Math 302

Chapter 2

Section 2.1: Linear First Order Differential Equation

Contents

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What Is A LFODE?

A first order ODE is said to be **linear** (abbreviated LFODE) if it can be written as

$$y' + p(x)y = f(x). \tag{1}$$

- Example: $y' + 3y/x^2 = 1$.
- Example: $xy' - 8x^2y = \sin x$.

More Terminology

- A first order ODE that is not of the form (1), then the ODE is said to be **nonlinear**.
  - Example: $xy' + 3y^2 = 2x$.
  - Example: $yy' + e^y = \tan(xy)$.
- When $f(x) = 0$ for any x, then $y' + p(x)y = 0$ is said to be **homogeneous**.
  - Example: $y' + 3y/x^2 = 0$.
  - Example: $xy' - 8x^2y = 0$.
- When $f(x)$ is not zero, then the LODE is said to be **nonhomogeneous**.

Example 1. Find all the solutions to

$$y' = \frac{1}{x^2}$$

Integrating

$$\Rightarrow \quad y(x) = y(x,c) = -\frac{1}{x} + c \quad (*)$$

$$x \in (-\infty, 0) \text{ or } x \in (0, \infty).$$

$\underbrace{(*)}_{y(x,c)}$ is called a <u>one-parameter family</u> of functions.

General Solution

We say that a function $y = y(x,c)$ is a **general solution** to (1) if

- For each fixed parameter c, the resulting function $y = y(x,c)$ is a solution to (1) on an an open interval (a,b).
- If $y_1 = y_1(x)$ is a solution to (1) on (a,b), then y_1 can be obtained from the formula $y = y(x,c)$ by choosing c appropriately.

We now find the general solution to

$$y' + p(x)y = 0 \tag{2}$$

where p is continuous on an interval (a, b).

Example 2. Let a be a constant (fixed).

1. Find the general solution of $y' - ay = 0$.
2. Solve the initial value problem

$$y' - ay = 0, \quad y(x_0) = y_0.$$

1) Writing $y' = ay \rightarrow y(x) = ce^{ax}$, c is a const.

Another approach:

- $y = 0$ is a solution.
- $y \neq 0$. So there is an interval I on which $y(x) \neq 0$, $\forall x \in I$.

$$\Rightarrow \frac{y'}{y} = a$$

But $\dfrac{y'}{y} = (\ln|y|)'$

$$\Rightarrow (\ln|y|)' = a \overset{\text{integrate}}{\Rightarrow} \ln|y| = ax + k$$

k constant.

Therefore $|y| = e^{ax+k} = e^{ax}e^{k}$.

Since e^{ax} is never zero, y has no zero and so y must me negative on I or positive on I.

Set
$$c := \begin{cases} e^k, & \text{if } y > 0 \text{ on } I \\ -e^k, & \text{if } y < 0 \text{ on } I. \end{cases}$$

$$\Rightarrow \quad y = ce^{ax} = c\, e^{-\int (-a)\,dx}, \quad c \text{ arbitrary}$$
$(p(x) = -a)$. (includes 0)

Conversely, $y = ce^{ax}$ satisfies $y' = ay$.

$\Rightarrow$ $y = ce^{ax}$ is general solution.

(b) The general solution is $y = ce^{ax}$.

Figure out the value of c.

$$y(x_0) = y_0 \quad \Rightarrow \quad ce^{ax_0} = y_0$$
$$\Rightarrow \quad c = \frac{y_0}{e^{ax_0}}$$

So,
$$y(x) = \frac{y_0}{e^{ax_0}} e^{ax} = y_0 e^{a(x - x_0)}$$

<u>Remark:</u> $a(x - x_0) = \int_{x_0}^{x} a\, dx$

$$\Rightarrow \quad y(x) = y_0\, e^{-\int_{x_0}^{x} (-a)\,dx} \qquad (p(x) = -a)$$

EXAMPLE 3.

1. Find the general solution of $xy' + y = 0$.
2. Solve the initial value problem

$$xy' + y = 0, \quad y(1) = 3.$$

$\div x \Rightarrow y' + \frac{y}{x} = 0$ (provided $x \neq 0$).

So, $p(x) = \frac{1}{x}$.

1) Quick approach: $y' = \frac{dy}{dx}$.

- $y = 0$ is a solution.
- Suppose $y \neq 0$. Then

$$x\frac{dy}{dx} = -y \Rightarrow \frac{dy}{y} = -\frac{1}{x}\,dx$$

$$\Rightarrow \underbrace{\int \frac{1}{y}\,dy}_{\text{int. w.r.t. } y} = \underbrace{-\int \frac{1}{x}\,dx}_{\text{int. w.r.t. } x} + k$$

$$\Rightarrow \ln|y| = -\ln|x| + k$$

$$\overset{\text{exp.}}{\Rightarrow} |y| = e^{\ln|x|^{-1}}e^{k}.$$

$$\Rightarrow |y| = \frac{1}{|x|}e^{k}$$

Capture the sign of $|\cdot|$ in c by setting

$$c = \begin{cases} e^{k}, & \text{if positive} \\ -e^{k}, & \text{if negative} \end{cases}$$

$$\Rightarrow y(x) = \frac{c}{x} = c\,e^{-\int \frac{1}{x}dx}$$ (c const) & includes 0

$p(x)$

2) $y(1) = 3$ & $y(x) = \frac{c}{x}$

$\Rightarrow \quad 3 = c$

Therefore, $y(x) = \frac{3}{x}$ is the solution to the IVP on $(0, \infty)$ or $(-\infty, 0)$.

