# Math 302

## Chapter 7

### Section 7.2: Series Solutions Near an Ordinary Point

Contents



Created by: Pierre-Olivier Parisé
Fall 2022

Main goal:

- Solve a second order ODE
$$A(x)y'' + B(x)y' + C(x)y = 0$$
where $A(x)$, $B(x)$, and $C(x)$ are polynomials.
- Use power series to obtain the solution $y(x)$. Such a solution is called a **power series solution** to the ODE.

Recall from the previous section that

- $y(x) = \sum_{n=0}^{\infty} a_n x^n$.
- $y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$.
- $y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$.

Remark:

- We denote the left-hand side by
$$L(y) := A(x)y'' + B(x)y' + C(x)y.$$
- The application $y \mapsto L(y)$ is called a **differential operator** in the litterature.

Example 1. Find a power series solution to $y'' + y = 0$.

① Left-hand side as a Power Series.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad , \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\Rightarrow y'' + y = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} \big((n+2)(n+1) a_{n+2} + a_n\big) x^n$$

## ② Find Recurrence Relation.

$$y'' + y = 0 \Rightarrow \sum_{n=0}^{\infty} \left( (n+2)(n+1) a_{n+2} + a_n \right) x^n = \sum_{n=0}^{\infty} 0 x^n$$

$$\Rightarrow (n+2)(n+1) a_{n+2} + a_n = 0 \quad (n \geq 0)$$

$$\Rightarrow a_{n+2} = -\frac{a_n}{(n+2)(n+1)} \quad (n \geq 0)$$

## ③ Find an expression for $a_n$.

$\underline{n=0}$ $a_0$ is arbitrary

$2 \cdot 1 \Rightarrow \underline{n=2}$ $a_2 = a_{0+2} = -\dfrac{a_0}{(0+2)(0+1)} = \overset{(-1)^1}{-}\dfrac{a_0}{\underbrace{2 \cdot 1}_{2!}}$

$2 \cdot 2 \leftarrow \underline{n=4}$ $a_4 = a_{2+2} = -\dfrac{a_2}{(2+2)(2+1)} = -\dfrac{a_2}{4 \cdot 3} = \dfrac{a_0}{\underbrace{4 \cdot 3 \cdot 2 \cdot 1}_{4!}}$ $(-1)^2$

$\vdots$

$\underline{n=2k} \rightarrow a_{2k} = (-1)^k \dfrac{a_0}{(2k)!}$

$\underline{n=1}$ $a_1$ arbitrary.

$\Rightarrow \underline{n=3}$ $a_3 = a_{1+2} = -\dfrac{a_1}{(1+2)(1+1)} = \overset{(-1)^1}{-}\dfrac{a_1}{\underbrace{3 \cdot 2 \cdot 1}_{3!}}$

$2 \cdot \underset{k}{1} + 1$

$2 \cdot \underset{k}{2} + 1$ $\underline{n=5}$ $a_5 = a_{3+2} = -\dfrac{a_3}{(3+2)(3+1)} = -\dfrac{a_3}{5 \cdot 4}$

$(-1)^2 = \dfrac{a_1}{\underbrace{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}_{5!}}$

$\underline{n=2k+1}$ $a_{2k+1} = (-1)^k \dfrac{a_1}{(2k+1)!}$

③ General Solution.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

$$= a_0 + a_1 x - \frac{a_0}{2!} x^2 - \frac{a_1}{3!} x^3 + \frac{a_0}{4!} x^4 + \frac{a_1}{5!} x^5 + \cdots$$

$$= a_0 - \frac{a_0}{2!} x^2 + \frac{a_0}{4!} x^4 + \cdots + a_1 x - \frac{a_1}{3!} x^3 + \frac{a_1}{5!} x^5 + \cdots$$

$$= a_0 \left( 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 + \cdots \right) + a_1 \left( x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 + \cdots \right)$$

$$= a_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$= a_0 \cos(x) + a_1 \sin(x).$$

**Recurrence Relation:**

Solving ODE with power series involves a lot of recurrence relations. In the above problems, we encountered:

$$a_{n+2} = -\frac{a_n}{(n+2)(n+1)}$$

Example 2. Find a power series solution to $x^2y'' + y = 0$.

