MATH-331 Introduction to Real Analysis — Victor Ho
Homework 03 — Fall 2021

Due date: October 11$^{\text{th}}$ 1:20pm — Total: /70.

Exercise	1 (5)	2 (5)	3 (5)	4 (5)	5 (10)	6 (10)	7 (5)	8 (5)	9 (5)	10 (10)
Score										

Table 1: Scores for each exercises

Instructions: You must answer all the questions below and send your solution by email (to `parisepo@hawaii.edu`). If you decide to not use LaTeX to hand out your solutions, please be sure that after you scan your copy, it is clear and readable. Make sure that you attached a copy of the homework assignment to your homework.
If you choose to use LaTeX, you can use the template available on the course website.
No late homework will be accepted. No format other than PDF will be accepted. Name your file as indicated in the syllabus.

1 Writing problems

For each of the following problems, you will be asked to write a clear and detailed proof. You will have the chance to rewrite your solution in your semester project after receiving feedback from me.

Exercise 1. (5 pts) Let $(a_n)_{n=1}^{\infty}$ be an increasing sequence and $(b_n)_{n=1}^{\infty}$ be a decreasing sequence. Let $(c_n)_{n=1}^{\infty}$ be the sequence defined by $c_n = b_n - a_n$. Show that if $\lim_{n\to\infty} c_n = 0$, then the sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ converges and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

Solution: First note that $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ converge as they are sub sequences of $(c_n)_{n=1}^{\infty}$, which is convergent, and from theorem 1.14 in the textbook, all sub sequences of a convergent sequence converge. This is because $(-a_n)_{n=1}^{\infty}$ is a sub sequence of $(c_n)_{n=1}^{\infty}$ when $b_n = 0$, so it converges to some value L, and by the multiplication rule for sequences, $(a_n)_{n=1}^{\infty}$ converges to $-L$. In addition, $(b_n)_{n=1}^{\infty}$ is also a sub sequence of $(c_n)_{n=1}^{\infty}$ when $a_n = 0$. Also note that since $c_n = b_n - a_n$, $\lim_{n\to\infty} c_n = 0 = \lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n$, so $\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n$. Therefore if $c_n = b_n - a_n$, and $\lim_{n\to\infty} c_n = 0$, we can say that the sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ converges and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$ $\square$

Exercise 2. (5 pts) Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$, and suppose that x_0 is an accumulation point of D. Suppose that for each sequence $(x_n)_{n=1}^{\infty}$ converging to x_0 with $x_n \in D\backslash\{x_0\}$ for each $n \geq 1$, then the sequence $(f(x_n))_{n=1}^{\infty}$ is Cauchy. Show that f has a limit at x_0.

[Hint: For two sequences (x_n) and (y_n) that satisfy the assumption, define the sequence (z_n) to be $z_{2n} = x_n$ and $z_{2n-1} = y_n$. Show that $(f(z_n))$ converges and the sequence $(f(x_n))$ and $(f(y_n))$ converges to the same limit as $(f(z_n))$. Conclude by a theorem in the lecture notes.]

Solution: Let x_n and y_n be two sequences that satisfy the above assumptions, where $x_n \neq y_n$, and let z_n be defined by $z_{2n} = x_n$ and $z_{2n-1} = y_n$. Since $x_n, y_n \in D\backslash\{x_0\}$, we can say that $z_n \in D\backslash\{x_0\}$, and therefore $f(z_n))_{n=1}^{\infty}$ is Cauchy. We know from a theorem in the lecture notes that every Cauchy sequence is convergent, so $f(z_n))$ is convergent. Since $f(x_n)$ and $f(y_n)$ are sub sequences of $f(z_n))$, they both converge to the same limit as $f(z_n))$. This implies that for all the sequences, $(x_n)_{n=1}^{\infty}$, that satisfy the above assumptions, $f((x_n)_{n=1}^{\infty}$ converges. Therefore we have that $f : D \subseteq \mathbb{R} \to \mathbb{R}$ and $x_0 \in acc(D)$, where $\forall (x_n)_{n=1}^{\infty}$ such that $\forall n \geq 1, x_n \to x_0, x_n \neq x_0, x_n \in D$, then $f(x_n) \to L$ for some $L \in \mathbb{R}$, which by a theorem in the lecture notes of 2.2 implies that f has a limit at x_0. $\square$

Exercise 3. (5 pts) Prove that if $f : D \subseteq \mathbb{R} \to \mathbb{R}$ has a limit at $x_0 \in \operatorname{acc} D$, then the limit is unique.

