Math 307

Chapter 1

Section 1.5: Determinants

Contents

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EXAMPLE 1. Find the equation of the parabola $ax^2 + bx + 1$ passing through the points $(1, 1)$ and $(2, 4)$.

$$x=1 \rightarrow a1^2 + b1 + 1 = 1$$
$$x=2 \rightarrow a2^2 + b2 + 1 = 4$$
$$\rightarrow \begin{cases} a + b = 0 \\ 4a + 2b = 4 \end{cases}$$

So,
$$\begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 2 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 4 \end{bmatrix} \quad \underset{a_{11}}{1}R_2 - \underset{a_{21}}{4}R_1 \rightarrow R_2$$

$$\hookrightarrow -2 = 1 \cdot 2 - 4 \cdot 1$$
$$= a_{11}a_{22} - a_{21}a_{12}.$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \end{bmatrix} \quad -\frac{1}{2}R_2 \rightarrow R_2$$

$$\hookrightarrow -2 = \frac{4}{1 \cdot 2 - 4 \cdot 1} = \frac{4}{a_{11}a_{22} - a_{21}a_{12}}$$

determinant

So, if $a_{11}a_{22} - a_{21}a_{12} = 0 \rightarrow$ system would not have any solution!
(it <u>determines</u> if a system is solvable or not!).

Historical Notes:

- Chinese scholars were the first to use determinants to solve systems of linear equations (3rd century BCE!).
- Cramer (1779) and Bezout (1779 also) used determinant to find a plane curve passing through a set of points, like we did in the previous example.

DEFINITION

2 by 2 matrices

Given a 2×2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

the determinant of A, denoted by $\det(A)$ is

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}.$$

Remark: Another notation for the determinant is

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

EXAMPLE 2. Calculate the determinant of the following matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}.$$

$\det(A) = 1 \cdot 4 - 2 \cdot 3 = 4 - 6 = -2$

$\det(B) = (-1)(-2) - 2 \cdot 1 = 2 - 2 = 0$

3 by 3 matrices

Let A be a general 3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{32} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- Minor: The minor of an entry a_{ij} is the matrix M_{ij} obtained from A by removing row i and column j.
- Cofactor: The cofactor of an entry a_{ij} is the matrix C_{ij} given by

$$C_{ij} = (-1)^{i+j} \det(M_{ij}).$$

Example 3. Find the minor M_{11}, and the cofactor C_{32} of the following matrices:

$$A = \begin{bmatrix} 2 & 3 & -2 \\ -1 & 6 & 3 \\ 4 & -2 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -3 & -1 \\ 1 & -1 & -1 \end{bmatrix}.$$

$\boxed{A}$

$$M_{11} = \begin{bmatrix} 2 & 3 & -2 \\ -1 & 6 & 3 \\ 4 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ -2 & 1 \end{bmatrix}$$

$$C_{32} = (-1)^{3+2} \det(M_{32}) = (-1)^5 \begin{vmatrix} 2 & -2 \\ -1 & 3 \end{vmatrix} = -4$$

$\boxed{B}$

$$M_{11} = \begin{bmatrix} -3 & -1 \\ -1 & -1 \end{bmatrix}$$

$$C_{32} = (-1)^{3+2} \det(M_{32}) = (-1)^5 \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = 5$$

The determinant of A is given by

$$\begin{aligned}\det(A) &= a_{11}\det(M_{11}) - a_{12}\det(M_{12}) + a_{13}\det(M_{13}) \\ &= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33}\end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33}\end{vmatrix} + a_{13}\begin{vmatrix} a_{12} & a_{22} \\ a_{31} & a_{32}\end{vmatrix} \\ &= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{32} + a_{13}a_{12}a_{32} - a_{13}a_{22}a_{31}.\end{aligned}$$

Expression obtained in the Gauss Elimination.

EXAMPLE 4. Find the determinant of the following matrices:

$$A = \begin{bmatrix} 2 & 3 & -2 \\ -1 & 6 & 3 \\ 4 & -2 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -3 & -1 \\ 1 & -1 & -1 \end{bmatrix}.$$

<u>A</u>

$$\det A = 2\begin{vmatrix} 6 & 3 \\ -2 & 1 \end{vmatrix} - 3\begin{vmatrix} -1 & 3 \\ 4 & 1 \end{vmatrix} - 2\begin{vmatrix} -1 & 6 \\ 4 & -2 \end{vmatrix}$$

$$= 2(6+6) - 3(-1-12) - 2(2-24)$$

$$= 24 + 39 + 44$$

$$= 107$$

<u>B</u>

$$\det B = 2$$

For General Matrices

The determinant is defined recursively.

1. If A is an 2×2 matrix, then $\det(A) = a_{11}a_{22} - a_{12}a_{21}$.
2. If A is an $n \times n$ matrix, then

$$\begin{aligned} \det(A) &= a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n} \\ &= a_{11}\det(M_{11}) - a_{12}\det(M_{12}) + \cdots + (-1)^{1+n}a_{1n}\det(M_{1n}). \end{aligned}$$

EXAMPLE 5. Compute the determinant of the following matrix:

$$A = \begin{bmatrix} 7 & -3 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 2 & 1 & -2 & -5 \\ 0 & 4 & 0 & 6 \end{bmatrix}.$$

$\det(A) = 84$

Determinant from any row or column

Lagrange's Expansion Formula: If A is an $n \times n$ matrix with $n \geq 2$, then

- $\det(A) = \sum_{j=1}^{n} a_{ij} C_{ij}$ for any row indexed by i.
- $\det(A) = \sum_{i=1}^{n} a_{ij} C_{ij}$ for any column indexed by j.

