Math 311

Chapter 3

Section 3.3: Diagonalization and Eigenvalues

Contents

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$$\begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} (-1)^2 & 0 \\ 0 & 4^2 \end{bmatrix}$$

Example 1. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$. Compute A^{100}.

Solution. WAY to long to compute directly.

Instead, we find

$$P = \begin{bmatrix} 1 & 2/3 \\ -1 & 1 \end{bmatrix} \qquad P^{-1} = \frac{3}{5}\begin{bmatrix} 1 & -2/3 \\ 1 & 1 \end{bmatrix}$$

Diagonal matrix

then

$$P^{-1}AP = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} \Rightarrow A = P\begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}P^{-1}$$

So,

$$A^2 = AA = \left(P\begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}\cancel{P^{-1}}\right)\left(\cancel{P}\begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}P^{-1}\right)$$

$$= P\begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}^2 P^{-1} = P\begin{bmatrix} (-1)^2 & 0 \\ 0 & 4^2 \end{bmatrix}P^{-1}$$

$$\Rightarrow A^{100} = P\begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}^{100} P^{-1} = P\begin{bmatrix} (-1)^{100} & 0 \\ 0 & (4)^{100} \end{bmatrix}P^{-1}$$

Fact: If $A = PDP^{-1}$, then $A^k = PD^kP^{-1}$.

Goal: Find the matrix P such that $P^{-1}AP$ is a diagonal matrix.

Eigenvalues and Eigenvectors

Exploration: Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}.$$

Set $\mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix}$ a 2×1 vector. Then

$$A\mathbf{x} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a + 2b \\ 3a + 2b \end{bmatrix}$$

Use Desmos[1] to explore and answer the following questions:

- Can you find an exceptional behavior of $A\mathbf{x}$ and $\mathbf{x}$ for certain choices of $\mathbf{x}$?
- Can you find a relation between $A\mathbf{x}$ and $\mathbf{x}$?

Record your observations in the following blank space:

(1) Output and input lay on the same line.

(2) Output is a scalar multiple of the input.

[1] https://www.desmos.com/calculator/5xlrp9fd7g

Definition 1. Let A be an $n \times n$ matrix.

a) A number λ is called an **eigenvalue** of A if there is a non-zero $n \times 1$ vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$.

b) The vector $\mathbf{x}$ is called an **eigenvector** associated to λ.

Example 2. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ and let $\mathbf{x} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$. Then

$$A\mathbf{x} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -2+4 \\ -6+4 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} = (-1)\begin{bmatrix} -2 \\ 2 \end{bmatrix} = -\vec{x}.$$

-1 : eigen value for A & $\vec{x}$ eigenvector.

Finding eigenvalues

Notice that

$$\begin{aligned} \lambda \text{ is an eigenvalue of } A &\iff A\mathbf{x} = \lambda\mathbf{x} \text{ for some } \mathbf{x} \neq 0 \\ &\iff (\lambda I - A)\mathbf{x} = 0 \text{ for some } \mathbf{x} \neq 0. \end{aligned}$$

$$(\lambda I - A)^{-1}(\lambda I - A)\vec{x} = \vec{x} = \vec{0}$$

So

$$\begin{aligned} \lambda \text{ is an eigenvalue of } A &\iff (\lambda I - A) \text{ is not invertible} \\ &\iff \det(\lambda I - A) = 0 \end{aligned}$$

Definition 2. The **characteristic polynomial** of an $n \times n$ matrix A is defined by

$$c_A(x) = \det(xI - A).$$

Conclusion:

$$\lambda \text{ is an eigenvalue of } A \iff \lambda \text{ is a root of } c_A(x).$$

Example 3. Find all eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}.$$

Solution.

We have

$$c_A(x) = \det(xI - A)$$

$$= \det\left(\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} x-1 & -2 \\ -3 & x-2 \end{bmatrix}\right)$$

$$= (x-1)(x-2) - 6 = x^2 - 3x - 4$$

$$= (x+1)(x-4)$$

Hence

$$c_A(x) = 0 \iff (x+1)(x-4) = 0$$

$$\iff x = -1 \text{ or } x = 4$$

Eigen values: $\boxed{\lambda_1 = -1, \quad \lambda_2 = 4}$.

$\underline{\lambda = 4}$ Write

$$(\lambda I - A)\vec{x} = 0 \iff \left(\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\iff \begin{bmatrix} 3 & -2 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\iff \begin{cases} 3x - 2y = 0 \\ -3x + 2y = 0 \end{cases}$$

$$\Rightarrow \quad 3x - 2y = 0 \quad \Rightarrow \quad x = \frac{2}{3} y. \text{ So}$$

$$\vec{x}_2 = \begin{bmatrix} 2/3\, y \\ y \end{bmatrix} = y \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$$

For $\lambda = -1$, we can set $x = 1$

$$\Rightarrow \quad \vec{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

For $\lambda = 4$, we can set $y = 1$

$$\Rightarrow \quad \vec{x}_2 = \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}.$$

Finding Eigenvectors

For a given eigenvalue λ, the eigenvectors associated to λ are the solutions $\mathbf{x}$ to the system

$$(\lambda I - A)\mathbf{x} = \vec{0}.$$

EXAMPLE 4. Find the eigenvectors associated to the each eigenvalue of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}.$$

