Real Numbers

- Field Axioms
- Order Axioms
- Completeness Axiom
- Absolute Value

1- Real numbers.

We assume that everybody knows what is $\mathbb{R}$.

1.1. Fields axioms.

Axioms. We equip $\mathbb{R}$ with $+$ & $\cdot$ such that

- **F1** $(x+y)+z = x+(y+z)$ & $(x\cdot y)\cdot z = x\cdot(y\cdot z)$.
- **F2** $x+y = y+x$ & $x\cdot y = y\cdot x$.
- **F3** $x\cdot(y+z) = x\cdot y + x\cdot z$
- **F4** $\exists!\, 0\in\mathbb{R}$ s.t. $0+x=x$ $\forall x\in\mathbb{R}$.
- **F5.** $\forall x\in\mathbb{R}, \exists!\, y\in\mathbb{R}$ s.t. $x+y=0$.
- **F6** $\exists!\, 1\in\mathbb{R}$ s.t. $1\cdot x = x$ $\forall x\in\mathbb{R}$ & $0\neq 1$.
- **F7** $\forall x\in\mathbb{R}, x\neq 0, \exists!\, y\in\mathbb{R}$ s.t. $x\cdot y = 1$.

Notations. additive inverse: $-x$.
multiplicative inverse: x^{-1} or $1/x$.
multiplication: xy

Algebraic properties

1) $0\cdot x = 0$ $\forall x\in\mathbb{R}$. [Proof: $\boxed{\begin{aligned} 0\cdot x &= (0+0)\cdot x = 0\cdot x + 0\cdot x \\ \Rightarrow\quad 0\cdot x - 0\cdot x &= 0\cdot x \\ \Rightarrow\quad 0 &= 0\cdot x \end{aligned}}$

2) $-xy = (-x)\cdot y = x\cdot(-y)$. [Proof: $\boxed{\begin{aligned} &-(xz) + xz = 0 \\ &x(-z) + xz \\ &\quad = x\cdot((-z)+z) = 0 \end{aligned}}$

3) $-(-x) = x$. [Proof: $\boxed{\begin{aligned} &-(-x) + (-x) = 0 \\ &-(-x) + (-x) + x = x \Rightarrow -(-x) = x \end{aligned}}$

$(\mathbb{R}, +, \cdot)$ is called a field.

1.2. Order Axioms.

Axioms We define a relation $>$ on $\mathbb{R}$ with the prop:

- **O1** $x<y \Rightarrow x+z<y+z \quad \forall z\in\mathbb{R}$.
- **O2** $x<y$ and $y<z \Rightarrow x<z$.
- **O3** $\forall x,y\in\mathbb{R}$, $x<y$, $y<x$, or $x=y$.
- **O4** $x<y$ and $z>0 \Rightarrow xz<yz$.

Notations. (i) $x\leq y \Leftrightarrow x<y$ or $x=y$.
(ii) $x>y \Leftrightarrow y<x$ (iii) $x\geq y \Leftrightarrow y\leq x$

Properties

(i) $x<y \Rightarrow -y<-x$. (ii) $0<1$.
(iii) $0<x<y \Rightarrow \frac{1}{y}<\frac{1}{x}$ (iv) $x^2\geq 0$.
(v) $x<y$ and $z<0 \Rightarrow zy<zx$.

Proofs. (i) O1 $\Rightarrow x-x<y-x \Rightarrow 0<y-x$
O1 again $\Rightarrow -y<-x$.
(ii) O3 $\Rightarrow 0<1$, $0>1$ or $0=1$.
$0=1$ impossible because $0\neq 1$ (see F6).
Suppose $0>1$. Adding (-1)
O1 $\Rightarrow -1>0$
Then, O4 $\Rightarrow 0\cdot(-1)>1\cdot(-1) \Rightarrow 0>-1$. #.
(iii) Let $0<x<y$. First we prove $\frac{1}{x}>0$.
Suppose $\frac{1}{x}<0$. Then, (i) $\Rightarrow -\frac{1}{x}>0$. Multiply by $-\frac{1}{x}$

and use O4 $\Rightarrow \quad 0\cdot\left(\frac{-1}{x}\right) < x\cdot\left(-\frac{1}{x}\right)$

