MATH-331 Introduction to Real Analysis	Created by P.-O. Parisé
Homework 04	Fall 2021

Due date: October 25$^{\text{th}}$ 1:20pm Total: /70.

Exercise	1 (5)	2 (5)	3 (5)	4 (5)	5 (10)	6 (10)	7 (5)	8 (5)	9 (5)	10 (10)
Score										

Table 1: Scores for each exercises

Instructions: You must answer all the questions below and send your solution by email (to parisepo@hawaii.edu). If you decide to not use LaTeX to hand out your solutions, please be sure that after you scan your copy, it is clear and readable. Make sure that you attached a copy of the homework assignment to your homework.
If you choose to use LaTeX, you can use the template available on the course website.
No late homework will be accepted. No format other than PDF will be accepted. Name your file as indicated in the syllabus.

1
Writing problems

For each of the following problems, you will be asked to write a clear and detailed proof. You will have the chance to rewrite your solution in your semester project after receiving feedback from me.

Exercise 1. (5 pts) Prove that, if $0 < x < \pi/2$, then $0 \leq \sin x \leq x$ with a geometric argument. [Hint: View $\sin x$ as a point on the unit circle in the first quadrant.]

Exercise 2. (5 pts) Let $f : A \to \mathbb{R}$ and $g : B \to A$ be two functions where $A, B \subset \mathbb{R}$. Let a be an accumulation point of A and b be an accumulation point of B. Suppose that

- $\lim_{t\to b} g(t) = a$.
- there is a $\eta > 0$ such that for any $t \in B \cap (b-\eta, b+\eta)$, $g(t) \neq a$.
- f has a limit at a.

Prove that $f \circ g$ has a limit at b and $\lim_{x\to a} f(x) = \lim_{t\to b} f(g(t))$. [This is the change of variable rule for limits.]

Exercise 3. (5 pts) Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and suppose that $f(x) = 0$ for each rational number x in $[a, b]$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

Exercise 4. (5 pts) Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and suppose that $f(c) > 0$ for some $c \in [a,b]$. Prove that there exist a number η and an interval $[u,v] \subset [a,b]$ such that $f(x) \geq \eta$ for all $x \in [u,v]$.

Exercise 5. (10 pts) Let $f : \mathbb{R} \to \mathbb{R}$ be a function that satisfies $f(x+y) = f(x) + f(y)$ for any real number x and y.

a) Suppose that f is continuous at some point c. Prove that f is continuous on $\mathbb{R}$.

b) Suppose that f is continuous on $\mathbb{R}$ and that $f(1) = k$. Prove that $f(x) = kx$ for all $x \in \mathbb{R}$. [Hint: start with x integer, then x rational, and finally use Exercise 3.]

2

Homework problems

Answer all the questions below. Make sure to show your work.

Exercise 6. (10pts) For each of the functions below, say if the limit exists or doesn't exist at the given point. Justify your answer (in other words, prove it!)

a) $f(x) = \sin(1/x)$ if $x \neq 0$ and $x_0 = 0$.

b) $f(x) = x\sin(1/x)$ and $x_0 = 0$.

Exercise 7. (5 pts) Let $c \in (a,b)$ and let f be a function defined on (a,b) except at c. Suppose that $f(x) > 0$ for any $x \in (a,b)\backslash\{c\}$, that $\lim_{x\to c} f(x)$ exists, and that

$$\lim_{x\to c} f(x) = \lim_{x\to c} \left[(f(x))^2 - f(x) - 3\right].$$

Find the value of $\lim_{x\to c} f(x)$. Explain each step carefully.

Exercise 8. (5 pts) Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} x & , x \in \mathbb{Q} \\ -x & , x \notin \mathbb{Q}. \end{cases}$$

is discontinuous at any point of $\mathbb{R}\backslash\{0\}$ and continuous at 0.

Exercise 9. (5 pts) Let $p(x) = x^2 + 2$. Find an interval where p is strictly decreasing and find a formula for its inverse.

Exercise 10. (10 pts) Let $p(x) = ax^3 + bx^2 + cx + d$ be a polynomial of degree 3 and $a > 0$. Prove that p has at least one real root by following these steps:

a) Prove that $\lim_{x\to\infty} p(x) = \infty$.

b) Prove that $\lim_{x\to-\infty} p(x) = -\infty$.

c) Conclude.

