Math 307

Chapter 5

Section 5.5: Similar Matrices, Diagonalization, and Jordan Canonical Form

Contents

Created by: Pierre-Olivier Parisé
Summer 2022

Motivation

Example 1. Let A be the 3×3 matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

Then, (a) compute A^5 (b) find the eigenvalues of A (c) find a basis for each eigenspace.

(a) $A \cdot A \cdot A \cdot A \cdot A = \begin{bmatrix} 2^5 & 0 & 0 \\ 0 & 3^5 & 0 \\ 0 & 0 & 4^5 \end{bmatrix}$

(b) $\lambda I - A = \begin{bmatrix} \lambda-2 & 0 & 0 \\ 0 & \lambda-3 & 0 \\ 0 & 0 & \lambda-4 \end{bmatrix} \rightarrow \det(\lambda I - A) = (\lambda-2)(\lambda-3)(\lambda-4)$

So, $\det(\lambda I - A) = 0 \leftrightarrow \lambda = 2, \lambda = 3$ or $\lambda = 4$.

(c) $\underline{\lambda = 2}$ $\quad 2I - A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \rightarrow y = 0, z = 0$ & x free variable

$\rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ basis for E_2.

$\underline{\lambda = 3}$ $\quad 3I - A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \rightarrow$ $x = 0$, $z = 0$, y free vari. $\rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix} = y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ basis for E_3.

$\underline{\lambda = 4}$ $\quad 4I - A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow$ $x = 0$, $y = 0$, z free var. $\rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} = z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ basis for E_4.

Remarks

- It is pretty easy to deal with diagonal matrices.
- Our goal is to try to transform a general matrix into a diagonal matrix.

EXAMPLE 2. Let A be the following 3×3 matrix

$$A = \begin{bmatrix} 6 & -4 & -2 \\ 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Find (a) the eigenvalues of A (b) a basis for each eigenspace (c) compute A^5.

(a) $\det(\lambda I - A) = \lambda^3 - 9\lambda^2 + 26\lambda - 24 = (\lambda-2)(\lambda-3)(\lambda-4)$.

$\to$ Eigenvalues: $\lambda=2$, $\lambda=3$ & $\lambda=4$.

(b) $\underline{\lambda=2.}$ $(2I-A) = \begin{bmatrix} -4 & 4 & 2 \\ -1 & 0 & 1 \\ -1 & 0 & -1 \end{bmatrix}$. Solve $(2I-A)v=0$, $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

$$\begin{bmatrix} -4 & 4 & 2 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ z/2 \\ z \end{bmatrix} = z\begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix}$$

A basis for E_2 is $\begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix}$

$\underline{\lambda=3.}$ $3I-A = \begin{bmatrix} -3 & 4 & 2 \\ -1 & 1 & 1 \\ -1 & 0 & 2 \end{bmatrix}$ Solve $(3I-A)v=0$. $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$\begin{bmatrix} -3 & 4 & 2 & 0 \\ -1 & 1 & 1 & 0 \\ -1 & 0 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2z \\ z \\ z \end{bmatrix} = z\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

A basis for E_3 is $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.

$\underline{\lambda=4.}$ $4I-A = \begin{bmatrix} -2 & 4 & 2 \\ -1 & 2 & 1 \\ -1 & 0 & 3 \end{bmatrix}$ Solve $(4I-A)v=0$, $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

$$\begin{bmatrix} -2 & 4 & 2 & 0 \\ -1 & 2 & 1 & 0 \\ -1 & 0 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3z \\ z \\ z \end{bmatrix} = z\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}.$$

So $\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$ is a basis for E_4.

c) We could compute directly A^5. This will be long though. Instead, we will try to transform A into a diagonal matrix D: we have to represent A in another basis.

The eigen vectors are the key!

