MATH 311

Chapter 2

SECTION 2.3: MATRIX MULTIPLICATION

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Composition of Transformations

EXAMPLE 1. Let $f(x) = \sin(x)$, $g(x) = x^2$, and $k(x) = \sqrt{x}$.

- a) Find $h = f \circ g$.
- b) Find $h = g \circ f$.
- c) Is $h = k \circ f$ well-defined?

SOLUTION.

DEFINITION 1. Let A be an $m \times n$ matrix and B be an $n \times k$ matrix. We define the composition of $T_A : \mathbb{R}^n \to \mathbb{R}^m$ with $T_B : \mathbb{R}^k \to \mathbb{R}^n$ as the function $T : \mathbb{R}^k \to \mathbb{R}^m$ defined by

$$T(\mathbf{x}) = (T_A \circ T_B)(\mathbf{x}) := T_A(T_B(\mathbf{x}))$$

for every $\mathbf{x} \in \mathbb{R}^k$.

Note: The order is very important! If $k \neq m$, then $T_B \circ T_A$ is not even defined!

Composing Two Matrix Transformation

Let
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 2 \end{bmatrix}$$
 and $B = \begin{bmatrix} 1 & 2 \\ 1 & -1 \\ -2 & 1 \end{bmatrix}$. Then, for $\mathbf{x} \in \mathbb{R}^2$, $(T_A \circ T_B)(\mathbf{x}) =$

In general:

$$(T_A \circ T_B)(\mathbf{x}) = T_A(T_B(\mathbf{x}))$$

$$= A(B\mathbf{x})$$

$$= A(x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \dots + x_k\mathbf{b}_k)$$

$$= A(x_1\mathbf{b}_1) + A(x_2\mathbf{b}_2) + \dots + A(x_k\mathbf{b}_k)$$

$$= x_1(A\mathbf{b}_1) + x_2(A\mathbf{b}_2) + \dots + x_k(A\mathbf{b}_k)$$

$$= [A\mathbf{b}_1 \ A\mathbf{b}_2 \ \dots \ A\mathbf{b}_k]\mathbf{x}.$$

Matrix Product

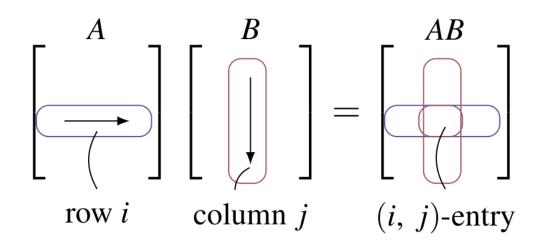
DEFINITION 2. Let A be an $m \times n$ matrix and B be an $n \times k$ matrix with $B = [\mathbf{b}_1 \ \mathbf{b}_2 \ \cdots \ \mathbf{b}_k]$, where \mathbf{b}_j is the column j of B. The **product matrix** AB is the $m \times k$ matrix defined as follows:

$$AB = A[\mathbf{b}_1 \ \mathbf{b}_2 \ \cdots \ \mathbf{b}_k] = [A\mathbf{b}_1 \ A\mathbf{b}_2 \ \cdots \ A\mathbf{b}_k]$$

Notes: The composite transformation $T_A \circ T_B$ is a matrix transformation induced by the matrix AB.

EXAMPLE 2. Compute the product
$$\begin{bmatrix} 5 & 0 & -7 \\ 1 & 5 & 9 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 0 \\ -1 & 3 \end{bmatrix}$$
.

Dot Product Rule



EXAMPLE 3. If
$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$
 and $B = \begin{bmatrix} 3 & 0 \\ -2 & 1 \\ 0 & 6 \end{bmatrix}$, find AB .

SOLUTION.

Compatibility Rule: The product of matrices A and B is only defined when the number of columns of A is equal to the number of rows of B.

EXAMPLE 4. (a) Compute the (2,4)-entry of AB if

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 & 6 & 0 \\ 0 & 2 & 3 & 4 \\ -1 & 0 & 5 & 8 \end{bmatrix}.$$

(b) Is BA well defined?

EXAMPLE 5. Let $A = \begin{bmatrix} 6 & 9 \\ -4 & -6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$. Compute A^2 , AB, BA, $(AB)^{\mathsf{T}}$ and $B^{\mathsf{T}}A^{\mathsf{T}}$.

SOLUTION.

Note: In general, $AB \neq BA$. If AB = BA, then we say that A and B commute.

THEOREM 1. Let a be a real number, and A, B, C are matrices of sizes such that the indicated matrix products are defined. Then:

- 1) IA = A and AI = A, where I denotes the identity matrix of proper size.
- $2) \ A(BC) = (AB)C.$
- 3) A(B+C) = AB + AC.
- 4) (B + C)A = BA + CA.
- 5) a(AB) = (aA)B = A(aB).
- $6) (AB)^{\top} = B^{\top} A^{\top}.$

Proof.

1) Assume that $A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_n]$ is of dimension $m \times n$ and I is the $m \times m$ identity matrix. Then

$$IA = [I\mathbf{a}_1 \ I\mathbf{a}_2 \ \cdots \ I\mathbf{a}_k]$$

= $[\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_k]$

where we used that $I\mathbf{x} = \mathbf{x}$ from Example 4 in Section 2.2.

2) If we write A in terms of its columns:

$$(B+C)A = [(B+C)\mathbf{a}_1 \cdots (B+C)\mathbf{a}_n]$$

$$= [B\mathbf{a}_1 + C\mathbf{a}_1 \cdots B\mathbf{a}_n + C\mathbf{a}_n]$$

$$= [B\mathbf{a}_1 \cdots B\mathbf{a}_n] + [C\mathbf{a}_1 \cdots C\mathbf{a}_n]$$

$$= BA + CA.$$

EXAMPLE 6. Simplify the following expression:

$$A(3B-C) + (A-2B)C + 2B(C+2A)$$

where A, B, C represent matrices.

EXAMPLE 7. Show that AB = BA if and only if $(A - B)(A + B) = A^2 - B^2$.

BLOCK MULTIPLICATION

DEFINITION 3. A matrix is said to be **partitioned into blocks** if the entries of the matrix are themselves matrices.

EXAMPLE 8. Writing $A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_n]$ in terms of its columns.

Matrix Product with Blocks

EXAMPLE 9. (a) Find a "nice" partition into blocks for the following matrices

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 2 & -1 & 4 & 2 & 1 \\ 3 & 1 & -1 & 7 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & -2 \\ 5 & 6 \\ 7 & 3 \\ -1 & 0 \\ 1 & 6 \end{bmatrix}.$$

(b) Use that to compute AB.

EXAMPLE 10. Obtain a formula for A^5 where $A = \begin{bmatrix} I & X \\ 0 & 0 \end{bmatrix}$ is a square matrix and I is an identity matrix.

SOLUTION.

Notes:

- Block Multiplication is useful in theory.
- It is also usuful in computing products of large matrices in a computer with limited memory capacity.