Section 7.2 — Problem A 15 Points

Solve the following IVP using power series:

$$(x^2 - 4)y'' - xy' - 3y = 0, \quad y(0) = -1,\, y'(0) = 2.$$

Section 7.2 — Problem B 10 Points

Solve the following IVP using power series:

$$y'' + (x-3)y' + 3y = 0, \quad y(3) = -2,\, y'(3) = 3,$$

given that the general solution to

$$y''(t) + ty' + 3y(t) = 0$$

is

$$y(t) = a_0 + a_1 t - \frac{3}{2}a_0 t^2 - \frac{2}{3}a_1 t^3 + \frac{5}{8}a_0 t^4 + \frac{3}{10}a_1 t^5 - \frac{7}{48}a_0 t^6 - \frac{2}{35}a_1 t^7 + \cdots .$$

Section 8.1 — Problem C 25 Points

Find the Laplace transform of the following functions (you can use the table)

1) $\cosh(t)\sin(t)$.
2) $\cosh^2(t)$.
3) $t\sinh(2t)$.
4) $\sin(2t) + \cos(4t)$.
5) $\sin(2t)\cos(3t)$.

Total (Points): 50.

Complete Solutions

Section 7.2 — Problem A — 15 Points

The initial condition are given at $x = 0$ and $x = 0$ is not a singular point of the ODE. So, we let $y(x) = \sum_{n=0}^{\infty} a_n x^n$ and therefore

$$\begin{aligned}
(x^2-4)y'' &= \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=2}^{\infty} n(n-1)4a_n x^{n-2} \\
xy' &= \sum_{n=1}^{\infty} n a_n x^n \\
3y &= \sum_{n=0}^{\infty} 3a_n x^n.
\end{aligned}$$

The left-hand side then becomes

$$\begin{aligned}
LHS &= \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty}(n+2)(n+1)4a_{n+2}x^n - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} 3a_n x^n \\
&= (-8a_2 - 3a_0) + (-24a_3 - 4a_1)x + \sum_{n=2}^{\infty}\Big[n(n-1)a_n - (n+3)a_n - (n+2)(n+1)4a_{n+2}\Big]x^n \\
&= (-8a_2 - 3a_0) + (-24a_3 - 4a_1)x + \sum_{n=2}^{\infty}\Big[(n^2-2n-3)a_n - 4(n^2+3n+2)a_{n+2}\Big]x^n
\end{aligned}$$

Setting $LHS = 0$, we obtain the following relations:

$$a_2 = -\frac{3}{8}a_0, \quad a_3 = -\frac{1}{6}a_1 \quad \text{and} \quad a_{n+2} = \frac{(n-3)(n+1)}{4(n+2)(n+1)}a_n = \frac{n-3}{4(n+2)}a_n.$$

Let's try to find an explicit expression for a_n.

- $a_4 = a_{2+2} = \frac{2-3}{4(2+2)}a_2 = \frac{-1}{16}\frac{-3}{8}a_0 = \frac{3}{128}a_0 = \frac{3}{2^7}a_0.$
- $a_5 = a_{3+2} = \frac{3-3}{4(5)}a_3 = 0.$
- $a_6 = a_{4+2} = \frac{4-3}{4(4+2)}a_4 = \frac{1}{24}\frac{3}{128}a_0 = \frac{1}{1024}a_0 = \frac{1}{2^{10}}a_0.$
- $a_7 = a_{5+2} = \frac{5-3}{4(5+2)}a_5 = 0.$
- $a_8 = a_{6+2} = \frac{6-3}{4(6+2)}a_6 = \frac{3}{32}\frac{1}{1024}a_0 = \frac{3}{2^{15}}a_0.$
- $a_9 = 0.$
- $a_{10} = a_{8+2} = \frac{8-3}{4(8+2)}a_8 = \frac{5}{4\cdot 10}\frac{3}{2^{15}}a_0 = \frac{3}{2^{18}}a_0.$
- $a_{11} = 0.$
- $a_{12} = a_{10+2} = \frac{10-3}{4(10+2)}a_{10} = \frac{7}{48}\frac{3}{2^{18}}a_0 = \frac{7}{2^{22}}a_0.$

