Section 5.4 — Problem 2 — 5 points

The eigenvalues are the solutions to the equation

$$\det\left(\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix} - \begin{bmatrix}4 & -4\\ 1 & 0\end{bmatrix}\right) = 0.$$

We have

$$\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix} - \begin{bmatrix}4 & -4\\ 1 & 0\end{bmatrix} = \begin{bmatrix}\lambda - 4 & 4\\ -1 & \lambda\end{bmatrix}$$

and therefore

$$\det\left(\begin{bmatrix}\lambda - 4 & 4\\ -1 & \lambda\end{bmatrix}\right) = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2.$$

The eigenvalue is $\lambda = 2$ with an algebraic multiplicity of 2.

The eigenspace E_2 is the nullspace (or kernel) of the matrix $2I - A$. We have

$$2I - A = \begin{bmatrix}-2 & 4\\ -1 & 2\end{bmatrix}.$$

We therefore have to solve the system

$$\begin{bmatrix}-2 & 4\\ -1 & 2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}.$$

This corresponds to a system of linear equations. We solve it to find

$$\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}2y\\ y\end{bmatrix} = y\begin{bmatrix}2\\ 1\end{bmatrix}.$$

Therefore, a basis for the eigenspace E_2 is

$$\begin{bmatrix}2\\ 1\end{bmatrix}.$$

Section 5.4 — Problem 10 — 5 points

We have to find the eigenvalues first. Those are the solutions to the characteristic equation

$$\det(\lambda I - A) = 0$$

with A as in the problem. We have

$$\lambda I - A = \begin{bmatrix} \lambda+6 & 0 & 8 \\ 4 & \lambda-2 & 4 \\ -4 & 0 & \lambda-6 \end{bmatrix}$$

and therefore

$$\det(\lambda I - A) = \lambda^3 - 2\lambda^2 - 4\lambda + 8 = (\lambda-2)^2(\lambda+2).$$

The eigenvalues are $\lambda_1 = 2$ (algebraic multiplicity 2) and $\lambda_2 = -2$ (algebraic multiplicity 1).

We have to find a basis for each eigenspace.

$\underline{E_2}$ We have to find all the solutions v to $(2I - A)v = 0$. In matrix form, this is

$$(2I - A)v = \begin{bmatrix} 8 & 0 & 8 \\ 4 & 0 & 4 \\ -4 & 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

After solving the system, we see that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -z \\ y \\ z \end{bmatrix} = y\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore a basis for the eigenspace E_2 is the following list:

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

$\underline{E_{-2}}$ We have to find all the solutions v to $(-2I - A)v = 0$. In matrix form, this is

$$(-2I - A)v = \begin{bmatrix} 4 & 0 & 8 \\ 4 & -4 & 4 \\ -4 & 0 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

After solving the system, we obtain $x = -2z$, $y = -z$ and z is a free variable. Therefore, we obtain

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2z \\ -z \\ z \end{bmatrix} = z\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}.$$

Therefore, a basis for E_{-2} is $\begin{bmatrix} -2 & -1 & 1 \end{bmatrix}^\top$.

Section 5.4 — Problem 16 — 10 points

The first step is to find the eigenvalues. Those are the solution to the characteristic equation:

$$\det(\lambda I - A) = 0.$$

With the data from the problem, we obtain

$$\det(\lambda I - A) = \lambda^2 + 4 = 0.$$

We find the roots to be $\lambda = \pm 2i$ where $i = \sqrt{-1}$, the imaginary unit in the complex numbers.

The second step is to find bases for the eigenspaces.

$\underline{E_{2i}}$ We have to find all the solutions to $(2iI - A)v = 0$. In matrix form, we have

$$(2iI - A)v = \begin{bmatrix} 4+2i & -5 \\ 4 & -4+2i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We can solve this system by finding the RREF of the system:

$$\begin{bmatrix} 4+2i & -5 & 0 \\ 4 & -4+2i & 0 \end{bmatrix} \sim \begin{bmatrix} 4+2i & -5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We therefore find that $5y = (4+2i)x$ which gives

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ (4+2i)x/5 \end{bmatrix} = x/5 \begin{bmatrix} 5 \\ 4+2i \end{bmatrix}.$$

A basis for E_{2i} is therefore

$$\begin{bmatrix} 5 \\ 4+2i \end{bmatrix}.$$

Another possibility for the basis is the vector $\begin{bmatrix} 1-i/2 & 1 \end{bmatrix}^\top$. The relation between the two vectors is

$$\begin{bmatrix} 5 \\ 4+2i \end{bmatrix} = (4+2i) \begin{bmatrix} 1-i/2 \\ 1 \end{bmatrix}.$$

