Math 331 Homework 2

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Exercise 1.
First, we'll prove that for any $n, m \geq 1$, $a_n \leq b_m$. We have three choices.
(1) $a_n < b_m$
(2) $a_n = b_m$
(3) $a_n > b_m$
We just need to prove that case 3 is false, and we'll get that the statement is true. Let $M = max\{n, m\}$ such that the set $[a_M, b_M] \subset [a_n, b_m]$. Now suppose toward a contradiction that $a_n > b_m$. If so, we get that the set $[a_n, b_m] = \emptyset$. This is a contradiction because from our initial assumption, we know that $[a_M, b_M] \subset [a_n, b_m]$. But if the second set is the empty set, this cannot be true.
Next we'll show that $sup(a_n)$ exists. We know for all $n \in \mathbb{N}$, $a_n \leq b_n$. Now say $a_n \to A$ and $b_n \to B$, then $A \leq B$ from the proof done in class. Therefore, we know a_n is bounded from above. By the Axiom of Completeness, we know a_n has a supremum.
We know that $sup(a_n)$ exists, so let's define $c = sup(a_n)$ and show that c is the element we are looking for such that $c \in [a_n, b_n] \forall n \geq \mathbb{N}$. We need first show that a_n converges to c
$\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that if $n > N$,

$$|a_n - c| < \epsilon$$

The expression inside the absolute value symbols will always be negative, so this expression is equal to

$$-(a_n - c) < \epsilon$$

$$c - a_n < \epsilon$$

Now if a_n is increasing, we will always find an N such that if $n \geq N$, the difference between c and a_n will be $< \epsilon$. So let's prove that a_n is increasing.

Assume towards a contradiction that a_n is decreasing, such that $a_n > a_{n+1}$. Then $[a_n, b_n] \subset [a_{n+1}, b_{n+1}]$. This is a contradiction because in the assumption, we must have that $[a_n, b_n] \supset [a_{n+1}, b_{n+1}]$. Therefore, because we only have three options, $a_n < a_{n+1}, a_n = a_{n+1}$, or $a_n > a_{n+1}$. Since the last one was proven contradictory, we are left with $a_n \leq a_{n+1} \forall n \in \mathbb{N}$

So since a_n is increasing and ϵ was arbitrary, $a_n \to c$. Therefore, referring back to having $b_n \to B$, we know $c \leq B$.
Next we need to prove that $b_n \geq B \forall n \in \mathbb{N}$. We do this by proving that b_n is decreasing.
Assume that b_n is increasing such that $b_n < b_{n+1}$. Then that means

$$[a_n, b_n] \subset [a_{n+1}, b_{n+1}]$$

This is a contradiction because according to the assumption, $[a_n, b_n] \supset [a_{n+1}, b_{n+1}]$. Therefore $b_n \geq b_{n+1}$.
Putting all this together,

$$a_n \leq c \leq B \leq b_n$$

$$a_n \leq c \leq b_n$$

We needed to show that no matter how large $n \in \mathbb{N}$ gets, c will always be contained in a subset. c must be both greater than a_n and less than b_n. Through these inequalities, we can see that this will be true. Therefore $c \in [a_n, b_n] \forall n \in \mathbb{N}$.

Part b We can use this result to prove that the set $\mathbb{R}$ is uncountable. Let's construct a function $f : \mathbb{N} \to \mathbb{R}$ of closed intervals such that $f(n) \notin [a_n, b_n]$ with $a_n < b_n$, and show that this function cannot be surjective.
As we saw in the first part of this exercise, we've proven the existence of an element $c = sup(a_n)$ that will always be contained in the set. Therefore, because we defined $f(n)$ as all elements outside of the set, it is impossible for there to be n such that $f(n) = c$. Now because there's an element $c \in \mathbb{R}$ without a pre-image, this function isn't surjective since the entire range is not mapped to.

Exercise 2. Prove that if $a_n \to A$, then $|a_n| \to |A|$.

