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1. a) We first check the base case, $n=1$. The sum is 1 and the formula evaluates to 1 as well so we are good.

Induction step: Assume that for some $n \in \mathbb{N}$ we have:

$$1+2+\ldots+n = \frac{n(n+1)}{2}$$

We add $n+1$ to each side and get:

$$1+2+\ldots+n = (n+1) + \frac{n(n+1)}{2}$$

The right hand side can now be rewritten as:

$$1+2+\ldots+n = \frac{(n+1)([n+1]+1)}{2}$$

Thus we have proved that if the formula holds for n, it holds for $n+1$. By the the principle of mathematical induction, the identity is true for all integers $n \in \mathbb{N}$ ■

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2. We will prove this by induction. For the base case we have
Let $n=1$, then

$$f(1) \Rightarrow 1 \leq 2^{1-1}$$
$$f(1) \Rightarrow 1 \leq 1$$

So the result is true for $n=1$

Now we will assume the result is true for $n=k$ and $f(k) \leq 2^{k-1}$, and let $n=k+1$ then,

$$f(k+1) = f(k+1-1) + f(k+1-2) + f(k+1-3)$$
$$\Rightarrow f(k+1) = f(k) + f(k-1) + f(k-2)$$

we then plug this into 2^{n-1}

$$< 2^{k-1} + 2^{k-2} + 2^{k-3}$$
$$\Rightarrow 2^k \cdot 2^{-1} + 2^k \cdot 2^{-2} + 2^k \cdot 2^{-3}$$
$$\Rightarrow 2^k \cdot \tfrac{1}{2} + 2^k \cdot \tfrac{1}{4} + 2^k \cdot \tfrac{1}{8}$$
$$\Rightarrow 2^k \left(\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8}\right)$$
$$\Rightarrow 2^k \left(\tfrac{7}{8}\right)$$

so we get

$$f(k+1) < 2^k$$
$$= 2^{(k+1)-1}$$

Therefore, the results are true for $n=k+1$, and is then true for all $n \in \mathbb{N}$. ∎

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3. a) To show $A \sim A$, we must exibit a 1-1 function f from A onto A. It seems reasonable to try 1_A. Now if

$$1_A(a_1) = 1_A(a_2) \quad \text{then}$$
$$a_1 = 1_A(a_1) = 1_A(a_2) = a_2$$

hence, 1_A is 1-1. It is clear that $1_A = A$ since, for any $a \in A$, $1_A(a) = a$. Thus, $A \sim A$

b) Suppose $A \sim B$. Then there is a 1-1 function f from A onto B. To show $B \sim A$, one must find a 1-1 function g from B onto A. The discerning reader should now observe that f^{-1} is the logical canidate. It has already been shown that f^{-1} is 1-1, $\text{dom } f^{-1} = \text{im } f = B$, and $\text{im } f^{-1} = \text{dom} f = A$; hence, f^{-1} is a 1-1 function from B onto A. Therefor $B \sim A$.

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3. c) Assume $A \sim B$ and $B \sim C$. There are 1-1 functions f from A onto B and g from B onto C. We seek a 1-1 function from A onto C. The only reasonable way to obtain a function from A onto C is to consider the composition of g by f, namely $g \circ f$. We know that $\mathrm{dom}(g \circ f) = A$, and it remains to be proved that $(g \circ f)$ is 1-1 and that $\mathrm{im}(g \circ f) = C$. ∎

4. Suppose $a_1, a_2, a_3, \ldots$ is an enumeration of the countable set A and B is any nonempty subset of A. If, for some $n \in \mathbb{N}$, the element a_n belongs to B, then we assign the natural number n to it. For each $n \in \mathbb{N}$ let $k(n)$ denote the number of elements among $a_1, a_2, \ldots, a_n$, which belong to the subset B. Then $0 \leq k(n) \leq n$. Therefore, B is countable by the countability Lemma. ∎

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5. a) $a^2 - b^2 = (a-b)(a+b)$
Since $a \geq 0$ and $b > 0$, $a+b > 0$
and $a-b < 0$ since $a < b$, thus:
$(a-b)(a+b) < 0$. so, $a^2 - b^2 < 0$. Thus $a^2 < b^2$.