[Hint for a): write your polynomial $p(x) = ax^3 + bx^2 + cx + d$ as $x^3(a + b/x + c/x^2 + d/x^3)$ and use the fact that $\lim_{x\to\infty} 1/x^n = 0$ for every $n \geq 1$.]

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1. Let C be the set of points (x,y) in plane which satisfy,

$$x^2+y^2=1 \qquad (1)$$

and consider a part of it, namely the smaller part, from point $A=(1,0)$ to $B=(0,1)$ and let $P=(p,q)$ be an intermediate point on this arc AB. Let $C=(p,0)$ and clearly we can see that the length of line segment PC, i.e. q is less than the length of line segment AP, i.e. $\sqrt{(p-1)^2+q^2}$. Now line segment AP of length $\sqrt{(p-1)^2+q^2}$ is also one such polygonal arc and hence by definition of supremum we have

$$x \geq \sqrt{(p-1)^2+q^2} > q$$

By the definition of circular functions, the coordinates of point P are $(\cos L, \sin L)$ and hence $p=\cos x$, $q=\sin x$. We have thus proved that

$$\sin x < x \qquad (2)$$

for values of L such that the point P lies in the smaller part of the curve C between A and B. This means that $x>0$ and

$$x < \int_0^1 \frac{dx}{\sqrt{1-x^2}}$$

where the integral on right is the length of the whole arc AB based on formula of arc-length in terms of integral. If we define circular functions on the basis of arc-length, then the constant π is defined to be twice the above integral

$$\pi = 2\int_0^1 \frac{dx}{\sqrt{1-x^2}}$$

Thus we have proved that $0 \leq \sin x \leq x$ for $0 \leq x < \pi/2$ ∎

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2. Now $f \circ g : B \to \mathbb{R}$ defined by $(f\circ g)(t) = f(g(t))$.
Since b is an accumulation point of B, $f\circ g$ has a limit at b exists, and

$$\lim_{t\to b} g(t) = a$$

Since, f has a limit at a and a is an accumulation point of A, then

$$\lim_{x\to a} f(x) \text{ exists}$$

then

$$\lim_{t\to g} (f\circ g)(t) = \lim_{t\to b} f(g(t)).$$

Now,

$$\lim_{t\to b} g(t) = a$$

implies that there is a neighborhood Q of b such that for $t \in Q\cap B$, $g(t) \neq a$ and $g(t) \in N(a)$, the neighborhood of a. Also,

$$\lim_{x\to a} f(x) = l$$

Implies that there is a neighborhood P of a such that $N(a) \subseteq P$, for $x \in P\cap A$,

$$f(x) \in (l-\varepsilon, l+\varepsilon), \text{ for } \varepsilon > 0.$$

Then we get, for $t \in Q\cap B$, and $g(t) \in N_a \subset P\cap A$ such that $f\circ g(t) \in (l-\varepsilon, l+\varepsilon)$, for $\varepsilon > 0$.
Then,

$$\lim_{t\to b} f\circ g(t) = \lim_{x\to a} f(x) \quad \blacksquare$$

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3. Suppose $a<b$. Suppose toward a contradiction, that there is an $x \in [a,b]$ with $f(x) \neq 0$. As f is continuous, there is for $\varepsilon = \frac{|f(x)|}{2}$ some $\delta > 0$ such that for all $x' \in [a,b]$ with $|x - x'| < \delta$, it follows that $|f(x) - f(x')| < \varepsilon$. As $a<b$, there is however a rational $x' \in [a,b]$ with $|x - x'| < \delta$. But now
$|f(x) - f(x')| = |f(x)| > \frac{|f(x)|}{2} = \varepsilon$