By our assumption that $a_n \to A$, we know that $\forall \epsilon > 0, \exists N_0 \in \mathbb{N}$ such that if $n \geq N_0$ then

$$|a_n - A| < \epsilon$$

Now our goal is to show that $\forall \epsilon > 0, \exists N_1 \in \mathbb{N}$ such that if $n \geq N_1$ then

$$||a_n| - |A|| < \epsilon$$

If we can show this, we know that $|A|$ is the correct limit for $|a_n|$.
Let's set $N = max\{N_0, N_1\}$. Now for $n \geq N$, we have the expression

$$||a_n| - |A||$$

Now by the 6th property of absolute values discussed in class,

$$||a_n| - |A|| \leq |a_n - A|$$

As you can see, the RHS is actually the same one from our assumption, and so we know that since $n \geq N$,

$$||a_n| - |A|| \leq |a_n - A| < \epsilon$$

$$||a_n| - |A|| < \epsilon$$

Therefore, since ϵ was arbitrary, $|a_n| \to |A|$. $\square$

Exercise 3. Let $(a_n), (b_n), (c_n)$, be sequences of real numbers. Prove that if $a_n \to L, b_n \to L$, and $a_n \leq c_n \leq b_n$, then $c_n \to L$.

Let $c_n \to C$. We will first prove that if $c_n \leq b_n$, then $C \leq L$. And by the same logic since $a_n \leq c_n$, $L \leq C$. Therefore since we're left with $L \leq C \leq L$, the only possible value that C can be is L.
Let's start with proving $C \leq L$.
If $c_n \leq b_n, \forall n \geq 1$, we have three cases.
(1) $C < L$
(2) $C = L$
(3) $C > L$
By disproving case 3, we will know either of cases (1) and (2) must be true, which is the definition of $\leq$.
Assume towards a contradiction that $C > L$ such that $C - L > 0$. Now let $\epsilon = \frac{C-L}{2}$. Since $c_n \to C \quad b_n \to L$, $\exists N_c, N_b$ such that

$$\begin{aligned} n \geq N_c &\Rightarrow |c_n - C| < \frac{C-L}{2} \\ n \geq N_b &\Rightarrow |b_n - L| < \frac{C-L}{2} \end{aligned}$$

We take $N := max\{N_c, N_b\}$ so that

$$-\frac{C-L}{2} < c_n - C < \frac{C-L}{2} \qquad (i)$$
$$-\frac{C-L}{2} < b_n - L < \frac{C-L}{2} \qquad (ii)$$

From (i),

$$\begin{aligned} c_n &> -\frac{C+L}{2} + \frac{2C}{2} \\ &> \frac{C+L}{2} \end{aligned}$$

From (ii),

$$\begin{aligned} b_n &< \frac{C-L}{2} + \frac{2L}{2} \\ &< \frac{C+L}{2} \end{aligned}$$

By combining these inequalities, we get that

$$b_n < \frac{C+L}{2} < c_n \qquad \#$$

This is a contradiction to our original assumption of $c_n \leq b_n\ \forall n \geq 1$, therefore $C \leq L$. Now by the same logic, we can prove that if $a_n \leq c_n, \forall n \geq 1$, and $a_n \to L, c_n \to C$, then $L \leq C$.
So far, we have that if $a_n \leq c_n \leq b_n\ \forall n \geq 1$, and $a_n \to L, b_n \to L, c_n \to C$, then

$$L \leq C \leq L$$

The only value of C that can logically keep this equation true is by having $C = L$. Therefore, if $a_n \to L, b_n \to L$, and $a_n \leq c_n \leq b_n$, then $c_n \to L$. $\square$

Exercise 4.

