Math 311

Chapter 2

Section 2.2: Matrix-Vector Multiplication

Contents

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MATRIX-VECTOR MULTIPLICATION

EXAMPLE 1. Write the system

$$(*) \quad \begin{aligned} 3x_1 + 2x_2 - 4x_3 &= 0 \\ x_1 - 3x_2 + \ x_3 &= 3 \\ x_2 - 5x_3 &= -1 \end{aligned} \quad \Leftrightarrow \begin{bmatrix} 3 & 2 & -4 \\ 1 & -3 & 1 \\ 0 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

in a compact form using a linear combination of vectors.

SOLUTION.

$$(*) \Leftrightarrow \begin{bmatrix} 3x_1 + 2x_2 - 4x_3 \\ x_1 - 3x_2 + x_3 \\ x_2 - 5x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} 3x_1 \\ x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ -3x_2 \\ x_2 \end{bmatrix} + \begin{bmatrix} -4x_3 \\ x_3 \\ -5x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}$$

$$\Leftrightarrow x_1 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ 1 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}$$

Note: Any system of linear equations can be rewritten as $A\mathbf{x} = \mathbf{b}$, where A is the matrix of coefficients, $\mathbf{x}$ is the n-vector containing the unknown, and $\mathbf{b}$ is the m-vector containing the constant terms of each equation.

DEFINITION 1.

- Let $A = [\mathbf{a_1}\ \mathbf{a_2}\ \cdots\ \mathbf{a_n}]$ be an $m \times n$ matrix, where the m-vectors $\mathbf{a_1}$, $\mathbf{a_2}$, ..., $\mathbf{a_n}$ represent the columns.
- Let $\mathbf{x}$ be any n-vector.

The **product** $A\mathbf{x}$ is defined to be the m-vector:

Result is a $m \times 1$ vector.

$$A\mathbf{x} = x_1\mathbf{a_1} + x_2\mathbf{a_2} + \cdots + x_n\mathbf{a_n}.$$

EXAMPLE 2. If $A = \begin{bmatrix} 2 & -1 & 3 & 5 \\ 0 & 2 & -3 & 1 \\ -3 & 4 & 1 & 2 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ -2 \end{bmatrix}$, then compute $A\mathbf{x}$.

SOLUTION.

$$A\vec{x} = 2\begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} + 1\begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} + 0\begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix} + (-2)\begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -7 \\ 0 \\ -6 \end{bmatrix} \swarrow \text{ 3x1 vector.}$$

REMARK: Nb. of colums of A should be equal to the # of rows of $\vec{x}$.

Properties:

- $A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y}$.
- $A(a\mathbf{x}) = a(A\mathbf{x}) = (aA)\mathbf{x}$, for any scalar a.
- $(A+B)\mathbf{x} = A\mathbf{x} + B\vec{\mathbf{x}}$.

THE DOT PRODUCT

DEFINITION 2. If $\mathbf{x}$ is an $1 \times n$ vector and $\mathbf{y}$ is an $n \times 1$ vectors, their **dot product** is defined to be the number

$$\mathbf{x} \cdot \mathbf{y} := x_1 y_1 + x_2 y_2 + \ldots + x_n y_n.$$

EXAMPLE 3. Use the dot product to compute $A\mathbf{x}$ where A and $\mathbf{x}$ are as in Example 2.

SOLUTION.

The 1^{st} entry of $A\vec{x}$ is

$$-7 = 2 \cdot 2 + (-1)(1) + (3)(0) + (5)(-2)$$

$$= \underbrace{\begin{bmatrix} 2 & -1 & 3 & 5 \end{bmatrix}}_{1^{st} \text{ row of } A} \cdot \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \\ -2 \end{bmatrix}}_{\vec{x}}$$

The 2^{nd} entry of $A\vec{x}$:

$$0 = \underbrace{\begin{bmatrix} 0 & 2 & -3 & 1 \end{bmatrix}}_{2^{nd} \text{ row of } A} \bullet \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \\ -2 \end{bmatrix}}_{\vec{x}}$$

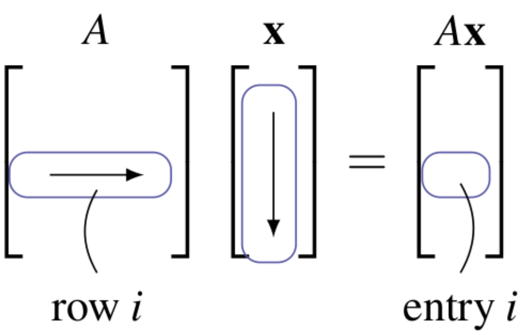
Finally, 3^{rd} entry of $A\vec{x}$:

$$-6 = \underbrace{\begin{bmatrix} -3 & 4 & 1 & 2 \end{bmatrix}}_{3^{rd} \text{ row of } A.} \bullet \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \\ -2 \end{bmatrix}}_{\vec{x}}$$

Now

$$A\vec{x} = \begin{bmatrix} 2 & -1 & 3 & 5 \\ 0 & 2 & -3 & 1 \\ -3 & 4 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} -7 \\ 0 \\ -6 \end{bmatrix}.$$

<u>The Dot Product Rule.</u>



To obtain the entry i of $A\mathbf{x}$, take the dot product of row i of A with the vector $\mathbf{x}$.

