Section 7.2 — Problem 1 — 25 points

We let $y(x) = \sum_{n=0}^{\infty} a_n x^n$. We have

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Therefore, we can rewrite

$$(1+x^2)y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x^2 \sum_{n=2}^{\infty} n(n-1)x^{n-2} = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n x^n$$

and

$$6xy' = 6x \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} 6n a_n x^n.$$

The expression of the left-hand side of the ODE is then

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=1}^{\infty} 6n a_n x^n + \sum_{n=0}^{\infty} 6 a_n x^n.$$

Shifting the first summation, we get

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=1}^{\infty} 6n a_n x^n + \sum_{n=0}^{\infty} 6 a_n x^n.$$

Combining similar powers together, we obtain

$$2a_2 + 6a_3 x + 6a_1 x + 6a_0 + 6a_1 x + \sum_{n=2}^{\infty} \Big[(n+2)(n+1)a_{n+2} + (n(n-1) + 6n + 6)a_n \Big] x^n$$

and therefore

$$(6a_0 + 2a_2) + (12a_1 + 6a_3)x + \sum_{n=2}^{\infty} \Big[(n+2)(n+1)a_{n+2} + (n(n-1) + 6n + 6)a_n \Big] x^n = 0 = \sum_{n=0}^{\infty} 0x^n.$$

Equating each coefficients with the same powers, we obtain

$$a_2 = -3a_0, \quad a_3 = -2a_1 \quad \text{and} \quad a_{n+2} = -\frac{n^2 + 5n + 6}{n^2 + 3n + 2} a_n$$

We can get a sample of the list of a_n for even indexed coefficients:

- a_0 arbitrary;
- $a_2 = -3a_0$;

- $a_4 = -\frac{2^2+5\cdot 2+6}{2^2+3\cdot 2+2} a_2 = -\frac{20}{12}(-3a_0) = 5a_0$;
- $a_6 = -\frac{4^2+5\cdot 4+6}{4^2+3\cdot 4+2} a_4 = -\frac{42}{30} 5a_0 = -7a_0$;
- $a_8 = -\frac{6^2+5\cdot 6+6}{6^2+3\cdot 6+2} a_6 = -\frac{72}{56}(-7a_0) = 9a_0$.

We see a pattern. The general pattern is $a_{2n} = (-1)^n(2n+1)a_0$, for $n \geq 1$. We can do the same for the coefficients indexed by odd integers:

- a_1 arbitrary;
- $a_3 = -2a_1$;
- $a_5 = -\frac{3^2+5\cdot 3+6}{3^2+3\cdot 3+2} a_3 = -\frac{30}{20}(-2a_1) = 3a_1$;
- $a_7 = -\frac{5^2+5\cdot 5+6}{5^2+3\times 5+2} a_5 = -\frac{56}{42} 3a_1 = -4a_1$;
- $a_9 = -\frac{7^2+5\cdot 7+6}{7^2+3\cdot 7+2} a_7 = -\frac{90}{72}(-4a_1) = 5a_1$.

We see a pattern. The general pattern is $a_{2n+1} = (-1)^n(n+1)a_1$. Therefore, our final answer is

$$\begin{aligned} y(x) &= a_0 + a_1 x + \sum_{n=1}^{\infty}(-1)^n(2n+1)a_0 x^{2n} + \sum_{n=1}^{\infty}(-1)^n(n+1)a_1 x^{2n+1} \\ &= a_0 \sum_{n=0}^{\infty}(-1)^n(2n+1)x^{2n} + a_1 \sum_{n=0}^{\infty}(-1)^n(n+1)x^{2n+1}. \end{aligned}$$

Remark: For those interested, we can find an explicit expression of the power series solution. First, notice that integrating term-by-term the first series in front of a_0 gives

$$\int_0^x \sum_{n=0}^{\infty}(-1)^n(2n+1)t^{2n}\,dt = \sum_{n=0}^{\infty}(-1)^n(2n+1)\left.\frac{t^{2n+1}}{2n+1}\right|_{t=0}^{t=x} = \sum_{n=0}^{\infty}(-1)^n x^{2n+1}.$$

Rewrite the last series as followed:

$$\sum_{n=0}^{\infty}(-1)^n x^{2n+1} = x\sum_{n=0}^{\infty}(-x^2)^n$$

and then use the power series of $(1-x)^{-1}$ with $-x^2$ in place of x to get

$$x\sum_{n=0}^{\infty}(-x^2)^n = \frac{x}{1+x^2}.$$

This is valid only when $-1 < x < 1$. To obtain the original series, we use the Fundamental Theorem of Calculus and take the derivative of the last expression we obtained:

$$\sum_{n=0}^{\infty}(-1)^n(2n+1)x^{2n} = \frac{d}{dx}\left(\frac{x}{1+x^2}\right) = \frac{1-x^2}{(1+x^2)^2}.$$

For the power series in front of a_1, we will manipulate algebraically the expression. A simple algebra trick leads to

$$\sum_{n=0}^{\infty}(-1)^n(n+1)x^{2n+1} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(2n+2)x^{2n+1} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(2n+1)x^{2n+1} + \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n x^{2n+1}.$$

