Differentiation Exercises

#3, 4, 6, 7, 8, 10, 14, 15, 16, 18, 19, 21, 22, 23, 24, 25, 28, 29, 31, ~~32~~, 34

#3 Let $x>0$. By def., we have

$$\frac{\sqrt{x+h}-\sqrt{x}}{h} = \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}$$

for $h\in\mathbb{R}$ s.t. $x+h>0$. So,

$$\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h} = \lim_{h\to 0}\frac{\cancel{h}}{\cancel{h}(\sqrt{x+h}+\sqrt{x})}$$

$$= \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

from the limit laws. thus,

$$f'(x) = \frac{1}{2\sqrt{x}}.$$

When $x=0$, we have

$$\lim_{h\to 0^+}\frac{\sqrt{h}}{h} = \lim_{h\to 0^+}\frac{1}{\sqrt{h}}$$

since $\lim_{h\to 0^+}\sqrt{h}=0$, the limit doesn't exist.

So, $f'(0)$ doesn't exist and $f(x)=\sqrt{x}$ is not differentiable at $x=0$. $\square$

#4 We have, for $h \in \mathbb{R}$ $(h \neq 0)$

$$\lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \lim_{h \to 0} 2x + h$$

$$= 2x$$

So, $f'(x) = 2x$. $\square$

#6. Homework 5.

#7. Homework 5.

#8. Suppose that f is uniformly differentiable on (a,b). Then f' is differentiable and $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. $\forall x, y \in (a,b)$

$$|x-y| < \delta \implies \left| \frac{f(x)-f(y)}{x-y} - f'(x) \right| < \varepsilon \quad (*)$$

Let $\varepsilon > 0$ & $x, y \in (a,b)$ s.t. $|x-y| < \delta$. Then

$$|f'(x) - f'(y)| = \left| f'(x) - \frac{f(x)-f(y)}{x-y} + \frac{f(x)-f(y)}{x-y} - f'(y) \right|$$

$$\leq \left| f'(x) - \frac{f(x)-f(y)}{x-y} \right| + \left| \frac{f(x)-f(y)}{x-y} - f'(y) \right|$$

Now, we see that $\frac{f(x)-f(y)}{x-y} = \frac{f(y)-f(x)}{y-x}$. So in $(*)$, interchanging the role of x & y (it is true for any x, y), then

$$|y-x| < \delta \implies \left|\frac{f(y)-f(x)}{y-x} - f'(y)\right| < \varepsilon.$$

thus, we get

$$|f'(x) - f'(y)| \le \left|\frac{f(x)-f(y)}{x-y} - f'(x)\right| + \left|\frac{f(y)-f(x)}{y-x} - f'(y)\right|$$

$$< \varepsilon + \varepsilon = 2\varepsilon.$$

So, f' is continuous on (a,b). □

#10 We remark that for any $x \in \mathbb{R}$

$$f(x) \le g(x) \le h(x). \quad (*)$$

Since $f(x_0) = g(x_0)$ and from $(*)$

$$f(x_0) \le g(x_0) \le h(x_0) = f(x_0)$$

$$\implies f(x_0) = g(x_0) = h(x_0).$$

So, $\forall x \in \mathbb{R}$

$$f(x) - f(x_0) \le g(x) - g(x_0) \le h(x) - g(x_0).$$

If $x > x_0$, then

$$\frac{f(x)-f(x_0)}{x-x_0} \le \frac{g(x)-g(x_0)}{x-x_0} \le \frac{h(x)-h(x_0)}{x-x_0}$$

and so $\lim_{x \to x_0^+} \frac{g(x)-g(x_0)}{x-x_0}$ exists and

$$\lim_{x \to x_0^+} \frac{f(x)-f(x_0)}{x-x_0} \le \lim_{x \to x_0^+} \frac{g(x)-g(x_0)}{x-x_0} \le \lim_{x \to x_0^+} \frac{h(x)-h(x_0)}{x-x_0}.$$

So, $$f'(x_0) \le \lim_{x \to x_0^+} \frac{g(x)-g(x_0)}{x-x_0} \le h'(x_0) \quad (*)$$

Also, if $x < x_0$, then

$$\frac{h(x)-h(x_0)}{x-x_0} \le \frac{g(x)-g(x_0)}{x-x_0} \le \frac{f(x)-f(x_0)}{x-x_0}.$$

Then $\lim_{x \to x_0^-} \frac{g(x)-g(x_0)}{x-x_0}$ exists and

$$\lim_{x \to x_0^-} \frac{h(x)-h(x_0)}{x-x_0} \le \lim_{x \to x_0^-} \frac{g(x)-g(x_0)}{x-x_0} \le \lim_{x \to x_0^-} \frac{f(x)-f(x_0)}{x-x_0}.$$

