MATH-307
Final

Created by Pierre-O. Parisé
Summer 2022

Last name: Solution.
First name: —

Question:	1	2	3	4	5	6	Total
Points:	20	20	20	20	10	10	100
Score:	—	—	—	—	—	—	—

Instructions: Make sure to write your complete name on your copy. You must answer all the questions below and write your answers directly on the questionnaire. At the end of the 80 minutes, return your copy.

No devices such as a smart phone, cell phone, laptop, or tablet can be used during the exam. You are not allowed to use the lecture notes or the textbook. You may bring one 2-sided cheat sheet of handwriting notes. You may use a digital calculator (no graphical calculators or symbolic calculators will be allowed).

You must show ALL your work to have full credit. An answer without justification is worth no point.

May the Force be with you!

Pierre-Olivier Parisé

Your Signature: ______________________

University of Hawai'i

QUESTION 1 **(20 pts)**

Verify if the following vector functions are solutions to the homogeneous system $Y' = AY$ with the given matrix A.

(a) (10 points) $A = \begin{bmatrix} 0 & -1 \\ 4 & 0 \end{bmatrix}$ and $Y(x) = \begin{bmatrix} \sin(2t) + \cos(2t) \\ -2\cos(2t) + 2\sin(2t) \end{bmatrix}$.

(b) (10 points) $A = \begin{bmatrix} 0 & -3 \\ -12 & 0 \end{bmatrix}$ and $Y(x) = \begin{bmatrix} e^{6t} + e^{-6t} \\ -2e^{6t} + 2e^{-6t} \end{bmatrix}$.

(a) $Y' = \begin{bmatrix} 2\cos(2t) - 2\sin(2t) \\ 4\sin(2t) + 4\cos(2t) \end{bmatrix}$

$AY = \begin{bmatrix} 2\cos(2t) - 2\sin(2t) \\ 4\sin(2t) + 4\cos(2t) \end{bmatrix}$

$\rightarrow$ Therefore, we see $Y' = AY$. ✓

(b) $Y' = \begin{bmatrix} 6e^{6t} - 6e^{-6t} \\ -12e^{6t} - 12e^{-6t} \end{bmatrix}$

$AY = \begin{bmatrix} 6e^{6t} - 6e^{-6t} \\ -12e^{6t} - 12e^{-6t} \end{bmatrix}$

$\rightarrow$ Therefore, we see $Y' = AY$. ✓

Question 2 **(20 pts)**

Consider the following matrix:

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

(a) (5 points) Find the eigenvalues of the matrix A. *(Hint: Use the last row to compute the determinant.)*

(b) (10 points) Find a basis and the dimension of each eigenspace.

(c) (5 points) Is A diagonalizable? If so, find the diagonal matrix D and the change of basis P such that $D = P^{-1}AP$.

(a) $\det(\lambda I - A) = \begin{vmatrix} \lambda-2 & 0 & -1 \\ -1 & \lambda-1 & -1 \\ 0 & 0 & \lambda-1 \end{vmatrix} = (\lambda-1)\begin{vmatrix} \lambda-2 & 0 \\ -1 & \lambda-2 \end{vmatrix} = (\lambda-1)(\lambda-2)^2$

Therefore, $\det(\lambda I - A) = 0 \leftrightarrow (\lambda-1)(\lambda-2)^2 = 0 \longleftrightarrow \lambda = 1$ or $\lambda = 2$ (mult. 2).

(b) $\underline{E_1}$ Solve $(I-A)v = 0$ with $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

$\rightarrow \begin{bmatrix} -1 & 0 & -1 \\ -1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow -x - z = 0 \rightarrow z = -x$, x, y free.

Therefore $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ -x \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ & $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ basis for E_1

$\underline{E_2}$. Solve $(2I - A)v = 0$ with $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$\rightarrow \begin{bmatrix} 0 & 0 & -1 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \leftrightarrow \begin{matrix} z = 0 \\ -x + y = 0 \end{matrix} \leftrightarrow z = 0$ & $y = x$, x free

Therefore, $v = \begin{bmatrix} x \\ x \\ 0 \end{bmatrix} = x\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ basis for E_2.

(c) The matrix D is

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

and the matrix P is

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 0 & 0 \end{bmatrix}.$$

QUESTION 3 **(20 pts)**

Solve the following homogeneous system of differential equations.

$$Y' = \begin{bmatrix} 0 & -1 \\ 9 & 0 \end{bmatrix} Y.$$

The diagonal matrix D and the change of basis P such that $A = PDP^{-1}$ are

$$D = \begin{bmatrix} 3i & 0 \\ 0 & -3i \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} i/3 & -i/3 \\ 1 & 1 \end{bmatrix}.$$

Solve Diagonal System.

