1. a) Let $\mathcal{P}_1$ be defined as a t.p. of $[a,b]$ s.t.

$$\mathcal{P}_1 := \{c_i, [x_i, x_{i-1}]\}.$$

By AP, $\frac{b-a}{n} < \delta$ for some $n \in \mathbb{R}$.

Then we have that $\mathcal{P} = [a, a+\frac{b-a}{n}], [a+\frac{b-a}{n}, a+\frac{2(b-a)}{n}], \ldots, [a+\frac{n(b-a)}{n}, b]$ is such a partition, for some $k \in \mathbb{R}$.

b) By definition, if f is R.I. then $\exists L \in \mathbb{R}$ s.t. $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. if $\mathcal{P}$ is a t.p. of $[a,b]$ and $\|\mathcal{P}\| < \delta$, then

$$|S(f,\mathcal{P}) - L| < \varepsilon.$$

Assume by contradiction that $\int_a^b f = L_1$ and $\int_a^b f = L_2$ with $L_1 \neq L_2$. By definition also, $\int_a^b f = \lim_{\|\mathcal{P}\| \to 0} S(f,\mathcal{P})$.

So $\forall \varepsilon > 0$, $\exists \delta_1 > 0$ s.t. if $|\|\mathcal{P}\| - 0| < \delta_1$, then

$$|S(f,\mathcal{P}) - L_1| < \frac{\varepsilon}{2}$$

and $\forall \varepsilon > 0$, $\exists \delta_2 > 0$ s.t. if $|\|\mathcal{P}\| - 0| < \delta$ then

$$|S(f,\mathcal{P}) - L_2| < \frac{\varepsilon}{2}.$$

Let $\delta := \min\{\delta_1, \delta_2\}$. We know if $L_1 \neq L_2$, then $|L_1 - L_2| > 0$, so by T.I.,

$$0 < |L_1 - L_2| = |L_1 - S(f,\mathcal{P}) + S(f,\mathcal{P}) - L_2|$$
$$\leq |L_1 - S(f,\mathcal{P})| + |S(f,\mathcal{P}) - L_2|$$

But since ε was arbitrary, L_1 must equal L_2.

2. a) By statement, f and g are R.I. on $[a,b]$. Then by definition, $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. if P is a t.p. of $[a,b]$ and $\|P\| < \delta$, then $|S(f,P) - \int_a^b f| < \varepsilon$ and $|S(g,P) - \int_a^b g| < \varepsilon$.

Proof goals:

1) Prove $f+g$ is R.I.

2) Prove $\int_a^b f+g = \int_a^b f + \int_a^b g$

1) Let $\varepsilon > 0$. Then, $\exists \delta_1 > 0$ s.t. if P_1 is a t.p. of $[a,b]$ and $\|P_1\| < \delta_1$, then $|S(f,P_1) - L_1| < \varepsilon/2$ for $L_1 = \int_a^b f$.
Further, $\exists \delta_2 > 0$ s.t. if P_2 is a t.p. of $[a,b]$ and $\|P_2\| < \delta_2$, then $|S(g,P_2) - L_2| < \varepsilon/2$ where $L_2 = \int_a^b g$.
So,
$$|S(f,P_1) - L_1| + |S(g,P_2) - L_2| < \varepsilon$$
So $f+g$ is R.I. on $[a,b]$.

2) Rewrite the integrals as sums:
$$\int_a^b f+g = \sum_{i=1}^{N} (f+g)(c_i)(x_i - x_{i-1})$$
and
$$\int_a^b f = \sum_{i=1}^{N} (f)(c_i)(x_i - x_{i-1}) \text{ and } \int_a^b g = \sum_{i=1}^{N} (g)(c_i)(x_i - x_{i-1})$$

Then, $\int_a^b f + \int_a^b g = \sum_{i=1}^{N} f(c_1)(x_1 - x_0) + g(c_1)(x_1 - x_0) \ldots$
$$= \sum_{i=1}^{N} (x_i - x_{i-1})(f+g)(c_i).$$
By addibility of functions.
Now, by definition of integrals as limits, $\int_a^b f = \lim_{\|P\| \to 0} S(f,P)$, $\int_a^b g = \lim_{\|P\| \to 0} S(g,P)$, and $\int_a^b (f+g) = \lim_{\|P\| \to 0} S((f+g),P)$.

