Math 311

Last Chapter

Section 8.1: Orthogonal Complements and Projections

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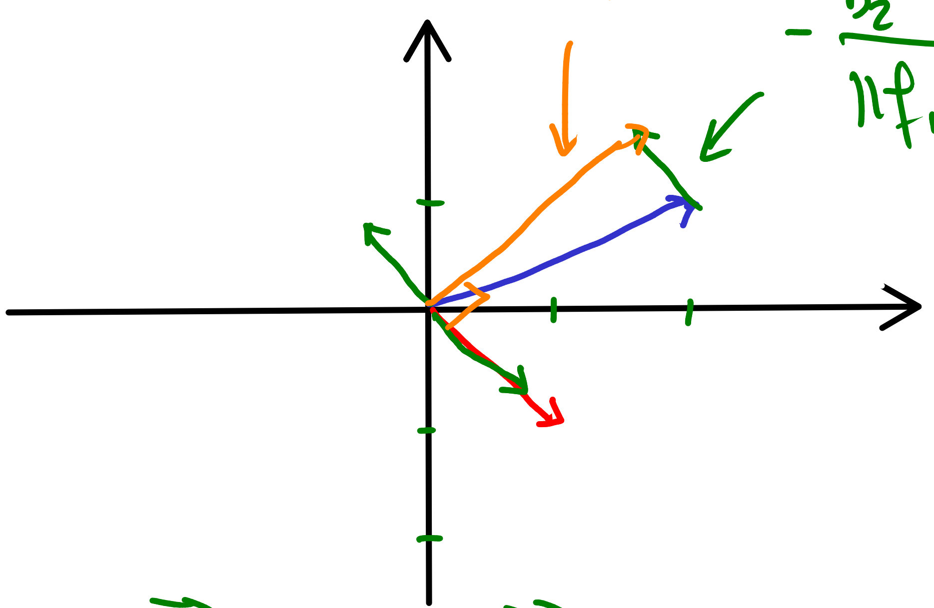
EXAMPLE 1. Let $V = \mathbb{R}^2$ and $B = \{\underbrace{(1,-1)}_{\vec{b_1}}, \underbrace{(2,1)}_{\vec{b_2}}\}$. Notice that B is not an orthogonal basis. Using the vectors from B, construct an orthogonal basis F.

SOLUTION. Geometric intuition: https://www.desmos.com/geometry/e9mrgozxmb.

$$F = \{\vec{f_1}, \vec{f_2}\}$$

Set $\vec{f_1} = \vec{b_1}$

f_2 $\quad -\dfrac{\vec{b_2}\cdot\vec{f_1}}{\|f_1\|^2}\vec{f_1}$



Set $\vec{f_2} = \vec{b_2} - \dfrac{\vec{b_2}\cdot\vec{f_1}}{\|\vec{f_1}\|^2}\vec{f_1}$.

We can show that $\vec{f_1}\cdot\vec{f_2} = 0$.

The Gram-Schmidt Orthogonalization Algorithm

Let $B = \{\mathbf{b_1}, \mathbf{b_2}, \ldots, \mathbf{b_m}\}$ be a basis of a subspace U of $V = \mathbb{R}^n$.

To transform B into an orthogonal basis $F = \{\mathbf{f_1}, \mathbf{f_2}, \ldots, \mathbf{f_m}\}$, we define

- $\mathbf{f_1} = \mathbf{b_1}$.
- $\mathbf{f_2} = \mathbf{b_2} - \dfrac{\mathbf{b_2} \cdot \mathbf{f_1}}{\|\mathbf{f_1}\|^2}\mathbf{f_1}$.
- $\mathbf{f_3} = \mathbf{b_3} - \dfrac{\mathbf{b_3} \cdot \mathbf{f_1}}{\|\mathbf{f_1}\|^2}\mathbf{f_1} - \dfrac{\mathbf{b_3} \cdot \mathbf{f_2}}{\|\mathbf{f_2}\|^2}\mathbf{f_2}$.
- $\cdots$
- $\mathbf{f_k} = \mathbf{b_k} - \dfrac{\mathbf{b_k} \cdot \mathbf{f_1}}{\|\mathbf{f_1}\|^2}\mathbf{f_1} - \dfrac{\mathbf{b_k} \cdot \mathbf{f_2}}{\|\mathbf{f_2}\|^2}\mathbf{f_2} - \cdots - \dfrac{\mathbf{b_k} \cdot \mathbf{f_{k-1}}}{\|\mathbf{f_{k-1}}\|^2}\mathbf{f_{k-1}}$.
- $\cdots$
- $\mathbf{f_m} = \mathbf{b_m} - \dfrac{\mathbf{b_m} \cdot \mathbf{f_1}}{\|\mathbf{f_1}\|^2}\mathbf{f_1} - \dfrac{\mathbf{b_m} \cdot \mathbf{f_2}}{\|\mathbf{f_2}\|^2}\mathbf{f_2} - \cdots - \dfrac{\mathbf{b_m} \cdot \mathbf{f_{m-1}}}{\|\mathbf{f_{m-1}}\|^2}\mathbf{f_{m-1}}$.