Sequences

- Convergences.
- Operations.
- Cauchy Sequences.
- Subsequences & Monotone sequences

2- Sequences

2.1. Convergence

Def. A sequence is a function $f: \mathbb{N} \to \mathbb{R}$.

We denote usually the element $f(n)$ by x_n and use $(a_n)_{n=1}^{\infty} = (a_0, a_1, a_2, \ldots)$ to denote all the members of the sequence.

⚠ The range of a sequence is $\{a_n \; n \in \mathbb{N}\}$ and is different from the list $(a_1, a_2, a_3, \ldots)$

Examples.

(i) the sequence $(a_n)_{n=1}^{\infty}$ defined by $a_n = 1 \; \forall n \geq 1$

(ii) the sequence $(a_n)_{n=1}^{\infty}$ defined recursively by

$$a_1 = r \quad , \quad a_n = r\, a_{n-1} \quad (n \geq 2).$$

We can show that $a_n = r^n$ (by induction).

(iii) the sequence $(p_n)_{n=1}^{\infty}$ of all prime numbers:

$$(p_n)_{n=1}^{\infty} = (2, 3, 5, 7, 11, 13, 17, 19, \ldots)$$

(iv) the sequence $(a_n)_{n=1}^{\infty}$ where $a_n = \frac{1}{n}$.

(v) the sequence $(a_n)_{n=1}^{\infty}$ where $a_n = (-1)^n$.

In the last two examples, by writing the element explicitly, we have

1) $\left(\frac{1}{n}\right)_{n=1}^{\infty} = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots, \frac{1}{1000}, \dots\right)$

[We see that it goes to 0 as n is bigger.]

2) $\left((-1)^n\right)_{n=1}^{\infty} = (-1, 1, -1, 1, -1, 1, \dots)$

[we see that it oscillates and we can't guess to where it will goes when n is bigger].

<u>Def.</u> A sequence $(a_n)_{n=1}^{\infty}$ is <u>convergent</u> if $\exists A \in \mathbb{R}$, $\forall \varepsilon > 0$, $\exists N = N(\varepsilon, L)$ s.t.

$$n \geq N(\varepsilon, L) \Rightarrow |a_n - A| < \varepsilon.$$

<u>Notation.</u> $A = \lim_{n \to \infty} a_n$, $a_n \xrightarrow[n \to \infty]{} A$

or simply $a_n \longrightarrow A$.

<u>Example.</u> $\frac{1}{n} \longrightarrow 0$.

Let $\varepsilon > 0$. By AP, $\exists N$ st. $N\varepsilon > 1$. So, $\frac{1}{N} < \varepsilon$. Then, if $n \geq N \Rightarrow \frac{1}{n} \leq \frac{1}{N} < \varepsilon$.

So $\frac{1}{n} \longrightarrow 0$.

Example. Let $(a_n) := (1)_{n=1}^{\infty}$ $(a_n = 1 \ \forall n)$. Then $a_n \to 1$. Indeed, whenever $\varepsilon > 0$, we have

$$|a_n - 1| = |1-1| = 0 < \varepsilon .$$

So, we can take any N.

Thm. If $a_n \to A$ and $a_n \to B$, then $A = B$.

Proof. Suppose $A \neq B$. Then $|A-B| > 0$. Take $\varepsilon := |A-B|/4$. Then $\exists N_A, N_B$ s.t.

$$n \geq N_A \Rightarrow |a_n - A| < \varepsilon$$

$$n \geq N_B \Rightarrow |a_n - B| < \varepsilon .$$

Let $N = \max\{N_A, N_B\}$. Then

$$\begin{aligned} |A-B| &= |A - a_n + a_n - B| \\ &\leq |A-a_n| + |a_n - B| \\ &< \varepsilon + \varepsilon = |A-B|/2 . \end{aligned}$$

$$\Rightarrow \quad |A-B| < |A-B|/2 \quad \Rightarrow \quad 2 < 1 .$$

This is a contradiction and $A = B$. $\square$

Def. A sequence $(a_n)_{n=1}^{\infty}$ is

- **bounded from above** if $\exists M \in \mathbb{R}$ s.t. $a_n \le M \ \forall n$.
- **bounded from below** if $\exists M \in \mathbb{R}$ s.t. $a_n \ge M \ \forall n$.
- **bounded** if it is bounded from below and from above.

