Section 9.1 — Problem 1 (10 Pts)

a. We want to find a, b, and c such that

$$a(x+1) + b(x^2) + c(3) = \mathbf{v} = 2x^2 + x - 1.$$

Therefore

$$bx^2 + ax + (a + 3c) = 2x^2 + x - 1.$$

and $b = 2$, $a = 1$ and $1 + 3c = -1$. Therefore

$$b = 2, \ a = 1 \text{ and } c = -2/3.$$

Hence

$$C_B(\mathbf{v}) = \begin{bmatrix} 1 \\ 2 \\ -2/3 \end{bmatrix}.$$

c. We want to find a, b, and c such that

$$\mathbf{v} = (1, -1, 2) = a(1, -1, 0) + b(1, 1, 1) + c(0, 1, 1).$$

Therefore

$$(1, -1, 2) = (a + b, -a + b + c, b + c)$$

and $a + b = 1$, $-a + b + c = -1$, and $b + c = 2$. Adding the first equation to the second equation, we get $2b + c = 0$ and subtracting the third equation to this last equation:

$$2b + c - b - c = -2 \quad \Rightarrow \quad b = -2.$$

We then find $a = 3$ and $c = 4$. Hence

$$C_B(\mathbf{v}) = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}.$$

Section 9.1 — Problem 4 (10 Pts)

a. Since B and D are the standard basis, it is more easy to find the matrix representation.

We have

$$T(1, 0, 0) = (1, 0, 0, 1) \quad \Rightarrow \quad C_D(T(1, 0, 0)) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

then

$$T(0,1,0) = (0,0,1,2) \quad \Rightarrow \quad C_D(T(0,1,0)) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix},$$

and then

$$T(0,0,1) = (1,2,-1,0) \quad \Rightarrow \quad C_D(T(0,0,1)) = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}.$$

Therefore

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix}$$

is the matrix representing T on the standard basis B and D.

We have

$$C_B(\mathbf{v}) = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \quad \Rightarrow \quad C_D(T(\mathbf{v})) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ -4 \\ -1 \end{bmatrix}.$$

c. Let $\mathbf{b_1} = 1$, $\mathbf{b_2} = x$, and $\mathbf{b_3} = x^2$. Let $\mathbf{d_1} = (1,0)$ and $\mathbf{d_2} = (1,-1)$.

We have

$$T(\mathbf{b_1}) = T(1 + 0x + 0x^2) = (1,0) \quad \Rightarrow \quad C_D(T(\mathbf{b_1})) = (1)\mathbf{d_1} + (0)\mathbf{d_2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

then

$$T(\mathbf{b_2}) = T(0 + 1x + 0x^2) = (0,2) \quad \Rightarrow \quad C_D(T(\mathbf{b_2})) = (2)\mathbf{d_1} + (-2)\mathbf{d_2} = \begin{bmatrix} 2 \\ -2 \end{bmatrix},$$

and

$$T(\mathbf{b_3}) = T(0 + 0x + 1x^2) = (1,0) \quad \Rightarrow \quad C_D(T(\mathbf{b_3})) = (1)\mathbf{d_1} + (0)\mathbf{d_2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Therefore

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 0 \end{bmatrix}$$

is the matrix representing T on the basis B and D.

We have $C_B(\mathbf{v}) = \begin{bmatrix} a & b & c \end{bmatrix}^\top$ and therefore

$$C_D(T(\mathbf{v})) = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a + 2b + c \\ -2b \end{bmatrix}.$$

Section 9.1 — Problem 21a (5 Pts)

Asssume that S and T are linear transformations. Let $\mathbf{v} \in \ker S \cap \ker T$. This means $\mathbf{v} \in \ker S$ and $\mathbf{v} \in \ker T$. Therefore, $S(\mathbf{v}) = \mathbf{0}$ and $T(\mathbf{v}) = \mathbf{0}$. Hence

$$(S+T)(\mathbf{v}) = S(\mathbf{v}) + T(\mathbf{v}) = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

and $\mathbf{v} \in \ker(S+T)$.

Section 9.2 — Problem 1a (10 Pts)

We have

$$(0,-1) = (-1)(0,1) + (0)(1,1) \quad \Rightarrow \quad C_D((0,-1)) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

and

$$(2,1) = (-1)(0,1) + (2)(1,1) \quad \Rightarrow \quad C_D((2,1)) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

Hence, we obtain

$$P_{D \leftarrow B} = \begin{bmatrix} -1 & -1 \\ 0 & 2 \end{bmatrix}.$$

For $\mathbf{v} = (3,-5)$, we have

$$C_B(\mathbf{v}) = \begin{bmatrix} 13/2 \\ 3/2 \end{bmatrix} \text{ and } C_D(\mathbf{v}) = \begin{bmatrix} -8 \\ 3 \end{bmatrix}.$$

We therefore have

$$P_{D \leftarrow B} C_B(\mathbf{v}) = \begin{bmatrix} -1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 13/2 \\ 3/2 \end{bmatrix} = \begin{bmatrix} -8 \\ 3 \end{bmatrix} = C_D(\mathbf{v}).$$

Section 9.2 — Problem 7b (10 Pts)

In the basis B_0, we have

$$C_{B_0}(T(1)) = C_{B_0}(1+x^2) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad C_{B_0}(T(x)) = C_{B_0}(1+x) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad C_{B_0}(T(x^2)) C_{B_0}(x+x^2) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Therefore

$$M_{B_0}(T) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

In the basis B, we have

$$C_B(T(1-x^2)) = C_B(1-x) = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}, \quad C_B(T(1+x)) = C_B(2+x+x^2) = \begin{bmatrix} -3 \\ 5 \\ -2 \end{bmatrix}$$

and

$$C_B(T(2x+x^2)) = C_B(2+3x+x^2) = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}.$$

Therefore

$$M_B(T) = \begin{bmatrix} -2 & -3 & -1 \\ 3 & 5 & 3 \\ -2 & -2 & 0 \end{bmatrix}.$$

It is straightforward to obtain

$$P = P_{B_0 \leftarrow B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}.$$

Hence, we get

$$P^{-1} = \begin{bmatrix} -1 & 1 & -2 \\ 2 & -1 & 2 \\ -1 & 1 & -1 \end{bmatrix}.$$

Hence, we get

$$P^{-1}M_{B_0}(T)P = \begin{bmatrix} -1 & 1 & -2 \\ 2 & -1 & 2 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -3 & -1 \\ 3 & 5 & 3 \\ -2 & -2 & 0 \end{bmatrix} = M_B(T).$$

Section 9.2 — Problem 8b (5 Pts)

Using python, we get

$$P^{-1} = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$$

and so

$$P^{-1}AP = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix} \begin{bmatrix} 29 & -12 \\ 70 & -29 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = D.$$

Using the columns of P, we define

$$B = \left\{ \begin{bmatrix} 3 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\}.$$