MATH-331 Introduction to Real Analysis | Jeselle-Paige Guillermo
Homework 03 | Fall 2021

Due date: October 11th 1:20pm | Total: /70.

Exercise	1 (5)	2 (5)	3 (5)	4 (5)	5 (10)	6 (10)	7 (5)	8 (5)	9 (5)	10 (10)
Score										

Table 1: Scores for each exercises

Instructions: You must answer all the questions below and send your solution by email (to `parisepo@hawaii.edu`). If you decide to not use LaTeX to hand out your solutions, please be sure that after you scan your copy, it is clear and readable. Make sure that you attached a copy of the homework assignment to your homework.
If you choose to use LaTeX, you can use the template available on the course website.
No late homework will be accepted. No format other than PDF will be accepted. Name your file as indicated in the syllabus.

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Writing problems

For each of the following problems, you will be asked to write a clear and detailed proof. You will have the chance to rewrite your solution in your semester project after receiving feedback from me.

Exercise 1. (5 pts) Let $(a_n)_{n=1}^{\infty}$ be an increasing sequence and $(b_n)_{n=1}^{\infty}$ be a decreasing sequence. Let $(c_n)_{n=1}^{\infty}$ be the sequence defined by $c_n = b_n - a_n$. Show that if $\lim_{n\to\infty} c_n = 0$, then the sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ converges and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

Exercise 2. (5 pts) Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$, and suppose that x_0 is an accumulation point of D. Suppose that for each sequence $(x_n)_{n=1}^{\infty}$ converging to x_0 with $x_n \in D\setminus\{x_0\}$ for each $n \geq 1$, then the sequence $(f(x_n))_{n=1}^{\infty}$ is Cauchy. Show that f has a limit at x_0.

[Hint: For two sequences (x_n) and (y_n) that satisfy the assumption, define the sequence (z_n) to be $z_{2n} = x_n$ and $z_{2n-1} = y_n$. Show that $(f(z_n))$ converges and the sequence $(f(x_n))$ and $(f(y_n))$ converges to the same limit as $(f(z_n))$. Conclude by a theorem in the lecture notes.]

Exercise 3. (5 pts) Prove that if $f : D \subseteq \mathbb{R} \to \mathbb{R}$ has a limit at $x_0 \in \operatorname{acc} D$, then the limit is unique.

Exercise 4. (5 pts) Suppose $f : D \subseteq \mathbb{R} \to \mathbb{R}$, $g : D \subseteq \mathbb{R} \to \mathbb{R}$ and $h : D \subseteq \mathbb{R} \to \mathbb{R}$ are three functions such that

$$f(x) \leq h(x) \leq g(x) \quad (\forall x \in D).$$

Suppose that f and g have limits at x_0 with $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x)$. Prove that h has a limit at x_0 and

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} h(x) = \lim_{x\to x_0} g(x).$$

Exercise 5. (10 pts) Let $f : (0, \infty) \to \mathbb{R}$ be a function. We say that f has a limit at ∞ if there exists a $L \in \mathbb{R}$ such that for any $\varepsilon > 0$, there is a real number $M > 0$ such that if $x > M$, then $|f(x) - L| < \varepsilon$.

- **a)** Show that if $g : (0, \infty) \to \mathbb{R}$ is bounded and $\lim_{x\to\infty} f(x) = 0$, then $\lim_{x\to\infty} f(x)g(x) = 0$.
- **b)** Let $a > 0$ and suppose that $f : (a, \infty) \to \mathbb{R}$ and define $g : (0, 1/a) \to \mathbb{R}$ by $g(x) = f(1/x)$. Show that f has a limit at ∞ if and only if g has a limit at 0.

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Homework problems

Answer all the questions below. Make sure to show your work.

Exercise 6. (10pts) For each of the sequences below, determine its nature (converges or diverges)[1]:

- **a)** (a_n) where $a_n = \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n}$.
- **b)** (a_n) where $a_n = \frac{1+2+\cdots+n}{n^2}$.

Exercise 7. (5 pts) Define $g : (0, 1) \to \mathbb{R}$ by $f(x) = \frac{\sqrt{1+x}-1}{x}$. Prove that g has a limit at 0 and find it.