So $\int_a^b f + \int_a^b g = \lim_{\|P\| \to 0} S(f,P) + \lim_{\|P\| \to 0} S(g,P)$ which we previously showed is equal to
$$\lim_{\|P\| \to 0} S((f+g),P)$$

and this is exactly $\int_a^b (f+g)$. So $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.

b) Since $g \geq f$, $\forall x \in [a,b]$, $g - f \geq 0$. If $g = f$, then $\int_a^b f = \int_a^b g$ trivially.
So we consider only $g > f$.
By definition of the Riemann Integral, $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. if P is a t.p. of $[a,b]$, and $\|P\| < \delta$, then

$$|S(f,P) - \int_a^b f| < \varepsilon$$

and same for g; $\forall \varepsilon > 0$, $\exists \delta_2 > 0$ s.t. if P is a t.p. of $[a,b]$, and $\|P\| < \delta_2$ then

$$|S(g,P) - \int_a^b g| < \varepsilon.$$

We then have

$$S(f,P) = \sum_{i=1}^{N} f(c_i)(x_i - x_{i-1})$$
$$S(g,P) = \sum_{i=1}^{N} g(c_i)(x_i - x_{i-1})$$

and we see that for each term $g(c_i)$, this will still be greater than each term $f(c_i)$, so $S(g,P) > S(f,P)$.
Now, let $\delta := \min\{\delta_1, \delta_2\}$.
We have

$$-\varepsilon < S(f,P) - \int_a^b f < \varepsilon \qquad (\star)$$

and

$$-\varepsilon < S(g,P) - \int_a^b g < \varepsilon. \qquad (\star\star)$$

Then $\int_a^b f - \varepsilon < S(f,P) < \varepsilon + \int_a^b f$ and $\int_a^b g - \varepsilon < S(g,P) < \varepsilon + \int_a^b g$.
Since $S(g,P) > S(f,P)$, we know $S(g,P) > \int_a^b f - \varepsilon$ and $\varepsilon + \int_a^b g > S(f,P)$, and $\int_a^b f - \varepsilon < \varepsilon + \int_a^b g$.
We then have $\int_a^b f < 2\varepsilon + \int_a^b g$. Since ε was arbitrary, we have $2\varepsilon = \varepsilon$.
So $\int_a^b f < \int_a^b g$.

3. From statement, if f is R.I. on $[a,b]$, then $\forall \varepsilon > 0$, $\exists \delta > 0$, s.t. if P is a t.p. of $[a,b]$ and $\|P\| < \delta$, then

$$|S(f,P) - \int_a^b f| < \varepsilon.$$

If $|f(x)| \leq M$ for all $x \in [a,b]$, then we know f is bounded above by M.

So, rewriting the sum we have

$$S(f,P) = \sum_{i=1}^{N} f(c_i)(x_i - x_{i-1})$$

and every $f(c_i) \leq M$.

The greatest this sum can be, then, is when, $\forall c_i$, $f(c_i) = M$. Which gives

$$\sum_{i=1}^{N} M(x_i - x_{i-1}).$$

Similarly, the maximum value for $x_i - x_{i-1}$ is when $x_i = b$ and $x_{i-1} = a$. So the maximum possible value for $S(f,P)$ is $M(b-a)$.

Since $\int_a^b f = \lim_{\|P\| \to 0} S(f,P)$, and M, b, and a are constants, we have

$$\int_a^b f \leq M(b-a).$$

4. Since f is Riemann integrable, we know $\forall \varepsilon > 0, \exists \delta > 0$ s.t. if P is a tagged partition of $[a,b]$ and $\|P\| < \delta$, then $|S(f,P) - \int_a^b f| < \varepsilon$.

And by statement, we know $\lim_{n\to\infty} \|P_n\| = 0$. So, let $\|P_n\| < \delta$.

By definition of a limit, $\forall \varepsilon > 0, \exists N > 0$ s.t. $n \geq N$, then $|\|P_n\| - L| < \varepsilon$.

We know $L = 0$ so $|P_n| < \varepsilon$. Since $\|P_n\|$ is always positive, $P_n < \varepsilon$.

Let $\delta = \varepsilon$. Then $P_n < \delta$, so $\|P_n\| < \delta$. Then, since we know $\lim_{n\to\infty} \|P_n\| = 0$, $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\delta > \|P_n\|$ and

$$\lim_{\|P_n\| \to 0} S(f, P_n) = \int_a^b f$$

by the definition of a Riemann integral.