$\Rightarrow \quad 0 < -1$

$\Rightarrow \quad 0 > 1$ (by (i) $-(-1)=1$). #

Apply two times O4 :

$$x<y \Rightarrow \frac{1}{x}\cdot x < \frac{1}{x}\cdot y$$

$$\Rightarrow 1\cdot\frac{1}{y} < \frac{1}{x}\cdot y\cdot\frac{1}{y}$$

$$\Rightarrow \frac{1}{y} < \frac{1}{x}.$$

(iv) Let $x \in \mathbb{R}$. By O3, $x>0$, $x=0$, $x<0$.

$\underline{x=0}$ $\quad x\cdot x = x^2 = 0$.

$\underline{x>0}$ $\quad$ O4 $\Rightarrow \quad x\cdot x > x\cdot 0 = 0 \Rightarrow x^2>0$.

$\underline{x<0}$ $\quad$ (i) $\Rightarrow \quad -x>0$. O4 $\Rightarrow (-x)\cdot(-x) > 0$.

Now, $(-x)\cdot(-x) = -(x\cdot(-x)) = -(-x\cdot x) = x\cdot x$.

(v) (i) $\Rightarrow \quad -z>0$. By O4 $\Rightarrow \quad x\cdot(-z) < y\cdot(-z)$

$\Rightarrow \quad -xz < -yz$.

Use (i) $\Rightarrow \quad -(-xz) > -(-yz) \Rightarrow xz > yz$. $\quad$ II

We say that $(\mathbb{R}, +, \cdot, <)$ is an ordered field.

1.3 Completeness axiom.

We could try to do analysis with $\mathbb{Q}$. But, we will hit some limitations.

Thm. There is no rational number x s.t. $x^2 = 2$.

Proof. Suppose, on the contrary, that $\exists x \in \mathbb{Q}$ s.t. $x^2 = 2$. Write $x = \frac{a}{b}$ with $\gcd(a,b) = 1$.

So, $a^2 = 2b^2$. This means that a is even

$$\Rightarrow \quad a = 2k \quad \text{for some } k \in \mathbb{Z}.$$

So, $4k^2 = 2b^2 \Rightarrow 2k^2 = b^2$.

So, b is also even. Thus a & b share a common factor (2), contradicting the fact that $\gcd(a,b) = 1$. $\square$

We could guess that $\sqrt{2}$ is the number seeking. We will prove that using the completeness axiom.

Def. A set $E \subseteq \mathbb{R}$ is <u>bounded from above</u> (resp. below) if $\exists M \in \mathbb{R}$ s.t. $x \leq M$ (resp. $x \geq M$) $\forall x \in E$. M is called an upper (resp. lower) bound for E. E is <u>bounded</u> if it is bounded from above & below.

Examples ① $E = \{ \frac{1}{n} : n \geq 1 \}$. Then, $n \geq 1 \Rightarrow \frac{1}{n} \leq 1$. So E is above by 1. It is also bounded below by 0.

② $E = \{ \frac{3n+1}{2n+2} : n \geq 1 \}$. Bounded above by $\frac{3}{2}$ and below by 1.

Def. (i) Let $E \subseteq \mathbb{R}$ be bounded from below. A $a \in \mathbb{R}$ is a <u>greatest lower bound</u> for E if

- a is a lower bound for E.
- for all lower bounds M of E, we have $M \leq a$.

(ii) Let $E \subseteq \mathbb{R}$ be bounded above. A $b \in \mathbb{R}$ is a <u>least upper bound</u> for E if

- b is an upper bound for E.
- for all upper bounds M of E, we have $b \leq M$.

Thm. If $E \subseteq \mathbb{R}$ and has a least upper bound, then it is unique.

