Math 307

Chapter 6

Section 6.4: Nonhomogeneous ~~Linear~~ Systems of ODEs.

Contents

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Consider a nonhomogeneous system of ODEs

$$Y' = AY + G.$$

$Y = MC$
$\downarrow$
$V(x)$

The trick is to use a method called **variation of parameter**.

$G=0$

Let M be the fundamental matrix of $Y' = AY$. We suppose we have the matrix M at hand.

Goal: Determine a vector function V such that $Y_P = MV$ is a particular solution to $Y' = AY + G$.

1) Take the derivative of Y_p.

$$Y_p' = M'V + MV'$$

2) Replace in $Y' = AY + G$

$$M'V + MV' = A(MV) + G$$

$$\Rightarrow \quad M'V - AMV + MV' = G$$

$$\Rightarrow \quad \underbrace{(M' - AM)}_{=0} V + MV' = G$$

because M is the foundamental Matrix of $Y' = AY \Leftrightarrow Y' - AY = 0$

So, $\quad 0V + MV' = G \quad \Rightarrow \quad MV' = G.$

Because $\quad \det(M) = \omega(Y_1(x), \dots, Y_n(x)) \neq 0$, M is invertible

$$\Rightarrow \quad V' = M^{-1}G$$

$$\overset{\text{integrate}}{\Rightarrow} \quad V = \int M^{-1} G \, dx.$$

6.3. $y' = ay + g(x)$
$y_p(x) = e^{ax} \int e^{-ax} g \, dx$

therefore,

$$Y_p(x) = M(x)V(x) = M(x) \int M^{-1} G \, dx.$$

EXAMPLE 1. Find the general solution to the system of ODEs

$$Y' = \underbrace{\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}}_{A} Y + \underbrace{\begin{bmatrix} 2 \\ x \end{bmatrix}}_{G}.$$

$$\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 2 \\ x \end{bmatrix}$$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} y_1 + 2y_2 + 2 \\ -y_1 + 4y_2 + x \end{bmatrix}$$

1) Solve $Y' = AY$.

1.1) Solve diagonal system.

A is diagonalizable $\to$ $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ & $P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.

$Y' = AY$ becomes $Z' = DZ$ $(Z = P^{-1}Y)$.

$\to$ $Z(x) = \begin{bmatrix} c_1 e^{2x} \\ c_2 e^{3x} \end{bmatrix}$.

1.2) Find Y.

So $Y_H = PZ = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 e^{2x} \\ c_2 e^{3x} \end{bmatrix} = \begin{bmatrix} 2c_1 e^{2x} + c_2 e^{3x} \\ c_1 e^{2x} + c_2 e^{3x} \end{bmatrix}$

$= \underbrace{\begin{bmatrix} 2e^{2x} & e^{3x} \\ e^{2x} & e^{3x} \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{C}$

2) Find Y_P.

$Y_P = M \int M^{-1} G \, dx$ with $G(x) = \begin{bmatrix} 2 \\ x \end{bmatrix}$

& $M = \begin{bmatrix} 2e^{2x} & e^{3x} \\ e^{2x} & e^{3x} \end{bmatrix}$.

we have $M^{-1}(x) = \dfrac{1}{\det(M)} \begin{bmatrix} e^{3x} & -e^{2x} \\ -e^{3x} & 2e^{2x} \end{bmatrix}^T = \dfrac{1}{e^{5x}} \begin{bmatrix} e^{3x} & -e^{3x} \\ -e^{2x} & 2e^{2x} \end{bmatrix}$

$\Rightarrow$ $M^{-1}(x) = \begin{bmatrix} e^{-2x} & -e^{-2x} \\ -e^{-3x} & 2e^{-3x} \end{bmatrix}$

$$Y_P = M \int M^{-1} G \, dx = \begin{bmatrix} 2e^{2x} & e^{3x} \\ e^{2x} & e^{3x} \end{bmatrix} \int \begin{bmatrix} e^{-2x} & -e^{-2x} \\ -e^{-3x} & 2e^{-3x} \end{bmatrix} \begin{bmatrix} 2 \\ x \end{bmatrix} dx$$

$$= \begin{bmatrix} 2e^{2x} & e^{3x} \\ e^{2x} & e^{3x} \end{bmatrix} \int \begin{bmatrix} 2e^{-2x} - xe^{-2x} \\ 2e^{-3x} + 2xe^{-3x} \end{bmatrix} dx$$

$$= \begin{bmatrix} 2e^{2x} & e^{3x} \\ e^{2x} & e^{3x} \end{bmatrix} \begin{bmatrix} \int 2e^{-2x} - xe^{-2x} \, dx \\ \int 2e^{-3x} + 2xe^{-3x} \, dx \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{2x} & e^{3x} \\ e^{2x} & e^{3x} \end{bmatrix} \begin{bmatrix} \frac{e^{-2x}}{4}(2x-3) \\ \frac{e^{-3x}}{9}(3x+7) \end{bmatrix}$$

$$\Rightarrow \quad Y_P = \begin{bmatrix} \frac{4x}{3} - \frac{13}{18} \\ \frac{5x}{6} + \frac{1}{36} \end{bmatrix}$$

3) <u>State the result.</u>

$$Y = Y_H + Y_P = \begin{bmatrix} 2c_1 e^{2x} + c_2 e^{3x} + \frac{x}{3} - \frac{19}{18} \\ c_1 e^{2x} + c_2 e^{3x} - \frac{x}{6} - \frac{11}{36} \end{bmatrix}. \quad \square$$