2. $T:\mathbb{R}^2\to\mathbb{R}^2$ by

$$T\begin{bmatrix}x_1^{2}\\x_2^{1}\end{bmatrix}=\begin{bmatrix}5x_1+3x_2\\-6x_1-4x_2\end{bmatrix};$$

α the standard basis for $\mathbb{R}^2$; β the basis consisting of

$$\alpha=\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}\qquad \begin{bmatrix}2\\1\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix};\leftarrow\beta$$

$$v=\begin{bmatrix}5\\-4\end{bmatrix}$$

a) Find $[T]_{\alpha}^{\alpha}$.

b) Find the change of basis matrix from α to β.

c) Find the change of basis matrix from β to α.

d) Find $[T]_{\beta}^{\beta}$.

e) Find $[v]_{\beta}$.

f) Find $[T(v)]_{\beta}$.

g) Use the result of part (f) to find $T(v)$.

a) $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}5\\-6\end{bmatrix}=c_1\begin{bmatrix}1\\0\end{bmatrix}+c_2\begin{bmatrix}0\\1\end{bmatrix}\to\ c_1=5,\ c_2=-6$

$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}3\\-4\end{bmatrix}=c_1\begin{bmatrix}1\\0\end{bmatrix}+c_2\begin{bmatrix}0\\1\end{bmatrix}\to\ c_1=3,\ c_2=-4$

$$[T]_{\alpha}^{\alpha}=\begin{bmatrix}5&3\\-6&-4\end{bmatrix}.$$

b) $I\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}x\\y\end{bmatrix}$. P, change of basis from α to β then

$P=[I]_{\beta}^{\alpha}$

$I\left(\begin{bmatrix}2\\1\end{bmatrix}\right)=\begin{bmatrix}2\\1\end{bmatrix}=c_1\begin{bmatrix}1\\0\end{bmatrix}+c_2\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}c_1\\c_2\end{bmatrix}\to\ c_1=2,\ c_2=1$

$I\left(\begin{bmatrix}1\\1\end{bmatrix}\right)=\begin{bmatrix}1\\1\end{bmatrix}=c_1\begin{bmatrix}1\\0\end{bmatrix}+c_2\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}c_1\\c_2\end{bmatrix}\to\ c_1=1,\ c_2=1$

$$P=[I]_{\beta}^{\alpha}=\begin{bmatrix}2&1\\1&1\end{bmatrix}$$

c) ① change of basis from β to α : inverse of P : $\boxed{P^{-1}}$.

② Find $[I]_{\alpha}^{\beta}$.

① $$\left[\begin{array}{cc|cc}2&1&1&0\\1&1&0&1\end{array}\right]\sim\left[\begin{array}{cc|cc}2&1&1&0\\0&-1&1&-2\end{array}\right]\sim\left[\begin{array}{cc|cc}2&0&2&-2\\0&-1&1&-2\end{array}\right]\sim\left[\begin{array}{cc|cc}1&0&1&-1\\0&1&-1&2\end{array}\right]$$ with $\underbrace{\begin{matrix}1&-1\\-1&2\end{matrix}}_{P^{-1}}$

So, $P^{-1}=\begin{bmatrix}1&-1\\-1&2\end{bmatrix}$.

② $I\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}1\\0\end{bmatrix}=c_1\begin{bmatrix}2\\1\end{bmatrix}+c_2\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}2c_1+c_2\\c_1+c_2\end{bmatrix}\to\begin{cases}2c_1+c_2=1\\c_1+c_2=0\end{cases}$

$c_1=-c_2\to-2c_2+c_2=1\to-c_2=1\to c_2=-1$

then $c_1=1$

$\to\begin{bmatrix}1\\0\end{bmatrix}_{\beta}=\begin{bmatrix}1\\-1\end{bmatrix}$

$I\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\1\end{bmatrix} = c_1\begin{bmatrix}2\\1\end{bmatrix} + c_2\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}2c_1+c_2\\c_1+c_2\end{bmatrix} \to \begin{cases}2c_1+c_2=0\\c_1+c_2=1\end{cases}$

$2c_1+c_2=0 \to c_2=-2c_1 \to c_1-2c_1=1 \to c_1=-1$

$\hookrightarrow c_2=-2(-1)=2.$

$\to \begin{bmatrix}0\\1\end{bmatrix}_\beta = \begin{bmatrix}-1\\2\end{bmatrix}$

So, $[I]_\alpha^\beta = P^{-1} = \begin{bmatrix}1 & -1\\-1 & 2\end{bmatrix}.$

d) $T\left(\begin{bmatrix}2\\1\end{bmatrix}\right) = \begin{bmatrix}13\\-16\end{bmatrix} = c_1\begin{bmatrix}2\\1\end{bmatrix} + c_2\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}2c_1+c_2\\c_1+c_2\end{bmatrix}$

