Math 302

Chapter 4

Section 4.4: Autonomous Second Order Equations

Contents

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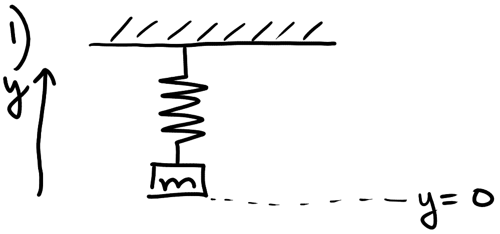
Fall 2022

Example 1. Consider an object with mass m suspended from a spring and moving vertically freely (in the void). Let y be the displacement of the object from the position it occupies when suspended at rest from the spring.

1. Use Newton's Second Law of motion and Hook's Law for springs to find a differential equation describing $y(t)$.
2. Solve this differential equation.

1)



Hook's law: $F_s = k\,\Delta L = ky$

Second Law of Motion: $ma = \Sigma F$

$$\Rightarrow \quad ma = -ky$$

$$\Rightarrow \quad my'' = -ky \quad \Rightarrow \quad \boxed{my'' + ky = 0} \quad (*)$$

2) The trick: $y'' = \underbrace{v'}_{\frac{dv}{dt}}$ & $v = y'$

$(*)$ becomes $mv' + ky = 0$

fct. of y

Chain rule $\Rightarrow$ $v' = \dfrac{dv}{dt} = \dfrac{dv}{dy}\dfrac{dy}{dt} = \dfrac{dv}{dy} \cdot v$

$$\Rightarrow \quad m\frac{dv}{dy} \cdot v + ky = 0 \qquad \text{(ODE for } v \text{ in } y\text{)}.$$

Seperable ODE: $m\dfrac{dv}{dy}\, v = -ky$

$$\Rightarrow \quad mv\,dv = -ky\,dy$$

$$\Rightarrow \quad \frac{mv^2}{2} = -\frac{ky^2}{2} + C$$

So, $\boxed{\dfrac{mv^2}{2} + \dfrac{ky^2}{2} = C}$ → implicit solution in terms of v & y.

Isolate v:

$$\frac{mv^2}{2} = C - \frac{ky^2}{2}$$

$$\Rightarrow \quad v^2 = \frac{2C - ky^2}{m}$$

$$\Rightarrow \quad v = \pm \sqrt{\frac{2C - ky^2}{m}}$$

Write $v = \dfrac{dy}{dt}$ $\qquad (c = 2C)$

$$\Rightarrow \quad \frac{dy}{dt} = \pm \sqrt{\frac{c - ky^2}{m}}$$

Use the + sign:

$$\frac{\sqrt{m}}{\sqrt{c - ky^2}}\, dy = dt$$

A u-sub: $u = \sqrt{k}\, y$ & integrate

$$\Rightarrow \quad \arcsin\left(\sqrt{\frac{k}{c}}\, y\right) = \sqrt{k/m}\; t + \phi$$

$$\Rightarrow \quad \sqrt{\frac{k}{c}}\, y = \sin\left(\sqrt{\frac{k}{m}}\, t + \phi\right)$$

$$\Rightarrow \quad y = \sqrt{\frac{c}{k}}\, \sin\left(\sqrt{\frac{k}{m}}\, t + \phi\right)$$

A second order ODE that can be written as

$$y'' = F(t, y, y')$$

$$y'' = F(y, y') \qquad (1)$$

$$= y + y'$$

where F is independent of t, is said to be **autonomous**.

Trick to convert to a first order ODE:

Write $y'' = v'$ & $v = y'$

$$\Rightarrow \quad v' = F(y, v)$$

Now, chain rule: $v' = \frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = \frac{dv}{dy} \cdot v$

$$\Rightarrow \quad \frac{dv}{dy} \cdot v = F(y, v)$$

Undamped Autonomous ODE

We will be interested in this particular **undamped autonomous ODE**:

$$y'' + p(y) = 0 \qquad (2)$$

which can be transformed, with the trick, into the first order ODE

$$v\frac{dv}{dy} + p(y) = 0. \qquad (3)$$

Solution: $v\frac{dv}{dy} = -p(y) \Rightarrow \frac{v^2}{2} = -P(y) + c \Rightarrow \boxed{\frac{v^2}{2} + P(y) = c}$

$P(y) = \int p(y)\,dy$

General Terminology

- The ODE (3) is called the **phase plane equivalent** of (2).
- The plane with axes y and v is called the **Poincaré phase plane** of the ODE (3)
- The integral curves of the ODE (3) are called **trajectories**.
- If a constant c is such that $p(c) = 0$, then
  - We say that $\boxed{y = c}$ is an **equilibrium** of (2).
  - We say that $(c, 0)$ is a **critical point** of (3).

