MATH-331 Introduction to Real Analysis | YOUR FULL NAME
Homework 05 | Fall 2021

Due date: November 8^{th} 1:20pm | Total: /70.

Exercise	1 (5)	2 (5)	3 (5)	4 (5)	5 (10)	6 (10)	7 (5)	8 (5)	9 (5)	10 (10)
Score										

Table 1: Scores for each exercises

Instructions: You must answer all the questions below and send your solution by email (to `parisepo@hawaii.edu`). If you decide to not use LaTeX to hand out your solutions, please be sure that after you scan your copy, it is clear and readable. Make sure that you attached a copy of the homework assignment to your homework.
If you choose to use LaTeX, you can use the template available on the course website.
No late homework will be accepted. No format other than PDF will be accepted. Name your file as indicated in the syllabus.

1 Writing problems

For each of the following problems, you will be asked to write a clear and detailed proof. You will have the chance to rewrite your solution in your semester project after receiving feedback from me.

Exercise 1. (5 pts) Let $f : \mathbb{R} \to \mathbb{R}$ and suppose that there exists a positive constant M such that $|f(y) - f(x)| \leq M|y - x|$ for all $x, y \in \mathbb{R}$. Prove that f is uniformly continuous on $\mathbb{R}$.

Solution:

Exercise 2. (5 pts) Let $f : [0, \infty) \to \mathbb{R}$ be nonnegative and continuous such that $\lim_{x\to\infty} f(x) = 0$. Prove that f attains its maximum at some point in $[0, \infty)$.

Solution:

Exercise 3. Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function such that $f([a, b]) \subseteq [a, b]$. Prove that there is a $c \in [a, b]$ such that $f(c) = c$. [This one of the many fixed point Theorem.]

Solution:

Exercise 4. (5 pts) Suppose that $f : (a, b) \to \mathbb{R}$ is twice differentiable on (a, b) and there are two points $c < d$ in (a, b) such that $f'(c) = f'(d)$. Show that there is a point $x \in (c, d)$ such that $f''(x) = 0$.

Solution:

Exercise 5. (10 pts) Suppose that $f : (a, b) \to \mathbb{R}$ is differentiable at $x_0 \in (a, b)$.

a) Prove that

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0 - h)}{2h} \tag{$\star$}$$

exists and equals $f'(x_0)$.

b) Find a continuous function $f : \mathbb{R} \to \mathbb{R}$ and a point $x_0 \in \mathbb{R}$ such that f is not differentiable at x_0, but the limit $(\star)$ exists.

Solution:

-2- Homework problems

Answer all the questions below. Make sure to show your work.

Exercise 6. (10pts)

a) Suppose $r > 0$. Prove that $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = x^r$ is differentiable on $(0, \infty)$ and compute its derivative. [Hint: take for granted that e^x and $\ln x$ are differentiable with $(e^x)' = e^x$ and $(\ln x)' = 1/x$. Rewrite then x^r in terms of a composition of these two differentiable functions.]

b) Define $f(x) = \sqrt{x^2 + \sin x + \cos x}$ where $x \in [0, \pi/2]$. Show that f is a differentiable function.

Solution:

Exercise 7. (5 pts) Show that $S \subseteq \mathbb{R}$ is closed if and only if $\mathbb{R} \setminus S$ is open.

Solution:

Exercise 8. (5 pts) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function and define $g(x) = x^2 f(x^3)$. Show that g is differentiable and compute its derivative.

Solution:

Exercise 9. (5 pts) Prove that $f(x) = \arcsin x$ is differentiable on its domain and find a formula for the derivative of f (justify all your steps!).

Solution:

Exercise 10. (10 pts) Use the Mean-Value Theorem to show the following inequalities.

a) $ny^{n-1}(x-y) \leq x^n - y^n \leq nx^{n-1}(x-y)$ if $n \in \mathbb{N}$ and $0 \leq y \leq x$.

b) $\sqrt{1+x} < 1 + \frac{1}{2}x$ for $x > 0$.

Solution: