Math 302

Chapter 5

Section 5.4: The Method of Undetermined Coefficient I

Contents

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We consider the following basic case:

$$ay'' + by' + cy = ke^{\alpha x}$$

where a, b, c, α, and k are fixed real numbers.

Case I

When $e^{\alpha x}$ is not a solution to the complementary equation $ay'' + by' + cy = 0$.

EXAMPLE 1. Find the general solution of

$$y'' - 7y' + 12y = 4e^{2x}.$$

2) Guess y_{par}

$y_{par}(x) = Ae^{2x} \Rightarrow y'_{par} = 2Ae^{2x}$ & $y'' = 4Ae^{2x}$

$\Rightarrow 4Ae^{2x} - 14Ae^{2x} + 12Ae^{2x} = 4e^{2x}$

$\Rightarrow 2Ae^{2x} = 4e^{2x} \Rightarrow A = 2$

So, $y_{par}(x) = 2e^{2x}$.

1) Solution Compl. Eq.

$y'' - 7y' + 12y = 0 \longrightarrow r^2 - 7r + 12 = (r-4)(r-3)$

$\longrightarrow$ roots are $r = 4$ & $r = 3$

$\longrightarrow$ Fund. set sol.: $y_1 = e^{4x}$ & $y_2 = e^{3x}$

not e^{2x}

3) General soln:

$$\boxed{\begin{aligned} y(x) &= y_{par} + c_1 y_1 + c_2 y_2 \\ &= 2e^{2x} + c_1 e^{4x} + c_2 e^{3x} \end{aligned}}$$

Case II

When $e^{\alpha x}$ is a solution to the complementary equation.

Example 2. Find the general solution of

$$y'' - 7y' + 12y = 5e^{4x}.$$

2) <u>Guess ypar:</u>

$y_{par} = Ae^{4x} \Rightarrow y' = 4Ae^{4x}$ & $y'' = 16Ae^{4x}$

$\Rightarrow 16Ae^{4x} - 28Ae^{4x} + 12Ae^{4x} = 5e^{4x}$

$\Rightarrow 0e^{4x} = 5e^{4x}$ ... because e^{4x} is solution to compl. eq.!

Use Variation of parameter.

$y_{par} = u(x)e^{4x} \Rightarrow y_{par}' = u'e^{4x} + 4ue^{4x}$

$y_{par}'' = u''e^{4x} + 4u'e^{4x} + 4u'e^{4x} + 16ue^{4x}$

$\Rightarrow u''e^{4x} + 8u'e^{4x} + 16ue^{4x} - 7u'e^{4x} - 28ue^{4x} + 12ue^{4x} = 5e^{4x}$

$\Rightarrow u''e^{4x} + 8u'e^{4x} + \cancel{16ue^{4x}} - 7u'e^{4x} \cancel{- 16ue^{4x}} = 5e^{4x}$

$\Rightarrow u''\cancel{e^{4x}} + 8u'\cancel{e^{4x}} - 7u'\cancel{e^{4x}} = 5\cancel{e^{4x}}$

$\Rightarrow u'' + 8u' - 7u' = 5$

$\Rightarrow u'' + u' = 5$

We want particular solution:

$u = A$ not working ✓ $u = Ax + B \rightarrow u'' = 0$, $u' = A$

$\Rightarrow u'' + u' = A = 5$

So, $u(x) = 5x + B \xrightarrow[\text{one}]{\text{simplest}} u(x) = 5x.$

Therefore, $y_{par}(x) = 5xe^{4x}.$

1) Solution complementary equation

$$y'' - 7y' + 12y = 0 \rightarrow r^2 - 7r + 12 = (r-4)(r-3)$$

$$\rightarrow y_1 = e^{4x} \quad \& \quad y_2 = e^{3x}$$

3) General Solo.

$$y(x) = 5xe^{4x} + c_1 e^{4x} + c_2 e^{3x}.$$

$$= (5x + c_1)e^{4x} + c_2 e^{3x}.$$

Case III

When $e^{\alpha x}$, and $xe^{\alpha x}$ are solutions to the complementary equation.

Example 3. Find the general solution of

$$y'' - 8y' + 16y = 2e^{4x}.$$

1) Compl. Eq.

$$y'' - 8y' + 16y = 0 \to r^2 - 8r + 16 = (r-4)^2$$

$$\to \text{root is } 4 \text{ (repeated)}$$

$$\to y_1 = e^{4x}, \quad y_2 = xe^{4x}.$$

2) Part. Sol.

