Math 302

Chapter 8

Section 8.4: Convolution

Contents

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Fall 2022

The Story of The Matches

- Suppose we have a number of matches we need to light.
- At each second, so at $t = 0$, $t = 1$, $t = 2$, $t = 3$, ..., $t = n$, we light a certain number of matches. Denote by $f(t)$ the number of matches lit at time t.
- Each matches give off smoke. Denote by $g(t)$ the smoke produced by a match after t seconds.

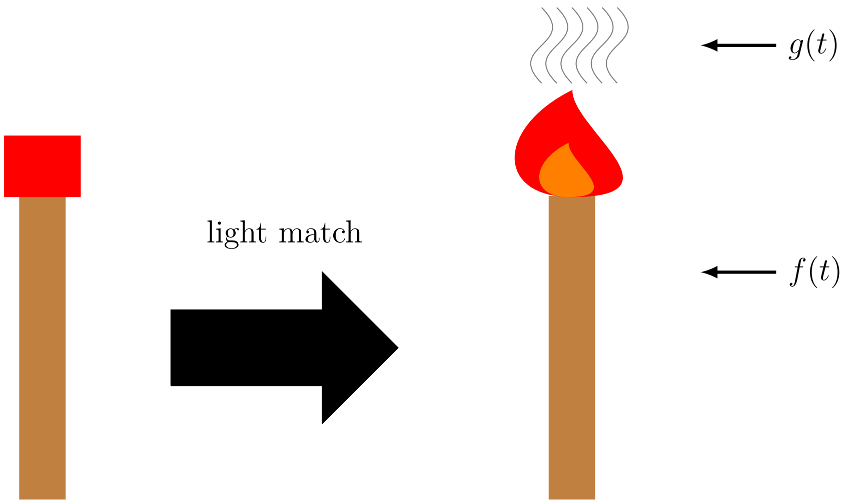


Figure 1: The Matches Problem

Question: What is the total quantity of smoke in the air after a certain time t?

Times (t)	$Q(t)$
0	$f(0)g(0)$
1	$f(1)g(0) + f(0)g(1)$
2	$f(2)g(0) + f(1)g(1) + f(0)g(2)$
n	???

The total contribution of the matches after n seconds:

$$Q(t) = f(0)g(n) + f(1)g(n-1) + \dots + f(n)g(0)$$

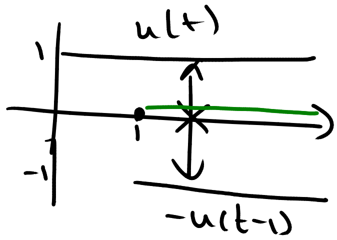
$$= \sum_{i=0}^{n} \underbrace{f(i)\,g(n-i)}_{f(x)\,g(n-x)}$$

What if we have a continuous phenomena?

Definition

The convolution of a function $f(t)$ with another function $g(t)$ is the new function $(f * g)(t)$ defined by

$$\boxed{(f * g)(t) = \int_0^t f(x)g(t-x)\,dx.}$$



Example 1. Let

$$f(t) = u(t) - u(t-1) \quad \text{and} \quad g(t) = u(t) - u(t-1).$$

Compute $f * g$.

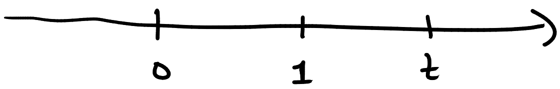
Explicitly: $f(t) = g(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & t \ge 1 \\ 0, & t < 0 \end{cases}$

1) $\underline{0 < t \le 1}$

$$f*g(t) = \int_0^t f(x)g(t-x)\,dx \qquad 0 < t - x < 1$$

$$= \int_0^t 1 \cdot 1 \, dx = t$$

2) $\underline{t > 1}$



$$f*g(t) = \int_0^t f(x)g(t-x)\,dx$$

Know: $0 \le x \le t \Rightarrow f(x) = 0$ if $x > 1$

$$\Rightarrow \int_0^t f(x)g(t-x)\,dx = \int_0^1 f(x)g(t-x)\,dx + \cancel{\int_1^t f(x)g(t-x)\,dx} \to 0$$

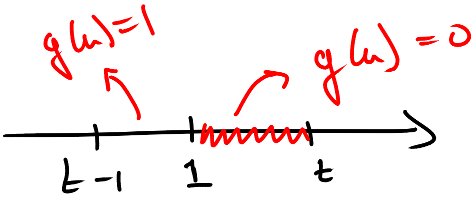
$$= \int_0^1 g(t-x)\,dx$$

$$u = t - x \quad \Rightarrow \quad \int_t^{t-1} g(u)\,(-du) = \int_{t-1}^{t} g(u)\,du$$

(a) $t-1 \geq 1 \Rightarrow t \geq 2 \Rightarrow g(u) = 0 \quad (t-1 < u < t)$

$$\Rightarrow \int_{t-1}^{t} g(u)\,du = 0$$

(b) $t-1 < 1$ & $t > 1$



$$\Rightarrow \int_{t-1}^{t} g(u)\,du = \int_{t-1}^{1} g(u)\,du$$

$$= \int_{t-1}^{1} 1\,du = 2-t$$

So,

$$f*g(t) = \begin{cases} t, & t \leq 1 \\ 2-t, & 1 < t < 2 \\ 0, & t \geq 2 \end{cases}$$

Desmos: https://www.desmos.com/calculator/h50sct4xeq

Laplace Transform

The nice properties of the convolution is a direct connection with the Laplace transform.

Example 2. Let $f(t) = e^t$ and $g(t) = e^{-t}$.

(a) Compute $f * g$.

(b) Find $L(f * g)$.

(c) Compare with $L(f)L(g)$.

