Math 307

Chapter 2

Section 2.3: Linear Independence and Bases

Contents

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Definition

Suppose that $v_1, v_2, \ldots, v_n$ are vectors in a vector space V.

- The vectors $v_1, v_2, \ldots, v_n$ are **linearly dependent** if there are scalars $c_1, c_2, \ldots, c_n$, not all zero, so that

$v_2 + 2v_3 = v_4$

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0.$$

$\rightarrow v_2 + 2v_3 - v_4 = 0$

- If $v_1, v_2, \ldots, v_n$ are not linearly dependent, then the vectors are **linearly independent**.

EXAMPLE 1. Are the vectors

$V = \mathbb{R}^3$

$$\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}3\\2\\1\end{bmatrix}, \begin{bmatrix}-1\\2\\5\end{bmatrix}$$

linearly dependent or linearly independent?

Goal: Find c_1, c_2, c_3, not all zero s.t.

$$c_1 \begin{bmatrix}1\\2\\3\end{bmatrix} + c_2 \begin{bmatrix}3\\2\\1\end{bmatrix} + c_3 \begin{bmatrix}-1\\2\\5\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

$$\equiv \begin{cases} c_1 + 3c_2 - c_3 = 0 \\ 2c_1 + 2c_2 + 2c_3 = 0 \\ 3c_1 + c_2 + 5c_3 = 0 \end{cases}$$

$$\begin{bmatrix} 1 & 3 & -1 & 0 \\ 2 & 2 & 2 & 0 \\ 3 & 1 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{array}{l} c_1 = -2c_3 \\ c_2 = c_3 \\ c_3 \text{ free} \end{array}$$

$c_3 = 1 \rightarrow c_1 = -2$ & $c_2 = 1$

$$\rightarrow -2\begin{bmatrix}1\\2\\3\end{bmatrix} + 1\begin{bmatrix}3\\2\\1\end{bmatrix} + 1\begin{bmatrix}-1\\2\\5\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \rightarrow \text{lin. dependent!}$$

c_1, c_2, c_3 are not all zeros!

$V = P_2$

EXAMPLE 2. Are x^2+1, x^2-x+1, $x+2$ linearly dependent or linearly independent?

Goal: Find c_1, c_2, c_3 s.t. c_1, c_2, c_3 not all zeros &

$$(*)\quad c_1(x^2+1) + c_2(x^2-x+1) + c_3(x+2) = 0x^2 + 0x + 0 = 0$$

$$\Leftrightarrow\quad (c_1+c_2)x^2 + (-c_2+c_3)x + (c_1+c_2+2c_3) = 0x^2+0x+0$$

$$\Leftrightarrow\quad \begin{cases} c_1+c_2=0 \\ -c_2+c_3=0 \\ c_1+c_2+2c_3=0 \end{cases}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 1 & 2 & 0 \end{bmatrix} \sim \begin{array}{cccc} c_1 & c_2 & c_3 & \\ \hline \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \rightarrow \begin{cases} c_1=0 \\ c_2=0 \\ c_3=0 \end{cases}$$

$\rightarrow$ we can't find non-zero scalars c_1, c_2, c_3 s.t. $(*)$ is satisfied.

$\rightarrow$ x^2+1, x^2-x+1, $x+2$ are lin. independent.

Remark: To show that the vectors $v_1, v_2, \ldots, v_n$ are linearly independent, we can verify that the following implication is true:

$$\text{If } c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0, \text{ then } c_1 = c_2 = \cdots = c_n = 0.$$

Dependence and Linear Combination

A way to check if a bunch of vectors are linearly dependent is outlined in the following statement.

Theorem 3. Suppose $v_1, v_2, \ldots, v_n$ are vectors in a vector space V. Then $v_1, v_2, \ldots, v_n$ are linearly dependent if and only if one of $v_1, v_2, \ldots, v_n$ is a linear combination of the others.

Example 4. Apply the last Theorem to show that the vectors

$$3\underset{v_1}{\begin{bmatrix}1\\-2\\3\end{bmatrix}} \text{ and } \underset{v_2}{\begin{bmatrix}3\\-6\\9\end{bmatrix}} \text{ and } \underset{v_3}{\begin{bmatrix}-1\\3\\0\end{bmatrix}}.$$

$$\underbrace{v_2 = 3v_1 + 0v_3}$$

$$3v_1 - v_2 + 0v_3 = 0$$

are linearly dependent.

$$c_1v_1 + c_2v_2 + c_3v_3 = 0$$

Trick.

$$\begin{bmatrix}1 & 3 & -1\\-2 & -6 & 3\\3 & 9 & 0\end{bmatrix} \sim \begin{array}{c} \begin{array}{ccc} c_1 & c_2 & \end{array} \\ \begin{bmatrix}\boxed{1} & \boxed{3} & 0\\0 & 0 & \boxed{1}\\0 & 0 & 0\end{bmatrix} \end{array}$$

- Pivots $\to$ lin. ind. vectors.
- vectors linearly dep.

Definition

The vectors v_1, v_2, ..., v_n of a vector space V are a **basis** if the two following conditions are satisfied:

- v_1, v_2, ..., v_n are linearly independent. [Indepence Condition or IC]
- v_1, v_2, ..., v_n span V. [Spanning condition, or SC]

EXAMPLE 5. Show that the vectors

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

forms a basis for $\mathbb{R}^3$. $= V$

1) $\begin{array}{cccc} c_1 & c_2 & c_3 & \\ \hline \end{array}$
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \rightarrow \begin{cases} c_1 = 0 \\ c_2 = 0 \\ c_3 = 0 \end{cases} \rightarrow \text{lin. independent.}$$

2) $\begin{array}{ccc} e_1 & e_2 & e_3 \end{array}$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3 \rightarrow \text{span } \mathbb{R}^3.$$

$\rightarrow$ consistent

$\rightarrow e_1, e_2, e_3$ form a basis of $\mathbb{R}^3$.

