Math 331: Homework 1

Liliana Kershner

1) Proof by Induction: Let n = 1 serve as the base case. Then

$$1 = \frac{(1+1)1}{2} = \frac{2}{2} = 1.$$

Therefore the base case is true. Assume P(n) is our expression $\frac{(n+1)n}{2}$, and P(n) is true. We then prove P(n+1). So

$$P(n+1) = 1+2+3+...n+(n+1) = \frac{(n+1)((n+1)+1)}{2}$$

Replace the left side $1+2+3+...+n$ by the expression $P(n)$:

$$\begin{aligned} P(n+1) = \frac{(n+1)n}{2} + (n+1) &= \frac{(n+1)((n+1)+1)}{2} \\ &= \frac{(n+1)(n+2)}{2} \\ &= \frac{n(n+1)+2(n+1)}{2} \\ &= \frac{n(n+1)}{2} + \frac{2(n+1)}{2} \\ &= \frac{n(n+1)}{2} + (n+1) \end{aligned}$$

The left and right sides are equivalent so this is true.

2) Proof by Induction: We first confirm the base cases satisfy the inequality:

$$\begin{aligned} f(1) = 1 &\implies f(1) \leq 2^{(1-1)} = 1 \\ f(2) = 2 &\implies f(2) \leq 2^{(2-1)} = 2 \\ f(3) = 3 &\implies f(3) \leq 2^{(3-1)} = 4. \end{aligned}$$

The base cases satisfy. We assume that $f(i)$ is true for $i : 1, 2, 3, ..., n$.

$$\begin{aligned} f(n+1) &= f((n+1)-1) + f((n+1)-2) + f((n+1)-3) \leq 2^{((n+1)-1)} \\ f(n+1) &= f(n) + f(n-1) + f(n-2) \leq 2^n \end{aligned}$$

By the induction hypothesis we know that $f(i) \leq 2^{i-1}$ is true, making $f(i-1)+f(i-2)+f(i-3)$, so when $i = n$, we have $f(n-1)+f(n-2)+f(n-3) \leq 2^{n-1}$. Set equal and we have

$$2^{n-1} - f(n-3) \leq f(n) + f(n-1) + f(n-2) \leq 2^n$$

Already we know that 2^{n-1} is less than 2^n for $n > 0$ and exactly equal when $n = 0$. Therefore, $f(n+1)$ must be true and $f(n)$ must be true.

3) Let $\exists a \in A; A \neq \emptyset; \forall a \in \mathbb{R}$. Suppose there is a function which maps $A \to A, \forall a \in A$. We test the identity function of A, denoted by $\amalg_A$. This function is one-to-one if $\amalg_A(a_1) = \amalg_A(a_2)$. The definition of the identity function is that this is true, and $\amalg_A(a_1) = \amalg_A(a_2) = a_1 = a_2$.

So for any $a \in A, \amalg_A(a) = a$, so $A \sim A$.

b) We assume $A \sim B$ is true but must prove in the opposite direction, $B \sim A$. We call the function which maps $A \to B$ as f, which is a one-to-one function. We notice that the inverse of f, denoted as f^{-1}, exists since f is a function and one-to-one. Further f^{-1} is also one-to-one. The domain of f is the range of f^{-1}, that is the elements of A which are mapped to B by f, are mapped from B to A by f^{-1}. Therefore, there exists a function that maps $A \to B$ and a function which maps $B \to A$, so $B \sim A$.

c) Assume there is a function f which satisfies $A \sim B$ and a function g that satisfies $B \sim C$. To find a function that maps $A \to B$, we consider the composition of f and g: $g \circ f$.

We define $g \circ f$ as the function $g(f(x))$. We know f and g are both one-to-one functions respectively, so $g(f(x))$ must also be a one-to-one function - if $g(f(a_1)) = g(f(a_2)), \forall a_1, a_2 \in A$ then $f(a_1) = f(a_2)$, which we saw was true in part a.

