Math 311

Chapter 6

Section 6.2: Linear Combination and Subspaces

Contents

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Spring 2024

Example 1. The solution to the homogeneous system

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 1 \\ 2 & 3 & 4 & 1 & 2 & -2 \\ 1 & 2 & 4 & 5 & 3 & -1 \\ 3 & 1 & 2 & 4 & 5 & 1 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

is

$$\mathbf{x} = t \begin{bmatrix} 1 \\ 9 \\ -7 \\ 3 \end{bmatrix} + s \begin{bmatrix} 0 \\ 3 \\ -3 \\ 1 \end{bmatrix} = t\mathbf{x_1} + s\mathbf{x_2}, \quad s, t \in \mathbb{R}.$$

Notice that

S1. $t = s = 0 \Rightarrow \vec{x} = \vec{0}$ is a solution.

S2. $\vec{x} = 5\vec{x}_1 + 3\vec{x}_2$ and $\vec{y} = -3\vec{x}_1 + \vec{x}_2$

$\Rightarrow \vec{x} + \vec{y} = 2\vec{x}_1 + 4\vec{x}_2$

$\Rightarrow \vec{x} + \vec{y}$ is still a solution.

S3. $\vec{x} = 5\vec{x}_1 + 3\vec{x}_2$

$\Rightarrow 2\vec{x} = 10\vec{x}_1 + 6\vec{x}_2$ still a solution.

If $U = \{t\mathbf{x_1} + s\mathbf{x_2} \,:\, s, t \in \mathbb{R}\}$ is the set of all solutions, then U is called a **subspace**.

DEFINITION 1. A subset U of a vector space V is called a **subspace** of V if it satisfies the following properties:

$\boxed{\text{S1.}}$ The zero vector $\mathbf{0} \in U$.

$\boxed{\text{S2.}}$ If $\mathbf{u_1} \in U$ and $\mathbf{u_2} \in U$, then $\mathbf{u_1} + \mathbf{u_2} \in U$.

$\boxed{\text{S3.}}$ If $\mathbf{u} \in U$ and a is a scalar, then $a\mathbf{u} \in U$.

Remarks:

① S2: U is said to be **closed under addition**.

② S3: U is said to be **closed under scalar multiplication**.

③ A subspace is a vector [redacted]space itself.

EXAMPLE 2. Let V be a vector space. Show that $U = \{\mathbf{0}\}$ is a subspace of V. This space is called the **zero subspace**.

SOLUTION.

S1. $\vec{0}$ is in $\{\vec{0}\}$.

S2. $\vec{u}_1 = \vec{0}$ and $\vec{u}_2 = \vec{0}$ $\Rightarrow$ $\vec{0} + \vec{0} = \vec{0} \in U$.

S3. $\vec{u} = \vec{0}$ and $a \in \mathbb{R}$ $\Rightarrow$ $a\vec{0} = \vec{0} \in U$.

Here S1-S3 are satisfied $\Rightarrow$ U is a subspace.

Note: Any subspace U of V such that $U \neq \{\mathbf{0}\}$ and $U \neq V$ is called a **proper subspace**.

Important Examples

EXAMPLE 3. Given an $m \times n$ matrix A, define

$$U = \text{null}A := \{\mathbf{x} \in \mathbb{R}^n \ : \ A\mathbf{x} = \underset{}{\vec{\mathbf{0}}}\}. \quad \vec{0}_{m\times 1}$$

Show that null$A$ is a subspace of $\mathbb{R}^n$.

SOLUTION. null A is a subset of $\mathbb{R}^n$.

S1. $A\vec{0}_{n\times 1} = \vec{0}_{m\times 1} \Rightarrow \vec{0}_{n\times 1} \in$ null A.

S2. Let $\vec{x}, \vec{y} \in$ null$A$. then

$$\begin{aligned} A(\vec{x}+\vec{y}) &= A\vec{x} + A\vec{y} \quad \text{(Matrix mult. prop.)} \\ &= \vec{0}_{m\times 1} + \vec{0}_{m\times 1} \\ &= \vec{0}_{m\times 1} \quad \Rightarrow \vec{x}+\vec{y} \in \text{null}A. \end{aligned}$$

S3. Let $\vec{x} \in$ null A and $a \in \mathbb{R}$. then

$$A(a\vec{x}) = a(A\vec{x}) = a\,\vec{0}_{m\times 1} = \vec{0}_{m\times 1}.$$

So, $a\vec{x} \in$ null A. Conclusion: null A a subspace.

Note: The subspace null$A$ is called the **null space** of a matrix A. It is the set of all solutions to the homogeneous system $A\mathbf{x} = \mathbf{0}$.

