MATH-331 Introduction to Real Analysis | Ian Oga
Homework 03 | Fall 2021

Due date: October 11$^{\text{th}}$ 1:20pm | Total: /70.

Exercise	1 (5)	2 (5)	3 (5)	4 (5)	5 (10)	6 (10)	7 (5)	8 (5)	9 (5)	10 (10)
Score										

Table 1: Scores for each exercises

Instructions: You must answer all the questions below and send your solution by email (to `parisepo@hawaii.edu`). If you decide to not use LaTeX to hand out your solutions, please be sure that after you scan your copy, it is clear and readable. Make sure that you attached a copy of the homework assignment to your homework.
If you choose to use LaTeX, you can use the template available on the course website.
No late homework will be accepted. No format other than PDF will be accepted. Name your file as indicated in the syllabus.

1
Writing problems

For each of the following problems, you will be asked to write a clear and detailed proof. You will have the chance to rewrite your solution in your semester project after receiving feedback from me.

Exercise 1. (5 pts) Let $(a_n)_{n=1}^{\infty}$ be an increasing sequence and $(b_n)_{n=1}^{\infty}$ be a decreasing sequence. Let $(c_n)_{n=1}^{\infty}$ be the sequence defined by $c_n = b_n - a_n$. Show that if $\lim_{n\to\infty} c_n = 0$, then the sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ converges and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

Solution: Since c_n converges, c_n is bounded. As c_n is bounded, there exists M such that $-M < c_n < M$ for all n. As a_n is increasing and b_n is decreasing, we also know that $a_n > a_1$ and $b_n < b_1$ for all n. Note that:
$a_n = b_n + (-c_n) < b_1 + M$
$b_n = c_n + a_n > -M + a_1$
Therefore a_n is bounded from above and b_n is bounded from below. Since a_n is increasing and bounded from above and b_n is decreasing and bounded from below, both a_n and b_n must converge.
Now suppose $\lim\limits_{n\to\infty} c_n = 0$. Then:
$\lim\limits_{n\to\infty} (b_n - a_n) = 0$
$\lim\limits_{n\to\infty} b_n - \lim\limits_{n\to\infty} a_n = 0$
$\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} a_n$

Exercise 2. (5 pts) Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$, and suppose that x_0 is an accumulation point of D. Suppose that for each sequence $(x_n)_{n=1}^{\infty}$ converging to x_0 with $x_n \in D \backslash \{x_0\}$ for each $n \geq 1$, then the sequence $(f(x_n))_{n=1}^{\infty}$ is Cauchy. Show that f has a limit at x_0.

[Hint: For two sequences (x_n) and (y_n) that satisfy the assumption, define the sequence (z_n) to be $z_{2n} = x_n$ and $z_{2n-1} = y_n$. Show that $(f(z_n))$ converges and the sequence $(f(x_n))$ and $(f(y_n))$ converges to the same limit as $(f(z_n))$. Conclude by a theorem in the lecture notes.]

Solution: Define x_n, y_n, and z_n as given in the hint. As z_n also satisfies the condition of converging to x_0 with $z_n \in D \backslash x_0$ for all $n \geq 1$, the sequence $f(z_n)$ is Cauchy and must converge. Any subsequence of a sequence that converges must also converge to the same limit, so $f(x_n)$ and $f(y_n)$ also converge to the same limit as $f(x_n)$. By Theorem 2.1, since any sequence converging to x_0 and not containing x_0 implies that the sequence $f(x_n)$ also converges, f has a limit at x_0. □

Exercise 3. (5 pts) Prove that if $f : D \subseteq \mathbb{R} \to \mathbb{R}$ has a limit at $x_0 \in \operatorname{acc} D$, then the limit is unique.

