Problem A

Find the inverse Laplace transform of the following functions. You may leave your answer as a convolution of two functions or as an integral.

1) $\dfrac{1}{s^2(s^2+4)}$.

2) $\dfrac{1}{s(s-2)}$.

3) $\dfrac{s}{(s+2)(s^2+9)}$.

4) $\dfrac{1}{(s-1)^3(s+2)^2}$.

Problem B

Solve the following integral equation:

$$y(t) = 1 + \int_0^t y(\tau)\, d\tau.$$

Problem C

Solve the following integro-differential equations:

1) $y(t) = 1 - \displaystyle\int_0^t (t-\tau)y(\tau)\, d\tau$.

2) $y'(t) = \sin t + \displaystyle\int_0^t y(t-\tau)\cos\tau\, d\tau$ avec $y(0) = 1$.

3) $y(t) = te^t - 2e^t \displaystyle\int_0^t e^{-\tau} y(\tau)\, d\tau$.

Complete Solutions

Problem A

1) We have a product of two functions $\frac{1}{s^2}$ and $\frac{1}{s^2+4}$. Therefore, the original function (inverse Laplace transform) is given by the convolution of the inverse of $\frac{1}{s^2}$ and $\frac{1}{s^2+4}$. We have

$$L^{-1}\left(\frac{1}{s^2}\right) = -t \quad \text{and} \quad L^{-1}\left(\frac{1}{s^2+4}\right) = \sin(2t).$$

Therefore, we get

$$h(t) = (-t) * \sin(2t).$$

We can leave our answer like this. If you computed the integral, then you should have

$$h(t) = \frac{1}{4}\Big(\sin(2t) - 2t\Big).$$

2) We have a product of $\frac{1}{s}$ and $\frac{1}{s-2}$. We could use the method of partial fractions decomposition, but it is more straightforward to use the convolution. We have

$$L^{-1}\left(\frac{1}{s}\right) = 1 \quad \text{and} \quad L^{-1}\left(\frac{1}{s-2}\right) = e^{2t}.$$

Therefore, the convolution Theorem tells us that

$$f * g = L^{-1}\Big(F(s)G(s)\Big).$$

We get

$$h(t) = 1 * e^{2t}.$$

The answer is correct in this form, but if you want the expression of $h(t)$, here it is:

$$h(t) = \frac{1}{2}(e^{2t} - 1) = e^t \sinh(t).$$

3) We will combine s with $s^2 + 9$:

$$H(s) = \frac{s}{(s+2)(s^2+9)} = \left(\frac{1}{s+2}\right)\left(\frac{s}{s^2+9}\right).$$

We have

$$L^{-1}\left(\frac{1}{s+2}\right) = e^{-2t} \quad \text{and} \quad L^{-1}\left(\frac{s}{s^2+9}\right) = \cos(3t).$$

Using the convolution, we obtain

$$h(t) = e^{-2t} * \cos(3t).$$

The answer is correct in this form, but we can integrate and get the exact expression of $h(t)$:

$$h(t) = \frac{1}{13}\Big(2e^{2t} + 3\sin(3t) - 2\cos(3t)\Big)$$

4) We can rewrite the function as

$$H(s) = \Big(\frac{1}{(s-1)^3}\Big)\Big(\frac{1}{(s+2)^2}\Big).$$

We have

$$L^{-1}\Big(\frac{1}{(s-1)^3}\Big) = t^2 e^t \quad \text{and} \quad L^{-1}\Big(\frac{1}{(s+2)^2}\Big) = -te^{-2t}.$$

Using the convolution, we get

$$h(t) = (t^2 e^t) * (-te^{-2t}).$$

If you computed the exact solution, you should find

$$h(t) = \frac{1}{27} e^{-2t}\Big(e^{3t}(-3t^2 + 4t - 2) + 2(t+1)\Big).$$

Problem B

Apply the Laplace transform on each side of the equation. We obtain

$$Y = \frac{1}{s} + \frac{Y}{s}.$$

Multiplying by s the equation and substracting by Y, we obtain

$$sY - Y = 1$$

which can be rewritten as

$$Y = \frac{1}{s-1}.$$

Therefore, we get

$$y(t) = e^t.$$

Remark: We can also transform this integral equation into an ODE. Take the derivative (by assuming that y is differentiable), then

$$y'(t) = 0 + y(t) = y(t).$$

The integral on the right-hand side becomes $y(t)$ because of the Fundamental Theorem of Calculus:

$$\frac{d}{dt}\Big(\int_0^t y(\tau)\, d\tau\Big) = y(t).$$

Notice also that since $\displaystyle\int_0^0 y(\tau)\, d\tau = 0$, we have $y(0) = 1$. So we have to solve the following IVP:

$$y' = y, \quad y(0) = 1.$$

The solution is $y(t) = e^t$. This is the same solution that we obtained using the Laplace transform. The advantage of the Laplace transform is that we don't need to assume necessarily that $y(t)$ is differentiable. We may only assume that $y(t)$ has a Laplace transform and this is a weaker assumption than a differentiability condition.

Problem C

1) The idea is to apply the Laplace transform on each side of the equation. The left-hand side is simply Y. To obtain the expression of the right-hand side, we use the convolution. We have

$$\int_0^t (t-\tau)y(\tau)\,d\tau = y * t.$$

Therefore, the Laplace transform of the right-hand side is

$$\frac{1}{s} - L(y*t) = \frac{1}{s} - \frac{Y}{s^2}.$$

The expression of the transformed equation is

$$Y = \frac{1}{s} - \frac{Y}{s^2} \Rightarrow Y\left(1+\frac{1}{s^2}\right) = \frac{1}{s}.$$

After isolating Y, we get

$$Y = \frac{s}{s^2+1}$$

and finding the inverse transform, we obtain the following solution:

$$y(t) = \cos(t).$$

2) We apply the Laplace transform on each side of the equation and we consider the following facts:

$$\int_0^t y(t-\tau)\cos(\tau)\,d\tau = \cos(t) * y(t).$$

The expression of the transformed equation is

$$sY - 1 = \frac{1}{s^2+1} + \frac{sY}{s^2+1}$$

and after isolating Y, we get

$$sY\left(1-\frac{1}{s^2+1}\right) = 1 + \frac{1}{s^2+1} \Rightarrow sY = \frac{s^2+2}{s^2} \Rightarrow Y = \frac{s^2+2}{s^3}.$$

Taking the inverse transform, we get

$$y(t) = 1 + t^2.$$

3) We notice that

$$e^t \int_0^t e^{-\tau} y(\tau)\,d\tau = \int_0^t e^{t-\tau}y(\tau) = y(t) * e^t.$$

Therefore, applying the Laplace transform, we get

$$Y = \frac{1}{(s-1)^2} - 2\frac{Y}{s-1} \Rightarrow Y\left(1+\frac{2}{s-1}\right) = \frac{1}{(s-1)^2}.$$

After isolating Y, we obtain the following equation:

$$Y = \frac{1}{(s-1)(s+1)} = \frac{1}{s^2-1}.$$

Finally, taking the inverse Laplace transform, we obtain the following solution:

$$y(t) = \sinh(t).$$