Math 307

Chapter 1

Section 1.5: Determinants

Contents

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Example 1. Find the equation of the parabola $ax^2 + bx + 1$ passing through the points $(1, 1)$ and $(2, 4)$.

$$x = 1 \rightarrow \quad a \cdot 1^2 + b \cdot 1 + 1 = 1 \qquad \rightarrow \quad \begin{cases} a + b = 0 \\ 4a + 2b = 3 \end{cases}$$
$$x = 2 \rightarrow \quad a \cdot 4 + b \cdot 2 + 1 = 4$$

So,

$$\begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 2 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 3 \end{bmatrix} \quad R_2 - 4R_1 \to R_2$$

$$\hookrightarrow -2 = 1 \cdot 2 - 4 \cdot 1$$
$$= a_{11}a_{22} - a_{21}a_{12}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -3/2 \end{bmatrix} \quad \tfrac{1}{2}R_2 \to R_2$$

$$\hookrightarrow -\frac{3}{2} = \frac{3}{1 \cdot 2 - 4 \cdot 1} = \frac{3}{a_{11}a_{22} - a_{21}a_{12}}$$

Since $a_{11}a_{22} - a_{21}a_{12} \neq 0$, I could find the solution to the problem.

Historical Notes:

- Chinese scholars were the first to use determinants to solve systems of linear equations (3rd century BCE!).
- Cramer (1779) and Bezout (1779 also) used determinant to find a plane curve passing through a set of points, like we did in the previous example.

DEFINITION

2 by 2 matrices

Given a 2×2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

the determinant of A, denoted by $\det(A)$ is

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}.$$

<u>Remark</u>: Another notation for the determinant is

$$\boxed{\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.}$$

EXAMPLE 2. Calculate the determinant of the following matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}.$$

$\det(A) = 1 \cdot 4 - 2 \cdot 3 = 4 - 6 = -2$

$\det(B) = (-1)(-2) - 1 \cdot 2 = 0.$

3 by 3 matrices

Let A be a general 3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{32} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- <u>Minor</u>: The minor of an entry a_{ij} is the matrix M_{ij} obtained from A by removing row i and column j.
- <u>Cofactor</u>: The cofactor of an entry a_{ij} is the matrix C_{ij} given by

$$C_{ij} = (-1)^{i+j} \det(M_{ij}).$$

EXAMPLE 3. Find the minor M_{11}, and the cofactor C_{32} of the following matrices:

$$A = \begin{bmatrix} 2 & 3 & -2 \\ -1 & 6 & 3 \\ 4 & -2 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 2 \\ 2 & -3 & -1 \\ 1 & -1 & -1 \end{bmatrix}.$$

A)

$$M_{11} = \begin{bmatrix} 6 & 3 \\ -2 & 1 \end{bmatrix}$$

$$C_{32} = \begin{vmatrix} 2 & -2 \\ -1 & 3 \end{vmatrix} \cdot (-1)^{3+2}$$

$$= -4$$

B)

$$C_{32} = (-1)^{3+2} \det(M_{32})$$

$$M_{32} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

$$= (-1)(1 \cdot (-1) - 2 \cdot 2)$$

$$= 5$$

$$M_{11} = \begin{bmatrix} -3 & -1 \\ -1 & -1 \end{bmatrix}$$

The determinant of A is given by

$$
\begin{aligned}
\det(A) &= \overbrace{a_{11}\det(M_{11})}^{C_{11}} \overbrace{- a_{12}\det(M_{12})}^{+\,C_{12}} \overbrace{+ a_{13}\det(M_{13})}^{+\,C_{13}} \\
&= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{12} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\
&= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{32} + a_{13}a_{12}a_{32} - a_{13}a_{22}a_{31}.
\end{aligned}
$$

