Section 2.4 — Problem 1 (4 Pts)

b. We have
$$\begin{bmatrix} 3 & 0 \\ 1 & -4 \end{bmatrix} \left(\tfrac{1}{2}\right) \begin{bmatrix} 4 & 0 \\ 1 & -3 \end{bmatrix} = \left(\tfrac{1}{2}\right) \begin{bmatrix} 12 & 0 \\ 0 & 12 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}.$$
Here, the matrix are not inverse of each other because we don't have $AB = I$. We can stop here and we don't have to calculate BA.

d. We have
$$\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
and
$$\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$
Therefore the two matrices are inverses of each other.

Section 2.4 — Problem 3b (6 Pts)

The system can be put in matrix form:
$$\begin{bmatrix} 2 & -3 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
Using the formula for the inverse of a 2×2 matrix with $a = 2$, $b = -3$, $c = 1$, and $d = -4$, we have
$$\begin{bmatrix} 2 & -3 \\ 1 & -4 \end{bmatrix}^{-1} = -\tfrac{1}{5} \begin{bmatrix} -4 & 3 \\ -1 & 2 \end{bmatrix}$$
Hence the solution is given by
$$\begin{bmatrix} x \\ y \end{bmatrix} = -\tfrac{1}{5} \begin{bmatrix} -4 & 3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -3/5 \\ -2/5 \end{bmatrix}.$$

Section 2.4 — Problem 5 (10 Pts)

d. The inverse does not distribute on the addition nor the substraction. We first take the inverse on each side to get
$$((I - 2A^\top)^{-1})^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} \iff ((I - 2A^\top)^{-1})^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}.$$

The left-hand side becomes simply $I - 2A^\top$ and therefore

$$I - 2A^\top = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \iff \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} = 2A^\top \iff \begin{bmatrix} 0 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} = A^\top.$$

We take the transpose on both side and since $(A^\top)^\top = A$, we get

$$\begin{bmatrix} 0 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}^\top = A \iff A = \begin{bmatrix} 0 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}.$$

g. We apply the same stategy. We start by taking the inverse on each side:

$$((A^\top - 2I)^{-1})^{-1} = \left(2\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\right)^{-1} \iff A^\top - 2I = \tfrac{1}{2}\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}.$$

After multiplying by 1/2 on the right and move the $2I$ on the other side, we get

$$A^\top = \begin{bmatrix} 3/2 & -1/2 \\ -1 & 1/2 \end{bmatrix} + 2I \iff A^\top = \begin{bmatrix} 7/2 & -1/2 \\ -1 & 5/2 \end{bmatrix} \iff A = \begin{bmatrix} 7/2 & -1 \\ -1/2 & 5/2 \end{bmatrix}.$$

Section 2.4 — Problem 9 (4 Pts)

b. This is false. For example, $I - I = O$ is not inversibe, but I is invertible.

c. This is true. If A and B are invertible, then A^{-1} and B are invertible. Therefore, from the properties of inverses, $A^{-1}B$ is invertible. Again, from the properties of the inverse, we know that the conjugate of an invertible matrix will be invertible, hence $(A^{-1}B)^\top$ is invertible.

Section 2.4 — Problem 39a (5 Pts)

Assume that P is idempotent and invertible, but $P \neq I$. We have $P^2 = P$, which can be rewritten as $P^2 - P = 0$. Factoring one P on the left, we get

$$P(P - I) = O \iff P^{-1}P(P - I) = P^{-1}O \iff P - I = O \iff P = I.$$

We get $P \neq I$ and $P = I$. This is a contradiction and the only invertible idempotent is I.

Section 2.5 — Problem 1 (6 Pts)

b. Let R_1, R_2, and R_3 be the rows of an arbitrary 3×3 matrix A. The elementary matrix E interchanges R_1 with R_3. The inverse E^{-1} must therefore undo what E does, so it must interchange R_1 and R_3 again:

$$E^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

d. Let R_1, R_2, and R_3 be the rows of an arbitrary 3 matrix A. The elementary matrix E replace the second row of A by $-2R_1 + R_2$. The inverse E^{-1} must therefore undo what E does, so it must replace the second row by $2R_1 + R_2$. Therefore the inverse of E is

$$E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Section 2.5 — Problem 6b (15 Pts)

The first operation is $R_2 - 5R_1$, so the elementary matrix corresponding to that operation is

$$E_1 = \begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix}.$$

Multiplying by E_1 to the left of A, we get

$$E_1 A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -6 \end{bmatrix}$$

The second operation is $R_1 - R_2$, so the elementary matrix corresponding to that second operation is

$$E_2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

Multiplying by E_2 to the left of $E_1 A$, we get

$$E_2 E_1 A = \begin{bmatrix} 1 & 0 & 7 \\ 0 & 2 & -6 \end{bmatrix}$$

The third operation is $\frac{1}{2} R_2$, so the elementary matrix corresponding to that third operation is

$$E_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}.$$

Multiplying by E_3 to the left of $E_2 E_1 A$, we get

$$E_3 E_2 E_1 A = \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -3 \end{bmatrix}.$$

Hence, we get that

$$U = E_3 E_2 E_1 \quad \text{and} \quad R = \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -3 \end{bmatrix}.$$