MATH-302 | Created by Pierre-O. Parisé
Midterm 01 | Fall 2022

Last name: —

First name: —

Question:	1	2	3	4	5	6	Total
Points:	20	20	20	20	10	10	100
Score:	—	—	—	—	—	—	—

Instructions: Make sure to write your complete name on your copy. You must answer all the questions below and write your answers directly on the questionnaire. At the end of the 75 minutes, return your copy.

No devices such as a smart phone, cell phone, laptop, or tablet can be used during the exam. You are not allowed to use the lecture notes or the textbook. You may bring one 2-sided cheat sheet of handwriting notes. You may use a digital calculator (no graphical calculators or symbolic calculators will be allowed).

You must show ALL your work to have full credit. An answer without justification is worth no point.

May the Force be with you!

Pierre-Olivier Parisé

Your Signature: ______

University of Hawai'i

Question 1 **(20 pts)**

Find the solution to the following IVP:

$$y' + \left(\frac{1+x}{x}\right) y = 0 \quad \text{and} \quad y(1) = 1.$$

Separable equation $\Rightarrow$ $\frac{y'}{y} = -\frac{1+x}{x} = -\frac{1}{x} + 1$

$\Rightarrow$ $\ln|y| = -\ln|x| - x + K$

$\Rightarrow$ $|y| = e^{-x}e^{K}/|x|$

$\Rightarrow$ $y(x) = \frac{ce^{-x}}{x}$ $(c = \pm e^{K})$

We have $y(1) = 1$ $\Rightarrow$ $1 = c \cdot e^{-1}$ $\Rightarrow$ $c = e$

Therefore, $\boxed{y(x) = \frac{e^{1-x}}{x}}$

Question 2 **(20 pts)**

Find the solutions to $3x^2ydx + 2x^3dy = 0$.

$$M_y = 3x^2, \quad N_x = 6x^2 \implies \text{Not exact.}$$

Int. Factor: $\dfrac{M_y - N_x}{N} = \dfrac{3x^2 - 6x^2}{2x^3} = -\dfrac{3}{2x}$ (ind. of y)

$$\implies \mu = e^{-\frac{3}{2}\int \frac{1}{x}dx} = x^{-3/2}$$

The new ODE is

$$3x^2y\,x^{-3/2}\,dx + 2x^3x^{-3/2}\,dy = 0$$

$$\iff 3x^{1/2}y\,dx + 2x^{3/2}\,dy = 0.$$

So, $F_x = 3x^{1/2}y$ & $F_y = 2x^{3/2} \implies F = 2x^{3/2}y$ & $F = 2x^{3/2}y$

The solution is $F = c \implies 2x^{3/2}y = c$

$$\implies \boxed{y(x) = \frac{c}{2}x^{-3/2}}$$

Question 3 **(20 pts)**

Solve the Bernoulli's equation $y' - xy = xy^{3/2}$.

① Comp. equations. $y' - xy = 0 \Rightarrow \frac{y'}{y} = x$

$\Rightarrow \ln|y| = \frac{x^2}{2} + k$

$\Rightarrow y(x) = c\, e^{x^2/2}$.

② Var. of Param: $y = u e^{x^2/2} \Rightarrow y' = u' e^{x^2/2} + x u e^{x^2/2}$

$\Rightarrow u' e^{x^2/2} + x u e^{x^2/2} - x u e^{x^2/2} = x u^{3/2} e^{3x^2/4}$

$\Rightarrow u' e^{x^2/2} = x u^{3/2} e^{3x^2/4}$

$\Rightarrow \frac{u'}{u^{3/2}} = x e^{x^2/4}$ $\left(\text{sub: } \begin{array}{l} v = x^2 \\ dv = 2x\,dx \end{array} \right)$

$\Rightarrow 2u^{1/2} = \int \frac{e^{v/4}}{2} dv = 2 e^{x^2/4} + c$

$\Rightarrow u = \left(e^{x^2/4} + c\right)^2$ ($c/2$ became c)

So,

$$\boxed{y(x) = e^{x^2/2}\left(e^{x^2/4} + c\right)^2 .}$$

Question 4 (20 pts)

A tank with a maximal capacity of 1200 gallons initially contains 40 pounds of salt dissolved in 600 gallons of water. Starting at $t_0 = 0$, water that contains 1/2 pound of salt per gallon is added to the tank at the rate of 6 gal/min and the resulting mixture is drained from the tank at 6gal/min.

(a) (5 points) Find a differential equation for $Q(t)$, the quantity of salt in the tank at time t (time is in minutes).

(b) (12 points) Solve the equation obtained from part (a).

(c) (3 points) Compute $\lim_{t\to\infty} Q(t)$. What does it represent physically?

(a) Volume is constant $\to$ 600 gal.

$$\frac{dQ}{dt} = \text{rate in} - \text{rate out} = \frac{1}{2}\frac{\text{lb}}{\text{gal}} \cdot \frac{6\text{gal}}{\text{min}} - \frac{Q\,\text{lb}}{600\text{gal}} \cdot \frac{6\text{gal}}{\text{min}}$$

$$= 3 - \frac{Q}{100}.$$

$$\Rightarrow \boxed{Q' = 3 - \frac{Q}{100}}.$$

(b) We have $Q' = \dfrac{300 - Q}{100}$

$$\Rightarrow \frac{Q'}{300-Q} = \frac{1}{100}$$

$$\Rightarrow -\ln|300-Q| = \frac{t}{100} + K$$

$$\Rightarrow \frac{1}{|300-Q|} = e^{t/100} e^{K}$$

$$\Rightarrow e^{-K} e^{-t/100} = |300-Q|$$

$$\Rightarrow Q = 300 - c e^{-t/100} \qquad (c = \pm e^{-K})$$

Init. Cond: $Q(0) = 40 \Rightarrow 40 = 300 - c \Rightarrow c = 260$

Therefore, $Q(t) = 300 - 260 e^{-t/100}$.

