Math 302

Chapter 5

Section 5.1: Homogeneous Linear Equations

Contents

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What Is A Second Order Linear ODE?

We will be mainly interested in the following specific ODEs:

$$y'' + p(x)y' + q(x)y = f(x) \tag{1}$$

where p, q, and f are continuous functions of the variable x.

- When $f(x) = 0$ for any x, the ODEs is called **homogeneous**.
- When $f(x) \neq 0$, the ODEs is called **non-homogeneous**.
- The function f is called the **forcing function** .
- The IVP associated to a second order ODE of the form (1) is

$$y'' + p(x)y' + q(x)y = f(x), \quad y(x_0) = k_0,\, y'(x_0) = k_1$$

for some point x_0 in an interval (a, b) and k_0, k_1 are arbitrary numbers.

Goal: To solve the homogeneous equation

$$y'' + p(x)y' + q(x)y = 0.$$

Example 1. Consider the ODE

$$y'' - y = 0.$$

a) Identify the functions p and q.

b) Verify that $y_1(x) = e^x$ and $y_2(x) = e^{-x}$ are solutions of the ODE on $(-\infty, \infty)$.

c) Verify that if c_1 and c_2 are arbitrary constants, then $y(x) = c_1e^x + c_2e^{-x}$ is a solution to the ODE on $(-\infty, \infty)$.

d) Solve the initial value problem

$$y'' - y = 0, \quad y(0) = 1,\, y'(0) = 3.$$

a) $p(x) = 0$ & $q(x) = -1$

b) $y_1' = e^x$, $y_1'' = e^x$ & $y_2' = -e^{-x}$ & $y_2'' = e^{-x}$

$\Rightarrow$ $y_1'' - y_1 = e^x - e^x = 0$ $\Rightarrow$ y_1, y_2 are solutions.

& $y_2'' - y_2 = e^{-x} - e^{-x} = 0$

c) We see that $y(x) = c_1y_1(x) + c_2y_2(x)$

$\Rightarrow$ $y'' - y = c_1y_1'' + c_2y_2'' - c_1y_1 - c_2y_2$

$= c_1(y_1'' - y_1) + c_2(y_2'' - y_2) = c_1 \cdot 0 + c_2 \cdot 0 = 0!$

d) The general solution is $y(x) = c_1 e^{x} + c_2 e^{-x}$.

$y(0) = 1 \Rightarrow c_1 + c_2 = 1$

We have $y'(x) = c_1 e^{x} - c_2 e^{-x}$ &

$y'(0) = 3 \Rightarrow c_1 - c_2 = 3$

Solve $\begin{cases} c_1 + c_2 = 1 \\ c_1 - c_2 = 3 \end{cases} \Rightarrow \boxed{\begin{array}{l} c_1 = 2 \\ c_2 = -1 \end{array}}$

Therefore,

$$\boxed{y(x) = 2e^{x} - e^{-x}}.$$

Example 2. Let ω be a positive number. Consider

$$y'' + \omega^2 y = 0.$$

a) Identify the functions $p(x)$ and $q(x)$.

b) Verify that $y_1(x) = \cos(\omega x)$ and $y_2(x) = \sin(\omega x)$ are solutions to the ODE.

c) Verify that $y(x) = c_1 \cos(\omega x) + c_2 \sin(\omega x)$ is a solution to the ODE.

a) $p(x) = 0$ & $q(x) = \omega^2$.

b) $y_1' = -\omega \sin(\omega x)$ & $y_1'' = -\omega^2 \cos(\omega x)$

$\Rightarrow y_1'' + \omega^2 y_1 = -\omega^2 \cos(\omega x) + \omega^2 \cos(\omega x) = 0$ ✓

$y_2' = \omega \cos(\omega x)$ & $y_2'' = -\omega^2 \cos(\omega x)$

$\Rightarrow y_2'' + \omega^2 y_2'' = -\omega^2 \cos(\omega x) + \omega^2 \cos(\omega x) = 0$ ✓

Therefore, y_1 & y_2 are solutions to the ODE.

c) We have $y(x) = c_1 y_1 + c_2 y_2$. Therefore

$$
\begin{aligned}
y'' - \omega^2 y &= c_1 y_1'' + c_2 y_2'' + \omega^2 c_1 y_1 + \omega^2 c_2 y_2 \\
&= c_1 \underbrace{\left(y_1'' + \omega^2 y_1\right)}_{=0} + c_2 \underbrace{\left(y_2'' + \omega^2 y_2\right)}_{=0} \\
&= c_1 \cdot 0 + c_2 \cdot 0 \\
&= 0
\end{aligned}
$$

Yes, y is a solution to the ODE!

