MATH-331 Introduction to Real Analysis — Victor Ho

Homework 07 — Fall 2021

Due date: December, 6^{th} 1:20pm — Total: 56/65.

Exercise	1 (10)	2 (5)	3 (10)	4 (5)	5 (5)	6 (10)	7 (5)	8 (5)	9 (5)	10 (5)
Score	10	5	6	5	5	8	2	5	5	5

Table 1: Scores for each exercises

Instructions: You must answer all the questions below and send your solution by email (to `parisepo@hawaii.edu`). If you decide to not use LaTeX to hand out your solutions, please be sure that after you scan your copy, it is clear and readable. Make sure that you attached a copy of the homework assignment to your homework.
If you choose to use LaTeX, you can use the template available on the course website.
No late homework will be accepted. No format other than PDF will be accepted. Name your file as indicated in the syllabus.

1 Writing problems

For each of the following problems, you will be asked to write a clear and detailed proof. All the exercises below can be solve without using the definition with partitions. Try to go back to homework 6 and use some of the exercises there to solve the following problems.

You will have the chance to rewrite your solution in your semester project after receiving feedback from me.

Exercise 1. (10 pts) Prove that a step function is Riemann integrable on $[a, b]$. Follow the steps below.

a) Let I be a subinterval of $[a, b]$ and put $\phi = c\chi_I$. Prove that ϕ is Riemann integrable and that $\int_a^b \phi = c\ell(I)$. [There are three cases to consider: $I = [u, v]$, $I = (u, v]$, and $I = \{u\} = [u, u]$.]

b) Prove by induction that if f_1, f_2, $\ldots$, f_n are Riemann integrable functions on $[a, b]$, then $f_1 + f_2 + \cdots + f_n$ is Riemann integrable and

$$\int_a^b (f_1 + f_2 + \cdots + f_n) = \int_a^b f_1 + \int_a^b f_2 + \cdots + \int_a^b f_n.$$

c) Write $\phi = \sum_{k=1}^n c_k \chi_{I_k}$. Use the second part of this exercise to show that ϕ is Riemann integrable.

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Solution: a.

There are 3 different possible cases for I, $I = [u, v]$, $I = (u, v]$, and $I = \{u\} = [u, u]$. Let $I = [u, v]$. For this $I, \ell(I) = v - u$. Note that since $\phi = c\chi_I$, and $\chi_I = 1$ for $x \in I$ and $\chi_I = 0$ for $x \notin I$, $\phi = c\chi_I = c$ for $x \in I = [u, v]$ and $\phi = 0$ for $x \in [a, u]$ and $x \in [v, b]$. Therefore since constants are R.I, and $[u, v] \subset [a, b]$, we can define three separate integrals $\int_a^u \phi, \int_u^v \phi, \int_v^b \phi$, and we know that they exist as ϕ is constant in $[a, u], [u, v]$ and $[v, b]$. Now since ϕ is R.I on $[a, u], [u, v], [v, b], u \in [a, v]$ and $v \in [a, b]$, $\int_a^b \phi = \int_a^u \phi, \int_u^v \phi, \int_v^b \phi$. Since $\int_a^u \phi = \int_a^u 0 = 0, \int_u^v \phi = \int_u^v c = c(u - v), \int_v^b \phi = \int_v^b 0 = 0$, we can say that $\int_a^b \phi = c(v - u) = c\ell(I)$.

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Now if $I = (u, v]$, then we know from the definition of χ that for the interval $[a, u]$, since $I = (u, v], \chi_I = 0$ for $x \in [a, u]$, and $\chi_I = 0$ for $x \in [v, b]$. Therefore ϕ is R.I on $[a, u]$ and $[v, b]$ and $\int_a^u \phi = 0 = \int_v^b$. Now define k so that $u < k < v$. Note that $\phi = c\chi_I$ is R.I on $[k, v]$ as any $x \in [k, v]$ is also in $(u, v]$, so $\phi = c$, so $\int_k^v \phi = \int_k^v c = c(v - k)$. Since ϕ is $R.I$ on $[k, v]$ for any $k \in (u, v]$, we can use a backwards version of exercise 5 from homework 6 to state that ϕ is $R.I$ on $[u, v]$. Therefore ϕ is R.I on $[a, u]$ $[u, v]$ and $[v, b]$ so using a theorem from class, ϕ is R.I on $[a, b]$ and $\int_a^b = \int_a^u \phi + \int_u^v \phi + \int_v^b \phi = 0 + \int_u^v c + 0$ since $\chi_I = 1$ for $x \in I = (u, v]$. So $\int_a^b = c(v - u) = c\ell(I)$.

