Math 307

Chapter 5

Section 5.5: Similar Matrices, Diagonalization, and Jordan Canonical Form

Contents

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Motivation

Example 1. Let A be the 3×3 matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

Then, (a) compute A^5 (b) find the eigenvalues of A (c) find a basis for each eigenspace.

(a) $$A \cdot A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 2^2 & 0 & 0 \\ 0 & 3^2 & 0 \\ 0 & 0 & 4^2 \end{bmatrix}$$

$$A^5 = A \cdot A \cdot A \cdot A \cdot A = \begin{bmatrix} 2^5 & 0 & 0 \\ 0 & 3^5 & 0 \\ 0 & 0 & 4^5 \end{bmatrix}$$

(b) $$\det(\lambda I - A) = 0 \iff \begin{vmatrix} \lambda-2 & 0 & 0 \\ 0 & \lambda-3 & 0 \\ 0 & 0 & \lambda-4 \end{vmatrix} = 0 \iff \underbrace{(\lambda-2)(\lambda-3)(\lambda-4)}_{=0}$$

$$\iff \lambda = 2, \lambda = 3, \lambda = 4.$$

(c) $E_2 \ (\lambda = 2)$ $\quad v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \to (\lambda I - A)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$

$$\begin{array}{c} \begin{array}{cccc} & y & z & \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{bmatrix} \end{array} \to \begin{array}{l} x \text{ free} \\ y = 0 \\ z = 0 \end{array} \to \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

basis for E_2

$E_3 \ (\lambda = 3)$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \to \begin{array}{l} x = 0 \\ z = 0 \\ y \text{ free} \end{array}$$

$$\to \begin{bmatrix} x \\ y \\ z \end{bmatrix} = y\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \leftarrow \text{basis for } E_3$$

$E_4 \ (\lambda = 4)$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{array}{l} x = 0 \\ y = 0 \\ z \text{ free} \end{array}$$

$$\to \begin{bmatrix} x \\ y \\ z \end{bmatrix} = z\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \leftarrow \text{basis for } E_4.$$

Remarks

- It is pretty easy to deal with diagonal matrices.
- Our goal is to try to transform a general matrix into a diagonal matrix.

Example 2. Let A be the following 3×3 matrix

$$A = \begin{bmatrix} 6 & -4 & -2 \\ 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Find (a) the eigenvalues of A (b) a basis for each eigenspace (c) compute A^5.

(a) $\det(\lambda I - A) = \lambda^3 - 9\lambda^2 + 26\lambda - 24 = (\lambda-2)(\lambda-3)(\lambda-4) = 0$

$\to \lambda = 2, \lambda = 3$, or $\lambda = 4$.

(b) $\underline{E_2 (\lambda=2)}$ $2I - A = \begin{bmatrix} -4 & 4 & 2 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix}$, solve $(2I - A)v = 0$, $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

$\to v = z \begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix}$ (z free var.) $\to w_1$

$\underline{E_3 (\lambda=3)}$ $3I - A = \begin{bmatrix} -3 & 4 & 2 \\ -1 & 1 & 1 \\ -1 & 0 & 2 \end{bmatrix}$, solve $(3I-A)v = 0$, $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

$\to v = z \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ (z free var.). $\to w_2$

$\underline{E_4 (\lambda=4)}$ $4I - A = \begin{bmatrix} -2 & 4 & 2 \\ -1 & 2 & 1 \\ -1 & 0 & 3 \end{bmatrix}$, solve $(4I - A)v = 0$, $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

$\to v = z \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \to w_3$

Because eigenvectors associated to different eigenvalues, we have that w_1, w_2, w_3 are lin. independent

$\to w_1, w_2, w_3$ form a basis for $\mathbb{R}^3$!

