MATH-331 Introduction to Real Analysis | Victor Ho
Homework 05 | Fall 2021

Due date: November 8th 1:20pm | Total: 58.5 /70.

| Exercise | 1 (5) | 2 (5) | 3 (5) | 4 (5) | 5 (10) | 6 (10) | 7 (5) | 8 (5) | 9 (5) | 10 (10) |
|---|---|---|---|---|---|---|---|---|---|---|
| Score | 5 | 5 | 5 | 4 | 10 | 10 | 4.5 | 2 | 5 | 10 |

Table 1: Scores for each exercises

**Instructions:** You must answer all the questions below and send your solution by email (to `parisepo@hawaii.edu`). If you decide to not use LaTeX to hand out your solutions, please be sure that after you scan your copy, it is clear and readable. Make sure that you attached a copy of the homework assignment to your homework.
If you choose to use LaTeX, you can use the template available on the course website.
No late homework will be accepted. No format other than PDF will be accepted. Name your file as indicated in the syllabus.

## 1 — Writing problems

For each of the following problems, you will be asked to write a clear and detailed proof. You will have the chance to rewrite your solution in your semester project after receiving feedback from me.

**Exercise 1.** (5 pts) Let $f : \mathbb{R} \to \mathbb{R}$ and suppose that there exists a positive constant $M$ such that $|f(y) - f(x)| \leq M|y - x|$ for all $x, y \in \mathbb{R}$. Prove that $f$ is uniformly continuous on $\mathbb{R}$.

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**Solution:** For $\epsilon > 0$, let $\delta = \frac{\epsilon}{M}$. Now if $|y-x| < \delta$, then since $M > 0$ we have $M|x-y| < \delta M$. Since $|f(y)-f(x)| \leq M|y-x|$ for all $x, y \in \mathbb{R}$, we can then say $|f(y)-f(x)| \leq |y-x| < \delta M = \frac{\epsilon}{M}M = \epsilon$, which by order axioms imply that $|f(y) - f(x)| < \epsilon$. Therefore we have for all $\epsilon > 0$, a $\delta$ such that if $|y - x| < \delta$ then $|f(y) - f(x)| < \epsilon$ for $x, y \in \mathbb{R}$. This is the definition of uniform continuity, so we can say that $f$ is uniformly continuous on $\mathbb{R}$. $\square$

**Exercise 2.** (5 pts) Let $f : [0, \infty) \to \mathbb{R}$ be nonnegative and continuous such that $\lim_{x\to\infty} f(x) = 0$. Prove that $f$ attains its maximum at some point in $[0, \infty)$.

**Solution:** First note that if $f$ is constant and $f(x) = 0$ for all $x \in [0, \infty]$, then we can just say that 0 is its maximum. Now assume that $f$ is not constant. Since $f$ is not constant, and non-negative, there must be a $n \in [0, \infty)$ such that $f(n) = c > 0$. Now since $\lim_{x\to\infty} f(x) = 0$ and $c > 0$, there must be a value $N \in [0, \infty)$ such that if $x \geq N$, then $f(x) < c$. Since $f(x) < c = f(n)$, we can say that $n < N$. Now consider the interval $[0, N] \subset [0, \infty]$. Since $f : [0, \infty) \to \mathbb{R}$ is

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This is because $f(x) < f(n)$ $\forall x > N$.

continuous on $[0, \infty)$ and $[0, N] \subset [0, \infty]$, by the extreme value theorem, $\exists a \in [0, N]$ such that $sup\{f(x), x \in [0, N]\} = f(a)$. By the definition of the supremum, since $n < N$ and $n > 0$, we can say that $n \in [0, N]$, so by the definition of the supremum, $f(n) = c \leq f(a)$. Since $f(x) < c$ for $x \geq N$, by order axioms, $f(x) < f(a)$ for $x \geq N$. Therefore for $x \in [0, N], f(x) \leq f(a)$ by the definition of the supremum, and for $x \geq N$ we have $f(x) \leq f(a)$ by construction, and therefore we can say that $f(a) \geq f(x)$ for $x \in [0, \infty)$. Therefore since $a \in [0, \infty)$, we can say that $f$ has a maximum in $[0, \infty)$ and it is equal to $f(a)$.

