Math 302

Chapter 1

Section 1.2: Basic Concepts

Contents

Created by: Pierre-Olivier Parisé
Fall 2022

- A **differential equation** (abbreviated by DE) is an equation that contains one or more derivatives of an unknown function.
  - Examples: $T' = -k(T - T_m)$, $y' = x^2$, $x^2y'' + xy' + 2 = 0$.
- The **order** of a DE is the order of the highest derivatives that it contains.
  - Example: $y' = x^2$ is of order $\underline{\ 1\ }$.
  - Example: $x^2y'' + xy' + 2 = 0$ is of order $\underline{\ 2\ }$.
- An **Ordinary Differential Equation** (abbreviated ODE) is a DE involving an unknown function of only one variable.
- An **Partial Differential Equation** (abbreviated PDE) is a DE involving an unknown function of more than one variable.

The simplest ODE is of the form

$$y' = f(x) \quad \text{or} \quad y^{(n)} = f(x)$$

where f is a known function of x.

Example 1. Find functions $y = y(x)$ satisfying

1. $y' = x^2$.
2. $y'' = \cos(x)$.

1) Integrate: $\int y'\,dx = \int x^2\,dx = \frac{x^3}{3} + c$

$\rightarrow \boxed{y(x) = \frac{x^3}{3} + c}$.

2) Integrate twice.

Write $g = y'$. So, $g' = \cos(x) \rightarrow \int g'\,dx = \int \cos x\,dx$

$\rightarrow g(x) = \sin x + c_1$

Now, $g = y' \rightarrow y'(x) = \sin x + c_1 \rightarrow \boxed{y(x) = -\cos x + c_1 x + c_2}$

Our goal is to study general ODEs of the form

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}).$$

A **solution** to the ODE

$$y^{(n)}(x) = f(x, y(x), y'(x), \dots, y^{(n-1)}(x))$$

is a function $y = y(x)$ that verifies the ODE for any x in some open interval (a, b).

Remark:

- Functions that satisfy an ODE at isolated points are not considered solutions.

Example 2. Verify that

$$y = \frac{x^2}{3} + \frac{1}{x}$$

is a solution of

$$xy' + y = x^2$$

on $(-\infty, 0)$ and $(0, \infty)$.

$$y' = \frac{2x}{3} - \frac{1}{x^2} \qquad (x \neq 0)$$

$$\rightarrow \quad xy' + y = x\left(\frac{2x}{3} - \frac{1}{x^2}\right) + \frac{x^2}{3} + \frac{1}{x}$$

$$= \frac{2x^2}{3} - \frac{1}{x} + \frac{x^2}{3} + \frac{1}{x}$$

$$= x^2 \quad \checkmark$$

Solution and Integral Curves

- The graph of a solution of an ODE is a **solution curve**.
- More generally, a curve C in the plane is said to be an **integral curve** of an ODE if every function $y = y(x)$ whose graph is a segment of C is a solution of the ODE.

EXAMPLE 3. Plot the solutions obtained in Example 2. Are they solution curves of the ODE?

Yes, graph of a function.

EXAMPLE 4. If a is any positive constant, check that the circle

$$x^2 + y^2 = a^2$$

is an integral curve of $y' = -x/y$.

$x^2+y^2 = a^2$ describe circles centered at the origin of radius a.

By isolating y, we find

$$y_1(x) = \sqrt{a^2-x^2} \quad \& \quad y_2(x) = -\sqrt{a^2-x^2}$$

Here $-a \leq x \leq a$.

- $y_1' = \dfrac{-x}{\sqrt{a^2-x^2}} = -\dfrac{x}{y_1}$ ✓

- $y_2' = \dfrac{x}{\sqrt{a^2-x^2}} = -\dfrac{x}{-\sqrt{a^2-x^2}} = -\dfrac{x}{y_2}$ ✓

Therefore, y_1 & y_2 are sol. to $y' = -x/y$ on $(-a,a)$.

$\Rightarrow$ $x^2+y^2 = a^2$ is an integral curve.

EXAMPLE 5. Find a solution of

$$y' = x^3$$

satisfying the additional condition $y(1) = 2$.

$y(x) = \frac{x^4}{4} + c.$

$y(1) = \frac{1}{4} + c$ & $y(1) = 2$

$\Rightarrow \frac{1}{4} + c = 2 \Leftrightarrow c = \frac{7}{4}$

So, $\boxed{y(x) = \frac{x^4}{4} + \frac{7}{4}}$.

EXAMPLE 6. All the solutions to

$$y'' - 2y' + 3y = 0$$

are the functions

$$y(x) = c_1 e^x + c_2 e^{-3x}$$

where c_1, c_2 are arbitrary constants. Find the solution that satisfies $y(0) = 1$ and $y'(0) = 0$.

$y(0) = c_1 e^0 + c_2 e^0 = c_1 + c_2$ & $y(0) = 1$

$y' = c_1 e^x - 3c_2 e^{-3x} \Rightarrow y'(0) = c_1 - 3c_2$

& $y'(0) = 0$

So

$\begin{cases} c_1 + c_2 = 1 \\ c_1 - 3c_2 = 0 \end{cases} \rightarrow \begin{aligned} c_2 &= \frac{1}{4} \\ c_1 &= \frac{3}{4} \end{aligned}$

$\Rightarrow \boxed{y(x) = \frac{3}{4} e^x + \frac{e^{-3x}}{4}}$

An **Initial Value Problem** (abbreviated by IVP) is an ODE with additional **Initial conditions**. The general form of an IVP is

$$y^{(n)}(x) = f(x, y(x), y'(x), \dots, y^{(n-1)}(x)), \quad y(x_0) = k_0,\ y'(x_0) = k_1, \dots,\ y^{(n-1)}(x_0) = k_{n-1}.$$

- The largest open interval that contains x_0 on which $y(x)$ is defined and satisfies the ODE is called the **interval of validity** of y.

EXAMPLE 7. Find the interval of validity of the solution to

$$y' = x^3,\ y(1) = 2.$$

We saw that $y(x) = \dfrac{x^4 + 7}{4}$.

This function is defined for all x in $(-\infty, \infty)$ & $(-\infty, \infty)$ contains 1.

$\rightarrow$ $(-\infty, \infty)$ is the interval of validity.

Example 8. Find the interval of validity of the solution to the following IVPs:

1. $xy' + y = x^2$, $y(1) = 4/3$.
2. $xy' + y = x^2$, $y(-1) = -2/3$.

1. We saw that $y(x) = \frac{x^2}{3} + \frac{1}{x}$ satisfied the DE.

Also, $y(1) = \frac{1}{3} + 1 = \frac{4}{3}$.

So y satisfies the IVP.

the domain of y is $(-\infty, 0) \cup (0, \infty)$.

$\Rightarrow$ $(0, \infty)$ is the interval of validity because $1 \in (0, \infty)$.

2. $y(x) = \frac{x^2}{3} + \frac{1}{x}$ satisfies the DE &

$$y(-1) = \frac{1}{3} - 1 = -\frac{2}{3}$$

$\Rightarrow$ y satisfies the IVP.

the domain of y is $(-\infty, 0)$

$\Rightarrow$ $(-\infty, 0)$ is the interval of validity.