Math 331 HW 2

9.) $a_n = (-1)^n$ - does not converge $b_n = (-1)^{n+1}$ - does not converge

$= \{-1, 1, -1, 1 \ldots\}$ $\{1, -1, 1, -1 \ldots\}$

$(a_n + b_n) = (-1)^n + (-1)^{n+1} := \{(-1+1), (1+-1), (-1+1) \ldots\}$

$a_n + b_n$ converges to zero $= \{0, 0, 0 \ldots\}$

10.) a.) $\frac{n^2+4n}{n^2-5} = \frac{n^2(1+\frac{4}{n})}{n^2(1-\frac{5}{n^2})} \begin{matrix} \to 1+0 \\ \to 1-0 \end{matrix}$ $\lim_{n\to\infty} \frac{n^2+4n}{n^2-5} = 1$

b.) $\frac{n}{n^2-3} = \frac{n(1)}{n(n-\frac{3}{n})} \begin{matrix} \to 1 \\ \to \infty \end{matrix} \Rightarrow \lim_{n\to\infty} \frac{n}{n^2-3} = 0$

c.) $\frac{\cos n}{n}$ since $\cos x \leq 1$ for $x \in \mathbb{R}$

$0 \leq \left|\frac{\cos n}{n}\right| \leq \frac{1}{n}$ for each $n \in \mathbb{N}$

because $\frac{\cos n}{n}$ is between 0 and $\frac{1}{n}$ which both converge to 0, $\frac{\cos n}{n}$ also converges to 0.

therefore, $\lim_{n\to\infty} \frac{\cos n}{n} = 0$

d.) $(\sqrt{4-\frac{1}{n}} - 2)n \cdot \frac{-\frac{1}{n}}{\sqrt{4-\frac{1}{n}}+2}$ * multiply by conjugate

$= \frac{-\frac{1}{n}}{\sqrt{4-\frac{1}{n}}+2} n = \frac{1}{\sqrt{\frac{4n-1}{n}}+2} \begin{matrix} \to \\ \to \end{matrix} -\frac{1}{4} \Rightarrow \lim_{n\to\infty} (\sqrt{4-\frac{1}{n}} - 2)n = \frac{1}{4}$

7.) Prove that: $(a_n)_{n=1}^{\infty} = \left(\frac{2n+1}{n}\right)_{n=1}^{\infty}$ is Cauchy

$\lim_{n\to\infty} \frac{2n+1}{n} \longrightarrow \frac{n(2+\frac{1}{n})}{n(1)} = 2 + \frac{1}{n}$ therefore $a_n \to 2 + \overset{\nearrow 0}{\frac{1}{n}} \Rightarrow a_n \to 2$

Using the theorem from class, we know that if a sequence $a_n \to A$, then (a_n) is Cauchy. $a_n \to 2 + \frac{1}{n}$, and we know that $\frac{1}{n}$ converges according to in class notes. Therefore $a_n \to 2 + 0 = 2$. Thus, a_n is a Cauchy sequence.

$\Longrightarrow$

1, 2, 3, 4, 5, 6, 8

8a. Prove $(a_n)_{n=1}^{\infty} = ((-1)^n)_{n=1}^{\infty}$ diverges

Assume towards contradiction that a_n has some limit L to which it converges. Then given $\varepsilon > 0$ we can find a positive integer N s.t. $|(-1)^n - L| < \varepsilon \quad \forall n > N$.

By the definition of a convergent sequence, $-\varepsilon < |a_n - L| < \varepsilon$.

So, for the n is even case, we get: $|(-1)^{2n} - L| < \varepsilon$

$|1 - L| < \varepsilon$

Setting $\varepsilon = 1$ we get: $-1 < 1 - L < 1$

$-2 < -L < 0$

for the n is odd case, we get: $|(-1)^{n+1} - L| < \varepsilon$

$|-1 - L| < \varepsilon$

$-1 < -1 - L < 1$

$0 < -L < 2$

So, $-L \in (-2,0)$ and $-L \in (0,2)$.

Therefore $-L \in (-2,0) \cap (0,2) = \emptyset$ $\times\times$

Since $-L$ is in the empty set, this is a contradiction.

