Section 1.3 — Problem 3 (10 Pts)

a. We want to know if $\mathbf{v} = a\mathbf{x} + b\mathbf{y} + c\mathbf{z}$. Using operations on n-vectors, we can rewrite this as the following system of linear equations:

$$\begin{cases} 2a + b + c = 0 \\ a + c = 1 \\ -a + b - 2c = -3 \end{cases} \longrightarrow \left[\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ -1 & 1 & -2 & -3 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Hence, by setting $c = t$, we get $a = 1 - t$, $b = -2 + t$, and $c = t$. Letting $t = 0$, we get that $\mathbf{v} = \mathbf{x} - 2\mathbf{y}$.

c. We want to know if $\mathbf{v} = a\mathbf{x} + b\mathbf{y} + c\mathbf{z}$. Using operations of n-vectors, we can rewrite this as the following system of linear equations:

$$\begin{cases} 2a + b + c = 3 \\ a + c = 1 \\ -a + b - 2c = 0 \end{cases} \longrightarrow \left[\begin{array}{ccc|c} 2 & 1 & 1 & 3 \\ 1 & 0 & 1 & 1 \\ -1 & 1 & -2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Hence, setting $c = t$, we get $a = 1 - t$, $b = 1 + t$ and $c = t$. Letting $t = 0$, we get that $\mathbf{v} = \mathbf{x} + \mathbf{y}$.

Section 1.3 — Problem 4a (5 Pts)

We want to know if $\mathbf{y} = x_1\mathbf{a_1} + x_2\mathbf{a_2} + x_3\mathbf{a_3}$. Then, we get the system

$$\begin{cases} -x_1 + 3x_2 + x_3 = 1 \\ 3x_1 + x_2 + x_3 = 2 \\ 2x_2 + x_3 = 4 \\ x_1 + x_3 = 0 \end{cases} \longrightarrow \left[\begin{array}{ccc|c} -1 & 3 & 1 & 1 \\ 3 & 1 & 1 & 2 \\ 0 & 2 & 1 & 4 \\ 1 & 0 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

We get $0 = 1$ and hence the system is inconsistent. This means $\mathbf{y}$ is not a linear combination of $\mathbf{a_1}$, $\mathbf{a_2}$, and $\mathbf{a_3}$.

Section 1.3 — Problem 5a (5 Pts)

The augmented matrix and the RREF of the system is

$$\left[\begin{array}{ccccc|c} 1 & 2 & -1 & 2 & 1 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 2 & 4 & -2 & 3 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

Hence, setting $x_2 = t$ and $x_5 = s$, then $x_1 = -2t + \frac{s}{3}$, $x_2 = t$, $x_3 = -\frac{2s}{3}$, $x_4 = -s$, and $x_5 = s$. We can rewrite this as

$$\mathbf{x} = \begin{bmatrix} -2t + \frac{s}{3} \\ t \\ -\frac{2s}{3} \\ -s \\ s \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1/3 \\ 0 \\ -2/3 \\ -1 \\ 1 \end{bmatrix}$$

with $t, s \in \mathbb{R}$.

Section 2.1 — Problem 3 (10 Pts)

a. Notice that $3A$ is a 2×2 matrix, but $-2B$ is an 2×3 matrix. The dimensions don't match and therefore $3A - 2B$ is undefined.

e. We have

$$4A^\top - 3C = 4 \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} 9 & -3 \\ 6 & 0 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 4 & -4 \end{bmatrix} + \begin{bmatrix} -9 & 3 \\ -6 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ -2 & -4 \end{bmatrix}.$$

Section 2.1 — Problem 13a (5 Pts)

Assume that $A = [a_{ij}]$, with $a_{ij} = 0$ if $i \neq j$ and $B = [b_{ij}]$, with $b_{ij} = 0$ if $i \neq j$. Then $A + B = C = [c_{ij}]$ with $c_{ij} = a_{ij} + b_{ij}$. So we have to show that $c_{ij} = 0$ if $i \neq j$. Assume that $i \neq j$. Then $a_{ij} = 0$ and $b_{ij} = 0$ form the assumptions and therefore

$$c_{ij} = a_{ij} + b_{ij} = 0 + 0 = 0.$$

Hence, $c_{ij} = 0$ for $i \neq j$ and $A + B$ is a diagonal matrix.

Section 2.1 — Problem 15 (10 Pts)

a. Let's do some algebra first:

$$A^\top + 3 \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 4 \end{bmatrix}^\top = A^\top + 3 \begin{bmatrix} 1 & 1 \\ -1 & 2 \\ 0 & 4 \end{bmatrix} = A^\top + \begin{bmatrix} 3 & 3 \\ -1 & 2 \\ 0 & 4 \end{bmatrix}.$$

Plugging this in the original equation, we obtain

$$A^\top + \begin{bmatrix} 3 & 3 \\ -1 & 2 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 5 \\ 3 & 8 \end{bmatrix} \iff A^\top = \begin{bmatrix} 2 & 1 \\ 0 & 5 \\ 3 & 8 \end{bmatrix} - \begin{bmatrix} 3 & 3 \\ -1 & 2 \\ 0 & 4 \end{bmatrix} \iff A^\top = \begin{bmatrix} -1 & -2 \\ 1 & 3 \\ 3 & 4 \end{bmatrix}.$$

Recall that $(A^\top)^\top = A$, so that

$$A = \begin{bmatrix} -1 & -2 \\ 1 & 3 \\ 3 & 4 \end{bmatrix}^\top = \begin{bmatrix} -1 & 1 & 3 \\ -2 & 3 & 4 \end{bmatrix}.$$

b. The left hand side can be rearranged, after some algebra, as

$$3A + \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}.$$

Hence, plugging that back in the original equation, we get

$$3A + \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 3 & 1 \end{bmatrix} \iff 3A = \begin{bmatrix} 6 & 0 \\ 3 & -3 \end{bmatrix} \iff A = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}.$$

Section 2.1 — Problem 17 (5 Pts)

Assume that A is a square matrix, say $n \times n$. Then $A^\top$ is also an $n \times n$ matrix and therefore $A + A^\top$ is well-defined. We then have

$$(A + A^\top)^\top = A^\top + (A^\top)^\top = A^\top + A = A + A^\top.$$

Hence, $A + A^\top$ is a symmetric matrix. $\square$