EXAMPLE 6. Compute again the determinant of the matrix A in Example 4 by

1. expanding with respect to another row.
2. expanding with respect to one of the column.

1) According to second row.

$$\det(A) = a_{21} C_{21} + a_{22} C_{22} + a_{23} C_{23} + a_{24} C_{24}$$

$$= -a_{21} \det(M_{21}) + a_{22} \det(M_{22}) - a_{23} \det(M_{23}) + a_{24} \det(M_{24})$$

$$= \cancel{0 \begin{vmatrix} -3 & 0 & 4 \\ 1 & -2 & -5 \\ 4 & 0 & 6 \end{vmatrix}} \to 0 + 1 \begin{vmatrix} 7 & 0 & 4 \\ 2 & -2 & -5 \\ 0 & 0 & 6 \end{vmatrix} - \cancel{0 \begin{vmatrix} 7 & -3 & 4 \\ 2 & 1 & -5 \\ 0 & 4 & 6 \end{vmatrix}} \to 0 + 3 \begin{vmatrix} 7 & -3 & 0 \\ 2 & 1 & -2 \\ 0 & 4 & 0 \end{vmatrix}$$

$$= 1 \left(6 \begin{vmatrix} 7 & 0 \\ 2 & -2 \end{vmatrix} \right) + 3 \left(2 \begin{vmatrix} 7 & -3 \\ 0 & 4 \end{vmatrix} \right)$$

$$= 6(-14) + 6(28) = 6 \cdot 14 = \boxed{84}$$

2) According to column 3.

$$\det(A) = \cancel{0 \begin{vmatrix} 0 & 1 & 3 \\ 2 & 1 & -5 \\ 0 & 4 & 6 \end{vmatrix}} \to 0 - \cancel{0 \begin{vmatrix} 7 & -3 & 4 \\ 2 & 1 & -5 \\ 0 & 4 & 6 \end{vmatrix}} \to 0 - 2 \begin{vmatrix} 7 & -3 & 4 \\ 0 & 1 & 3 \\ 0 & 4 & 6 \end{vmatrix} - \cancel{0 \begin{vmatrix} 7 & -3 & 4 \\ 0 & 1 & 3 \\ 2 & 1 & -5 \end{vmatrix}} \to 0$$

$$= -2 \begin{vmatrix} 7 & -3 & 4 \\ 0 & 1 & 3 \\ 0 & 4 & 6 \end{vmatrix}$$

$$= -2 \left(7 \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} \right) = -14 \cdot -6 = \boxed{84}$$

Advice: It would be clever to choose the row or column containing the greatest number of zeros.

When there too many zeros...

Example 7. Find the determinant

$$A = \begin{vmatrix} 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 2 & 0 \\ -1 & 2 & 3 & 0 & 4 & 2 & -1 \\ -2 & 2 & -1 & 0 & 3 & -3 & -1 \\ 1 & -1 & 1 & 0 & 3 & -2 & -1 \\ -2 & -1 & 1 & 0 & -5 & 1 & -6 \end{vmatrix}.$$

Develop det(A) w.r.t. the 4th column.
Only zeros!

$\to \det(A) = \boxed{0}$

Fact: If a matrix A has a row or a column of zeros, then $\det(A) = 0$.

When the type matters!

EXAMPLE 8. Find the determinant

$$A = \begin{vmatrix} 1 & 4 & 10 & 123 \\ 0 & 2 & 124 & \pi \\ 0 & 0 & 3 & \sqrt{2} \\ 0 & 0 & 0 & 4 \end{vmatrix}.$$

$$\det(A) = 1 \begin{vmatrix} 2 & 124 & \pi \\ 0 & 3 & \sqrt{2} \\ 0 & 0 & 4 \end{vmatrix}$$

$$= 1 \cdot 2 \begin{vmatrix} 3 & \sqrt{2} \\ 0 & 4 \end{vmatrix}$$

$$= 1 \cdot 2 \cdot 3 \cdot 4 = \boxed{24}$$

Fact: The determinant of a triangular (upper or lower) is the product of its diagonal entries.

When Operations matter!

When E is an elementary matrix,

- If E switches row i with row j, then $\det(E) = -1$.
- If E is obtained from I by multiplying a row by some scalar c, then $\det(E) = c$.
- If E is obtained from I by replacing a row of I by itself plus a multiple of another row of I, then $\det(E) = 1$.

This implies the following general facts: Suppose that $A = [a_{ij}]$ is an $n \times n$ matrix with $n \geq 2$.

- If B is a matrix obtained from A by interchanging two rows of A, then $\det(B) = -\det(A)$.
- If B is a matrix obtained from A by multiplying a row of A by a scalar c, then $\det(B) = c\det(A)$.
- If B is a matrix obtained from A by replacing a row of A by itself plus a mlutiple of another row of A, then $\det(B) = \det(A)$.