Power series = 0.

Suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ fn $0 \le |x| < a$

& suppose also that $\sum_{n=0}^{\infty} a_n x^n = 0 \quad \forall x \in (-a, a)$.

Simple trick:

$$a_n = \frac{f^{(n)}(0)}{n!} \quad (n \ge 0).$$

How do you get this formula?

- $f(0) = a_0 + 0 + 0 + 0 + \cdots = a_0$ ✓
- $f'(0) = 1 \cdot a_1 + 2 \cdot a_2 0 + 3 \cdot a_3 \cdot 0 + \cdots = a_1$
- $f''(0) = 2 \cdot 1\, a_2 + 3 \cdot 2\, a_3 \cdot 0 + \cdots = 2!\, a_2$

$\Rightarrow \quad a_2 = \frac{f''(0)}{2!} =$

$\vdots$

- $f^{(n)}(x) = \sum_{k=n}^{\infty} k(k-1)(k-2)\cdots(k-n+1)\, x^{k-n} a_k$

$\Rightarrow f^{(n)}(0) = \sum_{k=n+1}^{\infty} 0 + \underbrace{n(n-1)(n-2)\cdots 1}_{n!}\, a_n$

$\Rightarrow \quad a_n = \frac{f^{(n)}(0)}{n!}$

So, you have $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 0$

But, in this case, $f(x) = 0 \quad \Rightarrow \quad f^{(n)}(0) = 0$

Therefore, $a_n = \frac{f^{(n)}(0)}{n!} = 0 \quad (\forall n \ge 0)$.