(c) Change of basis from $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ to $\underbrace{w_1, w_2, w_3}_{\beta}$:

$$P = \begin{bmatrix} 1 & 2 & 3 \\ 1/2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} 0 & -2 & 2 \\ -1 & 4 & -1 \\ 1 & -2 & 0 \end{bmatrix}$$

Now,

$$[A]_\beta^\beta = P^{-1}AP = \begin{bmatrix} 0 & -2 & 2 \\ -1 & 4 & -1 \\ 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 6 & -4 & 2 \\ -1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1/2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \quad \text{diag. matrix!}$$

So, $A = P[A]_\beta^\beta P^{-1}$

$$\Rightarrow A^5 = (P[A]_\beta^\beta P^{-1})(P[A]_\beta^\beta P^{-1})(P[A]_\beta^\beta P^{-1})(P[A]_\beta^\beta P^{-1})(P[A]_\beta^\beta P^{-1})$$

$=I \quad =I \quad =I \quad I$

$$= P[A]_\beta^\beta \, I \, [A]_\beta^\beta \, I \, [A]_\beta^\beta \, I \, [A]_\beta^\beta \, I \, [A]_\beta^\beta P^{-1}$$

$$= P \left([A]_\beta^\beta\right)^5 P^{-1}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 1/2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2^5 & 0 & 0 \\ 0 & 3^5 & 0 \\ 0 & 0 & 4^5 \end{bmatrix} \begin{bmatrix} 0 & -2 & 2 \\ -1 & 4 & -1 \\ 1 & -2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2586 & -4264 & -422 \\ 781 & -1108 & -211 \\ 781 & -1140 & -179 \end{bmatrix}.$$

Definition

Diagonalizable Matrices:

An $n \times n$ matrix A is *diagonalizable* if there is a matrix D and an invertible matrix P such that

diagonal matrix.

$$A = \cancel{P^{-1}DP}$$

$$PDP^{-1}$$

Facts:

- Let A be an $n \times n$ matrix.
- Let λ_1, λ_2, ..., λ_k be the eigenvalues of A.
- Let E_{λ_1}, E_{λ_2}, ..., E_{λ_k} be the eigenspaces associated to each eigenvalue.

If $\dim(E_{\lambda_1}) + \dim(E_{\lambda_2}) + \cdots + \dim(E_{\lambda_k}) = n$, then A is diagonalizable.

EXAMPLE 3. Is the matrix from Example 2 diagonalizable?

A: was a 3×3 matrix $\to$ $n=3$

$\lambda_1 = 2$, $\lambda_2 = 3$, $\lambda_3 = 4$.

Also,

$\begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix}$ basis for E_2 $\to$ $\dim(E_2) = 1$

$+$

$\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ basis for E_3 $\to$ $\dim(E_3) = 1$

$+$

$\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$ basis for E_4 $\to$ $\dim(E_4) = 1$

$\overline{3}$

So, $\dim(E_1) + \dim(E_2) + \dim(E_3) = 3 = n$

$\to$ A is diagonalizable.

Example 4. Is the matrix

$$A = \begin{bmatrix} 1 & -2 & -6 \\ -2 & 2 & -5 \\ 2 & 1 & 8 \end{bmatrix}$$

diagonalizable? If so, determine the invertible matrix P such that PAP^{-1} $\cancel{P^{-1}AP}$ is a diagonal matrix.

1) Eigenvalues.

$\det(\lambda I - A) = \lambda^3 - 11\lambda^2 + 39\lambda - 45 = (\lambda-5)(\lambda-3)^2$

$\to \lambda = 5 \quad , \lambda = 3$ (mult. is 2)

2) Eigen Spaces.

E_5 $(\lambda=5)$ Solve $(5I - A)v = 0$ $\quad v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

$\to \quad v = z\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \to \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ basis for E_5.

$\to \dim(E_5) = 1$

E_5 $(\lambda=3)$ Solve $(3I - A)v = 0$, $\quad v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

$\to \quad v = z\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \to \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$ basis for E_3

$\to \quad \dim(E_3) = 1$

So,

$\dim(E_5) + \dim(E_3) = 2 \neq 3 (= n)$

$\to$ A is not diagonalizable

Example 5. Is the matrix

$1+0i \quad -1+0i$

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$1+0i \quad 1+0i$

diagonalizable? If so, determine the invertible matrix P such that ~~$P^{-1}AP$~~ PAP^{-1} is a diagonal matrix.

