# Exercises on Limits

No.: 1, 6, 7, 14, 19, 21, 24, 25.

#1 We have

$$\frac{x^2-4}{x+2} = \frac{(x-2)(x+2)}{x+2} = x-2.$$

The limit would be $L=-4$. Let $\varepsilon>0$.

Let $|x+2| = |x-(-2)| < \delta$. Then,

$$\left|f(x)-L\right| = \left|x-2+4\right| = |x+2|$$
$$< \delta = \varepsilon.$$

Choose $\delta := \varepsilon$.

#6 The function $f$ doesn't have a limit.

Suppose that it does, call it $L$.

Then, $\exists \delta>0$ s.t. $\forall x\in(0,1)$, $|x-0|<\delta$,

$$\left|\cos\left(\frac{1}{x}\right)-L\right| < 1$$

By AP, there is a integer $N$ s.t.

$$2N\delta > \frac{1}{\pi}$$

So, $\frac{1}{2N\pi} < \delta$ & $\frac{1}{(2N+1)\pi} < \delta$. So,

$$\left|\cos\left(\frac{1}{1/2N\pi}\right)-L\right| < \varepsilon \Rightarrow |1-L| < \varepsilon.$$

& $\left|\cos\left(1/(2N+1)\pi\right)-L\right| < \varepsilon \Rightarrow |1+L| < \varepsilon$

Thus, $1-L<1$ & $1+L<1$

$\Rightarrow$ $-L<0$ & $L<0$

$\Rightarrow$ $L>0$ & $L<0$ $\#$.

So, $f$ doesn't have a limit at $0$.

#7. Yes, because if $\varepsilon>0$, then

$$\left|x\cos\frac{1}{x}\right| \le |x| < \delta = \varepsilon.$$

Just take $\delta=\varepsilon$ & $x\cos\frac{1}{x} \to 0$.

#14. if $x_n \in \mathbb{Q}$ & $x_n \to x_0 \notin \mathbb{Q}$ then

$$f(x_n) = 8x_n \longrightarrow 8x_0$$

But, $x_n \notin \mathbb{Q}$ & $x_n \to x_0 \notin \mathbb{Q}$, then

$$f(x_n) = 2x_n^2+8 \longrightarrow 2\cdot x_0^2+8$$

To have a limit, we must have that

$$8x_0 = 2x_0^2+8$$

$$\Leftrightarrow \quad x_0^2-4x_0+4=0$$

$$\Leftrightarrow \quad (x_0-2)^2=0 \quad \Leftrightarrow \quad x_0=2.$$

The function has a limit at $x_0$.

If $(x_n)_{n=1}^{\infty} \subseteq \mathbb{R}$ s.t. $x_n \to 2$. Create the subsequences

$$a_{n_k} = x_k \quad \text{if} \quad x_k \in \mathbb{Q}$$

&

$$b_{n_k} = x_k \quad \text{if} \quad x_k \notin \mathbb{Q}.$$

So, $a_{n_k} \in \mathbb{Q}$ $\forall k$ & $b_{n_k} \in \mathbb{Q}$ $\forall k$.

We have

$$f(a_{n_k}) = 8a_{n_k} \longrightarrow 8 \cdot 2 = 16$$

&

$$f(b_{n_k}) = 2 \cdot b_{n_k}^2 + 8 \longrightarrow 8 + 8 = 16.$$

These are the only possibilities and from exercise 39 page 57, we have

$$f(x_n) \longrightarrow 16.$$

When $x_0 \neq 2$, $f$ doesn't have a limit.

<u>#19.</u> We have, if $x \neq 0$,

$$f(x) = \frac{\sqrt{9-x} - 3}{x} \cdot \frac{\sqrt{9-x} + 3}{\sqrt{9-x} + 3} = \frac{-x}{x\left(\sqrt{9-x} + 3\right)}$$

$$\Rightarrow \quad f(x) = \frac{-1}{\sqrt{9-x} + 3}, \quad x \neq 0.$$

Thus, since the limit of $1$ & $\sqrt{9-x}+3$ exists and

$$\lim_{x\to 0} \sqrt{9-x}+3 = 6 \neq 0,$$

from the product rule, we get

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{-1}{\sqrt{9-x}+3} = -\frac{1}{6}.$$

**#21.** Let $L = \lim_{x\to x_0} g(x)$. We know that $L \neq 0$.

From the definition, with $\varepsilon = \frac{|L|}{2} > 0$, $\exists\, \delta > 0$ s.t.

$$\forall x \in D, \quad |x - x_0| < \delta \implies |g(x) - L| < \frac{|L|}{2}.$$

Then, for $x \in (x_0 - \delta, x_0 + \delta)$, we find

$$|L| - |g(x)| < \frac{|L|}{2} \implies \frac{|L|}{2} < |g(x)|.$$

Statement. $\exists\, M > 0$ & $\exists\, \delta > 0$ s.t.
$|g(x)| > M$ for all $x \in (x_0 - \delta, x_0 + \delta)$.

