Math 311

Chapter 3

Section 3.2: Determinants and Matrix Inverses

Contents

Created by: Pierre-Olivier Parisé
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Product Rule

Example 1. Show that for any number a, b, c, d, we have the following identity

$$(a^2+b^2)(c^2+d^2) = (ac-bd)^2 + (ad+bc)^2.$$

Solution.

Theorem 1. If A and B are $n \times n$ matrices, then

$$\det(AB) = \det(A)\det(B).$$

Facts:

- For three matrices, $\det(ABC) = \det(A)\det(B)\det(C)$.
- For n matrices,

$$\det(A_1 A_2 \cdots A_n) = \det(A_1)\det(A_2)\cdots\det(A_n).$$

- For powers of a matrix, $\det(A^k) = (\det(A))^k$ (here, $k \geq 1$).

Example 2. Assume that $\det(A) = 2$, $\det(B) = 3$, and $\det(C) = -2$. Compute

$$\det(A^2 BCBC^2).$$

Solution.

Recall that

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ is invertible } \iff \det(A) = ad - bc \neq 0.$$

THEOREM 2. Let A be an $n \times n$ matrix. The matrix A is invertible if and only if $\det(A) \neq 0$. In this case, we have $\det(A^{-1}) = \frac{1}{\det(A)}$.

PROOF. See page 156 in the textbook for the complete proof.

EXAMPLE 3. For which real value(s) of c is the matrix $A = \begin{bmatrix} 0 & c & -c \\ -1 & 2 & 1 \\ c & -c & c \end{bmatrix}$ invertible?

SOLUTION.

Example 4. Let $A = \begin{bmatrix} 5 & 1 & 3 \\ -1 & 2 & 3 \\ 1 & 4 & 8 \end{bmatrix}$. Find $\det(A)$ and $\det(A^\top)$ and compare their values.

Solution.

Theorem 3. If A is an $n \times n$ matrix, then $\det(A) = \det(A^\top)$.

EXAMPLE 5. Assume that $\det(A) = 2$ and $\det(B) = 4$. Find the value of $\det(AA^T(B^T)^2)$.

SOLUTION.

EXAMPLE 6. A square matrix is called **orthogonal** if $A^{-1} = A^\top$. What are the possible values of $\det(A)$ if A is orthogonal?

SOLUTION.