# Math 302

## Chapter 2

### Section 2.4: Transformation of Nonlinear Equations Into Separable Equations

Contents



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We were able to solve

$$y' + p(x)y = f(x)$$

by

- finding a solution $y_1$ to the complementary equation and
- setting $y = uy_1$ where $u$ is the solution to the separable equation

$$u' = \frac{f(x)}{y_1(x)}.$$

## Bernoulli Equation

A **Bernoulli equation** is an equation of the form

$$y' + p(x)y = f(x)y^r \qquad (*)$$

where $r$ is any real number different from 0 and 1.

Trick to solve it:

1) Find a solution to $y' + p(x)y = 0$ (complementary eq.).
Call $y_1$ this solution.

2) Variation of parameter

$$y(x) = u(x)\,y_1(x) \longrightarrow y'(x) = u'y_1 + uy_1'$$

subst. in $(*)$ $\Rightarrow u'y_1 + uy_1' + p(x)uy_1 = f(x)u^r y_1^r$

$$\Rightarrow u'y_1 + u\underbrace{(y_1' + p(x)y_1)}_{=0} = f(x)u^r y_1^r$$

$$\Rightarrow u'y_1 = f(x)u^r y_1^r$$

So, $u$ is solution to

$$\frac{u'}{u^r} = f(x)\,y_1^{r-1} \qquad \text{(just integrate)}$$

Example 1. Solve the Bernoulli equation

$$y' - y = xy^2.$$

1) Solve complementary eq.

$$y' - y = 0 \quad \Rightarrow \quad y = ce^{x} \quad \Rightarrow \quad y_1 = e^{x}$$

2) Variation of parameter.

$$y(x) = ue^{x} \quad \Rightarrow \quad y'(x) = u'e^{x} + ue^{x}$$

$$\Rightarrow \quad u'e^{x} + \cancel{ue^{x}} - \cancel{ue^{x}} = x u^2 e^{2x}$$

$$\Rightarrow \quad u' e^{x} = x u^2 e^{2x}$$

$$\Rightarrow \quad \frac{u'}{u^2} = xe^{x}$$

$$\overset{\text{integrate}}{\Rightarrow} \quad -\frac{1}{u} = (x-1)e^{x} + c$$

$$\Rightarrow \quad u(x) = \frac{-1}{(x-1)e^{x} + c}$$

Therefore

$$\boxed{y(x) = \frac{-e^{x}}{(x-1)e^{x} + c}}$$

The first order ODE

$$y' = f(x, y)$$

is said to be **homogeneous of the second kind** if it takes the form

$$y' = q(y/x) \qquad (**)$$

where $q = q(u)$ is a function of a single variable.

EXAMPLE 2. The following ODEs are homogeneous of the second kind. Explain why.

1. $y' = \dfrac{y + xe^{-y/x}}{x}$. $\longrightarrow = \dfrac{y}{x} + e^{-y/x}$ , $q(u) = u + e^{-u}$
2. $x^2 y' = y^2 + xy - x^2$. $\xrightarrow{\div x^2}$ $y' = (y/x)^2 + \dfrac{y}{x}$

The trick:

Set $u = \dfrac{y}{x} \Rightarrow y = xu$.

Therefore $y' = u + xu'$ and substituing in $(**)$

$$\Rightarrow u + xu' = q(u)$$

Separating variables:

$$xu' = q(u) - u$$

$$\Rightarrow \frac{u'}{q(u) - u} = \frac{1}{x}$$

Then integrate to obtain $u$ & replace it in

$$y = xu.$$

**Example 3.**

1. Solve
$$y' = \frac{y + xe^{-y/x}}{x}.$$

2. Solve the boundary value problem
$$y' = \frac{y + xe^{-y/x}}{x}, \quad y(1) = 0.$$

1) We see that $y' = \frac{y}{x} + e^{-y/x}$ $(x \neq 0)$.

Set $u = \frac{y}{x}$ $\Rightarrow$ $y = xu$. $\Rightarrow$ $y' = u + xu'$

So, $\cancel{u} + xu' = \cancel{u} + e^{-u}$

$$\Rightarrow \quad xu' = e^{-u}$$
$$\Rightarrow \quad e^{u}u' = \frac{1}{x}$$
$$\Rightarrow \quad e^{u} = \ln|x| + c$$
$$\Rightarrow \quad u(x) = \ln(\ln|x| + c).$$

Since $y = xu$ $\Rightarrow$ $\boxed{y(x) = x\ln(\ln|x| + c)}$

2) We know $y(x) = x\ln(\ln|x| + c)$

& $y(1) = 0$

$$\Rightarrow \quad 0 = 1\ln(\ln 1 + c)$$
$$\Rightarrow \quad 0 = \ln(c) \overset{\text{exp.}}{\Rightarrow} c = 1$$

Thus, $\boxed{y(x) = x\ln(\ln|x| + 1)}$ $x \neq 0$.