Thus, $((-1)^n)_{n=1}^{\infty}$ diverges.

8b. Prove $(a_n)_{n=1}^{\infty} = \left(\sin\left(\frac{2n+1}{2}\pi\right)\right)_{n=1}^{\infty}$ diverges

Using the same reasoning as above, we can assume that a_n has some limit L to which it converges s.t. $-\varepsilon < |a_n - L| < \varepsilon$. Since the period of $a_n = 2$, the graph has a max/min at every $n \in \mathbb{N}$. ∴ So, we can split into two cases.

for the maximum, set $n = 2n$: $\left|\sin\left(\frac{2(2n+1)}{2}\pi\right) - L\right| < \varepsilon$

setting $\varepsilon = 1$ we get: $|1 - L| < \varepsilon$

$-1 < 1 - L < 1 \Leftrightarrow -2 < -L < 0$

for the minimum, set $n = n+1$: $\left|\sin\left(\frac{2((n+1)+1)}{2}\pi\right) - L\right| < \varepsilon$

$|-1 - L| < \varepsilon$

$-1 < -1 - L < 1 \Leftrightarrow 0 < -L < 2$

So, $-L \in (-2,0) \cap (0,2)$ $\times\times$ Thus $\left(\sin\left(\frac{2n+1}{2}\pi\right)\right)_{n=1}^{\infty}$ diverges

→

#, 3, 4, 5, 1

6a.) Prove $(a_n)_{n=1}^{\infty}$ given by $a_n = 5 + \frac{1}{n}$ for $n \geq 1$ converges

$A = 5$

for any $m, n \in \mathbb{N}$ with $n > m$

$$|a_m - a_n| = \left|\left(5 + \frac{1}{m}\right) - \left(5 + \frac{1}{n}\right)\right|$$

$$|a_m - a_n| = \left|5 + \frac{1}{m} - 5 - \frac{1}{n}\right|$$

$$|a_m - a_n| = \left|\frac{1}{m} - \frac{1}{n}\right| \quad \text{as } n > m \Rightarrow \frac{1}{n} < \frac{1}{m}$$

$$|a_m - a_n| = \frac{1}{m} - \frac{1}{n}$$

as $\frac{1}{n} > 0$ - $|a_m - a_n| < \frac{1}{m}$

Set $\varepsilon > 0$, choos $N > \frac{1}{\varepsilon}$

then, for any integers $m, n > N$ we have:

$|a_m - a_n| < \frac{1}{N}$

$|a_m - a_n| < \varepsilon$ This proves the Cauchy definition of a convergent sequence.

6b.) Prove $(a_n)_{n=1}^{\infty}$ given by $a_n = \frac{3n}{2n+1}$ for $n \geq 1$

$A = \frac{3}{2}$ for any $\varepsilon > 0$ $\exists N > 0$ st. if $n \geq N$ then $|a_n - A| < \varepsilon$

So, $\left|a_n - \frac{3}{2}\right| < \varepsilon$

$$\left|\frac{3n}{2n+1} - \frac{3}{2}\right| < \varepsilon$$

$$\left|\frac{2(3n)}{2(2n+1)} - \frac{3(2n+1)}{2(2n+1)}\right| < \varepsilon$$

$$\left|\frac{6n}{4n+2} - \frac{6n+3}{4n+2}\right| < \varepsilon$$

$$\left|\frac{-3}{4n+2}\right| < \varepsilon$$

$$\frac{3}{4n+2} < \varepsilon$$

$$\frac{4n+2}{3} > \frac{1}{\varepsilon}$$

$$4n+2 > \frac{3}{\varepsilon}$$

$$4n > \frac{3}{\varepsilon} - 2$$

$$n > \frac{\frac{3}{\varepsilon} - 2}{4}$$

So, there is an $N > 0$ st. if $n > \frac{\frac{3}{\varepsilon} - 2}{4}$ then $\left|a_n - 3/2\right| < \varepsilon$. Therefore, the sequence is convergent.

2.) Prove if $a_n \to A$, then $|a_n| \to |A|$

If $a_n \to A$ then, $|a_n - A| < \varepsilon$

If $|a_n| \to |A|$ then $||a_n| - |A|| < \varepsilon$

using the reverse triangle inequality we have that $||a_n| - |A|| \leq |a_n - A|$

so, $||a_n| - |A|| < \varepsilon$. Thus $\lim a_n = A$ therefore $\lim |a_n| = |A|$.

