MATH-307
Midterm 01

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Last name: Solutions.
First name: —

Question:	1	2	3	4	5	6	Total
Points:	20	20	20	20	10	10	100
Score:	—	—	—	—	—	—	—

Instructions: Make sure to write your complete name on your copy. You must answer all the questions below and write your answers directly on the questionnaire. At the end of the 80 minutes, hand out your copy.

No devices such as a smart phone, cell phone, laptop, or tablet can be used during the exam. You are not allowed to use the lecture notes, the textbook. You may bring one 2-sided cheat sheet of handwriting notes. You may use a digital calculator (no graphical calculator or symbolic calculator will be allowed).

You must show ALL your work to have full credit. An answer without justification worth no point.

Good luck!

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Your Signature: Solutions

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Question 1 **(20 pts)**

Using the Gauss-Jordan Elimination Method, say if the following systems of linear equations has one solution, more than one solution, or no solution. If the system has solution(s), write the solution(s) explicitly.

(a) (/ 10) $\begin{cases} 2x + 3y - 4z = 3 \\ 2x + 3y - 2z = 3 \\ 4x + 6y - 2z = 7 \end{cases}$ (b) (/ 10) $\begin{cases} 4x - 2y + 3z = 0 \\ 2x + 2y - 4z = 0 \end{cases}$.

(a) $$\begin{bmatrix} 2 & 3 & -4 & 3 \\ 2 & 3 & -2 & 3 \\ 4 & 6 & -2 & 7 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & -4 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 6 & 1 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \to R_2 \\ R_3 - 2R_1 \to R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 2 & 3 & -4 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} R_3 - 3R_2 \to R_3$$

$\Rightarrow$ $0=1$ $\to$ system not consistent $\to$ no solution.

(b) $$\begin{bmatrix} 4 & -2 & 3 & 0 \\ 2 & 2 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 4 & -2 & 3 & 0 \\ 0 & 6 & -11 & 0 \end{bmatrix} 2R_2 - R_1 \to R_2$$

$$\sim \begin{bmatrix} 12 & 0 & -2 & 0 \\ 0 & 6 & -11 & 0 \end{bmatrix} 3R_1 + R_2 \to R_1$$

$$\sim \begin{bmatrix} 1 & 0 & -1/6 & 0 \\ 0 & 1 & -11/6 & 0 \end{bmatrix} \begin{matrix} 1/12 R_1 \to R_1 \\ 1/6 R_2 \to R_2 \end{matrix}$$

$\to$ $\boxed{x = z/6 \quad \& \quad y = 11z/6 \,.}$

QUESTION 2 **(20 pts)**

Suppose we have the following system of linear equations:

$$\begin{cases} 2x - y + 3z = 42 \\ x + y - 2z = 42 \\ x + y + 5z = 21 \end{cases}.$$

(a) (/ 5) Write the system in its matrix form.

(b) (/ 10) Find the inverse of the matrix of coefficients.

(c) (/ 5) Find the solution to the system using the inverse.

(a) let $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & -2 \\ 1 & 1 & 5 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ & $B = \begin{bmatrix} 42 \\ 42 \\ 21 \end{bmatrix}$

Then $AX = B$ is the matrix form of the system.

(b) $$\left[\begin{array}{ccc|ccc} 2 & -1 & 3 & 1 & 0 & 0 \\ 1 & 1 & -2 & 0 & 1 & 0 \\ 1 & 1 & 5 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 2 & -1 & 3 & 1 & 0 & 0 \\ 0 & 3 & -7 & -1 & 2 & 0 \\ 0 & 3 & 7 & -1 & 0 & 2 \end{array}\right] \begin{array}{l} \\ 2R_2 - R_1 \to R_2 \\ 2R_3 - R_1 \to R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 6 & 0 & 2 & 2 & 2 & 0 \\ 0 & 3 & -7 & -1 & 2 & 0 \\ 0 & 0 & 14 & 0 & -2 & 2 \end{array}\right] \begin{array}{l} 3R_1 + R_2 \to R_1 \\ \\ R_3 - R_2 \to R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 42 & 0 & 0 & 14 & 16 & -2 \\ 0 & 6 & 0 & -2 & 2 & 2 \\ 0 & 0 & 14 & 0 & -2 & 2 \end{array}\right] \begin{array}{l} 7R_1 - R_3 \to R_1 \\ 2R_2 + R_3 \to R_2 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/6 & 8/21 & -1/21 \\ 0 & 1 & 0 & -1/3 & 1/3 & 1/3 \\ 0 & 0 & 1 & 0 & -1/7 & 1/7 \end{array}\right] \begin{array}{l} 1/42\, R_1 \to R_1 \\ 1/6\, R_2 \to R_2 \\ 1/14\, R_3 \to R_3 \end{array}$$

So, $A^{-1} = \begin{bmatrix} 7/6 & 8/21 & -1/21 \\ -1/3 & 1/3 & 1/3 \\ 0 & -1/7 & 1/7 \end{bmatrix}$

(c) $X = A^{-1} B = \begin{bmatrix} 7/6 & 8/21 & -1/21 \\ -1/3 & 1/3 & 1/3 \\ 0 & -1/7 & 1/7 \end{bmatrix} \begin{bmatrix} 42 \\ 42 \\ 21 \end{bmatrix}$

$= \begin{bmatrix} 64 \\ 7 \\ -3 \end{bmatrix}$

Question 3 **(20 pts)**

Suppose we have the following matrices:

$$A = \begin{bmatrix} -3 & 0 & 4 \\ 2 & -1 & 3 \\ 4 & 0 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 0 \\ -3 & 2 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & -1 & 0 & 6 \\ 0 & 0 & 0 & 5 \\ 0 & -5 & 0 & 1 \\ 3 & 4 & 0 & 3 \end{bmatrix}.$$

(a) (/ 5) Compute $2A$.

