# Question 1.

(a) $\begin{bmatrix} 3 & -4 & 2 & -4 \\ -3 & 4 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -4/3 & 0 & -7/6 \\ 0 & 0 & 1 & -1/4 \end{bmatrix}$

The system is consistent, therefore the vector is in the span.

(b) $\begin{array}{c} x^2 \\ x \\ 1 \end{array} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 2 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Not equal to the identity matrix

$\Rightarrow$ linearly dependent.

Question 2.

(a) $$\begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 1 \\ -1 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore, the set $\alpha$ is a basis.

(b) $$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ -1 & 2 & 3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1/5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2/5 \end{bmatrix}$$

Therefore $$[v]_\alpha = \begin{bmatrix} 1/5 \\ -1 \\ 2/5 \end{bmatrix}.$$

(c) $$T\left(\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}\right) = T\left( (1/5)\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} + (2/5)\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right)$$

$$= (1/5)\,T\left(\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}\right) - T\left(\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}\right) + (2/5)\,T\left(\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}\right)$$

$$= (1/5)(x+1) - (x^2+2) + (2/5)\,2x$$

$$= \boxed{-x^2 + x - \frac{9}{5}}.$$

## Question 3

(a) $$I\left(\begin{bmatrix}1\\3\\-1\end{bmatrix}\right) = (-9/5)\begin{bmatrix}1\\-1\\0\end{bmatrix} - 3\begin{bmatrix}1\\0\\1\end{bmatrix} + (7/5)\begin{bmatrix}0\\1\\-1\end{bmatrix}$$

$$I\left(\begin{bmatrix}0\\-1\\2\end{bmatrix}\right) = (-3/5)\begin{bmatrix}1\\-1\\0\end{bmatrix} - \begin{bmatrix}1\\0\\1\end{bmatrix} + (4/5)\begin{bmatrix}0\\1\\-1\end{bmatrix}$$

$$I\left(\begin{bmatrix}2\\1\\3\end{bmatrix}\right) = (2/5)\begin{bmatrix}1\\-1\\0\end{bmatrix} + (0)\begin{bmatrix}1\\0\\1\end{bmatrix} - (1/5)\begin{bmatrix}0\\1\\-1\end{bmatrix}$$

Therefore, $$[I]_\alpha^\beta = \begin{bmatrix}-9/5 & -3/5 & 2/5\\ -3 & -1 & 0\\ 7/5 & 4/5 & -1/5\end{bmatrix}.$$

(b) $[I]_\alpha^\beta$ is $P^{-1}$, so it represents the change of basis matrix from $\beta$ to $\alpha$.

(c) $$I\left(\begin{bmatrix}1\\-1\\0\end{bmatrix}\right) = (-1/2)\begin{bmatrix}1\\3\\-1\end{bmatrix} + (3/2)\begin{bmatrix}0\\-1\\2\end{bmatrix} + (5/2)\begin{bmatrix}2\\1\\3\end{bmatrix}$$

$$I\left(\begin{bmatrix}0\\-1\\2\end{bmatrix}\right) = (-1/2)\begin{bmatrix}1\\3\\-1\end{bmatrix} + (1/2)\begin{bmatrix}0\\-1\\2\end{bmatrix} - 3/2\begin{bmatrix}2\\1\\3\end{bmatrix}$$

$$I\left(\begin{bmatrix}0\\1\\-1\end{bmatrix}\right) = (-1)\begin{bmatrix}1\\3\\-1\end{bmatrix} + 3\begin{bmatrix}0\\-1\\2\end{bmatrix} + (0)\begin{bmatrix}2\\1\\3\end{bmatrix}.$$

Therefore, $$[I]_\beta^\alpha = \begin{bmatrix}-1/2 & -1/2 & -1\\ 3/2 & 1/2 & 3\\ 5/2 & -3/2 & 0\end{bmatrix}.$$

(d) $[I]_\beta^\alpha$ is $P$, so it's the change of basis from $\alpha$ to $\beta$.

Question 4.

(a) $[T]_\beta^\beta = P^{-1}[T]_\alpha^\alpha P$

$$= \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & -1 \\ -1 & -4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ 1 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & -34 & -10 \\ -4 & -19 & -6 \\ 20 & 89 & 29 \end{bmatrix}$$

(b) 1) Transform $[v]_\alpha$ into $[v]_\beta$.

$[v]_\beta = [I]_\alpha^\beta [v]_\alpha = P^{-1}[v]_\alpha$

$$= \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & -1 \\ -1 & -4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 6 \end{bmatrix}$$

2) Find $[T(v)]_\beta$.

$[T(v)]_\beta = [T]_\beta^\beta [v]_\beta = \begin{bmatrix} -5 & -34 & -10 \\ -4 & -19 & -6 \\ 20 & 89 & 29 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 6 \end{bmatrix}$

$$= \begin{bmatrix} -21 \\ -13 \\ 65 \end{bmatrix}.$$

Question 5.

(a) 1) $u = a_1x^2 + b_1x + c_1$ & $v = a_2x^2 + b_2x + c_2$

$\Rightarrow T(u+v) = T\left((a_1+a_2)x^2 + (b_1+b_2)x + (c_1+c_2)\right)$

$= x\left((a_1+a_2)x^2 + (b_1+b_2)x + (c_1+c_2)\right)$

$= (a_1+a_2)x^3 + (b_1+b_2)x^2 + (c_1+c_2)x$

$= a_1x^3 + a_2x^3 + b_1x^2 + b_2x^2 + c_1x + c_2x$

$= a_1x^3 + b_1x^2 + c_1x + a_2x^3 + b_2x^2 + c_2x$

$= T(u) + T(v)$ ✓

2) $T(du) = T\left((a_1d)x^2 + (b_1d)x + (c_1d)\right)$

$= x\left(a_1dx^2 + b_1dx + c_1d\right)$

$= a_1dx^3 + b_1dx^2 + c_1dx$

$= d\left(a_1x^3 + b_1x^2 + c_1x\right)$

$= d\,T(u)$.

(b) If $v \in \ker(T)$, then $T(v) = 0$. So

$T(u+v) = T(u) + T(v) = T(u) + 0 = T(u)$. ✓

Question 6.

(a) Nullity-Rank Theorem:

$$\dim(\ker(T)) + \dim(\text{Range}(T)) = \dim(V)$$

$$\Rightarrow \quad 1 + \text{rank}(T) = 5$$

$$\Rightarrow \quad \text{rank}(T) = 4.$$

$\boxed{\text{True}}$

(b) $\{1, x, x^2, \dots, x^n, \dots\}$ is a basis for $P$.

$\boxed{\text{False.}}$

(c) We need at least five vectors in a basis because $\dim(V)=5$.

$\boxed{\text{False}}$

(d)
$$(2T)\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = 2\,T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = 2\begin{bmatrix} 2x \\ 2x+2y \end{bmatrix} = \begin{bmatrix} 4x \\ 4x+4y \end{bmatrix}$$

$\boxed{\text{False}}$

(e) 1) $$T(f+g) = \int_a^b (f+g)'(x)\,dx = \int_a^b f'(x)+g'(x)\,dx = \int_a^b f'(x) + \int_a^b g'(x) = T(f) + T(g). \checkmark$$

2) $$T(cf) = \int_a^b (cf)'(x)\,dx = \int_a^b cf'(x)\,dx = cT(f). \checkmark$$

$\boxed{\text{True}}$