We have reached a contradiction and so $f(x) = 0$ for all $x \in [a,b]$ ∎

4. Let $f(c) > 0$ and $f: [a,b] \to \mathbb{R}$. As the function is continuous, we have, given $\varepsilon > 0$ there exists a $\delta > 0$ such that $|x - c| < \delta$ implies $|f(x) - f(c)| < \varepsilon$. We choose $\varepsilon = f(c)/2$ so that $|f(x) - f(c)| < f(c)/2$ for all $|x - c| < \delta$. That is, $-f(c)/2 < f(x) - f(c) < f(c)/2$ for all $-\delta < x - c < \delta$. So we have $f(c)/2 < f(x) < 3f(c)/2$ for all $c - \delta < x < c + \delta$.
Thus we have $f(x) \geq \eta$ where $\eta = \frac{f(c)}{2}$ for all $x \in (c-\delta, c+\delta)$. Thus we can simply take $u = c - \delta/2$, $v = c + \delta/2$ and so $c \in [u,v] \subseteq (c-\delta, c+\delta)$ and $f(x) \geq \eta$ where $\eta = \frac{f(c)}{2}$ as required. ∎

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5 a) If there exists $c \in \mathbb{R}$ such that $f(c) = 0$, then
$$f(x+c) = f(x)f(c) = 0$$
As every real number y can be written as $y = x+c$ for some real x, this function is either everywhere zero or nowhere zero. To prove the latter, let's consider the case that f is not the constant function $f=0$.
To prove continuity in this case, note for any $x \in \mathbb{R}$
$$f(x) = f(x+0) = f(x)f(0) \Rightarrow f(0) = 1$$
Continuity at 0 tells us that given any $\varepsilon_0 > 0$, we can find $\delta_0 > 0$ such that $|x| < \delta_0$ implies
$$|f(x) - 1| < \varepsilon_0$$
So let $c \in \mathbb{R}$ be fixed arbitrarily. Let $\varepsilon > 0$.
By continuity of f at 0, we choose $\delta > 0$ such that
$$|x-c| < \delta \Rightarrow |f(x-c) - 1| < \frac{\varepsilon}{|f(c)|}$$
Now notice that for all x such that $|x - c| < \delta$, we have
$$\begin{aligned} |f(x) - f(c)| &= |f(x-c+c) - f(c)| \\ &= |f(x-c)f(c) - f(c)| \\ &= |f(c)||f(x-c) - 1| \\ &< |f(c)| \frac{\varepsilon}{|f(c)|} = \varepsilon \end{aligned}$$
Hence f is continuous at c. Since c was arbitrary, f is continuous on all of $\mathbb{R}$. ∎

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5. b) We claim that, for $x \in \mathbb{R}$, and $n \in \mathbb{N}$, $f(nx) = nf(x)$. The claim is trivially true when $n=1$. Suppose, for $n \geq 1$, that $f(nx) = nf(x)$. Then,

$$\begin{aligned} f((n+1)x) &= f(nx+x) \\ &= f(nx)+f(x) \\ &= nf(x)+f(x) \\ &= (n+1)f(x) \quad (1) \end{aligned}$$

We now show that, for all $n \in \mathbb{N}$ and $y \in \mathbb{R}$, $f(\frac{y}{n}) = \frac{f(y)}{n}$. This follows by letting y/n take the role of x in (1). Next we show that, for all $r \in \mathbb{Q}^+$, $f(r) = f(1)r = kr$. Let $r \in \mathbb{Q}^+$. Then there exists $m, n \in \mathbb{N}$ such that $r = m/n$. Thus

$$\begin{aligned} f(r) = f(m/n) &= f(m \cdot \tfrac{1}{n}) \\ &= mf(\tfrac{1}{n}) \\ &= \tfrac{m}{n} f(1) \\ &= rk \end{aligned}$$

Next we observe that, for $x \in \mathbb{R}^+$, $f(x) = kx$. Let $x \in \mathbb{R}^+$, and let (r_n) be a sequence of positive rationals that converges to x. Then

$$\begin{aligned} f(x) &= \lim_{n\to+\infty} f(r_n) \\ &= \lim_{n\to+\infty} kr_n \\ &= k \lim_{n\to+\infty} r_n \\ &= kx \end{aligned}$$

