MATH-331 Introduction to Real Analysis | Jeselle-Paige Guillermo
Homework ~~05~~ 06 | Fall 2021

Due date: November, ~~22th 1:20pm~~ 24th 5:00pm | Total: /65.

Exercise	1 (10)	2 (10)	3 (5)	4 (5)	5 (5)	6 (10)	7 (5)	8 (5)	9 (5)	10 (5)
Score										

Table 1: Scores for each exercises

Instructions: You must answer all the questions below and send your solution by email (to `parisepo@hawaii.edu`). If you decide to not use LaTeX to hand out your solutions, please be sure that after you scan your copy, it is clear and readable. Make sure that you attached a copy of the homework assignment to your homework.
If you choose to use LaTeX, you can use the template available on the course website.
No late homework will be accepted. No format other than PDF will be accepted. Name your file as indicated in the syllabus.

1 — Writing problems

For each of the following problems, you will be asked to write a clear and detailed proof. You will have the chance to rewrite your solution in your semester project after receiving feedback from me.

Exercise 1. (10 pts)

a) Fix any $\delta > 0$ and let $[a, b]$ be an interval with $a < b$. Find a tagged partition $\mathcal{P}$ of $[a, b]$ such that $\|\mathcal{P}\| < \delta$.

b) Suppose that f is Riemann integrable. Show that in the definition of the Riemann integral, the number L is unique. [Remark: This is why we gave it the name $\int_a^b f$.]

Exercise 2. (10 pts) Suppose that f and g are Riemann integrable on the interval $[a, b]$.

a) Show that $\int_a^b (f + g) = \int_a^b f + \int_a^b g$.

b) Show that if $f(x) \leq g(x)$ for any $x \in [a, b]$, then $\int_a^b f \leq \int_a^b g$.

Exercise 3. (5 pts) Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable on $[a, b]$ and suppose that $|f(x)| \leq M$ for all $x \in [a, b]$. Show that $\int_a^b f \leq M(b - a)$.

Exercise 4. (5 pts) Suppose that f is Riemann integrable on $[a, b]$. Let $(\mathcal{P}_n)_{n=1}^{\infty}$ be a sequence of tagged partitions of $[a, b]$ such that the sequence $\lim_{n\to\infty} \|\mathcal{P}_n\| = 0$. Prove that the sequence $(S(f, \mathcal{P}_n))_{n=1}^{\infty}$ converges to $\int_a^b f$.

Exercise 5. (5 pts) Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Suppose that f is Riemann integrable on $[a, c]$ for any $c \in (a, b)$. Show that f is Riemann integrable on $[a, b]$. [Hint: Use the Cauchy criterion for integrals.]

2
Homework problems

Answer all the questions below. Make sure to show your work. When we are asking to show that a function is Riemann integrable on an interval $[a, b]$, you must use the definition or the properties of the Riemann integral presented in sections 6.1 and 6.2 respectively.

Exercise 6. (10pts)

a) Define the function $f : [a, b] \to \mathbb{R}$ by $f(x) = k$ for every $x \in [a, b]$ where $k \in \mathbb{R}$ is a fixed constant. Show that f is Riemann integrable on $[a, b]$ and that $\int_a^b k\,dx = k(b - a)$.

b) Let $f(x) = \sin^2(x)$ where $x \in [a, b]$ and assume that the function $g(x) := \cos(kx)$ is integrable on $[a, b]$ for any $k \in \mathbb{R}$. Show that f is Riemann integrable on $[a, b]$.

Exercise 7. (5 pts) Show that the function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} 1 & , \text{ if } 0 \le x < 1/2 \\ 0 & , \text{ if } 1/2 \le x \le 1 \end{cases}$$

is Riemann integrable on $[0, 1]$.

Exercise 8. (5 pts) Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = 1$ if $x = 1/n$ where $n \in \mathbb{N}$, and by $f(x) = 0$ if $x \neq 1/n$, $n \in \mathbb{N}$. Show that f is Riemann integrable on $[0, 1]$.

