Math 307

Chapter 2

Section 2.2: Subspaces and Spanning Sets

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Definition

In loose terms, a **subspace** is simply a vector space inside another vector space. Precisely, a subspace is a subset W of another vector space V such that W is itself a vector space under the same addition and scalar multiplication operations of V restricted to W.

The next result tells us that we only need to verify if the operations are closed.

THEOREM 1. Let W be a nonempty subset of a vector space V. Then W is a subspace of V if and only if for all vectors u and w in W and for all scalar c, we have

- $u + w$ is in W;
- cu is in W.

EXAMPLE 2. Let W be the set of all column vectors of the form

$$\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}.$$

All column vectors: $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Show that W is a subspace of the vector space of all column vectors.

1) $\underset{u}{\begin{bmatrix} x_1 \\ y_1 \\ 0 \end{bmatrix}} + \underset{w}{\begin{bmatrix} x_2 \\ y_2 \\ 0 \end{bmatrix}} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ 0 + 0 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$ ✓

$x = x_1 + x_2$
$y = y_1 + y_2$

2) $c \cdot \begin{bmatrix} x_1 \\ y_1 \\ 0 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cy_1 \\ c0 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cy_1 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$ ✓

$x = cx_1$
$y = cy_1$

EXAMPLE 3. Do the set of vectors of the form

$$\begin{bmatrix} x \\ 1 \end{bmatrix}$$

forms a subspace of $\mathbb{R}^2$?

1) $\underset{u}{\begin{bmatrix} x_1 \\ 1 \end{bmatrix}} + \underset{w}{\begin{bmatrix} x_2 \\ 1 \end{bmatrix}} = \begin{bmatrix} x_1 + x_2 \\ 1 + 1 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ \underbrace{2}_{\neq 1} \end{bmatrix}$ ✗

Not a subspace.

Example 4. Do the set of vectors of the form

$$\begin{bmatrix} x \\ y \\ x-2y \end{bmatrix}$$

forms a subspace of $\mathbb{R}^3$?

1)
$$\underset{u}{\begin{bmatrix} x_1 \\ y_1 \\ x_1-2y_1 \end{bmatrix}} + \underset{v}{\begin{bmatrix} x_2 \\ y_2 \\ x_2-2y_2 \end{bmatrix}} = \begin{bmatrix} x_1+x_2 \\ y_1+y_2 \\ x_1-2y_1+x_2-2y_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1+x_2 \\ y_1+y_2 \\ x_1+x_2-2(y_1+y_2) \end{bmatrix} \qquad x = x_1+x_2, \quad y = y_1+y_2$$

$$= \begin{bmatrix} x \\ y \\ x-2y \end{bmatrix} \quad \checkmark$$

2)
$$c \cdot \begin{bmatrix} x_1 \\ y_1 \\ x_1-2y_1 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cy_1 \\ c(x_1-2y_1) \end{bmatrix} = \begin{bmatrix} cx_1 \\ cy_1 \\ cx_1-c2y_1 \end{bmatrix}$$

$$= \begin{bmatrix} cx_1 \\ cy_1 \\ cx_1-2cy_1 \end{bmatrix} \qquad x = cx_1, \quad y = cx_2$$

$$= \begin{bmatrix} x \\ y \\ x-2y \end{bmatrix} \quad \checkmark$$

So, we have a subspace of $\mathbb{R}^3$.

$$W = \left\{ \begin{bmatrix} x \\ y \\ x-2y \end{bmatrix} : x, y \in \mathbb{R} \right\}$$

Important Examples: Set of Polynomials

Let n be a nonnegative integer and let P_n denote the set of polynomials of degree less than or equal to n on (a, b); that is the set of expressions $p(x)$ of the form

$$p(x) = a_k x^k + a_{k-1} x^{k-1} + \cdots + a_1 x + a_0$$

for k an integer such that $k \leq n$.

EXAMPLE 5. Let P_2 denote the set of polynomials of degree less than or equal to 2 on (a, b); that is the set of expressions $p(x)$ of the form

$$p(x) = ax^2 + bx + c.$$

Show that P_2 is a subspace of the vector space of functions $F(a, b)$.