Solution: Suppose towards a contradiction that the limit at x_0 is not unique. Then there are two limits L_1 and L_2 where $L_1 \neq L_2$. Without loss of generality, let $L_1 < L_2$. Then:

$$\exists \delta_1, \forall x \in D, |x - x_0| < \delta_1 \to |f(x) - L_1| < \frac{L_2 - L_1}{2}$$
$$\exists \delta_2, \forall x \in D, |x - x_0| < \delta_2 \to |f(x) - L_2| < \frac{L_2 - L_1}{2}$$

By defining $\delta = \min(\delta_1, \delta_2)$, we get that:

$$|x - x_0| < \delta \to |f(x) - L_1| < \frac{L_2 - L_1}{2}$$
$$|x - x_0| < \delta \to |f(x) - L_2| < \frac{L_2 - L_1}{2}$$

We now have the following:
$\frac{L_1 - L_2}{2} < f(x) - L_1 < \frac{L_2 - L_1}{2}$
$\frac{3L_1 - L_2}{2} < f(x) < \frac{L_2 + L_1}{2}$
$f(x) < \frac{L_2 + L_1}{2}$
And similarly:
$\frac{L_1 - L_2}{2} < f(x) - L_2 < \frac{L_2 - L_1}{2}$
$\frac{L_1 + L_2}{2} < f(x) < \frac{3L_2 - L_1}{2}$
$\frac{L_1 + L_2}{2} < f(x)$
Combining these inequalities, $f(x) < \frac{L_2 + L_1}{2} < f(x)$. This is a contradiction, so the limit at x_0 must be unique. □

Exercise 4. (5pts) Suppose $f : D \subseteq \mathbb{R} \to \mathbb{R}$, $g : D \subseteq \mathbb{R} \to \mathbb{R}$, and $h : D \subseteq \mathbb{R} \to \mathbb{R}$ are three functions such that

$$f(x) \leq h(x) \leq g(x) \quad (\forall x \in D)$$

Suppose that f and g have limits at x_0 with $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x)$. Prove that h has a limit at x_0 and

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = \lim_{x \to x_0} g(x)$$

Solution: Call $L = \lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x)$. Since f and g have a limit at x_0, we have the following for an arbitrary ε:

$$\exists \delta_1, \forall x \in D, |x - x_0| < \delta_1 \to |f(x) - L| < \varepsilon$$
$$\exists \delta_2, \forall x \in D, |x - x_0| < \delta_2 \to |g(x) - L| < \varepsilon$$

By defining $\delta = \min(\delta_1, \delta_2)$, we get that:

$$\exists \delta, \forall x \in D, |x - x_0| < \delta \to |f(x) - L| < \varepsilon$$
$$\exists \delta, \forall x \in D, |x - x_0| < \delta \to |g(x) - L| < \varepsilon$$

We then have:
$-\varepsilon < f(x) - L < \varepsilon$
$-\varepsilon - L < f(x) < \varepsilon - L$
and similarly:
$-\varepsilon - L < g(x) < \varepsilon - L$
Combining these inequalities:
$-\varepsilon - L < f(x) \leq h(x) \leq g(x) < \varepsilon - L$
$-\varepsilon - L < h(x) < \varepsilon - L$
$-\varepsilon < h(x) - L < \varepsilon$
$|h(x) - L| < \varepsilon$
In total,

$$\forall x \in D, |x - x_0| < \delta \to |h(x) - L| < \varepsilon$$

As ε was arbitrary, h has a limit of $L = \lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x)$ at x_0. $\square$

Exercise 5. (10 pts) Let $f : (0, \infty) \to \mathbb{R}$ be a function. We say that f has a limit at ∞ if there exists a $L \in \mathbb{R}$ such that for any $\varepsilon > 0$, there is a real number $M > 0$ such that if $x > M$, then $|f(x) - L| < \varepsilon$.

a) Show that if $g : (0, \infty) \to \mathbb{R}$ is bounded and $\lim_{x \to \infty} f(x) = 0$, then $\lim_{x \to \infty} f(x)g(x) = 0$.

b) Let $a > 0$ and suppose that $f : (a, \infty) \to \mathbb{R}$ and define $g : (0, 1/a) \to \mathbb{R}$ by $g(x) = f(1/x)$. Show that f has a limit at ∞ if and only if g has a limit at 0.

