Math 302

Chapter 5

Section 5.4: The Method of Undetermined Coefficient I

Contents

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We consider the following basic case:

$$ay'' + by' + cy = ke^{\alpha x}$$

where a, b, c, α, and k are fixed real numbers.

Case I

$\to$ α is not a root of $ar^2 + br + c = 0$.

When $e^{\alpha x}$ is not a solution to the complementary equation $ay'' + by' + cy = 0$.

Example 1. Find the general solution of

$$y'' - 7y' + 12y = 4e^{2x}. \longrightarrow \alpha = 2$$

1) Solve complementary Eq.

ch. eq. $\to$ $r^2 - 7r + 12 = 0$

$\to$ $(r-4)(r-3) = 0$

$\to$ $r = 4$ & $r = 3$.

So, $y_h(x) = c_1 e^{3x} + c_2 e^{4x}$.

2) Find A Particular Sol.

$y_{par}(x) = Ae^{2x}$ $\to$ $y' = 2Ae^{2x}$

$y'' = 4Ae^{2x}$

$y'' - 7y' + 12y = 4Ae^{2x} - 7(2Ae^{2x}) + 12Ae^{2x} = 2Ae^{2x}$

$\Rightarrow$ $2Ae^{2x} = 4e^{2x}$

$\Rightarrow$ $2A = 4$ $\Rightarrow$ $A = 2$

So, $y_{par}(x) = 2e^{2x}$

3) General Solution.

$y(x) = y_h(x) + y_{par}(x) = \boxed{c_1 e^{3x} + c_2 e^{4x} + 2e^{2x}}$

Case II $\to$ when α is a root of $ar^2 + br + c = 0$.

When $e^{\alpha x}$ is a solution to the complementary equation.

EXAMPLE 2. Find the general solution of

$$y'' - 7y' + 12y = 5e^{4x}.$$

1) <u>Solution to the complementary equation</u>

Same as Example 1 $\Rightarrow$ $y_h(x) = c_1 e^{3x} + c_2 e^{4x}$.

2) <u>Find a particular solution.</u>

$y_{par}(x) = \cancel{Ae^{4x}}$ $\to$ already have this solution.

$y_{par}(x) = u(x)\, e^{4x}$ $\to$ $y' = u' e^{4x} + 4u e^{4x}$

$y'' = u'' e^{4x} + 4u' e^{4x} + 4u' e^{4x} + 16 u e^{4x}$

$= u'' e^{4x} + 8u' e^{4x} + 16 u e^{4x}$.

$\Rightarrow$ $y'' - 7y' + 12y = u'' e^{4x} + u' e^{4x}$

So, $u'' \cancel{e^{4x}} + u' \cancel{e^{4x}} = 5 \cancel{e^{4x}}$

$\Rightarrow$ $u'' + u' = 5$

<u>Guesses:</u> $\cancel{u = A}$ $\to$ $u' = 0$, $u'' = 0$ $\Rightarrow$ $\cancel{0 = 5}$

$u = Ax + B$ $\to$ $u' = A$, $u'' = 0$ $\Rightarrow$ $A = 5$

Here, B is free $\Rightarrow$ $B = 0$.

$\Rightarrow$ $u(x) = 5x$ $\Rightarrow$ $y_{par}(x) = 5xe^{4x}$

(3) General Solution.

$$y(x) = y_h(x) + y_{par}(x) = \boxed{c_1 e^{3x} + c_2 e^{4x} + 5x e^{4x}}$$

In general: guess $y_{par}(x) = Axe^{4x}$

$$y' = Ae^{4x} + 4Axe^{4x}$$

$$y'' = 4Ae^{4x} + 4Ae^{4x} + 16Axe^{4x} = 8Ae^{4x} + 16Axe^{4x}$$

$$y'' - 7y' + 12y = 8Ae^{4x} + 16Axe^{4x} - 7Ae^{4x} - 28Axe^{4x} + 12Axe^{4x}$$

$$= Ae^{4x} + \cancel{0\, Axe^{4x}} = Ae^{4x}$$

$$\Rightarrow \quad Ae^{4x} = 5e^{4x} \quad \Rightarrow \quad \boxed{A = 5}$$

Case III

When $e^{\alpha x}$, and $xe^{\alpha x}$ are solutions to the complementary equation.

EXAMPLE 3. Find the general solution of

$$y'' - 8y' + 16y = 2e^{4x}.$$

1) Sol. to compl. Eq.

$y'' - 8y' + 16y = 0 \longrightarrow r^2 - 8r + 16 = 0$

$\longrightarrow (r-4)^2 = 0$

$\longrightarrow r = 4$ (repeated root).

$\Rightarrow y_h(x) = c_1 e^{4x} + c_2 x e^{4x}$

2) Find Particular Solution.

$y_{par}(x) = Ax^2 e^{4x}$.

$\Rightarrow y' = 2Axe^{4x} + 4Ax^2e^{4x}$

$y'' = 2Ae^{4x} + 8Axe^{4x} + 8Axe^{4x} + 16Ax^2e^{4x}$

$= 2Ae^{4x} + 16Axe^{4x} + 16Ax^2e^{4x}$.

