Math 302

Chapter 5

Section 5.1: Homogeneous Linear Equations

Contents

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We will be mainly interested in the following specific ODEs:

$$y'' + p(x)y' + q(x)y = f(x) \tag{1}$$

where p, q, and f are continuous functions of the variable x.

$y''(x) = x$

$\to \frac{x^2}{2} + c_1$

$\to \frac{x^3}{6} + c_1 x + c_2$

- When $f(x) = 0$ for any x, the ODEs is called **homogeneous**.
- When $f(x) \neq 0$, the ODEs is called **non-homogeneous**.
- The function f is called the **forcing function** .
- The IVP associated to a second order ODE of the form (1) is

$$y'' + p(x)y' + q(x)y = f(x), \quad y(x_0) = k_0, \, y'(x_0) = k_1$$

  for some point x_0 in an interval (a, b) and k_0, k_1 are arbitrary numbers.

Goal: To solve the homogeneous equation

$$y'' + p(x)y' + q(x)y = 0.$$

EXAMPLE 1. Consider the ODE

$$y'' + 0y' \overset{+(-y)}{-y} = 0$$

$$y'' - y = 0.$$

a) Identify the functions p and q.

b) Verify that $y_1(x) = e^x$ and $y_2(x) = e^{-x}$ are solutions of the ODE on $(-\infty, \infty)$.

c) Verify that if c_1 and c_2 are arbitrary constants, then $y(x) = c_1e^x + c_2e^{-x}$ is a solution to the ODE on $(-\infty, \infty)$.

d) Solve the initial value problem

$$y'' - y = 0, \quad y(0) = 1, \, y'(0) = 3.$$

a) $p(x) = 0 \qquad q(x) = -1$

b) $y_1'(x) = e^x$ & $y_1'' = e^x \to y'' - y = e^x - e^x$
$= 0$ ✓

$y_2' = -e^{-x}$ & $y_2'' = e^{-x} \to y'' - y = e^{-x} - e^{-x}$
$= 0$ ✓

c) $y' = c_1e^x - c_2e^{-x}$, $y'' = c_1e^x + c_2e^{-x}$.

$y'' - y = \cancel{c_1e^x} + \cancel{c_2e^{-x}} - (\cancel{c_1e^x} + \cancel{c_2e^{-x}}) = 0$ ✓

$$y = c_1 y_1 + c_2 y_2 \longrightarrow y'' = c_1 y_1'' + c_2 y_2''$$

$$\Rightarrow y'' - y = c_1 y_1'' + c_2 y_2'' - (c_1 y_1 + c_2 y_2)$$

$$= c_1 y_1'' - c_1 y_1 + c_2 y_2'' - c_2 y_2$$

$$= c_1 (\underbrace{y_1'' - y_1}_{=0}) + c_2 (\underbrace{y_2'' - y_2}_{=0})$$

$$= c_1 \cdot 0 + c_2 \cdot 0 = 0$$

d) General solution: $y(x) = c_1 e^{x} + c_2 e^{-x}$

$$\begin{cases} y(0) = 1 \\ y'(0) = 3 \end{cases} \longrightarrow \begin{cases} 1 = c_1 + c_2 & (1) \\ 3 = c_1 - c_2 & (2) \end{cases}$$

$$\begin{array}{rl} (1) & 1 = c_1 + c_2 \\ + (2) & 3 = c_1 - c_2 \\ \hline & 4 = 2c_1 \longrightarrow c_1 = 2 \end{array}$$

From (2), $3 = 2 - c_2 \longrightarrow c_2 = -1$

So, $\boxed{y(x) = 2e^{x} - e^{-x}}$.

Example 2. Let ω be a positive number. Consider

$$y'' + \omega^2 y = 0.$$

a) Identify the functions $p(x)$ and $q(x)$.

b) Verify that $y_1(x) = \cos(\omega x)$ and $y_2(x) = \sin(\omega x)$ are solutions to the ODE.

c) Verify that $y(x) = c_1 \cos(\omega x) + c_2 \sin(\omega x)$ is a solution to the ODE.

a) $p(x) = 0 \qquad q(x) = \omega^2$

b) $y_1' = -\omega \sin(\omega x) \quad , \quad y_1'' = -\omega^2 \cos(\omega x)$

$$\Rightarrow \quad y_1'' + \omega^2 y_1 = -\omega^2 \cos(\omega x) + \omega^2 \cos(\omega x) = 0 \ \checkmark$$

$y_2' = \omega \cos(\omega x) \quad , \quad y_2'' = -\omega^2 \sin(\omega x)$

$$\Rightarrow \quad y_2'' + \omega^2 y_2 = -\omega^2 \sin(\omega x) + \omega^2 \sin(\omega x) = 0 \ \checkmark$$

c) $y(x) = c_1 y_1 + c_2 y_2 \qquad (y_1 = \cos(\omega x) \ \& \ y_2 = \sin(\omega x))$

$y''(x) = c_1 y_1'' + c_2 y_2''$

$$\Rightarrow \quad y'' + \omega^2 y = c_1 y_1'' + c_2 y_2'' + \omega^2 (c_1 y_1 + c_2 y_2)$$

$$= c_1 y_1'' + c_2 y_2'' + \omega^2 c_1 y_1 + \omega^2 c_2 y_2$$

$$= c_1 \underbrace{(y_1'' + \omega^2 y_1)}_{=0} + c_2 \underbrace{(y_2'' + \omega^2 y_2)}_{=0}$$

$$= 0 \ \checkmark$$

$\rightarrow$ $y(x) = c_1 \cos(\omega x) + c_2 \sin(\omega x)$ is the general solution to the ODE.

