MATH-331 Introduction to Real Analysis | Ian Oga (24445161)
Homework 02 | Fall 2021

Due date: 20-09-2021 1:20pm | Total: /70.

Exercise	1 (10)	2 (5)	3 (5)	4 (5)	5 (5)	6 (10)	7 (5)	8 (10)	9 (5)	10 (10)
Score										

Table 1: Scores for each exercises

Instructions: You must answer all the questions below and send your solution by email (to `parisepo@hawaii.edu`). If you decide to not use LaTeX to hand out your solutions, please be sure that after you scan your copy, it is clear and readable. Make sure that you attached a copy of the homework assignment to your homework. No late homework will be accepted. No format other than PDF will be accepted. Name your file as indicated in the syllabus.

1 Writing problems

For each of the following problems, you will be asked to write a clear and detailed proof. You will have the chance to rewrite your solution in your semester project after receiving feedback from me.

Exercise 1. (10 pts)

a) Let $\{[a_n, b_n] : n \geq 1\}$ be a family of closed intervals such that $[a_1, b_1] \supset [a_2, b_2] \supset [a_3, b_3] \supset \cdots$. Show that there is a $c \in \mathbb{R}$ such that $c \in [a_n, b_n]$ for all $n \geq \mathbb{N}$. Follow the following steps to prove it:

- **(i)** Prove that for any $n, m \geq 1$, $a_n \leq b_m$. [hint: put $M := \max\{n, m\}$.]
- **(ii)** Show that $\sup\{a_n : n \geq 1\}$ exists.
- **(iii)** Show that $c = \sup\{a_n : n \geq 1\}$ satisfies the requirement.

b) Use this last result to prove that the set $\mathbb{R}$ is uncountable. [Hint: Show that any function $f : \mathbb{N} \to \mathbb{R}$ can't be surjective. To do so, construct a sequence of closed intervals such that $f(n) \notin [a_n, b_n]$ with $a_n < b_n$.]

Solution:

a) **(i)** Define $M = \max\{n, m\}$. Note that for $y \geq x$, $[a_x, b_x] \supseteq [a_y, b_y]$. This implies that $a_y, b_y \in [a_x, b_x]$ and $a_x \leq a_y \leq b_y \leq b_x$. Since $M \geq n$ and $M \geq m$, $a_n \leq a_M$, $b_M \leq b_m$, and $a_n \leq a_M \leq b_M \leq b_m$. Therefore $a_n \leq b_m$.

(ii) Since $a_n \leq b_m$ for all $n, m \geq 1$, $a_n \leq b_1$ for all $n \geq 1$ This proves that $\{a_n : n \geq 1\}$ is bounded from above. As $\{a_n : n \geq 1\}$ is non-empty, this set must have a supremum by the Axiom of Completeness.

(iii) By the definition of supremum, $c \geq a_n$ for all $n \geq 1$. Now suppose towards a contradiction that there exists $b_x < c$ for $x \geq 1$. Since $a_n \leq b_m$ for all $n, m \geq 1$, $a_n \leq b_x$ for all $n \geq 1$. This would make b_x an upper bound for $\{a_n : n \geq 1\}$ that is less than c, which is a contradiction. Therefore $c \leq b_n$ for all $n \geq 1$. As $a_n \leq c \leq b_n$ for all $n \geq 1$, $c \in [a_n, b_n]$ for all $n \geq 1$ which satisfies the requirement.

