Section 5.5 — Problem 1 — 10 points

Find the general solutions to the complementary equation.

The complementary equation is

$$y'' + 3y' + 2y = 0$$

The characteristic equation associated to the complementary equation is $r^2 + 3r' + 2 = 0$. Therefore, the roots are $r_1 = -1$ and $r_2 = -2$. The general solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^{-2x}.$$

Find a particular solution.

The right-hand side contains $\cos x$ and $\sin x$. Those functions are not contained in the complementary equation. We therefore suggest

$$y_{par}(x) = A \cos x + B \sin x.$$

We have

$$y' = -A \sin x + B \cos x$$
$$y'' = -A \cos x - B \sin x.$$

Replacing in the ODE, we get

$$y'' + 3y' + 2y = -A \cos x - B \sin x - 3A \sin x + 3B \cos x + 2A \cos x + 2B \sin x$$
$$= (A + 3B) \cos x + (-3A + B) \sin x.$$

This last expression should be equal to the right-hand side

$$(A + 3B) \cos x + (-3A + B) \sin x = 7 \cos x - \sin x.$$

We therefore have to find A, B satisfying

$$\begin{cases} A + 3B = 7 \\ -3A + B = -1 \end{cases}$$

The solution is $A = 1$ and $B = 2$. The expression of the particular solution is

$$y_{par}(x) = \cos x + 2 \sin x.$$

General solution.

Combining y_h and y_{par}, we get

$$y(x) = y_h(x) + y_{par}(x) = c_1 e^{-x} + c_2 e^{-2x} + \cos x + 2 \sin x.$$

Find the general solution to the complementary equation.

The complementary equation is

$$y'' + 2y' + y = 0.$$

The characteristic equation associated to the complementary equation is $r^2 + 2r + 1 = 0$. There is only one root: $r_1 = -1$. The solution to the complementary equation is therefore

$$y_h(x) = c_1 e^{-x} + c_2 x e^{-x}.$$

Find a particular solution.

The right-hand side contains the functions $e^x \cos x$ and $e^x \sin x$. Those are not contained in the solution to the complementary equation. Therefore, we suggest

$$y_{par}(x) = e^x (A \cos x + B \sin x).$$

We have

$$y'(x) = e^x (A \cos x + B \sin x) + e^x (-A \sin x + B \cos x)$$
$$y''(x) = e^x (A \cos x + B \sin x) + 2e^x (-A \sin x + B \cos x) + e^x (-A \cos x - B \sin x).$$

Replacing in the ODE, we get

$$y'' + 2y' + y = e^x (A \cos x + B \sin x) + 2e^x (-A \sin x + B \cos x) + e^x (-A \cos x - B \sin x)$$
$$+ 2e^x (A \cos x + B \sin x) + 2e^x (-A \sin x + B \cos x) + e^x (A \cos x + B \sin x)$$
$$= e^x \Big((3A + 3B) \cos x + (3B - 4A) \sin x \Big)$$

The right-hand side is $e^x (6 \cos x + 17 \sin x)$. Therefore, we must have

$$3A + 3B = 6 \quad \text{and} \quad 3B - 4A = 17.$$

The solution is $A = -11/7$ and $B = 25/7$. The particular solution is therefore

$$y_{par}(x) = -\frac{11e^x}{7} \cos x + \frac{25e^x}{7} \sin x.$$

General solution.

The general solution is therefore

$$y(x) = y_h(x) + y_{par}(x) = c_1 e^{-x} + c_2 x e^{-x} - \frac{11e^x}{7} \cos x + \frac{25e^x}{7} \sin x.$$

Complementary Equation.
The complementary equation is

$$y'' - 2y' + 5y = 0.$$

The characteristic polynomial associated to the complementary equation is $r^2 - 2r + 5 = 0$. The roots are complex numbers and they are

$$r_1 = 1 + 2i \quad \text{and} \quad r_2 = 1 - 2i.$$

Therefore, the solution is

$$y_h(x) = c_1 e^x \cos(2x) + c_2 e^x \sin(2x).$$

Find a particular solution.
The right-hand side unfortunately contains $e^x \cos(2x)$ and $e^x \sin(2x)$. These functions are also multiplied by a polynomial of degree 1. We therefore suggest

$$y_{par}(x) = xe^x\Big((Ax + B)\cos(2x) + (Cx + D)\sin(2x)\Big).$$

Following the hint, we have

$$A = 1,\ B = -1\,, C = 1 \text{ and } D = 1.$$

Therefore, the solution is

$$y_{par}(x) = xe^x\Big((x - 1)\cos(2x) + (x + 1)\sin(2x)\Big).$$

General solution.
Combining y_h and y_{par}, we obtain

$$y(x) = y_h(x) + y_{par}(x) = c_1 e^x \cos(2x) + c_2 e^x \sin(2x) + xe^x\Big((x - 1)\cos(2x) + (x + 1)\sin(2x)\Big).$$