Solution: Suppose towards a contradiction that the limit is not unique. Let $\lim_{x\to x_0} f(x) = L$ and $\lim_{x\to x_0} f(x) = M$ where $L \neq M$. Since $L \neq M$, we can say that $\frac{|L-M|}{2} > 0$. Now set $\epsilon = \frac{|L-M|}{2}$. By the definition of the limit, $\exists \delta_1$ such that if $|x - x_0| < \delta_1$, $|f(x) - L| < \frac{|L-M|}{2}$. Similarly, we can say that $\exists \delta_2$ such that if $|x - x_0| < \delta_2$, $|f(x) - M| < \frac{|L-M|}{2}$. Now let $\delta_3 = min\{\delta_1, \delta_2\}$. Now let $\delta_3 = min\{\delta_1, \delta_2\}$. Since $|f(x)-L| < \frac{|L-M|}{2}$ for $|x-x_0| < \delta_1$, and $|f(x)-M| < \frac{|L-M|}{2}$ for $|x-x_0| < \delta_2$, we can say that $|f(x) - L| + |f(x) - M| < \frac{|L-M|}{2} + \frac{|L-M|}{2} = |L - M|$ for $|x - x_0| < \delta_3$. Note that $|f(x)-L| = |L-f(x)|$, so $|f(x)-L|+|f(x)-M| < |L-M| = |L-f(x)|+|f(x)-M| < |L-M|$. By the triangle inequality, $|L-f(x)+f(x)-M| \leq |L-f(x)|+|f(x)-M| < |L-M|$ which simplifies to $|L-M| \leq |L-f(x)|+|f(x)-M| < |L-M|$, which by transitivity implies $|L-M| < |L-M|$, which is a contradiction, contradicting our claim that the limit is not unique. Therefore if $f : D \subseteq \mathbb{R} \to \mathbb{R}$ has a limit at $x_0 \in \operatorname{acc} D$, we can say that the limit is unique. $\square$

Exercise 4. (5 pts) Suppose $f : D \subseteq \mathbb{R} \to \mathbb{R}$, $g : D \subseteq \mathbb{R} \to \mathbb{R}$ and $h : D \subseteq \mathbb{R} \to \mathbb{R}$ are three functions such that

$$f(x) \leq h(x) \leq g(x) \quad (\forall x \in D).$$

Suppose that f and g have limits at x_0 with $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x)$. Prove that h has a limit at x_0 and

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} h(x) = \lim_{x\to x_0} g(x).$$

Solution: Suppose that the limit of $f(x), h(x)$ at x_0 is L. By the definition of the limit, since $\lim_{x\to x_0} f(x)$ exists, for all $\epsilon > 0$, $\exists \delta_1$ such that if $|x - x_0| < \delta_1$, then $|f(x) - L| < \epsilon$. This can be simplified to $-\epsilon < f(x) - L < \epsilon$ into $= \epsilon + L < f(x) < \epsilon + L$. Similarly, since $\lim_{x\to x_0} g(x)$ exists and is equal to $\lim_{x\to x_0} f(x)$, for all $\epsilon > 0$, $\exists \delta_2$ such that if $|x - x_0| < \delta_2$, then $|g(x) - L| < \epsilon$, so $-\epsilon + L < g(x) < \epsilon + L$. Now let $\delta_3 = min\{\delta_1, \delta_2\}$. Therefore whenever $0 < |x - x_0| < \delta_3$, $-\epsilon + L < f(x) \leq h(x) \leq g(x) < \epsilon + L$, which by transitivity order axioms imply $-\epsilon + L < h(x) < \epsilon + L$, which can be simplified to $-\epsilon < h(x) - L < \epsilon$ into $|h(x) - L| < \epsilon$ whenever $0 < |x - x_0| < \delta_3$. This is the definition of the limit, and therefore we can say that $\lim_{x\to x_0} h(x) = L = \lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x)$.

