Math 307

Chapter 1

Section 1.2: Matrices and Matrix Operations

Contents

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Definition

A matrix is a bunch of numbers arranged in m rows and n columns like an array:

$$A = \overbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}}^{n \text{ columns}} . \left.\vphantom{\begin{matrix} a \\ a \\ a \\ \vdots \\ a \end{matrix}}\right\} m \text{ rows}$$

- a_{ij} are the elements/entries of the matrix. Here are some notations to describe the entries of a matrix:
  - $\mathrm{ent}_{ij}(A)$.
  - $A = [a_{ij}]$. $m \times n$
- The dimensions of a matrix is the number of rows (m) and of columns (n). To specify the dimensions, we say *an $m \times n$ matrix A*.
- The set of all matrices of dimensions $m \times n$ is denoted by $M_{m\times n}(\mathbb{R})$.

EXAMPLE 1. Here are some examples of matrices:

$a_{12} = 2$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ -2 \\ 4 \\ 0 \end{bmatrix}_{4\times 1}, \quad D = \begin{bmatrix} -1 & 3 & 0 & 0.5 & \pi \end{bmatrix}. \; 1\times 5$$

dim : 2×3

Some Types of Matrices

- A row matrix is a matrix a dimensions $1 \times n$.
  - Remark: row matrices are models for row vectors.
- A column matrix is a matrix of dimensions $m \times 1$.
  - Remark: column matrices are models for column vectors.
- Square matrices are matrices of dimensions $n \times n$ (same number of rows and number of columns).
  - The elements $a_{11}, a_{22}, \ldots, a_{nn}$ of a square matrix are called diagonal entries.

EXAMPLE 2. For each matrix in Example 1,

1. Give the dimensions;
2. If possible, give the type of the matrix.

A) dim = 2×3.
No type

B) dim = 3×3.
Square

C) 4×1, column vector.

D) 1×5, row vector.

Equality

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are equal, written $A = B$, if

- they have the same dimensions and;
- they have the same entries, that is $a_{ij} = b_{ij}$ for all indices i, j.

EXAMPLE 3. Determine if the matrices A and B are equal.

1. $A = \begin{bmatrix} -1 & 2 \\ 1 & 12 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 2 \\ 1 & 12 \end{bmatrix}$.

2. $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \end{bmatrix}$.

1. Same dimensions ✓

$a_{11} = -1 \neq 5 = b_{11}$ ✗

Not equal

2. A : 2×1 & B: 1×2

not same dimensions

$\Rightarrow \quad A \neq B$

Addition

A matrix $A = [a_{ij}]_{m\times n}$ is added to another matrix $B = [b_{ij}]_{m\times n}$ in the following way:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2n} \\ b_{31} & b_{32} & b_{33} & \cdots & b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & b_{m3} & \cdots & b_{mn} \end{bmatrix} := \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{2n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{bmatrix}$$

<u>Remark:</u> To add two matrices together, they must have the same dimensions! Otherwise, it doesn't make sense!

EXAMPLE 4. Add together the given matrices, if possible.

a) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}_{3\times 2}$ ✓ and $B = \begin{bmatrix} 8 & 9 \\ 10 & 11 \\ 12 & 13 \end{bmatrix}_{3\times 2}$ ✓.

b) $A = \begin{bmatrix} -2 & 5 \\ 3 & 1 \end{bmatrix}_{2\times 2}$ and $B = \begin{bmatrix} 3 & 0 & -2 \end{bmatrix}_{1\times 3}$.

a) $A+B = \begin{bmatrix} 1+8 & 2+9 \\ 3+10 & 4+11 \\ 5+12 & 6+13 \end{bmatrix}$

$= \begin{bmatrix} 9 & 11 \\ 13 & 15 \\ 17 & 19 \end{bmatrix}$

b) Not same dimensions,

A+B is not defined

Scalar Multiplication

We can multiply a matrix by a number. Given a number t, we define

$$t\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} := \begin{bmatrix} ta_{11} & ta_{12} & ta_{13} & \cdots & ta_{1n} \\ ta_{21} & ta_{22} & ta_{23} & \cdots & ta_{2n} \\ ta_{31} & ta_{32} & ta_{33} & \cdots & ta_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ ta_{m1} & ta_{m2} & ta_{m3} & \cdots & ta_{mn} \end{bmatrix}.$$

Remark: A number will be called, in many circumstances, a scalar.

