Math 331: Homework 3

Liliana Kershner

1. We have that $c_n = b_n - a_n$. Since c_n has a limit, it is convergent and bounded. Since a_n is an increasing sequence $a_{n+1} \geq a_n$. Since b_n is decreasing, $b_n \geq b_{n+1}$. We need to know whether $b_n - a_n$ is an increasing or decreasing sequence. If $a_{n+1} \geq a_n$, this implies $-a_{n+1} \leq -a_n$ and $-b_n \leq -b_{n+1}$, which would mean $-a_n$ is a decreasing sequence and $-b_n$ is an increasing sequence. So c_n must also be a decreasing sequence. Take the limit of both sides as n tends to infinity: $\lim_{n\to\infty} c_n = \lim_{n\to\infty}(b_n - a_n)$.
Then

$$\begin{aligned}\lim_{n\to\infty} c_n &= \lim_{n\to\infty}(b_n - a_n)\\ \lim_{n\to\infty} c_n &= \lim_{n\to\infty}(b_n) - \lim_{n\to\infty}(a_n)\\ 0 &= \lim_{n\to\infty}(b_n) - \lim_{n\to\infty}(a_n)\\ \lim_{n\to\infty}(a_n) &= \lim_{n\to\infty}(b_n)\end{aligned}$$

Therefore the sequences (a_n) and (b_n) converge and share the same limit.

2. If $f(x_n)$ is Cauchy, then it is convergent to some L.
Let us define two new sequences, x_n and y_n as two sequences which satisfy our problem requirements. Then we know that $f(x_n)$ converges to some L_x and $f(y_n)$ converges to some L_y. Define again a new sequence z_n where $z_{2n} = x_n$ and $z_{2n-1} = y_n$. The sequence $f(z_n)$ converges to some limit L_z.
Notice that x_n and y_n are subsequences of z_n. Therefore if z_n converges to L_z then x_n and y_n must also converge to L_z, and $L_z = L_x = L_y$.
Thus, the sequence $f(x_n)$ as well as all its subsequences have a limit at x_0.

3. Assume to a contradiction that there are two limits which exist for the function $f(x)$ as x goes to x_0.
Then we have three conditions:

$$\begin{aligned}x_n &\neq x_0\\ x_n &\to x_0\\ x_n &\in D\end{aligned}$$

where D is the domain of $f(x)$. Then, $f(x)$ has a limit L_1 as $x \to x_0$ if and only if $\forall \varepsilon > 0, \exists \delta > 0$ s.t.
$|x - x_0| < \delta \implies |f(x) - L_1| < \varepsilon$. For the same conditions, $f(x)$ also has a limit at L_2.
Let $\varepsilon > 0$. By the method in the lecture notes from 10/6, let $\varepsilon = \frac{|L_1 - L_2|}{2}$ and $\delta := \min\{\frac{|L_1-L_2|}{2}, \frac{|L_1-L_2|}{2}\}$. Knowing then that $|f(x) - L_1| < \frac{|L_1-L_2|}{2}$ and $|f(x) - L_2| < \frac{|L_1-L_2|}{2}$, we have

$$\begin{aligned}&|f(x) - L_1| + |f(x) - L_2| < |L_1 - L_2|\\ &-|L_1 - L_2| < 2f(x) - L_1 - L_2 < |L_1 - L_2|\\ &-|L_1 - L_2| + L_1 + L_2 < 2f(x) < |L_1 - L_2| + L_1 + L_2\\ &\frac{-|L_1 - L_2| + L_1 + L_2}{2} < f(x) < \frac{|L_1 - L_2| + L_1 + L_2}{2}\end{aligned}$$

However, we know that $f(x) < \frac{|L_1-L_2|}{2} - L_1$ and $f(x) < \frac{|L_1-L_2|}{2} - L_2$ which presents a contradiction.
4. Let L be defined as the limit of $f(x)$ and $g(x)$ as $x \to x_0$ and let x_n be defined as a sequence such that $x_n \to x_0$, and $x_n \neq x_0$.
By the squeeze theorem which we covered in previous lectures, the limit of $h(x)$ as $x \to x_0$ exists and is exactly equal to L.
5.a) Let the function g be a bounded function with domain $(0, \infty)$ and the limit of $f(x)$ is 0. By limit arithmetic properties, regardless of the limit of g, call it L, we have $L \cdot 0$ for the limit of $f(x)g(x)$, so the limit is 0.

b) We first prove this forwards, that if $g(x)$ has a limit at 0, then $f(x)$ has a limit at ∞. If $g(x)$ has a limit at 0, then $\exists L \in \mathbb{R}$ s.t. $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\delta > |x| \implies |g(x) - L| < \varepsilon$. Based on our knowledge that $g(x) = f(\frac{1}{x})$ this can be written as $|f(\frac{1}{x}) - L| < \varepsilon$.