Exercise 5. (10 pts) Let $f : (0, \infty) \to \mathbb{R}$ be a function. We say that f has a limit at ∞ if there exists a $L \in \mathbb{R}$ such that for any $\varepsilon > 0$, there is a real number $M > 0$ such that if $x > M$, then $|f(x) - L| < \varepsilon$.

a) Show that if $g : (0, \infty) \to \mathbb{R}$ is bounded and $\lim_{x\to\infty} f(x) = 0$, then $\lim_{x\to\infty} f(x)g(x) = 0$.

b) Let $a > 0$ and suppose that $f : (a, \infty) \to \mathbb{R}$ and define $g : (0, 1/a) \to \mathbb{R}$ by $g(x) = f(1/x)$. Show that f has a limit at ∞ if and only if g has a limit at 0.

Solution: a. By AC, since g is bounded, by the definition of being bounded we can say that $|g(x)| \leq M$ for some $M \in \mathbb{R}$. Now let $\epsilon_1 = \frac{\epsilon}{M}$. Since the limit of $f(x)$ at ∞ is 0, by the definition of the limit, $\exists \delta$ such that if $0 < |x - x_0| < \delta$, $|f(x)| < \epsilon_1 = \frac{\epsilon}{M}$. Now consider $f(x)g(x)$. Note that by the properties of the absolute value function, $|f(x)||g(x)| = |f(x)g(x)|$. Therefore since $g(x) \leq M$, and $|f(x)| < \frac{\epsilon}{a}$ for δ such that $0 < |x - x_0| < \delta$, we can say that $|f(x)||g(x)| = |f(x)g(x)| < M(\frac{\epsilon}{M})$. This simplifies to $|f(x)g(x)| < \epsilon$ for δ such that $0 < |x - x_0| < \delta$, which is the definition of the limit, and therefore we can say that if $\lim_{x\to\infty} f(x) = 0$ and g is bounded, then $\lim_{x\to\infty} f(x)g(x) = 0$.
b.
I wrote two proofs since I'm not sure if the first one is rigorous enough
Proof 1 -
Note that since $g(x) = f(1/x)$, $\lim_{x\to x_0} g(x) = \lim_{x\to x_0} f(1/x)$. Therefore if f has a limit at ∞ then since $g(x) = f(1/x)$, since $1/0 = \infty$ at $x = 0$ we have $g(0) = f(\infty)$. Therefore since f has a limit at ∞, g must have a limit at 0 since $\lim_{x\to 0} g(x) = \lim_{x\to\infty} f(1/x)$. In the other direction, if g has a limit at 0, then since $g(x) = f(1/x)$, $g(0) = f(1/0) = f(\infty)$. Therefore we have $\lim_{x\to 0} g(x) = \lim_{x\to\infty} f(1/x)$ so f must have a limit at infinity if g has a limit at 0.
Proof 2 -
($\to$) Note that if f has a limit at ∞ we can say that $L \in \mathbb{R}$ such that for any $\varepsilon > 0$, there is a real number $M > 0$ such that if $x > M$, then $|f(x) - L| < \varepsilon$. Note that since $x > M$, we can say that $1/x < M$. Therefore if $1/x < M$, we have $|f(1/x) - L| < \epsilon$ for some real number M. Since $g(x) = f(1/x)$, we can substitute in $g(x)$ to get $|g(x) - L| < \epsilon$ for $1/x < M$. Now take $\delta = M$, $x_1 = 1/x$ and $x_0 = 0$. The previous statement now implies that for $L \in bR$ where $\forall \epsilon > 0$, we have $|x_1 - 0| < \delta$ implies $|g(x) - L| < \epsilon$, which is the definition of a limit at x_0, therefore $g(x)$ has a limit at 0 if f has a limit at ∞.
($\leftarrow$) By the definition of the limit, since $g(x)$ has a limit at 0, we can say that for $|x - 0| = |x| < \delta$, $|g(x) - L| < \epsilon$. Since $g : (0, 1/a) \to \mathbb{R}$, and $a > 0$ we can say that $x > 0$. Therefore $|x| < \delta$ implies $x < \delta$, and $1/x > 1/\delta$. Also since $g(x) = f(1/x)$, we can say that $|g(x) - L| = |f(1/x) - L|$. Therefore we can say that $|x| < \delta, |g(x) - L| < \epsilon$, implies that if $1/x > 1/\delta$, $|f(1/x| - L| < \epsilon$, which by the definition given to us at exercise 5 in this homework, since $\delta \in \mathbb{R}$, this implies that f has a limit at ∞. Therefore if g has a limit at 0, we can say that f has a limit at ∞. $\square$

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Homework problems

Answer all the questions below. Make sure to show your work.

