Math 331: Homework 5

1. Let $f: \mathbb{R} \to \mathbb{R}$ be a function. Suppose $\exists$ a positive constant function on $M \ni |f(y) - f(x)| \leq M|y-x|$, for all x, y in $\mathbb{R}$.
Now we have to show that f is uniformaly continuous on $\mathbb{R}$. Let $\varepsilon > 0$ and let $x, y \in \mathbb{R}$. Let $\delta > 0$ such that $|y-x| < \delta$. Now consider $|f(y) - f(x)| \leq M|y-x| < M\delta$
Choose $\delta = \frac{\varepsilon}{M}$ then $\delta > 0$

$$|f(y) - f(x)| \leq M|y-x| < M\delta = M\frac{\varepsilon}{M} = \varepsilon$$

i.e. for $\varepsilon > 0$ $\exists$ $\delta > 0$ $\ni |y-x| < \delta \Rightarrow |f(y) - f(x)| < \varepsilon$ for all x, y in $\mathbb{R}$. Hence f is uniformly continuous on $\mathbb{R}$. ∎

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2. Assume that $f(x)$ is not identically 0. Then there are numbers a, c such that $a \geq 0$, $c > 0$ such that $f(a) = c$. Now we also know that $\lim_{x \to \infty} f(x) = 0$. Hence, there is a number $N > 0$ such that $f(x) < c$ for all $x \geq N$. Moreover we can always choose $N > a$. Thus we have $f(x) < c$ for $x \geq N$ and $f(x) = c$ for at least one value $a \in [0, N]$. Let M be the maximum value of $f(x)$ in $[0, N]$. Clearly this is guaranteed because $f(x)$ is continuous in closed interval $[0, N]$. It is obvious that $M \geq f(a) = c$. And since $f(x) < c \leq M$ for all $x \geq N$. It follows that M is the maximum value of $f(y)$ on $[0, \infty)$. ∎

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3. Suppose that $f:[a,b]\to\mathbb{R}$ is a continuous function such that $f([a,b])\subseteq[a,b]$. So we have

$$f(a)\geq a \quad , f(b)\leq b$$
$$\Rightarrow f(a)-a\geq 0, \; f(b)-b\leq 0 \quad ①$$

Now we consider $h(x)=f(x)-x$. We get

$$h(a)=f(a)-a$$
$$h(b)=f(b)-b$$

From ①, we get $h(a)\geq 0, \; h(b)\leq 0$

$$\Rightarrow 0\in[h(b),h(a)] \quad ②$$

By the IVT, since h is continuous and $0\in[h(b),h(a)]$ there exist $c\in[a,b]$, such that

$$h(c)=f(c)-c=0$$

So $f(c)=c$ for some $c\in[a,b]$ ■

4. Given $f:(a,b)\to\mathbb{R}$ is twice differentiable we know that f is continuous and differentiable and also f' is continuous and differentiable.
Now given $\exists\, c<d\in(a,b)$ such that $f'(c)=f'(d)$, $f'(x)$ is continuous and differentiable in (a,b)

$\Rightarrow f'(x)$ is continuous and differentiable in (c,d)

Hence we have,

① $f'(x)$ is continuous in (c,d)

② $f'(x)$ is differentiable in (c,d)

③ $f'(c)=f'(d)$

Hence by Rolle's Theorem $\exists$ an $x\in(c,d)$ such that $f''(x)=0$. ■

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5. a) We are given,

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0-h)}{2h} = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)+f(x_0)-f(x_0-h)}{2h}$$

$$= \frac{1}{2}\lim_{h\to 0} \frac{f(x_0+h)-f(x)}{h} + \frac{1}{2}\lim_{h\to 0} \frac{f(x_0-h)-f(x_0)}{-h}$$

$$= \frac{1}{2} f'(x_0) + \frac{1}{2} f'(x_0) = f'(x)$$

Therefore $\lim_{h\to 0} \frac{f(x_0+h)-f(x_0-h)}{2h}$ exists and equals $f'(x_0)$. ■

b) Let $f(x) = |x|$. Then f is not differentiable at $x=0$. However,

$$\frac{f(h)-f(-h)}{2h} = \frac{|h|-|-h|}{2h} = 0$$

is convergent. ■

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6. a) We know $\ln(x) = g(x)$ is differentiable on $(0, \infty)$, and hence, a constant multiple of $g(x)$ is also differentiable, say

$$h(x) = r\,g(x) = r\ln(x), \text{ in } (0,\infty)$$

$$\Rightarrow r\ln(x) = \ln(x^r) \text{ is differentiable.}$$

Again, $k(x) = e^x$ is differentiable, so we will use the fact that composition of differentiable is differentiable. So we have

