MATH-331 Introduction to Real Analysis | YOUR FULL NAME
Homework 05 | Fall 2021

Due date: November 8^{th} 1:20pm | Total: 52/70.

Exercise	1 (5)	2 (5)	3 (5)	4 (5)	5 (10)	6 (10)	7 (5)	8 (5)	9 (5)	10 (10)
Score	4.5	3	5	5	7	10	3	5	5	4.5

Table 1: Scores for each exercises

Instructions: You must answer all the questions below and send your solution by email (to `parisepo@hawaii.edu`). If you decide to not use LaTeX to hand out your solutions, please be sure that after you scan your copy, it is clear and readable. Make sure that you attached a copy of the homework assignment to your homework.
If you choose to use LaTeX, you can use the template available on the course website.
No late homework will be accepted. No format other than PDF will be accepted. Name your file as indicated in the syllabus.

1 Writing problems

For each of the following problems, you will be asked to write a clear and detailed proof. You will have the chance to rewrite your solution in your semester project after receiving feedback from me.

Exercise 1. (5 pts) Let $f : \mathbb{R} \to \mathbb{R}$ and suppose that there exists a positive constant M such that $|f(y) - f(x)| \leq M|y - x|$ for all $x, y \in \mathbb{R}$. Prove that f is uniformly continuous on $\mathbb{R}$.

4.5/5

Solution: When trying to prove that f is uniformly continuous, we must show that for every $\epsilon > 0$, there exists $\delta > 0$ such that if $x, y \in \mathbb{R}$ with $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$. To do this, we can simply set $\delta = \frac{\epsilon}{2}$. Then we have

$\frac{\varepsilon}{M}$

$$|f(y) - f(x)| \leq M|x - y| \qquad \text{by the assumption}$$
$$|f(y) - f(x)| \leq M|x - y| < \delta M = \epsilon$$

Therefore since δ and ϵ were arbitrary, f is uniformly continuous. $\square$

Exercise 2. (5 pts) Let $f : [0, \infty) \to \mathbb{R}$ be nonnegative and continuous such that $\lim_{x\to\infty} f(x) = 0$. Prove that f attains its maximum at some point in $[0, \infty)$.

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M varies in term of ε. So, if $\varepsilon = f(M)$ then, $\exists M' > 0$ ...

Solution: By the definition of a limit as $x \to \infty$, we have here that for $L = 0$, $\forall \epsilon > 0$, $\exists M > 0$ such that if $x > M$,

$$|f(x)| < \epsilon$$

Now lets set $\epsilon = f(M)$. So that we have

oups: problem because M is defined is terms of ε. So you can't do this.

$$f(x) < f(M)$$

if $x > M$. We know this to be true because by our assumption, $\lim_{x\to\infty} f(x) = 0$. Now consider the closed interval $[0, M]$. Because it is closed and our function f is continuous by our assumption, we can apply the EVT. It states that $\exists c \in [0, M]$ such that $f(c) \geq f(x)$ for all $x \in [0, M]$. So we have the following inequality:

$$f(c) \geq f(M) = \epsilon > f((M, \infty))$$

Therefore, since ϵ was arbitrary, and $f(c)$ is greater than or equal to all $f(x)$ as $x \in [0, \infty]$, $f(c)$ is the functions maximum that it attains. $\square$

Exercise 3. Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function such that $f([a, b]) \subseteq [a, b]$. Prove that there is a $c \in [a, b]$ such that $f(c) = c$. [This one of the many fixed point Theorem.]

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Solution: Let's first define a function $g(x) = f(x) - x$. Now if we can find that there exists a $c \in [a, b]$ such that $g(c) = 0$, then we are done. Given $g(x)$, because the range of $f([a, b]) \in [a, b]$, we know that

$$g(a) \leq 0$$
$$g(b) \geq 0$$

oups: we have $f(a) \geq a \Rightarrow g(a) \geq 0$ & $f(b) \leq b \Rightarrow g(b) \leq 0$.