(c) $\lim_{t\to\infty} Q(t) = 300 - 260 \cdot 0 = 300$ (equilibrium solution).

QUESTION 5 **(10 pts)**

Answer the following questions.

(a) (5 points) Find **all** functions M such that the following ODE is exact:

$$M(x, y)dx + (x^2 - y^2)dy = 0.$$

<u>Exactness condition:</u> $M_y = N_x = 2x$

integrate $\Rightarrow$ $M(x,y) = \int 2x \, dy + g(x)$

$= \boxed{2xy + g(x)}$

(b) (5 points) The picture below represents a direction field for a certain differential equation. Draw five different integral curves on the picture below. Explain, with a short paragraph, how you drew the integral curves.

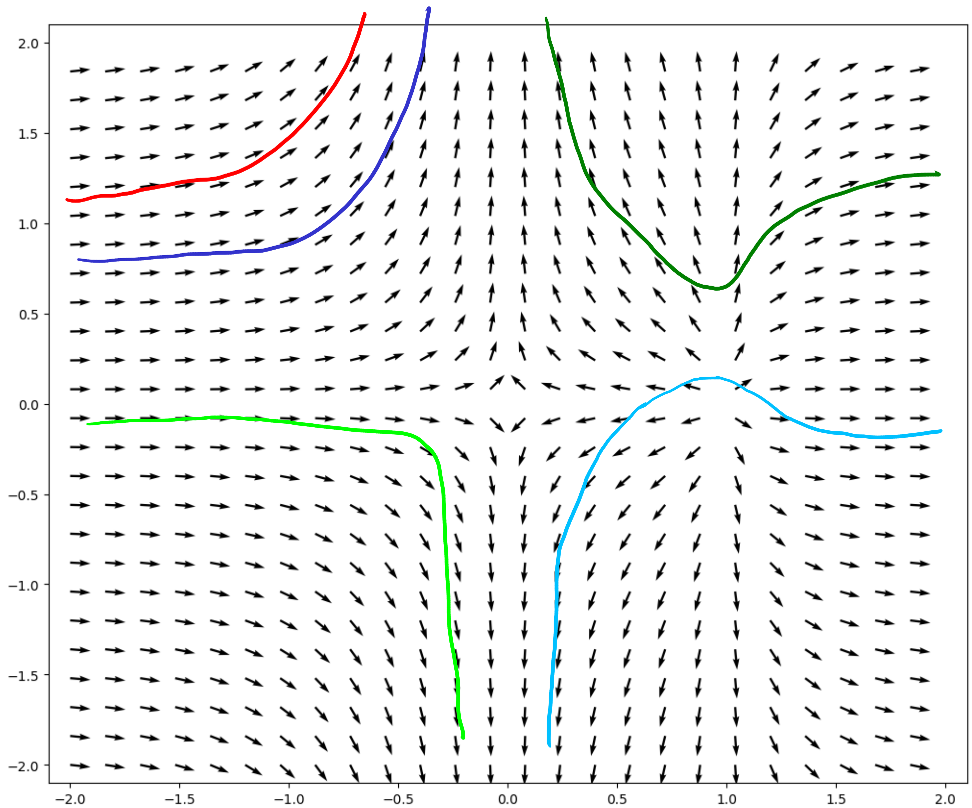

Figure 1: Direction field of some ODE

The integral curves follow the direction of the arrow in the direction line. There are "stream lines".

Question 6 **(10 pts)**

Answer the following statements with **True** or **False**. Write your answer on the horizontal line at the end of each statement. Justify your answer in the white space underneath each statement.

(a) (/ 2) The DE $(x^2 + xy)dx + (y^3 + x)dy = 0$ is exact.

$M_y = x$ & $N_x = 1$ $\Rightarrow$ Not exact.

(a) False.

(b) (/ 2) The order of the DE $x^2 y^{(3)} + y^4 = \tan(x)$ is four.

Highest derivative : 3

(b) False.

(c) (/ 2) If $y_1(x)$ is a solution to $y^2 + y' = 0$, then $2y_1$ is a solution to $y^2 + y' = 0$.

$(2y_1)^2 + (2y_1)' = 4y_1^2 + 2y_1' = 2(2y_1^2 + y_1') \neq 0$.

(c) False.

(d) (/ 2) Any solution to $y' + p(x)y = 0$ is of the form $y(x) = ce^{-\int p(x)\,dx}$.

$\frac{y'}{y} = -p(x) \Rightarrow \ln|y| = -\int p(x)\,dx$

$\Rightarrow y = c\,e^{-\int p\,dx}$ ✓

(d) True.

(e) (/ 2) The function $y(x) = \frac{1}{x-1}$ is a solution to the following IVP: $y' + y^2 = 0$ and $y(0) = -1$.

$y' = \frac{-1}{(x-1)^2} \rightarrow y' + y^2 = \frac{-1}{(x-1)^2} + \frac{1}{(x-1)^2} = 0$. ✓

$y(0) = \frac{1}{0-1} = -1$ ✓

(e) True.