MATH-331 Introduction to Real Analysis — Victor Ho
Homework 04 — Fall 2021

Due date: October 25th 1:20pm — Total: /70.

Exercise	1 (5)	2 (5)	3 (5)	4 (5)	5 (10)	6 (10)	7 (5)	8 (5)	9 (5)	10 (10)
Score										

Table 1: Scores for each exercises

Instructions: You must answer all the questions below and send your solution by email (to `parisepo@hawaii.edu`). If you decide to not use LaTeX to hand out your solutions, please be sure that after you scan your copy, it is clear and readable. Make sure that you attached a copy of the homework assignment to your homework.
If you choose to use LaTeX, you can use the template available on the course website.
No late homework will be accepted. No format other than PDF will be accepted. Name your file as indicated in the syllabus.

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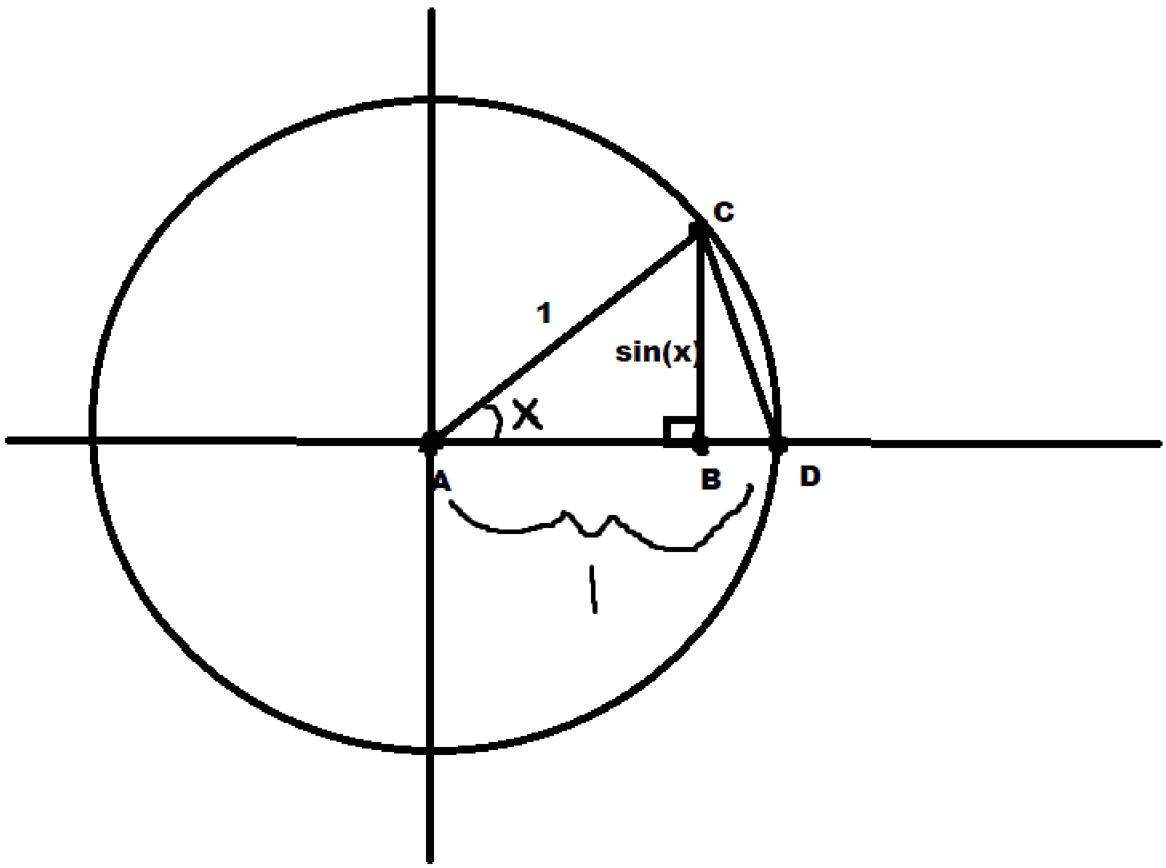
Writing problems

For each of the following problems, you will be asked to write a clear and detailed proof. You will have the chance to rewrite your solution in your semester project after receiving feedback from me.

Exercise 1. (5 pts) Prove that, if $0 < x < \pi/2$, then $0 \leq \sin x \leq x$ with a geometric argument. [Hint: View $\sin x$ as a point on the unit circle in the first quadrant.]

