General Definition

Let V be a set of objects called **vectors**. Assume

1. **Vector Addition:** Two vectors $\mathbf{v}$ and $\mathbf{w}$ can be added and denote this operation by $\mathbf{v} + \mathbf{w}$.
2. **Scalar Multiplication:** Any vector $\mathbf{v}$ can be multiplied by any number (scalar) a and denote this operation by $a\mathbf{v}$.

The set V is called a **vector space** if

1. Axioms for the vector addition:

   $\boxed{\text{A1.}}$ Closed: $\mathbf{v}, \mathbf{w} \in V \Rightarrow \mathbf{v} + \mathbf{w} \in V$.

   $\boxed{\text{A2.}}$ Commutativity: $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$.

   $\boxed{\text{A3.}}$ Associativity: $\mathbf{v} + (\mathbf{w} + \mathbf{z}) = (\mathbf{v} + \mathbf{w}) + \mathbf{z}$.

   $\boxed{\text{A4.}}$ Existence of a zero vector: $\mathbf{v} + \mathbf{0} = \mathbf{v} = \mathbf{0} + \mathbf{v}$.

   $\boxed{\text{A5.}}$ Existence of a negative: For each $\mathbf{v}$, there is a $\mathbf{w}$ such that $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v} = \mathbf{0}$.

2. Axioms for the scalar multiplication:

   $\boxed{\text{S1.}}$ $\mathbf{v} \in V \Rightarrow a\mathbf{v} \in V$.

   $\boxed{\text{S2.}}$ $a(\mathbf{v} + \mathbf{w}) = a\mathbf{v} + a\mathbf{w}$.

   $\boxed{\text{S3.}}$ $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.

   $\boxed{\text{S4.}}$ $a(b\mathbf{v}) = (ab)\mathbf{v}$.

   $\boxed{\text{S5.}}$ $1\mathbf{v} = \mathbf{v}$.

MATH-311
Midterm 03 (50min)

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Question:	1	2	3	4	5	Total
Points:	20	10	10	5	5	50
Score:	0	10	0	2	1	13

Instructions:

- Write your complete name on your copy.
- Answer all 5 questions below.
- Write your answers directly on the questionnaire.
- Show ALL your work to have full credit.
- Draw a square around your final answer.
- Return your copy when you're done or at the end of the 50min period.
- No electronic devices allowed during the exam.
- Scientific calculator allowed only (no graphical calculators).
- **Turn off your cellphone(s) during the exam.**
- Lecture notes and the textbook are not allowed during the exam.

Your Signature:

May the Force be with you!
Pierre

Question 1 (20 pts)

Let $A = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}$.

(a) (5 Pts) Find the eigenvalues of the matrix A.

(b) (10 Pts) Find the eigenvectors associated to each eigenvalue.

(c) (5 Pts) Is A diagonalizable? If so, find the matrix P such that $P^{-1}AP$ is a diagonal matrix.

QUESTION 2 (10 pts)

Let $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}$ be vectors in a vector space V. Simplify the following expression:

$$2(\mathbf{x} - \mathbf{y}) + 4(\mathbf{z} - \mathbf{y}) + 4(\mathbf{w} - \mathbf{z}) + (\mathbf{x} - 4\mathbf{w}).$$

$2x - 2y + 4z - 4y + 4w - 4z + x - 4w.$

$2x + x - 2y - 4y + 4z - 4z + 4w - 4w.$

$3x - 6y.$

$\boxed{3(x - 2y)}$

10/10

QUESTION 3 (10 pts)

Which of the following are subspaces of $\mathbf{M}_{22}$, the vector space of all 2×2 matrices with usual addition and scalar multiplication of matrices.

(a) (5 Pts) $U = \{A : A \in \mathbf{M}_{22} \text{ and } A = -A^{\top}\}$.

(b) (5 Pts) $U = \{A : A \in \mathbf{M}_{22} \text{ and } A^2 = I\}$.

QUESTION 4 (5 pts)

Answer the following questions:

(a) (3 Pts) Assume that A is an 3×3 matrix and that $c_A(x)$ is the characteristic polynomial of A. Show that

$$c_{A^2}(x^2) = (-1)c_A(x)c_A(-x).$$

[Hint: Use the following property of determinants: $\det(XY) = \det(X)\det(Y)$.*]*

(b) (2 Pts) What does the word "eigen" in "eigen-vectors" and "eigen-values" mean in English?

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b) eigen means awn in german

QUESTION 5 (5 pts)

Answer the following questions with **True** or **False**. Write down you answers on the line at the end of each question. Justify briefly your answer in the space after the statement of the problem.

(a) The set $U = \{p : p \in \mathbf{P_3} \text{ and } p(0) = 1\}$ is a subspace of $\mathbf{P_3}$. (0/ 1)

(a) ~~True~~

(b) If the solution to $A\mathbf{x} = \lambda\mathbf{x}$ is only $\mathbf{x} = \mathbf{0}$, then λ is an eigenvalue. (0/ 1)

(b) ~~True~~

(c) If a matrix A has $\lambda = 0$ as an eigenvalue, then A is not invertible. (0.5/ 1)

Justification

(c) True

(d) If A is a 2×2 matrix with two distinct eigenvectors, then A is diagonalizable. (0.5/ 1)

Justification

(d) True

(e) If A is a 2×2 matrix with eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$, then $P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. (0/ 1)

1/5