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Math 331 Homework 2

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Exercise 1.

First, we'll prove that for any $n, m \geq 1$, $a_n \leq b_m$. We have three choices.
(1) $a_n < b_m$
(2) $a_n = b_m$
(3) $a_n > b_m$

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We just need to prove that case 3 is false, and we'll get that the statement is true. Let $M = max\{n, m\}$ such that the set $[a_M, b_M] \subset [a_n, b_m]$. Now suppose toward a contradiction that $a_n > b_m$. If so, we get that the set $[a_n, b_m] = \emptyset$. This is a contradiction because from our initial assumption, we know that $[a_M, b_M] \subset [a_n, b_m]$. But if the second set is the empty set, this cannot be true.

-1 How can you be sure that $[a_M, b_M]$ is not also empty?

Next we'll show that $sup(a_n)$ exists. We know for all $n \in \mathbb{N}$, $a_n \leq b_n$. Now say $a_n \to A$ and $b_n \to B$, then $A \leq B$ from the proof done in class. Therefore, we know a_n is bounded from above. By the Axiom of Completeness, we know a_n has a supremum.

we know that $a_n \leq b_m \, \forall n$ -1 so b_m is an upperbound $\Rightarrow$ sup exists.

We know that $sup(a_n)$ exists, so let's define $c = sup(a_n)$ and show that c is the element we are looking for such that $c \in [a_n, b_n] \forall n \geq \mathbb{N}$. We need first show that a_n converges to c → You don't have to show that.
$\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that if $n > N$,

$$|a_n - c| < \epsilon$$

The expression inside the absolute value symbols will always be negative, so this expression is equal to

$$-(a_n - c) < \epsilon$$

$$c - a_n < \epsilon$$

Now if a_n is increasing, we will always find an N such that if $n \geq N$, the difference between c and a_n will be $< \epsilon$. So let's prove that a_n is increasing.

-1 Go check the solution on the course website.

Assume towards a contradiction that a_n is decreasing, such that $a_n > a_{n+1}$. Then $[a_n, b_n] \subset [a_{n+1}, b_{n+1}]$. This is a contradiction because in the assumption, we must have that $[a_n, b_n] \supset [a_{n+1}, b_{n+1}]$. Therefore, because we only have three options, $a_n < a_{n+1}, a_n = a_{n+1}$, or $a_n > a_{n+1}$. Since the last one was proven contradictory, we are left with $a_n \leq a_{n+1} \forall n \in \mathbb{N}$

So since a_n is increasing and ϵ was arbitrary, $a_n \to c$. Therefore, referring back to having $b_n \to B$, we know $c \leq B$.
Next we need to prove that $b_n \geq B \forall n \in \mathbb{N}$. We do this by proving that b_n is decreasing.
Assume that b_n is increasing such that $b_n < b_{n+1}$. Then that means

$$[a_n, b_n] \subset [a_{n+1}, b_{n+1}]$$

This is a contradiction because according to the assumption, $[a_n, b_n] \supset [a_{n+1}, b_{n+1}]$.
Therefore $b_n \geq b_{n+1}$.
Putting all this together,

$$a_n \leq c \leq B \leq b_n$$

$$a_n \leq c \leq b_n$$

We needed to show that no matter how large $n \in \mathbb{N}$ gets, c will always be contained in a subset. c must be both greater than a_n and less than b_n. Through these inequalities, we can see that this will be true. Therefore $c \in [a_n, b_n] \forall n \in \mathbb{N}$.

1/5 **Part b** We can use this result to prove that the set $\mathbb{R}$ is uncountable. Let's construct a function $f : \mathbb{N} \to \mathbb{R}$ of closed intervals such that $f(n) \notin [a_n, b_n]$ with $a_n < b_n$, and show that this function cannot be surjective. You have to construct the intervals...
As we saw in the first part of this exercise, we've proven the existence of an element $c = sup(a_n)$ that will always be contained in the set. Therefore, because we defined $f(n)$ as all elements outside of the set, it is impossible for there to be n such that $f(n) = c$. Now because there's an element $c \in \mathbb{R}$ without a pre-image, this function isn't surjective since the entire range is not mapped to.

