MATH 307

CHAPTER 5

SECTION 5.3: MATRICES FOR LINEAR TRANSFORMATIONS

CONTENTS

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Example 1. Let $T : \underset{V}{\mathbb{R}^3} \to \underset{W}{\mathbb{R}^3}$ be the linear transformation

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 5x + z \\ 3x + 2y - 3z \\ 5x \end{bmatrix}.$$

Give a matrix representing the linear transformation T.

$$A = \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 5 & 0 & 1 \\ 3 & 2 & -3 \\ 5 & 0 & 0 \end{bmatrix} \end{array} \longrightarrow T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \underbrace{\begin{bmatrix} 5 & 0 & 1 \\ 3 & 2 & -3 \\ 5 & 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Identify T with A.

Behind the scene:

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 5x \\ 3x \\ 5x \end{bmatrix} + \begin{bmatrix} 0 \\ 2y \\ 0 \end{bmatrix} + \begin{bmatrix} z \\ -3z \\ 0 \end{bmatrix} = x\begin{bmatrix} 5 \\ 3 \\ 5 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$$

$$T\left(\underset{(*)}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}\right) = \begin{bmatrix} 5 \\ 3 \\ 5 \end{bmatrix}, \quad T\left(\underset{(*)}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}\right) = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \quad T\left(\underset{(*)}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}\right) = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$$

(*) The vectors for a basis of $\mathbb{R}^3$.

(**)

$$\begin{bmatrix} 5 \\ 3 \\ 5 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 5\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

$$\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (-3)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The matrix A is the matrix representation of T w.r.t. the standard basis.

α: standard

$\longrightarrow A = [T]_\alpha^\alpha$

General Process:

Suppose $T : V \to W$ is a linear transformation.

- Let v_1, v_2, …, v_n form a basis α for V.
- Let w_1, w_2, …, w_m form a basis β for W.

Since 1) $T(v_1)$, $T(v_2)$, …, $T(v_n)$ belongs to W and β is a basis for W, we have

2)

$$
\begin{aligned}
T(v_1) &= a_{11}w_1 + a_{21}w_2 + \cdots + a_{m1}w_m \\
T(v_2) &= a_{12}w_1 + a_{22}w_2 + \cdots + a_{m2}w_m \\
&\vdots \\
T(v_n) &= a_{1n}w_1 + a_{2n}w_2 + \cdots + a_{mn}w_m.
\end{aligned}
$$

$T = I$

$\to [I]_\beta^\alpha$

We call the **matrix of T with respect to the bases α and β** the matrix $[T]_\alpha^\beta$ formed from the previous coefficients $a_{11}, a_{22}, \ldots, a_{mn}$:

$$
[T]_\alpha^\beta = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.
$$

Remarks:

- With the notation introduced in Chapter 2 on basis, we have

$$
[T]_\alpha^\beta = \Big[[T(v_1)]_\beta \quad [T(v_2)]_\beta \quad \cdots [T(v_n)]_\beta \Big].
$$

- When $T : V \to V$ is a linear transformation of V into itself and α is used for both the domain and the codomain, then we simply say **the matrix of T with respect to α** and we denote it by $[T]_\alpha^\alpha$.

Example 2. Let T be the linear transformation in Example 1. Let β be the basis given by

$T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$
$\quad V \qquad W$

$$\beta = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Find

1. the matrix of T with respect to the standard basis α of $\mathbb{R}^3$.
2. the matrix of T with respect to the basis β.
3. $[T]_\alpha^\beta$.

1) $[T]_\alpha^\alpha = \begin{bmatrix} 5 & 0 & 1 \\ 3 & 2 & -3 \\ 5 & 0 & 0 \end{bmatrix}$

3) (*) $T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 3 \\ 5 \end{bmatrix}$, $T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$, $T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$.

