MATH 307

CHAPTER 5

SECTION 5.2: THE ALGEBRA OF LINEAR TRANSFORMATIONS

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Addition

If $T : V \to W$ and $S : V \to W$ are two linear transformations, then their sum $T + S$ is the new linear transformation defined by

$$(T + S)(v) = T(v) + S(v) \quad v \text{ in } V.$$

EXAMPLE 1. Let T and S be the following linear transformations:

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x - y \\ x + 2y \end{bmatrix} \quad \text{and} \quad S \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 3y \\ x - y \end{bmatrix}.$$

Find $T + S$.

$$(T+S)\begin{bmatrix} x \\ y \end{bmatrix} = T\begin{bmatrix} x \\ y \end{bmatrix} + S\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x-y \\ x+2y \end{bmatrix} + \begin{bmatrix} x+3y \\ x-y \end{bmatrix}$$

$$= \begin{bmatrix} 2x - y + x + 3y \\ x + 2y + x - y \end{bmatrix} = \begin{bmatrix} 3x + 2y \\ 2x + y \end{bmatrix}$$

Scalar Multiplication

If $T : V \to W$ is a linear transformation and c is a real number, then the function cT is the linear transformation defined by

$$(cT)(v) = cT(v) \quad v \text{ in } V.$$

EXAMPLE 2. With T and S as in the previous example, find $S + 4T$.

$$(S+4T)\begin{bmatrix} x \\ y \end{bmatrix} = S\begin{bmatrix} x \\ y \end{bmatrix} + (4T)\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} x+3y \\ x-y \end{bmatrix} + 4\,T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+3y \\ x-y \end{bmatrix} + 4\begin{bmatrix} 2x-y \\ x+2y \end{bmatrix}$$

$$= \begin{bmatrix} x+3y \\ x-y \end{bmatrix} + \begin{bmatrix} 8x-4y \\ 4x+8y \end{bmatrix} = \begin{bmatrix} 9x - y \\ 5x + 7y \end{bmatrix}$$

Let $B(V, W)$ be the set of all linear transformations $T : V \to W$.

THEOREM 3. The set $B(V, W)$ equipped with the addition and scalar multiplication is a vector space.

Composition or Multiplication of Operators

If $T : V \to W$ and $S : W \to U$ are two linear transformations, then the composite $ST : V \to U$ is the linear transformation defined by

$$ST(v) = S(T(v)) \quad v \text{ in } V.$$

Example 4. Find ST with S and T as in example 1.

$$ST\begin{bmatrix} x \\ y \end{bmatrix} = S\left(T\begin{bmatrix} x \\ y \end{bmatrix}\right) = S\begin{bmatrix} \underline{2x-y} \\ \underbrace{x+2y}_{y} \end{bmatrix}^{x}$$

$$= \begin{bmatrix} (2x-y) + 3(x+2y) \\ 2x-y - x-2y \end{bmatrix}$$

$$= \begin{bmatrix} 5x + 5y \\ x - 3y \end{bmatrix}.$$