Natural guesses: $\underbrace{y_{par} = Ae^{4x} \quad \text{or} \quad y_{par} = Axe^{4x}}_{\text{won't work !!}}$

Again, let $y_{par} = u(x)e^{4x}$.

$$\Rightarrow \begin{cases} y_{par}' = u'e^{4x} + 4ue^{4x} \\ y_{par}'' = u''e^{4x} + 4u'e^{4x} + 4u'e^{4x} + 16ue^{4x} \end{cases}$$

$$\Rightarrow u''\cancel{e^{4x}} + \cancel{8u'}\cancel{e^{4x}} + \cancel{16u}\cancel{e^{4x}} - \cancel{8u'}\cancel{e^{4x}} - \cancel{32u}\cancel{e^{4x}} + \cancel{16u}\cancel{e^{4x}} = 2\cancel{e^{4x}}$$

$$\Rightarrow u'' = 2.$$

$\cancel{u=A}$, $u = \cancel{Ax} + B$, $u = Ax^2$ ✓ $\Rightarrow u(x) = x^2$

So,

$$y_{par}(x) = x^2 e^{4x}$$

3) General solution.

$$\boxed{\begin{aligned} y(x) &= x^2 e^{4x} + c_1 e^{4x} + c_2 x e^{4x} \\ &= (c_1 + c_2 x + x^2)\, e^{4x} \end{aligned}}$$

Recap

To find a particular solution to

$$ay'' + by' + cy = ke^{\alpha x}$$

where k is a fixed real number, we follow the following tips:

- If $e^{\alpha x}$ is not a solution of the complementary equation, then we take $y_{par}(x) = Ae^{\alpha x}$, where A is a constant.
- If $e^{\alpha x}$ is a solution of the complementary equation, then we take $y_{par}(x) = xAe^{\alpha x}$, where A is a constant.
- If both $e^{\alpha x}$ and $xe^{\alpha x}$ are solutions of the complementary equation, then we take $y_{par}(x) = Ax^2e^{\alpha x}$, where A is a constant.

We now consider a more general case:

$$ay'' + by' + cy = e^{\alpha x} G(x)$$

where a, b, c, α are fixed real numbers and $G(x)$ is a polynomial.

Case I

When $e^{\alpha x}$ is not a solution to the complementary equation $ay'' + by' + cy = 0$.

Example 4. Find the general solution to

$$y'' - 3y' + 2y = e^{3x}(x^2 + 2x - 1).$$

1) Compl. Eq.

$y'' - 3y' + 2y = 0 \rightarrow r^2 - 3r + 2 = (r-2)(r-1)$

$\rightarrow$ roots are 2 & 1

so, $y_1(x) = e^{2x}$ & $y_2(x) = e^{x}$ fund. sol.

2) Guess y_{par}.

Observation: e^{2x} & e^{x} is not in the R-hand side !!

$y_{par}(x) = e^{3x}(Ax^2 + Bx + C)$

$\Rightarrow y' = 3e^{3x}(Ax^2 + Bx + C) + e^{3x}(2Ax + B)$

& $y'' = 9e^{3x}(Ax^2 + Bx + C) + 3e^{3x}(2Ax + B)$
$+ 3e^{3x}(2Ax + B) + e^{3x}\, 2A$
$= 9e^{3x}(Ax^2 + Bx + C) + 6e^{3x}(2Ax + B) + e^{3x}\, 2A$

So, ODE becomes

$$\cancel{9e^{3x}(Ax^2+Bx+C)} + 6e^{3x}(2Ax+B) + 2Ae^{3x} - \cancel{9e^{3x}(Ax^2+Bx+C)}$$
$$-3e^{3x}(2Ax+B) + 2e^{3x}(Ax^2+Bx+C) = e^{3x}(x^2+2x-1)$$

$$\Rightarrow 2\cancel{e^{3x}}(Ax^2+Bx+C) + 3\cancel{e^{3x}}(2Ax+B) + 2A\cancel{e^{3x}} = \cancel{e^{3x}}(x^2+2x-1)$$

$$\Rightarrow \quad 2Ax^2 + (2B+6A)x + (2C+3B+2A) = x^2+2x-1$$

$$\Rightarrow \quad 2A=1, \quad 2B+6A=2, \quad 2C+3B+2A=-1$$

$$\Rightarrow \quad A = 1/2, \quad B = \frac{-1}{2}, \quad C = \frac{-1}{4}$$

Therefore, $\quad y_{par}(x) = e^{3x}\left(\frac{x^2}{2} - \frac{x}{2} - \frac{1}{4}\right)$

3) General sol.

$$\boxed{y(x) = e^{3x}\left(\frac{x^2}{2} - \frac{x}{2} - \frac{1}{4}\right) + c_1e^{2x} + c_2e^{x}}$$

Case II

When $e^{\alpha x}$ is a solution to the complementary equation.

