

## HW #1 Math 331

Ex: 1 Prove that for any  $n \in \mathbb{N}$ ,  $1+2+\dots+n = \frac{n(n+1)}{2}$

- For the base case 1, we observe this statement to be true:  $f(1) = \frac{1(1+1)}{2} = \frac{1(2)}{2} = 1 \checkmark$

So, we can assume this is true for all  $k$  such that

$$f(k) = \frac{k(k+1)}{2}$$

Thus we can use induction to prove this is also true for every  $k+1$ .

$$\begin{aligned} \text{Be careful, PMI reads: } f(k+1) &= 1 + 2 + \dots + k + (k+1) \\ \text{If } P(k) \text{ true. } &= \underbrace{k(k+1)}_2 + k+1 = \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\ P(k) \Rightarrow P(k+1) \text{ true } &= \frac{(k+1)(k+2)}{2} = \frac{(k+1)((k+1)+1)}{2} \end{aligned}$$

Substituting an integer 'n' for 'k+1' in the above equation yields  $\frac{n(n+1)}{2}$ , thus proving the case is true for any  $n \in \mathbb{N}$ . 4/5

E. 3. Prove that if A, B, and C are sets then

a.)  $A \sim A$  for all elements  $a$ , in set A, there exists an equivalent element in  $A$  such that for  $a \in A$  and  $a \sim a \Rightarrow a-a=0$

Thus, A is reflexive. You just need to use  $f(x)=x$ , the identity.

$\hookrightarrow \sim$  is reflexive.

b.) If  $A \sim B$ , then  $B \sim A \rightarrow$  You have to use the definition.

for any  $x \in A$  there is also an  $x \in B$  such that  $x \sim x$  because  $A \sim B$ . So,  $x \sim a \in A$  and  $a \sim b \in B$ , therefore  $x \sim b \in B$ .

Using the same reasoning, for any  $x \in B$  then  $x \in A$ . So,  $x \sim b \in B$  and  $x \sim a \in A$ . Therefore  $b \sim a$  and  $B \sim A$ .

c.) If  $A \sim B$  and  $B \sim C$ , then  $A \sim C \rightarrow$  You have to use the def. of  $\sim$ .

Let  $a \in A$ . Then  $a \in B$  because  $A \sim B$ . Since  $b \in B, b \in C$ , and  $a \in B$ , this implies that  $a \in C$ . Therefore every element of

$A$  is also an element of  $C$ . Thus,  $A \sim C$ .

Ex 4

Show any subset of a countable set is countable

what about the infinite case?

Suppose  $a_1, a_2, a_3, \dots, a_n$  are elements of set  $A$ , then because  $B$  is a subset of set  $A$  an  $\in B$ . Assigning the natural number  $n$  to  $a_n$ , we say  $f(n)$  denotes the number of elements in  $A$  belonging to set  $B$ .

Then,  $0 \leq f(n) \leq n$ . Therefore the number of elements in set  $B$  is less than or equal to the number of elements in  $A$ . Thus,  $B$  is countable by the Countability Lemma.

0/5

You already supposed that  $a_n = f(n)$  by giving them indices.

Ex 5 Let  $0 < a < b$ . Prove that  $a^2 < b^2$

a) ~~1)~~  $a=0$  Assume  $b$  is a positive number greater than  $a$ .  
 $a=0 < b \Rightarrow 0 \cdot b < b \cdot b \Rightarrow 0 < b^2$  therefore  $a^2 < b^2$  when  $a=0$ .

2.)  $a > 0$  by hypothesis  $a > 0$ .

what is  $x$ ? justify  $a < b \Rightarrow a+x=b$

$$a^2 < b^2 \Rightarrow a^2 < (a+x)^2$$

$$\text{here, you } \Rightarrow a^2 < a^2 + 2ax + x^2$$

$$\text{need } \Rightarrow \Rightarrow 0 < 2ax + x^2 \checkmark$$

Therefore, when  $a$  is greater than  $0$   $a^2 < b^2$ . Thus, proving that  $a^2 < b^2$  when  $0 < a < b$ .

