MATH-331 Introduction to Real Analysis — Victor Ho
Homework 06 — Fall 2021

Due date: November, 22$^{\text{th}}$ 1:20pm — Total: /65.

Exercise	1 (10)	2 (10)	3 (5)	4 (5)	5 (5)	6 (10)	7 (5)	8 (5)	9 (5)	10 (5)
Score										

Table 1: Scores for each exercises

Instructions: You must answer all the questions below and send your solution by email (to `parisepo@hawaii.edu`). If you decide to not use LaTeX to hand out your solutions, please be sure that after you scan your copy, it is clear and readable. Make sure that you attached a copy of the homework assignment to your homework.
If you choose to use LaTeX, you can use the template available on the course website.
No late homework will be accepted. No format other than PDF will be accepted. Name your file as indicated in the syllabus.

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Writing problems

For each of the following problems, you will be asked to write a clear and detailed proof. You will have the chance to rewrite your solution in your semester project after receiving feedback from me.

Exercise 1. (10 pts)

a) Fix any $\delta > 0$ and let $[a, b]$ be an interval with $a < b$. Find a tagged partition $\mathcal{P}$ of $[a, b]$ such that $\|\mathcal{P}\| < \delta$.

b) Suppose that f is Riemann integrable. Show that in the definition of the Riemann integral, the number L is unique. [Remark: This is why we gave it the name $\int_a^b f$.]

Solution: a. Let $\delta > 0$ and $m \in \mathbb{R}$. Then by AP $\exists n \in \mathbb{N}$ such that $n\delta > m$, and therefore $\delta > \frac{m}{n}$. Therefore we can just make $\mathcal{P}$ of $[a, b]$ such that $||\mathcal{P}|| = \frac{m}{n}$, and we'll have $||\mathcal{P}|| < \delta$.
b. Let $\epsilon > 0$ be arbitrary. Therefore by the definition of the R.I, $\exists L_1 \in \mathbb{R}$ such that $\exists \delta_1 > 0$ such that if $\mathcal{P}$ is a T.P with $||\mathcal{P}|| < \delta_1$ then $|S(f, P) - L_1| < \epsilon/2$. Now assume towards a contradiction that the R.I is not unique, so $\exists L_2 \in \mathbb{R}$, $L_2 \neq L_1$ such that $\exists \delta_2 > 0$ such that if $\mathcal{P}$ is a T.P with $\mathcal{P} < \delta_2$ then $|S(f, P) - L_2| < \epsilon/2$. Now let $\delta = min\{\delta_1, \delta_2\}$, and choose $\mathcal{P}$ such that $||\mathcal{P}|| < \delta$. Therefore with this δ we have $|S(f, P) - L_1| < \epsilon/2$ and $|S(f, P) - L_2| < \epsilon/2$, so we can add them both together to get $|S(f, P) - L_1| + |S(f, P) - L_2| < \epsilon/2 + \epsilon/2 = \epsilon$. By the triangle inequality and properties of the absolute value function, $|S(f, P) - L_1| + |S(f, P) - L_2| = |S(f, P) - L_1| + |-S(f, P) + L_2| \geq |S(f, P) - L_1 - S(f, P) + L_2| = |L_2 - L_1|$. We can then plug

this into the previous equation to get $|L_2 - L_1| < \epsilon$. Since $\epsilon > 0$ is arbitrary, the only way for $|L_2 - L_1| < \epsilon$ to be true for all ϵ is if $|L_2 - L_1| = 0$, so $L_2 = L_1$. This contradicts our claim that the R.I is not unique, and therefore the R.I of f is unique. $\square$

Exercise 2. (10 pts) Suppose that f and g are Riemann integrable on the interval $[a, b]$.

a) Show that $\int_a^b (f + g) = \int_a^b f + \int_a^b g$.

b) Show that if $f(x) \leq g(x)$ for any $x \in [a, b]$, then $\int_a^b f \leq \int_a^b g$.

