Math 311

Chapter 7

Section 7.2: Kernel and Image of a Linear Transformation

Contents

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For this section, $T : V \to W$ is assumed to be a linear transformation.

Definition 1.

① The **kernel** of T is the set

$$\ker T := \{\mathbf{v} \in V \ : \ T(\mathbf{v}) = \mathbf{0}\}.$$

② The **image** of T is the set

$$\operatorname{im} T := \{T(\mathbf{v}) \ : \ \mathbf{v} \in V\}.$$

Example 1. Let A be an $m \times n$ matrix and consider $T_A(\mathbf{x}) = A\mathbf{x}$, where $\mathbf{x} \in \mathbb{R}^n$.

a) We have $\ker T_A = \{\mathbf{x} \in \mathbb{R}^n \ : \ A\mathbf{x} = \mathbf{0}\} = \operatorname{null} A$.

b) We have $\operatorname{im} T_A = \{A\mathbf{x} \ : \ \mathbf{x} \in \mathbb{R}^n\} = \operatorname{im} A$.

Theorem 1. We have that $\ker T$ is a subspace of V and $\operatorname{im} T$ is a subspace of W.

Notes:

① We set $\operatorname{nullity} T := \dim(\ker T)$.

② We set $\operatorname{rank} T := \dim(\operatorname{im} T)$.

EXAMPLE 2. Define $T : \mathbf{M_{nn}} \to \mathbf{M_{nn}}$ by $T(A) := A - A^\top$. Find (a) $\ker T$ (b) $\operatorname{im} T$.

SOLUTION.

(a) Let $A \in M_{nn}$. By def.

$$A \in \ker T \iff T(A) = \vec{0}$$
$$\iff A - A^T = \vec{0}$$
$$\iff A = A^T$$
$$\iff A \text{ is symmetric.}$$

So, $\ker T = \{ A \in M_{nn} : A \text{ is symmetric} \}$.

(b) <u>Goal</u>: Describe all matrices $B \in M_{nn}$ such that there is a matrix A with

$$T(A) = B \iff A - A^T = B.$$

Notice that if $B \in \operatorname{im} T$, then

$$B = A - A^T \quad \text{for some } A \in M_{nn}.$$

So, $B^T = A^T - A$

$= -(A - A^T)$

$= -B$

$\Rightarrow B^T = -B \Rightarrow B$ is skew-symmetric.

In fact,

$\operatorname{Im} T = \{ B \in M_{nn} : B \text{ is skew-symm.} \}$.

[To prove once it for all, take

$A = \dfrac{B - B^T}{4}$, B skew-symm.

<u>You</u> can show that $T(A) = B$.]

DEFINITION 2. Assume that $T : V \to W$ is a linear transformation.

① T is said to be **onto** if $\operatorname{im} T = W$.

② T is said to be **one-to-one** if $\ker T = \{\mathbf{0}\}$.

EXAMPLE 3. Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ defined by $T(x, y, z) = (x, x, y, y)$.

a) Is T onto?

b) Is T one-to-one? If not, find nullity T.

a) No, because

$$\operatorname{im} T = \{ x(1,1,0,0) + y(0,0,1,1) : x, y \in \mathbb{R} \}.$$

$$= \operatorname{span}\{ (1,1,0,0), (0,0,1,1) \}.$$

$$\Rightarrow \dim(\operatorname{im} T) = 2.$$

But $\dim \mathbb{R}^4 = 4 > 2 \Rightarrow \operatorname{im} T \neq \mathbb{R}^4$

$\Rightarrow T$ is not onto.

(b) By definition

$$(x,y,z) \in \ker T \iff T(x,y,z) = (0,0,0,0)$$

$$\iff (x,x,y,y) = (0,0,0,0)$$

$$\iff x=0, \quad y=0, \quad z \in \mathbb{R}.$$

So,

$$\ker T = \{(0,0,z) : z \in \mathbb{R}\}$$

$$= \{z(0,0,1) : z \in \mathbb{R}\}$$

$$= \operatorname{span}\{(0,0,1)\}.$$

<u>1st</u>: T is not one-to-one because $\ker T \neq \{\vec{0}\}$.

<u>2nd</u>: $\operatorname{nullity} T = \dim(\ker T) = 1$.

Theorem 2. Let $T : V \to W$ be any linear transformation with $n = \dim V < \infty$. Then

$$\dim V = \operatorname{nullity} T + \operatorname{rank} T.$$

Idea of the Proof. We let

- $r = \operatorname{rank} T = \dim(\operatorname{im} T)$;
- $k = \operatorname{nullity} T = \dim(\ker T)$.

Let $\{\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_r}\}$ be a basis for $\operatorname{im} T$. Then there are vectors $\mathbf{e_1}$, $\mathbf{e_2}$, ..., $\mathbf{e_r}$ such that $T(\mathbf{e_i}) = \mathbf{w_i}$.

Let $\{\mathbf{f_1}, \mathbf{f_2}, \ldots, \mathbf{f_k}\}$ be a basis for $\ker T$.

Then the idea is to show that $\{\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_r}, \mathbf{f_1}, \ldots, \mathbf{f_k}\}$ is a basis for V, so that we get

$$n = k + r$$

showing the claim. □