Math 311

Chapter 1

Section 1.2: Gaussian Elimination

Contents

Created by: Pierre-Olivier Parisé
Spring 2024

Matrix and Augmented Matrix

The system

$$\begin{aligned} x + y + z &= 1 \\ 2x + 2y + z &= 3 \end{aligned}$$

can be put in **matrix form** (array of numbers):

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 3 \end{array}\right]$$

Definition 1.

- The matrix $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix}$ is called the **coefficient matrix** of the system.
- The matrix $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is called the **constant matrix** of the system.

Solving a system of linear equations requires to transform the coefficient matrix in the following form:

$$\begin{bmatrix} 1 & * & * \\ 0 & 1 & * \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & * & * \\ 0 & 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \end{bmatrix}$$

where $*$ denotes any real number.

Row-Echelon Form of a matrix

Definition 2. A matrix is said to be in **row-echelon form** (REF for short) if it satisfies the following three conditions:

1. All **zero rows** are at the bottom.
2. The first nonzero entry from the left in each nonzero row is 1, called the **leading 1** for that row.
3. Each leading 1 is to the right of all leading 1s in the rows above it

Definition 3. A row-echelon matrix is said to be in **reduced row-echelon form** (RREF for short) if, in addition, it satisfies the following condition:

4. Each leading 1 is the only nonzero entry in its column.

Example 1. Circle the matrices in REF. Draw a square around the matrices in RREF.

$$\begin{bmatrix} 1 & * & * \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \boxed{\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}} \quad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 0 & 3 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Theorem 1. Every matrix can be transformed, with row operations, to a REF (or RREF).

Algorithm 4. **Gaussian Elinination.** To solve a system of linear equations proceed as follows.

1. Carry the augmented matrix to a reduced row-echelon form using elementary row operations.
2. If a row $\begin{bmatrix} 0 & 0 & \ldots & 0 & 1 \end{bmatrix}$ occurs, the system is inconsistent.
3. Otherwise, find the parametric form of the solution set to the system of equations.

Example 2. Solve the following system:

$$\begin{aligned} 3x + y - \ \ 4z &= -1 \\ x \qquad + 10z &= 5 \\ 4x + y + \ \ 6z &= 1 \end{aligned}$$

Solution.

1. RREF

$$\left[\begin{array}{ccc|c} 3 & 1 & -4 & -1 \\ 1 & 0 & 10 & 5 \\ 4 & 1 & 6 & 1 \end{array}\right] \xrightarrow{\cdots} \left[\begin{array}{ccc|c} 1 & 0 & 10 & 5 \\ 0 & 1 & -34 & -16 \\ 0 & 0 & 0 & -3 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 10 & 5 \\ 0 & 1 & -34 & -16 \\ 0 & 0 & 0 & 1 \end{array}\right] \quad -\tfrac{1}{3}R_3 .$$

$$\Rightarrow \quad 0x + 0y + 0z = 1 \quad \text{(third row)}$$

$$\Rightarrow \quad 0 = 1$$

2. there is not solution.

EXAMPLE 3. Solve the following equation:

$$\begin{aligned} x_1 - 2x_2 - x_3 + 3x_4 &= 1 \\ 2x_1 - 4x_2 + x_3 \phantom{{}+3x_4} &= 5 \\ x_1 - 2x_2 + 2x_3 - 3x_4 &= 4 \end{aligned}$$

SOLUTION.

$\boxed{1.}$ $$\begin{bmatrix} 1 & -2 & -1 & 3 & 1 \\ 2 & -4 & 1 & 0 & 5 \\ 1 & -2 & 2 & -3 & 4 \end{bmatrix} \xrightarrow{\cdots} \left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\boxed{2}$ Write the new system:

(1) $x_1 - 2x_2 + x_4 = 2$

(2) $x_3 - 2x_4 = 1$

(1) $\Rightarrow$ $x_1 = 2 + 2\boxed{x_2} - \boxed{x_4}$

(2) $\Rightarrow$ $x_3 = 1 + 2\boxed{x_4}$

Set $x_2 = s$ and $x_4 = t$, then

$$x_1 = 2 + 2s - t \qquad x_3 = 1 + 2t$$
$$x_2 = s \qquad x_4 = t \qquad , s, t \in \mathbb{R}$$

Note: The variable corresponding to the leading ones are called **leading variables**.

It can be proved that the number r of leading 1s must be the same in each of the row-echelon matrices.

Definition 5. The **rank** of a matrix A is the number of leading 1s in any REF to which A can be carried by row operations.

Example 4. Compute the rank of

$$A = \begin{bmatrix} 1 & 1 & -1 & 4 \\ 2 & 1 & 3 & 0 \\ 0 & 1 & -5 & 8 \end{bmatrix}$$

Solution.

$$A \to \begin{bmatrix} 1 & 1 & -1 & 4 \\ 0 & 1 & -5 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \operatorname{rank} A = 2.$$

THEOREM 2. Suppose a system of m equations in n variables is consistent, and that the rank of the coefficient matrix is r.

1. The set of solutions involves $n-r$ parameters.
2. If $r<n$, the system has infinitely many solutions.
3. If $r=n$, the system has a unique solution.

Three situations occur:

- No solution.
- Unique solution.
- Infinitely many solutions.

EXAMPLE 5. A system of equation with $m=4$ linear equations and $n=5$ variables has been carried to the following REF by row operations:

$$\left[\begin{array}{ccccc|c} \multicolumn{5}{c}{\overbrace{\hphantom{1\quad 2\quad -1\quad 3\quad -1}}^{A}} & \\ 1 & 2 & 1 & 3 & 1 & 1 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(a) Find the rank of the coefficient matrix. (b) Is there no solution, unique solution, or infinitely many solutions?

SOLUTION.

(a) rank $A = 3$

(b) Consistent.

So, $3 < 5 \overset{\text{THM2}}{\Rightarrow} \infty$ many solutions.