1) $\underbrace{(a_1x^2 + b_1x + c_1)}_{u} + \underbrace{(a_2x^2 + b_2x + c_2)}_{w}$

$= a_1x^2 + b_1x + c_1 + a_2x^2 + b_2x + c_2$

$= \underbrace{(a_1 + a_2)}_{= a}x^2 + \underbrace{(b_1 + b_2)}_{= b}x + \underbrace{(c_1 + c_2)}_{= c}$

$= ax^2 + bx + c$ ✓

2) $d \cdot (a_1x^2 + b_1x + c_1) = d(a_1x^2 + b_1x + c_1)$

$= \underbrace{da_1}_{= a}x^2 + \underbrace{db_1}_{= b}x + \underbrace{dc_1}_{= c}$

$= ax^2 + bx + c.$ ✓

So, P_2 is a subspace of $F(a,b)$.
↳ vector space.

<u>In general</u>: P_n is a subspace of $F(a,b)$.

<u>Fact</u>: Let P denote the set of all polynomials on (a, b). This means P is the set of expressions $p(x)$ of the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

Show that P is a subspace of the vector space of functions $F(a, b)$.

Linear Combinations

Given a bunch of vectors v_1, v_2, …, v_n in a vector space V, a **linear combination** of these vectors is

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

for some scalars c_1, c_2, …, c_n.

EXAMPLE 6. Is the polynomial $v(x) = 2x^2 + x + 1$ a linear combination of the polynomials $v_1(x) = x^2 + 1$, $v_2(x) = x^2 - 1$, $v_3(x) = x + 1$?

Goal: Find c_1, c_2, c_3 s.t.

$(*)$ $2x^2 + x + 1 = c_1 v_1(x) + c_2 v_2(x) + c_3 v_3(x)$.

$c_1 = 1, c_2 = 1, c_3 = 2$

$1 \cdot (x^2+1) + 1 \cdot (x^2 - 1) + 2 \cdot (x+1) = 1(x^2+1) + 1(x^2-1) + 2(x+1)$

$= x^2 + 1 + x^2 - 1 + 2x + 2$

$= 2x^2 + 2x + 2$

$(*)$ $2x^2 + x + 1 = c_1(x^2+1) + c_2(x^2-1) + c_3(x+1)$

$= (c_1 + c_2)x^2 + c_3 x + (c_1 - c_2 + c_3)$

$\Leftrightarrow$ $2 = c_1 + c_2$, $\boxed{1 = c_3}$, $1 = c_1 - c_2 + c_3$

$\to$ ① $c_1 + c_2 = 2$ & ② $0 = c_1 - c_2$

① + ② $\to$ $2c_1 = 2 \to c_1 = 1$

① − ② $\to$ $2c_2 = 2 \to c_2 = 1$

So, $(2x^2 + x + 1) = 1(x^2+1) + 1(x^2-1) + 1(x+1)$ ✓

Spanning set

The set of all linear combinations of vectors v_1, v_2, …, v_n of V is called the **spanning set of $\mathbf{v_1}$, $\mathbf{v_2}$, …, $\mathbf{v_n}$**.

The notation for the spanning set of the subspace of V generated by the vectors v_1, v_2, …, v_n is

$$\mathrm{Span}\,\{v_1, v_2, \ldots, v_n\}.$$ $\to$!subspace!

EXAMPLE 7. Is the vector

$$\begin{bmatrix} 2 \\ -5 \\ 1 \\ 10 \end{bmatrix} \text{ in the Span } \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\} ?$$

Question.

Can we write

$$\begin{bmatrix} 2 \\ -5 \\ 1 \\ 10 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 3 \end{bmatrix}$$ for some c_1, c_2, c_3?

$$\to \begin{bmatrix} 2 \\ -5 \\ 1 \\ 10 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 - c_3 \\ -c_1 - 2c_2 \\ 2c_1 - c_2 + c_3 \\ 3c_1 + 2c_2 + 3c_3 \end{bmatrix} \leftrightarrow \begin{cases} 2 = c_1 + c_2 - c_3 \\ -5 = -c_1 - 2c_2 \\ 1 = 2c_1 - c_2 + c_3 \\ 10 = 3c_1 + 2c_2 + 3c_3 \end{cases}$$

Solve the system:

