MATH 307

CHAPTER 1

SECTION 1.3: INVERSES OF MATRICES

CONTENTS

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WHAT IS AN INVERSE?

For Real Numbers

EXAMPLE 1. Find the value of x if

1. $2x - 1 = 0$.
2. $x^2 - x = 0$.

1) $\frac{2x}{2} = \frac{1}{2}$

$x = 1/2$

2) $\frac{x^2 - x}{x} = 0 \quad x \neq 0$

$x - 1 = 0 \quad \to \quad x = 1$

Secretly:

2^{-1} $\uparrow$

- In the first equation, we multiplied by the inverse of 2, which is $1/2$, because $(1/2)2 = 1$.
- In the second equation, we examined the values of x and made sure we avoid the value 0 because 0 is not "divisible". In other words, it doesn't have an inverse.

For Matrices

We say that a square matrix A is invertible if there is another matrix B such that

$$AB = BA = I.$$

Remarks:

- Not all non-zero square matrices are invertible.
- Matrices that are invertible are called **nonsingular** and matrices that are not invertible are called **singular**.
- If the inverse exists, then there is only one inverse and we denote it by A^{-1}.

EXAMPLE 2. Verify that the matrix B is the inverse of A if

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}_{2\times 2} \quad B = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}_{2\times 2}.$$

$\underline{AB}$

$$\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \checkmark$$

$\underline{BA}$

$$\begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \checkmark$$

So B is the inverse of A!

Is $B^{-1} = A$? find C s.t. $CB = BC = I \to C = \textcircled{A} \to B^{-1} = A$

$AB = BA = I \checkmark$

Properties of Inverses

Example 3. Find the inverse of the product

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 2 & 1 & 1 \\ 4 & 5 & 1 \end{bmatrix}.$$

1) Proof that $(AB)^{-1} = B^{-1}A^{-1}$.

$$AB(B^{-1}A^{-1}) = A\underbrace{(BB^{-1})}_{=I}A^{-1} = (AI)A^{-1} = AA^{-1} = I$$

2) By the property,

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 2 & 1 & 1 \\ 4 & 5 & 1 \end{bmatrix}\right)^{-1} = \begin{bmatrix} 2 & 1 & 3 \\ 2 & 1 & 1 \\ 4 & 5 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} -1/3 & 7/6 & -1/6 \\ 1/6 & -5/6 & 1/3 \\ 1/2 & -1/2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1/3 & 7/6 & -5/6 \\ 1/6 & -5/6 & 2 \\ 1/2 & -1/2 & 1 \end{bmatrix}$$

General Facts: Let A and B be matrices of the same size and let m be a positive integer.

- If A and B are invertible, then AB is invertible with $(A\underset{\curvearrowleft}{B})^{-1} = B^{-1}A^{-1}$. $(AB)^{-1} \neq A^{-1}B^{-1}$.
- If A is invertible, then A^{-1} is also invertible and $(A^{-1})^{-1} = A$. $A \cdot A^{-1} = I$
- If A is invertible, then A^m is also invertible and $(A^{\overset{\curvearrowright}{m}})^{-1} = (A^{-1})^m$. $(2^{-2})^4 = (2^4)^{-2}$
- Suppose that A and B are $n \times n$ matrices such that $\boxed{AB = I}$ or $\boxed{BA = I.}$ Then A has an inverse and $A^{-1} = B$.

HOW DO WE FIND THE INVERSE?

For numbers, finding the inverses is quite straightforward, or should we say "we are used to divide with numbers".

$$AX = B \to \cancel{A^{-1}A}X = A^{-1}B$$

$$\frac{ax}{a} = \frac{b}{a} \to x = \frac{b}{a}$$

Little Warm-up

For matrices, it is not that obvious.

