# Math 307

## Chapter 5

### Section 5.1: Linear Transformations

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Convention:

The addition and scalar multiplication on the set of column vectors $\mathbb{R}^n$ are the usual ones that make $\mathbb{R}^n$ a vector space. If the addition is changed, it will be mentioned explicitly in the text.

## Definition

If $V$ and $W$ are vector spaces, a function $T : V \to W$ is called a **linear transformation** if, for all vectors $u$ and $v$ in $V$ and all scalars $c$, the following two properties are satisfied:

1. $T(u + v) = T(u) + T(v)$;
2. $T(cv) = cT(v)$.

EXAMPLE 1. Let $A$ be an $m \times n$ matrix. We define $T : \mathbb{R}^n \to \mathbb{R}^m$ by

$$T(X) := AX$$

where $X$ is an $n \times 1$ column vector. Verify that the function $T$ is a linear transformation.

1) $X$ & $Y$ be two $n\times 1$ column vectors.

$T(X+Y) = A(X+Y) = AX + AY = T(X) + T(Y)$. ✓

2) $c$ be a number.

$T(cX) = A(cX) = c\,AX = c\,T(X)$. ✓

So $T$ is linear.

**Example 2.** Verify if the given function $T:\mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y-z \\ x+2y+z \end{bmatrix}$$

is a linear transformation.

1) $u = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ & $v = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$.

then $T(u+v) = T\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{bmatrix} = \begin{bmatrix} x_1+x_2+y_1+y_2-(z_1+z_2) \\ x_1+x_2+2(y_1+y_2)+z_1+z_2 \end{bmatrix}$

$$= \begin{bmatrix} x_1+y_1-z_1+x_2+y_2-z_2 \\ x_1+2y_1+z_1+x_2+2y_2+z_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1+y_1-z_1 \\ x_1+2y_1+z_1 \end{bmatrix} + \begin{bmatrix} x_2+y_2-z_2 \\ x_2+2y_2+z_2 \end{bmatrix}$$

$$= T(u) + T(v). \checkmark$$

2) $c$ is a number.

$T(cu) = T\begin{bmatrix} cx_1 \\ cy_1 \\ cz_1 \end{bmatrix} = \begin{bmatrix} cx_1+cy_1-cz_1 \\ cx_1+2cy_1+cz_1 \end{bmatrix}$

$$= \begin{bmatrix} c(x_1+y_1-z_1) \\ c(x_1+2y_1+z_1) \end{bmatrix}$$

$$= c\begin{bmatrix} x_1+y_1-z_1 \\ x_1+2y_1+z_1 \end{bmatrix} = cT(u) \checkmark$$

So $T$ is linear.

EXAMPLE 3. Let $D(a,b)$ be the subspace of $F(a,b)$ of differentiable function on the interval $(a,b)$. Define the function $T : D(a,b) \to F(a,b)$ by

$$T(f) := f'$$

meaning that $T(f)(x) = f'(x)$ for every $x$ in $(a,b)$. Verify that $T$ is a linear transformation.

1) Let $f$ & $g$ be two diff. fcts. then

$T(f+g) = (f+g)' = f'+g'$ (Calculus I)

$= T(f) + T(g)$. ✓

2) $c$ a number

$T(cf) = (cf)' = cf'$ (Calculus I)

$= cT(f)$ ✓.

So $T$ is a linear transformation.

Remark: The linear transformation in the previous example is called a differential operator and is quite useful in the theory of ODE and PDE.

If $T : V \to W$ is a linear transformation, then we can prove that

- $T(0) = 0$;
- $T(-v) = -T(v)$ for any vector $v$ in $V$;
- $T(u - v) = T(u) - T(v)$ for any vector $u$, $v$ in $V$.

There is another important property of a linear transformation which we shall illustrate by an example.

Example 4. Suppose that $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation so that

$$T\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}2\\3\end{bmatrix}, \quad T\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{pmatrix}0\\3\end{pmatrix} \quad \text{and} \quad T\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}0\\2\end{bmatrix}.$$

Find the value of $T\begin{bmatrix}1\\3\\0\end{bmatrix}$.

We can verify that $\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}$ & $\begin{bmatrix}1\\0\\1\end{bmatrix}$ forms a basis for $\mathbb{R}^3$.

