MATH 307

CHAPTER 6

SECTION 6.2: HOMOGENEOUS SYSTEMS WITH CONSTANT COEFFICIENTS
THE DIAGONALIZABLE CASE

CONTENTS

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Example 1. Determine the general solution to

$$Y' = \underbrace{\begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}}_{A} Y.$$

1) Transform A into a diagonal matrix.

- $\lambda = -1$ & $\lambda = 4$.
- $\dim(E_{-1}) = 1$ & $\dim(E_4) = 1$
- $\dim(E_{-1}) + \dim(E_4) = 2$ ✓ $\rightarrow$ A is diagonalizable.

$$D = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} \quad \& \quad P = \begin{bmatrix} 3/2 & -1 \\ 1 & 1 \end{bmatrix}$$

$\Rightarrow \quad A = PDP^{-1}$.

2) Solve the diagonal system.

$(af)' = af'$

$(AY)' = AY'$ (*)

$Y' = AY \rightarrow Y' = PDP^{-1}Y \rightarrow P^{-1}Y' = DP^{-1}Y$

used (*) $\rightarrow$ $\rightarrow \underbrace{(P^{-1}Y)}_{Z}' = D \underbrace{(P^{-1}Y)}_{Z}$

Let $Z = P^{-1}Y \Rightarrow Z' = DZ = \begin{bmatrix} \boxed{-1} & 0 \\ 0 & \boxed{4} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$.

d_1, c_2

$$\Rightarrow Z(x) = \begin{bmatrix} c_1 e^{-x} \\ c_2 e^{4x} \end{bmatrix}.$$

So, $Y = PZ = \begin{bmatrix} 3/2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{-x} \\ c_2 e^{4x} \end{bmatrix} = \begin{bmatrix} \frac{3c_1}{2} e^{-x} - c_2 e^{4x} \\ c_1 e^{-x} + c_2 e^{4x} \end{bmatrix}$.

Fact: Suppose A and B are $n \times n$ matrices with $B = P^{-1}AP$ for some invertible $n \times n$ matrix P. Then

- If Z is a solution to $Y' = BY$ ($Z' = BZ$), then PZ is a solution to $Y' = AY$.
- If $Z_1, Z_2, \ldots, Z_n$ is a fundamental set of solutions of $Y' = BY$, then $PZ_1, PZ_2, \ldots, PZ_n$ is a fundamental set of solutions to $Y' = AY$.

EXAMPLE 2. Solve the initial value problem

$$Y' = \underbrace{\begin{bmatrix} 2 & -3 & -3 \\ 2 & -2 & -2 \\ -2 & 1 & 1 \end{bmatrix}}_{A} Y, \quad Y(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

1) Transform into D.

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \& \quad P = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}. \qquad A = PDP^{-1}.$$

2) Solve diagonal system.

$Z = P^{-1}Y \rightarrow Y' = AY$ becomes $Z' = DZ$.

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} Z$$

From 6.1, $Z(x) = \begin{bmatrix} c_1 e^{2x} \\ c_2 e^{-x} \\ c_3 \end{bmatrix}$

$$\rightarrow Y_H = PZ = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{2x} \\ c_2 e^{-x} \\ c_3 \end{bmatrix} = \begin{bmatrix} c_1 e^{2x} - c_3 \\ -c_2 e^{-x} - c_3 \\ c_1 e^{2x} + c_2 e^{-x} + c_3 \end{bmatrix}$$

3) Initial conditions.

$$Y(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 - c_3 \\ -c_2 - c_3 \\ c_1 + c_2 + c_3 \end{bmatrix} \rightarrow \begin{cases} c_1 = c_3 + 1 \\ c_2 = -c_3 \\ 0 = c_1 + c_2 + c_3 \end{cases}$$

$$0 = c_3 + 1 - c_3 + c_3 \Rightarrow c_3 = -1$$

$$c_1 = 0$$
$$c_2 = 1$$

So, $Y(x) = \begin{bmatrix} 1 \\ -e^{-x} + 1 \\ e^{-x} - 1 \end{bmatrix}$

Complex Exponential Function $x=b$

For a complex number $z = a + ib$, we define

$e^{ix} = \cos(x) + i\sin(x)$.

$$e^z = e^{a+ib} = e^a \cos(b) + ie^a \sin(b).$$

$e^{a+ib} = e^a(\cos b + i\sin b)$.