EXAMPLE 4. Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}.$$

$\underline{\text{Goal}}$ Find B s.t. $AB = I_2$

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} b_{11}+2b_{21} & b_{12}+2b_{22} \\ 3b_{11}+5b_{21} & 3b_{12}+5b_{22} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow ① \begin{cases} \overset{\downarrow}{b_{11}} + 2\overset{\downarrow}{b_{21}} = 1 \\ 3b_{11} + 5b_{21} = 0 \end{cases} \qquad ② \begin{cases} \overset{\downarrow}{b_{12}} + 2\overset{\downarrow}{b_{22}} = 0 \\ 3b_{12} + 5b_{22} = 1 \end{cases}$$

$$① \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 5 & 0 \end{array}\right] \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \end{bmatrix} \Rightarrow \boxed{b_{11} = -5} \quad \boxed{b_{21} = 3}$$

$$② \begin{bmatrix} 1 & 2 & 0 \\ 3 & 5 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix} \Rightarrow \boxed{b_{12} = 2} \quad \boxed{b_{22} = -1}$$

$$B = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix} \qquad \hookrightarrow \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 5 & 0 & 1 \end{array}\right] \sim \begin{bmatrix} 1 & 0 & -5 & 2 \\ 0 & 1 & 3 & -1 \end{bmatrix}$$

Systematic method with Augmented Matrices

Given a square matrix $A = [a_{ij}]$, we "augment" A with the identity matrix:

$$[A \,\vdots\, I] = \left[\begin{array}{cccc|cccc} a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0 & 0 & \cdots & 1 \end{array}\right].$$

Now, the goal, if possible, is to perform row operations to change the left-side (the matrix A) into the identity matrix, that is:

$$\begin{bmatrix} I & B \end{bmatrix} = \left[\begin{matrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{matrix}\right| \left.\underbrace{\begin{matrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{matrix}}\right].$$

Remark:

- When it's possible to transform the augmented matrix $[A \quad I]$ into the augmented matrix $[I \quad B]$, then B is the inverse of A.
- When it's not possible to transform $[A \quad I]$ into $[I \quad B]$, then A is singular.

EXAMPLE 5. If possible, find the inverse of the following matrix:

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 2 & 1 & 1 \\ 4 & 5 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} 2 & 1 & 3 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 4 & 5 & 1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 3 & 1 & 0 & 0 \\ 0 & 0 & -2 & -1 & 1 & 0 \\ 0 & 3 & -5 & -2 & 0 & 1 \end{bmatrix} \begin{array}{l} \\ R_2 - R_1 \to R_2 \\ R_3 - 2R_1 \to R_3 \end{array}$$

$$\sim \begin{bmatrix} 2 & 1 & 3 & 1 & 0 & 0 \\ 0 & 3 & -5 & -2 & 0 & 1 \\ 0 & 0 & -2 & -1 & 1 & 0 \end{bmatrix} \begin{array}{l} \\ R_3 \to R_2 \\ R_2 \to R_3 \end{array}$$

$$\sim \begin{bmatrix} 6 & 0 & 14 & 5 & 0 & -1 \\ 0 & 3 & -5 & -2 & 0 & 1 \\ 0 & 0 & -2 & -1 & 1 & 0 \end{bmatrix} \begin{array}{l} 3R_1 - R_2 \to R_1 \\ \\ \\ \end{array}$$

$$\sim \begin{bmatrix} 6 & 0 & 0 & -2 & 7 & -1 \\ 0 & 6 & 0 & 1 & -5 & 2 \\ 0 & 0 & -2 & -1 & 1 & 0 \end{bmatrix} \begin{array}{l} R_1 + 7R_3 \to R_1 \\ 2R_2 - 5R_3 \to R_2 \\ \end{array}$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & -1/3 & 7/6 & -1/6 \\ 0 & 1 & 0 & 1/6 & -5/6 & 1/3 \\ 0 & 0 & 1 & 1/2 & -1/2 & 0 \end{array}\right] \begin{array}{l} 1/6 R_1 \to R_1 \\ 1/6 R_2 \to R_2 \\ -1/2 R_3 \to R_3 \end{array}$$

$$\underbrace{}_{A^{-1}}$$

So, A is invertible &

$$A^{-1} = \begin{bmatrix} -1/3 & 7/6 & -1/6 \\ 1/6 & -5/6 & 1/3 \\ 1/2 & -1/2 & 0 \end{bmatrix}$$

Example 6. If possible, find the inverse of the following matrix:

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -3 & 1 \\ 1 & -1 & -1 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & -2 & 2 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & \boxed{-1} & 3 & 2 & -1 & 0 \\ 0 & -1 & 3 & 1 & 0 & -1 \end{bmatrix} \begin{array}{l} \\ 2R_1 - R_2 \to R_2 \\ R_1 - R_3 \to R_3 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & -4 & -3 & 1 & 0 \\ 0 & -1 & 3 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 \end{bmatrix} \begin{array}{l} R_1 - 2R_2 \to R_1 \\ \\ R_3 - R_2 \to R_3 \end{array}$$

Because of the line of zeros,

A^{-1} doesn't exist.

