MATH 311

CHAPTER 6

SECTION 6.1: VECTOR SPACES

CONTENTS

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Column Vectors

Recall that

$$\mathbb{R}^n = \{\mathbf{x} : \mathbf{x} \text{ is an } n \times 1 \text{ vector}\}.$$

① For addition:

A1. $\vec{x}, \vec{y} \Rightarrow \vec{x} + \vec{y} \in \mathbb{R}^n$.

A2. $\vec{x} + \vec{y} = \vec{y} + \vec{x}$ (Commutativity).

A3. $(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$ (Assoc.)

A4. $\vec{x} + \vec{0} = \vec{x} = \vec{0} + \vec{x}$

A5. For any $\vec{x}$, there is a $\vec{y}$ s.t.
$\vec{x} + \vec{y} = \vec{y} + \vec{x} = \vec{0}$ (here $\vec{y} = -\vec{x}$)

② For scalar multiplication:

S1. $\vec{x}$ and $a \in \mathbb{R} \Rightarrow a\vec{x} \in \mathbb{R}^n$.

S2. $a(\vec{x} + \vec{y}) = a\vec{x} + a\vec{y}$.

S3. $(a+b)\vec{x} = a\vec{x} + b\vec{x}$

S4. $a(b\vec{x}) = (ab)\vec{x}$

S5. $1\vec{x} = \vec{x}$

Conclusion: $\mathbb{R}^n$ is a vector space.

General Definition

Let V be a set of objects called **vectors**. Assume

1. **Vector Addition:** Two vectors $\mathbf{v}$ and $\mathbf{w}$ can be added and denote this operation by $\mathbf{v} + \mathbf{w}$.
2. **Scalar Multiplication:** Any vector $\mathbf{v}$ can be multiplied by any number (scalar) a and denote this operation by $a\mathbf{v}$.

The set V is called a **vector space** if

1. Axioms for the vector addition:

   $\boxed{\text{A1.}}$ Closed: $\mathbf{v}, \mathbf{w} \in V \Rightarrow \mathbf{v} + \mathbf{w} \in V$.

   $\boxed{\text{A2.}}$ Commutativity: $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$.

   $\boxed{\text{A3.}}$ Associativity: $\mathbf{v} + (\mathbf{w} + \mathbf{z}) = (\mathbf{v} + \mathbf{w}) + \mathbf{z}$.

   $\boxed{\text{A4.}}$ Existence of a zero vector: $\mathbf{v} + \mathbf{0} = \mathbf{v} = \mathbf{0} + \mathbf{v}$.

   $\boxed{\text{A5.}}$ Existence of a negative: For each $\mathbf{v}$, there is a $\mathbf{w}$ such that $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v} = \mathbf{0}$.

2. Axioms for the scalar multiplication:

   $\boxed{\text{S1.}}$ $\mathbf{v} \in V \Rightarrow a\mathbf{v} \in V$.

   $\boxed{\text{S2.}}$ $a(\mathbf{v} + \mathbf{w}) = a\mathbf{v} + a\mathbf{w}$.

   $\boxed{\text{S3.}}$ $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.

   $\boxed{\text{S4.}}$ $a(b\mathbf{v}) = (ab)\mathbf{v}$.

   $\boxed{\text{S5.}}$ $1\mathbf{v} = \mathbf{v}$.

Spaces of Matrices

m, n are fixed

EXAMPLE 1. Let $\mathbf{M_{mn}}$ be the set of all $m \times n$ matrices, that is

$$\mathbf{M_{mn}} := \{A : A \text{ is an } m \times n \text{ matrix.}\}$$

Consider the addition and scalar multiplication for matrices. Show that $\mathbf{M_{mn}}$ is a vector space.

SOLUTION.

M_{mn} is a vector space with addition and scalar multiplication as defined in Chapter 2.