The domain of f is A, and its range is B. The domain of g is B, and its range is C. Therefore, the domain of $g \circ f$ is A, and its range is C. This is because the input of f must be an element of A, and it maps to B. The input of g must be an element of B, and it maps to C. Therefore the only viable function that maps $A \sim C$ is $g \circ f$.

4) Let our countable set be A. If A is empty, it is countable, and any subset of A is also empty and countable. Let there be a second set B. By the definition of countability, there exists a surjective function f which maps A to B, with A and B countable. Or, if B is countable, and there is an injective function f from A to B, then A is also countable.

If we take a subset of A, the function f still applies, and still maps to a subset of B. Its injectivity and surjectivity are not changed by size of the set.

5) a) We have $0 < a < b$. Axiom 8 states that for $x < y$ and $z > 0$, $xz < yz$. Therefore, multiply both sides by a, we have $a^2 < ab$. Multiply both sides of the original by b, we have $ab < b^2$. So $a^2 < ab < b^2$.
b) We seek to prove that if $a^2 < b^2$ then $a < b$. We are given that $a < b = (\sqrt{a})^2 < (\sqrt{b})^2, \forall a, b \in \mathbb{R}$ such that $0 < a < b$.

We have three cases: $\sqrt{a} < \sqrt{b}$, $\sqrt{a} = \sqrt{b}$, and $\sqrt{a} > \sqrt{b}$.

Case 1: For $\sqrt{a} > \sqrt{b}$ with the condition that $a < b$, we have

$$\begin{gathered}
\sqrt{a} > \sqrt{b} \\
\sqrt{a} \cdot \sqrt{b} > \sqrt{b} \cdot \sqrt{b} \\
\sqrt{ab} > b \\
\text{and} \\
\sqrt{a} > \sqrt{b} \\
\sqrt{a} \cdot \sqrt{a} > \sqrt{b} \cdot \sqrt{a} \\
a > \sqrt{ab}.
\end{gathered}$$

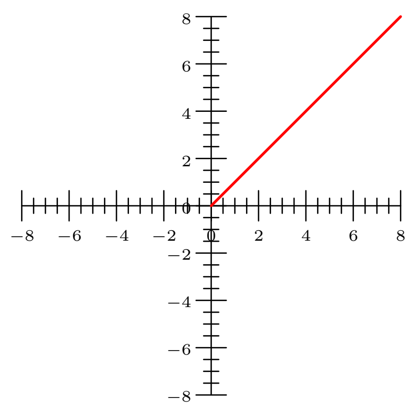
We get $a > \sqrt{ab} > b$ which is clearly a contradiction with $a < b$.

Case 2: For $\sqrt{a} = \sqrt{b}$ with $a < b$, we have

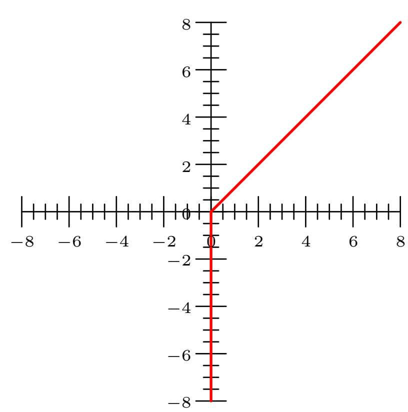
$$\begin{aligned}
\sqrt{a} &= \sqrt{b} \\
\sqrt{a} \cdot \sqrt{a} &= \sqrt{b} \cdot \sqrt{a} \\
a &= \sqrt{ab} \\
&\text{and} \\
\sqrt{a} \cdot \sqrt{b} &= \sqrt{b} \cdot \sqrt{b} \\
\sqrt{ab} &= b.
\end{aligned}$$

This implies $a = b$, which is impossible with our assumption $a < b$. The only true case must be $\sqrt{a} < \sqrt{b}$.