Complementary Equation.
The complementary equation is

$$y'' - 2y' + 2y = 0.$$

The characteristic polynomial associated to the complementary equation is $r^2 - 2r + 2 = 0$. The roots are complex and they are

$$r_1 = 1 + i \quad \text{and} \quad r_2 = 1 - i.$$

Therefore, the solution is

$$y_h(x) = c_1 e^x \cos(x) + c_2 e^x \sin(x).$$

Find a particular solution.
The right-hand side unfortunately contains $e^x \cos(x)$ and $e^x \sin(x)$. Since these functions are only multiplied by constants, we therefore suggest

$$y_{par}(x) = xe^x \Big(A\cos(x) + B\sin(x) \Big).$$

We find that

$$y'(x) = e^x \Big(A\cos x + B\sin x \Big) + xe^x \Big(A\cos x + B\sin x \Big) + xe^x \Big(-A\sin x + B\cos x \Big)$$
$$= e^x \Big(\big((A+B)x + A \big) \cos(x) + \big((B-A)x + B \big) \sin(x) \Big)$$

and

$$y''(x) = e^x \Big(\big((A+B)x + A \big) \cos(x) + \big((B-A)x + B \big) \sin(x) \Big)$$
$$+ e^x \Big((A+B)\cos(x) - \big((A+B)x + A \big) \sin x + (B-A)\sin(x) + \big((B-A)x + B \big) \cos(x) \Big)$$
$$= e^x \Big(\big(2Bx + 2(A+B) \big) \cos(x) + \big(-2Ax + 2(B-A) \big) \sin(x) \Big)$$

We replace in the left-hand side of the ODE to get

$$y'' - 2y' + 2y = e^x \Big(2B\cos(x) - 2A\sin(x) \Big).$$

The right-hand side is $e^x(-6\cos x - 4\sin x)$. Therefore, we must have

$$2B = -6 \quad \text{and} \quad -2A = -4.$$

We conclude that $B = -3$ and $A = 2$ and the solution is

$$y_{par}(x) = xe^x \Big(2\cos x - 3\sin x \Big).$$

General solution.
Combining y_h and y_{par}, we obtain

$$y(x) = y_h(x) + y_{par}(x) = c_1 e^x \cos(x) + c_2 e^x \sin(x) + x e^x \Big(2 \cos x - 3 \sin x \Big).$$

Initial Value Problem.
We have $y(0) = 1$, so

$$c_1 = 1.$$

The expression of the derivative of the particular solution was computed in the second step. Replacing A and B by their values, we have

$$y'(x) = c_1 e^x \cos x + c_2 e^x \sin x - c_1 e^x \sin x + c_2 e^x \cos x + e^x \Big(\big(2 - x\big) \cos x + \big(-5x - 3 \big) \sin x \Big).$$

Since $y'(0) = 4$, we obtain

$$c_1 + c_2 + 2 = 4 \iff c_1 + c_2 = 2.$$

Solving for c_1 and c_2, we get

$$c_2 = 1.$$

Therefore, the solution to the IVP is

$$y(x) = e^x \cos(x) + e^x \sin(x) + x e^x \Big(2 \cos x - 3 \sin x \Big).$$

We set
$$y = uy_1 = ux.$$

Then the derivative and second derivative are
$$y' = u'x + u$$
$$y'' = u''x + 2u'.$$

Substituting in the ODE, we obtain
$$x^2(u''x + 2u') - x(u'x + u) + ux = x$$

This simplifies to
$$x^3u'' + x^2u' = x.$$

Dividing through x, we get
$$x^2u'' + xu' = 1.$$

Set $v = u'$ so that $v' = u''$. Therefore, the ODE is reduced to
$$x^2v' + xv = 1.$$

Dividing again by x, we get
$$xv' + v = \frac{1}{x}.$$

The left-hand side can be rewritten as
$$\frac{d}{dx}(xv) = \frac{1}{x}.$$

Integrating with respect to x, we obtain
$$xv = \ln|x| + c_1 \quad \Rightarrow \quad v(x) = \frac{\ln|x|}{x} + \frac{c_1}{x}.$$

Since $v(x) = u'(x)$, integrating again with respect to x the expression of v, we obtain
$$u(x) = \frac{(\ln|x|)^2}{2} + c_1\ln|x| + c_2.$$

Replacing in $y(x)$, we conclude that the general solution is
$$y(x) = \frac{x(\ln|x|)^2}{2} + c_1 x\ln|x| + c_2 x.$$

From this, we see that a foundamental set of solution for the complementary equation is $\{x, x\ln|x|\}$.

TOTAL (POINTS): 50.

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