1) $\begin{bmatrix}1\\3\\0\end{bmatrix} = 2\begin{bmatrix}1\\1\\0\end{bmatrix} + \begin{bmatrix}0\\1\\1\end{bmatrix} - \begin{bmatrix}1\\0\\1\end{bmatrix}$. Since $T$ is linear:

$$T\begin{bmatrix}1\\3\\0\end{bmatrix} = T\left(2\begin{bmatrix}1\\1\\0\end{bmatrix} + \begin{bmatrix}0\\1\\1\end{bmatrix}\right) - T\begin{bmatrix}1\\0\\1\end{bmatrix}$$

$$= 2T\begin{bmatrix}1\\1\\0\end{bmatrix} + T\begin{bmatrix}0\\1\\1\end{bmatrix} - T\begin{bmatrix}1\\0\\1\end{bmatrix}$$

$$= 2\begin{bmatrix}2\\3\end{bmatrix} + \begin{bmatrix}0\\3\end{bmatrix} - \begin{bmatrix}0\\2\end{bmatrix} = \begin{bmatrix}4\\7\end{bmatrix}$$

<u>Fact</u>: If $v_1, v_2, \ldots, v_n$ form a basis, then the values of a linear transformation $T$ is determined by its value on $v_1, v_2, \ldots, v_n$ because for any $v \in V$, we have

$$T(v) = T(c_1v_1 + c_2v_2 + \cdots + c_nv_n) = c_1T(v_1) + c_2T(v_2) + \cdots + c_nT(v_n).$$

## Kernel

If $T : V \to W$ is a linear transformation, then the **kernel** of $T$ is the set of all vectors $v$ in $V$ such that $T(v) = 0$. In set notation:

$$\ker(T) = \{v \in V \ : \ T(v) = 0\}.$$

This is in general a subspace of $V$.

EXAMPLE 5. Find a basis for the kernel of the linear transformation

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + y - z \\ x + 2y + z \end{bmatrix}.$$

We want to find all $u = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ s.t. $T(u) = 0$.

$$\Leftrightarrow \begin{bmatrix} x+y-z \\ x+2y+z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{cases} x+y-z = 0 \\ x+2y+z = 0 \end{cases}$$

Syst. of lin. equations!

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & 2 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix} \to \begin{matrix} x = -3z \\ y = -2z \end{matrix}$$

So,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3z \\ -2z \\ z \end{bmatrix} = z\begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$$

Thus, $\ker T = \left\{ z\begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} : z \in \mathbb{R} \right\}$

and a basis is $\begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$.

Remark: The kernel of a transformation is related to the solutions of the system of linear equations $AX = 0$ when $T(X) = AX$ with $A$ an $m \times n$ matrix. In this particular situation, the kernel $\ker(T)$ is called the **null space** of $A$ also denoted by $NS(A)$. In other words, we have

$$NS(A) = \ker(T).$$

## Range

If $T : V \to W$ is a linear transformation, then the **range** of $T$ is the set of all vectors $T(v)$ where $v$ is in $V$. In set notation:

$$\text{range}\,(T) = \{T(v)\,:\, v \in V\}.$$

This is in general a subspace of $W$.

Facts:

- Finding a basis for the range of a tranformation $T$ given by $T(X) = AX$ where $A$ is an $m \times n$ matrix is equivalent to finding a basis for the spanning set of the columns of the matrix $A$.
- The subspace spans by the column of a matrix $A$ is called the **column space** and is denoted by $CS(A)$.

EXAMPLE 6. Find a basis for the range of the linear transformation of Example 5 using the column space of a certain matrix.

We can express $T$ as $T(X) = AX$ where

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix}$$

Now, we see that

$$T(X) = A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\begin{bmatrix} 1 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ 2 \end{bmatrix} + z\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

So, the range$(T)$ is determined by the

$$\text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} = U$$

We simply need to find a basis for $U$.

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -3\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

& $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ are lin. indep.

So a basis for range$(T)$ or $CS(A)$ is

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

In summary, to find range$(T)$ or $CS(A)$ for a linear transformation of the form $T(X) = AX$, we follow these steps:

- express $T(v)$ as a linear combination of column vectors $v_1, v_2, \ldots, v_n$.
- Write each vector in a matrix $A = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$.
- Find the RREF of $A$.
- The column with the first 1 in a row will be a pivot and the vector corresponding to the column will be part of the basis.

Fact: We call $\dim(CS(A)) = \dim(\text{range}(T))$ the **rank** of the matrix $A$ or transformation $T$.

## Rank-Nullity Identity

We define

- the **nullity** of a linear transformation $T$ as the dimension of $\ker(T)$.
- the **rank** of a linear transformation $T$ as the dimension of range$(T)$.

Here is an important identity relating the rank and the nullity of a linear transformation.

THEOREM 7. If $T : V \to W$ is a linear transformation, then

$$\dim(\ker(T)) + \dim(\text{range}(T)) = \dim(V).$$

Remark: For an $m \times n$ matrix, we obtain

$$\dim(NS(A)) + \dim(CS(A)) = n.$$

EXAMPLE 8. Verify the Rank-Nullity Identity for the matrix in Example 5 and Example 6.

We found that $\begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$ was a basis for ker(T) or NS(A). So, $\dim(\ker T) = 1$.

We also found that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ & $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ form a basis for range (T) or CS(A). So

$\dim(\text{range}(T)) = 2$

$\Rightarrow \quad \dim(\ker(T)) + \dim(\text{range}(T)) = 1 + 2 = 3$ ✓