Math 302

Chapter 5

Section 5.5: The Method of Undetermined Coefficient II

Contents

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We consider the following first basic case:

$$ay'' + by' + cy = F\cos\omega x + G\sin\omega x$$

where F, G and $\cancel{\alpha}$ ω are fixed real numbers.

Case I

When $\cos\omega x$ and $\sin\omega x$ are not solution to the complementary equation $ay'' + by' + cy = 0$.

EXAMPLE 1. Find the general solution to

$$y'' - 2y' + y = 5\cos 2x + 10\sin 2x.$$

1 Comple. Eq.

$$y'' - 2y' + y = 0 \rightarrow r^2 - 2r + 1 = 0$$

$$\rightarrow (r-1)^2 = 0$$

$$\rightarrow r = 1 \quad \text{(repeated root)}.$$

$$\Rightarrow \quad y_h(x) = c_1 e^{x} + c_1 x e^{x}$$

$\cos(2x)$ & $\sin(2x)$ don't appear

2 Part. Sol.

$$y_p(x) = A\cos(2x) + B\sin(2x)$$

$$\Rightarrow y'(x) = -2A\sin(2x) + 2B\cos(2x)$$

$$y''(x) = -4A\cos(2x) - 4B\sin(2x).$$

Replace in the ODE:

$$-4A\cos(2x) - 4B\sin(2x) + 4A\sin(2A) - 4B\cos(2x)$$
$$+ A\cos(2x) + B\sin(2x) = 5\cos(2x) + 10\sin(2x)$$

$$\Rightarrow (-3A - 4B)\cos(2x) + (4A - 3B)\sin(2x)$$
$$= 5\cos(2x) + 10\sin(2x)$$

$$\Rightarrow \begin{cases} -3A - 4B = 5 & (1) \\ 4A - 3B = 10 & (2) \end{cases} \longrightarrow \quad 4A = 3B + 10$$
$$\longrightarrow \quad A = \frac{3}{4}B + \frac{10}{4}$$

Replace in (1) $\Rightarrow$ $-3\left(\frac{3}{4}B + \frac{10}{4}\right) - 4B = 5$

$$\Rightarrow \left(-\frac{9}{4} - 4\right)B - \frac{30}{4} = 5$$

$$\Rightarrow \frac{-25}{4}B = \frac{50}{4}$$

$$\Rightarrow B = -2$$

From (2) $\Rightarrow$ $4A + 6 = 10 \Rightarrow A = 1$

So, $y_{par}(x) = \cos(2x) - 2\sin(2x)$.

(3) General Solution

$$y(x) = y_h(x) + y_{par}(x) = c_1e^x + c_2xe^x + \cos(2x) - 2\sin(2x).$$

Case II

When $\cos\omega x$ or $\sin\omega x$ are solutions to the complementary equation.

EXAMPLE 2. Find the general solution to

$$y'' + 4y = 8\cos 2x + 12\sin 2x.$$

① Comple. Equation.

$$y'' + 4y = 0 \longrightarrow r^2 + 4 = 0$$
$$\longrightarrow r^2 = -4$$
$$\longrightarrow r = \pm\sqrt{-4}$$
$$\longrightarrow r = \pm\sqrt{4}\sqrt{-1}$$
$$\longrightarrow r = \pm 2i$$

So, $y_h(x) = c_1\cos(2x) + c_2\sin(2x)$.

② Part. solution.

$$y_{par}(x) = x\left[A\cos(2x) + B\sin(2x)\right]$$

$$\Rightarrow y'(x) = A\cos(2x) + B\sin(2x) + x\left[-2A\sin(2x) + 2B\cos(2x)\right]$$

&

$$y''(x) = -2A\sin(2x) + 2B\cos(2x) - 2A\sin(2x) + 2B\cos(2x) - x(4A\cos(2x) + 4B\sin(2x)).$$

$$= -4A\sin(2x) + 4B\cos(2x) - x(4A\cos(2x) + 4B\sin 2x)$$

Replace in the ODE:

$$\Rightarrow \quad 4B\cos(2x) - 4A\sin(2x) = 8\cos(2x) + 12\sin(2x)$$

$$\Rightarrow \quad 4B = 8 \quad \& \quad -4A = 12$$

$$\Rightarrow \quad B = 2 \quad \& \quad A = -3$$

So, $$y_{par}(x) = x\left(-3\cos 2x + 2\sin 2x\right)$$

③ <u>General Solution:</u>

$$y(x) = y_h + y_{par}$$

$$= c_1\cos 2x + c_2\sin 2x + x\left(2\sin 2x - 3\cos 2x\right).$$

We consider the following second basic case:

$$ay'' + by' + cy = F(x)\cos\omega x + G(x)\sin\omega x$$

where ω is a fixed real number and F, G are two polynomials.

