Section 4.2 — Problem 5 — 25 points

(a) (20 points) We have $T_m = 35$ and $T_0 = 150$. Threfore, the expression of $T(t)$ is

$$T(t) = 35 + 115e^{-kt}.$$

Let t_0 be the time taking for the object to drop to the temperature of 120 and let t_1 be the time for the object to drop to the temperature of 90. Then, using the formula of T, we have

$$\begin{aligned} 120 &= T(t_0) = 35 + 115e^{-kt_0}, \\ 90 &= T(t_1) = 35 + 115e^{-kt_1}. \end{aligned}$$

Therefore, after substracting 35, dividing by 115 and taking the ln on each side, we obtain

$$\begin{aligned} \ln(17/23) &= -kt_0, \\ \ln(12/23) &= -kt_1. \end{aligned}$$

Taking the difference of the first equation with the second equation, we get

$$k(t_1 - t_0) = \ln(17/23) - \ln(12/23) = \ln(17/12).$$

However, we know that $t_1 - t_0 = 5$ because t_0 and t_1 represents the time between the beginning and 12:15PM and 12:20PM respectively. Therefore, we obtain

$$k = \frac{\ln(17/12)}{5}.$$

Plugging this in the expression of T, we obtain

$$T(t) = 35 + 115e^{-\frac{\ln(17/12)}{5}t}.$$

Now that we have the constant k, we can now find one of the values of t_0 or t_1. We can then substract t_0 from the time 12:15PM to obtain the time the object was moved outside. From the equation $\ln(17/23) = -kt_0$, we can find t_0:

$$t_0 = -\frac{5\ln(17/23)}{\ln(17/12)} \approx 4.33929 \approx 4\tfrac{1}{3}.$$

The units of t_0 are min and therefore the time that the object was moved outside is approximately 12:10:40PM.

(b) (5 points) We have to solve

$$40 = 35 + 115e^{-\frac{\ln(17/12)}{5}t}.$$

We find $t \approx 45$ and therefore at 12:55:40PM.

Section 4.2 — Problem 11 — 25 points

The initial volume of water is $V_0 = 100$ gallons and it contains initially $Q_0 = 20$lb of salt. Let $Q(t)$ be the quantity of salt in the tank and let $V(t)$ be the volume of water in the tank.

There are 4 gallons of product per minute coming in the tank and there are 2 gallons of product per minute coming out the tank. So, the differential equation modeling the volume of stuff in the tank is

$$\frac{dV}{dt} = 4 - 2 = 2.$$

Therefore, we have $V(t) = 2t + c$ for some constant c. Initially, we have $V_0 = 100$ and $c = 100$. Therefore, we obtain

$$V(t) = 2t + 100.$$

The concentration of salt coming in the tank is $(1/4) \cdot 4 = 1\,\text{lb/min}$. The concentration of salt coming out of the tank is $\frac{Q(t)}{V(t)} \cdot 2 = 2Q(t)/(2t + 100)\,\text{lb/min}$. The differential equation modeling $Q(t)$ is then

$$\frac{dQ}{dt} = 1 - 2\frac{Q}{2t + 100} \iff \frac{dQ}{dt} + 2\frac{Q}{2t + 100} = 1.$$

To solve this differential equation, we first solve the complementary equation $Q' + 2Q/(2t + 100) = 0$. We separate the variables:

$$\frac{Q'}{Q} = -\frac{2}{2t + 100} \quad \Rightarrow \quad \ln Q = -\ln(2t + 100) + K.$$

Therefore, $Q(t) = c/(2t + 100)$ where $c = e^K$. We didn't use the absolute value because $Q > 0$ and $t \geq 0$ so that $2t + 100 \geq 100$.

We use variation of parameter to solve the initial EDO. Let $Q(t) = u(t)/(2t + 100)$. Therefore,

$$Q' = \frac{u'}{2t + 100} - \frac{2u}{(2t + 100)^2}.$$

Replacing Q and Q' in the initial differential equation, we get

$$\begin{aligned} &\frac{u'}{2t + 100} - \frac{2u}{(2t + 100)^2} + \frac{2u}{(2t + 100)^2} = 1 \\ \iff\ & u' = 2t + 100 \\ \iff\ & u(t) = t^2 + 100t + c. \end{aligned}$$

Therefore, we find that

$$Q(t) = \frac{t^2 + 100t}{2t + 100} + \frac{c}{2t + 100}.$$

At $t = 0$, we know that $Q_0 = 20$. Therefore, we get

$$20 = \frac{c}{100} \quad \Rightarrow \quad 2000 = c.$$

So the quantity of salt at time t is

$$Q(t) = \frac{t^2 + 100t}{2t + 100} + \frac{2000}{2t + 100}.$$

Using the function for the volume, we know that the maximum it can reach is 200gal, we find $200 = 2t + 100$ which is $t = 50$. After 50min, the tank will overflow and so we replace this value of t in the expression of $Q(t)$ to get

$$Q(50) = 47.5\,\text{lb}.$$