MATH-302 Pierre-Olivier Parisé
Homework 13 Solutions Fall 2022

Section 8.2 — Problem A **20 Points**

Solve the following IVP using the Laplace transform:

$$2y'' - 3y' - 2y = 4e^t, \quad y(0) = 1,\ y'(0) = -2.$$

Section 8.3 — Problem B **10 Points**

Express the given function f in terms of the unit step functions.

1) $f(t) = \begin{cases} t & , 0 \leq t < 1 \\ 1 & , t \geq 1. \end{cases}$

2) $f(t) = \begin{cases} t^2 & , 0 \leq t < 1 \\ \sin(t) & , t \geq 1. \end{cases}$

Section 8.3 — Problem C **10 Points**

Find the Laplace tranform of the given function.

1) $f(t) = \begin{cases} te^t & , 0 \leq t < 1 \\ e^t & , t \geq 1. \end{cases}$

2) $f(t) = \begin{cases} 3 & , 0 \leq t < 2 \\ 3t + 2 & , 2 \leq t < 4 \\ 4t & , t \geq 4. \end{cases}$

Section 8.3 — Problem D **10 Points**

Find the inverse Laplace transform of the given function.

1) $H(s) = \dfrac{e^{-s}}{s^3} + \dfrac{e^{-2s}}{s^2}$.

2) $H(s) = \dfrac{5}{s} - \dfrac{1}{s^2} + e^{-3s}\Big(\dfrac{6}{s} + \dfrac{7}{s^2}\Big) + \dfrac{3e^{-6s}}{s^3}$.

TOTAL (POINTS): 50.

Complete Solutions

Section 8.2 — Problem A — 20 Points

Apply the Laplace transform to the ODE to get

$$2(s^2Y - sy(0) - y'(0)) - 3(sY - y(0)) - 2Y = \frac{4}{s-1}.$$

Using the initial condition and collecting the terms, we obtain

$$(2s^2 - 3s - 2)Y = \frac{4}{s-1} + 2s - 7$$

Since $2s^2 - 3s - 2 = (2s+1)(s-2)$, we obtain

$$Y(s) = \frac{4}{(2s+1)(s-2)(s-1)} + \frac{2s-7}{(2s+1)(s-2)}.$$

We can rewrite each term in the RHS using the partial fraction decomposition:

$$\frac{4}{(2s+1)(s-2)(s-1)} = -\frac{4/3}{s-1} + \frac{16/15}{2s+1} + \frac{4/5}{s-2}$$

and

$$\frac{2s-7}{(2s+1)(s-2)} = \frac{16/5}{2s+1} - \frac{3/5}{s-2}.$$

Therefore, the expression of $Y(s)$ becomes

$$Y(s) = -\frac{4/3}{s-1} + \frac{64/15}{2s+1} + \frac{1/5}{s-2} = -\frac{4/3}{s-1} + \frac{32/15}{s+1/2} + \frac{1/5}{s-2}.$$

Taking the inverse Laplace transform, we obtain

$$y(t) = -\frac{4}{3}e^t + \frac{32}{15}e^{-t/2} + \frac{1}{5}e^{2t}.$$

Section 8.3 — Problem B **10 Points**

1) To deal with the first part, we use the function $tu(t)$. To deal the second part, we use $u(t-1) - tu(t-1)$. The expression $tu(t-1)$ is present because we want to cancel out the first $tu(t)$. Therefore, the expression of the function is

$$f(t) = tu(t) + u(t-1) - tu(t-1) = t(u(t) - u(t-1)) + u(t-1).$$

2) To deal with the first part, we use the function $t^2u(t)$. To deal with the second part, we use $u(t-1)\sin(t)$ and to cancel out the term $t^2u(t)$, we substract $t^2u(t-1)$. Therefore, the expression of the function is

$$f(t) = t^2u(t) + \sin(t)u(t-1) - t^2u(t-1) = t^2(u(t) - u(t-1)) + \sin(t)u(t-1).$$

Section 8.3 — Problem C **10 Points**

1) We rewrite the function with the unit step function:

$$f(t) = te^t u(t) + e^t u(t-1) - te^t u(t-1).$$

We will use the first formula:

$$L(u(t-a)f(t)) = e^{-sa}L(f(t+a)).$$

We first have that

$$L(u(t)te^t) = e^{-s(0)}L((t+0)e^{t+0}) = L(te^t) = \frac{1}{(s-1)^2}.$$

Secondly, we have

$$L(e^t u(t-1)) = e^{-s}L(e^{t+1}).$$

But $e^{t+1} = ee^t$ and therefore

$$L(e^t u(t-1)) = e^{1-s}L(e^t) = \frac{e^{1-s}}{s-1}.$$

Thirdly, we have

$$L(te^t u(t-1)) = e^{-s}L((t+1)e^{t+1}).$$

But $(t+1)e^{t+1} = e(te^t + e^t)$ and therefore

$$L(te^t u(t-1)) = e^{1-s}\Big(L(te^t) + L(e^t)\Big) = \frac{e^{1-s}}{(s-1)^2} + \frac{e^{1-s}}{s-1}.$$

