Math 311

Chapter 1

Section 1.1: Systems of Linear Equations

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Spring 2024

Terminology

Definition 1.

- An equation of the form $a_1x_1 + a_2x_2 + \ldots + a_nx_n$ is called a **linear equation** in the n variables x_1, x_2, $\ldots$, x_n.
- a_1, $\ldots$, a_n are fixed real numbers called the **coefficients**.
- b is a fixed real number called the **constant term**.
- A *finite* collection of linear equations is called a **system of linear equations**.

Definition 2.

- Given a linear equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$, a list s_1, s_2, $\ldots$, s_n of n numbers is called a **solution** to the equation if
$$a_1s_1 + a_2s_2 + \ldots + a_ns_n = b.$$
- A list s_1, s_2, $\ldots$, s_n of n numbers is a **solution to a system** of linear equations if it is a solution of every linear equation of the system.
- Two systems of linear equations are **equivalent** if they have the same set of solutions.

Example 1. Consider the following system:

$$\begin{aligned} ①\quad x_1 - 2x_2 + 3x_3 + x_4 &= -3 \\ ②\quad 2x_1 - x_2 + 3x_3 - x_4 &= 0 \end{aligned}.$$

(a) Show that $x_1 = 1$, $x_2 = 2$, $x_3 = 0$, and $x_4 = 0$ is a solution.

(b) Show that, for arbitrary values of s and t, $x_1 = t - s + 1$, $x_2 = t + s + 2$, $x_3 = s$, and $x_4 = t$ is a solution.

Solution.

(a) ① $1 - 2(2) + 3(0) + (0) = -3$ ✓

② $2(1) - 2 + 3(0) - 0 = 0$ ✓

(b) ① $(t - s + 1) - 2(t + s + 2) + 3(s) + t$

$= \cancel{t} - \cancel{s} + 1 - \cancel{2t} - \cancel{2s} - 4 + \cancel{3s} + \cancel{t} = -3$ ✓

② $2(t - s + 1) - (t + s + 2) + 3s - t$

$= \cancel{2t} - \cancel{2s} + \cancel{2} - \cancel{t} - \cancel{s} - \cancel{2} + \cancel{3s} - \cancel{t} = 0$ ✓

Definition 3.

- The quantities s and t are called **parameters**.
- The set of solutions, described with parameters, is said to be given in **parametric form** and is called the **general solution**.

Geometric interpretations

An equation in 2 variables (namely $x_1 = x$ and $x_2 = y$) can be drawn in a cartesian plane.

Check out Desmos:
https://www.desmos.com/calculator/dbnumvofgs.

Three alternatives:

- The system has a unique solution (the lines intersect at a single point).
- The system has no solution.
- The system has infinitely many solutions (the lines are identical).

In general:

- If the system has at least one solution, the system is called **consistent**.
- If the system has no solution, the system is called **inconsistent**.

General Presentation

A system of m linear equations in n variables:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

Goal: Manipulate the equations in the system to reduce it to another simpler system with the same set of solutions.

EXAMPLE 2. Solve the system $2x + y = 7$, $x + 2y = -2$.

$$S_1 \begin{cases} E_1: 2x + y = 7 \\ E_2: x + 2y = -2 \end{cases} \longrightarrow \begin{array}{l} E_1: x + 2y = -2 \\ E_2: 2x + y = 7 \end{array} \quad E_1 \leftrightarrow E_2$$

$$\begin{array}{r} 2x + y = 7 \\ -2x - 4y = 4 \\ \hline -3y = 11 \end{array} \qquad \longrightarrow \begin{array}{l} E_1: x + 2y = -2 \\ E_2: \quad -3y = 11 \end{array} \quad E_2 - 2E_1$$

$$\longrightarrow \begin{array}{l} E_1: x + 2y = -2 \\ E_2: \quad y = -11/3 \end{array} \quad -\frac{1}{3}E_2$$

$$\begin{array}{r} x + 2y = -2 \\ -2y = 22/3 \\ \hline x \quad = 16/3 \end{array} \qquad \longrightarrow S_2 \begin{cases} x = 16/3 & E_1 - 2E_2 \\ y = -11/3 \end{cases}$$

Then, $\boxed{x = 16/3 \,,\, y = -11/3}$ is the solution to the system.

Three Types of Elementary Operations

I. Interchange two equations.

II. Multiply one equation by a nonzero number.

III. Add a multiple of one equation to a different equation.

THEOREM 1. Suppose that a sequence of elementary operations is performed on a system of linear equations. Then the resulting system has the same set of solutions as the original, so the two systems are equivalent.

A Little Shortcut

DEFINITION 4. The **augmented matrix** of a system is an array of numbers where each row is obtained from each equation by removing the variable.

EXAMPLE 3. Find the augmented matrix associated to the system in Example 2.

SOLUTION.

$$\begin{aligned} 2x + y &= 7 \\ x + 2y &= -2 \end{aligned} \longrightarrow \begin{array}{c} \begin{array}{ccc} x & y & \end{array} \\ \left[\begin{array}{cc|c} 2 & 1 & 7 \\ 1 & 2 & -2 \end{array}\right] \end{array}$$

<u>Elementary operations translate to:</u>

I. Interchange two rows.

II. Multiply one row by a nonzero number.

III. Add a multiple of one row to a different row.

EXAMPLE 4. Find all solutions to the following system of equations using elementary operations on the augmented matrix.

$$\begin{aligned} 3x + 4y + z &= 1 \\ 2x + 3y \phantom{{}+z} &= 0 \\ 4x + 3y - z &= -2 \end{aligned}$$

SOLUTION.

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 3 & 4 & 1 & 1 \\ 2 & 3 & 0 & 0 \\ 4 & 3 & -1 & -2 \end{array}\right] \end{array} \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 3 & 0 & 0 \\ 4 & 3 & -1 & -2 \end{array}\right] \begin{array}{l} R_1 - R_2 \\ \\ \\ \end{array}$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & -1 & -5 & -6 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 4R_1 \end{array}$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & -7 & -8 \end{array}\right] \begin{array}{l} R_1 - R_2 \\ \\ R_3 + R_2 \end{array}$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 1 & 8/7 \end{array}\right] \begin{array}{l} \\ \\ -\frac{1}{7} R_3 \end{array}$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -3/7 \\ 0 & 1 & 0 & 2/7 \\ 0 & 0 & 1 & 8/7 \end{array}\right] \begin{array}{l} R_1 - 3R_3 \\ R_2 + 2R_3 \\ \end{array}$$

So,
$$\boxed{\begin{aligned} x &= -3/7 \\ y &= 2/7 \\ z &= 8/7 \end{aligned}}$$

Note: Any row operation can be reversed.

I. Interchanging two rows is reversed by interchanging them again.

II. Multiplying a row by $k \neq 0$ is reversed by multiplying by $1/k$.

III. Adding k times row p to a different row q is reversed by adding $-k$ times row p to the new row q.