Math 307

Chapter 5

Section 5.4: Eigenvalues and Eigenvectors of Matrices

Contents

Created by: Pierre-Olivier Parisé
Summer 2022

Example: Markov Processes

A market research company has predicted the future market share proportion of three compagnies A, B, and C. The future market is determined by a transition matrix P:

$$\begin{bmatrix} 0.8 & 0.03 & 0.2 \\ 0.1 & 0.95 & 0.05 \\ 0.1 & 0.02 & 0.75 \end{bmatrix}$$

We have the following interpretation of each row of P:

- the first column of P represents the share of Company A that will pass to Company A, Company B, and Company C respectively after a month.
- the second column of P represents the share of Company B that will pass to Company A, Company B and Company C respectively after a month.
- the third column of P represents the share of Company C that will pass to Company A, Company B and Company C.

EXAMPLE 1. If the initial market for the three companies is represented by $s_0 = \begin{bmatrix} 30 & 15 & 55 \end{bmatrix}$, that is, Company A has 30% share, Company B has 15% share, and Company C has 55% share. Calculate the predicted market

1. after 1 month, that is s_1.
2. after 2 month, that is s_2.
3. Using Python, find s_5 up to s_{40}.

1) $s_1 = Ps_0 = \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.03 & 0.95 & 0.02 \\ 0.2 & 0.05 & 0.75 \end{bmatrix} \begin{bmatrix} 30 \\ 15 \\ 55 \end{bmatrix} = \begin{bmatrix} 31 \\ 16.25 \\ 48 \end{bmatrix}$

2) $s_2 = Ps_1 = P^2 s_0 = \begin{bmatrix} 31.225 \\ 17.3275 \\ 43.0125 \end{bmatrix}$

3) See <u>Python File</u>.

Remark: After a period of time, the market share seems to stabilize. In fact, there are tools to compute this asymptotic values: eigenvalues and eigenvectors!

Definition

Let A be an $n \times n$ matrix.

- An **eigenvalue** of A is a scalar λ such that there is a nonzero column vector v in $\mathbb{R}^n$ so that

$$Av = \lambda v.$$

- The nonzero vector v is called an **eigenvector** of A associated to the eigenvalue λ.

EXAMPLE 2. Verify that the vector $v = \begin{bmatrix} -1 & 1 \end{bmatrix}^\top$ is an eigenvector of the matrix

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}.$$

$$A \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

So, $\lambda = 4$ & $v = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector of A.

Method to Find Eigenvalues

If v is a $n \times 1$ column vector and A is an $n \times n$ matrix, then the equation

$$Av = \lambda v$$

for some scalar λ can be rewritten as

$$(\lambda I - A)v = 0.$$

There are two cases to consider:

- If the matrix $\lambda I - A$ is invertible, then the only solution is $v = 0$. This is not valid because an eigenvector should not be zero.
- If the matrix $\lambda I - A$ is not invertible, then the system has non-trivial solutions. In particular, this means that $\det(\lambda I - A) = 0$.

Characteristic Equation: The eigenvalues are obtained from solving the characteristic equation of the matrix A given by

$$\det(\lambda I - A) = 0.$$

EXAMPLE 3. Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}.$$

$$\lambda I - A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} \lambda-1 & 3 \\ 2 & \lambda-2 \end{bmatrix}$$

So,

$$\begin{aligned} \det(\lambda I - A) &= (\lambda-1)(\lambda-2) - 6 \\ &= \lambda^2 - 3\lambda + 2 - 6 \\ &= \lambda^2 - 3\lambda - 4 \\ &= (\lambda-4)(\lambda+1) \end{aligned}$$

$$\Rightarrow \det(\lambda I - A) = 0$$

$$\Leftrightarrow (\lambda-4)(\lambda+1) = 0$$

$$\Leftrightarrow \boxed{\lambda = 4} \text{ or } \boxed{\lambda = -1}$$

Remark: The expression $\det(\lambda I - A)$ is also called the characteristic polynomial (a polynomial of degree n in the variable λ).

EXAMPLE 4. Find the eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}.$$

$\underline{\lambda = 4}$ Solve the system $(4I - A)v = 0$ $\quad v = \begin{bmatrix} x \\ y \end{bmatrix}$.

$$4I - A = \begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix}.$$

$$\rightarrow \begin{bmatrix} 3 & 3 & 0 \\ 2 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ y \end{bmatrix} = y\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ will be a basis fn the null space of $4I - A$.

$\underline{\lambda = -1.}$ Solve the system $(-I - A)v = 0$ $\quad v = \begin{bmatrix} x \\ y \end{bmatrix}$.

$$(-1)I - A = \begin{bmatrix} -2 & 2 \\ 1 & -3 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} -2 & 2 & 0 \\ 1 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{3y}{2} \\ y \end{bmatrix} = y\begin{bmatrix} -3/2 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} -3/2 \\ 1 \end{bmatrix}$ is a basis for the null space of $-I - A$.

Remark: When λ is an eigenvalue of A, then the steps to find the eigenvectors associated to λ are the following:

1. Form the matrix $\lambda I - A$.
2. Find a basis for the nullspace of $\lambda I - A$, or equivalently find all non-trivial solutions to the system $(\lambda I - A)v = 0$ where $v = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}^\top$.

Terminology:

- The set of eigenvectors associated to an eigenvalue λ is called the eigenspace and is denoted by E_λ.
- We call $\dim(E_\lambda)$ the **geometric multiplicity** of λ.
- The **algebraic multiplicity** of λ is the number of times the factor $(x - \lambda)$ appears in the characteristic polynomial.

