Math 311

Chapter 3

Section 3.2: Determinants and Matrix Inverses

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Created by: Pierre-Olivier Parisé

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Example 1. Show that for any number a, b, c, d, we have the following identity

$$(a^2+b^2)(c^2+d^2) = (ac-bd)^2 + (ad+bc)^2.$$

Solution. Checked with <u>Python</u>!

Set $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ and $B = \begin{bmatrix} c & d \\ -d & c \end{bmatrix}$.

Then,

$$AB = \begin{bmatrix} ac-bd & ad+bc \\ -bc-ad & -bd+ac \end{bmatrix}$$

$$\Rightarrow \det AB = (ac-bd)(-bd+ac) - (ad+bc)(-bc-ad)$$
$$= (ac-bd)^2 + (ad+bc)(bc+ad)$$
$$= (ac-bd)^2 + (ad+bc)^2$$
$$= (a^2+b^2)(c^2+d^2) \quad \text{(From the identity)}$$
$$= \det A \det B.$$

Theorem 1. If A and B are $n \times n$ matrices, then

$$\det(AB) = \det(A)\det(B).$$

Facts:

- For three matrices, $\det(ABC) = \det(A)\det(B)\det(C)$.
- For n matrices,

$$\det(A_1 A_2 \cdots A_n) = \det(A_1)\det(A_2)\cdots\det(A_n).$$

- For powers of a matrix, $\det(A^k) = (\det(A))^k$ (here, $k \geq 1$).

Example 2. Assume that $\det(A) = 2$, $\det(B) = 3$, and $\det(C) = -2$. Compute

$$D = \det(A^2BCBC^2).$$

Solution.

$$D = \det(A^2)\det(B)\det(C)\det(B)\det(C^2)$$

$$= (\det A)^2 \det B \det C \det B (\det C)^2$$

$$= (2^2)(3)(-2)(3)(-2)^2$$

$$= 4 \cdot 9(-8)$$

$$= \boxed{-288}$$

Recall that

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ is invertible } \iff \det(A) = ad - bc \neq 0.$$

Theorem 2. Let A be an $n \times n$ matrix. The matrix A is invertible if and only if $\det(A) \neq 0$. In this case, we have $\det(A^{-1}) = \frac{1}{\det(A)}$.

$AA^{-1} = I \Rightarrow \det(A)\det(A^{-1}) = 1$

Proof. See page 156 in the textbook for the complete proof.

Example 3. For which real value(s) of c is the matrix $A = \begin{bmatrix} 0 & c & -c \\ -1 & 2 & 1 \\ c & -c & c \end{bmatrix}$ invertible?

Solution.

We have

$$\det A = -c \begin{vmatrix} -1 & 1 \\ c & c \end{vmatrix} + (-c) \begin{vmatrix} -1 & 2 \\ c & -c \end{vmatrix}$$

$$= -c(-2c) - c(-c)$$

$$= 3c^2$$

A is invertible

$\iff \det A \neq 0$

$\iff 3c^2 \neq 0$

$\iff \boxed{c \neq 0}$

EXAMPLE 4. Let $A = \begin{bmatrix} 5 & 1 & 3 \\ -1 & 2 & 3 \\ 1 & 4 & 8 \end{bmatrix}$. Find $\det(A)$ and $\det(A^\top)$ and compare their values.

SOLUTION.

$$\det A = (5)\begin{vmatrix} 2 & 3 \\ 4 & 8 \end{vmatrix} - (1)\begin{vmatrix} -1 & 3 \\ 1 & 8 \end{vmatrix} + (3)\begin{vmatrix} -1 & 2 \\ 1 & 4 \end{vmatrix}$$

$$= 13$$

$$\det A^T = \begin{vmatrix} 5 & -1 & 1 \\ 1 & 2 & 4 \\ 3 & 3 & 8 \end{vmatrix}$$

$$= (5)\begin{vmatrix} 2 & 4 \\ 3 & 8 \end{vmatrix} - (1)\begin{vmatrix} -1 & 1 \\ 3 & 8 \end{vmatrix} + (3)\begin{vmatrix} -1 & 1 \\ 2 & 4 \end{vmatrix}$$

$$= 13$$

THEOREM 3. If A is an $n \times n$ matrix, then $\det(A) = \det(A^\top)$.

Example 5. Assume that $\det(A) = 2$ and $\det(B) = 4$. Find the value of $\det(AA^T(B^T)^2)$. $= D$

Solution.

$$D = \det(A)\det(A^T)\det((B^T)^2)$$

$$= \det A \det A \,(\det B^T)^2$$

$$= \det A \det A \,(\det B)^2$$

$$= (2)(2)(4)^2 = \boxed{64}$$

Example 6. A square matrix is called **orthogonal** if $A^{-1} = A^{\top}$. What are the possible values of $\det(A)$ if A is orthogonal?

Solution. Assume that A is orthogonal.

$\Rightarrow A^{-1} = A^T$.

So, $\det(AA^{-1}) = \det(I) = 1$

$\Rightarrow \det(A)\det(A^{-1}) = 1$

$\Rightarrow \det(A)\det(A^T) = 1$

$\Rightarrow \det(A)\det(A) = 1 \Rightarrow (\det A)^2 = 1$

$\Rightarrow \det A = \pm 1$. $\square$