Remark. A sequence is bounded iff. $\exists M > 0$ s.t.

$$|a_n| \le M.$$

Thm. If $a_n \to A$, then $(a_n)_{n=1}^{\infty}$ is bounded.

Proof. Let $\varepsilon = 1$. Then, $\exists N \in \mathbb{N}$ s.t.

$$n \ge N \Rightarrow |a_n - A| < 1.$$

So, $\forall n \ge N$,

$$|a_n| - |A| \le |a_n - A| < 1$$

$$\Rightarrow \quad |a_n| \le |A| + 1 \quad \forall n \ge N.$$

Take $M := \max\{|a_1|, \dots, |a_{N-1}|, |A|+1\}$. Then

$$|a_n| \le M \quad \forall n \ge 1. \qquad \square$$

Example. Let $(a_n)_{n=1}^{\infty}$ be defined by

$$a_n = \sum_{k=1}^{n} \frac{1}{k}.$$

So, $a_1 = 1$, $a_2 = 1 + \frac{1}{2} = \frac{3}{2}$, ...

We see that

$$a_{2^n} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \dots + \left(\frac{1}{2^{n-1}+1} + \dots + \frac{1}{2^n}\right)$$

$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \dots + \left(\frac{1}{2^n} + \dots + \frac{1}{2^n}\right)$$

$$> 1 + \frac{n+1}{2}.$$

Then, the sequence is unbounded, so is not convergent.
[If $M \in \mathbb{R}$, By AP, there is an integer $N = n+1$ s.t. $\frac{(n+1)}{2} > M$.]

<u>Def.</u> $(a_n)_{n=1}^{\infty}$ is <u>convergent</u> iff $\exists A \in \mathbb{R}$ s.t. $a_n \to A$.
If $(a_n)_{n=1}^{\infty}$ is not convergent, then it is <u>divergent</u>.

<u>Example</u> Let $a_n = 1 + \frac{1}{n}$. Does (a_n) converges?

2.2. Operations

Thm. If $a_n \to A$ and $b_n \to B$, then $a_n + b_n \to A+B$.

Proof. Suppose that $a_n \to A$ & $b_n \to B$. Let $\varepsilon > 0$. Then $\exists N_A$ and $\exists N_B$ s.t.

- $n \geq N_A \Rightarrow |a_n - A| < \frac{\varepsilon}{2}$
- $n \geq N_B \Rightarrow |b_n - B| < \frac{\varepsilon}{2}$.

Take $N := \max\{N_A, N_B\}$. Then, if $n \geq N$,

$$|a_n + b_n - (A+B)| \leq |a_n - A| + |b_n - B| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So, $a_n + b_n \to A+B$.

Thm. If $a_n \to A$ & $b_n \to B$, then $a_n b_n \to AB$.

Proof. Suppose $a_n \to A$ and $b_n \to B$. We have

$$|a_n b_n - AB| = |a_n b_n - b_n A + b_n A - AB| \leq |(a_n - A) b_n| + |(b_n - B) A|.$$

Now, (b_n) is convergent, so it is bounded:

$$\exists M > 0 \text{ s.t. } |b_n| < M, \ \forall n \geq 1.$$

So,

$$|a_nb_n - AB| \leq |a_n - A|\cdot|b_n| + |b_n - B|\cdot|A|$$
$$\leq |a_n - A|\cdot M + |b_n - B|\cdot|A|. \quad (*)$$

Let $\varepsilon > 0$. Then $\exists N_A, \exists N_B$ s.t.