6) Our graph has two main portions: for $x, y \geq 0$ and $x, y \leq 0$. When $x, y \geq 0$, we have $x = |x|, y = |y|$ so $x + |x| = y + |y| \implies 2x = 2y \implies x = y, \forall x, y \geq 0$. That graph looks like this:



However, we must also look at the x or y is 0 case and the $x, y < 0$ cases. Take $x = 0$, then any $y \in (-\infty, 0]$ satisfies as the corresponding y coordinate, since $y = -|y|$ for negative y.



7) We have

$$\begin{aligned}
\sqrt{xy} &\leq \frac{x+y}{\sqrt{2}} \\
\sqrt{xy}\cdot\sqrt{2} &\leq x+y \\
\sqrt{2xy} &\leq x+y \\
(\sqrt{2xy})^2 &\leq (x+y)^2 \\
2xy &\leq (x+y)(x+y) \\
2xy &\leq x^2+2xy+y^2
\end{aligned}$$

For $y = 0$

$$0 \leq x^2.$$

Since x is a positive number, this is true. For $x = 0$:

$$0 \leq y^2.$$

and by the same reasoning this is true. When $x, y > 0$, subtract $2xy$ from either side to get $0 \leq x^2 + y^2$ which must be true. Therefore this expression must be true.

8) a) Since x is bounded below by 0, the infimum of the set is 0. Since x is bounded above by $x^2 = 9$, its supremum is 3.

b) The set E is bounded on either side as $4 < \frac{4n+5}{n+1} \leq \frac{9}{2}$. We will then show that 4 is the greatest lower bound for E. Consider x to be $inf(E)$. We then show that $4 = inf(E)$. Consider the following cases:

$$1) x < 4$$

$$2) x = 4$$

$$3) x > 4$$

Since we defined x as the infimum, it cannot be greater than 4 regardless of whether 4 is an infimum or not, so case 3 already presents a contradiction. Suppose $x > 4$. Then $4 - x > 0$. We apply the Archimedean Property with $x_1 = 4 - x$ and $y_1 = x - 5$. Then

$$\begin{aligned}
n(4-x) &< (x-5) \\
4n - nx &< x - 5 \\
4n &< nx + x - 5 \\
4n + 5 &< x(n+1) \\
\frac{4n+5}{n+1} &< x
\end{aligned}$$

Which is a contradiction, and gives x = 4 as our infimum.

9) A and $P(A)$ are equal since they contain the same elements, but we will show they are not equivalent, since they have different amounts of elements. Assume to a contradiction there is a bijective function f from A to $P(A)$. A bijective function must map both sets one-to-one, as in, every element in the set of A must map to an element in the set $P(A)$ exactly. Since $P(A)$ is a power set, this is impossible–there will be terms in $P(A)$ that will not have a corresponding term in A. Therefore the sets are non-equivalent. This can be applied more broadly to $P(\mathbb{N})$. Assume the set $A = \mathbb{N}$. There is no bijective function which can map $\mathbb{N}$ to $P(\mathbb{N})$ for the same reason as there is no function for A and $P(A)$.

10) a) We wish to prove that the supremum of the set rE for $r > 0$ is the same as the supremum of the set E, multiplied by r. Let the supremum of E be called x. On one side of the equation we have rx, and we will prove this is equivalent to the supremum of the set rE. If every element in the set of E is mutliplied by r, then the supremum increases by a factor of r. If x is the supremum and n is the next smallest element, then we have $xr, nr \in rE$. If x

and n are both elements of E, and x is multiplied by r, we have $xr, n \in E$. The values xr are equivalent in both sets. However this does not guarantee that xr is still the supremum of E.

b) In this case, we examine $r = 0$ and $r \neq 0$. When $r = 0$ we know by the identity field axiom that $x + 0 = x$. So this is true. Assume $r \neq 0$. We want to prove that $x + r$ where $x = sup(E)$ is the same as $sup(E + r)$. By similar reasoning in part a, we know that $sup(E + r)$ is the supremum of the set when every element of E has r added to it. Therefore, the supremum of this set is still $x + r$. The supremum of the set E is x. When added to r, we get $x + r$. Therefore these two elements are equal (but again not necessarily still supremums).