Section 2.1 — Problem 3 — 5 points

We separate the variable:

$$xy' = -(\ln x)y \quad \Rightarrow \quad \frac{y'}{y} = -\frac{\ln x}{x}.$$

Then we integrate, to obtain

$$\ln|y| = -\int \frac{\ln x}{x}\,dx + K.$$

To simplify the right-hand side, let $u = \ln x$, then $du = \frac{dx}{x}$ and

$$\int \frac{\ln x}{x}\,dx = \int u\,du = \frac{u^2}{2} = \frac{(\ln x)^2}{2}.$$

We can therefore write

$$\ln|y| = -\frac{(\ln x)^2}{2} + K$$

and by taking the exponential on each side, we obtain

$$|y| = \exp\left(\frac{-(\ln x)^2}{2}\right)\exp(K) = e^{-(\ln x)^2/2}e^K.$$

Since the exponential function is positive, then y can be only strictly positive or strictly negative. By letting $c = \pm e^K$, we then obtain

$$y(x) = ce^{-(\ln x)^2/2}. \tag{1}$$

Remarks:

- We devided by y and x. Therefore, y can't be zero and x can't be zero.
- We can check that $y = 0$ is a solution. We can incorporate this solution in our solution (1) by including $c = 0$.

Section 2.1 — Problem 7 — 5 points

We separate the variables:

$$xy' + \left(1 + \frac{1}{\ln x}\right) y = 0 \quad \Rightarrow \quad \frac{y'}{y} = -\frac{1}{x}\left(1 + \frac{1}{\ln x}\right).$$

We have to add the assumptions that $x > 0$ (because we have a $\ln x$), $x \neq 1$ (because $\ln 1 = 0$) and y is not the zero function. We then integrate:

$$\ln|y| = -\int \frac{1}{x}\left(1 + \frac{1}{\ln x}\right) dx + K.$$

To simplify the right-hand side, we let $u = \ln x$. We have $du = dx/x$ and therefore

$$\int \frac{1}{x}\left(1 + \frac{1}{\ln x}\right) dx = \int 1 + \frac{1}{u}\, du = u + \ln|u| = \ln x + \ln|\ln x|.$$

So we obtain

$$\ln|y| = -(\ln x + \ln|\ln x|) + K.$$

Taking the exponential, we obtain

$$|y| = e^{-\ln x - \ln|\ln x|} e^K = \frac{e^K}{x|\ln x|} = \frac{\pm e^K}{x \ln x}.$$

Now, since $x \ln x$ is always negative on $(0,1)$ and always positive on $(1,\infty)$, the overall sign of the function y won't change in those intervals, respectively. We can therefore absorb the signs by letting $c = \pm e^K$ and therefore

$$y = \frac{c}{x \ln x}.$$

Here, the function y is well-defined on $(0,1)$ and on $(1,\infty)$.

We have to determine the constant c that satisfies the IVP. We have $y(e) = 1$ and therefore

$$1 = \frac{c}{e \ln e} \quad \Rightarrow \quad 1 = \frac{c}{e} \quad \Rightarrow \quad c = e.$$

We then have

$$y(x) = \frac{e}{x \ln x}.$$

Remark:

- The interval of validity of the solution to the IVP is $(1,\infty)$ because $e \in (1,\infty)$.

Section 2.1 — Problem 19 — 10 points

Complementary Equation.

We first solve the complementary equation. The complementary equation is $xy' + 2y = 0$. We separate the variables:

$$\frac{y'}{y} = -\frac{2}{x}$$

where $y \neq 0$ and $x \neq 0$. We integrate to get

$$\ln|y| = -2\ln|x| + K$$

and therefore, taking the exponential, we get

$$|y| = \frac{e^K}{|x|^2} = \frac{e^K}{x^2}.$$

Since x^2 is always positive for $x \neq 0$, we can write $c = \pm e^K$ and

$$y(x) = \frac{c}{x^2}.$$

Variation of parameter.

Let $y_1 = 1/x^2$ (pick one solution, here $c = 1$). Set $y = uy_1 = u/x^2$. We have $y' = u'/x^2 - 2u/x^3$ and replace the expression of y' and y in the DE:

$$x\left(\frac{u'}{x} - \frac{2u}{x^3}\right) + 2\frac{u}{x^2} = \frac{2}{x^2} + 1 \qquad \Longleftrightarrow \qquad u' = \frac{2}{x^2} + 1.$$

We integrate and get $u(x) = -2/x + x + c$. Therefore the general solution is

$$y(x) = u(x)y_1(x) = \frac{\left(-\frac{2}{x} + x + c\right)}{x^2} = \frac{1}{x} - \frac{2}{x^3} + \frac{c}{x^2}.$$

Section 2.1 — Problem 31 — 10 points

Complementary Equation.

The complementary equation is $xy' + 2y = 0$. We solved this ODE in the previous problem. The solution was

$$y(x) = \frac{c}{x^2}.$$

Variation of Parameter.