Example 4. Given an $m \times n$ matrix A, define

$$\operatorname{im}A := \{\overbrace{A\mathbf{x}}^{m\times 1} : \mathbf{x} \in \mathbb{R}^n\}.$$

Show that $\operatorname{im}A$ is a subspace of $\mathbb{R}^m$.

Solution. $\operatorname{im} A$ is a subset of $\mathbb{R}^m$.

S1. If $\vec{x} = \vec{0} \in \mathbb{R}^n$, then

$$A\vec{x} = A\vec{0} = \vec{0} \in \mathbb{R}^m$$

$\Rightarrow \quad \vec{0}_{m\times 1} \in \operatorname{im} A$.

S2. Assume $\vec{y}_1, \vec{y}_2 \in \operatorname{im} A$.

Goal: $\vec{y}_1 + \vec{y}_2 \in \operatorname{im} A$. $(\vec{y}_1 + \vec{y}_2 = A\vec{x})$.

Here, $\vec{y}_1 = A\vec{x}_1$ and $\vec{y}_2 = A\vec{x}_2$

$$\Rightarrow \quad \vec{y}_1 + \vec{y}_2 = A\vec{x}_1 + A\vec{x}_2 = A(\vec{x}_1 + \vec{x}_2)$$

So, if $\vec{x} = \vec{x}_1 + \vec{x}_2 \Rightarrow \vec{y}_1 + \vec{y}_2 = A\vec{x}$. ✓

S3. Assume $\vec{y} \in \operatorname{im} A$ and $a \in \mathbb{R}$.

$$a\vec{y} = aA\vec{x} = A(\underbrace{a\vec{x}}_{=\vec{u}}) = A\vec{u} \Rightarrow a\vec{y} \in \operatorname{im} A. ✓$$

Note: The subspace $\operatorname{im}A$ is called the **image space** (or **range space**) of the matrix A. It is the set of all vectors $\mathbf{b}$ such that $A\mathbf{x} = \mathbf{b}$ has a solution.

EXAMPLE 5. For an $n \times n$ matrix A and a number λ, define

$$E_\lambda(A) := \{\mathbf{x} \in \mathbb{R}^n \ : \ A\mathbf{x} = \lambda\mathbf{x}\}.$$

Show that $E_\lambda(A)$ is a subspace of $\mathbb{R}^n$.

SOLUTION. We see that $E_\lambda(A) \subseteq \mathbb{R}^n$

S1. $A\vec{0} = \vec{0}$, $\lambda\vec{0} = \vec{0}$ $\Rightarrow$ $A\vec{0} = \lambda\vec{0}$

$\Rightarrow$ $\vec{0} \in E_\lambda(A)$ ✓.

S2. $\vec{x}, \vec{y} \in E_\lambda(A)$. Goal: $\vec{x} + \vec{y} \in E_\lambda(A)$.

So, $A\vec{x} = \lambda\vec{x}$ and $A\vec{y} = \lambda\vec{y}$.

$$A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y} = \lambda\vec{x} + \lambda\vec{y}$$
$$= \lambda(\vec{x}+\vec{y}) \checkmark.$$

Then, $\vec{x}+\vec{y} \in E_\lambda(A)$.

S3. Let $\vec{x} \in E_\lambda(A)$ and $\mu \in \mathbb{R}$.

Goal: $\mu\vec{x} \in E_\lambda(A)$.

$\Rightarrow A(\mu\vec{x}) = \mu(A\vec{x}) = \mu(\lambda\vec{x}) = \lambda(\mu\vec{x})$ ✓.

$\Rightarrow \mu\vec{x} \in E_\lambda(A)$. Conclusion: $E_\lambda(A)$ subspace.

Note: When λ is an eigenvalue of A, the subspace $\mathrm{E}_\lambda(A)$ is called the **eigenspace** associated to λ.

More Examples

EXAMPLE 6. Let $\mathbf{M}_{nn}$ be the vector space of $n \times n$ matrices. Show that $U = \{A : A^\top = A\}$ is a subspace of $\mathbf{M}_{nn}$.

SOLUTION. For sure $U \subseteq M_{nn}$.

S1. $0^T = 0 \Rightarrow 0 \in U$.

S2. $A, B \in U$. Goal: $(A+B)^T = A + B$.

From properties in Chapter 2:

$$(A+B)^T = A^T + B^T$$
$$= A + B \Rightarrow A+B \in U.$$

S3. $A \in U$, $\lambda \in \mathbb{R}$. Goal: $(\lambda A)^T = \lambda A$.

From properties in Chapter 2:

$$(\lambda A)^T = \lambda A^T = \lambda A \Rightarrow \lambda A \in U.$$

Conclusion: U is a subspace.

