

Q.

1	2	3	4	5	6	7	8
0	5	5	5	5	0	3	5

Score

Q.

9	10	Tot
4	2	34

Score

(/70).

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0/5

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2. Let $f: D \subset \mathbb{R} \rightarrow \mathbb{R}$, and suppose that x_0 is an accumulation point of D . Suppose that for each sequence $(x_n)_{n=1}^{\infty}$ converging to x_0 with $x_n \in D \setminus \{x_0\}$ for each $n \geq 1$, then the sequence $(f(x_n))_{n=1}^{\infty}$ is Cauchy. Show that f has a limit at x_0 .

Proof: Because the sequence $(f(x_n))_{n=1}^{\infty}$ is Cauchy we know that it converges and has a limit L . Suppose that L is not a limit of f at x_0 . Thus, there is $\epsilon > 0$ such that for every $\delta > 0$, there is $x \in D$, with $0 < |x - x_0| < \delta$ and such that $|f(x) - L| \geq \epsilon$. In particular, for each positive integer n , there is $x_n \in D$ with $0 < |x_n - x_0| < \frac{1}{n}$ such that if $|x_n - L| \geq \epsilon$, the sequence $\{x_n\}_{n=1}^{\infty}$ converges to x_0 and is a sequence of members of D distinct from x_0 ; hence, $\{f(x_n)\}_{n=1}^{\infty}$ converges to L , contrary to the fact that $|f(x_n) - L| \geq \epsilon > 0$ for all n . Thus, L must be the limit of f at x_0 .

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It is working.

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3. To show that a limit is unique we aim to prove that if $\lim_{x \rightarrow a} f(x) = x$ and $\lim_{x \rightarrow a} f(x) = x_0$, then $x_0 = x$, and therefore the limit is unique.

Proof: Suppose $\lim_{x \rightarrow a} f(x) = x_0$ and $\lim_{x \rightarrow a} f(x) = x_1$. Let $\epsilon > 0$ be arbitrary. There exists δ_1, δ_2 such that if $0 < |x - a| < \delta_1$, then $|f(x) - x_0| < \epsilon/2$ and if $0 < |x - a| < \delta_2$, then $|f(x) - x_1| < \epsilon/2$. Pick $\delta = \min\{\delta_1, \delta_2\}$, then if $0 < |x - a| < \delta$, we have:

$$|x_0 - x_1| = |x_0 - x_1 + f(x) - f(x)| \\ \leq |f(x) - x_0| + |f(x) - x_1| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Because $\epsilon > 0$ was arbitrary, we have $x_0 = x_1$ and therefore f has a limit at x_0 and that is unique. SIS

4. Denote $L = \lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} h(x)$

Then, $\forall \epsilon > 0, \exists \delta_1 > 0$, when $0 < |x - x_0| < \delta_1$, we have $|f(x) - L| < \epsilon$, and hence

$$L - \epsilon < f(x) < L + \epsilon$$

Similarly, $\forall \epsilon > 0, \exists \delta_2 > 0$, when $0 < |x - x_0| < \delta_2$, we have

$$L - \epsilon < h(x) < L + \epsilon$$

Let $\delta = \min(\delta_1, \delta_2)$. When $0 < |x - x_0| < \delta$, we have

$$L - \epsilon < f(x) \leq g(x) \leq h(x) < L + \epsilon$$

namely,

$$|g(x) - L| < \epsilon$$

Thus $\lim_{x \rightarrow x_0} g(x) = L$. So $\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} h(x) = \lim_{x \rightarrow x_0} g(x)$. SIS

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5. a) If g is bounded, it means that $|g(x)| < c$ for some c , for all x . Now

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$$0 \leq |f(x)g(x)| = |f(x)||g(x)| \leq c|f(x)|$$

Since $\lim_{x \rightarrow \infty} f(x) = 0$, we have $\lim_{x \rightarrow \infty} |f(x)| = 0$, and so by the squeeze theorem **4/5**

$$\lim_{x \rightarrow \infty} f(x)g(x) = 0$$

\hookrightarrow squeeze thm was not prove for $x \rightarrow \infty$

b) Let $f: (a, \infty) \rightarrow \mathbb{R}$ for some $a > 0$, and let $g: (0, \frac{1}{a}) \rightarrow \mathbb{R}$ be defined by $g(x) = f(1/x)$

\Rightarrow Suppose f has a limit at ∞

$$\text{if } x \rightarrow \infty \Rightarrow \frac{1}{x} \rightarrow 0$$

$$f\left(\frac{1}{x}\right) \rightarrow 0 \Rightarrow g(x) \rightarrow 0$$

so $g(x)$ has a limit at 0

c) Suppose $g(x)$ has a limit at 0

$$g(x) \xrightarrow{x \rightarrow 0} 0$$

$$f\left(\frac{1}{x}\right) \rightarrow 0$$

If $\frac{1}{x} \rightarrow 0$ as $x \rightarrow \infty$, then f has a limit at ∞

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You have to use the definition provided in the exercise.

