Math 307

Chapter 1

Section 1.1: Systems of Linear Equations

Contents

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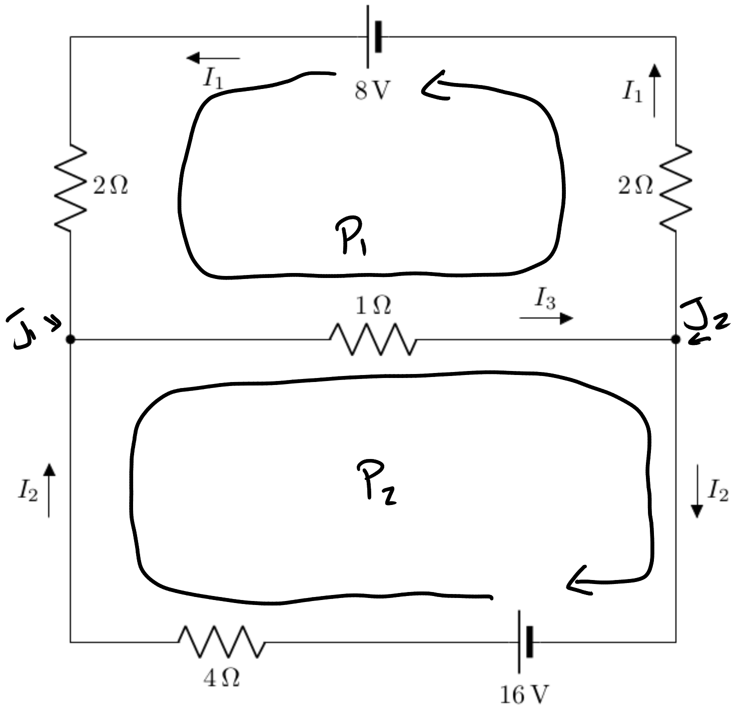

Ohm's Law

- Voltage drop at a resistor is given by $V = IR$.

Kirchhorff's Laws

- Junction: Current flowing into a junction must flow out of it.
- Path: Sum of IR terms in any direction around a closed path is equal to the total voltage in the path in that direction.

Goal: Find the values of I_1, I_2, I_3

J1) $I_1 + I_2 = I_3$

J2) $I_3 = I_1 + I_2$

$\longrightarrow I_1 + I_2 - I_3 = 0$

PATH. P1) $2I_1 + I_3 + 2I_1 = 8 \rightarrow 4I_1 + I_3 = 8$

P2) $4I_2 + I_3 = 16$

To find I_1, I_2, I_3, we must solve the system of lin. eqs.:

$$\begin{cases} I_1 + I_2 - I_3 = 0 \\ 4I_1 \qquad + I_3 = 8 \\ \qquad 4I_2 + I_3 = 16 \end{cases}$$

Linear Equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where

- $a_1, a_2, \ldots, a_n$ are constants.
- n is the number of variables.
- $x_1, x_2, \ldots, x_n$ are the variables (unknowns).
- b is the right-hand side constant term.

Systems of Linear Equations

last example
$n=3$
$m=3$

$$\begin{array}{ll} \text{no 1)} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ \text{no 2)} & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ & \vdots \\ \text{no m)} & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}$$

where

- m is the number of linear equations.
- n is the number of variables.
- $a_{11}, \ldots, a_{mn}$ are constants.
- $b_1, b_2, \ldots, b_m$ are the right-hand side constant terms.
- $x_1, \ldots, x_n$ are the variables (unknowns).

Solution of a System of Linear Equations

A list $(x_1^*, x_2^*, \ldots, x_n^*)$ is a solution to a system of linear equations if it satisfies each equation of the system.

Going back to our previous example

$(1, 3, 4)$ is a solution to our system in the last example.

$\uparrow x_1 \quad \uparrow x_2 \quad \uparrow x_3$

$1 + 3 - 4 = 0$ ✓

$4 \cdot 1 + 4 = 8$ ✓

$4 \cdot 3 + 4 = 16$ ✓

Systems of two linear equations with two variables

$$\begin{aligned} x_1 + x_2 &= 0 \\ 2x_1 + x_2 &= 1. \end{aligned}$$

Method 1 (Isolate)

1) $x_2 = 1 - 2x_1$

2) $x_1 + 1 - 2x_1 = 0$

$\rightarrow 1 - x_1 = 0$

$\rightarrow 1 = x_1$

3) $x_2 = 1 - 2 \cdot 1 = -1$

solution: $(1, -1)$

$\uparrow \quad \uparrow$

$x_1 \quad x_2$

Method 2 (Operations)