3/5 **Exercise 2.** Prove that if $a_n \to A$, then $|a_n| \to |A|$.

By our assumption that $a_n \to A$, we know that $\forall \epsilon > 0, \exists N_0 \in \mathbb{N}$ such that if $n \geq N_0$ then

$$|a_n - A| < \epsilon$$

Now our goal is to show that $\forall \epsilon > 0, \exists N_1 \in \mathbb{N}$ such that if $n \geq N_1$ then

$$||a_n| - |A|| < \epsilon$$

You have to find the integer N_1!

If we can show this, we know that $|A|$ is the correct limit for $|a_n|$.
Let's set $N = max\{N_0, N_1\}$. Now for $n \geq N$, we have the expression

$$||a_n| - |A||$$

Now by the 6th property of absolute values discussed in class,

$$||a_n| - |A|| \leq |a_n - A|$$

As you can see, the RHS is actually the same one from our assumption, and so we know that since $n \geq N$,

$$||a_n| - |A|| \leq |a_n - A| < \epsilon$$

$$||a_n| - |A|| < \epsilon$$

Therefore, since ϵ was arbitrary, $|a_n| \to |A|$. $\square$

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Exercise 3. Let $(a_n), (b_n), (c_n)$, be sequences of real numbers. Prove that if $a_n \to L, b_n \to L$, and $a_n \leq c_n \leq b_n$, then $c_n \to L$.

You have to prove it. You don't know that yet.

Let $c_n \to C$. We will first prove that if $c_n \leq b_n$, then $C \leq L$. And by the same logic since $a_n \leq c_n$, $L \leq C$. Therefore since we're left with $L \leq C \leq L$, the only possible value that C can be is L.
Let's start with proving $C \leq L$.
If $c_n \leq b_n, \forall n \geq 1$, we have three cases.
(1) $C < L$
(2) $C = L$
(3) $C > L$

→ You can use a theorem from the lecture notes to prove that $C \leq L$ & $L \leq C$.

By disproving case 3, we will know either of cases (1) and (2) must be true, which is the definition of $\leq$.
Assume towards a contradiction that $C > L$ such that $C - L > 0$. Now let $\epsilon = \frac{C-L}{2}$. Since $c_n \to C \quad b_n \to L$, $\exists N_c, N_b$ such that

$$n \geq N_c \Rightarrow |c_n - C| < \frac{C-L}{2}$$
$$n \geq N_b \Rightarrow |b_n - L| < \frac{C-L}{2}$$

We take $N := max\{N_c, N_b\}$ so that

$$-\frac{C-L}{2} < c_n - C < \frac{C-L}{2} \qquad (i)$$
$$-\frac{C-L}{2} < b_n - L < \frac{C-L}{2} \qquad (ii)$$

From (i),

$$\begin{aligned} c_n &> -\frac{C+L}{2} + \frac{2C}{2} \\ &> \frac{C+L}{2} \end{aligned}$$

From (ii),

$$\begin{aligned} b_n &< \frac{C-L}{2} + \frac{2L}{2} \\ &< \frac{C+L}{2} \end{aligned}$$

By combining these inequalities, we get that

$$b_n < \frac{C+L}{2} < c_n \qquad \#$$

This is a contradiction to our original assumption of $c_n \leq b_n\ \forall n \geq 1$, therefore $C \leq L$. Now by the same logic, we can prove that if $a_n \leq c_n, \forall n \geq 1$, and $a_n \to L, c_n \to C$, then $L \leq C$.
So far, we have that if $a_n \leq c_n \leq b_n\ \forall n \geq 1$, and $a_n \to L, b_n \to L, c_n \to C$, then

$$L \leq C \leq L$$

The only value of C that can logically keep this equation true is by having $C = L$. Therefore, if $a_n \to L, b_n \to L$, and $a_n \leq c_n \leq b_n$, then $c_n \to L$. $\square$

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Exercise 4.

Exercise 5.