We also know that f is continuous on $[a, b]$, therefore we can use the IVT. By doing so, since $g(a) \leq 0 \leq g(b)$, we know that there exists $c \in [a, b]$ such that $g(c) = 0$. Therefore, plugging this in, we get

To use IVT, you need $g(a) < 0 < g(b)$. But the cases $g(a) = 0$ or $g(b) = 0$ are simple: just use $c = a$ or $c = b$.

$$g(c) = f(c) - c$$
$$= 0$$
$$f(c) - c = 0$$
$$f(c) = c$$

Therefore, there exists a $c \in [a, b]$ such that $f(c) = c$. $\square$

Exercise 4. (5 pts) Suppose that $f : (a, b) \to \mathbb{R}$ is twice differentiable on (a, b) and there are two points $c < d$ in (a, b) such that $f'(c) = f'(d)$. Show that there is a point $x \in (c, d)$ such that $f''(x) = 0$.

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Solution: Let's look at the closed interval $[c, d]$ and define a function $g(x) = f'(x)$ such that our assumption turns into

$$g(c) = g(d)$$

We know g is differentiable on $[c, d]$ because f is twice differentiable, and $f'' = g'$. It's also continuous because any function that is differentiable on an interval is also continuous on that interval. And so g is continuous on (a, b). g is then also continuous on $[c, d]$ because $[c, d] \in (a, b)$. Therefore, we can apply the Mean Value Theorem. For this case, it states that $\exists\, x \in [c, d]$ such that

$$g'(x) = \frac{g(d) - g(c)}{d - c}$$

From our assumption, we know that $g(c) = g(d)$, and thus

$$\frac{g(d) - g(c)}{d - c} = 0$$

You can also simply use Rolle's theorem.

Therefore, $\exists\, x \in [c, d]$ such that $g'(x) = 0$. Now we have that

$$x \in [c, d] \in (a, b)$$

So $\exists\, x \in (a, b)$ such that $g'(x) = f''(x) = 0$. $\square$

Exercise 5. (10 pts) Suppose that $f : (a, b) \to \mathbb{R}$ is differentiable at $x_0 \in (a, b)$.

a) Prove that

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0 - h)}{2h} \tag{$\star$}$$

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exists and equals $f'(x_0)$.

b) Find a continuous function $f : \mathbb{R} \to \mathbb{R}$ and a point $x_0 \in \mathbb{R}$ such that f is not differentiable at x_0, but the limit $(\star)$ exists.

Solution: a) First let's show that this limit exists with some changing of variables. We can have

$$2h = l$$
$$x_0 + h = y_0$$
$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0 - h)}{2h} = \lim_{l \to 0} \frac{f(y_0) - f(y_0 - l)}{l}$$

$x_0 - h = y_0 - 2h$
$= y_0 - l$ ✓

Therefore, we have the standard definition of a derivative, and it shows that it is differentiable at the point $y_0 = x_0 + h$. Now that we've shown that it exists, we can apply some algebra using the algebra of limits rules from the lecture notes. Let's first look at the limit

$$\lim_{h \to 0} \frac{f(x_0) - f(x_0 - h)}{h} = f'(x_0)$$

→ here this is not exactly the definition of $f'(x_0)$: $f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$

you have to change l by $-h$ so that you get this

We can show this by defining another function $g(x)$ such that $f(x) = -g(x)$ and so $f'(x) = -g'(x)$. We'll also have a variable $k = -h$. If we can show that $g'(x)$ exists, then $f(x)$ exists. The negative doesn't affect the differentiability. Thus we have,

$$\begin{aligned}\lim_{h\to 0} \frac{f(x_0) - f(x_0 - h)}{h} &= \lim_{k\to 0} \frac{g(x_0 + k) - g(x_0)}{-k} \\ &= \lim_{k\to 0} -\left(\frac{g(x_0 + k) - g(x_0)}{k}\right)\end{aligned}$$

by the assumption, f is diff. at x_0.

Now we have, from the product rule of limits,

$$\begin{aligned}\lim_{h\to 0} \frac{f(x_0) - f(x_0 - h)}{h} &= -\lim_{k\to 0} \frac{g(x_0 + k) - g(x_0)}{k} \\ &= -g'(x_0) \\ &= f'(x_0)\end{aligned}$$

Applying this, we then have the following relationships:

$$\begin{aligned}\lim_{h\to 0} \frac{f(x_0 + h) - f(x_0 - h)}{2h} &= \frac{1}{2}\left(\lim_{h\to 0} \frac{f(x_0) - f(x_0 - h)}{h} + \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h}\right) \\ &= \frac{f'(x_0) + f'(x_0)}{2} \\ &= f'(x_0)\end{aligned}$$

Therefore,

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$$\lim_{h\to 0} \frac{f(x_0 + h) - f(x_0 - h)}{2h}$$

is defined and is equal to $f'(x_0)$.

b) Let's define our function to be $f(x) = |x|$. Then show that it is uniformly continuous at $x_0 = 0$, and that there exists a limit at $x_0 = 0$, but that it is not differentiable.

