Math 311

Chapter 6

Section 6.3: Linear Independence and Dimension

Contents

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EXAMPLE 1. Let $\mathbf{v} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$. Let $\mathbf{u_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u_2} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u_3} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{v_3} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$

a) Consider the vectors. Can you write $\mathbf{v}$ as a unique linear combination of the vectors $\mathbf{u_1}$, $\mathbf{u_2}$, and $\mathbf{u_3}$?

b) Consider the vectors. Can you write the vector $\mathbf{v}$ as a unique linear combination of the vectors $\mathbf{v_1}$, $\mathbf{v_2}$, $\mathbf{v_3}$?

SOLUTION.

(a) Let $\vec{v} = a_1\vec{u}_1 + a_2\vec{u}_2 + a_3\vec{u}_3$. The solution is $a_1 = 2-t$, $a_2 = -1-t$ and $a_3 = t$.

$$\left.\begin{array}{l} t=0 \to \vec{v} = 2\vec{u}_1 + (-1)\vec{u}_2 + 0\vec{u}_3 \\ t=1 \to \vec{v} = 1\vec{u}_1 + (-2)\vec{u}_2 + 1\vec{u}_3 \end{array}\right\} \text{not unique.}$$

(b) Let $\vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3$. The solution

$a_1 = 3$, $a_2 = 2$, $a_3 = -3$.

$\to \vec{v} = 3\vec{v}_1 + 2\vec{v}_2 + (-3)\vec{v}_3$ $\boxed{\text{Unique!}}$

Definition 1. A set of vectors $\{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}\}$ in a vector space V is called **linearly independent** (or simply **independent**) if

$$s_1\mathbf{v_1} + s_2\mathbf{v_2} + \cdots + s_n\mathbf{v_n} = \mathbf{0} \quad \Rightarrow \quad s_1 = s_2 = \cdots = s_n = 0.$$

A set of vectors that is not independent is said to be **linearly dependent** (or simply **dependent**).

Note:

- The **trivial linear combination** of the vectors $\mathbf{v_1}$, $\mathbf{v_2}$, …, $\mathbf{v_n}$ is the one with every coefficient zero:

$$0\mathbf{v_1} + 0\mathbf{v_2} + \cdots + 0\mathbf{v_n}.$$

- So the vectors $\mathbf{v_1}$, $\mathbf{v_2}$, …, $\mathbf{v_n}$ are linearly independent if and only if the only way to write $\mathbf{0}$ is using the trivial linear combination.

Example 2. Show that the set

$$\left\{ \overset{v_1}{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}, \overset{v_2}{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}}, \overset{v_3}{\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}}, \overset{v_4}{\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}} \right\}$$

is independent. in M_{22} . $\quad \vec{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Solution.

Write

$$s_1\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + s_2\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + s_3\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} + s_4\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} s_1 + s_2 & s_1 + s_4 \\ s_2 + s_3 & -s_3 + s_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} s_1 + s_2 = 0 \; ① & s_1 + s_4 = 0 \; ② \\ s_2 + s_3 = 0 & -s_3 + s_4 = 0 \end{cases}$$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right] \xrightarrow{\cdots} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\Rightarrow \quad s_1 = s_2 = s_3 = s_4 = 0$$

$\Rightarrow$ matrices are linearly independent.

Example 3. Let $\{\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}\}$ be an independent set in a vector space V. Which of the following set is indepedent?

a) $\{\underbrace{\mathbf{x}-\mathbf{y}}_{v_1}, \underbrace{\mathbf{y}-\mathbf{z}}_{v_2}, \underbrace{\mathbf{z}-\mathbf{x}}_{v_3}\}$.

b) $\{\mathbf{x}-\mathbf{y}, \mathbf{y}-\mathbf{z}, \mathbf{z}-\mathbf{w}, \mathbf{w}-\mathbf{x}\}$.

