Section 1.2, Problem 2

We have

$$2B = 2\begin{bmatrix} 2 & -1 \\ -3 & -2 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -6 & -4 \\ 0 & 8 \end{bmatrix}.$$

Section 1.2, Problem 9

We will first multiply E by the first column of F:

$$EF_1 = \begin{bmatrix} 1 & -3 & 5 \\ 2 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}.$$

Then we multiply E by the second column of F:

$$EF_2 = \begin{bmatrix} 1 & -3 & 5 \\ 2 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ -5 \\ -4 \end{bmatrix}.$$

Finally, we multiply E by the third column of F:

$$EF_3 = \begin{bmatrix} 1 & -3 & 5 \\ 2 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \\ 1 \end{bmatrix} = \begin{bmatrix} -9 \\ 13 \\ 10 \end{bmatrix}.$$

Putting each new columns together, we obtain

$$EF = \begin{bmatrix} 1 & -3 & 5 \\ 2 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 4 \\ 2 & -3 & 6 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 8 & -9 \\ 3 & -5 & 13 \\ 3 & -4 & 10 \end{bmatrix}.$$

Section 1.2, Problem 23

1. Using the properties of the product over the addition:

$$(A+B)^2 = (A+B)(A+B) = AA + AB + BA + BB = A^2 + AB + BA + B^2.$$

2. Consider the following two matrices:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

Then we obtain

$$(A+B)^2 = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \right)^2 = \begin{bmatrix} 33 & 41 & 47 \\ 77 & 96 & 110 \\ 113 & 140 & 165 \end{bmatrix}.$$

However, we have

$$\begin{aligned} A^2 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ B^2 &= \begin{bmatrix} 30 & 36 & 42 \\ 66 & 81 & 96 \\ 102 & 126 & 150 \end{bmatrix} \\ 2AB &= \begin{bmatrix} 2 & 4 & 6 \\ 14 & 16 & 18 \\ 8 & 10 & 12 \end{bmatrix}. \end{aligned}$$

So adding up together the previous results, we obtain

$$A^2 + B^2 + 2AB = \begin{bmatrix} 33 & 40 & 48 \\ 80 & 98 & 114 \\ 110 & 136 & 163 \end{bmatrix}.$$

If we compare with $(A+B)^2$, then we see that

$$(A+B)^2 \neq A^2 + B^2 + 2AB.$$

Section 1.3, Problem 3

To find the inverse of A, we augment A with the 3×3 matrix. Then, we reduce the left-hand side to the identity.

$$\begin{aligned}
\begin{bmatrix} 1 & -2 & 3 & 1 & 0 & 0 \\ 2 & -1 & 4 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} &\sim \begin{bmatrix} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & -3 & 2 & 2 & -1 & 0 \\ 0 & 3 & -2 & -1 & 0 & 1 \end{bmatrix} \\
&\sim \begin{bmatrix} 3 & 0 & 5 & -1 & 2 & 0 \\ 0 & -3 & 2 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{bmatrix}
\end{aligned}$$

We can see that there is a line starting with three zeros, but with 1, -1, and 1. Thus, the matrix A has no inverse.

Section 1.3, Problem 5

To find the inverse of A, we augment A with the 3×3 matrix. Then, we reduce the left-hand side to the identity.

$$\begin{aligned}
\begin{bmatrix} 0 & -2 & 1 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{bmatrix} &\sim \begin{bmatrix} 2 & 1 & 2 & 0 & 0 & 1 \\ 0 & 3 & -3 & 0 & 1 & -1 \\ 0 & -2 & 1 & 1 & 0 & 0 \end{bmatrix} \\
&\sim \begin{bmatrix} 6 & 0 & 9 & 0 & -1 & 4 \\ 0 & 3 & -3 & 0 & 1 & -1 \\ 0 & 0 & -3 & 3 & 2 & -2 \end{bmatrix} \\
&\sim \begin{bmatrix} 6 & 0 & 0 & 9 & 5 & -2 \\ 0 & 3 & 0 & -3 & -1 & 1 \\ 0 & 0 & -3 & 3 & 2 & -2 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 0 & 0 & 3/2 & 5/6 & -1/3 \\ 0 & 1 & 0 & -1 & -1/3 & 1/3 \\ 0 & 0 & 1 & -1 & -2/3 & 2/3 \end{bmatrix}.
\end{aligned}$$

We can conclude that the matrix is invertible and its inverse is

$$A^{-1} = \begin{bmatrix} 3/2 & 5/6 & -1/3 \\ -1 & -1/3 & 1/3 \\ -1 & -2/3 & 2/3 \end{bmatrix}.$$

Section 1.3, Problem 16

If A has a row of zeros, this means the augmented matrix created out of A and the identity matrix has the following form:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}$$

Thus, there is a row with zeros which equals 1. Thus the matrix can't have an inverse.

If A has a column of zeros, then $A^\top$ has a row of zeros. Since A is invertible if and only if $A^\top$ is invertible, this implies that we can check if $A^\top$ is invertible. From the previous paragraph, since $A^\top$ has a row of zeros, then $A^\top$ is not invertible. So A is not invertible.