Math 307

Chapter 5

Section 5.4: Eigenvalues and Eigenvectors of Matrices

Contents

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Example: Markov Processes

A market research company has predicted the future market share proportion of three compagnies A, B, and C. The future market is determined by a transition matrix P:

$$P = \begin{bmatrix} 0.8 & 0.03 & 0.2 \\ 0.1 & 0.95 & 0.05 \\ 0.1 & 0.02 & 0.75 \end{bmatrix}$$

We have the following interpretation of each row of P:

- the first column of P represents the share of Company A that will pass to Company A, Company B, and Company C respectively after a month.
- the second column of P represents the share of Company B that will pass to Company A, Company B and Company C respectively after a month.
- the third column of P represents the share of Company C that will pass to Company A, Company B and Company C.

Example 1. If the initial market for the three companies is represented by $s_0 = \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 30 & 15 & 55 \end{bmatrix}^T \end{array}$, that is, Company A has 30% share, Company B has 15% share, and Company C has 55% share. Calculate the predicted market

1. after 1 month, that is s_1.
2. after 2 month, that is s_2.
3. Using Python, find s_5 up to s_{40}.

1) $$s_1 = Ps_0 = \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.03 & 0.95 & 0.02 \\ 0.2 & 0.05 & 0.75 \end{bmatrix} \begin{bmatrix} 30 \\ 15 \\ 55 \end{bmatrix} = \begin{bmatrix} 31 \\ 16.25 \\ 48 \end{bmatrix}$$

2) $$s_2 = Ps_1 = P^2 s_0 = \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.03 & 0.95 & 0.02 \\ 0.2 & 0.05 & 0.75 \end{bmatrix} \begin{bmatrix} 31 \\ 16.25 \\ 48 \end{bmatrix} = \begin{bmatrix} 31.225 \\ 17.3275 \\ 43.0125 \end{bmatrix}.$$

3) $\underbrace{s_n}_{n \text{ month}} = P^n p_0$, $\quad \underbrace{p_m}_{\substack{m \text{ months} \\ m>n}} = P^{m-n} s_n \approx 1\, p_n$

Remark: After a period of time, the market share seems to stabilize. In fact, there are tools to compute this asymptotic values: eigenvalues and eigenvectors!

Definition

Let A be an $n \times n$ matrix.

- An **eigenvalue** of A is a scalar λ such that there is a nonzero column vector v in $\mathbb{R}^n$ so that

$$Av = \lambda v.$$

v eigen vector

$\Rightarrow$ 5 0 v eigenvector.

- The nonzero vector v is called an **eigenvector** of A associated to the eigenvalue λ.

EXAMPLE 2. Verify that the vector $v = \begin{bmatrix} -1 & 1 \end{bmatrix}^\top$ is an eigenvector of the matrix

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}.$$

$$Av = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 4v.$$

So, $\lambda = 4$ is an eigenvalue & $v = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector.

take $w = \begin{bmatrix} -5 \\ 5 \end{bmatrix}$.

$$Aw = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} -5 \\ 5 \end{bmatrix} = \begin{bmatrix} -20 \\ 20 \end{bmatrix}$$

$$= 20 \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}$$

$$= 4 \begin{bmatrix} -5 \\ 5 \end{bmatrix} = 4w$$

$\Rightarrow$ w is also an eigenvector for A (associated to 4).

Method to Find Eigenvalues

If v is a $n \times 1$ column vector and A is an $n \times n$ matrix, then the equation

$$Av = \lambda v$$

for some scalar λ can be rewritten as

$$AX = B \leftarrow \underbrace{(\lambda I - A)}_{\text{matrix.}} v = 0. \rightarrow \text{system of linear equations.}$$

There are two cases to consider:

- ✗ If the matrix $\lambda I - A$ is invertible, then the only solution is $v = 0$. This is not valid because an eigenvector should not be zero.
- → ✓ If the matrix $\lambda I - A$ is not invertible, then the system has non-trivial solutions. In particular, this means that $\det(\lambda I - A) = 0$.

Characteristic Equation: The eigenvalues are obtained from solving the characteristic equation of the matrix A given by

$$\det(\lambda I - A) = 0.$$

EXAMPLE 3. Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}.$$

1) $\lambda I - A = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} \lambda - 1 & 3 \\ 2 & \lambda - 2 \end{bmatrix}$

2) $\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & 3 \\ 2 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 2) - 2 \cdot 3$

$= \lambda^2 - 3\lambda + 2 - 6$

$= \lambda^2 - 3\lambda - 4.$

$= (\lambda - 4)(\lambda + 1)$

3) Solve $\det(\lambda I - A) = 0$

→ $(\lambda - 4)(\lambda + 1) = 0$ → $\lambda = 4$ or $\lambda = -1$.

Remark: The expression $\det(\lambda I - A)$ is also called the characteristic polynomial (a polynomial of degree n in the variable λ).