Exercise 8. (5 pts) Suppose that $f : (0, 1) \to \mathbb{R}$ has a limit at $x_0 = 1$ and $\lim_{x\to 1} f(x) = 1$. Compute the value of the limit

$$\lim_{x\to 1} \frac{f(x)(1 - f(x)^2)}{1 - f(x)}.$$

Exercise 9. (5 pts) Prove that if $f : D \to \mathbb{R}$ has a limit at x_0, then $|f|(x) := |f(x)|$ has a limit at x_0.

Exercise 10. (10 pts) Using the link between sequences and limits of functions, show the following.

- **a)** If $f(x) = x^n$ $(n \geq 0)$, then $\lim_{x\to x_0} f(x) = x_0^n$ for any $x_0 \in \mathbb{R}$.
- **b)** If $x_0 \in [0, \infty)$, then $\lim_{x\to x_0} \sqrt{x} = \sqrt{x_0}$.

[1] You don't need to compute the limit.

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Math 331- HW#3

1) (a_n) is increasing, so if $x \leq y$, then $f(x) \leq f(y)$
(b_n) is decreasing, so if $x \geq y$, then $f(x) \geq f(y)$
$c_n = b_n - a_n$ so $a_n < c_n < b_n$

Let $\lim_{n\to\infty} c_n = 0$
For some $\varepsilon > 0$ $\exists \delta > 0$ s.t $0 < |x - x_0| < \delta$.
Then $|c_n - 0| < \varepsilon$
and so, $|b_n - a_n| < \varepsilon$ OR $|a_n - b_n| < \varepsilon$, b_n and a_n converges

$\lim_{n\to\infty} c_n = 0$, so $\lim_{n\to\infty}(b_n - a_n) = 0$
$\lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n = 0$
$\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n$

2) f has a limit at $x_0 \Leftrightarrow \forall (x_n)(x_n \to x_0, x_n \in D \setminus \{x_0\})$, $f(x_n)$ converges
Hence, $f(x_n) \to L(x)$, when $x = x_n$

consider (x_n) & (y_n) s.t $x_n, y_n \in D$, $x_n \neq x_0$, $y_n \neq x_0$ for $n = 1, 2 \ldots$
and both (x_n) & (y_n) converge to x_0
Assume $f(x_n) \to L_1$ $f(y_n) \to L_2$
Define a new sequence (z_n) where $z_{2n} = x_n$ & $z_{2n-1} = y_n$
where $x_n \in D \setminus \{x_0\}$ and $f(z_n)$ converges to x_0
$f(x_n)$ & $f(y_n)$ are subsequences of $f(z_n)$
thus have the same limit as $f(z_n)$
$\lim_{x\to x_0} f(z_n) = L_1 = L_2$

Proof: suppose f has a limit L at x_0.
So, for some $\varepsilon > 0$ $\exists \delta > 0$ s.t. $0 < |x - x_0| < \delta$ with $x \in D$
Then, $|f(x) - L| < \varepsilon$

$\{x_n\}_{n=1}^{\infty}$ converges to x_0
So, $\exists N$ s.t for $n \geq N$, $|x_n - x_0| < \delta$
Then $|f(x_n) - L| < \varepsilon$,
and $\{f(x_n)\}_{n=1}^{\infty}$ converges L at x_0

3) Let $f: D \subseteq \mathbb{R} \to \mathbb{R}$ be a function an $x_0 \in acc(D)$
Let the $\lim_{x \to x_0} f(x)$ exist
Suppose $\lim_{x \to x_0} f(x) = L_1$ and $\lim_{x \to x_0} f(x) = L_2$
Goal: Prove $L_1 = L_2$
Let $\varepsilon > 0$ be arbitrary
Since $\lim_{x \to x_0} f(x) = L_1$, $\exists \delta_1 > 0$ s.t. $0 < |x - x_0| < \delta_1$
$\Rightarrow |f(x) - L_1| < \varepsilon/2$ (*)
Also, since $\lim_{x \to x_0} f(x) = L_2$, $\exists \delta_2 > 0$ s.t. $0 < |x - x_0| < \delta_2$
$\Rightarrow |f(x) - L_2| < \varepsilon/2$ (**)
Let $\delta = \min\{\delta_1, \delta_2\}$
Then * & ** $\forall |x - x_0| < \delta$
Now

$$
\begin{aligned}
|L_1 - L_2| &= |L_1 - f(x) + f(x) - L_2| \\
&\leq |L_1 - f(x)| + |f(x) - L_2| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}
$$