Exercise 6. (10pts) For each of the sequences below, determine its nature (converges or diverges)[1]:

[1] You don't need to compute the limit.

a) (a_n) where $a_n = \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n}$.

b) (a_n) where $a_n = \frac{1+2+\cdots+n}{n^2}$.

Solution: a. Note that a_n is decreasing, as $\frac{1}{2n} \leq \cdots \leq \frac{1}{n+1} \leq \frac{1}{n}$. Therefore the first term of the sequence, will be the upper bound. For the lower bound, note that $\frac{1}{n} \to 0$ as n goes to infinity. Also note that all the terms of the sequence converge to 0 as n goes to infinity as since we have $\frac{1}{n} \to 0$, $|\frac{1}{n}| < \epsilon$ for $n \geq N$. Therefore $|\frac{1}{2n}| \leq \cdots \leq |\frac{1}{n+1}| \leq |\frac{1}{n}| < \epsilon$ for $n \geq N$, and since the sum of sequences converge to the sum of their limit, we can say that the entire sequence converges to 0 and therefore 0 is a lower bound. Since the sequence has an upper and lower bound, we can say that is it bounded. Therefore because the sequence is bounded, and is monotone as it is decreasing, by theorem 1.16 in the textbook, the sequence is convergent.
b. Note that a_n can be rewritten as $\frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{n}{n^2}$. Also note that a_n is increasing, as $\frac{1}{n^2} \leq \frac{2}{n^2} \leq \cdots \leq \frac{n}{n^2}$. Therefore the first term of the sequence is the lower bound. Also note that as n goes to infinity, all the terms of the sequence go to 0 since $\frac{1}{n^2}, \frac{2}{n^2}, \cdots, \frac{1}{n}$ are multiples of $\frac{1}{n}$, and $\frac{1}{n} \to 0$. Therefore the sum of all the terms of the sequence is 0, and so 0 is an upper bound. Since a_n has both an upper and lower bound, we can say that it is bounded. Since the a_n is bounded and monotone as it is increasing, by theorem 1.16 in the textbook, the sequence is convergent. $\square$

Exercise 7. (5 pts) Define $g : (0,1) \to \mathbb{R}$ by $f(x) = \frac{\sqrt{1+x}-1}{x}$. Prove that g has a limit at 0 and find it.

Solution: $f(x) = \frac{\sqrt{1+x}-1}{x}$ can be simplified to $f(x) = \frac{\sqrt{1+x}-1}{x} * \frac{\sqrt{1+x}+1}{\sqrt{1+x}+1} = \frac{1+x-1}{x(\sqrt{1+x}+1)} = \frac{x}{x(\sqrt{1+x}+1)} = \frac{1}{\sqrt{1+x}+1}$ Since $\lim_{x\to 0} \sqrt{1+x}+1 = 1+0+1 = 2$, and $\lim_{x\to 0} 1 = 1$, $\lim_{x\to 0} \frac{1}{\sqrt{1+x}+1} = \frac{1}{2}$. $\square$

Exercise 8. (5 pts) Suppose that $f : (0,1) \to \mathbb{R}$ has a limit at $x_0 = 1$ and $\lim_{x\to 1} f(x) = 1$. Compute the value of the limit

$$\lim_{x\to 1} \frac{f(x)(1-f(x)^2)}{1-f(x)}.$$

Solution: $\lim_{x\to 1} \frac{f(x)(1-f(x)^2)}{1-f(x)}$ can be expressed as $\lim_{x\to 1} \frac{f(x)(1+f(x))(1-f(x))}{1-f(x)}$. Now we can factor out $1 - f(x)$ to get $\lim_{x\to 1} \frac{f(x)(1+f(x))}{1}$. Since $\lim x \to 1 f(x) = 1, \lim_{x\to 1} \frac{f(x)(1+f(x))}{1} = \frac{1(1+1)}{1} = 2$. Therefore $\lim_{x\to 1} \frac{f(x)(1-f(x)^2)}{1-f(x)} = 2$ $\square$

Exercise 9. (5 pts) Prove that if $f : D \to \mathbb{R}$ has a limit at x_0, then $|f|(x) := |f(x)|$ has a limit at x_0.