1) $x_1 + x_2 = 0 \quad E_1$

$2x_1 + x_2 = 1 \quad E_2$

2) $E_1 - E_2 \rightarrow E_1$

$$\begin{array}{rl} & x_1 + x_2 = 0 \\ - & 2x_1 + x_2 = 1 \\ \hline & -x_1 + \quad = -1 \end{array}$$

$\rightarrow -x_1 = -1$

$$\begin{cases} -x_1 = -1 & E_1 \\ 2x_1 + x_2 = 1 & E_2 \end{cases}$$

3) $-E_1 \rightarrow E_1$

$-(-x_1) = -1 \quad \rightarrow x_1 = 1$

$$\begin{cases} x_1 = 1 & E_1 \\ 2x_1 + x_2 = 1 & E_2 \end{cases}$$

4) $2E_1 - E_2 \rightarrow E_2$

$$\begin{array}{rl} & 2x_1 \quad = 2 \\ - & 2x_1 + x_2 = 1 \\ \hline & -x_2 = 1 \end{array}$$

$$\begin{cases} x_1 = 1 & E_1 \\ -x_2 = 1 & E_2 \end{cases}$$

$\downarrow$

$$\boxed{\begin{cases} x_1 = 1 \\ x_2 = -1 \end{cases}}$$

Gauss-Jordan Elimination

Based on three *elementary operations* on the equations:

- Interchange two equations in the system.
- Replace an equation by a multiple of itself.
- Replace an equation by itself plus a multiple of another equation.

Main GOAL: transform our system into

$$\begin{aligned} x + 0y + 0z &= \tilde{b}_1 \\ 0x + y + 0z &= \tilde{b}_2 \\ 0x + 0y + z &= \tilde{b}_3. \end{aligned}$$

Example 1. Find the solution(s) to the following system of linear equations:

$$
\begin{aligned}
1x - y + z &= 0 & E_1\\
2x - 3y + 4z &= -2 & E_2\\
-2x - y + z &= 7. & E_3
\end{aligned}
$$

1) $-2E_1 + E_2 \to E_2$

$$
\begin{array}{rl}
 & -2x + 2y - 2z = 0\\
+ & 2x - 3y + 4z = -2\\
\hline
 & -y + 2z = -2
\end{array}
$$

$2E_1 + E_3 \to E_3$

$$
\begin{array}{rl}
 & 2x - 2y + 2z = 0\\
+ & -2x - y + z = 7\\
\hline
 & -3y + 3z = 7
\end{array}
$$

$$
\begin{cases}
x - y + z = 0 & E_1\\
-y + 2z = -2 & E_2\\
-3y + 3z = 7 & E_3
\end{cases}
$$

2) $E_1 - E_2 \to E_1$

$$
\begin{array}{rl}
 & x - y + z = 0\\
- & -y + 2z = -2\\
\hline
 & x - z = 2
\end{array}
$$

$-3E_2 + E_3 \to E_3$

$$
\begin{array}{rl}
 & 3y - 6z = 6\\
+ & -3y + 3z = 7\\
\hline
 & -3z = 13
\end{array}
$$

$$
\begin{cases}
x - z = 2 & E_1\\
-y + 2z = -2 & E_2\\
-3z = 13 & E_3
\end{cases}
$$

3). $2E_3 + 3E_2 \to E_2$

$$
\begin{array}{rl}
 & -6z = 26\\
+ & -3y + 6z = -6\\
\hline
 & -3y = 20
\end{array}
$$

$3E_1 - E_3 \to E_1$

$$
\begin{array}{rl}
 & 3x - 3z = 6\\
- & -3z = 13\\
\hline
 & 3x = -7
\end{array}
$$

$$
\begin{cases}
3x = -7\\
-3y = 20\\
-3z = 13
\end{cases}
$$

4)

$$\frac{3x}{3} = \frac{-7}{3} \to$$

$$\frac{-3y}{-3} = \frac{20}{-3} \to$$

$$\frac{-3z}{-3} = \frac{13}{-3} \to$$

$$\boxed{\begin{aligned} x &= -\frac{7}{3}\\ y &= -\frac{20}{3}\\ z &= -\frac{13}{3}\end{aligned}}$$

Augmented Matrix

More efficient way: transform the system in an **augmented matrix**.