b) Suppose towards a contradiction that $\mathbb{R}$ is countable. Then there exists a bijection $f : \mathbb{N} \to \mathbb{R}$. Define interval $[a_1, b_1] = [f(1) - 2, f(1) - 1]$. We now define a sequence of intervals $[a_n, b_n]$ as follows for $n > 1$:
$[a_n, b_n] = [a_{n-1}, b_{n-1}]$ for $f(n) \notin [a_{n-1}, b_{n-1}]$
$[a_n, b_n] = [\frac{f(n)+b_{n-1}}{2}, b_{n-1}]$ for $f(n) \in [a_{n-1}, b_{n-1}]$ and $f(n) \neq b_{n-1}$
$[a_n, b_n] = [a_{n-1}, \frac{a_{n-1}+b_{n-1}}{2}]$ for $f(n) = b_{n-1}$
Note how the following properties are true for all n:
$f(n) \notin [a_n, b_n]$
$a_n < b_n$
$[a_n, b_n] \supseteq [a_m, b_m]$ for $n > m$
From part a, we know that there exists some $c \in [a_n, b_n]$ for all $n \geq \mathbb{N}$. Now suppose towards a contradiction that there exists a $f(N) = c$. We know that $f(N) \notin [a_N, b_N]$, but $c \in [a_N, b_N]$, which is a contradiction. Since no $f(N) = c$, $f : \mathbb{N} \to \mathbb{R}$ is not surjective, which is a contradiction to f being a bijection. Therefore $\mathbb{N}$ is not countable. $\square$

Exercise 2. (5 pts) Prove that if $a_n \to A$, then $|a_n| \to |A|$.

Solution: Let $\varepsilon > 0$ be arbitrary. Since $a_n \to A$, there exists $N \in \mathbb{N}$ such that $|a_n - A| < \varepsilon$ for all $n \geq N$. By the reverse triangle inequality, $||a_n| - |A|| < \varepsilon$ for all $n \geq N$. Since $\varepsilon > 0$ is arbitrary, $|a_n| \to |A|$. $\square$

Exercise 3. (5 pts) Let (a_n), (b_n), and (c_n) be sequences of real numbers. Prove that if $a_n \to L$, $b_n \to L$, and $a_n \leq c_n \leq b_n$, then $c_n \to L$.

Solution: Let $\varepsilon > 0$ be arbitrary. As $a_n \to L$ and $b_n \to L$, there exists $N_1, N_2 \in \mathbb{N}$ such that $|a_n - L| < \varepsilon$ for all $n \geq N_1$ and $|b_n - L| < \varepsilon$ for all $n \geq N_2$. Define $N = \max(N_1, N_2)$. Then $|a_n - L| < \varepsilon$ and $|b_n - L| < \varepsilon$ for all $n \geq N$. We then have the following for $n \geq N$:
$-\varepsilon < a_n - L < \varepsilon$ and $-\varepsilon < b_n - L < \varepsilon$
As $a_n \leq c_n \leq b_n$, we have $a_n - L \leq c_n - L \leq b_n - L$
$-\varepsilon < a_n - L \leq c_n - L \leq b_n - L < \varepsilon$
$-\varepsilon < c_n - L < \varepsilon$
$|c_n - L| < \varepsilon$
As $\varepsilon > 0$ is arbitrary and $|c_n - L| < \varepsilon$ for all $n \geq N$, $c_n \to L$. $\square$

Exercise 4. (5 pts) Prove that if $a_n \to A$ and $a_n \geq 0$ for all $n \geq 1$, then $\sqrt{a_n} \to \sqrt{A}$. Follow the following steps to prove it:

1. Consider the case $A = 0$.

2. Suppose that $A \neq 0$. Show that there is a $N_1 \in \mathbb{N}$ such that if $n \geq N_1$, then $\sqrt{a_n} \geq \sqrt{|A|/2}$. [Hint: use the definition of convergence of $(a_n)_{n\geq 0}$ with a clever choice of ε and use the properties of the absolute value.]

3. Use the convergence of (a_n) again to find a N_2 such that $|a_n - A| < \frac{3}{4}\frac{\varepsilon}{\sqrt{|A|}}$.

4. Express $\sqrt{a_n} - A$ as $\frac{a_n - A}{\sqrt{a_n}+\sqrt{A}}$ and put $N = \max\{N_1, N_2\}$. Conclude.

Solution:

1. Suppose $A = 0$. Let $\varepsilon > 0$ be arbitrary. Since $a_n \to 0$, there exists $N \in \mathbb{N}$ such that $|a_n| < \varepsilon$ for all $n \geq N$. Since a_n and ε are positive, $a_n < \varepsilon$ and $\sqrt{a_n} < \sqrt{\varepsilon}$. Then $|\sqrt{a_n} - 0| < \sqrt{\varepsilon}$ for all $n \geq N$ and $\sqrt{a_n} \to 0$.