$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ -1 & -2 & 0 & -5 \\ 2 & -1 & 1 & 1 \\ 3 & 2 & 3 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\to$ $c_1 = 1$, $c_2 = 2$, $c_3 = 1$

$$\to \begin{bmatrix} 2 \\ -5 \\ 1 \\ 10 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 3 \end{bmatrix} \to \begin{bmatrix} 2 \\ -5 \\ 1 \\ 10 \end{bmatrix} \text{ is in Span...}$$

Spanning a whole vector space

We say that the vectors v_1, v_2, ..., v_n of a vector space V span V if

$$\operatorname{Span}\{v_1, v_2, \ldots, v_n\} = V. \quad \Rightarrow v = c_1v_1 + c_2v_2 + \cdots + c_nv_n$$

In other words, each vector in V is a linear combination of the vectors v_1, v_2, ..., v_n.

Example 8. Do

$$v_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

span $\mathbb{R}^2$? $\leftarrow \begin{bmatrix} x \\ y \end{bmatrix}$

$\Downarrow$

Question: If $\begin{bmatrix} x \\ y \end{bmatrix}$, can we find c_1, c_2 such that

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1\begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ -4 \end{bmatrix} \;?$$

$$\rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 + 2c_2 \\ -2c_1 - 4c_2 \end{bmatrix} \leftrightarrow \begin{cases} c_1 + 2c_2 = x \\ -2c_1 - 4c_2 = y \end{cases}.$$

$$\left[\begin{array}{cc|c} 1 & 2 & x \\ -2 & -4 & y \end{array}\right] \sim \begin{array}{c} \begin{array}{ccc} c_1 & c_2 & \end{array} \\ \begin{bmatrix} 1 & 2 & x \\ 0 & 0 & 2x+y \end{bmatrix} \end{array} \rightarrow (*)\; 0 = 2x + y$$

For example, $x=0$ & $y=1$ $\rightarrow$ $(*)$ $0 = 1$ ✗

the system is not consistent! $\rightarrow$ not possible!

$\rightarrow \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ & $\begin{bmatrix} 2 \\ -4 \end{bmatrix}$ doesn't span $\mathbb{R}^2$ or $\operatorname{Span}\left\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \end{bmatrix}\right\} \neq \mathbb{R}^2$

Quicker way:

1) $\begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix}$ if $\sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\rightarrow$ consistent $\rightarrow$ v_1, v_2 span V

2) $\begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix}$ if $\sim \begin{bmatrix} * & * & * \\ 0 & 0 & c \end{bmatrix}$ $c \neq 0$ $\rightarrow$ inconsistent $\rightarrow$ v_1, v_2 ~~span~~ V

$v_2 + 2v_3 = x - 5 + 6 = x + 1 = v_4$

Example 9. Let $v_1(x) = x^2 + x - 3$, $v_2(x) = x - 5$, $v_3(x) = 3$, and $v_4(x) = x + 1$.

1. Do v_1, v_2, v_3 span P_2? $V = P_2$
2. Do v_2, v_3, v_4 span P_1? $V = P_1$
3. Do v_1, v_2, v_3 span P_3? $V = P_3$

1) Goal: c_1, c_2, c_3 s.t. $ax^2 + bx + c = c_1 v_1 + c_2 v_2 + c_3 v_3$

$$\begin{array}{c} \\ x^2 \\ x \\ \text{cst} \end{array}\begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -3 & -5 & 3 \end{bmatrix} \end{array} \sim \begin{bmatrix} -3 & -5 & 3 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & -5 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \sim \begin{bmatrix} 1 & -5/3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \text{consistent}$$

So, v_1, v_2, v_3 span P_2

2) $$\begin{array}{c} \\ x \\ 1 \end{array}\begin{array}{c} \begin{array}{ccc} v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ -5 & 3 & 1 \end{bmatrix} \end{array} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 6 \end{bmatrix} \sim \begin{array}{c} \begin{array}{ccc} c_1 & c_2 & \cancel{c_3} = 0 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \end{array} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

consistent $\rightarrow$ v_2, v_3, v_4 span P_1.

3) Goal: $ax^3 + bx^2 + cx + d = c_1 v_1 + c_2 v_2 + c_3 v_3$.

$\rightarrow$ there is no x^3 in v_1, v_2 & v_3 $\rightarrow$ impossible

$\rightarrow$ $\operatorname{Span}\{v_1, v_2, v_3\} \neq P_3$