Math 307

Chapter 6

Section 6.3: Homogeneous Systems With Constant Coefficients
The Nondiagonalizable Case

Contents

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EXAMPLE 1. Find the general solution to the system

$$Y' = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} Y.$$

$Y' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix}$ & $Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ $\to$ $\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 3y_1 + y_2 \\ 2y_2 \end{bmatrix}$

$\to$ $\underline{y_1' = 3y_1 + y_2}$ (**) & $\underline{y_2' = 2y_2}$ (*)

(*) $y_2(x) = c_2 e^{2x}$

(**) $y_1' = 3y_1 + c_2 e^{2x}$

Non-homogeneous ODE

Solving One Nonhomogeneous ODE

Given an nonhomogeneous ODE

$$y' = ay + g \qquad (\star)$$

the general solution is given by

$$y = y_H + y_P$$

where

- y_H is the general solution to the homogeneous ODE $y' = ay$.
- y_P is a particular solution to the ODE $(\star)$ and it has the following form:

$$(*) \to \boxed{y_P(x) = e^{ax} \int e^{-ax} g(x)\, dx.}$$

EXAMPLE 2. Complete the previous example.

(**) $y_1' = 3y_1 + c_2 e^{2x}$

1) $y_1' = 3y_1$ $\to$ $y_H(x) = c_1 e^{3x}$

2) $a = 3$, $g(x) = c_2 e^{2x}$ $\to$ $y_P(x) = e^{3x} \int e^{-3x} c_2 e^{2x}\, dx$
$= -c_2 e^{2x}$

$y_1 = y_H + y_P = c_1 e^{3x} - c_2 e^{2x}$

When A in $Y' = AY$ is not diagonalizable, we can use the Jordan Canonical Form B of A.

EXAMPLE 3. Find the general solution of $Y' = AY$ for

$$A = \underbrace{\begin{bmatrix} 1 & -2 & -6 \\ -2 & 2 & -5 \\ 2 & 1 & 8 \end{bmatrix}}_{\text{not diag.}}.$$

1) Jordan Canonical Form.

$$B = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \quad P = \begin{bmatrix} -2 & 6 & -1 \\ -1 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

2) Solve upper-triangular system:

$Z = P^{-1}Y \to Y' = AY$ becomes $Z' = BZ \; (*)$

$$(*) \to \begin{cases} z_1' = 3z_1 + z_2 \\ z_2' = 3z_2 \\ z_3' = 5z_3 \end{cases}$$

$\to z_2 = c_2 e^{3x}$ & $z_3 = c_3 e^{5x}$

Now, becomes $z_1' = \underbrace{3}_{a} z_1 + \underbrace{c_2 e^{3x}}_{g(x)}$

$$z_1 = z_H + z_p = c_1 e^{3x} + e^{3x} \int e^{-3x} c_2 e^{3x} \, dx$$
$$= c_1 e^{3x} + c_2 x e^{3x}$$

So, $Z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \to Y = PZ$.