Math 311

Chapter 2

Section 2.5: Elementary Matrices

Contents

Basics 2

Inverses and Elementary Matrices 4

Created by: Pierre-Olivier Parisé
Spring 2024

Example 1. Let $E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$, and $E_3 = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$. Let $A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$.

a) Find $E_1 A$ and interprete the result.

b) Find $E_2 A$ and interprete the result.

c) Find $E_3 A$ and interprete the result.

Solution.

DEFINITION 1. An $n \times n$ matrix E is called an **elementary matrix** if it can be obtained from the identity matrix I_n by a single elementary row operation. We say that E is of type I, II, or III if the operation used to obtain E is of that type.

THEOREM 1.

1. If an elementary row operation is performed on an $m \times n$ matrix A, then the result is EA, where E is the associated elementary matrix.

2. Every elementary matrix E is invertible, and E^{-1} correspond to the inverse of the row operation that produces E.

Reminder:

Type	Operation	Inverse Operation
I	Interchange rows p and q	Interchange rows p and q
II	Multiply row p by $k \neq 0$	Multiply row p by $1/k$, $k \neq 0$
III	Add k times row p to row $q \neq p$	Subtract k times row p from row q, $q \neq p$

EXAMPLE 2. For each of the following matrices, describe the corresponding row operation and write the inverse.

$$E_1 = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Example 3. By recording each row operation as an elementary matrix, show that the invertible matrix $A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$ is a product of elementary matrices.

Solution.

Theorem 2. A square matrix is invertible if and only if it is a product of elementary matrices.

Assume that an $m \times n$ matrix A is carried to a matrix B (written $A \to B$) by a series of k elementary row operations.

Let E_1, E_2, $\ldots$, E_k be the corresponding elementary matrices. Then

$$AI_m \to E_1 A \to E_2 E_1 A \to \cdots \to E_k E_{k-1} \cdots E_2 E_1 A = B.$$

Writing $U = E_k E_{k-1} \cdots E_2 E_1$, then U is invertible and $B = UA$.

DEFINITION 2. We say that two matrices A and B are **row-equivalent** if there is an invertible matrix U such that $B = UA$.

EXAMPLE 4. Express the RREF of the matrix $A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & 0 \end{bmatrix}$ and a product UA, with U a 2×2 invertible matrix.

SOLUTION.