Note: The set U is the subspace of all symmetric matrices.

Non-Examples

EXAMPLE 7. Show that the set

$$U = \{p : p \in \mathbf{P}_3 \text{ and } p(2) = 1\}$$

is not a subspace of $\mathbf{P}_3$.

SOLUTION.

S1. Recall that

$$0(x) = 0x^3 + 0x^2 + 0x + 0 = 0$$

$$\Rightarrow 0(2) = 0 \neq 1 \Rightarrow 0(x) \notin U.$$

Conclusion: U is not a subspace of P_3.

EXAMPLE 8. The solutions set to the system $A\mathbf{x} = \mathbf{0}$ given in Example 1 is given by the linear combination

$$t\mathbf{x_1} + s\mathbf{x_2}, \quad t, s \in \mathbb{R}.$$

The set $\{t\mathbf{x_1} + s\mathbf{x_2} \ : \ t, s \in \mathbb{R}\}$ is called the **span** of $\mathbf{x_1}$ and $\mathbf{x_2}$.

$\hookrightarrow 2\vec{x}_1 + 2\vec{x}_2, \quad 0\vec{x}_1 + 2\vec{x}_2, \quad -\vec{x}_1 - \vec{x}_2, \ldots$

DEFINITION 2. Let $\{\mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v_n}\}$ be a collection of vectors in a vector space V.

① a **linear combination** of the vectors $\mathbf{v_1}$, $\mathbf{v_2}$, ..., $\mathbf{v_n}$ is an expression of the form

$$a_1\mathbf{v_1} + a_2\mathbf{v_2} + \cdots + a_n\mathbf{v_n}$$

where a_1, a_2, ..., a_n are scalars called the **coefficients** of each vector.

② The set of all linear combinations of these vectors is called their **span**.

③ If it happens that $V = \operatorname{span}\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$, then the vectors are called a **spanning set** for V.

Remarks:

- $\operatorname{span}\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ is a subspace of V.

Example 9. Consider $p_1 = 1 + x + 4x^2$ and $p_2 = 1 + 5x + x^2$, two polynomials in $\mathbf{P}_2$.

a) Is p_1 in the span$\{1 + 2x - x^2, 3 + 5x + 2x^2\}$.

b) Is p_2 in the span$\{1 + 2x - x^2, 3 + 5x + 2x^2\}$.

Solution.

a) <u>Goal:</u> $\vec{v} = a_1\vec{v_1} + a_2\vec{v_2}$

$\Longleftrightarrow p_1(x) = a_1(1+2x-x^2) + a_2(3+5x+2x^2)$

$\Longleftrightarrow p_1(x) = (a_1 + 3a_2) + (2a_1 + 5a_2)x + (-a_1 + 2a_2)x^2$

$\Longleftrightarrow 1 + x + 4x^2 = (a_1 + 3a_2) + (2a_1 + 5a_2)x + (-a_1 + 2a_2)x^2$

$\Longleftrightarrow 1 = a_1 + 3a_2, \quad 1 = 2a_1 + 5a_2$

$4 = -a_1 + 2a_2$

$$\Longleftrightarrow \left[\begin{array}{cc|c} 1 & 3 & 1 \\ 2 & 5 & 1 \\ -1 & 2 & 4 \end{array}\right] \longrightarrow a_1 = -2, \; a_2 = 1$$

$\rightarrow p_1(x) = -2(1+2x-x^2) + 1(3+5x+2x^2)$.

(b) $p_2(x) = a_1(1+2x-x^2) + a_2(3+5x+2x^2)$

$$\Leftrightarrow 1+5x+x^2 = (a_1+3a_2) + (2a_1+5a_2)x + (-a_1+2a_2)x^2$$

$\Leftrightarrow$ $a_1+3a_2 = 1$, $2a_1+5a_2 = 5$

$-a_1+2a_2 = 1$

$$\Leftrightarrow \left[\begin{array}{cc|c} 1 & 3 & 1 \\ 2 & 5 & 5 \\ -1 & 2 & 1 \end{array}\right] \longrightarrow \cdots \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

$\to$ system is inconsistent.

$\to$ $p_2 \notin \text{Span}\{1+2x+x^2, 3+5x+2x^2\}$.

<u>span vs spanning set</u>

$V = \mathbb{R}^3$, $\vec{v}_1 = (1,0,0)$, $\vec{v}_2 = (0,1,0)$.

$\text{span}\{\vec{v}_1, \vec{v}_2\} = \{a_1\vec{v}_1 + a_2\vec{v}_2\} = \{(a_1, a_2, 0)\}$

$\text{span}\{\vec{v}_1, \vec{v}_2\} \neq V$