Math 307

Chapter 6

Section 6.3: Homogeneous Systems With Constant Coefficients
The Nondiagonalizable Case

Contents

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Example 1. Find the general solution to the system

$$Y' = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} Y.$$

We see that the system is

$$\begin{cases} y_1' = 3y_1 + y_2 \\ y_2' = 2y_2 \end{cases}$$

We can solve $y_2' = 2y_2$

$\to y_2(x) = c_2 e^{2x}$.

But, how do we solve
$y_1' = 3y_1 + c_2 e^{2x}$ (non-homogeneous ODE).

Solving One Nonhomogeneous ODE

Given an nonhomogeneous ODE

$$y' = ay + g \qquad (\star)$$

the general solution is given by

$$y = y_H + y_P$$

where

- y_H is the general solution to the homogeneous ODE $y' = ay$.
- y_P is a particular solution to the ODE $(\star)$ and it has the following form:

$$y_P(x) = e^{ax} \int e^{-ax} g(x)\, dx.$$

Example 2. Complete the previous example.

1) Homogenous part:

$y' = 3y \to y_H(x) = c_1 e^{3x}$

2) Non homogenous part:

$g(x) = c_2 e^{2x}$

$\to y_P(x) = e^{3x} \int e^{-3x} c_2 e^{2x}\, dx$

$= c_2 e^{3x} (-e^{-x})$

$= -c_2 e^{2x}$

Therefore $y_1(x) = y_H(x) + y_P(x) = c_1 e^{3x} - c_2 e^{2x}$

When A in $Y' = AY$ is not diagonalizable, we can use the Jordan Canonical Form B of A.

EXAMPLE 3. Find the general solution of $Y' = AY$ for

$$A = \begin{bmatrix} 1 & -2 & -6 \\ -2 & 2 & -5 \\ 2 & 1 & 8 \end{bmatrix}.$$

1) Find the Jordan Canonical form:

Eigen values of A: $\lambda = 5$, $\lambda = 3$ (alg. mult. = 2).

We can show that $\dim(E_?) + \dim(E_3) = 2 \neq 3 \rightarrow A$ not diag.

Jordan Canonical form of A:

$$B = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \quad \& \quad P = \begin{bmatrix} -2 & 0 & -1 \\ -1 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

$\hookrightarrow$ $A = PBP^{-1}$.

2) Solve diagonal system: $Z = P^{-1}Y$

$$\rightarrow \quad Z' = BZ \quad \rightarrow \quad \begin{cases} z_1' = 3z_1 + z_2 \\ z_2' = 3z_2 \\ z_3' = 5z_3 \end{cases}$$

We find $z_3 = c_3 e^{5x}$ & $z_2(x) = c_2 e^{3x}$.

$$\Rightarrow \quad z_1' = 3z_1 + \underbrace{c_2 e^{3x}}_{g(x)}$$

So, $z_H = c_1 e^{3x}$ & $z_p = e^{3x} \int e^{-3x} c_2 e^{3x}\,dx = c_2 x e^{3x}$

Therefore $z_1(x) = z_H + z_p = c_1 e^{3x} + c_2 x e^{3x}$

So,
$$Z = \begin{bmatrix} c_1 e^{3x} + c_2 x e^{3x} \\ c_2 e^{3x} \\ c_3 e^{5x} \end{bmatrix}$$

3) <u>Find Y</u>

We know that $\quad Z = P^{-1}Y \Rightarrow Y = PZ$.

Therefore

$$Y = \begin{bmatrix} -2 & 0 & -1 \\ -1 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{3x} + c_2 x e^{3x} \\ c_2 e^{3x} \\ c_3 e^{5x} \end{bmatrix}$$

$$= \begin{bmatrix} -2c_1 e^{3x} - 2c_2 x e^{3x} - c_3 e^{5x} \\ -c_1 e^{3x} + c_2 (e^{3x} - xe^{3x}) - c_3 e^{5x} \\ c_1 e^{3x} + c_2 x e^{3x} + c_3 e^{5x} \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} -2e^{3x} & xe^{3x} & -e^{5x} \\ -e^{3x} & e^{3x} - xe^{3x} & -e^{5x} \\ e^{3x} & xe^{3x} & e^{5x} \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}}_{C}$$