Math 311

Chapter 2

Section 2.4: Matrix Inverses

Contents

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Numbers: To solve the equation $2x + 1 = 0$:

$$2x + 1 = 0 \iff 2x = -1 \iff \frac{2x}{2} = -\frac{1}{2} \iff x = -\frac{1}{2}.$$

The number $2^{-1} = \frac{1}{2}$ is called the **inverse** of 2 because $2(2^{-1}) = 1$.

Definition 1. If A is a square matrix, a matrix B is called an **inverse** of A if and only if

$$AB = I \quad \text{and} \quad BA = I.$$

If A has an inverse, then A is called an **invertible matrix**.

Example 1. Show that $B = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$ is an inverse of $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$.

Solution.

$$AB = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \checkmark$$

and

$$BA = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \checkmark$$

EXAMPLE 2. Show that $A = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix}$ has no inverse.

SOLUTION. Assume $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an inverse of A.

$\Rightarrow AB = I_2 \Rightarrow \begin{bmatrix} c & d \\ 3c & 3d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow c=1,\ d=0,\ 3c=0,\ 3d=1$

$\Rightarrow c=1$ and $c=0$ not possible!

$\Rightarrow A$ has no inverse!

Note: There are non-zero matrices that do not have an inverse!

THEOREM 1. If B and C are both inverses of a matrix A, then $B = C$.

PROOF. Since B and C are both inverses of A, we have $AC = I = CA$ and $AB = I = BA$. Therefore,

$$B = IB = (CA)B = C(AB) = CI = C. \qquad \square$$

Note:

- The last result tells us that when A has an inverse, it is unique (there is only one inverse).
- So, we denote the inverse of A by A^{-1}. (2^{-1})
- If B satisfies $AB = I$ and $BA = I$, then $B = A^{-1}$ (Inverse Criterion).

Inverses of 2×2 matrices

EXAMPLE 3. Find the inverse of $A = \begin{bmatrix} 5 & -3 \\ 7 & 4 \end{bmatrix}$.

SOLUTION. Let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then

$$AB = I \iff \begin{bmatrix} 5a-3c & 5b-3d \\ 7a+4c & 7b+4d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\iff \begin{cases} 5a-3c=1 & 5b-3d=0 \\ 7a+4c=0 & 7b+4d=1 \end{cases}$$

$$\iff \left[\begin{array}{cccc|c} 5 & 0 & -3 & 0 & 1 \\ 0 & 5 & 0 & -3 & 0 \\ 7 & 0 & 4 & 0 & 0 \\ 0 & 7 & 0 & 4 & 1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 4/41 \\ 0 & 1 & 0 & 0 & 3/41 \\ 0 & 0 & 1 & 0 & -7/41 \\ 0 & 0 & 0 & 1 & 5/41 \end{array}\right]$$

$$\Rightarrow B = \begin{bmatrix} 4/41 & 3/41 \\ -7/41 & 5/41 \end{bmatrix} = \frac{1}{41}\begin{bmatrix} 4 & 3 \\ -7 & 5 \end{bmatrix}. \quad (B = A^{-1}).$$

In General:

If $ad-bc \neq 0$, then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Recall: A system of linear equations can be written in matrix form

$$A\mathbf{x} = \mathbf{b}.$$

THEOREM 2. If the $n \times n$ matrix A is invertible, then the system has the unique solution

$$\mathbf{x} = A^{-1}\mathbf{b}.$$

PROOF. Start from

$$A\mathbf{x} = \mathbf{b} \iff A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{b} \iff (A^{-1}A)\mathbf{x} = A^{-1}\mathbf{b}.$$

We know that $A^{-1}A = I$. Hence $I\mathbf{x} = A^{-1}\mathbf{b}$. $\square$

EXAMPLE 4. Solve the system $\begin{cases} 5x_1 - 3x_2 = -4 \\ 7x_1 + 4x_2 = 8 \end{cases}$.

SOLUTION. We have

$$\begin{bmatrix} 5 & -3 \\ 7 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4 \\ 8 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 & -3 \\ 7 & 4 \end{bmatrix}^{-1} \begin{bmatrix} -4 \\ 8 \end{bmatrix} = \frac{1}{41} \begin{bmatrix} 4 & 3 \\ -7 & 5 \end{bmatrix} \begin{bmatrix} -4 \\ 8 \end{bmatrix}$$

$$\Rightarrow \vec{x} = \begin{bmatrix} 8/41 \\ 68/41 \end{bmatrix}$$

ALGORITHM 1. If A is an invertible (square) matrix, there exists a sequence of elementary row operations that

- carry A to the identity matrix I;
- carry I to the inverse A^{-1}.

