Math 302

Chapter 2

Section 2.2: Separable Equations

Contents

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What Is a Separable First Order ODE

A first order differential equation is separable if it can be written as

$$h(y)y' = g(x) \tag{1}$$

where

- the left-hand side is a product of a function h of y with the derivative y'.
- the right-hand side is a function g of the variable x.

Example 1. Solve the equation

$$y' = x(1 + y^2).$$

Trick:

- Write the derivative y' as $\frac{dy}{dx}$.
- Write the ODE in the form $h(y)dy = g(x)dx$.
- Integrate both sides.

Example 2.

1. Solve the equation
$$y' = -x/y.$$

2. Solve the initial value problem
$$y' = -x/y, \quad y(1) = 1.$$

Implicit Solutions of Separable Equations

In the previous examples, we could find an explicit function $y = y(x)$ that is a solution to the ODE. It not always the case though...

Example 3. If possible, find a solution to

$$y' = \frac{2x+1}{5y^4+1}.$$

Terminology: Let the functions $h(y)$ and $g(x)$ be continuous on (c, d) and (a, b) respectively. Suppose

- $H(y)$ is an antiderivative of $h(y)$ on (c, d).
- $G(x)$ is an antiderivative of $h(x)$ on (a, b).
- c is a constant.

Then the implicit equation

$$H(y) = G(x) + c$$

is called an *implicit solution* to (1).

Example 4. Find an implicit solution of

$$y' = \frac{2x+1}{5y^4+1}, \quad y(2) = 1.$$

Terminology:

Let the functions $h(y)$ and $g(x)$ be continuous on (c, d) and (a, b) respectively. Suppose

- $H(y)$ is an antiderivative of $h(y)$ on (c, d).
- $G(x)$ is an antiderivative of $h(x)$ on (a, b).
- $c = H(y_0) - G(x_0)$.

Then the implicit equation

$$H(y) = G(x) + H(y_0) - G(x_0)$$

is called an *implicit solution of the initial value problem.*

Implicit Solutions and Integral Curves

The graph of an implicit solution to

$$h(y)y' = g(x)$$

is an integral curve.

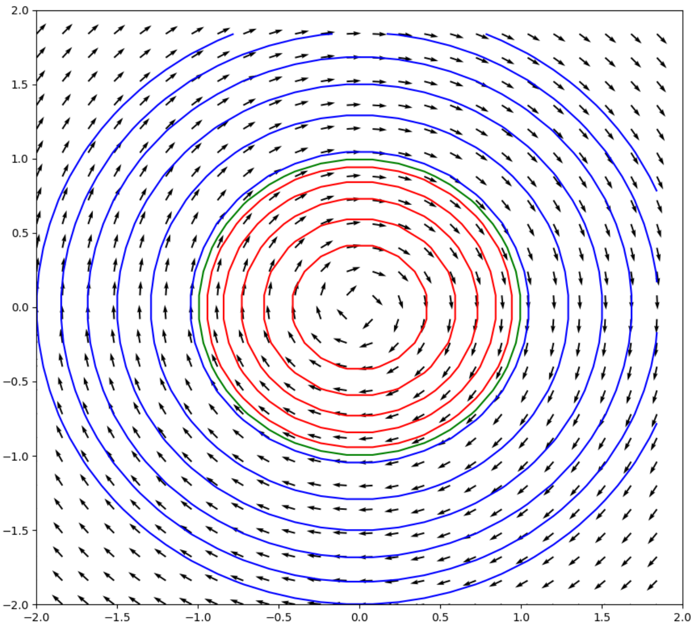

Figure 1: Direction field and implicit solutions of $y' = -\frac{x}{y}$.

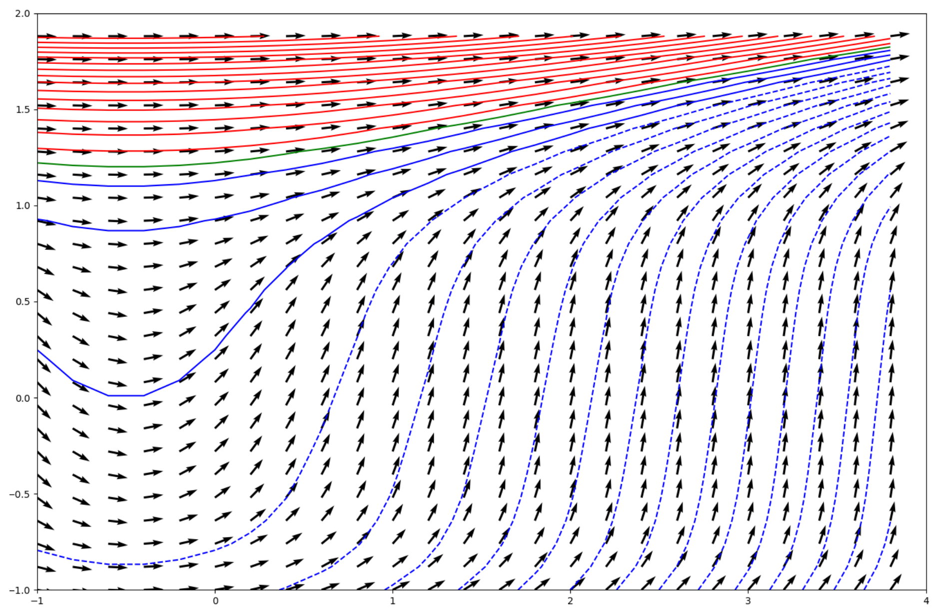

Figure 2: Direction field and implicit solutions of $y' = \frac{2x+1}{5y^4+1}$. In green you can see the implicit solution that satisfies $y(2) = 1$

Constant Solutions of Separable Equations

An equation of the form

$$y' = g(x)p(y)$$

is separable because it can be put in the following forms:

$$\frac{y'}{p(y)} = g(x).$$

Problem:

- The division by $p(y)$ is not possible if $p(y) = 0$.

Example 5. Find all solutions to

$$y' = 2xy^2.$$

Example 6. Find all solutions of

$$y' = \frac{1}{2}x(1 - y^2).$$