MATH 307

CHAPTER 5

SECTION 5.3: MATRICES FOR LINEAR TRANSFORMATIONS

CONTENTS

CREATED BY: PIERRE-OLIVIER PARISÉ
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Example 1. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5x + z \\ 3x + 2y - 3z \\ 5x \end{bmatrix}.$$

Give a matrix representing the linear transformation T.

Try to solve $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$, meaning

$$\begin{bmatrix} 5x+z \\ 3x+2y-3z \\ 5x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Longleftrightarrow \begin{cases} 5x+z=0 \\ 3x+2y-3z=0 \\ 5x=0 \end{cases}$$

This system can be rewritten as

$$\underbrace{\begin{bmatrix} 5 & 0 & 1 \\ 3 & 2 & -3 \\ 5 & 0 & 0 \end{bmatrix}}_{= A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{= X} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

But, $AX = T(X)$ and so A a matrix representing T.

<u>Behind the scene:</u>

We know that $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is basis for $\mathbb{R}^3$.

If we evalute T at $u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $u = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ & $u = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, we obtain:

$$T\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 5 \end{bmatrix}, \quad T\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \quad T\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$$

Then,

$$A = \begin{bmatrix} T\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} & T\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} & T\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 5 & 0 & 1 \\ 3 & 2 & -3 \\ 5 & 0 & 0 \end{bmatrix}$$

So, A is called the matrix of T w.r.t. the standard basis.

General Process:

Suppose $T : V \to W$ is a linear transformation.

- Let v_1, v_2, …, v_n form a basis α for V.
- Let w_1, w_2, …, w_m form a basis β for W.

Since $T(v_1)$, $T(v_2)$, …, $T(v_n)$ belongs to W and β is a basis for W, we have

$$\begin{aligned} T(v_1) &= a_{11}w_1 + a_{21}w_2 + \cdots + a_{m1}w_m \\ T(v_2) &= a_{12}w_1 + a_{22}w_2 + \cdots + a_{m2}w_m \\ &\vdots \\ T(v_n) &= a_{1n}w_1 + a_{2n}w_2 + \cdots + a_{mn}w_m. \end{aligned}$$

We call the **matrix of T with respect to the bases** α **and** β the matrix $[T]_\alpha^\beta$ formed from the previous coefficients $a_{11}, a_{22}, \ldots, a_{mn}$:

$$[T]_\alpha^\beta = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Remarks:

- With the notation introduced in Chapter 2 on basis, we have
$$[T]_\alpha^\beta = \left[[T(v_1)]_\beta \quad [T(v_1)]_\beta \quad \cdots [T(v_n)]_\beta \right].$$
- When $T : V \to V$ is a linear transformation of V into itself and α is used for both the domain and the codomain, then we simply say **the matrix of T with respect to** α and we denote it by $[T]_\alpha^\alpha$.

Example 2. Let T be the linear transformation in Example 1. Let β be the basis given by

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Find

1. the matrix of T with respect to the standard basis α of $\mathbb{R}^3$.
2. the matrix of T with respect to the basis β.
3. $[T]_\alpha^\beta$.

1) Example 1 $\to$ $[T]_\alpha^\alpha = \begin{bmatrix} 5 & 0 & 1 \\ 3 & 2 & -3 \\ 5 & 0 & 0 \end{bmatrix}$

2) We have

$$T\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \\ 5 \end{bmatrix}, \quad T\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ 5 \end{bmatrix}, \quad T\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 7 \\ 1 & -1 & 1 & -1 \\ 2 & 1 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{bmatrix} \to \begin{bmatrix} 7 \\ -1 \\ 5 \end{bmatrix} = -2\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + 4\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + 5\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}_\beta$$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & -1 & 1 & -2 \\ 2 & 1 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 6 \\ -2 \\ 5 \end{bmatrix} = -1\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + 4\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}_\beta$$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & -1 & 1 & 2 \\ 2 & 1 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix} \to \begin{bmatrix} 6 \\ 2 \\ 5 \end{bmatrix} = -1\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + 5\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}_\beta$$

So, $[T]_\beta^\beta = \begin{bmatrix} -2 & -1 & -1 \\ 4 & 4 & 2 \\ 5 & 3 & 5 \end{bmatrix}$

3) $$T\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 5 \end{bmatrix}, \quad T\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \quad T\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 5 \\ 1 & -1 & 1 & 3 \\ 2 & 1 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 5 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}_\beta$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 2 \\ 2 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}_\beta$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -3 \\ 2 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}_\beta .$$

Thus,

$$[T]_\alpha^\beta = \begin{bmatrix} 0 & 0 & -1 \\ 4 & -1 & 2 \\ 1 & 1 & 0 \end{bmatrix}$$

$\boxed{\text{Matrix of the Composition}}$

Let $T : V \to W$ and $S : W \to U$ be linear transformations. Suppose that

- α is a basis for V;
- β is a basis for W;
- γ is a basis for U.

Then we have

$$[ST]_\alpha^\gamma = [S]_\beta^\gamma [T]_\alpha^\beta.$$

Given a transformation $T : V \to W$, a basis α for V and a basis β for W, we then have

$$[T(v)]_\beta = [T]_\alpha^\beta [v]_\alpha.$$

Remark: The last equality means that the vector $T(v)$ is obtained by multiplying the matrix of T with respect to α and β by the vector of the coordinates of v in the basis α.

Example 3. Let T, α and β be as in Example 2.

1. Find the coordinate vector of $v = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^\top$ with respect to the basis α.
2. Find coordinate vector of $T(v)$ with respect to the basis β.
3. Use the result in part (b) to find $T(v)$ in the standard basis.