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a) Take $\lim_{n \rightarrow \infty} \frac{1}{n} + \frac{1}{n+1} + \dots + \frac{1}{2n}$. By the sum rule, we have $\lim_{n \rightarrow \infty} \frac{1}{n} + \lim_{n \rightarrow \infty} \frac{1}{n+1} + \dots + \lim_{n \rightarrow \infty} \frac{1}{2n}$. Each of the individual limits goes to 0 so $\lim_{n \rightarrow \infty} \frac{1}{n} + \frac{1}{n+1} + \dots + \frac{1}{2n} = 0$

Not correct.
sum rule
can't be applied

b) With some algebra and the sum rule we have

$$\lim_{n \rightarrow \infty} \frac{\frac{1}{n^2} + \frac{2}{n^2} + \dots + \frac{n}{n^2}}{n^2} = \lim_{n \rightarrow \infty} \frac{1}{n^2} + \lim_{n \rightarrow \infty} \frac{2}{n^2} + \dots + \lim_{n \rightarrow \infty} \frac{n}{n^2} = 0$$

Each of these limits goes to zero and so $\lim_{n \rightarrow \infty} \frac{\frac{1}{n^2} + \frac{2}{n^2} + \dots + \frac{n}{n^2}}{n^2} = 0$

Not correct! Sum rule
doesn't apply

Math 331 : Homework 3 Due.

7. The $\lim_{x \rightarrow 0} \frac{\sqrt{1+x}-1}{x} = \frac{1}{2}$. To prove this, let $\epsilon > 0$ be given. We seek $\delta > 0$, so that whenever $0 < |x| < \delta$, $|g(x) - \frac{1}{2}| < \epsilon$.
- Choose $\delta = \epsilon$. Suppose $0 < |x| < \delta$ with $0 < x < 1$. Then,

$$\begin{aligned} |g(x) - \frac{1}{2}| &= \left| \frac{\sqrt{1+x}-1}{x} - \frac{1}{2} \right| = \left| \frac{2\sqrt{1+x}-2-x}{2x} \right| \\ &= \left| \frac{2\sqrt{1+x}-(2+x)}{2x} \cdot \frac{2\sqrt{1+x}+(2+x)}{2\sqrt{1+x}+(2+x)} \right| = \left| \frac{-x^2}{2x(2\sqrt{1+x}+(2+x))} \right| \\ &< \left| \frac{-x}{8} \right| < |x| < \delta = \epsilon \end{aligned}$$

justify this

8. $\lim_{x \rightarrow 1} f(x) = 1$

$$\begin{aligned} \lim_{x \rightarrow 1} f(x)(1-f(x)^2) &= \lim_{x \rightarrow 1} \frac{(f(x)-f(x)^3)(f(x)+f(x)^2)}{1-f(x)} \quad \text{Take } \delta < \frac{1}{2} \Rightarrow \sqrt{1+x} > 1/\sqrt{2} \quad \& \quad 2+x > 3/2 \\ &\Rightarrow \sqrt{1+x} + 2+x > 2\sqrt{1/\sqrt{2}} + 3/2 \geq 1+1-2 \Rightarrow (*) \leq \frac{|x-1|}{4} \\ &= \lim_{x \rightarrow 1} \frac{(f(x)-f(x)^3)(f(x)+f(x)^2)}{f(x)-f(x)^2+f(x)^2-f(x)^3} \\ &= \lim_{x \rightarrow 1} f(x) + f(x)^2 \\ &= \lim_{x \rightarrow 1} f(x) + \lim_{x \rightarrow 1} f(x)^2 \\ &= 1+1 \\ &= 2 \end{aligned}$$

(by)

$f(x) = x^n$ then $\lim_{x \rightarrow x_0} f(x) = x_0^n$ for any $x_0 \in \mathbb{R}$
 If $x_0 \in [0, \infty)$, then $\lim_{x \rightarrow x_0} \sqrt{x} = \sqrt{x_0}$

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9. Since f has a limit at x_0 , $\forall \epsilon > 0$,
 $\exists \delta > 0$, for $x \in D$ with $|x - x_0| < \delta$
 we have

$$|f(x) - f(x_0)| < \epsilon$$

Hence

$$|f(x) - f(x_0)| \leq |f(x) - f(x_0)| < \epsilon$$

Thus, f has a limit at x_0 .

what did you use
 here?
 $|x-x_0| =$

10. a) Assume $f(x)$ is increasing. Let
 $D = \{x : x \in (\alpha, \beta), f \text{ does not have a limit at } x\}$. By Lemma 2.7 in the book,
 $x \in D \iff U(x) - L(x) \neq 0$. and, since f is increasing, $\iff U(x) - L(x) > 0$. Let

$$D_n = \{x : U(x) - L(x) > \frac{1}{n}\}$$

It is clear that $D = \bigcup_{n=1}^{\infty} D_n$. Now, suppose $\{x_1, \dots, x_r\} \subset D_n$ with $\alpha < x_1 < x_2 < \dots < x_r < \beta$.