There is no easy pattern. Therefore, we will only give a finite number of terms of the power series solution:

$$y(x) = a_0 + a_1 x - \frac{3}{8}a_0 x^2 - \frac{1}{6}a_1 x^3 + \frac{3}{2^7}a_0 x^4 + \frac{1}{2^{10}}a_0 x^6 + \frac{3}{2^{15}}a_0 x^8 + \frac{3}{2^{18}}a_0 x^{10} + \frac{7}{2^{22}}a_0 x^{12} + \cdots.$$

We know that $y(0) = -1$. So $a_0 = -1$. Also, we know that $y'(0) = 2$, so $a_1 = 2$. Replacing this in the general solution, we obtain

$$y(x) = -1 + 2x + \frac{3}{8}x^2 - \frac{1}{3}x^3 - \frac{3}{2^7}x^4 - \frac{1}{2^{10}}x^6 - \frac{3}{2^{15}}x^8 - \frac{3}{2^{18}}x^{10} - \frac{7}{2^{22}}x^{12} - \cdots.$$

<u>Remark:</u> Notice that there is a finite number of powers of x multiplying the arbitrary constant a_1. In other words, the function $a_1(x - \frac{x^3}{6})$ is a solution to the ODE (you can check this). So we can use the method of variation of parameters to find a second solution.

Let $y(x) = u(x)y_1(x)$. Then, we have

$$\begin{aligned} y'(x) &= u'y_1 + uy_1' \\ y''(x) &= u''y_1 + 2u'y_1 + uy_1'' \end{aligned}$$

and replacing this in the ODE:

$$(x^2-4)(u''y_1 + 2u'y_1 + uy_1'') - x(u'y_1 + uy_1') - 3uy_1 = (x^2-4)(u''y_1 + 2u'y_1) - xu'y_1$$

and therefore

$$(x^2-4)y_1u'' + \Big(2(x^2-4)y_1 - xy_1\Big)u' = 0.$$

After dividing by $(x^2-4)y_1$, we get

$$u'' + \Big(2 - \frac{x}{x^2-4}\Big)u' = 0.$$

Letting $z = u'$, we have to solve the following first order ODE:

$$z' + \Big(2 - \frac{x}{x^2-4}\Big)z = 0.$$

The solution is given by

$$z(x) = a_0 e^{-2x}\sqrt{|x^2-4|}.$$

Therefore, integrating leads to

$$u(x) = a_0 \int e^{-2x}\sqrt{|x^2-4|}\,dx + a_1$$

Finally, replacing the expression of $u(x)$ in $y(x)$, we get

$$y(x) = a_0\Big(x - \frac{x^3}{6}\Big)\int e^{-2x}\sqrt{|x^2-4|}\,dx + a_1\Big(x - \frac{x^3}{6}\Big).$$

Section 7.2 — Problem B **10 Points**

We translate the problem, so that the initial condition are given at 0. We let $t = x - 3$ and set $y(t) = y(x-3)$. Therefore, we see that

$$y''(t) = y'',\ y'(t) = y'.$$

Therefore, the ODE becomes

$$y''(t) + ty'(t) + 3y(t) = 0.$$

From the assumptions of the problem, we have that

$$y(t) = a_0 + a_1 t - \frac{3}{2}a_0 t^2 - \frac{2}{3}a_1 t^3 + \frac{5}{8}a_0 t^4 + \frac{3}{10}a_1 t^5 - \frac{7}{48}a_0 t^6 - \frac{2}{35}a_1 t^7 + \cdots .$$

and changing t for $x - 3$ gives

$$\begin{aligned} y(x) = a_0 + a_1(x-3) - \frac{3}{2}a_0(x-3)^2 - \frac{2}{3}a_1(x-3)^3 + \frac{5}{8}a_0(x-3)^4 + \frac{3}{10}a_1(x-3)^5 \\ - \frac{7}{48}a_0(x-3)^6 - \frac{2}{35}a_1(x-3)^7 + \cdots . \end{aligned}$$

