Section 2.6 — Problem 1a (10 Pts)

First we write

$$\begin{bmatrix} 8 \\ 3 \\ 7 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}.$$

Therefore, using the fact that T is a linear transformation, we get

$$\begin{aligned} T \begin{bmatrix} 8 \\ 3 \\ 7 \end{bmatrix} = T \left(2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right) &= 2 \left(T \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right) + 3 \left(T \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right) \\ &= 2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 6 \end{bmatrix}. \end{aligned}$$

Section 2.6 — Problem 7a (5 Pts)

Notice that

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} u \\ v \end{bmatrix} \right) = T \left(\begin{bmatrix} x+u \\ y+v \end{bmatrix} \right) = \begin{bmatrix} (x+u)(y+v) \\ 0 \end{bmatrix}$$

But,

$$T \begin{bmatrix} x \\ y \end{bmatrix} + T \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} xy \\ 0 \end{bmatrix} + \begin{bmatrix} uv \\ 0 \end{bmatrix} = \begin{bmatrix} xy + uv \\ 0 \end{bmatrix}.$$

We have $(x+u)(y+v) = xy + xv + yu + yv \neq xy + uv$ in general. Therefore,

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} u \\ v \end{bmatrix} \right) \neq T \begin{bmatrix} x \\ y \end{bmatrix} + T \begin{bmatrix} u \\ v \end{bmatrix}$$

The transformation T does not satisfy Axiom T1, and is therefore not linear.

Section 2.6 — Problem 14a (5 Pts)

There are many ways of doing that. Here are two ways:

1. Since $\mathbf{0} = \mathbf{0} + \mathbf{0}$ from the property of the zero vector, applying T on both sides and use the linearity of T gives
$$T(\mathbf{0}) = T(\mathbf{0}) + T(\mathbf{0}) = 2T(\mathbf{0}) \quad \Rightarrow \quad T(\mathbf{0}) - T(\mathbf{0}) = 2T(\mathbf{0}) - T(\mathbf{0}) \quad \Rightarrow \quad \mathbf{0} = T(\mathbf{0}).$$

2. Recall the property $0\mathbf{v} = \mathbf{0}$ for any vector $\mathbf{v}$. We therefore have $0\mathbf{0} = \mathbf{0}$ and applying T and using linearity of T:
$$0T(\mathbf{0}) = T(\mathbf{0}).$$
Again, using the property $0\mathbf{v} = \mathbf{0}$, we get that $0T(\mathbf{0}) = \mathbf{0}$. Hence, $\mathbf{0} = T(\mathbf{0})$.

Section 7.1 — Problem 4c (10 Pts)

First, notice that $\{x^2, x+1, x-1\}$ is a basis for $\mathbf{P_2}$. Notice that any polynomial $a+bx+cx^2$ can be written as

$$a+bx+cx^2 = cx^2 + \left(\frac{a+b}{2}\right)(x+1) + \left(\frac{a+b}{1}\right)(x-1)$$

Hence,

$$T(a+bx+cx^2) = \left(\frac{a+b}{2}\right)T(x+1) + \left(\frac{a-b}{2}\right)T(x-1) + cT(x^2) = \left(\frac{a-b}{2}\right)x + cx^3$$

For $\mathbf{v} = x^2+x+1$, we get

$$T(\mathbf{v}) = \left(\frac{1-1}{2}\right)x + (1)x^3 = x^3.$$

Additional notes: Notice that T is not injective! We can show that $\ker T = \operatorname{span}\{x+1\}$.

Section 7.2 — Problem 1a (15 Pts)

Finding the kernel. By definition $\ker T_A = \{\mathbf{x} : T_A\mathbf{x} = \mathbf{0}\}$. We have

$$\mathbf{x} \in \ker T_A \iff T_A\mathbf{x} = \mathbf{0} \iff \begin{bmatrix} 1 & 2 & -1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & -3 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We find the RREF of A:

$$A \to \begin{bmatrix} 1 & 0 & \frac{1}{5} & \frac{3}{5} \\ 0 & 1 & -\frac{3}{5} & \frac{1}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore we have $x = -z/5 - 3w/5$ and $y = 3z/5 - w/5$, with $z, w \in \mathbb{R}$.