Now, we see that

$$\sum_{n=0}^{\infty}(-1)^n(2n+1)x^{2n+1} = x\sum_{n=0}^{\infty}(2n+1)x^{2n} = x\left(\frac{1-x^2}{(1+x^2)^2}\right) = \frac{x-x^3}{(1+x^2)^2}.$$

We also see, using the power series of $(1-x)^{-1}$ with $-x^2$ in place of x that

$$\sum_{n=0}^{\infty}(-1)^n x^{2n+1} = x\sum_{n=0}^{\infty}(-x^2)^n = \frac{x}{1+x^2}.$$

Combining everything together, we get

$$\begin{aligned}\sum_{n=0}^{\infty}(-1)^n(n+1)x^{2n+1} &= \frac{1}{2}\left(\frac{x-x^3}{(1+x^2)^2}\right) + \frac{1}{2}\left(\frac{x}{1+x^2}\right)\\ &= \frac{1}{2}\left(\frac{x-x^3+x(1+x^2)}{(1+x^2)^2}\right)\\ &= \frac{x}{(1+x^2)^2}.\end{aligned}$$

Finally, we can rewrite the power series solution (valid for $-1 < x < 1$) as

$$y(x) = a_0\frac{1-x^2}{(1+x^2)^2} + a_1\frac{x}{(1+x^2)^2}.$$

You can check that $y_1(x) := \frac{1-x^2}{(1+x^2)^2}$ is a solution to the ODE of the problem and $y_2(x) := \frac{x}{(1+x^2)^2}$ is also a solution to the ODE. We see that

$$\frac{y_1}{y_2} = \frac{1-x^2}{x} = \frac{1}{x} - x$$

which is not constant. Therefore, $\{y_1, y_2\}$ is a fundamental set of solutions for the ODE! Isn't it beautiful ;)

Section 7.2 — Problem 7 — 25 points

Write $y(x) = \sum_{n=0}^{\infty} a_n x^n$. Using the rules to differentiate power series, we get

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Therefore, we get

$$(1-x^2)y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n$$

and

$$5xy'(x) = \sum_{n=1}^{\infty} 5na_n x^n.$$

Putting that into the left-hand side of the ODE, we get

$$\begin{aligned}
&\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} 5na_n x^n - \sum_{n=0}^{\infty} 4a_n x^n \\
&= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} 5na_n x^n - \sum_{n=0}^{\infty} 4a_n x^n \\
&= a_2 + a_3 x - 5a_1 x - 4a_0 - 4a_1 x + \sum_{n=2}^{\infty} \Big((n+2)(n+1)a_{n+2} - \Big[n(n-1) + 5n + 4\Big] a_n \Big) x^n \\
&= (a_2 - 4a_0) + (a_3 - 9a_1)x + \sum_{n=2}^{\infty} \Big((n+2)(n+1)a_{n+2} - \Big[n(n-1) + 5n + 4\Big] a_n \Big) x^n.
\end{aligned}$$

Equating the left-hand side with the right-hand side, we get

$$a_2 = 4a_0, \quad a_3 = 9a_1 \quad \text{and} \quad a_{n+2} = \frac{n^2+4n+4}{(n+2)(n+1)} a_n.$$

Since $n^2 + 4n + 4 = (n+2)^2$, we get

$$a_2 = 4a_0, \quad a_3 = 9a_1 \quad \text{and} \quad a_{n+2} = \frac{n+2}{n+1} a_n.$$

For even integers, we see that

- $a_4 = \frac{4}{3} a_2 = \frac{4}{3} 4a_0 = \frac{4 \cdot 2}{3 \cdot 1} 2a_0$;
- $a_6 = \frac{6}{5} a_4 = \frac{6 \cdot 4 \cdot 2}{5 \cdot 3 \cdot 1} 2a_0$;
- $a_8 = \frac{8}{7} a_6 = \frac{8 \cdot 6 \cdot 4 \cdot 2}{7 \cdot 5 \cdot 3 \cdot 1} 2a_0$;
- In general $a_{2n+2} = \frac{(2n+2)!!}{(2n+1)!!}(2a_0)$ where $k!!$ is the double factorial of an integer k (look it up on Google ;)).

It is more appropriate to write the general rule as $a_{2n} = \frac{(2n)!!}{(2n-1)!!}(2a_0)$. For odd indexes, we have

- $a_5 = \frac{5}{4} a_3 = \frac{5}{4} 9a_1 = \frac{5 \cdot 3}{4 \cdot 2}(6a_1)$;

- $a_7 = \frac{7}{6}a_5 = \frac{7 \cdot 5 \cdot 3}{6 \cdot 4 \cdot 2}(6a_1)$;
- $a_9 = \frac{9}{8}a_7 = \frac{9 \cdot 7 \cdot 5 \cdot 3}{8 \cdot 6 \cdot 4 \cdot 2}(6a_1)$;
- In general $a_{2n+3} = \frac{(2n+3)!!}{(2n+2)!!}(6a_1)$.