From there, we see that $y(3) = a_0$ and $y'(3) = a_1$. From the initial conditions, we get $a_0 = -2$ and $a_1 = 3$. The solution to the IVP is therefore

$$\begin{aligned} y(x) = -2 + 3(x-3) + 3(x-3)^2 - 2(x-3)^3 - \frac{5}{4}(x-3)^4 + \frac{9}{10}(x-5)^5 + \frac{7}{24}(x-3)^6 \\ - \frac{6}{35}(x-3)^7 + \cdots . \end{aligned}$$

Section 8.1 — Problem C   25 Points

1) We have $\cosh(t) = \frac{e^t+e^{-t}}{2}$ and therefore

$$\cosh(t)\sin(t) = \frac{1}{2}e^t\sin(t) + \frac{1}{2}e^{-t}\sin(t).$$

Using the linearity, we see that

$$L\Big(\cosh(t)\sin(t)\Big) = \frac{1}{2}L(e^t\sin(t)) + \frac{1}{2}L(e^{-t}\sin(t)).$$

From the table and the property that $L(e^{at}f(t)) = F(s-a)$, we obtain

$$L(e^t\sin t) = \frac{1}{(s-1)^2+1} \quad \text{and} \quad L(e^{-t}\sin(t)) = \frac{1}{(s+1)^2+1}.$$

Therefore, we obtain

$$L\Big(\cosh(t)\sin(t)\Big) = \frac{1/2}{(s-1)^2+1} + \frac{1/2}{(s+1)^2+1}.$$

2) We have

$$\cosh^2(t) = \Big(\frac{e^t+e^{-t}}{2}\Big)^2 = \frac{e^{2t}+2+e^{-2t}}{4}.$$

Using the linearity, we obtain

$$L\Big(\cosh^2(t)\Big) = \frac{1}{4}L(e^{2t}) + \frac{1}{2}L(1) + \frac{1}{4}L(e^{-2t}).$$

From the table, we have

$$L(e^{2t}) = \frac{1}{s-2}, \quad L(1) = \frac{1}{s} \quad \text{and} \quad L(e^{-2t}) = \frac{1}{s+2}.$$

Therefore, we get

$$L\Big(\cosh^2(t)\Big) = \frac{1}{4(s-2)} + \frac{1}{2s} + \frac{1}{4(s+1)}.$$

3) We have that

$$L(\sinh(2t)) = \frac{2}{s^2+4}.$$

From the fact that $L(tf(t)) = -\frac{d}{ds}F(s)$, we get that

$$L\Big(t\sinh(2t)\Big) = -\frac{d}{ds}\Big(\frac{2}{s^2+4}\Big) = \frac{4s}{(s^2+4)^2}.$$

4) Using the linearity, we have

$$L\Big(\sin(2t)+\cos(4t)\Big) = L\Big(\sin(2t)\Big) + L\Big(\cos(4t)\Big).$$

From the table, we have

$$L\Big(\sin(2t)\Big) = \frac{2}{s^2+4} \quad \text{and} \quad L\Big(\cos(4t)\Big) = \frac{s}{s^2+16}.$$

Therefore, we obtain

$$L\Big(\sin(2t)+\cos(4t)\Big) = \frac{2}{s^2+4} + \frac{s}{s^2+16}.$$

5) Using a trigonometric identity, we have

$$\sin(2t)\cos(3t) = \frac{1}{2}\sin(-t) + \frac{1}{2}\sin(5t) = \frac{1}{2}\sin(5t) - \frac{1}{2}\sin(t).$$

Using the linearity, we see that

$$L\Big(\sin(2t)\cos(3t)\Big) = \frac{1}{2}L\Big(\sin(5t)\Big) - \frac{1}{2}L\Big(\sin(t)\Big).$$

Using the table, we obtain

$$L\Big(\sin(2t)\cos(3t)\Big) = \frac{5}{2(s^2+25)} - \frac{1}{2(s^2+1)}.$$

TOTAL (POINTS): 50.