Math 311

Last Chapter

Section 10.1: Inner Product Spaces

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For $\mathbb{R}^n$, if we define

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n,$$

then the following properties are satisfied:

- P1 $\langle \mathbf{x}, \mathbf{y} \rangle$ is real number;
- P2 $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$;
- P3 $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$;
- P4 $\langle a\mathbf{x}, \mathbf{y} \rangle = a \langle \mathbf{x}, \mathbf{y} \rangle$;
- P5 $\mathbf{x} \neq 0$ if and only if $\langle \mathbf{x}, \mathbf{x} \rangle > 0$.

When P1-P5 are satisfied, we say that the dot product is an inner product and $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ is an inner product space.

DEFINITION 1. Let V be a vector space. If a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ satisfies P1-P5, then we say that $\langle \cdot, \cdot \rangle$ is an **inner product** defined on V and $(V, \langle \cdot, \cdot \rangle)$ is an **inner product space**.

Remarks:

1. for $\mathbf{v} \in V$, we define $\|\mathbf{v}\| := \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$.
2. $\mathbf{v}, \mathbf{w} \in V$ are orthogonal if and only if $\langle \mathbf{v}, \mathbf{w} \rangle = 0$.
3. All notions from 5.3 and 8.1 extends to a general inner product space.

Vectors

Example 1. We can show that

$$\langle \mathbf{x}, \mathbf{y} \rangle := 5x_1y_1 + 7x_1y_2 + 7x_2y_1 + 10x_2y_2$$

is an inner product on $\mathbb{R}^2$. Show that

a) $\mathbf{x} = \begin{bmatrix} 1 & 1 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} -1 & 1 \end{bmatrix}$ are not orthogonal.

b) $\mathbf{x} = \begin{bmatrix} 2 & 1 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 24 & -17 \end{bmatrix}$ are orthogonal.

Solution.

a)

$$\langle \vec{x}, \vec{y} \rangle = 5(1)(-1) + 7(1)(1) + 7(1)(-1) + 10(1)(1)$$

$$= 5 \neq 0 .$$

b) $\langle \vec{x}, \vec{y} \rangle = 5(2)(24) + 7(2)(-17) + 7(1)(24) + 10(1)(-17)$

$= 240 - 238 + 168 - 170$

$= 0$

Matrices

Example 2. For a matrix $A \in \mathbf{M_{nn}}$, we define its **trace** to be

$$\operatorname{tr}(A) := a_{11} + a_{22} + \cdots + a_{nn}.$$

Then the function

$$\langle A, B \rangle = \operatorname{tr}(AB^{\top})$$

defines an inner product on $\mathbf{M_{nn}}$.

Space of Continuous Functions

Example 3. Let $\mathbf{C}[a,b]$ be the vector space of **real-valued continuous functions** on the interval $[a,b]$. The application

$$\langle f, g \rangle = \int_a^b f(x)g(x)\,dx$$

is an inner product on $\mathbf{C}[a,b]$.

It is possible to have a theory of vector spaces using complex numbers. We simply replace $\mathbb{R}$ by $\mathbb{C}$, the set of complex numbers, everywhere in the definitions.

However, we have to modify the definition of an inner product.

Definition 2. Let V be a **complex vector space**. An application $\langle \cdot, \cdot \rangle\, V \times V \to \mathbb{C}$ is a **complex inner product** if

- P1 $\langle \mathbf{x}, \mathbf{y} \rangle$ is a complex number;
- P2 $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$, where $\overline{w} = u - iv$ is the complex conjugate of $w = u + iv$;
- P3 $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$;
- P4 $\langle a\mathbf{x}, \mathbf{y} \rangle = a \langle \mathbf{x}, \mathbf{y} \rangle$ for any complex number a;
- P5 $\mathbf{x} \neq 0$ if and only if $\langle \mathbf{x}, \mathbf{x} \rangle > 0$.

Remarks: The extension of vector space and inner product to complex numbers is used, for instance, in the foundations of Quantum Mechanics.