Spaces of Polynomials

EXAMPLE 2. Consider the space $\mathbf{P_3}$ of all polynomials of degree at most 3, that is

$$\mathbf{P} := \{a_3x^3 + a_2x^2 + a_1x + a_0 \; : \; a_i \in \mathbb{R}\}.$$

Define 0. $p(x)=q(x)$ iff p, q have same coef.

1. Addition: for two polynomials $p(x) = a_3x^3+a_2x^2+a_1x+a_0$ and $q(x) = b_3x^3 + b_2x^2 + b_1x + b_0$, define $p + q$ as the polynomial

$$\begin{aligned}(p+q)(x) &= p(x) + q(x)\\ &= (a_3+b_3)x^3 + (a_2+b_2)x^2 + (a_1+b_1)x + (a_0+b_0).\end{aligned}$$

2. Scalar multiplicatoin: for a polynomial $p(x) = a_3x^3+a_2x^2+a_1x + a_0$, define ap as the polynomial

$$(ap)(x) = ap(x) = (aa_3)x^3 + (aa_2)x^2 + (aa_1)x + (aa_0).$$

Show that $\mathbf{P_3}$, with this addition and scalar multiplication, is a vector space.

SOLUTION.

① Addition:

A1 $p+q$ has degree 3 by definition.

So, $p+q \in P_3$.

A2

$$(p+q)(x) = (a_3+b_3)x^3 + (a_2+b_2)x^2 + (a_1+b_1)x + (a_0+b_0)$$

$$= a_3x^3 + b_3x^3 + a_2x^2 + b_2x^2 + a_1x + b_1x + a_0 + b_0$$

$$(q+p)(x) = (b_3+a_3)x^3 + (b_2+a_2)x^2 + (b_1+a_1)x + (b_0+a_0)$$

$$= b_3x^3 + a_3x^3 + b_2x^2 + a_2x^2 + b_1x + a_1x + b_0 + a_0$$

The expressions $a_i x^i$ and $b_j x^j$ are $\mathbb{R}$ numbers, so $\mathbb{R}$ number commute

$$\Rightarrow (p+q)(x) = (q+p)(x).$$

A3 Let $r(x) = c_3x^3 + c_2x^2 + c_1x + c_0$.

$$(p+q)+r = \left((a_3+b_3)x^3 + (a_2+b_2)x^2 + (a_1+b_1)x + (a_0+b_0)\right) + r$$

$$= ((a_3+b_3)+c_3)x^3 + ((a_2+b_2)+c_2)x^2 + ((a_1+b_1)+c_1)x + ((a_0+b_0)+c_0)$$

$$= (a_3+(b_3+c_3))x^3 + (a_2+(b_2+c_2))x^2 + (a_1+(b_1+c_1))x + (a_0+(b_0+c_0))$$

$$= p + (q+r). \quad \checkmark$$

<u>A4</u>. Define: $0(x) = 0x^3 + 0x^2 + 0x + 0 = 0$.

$$\Rightarrow (p+0) = (a_3+0)x^3 + (a_2+0)x^2 + (a_1+0)x + (a_0+0)$$

$$= a_3x^3 + a_2x^2 + a_1x + a_0.$$

$$= p. \quad \checkmark$$

<u>A5</u>. Define $q(x) = (-a_3)x^3 + (-a_2)x^2 + (-a_1)x + (-a_0)$

$$\Rightarrow p+q = (a_3-a_3)x^3 + (a_2-a_2)x^2 + (a_1-a_1)x + (a_0-a_0)$$

$$= 0x^3 + 0x^2 + 0x + 0 = 0 .$$

<u>Scalar Multiplication</u>. S3–S5 are verified also.

<u>S1</u>. By definition, ap is a polynomial of degree at most 3.