Take $M = \frac{1}{\delta}$. Then since $\delta > 0$, $M > 0$. By the reciprocal rule, we know that if $\delta > |x|$ then $\frac{1}{x} > \frac{1}{\delta}$ and $x > M$. Therefore the limit of $f(x)$ exists at ∞.
Proving it the oppsite way, take that $f(x)$ has a limit at infinity. Therefore there exists an $L \in \mathbb{R}$ s.t. for any $\varepsilon > 0$, there is a real number $M > 0$ s.t if $x > M$, then $|f(x) - L| < \varepsilon$. Take $M = \frac{1}{\delta}$ again. Then $\delta > 0$ and $x > \frac{1}{\delta}$ and by the reciprocal rule, $\delta > \frac{1}{x}$. Since $g(x) = f(\frac{1}{x})$, we can then say that $|g(x) - L| = |f(\frac{1}{x}) - L| < \varepsilon$ so the limit of $g(x)$ exists at 0.

6. a) We see that $\frac{1}{n} + \frac{1}{n+1} + ... + \frac{1}{2n} > \frac{1}{2n} + \frac{1}{2n} + ... + \frac{1}{2n}$, which can be written as $\frac{1}{n} + \frac{1}{n+1} + ... + \frac{1}{2n} > \frac{1}{2n}(n) = \frac{1}{2}$. Therefore this is a bounded sequence. Since each denominator is growing faster than the numerators, this is a monotone, decreasing sequence. Therefore it will converge by BWT.

b) Take $\lim_{n\to\infty} \frac{1+2+3+...+n}{n^2} = \lim_{n\to\infty} \frac{\sum_{n=1}^{\infty} n}{n^2}$.
Divide the numerator and denominator by n^2. We have $\lim_{n\to\infty} \frac{\sum_{n=1}^{\infty} \frac{n}{n^2}}{\frac{n^2}{n^2}}$ and $\lim_{n\to\infty} \frac{\sum_{n=1}^{\infty} \frac{1}{n}}{1} = \lim_{n\to\infty} \sum_{n=1}^{\infty} \frac{1}{n}$.
We know already from previous knowledge that this sequence converges to 0.
7. We have

$$\begin{aligned}\frac{\sqrt{1+x}-1}{x} \cdot \frac{\sqrt{1+x}+1}{\sqrt{1+x}+1} &= \frac{(1+x)-1}{x(\sqrt{1+x}+1)} \\ &= \frac{1}{\sqrt{1+x}+1}\end{aligned}$$

Now we can take the limit since the denominator is non-zero. By quotient rule we have

$$\begin{aligned}\lim_{x\to 0} \frac{1}{\sqrt{1+x}+1} &= \frac{\lim_{x\to 0}(1)}{\lim_{x\to 0}(\sqrt{1+x}+1)} \\ &= \frac{1}{\sqrt{1}+1} \\ &= \frac{1}{2}.\end{aligned}$$

Therefore the limit at 0 is $\frac{1}{2}$.
8. We have

$$\lim_{x\to 1} \frac{f(x)(1-f(x)^2)}{1-f(x)}.$$

We first simplify so the denominator is defined.

$$\begin{aligned}\frac{f(x)(1-f(x)^2)}{1-f(x)} \cdot \frac{1+f(x)}{1+f(x)} &= \frac{f(x)(1-f(x)^2)(1+f(x))}{1-f(x)^2} \\ &= f(x)(1+f(x))\end{aligned}$$

Apply the limit:

$$\begin{aligned}\lim_{x\to 1} f(x)(1+f(x)) &= 1(1+1) \\ &= 2.\end{aligned}$$

9. Let the limit of the function $f(x)$ be equal to L. We know also that x_0 is an accumulation point of the domain of $f(x)$ since $\lim_{x\to x_0} f(x) = L$. Then $\exists \varepsilon, N \in \mathbb{N}$ s.t. $n \geq N \implies |f(x) - L| < \varepsilon$. By the triangle inequality, $||f(x)| - |L|| < |f(x) - L|$, so $||f(x)| - |L|| < \varepsilon$ and the limit exists at x_0 when we take the absolute value of $f(x)$.

10.
a) Since $f(x)$ is a continuous function for $n \geq 0$, then we can directly plug the value of x_0 into the function for x. Therefore its limit is x_0^n.
b) From the lecture on 10/4, we know that for $x_0 \geq 0$, $\lim_{x \to x_0} \sqrt{x} = \sqrt{x_0}$. Notice this would not be true for $x_0 < 0$.