# Math 307

$$T(f) = f'$$

$$\begin{cases} (f+g)' = f' + g' \\ (cf)' = cf' \end{cases}$$

# Chapter 5

## Section 5.1: Linear Transformations

Contents




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Convention:

- The addition and scalar multiplication on the set of column vectors $\mathbb{R}^n$ are the usual ones that make $\mathbb{R}^n$ a vector space. If the addition is changed, it will be mentioned explicitly in the text.
- Same convention for $M_{m\times n}(\mathbb{R})$ & $F(a,b)$.

## Definition

If $V$ and $W$ are vector spaces, a function $T : V \to W$ is called a **linear transformation** if, for all vectors $u$ and $v$ in $V$ and all scalars $c$, the following two properties are satisfied:

1. $T(u+v) = T(u) + T(v)$;
2. $T(cv) = cT(v)$.

$AX = \cancel{B}$

EXAMPLE 1. Let $A$ be an $m \times n$ matrix. We define $T : \underset{V}{\mathbb{R}^n} \to \underset{W}{\mathbb{R}^m}$ by

$$T(X) := AX$$

where $X$ is an $n \times 1$ column vector. Verify that the function $T$ is a linear transformation.

$u = X$, $v = Y$ $\to$ $n\times 1$ column vectors

1) $T(X+Y) = A(X+Y) = AX + AY = T(X) + T(Y)$. ✓

2) scalar $c$.

$T(cX) = A(cX) = c\,AX = c\,T(X)$. ✓

So, $T$ is a linear Transformation.

$T : \mathbb{R} \to \mathbb{R}$, $T(x) := ax$ ($a$ is real number).

Example 2. Verify if the given function $T:\mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y-z \\ x+2y+z \end{bmatrix}$$

is a linear transformation.

1) $u = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ & $v = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$

$$T(u+v) = T\left(\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{bmatrix}\right) = \begin{bmatrix} (x_1+x_2)+(y_1+y_2)-(z_1+z_2) \\ (x_1+x_2)+2(y_1+y_2)+(z_1+z_2) \end{bmatrix}$$

$$= \begin{bmatrix} (x_1+y_1-z_1)+(x_2+y_2-z_2) \\ (x_1+2y_1+z_1)+(x_2+2y_2+z_2) \end{bmatrix}$$

$$= \begin{bmatrix} x_1+y_1-z_1 \\ x_1+2y_1+z_1 \end{bmatrix} + \begin{bmatrix} x_2+y_2-z_2 \\ x_2+2y_2+z_2 \end{bmatrix}$$

$$= T\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) = T(u)+T(v) \checkmark$$

2) $c$ a scalar.

$$T(cu) = T\left(\begin{bmatrix} cx_1 \\ cy_1 \\ cz_1 \end{bmatrix}\right) = \begin{bmatrix} cx_1+cy_1-cz_1 \\ cx_1+2cy_1+cz_1 \end{bmatrix} = \begin{bmatrix} c(x_1+y_1-z_1) \\ c(x_1+2y_1+z_1) \end{bmatrix}$$

$$= c\begin{bmatrix} x_1+y_1-z_1 \\ x_1+2y_1+z_1 \end{bmatrix}$$

$$= cT\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\right) = cT(u). \checkmark$$

2nd way.

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y-z \\ x+2y+z \end{bmatrix} \overset{\text{say}}{=} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow (*)\begin{cases} x+y-z=0 \\ x+2y+z=0 \end{cases}$$

$(*)$ becomes $\boxed{AX} = B$.

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So, $T(X) = AX$

From Example 1, $T$ is a lin. Transf.

$\to D(a,b): \{ f : f'(x) \text{ exists for any } \underset{a<x<b}{x \in (a,b)} \}$

Example 3. Let $D(a, b)$ be the subspace of $F(a, b)$ of differentiable function on the interval $(a, b)$. Define the function $T : \underset{V}{D(a, b)} \to \underset{W}{F(a, b)}$ by

$$T(f) := f'$$

meaning that $T(f)(x) = f'(x)$ for every $x$ in $(a, b)$. Verify that $T$ is a linear transformation.

