Math 302

Chapter 7

Section 7.2: Series Solutions Near an Ordinary Point

Contents

Created by: Pierre-Olivier Parisé

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Main goal:

- Solve a second order ODE

$$A(x)y'' + B(x)y' + C(x)y = 0$$

where $A(x)$, $B(x)$, and $C(x)$ are polynomials.

- Use power series to obtain the solution $y(x)$. Such a solution is called a **power series solution** to the ODE.

Recall from the previous section that

- $y(x) = \sum_{n=0}^{\infty} a_n x^n$.
- $y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$.
- $y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$.

Remark:

- We denote the left-hand side by

$$L(y) := A(x)y'' + B(x)y' + C(x)y.$$

- The application $y \mapsto L(y)$ is called a **differential operator** in the litterature.

EXAMPLE 1. Find a power series solution to $y'' + y = 0$.

① Left-hand as a power series

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad \& \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\Rightarrow \quad y'' + y = \sum_{n=0}^{\infty} a_n x^n + \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$= \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$= \sum_{n=0}^{\infty} \left[a_n + (n+2)(n+1) a_{n+2}\right] x^n$$

② Find recurrence relation

$$\sum_{n=0}^{\infty} [a_n + (n+2)(n+1)a_{n+2}]x^n = 0 = \sum_{n=0}^{\infty} 0 \cdot x^n$$

$$\Rightarrow \quad a_n + (n+2)(n+1)a_{n+2} = 0 \qquad (n \geq 0)$$

$$\Rightarrow \quad a_{n+2} = \frac{-a_n}{(n+2)(n+1)} \qquad (n \geq 0)$$

③ Find coefficients.

$\underline{n=0}$, a_0 arbitrary $\Rightarrow$ $a_2 = \frac{-a_0}{(0+2)(0+1)} = \frac{-a_0}{2\cdot 1}$

$\boxed{\text{Here, } k! = k(k-1)\cdots 2\cdot 1.}$

$$a_4 = \frac{-a_2}{(2+2)(2+1)} = -\frac{a_2}{4\cdot 3}$$

$$= \frac{a_0}{4\cdot 3\cdot 2\cdot 1}$$

$$\vdots$$

$$a_{2n} = (-1)^n \frac{a_0}{(2n)!}$$

$\underline{n=1}$ a_1 arbitrary $\Rightarrow$ $a_3 = \frac{-a_1}{(1+2)(1+1)} = \frac{-a_1}{3\cdot 2}$

$$a_5 = \frac{-a_3}{(3+2)(3+1)} = \frac{-a_3}{5\cdot 4}$$

$$= \frac{a_1}{5\cdot 4\cdot 3\cdot 2}$$

$$\vdots$$

$$a_{2n+1} = (-1)^n \frac{a_1}{(2n+1)!}$$

③ General Solution

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots$$

$$= a_0 - \frac{a_0}{2!}x^2 + \frac{a_0}{4!}x^4 - \frac{a_0}{6!}x^6 + \dots$$

$$+ a_1 x - \frac{a_1}{3!}x^3 + \frac{a_1}{5!}x^5 - \frac{a_1}{7!}x^7 + \dots$$

$$= a_0\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots\right)$$

$$+ a_1\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots\right)$$

$$= a_0 \cos(x) + a_1 \sin(x) \text{ !}.$$

Recurrence Relation:

Solving ODE with power series involves a lot of recurrence relations. In the above problems, we encountered:

$$a_{n+2} = \frac{-a_n}{(n+2)(n+1)}$$

start at $n=0$ $\to$ a_{2n}.

start at $n=1$ $\to$ a_{2n+1}.

Example 2. Find a power series solution to $x^2y'' + y = 0$.

