FINAL
ANSWERS KEY

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M311, SPRING 24'

Question 1

(a) (T1) $T(\vec{u}+\vec{v}) = T(x_1+x_2, y_1+y_2, z_1+z_2)$

$= (2(y_1+y_2), z_1+z_2, z_1+z_2)$

$= (2y_1+2y_2, z_1+z_2, z_1+z_2)$

$= (2y_1, z_1, z_1) + (2y_2, z_2, z_2)$

$= T(x_1, y_1, z_1) + T(x_2, y_2, z_2) = T(\vec{u})+T(\vec{v})$ ✓

(T2) $T(a\vec{u}) = T(ax, ay, az)$

$= (2(ay), az, az)$

$= (a\,2y, az, az) = a(2y, z, z) = aT(\vec{u})$ ✓

So T is a linear transformation.

(b) $T(x,y,z) = \vec{0} \iff (2y, z, z) = (0,0,0)$

$\iff y = z = 0, \quad x \in \mathbb{R}$

$\iff u = (x, 0, 0), \quad x \in \mathbb{R}$

Hence,

$\ker T = \{(x,0,0) : x \in \mathbb{R}\} = \text{Span}\{(1,0,0)\} \Rightarrow$ **$\dim \ker T = 1$**

(c) $V = \mathbb{R}^3 \Rightarrow \dim \mathbb{R}^3 = \text{nullity } T + \text{rank } T$

$\Rightarrow 3 = 1 + \text{rank } T$

$\Rightarrow$ **$\text{rank } T = 2.$**

Question 2

(a) $C_D(1,0,0) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$C_B(0,1,0) = \begin{bmatrix} 0 \\ 0 \\ 1/2 \end{bmatrix} \Rightarrow P_{D \leftarrow B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1/2 \\ -1 & 1/2 & 1/4 \end{bmatrix}$

$C_B(0,0,1) = \begin{bmatrix} 0 \\ 1/2 \\ 1/4 \end{bmatrix}$

(b) $T(1,0,0) = (0,0,0) \Rightarrow C_B(T(1,0,0)) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$T(0,1,0) = (2,0,0) \Rightarrow C_B(T(0,1,0)) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

$T(0,0,1) = (0,1,1) \Rightarrow C_B(T(0,0,1)) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Hence,

$$M_B(T) = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Question 3

(a) $(-1,0,3)\cdot(0,-3,2) = 0+0+6 = 6$.

(b) $\begin{bmatrix} -1 & 0 \\ 0 & -3 \\ 3 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & -3 \\ 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & -3 \\ 0 & 1 \end{bmatrix}$

Two pivots $\Rightarrow$ $\dim U = 2$.

(c) Set $\vec{f}_1 = (-1,0,3)$.

$$\vec{f}_2 = (0,-3,2) - \frac{(0,-3,2)\cdot(-1,0,3)}{\|(-1,0,3)\|^2}(-1,0,3)$$

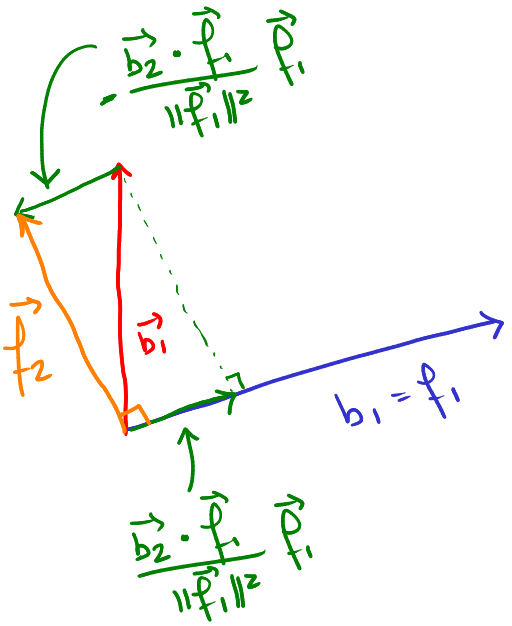
$$= (0,-3,2) - \frac{6}{10}(-1,0,3)$$

$$= (0,-3,2) - \frac{3}{5}(-1,0,3)$$

$$\Rightarrow \vec{f}_2 = \left(\frac{3}{5}, -3, \frac{1}{5}\right)$$

Hence, $F = \{(-1,0,3), (3/5, -3, 1/5)\}$.

(d)



Two vectors $\vec{b}_1, \vec{b}_2$ in $\mathbb{R}^2$

Question 4

(a) $T(-3\vec{v_1} + 2\vec{v_2} + 2\vec{v_3}) = (-3)\,T(\vec{v_1}) + 2T(\vec{v_2}) + 2T(\vec{v_3})$

$= -3(2,2) + 2(1,2) + 2(2,1)$

$= (-6+2+4, -6+4+2)$

$= (0,0)$

Hence, $-3\vec{v_1} + 2\vec{v_2} + 2\vec{v_3} \in \ker T$.

(b) Let $\vec{v} = x\vec{v_1} + y\vec{v_2} + z\vec{v_3}$.

Then $T(\vec{v}) = x\,T(\vec{v_1}) + y\,T(\vec{v_2}) + z\,T(\vec{v_3})$

$= x(2,2) + y(1,2) + z(2,1)$

$= (2x+y+2z,\ 2x+2y+z)$

Hence

$T(\vec{v}) = (0,0) \iff 2x+y+2z=0,\ 2x+2y+z=0$

$\iff 2x+y+2z=0,\ -y+z=0$

$\iff 2x+y+2z=0,\ z=y$

$\iff x = -\frac{3}{2}y,\ z=y,\ y \in \mathbb{R}$

Hence,

$\ker T = \{ y(-\frac{3}{2}, 1, 1) : y \in \mathbb{R} \}$

$= \text{Span}\{(-3/2, 1, 1)\}$

$\therefore \text{nullity } T = \dim \ker T = 1$. □