Section 2.4 — Problem 3 — 5 points

This is a Bernoulli equation with $r = 1/2$.

First step is to find a solution to the complementary equation. We have

$$x^2y' + 2y = 0.$$

This is a separable equation and the solution is

$$y(x) = ce^{\frac{2}{x}}.$$

Second step is to use the variation of parameter. Let $y(x) = u(x)e^{\frac{2}{x}}$. Then we obtain

$$y' = u'e^{\frac{2}{x}} - 2u\frac{e^{\frac{2}{x}}}{x^2}.$$

We replace y and y' in the differential equation to get

$$x^2\left(u'e^{\frac{2}{x}} - 2u\frac{e^{\frac{2}{x}}}{x^2}\right) + 2ue^{\frac{2}{x}} = 2e^{1/x}u^{1/2}e^{1/x}.$$

After simplifying the left-hand side and the right-hand side, we obtain

$$x^2u' = 2u^{1/2}$$

and this is a separable equation. Rewrite the last equation as

$$\frac{du}{u^{1/2}} = \frac{dx}{x^2}$$

and integrate to obtain

$$2u^{1/2} = -1/x + k.$$

Divide by 2 and then square both sides to obtain

$$u(x) = \left(c - \frac{1}{2x}\right)^2$$

with $c = k/2$. Therefore the solution is

$$y(x) = u(x)e^{-2/x} = \left(c - \frac{1}{2x}\right)^2 e^{2/x}.$$

Section 2.4 — Problem 9 — 5 points

This is again a Bernouilli equation with $r = 4$.

We first find a solution to the complementary equation. We have

$$xy' + y = 0$$

which is separable and we find

$$y(x) = \frac{c}{x}.$$

We now use the variation of parameter. We take $y(x) = u(x)/x$ for some function u. The derivative of y is

$$y'(x) = \frac{u'}{x} - \frac{u}{x^2}.$$

We then replace y and y' into the DE:

$$x\left(\frac{u'}{x} - \frac{u}{x^2}\right) + \frac{u}{x} = x^3 u.$$

After simplifying the left-hand side, we get

$$u' = x^3 u$$

which is separable. We separate the variable and integrate to get

$$\ln|u| = \frac{x^4}{4} + k.$$

Setting $c = \pm e^k$, we obtain

$$u(x) = ce^{x^4/4}.$$

Therefore, the general solution is

$$y(x) = \frac{u(x)}{x} = \frac{ce^{x^4/4}}{x}.$$

We have to find c such that $y(1) = 1/2$. We replace this condition in our function and we get

$$\frac{1}{2} = ce^{1/4} \quad \Rightarrow \quad c = 0.5e^{-1/4}.$$

The solution to the IVP is then

$$y(x) = 0.5\frac{e^{(x^4-1)/4}}{x}.$$

Section 2.4 — Problem 17 — 10 points

The ODE can be rewritten as

$$y' = \frac{y^4 + x^4}{xy^3} = \frac{y}{x} + \left(\frac{x}{y}\right)^3.$$

Let $u = y/x$ so that $y = xu$. Taking the derivative with respect to x, we find that

$$y' = u + xu'.$$

Replace y by xu and y' by the last expression in the ODE to get

$$u + xu' = u + \frac{1}{u^3}.$$

After substracting by u on each side, we then obtain the following ODE in u:

$$xu' = \frac{1}{u^3}$$

which is a separable equation. The solution is

$$\frac{u^4}{4} = \ln|x| + c.$$

Since $u = y/x$, we have $y^4/x^4 = u^4$. Replacing u^4 by $4\ln|x| + 4c$, we obtain

$$\frac{y^4}{4x^4} = \ln|x| + c.$$

We can leave our answer in an implicit form.

Or you can find explicitly the solution for y by taking the 4-th root. We first multiply by x^4:

$$y^4 = 4x^4 \ln|x| + 4cx^4$$

and then take the 4-th root to obtain

$$y(x) = \pm\sqrt[4]{4x^4 \ln|x| + 4cx^4}.$$

<u>Remarks:</u>

- We can't remove the absolute value in $\ln|x|$. If we want to do so, we then have to consider wheither $x > 0$ or $x < 0$. Therefore, the implicit solution then takes the following form

$$\begin{cases} \frac{y^4}{4x^4} = \ln x + c & \text{, if } x > 0; \\ \frac{y^4}{4x^4} = \ln(-x) + c & \text{, if } x < 0. \end{cases}$$

- We can decide between the two forms of the implicit solution if we have an initial condition $y(x_0) = y_0$. When $x_0 > 0$, the interval of validity is $(0, \infty)$ and the implicit solution will be

$$\frac{y^4}{4x^4} = \ln x + c.$$

  When $x_0 < 0$, the interval of validity is $(-\infty, 0)$ and the implicit solution will be

$$\frac{y^4}{4x^4} = \ln(-x) + c.$$

- On the next page, you can see a direction field together with some implicit solutions.