Inverses to Solve Systems

If you have a given system of linear equations

$$AX = B \qquad\qquad AXA^{-1} \neq AA^{-1}X$$

where A is a nonsingular matrix, then you can find X (the vector of solutions) by multiplying on the left the whole equation by the inverse A^{-1}:

$$\underbrace{A^{-1}A}_{I}X = A^{-1}B \quad \Rightarrow \quad \underline{X} = \underline{A^{-1}B}. \qquad\qquad \cancel{\frac{X}{A}}$$

Example 7. Solve the system

$$\begin{aligned} 2x + y + 3z &= 6 \\ 2x + y + z &= -12 \\ 4x + 5y + z &= 3. \end{aligned}$$

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 2 & 1 & 1 \\ 4 & 5 & 1 \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad B = \begin{bmatrix} 6 \\ -13 \\ 3 \end{bmatrix}$$

To solve: $AX = B$

From Ex. 5,

$$A^{-1} = \begin{bmatrix} -1/3 & 7/6 & -1/6 \\ 1/3 & -5/3 & 2/3 \\ 1/2 & -1/2 & 0 \end{bmatrix}$$

So, $$X = A^{-1}B = \begin{bmatrix} -33/2 \\ 12 \\ 9 \end{bmatrix}$$

When we are performing row operations, we are in fact performing matrix multiplication with special matrices that we call elementary matrices.

Three types

- An elementary matrix obtained by interchanging two rows of I.
- An elementary matrix obtained by multiplying a row I by a nonzero number.
- An elementary matrix obtained by replacing a row of I by itself plus a multiple of another row of I.

EXAMPLE 8. Here are some examples of dimensions 3×3:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_1 = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R1 \leftrightarrow R2} \quad E_2 = \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{\& \; R_2 \leftrightarrow R_3, \; \times 2 \; R_1} \quad E_3 = \underbrace{\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R_1 + 2R_2 \to R_1}.$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$$

- $$E_1 A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 1 \\ 3 & 3 & 3 \end{bmatrix}$$

- $$E_2 A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 3 & 3 & 3 \\ 2 & 2 & 2 \end{bmatrix}$$

- $$E_3 A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 5 & 5 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$$

Some mysteries Unraveled!

When we were performing row operations on a matrix A, we were in fact performing a multiplication of an elementary matrix with A. Here are some facts related to this:

- If E is obtained by interchanging rows i and j of I, then EA is the matrix obtained from A by interchanging rows i and j of A.
- If E is obtained by multiplying row i of I by a scalar c, then EA is the matrix obtained from A by multiplying row i of A by c.
- If E is obtained by replacing row i of I by itself plus c times the row j of I, then EA is the matrix obtained from A by replacing row i of A by itself plus c times row j of A.

Example 9. Give the elementary matrices used in Example 5. At each step, using the elementary matrices, give the expression of the matrix resulting from the row operations.

Inverses of elementary matrices

Example 10. Consider the following elementary matrices

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

For each of them, find the inverse.

Remarks: In general, if E is an elementary matrix, then E is invertible and:

- If E is obtained by interchanging two rows of I, then $E^{-1} = E$;
- If E is obtained by multiplying row i of I by a nonzero scalar c, then E^{-1} is the matrix obtained by multiplying row i of I by $1/c$;
- If E is obtained by replacing row i of I by itself plus c times row j of I, then E^{-1} is the matrix obtained by replacing row i of I by itself plus $-c$ times row j of I.

Consequences:

- A square matrix A is invertible if and only if A is a product of elementary matrices.