Math 311

Chapter 9

Section 9.1: The Matrix of a Linear Transformation

Contents

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Coordinate Vector

Let V be a vector space with $\dim V = n$ and $\mathbf{v} \in V$.

Given a basis $B = \{\mathbf{b_1}, \mathbf{b_2}, \ldots, \mathbf{v_n}\}$ of V, recall that $C_B : V \to \mathbb{R}^n$ is given by

$$C_B(\mathbf{v}) = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

Example 1. Let $\mathbf{x} = (2, 1, 3)$ and

$$B = \{(1, 0, 1), (1, 1, 0), (0, 1, 1)\}$$

be a basis of $\mathbb{R}^3$. Find $C_B(\mathbf{x})$.

Solution.

Matrix of a Linear transformation

Suppose we have the transformation

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+z \\ 2z \\ y-z \\ x+2y \end{bmatrix}.$$

Notice that, if we apply T to the standard basis of $\mathbb{R}^3$, we get

$$T\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \mathbf{a_1}, \quad T\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} = \mathbf{a_2}, \quad T\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} = \mathbf{a_3}.$$

Then, setting

$$A = \begin{bmatrix} \mathbf{a_1} & \mathbf{a_2} & \mathbf{a_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix} \quad \Rightarrow \quad T(\mathbf{x}) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

The matrix A is called the **matrix representation of the linear transformation** in term of the standard basis of $\mathbb{R}^3$ and $\mathbb{R}^4$.

What if we change basis?

Example 2. Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation defined by

$$T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+z \\ 2z \\ y-z \\ x+2y \end{bmatrix}.$$

We assume we have two basis:

- a basis $B = \{ \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^\top, \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^\top, \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^\top \}$ of $\mathbb{R}^3$.
- a basis $D = \{ \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}^\top, \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}^\top, \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}^\top, \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}^\top \}$ of $\mathbb{R}^4$.

Find a matrix representing T on these basis.

Solution.

General Procedure

To find the **matrix representation** of $T : \mathbb{R}^n \to \mathbb{R}^m$ on a basis B of $\mathbb{R}^n$ and on a basis D of $\mathbb{R}^m$, we follow these steps:

① Evaluate $\mathbf{t_1} = T(\mathbf{b_1})$, $\mathbf{t_2} = T(\mathbf{b_2})$, ..., $\mathbf{t_n} = T(\mathbf{b_n})$.

② Find $C_D(\mathbf{t_1})$, $C_D(\mathbf{t_2})$, ..., $C_D(\mathbf{t_n})$.

③ Set the $m \times n$ matrix

$$A = \left[C_D(\mathbf{t_1}) \;\; C_D(\mathbf{t_2}) \;\; \cdots \;\; C_D(\mathbf{t_n})\right].$$

④ Then we have, for any $\mathbf{x} \in \mathbb{R}^n$,

$$C_D T(\mathbf{x}) = T_A C_B(\mathbf{x}) = A C_B(\mathbf{x}).$$