If $I = \{u\} = [u, u]$, first note that $\ell(I) = u - u = 0$, so $c\ell(I) = 0$. Now define k so that $a < k < u$. Therefore $\chi_I = 0$ for $[a, k]$, so $\phi = c\chi_I = 0$ for $x \in [a, k]$ so ϕ is R.I for any $k \in (a, u)$, and so from exercise 5 from homework 6, ϕ is R.I on $[a, u]$. WLOG replacing u with v, defining $k \in (u, v)$ and using a backwards version of exercise 5 from homework 6, we also know that ϕ is R.I on $[u, v]$. Therefore since ϕ is R.I on $[a, u], [u, b]$ and $u \in [a, b]$ we know that $\int_a^b \phi = \int_a^u \phi + \int_u^b \phi = 0 + 0 = 0 = c\ell(I)$.

b. Note that at $n = 1$, $\int_a^b f_1 = \int_a^b f_1$. At n = 2, since $f_1, f_2, ...$ are R.I on $[a, b]$, by the sum rule for integrals, $\int_a^b (f_1 + \cdots + f_n) = \int_a^b f_1 + f_2 = \int_a^b f_1 + \int_a^b f_2$, proving our base case. Now assume through induction that $\int_a^b (f_1 + f_2 + \cdots + f_{n-1}) = \int_a^b f_1 + \int_a^b f_2 + \cdots + \int_a^b f_{n-1}$ at $n - 1$. Since f_n is R.I on $[a, b]$ by assumption, $\int_a^b f_n$ exists, so we can add it to both sides of the equation to get $\int_a^b (f_1 + f_2 + \cdots + f_{n-1}) + \int_a^b f_n = \int_a^b f_1 + \int_a^b f_2 + \cdots + \int_a^b f_{n-1} + \int_a^b f_n$. Since $\int_a^b (f_1 + f_2 + \cdots + f_{n-1})$ is R.I on $[a, b]$ by the inductive hypothesis, and $\int_a^b f_n$ is also R.I on $[a, b]$, by the sum rule for integrals, $\int_a^b (f_1 + f_2 + \cdots + f_{n-1}) + \int_a^b f_n = \int_a^b (f_1 + f_2 + \cdots + f_{n-1} + f_n)$. Therefore we have $\int_a^b (f_1 + f_2 + \cdots + f_{n-1} + f_n) = \int_a^b f_1 + \int_a^b f_2 + \cdots + \int_a^b f_{n-1} + \int_a^b f_n$, proving that if it is true for $n - 1$, it is true for n, and therefore by the principle of mathematical induction, this is true for all $n \in \mathbb{N}$.

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c. Let $\phi = \sum_{k=1}^n c_k \chi_{I_k}$. We know from 1a that if I is a sub interval of $[a, b]$ then $\phi = c\chi_I$ is R.I on $[a, b]$. Therefore $c_k \chi_{I_k}$ is R.I on $[a, b]$ for any k. We also know from 1b that if $f_1 \ldots, f_n$ are R.I, then their sum is R.I. Therefore since $c_k \chi_{I_k}$ is R.I on $[a, b]$ for any k, $c\chi_1, \ldots, c\chi_k$ are R.I functions on $[a, b]$, so $c\chi_1 + \ldots + c\chi_k = \sum_{k=1}^n c_k \chi_{I_k}$ is also R.I on $[a, b]$. $\square$

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Exercise 2. (5 pts) Suppose that f is Riemann integrable on $[a, b]$ and that f is nonnegative (means that $f(x) \geq 0$ for $x \in [a, b]$). Let $u, v \in \mathbb{R}$. Show that if $a \leq u < v \leq b$, then

$$\int_u^v f \leq \int_a^b f.$$

[Hint: Use the following property of the Riemann Integral multiple times: $\int_a^b f = \int_a^c f + \int_c^b f$.]