Solution: If $f : D \to \mathbb{R}$ has a limit at x_0, then by the definition of the limit, for all $\epsilon > 0$, $\exists\delta$ such that if $|x - x_0| < \delta$, then $|f(x) - L| < \epsilon$. Note that by the triangle inequalities, $||f(x)| - |L|| \leq |f(x) - L| < \epsilon$ which by transitivity implies $||f(x)| - |L|| < \epsilon$, so therefore we can say that for all $\epsilon > 0$, $\exists\delta$ such that if $|x - x_0| < \delta$, $||f(x)| - |L|| < \epsilon$, which is the definition of the limit. Therefore we can say that if $f(x)$ has a limit, L, at x_0, $|f(x)|$ has a limit at x_0, and it is equal to $|L|$. $\square$

Exercise 10. (10 pts) Using the link between sequences and limits of functions, show the following.

a) If $f(x) = x^n$ $(n \geq 0)$, then $\lim_{x \to x_0} f(x) = x_0^n$ for any $x_0 \in \mathbb{R}$.

b) If $x_0 \in [0, \infty)$, then $\lim_{x \to x_0} \sqrt{x} = \sqrt{x_0}$.

Solution: a. First note that $x \to x_0$ for $x_0 \in \mathbb{R}$. Since the multiple of sequences converges to the multiple of their limits, we can multiply x n times to get $x^n \to x_0^n$. Therefore since $f(x) = x^n$, we can say that $\lim_{x \to x_0} f(x) = x_0^n$ for any $x_0 \in \mathbb{R}$.
b. First note that $x \to x_0$ for $x_0 \in \mathbb{R}$. We know from the proof in homework 2 number 4 that if $a_n \to A$, $\sqrt{a_n} \to \sqrt{A}$, as long as A is non negative. Since $x_0 \in [0, \infty)$ implies that it's positive, we can say that $\sqrt{x} \to \sqrt{x_0}$, and therefore $\lim_{x \to x_0} \sqrt{x} = \sqrt{x_0}$. $\square$

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Bonus

Exercise 11. Assume that $f : \mathbb{R} \to \mathbb{R}$ such that $f(x + y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$.

a) Show that f has a limit at every point of $\mathbb{R}$.

b) Show that either $\lim_{x \to 0} f(x) = 1$ or $f(x) = 0$ for any $x \in \mathbb{R}$.

Solution: a. Note from b that either $\lim_{x \to 0} f(x) = 1$ or $f(x) = 0$ for any $x \in \mathbb{R}$. If $f(x) = 0$, then $f(x + y) = f(x)f(y) = 0f(y) = 0$, which would mean the limit of $f(x + y)$ will be 0. WLOG we can say the same if $f(y) = 0$. If instead $\lim_{x \to 0} f(x) = 1$ and $\lim_{y \to 0} f(y) = 1$, then $f(x + y)$ will have a limit of 1 as $\lim_{x \to 0} f(x) \lim_{y \to 0} f(y) = 1(1) = 1$.
b. Suppose $\lim_{x \to 0} f(x) = a$, where $a \neq 1$. Then by the definition of f, we can say that $f(x) = f(x + 0) = f(x)f(0) = af(x)$. Since $a \neq 1$, we can say that $f(x) = af(x)$ is only true if $f(x) = 0$. Therefore if $\lim_{x \to 0} f(x) = a$, where $a \neq 1$, $f(x) = 0$. Now suppose in the opposite direction that $f(x) \neq 0$. We can use a similar argument to show that $f(x) = f(x + 0) = f(x)f(0)$, so $f(x) = f(x)f(0)$ which is only true if $\lim_{x \to 0} f(x) = 1$. Therefore we can say that for the definition of f provided, for any $x \in \mathbb{R}$ we have either $\lim_{x \to 0} f(x) = 1$ or $f(x) = 0$. $\square$