If $x=0$, then we also claim $f(x)=kx$, i.e., we claim $f(0)=0$. This follows from the given statements, because $f(0) = f(0+0) = f(0)+f(0)$. Finally we show that, for $x \in \mathbb{R}^-$, $f(x) = kx$. Clearly,

$$f(0) = 0 = f(x-x) = f(x+(-x)) = f(x) + f(-x).$$

Thus, for $x \in \mathbb{R}^-$

$$\begin{aligned} f(x) &= -f(-x) \\ &= -k(-x) \\ &= kx \end{aligned}$$

∎

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8. $f : \mathbb{R} \to \mathbb{R}$, $f(x) := \begin{cases} x, & x \in \mathbb{Q} \\ -x, & x \in \mathbb{Q} \end{cases}$

let $\left.\begin{array}{l} a_n \in \mathbb{Q} \xrightarrow{a_n \neq a} a \\ b_n \in \mathbb{Q} \longrightarrow a \end{array}\right\rangle \begin{array}{l} f(a_n) = a_n \to a \\ f(b_n) = 0 \to 0 \end{array}$

$\Rightarrow \lim f(a_n) = a$ and $\lim f(b_n) = 0$

i.e. $\boxed{a=0}$

i.e. limit exists, iff $a = 0$

Therefore $f(x)$ is discontinuous at any point of $\mathbb{R} \setminus \{0\}$ and continuous at 0.

$ax^3 + bx^2 + cx + d$

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9. $p(x) = x^2 + 2$

$f'(x) < 0$

$2x < 0$

$p(x)$ is strictly decreasing: $(-\infty, 0)$

to find the inverse we have,

$y = x^2 + 2$

$x = y^2 + 2$

$\sqrt{y^2} = \sqrt{x-2}$

$y = \sqrt{x-2}$

so $p(x)^{-1} = \sqrt{x-2}$

10. a) Let $p(x) = ax^3 + bx^2 + cx + d$

$\Rightarrow p(x) = x^3(a + b/x + c/x^2 + d/x^3)$

so

$\lim_{x \to +\infty} x^3(a + b/x + c/x^2 + d/x^3) = \lim_{x \to +\infty} x^3(a + 0 + 0 + 0)$

$= \lim_{x \to +\infty} x^3(a) = \infty$

b) also we have

$\lim_{x \to -\infty} x^3(a + b/x + c/x^2 + d/x^3) = \lim_{x \to -\infty} x^3(a + 0 + 0 + 0)$

$= \lim_{x \to -\infty} x^3(a) = -\infty$

c) That is, one side of the graph goes up forever and the other side goes down forever. Since the function is continuous, by IVT, it must hit all values between $-\infty$ and ∞ for some x. Hence, in particular it must hit 0, and so the function $p(x)$ has at least one real root. ∎

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6. a) The limit does not exist. Let $f(x) = |\sin(1/x)|$ and consider a sequence (S_n), $S_n = 1/(n\pi/2)$, which converges to 0^+. However $f(S_n) = |\sin(1/S_n)| = |\sin(\pi/2)|$, which does not converge. So $\lim_{x \to 0} \sin(1/x)$ does not exist. ∎

b) Consider the function $f:(0,1) \to \mathbb{R}$, defined by $f(x) = x\sin\frac{1}{x}$. It has been shown above that $\sin\frac{1}{x}$ fails to have a limit at zero. However, $\sin\frac{1}{x}$ is bounded above by 1 and below by -1. Now it is clear that

$$|f(x)| = \left|x\sin\left(\frac{1}{x}\right)\right| \leq |x|$$

hence, f has a limit at 0; in fact $\lim_{x_0 \to 0} f(x) = 0$ ∎

7. $\lim_{x \to c} f(x) = \lim_{x \to c} [(f(x))^2 - f(x) - 3]$

Say $\lim_{x \to c} f(x) = L$

Now, $\lim_{x \to c} f(x) = \lim_{x \to c} [(f(x))^2 - f(x) - 3]$

$\Rightarrow \lim_{x \to c} f(x) = \lim_{x \to c} (f(x))^2 - \lim_{x \to c} f(x) - \lim_{x \to c} 3$

$\Rightarrow \lim_{x \to c} f(x) = (\lim_{x \to c} f(x))^2 - \lim_{x \to c} f(x) - 3$

$L = L^2 - L - 3$

$0 = L^2 - 2L - 3$

$(L+1)(L-3) = 0$

$L = 3, -1$

Hence $\lim_{x \to c} f(x) = 3$ since $f(x) > 0$