Using block matrices, the algorithm can be rewritten as followed:

$$[\,A \quad I\,] \longrightarrow \cdots \longrightarrow [\,I \quad A^{-1}\,].$$

EXAMPLE 5. Find the inverse of the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 2 & 0 \\ -1 & -1 & 0 \end{bmatrix}$.

SOLUTION.

$$[A \quad I] = \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 3 & 2 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 2 & 3 & -3 & 1 & 0 \\ 0 & -1 & -1 & 1 & 0 & 1 \end{array}\right] \begin{array}{l} R_2 - 3R_1 \\ R_3 + R_1 \end{array}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 2 & 3 & -3 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}\right] 2E_3 + E_2$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & 2 & 0 & 0 & -2 & -6 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}\right] \begin{array}{l} E_3+E_1 \\ E_2-3E_3 \\ \\ \end{array}$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & -1 & -3 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}\right] \ \tfrac{1}{2}E_2$$

$$\Rightarrow \boxed{A^{-1} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & -1 & -3 \\ -1 & 1 & 2 \end{bmatrix}}$$

Note: If A is an $n \times n$ matrix, either

- A can be reduced to I and then the algorithm produces A^{-1};
- or A can't be reduced to I and then A^{-1} does not exist.

Properties of the Inverse

Theorem 3. All the matrices in this statement are square matrices of the same size.

1. I is invertible and $I^{-1} = I$.
2. If A is invertible, then A^{-1} is invertible and $(A^{-1})^{-1} = A$.
3. If A and B are invertible, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.
4. If A is invertible and $a \neq 0$ is a number, then aA is invertible and $(aA)^{-1} = \frac{1}{a}A^{-1}$.
5. If A is invertible, then $A^{\top}$ is invertible and $(A^{\top})^{-1} = (A^{-1})^{\top}$.
   A invertible
6. If $AB = AC$, then $B = C$ (left cancellation law).
   A invertible
7. If $BA = CA$, then $B = C$ (right cancellation law).

Warning!

- The statement "If A and B are both invertible, then $A + B$ is invertible" is not true.
- Cross cancelling is wrong. This means "If $AB = CA$, then $B = C$" is a false statement.

EXAMPLE 6. Find A if $\left(\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix} A\right)^{-1} = \begin{bmatrix}1 & 0\\ 2 & 2\end{bmatrix}$.

SOLUTION.

$$\Leftrightarrow \quad A^{-1}\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}^{-1} = \begin{bmatrix}1 & 0\\ 2 & 2\end{bmatrix}$$

$$\Leftrightarrow \quad AA^{-1}\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}^{-1} = A\begin{bmatrix}1 & 0\\ 2 & 2\end{bmatrix}$$

$$\Leftrightarrow \quad \begin{bmatrix}1 & 0\\ -2 & 1\end{bmatrix} = A\begin{bmatrix}1 & 0\\ 2 & 2\end{bmatrix}$$

$$\Leftrightarrow \quad \begin{bmatrix}1 & 0\\ -2 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 2 & 2\end{bmatrix}^{-1} = A$$

$$\begin{bmatrix}1 & 0\\ -2 & 1\end{bmatrix}\frac{1}{2}\begin{bmatrix}2 & 0\\ -2 & 1\end{bmatrix} = A$$

$$\Leftrightarrow \quad A = \begin{bmatrix}1 & 0\\ -5/2 & 1/2\end{bmatrix}.$$

EXAMPLE 7. If A, B, and C are $n \times n$ invertible matrices, show that ABC is invertible with $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

SOLUTION.

① $ABC(C^{-1}B^{-1}A^{-1}) = AB(CC^{-1})B^{-1}A^{-1} = ABIB^{-1}A^{-1}$
$= A(BB^{-1})A^{-1} = AA^{-1} = I$ ✓

② $C^{-1}B^{-1}A^{-1}(ABC) = C^{-1}B^{-1}(A^{-1}A)BC$
$= C^{-1}(B^{-1}B)C = C^{-1}C = I.$ ✓ □

<u>Note:</u>

- If $A_1, A_2, \ldots, A_k$ are invertible, then $(A_1A_2\cdots A_k)^{-1} = A_k^{-1}\cdots A_2^{-1}A_1^{-1}$.
- If A is invertible and $k \geq 0$, then $(A^k)^{-1} = (A^{-1})^k$.