$\boxed{\text{Question 1}}$ $T(x,y,z) = (x,-y,x)$.

(a)

T1 $\vec{u} = (x_1, y_1, z_1)$, $\vec{v} = (x_2, y_2, z_2)$

Goal: $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$.

On one hand, we have

$$T(\vec{u}+\vec{v}) = T(\underbrace{x_1+x_2}_{x}, \underbrace{y_1+y_2}_{y}, \underbrace{z_1+z_2}_{z})$$

$$= (x_1+x_2, -(y_1+y_2), x_1+x_2)$$

$$= (x_1+x_2, -y_1-y_2, x_1+x_2).$$

On the other hand, we have

$$T(\vec{u})+T(\vec{v}) = T(x_1, y_1, z_1) + T(x_2, y_2, z_2)$$

$$= (x_1, -y_1, x_1) + (x_2, -y_2, x_2)$$

$$= (x_1+x_2, -y_1-y_2, x_1+x_2).$$

Hence $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$.

(T2) $\vec{u} = (x, y, z)$ and $a \in \mathbb{R}$.

Goal: $T(a\vec{u}) = aT(\vec{u})$

On one hand, we have:

$$T(a\vec{u}) = T(ax, ay, az) = (ax, -ay, ax).$$

On the other hand, we have

$$aT(\vec{u}) = aT(x, y, z) = a(x, -y, x)$$
$$= (ax, -ay, ax).$$

Hence, $T(a\vec{u}) = aT(\vec{u})$.

Since T satisfies (T1) and (T2), T is a linear transformation.

(b) (1) Kernel

$$\vec{u} = (x, y, z) \in \ker T \Leftrightarrow T(x, y, z) = (0, 0, 0)$$
$$\Leftrightarrow (x, -y, x) = (0, 0, 0)$$
$$\Leftrightarrow x = 0,\ -y = 0,\ x = 0,\ z \in \mathbb{R}$$
$$\Leftrightarrow \vec{u} = (0, 0, z),\ z \in \mathbb{R}$$

Hence $\ker T = \{(0,0,z) : z \in \mathbb{R}\}$.

② Image.

We have

$$T(x,y,z) = (x,-y,x)$$

$$= (x,0,x) + (0,-y,0)$$

$$= \underbrace{x(1,0,1) + y(0,-1,0)}_{\text{linear combination!}}$$

Hence

$$\operatorname{Im} T = \{T(x,y,z) : (x,y,z) \in \mathbb{R}\}$$

$$= \{x(1,0,1) + y(0,-1,0) : x,y \in \mathbb{R}\}$$

$$= \operatorname{span}\{(1,0,1),(0,-1,0)\}.$$

(c) ① Nullity.

We have $\operatorname{nullity}(T) = \dim(\ker T)$.

We know that

$$\ker T = \{(0,0,z) : z \in \mathbb{R}\}$$
$$= \{z(0,0,1) : z \in \mathbb{R}\}$$
$$= \operatorname{span}\{(0,0,1)\}.$$

Hence $\dim(\ker T) = 1$.

(2) <u>Rank.</u>

We have $\operatorname{im} T = \operatorname{span}\{(1,0,1),(0,-1,0)\}$.

Since $(1,0,1)\cdot(0,-1,0) = 0+0+0 = 0$, then $(1,0,1)$ and $(0,-1,0)$ are orthogonal and therefore linearly independent.

This means $\{(1,0,1),(0,-1,0)\}$ is a basis for $\operatorname{im} T$. By definition:

$$\operatorname{rank}(T) = \dim(\operatorname{im} T) = 2.$$

Verification: Using the dimension theorem.

$$\text{nullity}(T) + \text{rank}(T) = \dim \underbrace{(\mathbb{R}^3)}_{\text{input space}}$$

$$\Rightarrow \quad 1 \quad + \quad 2 \quad = \quad 3 \checkmark$$

$\boxed{\text{Question 2}}$ $U = \text{span}\{(1,1,1), (-1,0,2)\}$.

(a) We have

$$(1,1,1)\cdot(-1,0,2) = (1)(-1) + (1)(0) + (1)(2)$$

$$= -1 + 0 + 2$$

$$= 1 \neq 0$$

Hence $(1,1,1)$ and $(-1,0,2)$ are not orthogonal.

(b) $(2,-3,1) = a(1,1,1) + b(-1,0,2)$

$$= (a-b, \ a, \ a+2b)$$

$$\Rightarrow \quad 2 = a-b, \quad -3 = a, \quad a+2b = 1$$

$\Rightarrow \quad a = -3, \qquad -3 - b = 2, \qquad -3 + 2b = 1$

$\Rightarrow \quad a = -3, \qquad b = -5 \quad \text{and} \quad b = 2$

Impossible!

Hence, $(2,-3,1) \notin U$.

<u>Second method</u>

$(2,-3,1) \cdot (1,1,1) = 2 - 3 + 1 = 0$

$(2,-3,1) \cdot (-1,0,2) = -2 + 0 + 2 = 0.$

So, $(2,-3,1) \perp (1,1,1)$

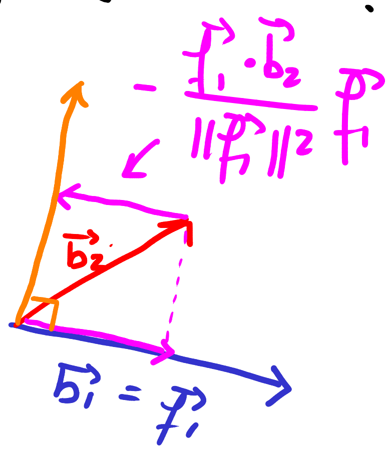
and $(2,-3,1) \perp (-1,0,2)$

$\Rightarrow \quad (2,-3,1) \notin U.$

(c) Set $\vec{b}_1 = (1,1,1)$ and $\vec{b}_2 = (-1,0,2)$.

