Math 302

Chapter 7

Section 7.2: Series Solutions Near an Ordinary Point

Contents

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Setup

Main goal:

- Solve a second order ODE

$$A(x)y'' + B(x)y' + C(x)y = 0$$

where $A(x)$, $B(x)$, and $C(x)$ are polynomials.

- Use power series to obtain the solution $y(x)$. Such a solution is called a **power series solution** to the ODE.

Recall from the previous section that

- $y(x) = \sum_{n=0}^{\infty} a_n x^n$.
- $y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$.
- $y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$.

Remark:

- We denote the left-hand side by

$$L(y) := A(x)y'' + B(x)y' + C(x)y.$$

- The application $y \mapsto L(y)$ is called a **differential operator** in the litterature.

Example 1. Find a power series solution to $y'' + y = 0$.

Recurrence Relation:

Solving ODE with power series involves a lot of recurrence relations. In the above problems, we encountered:

Example 2. Find a power series solution to $x^2y'' + y = 0$.

- A number x_0 is called an **ordinary point** if $A(x_0) \neq 0$.
- A number x_0 is called a **singular point** if $A(x_0) = 0$.

We will mainly focuss on power series solutions centered at ordinary points.

EXAMPLE 3. For each of the following ODEs, find the singular points.

(a) $(1 - x^2)y'' + y = 0$.

(b) $(1 + 2x + x^2)y'' + y' + (2 + x)y = 0$.

(c) $(2x + 3x^2 + x^3)y'' + (x + 1)y' + (x^2 + 1)y = 0$.

Remark:

- A power series solution must be centered at an ordinary point, that is, if x_0 is an ordinary point, then the form of the solution is

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

- In Example 2, we see why we can't solve: The power series used was centered at $x_0 = 0$, a singular point.
- In the case of a singular points, we need the Frobenius method. This is covered in a second class in ODE.

Example 4.

(a) Find a power series solution of

$$(x^2 - 4)y'' + 3xy + y = 0.$$

(b) Find the solution to the IVP

$$(x^2 - 4)y'' + 3xy + y = 0, \quad y(0) = 4,\ y'(0) = 1.$$

Example 5. Find a power series solution to the following IVP:

$$(t^2 - 2t - 3)\frac{d^2y}{dt^2} + 3(t-1)\frac{dy}{dt} + y = 0, \quad y(1) = 4,\ y'(1) = -1.$$

It is important to know where our solution is valid.

- The **radius of convergence** of a power series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ is the number R such that
  - $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ converges for any x such that $|x - x_0| < R$.
  - $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ diverges for all x such that $|x - x_0| > R$.
- If the limit

$$L := \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

  exists, then the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$ is $R = \frac{1}{L}$.

Example 6. Find the radius of convergence of

(a) $f(x) = \sum_{n=0}^{\infty} x^n$.

(b) $g(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$.

Theorem 7. Suppose that x_0 is an ordinary point of the ODE

$$A(x)y'' + B(x)y' + C(x)y = 0.$$

Then the ODE has a general solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

The radius of convergence of any such series solution is at least as large as the distance from x_0 to the nearest (real or complex) singular point of the ODE.

Example 8. Determine the radius of convergence guaranteed by the last Theorem of a series solution of

$$(x^2 + 9)y'' + xy' + x^2 y = 0$$

(a) in powers of x.

(b) in powers of $x - 4$.

When we have a solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

of an ODE

$$A(x)y'' + B(x)y' + C(x)y = 0,$$

we can draw an approximation of the solution.

- The **Taylor polynomial** $T_N(x)$, where $N \geq 0$ is an integer, is given by the expression

$$T_N(x) = \sum_{n=0}^{N} a_n(x - x_0)^n = a_0 + a_1(x - x_0) + \cdots + a_N(x - x_0)^N.$$

- When the power series of $y(x)$ converges on a given interval I, we have

$$y(x) \approx T_N(x)$$

  for a sufficiently large integer N.

EXAMPLE 9.

(a) Plot the graph of $T_4(x)$, $T_{10}(x)$, and $T_{20}(X)$ of the power series representation of $f(x) = \cos(x)$.

(b) Plot the graph of $T_4(x)$, $T_{10}(x)$, $T_{20}(x)$ for the power series solution of Example 5.