MATH 331 Homework 2

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1. a)
i) We will prove a_n and b_m are both bounded sets. Both sets are bounded on the bottom by $n, m \geq 1$. Define $M := \max n, m$ and we consider the set $[a_M, b_M]$. Then $a_n \leq a_M$ and $b_m \geq b_M$. Therefore there is an upper bound for the set a_n and an upper bound for the set b_m.
ii) Since in part i we proved a_n and b_m are bounded sets, by the axiom of completeness, the supremum of a_n also exists.
iii) Take c to be the supremum of a_n, which then means c must be less than all b_n. Let $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $n \geq N$:

$$|a_n - A| < \varepsilon.$$

We want to prove

$$|b_n - A| > c.$$

Then

$$\begin{aligned}|b_n - A| &> c\\ |b_n - A| &> b_n\\ A &> 0\end{aligned}$$

b) The set $\mathbb{R}$ is uncountable because there will always exist a value c which is not mapped by $f(n) = [a_n, b_n]$. We showed in part a) that the value c exists between a_n and b_n, making $f(n)$ not a surjective function.
2. By definition, if $a_n \to A$ then $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that for $n \geq N$,

$$|a_n - A| < \varepsilon.$$

By the triangle inequality, $||a_n| - |A|| \leq |a_n - A|$, so

$$\begin{aligned}||a_n| - |A|| \leq |a_n - A| &< \varepsilon\\ ||a_n| - |A|| &< \varepsilon\end{aligned}$$

So $||a_n| - |A|| < \varepsilon$ and $|a_n| \to |A|$.
3. We want to prove $c_n \to L$. We know that $\forall \varepsilon > 0, \exists N_A, N_B \in \mathbb{N}$, for $n \geq N_A, N_B$, $-\varepsilon < |a_n - L| < \varepsilon$ and $-\varepsilon < |b_n - L| < \varepsilon$ by definition.
Take $N := \max\{N_A, N_B\}$. If $n \geq N$, then
If $a_n \leq c_n \leq b_n$, then $a_n - L \leq c_n - L \leq b_n - L$.
4.
1. Assume $A = 0$. We know the square root of 0 is still 0. Therefore we must prove
2. If $a_n \to A$ then $\forall \varepsilon > 0, \exists N \in \mathbb{N}, n \leq N$:

$$|a_n - A| < \varepsilon$$

Let $\varepsilon = \frac{\sqrt{A}}{\sqrt{2}}$. Then

$$\begin{aligned}\frac{a_n}{\sqrt{a_n}} - \frac{A}{\sqrt{A}} &< \varepsilon\\ \frac{a_n\sqrt{A} - A\sqrt{a_n}}{\sqrt{a_n A}} &< \varepsilon\\ \frac{a_n\sqrt{A} - A\sqrt{a_n} - A\sqrt{A} + A\sqrt{A}}{\sqrt{a_n A}} &< \varepsilon\\ \frac{(a_n - A)\sqrt{A} - A(\sqrt{a_n} - \sqrt{A})}{\sqrt{a_n A}} &< \varepsilon\\ \frac{(a_n - A)\sqrt{A}}{\sqrt{a_n A}} - \frac{A(\sqrt{a_n} - \sqrt{A})}{\sqrt{a_n A}} &< \varepsilon\end{aligned}$$

There is a natural number N_1 such that for $n \geq N, |\sqrt{a_n} - \sqrt{A}| < \frac{\sqrt{A}}{\sqrt{2}}$ and $\frac{\sqrt{A}}{\sqrt{2}} < |\sqrt{a_n}|$ by the triangle inequality.

3. Let $\sqrt{a_n} - A = \frac{a_n - A}{\sqrt{a_n}+\sqrt{A}}$. From part 2 we know that $\frac{\sqrt{A}}{\sqrt{2}} < |\sqrt{a_n}|$. So

$$\sqrt{a_n} + \sqrt{A} > \sqrt{\frac{A}{2}} + \sqrt{A}$$
$$\frac{1}{\sqrt{a_n} + \sqrt{A}} < \frac{1}{\sqrt{\frac{A}{2}} + \sqrt{A}}$$
$$\frac{1}{\sqrt{a_n} + \sqrt{A}} < \frac{\sqrt{2}}{\sqrt{A}(1 + \sqrt{2})}$$

