MIDTERM 01 SOLUTIONS

M311 SPRING 24

Question 1

$$\left[\begin{array}{cccc|c} 1 & 1 & 3 & -1 & 3 \\ 2 & -2 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 3 & -1 & 3 \\ 0 & -4 & -5 & 3 & -6 \end{array}\right] R_2 - 2R_1$$

$$\to \left[\begin{array}{cccc|c} 1 & 1 & 3 & -1 & 3 \\ 0 & 1 & 5/4 & -3/4 & 3/2 \end{array}\right] -\tfrac{1}{4}R_2$$

$$\to \left[\begin{array}{cccc|c} 1 & 0 & 7/4 & -1/4 & 3/2 \\ 0 & 1 & 5/4 & -3/4 & 3/2 \end{array}\right] R_1 - R_2$$

So, $\quad x_1 + \frac{7}{4}x_3 - \frac{x_4}{4} = \frac{3}{2}$

$\quad x_2 + \frac{5}{4}x_3 - \frac{3}{4}x_4 = \frac{3}{2}$

Let $x_3 = t$ and $x_4 = s$. Then

$x_1 = \frac{3}{2} - \frac{7}{4}t + \frac{s}{4} \qquad x_3 = t$

$x_2 = \frac{3}{2} - \frac{5}{4}t + \frac{3}{4}s \qquad x_4 = s$

Question 2

(a) Let a, b, c, d be such that

$$a\vec{x} + b\vec{y} + c\vec{z} + d\vec{v} = \vec{w}$$

$$\Leftrightarrow a\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + b\begin{bmatrix}1\\-1\\2\\-1\end{bmatrix} + c\begin{bmatrix}-2\\3\\2\\0\end{bmatrix} + d\begin{bmatrix}-1\\1\\0\\1\end{bmatrix} = \begin{bmatrix}9\\-8\\0\\1\end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix}a+b-2c-d\\a-b+3c+d\\a+2b+2c\\a-b+d\end{bmatrix} = \begin{bmatrix}9\\-8\\0\\1\end{bmatrix}$$

$$\Leftrightarrow \begin{cases} a+b-2c-d = 9 \\ a-b+3c+d = -8 \\ a+2b+2c = 0 \\ a-b+d = 1 \end{cases}$$

(b) Yes, the solution is

$$a = 2,\ b = 2,\ c = -3,\ d = 1.$$

Hence,

$$\vec{w} = 2\vec{x} + 2\vec{y} - 3\vec{z} + \vec{v}.$$

Question 3

(a) $\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 2 & -1 & 1 & -2 & 0 \\ 3 & -1 & 2 & -1 & 0 \end{array}\right]$

(b) Since the # eq = 3 and # var. = 4 and 4>3, then there are infinitely many solution.

(c) The equation are

$x_1 - 3x_4 = 0$ $\qquad x_3 + 4x_4 = 0$

$x_2 = 0$

Set $x_4 = t$, then

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3t \\ 0 \\ -4t \\ t \end{bmatrix} = t \underbrace{\begin{bmatrix} 3 \\ 0 \\ -4 \\ 1 \end{bmatrix}}_{\text{basic solution.}}$$

Question 4

$$\Leftrightarrow \quad 2(A^T)^T - 5\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}^T = 4A - \begin{bmatrix} 9 & 9 \\ -9 & 0 \end{bmatrix}$$

$$\Leftrightarrow \quad 2A - 5\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = 4A - \begin{bmatrix} 9 & 9 \\ -9 & 0 \end{bmatrix}$$

$$\Leftrightarrow \quad 2A - \begin{bmatrix} 5 & -5 \\ 0 & 10 \end{bmatrix} = 4A - \begin{bmatrix} 9 & 9 \\ -9 & 0 \end{bmatrix}$$

$$\Leftrightarrow \quad \begin{bmatrix} 9 & 9 \\ -9 & 0 \end{bmatrix} - \begin{bmatrix} 5 & -5 \\ 0 & 10 \end{bmatrix} = 4A - 2A$$

$$\Leftrightarrow \quad \begin{bmatrix} 4 & 14 \\ -9 & -10 \end{bmatrix} = 2A$$

$$\Leftrightarrow \quad \boxed{A = \begin{bmatrix} 2 & 7 \\ -9/2 & -5 \end{bmatrix}}$$

Question 5

Assume A, B are skew symmetric.

Then

$$
\begin{aligned}
(A-B)^T &= A^T - B^T && \text{(Prop. transpose)} \\
&= -A - (-B) && \text{(}A, B \text{ skew-sym.)} \\
&= -A + B \\
&= -(A-B)
\end{aligned}
$$

Hence, $A-B$ is skew symmetric. $\square$

Question 6

(a) True, rank = # leading ones = 2.

(b) False, it always has the trivial solution.

(c) False.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{ has solution } \begin{array}{l} x_1 = 1-t \\ x_2 = 2 \\ x_3 = t \end{array}$$

For $t=0$, $\vec{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ is a solution

For $t=1$, $\vec{x}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$ is also a solution. But $\vec{x_1} + \vec{x_2} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$ is not a solution because the second entry is 4, not 2.

(d) False. The conclusion is valid if the system is assumed to be consistent.