# Math 307

## Chapter 1

Section 1.4: Special Matrices and Additional Properties

Contents




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## Diagonal Matrices

A diagonal matrix is a square matrix whose off diagonal entries are zero.

- Remark: We denote a diagonal matrix by $\mathrm{diag}\,(d_1, d_2, \ldots, d_n)$.

**Example 1.** Give some examples of diagonal matrices.

**Example 2.** Suppose $A = \mathrm{diag}(1, 2, -4, 3, 5)$ and $B = \mathrm{diag}(-1, 2, 0, 4, 3)$.

1) Is $A$ invertible? 2) Is $A + B$ invertible? 3) Is $AB$ invertible?

General Facts: Suppose $A$ and $B$ are diagonal matrices

$$A = \operatorname{diag}(a_1, a_2, \ldots, a_n) \quad \text{and} \quad B = \operatorname{diag}(b_1, b_2, \ldots, b_n).$$

- $A + B = \operatorname{diag}(a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$.
- $AB = \operatorname{diag}(a_1 b_1, a_2 b_2, \ldots, a_n b_n)$.
- $A$ is invertible if and only if $a_i \neq 0$ for each $i$. In this case, we have

$$A^{-1} = \operatorname{diag}(1/a_1, 1/a_2, \ldots, 1/a_n).$$

# Triangular Matrices

- Upper Triangular: Square matrices whose entries below the diagonal are zero.
- Lower Triangular: Square matrices whose entries above the diagonal are zero.

**Example 3.** Give an example of an upper triangular matrix and an example of a lower triangular matrix.

General Facts:

- If $A$ and $B$ are both upper triangular, then so is $A+B$; similarly if $A$ and $B$ are both lower triangular, then so is $A+B$.
- If $A$ and $B$ are both upper triangular, then so is $AB$; similarly if $A$ and $B$ are both lower triangular, then so is $AB$.
- $A$ is invertible if and only if each of the diagonal entries of $A$ is nonzero.

**Example 4.** Let $A$ and $B$ be the two following $3 \times 3$ matrices:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 7 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 4 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

1. Is $A$ invertible?
2. Is $B$ invertible?
3. Is $AB$ upper or lower triangular matrix?

# Symmetric Matrices

- Transpose: The transpose of a matrix $A$ of dimensions $m \times n$, denoted $A^\top$, is the matrix obtained by interchanging the rows and columns of $A$.
- Symmetric: A matrix $A$ is said to be symmetric if $A = A^\top$.

**Example 5.** Let $A$ and $B$ be the following matrices.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}.$$

1. Find $B^\top$.
2. Is $A$ symmetric?

General facts about transpose:

- $(A^\top)^\top = A$.
- $(cA)^\top = cA^\top$.
- $(A^T)^{-1} = (A^{-1})^\top$.
- $(A+B)^\top = A^\top + B^\top$.
- $(AB)^\top = B^\top A^\top$.

General Facts about symmetric: Suppose $A$ and $B$ are matrices of the same size.

- If $A$ and $B$ are symmetric matrices, then so is $A + B$.
- If $A$ is symmetric, then $A$ is a square matrix and $cA$ is symmetric for any scalar $c$.
- $A^\top A$ and $AA^\top$ are symmetric matrices.
- If $A$ is an invertible symmetric matrix, then $A^{-1}$ is a symmetric matrix.

**Example 6.** Is the matrix

$$A = \begin{bmatrix} 1 & 0 & 4 \\ -4 & 0 & 6 \\ 2 & 0 & -10 \end{bmatrix}$$

invertible?

Remarks:

- Using row operations on the transposed matrix is equivalent to applying column operations to the original matrix.
- So, in general, what we learned to do with the rows of a matrix can also be done with the columns of a matrix.
- Taking column operations will be important when we will find the row space and column space of a matrix.