Math 307

Chapter 5

Section 5.1: Linear Transformations

Contents

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What is a Linear Transformation?

Convention:

The addition and scalar multiplication on the set of column vectors $\mathbb{R}^n$ are the usual ones that make $\mathbb{R}^n$ a vector space. If the addition is changed, it will be mentioned explicitly in the text.

Definition

If V and W are vector spaces, a function $T : V \to W$ is called a **linear transformation** if, for all vectors u and v in V and all scalars c, the following two properties are satisfied:

1. $T(u + v) = T(u) + T(v)$;
2. $T(cv) = cT(v)$.

Example 1. Let A be an $m \times n$ matrix. We define $T : \mathbb{R}^n \to \mathbb{R}^m$ by

$$T(X) := AX$$

where X is an $n \times 1$ column vector. Verify that the function T is a linear transformation.

Example 2. Verify if the given function $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + y - z \\ x + 2y + z \end{bmatrix}$$

is a linear transformation.

Example 3. Let $D(a, b)$ be the subspace of $F(a, b)$ of differentiable function on the interval (a, b). Define the function $T : D(a, b) \to F(a, b)$ by

$$T(f) := f'$$

meaning that $T(f)(x) = f'(x)$ for every x in (a, b). Verify that T is a linear transformation.

Remark: The linear transformation in the previous example is called a differential operator and is quite useful in the theory of ODE and PDE.

If $T : V \to W$ is a linear transformation, then we can prove that

- $T(0) = 0$;
- $T(-v) = -T(v)$ for any vector v in V;
- $T(u - v) = T(u) - T(v)$ for any vector u, v in V.

There is another important property of a linear transformation which we shall illustrate by an example.

Example 4. Suppose that $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation so that

$$T\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right) = \begin{bmatrix}2\\3\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right) = \begin{bmatrix}0\\3\end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\2\end{bmatrix}.$$

Find the value of $T\left(\begin{bmatrix}1\\3\\0\end{bmatrix}\right)$.

Fact: If $v_1, v_2, \ldots, v_n$ form a basis, then the values of a linear transformation T is determined by its value on $v_1, v_2, \ldots, v_n$ because for any $v \in V$, we have

$$T(v) = T(c_1v_1 + c_2v_2 + \cdots + c_nv_n) = c_1T(v_1) + c_2T(v_2) + \cdots + c_nT(v_n).$$

Kernel

If $T : V \to W$ is a linear transformation, then the **kernel** of T is the set of all vectors v in V such that $T(v) = 0$. In set notation:

$$\ker(T) = \{v \in V \ : \ T(v) = 0\}.$$

This is in general a subspace of V.

EXAMPLE 5. Find a basis for the kernel of the linear transformation

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + y - z \\ x + 2y + z \end{bmatrix}.$$

Remark: The kernel of a transformation is related to the solutions of the system of linear equations $AX = 0$ when $T(X) = AX$ with A an $m \times n$ matrix. In this particular situation, the kernel $\ker(T)$ is called the **null space** of A also denoted by $NS(A)$. In other words, we have

$$NS(A) = \ker(T).$$

Range

If $T : V \to W$ is a linear transformation, then the **range** of T is the set of all vectors $T(v)$ where v is in V. In set notation:

$$\text{range}\,(T) = \{T(v) \,:\, v \in V\}.$$

This is in general a subspace of W.

Facts:

- Finding a basis for the range of a tranformation T given by $T(X) = AX$ where A is an $m \times n$ matrix is equivalent to finding a basis for the spanning set of the columns of the matrix A.
- The subspace spans by the column of a matrix A is called the **column space** and is denoted by $CS(A)$.

Example 6. Find a basis for the range of the linear transformation of Example 5 using the column space of a certain matrix.

In summary, to find range(T) or $CS(A)$ for a linear transformation of the form $T(X) = AX$, we follow these steps:

- express $T(v)$ as a linear combination of column vectors v_1, v_2, $\ldots$, v_n.
- Write each vector in a matrix $A = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$.
- Find the RREF of A.
- The column with the first 1 in a row will be a pivot and the vector corresponding to the column will be part of the basis.

Fact: We call $\dim(CS(A)) = \dim(\text{range}\,(T))$ the **rank** of the matrix A or transformation T.

Rank-Nullity Identity

We define

- the **nullity** of a linear transformation T as the dimension of $\ker(T)$.
- the **rank** of a linear transformation T as the dimension of range(T).

Here is an important identity relating the rank and the nullity of a linear transformation.

THEOREM 7. If $T : V \to W$ is a linear transformation, then

$$\dim(\ker(T)) + \dim(\text{range}\,(T)) = \dim(V).$$

Remark: For an $m \times n$ matrix, we obtain

$$\dim(NS(A)) + \dim(CS(A)) = n.$$

EXAMPLE 8. Verify the Rank-Nullity Identity for the matrix in Example 5 and Example 6.