1) Eigenvalues.

$\det(\lambda I - A) = \lambda^2 - 2\lambda + 2 \to \lambda_1 = 1+i.$ $\lambda_2 = 1-i.$ (Ex. 7 in Sec. 5.4)

2) Eigenspaces. (Ex. 7).

E_{1+i} $\quad v = x\begin{bmatrix} 1 \\ -i \end{bmatrix} \to \begin{bmatrix} 1 \\ -i \end{bmatrix}$ basis for E_{1+i}.

is complex

E_{1-i} $\quad v = x\begin{bmatrix} 1 \\ i \end{bmatrix} \to \begin{bmatrix} 1 \\ i \end{bmatrix}$ basis for E_{1-i}

is complex.

$$\dim(E_{1+i}) + \dim(E_{1-i}) = 1 + 1 = 2 = n$$

$\to$ A is diagonalizable.

3) Change of basis.

$$P = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \quad \& \quad P^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ i/2 & -i/2 \end{bmatrix}$$

$$D = [A]_\beta^\beta = P^{-1}AP = \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix}.$$

In general:

An $n \times n$ matrix A is *similar* to an $n \times n$ matrix B if there is an invertible $n \times n$ matrix P such that

$$B = P^{-1}AP.$$

Notation: $A \sim B$ means that A is similar to B.

Facts:

- If A is similar to B and B is similar to C, then A is similar to C.
- If P is the change of bases matrix from α to β and T is a linear transformation, then $[T]_\beta^\beta = P^{-1}[T]_\alpha^\alpha P$. So $[T]_\beta^\beta \sim [T]_\alpha^\alpha$.

Question:

For non-diagonalizable matrices, can we reduce them to a simple form?

In other words, can we find a matrix B, as simple as possible, such that $B \sim A$?

Answer: Yes! We will replace the diagonal form by the Jordan canonical form.

Jordan blocks

A Jordan block is a square matrix A taking the following shape:

$$A = \begin{bmatrix} \mu & 1 & 0 & \cdots & 0 & 0 \\ 0 & \mu & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \mu & 1 \\ 0 & 0 & 0 & \cdots & 0 & \mu \end{bmatrix}.$$

1x1 Jordan block

$A = [\mu]$

Why are these type of matrices important?

EXAMPLE 6. Let A be the matrix

$$A = \begin{bmatrix} \mu & 1 & 0 \\ 0 & \mu & 1 \\ 0 & 0 & \mu \end{bmatrix}.$$

(a) Compute $\det(\lambda I - A)$. (b) Find the dimension of the eigenspaces.

(a) $\det(\lambda I - A) = \begin{vmatrix} \lambda-\mu & -1 & 0 \\ 0 & \lambda-\mu & -1 \\ 0 & 0 & \lambda-\mu \end{vmatrix} = (\lambda-\mu)^3 = 0$

$\to \lambda = \mu$ (alg. mult. = 3)

(b) $\underline{E_\mu}$. $(\mu I - A)v = 0$, $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$\mu I - A = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \left[\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to$ $y=0$, $z=0$, x free

$\to v = x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. $\to \dim(E_\mu) = 1$.

Remark:

- a $n \times n$ Jordan block associated to a number μ has only one eigenvalue.
- The algebraic multiplicity of this eigenvalue is necessarily equal to n.
- We always have $\dim(E_\mu) = 1$ for an $n \times n$ Jordan block. geo. mult.
- Jordan blocks are the building blocks for the set of matrices that can't be diagonalizable.

Reduction to Jordan Blocks

Example 7. We know that the matrix

$$A = \begin{bmatrix} 1 & -2 & -6 \\ -2 & 2 & -5 \\ 2 & 1 & 8 \end{bmatrix}$$

is not diagonalizable. Find a matrix B, not necessarily a diagonal matrix, such that A is similar to B.

1) Eigenvalues.