<u>S2</u>. $a(p+q) \stackrel{?}{=} ap + aq$

$$a(p+q) = a\left((a_3+b_3)x^3 + (a_2+b_2)x^2 + (a_1+b_1)x + (a_0+b_0)\right)$$

$$= a(a_3+b_3)x^3 + a(a_2+b_2)x^2 + a(a_1+b_1)x + a(a_0+b_0)$$

$$= a(a_3)x^3 + aa_2x^2 + aa_1x + aa_0 + ab_3x^3 + ab_2x^2 + ab_1x + ab_0 = ap+aq \quad ✓$$

<u>Note:</u>

① The space of polynomial of degree at most n is denoted by $\mathbf{P_n}$ and is a vector space using the addition and scalar multiplication introduced above.

② The space of all polynomial of any degree is denoted by $\mathbf{P}$ and it is a vector space using the addition and scalar multiplication introduced above.

Weird Example

Example 3. Consider the set of all 2×1 vectors $\mathbb{R}^2$. Define the addition and scalar multiplication:

1. $\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 + 1 \end{bmatrix}$.
2. $a\mathbf{x} = a \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_1 \\ ax_2 + a - 1 \end{bmatrix}$.

Show that $\mathbb{R}^2$, with these operations, is a vector space.

Solution.

① Addition

A1. Since $\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 + 1 \end{bmatrix}$ is a 2×1 vector,

$\vec{x} + \vec{y} \in \mathbb{R}^2$. ✓

A2. $\vec{x} + \vec{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 + 1 \end{bmatrix}$

same ✓
commutativity
of $\mathbb{R}$ numbers.

$\vec{y} + \vec{x} = \begin{bmatrix} y_1 + x_1 \\ y_2 + x_2 + 1 \end{bmatrix}$

A3. $(\vec{x} + \vec{y}) + \vec{z} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 + 1 \end{bmatrix} + \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ ✓

$$= \begin{bmatrix} x_1 + y_1 + z_1 \\ x_2 + y_2 + 1 + z_2 + 1 \end{bmatrix}$$

$$\vec{x} + (\vec{y} + \vec{z}) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 + z_1 \\ y_2 + z_2 + 1 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + y_1 + z_1 \\ x_2 + y_2 + z_2 + 1 + 1 \end{bmatrix}$$

Same so equal. ✓

<u>A4</u> $\vec{x} + \vec{y} = \vec{x}$ Goal: Find that $\vec{y}$!

$$\Rightarrow \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 + 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Leftrightarrow \begin{array}{l} x_1 + y_1 = x_1 \\ x_2 + y_2 + 1 = x_2 \end{array}$$

$$\Rightarrow \begin{cases} y_1 = 0 \\ y_2 = -1 \end{cases} \Rightarrow \vec{0} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$ ✓

<u>A5</u>. $\vec{x} + \vec{y} = \vec{0} \Leftrightarrow \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 + 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

Hence, $\vec{y} = \begin{bmatrix} -x_1 \\ -2 - x_2 \end{bmatrix} \neq -\vec{x}$ ✓

② Scalar Multiplication (All satisfied)

S1. $a\vec{x}$ is a 2×1 vector from the definition. ✓

S2. $a(\vec{x}+\vec{y}) = a\left(\begin{bmatrix} x_1+y_1 \\ x_2+y_2+1 \end{bmatrix}\right)$

$$= \begin{bmatrix} a(x_1+y_1) \\ a(x_2+y_2+1)+a-1 \end{bmatrix}$$

$$= \begin{bmatrix} ax_1+ay_1 \\ ax_2+ay_2+a+a-1 \end{bmatrix}$$

$$= \begin{bmatrix} ax_1+ay_1 \\ ax_2+ay_2+2a-1 \end{bmatrix}$$

$$a\vec{x}+a\vec{y} = \begin{bmatrix} ax_1 \\ ax_2+a-1 \end{bmatrix} + \begin{bmatrix} ay_1 \\ ay_2+a-1 \end{bmatrix}$$

