MATH-311
Midterm 01 (50min)

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First name: Lillie

Question:	1	2	3	4	5	6	Total
Points:	10	10	10	10	6	4	50
Score:	10	10	8	10	4	2	44

Good!

Instructions:

- Write your complete name on your copy.
- Answer all 6 questions below.
- Write your answers directly on the questionnaire.
- Show ALL your work to have full credit.
- Draw a square around your final answer.
- Return your copy when you're done or at the end of the 50min period.
- No electronic devices allowed during the exam.
- Scientific calculator allowed only (no graphical calculators).
- **Turn off your cellphone(s) during the exam.**
- Lecture notes and the textbook are not allowed during the exam.

Your Signature: Lillie Jones

MAY THE FORCE BE WITH YOU!
PIERRE

$\frac{1}{2} - \frac{6}{2}$ $-\frac{5}{2}$

QUESTION 1 (10 pts)

Find the solution to the following system of linear equations:

$$\begin{cases} x_1 + \ \ x_2 + 3x_3 - x_4 = 3 \\ 2x_1 - 2x_2 + \ \ x_3 + x_4 = 0 \end{cases}$$

Does it have one solution, or infinitely many solutions?

$$\left[\begin{array}{cccc|c} 1 & 1 & 3 & -1 & 3 \\ 2 & -2 & 1 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 3 & -1 & 3 \\ 1 & -1 & 1/2 & 1/2 & 0 \end{array}\right] r_2/2 \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 3 & -1 & 3 \\ 0 & 2 & 5/2 & -3/2 & 3 \end{array}\right] r_1 - r_2 \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 3 & -1 & 3 \\ 0 & 1 & 5/4 & -3/4 & 3/2 \end{array}\right] \frac{R_2}{2}$$

$x_1 \ x_2 \ x_3 \ x_4$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 7/4 & -1/4 & 3/2 \\ 0 & 1 & 5/4 & -3/4 & 3/2 \end{array}\right] R_1 - R_2$$

$$x_1 + \frac{7}{4}x_3 - \frac{1}{4}x_4 = 3/2$$
$$x_2 + \frac{5}{4}x_3 - \frac{3}{4}x_4 = 3/2$$

$$\Rightarrow \boxed{\begin{aligned} x_1 &= 3/2 - \tfrac{7}{4}s + \tfrac{1}{4}t \\ x_2 &= 3/2 - \tfrac{5}{4}s + \tfrac{3}{4}t \\ x_3 &= s \\ x_4 &= t \end{aligned}}$$

has infinitely many solutions

10/10

QUESTION 2 (10 pts)

Consider the following vectors:

$$\mathbf{x} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \mathbf{y} = \begin{bmatrix}1\\-1\\2\\-1\end{bmatrix}, \mathbf{z} = \begin{bmatrix}-2\\3\\2\\0\end{bmatrix}, \mathbf{v} = \begin{bmatrix}-1\\1\\0\\1\end{bmatrix}, \mathbf{w} = \begin{bmatrix}9\\-8\\0\\1\end{bmatrix}.$$

We would like to know if $\mathbf{w}$ is a linear combination of $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ and $\mathbf{v}$.

(a) (5 points) Write down the system of linear equations corresponding to this problem. **DO NOT SOLVE THE SYSTEM.**

(b) (5 points) If the RREF of the augmented matrix of the system from part (a) is

$$\left[\begin{array}{cccc|c}1&0&0&0&2\\0&1&0&0&2\\0&0&1&0&-3\\0&0&0&1&1\end{array}\right],$$

can you express $\mathbf{w}$ as a linear combination of $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, and $\mathbf{v}$? If so, write $\mathbf{w}$ as a linear combination of the other vectors.

$\vec{w} = a\vec{x} + b\vec{y} + c\vec{z} + d\vec{v}$

$\vec{w} = a\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + b\begin{bmatrix}1\\-1\\2\\-1\end{bmatrix} + c\begin{bmatrix}-2\\3\\2\\0\end{bmatrix} + d\begin{bmatrix}-1\\1\\0\\1\end{bmatrix}$

$$\begin{bmatrix}9\\-8\\0\\1\end{bmatrix} = \begin{bmatrix}a+b-2c-d\\a-b+3c+d\\a+2b+2c+0\\a-b+0-d\end{bmatrix} \Leftrightarrow \left[\begin{array}{cccc|c}1&1&-2&-1&9\\1&-1&3&1&-8\\1&2&2&0&0\\1&-1&0&1&1\end{array}\right] \cdots \rightarrow \begin{array}{c}\begin{array}{cccc}a&b&c&d\end{array}\\ \left[\begin{array}{cccc|c}1&0&0&0&2\\0&1&0&0&2\\0&0&1&0&-3\\0&0&0&1&1\end{array}\right]\end{array}$$

$a=2$
$b=2$
$c=-3$
$d=1$

$\vec{w} = 2\vec{x} + 2\vec{y} - 3\vec{z} + \vec{v}$

10/10

8/10

QUESTION 3 (10 pts)

Consider the following homogeneous system of linear equations:

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ 2x_1 - x_2 + x_3 - 2x_4 = 0 \\ 3x_1 - x_2 + 2x_3 - x_4 = 0 \end{cases}.$$

(a) (2 points) Write the augmented matrix of the system. 2/2

(b) (2 points) Are there one solution or infinitely many solutions? Justify your answer. 1/2

(c) (6 points) The RREF of the augmented matrix of the system is

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -3 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 & 0 \end{array}\right]$$

Express the solution as a linear combination of basic solution(s). 5/6

a) $\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 2 & -1 & 1 & -2 & 0 \\ 3 & -1 & 2 & -1 & 0 \end{array}\right] \dashrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -3 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 & 0 \end{array}\right]$

there are many solutions b/c x_4 doesn't have a leading one and can be anything which affects the values of all the other variables.