So, $$h'(x_0) \le \lim_{x \to x_0^-} \frac{g(x)-g(x_0)}{x-x_0} \le f'(x_0) \quad (**).$$

From $(*)$ & $(**)$, we get that

$$\begin{aligned} f'(x_0) \le \lim_{x \to x_0^+} \frac{g(x)-g(x_0)}{x-x_0} &\le h'(x_0) \\ &\le \lim_{x \to x_0^-} \frac{g(x)-g(x_0)}{x-x_0} \\ &\le f'(x_0). \end{aligned}$$

$$\Rightarrow \lim_{x \to x_0^-} \frac{g(x)-g(x_0)}{x-x_0} = \lim_{x \to x_0^+} \frac{g(x)-g(x_0)}{x-x_0}.$$

$\Rightarrow \lim_{x \to x_0} \frac{g(x)-g(x_0)}{x-x_0}$ exists $\Rightarrow g'(x_0)$ exists.

From the last inequalities, we get

$$f'(x_0) \leq g'(x_0) \leq h'(x_0) \leq f'(x_0)$$

$$\Rightarrow \quad f'(x_0) = g'(x_0) = h'(x_0). \qquad \square$$

#14 Homework 5.

#15. Let $f(x) = \sqrt{x + \sqrt{x + \sqrt{x}}}$, $x \geq 0$.

put $g(x) = x$, $x \geq 0$ &
$h(x) = \sqrt{x}$, $x \geq 0$

Then

$$f(x) = h\big(g(x) + h(g(x) + h(x))\big)$$

h & g are differentiable on $(0, \infty)$, so by the chain rule and the sum rule, f is differentiable on $(0, \infty)$. We have

$$f'(x) = h'\big(g(x) + h(g(x) + h(x))\big) \cdot \Big(g'(x) + h'(g(x) + h(x)) \cdot (g'(x) + h'(x))\Big)$$

$$= \frac{1}{2\sqrt{x + \sqrt{x + \sqrt{x}}}} \cdot \left(1 + \frac{1}{2\sqrt{x + \sqrt{x}}} \cdot \left(x + \frac{1}{2\sqrt{x}}\right)\right).$$

$\square$

#16 We have
$$f(0) = 0 = f(2).$$
So, by Rolle's theorem, there is a $c \in (0,2)$ s.t.
$$f'(c) = 0.$$
We have $f'(x) = \dfrac{1}{2\sqrt{2x - x^2}} \cdot (2-2x)$

So,
$$f'(c) = 0 \iff c = 1. \qquad \square$$

#18. First, we see that
$$f(-1) = -1 + 3 + b = b+2$$
&
$$f(1) = 1 - 3 + b = b-2.$$
So, $f(1) < f(-1)$.

If $f(-1) < 0$ (equivalent to $b < -2$), then f has no root in $[-1,1]$.

If $f(1) > 0$ (equivalent to $b > 2$), then f has no root in $[-1,1]$.

Suppose $f(1) < 0$ & $f(-1) > 0$. In this case
$$b < 2 \text{ and } b > -2 \implies b \in (-2,2).$$

By the Mean-Value theorem, there exists $c \in (-1,1)$ s.t. $f(c) = 0$. We will show that there is no other root by argument by contradiction.

Suppose $\exists\, c_1, c_2 \in (-1,1)$ with $c_1 \neq c_2$ s.t.

$$f(c_1) = 0 = f(c_2).$$

By Rolle's theorem, there is a t between c_1 & c_2 s.t.

$$f'(t) = 0.$$

But $f'(t) = 3t^2 - 3 = 3(t+1)(t-1)$.

Since $t \in (-1,1)$, $t+1 > 0$ & $t-1 < 0$, and so

$$f'(t) < 0.$$

This is a contradiction, and we conclude that f has exactly one root in $[-1,1]$. $\square$

#21 Suppose $f(0) = g(0)$ & $f'(x) > g'(x)$. $\forall x \in (0,1)$

This means that the function $h(x) := f(x) - g(x)$ is strictly increasing on $(0,1)$. Let $0 < x < y < z < w < 1$

since h is strictly increasing, then

$$f(x)-g(x) < f(y)-g(y) < f(z)-g(z) < f(w)-g(w).$$

Taking the limit as $x \to 0$ & $w \to 1$, by the continuity of f, we see that

$$f(z)-g(z) > f(y)-g(y) \geq f(0)-g(0) = 0$$

$$\Rightarrow \quad f(z)-g(z) \qquad \forall z \in (0,1). \quad (*)$$

and

$$f(y)-g(y) < f(z)-g(z) \leq f(1)-g(1)$$

From $(*)$ and the fact that $y \in (0,1)$, we conclude that

$$f(1)-g(1) > f(y)-g(y) > 0$$

Thus, we deduce that $f(x) > g(x)$ for any $x \in (0,1]$. $\square$

#22 Homework 5.

