Math 311

Chapter 1

Section 1.3: Homogeneous Equations

Contents

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Spring 2024

$x + y = 0$

$x + 2y = 0$

TERMINOLOGY

DEFINITION 1. A system of linear equations in $x_1, \ldots, x_n$ is called **homogeneous** if all the constant terms are zero.

- **Trivial solution:** $x_1 = 0, x_2 = 0, \ldots, x_n = 0$.
- **Non trivial solution:** Any solution in which at least one variable has a nonzero value.

EXAMPLE 1. Show that the following homogeneous system has nontrivial solutions.

$$\begin{aligned} x_1 - x_2 + 2x_3 - x_4 &= 0 \\ 2x_1 + 2x_2 \qquad + x_4 &= 0 \\ 3x_1 + x_2 + 2x_3 - x_4 &= 0 \end{aligned}$$

SOLUTION. RREF is $\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$

So, $x_4 = 0$ and

$x_2 - x_3 = 0 \Rightarrow x_2 = x_3$

Also, $x_1 + x_3 = 0$

$\Rightarrow x_1 = -x_3$

$\Rightarrow$ $x_1 = -s$, $x_2 = s$, $x_3 = s$, $x_4 = 0$ $\xRightarrow{s=1}$ $\boxed{x_1 = -1,\ x_2 = 1,\ x_3 = 1,\ x_4 = 0}$

THEOREM 1. If a homogeneous system of linear equations has more variables than equations, then it has a nontrivial solution (in fact, infinitely many).

Linear Combinations

Definition 2.

- An **n-column vector**: $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$.
- Set of all n-column vectors is denoted by $\mathbb{R}^n$.
- **Equality**: $\mathbf{x} = \mathbf{y}$ if $\mathbf{x}$ and $\mathbf{y}$ are of the same size and all entries are the same. $\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Leftrightarrow \begin{matrix} 1 = x_1 \\ 2 = x_2 \end{matrix}$
- **Sum** of two n-column vectors $\mathbf{x}$,$\mathbf{y}$ is the new n-column vector $\mathbf{x}+\mathbf{y}$ obtained by adding corresponding entries. $\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$
- **Scalar multiplication** $k\mathbf{x}$ of a n-vector $\mathbf{x}$ with a scalar k is obtained by multiplying each entry of $\mathbf{x}$ by k. $2\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$.
- **Linear combination**: A sum of scalar multiples of several column vectors.

Example 2. If $\mathbf{x} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, then

$$\mathbf{x}+\mathbf{y} = \begin{bmatrix} 3-1 \\ -2+1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \text{ and } 2\mathbf{x} = \begin{bmatrix} (2)(3) \\ (2)(-2) \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \end{bmatrix}$$

$\vec{x} + 3\vec{y} + (-2)\vec{z} \leftarrow$ linear combination.

Example 3. Let

$$\mathbf{x} = \begin{bmatrix}1\\0\\1\end{bmatrix}, \mathbf{y} = \begin{bmatrix}2\\1\\0\end{bmatrix}, \mathbf{z} = \begin{bmatrix}3\\1\\1\end{bmatrix}, \mathbf{v} = \begin{bmatrix}0\\-1\\2\end{bmatrix}, \mathbf{w} = \begin{bmatrix}1\\1\\1\end{bmatrix}.$$

Determine weither $\mathbf{v}$ and $\mathbf{w}$ are linear combinations of $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$.

Solution.

① $\vec{v} = a\vec{x} + b\vec{y} + c\vec{z}$ (Goal)

$$\Leftrightarrow \begin{bmatrix}0\\-1\\2\end{bmatrix} = a\begin{bmatrix}1\\0\\1\end{bmatrix} + b\begin{bmatrix}2\\1\\0\end{bmatrix} + c\begin{bmatrix}3\\1\\1\end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix}0\\-1\\2\end{bmatrix} = \begin{bmatrix}a\\0\\a\end{bmatrix} + \begin{bmatrix}2b\\b\\0\end{bmatrix} + \begin{bmatrix}3c\\c\\c\end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix}0\\-1\\2\end{bmatrix} = \begin{bmatrix}a+2b+3c\\b+c\\a+c\end{bmatrix} \Leftrightarrow \begin{aligned} a+2b+3c &= 0\\ b+c &= -1\\ a+c &= 2\end{aligned}$$

So
$$\left[\begin{array}{ccc|c}1&2&3&0\\0&1&1&-1\\1&0&1&2\end{array}\right] \xrightarrow{\cdots} \left[\begin{array}{ccc|c}1&0&1&-2\\0&1&1&-1\\0&0&0&0\end{array}\right]$$

System is consistent $\rightarrow$ $a = -2-s$, $b = -1-s$, $c = s$.

