MATH 302

CHAPTER 5

SECTION 5.2: CONSTANT COEFFICIENT HOMOGENEOUS EQUATIONS

CONTENTS

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FALL 2022

What is a Constant Coefficient Homogeneous ODE?

We restrict even further the second order ODE. A **second order constant coefficient ODE** is an ODE of the form

$$ay'' + by' + cy = f(x) \tag{1}$$

where a, b, c are fixed numbers and f is a continuous function.

Goal:

Find the solutions to

$$ay'' + by' + cy = 0.$$

We call this the **constant coefficient homogeneous ODE**.

Trick:

Guess that $y(x) = e^{rx}$, for some r.

So, $y' = re^{rx}$ & $y'' = r^2 e^{rx}$

Replace in the ODE

$$\Rightarrow a r^2 e^{rx} + b r e^{rx} + c e^{rx} = 0$$

$$\Rightarrow (ar^2 + br + c) e^{rx} = 0$$

$$\Rightarrow ar^2 + br + c = 0$$

Therefore, $y(x) = e^{rx}$ is a solution iff. r is a solution to $ar^2 + br + c = 0$.

Terminology:

- $ar^2 + br + c$ is called the characteric polynomial
- $ar^2 + br + c = 0$ is called the characteristic equation.

Roots:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \rightarrow \underline{3 \text{ cases}}.$$

Distinct Real Roots: $\cancel{\sqrt{}}b^2 - 4ac > 0$

Example 1. Find the general solution of

$$y'' + 6y' + 5y = 0.$$

$y = e^{rx} \longrightarrow r^2 e^{rx} + 6re^{rx} + 5e^{rx} = 0$

$\Rightarrow r^2 + 6r + 5 = 0$

So, $r^2 + 6r + 5 = (r+1)(r+5)$

So, $r = -1$ & $r = -5$ are the roots.

Therefore, we have two functions:

$y_1(x) = e^{-x}$ & $y_2(x) = e^{-5x}$

So, $\dfrac{y_2}{y_1} = \dfrac{e^{-5x}}{e^{-x}} = e^{-4x}$ (not a constant).

So, $\{y_1, y_2\}$ is a fund. set of solutions

$\Rightarrow$ general solution:

$$\boxed{y(x) = c_1 e^{-x} + c_2 e^{-5x}}$$

General Fact:

- If the roots of the characteristic polynomial are r_1 and r_2, then $y_1(x) = e^{r_1 x}$ and $y_2 = e^{r_2 x}$ are solutions to the ODE.
- The general solutions is given by

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

Example 2.

a) Find the general solution of

$$y'' + 6y' + 9y = 0.$$

~~b) Solve the following IVP:~~

$$\cancel{y'' + 6y' + 9y = 0, \quad y(0) = 3, y'(0) = -1.}$$

① Polynomial:

$$r^2 + 6r + 9 = 0 \longrightarrow r = \frac{-6 \pm \sqrt{36-36}}{2}$$

$$r^2 + 6r + 9 = (r+3)^2 \qquad r = \frac{-6}{2} = -3$$

Apparently, $y_1(x) = e^{-3x}$

② Varia. of Params.

$$y = ue^{-3x} \Rightarrow y' = u'e^{-3x} - 3ue^{-3x}$$

$$y'' = u''e^{-3x} - 3u'e^{-3x} - 3u'e^{-3x} + 9ue^{-3x}$$

Replace y', y'' & y in the ODE.

$$y'' + 6y' + 9y = u''e^{-3x}$$

$$\Rightarrow u''e^{-3x} = 0 \Rightarrow u'' = 0$$

So, $u' = A \Rightarrow u = Ax + B$

Therefore, $y(x) = (Ax+B)e^{-3x}$

$A = 0$ & $B = 1$ $\Rightarrow$ $y(x) = e^{-3x}$

$A = 1$ & $B = 0$ $\Rightarrow$ $y(x) = xe^{-3x}$

Take $y_1 = e^{-3x}$ & $y_2 = xe^{-3x}$

$\frac{y_2}{y_1} = x$ (not constant).

