General Definition

Let V be a set of objects called **vectors**. Assume

1. **Vector Addition:** Two vectors $\mathbf{v}$ and $\mathbf{w}$ can be added and denote this operation by $\mathbf{v} + \mathbf{w}$.
2. **Scalar Multiplication:** Any vector $\mathbf{v}$ can be multiplied by any number (scalar) a and denote this operation by $a\mathbf{v}$.

The set V is called a **vector space** if

1. Axioms for the vector addition:

   $\boxed{\text{A1.}}$ Closed: $\mathbf{v}, \mathbf{w} \in V \Rightarrow \mathbf{v} + \mathbf{w} \in V$.

   $\boxed{\text{A2.}}$ Commutativity: $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$.

   $\boxed{\text{A3.}}$ Associativity: $\mathbf{v} + (\mathbf{w} + \mathbf{z}) = (\mathbf{v} + \mathbf{w}) + \mathbf{z}$.

   $\boxed{\text{A4.}}$ Existence of a zero vector: $\mathbf{v} + \mathbf{0} = \mathbf{v} = \mathbf{0} + \mathbf{v}$.

   $\boxed{\text{A5.}}$ Existence of a negative: For each $\mathbf{v}$, there is a $\mathbf{w}$ such that $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v} = \mathbf{0}$.

2. Axioms for the scalar multiplication:

   $\boxed{\text{S1.}}$ $\mathbf{v} \in V \Rightarrow a\mathbf{v} \in V$.

   $\boxed{\text{S2.}}$ $a(\mathbf{v} + \mathbf{w}) = a\mathbf{v} + a\mathbf{w}$.

   $\boxed{\text{S3.}}$ $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.

   $\boxed{\text{S4.}}$ $a(b\mathbf{v}) = (ab)\mathbf{v}$.

   $\boxed{\text{S5.}}$ $1\mathbf{v} = \mathbf{v}$.