Math 302

Chapter 4

Section 4.2: Cooling and Mixing

Contents

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Recall that Newton's law of cooling is given by

$$T' = -k(T - T_m) \tag{1}$$

where $k > 0$ is a constant, T is the temperature of the object, and T_m is the temperature of the medium (surrounding).

Example 1. Find the solution to (1) subject to the additional condition $T_0 = T(0)$.

Since T_m is constant, (1) is separable:

$$\frac{T'}{T-T_m} = -k \qquad (T \neq T_m).$$

Integrating gives

$$\ln|T-T_m| = -kt + C$$

$$\Rightarrow \quad |T-T_m| = e^{-kt} e^{C}$$

$$\Rightarrow \quad T - T_m = c e^{-kt} \qquad \left(c = \pm e^{C}\right)$$

$$\Rightarrow \quad T = T_m + c e^{-kt}.$$

We know $T(0) = T_0$

$$\Rightarrow \quad T_0 = T_m + c$$

$$\Rightarrow \quad c = T_0 - T_m$$

Therefore:

$$\boxed{T(t) = T_m + (T_0 - T_m) e^{-kt}}$$

Example 2. A ceramic insulator is baked at 400°C and cooled in a room in which the temperature is 25°C. After 4 minutes the temperature of the insulator is 200°C. What is its temperature after 8 minutes?

$T_0 = 400°C$ $\Rightarrow$ $T(t) = 25 + 375 e^{-kt}$ (t : minutes)

$T_m = 25°C$

Find value of k:

$$T(4) = 200°C \Leftrightarrow 200 = 25 + 375 e^{-k4}$$

$$\Leftrightarrow \frac{175}{375} = e^{-4k}$$

$$\Leftrightarrow \ln\left(\frac{175}{375}\right) = -4k$$

$$\Leftrightarrow k = -\frac{1}{4} \ln\left(\frac{175}{375}\right) \approx 0.190535$$

Therefore:

$$T(t) = 25 + 375 e^{\frac{t}{4} \ln\left(\frac{175}{375}\right)} \approx \boxed{25 + 375 e^{-0.190535 t}}.$$

or

$$T(t) = 25 + 375 \left(\frac{175}{375}\right)^{t/4} = 25 + 375 \left(\frac{7}{15}\right)^{t/4}$$

So, after 8 min.:

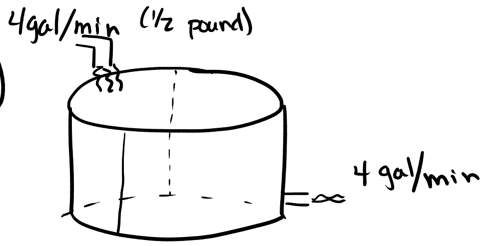
$$T(8) = 25 + 375 \left(7/15\right)^{8/4} = 106.66\ldots °C \approx \boxed{107°C}$$

Example 3. A tank initially contains 40 pounds of salt dissolved in 600 gallons of water. Starting at $t_0 = 0$, water that contains $1/2$ pound of salt per gallon is poured into the tank at the rate of 4gal/min and the mixture is drained from the tank at the same rate. We assume that the mixture is stirred instantly so that the salt is always uniformly distributed throughout the mixture.

1. Find a differential equation for the quantity $Q(t)$ of salt in the tank at time $t > 0$, and solve the equation to determine $Q(t)$.

2. Find $\lim_{t\to\infty} Q(t)$.

1)



$Q(t)$: Quantity of salt in pounds.
$Q'(t)$: rate of change of Q
t : time in minutes.

Q' = rate in − rate out

Rate in: 4 gal/min containing ½ pounds of salt per gallon

$$\Rightarrow \quad \frac{1}{2}\frac{\text{lb}}{\text{gal}} \cdot 4\,\frac{\text{gal}}{\text{min}} = 2\ \text{lb/min}.$$

Rate out: 4 gal/min containing $Q(t)$ pounds of salt.

Information: Volume of mixture always constant.

$$\frac{Q(t)}{600}\,\frac{\text{lb}}{\text{gal}} \cdot 4\,\frac{\text{gal}}{\text{min}} = \frac{Q(t)}{150}\,\frac{\text{lb}}{\text{min}}$$

Therefore:

$$Q'(t) = 2 - \frac{Q(t)}{150} = \frac{300 - Q(t)}{150}.$$

$$\Rightarrow \quad \boxed{Q' = \frac{300 - Q}{150}}$$

This is a seperable equation:

$$\frac{Q'}{300-Q} = 150 \qquad (Q \neq 250)$$

$$\Rightarrow \quad -\ln|300-Q| = 150t + k$$

$$\Rightarrow \quad \frac{1}{|300-Q|} = e^{150t} e^{k}$$

$$\Rightarrow \quad |300-Q| = e^{-150t} e^{-k}$$

Write $c = \pm e^{-k}$ $\Rightarrow$ $Q = 300 - ce^{-150t}$

We know $Q(0) = 40 \Rightarrow c = 300 - 40 = 260$.

$$\Rightarrow \quad Q(t) = 300 - 260e^{-150t}.$$

2) $$\lim_{t\to\infty} Q(t) = 300 - 260 \lim_{t\to\infty} e^{-150t}$$

$$= 300 - 0$$

$$= \boxed{300}$$

$\longrightarrow$ Remark: this is the constant solution

$Q = 300$

This represents the expected long time behavior of $Q(t)$!