① Left-hand Side as a series

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad \& \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\Rightarrow \quad x^2 y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^n$$

So,

$$x y'' + y = \sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=2}^{\infty} \left(n(n-1) a_n - a_n\right) x^n$$

$$+ \quad a_0 + a_1 x$$

② Recurrence Relation

$$x y'' + y = 0 = \sum_{n=0}^{\infty} 0 x^n \quad \Rightarrow \quad \begin{cases} a_0 = a_1 = 0 \\ n(n-1) a_n - a_n = 0, \quad n \geq 2 \end{cases}$$

From the second relation, we see that

$\underline{n=2}$ $\quad 2 \cdot 1 \, a_2 - a_2 = 0 \quad \Rightarrow \quad a_2 = 0$

$\underline{n=3}$ $\quad 3 \cdot 2 \, a_3 - a_3 = 0 \quad \Rightarrow \quad a_3 = 0$

In general, $\quad a_n = 0 \quad , n \geq 2 \ldots$

③ Solution

$$y(x) = \sum_{n=0}^{\infty} 0 \cdot x^n = 0 \quad ??\ ?$$

No solutions as power series $\sum_{n=0}^{\infty} a_n x^n$.

Maybe as $\sum_{n=0}^{\infty} a_n (x-1)^n$ (see later).

The solution is

$$y(x) = c_1 \sqrt{x} \cos\left(\frac{\sqrt{3}}{2} \ln(x)\right) + c_2 \sqrt{x} \sin\left(\frac{\sqrt{3}}{2} \ln(x)\right).$$

- A number x_0 is called an **ordinary point** if $A(x_0) \neq 0$.
- A number x_0 is called a **singular point** if $A(x_0) = 0$.

We will mainly focuss on power series solutions centered at ordinary points.

EXAMPLE 3. For each of the following ODEs, find the singular points.

(a) $(1 - x^2)y'' + y = 0$.

(b) $(1 + 2x + x^2)y'' + y' + (2 + x)y = 0$.

(c) $(2x + 3x^2 + x^3)y'' + (x + 1)y' + (x^2 + 1)y = 0$.

(a) $(1-x^2)=0 \iff x=-1$ or $x=1$

$\Rightarrow$ Sing. pt : $\boxed{-1 \,\&\, 1}$.

(b) $x^2+2x+1=0 \iff (x+1)^2=0$

$\iff x=-1 \rightarrow$ Sing. pt. : $\boxed{x=-1.}$

(c) $(2x+3x^2+x^3)=0 \iff (2+3x+x^2)x=0$

$\iff (x+2)(x+1)x=0$

$\iff x=-2,\ x=-1,\ x=0$

Sing. pts: $\boxed{-2, -1, 0.}$

Remark:

- A power series solution must be centered at an ordinary point, that is, if x_0 is an ordinary point, then the form of the solution is

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

- In Example 2, we see why we can't solve: The power series used was centered at $x_0 = 0$, a singular point.
- In the case of a singular points, we need the Frobenius method. This is covered in a second class in ODE.

EXAMPLE 4.

(a) Find a power series solution of

$$(x^2-4)y''+3xy+y=0.$$

(b) Find the solution to the IVP

$$(x^2-4)y''+3xy+y=0, \quad y(0)=4,\ y'(0)=1.$$

(a) ① Sing. Pts: $-2, 2$. So $x=0$ ord. pt. ✓

② $y(x)=\sum_{n=0}^{\infty} a_n x^n$

$y'(x)=\sum_{n=1}^{\infty} n a_n x^{n-1} \longrightarrow 3xy'=\sum_{n=1}^{\infty} 3n a_n x^n$

$y''(x)=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} \longrightarrow x^2y''=\sum_{n=2}^{\infty} n(n-1)a_n x^n$

$-4y''=\sum_{n=2}^{\infty} -4n(n-1)a_n x^{n-2}$

So

$$LHS = \sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=2}^{\infty} -4n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} 3n a_n x^n + \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} -4(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty} 3n a_n x^n + \sum_{n=0}^{\infty} a_n x^n$$

$$= (a_0-8a_2) + (4a_1-24a_3)x + \sum_{n=2}^{\infty}\left(n(n-1)a_n - 4(n+2)(n+1)a_{n+2} + 3na_n + a_n\right)x^n$$