Since ε is arbitrary, $L_1 = L_2$ $\square$

4) Let $\varepsilon > 0$, $\lim_{x \to x_0} f(x) = L$ exist for some $\delta_1 > 0$ s.t.
$|f(x) - L| < \varepsilon$, $0 < |x - x_0| < \delta_1$
That is $L - \varepsilon < f(x) < L + \varepsilon$, $0 < |x - x_0| < \delta_1$

Then, $\lim_{x \to x_0} g(x) = L \Rightarrow L - \varepsilon < g(x) < L + \varepsilon$, $0 < |x - x_0| < \delta_2$

Let $\delta = \min\{\delta_1, \delta_2\}$ then
$L - \varepsilon < f(x) \leq h(x) \leq g(x) < L + \varepsilon$, $0 < |x - x_0| < \delta$

So $L - \varepsilon < h(x) < L + \varepsilon$, $0 < |x - x_0| < \delta$
$\Rightarrow |h(x) - L| < \varepsilon$
$\Rightarrow \lim_{x \to x_0} h(x) = L$ exists for some $\delta > 0$

$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = \lim_{x \to x_0} g(x)$

5a) Proof:
Suppose g is bounded by M and $\lim_{x \to \infty} f(x) = 0$
$|g(x)| \leq M$ $\forall x$
Then $0 \leq |f(x) g(x)| = |f(x) \cdot g(x)| \leq |f(x)| \cdot M$
Then $\lim_{x \to \infty} 0 = 0$, $\lim_{x \to \infty} |f(x)| \cdot M = |0| \cdot M = 0$
So by squeeze thm, $\lim_{x \to \infty} |f(x) g(x)| = 0 \Rightarrow \lim_{x \to \infty} f(x) g(x) = 0$

5b) If f has a limit at ∞

then as $x \to \infty$, $\frac{1}{x} \to \infty$

so $f(\frac{1}{x}) \to 0$ and $g(x) \to 0$

Thus, $g(x)$ has a limit at 0

If $g(x)$ has a limit at 0

$g(x) \to 0$

$f(1/x) \to 0$

$1/x \to 0$ as $x \to \infty$

f has a limit at ∞

6a) Take $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (\frac{1}{n} + \frac{1}{n+1} + \ldots + \frac{1}{2n})$

By sum rule,

$$\lim_{n\to\infty} (\frac{1}{n} + \frac{1}{n+1} + \ldots + \frac{1}{2n}) = \lim_{n\to\infty} \frac{1}{n} + \lim_{n\to\infty} \frac{1}{n+1} + \ldots + \lim_{n\to\infty} \frac{1}{2n}$$

All limits approach 0, so $\lim_{n\to\infty} a_n = 0$ and a_n converges

6b) Take $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (\frac{1+2+\ldots+n}{n^2}) = \lim_{n\to\infty} (\frac{1}{n^2} + \frac{2}{n^2} + \ldots + \frac{1}{n})$

Again by sum rule,

$$\lim_{n\to\infty} (\frac{1}{n^2} + \frac{2}{n^2} + \ldots + \frac{1}{n}) = \lim_{n\to\infty} (\frac{1}{n^2}) + \lim_{n\to\infty} (\frac{2}{n^2}) + \ldots + \lim_{n\to\infty} (\frac{1}{n})$$

All limits approach 0, so $\lim_{n\to\infty} a_n = 0$ and a_n converges

7) $\lim_{x\to 1} f(x) = 1$

$$g(x) = \frac{\sqrt{1+x} - 1}{x} \cdot \frac{\sqrt{1+x} + 1}{\sqrt{1+x} + 1}$$

$$= \frac{(1+x) - 1}{x(\sqrt{1+x} + 1)}$$

$$= \frac{x}{x(\sqrt{1+x} + 1)}$$

$$= \frac{1}{\sqrt{1+x} + 1}$$

Let $\varepsilon > 0$, $\delta > 0$ st. $|g(x) - \frac{1}{2}| < \varepsilon$ when $0 < |x - 0| < \delta$

$$|g(x) - \tfrac{1}{2}| = \left| \frac{1}{\sqrt{1+x} + 1} - \frac{1}{2} \right|$$

