Section 5.1 — Problem 1 **(10 Pts)**

a. This is not a subspace because the vector $\mathbf{0} = (0, 0, 0)$ is not in U.

b. This is a subspace because it satisfies S1-S3.

S1. Set $s = t = 0$, then $(0, 0, 0) \in U$.

S2. Let $\mathbf{u_1} = (0, s_1, t_1)$ and $\mathbf{u_2} = (0, s_2, t_2)$. Then, we get

$$\mathbf{u_1} + \mathbf{u_2} = (0 + 0, s_1 + s_2, t_1 + t_2) = (0, s, t)$$

where $s := s_1 + s_2$ and $t := t_1 + t_2$. Hence, $\mathbf{u_1} + \mathbf{u_2} \in U$.

S3. Let $\mathbf{u_1} = (0, s_1, t_1)$ and let $a \in \mathbb{R}$. Then

$$a\mathbf{u_1} = (a(0), as_1, at_1) = (0, s, t),$$

where $s = as_1$ and $t = at_1$. Hence $a\mathbf{u_1} \in U$.

Section 5.1 — Problem 17a **(10 Pts)**

Notice that

$$A\mathbf{x} = B\mathbf{x} \iff A\mathbf{x} - B\mathbf{x} = \mathbf{0} \iff (A - B)\mathbf{x} = \mathbf{0}.$$

Hence,

$$U = \{\mathbf{x} \in \mathbb{R}^n \,:\, A\mathbf{x} = B\mathbf{x}\} = \{\mathbf{x} \in \mathbb{R}^n \,:\, (A - B)\mathbf{x} = \mathbf{0}\} = \text{null}\,(A - B).$$

Since $C = A - B$ is a $m \times n$ matrix, from the lecture notes, we know that null$(A - B)$ is a subspace of $\mathbb{R}^n$.

Section 6.2 — Problem 1 **(15 Pts)**

b. This is a subset of $\mathbf{P_3}$ because x times a polynomial of degree at most 2 results in a polynomial of degree at most 3.

S1. Set $g(x) = 0$ for any x, then $xg(x) = 0$. Therefore, the zero polynomial is in U.

S2. Let $p_1(x) = xg_1(x)$ and $p_2(x) = xg_2(x)$, where g_1 and g_2 are in $\mathbf{P_2}$. Then

$$p_1(x) + p_2(x) = xg_1(x) + xg_2(x) = x(g_1(x) + g_2(x)) = xg(x)$$

where $g(x) = g_1(x) + g_2(x) \in \mathbf{P_2}$. Hence, $p_1 + p_2 \in U$.

S3. Let $p(x) = xg_1(x)$, where $g_1 \in \mathbf{P_2}$. Then

$$ap(x) = axg_1(x) = x(ag_1(x)) = xg(x)$$

with $g(x) = ag_1(x) \in \mathbf{P_2}$. Hence, $ap \in U$.

Since S1, S2, S3 are satisfied, we conclude that U is a subspace of $\mathbf{P_3}$.

f. Since the zero polynomial is of degree 0, it is not in U. Therefore, U is not a subspace.

Section 6.2 — Problem 2 (15 Pts)

d. Since the product of 2×2 matrices stay a 2×2 matrix, the set U is a subset of $\mathbf{M_{22}}$.

S1. If $A = \mathbf{0}$, then $\mathbf{0}B = \mathbf{0}$. Hence, $\mathbf{0} \in U$.

S2. Assume that A and C are in U. Then $AB = \mathbf{0}$ and $CB = \mathbf{0}$. Therefore

$$(A + C)B = AB + CB = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

Hence, $A + C \in U$.

S3. Assume that $A \in U$ and $a \in \mathbb{R}$. Therefore, $AB = 0$ and

$$(aA)B = a(AB) = a\mathbf{0} = \mathbf{0}.$$

Hence, $aA \in U$.

Since S1, S2, and S3 are satisfied, we get that U is a subspace.

e. U is a subset of $\mathbf{M_{22}}$.

S1. Notice that $\mathbf{0}^2 = \mathbf{0}$ and therefore $\mathbf{0} \in U$.

S2. Let $A, B \in U$, so that $A^2 = A$ and $B^2 = B$. Now, we have

$$(A + B)^2 = A^2 + AB + BA + B^2 = A + AB + BA + B.$$

Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$. We get check that $A^2 = A$ and $B^2 = B$. However,

$$A + B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \Rightarrow \quad (A + B)^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \neq A + B.$$

Hence, $A + B \notin U$.

Since S2 is not satisfied, U is not a subspace.