Math 311

Chapter 2

Section 2.5: Elementary Matrices

Contents

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Example 1. Let $E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$, and $E_3 = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$. Let $A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$.

a) Find $E_1 A$ and interprete the result.

b) Find $E_2 A$ and interprete the result.

c) Find $E_3 A$ and interprete the result.

Solution.

(a) $E_1 A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$

(b) $E_2 A = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 9\cdot 1 & 9\cdot 2 & 9\cdot 3 \end{bmatrix}$

(c) $E_3 A = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

$= \begin{bmatrix} 3+5\cdot 1 & 2+5\cdot 2 & 1+5\cdot 3 \\ 1 & 2 & 3 \end{bmatrix}$

R_1 was replaced by $R_1 + 5R_2$.

Definition 1. An $n \times n$ matrix E is called an **elementary matrix** if it can be obtained from the identity matrix I_n by a single elementary row operation. We say that E is of type I, II, or III if the operation used to obtain E is of that type.

Theorem 1.

1. If an elementary row operation is performed on an $m \times n$ matrix A, then the result is EA, where E is the associated elementary matrix.

2. Every elementary matrix E is invertible, and E^{-1} correspond to the inverse of the row operation that produces E.

Reminder:

Type	Operation	Inverse Operation
I	Interchange rows p and q	Interchange rows p and q
II	Multiply row p by $k \neq 0$	Multiply row p by $1/k$, $k \neq 0$
III	Add k times row p to row $q \neq p$	Subtract k times row p from row q, $q \neq p$

Example 2. For each of the following matrices, describe the corresponding row operation and write the inverse.

$$E_1 = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

1) $R_1 + 3R_3$ ✓

2) Interchange R_1 & R_2

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_2^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

EXAMPLE 3. By recording each row operation as an elementary matrix, show that the invertible matrix $A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$ is a product of elementary matrices.

SOLUTION.

$$[A \mid I] = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right]_{R_2 - 2R_1}$$

$$\text{Set } E_1 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \Rightarrow E_1 A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = B_1$$

$$\text{Set } E_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow E_2 B_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = B_2$$

$$\text{Set } E_3 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \Rightarrow E_3 B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

Then, $E_3 E_2 E_1 A = I_2$

$$\to E_3^{-1} E_3 E_2 E_1 A = E_3^{-1} I_2 \to E_2 E_1 A = E_3^{-1}$$

$$\to A = E_1^{-1} E_2^{-1} E_3^{-1}$$

THEOREM 2. A square matrix is invertible if and only if it is a product of elementary matrices.

Assume that an $m \times n$ matrix A is carried to a matrix B (written $A \to B$) by a series of k elementary row operations.

Let E_1, E_2, ..., E_k be the corresponding elementary matrices. Then

$$AI_m \to E_1A \to E_2E_1A \to \cdots \to E_kE_{k-1}\cdots E_2E_1A = B.$$

Writing $U = E_kE_{k-1}\cdots E_2E_1$, then U is invertible and $B = UA$.

Definition 2. We say that two matrices A and B are **row-equivalent** if there is an invertible matrix U such that $B = UA$.

Example 4. Express the RREF of the matrix $A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & 0 \end{bmatrix}$ and a product UA, with U a 2×2 invertible matrix.

Solution.

$$\begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & 0 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -4 \end{bmatrix}$$

(1) $R_2 \leftarrow R_2 + 2R_1 \to E_1 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

(2) $R_2 \leftarrow -R_2 \to E_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(3) $R_1 \leftarrow R_1 + R_2 \to E_3 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$$B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -4 \end{bmatrix}$$