b) We have $b > a$. From this
we know that $b-a > 0$.
Breaking this up into a
difference of squares we get
$(\sqrt{b} - \sqrt{a})(\sqrt{b} + \sqrt{a}) > 0$. Then by
dividing each side by $(\sqrt{b} + \sqrt{a})$
we are left with:

$$\sqrt{b} - \sqrt{a} > 0$$
$$\Rightarrow \sqrt{b} > \sqrt{a}$$

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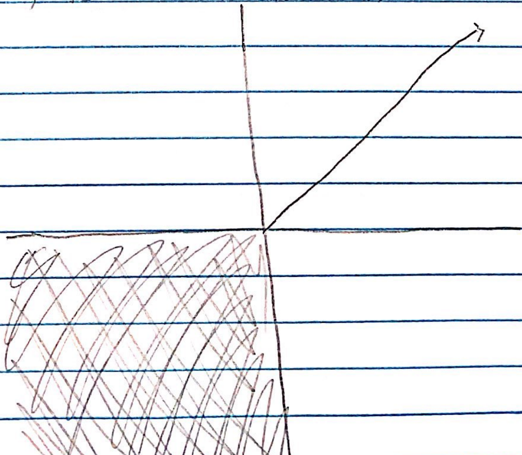
6. We look at all 4 possible cases and see what it produces

1) $x \geq 0, y \geq 0 \Rightarrow 2y = 2x \Rightarrow y = x$
This when plotted is the bisector of 1st quadrant

2) $x \geq 0, y \leq 0, x = 0$. so any pair $(0, y \leq 0)$ is a solution

3) $x \leq 0, y \leq 0 \Rightarrow 0 = 0$. so any pair $(x \leq 0, y \leq 0)$ is a solution

4) $x \leq 0, y \geq 0 \Rightarrow y = 0$. So any pair $(x \leq 0, 0)$ is a solution



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7. We have the following argument

$$0 \le (x-y)^2 \iff 0 \le x^2 - 2xy + y^2$$
$$\iff 4xy \le x^2 + 2xy + y^2$$
$$\iff xy \le \left(\frac{x+y}{2}\right)^2$$
$$\iff \sqrt{xy} \le \left(\frac{x+y}{2}\right)$$

In regards to equality, notice that $\sqrt{xy} \le \frac{x+y}{2} \iff 2\sqrt{xy} \le x+y$, and it becomes clear that equality holds if and only if $x=y$ ∎

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8. a) $E := \{x \in \mathbb{R} : x \geq 0 \text{ and } x^2 \leq 9\}$

infimum: the greatest lower bound is 0 because $x \geq 0$

supremum: the lowest upper bound is 3 because $x \leq 3$

b) $E := \{\frac{4n+5}{n+1} : n \in \mathbb{N}\}$

The least possible value that n can be is 1. After we input 1 we get

$$\frac{4(1)+5}{1+1} = \frac{4+5}{2} = \frac{-9}{2}$$

So the supremum is $9/2$;

This set does not have an inf (E) because we cannot plug in a lower number than 1 because any number below 1 is not in the set of natural numbers anymore

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9. Let us consider there are n total elements in set A. From the definition of power set $P(A)$ the total number of elements in $P(A) = 2^n$. From the definition of equivalent set we know that two sets are called equivalent if they have the same number of elements. So because $2^n \neq n$, A is not equivalent to $P(A)$. $\blacksquare$

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10. a) Given $r \in \mathbb{R}$ and $r > 0$ and $E \subseteq \mathbb{R}$ then $rE = \{rx \mid x \in E\}$
Let $M = \sup(E)$ then $x \leq M$ $\forall x \in E$
$\Rightarrow rx \leq rM$ $\forall r \in E$ [$\because r > 0$]
which shows that rM is an upper bound of rE. Let $\epsilon > 0$ then $M = \sup(E)$ implies there exists $y \in E$ such that $y > M - \frac{\epsilon}{r}$
$\Rightarrow ry > rM - \epsilon$
Since $\frac{\epsilon}{r} > 0$ and $r > 0$ so $\epsilon > 0$.
We see that for $\epsilon > 0$ $\exists ry$ such that $ry > rM - \epsilon$, $ry \in rE$.
Which shows that rM is the least upper bound of rE
$\therefore \sup(rE) = rM = r\sup(E)$ ∎

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10. b)