Example 5. Give the resulting matrix after performing the following operations:

$$\underbrace{2\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{1} - \underbrace{\begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix}}_{2}$$

1 $2\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$

2 $-1\begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$

3 $2\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + (-1)\begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 5 & 6 \end{bmatrix}$

Remarks:

- In general, the operation $A - B$ is defined by $A + (-B)$ where $-B$ is B multiplied by the scalar -1.
- The zero matrix $O_{m\times n}$ of dimensions $m \times n$ is the matrix containing only zeros:

$$O_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad O_{3\times 1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

- Adding the zero matrix doesn't change anything: $O_{m\times n} + A = A + O_{m\times n} = A$.
- Multiplying by the number 0 changes all the entries of the matrix A: $0A = O_{m\times n}$.
- Here are basic arithmetic with the addition and scalar multiplication: If A, B, and C are matrices of the same size and if s and t are numbers, then

1. $A + B = B + A$.
2. $A + (B + C) = (A + B) + C$.
3. $s(tA) = (st)A$.
4. $s(A + B) = sA + sB$.
5. $(s + t)A = sA + tA$.

Matrix Multiplication

Row Matrix times a Column Vector

EXAMPLE 6. In a hale'kū'ai, there are selling oranges, pineaples, and mangos. A pound of oranges cost \$2, a pound of pineaples is \$3, and a pound of mangos is \$4. You buy 2 pounds of oranges, 3 pounds of pineaples, and 2 pounds of mangos. What is the total cost of your purchase?

Trad. Way: $2 \times 2 + 3 \times 3 + 2 \times 4 = \$19$

Amount things $\to$ $\begin{array}{ccc} \text{oranges} & \text{Pinea.} & \text{mangos} \\ \end{array}$ $\begin{bmatrix} 2 & 3 & 2 \end{bmatrix}$

Prices fruit $\to$ $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \begin{array}{l} \leftarrow \text{oranges} \\ \leftarrow \text{Pinea.} \\ \leftarrow \text{mangos.} \end{array}$

$\begin{bmatrix} 2 & 3 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \text{total cost} = \underline{2 \times 2} + \underline{3 \times 3} + \underline{2 \times 4}$

In General: Given a row vector $v = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$ and a column vector

$$u = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix},$$

we define the product vu by

$$vu := a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

Remark:

- We see that multiplying a matrix of dimensions $1 \times n$ with a matrix of dimensions $n \times 1$ gives a matrix of dimensions 1×1 (a number).
- If the number of elements in the vector v would be different from the number of elements in the vector u, then the product vu doesn't make sense. In this case, we can't perform the operation!

Matrix times a Column Vector

EXAMPLE 7. Your friend and you are in the same hale'kū'ai. You decide to buy 5 pounds of oranges, 2 pounds of pineaples and 1 pound mangos. Your friend, prefering mangos over the other fruits, buys 2 pounds of oranges, 2 pounds of pineaples, and 10 pounds of mangos. How much did it cost to you and your friend?

You: $\begin{bmatrix} 5 & 2 & 1 \end{bmatrix}$

Friend: $\begin{bmatrix} 2 & 2 & 10 \end{bmatrix}$

Prices: $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$

$$\text{You\_cost} = \begin{bmatrix} 5 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \$20$$

$$\text{Friend\_cost} = \begin{bmatrix} 2 & 2 & 10 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \$50$$

$$\begin{bmatrix} \text{You\_cost} \\ \text{Friend\_cost} \end{bmatrix} = \begin{bmatrix} \boxed{20} \\ \boxed{50} \end{bmatrix} = \begin{bmatrix} \boxed{5 \quad 2 \quad 1} \\ \boxed{2 \quad 2 \quad 10} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

In General: Given a $m \times n$ matrix A and a $n \times 1$ column vector u, the product Au is the new matrix $C = [c_j]_{1 \leq j \leq m}$ of dimensions $m \times 1$ where

resulting dimensions of the matrix

$$c_1 = a_{11}b_1 + a_{12}b_2 + \cdots + a_{1n}b_n$$
$$c_2 = a_{21}b_1 + a_{22}b_2 + \cdots + a_{2n}b_n$$
$$\vdots$$
$$c_m = a_{m1}b_1 + a_{m2}b_2 + \cdots + a_{mn}b_n.$$

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix} = \begin{bmatrix} a_{11}b_1 + \cdots + a_{1n}b_n \\ a_{21}b_1 + \cdots + a_{2n}b_n \\ \vdots \\ a_{m1}b_1 + \cdots + a_{mn}b_n \end{bmatrix}$$

Remarks:

- Multiplying a matrix of dimensions $m \times n$ with a matrix of dimensions $n \times 1$ results in a matrix of dimensions $m \times 1$.
- To multiplying a matrix with a column vector, the number of columns in the matrix must be the same as the number of elements in the column vector.

