MATH-331 Introduction to Real Analysis — Jeselle Guillermo

Homework 02 — Fall 2021

Due date: 20-09-2021 1:20pm — Total: /70.

Exercise	1 (10)	2 (5)	3 (5)	4 (5)	5 (5)	6 (10)	7 (5)	8 (10)	9 (5)	10 (10)
Score										

Table 1: Scores for each exercises

Instructions: You must answer all the questions below and send your solution by email (to `parisepo@hawaii.edu`). If you decide to not use LaTeX to hand out your solutions, please be sure that after you scan your copy, it is clear and readable. Make sure that you attached a copy of the homework assignment to your homework. No late homework will be accepted. No format other than PDF will be accepted. Name your file as indicated in the syllabus.

1 Writing problems

For each of the following problems, you will be asked to write a clear and detailed proof. You will have the chance to rewrite your solution in your semester project after receiving feedback from me.

Exercise 1. (10 pts)

a) Let $\{[a_n, b_n] : n \geq 1\}$ be a family of closed intervals such that $[a_1, b_1] \supset [a_2, b_2] \supset [a_3, b_3] \supset \cdots$. Show that there is a $c \in \mathbb{R}$ such that $c \in [a_n, b_n]$ for all $n \geq \mathbb{N}$. Follow the following steps to prove it:

(i) Prove that for any $n, m \geq 1$, $a_n \leq b_m$. [hint: put $M := \max\{n, m\}$.]

(ii) Show that $\sup\{a_n : n \geq 1\}$ exists.

(iii) Show that $c = \sup\{a_n : n \geq 1\}$ satisfies the requirement.

b) Use this last result to prove that the set $\mathbb{R}$ is uncountable. [Hint: Show that any function $f : \mathbb{N} \to \mathbb{R}$ can't be surjective. To do so, construct a sequence of closed intervals such that $f(n) \notin [a_n, b_n]$ with $a_n < b_n$.]

Solution:

Exercise 2. (5 pts) Prove that if $a_n \to A$, then $|a_n| \to |A|$.

Solution: If $(a_n)_{n=1}^{\infty}$ is convergent, then $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $|a_n - A| < \varepsilon$ for all $n \geq N$.
Now, $||a_n| - |A|| \leq |a_n - A|$ (Since $||x| - |y|| \leq |x - y|$).
So, $||a_n| - |A|| < \varepsilon$. This is true for every $\varepsilon > 0$.
Then, by the definition of convergent sequences, $|a_n| \to |A|$. $\square$

Exercise 3. (5 pts) Let (a_n), (b_n), and (c_n) be sequences of real numbers. Prove that if $a_n \to L$, $b_n \to L$, and $a_n \leq c_n \leq b_n$, then $c_n \to L$.

Solution: Since $a_n \to L$, then $|a_n - L| < \varepsilon$ for all $n \geq N_1$.
This means $L - \varepsilon < a_n < L + \varepsilon$.
Similarly for $b_n \to L$, $|b_n - L| < \varepsilon$ for all $n \geq N_2$.
This means $L - \varepsilon < b_n < L + \varepsilon$.
Let $N = max\{N_1, N_2\}$
Given, $a_n \leq c_n \leq b_n$
Then, $L - \varepsilon < a_n < c_n < b_n < L + \varepsilon$ for all $n \geq N$.
$L - \varepsilon < c_n < L + \varepsilon$
$|c_n - L| < \varepsilon$ for all $n \geq N$. Therefore, the $c_n \to L$. $\square$

Exercise 4. (5 pts) Prove that if $a_n \to A$ and $a_n \geq 0$ for all $n \geq 1$, then $\sqrt{a_n} \to \sqrt{A}$. Follow the following steps to prove it:

1. Consider the case $A = 0$.
2. Suppose that $A \neq 0$. Show that there is a $N_1 \in \mathbb{N}$ such that if $n \geq N_1$, then $\sqrt{a_n} \geq \sqrt{|A|/2}$. [Hint: use the definition of convergence of $(a_n)_{n\geq 0}$ with a clever choice of ε and use the properties of the absolute value.]
3. Use the convergence of (a_n) again to find a N_2 such that $|a_n - A| < \frac{3}{4}\frac{\varepsilon}{\sqrt{|A|}}$.
4. Express $\sqrt{a_n} - A$ as $\frac{a_n - A}{\sqrt{a_n}+\sqrt{A}}$ and put $N = \max\{N_1, N_2\}$. Conclude.