Set $\vec{f}_1 = \vec{b}_1 = (1,1,1)$

Set $\vec{f}_2 = \vec{b}_2 - \dfrac{\vec{f}_1 \cdot \vec{b}_2}{\|\vec{f}_1\|^2} \vec{f}_1$



$$= (-1, 0, 2) - \frac{(1)}{3}(1,1,1)$$

$$= (-1, 0, 2) - \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right)$$

$$= \left(-\frac{4}{3}, \frac{-1}{3}, \frac{5}{3}\right)$$

Hence, $F = \left\{ (1,1,1), \left(-\frac{4}{3}, \frac{-1}{3}, \frac{5}{3}\right) \right\}$.

$\boxed{\text{Question 3}}$

$V = \mathbb{R}^3$

$$B = \{ \overbrace{(1,0,0)}^{\vec{b}_1}, \overbrace{(0,1,0)}^{\vec{b}_2}, \overbrace{(0,0,1)}^{\vec{b}_3} \}$$

$$D = \{ \underbrace{(1,1,0)}_{\vec{d}_1}, \underbrace{(1,0,1)}_{\vec{d}_2}, \underbrace{(0,1,0)}_{\vec{d}_3} \}$$

$$T(a,b,c) = (2a-b, b+c, c-3a)$$

(a) Goal: Find $P_{B \leftarrow D}$.

We have

$$\vec{d}_1 = (1,1,0) = (1)(1,0,0) + (1)(0,1,0) + (0)(0,0,1)$$

$$\Rightarrow \quad C_B(\vec{d_1}) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Similarly, we get

$$C_B(\vec{d_2}) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ and } C_B(\vec{d_3}) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Hence

$$P_{B \leftarrow D} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

(b) We have

$$T(\vec{b_1}) = T(1,0,0) = (2,0,-3)$$

$$\Rightarrow \quad C_B(T(\vec{b_1})) = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$$

$$T(\vec{b_2}) = T(0,1,0) = (-1,1,0)$$

$$\Rightarrow \quad C_B(T(\vec{b_2})) = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$T(\vec{b_3}) = T(0,0,1) = (0,1,1)$$

$$\Rightarrow C_B(T(\vec{b_3})) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Hence,

$$M_B(T) = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 1 \\ -3 & 0 & 1 \end{bmatrix}$$

(c) What is given:

① $P_{D \leftarrow B} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}.$

② $P_{D \leftarrow B}^{-1} = P_{B \leftarrow D}$

③ $P_{D \leftarrow B}^{-1} M_D(T) P_{D \leftarrow B} = M_B(T)$

From ③ $\Rightarrow M_D(T) = P_{D \leftarrow B} M_B(T) P_{D \leftarrow B}^{-1}$

$$\Rightarrow M_D(T) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 1 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow M_D(T) = \begin{bmatrix} 4 & 4 & -1 \\ -3 & -2 & 0 \\ -3 & -3 & 2 \end{bmatrix}.$$

PYTHON IS OUR FRIEND!

$\boxed{\text{Question 4}}$ $T : P_n \to P_n$, $T(p(x)) = p(x) - p(-x)$.

(a) Kernel

$$p(x) \in \ker T \iff T(p(x)) = 0$$
$$\iff p(x) - p(-x) = 0$$
$$\iff p(x) = p(-x)$$

Hence, $\ker T = \{ p : p(x) = p(-x) \}$

Image

WTS: $\operatorname{im} T = \{ q : q(-x) = -q(x) \}$.

If $q \in \operatorname{im} T$, then

$$q(x) = T(p(x)) = p(x) - p(-x).$$

Then $q(-x) = p(-x) - p(x)$

$$= -(\underbrace{p(x) - p(-x)}_{q(x)})$$

$$= -q(x)$$

Hence $q \in \operatorname{im} T \Rightarrow q(-x) = -q(x)$.

Now, if $q(-x) = -q(x)$.

Set $p(x) = \frac{q(x)}{2}$. then

$$T(p(x)) = p(x) - p(-x)$$

$$= \frac{q(x)}{2} - \frac{q(-x)}{2}$$

$$= \frac{q(x)}{2} - \left(\frac{-q(x)}{2}\right) \qquad [q(-x) = -q(x)]$$

$$= \frac{q(x)}{2} + \frac{q(x)}{2}$$

$$= q(x)$$

$\Rightarrow q \in \operatorname{im} T$.

Hence, $\operatorname{im} T = \{ q : q(-x) = -q(x) \}$.

(b) U: subspace of odd polynomials.

V: subspace of even polymials.

Ker ingredient: dimension Theorem.

We know that $\dim \mathcal{P}_n = n+1$

Dimension $\Rightarrow$ Theorem $\quad \dim \mathcal{P}_n = \text{nullity}(T) + \text{rank}(T)$

$$\Rightarrow \quad n+1 = \underbrace{\dim(\ker T)}_{\text{space of even poly} = U} + \underbrace{\dim(\operatorname{im} T)}_{\text{space of odd poly.} = V}$$

$$\Rightarrow \quad n+1 = \dim U + \dim V. \qquad \square$$