Math 311

Chapter 2

Section 2.3: Matrix Multiplication

Contents

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EXAMPLE 1. Let $f(x) = \sin(x)$, $g(x) = x^2$, and $k(x) = \sqrt{x}$.

a) Find $h = f \circ g$.

b) Find $h = g \circ f$.

c) Is $h = k \circ f$ well-defined?

SOLUTION.

(a) $h(x) = f(g(x)) = f(x^2) = \sin(x^2)$.

(b) $h(x) = g(f(x)) = g(\sin(x)) = \sin^2(x)$

(c) $h(x)$ = undefined for certain $x \in \mathbb{R}$.

DEFINITION 1. Let A be an $m \times n$ matrix and B be an $n \times k$ matrix. We define the composition of $T_A : \mathbb{R}^n \to \mathbb{R}^m$ with $T_B : \mathbb{R}^k \to \mathbb{R}^n$ as the function $T : \mathbb{R}^k \to \mathbb{R}^m$ defined by

$$T(\mathbf{x}) = (T_A \circ T_B)(\mathbf{x}) := T_A(T_B(\mathbf{x}))$$

for every $\mathbf{x} \in \mathbb{R}^k$.

<u>Note:</u> The order is very important! If $k \neq m$, then $T_B \circ T_A$ is not even defined!

Composing Two Matrix Transformation

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 1 & -1 \\ -2 & 1 \end{bmatrix}$. Then, for $\mathbf{x} \in \mathbb{R}^2$,

$$(T_A \circ T_B)(\mathbf{x}) = T_A(T_B(\vec{x})) \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

$$= A(B\vec{x})$$

$$= A\left(x_1 \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}\right)$$

$$= A\left(x_1 \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}\right) + A\left(x_2 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}\right)$$

$$= x_1 A\vec{b}_1 + x_2 A\vec{b}_2$$

$$= \boxed{\begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\boxed{\begin{aligned} A\vec{b}_1 &= \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix} \\ A\vec{b}_2 &= \begin{bmatrix} 3 \\ 2 \end{bmatrix} \\ \text{so } &\begin{bmatrix} -3 & 3 \\ 2 & 2 \end{bmatrix} \end{aligned}}$$

In general:

$$\begin{aligned}
(T_A \circ T_B)(\mathbf{x}) &= T_A(T_B(\mathbf{x})) \\
&= A(B\mathbf{x}) \\
&= A(x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_k\mathbf{b}_k) \\
&= A(x_1\mathbf{b}_1) + A(x_2\mathbf{b}_2) + \cdots + A(x_k\mathbf{b}_k) \\
&= x_1(A\mathbf{b}_1) + x_2(A\mathbf{b}_2) + \cdots + x_k(A\mathbf{b}_k) \\
&= [A\mathbf{b}_1 \ A\mathbf{b}_2 \ \cdots \ A\mathbf{b}_k]\mathbf{x}.
\end{aligned}$$

DEFINITION 2. Let A be an $m \times n$ matrix and B be an $n \times k$ matrix with $B = [\mathbf{b}_1\ \mathbf{b}_2\ \cdots\ \mathbf{b}_k]$, where $\mathbf{b}_j$ is the column j of B. The **product matrix** AB is the $m \times k$ matrix defined as follows:

$$AB = A[\mathbf{b}_1\ \mathbf{b}_2\ \cdots\ \mathbf{b}_k] = [A\mathbf{b}_1\ A\mathbf{b}_2\ \cdots\ A\mathbf{b}_k]$$

Notes: The composite transformation $T_A \circ T_B$ is a matrix transformation induced by the matrix AB.

EXAMPLE 2. Compute the product $\underbrace{\begin{bmatrix} 5 & 0 & -7 \\ 1 & 5 & 9 \end{bmatrix}}_{=A} \underbrace{\begin{bmatrix} 3 & 2 \\ 1 & 0 \\ -1 & 3 \end{bmatrix}}_{=B}$.

SOLUTION.

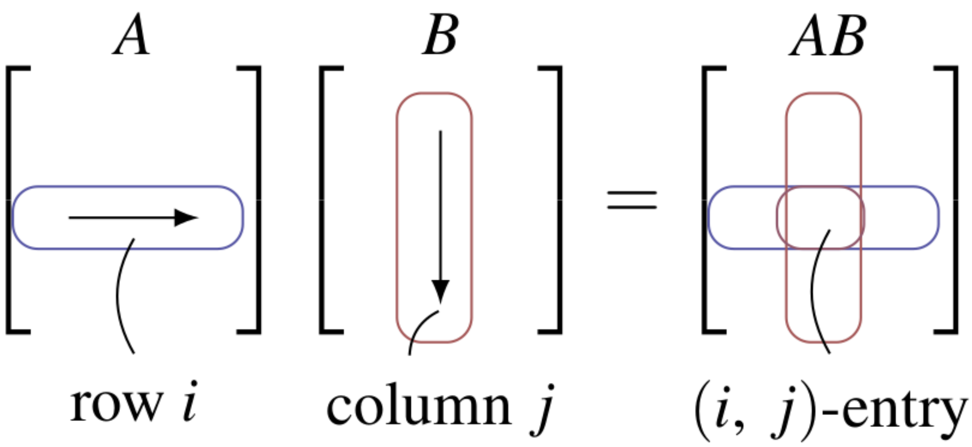
$$AB = [\, A\vec{b}_1 \quad A\vec{b}_2 \,]$$