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To be uniformly continuous, for every $\epsilon > 0$, there exists $\delta > 0$ such that if $x, y \in \mathbb{R}$, and $|x-y| < \delta$, then

$$\begin{aligned}|f(x) - f(y)| &< \epsilon \\ ||x| - |y|| &< \epsilon\end{aligned}$$

We have that

$$\begin{aligned}|x| &= |x - y + y| \leq |x - y| + |y| \\ |y| &= |y - x + x| \leq |x - y| + |x| \\ -|x - y| &\leq |x| - |y| \leq |x - y| \\ \Rightarrow ||x| - |y|| &\leq |x - y|\end{aligned}$$

So if we set $\delta = \epsilon$, then $||x| - |y|| \leq |x - y| < \delta = \epsilon$. Thus $f(x) = |x|$ is uniformly continuous on $\mathbb{R}$. Next we show that it has a limit at $x_0 = 0$. At $x_0 = 0$, we suspect $L = 0$. So then $\forall\, \epsilon > 0$, there exists $\delta > 0$ such that if $|x - x_0| < \delta$, then

$$\begin{aligned}|f(x) - L| &< \epsilon \\ |x| &< \epsilon\end{aligned}$$

We have that $x_0 = 0$ so $|x - x_0| < \delta \Rightarrow |x| < \delta$. Therefore we set $\epsilon = \delta$, then

$$|f(x)| = |x| < \delta = \epsilon$$

Therefore $f(x) = |x|$ has a limit at $x_0 = 0$ with $L = 0$.
Next we show that it is not differentiable at $x_0 = 0$. Using our limit definition of a derivative, we have

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

$$\lim_{h \to 0} \frac{|h|}{h}$$

Now that we have this, the left hand limit and the right hand limit must exist and be equal for this limit to exist. If not, then this function is not differentiable at x_0. From the positives, since $h > 0$, we have that $|h| = h$.

$$\lim_{h \to 0^+} \frac{h}{h} = 1$$

Now from the negatives, since $h < 0$, we have

$$\lim_{h \to 0^-} \frac{h}{-h} = -1$$

You had to show that the $\lim_{h \to 0} \frac{f(x_0+h) - f(x_0-h)}{2h}$ for $f(x) = |x|$ and $x_0 = 0$ exists...

Since the left and right limits are not equal to each other, the function $f(x) = |x|$ is not differentiable at $x_0 = 0$. Although it is continouous and has a limit at $x_0 = 0$. ☐

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Homework problems

Answer all the questions below. Make sure to show your work.

Exercise 6. (10pts)

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a) Suppose $r > 0$. Prove that $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = x^r$ is differentiable on $(0, \infty)$ and compute its derivative. [Hint: take for granted that e^x and $\ln x$ are differentiable with $(e^x)' = e^x$ and $(\ln x)' = 1/x$. Rewrite then x^r in terms of a composition of these two differentiable functions.]

b) Define $f(x) = \sqrt{x^2 + \sin x + \cos x}$ where $x \in [0, \pi/2]$. Show that f is a differentiable function.

Solution: a) We can define the function $f(x) = x^r$ in this way:

$$f(x) = x^r = e^{r \ln x}$$

Next, we can then define two functions $g : (0, \infty) \to \mathbb{R}$ and $h : (0, \infty) \to \mathbb{R}$ such that

$$g(x) = e^x$$
$$h(x) = r \ln x$$

We know from our assumption in the hint that $g(x)$ is differentiable on $(0, \infty)$. We also know that $h(x)$ is differentiable on $(0, \infty)$ because r is a positive constant. If we define a constant function y such that $y(x) = r$ for all $x \in (0, \infty)$, y is trivially differentiable. We also know the function $\ln x$ is differentiable on $(0, \infty)$ by our assumption in the hint. Then by the product rule of differentiable functions, $r \cdot \ln x$ is differentiable on $(0, \infty)$. Therefore $h(x)$ is differentiable on $(0, \infty)$.
Then from here, we can define the function f as

