Math 311

Chapter 9

Section 9.1: The Matrix of a Linear Transformation

Contents

Created by: Pierre-Olivier Parisé

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Coordinate Vector

Let V be a vector space with $\dim V = n$ and $\mathbf{v} \in V$.

Given a basis $B = \{\mathbf{b_1}, \mathbf{b_2}, \ldots, \vec{\mathbf{b}}_{\mathbf{n}}\}$ of V, recall that $C_B : V \to \mathbb{R}^n$ is given by

$$C_B(\mathbf{v}) = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

Example 1. Let $\mathbf{x} = (2, 1, 3)$ and

$$B = \{\overset{\vec{b_1}}{(1, 0, 1)}, \overset{\vec{b_2}}{(1, 1, 0)}, \overset{\vec{b_3}}{(0, 1, 1)}\}$$

be a basis of $\mathbb{R}^3$. Find $C_B(\mathbf{x})$.

Solution.

Here $\vec{x} = x_1 \vec{b_1} + x_2 \vec{b_2} + x_3 \vec{b_3}$

$$\Rightarrow C_B(\vec{x}) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

We have $(2,1,3) = x_1 \vec{b_1} + x_2 \vec{b_2} + x_3 \vec{b_3}$

$\Leftrightarrow x_1 = 2, \quad x_2 = 0, \quad x_3 = 1$

$$\Rightarrow C_B(2,1,3) = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Matrix of a Linear transformation

Suppose we have the transformation

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+z \\ 2z \\ y-z \\ x+2y \end{bmatrix}.$$

Notice that, if we apply T to the standard basis of $\mathbb{R}^3$, we get

$$T\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \mathbf{a_1}, \quad T\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} = \mathbf{a_2}, \quad T\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} = \mathbf{a_3}.$$

Then, setting

$$A = \begin{bmatrix} \mathbf{a_1} & \mathbf{a_2} & \mathbf{a_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix} \quad \Rightarrow \quad T(\mathbf{x}) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

The matrix A is called the **matrix representation of the linear transformation** in term of the standard basis of $\mathbb{R}^3$ and $\mathbb{R}^4$.

What if we change basis?

Example 2. Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation defined by

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+z \\ 2z \\ y-z \\ x+2y \end{bmatrix}.$$

We assume we have two basis:

- a basis $B = \{\begin{bmatrix}1 & 0 & 1\end{bmatrix}^\top, \begin{bmatrix}1 & 1 & 0\end{bmatrix}^\top, \begin{bmatrix}0 & 1 & 1\end{bmatrix}^\top\}$ of $\mathbb{R}^3$.
- a basis $D = \{\underbrace{\begin{bmatrix}1 & 0 & 1 & 0\end{bmatrix}^\top}_{\vec{d}_1}, \underbrace{\begin{bmatrix}0 & 1 & 0 & 1\end{bmatrix}^\top}_{\vec{d}_2}, \underbrace{\begin{bmatrix}1 & 1 & 0 & 0\end{bmatrix}^\top}_{\vec{d}_3}, \underbrace{\begin{bmatrix}1 & 0 & 0 & 1\end{bmatrix}^\top}_{\vec{d}_4}\}$ of $\mathbb{R}^4$.

Find a matrix representing T on these basis.

Solution.

① Evaluate T on B

$$T\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+1 \\ 2(1) \\ 0-1 \\ 1+2(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -1 \\ 1 \end{bmatrix} = \vec{t}_1$$

$$T\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \vec{t}_2$$

$$T\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix} = \vec{t}_3$$

② Write $\vec{t}_1, \vec{t}_2, \vec{t}_3$ in the basis D

$$\vec{t}_1 = \begin{bmatrix} 2 \\ 2 \\ -1 \\ 1 \end{bmatrix} = (-1)\vec{d}_1 + (0)\vec{d}_2 + (2)\vec{d}_3 + (1)\vec{d}_4$$

$$\Rightarrow C_D(\vec{t}_1) = \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}.$$

$$\vec{t}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = (1)\vec{d}_1 + (1/2)\vec{d}_2 + (-1/2)\vec{d}_3 + (1/2)\vec{d}_4$$

$$\Rightarrow C_D(\vec{t}_2) = \begin{bmatrix} 1 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$$

$$\vec{t}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix} = (0)\vec{d}_1 + (3/2)\vec{d}_2 + (1/2)\vec{d}_3 + (1/2)\vec{d}_4$$

$$\Rightarrow C_D(\vec{t}_3) = \begin{bmatrix} 0 \\ 3/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

③ Construct the matrix A

$$A = \left[C_D(T_1) \ \ C_D(T_2), C_D(T_3) \right] = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1/2 & 3/2 \\ 2 & -1/2 & 1/2 \\ 1 & 1/2 & 1/2 \end{bmatrix}$$

④ Property. $\forall \vec{x} \in \mathbb{R}^3$

$$C_D\, T(\vec{x}) = A\, C_B(\vec{x})$$

test: $\vec{x} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \rightarrow C_B(\vec{x}) = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

$$\Rightarrow A C_B(\vec{x}) = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1/2 & 3/2 \\ 2 & -1/2 & 1/2 \\ 1 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 3/2 \\ 9/2 \\ 5/2 \end{bmatrix}$$

we have

$$T\vec{x} = T \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ -2 \\ 4 \end{bmatrix}$$

$$\Rightarrow C_D(T\vec{x}) = \begin{bmatrix} -2 \\ 3/2 \\ 9/2 \\ 5/2 \end{bmatrix}$$

Same!

General Procedure

To find the **matrix representation** of $T : \mathbb{R}^n \to \mathbb{R}^m$ on a basis B of $\mathbb{R}^n$ and on a basis D of $\mathbb{R}^m$, we follow these steps:

① Evaluate $\mathbf{t_1} = T(\mathbf{b_1})$, $\mathbf{t_2} = T(\mathbf{b_2})$, …, $\mathbf{t_n} = T(\mathbf{b_n})$.

② Find $C_D(\mathbf{t_1})$, $C_D(\mathbf{t_2})$, …, $C_D(\mathbf{t_n})$.

③ Set the $m \times n$ matrix

$$A = \left[C_D(\mathbf{t_1}) \;\; C_D(\mathbf{t_2}) \;\; \cdots \;\; C_D(\mathbf{t_n})\right].$$

④ Then we have, for any $\mathbf{x} \in \mathbb{R}^n$,

$$C_D T(\mathbf{x}) = T_A C_B(\mathbf{x}) = A C_B(\mathbf{x}).$$