Math 311

Chapter 9

Section 9.2: Operators and Similarity

Contents

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DEFINITION 1. A linear transformation $T : V \to W$ is called an **linear operator** if $V = W$. We will therefore write $T : V \to V$, where V is a vector space.

B-matrix

Recall that if $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear operator and $E = \{\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}\}$ is the standard basis, then the matrix representing T on the basis E is

$$A = \begin{bmatrix} T(\mathbf{e_1}) & T(\mathbf{e_2}) & \cdots & T(\mathbf{e_n}) \end{bmatrix}.$$

DEFINITION 2. Let

- V be a vector space;
- $T : V \to V$ be a linear operator;
- $B = \{\mathbf{b_1}, \mathbf{b_2}, \ldots, \mathbf{b_n}\}$ be a basis.

The **B-matrix** of T is the matrix representing T on the basis B:

$$M_B(T) := \begin{bmatrix} C_B(T(\mathbf{b_1})) & C_B(T(\mathbf{b_2})) & \cdots & C_B(T(\mathbf{b_n})) \end{bmatrix}.$$

Properties:

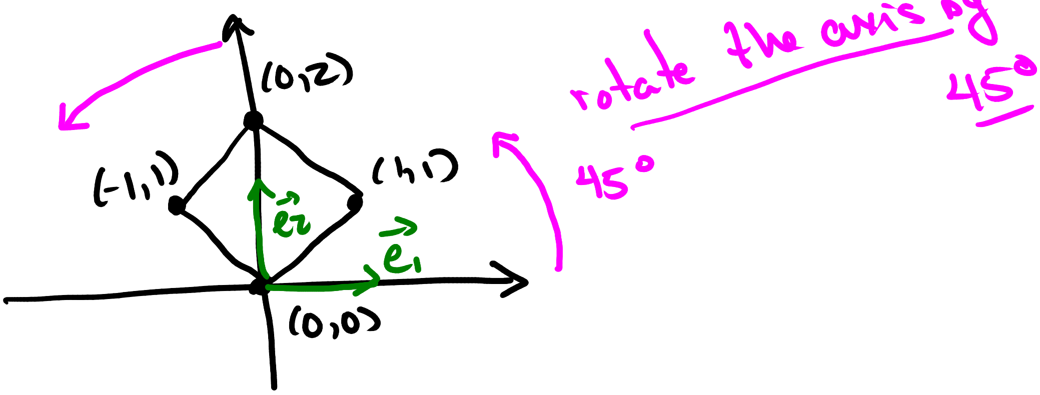
① $C_B(T(\mathbf{v})) = M_B(T)C_B(\mathbf{v})$ for all $\mathbf{v} \in V$.

② T is an isomorphism if and only if $M_B(T)$ is invertible. Moreover, $M_B(T^{-1}) = (M_B(T))^{-1}$.

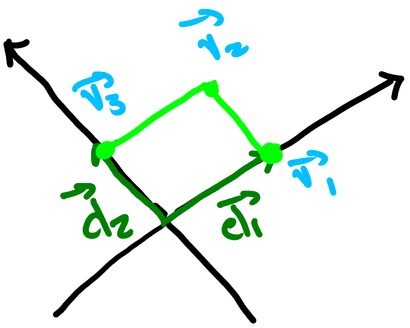
EXAMPLE 1. Consider the square with vertices $(0,0)$, $(1,1)$, $(0,2)$, $(-1,1)$. Find a basis D on which the coordinates of the vertices become $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$.

SOLUTION. $B = \{\vec{e}_1, \vec{e}_2\}$ be the standard basis

Picture:



therefore $D = \{\vec{d}_1 = (1,1), \ \vec{d}_2 = (-1,1)\}$ will be the new basis because:



$C_D(\vec{v}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$C_D(\vec{v}_2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$C_D(\vec{v}_3) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Goal: Given two basis

- $B = \{\mathbf{b_1}, \mathbf{b_2}, \dots, \mathbf{b_n}\}$;
- $D = \{\mathbf{d_1}, \mathbf{d_2}, \dots, \mathbf{d_n}\}$;

how do we get $C_D(\mathbf{v})$ from $C_B(\mathbf{v})$?

EXAMPLE 1. [Continued]

Trick: Transform any $\vec{v}$ in $C_B(\vec{v})$ into $\vec{v}$ in $C_D(\vec{v})$.