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We're trying to show that if $a_n \to A$, then $\sigma_n \to A$. To write it a little simpler, let's rewrite σ_n as

$$\frac{\sum_{n=1}^{\infty} a_n}{n}$$

Now we'll use the definition of convergence. For all $\epsilon > 0$, there exists an $N_1 \in \mathbb{N}$ such that if $n \geq N_1$, then

$$\left| \frac{\sum_{n=1}^{\infty} a_n}{n} - A \right| < \epsilon$$

$$\left| \frac{\sum_{n=1}^{\infty} a_n}{n} - \frac{nA}{n} \right| < \epsilon$$

$$\left| \frac{\sum_{n=1}^{\infty} a_n - A}{n} \right| < \epsilon$$

$$\frac{\left|\sum_{n=1}^{\infty} a_n - A\right|}{|n|} < \epsilon$$

by property of the absolute value

$$\frac{\left|\sum_{n=1}^{\infty} a_n - A\right|}{n} \leq \frac{\sum_{n=1}^{\infty} |a_n - A|}{n}$$

by the Triangle Inequality

Now we know that the inner expression of the summation, $|a_n - A| < \epsilon \, \forall n \geq N$ because of our initial assumption that $a_n \to A$. With this, we are saying that for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that if $n \geq N_0$,

(*)
$$|a_n - A| < \epsilon$$

where does N_1 comes from??

Therefore, if we set $N = max\{N_1, N_0\}$, then for $n \geq N$,

$$\frac{\sum_{n=1}^{\infty} |a_n - A|}{n} < \frac{\sum_{n=1}^{\infty} \epsilon}{n}$$

$$\frac{\sum_{n=1}^{\infty} \epsilon}{n} = \frac{\epsilon \cdot n}{n}$$

$$= \epsilon$$

You can't apply (*) for index $k < N$. You have to split the sum in two: from $k=1$ to $N-1$ and from $k=N$ to $k=n$.

Putting it all together, we get that

$$\left| \frac{\sum_{n=1}^{\infty} a_n}{n} - A \right| \leq \frac{\sum_{n=1}^{\infty} |a_n - A|}{n} < \frac{\sum_{n=1}^{\infty} \epsilon}{n} = \epsilon$$

$$\left| \frac{\sum_{n=1}^{\infty} a_n}{n} - A \right| < \epsilon$$

Therefore, because ϵ was arbitrary, $\sigma_n \to A$. $\square$

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Exercise 6. Prove these sequences converge.
(a) $a_n = 5 + \frac{1}{n}$
It is reasonable to assume that $a_n \to 5$
$\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that if $n \geq N$,

$$|(5 + \frac{1}{n}) - 5| < \epsilon$$

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$$|\frac{1}{n}| < \epsilon$$

$$\frac{|1|}{|n|} < \epsilon$$

by property of absolute values

$$\frac{1}{n} < \epsilon$$

We must show the last statement to be true. We will prove this using the Archimedes Principle.
By the Archimedean Principle, we assign $x = \epsilon$ and $y = 1$. $\exists N_0 \in \mathbb{N}$ such that

$$N_0\epsilon > 1$$

Take $N = N_0$, so that if $n \geq N_0$, then

$$n\epsilon \geq N_0\epsilon > 1$$

So $n\epsilon > 1 (\forall n \geq N_0)$. And so therefore $\frac{1}{n} < \epsilon$. Since ϵ was arbitrary, $a_n \to 5$. ✓ $\square$

(b) $a_n = \frac{3n}{2n+1}$ for $n \geq 1$.
It is reasonable to assume that $a_n \to \frac{3}{2}$
$\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that if $n \geq N$, then

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$$|a_n - \frac{3}{2}| < \epsilon$$

→ this is the goal. Be clear about it by writing a sentence like: "The goal is to show that [...]"

$$\left|\frac{3n}{2n+1} - \frac{3}{2}\right| < \epsilon$$

$$\left|\frac{3}{4n+2}\right| < \epsilon$$

$$\frac{|3|}{|4n+2|} < \epsilon$$

by property of absolute values

$$\frac{3}{4n+2} < \epsilon$$

$$\frac{4n+2}{3} > \frac{1}{\epsilon}$$

by the third property of the order axioms

Use the AP with $y = \frac{3}{\varepsilon} - 2 / 4$ & $x = 1$.