$\underline{E_{-2i}}$ We have to find all the solutions to $(-2iI - A)v = 0$. In matrix form, we have

$$(-2iI - A)v = 0 = \begin{bmatrix} 4-2i & -5 \\ 4 & -4-2i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We can see that the first row is a multiple of the second row, Therefore, the system has infinitely many solutions and we have, from the first equation,

$$(4-2i)x - 5y = 0 \quad \Rightarrow \quad y = (4-2i)x/5.$$

Therefore, we obtain

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ (4-2i)x/5 \end{bmatrix} = x/5 \begin{bmatrix} 5 \\ 4-2i \end{bmatrix}.$$

A basis for E_{-2i} is $\begin{bmatrix} 5 & 4-2i \end{bmatrix}^\top$. Another possibility is the vector $\begin{bmatrix} 1+i/2 & 1 \end{bmatrix}^\top$.

Section 5.4 — Problem 25 — 5 points

Suppose that v is an eigenvector for the square matrix A with the associated eigenvalue $\lambda = r$. We want to show that v is also an eigenvector for A^2, but for the eigenvalue $\lambda = r^2$. Since v is an eigenvector associated to $\lambda = r$, we have

$$Av = rv.$$

If we apply a second time the matrix A on the last equation, we get

$$A^2 v = AAv = A(rv) = r(Av)$$

where the last equality comes from the associativity of the matrix multiplication and the scalar multiplication. Since v is an eigenvector of A associated to $\lambda = r$, we have

$$r(Av) = r(rv) = r^2 v.$$

Therefore, we obtain

$$A^2 v = r^2 v.$$

This is what we wanted to prove because it shows that v is an eigenvector of A^2 with the eigenvalue $\lambda = r^2$.

Section 5.5 — Problem 2 — 5 points

In Exercise 2, Section 5.4, the matrix A was

$$A = \begin{bmatrix} 4 & -4 \\ 1 & 0 \end{bmatrix}.$$

The number $\lambda = 2$ is the only eigenvalue of A. Also, we found that $\begin{bmatrix} 2 & 1 \end{bmatrix}^\top$ was a basis for the eigenspace E_2. We therefore have $\dim(E_2) = 1 \neq 2 = \dim(\mathbb{R}^2)$ and the matrix A is not diagonalizable.

Section 5.5 — Problem 10 — 10 points

The matrix in Exercise 10, Section 5.4 was

$$A = \begin{bmatrix} -6 & 0 & -8 \\ -4 & 2 & -4 \\ 4 & 0 & 6 \end{bmatrix}.$$

We found that the numbers $\lambda_1 = 2$ and $\lambda_2 = -2$ were the eigenvalues of A. We also found that the vectors $\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^\top$, $\begin{bmatrix} -1 & 0 & 1 \end{bmatrix}^\top$ form a basis for E_2 and the vector $\begin{bmatrix} -1/2 & 1 & 1 \end{bmatrix}^\top$ form a basis for E_{-2}. Therefore, we have

$$\dim(E_2) + \dim(E_{-2}) = 2 + 1 = 3 = \dim(\mathbb{R}^3).$$

The condition is satisfied and the matrix A is diagonalizable.

To find P, we have to put the basis of the eigenspaces in a column. To ovoid fractions in the matrix P, we will scale by 2 the vector $\begin{bmatrix} -1/2 & 1 & 1 \end{bmatrix}$. Therefore, we obtain

$$P = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}.$$

Using Python, we compute the inverse P^{-1}:

$$P^{-1} = \begin{bmatrix} -2 & 1 & -2 \\ -2 & 0 & -1 \\ 1 & 0 & 1 \end{bmatrix}.$$

The matrix D similar to A is

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

We therefore have

$$A = PDP^{-1}.$$

Section 5.5 — Problem 16 — 10 points

From the work done in Exercise 16, Section 5.4, the eigenvalues of A were $2i$ and $-2i$. We found that the vector $\begin{bmatrix}5 & 4+2i\end{bmatrix}^\top$ forms a basis for E_{2i} and the vector $\begin{bmatrix}5 & 4-2i\end{bmatrix}^\top$ forms a basis for E_{-2i}. Therefore, we have

$$\dim(E_{2i}) + \dim(E_{-2i}) = 1 + 1 = \dim(\mathbb{C}^2).$$

The condition is satisfied and therefore the matrix A is diagonalizable (over the complex numbers!).

