Math 307

Chapter 2

Section 2.2: Subspaces and Spanning Sets

Contents

Created by: Pierre-Olivier Parisé

Summer 2022

Definition

In loose terms, a **subspace** is simply a vector space inside another vector space. Precisely, a subspace is a subset W of another vector space V such that W is itself a vector space under the same addition and scalar multiplication operations of V restricted to W.

The next result tells us that we only need to verify if the operations are closed.

THEOREM 1. Let W be a nonempty subset of a vector space V. Then W is a subspace of V if and only if for all vectors u and w in W and for all scalar c, we have

- $u + w$ is in W;
- cu is in W.

EXAMPLE 2. Let W be the set of all column vectors of the form

$$\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}.$$

Show that W is a subspace of the vector space of all column vectors.

EXAMPLE 3. Do the set of vectors of the form

$$\begin{bmatrix} x \\ 1 \end{bmatrix}$$

forms a subspace of $\mathbb{R}^2$?

Example 4. Do the set of vectors of the form

$$\begin{bmatrix} x \\ y \\ x - 2y \end{bmatrix}$$

forms a subspace of $\mathbb{R}^3$?

Important Examples: Set of Polynomials

Let n be a nonnegative integer and let P_n denote the set of polynomials of degree less than or equal to n on (a, b); that is the set of expressions $p(x)$ of the form

$$p(x) = a_k x^k + a_{k-1} x^{k-1} + \cdots + a_1 x + a_0$$

for k an integer such that $k \leq n$.

Example 5. Let P_2 denote the set of polynomials of degree less than or equal to 2 on (a, b); that is the set of expressions $p(x)$ of the form

$$p(x) = ax^2 + bx + c.$$

Show that P_2 is a subspace of the vector space of functions $F(a, b)$.

Fact: Let P denote the set of all polynomials on (a, b). This means P is the set of expressions $p(x)$ of the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

Show that P is a subspace of the vector space of functions $F(a, b)$.

Linear Combinations

Given a bunch of vectors v_1, v_2, …, v_n in a vector space V, a **linear combination** of these vectors is

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

for some scalars c_1, c_2, …, c_n.

Example 6. Is the polynomial $v(x) = 2x^2 + x + 1$ a linear combination of the polynomials $v_1(x) = x^2 + 1$, $v_2(x) = x^2 - 1$, $v_3(x) = x + 1$?

Spanning set

The set of all linear combinations of vectors v_1, v_2, …, v_n of V is called the **spanning set of $\mathbf{v_1}$, $\mathbf{v_2}$, …, $\mathbf{v_n}$**.

The notation for the spanning set of the subspace of V generated by the vectors v_1, v_2, …, v_n is

$$\text{Span}\,\{v_1, v_2, \ldots, v_n\}.$$

EXAMPLE 7. Is the vector

$$\begin{bmatrix} 2 \\ -5 \\ 1 \\ 10 \end{bmatrix} \text{ in the Span } \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\} ?$$

Spanning a whole vector space

We say that the vectors v_1, v_2, ..., v_n of a vector space V span V if

$$\text{Span}\{v_1, v_2, \ldots, v_n\} = V.$$

In other words, each vector in V is a linear combination of the vectors v_1, v_2, ..., v_n.

EXAMPLE 8. Do

$$v_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

span $\mathbb{R}^2$?

Example 9. Let $v_1(x) = x^2 + x - 3$, $v_2(x) = x - 5$, $v_3(x) = 3$, and $v_4(x) = x + 1$.

1. Do v_1, v_2, v_3 span P_2?
2. Do v_2, v_3, v_4 span P_1?
3. Do v_1, v_2, v_3 span P_3?