Write the change of basis from $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ to the set of eigen vectors which form a basis Λ of $\mathbb{R}^3$:

$$P = \begin{bmatrix} 1 & 2 & 3 \\ 1/2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Now, the matrix A in the basis Λ is

$$[A]_\Lambda = P^{-1} A P$$

$$\rightarrow \quad A = P\, [A]_\Lambda^\Lambda P^{-1}$$

Now, we have

$$A^5 = (P\,[A]_\Lambda^\Lambda \overbrace{P^{-1})(P}^{=I}\,[A]_\Lambda^\Lambda \overbrace{P^{-1})(P}^{=I}\,[A]_\Lambda^\Lambda \overbrace{P^{-1})(P}^{=I}\,[A]_\Lambda^\Lambda \overbrace{P^{-1})(P}^{=I}\,[A]_\Lambda^\Lambda P^{-1})$$

$$= P\,[A]_\Lambda^\Lambda\, I\, [A]_\Lambda^\Lambda\, I\, [A]_\Lambda^\Lambda\, I\, [A]_\Lambda^\Lambda\, I\, [A]_\Lambda^\Lambda P^{-1}$$

$$= P\left([A]_\Lambda^\Lambda\right)^5 P^{-1}.$$

Also, $[A]_\Lambda^\Lambda = P^{-1} A P = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ A diag. Matrix!

$$\Rightarrow \quad A^5 = P^{-1}\left([A]_\Lambda^\Lambda\right)^5 P = \begin{pmatrix} 0 & -2 & 2 \\ -1 & 4 & -1 \\ 1 & -2 & 0 \end{pmatrix}\begin{pmatrix} 2^5 & 0 & 0 \\ 0 & 3^5 & 0 \\ 0 & 0 & 4^5 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 1/2 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{bmatrix} 2586 & -4264 & -422 \\ 781 & -1108 & -211 \\ 781 & -1140 & -179 \end{bmatrix}.$$

Definition

Diagonalizable Matrices:

An $n \times n$ matrix A is *diagonalizable* if there is a matrix D and an invertible matrix P such that

$$A = P^{-1}DP$$

Facts:

- Let A be an $n \times n$ matrix.
- Let λ_1, λ_2, $\ldots$, λ_k be the eigenvalues of A.
- Let E_{λ_1}, E_{λ_2}, $\ldots$, E_{λ_k} be the eigenspaces associated to each eigenvalue.

If $\dim(E_{\lambda_1}) + \dim(E_{\lambda_2}) + \cdots + \dim(E_{\lambda_k}) = n$, then A is diagonalizable.

EXAMPLE 3. Is the matrix from Example 2 diagonalizable?

A is a 3×3 matrix $\rightarrow$ $n = 3$.

$\lambda_1 = 2$, $\lambda_2 = 3$, $\lambda_3 = 4$

We have

$$\dim(E_2) = 1, \quad \dim(E_3) = 1, \quad \dim(E_4) = 1$$

$$\rightarrow \quad \dim(E_2) + \dim(E_3) + \dim(E_4) = 1 + 1 + 1 = 3$$

$\rightarrow$ A is diagonalizable.

Example 4. Is the matrix

$$A = \begin{bmatrix} 1 & -2 & -6 \\ -2 & 2 & -5 \\ 2 & 1 & 8 \end{bmatrix}$$

diagonalizable? If so, determine the invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

<u>Eigen Values.</u>

$$\det(\lambda I - A) = \lambda^3 - 11\lambda^2 + 39\lambda - 45 = (\lambda-5)(\lambda-3)^2$$

$\rightarrow \quad \lambda = 5, \quad \lambda = 3$ (multiplicity 2).

<u>Eigen Spaces.</u>

<u>E_5 ($\lambda=5$)</u> Solve $(5I-A)v = 0 \qquad v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

$$5I - A = \begin{bmatrix} 4 & 2 & 6 \\ 2 & 3 & 5 \\ -2 & -1 & -3 \end{bmatrix} \rightarrow \left[\begin{array}{ccc|c} 4 & 2 & 6 & 0 \\ 2 & 3 & 5 & 0 \\ -2 & -1 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \begin{matrix} x = -z \\ y = -z \\ z = z \end{matrix}$$

So, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -z \\ -z \\ z \end{bmatrix} = z \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ basis for E_5 & $\dim(E_5) = 1$

<u>E_3 ($\lambda=3$)</u> Solve $(3I-A)v = 0 \qquad v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$3I - A = \begin{bmatrix} 2 & 2 & 6 \\ 2 & 1 & 5 \\ -2 & -1 & -5 \end{bmatrix} \rightarrow \left[\begin{array}{ccc|c} 2 & 2 & 6 & 0 \\ 2 & 1 & 5 & 0 \\ -2 & -1 & -5 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \begin{matrix} x = -2z \\ y = -z \\ z = z \end{matrix}$$

So $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2z \\ -z \\ z \end{bmatrix} = z \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$ basis for E_3 & $\dim(E_3) = 1$

We have $\dim(E_5) + \dim(E_3) = 1+1 = 2 \neq 3$.