**Exercise 3.** Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function such that $f([a, b]) \subseteq [a, b]$. Prove that there is a $c \in [a, b]$ such that $f(c) = c$. [This one of the many fixed point Theorem.]

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**Solution:** If $f(a) = a$ or $f(b) = b$, then we can take $c = a$ or $c = b$ respectively, and we will have $f(c) = c$ for $c \in [a, b]$. Now consider $f(a) \neq a$ and $f(b) \neq b$. Therefore $f(a) > a$ and $f(b) < b$. Now let $g(x) = f(x) - x$. Note that since $f(x)$ is continuous on $[a, b]$ and $x$ is also continuous on $[a, b]$, we can say that by the sum rule for continuity, $g(x)$ is continuous. At $x = a$, $g(a) = f(a) - a$, and since $f(a) > a$, we can say that $g(a) > 0$. At $x = b$, $g(b) = f(b) - b$ and since $f(b) < b$, we can say that $g(b) < 0$. Therefore since $g(x)$ is continuous on $[a, b]$, and $g(a) \neq g(b)$ and $g(a) < 0 < g(b)$, by the intermediate value theorem there exists a $c \in [a, b]$ such that $g(c) = 0$. Since $g(c) = f(c) - c = 0$, we can add $c$ to both sides to get $f(c) = c$. □

**Exercise 4.** (5 pts) Suppose that $f : (a, b) \to \mathbb{R}$ is twice differentiable on $(a, b)$ and there are two points $c < d$ in $(a, b)$ such that $f'(c) = f'(d)$. Show that there is a point $x \in (c, d)$ such that $f''(x) = 0$.

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Continuity of $f'$ is also required.

**Solution:** Since $f : (a, b) \to \mathbb{R}$ is twice differentiable in $\mathbb{R}$ and $c, d \in (a, b)$ where $c < d$, we can say that $[c, d] \subset (a, b)$, so $f : [c, d] \to \mathbb{R}$ is also twice differentiable in $\mathbb{R}$, and therefore $f' : [c, d] \to \mathbb{R}$ is differentiable in $\mathbb{R}$. Since $f'(c) = f'(d)$, and $f'$ is differentiable on $[c, d]$, by Rolle's theorem, $\exists x \in (c, d)$ such that $f''(x) = 0$. □

**Exercise 5.** (10 pts) Suppose that $f : (a, b) \to \mathbb{R}$ is differentiable at $x_0 \in (a, b)$.

**a)** Prove that

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$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0 - h)}{2h} \qquad (\star)$$

exists and equals $f'(x_0)$.

**b)** Find a continuous function $f : \mathbb{R} \to \mathbb{R}$ and a point $x_0 \in \mathbb{R}$ such that $f$ is not differentiable at $x_0$, but the limit $(\star)$ exists.

**Solution:** a. First note that by the definition of the derivative, since $f$ is differentiable at $x_0 \in (a, b)$, we know that $f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$. Now we can substitute $h$ for $-h$ in that definition to get $f'(x_0) = \lim_{h \to 0} \frac{f(x_0-h) - f(x_0)}{-h} = \frac{f(x_0) - f(x_0-h)}{h}$. Now note that $\lim_{h \to 0} \frac{f(x_0+h) - f(x_0-h)}{2h} = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0-h) + f(x_0) - f(x_0)}{2h} = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{2h} + \frac{f(x_0) - f(x_0-h)}{2h} = \frac{1}{2}(\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} +$

5/5 So the limit (*) exists.