- $n \geq N_A \Rightarrow |a_n - A| < \dfrac{\varepsilon}{2M}$.

- $n \geq N_B \Rightarrow |b_n - B| < \dfrac{\varepsilon}{2(|A|+1)}$.

Take $N := \max\{N_A, N_B\}$. Then, from $(*)$,

$$|a_nb_n - AB| < \frac{\varepsilon}{2M}\cdot M + \frac{\varepsilon}{2(|A|+1)}\cdot|A|$$

$$= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So, $a_nb_n \longrightarrow AB$. $\square$.

<u>Remark</u>. • Since $(\alpha)_{n=1}^{\infty}$ converges to α, we get that $\alpha a_n \longrightarrow \alpha A$ if $a_n \to A$.

• In particular, $a_n \to A$, then $-a_n \to -A$.

• Also, $a_n \to A$ & $b_n \to B$, then $a_n - b_n \to A - B$.

• Also, $a_n \to A$ & $b_n \to B$, then $\alpha a_n + \beta b_n \to \alpha A + \beta B$.

- $a_n + b_n \longrightarrow L$, it does not imply that $(a_n)_{n=1}^{\infty}$ or $(b_n)_{n=1}^{\infty}$ converges.
- Same for $a_n - b_n$ and $a_n b_n$.

Division must be approach with some caution.

<u>Thm.</u> If $a_n \to A$ & $b_n \to B$ with $b_n \neq 0 \ \forall n$ & $B \neq 0$ then $\dfrac{a_n}{b_n} \longrightarrow \dfrac{A}{B}$.

<u>Proof.</u> Suppose $a_n \to A$ & $b_n \to B$. We have

$$\left|\frac{a_n}{b_n} - \frac{A}{B}\right| = \left|\frac{a_n B - A b_n}{b_n B}\right|$$

$$= \frac{|a_n B - AB + AB - A b_n|}{|b_n B|}$$

$$\leq \frac{|a_n - A||B| + |A||B - b_n|}{|b_n||B|}.$$

can't use (b_n) bounded

We know that there is a $N_1 \in \mathbb{N}$ s.t.

$$n \geq N_1 \Rightarrow |b_n - B| < |B|/2.$$

So,

$$n \geq N_1 \Rightarrow |B| - |b_n| < |B|/2$$

$$\Rightarrow \frac{|B|}{2} < |b_n|.$$

Now, if $n \geq N_1$, then $\frac{1}{|b_n|} < \frac{2}{|B|}$

and so,

$$\left|\frac{a_n}{b_n} - \frac{A}{B}\right| < \frac{|a_n - A||B| + |A||b_n - B|}{\left|\frac{2}{B}\right| \cdot |B|}$$

$$= \frac{|a_n - A||B| + |A||b_n - B|}{2}.$$

Let $\varepsilon > 0$. Then $\exists N_A, N_B \in \mathbb{N}$ s.t.

- $n \geq N_A \implies |a_n - A| < \frac{\varepsilon}{|B|+1}$.
- $n \geq N_B \implies |b_n - B| < \frac{\varepsilon}{|A|+1}$.

Let $N := \max\{N_A, N_B, N_1\}$. Then, if $n \geq N$,

$$\left|\frac{a_n}{b_n} - \frac{A}{B}\right| < \frac{\left(\frac{\varepsilon}{|B|+1}\right)|B| + \left(\frac{\varepsilon}{|A|+1}\right) \cdot |A|}{2}$$

$$= \frac{\varepsilon + \varepsilon}{2} = \varepsilon.$$

So, $\frac{a_n}{b_n} \longrightarrow \frac{A}{B}$. $\square$

<u>Remark</u> if $b_n \to B$, $b_n \neq 0$ $\forall n$ & $B \neq 0$, then $\frac{1}{b_n} \to \frac{1}{B}$.
[Apply with $a_n = 1$ $\forall n$].

