Section 3.3 — Problem 1b (15 Pts)

Characteristic polynomial: We have

$$c_A(x) = \det(xI - A) = \begin{vmatrix} x-2 & 4 \\ 1 & x+1 \end{vmatrix} = (x-2)(x+1) - 4 = x^2 - x - 6.$$

Eigenvalues: We have

$$c_A(x) = 0 \iff x^2 - x - 6 = 0 \iff (x+2)(x-3) = 0.$$

Hence $\lambda_1 = -2$ and $\lambda_2 = 3$.
Eigenvectors: We have two eigenvalues.

- $\lambda = -2$. We have to solve the system
$$\left(-2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \begin{bmatrix} -4 & 4 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$
The system is $-4x + 4y = 0$ and $x - y = 0$. This reduces to the single equation
$$x - y = 0 \iff x = y.$$
Hence, we have
$$\mathbf{x_1} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ x \end{bmatrix} = x\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$
- $\lambda = 3$. We have to solve the system
$$\begin{bmatrix} 1 & 4 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$
The system of equations is $x + 4y = 0$ and $x + 4y = 0$. This can be rewritten as the single equation
$$x + 4y = 0 \iff x = -4y.$$
Hence
$$\mathbf{x_2} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4y \\ y \end{bmatrix} = y\begin{bmatrix} -4 \\ 1 \end{bmatrix}.$$

Diagonalization: Since the eigenvalues are all different, the matrix P exists. In the expression of $\mathbf{x_1}$, we let $x = 1$, so that

$$\mathbf{x_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

In the expression of $\mathbf{x_2}$, we let $y = 1$, so that

$$\mathbf{x_2} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}.$$

Hence, $P = \begin{bmatrix} \mathbf{x_1} & \mathbf{x_2} \end{bmatrix}$, so that

$$P = \begin{bmatrix} 1 & -4 \\ 1 & 1 \end{bmatrix}.$$

Section 3.3 — Problem 3 (5 Pts)

Assume that A has $\lambda = 0$ as an eigvalue. The caracteristic polynomial is $c_A(x) = \det(xI - A)$. Since $\lambda = 0$ is an eigenvalue, then $c_A(0) = 0$. This means that $\det(0I - A) = \det(A) = 0$. Hence, A is not invertible.

Now assume that A is not invertible. Then $\det(A) = 0$. Using the characteristic polynomial, we see that $c_A(0) = \det(A) = 0$, hence $\lambda = 0$ is an eigenvalue of A.

Section 6.1 — Problem 1 (10 Pts)

a. Since the addition is defined as the one we define on $\mathbb{R}^3$, the axioms A1-A5 are satisfied. We have to check if S1-S5 are satisfied.

S1. From the definition $a(x_1, x_2, x_3) = (ax_1, x_2, ax_3)$ is a vector in $\mathbb{R}^3$. We're good!

S2. We have to check if $a(\mathbf{x} + \mathbf{y}) = a\mathbf{x} + a\mathbf{y}$. We have

$$a(\mathbf{x} + \mathbf{y}) = a(x_1 + y_1, x_2 + y_2, x_3 + y_3) = (a(x_1 + y_1), x_2 + y_2, a(x_3 + y_3))$$

and

$$a\mathbf{x} + b\mathbf{y} = (ax_1, x_2, ax_3) + (ay_1, y_2, ay_3) = (ax_1 + ay_1, x_2 + y_2, ax_3 + ay_3)$$

Factoring a from the first and third entries, we see that

$$a\mathbf{x} + a\mathbf{y} = a(\mathbf{x} + \mathbf{y}).$$

S3. We have now to check that $(a + b)\mathbf{x} = a\mathbf{x} + b\mathbf{x}$. We have

$$(a + b)\mathbf{x} = ((a + b)x_1, x_2, (a + b)x_3) = (ax_1 + bx_1, x_2, ax_3 + bx_3)$$

and

$$a\mathbf{x} + b\mathbf{x} = (ax_1, x_2, ax_3) + (bx_1, x_2, bx_3) = (ax_1 + bx_1, x_2 + x_2, ax_3 + bx_3)$$

We see that the second entry is now $2x_2$ which is different from x_2! In general, we won't have $(a + b)\mathbf{x} = a\mathbf{x} + b\mathbf{x}$.
To support that claim, let's give an example. Let $\mathbf{x} = (1, 1, 1)$. Then

$$(1 + 2)\mathbf{x} = (3)\mathbf{x} = (3, 1, 3)$$

but

$$(1)\mathbf{x} + (2)\mathbf{x} = (1, 1, 1) + (2, 1, 2) = (3, 2, 3).$$

We see that $(3, 1, 3) \neq (3, 2, 3)$.