Solution: a. Since f is R.I on $[a, b]$, $\exists L_1 \in \mathbb{R}$ such that $\forall \epsilon > 0, \exists \delta_1 > 0$ such that if P is a T.P of $[a, b]$ with $||P|| < \delta_1$ then $|S(f, P) - L_1| < \epsilon/2$. Similarly, since g is R.I on $[a, b]$, $\exists L_2 \in \mathbb{R}$ such that $\forall \epsilon > 0, \exists \delta_2 > 0$ such that if P is a T.P of $[a, b]$ with $||P|| < \delta_2$ then $|S(g, P) - L_2| < \epsilon/2$. Now let $\delta = min\{\delta_1, \delta_2\}$, so choose P where $||P|| < \delta$. With this δ we have $|S(f, P) - L_1| < \epsilon/2$ and $|S(g, P) - L_2| < \epsilon/2$, so we can add them together to get $|S(f, P) - L_1| + |S(g, P) - L_2| < \epsilon/2 + \epsilon/2 = \epsilon$. By the triangle inequality, $|S(f, P) - L_1| + |S(g, P) - L_2| \geq |S(f, P) - L_1 + S(g, P) - L_2| = |S(f, P) + S(g, P) - (L_1 + L_2)|$. We can then plug this into the previous equation to get $|S(f, P) + S(g, P) - (L_1 + L_2)| < \epsilon$ where P is a T.P of $[a, b]$ with $||P|| < \delta$, $\epsilon > 0, \delta > 0$. This is the definition of the R.I, so $f + g$ is R.I, with $\int_a^b (f + g) = L_1 + L_2$. Since we know that the R.I is unique from 1a, $L_1 = \int_a^b f, L_2 = \int_a^b g$ so $L_1 + L_2 = \int_a^b f + \int_a^b g$, so $\int_a^b (f + g) = \int_a^b f + \int_a^b g$.
b. Since $f(x) \leq g(x)$ for any $x \in [a, b]$, we can say that $S(f, P) \leq S(g, P)$ for a T.P P of $[a, b]$. Since $S(f, P) \leq S(g, P)$ for any $x \in [a, b]$, we can say that $S(g, P) - S(f, P) \geq 0$ for any $x \in [a, b]$. From 2a and a theorem from class, we know that since f, g are R.I, $-f, g$ are R.I and so $g - f$ is R.I, and so by the definition of the R.I, $\exists L = (\int_a^b g - \int_a^b f)$ where $\forall \epsilon > 0$, $\exists \delta > 0$ such that if P is a T.P of $[a, b]$ with $||P|| < \delta$, $|S(g, P) - S(f, P) - (\int_a^b g - \int_a^b f)| < \epsilon$. Since $S(g, P) - S(f, P) \geq 0$, it must be the case that $(\int_a^b f + \int_a^b g) \geq 0$. This is because if $(\int_a^b f + \int_a^b g) < 0$, then $-(\int_a^b f + \int_a^b g) > 0$, so $|S(g, P) - S(f, P) - (\int_a^b g - \int_a^b f)| > 0$, so there will be a value of ϵ where $|S(g, P) - S(f, P) - (\int_a^b g - \int_a^b f)| \not< \epsilon$, which contradicts the fact that $g - f$ is R.I., and therefore it must be the case that $\int_a^b g - \int_a^b f \geq 0$. Since $(\int_a^b g - \int_a^b f) \geq 0$, we can add $\int_a^b f$ to both sides to get $\int_a^b g \geq \int_a^b f$. Therefore if $f(x) \leq g(x)$ for any $x \in [a, b]$ then we have $\int_a^b f \leq \int_a^b g$. $\square$

Exercise 3. (5 pts) Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable on $[a, b]$ and suppose that $|f(x)| \leq M$ for all $x \in [a, b]$. Show that $\int_a^b f \leq M(b - a)$.