$$k \circ h(x) \text{ is differentiable in } (0,\infty)$$

$$\Rightarrow k \circ h(x) = k(\ln(x^r)) = e^{\ln(x^r)} = x^r = f(x)$$

So, $f(x) = x^r$ is also differentiable in $(0,\infty)$ and its derivative is,

$$f'(x) = r x^{r-1}$$

from the general formula. ■

b) $f(x) = \sqrt{x^2 + \sin x + \cos x}$

We can find $f'(x)$ using the chain rule and show that $f'(x)$ exists for all x i.e. f is differentiable. By the chain rule,

$$f'(x) = \frac{d}{dx}\left(\sqrt{x^2+\sin x+\cos x}\right) = \frac{1}{2\sqrt{x^2+\sin x+\cos x}} \times (2x + \cos x - \sin x)$$

$$\Rightarrow f'(x) = \frac{2x+\cos x - \sin x}{2\sqrt{x^2+\sin x+\cos x}}$$

Clealy, $2\sqrt{x^2+\sin x+\cos x} \neq 0$, $\forall x \in \mathbb{R}$. So $f'(x)$ is defined for all $x \in \mathbb{R}$ and therefore f is differentiable on $\mathbb{R}$. ■

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7. Suppose that S is closed and $x \in \mathbb{R}/S$. Note that x is not a cluster point of S; Thus every neighborhood of x contains no point of S, & there is a neighborhood of x. Say $(x, s) \subset \mathbb{R}/S$. Hence $\mathbb{R}/S$ is open. Conversely, suppose that $\mathbb{R}/S$ is open. We shall show that S is closed. Suppose that x is a cluster point of S. Then we claim that $x \in S$. If $x \notin S$ then x is a cluster point of S. Then we claim x is in $\mathbb{R}/S$, Thus $\mathbb{R}/S$ contains a neighborhood of. Therefore x is not a cluster point of S, and we have a contradiction. Hence S is closed. ∎

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8. Since x^3 is differentiable, because every polynomial is differentiable, by the chain, $f(x^3)$ is differentiable and,

$$(f(x^3))' = f'(x^3)(x^3)' = 3x^2 f'(x^3)$$

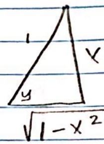
By the multiplication rule, g is differentiable and

$$g'(x) = 2x f(x^3) + x^2 3x^2 f'(x^3)$$
$$= 2x f(x^3) + 3x^4 f'(x^3) \quad \blacksquare$$

9. Let $y = \arcsin(x)$, then it follows that

① $\sin(y) = x$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

Notice from the triangle if $y = \arcsin(x)$ the $\sin(y) = \frac{x}{1} = x$ & $\cos(y) = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$



Taking the derivative of ① with respect to x, we have

$$\cos(y)\frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos(y)} = \sec(y) = \frac{1}{\sqrt{1-x^2}}$$

so the derivative $y = \arcsin(x) = \frac{1}{\sqrt{1-x^2}}$ $\blacksquare$

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10. a) Consider a function $f(x) = x^n$, $n \geq 1$. Since $f(x)$ is a polynomial, it is continuous and differentiable everywhere. In particular $f(x)$ is continuous on $[y, x]$ and differentiable on (y, x).
By the Mean Value Theorem there exists c in (y, x) such that

$$f'(c) = \frac{f(x) - f(y)}{x - y}$$

$$\Rightarrow n c^{n-1} = \frac{x^n - y^n}{x - y} \rightarrow (1)$$

$$\Rightarrow y^{n-1} < c^{n-1} < x^{n-1}$$

$$\Rightarrow n y^{n-1} < n c^{n-1} < n x^{n-1}$$

$$\Rightarrow n y^{n-1} < \frac{x^n - y^n}{x - y} < n x^{n-1} \quad \text{(from 1)}$$

$$\Rightarrow n y^{n-1} (x - y) < x^n - y^n < n x^{n-1} (x - y)$$

b) If $x > 0$, apply the Mean Value Theorem to $f(x) = \sqrt{1+x}$ on the interval $[0, x]$.
There exist $c \in [0, x]$ such that

$$\frac{\sqrt{1+x} - 1}{x} = \frac{f(x) - f(0)}{x - 0} = f'(c) = \frac{1}{2\sqrt{1+c}} < \frac{1}{c}$$

The last inequality holds because $c > 0$.
Multiplying by the positive number x and transposing the -1 gives

$$\sqrt{1+x} < 1 + \frac{1}{2}$$

for $x > 0$.