We let $y = uy_1$ for some particular solution y_1 of the complementary equation. We choose $y_1(x) = 1/x^2$ (so $c = 1$). Therefore, $y = u/x^2$ and $y' = u'/x^2 - 2u/x^3$. We replace these information in the ODE:

$$x\left(\frac{u'}{x^2} - 2\frac{u}{x^3}\right) + 2\frac{u}{x^2} = 8x^2 \quad \Rightarrow \quad u' = 8x^2.$$

We integrate to get $u(x) = (8/3)x^3 + c$. Therefore, we obtain

$$y(x) = \frac{\left(\frac{8x^3}{3} + c\right)}{x^2} = \frac{8}{3}x + \frac{c}{x^2}.$$

IVP.

We have $y(1) = 3$. Therefore

$$3 = \frac{8}{3} + c \quad \Rightarrow \quad c = 1/3.$$

Thus, the solution to the IVP is

$$y(x) = \frac{8}{3}x + \frac{1}{3x^2}.$$

Remark:

- It is not necessary to mention it, but the interval of validity is $(0, \infty)$ since $1 \in (0, \infty)$.

Section 2.2 — Problem 3 — 10 points

We rewrite the ODE so that the variables are separated:

$$\frac{y'}{y^2+y} = -\frac{1}{x}. \tag{2}$$

This is a valid equation if y^2+y is not zero.

Find the Constant Solutions:

We have $y^2+y=0$ when $y=0$ or $y=-1$. Those are the constant solutions of the ODE.

Find the Non-Constant Solutions:

We now suppose that y is not always 0 and -1. Therefore the form (2) is valid (with the additional detail that $x \neq 0$).

We have to integrate both sides. The integral in y is dealt with partial fractions. We have

$$\frac{1}{y^2+y} = \frac{1}{y(y+1)} = \frac{1}{y} - \frac{1}{y+1}$$

and therefore

$$\int \frac{dy}{y^2+y} = \int \frac{dy}{y(y+1)} == \int \frac{1}{y} - \frac{1}{y+1}\, dy = \ln|y| + \ln|y+1|.$$

The integral in x in simply $-\ln|x| + K$. So, putting everything together, we get

$$\ln|y| + \ln|y+1| = -\ln|x| + K.$$

Taking the exponential gives us now

$$|y||y-1| = \frac{e^K}{|x|}.$$

Now, the function y can't change sign and therefore, we can write

$$y(y-1) = \frac{c}{|x|}$$

where $c = \pm e^K$. We can leave the solution as

$$|x|y^2 - |x|y = c$$

which gives us an implicit solution for $x \neq 0$ and $y \neq 0, -1$. We can also find explicitly the solution by using the quadratic formula. The polynomial in question is

$$|x|y^2 - |x|y - c = 0$$

and we solve for y:

$$y(x) = \frac{|x| \pm \sqrt{|x|^2 + 4|x|c}}{2|x|} = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 4c/|x|}.$$

There are therefore two possibles explicit solutions

$$y_1(x) = \frac{1}{2} + \frac{1}{2}\sqrt{1+4c/|x|} \quad \text{or} \quad y_2(x) = \frac{1}{2} - \frac{1}{2}\sqrt{1+4c/|x|}$$

Section 2.2 — Problem 19 — 10 points

The ODE can be separated:

$$(1+2y)y' = 2x.$$

After integrating, we find

$$y + y^2 = x^2 + c.$$

We can leave our solution like this; this is the implicit solution to the ODE.

Let's find the value of c. We have $y(2) = 0$ and therefore

$$0 + 0^2 = 2^2 + c \quad \Rightarrow \quad c = -4.$$

The implicit solution to the IVP is

$$y + y^2 = x^2 - 4.$$

We can also find an explicit expression for y. We consider the implicit equation as a polynomial in y:

$$y^2 + y - x^2 - c = 0.$$

From the quadratic formula, we get

$$y(x) = \frac{-1 \pm \sqrt{1 + 4(x^2 + c)}}{2}.$$

This leads to the following two possible solutions:

$$y_1(x) = \frac{-1 + \sqrt{1 + 4(x^2 + c)}}{2} \quad \text{and} \quad y_2(x) = \frac{-1 - \sqrt{1 + 4(x^2 + c)}}{2}.$$

Since y_2 is always smaller than -1, we must use y_1 for the solution of the IVP. We have $y_1(2) = 0$ and therefore

$$0 = \frac{-1 + \sqrt{1 + 16 + 4c}}{2} \quad \Longleftrightarrow \quad c = -4.$$