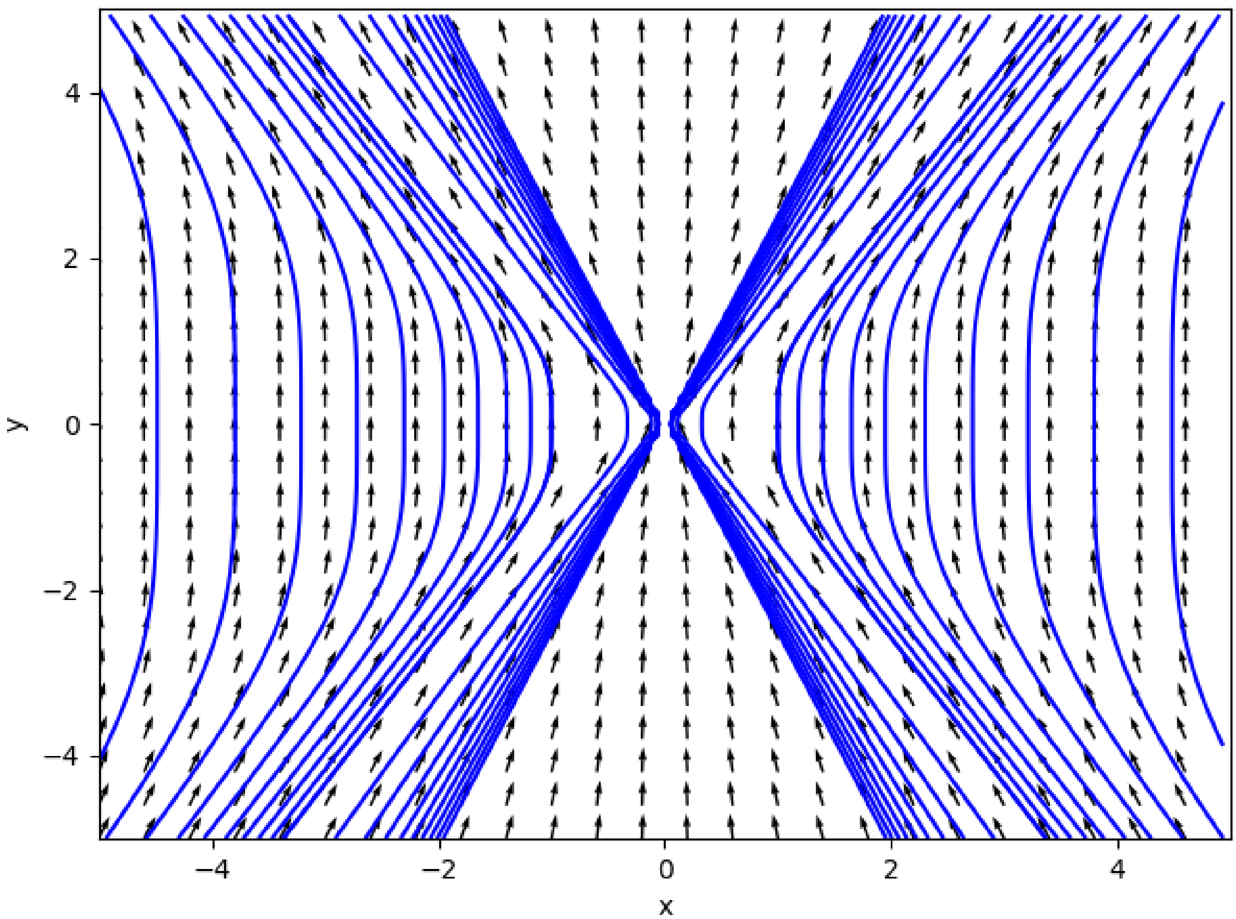

Figure 1: Direction field of $xy^3y' = y^4 + x^4$ and some implicit solutions

Section 2.4 — Problem 23 — 5 points

We first find the general solution. We rewrite the ODE as

$$y' = \frac{x^3 + y^3}{xy^2} = \left(\frac{x}{y}\right)^2 + \frac{y}{x}.$$

Let $u = y/x$ and so $y = xu$. Therefore, $y' = u + xu'$. Replacing this in the ODE, we get