Combining everything, we obtain

$$\begin{aligned} L(f(t)) &= \frac{1}{(s-1)^2} + \frac{e^{1-s}}{s-1} - \frac{e^{1-s}}{(s-1)^2} - \frac{e^{1-s}}{s-1} \\ &= \frac{1}{(s-1)^2} - \frac{e^{1-s}}{(s-1)^2}. \end{aligned}$$

2) The function can be rewritten as followed:

$$\begin{aligned} f(t) &= 3u(t) + (3t+2)u(t-2) - 3u(t-2) + 4tu(t-4) - (3t+2)u(t-4) \\ &= 3u(t) + (3t-1)u(t-2) + (t-2)u(t-4). \end{aligned}$$

We will use the second formula:

$$L(u(t-a)f(t-a)) = e^{-sa}F(s).$$

To apply this formula, we rewrite the function $f(t)$ in the following way:

$$\begin{aligned} f(t) &= 3u(t) + (3t - 6 + 5)u(t-2) + (t - 4 + 2)u(t-4) \\ &= 3u(t) + 3(t-2)u(t-2) + 5u(t-2) + (t-4)u(t-4) + 2u(t-4). \end{aligned}$$

Therefore, we have

$$\begin{aligned} L(f(t)) &= 3L(u(t)) + 3L((t-2)u(t-2)) + 5L(u(t-2)) + L((t-4)u(t-4)) + 2L(u(t-4)) \\ &= \frac{3}{s} + 3e^{-2s}L(t) + \frac{5e^{-2s}}{s} + e^{-4s}L(t) + \frac{2e^{-4s}}{s} \\ &= \frac{3}{s} + \frac{3e^{-2s}}{s^2} + \frac{5e^{-2s}}{s} + \frac{e^{-4s}}{s^2} + \frac{2e^{-4s}}{s}. \end{aligned}$$

Section 8.3 — Problem D — 10 Points

1) From the formula

$$L(u(t-a)f(t-a)) = e^{-sa}F(s),$$

we extract the following information from the first term:

$$F(s) = \frac{1}{s^3} \quad \text{and} \quad a = 1$$

and therefore

$$\begin{aligned} L^{-1}\Big(\frac{e^{-s}}{s^3}\Big) &= u(t-1)\Big(\frac{1}{2}(t-1)^2\Big) \\ &= \frac{1}{2}(t-1)^2 u(t-1). \end{aligned}$$

From the same formula, we extract the following information from the second term:

$$F(s) = \frac{1}{s^2} \quad \text{and} \quad a = 2$$

and therefore

$$L^{-1}\Big(\frac{e^{-2s}}{s^2}\Big) = u(t-2)(t-2).$$

Therefore, the final answer is

$$L^{-1}(H(s)) = \frac{1}{2}(t-1)^2 u(t-1) + (t-2)u(t-2).$$

2) We have

$$L^{-1}\Big(\frac{5}{s}\Big) = 5 \quad \text{and} \quad L^{-1}\Big(\frac{1}{s^2}\Big) = t.$$

Using the formula

$$L(u(t-a)f(t-a)) = e^{-sa}F(s),$$

we see that

$$L^{-1}\Big(\frac{e^{-3s}}{s}\Big) = u(t-3), \quad L^{-1}\Big(\frac{e^{-3s}}{s^2}\Big) = (t-3)u(t-3)$$

and

$$L^{-1}\Big(\frac{e^{-6s}}{s^3}\Big) = \frac{1}{2}(t-6)^2 u(t-6).$$

The final answer is then

$$\begin{aligned} L^{-1}(H(s)) &= 5L^{-1}\Big(\frac{1}{s}\Big) - L^{-1}\Big(\frac{1}{s^2}\Big) + 6L^{-1}\Big(\frac{e^{-3s}}{s}\Big) + 7L^{-1}\Big(\frac{e^{-3s}}{s^2}\Big) + 3L^{-1}\Big(\frac{e^{-6x}}{s^3}\Big) \\ &= 5 - t + 6u(t-3) + 6u(t-3) + 7(t-3)u(t-3) + \frac{3}{2}(t-6)^2 u(t-6). \end{aligned}$$

TOTAL (POINTS): 50.