$$f(x) = (g \circ h)(x)$$

By the composition rule of differentiable functions, since g and h are differentiable, $f(x)$ is also differentiable.
To compute the differential, we will make use of the assumption in the hint that

$$(e^x)' = e^x$$
$$(\ln x)' = \frac{1}{x}$$

And also the chain rule. Using these, we have the following computations:

$$\begin{aligned}
f(x) &= (g \circ h)(x) \\
f'(x) &= g'(h(x))h'(x) \\
&= g'(r \ln x) \cdot \frac{r}{x} \\
&= e^{r \ln x} \cdot \frac{r}{x} \\
&= \frac{re^{\ln x^r}}{x} \\
&= \frac{rx^r}{x} \\
&= \frac{rx^{r-1}x}{x} \\
&= rx^{r-1}
\end{aligned}$$

b) Let's define the following functions

$$\begin{aligned}
a(x) &= x^2 \\
b(x) &= \sin x \\
c(x) &= \cos x \\
g(x) &= \sqrt{x}
\end{aligned}$$

The functions $a(x), b(x), c(x)$ we know are differentiable for all $x \in \mathbb{R}$. Therefore by the sum rule of differentiable functions,

$$h(x) = a(x) + b(x) + c(x)$$

is differentiable. Now we also know that the function $g(x)$ is differentiable for all $x \in [0, \infty)$. Therefore by the composition rule of differentiable functions,

$$g(h(x)) = f(x)$$

is differentiable for all $x \in [0, \infty]$. Now since the interval $[0, \pi/2] \in [0, \infty)$, $f(x)$ is also differentiable on $[0, \pi/2]$. $\square$

because the concept of accumulation points depends heavily on the set.

x_0 would be an accumulation pt. of $\mathbb{R}$ but not necessarily of S.

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Exercise 7. (5 pts) Show that $S \subseteq \mathbb{R}$ is closed if and only if $\mathbb{R}\backslash S$ is open.

which set? This is important information!

Solution: ($\Rightarrow$) Suppose that S is closed, and assume we have a point $x_0 \in \mathbb{R}\backslash S$. Since x_0 is a point in $\mathbb{R}$, from a theorem in the homework, any point in $\mathbb{R}$ is an accumulation point of $\mathbb{R}$. Therefore x_0 itself is an accumulation point. Now S is closed, so from a theorem in the textbook, we know that S contains all of its accumulation points. Therefore $x_0 \notin S$.
There is therefore a neighborhood Q around x_0 that contain no points of S. But now, we observe that the neighborhood $Q \subset \mathbb{R}\backslash S$, and by definition, $\mathbb{R}\backslash S$ is open.

($\Leftarrow$) Now we supposed that $\mathbb{R}\backslash S$ is open. Our goal is to show that for any accumulation point x_0 of S, that $x_0 \in S$. And therefore S will be closed.
Let $x_0 \in \operatorname{acc}(S)$. We have two cases. Either $x_0 \in S$ or $x_0 \in \mathbb{R}\backslash S$. If $x_0 \in \mathbb{R}\backslash S$, then $\mathbb{R}\backslash S$ would be open. Then by definition, there exists a neighborhood Q of x_0 such that $Q \subset \mathbb{R}\backslash S$. This contradicts the fact that $x_0 \in \operatorname{acc}(S)$. Therefore, $x_0 \in S$ and therefore S is closed. ✓ $\square$

Exercise 8. (5 pts) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function and define $g(x) = x^2 f(x^3)$. Show that g is differentiable and compute its derivative.