So, $\{y_1, y_2\}$ is a fund. set of solutions

$\Rightarrow$ $\boxed{\text{general solution is: } y(x) = c_1 e^{-3x} + c_2 x e^{-3x}}$

<u>General Facts:</u>

- If the root of the characteristic polynomial is r_1, then $y_1(x) = e^{r_1 x}$ and $y_2(x) = xe^{r_1 x}$ are solutions to the ODE.
- The general solution is given by

$$y(x) = e^{r_1 x}(c_1 + c_2 x).$$

EXAMPLE 3.

a) Find the general solution of

$$y'' + 4y' + 13y = 0.$$

b) Solve the following IVP:

$$y'' + 4y + 13y = 0, \quad y(0) = 2,\ y'(0) = -3.$$

(a) ① Polynomial

$$r^2 + 4r + 13 = 0$$

$$\Rightarrow \quad r = \frac{-4 \pm \sqrt{16 - 52}}{2} = \frac{-4 \pm \sqrt{-36}}{2}$$

Complex Numbers

A complex number is an expression of the form

real part

imaginary part.

$$z = \alpha + i\beta$$

where α, β are real numbers and $i^2 = -1$ $(i = \sqrt{-1})$.

Consider $z = \alpha + i\beta$ and $w = \gamma + i\mu$.

- $z = w$ if and only if $\alpha = \gamma$ and $\beta = \mu$.
- $z + w = (\alpha + \gamma) + i(\beta + \mu)$.
- $zw = (\alpha\gamma - \beta\mu) + i(\alpha\mu + \beta\gamma)$.
- $z/w = \frac{(\alpha+i\beta)(\gamma-i\mu)}{(\gamma+i\mu)(\gamma-i\mu)}$, if $w \neq 0$.

EXAMPLE 4. If $z = 1 + i$ and $w = 1 - i$, find

a) $z + w$.

b) zw.

c) z/w.

(a) $z + w = (1+1) + (1-1)i = 2 + 0 \cdot i = 2$

(b) $zw = (1+i)(1-i) = 1 - i + i - i^2 = 1 - i + i - (-1)$
$= 2$

(c) $\frac{z}{w} = \frac{1+i}{1-i} \cdot \frac{1+i}{1+i} = \frac{(1+i)(1+i)}{(1-i)(1+i)} = \frac{1 + i + i + i^2}{2} = i$

Example 5. Complete the previous example.

(a) $$r = \frac{-4 \pm \sqrt{-36}}{2} = \frac{-4 \pm \sqrt{(-1)36}}{2} = \frac{-4 \pm \sqrt{-1}\sqrt{36}}{2}$$
$$= \frac{-4 \pm i6}{2}$$

$$\Rightarrow \quad r = -2+3i \quad \& \quad r = -2-3i$$

The solutions:
$$y_1(x) = e^{(-2+3i)x} \quad \& \quad y_2(x) = e^{(-2-3i)x}$$
$$\Rightarrow \quad y_1(x) = e^{-2x} e^{3ix} \quad \& \quad y_2(x) = e^{-2x} e^{-3ix}$$

② Var. of Params.

$$y(x) = e^{-2x} \cdot u(x)$$

$$\Rightarrow \quad y'(x) = -2e^{-2x} u + e^{-2x} u'$$
$$\& \quad y''(x) = 4e^{-2x} u - 2e^{-2x} u' - 2e^{-2x} u' + e^{-2x} u''$$

Replace in the ODE

$$\Rightarrow \quad y'' + 4y' + 13y = \underbrace{u'' e^{-2x} + 9ue^{-2x}}_{} = 0$$

$$\Rightarrow \quad u'' + \underset{\uparrow \atop 3^2}{9}u = 0$$

Solutions: $$u_1(x) = \cos(3x) \quad \& \quad u_2(x) = \sin(3x).$$

So, $$y_1(x) = e^{-2x}\cos(3x) \quad \& \quad y_2(x) = e^{-2x}\sin(3x)$$
$$\frac{y_2}{y_1} = \frac{\sin(3x)}{\cos(3x)} = \tan(3x) \quad \left(\text{not constant}\right).$$

General solution: $y(x) = c_1 e^{-2x} \cos(3x) + c_2 e^{-2x} \sin(3x)$

(b) $y(0) = 2$

$y'(0) = -3$

$$y'(x) = -2c_1 e^{-2x} \cos(3x) - 3c_1 e^{-2x} \sin(3x) - 2c_2 e^{-2x} \sin(3x) + 3c_2 e^{-2x} \cos(3x)$$

$y(0) = c_1 = 2$

$y'(0) = -2c_1 + 3c_2 = -3 \Rightarrow -4 + 3c_2 = -3$

$\Rightarrow c_3 = \frac{1}{3}$

So, $y(x) = 2e^{-2x} \cos(3x) + \frac{1}{3} e^{-2x} \sin(3x)$.

General Facts:

- If $r_1 = \alpha + \beta i$ and $r_2 = \alpha - \beta i$ are the roots of the characteristic polynomial, then $y_1(x) = e^{\alpha x} \cos(\beta x)$ and $y_2(x) = e^{\alpha x} \sin(\beta x)$ are solutions to the ODE.
- The general solution has the form

$$y(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x)).$$