ODE Second order

Basics

$$y'' + p(x)y' + q(x)y = f(x)$$

forcing fct.

- $f \neq 0$ non-homogeneous.
- $f = 0$ homogeneous
- $\{y_1, y_2\}$ fundamental set sol. if y_2/y_1 or y_1/y_2 not constant
- general sol.: $y = c_1 y_1 + c_2 y_2$
- solutions exist always if p & q continuous.
- Particular solution y_{par}.
- Superposition principle: $y_c + y_{par}$ general solution.

Constant coefficient homogeneous.

- $ay'' + by' + cy = 0$
  - $ar^2 + br + c = 0$ (char. Eq.)
    - ① Real-roots: $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$
    - ② Repeated roots: $y = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$
    - ③ Complex roots: $r_1 = \alpha + \beta i$ & $r_2 = \alpha - \beta i$

$$\Rightarrow y = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x).$$

Constant Coefficient General

$$ay'' + by' + cy = e^{\alpha x}\left(F(x)\cos(\omega x) + G(x)\sin(\omega x)\right) \quad (\omega \neq 0)$$

I) $e^{\alpha x}\cos(\omega x)$ & $e^{\alpha x}\sin(\omega x)$ are not solution.
- $y_{par}(x) = e^{\alpha x}\left(A(x)\cos(\omega x) + B(x)\sin(\omega x)\right)$
  - A, B: poly. degree is the highest of F & G.

II) $e^{\alpha x}\cos(\omega x)$ & $e^{\alpha x}\sin(\omega x)$ are solution
- $y_{par}(x) = x e^{\alpha x}\left(A(x)\cos(\omega x) + B(x)\sin(\omega x)\right)$
  - A, B: poly. degree is the highest of F & G.

$\omega = 0$

Constant Coefficient trig.

$$ay'' + by' + cy = F(x)\cos(\omega x) + G(x)\sin(\omega x)$$

I) $\cos(\omega x)$ & $\sin(\omega x)$ not sol.
- $y_{par}(x) = A(x)\cos(\omega x) + B(x)\sin(\omega x)$
  - A, B: highest degree of F & G.

II) $\cos(\omega x)$ & $\sin(\omega x)$ are sol.
- $y_{par}(x) = x\left(A(x)\cos(\omega x) + B(x)\sin(\omega x)\right)$, A, B: highest degree of F & G.

Constant Coefficient expo.

$$ay'' + by' + cy = e^{\alpha x}F(x)$$

I) $e^{\alpha x}$ not sol.
- $y_{par} = e^{\alpha x}(A(x))$
  - A: poly. same degree as $F(x)$

II) $e^{\alpha x}$ is solution.
- $y_{par} = x e^{\alpha x} A(x)$
  - A: poly. same degree as $F(x)$

III) $e^{\alpha x}$ & $x e^{\alpha x}$ sol.
- $y_{par} = x^2 e^{\alpha x} A(x)$
  - A: poly. same degree as $F(x)$

Change of Order

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x)$$

given $y_1(x)$ sol. compl. Eq.

1) Var. of param: $y = u \cdot y_1$
2) Change order: $z = u'$, $z' = u''$
3) Solve the ODE obtained fn z
4) Integrate to get u.
5) Replace u in y.