It is more appropriate to write the general rule as $a_{2n+1} = \frac{(2n+1)!!}{(2n)!!}(6a_1)$.

Therefore, the general solution looks like

$$\begin{aligned} y(x) &= a_0 + a_1 x + 2a_0 \sum_{n=1}^{\infty} \frac{(2n)!!}{(2n-1)!!} x^{2n} + 6a_1 \sum_{n=1}^{\infty} \frac{(2n+1)!!}{(2n)!!} x^{2n+1} \\ &= a_0\left(1 + 2\sum_{n=1}^{\infty} \frac{(2n)!!}{(2n-1)!!} x^{2n}\right) + a_1\left(x + 6\sum_{n=1}^{\infty} \frac{(2n+1)!!}{(2n)!!} x^{2n+1}\right). \end{aligned}$$

<u>More details</u>: We can rewrite the second series as a closed expression. By integrating terms-by-terms, we get

$$\sum_{n=1}^{\infty} \frac{(2n+1)!!}{(2n)!!} \frac{x^{2n+2}}{2n+2} = \sum_{n=1}^{\infty} \frac{(2n+1)!!}{(2n+2)!!} x^{2n+2} = \sum_{n=2}^{\infty} \frac{(2n-1)!!}{(2n)!!} x^{2n}.$$

The power series representation of $(1-x^2)^{-1/2}$ (for $-1 < x < 1$) is

$$1 + \frac{1}{2}x^2 + \frac{1 \cdot 3}{2 \cdot 4}x^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^6 + \cdots = 1 + \frac{1}{2}x^2 + \sum_{n=2}^{\infty} \frac{(2n-1)!!}{(2n)!!} x^{2n}$$

and therefore, we obtain

$$\sum_{n=2}^{\infty} \frac{(2n-1)!!}{(2n)!!} x^{2n} = (1-x^2)^{-1/2} - 1 - \frac{1}{2}x^2.$$

Since we took the integral, undoing this process is differentiation. So, after differentiating, we obtain

$$\sum_{n=1}^{\infty} \frac{(2n+1)!!}{(2n)!!} x^{2n+1} = \frac{2x}{(1-x^2)^{3/2}} - x.$$

Therefore, setting $y_2(x) = x + 6\sum_{n=1}^{\infty} \frac{(2n+1)!!}{(2n)!!} x^{2n+1}$, we get

$$y_2(x) = \frac{12x}{(1-x^2)^{3/2}} - 5x.$$

For the power series in front for a_0, we will use another strategy: Variation of parameters. We know one solution of the ODE. So, let

$$y(x) = u(x) y_2(x).$$

Therefore, we get

$$y'(x) = u'(x)y_2(x) + u(x)y_2'(x) \quad \text{and} \quad y''(x) = u''(x)y_2 + 2u'(x)y_2(x) + u(x)y_2''(x).$$

Replacing in the ODE, we see that

$$(1-x^2)(u''y_2 + 2u'y_2 + uy_2'') - 5x(u'y_2 + uy_2') - 4uy_2$$

which can be simplified to

$$(1-x^2)(u''y_2 + 2u'y_2) - 5xu'y_2 + u\Big((1-x^2)y_2'' - 5xy_2' - 4y_2)\Big).$$

Because y_2 is a solution, the last expression can be simplified to

$$(1-x^2)u''y_2 + ((1-x^2)2y_2 - 5xy_2)u'.$$

Letting $z = u'$, then the ODE becomes

$$(1-x^2)y_2z' + ((1-x^2)2y_2 - 5xy_2)z = 0$$

and solving for z, we obtain

$$\frac{z'}{z} = -\frac{(1-x^2)2y_2 - 5xy_2}{(1-x^2)y_2} = -\frac{2(1-x^2) - 5x}{1-x^2} = -2 + \frac{5x}{1-x^2}.$$

We can then integrate to get

$$\ln|z| = -2x + \int \frac{5x}{1-x^2}\, dx = -2x - \frac{5}{2}\ln|1-x^2| + k_1.$$

Taking the exponential and changing the name of the constant, we get

$$z = c_1 \frac{e^{-2x}}{(1-x^2)^{5/2}}.$$

Since $z = u'$, we get

$$u(x) = c_1 \int \frac{e^{-2x}}{(1-x^2)^{5/2}}\, dx + c_2.$$

Therefore, we obtain the general solution as

$$\begin{aligned} y(x) &= \Big(c_1 \int \frac{e^{-2x}}{(1-x^2)^{5/2}}\, dx + c_2\Big)y_2(x) \\ &= c_1 y_2(x) \int \frac{e^{-2x}}{(1-x^2)^{5/2}}\, dx + c_2 y_2(x). \end{aligned}$$

From these last calculations, we can conclude that

$$1 + 2\sum_{n=1}^{\infty} \frac{(2n)!!}{(2n-1)!!}x^{2n} = \Big(\frac{12x}{(1-x^2)^{3/2}} - 5x\Big) \int \frac{e^{2x}}{(1-x^2)^{5/2}}\, dx.$$

Isn't beautiful? ;)

Total (Points): 50.