Math 307

Chapter 1

Section 1.6: Further Properties of Determinants

Contents

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Invertibility

A square matrix A is invertible if and only if $\det(A) \neq 0$.

Remark:

- If A is an invertible matrix, then $\det(A^{-1}) = 1/\det(A)$.
- Determinant can help to determine if a system of linear equations has a solution or not.

EXAMPLE 1. Which of the following matrices are invertible:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -4 & -3 & -2 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 4 & 5 & 9 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -1 \\ 0 & 0 & 4 \end{bmatrix}.$$

Matrix Multiplication

If A and B are two $n \times n$ matrices, then

$$\det(AB) = \det(A)\det(B).$$

EXAMPLE 2. Knowing that $\det(A) = 2$ and $\det(AB) = 32$, find the determinant of the matrix B.

Transpose

If A is a square matrix, then $\det(A^\top) = \det(A)$.

EXAMPLE 3. If A is the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & -2 \\ 0 & 0 & 1 \end{bmatrix},$$

then find the determinant of $AA^\top$.

Matrix of Cofactors

The cofactor matrix is the matrix C of all the cofactors of a given matrix A. If A has dimensions $n \times n$, then

$$C = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

where each C_{ij} is the cofactor obtained from A.

Example 4. Find the matrix of cofactors of the following matrix

$$A = \begin{bmatrix} -2 & 3 & 0 \\ 4 & 10 & 2 \\ -5 & 7 & 0 \end{bmatrix}.$$

Definition of the Adjoint

The adjoint of a matrix A of dimensions $n \times n$ is the transpose of the cofactor matrix.

Explicitly, we denote the adjoint of A by $\operatorname{adj}(A)$ and its expression is

$$\operatorname{adj}(A) = C^\top = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}.$$

EXAMPLE 5. Find the adjoint of the matrix in Example 4.

Another Way to Find the Inverse

If A is an invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A).$$

EXAMPLE 6. Find the inverse of the matrix in Example 4.

Cramer's Rule

Suppose that $AX = B$ is a system of n linear equations in n unknowns such that $\det(A) \neq 0$. Let

- A_1 be the matrix obtained from A by replacing the first column of A by B;
- A_2 be the matrix obtained from A by replacing the second column of A by B.
- etc.

Then the solutions to the system are

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \quad \ldots, \quad x_n = \frac{\det(A_n)}{\det(A)}.$$

Example 7. Use Cramer's rule to solve the system

$$\begin{aligned} -2x + 3y &= 2 \\ 4x + 10y + 2z &= 3 \\ -5x + 7y &= 1. \end{aligned}$$