$$u + xu' = \frac{1}{u^2} + u.$$

After substracting by u both sides, we obtain the following separable equation:

$$xu' = \frac{1}{u^2}.$$

The solution is

$$\frac{u^3}{3} = \ln|x| + c.$$

Since $u = y/x$, we then get

$$\frac{y^3}{3x^3} = \ln|x| + c$$

and therefore

$$y^3 = 3x^3 \ln|x| + 3cx^3.$$

Taking the cube root, we obtain

$$y(x) = \sqrt[3]{3x^3 \ln|x| + 3cx^3}.$$

We have $y(1) = 3$. We then get

$$3 = \sqrt[3]{3\ln(1) + 3c} \quad \Rightarrow \quad 3 = \sqrt[3]{3c} \quad \Rightarrow \quad 9 = c.$$

Since $x_0 = 1 > 0$, we can write our solution to the IVP as

$$y(x) = \sqrt[3]{3x^3 \ln x + 27x^3} \quad (\ x > 0\ ).$$

On the next page, you will see a plot of the direction field and the solution curve to the IVP.

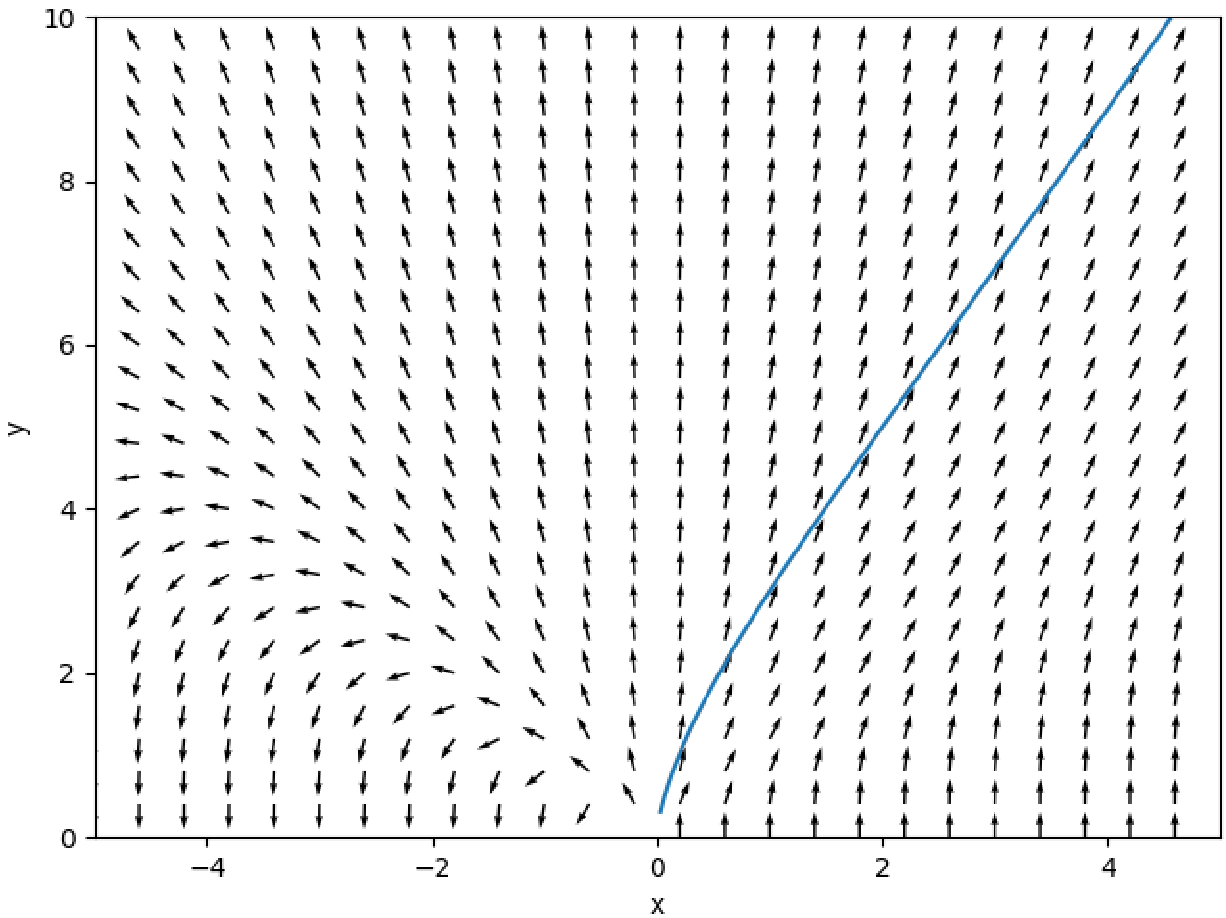

Figure 2: Direction field of $y' = \frac{x^3+y^3}{xy^2}$ and the solution curve to the IVP

