Math 307

Chapter 2

Section 2.1: Vector Spaces

Contents

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Solutions to System of Linear Equations

EXAMPLE 1. Describe the set of solutions of the following system of linear equations.

$$\begin{aligned} 2x + 3y - z &= \cancel{3}\,0 \\ -x - y + 3z &= 0 \\ x + 2y + 2z &= \cancel{8}\,0 \\ y + 5z &= \cancel{8}.\,0 \end{aligned}$$

$$\begin{bmatrix} 2 & 3 & -1 & 0 \\ -1 & -1 & 3 & 0 \\ 1 & 2 & 2 & 0 \\ 0 & 1 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -8 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \quad x = 8z$ & $y = -5z$, $\quad z$ free parameter.

Structure.

1) $z = 1 \to X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -5 \\ 1 \end{bmatrix}$ & $z = -1 \to X_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -8 \\ 5 \\ -1 \end{bmatrix}$

$X_1 + X_2 = \begin{bmatrix} 8-8 \\ -5+5 \\ 1-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \to$ solution from $z = 0$

2) $2X_1 = \begin{bmatrix} 16 \\ -10 \\ 2 \end{bmatrix} \to$

$$\begin{aligned} x &= 2\cdot 8 \cdot \underbrace{1}_{z} = 8\cdot\underbrace{2}_{z} \\ y &= 2\cdot -5\cdot 1 = -5\cdot\underbrace{2}_{z} \\ z &= 2\cdot 1 = 2 \end{aligned}$$

$\to$ solution from $z = 2$.

3) General solution

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8z \\ -5z \\ z \end{bmatrix} = z\begin{bmatrix} 8 \\ -5 \\ 1 \end{bmatrix}, \quad z \in \mathbb{R}$$

$$GS = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x = 8z,\ y = -5z,\ z \in \mathbb{R} \right\}$$

GS with $+$ (matrix add.) & $\cdot$ (scalar multipli.) is called a vector space. This works only for homogeneous systems.

Precise Definition

A nonempty set V is a **vector space** if there are operations of addition (denoted by $+$) and scalar multiplication (denoted by $\cdot$) on V such that the following eight properties are satisfied:

1. $u + v = v + u$ for any u and v in V;
2. $u + (v + w) = (u + v) + w$ for any u, v, w in V;
3. There is an element denoted 0 in V so that $v + 0 = v$ for any v in V.
4. For each v in V there is an element denoted $-v$ so that $v + (-v) = 0$.
5. $c \cdot (u + v) = c \cdot u + c \cdot v$ for all real number c and for all u and v in V;
6. $(c + d) \cdot v = c \cdot v + d \cdot v$ for all real numbers c and d and for all v in V;
7. $c \cdot (d \cdot v) = (cd) \cdot v$ for all real numbers c and d and for all v in V;
8. $1 \cdot v = v$ for all v in V.
   number

Remarks:

- The eight above properties are called *axioms*, *postulates*, or *laws* of a vector spaces.
- Don't confuse the abstract vectors from the more concrete column-vectors or row-vectors.
- The elements of the set V are called *vectors*.
- The real numbers are called *scalars*.

Column Vectors as a Vector space

$V = \mathbb{R}^3$

EXAMPLE 2. The set (V) of all 3×1 column vectors, denoted by $\mathbb{R}^3$, is a vector space if we define

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} := \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix} \quad \text{and} \quad c \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} := \begin{bmatrix} cx_1 \\ cx_2 \\ cx_3 \end{bmatrix}.$$

1) $u = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ & $v = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \Rightarrow u + v = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix} = \begin{bmatrix} y_1 + x_1 \\ y_2 + x_2 \\ y_3 + x_3 \end{bmatrix} = v + u$

2) $w = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \to (u+v) + w = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix} + \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} x_1 + (y_1 + z_1) \\ x_2 + (y_2 + z_2) \\ x_3 + (y_3 + z_3) \end{bmatrix}$

$= \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{= u} + \begin{bmatrix} y_1 + z_1 \\ y_2 + z_2 \\ y_3 + z_3 \end{bmatrix} = u + \left(\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \right)$

$= u + (v + w)$

3) $0 := \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow u + 0 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 0 = \begin{bmatrix} x_1 + 0 \\ x_2 + 0 \\ x_3 + 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = u$

4) $u = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, what would $-u$ s.t. $u + (-u) = 0$

$-u := \begin{bmatrix} -x_1 \\ -x_2 \\ -x_3 \end{bmatrix} \rightarrow u + (-u) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} -x_1 \\ -x_2 \\ -x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_1 \\ x_2 - x_2 \\ x_3 - x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0$