(**) $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 1 & -1 & 1 & 3 \\ 2 & 1 & 1 & 5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right] \to \begin{bmatrix} 5 \\ 3 \\ 5 \end{bmatrix}_\beta = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}$

$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 2 \\ 2 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right] \to \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}_\beta = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -3 \\ 2 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right] \to \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}_\beta = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$

$\to [T]_\alpha^\beta = \begin{bmatrix} 0 & 0 & -1 \\ 4 & -1 & 2 \\ 1 & 1 & 0 \end{bmatrix}$

$\begin{pmatrix} 7 \\ -1 \\ 5 \end{pmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$

2) Goal: Find $[T]_\beta^\beta$

(*) $T\left(\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}\right) = \begin{pmatrix} 7 \\ -1 \\ 5 \end{pmatrix}$, $T\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 6 \\ -2 \\ 5 \end{bmatrix}$, $T\left(\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 6 \\ 2 \\ 5 \end{bmatrix}$

(**) $\begin{bmatrix} 7 \\ -1 \\ 5 \end{bmatrix}_\beta = \begin{bmatrix} -2 \\ 4 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 6 \\ -2 \\ 5 \end{bmatrix}_\beta = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$, $\begin{pmatrix} 6 \\ 2 \\ 5 \end{pmatrix}_\beta = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$.

$$[T]_\beta^\beta = \begin{bmatrix} -2 & -1 & -1 \\ 4 & 4 & 2 \\ 3 & 3 & 5 \end{bmatrix}.$$

Matrix of the Composition

Let $T : V \to W$ and $S : W \to U$ be linear transformations. Suppose that

- α is a basis for V;
- β is a basis for W;
- γ is a basis for U.

$ST : V \to U$.

Then we have

$$[ST]_\alpha^\gamma = [S]_\beta^\gamma [T]_\alpha^\beta.$$

Given a transformation $T : V \to W$, a basis α for V and a basis β for W, we then have

$$[T(v)]_\beta = [T]_\alpha^\beta [v]_\alpha.$$

Remark: The last equality means that the vector $T(v)$ is obtained by multiplying the matrix of T with respect to α and β by the vector of the coordinates of v in the basis α.

Example 3. Let T, α and β be as in Example 2.

1. Find the coordinate vector of $v = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^\top$ with respect to the basis α.
2. Find coordinate vector of $T(v)$ with respect to the basis β.
3. Use the result in part (b) to find $T(v)$ in the standard basis.

1) $$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_\alpha = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

2) $$\left[T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right) \right]_\beta = [T]_\alpha^\beta [v]_\alpha$$

$$= \begin{bmatrix} 0 & 0 & -1 \\ 4 & -1 & 2 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 8 \\ 3 \end{bmatrix}$$

3) $$T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right) = -3 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + 8 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 \\ -8 \\ 5 \end{bmatrix}.$$

Matrix of a Change of Basis

Example 4. Let α be the standard basis for $\mathbb{R}^3$ and let β be the basis in Example 2. Find a matrix that will send each vector in the basis α to the vectors in the basis β.

$$\alpha = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \qquad \beta = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Finding a linear transformation.

Goal: Find a matrix A s.t.

$$① \; A\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad ② \; A\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad ③ \; A\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Write $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

$$① \longrightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$② \longrightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

← coordinate vector w.r.t. the standard basis.

$$③ \longrightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

← third column of the matrix A.

the matrix A : change of basis from α to β.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \ldots$$

General Procedure:

Let α and β be two bases of V:

- α be a basis with vectors $v_1, v_2, \ldots, v_n$.
- β be a basis with vectors $w_1, w_2, \ldots, w_n$.

Write

$[w_1]_\alpha$

$$\begin{aligned} w_1 &= p_{11}v_1 + p_{21}v_2 + \cdots + p_{n1}v_n \\ w_2 &= p_{12}v_1 + p_{22}v_2 + \cdots + p_{n2}v_n \\ &\vdots \\ w_n &= p_{1n}v_1 + p_{2n}v_2 + \cdots + p_{nn}v_n. \end{aligned}$$

$[w_2]_\alpha$

Then the matrix

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}$$

is called the **change of basis matrix from** α **to** β.

Fact:

- If we define $I(v) = v$ to be the identity transformation, then in fact $P = \boxed{[I]^\alpha_\beta}$ So, $[v]_\alpha = P[v]_\beta$.