The solution to the differential equation $y' = (a + ib)y$ is

$$y(x) = e^{(a+ib)x}.$$

Finding solutions with complex numbers

EXAMPLE 3. Find the general solution to

$$Y' = \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}_{A} Y.$$

1) Transform A into D.

$D = \begin{bmatrix} 1-i & 0 \\ 0 & 1+i \end{bmatrix}$ & $P = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}$ $\quad (A = PDP^{-1})$.

2) Solve the diagonal system.

$Z = P^{-1}Y \to Y' = AY$ becomes $Z' = DZ$.

$= \begin{bmatrix} 1-i & 0 \\ 0 & 1+i \end{bmatrix} Z$.

From 6.1, $Z(x) = \begin{bmatrix} c_1 e^{\overbrace{(1-i)}^{d_1} x} \\ c_2 e^{\underbrace{(1+i)}_{d_2} x} \end{bmatrix}$.

$\overline{1+i} = 1-i$

$(*)\ e^{(1-i)x} = e^{x-ix} = e^x \cdot \underbrace{e^{-ix}}_{??} = e^x(\cos(-x) + i\sin(-x))$

$= e^x(\cos(x) - i\sin(x))$

$= \underbrace{e^x\cos(x)} - i\,\underbrace{e^x\sin x}$

Apply P to $(*)$ to get

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{(Taylor series of } e^x\text{)}.$$

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{n=0}^{\infty} \frac{i^n x^n}{n!} = \frac{i^0 x^0}{1} + \frac{ix}{1} + \frac{i^2 x^2}{2!} + \frac{i^3 x^3}{3!} + \dots$$

$$e^n = \underbrace{e \cdot e \dots e}_{n \text{ times}}.$$

$$e^\pi = \sum_{n=0}^{\infty} \frac{\pi^n}{n!}$$

$(n! = n \cdot (n-1) \cdot (n-2) \dots 1$

$3! = 3 \cdot 2 \cdot 1 = 6.$

$i^0 = 1$

$i^1 = i$

$i^2 = -1$

$i^3 = i^2 \cdot i = -i$

$i^4 = i^2 \cdot i^2 = 1$

$i^5 = i^4 \cdot i = i$

$i^6 = i^5 \cdot i = -1$

$i^7 = i^6 \cdot i = -i$

$$= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} + \dots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right)$$

$$= \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}}_{\cos(x)} + i \underbrace{\left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right]}_{\sin(x)}$$

$$= \cos(x) + i \sin(x). \quad \text{(Euler's Identity)}$$

$$Y_1 = P\underbrace{\begin{bmatrix} e^{(1-i)x} \\ 0 \end{bmatrix}}_{Z_1} = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{x}\cos x - ie^{x}\sin x \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -ie^{x}\cos x + i^2 e^{x}\sin x \\ e^{x}\cos x - ie^{x}\sin x \end{bmatrix}$$

$$= \begin{bmatrix} -e^{x}\sin x - ie^{x}\cos x \\ e^{x}\cos x - ie^{x}\sin x \end{bmatrix}$$

$$= \begin{bmatrix} -e^{x}\sin x \\ e^{x}\cos x \end{bmatrix} + \begin{bmatrix} -ie^{x}\cos x \\ -ie^{x}\sin x \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} -e^{x}\sin x \\ e^{x}\cos x \end{bmatrix}}_{U} + i\underbrace{\begin{bmatrix} -e^{x}\cos x \\ -e^{x}\sin x \end{bmatrix}}_{V}$$

Here, we have $U' = \begin{bmatrix} -e^{x}\sin x - e^{x}\cos x \\ e^{x}\cos x - e^{x}\sin x \end{bmatrix}$ are $=$

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} U = \begin{bmatrix} -e^{x}\sin x - e^{x}\cos x \\ -e^{x}\sin x + e^{x}\cos x \end{bmatrix}$$

$\to$ U is a solution to $Y' = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} Y$.

