Math 311

Chapter 7

Section 7.3: Coordinates Isomorphism and Composition

Contents

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Coordinates of a Vector

Assume that V is a vector space with $n = \dim V < \infty$.

Let

$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

① $B = \{\mathbf{b_1}, \mathbf{b_2}, \ldots, \mathbf{b_n}\}$ be a basis of V

② $E = \{\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}\}$ be the standard basis for $\mathbb{R}^n$.

Given any $\mathbf{v} \in V$ with $\mathbf{v} = v_1 \mathbf{b_1} + v_2 \mathbf{b_2} + \cdots + v_n \mathbf{b_n}$, define the linear transformation $C_B : V \to \mathbb{R}^n$ as

$$C_B(\mathbf{v}) = v_1 \mathbf{e_1} + v_2 \mathbf{e_2} + \cdots + v_n \mathbf{e_n} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

Definition 1. The linear transformation C_B is called the **coordinates isomorphism**.

Notes:

① The coordinates isomorphism C_B gives a way to regard any vector space V of dimension n as $\mathbb{R}^n$.

② In mathematical jargon, we say that V ($\dim V = n$) and $\mathbb{R}^n$ are **isomorphic**.

③ More generally, two vector spaces V and W are isomorphic if there is a linear transformation $T : V \to W$ which is onto and one-to-one.

Composition

Definition 2. Let V, W and U be vector spaces. Let $T : V \to W$ and $S : W \to U$ be linear transformations. The **composite transformation** $ST : V \to U$ of T and S is defined by

$$ST(\mathbf{v}) = S(T(\mathbf{v})) \quad \mathbf{v} \in V.$$

Notes:

① ST is a linear transformation.

② TS might not be defined unless $U = V$.

③ We say that $T : V \to W$ is an **isomorphism** if there exists a linear transformation $S : W \to V$ such that:

- $ST = 1_V$. $\to 1_V(\vec{v}) = \vec{v}$
- $TS = 1_W$. $\to 1_W(\vec{w}) = \vec{w}$.

In this case, $S = T^{-1}$ is called the inverse of T.

Example 1. Let $B = \{1, x, x^2\}$ be a basis for $\mathbf{P_2}$.

a) Find the coordinate transformation C_B.

b) Find C_B^{-1}.

Solution.

a) $p(x) = 1 - 2x + 3x^2$

$$\Rightarrow C_B(p(x)) = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

In general, $p(x) = a_0 + a_1 x + a_2 x^2$

$$\Rightarrow \quad C_B(p(x)) = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}.$$

b) The inverse $C_B^{-1} : \mathbb{R}^3 \to P_2$

for example:

$$C_B^{-1}\left(\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}\right) = 1 - 2x + 3x^2$$

In general

$$C_B^{-1}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = x_1 + x_2 x + x_3 x^2 .$$