Sometimes, the ODE will be given in the following form:

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0$$

where P_0, P_1, and P_2 are continuous functions.

EXAMPLE 3. Consider the equation

$$x^2y'' + xy' - 4y = 0.$$

a) Identify the functions $p(x)$ and $q(x)$.

b) Verify that $y_1(x) = x^2$ and $y_2(x) = 1/x^2$ are solutions to the ODE.

c) Verify that if c_1 and c_2 are arbitrary numbers, then $y(x) = c_1x^2 + c_2/x^2$ is a solution of the ODE.

d) Solve the IVP

$$x^2y'' + xy' - 4y = 0, \quad y(1) = 2,\ y'(1) = 0.$$

a) $p(x) = \frac{1}{x}$ $\quad q(x) = -\frac{4}{x^2}$ $\quad y'' + \frac{1}{x}y' - \frac{4}{x^2}y = 0$

b) $y_1' = 2x$
$y_1'' = 2$
$\rightarrow x^2(2) + x(2x) - 4x^2$
$= 4x^2 - 4x^2 = 0$ ✓

$y_2' = \frac{-2}{x^3}$
$y'' = \frac{6}{x^4}$
$\rightarrow x^2 \cdot \frac{6}{x^4} + x\left(-\frac{2}{x^3}\right) - 4\left(\frac{1}{x^2}\right)$
$= \frac{6}{x^2} - \frac{2}{x^2} - \frac{4}{x^2} = 0$ ✓

c) $y(x)$ is a linear combination of y_1 & y_2
$\Rightarrow y(x)$ is a solution. (in fact the general sol.)

d) $y(1) = 2 = c_1 + c_2$

$y'(x) = 2c_1x - \frac{2c_2}{x^3} \Rightarrow y'(1) = 0 = 2c_1 - 2c_2$

$$\begin{cases} c_1 + c_2 = 2 & (1) \\ 2c_1 - 2c_2 = 0 & (2) \end{cases}$$

(1) + (2) $\rightarrow 4c_1 = 4 \rightarrow c_1 = 1$

From (2) $\rightarrow c_2 = 1 \Rightarrow \boxed{y(x) = x^2 + \frac{1}{x^2}}$

$2y_1 + \left(100^{100^{100}}\right) y_2$

Linear combinations

If y_1 and y_2 are functions, we say that the function

$$y(x) = c_1 y_1(x) + c_2 y_2(x),$$

where c_1 and c_2 are numbers, is a **linear combination** of y_1 and y_2.

Fact:

- If y_1 and y_2 are solutions to (1), then any linear combinations of y_1 and y_2 is a solution to (1).

Fundamental Set of Solutions

We say that $\{y_1, y_2\}$ is a **fundamental set of solutions** for (1) if every solutions of the ODE is a linear combination of y_1 and y_2.

Facts:

- $\{y_1, y_2\}$ is a fundamental set of solutions for (1) if and only if neither y_2/y_1 or y_1/y_2 is a constant.

Example 4. Show that

a) • The functions $\{y_1, y_2\}$ where y_1, y_2 are as in Example 1 is a foundamental set of solutions.

b) • Same question for y_1, y_2 from Example 2.

c) • Same question for y_1, y_2 from Example 3.

a) $y_1 = e^x$, $y_2 = e^{-x}$ $\rightarrow$ $\dfrac{y_1}{y_2} = \dfrac{e^x}{e^{-x}} = e^{2x}$ not constant

$\rightarrow$ $\{y_1, y_2\}$ is a Fund. Set of Sols.

b) $y_1 = \cos(\omega x)$, $y_2 = \sin(\omega x)$ $\rightarrow$ $\dfrac{y_2}{y_1} = \dfrac{\sin(\omega x)}{\cos(\omega x)} = \tan(\omega x)$ not constant

$\rightarrow$ $\{y_1, y_2\}$ is an FSS.

c) $y_1 = x^2$, $y_2 = 1/x^2$ $\rightarrow$ $\dfrac{y_1}{y_2} = \dfrac{x^2}{1/x^2} = x^4$ not constant. $\rightarrow$ $\{y_1, y_2\}$ is an FSS.

General Solutions

If $\{y_1, y_2\}$ is a fundamental set of solutions for (1), then we call the linear combination $y(x) = c_1 y_1 + c_2 y_2$ the **general solution** to (1).

It is always clever to verify if an ODE has solutions. Here are some important facts about existence and uniqueness of solutions to an ODE of the form (1).

Existence

Assume that p and q are continuous on an open interval (a, b). Then the ODE

$$y'' + p(x)y' + q(x)y = 0$$

has at least one solution on the interval (a, b).

Uniqueness

Assume again that p and q are continuous on an open interval (a, b) and let x_0 be any point in (a, b). Then the IVP

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = k_0,\, y'(x_0) = k_1$$

has a unique solution on (a, b).