<u>Conclusion:</u> Axiom S3 is not satisfied with this scalar multiplication and $\mathbb{R}^3$ is not a vector space if we consider this scalar multiplication.

b. Since the addition is defined as the one defined on $\mathbb{R}^3$, the axioms A1-A5 are satisfied. We will check if axioms S1-S5 are satisfied. In fact, we can go straight to the last one! Indeed, to satisfy S5, we must show that

$$1\mathbf{x} = \mathbf{x}.$$

However, based on the definition of the scalar multiplication, we have

$$1(x_1, x_2, x_3) = ((1)x_1, 0, (1)x_3) = (x_1, 0, x_3).$$

However, $(x_1, 0, x_3) \neq (x_1, x_2, x_3)$ in general. To support that claim, let's consider the following example. Let $\mathbf{x} = (1, 1, 1)$, then

$$1\mathbf{x} = (1, 0, 1) \neq (1, 1, 1).$$

Conclusion: Axiom S5 is not satisfied with this scalar multiplication and $\mathbb{R}^3$ is not a vector space if we consider this scalar multiplication.

Section 6.1 — Problem 2 (10 Pts)

e. We'll check if all the axioms are satisfied. The addition and scalar multiplication is the same as the ones used for 2×2 matrices.

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{bmatrix}$, and $C = \begin{bmatrix} c_{11} & c_{12} \\ 0 & c_{22} \end{bmatrix}$ be 2×2 matrices from V.

We first start with the axioms A1-A5.

A1. We have

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ 0 + 0 & a_{22} + b_{22} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ 0 & a_{22} + b_{22} \end{bmatrix}.$$

We see that $A + B$ is of the same type as the matrices in V. We're good!

A2. We have

$$A+B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ 0 & a_{22} + b_{22} \end{bmatrix} = \begin{bmatrix} b_{11} + a_{11} & b_{12} + a_{12} \\ 0 + 0 & b_{22} + a_{22} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} = B+A.$$

So we're good!

A3. Now we have

$$(A + B) + C = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ 0 & a_{22} + b_{22} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ 0 & c_{22} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} + c_{11} & a_{12} + b_{12} + c_{12} \\ 0 + 0 & a_{22} + b_{22} + c_{22} \end{bmatrix}$$

and

$$A + (B + C) = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} + c_{11} & b_{12} + c_{12} \\ 0 & b_{22} + c_{22} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} + c_{11} & a_{12} + b_{12} + c_{12} \\ 0 + 0 & a_{22} + b_{22} + c_{22} \end{bmatrix}.$$

Hence, $(A + B) + C = A + (B + C)$ and we're good!

A4. Let $\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then, we see that

$$A + \mathbf{0} = \begin{bmatrix} a_{11} + 0 & a_{12} + 0 \\ 0 + 0 & a_{22} + 0 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} = A.$$

We're still good!

A5. Let $-A = \begin{bmatrix} -a_{11} & -a_{12} \\ 0 & -a_{22} \end{bmatrix}$. Then

$$A + (-A) = \begin{bmatrix} a_{11} - a_{11} & a_{12} - a_{12} \\ 0 - 0 & a_{22} - a_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0}.$$

We are doing pretty great so far!

We now verify if S1-S5 are satisfied.

S1. We have

$$aA = \begin{bmatrix} aa_{11} & aa_{12} \\ (a)(0) & aa_{22} \end{bmatrix} = \begin{bmatrix} aa_{11} & aa_{12} \\ 0 & aa_{22} \end{bmatrix}.$$

Then we see that aA is a matrix from V. We're good!

S2. We have

$$\begin{aligned} a(A + B) = a \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ 0 & a_{22} + b_{22} \end{bmatrix} &= \begin{bmatrix} a(a_{11} + b_{11}) & a(a_{12} + b_{12}) \\ (a)(0) & a(a_{22} + b_{22}) \end{bmatrix} \\ &= \begin{bmatrix} aa_{11} + ab_{11} & aa_{12} + ab_{12} \\ 0 & aa_{22} + ab_{22} \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} aA + aB = \begin{bmatrix} aa_{11} & aa_{12} \\ (a)(0) & aa_{22} \end{bmatrix} + \begin{bmatrix} ab_{11} & ab_{12} \\ (a)(0) & ab_{22} \end{bmatrix} &= \begin{bmatrix} aa_{11} + ab_{11} & aa_{12} + ab_{12} \\ 0 + 0 & aa_{22} + ab_{22} \end{bmatrix} \\ &= \begin{bmatrix} aa_{11} + ab_{11} & aa_{12} + ab_{12} \\ 0 & aa_{22} + ab_{22} \end{bmatrix} \end{aligned}$$

Hence $a(A + B) = aA + aB$. We're good!