Solution:

a) Since g is bounded, there exists B such that $|g(x)| < B$ for all x. Since the limit of f at ∞ is 0, there exists $M > 0$ such that for all $x > M, |f(x)| < \frac{\varepsilon}{B}$ for an arbitrary ε. Then for all $x > M$:
$|f(x)g(x)| = |f(x)||g(x)|$
$|f(x)g(x)| < B|f(x)|$
$|f(x)g(x)| < \varepsilon$
Therefore the limit of $f(x)g(x)$ at ∞ is 0.

b) ($\rightarrow$) Suppose g has a limit at 0. Then for arbitrary ε, there exists δ and L such that $|x| < \delta \rightarrow |g(x) - L| < \varepsilon$. Substituting $\frac{1}{y}$ for x:
$|\frac{1}{y}| < \delta \rightarrow |g(\frac{1}{y}) - L| < \varepsilon$
$\frac{1}{y} < \delta \rightarrow |f(y) - L| < \varepsilon$
$y > \frac{1}{\delta} \rightarrow |f(y) - L| < \varepsilon$
Since $\delta > 0$, $\frac{1}{\delta} > 0$. This altogether proves that f has a limit at ∞

($\leftarrow$) Suppose f has a limit at ∞. Then for arbitrary ε, there exists $M > 0$ and L such that for all $x > M, |f(x) - L| < \varepsilon$ Substituting $\frac{1}{y}$ for x:
$\frac{1}{y} > M \rightarrow |f(\frac{1}{y}) - L| < \varepsilon$
$y < \frac{1}{M} \rightarrow |g(y) - L| < \varepsilon$
As $x > 0$, $\frac{1}{y} > 0$ and $y > 0 > \frac{-1}{M}$ Therefore:
$\frac{-1}{M} < y < \frac{1}{M} \rightarrow |g(y) - L| < \varepsilon$
$|y| < \frac{1}{M} \rightarrow |g(y) - L| < \varepsilon$
Since $M > 0$, $\frac{1}{M} > 0$. This altogether proves that g has a limit at 0. $\square$

2
Homework problems

Answer all the questions below. Make sure to show your work.

Exercise 6. (10pts) For each of the sequences below, determine its nature (converges or diverges)[1]:

a) (a_n) where $a_n = \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n}$.

b) (a_n) where $a_n = \frac{1+2+\cdots+n}{n^2}$.

Solution:

a) We will show that this sequence is decreasing. Consider a_{n+1}:
$a_{n+1} = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2}$
$a_{n+1} = a_n - \frac{1}{n} + \frac{1}{2n+1} + \frac{1}{2n+2}$
$a_{n+1} - a_n = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n}$
Note that $\frac{1}{2n+1}, \frac{1}{2n+2} < \frac{1}{2n}$. Therefore $\frac{1}{2n+1} + \frac{1}{2n+2} < \frac{1}{n}$ and $\frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n} < 0$
$a_{n+1} - a_n < 0$
Therefore the sequence a_n is decreasing. Since n is always positive, every term of a_n is positive and the sequence is bounded from below by 0. Since a_n is decreasing and bounded from below, the sequence must converge.

b) $1 + 2 + \cdots + n$ forms the triangular numbers, and is equal to $0.5(n^2 + n)$. We then have that

$$a_n = \frac{0.5(n^2 + n)}{n^2}$$
$$a_n = 0.5 + \frac{0.5}{n}$$

Since $\frac{0.5}{n}$ will converge, a_n will converge. $\square$

[1]You don't need to compute the limit.

Exercise 7. (5 pts) Define $g : (0,1) \to \mathbb{R}$ by $f(x) = \frac{\sqrt{1+x}-1}{x}$. Prove that g has a limit at 0 and find it.

