MATH 302

Chapter 8

SECTION 8.1: LAPLACE TRANSFORMS

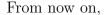
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THE LAPLACE TRANSFORM



• the variable t stands for the independent variable (time).

Discrete Process: Power series

Continuous Process

Remark:

- Recall that, with power series, we were able to solve a differential equation by solving a recurrence relation (so, basically, doing some algebra with a discrete number of data).
- With the Laplace transform, we will also be able to reduce an ODE problem into an algebra one.
- We use the symbol L(f(t)) to also denote the Laplace transform F(s).

EXAMPLE 1. Compute the Laplace transform of the function f(t) = t.

Here is a sample table of Laplace Transforms.

Function	Transform	Function	Transform
1	$\frac{1}{s}$	$\sin(\omega t)$	$\frac{\omega}{s^2 + \omega^2}$
t	$\frac{1}{s^2}$	$\cos(\omega t)$	$\frac{s}{s^2 + \omega^2}$
t^n	$\frac{n!}{s^{n+1}}$	$\sinh(\omega t)$	$\frac{\omega}{s^2 - \omega^2}$
e^{at}	$\frac{1}{s-a}$	$\cosh(\omega t)$	$\frac{s}{s^2 - \omega^2}$

Table 1: Laplace Transforms (sample)

Some Comments on Existence

It is important to check if a function possesses a Laplace transform.

Exponential Order Criterion.

If f(t) is a function satisfying

$$|f(t)| \le Me^{s_0 t}, \ t \ge t_0$$

for some numbers s_0 , t_0 , and M, then F(s) exists for $s > s_0$.

Remarks:

- Later on, we will see that the Laplace transform exists for discontinuous functions.
- Even more than that, we will apply the Laplace transform on functions taking ∞ as values!

EXAMPLE 2. The function $f(t) = e^{t^2}$ doesn't have a Laplace transform.

LINEARITY OF LAPLACE TRANSFORM

EXAMPLE 3. Justify that

$$L(\sinh(\omega t)) = \frac{\omega}{s^2 - \omega^2}.$$

Linearity of Laplace transform:

If f and g are two functions, and a, b are two real numbers, then

$$L(af(t)+bg(t))=aL(f(t))+bL(g(t))=aF(s)+bG(s).$$

You can apply this repeatedly to more than two functions.

FIRST SHIFT THEOREM

Did you notice that

$$L(e^{at}) = \frac{1}{s-a}?$$

- This is L(1), but with a shift s a!!!
- Since $e^{at} = 1 \cdot e^{at}$, we have the following shifting result.

Shifting Theorem:

If f(t) is a function with a Laplace transform F(s), then

$$L(e^{at}f(t)) = F(s-a).$$

EXAMPLE 4. Find the Laplace transform of

(a)
$$f(t) = e^{at} \sin(\omega t)$$
.

(b)
$$f(t) = e^{at} \cos(\omega t)$$
.

POWERS AND DERIVATIVES

Did you notice that

$$L(t) = \frac{1}{s^2} = -\frac{d}{ds} \left(\frac{1}{s}\right)?$$

- This is the derivative of L(1), but with a different sign.
- Since $t = 1 \cdot t$, we have the following result.

Powers Transformed in Derivatives.

If f has a Laplace transform and n is a positive integer, then

$$L(t^n f(t)) = (-1)^n F^{(n)}(s).$$

EXAMPLE 5. Find the Laplace transform of

(a) $f(t) = t \cos(\omega t)$.

(c) $f(t) = te^{at}$.

(b) $f(t) = t \sinh(\omega t)$.

(d) $f(t) = t\sin(2t) + t^2\cos(t)\sin(t)$.

Did you notice that

$$L(\cos(t)) = \frac{s}{s^2 + 1} =$$
?

Derivatives Transformed in Powers:

If $f, f', \ldots, f^{(n)}$ have a Laplace transform for $n \geq 1$, then

$$L(f^{(n)}(t)) = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s f^{(n-2)}(0) - f^{(n-1)}(0).$$

Most relevant formulas:

- n = 1: L(f'(t)) = sF(s) f(0).
- n = 2: $L(f''(t)) = s^2 F(s) sf(0) f'(0)$.
- n = 3: $L(f^{(3)}(t)) = s^3 F(s) s^2 f(0) sf'(0) f''(0)$.

EXAMPLE 6. Find the Laplace transform of

- (a) $f(t) = \cos^2(t)$.
- **(b)** $g(t) = \sin^2(t)$.