Solution: Fix M as a constant, and let $g(x) = M$ for all $x \in [a, b]$. From the assumption we have $|f(x)| \leq M$ so $|f(x)| \leq g(x)$, or $f(x) \leq g(x)$ for all $x \in [a, b]$. From 2b, we know that if $f(x) \leq g(x)$ for all $x \in [a, b]$ then $\int_a^b f \leq \int_a^b g$. Also from 6a, since $g(x) = M$ where M is a fixed constant fir every $x \in [a, b]$, then $\int_a^b g = \int_a^b M = M(b - a)$. Therefore $\int_a^b f \leq \int_a^b g = M(b - a)$, which can be simplified to $\int_a^b f \leq M(b - a)$. $\square$

Exercise 4. (5 pts) Suppose that f is Riemann integrable on $[a, b]$. Let $(\mathcal{P}_n)_{n=1}^{\infty}$ be a sequence of tagged partitions of $[a, b]$ such that the sequence $\lim_{n\to\infty} \|\mathcal{P}_n\| = 0$. Prove that the sequence $(S(f, \mathcal{P}_n))_{n=1}^{\infty}$ converges to $\int_a^b f$.

Solution: Since $\lim_{n\to\infty} \|\mathcal{P}_n\| = 0$ for $(\mathcal{P}_n)_{n=1}^{\infty}$ of $[a,b]$, by the sequence definition of the limit, $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that for $n \geq N, n \in \mathbb{N}$ then $||\mathcal{P}_n - 0|| = ||\mathcal{P}_n|| < \epsilon$. Now let $\delta = \epsilon$, so we have $||\mathcal{P}_n|| < \delta$. Therefore since f is R.I on $[a,b]$, and $||\mathcal{P}_n|| < \delta$, then by the definition of the R.I, since $(\mathcal{P}_n)_{n=1}^{\infty}$ is a sequence of tagged partitions of $[a,b]$, $|(S(f,\mathcal{P}_n)) - \int_a^b f| < \epsilon$ for any n. Therefore we have $|(S(f,\mathcal{P}_n))_{n=1}^{\infty} - \int_a^b f| < \epsilon$. To summarize, we have if f is R.I on $[a,b]$, $(\mathcal{P}_n)_{n=1}^{\infty}$ is a sequence of tagged partitions of $[a,b]$ and $\lim_{n\to\infty} \|\mathcal{P}_n\| = 0$, then $\forall \epsilon > 0$, $\exists N$ such that if $n \geq N$ then $|(S(f,\mathcal{P}_n))_{n=1}^{\infty} - \int_a^b f| < \epsilon$. Since $\int_a^b f \in \mathbb{R}$ is required by the definition of the R.I, by the definition of convergence we can say that $(S(f,\mathcal{P}_n))_{n=1}^{\infty}$ converges to $\int_a^b f$. $\square$

Exercise 5. (5 pts) Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Suppose that f is Riemann integrable on $[a,c]$ for any $c \in (a,b)$. Show that f is Riemann integrable on $[a,b]$. [Hint: Use the Cauchy criterion for integrals.]