For ex.: $2\vec{e_1} + 3\vec{e_2} = 2(a_1\vec{b_1} + a_2\vec{b_2}) + 3(c_1\vec{b_1} + c_2\vec{b_2})$

Let $\vec{v} = (a,b) \in \mathbb{R}^2$, then

$$C_D(\vec{v}) = \left(\frac{a+b}{2}\right)\begin{bmatrix}1\\1\end{bmatrix} + \left(\frac{a-b}{2}\right)\begin{bmatrix}1\\-1\end{bmatrix}$$

$$= \begin{bmatrix}\frac{a+b}{2}\\ \frac{a-b}{2}\end{bmatrix} = \begin{bmatrix}1/2 & 1/2\\ 1/2 & -1/2\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} \leftarrow C_B(\vec{v})$$

$C_D(\vec{e_1})$ $\boxed{\text{Matrix of } \mathbb{1}_V \text{ on } B \text{ and } D}$ $C_D(\vec{e_2})$

DEFINITION 3. We define the **change matrix** from B to D as

$$P_{D\leftarrow B} = \left[C_D(\mathbf{b_1}) \;\; C_D(\mathbf{b_2}) \;\; \cdots \;\; C_D(\mathbf{b_n})\right].$$

Properties:

① For any vector $\mathbf{v} \in V$, we have $C_D(\mathbf{v}) = P_{D\leftarrow B}C_B(\mathbf{v})$.

② $P_{B\leftarrow B} = I_n$.

③ $P_{D\leftarrow B}$ is invertible and $(P_{D\leftarrow B})^{-1} = P_{B\leftarrow D}$.

EXAMPLE 2. Let $V = \mathbb{R}^2$ and $B = \{(1,2),(0,1)\}$, $D = \{(1,1),(-1,1)\}$.

a) Find $P_{D\leftarrow B}$.

b) Verify that $C_D(\mathbf{x}) = P_{D\leftarrow B}C_B(\mathbf{x})$.

c) Find $P_{B\leftarrow D}$, verify that $C_B(\mathbf{x}) = P_{B\leftarrow D}C_D(\mathbf{x})$.

SOLUTION.

a) $(1,2) = (3/2)(1,1) + (1/2)(-1,1)$

$$\Rightarrow C_D((1,2)) = \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}$$

$(0,1) = (1/2)(1,1) + (1/2)(-1,1)$

$$\Rightarrow C_D((0,1)) = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

So, $P_{D\leftarrow B} = \begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$.

b) Choose $\vec{x} = (2,3)$

$$C_B(\vec{x}) = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \text{and} \quad C_D(\vec{x}) = \begin{bmatrix} 5/2 \\ 1/2 \end{bmatrix}$$

then

$$P_{D \leftarrow B}\, C_B(\vec{x}) = \begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 5/2 \\ 1/2 \end{bmatrix} = C_D(\vec{x}) .$$

c) $P_{B \leftarrow D} = \left(P_{D \leftarrow B}\right)^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}.$

Let $\vec{x} = (2,3)$, then

$$P_{B \leftarrow D}\, C_D(\vec{x}) = \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 5/2 \\ 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -1 \end{bmatrix} = C_B(\vec{x}) .$$

<u>$C_B(\vec{x})$</u>

$(2,3) = a(1,2) + b(0,1)$

$\Leftrightarrow \begin{cases} 2 = a + 0b \\ 3 = 2a + b \end{cases} \Leftrightarrow \begin{cases} a = 2 \\ 3 = 4 + b \end{cases} \Leftrightarrow \begin{cases} a = 2 \\ b = -1 \end{cases}$

Example 3. Let $A = \begin{bmatrix} 11 & -6 \\ 12 & -6 \end{bmatrix}$, $P = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$, and $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

a) Verify that $P^{-1}AP = D$.

b) Find a basis B such that $M_B(T_A) = D$.

(a)

$$P^{-1} = \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix} \Rightarrow P^{-1}AP \overset{\substack{\text{Python} \\ \downarrow}}{=} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = D \checkmark$$

(b) Recall that $T_A \vec{x} = A\vec{x}$.

Let $\vec{b_1} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\vec{b_2} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

$$T_A(\vec{b_1}) = \begin{bmatrix} 11 & -6 \\ 12 & -6 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix} = (2)\vec{b_1} + 0\vec{b_2}$$

$$T_A(\vec{b_2}) = \begin{bmatrix} 11 & -6 \\ 12 & -6 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ 12 \end{bmatrix} = (0)\vec{b_1} + (3)\vec{b_2}$$

$$\Rightarrow M_B(T_A) = \begin{bmatrix} C_B(T_A(\vec{b_1})) & C_B(T_A(\vec{b_2})) \end{bmatrix} = \overbrace{\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}}^{D}$$

Theorem 1.

① Let A be an $n \times n$ matrix and E be standard basis of $\mathbb{R}^n$.

② Let B be a basis of $\mathbb{R}^n$.

③ Let P be the invertible matrix whose columns are the vectors in B in order.

Then

$$M_B(T_A) = P^{-1}AP.$$