$\to \begin{cases}2c_1+c_2=13\\c_1+c_2=-16\end{cases}$

$c_2=-16-c_1$

$\hookrightarrow 2c_1-16-c_1=13$

$\hookrightarrow c_1=29$

$\hookrightarrow c_2=-45$

$\to \begin{bmatrix}13\\-16\end{bmatrix}_\beta = \begin{bmatrix}29\\-45\end{bmatrix}.$

$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}8\\-10\end{bmatrix} = c_1\begin{bmatrix}2\\1\end{bmatrix} + c_2\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}2c_1+c_2\\c_1+c_2\end{bmatrix}$

$\to \begin{cases}2c_1+c_2=8\\c_1+c_2=-10\end{cases}$

$c_2=-10-c_1$

$\hookrightarrow 2c_1-10-c_1=8$

$\hookrightarrow c_1=18$

$\hookrightarrow c_2=-10-18=-28$

$\to \begin{bmatrix}8\\-10\end{bmatrix}_\beta = \begin{bmatrix}18\\-28\end{bmatrix}$

$[T]_\beta^\beta = \begin{bmatrix}29 & 18\\-45 & -28\end{bmatrix}$

e) ① $[v]_\beta = \begin{bmatrix}c_1\\c_2\end{bmatrix}$ where $\begin{bmatrix}5\\-4\end{bmatrix} = c_1\begin{bmatrix}2\\1\end{bmatrix} + c_2\begin{bmatrix}1\\1\end{bmatrix}.$

$= \begin{bmatrix}2c_1+c_2\\c_1+c_2\end{bmatrix}$

$\to \begin{cases}2c_1+c_2=5\\c_1+c_2=-4\end{cases} \to c_2=-4-c_1$

$\hookrightarrow 2c_1-4-c_1=5$

$\hookrightarrow c_1=9$

$\hookrightarrow c_2=-13$

Therefore, $[v]_\beta = \begin{bmatrix}9\\-13\end{bmatrix}.$

② $[v]_\alpha = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$, $[v]_\beta = [I]_\alpha^\beta [v]_\alpha = P^{-1} [v]_\alpha$

$$[v]_\beta = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ -4 \end{bmatrix} = \begin{bmatrix} 9 \\ -13 \end{bmatrix} .$$

f) $[T(v)]_\beta = [T]_\beta^\beta \; [v]_\beta$ $\left([T(v)]_\beta = [T]_\alpha^\beta [v]_\alpha \right.$ $\left. \alpha = \beta . \right)$

$$= \begin{bmatrix} 29 & 18 \\ -45 & -28 \end{bmatrix} \begin{bmatrix} 9 \\ -13 \end{bmatrix}$$

$$= \begin{bmatrix} 27 \\ -41 \end{bmatrix} .$$

g) $T(x) = (27) \begin{bmatrix} 2 \\ 1 \end{bmatrix} + (-41) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$= \begin{bmatrix} 13 \\ -14 \end{bmatrix} .$$

9. Suppose that v_1, v_2, v_3 form a basis α for a vector space V and $T : V \to V$ is a linear transformation such that

$$T(v_1) = v_1 - v_2, \qquad T(v_2) = v_2 - v_3,$$
$$T(v_3) = v_3 - v_1.$$

a) Find $[T]_\alpha^\alpha$.

b) Find $[T(v)]_\alpha$ if $v = v_1 - 2v_2 + 3v_3$.

c) Use the result of part (b) to find $T(v)$ in terms of v_1, v_2, v_3.

a) $T(v_1) = \underset{\substack{\uparrow\\ c_1}}{v_1} - \underset{\substack{\uparrow\\ c_2}}{v_2} + \underset{\substack{\uparrow\\ c_3}}{0v_3}$

$T(v_2) = \underset{\substack{\uparrow\\ c_1}}{0v_1} + \underset{\substack{\uparrow\\ c_2}}{v_2} - \underset{\substack{\uparrow\\ c_3}}{v_3}$

$T(v_3) = \underset{\substack{\uparrow\\ c_1}}{-v_1} + \underset{\substack{\uparrow\\ c_2}}{0v_2} + \underset{\substack{\uparrow\\ c_3}}{v_3}$

$$[T]_\alpha^\alpha = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

b) $[T(v)]_\alpha = [T]_\alpha^\alpha [v]_\alpha$

$$[v]_\alpha = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ 5 \end{bmatrix} \begin{matrix} \to c_1 \\ \to c_2 \\ \to c_3 \end{matrix}$$

c) $T(v) = c_1 v_1 + c_2 v_2 + c_3 v_3$

$= -2v_1 - 3v_2 + 5v_3$