Solution: Since $0 < x < \pi/2$, we can restrict the unit circle to the first quadrant. Then we can draw a right triangle $\triangle ABC$ similar to what I drew below by using the radius of a unit circle as the hypotenuse. Note that side BC is equal to $\sin(x)$ as $\sin(x)$ is equal to opposite over hypotenuse so $\sin(x) = BC/AC$, but $AC = 1$ because the radius of a unit circle is 1, so $\sin(x) = BC$. Also since we are in the first quadrant of the unit circle, $0 \leq sin(x) \leq 1$. Now we can draw another triangle $\triangle ADC$ over $\triangle ABC$ using the radius of the unit circle as a base. This triangle will have a base of 1 and height of $\sin(x)$, so by the area of a triangle formula, it has an area of $\sin(x)/2$. Now consider the circle sector of ADC. By the sector of a circle formula, it has an area of $x/2 * 1^2 = x/2$. Since $\triangle ADC$ is a triangle inside the sector ADC, it has a area less than or equal to the sector ADC. Therefore $sin(x)/2 \leq x/2$, and we can multiply both sides by 2 to get $sin(x) \leq x$. Therefore since $0 \leq \sin(x)$, if $0 < x < \pi/2$ we have $0 \leq sin(x) \leq x$.



Exercise 2. (5 pts) Let $f : A \to \mathbb{R}$ and $g : B \to A$ be two functions where $A, B \subset \mathbb{R}$. Let a be an accumulation point of A and b be an accumulation point of B. Suppose that

- $\lim_{t\to b} g(t) = a$.

- there is a $\eta > 0$ such that for any $t \in B \cap (b - \eta, b + \eta)$, $g(t) \neq a$.
- f has a limit at a.

Prove that $f \circ g$ has a limit at b and $\lim_{x \to a} f(x) = \lim_{t \to b} f(g(t))$. [This is the change of variable rule for limits.]

Solution: By theorem 3.1 in the textbook, since $f : A \to \mathbb{R}$ with $a \in A$ is an accumulation point of A, $\lim_{x \to a} f(x) = f(a)$, and f is continuous at a. By the definition of continuity, since f is continuous at a, for all $\epsilon > 0, \exists \delta > 0$ such that if $|x - a| < \delta, x \in A, |f(x) - f(a)| < \epsilon$. Note that since $t \in B \cap (b - \eta, b + \eta)$, $g(t) \neq a$, and $g : B \to A$, we can say that $g(t) \in A$. Therefore we can substitute in $g(t)$ for x in the definition of continuity for f to get for all $\epsilon > 0, \exists \delta > 0$ such that if $|g(t) - a| < \delta, g(t) \in A, |f(g(t)) - f(a)| < \epsilon$. Since $\lim_{t \to b} g(t) = a$, we can substitute in $g(b)$ for a as well. Therefore we have for all $\epsilon > 0, \exists \delta > 0$ such that if $|g(t) - g(b)| < \delta, g(t) \in A, |f(g(t)) - f(g(b))| < \epsilon$. This is the definition of the limit of $f \circ g$ at b, and it is equal to $f(g(b)) = f(a)$. Therefore we have $\lim_{x \to a} f(x) = f(a) = \lim_{t \to b} f(g(t))$, or $\lim_{x \to a} f(x) = \lim_{t \to b} f(g(t))$.

Exercise 3. (5 pts) Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and suppose that $f(x) = 0$ for each rational number x in $[a, b]$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

Solution: Let Q be the set of rational numbers in $[a, b]$. From a theorem in class, rational numbers are dense in $\mathbb{R}$, so there is a rational number between any two real numbers so using this fact, we can say that any $x \in [a, b]$ is an accumulation point of Q. Therefore by theorem 1.17 in the textbook, since x is an accumulation point of Q, there is a sequence of rational numbers $\{p_n\}_{n=1}^{\infty}$ such that $\{p_n\}_{n=1}^{\infty} \to x$. Therefore $\lim_{n \to \infty} f(p_n) = f(x)$. Since we know that $f(p) = 0$ for any rational number p in $[a, b]$, we can say that $\lim_{n \to \infty} f(p_n) = 0 = f(x)$, and therefore $f(x) = 0$ for any $x \in [a, b]$. $\square$

Exercise 4. (5 pts) Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and suppose that $f(c) > 0$ for some $c \in [a, b]$. Prove that there exist a number η and an interval $[u, v] \subset [a, b]$ such that $f(x) \geq \eta$ for all $x \in [u, v]$.