Solution: Let $\epsilon > 0$ be arbitrary. Now let $c \in (a,b)$ such that $b - c < \epsilon$. Now let P_1, P_2 be a t.p of $[a,b]$ with $||P_1|| < \delta$ and $||P_2|| < \delta$. Now let $P_{1a} = \{(c_i, [x_{i-1}, x_i] \in P_1 : [x_{i-1}, x_i] \subset [a,c]\}$. Let N_1 be equal to $card(P_{1a})$. Now let $\hat{P_{1a}} = P_{1a} \cup \{(c, [x_{N_1}, c])\}$. Now define $P_{1b} = \{(c_i, [x_{i-1}, x_i]) \in P_1 : [x_{i-1}, x_i] \subset [c,b]\}$, and let $\hat{P_{1b}} = P_{1b} \cup \{(c, [c, x_{N_1+1]})\}$. Now define $P_{2a}, P_{2b}, \hat{P_{2b}}, \hat{P_{2b}}$ in a similar way. So let $P_{2a} = \{(c_i, [x_{i-1}, x_i]) \in P_2 : [x_{i-1}, x_i] \subset [a,c]\}$. Let N_2 be equal to $card(P_{2a})$. Now let $\hat{P_{2a}} = P_{2a} \cup \{(c, [x_{N_2}, c])\}$. Now define $P_{2b} = \{(c_i, [x_{i-1}, x_i]) \in P_2 : [x_{i-1}, x_i] \subset [c,b]\}$, and let $\hat{P_{2b}} = P_{2b} \cup \{(c, [c, x_{N_2+1]})\}$. Note that $P_1 = \hat{P_{1a}} \cup \hat{P_{1b}}$ and $P_2 = \hat{P_{2a}} \cup \hat{P_{2b}}$. Since $\hat{P_{1a}} \subset P_1$ and $\hat{P_{2a}} \subset P_2$, $||\hat{P_{1a}}|| < \delta$, $\hat{P_{2a}} < \delta$ and f is integrable on $[a,c]$, and P_1, P_2 are t.p of $[a,c]$, by the Cauchy Criterion, we can say $|S(f,\hat{P_{1a}}) - S(f,\hat{P_{2a}})| < \epsilon$. Also since f is bounded on $[a,b]$, we can say that $\exists M \in \mathbb{R}$ such that $f(x) \leq M$ for $x \in [a,b]$. Therefore $S(f,\hat{P_{1b}}) = \sum_{i=1}^{card(\hat{P_{1b}})} f(c_i)(x_i - x_{i-1}) \leq \sum_{i=1}^{card(\hat{P_{1b}})} M(x_i - x_{i-1}) = M\sum_{i=1}^{card(\hat{P_{1b}})}(x_i - x_{i-1}) = M(b-c) < M\epsilon$, by the ϵ we defined earlier. Therefore $S(f,\hat{P_{1b}}) < M\epsilon$. Similarly, $S(f,\hat{P_{2b}}) = \sum_{i=1}^{card(\hat{P_{2b}})} f(c_i)(x_i - x_{i-1}) \leq \sum_{i=1}^{card(\hat{P_{2b}})} M(x_i - x_{i-1}) = M\sum_{i=1}^{card(\hat{P_{2b}})}(x_i - x_{i-1}) = M(b-c) < M\epsilon$, so $S(f,\hat{P_{2b}}) < M\epsilon$. Now consider $|S(f,P_1) - S(f,P_2)|$. Since $P_1 = \hat{P_{1a}} \cup \hat{P_{1b}}$ and $P_2 = \hat{P_{2a}} \cup \hat{P_{2b}}$, $S(f,P_1) = S(f,\hat{P_{1a}}) + S(f,\hat{P_{1b}})$ and $S(f,P_2) = S(f,\hat{P_{2a}}) + S(f,\hat{P_{2b}})$. Therefore $|S(f,P_1) - S(f,P_2)| = |S(f,\hat{P_{1a}}) + S(f,\hat{P_{1b}}) - S(f,\hat{P_{2a}}) - S(f,\hat{P_{2b}})|$, which by the triangle inequality, $|S(f,\hat{P_{1a}}) + S(f,\hat{P_{1b}}) - S(f,\hat{P_{2a}}) - S(f,\hat{P_{2b}})| \leq |S(f,\hat{P_{1a}}) - S(f,\hat{P_{2a}})| + |S(f,\hat{P_{1b}}) - S(f,\hat{P_{2b}})|$. From earlier, we know that $|S(f,\hat{P_{1a}}) - S(f,\hat{P_{2a}})| < \epsilon$ and $S(f,\hat{P_{1b}}) < M(b-c)$ and $S(f,\hat{P_{2b}}) < M(b-c)$. Therefore $|S(f,\hat{P_{1a}}) - S(f,\hat{P_{2a}})| + |S(f,\hat{P_{1b}}) - S(f,\hat{P_{2b}})| < \epsilon + M\epsilon - M\epsilon = \epsilon$. Therefore by order axioms, we can say that $|S(f,P_1) - S(f,P_2)| < \epsilon$. Since $\epsilon > 0$ was arbitrary, we have $\forall \epsilon > 0$, with P_1, P_2 being a t.p of $[a,b]$ where $||P_1|| < \delta$ and $||P_2|| < \delta$ for some $\delta > 0$, then $|S(f,P_1) - S(f,P_2)| < \epsilon$. Therefore by the Cauchy Criterion, f is integrable on $[a,b]$. $\square$

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Homework problems

Answer all the questions below. Make sure to show your work.