Example 4. Find the eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}.$$

Know: $\lambda = 4$ & $\lambda = -1$ are eigenvalues.

we have to find $v = \begin{bmatrix} x \\ y \end{bmatrix}$ s.t.

1) $(4I - A)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ 2) $(-1I - A)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

1) $\lambda = 4$. $4I - A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix}$

So, have to solve $\begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$\rightarrow \begin{cases} 3x + 3y = 0 \\ 2x + 2y = 0 \end{cases} \rightarrow x + y = 0 \rightarrow y = -x.$

infinitely many solutions & x is a free parameter

$\rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -x \end{bmatrix} = x\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

So, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is a basis for the $\underbrace{\text{null space}}_{\text{kernel}}$ of $\begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix}$.

Remark: When λ is an eigenvalue of A, then the steps to find the eigenvectors associated to λ are the following:

1. Form the matrix $\lambda I - A$.
2. Find a basis for the nullspace of $\lambda I - A$, or equivalently find all non-trivial solutions to the system $(\lambda I - A)v = 0$ where $v = \begin{bmatrix} \cancel{x}_1 & \cancel{x}_2 & \cdots & \cancel{x}_n \end{bmatrix}^\top$.

Terminology:

- The set of eigenvectors associated to an eigenvalue λ is called the eigenspace and is denoted by E_λ.
- We call $\dim(E_\lambda)$ the **geometric multiplicity** of λ.
- The **algebraic multiplicity** of $\underset{=r}{\lambda}$ is the number of times the factor $\overset{(\lambda - r)}{\cancel{(x - \lambda)}}$ appears in the characteristic polynomial.

Exemple 4 (cont'd)

2) $\lambda = -1$ $\quad -I - A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix}$

solve $\begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \to \begin{cases} -2x + 3y = 0 \\ 2x - 3y = 0 \end{cases} \to -2x+3y=0$

$\to y = \frac{2}{3}x$.

$\to \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2/3\, x \end{bmatrix} = x \begin{bmatrix} 1 \\ 2/3 \end{bmatrix}$.

A basis for the space of eigen vectors is $\begin{bmatrix} 1 \\ 2/3 \end{bmatrix}$.

Alternative: $\quad x\begin{bmatrix} 1 \\ 2/3 \end{bmatrix} = \frac{x}{3}\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

I can choose $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ as a basis.

Fact: $\underbrace{\overset{\lambda=4}{\begin{bmatrix} 1 \\ -1 \end{bmatrix}} \;\&\; \overset{\lambda=-1}{\begin{bmatrix} 3 \\ 2 \end{bmatrix}}}_{\text{basis for } \mathbb{R}^2}$ are lin. independent.

- If v_1 is an eigen vector for λ_1 and v_2 is an eigen vector for λ_2

  $\Rightarrow$ v_1, v_2 are lin. independent.

EXAMPLE 5. Find the eigenvalues and bases for the eigenspace associated to each eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

1) Find eigenvalues.

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-2 & 1 & -3 \\ 0 & \lambda+1 & 0 \\ 0 & 0 & \lambda+1 \end{vmatrix} = (\lambda-2)(\lambda+1)(\lambda+1) = (\lambda-2)(\lambda+1)^2$$

$$\det(\lambda I - A) = 0 \iff (\lambda-2)(\lambda+1)^2 = 0 \iff \begin{array}{ll} \lambda = 2 & (\text{mult. } 1) \\ \text{or} & \\ \lambda = -1 & (\text{mult. } 2) \end{array}$$

2) Find bases for eigenspaces.

E_2 $(\lambda=2)$ Solve $(2I-A)v = 0$ $\quad v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$

$$2I - A = \begin{bmatrix} 0 & 1 & -3 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \to \left[\begin{array}{ccc|c} 0 & 1 & -3 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right] \to \begin{array}{l} 3y = 0 \\ 3z = 0 \end{array}$$

$\to y = z = 0$
x free

$$\to v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\to \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ basis for E_2.

E_{-1} $(\lambda=-1)$ Solve $(-I-A)v = 0$, $\quad v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$

$$-I - A = \begin{bmatrix} -3 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \left[\begin{array}{ccc|c} -3 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \begin{array}{l} -3x + y - 3z = 0 \\ y, z \text{ free} \end{array}$$

$\to x = -\frac{y}{3} + z$
y, z free.

$$\to v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -y/3 + z \\ y \\ z \end{bmatrix} = \begin{bmatrix} -y/3 \\ y \\ 0 \end{bmatrix} + \begin{bmatrix} z \\ 0 \\ z \end{bmatrix} = y \begin{bmatrix} -1/3 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$\to \begin{bmatrix} -1/3 \\ 1 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is a basis for E_{-1}.

Fact: If λ and μ are two different eigenvalues for a matrix A and if v and w are two eigenvectors associated to the eigenvalues λ and μ respectively, then v and w are linearly independent.