2. Suppose $A \neq 0$. Let $\varepsilon > 0$ be $\frac{|A|}{2}$. There exists $N_1 \in \mathbb{N}$ such that $|a_n - A| < \frac{|A|}{2}$ for all $n \geq N$. Then for all $n \geq N_1$:
$||a_n| - |A|| < \frac{|A|}{2}$
$-\frac{|A|}{2} < |a_n| - |A| < \frac{|A|}{2}$
$|a_n| - |A| > -\frac{|A|}{2}$
$|a_n| > \frac{|A|}{2}$
$\sqrt{a_n} > \sqrt{\frac{|A|}{2}}$

3. Let $\varepsilon > 0$ be arbitrary. Since $a_n \to A$, there exists $N_2 \in \mathbb{N}$ such that $|a_n - A| < \frac{3}{4}\frac{\varepsilon}{\sqrt{|A|}}$ for all $n \geq N_2$.

4. Note that $\sqrt{a_n} - \sqrt{A} = \frac{a_n - A}{\sqrt{a_n}+\sqrt{A}}$ and define $N = \max(N_1, N_2)$. Since $\sqrt{a_n} > \sqrt{\frac{|A|}{2}}$, we have the following for $n \geq N$:
$\sqrt{a_n} + \sqrt{A} > \sqrt{|A|}(1 + \frac{1}{\sqrt{2}})$
$\frac{1}{\sqrt{a_n}+\sqrt{A}} < \frac{1}{\sqrt{|A|}(1+\frac{1}{\sqrt{2}})}$
Then for $n \geq N$:
$|\sqrt{a_n} - \sqrt{A}| = |\frac{a_n - A}{\sqrt{a_n}+\sqrt{A}}|$
$|\sqrt{a_n} - \sqrt{A}| = \frac{1}{\sqrt{a_n}+\sqrt{A}}|a_n - A|$
$|\sqrt{a_n} - \sqrt{A}| < \frac{1}{(1+\frac{1}{\sqrt{2}})\sqrt{|A|}}(\frac{3}{4}\frac{\varepsilon}{\sqrt{|A|}})$
$|\sqrt{a_n} - \sqrt{A}| < (\frac{3\sqrt{2}}{4\sqrt{2}+4})\frac{\varepsilon}{|A|}$
I do not understand why we made the choice to have $|a_n - A| < \frac{3}{4}\frac{\varepsilon}{\sqrt{|A|}}$ in part 3 since it leaves $|A|$ in the denominator. You could define N_2 instead as being such that $|a_n - A| < \varepsilon(1 + \frac{1}{\sqrt{2}})\sqrt{|A|}$ for all $n \geq N_2$ and get the following for $n \geq N = \max(N_1, N_2)$:
$|\sqrt{a_n} - \sqrt{A}| = \frac{1}{\sqrt{a_n}+\sqrt{A}}|a_n - A|$
$|\sqrt{a_n} - \sqrt{A}| < \frac{1}{(1+\frac{1}{\sqrt{2}})\sqrt{|A|}}(\varepsilon(1 + \frac{1}{\sqrt{2}})\sqrt{|A|})$
$|\sqrt{a_n} - \sqrt{A}| < \varepsilon$
This proves that $\sqrt{a_n} \to \sqrt{A}$. $\square$

Exercise 5. (5 pts) For each sequence $(a_n)_{n=1}^{\infty}$, define the sequence $(\sigma_n)_{n=1}^{\infty}$ by

$$\sigma_n := \frac{a_1 + a_2 + \cdots + a_n}{n} \quad (n \geq 1).$$

Prove that if $a_n \to A$, then $\sigma_n \to A$. Find an example of a divergent sequence (a_n) such that $(\sigma_n)_{n=1}^{\infty}$ converges.