Proof. Let a_1 & a_2 be two lup of E. Then

1) $\forall x \in E$, $x \leq a_1$.
2) $\forall a$ u.b., $a_1 \leq a$.
3) $\forall x \in E$, $x \leq a_2$.
4) $\forall a$ u.b., $a_2 \leq a$.

By 2) and 4), so $a_1 \leq a_2$ and $a_2 \leq a_1$. Then,
$(a_1 < a_2$ or $a_1 = a_2)$ and $(a_2 < a_1$ or $a_2 = a_1)$.
So, doing all the logic, $a_1 = a_2$. $\square$

Notation glb E or $\inf E$
lup E or $\sup E$.

Problem with $\mathbb{Q}$: $\{x \in \mathbb{R} : x^2 < 2\}$ has not upper bound in $\mathbb{Q}$.

<u>Axiom completeness.</u> (AC)
Every non-empty set $E \subseteq \mathbb{R}$ which is bounded from above has a supremum.

Thm. If $S \subseteq \mathbb{R}$ $(S \neq \emptyset)$ is bounded below, then it has an infimum.

Proof. Suppose $E \neq \emptyset$ and is bounded from below.
Consider the set $-E := \{-x : x \in E\}$. So, $-E$ is bounded from above because

$$\exists M \in \mathbb{R} \text{ s.t. } x \geq M \ \forall x \in E \implies -x \leq -M \ \forall x \in E$$
$$\implies -M \text{ upper bound for } -E.$$

Also, $-E$ is not empty because $E \neq \emptyset$. Then, by Ax 12, $-E$ has a supremum, $\sup(-E)$.
We will show that $-\sup(-E)$ is the infimum of E.

- $\forall x \in E$, we have $-x \leq \sup(-E)$
$$\implies x \geq -\sup(-E).$$
- Let M be a lower bound for E. Then $-M$ is an upper bound for $-E$ and we have
$$\sup(-E) \leq -M$$
$$\implies -\sup(-E) \geq M$$

Then $-\sup(-E)$ is the infimum of E. $\square$

Thm (Archimedean property) (AP)
Let $x > 0$ and $y \in \mathbb{R}$. Then $\exists n \in \mathbb{N}$ s.t. $nx > y$.
Proof. If $y \leq 0$. then $n = 1$ does the job. Let $y > 0$.

Suppose that it's not the case, so

$$\exists x>0, \exists y>0 \text{ s.t. } \quad nx \le y \quad \forall n \in \mathbb{N}.$$

Let $E := \{ nx : n \in \mathbb{N} \}$. Then it is bounded from above by y. By AC, $\sup E$ exists.

Now, $\sup E$ is an upper Bound, so

$$nx \le \sup E \quad \forall n \in \mathbb{N}.$$

In particular, $(n+1)x \le \sup E \quad \forall n \in \mathbb{N}$

$$\Rightarrow \quad nx \le \sup E - x \quad \forall n \in \mathbb{N}.$$

Now, since $x>0$, we have

$$x + \sup E > \sup E \Rightarrow \sup E > \sup E - x.$$

So $\sup E - x$ is an upper bound of E & is smaller than $\sup E$, contradicting the def of $\sup E$.

$\square$

<u>Example</u>. Let $E := \{ \frac{3n+1}{2n+2} : n \ge 1 \}$. We saw that

$$1 \le \frac{3n+1}{2n+2} \le \frac{3}{2}.$$

Let show that $\inf E = 1$ and $\sup E = \frac{3}{2}$.

- $1 \in E$ so it must be the $\inf E$.
- Let $x := \sup E$. Then we have $x \le \frac{3}{2}$. There are two cases: $x < 3/2$ or $x = 3/2$. Suppose $x < 3/2$.

Remark that $3-2x>0$. In the AP, take x to be $3-2x$ and $y=2x-1$, then there is an integer $n\in\mathbb{N}$ s.t.

$$n(3-2x)>2x-1$$

After some algebra $\Rightarrow$ $\frac{3n+1}{2n+2}>x$.

But this contradicts the definition of x!

So $\sup E=\frac{3}{2}$.