Example. Let $a_n := \dfrac{n^3+2n^2+4}{2n^3+1}$.

We can't apply directly the previous theorems.

But,

$$a_n = \frac{n^3\left[1+ 2/n + 4/n^3\right]}{n^3\left[2+1/n^3\right]} = \frac{1+2/n+4/n^3}{2+1/n^3}.$$

Now, we know that $\dfrac{1}{n} \to 0$, so

$$\frac{1}{n^2} \to 0 \quad \text{and} \quad \frac{1}{n^3} \to 0.$$

So, we get

$$1+2\left(\frac{1}{n}\right)+4\cdot\left(\frac{1}{n^3}\right) \to 1+2\cdot 0+4\cdot 0 = 1.$$

&

$$2+\frac{1}{n^3} \longrightarrow 2+0 = 2.$$

Thus,

$$a_n \longrightarrow \frac{1}{2}. \qquad \square$$

Thm. If $a_n \to A$, $b_n \to B$ & $a_n \le b_n$ $\forall n \ge 1$, then $A \le B$.

Proof. Suppose that $B < A$. Then $A-B>0$. $\exists N_A$ and N_B s.t.

- $n \ge N_A \Rightarrow |a_n - A| < \dfrac{A-B}{2}$
- $n \ge N_B \Rightarrow |b_n - B| < \dfrac{A-B}{2}$.

Take $N := \max\{N_A, N_B\}$. Then

$$\frac{B-A}{2} < a_N - A < \frac{A-B}{2} \implies a_N > \frac{A+B}{2}$$

&

$$\frac{B-A}{2} < b_N - B < \frac{A-B}{2} \implies b_N < \frac{A+B}{2}.$$

Thus, $\exists N$ s.t. $b_N < a_N$, contradiction. $\square$

$\boxed{!}$ $a_n < b_n \ \forall n \in \mathbb{N} \not\Rightarrow A < B$ in general.
[take $a_n = \frac{1}{2n}$ and $b_n = \frac{1}{n}$].

Example Let $a_n := \frac{(-1)^n}{n}$. It seems reasonable that $a_n \to 0$. Indeed, it does:

$$\left|\frac{(-1)^n}{n}\right| \leq \frac{1}{n}.$$

Now, by AP, $\exists N \in \mathbb{N}$ s.t. $\frac{1}{N} < \varepsilon$. So,

$$n \geq N \implies \left|\frac{(-1)^n}{n}\right| \leq \frac{1}{n} < \varepsilon.$$

Thm If $a_n \to 0$ and $(b_n)_{n=1}^{\infty}$ is bounded, then $a_n b_n \to 0$.

Proof $\exists M > 0$ s.t. $|b_n| < M$. Let $\varepsilon > 0$. Choose N s.t. $n \geq N \implies |a_n| < \varepsilon/M$. Then, $n \geq N$,

$$|a_n b_n| \leq |a_n| M < (\varepsilon/M) M = \varepsilon.$$

$\square$

2.3 Cauchy sequences.

Def. A sequence $(a_n)_{n=1}^{\infty}$ is a Cauchy seq. if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t.

$$m, n \geq N \Rightarrow |a_n - a_m| < \varepsilon.$$

Thm. Every Cauchy sequence is bounded.

Thm. Every convergent sequence is Cauchy.

Proof. Let $a_n \to A$. Let $\varepsilon > 0$. $\exists N \in \mathbb{N}$ s.t.

$$n \geq N \Rightarrow |a_n - A| < \frac{\varepsilon}{2}.$$

Then, if $n, m \geq N$,

$$|a_n - a_m| \leq |a_n - A| + |A - a_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \square$$

Example. Let $a_n = (-1)^n$. Then,

$$|a_n - a_{n+1}| = 2$$

and so $(a_n)_{n=1}^{\infty}$ is not Cauchy. So it is not convergent.