Exercise 6. (10pts)

a) Define the function $f : [a,b] \to \mathbb{R}$ by $f(x) = k$ for every $x \in [a,b]$ where $k \in \mathbb{R}$ is a fixed constant. Show that f is Riemann integrable on $[a,b]$ and that $\int_a^b k\,dx = k(b-a)$.

b) Let $f(x) = \sin^2(x)$ where $x \in [a, b]$ and assume that the function $g(x) := \cos(kx)$ is integrable on $[a, b]$ for any $k \in \mathbb{R}$. Show that f is Riemann integrable on $[a, b]$.

Solution: a. Let $\epsilon > 0$ and P be a a T.P of $[a, b]$ with $||P|| < \delta$ for $\delta > 0$. By the definition of a T.P, $P = \{(c_i, x_i - x_{i-1}]) : i = 1...N\}$ and so $S(f, P) = \sum_{i=1}^{N} f(c_i)(x_i - x_{i-1})$. Since $f(x) = k$ for every $x \in [a, b]$, and P is a T.P of $[a, b]$, the summation can be simplified to $k \sum_{i=1}^{N}(x_i - x_{i-1}) = k(x_N - x_1)$. Since $\cup_1^N [x_{i-1}, x_i] = [a, b]$, we know that $x_N = b$ and $x_1 = a$, so $S(f, P) = k(b - a)$. Therefore $|S(f, P) - k(b - a)| = 0$, so $\int_a^b k\,dx = k(b - a)$ will satisfy the expression $|S(f, P) - \int_a^b k\,dx| < \epsilon$, and from 1b, we know that this solution is unique. To summarize, we have for $\epsilon > 0$, if P is a T.P of $[a, b]$ such that $||P|| < \delta$ for $\delta > 0$, $|S(f, P) - k(b - a)| < \epsilon$, and therefore by the definition of the R.I we can say that f is R.I on $[a, b]$ and $\int_a^b k\,dx = k(b - a)$.
b. By trig identities, $\sin^2(x) = \frac{1-\cos(2x)}{2}$. Therefore $\int_a^b \sin^2(x)dx = \int_a^b \frac{1-\cos(2x)}{2}dx$. From the sum rule for integrals, we know that this is equal to $\int_a^b 1/2dx + \int_a^b -\cos(2x)/2dx$. From 6a, we know that $\int_a^b 1/2dx$ is R.I and equal to $1/2(b - a)$. Also since $g(x) = cos(kx)$, for $k = 2, g(x) = cos(2x)$ is R.I on $[a, b]$ by the assumption. By a theorem in class, since $-1/2 \in \mathbb{R}$, $\int_a^b -\cos(2x)/2$ is R.I on $[a, b]$ and equal to $\frac{-1}{2} \int_a^b \cos(2x)$. Therefore we have $\int_a^b 1/2dx$ and $\int_a^b -\cos(2x)/2$ is R.I on $[a, b]$, so by the sum rule for integrals, $\int_a^b 1/2dx + \int_a^b -\cos(2x)/2 = \int_a^b \frac{1-\cos(2x)}{2}dx = \int_a^b \sin^2(x)dx$ is R.I on $[a, b]$. Finding the exact value of the integral isn't required, but we can find it by using the fundamental theorem of calculus with $h(x) = sin(2x)/4$ since $h'(x) = cos(2x)/2$, which will allow us to use the theorem to solve $\int_a^b -\cos(2x)/2$.

Exercise 7. (5 pts) Show that the function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} 1 & \text{, if } 0 \le x < 1/2 \\ 0 & \text{, if } 1/2 \le x \le 1 \end{cases}$$

is Riemann integrable on $[0, 1]$.

