

MATH 307

CHAPTER 2

SECTION 2.4: DIMENSION; NULLSPACE, ROW SPACE, AND COLUMN SPACE

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Definition

If a vector space V has a basis α of n vectors v_1, v_2, \dots, v_n , then the **dimension** of V is the number of elements in the basis α .

The definition comes from the following result telling us that all basis have the same number of elements.

THEOREM 1. If v_1, v_2, \dots, v_n and w_1, w_2, \dots, w_m both form bases for a vector space V , then $n = m$.

Remark:

- We use the symbol $\dim(V)$ to denote the dimension of a vector space V .
- If, for a vector space V , $\dim(V)$ is a finite number, then V is said to be **finite dimensional**.
- If, for a vector space V , $\dim(V)$ is infinite (so equal to $+\infty$), then V is said to be **infinite dimensional**.

EXAMPLE 2. Which of the following vector space is finite dimensional and which one is infinite dimensional?

- | | |
|---------------------|------------|
| 1. \mathbb{R}^3 . | 3. P_n . |
| 2. \mathbb{R}^n . | 4. P . |

Basic Facts on Basis and Dimension

For a set of vectors to be a basis, it must have the same number of elements as the dimension of the vector space.

When there are not enough vectors but we know that there are linearly independent, we can add a set of new vectors so that the new set composed of the old vectors and the new vectors become a basis.

EXAMPLE 3. Let α be the set of vectors

$$\alpha = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Complete the set α with a certain number of vectors so that it becomes a basis of \mathbb{R}^3 .

Remark: Given a vector space V of dimension n ,

- if v_1, v_2, \dots, v_k are linearly independent vectors in V , then there exist vectors v_{k+1}, \dots, v_n so that $v_1, v_2, \dots, v_k, v_{k+1}, \dots, v_n$ form a basis for V .

When there are too many vectors, but we know that the set of vectors spans V , we can extract a basis from it.

EXAMPLE 4. Let α be the set

$$\alpha = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Extract a basis from the set α .

Remark: Given a vector space V of dimension n ,

- if v_1, v_2, \dots, v_k span V , then there exists a subset of v_1, v_2, \dots, v_k that forms a basis of V .

THEOREM 5. Suppose that V is a vector space of dimension n .

- If the vectors v_1, v_2, \dots, v_n are linearly independent, then v_1, v_2, \dots, v_n form a basis for V .
- If the vectors v_1, v_2, \dots, v_n span V , then v_1, v_2, \dots, v_n form a basis for V .

EXAMPLE 6. Show that $x^2 - 1, x^2 + 1, x + 1$ form a basis for P_2 .