~~What~~ How did you find x & y? You have to start from you want and go backward:

$$\frac{3n+1}{2n+1}>x \iff 3n+1>2nx+2x \iff (3-2x)n>2x-1.$$

We are now ready for the proof of existence of $\sqrt{2}$

<u>Thm</u> If $p>0$, then $\exists x>0$ s.t. $x^2=p$.

<u>Proof.</u> First suppose that $p\geq 1$. Define $E\subseteq\mathbb{R}$ as

$$E:=\{y\in\mathbb{R}:\ y>0 \text{ and } y^2\leq p\}.$$

$E\neq\emptyset$ since $1\in E$ and is bounded from above by p since $y>p$ implies that $y^2>yp>p^2$ $(y>p)$ and so $y\notin E$. So by the AC, $\sup E$ exists. Let $x:=\sup E$ and we will prove that $x^2=p$. There are three possibilities:

$$x^2<p,\qquad x^2=p,\qquad x^2>p.$$

Suppose $x^2 < p$. Let $\delta := \min\{1, \frac{p-x^2}{2x+1}\}$.

Then,

$$(x+\delta)^2 = x^2 + 2\delta x + \delta^2 \le x^2 + 2\delta x + \delta$$
$$\le x^2 + p - x^2$$
$$= p$$

So, $(x+\delta)^2 \le p \Rightarrow x+\delta \in E$. But $x < x+\delta$, $\#$.

Suppose $x^2 > p$. Let $\delta := \frac{x^2-p}{2x}$, then

$$(x-\delta)^2 = x^2 - 2\delta x + \delta^2 \ge x^2 - 2\delta x$$
$$= x^2 + p - x^2 = p.$$

Thus, since $\forall y \in E$, $y \le p$, $x-\delta$ is an upper bound for E, contradicting the def. of x.

Thus, we must conclude that $x^2 = p$.

The case $0 < p < 1$ is a consequence of the last part because $\frac{1}{p} > 1$. $\square$

Remarks.

① The positive square root of $p>0$ is denoted by $\sqrt{p}$.

② The negative square root of $p>0$ is denoted by $-\sqrt{p}$.

③ We always have that $\sqrt{xy} = \sqrt{x}\sqrt{y}$ $(x, y \ge 0)$.

④ We always have that $\sqrt{x/y} = \sqrt{x}/\sqrt{y}$ $(x \ge 0, y > 0)$

Notations. • $[a,b] := \{x \in \mathbb{R} : a \le x \le b\}$. (closed)

- $(a,b) := \{x \in \mathbb{R} : a < x < b\}$. (open).
- $[a,b) := \{x \in \mathbb{R} : a \le x < b\}$.
- $(a,b] := \{x \in \mathbb{R} : a < x \le b\}$.

thm. If $x \in \mathbb{R}$, then $\exists n \in \mathbb{Z}$ s.t.

$$n \le x < n+1$$

Proof. Let $A := \{n : n \in \mathbb{Z}$ & $n \le x\}$. Two cases:

- $\underline{A = \emptyset}$. Then $\forall n \in \mathbb{Z}$, $n > x$. If $x \ge 0$, then this is impossible because $n = -1$ is less then x. If $x < 0$, then $-x > 0$ and $-\frac{1}{x} > 0$. By AP, $\exists N \in \mathbb{N}$ s.t.

$$N \cdot \left(-\frac{1}{x}\right) > 1$$

$$\iff \quad N < x.$$

Contradiction with $n > x$ $\forall n \in \mathbb{Z}$. So $A \ne \emptyset$.

- $\underline{A \ne \emptyset}$ A is bounded from above by x

$$\overset{AC}{\Longrightarrow} \quad b = \sup A \quad \text{exists.}$$

Now, $b-1$ is not an upper bound for A. There is a $m \in A$ s.t. $b-1 < m \le b$ and

$$b < m+1 \leq b+1$$

Since $m \in A \Rightarrow m \leq x \leq b < m+1$. □

Thm. Between any two real numbers, there is a rational number.

Proof. Let $x < y$ with $x, y \in \mathbb{R}$.