We have:

$$n^2 a_n - n a_n - 4(n^2+3n+2)a_{n+2} + 3na_n + a_n$$

$$= (n^2 - n + 3n + 1)a_n - 4(n+2)(n+1)a_{n+2}$$

$$= (n^2 + 2n + 1)a_n - 4(n+2)(n+1)a_{n+2}$$

$$= (n+1)^2 a_n - 4(n+2)(n+1)a_{n+2}.$$

So,

$$LHS = (a_0 - 8a_2) + (4a_1 - 24a_3)x + \sum_{n=2}^{\infty} \left((n+1)^2 a_n - 4(n+2)(n+1)a_{n+2}\right)x^n$$

$$= 0 = \sum_{n=1}^{\infty} 0 \cdot x^n$$

$$\Rightarrow \begin{cases} a_0 - 8a_2 = 0 \\ 4a_1 - 24a_3 = 0 \\ (n+1)^2 a_n - 4(n+2)(n+1)a_{n+2} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} a_2 = \dfrac{a_0}{8} \\ a_3 = \dfrac{a_1}{6} \\ a_{n+2} = \dfrac{n+1}{4(n+2)} a_n \end{cases} \qquad \begin{matrix} a_0 \\ a_1 \end{matrix} \text{ arbitrary.}$$

Start recurence.

a_0, a_1 arbitrary (no conditions).

$$a_2 = \frac{a_0}{8}, \qquad a_3 = \frac{a_1}{6}$$

$$a_4 = \frac{(2+1)}{4(2+2)} a_2 = \frac{3}{16} \cdot \frac{a_0}{8} = \frac{3}{128} a_0$$

$$a_5 = \frac{3+1}{4(3+2)} a_3 = \frac{1}{5} \cdot \frac{a_1}{6} = \frac{a_1}{30}$$

$$a_6 = \frac{4+1}{4(4+2)} a_4 = \frac{5}{4 \cdot 6} \cdot \frac{3}{128} \cdot a_0 = \frac{5}{1024} a_0$$

$$a_7 = \frac{5+1}{4(5+2)} a_5 = \frac{3}{7} \cdot \frac{a_1}{30} = \frac{1}{70} a_1$$

$$\vdots$$

③ <u>General solution</u>

$$\begin{aligned} y(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 \\ &\qquad + a_6 x^6 + a_7 x^7 + \cdots \\ &= a_0 + a_1 x + \frac{a_0}{8} x^2 + \frac{a_1}{6} x^3 + \frac{3}{128} a_0 x^4 \\ &\qquad + \frac{a_1}{30} x^5 + \frac{5 a_0}{1024} x^6 + \frac{a_1}{70} x^7 + \cdots \end{aligned}$$

(b) $y(0) = 4 \Rightarrow a_0 = 4$

$$y'(x) = a_1 + \frac{a_0}{4} x + \frac{a_1}{2} x^2 + \frac{3}{32} a_0 x^3 + \cdots$$

$$\Rightarrow y'(0) = 1 = a_1$$

Thus,

$$\begin{aligned} y(x) &= 4 + x + \frac{x^2}{2} + \frac{1}{6} x^3 + \frac{3}{32} x^4 \\ &\qquad + \frac{1}{30} x^5 + \frac{5}{256} x^6 + \frac{1}{70} x^7 + \cdots \end{aligned}$$

EXAMPLE 5. Find a power series solution to the following IVP:

$$(t^2-2t-3)\frac{d^2y}{dt^2}+3(t-1)\frac{dy}{dt}+y=0, \quad y(1)=4,\ y'(1)=-1.$$

Problem: $y(t) = \sum_{n=0}^{\infty} a_n t^n \Rightarrow y(1) = \sum_{n=0}^{\infty} a_n$ Diff. to compute!

Solution: $y(t) = \sum_{n=0}^{\infty} a_n (t-1)^n \Rightarrow y(1) = a_0$ Better

① Translate.

$x^2 + \cancel{2x} + 1$
$- \cancel{2x} - 2 \quad -3$

Put $x = t-1$. Then

- $t^2 - 2t - 3 = (x+1)^2 - 2(x+1) - 3 = x^2 - 4$
- $3(t-1) = 3x$
- $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = y'$
- $\frac{d^2y}{dt^2} = \frac{d}{dt}(y') = \frac{d}{dx}(y')\cdot\frac{dx}{dt} = y''$

So,

$$(t^2-2t-3)\frac{d^2y}{dt^2} + 3(t-1)\frac{dy}{dt} + y(t)$$
$$= (x^2-4)y'' + 3xy' + y$$

ODE is

$$(x^2-4)y'' + 3xy' + y = 0$$

② Use the preceding problem.

Write $y(x) = \sum_{n=0}^{\infty} a_n x^n$.

