Math 307

Chapter 1

Section 1.6: Further Properties of Determinants

Contents

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Invertibility

A square matrix A is invertible if and only if $\det(A) \neq 0$.

Remark:

- If A is an invertible matrix, then $\det(A^{-1}) = 1/\det(A)$.
- Determinant can help to determine if a system of linear equations has a solution or not.

EXAMPLE 1. Which of the following matrices are invertible:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -4 & -3 & -2 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 4 & 5 & 9 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -1 \\ 0 & 0 & 4 \end{bmatrix}.$$

A) $\det(A) = -1 \neq 0 \Rightarrow$ A is invertible.

B) $\det(B) = 0 \Rightarrow$ B is not invertible.

c) $\det(C) = 12 \neq 0 \Rightarrow$ C is invertible

$$1 = \det(I) = \det(AA^{-1}) = \det(A)\det(A^{-1})$$

$$\det(AB^{-1}) = \det(A)\det(B^{-1})$$

$$\Rightarrow \det(AB^{-1}) = \frac{\det(A)}{\det(B)}$$

Matrix Multiplication

If A and B are two $n \times n$ matrices, then

$$\det(AB) = \det(A)\boxed{\det(B)}.$$

Example 2. Knowing that $\det(A) = 2$ and $\det(AB) = 32$, find the determinant of the matrix B.

$$\det(AB) = \det(A)\det(B) \Rightarrow \det(B) = \frac{\det(AB)}{\det(A)}.$$

$$\Rightarrow \det(B) = \frac{32}{2} = \boxed{16}$$

Transpose

If A is a square matrix, then $\det(A^\top) = \det(A)$.

Example 3. If A is the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & -2 \\ 0 & 0 & 1 \end{bmatrix},$$

then find the determinant of $AA^\top$.

$$\det(A+B) \neq \det(A) + \det(B)$$

We have $\det(AA^T) = \det(A)\det(A^T)$

$$= \det(A)\det(A) = \det(A)^2$$

$$\det(A) = 5 \Rightarrow \det(AA^T) = 5^2 = \boxed{25}$$

Matrix of Cofactors

The cofactor matrix is the matrix C of all the cofactors of a given matrix A. If A has dimensions $n \times n$, then

$$C = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

where each C_{ij} is the cofactor obtained from A.

EXAMPLE 4. Find the matrix of cofactors of the following matrix

$$A = \begin{bmatrix} -2 & 3 & 0 \\ 4 & 10 & 2 \\ -5 & 7 & 0 \end{bmatrix}.$$

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 10 & 2 \\ 7 & 0 \end{vmatrix} = -14$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 4 & 2 \\ -5 & 0 \end{vmatrix} = -10$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 4 & 10 \\ -5 & 7 \end{vmatrix} = 78$$

$$C_{21} = (-1)^{2+1} \begin{vmatrix} 3 & 0 \\ 7 & 0 \end{vmatrix} = 0$$

$$C_{22} = (-1)^{2+2} \begin{vmatrix} -2 & 0 \\ -5 & 0 \end{vmatrix} = 0$$

$$C_{23} = (-1)^{2+3} \begin{vmatrix} -2 & 3 \\ -5 & 7 \end{vmatrix} = -1$$

$$C_{31} = (-1)^{3+1} \begin{vmatrix} 3 & 0 \\ 10 & 2 \end{vmatrix} = 6$$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} -2 & 0 \\ 4 & 2 \end{vmatrix} = 4$$

$$C_{33} = (-1)^{3+3} \begin{vmatrix} -2 & 3 \\ 4 & 10 \end{vmatrix} = -32$$

$$C = \begin{bmatrix} -14 & -10 & 78 \\ 0 & 0 & -1 \\ 6 & 4 & -32 \end{bmatrix}.$$

Definition of the Adjoint

The adjoint of a matrix A of dimensions $n \times n$ is the transpose of the cofactor matrix.

Explicitly, we denote the adjoint of A by $\mathrm{adj}\,(A)$ and its expression is

$$\mathrm{adj}(A) = C^\top = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}.$$

EXAMPLE 5. Find the adjoint of the matrix in Example 4.

$$\mathrm{adj}(A) = C^T = \begin{bmatrix} -14 & 0 & 6 \\ -10 & 0 & 4 \\ 78 & -1 & -32 \end{bmatrix}$$

Another Way to Find the Inverse

If A is an invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)} \mathrm{adj}\,(A).$$

EXAMPLE 6. Find the inverse of the matrix in Example 4.

$$\mathrm{adj}(A) = \begin{bmatrix} -14 & 0 & 6 \\ -10 & 0 & 4 \\ 78 & -1 & -32 \end{bmatrix} \qquad \det(A) = -2.$$

$$\Rightarrow \quad A^{-1} = \frac{1}{-2} \begin{bmatrix} -14 & 0 & 6 \\ -10 & 0 & 4 \\ 78 & -1 & -32 \end{bmatrix} = \begin{bmatrix} 7 & 0 & -3 \\ 5 & 0 & -2 \\ -39 & 1/2 & 16 \end{bmatrix}$$

Suppose that $AX = B$ is a system of n linear equations in n unknowns such that $\det(A) \neq 0$. Let

- A_1 be the matrix obtained from A by replacing the first column of A by B;
- A_2 be the matrix obtained from A by replacing the second column of A by B.
- etc.

Then the solutions to the system are

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \quad \ldots, \quad x_n = \frac{\det(A_n)}{\det(A)}.$$

Example 7. Use Cramer's rule to solve the system

$$\begin{aligned} -2x + 3y &= 2 \\ 4x + 10y + 2z &= 3 \\ -5x + 7y &= 1. \end{aligned} \quad \longrightarrow \quad AX = B.$$

$$A = \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} -2 & 3 & 0 \\ 4 & 10 & 2 \\ -5 & 7 & 0 \end{bmatrix} \end{array} \qquad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad B = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

From Ex. 1 $\longrightarrow$ $\det(A) = -2$.

$$x = \frac{\begin{vmatrix} 2 & 3 & 0 \\ 3 & 10 & 2 \\ 1 & 7 & 0 \end{vmatrix}}{-2} = \frac{-\cancel{2}\begin{vmatrix} 2 & 3 \\ 1 & 7 \end{vmatrix}}{-\cancel{2}} = \boxed{11}$$

$$y = \frac{\begin{vmatrix} -2 & 2 & 0 \\ 4 & 3 & 2 \\ -5 & 1 & 0 \end{vmatrix}}{-2} = \frac{\cancel{-2}\begin{vmatrix} -2 & 2 \\ -5 & 1 \end{vmatrix}}{\cancel{-2}} = \boxed{8}$$

$$z = \frac{\begin{vmatrix} -2 & 3 & 2 \\ 4 & 10 & 3 \\ -5 & 7 & 1 \end{vmatrix}}{-2} = \boxed{-\frac{121}{2}}$$