Math 302

Chapter 7

Section 7.1: Review of Power Series

Contents

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Why Power Series?

Most of the differential equation of order 2 we have encountered are constant coefficients ODE. In most real-life application, the coefficients will be **variable coefficients** such as

- **Bessel's equation** of order n:
$$x^2y'' + xy' + (x^2 - n^2)y = 0.$$
- **Legendre's equation** of order n:
$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0.$$

The methods we used in chapter 5won't be of use in those situations. This is why we need power series and the **power series method**.

Basic Definitions

- A **Power series** centered at a number a is an expression involving an infinite sum of powers of $(x - a)$:
$$\sum_{n=0}^{\infty} a_n(x-a)^n.$$
- If $a = 0$, we simply write
$$\sum_{n=0}^{\infty} a_n x^n.$$

We will confine ourselves to power series centered at $a = 0$.

- A power series **converges** on an interval I provided that for any x in this interval I, the following limit exists

$$\lim_{N\to\infty} \sum_{n=0}^{N} a_n x^n.$$

- If a function f is expressed as a power series on I, we then write

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

and called this a **power series representation** of f.

Some examples of power series representations of some famous[1] function

- $e^x =$

- $\cos x =$

- $\sin x =$

- $\cosh x =$

- $\sinh x =$

- $\ln(1+x) =$

- $\dfrac{1}{1-x} =$

- $(1+x)^\alpha =$

<u>Remark:</u>

- The **Taylor series** of f is

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

- The **Maclaurin series** of f is

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n.$$

[1]They made the coverpage of New York Times magazine several times for their influence on the world.

Differentiation

If $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

and in general

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) a_n x^{n-k}.$$

EXAMPLE 1. Differentiate the power series representation of $\sin x$.

Identity Principle or Uniqueness of Power series

If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$, then

$$f(x) = g(x) \quad \iff \quad a_n = b_n, \text{ for all } n \geq 0.$$

Consequence: We have

$$\sum_{n=0} a_n x^n = 0$$

if, and only if, $a_n = 0$ for all $n \geq 0$.

Example 2. Find $y(x)$ if

$$y' = \sum_{n=1}^{\infty} x^n \quad \text{and} \quad y(0) = 0.$$

Sum, Difference and Multiplication by A Constant

If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ are two power series, then

- $f(x) + g(x) = \displaystyle\sum_{n=0}^{\infty} (a_n + b_n)x^n$.
- $f(x) - g(x) = \displaystyle\sum_{n=0}^{\infty} (a_n - b_n)x^n$.
- $cf(x) = \displaystyle\sum_{n=0}^{\infty} (ca_n)x^n$.

EXAMPLE 3. Use the definition of $\cosh(x)$ and $\sinh(x)$ to find its power series representation.

Product with Polynomials

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$.

- $g(x) = cx$, then

$$f(x)g(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right) cx$$
$$=$$

- $g(x) = cx^2$, then

$$f(x)g(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right) cx^2$$
$$=$$

- $g(x) = cx^3$, then

$$f(x)g(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right) cx^3$$
$$=$$

EXAMPLE 4. If $f(x) = \sum_{n=0}^{\infty} a_n x^n$, find the expression of

(a) xf'.

(b) $(2-x)f''$.

Shifting

For any integer k, if

$$y(x) = \sum_{n=n_0}^{\infty} a_n x^{n-k}$$

then

$$y(x) = \sum_{n=n_0-k}^{\infty} a_{n+k} x^n$$

Example 5. Complete Example 4.

Example 6. Express $2y - xy''$ as a power series $\sum_{n=0}^{\infty} c_n x^n$.