5. We know by statement that f is R.I on $[a,c]$.
Now, let P_1 and P_2 be two t.p. of $[a,b]$. Let c be close enough to b s.t. $b-c<\varepsilon$. The Cauchy Criterion states that $\forall \varepsilon>0, \exists \delta>0$ s.t. $\|P_1\|<\delta$ and $\|P_2\|<\delta$ then $|S(f,P_1)-S(f,P_2)|<\varepsilon$.
Now we separate P_1 and P_2 s.t.

$$P_{1a}:=\{(c_i,[x_{i-1},x_i])\in P_1 ; [x_{i-1},x_i]\subseteq[a,c]\}$$
$$P_{1b}:=\{(c_i,[x_{i-1},x_i])\in P_2 ; [x_{i-1},x_i]\subseteq[c,b]\}$$

Then, define $\tilde{P}_{1a}=P_{1a}\cup\{(c,[x_{N_{1a}},c])\}$ to include the section around c within $[a,c]$. And define $\tilde{P}_{1b}=\{(c,[c,x_{N_{1a}+1}])\}$.
Further

$$P_{2a}:=\{(c_i,[x_{i-1},x_i]\in P_2 ; [x_{i-1},x_i]\subseteq[a,c]\} \Rightarrow \tilde{P}_{2a}=P_{2a}\cup\{(c,[x_{2Na},c])\}$$
$$P_{2b}:=\{(c_i,[x_{i-1},x_i]\in P_2 ; [x_{i-1},x_i]\subseteq[c,b]\} \Rightarrow \tilde{P}_{2b}=P_{2b}\cup\{(c,[x_{2Na+1},c])\}$$

Then, we know

$$|S(f,P_1)-S(f,P_2)|<\varepsilon$$

and from our partitions we know

$$|S(f,P_1)-S(f,P_2)|=|S(f,\tilde{P}_{1a})+S(f,\tilde{P}_{1b})-S(f,\tilde{P}_{2a})-S(f,\tilde{P}_{2b})|$$

and by triangle inequality this is

$$\text{"} \quad \text{"} \leq |S(f,\tilde{P}_{1a})-S(f,\tilde{P}_{2a})|+|S(f,\tilde{P}_{1b})+S(f,\tilde{P}_{2b})|$$

all by the cauchy criterion.
Since f is bounded, we can denote this bound as M, and

$$S(f,\tilde{P}_{1b})\leq\sum_{i=1}^{N}|f(c_i)|(x_i-x_{i-1})$$
$$\leq M\sum_{i=1}^{N}x_i-x_{i-1}$$
$$=M(b-c),$$

so $S(f,\tilde{P}_{1b})<\varepsilon M$. And by the same logic, $S(f,\tilde{P}_{2b})<\varepsilon M$.
Then $|S(f,\tilde{P}_{1a})-S(f,\tilde{P}_{2a})|<\varepsilon$, which we know because f is R.I. on $[a,c]$. Then, our whole expression is less than $\varepsilon+2\varepsilon M=\varepsilon(1+2M)$. So, define our original ε as $\frac{\varepsilon}{1+2M}$.

6. a) If f is R.I. on $[a,b]$ then $\forall \varepsilon>0, \exists \delta>0$ s.t. if P is a t.p. of $[a,b]$ and $\|P\|<\delta$ then

$$|S(f,P)-\int_a^b f|<\varepsilon.$$

Let $\varepsilon>0$ and replace f by k:

$$|S(k,P)-\int_a^b k|<\varepsilon.$$

By definition

$$\int_a^b k=\lim_{\|P\|\to 0} S(k,P)$$

And we have

$$\sum_{i=1}^{N} f(c_i)(x_{i-1}-x_i)$$

$$=\sum_{i=1}^{N} k(x_{i-1}-x_i)$$

$$=\sum_{i=1}^{N} k(b-a).$$

Then

$$\lim_{\|P\|\to 0} k(b-a)=\int_a^b k$$

and since $k, b,$ and a are constants, $\lim_{\|P\|\to 0} k(b-a)=k(b-a)$.
So, since each L is unique,

$$\int_a^b k=k(b-a).$$

b) If f is R.I. then $\forall \varepsilon>0, \exists \delta>0$ s.t. if P is a t.p. of $[a,b]$ and $\|P\|<\delta$, then

$$|S(f,P)-\int_a^b f|<\varepsilon.$$

By definition

$$\lim_{\|P\|\to 0} S(f,P)=\int_a^b f$$

and

$$S(f,P)=\sum_{i=1}^{N} f(c_i)(x_{i-1}-x_i)$$

$$=\sum_{i=1}^{N} \sin^2(c_i)(x_{i-1}-x_i)$$

and by a trig identity:

$$=\sum_{i=1}^{N} \frac{1-\cos(2c_i)}{2}(x_{i-1}-x_i)$$

$$= \frac{1}{2}\sum_{i=1}^{N}[1-\cos(2c_i)](x_{i-1}-x_i)$$

$$= \frac{1}{2}\sum_{i=1}^{N}[1-\cos(2c_i)](b-a).$$

Then

$$\int_a^b f = \frac{1}{2}\lim_{\|P\|\to 0}\sum_{i=1}^{N} b-a - \frac{1}{2}\lim_{\|P\|\to 0}\sum_{i=1}^{N}\cos(2c_i)(b-a)$$