1) $$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_\alpha .$$

2) $$[T(v)]_\beta = [T]_\alpha^\beta [v]_\alpha = \begin{bmatrix} 0 & 0 & -1 \\ 4 & -1 & 2 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$
$$= \begin{bmatrix} -3 \\ 8 \\ 3 \end{bmatrix}$$

3) $$T(v) = -3\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + 8\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} 8 \\ -8 \\ 5 \end{bmatrix} .$$

Matrix of a Change of Basis

Example 4. Let α be the standard basis for $\mathbb{R}^3$ and let β be the basis in Example 2. Find a matrix that will send each vector in the basis α to the vectors in the basis β.

Find a matrix A s.t.

(1) $A\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\1\\2\end{bmatrix}$, (2) $A\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}1\\-1\\1\end{bmatrix}$, (3) $A\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix}$.

Write $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

(1) $\rightarrow \begin{bmatrix}a_{11}\\a_{21}\\a_{31}\end{bmatrix} = \begin{bmatrix}1\\1\\2\end{bmatrix} = 1\begin{bmatrix}1\\0\\0\end{bmatrix} + 1\begin{bmatrix}0\\1\\0\end{bmatrix} + 2\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}1\\1\\2\end{bmatrix}_\alpha$

(2) $\rightarrow \begin{bmatrix}a_{12}\\a_{22}\\a_{31}\end{bmatrix} = \begin{bmatrix}1\\-1\\1\end{bmatrix} = 1\begin{bmatrix}1\\0\\0\end{bmatrix} + (-1)\begin{bmatrix}0\\1\\0\end{bmatrix} + 1\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}1\\-1\\1\end{bmatrix}_\alpha$

(3) $\rightarrow \begin{bmatrix}a_{13}\\a_{23}\\a_{33}\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix} = 1\begin{bmatrix}1\\0\\0\end{bmatrix} + 1\begin{bmatrix}0\\1\\0\end{bmatrix} + 1\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix}_\alpha$

In fact, A is the transformation $I(v) = v$ and thus,

$$A = [I]_\beta^\alpha = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$

General Procedure:

Let α and β be two bases of V:

- α be a basis with vectors $v_1, v_2, \ldots, v_n$.
- β be a basis with vectors $w_1, w_2, \ldots, w_n$.

Write

$$\begin{aligned} w_1 &= p_{11}v_1 + p_{21}v_2 + \cdots + p_{n1}v_n \\ w_2 &= p_{12}v_1 + p_{22}v_2 + \cdots + p_{n2}v_n \\ &\vdots \\ w_n &= p_{1n}v_1 + p_{2n}v_2 + \cdots + p_{nn}v_n. \end{aligned}$$

Then the matrix

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}$$

is called the **change of basis matrix from α to β**.

Fact:

- If we define $I(v) = v$ to be the identity transformation, then in fact $P = [I]_\beta^\alpha$. So, $[v]_\alpha = P[v]_\beta$.

In indeed, $I(w_1) = p_{11}v_1 + p_{21}v_2 + \cdots + p_{n1}v_n$

$$\vdots$$

$I(w_n) = p_{1n}v_1 + p_{2n}v_2 + \cdots + p_{nn}v_n$

From what we've seen, this implies that

$$[I]_\beta^\alpha = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix} = P$$

Also, $[v]_\alpha = [I]_\beta^\alpha [v]_\beta = P[v]_\beta$.

- If P is the change of basis matrix from a basis α to a basis β of a vector space, then the change of basis from β to α is P^{-1}. So $P^{-1} = [I]_\alpha^\beta$ and $[v]_\beta = P^{-1}[v]_\alpha$.

Consequence on the Matrix of a Linear Transformation

EXAMPLE 5. Let α be the standard basis and let β be the basis in Example 2. Suppose that a linear transformation T has the following matrix with respect to α:

$$[T]_\alpha^\alpha = \begin{bmatrix} 1 & 3 & -1 \\ 2 & -1 & 2 \\ 1 & 3 & -1 \end{bmatrix}.$$

Find $[T(v)]_\beta$ where $[v]_\beta = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^\top$.

Trick: Use Change of basis from Example 4.

We have

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & 1 \end{bmatrix}. \quad (\text{change } \alpha \to \beta)$$

We also know that for any vector w.

$$[w]_\alpha = P[w]_\beta \quad (*)$$

&

$$[w]_\beta = P^{-1}[w]_\alpha \quad (**)$$

Apply $(**)$ to $w = T(v)$:

$$[T(v)]_\beta = P^{-1}[T(v)]_\alpha.$$

But, $[T(v)]_\alpha = [T]_\alpha^\alpha [v]_\alpha$.

So,

$$[T(v)]_\beta = P^{-1}[T]_\alpha^\alpha [v]_\alpha$$

What is $[v]_\alpha$?

Apply (*) with $w = v$:

$$[v]_\alpha = P\,[v]_\beta .$$

Thus,

$$[T(v)]_\beta = P^{-1}[T]_\alpha^\alpha P[v]_\beta$$

$$= \begin{bmatrix} -1 & 0 & 1 \\ 1/2 & -1/2 & 0 \\ 3/2 & 1/2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 & -1 \\ 2 & -1 & 2 \\ 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -19/2 \\ 29/2 \end{bmatrix}$$

<u>Facts</u>:

- If $T : V \to V$ is a linear transformation, α and β are bases for V, and P is the change of basis matrix from α to β, then
$$[T]_\beta^\beta = P^{-1}[T]_\alpha^\alpha P.$$
- If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation and A is the matrix of T with respect to the standard basis of $\mathbb{R}^n$ and $\mathbb{R}^m$, then
$$T(X) = AX.$$