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Due date: 09/07/2021 1:20pm

Instructions: You must answer all the questions below and send your solution by email (to `parisepo@hawaii.edu`). If you decide to not use LaTeX to hand out your solutions, please be sure that after you scan your copy, it is clear and readable. No late homework will be accepted. No format other than PDF will be accepted. Name your file as indicated in the syllabus.

1 Homework problems

Exercise 1. Prove that for any $n \in N, 1 + 2 + ... + n = \frac{n(n+1)}{2}$.

Solution: The sum of numbers from 1 to n can be represented by:
$\sum = (n - 0) + (n - 1) + ... + (n - (n - 1)) + (n - n)$
This also means that $\sum = (n - n) + (n - (n - 1)) + ... + (n - 1) + (n - 0)$.
simplify the expression, add $\sum$ to itself.
This leaves us with $2\sum = n + n + n + n + ...$ The number of elements in the set is equal to n+1, since the set has values from 0 to n.
This makes $2\sum = n(n + 1)$.
Divide both sides by 2 to get the sum for 1 set of $\sum$
$\sum = \frac{n(n+1)}{2}$ □

Exercise 2. Define: $f : \mathbb{N} \to \mathbb{N}$ by $f(1) = 1, f(2) = 2$, and $f(3) = 3$ and

$$f(n) := f(n - 1) + f(n - 2) + f(n - 3)(n \geq 4).$$

Prove that $f(n) \leq 2^{n-1}$ $\forall n \in \mathbb{N}$.

Solution: -

Exercise 3. Prove that if A, B, and C are sets, then
a) $A \sim A$.
b) If $A \sim B$, then $B \sim A$.
c) If $A \sim B$ and $B \sim C$, then $A \sim C$.

Solution: If A, B, and C are sets:
a) A contains n elements $a_1, a_2, ..., a_n$. If A also contains n elements $a_1, a_2, ..., a_n$, and all numbers are equal to themselves, then $A \sim A$.
b) If A contains n elements $a_1, a_2, ..., a_n$, and B contains n elements $b_1, b_2, ..., b_n$, such that $\forall a_n \in A$, and $\forall b_n \in B$, $a_n = b_n$, then $\forall a_n \in A$, and $\forall b_n \in B$, $b_n = a_n$, due to the fact that

the

relation $=$ is a symmetric relation. Therefore if $A \sim B$ then $B \sim A$.

c) If A contains n elements $a_1, a_2, ..., a_n$, B contains n elements $b_1, b_2, ..., b_n$, and C contains elements $c_1, c_2, ..., c_n$, such that $\forall a_n \in A$, $\forall b_n \in B$, and $\forall c_n \in C$, $a_n = b_n$, and $b_n = c_n$, then $a_n = c_n$. This is because $\square$

Exercise 4. Show that any subset of a countable set is countable.

Solution: A subset of a set contains elements from that set and only that set. The subset must then have strictly less elements than the parent set, so if the parent set is countable, then a set with less elements will also be countable. $\square$

Exercise 5. Let $0 < a < b$ be positive real numbers. Prove that

a) $a^2 < b^2$

b) $\sqrt{a} < \sqrt{b}$

Solution:

a) $a < b$

$a^2 < ab$

$ab < b^2 \Leftrightarrow b^2 > ab$

If $a^2 < ab$ and $ab < b^2$, then $a^2 < b^2$ by the transitive property.

b) $\sqrt{a} < \sqrt{b}$

$a < \sqrt{ab}$

$\sqrt{ab} < b$

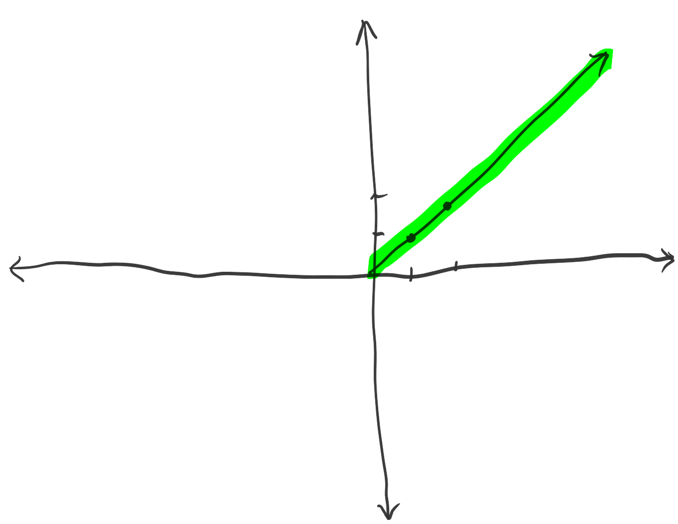
If $a < \sqrt{ab}$ and $\sqrt{ab} < b$, then $a < b$ by the transitive property. Therefore, we know $\sqrt{a} < \sqrt{b}$ is true. $\square$

Exercise 6. Sketch the region of the points (x, y) satisfying the following relation: $x+ \mid x \mid = y+ \mid y \mid$ (explain your answer).

Solution:

$x+|x| \geq 0$

$y+|y| \geq 0$



$\square$

Exercise 7. If $x \geq 0$ and $y \geq 0$, prove that $\sqrt{xy} \leq \frac{x+y}{\sqrt{2}}$

Solution:
$\sqrt{xy} \leq \frac{x+y}{\sqrt{2}}$
$\sqrt{2xy} \leq x + y$
$2xy \leq (x+y)^2$
$2xy \leq x^2 + 2xy + y^2$
$0 \leq x^2 + y^2$
$0 <$ positive $\Rightarrow$ True $\square$

Exercise 8. Find the infimum and supremum (if they exist) of the following sets. Make sure to justify all your answers.
a) $E := \{x \in \mathbb{R} : x \geq 0 \text{ and } x^2 \leq 9\}$
b) $E := \{\frac{4n+5}{n+1} : n \in \mathbb{N}\}$

Solution:
a) Supremum: 9
Infimum: 0

b) Supremum: 4
Infimum: 5 $\square$

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Writing problems

For each of the following problems, you will be ask to write a clear and detailed proof. You will have the chance to rewrite your solution in your semester project after receiving feedback from me.

Exercise 9. Let A be a non-empty set and $P(A)$ be its power set (the family of all subsets of A). Prove that A is not equivalent to $P(A)$. Deduce that $P(\mathbb{N})$ is not countable.

Solution: Suppose toward a contradiction that $f : A \to P(A)$ is a bijection $\square$

Exercise 10. Let $E \subseteq \mathbb{R}$ be bounded from above and $E \neq \emptyset$. For $r \in \mathbb{R}$, let
$$rE := \{rx : x \in E\} \text{ and } r + E := \{r + x : x \in E\}.$$

Show that
a) if $r > 0$, then $\sup(rE) = r \sup(E)$.
b) for any $r \in \mathbb{R}, \sup(r + E) = r + \sup E$.

Solution:

a) The supremum of rE is the supremum of the set rE in which every $x \in E$ is multiplied by r. This means that the supremum is some rx in rE. The supremum of E is some x in E, and since every element in rE is r times all x in E , the supremum of rE must be $r \sup(E)$.

b) The supremum of $r + E$ is the supremum of the set $r + E$ in which r is added to every $x \in E$. This means the supremum is some $r + x$ in $r + E$. The supremum of E is some x in E, and since every element in $r + E$ is r plus all x in E, the supremum of $r + E$ must be $r + \sup(E)$ $\square$