Math 307

Chapter 6

Section 6.1: The Theory of Systems of Linear Differential Equations

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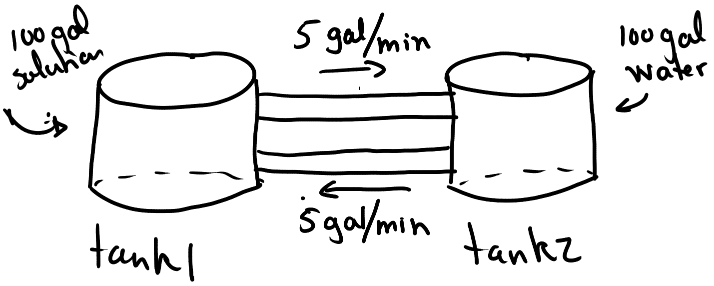
Mixing Problems

Example 1. Consider two tanks each with volume 100 gallons. The two tanks are connected together by two pipes. The first tank initially contains a well-mixed solution of 5lb salt in ~~50~~ 100 gal water. The second tank initially contains 100 gal salt-free water.

A pipe from tank 1 to tank 2 allows the solution in tank 1 to enter tank 2 at a rate of 5 gal/min. A second pipe from tank 2 to tank 1 allows the solution from tank 2 to enter tank 1 at a rate of 5 gal/min.

Assume that the salt mixture in each tank is well-stirred. Find a model describing the quantity of salt in each tank.



C_1: concentration of salt in tank 1 (lb/gal).

C_2: concentration of salt in tank 2 (lb/gal).

Q_1: salt in tank 1 (lb).

Q_2: salt in tank 2 (lb).

$$C_1 = \frac{Q_1}{\underset{\text{total vol.}}{100}}, \quad C_2 = \frac{Q_2}{100}.$$

rate of change of Q_1

$$\underset{\text{lb/min}}{Q_1'} = -5 \cdot C_1 + 5C_2$$

$$= -\frac{5 \cdot Q_1}{100} + \frac{5 \cdot Q_2}{100}$$

$$= \frac{Q_2}{20} - \frac{Q_1}{20}$$

$$\rightarrow \underset{\frac{dQ_1}{dt}}{Q_1'} = \underbrace{\frac{Q_2}{20} - \frac{Q_1}{20}}_{\text{diff. equation}}.$$

rate of change of Q_2

$$Q_2' = 5 \cdot C_1 - 5C_2$$

$$= \frac{5 Q_1}{100} - \frac{5 \cdot Q_2}{100}$$

$$= \frac{Q_1}{20} - \frac{Q_2}{20}$$

$$\rightarrow \underbrace{Q_2' = \frac{Q_1}{20} - \frac{Q_2}{20}}_{\text{diff. diff.}}.$$

$$\rightarrow \begin{cases} Q_1' = \dfrac{Q_2}{20} - \dfrac{Q_1}{20} \\ Q_2' = \dfrac{Q_1}{20} - \dfrac{Q_2}{20}. \end{cases} \leftarrow \text{syst. of diff. eqs.}$$

$Y_1 = Q_1$

$Y_2 = Q_2$

$$\rightarrow \begin{cases} Y_1' = \dfrac{Y_2}{20} - \dfrac{Y_1}{20} = -\dfrac{Y_1}{20} + \dfrac{Y_2}{20} \\ Y_2' = \dfrac{Y_1}{20} - \dfrac{Y_2}{20} \end{cases}$$

$$\underbrace{\begin{bmatrix} Y_1' \\ Y_2' \end{bmatrix}}_{Y'} = \underbrace{\begin{bmatrix} \frac{-1}{20} & \frac{1}{20} \\ \frac{1}{20} & \frac{-1}{20} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}}_{Y}$$

$$\rightarrow \quad Y' = AY \qquad G = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Definitions

System of ODEs

A **system of n first order linear differential equations** (system of n ODEs for short) is a vector-equation:

$$Y' = AY + G$$

where

- Y is an $n \times 1$ vector of unknown functions:

$$Y(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_n(x) \end{bmatrix}.$$

- Y' is the $n \times 1$ vector of derivatives of the unknown functions:

$$Y'(x) = \begin{bmatrix} y_1'(x) \\ y_2'(x) \\ \vdots \\ y_n'(x) \end{bmatrix}.$$

- A is an $n \times n$ matrix of ~~functions~~: numbers.

$$A = \begin{bmatrix} a_{11}(x) & a_{12}(x) & \cdots & a_{1n}(x) \\ a_{21}(x) & a_{22}(x) & \cdots & a_{2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(x) & a_{n2}(x) & \cdots & a_{nn}(x) \end{bmatrix}.$$

- G is an $n \times 1$ column vector of functions:

$$G(x) = \begin{bmatrix} g_1(x) \\ g_2(x) \\ \vdots \\ g_n(x) \end{bmatrix}.$$

If we add the additional conditions $Y(x_0) = B$ for some real number x_0 and an $n \times 1$ column vector B, the system of ODEs is called an **initial value problem**.