Section 2.5 — Problem 1 — 5 points

We have $M(x,y) = 6x^2y^2$ and $N(x,y) = 4x^3y$. We have

$$M_y = 12xy \quad \text{and} \quad N_x = 12x^2y.$$

We see that $M_y = N_x$ and therefore the ODE is exact.

To solve is, we have to find a function $F(x,y)$ such that

$$F_x = M = 6x^2y^2 \quad \text{and} \quad F_y = N = 4x^3y.$$

Integrating with respect to x the equation $F_x = 6x^2y^2$ yields

$$G(x,y) = 2x^3y^2.$$

Integrating with respect to y the equation $F_y = 4x^3y$ yields

$$H(x,y) = 2x^3y^2.$$

Therefore, we get

$$F(x,y) = 2x^3y^2$$

and the implicit solutions are

$$2x^3y^2 = c.$$

Section 2.5 — Problem 5 — 5 points

We have $M = (x+y)^2$ and $N = (x+y)^2$. We see that $M_y = x+y$ and $N_x = x+y$. Therefore, the equation is exact.

We have to find a function F such that $F_x = (x+y)^2$ and $F_y = (x+y)^2$. Integrating these equations with respect to x and y respectively, we obtain

$$G(x,y) = \frac{(x+y)^3}{3} \quad \text{and} \quad H(x,y) = \frac{(x+y)^3}{3}.$$

Therefore, $F(x,y) = (x+y)^3/3$ and the implicit solutions are

$$\frac{(x+y)^3}{3} = c.$$

We can find explicitly y in this case. Multiply by 3, take the cube root and substract by x to obtain

$$y(x) = \sqrt[3]{3c} - x = C - x$$

where $C = \sqrt[3]{3c}$. Therefore, the solutions are simply lines!