$\underline{\lambda = -1}$ Let $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, so that

$$((-1)I - A)\overbrace{\begin{bmatrix} x \\ y \end{bmatrix}}^{\vec{x}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow \left(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} -2 & -2 \\ -3 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} -2x - 2y = 0 \\ -3x - 3y = 0 \end{cases}$$

$$\Rightarrow x + y = 0 \Rightarrow y = -x \Rightarrow \vec{x}_1 = \begin{bmatrix} x \\ -x \end{bmatrix} = x\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Example 5. Find all eigenvalues and associated eigenvectors of the matrix

$$A = \begin{bmatrix} 7 & 0 & -4 \\ 0 & 5 & 0 \\ 5 & 0 & -2 \end{bmatrix}.$$

Solution.

① Eigenvalues

$$c_A(x) = \det(xI - A)$$

$$= \begin{vmatrix} x-7 & 0 & 4 \\ 0 & x-5 & 0 \\ -5 & 0 & x+2 \end{vmatrix}$$

$$\Rightarrow c_A(x) = (x-2)(x-3)(x-5) = 0$$

So, $\lambda_1 = 2$, $\lambda_2 = 3$, $\lambda_3 = 5$.

② Eigen vectors

$\lambda_1 = 2$ $\quad (2I - A)\vec{x} = \vec{0} \Rightarrow \begin{bmatrix} -5 & 0 & 4 \\ 0 & -3 & 0 \\ -5 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -5x + 4z = 0$ and $-3y = 0$

$\Rightarrow x = \frac{4}{5} z$ and $y = 0$. $\Rightarrow \vec{x}_1 = z \begin{bmatrix} 4/5 \\ 0 \\ 1 \end{bmatrix}$.

$\underline{\lambda_2 = 3}$ $(3I - A)\vec{x} = \vec{0} \Rightarrow \begin{bmatrix} -4 & 0 & 4 \\ 0 & -2 & 0 \\ -5 & 0 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Solution: $-4x + 4z = 0$, $-2y = 0$
$-5x + 5z = 0$

$\Rightarrow$ $-x + z = 0$ & $y = 0$
$-x + z = 0$

$\Rightarrow$ $z = x$, $y = 0 \Rightarrow \vec{x} = \begin{bmatrix} x \\ 0 \\ x \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

$\underline{\lambda_5 = 5}$

$(5I - A)\vec{x} = \vec{0} \Rightarrow \begin{bmatrix} -2 & 0 & 4 \\ 0 & 0 & 0 \\ -5 & 0 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Solution is $x = 0$, $y =$ parameter, $z = 0$

So, $\vec{x} = \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix} = y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

So, a set of eigenvectors is

$$\overset{\lambda=2}{\begin{bmatrix} -4/5 \\ 0 \\ 1 \end{bmatrix}}, \overset{\lambda=3}{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}, \overset{\lambda=5}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}.$$

EXAMPLE 6. Find a matrix P such that

$$P^{-1}AP$$

is a diagonal matrix, where A is from Example 1.

SOLUTION.

Recall that the eigen vectors are

$$\vec{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \vec{x}_2 = \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}.$$

Use $\vec{x}_1$ and $\vec{x}_2$ to create the columns of P:

$$P = [\vec{x}_1 \ \vec{x}_2] = \begin{bmatrix} 1 & 2/3 \\ -1 & 1 \end{bmatrix}.$$

$$\Rightarrow P^{-1} = \frac{3}{5}\begin{bmatrix} 1 & -2/3 \\ 1 & 1 \end{bmatrix}.$$

Hence

$$P^{-1}AP = \frac{3}{5}\begin{bmatrix} 1 & -2/3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2/3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \overset{\lambda_1 = -1}{-1} & 0 \\ 0 & \overset{\lambda_2 = 4}{4} \end{bmatrix}$$

THEOREM 1. Let A be an $n \times n$ matrix. Then if all eigenvalues of A are distinct, then A is diagonalizable.

Notice that if A is diagonalizable and if we let $P = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{bmatrix}$:

$$\begin{aligned} P^{-1}AP = D \iff & AP = PD \\ \iff & A\begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{bmatrix} D \\ \iff & A\mathbf{x}_1 = \lambda_1\mathbf{x}_1,\ A\mathbf{x}_2 = \lambda_2\mathbf{x}_2, \ldots, A\mathbf{x}_n = \lambda_n\mathbf{x}_n. \end{aligned}$$

ALGORITHM 1. Let A be an $n \times n$ matrix with distinct eigenvalues.

① Find all distinct eigenvalues of A.

② For each eigenvalue of A, find the corresponding set of eigenvectors.

③ If $\mathbf{x}_1$, $\mathbf{x}_2$, ..., $\mathbf{x}_n$ are a set of n distinct eigenvectors, then set
$$P = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{bmatrix}.$$

WARNING!

Not every matrix is diagonalizable. For instance, the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

is not diagonalizable.

For a more general algorithm, see *Jordan Canonical Form*, Chapter 11 from the textbook. Complex numbers are required.