Homogeneous and Non-homogeneous

- If $G(x) = 0$ for every x, the system of ODEs is called **homogeneous**.
- if $G(x)$ is not zero, then the system of ODEs is called **non-homogeneous**.

Example 2. Consider the following system of ODEs:

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} \leftarrow Y' = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} Y. \rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

1. Is this a homogeneous or non-homogeneous system of ODEs?
2. Show that

$$Y(x) = \begin{bmatrix} e^{2x} + e^{3x} \\ 2e^{2x} + e^{3x} \end{bmatrix} \begin{matrix} \rightarrow y_1 \\ \rightarrow y_2 \end{matrix}$$

is a solution to the system.

1. $G = 0 \rightarrow$ homogeneous.

2. a) $$y' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} (e^{2x} + e^{3x})' \\ (2e^{2x} + e^{3x})' \end{bmatrix} = \begin{bmatrix} 2e^{2x} + 3e^{3x} \\ 4e^{2x} + 3e^{3x} \end{bmatrix} \checkmark$$

b) $$\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} e^{2x} + e^{3x} \\ 2e^{2x} + e^{3x} \end{bmatrix} = \begin{bmatrix} 4(e^{2x} + e^{3x}) - (2e^{2x} + e^{3x}) \\ 2(e^{2x} + e^{3x}) + (2e^{2x} + e^{3x}) \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{2x} + 3e^{3x} \\ 4e^{2x} + 3e^{3x} \end{bmatrix} \checkmark$$

so, $y' = AY$ $\rightarrow Y$ is a solution!! :)

Example 3. Consider the following initial value problem:

$$Y' = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} Y \quad \text{and} \quad Y(0) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$x = t$

Show that

$$Y(x) = \begin{bmatrix} 2e^{2x} + e^{3x} \\ 4e^{2x} + e^{3x} \end{bmatrix}$$

is a solution to the initial value problem.

1) verify that Y satisfies $Y' = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} Y$.

$$y' = \begin{bmatrix} (2e^{2x} + e^{3x})' \\ (4e^{2x} + e^{3x})' \end{bmatrix} = \begin{bmatrix} 4e^{2x} + 3e^{3x} \\ 8e^{2x} + 3e^{3x} \end{bmatrix}, \quad \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2e^{2x} + e^{3x} \\ 4e^{2x} + e^{3x} \end{bmatrix}$$

$$= \begin{bmatrix} 4e^{2x} + 3e^{3x} \\ 8e^{2x} + 3e^{3x} \end{bmatrix} \checkmark$$

2) verify that $Y(0) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$$\rightarrow Y(0) = \begin{bmatrix} 2e^0 + e^0 \\ 4e^0 + e^0 \end{bmatrix} = \begin{bmatrix} 2+1 \\ 4+1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \checkmark.$$

Existence and Uniqueness Theorem

Consider the initial value problem

$$Y' = AY + G \quad \text{and} \quad \boxed{Y(x_0) = B}. \tag{$\star$}$$

If all the entries $a_{ij}(x)$ of A and all the entries $g_i(x)$ of G are continuous functions, then the initial value problem ($\star$) has a unique solution.

Solutions as a Subspace

EXAMPLE 4. Consider the following system of ODEs:

$$Y' = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} Y.$$

If the general solution to the system is

$$Y(x) = \begin{bmatrix} c_1 e^{2x} + c_2 e^{3x} \\ 2c_1 e^{2x} + c_2 e^{3x} \end{bmatrix},$$

c_1, c_2 can be any real numbers.

describe the structure of the set of solutions.

$$Y(x) = \begin{bmatrix} c_1 e^{2x} \\ 2c_1 e^{2x} \end{bmatrix} + \begin{bmatrix} c_2 e^{3x} \\ c_2 e^{3x} \end{bmatrix} = c_1 \underbrace{\begin{bmatrix} e^{2x} \\ 2e^{2x} \end{bmatrix}}_{=Y_1(x)} + c_2 \underbrace{\begin{bmatrix} e^{3x} \\ e^{3x} \end{bmatrix}}_{=Y_2(x)}.$$

→ Y is a linear comb. of Y_1 & Y_2

→ set of solutions is spanned by Y_1 & Y_2.
and we can show that Y_1 & Y_2 form a basis for the set of solutions to the system of 2 ODEs.

→ the dimension of the set of solutions is 2!

