Math 307

Chapter 1

Section 1.4: Special Matrices and Additional Properties

Contents

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Diagonal Matrices

A diagonal matrix is a square matrix whose off diagonal entries are zero.

- <u>Remark</u>: We denote a diagonal matrix by $\operatorname{diag}(d_1, d_2, \ldots, d_n)$.

Example 1. Give some examples of diagonal matrices.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Example 2. Suppose $A = \operatorname{diag}(1, 2, -4, 3, 5)$ and $B = \operatorname{diag}(-1, 2, 0, 4, 3)$.

1) Is A invertible? 2) Is $A + B$ invertible? 3) Is AB invertible?

1) $$\left[\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1/4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1/5 \end{array}\right]$$ YES!

$$A^{-1} = \operatorname{diag}(1, 1/2, -1/4, 1/3, 1/5).$$

2) $$A+B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 8 \end{bmatrix} = \operatorname{diag}(0, 4, -4, 7, 8).$$

NO

3) $AB = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -16 & 0 \\ 0 & 0 & 0 & 0 & 15 \end{bmatrix}$

$= \text{diag}(-1, 4, 0, -16, 15).$

There is a zero $\Rightarrow$ AB is not invertible.

General Facts: Suppose A and B are diagonal matrices

$$A = \text{diag}(a_1, a_2, \ldots, a_n) \quad \text{and} \quad B = \text{diag}(b_1, b_2, \ldots, b_n).$$

- $A + B = \text{diag}(a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$.
- $AB = \text{diag}(a_1 b_1, a_2 b_2, \ldots, a_n b_n)$.
- A is invertible if and only if $a_i \neq 0$ for each i. In this case, we have

$$A^{-1} = \text{diag}(1/a_1, 1/a_2, \ldots, 1/a_n).$$

- Upper Triangular: Square matrices whose entries below the diagonal are zero.
- Lower Triangular: Square matrices whose entries above the diagonal are zero.

EXAMPLE 3. Give an example of an upper triangular matrix and an example of a lower triangular matrix.

$$\begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

upper triangular.

lower triangular.

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 4 & 2 \end{bmatrix}$$

General Facts:

- If A and B are both upper triangular, then so is $A + B$; similarly if A and B are both lower triangular, then so is $A + B$.
- If A and B are both upper triangular, then so is AB; similarly if A and B are both lower triangular, then so is AB.
- A is invertible if and only if each of the diagonal entries of A is nonzero.

Example 4. Let A and B be the two following 3×3 matrices:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 7 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 4 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

1. Is A invertible?
2. Is B invertible?
3. Is AB upper or lower triangular matrix?

1) Yes because entries on diag are not zero.

2) No, B has an entry = 0.

3) AB is upper triangular because A & B are.

Verification:

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} -1 & 4 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 4 \\ 0 & -4 & 8 \\ 0 & 0 & 0 \end{bmatrix} \checkmark$$

Symmetric Matrices

- Transpose: The transpose of a matrix A of dimensions $m \times n$, denoted $A^\top$, is the matrix obtained by interchanging the rows and columns of A.
- Symmetric: A matrix A is said to be symmetric if $A = A^\top$.

Example 5. Let A and B be the following matrices.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}.$$

1. Find $B^\top$.
2. Is A symmetric?

1) $B^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

2) $A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} = A \to$ Yes!

General facts about transpose:

- $(A^\top)^\top = A$.
- $(cA)^\top = cA^\top$.
- $(A^T)^{-1} = (A^{-1})^\top$.
- $(A+B)^\top = A^\top + B^\top$.
- $(AB)^\top = B^\top A^\top$.

General Facts about symmetric: Suppose A and B are matrices of the same size.

- If A and B are symmetric matrices, then so is $A + B$.
- If A is symmetric, then A is a square matrix and cA is symmetric for any scalar c.
- $A^\top A$ and $AA^\top$ are symmetric matrices.
- If A is an invertible symmetric matrix, then A^{-1} is a symmetric matrix.

EXAMPLE 6. Is the matrix

$$A = \begin{bmatrix} 1 & 0 & 4 \\ -4 & 0 & 6 \\ 2 & 0 & -10 \end{bmatrix}$$

invertible?

The trick is to take A^T because

$$A \text{ invertible} \Leftrightarrow A^T \text{ is invertible.}$$

So,

$$A^T = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 0 & 0 \\ 4 & 6 & -10 \end{bmatrix}.$$

Find inverse.

$$\left[\begin{array}{ccc|ccc} 1 & -4 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 4 & 6 & -10 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & -4 & 2 & 1 & 0 & 0 \\ 4 & 6 & -10 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} R_2 \leftarrow R_3 \\ R_3 \leftarrow R_2 \end{array}$$

Line of zeros $\longrightarrow$ A^T singular $\longrightarrow$ A singular.

$R_2 \leftrightarrow R_3$ A^T $\longleftrightarrow$ $C_2 \leftrightarrow C_3$ of A

Remarks:

- Using row operations on the transposed matrix is equivalent to applying column operations to the original matrix.
- So, in general, what we learned to do with the rows of a matrix can also be done with the columns of a matrix.
- Taking column operations will be important when we will find the row space and column space of a matrix.