4A = 10}$

$$(\mathrm{I}) + (\mathrm{II}) \longrightarrow \begin{array}{r} -3A - 4B = 5 \\ + \quad 4A - 3B = 10 \\ \hline A - 7B = 15 \end{array} \longrightarrow A = 15 + 7B$$

Replace in $(\mathrm{I}) \longrightarrow -45 - 21B - 4B = 15$

$\longrightarrow -25B = 50 \longrightarrow B = -2$

$\longrightarrow A = 1$

So $y_{par}(x) = \cos 2x - 2\sin 2x$

3) General sol.

$$\boxed{y(x) = \cos 2x - 2\sin 2x + (c_1 + c_2 x)e^{x}}$$

Case II

When $\cos \omega x$ or $\sin \omega x$ are solutions to the complementary equation.

EXAMPLE 2. Find the general solution to

$$y'' + 4y = 8\cos 2x + 12\sin 2x.$$

Same !

1) Compl. Eq.

$y'' + 4y = 0 \rightarrow r^2 + 4 \rightarrow$ roots are $\pm 2i$

$\rightarrow \underbrace{y_1 = \cos 2x \ \& \ y_2 = \sin 2x}_{\text{oops}}$

2) Guess y_{par}.

$y_{par}(x) = x\,(A\cos 2x + B\sin 2x)$

$\Rightarrow \quad y' = A\cos 2x + B\sin 2x + x(-2A\sin 2x + 2B\cos 2x)$

$\& \quad y'' = -2A\sin 2x + 2B\cos 2x + (-2A\sin 2x + 2B\cos 2x)$

$+ x(-4A\cos 2x - 4B\sin 2x)$

Replace in the EDO :

$-2A\cancel{\sin} 2x + 2B\cancel{\cos} 2x - 2A\cancel{\sin} 2x + 2B\cancel{\cos} 2x$

$- \cancel{4Ax\cos 2x} - \cancel{4Bx\sin 2x} + \cancel{4Ax\cos 2x} + \cancel{4Bx\sin 2x}$

$= 8\cos 2x + 12\sin 2x$

$\Rightarrow \quad 4B\cos 2x + (-4A)\sin 2x = 8\cos 2x + 12\sin 2x$

$\Rightarrow \quad 4B = 8 \ \& \ -4A = 12 \Rightarrow B = 2 \ \& \ A = -3$

$\Rightarrow \quad y_{par}(x) = -3x\cos 2x + 2x\sin 2x$

3) <u>General sol.</u>

$$y(x) = -3x\cos 2x + 2x\sin 2x + c_1\cos 2x + c_2\sin 2x$$

$$= \boxed{(-3x+c_1)\cos 2x + (2x+c_2)\sin 2x}$$

We consider the following second basic case:

$$ay'' + by' + cy = F(x)\cos\omega x + G(x)\sin\omega x$$

where ω is a fixed real number and F, G are two polynomials.

There are still two cases: weither $\cos\omega x$ and $\sin\omega x$ are or are not solutions to the complementary equation.

Example 3. Find the general solution to

$$y'' + 3y' + 2y = \underbrace{(16 + 20x)}_{\text{poly degré 1}}\cos x + \underbrace{10}_{\text{poly degree 0.}}\sin x.$$

1) <u>Compl. Eq.</u>

$$y'' + 3y' + 2y = 0 \to r^2 + 3r + 2 = 0$$

$$\to (r+2)(r+1) = 0$$

$\to$ roots are $r = -2$ & $r = -1$.

So, $y_1 = e^{-2x}$ & $y_2 = e^{-x}$.

