Math 302

Chapter 5

Section 5.3: Nonhomogeneous Linear Equations

Contents

Created by: Pierre-Olivier Parisé
Fall 2022

Our goal is to find the solutions to

$$y'' + p(x)y' + q(x)y = f(x). \tag{1}$$

Nomenclature:

- the equation $y'' + p(x)y' + q(x)y = 0$ is the **complementary equation** for (1).
- a **particular solution** is a solution y_{par} of (1).

Example 1. Find a particular solution to the following ODE:

$$y'' - 2y' + y = 4x.$$

Trick: Guessing!

What we know:

$y(x) = A \to y'(x) = 0$

$y(x) = Ax + B \to y'(x) = A$ & $y''(x) = 0$

$y(x) = Ax^2 + Bx + C \to y'(x) = 2Ax + B$, $y'' = 2A$

Suppose $y(x) = Ax + B \to y' = A$

Replace y, y' & y'' in the ODE:

$$0 - 2A + Ax + B = 4x$$

$$\Leftrightarrow \quad Ax + (B - 2A) = 4x$$

$$\Leftrightarrow \quad A = 4 \quad \& \quad B - 2A = 0$$

$$\Leftrightarrow \quad A = 4 \quad \& \quad B = 8$$

Therefore $y(x) = 4x + 8$ is a part. sol. to the ODE! ☺

Assumptions:

1) Suppose $\{y_1, y_2\}$ is a fundamental set of solutions to
$$y'' + p(x)y' + q(x)y = 0.$$

2) Suppose y_{par} is a particular solution to
$$y'' + p(x)y' + q(x)y = f(x).$$

Conclusion:

- Then the $y = y_{par} + c_1 y_1 + c_2 y_2$ is the general solution of
$$y'' + p(x)y' + q(x)y = f(x).$$

Example 2.

a) Find the general solution of
$$y'' - 2y' + y = -3 - x + x^2.$$

b) Solve the following IVP:
$$y'' - 2y' + y = -3 - x + x^2, \quad y(0) = -2,\ y'(0) = 1.$$

(a) 2) Find y_{par}

guess: $y_{par} = Ax^2 + Bx + C$

$\Rightarrow y'_{par} = 2Ax + B$ & $y''_{par} = 2A$

So,
$$2A - 4Ax - 2B + Ax^2 + Bx + C = -3 - x + x^2$$
$$\Leftrightarrow 2A - 2B + C + (B - 4A)x + Ax^2 = -3 - x + x^2$$
$$\Leftrightarrow 2A - 2B + C = -3, \quad B - 4A = -1 \quad \& \quad A = 1$$
$$\Leftrightarrow 2 - 2B + C = -3, \quad B - 4 = -1 \quad \& \quad A = 1$$
$$\Leftrightarrow 2 - 6 + C = -3 \quad B = 3 \quad \& \quad A = 1$$
$$\Leftrightarrow C = 1, \quad B = 3 \quad \& \quad A = 1$$

So, $y_{par}(x) = x^2 + 3x + 1$.

1) Find solution compl. eq.

$y'' - 2y' + y = 0 \longrightarrow r^2 - 2r + 1 = (r-1)^2 \longrightarrow$ root is $r=1$ (repeated).

$\Rightarrow \quad y(x) = (c_1 + c_2 x)e^x \longrightarrow y_1 = e^x$ & $y_2 = xe^x$

3) General solution

$$\boxed{\begin{aligned} y(x) &= y_{par} + c_1 y_1 + c_2 y_2 \\ &= x^2 + 3x + 1 + (c_1 + c_2 x) e^x \end{aligned}}.$$

(b) $y(0) = -2$.
$y'(0) = 1$.

We have $y(x) =$
& $y'(x) = 2x + 3 + c_2 e^x + (c_1 + c_2 x) e^x$

$\Rightarrow \quad -2 = y(0) = 1 + c_1$
& $\quad 1 = y'(0) = 3 + c_2 + c_1$

$\Leftrightarrow \begin{cases} c_1 = -3 \\ c_1 + c_2 = -2 \end{cases} \Leftrightarrow \begin{cases} c_1 = -3 \\ c_2 = 1 \end{cases}$

Therefore,

$$\boxed{y(x) = x^2 + 3x + 1 + (x-3) e^x}$$

EXAMPLE 3. Suppose that we know that $y_1(x) = x^4/15$ is a particular solution to

$$x^2y'' + 4xy' + 2y = 2x^4$$

and that $y_2(x) = x^2/3$ is a particular solution to

$$x^2y'' + 4xy' + 2y = 4x^2.$$

Find a particular solution to

$$x^2y'' + 4xy' + 2y = 2x^4 + 4x^2.$$

Let $y(x) = \frac{x^4}{15} + \frac{x^2}{3}$.

Then $4x\,y' = \frac{16x^4}{15} + \frac{8x^2}{3}$

$x^2y'' = \frac{12x^4}{15} + \frac{2x^2}{3} = \frac{4}{5}x^4 + \frac{2}{3}x^2$

(*)

$$\Rightarrow x^2y'' + 4xy' + 2y = \frac{12x^4}{15} + \frac{2x^2}{3} + \frac{16x^4}{15} + \frac{8}{3}x^2 + \frac{2x^4}{15} + \frac{2}{3}x^2$$

$$= \frac{30}{15}x^4 + \frac{12}{3}x^2 = 2x^4 + 4x^2 \checkmark$$

But, $y = y_1 + y_2 \Rightarrow$

$$\begin{aligned} & x^2y'' = x^2y_1'' + x^2y_2'' \\ + \;& 4xy' = 4xy_1' + 4xy_2' \\ + \;& 2y = 2y_1 + 2y_2 \end{aligned}$$

$$x^2y'' + 4xy' + 2y = \boxed{x^2y_1'' + 4xy_1' + 2y_1} \; \boxed{+ x^2y_2'' + 4xy_2' + 2y_2}$$

$$= 2x^4 - 4x^2$$

So, I don't have to do calculations (*).

General Fact: If y_1 is a particular solution to

$$y'' + p(x)y' + q(x)y = f_1(x)$$

and y_2 is a particular solution to

$$y'' + p(x)y' + q(x)y = f_2(x)$$

then $y_{par} = y_1 + y_2$ is a particular solution to

$$y'' + p(x)y' + q(x)y = f_1(x) + f_2(x).$$