Math 307

Chapter 5

Section 5.3: Matrices For Linear Transformations

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EXAMPLE 1. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 5x + z \\ 3x + 2y - 3z \\ 5x \end{bmatrix}.$$

Give a matrix representing the linear transformation T.

General Process:

Suppose $T : V \to W$ is a linear transformation.

- Let v_1, v_2, ..., v_n form a basis α for V.
- Let w_1, w_2, ..., w_m form a basis β for W.

Since $T(v_1)$, $T(v_2)$, ..., $T(v_n)$ belongs to W and β is a basis for W, we have

$$\begin{aligned} T(v_1) &= a_{11}w_1 + a_{21}w_2 + \cdots + a_{m1}w_m \\ T(v_2) &= a_{12}w_1 + a_{22}w_2 + \cdots + a_{m2}w_m \\ &\vdots \\ T(v_n) &= a_{1n}w_1 + a_{2n}w_2 + \cdots + a_{mn}w_m. \end{aligned}$$

We call the **matrix of T with respect to the bases** α **and** β the matrix $[T]_\alpha^\beta$ formed from the previous coefficients $a_{11}, a_{22}, \ldots, a_{mn}$:

$$[T]_\alpha^\beta = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Remarks:

- With the notation introduced in Chapter 2 on basis, we have
$$[T]_\alpha^\beta = \begin{bmatrix} [T(v_1)]_\beta & [T(v_1)]_\beta & \cdots [T(v_n)]_\beta \end{bmatrix}.$$
- When $T : V \to V$ is a linear transformation of V into itself and α is used for both the domain and the codomain, then we simply say **the matrix of T with respect to** α and we denote it by $[T]_\alpha^\alpha$.

Example 2. Let T be the linear transformation in Example 1. Let β be the basis given by

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Find

1. the matrix of T with respect to the standard basis α of $\mathbb{R}^3$.
2. the matrix of T with respect to the basis β.
3. $[T]_\alpha^\beta$.

Matrix of the Composition

Let $T : V \to W$ and $S : W \to U$ be linear transformations. Suppose that

- α is a basis for V;
- β is a basis for W;
- γ is a basis for U.

Then we have

$$[ST]^{\gamma}_{\alpha} = [S]^{\gamma}_{\beta}[T]^{\beta}_{\alpha}.$$

Given a transformation $T : V \to W$, a basis α for V and a basis β for W, we then have

$$[T(v)]_\beta = [T]_\alpha^\beta [v]_\alpha.$$

Remark: The last equality means that the vector $T(v)$ is obtained by multiplying the matrix of T with respect to α and β by the vector of the coordinates of v in the basis α.

Example 3. Let T, α and β be as in Example 2.

1. Find the coordinate vector of $v = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^\top$ with respect to the basis α.
2. Find coordinate vector of $T(v)$ with respect to the basis β.
3. Use the result in part (b) to find $T(v)$ in the standard basis.

Matrix of a Change of Basis

Example 4. Let α be the standard basis for $\mathbb{R}^3$ and let β be the basis in Example 2. Find a matrix that will send each vector in the basis α to the vectors in the basis β.

General Procedure:

Let α and β be two bases of V:

- α be a basis with vectors v_1, v_2, ..., v_n.
- β be a basis with vectors w_1, w_2, ..., w_n.

Write

$$\begin{aligned} w_1 &= p_{11}v_1 + p_{21}v_2 + \cdots + p_{n1}v_n \\ w_2 &= p_{12}v_1 + p_{22}v_2 + \cdots + p_{n2}v_n \\ &\vdots \\ w_n &= p_{1n}v_1 + p_{2n}v_2 + \cdots + p_{nn}v_n. \end{aligned}$$

Then the matrix

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}$$

is called the **change of basis matrix from α to β**.

Fact:

- If we define $I(v) = v$ to be the identity transformation, then in fact $P = [I]_\beta^\alpha$. So, $[v]_\alpha = P[v]_\beta$.

- If P is the change of basis matrix from a basis α to a basis β of a vector space, then the change of basis from β to α is P^{-1}. So $P^{-1} = [I]_\alpha^\beta$ and $[v]_\beta = P^{-1}[v]_\alpha$.

Consequence on the Matrix of a Linear Transformation

Example 5. Let α be the standard basis and let β be the basis in Example 2. Suppose that a linear transformation T has the following matrix with respect to α:

$$[T]_\alpha^\alpha = \begin{bmatrix} 1 & 3 & -1 \\ 2 & -1 & 2 \\ 1 & 3 & -1 \end{bmatrix}.$$

Find $[T(v)]_\beta$ where $[v]_\beta = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^\top$.

Facts:

- If $T : V \to V$ is a linear transformation, α and β are bases for V, and P is the change of basis matrix from α to β, then

$$[T]_\beta^\beta = P^{-1}[T]_\alpha^\alpha P.$$

- If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation and A is the matrix of T with respect to the standard basis of $\mathbb{R}^n$ and $\mathbb{R}^m$, then

$$T(X) = AX.$$