<u>Remark:</u> $y(x) = \underset{\uparrow \atop y_0}{3}\, e^{-\int_{\underset{\uparrow \atop x_0}{1}}^{x} \frac{1}{x}\,dx}$ $\nwarrow$ $p(x)$.

<u>General facts:</u>

- The general solution to (2) is given by

$$y = ce^{-P(x)}$$

where $P(x) = \int p(x)\,dx$ is any antiderivative of $p(x)$.

- The solution to the IVP

$$y' + p(x)y = 0, \quad y(x_0) = y_0$$

is given by

$$y(x) = y_0 e^{-\int_{x_0}^{x} p(x)\,dx}.$$

We now want to find the general solution to

$$y' + p(x)y = f(x)$$

where the functions $p(x)$ and $f(x)$ are continuous on an open interval (a, b).

Remark:

- The homogeneous part $y' + p(x)y = 0$ is called the **complementary equation**.

Example 4. Find the general solution of

$$y' + 2y = x^3 e^{-2x}.$$

1) Find a solution to complementary equation.

$$y' + 2y = 0 \quad \Rightarrow \quad y_{cp}(x) = ce^{-2x}$$

2) Make c a function of x! (Variation of par.)

Let $c = u(x)$ & $y(x) = ue^{-2x}$.

$$\Rightarrow \quad y' = u'e^{-2x} - 2ue^{2x}$$

$$\Rightarrow \quad u'e^{-2x} + \cancel{2ue^{2x}} - \cancel{2ue^{2x}} = x^3 e^{-2x}$$

$$\Rightarrow \quad u'\cancel{e^{-2x}} = x^3 \cancel{e^{-2x}}$$

$$\Rightarrow \quad u' = x^3$$

So $\quad u = \int x^3 \, dx = x^4 + c$

$$\Rightarrow \quad y(x) = (x^4 + c)e^{-2x}$$

$$= \boxed{ce^{-2x} + x^4 e^{-2x}}$$

General solution!

Summary of The Method

- Find a function y_1 such that $y_1' + p(x)y_1' = 0$
- Write $y = uy_1$ where u is an unknown function.
- Solve $u'y_1 = f(x)$.
- Substitute u in y.

EXAMPLE 5.

1. Find the general solution
$$y' + (\cot x)y = x \csc x.$$
2. Solve the initial value problem
$$y' + (\cot x)y = x \csc x, \quad y(\pi/2) = 1.$$

1) • Complementary equation:

$$y' + (\cot x)\, y = 0 \Rightarrow \frac{dy}{y} = -\cot x \, dx$$

$v = \sin x$

$\to \int \frac{-1}{v}\, dv$

$$\overset{\text{int.}}{\Rightarrow} \ln|y| = \int -\frac{\cos x}{\sin x}\, dx + k$$

$$\Rightarrow \ln|y| = -\ln|\sin x| + k$$

$$\Rightarrow |y| = \frac{e^k}{|\sin x|}$$

Gathering the sign:

$$y_{cpl}(x) = \frac{c}{\sin x}.$$

• Variation of the parameter

Write $y(x) = \dfrac{u}{\sin x} \Rightarrow y' = \dfrac{u'}{\sin x} - \dfrac{u \cos x}{\sin^2 x}$

So, $\dfrac{u'}{\sin x} - \cancel{\dfrac{u\cos x}{\sin^2 x}} + \cancel{\dfrac{\cos x}{\sin x}\dfrac{u}{\sin x}} = x\csc x$

$$\Rightarrow \frac{u'}{\sin x} = x \csc x$$

$$\Rightarrow u' = x \quad \Rightarrow \quad u(x) = x^2 + c$$

Therefore $y(x) = \dfrac{c}{\sin x} + \dfrac{x^2}{\sin x}$.

2) $y(\pi/2) = 1 \Rightarrow \dfrac{c}{1} + \dfrac{\pi^2/4}{1} = 1$

$$\Rightarrow c = 1 - \frac{\pi^2}{4}$$

Therefore

$$y(x) = \boxed{\frac{4-\pi^2}{4\sin x} + \frac{x^2}{\sin x}}$$

General Theorem

Suppose

- $p(x)$ and $f(x)$ are continuous on an interval (a, b)
- y_1 is a solution to the complementary equation.

Then the general solution to $y' + p(x)y = f(x)$ is

$$y(x) = y_1(x)\left(c + \int \frac{f(x)}{y_1(x)}\, dx\right)$$

for each x in (a, b).

Existence Theorem

Suppose

- $p(x)$ and $f(x)$ are continuous on an interval (a, b).
- y_1 is a solution to the complementary equation.
- x_0 is an arbitrary number in (a, b) and y_0 is an arbitrary number.

Then the boundary value problem

$$y' + p(x)y + f(x), \quad y(x_0) = y_0$$

has a unique solution which is of the form

$$y(x) = y_1(x)\left(\frac{y_0}{y_1(x_0)} + \int_{x_0}^{x} \frac{f(t)}{y_1(t)}\, dt\right)$$

for each x in (a, b).