Math 307

Chapter 1

Section 1.4: Special Matrices and Additional Properties

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A diagonal matrix is a square matrix whose off diagonal entries are zero.

- Remark: We denote a diagonal matrix by diag $(d_1, d_2, \ldots, d_n)$.

Example 1. Give some examples of diagonal matrices.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix} = \operatorname{diag}(1, 2, -3)$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} = \operatorname{diag}(1, 0, -5, 3)$$

Example 2. Suppose $A = \operatorname{diag}(1, 2, -4, 3, 5)$ and $B = \operatorname{diag}(-1, 2, 0, 4, 3)$.

1) Is A invertible? 2) Is $A + B$ invertible? 3) Is AB invertible?

1)
$$\left[\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1/4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1/5 \end{array}\right] \begin{array}{l} \\ 1/2\, R_2 \to R_2 \\ -1/4\, R_3 \to R_3 \\ 1/3\, R_4 \to R_4 \\ 1/5\, R_5 \to R_5 \end{array}$$

$$A^{-1} = \operatorname{diag}(1, 1/2, -1/4, 1/3, 1/5)$$

2) $A+B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 8 \end{bmatrix}$

$A+B$ is not invertible. $A+B$ is still a diag. matrix.

3) $AB = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 0 & 15 \end{bmatrix}$

AB is not invertible.

General Facts: Suppose A and B are diagonal matrices

$$A = \operatorname{diag}(a_1, a_2, \ldots, a_n) \quad \text{and} \quad B = \operatorname{diag}(b_1, b_2, \ldots, b_n).$$

- $A + B = \operatorname{diag}(a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$.
- $AB = \operatorname{diag}(a_1 b_1, a_2 b_2, \ldots, a_n b_n)$.
- A is invertible if and only if $a_i \neq 0$ for each i. In this case, we have

$$A^{-1} = \operatorname{diag}(1/a_1, 1/a_2, \ldots, 1/a_n).$$

A & B diag, $AB \stackrel{?}{=} BA$

- Upper Triangular: Square matrices whose entries below the diagonal are zero.
- Lower Triangular: Square matrices whose entries above the diagonal are zero.

EXAMPLE 3. Give an example of an upper triangular matrix and an example of a lower triangular matrix.

$$\begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

upper triangular

lower triangular.

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 4 & 2 \end{bmatrix}$$

upper triangular.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \checkmark$$

$$\left[\begin{array}{ccc|ccc} \boxed{2} & 3 & 4 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

General Facts:

- If A and B are both upper triangular, then so is $A + B$; similarly if A and B are both lower triangular, then so is $A + B$.
- If A and B are both upper triangular, then so is AB; similarly if A and B are both lower triangular, then so is AB.
  → upper triangular matrix
- A is invertible if and only if each of the diagonal entries of A is nonzero.

Example 4. Let A and B be the two following 3×3 matrices:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 7 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 4 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

1. Is A invertible?
2. Is B invertible?
3. Is AB upper or lower triangular matrix?

1) Yes, the entries on the diag. are not zero.

2) No, there is an entry on the main diag.

3) AB is upper triangular since A & B are.

Check.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 7 \end{bmatrix}\begin{bmatrix} -1 & 4 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 5 \\ 0 & -4 & 8 \\ 0 & 0 & 0 \end{bmatrix} \checkmark$$

- Transpose: The transpose of a matrix A of dimensions $m \times n$, denoted $A^\top$, is the matrix obtained by interchanging the rows and columns of A.
- Symmetric: A matrix A is said to be symmetric if $A = A^\top$.

EXAMPLE 5. Let A and B be the following matrices.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}.$$

1. Find $B^\top$.
2. Is A symmetric?

1) $B^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$ same. $\longleftrightarrow$ $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

2) $A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} = A$ ✓

Fact: symmetric matrices should square matrix.

A is symmetric.

General facts about transpose: A inv. $\leftrightarrow$ A^T inv.

- $(A^\top)^\top = A$.
- $(cA)^\top = cA^\top$.
- $(A^T)^{-1} = (A^{-1})^\top$.
- $(A+B)^\top = A^\top + B^\top$.
- $(AB)^\top = B^\top A^\top$.

General Facts about symmetric: Suppose A and B are matrices of the same size.

- If A and B are symmetric matrices, then so is $A+B$. $(A+B)^T = A^T + B^T = A + B$
- If A is symmetric, then A is a square matrix and cA is symmetric for any scalar c. $\to (cA)^T = cA^T = cA$
- $A^\top A$ and $AA^\top$ are symmetric matrices. $(A^T A)^T = A^T (A^T)^T = A^T A$
- If A is an invertible symmetric matrix, then A^{-1} is a symmetric matrix.

Symmetric Matrices

- Transpose: The transpose of a matrix A of dimensions $m \times n$, denoted $A^\top$, is the matrix obtained by interchanging the rows and columns of A.
- Symmetric: A matrix A is said to be symmetric if $A = A^\top$.

Example 5. Let A and B be the following matrices.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}.$$

1. Find $B^\top$.
2. Is A symmetric?

General facts about transpose:

- $(A^\top)^\top = A$.
- $(cA)^\top = cA^\top$.
- $(A^T)^{-1} = (A^{-1})^\top$.
- $(A+B)^\top = A^\top + B^\top$.
- $(AB)^\top = B^\top A^\top$.

General Facts about symmetric: Suppose A and B are matrices of the same size.

- If A and B are symmetric matrices, then so is $A + B$.
- If A is symmetric, then A is a square matrix and cA is symmetric for any scalar c.
- $A^\top A$ and $AA^\top$ are symmetric matrices.
- If A is an invertible symmetric matrix, then A^{-1} is a symmetric matrix.

Example 6. Is the matrix

$$A = \begin{bmatrix} 1 & 0 & 4 \\ -4 & 0 & 6 \\ 2 & 0 & -10 \end{bmatrix}$$

invertible?

Remarks:

- Using row operations on the transposed matrix is equivalent to applying column operations to the original matrix.
- So, in general, what we learned to do with the rows of a matrix can also be done with the columns of a matrix.
- Taking column operations will be important when we will find the row space and column space of a matrix.