So the solution is

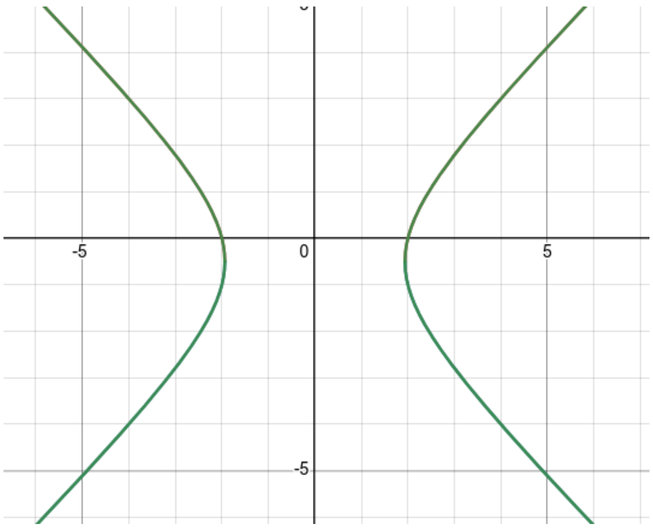
$$y_1(x) = \frac{-1 + \sqrt{1 + 4(x^2 - 4)}}{2}$$

with the interval of validity being $[2, \infty)$.

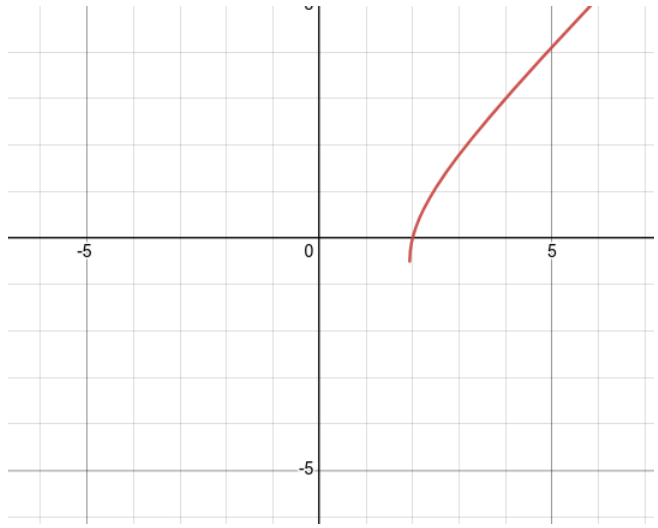
Remarks:

- the value of the constant $c = -4$ is the same in the implicit solution and the solutions y_1, y_2. This makes sense if you obtain y_1 and y_2 directly from the implicit solution to the IVP. However, y_2 is not a solution to the IVP because $y_2(2) \neq 0$.
- The graphs of the implicit solution with y_1 and y_2 are displayed on the next page. We can see that y_1, y_2 are parts of the curve $y + y^2 = x^2 - 4$ (defined implicitly).

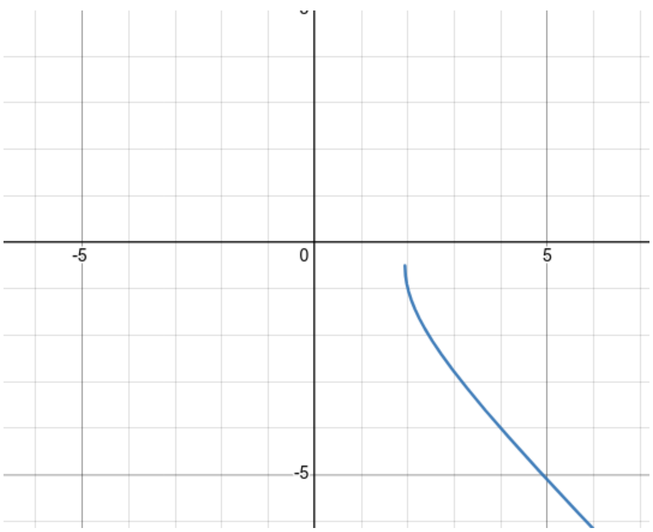
Total (Points): 50.



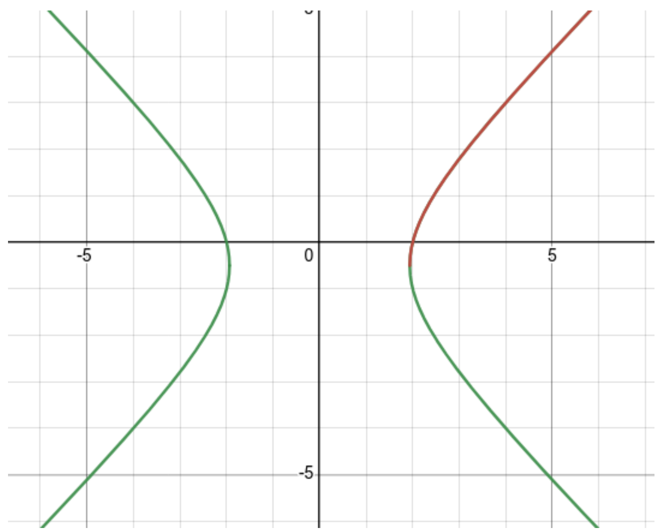
(a) Graph of $y + y^2 = x^2 - 4$



(b) Graph of $y_1(x)$



(c) Graph of $y_2(x)$



(d) Graphs of $y + y^2 = x^2 - 4$ and $y_1(x)$