Solution: Suppose $a_n \to A$ and let ε be arbitrary. Then there exists $N_1 \in \mathbb{N}$ such that $|a_n - A| < \frac{\varepsilon}{2}$ for all $n \geq N_1$. Since a_n converges, $a_n - A$ and $|a_n - A|$ converge by Exercise 2. Sequence $|a_n - A|$ is then bounded. Define $M > 0$ so that $\forall n \geq 1, |a_n - A| \leq M$ for all n. Then for all $n \geq N_1$:
$|\sigma_n - A| = |\frac{a_1+a_2+\cdots+a_n}{n} - A|$
$|\sigma_n - A| = |\frac{a_1+a_2+\cdots+a_n-An}{n}|$
$|\sigma_n - A| = |\frac{(a_1-A)+(a_2-A)+\cdots+(a_n-A)}{n}|$
$|\sigma_n - A| \leq \frac{1}{n}(|a_1 - A| + |a_2 - A| + \cdots + |a_n - A|)$
$|\sigma_n - A| \leq \frac{1}{n}(|a_1 - A| + |a_2 - A| + \cdots + |a_n - A|)$
$|\sigma_n - A| \leq \frac{1}{n}(M(N-1) + |a_{N_1} - A| + |a_{N_1+1} - A| + \cdots + |a_n - A|)$
$|\sigma_n - A| < \frac{1}{n}(M(N_1 - 1) + \frac{\varepsilon}{2}(n - N_1))$
$|\sigma_n - A| < \frac{M(N_1-1)}{n} + \frac{\varepsilon}{2}(\frac{n-N_1}{n})$
Since $N_1, n > 0$, $n - N_1 < n$ and $\frac{n-N_1}{n} < 1$. Therefore $\frac{\varepsilon}{2}(\frac{n-N_1}{n}) < \frac{\varepsilon}{2}$. By the Archimedean Principle, we also know that there exists $N_2 \in \mathbb{N}$ such that $N_2\frac{\varepsilon}{2} > M(N_1 - 1)$. We can also choose N_2 such that $N_2 \geq N_1$. Therefore $\frac{M(N_1-1)}{n} < \frac{\varepsilon}{2}$ for all $n \geq N_2$. Combining these, for all $n \geq N_2 \geq N_1$:
$|\sigma_n - A| < \frac{M(N_1-1)}{n} + \frac{\varepsilon}{2}(\frac{n-N_1}{n})$
$|\sigma_n - A| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$
$|\sigma_n - A| < \varepsilon$
As ε is arbitrary, $\sigma_n \to A$. □

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Homework problems

Exercise 6. (10 pts) Use the definition of convergence to prove that each of the following sequences converges.

a) $(a_n)_{n=1}^{\infty}$ given by $a_n = 5 + 1/n$ for $n \geq 1$.

b) $(a_n)_{n=1}^{\infty}$ given by $a_n = \frac{3n}{2n+1}$ for $n \geq 1$.

Solution:

a) Let $A = 5$ and ε be arbitrary. By the Archimedean Principle, we know that there exists an $N \in \mathbb{N}$ such that $N\varepsilon > 1$ and therefore $\varepsilon > \frac{1}{N}$. We want to prove that $|a_n - A| < \varepsilon$ for all $n \geq N$. Note that for all $n \geq N$:
$|a_n - A| = |(5 + \frac{1}{n}) - 5|$
$|a_n - A| = |\frac{1}{n}|$
$|a_n - A| = \frac{1}{n}$
$|a_n - A| \leq \frac{1}{N}$
$|a_n - A| < \varepsilon$
Therefore $a_n \to 5$.