$$= \left| \frac{2 - \sqrt{1+x} - 1}{2(\sqrt{1+x} + 1)} \right|$$

$$= \left| \frac{1 - \sqrt{1+x}}{2(\sqrt{1+x} + 1)} \right|$$

$$= \left| \frac{\sqrt{1+x} - 1}{2(\sqrt{1+x} + 1)} \right|$$

$$= \left| \frac{\sqrt{1+x} - 1}{2(\sqrt{1+x} + 1)} \cdot \frac{\sqrt{1+x} + 1}{\sqrt{1+x} + 1} \right|$$

$$= \left| \frac{1+x-1}{2(\sqrt{1+x} + 1)^2} \right|$$

$$= \left| \frac{x}{2(\sqrt{1+x} + 1)^2} \right|$$

Since $x \in (0,1)$, $x > 0$

$1 + x > 1$

$1 + \sqrt{1+x} > 2$

$(1 + \sqrt{1+x})^2 > 4$

$2(1 + \sqrt{1+x})^2 > 8$

$$|g(x) - \tfrac{1}{2}| = \left| \frac{x}{2(\sqrt{1+x} + 1)^2} \right| < \frac{|x|}{8}$$

Let $\varepsilon > 0$ choose 8ε for $0 < |x - x_0| < \delta$

$$|g(x) - \tfrac{1}{2}| < \frac{|x|}{8} < \frac{\delta}{8} = \frac{8\varepsilon}{8} = \varepsilon$$

$|g(x) - \frac{1}{2}| < \varepsilon$

So, $g(x) = \frac{\sqrt{1+x} - 1}{x}$ has a limit at $x = 0$

$\lim_{x\to 0} g(x) = \frac{1}{2}$

8) $\lim_{x\to 1} \frac{f(x)(1-f(x)^2)}{1-f(x)} = \lim_{x\to 1} \frac{(f(x)-f(x)^3)}{1-f(x)} \cdot \frac{(f(x)+f(x^2))}{(f(x)+f(x)^2)}$

$= \lim_{x\to 1} \frac{(f(x)-f(x)^3)(f(x)+f(x)^2)}{f(x)-f(x)^2+f(x)^2-f(x)^3}$

$= \lim_{x\to 1} f(x)+f(x)^2$

$= \lim_{x\to 1} f(x) + \lim_{x\to 1} f(x)^2$

$= 1 + 1$

$= 2$

9) $f: D \to \mathbb{R}$ has a limit at $x = x_0$

choose $\varepsilon > 0$

$\lim_{x\to x_0} f(x) = L$

So, $\exists \delta > 0$ for $0 < |x - x_0| < \delta$

$|f(x) - L| < \varepsilon$

consider $|f(x)| - |L|$

then $|f(x)| - |L| \leq |f(x) - L|$ by triangle inequality

$< \varepsilon$

$\lim_{x\to x_0} |f(x)| = |\lim_{x\to x_0} f(x)|$

10a) we want to prove x^n has a limit at $x_0 \in (\alpha, \beta)$

x^n is increasing

we can use Lemma 2.7 from the book

If $f(x)$ has a limit at $x_0 \in (\alpha, \beta)$, then $\lim_{x\to x_0} f(x) = f(x_0)$

10b) we want to prove for any $x_0 \in [0, \infty)$ for every $\varepsilon > 0$ $\exists \delta > 0$ s.t. $|x - x_0| < \delta$

let $\varepsilon > 0$ and $\delta = \sqrt{x_0} \cdot \varepsilon > 0$

$|\sqrt{x} - \sqrt{x_0}| = \left| \frac{(\sqrt{x}-\sqrt{x_0})(\sqrt{x}+\sqrt{x_0})}{(\sqrt{x}+\sqrt{x_0})} \right|$

$= \frac{|x - x_0|}{\sqrt{x}+\sqrt{x_0}}$

$< \frac{|x - x_0|}{\sqrt{x_0}}$

$< \frac{\delta}{\sqrt{x_0}} = \frac{\sqrt{x_0} \cdot \varepsilon}{\sqrt{x_0}}$

$< \varepsilon$

So $\forall x [0, \infty)$ with $0 < |x_1 - x_0| < \delta$ $|\sqrt{x} - \sqrt{x_0}| < \varepsilon$

$\lim_{x\to x_0} \sqrt{x} = \sqrt{x_0}$