1/5

b.) Prove that  $\sqrt{a} < \sqrt{b}$

Let  $0 < a < b$

$$\Rightarrow 0 < b-a$$

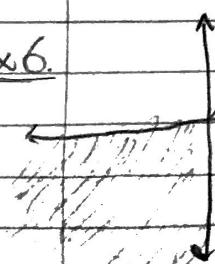
$$\sqrt{0} < \sqrt{b-a}$$

$$\sqrt{a} < \sqrt{b} \checkmark$$

Therefore, when  $b$  is greater than  $a$ ,  $\sqrt{b}$  is also greater than  $\sqrt{a}$ .

0/5

Ex 6.



The shaded region represents when the points  $(x,y)$  satisfy the equation  $x+|x|=y+|y|$ . This includes every point in the negative region because when  $(x,y)$  are negative, both sides of the equation are 0, thus they are equivalent. The line  $y=x$  is also included because the equation will be equal when  $x+|x|=2x$ , or  $2x=2y$ . This occurs whenever  $y=x$ .

You missed two parts

2/5

2, 3c, 8a, b, 9, 10

$$\sqrt{2} + z = 2 \\ z = 2 - \sqrt{2}$$

$$2xy - \sqrt{2}xy \\ 3 > x \Leftrightarrow 3-x > 0 \\ n(3-x) > 0$$

HW 1 Cont'd

Ex 7. If  $x \geq 0$  and  $y \geq 0$  prove that  $\sqrt{xy} \leq \frac{x+y}{\sqrt{2}}$  since  $x$  and  $y > 0$ , we can substitute  $x=u^2$  and  $y=v^2$  good

2/5

$$\Rightarrow \sqrt{u^2 v^2} \leq \frac{u^2 + v^2}{\sqrt{2}} \Rightarrow uv \leq \frac{u^2 + v^2}{\sqrt{2}} \\ \sqrt{2}uv \leq u^2 + v^2$$

by  $u^2 = x$  not  
 $u^2 = x^2$  ..

$$0 \leq u^2 - \sqrt{2}uv + v^2 * \text{Substitute } x \text{ and } y$$

$$0 \leq x^2 - \sqrt{2}xy + y^2 \\ 2\sqrt{2}xy \leq x^2 + 2xy + y^2$$

back into the equation  
\* add  $2\sqrt{2}(xy)$  both sides

$$xy \leq \frac{x^2 + 2xy + y^2}{2 - \sqrt{2}} * \text{Divide both sides by } 2 - \sqrt{2}$$

$$xy \leq \frac{(x+y)^2}{2 - \sqrt{2}} \Rightarrow \sqrt{xy} \leq \frac{x+y}{\sqrt{2}}$$

Why

$$-\sqrt{2}xy - \sqrt{2}xy = 2xy?$$

This is not true!

You should get

$$-2\sqrt{2}xy$$

7/10

8.) Find the infimum and supremum of the following sets

a)  $E := \{x \in \mathbb{R} : x \geq 0 \text{ and } x^2 \leq 9\}$

$$x \geq 0 \quad x^2 \leq 9 \Rightarrow x \leq 3$$

(justify why?)

$E \neq \emptyset$  because  $3 \in E$

5/5

If  $x > 3$  then  $x \cdot x > 3 \cdot x \Rightarrow x^2 > 3^2 \Rightarrow x^2 > 9$

$x \cdot 3 > 3 \cdot 3 \rightarrow$  you have used this for

$$\text{so, } x^2 > 9 \Rightarrow x \notin E$$

Exercise 5.

so,  $x \in E \Rightarrow x \leq 3$  Therefore, 3 is an upper bound for  $E$ . By AC the supremum exists.

not necessary.

