Math 302

Chapter 2

Section 2.4: Transformation of Nonlinear Equations Into Separable Equations

Contents

Created by: Pierre-Olivier Parisé

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We were able to solve

$$y' + p(x)y = f(x)$$

by

- finding a solution y_1 to the complementary equation and
- setting $y = uy_1$ where u is the solution to the separable equation

$$u' = \frac{f(x)}{y_1(x)}.$$

Bernoulli Equation

A **Bernoulli equation** is an equation of the form

$$y' + p(x)y = f(x)y^r$$

where r is any real number different from 0 and 1.

Trick to solve it:

1) Solve the complementary equation, that is find y_1 s.t. $y_1' + p(x) y_1 = 0$

2) Use variation of parameter.

$y(x) = u(x) y_1$.

So, $y' = u'y_1 + uy_1'$ & $y = uy_1$

$\Rightarrow u'y_1 + uy_1' + p(x)uy_1 = f(x) u^r y_1^r$

$\Rightarrow u'y_1 + u\underbrace{(y_1' + p(x)y_1)}_{=0} = f(x) u^r y_1^r$

$\Rightarrow u'y_1 = f(x) u^r y_1^r$ $\rightarrow$ separable equation

$\Rightarrow \dfrac{u'}{u^r} = f(x) y_1^{r-1}$ $\rightarrow$ just integrate to find u.

Example 1. Solve the Bernoulli equation

$$y' - y = xy^2. \quad \swarrow r = 2$$

1) Complementary Equation

$$y' - y = 0 \quad \rightarrow \quad y(x) = ce^{x} \quad \rightarrow \quad y_1 = e^{x}$$

$$(y' = y)$$

2) Variation of Parameter:

Let $y = u y_1 = u e^{x}$

$$y' = u' e^{x} + u e^{x} \quad \& \quad y = u e^{x}$$

$$\Rightarrow \quad u' e^{x} + \cancel{u e^{x}} - \cancel{u e^{x}} = x y^2$$

$$\Rightarrow \quad u' e^{x} = x y^2$$

$$\Rightarrow \quad u' = \frac{x}{e^{x}} u^2 e^{2x}$$

$$\Rightarrow \quad \frac{u'}{u^2} = x e^{x} \quad \Rightarrow \quad \frac{-1}{u} = (x-1)e^{x} + c$$

$$\Rightarrow \quad u(x) = \frac{-1}{(x-1)e^{x} + c}$$

3) Replace u in y:

$$\boxed{y(x) = u(x)\, e^{x} = \frac{-e^{x}}{(x-1)e^{x} + c}}$$

The first order ODE

$$y' = f(x, y)$$

is said to be **homogeneous of the second kind** if it takes the form

$$y' = q(y/x)$$

where $q = q(u)$ is a function of a single variable.

EXAMPLE 2. The following ODEs are homogeneous of the second kind. Explain why.

1. $y' = \dfrac{y + xe^{-y/x}}{x}$. $= \dfrac{y}{x} + e^{-y/x} = u + e^{-u}$ $(u = y/x)$
2. $x^2y' = y^2 + xy - x^2$. $\Rightarrow y' = \dfrac{y^2}{x^2} + \dfrac{xy}{x^2} - \dfrac{x^2}{x^2} = \left(\dfrac{y}{x}\right)^2 + \dfrac{y}{x} - 1$

$q(u) = u^2 + u - 1 \leftarrow \boxed{u = y/x} \leftarrow$

The trick:

Let $u = y/x \Rightarrow y = xu$

we get $y' = u + xu'$

$\Rightarrow u + xu' = q(y/x) = q(u)$

$\Rightarrow u + xu' = q(u)$

$\Rightarrow xu' = q(u) - u \rightarrow$ separable equation!

$\Rightarrow \dfrac{u'}{q(u) - u} = \dfrac{1}{x} \rightarrow$ integrate to find u.

Example 3.

1. Solve
$$y' = \frac{y + xe^{-y/x}}{x}.$$

2. Solve the boundary value problem
$$y' = \frac{y + xe^{-y/x}}{x}, \quad y(1) = 0.$$

1) Here: $y' = \frac{y}{x} + e^{-y/x} = g(y/x)$ with $g(u) = u + e^{-u}$.

Let $u = y/x$ $\rightarrow$ $y = xu$

$\rightarrow$ $y' = u + xu'$

So, $u + xu' = u + e^{-u}$

$\Rightarrow$ $xu' = e^{-u}$

$\Rightarrow$ $e^{u}u' = \frac{1}{x}$ $\left(e^{u}du = \frac{dx}{x} \right)$

$\Rightarrow$ $e^{u} = \ln|x| + c$

$\Rightarrow$ $u(x) = \ln(\ln|x| + c)$

Replace u in y: $y(x) = u(x)\,x = x\ln(\ln|x| + c)$.

$\rightarrow$ defined on $(-\infty, 0) \cup (0, \infty)$.

2) Satisfy $y(1) = 0$.

$$0 = 1\left(\ln\left(\cancel{\ln|1|} + c \right) \right)$$

$$\Rightarrow \quad 0 = \ln(c)$$

$$\Rightarrow \quad 1 = c$$

Here, $x_0 = 1 \in (0, \infty)$ $\Rightarrow$ $(0, \infty)$ is the interval of validity

$$\Rightarrow \quad \boxed{y(x) = x \ln(\ln(x) + 1)}$$ (because $x > 0$ $\Downarrow$ $|x| = x$)