EXAMPLE 2. Consider the motion of a pendulum with mass m, attached to the end of a weightless rod with length L rotating on a frictionless axle. We assume there's no air resistance. The ODE describing the angle y is

$$mLy'' = -mg\sin y.$$

1. Solve this ODE with the additional assumption that $v = v_0$ at $y = \cancel{0}\, y_0$.
2. Find the critical points of this ODE.
3. Study the behavior when $|v_0| > 2\sqrt{g/L}$.
4. Study the behavior when $0 < |v_0| < 2\sqrt{g/L}$.

1) Write $y'' = v'$ & $v = y'$.

$$\Rightarrow mL\,v' = -mg\sin(y)$$

$v' = \frac{dv}{dy}\cdot\frac{dy}{dt}$ (Chain rule) $\Rightarrow v' = \frac{dv}{dy}\cdot v$

$$\Rightarrow mL\frac{dv}{dy}\cdot v = -mg\sin(y)$$

$$\Rightarrow mLv\,dv = -mg\sin(y)\,dy$$

$$\Rightarrow \frac{\cancel{m}Lv^2}{2} = \cancel{m}g\cos(y) + c$$

$$\Rightarrow \frac{Lv^2}{2} = g\cos(y) + c$$

We know that

$$\frac{Lv_0^2}{2} = g\cos(y_0) + c \Rightarrow c = \frac{Lv_0^2}{2} - g\cos(y_0)$$

$$\Rightarrow \frac{Lv^2}{2} = g\cos(y) + \frac{Lv_0^2}{2} - g\cos(y_0)$$

$$\Rightarrow \frac{v^2}{2} = \frac{g}{L}\cos(y) + \frac{v_0^2}{2} - \frac{g}{L}\cos(y_0)$$

$$\Rightarrow \quad \frac{v^2}{2} = \frac{g}{L}\left(\cos y - \cos y_0\right) + \frac{v_0^2}{2}$$

$$\Rightarrow \quad \frac{v^2}{2} = 2\frac{g}{L} \sin^2\left(\frac{y}{2}\right) + \frac{v_0^2}{2}$$

$$\Rightarrow \quad v^2 = v_0^2 + 4\frac{g}{L}\sin^2(y/2)$$

$$\Rightarrow \quad \boxed{v^2 = v_0^2 + v_c^2 \sin^{-2}(y/2)} \qquad v_c = 2\sqrt{g/L}$$

2) $mLy'' = -mg\sin(y) \longrightarrow y'' + \underbrace{\frac{g}{L}\sin(y)}_{p(y)} = 0$

$p(y) = 0 \longleftrightarrow \frac{g}{L}\sin(y) = 0 \longleftrightarrow \sin(y) = 0$

$\longleftrightarrow y = n\pi$ (n any integer)

3) Suppose $|v_0| > 2\sqrt{g/L} \Rightarrow v_0^2 > \frac{4g}{L} = v_c^2$

& $v_c^2 \geq v_c \sin^{-2}(y/2)$

$$\Rightarrow \quad v_0^2 > v_c^2 \sin^2(y/2)$$

$$\Rightarrow \quad v_0^2 - v_c^2 \sin^{-2}(y/2) > 0$$

& $v_0^2 + v_c^2 \sin^2(y/2) > 0$

So, $v^2 = v_0^2 + v_c^2 \sin^{-2}(y/2) > 0$

$\Rightarrow \quad v^2 > 0$

$\Rightarrow$ there is always velocity
so pendulum $\boxed{\text{turns forever}}$.

