MATH-302
Midterm 02

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Last name: ______________________________
First name: ______________________________

Instructions:

- Make sure to write your complete name on your copy.
- You must answer all the questions below and write your answers directly on the questionnaire.
- You have 75 minutes to complete the exam.
- When you are done (or at the end of the 75min period), return your copy.
- No devices such as a smart phone, cell phone, laptop, or tablet can be used during the exam.
- **Turn your cellphone off during the exam.**
- You may use a digital calculator (no graphical calculators or symbolic calculators will be allowed).
- You are not allowed to use the lecture notes or the textbook.
- You may bring one 2-sided cheat sheet of handwriting notes.
- You must show ALL your work to have full credit. An answer without justification is worth no point.

Your Signature: ______________________________

May the Force be with you!

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QUESTION 1 (20 pts)

For the given ODE, find the general solution.

(a) (10 points) $y'' + 2y' + y = 0$.

Characteristic ~~poly~~ eq: $r^2 + 2r + 1 = 0 \iff (r+1)^2 = 0$

$\iff r = -1$ mult. roots

So,

$$\boxed{y(x) = c_1 e^{-x} + x c_2 e^{-x}}$$

(b) (10 points) $y'' + 6y' + 10y = 0$.

Char. eq. $r^2 + 6r + 10 = 0 \Rightarrow r = \frac{-6 \pm \sqrt{36 - 40}}{2}$

$\Rightarrow r_1 = \frac{-3 + \sqrt{-4}}{2}, \quad r_2 = -3 - \frac{\sqrt{-4}}{2}$

$\Rightarrow r_1 = -3 + \frac{2i}{2}, \quad r_2 = -3 - \frac{2i}{2}$

$\Rightarrow r_1 = -3 + i$ & $r_2 = -3 - i$.

So,

$$\boxed{y(x) = c_1 e^{-3x} \cos(x) + c_2 e^{-3x} \sin(x).}$$

QUESTION 2 (20 pts)

For the following ODEs, give the form of the particular solution. **Don't solve for the constants.**

(a) (10 points) $y'' + 5y' - 6y = 22 + 18x - 18x^2$.

① Compl. equat. $r^2 + 5r - 6 = 0 \Rightarrow (r+6)(r-1) = 0$

$\Rightarrow r = -6$ or $r = 1$

So, $y(x) = c_1 e^{x} + c_2 e^{-6x}$.

② Part. solu.

So exponentials $\Rightarrow$ $\boxed{y_{par}(x) = Ax^2 + Bx + C.}$

(b) (10 points) $y'' - 2y' + 5y = e^x\big((6+8x)\cos(2x) + (6-8x)\sin(2x)\big)$.

① Compl. Eq. $r^2 - 2r + 5 = 0 \Rightarrow r = \frac{2 \pm \sqrt{4-20}}{2}$

$\Rightarrow r_1 = 1 + \frac{\sqrt{-16}}{2}$, $r_2 = 1 - \frac{\sqrt{-16}}{2}$

$\Rightarrow r_1 = 1 + \frac{4i}{2}$, $r_2 = 1 - \frac{4i}{2}$

$\Rightarrow r_1 = 1+2i$, $r_2 = 1-2i$.

So, $y(x) = c_1 e^{x}\cos(2x) + c_2 e^{x}\sin(2x)$.

② Part. Sol. $e^{x}\cos(2x)$ & $e^{x}\sin(2x)$ are in Right-hand sides:

$\boxed{y_{par}(x) = xe^{x}\big((Ax+B)\cos(2x) + (Cx+D)\sin(2x)\big).}$

QUESTION 3 (20 pts)

Find the general solution to the following ODE:

$$y'' - 4y' - 5y = -6e^{-x}.$$

① Complement Equations.

$y'' - 4y' - 5y = 0 \Rightarrow r^2 - 4r - 5 = 0$

$\Rightarrow (r+1)(r-5) = 0$

$\Rightarrow r = -1$ & $r = 5$.

So, $y_c(x) = c_1 e^{-x} + c_2 e^{5x}$.

② Parti. Solut.

e^{-x} is in the right-hand side

$\Rightarrow y_{par}(x) = Axe^{-x}$.

We have $y' = Ae^{-x} - Axe^{-x}$

$y'' = -Ae^{-x} - Ae^{-x} + Axe^{-x} = -2Ae^{-x} + Axe^{-x}$.

So,

$-2Ae^{-x} \cancel{+ Axe^{-x}} - 4Ae^{-x} \cancel{+ 4Axe^{-x}} \cancel{- 5Axe^{-x}} = -6e^{-x}$

$\Rightarrow -6Ae^{-x} = -6e^{-x}$

$\Rightarrow A = 1$. So, $y_{par}(x) = xe^{-x}$.

③ Answer, General Solution

$y(x) = y_c(x) + y_{par}(x) = \boxed{c_1 e^{-x} + c_2 e^{5x} + xe^{-x}}$.

