Math 311

Chapter 7

Section 7.1: Linear Transformations

Contents

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Given an $m \times n$ matrix A, we introduced the transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$T_A(\mathbf{x}) = A\mathbf{x} \quad (\mathbf{x} \in \mathbb{R}^n).$$

From the properties of matrix multiplication, we have

(T1) $T_A(\mathbf{x} + \mathbf{y}) = A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = T_A(\mathbf{x}) + T_A(\mathbf{y})$.

(T2) $T_A(a\mathbf{x}) = A(a\mathbf{x}) = a(A\mathbf{x}) = aT_A(\mathbf{x})$.

The transformations satisfying (T1) and (T2) are very special and play an important role in linear algebra.

DEFINITION 1. Let V and W be two vector spaces. A transformation $T : V \to W$ is called a **linear transformation** if it satisfies the following two conditions for any vectors $\mathbf{v_1}$ and $\mathbf{v_2}$ in V and any scalars a:

(T1) $T(\mathbf{v_1} + \mathbf{v_2}) = T(\mathbf{v_1}) + T(\mathbf{v_2})$.

(T2) $T(a\mathbf{v_1}) = aT(\mathbf{v_1})$.

Notations:

① The **identity transformation** is the transformation $1_V : V \to V$ given by $1_V(\mathbf{v}) = \mathbf{v}$, for any $\mathbf{v} \in V$.

② The **zero transformation** is the transformation $0 : V \to W$ given by $0(\mathbf{v}) = \mathbf{0}$, for any $\mathbf{v} \in V$.

Example 1. Show that the following transformation is a linear transformation.

$$D : \mathbf{P}_n \to \mathbf{P_{n-1}}, \qquad D(p(x)) = p'(x).$$

Solution.

(T1) Let $\overset{v_1}{\overset{\shortparallel}{p}}, \overset{v_2}{\overset{\shortparallel}{q}} \in P_n$

$$D(p(x)+q(x)) = (p(x)+q(x))'$$

$$= p'(x) + q'(x) \quad \text{(Calculus)}$$

$$= D(p(x)) + D(q(x)) \checkmark$$

(T2) Let $\overset{v_1}{\overset{\shortparallel}{p}} \in P_n$ and $a \in \mathbb{R}$

$$D(ap(x)) = (a\,p(x))'$$

$$= a\,p'(x) \quad \text{(Calculus)}$$

$$= a\,D(p(x)) \checkmark$$

So, D is a linear transformation.

$T(\vec{0}) = \vec{0}$
idea: $\vec{0} + \vec{0} = \vec{0}$

EXAMPLE 2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that

$$T\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}2\\-3\end{bmatrix} \quad \text{and} \quad T\begin{bmatrix}1\\-2\end{bmatrix} = \begin{bmatrix}5\\1\end{bmatrix}.$$

Find $T\begin{bmatrix}4\\3\end{bmatrix}$.

SOLUTION. $\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-2\end{bmatrix}\right\}$ is a basis.

Trick: $\begin{bmatrix}4\\3\end{bmatrix} = \frac{11}{3}\begin{bmatrix}1\\1\end{bmatrix} + \frac{1}{3}\begin{bmatrix}1\\-2\end{bmatrix}$.

$$\Rightarrow T\begin{bmatrix}4\\3\end{bmatrix} = T\left(\frac{11}{3}\begin{bmatrix}1\\1\end{bmatrix} + \frac{1}{3}\begin{bmatrix}1\\-2\end{bmatrix}\right)$$

$$\overset{(T1)}{=} T\left(\frac{11}{3}\begin{bmatrix}1\\1\end{bmatrix}\right) + T\left(\frac{1}{3}\begin{bmatrix}1\\-2\end{bmatrix}\right)$$

$$= \frac{11}{3}T\begin{bmatrix}1\\1\end{bmatrix} + \frac{1}{3}T\begin{bmatrix}1\\-2\end{bmatrix}$$

$$= \frac{11}{3}\begin{bmatrix} 2 \\ -3 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 \\ -32/3 \end{bmatrix}.$$

THEOREM 1. If $T : V \to W$ is a linear transformation and $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k} \in V$ and $v_1, v_2, \ldots, v_k \in \mathbb{R}$, then

$$T(v_1\mathbf{v_1} + v_2\mathbf{v_2} + \cdots + v_k\mathbf{v_k}) = v_1T(\mathbf{v_1}) + v_2T(\mathbf{v_2}) + \cdots + v_kT(\mathbf{v_k}).$$

EXAMPLE 3. Find the expression of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that

$$T\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1\\0\\1\end{bmatrix}, \quad \text{and} \quad T\begin{bmatrix}-1\\0\end{bmatrix} = \begin{bmatrix}0\\1\\1\end{bmatrix}.$$

SOLUTION.

Goal: $T\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}??\\??\\??\end{bmatrix}_{3\times 1}$

Trick: $\begin{bmatrix}x\\y\end{bmatrix} = (y/2)\begin{bmatrix}1\\2\end{bmatrix} + (-x+\frac{y}{2})\begin{bmatrix}-1\\0\end{bmatrix}$

$$\Rightarrow T\begin{bmatrix}x\\y\end{bmatrix} = T\left(\frac{y}{2}\begin{bmatrix}1\\2\end{bmatrix} + \left(-x+\frac{y}{2}\right)\begin{bmatrix}-1\\0\end{bmatrix}\right)$$

$$= \frac{y}{2}T\begin{bmatrix}1\\2\end{bmatrix} + \left(-x+\frac{y}{2}\right)T\begin{bmatrix}-1\\0\end{bmatrix}$$

$$= \frac{y}{2}\begin{bmatrix}1\\0\\1\end{bmatrix} + (-x+y/2)\begin{bmatrix}0\\1\\1\end{bmatrix}$$

$$= \begin{bmatrix}y/2\\-x+y/2\\-x+y\end{bmatrix}$$

other expression:

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1/2 \\ -1 & 1/2 \\ -1 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1/2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = y/2$$

Claim:

$T : \mathbb{R}^n \to \mathbb{R}^m \Rightarrow T(\vec{x}) = A\vec{x}$ for an $m \times n$ matrix A.

THEOREM 2. Let

① V and W be vector spaces

② $\{\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}\}$ be a basis for V.

③ $\mathbf{w_1}$, $\mathbf{w_2}$, ..., $\mathbf{w_n}$ be vectors in W

Then there exists a unique linear transformation $T : V \to W$ satisfying $T(\mathbf{e_i}) = \mathbf{w_i}$, for any $i = 1, 2, \ldots, n$. In particular, the action of T on a given $\mathbf{v} = v_1\mathbf{e_1} + v_2\mathbf{e_2} + \cdots + v_n\mathbf{v_n}$ is

$$T(\mathbf{v}) = v_1\mathbf{w_1} + v_2\mathbf{w_2} + \cdots + v_n\mathbf{w_n}.$$