

MATH 307

CHAPTER 5

SECTION 5.5: SIMILAR MATRICES, DIAGONALIZATION, AND JORDAN CANONICAL FORM

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Motivation

EXAMPLE 1. Let A be the 3×3 matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

Then, (a) compute A^5 (b) find the eigenvalues of A (c) find a basis for each eigenspace.

Remarks

- It is pretty easy to deal with diagonal matrices.
- Our goal is to try to transform a general matrix into a diagonal matrix.

EXAMPLE 2. Let A be the following 3×3 matrix

$$A = \begin{bmatrix} 6 & -4 & -2 \\ 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Find (a) the eigenvalues of A (b) a basis for each eigenspace (c) compute A^5 .

Definition

Diagonalizable Matrices:

An $n \times n$ matrix A is *diagonalizable* if there is a matrix D and an invertible matrix P such that

$$A = P^{-1}DP$$

Facts:

- Let A be an $n \times n$ matrix.
- Let $\lambda_1, \lambda_2, \dots, \lambda_k$ be the eigenvalues of A .
- Let $E_{\lambda_1}, E_{\lambda_2}, \dots, E_{\lambda_k}$ be the eigenspaces associated to each eigenvalue.

If $\dim(E_{\lambda_1}) + \dim(E_{\lambda_2}) + \dots + \dim(E_{\lambda_k}) = n$, then A is diagonalizable.

EXAMPLE 3. Is the matrix from Example 2 diagonalizable?

EXAMPLE 4. Is the matrix

$$A = \begin{bmatrix} 1 & -2 & -6 \\ -2 & 2 & -5 \\ 2 & 1 & 8 \end{bmatrix}$$

diagonalizable? If so, determine the invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

EXAMPLE 5. Is the matrix

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

diagonalizable? If so, determine the invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

In general:

An $n \times n$ matrix A is *similar* to an $n \times n$ matrix B if there is an invertible $n \times n$ matrix P such that

$$B = P^{-1}AP.$$

Notation: $A \sim B$ means that A is similar to B .

Facts:

- If A is similar to B and B is similar to C , then A is similar to C .
- If P is the change of bases matrix from α to β and T is a linear transformation, then $[T]_{\beta}^{\beta} = P^{-1}[T]_{\alpha}^{\alpha}P$. So $[T]_{\beta}^{\beta} \sim [T]_{\alpha}^{\alpha}$.

Question:

For non-diagonalizable matrices, can we reduce them to a simple form?

In other words, can we find a matrix B , as simple as possible, such that $B \sim A$?

Answer: Yes! We will replace the diagonal form by the Jordan canonical form.

Jordan blocks

A Jordan block is a square matrix A taking the following shape:

$$A = \begin{bmatrix} \mu & 1 & 0 & \cdots & 0 & 0 \\ 0 & \mu & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \mu & 1 \\ 0 & 0 & 0 & \cdots & 0 & \mu \end{bmatrix}.$$

Why are these type of matrices important?

EXAMPLE 6. Let A be the matrix

$$A = \begin{bmatrix} \mu & 1 & 0 \\ 0 & \mu & 1 \\ 0 & 0 & \mu \end{bmatrix}.$$

(a) Compute $\det(\lambda I - A)$. (b) Find the dimension of the eigenspaces.

Remark:

- a $n \times n$ Jordan block associated to a number μ has only one eigenvalue.
- The algebraic multiplicity of this eigenvalue is necessarily equal to n .
- We always have $\dim(E_\mu) = 1$ for an $n \times n$ Jordan block.
- Jordan blocks are the building blocks for the set of matrices that can't be diagonalizable.

Reduction to Jordan Blocks

EXAMPLE 7. We know that the matrix

$$A = \begin{bmatrix} 1 & -2 & -6 \\ -2 & 2 & -5 \\ 2 & 1 & 8 \end{bmatrix}$$

is not diagonalizable. Find a matrix B , not necessarily a diagonal matrix, such that A is similar to B .

General Procedure: Suppose A is an $n \times n$ matrix.

- Express $\det(\lambda I - A)$ as

$$\det(\lambda I - A) = (\lambda - \lambda_1)^{m_1} (\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_k)^{m_k}$$

where m_1 is the multiplicity of λ_1 , m_2 is the multiplicity of λ_2 , \dots , m_k is the multiplicity of λ_k .

- For each λ_j , write

$$A_j = \begin{bmatrix} J_{m_{j-1}+1} & 0 & \cdots & 0 \\ 0 & J_{m_{j-1}+2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_{m_j} \end{bmatrix}.$$

where each J_p , for $p = m_{j-1} + 1, \dots, m_j$, is a Jordan block

$$J_p = \begin{bmatrix} \lambda_j & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_j & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_j & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_j \end{bmatrix}.$$

- Then the Jordan Canonical Form (JCF) is

$$B = \begin{bmatrix} A_1 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ 0 & 0 & A_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_k \end{bmatrix}$$

- The invertible matrix P such that $B = P^{-1}AP$ is more complicated to find. In theory, the method to find P uses the notion of a **generalized eigenvector**. In our situation, we will use Python to find this matrix P .

If you want to know more on the generalized eigenvectors and the Jordan Canonical Form, I suggest to take a look at the following references:

- A more math article: *Down With Determinants!* by Sheldon Axler, <https://www.maa.org/sites/default/files/pdf/awards/Axler-Ford-1996.pdf>.
- A Youtube video: <https://www.youtube.com/watch?v=GVixvieNnyc>.

EXAMPLE 8. Let A be an 7×7 matrix with the following eigenvalues:

$$\{1, 1, 1, 1, 2, 2, 3\}.$$

Give the possible Jordan canonical form B of the matrix A .