S3. We have

$$(a + b)A = \begin{bmatrix} (a + b)a_{11} & (a + b)a_{12} \\ 0 & (a + b)a_{22} \end{bmatrix} = \begin{bmatrix} aa_{11} + ba_{11} & aa_{12} + ba_{12} \\ 0 & aa_{22} + ba_{22} \end{bmatrix}$$

and

$$aA + bA = \begin{bmatrix} aa_{11} & aa_{12} \\ 0 & aa_{22} \end{bmatrix} + \begin{bmatrix} ba_{11} & ba_{12} \\ 0 & ab_{22} \end{bmatrix} = \begin{bmatrix} aa_{11} + ba_{11} & aa_{12} + ba_{12} \\ 0 & aa_{22} + ba_{22} \end{bmatrix}.$$

Hence $(a + b)A = aA + bA$. We're still good!

S4. We have

$$a(bA) = a \begin{bmatrix} ba_{11} & ba_{12} \\ (b)(0) & ba_{22} \end{bmatrix} = \begin{bmatrix} aba_{11} & aba_{12} \\ (a)(0) & aba_{22} \end{bmatrix} = \begin{bmatrix} aba_{11} & aba_{12} \\ 0 & aba_{22} \end{bmatrix}$$

and

$$(ab)A = \begin{bmatrix} aba_{11} & aba_{12} \\ (ab)(0) & aba_{22} \end{bmatrix} = \begin{bmatrix} aba_{11} & aba_{12} \\ 0 & aba_{22} \end{bmatrix}.$$

We see that $a(bA) = (ab)A$. Hey, you know what? We're still good!

S5. Finally, we have

$$1A = \begin{bmatrix} (1)a_{11} & (1)a_{12} \\ (1)(0) & (1)a_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} = A.$$

Hence, we are good!

Conclusion: The set V equipped with the addition and scalar multiplication of 2×2 matrices is a vector space because A1-A5 and S1-S5 are satisfied.

g. Here, it's gonna be pretty quick. Let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Then, after calculations, we get $\det(A) = \det(B) = 0$.

But $A + B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\det(A + B) = -1$. So $A + B$ is not part of V. Axiom A1 is not satisfied!

Conclusion: The set V is not a vector space if we use the usual matrix addition and scalar multiplication as operations.

Section 6.1 — Problem 5a (5 Pts)

From the first equation, we find

$$2\mathbf{x} + \mathbf{y} + (-\mathbf{y}) = \mathbf{u} + (-\mathbf{y}) \iff 2\mathbf{x} = \mathbf{u} - \mathbf{y} \iff \mathbf{x} = \tfrac{1}{2}\mathbf{u} - \tfrac{1}{2}\mathbf{y}.$$

Plugging $\mathbf{x}$ in the second equation, we find that

$$5(\tfrac{1}{2}\mathbf{u} - \tfrac{1}{2}\mathbf{y}) + 3\mathbf{y} = \mathbf{v} \iff \tfrac{5}{2}\mathbf{u} - \tfrac{5}{2}\mathbf{y} + 3\mathbf{y} = \mathbf{v} \iff \tfrac{5}{2}\mathbf{u} + \tfrac{1}{2}\mathbf{y} = \mathbf{v}$$

moving $\frac{5}{2}\mathbf{u}$ on the other side, we get

$$\tfrac{1}{2}\mathbf{y} = \mathbf{v} - \tfrac{5}{2}\mathbf{u} \iff \mathbf{y} = 2\mathbf{v} - 5\mathbf{u}.$$

Plugging the expression of $\mathbf{y}$ back in the expression of $\mathbf{x}$, then

$$\mathbf{x} = \tfrac{1}{2}\mathbf{u} - \tfrac{1}{2}(2\mathbf{v} - 5\mathbf{u}) = \tfrac{1}{2}\mathbf{u} - \mathbf{v} + \tfrac{5}{2}\mathbf{u} = 3\mathbf{u} - \mathbf{v}.$$

Section 6.1 — Problem 7b (5 Pts)

We have

$$4(3\mathbf{u} - \mathbf{v} + \mathbf{w}) - 2[(3\mathbf{u} - 2\mathbf{v}) - 3(\mathbf{v} - \mathbf{w})]$$

which is equal to

$$\begin{aligned} 12\mathbf{u} - 4\mathbf{v} + 4\mathbf{w} - 2(3\mathbf{u} - 2\mathbf{v}) + 6(\mathbf{v} - \mathbf{w}) &= 12\mathbf{u} - 4\mathbf{v} + 4\mathbf{w} - 6\mathbf{u} + 4\mathbf{v} + 6\mathbf{v} - 6\mathbf{w} \\ &= 6\mathbf{u} + 6\mathbf{v} - 2\mathbf{w}. \end{aligned}$$

and

$$6(\mathbf{w} - \mathbf{u} - \mathbf{v}) = 6\mathbf{w} - 6\mathbf{u} - 6\mathbf{v}.$$

Hence, the answer is $4\mathbf{w}$.