There are still two cases: weither $\cos\omega x$ and $\sin\omega x$ are or are not solutions to the complementary equation.

EXAMPLE 3. Find the general solution to

$$y'' + 3y' + 2y = (16 + 20x)\cos x + 10\sin x.$$

① Compl. Eq.

$y'' + 3y' + 2y = 0 \Rightarrow r^2 + 3r + 2 = 0$

$\Rightarrow (r+2)(r+1) = 0$

$\Rightarrow r = -1 \;\&\; r = -2.$

So, $y_h(x) = c_1 e^{-x} + c_2 e^{-2x}$

② Particular solutions.

$y_{par}(x) = (Ax+B)\cos(x) + (Cx+D)\sin(x).$

$y' = A\cos x - (Ax+B)\sin(x)$
$+ C\sin x + (Cx+D)\cos(x)$

$y'' = -A\sin x - A\sin x - (Ax+B)\cos x$
$+ C\cos x + C\cos x - (Cx+D)\sin x$
$= -2A\sin x + 2C\cos x - (Ax+B)\cos x$
$- (Cx+D)\sin x$

Replace in the ODE:

$$-2A\sin x - 2C\cos x - (Ax+B)\cos x - (Cx+D)\sin x$$

$$-3A\cos x + 3(Ax+B)\sin(x) - 3C\sin x$$
$$-3((Cx+D)\cos(x)$$

$$+2(Ax+B)\cos(x) + 2((Cx+D))\sin(x)$$

$$= [B+2A+3D+2C+(A+3C)x]\cos(x)$$
$$+[D+3C-3B-2A+(C-3A)x]\sin x.$$

$$= (16+20x)\cos(x) + 10\sin x.$$

$$\Rightarrow \quad B+2A+3D+2C=16 \quad \& \quad A+3C=20$$
$$D+3C-3B-2A=10 \quad \& \quad C-3A=0$$

$$\Rightarrow \quad A=2, \quad B=1, \quad C=6 \quad \& \quad D=-1.$$

Therefore:
$$y_{par}(x) = (2x+1)\cos x + (6x-1)\sin x.$$

③ General Solution:

$$y(x) = y_h(x) + y_{par}(x)$$
$$= c_1e^{-x} + c_2e^{-2x} + (2x+1)\cos(x) + (6x-1)\sin x.$$

We now consider the more general case

$$ay'' + by' + c = e^{\alpha x}\left(F(x)\cos\omega x + G(x)\sin\omega x\right)$$

where α, ω are real numbers with $\omega \neq 0$ and F, G are polynomials.

There are also two cases: weither $e^{\alpha x}\cos\omega x$ and/or $e^{\alpha x}\sin\omega x$ are or are not solutions to the complementary equation.

Example 4. Find the general solution of

$$y'' + 2y' + 5y = e^{-x}\left((6-16x)\cos 2x - (8+8x)\sin 2x\right).$$

$e^{-x}\cos 2x$ $\quad$ $e^{-x}\sin 2x$

① Compl. Eq.

$$y'' + 2y' + 5y = 0 \implies r^2 + 2r + 5 = 0$$

$$\implies r = \frac{-2 \pm \sqrt{4-20}}{2}$$

$$= -1 \pm \frac{\sqrt{-16}}{2}$$

$$= -1 \pm \frac{\sqrt{-1}\sqrt{16}}{2}$$

$$\implies r_1 = -1 + 2i$$

& $$r_2 = -1 - 2i$$

So,

$$y_h(x) = c_1 e^{-x}\cos(2x) + c_2 e^{-x}\sin(2x)$$

$$= e^{-x}\left(c_1\cos(2x) + c_2\sin(2x)\right)$$

② Part. solution:

$$y_{par}(x) = x e^{-x}\left[(Ax+B)\cos 2x + (Cx+B)\sin 2x\right]$$

Instead use var. of param:

$$y(x) = u(x)\, e^{-x}$$

$$\Rightarrow \quad y'(x) = u' e^{-x} - u e^{-x}$$

$$\& \quad y''(x) = u'' e^{-x} - 2u' e^{-x} + u e^{-x}$$

Replace in the ODE:

