

# MATH 311

## CHAPTER 1

### SECTION 1.2: GAUSSIAN ELIMINATION

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# MATRIX AND AUGMENTED MATRIX

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The system

$$x + y + z = 1$$

$$2x + 2y + z = 3$$

can be put in **matrix form** (array of numbers):

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 3 \end{array} \right]$$

## DEFINITION 1.

- The matrix  $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix}$  is called the **coefficient matrix** of the system.
- The matrix  $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$  is called the **constant matrix** of the system.

Solving a system of linear equations requires to transform the coefficient matrix in the following form:

$$\begin{bmatrix} 1 & * & * \\ 0 & 1 & * \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & * & * \\ 0 & 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \end{bmatrix}$$

where  $*$  denotes any real number.

# ROW-ECHELON FORM OF A MATRIX

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**DEFINITION 2.** A matrix is said to be in **row-echelon form** (REF for short) if it satisfies the following three conditions:

1. All **zero rows** are at the bottom.
2. The first nonzero entry from the left in each nonzero row is 1, called the **leading 1** for that row.
3. Each leading 1 is to the right of all leading 1s in the rows above it

**DEFINITION 3.** A row-echelon matrix is said to be in **reduced row-echelon form** (RREF for short) if, in addition, it satisfies the following condition:

4. Each leading 1 is the only nonzero entry in its column.

**EXAMPLE 1.** Circle the matrices in REF. Draw a square around the matrices in RREF.

$$\begin{bmatrix} 1 & * & * \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 0 & 3 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

**THEOREM 1.** Every matrix can be transformed, with row operations, to a REF (or RREF).

**ALGORITHM 4. Gaussian Elimination.** To solve a system of linear equations proceed as follows.

1. Carry the augmented matrix to a reduced row-echelon form using elementary row operations.
2. If a row  $[0 \ 0 \ \dots \ 0 \ 1]$  occurs, the system is inconsistent.
3. Otherwise, find the parametric form of the solution set to the system of equations.

**EXAMPLE 2.** Solve the following system:

$$3x + y - 4z = -1$$

$$x + 10z = 5$$

$$4x + y + 6z = 1$$

**SOLUTION.**



**EXAMPLE 3.** Solve the following equation:

$$\begin{aligned}x_1 - 2x_2 - x_3 + 3x_4 &= 1 \\2x_1 - 4x_2 + x_3 &= 5 \\x_1 - 2x_2 + 2x_3 - 3x_4 &= 4\end{aligned}$$

**SOLUTION.**

Note: The variable corresponding to the leading ones are called **leading variables**.

## RANK

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It can be proved that the number  $r$  of leading 1s must be the same in each of the row-echelon matrices.

**DEFINITION 5.** The **rank** of a matrix  $A$  is the number of leading 1s in any REF to which  $A$  can be carried by row operations.

**EXAMPLE 4.** Compute the rank of

$$A = \begin{bmatrix} 1 & 1 & -1 & 4 \\ 2 & 1 & 3 & 0 \\ 0 & 1 & -5 & 8 \end{bmatrix}$$

**SOLUTION.**



**THEOREM 2.** Suppose a system of  $m$  equations in  $n$  variables is consistent, and that the rank of the coefficient matrix is  $r$ .

1. The set of solutions involves  $n - r$  parameters.
2. If  $r < n$ , the system has infinitely many solutions.
3. If  $r = n$ , the system has a unique solution.

Three situations occur:

- No solution.
- Unique solution.
- Infinitely many solutions.

**EXAMPLE 5.** A system of equation with  $m = 4$  linear equations and  $n = 5$  variables has been carried to the following REF by row operations:

$$\left[ \begin{array}{ccccc|c} 1 & 2 & 1 & 3 & 1 & 1 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

(a) Find the rank of the coefficient matrix. (b) Is there no solution, unique solution, or infinitely many solutions?

**SOLUTION.**