① $y(x) = \sum_{n=0}^{\infty} a_n x^n \to y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

$$x^2 y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^n$$

$$\Rightarrow x^2 y'' + y = \sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=2}^{\infty} a_n x^n + a_0 + a_1 x$$

$$= \sum_{n=2}^{\infty} \left(n(n-1) a_n + a_n\right) x^n + a_0 + a_1 x$$

② $x^2 y'' + y = 0$

$$\Rightarrow a_0 + a_1 x + \sum_{n=2}^{\infty} [n(n-1) a_n + a_n] x^n = 0$$

$$\Rightarrow \begin{cases} a_0 = 0 \\ a_1 = 0 \\ n(n-1)a_n + a_n = 0 & (n \geq 2) \end{cases}$$

$$\Rightarrow \begin{cases} a_0 = 0 \\ a_1 = 0 \\ (n^2 - n + 1) a_n = 0 & (n \geq 2) \end{cases}$$

$$\Rightarrow a_0 = a_1 = 0, \quad a_n = 0 \quad (n \geq 2).$$

③ General Solu.

$$y(x) = \sum_{n=0}^{\infty} a_n x = 0$$

No solution.

$$y(x) = c_1 \sqrt{x} \cos\left(\frac{\sqrt{3}}{2} \ln|x|\right) + c_2 \sqrt{x} \sin\left(\frac{\sqrt{3}}{2} \ln|x|\right).$$

$A(x)y'' + \cdots$

## Ordinary and Singular Points

- A number $x_0$ is called an **ordinary point** if $A(x_0) \neq 0$.
- A number $x_0$ is called a **singular point** if $A(x_0) = 0$.

We will mainly focus ~~s~~ on power series solutions centered at ordinary points.

Example 3. For each of the following ODEs, find the singular points.

**(a)** $(1 - x^2)y'' + y = 0$.

**(b)** $(1 + 2x + x^2)y'' + y' + (2 + x)y = 0$.

**(c)** $(2x + 3x^2 + x^3)y'' + (x + 1)y' + (x^2 + 1)y = 0$.

(a) $1 - x^2 = 0 \iff x = \pm 1$

sing. Pts. are $x = -1$ & $x = 1$.

(b) $1 + 2x + x^2 = 0 \iff (x+1)^2 = 0$

$\iff x = -1$. ← sing Pts.

(c) $2x + 3x^2 + x^3 = 0 \iff (2 + 3x + x^2)x = 0$

$\iff (x+2)(x+1)x = 0$

$\iff x = -2, \ x = -1, \ x = 0$

Sing. Pts: $-2, -1, 0$.

Remark:

- A power series solution must be centered at an ordinary point, that is, if $x_0$ is an ordinary point, then the form of the solution is

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

- In Example 2, we see why we can't solve: The power series used was centered at $x_0 = 0$, a singular point.
- In the case of a singular points, we need the Frobenius method. This is covered in a second class in ODE.

Example 4.

$\to \sum a_n x^n$

**(a)** Find a power series solution of

$$(x^2-4)y'' + 3xy' + y = 0.$$

**(b)** Find the solution to the IVP

$$(x^2-4)y'' + 3xy' + y = 0, \quad y(0) = 4,\ y'(0) = 1.$$

(a) ① Singular Points $\quad x^2-4=0 \iff x=-2$ or $x=2$.

② $y(x) = \sum_{n=0}^{\infty} a_n x^n \to y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$

$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$.

• $(x^2-4)y'' = x^2 y'' - 4y''$

$= \sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=2}^{\infty} 4n(n-1) a_n x^{n-2}$

• $3xy' = \sum_{n=1}^{\infty} 3n a_n x^n$

$$\text{LHS} = \sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=2}^{\infty} 4n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} 3n a_n x^n + \sum_{n=0}^{\infty} a_n x^n$$

$$= (a_0 - 8a_2) + (4a_1 - 24a_3)x + \sum_{n=2}^{\infty} \Big[n(n-1)a_n - 4(n+2)(n+1)a_{n+2} + 3na_n + a_n\Big] x^n$$

$$n(n-1)a_n - 4(n+2)(n+1)a_{n+2} + 3na_n + a_n$$

$$= (n^2 - n + 3n + 1)a_n - 4(n+2)(n+1)a_{n+2}$$

$$= (n^2 + 2n + 1)a_n - 4(n+2)(n+1)a_{n+2}$$

$$= (n+1)^2 a_n - 4(n+2)(n+1)a_{n+2}.$$

③ <u>LHS = 0</u>

$$\text{LHS} = 0 \iff \begin{aligned} a_0 - 8a_2 &= 0 \\ 4a_1 - 24a_3 &= 0 \\ (n+1)^2 a_n - 4(n+2)(n+1)a_{n+2} &= 0 \end{aligned}$$

