Math 311

Chapter 2

Section 2.1: Matrix Addition, Scalar Multiplication, and Transposition

Contents

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Matrices

Definition 1.

- A **matrix** is an array of numbers and the numbers are called **entries**.
- A matrix has m **rows** and n **columns** and is called an $\mathbf{m \times n}$ **matrix**. The numbers of rows and columns, $m \times n$, are called the **dimensions** of the matrix.
- A **column matrix**, or n-vector or column vector is an $n \times 1$ matrix.
- A **row matrix**, or row vector, is an $1 \times n$ matrix.
- The $(\mathbf{i}, \mathbf{j})$**-entry** of a matrix is the number lying in row i and column j.

Example 1. Below are matrices.

(a) Identify the 2×3 matrix. (b) Identify the 3×2 matrix.
(c) Identify the column vector and indicate its dimensions.
(d) Identify the row vector and indicate its dimensions.
(e) What is the (1, 2)-entry of the matrix B.

$$\underset{3\times 1}{A} \overset{(c)}{=} \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}_{3\times 1}, \quad B \overset{(a)}{=} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 5 & 6 \end{bmatrix}, \quad C \underset{1\times 3}{\overset{(d)}{=}} \begin{bmatrix} 1 & 3 & 4 \end{bmatrix}, \quad D \overset{(b)}{=} \begin{bmatrix} 1 & 4 \\ 4 & 10 \\ -3 & -1 \end{bmatrix}.$$

Notations and Conventions

The general notations for an $m \times n$ matrix A:

- $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$
- For example, a generic 3×5 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \end{bmatrix}$$

- A shorcut: $A = [a_{ij}]$.
- Another shortcut: $A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_\mathbf{n}]$, where $\mathbf{a}_1$, $\mathbf{a}_2$, ..., $\mathbf{a}_n$ represents the columns of the matrix A.

Here are conventions to keep in mind:

- If a matrix has size $m \times n$, it has m rows and n columns.
- If we speak of the (i, j)-entry of a matrix, it lies in row i and column j.
- If an entry is denoted by a_{ij}, the first subscript i refers to the row and the second subscript j to the column in which the number a_{ij} lies.

Matrix Equality

Definition 2. Two matrices A and B are **equal** (denoted as $A = B$) if the following conditions are met:

- They have the same size.
- Corresponding entries are equal.

Example 2. Let a be a real number and

$$A = \begin{bmatrix} a^2 & (a-1)^2 \\ 2(a-2) & a^2 - 3a + 2 \end{bmatrix}$$

and

$$B = \begin{bmatrix} a^2 & a^2 - 2a + 1 \\ 2a - 4 & (a-2)(a-1) \end{bmatrix}.$$

and

$$C = \begin{bmatrix} a^2 & (a-1)^2 \\ 2a - 5 & (a-2)(a-1) \end{bmatrix}$$

(a) Do we have $A = B$? (b) Do we have $A = C$?

Solution.

(a) A is 2×2 & B is 2×2.

$a^2 = a^2$ ✓ $\quad (a-1)^2 = a^2 - 2a + 1$ ✓

$2(a-2) = 2a - 4$ ✓ $\quad a^2 - 3a + 2 = (a-2)(a-1)$ ✓

$\Rightarrow A = B$ $\quad \to A \neq C$

(b) Dimensions ✓ but $2(a-2) = 2a-4 \neq 2a-5$ ✗

Matrix Addition

Definition 3. If A and B are matrices of the same size, their **sum** $A + B$ is the matrix formed by adding corresponding entries.

Example 3. If $A = \begin{bmatrix} -2 & 3 & 2 \\ 3 & 4 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 6 \end{bmatrix}$, then

$$A + B = \begin{bmatrix} -2+1 & 3+1 & 2+(-1) \\ 3+2 & 4+0 & (-1)+6 \end{bmatrix} = \begin{bmatrix} -1 & 4 & 1 \\ 5 & 4 & 5 \end{bmatrix}.$$

Example 4. Find the values of a, b, and c if

$$\begin{bmatrix} a & b & c \end{bmatrix} + \begin{bmatrix} c & a & b \end{bmatrix} = \begin{bmatrix} 3 & 2 & -1 \end{bmatrix}.$$

Solution.

$$\Leftrightarrow \begin{bmatrix} a+c & b+a & c+b \end{bmatrix} = \begin{bmatrix} 3 & 2 & -1 \end{bmatrix}$$

$$\Leftrightarrow a+c=3, \quad b+a=2, \quad c+b=-1$$

$$\Leftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & -1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\Rightarrow \boxed{a=3, \; b=-1, \; c=0}.$$

Theorem 1. Let A, B, and C be arbitrary $m \times n$ matrices. Then the following holds:

1. $A + B = B + A$ (commutative law).
2. $A + (B + C) = (A + B) + C$ (associative law).

Proof. We will prove property 1. Let $A = [a_{ij}]$ and $B = [b_{ij}]$. Then

$$A + B = [a_{ij} + b_{ij}] = [b_{ij} + a_{ij}] = B + A. \qquad \square$$

Definition 4. Let A and B be two $m \times n$ matrix.

- **Zero matrix**: The $m \times n$ matrix O in which every entry is zero.
- **Negative**: The $m \times n$ matrix $-A$ in which every entry is obtained by multiplying entries of A by -1.
- **Difference:** It is defined by $A - B = A + (-B)$.

Example 5. Let $A = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ -2 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$. Compute (a) $-A$; (b) $A + B - C$.

Solution.