5) c real generic number

$c \cdot (u + v) = c \cdot \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix} = \begin{bmatrix} c(x_1 + y_1) \\ c(x_2 + y_2) \\ c(x_3 + y_3) \end{bmatrix} = \begin{bmatrix} cx_1 + cy_1 \\ cx_2 + cy_2 \\ cx_3 + cy_3 \end{bmatrix}$

$= \begin{bmatrix} cx_1 \\ cx_2 \\ cx_3 \end{bmatrix} + \begin{bmatrix} cy_1 \\ cy_2 \\ cy_3 \end{bmatrix}$

$= c \cdot u + c \cdot v$

6) $(c + d) \cdot u = \begin{bmatrix} (c+d) x_1 \\ (c+d) x_2 \\ (c+d) x_3 \end{bmatrix} = \begin{bmatrix} cx_1 + dx_1 \\ cx_2 + dx_2 \\ cx_3 + dx_3 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \\ cx_3 \end{bmatrix} + \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} = c \cdot u + d \cdot u$

c, d are generic.

7) $c \cdot (d \cdot u) = c \cdot \left(\begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} \right) = \begin{bmatrix} cdx_1 \\ cdx_2 \\ cdx_3 \end{bmatrix} = (cd) \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = (cd) \cdot u.$

8) $1 \cdot u = 1 \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = u.$

So, 1-8 are satisfied $\rightarrow$ $(V, +, \cdot)$ is a vector space.

or V is a vector space.

Example 3. More generally, the set of all $n \times 1$ column vectors, denoted by $\mathbb{R}^n$ is a vector space if we define

$\mathbb{R}^n$

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} := \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix} \quad \text{and} \quad c \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} := \begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{bmatrix}.$$

Remark:

- The set of $1 \times n$ row vectors is also a vector space with the addition and scalar multiplication defined component-wise in a similar way.

Matrices as a Vector Space

coeff. are numbers.

EXAMPLE 4. The set of $m \times n$ matrices $M_{m\times n}(\mathbb{R})$ is a vector space if we define the addition of two matrices and the scalar multiplication of a real number with a matrix by the matrix addition and matrix scalar multiplication defined in the previous chapter (see section 1.2).

1-8 are satisfied because of section 1.2.

Functions as a Vector Space

EXAMPLE 5. Let $F(a, b)$ denote the set of all real-valued functions defined on (a, b). Some examples are $f(x) = x^2$, $f(x) = \sin x$, $f(x) = |x|$, etc.

We define the addition of two functions f and g to be the new function $(f + g)$ defined on (a, b) by

$$(f + g)(x) := f(x) + g(x).$$

We define the scalar multiplication of a function f with a real number c to be the new function (cf) defined on (a, b) by

$$(\underbrace{cf}_{c\cdot f})(x) := cf(x).$$

Show that $F(a, b)$ is a vector space.

1) f, g generic functions. Goal: $f+g = g+f$. $\Longleftrightarrow \overbrace{(f+g)}^{h}(x) = \overbrace{(g+f)}^{k}(x)$.

$(f+g)(x) = f(x) + g(x) = g(x) + f(x)$ ($f(x), g(x)$ are numbers)

$= (g+f)(x)$ $\checkmark$

2) Goal: $(f+g)+h = f+(g+h)$ $\Longleftrightarrow ((f+g)+h)(x) = (f+(g+h))(x)$

$((f+g)+h)(x) = (f+g)(x) + h(x) = (f(x) + \underbrace{g(x)) + h(x)}$

$= f(x) + (g+h)(x)$

$= (f+(g+h))(x)$ $\checkmark$

3) Find 0 s.t. $f + 0 = f$. $\Leftrightarrow$ $f(x) + 0(x) = f(x)$

$0(x) = 0$ $\rightarrow$ $f(x) + 0(x) = f(x) + 0 = f(x)$ ✓

4) Find $-f$ s.t. $f + (-f) = 0$ $\Leftrightarrow$ $f(x) + (-f)(x) = 0(x)$
$= 0$

$(-f)(x) = -f(x)$ $\rightarrow$ $f(x) + (-f)(x) = f(x) + (-f(x)) = 0$
$= 0(x)$. ✓

5) <u>Goal</u>: $c \cdot (f+g) = c \cdot f + c \cdot g$ $\Leftrightarrow$ $(c \cdot (f+g))(x)$
$= (c \cdot f)(x) + (c \cdot g)(x)$

$(c \cdot (f+g))(x) = c\,(f+g)(x) = c(f(x) + g(x)) = cf(x) + cg(x)$
$= (c \cdot f)(x) + (c \cdot g)(x)$ ✓

7) <u>Goal</u>: $c \cdot (d \cdot f) = (cd) \cdot f$ $\leftrightarrow$ $(c \cdot (d \cdot f))(x) = ((cd) \cdot f)(x)$.