Choose z_1, \dots, z_{r+1} such that $\alpha < z_1 < x_1, x_i < z_{i+1} < x_{i+1}$ for $i = 1, 2, \dots, r-1$, and $x_r < z_{r+1} < \beta$. Now for each i , $f(z_i) \leq L(x_i)$ and $U(x_i) \leq f(z_{i+1})$

hence

$$f(z_{i+1}) - f(z_i) \geq U(x_i) - L(x_i) > \frac{1}{n} \quad \text{now}$$

$$f(\beta) - f(\alpha) = f(\beta) - f(z_{r+1})$$

$$+ \sum_{k=2}^{r+1} [f(z_k) - f(z_{k-1})] + f(z_1) - f(\alpha) \geq r \left(\frac{1}{n} \right)$$

$$f(\beta) - f(\alpha) = f(\beta) - f(z_{r+1})$$

$$+ \sum_{k=2}^{r+1} [f(z_k) - f(z_{k-1})] + f(z_1) - f(\alpha) \geq r \left(\frac{1}{n} \right)$$

Since $f(\beta) - f(\alpha) > 0$ is a fixed real number, it is necessary that $r \leq n[f(\beta) - f(\alpha)]$. Therefore, D_n is finite for each n ; hence whenever the limit exists $\lim_{x \rightarrow x_0} f(x) = x_0$

?
 ↗

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If $x_0 \in [0, \infty)$, then $\lim_{x \rightarrow x_0} \sqrt{x} = \sqrt{x_0}$

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10. b) We need to prove that for any point $x_0 \in [0, \infty)$, for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|x - x_0| < \delta \Rightarrow |\sqrt{x} - \sqrt{x_0}| < \epsilon$$

So, to find a δ , we turn to the inequality $|\sqrt{x} - \sqrt{x_0}| < \epsilon$. Since we want an expression involving $|x - x_0|$, multiply by the conjugate to remove the square roots.

$$|\sqrt{x} - \sqrt{x_0}| < \epsilon \Rightarrow |\sqrt{x} - \sqrt{x_0}| \cdot |\sqrt{x} + \sqrt{x_0}| < \epsilon \cdot |\sqrt{x} + \sqrt{x_0}|$$
$$|x - x_0| < \epsilon \cdot |\sqrt{x} + \sqrt{x_0}| \quad (1)$$

Now if you require that $|x - x_0| < 1$, then it follows that $x - x_0 < 1$, so $x_0 - 1 < x < x_0 + 1$, and therefore that $\sqrt{x} < \sqrt{x_0 + 1}$. Therefore, $\sqrt{x} + \sqrt{x_0} < \sqrt{x_0 + 1} + \sqrt{x_0}$, which combined with (1) tells us that

$$|x - x_0| < \epsilon (\sqrt{x_0 + 1} + \sqrt{x_0})$$

So let $\delta = \min(1, \epsilon(\sqrt{x_0 + 1} + \sqrt{x_0}))$. This proves that $\lim_{x \rightarrow x_0} \sqrt{x} = \sqrt{x_0}$.

I'm little bit confused...

You have

$$|\sqrt{x} - \sqrt{x_0}| = (\sqrt{x} - \sqrt{x_0})(\sqrt{x} + \sqrt{x_0})$$

$$= \frac{x - x_0}{(\sqrt{x} + \sqrt{x_0})}$$

So, you have to bound $\frac{1}{(\sqrt{x} + \sqrt{x_0})}$. and not $\sqrt{x} + \sqrt{x_0}$.

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II.a) To prove that f has a limit at every point, I interpret this to prove that $\lim_{x \rightarrow a} f(x)$ exists for every a , given that it exists for 0 .

We have:

$$\lim_{x \rightarrow a} f(x) = \lim_{h \rightarrow 0} f(a+h) = \lim_{h \rightarrow 0} f(a)f(h) = f(a) \cdot \lim_{h \rightarrow 0} f(h).$$

Since we know that the expression on the right exists, you can go backwards to see that the expression on the left also exists

b) Assume that $\lim_{x \rightarrow 0} f(x) \neq 0$, then we must show that $f(x)$ is zero for all x .

$f(x) = f(x-h) \cdot f(h)$, then h goes to zero, and use the explanation above that f is continuous, and part (a) that the limit exists in every point. Then we get $f(x) = \lim_{h \rightarrow 0} f(x-h) \cdot \lim_{h \rightarrow 0} f(h) = f(x) \cdot \lim_{h \rightarrow 0} f(h)$. And since we know that the last limit is not 1, $f(x)$ must be zero.