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = z \begin{bmatrix} -1/5 \\ 3/5 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} -3/5 \\ -1/5 \\ 0 \\ 1 \end{bmatrix} = z\mathbf{x_1} + w\mathbf{x_2}$$

where $z, w \in \mathbb{R}$. Hence

$$\ker T_A = \operatorname{span}\{\mathbf{x_1}, \mathbf{x_2}\} = \operatorname{span}\left\{ \begin{bmatrix} -\frac{1}{5} & \frac{3}{5} & 1 & 0 \end{bmatrix}^\top, \begin{bmatrix} -\frac{3}{5} & -\frac{1}{5} & 0 & 1 \end{bmatrix}^\top \right\}.$$

This is a basis for $\ker T_A$ and therefore nullity $T_A = 2$.

Finding the image. By definition $\operatorname{Im} T_A = \{T_A(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^4\} = \{A\mathbf{x} : \mathbf{x} \in \mathbb{R}^4\}$. We have

$$A\mathbf{x} = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & -3 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = x \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + y \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} + w \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}.$$

Since x, y, z, w are arbitrary scalars, we see that

$$\operatorname{Im} T_A = \operatorname{span}\left\{ \begin{bmatrix}1\\3\\1\end{bmatrix}, \begin{bmatrix}2\\1\\-3\end{bmatrix}, \begin{bmatrix}-1\\0\\2\end{bmatrix}, \begin{bmatrix}1\\2\\0\end{bmatrix} \right\}.$$

From the RREF of A, there is a pivot in the first and second columns of the RREF. Therefore, the first and second columns of A forms a basis for $\operatorname{Im} T_A$. Hence

$$\operatorname{Im} T_A = \operatorname{span}\left\{ \begin{bmatrix}1\\3\\1\end{bmatrix}, \begin{bmatrix}2\\1\\-3\end{bmatrix} \right\}.$$

These two vectors form a basis for $\operatorname{Im} T_A$ and therefore $\operatorname{rank} T_A = 2$.

Additional Notes: There is a quicker way to find the rank of T_A by using the Dimension Theorem. Since we know that $\text{nullity} T_A = 2$ and $V = \mathbb{R}^4$, the Dimension Theorem tells us that

$$\text{nullity}\, T_A + \operatorname{rank} T_A = \dim V \quad \Rightarrow \quad 2 + \operatorname{rank} T_A = 4 \quad \Rightarrow \quad \operatorname{rank} T_A = 2.$$

Section 7.2 — Problem 20 (5 Pts)

Let $T : \mathbf{M_{nn}} \to \mathbf{M_{nn}}$ be the linear transformation

$$T(A) = A - A^{\top}.$$

From the Example done in the lecture notes, we know that

$$\ker T = \{A \in \mathbf{M_{nn}} \,:\, A \text{ is symmetric}\} = U \quad \text{and} \quad \operatorname{Im} T = \{A \in \mathbf{M_{nn}} \,:\, A \text{ is skew-symmetric}\} = V.$$

Therefore, by the Dimension Theorem, with $V = \mathbf{M_{nn}}$ we get

$$\text{nullity}\, T + \operatorname{rank} T = \dim(\mathbf{M_{nn}}).$$

We have $\text{nullity}\, T = \dim(\ker T) = \dim U$, $\mathbf{rank}\, T = \dim(\operatorname{Im} T)$, and $\dim(\mathbf{M_{nn}}) = n^2$. Therefore, replacing all the data in the formula from the Dimension Theorem, we get

$$\dim U + \dim V = n^2.$$