MATH 311

LAST CHAPTER

SECTION 5.3: ORTHOGONALITY

CONTENTS

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Dot Product

If $\mathbf{x}$ is an $1 \times n$ column vector and $\mathbf{y}$ is an $n \times 1$ column vector, then recall that

$$\begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1y_1 + x_2y_2 + \cdots + x_ny_n \end{bmatrix}.$$

The result is a 1×1 matrix that we treat as a number.

Definition 1. Let $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \ldots & x_n \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 & y_2 & \ldots & y_n \end{bmatrix}$ be two $1 \times n$ row vectors in $\mathbb{R}^n$. Their **dot product** is defined as followed:

$$\mathbf{x} \cdot \mathbf{y} := \mathbf{x}\mathbf{y}^\top = x_1y_1 + x_2y_2 + \cdots + x_ny_n.$$

Example 1. If $\mathbf{x} = \begin{bmatrix} 1 & -1 & -3 & 1 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 2 & 1 & 1 & 0 \end{bmatrix}$. Then

$$\mathbf{x} \cdot \mathbf{y} = (1)(2) + (-1)(1) + (-3)(1) + (1)(0) = -2.$$

Notes:

① We can use other representations of vectors in $\mathbb{R}^n$.

② For instance, if $\mathbf{x}$ and $\mathbf{y}$ are $n \times 1$ column vectors, then

$$\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n = \mathbf{x}^\top \mathbf{y}.$$

Length

Definition 2. Let $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}$. The **length** $\|\mathbf{x}\|$ is defined by

$$\|\mathbf{x}\| := \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

Example 2. If $\mathbf{x} = \begin{bmatrix} 1 & 3 & -2 & 0 \end{bmatrix}$, then

$$\|\mathbf{x}\| = \sqrt{(1)^2 + (3)^2 + (-2)^2 + (0)^2} = \sqrt{1 + 9 + 4} = \sqrt{14}.$$

Properties:

① $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$.

② $\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) = \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z}$.

③ $(a\mathbf{x}) \cdot \mathbf{y} = a(\mathbf{x} \cdot \mathbf{y}) = \mathbf{x} \cdot (a\mathbf{y})$.

④ $\|\mathbf{x}\|^2 = \mathbf{x} \cdot \mathbf{x}$.

⑤ $\|\mathbf{x}\| \geq 0$, and $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$.

⑥ $\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + 2(\mathbf{x} \cdot \mathbf{y}) + \|\mathbf{y}\|^2$.

Cauchy-Schwarz Inequality

EXAMPLE 3. Let $\mathbf{x} = (a, b)$ and $\mathbf{y} = (c, d)$. Show that

$$|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\|.$$

SOLUTION.

THEOREM 1. If $\mathbf{x}$ and $\mathbf{y}$ are in $\mathbb{R}^n$, then

$$|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\|.$$

Triangle Inequality

THEOREM 2. If $\mathbf{x}$ and $\mathbf{y}$ are in $\mathbb{R}^n$, then $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\|+\|\mathbf{y}\|$.

Illustration in $\mathbb{R}^2$.

Distance

DEFINITION 3. If $\mathbf{x}$ and $\mathbf{y}$ are two vectors in $\mathbb{R}^n$, the **distance** $d(\mathbf{x}, \mathbf{y})$ is defined by

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|.$$

Illustration in $\mathbb{R}^2$.

DEFINITION 4. Two vectors $\mathbf{x}$ and $\mathbf{y}$ are **orthogonal** if

$$\mathbf{x} \cdot \mathbf{y} = 0.$$

If $\mathbf{x}$ and $\mathbf{y}$ are orthogonal, we write $\mathbf{x} \perp \mathbf{y}$.

EXAMPLE 4. Let $\mathbf{x} = \begin{bmatrix} 1 & 1 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 1 & -1 \end{bmatrix}$.

a) Are $\mathbf{x}$, $\mathbf{y}$ orthogonal?

b) If they are orthogonal, then draw the vectors in a coordinates plane and give one special geometric properties.

Notes: In $\mathbb{R}^2$, we can show that

$$\mathbf{x} \cdot \mathbf{y} = \|x\| \|y\| \cos \theta$$

where θ is the angle between the vectors $\mathbf{x}$ and $\mathbf{y}$.

Orthogonal Sets

DEFINITION 5. A collection of vectors $\{\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_k}\}$ is an **orthogonal set** if

① $\mathbf{x_i} \cdot \mathbf{x_j} = 0$ for any $i \neq j$.

② $\mathbf{x_i} \neq 0$ for any i.

EXAMPLE 5. Let

a) $S_1 = \{(0,0,0), (1,2,3), (-1,-1,-1)\}$.

b) $S_2 = \{(1,2,3), (-1,-1,-1), (1,1,1)\}$.

c) $S_3 = \{(3,4,5), (-4,3,0), (-3,-4,5)\}$.

Which one of these sets is an orthogonal set?

SOLUTION.

Orthonormal Sets

DEFINITION 6. A collection of vectors $\{\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_k}\}$ is an **orthonormal set** if

① it is an orthogonal set.

② $\|\mathbf{x_i}\| = 1$ for every index i.

EXAMPLE 6. The standard basis $\{\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}\}$ is an orthonormal set in $\mathbb{R}^n$.

We can always obtain an orthonormal set from an orthogonal set by **normalizing** the vectors in the orthogonal set.

EXAMPLE 7. Obtain an orthonormal set by normalizing the following orthogonal set:

$$\{(1, -1, 2), (0, 2, 1), (5, 1, -2)\}.$$

SOLUTION.

Pythagoras' Theorem

THEOREM 3. If $\{\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_k}\}$ is an orthogonal set in $\mathbb{R}^n$, then

$$\|\mathbf{x_1} + \mathbf{x_2} + \cdots + \mathbf{x_k}\|^2 = \|\mathbf{x_1}\|^2 + \|\mathbf{x_2}\|^2 + \cdots + \|\mathbf{x_k}\|^2.$$

Illustration in $\mathbb{R}^2$.

Linearly Independent

THEOREM 4. If $S = \{\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_k}\}$ is an orthogonal set in $\mathbb{R}^n$, then S is linearly independent.

Fourier Expansion

Example 8. Let $U = \text{span}\{(1,-2,3),(-1,1,1)\}$ and $\mathbf{x} = (13,-20,15) \in U$.

a) Show $\{(1,-2,3),(-1,1,1)\}$ is an orthogonal basis of U.

b) Express $\mathbf{x}$ as a linear combination of the basis of U.

Solution.

Theorem 5. Let $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_m}\}$ be an orthogonal basis of a subspace U of $\mathbb{R}^n$. For any $\mathbf{x} \in U$, we have

$$\mathbf{x} = \Big(\frac{\mathbf{x} \cdot \mathbf{u_1}}{\|\mathbf{u_1}\|^2}\Big)\mathbf{u_1} + \Big(\frac{\mathbf{x} \cdot \mathbf{u_2}}{\|\mathbf{u_2}\|^2}\Big)\mathbf{u_2} + \cdots + \Big(\frac{\mathbf{x} \cdot \mathbf{u_m}}{\|\mathbf{u_m}\|^2}\Big)\mathbf{u_m}.$$

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EXAMPLE 9. Let $U = \text{span}\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_m}\}$ and let $\mathbf{x} \in \mathbb{R}^n$. Show that if $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{x} \perp \mathbf{u_k}$ for each $1 \leq k \leq m$, then $\mathbf{x} \notin U$.

SOLUTION.