$\Rightarrow y'' - 8y' + 16y = 2Ae^{4x} + \cancel{16Axe^{4x}} + \cancel{16Ax^2e^{4x}}$
$- \cancel{16Axe^{4x}} - \cancel{32Ax^2e^{4x}} + \cancel{16x^2e^{4x}}$

$= 2Ae^{4x}$

So, $\cancel{2}A\cancel{e^{4x}} = \cancel{2}\cancel{e^{4x}} \Rightarrow \boxed{A = 1}$

$\Rightarrow y_{par}(x) = x^2 e^{4x}$

3) <u>General solution</u>

$$y(x) = y_h(x) + y_{par}(x) = \boxed{c_1 e^{4x} + c_2 x e^{4x} + x^2 e^{4x}.}$$

Recap

To find a particular solution to

$$ay'' + by' + cy = ke^{\alpha x}$$

where k is a fixed real number, we follow the following tips:

- If $e^{\alpha x}$ is not a solution of the complementary equation, then we take $y_{par}(x) = Ae^{\alpha x}$, where A is a constant.
- If $e^{\alpha x}$ is a solution of the complementary equation, then we take $y_{par}(x) = xAe^{\alpha x}$, where A is a constant.
- If both $e^{\alpha x}$ and $xe^{\alpha x}$ are solutions of the complementary equation, then we take $y_{par}(x) = Ax^2e^{\alpha x}$, where A is a constant.

We now consider a more general case:

$$ay'' + by' + cy = e^{\alpha x} G(x)$$

where a, b, c, α are fixed real numbers and $G(x)$ is a polynomial.

Case I

When $e^{\alpha x}$ is not a solution to the complementary equation $ay'' + by' + cy = 0$.

EXAMPLE 4. Find the general solution to

$$y'' - 3y' + 2y = e^{3x}(x^2 + 2x - 1).$$

1) Sol. to compl. Eq.

$y'' - 3y' + 2y = 0 \quad \Rightarrow \quad r^2 - 3r + 2 = 0$

$\Rightarrow \quad (r-1)(r-2) = 0$

$\Rightarrow \quad r = 1 \;\&\; r = 2.$

So, $\quad y_h(x) = c_1 e^{x} + c_2 e^{2x}.$

2) Find a part. Solution.

Right-hand side: $\quad x^2 e^{3x} + 2x e^{3x} - e^{3x}$

$\cancel{y_{par}(x) = Ae^{3x}} \quad \rightarrow \quad y' = 3Ae^{3x} \;\&\; y'' = 9Ae^{3x}$

$\cancel{y_{par}(x) = (Ax+B)e^{3x}}$

$y_{par}(x) = (Ax^2 + Bx + C)e^{3x}.$

$\Rightarrow \quad y' = (2Ax + B)e^{3x} + 3(Ax^2 + Bx + C)e^{3x}$

$y'' = 2Ae^{3x} + 3(2Ax+B)e^{3x} + 3(2Ax+B)e^{3x}$

$\quad + 9(Ax^2 + Bx + C)e^{3x}$

$$= 2Ae^{3x} + 6(2Ax+B)e^{3x} + 9(Ax^2+Bx+C)e^{3x}$$

Replace in the ODE:

$$y'' - 3y' + 2y = 2Ae^{3x} + 6(2Ax+B)e^{3x} + \cancel{9(Ax^2+Bx+C)}e^{3x}$$
$$- 3(2Ax+B)e^{3x} - \cancel{9(Ax^2+Bx+C)}e^{3x}$$
$$+ 2(Ax^2+Bx+C)e^{3x}$$

$$= 2Ae^{3x} + 3(2Ax+B)e^{3x} + 2(Ax^2+Bx+C)e^{3x}$$

$$= 2Ae^{3x} + 6Axe^{3x} + 3Be^{3x}$$
$$+ 2Ax^2e^{3x} + 2Bxe^{3x} + 2Ce^{3x}$$

$$= (2A+3B+2C)e^{3x} + (6A+2B)xe^{3x} + 2Ax^2e^{3x}$$

$$= x^2e^{3x} + 2xe^{3x} - e^{3x}$$

$$\Rightarrow \begin{cases} 2A = 1 \\ 6A + 2B = 2 \\ 2A + 3B + 2C = -1 \end{cases} \Rightarrow \begin{cases} A = 1/2 \\ 3 + 2B = 2 \\ 1 + 3B + 2C = -1 \end{cases}$$

$$\Rightarrow \begin{cases} A = 1/2 \\ B = -1/2 \\ 1 - \frac{3}{2} + 2C = -1 \end{cases}$$

$$y'' + y' + y = (5+x)e^x$$

$$y_{par} = (Ax+B)e^x$$

$$\Rightarrow \begin{cases} A = 1/2 \\ B = -1/2 \end{cases} \quad C = -1/4$$

③ <u>General Solution.</u>

$$y(x) = y_h(x) + y_{par}(x) = c_1e^x + c_2e^{2x} + \left(\frac{x^2}{2} - \frac{x}{2} - \frac{1}{4}\right)e^{3x}$$

Case II

When $e^{\alpha x}$ is a solution to the complementary equation.

