Math 307

Chapter 6

Section 6.4: Nonhomogeneous Linear Systems

Contents

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What's The Trick?

Consider a nonhomogeneous system of ODEs

$$Y' = AY + G.$$

The trick is to use a method called **variation of parameter**.

Let M be the fundamental matrix of $Y' = AY$. We suppose we have the matrix M at hand.

Goal: Determine a vector function V such that $Y_P = MV$ is a particular solution to $Y' = AY + G$.

1) Derivative of Y_P

$$Y_P' = M'V + MV'$$

2) Replace in $Y' = AY + G$

$$M'V + MV' = AMV + G$$

$$\Rightarrow M'V - AMV + MV' = G$$

$$\Rightarrow (M' - AM)V + MV' = G.$$

But M is a solution to $Y' = AY \Rightarrow M' - AM = 0$.

therefore

$$0M + MV' = G$$

$$\Rightarrow MV' = G$$

Since $\det(M) = W(Y_1(x), \dots, Y_n(x)) \neq 0$, M is invertible

$$\Rightarrow M^{-1}MV' = M^{-1}G$$

$$\Rightarrow V' = M^{-1}G \Rightarrow V = \int M^{-1}(x)G(x)\,dx$$

therefore

$$\boxed{Y_p(x) = M(x)V(x) = M(x)\int M^{-1}(x)\,G(x)\,dx}$$

EXAMPLE 1. Find the general solution to the system of ODEs

$$Y' = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} Y + \begin{bmatrix} 2 \\ x \end{bmatrix}.$$

1) Homogeneous part.

$$Y' = \underbrace{\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}}_{A.} Y$$

i) Eigenvalues: $\lambda = 2$, $\lambda = 3$

$\hookrightarrow \dim(E_2) + \dim(E_3) = 2 \to A$ diag.

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \ \& \ P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \ (A = PDP^{-1}).$$

ii) Diag. system $Z = P^{-1}Y$

$$Z' = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} Z \longrightarrow Z = \begin{bmatrix} c_1 e^{2x} \\ c_2 e^{3x} \end{bmatrix} \longrightarrow Y_H = \begin{bmatrix} 2c_1 e^{2x} + c_2 e^{3x} \\ c_1 e^{2x} + c_2 e^{3x} \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 2e^{2x} & e^{3x} \\ e^{2x} & e^{3x} \end{bmatrix}}_{M} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

2) Non-homogeneous part.

$Y = Y_H + Y_P$.

$$G(x) = \begin{bmatrix} 2 \\ x \end{bmatrix}$$

$$Y_P = M \int M^{-1} G \, dx$$

We have $M^{-1}(x) = \dfrac{1}{e^{5x}} \begin{bmatrix} e^{3x} & -e^{3x} \\ -e^{2x} & 2e^{2x} \end{bmatrix} = \begin{bmatrix} e^{-2x} & -e^{-2x} \\ -e^{-3x} & 2e^{-3x} \end{bmatrix}$

$$\Rightarrow M^{-1}G = \begin{bmatrix} e^{-2x} & -e^{-2x} \\ -e^{-3x} & 2e^{-3x} \end{bmatrix} \begin{bmatrix} 2 \\ x \end{bmatrix} = \begin{bmatrix} 2e^{-2x} - xe^{-2x} \\ -2e^{-3x} + 2xe^{-3x} \end{bmatrix}$$

Now, $\int M^{-1} G \, dx = \begin{bmatrix} \int 2e^{-2x} - xe^{-2x} \, dx \\ \int -2e^{-3x} + 2xe^{-3x} \, dx \end{bmatrix}$

$$= \begin{bmatrix} \frac{e^{-2x}}{4}(2x-3) \\ \frac{e^{-3x}}{9}(3x+7) \end{bmatrix}$$

Finally, $Y_p = M \begin{bmatrix} \frac{e^{-2x}}{4}(2x-3) \\ \frac{e^{-3x}}{9}(3x+7) \end{bmatrix} = \begin{bmatrix} \frac{4x}{3} - \frac{13}{18} \\ \frac{5x}{6} + \frac{1}{36} \end{bmatrix}$

and so

$$Y = Y_H + Y_p = \begin{bmatrix} 2c_1 e^{2x} + c_2 e^{3x} + \frac{4x}{3} - \frac{13}{18} \\ c_1 e^{2x} + c_2 e^{3x} + \frac{5x}{6} + \frac{1}{36} \end{bmatrix}.$$