RHS is not an integer.

$$n > \frac{\frac{3}{\epsilon} - 2}{4}$$

If we just set the RHS equal to N, we know that there exists N such that if $n > N$, the above expression holds true. Therefore since ϵ was arbitary and

$$\left|\frac{3n}{2n+1} - \frac{3}{2}\right| < \epsilon$$

$a_n \to \frac{3}{2}$
$\square$

Exercise 7.

$a_n = \frac{2n+1}{n}$
From class, we know that if $a_n \to A$, then a_n is Cauchy, so we will do that.
It is reasonable to assume that $a_n \to 2$. Then $\forall \epsilon > 0$, there exists an $N \in \mathbb{N}$ such that if $n \geq N$ then

$$|a_n - 2| < \epsilon$$
$$\left|\frac{2n+1}{n} - \frac{2n}{n}\right| < \epsilon$$
$$\left|\frac{1}{n}\right| < \epsilon$$
$$\frac{|1|}{|n|} < \epsilon$$

Make sure that you write your goal clearly. This is what you want to prove, not what you have as an assumption

by the property of absolute values

$$\frac{1}{n} < \epsilon$$

As shown previously in this homework assignment and in class, we've proven that we can show $\frac{1}{n} < \epsilon$ using the Archimedean Property. Now therefore, becuase ϵ was arbitrary, $a_n \to 2$.
So, according to the Theorem from class, if $a_n \to A$, then a_n is Cauchy. Therefore, since $a_n \to 2$, it is Cauchy. $\square$

Exercise 8.

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(a) $a_n = (-1)^n$
Suppose toward a contradiction that $a_n \to A$. Then following after the definition of a convergent sequence, $\forall \epsilon > 0$, there exists an $N \in \mathbb{N}$ such that if $n \geq N$, then

$$|a_n - A| < \epsilon$$

Now set $\epsilon = 1$ and let's rewrite this sequence as a piece-wise function.
$(-1)^n = \{1 \text{ if n is even}, -1 \text{ if n is odd}\}$.
We have two cases. For the first, say $n = 2N$. Obviously, $2N \geq N$ and is even.

$$|(-1)^{2N} - A| < 1$$
$$|1 - A| < 1$$
$$-1 < 1 - A < 1$$

by the definition of absolute values

$$-2 < -A < 0$$

Now for the second case, say $n = 2N + 1$.

$$|(-1)^{2N+1} - A| < 1$$
$$|-1 - A| < 1$$
$$-1 < -1 - A < 1$$

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by the definition of absolute values

$$0 < -A < 2$$

So combining these two cases, we have that $-A \in (-2, 0) \cup (0, -2) = \emptyset$. Therefore, this is a contradiction. So a_n does not converge, and is therefore divergent. $\square$

(b) $a_n = \sin(\frac{2n+1}{2}\pi)$
If we look closely, we can see this yields the exact same sequence as the previous problem. This can be easier to see by rearranging some terms.

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$$\sin(\frac{2n+1}{2}\pi) = \sin(n\pi + \frac{\pi}{2})$$

As you can see, we start at $\frac{\pi}{2}$ and increments of π. On the unit circle, this starts the cycle at 90 degrees, and has us move π radians to the other side repeatedly. As a function of $\sin(x)$, this yields $\{1$ if n is even, -1 if n is odd$\}$. This is the exact same situation as before and can be proven identically. Therefore, this sequence does not converge but rather leads to a contradiction, deeming it divergent.

Exercise 9.

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$a_n = (-1)^n$
$b_n = (-1)^{n+1}$

These two sequences diverge. For a_n, we've already proved that this is divergent previously for Exercise 8a. For b_n, we will show its proof but it is very similar to the proof for a_n.

