Math 311

Last Chapter

Section 5.3: Orthogonality

Contents

Created by: Pierre-Olivier Parisé

Spring 2024

Dot Product

If $\mathbf{x}$ is an $1 \times n$ ~~column~~ row vector and $\mathbf{y}$ is an $n \times 1$ column vector, then recall that

$$\begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1y_1 + x_2y_2 + \cdots + x_ny_n \end{bmatrix}.$$

The result is a 1×1 matrix that we treat as a number.

DEFINITION 1. Let $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \ldots & x_n \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 & y_2 & \ldots & y_n \end{bmatrix}$ be two $1 \times n$ row vectors in $\mathbb{R}^n$. Their **dot product** is defined as followed:

$$\mathbf{x} \cdot \mathbf{y} := \mathbf{x}\mathbf{y}^\top = x_1y_1 + x_2y_2 + \cdots + x_ny_n.$$

EXAMPLE 1. If $\mathbf{x} = \begin{bmatrix} 1 & -1 & -3 & 1 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 2 & 1 & 1 & 0 \end{bmatrix}$. Then

$$\mathbf{x} \cdot \mathbf{y} = (1)(2) + (-1)(1) + (-3)(1) + (1)(0) = -2.$$

Notes:

① We can use other representations of vectors in $\mathbb{R}^n$.

② For instance, if $\mathbf{x}$ and $\mathbf{y}$ are $n \times 1$ column vectors, then

$$\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n = \mathbf{x}^\top \mathbf{y}.$$

Length

Definition 2. Let $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}$. The **length** $\|\mathbf{x}\|$ is defined by

$$\|\mathbf{x}\| := \sqrt{\mathbf{x}\cdot\mathbf{x}} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

Example 2. If $\mathbf{x} = \begin{bmatrix} 1 & 3 & -2 & 0 \end{bmatrix}$, then

$$\|\mathbf{x}\| = \sqrt{(1)^2 + (3)^2 + (-2)^2 + (0)^2} = \sqrt{1+9+4} = \sqrt{14}.$$

Properties:

① $\mathbf{x}\cdot\mathbf{y} = \mathbf{y}\cdot\mathbf{x}$.

② $\mathbf{x}\cdot(\mathbf{y}+\mathbf{z}) = \mathbf{x}\cdot\mathbf{y} + \mathbf{x}\cdot\mathbf{z}$.

③ $(a\mathbf{x})\cdot\mathbf{y} = a(\mathbf{x}\cdot\mathbf{y}) = \mathbf{x}\cdot(a\mathbf{y})$.

④ $\|\mathbf{x}\|^2 = \mathbf{x}\cdot\mathbf{x}$.

⑤ $\|\mathbf{x}\| \geq 0$, and $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$.

⑥ $\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + 2(\mathbf{x}\cdot\mathbf{y}) + \|\mathbf{y}\|^2$.

Proof:

$$\begin{aligned}
\|\vec{x}+\vec{y}\|^2 &\overset{4}{=} (\vec{x}+\vec{y})\cdot(\vec{x}+\vec{y}) \\
&\overset{2}{=} (\vec{x}+\vec{y})\cdot\vec{x} + (\vec{x}+\vec{y})\cdot\vec{y} \\
&\overset{1}{=} \vec{x}\cdot(\vec{x}+\vec{y}) + \vec{y}\cdot(\vec{x}+\vec{y}) \\
&\overset{2}{=} \overset{4}{\vec{x}\cdot\vec{x}} + \vec{x}\cdot\vec{y} + \underbrace{\vec{y}\cdot\vec{x}}_{\vec{x}\cdot\vec{y}\ (1)} + \overset{4}{\vec{y}\cdot\vec{y}}
\end{aligned}$$

Cauchy-Schwarz Inequality

EXAMPLE 3. Let $\mathbf{x} = (a, b)$ and $\mathbf{y} = (c, d)$. Show that

$$|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \|\mathbf{y}\|.$$

SOLUTION.

$$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\| \iff |\vec{x} \cdot \vec{y}|^2 \leq \|\vec{x}\|^2 \|\vec{y}\|^2$$