EXAMPLE 5. Find the general solution to

$$y'' - 4y' + 3y = e^{3x}(12x^2 + 8x + 6).$$

1) Sol. to comple. eq.

$$r^2 - 4r + 3 = 0 \Rightarrow (r-3)(r-1) = 0$$
$$\Rightarrow r = 3 \ \& \ r = 1$$

So

$$y_h(x) = c_1 e^x + c_2 e^{3x}$$

2) Find a particular sol.

$$y_{par}(x) = x e^{3x}(Ax^2 + Bx + C) = e^{3x}(Ax^3 + Bx^2 + Cx)$$

$$y' = 3e^{3x}(Ax^3 + Bx^2 + Cx) + e^{3x}(3Ax^2 + 2Bx + C)$$

$$y'' = 9e^{3x}(Ax^3 + Bx^2 + Cx) + 6e^{3x}(3Ax^2 + 2Bx + C) + e^{3x}(6Ax + 2B).$$

Replace in the ODE:

$$y'' - 4y' + 3y = [2(3Ax^2 + 2Bx + C) + (6Ax + 2B)]e^{3x}$$

So,

$$(6Ax^2 + (6A + 4B)x + 2C + 2B)\cancel{e^{3x}} = (12x^2 + 8x + 6)\cancel{e^{3x}}$$

$$\Rightarrow 6Ax^2 + (6A+4B)x + 2C+2B = 12x^2 + 8x + 6$$

$$\Rightarrow 6A = 12, \quad 6A + 4B = 8 \quad \& \quad 2C + 2B = 6$$

$$\Rightarrow A = 2, \quad 12 + 4B = 8 \quad \& \quad 2C + 2B = 6$$

$$\Rightarrow A = 2, \quad B = -1 \quad \& \quad 2C - 2 = 6$$

$$\Rightarrow A = 2, \quad B = -1 \quad \& \quad C = 4$$

So,

$$y_{par}(x) = xe^{3x}(2x^2 - x + 4)$$

3 <u>General solution</u>:

$$\begin{aligned} y(x) &= y_h(x) + y_{par}(x) \\ &= C_1 e^{x} + C_2 e^{3x} + xe^{3x}(2x^2 - x + 4). \end{aligned}$$

Case III

When $e^{\alpha x}$ and $xe^{\alpha x}$ are solutions to the complementary equation.

EXAMPLE 6. Find the general solution to

$$4y'' + 4y' + y = e^{-x/2}(144x^2 + 48x - 8).$$

① Complementary Equations.

$$4y'' + 4y' + y = 0 \longrightarrow 4r^2 + 4r + 1 = 0$$
$$\longrightarrow (2r+1)^2 = 0$$
$$\longrightarrow r = -\frac{1}{2}.$$

So,

$$y_h(x) = c_1 e^{-\frac{x}{2}} + c_2 x e^{-\frac{x}{2}}.$$

② Particular Solution.

$$y_{par}(x) = x^2 e^{-x/2}(Ax^2 + Bx + C)$$
$$= e^{-x/2}(Ax^4 + Bx^3 + Cx^2).$$

$$\Rightarrow \quad y' = -\frac{1}{2} e^{-x/2}(Ax^4 + Bx^3 + Cx^2) + e^{-x/2}(4Ax^3 + 3Bx^2 + 2Cx)$$

$$y'' = \ldots$$

Replace y' & y'' in the ODE and find A, B, C:

$$A = 3, \quad B = 2 \quad \& \quad C = -1.$$

$$\Rightarrow \quad y_{par}(x) = x^2 e^{-x/2}(3x^2 + 2x - 1)$$

③ General Solution: $y(x) = \boxed{c_1 e^{-x/2} + c_2 x e^{-x/2} + x^2 e^{-x/2}(3x^2 + 2x - 1)}$

Recap

To find a particular solution to

$$ay'' + by' + cy = ke^{\alpha x}G(x)$$

where k is a fixed real number and $G(x)$ is a polynomial, we follow the following tips:

- If $e^{\alpha x}$ is not a solution of the complementary equation, then we take $y_{par}(x) =$ [redacted] $e^{\alpha x}Q(x)$, where A is a constant and $Q(x)$ is a polynomial of the same degree as $G(x)$.
- If $e^{\alpha x}$ is a solution of the complementary equation, then we take $y_{par}(x) =$ [redacted] $xe^{\alpha x}Q(x)$, where A is a constant and $Q(x)$ is a polynomial of the same degree as $G(x)$.
- If $e^{\alpha x}$ and $xe^{\alpha x}$ are solutions to the complementary equation, then we take $y_{par}(x) =$ [redacted] $x^2e^{\alpha x}Q(x)$, where A is a constant and $Q(x)$ is a polynomial of the same degree as $G(x)$.