4) $0 < |v_0| < 2\sqrt{g/L} \rightarrow v_0^2 < 4g/L = v_c^2$

$\Rightarrow v^2 = v_0^2 + v_c^2 \sin^{-2}(y/2)$ is zero

$\Rightarrow$ velocity will be zero at some point. $\Rightarrow$ $\boxed{\text{pendulum oscillates}}$

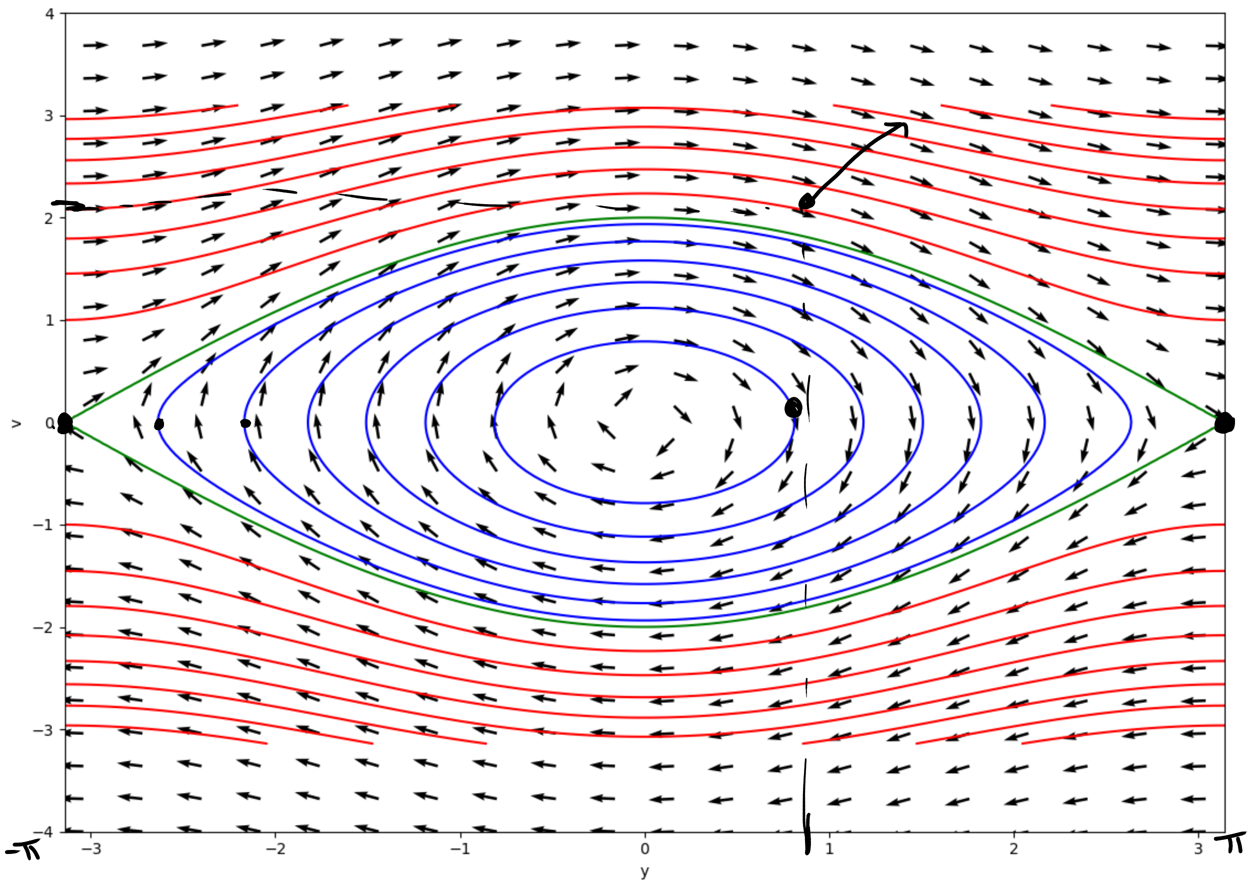


Figure 1: Phase space of the undamped pendulum ODE and some trajectories

<u>Remark:</u>

- the curves in the phase plane that separates trajectories of whirling solutions (in red) from the trajectories of oscillating solutions (in blue) are called **separatrix** (in green).
- For a detail study of the stability/unstability behavior of the undamped equation (3), you may read the pages 170-172 of the textbook.
- For a study of the damped ODE

$$y'' + q(y, y')y' + p(y) = 0,$$

you may read pages 172-175 of the textbook.