#23 Homework 6.

#24 Repeat the technic of question 10 in homework 5.

#25. $f : (a,b) \to \mathbb{R}$ s.t. $|f'(x)| \leq M$ $\forall x \in (a,b)$.

By definition of the derivative, we see that $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. if $|x-y| < \varepsilon$

$$\left| \frac{f(x) - f(y)}{x-y} - f'(y) \right| \leq \varepsilon$$

$$\Leftrightarrow \quad \left| \frac{f(x) - f(y)}{x-y} \right| < |f'(y)| + \varepsilon \leq M + \varepsilon.$$

Take $\varepsilon = 1$. So, $\exists \delta_1 > 0$ s.t. if $|x-y| < \delta_1$

then $\left| \frac{f(x) - f(y)}{x-y} \right| < |f'(y)| + 1 \leq M + 1$.

Let $\eta > 0$ be arbitrary. Let $\delta := \min\{ \delta_1, \varepsilon/M+1 \}$.

If $|x-y| < \delta$, then

$$|f(x) - f(y)| < (M+1)\, |x-y| \leq \varepsilon$$

$\Rightarrow$ $|f(x) - f(y)| < \varepsilon$ $\forall x, y \in (a,b)$ with $|x-y| < \delta$.

So, f is uniformly continuous on (a,b). $\square$

Example: $f(x) = \sqrt{x}$ uniformly continuous on $(0,1)$ but $f'(x) = \frac{1}{2\sqrt{x}}$ is unbounded on $(0,1)$.

#28. the derivative is

$$f'(x) = 6x^2 + 6x - 36 = 6(x^2 + x - 6)$$

$$\Rightarrow \quad f'(x) = 6(x+3)(x-2).$$

For $x \in [-1,1]$, we have

$$x+3 > 0 \quad \& \quad x-2 < 0$$

$$\Rightarrow \quad f'(x) < 0.$$

So the function is stricly decreasing on $[-1,1]$. From what we know on strictly increasing function, we know that f is 1-1. $\square$

#29. On $[0,1]$, we have

$$f(0) = 0 \quad \& \quad f(1) = 1 - 3 + 17 = 15$$

& on $[-1,0]$, we have

$$f(-1) = -1 - 3 + 17 = 14 \quad \& \quad f(0) = 0.$$

So, by the MVT, with $[0,1]$ & $L = 10$,

$\exists\, c_1 \in (0,1)$ s.t. $f(c_1) = 10$.

Also, by the MVT, with $[-1,0]$ & $L = 10$,

$\exists\, c_2 \in (-1,0)$ s.t. $f(c_2) = 10$.

So, f is not injective. $\square$

#31 Suppose $a < c < b$ & $f'(c) > 0$.

By the definition of $f'(c)$, $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall x \in (a,b)$

$$|x-c| < \delta \implies \left| \frac{f(x)-f(c)}{x-c} - f'(c) \right| < \varepsilon .$$

Put $\varepsilon := \frac{f'(c)}{2}$. Then for $|x-c| < \delta$

$$-\frac{f(x)-f(c)}{x-c} + f'(c) < \frac{f'(c)}{2}$$

$$\implies \quad \frac{f'(c)}{2} < \frac{f(x)-f(c)}{x-c} .$$

Since $a < c < b$, take a rational r s.t.

$$a < c < r < b.$$

Put $x := \min\{ c + \frac{\delta}{2}, r \}$. Then,

$|x-c| < \delta$, $x - c > 0$ and

$$f(x) - f(c) > \frac{f'(c)}{2} \cdot (x-c) > 0$$

$\implies$ $f(x) > f(c)$ for that choice of x.

$\square$

#33 Don't do it.

#34. Let $g := f^{-1}$, the inverse of f on $[a,b]$ where $f : [a,b] \longrightarrow f([a,b]) =: J$.

So, $g : J \to [a,b]$.

Let $d = f(c)$. Let $y \in J$, $y \neq d$. Since f is a bijection, for such y, $\exists x \neq c$ s.t.

$$y = f(x).$$

So, $\forall y \in J$,

$$\frac{g(y) - g(d)}{y - d} = \frac{g(f(x)) - g(f(c))}{f(x) - f(c)}$$

$$= \frac{x - c}{f(x) - f(c)} = \frac{1}{\frac{f(x) - f(c)}{x - c}}.$$

By the change of variable rule & the quotient rule ($f'(c) \neq 0$), then

$$g'(d) = \lim_{y \to d} \frac{g(y) - g(d)}{y - d} = \lim_{x \to c} \frac{1}{\frac{f(x) - f(c)}{x - c}} = \frac{1}{f'(c)}.$$

$\square$