$$\Rightarrow \quad a_2 = \frac{a_0}{8}, \quad a_3 = \frac{a_1}{6}$$

$$\& \quad a_{n+2} = \frac{n+1}{4(n+2)} a_n$$

<u>Start recurrence.</u>

$a_0$ arbitrary $\longrightarrow$ $a_2 = \frac{a_0}{8}$

$$a_4 = a_{2+2} = \frac{2+1}{4(2+2)} a_2 = \frac{3}{16} \cdot \frac{a_0}{8} = \frac{3a_0}{128}$$

$$a_6 = a_{4+2} = \frac{4+1}{4(4+2)} a_4 = \frac{5}{4(6)} \frac{3a_0}{128} = \frac{15}{1024} a_0$$

$\underline{a_1 \text{ arbitrary}}$. $\to$ $a_3 = \dfrac{a_1}{6}$

$\to a_5 = a_{3+2} = \dfrac{3+1}{4(3+2)} a_3 = \dfrac{4}{4(5)} \cdot \dfrac{a_1}{6}$

$= \dfrac{a_1}{30}$

$\to a_7 = a_{5+2} = \dfrac{5+1}{4(5+2)} a_5 = \dfrac{6}{4 \cdot 7} \cdot \dfrac{a_1}{30}$

$= \dfrac{a_1}{140}$

## 4) General Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$$

$$= a_0 + a_1 x + \frac{a_0}{8} x^2 + \frac{a_1}{6} x^3 + \frac{3a_0}{128} x^4$$

$$+ \frac{a_1}{30} x^5 + \cdots$$

(b) Solution to the IVP.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

$$\Rightarrow y(0) = a_0 \Rightarrow a_0 = 4$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

$$\Rightarrow y'(0) = a_1 \Rightarrow a_1 = 1$$

EXAMPLE 5. Find a power series solution to the following IVP:

$$(t^2-2t-3)\frac{d^2y}{dt^2}+3(t-1)\frac{dy}{dt}+y=0, \quad y(1)=4,\ y'(1)=-1.$$

Assume $y(t)=\sum_{n=0}^{\infty} a_n t^n \to \sum_{n=0}^{\infty} a_n = 4$.

$y'(t)=\sum_{n=1}^{\infty} n a_n t^{n-1} \to \sum_{n=0}^{\infty} n a_n = -1$.

Modify: $y(t) = \sum_{n=0}^{\infty} a_n (t-1)^n \to a_0 = 4$

Change of variable: $x = t-1 \Rightarrow y(x) = \sum_{n=0}^{\infty} a_n x^n$.

$\Rightarrow y(0) = 4$ & $y'(0) = -1$

- $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = y' \cdot 1 = y'$
- $\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}(y') = \frac{d}{dx}(y') \cdot \frac{dx}{dt} = y''$
- $t^2-2t-3 = (x+1)^2 - 2(x+1) - 3 = x^2-4$.
- $t-1 = x$

Therefore, IVP becomes.

$(x^2-4)y'' + 3xy' + y = 0, \quad y(0)=4$ & $y'(0)=-1$.

This is Example 4!

$$y(x) = a_0 + a_1 x + \frac{a_0}{8} x^2 + \frac{a_1}{6} x^3 + \frac{3a_0}{128} x^4 + \frac{a_1}{30} x^5 + \cdots$$

Replace $x = t-1$ :

$$y(t) = a_0 + a_1(t-1) + \frac{a_0}{8}(t-1)^2 + \frac{a_1}{6}(t-1)^3 + \frac{3a_0}{128}(t-1)^4 + \frac{a_1}{30}(t-1)^5 + \cdots$$

<u>Find $a_0$ & $a_1$</u>

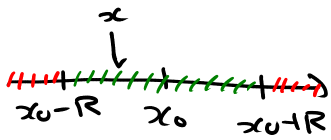
$y(1) = a_0 = 4$ & $y'(1) = a_1 = -1$

So,

$$y(t) = 4 - (t-1) + \frac{1}{2}(t-1)^2 - \frac{1}{6}(t-1)^3 + \frac{3}{32}(t-1)^4 - \frac{1}{30}(t-1)^5 + \cdots$$

It is important to know where our solution is valid.