Exercise 9. (5 pts) Show that the function $f : [0, 1] \to \mathbb{R}$ defined by $f(x) = 0$ if $x \neq 0$ and $f(x) = 4$ if $x = 0$ is Riemann integrable on $[0, 1]$.

Exercise 10. (5 pts) Let $\mathcal{P}$ be the following tagged partition of $[-1, 2]$:

$$\begin{aligned}\mathcal{P} := \{&(-9, [-1, -.8]), (-.7, [-.8, -.3]), (-.1, [-.3, 0]), (.2, [0, 0.2]), (.2, [.2, .4]), (.8, [.4, 1]), \\ &(1.42, [1, 1.5]), (1.9, [1.5, 2])\}.\end{aligned}$$

Find another partition $\mathcal{P}_0$ such that $\|\mathcal{P}_0\| \le \|\mathcal{P}\|/3$.

Exercise 1

a) Fix any $\delta > 0$ and let $[a,b]$ be an interval with $a<b$. Find a tagged partition $\mathcal{P}$ of $[a,b]$ such that $\|\mathcal{P}\| < \delta$

b) Suppose that f is Riemann integrable. Show that in the definition of the Riemann integral, the number L is unique. [Remark: this is why we gave it the name ${}_a\int^b f$]

a) Given $a<b \Rightarrow b-a>0$

$\Rightarrow \exists n \in \mathbb{N}$ s.t. $\frac{b-a}{n} < \delta$

consider $\mathcal{P}$ a partition of $[a,b]$ defined

$\{a = x_0, x_1, \dots, x_{n-1}, x_n = b\}$ where

$p_i = a + \frac{i(b-a)}{n}$

$p_1 - a = \frac{b-a}{n} < \delta$

$$p_k - p_{k-1} = a + \frac{k(b-a)}{n} - a - \frac{(k-1)(b-a)}{n}$$
$$= \frac{k(b-a)}{n} - \frac{(k-1)(b-a)}{n}$$
$$= \frac{b-a}{n}$$
$$< \delta$$

$\|\mathcal{P}\| < \delta$

b) f is integrable in $[a,b]$ iff $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. for each P where $\|P\| < \delta$ then $|S(f,P) - L| < \varepsilon$

We know $L = {}_a\int^b f(x)$

Goal: show that L is unique

Proof:

Assume L_1 & L_2 are RI. of f on $[a,b]$

Let $\varepsilon > 0$. Then for each $i = 1, 2$ $\exists \delta_i > 0$ s.t

$\|\mathcal{P}\| < \delta_j \Rightarrow |\beta - L_j| < \frac{\varepsilon}{2}$

Take $\delta = \min\{\delta_1, \delta_2\}$

$0 \le |L_1 - L_2| \le |\beta - L_1| + |\beta - L_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

Since $\varepsilon > 0$ was arbitrary

$0 \le |L_1 - L_2| < \varepsilon$

$\therefore |L_1 - L_2| = 0 \Rightarrow L_1 = L_2$

Exercise 2

Suppose that f and g are Riemann integrable on the interval $[a,b]$.

a) Show that ${}_a\int^b (f+g) = {}_a\int^b f + {}_a\int^b g$

b) Show that if $f(x) \le g(x)$ for any $x \in [a,b]$, then ${}_a\int^b f \le {}_a\int^b g$

a) Let $I(f) := {}_a\int^b f$ and $I(g) := {}_a\int^b g$

Let $\varepsilon > 0$ Then there exist some $\delta > 0$ s.t.

$\left|\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) - I(f)\right| \le \varepsilon$ and $\left|\sum_{i=1}^{n} g(c_i)(x_i - x_{i-1}) - I(g)\right| \le \varepsilon$

If $\mathcal{P} = \{x_0, x_1, \dots, x_n\}$ is a partition of $[a,b]$ and $\|\mathcal{P}\| < \delta$ and $c_i \in [x_{i-1}, x_i]$ for $i = 1, 2, \dots, n$. Then,

$\left|\sum_{i=1}^{n} (kf)(c_i)(x_i - x_{i-1}) - kI(f)\right| = |k| \left|\sum_{i=1}^{n} (f)(c_i)(x_i - x_{i-1}) - I(f)\right| \le |c|\varepsilon$