$\lim_{h\to 0}\frac{f(x_0)-f(x_0-h)}{h}$), which by the definition of the derivative equals to $\frac{1}{2}(f'(x_0)+f'(x_0)=\frac{2f'(x_0)}{2}=f'(x_0)$. ✓

5/5 b. Let $f:\mathbb{R}\to\mathbb{R}, f(x)=|x|$. We know that $|x|$ is continuous as we can take $\delta=\epsilon>0$, so that if $|x-x_0|<\delta$, $||x|-|x_0||\le|x-x_0|<\delta=\epsilon$ by the triangle inequality, so by order axioms, $||x|-|x_0||<\epsilon$. Consider $x_0=0$. We know from a proof in class that $|x_0|$ is not differentiable at $x_0=0$. However in the above limit, at $x_0=0, \lim_{h\to 0}\frac{f(|0+h|)-f(|0-h|)}{2h}=\lim_{h\to 0}\frac{f(|h|)-f(|-h|)}{2h}=\lim_{h\to 0}\frac{f(h)-f(h)}{2h}=\lim_{h\to 0}\frac{0}{2h}=\lim_{h\to 0}0=0$, so the limit exists. ☐

$f(|h|)-f(|h|)$

## 2

## Homework problems

Answer all the questions below. Make sure to show your work.

**Exercise 6.** (10pts)

**a)** Suppose $r>0$. Prove that $f:(0,\infty)\to\mathbb{R}$ defined by $f(x)=x^r$ is differentiable on $(0,\infty)$ and compute its derivative. [Hint: take for granted that $e^x$ and $\ln x$ are differentiable with $(e^x)'=e^x$ and $(\ln x)'=1/x$. Rewrite then $x^r$ in terms of a composition of these two differentiable functions.]

10/10 **b)** Define $f(x)=\sqrt{x^2+\sin x+\cos x}$ where $x\in[0,\pi/2]$. Show that $f$ is a differentiable function.

**Solution:** a. Since $e^{\ln(x)}=x, x^r=e^{\ln x^r}$, which by properties of ln simplifies to $e^{r\ln x}$. Since $e^x$ and $\ln x$ are differentiable, we can then define $g(x)=e^x$ and $h(x)=\ln(x)$, so that $f(x)=e^{r\ln x}=(g\circ h)(x^r)$, and since $h$ is differentiable for $x>0$ and $g$ is differentiable for any $h(x)$, we can use the chain rule, so $f=(g\circ h)$ is differentiable and is equal to $f'(x)=(g\circ h)'(x^r)=(e^{r\ln x})'=e^{r\ln x}*\frac{r}{x}$. 5/5 We can then convert $e^{r\ln x}$ back into $e^{\ln x^r}$ and then into $x^r$ to get $x^r*\frac{r}{x}=\frac{rx^r}{x}=rx^{r-1}$. Therefore the derivative of $f(x)=x^r$ is $rx^{r-1}$.

b. Let $g(x)=\sqrt{x}$ and $h(x)=x^2+\sin x+\cos x$. Therefore $g\circ h=\sqrt{x^2+\sin x+\cos x}=f(x)$. Note that since $x^2,\sin x,\cos x$ are all differentiable at $x\in[0,\pi/2]$, $h(x)=x^2+\sin x+\cos x$ is differentiable 5/5 at $x\in[0,\pi/2]$ by the sum rule for derivatives, and is equal to $h'(x)=2x+\cos x-\sin x$. Also note that since $g(x)=\sqrt{x}=x^{1/2}$, and $1/2>0$, by the result of 6a, $g(x)$ is differentiable for $x>0$ and equal to $\frac{1}{2}x^{-1/2}$. Note that $h(x)>0$ for $x\in[0,\pi/2]$ since in that interval $x^2,\sin(x)$, and $\cos(x)$ are always non-negative. Therefore $g(x)$ will be differentiable at $h(x)$ for $x\in[0,\pi/2]$. Therefore since $h$ is differentiable at $x\in[0,\pi/2]$, and $g$ is differentiable at $h(x)$ for $x\in[0,\pi/2]$, by the chain rule $(g\circ h)(x)=f(x)$ is differentiable at $x\in[0,\pi/2]$, so $f$ is differentiable at $x\in[0,\pi/2]$ and is equal to $f'(x)=(g\circ h)'(x)=(\sqrt{x^2+\sin x+\cos x})'=\frac{1}{2\sqrt{x^2+\sin x+\cos x}}*2x+\cos x-\sin x=\frac{2x+\cos x-\sin x}{2\sqrt{x^2+\sin x+\cos x}}$.☐