EXAMPLE 5. Find the general solution to

$$y'' - 4y' + 3y = e^{3x}(12x^2 + 8x + 6).$$

1) Compl. Eq.

$$y'' - 4y' + 3y = 0 \longrightarrow r^2 - 4r + 3 = (r-3)(r-1)$$

$\longrightarrow$ roots are 3 & 1

$\longrightarrow \underbrace{y_1(x) = e^{3x}}_{\text{oops ...}}$ & $y_2(x) = e^{x}$

2) Guess y_{par}.

$$y_{par}(x) = x e^{3x}(Ax^2 + Bx + C) = e^{3x}(Ax^3 + Bx^2 + Cx)$$

$$y' = 3e^{3x}(Ax^3 + Bx^2 + Cx) + e^{3x}(3Ax^2 + 2Bx + C)$$

$\Rightarrow$ &

$$y'' = 9e^{3x}(Ax^3 + Bx^2 + Cx) + 3e^{3x}(3Ax^2 + 2Bx + C) + 3e^{3x}(3Ax^2 + 2Bx + C) + e^{3x}(6Ax + 2B)$$

$$= 9e^{3x}(Ax^3 + Bx^2 + Cx) + 6e^{3x}(3Ax^2 + 2Bx + C) + e^{3x}(6Ax + 2B)$$

Replace in the ODE:

$$\Rightarrow 9e^{3x}(\cancel{Ax^3 + Bx^2} + Cx) + 6\cancel{e^{3x}}(3Ax^2 + 2Bx + C) + \cancel{e^{3x}}(6Ax + 2B)$$
$$- 12e^{3x}(\cancel{Ax^3 + Bx^2} + Cx) - 4\cancel{e^{3x}}(3Ax^2 + 2Bx + C) + 3\cancel{e^{3x}}(\cancel{Ax^3 + Bx^2} + Cx)$$
$$= \cancel{e^{3x}}(12x^2 + 8x + 6)$$

$$\Rightarrow 2(3Ax^2 + 2Bx + C) + (6Ax + 2B) = 12x^2 + 8x + 6$$

$$\Rightarrow 6Ax^2 + (4B + 6A)x + 2C + 2B = 12x^2 + 8x + 6$$

$$\Rightarrow 6A = 12, \quad 4B + 6A = 8, \quad 2C + 2B = 6$$

$$\Rightarrow A = 2, \quad B = -1, \quad C = 4$$

So, $\quad y_{par}(x) = e^{3x}(2x^3 - x^2 + 4x)$

3) <u>General Solution.</u>

$$\boxed{y(x) = e^{3x}(2x^3 - x^2 + 4x) + c_1 e^{3x} + c_2 e^{x}}$$

Case III

When $e^{\alpha x}$ and $xe^{\alpha x}$ are solutions to the complementary equation.

EXAMPLE 6. Find the general solution to

$$4y'' + 4y' + y = e^{-x/2}(144x^2 + 48x - 8).$$

1) Compl. Eq.

$4y'' + 4y' + y = 0 \longrightarrow 4r^2 + 4r + 1 = (2r+1)^2$

$\longrightarrow$ root is $\frac{-1}{2}$ (repeated)

$\longrightarrow y_1 = e^{-x/2}$ & $y_2 = xe^{-x/2}$.

2) Guess y_{par}.

$y_{par} = x^2 e^{-x/2}(Ax^2 + Bx + C)$

$= e^{-x/2}(Ax^4 + Bx^3 + Cx^2)$

Compute y' & y'' & replace in ODE to find A, B, C:

$A = 3, \quad B = 2, \quad C = -1$

$\Rightarrow \quad y_{par}(x) = e^{-x/2}(3x^4 + 2x^3 - x^2)$

3) General Sol.

$$\boxed{y(x) = e^{-x/2}(3x^4 + 2x^3 - x^2) + c_1 e^{-x/2} + c_2 x e^{-x/2}}$$

Recap

To find a particular solution to

$$ay'' + by' + cy = ke^{\alpha x} G(x)$$

where k is a fixed real number and $G(x)$ is a polynomial, we follow the following tips:

- If $e^{\alpha x}$ is not a solution of the complementary equation, then we take $y_{par}(x) = Ae^{\alpha x} Q(x)$, where A is a constant and $Q(x)$ is a polynomial of the same degree as $G(x)$.
- If $e^{\alpha x}$ is a solution of the complementary equation, then we take $y_{par}(x) = Axe^{\alpha x} Q(x)$, where A is a constant and $Q(x)$ is a polynomial of the same degree as $G(x)$.
- If $e^{\alpha x}$ and $xe^{\alpha x}$ are solutions to the complementary equation, then we take $y_{par}(x) = Ax^2 e^{\alpha x} Q(x)$, where A is a constant and $Q(x)$ is a polynomial of the same degree as $G(x)$.