(b) (/ 5) Compute $AB^\top$.

(c) (/ 5) Compute $\det(C)$.

(d) (/ 5) Compute $\det(A^\top B)$.

(a) $2A = \begin{bmatrix} -6 & 0 & 8 \\ 4 & -2 & 6 \\ 8 & 0 & 10 \end{bmatrix}$

(b) $B^T = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 1 & 2 \\ 3 & 0 & 1 \end{bmatrix}$ $\quad AB^T = \begin{bmatrix} -3 & 0 & 4 \\ 2 & -1 & 3 \\ 4 & 0 & 5 \end{bmatrix} \begin{bmatrix} 2 & 0 & -3 \\ -1 & 1 & 2 \\ 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 13 \\ 12 & -1 & -5 \\ 23 & 0 & -7 \end{bmatrix}$

(c) column of zeros $\rightarrow$ $\det(C) = 0$.

(d) $BA^T = (AB^T)^T \rightarrow \det(BA^T) = \det(AB^T)$.

From (b),

$$\det(AB^T) = \begin{vmatrix} 6 & 0 & 13 \\ 12 & -1 & -5 \\ 23 & 0 & -7 \end{vmatrix} = -1 \begin{vmatrix} 6 & 13 \\ 23 & -7 \end{vmatrix} = \boxed{-341}$$

QUESTION 4 **(20 pts)**

Use Cramer's rule to solve the following system of linear equations.

(a) (/ 10) $\begin{cases} 3x - 4y = 1 \\ 2x - 3y = 2 \end{cases}$.

(b) (/ 10) $\begin{cases} 3x - y = 1 \\ y - 3z = 1 \\ 2x + z = 1 \end{cases}$.

(a)

$$x = \frac{\begin{vmatrix} 1 & -4 \\ 2 & -3 \end{vmatrix}}{\begin{vmatrix} 3 & -4 \\ 2 & -3 \end{vmatrix}} = \frac{-3+8}{-9+8} = \boxed{-5}$$

$$y = \frac{\begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix}}{-1} = \frac{6-2}{-1} = \boxed{-4}$$

(b) $A = \begin{bmatrix} 3 & -1 & 0 \\ 0 & 1 & -3 \\ 2 & 0 & 1 \end{bmatrix} \longrightarrow \det(A) = 2\begin{vmatrix} -1 & 0 \\ 1 & -3 \end{vmatrix} + 1\begin{vmatrix} 3 & -1 \\ 0 & 1 \end{vmatrix} = -3$

So,

$$x = \frac{\begin{vmatrix} 1 & -1 & 0 \\ 1 & 1 & -3 \\ 1 & 0 & 1 \end{vmatrix}}{-3} = \frac{1\begin{vmatrix} -1 & 0 \\ 1 & -3 \end{vmatrix} + 1\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}}{-3} = \boxed{-\frac{5}{3}}$$

$$y = \frac{\begin{vmatrix} 3 & 1 & 0 \\ 0 & 1 & -3 \\ 2 & 1 & 1 \end{vmatrix}}{-3} = \frac{3\begin{vmatrix} 1 & -3 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 0 & -3 \\ 2 & 1 \end{vmatrix}}{-3} = \boxed{-6}$$

$$z = \frac{\begin{vmatrix} 3 & -1 & 1 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix}}{-3} = \frac{1\begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix}}{-3} = \boxed{\frac{1}{3}}$$

Question 5 (10 pts)

Answer the following following

(a) (/ 5) Suppose A and B are $n \times n$ symmetric matrices. Show that $(AB)^\top = BA$.

(b) (/ 5) Find two matrices A and B such that $AB \neq BA$.

(a) $(AB)^T = B^T A^T$ (prop. of transpose).

But, A, B symmetric $\rightarrow$ $A^T = A$ & $B^T = B$

$\rightarrow$ $B^T A^T = BA$.

So, $(AB)^T = BA$.

(b) $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$AB = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$ $BA = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$

We see that $AB \neq BA$.

Question 6 (10 pts)

Answer with **True** or **False** the following statements. Write your answer on the horizontal line at the end of each statement. Justify your answer in the white space underneath the statement.

(a) (/ 2) If A is a 2×2 upper triangular matrix and B is a 2×2 lower triangular matrix, then AB is upper triangular.

$A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$ $\rightarrow$ $AB = \begin{pmatrix} 5 & 6 \\ 6 & 9 \end{pmatrix}$ which is not upper triangular.

$B = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$

(a) False

(b) (/ 2) If A is a 5×3 matrix and B is a 5×5 matrix, then AB is well-defined.

nb of col. of A $\neq$ nb. of rows of B.

(b) False

(c) (/ 2) If A is a $n \times n$ matrix, then $A^\top A$ is a symmetric matrix.

1) $A^T A$ is a square matrix.

2) $(A^T A)^T = A^T (A^T)^T = A^T A$.

(c) True.

(d) (/ 2) Suppose A and B are $n \times n$ matrices. If A is invertible and B is invertible, then $(AB)^{-1} = A^{-1}B^{-1}$.

Good formula should be : $(AB)^{-1} = B^{-1}A^{-1}$.

(d) False.

(e) (/ 2) Prof. Parisé is surfing at Ala Moana. (No justification needed)

Told you on the first day of instruction!

(e) True.