Exercise 5.
We're trying to show that if $a_n \to A$, then $\sigma_n \to A$. To write it a little simpler, let's rewrite σ_n as

$$\frac{\displaystyle\sum_{n=1}^{\infty} a_n}{n}$$

Now we'll use the definition of convergence. For all $\epsilon > 0$, there exists an $N_1 \in \mathbb{N}$ such that if $n \geq N_1$, then

$$\left| \frac{\displaystyle\sum_{n=1}^{\infty} a_n}{n} - A \right| < \epsilon$$

$$\left| \frac{\displaystyle\sum_{n=1}^{\infty} a_n}{n} - \frac{nA}{n} \right| < \epsilon$$

$$\left| \frac{\displaystyle\sum_{n=1}^{\infty} a_n - A}{n} \right| < \epsilon$$

$$\frac{\left|\displaystyle\sum_{n=1}^{\infty} a_n - A\right|}{|n|} < \epsilon$$

by property of the absolute value

$$\frac{\left|\displaystyle\sum_{n=1}^{\infty} a_n - A\right|}{n} \leq \frac{\displaystyle\sum_{n=1}^{\infty} |a_n - A|}{n}$$

by the Triangle Inequality

Now we know that the inner expression of the summation, $|a_n - A| < \epsilon \, \forall n \geq N$ because of our initial assumption that $a_n \to A$. With this, we are saying that for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that if $n \geq N_0$,

$$|a_n - A| < \epsilon$$

Therefore, if we set $N = max\{N_1, N_0\}$, then for $n \geq N$,

$$\frac{\sum_{n=1}^{\infty} |a_n - A|}{n} < \frac{\sum_{n=1}^{\infty} \epsilon}{n}$$
$$\frac{\sum_{n=1}^{\infty} \epsilon}{n} = \frac{\epsilon \cdot n}{n}$$
$$= \epsilon$$

Putting it all together, we get that

$$\left| \frac{\sum_{n=1}^{\infty} a_n}{n} - A \right| \leq \frac{\sum_{n=1}^{\infty} |a_n - A|}{n} < \frac{\sum_{n=1}^{\infty} \epsilon}{n} = \epsilon$$

$$\left| \frac{\sum_{n=1}^{\infty} a_n}{n} - A \right| < \epsilon$$

Therefore, because ϵ was arbitrary, $\sigma_n \to A$. $\square$

Exercise 6. Prove these sequences converge.
(a) $a_n = 5 + \frac{1}{n}$
It is reasonable to assume that $a_n \to 5$
$\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that if $n \geq N$,

$$|(5 + \frac{1}{n}) - 5| < \epsilon$$

$$|\frac{1}{n}| < \epsilon$$
$$\frac{|1|}{|n|} < \epsilon$$

by property of absolute values

$$\frac{1}{n} < \epsilon$$

We must show the last statement to be true. We will prove this using the Archimedes Principle.
By the Archimedean Principle, we assign $x = \epsilon$ and $y = 1$. $\exists N_0 \in \mathbb{N}$ such that

$$N_0\epsilon > 1$$

Take $N = N_0$, so that if $n \geq N_0$, then

$$n\epsilon \geq N_0\epsilon > 1$$

So $n\epsilon > 1 (\forall n \geq N_0)$. And so therefore $\frac{1}{n} < \epsilon$. Since ϵ was arbitrary, $a_n \to 5$. $\square$

(b) $a_n = \frac{3n}{2n+1}$ for $n \geq 1$.
It is reasonable to assume that $a_n \to \frac{3}{2}$
$\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that if $n \geq N$, then

$$|a_n - \frac{3}{2}| < \epsilon$$

$$\left|\frac{3n}{2n+1} - \frac{3}{2}\right| < \epsilon$$
$$\left|\frac{3}{4n+2}\right| < \epsilon$$
$$\frac{|3|}{|4n+2|} < \epsilon$$

by property of absolute values

$$\frac{3}{4n+2} < \epsilon$$
$$\frac{4n+2}{3} > \frac{1}{\epsilon}$$

by the third property of the order axioms

$$n > \frac{\frac{3}{\epsilon} - 2}{4}$$

If we just set the RHS equal to N, we know that there exists N such that if $n > N$, the above expression holds true. Therefore since ϵ was arbitary and

$$\left|\frac{3n}{2n+1} - \frac{3}{2}\right| < \epsilon$$

$a_n \to \frac{3}{2}$
$\square$

Exercise 7.

