MATH-311
Final (2h)

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# UNIVERSITY OF HAWAIʻI

**Last name:** Jones

**First name:** Lillie

| Question: | 1 | 2 | 3 | 4 | Total |
|---|---|---|---|---|---|
| Points: | 15 | 15 | 15 | 5 | 50 |
| Score: | 11 | 10 | 11 | 0 | 32 |

**Instructions:**

- Write your complete name on your copy.
- Answer all 4 questions below.
- Write your answers directly on the questionnaire.
- Show ALL your work to have full credit.
- Draw a square around your final answer.
- Return your copy when you're done or at the end of the 50min period.
- No electronic devices allowed during the exam.
- Scientific calculator allowed only (no graphical calculators).
- **Turn off your cellphone(s) during the exam.**
- Lecture notes and the textbook are not allowed during the exam.

**Your Signature:** Lillie

MAY THE FORCE BE WITH YOU!
PIERRE

ker $T = 0$ $\mathbb{R}^3 = 3$

QUESTION 1 (15 pts)

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x, y, z) = (2y, z, z)$.

(a) (5 Pts) Show that $T$ is a linear transformation.

(b) (5 Pts) Find the kernel of $T$ and its dimension.

(c) (5 Pts) Using the Dimension Theorem, deduce the rank of $T$.

Ker + Rank = $\mathbb{R}^3$

a) T1 $T(u+v) = T(u) + T(v)$

$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}\right) = T$

1/5

T2 $T(aU) = aT(u)$

11/15

b) $T(x, y, z) = (2y, z, z) = \vec{0}$

$a(2y) + b(z) + c(z) = \vec{0}$

$x(0) = 0$
$y(2a) = 0$
$z(b+c) = 0$

$\Rightarrow$ $2a = 0$, $b + c = 0$ $\Rightarrow$ $a = 0$, $b = -t$, $c = t$ $\Rightarrow$ $\begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} \Rightarrow t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

dim = 1 ✓

c) $1 + \text{Rank}\,T = 3$
$-1 \qquad -1$
Rank $T = 2$ ✓

## QUESTION 2 $\vec{b_1}$ $\vec{b_2}$ $\vec{b_3}$ (15 pts)

Let $V = \mathbb{R}^3$. Let $B$ be the standard basis $\{(1,0,0),(0,1,0),(0,0,1)\}$ and let the set $D = \{(1,2,0),(0,-1,2),(0,2,0)\}$ be another basis of $\mathbb{R}^3$.

(a) (10 Pts) Knowing that

- $(1,0,0) = a(1,2,0) + b(0,-1,2) + c(0,2,0)$, where $a = 1$, $b = 0$, $c = -1$;
- $(0,1,0) = a(1,2,0) + b(0,-1,2) + c(0,2,0)$, where $a = 0$, $b = 0$, $c = 1/2$;
- $(0,0,1) = a(1,2,0) + b(0,-1,2) + c(0,2,0)$, where $a = 0$, $b = 1/2$, $c = 1/4$;

Find the change matrix $P_{D\leftarrow B}$. Justify carefully your answer.

(b) (5 Pts) Let $T(x,y,z) = (2y,z,z)$. Find the matrix representation of $T$ on the basis $B$, that is $M_B(T)$.

$C_D(\vec{b_1}) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ $(1,0,0) = 1(1,2,0) + 0(0,-1,2) + -1(0,2,0)$

$= (1,2,0) + (0,-2,0)$

$\stackrel{\checkmark}{=} (1,0,0)$

$C_D(\vec{b_2}) = \begin{bmatrix} 0 \\ 0 \\ 1/2 \end{bmatrix}$ $(0,1,0) = 0(1,2,0) + 0(0,-1,2) + 1/2(0,2,0)$

$\stackrel{\checkmark}{=} (0,1,0)$

$C_D(\vec{b_3}) = \begin{bmatrix} 0 \\ 1/2 \\ 1/4 \end{bmatrix}$

$(0,0,1) = 0(1,2,0) + 1/2(0,-1,2) + 1/4(0,2,0)$

$(0,0,1) \stackrel{\checkmark}{=} (0,0,1)$

$P_{D\leftarrow B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1/2 \\ -1 & 1/2 & 1/4 \end{bmatrix}$ ✓

b) $C_D(\vec{b_1}(T)) = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$ $C_D(\vec{b_2}(T)) = \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \end{bmatrix}$ $C_D(b_3(T)) = \begin{bmatrix} 1 \\ 1/4 \\ 1/4 \end{bmatrix}$

