Section 3.2 — Problem 2 (10 Pts)

a. Denote by A the matrix. We compute

$$\det A = 37 - 5c.$$

So A is invertible $\iff \det A \neq 0 \iff 37 - 5c = 0 \iff c \neq 37/5$.

d. Denote by D the matrix. Using column and row operations, we see that

$$\det D = \begin{vmatrix} 1 & c & 3 \\ 0 & 2 & c \\ 1 & c & 4 \end{vmatrix} = \begin{vmatrix} 1 & c & 3 \\ 0 & 2 & c \\ 0 & 0 & 1 \end{vmatrix} = 2.$$

The first equality comes from replacing C_1 of A by $C_1 - C_3$ and the second equality comes form replace R_3 by $R_3 - R_1$.

Hence, $\det D \neq 0$ and the matrix D is invertible for any values of c.

Section 3.2 — Problem 3 (10 Pts)

a. From the properties of determinant, we compute

$$\det(A^3BC^\top B^{-1}) = \det(A^3)\det(B)\det(C^\top)\det(B^{-1}) = (\det A)^3 \det B \det C\big(\tfrac{1}{\det B}\big)$$

Notice that $\det B(\frac{1}{\det B}) = 1$ and therefore, replacing the values for each determinant, we get

$$\det(A^3BC^\top B^{-1}) = (\det A)^3 \det C = (-1)^3(3) = -3$$

b. From the properties of determinant again, we compute

$$\begin{aligned}\det(B^2C^{-1}AB^{-1}C^\top) &= \det(B^2)\det(C^{-1})\det(A)\det(B^{-1})\det(C^\top) \\ &= (\det B)^2(\tfrac{1}{\det C})\det(A)(\tfrac{1}{\det B})\det C \\ &= \det B \det A\end{aligned}$$

Plugging in the values for the determinants, we find that

$$\det(B^2C^{-1}AB^{-1}C^\top) = (2)(-1) = -2.$$

Section 3.2 — Problem 7a (10 Pts)

Denote the matrix by A. We can extract a 2 from the last column and the first row so that we get

$$\det A = 4\begin{vmatrix} 1 & -1 & 0 \\ c+1 & -1 & a \\ d-2 & 2 & b \end{vmatrix}$$

Now, we will replace R_2 by $R_2 - R_1$ and R_3 by $R_3 + 2R_1$. These two elementary operations don't change the value of the determinant and therefore

$$\begin{vmatrix} 1 & -1 & 0 \\ c+1 & -1 & a \\ d-2 & 2 & b \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 \\ c & 0 & a \\ d & 0 & b \end{vmatrix} = (-1)(-1)\begin{vmatrix} c & a \\ d & b \end{vmatrix}.$$

Thus,

$$\det A = 4\begin{vmatrix} c & a \\ d & b \end{vmatrix}.$$

We are almost there! Notice that, by interchanging the first column and the second column, we get

$$\begin{vmatrix} c & a \\ d & b \end{vmatrix} = (-1)\begin{vmatrix} a & c \\ b & d \end{vmatrix}.$$

Now notice again that

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^\top \quad \Rightarrow \quad \begin{vmatrix} a & c \\ b & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

Therefore, we get

$$\det A = (4)(-1)\begin{vmatrix} a & b \\ c & d \end{vmatrix} = (-4)\begin{vmatrix} a & b \\ c & d \end{vmatrix} = (-4)(-2) = 8.$$

Section 3.2 — Problem 8a (10 Pts)

The matrix of coefficients and the constant vector are

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 7 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

We have $\det A = 14 - 3 = 11$.
To get the value of x, we replace the first column of A and set

$$x = \frac{\begin{vmatrix} 1 & 1 \\ -2 & 7 \end{vmatrix}}{\det A} = \frac{7+2}{11} = \frac{9}{11}.$$

To get the value of y, we replace the first column of A and set

$$y = \frac{\begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix}}{\det A} = \frac{-4-3}{11} = -\frac{7}{11}.$$

Section 3.2 — Problem 13 (10 Pts)

Assume that A and B are two $n \times n$ matrices. From the product rule, we get

$$\det(AB) = \det(A)\det(B).$$

Using the product rule on BA, we get

$$\det(BA) = \det(B)\det(A).$$

Using the commutativity of the multiplication of real numbers, we get

$$\det(AB) = \det(A)\det(B) = \det(B)\det(A) = \det(BA).$$

This completes the proof. $\square$