Therefore, A is not diagonalizable.

Example 5. Is the matrix

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

diagonalizable? If so, determine the invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

Eigen Values.

$$\det(\lambda I - A) = (\lambda - 1)^2 + 1 = \lambda^2 - 2\lambda + 2$$

$$\lambda = \frac{-(-2) \pm \sqrt{4 - 4\cdot 2}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

Eigen Spaces.

E_{1+i} $(\lambda = 1+i)$ Solve $((1+i)I - A)v = 0$ $v = \begin{bmatrix} x \\ y \end{bmatrix}$, $x, y \in \mathbb{C}$.

$$(1+i)I - A = \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \to \left[\begin{array}{cc|c} i & 1 & 0 \\ -1 & i & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \to ix + y = 0 \Leftrightarrow y = -ix$$

Thus, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -ix \end{bmatrix} = x\begin{bmatrix} 1 \\ -i \end{bmatrix} \to \begin{bmatrix} 1 \\ -i \end{bmatrix}$ basis for E_{1+i} & $\dim(E_{1+i}) = 1$.

E_{1-i} $(\lambda = 1-i)$ Solve $((1-i)I - A)v = 0$, $v = \begin{bmatrix} x \\ y \end{bmatrix}$, $x, y \in \mathbb{C}$.

$$(1-i)I - A = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \to \left[\begin{array}{cc|c} -i & 1 & 0 \\ -1 & -i & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \to -ix + y = 0 \Leftrightarrow y = ix$$

Thus, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ ix \end{bmatrix} = x\begin{bmatrix} 1 \\ i \end{bmatrix} \longrightarrow \begin{bmatrix} 1 \\ i \end{bmatrix}$ basis for E_{1-i} & $\dim(E_{1-i}) = 1$

We have $\dim(E_{1+i}) + \dim(E_{1-i}) = 1 + 1 = 2 \to A$ is diag.

$$P = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \quad \& \quad P^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ i/2 & -i/2 \end{bmatrix}$$

$$\to D = P^{-1}AP = \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix}$$

In general:

An $n \times n$ matrix A is *similar* to an $n \times n$ matrix B if there is an invertible $n \times n$ matrix P such that

$$B = P^{-1}AP.$$

Notation: $A \sim B$ means that A is similar to B.

Facts:

- If A is similar to B and B is similar to C, then A is similar to C.
- If P is the change of bases matrix from α to β and T is a linear transformation, then $[T]_\beta^\beta = P^{-1}[T]_\alpha^\alpha P$. So $[T]_\beta^\beta \sim [T]_\alpha^\alpha$.

Question:

For non-diagonalizable matrices, can we reduce them to a simple form?

In other words, can we find a matrix B, as simple as possible, such that $B \sim A$?

Answer: Yes! We will replace the diagonal form by the Jordan canonical form.

Jordan blocks

A Jordan block is a square matrix A taking the following shape:

$$A = \begin{bmatrix} \mu & 1 & 0 & \cdots & 0 & 0 \\ 0 & \mu & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \mu & 1 \\ 0 & 0 & 0 & \cdots & 0 & \mu \end{bmatrix}.$$

Why are these type of matrices important?

Example 6. Let A be the matrix

$$A = \begin{bmatrix} \mu & 1 & 0 \\ 0 & \mu & 1 \\ 0 & 0 & \mu \end{bmatrix}.$$

(a) Compute $\det(\lambda I - A)$. (b) Find the dimension of the eigenspaces.

(a) $\det(\lambda I - A) = (\lambda - \mu)^3 \rightarrow \lambda = \mu$.

(b) $\underline{E_\mu}$ Solve $(\mu I - A)v = 0$ $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

$$\mu I - A = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rightarrow -y = 0$ & $-z = 0$, x free

$\rightarrow y = z = 0$ & x free var.