Exo 1 (a)

$$(3+2i)+(3+\overset{1}{i}) = (3+3) + (2+1)i = 6+3i \,.$$

Exo 2 (a)

$$(3+2i)(3+i) = 3\cdot 3 + 3\cdot i + 2i\cdot 3 + 2i^2 = 9 + 3i + 6i - 2$$

$$= 7 + 9i$$

Exo. 4 (a)

$$\frac{2+4i}{i} = \frac{2+4i}{0+i} \cdot \frac{0-i}{0-i} = \frac{(2+4i)(-i)}{i\,(-i)} = \frac{-2i+4}{-(-1)} = \frac{4-2i}{1}$$

$$= 4-2i$$

Example 6. Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda & 1 \\ -1 & \lambda \end{vmatrix} = \lambda^2 + 1$$

$$\det(\lambda I - A) = 0 \iff \lambda^2 + 1 = 0 \iff \lambda^2 = -1$$

$$\iff \lambda = \pm \underbrace{\sqrt{-1}}_{i}$$

i: imaginary numbers
or
complex numbers

Complex numbers

We define $i = \sqrt{-1}$ such that $i^2 = -1$. A complex number z is

$$z = \underbrace{a}_{\text{Real part}} + b\overset{i}{i}. \quad b \to \text{imaginary part}$$

a, b real numbers.

Arithmetic Operations:

$2 + 3i \neq 2 + 4i$

- Equality: $a + bi = c + di$ if $a = c$ and $b = d$.
- Addition: $(a + bi) + (c + di) = (a + c) + (b + d)i$.
- Multiplication: Like multiplying two polynomials

$$(a + bi)(c + di) = ac + adi + cbi + bd\,i^2$$
$$= (ac - bd) + (ad + cb)i$$

- Conjugate: $\overline{a + bi} = a - bi$.

Complex Eigenvalues

To deal with the complex eigenvalues, we have to consider vectors with complex entries.

All the definitions associated to matrix addition, scalar multiplication, matrix multiplication, and the determinant of a matrix are all the same: Instead of using the addition and multiplication of the real numbers, we use the addition and multiplication of complex numbers.

EXAMPLE 7. Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

1) Find eigenvalues

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-1 & 1 \\ -1 & \lambda-1 \end{vmatrix} = (\lambda-1)^2 + 1 = \lambda^2 - 2\lambda + 2 = 0$$

$$\Leftrightarrow \quad \lambda = \frac{-(-2) \pm \sqrt{(-2)^2 - 4\cdot 1\cdot 2}}{2\cdot 1} \qquad \left(\frac{-b \pm \sqrt{b^2-4ac}}{2a} \quad ax^2+bx+c \right)$$

$$= \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = 2 \pm \sqrt{(-1)4}/2$$

$$= \frac{2 \pm (\sqrt{-1})\sqrt{4}}{2}$$

$$= 1 \pm i.$$

$\lambda = 1+i$ & $\lambda = 1-i$.

2) Eigenspaces.

E_{1+i} $(\lambda = 1+i)$ Solve $((1+i)I - A)v = 0$, $v = \begin{bmatrix} x \\ y \end{bmatrix}$.

$$(1+i)I - A = \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \to \left[\begin{array}{cc|c} i & 1 & 0 \\ -1 & i & 0 \end{array}\right] \xrightarrow{R_1 + iR_2 \to R_2} \sim \left[\begin{array}{cc|c} i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\to ix + y = 0 \to y = -ix$

x free variable

$$\to \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -ix \end{bmatrix} = x\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$\to \left\{\begin{bmatrix} 1 \\ -i \end{bmatrix}\right\}$ is basis for E_{1+i}.

$\underline{E_{1-i}}$ $(\lambda = 1-i)$ $\quad ((1-i)I - A)v = 0$, $v = \begin{bmatrix} x \\ y \end{bmatrix}$ (x, y complex).

$$(1-i)I - A = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \to \begin{bmatrix} -i & 1 & 0 \\ -1 & -i & 0 \end{bmatrix} R_1 - iR_2 \to R_2$$

$$\sim \begin{bmatrix} -i & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\to$ $-ix + y = 0$ $\to$ $y = ix$, x free variable.

$\to$ $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ ix \end{bmatrix} = x \begin{bmatrix} 1 \\ i \end{bmatrix}$

$\to$ $\begin{bmatrix} 1 \\ i \end{bmatrix}$ basis for E_{1-i}

<u>Fact</u>: Let A be an $n \times n$ matrix with real numbers as entries. Suppose

- λ is an eigenvalue of A;
- $v_1, v_2, \ldots, v_k$ are basis of the eigenspace E_λ, then $\overline{\lambda}$ is also an eigenvalue of A and $\overline{v}_1, \overline{v}_2, \ldots, \overline{v}_k$ form a basis for the eigenspace $E_{\overline{\lambda}}$.

If v is a column vector, the notation $\overline{v}$ means we are taking the conjugate of each component of v.