Math 307

Chapter 2

Section 2.3: Linear Independence and Bases

Contents

Created by: Pierre-Olivier Parisé
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Definition

Suppose that v_1, v_2, …, v_n are vectors in a vector space V.

- The vectors v_1, v_2, …, v_n are **linearly dependent** if there are scalars c_1, c_2, …, c_n, not all zero, so that

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0.$$

- If v_1, v_2, …, v_n are not linearly dependent, then the vectors are **linearly independent**.

EXAMPLE 1. Are the vectors

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$$

linearly dependent or linearly independent?

Example 2. Are x^2+1, x^2-x+1, $x+2$ linearly dependent or linearly independent?

Remark: To show that the vectors v_1, v_2, …, v_n are linearly independent, we can verify that the following implication is true:

$$\text{If } c_1v_1+c_2v_2+\cdots+c_nv_n=0, \text{ then } c_1=c_2=\cdots=c_n=0.$$

Dependence and Linear Combination

A way to check if a bunch of vectors are linearly dependent is outlined in the following statement.

Theorem 3. Suppose $v_1, v_2, \ldots, v_n$ are vectors in a vector space V. Then $v_1, v_2, \ldots, v_n$ are linearly dependent if and only if one of $v_1, v_2, \ldots, v_n$ is a linear combination of the others.

Example 4. Apply the last Theorem to show that the vectors

$$\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \text{ and } \begin{bmatrix} 3 \\ -6 \\ 9 \end{bmatrix} \text{ and } \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}.$$

are linearly dependent.

Definition

The vectors v_1, v_2, …, v_n of a vector space V are a **basis** if the two following conditions are satisfied:

- v_1, v_2, …, v_n are linearly independent. [Indepence Condition or IC]
- v_1, v_2, …, v_n span V. [Spanning condition, or SC]

Example 5. Show that the vectors

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

forms a basis for $\mathbb{R}^3$.

Remark: The basis in the last example is called the standard basis for $\mathbb{R}^3$. More generally, the vectors

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad \ldots, \quad e_{n-1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix}, \quad e_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

forms a basis for the vector space $\mathbb{R}^n$ of column vectors of dimensions $n \times 1$.

Basis for Matrices and Polynomials

Example 6. The vectors

$$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

form a basis for the vector space of 2×2 matrices $M_{2\times 2}(\mathbb{R})$.

Remark: The basis in the last example is called the standard basis for the vector space $M_{2\times 2}(\mathbb{R})$. More generally, the vectors E_{ij} with a 1 in the entry ij and 0 elsewhere forms a basis for the space of matrices $M_{m\times n}(\mathbb{R})$.

Example 7. The vectors

$$1,\, x,\, x^2$$

form a basis for the set of polynomials P_2.

Remark: The basis in the last example is also called the standard basis for the vector space P_2. More generally, for a nonnegative integer n, the vectors

$$x^n,\, x^{n-1},\, \ldots,\, x,\, 1$$

form a basis for the vector space P_n of polynomials of degree less than or equal to n.

Example 8. Do the vectors

$$\begin{bmatrix}1\\0\\1\end{bmatrix}, \quad \begin{bmatrix}1\\1\\1\end{bmatrix}, \quad \begin{bmatrix}-1\\1\\1\end{bmatrix}$$

form a basis for the vector space of 3×1 column vectors?

Coordinates relative to a basis

In many applications, like robotics, it is really important to be able to represent the position of a moving part of a robot in terms of a new coordinates system.

Basis are an essential tools to do that. Given a basis v_1, v_2, $\ldots$, v_n of a vector space V, each vector v in V can be expressed as a linear combination of the vectors in the basis:

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n. \tag{1}$$

Moreover, the scalars c_1, c_2, $\ldots$, c_n in the Equation (1) are unique. This means that there is only one list of scalars c_1, c_2, $\ldots$, c_n that satisfies Equation (1).

- The list of scalars c_1, c_2, $\ldots$, c_n are called the **coordinates of v relative to the basis $\mathbf{v_1}$, $\mathbf{v_2}$, $\ldots$, $\mathbf{v_n}$.**
- If α denotes the basis v_1, v_2, $\ldots$, v_n, then the column vector

  $$[v]_\alpha = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

  is called the **coordinate vector of v relative to the basis** α.

Remarks:

- Coordinates relative to the standard basis:

- It is important to not confuse the column vectors representing the vector in a certain basis with the column vectors representing the vector in the standard basis.

Example 9. Find the coordinate vector of

$$v = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}$$

relative to the basis α for $\mathbb{R}^3$ presented in Example 8.

EXAMPLE 10. Find the coordinate in the standard basis of the vector v in $\mathbb{R}^3$ if

$$[v]_\alpha = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

where α is the basis for $\mathbb{R}^3$ in Example 8.