Hence kf is integrable on $[a,b]$ and ${}_a\int^b (kf)(x) = k\,{}_a\int^b f(x)$

Moreover, $\left|\sum_{i=1}^{n} (f+g)(c_i)(x_i - x_{i-1}) - [I(f) + I(g)]\right|$

$\le \left|\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) - I(f)\right| + \left|\sum_{i=1}^{n} g(c_i)(x_i - x_{i-1}) - I(g)\right|$

$\le 2\varepsilon$

Therefore $f+g$ is integrable on $[a,b]$ and ${}_a\int^b (f+g) = {}_a\int^b f + {}_a\int^b g$ $\square$

b) let $h = g - f$ be integrable on $[a,b]$

Since $h(x) \ge 0$ for all $x \in [a,b]$ then $L(h,\mathcal{P}) > 0$ for any partition $\mathcal{P}$ of $[a,b]$

Hence ${}_a\int^b h = L(h) \ge 0$

Then, we see ${}_a\int^b g - {}_a\int^b f = {}_a\int^b h \ge 0$

So, ${}_a\int^b g \ge {}_a\int^b f$

Exercise 3

Let $f:[a,b]\to\mathbb{R}$ be Riemann integrable on $[a,b]$ and suppose that $|f(x)|\le M$ for all $x\in[a,b]$. Show that ${}_a\int^b f \le M(b-a)$

Let $\mathcal{P}=\{c_i,[x_{i-1},x_i]\}$ be a tagged partition of $[a,b]$

$$|S(f,\mathcal{P})| = \left|\sum_{i=1}^{n} f(c_i)(x_i-x_{i-1})\right|$$
$$\le \sum_{i=1}^{n} |f(c_i)||(x_i-x_{i-1})|$$
$$\le M|(x_i-x_{i-1})|$$
$$\le M(b-a)$$

since f is Riemann integral

$$\left|{}_a\int^b f(x) - S(f,\mathcal{P})\right| < \varepsilon$$

$$\left|{}_a\int^b f(x)\right| \le \left|{}_a\int^b f(x) - S(f,\mathcal{P}) + S(f,\mathcal{P})\right|$$
$$\le \left|{}_a\int^b f(x) - S(f,\mathcal{P})\right| + |S(f,\mathcal{P}|$$
$$\le \varepsilon + m(b-a)$$

since $\varepsilon>0$ is arbitrary

$${}_a\int^b f(x) \le m(b-a)$$

Exercise 4

Suppose that f is Riemann Integrable on $[a,b]$.
Let $(\mathcal{P}_n)_{n=1}^{\infty}$ be a sequence of tagged partitions of $[a,b]$ such that the sequence $\lim_{n\to\infty}\|\mathcal{P}_n\|=0$. Prove that the sequence $(S(f,\mathcal{P}_n))_{n=1}^{\infty}$ converges to ${}_a\int^b f$

f is Riemann Integrable on $[a,b]$

$\Rightarrow {}_a\int^b f = L$

consider for each $n\in\mathbb{N}$ $x_n = S(f;\mathcal{P}_n)$

$\Rightarrow$ we have the sequence x_n

Now we show that $x_n\to L$ or $\lim_{n\to\infty} x_n = L$

For $\varepsilon>0$ $\exists N\in\mathbb{N}$ s.t. $|x_n-L|<\varepsilon$ $\forall n\ge N$

since f is R.I. given $\varepsilon>0$ $\exists\delta>0$ s.t. if $\mathcal{P}$ is a tagged partition of $[a,b]$ with $\|\mathcal{P}\|<\delta$ then