Then, $\frac{1}{y-x} > 0$ and by AP:

$$\exists N \in \mathbb{N}, \quad N(y-x) > 1.$$

From the previous thm, $\exists n \in \mathbb{N}$ s.t.

$$n \leq Nx < n+1.$$

So,
$$n+1 \leq Nx+1 < Nx + N(y-x) = Ny$$

$$\Rightarrow \quad Nx < n+1 < Ny$$

$$\Rightarrow \quad x < \frac{n+1}{N} < y \quad (N \geq 1).$$

Take $r = \frac{n+1}{N} \in \mathbb{Q}$. □

The last theorem means that every open interval (x, y) $(x < y)$ contains a rational number. This means that $\mathbb{Q}$ is <u>dense</u> in $\mathbb{R}$, noted $\overline{\mathbb{Q}} = \mathbb{R}$.

1.4. Absolute value.

Def. If $x \in \mathbb{R}$, then $|x| = \begin{cases} x & , x \geq 0 \\ -x & , x < 0. \end{cases}$

Properties

- $|x| = \sqrt{x^2}$ $\forall x \in \mathbb{R}$. [$x \in \mathbb{R}$, $x \geq 0$, $\sqrt{x^2} = x = |x|$
  $x < 0$, $\sqrt{x^2} = \sqrt{(-x)^2} = -x = |x|$.]

- $\pm x \leq |x|$ $\forall x \in \mathbb{R}$. [$x \in \mathbb{R}$, $x \geq 0$, $x = |x|$
  Same fn $-x$. $\leftarrow$ $x < 0$, $x < 0 < -x = |x|$.]

- $|xy| = |x| \cdot |y|$, $\forall x, y \in \mathbb{R}$. [$x, y \in \mathbb{R}$, then
  $|xy| = \sqrt{(xy)^2} = \sqrt{x^2 y^2} = \sqrt{x^2}\sqrt{y^2} = |x||y|$.]

- If $\varepsilon > 0$, then $|a| \leq \varepsilon$ iff. $-\varepsilon \leq a \leq \varepsilon$.
  [Proof: $a \geq 0 \Rightarrow a \leq \varepsilon$. Comme $-\varepsilon < 0 \Rightarrow -\varepsilon \leq |a| \leq \varepsilon$.
  $a < 0 \Rightarrow -a \leq \varepsilon \Rightarrow -\varepsilon \leq a < 0 < \varepsilon$.
  if $-\varepsilon \leq a \leq \varepsilon$. If $a \geq 0 \Rightarrow a = |a| \leq \varepsilon$.
  If $a < 0 \Rightarrow -\varepsilon \leq a \Rightarrow -a \leq \varepsilon \Rightarrow |a| \leq \varepsilon$.]

- $|x+y| \leq |x| + |y|$ $\forall x, y \in \mathbb{R}$. [Proof: we have $\pm x \leq |x|$ et $\pm y \leq |y|$. Ainsi $x + y \leq |x| + |y|$. De plus, $x \geq -|x|$ et $y \geq -|y|$
  $\Rightarrow$ $x + y \geq -|x| - |y|$ $\Rightarrow$ $|x+y| \leq |x| + |y|$.]

- $||x| - |y|| \leq |x - y|$. [$x = x - y + y$ and $y = y - x + x$].

1.5 $\mathbb{R}$ is uncountable

Lemma. Let $[a_1,b_1] \supseteq [a_2,b_2] \supseteq \cdots \supseteq [a_n,b_n] \supseteq \cdots$ be closed intervals. Then $\exists x \in \mathbb{R}$ s.t. $x \in [a_n,b_n]\ \forall n$.

Proof. Let $A := \{a_1, a_2, \ldots, a_n \ldots\}$.

We see that

$$a_1 \le a_2 \le a_3 \le \ldots \le a_n \le \ldots \quad \&$$
$$b_1 \ge b_2 \ge b_3 \ge \ldots \ge b_n \ge \ldots$$

But also, we have $a_n \le b_m\ \forall n, m$.
Indeed, let $M := \max\{n, m\}$. Then,

$$a_n \le a_M \le b_M \le b_m$$

This implies that b_n is an upper bound for A (for every $n \ge 1$). By AC, $\sup A$ exists. Let $x := \sup A$. We will show that x satisfies all the requirements.