Since we know b and a are constants, $\lim_{\|P\|\to 0}\sum_{i=1}^{N}(b-a) = (b-a)$ and by assumption $\cos(2x)$ is R.I. So, f is R.I on $[a,b]$.

7. Show that the function $f:[0,1]\to\mathbb{R}$ defined by

$$f(x):=\begin{cases}1, & 0\le x\le 1/2\\ 0, & 1/2\le x\le 1\end{cases}$$

is R.I. on $[0,1]$.

If f is R.I then $\forall \varepsilon>0$, $\exists \delta>0$ s.t. if $\mathcal{P}$ is a t.p. of $[0,1]$, and $\|\mathcal{P}\|<\delta$, then

$$|S(f,\mathcal{P})-\int_0^1 f|<\varepsilon$$

Let there be two tagged partitions of $[0,1]$, denoted by $\mathcal{P}_1$ and $\mathcal{P}_2$, for the intervals $[0,1/2]$ and $[1/2,1]$ respectively.
So $\mathcal{P}_1:=\{(c_i,[x_{i-1},x_i]);\ c\in[0,1/2]\}$ and $\mathcal{P}_2:=\{(c_i,[x_i,x_{i-1}]);\ c\in[1/2,1]\}$.
Then $S(f,\mathcal{P})=S(f,\mathcal{P}_1)+S(f,\mathcal{P}_2)$. Denote N_1 by card $(\mathcal{P}_1)$, and N_2 by card $(\mathcal{P}_2)$.

$$S(f,\mathcal{P}_1)=\sum_{i=1}^{N_1} f(c_i)(x_i-x_{i-1})$$

Since on this interval $f(x)=1$, we have

$$S(f,\mathcal{P}_1)=\sum_{i=1}^{N_1} 1(x_i-x_{i-1})$$

$$=x_{N_1}$$

and $$S(f,\mathcal{P}_2)=\sum_{i=1}^{N_2} f(c_i)(x_i-x_{i-1})$$

and since $f(x)$ is 0 on this interval,

$$S(f,\mathcal{P}_2)=\sum_{i=1}^{N_2} 0(x_i-x_{i-1})$$

$$=0.$$

We know by general calculus that $\int_0^{1/2} f(x)\,dx=1/2$, and $\int_{1/2}^1 f(x)\,dx=0$. So $S(f,\mathcal{P})=1/2\,(x_{N_1})$.

We can rewrite this as $S(f,\mathcal{P})-\frac{1}{2}=\frac{1}{2}(x_{N_1})$, and

$|S(f,P) - 1/2| = |x_{N_i} - 1/2|.$

Since $\|P\| < \delta$, $|S(f,P) - 1/2| < \delta$ so $\delta = \varepsilon$.

8. The cauchy criterion states $\forall \varepsilon > 0, \exists \delta > 0$ s.t. P_1 and P_2 are t.p. of $[0,1]$ with $\|P_1\| < \delta$ and $\|P_2\| < \delta$ then

$$|S(f, P_1) - S(f, P_2)| < \varepsilon.$$

So, let P_1 and P_2 be t.p. of $[0,1]$ s.t.

$$P_{1a}: \{(c_i, [x_{i-1}, x_i]) \in P_1 ; [x_{i-1}, x_i] ; [x_i, x_{i-1}] \subseteq [0, c]\}$$
$$P_{1b}: \{(c_i, [x_{i-1}, x_i]) \in P_1 ; [x_{i-1}, x_i] ; [x_i, x_{i-1}] \subseteq [c, 1]\}$$

then

$$P_{2a}: \{(c_i, [x_{i-1}, x_i] \in P_2 ; [x_{i-1}, x_i] \subseteq [0, c]\}$$
$$P_{2b}: \{(c_i, [x_{i-1}, x_i] \in P_2 ; [x_{i-1}, x_i] \subseteq [c, 1]\}$$

Where $c \in [0,1]$ since f is 1 at $1/n$ and $0 < \frac{1}{n} \leq 1$ $\forall n \in \mathbb{N}$. Since $\frac{1}{n} \to 0$, $1/n < c$ for $1, 2, \ldots, 1/n$. We will show f is R.I. on $[c, 1]$. Let $c < \varepsilon$.