Def. Let $S \subseteq \mathbb{R}$. $x \in \mathbb{R}$ is an accumulation point for S iff $\forall \delta > 0$, $(x - \delta, x + \delta)$ contains infinitely many points of S.

Example $S = \{\frac{1}{n} : n \in \mathbb{N}\}$. Then 0 is an accumulation point of S.

- Any finite set has no accumulation point.

- Each $x \in \mathbb{R}$ is an accumulation point of $\mathbb{Q}$

[If $\delta > 0$, then there is $r_1 \in \mathbb{Q}$ $x < r_1 < x+\delta$. Also, $\exists r_2 \in \mathbb{Q}$ s.t. $x < r_2 < r_1$. So, $\exists r_n$ s.t. $x < \dots < r_n < \dots < r_1 < x+\delta$.]

<u>Thm.</u> (Bolzano - Weierstrass)

If $S \subseteq \mathbb{R}$, S infinite and S is bounded, then E has an accumulation point.

<u>Proof.</u> Since S is bounded, then are $\alpha, \beta \in \mathbb{R}$ s.t. $S \subseteq [\alpha, \beta]$.

- Let $\alpha_1 = \frac{\alpha+\beta}{2}$ be the midpoint of α, β.

Then $[\alpha, \alpha_1] \cap S$ or $[\alpha_1, \beta] \cap S$ must be infinite. Denote it by $[a_1, b_1]$.

- Let $\alpha_2 := \frac{a_1+b_1}{2}$ be the midpoint of a_1, b_1.

Again, $[a_1, \alpha_2] \cap S$ or $[\alpha_2, b_1] \cap S$ is infinite. Denote it by $[a_2, b_2]$.

Continuing in this fashion, we obtain $[a_n, b_n]$

- $b_n - a_n = 2^{-n}(\beta - \alpha)$
- $[a_n, b_n] \subseteq \dots \subseteq [a_1, b_1] \subseteq [\alpha, \beta]$
- $[a_n, b_n] \cap S$ is infinite

Let $Q := \{a_n : n \geq 1\}$. Since $Q \subseteq [a_n, b_n] \subseteq [\alpha, \beta]$, Q is bounded. By AC, $\sup Q$ exists and let

$$x := \sup Q.$$

We want to show that x is an accumulation point of S. Let $\delta > 0$.

By definition of the sup, there is a n s.t.

$$x - \delta < a_n \leq x.$$

Also, by definition of the sup, $\forall m \geq n$

$$x - \delta < a_n \leq a_m \leq x.$$

Now, if we can show that for some m

$$[a_m, b_m] \subseteq (x - \delta, x + \delta)$$

we will win.

Take $m \geq n$ sufficiently large (garantee by AP)

$$2^{-m}(\beta - \alpha) < \delta.$$

Then,

$$b_m - a_m = 2^{-m}(\beta - \alpha) < \delta$$

$$\Rightarrow \quad b_m < a_m + \delta \leq x + \delta$$

$$\Rightarrow \quad x - \delta < a_m \leq b_m < x + \delta.$$

Thus, $[a_m, b_m] \subseteq (x-\delta, x+\delta)$. Since $[a_m, b_m]$ contains infinitely many elements of S, $(x-\delta, x+\delta)$ also contains infinitely many elements.

<u>Thm</u> Every Cauchy sequence is convergent.

<u>Proof</u>. Let $(a_n)_{n=1}^{\infty}$ be a Cauchy sequence.

Suppose $S := \{a_n : n \geq 1\}$.

- Suppose than $\{a_n : n \geq 1\}$ is finite. So,

$$\{a_n : n \geq 1\} = \{a_1, a_2, \dots, a_k\}.$$

Choose $\varepsilon := \min\{|a_i - a_j| : i = 1, 2, \dots, k\}$. Then $\exists N \in \mathbb{N}$ s.t.

$$|a_n - a_m| < \frac{\varepsilon}{2} \quad \forall n, m \geq N.$$

Suppose $\exists n, m$ s.t. $|a_n - a_m| > 0$. Then, there are $i, j \in \{1, 2, \dots, k\}$ s.t. $a_n = a_i$ & $a_m = a_j$

$$\Rightarrow \quad |a_i - a_j| < \frac{\varepsilon}{2} < |a_i - a_j| \quad \#.$$

So, $a_n = a_m = s \quad \forall n \geq N$ (constant sequence).

Thus, $a_n \longrightarrow s$.

- Suppose S is infinite. Since (a_n) is a Cauchey sequence, S must be bounded.

By BWT, S must has an acc. pt., say A.

Let $\varepsilon > 0$. Then

- $\exists N \in \mathbb{N}$ s.t.

$$n, m \geq N \Rightarrow |a_n - a_m| < \frac{\varepsilon}{2}.$$

- $(A-\varepsilon, A+\varepsilon) \cap S$ must be infinite. So, in particular, $\exists n_0 \in \mathbb{N}$ s.t.

$$n_0 \geq N \text{ and } a_{n_0} \in \left(A - \frac{\varepsilon}{2}, A + \frac{\varepsilon}{2}\right)$$

[Otherwise $(A-\varepsilon, A+\varepsilon) \cap (S \setminus \{a_1, \dots, a_N\}) = \phi$].

Now,

$$n \geq n_0 \Rightarrow |a_n - A| \leq |a_n - a_{n_0}| + |a_{n_0} - A|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So, $a_n \to A$.

2.4 Subsequences and monotonic seq.

Def. $(a_n)_{n=1}^{\infty}$ and $(n_k)_{k=1}^{\infty}$ be any sequence of positive integers s.t.

$$1 \leq n_1 < n_2 < n_2 < n_3 < \cdots$$

The sequence $(a_{n_k})_{k=1}^{\infty}$ is called a subsequence of $(a_n)_{n=1}^{\infty}$.

Examples

① $a_n = \frac{1+(-1)^n}{2}$. Take $n_k = 2k$. Then

$$a_{n_k} = a_{2k} = \frac{1+(-1)^{2k}}{2} = 1.$$

Take $n_k = 2k+1 \Rightarrow a_{n_k} = a_{2k+1} = \frac{1+(-1)^{2k+1}}{2} = 0$.

[Now, we see that (a_n) has two subsequences that converges to different values. This is why $\left(\frac{1+(-1)^n}{2}\right)_{n=1}^{\infty}$ diverges. This is a general principle!]

<u>Thm.</u> $(a_n)_{n=1}^{\infty}$ converges iff all its subsequence converges and has the same limit.

<u>Proof.</u> ($\Longrightarrow$) Suppose $a_n \longrightarrow A$ and let $(a_{n_k})_{k=1}^{\infty}$ be a subsequence of (a_n). Let $\varepsilon > 0$. $\exists N \in \mathbb{N}$ s.t.

$$n \geq N \Rightarrow |a_n - A| < \varepsilon.$$

Now, $k \geq N \Rightarrow n_k \geq N$

$$\Rightarrow |a_{n_k} - A| < \varepsilon.$$

So, $\lim\limits_{k\to\infty} a_{n_k} = A$.

($\Longleftarrow$) Since (a_n) is a subsequence of itself, (a_n) must converge.

<u>Remark:</u> $n_k = n+j \ (j \geq 1) \Rightarrow \lim\limits_{n\to\infty} a_{n+j} = A$. $\square$

Def. A sequence $(a_n)_{n=1}^{\infty}$ is

- <u>increasing</u>. if $a_n \le a_{n+1}$ $(\forall n \ge 1)$.
- <u>decreasing</u>. if $a_n \ge a_{n+1}$ $(\forall n \ge 1)$.
- <u>monotone</u>. if increasing or decreasing.