b) Let $A = \frac{3}{2}$ and ε be arbitrary. Let $X = \frac{1}{\varepsilon} - 0.5$. We know from Theorem 0.21 that there exists $N \in \mathbb{N}$ such that $N \geq X$. Note that for all $n \geq N$:
$n \geq X$
$2n + 1 \geq 2X + 1$
$\frac{1}{2n+1} \leq \frac{1}{2X+1}$
$\frac{1.5}{2n+1} \leq \frac{1.5}{2X+1}$
$\frac{1.5}{2n+1} \leq \frac{1.5}{2*(\frac{1}{\varepsilon}-0.5)+1}$
$\frac{1.5}{2n+1} \leq 0.75 * \varepsilon$
$\frac{1.5}{2n+1} < \varepsilon$
We want to prove that $|a_n - A| < \varepsilon$ for all $n \geq N$. Note that for all $n \geq N$:
$|a_n - A| = |\frac{3n}{2n+1} - \frac{3}{2}|$
$|a_n - A| = |\frac{3n}{2n+1} - \frac{3n+1.5}{2n+1}|$
$|a_n - A| = |\frac{-1.5}{2n+1}|$
$|a_n - A| = \frac{1.5}{2n+1}$
$|a_n - A| < \varepsilon$
Therefore $a_n \to \frac{3}{2}$. □

Exercise 7. (5 pts) Prove that the sequence $(a_n)_{n=1}^{\infty} = \left(\frac{2n+1}{n}\right)_{n=1}^{\infty}$ is a Cauchy sequence.

Solution: We want to prove that a_n converges since all converging sequences are Cauchy by Theorem 1.3. Let $A = 2$ and ε be arbitrary. By the Archimedean Principle, we know that there exists an $N \in \mathbb{N}$ such that $N\varepsilon > 1$ and therefore $\varepsilon > \frac{1}{N}$. We want to prove that $|a_n - A| < \varepsilon$ for all $n \geq N$. Note that for all $n \geq N$:
$|a_n - A| = |\frac{2n+1}{n} - 2|$
$|a_n - A| = |\frac{2n+1-2n}{n}|$
$|a_n - A| = |\frac{1}{n}|$
$|a_n - A| = \frac{1}{n}$
$|a_n - A| \leq \frac{1}{N}$
$|a_n - A| < \varepsilon$
Therefore a_n converges and is Cauchy. □

Exercise 8. (10 pts) Prove that each of the following sequence diverges.

a) $(a_n)_{n=1}^{\infty} = ((-1)^n)_{n=1}^{\infty}$.

b) $(a_n)_{n=1}^{\infty} = (\sin(\frac{2n+1}{2}\pi))_{n=1}^{\infty}$.

Solution:

a) Suppose towards a contradiction that a_n converges. This means that $\exists A \in \mathbb{R}, \forall \varepsilon \in \mathbb{R}, \exists N \in \mathbb{N}, \forall n \geq N, |a_n - A| < \varepsilon$. Let A and N be arbitrary and $\varepsilon = 1$. Note that if N is even, $a_N = 1$ and $a_{N+1} = -1$. Similarly, if N is odd, $a_N = -1$ and $a_{N+1} = 1$. Regardless of N, we must have $|1 - A| < \varepsilon$ and $|-1 - A| < \varepsilon$. Therefore:
$|1 - A| < 1$ and $|-1 - A| < 1$
$-1 < 1 - A < 1$ and $-1 < -1 - A < 1$

$-2 < -A < 0$ and $0 < -A < 2$
$0 < A < 2$ and $-2 < A < 0$
A cannot satisfy both of these requirements at once. This is a contradiction. Therefore a_n does not converge and must diverge.

b) If n is even we can express n as $2k$ for integer k. Then $a_n = \sin(\pi\frac{4k+1}{2}) = \sin(2k\pi + \frac{\pi}{2}) = 1$. If n is odd we can express n as $2k+1$ for integer k. Then $a_n = \sin(\pi\frac{4k+2+1}{2}) = \sin(2k\pi + \frac{3\pi}{2}) = -1$. This means that for all integers n, $a_n = (-1)^n$ which we know diverges. Since convergence is only concerned with integer indices of sequences, a_n must diverge. □

Exercise 9. (5 pts) Give an examples of two sequences (a_n) and (b_n) such that (a_n) and (b_n) don't converge, but $(a_n + b_n)$ converge.

