Math 302

Chapter 5

Section 5.2: Constant Coefficient Homogeneous Equations

Contents

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What is a Constant Coefficient Homogeneous ODE?

We restrict even further the second order ODE. A **second order constant coefficient ODE** is an ODE of the form

$$ay'' + by' + cy = f(x) \tag{1}$$

where a, b, c are fixed numbers and f is a continuous function.

Goal:

Find the solutions to

$$ay'' + by' + cy = 0.$$

We call this the **constant coefficient homogeneous ODE**.

Trick:

Guess that the solution is of the form

$$y(x) = e^{rx} \quad , \text{ for some } r.$$

We have $y'(x) = re^{rx}$ & $y''(x) = r^2 e^{rx}$

$$\Rightarrow ay'' + by' + cy = ar^2e^{rx} + bre^{rx} + ce^{rx}$$
$$= e^{rx}(ar^2 + br + c)$$

Therefore, $y(x) = e^{rx}$ is a solution to the ODE

iff $ar^2 + br + c = 0$

iff r root of the polynomial

$ar^2 + br + c$.

We call

- $ar^2 + br + c$ the characteristic polynomial.
- $ar^2 + br + c = 0$ the characteristic equation.

The roots are given by

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow 3 \text{ cases!}.$$

Example 1. Find the general solution of

$$y'' + 6y' + 5y = 0.$$

a) We have $r^2 + 6r + 5 = (r+5)(r+1)$

So, $(r+5)(r+1) = 0 \iff r = -5$ or $r = -1$

Two solutions are

$y_1(x) = e^{-5x}$ & $y_2(x) = e^{-x}$.

Now,

$\frac{y_1}{y_2} = \frac{e^{-5x}}{e^{-x}} = e^{-4x} \rightarrow$ not constant

$\Rightarrow \{y_1, y_2\}$ is a fund. set of sols.

$\Rightarrow y(x) = c_1 e^{-5x} + c_2 e^{-x}$ is the gen. sol.

General Fact:

- If the roots of the characteristic polynomial are r_1 and r_2, then $y_1(x) = e^{r_1 x}$ and $y_2 = e^{r_2 x}$ are solutions to the ODE.
- The general solutions is given by

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

Example 2.

a) Find the general solution of

$$y'' + 6y' + 9y = 0.$$

b) Solve the following IVP:

$$y'' + 6y' + 9y = 0, \quad y(0) = 3,\ y'(0) = -1.$$

a) We have $r^2 + 6r + 9 = (r+3)^2$

So, $r^2+6r+9=0 \Leftrightarrow r=-3$ (repeated roots)

A solution is $y_1(x) = e^{-3x}$.

Question : How do we find the other solution such that $\{y_1, y_2\}$ is a fund. set of solutions ?

Use variation of Parameter :

Let $y_2(x) = u(x)e^{-3x} \Rightarrow y' = u'e^{-3x} - 3ue^{-3x}$

$y'' = u''e^{-3x} - 3u'e^{-3x} - 3u'e^{-3x} + 9ue^{-3x}$

Therefore, the ODE becomes

$$u''e^{-3x} - 3u'e^{-3x} \cancel{- 3u'e^{-3x}} + \cancel{9ue^{-3x}} + \cancel{6u'e^{-3x}} - \cancel{18ue^{-3x}} + 9\cancel{ue^{-3x}} = 0$$

$\Rightarrow \quad u''e^{-3x} = 0$

$\Rightarrow \quad u'' = 0 \quad \Rightarrow \quad u(x) = Ax + B.$

Let $A=1 \Rightarrow y_2(x) = u(x)e^{-3x} = xe^{-3x}$.

We see that $y_1/y_2 = \dfrac{e^{-3x}}{xe^{-3x}} = \dfrac{1}{x}$ not constant

$\Rightarrow y(x) = c_1e^{-3x} + c_2xe^{-3x}$ is the gen. sol.

(b) $y(0) = 3$ & $y'(0) = -1$

Then $\quad y(0) = 3 \Rightarrow c_1 = 3.$

Also, $\quad y'(x) = -3c_1 e^{-3x} + c_2 e^{-3x} - 3c_2 x e^{-3x}$

$$\Rightarrow y'(0) = -3c_1 + c_2$$

$$\Rightarrow -3c_1 + c_2 = -1$$

$$\Rightarrow c_2 = -1 + 9 = 8$$

Therefore

$$\boxed{y(x) = 3e^{-3x} + 8xe^{-3x} = (3+8x)e^{-3x}}.$$

General Facts:

- If the root of the characteristic polynomial is r_1, then $y_1(x) = e^{r_1 x}$ and $y_2(x) = xe^{r_1 x}$ are solutions to the ODE.
- The general solution is given by

$$y(x) = e^{r_1 x}(c_1 + c_2 x).$$

EXAMPLE 3.

a) Find the general solution of

$$y'' + 4y' + 13y = 0.$$

b) Solve the following IVP:

$$y'' + 4y + 13y = 0, \quad y(0) = 2,\ y'(0) = -3.$$

a) We have $r^2 + 4r + 13$ & the roots are

$$r = \frac{-4 \pm \sqrt{16 - 52}}{2} = \frac{-4 \pm \sqrt{-36}}{2}$$

What? $\sqrt{-36} = ????$

Complex Numbers

A complex number is an expression of the form

$$z = \alpha + i\beta$$

where α, β are real numbers and $i^2 = -1$ ($i = \sqrt{-1}$).