$|S(f,\mathcal{P})-L|<\varepsilon$

Given $\lim_{n\to\infty}\|\mathcal{P}_n\|=0 \Rightarrow \|\mathcal{P}_n\|\to 0$

there exist an N s.t. $\|\mathcal{P}_n\|<\delta$ $\forall n\ge N$

so,

$|S(f,\mathcal{P}_n)-L|<\varepsilon$

$|x_n-L|<\varepsilon$

$\lim_{n\to\infty} x_n = L$

$\lim_{n\to\infty} S(f,\mathcal{P}_n) = {}_a\int^b f$

$S(f,\mathcal{P}_n)\to {}_a\int^b f$

Exercise 5

Let $f:[a,b]\to\mathbb{R}$ be a bounded function. Suppose that f is Riemann integrable on $[a,c]$ for any $c\in(a,b)$. Show that f is Riemann integrable on $[a,b]$. [Hint: use the cauchy criterion for integrals]

let $P:=\{a=x_0<x_1<\dots<c<\dots<x_n=b\}$

$S(P,f)=\sum_{i=1}^{n} f(c_i)(x_i-x_{i-1})$

$|{}_a\int^b f - S(P,f)| = |{}_a\int^b f - \sum_{i=1}^{n} f(c_i)(x_i-x_{i-1})|$

$< \varepsilon/3$

$|{}_a\int^c f - S(P_2,f)| = |{}_a\int^c f - \sum_{i=1}^{n} f(c_i)(x_i-x_{i-1})|$

$< \varepsilon/3$

let $M=\sup|f(x)|$

$|{}_a\int^c f - {}_a\int^b f| \le |{}_a\int^b f - S(P,f)| + |S(P,f)-S(P_2,f)| + |{}_a\int^c f - S(P_2,f)|$

$\le \frac{2\varepsilon}{3} + |f(c_i)|(a-x_i)$

$\le \frac{2\varepsilon}{3} + M(a-x_1)$

$<\varepsilon$

$\lim_{c\to b} {}_a\int^c f = {}_a\int^b f$

$\therefore$ f is R.I on $[a,b]$

Exercise 6

a) define the function $f:[a,b]\to\mathbb{R}$ by $f(x)=k$ for every $x\in[a,b]$ where $k\in\mathbb{R}$ is a fixed constant. show that f is Riemann Integral on $[a,b]$ and that ${}_a\int^b k\,dx = k(b-a)$

b) Let $f(x)=\sin^2(x)$ where $x\in[a,b]$ and assume that the function $g(x):=\cos(kx)$ is integrable on $[a,b]$ for any $k\in\mathbb{R}$. show that f is Riemann Integrable on $[a,b]$

a) $f:[a,b]\to\mathbb{R}$ be a constant function $\ni f(x)=c$ $\forall x\in[a,b]$

let $\{a=x_0,x_1,\dots,x_{n-1},x_n=b\}$ be a partition of $[a,b]$

By the extreme value theorem $\exists u_j, s_j\in[x_{j-1},x_j]$ s.t.

$\sup\{f(x): x\in[x_{j-1},x_j]\}=f(s_j)$

$\inf\{f(x): x\in[x_{j-1},x_j]\}=f(u_j)$

Now define

$h=\sum_{j=1}^{N} f(s_j)$ $x\in[x_{j-1},x_j]$

$h(x_j-x_{j-1})=\sum_{i=1}^{N} k(x_j-x_{j-1})$

$=k(x_n-x_0)$

$=k(b-a)$

$g=\sum_{j=1}^{N} f(u_j)$ $x\in[x_{j-1},x_j]$

$g(x_j-x_{j-1})=\sum_{i=1}^{N} k(x_j-x_{j-1})$

$=k(x_n-x_0)$

$=k(b-a)$

so, ${}_a\int^b f = f(s_j)=f(u_j)=k(b-a)$

${}_a\int^b f = {}_a\int^b k = k(b-a)$

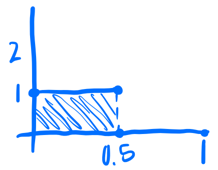
b) $f(x)=\sin^2 x$ is continuous on $[a,b]$ so f is riemann integrable

Exercise 7

Show that the function $f:[0,1]\to\mathbb{R}$ defined by

$$f(x) := \begin{cases} 1, & \text{if } 0 \le x < 1/2 \\ 0, & \text{if } 1/2 \le x \le 1 \end{cases}$$

is Riemann integrable on $[0,1]$



$\int_0^1 f = 1/2$

Let $\varepsilon > 0$

Suppose $\delta > 0$ and $\mathcal{P}$ is a tagged partition st $\|\mathcal{P}\| < \delta$