- The **radius of convergence** of a power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is the number $R$ such that
  - $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges for any $x$ such that $|x-x_0| < R$.
  - $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ diverges for all $x$ such that $|x-x_0| > R$.




- If the limit

$$L := \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

exists, then the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$ is $R = \frac{1}{L}$.

EXAMPLE 6. Find the radius of convergence of

**(a)** $f(x) = \sum_{n=0}^{\infty} x^n$. $\to (1-x)^{-1}$

**(b)** $g(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$. $\to e^x$

(a) $a_n = 1 \Rightarrow L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{1} = 1$.

$R = \frac{1}{L} = \boxed{1}$

(b) $a_n = \frac{1}{n!} \Rightarrow L = \lim_{n\to\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n\to\infty} \frac{1}{n+1} = 0$

$R = \frac{1}{L} = \frac{1}{0}$ "=" $\infty$

**Theorem 7.** Suppose that $x_0$ is an ordinary point of the ODE

$$A(x)y'' + B(x)y' + C(x)y = 0.$$

Then the ODE has a general solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

The radius of convergence of any such series solution is at least as large as the distance from $x_0$ to the nearest (real or complex) singular point of the ODE.

**Example 8.** Determine the radius of convergence guaranteed by the last Theorem of a series solution of

$$(x^2 + 9)y'' + xy' + x^2 y = 0$$

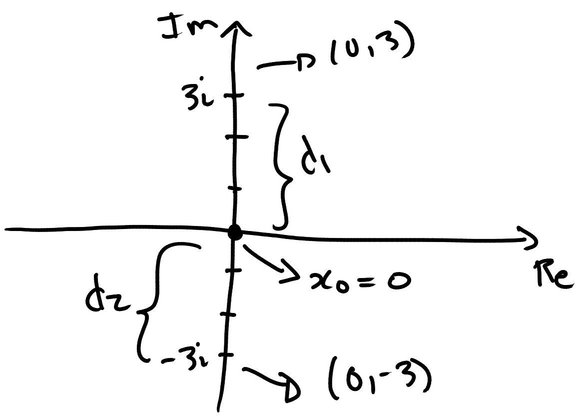
**(a)** in powers of $x$. $\to x_0 = 0$

**(b)** in powers of $x - 4$.

(a) $x^2 + 9 = 0 \Rightarrow x^2 = -9 \Rightarrow x = \pm\sqrt{-9}$

$\Rightarrow x = \pm 3i$

$z = a + bi$ (as a vector)




$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

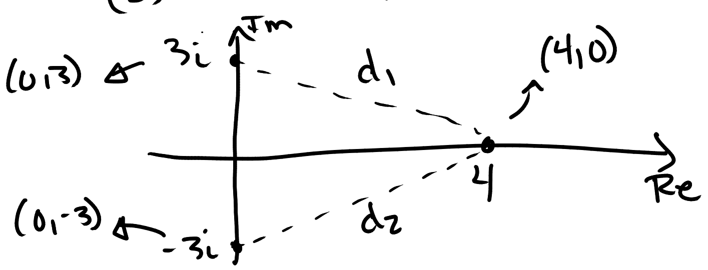
$d_1 = \sqrt{(0-0)^2 + (3-0)^2}$

$= \sqrt{3^2} = 3$

$d_2 = 3$

$R = \min\{3, 3\} = 3$

(b) $x_0 = 4$




$d_1 = \sqrt{(3-0)^2 + (0-4)^2}$

$= \sqrt{9 + 16} = 5$

$d_2 = 5$

$\boxed{R = 5}$

When we have a solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

of an ODE

$$A(x)y'' + B(x)y' + C(x)y = 0,$$

we can draw an approximation of the solution.

- The **Taylor polynomial** $T_N(x)$, where $N \geq 0$ is an integer, is given by the expression

$$T_N(x) = \sum_{n=0}^{N} a_n (x - x_0)^n = a_0 + a_1(x - x_0) + \cdots + a_N (x - x_0)^N.$$

- When the power series of $y(x)$ converges on a given interval $I$, we have

$$y(x) \approx T_N(x)$$

  for a sufficiently large integer $N$.

Example 9.

**(a)** Plot the graph of $T_4(x)$, $T_{10}(x)$, and $T_{20}(X)$ of the power series representation of $f(x) = \cos(x)$.

**(b)** Plot the graph of $T_4(x)$, $T_{10}(x)$, $T_{20}(x)$ for the power series solution of Example 5.