$\sin(0)=0$
$\cos(\frac{\pi}{2})=0$

**Exercise 7.** (5 pts) Show that $S\subseteq\mathbb{R}$ is closed if and only if $\mathbb{R}\backslash S$ is open.

in LaTeX \Rightarrow

**Solution:** (->) By the definition of a closed set, because $S$ is a closed set, $S$ contains all of its accumulation points. Therefore for $x\in\mathbb{R}\setminus S$, $x$ cannot be an accumulation point of $S$, so by the definition of an accumulation point, $\forall\delta>0(x-\delta,x+\delta)\cap S$ does not contain infinitely many points, and therefore $\exists\delta>0$ where $(x-\delta,x+\delta)\cap\mathbb{R}\setminus S$ contains points which will make $(x-\delta,x+\delta)\subset\mathbb{R}\setminus S$. ✓ Therefore $\exists\delta>0$ where $(x-\delta,x+\delta)\subset\mathbb{R}\setminus S$, which is the definition of an open set, so if $S\subseteq\mathbb{R}$, 4.5/5

It only contains a finite number of points. Take a new $\delta'$ as $\delta':=\min\{|x_1-x|,\dots,|x_n-x|,\delta\}$. then $(x-\delta',x+\delta')\subseteq\mathbb{R}\setminus S$.

$\mathbb{R} \backslash S$ is open.
(<-) Let $x$ be an accumulation point of $S$. Assume towards a contradiction that $x \in \mathbb{R} \setminus S$. Since $x$ is an accumulation point of $S$, by the definition of an accumulation point, $\forall \delta > 0$, $(x-\delta, x+\delta) \cap S$ contains infinitely many points. However since $x \in \mathbb{R} \setminus S$, which is an open set, $\exists \delta > 0$ such that $(x-\delta, x+\delta) \subset \mathbb{R} \setminus S$. This is impossible as since $(x-\delta, x+\delta) \cap S$ contains infinitely many points, $(x-\delta, x+\delta) \not\subset \mathbb{R} \setminus S$, contradicting our claim that $x \in \mathbb{R} \setminus S$. Therefore if $\mathbb{R} \setminus S$ is open, all $acc(S) \in S$, so all $acc(S) \subset S$, which is the definition of a closed set, so $S$ is closed. ✓

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**Exercise 8.** (5 pts) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function and define $g(x) = x^2 f(x^3)$. Show that $g$ is differentiable and compute its derivative.

**Solution:** Since $2 > 0$, we know from lecture notes that $x^2$ is differentiable on $\mathbb{R}$ and equal to $2x$. Since $f : \mathbb{R} \to \mathbb{R}$ is also differentiable, $(x^2 f(x^3))$ is differentiable, and since $g(x) = (x^2 f(x^3))$, we can say $g$ is differentiable and is equal to $(x^2 f(x^3))' = x^2 f'(x^3) + 2x f(x^3)$. □

use chain rule here
$x^2 f'(x^3) \cdot 3x^2$

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**Exercise 9.** (5 pts) Prove that $f(x) = \arcsin x$ is differentiable on its domain and find a formula for the derivative of $f$ (justify all your steps!).