Solution: There are 4 cases, $u = a$ and $v = b$, $u = a$ but $v < b$, $a < u$ but $v = b$ and $a < u < v < b$.
If $u = a$ and $v = b$, we can say that $\int_a^b f = \int_u^v f$, therefore $\int_u^v f \leq \int_a^b f$ is valid.
If $u = a$ but $v < b$, then we can first say that $\int_a^b f = \int_u^b f$. Then since $a = u < v < b$, $v \in (u, b)$ and f is R.I on $[a, b]$ so since $u = a$, it is R.I on $[u, b]$, so by a theorem in class $\int_u^b f = \int_u^v f + \int_v^b f$. Since $f(x) \geq 0$ for $x \in [a, b]$, we can say that $\int_v^b f \geq 0$, and therefore since $\int_u^b f = \int_u^v f + \int_v^b f$, we can say that $\int_u^b f = \int_a^b f \geq \int_u^v f$.

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Similarly, if $a < u$ but $v = b$ then we can first first $\int_a^b f = \int_a^v f$. Then since $u \in (a, v)$ and f is R.I on $[a, b]$ we can say that $\int_a^v f = \int_a^u f + \int_u^v f$. Then since $f(x) \geq 0$ for $x \in [a, b]$, we can say that $\int_a^u f \geq 0$, and therefore $\int_a^v f = \int_a^b f \geq \int_u^v f$.
If instead we have $a < u < v < b$, then we can first note that since $u \in (a, b)$, and f is R.I on $[a, b]$, $\int_a^b f = \int_a^u f + \int_u^b f$. Then since $u < v < b$, we can say that $v \in (u, b)$, and since f is R.I on $(u, b) \subset (a, b)$, we can say that $\int_u^b f = \int_u^v f + \int_v^b f$. Therefore we have $\int_a^b f = \int_a^u f + \int_u^v f + \int_v^b f$. Since $f(x) \geq 0$ for $x \in [a, b]$, we can then say that $\int_a^u f \geq 0$ and $\int_v^b f \geq 0$. Therefore since $\int_a^b f = \int_a^u f + \int_u^v f + \int_v^b f$, we can say that $\int_a^b f \geq \int_u^v f$.

Exercise 3. (10 pts) Use the Fundamental Theorem of Calculus to solve the following problems:

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a) Suppose that f is continuous on $[a, b]$ and that f is nonnegative on $[a, b]$. Show that if $\int_a^b f = 0$, then $f(x) = 0$ for any $x \in [a, b]$.

b) Suppose that f and g are continuous on $[a, b]$ such that $\int_a^b f = \int_a^b g$. Show that there exists a point $c \in (a, b)$ such that $f(c) = g(c)$.

$\to$ because F is increasing $\Rightarrow$ $F(x) \leq F(b) = 0$.

Solution: a. If $\int_a^b f = 0$, we can say that f is R.I on $[a, b]$. Now let $F(x) = \int_a^b f$. Since f is $R.I$ on $[a, b]$, by the fundamental theorem on calculus F is differentiable for $x \in [a, b]$, and $F'(x) = f(x)$. Note that since $\int_a^b f = 0$, we can say that $F(x) = 0$ for $x \in [a, b]$. Since the derivative of 0 is 0 by the constants rule for derivatives, $F'(x) = 0$ for $x \in [a, b]$. Since $F'(x) = f(x)$, we can therefore say that $f(x) = 0$ for $x \in [a, b]$.

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b. Since f and g are continuous on $[a, b]$, by the sum rule for continuity, $f - g$ is continuous on $[a, b]$. From exercise 4 below, we know that since $f - g$ is continuous on $[a, b]$, $\exists c \in [a, b]$ such that $((f - g)(c))(b - a) = f(c) - g(c)(b - a) = \int_a^b f - g$. From the sum rule of integrals, we know that $\int_a^b f - g = \int_a^b f - \int_a^b g$. Since $\int_a^b f = \int_a^b g$, $\int_a^b f - g = \int_a^b f - \int_a^b g = 0$. Therefore $f(c) - g(c)(b - a) = 0$, so $f(c) - g(c) = 0$, so $f(c) = g(c)$ for some $c \in [a, b]$. $\to$ you can't use Exo. 4 ...

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Exercise 4. (5 pts) Let f be a continuous function on $[a, b]$. Prove that there exists a number $c \in [a, b]$ such that $f(c)(b - a) = \int_a^b f$.