1) $f$ & $g$ two differentiable fcts. ($f, g \in D(a,b)$).

$$T(f+g) = (f+g)' = f' + g' \quad (\text{Calc I})$$
$$= T(f) + T(g) \quad \checkmark$$

2) $c$ scalar.

$$T(cf) = (cf)' = c f' \quad (\text{Calc I})$$
$$= c\,T(f). \quad \checkmark$$

Remark: The linear transformation in the previous example is called a differential operator and is quite useful in the theory of ODE and PDE.

If $T : V \to W$ is a linear transformation, then we can prove that

- $T(0_V) = 0_W$;
- $T(-v) = -T(v)$ for any vector $v$ in $V$;
- $T(u - v) = T(u) - T(v)$ for any vector $u$, $v$ in $V$.

There is another important property of a linear transformation which we shall illustrate by an example.

Example 4. Suppose that $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation so that

$$T\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right) = \begin{bmatrix}2\\3\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right) = \begin{bmatrix}0\\3\end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\2\end{bmatrix}.$$

Find the value of $T\left(\begin{bmatrix}1\\3\\0\end{bmatrix}\right)$.

$T(u+v) = T(u) + T(v)$
$T(cu) = cT(u)$

$\to T(au+bv)$
$= T(au) + T(bv)$
$= aT(u) + bT(v)$.

$$c_1\begin{bmatrix}1\\1\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\\1\end{bmatrix} + c_3\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}1\\3\\0\end{bmatrix} \qquad \begin{bmatrix}c_1+c_3\\c_1+c_2\\c_2+c_3\end{bmatrix} = \begin{bmatrix}1\\3\\0\end{bmatrix}$$

Fact: $\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}$ basis for $\mathbb{R}^3$.

$$\to \begin{bmatrix}1\\3\\0\end{bmatrix} = 2\begin{bmatrix}1\\1\\0\end{bmatrix} + \begin{bmatrix}0\\1\\1\end{bmatrix} - \begin{bmatrix}1\\0\\1\end{bmatrix}$$

$$\to T\left(\begin{bmatrix}1\\3\\0\end{bmatrix}\right) = T\Bigg(\underbrace{2\begin{bmatrix}1\\1\\0\end{bmatrix} + \begin{bmatrix}0\\1\\1\end{bmatrix}}_{u} - \underbrace{\begin{bmatrix}1\\0\\1\end{bmatrix}}_{v}\Bigg)$$

$$= T\left(2\begin{bmatrix}1\\1\\0\end{bmatrix} + \begin{bmatrix}0\\1\\1\end{bmatrix}\right) - T\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right)$$

$$= 2T\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right) + T\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right) - T\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right)$$

$$= 2\begin{bmatrix}2\\3\end{bmatrix} + \begin{bmatrix}0\\3\end{bmatrix} - \begin{bmatrix}0\\2\end{bmatrix} = \begin{bmatrix}4\\7\end{bmatrix}$$

Fact: If $v_1$, $v_2$, …, $v_n$ form a basis, then the values of a linear transformation $T$ is determined by its value on $v_1$, $v_2$, …, $v_n$ because for any $v \in V$, we have

$$T(v) = T(c_1v_1 + c_2v_2 + \cdots + c_nv_n) = c_1T(v_1) + c_2T(v_2) + \cdots + c_nT(v_n).$$

## Kernel

If $T : V \to W$ is a linear transformation, then the **kernel** of $T$ is the set of all vectors $v$ in $V$ such that $T(v) = 0$. In set notation:

$$\ker(T) = \{v \in V \, : \, T(v) = 0_W\}.$$

This is in general a subspace of $V$.

