Solutions MT03 P.-O. Parisé

M311 Spring 2024

Q1 | Q2 | Q3 | Q4 | Q5

Question 1

(a) $\det(\lambda I - A) = \det\begin{bmatrix} \lambda-2 & 4 \\ 1 & \lambda+1 \end{bmatrix} = (\lambda-2)(\lambda+1)-4$

$\Rightarrow \quad \det(\lambda I - A) = \lambda^2 - \lambda - 6 = (\lambda-3)(\lambda+2)$

Hence $\lambda_1 = -2, \ \lambda_2 = 3.$

(b) $\underline{\lambda_1 = -2}$

$((-2)I - A)\vec{x} = \vec{0} \Leftrightarrow \begin{bmatrix} -4 & 4 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Leftrightarrow \begin{array}{l} -4x + 4y = 0 \\ x - y = 0 \end{array} \Leftrightarrow \quad x = y \quad \Rightarrow \vec{x}_1 = x\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$

$\underline{\lambda_2 = 3}$

$(3I - A)\vec{x} = \vec{0} \Leftrightarrow \begin{bmatrix} 1 & 4 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Leftrightarrow \ x + 4y = 0 \ \Leftrightarrow \ x = -4y.$

$\Rightarrow \quad \vec{x}_2 = y\begin{bmatrix} -4 \\ 1 \end{bmatrix}.$

(d) Since $\lambda_1 \neq \lambda_2$, A is diagonalisable. We have

$$\lambda_1 = -2 \longrightarrow \vec{x_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (x=1)$$

$$\lambda_2 = 3 \longrightarrow \vec{x_2} = \begin{bmatrix} -4 \\ 1 \end{bmatrix} \quad (y=1)$$

Thus,

$$P = \begin{bmatrix} 1 & -4 \\ 1 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Question 2.

$$2(\vec{x}-\vec{y}) + 4(\vec{z}-\vec{y}) + 4(\vec{w}-\vec{z}) + (\vec{x}-4\vec{w})$$

$$= 2\vec{x} - 2\vec{y} + \cancel{4\vec{z}} - 4\vec{y} + \cancel{4\vec{w}} - \cancel{4\vec{z}} + \vec{x} - \cancel{4\vec{w}}$$

$$= \boxed{3\vec{x} - 6\vec{y}}$$

Question 3

(a) S1 $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in U.$$

S2. Let $A, B \in U$. Then

$$(A+B)^T = A^T + B^T = -A + (-B) = -(A+B)$$

$$\Rightarrow \quad A+B \in U.$$

S3. Let $A \in U$ and $a \in \mathbb{R}$. Then

$$(aA)^T = a A^T = a(-A) = -(aA)$$

$$\Rightarrow \quad aA \in U.$$

So, U is a subspace. $\left(\text{Note } U = \left\{ \begin{bmatrix} 0 & x \\ -x & 0 \end{bmatrix} : x \in \mathbb{R} \right\} \right)$

(b) S1 fails.

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Question 4

(a) Notice that $c_{A^2}(x) = \det(xI - A^2)$

$$\Rightarrow c_{A^2}(x^2) = \det(x^2 I - A^2)$$

$$= \det((xI - A)(xI + A))$$

$$= \det(xI - A) \det(xI + A)$$

$$= \det(xI - A)\,(-1)^3 \det(-xI - A)$$

$$= -c_A(x)\, c_A(-x)$$

(b) Eigen means "characteristic" or "clean".

Question 5

(a) False. $0 \notin U$ because $0(0) = 0 \neq 1$.

(b) False. Need a non trivial solution $\vec{x}$.

(c) True. $\lambda = 0$ is an eigenvalue of A

$$\Rightarrow \det(\lambda I - A) = 0 \Rightarrow \det(-A) = 0.$$

(d) True. Result in lecture notes.

(e) True.

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$