Math 307

Chapter 2

Section 2.1: Vector Spaces

Contents

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Solutions to System of Linear Equations

EXAMPLE 1. Describe the set of solutions of the following system of linear equations.

$$\begin{aligned} 2x + 3y - z &= 3 \\ -x - y + 3z &= 0 \\ x + 2y + 2z &= 3 \\ y + 5z &= 3. \end{aligned}$$

Precise Definition

A nonempty set V is a **vector space** if there are operations of addition (denoted by $+$) and scalar multiplication (denoted by $\cdot$) on V such that the following eight properties are satisfied:

1. $u + v = v + u$ for any u and v in V;
2. $u + (v + w) = (u + v) + w$ for any u, v, w in V;
3. There is an element denoted 0 in V so that $v + 0 = v$ for any v in V.
4. For each v in V there is an element denoted $-v$ so that $v + (-v) = 0$.
5. $c \cdot (u + v) = c \cdot u + c \cdot v$ for all real number c and for all u and v in V;
6. $(c + d) \cdot v = c \cdot v + d \cdot v$ for all real numbers c and d and for all v in V;
7. $c \cdot (d \cdot v) = (cd) \cdot v$ for all real numbers c and d and for all v in V;
8. $1 \cdot v = v$ for all v in V.

Remarks:

- The eight above properties are called *axioms*, *postulates*, or *laws* of a vector spaces.
- Don't confuse the abstract vectors from the more concrete column-vectors or row-vectors.
- The elements of the set V are called *vectors*.
- The real numbers are called *scalars*.

Column Vectors as a Vector space

EXAMPLE 2. The set of all 3×1 column vectors, denoted by $\mathbb{R}^3$, is a vector space if we define

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} := \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix} \quad \text{and} \quad c \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} := \begin{bmatrix} cx_1 \\ cx_2 \\ cx_3 \end{bmatrix}.$$

Example 3. More generally, the set of all $n \times 1$ column vectors, denoted by $\mathbb{R}^n$ is a vector space if we define

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} := \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix} \quad \text{and} \quad c \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} := \begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{bmatrix}.$$

Remark:

- The set of $1 \times n$ row vectors is also a vector space with the addition and scalar multiplication defined component-wise in a similar way.

Matrices as a Vector Space

EXAMPLE 4. The set of $m \times n$ matrices $M_{m\times n}(\mathbb{R})$ is a vector space if we define the addition of two matrices and the scalar multiplication of a real number with a matrix by the matrix addition and matrix scalar multiplication defined in the previous chapter (see section 1.2).

Functions as a Vector Space

EXAMPLE 5. Let $F(a, b)$ denote the set of all real-valued functions defined on (a, b). Some examples are $f(x) = x^2$, $f(x) = \sin x$, $f(x) = |x|$, etc.

We define the addition of two functions f and g to be the new function $(f + g)$ defined on (a, b) by

$$(f + g)(x) := f(x) + g(x).$$

We define the scalar multiplication of a function f with a real number c to be the new function (cf) defined on (a, b) by

$$(cf)(x) := cf(x).$$

Show that $F(a, b)$ is a vector space.

A nonexample

Example 6. Let V be the set of 1×2 row vectors. We define an addition and a scalar multiplication by

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} + \begin{bmatrix} y_1 & y_2 \end{bmatrix} := \begin{bmatrix} x_1 + y_1 + 1 & x_2 + y_2 \end{bmatrix} \quad \text{and} \quad c \begin{bmatrix} x_1 & x_2 \end{bmatrix} := \begin{bmatrix} cx_1 & cx_2 \end{bmatrix}.$$

Is V equipped with these operations a vector space?

Simple Properties of Vector Spaces

Uniqueness

Suppose that V is a vector space.

- There is only one zero vector in V.
- If v is a vector in V, there is only one negative (denoted by $-v$) of v.

Multiplying by Zero

Let V be a vector space.

- For any vector v in V, we have $0 \cdot v = 0$.
- For any real number c, we have $c \cdot 0 = 0$.

Subtraction in Vector space

Let V be a vector space. Then for any vector v in V, we have

$$(-1) \cdot v = -v.$$

Remarks:

- We usually write cv instead of $c \cdot v$ for the scalar multiplication. It simplifies the notation.
- Substracting two vectors is done in the following way:

$$u - v := u + (-v).$$