Math 302

Chapter 2

Section 2.5: Exact Equations

Contents

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Another Way to Present an ODE

Example 1. Consider $y' = dy/dx$ and use this to rewrite the ODE

$$y' = \frac{y + xe^{-y/x}}{x}$$

in terms of dx and dy.

Convenient form:

We will now consider an homogeneous first order ODE in the form

$$M(x,y)dx + N(x,y)dy = 0 \tag{1}$$

where M and N are two functions of the variables x and y.

Two interpretations:

- the equation (1) can be interpreted as

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{2}$$

  where x is the independent variable and y is the dependent variable.

- the equation (1) can be interpreted as

$$M(x,y)\frac{dx}{dy} + N(x,y) = 0 \tag{3}$$

  where x is the dependent variable and y is the independent variable.

- An implicit equation $F(x,y) = c$ is said to be an **implicit solution** to (1) if
  - every function $y = y(x)$ satisfying $F(x, y(x)) = c$ is a solution to (2).
  - every function $x = x(y)$ satisfying $F(x(y), y) = c$ is a solution to (3)

EXAMPLE 2. Show that

$$x^4y^3 + x^2y^5 + 2xy = c$$

is an implicit solution of

$$(4x^3y^3 + 2xy^5 + 2y)dx + (3x^4y^2 + 5x^2y^4 + 2x)dy = 0.$$

<u>General Fact:</u>

If $F(x, y) = c$ with F having continuous partial derivatives F_x and F_y, then

$$F(x, y) = c$$

is an implicit solution to the differential equation

$$F_x(x, y)dx + F_y(x, y)dy = 0.$$

So, a differential equation is said to be **exact** on an open rectangle R if there is a function $F = F(x, y)$ such that

$$F_x(x, y) = M(x, y) \quad \text{and} \quad F_y = N(x, y).$$

Useful fact (the exactness condition):

A differentiel equation is exact if and only if

$$M_y(x, y) = N_x(x, y).$$

Example 3. Check if the following ODEs are exact or not.

1. $3x^2 y dx + 4x^3 dy = 0$.
2. $(4x^3y^3 + 3x^2)dx + (3x^4y^2 + 6y^2)dy = 0$.

Example 4. Solve

$$y' = -\frac{4x^3y^3 + 3x^2}{3x^4y^2 + 6y^2}.$$

Non Rigorous but "Fast" Procedure to Solve An Exact ODE

[I] Check that the equation

$$M(x, y)dx + N(x, y)dy = 0$$

satisfies the exactness condition.

[II] Integrate the equation $F_x = M(x, y)$ with respect to x to get

$$F(x, y) = G(x, y).$$

[III] Integrate the equation $F_y = N(x, y)$ with respect to y to get

$$F(x, y) = H(x, y).$$

[IV] Identity what is in common in the expressions of the functions G and H. Call this common part $F_1(x, y)$.

[V] Identity what is not in common in the expressions of the functions G and H. Gather the uncommon part in a function $F_2(x, y)$.

[VI] Write $F(x, y) = F_1(x, y) + F_2(x, y)$.

Remarks:

- This shortcut may not work if one of the function G or H has an integral that can't be simplified.
- Sometimes, the rigorous procedure is faster (see next section).
- For the step-by-step rigorous procedure, see Example 2.5.3 (p.75) and p.77 of the textbook.