Remark: The basis in the last example is called the standard basis for $\mathbb{R}^3$. More generally, the vectors

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad \ldots, \quad e_{n-1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix}, \quad e_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

forms a basis for the vector space $\mathbb{R}^n$ of column vectors of dimensions $n \times 1$.

Basis for Matrices and Polynomials

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = aE_{11} + bE_{12} + cE_{21} + dE_{22}$$

Example 6. The vectors

$$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

form a basis for the vector space of 2×2 matrices $M_{2\times 2}(\mathbb{R})$.

Remark: The basis in the last example is called the standard basis for the vector space $M_{2\times 2}(\mathbb{R})$. More generally, the vectors E_{ij} with a 1 in the entry ij and 0 elsewhere forms a basis for the space of matrices $M_{m\times n}(\mathbb{R})$.

Example 7. The vectors

$$1,\ x,\ x^2$$

form a basis for the set of polynomials P_2.

$$\begin{array}{cc} & \begin{array}{ccc} v_1 & v_2 & v_3 \\ 1 & x & x^2 \end{array} \\ \begin{array}{c} x^2 \\ x \\ 1 \end{array} & \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to$$

lin. independent & span P_2

$\to$ form a basis for P_2.

(condition 1) + 2) together

Remark: The basis in the last example is also called the standard basis for the vector space P_2. More generally, for a nonnegative integer n, the vectors

$$x^n,\ x^{n-1},\ \ldots,\ x,\ 1$$

form a basis for the vector space P_n of polynomials of degree less than or equal to n.

Example 8. Do the vectors

$$\begin{array}{ccc} v_1 & v_2 & v_3 \\ \begin{bmatrix}1\\0\\1\end{bmatrix}, & \begin{bmatrix}1\\1\\1\end{bmatrix}, & \begin{bmatrix}-1\\1\\1\end{bmatrix} \end{array}$$

form a basis for the vector space of $\underbrace{3 \times 1 \text{ column vectors}}_{\mathbb{R}^3}$?

Combine vectors in a matrix:

$$\begin{array}{ccc} v_1 & v_2 & v_3 \\ \hline \end{array}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ v_1, v_2, v_3 are lin. ind.

& $\text{Span}\{v_1, v_2, v_3\} = \mathbb{R}^3$

$\Rightarrow$ v_1, v_2, v_3 form a basis for $\mathbb{R}^3$

Coordinates relative to a basis

In many applications, like robotics, it is really important to be able to represent the position of a moving part of a robot in terms of a new coordinates system.

Basis are an essential tools to do that. Given a basis $v_1, v_2, \ldots, v_n$ of a vector space V, each vector v in V can be expressed as a linear combination of the vectors in the basis:

$$v = c_1v_1 + c_2v_2 + \cdots + c_nv_n. \tag{1}$$

Moreover, the scalars $c_1, c_2, \ldots, c_n$ in the Equation (1) are unique. This means that there is only one list of scalars $c_1, c_2, \ldots, c_n$ that satisfies Equation (1).

- The list of scalars $c_1, c_2, \ldots, c_n$ are called the **coordinates of v relative to the basis $\mathbf{v_1, v_2, \ldots, v_n}$.**
- If α denotes the basis $v_1, v_2, \ldots, v_n$, then the column vector

$$[v]_\alpha = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

$$[v]_\alpha = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

is called the **coordinate vector of v relative to the basis α.**

Remarks:

- Coordinates relative to the $\overbrace{\text{standard basis}}^{\mathbb{R}^n}$:

$\underline{n=3}$. Let α be the standard basis for $\mathbb{R}^3$: $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$.

$$v = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix}$$

$$= \underline{x_1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \underline{x_2}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \underline{x_3}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$[v]_\alpha = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \rightarrow$$

coord. of v.

When α is the standard basis, we will drop the brackets & α & simply write v (instead of $[v]_\alpha$).

Ex: $v = 1 + x + x^2$
α standard basis for P_2 $\rightarrow$ $v = 1 + 1 \cdot x + 1 \cdot x^2 \rightarrow [v]_\alpha = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

- It is important to not confuse the column vectors representing the vector in a certain basis with the column vectors representing the vector in the standard basis.

Example 9. Find the coordinate vector of

$$v = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix} \text{ standard basis.}$$

relative to the basis α for $\mathbb{R}^3$ presented in Example 8.

$$\alpha : \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

$$v = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

$$\rightarrow \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 - c_3 \\ c_2 + c_3 \\ c_1 + c_2 + c_3 \end{bmatrix} \leftrightarrow \begin{cases} c_1 + c_2 - c_3 = 1 \\ c_2 + c_3 = 3 \\ c_1 + c_2 + c_3 = 7 \end{cases}$$

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 3 \\ 1 & 1 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3/2 \\ 0 & 0 & 1 & 3/2 \end{bmatrix} \rightarrow \begin{matrix} c_1 = 4 \\ c_2 = 3/2 \\ c_3 = 3/2 \end{matrix}$$

$$\rightarrow [v]_\alpha = \begin{bmatrix} 4 \\ 3/2 \\ 3/2 \end{bmatrix}.$$

Example 10. Find the coordinate in the standard basis of the vector v in $\mathbb{R}^3$ if

$$[v]_\alpha = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \begin{array}{l} \to c_1 \\ \to c_2 \\ \to c_3 \end{array}$$

where α is the basis for $\mathbb{R}^3$ in Example 8.

$$v = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$c_1 = 1$
$c_2 = 2$
$c_3 = -1$

$$= 1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} \checkmark$$