Math 307

Chapter 2

Section 2.4: Dimension; ~~Nullspace, Row Space, and Column Space~~

Contents

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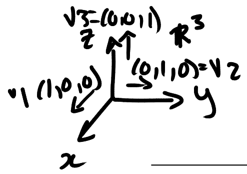
↦ v_1, v_2, v_3 is the standard basis for $\mathbb{R}^3 \to \dim(\mathbb{R}^3)=3$



What is the Dimension of a Vector Space?

Definition

If a vector space V has a basis α of n vectors $v_1, v_2, \dots, v_n$, then the **dimension** of V is the number of elements in the basis α.

The definition comes from the following result telling us that all basis have the same number of elements.

Theorem 1. If $\overbrace{v_1, v_2, \dots, v_n}^{\alpha}$ and $\overbrace{w_1, w_2, \dots, w_m}^{\beta}$ both form bases for a vector space V, then $n = m$.

Remark:

- We use the symbol $\dim(V)$ to denote the dimension of a vector space V.
- If, for a vector space V, $\dim(V)$ is a finite number, then V is said to be **finite dimensional**.
- If, for a vector space V, $\dim(V)$ is infinite (so equal to $+\infty$), then V is said to be **infinite dimensional**.

Example 2. Which of the following vector space is finite dimensional and which one is infinite dimensional?

1. $\mathbb{R}^3$.
2. $\mathbb{R}^n$.
3. P_n.
4. P.

1) $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$ basis for $\mathbb{R}^3 \to \dim(\mathbb{R}^3) = 3$.
↳ finite dimensional.

2) $\underbrace{\left\{ \begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}, \begin{bmatrix}0\\1\\\vdots\\0\end{bmatrix}, \dots, \begin{bmatrix}0\\0\\\vdots\\1\end{bmatrix} \right\}}_{n \text{ vectors}}$ basis for $\mathbb{R}^n \to \dim(\mathbb{R}^n) = n$
↳ finite dimensial.

3) $\underbrace{\{1, x, x^2, \dots, x^n\}}_{n+1 \text{ vectors}}$ basis for $P_n \to \dim(P_n) = n+1$
↳ finite dimensional.

4) $\underbrace{\{1, x, x^2, x^3, \dots, x^n, \dots\}}_{+\infty}$ basis for $P \to \dim(P) = \infty$
↳ infinite dimensional.

Basic Facts on Basis and Dimension

For a set of vectors to be a basis, it must have the same number of elements as the dimension of the vector space.

When there are not enough vectors but we know that there are linearly independant, we can add a set of new vectors so that the new set composed of the old vectors and the new vectors become a basis.

EXAMPLE 3. Let α be the set of vectors

$$\alpha = \left\{ \overset{v_1}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}, \overset{v_2}{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}} \right\}.$$

Complete the set α with a certain number of vectors so that it becomes a basis of $\mathbb{R}^3$.

lin. ind. $\quad c_1 v_1 + c_2 v_2 = 0$

$$\begin{array}{c} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\ c_1 \quad c_2 \quad \end{array} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{matrix} c_1 = 0 \\ c_2 = 0 \end{matrix} \to \text{lin. ind.}$$

Goal: Find $v_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ s.t. $\{v_1, v_2, v_3\}$ forms a basis.

Find a, b, c s.t.

$$\begin{bmatrix} 1 & 0 & a \\ 1 & 1 & b \\ 0 & 1 & c \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \not\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

compare

$$\begin{bmatrix} 1 & 0 & a \\ 1 & 1 & b \\ 0 & 1 & c \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b-a \\ 0 & 0 & c-b+a \end{bmatrix} \to \begin{matrix} a = 0 \\ b - a = 0 \\ c - b + a = 1 \end{matrix} \to \begin{matrix} a = 0 \\ b = 0 \\ c = 1 \end{matrix}$$

Now, $v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

So, $\{v_1, v_2, v_3\}$ forms a basis for $\mathbb{R}^3$.

Remark: Given a vector space V of dimension n,

- if $v_1, v_2, \ldots, v_k$ are linearly independent vectors in V, then there exist vectors $v_{k+1}, \ldots, v_n$ so that $v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_n$ form a basis for V.

When there are to much vectors, but we know that the set of vectors span V, we can extract a basis from it.

Example 4. Let α be the set

$$\alpha = \left\{ \begin{array}{c} v_1 \\ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \end{array}, \begin{array}{c} v_2 \\ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \end{array}, \begin{array}{c} v_3 \\ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \end{array}, \begin{array}{c} v_4 \\ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \end{array} \right\}.$$

for $\mathbb{R}^3$

Extract a basis from the set α.

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \sim \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \\ \downarrow & \downarrow & \downarrow & \downarrow \end{array} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Here, extract v_1, v_2, v_3

$$\to \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$ a basis. because

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Remark: Given a vector space V of dimension n,

- if $v_1, v_2, \ldots, v_k$ span V, then there exists a subset of $v_1, v_2, \ldots, v_k$ that forms a basis of V.

Theorem 5. Suppose that V is a vector space of dimension n.

- If the vectors $v_1, v_2, \ldots, v_n$ are linearly independent, then $v_1, v_2, \ldots, v_n$ form a basis for V.
- If the vectors $v_1, v_2, \ldots, v_n$ span V, then $v_1, v_2, \ldots, v_n$ form a basis for V.

Example 6. Show that $\underbrace{x^2 - 1,\ x^2 + 1,\ x + 1}_{\text{3 vectors.}}$ form a basis for P_2. $\dim(P_2) = 3$

$$\begin{array}{c} x^2 \\ x \\ \text{cst} \end{array} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \text{basis for } P_2.$$