MATH 311

CHAPTER 9

SECTION 9.2: OPERATORS AND SIMILARITY

CONTENTS

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DEFINITION 1. A linear transformation $T : V \to W$ is called an **linear operator** if $V = W$. We will therefore write $T : V \to V$, where V is a vector space.

B-matrix

Recall that if $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear operator and $E = \{\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}\}$ is the standard basis, then the matrix representing T on the basis E is

$$A = \begin{bmatrix} T(\mathbf{e_1}) & T(\mathbf{e_2}) & \cdots & T(\mathbf{e_n}) \end{bmatrix}.$$

DEFINITION 2. Let

- V be a vector space;
- $T : V \to V$ be a linear operator;
- $B = \{\mathbf{b_1}, \mathbf{b_2}, \ldots, \mathbf{b_n}\}$ be a basis.

The **B-matrix** of T is the matrix representing T on the basis B:

$$M_B(T) := \begin{bmatrix} C_B(T(\mathbf{b_1})) & C_B(T(\mathbf{b_2})) & \cdots & C_B(T(\mathbf{b_n})) \end{bmatrix}.$$

Properties:

① $C_B(T(\mathbf{v})) = M_B(T)C_B(\mathbf{v})$ for all $\mathbf{v} \in V$.

② T is an isomorphism if and only if $M_B(T)$ is invertible. More over, $M_B(T^{-1}) = (M_B(T))^{-1}$.

Example 1. Consider the square with vertices $(0,0)$, $(1,1)$, $(0,2)$, $(-1,1)$. Find a basis D on which the coordinates of the vertices become $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$.

Solution.

Goal: Given two basis

- $B = \{\mathbf{b_1}, \mathbf{b_2}, \ldots, \mathbf{b_n}\}$;
- $D = \{\mathbf{d_1}, \mathbf{d_2}, \ldots, \mathbf{d_n}\}$;

how do we get $C_D(\mathbf{v})$ from $C_B(\mathbf{v})$?

Example 1. [Continued]

Definition 3. We define the **change matrix** from B to D as

$$P_{D\leftarrow B} = \left[C_D(\mathbf{b_1}) \;\; C_D(\mathbf{b_2}) \;\; \cdots \;\; C_D(\mathbf{b_n})\right].$$

Properties:

① For any vector $\mathbf{v} \in V$, we have $C_D(\mathbf{v}) = P_{D\leftarrow B} C_B(\mathbf{v})$.

② $P_{B\leftarrow B} = I_n$.

③ $P_{D\leftarrow B}$ is invertible and $(P_{D\leftarrow B})^{-1} = P_{B\leftarrow D}$.

Example 2. Let $V = \mathbb{R}^2$ and $B = \{(1,2),(0,1)\}$, $D = \{(1,1),(-1,1)\}$.

a) Find $P_{D\leftarrow B}$.

b) Verify that $C_D(\mathbf{x}) = P_{D\leftarrow B}C_B(\mathbf{x})$.

c) Find $P_{B\leftarrow D}$, verify that $C_B(\mathbf{x}) = P_{B\leftarrow D}C_D(\mathbf{x})$.

Solution.

EXAMPLE 3. Let $A = \begin{bmatrix} 11 & -6 \\ 12 & -6 \end{bmatrix}$, $P = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$, and $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

a) Verify that $P^{-1}AP = D$.

b) Find a basis B such that $M_B(T_A) = D$.

Theorem 1.

① Let A be an $n \times n$ matrix and E be standard basis of $\mathbb{R}^n$.

② Let B be a basis of $\mathbb{R}^n$.

③ Let P be the invertible matrix whose columns are the vectors in B in order.

Then

$$M_B(T_A) = P^{-1}AP.$$