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Solution: Let $F(x) = \int_a^x f$. Since f is continuous on $[a, b]$, by the FTC, F is differentiable and continuous for $x \in [a, b]$ and $F'(x) = f(x)$. Since F is continuous and differentiable on $[a, b]$, by the MVP, $\exists c \in (a, b)$ such that $F'(c) = \frac{F(b) - F(a)}{b - a}$. Note that since $F'(x) = f(x)$, $F'(c) = f(c)$, and since $F(x) = \int_a^x f$, $F(b) = \int_a^b f$, $F(a) = \int_a^a f = 0$. Therefore $F'(c) = f(c) = \frac{\int_a^b f - 0}{b - a}$. We can then simplify by multiplying $b - a$ to both sides to get $f(c)(b - a) = \int_a^b f$ for some $c \in [a, b]$. $\square$

Exercise 5. (5 pts) Suppose that f is Riemann integrable on $[a, b]$ and is strictly increasing there. Prove that there exists a point $c \in (a, b)$ such that

$$\int_a^b f = f(a)(c-a) + f(b)(b-c).$$

[Hint: Define the function $g(x) = f(a)(x-a) + f(b)(b-x)$. Show that $\int_a^b f$ is between the numbers $f(a)(b-a)$ and $f(b)(b-a)$ and use the Intermediate Value Theorem.]

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Solution: Let $g(x) = f(a)(x-a) + f(b)(b-x)$. Note that because f is strictly increasing, $f(a) < f(x) < f(b)$ for $x \in (a, b)$. Therefore by a theorem in the lecture notes, $\int_a^b f(a)dx < \int_a^b f(x)dx < \int_a^b f(b)dx$. Since $f(a)$ and $f(b)$ are fixed, they are a constant, and therefore $\int_a^b f(a)dx = f(a)(b-a)$ and $\int_a^b f(b)dx = f(b)(b-a)$. Note that $g(a) = f(a)(b-a)$ and $g(b) = f(b)(b-a)$. Now we can substitute this in to the earlier inequality to get $g(a) < \int_a^b f(x)dx < g(b), x \in (a, b)$. Also note that g is a continuous function on $[a, b]$ as $f(a)$ is a constant, and $x - a$ is a linear function, so $f(a)(x-a)$ is continuous. Similarly, since $f(b)$ is a constant, and $b - x$ is linear, $f(b)(b-x)$ is continuous, and therefore their sum, $f(a)(x-a) + f(b)(b-x) = g(x)$ is also continuous. Since $g(a) < g(b)$ by order axioms, $g(a) \neq g(b)$, and since g is continuous on $[a, b]$, by IVT, since we have $g(a) < \int_a^b f(x)dx < g(b)$, there $\exists c \in (a, b)$ such that $g(c) = \int_a^b f(x), x \in (a, b)$. From the definition of $g, g(c) = f(a)(x-c) + f(c)(b-c)$. Therefore there $\exists c \in (a, b)$ such that $\int_a^b f = f(a)(x-c) + f(c)(b-c)$. $\square$

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Homework problems

Answer all the questions below. Make sure to show your work.

Exercise 6. (10pts)

a) Show that the function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & , x \in \mathbb{Q} \\ 0 & , x \notin \mathbb{Q} \end{cases}$$

is not Riemann integrable on $[0, 1]$. [Hint: Use exercise 4 from Homework 6.]

b) Define the two functions $g : [0, 1] \to \mathbb{R}$ and $h : [0, 1] \to \mathbb{R}$ by $g = \chi_{(0,1]}$ and

$$h(x) = \begin{cases} 0 & , x \notin \mathbb{Q} \\ \frac{1}{q} & , x = p/q \in \mathbb{Q}. \end{cases}$$

Use the first part to show that $g \circ h$ is not Riemann integrable on $[0, 1]$. What can you say about the composition of two Riemann integrable functions in light of this last examples?

* Not sure about that, you should get 1
** Not clear... Are taking a sub partition with the tags in $\mathbb{Q}$?

Solution: a. Assume towards a contradiction that f is R.I on [0, 1]. From exercise 4 in problem set 6, this implies that if $(P_n)_{n=1}^{\infty}$ is a sequence of T.P of [0, 1] such that $\lim_{n\to\infty} ||P_n|| = 0$, then $(S(f,P_n))_{n=1}^{\infty} \to \int_a^b f$. Now let $(S_1(f,P_n))_{n=1}^{\infty}$ be a subsequence of $(S(f,P_n))_{n=1}^{\infty}$ for $x \in \mathbb{Q}$,
** and let $(S_2(f,P_n))_{n=1}^{\infty}$ be a sequence of $(S(f,P_n))_{n=1}^{\infty}$ for $x \notin \mathbb{Q}$. Note that for $N = card(P_n)$, $(S_1(f,P_n))_{n=1}^{\infty} = (\sum_1^N f(c)(x_i - x_{i-1}))_1^{\infty} = (\sum_1^N (x_i - x_{i-1}))_1^{\infty}$ by the definition of f for $x \in \mathbb{Q}$. Since
* $(\sum_1^N f(x_i - x_{i-1}))_1^{\infty} = (x_N - x_1)_1^{\infty} \to x_n - x_1$ by the constant rule for convergence, $(S_1(f,P_n))_{n=1}^{\infty} \to (x_N - x_1)$. Now note that $(S_2(f,P_n))_{n=1}^{\infty} = (\sum_1^N f(c)(x_i - x_{i-1}))_1^{\infty} = (\sum_1^N 0(x_i - x_{i-1}))_1^{\infty} = (0)_1^{\infty}$ by the definition of f for $x \notin \mathbb{Q}$. Since $0 \to 0$, $(S_2(f,P_n))_{n=1}^{\infty} \to 0$. By a theorem in the textbook, if a sequence is convergent, then its sub sequences converges to the same limit. However $(S_1(f,P_n))_{n=1}^{\infty}$
3/5 and $(S_2(f,P_n))_{n=1}^{\infty}$ are two sub sequences of $(S_(f,P_n))_{n=1}^{\infty}$ that converge to different limits, so $(S_2(f,P_n))_{n=1}^{\infty}$ can't be convergent, contradicting our claim that $(S(f,P_n))_{n=1}^{\infty} \to \int_a^b f$. Therefore f is not R.I on [0, 1].
b.

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$$(g \circ h)(x) = \begin{cases} 0 & , h(x) \notin (0,1] \\ 1 & , h(x) \in (0,1]. \end{cases}$$

If $x \notin \mathbb{Q}$, $h(x) = 0$, so $(g \circ h)(x) = 0$ for $x \notin \mathbb{Q}$. If $x = p/q \in \mathbb{Q}$, then $h(x) = 1/q$, and since g has a domain of $[0,1]$, and $1/q \neq 0$ for any integer q, $1/q \in (0,1]$, so $(g \circ h)(x) = 1$ for $x \in \mathbb{Q}$. Therefore

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$$(g \circ h)(x) = \begin{cases} 0 & , x \notin \mathbb{Q} \\ 1 & , x \in \mathbb{Q}. \end{cases}$$

We know from 6a that this function, $(g \circ h)$, is not R.I on [0, 1]. From these two examples, I can say that the composition of two R.I functions are not necessarily R.I. ✓ □

Exercise 7. (5 pts) Show that if f is continuous on $[a,b]$, then $|f|$ is Riemann integrable on $[a,b]$ and

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$$\left| \int_a^b f \right| \leq \int_a^b |f|.$$

[Hint: There is a clever way to show that $|f|$ is Riemann integrable without using the definition with the partitions.]

this idea doesn't cover all the cases...

Solution: If $f(x) > 0$ for $x \in [a,b]$, then $\int_a^b |f| = \int_a^b f$. Since $f > 0$, from a theorem in class, $\int_a^b f > 0$, so $\left|\int_a^b f\right| = \int_a^b f$. Therefore if $f > 0$, $\left|\int_a^b f\right| = \int_a^b |f|$. If $f(x) = 0$ for $x \in [a,b]$, then $\left|\int_a^b f\right| = 0 = \int_a^b |f|$, so $\left|\int_a^b f\right| = \int_a^b |f|$ for $f = 0$. If $f(x) < 0, x \in [a,b]$ then $|f| > 0$. Therefore we can say that $f \leq |f|$. From the properties of the integral, since $f \leq |f|$. $\int_a^b f \leq \int_a^b |f|$. Therefore $|\int_a^b f| \leq |\int_a^b |f||$. Since $|f| > 0$, $\int_a^b |f| > 0$, so $|\int_a^b |f|| = \int_a^b |f|$, and therefore $|\int_a^b f| \leq \int_a^b |f|$. Since $\left|\int_a^b f\right|$ is either equal or less than $\int_a^b |f|$ for any $x \in [a,b]$, we can say that $\left|\int_a^b f\right| \leq \int_a^b |f|$. □

Exercise 8. (5 pts) Find $f'(x)$ if $f(x) = \displaystyle\int_{\sqrt{x}}^{\sqrt[3]{x}} \frac{1}{1+t^3}\, dt$ where $x \in [0,1]$.

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Solution: Let G be the anti derivative of $\frac{1}{1+t^3}$ for $[\sqrt{x}, \sqrt[3]{x}]$. Since we are given that $\frac{1}{1+t^3}$ is $R.I$ on $[\sqrt{x}, \sqrt[3]{x}]$, we know by the FTC that $\int_{\sqrt{x}}^{\sqrt[3]{x}} \frac{1}{1+t^3}\, dt = G(\sqrt[3]{x}) - G(\sqrt{x})$. Therefore since $f(x) = \int_{\sqrt{x}}^{\sqrt[3]{x}} \frac{1}{1+t^3}\, dt$ for $x \in [0,1]$, $f(x) = G(\sqrt[3]{x}) - G(\sqrt{x})$. Note that since G is the anti-derivative of $\frac{1}{1+t^3}$, $G' = \frac{1}{1+t^3}$. Now since $f(x) = G(\sqrt[3]{x}) - G(\sqrt{x})$, $f'(x) = G'(\sqrt[3]{x}) - G'(\sqrt{x})$. Note that since $\sqrt[3]{x}$ and $\sqrt{x}$ are differentiable for $x \in [0,1]$, and G is differentiable at $\sqrt[3]{x}$ and $\sqrt{x}$ for $x \in [0,1]$, we can use the chain rule to evaluate $G'(\sqrt[3]{x})$ and $G'(\sqrt{x})$. Therefore by the chain rule, $G'(\sqrt[3]{x}) = (\frac{1}{3x^{2/3}})(\frac{1}{1+(\sqrt[3]{x})^3}) = (\frac{1}{3x^{2/3}})(\frac{1}{1+x})$, and $G'(\sqrt{x}) = (\frac{1}{2\sqrt{x}})(\frac{1}{1+(\sqrt{x})^3})$.
Therefore $f'(x) = G'(\sqrt[3]{x}) - G'(\sqrt{x}) = (\frac{1}{3x^{2/3}})(\frac{1}{1+x}) - (\frac{1}{2\sqrt{x}})(\frac{1}{1+(\sqrt{x})^3})$. □

Exercise 9. (5 pts) Find a function $f : [1,\infty) \to \mathbb{R}$ such that $f(1) = 0$ and $f'(x) = 1 + \sin(x^2)$ for all $x > 1$.

Solution: Let

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$$f(x) = \begin{cases} 0 & , x = 1 \\ \int_1^{x^2} 1 + \sin(t)dt & , x > 1 \end{cases}$$

Exercise 10. (5 pts) By thinking the following sum as a Riemann sum, evaluate

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$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{k^2+n^2}.$$

Solution: First factor out n so that $\frac{n}{k^2+n^2} = \frac{1}{\frac{k^2}{n}+n}$. Then we can factor out $\frac{1}{n}$ to get $\frac{1}{\frac{k^2}{n}+n} = \frac{1}{\frac{k^2}{n^2}+1}(\frac{1}{n}) = \frac{1}{(\frac{k}{n})^2+1}(\frac{1}{n})$. Therefore $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{k^2+n^2} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{(\frac{k}{n})^2+1}(\frac{1}{n})$ Now let $f(x) = \frac{1}{\frac{k}{n}+1}$. Therefore $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{(\frac{k}{n})^2+1}(\frac{1}{n}) = \lim_{n\to\infty} \sum_{k=1}^{n} f(k/n)(\frac{1}{n})$. Note that $\lim_{n\to\infty} k/n$ is bounded by $[0,1]$ for $k \in [1,n]$, as k/n is a strictly increasing function for $k \in [1,n]$, and a fixed n, so $k = 1$ and $k = n$ will give us the lower and upper bounds respectively. At $k = 1$, $\lim_{n\to\infty} k/n = \lim_{n\to\infty} 1/n = 0$, from lecture notes. At $k = n, \lim_{n\to\infty} k/n = \lim_{n\to\infty} n/n = \lim_{n\to\infty} 1 = 1$. Therefore $\lim_{n\to\infty} \sum_{k=1}^{n} f(k/n)(\frac{1}{n})$ can be viewed as a Riemann Sum of f over the interval $[0,1]$, with $\frac{1}{n}$ as the size of each interval, and k/n as the tag, so $\lim_{n\to\infty} \sum_{k=1}^{n} f(k/n)(\frac{1}{n}) = \int_0^1 \frac{1}{x^2+1}dx$. From trig identities, $\int_0^1 \frac{1}{x^2+1}dx = (\text{Arctan}\,(x))_0^1 = \text{Arctan}\,(1) - \text{Arctan}\,(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}$.