Let $\mathcal{P}_1 = \{c_i, [x_{i-1}, x_i] : c_i \in [0, 1/2)$

$\mathcal{P}_2 = \{c_i, [x_{i-1}, x_i] : c_i \in [1/2, 1]$

We have $S(f,\mathcal{P}) = S(f,\mathcal{P}_1) + S(f,\mathcal{P}_2)$

Now $N_1 = \text{card}(\mathcal{P}_1)$

$S(f,\mathcal{P}_1) = \sum_{i=1}^{N_1} f(c_i)(x_i - x_{i-1}) = 1 \sum_{i=1}^{N_1} (x_i - x_{i-1}) = x_{N_1}$

and $N_2 = \text{card}(\mathcal{P}_2)$

$S(f,\mathcal{P}_2) = 0(x_{N_1+N_2} - x_{N_1}) = 0$

so $S(f,\mathcal{P}) = x_{N_1} + 0 = x_{N_1}$

Since $\|\mathcal{P}\| < \delta$

then $x_{N_1} < \delta$

$S(f,\mathcal{P}) = x_{N_1} < \delta$

$S(f,\mathcal{P}) < \delta$

Choose $\delta = \varepsilon$

Then $|S(f,\mathcal{P}) - 1/2| < \varepsilon$

Exercise 8

Let $f:[0,1]\to\mathbb{R}$ be defined by $f(x)=1$ if $x=1/n$ where $n\in\mathbb{N}$ and by $f(x)=0$ if $x\ne 1/n$, $n\in\mathbb{N}$. Show that f is Riemann integrable on $[0,1]$

$$f(x) = \begin{Bmatrix} 1 & x = 1/n \\ 0 & x \ne 1/2 \end{Bmatrix} \text{ in } [0,1]$$

Exercise 9

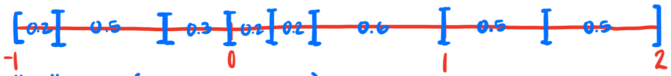
Show that the function $f:[0,1]\to\mathbb{R}$ defined by $f(x)=0$ if $x\neq 0$ and $f(x)=4$ if $x=0$ is Riemann Integrable on $[0,1]$

Exercise 10

Let P be the following tagged partition of $[-1,2]$

$$P:=\{(-.9,[-1,-.8]),(-0.7,[-0.8,-0.3]),(-0.1,[-0.3,0]),(0.2,[0,0.2]),$$
$$(0.2,[0.2,0.4]),(0.8,[0.4,1])(1.42,[1,1.5]),(1.9,[1.5,2])\}$$

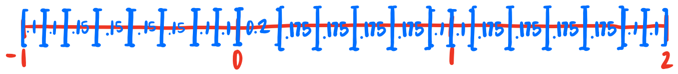
Find another partition P_0 s.t. $\|P_0\|\le\|P\|/3$



$\|P\|=\max\{x_i-x_{i-1}: i=1,2,\dots,N\}=0.6$

$\|P\|/3=0.6/3=0.2$

$$P_0:=\{(-.95,[-1,-.9]),(-.85[-.9,-.8])(-.75[-.8,-.65])(0.65,[-.65,-.5]),(-.45,[-.5,-.35])(-.25,[-.35,-.2]),(-.1,[-.2,-.1]),$$
$$(-.05,[-.1,0]),(0.1,[0,0.2]),(0.3,[0.2,0.375])(0.5,[0.375,.55])(0.6,[.55,.725])(0.8,[.725,.9])(.9,[.9,1]),$$
$$(1.05,[1,1.1]),(1.2,[1.1,1.275]),(1.3,[1.275,1.45]),(1.5,[1.45,1.625]),(1.7,[1.625,1.8]),(1.85,[1.8,1.9]),(1.95,[1.9,2])\}=0.2$$



$\|P_0\|=0.2$