2) <u>Guess y_{par}.</u>

$$y_{par}(x) = (Ax+B)\cos x + (Cx+D)\sin x$$

$$\Rightarrow \quad y' = A\cos 2x - (Ax+B)\sin 2x + C\sin 2x + ((x+D)\cos 2x$$

$$\& \quad y'' = -A\sin 2x - A\sin 2x - (Ax+B)\sin 2x$$
$$+ C\cos 2x - (Cx+D)\sin 2x + C\cos 2x$$

Replace in the EDO & simplify:

$$[B + 3A + 3D + 2C + (A+3C)x]\cos x$$

$$+ [D + 3C - 3B - 2A + (C-3A)x]\sin x = (16+20x)\cos x + 10\sin x$$

$$\Rightarrow \begin{cases} B + 3A + 3D + 2C = 16 & (\mathrm{I}) \\ A + 3C = 20 & (\mathrm{II}) \\ D + 3C - 3B - 2A = 10 & (\mathrm{III}) \\ C - 3A = 0 & (\mathrm{IV}) \end{cases}$$

$$\begin{array}{l} 3(\mathrm{II}) \\ + (\mathrm{IV}) \end{array} \rightarrow + \begin{array}{r} \cancel{3A} + 9C = 60 \\ C \cancel{-3A} = 0 \\ \hline 10C = 60 \end{array}$$

$$\Rightarrow C = 6$$

$$\Rightarrow A = 2$$

Replacing in I & III

$$\Rightarrow \begin{cases} B + 6 + 3D + 12 = 16 \\ D + 18 - 3B - 4 = 10 \end{cases}$$

$$\Rightarrow \begin{cases} B + 3D = -2 & (\mathrm{I}') \\ -3B + D = -4 & (\mathrm{II}'') \end{cases}$$

$$\begin{array}{l} 3(\mathrm{I}') \\ + (\mathrm{II}'') \end{array} \rightarrow + \begin{array}{r} \cancel{3B} + 9D = -6 \\ \cancel{-3B} + D = -4 \\ \hline 10D = -10 \end{array}$$

$$\Rightarrow D = -1$$

$$\Rightarrow B = 1$$

Therefore

$$y_{par}(x) = (2x+1)\cos x + (6x-1)\sin x$$

3) <u>General Sol.</u>

$$\boxed{y(x) = (2x+1)\cos x + (6x-1)\sin x + c_1 e^{-2x} + c_2 e^{-x}}.$$

We now consider the more general case

$$ay'' + by' + c = e^{\alpha x}\left(F(x)\cos\omega x + G(x)\sin\omega x\right)$$

where α, ω are real numbers with $\omega \neq 0$ and F, G are polynomials.

There are also two cases: weither $e^{\alpha x}\cos\omega x$ and/or $e^{\alpha x}\sin\omega x$ are or are not solutions to the complementary equation.

Example 4. Find the general solution of

$$y'' + 2y' + 5y = e^{-x}\left((6-16x)\cos 2x - (8+8x)\sin 2x\right).$$

1) Compl. eq.

$$y'' + 2y' - 5y = 0 \longrightarrow r^2 + 2r + 5 = 0$$

$$\longrightarrow r = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$$

So, $\underbrace{y_1 = e^{-x}\cos 2x \ \ \& \ \ y_2 = e^{-x}\sin 2x}_{\text{oops...}}$

2) Guess y_{par}.

Var. of Param: $y_{par}(x) = u(x)\,e^{-x}$

$$\longrightarrow y' = u'e^{-x} - ue^{-x} \quad \& \quad y'' = u''e^{-x} - 2u'e^{-x} + ue^{-x}$$

Replace in the EDO:

$$u''\cancel{e^{-x}} - \cancel{2u'e^{-x}} + u\cancel{e^{-x}} + \cancel{2u'e^{-x}} - 2u\cancel{e^{-x}} + 5u\cancel{e^{-x}}$$

$$= \cancel{e^{-x}}\left((6-16x)\cos 2x - (8-8x)\sin 2x\right)$$

$$\Rightarrow u'' + 4u = (6-16x)\cos 2x - (8-8x)\sin 2x$$

$\cos 2x$ & $\sin 2x$ are solutions to $u'' + 4u = 0$.