The matrix P is obtained from joining the basis of E_{2i} and E_{-2i} in a matrix:

$$P = \begin{bmatrix} 5 & 5 \\ 4+2i & 4-2i \end{bmatrix}.$$

The inverse P^{-1} is computed with Python and the result is

$$P^{-1} = \begin{bmatrix} \frac{1}{10} + \frac{i}{5} & -\frac{i}{4} \\ \frac{1}{10} - \frac{i}{5} & \frac{i}{4} \end{bmatrix}.$$

The matrix D similar to A is

$$D = \begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix}$$

and $A = PDP^{-1}$.

Section 5.5 — Problem 24 — 5 points

The eigenvalues of the matrix are $\lambda_1 = 3$ and $\lambda_2 = -4$. Therefore, the canonical Jordan form is of the following shape:

$$D = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}.$$

We now list the possibilities for A_1 and A_2 according to the geometric multiplicities of each eigenvalues.

1. The algebraic multiplicity of the eigen value 3 is 2 (because we have two times the factor $(\lambda - 3)$ in the characteristic polynomial). Therefore the matrix A_1 has dimension 2×2 with the number 3 on the main diagonal.

   Geometric Multiplicity $= 1$. In this case, we have

   $$A_1 = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}.$$

   Geometric Multiplicity $= 2$. In this case, we simply have a diagonal matrix:

   $$A_1 = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}.$$

2. The algebraic multiplicity of the eigenvalue -4 is 4 because the factor $(\lambda + 4)$ appears four times in the characteristic polynomial. Therefore the matrix A_2 has dimensions 4×4 with the number -4 on the main diagonal.

   Geometric Multiplicity $= 1$. In this case, we have

   $$A_2 = \begin{bmatrix} -4 & 1 & 0 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & -4 \end{bmatrix}.$$

   Geometric Multiplicity $= 2$. In this case, we have

   $$A_2 = \begin{bmatrix} -4 & 1 & 0 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}.$$

   Geometric Multiplicity $= 3$. In this case, we have

   $$A_2 = \begin{bmatrix} -4 & 1 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}.$$

Geometric Multiplicity = 4. In this case, we simply obtain a diagonal matrix:

$$A_2 = \begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

.

Therefore, there are 8 possibilities for the Jordan Canonical Form:

1. $D = \begin{bmatrix} 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 1 & 0 & 0 \\ 0 & 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & -4 \end{bmatrix}.$

2. $D = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 1 & 0 & 0 \\ 0 & 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & -4 \end{bmatrix}.$

3. $D = \begin{bmatrix} 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 1 & 0 & 0 \\ 0 & 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 \end{bmatrix}.$

4. $D = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 1 & 0 & 0 \\ 0 & 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 \end{bmatrix}.$

5. $D = \begin{bmatrix} 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 1 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 \end{bmatrix}.$

6. $D = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 1 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 \end{bmatrix}.$

7. $D = \begin{bmatrix} 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 \end{bmatrix}.$

8. $D = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 \end{bmatrix}.$

Section 5.5 — Problem 32 — 5 points

a) We have to find an invertible matrix P such that $A = PAP^{-1}$. We can take $P = I$ where I is the identity matrix. We therefore have $P^{-1} = I$ and $A = IAI$. So $A \sim A$.

b) Suppose that $A \sim B$. Then there is an invertible matrix P such that $A = PBP^{-1}$. If we multiply by P^{-1} on the left and by P on the right, we obtain

$$P^{-1}AP = P^{-1}PBP^{-1}P = IBI = B.$$

Therefore, using the matrix P^{-1} in the definition of similarity between matrices, we see that $B \sim A$.

c) Suppose that $A \sim B$ and $B \sim C$. Therefore, there are invertible matrices P and Q such that

$$A = PBP^{-1} \quad \text{and} \quad B = QCQ^{-1}.$$

We therefore have

$$A = P(QCQ^{-1})P^{-1} = PQCQ^{-1}P^{-1}.$$

Since $Q^{-1}P^{-1} = (PQ)^{-1}$, we can rewrite the previous equation as

$$PQCQ^{-1}P^{-1} = PQC(PQ)^{-1}.$$

Since P and Q are invertible, then PQ is an invertible matrix. Using PQ as the matrix P in the definition of similar matrices, we see that $A \sim C$.

TOTAL (POINTS): 60.