$$
\begin{aligned}
a_{11}\cancel{x_1} + a_{12}\cancel{x_2} + \cdots + a_{1n}\cancel{x_n} &= b_1 \\
a_{21}\cancel{x_1} + a_{22}\cancel{x_2} + \cdots + a_{2n}\cancel{x_n} &= b_2 \\
&\vdots \\
a_{m1}\cancel{x_1} + a_{m2}\cancel{x_2} + \cdots + a_{mn}\cancel{x_n} &= b_m
\end{aligned}
\quad \Longrightarrow \quad
\begin{array}{c}
\begin{array}{ccccc} x_1 & x_2 & & x_n & \overset{b}{\downarrow} \end{array} \\
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn} & b_m
\end{bmatrix}
\end{array}
$$

Example 2. Find the augmented matrix of the system of Example 1.

$$
\begin{aligned}
1x - y + z &= 0 \\
2x - 3y + 4z &= -2 \\
-2x - y + z &= 7
\end{aligned}
\quad \Longrightarrow \quad
\begin{array}{c}
\begin{array}{cccc} x & y & z & b \end{array} \\
\begin{bmatrix}
1 & -1 & 1 & 0 \\
2 & -3 & 4 & -2 \\
-2 & -1 & 1 & 7
\end{bmatrix}
\end{array}
$$

Elementary operations revisited

Elementary operations on linear equations become elementary operations on the rows of the augmented matrix:

- Interchange two rows.
- Replace a row by a multiple of itself.
- Replace a row by itself plus a multiple of another row.

Example 3. Solve the system:

$$
\begin{aligned}
2x + 3y - z &= 3\\
-x - y + 3z &= 0\\
x + 2y + 2z &= 3\\
y + 5z &= 3.
\end{aligned}
$$

$$
\begin{bmatrix} 2 & 3 & -1 & 3 \\ -1 & -1 & 3 & 0 \\ 1 & 2 & 2 & 3 \\ 0 & 1 & 5 & 3 \end{bmatrix}
\sim
\begin{bmatrix} 2 & 3 & -1 & 3 \\ \boxed{0} & 1 & 5 & 3 \\ 0 & -1 & -5 & -3 \\ 0 & 1 & 5 & 3 \end{bmatrix}
\begin{matrix} \\ R_1 + 2R_2 \to R_2 \\ R_1 - 2R_3 \to R_3 \\ \\ \end{matrix}
$$

$$
\sim
\begin{bmatrix} 2 & 0 & -16 & -6 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\begin{matrix} R_1 - 3R_2 \to R_1 \\ \\ R_2 + R_3 \to R_3 \\ R_2 - R_4 \to R_4 \end{matrix}
$$

$$
\sim
\begin{array}{cccc}
x & y & z & b \\
\end{array}
\begin{bmatrix} 1 & 0 & -8 & -3 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\begin{matrix} \tfrac{1}{2}R_1 \to R_1 \\ \\ \\ \\ \end{matrix}
$$

$x - 8z = -3$ & $y + 5z = 3$

$\Rightarrow$ $x = -3 + 8z$ & $y = 3 - 5z$.

- **Remarks.**
  - z is a free variable or free parameter.
  - x & y are called dependent variables.
  - Geometry: solutions is a line in 3D

    $z = t$ $\quad x = -3 + 8t$ $\qquad$ direction is

    $y = 3 - 5t$ $\qquad \langle 8, -5, 1 \rangle$.

    $z = t$
  - There are infinitely many solutions!

    $\langle 5, -2, 1 \rangle_{z=1}$ & $\langle 13, -7, 2 \rangle_{z=2}$ are solutions

Example 4. Solve the system:

$$n = 5 \qquad m = 3$$

$$\begin{aligned} 4x_1 - 8x_2 - x_3 + x_4 + 3x_5 &= 0 \\ 5x_1 - 10x_2 - x_3 + 2x_4 + 3x_5 &= 0 \\ 3x_1 - 6x_2 - x_3 + x_4 + 2x_5 &= 0. \end{aligned}$$

$$\begin{array}{c c} & \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & b \end{array} \\ \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} & \left[\begin{array}{cccccc} 4 & -8 & -1 & 1 & 3 & 0 \\ 5 & -10 & -1 & 2 & 3 & 0 \\ 3 & -6 & -1 & 1 & 2 & 0 \end{array}\right] \end{array} \sim \left[\begin{array}{cccccc} 4 & -8 & -1 & 1 & 3 & 0 \\ 0 & 0 & -1 & -3 & 3 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right] \begin{array}{l} \\ 5R_1 - 4R_2 \to R_2 \\ 3R_1 - 4R_3 \to R_3 \end{array}$$