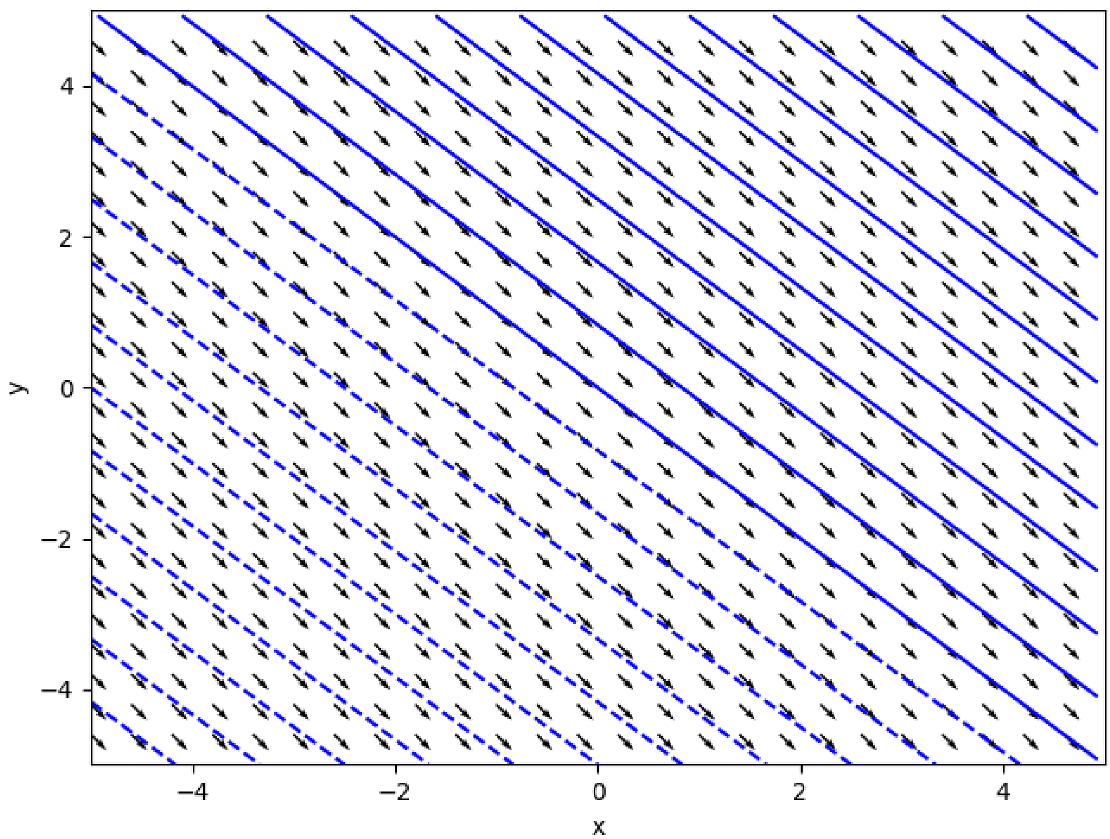

Figure 3: Direction field of $(x+y)^2dx + (x+y)^2dy = 0$ and some solution curves

Section 2.6 — Problem 3 — 5 points

We see here that the equation is not exact because $M_y = 1$ and $N_x = -1$. We therefore have to find an integrating factor.

- $(M_y - N_x)/N = (1+1)/(-x) = -2/x$ which is independent of y. Therefore, we have

  $$e^{\int -2/x\, dx} = e^{-2\ln|x|} = -\frac{2}{|x|^2} = -\frac{2}{x^2}.$$

  We then chose

  $$\mu(x) = -e^{\int p(x)\, dx} = \frac{2}{x^2}.$$

- We could also use the other equation because it's gonna be independent of x.

We multiply the ODE by $\mu(x)$ and obtain

$$\frac{2y}{x^2}\, dx - \frac{2}{x}\, dy = 0.$$

We know that this should be exact. So we are searching for F such that $F_x = 2y/x^2$ and $F_y = -2/x$. We integrate with respect to x the first equation to get

$$G(x, y) = -\frac{2y}{x}.$$

We integrate with respect to y the second equation to get

$$H(x, y) = \frac{-2y}{x}.$$

Therefore, we get

$$F(x, y) = -\frac{y}{x}$$

and the implicit solutions are given by

$$-\frac{y}{x} = c.$$

Rearraging this last equation, we get

$$y = -cx$$

which are lines! It's possible to replace $-c$ by the more simpler constant C where $C = -c$.

Section 2.6 — Problem 11 — 10 points

This ODE is not exact because

$$M_y = 12x^3 + 48x^2y \neq 36x^3 + 96x^2y = N_x.$$

We therefore have to find an integrating factor.

- The quotient $(M_y - N_x)/N$ is

$$\frac{M_y - N_x}{N} = \frac{12x^3 + 48x^2y - 36x^3 - 96x^2y}{9x^4 + 32x^3y + 4y} = \frac{-24x^3 - 48x^2y}{9x^4 + 32x^3y + 4y}.$$

  There is no way to simplify this function into an expression independent of y...

- The quotient $(N_x - M_y)/M$ is

$$\frac{N_x - M_y}{M} = \frac{36x^3 + 96x^2y - 12x^3 - 48x^2y}{12x^3y + 24x^2y^2} = \frac{24x^3 + 48x^2y}{(12x^3 + 24x^2y)y} = \frac{2}{y}.$$

  This expression is independent of x! Therefore we have

$$e^{\int 2/y\,dy} = e^{2\ln|y|} = y^2.$$

  We then choose $\mu(y) = y^2$.

We multiply the ODE by y^2. We obtain

$$(12x^3y^3 + 24x^2y^2)dx + (9x^4y^2 + 32x^3y^3 + 4y^3)dy = 0.$$

This ODE should now be exact. We therefore want to find a function F such that

$$F_x = 12x^3y^3 + 24x^2y^2 \quad \text{and} \quad F_y = 9x^4y^2 + 32x^3y^3 + 4y^3.$$

Integrating the first equation with respect to x yields

$$G(x,y) = 3x^4y^3 + 8x^3y^2.$$

Integrating the second equation with respect to y yields

$$H(x,y) = 3x^4y^3 + 9x^3y^4 + y^4.$$

Therefore, we obtain

$$F(x,y) = 3x^4y^3 + 8x^3y^2 + 9x^3y^4 + y^4.$$

The implicit solutions are then

$$3x^4y^3 + 8x^3y^2 + 9x^3y^4 + y^4 = c.$$

Total (Points): 50.