Example 4. Find the determinant of the following matrices:

$$
A = \begin{bmatrix} 2 & 3 & -2 \\ -1 & 6 & 3 \\ 4 & -2 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \cancel{1} & \cancel{-2} & \cancel{2} \\ 2 & -3 & \cancel{-1} \\ 1 & -1 & \cancel{-1} \end{bmatrix}.
$$

$$\det(A) = 2\begin{vmatrix} 6 & 3 \\ -2 & 1 \end{vmatrix} - 3\begin{vmatrix} -1 & 3 \\ 4 & 1 \end{vmatrix} + (-2)\begin{vmatrix} -1 & 6 \\ 4 & -2 \end{vmatrix}$$

$$= 2(6\cdot 1 + 6) - 3(-1\cdot 1 - 12) - 2(2 - 24)$$

$$= 107$$

$$\det(B) = 1\begin{vmatrix} -3 & -1 \\ -1 & -1 \end{vmatrix} - (-2)\begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} + 2\begin{vmatrix} 2 & -3 \\ 1 & -1 \end{vmatrix}$$

$$= 1\cdot(2) - 2(1) + 2(1)$$

$$= 2$$

For General Matrices

The determinant is defined recursively.

1. If A is an 2×2 matrix, then $\det(A) = a_{11}a_{22} - a_{12}a_{21}$.
2. If A is an $n \times n$ matrix, then

$$\begin{aligned}\det(A) &= a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n} \\ &= a_{11}\det(M_{11}) - a_{12}\det(M_{12}) + \cdots + (-1)^{1+n}a_{1n}\det(M_{1n}).\end{aligned}$$

Development w.r.t. first row.

EXAMPLE 5. Compute the determinant of the following matrix:

$$A = \begin{bmatrix} 7 & -3 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 2 & 1 & -2 & -5 \\ 0 & 4 & 0 & 6 \end{bmatrix}.$$

$$\det(A) = 7\begin{vmatrix} 1 & 0 & 3 \\ 1 & -2 & -5 \\ 4 & 0 & 6 \end{vmatrix} - (-3)\begin{vmatrix} 0 & 0 & 3 \\ 2 & -2 & -5 \\ 0 & 0 & 6 \end{vmatrix} + 0\begin{vmatrix} 0 & 1 & 3 \\ 2 & 1 & -5 \\ 0 & 4 & 6 \end{vmatrix} - 4\begin{vmatrix} 0 & 1 & 0 \\ 2 & 1 & -2 \\ 0 & 4 & 0 \end{vmatrix}$$

(1) $\xrightarrow{\quad}$

(1) $$7\begin{vmatrix} 1 & 0 & 3 \\ 1 & -2 & -5 \\ 4 & 0 & 6 \end{vmatrix} = 7\left(1\begin{vmatrix} -2 & -5 \\ 0 & 6 \end{vmatrix} - 0\begin{vmatrix} 1 & -5 \\ 4 & 6 \end{vmatrix} + 3\begin{vmatrix} 1 & -2 \\ 4 & 0 \end{vmatrix}\right)$$

$$= 7\left(-12 - 0 + 3 \cdot 8\right)$$

$$= 7 \cdot 12$$

$$= 84$$

Answer $\det(A) = 84$

Determinant from any row or column

Lagrange's Expansion Formula: If A is an $n \times n$ matrix with $n \geq 2$, then

- $\det(A) = \sum_{j=1}^{n} a_{ij} C_{ij}$ for any row indexed by i.
- $\det(A) = \sum_{i=1}^{n} a_{ij} C_{ij}$ for any column indexed by j.

EXAMPLE 6. Compute again the determinant of the matrix A in Example 4 by

1. expanding with respect to another row.
2. expanding with respect to one of the column.