Solution: From the extreme value theorem, since $f : [a, b] \to \mathbb{R}$ is continuous, and $[u, v] \subset [a, b]$, f is bounded and $\exists d \in [u, v]$ such that $\inf\{f(x), x \in [u, v]\} = f(d)$. Therefore we can take $\eta = f(d)$ and we'll have $f(x) \geq \eta$ for all $x \in [u, v]$ by the definition of the infimum. $\square$

Exercise 5. (10 pts) Let $f : \mathbb{R} \to \mathbb{R}$ be a function that satisfies $f(x + y) = f(x) + f(y)$ for any real number x and y.

a) Suppose that f is continuous at some point c. Prove that f is continuous on $\mathbb{R}$.

b) Suppose that f is continuous on $\mathbb{R}$ and that $f(1) = k$. Prove that $f(x) = kx$ for all $x \in \mathbb{R}$. [Hint: start with x integer, then x rational, and finally use Exercise 3.]

Solution: **a)** Let x, a, b be any real number such that $x, a, b \in \mathbb{R}$, and $f(x) = f(a) + f(b)$. Since f is continuous at c, we can say that $f(a)$ and $f(b)$ are continuous at c. Since $f(x) = f(a) + f(b)$, by theorem 3.2 in the textbook we can say that $f(x)$ is continuous at c. Since x, a, b are arbitrary, we can apply this logic to any $x \in \mathbb{R}$ so f is continuous on $\mathbb{R}$ if f is continuous at some point c.

b) If x is an integer, we can express $f(x)$ as $f(1) + ... + f(1)$, or $\sum_{n=1}^{x} f(1) = xf(1)$. Since $f(1) = k, xf(1) = kx, sof(x) = kx$ if x is an integer. If x is a rational number, it can be expressed as $\frac{p}{q}$ for 2 integers p, q. From the previous part of the problem, we know that $f(p) = kp$ and $f(q) = kq$, therefore $f(x) = kx$ can be expressed as $\frac{f(p)}{f(q)} = \frac{kp}{kq} = k\frac{p}{q} = kx$. Therefore if x is rational and $f(1) = k$, then $f(x) = kx$. Then since $f(x) = kx$ for any rational number in $\mathbb{R}$, WLOG by exercise 3 replacing 0 with kx, we can say that $f(x) = kx$ for all $x \in bR$.

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Homework problems

Answer all the questions below. Make sure to show your work.

Exercise 6. (10pts) For each of the functions below, say if the limit exists or doesn't exist at the given point. Justify your answer (in other words, prove it!)

a) $f(x) = \sin(1/x)$ and $x_0 = 0$.

b) $f(x) = x\sin(1/x)$ and $x_0 = 0$.

Solution: **a)** Assume towards a contradiction that $f(x) = \sin(1/x)$ has a limit, l, at $x_0 = 0$. Therefore $\forall \epsilon > 0$, there exists $\delta > 0$ such that $|\sin\frac{1}{x} - l| < \epsilon$ if $0 < |x - 0| = |x| < \delta$. Fix $\epsilon = 1/2$ and choose x_1, x_2, δ such that $0 < |x_1| < \delta$ and $0 < |x_2| < \delta$. Therefore $|\sin\frac{1}{x_1} - l| < 1/2$ and $|\sin\frac{1}{x_2} - l| < 1/2$. Therefore we can say that $|\sin\frac{1}{x_1} - l| + |\sin\frac{1}{x_2} - l| < 1/2 + 1/2 = 1$. By the triangle inequality, $|\sin\frac{1}{x_1} - l| + \sin\frac{1}{x_2} - l| \geq \sin\frac{1}{x_1} - l + (sin\frac{1}{x_2} - l)| \geq \sin\frac{1}{x_2} - l| \geq \sin\frac{1}{x_1} - l - (sin\frac{1}{x_2} - l)| = \sin\frac{1}{x_1} - \sin\frac{1}{x_2})|$. Therefore by transitivity, $|\sin\frac{1}{x_1} - \sin\frac{1}{x_2})| \leq 1$. Now let $x_1 = \frac{1}{y\pi/2}$ and $x_2 = \frac{1}{y\pi/2+\pi}$ with y being large enough such that $0 < |x_1| < \delta, 0 < |x_2| < \delta$, and $sin(y\pi/2) = \sin(\pi/2) = 1, \sin(y\pi/2 + \pi) = \sin(\pi/2 + \pi) = -1$. Therefore using our previous result and the definition of the limit, we have $|\sin\frac{1}{x_1} - \sin\frac{1}{x_2})| = \sin\frac{1}{\frac{1}{y\pi/2}} - \sin\frac{1}{\frac{1}{y\pi/2+\pi}}| = \sin(y\pi/2) - \sin(y\pi/2 + \pi)| < 1$, and therefore $|1 - (-1)| = 2 < 1$, which is a contradiction. Therefore $f(x) = \sin(1/x)$ does not have a limit at $x_0 = 0$.