$$A\vec{b}_1 = \begin{bmatrix} 5 & 0 & -7 \\ 1 & 5 & 9 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 22 \\ -1 \end{bmatrix}$$

$$A\vec{b}_2 = \begin{bmatrix} 5 & 0 & -7 \\ 1 & 5 & 9 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -11 \\ 29 \end{bmatrix}$$

$$\rightarrow \boxed{AB = \begin{bmatrix} 22 & -11 \\ -1 & 29 \end{bmatrix}}$$

Dot Product Rule



Example 3. If $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}_{3\times 3}$ and $B = \begin{bmatrix} 3 & 0 \\ -2 & 1 \\ 0 & 6 \end{bmatrix}_{3\times 2}$, find AB.

Solution.

$$AB = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ -2 & 1 \\ 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 3-2+0 & 0+1+0 \\ 0-2+0 & 0+1-6 \\ -3+0+0 & 0+0+6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ -2 & -5 \\ -3 & 6 \end{bmatrix}.$$

Compability Rule: The product of matrices A and B is only defined when the number of columns of A is equal to the number of rows of B.

Example 4. (a) Compute the (2, 4)-entry of AB if

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix}_{2\times 3} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 & 6 & 0 \\ 0 & 2 & 3 & 4 \\ -1 & 0 & 5 & 8 \end{bmatrix}_{3\times 4}.$$

(b) Is BA well defined?

Solution.

(a) AB is well-defined. Let $C = AB = [c_{ij}]$.

$$c_{24} = \begin{bmatrix} 0 & 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 4 \\ 8 \end{bmatrix} = 0 + 4 + 32 = 36$$

(b) $A: 2\times 3$ $\quad$ BA is defined or not?

$B: 3\times 4$

Nb. colums of $B = 4$ $\Big\}$ Don't match.

Nb. rows of $A = 2$

$\Rightarrow$ BA is not defined.

Example 5. Let $A = \begin{bmatrix} 6 & 9 \\ -4 & -6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$. Compute A^2, AB, BA, $(AB)^\top$ and $B^\top A^\top$.

Solution.

$$A^2 = \underbrace{A}_{2\times 2}\underbrace{A}_{2\times 2} = \begin{bmatrix} 6 & 9 \\ -4 & -6 \end{bmatrix}\begin{bmatrix} 6 & 9 \\ -4 & -6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\underbrace{A}_{2\times 2}\underbrace{B}_{2\times 2} = \begin{bmatrix} 6 & 9 \\ -4 & -6 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 12 \\ 2 & -8 \end{bmatrix}$$

$AB \neq BA$

$$\underbrace{B}_{2\times 2}\underbrace{A}_{2\times 2} = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 6 & 9 \\ -4 & -6 \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ -6 & -9 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} -3 & 12 \\ 2 & -8 \end{bmatrix}^T = \begin{bmatrix} -3 & 2 \\ 12 & -8 \end{bmatrix}$$

$(AB)^T = B^T A^T$

$$B^T A^T = \underbrace{\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}}_{2\times 2}\underbrace{\begin{bmatrix} 6 & -4 \\ 9 & -6 \end{bmatrix}}_{2\times 2} = \begin{bmatrix} -3 & 2 \\ 12 & -8 \end{bmatrix}$$

$(AB)^T \neq A^T B^T$

$ab = ba, \; a,b \in \mathbb{R}$

<u>Note:</u> In general, $AB \neq BA$. If $AB = BA$, then we say that A and B **commute**.

Theorem 1. Let a be a real number, and A, B, C are matrices of sizes such that the indicated matrix products are defined. Then:

1) $\underset{m\times m}{I}\overset{m\times n}{A} = A$ and $A\overset{n\times n}{I} = A$, where I denotes the identity matrix of proper size.
2) $A(BC) = (AB)C$.
3) $A(B + C) = AB + AC$.
4) $(B + C)A = BA + CA$.
5) $a(AB) = (aA)B = A(aB)$.
6) $(AB)^\top = B^\top A^\top$.

Proof.

1) Assume that $A = [\mathbf{a}_1\ \mathbf{a}_2\ \cdots\ \mathbf{a}_n]$ is of dimension $m \times n$ and I is the $m \times m$ identity matrix. Then

$$\begin{aligned} IA &= [I\mathbf{a}_1\ I\mathbf{a}_2\ \cdots\ I\mathbf{a}_k] \\ &= [\mathbf{a}_1\ \mathbf{a}_2\ \cdots\ \mathbf{a}_k] = A \end{aligned}$$

where we used that $I\mathbf{x} = \mathbf{x}$ from Example 4 in Section 2.2.