- $x \ge a_n$ because x is $\sup A$.
- $x \le b_n$ because each b_n is an upper bound for A and x is $\sup A$.

Thus, $a_n \le x \le b_n \quad \forall n \ge 1$. $\square$

Thm. $\mathbb{R}$ is uncountable.

Proof. Let $f: \mathbb{N} \to \mathbb{R}$ be a function. We will show that f can't be a bijection by showing that f can't be a surjection.

Let $a_1, b_1 \in \mathbb{R}$, $a_1 < b_1$, s.t. $f(1) \notin [a_1, b_1]$. This is possible by the AP.

Let $a_2, b_2 \in \mathbb{R}$, $a_2 < b_2$, $[a_2, b_2] \subseteq [a_1, b_2]$ s.t. $f(2) \notin [a_2, b_2]$. This is possible.

- If $f(2) \notin [a_1, b_1]$, then take
$$a_2 = \frac{a_1}{2}, \quad b_2 = \frac{b_1}{2}$$

- If $f(2) \in [a_1, b_1]$, there are rational numbers $r, \tilde{r}$ s.t.
$$a_1 < r < f(2) < \tilde{r} < b_1.$$
Take $a_2 = r$ and $b_2 = \tilde{r}$.

Let $a_1, b_1, \dots, a_k, b_k$ be given, then chose a_{k+1}, b_{k+1} s.t. $a_{k+1} < b_{k+1}$, $[a_{k+1}, b_{k+1}] \subseteq [a_k, b_k]$ and $f(k+1) \notin [a_{k+1}, b_{k+1}]$.

By the lemma, there is a $x \in [a_n, b_n]$ for any $n \geq 1$. Since $f(n) \notin [a_n, b_n]$, $\forall n$

$$\Rightarrow \quad f(n) \neq x \quad \forall n.$$

So, f is not surjective. $\square$

Exercises.

#1 If $x<y$, prove that $x< \frac{x+y}{2} < y$.

#2 If $x\geq 0$ and $y\geq 0$, prove that $xy \leq \left(\frac{x+y}{2}\right)^2$

#3 If $0<a<b$, prove that $0<a^2<b^2$ (HW01)

#4 (HW01) If $0<a<b$, prove that $0<\sqrt{a}<\sqrt{b}$.

#5 Prove that if $E\subseteq \mathbb{R}$ has a g.l.b., then it is unique.

#6. If $E\subseteq\mathbb{R}$ is bounded from above and $x=\sup E$, prove that for each $\varepsilon>0$, there is $a\in E$ such that $x-\varepsilon < a \leq x$.

#7. (HW01) Prove that if $p>0$ and $n\in\mathbb{N}$, then there is a unique positive real number x such that $x^n=p$

#8 Find the inf and sup of the following sets. Make sure to justify all your answers:

1) $E:=\left\{\frac{1}{n} : n\in\mathbb{N}\right\}$ 2) $E=\left\{\frac{4n+5}{3n+3} : n\in\mathbb{N}\right\}$.

3) $E:=\{x\in\mathbb{R} : x>0 \text{ and } x^2\leq 9\}$.

#9. Let $E\subseteq\mathbb{R}$ and $E\neq\emptyset$. Let $-E:=\{-x : x\in E\}$. If E is bounded, show that

(a) $-\sup E = \inf(-E)$ (b) $-\inf E = \sup(-E)$.

#10 Let $E \subseteq \mathbb{R}$ and $E \neq \emptyset$. For $r \in \mathbb{R}$, let

$rE := \{rx : x \in E\}$ and $r + E := \{r + x : x \in E\}$.

Show that

(a) if $r > 0$, $\sup(rE) = r \sup E$.

(b) $\sup(r + E) = r + \sup E$.

From the book

0.5 : 4f