$M_B(T) = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1/2 & 1/4 \\ -1 & 1/2 & 1/4 \end{bmatrix}$ 0/5

Find $T(1,0,0)$ and write those
$T(0,1,0)$ in the basis $B$.
$T(0,0,1)$

10/15

$\frac{6}{10} = \frac{3}{5}$ $\frac{3}{5} \cdot \frac{3}{1} = \frac{9}{5}$

## QUESTION 3 (15 pts)

Let $U = \text{span}\{(-1,0,3),(0,-3,2)\}$.

(a) (3 Pts) Are $(-1,0,3)$ and $(0,-3,2)$ orthogonal?

(b) (5 Pts) What is the dimension of $U$. Justify carefully your answer.

(c) (5 Pts) Using the Gram-Schmidt process, transform the set of vectors $\{(-1,0,3),\underline{(0,-3,2)}\}$ in a set of orthogonal vectors $F$. $v_2$

(d) (2 Pts) Illustrate visually the Gram-Schmidt orthogonalization process for two vectors in $\mathbb{R}^2$.

a) $(-1)(0)+(0)(-3)+(3)(2) = 6 \neq 0 \Rightarrow$ not orthogonal ✓ 3/3

b) $\dim U = 2$ ✓ More justification 3/5

c) $$V_2 = (0,-3,2) - \frac{(-1,0,3)(0,-3,2)}{(-1)^2+(0)+(3)^2} \cdot (-1,0,3)$$

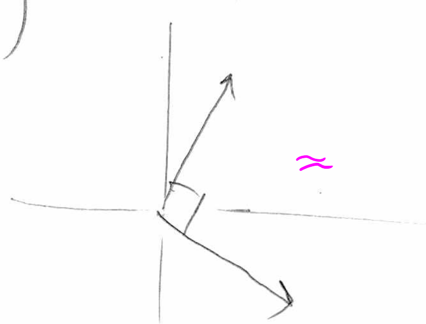
$$= (0,-3,2) - \left(\frac{3}{5}\right)\cdot(-1,0,3)$$

$$= \left(0+\frac{3}{5}, -3-0, 2-\frac{9}{5}\right) \qquad \frac{10}{5}-\frac{9}{5} \qquad 5/5$$

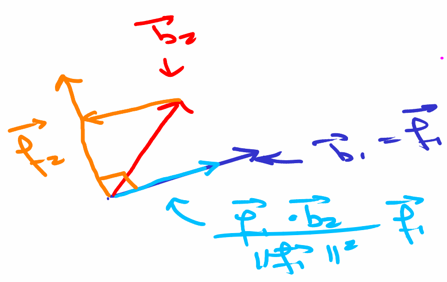
$$= \left(\frac{3}{5}, -3, \frac{1}{5}\right)$$

$$\vec{F} = \left\{(-1,0,3), \left(\frac{3}{5},-3,\frac{1}{5}\right)\right\}$$ ✓

d)



$\approx$




0/2

11/15

## QUESTION 4 (5 pts)

Assume that $T : V \to \mathbb{R}^2$ is a linear transformation where $V$ is a vector space and $\mathbb{R}$ is the vector space of real numbers. Let $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ be a basis of $V$.

(a) (1 Pt) If $T(\mathbf{v_1}) = (2,2)$, $T(\mathbf{v_2}) = (1,2)$ and $T(\mathbf{v_3}) = (2,1)$, show that $-3\mathbf{v_1}+2\mathbf{v_2}+2\mathbf{v_3} \in \ker T$. Explain carefully your answer.

(b) (4 Pts) If $T(\mathbf{v_1})$, $T(\mathbf{v_2})$ and $T(\mathbf{v_3})$ are defined as in part (a), find the nullity of $T$. Explain carefully your answer.

a) $T(v_1+v_2+v_3) = T(v_1)+T(v_2)+T(v_3)$

$T($ $\quad = (2,2)+((1,2))+(2,1)$

almost! on the right track!

0/1

Assume linear independence

$-3v_1+2v_2+2v_3=\vec{0}$

$-3(v_1)=$

0/5