Example 5. Find a basis for the kernel of the linear transformation

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y-z \\ x+2y+z \end{bmatrix}. \quad \to 0_W = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Goal: Find $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ s.t. $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

becomes $\begin{bmatrix} x+y-z \\ x+2y+z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \to \begin{cases} x+y-z=0 \\ x+2y+z=0 \end{cases}$.

$\to$ $x = -3z + w$
$y = -2z$.

$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -3z \\ -2z \\ z \\ 0 \end{bmatrix} + \begin{bmatrix} w \\ 0 \\ 0 \\ w \end{bmatrix} = z\begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix} + w\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$\to \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3z \\ -2z \\ z \end{bmatrix}$ (vectors in the $\ker(T)$)

or $\ker(T) = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3z \\ -2z \\ z \end{bmatrix} \right\}$.

Here, $\begin{bmatrix} -3z \\ -2z \\ z \end{bmatrix} = z\underbrace{\begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}}$. $\to$ basis for $\ker(T)$ is $\begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$.

Remark: The kernel of a transformation is related to the solutions of the system of linear equations $AX = 0$ when $T(X) = AX$ with $A$ an $m \times n$ matrix. In this particular situation, the kernel $\ker(T)$ is called the **null space** of $A$ also denoted by $NS(A)$. In other words, we have

$$NS(A) = \ker(T).$$

## Range

If $T : V \to W$ is a linear transformation, then the **range** of $T$ is the set of all vectors $T(v)$ where $v$ is in $V$. In set notation:

$$\operatorname{range}(T) = \{T(v) \, : \, v \in V\}.$$

This is in general a subspace of $W$.

Facts:

- Finding a basis for the range of a tranformation $T$ given by $T(X) = AX$ where $A$ is an $m \times n$ matrix is equivalent to finding a basis for the spanning set of the columns of the matrix $A$.
- The subspace spans by the column of a matrix $A$ is called the **column space** and is denoted by $CS(A)$.

EXAMPLE 6. Find a basis for the range of the linear transformation of Example 5 using the column space of a certain matrix.

$T$ as $T(X) = AX$ with

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix} \quad \& \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

We see :

$$T(X) = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ 2 \end{bmatrix} + z \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

$$\rightarrow \operatorname{range}(T) = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} \subseteq \mathbb{R}^2$$

So,

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \end{bmatrix} \rightarrow \text{extract } \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\text{basis of range}(T)}$$

In summary, to find range $(T)$ or $CS(A)$ for a linear transformation of the form $T(X) = AX$, we follow these steps:

- express $T(v)$ as a linear combination of column vectors $v_1, v_2, \ldots, v_n$.
- Write each vector in a matrix $A = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$.
- Find the RREF of $A$.
- The column with the first 1 in a row will be a pivot and the vector corresponding to the column will be part of the basis.

Fact: We call $\dim(CS(A)) = \dim(\text{range}\,(T))$ the **rank** of the matrix $A$ or transformation $T$.

## Rank-Nullity Identity

We define

- the **nullity** of a linear transformation $T$ as the dimension of $\ker(T)$.
- the **rank** of a linear transformation $T$ as the dimension of range $(T)$.

Here is an important identity relating the rank and the nullity of a linear transformation.

THEOREM 7. If $T : V \to W$ is a linear transformation, then

$$\dim(\ker(T)) + \dim(\text{range}\,(T)) = \dim(V).$$

Remark: For an $m \times n$ matrix, we obtain

applied to $n \times 1$ vectors.

$$\dim(NS(A)) + \dim(CS(A)) = n.$$

EXAMPLE 8. Verify the Rank-Nullity Identity for the matrix in Example 5 and Example 6.

$\begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$ is a basis for $\ker(T)$

$\to \dim(\ker(T)) = 1$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ & $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a basis for range $(T)$.

$\to \dim(\text{range}\,(T)) = 2$.

$V = \mathbb{R}^3$

$1 + 2 = 3 = \dim(V)$ ✓.