EXAMPLE 4. Find an $n \times n$ matrix A such that $A\mathbf{x} = \mathbf{x}$, for any $\mathbf{x} \in \mathbb{R}^n$.

SOLUTION.

Start with 2×2 : $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. So

$$A\vec{x} = \vec{x} \iff \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \forall \vec{x} \in \mathbb{R}^2$$

$$\iff \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\underline{x_1 = 1, x_2 = 0}$: $\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow a = 1, \; c = 0$

$\underline{x_1 = 0, x_2 = 1}$: $\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow b = 0, \; d = 1$

So, $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow$ 2×2 Identity matrix.

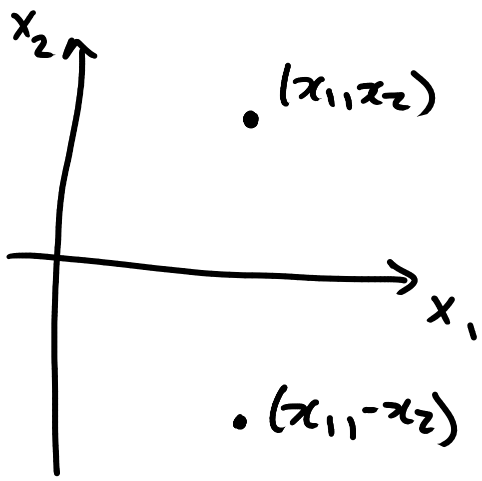
$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow$ 3×3 Identity matrix

$\boxed{n\times n \text{ identity matrix}}$: $I = [a_{ij}]$, $a_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$

THEOREM 1. Let A and B be two $m \times n$ matrices. If $A\mathbf{x} = B\mathbf{x}$ for any $\mathbf{x} \in \mathbb{R}^n$, then $A = B$.

EXAMPLE 5. A function is defined as follows: it reflects a 2×1 vector accross the x-axis in the 2D space. Illustrate graphically the **action** of this function and find a formula to describe it.

SOLUTION.



So, the transformation T is

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$$

Here

$$\begin{bmatrix} x_1 \\ -x_2 \end{bmatrix} = \begin{bmatrix} 1 \cdot x_1 + 0 \cdot x_2 \\ 0 \cdot x_1 + (-1)\, x_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \left(A\vec{x} \text{ for } A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right)$$

DEFINITION 3. Given an $m \times n$ matrix A, the **matrix transformation induced** by the matrix A denoted by T_A is defined by

Ref. accross y:

$$T_A(\mathbf{x}) = A\mathbf{x} \quad \forall \mathbf{x} \in \mathbb{R}^n.$$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ x_2 \end{bmatrix}$$

Note:

- For each $\mathbf{x} \in \mathbb{R}^n$, we have $T_A(\mathbf{x}) \in \mathbb{R}^m$. In this case, the expression of $T_A(\mathbf{x})$ is called the **action** of T_A.
- Therefore, $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is a function.
- For two matrices A and B, we say that T_A and T_B are **equal** if they have the same action, meaning $T_A(\mathbf{x}) = T_B(\mathbf{x})$, for any $\mathbf{x} \in \mathbb{R}^n$.

EXAMPLE 6. Let A be the $m \times n$ zero matrix. Then T_A is called the **zero matrix-transformation**. Show that $T_A(\mathbf{x}) = \mathbf{0}$, where $\mathbf{0}$ is the m-vector with 0 in all its entries.

SOLUTION.

Write $A = [\underbrace{\vec{0} \quad \vec{0} \quad \cdots \quad \vec{0}}_{n \text{ times}}]$ $\qquad \vec{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \Big\} m \text{ zeros}$

$$\begin{aligned} T_A(\vec{x}) &= A\vec{x} \\ &= x_1 \vec{0} + x_2 \vec{0} + \cdots + x_n \vec{0} \\ &= \vec{0} + \vec{0} + \cdots + \vec{0} \\ &= \vec{0} \end{aligned}$$

$\square$