$\rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ basis for E_μ & $\dim(E_\mu) = 1$.

We see that: $\left.\begin{array}{l} \text{alg. mult } (\lambda = \mu) = 3 \\ \text{geom. mult } (\lambda = \mu) = 1 \end{array}\right\}$ play important role later.

<u>Remark</u>:

- a $n \times n$ Jordan block associated to a number μ has only one eigenvalue.
- The algebraic multiplicity of this eigenvalue is necessarily equal to n.
- We always have $\dim(E_\mu) = 1$ for an $n \times n$ Jordan block.
- Jordan blocks are the building blocks for the set of matrices that can't be diagonalizable.

Reduction to Jordan Blocks

EXAMPLE 7. We know that the matrix

$$A = \begin{bmatrix} 1 & -2 & -6 \\ -2 & 2 & -5 \\ 2 & 1 & 8 \end{bmatrix}$$

is not diagonalizable. Find a matrix B, not necessarily a diagonal matrix, such that A is similar to B.

We know that

geom. mult $(\lambda = 5) = \dim(E_5) = 1$
alg. mult $(\lambda = 5) = 1$

&

geom. mult $(\lambda = 3) = \dim(E_3) = 1$
alg. mult $(\lambda = 3) = 2$.

So, the matrix B will be contructed as followed

$$B = \begin{bmatrix} \boxed{5} & 0 & 0 \\ 0 & 3 & * \\ 0 & 0 & 3 \end{bmatrix}$$

Nothing to do with block 5.

Block $\begin{bmatrix} 3 & * \\ 0 & 3 \end{bmatrix}$:

$\to \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ geom. mult $= 1$

$\to \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ geom. mult $= 2$

our case $\to$ geom mult $= 1$

$\to \begin{bmatrix} 3 & * \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$.

Therefore:

$$B = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

General Procedure: Suppose A is an $n \times n$ matrix.

- Express $\det(\lambda I - A)$ as

$$\det(\lambda I - A) = (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_k)^{m_k}$$

  where m_1 is the multiplicity of λ_1, m_2 is the multiplicity of λ_2, ..., m_k is the multiplicity of λ_k.

- For each λ_j, write

$$A_j = \begin{bmatrix} J_{m_{j-1}+1} & 0 & \cdots & 0 \\ 0 & J_{m_{j-1}+2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_{m_j} \end{bmatrix}.$$

  where each J_p, for $p = m_{j-1} + 1, \dots m_j$, is a Jordan block

$$J_p = \begin{bmatrix} \lambda_j & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_j & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_j & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_j \end{bmatrix}.$$

- Then the Jordan Canonical Form (JCF) is

$$B = \begin{bmatrix} A_1 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ 0 & 0 & A_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_k \end{bmatrix}$$

- The invertible matrix P such that $B = P^{-1}AP$ is more complicated to find. In theory, the method to find P uses the notion of a **generalized eigenvector**. In our situation, we will use Python to find this matrix P.

If you want to know more on the generalized eigenvectors and the Jordan Canonical Form, I suggest to take a look at the following references:

- A more math article: *Down With Determinants!* by Sheldon Axler, https://www.maa.org/sites/default/files/pdf/awards/Axler-Ford-1996.pdf.
- A Youtube video: https://www.youtube.com/watch?v=GVixvieNnyc.

Example 8. Let A be an 7×7 matrix with the following eigenvalues:

$$\{1, 1, 1, 1, 2, 2, 3\}.$$

Give the possible Jordan canonical form B of the matrix A.

Write $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$

then $B = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix}$

Shape of A_1

alg mult = 1

$$A_1 = \begin{bmatrix} 1 & * & 0 & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix} :$$

$\to \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ geom. mult = 1

$\to \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ geom. mult = 2

$\to \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ geom mult = 3

$\to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ geom mult = 4

Shape of A_2.

$$A_2 = \begin{bmatrix} 2 & * \\ 0 & 2 \end{bmatrix} :$$

$\to \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ geom. mult = 1

$\to \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ geom mult = 2

Shape of A_3.

$$A_3 = [3]$$

So, there are 8 possibilities for B. For example:

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$