Math 302

Chapter 1

Section 1.1: Applications Leading to DEs

Contents

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Little experiment

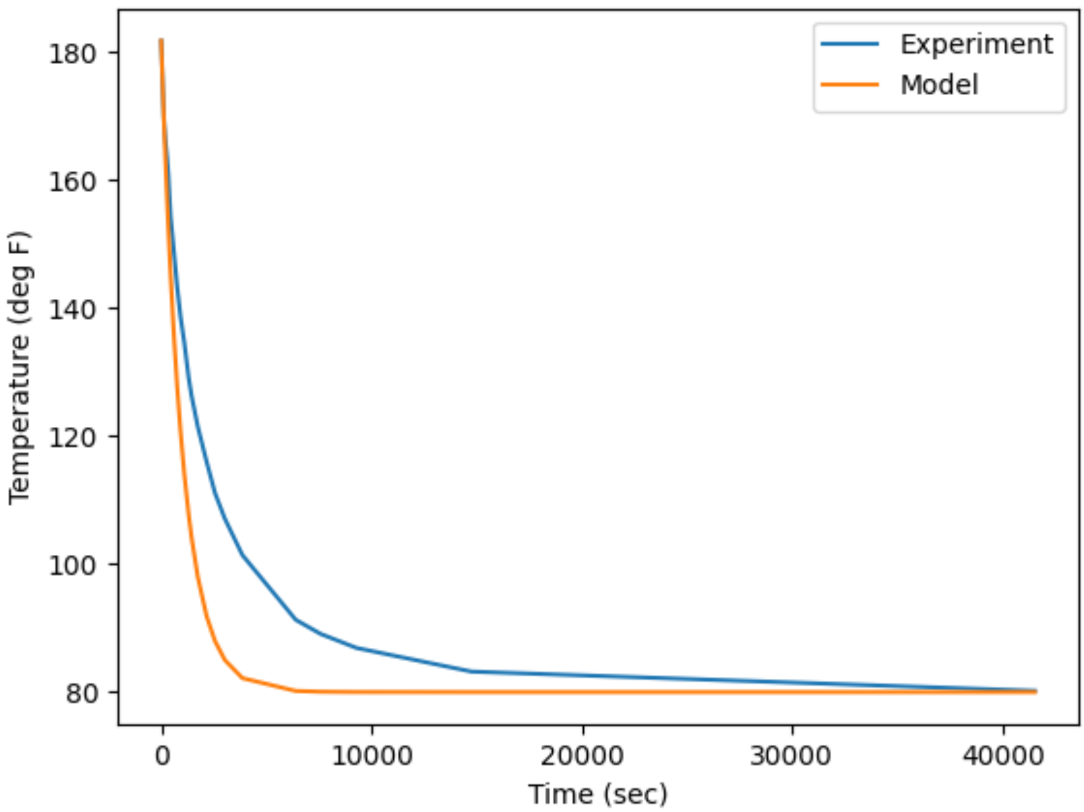
Example 1. Poor some hot water in a teapod and take its temperature with a thermometer. Take the temperature every 5 minutes. Record your data in a table and plot them in a Times VS Temperature graph.

Tables

	Time	Temperature
0 sec.	19:00:00	181.8 °F
126 sec.	19:02:06	169.8 °F
200 sec.	19:03:20	166.3 °F
280 sec.	19:04:40	163.6 °F
340 sec.	19:05:40	160.9 °F
443 sec.	19:07:23	154.5 °F
600 sec.	19:10:00	149.0 °F
760 sec.	19:12:40	143.3 °F
918 sec.	19:15:18	138.7 °F
1100 sec.	19:18:20	134.6 °F
1312 sec.	19:21:52	129.0 °F
1450 sec.	19:24:10	126.2 °F
1740 sec.	19:29:00	121.5 °F
2160 sec.	19:36:00	116.0 °F

	Time	Temperature
2560 sec.	19:42:40	111.1 °F
3026 sec.	19:50:26	107.1 °F
3863 sec.	20:04:23	101.4 °F
6396 sec.	20:46:36	91.3 °F
7574 sec.	21:06:14	89.1 °F
9280 sec.	21:34:40	86.9 °F
14,755 sec.	23:05:55	83.2 °F
41,526 sec.	6:32:06	80.2 °F

Plots



EXAMPLE 2. Let $T = T(t)$ be the temperature of a body at time t and let T_m be the temperature of its surrounding. Assuming that

- the rate of cooling of the body is directly proportial to the temperature difference of the surface area exposed
- the temperature of the surrounding does not change

deduce a model describing the evolution of the temperature $T(t)$ of the body.

Information: rate of change of T

$$\rightarrow \quad \frac{dT}{dt} \propto T - T_m$$

$\frac{dT}{dt}$ should be negative if $T > T_m$ (cooling).

$\frac{dT}{dt}$ should be positive if $T < T_m$ (warming).

This is a DE.

Therefore

$$\frac{dT}{dt} = -k\,(T - T_m) \qquad (*)$$

where $k > 0$ is a positive constant depending on the properties of the object and the medium.

T_m remains constant, so we'll see that a function satisfying the DE $(*)$ is

$$T(t) = T_m + (T_0 - T_m)e^{-kt} \qquad t \geq 0$$

where T_0 is the temperature of the body at time $t=0$ (start).

Assuming that the medium (surrounding) remains at constant temperature seems reasonable if we're considering a cup of tea/coffee cooling in a room.

What if the body warms or cools its surrounding, resulting in changing drastically the surrounding temperature?

EXAMPLE 3. Let $T = T(t)$ be the temperature of the body at time t and let $T_m = T_m(t)$ be the temperature of its surrounding. Assuming that

- the rate of cooling of the body is directly proportial to the temperature difference of the surface area exposed
- the energy is preserved

deduce a model describing the evolution of the temperature $T(t)$ of the body.

Let $T_0 = T(0)$ and $T_{m0} = T_m(0)$.

The DE from Ex. 2 is still valid:

$$T' = -k(T - T_m). \quad (*)$$

However, we have to replace T_m by something simpler. We assume further that

- change in heat of the object as its temperature increase from T_0 to T is $a(T - T_0)$ $(a > 0)$.
- change in heat of the medium as its temperature increase from T_{m0} to T_m is $a_m(T_m - T_{m0})$ $(a_m > 0)$

By conservation:

$$a(T - T_0) + a_m(T_m - T_{m0}) = 0$$

$$\Rightarrow \quad T_m = -\frac{a}{a_m}(T - T_0) + T_{m0}.$$

Replace this in the DE $(*)$:

$$T' = -k\left(1+\frac{a}{a_m}\right)T + k\left(T_{m0} + \frac{a}{a_m}T_0\right). \quad (**)$$

We'll be able to show that the functions that satisfy $(**)$ are

$$T(t) = \frac{aT_0 + a_mT_{m0}}{a + a_m} + \frac{a_m(T_0 - T_{m0})}{a + a_m}e^{-k\left(1+\frac{a}{a_m}\right)t}.$$