Sometimes, the ODE will be given in the following form:

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0$$

where P_0, P_1, and P_2 are continuous functions.

EXAMPLE 3. Consider the equation

$$x^2y'' + xy' - 4y = 0.$$

a) Identify the functions $p(x)$ and $q(x)$.

b) Verify that $y_1(x) = x^2$ and $y_2(x) = 1/x^2$ are solutions to the ODE.

c) Verify that if c_1 and c_2 are arbitrary numbers, then $y(x) = c_1x^2 + c_2/x^2$ is a solution of the ODE.

d) Solve the IVP

$$x^2y'' + xy' - 4y = 0, \quad y(1) = 2,\ y'(1) = 0.$$

a) Divide by x^2

$$\Rightarrow \quad y'' + \frac{1}{x} y' - \frac{4}{x^2} y = 0$$

$$\Rightarrow \quad p(x) = 1/x \quad \& \quad q(x) = \frac{-1}{x^2}.$$

b) $y_1' = 2x$ & $y_1'' = 2$

$$\Rightarrow \quad y_1'' + \frac{1}{x} y_1' - \frac{4}{x^2} y_1 = 2 + \frac{2x}{x} - \frac{4x^2}{x^2}$$

$$= 2+2-4 = 0 \checkmark$$

$y_1' = \frac{-2}{x^3}$ & $y_2'' = +\frac{6}{x^4}$

$$\Rightarrow \quad y_2'' + \frac{1}{x} y_2' - \frac{4}{x^2} y_2 = \frac{+6}{x^4} - \frac{2}{x^4} - \frac{4}{x^4} = 0 \checkmark$$

Therefore, y_1 & y_2 are solutions to the ODE.

(c) We know y_1, y_2 satisfies the ODE

$\Rightarrow$ $y(x) = c_1 y_1 + c_2 y_2$ also satisfies the ODE. $\checkmark$

Linear combinations

If y_1 and y_2 are functions, we say that the function

$$y(x) = c_1 y_1(x) + c_2 y_2(x),$$

where c_1 and c_2 are numbers, is a **linear combination** of y_1 and y_2.

Fact:

- If y_1 and y_2 are solutions to (1), then any linear combinations of y_1 and y_2 is a solution to (1).

Fundamental Set of Solutions

We say that $\{y_1, y_2\}$ is a **fundamental set of solutions** for (1) if every solutions of the ODE is a linear combination of y_1 and y_2.

Facts:

- $\{y_1, y_2\}$ is a fundamental set of solutions for (1) if and only if neither y_2/y_1 or y_1/y_2 is a constant.

EXAMPLE 4. Show that

- The functions $\{y_1, y_2\}$ where y_1, y_2 are as in Example 1 is a foundamental set of solutions.
- Same question for y_1, y_2 from Example 2.
- Same question for y_1, y_2 from Example 3.

a) $y_1(x) = e^x$, $y_2(x) = e^{-x}$ $\Rightarrow$ $\frac{y_1}{y_2} = \frac{e^x}{e^{-x}} = e^{2x}$ which is not constant.

$\Rightarrow$ $\{y_1, y_2\}$ is a fund. set of sols.

b) $y_1(x) = \cos(\omega x)$, $y_2(x) = \sin(\omega x)$ $\Rightarrow$ $\frac{y_2}{y_1} = \frac{\sin(\omega x)}{\cos(\omega x)} = \tan(\omega x)$ not const.

$\Rightarrow$ $\{y_1, y_2\}$ fundamental set of sols.

c) $y_1(x) = x^2$, $y_2(x) = 1/x^2$ $\Rightarrow$ $\frac{y_1}{y_2} = \frac{x^2}{1/x^2} = x^4$ not constant

$\Rightarrow$ $\{y_1, y_2\}$ fund. set of sols.

General Solutions

If $\{y_1, y_2\}$ is a fundamental set of solutions for (1), then we call the linear combination $y(x) = c_1 y_1 + c_2 y_2$ the **general solution** to (1).

Existence and Uniqueness of Solutions

It is always clever to verify if an ODE has solutions. Here are some important facts about existence and uniqueness of solutions to an ODE of the form (1).

Existence

Assume that p and q are continuous on an open interval (a, b). Then the ODE

$$y'' + p(x)y' + q(x)y = 0$$

has at least one solution on the interval (a, b).

Uniqueness

Assume again that p and q are continuous on an open interval (a, b) and let x_0 be any point in (a, b). Then the IVP

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = k_0,\ y'(x_0) = k_1$$

has a unique solution on (a, b).