$s = 0 \Rightarrow a = -2, b = -1, c = 0 \Rightarrow \boxed{\vec{v} = -2\vec{x} + (-\vec{y})}$.

(2) $\vec{w} = a\vec{x} + b\vec{y} + c\vec{z}$ (Goal)

$\vdots$

$$\Leftrightarrow \quad \begin{aligned} a + 2b + 3c &= 1 \\ b + c &= 1 \\ a + \phantom{2b+{}} c &= 1 \end{aligned}$$

$$\Leftrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{array}\right] \xrightarrow{\cdots} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

So, last row is $0 = 1$ (contradiction).

Conclusion: $\vec{w}$ is not a linear comb. of $\vec{x}, \vec{y}, \vec{z}$.

Notation:

- Write n variables x_1, x_2, $\ldots$, x_n as $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$.

The solution in Example 1 can be written as

$$\mathbf{x} = \begin{bmatrix} -t \\ t \\ t \\ 0 \end{bmatrix} = +t \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

Theorem 2. Any linear combination of solutions to a homogeneous system is again a solution.

Proof. Let $\mathbf{x}$ and $\mathbf{y}$ be two different solutions to a homogeneous system. Let $\mathbf{z} = c\mathbf{x} + d\mathbf{y}$. Then, by definition, each component of $\mathbf{z}$ is $cx_j + dy_j$, for each j. Plugging that in each equation of the system:

$$\begin{aligned} a_{i1}(cx_1 + dy_1) + a_{i2}(cx_2 + dy_2) + \cdots + a_{in}(cx_n + dy_n) \\ &= c(a_{i1}x_1 + \cdots + a_{in}x_n) + d(a_{i1}y_1 + \cdots + a_{in}y_n) \\ &= c(0) + d(0) \\ &= 0 \end{aligned}$$

Therefore, $\mathbf{z}$ is a solution to the homogeneous system.

Example 4. Solve the homogeneous system with coefficient matrix

$$A = \begin{bmatrix} 1 & -2 & 3 & -2 \\ -3 & 6 & 1 & 0 \\ -2 & 4 & 4 & -2 \end{bmatrix}$$

and express the solution as a linear combination of particular solutions.

Solution.

$$\left[\begin{array}{cccc|c} 1 & -2 & 3 & -2 & 0 \\ -3 & 6 & 1 & 0 & 0 \\ -2 & 4 & 4 & -2 & 0 \end{array}\right] \xrightarrow{\cdots} \left[\begin{array}{cccc|c} 1 & -2 & 0 & -1/5 & 0 \\ 0 & 0 & 1 & -3/5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

So, $x_1 = 2s + t/5 \qquad x_3 = \frac{3}{5}t$

$x_2 = s \qquad x_4 = t$

General solution:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2s + t/5 \\ s \\ 3/5\, t \\ t \end{bmatrix} = \begin{bmatrix} 2s \\ s \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} t/5 \\ 0 \\ 3/5\, t \\ t \end{bmatrix}$$

$$\Rightarrow \vec{x} = s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1/5 \\ 0 \\ 3/5 \\ 1 \end{bmatrix}, \quad s, t \in \mathbb{R}.$$

You can always rescale:

$$\vec{x} = s\begin{bmatrix}2\\1\\0\\0\end{bmatrix} + T\begin{bmatrix}1\\0\\3\\5\end{bmatrix} \quad (T = t/5).$$

Definition 3. The gaussian algorithm systematically produces solutions to any homogeneous systems of linear equations, called **basic solutions**, one for every parameter.

Hence, the basic solutions in the previous example are

$$\mathbf{x_1} = \begin{bmatrix}2\\1\\0\\0\end{bmatrix} \quad \text{and} \quad \mathbf{x_2} = \begin{bmatrix}\frac{1}{5}\\0\\\frac{3}{5}\\1\end{bmatrix}.$$

Theorem 3. Let A be the coefficient matrix of a homogeneous system of m linear equations in n variables. If A has rank r, then

1. The system has exactly $n - r$ basic solutions, one for each parameter.
2. Every solution is a linear combination of these basic solutions.