$(c \cdot (d \cdot f))(x) = c\,(d \cdot f)(x) = c\,(d\,f(x)) = (cd)\,f(x)$
$= ((cd) \cdot f)(x)$ ✓

6) <u>Goal</u>: $(c+d) \cdot f = c \cdot f + d \cdot f$ $\leftrightarrow$ $((c+d) \cdot f)(x) =$
$(c \cdot f)(x) + (d \cdot f)(x)$

$((c+d) \cdot f)(x) = (c+d)f(x) = cf(x) + df(x) = (c \cdot f)(x) + (d \cdot f)(x)$ ✓

8) <u>Goal</u>: $(1 \cdot f) = f$ $\leftrightarrow$ $(1 \cdot f)(x) = f(x)$.

$(1 \cdot f)(x) = 1\,f(x) = f(x)$ ✓

1-8 are satisfied $\Rightarrow$ $F(a,b)$ is a vector space.

A nonexample

Example 6. Let V be the set of 1×2 row vectors. We define an addition and a scalar multiplication by

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} + \begin{bmatrix} y_1 & y_2 \end{bmatrix} := \begin{bmatrix} x_1 + y_1 + 1 & x_2 + y_2 \end{bmatrix} \quad \text{and} \quad c \cdot \begin{bmatrix} x_1 & x_2 \end{bmatrix} := \begin{bmatrix} cx_1 & cx_2 \end{bmatrix}.$$

Is V equipped with these operations a vector space?

1) $X = [x_1 \quad x_2]$ $\rightarrow$ $X + Y = [x_1 + y_1 + 1 \quad x_2 + y_2]$
$Y = [y_1 \quad y_2]$ $\quad = [\underline{y_1} + \underline{x_1} + 1 \quad \underline{y_2} + \underline{x_2}]$
$= Y + X$ ✓

2) $Z = [z_1 \quad z_2]$ $\rightarrow$ $X + (Y + Z) = X + [y_1 + z_1 + 1 \quad y_2 + z_2]$
$= [x_1 + (y_1 + z_1 + 1) + 1 \quad x_2 + y_2 + z_2]$
$= [(x_1 + y_1 + 1) + z_1 + 1 \quad (x_2 + y_2) + z_2]$
$= [x_1 + y_1 + 1 \quad x_2 + y_2] + [z_1 \quad z_2]$
$= (X + Y) + Z$ ✓

3) Goal: $X + \overset{[y_1 \quad y_2]}{0} = X$ $\leftrightarrow$ $[x_1 + y_1 + 1 \quad x_2 + y_2] = [x_1 \quad x_2]$
$\rightarrow$ $\cancel{x_1} + y_1 + 1 = \cancel{x_1}$ & $\cancel{x_2} + y_2 = \cancel{x_2}$ $\rightarrow$ $0 = [-1 \quad 0]$
$\rightarrow$ $y_1 = -1$ & $y_2 = 0$

4) Goal Find $-X$ s.t. $X + (-X) = 0$.
$-X = [-2 - x_1 \quad -x_2]$ $\rightarrow$ $X + (-X) = [\cancel{x_1} + (-2 - \cancel{x_1}) + 1 \quad x_2 - x_2]$
$= [-1 \quad 0] = 0$

5) $c\cdot(X+Y) = c\cdot[x_1+y_1+1 \quad x_2+y_2] = [cx_1+cy_1+c \quad cx_2+cy_2]$

$c\cdot X + c\cdot Y = [cx_1 \quad cx_2] + [cy_1 \quad cy_2]$

$= [cx_1+cy_1+1 \quad cx_2+cy_2]$

so, $c\cdot(X+Y) \overset{?}{=} c\cdot X + c\cdot Y$

$\hookrightarrow [cx_1+cy_1+c \quad cx_2+cy_2] \overset{?}{=} [cx_1+cy_1+1 \quad cx_2+cy_2]$

$\rightarrow \cancel{cx_1}+\cancel{cy_1}+c = \cancel{cx_1}+\cancel{cy_1}+1$

$\rightarrow \cancel{c=1}$ But c can be 2, 3, any real number, not just c=1.

Property 5 is not satisfied!

V is not a vector space.

Simple Properties of Vector Spaces

Uniqueness

Suppose that V is a vector space.

- There is only one zero vector in V.
- If v is a vector in V, there is only one negative (denoted by $-v$) of v.

Multiplying by Zero

Let V be a vector space.

- For any vector v in V, we have $0 \cdot v = 0$.
- For any real number c, we have $c \cdot 0 = 0$.

Subtraction in Vector space

Let V be a vector space. Then for any vector v in V, we have

$$(-1) \cdot v = -v.$$

Remarks:

- We usually write cv instead of $c \cdot v$ for the scalar multiplication. It simplifies the notation.
- Substracting two vectors is done in the following way:

$$u - v := u + (-v).$$