Section 5.3 — Problem 1b (5 Pts)

Normalizing means to divide each vector in the basis by their length, so that they have length 1. Therefore, we get

- $\mathbf{f_1} = \frac{(1,1,1)}{\|(1,1,1)\|} = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$.
- $\mathbf{f_2} = \frac{(4,1,-5)}{\|(4,1,-5)\|} = (4/\sqrt{42}, 1/\sqrt{42}, -5/\sqrt{42})$.
- $\mathbf{f_3} = \frac{(2,-3,1)}{\|(2,-3,1)\|} = (2\sqrt{14}, -3/\sqrt{14}, 1/\sqrt{14})$.

Section 5.3 — Problem 2a (5 Pts)

We have

- $(1,-1,2,5)\cdot(4,1,1,-1) = 4-1+2-5 = 6-6 = 0$.
- $(1,-1,2,5)\cdot(-7,28,5,5) = -7-28+10+25 = -35+35 = 0$.
- $(4,1,1,-1)\cdot(-7,28,5,5) = -28+28+5-5 = 0$.

Hence, the set of vectors is orthogonal.

Section 5.3 — Problem 4a (5 Pts)

Using the Expansion Theorem, we get

$$\begin{aligned}\mathbf{x} &= \frac{(13,-20,15)\cdot(1,-2,3)}{\|(1,-2,3)\|^2}(1,-2,3) + \frac{(13,-20,15)\cdot(-1,1,1)}{\|(-1,1,1)\|^2}(-1,1,1) \\ &= \frac{98}{14}(1,-2,3) - \frac{18}{3}(-1,1,1) \\ &= 7(1,-2,3) - 6(-1,1,1).\end{aligned}$$

Hence, $\mathbf{x} = 7(1,-2,3) - 6(-1,1,1)$.

Section 5.3 — Problem 8 (10 Pts)

a. We have $\mathbf{0}\cdot\mathbf{v} = 0$, hence $\mathbf{0} \in P$. Also, if $\mathbf{x}, \mathbf{y} \in P$, then $\mathbf{x}\cdot\mathbf{v} = 0$ and $\mathbf{y}\cdot\mathbf{v} = 0$. Therefore

$$(\mathbf{x}+\mathbf{y})\cdot\mathbf{v} = \mathbf{x}\cdot\mathbf{v} + \mathbf{y}\cdot\mathbf{v} = 0 + 0 = 0.$$

Hence, $\mathbf{x}+\mathbf{y} \in P$. Finally, if $a \in \mathbb{R}$ and $\mathbf{x} \in P$, then

$$(a\mathbf{x})\cdot\mathbf{v} = a(\mathbf{x}\cdot\mathbf{v}) = a(0) = 0.$$

Hence, $a\mathbf{x} \in P$. Conclusion: P is a subspace of $\mathbb{R}^n$.

b. We have $0\mathbf{v} = \mathbf{0}$ for $t = 0$ and so $\mathbf{0} \in \mathbb{R}\mathbf{v}$. Also, let $\mathbf{x}, \mathbf{y} \in \mathbb{R}\mathbf{v}$ and $a \in \mathbb{R}$. Then $\mathbf{x} = t_1\mathbf{v}$ and $\mathbf{y} = t_2\mathbf{v}$, for some $t_1, t_2 \in \mathbb{R}$.

- We have $\mathbf{x} + \mathbf{y} = t_1\mathbf{v} + t_2\mathbf{v} = (t_1 + t_2)\mathbf{v} = t\mathbf{v}$, where $t = t_1 + t_2$. Hence, $\mathbf{x} + \mathbf{y} \in \mathbb{R}\mathbf{v}$.
- We have $a\mathbf{x} = a(t_1\mathbf{x}) = (at_1)\mathbf{x} = t\mathbf{x}$, where $t = at_1$. Hence, $a\mathbf{x} \in \mathbb{R}\mathbf{v}$.

<u>Conclusion:</u> $\mathbb{R}\mathbf{v}$ is a subspace of $\mathbb{R}^n$.

c. The subspace P is a plane in $\mathbb{R}^3$ passing through the origin with normal vector $\mathbf{v}$.

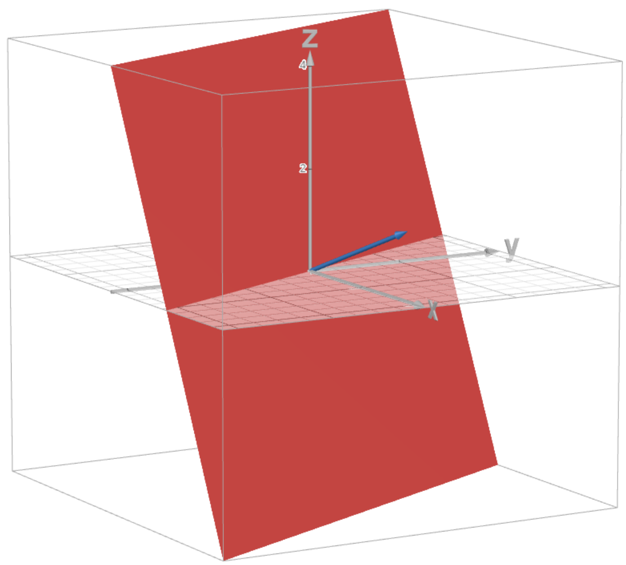

Figure 1: The set P with $\mathbf{v} = (2, 1, 1)$. Desmos Link

The subspace $\mathbb{R}\mathbf{v}$ is a line passing through the origin with direction vector $\mathbf{v}$.