Now define

$$\tilde{P}_{1a} := P_{1a} \cup \{c, [x_{N_{a}}, c]\}$$
$$\tilde{P}_{1b} := P_{1b} \cup \{c, [c, x_{N_{a+1}}]\}$$
$$\tilde{P}_{2a} := P_{2a} \cup \{c, [x_{2N_a}, c]\}$$
$$\tilde{P}_{2b} := P_{2b} \cup \{c, [c, x_{2N_{a+1}}]\}.$$

Then, splitting the partitions:

$$|S(f, P_1) - S(f, P_2)| \Rightarrow |S(f, \tilde{P}_{1a}) + S(f, \tilde{P}_{1b}) - S(f, \tilde{P}_{2a}) - S(f, \tilde{P}_{2b})| \quad \text{and by T.I:}$$
$$\leq |S(f, \tilde{P}_{1a}) - S(f, \tilde{P}_{2a})| + |S(f, \tilde{P}_{1b}) - S(f, \tilde{P}_{2b})|$$

We know f is bounded; call the upper bound of f as M.

Then by the Cauchy criterion,

$$|S(f, \tilde{P}_{1a}) - S(f, \tilde{P}_{2a})| < \varepsilon$$
$$|S(f, \tilde{P}_{1b}) - S(f, \tilde{P}_{2b})| < 2M\varepsilon$$

So we define our original ε as $\frac{\varepsilon}{1+2M}$. By the same logic as 5, f is R.I. over $[0,1]$.

9. Our theorem for step functions states if $\varphi: \sum_{i=1}^{n} c_k I_k$ is a step function on $[a,b]$ then φ is R.I. on $[a,b]$ and $\int_a^b \varphi$ is the sum $\sum_{i=1}^{n} c_k \, l(I_k)$ where $l(I_k)$ is the length of I_k and c_k is the value of φ on I_k.

Since our function f is 4 for $l(I_k)=0$ and 0 for $l(I_k)=1$, $\int_a^b f(x)=0$.
We will use the definition of R.I to prove f is R.I and that $L=0$.
$\forall \varepsilon>0, \exists \delta>0$ s.t. if P is a t.p. of $[0,1]$, and $\|P\|<\delta$, then

$$|S(f,P)-L|<\varepsilon.$$

By definition also, $\lim_{\|P\|\to 0} S(f,P) = \int_a^b f$ which we claim is 0.

$S(f,P)$ can be rewritten as

$$\sum_{i=1}^{n} f(c_i)(x_{i-1}-x_i)$$

and $x_{i-1}-x_i = 1$ for $x_i=0$ and $x_{i-1}=1$ when $f(x)=0$. so

$$\sum_{i=1}^{n} (0)(1)=0.$$

When $f(x)=4$, we get that $x=0$, so

$$\sum_{i=1}^{n} (4)(0-0)=0.$$

Either way, we have

$$\lim_{\|P\|\to 0} 0 = \int_a^b f$$

and 0 is a constant so

$$\lim_{\|P\|\to 0} 0 = 0.$$

Since every L is unique, $\int_0^1 f(x)=0$, and f is R.I.

10. First we find $\|P\|$.

Testing each interval we have:

1) $|(-1)-(-0.8)| = 0.2$

2) $|(-0.8)-(-0.3)| = 0.5$

3) $|(-0.3) - 0| = 0.3$

4) $|0 - 0.2| = 0.2$

5) $|0.2 - 0.4| = 0.2$

6) $|0.4 - 1| = 0.6$

7) $|1 - 1.5| = 0.5$

8) $|1.5 - 2| = 0.5$

So the norm of P is 0.6. Thus $\|P_0\|$ must be less than or equal to 0.2.

So our new P_0 is

$$P_0 = [-0.9, [-1, -0.8]], [-0.7, [-0.8, -0.6]], [-0.5, [-0.6, -0.4]], [-0.3, [-0.4, -0.2]],$$

$$[-0.1, [-0.2, 0]], [0.1, [0, 0.2]], [0.3, [0.2, 0.4]], [0.5, [0.4, 0.6]],$$

$$[0.7, [0.6, 0.8]], [0.9, [0.8, 1]], [1.1, [1, 1.2]], [1.3, [1.2, 1.4]],$$

$$[1.5, [1.4, 1.6]], [1.7, [1.6, 1.8]], [1.9, [1.8, 2]].$$