Question 4 (20 pts)

Find the general solution to the following ODE:

$$x^2y'' + xy' - 4y = -6x - 4$$

knowing that $y_1(x) = x^2$ is a solution to the complementary equation.

① Var. Param. Let $y(x) = u(x)\,x^2$.

$\Rightarrow \quad y' = u'x^2 + 2xu$

$\quad\;\; y'' = u''x^2 + 4xu' + 2u$.

Insert in the ODE:

$$x^2(u''x^2 + 4xu' + 2u) + x(u'x^2 + 2xu) - 4ux^2 = -6x - 4$$

$$\Rightarrow u''x^4 + 4x^3u' + \cancel{2x^2u} + u'x^3 + \cancel{2x^2u} \cancel{-4ux^2} = -6x - 4$$

$$\Rightarrow u''x^4 + 5x^3u' = -6x - 4.$$

② Reduction Set $z = u'$ & so $z' = u''$

$\Rightarrow \quad z'x^4 + 5x^3z = -6x - 4$.

mult. x x $\Rightarrow \quad z'x^5 + 5x^4z = -6x^2 - 4x$

$\Rightarrow \quad (zx^5)' = -6x^2 - 4x$

$\Rightarrow \quad zx^5 = -2x^3 - 2x^2 + c_1$

$\Rightarrow \quad z = \dfrac{-2}{x^2} - \dfrac{2}{x^3} + \dfrac{c_1}{x^5}$

Since $z = u'$ $\Rightarrow$ $u(x) = \dfrac{2}{x} + \dfrac{1}{x^2} - \dfrac{c_1}{4x^4} + c_2$

③ Answer:

$$y(x) = u(x)\cdot x^2 = \boxed{2x + 1 - \frac{c_1}{4x^2} + c_2x^2}.$$

QUESTION 5 (10 pts)

(a) (5 points) If y_1 and y_2 are two functions, the Wronkians W of $\{y_1, y_2\}$ is

$$W = y_1 y_2' - y_1' y_2.$$

Show that if $\{y_1, y_2\}$ is **not** a set of fundamental solutions, then $W = 0$.

If $\{y_1, y_2\}$ is not a set of fundamental solution, then $\frac{y_1}{y_2} = c$ for some constant c.

Take the derivative:

$$\frac{y_1' y_2 - y_1 y_2'}{y_2^2} = 0$$

and multiplying by y_2^2 & by -1

$$\Rightarrow \quad y_1 y_2' - y_1' y_2 = 0 \quad \Rightarrow \quad W = 0.$$

(b) (5 points) Solve the following IVP:

$$y'' + y = 0, \quad y(0) = 0, y'(0) = 1.$$

The general solution is $y(x) = c_1 \cos(x) + c_2 \sin(x)$.

So,

$$y(0) = c_1 = 0$$

&

$$y'(x) = -c_1 \sin x + c_2 \cos(x)$$

$$\Rightarrow \quad y'(0) = c_2 = 1.$$

So, $\boxed{y(x) = \sin(x).}$

QUESTION 6 (10 pts)

Answer the following statements with **True** or **False**. Write your answer on the horizontal line at the end of each statement. Justify your answer in the white space underneath each statement.

(a) (/ 2) $\{x, 1\}$ is a fundamental of solutions to $y'' = 0$.

Integrate two times: $Ax + B = y(x)$.
& $\frac{x}{1} = x$ (not constant).

(a) True.

(b) (/ 2) If $y_1(x) = \cos(2x) + \sin(2x)$ and $y_2(x) = 2\cos(2x) + 2\sin(2x)$ are solutions to $y'' + 4y = 0$, then $y(x) = 3\cos(2x) + 3\sin(2x)$ is a solution to $y'' + 4y = 0$.

Principle of superposition: $y_1 + y_2$ solution to $y''+4y=0$.
We see that $y(x) = y_1(x) + y_2(x)$.

(b) True

(c) (/ 2) In the Spring-mass system model $y'' + (k/m)y = \frac{F_0}{m}\cos(\omega t)$, a resonance occurs when $\sqrt{k/m} = \omega$.

True, the sol. to the complementary eq. is
$y_c(x) = A\cos(\sqrt{k/m}\,x) + B\sin(\sqrt{k/m}\,x)$
$= A\cos(\omega x) + B\sin(\omega x)$.

(c) True.

(d) (/ 2) If $y_1 = x$ and $y_2 = e^x$ are solutions to the complementary equation $(x-1)y'' - xy' + y = (x-1)^2$, then the solution should have the form $y(x) = xu_1(x) + e^x u_2(x)$.

This is the method of Var. of Params for 2nd order diff. equa.

(d) True.

(e) (/ 2) The function $y(x) = \sin(x) + \cos(x)$ is a solution to the following IVP: $y'' + y = 0$, $y(0) = 1$, $y'(0) = 1$.

$y''(x) = -\sin(x) - \cos(x)$
$+\ y(x) = \sin(x) + \cos(x)$ ✓ & $y(0) = 1$
$y'' + y(x) = 0$ $\qquad y'(0) = \cos(0) = 1$ ✓

(e) True.