$Z = P^{-1}Y \rightarrow Y' = AY$ becomes $Z' = DZ$.

So, $Z(x) = \begin{bmatrix} c_1 e^{3ix} \\ c_2 e^{-3ix} \end{bmatrix} \rightarrow Z_1 = \begin{bmatrix} e^{3ix} \\ 0 \end{bmatrix}$

Find Y.

$$Y_1 = PZ_1 = \begin{bmatrix} i/3 & -i/3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{3ix} \\ 0 \end{bmatrix} = \begin{bmatrix} ie^{3ix}/3 \\ e^{3ix} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{i\cos(3x)}{3} - \dfrac{\sin(3x)}{3} \\ \cos(3x) + i\sin(3x) \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} -\dfrac{\sin(3x)}{3} \\ \cos(3x) \end{bmatrix}}_{U} + i \underbrace{\begin{bmatrix} \dfrac{\cos 3x}{3} \\ \sin(3x) \end{bmatrix}}_{V}$$

So U & V are solutions to $Y' = AY$ and they form a foundamental set of solutions for $Y' = AY$

$$\Rightarrow \quad Y(x) = c_1 U + c_2 V = \begin{bmatrix} -\dfrac{c_1}{3}\sin 3x + \dfrac{c_2}{3}\cos 3x \\ c_1 \cos 3x + c_2 \sin 3x \end{bmatrix}.$$

Question 4 **(20 pts)**

Solve the following initial value problem:

$$Y' = \begin{bmatrix} 2 & 1 \\ -3 & -2 \end{bmatrix} Y \quad \text{and} \quad Y(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The diagonal matrix D and the change of basis P such that $A = PDP^{-1}$ are

$$D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} -1 & -1/3 \\ 1 & 1 \end{bmatrix}.$$

Solve diagonal System.

$Z = P^{-1} Y \rightarrow Y' = AY$ becomes $Z' = DZ$.

Solution $\rightarrow Z(x) = \begin{bmatrix} c_1 e^{x} \\ c_2 e^{-x} \end{bmatrix}$.

Find Y.

We know $Y = PZ \rightarrow Y = \begin{bmatrix} -1 & -1/3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{x} \\ c_2 e^{-x} \end{bmatrix}$

$$\rightarrow Y = \begin{bmatrix} -c_1 e^{x} - \frac{c_2}{3} e^{-x} \\ c_1 e^{x} + c_2 e^{-x} \end{bmatrix}$$

Question 5 **(10 pts)**

Answer the following questions.

(a) (5 points) Let A be a square matrix. Suppose that λ is an eigenvalue for A. Show that λ^3 is an eigenvalue for A^3.

(b) (5 points) Suppose the characteristic polynomial of a square matrix A is

$$(\lambda-2)^4(\lambda+2)^2(\lambda-1).$$

Suppose further that $\dim(E_2)=2$, $\dim(E_{-2})=2$, and $\dim(E_1)=1$. Give the Jordan Canonical Form of A.

(a) We have $Av = \lambda v$.

So, $A^2 v = A(Av) = A(\lambda v) = \lambda(Av) = \lambda(\lambda v) = \lambda^2 v$

So, $A^3 v = A(A^2 v) = A(\lambda^2 v) = \lambda^2(Av) = \lambda^2(\lambda v) = \lambda^3 v$ ✓

(b) The Jordan Canonical Form of A is

$$B = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

QUESTION 6 (10 pts)

Answer the following statements with **True** or **False**. Write your answer on the horizontal line at the end of each statement. Justify your answer in the white space underneath each statement.

(a) (/ 2) For an eigenvalue λ of a square matrix, there is only one eigenvector v associated with λ.

v eigenvector $\Rightarrow$ $2v$ is also an eigenvector.

(a) False.

(b) (/ 2) If the characteristic polynomial of a square matrix A is $(\lambda-4)^2(\lambda+3)^3(\lambda+2)^2$, then there are 12 possible Jordan Canonical Forms of A.

$2 \times 3 \times 2 = 12$ possibilities

(b) True

(c) (/ 2) If 2 and 3 are the eigenvalue of a 3×3 matrix A and if $\dim(E_2) + \dim(E_3) = 2$, then A is diagonalizable.

$\dim(E_2) + \dim(E_3) = 2 < 3$.

(c) False

(d) (/ 2) If v is an eigenvector associated to the eigenvalue λ and if w is an eigenvector associated to the eigenvalue μ, with $\mu \neq \lambda$, then v and w are linearly dependent.

v & w are linearly independent.

(d) False.

(e) (/ 2) Eigenvalues and eigenvectors are important tools to solve applied scientific problems.

→ Mixing problems. → Page Rank from Google.

→ Markov Chains .

(e) True.