$$= \begin{bmatrix} ax_1+ay_1 \\ ax_2+a-1+ay_2+a-1+1 \end{bmatrix}$$

$$= \begin{bmatrix} ax_1+ay_1 \\ ax_2+ay_2+2a-1 \end{bmatrix}$$

Same! ✓

S3. $(a+b)\vec{x} = \begin{bmatrix} (a+b)x_1 \\ (a+b)x_2 + a+b-1 \end{bmatrix}$

$$= \begin{bmatrix} ax_1 + bx_1 \\ ax_2 + bx_2 + a+b-1 \end{bmatrix}$$

SAME !! ✓

$$a\vec{x} + b\vec{x} = \begin{bmatrix} ax_1 \\ ax_2 + a - 1 \end{bmatrix} + \begin{bmatrix} bx_1 \\ bx_2 + b - 1 \end{bmatrix}$$

$$= \begin{bmatrix} ax_1 + bx_1 \\ ax_2 + a - 1 + bx_2 + b - 1 + 1 \end{bmatrix}$$

$$= \begin{bmatrix} ax_1 + bx_1 \\ ax_2 + bx_2 + a + b - 1 \end{bmatrix}$$

S4. $a(b\vec{x}) = a\begin{bmatrix} bx_1 \\ bx_2 + b - 1 \end{bmatrix} = \begin{bmatrix} abx_1 \\ a(bx_2 + b - 1) + a - 1 \end{bmatrix}$

$$= \begin{bmatrix} abx_1 \\ abx_2 + ab - 1 \end{bmatrix}$$

$$(ab)\vec{x} = \begin{bmatrix} abx_1 \\ abx_2 + ab - 1 \end{bmatrix}$$ SAME !! ✓

S5. $1\vec{x} = \begin{bmatrix} 1x_1 \\ 1x_2 + 1 - 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{x}$ ✓.

Non-Example

Example 4. Consider the set of all 2×1 vectors $\mathbb{R}^2$. Define the addition and scalar multiplication:

1. $\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 + 1 \end{bmatrix}$.
2. $a\mathbf{x} = a \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_1 \\ ax_2 - 1 \end{bmatrix}$.

Goal: One of A1-A5 or S1-S5 is not satisfied

Show that $\mathbb{R}^2$, with these operations, is not a vector space.

Solution.

① Addition.

Same addition as in Example 3.

A1-A5 are satisfied.

② Scalar multiplication

S5: $1\vec{x} = \vec{x}$

We have $1\vec{x} = \begin{bmatrix} x_1 \\ x_2 - 1 \end{bmatrix} \neq \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{x}$

$\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow 1\vec{x} = \begin{bmatrix} 1 \\ 1-1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \neq \vec{x}$!

S5 is not satisfied! Not a vector space.

Consider a general vector space V.

① Cancellation: If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, then

$$\mathbf{v} + \mathbf{u} = \mathbf{v} + \mathbf{w} \Longrightarrow \mathbf{u} = \mathbf{w}.$$

② Multiplying by scalar 0:

$$0\mathbf{v} = \mathbf{0}.$$

③ Multiplying by the zero vector:

$$a\mathbf{0} = \mathbf{0}.$$

④ If $a\vec{v} = \vec{0}$, then $a = 0$ or $\mathbf{v} = \mathbf{0}$.

EXAMPLE 5. Simplify the following expression:

$$3(2(\mathbf{u} - 2\mathbf{v} - \mathbf{w}) + 3(\mathbf{w} - \mathbf{v}) - 7(\mathbf{u} - 3\mathbf{v} - \mathbf{w}).$$

SOLUTION.

$$= 3(2\vec{u} - 4\vec{v} - 2\vec{w}) + 3\vec{w} - 3\vec{v} - 7\vec{u} + 21\vec{v} + 7\vec{w}$$

$$= 6\vec{u} - 12\vec{v} - 6\vec{w} + 3\vec{w} - 3\vec{v} - 7\vec{u} + 21\vec{v} + 7\vec{w}$$

$$= \boxed{-\vec{u} + 6\vec{v} + 4\vec{w}}$$