Math 302

Chapter 2

Section 2.5: Exact Equations

Contents

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Example 1. Consider $y' = dy/dx$ and use this to rewrite the ODE

$$y' = \frac{y + xe^{-y/x}}{x}$$

in terms of dx and dy.

if $y' = dy/dx$, then

$$\frac{dy}{dx} = \frac{y + xe^{-y/x}}{x}$$

$$\Rightarrow \quad x\,dy = \left(y + xe^{-y/x}\right) dx$$

$$\Rightarrow \quad -\left(y + xe^{-y/x}\right) dx + x\,dy = 0$$

in differential form

Convenient form:

We will know consider an homogeneous first order ODE in the form

$$M(x,y)dx + N(x,y)dy = 0 \tag{1}$$

where M and N are two functions of the variables x and y.

Two interpretations:

- the equation (1) can be interpreted as

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{2}$$

  where x is the independent variable and y is the dependent variable.
- the equation (1) can be interpreted as

$$M(x,y)\frac{dx}{dy} + N(x,y) = 0 \tag{3}$$

  where x is the dependent variable and y is the independent variable.
- An implicit equation $F(x,y) = c$ is said to be an **implicit solution** to (1) if
  - every functions $y = y(x)$ satisfying $F(x, y(x)) = c$ is a solution to (2).
  - every functions $x = x(y)$ satisfying $F(x(y), y) = c$ is a solution to (3)

EXAMPLE 2. Show that

$$x^4y^3 + x^2y^5 + 2xy = c \qquad (*)$$

is an implicit solution of

$$\underbrace{(4x^3y^3 + 2xy^5 + 2y)}_{M(x,y)}dx + \underbrace{(3x^4y^2 + 5x^2y^4 + 2x)}_{N(x,y)}dy = 0.$$

① Have to show that

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \qquad (y = y(x)).$$

Diff. implicitly $(*)$ w.r.t. x

$$\Rightarrow 4x^3y^3 + 3x^4y^2y' + 2xy^5 + 5x^2y^4y' + 2y + 2xy' = 0$$

$$\Rightarrow \underbrace{4x^3y^3 + 2xy^5 + 2y}_{M(x,y)} + \underbrace{(3x^4y^2 + 5x^2y^4 + 2x)}_{N(x,y)}y' = 0 \checkmark$$

② Have to show that

$$M(x,y)\underbrace{\frac{dx}{dy}}^{x'} + N(x,y) = 0. \qquad (x = x(y))$$

Diff. implit. w.r.t. y :

$$\Rightarrow 4x^3x'y^3 + 3x^4y^2 + 2xx'y^5 + 5x^2y^4 + 2x'y + 2x = 0$$

$$\Rightarrow (\underbrace{4x^3y^3 + 2xy^5 + 2y}_{M(x,y)})x' + (\underbrace{3x^4y^2 + 5x^2y^4 + 2x}_{N(x,y)}) = 0 \checkmark$$

General Fact:

If $F(x,y) = c$ with F having continuous partial derivatives F_x and F_y, then

$$F(x,y) = c$$

is an implicit solution to the differential equation

$$F_x(x,y)dx + F_y(x,y)dy = 0.$$

So, a differential equation is said to be **exact** on an open rectangle R if there is a function $F = F(x, y)$ such that

$$F_x(x, y) = M(x, y) \quad \text{and} \quad F_y = N(x, y).$$

Useful fact (the exactness condition):

A differentiel equation is exact if and only if

$$M_y(x, y) = N_x(x, y).$$

EXAMPLE 3. Check if the following ODEs are exact or not.

1. $3x^2 y dx + 4x^3 dy = 0$.
2. $(4x^3y^3 + 3x^2)dx + (3x^4y^2 + 6y^2)dy = 0$.

1) $M(x,y) = 3x^2 y$, $N(x,y) = 4x^3$ $\Rightarrow$ $M_y = 3x^2$, $N_x = 12x^2$ $\Rightarrow$ $M_y \neq N_x$ $\boxed{\text{Not exact}}$

2) $M(x,y) = 4x^3y^3 + 3x^2$, $N(x,y) = 3x^4y^2 + 6y^2$ $\Rightarrow$ $M_y = 12x^3y^2$, $N_x = 12x^3y^2$ $\Rightarrow$ $M_y = N_x$ $\boxed{\text{exact}}$

Example 4. Solve

$$y' = -\frac{4x^3y^3 + 3x^2}{3x^4y^2 + 6y^2}.$$

1) Write as $M(x,y)\,dx + N(x,y)\,dy = 0$.

$$(3x^4y^2 + 6y^2)\,dy = -(4x^3y^3 + 3x^2)\,dx$$

$$\Rightarrow \underbrace{(3x^4y^2 + 6y^2)}_{N}dy + \underbrace{(4x^3y^3 + 3x^2)}_{M}\,dx = 0$$

2) Verify it's exact.

$M_y = 12x^3y^2$

$N_x = 12x^3y^2$

$\rightarrow M_y = N_x \Rightarrow$ Exact

3) Find F s.t. $F_x = M$ & $F_y = N$.

Ⓐ $F_x = 4x^3y^3 + 3x^2$ & Ⓑ $F_y = 3x^4y^2 + 6y^2$

Ⓐ Integrate w.r.t. x

$$F = x^4y^3 + x^3$$

Ⓑ Integrate w.r.t. y.

$$F = x^4y^3 + 2y^3$$

x^4y^3 is common to F $\rightarrow$ keep x^4y^3 once in F

x^3 not common $\rightarrow$ add x^3 in F

$2y^3$ not common $\rightarrow$ add $2y^3$ in F

So $F(x,y) = x^4y^3 + x^3 + 2y^3$

$$\Rightarrow \boxed{x^4y^3 + x^3 + 2y^3 = c}$$ implicit solution.

Non Rigorous but "Fast" Procedure to Solve An Exact ODE

[I] Check that the equation

$$M(x,y)dx + N(x,y)dy = 0$$

satisfies the exactness condition.

[II] Integrate the equation $F_x = M(x,y)$ with respect to x to get

$$F(x,y) = G(x,y).$$

[III] Integrate the equation $F_y = N(x,y)$ with respect to y to get

$$F(x,y) = H(x,y).$$

[IV] Identity what is in common in the expressions of the functions G and H. Call this common part $F_1(x,y)$.

[V] Identity what is not in common in the expressions of the functions G and H. Gather the uncommon part in a function $F_2(x,y)$.

[VI] Write $F(x,y) = F_1(x,y) + F_2(x,y)$.

Remarks:

- This shortcut may not work if one of the function G or H has an integral that can't be simplified.
- Sometimes, the rigorous procedure is faster (see next section).
- For the step-by-step rigorous procedure, see Example 2.5.3 (p.75) and p.77 of the textbook.