

MATH 311

CHAPTER 6

SECTION 6.3: LINEAR INDEPENDENCE AND DIMENSION

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LINEAR INDEPENDENCE

EXAMPLE 1. Let $\mathbf{v} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$. Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

and $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$

- a) Consider the vectors. Can you write \mathbf{v} as a unique linear combination of the vectors \mathbf{u}_1 , \mathbf{u}_2 , and \mathbf{u}_3 ?
- b) Consider the vectors. Can you write the vector \mathbf{v} as a unique linear combination of the vectors \mathbf{v}_1 , \mathbf{v}_2 , \mathbf{v}_3 ?

SOLUTION.

DEFINITION 1. A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ in a vector space V is called **linearly independent** (or simply **independent**) if

$$s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots + s_n\mathbf{v}_n = \mathbf{0} \quad \Rightarrow \quad s_1 = s_2 = \cdots = s_n = 0.$$

A set of vectors that is not independent is said to be **linearly dependent** (or simply **dependent**).

Note:

- The **trivial linear combination** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ is the one with every coefficient zero:

$$0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_n.$$

- So the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are linearly independent if and only if the only way to write $\mathbf{0}$ is using the trivial linear combination.

EXAMPLE 2. Show that the set

$$\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}$$

is independent.

SOLUTION.

EXAMPLE 3. Let $\{\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}\}$ be an independent set in a vector space V . Which of the following set is independent?

a) $\{\mathbf{x} - \mathbf{y}, \mathbf{y} - \mathbf{z}, \mathbf{z} - \mathbf{x}\}$.

b) $\{\mathbf{x} - \mathbf{y}, \mathbf{y} - \mathbf{z}, \mathbf{z} - \mathbf{w}, \mathbf{w} - \mathbf{x}\}$.

SOLUTION.

DEFINITION 2. A set $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ of vectors in a vector space V is called a basis of V if it satisfies the following two conditions:

- ① $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is linearly independent.
- ② $V = \text{span}\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$.

EXAMPLE 4. Let $V = \mathbb{R}^3$. Verify the following.

- a) If $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are the columns of I_3 , then $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a basis for \mathbb{R}^3 .
- b) $\{[1 \ -1 \ 0]^\top, [3 \ 2 \ -1]^\top, [3 \ 5 \ -2]^\top\}$ is a basis for \mathbb{R}^3 .

Observations:

- Invariance Theorem (p.347): If $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is a basis for a vector space V and if $\{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_m\}$ is a basis for a vector space V , then $m = n$.

DEFINITION 3. If $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is a basis of a nonzero vector space V , the number n of vectors in the basis is called the **dimension**, and we write

$$\dim V = n.$$

In the case of the zero vector space, we define $\dim\{\mathbf{0}\} = 0$.

Note:

- ① We have $\dim \mathbb{R}^m = m$ because the m columns of the identity matrix I_m is a basis.
- ② We have $\dim \mathbf{M}_{mn} = mn$. Let E_{ij} be the matrix with a 1 in the (i, j) -entry and 0 elsewhere. A basis for \mathbf{M}_{mn} is

$$B = \{M_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}.$$

This is called the **canonical basis** or **standard basis** of \mathbf{M}_{mn} . For instance, if $m = n = 2$, then a basis for \mathbf{M}_{22} is

$$\begin{aligned} B &= \{M_{11}, M_{12}, M_{21}, M_{22}\} \\ &= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \end{aligned}$$

- ③ We have $\dim \mathbf{P}_n =$.

- ④ Any subspace U of a vector space V is a vector space. Therefore, we can find the dimension of U .

Subspaces of \mathbb{R}^m

For subspaces of \mathbb{R}^m , there is a really nice way to determine a basis and the dimension of a spanning set. Let

$$U = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}.$$

Let

- $A = [\mathbf{v}_1 \ \mathbf{v}_2 \ \cdots \ \mathbf{v}_n]$.
- R be the RREF of A .

Then

- $\dim U =$ number of pivots in R .
- A basis for U is given by the vector in the same column as the pivots.

EXAMPLE 5. Find a basis and calculate the dimension for the following subspace of \mathbb{R}^4 :

$$U = \text{span}\{(1, -1, 2, 0), (2, 3, 0, 3), (1, 9, -6, 6)\}.$$

SOLUTION.

Note: This trick also works for subspaces of the space of polynomials \mathbf{P}_n .

Subspaces of Matrices

EXAMPLE 6. Define the subspace of \mathbf{M}_{22} as

$$U = \left\{ X \in \mathbf{M}_{22} : \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} X = X \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \right\}.$$

Find a basis of U and its dimension.

SOLUTION.