So,

$$(ac + bd)^2 \leq (a^2 + b^2)(c^2 + d^2)$$

$$\iff \cancel{a^2c^2} + 2acbd + \cancel{b^2d^2} \leq \cancel{a^2c^2} + a^2d^2 + b^2c^2 + \cancel{b^2d^2}$$

$$\iff 2acbd \leq a^2d^2 + b^2c^2$$

$$\iff 0 \leq a^2d^2 - 2adbc + b^2c^2$$

$$\iff 0 \leq (ad - bc)^2 \to \text{this is always true!!!}$$

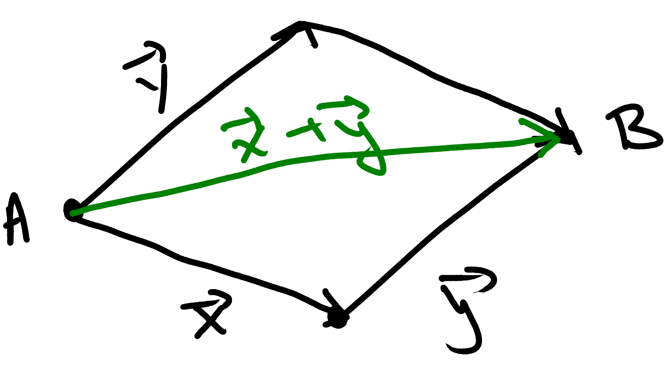
THEOREM 1. If $\mathbf{x}$ and $\mathbf{y}$ are in $\mathbb{R}^n$, then

$$|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \|\mathbf{y}\|.$$

Triangle Inequality

Theorem 2. If $\mathbf{x}$ and $\mathbf{y}$ are in $\mathbb{R}^n$, then $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.

Illustration in $\mathbb{R}^2$.



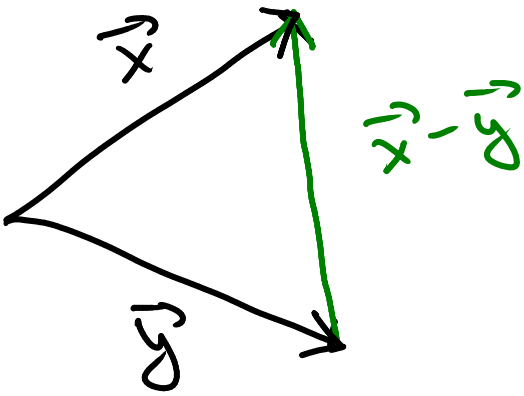
Shortest path from A to B is $\|\vec{x}+\vec{y}\|$

Distance

Definition 3. If $\mathbf{x}$ and $\mathbf{y}$ are two vectors in $\mathbb{R}^n$, the **distance** $d(\mathbf{x}, \mathbf{y})$ is defined by

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|.$$

Illustration in $\mathbb{R}^2$.



distance between $\vec{x}$ and $\vec{y}$ = how far the the tip of $\vec{x}$ is from $\vec{y}$.

ORTHOGONALITY

DEFINITION 4. Two vectors $\mathbf{x}$ and $\mathbf{y}$ are **orthogonal** if

$$\mathbf{x} \cdot \mathbf{y} = 0.$$

If $\mathbf{x}$ and $\mathbf{y}$ are orthogonal, we write $\mathbf{x} \perp \mathbf{y}$.

EXAMPLE 4. Let $\mathbf{x} = \begin{bmatrix} 1 & 1 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 1 & -1 \end{bmatrix}$.

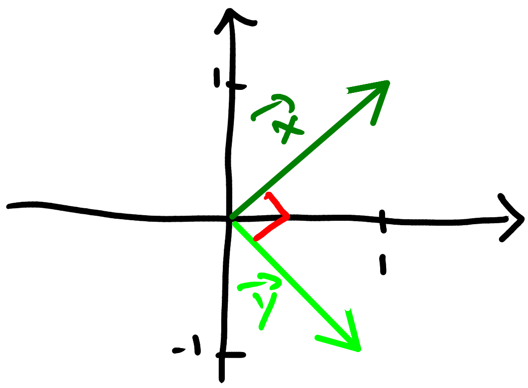
a) Are $\mathbf{x}$, $\mathbf{y}$ orthogonal?

b) If they are orthogonal, then draw the vectors in a coordinates plane and give one special geometric properties.

a) $\vec{x} \cdot \vec{y} = (1)(1) + (1)(-1) = 1 - 1 = 0$

$\Rightarrow \quad \vec{x} \perp \vec{y}$.

b)



the angle between $\vec{x}$ and $\vec{y}$ is 90°.