$b_n = \{1$ if n is even, -1 if n is odd$\}$. Now assume $b_n \to B$. That means that for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that if $n \geq N$, then

$$|b_n - B| < \epsilon$$

Let's set $\epsilon = 1$ and consider two cases. When $n = 2N$ and $n = 2N + 1$.
If $n = 2N$, we know that n will be even.

$$|(-1)^{2N} - B| < 1$$
$$|-1 - B| < 1$$
$$-1 < -1 - B < 1$$

by the definition of absolute values

$$0 < -B < 2$$

For the second case, let's set $n = 2N + 1$.

$$|(-1)^{2N+1} - B| < 1$$
$$|1 - B| < 1$$
$$-1 < 1 - B < 1$$

by the definition of absolute values

$$-2 < -B < 0$$

So combining these two cases, we get that $-B \in (-2, 0) \cup (0, 2) = \emptyset$. Therefore this is a contradiction. Therefore, b_n is divergent.
Now that we know a_n, b_n are divergent, we must prove that $a_n + b_n$ is convergent.
Let $c_n = a_n + b_n$

$$c_n = (-1)^n + (-1)^{n+1}$$

It is reasonable to assume that $c_n \to 0$. We must show that for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that if $n \geq N$,

$$|c_n - 0| < \epsilon$$
$$|(-1)^n + (-1)^{n+1}| < \epsilon$$

Let's take a look at this function in a piece-wise manner. $c_n = \{0$ if n even, 0 if n odd$\}$. Therefore, since n will either be even or odd $\forall n \in \mathbb{N}$, this sequence will always be 0.
From here, we can take $N = 1$. So if $n \geq 1$, then

$$|c_n - 0| = 0 < \epsilon$$

We just showed that if $\epsilon > 0$, then $\exists N = 1$ such that $\forall n \geq N$,

$$|c_n - 0| < \epsilon$$

So since ϵ was arbitrary, we conclude that $c_n \to 0$.
$\square$

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Exercise 10.
For problems (a) and (b), we are using the known proof from class that

$$lim_{n \to \infty}(\frac{1}{n}) = 0$$

(In LaTeX, we write \lim_{n \rightarrow n}

(a) $\frac{n^2+4n}{n^2-5}$

$$\frac{n^2 + 4n}{n^2 - 5} = \frac{n^2(1 + \frac{4}{n})}{n^2(1 - \frac{5}{n^2})}$$
$$= \frac{1 + \frac{1}{n}}{1 - \frac{5}{n^2}}$$
$$lim_{n \to \infty} \frac{1 + \frac{1}{n}}{1 - \frac{5}{n^2}} = \frac{lim_{n \to \infty}(1 + \frac{1}{n})}{lim_{n \to \infty}(1 - \frac{5}{n^2})}$$

by the quotient rule

Now we'll first take the limit of the numerator

$$lim_{n\to\infty}(1+\frac{1}{n}) = 1 + lim_{n\to\infty}(\frac{1}{n})$$

by linearity

$$= 1 + 0$$
$$= 1$$

Now we will take the limit of the denominator

$$lim_{n\to\infty}(1-\frac{5}{n^2}) = 1 - 5 \cdot lim_{n\to\infty}(\frac{1}{n}) \cdot lim_{n\to\infty}(\frac{1}{n})$$

by linearity and the product rule

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$$= 1 - 5 \cdot 0 \cdot 0$$
$$= 1$$

Therefore

$$lim_{n\to\infty}(\frac{n^2+4n}{n^2-5}) = \frac{1}{1} = 1$$

(b) $\frac{n}{n^2-3}$

$$\frac{n}{n^2-3} = \frac{n^2(\frac{1}{n})}{n^2(1-\frac{3}{n})}$$
$$= \frac{\frac{1}{n}}{1-\frac{3}{n}}$$

$$lim_{n\to\infty}(\frac{\frac{1}{n}}{1-\frac{3}{n}}) = \frac{lim_{n\to\infty}(\frac{1}{n})}{lim_{n\to\infty}(1-\frac{3}{n})}$$

by the quotient rule

First let's take the limit of the numerator

$$lim_{n\to\infty}(\frac{1}{n}) = 0$$

Now we will take the limit of the denominator

$$lim_{n\to\infty}(1-\frac{3}{n}) = 1 - 3 \cdot lim_{n\to\infty}(\frac{1}{n})$$

by linearity and the product rule

$$= 1 - 3 \cdot 0$$
$$= 1 - 0$$
$$= 1$$

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Now combining the numerator and denominator, we have that

$$lim_{n \to \infty}(\frac{n}{n^2 - 3}) = \frac{0}{1} = 0$$

(c) & (d)??