Solution: If $\{a_n \to A$, then given $\varepsilon > 0$ for some $N \in \mathbb{N}$ s.t. $n \geq N$, we have $|a_n - A| < \varepsilon$.
1) If $A = 0$, then for some $\varepsilon > 0$ there exist an $N \in \mathbb{N}$ s.t. $|a_n - 0| < \varepsilon^2$.
Then $a_n < \varepsilon^2$
$\Rightarrow \sqrt{a_n} < \sqrt{\varepsilon^2}$
$\Rightarrow \sqrt{a_n} < \varepsilon$
$|\sqrt{a_n} - 0| < \varepsilon$
Therefore, $\sqrt{a_n}$ converges to 0

Exercise 5. (5 pts) For each sequence $(a_n)_{n=1}^{\infty}$, define the sequence $(\sigma_n)_{n=1}^{\infty}$ by

$$\sigma_n := \frac{a_1 + a_2 + \cdots + a_n}{n} \quad (n \geq 1).$$

Prove that if $a_n \to A$, then $\sigma_n \to A$. Find an example of a divergent sequence (a_n) such that $(\sigma_n)_{n=1}^{\infty}$ converges.

Solution: Proof:

Example:
Let $(a_n)_{n=1}^{\infty} = (-1)^n$
(a_n) is divergent, but
$\sigma_n = \frac{-1+1-1+1....(-1)^n}{n} = 0$ or $\frac{-1}{n}$
Thus, σ_n converges to 0. $\square$

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Homework problems

Exercise 6. (10 pts) Use the definition of convergence to prove that each of the following sequences converges.

a) $(a_n)_{n=1}^{\infty}$ given by $a_n = 5 + 1/n$ for $n \geq 1$.

b) $(a_n)_{n=1}^{\infty}$ given by $a_n = \frac{3n}{2n+1}$ for $n \geq 1$.

Solution: **a)** If $(a_n)_{n=1}^{\infty}$ is convergent, then $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $|a_n - A| < \varepsilon$ for all $n \geq N$

Let $(a_n) = 5 + \frac{1}{n}$. The $\lim_{n\to\infty} 5 + \frac{1}{n} = 5$. Our goal is to show that a_n converges to 5.

$$\begin{aligned}
|(5 + \frac{1}{n}) - 5| &< \varepsilon \\
\frac{1}{n} &< \varepsilon \\
1 &< n\varepsilon \\
\frac{1}{\varepsilon} &< n \\
\frac{1}{\varepsilon} &< N < n (\text{Since } n \geq N) \\
\frac{1}{N} &< \varepsilon
\end{aligned}$$

By AP for any $\varepsilon > 0$ there is an $N \in \mathbb{N}$ s.t.$\frac{1}{N} < \varepsilon$

$$
\begin{aligned}
|a_n - 5| &= |(5 + \frac{1}{n}) - 5| \\
&= \frac{1}{n} \\
&\leq \frac{1}{N} (\text{Since } n \geq N \Rightarrow \frac{1}{n} \leq \frac{1}{N}) \\
&< \varepsilon
\end{aligned}
$$

Therefore, for every $\varepsilon > 0$, there exist an N s.t. $|a_n - 5| < \varepsilon$ for all $n \geq N$ and the sequence converges to 5.

b) $(a_n)_{n=1}^{\infty}$ given by $a_n = \frac{3n}{2n+1}$ for $n \geq 1$.
Let $(a_n) = \frac{3n}{2n+1}$. The $\lim\limits_{n\to\infty} \frac{3n}{2n+1} = \frac{3}{2}$. Our goal is to show that a_n converges to $\frac{3}{2}$.
By AP for any $\varepsilon > 0$ there is an $N \in \mathbb{N}$ s.t. $\frac{1}{N} < \varepsilon$

$$
\begin{aligned}
|a_n - \frac{3}{2}| &= |\frac{3n}{2n+1} - \frac{3}{2}| \\
&= |\frac{6n - 6n - 3}{2(2n+1)}| \\
&= |\frac{-3}{4n+2}| \\
&= \frac{3}{4n+2} \\
&\leq \frac{3}{4n} (\text{Since } \frac{1}{4n+2} \leq \frac{1}{4n}) \\
&= \frac{3}{4N} (\text{Since } n \geq N \Rightarrow \frac{1}{n} \leq \frac{1}{N}) \\
&< \frac{3\varepsilon}{4} \\
&< \varepsilon
\end{aligned}
$$

Therefore, for every $\varepsilon > 0$, there exist an N s.t. $|a_n - \frac{3}{2}| < \varepsilon$ for all $n \geq N$ and the sequence converges to $\frac{3}{2}$.

Exercise 7. (5 pts) Prove that the sequence $(a_n)_{n=1}^{\infty} = \left(\frac{2n+1}{n}\right)_{n=1}^{\infty}$ is a Cauchy sequence.

