Math 311

Chapter 7

Section 7.2: Kernel and Image of a Linear Transformation

Contents

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For this section, $T : V \to W$ is assumed to be a linear transformation.

Definition 1.

① The **kernel** of T is the set

$$\ker T := \{\mathbf{v} \in V \,:\, T(\mathbf{v}) = \mathbf{0}\}.$$

② The **image** of T is the set

$$\operatorname{im} T := \{T(\mathbf{v}) \,:\, \mathbf{v} \in V\}.$$

Example 1. Let A be an $m \times n$ matrix and consider $T_A(\mathbf{x}) = A\mathbf{x}$, where $\mathbf{x} \in \mathbb{R}^n$.

a) We have $\ker T_A = \{\mathbf{x} \in \mathbb{R}^n \,:\, A\mathbf{x} = \mathbf{0}\} = \operatorname{null} A$.

b) We have $\operatorname{im} T_A = \{A\mathbf{x} \,:\, \mathbf{x} \in \mathbb{R}^n\} = \operatorname{im} A$.

Theorem 1. We have that $\ker T$ is a subspace of V and $\operatorname{im} T$ is a subspace of W.

Notes:

① We set $\operatorname{nullity} T := \dim(\ker T)$.

② We set $\operatorname{rank} T := \dim(\operatorname{im} T)$.

Example 2. Define $T : \mathbf{M_{nn}} \to \mathbf{M_{nn}}$ by $T(A) := A - A^\top$. Find (a) $\ker T$ (b) $\operatorname{im} T$.

Solution.

Definition 2. Assume that $T : V \to W$ is a linear transformation.

① T is said to be **onto** if $\operatorname{im} T = W$.

② T is said to be **one-to-one** if $\ker T = \{\mathbf{0}\}$.

Example 3. Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ defined by $T(x, y, z) = (x, x, y, y)$.

a) Is T onto?

b) Is T one-to-one? If not, find nullity T.

Dimension Theorem

Theorem 2. Let $T : V \to W$ be any linear transformation with $n = \dim V < \infty$. Then

$$\dim V = \operatorname{nullity} T + \operatorname{rank} T.$$

Idea of the Proof. We let

- $r = \operatorname{rank} T = \dim(\operatorname{im} T)$;
- $k = \operatorname{nullity} T = \dim(\ker T)$.

Let $\{\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_r}\}$ be a basis for $\operatorname{im} T$. Then there are vectors $\mathbf{e_1}$, $\mathbf{e_2}$, …, $\mathbf{e_r}$ such that $T(\mathbf{e_i}) = \mathbf{w_i}$.

Let $\{\mathbf{f_1}, \mathbf{f_2}, \ldots, \mathbf{f_k}\}$ be a basis for $\ker T$.

Then the idea is to show that $\{\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_r}, \mathbf{f_1}, \ldots, \mathbf{f_k}\}$ is a basis for V, so that we get

$$n = k + r$$

showing the claim. $\square$