According to second row. $(i=2)$

1) $\det(A) = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} + a_{24}C_{24}$

$$= -a_{21}\det(M_{21}) + a_{22}\det(M_{22}) - a_{23}\det(M_{23}) + a_{24}\det(M_{24})$$

$$= -0 \cdot \cancel{\det(M_{21})} \to 0 + 1 \begin{vmatrix} 7 & 0 & 4 \\ 2 & -2 & -5 \\ 0 & 0 & -6 \end{vmatrix} - 0 \, \cancel{\det(M_{23})} \to 0$$

$$+ 3 \begin{vmatrix} 7 & -3 & 0 \\ 2 & 1 & -2 \\ 0 & 4 & 0 \end{vmatrix}$$

$$= 1 \cdot \left(6 \begin{vmatrix} 7 & 0 \\ 2 & -2 \end{vmatrix} \right) + 3 \left(0 \cancel{\begin{vmatrix} 2 & 1 \\ 0 & 4 \end{vmatrix}} \to 0 - (-2) \begin{vmatrix} 7 & -3 \\ 0 & 4 \end{vmatrix} + 0 \cancel{\begin{vmatrix} 7 & -3 \\ 0 & 4 \end{vmatrix}} \to 0 \right)$$

$$= 1 \cdot 6 (-14) + 3 \cdot 2 \cdot 28$$

$$= \boxed{84}$$

According to the 3rd column. $(j=3)$

2) $\det(A) = a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} + a_{43}C_{43}$

$$= 0 \cdot \cancel{\det(M_{13})} \to 0 - 0 \cdot \cancel{\det(M_{23})} \to 0 + (-2) \begin{vmatrix} 7 & -3 & 4 \\ 0 & 1 & 3 \\ 0 & 4 & 6 \end{vmatrix} - \cancel{0 \det(M_{43})} \to 0$$

$$= (-2) \left(7 \cdot \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} - 0 + 0 \right) = (-14)(-6) = \boxed{84}$$

Advice: It would be clever to choose the row or column containing the greatest number of zeros.

When there too many zeros...

EXAMPLE 7. Find the determinant

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 2 & 0 \\ -1 & 2 & 3 & 0 & 4 & 2 & -1 \\ -2 & 2 & -1 & 0 & 3 & -3 & -1 \\ 1 & -1 & 1 & 0 & 3 & -2 & -1 \\ -2 & -1 & 1 & 0 & -5 & 1 & -6 \end{bmatrix}.$$

Develop det(A) w.r.t. the 4th column.
Only column of zeros

$$\to \det(A) = -0 \cdot \det(M_{14}) + 0 \cdot \det(M_{24}) - \cdots + 0 \cdot \det(M_{84})$$
$$= \boxed{0}$$

Fact: If a matrix A has a row or a column of zeros, then $\det(A) = 0$.

When the type matters!

Example 8. Find the determinant

$$\cancel{A =} \begin{vmatrix} 1 & 4 & 10 & 123 \\ 0 & 2 & 124 & \pi \\ 0 & 0 & 3 & \sqrt{2} \\ 0 & 0 & 0 & 4 \end{vmatrix}.$$

$$\det(A) = 1 \cdot \begin{vmatrix} 2 & 124 & \pi \\ 0 & 3 & \sqrt{2} \\ 0 & 0 & 4 \end{vmatrix}$$

$$= 1 \cdot \left(2 \begin{vmatrix} 3 & \sqrt{2} \\ 0 & 4 \end{vmatrix} \right)$$

$$= 1 \cdot 2 \cdot \left(3 \cdot 4 - \cancelto{0}{\sqrt{2} \cdot 0} \right)$$

$$= 1 \cdot 2 \cdot 3 \cdot 4 \quad = \boxed{24}$$

Fact: The determinant of a triangular (upper or lower) is the product of its diagonal entries.

When Operations matter!

When E is an elementary matrix,

- If E switches row i with row j, then $\det(E) = -1$.
- If E is obtained from I by multiplying a row by some scalar c, then $\det(E) = c$.
- If E is obtained from I by replacing a row of I by itself plus a multiple of another row of I, then $\det(E) = 1$.

This implies the following general facts: Suppose that $A = [a_{ij}]$ is an $n \times n$ matrix with $n \geq 2$.

- If B is a matrix obtained from A by interchanging two rows of A, then $\det(B) = -\det(A)$.
- If B is a matrix obtained from A by multiplying a row of A by a scalar c, then $\det(B) = c\det(A)$.
- If B is a matrix obtained from A by replacing a row of A by itself plus a mlutiple of another row of A, then $\det(B) = \det(A)$.