Solution: a_n is a cauchy sequence if $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t $m, n \geq N \Rightarrow |a_m - a_n| < \varepsilon$
By AP, there exist an $N \in \mathbb{N}$ s.t. $\frac{1}{N} < \frac{\varepsilon}{2}$
Take $m, n \geq N$

$$\begin{aligned}|a_m - a_n| &= |\frac{2n+1}{n} - \frac{2m+1}{m}| \\ &= |2 + \frac{1}{n} - 2 - \frac{1}{m}| \\ &= |\frac{1}{n} - \frac{1}{m} \\ &\leq \frac{1}{n} + \frac{1}{m}\end{aligned}$$

Since $n, m \geq N$, then $\frac{1}{n}, \frac{1}{m} \leq \frac{1}{N}$

$$\begin{aligned}\frac{1}{n}, \frac{1}{m} &\leq \frac{1}{N} \\ &< \frac{\varepsilon}{2}\end{aligned}$$

Therefore,

$$\begin{aligned}|a_m - a_n| &\leq \frac{1}{n} + \frac{1}{m} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon\end{aligned}$$

Hence, $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t $|a_m - a_n| < \varepsilon$ for all $m, n \geq N$.
Thus, $(a_n)_{n=1}^{\infty} = \left(\frac{2n+1}{n}\right)_{n=1}^{\infty}$ is a Cauchy sequence.

Exercise 8. (10 pts) Prove that each of the following sequence diverges.

a) $(a_n)_{n=1}^{\infty} = ((-1)^n)_{n=1}^{\infty}$.

b) $(a_n)_{n=1}^{\infty} = (\sin(\frac{2n+1}{2}\pi))_{n=1}^{\infty}$.

Solution: **a)** $(a_n)_{n=1}^{\infty} = ((-1)^n)_{n=1}^{\infty}$.
When n is even $a_n = 1$.
When n is odd $a_n = -1$.
Thus, $(-1)^n$ is divergent.

b) $(a_n)_{n=1}^{\infty} = (\sin(\frac{2n+1}{2}\pi))_{n=1}^{\infty}$.
$\sin(\frac{2n+1}{2}\pi) = \sin(n\pi + \frac{\pi}{2})$
$= \sin(n\pi)\cos(\frac{\pi}{2}) + \sin(\frac{\pi}{2})\cos(n\pi)$
$= 0 + cos(n\pi)$
$= 1, -1, 1, -1....$ for n =0, 1, 2 ...
Thus, $\sin(\frac{2n+1}{2}\pi)$ is divergent. □

Exercise 9. (5 pts) Give an example of two sequences (a_n) and (b_n) such that (a_n) and (b_n) don't converge, but $(a_n + b_n)$ converge.

Solution: Let $(a_n) = n^2$ and $(b_n) = -n^2$, these two sequences do not converge.
$(a_n + b_n) = n^2 - n^2$=0, this sequence converges. □

Exercise 10. (10 pts) With the limit operations and the writing problems, find the limit of the following sequence with general term

a) $\frac{n^2+4n}{n^2-5}$.

b) $\frac{n}{n^2-3}$.

c) $\frac{\cos n}{n}$. [You can use what you know on the cosine function.]

d) $\left(\sqrt{4 - \frac{1}{n}} - 2\right)n$.

Solution: **a)** $\frac{n^2+4n}{n^2-5}$
$\frac{n^2+4n}{n^2-5} = \frac{1+\frac{4}{n}}{1-\frac{5}{n^2}}$
$1 + \frac{4}{n}$ converges to $1 + 0 = 1$
$1 - \frac{5}{n^2}$ converges to $1 - 0 = 1$
Thus, $\frac{n^2+4n}{n^2-5}$ converges to $\frac{1}{1} = 1$

b) $\frac{n}{n^2-3}$
$\frac{n}{n^2-3} = \frac{\frac{1}{n}}{1-\frac{3}{n^2}}$
$\frac{1}{n}$ converges to 0.
$1 - \frac{3}{n^2}$ converges to 1-0=1 Thus, $\frac{n}{n^2-3}$ converges to $\frac{0}{1} = 0$

c) $\frac{\cos n}{n}$. [You can use what you know on the cosine function.]
$\frac{\cos n}{n} = \frac{1}{n} \cdot \cos n$
Consider the theorem if $an \to 0$ and b_n is bounded then $a_n b_n \to 0$
Let $a_n = \frac{1}{n}$, this converges to 0
and $b_n = \cos n$, which is bounded.
Thus, $\frac{\cos n}{n}$ converges to 0

d) $\left(\sqrt{4-\frac{1}{n}}-2\right)n$.

$\left(\sqrt{4-\frac{1}{n}}-2\right)n = \frac{\left(\sqrt{4-\frac{1}{n}}-2\right)\left(\sqrt{4-\frac{1}{n}}+2\right)n}{\left(\sqrt{4-\frac{1}{n}}+2\right)} = \frac{-1}{\left(\sqrt{4-\frac{1}{n}}+2\right)}$

$\left(\sqrt{4-\frac{1}{n}}+2\right)$ converges to 4.

So, $\left(\sqrt{4-\frac{1}{n}}-2\right)n$ converges to $\frac{-1}{4}$.