

MATH 311

CHAPTER 2

SECTION 2.1: MATRIX ADDITION, SCALAR MULTIPLICATION, AND TRANSPOSITION

CONTENTS

Matrices	2
Matrix Equality	4
Matrix Addition	5
Scalar Multiplication	8
Transposition	10

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MATRICES

DEFINITION 1.

- A **matrix** is an array of numbers and the numbers are called **entries**.
- A matrix has m **rows** and n **columns** and is called an $\mathbf{m} \times \mathbf{n}$ **matrix**. The numbers of rows and columns, $m \times n$, are called the **dimensions** of the matrix.
- A **column matrix**, or n -vector or column vector is an $n \times 1$ matrix.
- A **row matrix**, or row vector, is an $1 \times n$ matrix.
- The **(i,j)-entry** of a matrix is the number lying in row i and column j .

EXAMPLE 1. Below are matrices.

- (a) Identify the 2×3 matrix. (b) Identify the 3×2 matrix.
(c) Identify the column vector and indicate its dimensions.
(d) Identify the row vector and indicate its dimensions.
(e) What is the $(1, 2)$ -entry of the matrix B .

$$A = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 5 & 6 \end{bmatrix}, C = [1 \ 3 \ 4], D = \begin{bmatrix} 1 & 4 \\ 4 & 10 \\ -3 & -1 \end{bmatrix}.$$

Notations and Conventions

The general notations for an $m \times n$ matrix A :

- $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$

- For example, a generic 3×5 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \end{bmatrix}$$

- A shortcut: $A = [a_{ij}]$.
- Another shortcut: $A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_n]$, where $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ represents the columns of the matrix A .

Here are conventions to keep in mind:

- If a matrix has size $m \times n$, it has m rows and n columns.
- If we speak of the (i, j) -entry of a matrix, it lies in row i and column j .
- If an entry is denoted by a_{ij} , the first subscript i refers to the row and the second subscript j to the column in which the number a_{ij} lies.

MATRIX EQUALITY

DEFINITION 2. Two matrices A and B are **equal** (denoted as $A = B$) if the following conditions are met:

- They have the same size.
- Corresponding entries are equal.

EXAMPLE 2. Let a be a real number and

$$A = \begin{bmatrix} a^2 & (a-1)^2 \\ 2(a-2) & a^2 - 3a + 2 \end{bmatrix}$$

and

$$B = \begin{bmatrix} a^2 & a^2 - 2a + 1 \\ 2a - 4 & (a-2)(a-1) \end{bmatrix}.$$

and

$$C = \begin{bmatrix} a^2 & (a-1)^2 \\ 2a - 5 & (a-2)(a-1) \end{bmatrix}$$

(a) Do we have $A = B$? (b) Do we have $A = C$?

SOLUTION.

MATRIX ADDITION

DEFINITION 3. If A and B are matrices of the same size, their **sum** $A + B$ is the matrix formed by adding corresponding entries.

EXAMPLE 3. If $A = \begin{bmatrix} -2 & 3 & 2 \\ 3 & 4 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 6 \end{bmatrix}$, then

$$A + B = \begin{bmatrix} -2 + 1 & 3 + 1 & 2 + (-1) \\ 3 + 2 & 4 + 0 & (-1) + 6 \end{bmatrix} = \begin{bmatrix} -1 & 4 & 1 \\ 5 & 4 & 5 \end{bmatrix}.$$

EXAMPLE 4. Find the values of a , b , and c if

$$\begin{bmatrix} a & b & c \end{bmatrix} + \begin{bmatrix} c & a & b \end{bmatrix} = \begin{bmatrix} 3 & 2 & -1 \end{bmatrix}.$$

SOLUTION.

THEOREM 1. Let A , B , and C be arbitrary $m \times n$ matrices. Then the following holds:

1. $A + B = B + A$ (commutative law).
2. $A + (B + C) = (A + B) + C$ (associative law).

PROOF. We will prove property 1. Let $A = [a_{ij}]$ and $B = [b_{ij}]$. Then

$$A + B = [a_{ij} + b_{ij}] = [b_{ij} + a_{ij}] = B + A. \quad \square$$

DEFINITION 4. Let A and B be two $m \times n$ matrix.

- **Zero matrix:** The $m \times n$ matrix O in which every entry is zero.
- **Negative:** The $m \times n$ matrix $-A$ in which every entry is obtained by multiplying entries of A by -1 .
- **Difference:** It is defined by $A - B = A + (-B)$.

EXAMPLE 5. Let $A = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ -2 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$. Compute (a) $-A$; (b) $A + B - C$.

SOLUTION.

THEOREM 2. For any $m \times n$ matrix A :

1. $O + A = A$.
2. $A + (-A) = O$.

EXAMPLE 6. Find the entries of the matrix X if it satisfies the following equation:

$$\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} + X = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}.$$

SOLUTION.

SCALAR MULTIPLICATION

DEFINITION 5. If k is a number and A a matrix, then the **scalar multiple** kA is the matrix obtained by multiplying each entry of A by k .

Note: the number k is called a **scalar**.

EXAMPLE 7. If $A = \begin{bmatrix} 3 & -1 & 4 \\ 2 & 0 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 2 \end{bmatrix}$.
Compute $3A - 2B$.

SOLUTION.

EXAMPLE 8. Show that if $kA = O$, then $k = 0$ or $A = O$.

SOLUTION.

THEOREM 3. Let A and B be two $m \times n$ matrices and k, l be two numbers. Then

1. $k(A + B) = kA + kB$ (distributive law I).
2. $(k + l)A = kA + lA$ (distributive law II).
3. $(kl)A = k(lA)$.
4. $1A = A$ and $(-1)A = -A$.

TRANSPOSITION

DEFINITION 6. If A is an $m \times n$ matrix, the **transpose** of A , written A^T , is the $n \times m$ matrix whose rows are the columns of A in the same order.

Note: Based on the definition, we can write

$$A^T = [a_{ij}]^T = [a_{ji}].$$

EXAMPLE 9. Find the transpose of each of the following matrices.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 2 \\ -1 & 2 & 1 \end{bmatrix}.$$

SOLUTION.

THEOREM 4. Let A and B denote matrices of the same size, and let k denote a scalar.

1. If A is an $m \times n$ matrix, then A^\top is an $n \times m$ matrix.
2. $(A^\top)^\top = A$.
3. $(kA)^\top = kA^\top$.
4. $(A + B)^\top = A^\top + B^\top$.

PROOF. We will prove property 4. Write $A = [a_{ij}]$ and $B = [b_{ij}]$ and $A + B = [c_{ij}]$ with $c_{ij} = a_{ij} + b_{ij}$. Therefore,

$$(A + B)^\top = [c_{ji}] = [a_{ji} + b_{ji}] = [a_{ji}] + [b_{ji}] = A^\top + B^\top. \quad \square$$

EXAMPLE 10. Find the values of the entries of the matrix A if

$$\left(A + 3 \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 4 \end{bmatrix} \right)^\top = \begin{bmatrix} 2 & 1 \\ 0 & 5 \\ 3 & 8 \end{bmatrix}.$$

DEFINITION 7. A matrix A is symmetric if $A = A^\top$.

EXAMPLE 11. Show that if A and B are symmetric $n \times n$ matrices, then $A + B$ is symmetric.

SOLUTION.