$I(w_1) = w_1 = p_{11}v_1 + p_{21}v_2 + \cdots + p_{n1}v_n$

$I(w_2) = w_2 = p_{12}v_1 + p_{22}v_2 + \cdots + p_{n2}v_n$

$\vdots$

$I(w_n) = w_n = p_{1n}v_1 + p_{2n}v_2 + \cdots + p_{nn}v_n$

- If P is the change of basis matrix from a basis α to a basis β of a vector space, then the change of basis from β to α is P^{-1}. So $P^{-1} = \underline{[I]^\beta_\alpha}$ and $[v]_\beta = P^{-1}[v]_\alpha$.

Consequence on the Matrix of a Linear Transformation

EXAMPLE 5. Let α be the standard basis and let β be the basis in Example 2. Suppose that a linear transformation T has the following matrix with respect to α:

$$[T]_\alpha^\alpha = \begin{bmatrix} 1 & 3 & -1 \\ 2 & -1 & 2 \\ 1 & 3 & -1 \end{bmatrix}.$$

Find $\underline{[T(v)]_\beta}$ where $[v]_\beta = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^\top$.

<u>Trick.</u> Use P found in Example 4:

$$[I]_\beta^\alpha = P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \quad (\text{change of basis } \alpha \longrightarrow \beta).$$

<u>know:</u> $[T(v)]_\alpha = [T]_\alpha^\alpha [v]_\alpha \quad (*)$

$$[v]_\alpha = [I]_\beta^\alpha [v]_\beta = P[v]_\beta$$

$(*)$ becomes $\quad [T(v)]_\alpha = [T]_\alpha^\alpha P [v]_\beta . \quad (**)$

$$[T(v)]_\beta = [I]_\alpha^\beta [T(v)]_\alpha = P^{-1} [T(v)]_\alpha .$$

From $(**)$:

$$[T(v)]_\beta = \underbrace{P^{-1} [T]_\alpha^\alpha P}_{\text{matrix}} [v]_\beta$$

$$= \begin{bmatrix} -1 & 0 & 1 \\ 1/2 & -1/2 & 0 \\ 3/2 & 1/2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 & -1 \\ 2 & -1 & 2 \\ 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -19/2 \\ 29/2 \end{bmatrix}$$

Facts:

- If $T : V \to V$ is a linear transformation, α and β are bases for V, and P is the change of basis matrix from α to β, then

$$[T]_\beta^\beta = P^{-1}[T]_\alpha^\alpha P.$$

- If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation and A is the matrix of T with respect to the standard basis of $\mathbb{R}^n$ and $\mathbb{R}^m$, then

$$T(X) = AX.$$

EXAMPLE 6 (Extra).

$$\alpha = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$$

$$\beta = \left\{ \underset{w_1}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}, \underset{w_2}{\begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix}}, \underset{w_3}{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}} \right\}.$$

Change of basis $\alpha \to \beta$ $\leftrightarrow$ Find the matrix $[I]_\beta^\alpha$

① $I(w_1) = w_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

$$= \begin{bmatrix} c_1 + c_2 + c_3 \\ c_1 + 2c_2 - c_3 \\ c_1 + 2c_3 \end{bmatrix}$$

$\to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & 1 \\ 1 & 0 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \to \begin{matrix} c_1 = 1 \\ c_2 = 0 \\ c_3 = 0 \end{matrix} \to [w_1]_\alpha = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

② $I(w_2) = w_2 = \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

$\to \begin{bmatrix} 1 & 1 & 1 & -2 \\ 1 & 2 & -1 & 3 \\ 1 & 0 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 20 \\ 0 & 1 & 0 & -13 \\ 0 & 0 & 1 & -9 \end{bmatrix} \to [w_2]_\alpha = \begin{bmatrix} 20 \\ -13 \\ -9 \end{bmatrix}$

③ $I(w_3) = w_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

$\to \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & -1 & 1 \\ 1 & 0 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix} \to [w_3]_\alpha = \begin{bmatrix} 5 \\ -3 \\ -2 \end{bmatrix}$

Now,

$$P = [I]_\beta^\alpha = \begin{bmatrix} 1 & 20 & 5 \\ 0 & -13 & -3 \\ 0 & -9 & -2 \end{bmatrix}.$$