From Ex.4, we know

$\lambda_1 = 5$ (alg. mult $= 1$)

$\lambda_2 = 3$ (alg. mult $= 2$)

2) Eigenspaces.

E_5 $(\lambda = 5)$ $\dim(E_5) = 1$ (geo. mult $= 1$)

E_3 $(\lambda = 5)$ $\dim(E_3) = 1$, (geo. mult $= 1$)

$1 + 1 = 2 \neq 3$

3) Jordan Canonical Form.

$$B = \begin{bmatrix} \boxed{5} & 0 & 0 \\ 0 & 3 & * \\ 0 & 0 & 3 \end{bmatrix}$$

J_5, J_3

2×2 Jordan block

$$\begin{bmatrix} \mu & 1 \\ 0 & \mu \end{bmatrix} \to \mu = 3$$

J_3: two choices.

1) geo. mult. $= 1$ $\to$ $J_3 = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$

2) geo. mult $= 2$ $\to$ $J_3 = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

so, we take the 2nd option

$$\Rightarrow B = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}.$$

General Procedure: Suppose A is an $n \times n$ matrix.

- Express $\det(\lambda I - A)$ as

$$\det(\lambda I - A) = (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_k)^{m_k}$$

  where m_1 is the multiplicity of λ_1, m_2 is the multiplicity of λ_2, ..., m_k is the multiplicity of λ_k.

- For each λ_j, write

$$A_j = \begin{bmatrix} J_{m_{j-1}+1} & 0 & \cdots & 0 \\ 0 & J_{m_{j-1}+2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_{m_j} \end{bmatrix}.$$

  $m_{j-1}+$

  Ex.: $m_1 = 2$

  $A_1 = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}$

  $m_2 = 2$

  $A_2 = \begin{bmatrix} J_3 & 0 \\ 0 & J_4 \end{bmatrix}$

  where each J_p, for $p = m_{j-1} + 1, \ldots m_j$, is a Jordan block

$$J_p = \begin{bmatrix} \lambda_j & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_j & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_j & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_j \end{bmatrix}.$$

- Then the Jordan Canonical Form (JCF) is

$$B = \begin{bmatrix} A_1 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ 0 & 0 & A_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_k \end{bmatrix}$$

- The invertible matrix P such that $B = P^{-1}AP$ is more complicated to find. In theory, the method to find P uses the notion of a **generalized eigenvector**. In our situation, we will use Python to find this matrix P.

If you want to know more on the generalized eigenvectors and the Jordan Canonical Form, I suggest to take a look at the following references:

- A more math article: *Down With Determinants!* by Sheldon Axler, `https://www.maa.org/sites/default/files/pdf/awards/Axler-Ford-1996.pdf`.
- A Youtube video: `https://www.youtube.com/watch?v=GVixvieNnyc`.

Example 8. Let A be an 7×7 matrix with the following eigenvalues:

$$\{\underbrace{1,1,1,1}, \underbrace{2,2}, 3\}. \qquad \det(\lambda I - A) = (\lambda-1)^4 (\lambda-2)^2 (\lambda-3)$$

Give the possible Jordan canonical form B of the matrix A.

Write $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$.

$$B = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix}$$

<u>shape of A_1</u>

alg mult = 4.

$$A_1 = \begin{bmatrix} 1 & * & 0 & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix} :$$

1) <u>geom. mult = 1</u> $\quad A_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

2) <u>geom. mult = 2.</u> $\quad A_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

3) <u>geom. mult = 3</u> $\quad A_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

4) <u>geom. mult = 4</u> $\quad A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

<u>shape of A_2.</u>

alg. mult = 2

$$A_2 = \begin{bmatrix} 2 & * \\ 0 & 2 \end{bmatrix} :$$

1) <u>geom. mult. = 1</u> $\quad A_2 = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

2) <u>geom. mult = 2</u> $\quad A_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

Shape of A_3

alg. mult. $=1$

$$A_3 = [3]$$

First possible B is:

$$B = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

In total, there are $4 \times 2 \times 1 = 8$ possible Jordan Canonical Forms.