Section 3.1 — Problem 1 (20 Pts)

e. We have

$$\det \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = (\cos\theta)(\cos\theta) - (-\sin\theta)(\sin\theta) = \cos^2\theta + \sin^2\theta = 1.$$

f. We develop along the first row. We get

$$\begin{aligned} \det \begin{bmatrix} 2 & 0 & -3 \\ 1 & 2 & 5 \\ 0 & 3 & 0 \end{bmatrix} &= 2\det \begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix} - (0)\det \begin{bmatrix} 1 & 5 \\ 0 & 0 \end{bmatrix} + (-3)\det \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \\ &= 2(-15) + 0 - 3(3) = -39. \end{aligned}$$

k. We develop the determinant along the first row again. We get

$$\begin{aligned} \det \begin{bmatrix} 0 & 1 & -1 & 0 \\ 3 & 0 & 0 & 2 \\ 0 & 1 & 2 & 1 \\ 5 & 0 & 0 & 7 \end{bmatrix} &= -(1)\det \begin{bmatrix} 3 & 0 & 2 \\ 0 & 2 & 1 \\ 5 & 0 & 7 \end{bmatrix} + (-1)\det \begin{bmatrix} 3 & 0 & 2 \\ 0 & 1 & 1 \\ 5 & 0 & 7 \end{bmatrix} \\ &= (-1)(2)\det \begin{bmatrix} 3 & 2 \\ 5 & 7 \end{bmatrix} + (-1)(1)\det \begin{bmatrix} 3 & 2 \\ 5 & 7 \end{bmatrix} \\ &= (-2)(11) - (21 - 10) \\ &= -33. \end{aligned}$$

n. Your answer should be -56.

Section 3.1 — Problem 7a (20 Pts)

Let A be the matrix such that $\det A = -1$. Let B be the matrix in part a. Here are the steps to go from $A \to B$.

1. Replaced row 1 of A by $R_1 + 3R_2$. Call this matrix A_1. We have $\det A_1 = \det A$.
2. Replaced R_3 of A_1 by $-R_3$. Call this new matrix A_2. Then $\det A_2 = -\det A_1 = -\det A$.
3. Replace R_2 of A_2 by $2R_2$. Call this new matrix A_3. Then $\det A_3 = 2\det A_2 = -2\det A$.
4. Swap R_1 with R_2 of A_3. Call this new matrix A_4. Then $\det A_4 = -\det A_3 = 2\det A$.
5. Swap R_2 with R_3 of A_4. Call this new matrix A_5 which is now B. Then $\det A_5 = -\det A_4 = -2\det A$.

Hence $\det B = -2\det A = 2$.
Below is the matrix obtained after each row operations:

$$\begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix} \longrightarrow \begin{bmatrix} a+3p & b+3q & c+3r \\ p & q & r \\ x & y & z \end{bmatrix} \longrightarrow \begin{bmatrix} a+3p & b+3q & c+3r \\ p & q & r \\ -x & -y & -z \end{bmatrix}$$
$$\longrightarrow \begin{bmatrix} a+3p & b+3q & c+3r \\ 2p & 2q & 2r \\ -x & -y & -z \end{bmatrix} \longrightarrow \begin{bmatrix} -x & -y & -z \\ 2p & 2q & 2r \\ a+3p & b+3q & c+3r \end{bmatrix}$$
$$\longrightarrow \begin{bmatrix} -x & -y & -z \\ a+3p & b+3q & c+3r \\ 2p & 2q & 2r \end{bmatrix}$$

Section 3.1 — Problem 15a (5 Pts)

The simple trick is to develop along the last column of the matrix and find that the coefficient b in front of y is

$$-\det \begin{bmatrix} 5 & -1 \\ -5 & 4 \end{bmatrix} = -(20-5) = -15.$$

Section 3.1 — Problem 17 (5 Pts)

Using row operations, we see that

$$\det \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & x & x \\ 1 & x & 0 & x \\ 1 & x & x & 0 \end{bmatrix} = \det \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & -x & 0 & x \\ 0 & 0 & -x & x \\ 1 & x & x & 0 \end{bmatrix} = -\det \begin{bmatrix} 1 & x & x & 0 \\ 0 & -x & 0 & x \\ 0 & 0 & -x & x \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

and

$$-\det \begin{bmatrix} 1 & x & x & 0 \\ 0 & -x & 0 & x \\ 0 & 0 & -x & x \\ 0 & 1 & 1 & 1 \end{bmatrix} = -x^2 \det \begin{bmatrix} 1 & x & x & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} = -x^2 \det \begin{bmatrix} 1 & x & x & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix} = -3x^2.$$

If A denote the matrix from the beginning, then $\det A = -3x^2$.