Solution: From Exercise 8, we know that the sequence $(a_n)_{n=1}^{\infty} = ((-1)^n)_{n=1}^{\infty}$ diverges. We know that sequence αa_n converges iff a_n converges for $\alpha \in \mathbb{R}$. By the contrapositive, since a_n diverges, $(b_n)_{n=1}^{\infty} = (-a_n)_{n=1}^{\infty}$ also diverges. We then have:
$(a_n + b_n)_{n=1}^{\infty} = (a_n + (-a_n))_{n=1}^{\infty}$
$(a_n + b_n)_{n=1}^{\infty} = (0)_{n=1}^{\infty}$
As this sequence is a constant at 0, $(a_n + b_n)$ clearly converges. □

Exercise 10. (10 pts) With the limit operations and the writing problems, find the limit of the following sequence with general term

a) $\frac{n^2+4n}{n^2-5}$.

b) $\frac{n}{n^2-3}$.

c) $\frac{\cos n}{n}$. [You can use what you know on the cosine function.]

d) $\left(\sqrt{4-\frac{1}{n}}-2\right)n$.

Solution: For these problems, call the sequence f_n

a) Note that $f_n = \frac{1+(4/n)}{1-(5/n^2)}$. Now define sequences $a_n = 1 + \frac{4}{n}$ and $b_n = 1 - \frac{5}{n^2}$. Since sequence $\frac{1}{n}$ converges to 0, $\frac{4}{n}$ converges to 0 and $a_n \to 1$. Similarly, since sequence $\frac{1}{n^2}$ converges to 0, $\frac{-5}{n^2}$ converges to 0 and $b_n \to 1$. We know that for $a_n \to A$ and $b_n \to B$, $\frac{a_n}{b_n} \to \frac{A}{B}$. Therefore $f_n \to 1$.

b) Note that $f_n = \frac{1/n}{1-(3/n^2)}$. Now define sequences $a_n = \frac{1}{n}$ and $b_n = 1 - \frac{3}{n^2}$. Similar to part a, we can see that $a_n \to 0$ and $b_n \to 1$. Therefore $f_n \to \frac{0}{1} = 0$.

c) We will prove that the limit is 0. To do this, we will prove that for arbitrary ε, there exists $N \in \mathbb{N}$ such that $|f_n| < \varepsilon$ for all $n \geq N$. Note that since $-1 \leq \cos(n) \leq 1$, $|\cos(n)| \leq 1$ and $|\frac{\cos(n)}{n}| \leq \frac{1}{n}$. Therefore:
$|f_n| = |\frac{\cos(n)}{n}|$
$|f_n| \leq \frac{1}{n}$
By the Archimedean Principle, we know that there exists N such that $1 < N\varepsilon$. Then $\frac{1}{n} < \varepsilon$

for all $n \geq N$. We then have that for all $n \geq N$:
$|f_n| \leq \frac{1}{n}$
$|f_n| < \varepsilon$
This proves that the limit of f_n is 0.

d) Note that for $f_n = \left(\sqrt{4 - \frac{1}{n}} - 2\right)n$:
$f_n = \frac{\sqrt{4-\frac{1}{n}}-2}{\frac{1}{n}}$
$f_n = \frac{\sqrt{4-\frac{1}{n}}-2}{\frac{1}{n}}(\frac{\sqrt{4-\frac{1}{n}}+2}{\sqrt{4-\frac{1}{n}}+2})(\frac{n}{n})$
$f_n = \frac{(4-\frac{1}{n}-4)n}{\sqrt{4-\frac{1}{n}}+2}$
$f_n = \frac{-1}{\sqrt{4-\frac{1}{n}}+2}$
Now consider the sequence $a_n = \sqrt{4 - \frac{1}{n}}+2$. Since sequence $\frac{1}{n}$ converges to 0, $4-\frac{1}{n}$ converges to 4. From Exercise 4, the sequence $\sqrt{4 - \frac{1}{n}}$ converges to $\sqrt{4} = 2$, and thus $\sqrt{4 - \frac{1}{n}} + 2$ converges to 4. Since the denominator of f_n converges to 4, $f_n \to \frac{-1}{4}$ $\square$