Fact: The set of solutions to a homogeneous system of n ODEs $Y' = A(x)Y$ form a vector space of dimension n.

Nomenclature

- A set of n linearly independent solutions $Y_1, Y_2, \ldots, Y_n$ to a homogeneous system of n ODEs is called a **fundamental set of solutions**.
- A **general solution**, denoted by Y_H, to a homogeneous system of n ODEs with fundamental set of solutions $Y_1, Y_2, \ldots, Y_n$ is a linear combination of $Y_1, Y_2, \ldots, Y_n$, that is

$$Y_H = c_1 Y_1 + c_2 Y_2 + \cdots + c_n Y_n.$$

- The **matrix of fundamental solutions**, denoted by M, is the matrix M form by the vector functions $Y_1, Y_2, \ldots, Y_n$ in the fundamental set of solutions:

$$M = \begin{bmatrix} Y_1 & Y_2 & \cdots & Y_n \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{bmatrix}.$$

$$Y_1 = \begin{bmatrix} y_{11}(x) \\ y_{21}(x) \\ \vdots \\ y_{n1}(x) \end{bmatrix}, \quad Y_2 = \begin{bmatrix} y_{12}(x) \\ y_{22}(x) \\ \vdots \\ y_{n2}(x) \end{bmatrix}, \ldots$$

Ex. 4:

$$M = \begin{bmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & e^{3x} \end{bmatrix}$$

$\uparrow$ Y_1 $\quad$ $\uparrow$ Y_2

Non-homogeneous Systems

Solutions to non-homogeneous systems and homogeneous system are related by one thing:

- A **particular solution** to a system $Y' = AY + \cancel{G}$, denoted by Y_P, is a specific solution to the system.

Therefore, every solution Y to the system $Y' = AY + G$ has the form

$$Y = Y_H + Y_P = c_1 Y_1 + c_2 Y_2 + \cdots + c_n Y_n + Y_P = MC + Y_P$$

Y_P must not equal one of the set of foundamental solutions.

where

- Y_H is the general solution to the system $Y' = AY$.
- Y_P is a particular solution to the system $Y' = AY + \cancel{B}.$ G
- $C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

Wronkian

Definition

Given n column vector functions

$$Y_1(x) = \begin{bmatrix} y_{11}(x) \\ y_{21}(x) \\ \vdots \\ y_{n1}(x) \end{bmatrix}, \quad Y_2(x) = \begin{bmatrix} y_{21}(x) \\ y_{22}(x) \\ \vdots \\ y_{n2}(x) \end{bmatrix}, \quad \cdots, \quad Y_n(x) = \begin{bmatrix} y_{n1}(x) \\ y_{n2}(x) \\ \vdots \\ y_{nn}(x) \end{bmatrix}$$

then the **Wronkian** of Y_1, Y_2, ..., Y_n is defined as

$$w(Y_1(x), Y_2(x), \cdots, Y_n(x)) := \begin{vmatrix} y_{11}(x) & y_{12}(x) & \cdots & y_{1n}(x) \\ y_{21}(x) & y_{22}(x) & \cdots & y_{2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1}(x) & y_{n2}(x) & \cdots & y_{nn}(x) \end{vmatrix}.$$

Example 5. Let Y_1 and Y_2 be the vector functions

$$Y_1(x) = \begin{bmatrix} e^{2x} \\ 2e^{2x} \end{bmatrix} \quad \text{and} \quad Y_2(x) = \begin{bmatrix} e^{3x} \\ e^{3x} \end{bmatrix}.$$

Compute $w(Y_1(x), Y_2(x))$.

$$w(Y_1(x), Y_2(x)) = \begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & e^{3x} \end{vmatrix} = e^{2x} e^{3x} - e^{3x}(2e^{2x})$$

$$= e^{2x+3x} - 2e^{3x+2x}$$

$$= e^{5x} - 2e^{5x}$$

$$= \boxed{-e^{5x}}$$

Linear Independence of Vector Functions

EXAMPLE 6. Show that the vector functions in Example 5 are linearly independent.

To show that Y_1 & Y_2 are lin. ind., we have to show that

$$c_1 Y_1(x) + c_2 Y_2(x) = 0 \Rightarrow c_1 = c_2 = 0.$$

Now,

$$c_1 \begin{bmatrix} e^{2x} \\ 2e^{2x} \end{bmatrix} + c_2 \begin{bmatrix} e^{3x} \\ e^{3x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} c_1 e^{2x} + c_2 e^{3x} \\ 2c_1 e^{2x} + c_2 e^{3x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \underbrace{\begin{bmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & e^{3x} \end{bmatrix}}_{A=} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ (should be true for any x)

If A is invertible, then $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = A^{-1} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

For A to be invertible, $\det(A) \neq 0$.

