Section 6.2 — Problem 6 (10 Pts)

a. We want to find a, b, and c such that

$$a(x+1)+b(x^2+x)+c(x^2+2)=x^2+3x+2$$

which can be rewritten as

$$(b+c)x^2+(a+b)x+(a+2c)=x^2+3x+2.$$

Therefore, a, b, c are solutions to the system

$$\begin{cases} \quad b+\ \ c=1 \\ a+b \qquad =3 \\ a \qquad +2c=2 \end{cases}$$

The solution to this system is $a=2$, $b=1$ and $c=0$. Therefore,

$$2(x+1)+(x^2+x)=x^2+3x+2.$$

c. We want to find a, b, and c such that

$$a(x+1)+b(x^2+x)+c(x^2+2)=x^2+1$$

which can be rewritten as

$$(b+c)x^2+(a+b)x+(a+2c)=x^2+1.$$

Therefore, a, b, c are solutions to the system

$$\begin{cases} \quad b+\ \ c=1 \\ a+b \qquad =0 \\ a \qquad +2c=1 \end{cases}$$

The solution to this system is $a=-1/3$, $b=1/3$, and $c=2/3$. Therefore

$$-\tfrac{1}{3}(x+1)+\tfrac{1}{3}(x^2+x)+\tfrac{2}{3}(x^2+2)=x^2+1.$$

Section 6.2 — Problem 9 (10 Pts)

a. Collect the vectors in a matrix
$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$
The goal is to write any vector (a, b, c) from $\mathbb{R}^3$ as a linear combination of the vectors in the collection, that is solving the system
$$A\mathbf{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$
We have $\det(A) = 2 \neq 0$. Therefore the matrix A is invertible and there is always a solution to the above system. Hence, the vectors span $\mathbb{R}^3$.

c. Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a 2×2 matrix. We want to know if
$$M = x_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + x_3 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$
Using the operations on matrices, we then get
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + x_4 & x_3 + x_4 \\ x_3 & x_2 + x_4 \end{bmatrix}.$$
Therefore, the vector $\mathbf{x}$ is solution to the system
$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}.$$
If A is the matrix of coefficients, then $\det(A) = -1 \neq 0$. Therefore, the matrix A is invertible and the above system always has a solution. Hence, the set of matrices span $\mathbf{M_{22}}$.

Section 6.2 — Problem 14 (5 Pts)

If it was possible, then a linear combination with the two given vectors will give $(c + d, 2c + d, d)$, for $c, d \in \mathbb{R}$. If $d \neq 0$, then it won't take the form of the vectors in U. So it is impossible.

Section 5.2 — Problem 3 (10 Pts)

a. We put the vectors in a matrix
$$A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 3 & 9 \\ 2 & 0 & -6 \\ 0 & 3 & 6 \end{bmatrix}.$$

We find the RREF of A:

$$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The dimension of U is 2 because the number of pivots in the RREF is 2 and a basis for U is the first two columns of A.

c. We put the vectors in a matrix

$$A = \begin{bmatrix} -1 & 2 & 4 & 3 \\ 2 & 0 & 4 & -2 \\ 1 & 3 & 11 & 2 \\ 0 & -1 & -3 & -1 \end{bmatrix}.$$

We find the RREF of A:

$$\begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The dimension of U is 2 because the number of pivots in the RREF is 2 and a basis for U is the first two columns of A.

Section 6.3 — Problem 1d (5 Pts)

Set

$$a \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + c \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + d \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

We obtain, after using the matrix operations,

$$\begin{bmatrix} a+c+d & a+b+d \\ a+b+c & b+c+d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

We therefore get the following system

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

If A is the matrix of coefficients, then $\det(A) = 3$ and hence the matrix A is invertible. Therefore, the unique solution to the system is $A^{-1}\mathbf{0} = \mathbf{0}$. Hence $a = b = c = d = 0$ and the matrices are linearly independent.

Section 6.3 — Problem 6 (10 Pts)

c. Let $p(x) = ax^2 + bx + c$ be an element of U. This means $p(1) = 0$, and therefore $a + b + c = 0$. Hence

$$\begin{aligned} U = \{ax^2 + bx + c \,:\, a + b + c = 0\} &= \{ax^2 + bx + (-b - a) \,:\, a, b \in \mathbb{R}\} \\ &= \{a(x^2 - 1) + b(x - 1) \,:\, a, b, c \in \mathbb{R}\}. \end{aligned}$$

Hence we have $U = \text{span}\{x^2 - 1, x - 1\}$ and $\{x^2 - 1, x - 1\}$ is linear independent. Therefore $\{x^2 - 1, x - 1\}$ is a basis for U and $\dim U = 2$.

d. Let $p(x) = ax^2 + bx + c$ be an element of U. This means $p(x) = p(-x)$ and therefore

$$ax^2 + bx + c = ax^2 - bx + c \iff 2bx = 0 \iff b = 0.$$

Hence

$$U = \{ax^2 + c \,:\, a, c \in \mathbb{R}\} = \text{span}\,\{x^2, 1\}.$$

Also $\{x^2, 1\}$ are linear independent and therefore form a basis for U. We then get $\dim U = 2$.