Solution: For an arbitrary $\varepsilon > 0$, let $\delta = \varepsilon$. To show that the limit at 0 exists and is equal to 0.5, we will show that $|x| < \delta \to |f(x) - 0.5| < \varepsilon$.
$|f(x) - 0.5| = |\frac{\sqrt{1+x}-1}{x} - 0.5|$
$|f(x) - 0.5| = |\frac{(\sqrt{1+x}-1)(\sqrt{1+x}+1)}{x(\sqrt{1+x}+1)} - 0.5|$
$|f(x) - 0.5| = |\frac{1+x-1}{x(\sqrt{1+x}+1)} - 0.5|$
$|f(x) - 0.5| = |\frac{1}{\sqrt{1+x}+1} - 0.5|$
Suppose $|x| < \delta$. Then $-\delta < x < \delta$. As the domain of g is only for positive x, $0 < x < \delta$. Then:
$2 < \sqrt{1+x} + 1 < \sqrt{1+\delta} + 1$
$0.5 > \frac{1}{\sqrt{1+x}+1} > \frac{1}{\sqrt{1+\delta}+1}$
$0 > \frac{1}{\sqrt{1+x}+1} - 0.5 > \frac{1}{\sqrt{1+\delta}+1} - 0.5$
$0 < 0.5 - \frac{1}{\sqrt{1+x}+1} < 0.5 - \frac{1}{\sqrt{1+\delta}+1}$
For $0 < a < b$, $|a| = a$. Therefore $|a| < b$. Applying that to this inequality:
$|0.5 - \frac{1}{\sqrt{1+x}+1}| < 0.5 - \frac{1}{\sqrt{1+\delta}+1}$
$|f(x) - 0.5| < 0.5 - \frac{1}{\sqrt{1+\delta}+1}$
$|f(x) - 0.5| < 0.5 - \frac{1}{\sqrt{1+\varepsilon}+1}$
We will now show that $0.5 - \frac{1}{\sqrt{1+\varepsilon}+1} \leq \varepsilon$ for $\varepsilon > 0$. Assume towards a contradiction that $0.5 - \frac{1}{\sqrt{1+\varepsilon}+1} > \varepsilon$. We now have the following:
$0 > \varepsilon - 0.5 + \frac{1}{\sqrt{1+\varepsilon}+1}$
$0 > (\varepsilon - 0.5)(\sqrt{1+\varepsilon} + 1) + 1$
Since the functions $\varepsilon - 0.5$ and $\sqrt{1+\varepsilon} + 1$ are increasing, function $h(\varepsilon) = (\varepsilon - 0.5)(\sqrt{1+\varepsilon} + 1) + 1$ is increasing. We then have that $h(\varepsilon) > h(0) = 0$ for $\varepsilon > 0$. Therefore $0 > h(\varepsilon) > 0$ which is a contradiction. Therefore $0.5 - \frac{1}{\sqrt{1+\varepsilon}+1} \leq \varepsilon$ for $\varepsilon > 0$ and $|f(x) - 0.5| < \varepsilon$, which proves that g has a limit at 0. $\square$

Exercise 8. (5 pts) Suppose that $f : (0,1) \to \mathbb{R}$ has a limit at $x_0 = 1$ and $\lim_{x\to 1} f(x) = 1$. Compute the value of the limit

$$\lim_{x\to 1} \frac{f(x)(1 - f(x)^2)}{1 - f(x)}.$$

Solution:
$\lim\limits_{x\to x_0} \frac{f(x)(1-f(x)^2)}{1-f(x)} = \lim\limits_{x\to 1} \frac{f(x)(1-f(x))(1+f(x))}{1-f(x)}$
$\lim\limits_{x\to x_0} \frac{f(x)(1-f(x)^2)}{1-f(x)} = \lim\limits_{x\to 1} f(x)(1 + f(x))$
$\lim\limits_{x\to x_0} \frac{f(x)(1-f(x)^2)}{1-f(x)} = \lim\limits_{x\to 1} f(x) \cdot \lim\limits_{x\to 1} (1 + f(x))$
$\lim\limits_{x\to x_0} \frac{f(x)(1-f(x)^2)}{1-f(x)} = \lim\limits_{x\to 1} f(x) \cdot (1 + \lim\limits_{x\to 1} f(x))$
$\lim\limits_{x\to x_0} \frac{f(x)(1-f(x)^2)}{1-f(x)} = 1 \cdot (1 + 1)$
$\lim\limits_{x\to x_0} \frac{f(x)(1-f(x)^2)}{1-f(x)} = 2$ $\square$

Exercise 9. (5 pts) Prove that if $f : D \to \mathbb{R}$ has a limit at x_0, then $|f|(x) := |f(x)|$ has a limit at x_0.

