Math 307

Chapter 5

Section 5.2: The Algebra of Linear Transformations

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Created by: Pierre-Olivier Parisé
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Operations on Linear Transformations

Addition

If $T : V \to W$ and $S : V \to W$ are two linear transformations, then their sum $T + S$ is the new linear transformation defined by

$$(T + S)(v) = T(v) + S(v) \quad v \text{ in } V.$$

Example 1. Let T and S be the following linear transformations:

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x - y \\ x + 2y \end{bmatrix} \quad \text{and} \quad S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + 3y \\ x - y \end{bmatrix}.$$

Find $T + S$.

$v = \begin{bmatrix} x \\ y \end{bmatrix}$, $(T+S)(v) = T(v) + S(v)$

$= \begin{bmatrix} 2x-y \\ x+2y \end{bmatrix} + \begin{bmatrix} x+3y \\ x-y \end{bmatrix}$

$= \begin{bmatrix} 2x-y+x+3y \\ x+2y+x-y \end{bmatrix} = \begin{bmatrix} 3x+2y \\ 2x+y \end{bmatrix}$

Scalar Multiplication

If $T : V \to W$ is a linear transformation and c is a real number, then the function cT is the linear transformation defined by

$$(cT)(v) = cT(v) \quad v \text{ in } V.$$

Example 2. With T and S as in the previous example, find $S + 4T$.

$(4T)\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = 4\,T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$

$= 4\begin{bmatrix} 2x-y \\ x+2y \end{bmatrix}$

$= \begin{bmatrix} 8x-4y \\ 4x+8y \end{bmatrix}$

$(S+4T)\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) + (4T)\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$

$= \begin{bmatrix} x+3y \\ x-y \end{bmatrix} + \begin{bmatrix} 8x-4y \\ 4x+8y \end{bmatrix}$

$= \begin{bmatrix} 9x-y \\ 5x+7y \end{bmatrix}$

Let $B(V, W)$ be the set of all linear transformations $T : V \to W$.

Theorem 3. The set $B(V, W)$ equipped with the addition and scalar multiplication is a vector space.

Composition or Multiplication of Operators

If $T : V \to W$ and $S : W \to U$ are two linear transformations, then the composite $ST : V \to U$ is the linear transformation defined by

$$ST(v) = S(T(v)) \quad v \text{ in } V.$$

$S \circ T$

ST

EXAMPLE 4. Find ST with S and T as in Example 1.

$$ST\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = S\left(T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)\right)$$

$$= S\left(\begin{bmatrix} 2x - y \\ x + 2y \end{bmatrix}\right)$$

$$= \begin{bmatrix} (2x - y) + 3(x + 2y) \\ (2x - y) - (x + 2y) \end{bmatrix}$$

$$= \begin{bmatrix} 5x + 5y \\ x - 3y \end{bmatrix}.$$