b) Note that $|x\sin\frac{1}{x} - 0| = |x\sin\frac{1}{x}|$. Since sin has a range of 0 to 1, we can say that $|x\sin\frac{1}{x}| \leq |x|$. Now let $\delta = \epsilon, \epsilon > 0$. Now if $|x| < \delta$, then $|x\sin\frac{1}{x}| \leq |x| < \delta = \epsilon$, which can be simplified to $|x\sin\frac{1}{x}| < \epsilon$. Since δ, ϵ was arbitrary, we can conclude that for all $\epsilon > 0$ there exists $\delta > 0$ where $|x - 0| < \delta$ implies that $|x\sin\frac{1}{x} - 0| < \epsilon$, and therefore $x\sin\frac{1}{x}$ has a limit at $x_0 = 0$, and it is 0. $\square$

Exercise 7. (5 pts) Let $c \in (a,b)$ and let f be a function defined on (a,b) except at c. Suppose that $f(x) > 0$ for any $x \in (a,b)\backslash\{c\}$, that $\lim_{x\to c} f(x)$ exists, and that

$$\lim_{x\to c} f(x) = \lim_{x\to c} \left[(f(x))^2 - f(x) - 3\right].$$

Find the value of $\lim_{x\to c} f(x)$. Explain each step carefully.

Solution: Since $\lim_{x\to c} f(x) = \lim_{x\to c}((f(x))^2 - f(x) - 3)$ we can subtract $\lim_{x\to c} f(x)$ from both sides to get $0 = \lim_{x\to c}(f(x)^2 - f(x) - 3) - \lim_{x\to c} f(x)$. By the sum rule of limits, this is equal to $0 = \lim_{x\to c}(f(x)^2 - 2f(x) - 3)$ which can be factored into $0 = \lim_{x\to c}((f(x)-3)(f(x)+1))$. Then by the sum and product rules, we have $0 = \lim_{x\to c}((f(x)-3)(f(x)+1)) = (\lim_{x\to c} f(x) - \lim_{x\to c} 3)(\lim_{x\to c} f(x) + \lim_{x\to c} 1)$. Since 3 and 1 are constants, this is equal to $0 = (\lim_{x\to c} f(x) - 3)(\lim_{x\to c} f(x) + 1)$. For this to be true, we must have either $\lim_{x\to c} f(x) = 3$ or $\lim_{x\to c} = -1$. Since we defined $f(x) > 0$ for any $x \in (a,b) \setminus \{c\}$ where $\lim_{x\to c} f(x)$ exists, $\lim_{x\to c} f(x)$ can't be equal to -1. Therefore $\lim_{x\to c} f(x) = 3$. $\square$

Exercise 8. (5 pts) Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} x & , x \in \mathbb{Q} \\ -x & , x \notin \mathbb{Q}. \end{cases}$$

is discontinuous at any point of $\mathbb{R}\backslash\{0\}$ and continuous at 0.

Solution: Let $x \in \mathbb{R}$ be an accumulation point of $\mathbb{R}$, by theorem 3.1 in the textbook, since $f : \mathbb{R} \to \mathbb{R}$ and $x \in \mathbb{R}$ is an accumulation point of $\mathbb{R}$, then the two statements, every sequence in $\mathbb{R}$ converging to x converges to $f(x)$, and f is continuous at x, are equivalent. Note that irrational numbers are also dense in $\mathbb{R}$ as since $\mathbb{Q}$ is dense in $\mathbb{R}$, for any irrational number, r, $\mathbb{Q} + r$ is also dense in $\mathbb{R} + r = \mathbb{R}$. Now since rational numbers are dense in $\mathbb{R}$, we can define a sequence of rational numbers, p_n that converges to an irrational number, r. Therefore from the previously mentioned theorem, $f(p_n) \to f(r)$. Since p_n are rational numbers, $f(p_n) = x$, and since r is irrational, $f(r) = -x$. Therefore $x \to -x$. This is only true if $x = 0$, so f can only be continuous at $x = 0$. $\square$

Exercise 9. (5 pts) Let $p(x) = x^2 + 2$. Find an interval where p is strictly decreasing and find a formula for its inverse.