Thm (a_n) monotone converges iff (a_n) is bounded.

Proof. $(\Rightarrow)$ (a_n) converges then it is bounded.

$(\Leftarrow)$ Suppose (a_n) is bounded and increasing. Let $S := \{a_n : n \ge 1\}$.

Then S is bounded and so $A := \sup S$ exists by the CA. We will show that

$$a_n \to A.$$

Let $\varepsilon > 0$. Then $A - \varepsilon$ is not an upper bound for S and so $\exists N \in \mathbb{N}$ s.t.

$$A - \varepsilon < a_N \le A.$$

Since a_n is increasing, $\forall n \ge N$

$$A - \varepsilon < a_N \le a_n \le A < A + \varepsilon$$

$$\Rightarrow \quad |a_n - A| < \varepsilon \quad (n \ge N.)$$

So $a_n \to A$.

For decreasing, we apply the previous step with $(b_n) = (-a_n)$. $\square$

Examples.

① Let $0<b<1$ and consider $a_n = b^n$ $(n \geq 1)$.

Then

- $b^{n+1} < b^n \Leftrightarrow b^n(b-1) < 0$.

  Since $b-1<0$ & $b^n>0 \Rightarrow b^n(b-1)<0$

  $\Rightarrow (b^n)_{n=1}^{\infty}$ is decreasing.

- Also, $0<b<1 \Rightarrow 0<b^n<1$.

  So it is bounded.

Thus, (b^n) converges say to B. What is B?

[We know that every subsequence must converge to the same limit].

We know

- $b^n \longrightarrow B$
- $b^{2n} \longrightarrow B$ but

$$b^{2n} = b^n \cdot b^n \longrightarrow B \cdot B = B^2.$$

By uniqueness,

$$B = B^2 \Leftrightarrow B(B-1) = 0$$
$$\Leftrightarrow B=0 \text{ or } B=1.$$

But $b^{n+1} < b < 1$ $\forall n \Rightarrow B < 1$.

Thus, $\lim_{n\to\infty} b^n = 0$. □

② Let $(a_n)_{n=1}^{\infty}$ be defined by

$$a_1 = 1, \qquad a_n = \sqrt{2a_{n-1}} \qquad n \geq 2.$$

- We see that

$$a_2 = \sqrt{2 \cdot a_1} = \sqrt{2} > 1 = a_1.$$

Suppose $a_n \leq a_{n+1}$. Then

$$a_{n+1} = \sqrt{2 \cdot a_n} \leq \sqrt{2a_{n+1}} = a_{n+2}.$$

By PMI, $a_n \leq a_{n+1}$ $\forall n \geq 1 \Rightarrow (a_n)$ is inc.

- We also have $a_n \leq 2$. [Indeed:

  - $a_1 = 1 \leq 2$
  - $a_n \leq 2$, then $a_{n+1} = \sqrt{2a_n}$
  $\leq \sqrt{2 \cdot 2} = 2$.

  By the PMI, $a_n \leq 2$]

So, by the previous thm., $a_n \to A$, some $A \in \mathbb{R}$.

Now, $$a_{n+1} \longrightarrow A$$

$$a_{n+1}^2 \longrightarrow A^2$$

But, $a_{n+1}^2 = 2a_n \longrightarrow 2A$. Thus,

$$A^2 = 2A \Rightarrow A = 2 \text{ or } A = 0$$

But $A \neq 0$ (because $0 < 1 \leq a_n$ $\forall n$). So

$$\lim_{n \to \infty} a_n = A.$$

Suggested problems from the book.

- Example 1.8, Theorem 1.15, Example 1.10.
- Section 1.1: 2, 4 - 11
- Section 1.2: 14, 17, 21
- Section 1.3: 25 - 28, 31, 32,
- Section 1.4: 35, 36, 37, 38, 39, 43, 47.