$$\sim \left[\begin{array}{cccccc} 4 & -8 & 0 & 4 & 0 & 0 \\ 0 & 0 & -1 & -3 & 3 & 0 \\ 0 & 0 & 0 & -4 & 4 & 0 \end{array}\right] \begin{array}{l} R_1 - R_2 \to R_1 \\ \\ R_2 + R_3 \to R_3 \end{array}$$

$$\sim \left[\begin{array}{cccccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & -3 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{array}\right] \begin{array}{l} 1/4\, R_1 \to R_1 \\ -1R_2 \to R_2 \\ -\frac{1}{4} R_3 \to R_3 \end{array}$$

$$\sim \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & b \end{array} \\ \left[\begin{array}{cccccc} \underline{1} & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & \underline{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \underline{1} & -1 & 0 \end{array}\right] \end{array} \begin{array}{l} R_1 - R_3 \to R_1 \\ R_2 - 3R_3 \to R_2 \\ \\ \end{array}$$

$$\begin{aligned} x_1 - 2x_2 + x_5 &= 0 \\ x_3 &= 0 \\ x_4 - x_5 &= 0 \end{aligned} \quad \longrightarrow \quad \boxed{\begin{aligned} x_1 &= 2x_2 - x_5 \\ x_3 &= 0 \\ x_4 &= x_5 \end{aligned}}$$

Here x_2 & x_5 are free parameters.

x_1, x_3 & x_4 are dependent variables.

Reduced row-echelon form (RREF)

Transformed augmented matrix after row operations:

- Any rows of zero (called zero rows) appear at the bottom.
- The first nonzero entry of a nonzero row is 1 (called a leading 1).
- The leading 1 of a nonzero row appears to the right of the leading 1 of any preceding row.
- All the other entries of a column containing a leading 1 are zero.

Consistent Systems vs Inconsistent Systems

- Consistent: means the system of equations has at least one solution.
  - How to recognize that a system is consistent?

(1) RREF has the form

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & & * \\ 0 & 1 & 0 & \cdots & & * \\ \vdots & \vdots & \vdots & & & \\ 0 & 0 & 0 & \cdots & 1 & * \end{bmatrix}$$

(2) RREF has the form

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & * \\ 0 & 1 & 0 & \cdots & 0 & * \\ \vdots & \vdots & \vdots & & & \\ 0 & 0 & 0 & \cdots & 1 & * \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \rightarrow$$ contain rows of zeros.

- Inconsistent: means the system of equations has no solution.
  - How to recognize that a system is inconsistent?

(1)

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & * \\ 0 & 1 & 0 & \cdots & 0 & * \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \tilde{b} \end{bmatrix} \rightarrow$$ line of zeros except the last entry which is $\neq 0$ ($\tilde{b} \neq 0$)

This means $0 = \tilde{b}$ with $\tilde{b} \neq 0$.

Homogeneous System

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

- Trivial solution: $x_1 = x_2 = \cdots = x_n = 0$.

THEOREM 5. A homogeneous system of m linear equations in n variables

- has infinitely many solutions if $m < n$.
- has only the trivial solution if $m = n$.

In the other case, when $m > n$, we have to do more work. To be more precised, we still have to find the RREF of the augmented matrix of the associated system and conclude from the RREF if the system has solutions or not.

Goal. Transform the augmented matrix into an new augmented matrix with the following properties:

- any zero rows appear at the bottom.
- The first nonzero entry of a nonzero row is 1.
- The leading 1 of a nonzero row appears to the right of the leading 1 of any preceding row.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$ Example of a valid reduced Gaussian Form.

Example 6. Determine the values of a, b, and c so that the system

$$x - y + 2z = a$$
$$2x + y - z = b$$
$$x + 2y - 3z = c$$

has solutions.

$$\begin{bmatrix} 1 & -1 & 2 & a \\ 2 & 1 & -1 & b \\ 1 & 2 & -3 & c \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & a \\ 0 & -3 & 5 & 2a-b \\ 0 & -3 & 5 & a-c \end{bmatrix} \begin{matrix} \\ 2R_1 - R_2 \to R_2 \\ R_1 - R_3 \to R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & a \\ 0 & -3 & 5 & 2a-b \\ 0 & 0 & 0 & a-b+c \end{bmatrix} \begin{matrix} \\ \\ R_2 - R_3 \end{matrix}$$

$a=1$
$b=1$ $\quad 1-1+0=0$ ✓ $\quad \to \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & -3 & 5 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ ✓
$c=0$

Here, if $\boxed{a-b+c=0}$, the system is consistent!