So we guess $u = x\left((Ax+B)\cos 2x + (Cx+D)\sin 2x\right)$

$$= (Ax^2+Bx)\cos 2x + (Cx^2+Dx)\sin 2x$$

We compute u'' & replace it in the EDO:

$$(2A + 4D + 8Cx)\cos 2x + (2C - 4B - 8Ax)\sin 2x$$
$$= (6-16x)\cos 2x - (8+8x)\sin 2x$$

$$\Rightarrow \begin{cases} 2A+4D=6 \\ 8C=-16 \\ 2C-4B=-8 \\ -8A=-8 \end{cases} \Rightarrow \begin{cases} 2+4D=6 \\ C=-2 \\ -4-4B=-8 \\ A=-1 \end{cases}$$

$$\Rightarrow \begin{cases} D=1 \\ C=-2 \\ B=1 \\ A=1 \end{cases}$$

So, $y_{par}(x) = e^{-x}x\left[(x+1)\cos 2x + (-2x+1)\sin 2x\right]$

3) <u>General Solution</u>

$$\boxed{y(x) = xe^{-x}\left[(x+1)\cos 2x + (1-2x)\sin 2x\right] + c_1e^{-x}\cos 2x + c_2e^{-2x}\sin 2x}$$

Recap

A particular solution of

$$ay'' + by' + cy = e^{\alpha x} \left(F(x) \cos \omega x + G(x) \sin \omega x\right)$$

where $\omega \neq 0$ has the form

- when $e^{\alpha x} \cos \omega x$ and $e^{\alpha x} \sin \omega x$ are not solutions to the complementary equation,

$$y_{par}(x) = e^{\alpha x} \left(A(x) \cos \omega x + B(x) \sin \omega x\right),$$

  with $A(x)$ and $B(x)$ are polynomials of the same degree as the biggest degree between $F(x)$ and $G(x)$

- When $e^{\alpha x} \cos \omega x$ and $e^{\alpha x} \sin \omega x$ are solutions to the complementary equation,

$$y_{par}(x) = xe^{\alpha x} \left(A(x) \cos \omega x + B(x) \sin \omega x\right),$$

  with $A(x)$ and $B(x)$ are polynomials of the same degree as the highest degree between the polynomials $F(x)$ and $G(x)$.

or let $y_{par} = u e^{\alpha x}$.

The

$$ay'' + by' + cy = au'' e^{\alpha x} + (a\alpha^2 + b\alpha + c) u e^{\alpha x}$$

So, find u s.t.

$$au'' + (a\alpha^2 + b\alpha + c) u = F(x) \cos(\omega x) + G(x) \sin(\omega x).$$

Guess: $u(x) = x \left(A(x) \cos(\omega x) + B(x) \sin(\omega x)\right)$

where A & B are polynomials.

$$y_{par}' = u'e^{\alpha x} + \alpha u e^{\alpha x}$$

$$y_{par}'' = u''e^{\alpha x} + \alpha u'e^{\alpha x} + \alpha u'e^{\alpha x} + \alpha^2 u e^{\alpha x}$$

$$= u'' e^{\alpha x} + 2\alpha u'e^{\alpha x} + \alpha^2 u e^{\alpha x}$$

$$au''e^{\alpha x} + 2a\alpha u'e^{\alpha x} + a\alpha^2 e^{\alpha x}$$

$$+ bu'e^{\alpha x} + b\alpha u e^{\alpha x} + cue^{\alpha x}$$

$\alpha + \beta i \quad \alpha - \beta i$
$= \alpha^2 + \beta^2$

$$= au'' e^{\alpha x} + \underbrace{(2a\alpha + b)}_{(*)} u'e^{\alpha x} + (a\alpha^2 + b\alpha + c)ue^{\alpha x}$$

$(*)$

$$ar^2 + br + c = a(r^2 + \tfrac{b}{a}r + c/a) = a(r - \alpha - \beta i)(r - \alpha + \beta i) = ar^2 - 2\alpha ar + a(\alpha^2 + \beta^2)$$

$$\Rightarrow \quad b = -2a\alpha$$

$$\Rightarrow \quad 2a\alpha + b = 0$$

So, $ay'' + by' + cy = au'' e^{\alpha x} + (a\alpha^2 + b\alpha + c)ue^{\alpha x}$