You can also check that V is a solution to $Y' = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} Y$.

So, U & V will form the foundamental of solutions &

$$\boxed{Y = c_1 U + c_2 V = \begin{bmatrix} -c_1 e^{x}\sin x - c_2 e^{x}\cos x \\ c_1 e^{x}\cos x - c_2 e^{x}\sin x \end{bmatrix}}$$

Fact: If $U(x) + iV(x)$ is a solution to $Y' = AY$, then $U(x)$ and $V(x)$ are solutions to $Y' = AY$.

Example 4. Find the general solution to

$$Y' = \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}}_{A} Y.$$

1) Diagonal form.

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1-i & 0 \\ 0 & 0 & 1+i \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 1+i & 1-i \\ 0 & -i & i \\ 0 & 1 & 1 \end{bmatrix}$$

2) Solve the diagonal system.

$Z = P^{-1}Y \to Y' = AY$ becomes $Z' \overset{(*)}{=} DZ$

solution to $(*)$ $\to$ $Z(x) = \begin{bmatrix} c_1 e^{x} \\ c_2 e^{(1-i)x} \\ c_3 e^{(1+i)x} \end{bmatrix}$.

$= c_1 Z_1 + c_2 Z_2 + c_3 Z_3$

$Z_1 = \begin{bmatrix} e^{x} \\ 0 \\ 0 \end{bmatrix}$ ✓

$Z_2 = \begin{bmatrix} 0 \\ e^{(1-i)x} \\ 0 \end{bmatrix}$ ✓

$Z_3 = \begin{bmatrix} 0 \\ 0 \\ e^{(1+i)x} \end{bmatrix}$ ✗

1) $\underline{Z_1}$ $\quad Y_1 = PZ_1 = \begin{bmatrix} e^{x} \\ 0 \\ 0 \end{bmatrix}$

2) $\underline{Z_2}$ $\quad Y_2 = PZ_2 = \begin{bmatrix} (1+i)\, e^{(1-i)x} \\ -i\, e^{(1-i)x} \\ e^{(1-i)x} \end{bmatrix} \begin{matrix} (*) \\ (**) \\ (***) \end{matrix}$

Here, $(*)$ $(1+i)e^{(1-i)x} = (1+i)e^{x}(\cos(x) - i\sin(x))$

$= e^{x}\cos x + e^{x}\sin x + i(e^{x}\cos x - e^{x}\sin x)$

$(**)$ $-i\,e^{(1-i)x} = -i\,e^{x}(\cos x - i\sin x)$

$= -e^{x}\sin x - i e^{x}\cos x$

$(***)$ $e^{(1-i)x} = e^{x}\cos x - i e^{x}\sin x$.

$$Y_2(x) = \underbrace{\begin{bmatrix} e^x\cos x + e^x\sin x \\ -e^x\sin x \\ e^x\cos x \end{bmatrix}}_{U} + i\underbrace{\begin{bmatrix} e^x\cos x - e^x\sin x \\ -e^x\cos x \\ -e^x\sin x \end{bmatrix}}_{V}$$

So, Y_1, U, V are the set of foundamental solutions.

therefore,

$$Y = c_1Y_1 + c_2U + c_3V$$

$$= \begin{bmatrix} c_1e^x + c_2(e^x\cos x + e^x\sin x) + c_3(e^x\cos x - e^x\sin x) \\ -c_2e^x\sin x - c_3e^x\cos x \\ c_2e^x\cos x - c_3e^x\sin x \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} e^x & e^x\cos x + e^x\sin x & e^x\cos x - e^x\sin x \\ 0 & -e^x\sin x & -e^x\cos x \\ 0 & e^x\cos x & -e^x\sin x \end{bmatrix}}_{M}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$