Solution.

a) Write $s_1(\vec{x}-\vec{y}) + s_2(\vec{y}-\vec{z}) + s_3(\vec{z}-\vec{x}) = \vec{0}$

$$\Rightarrow (s_1 - s_3)\vec{x} + (-s_1+s_2)\vec{y} + (-s_2+s_3)\vec{z} = \vec{0}$$
$$+ \ 0\vec{w}$$

lin. ind.

$$\Rightarrow \quad s_1 - s_3 = 0, \quad -s_1+s_2 = 0, \quad -s_2+s_3 = 0$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] \xrightarrow{\cdots} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} s_1 = t \\ s_2 = t \\ s_3 = t \end{array}$$

$\hookrightarrow$ <u>lin. dep.</u>

b) Write

$$\underset{1}{\underset{\parallel}{s_1}}(\vec{x}-\vec{y}) + \underset{1}{\underset{\parallel}{s_2}}(\vec{y}-\vec{z}) + \underset{1}{\underset{\parallel}{s_3}}(\vec{z}-\vec{w}) + \underset{1}{\underset{\parallel}{s_4}}(\vec{w}-\vec{x}) = \vec{0}$$

$$\Rightarrow \cancel{\vec{x}} - \cancel{\vec{y}} + \cancel{\vec{y}} - \cancel{\vec{z}} + \cancel{\vec{z}} - \cancel{\vec{w}} + \cancel{\vec{w}} - \cancel{\vec{x}} = \vec{0}$$

$\Rightarrow$ lin. dependent.

$$(s_1 - s_4)\vec{x} + (-s_1+s_2)\vec{y} + (-s_2+s_3)\vec{z} + (-s_3+s_4)\vec{w} = \vec{0}$$

$\Rightarrow \quad s_1 - s_4 = 0, \; -s_1+s_2 = 0, \; -s_2+s_3=0$

$-s_3+s_4=0$

$\Rightarrow \quad s_1 = s_4, \quad s_2 = s_4, \quad s_3 = s_4, \quad s_3 = s_4$

$\Rightarrow \quad s_4 = t, \quad s_1 = t, \; s_2 = t, \; s_3 = t, \; s_4 = t.$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right]$$

DEFINITION 2. A set $\{\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}\}$ of vectors in a vector space V is called a basis of V if it satisfies the following two conditions:

① $\{\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}\}$ is linearly independent.

② $V = \text{span}\{\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}\}$. $\to \vec{v} = s_1\vec{e_1} + \cdots + s_n\vec{e_n}$.

EXAMPLE 4. Let $V = \mathbb{R}^3$. Verify the following.

a) If $\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{e_3}$ are the columns of I_3, then $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ is a basis for $\mathbb{R}^3$.

b) $\{\overset{\vec{v}_1}{[1\ -1\ 0]^\top}, \overset{\vec{v}_2}{[3\ 2\ -1]^\top}, \overset{\vec{v}_3}{[3\ 5\ -2]^\top}\}$ is a basis for $\mathbb{R}^3$.

a) ① Lin. ind.

$$s_1\vec{e_1} + s_2\vec{e_2} + s_3\vec{e_3} = \vec{0}$$

$$\Rightarrow s_1\begin{bmatrix}1\\0\\0\end{bmatrix} + s_2\begin{bmatrix}0\\1\\0\end{bmatrix} + s_3\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}s_1\\s_2\\s_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \Rightarrow s_1 = 0 \quad s_2 = 0 \quad s_3 = 0$$

② $V = \text{span}\{\vec{e_1}, \vec{e_2}, \vec{e_3}\}$.

$$\vec{v} = \begin{bmatrix}a\\b\\c\end{bmatrix} = a\underbrace{\begin{bmatrix}1\\0\\0\end{bmatrix}}_{\vec{e_1}} + b\underbrace{\begin{bmatrix}0\\1\\0\end{bmatrix}}_{\vec{e_2}} + c\underbrace{\begin{bmatrix}0\\0\\1\end{bmatrix}}_{\vec{e_3}}$$

So, $V = \text{span}\{\vec{e_1}, \vec{e_2}, \vec{e_3}\}$.

Therefore, $\{\vec{e_1}, \vec{e_2}, \vec{e_3}\}$ is a basis of $\mathbb{R}^3$.

(b) ① linearly independent

$s_1\vec{v_1} + s_2\vec{v_2} + s_3\vec{v_3} = \vec{0}$

$$\left[\begin{array}{ccc|c} 1 & 3 & 3 & 0 \\ -1 & 2 & 5 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right] \xrightarrow{\text{Python}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$\Rightarrow s_1 = 0, \ s_2 = 0, \ s_3 = 0$

$\Rightarrow \vec{v}_1, \vec{v}_2, \vec{v}_3$ are lin. ind.

② Spanning set

$\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, $a, b, c \in \mathbb{R}$. So

$\vec{v} = s_1\vec{v_1} + s_2\vec{v_2} + s_3\vec{v_3}$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 3 & a \\ -1 & 2 & 5 & b \\ 0 & -1 & -2 & c \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -a/2 - 3b/2 - 9c/2 \\ 0 & 1 & 0 & a + b + 4c \\ 0 & 0 & 1 & -a/2 - b/2 - 5c/2 \end{array}\right]$$

$\Rightarrow$ there is a solution for every $a, b, c \in \mathbb{R}$.