From Example 4,

$$y(x) = a_0 + a_1 x + \frac{a_0}{8} x^2 + \frac{a_1}{6} x^3 + \frac{3}{128} a_0 x^4 + \frac{a_1}{30} x^5 + \frac{5a_0}{1024} x^6 + \frac{a_1}{70} x^7 + \dots$$

Replace x by $t-1$:

$$y(t) = a_0 + a_1(t-1) + \frac{a_0}{8}(t-1)^2 + \frac{a_1}{6}(t-1)^3 + \frac{3}{128} a_0 (t-1)^4 + \frac{a_1}{30}(t-1)^5 + \frac{5a_0}{1024}(t-1)^6 + \frac{a_1}{70}(t-1)^7 + \dots$$

③ Solve the IVP.

$y(1) = 4 \Rightarrow a_0 = 4$

$a_1 = y'(1) = -1 \Rightarrow a_0 = -1$

So,

$$y(x) = 4 - (t-1) + \frac{1}{2}(t-1)^2 - \frac{1}{6}(t-1)^3 + \frac{3}{32}(t-1)^4 - \frac{1}{30}(t-1)^5 + \frac{5}{256}(t-1)^6 + \dots$$

It is important to know where our solution is valid.

- The **radius of convergence** of a power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is the number R such that
  - $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges for any x such that $|x-x_0| < R$.
  - $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ diverges for all x such that $|x-x_0| > R$.
- If the limit
$$L := \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$
exists, then the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$ is $R = \frac{1}{L}$.

EXAMPLE 6. Find the radius of convergence of

(a) $f(x) = \sum_{n=0}^{\infty} x^n$.

(b) $g(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$.

(a) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{1}{1} = 1$

So $R = 1$

(b) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{1/(n+1)!}{1/n!}$

$= \lim_{n\to\infty} \frac{n!}{(n+1)n!}$

$= \lim_{n\to\infty} \frac{1}{n+1} = 0$

So, $R = \frac{1}{0} = +\infty$.

Theorem 7. Suppose that x_0 is an ordinary point of the ODE

$$A(x)y'' + B(x)y' + C(x)y = 0.$$

Then the ODE has a general solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

The radius of convergence of any such series solution is at least as large as the distance from x_0 to the nearest (real or complex) singular point of the ODE.

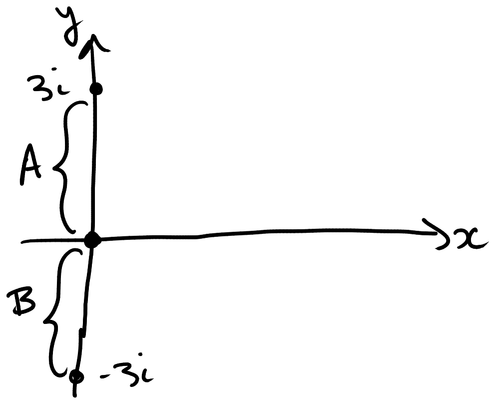
Example 8. Determine the radius of convergence guaranteed by the last Theorem of a series solution of

$$(x^2 + 9)y'' + xy' + x^2 y = 0$$

(a) in powers of x.

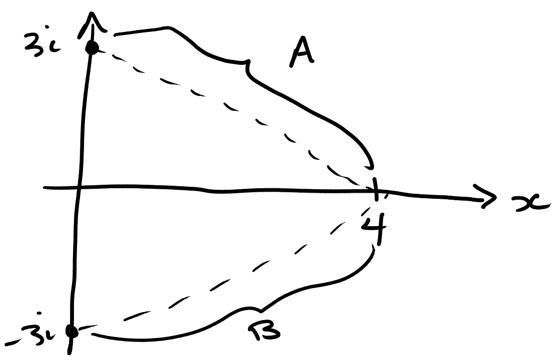
(b) in powers of $x - 4$.

(a) Sing pts are $x = 3i$ & $x = -3i$.



$R = \max\{A, B\} = \boxed{3}$

(b) Sing pts. are $x = 3i$ & $x = -3i$



$A = \sqrt{3^2 + 4^2} = 5$

$B = \sqrt{-3^2 + 4^2} = 5$

So

$R = \max\{A, B\} = \boxed{5}$