Solution: Let $\epsilon > 0$ and P be a T.P of $[0, 1]$ such that $||P|| < \delta$, $P = \{(c_i, [x_{i-1}, x_i] : c_i \in [0, 1]\}$. Let P_1, P_2 be the following collection of tagged intervals, $P_1 = \{(c_i, [x_{i-1}, x_i] : c_i \in [0, 1/2)\}$, $P_2 = \{(c_i, [x_{i-1}, x_i] : c_i \in [1/2, 1]\}$. Note that $P = P_1 \cup P_2$ and $P_1 \cap P_2 = \emptyset$, so $S(f, P) = S(f, P_1) + S(f, P_2)$. Now if $N_1 = card(P_1), N_2 = card(P_2), S(f, P_1) = \sum_{i=1}^{N_1} f(c_i)(x_i - x_{i-1})$. Since in $P_1, c_i \in [0, 1/2), f(c_i) = 1, soS(f, P_1) = \sum_{i=1}^{N_1}(x_i - x_{i-1}) = x_{N_1} - x_1$. Also $S(f, P_2) = \sum_{i=N_1+1}^{N_2} f(c_i)(x_i - x_{i-1})$, and since for $P_2, c_i \in [1/2, 1], f(c_i) = 0$, so $s(f, P_2) = 0(\sum_{i=N_1+1}^{N_2}(x_i - x_{i-1})) = 0$. Therefore $S(f, P) = S(f, P_1) + S(f, P_2) = x_{N_1} - x_1 = x_{N_1} - x_1 + 1/2 - 1/2$. So $|S(f, P_2) - 1/2| \le |x_{N_1} - x_1 - 1/2 + 0|$ which by the triangle inequality, $|x_{N_1} - x_1 - 1/2 + 0| \le |1/2 - x_{N_1}| + |0 - x_1|$. Since $||P|| < \delta, |1/2 - x_{N_1}| < \delta, |0 - x_1|. < \delta$, so $|1/2 - x_{N_1}| + |0 - x_1| < 2\delta$. Now put $\delta = \epsilon/2$. Therefore $|S(f, P) - 1/2| \le |1/2 - x_{N_1}| + |0 - x_1| < 2\delta = \epsilon$ so by transitivity, $|S(f, P) - 1/2| < \epsilon$. Therefore by the definition of the R.I f is R.I, and $\int_0^1 f = 1/2$. □

Exercise 8. (5 pts) Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = 1$ if $x = 1/n$ where $n \in \mathbb{N}$, and by $f(x) = 0$ if $x \neq 1/n$, $n \in \mathbb{N}$. Show that f is Riemann integrable on $[0, 1]$.

Solution: Let $\epsilon > 0$, P be a t.p of $[0,1]$ such that $||P|| < \delta, \delta > 0$. Now let $P_1 \subset P$ be a t.p of [0, 1] defined by $P_1 = \{(c_i, [x_{i-1}, x_i] : c_i = x = 1/n, n \in \mathbb{N}\}$. And let $P_2 = P \setminus P_1 = \{(c_i, [x_{i-1}, x_i]) : c_i = x \neq 1/n, n \in \mathbb{N}\}$. Therefore $P = P_1 \cup P_2$, so $S(f, P) = S(f, P_1) + S(f, P_2)$. Now note since $f(x) = 1$ if $x = 1/n, n \in \mathbb{N}$, for $N = card(P_1), S(f, P_1) = \sum_{i=1}^{N} c_i(x_i - x_{i-1}) = \sum_{i=1}^{N}(x_i - x_{i-1})$, and since $P_1 \subset P$ and $||P|| < \delta, ||P_1|| < \delta$, so $(x_i - x_{i-1}) < \delta$, and so $\sum_{i=1}^{N}(x_i - x_{i-1}) < \sum_{i=1}^{N} \delta = N\delta$. Therefore $S(f, P_1) < N\delta$. Also note that since $f(x) = 0$ if $x \neq 1/n, n \in \mathbb{N}$, $S(f, P_2) = \sum_{i=1}^{card(P_2)} c_i(x_i - x_{i-1}) = \sum_{i=1}^{card(P_2)} 0(x_i - x_{i-1}) = 0$. Now select $\delta = \epsilon/2N$, so if $||P|| < \delta$, then $|S(f, P)| = |S(f, P_1) + S(f, P_2)| < N\delta + 0 = \epsilon/2 \leq \epsilon$. Therefore with this δ, we have $\forall \epsilon > 0$, if P is a t.p of $[0,1]$ with $||P|| < \delta$, $|S(f, P) - 0| < \epsilon$, so by the definition of the R.I, f is R.I. and $\int_0^1 f = 0$. $\square$