(a) $f*g(t) = \int_0^t e^{x} e^{-(t-x)} \, dx$

$$= \frac{e^t - e^{-t}}{2} = \sinh(t).$$

(b) $L(f*g(t)) = L(\sinh(t)) = \dfrac{1}{s^2-1}$

(c) $L(f(t)) = L(e^t) = \dfrac{1}{s-1}$

$L(g(t)) = L(e^{-t}) = \dfrac{1}{s+1}$

$L(f(t))L(g(t)) = \dfrac{1}{s-1} \cdot \dfrac{1}{s+1} = \dfrac{1}{s^2-1}$

Same

<u>Tranform of Convolution:</u> If

- $f(t)$ is a function with Laplace transform $F(s)$;
- $g(t)$ is a function with Laplace transform $G(s)$;

then

$$L(f * g) = L(f)L(g) = F(s)G(s).$$

Example 3. Find the inverse Laplace transform of the following function:

$$H(s) = \frac{1}{s^2(s^2+4)}.$$

$$\frac{1}{s^2(s^2+4)} = \underbrace{\frac{1}{s^2}}_{F(s)} \cdot \underbrace{\frac{1}{s^2+4}}_{G(s)} \qquad s^2+4 \to 2^2$$

- $F(s) \xleftrightarrow{\text{table}} t$
- $G(s) = \frac{1}{2}\left(\frac{2}{s^2+2^2}\right) \xleftrightarrow{\text{table}} \frac{1}{2}\sin(2t)$

From convolution theorem:

$$h(t) = f(t) * g(t) = t * \frac{1}{2}\sin(2t)$$

$$= \int_0^t x\left(\frac{1}{2}\sin(2(t-x))\right)dx$$

$$= \frac{1}{2}\int_0^t x \sin(2(t-x))\,dx$$

$$= \underline{\frac{1}{2}(t - \sin t \cos t)}.$$

$$\frac{1}{(s-1)(s-2)} \to \frac{1}{s-1}\cdot\frac{1}{s-2} \to f * g$$

$$\frac{1}{(s-1)^2} = \frac{1}{s-1}\cdot\frac{1}{s-1} \to f * f$$

As a special case of the Laplace transform of a convolution, we can take the Laplace transform of an integral.

Example 4. Suppose f has a Laplace transform given by $F(s)$. Find the Laplace transform of

$$h(t) = \int_0^t f(x)\,dx.$$

$$h(t) = \int_0^t f(x) \cdot \underbrace{1}_{g(t-x)=1} \, dx$$

$$\Rightarrow \quad h(t) = f(t) * 1$$

$$\Rightarrow \quad L(h(t)) = L(f(t) * 1)$$

$$= L(f(t)) \cdot L(1)$$

$$= \frac{F(s)}{s}$$

$$\boxed{L\left(\int_0^t f(x)\,dx\right) = \frac{F(s)}{s}}$$

Other related results:

- For $\cancel{g}\,h(t) = \int_0^t \int_0^x f(u)\,du\,dx$, we have $\cancel{G}\,H(s) = F(s)/s^2$.
- For a function $\cancel{g}\,h(t)$ given as three integrals, then $\cancel{G}\,H(s) = F(s)/s^3$.
- For a function $\cancel{g}\,h(t)$ given as n integrals, then $\cancel{G}\,H(s) = F(s)/s^n$.

We can solve more than just an ODE!

EXAMPLE 5. Find the solution to the following integro-differential equation

$$\int_0^t y(u)\,du + y'(t) = t, \quad \longleftrightarrow \quad y(t) + y''(t) = 1$$

where $y(0) = 0$.

① Laplace Transform $\qquad Y(s) = L(y(t))$

$$L\left(\int_0^t y(u)\,du\right) + L\left(y'(t)\right) = L(t)$$

$$\frac{Y}{s} + sY - y(0) = \frac{1}{s^2}$$

$$\Rightarrow \quad \frac{Y}{s} + sY = \frac{1}{s^2}$$

$$\Rightarrow \quad Y\left(\frac{1+s^2}{s}\right) = \frac{1}{s^2}$$

$$\Rightarrow \quad Y = \frac{1}{s(1+s^2)}$$

② Inverse

$$Y(s) = \underbrace{\frac{1}{s}}_{L(1)} \cdot \underbrace{\frac{1}{s^2+1}}_{L(\sin t)}$$

Convo.

$$\Rightarrow \quad y(t) = \int_0^t 1 \cdot \sin(t-x)\,dx = \boxed{\cos(t) - 1}.$$

Example 6. Find the general solution to the following integral equation

$$\underline{y(t)} = \sin(t) - 2\int_0^t \underline{y(u)}\cos(t-u)\,du.$$

$$y(t) = \sin(t) - 2\int_0^t y(x)\cos(t-x)\,dx$$

$$\Rightarrow \quad y(t) = \sin t - 2\left(y * \cos(t)\right)$$

① Apply Laplace $\qquad Y(s) = L(y(t))$

$$Y = \frac{1}{s^2+1} - 2\,L(y)\cdot L(\cos t)$$

$$\Rightarrow \quad Y = \frac{1}{s^2+1} - 2Y\cdot\frac{s}{s^2+1}$$

$$\Rightarrow \quad Y\left(1 + \frac{2s}{s^2+1}\right) = \frac{1}{s^2+1}$$

$$\Rightarrow \quad Y\left(\frac{s^2+2s+1}{s^2+1}\right) = \frac{1}{s^2+1}$$

$$\Rightarrow \quad Y = \frac{1}{s^2+2s+1}$$

② Inverse.

$$s^2+2s+1 = (s+1)^2 \longrightarrow Y = \frac{1}{(s+1)^2}$$

From table $\longrightarrow \boxed{y(t) = e^{-t}t}$