Math 302

Chapter 8

Section 8.4: Convolution

Contents

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The Story of The Matches

- Suppose we have a number of matches we need to light.
- At each second, so at $t=0$, $t=1$, $t=2$, $t=3$, ..., $t=n$, we light a certain number of matches. Denote by $f(t)$ the number of matches lit at time t.
- Each matches give off smoke. Denote by $g(t)$ the smoke produced by a match after t seconds.

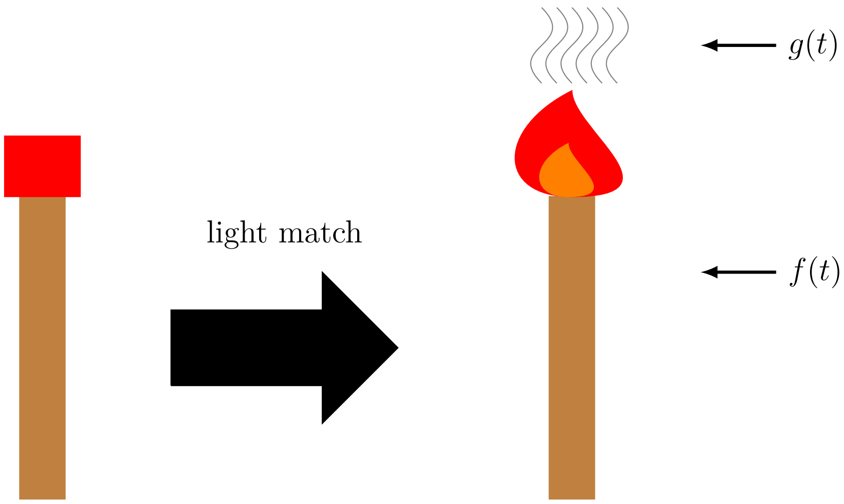


Figure 1: The Matches Problem

Question: What is the total quantity of smoke in the air after a certain time t?

Times (t)	$Q(t)$
0	$f(0)\,S(0)$
1	$f(0)\,S(1) + f(1)\,S(0)$
2	$f(0)\,S(2) + f(1)\,S(1) + f(2)\,S(0)$
n	??

The total contribution of the matches after n seconds:

$$Q(t) = \sum_{k=0}^{n} f(k)\,S(n-k)$$

What if we have a continuous phenomena?

Definition

The convolution of a function $f(t)$ with another function $g(t)$ is the new function $(f * g)(t)$ defined by

$$(f * g)(t) = \int_0^t f(x)g(t-x)\,dx.$$

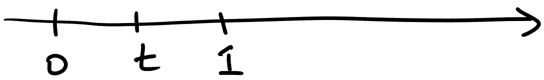
Example 1. Let

$$f(t) = u(t) - u(t-1) \quad \text{and} \quad g(t) = u(t) - u(t-1).$$

Compute $f * g$.

Explicitly: $f(t) = g(t) = \begin{cases} 0, & t < 0 \\ 1, & 0 \leq t < 1 \\ 0, & t \geq 1 \end{cases}$

1) $\underline{0 < t < 1}$



$f * g(t) = \int_0^t f(x)\, g(t-x)\, dx$

$0 \leq x \leq t \Rightarrow f(x) = 1$

$\Rightarrow f * g(t) = \int_0^t g(t-x)\, dx.$

Change variable $u = t - x \Rightarrow du = -dx$

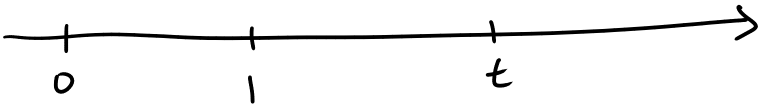
& $f * g(t) = \int_t^0 g(u)\,(-du)$

$= \int_0^t g(u)\, du$

Now, $0 \leq u \leq t \leq 1 \Rightarrow g(u) = 1$

$\Rightarrow f * g(t) = \int_0^t 1 \, du = t$

2) $\underline{t>1}$



$$f*g(t) = \int_0^t f(x)\, g(t-x)\, dx$$

$0 \le x \le t$, but $f(x) = 0$ when $x > 1$

$$\Rightarrow \quad f*g(t) = \int_0^1 g(t-x)\, dx$$

Set $u = t-x \Rightarrow f*g(t) = \int_{t-1}^{t} g(u)\, du$

$\underline{(a)}$ $t-1<1 \Rightarrow t<2$

$$\Rightarrow f*g(t) = \int_{t-1}^{1} 1\, du = 2-t$$

$\underline{(b)}$ $t-1>1 \Rightarrow t>2$

$$\Rightarrow f*g(t) = \int_{t-1}^{t} 0\, du = 0$$

So, we get

$$f*g(t) = \begin{cases} t, & 0 \le t \le 1 \\ 2-t, & 1 \le t \le 2 \\ 0, & \text{elsewhere.} \end{cases}$$

Desmos: https://www.desmos.com/calculator/h50sct4xeq

Laplace Transform

The nice properties of the convolution is a direct connection with the Laplace transform.