$x_1 - 3x_4 = 0$

$x_2 = 0$

$x_3 + x_4 = 0$

$$\boxed{\begin{aligned} x_1 &= 3s \\ x_2 &= 0 \\ x_3 &= \cancel{-s} \ \text{-4s} \\ x_4 &= s \end{aligned}}$$

Write down the part you are answering.
Is the above text answering b)?

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s\begin{bmatrix} 3 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

For b), the reason is #eq < #var and the system is homogeneous. Therefore, there will be infinitely many solution by a result from the lecture notes.

Question 4 (10 pts)

Find the entries of the matrix A if A satisfies the equation:

$$\left(2A^{\mathsf{T}} - 5\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}\right)^{\mathsf{T}} = 4A - 9\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}.$$

$$\left(2A^T - \begin{bmatrix} 5 & 0 \\ -5 & 10 \end{bmatrix}\right)^T = 4A - \begin{bmatrix} 9 & 9 \\ -9 & 0 \end{bmatrix}$$

$$(2A^T)^T - \begin{bmatrix} 5 & 0 \\ -5 & 10 \end{bmatrix}^T = 4A - \begin{bmatrix} 9 & 9 \\ -9 & 0 \end{bmatrix}$$

$$2A - \begin{bmatrix} 5 & -5 \\ 0 & 10 \end{bmatrix} = 4A - \begin{bmatrix} 9 & 9 \\ -9 & 0 \end{bmatrix}$$

$$-2A \qquad\qquad -2A$$

$$-\begin{bmatrix} 5 & -5 \\ 0 & 10 \end{bmatrix} = 2A - \begin{bmatrix} 9 & 9 \\ -9 & 0 \end{bmatrix}$$

$$+\begin{bmatrix} 9 & 9 \\ -9 & 0 \end{bmatrix} \qquad\qquad +\begin{bmatrix} 9 & 9 \\ -9 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 9 & 9 \\ -9 & 0 \end{bmatrix} - \begin{bmatrix} 5 & -5 \\ 0 & 10 \end{bmatrix} = 2A$$

$$\frac{1}{2}\begin{bmatrix} 4 & 14 \\ -9 & -10 \end{bmatrix} = 2A \cdot \frac{1}{2}$$

$$A = \begin{bmatrix} 4 & 14 \\ -9 & -10 \end{bmatrix} \cdot \frac{1}{2}$$

$$A = \begin{bmatrix} 2 & 7 \\ -9/2 & -5 \end{bmatrix}$$

10/10

QUESTION 5 **(6 pts)**

A square matrix A is **skew-symmetric** if $A^\top = -A$. Show that if A and B are skew-symmetric, then $A - B$ is skew-symmetric.

$A^T = -A$ $B^T = -B$

$A = -A^T$ $B = -B^T$

has to be true given A & B are skew-symmetric

4/6

$A - B \Rightarrow -A^T - -B^T \Rightarrow (-A^T - -B^T)^T \Rightarrow (-A^T)^T - (-B^T)^T \Rightarrow$

$(-A) - (-B)$ statement is skew-symmetric if $A^T = -A$ + $B^T = -B$

$A^T - B^T \Rightarrow$

Almost there. You were on the right track!!

$$(A - B)^\top = A^\top - B^\top = (-A) - (-B) = -A + B = -(A - B)$$

2/4

QUESTION 6 **(4 pts)**

Answer the following questions with **True** or **False**. Write down you answers on the line at the end of each question. Justify briefly your answer in the space after the statement of the problem.

(a) A matrix B with RREF $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ has rank $(A) = 2$. (0/ 1)

(a) False

(b) A homogeneous system can have no solution. (1/ 1)

even if variables = 0

$\begin{bmatrix} 0 & 1 & 0 \\ & & \end{bmatrix}$ - $x_2 = 0$ that is still a solution

Always has the trivial solution.

(b) False

(c) If $\mathbf{x_1}$ and $\mathbf{x_2}$ are solutions to a system of linear equations denoted by (S), then $2\mathbf{x_1} - \mathbf{x_2}$ is also a solution of the system (S). (0/ 1)

$(S) = x_1$

$= x_2$

(c) True

(d) A system of 3 linear equations in 2 variables with a coefficient matrix of rank 2 has a unique solution. (1/ 1)

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

when ref form bottom row should be all zeros which gives you infinite solutions.

(d) false