Math 302

Chapter 2

Section 2.5: Exact Equations

Contents

Created by: Pierre-Olivier Parisé

Fall 2022

EXAMPLE 1. Consider $y' = dy/dx$ and use this to rewrite the ODE

$$y' = \frac{y + xe^{-y/x}}{x}$$

in terms of dx and dy.

Let $y' = \frac{dy}{dx}$ $\to$ $\frac{dy}{dx} = \frac{y + xe^{-y/x}}{x}$

$\to$ $x\,dy = (y + xe^{-y/x})\,dx$

$\to$ $(y + xe^{-y/x})\,dx - x\,dy = 0$

$F = F(x,y) \Rightarrow dF = \underset{\downarrow}{F_x}\,dx + \underset{\downarrow}{F_y}\,dy$

$\frac{\partial F}{\partial x}$ $\qquad$ $\frac{\partial F}{\partial y}$

Convenient form:

We will now consider an homogeneous first order ODE in the form

$$M(x, y)dx + N(x, y)dy = 0 \tag{1}$$

where M and N are two functions of the variables x and y.

Two interpretations:

- the equation (1) can be interpreted as

$$M(x, y) + N(x, y)\overset{y'}{\frac{dy}{dx}} = 0 \tag{2}$$

  where x is the independent variable and y is the dependent variable.

- the equation (1) can be interpreted as

$$M(x, y)\overset{x'}{\frac{dx}{dy}} + N(x, y) = 0 \tag{3}$$

  where x is the dependent variable and y is the independent variable.

- An implicit equation $F(x, y) = c$ is said to be an **implicit solution** to (1) if
  - every function $y = y(x)$ satisfying $F(x, y(x)) = c$ is a solution to (2).
  - every function $x = x(y)$ satisfying $F(x(y), y) = c$ is a solution to (3)

Example 2. Show that

$$\overbrace{x^4y^3 + x^2y^5 + 2xy}^{F(x,y)} = c$$

$d($ $) \quad d(\ )$

is an implicit solution of

$$\underline{(4x^3y^3 + 2xy^5 + 2y)}dx + (3x^4y^2 + 5x^2y^4 + 2x)dy = 0.$$

$M(x,y) = 4x^3y^3 + 2xy^5 + 2y \qquad N(x,y) = 3x^4y^2 + 5x^2y^4 + 2x$

1) y satisfy $\quad x^4y^3 + x^2y^5 + 2xy = c$

$\Rightarrow 4x^3y^3 + 3x^4y^2y' + 2xy^5 + 5x^2y^4y' + 2y + 2xy' = 0$

$\Rightarrow \underbrace{(4x^3y^3 + 2xy^5 + 2y)}_{M} + \underbrace{(3x^4y^2 + 5x^2y^4 + 2x)}_{N}y' = 0$

Indeed, $F(x,y) = c$ is a implicit solution of the ODE.

2) $x = x(y)$ satisfies $\quad x^4y^3 + x^2y^5 + 2xy = c$

Use implicit differentiation to find $x' = \dfrac{dx}{dy}$.

Fact: $(4x^3y^3 + 2xy^5 + 2y)dx + (3x^4y^2 + 5x^2y^4 + 2x)dy$

$\quad = F_x\, dx + F_y\, dy$

$\quad = dF$

General Fact:

If $F(x,y) = c$ with F having continuous partial derivatives F_x and F_y, then

$$F(x,y) = c$$

is an implicit solution to the differential equation

$$\boxed{F_x(x,y)dx + F_y(x,y)dy = 0.}$$

So, a differential equation is said to be **exact** on an open rectangle R if there is a function $F = F(x, y)$ such that

$$F_x(x, y) = M(x, y) \quad \text{and} \quad F_y = N(x, y). \rightarrow$$

$$dF = F_x dx + F_y dy = M dx + N dy$$

Useful fact (the exactness condition):

A differentiel equation is exact if and only if

$$M_y(x, y) = N_x(x, y).$$

EXAMPLE 3. Check if the following ODEs are exact or not.

1. $3x^2 y dx + 4x^3 dy = 0$.
2. $(4x^3 y^3 + 3x^2) dx + (3x^4 y^2 + 6y^2) dy = 0$.

1) $M = 3x^2 y \rightarrow M_y = 3x^2$
$N = 4x^3 \rightarrow N_x = 12x^2$
$\Rightarrow M_y \neq N_x \Rightarrow$ $\boxed{\text{not exact}}$

2) $M = 4x^3 y^3 + 3x^2 \rightarrow M_y = 12x^3 y^2$
$N = 3x^4 y^2 + 6y^2 \rightarrow N_x = 12x^3 y^2$
$\Rightarrow M_y = N_x \Rightarrow$ $\boxed{\text{exact!}}$

Example 4. Solve

$$y' = -\frac{4x^3y^3 + 3x^2}{3x^4y^2 + 6y^2}.$$

$y' = \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = -\frac{4x^3y^3+3x^2}{3x^4y^2+6y^2} \rightarrow (4x^3y^3+3x^2)\,dx + (3x^4y^2+6y^2)\,dy = 0$

1) Verify exactness condition

$M = 4x^3y^3+3x^2$
$N = 3x^4y^2+6y^2$ $\Rightarrow$ exact from previous example.

2) Find F s.t. $F_x = M$ & $F_y = N$

(A) $F_x = 4x^3y^3 + 3x^2$

(B) $F_y = 3x^4y^2 + 6y^2$

(A) Integrate w.r.t. x

$F(x,y) = \int 4x^3y^3 + 3x^2\,dx = x^4y^3 + x^3$ → need one Anti-derivative.

(B) Integrate w.r.t. y.

$F(x,y) = \int 3x^4y^2 + 6y^2\,dy = x^4y^3 + 2y^3$

$\Rightarrow F(x,y) = $ (Common one) + (Not in common)

$= x^4y^3 + x^3 + 2y^3$

Implicit solution: $\boxed{x^4y^3 + x^3 + 2y^3 = c}$

Non Rigorous but "Fast" Procedure to Solve An Exact ODE

[I] Check that the equation

$$M(x, y)dx + N(x, y)dy = 0$$

satisfies the exactness condition.

[II] Integrate the equation $F_x = M(x, y)$ with respect to x to get

$$F(x, y) = G(x, y).$$

[III] Integrate the equation $F_y = N(x, y)$ with respect to y to get

$$F(x, y) = H(x, y).$$

[IV] Identity what is in common in the expressions of the functions G and H. Call this common part $F_1(x, y)$.

[V] Identity what is not in common in the expressions of the functions G and H. Gather the uncommon part in a function $F_2(x, y)$.

[VI] Write $F(x, y) = F_1(x, y) + F_2(x, y)$.

Remarks:

- This shortcut may not work if one of the function G or H has an integral that can't be simplified.
- Sometimes, the rigorous procedure is faster (see next section).
- For the step-by-step rigorous procedure, see Example 2.5.3 (p.75) and p.77 of the textbook.