Solution: From a theorem in class, we know that every polynomial is differentiable. Therefore, x^2 and x^3 are differentiable. Let's define a function $h(x) = x^3$ such that

$$g(x) = x^2 f(h(x))$$

We know $h(x) = x^3$ is differentiable, and $f(x)$ we know is differentiable from the assumption. Therefore, by the composition rule of differentiable functions, $f(h(x))$ is differentiable. Now, we know also that x^2 is differentiable. Then by the product rule of differentiable functions, we know that

$$x^2 f(h(x))$$

is differentiable. Therefore,

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$$x^2 f(h(x)) = x^2 f(x^3) = g(x)$$

is differentiable.
Now to find the derivative of $g(x)$, we use the chain rule and the product rule for derivatives.
Let's go back and define $g(x) = x^2 f(h(x))$. Then we have

$$\begin{aligned} g'(x) &= (x^2)'(f(h(x)) + x^2(f(h(x))' \\ &= 2x \cdot f(h(x)) + x^2 \cdot f'(h(x)) \cdot h'(x) \\ &= 2x \cdot f(x^3) + x^2 \cdot f'(x^3) \cdot 3x^2 \\ &= 2xf(x^3) + 3x^4 f'(x^3) \end{aligned}$$

Exercise 9. (5 pts) Prove that $f(x) = \arcsin x$ is differentiable on its domain and find a formula for the derivative of f (justify all your steps!).

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Solution: First, let's look at the function $\sin x$. We have that it is strictly increasing on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. We know this because the derivative, $\cos x > 0$ for all $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$. We also know that it is differentiable on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. Therefore, by the inverse rule from the lecture notes, we know that the inverse of $\sin x$ exists and is also differentiable on the domain $\sin((-\frac{\pi}{2}, \frac{\pi}{2})) = (-1, 1)$.
We have that $\arcsin(x) = \sin^{-1}(x)$, and therefore $\arcsin(x)$ is differentiable on its domain.
From the same invesre rule, we are also given a formula to compute the derivative. We have that it is

$c = \sin x$

$$(f^{-1})'(c) = \frac{1}{f'(f^{-1})(c))}$$
$$(\arcsin)'(c) = \frac{1}{\cos(\arcsin(c))}$$

$f(x) = \arcsin x$
$f'(c) = \frac{1}{(f^{-1})'(f(c))}$

On a reference triangle constructed in the unit circle, we have that one of the sides is of length 1, and the other is of length c. When we have the expression $\cos(\arcsin(c))$, it is equal to the lengt of the third side in this right triangle. Therefore, by using the Pythagorean Theorem, we know that the third side is of length $\sqrt{1-c^2}$. Therefore,

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$$

for all $x \in [-1, 1]$. □

Exercise 10. (10 pts) Use the Mean-Value Theorem to show the following inequalities.

a) $ny^{n-1}(x-y) \le x^n - y^n \le nx^{n-1}(x-y)$ if $n \in \mathbb{N}$ and $0 \le y \le x$.

b) $\sqrt{1+x} < 1 + \frac{1}{2}x$ for $x > 0$.

4.5/10

Solution: a) Let's first define a function $f(x) = x^n$ for $n \in \mathbb{N}$. Now let's look at the closed interval $[y, x]$ (since by the assumption, $y \le x$). Now for our function f, we know from a theorem in the lecture notes that any polynomial is continuous and differentiable. So we can apply the Mean Value Theorem for f on $[y, x]$. If we do that, we get that there exists $c \in [y, x]$ such that

$$f'(c) = \frac{f(x) - f(y)}{x - y}$$

$\exists c \in (y, x)$ so, you must suppose that $y < x$.

Now we look at $f'(c)$ compared to $f'(y)$ and $f'(x)$. If we take the second derivative of f, we get

$$f'(x) = nx^{n-1}$$
$$f''(x) = n(n-1)x^{n-2}$$

$f''(x) = 0$ if $n \le 1$ and $f''(x) > 0$ if $n > 1$. Therefore, if $n = 0, 1$, then $f'(y) = f'(x)$. If $n > 1$ then $f'(y) < f'(x)$. So for all $n \in \mathbb{N}$, $f'(y) \le f'(x)$. Now we have the point $c \in [y, x]$. Because of this, and the same reasoning above, we can conclude that

$$f'(y) \le f'(c) \le f'(x)$$
$$f'(y) \le \frac{f(x) - f(y)}{x - y} \le f'(x)$$
$$f'(y)(x-y) \le f(x) - f(y) \le f'(x)(x-y)$$
$$ny^{n-1}(x-y) \le x^n - y^n \le nx^{n-1}(x-y)$$

You must suppose that $y < x$ to avoid a division by 0.

for all $n \in \mathbb{N}$. □

What about b)??