Conclusion: ① & ② are satisfied $\Rightarrow$ basis.

Observations:

- Invariance Theorem (p.347): If $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is a basis for a vector space V and if $\{\mathbf{f_1}, \mathbf{f_2}, \ldots, \mathbf{f_m}\}$ is a basis for a vector space V, then $m = n$.

Definition 3. If $\{\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}\}$ is a basis of a nonzero vector space V, the number n of vectors in the basis is called the **dimension**, and we write

$$\dim V = n.$$

In the case of the zero vector space, we define $\dim\{\mathbf{0}\} = 0$.

Note:

① We have $\dim \mathbb{R}^m = m$ because the m columns of the identity matrix I_m is a basis.

② We have $\dim \mathbf{M_{mn}} = mn$. Let E_{ij} be the matrix with a 1 in the (i, j)-entry and 0 elsewhere. A basis for $\mathbf{M_{mn}}$ is

$$B = \{M_{ij} \; : \; 1 \leq i \leq m, \; 1 \leq j \leq n\}.$$

This is called the **canonical basis** or **standard basis** of $\mathbf{M_{mn}}$. For instance, if $m = n = 2$, then a basis for $\mathbf{M_{22}}$ is

$$\begin{aligned} B &= \{M_{11}, M_{12}, M_{21}, M_{22}\} \\ &= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \end{aligned}$$

③ We have $\dim \mathbf{P_n} = n+1$. A basis is

$$B = \{1, x, x^2, \ldots, x^n\}$$

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \quad .$$

④ Any subspace U of a vector space V is a vector space. Therefore, we can find the dimension of U.

Subspaces of $\mathbb{R}^m$

For subspaces of $\mathbb{R}^m$, there is a really nice way to determine a basis and the dimension of a spanning set. Let

$$U = \operatorname{span}\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}.$$

Let

- $A = \begin{bmatrix} \mathbf{v_1} & \mathbf{v_2} & \cdots & \mathbf{v_n} \end{bmatrix}$.
- R be the RREF of A.

Then

- $\dim U$ = number of pivots in R.
- A basis for U is given by the vector in the same column as the pivots.

Example 5. Find a basis and calculate the dimension for the following subspace of $\mathbb{R}^4$:

$$U = \operatorname{span}\{(1, -1, 2, 0), (2, 3, 0, 3), (1, 9, -6, 6)\}.$$

Solution.

$$\begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 3 \\ 0 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 9 \\ -6 \\ 6 \end{bmatrix}$$

We have

$$A = \left[\begin{array}{cc|c} 1 & 2 & 1 \\ -1 & 3 & 9 \\ 2 & 0 & -6 \\ 0 & 3 & 6 \end{array}\right]$$

The RREF is

$$R = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence,

- $\dim U = 2$
- A basis for U is

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 3 \end{bmatrix} \right\}.$$

$$1 + x + 2x^2 \longrightarrow \begin{array}{c} 1: \\ x: \\ x^2: \end{array} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

<u>Note:</u> This trick also works for subspaces of the space of polynomials $\mathbf{P_n}$.

Subspaces of Matrices

EXAMPLE 6. Define the subspace of $\mathbf{M}_{22}$ as

$$U = \left\{ X \in \mathbf{M}_{22} \; : \; \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} X = X \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \right\}.$$

Find a basis of U and its dimension.

SOLUTION.

Write $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. $X \in U$ iff

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a & b \\ a & b \end{bmatrix} = \begin{bmatrix} a+b & 0 \\ c+d & 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{matrix} a = a+b \\ a = c+d \end{matrix} \qquad \begin{matrix} b = 0 \\ b = 0 \end{matrix} \quad \Rightarrow \quad \begin{matrix} a = c+d \\ b = 0 \\ c, d \in \mathbb{R} \end{matrix}$$

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c+d & 0 \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} c & 0 \\ c & 0 \end{bmatrix} + \begin{bmatrix} d & 0 \\ 0 & d \end{bmatrix}$$

$$= c\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So,

$$U = \operatorname{Span}\left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

- $\left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is independent:

$$s_1 \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + s_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow \quad s_1 + s_2 = 0 \qquad\qquad 0 = 0$

$\boxed{s_1 = 0} \qquad\qquad \boxed{s_2 = 0}$

<u>Conclusion:</u>

$\left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for U.