EXAMPLE 5. Find the eigenvalues and bases for the eigenspace associated to each eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Eigenvalues.

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-2 & 1 & -3 \\ 0 & \lambda+1 & 0 \\ 0 & 0 & \lambda+1 \end{vmatrix} = (\lambda-2)(\lambda+1)^2$$

So $\det(\lambda I - A) = 0 \longleftrightarrow (\lambda-2)(\lambda+1)^2 = 0 \longleftrightarrow \boxed{\begin{matrix}\lambda = 2 \\ \lambda = -1\end{matrix}}$

Eigenspace E_2. $\lambda = 2$, so $2I - A = \begin{bmatrix} 0 & 1 & -3 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$\rightarrow \left[\begin{array}{ccc|c} 0 & 1 & -3 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right] \rightarrow$ $y = 0$, $z = 0$, x free variable $\rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

A basis for E_2 would be $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Eigen Space E_{-1}. $\lambda = -1 \rightarrow -I - A = \begin{bmatrix} -3 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

$\rightarrow \left[\begin{array}{ccc|c} -3 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1/3 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\rightarrow$ $x = -y/3 + z$, y, z free var. $\rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{y}{3} + z \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{y}{3} \\ y \\ 0 \end{bmatrix} + \begin{bmatrix} z \\ 0 \\ z \end{bmatrix}$

So, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = y \begin{bmatrix} -1/3 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Thus, a basis for E_{-1} is $\begin{bmatrix} -1/3 \\ 0 \\ 1 \end{bmatrix}$ & $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

EXAMPLE 6. Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda & +1 \\ -1 & \lambda \end{vmatrix} = \lambda^2 + 1 .$$

So, $\det(\lambda I - A) = 0 \iff \lambda^2 + 1 = 0$

$\iff \lambda^2 = -1$

$\iff \lambda = \pm \underbrace{\sqrt{-1}}_{i}$

i : complex number!

Complex numbers

We define $i = \sqrt{-1}$ such that $i^2 = -1$. A complex number z is

$$z = a + bi.$$

Arithmetic Operations:

- Equality: $a + bi = c + di$ if $a = c$ and $b = d$.
- Addition: $(a + bi) + (c + di) = (a + c) + (b + d)i$.
- Multiplication: Like multiplying two polynomials

$$(a + bi)(c + di) =$$

- Conjugate: $\overline{a + bi} = a - bi$.

Complex Eigenvalues

To deal with the complex eigenvalues, we have to consider vectors with complex entries.

All the definitions associated to matrix addition, scalar multiplication, matrix multiplication, and the determinant of a matrix are all the same: Instead of using the addition and multiplication of the real numbers, we use the addition and multiplication of complex numbers.

EXAMPLE 7. Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Eigen values.

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-1 & 1 \\ -1 & \lambda-1 \end{vmatrix} = (\lambda-1)^2 + 1 = \lambda^2 - 2\lambda + 1 + 1 = \lambda^2 - 2\lambda + 2.$$

So, $\lambda^2 - 2\lambda + 2 = 0 \longleftrightarrow \lambda = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$$= \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2}$$

But, $\sqrt{-4} = \sqrt{-1}\sqrt{4} = 2 \cdot i \rightarrow \lambda = 1 \pm i$

Basis for E_{1+i} $\quad \lambda = 1+i \rightarrow (1+i)I - A = \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}$

$$\begin{bmatrix} i & 1 & 0 \\ -1 & i & 0 \end{bmatrix} \sim \begin{bmatrix} i & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{R_1 + iR_2} \sim \begin{bmatrix} 1 & 1/i & 0 \\ 0 & 0 & 0 \end{bmatrix} \tfrac{1}{i}R_1 \to R_1$$

$$\rightarrow x + \frac{y}{i} = 0 \rightarrow x = -\frac{y}{i} = yi$$

$$\rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} yi \\ y \end{bmatrix} = y \begin{bmatrix} i \\ 1 \end{bmatrix}$$

Thus, $\begin{bmatrix} i \\ 1 \end{bmatrix}$ is a basis for E_{1+i}

Basis for E_{1-i} $\quad \lambda = 1-i \rightarrow (1-i)I - A = \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}$

$$\rightarrow \left[\begin{array}{cc|c} -i & 1 & 0 \\ -1 & -i & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]_{R_1 - iR_2 \to R_2}$$

$$\rightarrow -ix + y = 0 \rightarrow y = -ix$$

So, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -ix \end{bmatrix} = x \begin{bmatrix} 1 \\ -i \end{bmatrix}$

Therefore, $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ is a basis for E_{1-i}.

Remark: The textbook does not talk about this other approach to avoid complex numbers.
If you encounter a matrix with complex eigenvalues, you can transform your matrix into another matrix that will have real eigenvalues.
This comes from the fact that

$$a + bi \leftrightarrow \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

So you can replace each a_{ij} in the matrix A by $\begin{pmatrix} a_{ij} & 0 \\ 0 & a_{ij} \end{pmatrix}$.

Fact: Let A be an $n \times n$ matrix with real numbers as entries. Suppose

- λ is an eigenvalue of A;
- $v_1, v_2, \ldots, v_k$ are basis of the eigenspace E_λ, then $\overline{\lambda}$ is also an eigenvalue of A and $\overline{v}_1$, $\overline{v}_2$, $\ldots$, $\overline{v}_k$ form a basis for the eigenspace $E_{\overline{\lambda}}$.

If v is a column vector, the notation $\overline{v}$ means we are taking the conjugate of each component of v.