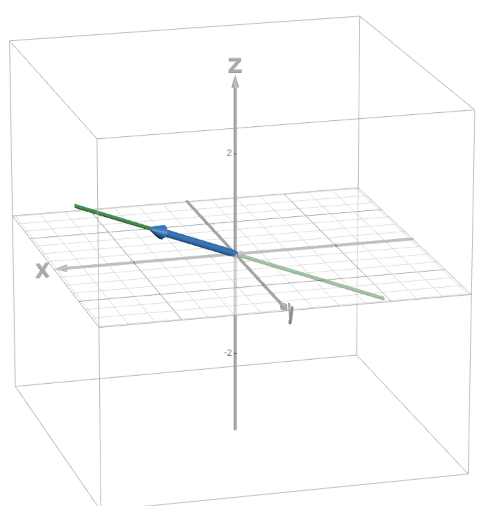

Figure 2: The set $\mathbb{R}\mathbf{v}$ with $\mathbf{v} = (2, 1, 1)$. Desmos Link

Section 8.1 — Problem 1 (10 Pts)

b. Set $\mathbf{f_1} = (2,1)$. Then

$$\mathbf{f_2} = (1,2) - \frac{(1,2)\cdot(2,1)}{\|(2,1)\|^2}(2,1) = (-3/5, 6/5).$$

We then have $\{\mathbf{f_1}, \mathbf{f_2}\}$ is a new basis of $\mathbb{R}^2$ that is orthogonal.

c. Set $\mathbf{f_1} = (1,-1,1)$. Then,

$$\mathbf{f_2} = (1,0,1) - \frac{(1,-1,1)\cdot(1,0,1)}{\|(1,-1,1)\|^2}(1,-1,1) = (1/3, 2/3, 1/3)$$

and

$$\begin{aligned}\mathbf{f_3} &= (1,1,2) - \frac{(1,1,2)\cdot(1,-1,1)}{\|(1,-1,1)\|^2}(1,-1,1) - \frac{(1,1,2)\cdot(1/3,2/3,1/3)}{\|(1/3,2/3,1/3)\|^2}(1/3,2/3,1/3) \\ &= (-1/2, 0, 1/2)\end{aligned}$$

Hence $\{(1,-1,1), (1/3,2/3,1/3), (-1/2,0,1/2)\}$ is a new basis for $\mathbb{R}^3$ that is orthogonal.

Section 8.1 — Problem 4a (10 Pts)

We notice that $(1,1,1)$ and $(0,1,1)$ are linearly independent. We can use the Gram-Schmidt Process to find an orthogonal basis for U.
We set $\mathbf{f_1} = (1,1,1)$ and

$$\mathbf{f_2} = (0,1,1) - \frac{(0,1,1)\cdot(1,1,1)}{3}(1,1,1) = (-2/3, 1/3, 1/3).$$

Hence, $\{(1,1,1), (-2/3,1/3,1/3)\}$ is a new orthogonal basis for U.

Section 10.1 — Problem 23 (5 Pts)

We have

$$\begin{aligned}\|\mathbf{v}+\mathbf{w}\|^2 &= \langle \mathbf{v}+\mathbf{w}, \mathbf{v}+\mathbf{w}\rangle \\ &= \langle \mathbf{v}+\mathbf{w}, \mathbf{v}\rangle + \langle \mathbf{v}+\mathbf{w}, \mathbf{w}\rangle \\ &= \langle \mathbf{v}, \mathbf{v}\rangle + \langle \mathbf{w}, \mathbf{v}\rangle + \langle \mathbf{v}, \mathbf{w}\rangle + \langle \mathbf{w}, \mathbf{w}\rangle \\ &= \|\mathbf{v}\|^2 + \langle \mathbf{v}, \mathbf{w}\rangle + \langle \mathbf{v}, \mathbf{w}\rangle + \|\mathbf{w}\|^2 \\ &= \|\mathbf{v}\|^2 + 2\langle \mathbf{v}, \mathbf{w}\rangle + \|\mathbf{w}\|^2.\end{aligned}$$

Similarly, we get

$$\begin{aligned}\|\mathbf{v}-\mathbf{w}\|^2 &= \langle \mathbf{v}-\mathbf{w}, \mathbf{v}-\mathbf{w}\rangle \\ &= \langle \mathbf{v}-\mathbf{w}, \mathbf{v}\rangle + \langle \mathbf{v}+\mathbf{w}, -\mathbf{w}\rangle \\ &= \langle \mathbf{v}, \mathbf{v}\rangle + \langle -\mathbf{w}, \mathbf{v}\rangle + \langle \mathbf{v}, -\mathbf{w}\rangle + \langle -\mathbf{w}, -\mathbf{w}\rangle \\ &= \|\mathbf{v}\|^2 - \langle \mathbf{w}, \mathbf{v}\rangle - \langle \mathbf{v}, \mathbf{w}\rangle + \langle \mathbf{w}, \mathbf{w}\rangle \\ &= \|\mathbf{v}\|^2 - \langle \mathbf{v}, \mathbf{w}\rangle - \langle \mathbf{v}, \mathbf{w}\rangle + \|\mathbf{w}\|^2 \\ &= \|\mathbf{v}\|^2 - 2\langle \mathbf{v}, \mathbf{w}\rangle + \|\mathbf{w}\|^2.\end{aligned}$$

Therefore,
$$\frac{1}{2}\Big(\|\mathbf{v}+\mathbf{w}\|^2+\|\mathbf{v}-\mathbf{w}\|^2\Big)=\frac{1}{2}\Big(2\|\mathbf{v}\|^2+2\|\mathbf{w}\|^2\Big)=\|\mathbf{v}\|^2+\|\mathbf{w}\|^2.$$