From example 5, $\det(A) = \begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & e^{3x} \end{vmatrix} = W(Y_1(x), Y_2(x))$

$$= -e^{5x}$$

$$\neq 0$$

$\rightarrow$ A is invertible $\Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow c_1 = c_2 = 0$ ✓

$\rightarrow$ Y_1, Y_2 are lin. independent.

Main Important Fact:

Given a list Y_1, Y_2, $\cdots$, Y_n of vector functions, if $w(Y_1(x), Y_2(x), \ldots, Y_n(x)) \neq 0$ for some x, then Y_1, Y_2, $\cdots$, Y_n are linearly independent.

Other Facts:

- If Y_1, Y_2, ..., Y_n are linearly dependent, then $w(Y_1(x), Y_2(x), \ldots, Y_n(x)) = 0$ for any x.
- If Y_1, Y_2, ..., Y_n are solutions to $Y' = AY$ and if $w(Y_1(x), Y_2(x), \ldots, Y_n(x)) = 0$ for some x, then Y_1, Y_2, ..., Y_n are linearly dependent.
- If Y_1, Y_2, ..., Y_n is a fundamental set of solutions to $Y' = AY$, then

$$w(Y_1(x), Y_2(x), \ldots, Y_n(x)) \neq 0$$

  for every x.

Our investigations in the next chapter will focus mainly on system of n ODEs with constant coefficients. This means:

The entries of the matrix A in the equation $Y' = AY + G$ are constants.

We begin with the case of a diagonal matrix A.

Example 7. Solve the system

$$Y' = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} Y.$$

$$Y' = \begin{bmatrix} y_1'(x) \\ y_2'(x) \end{bmatrix}, \quad Y = \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix}$$

AY

$$\Rightarrow \begin{bmatrix} y_1'(x) \\ y_2'(x) \end{bmatrix} = \begin{bmatrix} 3y_1(x) \\ -y_2(x) \end{bmatrix}$$

$$\Rightarrow \begin{cases} y_1'(x) = 3y_1(x) & (1) \\ y_2'(x) = -y_2(x) & (2) \end{cases}$$

(1) $y_1'(x) = 3y_1(x)$ $\to$ let $f(x) = y_1(x)$

$\to$ $f'(x) = 3f(x)$

$\to$ $f'(x) = 3\underbrace{e^{3x}}_{f(x)}$ $\to$ $\boxed{f(x) = c_1 e^{3x}}$

$\underline{y_1(x) = c_1 e^{3x}}$

(2) $y_2'(x) = -y_2(x)$

$\to$ $y_2(x) = c_2 e^{-x}$

The solution is

$$Y(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} c_1 e^{3x} \\ c_2 e^{-x} \end{bmatrix} = c_1 \underbrace{\begin{bmatrix} e^{3x} \\ 0 \end{bmatrix}}_{Y_1(x)} + c_2 \underbrace{\begin{bmatrix} 0 \\ e^{-x} \end{bmatrix}}_{Y_2(x)}$$

The general solution to a homogeneous system $Y' = AX$ where A is a diagonal matrix

$$A = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

is given by

$$Y_H = \begin{bmatrix} e^{d_1 x} & 0 & \cdots & 0 \\ 0 & e^{d_2 x} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{d_n x} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} c_1 e^{d_1 x} \\ c_2 e^{d_2 x} \\ \vdots \\ c_n e^{d_n x} \end{bmatrix}.$$

$$\begin{array}{cccc} Y_1 & Y_2 & \cdots & Y_n \end{array}$$

EXAMPLE 8. Solve the initial value problem

$$Y' = \begin{bmatrix} \underset{d_1}{-3} & 0 & 0 & 0 \\ 0 & \underset{d_2}{-2} & 0 & 0 \\ 0 & 0 & \underset{d_3}{2} & 0 \\ 0 & 0 & 0 & \underset{d_4}{5} \end{bmatrix} Y \quad \text{and} \quad Y(0) = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}.$$

(1) Find Y_H.

$$Y_H(x) = \begin{bmatrix} c_1 e^{-3x} \\ c_2 e^{-2x} \\ c_3 e^{2x} \\ c_4 e^{5x} \end{bmatrix}$$

(2) Find the values of c_1, c_2, c_3, c_4.

$$\begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} = Y_H(0) = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} \longrightarrow \begin{array}{l} c_1 = 2 \\ c_2 = 1 \\ c_3 = -1 \\ c_4 = 0 \end{array}$$

$$Y(x) = \begin{bmatrix} 2e^{-3x} \\ e^{-2x} \\ -e^{2x} \\ 0 \end{bmatrix}.$$