Matrix times a Matrix

Prices for different markets

Let A and B be two matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad B = \begin{array}{c} \begin{array}{ccccc} \text{St1} & \text{St2} & \text{St3} & \cdots & \text{St4} \end{array} \\ \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1k} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2k} \\ b_{31} & b_{32} & b_{33} & \cdots & b_{3k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \cdots & b_{nk} \end{bmatrix} \end{array}.$$

We will adopt the following notation

$$B = \begin{bmatrix} B_1 & B_2 & B_3 & \cdots & B_k \end{bmatrix}$$

where B_1, B_2, …, B_k are the columns of B:

$$B_1 = \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{m1} \end{bmatrix}, \quad B_2 = \begin{bmatrix} b_{21} \\ b_{22} \\ \vdots \\ b_{m2} \end{bmatrix}, \quad \cdots, \quad B_k = \begin{bmatrix} b_{1k} \\ b_{2k} \\ \vdots \\ b_{mk} \end{bmatrix}.$$

The multiplication of A with B is defined as the new matrix C of dimensions $m \times k$:

$$C = \begin{bmatrix} AB_1 & AB_2 & AB_3 & \cdots & AB_k \end{bmatrix}.$$

In other words, the columns C_1, C_2, …, C_k of the matrix C is the column vectors AB_1, AB_2, …, AB_k.

Remarks:

- To multiply a matrix A with a matrix B, the number of columns of A must agree with the number of rows of B. If this is not satisfied, the product AB is not defined!
- The identity matrix is the square matrix I_n with all 1 on the diagonal and zeros elsewhere:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

- The identity matrix is such that $I_m A = A$ and $AI_n = A$ for any matrix A of dimensions $m \times n$.
- We can multiply several times the same matrix with itself. This is the *powers of a matrix*. If A is a square matrix, then

$$A^1 = A, \quad A^2 = AA, \quad A^3 = A^2A = AAA, \cdots \quad A^n = A^{n-1}A = \underbrace{AA\cdots A}_{n \text{ times}}$$

- Here are some basic arithmetic with the matrix multiplication:

  1. $A(BC) = (AB)C$;
  2. $A(B + C) = AB + AC$;
  3. $(A + B)C = AC + BC$;
  4. $\lambda(AB) = (\lambda A)B = A(\lambda B)$. λ is scalar.

$AB \neq BA$

Example 8. If possible, compute the products AB and BA where

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & -3 & 1 \\ -2 & 2 & 1 \end{bmatrix}_{3\times 3} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}_{3\times 2}.$$

1) $AB = \left[A\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad A\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right]$

column vector — column vector.

$$A\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & -3 & 1 \\ -2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1\cdot 0 + 2\cdot 1 + (-1)\cdot 0 \\ 1\cdot 0 + (-3)\cdot 1 + 1\cdot 0 \\ -2\cdot 0 + 2\cdot 1 + 1\cdot 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$$

$$A\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & -3 & 1 \\ -2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

So $AB = \begin{bmatrix} 2 & -1 \\ -3 & 1 \\ 2 & 1 \end{bmatrix}$

2) BA not defined because

nb. col. of B $\neq$ nb. rows of A.

EXAMPLE 9. If possible, compute the products AB and BA where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -1 & -2 & -3 \end{bmatrix}_{3\times 3} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}_{3\times 3}.$$

$$AB = \left[A\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad A\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad A\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right] = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 6 & 2 \\ -2 & -3 & -1 \end{bmatrix}$$

c_{21}

$$A\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -1 & -2 & -3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} \qquad A\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$A\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -1 & -2 & -3 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix}$$

$$BA = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} \boxed{-1} & -2 & -3 \\ \boxed{1} & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$$

We see that $AB \neq BA$.

The matrix multiplacation is not commutative.

Remark: In general, the matrix multiplication does not satisfy the usual rule

$$AB = BA.$$

Consider

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

Augmented Matrix Notation

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

One side of the coin!

Rewriting a System in Matrix form

Letting

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}$$

we can rewrite the system as

$$\boxed{AX = B}$$

Example 10. Rewrite the following system in its matrix form:

$$\begin{aligned} 3x_1 + 4x_2 - 5x_3 &= 1 \\ 5x_1 + 5x_2 - 3x_3 &= 2 \\ -2x_1 - 5x_2 + 0.5x_3 &= 3. \end{aligned}$$

$$A = \begin{bmatrix} 3 & 4 & -5 \\ 5 & 5 & -3 \\ -2 & -5 & 0.5 \end{bmatrix} \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\rightarrow \quad AX = B.$