Solution: p is strictly decreasing in the interval $(-\infty, 0)$. Therefore by a theorem in our lecture notes, since p is strictly decreasing, p is bijective, so p has an inverse. To calculate it, we need to find $p(x) = y$, so $p^{-1}(y) = x$. Therefore $y = x^2 + 2 = y - 2 = x^2 = \pm\sqrt{y-2} = x$. Since our interval is $(-\infty, 0)$, $p^{-1}(x) = -\sqrt{x-2}$. $\square$

Exercise 10. (10 pts) Let $p(x) = ax^3 + bx^2 + cx + d$ be a polynomial of degree 3 and $a > 0$. Prove that p has at least one real root by following these steps:

a) Prove that $\lim_{x\to\infty} p(x) = \infty$.

b) Prove that $\lim_{x\to-\infty} p(x) = -\infty$.

c) Conclude.

[Hint for a): write your polynomial $p(x) = ax^3 + bx^2 + cx + d$ as $x^3(a + b/x + c/x^2 + d/x^3)$ and use the fact that $\lim_{x\to\infty} 1/x^n = 0$ for every $n \geq 1$.]

Solution: **a)** Note that $p(x) = ax^3+bx^2+cx+d$ can be rewritten as $p(x) = x^3(a+b/x+c/x^2+d/x^3)$. Since $\lim_{x\to\infty} 1/x^n = 0$ for every $n \geq 1$, we can use the sum and product rule for limits to get $\lim_{x\to\infty} p(x) = \lim_{x\to\infty} x^3(\lim_{x\to\infty} a+\lim_{x\to\infty} b/x+\lim_{x\to\infty} c/x^2+\lim_{x\to\infty} d/x^3$ which simplifies to $\lim_{x\to\infty} p(x) = \lim_{x\to\infty} x^3(a + 0 + 0 + 0) = a \lim_{x\to\infty} x^3$. Since $\lim_{x\to\infty} x = \infty$ we can say that $\lim_{x\to\infty} x^3 = \infty$ and therefore $a \lim_{x\to\infty} x^3 = \infty$. Therefore $\lim_{x\to\infty} p(x) = a \lim_{x\to\infty} x^3 = a(\infty) = \infty$, so $\lim_{x\to\infty} p(x) = \infty$.

b) Note that since $\lim_{x\to\infty} 1/x^n = 0$ for every $n \geq 1$, $\lim_{x\to-\infty} 1/x^n = 0$ for every $n \geq 1$. We can then rewrite $p(x)$ as $p(x) = x^3(a + b/x + c/x^2 + d/x^3)$. Since $\lim_{x\to-\infty} 1/x^n = 0$ for every $n \geq 1$, we can use the sum and product rule for limits to get $\lim_{x\to-\infty} p(x) = \lim_{x\to-\infty} x^3(\lim_{x\to-\infty} a + \lim_{x\to-\infty} b/x + \lim_{x\to-\infty} c/x^2 + \lim_{x\to-\infty} d/x^3$ which simplifies to $\lim_{x\to-\infty} p(x) = \lim_{x\to-\infty} x^3(a + 0 + 0 + 0) = a \lim_{x\to-\infty} x^3$. Since $\lim_{x\to-\infty} x = -\infty$ we can say that $\lim_{x\to-\infty} x^3 = -\infty$ and therefore $a \lim_{x\to-\infty} x^3 = -\infty$. Therefore $\lim_{x\to-\infty} p(x) = a \lim_{x\to-\infty} x^3 = a(-\infty) = -\infty$, so $\lim_{x\to-\infty} p(x) = -\infty$.

c) We know from our class notes that as a consequence of the sum rules, any polynomial is continuous, so $p(x)$ is continuous on $[-\infty, \infty] \to \mathbb{R}$. Since $\lim_{x\to-\infty} p(x) = -\infty \neq \lim_{x\to\infty} p(x) = \infty$, and $-\infty < 0 < \infty$, by the Intermediate Value Theorem there exists $c \in [-\infty, \infty]$ such that $p(c) = 0$, and that will be our real root. $\square$