Math 302

Chapter 5

Section 5.7: Variation Of Parameters

Contents

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Our goal in this section is to find the solutions to

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x)$$

using the method **variation of parameters**. Our assumption is

- We know at least two solutions to the complementary equation $P_0(x)y'' + P_1(x)y' + P_2(x)y = 0$.

EXAMPLE 1. Find the general solution to

$$x^2y'' - 2xy' + 2y = x^{9/2}$$

given that $y_1(x) = x$ and $y_2(x) = x^2$ are solutions to the complementary equation.

SOLUTION. Key idea: Use variation of parameters on the two constants.

We know that the general solution to the complementary equation is

$$y(x) = c_1x + c_2x^2.$$

We will change c_1 and c_2 by two functions u_1 and u_2 to find a particular solution to the ODE. So we let

$$y_{par}(x) = y(x) = u_1(x)x + u_2(x)x^2.$$

We can then compute y' and y''. We have

$$y' = u_1'x + u_1 + u_2'x^2 + 2u_2x.$$

We will impose one condition on u_1, u_2 to make things easier. We will assume that u_1 and u_2 are chosen so that

$$u_1'x + u_2'x^2 = 0. \tag{1}$$

Therefore, the derivative of y is simply given by

$$y' = u_1 + 2u_2x.$$

In other words, we have taken the derivative of x and x^2 in the expression of y and leaving u_1 and u_2 untouched.

With this expression of y', we can obtain the second derivative

$$y'' = u_1' + 2u_2'x + 2u_2.$$

We can then replace y, y' and y'' in the ODE and after simplifying we get

$$x^2u_1' + 2x^3u_2' = x^{9/2}.$$

We then have to find the functions u_1, u_2 satisfying the system of differential equations:

$$\begin{cases} u_1'x + u_2'x^2 = 0 \\ x^2u_1' + 2x^3u_2' = x^{9/2}. \end{cases}$$

From the first equation, we see that $u_1' = -u_2' x$. Replacing this in the second equation, we obtain

$$-x^3 u_2' + 2x^3 u_2' = x^{9/2}$$

and therefore $u_2' = -x^{3/2}$. This also means that $u_1' = x^{5/2}$. Integrating with respect to x, we obtain

$$u_1(x) = -\frac{2}{7} x^{7/2} \quad \text{and} \quad u_2(x) = \frac{2}{5} x^{5/2}.$$

Therefore, replacing in y_{par}, we obtain

$$y_{par}(x) = -\frac{2}{5} x^{9/2} + \frac{2}{7} x^{9/2} = \frac{4}{35} x^{9/2}.$$

Finally, the general solution to the original ODE is

$$y(x) = y_{par}(x) + c_1 x + c_2 x^2 = \frac{4}{35} x^{9/2} + c_1 x + c_2 x^2.$$

General Procedure

To find a particular solution to

$$P_0(x)y'' + P_1(x)y' + P_0(x)y = F(x)$$

knowing two solutions $y_1(x)$ and $y_2(x)$ to the complementary equation, we follow these steps:

- Write $y_{par}(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$.
- Write the system

$$\begin{aligned} u_1'y_1 + u_2'y_2 &= 0 \\ u_1'y_1' + u_2'y_2' &= \frac{F}{P_0}. \end{aligned}$$

- Solve the system for u_1' and u_2':

$$u_1' = -\frac{Fy_2}{P_0(y_1y_2' - y_1'y_2)} \quad \text{and} \quad u_2' = \frac{Fy_1}{P_0(y_1y_2' - y_1'y_2)}.$$

- Obtain u_1 and u_2 by integrating u_1' and u_2' respectively.
- Substitute u_1 and u_2 in $y_{par}(x)$ to obtain the particular solution.

EXAMPLE 2. Find a particular solution to

$$y'' + 3y' + 2y = \frac{1}{1 + e^x}.$$

SOLUTION. The general solution to the complementary equation $y'' + 3y' + 2y = 0$ is

$$y(x) = c_1e^{-2x} + c_2e^{-x}.$$

So we set

$$y_{par}(x) = y(x) = u_1(x)e^{-2x} + u_2(x)e^{-x}.$$

We add the restriction

$$u_1'e^{-2x} + u_2'e^{-x} = 0 \iff u_1' + u_2'e^x = 0.$$

Therefore, the derivative of y is

$$y' = -2u_1e^{-2x} - u_2e^{-x}$$

and the second derivative is

$$y'' = -2u_1'e^{-2x} + 4u_1e^{-2x} - u_2'e^{-x} + u_2e^{-x}.$$

Replacing this in the initial ODE and simplifying, we get

$$-2u_1' - u_2'e^x = \frac{e^{2x}}{1 + e^x}.$$

Therefore, u_1 and u_2 must satisfy the following system of ODEs:

$$\begin{cases} u_1' + u_2' e^x = 0 \\ -2u_1' - u_2' e^x = \frac{e^{2x}}{1+e^x}. \end{cases}$$

From the first equation, we see that $u_1' = -u_2' e^x$. Replacing this into the second equation, we see that

$$2u_2' e^x - u_2' e^x = \frac{e^{2x}}{1+e^x} \iff u_2' = \frac{e^x}{1+e^x}.$$

Plugging this in the expression of u_1', we see that

$$u_1' = -\frac{e^{2x}}{1+e^x}.$$

Integrating $u_2' = \frac{e^x}{1+e^x}$ with the change of variable $v = 1 + e^x$, we see that

$$u_2(x) = \ln(1 + e^x)$$

where the absolute value was removed because $1 + e^x$ is always positive for any x. To integrate $u_1' = -\frac{e^{2x}}{1+e^x}$, we make the change of variable $v = e^x$ and then integrate $v/(1 + v) = v + 1$. Therefore, we see that

$$u_1(x) = e^{2x} + e^x.$$

Plugging into y_{par}, we obtain

$$y_{par}(x) = e^{-2x}(e^{2x} + e^x) + e^{-x}\ln(1 + e^x) = 1 + e^{-x} + e^{-x}\ln(1 + e^x).$$

The general solution to the ODE is then

$$y(x) = y_{par}(x) + c_1 e^{-2x} + c_2 e^{-x} = 1 + e^{-x} + e^{-x}\ln(1 + e^x) + c_1 e^{-2x} + c_2 e^{-x}.$$