(a) $-A = \begin{bmatrix} -3 & 1 \\ -1 & -2 \end{bmatrix}$

(b) $= \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -2 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ -3 & -1 \end{bmatrix}$

$= \begin{bmatrix} 3 & -2 \\ -4 & 1 \end{bmatrix}$

Theorem 2. For any $m \times n$ matrix A:

1. $O + A = A$.
2. $A + (-A) = O$.

Example 6. Find the entries of the matrix X if it satisfies the following equation:

$$\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} + X = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}.$$

Solution.

$$\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} + X = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + X = \begin{bmatrix} -2 & -2 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \boxed{X = \begin{bmatrix} -2 & -2 \\ 0 & 1 \end{bmatrix}}$$

Scalar Multiplication

Definition 5. If k is a number and A a matrix, then the **scalar multiple** kA is the matrix obtained by multiplying each entry of A by k.

Note: the number k is called a **scalar**.

Example 7. If $A = \begin{bmatrix} 3 & -1 & 4 \\ 2 & 0 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 2 \end{bmatrix}$. Compute $3A - 2B$.

Solution.

$$3A - 2B = \begin{bmatrix} 9 & -3 & 12 \\ 6 & 0 & 18 \end{bmatrix} + \begin{bmatrix} -2 & -4 & 2 \\ 0 & -6 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -7 & 14 \\ 6 & -6 & 14 \end{bmatrix}.$$

EXAMPLE 8. Show that if $kA = O$, then $k = 0$ or $A = O$.

SOLUTION.

Assume $kA = O$. Then $[ka_{ij}] = [0]$.

So, $ka_{ij} = 0$, for any i,j.

If $k \neq 0$, then $\frac{\cancel{k}a_{ij}}{\cancel{k}} = \frac{0}{k} = 0$, $\forall i,j$

$\Rightarrow \quad a_{ij} = 0, \quad \forall i,j.$

$\Rightarrow \quad A = O.$

The other case left is $k=0$, but this the other case in the conclusion. $\square$

THEOREM 3. Let A and B be two $m \times n$ matrices and k, l be two numbers. Then

1. $k(A + B) = kA + kB$ (distributive law I).
2. $(k + l)A = kA + lA$ (distributive law II).
3. $(kl)A = k(lA)$.
4. $1A = A$ and $(-1)A = -A$.

DEFINITION 6. If A is an $m \times n$ matrix, the **transpose** of A, written A^T, is the $n \times m$ matrix whose rows are the columns of A in the same order.

Note: Based on the definition, we can write

$$A^\top = [a_{ij}]^\top = [a_{ji}].$$

EXAMPLE 9. Find the transpose of each of the following matrices.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 2 \\ -1 & 2 & 1 \end{bmatrix}.$$

SOLUTION.

$$A^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \to 2 \times 3 \text{ matrix.}$$

$$B^T = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 2 \\ -1 & 2 & 1 \end{bmatrix} \to 3 \times 3 \text{ matrix}$$

and Symmetric!

THEOREM 4. Let A and B denote matrices of the same size, and let k denote a scalar.

1. If A is an $m \times n$ matrix, then $A^\top$ is an $n \times m$ matrix.
2. $(A^\top)^\top = A$.
3. $(kA)^\top = kA^\top$.
4. $(A+B)^\top = A^\top + B^\top$.

PROOF. We will prove property 4. Write $A = [a_{ij}]$ and $B = [b_{ij}]$ and $A+B = [c_{ij}]$ with $c_{ij} = a_{ij} + b_{ij}$. Therefore,

$$(A+B)^\top = [c_{ji}] = [a_{ji}+b_{ji}] = [a_{ji}]+[b_{ji}] = A^\top + B^\top. \quad \square$$

EXAMPLE 10. Find the values of the entries of the matrix A if

$$\left(A + 3\begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 4 \end{bmatrix}\right)^\top = \begin{bmatrix} 2 & 1 \\ 0 & 5 \\ 3 & 8 \end{bmatrix}.$$

$$\Leftrightarrow \quad A^T + \left(3\begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 4 \end{bmatrix}\right)^T = \begin{bmatrix} 2 & 1 \\ 0 & 5 \\ 3 & 8 \end{bmatrix}$$

$$\Leftrightarrow \quad A^T + 3\begin{bmatrix} 1 & 1 \\ -1 & 2 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 5 \\ 3 & 8 \end{bmatrix}$$

$$\Leftrightarrow \quad A^T = \begin{bmatrix} 2 & 1 \\ 0 & 5 \\ 3 & 8 \end{bmatrix} - \begin{bmatrix} 3 & 3 \\ -3 & 6 \\ 0 & 12 \end{bmatrix}$$

$$(A^T)^T = \begin{bmatrix} -1 & -2 \\ 3 & -1 \\ 3 & -4 \end{bmatrix}^T$$

$$\Rightarrow \boxed{A = \begin{bmatrix} -1 & 3 & 3 \\ -2 & -1 & -4 \end{bmatrix}}$$

$$((A^T)^T)^T = A^T$$

DEFINITION 7. A matrix A is symmetric if $A = A^\top$.

EXAMPLE 11. Show that if A and B are symmetric $n \times n$ matrices, then $A + B$ is symmetric.

SOLUTION.

Assume A and B are symmetric $n \times n$ matrices.

Goal: to show $(A+B)^T = A+B$.

So, $$(A+B)^T = A^T + B^T \quad (\text{THM 4, Prop. 4})$$
$$= A + B \quad (A, B \text{ are symm.})$$

Conclusion: $A+B$ is symmetric. $\square$