$$u'' e^{-x} \cancel{- 2u' e^{-x}} + u e^{-x} \cancel{+ 2u' e^{-x}} - 2u e^{-x} + 5u e^{-x}$$

$$= u'' \cancel{e^{-x}} + 4u \cancel{e^{-x}}$$

$$= \cancel{e^{-x}} \left[(6 - 16x) \cos 2x - (8 + 8x) \sin 2x \right]$$

$$\Rightarrow \quad u'' + 4u = (6 - 16x) \cos 2x - (8 + 8x) \sin 2x$$

2.1 Complementary Eq.

$$u'' + 4u = 0 \quad \to \quad r^2 + 4 = 0 \quad \to \quad \begin{array}{l} r_1 = 2i \\ r_2 = -2i \end{array}$$

$$\Rightarrow \quad u_h(x) = c_1 \cos(2x) + c_2 \sin(2x).$$

2.2 Find part. sol. for u.

$$u_{par}(x) = x\left[(Ax + B) \cos(2x) + (Cx + D) \sin(2x)\right].$$

$$= (Ax^2 + Bx) \cos(2x) + (Cx^2 + Dx) \sin(2x)$$

$$u' = (2Ax + B) \cos(2x) - 2(Ax^2 + Bx) \sin(2x)$$

$$+ (2Cx + D) \sin(2x) + 2(Cx^2 + Dx) \cos(2x)$$

$$\& \quad u'' = 2A \cos(2x) - 4(2Ax + B) \sin(2x) - 4(Ax^2 + Bx) \cos(2x)$$

$$+ 2C \sin(2x) + 4(2Cx + D) \cos(2x) - 4(Cx^2 + Dx) \sin(2x)$$

$$
\begin{aligned}
u'' + 4u = {} & 2A\cos(2x) - 4(2Ax+B)\sin(2x) \\
& - 4\cancel{(Ax^2+Bx)\cos(2x)} + 2C\sin(2x) \\
& + 4(2Cx+D)\cos(2x) - 4\cancel{(Cx^2+Dx)\sin(2x)} \\
& + 4\cancel{(Ax^2+Bx)\cos(2x)} + 4\cancel{(Cx^2+Dx)\sin(2x)} \\
= {} & (8Cx + 2A + 4D)\cos(2x) \\
& + (-8Ax - 4B + 2C)\sin(2x) \\
= {} & (6 - 16x)\cos(2x) - (8 + 8x)\sin(2x).
\end{aligned}
$$

$$
\Rightarrow \begin{cases} 8C = -16 \\ 2A + 4D = 6 \\ -8A = -8 \\ -4B + 2C = -8 \end{cases} \Rightarrow \begin{cases} A = 1 \\ B = 1 \\ C = -2 \\ D = 1 \end{cases}
$$

So, $u_{par}(x) = x\left((x+1)\cos(2x) + (-2x+1)\sin(2x)\right)$

(set $c_1 = c_2 = 0$).

$$
\Rightarrow y_{par}(x) = e^{-x}x\left((x+1)\cos(2x) + (-2x+1)\sin(2x)\right).
$$

③ General Solution:

$$
\begin{aligned}
y(x) &= y_h(x) + y_{par}(x) \\
&= c_1 e^{-x}\cos(2x) + c_2 e^{-x}\sin(2x) \\
&\quad + x e^{-x}\left((x+1)\cos(2x) + (-2x+1)\sin(2x)\right).
\end{aligned}
$$

Recap

A particular solution of

$c = 0$

$ar^2 + br = 0$

$(ar + b)r = 0$

$$ay'' + by' + cy = e^{\alpha x}\left(F(x)\cos\omega x + G(x)\sin\omega x\right)$$

where $\omega \neq 0$ has the form

- when $e^{\alpha x}\cos\omega x$ and $e^{\alpha x}\sin\omega x$ are not solutions to the complementary equation,

$$y_{par}(x) = e^{\alpha x}\left(A(x)\cos\omega x + B(x)\sin\omega x\right),$$

  with $A(x)$ and $B(x)$ are polynomials of the same degree as the biggest degree between $F(x)$ and $G(x)$

- When $e^{\alpha x}\cos\omega x$ and $e^{\alpha x}\sin\omega x$ are solutions to the complementary equation,

$$y_{par}(x) = xe^{\alpha x}\left(A(x)\cos\omega x + B(x)\sin\omega x\right),$$

  with $A(x)$ and $B(x)$ are polynomials of the same degree as the highest degree between the polynomials $F(x)$ and $G(x)$.