MATH-331 Introduction to Real Analysis — YOUR FULL NAME

Homework 05 — Fall 2021

Due date: November, 22^{th} 1:20pm — Total: /65.

Exercise	1 (10)	2 (10)	3 (5)	4 (5)	5 (5)	6 (10)	7 (5)	8 (5)	9 (5)	10 (5)
Score										

Table 1: Scores for each exercises

Instructions: You must answer all the questions below and send your solution by email (to `parisepo@hawaii.edu`). If you decide to not use LaTeX to hand out your solutions, please be sure that after you scan your copy, it is clear and readable. Make sure that you attached a copy of the homework assignment to your homework.
If you choose to use LaTeX, you can use the template available on the course website.
No late homework will be accepted. No format other than PDF will be accepted. Name your file as indicated in the syllabus.

1 Writing problems

For each of the following problems, you will be asked to write a clear and detailed proof. You will have the chance to rewrite your solution in your semester project after receiving feedback from me.

Exercise 1. (10 pts)

a) Fix any $\delta > 0$ and let $[a, b]$ be an interval with $a < b$. Find a tagged partition $\mathcal{P}$ of $[a, b]$ such that $\|\mathcal{P}\| < \delta$.

b) Suppose that f is Riemann integrable. Show that in the definition of the Riemann integral, the number L is unique. [Remark: This is why we gave it the name $\int_a^b f$.]

Solution:

Exercise 2. (10 pts) Suppose that f and g are Riemann integrable on the interval $[a, b]$.

a) Show that $\int_a^b (f + g) = \int_a^b f + \int_a^b g$.

b) Show that if $f(x) \leq g(x)$ for any $x \in [a, b]$, then $\int_a^b f \leq \int_a^b g$.

Solution:

Exercise 3. (5 pts) Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable on $[a,b]$ and suppose that $|f(x)| \leq M$ for all $x \in [a,b]$. Show that $\int_a^b f \leq M(b-a)$.

Solution:

Exercise 4. (5 pts) Suppose that f is Riemann integrable on $[a,b]$. Let $(\mathcal{P}_n)_{n=1}^{\infty}$ be a sequence of tagged partitions of $[a,b]$ such that the sequence $\lim_{n\to\infty} \|\mathcal{P}_n\| = 0$. Prove that the sequence $(S(f,\mathcal{P}_n))_{n=1}^{\infty}$ converges to $\int_a^b f$.

Solution:

Exercise 5. (5 pts) Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Suppose that f is Riemann integrable on $[a,c]$ for any $c \in (a,b)$. Show that f is Riemann integrable on $[a,b]$. [Hint: Use the Cauchy criterion for integrals.]

Solution:

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Homework problems

Answer all the questions below. Make sure to show your work.

Exercise 6. (10pts)

a) Define the function $f : [a,b] \to \mathbb{R}$ by $f(x) = k$ for every $x \in [a,b]$ where $k \in \mathbb{R}$ is a fixed constant. Show that f is Riemann integrable on $[a,b]$ and that $\int_a^b k\,dx = k(b-a)$.

b) Let $f(x) = \sin^2(x)$ where $x \in [a,b]$ and assume that the function $g(x) := \cos(kx)$ is integrable on $[a,b]$ for any $k \in \mathbb{R}$. Show that f is Riemann integrable on $[a,b]$.

Solution:

Exercise 7. (5 pts) Show that the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} 1 & \text{, if } 0 \leq x < 1/2 \\ 0 & \text{, if } 1/2 \leq x \leq 1 \end{cases}$$

is Riemann integrable on $[0,1]$.

Solution:

Exercise 8. (5 pts) Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = 1$ if $x = 1/n$ where $n \in \mathbb{N}$, and by $f(x) = 0$ if $x \neq 1/n$, $n \in \mathbb{N}$. Show that f is Riemann integrable on $[0,1]$.

Solution:

Exercise 9. (5 pts) Show that the function $f : [0,1] \to \mathbb{R}$ defined by $f(x) = 0$ if $x \neq 0$ and $f(x) = 4$ if $x = 0$ is Riemann integrable on $[0,1]$.

Solution:

Exercise 10. (5 pts) Let $\mathcal{P}$ be the following tagged partition of $[-1,2]$:

$$\mathcal{P} := \{(-9,[-1,-.8]),(-.7,[-.8,-.3]),(-.1,[-.3,0]),(.2,[0,0.2]),(.2,[.2,.4]),(.8,[.4,1]),\\ (1.42,[1,1.5]),(1.9,[1.5,2])\}.$$

Find another partition $\mathcal{P}_0$ such that $\|\mathcal{P}_0\| \leq \|\mathcal{P}\|/3$.

Solution: