Math 331: HW 04

Liliana Kershner

1. Looking at a unit circle, we see that if $x < \frac{\pi}{2}$, then $\sin x$ will always be less than $\frac{\pi}{2}$. Namely, the sine function is equal to 1 when $x = \frac{\pi}{2}$, and $\frac{\pi}{2}$ is approximately equal to 1.6.
Therefore it is true that if $x < \frac{\pi}{2}$, $\sin x < x$.
Taking the other side of the inequality, if $x > 0$, then $\sin x > 0$ because at $x = 0$ is the only time in the interval $[0, \frac{\pi}{2}]$ that $\sin x$ will equal 0. Then, if $0 < x < \frac{\pi}{2}$, $0 < \sin x < \frac{\pi}{2}$.

2. Suppose the limit of $f(g(t))$ exists at b, call this L. We will first prove $\lim_{x \to a} f(x)$ exists and equals L. By the definition of a limit, there then exists some interval B containing b s.t. if $t \in I, |f(g(t)) - L| < \varepsilon, \forall \varepsilon > 0$.
Imagine if the infimum or supremum of the set given by the function $g(t)$ is equal to b. Then we can define a smaller, closed subinterval of B, call it B_1, s.t. $g(b) = a$ is either the supremum or the infimum of B_1, and by previous class theorems, we know that g is continuous on B_1.
Let the range of g in B_1 be the closed interval B_2. Then there $\exists x \in B_2$ s.t. $g(t) = x$, and since $B_2 \subset B_1 \subset B$, $|f(x) - L| = |f(g(t)) - L| < \varepsilon$. So the limit of $f(x)$ exists.
Now, knowing that $\lim_{x \to a} f(x)$ exists, we will prove the limit of $\lim_{t \to b} f(g(t))$ exists and is equal to L. Let $a, x \in A$ s.t. $|f(x) - L| < \varepsilon$. We know $g(t)$ is continuous at b, and if $t \in B$, then $g(t) \in A$, from which we know $|f(g(t)) - L| < \varepsilon$. So $\lim_{x \to a} f(x) = \lim_{t \to b} f(g(t))$.

3. Let $f : [a, b] \to \mathbb{R}$. We know from the problem statement that f is continuous on $[a, b]$, so by definition $f(x)$ is uniformly continuous. So $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x, y \in D, |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$.
We then construct a sequence. Define x_1 as the rational number which sits between $(x - 1, x + 1)$, by the density of rational numbers this is true. Define x_2 as the rational number which sits between $(x - \frac{1}{2}, x + \frac{1}{2})$. By the same property this is true. We then have a sequence $x_n := \{x_0, x_1, x_2, x_3, ...x_n\}$ of rational numbers such that $x_n \in (x - \delta, x + \delta)$ and $x_n \to x$. We know that $f(x)$ is equal to 0, so by construction, if $x_n \to x$ and $f(x) = 0$, then $f(x_n) \to f(x) = 0 \to 0$.
Therefore the limit of this sequence as n approaches ∞ is 0.
The density of rational numbers states that for any two irrational numbers y_1, y_2 with $y_1 < y_2$, there must exist between them a rational number x. Since the subsequence of rational numbers converges to 0, the sequence of irrational numbers must also converge to 0.

4. The extreme value theorem states that for f continuous on an interval $[a, b]$, f will have a maximum, define this v, and a minimum, define this u, within the interval $[a, b]$. Or, that $f(u) \leq f(c) \leq f(v)$ for some $c \in [a, b]$.
Therefore the interval $[u, v] \in [a, b]$. Since the set $[u, v]$ is a subset of $[a, b]$ there are elements in $[a, b]$ which are not in $[u, v]$ but every element of $[u, v]$ is in $[a, b]$. Define then the term $\eta \in [a, b]/\{[u, v]\}$ with $\eta \leq f(u)$.
By the same theorem then, knowing f is continuous on the entire interval of $[a, b]$ and is therefore continuous on the interval $[u, v]$, there exists a maximum of the set, call it m, and a minimum, call it n. Then for some $x \in [u, v], f(n) \leq f(x) \leq f(m)$. Since the values of $f(x)$ will always be between $f(n), f(m) \in [u, v]$, the value of η will never be reached on this interval. Therefore $f(x) \geq \eta$.

5. a) Let c be a point within the set $\mathbb{R}$ such that f is continuous at c. We will then prove that f is continuous at 0 and f is continuous at all $x \in \mathbb{R}$.
Let $h \in \mathbb{R}$ and $h \to 0$. Then, $f(h + c) = f(h) + f(c)$ and $f(h + c) - f(c) = f(h)$. We know that $f(c)$ is continuous, and as $h \to 0$, the left-hand limit exists due to the continuity of f. We see this limit is 0, and further that the right-hand limits exists and is 0. And, $f(0 + 0) = f(0) + f(0) = 0$.
We use the same strategy to prove for any $x \in \mathbb{R}$. We have $f(x + h) = f(x) + f(h)$. The limit at 0 exists and so the limit as h goes to 0 of the right-hand side exists. $f(x)$ is constant because x is fixed, so the limit as h goes to 0 of the left-hand side exists. Since the limit of $f(h)$ as $h \to 0$ is 0, then $\lim_{h \to 0} f(x + h) = f(x)$. So f is continuous for all $x \in \mathbb{R}$.
b) We know then from part a that f is continuous on $\mathbb{R}$. Let $f(1) = k$. We will then prove that $f(x) = kx, \forall x \in \mathbb{R}$ by induction.
Set the base case as $x = 0$. For $x = 0$ we have from the previous proof that $f(0) = 0$. $0 \cdot k = 0$ so we see this is true.
We check the $x + 1$ case. We know then that $f(1) + f(x) = f(x + 1)$ and $f(1) = f(x + 1) - f(x)$ which, by our assumption, is equal to k. Then $f(x + 1) = f(x) + k$.
If $f(x) = kx$, then we have $f(x + 1) = kx + k = k(x + 1)$ which follows the assumption.
Therefore this is true for all $x \in \mathbb{R}$.

6. a) $\lim_{x\to x_0} f(x) = \sin(\frac{1}{x})$ when $x_0 = 0$. So we find $\lim_{x\to 0}\sin(\frac{1}{x})$. By lecture notes, we can then take the limit of the inside function as $x_0 \to 0$.
So $\lim_{x\to 0}\frac{1}{x}$. This function diverges because the left-side limit and the right-side limit are not equal.
b) Using the squeeze theorem:
We know that for any value of $\sin(x)$, it must be between $[-1,1]$. So the limit as $x \to 0$ is in the same interval.
Then $-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$. And

$$\lim_{x\to 0} -x \leq \lim_{x\to 0} x\sin\left(\frac{1}{x}\right) \leq \lim_{x\to 0} x.$$

So $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right)$ must also go to 0.

7. We will find the value of $\lim_{x\to c} f(x) = \lim_{x\to c}((f(x))^2 - f(x) - 3)$.

$$\begin{aligned}
\lim_{x\to c} f(x) &= \lim_{x\to c}((f(x))^2 - f(x) - 3)\\
\lim_{x\to c} f(x) &= \lim_{x\to c}(f(x))^2 - \lim_{x\to c} f(x) - \lim_{x\to c} 3\\
0 &= \lim_{x\to c}(f(x))^2 - 2(\lim_{x\to c} f(x)) - 3.
\end{aligned}$$

Solving the quadratic, we find the roots are -1 and 3, so the limit must equal either -1 or 3. Since $f(x) > 0$ from the problem statement, $\lim_{x\to c} f(x) = 3$.

8. Notice that for positive x, we have a sequence of positve, rational numbers x, however for negative x, we have a sequence of negative, irrational numbers. We then construct two sequences: let $x_1, x_2, x_3, ...x_n$ be defined as the positive domain of f and $y_1, y_2, y_3, ...y_n$ be defined as the negative domain of f.
To prove discontinuity, it would be sufficient to show that $\lim_{x\to x_n} f(x) \neq \lim_{y\to y_n} f(y)$ since the left and right hand limits must be the same.
We have that $\lim_{x\to x_n} f(x)$ will always be positive, because there are no negative values of x in the set. Meanwhile, we have that $\lim_{y\to y_n} f(y)$ will be negative because there are only negative values in the set. Therefore the parity of the limits will not be the same, so f is discontinuous at every point in $\mathbb{R}$ except for 0.
To confirm continuity at 0, when $x = 0$ is is neither positive nor negative, so the right and left hand limits will exist in the same set.

9. If $p(x) = x^2 + 2$ then the function is only decreasing for $x \in [1, 0]$.
Find the inverse:

$$\begin{aligned}
y &= x^2 + 2\\
x &= y^2 + 2\\
x - 2 &= y^2\\
\sqrt{x-2} &= y.
\end{aligned}$$

So $p^{-1}(x) = \sqrt{x-2}$.

10. a) We rewrite $p(x) = ax^3 + bx^2 + cx + d$ as $p(x) = x^3(a + \frac{b}{x} + \frac{c}{x^2} + \frac{d}{x^3})$. Taking the limits and applying the sum rule and product rule, we have

$$\lim_{x\to\infty} p(x) = \lim_{x\to\infty} x^3(\lim_{x\to\infty} a + \lim_{x\to\infty}\frac{b}{x} + \lim_{x\to\infty}\frac{c}{x^2} + \lim_{x\to\infty}\frac{d}{x^3}).$$

Regardless of the value of a, b, c, d, so long as $a > 0$ the limit of this polynomial is a by a previous proof in lecture. Then, we have

$$\lim_{x\to\infty} x^3(a + 0 + 0 + 0)$$

$$(\infty)(a)$$

$$\infty.$$

Regardless of the value of a, it can never overcome the value of infinity.
b) We take a similar approach. Again factor $p(x)$ as $x^3(a + \frac{b}{x} + \frac{c}{x^2} + \frac{d}{x^3})$. Then we take the limits:

$$\lim_{x \to -\infty} p(x) = \lim_{x \to -\infty} x^3 (\lim_{x \to -\infty} a + \lim_{x \to -\infty} \frac{b}{x} + \lim_{x \to -\infty} \frac{c}{x^2} + \lim_{x \to -\infty} \frac{d}{x^3}).$$

We know already that again, by a previous proof, the limits of the form $\frac{1}{x}$ will go to 0. However, since our polynomial has an odd power, the sign of the limit is determined by the odd power. So:

$$\lim_{x \to -\infty} x^3 (a + 0 + 0 + 0)$$

$$(-\infty)(a)$$

$$-\infty.$$

c) Since $p(x)$ is continuous on $(-\infty, \infty)$, which means the polynomial is defined at 0. So there is at least one root which exists.