$x \in (-\frac{\pi}{2}, \frac{\pi}{2})$

**Solution:** Let $g(x) = \sin(x)$ on the interval $[-\pi/2, \pi/2]$. We know that $\sin(x)$ is differentiable and continuous on this interval, and also that $g'(x) = \cos(x)$, and $\cos(x) \neq 0$ on $(-\pi/2, \pi/2)$, so $g'(x) \neq 0$ for $x \in [-\pi/2, \pi/2]$. Also note that the range of $\sin(x)$ on $[-\pi/2, \pi/2]$ is $[-1, 1]$. Therefore by the inverse theorem for derivatives, $f(x) = \sin^{-1}(x) = \arcsin(x)$ is differentiable and continuous on $[-1, 1]$ and $f'(x) = f'(g(f(x))) = \frac{1}{g'(f(x))} = \frac{1}{\cos(\arcsin(x))}$. Now note that from trig identities, $\cos^2 x + \sin^2 x = 1$, so $\cos^2 x = 1 - \sin^2 x$ so $\cos x = \sqrt{1 - (\sin x)^2}$. Therefore we can substitute this into the previous equation to get $f'(x) = \frac{1}{\cos(\arcsin(x))} = \frac{1}{\sqrt{1-(\sin(\arcsin(x)))^2}} = \frac{1}{\sqrt{1-x^2}}$ for $x \in [-1, 1]$. To summarize, we have $f(x) = \arcsin(x)$ is differentiable on its domain of $[-1, 1]$ and is equal to $\frac{1}{\sqrt{1-x^2}}$. □

**Exercise 10.** (10 pts) Use the Mean-Value Theorem to show the following inequalities.

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**a)** $ny^{n-1}(x - y) \leq x^n - y^n \leq nx^{n-1}(x - y)$ if $n \in \mathbb{N}$ and $0 \leq y \leq x$.

**b)** $\sqrt{1 + x} < 1 + \frac{1}{2}x$ for $x > 0$.

**Solution:** a. Let $f(x) = x^n$, and we know from work done in class that $x_n$ is both continuous and differentiable on $\mathbb{R}$. Now consider the interval $[y, x] \in \mathbb{R}$. Since $[y, x] \in \mathbb{R}$, $f(x) = x^n$ is both
5/5 continuous and differentiable on $[y, x]$, so by MVP $\exists c \in (y, x)$ such that $f'(c) = nc^{n-1} = \frac{x^n - y^n}{x - y}$. We can then multiply both sides by $x - y$ to get $nc^{n-1}(x - y) = x^n - y^n$. Since $c \in (y, x)$, $c > y$ and $c < x$, and therefore we have $ny^{n-1}(x - y) \leq x^n - y^n$ and $nx^{n-1}(x - y) \geq x^n - y^n$, which by order axioms implies that $ny^{n-1}(x - y) \leq x^n - y^n \leq nx^{n-1}(x - y)$ for $n \in \mathbb{N}$ and $0 \leq y \leq x$.
b. Let $f(x) = \sqrt{1 + x}$ on the interval $[0, x]$. Note that $f$ is differentiable and continuous on the interval $[0, x]$. Let $g(x) = 1 + x$ and $h(x) = \sqrt{x}$. We know from the sum rule of derivatives and continuity that $g$ is differentiable and continuous for $x \geq 0$ and from work done in class, we know that $h$ is differentiable and continuous for $x \geq 0$. Therefore by chain rule $f$ is differentiable

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and continuous for $x \geq 0$, which means it's also differentiable and continuous at $[0, x]$ for $x > 0$. Therefore by MVP $\exists c \in (0, x)$ such that $f'(c) = \frac{1}{2\sqrt{1+c}} = \frac{\sqrt{1+x}-\sqrt{1}}{x} = \frac{\sqrt{1+x}-1}{x}$. We can then multiply both sides by 2 to get $\frac{1}{\sqrt{1+c}} = 2\frac{\sqrt{1+x}-1}{x}$. Note that since $c \in (0, x)$, we can say that $c > 0$ so $\frac{1}{\sqrt{1+c}} < 1$. Therefore by order axioms, since $\frac{1}{\sqrt{1+c}} = 2\frac{\sqrt{1+x}-1}{x}$, $2\frac{\sqrt{1+x}-1}{x} < 1$. We can then simplify $2\frac{\sqrt{1+x}-1}{x} < 1$ to $\frac{\sqrt{1+x}-1}{x} < \frac{1}{2}$ to $\sqrt{1+x} - 1 < \frac{1}{2}x$ to finally $\sqrt{1+x} < \frac{1}{2}x + 1$. Therefore we have $\sqrt{1+x} < \frac{1}{2}x + 1$ for $x > 0$. $\square$