Example 2. Let $f(t) = e^t$ and $g(t) = e^{-t}$.

(a) Compute $f * g$.

(b) Find $L(f * g)$.

(c) Compare with $L(f)L(g)$.

(a) $f*g(t) = \int_0^t e^{x} e^{x-t}\, dx$

$= \int_0^t e^{2x-t}\, dx = \left.\frac{e^{2x-t}}{2}\right|_0^t$

$\Rightarrow \quad f*g(t) = \frac{e^t - e^{-t}}{2} = \sinh(t)$

(b) $L(f*g) = L(\sinh(t)) = \frac{1}{s^2-1}$

$\shortparallel$

(c) $L(f) = L(e^t) = \frac{1}{s-1}$

$L(g) = L(e^{-t}) = \frac{1}{s+1}$

$\Rightarrow L(f)L(g) = \frac{1}{s^2-1}$

Tranform of Convolution: If

- $f(t)$ is a function with Laplace transform $F(s)$;
- $g(t)$ is a function with Laplace transform $G(s)$;

then

$$L(f * g) = L(f)L(g) = F(s)G(s).$$

Example 3. Find the inverse Laplace transform of the following functions.

(a) $\dfrac{1}{s^2(s^2+4)}$. $= H(s)$

(b) $\dfrac{s(s+3)}{(s^2+4)(s^2+6s+10)}$. $= H(s)$

(a) $$\frac{1}{s^2(s^2+4)} = \frac{1}{s^2} \cdot \frac{1}{s^2+4}$$

$$= \underbrace{L(t)}_{F(s)} \cdot \underbrace{L\left(\tfrac{1}{2}\sin 2t\right)}_{G(s)}$$

$$\Rightarrow \quad h(t) = \frac{1}{2}\int_0^t x \sin\left(2(t-x)\right) dx$$

$$= \frac{1}{2}\left(t - \sin t \cos t\right)$$

(b) $$H(s) = \frac{s}{s^2+4} \cdot \frac{s+3}{s^2+6s+9+1}$$

$$= \underbrace{\frac{s}{s^2+4}}_{F(s)} \cdot \underbrace{\frac{s+3}{(s+3)^2+1}}_{G(s)}$$

$$f(t) = \cos(2t)$$
$$g(t) = e^{-3t}\cos t$$

$$\Rightarrow \quad h(t) = \int_0^t \cos(2x)\, e^{-3(t-x)} \cos(t-x)\, dx$$

$$= \frac{e^{-3t}}{30}\left(\sin(t) - 7\cos(t)\right) + \frac{1}{30}\left(4\sin(2t) + 7\cos(2t)\right)$$

EXAMPLE 4. Find the solution $y(t)$ to the following IVP:

$$y'' + 3y' + y = f(t), \quad y(0) = 0,\ y'(0) = 0.$$

$$Y(s) = L(y)$$

$$\Rightarrow L(y') = sY - y(0) = sY$$

$$L(y'') = s^2Y - y(0)s - y'(0) = s^2Y$$

Write $F(s) = L(f)$

$$\Rightarrow s^2Y + 3sY + Y = F$$

$$\Rightarrow (s^2 + 3s + 1)Y = F$$

$$\Rightarrow Y = \frac{F}{s^2 + 3s + 1}$$

Now,

$$s^2 + 3s + 1 = \left(s + \frac{3}{2}\right)^2 + \frac{9}{4} + 1$$

$$= \left(s + \frac{3}{2}\right)^2 + \frac{13}{4}$$

$$= \left(s + \frac{3}{2}\right)^2 + \left(\frac{\sqrt{13}}{2}\right)^2$$

Therefore,

$$Y = \frac{1}{\left(s + \frac{3}{2}\right)^2 + \left(\frac{\sqrt{13}}{2}\right)^2} \cdot F$$

$$= \frac{2}{\sqrt{13}} \frac{\sqrt{13}/2}{(s + 3/2)^2 + \left(\frac{\sqrt{13}}{2}\right)^2} \cdot F$$

$$\mathcal{L}\left(e^{-3t/2}\sin\frac{\sqrt{13}}{2}t\right) = \frac{\sqrt{13}/2}{(s+3/2)^2 + (\sqrt{13}/2)^2}$$

$$\mathcal{L}(f) = F.$$

So,

$$\boxed{y(t) = \frac{2}{\sqrt{13}}\int_0^t e^{-3x/2}\sin\left(\frac{\sqrt{13}}{2}x\right) f(t-x)\, dx}$$