Just put $M := \frac{|L|}{2} > 0$. $\square$

#24. WLOG, $f$ is increasing on $[a,b]$.

Let $x_n \longrightarrow a$. By passing to a subsequence, we may suppose that $x_n$ decreases to $a$. So,

$$x_{n+1} < x_n \implies f(x_{n+1}) < f(x_n)$$

because $f$ is increasing. Thus, the sequence $(f(x_n))_{n=1}^{\infty}$ is decreasing. It is bounded below by $f(a)$. So, it must converge to some $L \in \mathbb{R}$.

So, by the characterization of limits in terms of sequences, $\lim_{x \to a} f(x)$ exists.

Use the same strategy for $b$.

#25. Suppose $f$ has a limit at $x_0$ and

$$\lim_{x \to x_0} f(x) = f(x_0).$$

Notice that $g$ is increasing: if $x < y$, then

$$[a,x] \subseteq [a,y] \implies \sup\{f(t) : a \le t \le x\}$$
$$\le \sup\{f(t) : a \le t \le y\}$$

Also, since $g(x) \in \mathbb{R}$, then $f$ is bounded from above on any interval $[a, x]$.

Also, we have

$$\sup(A \cup B) = \max\{\sup A, \sup B\}. \quad (*)$$

We will consider two cases. For any $x \in \mathbb{R}$:

- $\underline{x \le x_0}$. Then

$$|g(x) - g(x_0)| = g(x_0) - g(x)$$
$$= \sup\{f(t) : a \le t \le x_0\} - \sup\{f(t) : a \le t \le x\}$$

Put $A := [a, x]$ & $B = [x, x_0]$ in $(*)$. Then

$$|g(x) - g(x_0)| = \max\{\sup\{f(t) : t \in A\}, \sup\{f(t) : t \in B\}\}$$
$$- \sup\{f(t) : a \le t \le x\}$$

if max is $\sup\{f(t) : t \in A\}$, then

$$|g(x) - g(x_0)| = 0$$

if max is $\sup\{f(t) : t \in B\}$, then

$$|g(x) - g(x_0)| = \sup\{f(t) : x \le t \le x_0\}$$
$$- \sup\{f(t) : a \le t \le x\}$$

$$= \sup\{f(t) : x \le t \le x_0\} - f(x_0)$$
$$+ \underbrace{f(x_0) - \sup\{f(t) : a \le t \le x\}}_{\le f(x)}$$

• $\underline{x > x_0}$. We have, with $A = [a, x_0]$ & $B = [x_0, x]$, that

$$|g(x) - g(x_0)| = \sup\{f(t) : a \le t \le x\} - \sup\{f(t) : a \le t \le x_0\}$$

$$= \max\{\sup\{f(t) : t \in A\}, \sup\{f(t) : t \in B\}\} - \sup\{f(t) : a \le t \le x_0\}$$

If max. is $\sup\{f(t) : t \in A\}$, then $|g(x) - g(x_0)| = 0$.

If max. is $\sup\{f(t) : t \in B\}$, then

$$|g(x) - g(x_0)| = \sup\{f(t) : x_0 \le t \le x\} - \sup\{f(t) : a \le t \le x_0\}$$

$$\le \sup\{f(t) : x_0 \le t \le x\} - f(x_0).$$

Let $\varepsilon > 0$. By the assumptions, $\exists \delta > 0$ s.t. $\forall t$

$$|t - x_0| < \delta \Rightarrow |f(t) - f(x_0)| < \frac{\varepsilon}{2}.$$

Suppose $x \in (x_0 - \delta, x_0 + \delta)$. Two cases:

• $x_0 - \delta < x < x_0$. In this case, we have shown that

$$|g(x) - g(x_0)| \leq \sup\{f(t) : x \leq t \leq x_0\} - f(x_0) + f(x_0) - f(x)$$

Then, for all $t \in (x, x_0)$, we have

$$f(t) - f(x_0) \leq |f(t) - f(x_0)| < \frac{\varepsilon}{2}$$

$$\Rightarrow \sup\{f(t) : x \leq t \leq x_0\} - f(x_0) < \frac{\varepsilon}{2}.$$

Also, $f(x_0) - f(x) \leq |f(x) - f(x_0)| < \frac{\varepsilon}{2}$.

So, we get

$$|g(x) - g(x_0)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

• $x_0 < x < x_0 + \delta$. In this case, we've shown that

$$|g(x) - g(x_0)| \leq \sup\{f(t) : x_0 \leq t \leq x\} - f(x_0)$$

If $t \in (x_0, x)$, then

$$f(t) - f(x_0) \leq |f(t) - f(x_0)| < \frac{\varepsilon}{2}.$$

So, $\sup\{f(t) : x_0 \leq t \leq x\} - f(x_0) < \frac{\varepsilon}{2}$.

$\Rightarrow$ $|g(x) - g(x_0)| < \varepsilon/2 < \varepsilon$.

Thus in all the cases, $|g(x)-g(x_0)| < \varepsilon$.
this means that the limit exists and

$$\lim_{x \to x_0} g(x) = g(x_0) . \qquad \square$$