$$\text{Suppose } x^2 > 9 \Rightarrow x > 3. \text{ Let } \delta := \min\{3, \frac{p-x}{2x+1}\} * \delta \leq \delta^2 \Leftrightarrow \delta^2 \leq \delta \leq 3 \\ (x+\delta)^2 = x^2 + 2x\delta + \delta^2 \geq \underbrace{x^2 - 2x\delta}_{= x^2 + p - x^2 = p} = x^2 + p - x^2 = p$$

$$\text{So, } (x-\delta)^2 \geq p \geq y^2 \quad \forall y \in E$$

$x - \delta$  is an upper bound of  $E$  and  $x - \delta < x$   $\times$

This is a contradiction to the proof above which shows that  $x - \delta$  is an upper bound of  $E$ . Thus  $3 = \sup E$

8a) Cont'd. Using the previous argument, we know set  $E$  is nonempty and we can prove the existence of an infimum.

$$0 \leq x \quad x^2 \leq 9 \Rightarrow x \leq 3 \\ \Rightarrow 0 \leq x \leq 3$$

Suppose,  $0 \leq p \leq 3$  then

$$\frac{1}{p} \geq \frac{1}{3} \text{ st. } x^2 = \frac{1}{p} \text{ So, } p = \frac{1}{x^2} = \left(\frac{1}{x}\right)^2 \text{ thus } \frac{1}{x} \geq 0$$

Therefore the infimum of  $E$  exists and  $0 := \inf E$

it's not a rigorous proof.  $\rightarrow 1$   
 8b)  $E := \left\{ \frac{4n+5}{n+1} : n \in \mathbb{N} \right\}$

$$1) \text{ if } n=1, \text{ then } \frac{4n+5}{n+1} = \frac{9}{2} \Rightarrow \frac{9}{2} \in E \Rightarrow \frac{9}{2} \in E \Rightarrow \sup E = \frac{9}{2}$$

✓/S

Suppose  $x < \frac{9}{2} \Rightarrow 9 - 2x > 0 \quad \text{why do you do this part?}$

Using AP use  $9 - 2x$  (for  $x$ ) ; use  $x - 5$  (for  $y$ )

$$\frac{4n+5}{n+1} > x \Leftrightarrow 4n+5 > nx + x \\ \Leftrightarrow 4n - nx > x - 5 \\ n(4 - x) > x - 5$$

$$\exists n \in \mathbb{N} \text{ st. } n(9 - 2x) > x - 5 \Rightarrow \frac{4n+5}{n+1} > x \quad \times$$

This is a contradiction because  $\frac{9}{2}$  is the supremum.

What about 4?

9.)  $A \neq \emptyset$ ;  $P(A) :=$  "powerset A"

✓/S. We need to use the definition of countable sets!

pt. 1 ★ The set  $A$  is defined as a "nonempty" set. The powerset  $A$  contains the empty set along with all subsets of  $A$ , therefore  $A \neq \emptyset \neq P(A) := \{\emptyset, a, a_1, a_2, \dots, a_n : a \in P(A)\}$ .

Additionally, suppose we define  $C := \{x : x \in A \text{ and } x \notin f(x)\}$

To show that  $C$  (which represents  $P(A)$ ) is not equal to set  $A$ , we can consider the elements  $a \in A$ . Based on the definition of  $C$ ,  $a \in C$  iff  $a \notin f(a)$ . Therefore  $C \cap \{a\} \neq a \cap \{a\}$ , so,  $C$  is not equal to the set  $A$ .

2 ★ The set of all natural numbers  $\mathbb{N}$  is countably infinite,

so, there is a one-one correspondence from  $\mathbb{N}$  to every element in the set  $N$ . Assuming  $P(\mathbb{N})$  is also countable,

we can assign every subset  $x \in \mathbb{N}$  to a natural number  $n \in \mathbb{N}$ .

### For no. 9.

I would go along the following lines:

- Suppose, if possible, that  $A \sim P(A)$ .
- Use the definition:  $\exists f: A \rightarrow P(A)$  o.t.  $f$  is a bijection.
- Use the hint: define  
 $C := \{x : x \in A \text{ and } x \notin f(x)\}$ .
- By bijectivity,  $\exists a \in A$  s.t.  $f(a) = C$ .

Do we have  $x \in f(x)$ ? Try to answer this question and get a contradiction.