General Convolution Formula: The solution $y(t)$ to the following IVP

$$ay'' + by' + cy = f(t), \quad y(0) = k_0,\, y'(0) = k_1$$

is

$$y(t) = k_0 y_1(t) + k_1 y_2(t) + (w * f)(t)$$

where

- y_1 is a solution to the following IVP

$$ay_1'' + by_1'' + cy_1 = 0, \quad y_1(0) = 1,\, y_1'(0) = 0;$$

- y_2 is a solution to the following IVP

$$ay_2'' + by_2'' + cy_2 = 0, \quad y_2(0) = 0,\, y_2'(0) = 1;$$

- $w(t)$ satisfies

$$w(t) = \frac{1}{a}y_2(t).$$

As a special case of the Laplace transform of a convolution, we can take the Laplace transform of an integral.

EXAMPLE 5. Suppose f has a Laplace transform given by $F(s)$. Find the Laplace transform of

$$h(t) = \int_0^t f(x)\,dx.$$

Set $g(t) = 1$ for any $t \geq 0$.

then,

$$h(t) = \int_0^t f(x)\, g(t-x)\, dx$$
$$= \int_0^t f(x)\, dx$$

Convolution theorem:

$$H(s) = F(s) \cdot G(s) = \boxed{\frac{F(s)}{s}}$$

In other way:

$$\boxed{L\left(\int_0^t f(x)\,dx\right) = \frac{F(s)}{s}}$$

$$L\left(\int g(x)\,dx\right) = \frac{G(s)}{s} = \frac{F(s)}{s^2}$$

Other related results:

- For $g(t) = \int_0^t \int_0^x f(u)\,du\,dx$, we have $G(s) = F(s)/s^2$.
- For a function $g(t)$ given as three integrals, then $G(s) = F(s)/s^3$.
- For a function $g(t)$ given as n integrals, then $G(s) = F(s)/s^n$.

We can solve more than just an ODE!

EXAMPLE 6. Find the solution to the following integro-differential equation

$$\int_0^t y(u)\,du + y'(t) = t,$$

where $y(0) = 0$. Let $Y = L(y)$

① Laplace Transform

$$L\left(\int_0^t y(u)\,du\right) + L(y'(t)) = L(t)$$

$$\Rightarrow \quad \frac{Y}{s} + sY - y(0) = \frac{1}{s^2}$$

$$\Rightarrow \quad Y\left(\frac{1}{s} + s\right) = \frac{1}{s^2}$$

$$\Rightarrow \quad Y\left(\frac{1+s^2}{s}\right) = \frac{1}{s^2}$$

$$\Rightarrow \quad Y = \frac{1}{s(1+s^2)}$$

② Inverse

$$Y = \underbrace{\frac{1}{s}}_{L(1)} \cdot \underbrace{\frac{1}{s^2+1}}_{L(\sin t)}$$

$$\Rightarrow \quad y(t) = \int_0^t 1 \cdot \sin(t-x)\,dx = \boxed{\cos(t) - 1}$$

EXAMPLE 7. Find the general solution to the following integral equation

$$y(t) = \sin(t) - 2\int_0^t y(u)\cos(t-u)\,du.$$

Write $Y = L(y)$.

① $Y = L(\sin t) - 2L\Big(\underbrace{\int_0^t y(u)\cos(t-u)\,du}_{y * \cos(t)\,!}\Big)$

$\Rightarrow \quad Y = \dfrac{1}{s^2+1} - 2\,Y \cdot L(\cos t)$

$\Rightarrow \quad Y = \dfrac{1}{s^2+1} - 2Y \cdot \dfrac{s}{s^2+1}$

$\Rightarrow \quad Y\left(1 + \dfrac{2s}{s^2+1}\right) = \dfrac{1}{s^2+1}$

$\Rightarrow \quad Y\left(\dfrac{s^2+2s+1}{s^2+1}\right) = \dfrac{1}{s^2+1}$

$\Rightarrow \quad Y = \dfrac{1}{(s+1)^2}$.

② Inverse

$L^{-1}\left(\dfrac{1}{s^2}\right) = -t \quad$ (shifted) $\Rightarrow L^{-1}\left(\dfrac{1}{(s+1)^2}\right) = -te^{-t}$

So, $\boxed{y(t) = -te^{-t}}$