Math 311

Chapter 6

Section 6.2: Linear Combination and Subspaces

Contents

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Spring 2024

Example 1. The solution to the homogeneous system

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 1 \\ 2 & 3 & 4 & 1 & 2 & -2 \\ 1 & 2 & 4 & 5 & 3 & -1 \\ 3 & 1 & 2 & 4 & 5 & 1 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

is

$$\mathbf{x} = t \begin{bmatrix} 1 \\ 9 \\ -7 \\ 3 \end{bmatrix} + s \begin{bmatrix} 0 \\ 3 \\ -3 \\ 1 \end{bmatrix} = t\mathbf{x_1} + s\mathbf{x_2}, \quad s, t \in \mathbb{R}.$$

Notice that

S1.

S2.

S3.

If $U = \{t\mathbf{x_1} + s\mathbf{x_2} \, : \, s, t \in \mathbb{R}\}$ is the set of all solutions, then U is called a **subspace**.

Definition 1. A subset U of a vector space V is called a **subspace** of V if it satisfies the following properties:

$\boxed{\text{S1.}}$ The zero vector $\mathbf{0} \in U$.

$\boxed{\text{S2.}}$ If $\mathbf{u_1} \in U$ and $\mathbf{u_2} \in U$, then $\mathbf{u_1} + \mathbf{u_2} \in U$.

$\boxed{\text{S3.}}$ If $\mathbf{u} \in U$ and a is a scalar, then $a\mathbf{u} \in U$.

Remarks:

① S2: U is said to be **closed under addition**.

② S3: U is said to be **closed under scalar multiplication**.

③ A subspace is a vector subspace itself.

Example 2. Let V be a vector space. Show that $U = \{\mathbf{0}\}$ is a subspace of V. This space is called the **zero subspace**.

Solution.

Note: Any subspace U of V such that $U \neq \{\mathbf{0}\}$ and $U \neq V$ is called a **proper subspace**.

Important Examples

EXAMPLE 3. Given an $m \times n$ matrix A, define

$$\mathrm{null} A := \{\mathbf{x} \in \mathbb{R}^n \,:\, A\mathbf{x} = \mathbf{0}\}.$$

Show that $\mathrm{null} A$ is a subspace of $\mathbb{R}^n$.

SOLUTION.

Note: The subspace $\mathrm{null} A$ is called the **null space** of a matrix A. It is the set of all solutions to the homogeneous system $A\mathbf{x} = \mathbf{0}$.

Example 4. Given an $m \times n$ matrix A, define

$$\text{im} A := \{A\mathbf{x} \,:\, \mathbf{x} \in \mathbb{R}^n\}.$$

Show that $\text{im} A$ is a subspace of $\mathbb{R}^m$.

Solution.

Note: The subspace $\text{im} A$ is called the **image space** (or **range space**) of the matrix A. It is the set of all vectors $\mathbf{b}$ such that $A\mathbf{x} = \mathbf{b}$ has a solution.

Example 5. For an $n \times n$ matrix A and a number λ, define

$$E_\lambda(A) := \{\mathbf{x} \in \mathbb{R}^n \,:\, A\mathbf{x} = \lambda\mathbf{x}\}.$$

Show that $E_\lambda(A)$ is a subspace of $\mathbb{R}^n$.

Solution.

Note: When λ is an eigenvalue of A, the subspace $\mathrm{E}_\lambda(A)$ is called the **eigenspace** associated to λ.

More Examples

Example 6. Let $\mathbf{M}_{nn}$ be the vector space of $n \times n$ matrices. Show that $U = \{A : A^\top = A\}$ is a subspace of $\mathbf{M}_{nn}$.

Solution.

Note: The set U is the subspace of all symmetric matrices.

Non-Examples

Example 7. Show that the set

$$U = \{p : p \in \mathbf{P}_3 \text{ and } p(2) = 1\}$$

is not a subspace of $\mathbf{P}_3$.

Solution.

EXAMPLE 8. The solutions set to the system $A\mathbf{x} = \mathbf{0}$ given in Example 1 is given by the linear combination

$$t\mathbf{x_1} + s\mathbf{x_2}, \quad t, s \in \mathbb{R}.$$

The set $\{t\mathbf{x_1} + s\mathbf{x_2} \,:\, t, s \in \mathbb{R}\}$ is called the **span** of $\mathbf{x_1}$ and $\mathbf{x_2}$.

DEFINITION 2. Let $\{\mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v_n}\}$ be a collection of vectors in a vector space V.

① a **linear combination** of the vectors $\mathbf{v_1}$, $\mathbf{v_2}$, ..., $\mathbf{v_n}$ is an expression of the form

$$a_1\mathbf{v_1} + a_2\mathbf{v_2} + \cdots + a_n\mathbf{v_n}$$

where a_1, a_2, ..., a_n are scalars called the **coefficients** of each vector.

② The set of all linear combinations of these vectors is called their **span**.

③ If it happens that $V = \text{span}\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$, then the vectors are called a **spanning set** for V.

Remarks:

- $\text{span}\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ is a subspace of V.

Example 9. Consider $p_1 = 1 + x + 4x^2$ and $p_2 = 1 + 5x + x^2$, two polynomials in $\mathbf{P}_2$.

a) Is p_1 in the span$\{1 + 2x - x^2, 3 + 5x + 2x^2\}$.

b) Is p_2 in the span$\{1 + 2x - x^2, 3 + 5x + 2x^2\}$.

Solution.