Exercise 9. (5 pts) Show that the function $f : [0,1] \to \mathbb{R}$ defined by $f(x) = 0$ if $x \neq 0$ and $f(x) = 4$ if $x = 0$ is Riemann integrable on $[0,1]$.

Solution: Let $\epsilon > 0$, and P be a t.p of $[0,1]$ such that $||P|| < \delta$. Let $P_1 \subset P$ be a t.p of $[0,1]$ defined by $P_1 = \{(c_i, [x_{i-1}, x_i] : c_i = 0\}$ and let $P_2 = P \setminus P_1 = \{(c_i, [x_{i-1}, x_i]) : c_i \neq 0\}$. Therefore $P = P_1 \cup P_2$, so $S(f, P) = S(f, P_1) + S(f, P_2)$. Let $N = card(P_1)$. Note that since $f(x) = 4$ for $x = 0, \sum_{i=1}^{N} f(c_i)(x_i - x_{i-1}) = \sum_{i=1}^{N} 4(x_i - x_{i-1}) = 4\sum_{i=1}^{N}(x_i - x_{i-1})$. Since $P_1 \subset P$ and $||P|| < \delta, ||P_1|| < \delta$, so $4\sum_{i=1}^{N}(x_i - x_{i-1}) < 4\sum_{i=1}^{N} \delta = 4N\delta$. Now note that since $f(x) = 0$ for $x \neq 0$, $S(f, P_2) = \sum_{i=1}^{card(P_2)} c_i(x_i - x_{i-1}) = \sum_{i=1}^{card(P_2)} 0(x_i - x_{i-1}) = 0$. Now select $\delta = \epsilon/8N$, so if $||P|| < \delta$, then $|S(f, P)| = |S(f, P_1) + S(f, P_2)| < 4N\delta + 0 = \epsilon/2 \leq \epsilon$. Therefore with this δ, we have $\forall \epsilon > 0$, if P is a t.p of $[0,1]$ with $||P|| < \delta$, $|S(f, P) - 0| < \epsilon$, so by the definition of the R.I, f is R.I. and $\int_0^1 f = 0$. $\square$

Exercise 10. (5 pts) Let $\mathcal{P}$ be the following tagged partition of $[-1, 2]$:

$$\mathcal{P} := \{(-9, [-1, -.8]), (-.7, [-.8, -.3]), (-.1, [-.3, 0]), (.2, [0, 0.2]), (.2, [.2, .4]), (.8, [.4, 1]), (1.42, [1, 1.5]), (1.9, [1.5, 2])\}.$$

Find another partition $\mathcal{P}_0$ such that $\|\mathcal{P}_0\| \leq \|\mathcal{P}\|/3$.

Solution: Since $||\mathcal{P}|| = max\{x_i - x_{i-1} : i...N\} = 1 - .4 = .6, ||\mathcal{P}||/3 = 0.2$. So we just need $\mathcal{P}_0$ to be made up of intervals of length 0.2. Let

$$\mathcal{P}_0 := \{(1, [-1, -.8]), (2, [-.8, -.6]), (3, [-.6, -.4]), (4, [-.4, -.2]), (5, [-.2, 0]), (6, [0, .2]), (7, [.2, .4]), (8, [.4, .6]), (10, [.6, .8]), (11, [.8, 1]), (12, [1, 1.2]), (13, [1.2, 1.4]), (14, [1.4, 1.6]), (15, [1.6, 1.8]), (16, [1.8, 2])\}.$$

With this $\mathcal{P}_0, ||\mathcal{P}_0|| = 0.2 \leq 0.2 = ||\mathcal{P}||/3$.