Notes: In $\mathbb{R}^2$, we can show that

$$\mathbf{x} \cdot \mathbf{y} = \|x\|\|y\| \cos\theta$$

where θ is the angle between the vectors $\mathbf{x}$ and $\mathbf{y}$.

Orthogonal Sets

DEFINITION 5. A collection of vectors $\{\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_k}\}$ is an **orthogonal set** if

① $\mathbf{x_i} \cdot \mathbf{x_j} = 0$ for any $i \neq j$.

② $\mathbf{x_i} \neq 0$ for any i.

EXAMPLE 5. Let

a) $S_1 = \{(0,0,0), (1,2,3), (-1,-1,-1)\}$.

b) $S_2 = \{(1,2,3), (-1,-1,-1), (1,1,1)\}$.

c) $S_3 = \{(3,4,5), (-4,3,0), (-3,-4,5)\}$.

Which one of these sets is an orthogonal set?

SOLUTION.

a) S_1 is not orth. set because $(0,0,0) \in S_1$.

b) $(1,2,3) \cdot (-1,-1,-1) = -6 \neq 0$

$\Rightarrow$ S_2 is not an orth. set.

c) $(3,4,5) \cdot (-4,3,0) = -12 + 12 + 0 = 0$

$(3,4,5) \cdot (-3,-4,5) = -9 - 16 + 25 = 0$

$(-4,3,0) \cdot (-3,-4,5) = 12 - 12 = 0$

S_3 is an orth. set.

Orthonormal Sets

DEFINITION 6. A collection of vectors $\{\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_k}\}$ is an **orthonormal set** if

① it is an orthogonal set.

② $\|\mathbf{x_i}\| = 1$ for every index i.

EXAMPLE 6. The standard basis $\{\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}\}$ is an orthonormal set in $\mathbb{R}^n$.

We can always obtain an orthonormal set from an orthogonal set by **normalizing** the vectors in the orthogonal set.

EXAMPLE 7. Obtain an orthonormal set by normalizing the following orthogonal set:

$$\{(1, -1, 2), (0, 2, 1), (5, 1, -2)\}.$$

SOLUTION. Because $(a\vec{x}) \cdot \vec{y} = a(\vec{x} \cdot \vec{y})$

① $a = \|(1,-1,2)\| = \sqrt{6} \rightarrow \vec{y}_1 = \dfrac{(1,-1,2)}{\|(1,-1,2)\|}$

$\Rightarrow \vec{y}_1 = \left(\dfrac{1}{\sqrt{6}}, \dfrac{-1}{\sqrt{6}}, \dfrac{2}{\sqrt{6}}\right)$

② $\vec{y}_2 = \dfrac{(0,2,1)}{\|(0,2,1)\|} = \left(0, \dfrac{2}{\sqrt{5}}, \dfrac{2}{\sqrt{6}}\right)$

③ $\vec{y}_3 = \dfrac{(5,1,-2)}{\|(5,1,-2)\|} = \left(\dfrac{5}{\sqrt{30}}, \dfrac{1}{\sqrt{30}}, \dfrac{-2}{\sqrt{30}}\right)$

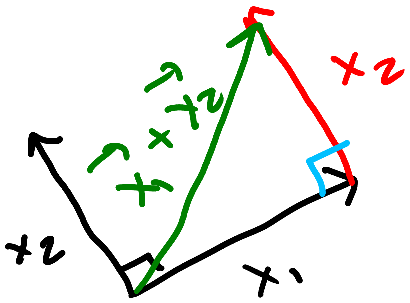
new orthonormal set.

Pythagoras' Theorem

THEOREM 3. If $\{\mathbf{x_1}, \mathbf{x_2}, \dots, \mathbf{x_k}\}$ is an orthogonal set in $\mathbb{R}^n$, then

$$\|\mathbf{x_1} + \mathbf{x_2} + \cdots + \mathbf{x_k}\|^2 = \|\mathbf{x_1}\|^2 + \|\mathbf{x_2}\|^2 + \cdots + \|\mathbf{x_k}\|^2.$$

Illustration in $\mathbb{R}^2$.



Linearly Independent

THEOREM 4. If $S = \{\mathbf{x_1}, \mathbf{x_2}, \dots, \mathbf{x_k}\}$ is an orthogonal set in $\mathbb{R}^n$, then S is linearly independent.