$a_n = \frac{2n+1}{n}$
From class, we know that if $a_n \to A$, then a_n is Cauchy, so we will do that.
It is reasonable to assume that $a_n \to 2$. Then $\forall \epsilon > 0$, there exists an $N \in \mathbb{N}$ such that if $n \geq N$ then

$$\begin{aligned} |a_n - 2| &< \epsilon \\ \left| \frac{2n+1}{n} - \frac{2n}{n} \right| &< \epsilon \\ \left| \frac{1}{n} \right| &< \epsilon \\ \frac{|1|}{|n|} &< \epsilon \end{aligned}$$

by the property of absolute values

$$\frac{1}{n} < \epsilon$$

As shown previously in this homework assignment and in class, we've proven that we can show $\frac{1}{n} < \epsilon$ using the Archimedean Property. Now therefore, becuase ϵ was arbitrary, $a_n \to 2$.
So, according to the Theorem from class, if $a_n \to A$, then a_n is Cauchy. Therefore, since $a_n \to 2$, it is Cauchy. □

Exercise 8.
(a) $a_n = (-1)^n$
Suppose toward a contradiction that $a_n \to A$. Then following after the definition of a convergent sequence, $\forall \epsilon > 0$, there exists an $N \in \mathbb{N}$ such that if $n \geq N$, then

$$|a_n - A| < \epsilon$$

Now set $\epsilon = 1$ and let's rewrite this sequence as a piece-wise function.
$(-1)^n = \{1 \text{ if n is even}, -1 \text{ if n is odd}\}$.
We have two cases. For the first, say $n = 2N$. Obviously, $2N \geq N$ and is even.

$$\begin{aligned} |(-1)^{2N} - A| &< 1 \\ |1 - A| &< 1 \\ -1 < 1 - A &< 1 \end{aligned}$$

by the definition of absolute values

$$-2 < -A < 0$$

Now for the second case, say $n = 2N + 1$.

$$\begin{aligned} |(-1)^{2N+1} - A| &< 1 \\ |-1 - A| &< 1 \\ -1 < -1 - A &< 1 \end{aligned}$$

by the definition of absolute values

$$0 < -A < 2$$

So combining these two cases, we have that $-A \in (-2, 0) \cup (0, -2) = \emptyset$. Therefore, this is a contradiction. So a_n does not converge, and is therefore divergent. $\square$

(b) $a_n = \sin(\frac{2n+1}{2}\pi)$
If we look closely, we can see this yields the exact same sequence as the previous problem. This can be easier to see by rearranging some terms.

$$\sin(\frac{2n+1}{2}\pi) = \sin(n\pi + \frac{\pi}{2})$$

As you can see, we start at $\frac{\pi}{2}$ and increments of π. On the unit circle, this starts the cycle at 90 degrees, and has us move π radians to the other side repeatedly. As a function of $\sin(x)$, this yields $\{1$ if n is even, -1 if n is odd$\}$. This is the exact same situation as before and can be proven identically. Therefore, this sequence does not converge but rather leads to a contradiction, deeming it divergent.

Exercise 9.
$a_n = (-1)^n$
$b_n = (-1)^{n+1}$

These two sequences diverge. For a_n, we've already proved that this is divergent previously for Exercise 8a. For b_n, we will show its proof but it is very similar to the proof for a_n.

$b_n = \{1$ if n is even, -1 if n is odd$\}$. Now assume $b_n \to B$. That means that for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that if $n \geq N$, then

$$|b_n - B| < \epsilon$$

Let's set $\epsilon = 1$ and consider two cases. When $n = 2N$ and $n = 2N + 1$.
If $n = 2N$, we know that n will be even.

$$\begin{aligned} |(-1)^{2N} - B| &< 1 \\ |-1 - B| &< 1 \\ -1 < -1 - B &< 1 \end{aligned}$$

by the definition of absolute values

$$0 < -B < 2$$

For the second case, let's set $n = 2N + 1$.