Consider $z = \alpha + i\beta$ and $w = \gamma + i\mu$.

- $z = w$ if and only if $\alpha = \gamma$ and $\beta = \mu$.
- $z + w = (\alpha + \gamma) + i(\beta + \mu)$.
- $zw = (\alpha\gamma - \beta\mu) + i(\alpha\mu + \beta\gamma)$.
- $z/w = \frac{(\alpha+i\beta)(\gamma-i\mu)}{(\gamma+i\mu)(\gamma-i\mu)}$, if $w \neq 0$.

EXAMPLE 4. If $z = 1 + i$ and $w = 1 - i$, find

a) $z + w$. b) zw. c) z/w.

a) $1+i + (1-i) = 1+1 + (1-1)i = \boxed{2}$

b) $(1+i)(1-i) = 1 - i + i - i^2 = 1 - i + i - (-1) = \boxed{2}$

c) $z/w = \frac{1+i}{1-i} \cdot \frac{1+i}{1+i} = \frac{1+i+i+i^2}{1+i-i-i^2} = \frac{2i}{2} = \boxed{i}$

Example 5. Complete the previous example.

a) We had
$$r = \frac{-4 \pm \sqrt{-36}}{2} = \frac{-4 \pm \sqrt{(-1)(36)}}{2}$$
$$= -2 \pm \frac{\sqrt{-1}\sqrt{36}}{2}$$
$$\Rightarrow \quad r_1 = -2 + i\,6/2$$
$$\text{or } r_2 = -2 - i\,6/2$$
$$\Rightarrow \quad r_1 = -2 + 3i$$
$$\text{or } r_2 = -2 - 3i$$

Therefore, our guess would be to take
$$y_1 = e^{(-2+3i)x} \quad \& \quad y_2 = e^{(-2-3i)x}.$$

But, what does it mean $e^{\text{complex number}}$?

Instead, we'll use variation of parameter.

We can rewrite
$$y_1 = \boxed{e^{-2x}}\, e^{3ix} \quad \& \quad y_2 = \boxed{e^{-2x}}\, e^{-3ix}$$

→ functions multiplying e^{-2x}.

So, we may suppose that
$$y(x) = u(x)e^{-2x}.$$

Then $y' = u'e^{-2x} - 2ue^{-2x}$

& $y'' = u''e^{-2x} - 2u'e^{-2x} - 2u'e^{-2x} + 4ue^{-2x}.$

$$\Rightarrow u''e^{-2x} \cancel{-2u'e^{-2x}} \cancel{-2u'e^{-2x}} + 4ue^{-2x} + \cancel{4u'e^{-2x}} - 8ue^{-2x} + 13ue^{-2x} = 0$$
$$\Rightarrow u''e^{-2x} + 9ue^{-2x} = 0$$

$$\Rightarrow \quad u'' + 9u = 0 \quad \Rightarrow \quad \begin{cases} u_1(x) = \cos 3x \\ u_2(x) = \sin 3x \end{cases}$$

Therefore, $y_1(x) = u_1(x) e^{-2x} = e^{-2x} \cos(3x)$

& $y_2(x) = u_2(x) e^{-2x} = e^{-2x} \sin(3x)$

Since $\frac{y_2}{y_1} = \tan 3x$ is not constant, then

$\{e^{-2x}\cos(3x), e^{-2x}\sin(3x)\}$ is a fund. set solo.

$$\Rightarrow \quad y(x) = c_1 e^{-2x}\cos(3x) + c_2 e^{-2x}\sin(3x) \text{ gen. sol.}$$

(b) $y(0) = 2 \Rightarrow c_1 = 2$

$$y'(x) = -2c_1 e^{-2x}\cos(3x) - 3c_1 e^{-2x}\sin(3x) - 2c_2 e^{-2x}\sin(3x) + 3c_2 e^{-2x}\cos(3x)$$

So $y'(0) = -3 \Rightarrow -2c_1 + 3c_2 = -3$

$\Rightarrow c_2 = 1/3$

therefore,

$$\boxed{\begin{aligned} y(x) &= 2e^{-2x}\cos(3x) + \frac{1}{3}e^{-2x}\sin(3x) \\ &= e^{-2x}\left(2\cos 3x + \frac{1}{3}\sin 3x\right) \end{aligned}}$$

General Facts:

- If $r_1 = \alpha + \beta i$ and $r_2 = \alpha - \beta i$ are the roots of the characteristic polynomial, then $y_1(x) = e^{\alpha x}\cos(\beta x)$ and $y_2(x) = e^{\alpha x}\sin(\beta x)$ are solutions to the ODE.
- The general solution has the form

$$y(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x)).$$