Proof: Let $\vec{x} = a_1\vec{x}_1 + a_2\vec{x}_2 + \cdots + a_k\vec{x}_k = \vec{0}$

Then $\vec{x}_1 \cdot \vec{x} = a_1 \vec{x}_1 \cdot \vec{x}_1 + a_2 \vec{x}_1 \cdot \vec{x}_2 + \cdots + a_k \vec{x}_1 \cdot \vec{x}_k$

$= a_1 \|\vec{x}_1\|^2$

$\Rightarrow 0 = \vec{x}_1 \cdot \vec{x} = a_1 \|\vec{x}_1\|^2$

$\Rightarrow 0 = a_1$. In this way, $a_2 = a_3 = \cdots = a_k = 0$. $\square$

Example 8. Let $U = \text{span}\{(1,-2,3),(-1,1,1)\}$ and $\mathbf{x} = (13,-20,15) \in U$.

a) Show $\{(1,-2,3),(-1,1,1)\}$ is an orthogonal basis of U.

b) Express $\mathbf{x}$ as a linear combination of the basis of U.

Solution.

a) $(1,-2,3)\cdot(-1,1,1) = -1-2+3 = 0$

So, $\{(1,-2,3),(-1,1,1)\}$ is an orthogonal set

$\Rightarrow \{(1,-2,3),(-1,1,1)\}$ is linearly independent.

Because U is spanned by these vectors

$\Rightarrow \{(1,-2,3),(-1,1,1)\}$ is an orthogonal basis for U.

(b) Let $\vec{x} = (13, -20, 15)$

$= a\vec{x}_1 + b\vec{x}_2$

where $\vec{x}_1 = (1, -2, 3)$, $\vec{x}_2 = (-1, 1, 1)$.

First:

$\vec{x} \cdot \vec{x}_1 = a\, \vec{x}_1 \cdot \vec{x}_1 = a \|\vec{x}_1\|^2$

$\Rightarrow \quad a = \dfrac{\vec{x} \cdot \vec{x}_1}{\|\vec{x}_1\|^2}$

$= \dfrac{1}{14}\Big((13, -20, 15) \cdot (1, -2, 3)\Big)$

$= 98/14 = \boxed{7}$

Second:

$\vec{x} \cdot \vec{x}_2 = b\|\vec{x}_2\|^2 \Rightarrow b = \dfrac{\vec{x} \cdot \vec{x}_2}{\|\vec{x}_2\|^2} = \dfrac{-18}{3} = \boxed{-6}$

Theorem 5. Let $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_m}\}$ be an orthogonal basis of a subspace U of $\mathbb{R}^n$. For any $\mathbf{x} \in U$, we have

$$\mathbf{x} = \left(\frac{\mathbf{x} \cdot \mathbf{u_1}}{\|\mathbf{u_1}\|^2}\right)\mathbf{u_1} + \left(\frac{\mathbf{x} \cdot \mathbf{u_2}}{\|\mathbf{u_2}\|^2}\right)\mathbf{u_2} + \cdots + \left(\frac{\mathbf{x} \cdot \mathbf{u_m}}{\|\mathbf{u_m}\|^2}\right)\mathbf{u_m}.$$

Criteria to be in the Span

EXAMPLE 9. Let $U = \text{span}\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_m}\}$ and let $\mathbf{x} \in \mathbb{R}^n$. Show that if $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{x} \perp \mathbf{u_k}$ for each $1 \leq k \leq m$, then $\mathbf{x} \notin U$.

SOLUTION.

By contradiction, assume $\vec{x} \in U$.

This means by Thm 5

$$\vec{x} = \frac{x \cdot \vec{u}_1}{\|\vec{u}_1\|^2} \vec{u}_1 + \frac{x \cdot \vec{u}_2}{\|\vec{u}_2\|^2} u_2 + \cdots + \frac{x \cdot \vec{u}_m}{\|\vec{u}_m\|^2} \vec{u}_m.$$

But, $\vec{x} \cdot \vec{u}_k = 0$, for any k

$$\Rightarrow \vec{x} = \frac{0}{\|\vec{u}_1\|^2} \vec{u}_1 + \frac{0}{\|\vec{u}_2\|^2} \vec{u}_2 + \cdots + \frac{0}{\|\vec{u}_m\|^2} \vec{u}_m$$

$$= 0\vec{u}_1 + 0\vec{u}_2 + \cdots + 0\vec{u}_m$$

$$= \vec{0}.$$

So, a contradiction. Hence $\vec{x} \notin U$. $\square$