$$\begin{aligned} |(-1)^{2N+1} - B| &< 1 \\ |1 - B| &< 1 \\ -1 < 1 - B &< 1 \end{aligned}$$

by the definition of absolute values

$$-2 < -B < 0$$

So combining these two cases, we get that $-B \in (-2, 0) \cup (0, 2) = \emptyset$. Therefore this is a contradiction. Therefore, b_n is divergent.
Now that we know a_n, b_n are divergent, we must prove that $a_n + b_n$ is convergent. Let $c_n = a_n + b_n$

$$c_n = (-1)^n + (-1)^{n+1}$$

It is reasonable to assume that $c_n \to 0$. We must show that for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that if $n \geq N$,

$$\begin{aligned} |c_n - 0| &< \epsilon \\ |(-1)^n + (-1)^{n+1}| &< \epsilon \end{aligned}$$

Let's take a look at this function in a piece-wise manner. $c_n = \{0$ if n even, 0 if n odd$\}$. Therefore, since n will either be even or odd $\forall n \in \mathbb{N}$, this sequence will always be 0.
From here, we can take $N = 1$. So if $n \geq 1$, then

$$|c_n - 0| = 0 < \epsilon$$

We just showed that if $\epsilon > 0$, then $\exists N = 1$ such that $\forall n \geq N$,

$$|c_n - 0| < \epsilon$$

So since ϵ was arbitrary, we conclude that $c_n \to 0$.
$\square$

Exercise 10.
For problems (a) and (b), we are using the known proof from class that

$$lim_{n\to\infty}(\frac{1}{n}) = 0$$

(a) $\frac{n^2+4n}{n^2-5}$

$$\begin{aligned} \frac{n^2 + 4n}{n^2 - 5} &= \frac{n^2(1 + \frac{4}{n})}{n^2(1 - \frac{5}{n^2})} \\ &= \frac{1 + \frac{1}{n}}{1 - \frac{5}{n^2}} \\ lim_{n\to\infty}\frac{1 + \frac{1}{n}}{1 - \frac{5}{n^2}} &= \frac{lim_{n\to\infty}(1 + \frac{1}{n})}{lim_{n\to\infty}(1 - \frac{5}{n^2})} \end{aligned}$$

by the quotient rule

Now we'll first take the limit of the numerator

$$lim_{n\to\infty}(1+\frac{1}{n}) = 1 + lim_{n\to\infty}(\frac{1}{n})$$

by linearity

$$\begin{aligned} &= 1 + 0 \\ &= 1 \end{aligned}$$

Now we will take the limit of the denominator

$$lim_{n\to\infty}(1-\frac{5}{n^2}) = 1 - 5 \cdot lim_{n\to\infty}(\frac{1}{n}) \cdot lim_{n\to\infty}(\frac{1}{n})$$

by linearity and the product rule

$$\begin{aligned} &= 1 - 5 \cdot 0 \cdot 0 \\ &= 1 \end{aligned}$$

Therefore

$$lim_{n\to\infty}(\frac{n^2+4n}{n^2-5}) = \frac{1}{1} = 1$$

(b) $\frac{n}{n^2-3}$

$$\begin{aligned} \frac{n}{n^2-3} &= \frac{n^2(\frac{1}{n})}{n^2(1-\frac{3}{n})} \\ &= \frac{\frac{1}{n}}{1-\frac{3}{n}} \end{aligned}$$

$$lim_{n\to\infty}(\frac{\frac{1}{n}}{1-\frac{3}{n}}) = \frac{lim_{n\to\infty}(\frac{1}{n})}{lim_{n\to\infty}(1-\frac{3}{n})}$$

by the quotient rule

First let's take the limit of the numerator

$$lim_{n\to\infty}(\frac{1}{n}) = 0$$

Now we will take the limit of the denominator

$$lim_{n\to\infty}(1-\frac{3}{n}) = 1 - 3 \cdot lim_{n\to\infty}(\frac{1}{n})$$

by linearity and the product rule

$$\begin{aligned}&= 1 - 3 \cdot 0\\&= 1 - 0\\&= 1\end{aligned}$$

Now combining the numerator and denominator, we have that

$$lim_{n\to\infty}(\frac{n}{n^2-3}) = \frac{0}{1} = 0$$