2) If we write A in terms of its columns:

$$\begin{aligned} (B + C)A &= [(B + C)\mathbf{a}_1\ \cdots\ (B + C)\mathbf{a}_n] \\ &= [B\mathbf{a}_1 + C\mathbf{a}_1\ \cdots\ B\mathbf{a}_n + C\mathbf{a}_n] \\ &= [B\mathbf{a}_1\ \cdots\ B\mathbf{a}_n] + [C\mathbf{a}_1\ \cdots\ C\mathbf{a}_n] \\ &= BA + CA. \end{aligned}$$

□

EXAMPLE 6. Simplify the following expression:

$$\text{Expr} = A(3B - C) + (A - 2B)C + 2B(C + 2A)$$

where A, B, C represent matrices.

SOLUTION.

$$\begin{aligned}
\text{Expr} &= A(3B) + A(-C) \\
&\quad + AC + (-2B)C \\
&\quad + (2B)C + (2B)(2A) \\
&= 3(AB) - \cancel{AC} + \cancel{AC} - \cancel{2(BC)} \\
&\quad + \cancel{2(BC)} + 4(BA) \\
&= 3AB + 4BA \neq 7AB
\end{aligned}$$

Example 7. Show that $AB = BA$ if and only if $(A - B)(A + B) = A^2 - B^2$. $\left((a-b)(a+b) = a^2 + \cancel{ab} \cancel{ba} - b^2 \right)$

Solution.

($\Rightarrow$) If $AB = BA$, then $(A-B)(A+B) = A^2 - B^2$

Assume $AB = BA$. So $\longrightarrow AB - BA = 0$

$$(A-B)\overbrace{(A+B)} = A(A+B) - B(A+B)$$
$$= AA + AB - BA - BB$$
$$= A^2 + \underbrace{AB - BA}_{=0} - B^2$$
$$= A^2 + 0 - B^2 = A^2 - B^2.$$

($\Leftarrow$) If $(A-B)(A+B) = A^2 - B^2$, then $AB = BA$.

Assume $(A-B)(A+B) = A^2 - B^2$.

$$\Rightarrow \quad A^2 + AB - BA - B^2 = A^2 - B^2$$
$$\Rightarrow \quad \cancel{A^2 - A^2}^{\,0} + AB - BA - \cancel{B^2 + B^2}^{\,0} = \cancel{A^2 - A^2}^{\,0} - \cancel{B^2 + B^2}^{\,0}$$
$$\Rightarrow \quad \cancel{+BA}\, AB - \cancel{BA} = 0 \qquad +BA$$
$$\Rightarrow \quad AB = BA \rightarrow A, B \text{ commute.} \qquad \square$$

DEFINITION 3. A matrix is said to be **partitioned into blocks** if the entries of the matrix are themselves matrices.

EXAMPLE 8. Writing $A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_n]$ in terms of its columns.

Matrix Product with Blocks

EXAMPLE 9. (a) Find a "nice" partition into blocks for the following matrices

$$A = \begin{array}{c} \begin{matrix} I_2 & \qquad O_{2\times 3} \end{matrix} \\ \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 2 & -1 & 4 & 2 & 1 \\ 3 & 1 & -1 & 7 & 5 \end{array}\right] \\ \begin{matrix} P & \qquad Q \end{matrix} \end{array} \quad \text{and} \quad B = \left[\begin{array}{cc} 4 & -2 \\ 5 & 6 \\ \hline 7 & 3 \\ -1 & 0 \\ 1 & 6 \end{array}\right] \begin{array}{l} \to X \\ \\ \\ \to Y \\ \\ \end{array}.$$

(b) Use that to compute AB.

SOLUTION.

(b) $$AB = \begin{bmatrix} I_2 & O_{2\times 3} \\ P & Q \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

$$= \begin{bmatrix} I_2 X + O_{2\times 3} Y \\ PX + QY \end{bmatrix} = \begin{bmatrix} X \\ PX + QY \end{bmatrix}$$

EXAMPLE 10. Obtain a formula for A^5 where $A = \begin{bmatrix} I & X \\ 0 & 0 \end{bmatrix}$ is a square matrix and I is an identity matrix.

SOLUTION.

$$A^2 = \begin{bmatrix} I & X \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I & X \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} II + X0 & XI + \cancel{X0}^{\,0} \\ 0I + 00 & 0X + 00 \end{bmatrix}$$

$$= \begin{bmatrix} I & X \\ 0 & 0 \end{bmatrix} = A$$

$$A^3 = AAA = AA^2 = AA = A^2 = A$$

$$A^4 = AAAA = A^2A^2 = AA = A^2 = A$$

$$A^5 = AA^4 = AA = A$$

Notes:

- Block Multiplication is useful in theory.
- It is also usuful in computing products of large matrices in a computer with limited memory capacity.