Solution: If f has a limit at x_0, then:

$$\exists L, \forall \varepsilon, \exists \delta, \forall x \in D, |x - x_0| < \delta \to |f(x) - L| < \varepsilon$$

By the reverse triangle inequality, $|f(x) - L| < \varepsilon \to ||f(x)| - |L|| < \varepsilon$. Therefore

$$\exists L, \forall \varepsilon, \exists \delta, \forall x \in D, |x - x_0| < \delta \to ||f(x)| - |L|| < \varepsilon$$

This proves that $|f(x)|$ will have a limit of $|L|$ at x_0. $\square$

Exercise 10. (10 pts) Using the link between sequences and limits of functions, show the following.

a) If $f(x) = x^n$ $(n \geq 0)$, then $\lim_{x \to x_0} f(x) = x_0^n$ for any $x_0 \in \mathbb{R}$.

b) If $x_0 \in [0, \infty)$, then $\lim_{x \to x_0} \sqrt{x} = \sqrt{x_0}$.

Solution: By Theorem 2.1, we will show that for any sequence x_m converging to x_0 where $x_m \in D \backslash x_0$ for all m, the sequence $f(x_m)$ converges to the given value.

a) Since x_m converges to x_0, x_m is bounded. There then exists M such that $x_m \leq M$ for all m. For n=0, $f(x) = 1$ and $\lim\limits_{x \to x_0} f(x) = x_0^0 = 1$. Now suppose that if $f(x) = x^i$ $(i \geq 0)$, $\lim\limits_{x \to x_0} f(x) = x_0^i$ for all $i \leq k$. We then have the following:
$\exists N_1, \forall m > N_1, |x_m - x_0| < \frac{\varepsilon}{2M^k}$
$\exists N_2, \forall m > N_2, |x_m^k - x_0^k| < \frac{\varepsilon}{2x_0}$
Now for all $m > N$ for $N = \max(N_1, N_2)$:
$|x_m^{k+1} - x_0^{k+1}| = |x_m^{k+1} - x_m^k x_0 + x_m^k x_0 - x_0^{k+1}|$
$|x_m^{k+1} - x_0^{k+1}| = |x_m^k(x_m - x_0) + x_0(x_m^k - x_0^k)|$
$|x_m^{k+1} - x_0^{k+1}| \leq |x_m^k(x_m - x_0)| + |x_0(x_m^k - x_0^k)|$
$|x_m^{k+1} - x_0^{k+1}| \leq x_m^k|x_m - x_0| + x_0|x_m^k - x_0^k|$
$|x_m^{k+1} - x_0^{k+1}| < x_m^k \frac{\varepsilon}{2M^k} + x_0 \frac{\varepsilon}{2x_0}$
As $x_m \leq M$, $x_m^k \leq M^k$
$|x_m^{k+1} - x_0^{k+1}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$
$|x_m^{k+1} - x_0^{k+1}| < \varepsilon$
Therefore $\lim\limits_{x \to x_0} x^{k+1} = x_0^{k+1}$. By strong induction, $\lim\limits_{x \to x_0} x^n = x_0^n$ for any integer $n \geq 0$.

b) Since x_m converges to x_0, we have:
$\exists N, \forall m > N, |x_m - x_0| < \varepsilon(\sqrt{x_0})$
Now for all $m > N$:
$|\sqrt{x_m} - \sqrt{x_0}| = |\frac{x_m - x_0}{\sqrt{x_m} + \sqrt{x_0}}|$
$|\sqrt{x_m} - \sqrt{x_0}| = \frac{|x_m - x_0|}{|\sqrt{x_m} + \sqrt{x_0}|}$
$|\sqrt{x_m} - \sqrt{x_0}| < \frac{\sqrt{x_0}}{\sqrt{x_m} + \sqrt{x_0}}\varepsilon$
$|\sqrt{x_m} - \sqrt{x_0}| < \varepsilon$
Therefore $\lim\limits_{x \to x_0} \sqrt{x_m} = \sqrt{x_0}$ $\square$

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Bonus

Exercise 11. Assume that $f : \mathbb{R} \to \mathbb{R}$ such that $f(x + y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$.

a) Show that f has a limit at every point of $\mathbb{R}$.

b) Show that either $\lim_{x\to 0} f(x) = 1$ or $f(x) = 0$ for any $x \in \mathbb{R}$.

Solution:

a)

b)