$\longrightarrow$

~~3~~,~~4~~,~~5~~,1

3.) Prove that if $a_n \to L$, $b_n \to L$ and $a_n \le c_n \le b_n$, then $c_n \to L$

Since $a_n \to L$, $|a_n - L| < \varepsilon$ for $\varepsilon > 0$
and $|b_n - L| < \varepsilon$

By the definition of a limit, L is equal to the least upper bound of the sequence. Since $b_n \ge a_n$ and $L = L$, we can write the inequality:
$|a_n - L| \le |b_n - L| < \varepsilon$ Since L is the least upper bound of b_n, and $a_n \le c_n \le b_n$, $|c_n - L| < \varepsilon$.
Therefore $c_n \to L$

worked on w/ classmates

4.) $\sqrt{a_n} - \sqrt{A} = \frac{a_n - A}{\sqrt{a_n} + \sqrt{A}}$ and define $N = \max(N_1, N_2)$. Since $\sqrt{a_n} > \sqrt{\frac{|A|}{2}}$
we have that for $n \ge N$: $\sqrt{a_n} + \sqrt{A} > \sqrt{|A|}\left(1 + \frac{1}{\sqrt{2}}\right)$

$$\frac{1}{\sqrt{a_n} + \sqrt{A}} < \frac{1}{\sqrt{|A|}\left(1 + \frac{1}{\sqrt{2}}\right)}$$

$$\frac{1}{\sqrt{a_n} + \sqrt{A}} < \frac{1}{2\sqrt{|A|}}$$

for $n \ge N$:

$$|\sqrt{a_n} - \sqrt{A}| = \left|\frac{a_n - A}{\sqrt{a_n} + \sqrt{A}}\right|$$
$$|\sqrt{a_n} - \sqrt{A}| = \frac{1}{\sqrt{a_n} + \sqrt{A}}|a_n - A|$$
$$|\sqrt{a_n} - \sqrt{A}| < \frac{1}{2\sqrt{|A|}}\left(\frac{3}{4}\frac{\varepsilon}{\sqrt{|A|}}\right)$$
$$|\sqrt{a_n} - \sqrt{A}| < \frac{3\varepsilon}{8|A|}$$

*Using $|a_n - A| < 2\varepsilon\sqrt{|A|}$ in part 3
$\forall n \ge N_2$ and, $n \ge N = \max(N_1, N_2)$:

$$|\sqrt{a_n} - \sqrt{A}| = \frac{1}{\sqrt{a_n} + \sqrt{A}}|a_n - A|$$
$$|\sqrt{a_n} - \sqrt{A}| < \frac{1}{2\sqrt{|A|}}\left(2\varepsilon\sqrt{|A|}\right)$$
$$|\sqrt{a_n} - \sqrt{A}| < \varepsilon$$

so, $\sqrt{a_n} \to \sqrt{A}$

5.) If $a_n \to A$, then by the proof done in class, we know that $a_n + a_n \to 2A$, and $|a_n - A| < \varepsilon$ for $\varepsilon > 0$, $n \ge 1$.
By the quotient rule we know that $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $n \ge N \Rightarrow \left|\frac{a_n}{b_n} - \frac{A}{B}\right| < \varepsilon$ So, if $|a_n - A| < \varepsilon'$, then $\left|\frac{a_n}{n} - \frac{A}{A}\right| < \varepsilon$ as well. In combination with the proof showing $a_n + b_n \to A + B$ and $a_n + a_n \to 2A$, the sequence $\sigma_n := \frac{a_1 + a_2 + \dots a_n}{n}$ $n \ge 1$ can be shown as $\left|\frac{a_1 + a_2 + \dots a_n}{n} - A\right| < \varepsilon$. Since $n \ge 1$ divides $a_1 + a_2 \dots + a_n$, we are left with $|a_n - A| < \varepsilon$, and $|\sigma_n - A| < \varepsilon$. Therefore $\sigma_n \to A$.

If $a_n = (-1)^n$ then σ_n converges.
(divergent)