To show these two sets  $\mathbb{N}$  and  $P(\mathbb{N})$  are equinumerous an attempt to pair their elements is shown below.

$$\begin{array}{c} \times \\ \mathbb{N} \end{array} \left\{ \begin{array}{l} 1 \leftrightarrow \{1, 7\} \\ 2 \leftrightarrow \{5, 6\} \\ 3 \leftrightarrow \{4, 2\} \\ 4 \leftrightarrow \{1, 3\} \\ \vdots \quad \vdots \end{array} \right. P(\mathbb{N})$$

So, some elements in  $\mathbb{N}$  are paired with subsets "selfish" containing the same number. (Ex 1)

The set  $P(\mathbb{N})$  contains this set

of "selfish" sets as an element as well. Thus, pairing this "selfish" element to a natural number is a contradiction.

If the set of selfish elements "A" is paired with some natural number "a" where  $a \in A$ , the definition of A is contradicted. Therefore, no element of  $P(\mathbb{N})$  which maps to  $\mathbb{N}$  can exist. Thus  $P(\mathbb{N})$  is uncountable.

6/10

10.) Let  $E \subseteq \mathbb{R}$  be bounded from above and  $E \neq \emptyset$  for  $r \in \mathbb{R}$ , let  $rE := \{rx : x \in E\}$  and  $r+E := \{r+x : x \in E\}$

a.) Show if  $r > 0$ , then  $\sup(rE) = r\sup(E)$

Let  $E := \{x : x \in E\}$ , so  $rE := \{rx : x \in E\}$  is the image of  $\{E\}$  after multiplying every image by r.

So,  $r(x) : x \in E = rx : x \in E$ . Therefore

$A = \sup(E)$  so,  $B = rA = \sup(rE)$ . The set  $rE$  contains an integer r, such that  $E \neq \emptyset$ , and has the upper bound B so,  $\sup(E)$  exists. Suppose by contradiction that  $\sup(rE) < B$ . For an element  $y \in rE$

there exists an element  $x \in E$  such that  $rx = y < B$  and  $x = \frac{y}{r} < A$ . So, there exists another value w such that  $\frac{y}{r} < w < A$  which maps to  $rE$ , but is greater than B. This contradicts that  $B = \sup(rE)$ , so,  $B = rA = \sup(rE)$ .

line you  
got the  
idea!

b.   
→

For 10a), I would suggest that you follow this path:  
Set  $s := \sup(E)$ .

- Show that  $r_s$  is an upperbound for  $r_E$ .  
This means to show  
 $\forall x \in E, \quad r_x \leq r_s.$
- Show that  $r_s$  is the best upperbound:  
that is if  $b$  is another upperbound for  $r_E$ , then  $r_s \leq b$ .

10 b.) For any  $r \in \mathbb{R}$ ,  $\sup(r+E) = r + \sup E$

- S/S
- \* Let  $x = r+E$  then  $x = r+y \quad \forall y \in E$ . By the definition of a supremum,  $y \leq \sup E$ , so  $x \leq r + \sup E$ . Since  $x \in r+E$  was arbitrary,  $r + \sup E$  is an upper bound for  $r+E$ . So,  $\sup(r+E) \leq r + \sup E$ . Great!
  - \* Let  $y \in E$ , then  $r+y \in r+E$ . So, since a supremum is an upper bound,  $r+y \leq \sup(r+E)$  and therefore  $y \leq \sup(r+E) - r$ . So,  $\sup(r+E) - r$  is an upper bound on  $E$ , and  $\sup E \leq \sup(r+E) - r$  therefore  $\sup(r+E) \geq r + \sup E$ . → Because of the Axiom on  $\leq$ .  
Using this in addition to the above argument shows definitively that  $\sup(r+E) = r + \sup(E)$  Great!

2.)  $N \rightarrow N$   $f(1)=1, f(2)=2, f(3)=3$  1, 2, 3, 6,  
 $f(n) := f(n-1) + f(n-2) + f(n-3)$

induction

1/5

$$f(1) = 1 \leq 2^{2-1} = 1 \checkmark$$

$$f(n+1) := f(n) + f(n-1) + f(n-2) \leq 2^n$$