Math 311

Chapter 2

Section 2.2: Matrix-Vector Multiplication

Contents

Created by: Pierre-Olivier Parisé

Spring 2024

Matrix-Vector Multiplication

Example 1. Write the system

$$\begin{aligned} 3x_1 + 2x_2 - 4x_3 &= 0 \\ x_1 - 3x_2 + x_3 &= 3 \\ x_2 - 5x_3 &= -1 \end{aligned}$$

in a compact form using a linear combination of vectors.

Solution.

Note: Any system of linear equations can be rewritten as $A\mathbf{x} = \mathbf{b}$, where A is the matrix of coefficients, $\mathbf{x}$ is the n-vector containing the unknown, and $\mathbf{b}$ is the m-vector containing the constant terms of each equation.

Definition 1.

- Let $A = [\mathbf{a_1}\ \mathbf{a_2}\ \cdots\ \mathbf{a_n}]$ be an $m \times n$ matrix, where the m-vectors $\mathbf{a_1}$, $\mathbf{a_2}$, ..., $\mathbf{a_n}$ represent the columns.
- Let $\mathbf{x}$ be any n-vector.

The **product** $A\mathbf{x}$ is defined to be the m-vector:

$$A\mathbf{x} = x_1\mathbf{a_1} + x_2\mathbf{a_2} + \cdots + x_n\mathbf{a_n}.$$

Example 2. If $A = \begin{bmatrix} 2 & -1 & 3 & 5 \\ 0 & 2 & -3 & 1 \\ -3 & 4 & 1 & 2 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ -2 \end{bmatrix}$, then compute $A\mathbf{x}$.

Solution.

Properties:

- $A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y}$.
- $A(a\mathbf{x}) = a(A\mathbf{x}) = (aA)\mathbf{x}$, for any scalar a.
- $(A+B)\mathbf{x} = A\mathbf{x} + B\mathbf{y}$.

THE DOT PRODUCT

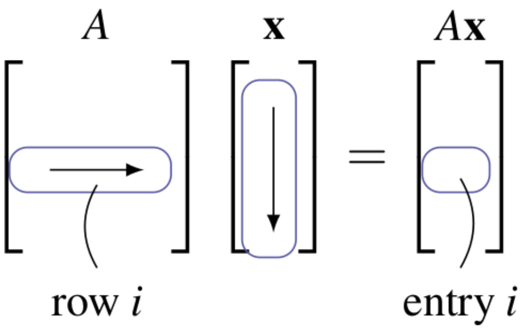
DEFINITION 2. If $\mathbf{x}$ is an $1 \times n$ vector and $\mathbf{y}$ is an $n \times 1$ vectors, their **dot product** is defined to be the number

$$\mathbf{x} \cdot \mathbf{y} := x_1 y_1 + x_2 y_2 + \ldots + x_n y_n.$$

EXAMPLE 3. Use the dot product to compute $A\mathbf{x}$ where A and $\mathbf{x}$ are as in Example 2.

SOLUTION.

The Dot Product Rule.



To obtain the entry i of $A\mathbf{x}$, take the dot product of row i of A with the vector $\mathbf{x}$.

Example 4. Find an $n \times n$ matrix A such that $A\mathbf{x} = \mathbf{x}$, for any $\mathbf{x} \in \mathbb{R}^n$.

Solution.

Theorem 1. Let A and B be two $m \times n$ matrices. If $A\mathbf{x} = B\mathbf{x}$ for any $\mathbf{x} \in \mathbb{R}^n$, then $A = B$.

Example 5. A function is defined as follows: it reflects a 2×1 vector accross the x-axis in the 2D space. Illustrate graphically the **action** of this function and find a formula to describe it.

Solution.

DEFINITION 3. Given an $m \times n$ matrix A, the **matrix transformation induced** by the matrix A denoted by T_A is defined by

$$T_A(\mathbf{x}) = A\mathbf{x} \quad \forall \mathbf{x} \in \mathbb{R}^n.$$

Note:

- For each $\mathbf{x} \in \mathbb{R}^n$, we have $T_A(\mathbf{x}) \in \mathbb{R}^m$. In this case, the expression of $T_A(\mathbf{x})$ is called the **action** of T_A.
- Therefore, $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is a function.
- For two matrices A and B, we say that T_A and T_B are **equal** if they have the same action, meaning $T_A(\mathbf{x}) = T_B(\mathbf{x})$, for any $\mathbf{x} \in \mathbb{R}^n$.

EXAMPLE 6. Let A be the $m \times n$ zero matrix. Then T_A is called the **zero matrix-transformation**. Show that $T_A(\mathbf{x}) = \mathbf{0}$, where $\mathbf{0}$ is the m-vector with 0 in all its entries.

SOLUTION.