Substitute:

$$(\sqrt{a_n} - \sqrt{A})(\frac{\sqrt{2}}{\sqrt{A}(1 + \sqrt{2})}) = |a_n - A|$$
$$(\sqrt{a_n} - \sqrt{A})(\frac{\sqrt{2}}{\sqrt{A}(1 + \sqrt{2})}) < \frac{3}{4} \cdot \frac{\varepsilon}{\sqrt{A}}$$
$$(\sqrt{a_n} - \sqrt{A}) \left(\frac{\sqrt{2}}{1 + \sqrt{2}} \right) < \frac{3}{4}\varepsilon$$
$$(\sqrt{a_n} - \sqrt{A}) < \frac{3(1 + \sqrt{2})}{4\sqrt{2}}\varepsilon$$

and $(\sqrt{a_n} - \sqrt{A}) < \varepsilon$.

5. We want to prove that $\sigma \to A$. That means, we hope to prove

$$|\frac{a_1 + a_2 + a_3 + ... + a_n}{n} - A| < \varepsilon$$

$\forall \varepsilon > 0, \exists N \in \mathbb{N}, n \geq N$. Then

$$|\frac{a_1 + a_2 + a_3 + ... + a_n}{n} - A| < \varepsilon$$
$$|\frac{a_1 + a_2 + a_3 + ... + a_n}{n} - \frac{An}{n}| < \varepsilon$$
$$|\frac{a_1 + a_2 + a_3 + ... + a_n - An}{n}| < \varepsilon$$
$$|\frac{(a_1 - A) + (a_2 - A) + (a_3 - A) + ... + (a_n - A)}{n}| < \varepsilon$$
$$|\frac{\sum_{k=1}^{n}(ak - A)}{n}| < \varepsilon$$
$$\frac{\sum_{k=1}^{n}(ak - A)}{n} + \frac{\sum_{k=N}^{N-1}(ak - A)}{n} < \varepsilon$$

We know the behavior of $\frac{\sum_{k=1}^{n}(ak-A)}{n}$, but we don't know how the sequence behaves for all $N - 1$. So we examine this case.

$$\frac{\sum_{k=N}^{N-1}(ak-A)}{n} < \varepsilon - \frac{\sum_{k=1}^{n}(ak-A)}{n}$$
$$\frac{1}{n} \cdot \sum_{k=N}^{N-1}(ak-A) < \varepsilon - \frac{\sum_{k=1}^{n}(ak-A)}{n}$$
$$\frac{1}{n} \cdot \sum_{k=N}^{N-1}(ak) - \sum_{k=N}^{N-1}(A) < \varepsilon - \frac{\sum_{k=1}^{n}(ak-A)}{n}$$
$$\frac{1}{n} \cdot \sum_{k=N}^{N-1}(ak) - A < \varepsilon - \frac{\sum_{k=1}^{n}(ak-A)}{n}$$

We then prove $\frac{a_{n_1}-A}{n} < \varepsilon$. We do this by contradiction: Assume $\exists \varepsilon > 0 \forall N \in \mathbb{N}$ s.t. $n \geq N$. Then

$$|a_{n-1} - A| > n\varepsilon.$$

and

$$-n\varepsilon < a_{n-1} - A < n\varepsilon$$

This is clearly a contradiction, since our inequality is of the form $+x < a_{n-1} - A < -x$, which makes no sense since we have positive x.

6. a) Our hypothesis is that a_n converges to 5, since as n becomes infinitely large, $\frac{1}{n}$ approaches 0, leaving $5+0=5$. So, by the definition of convergence, $\forall \varepsilon, \exists N$ such that $|a_n - A| < \varepsilon$ for $n \leq N$ where $A = 5$.
We have

$$|a_n - A| < \varepsilon$$
$$|5 + \frac{1}{n} - 5| < \varepsilon$$
$$|\frac{1}{n}| < \varepsilon.$$

Since a_n begins at $n=1$ we know the values of this sequence are always positive. Therefore $|\frac{1}{n}| = \frac{1}{n}$. Then

$$\frac{1}{n} < \varepsilon$$
$$n > \frac{1}{\varepsilon}.$$

Therefore $N = \frac{1}{\varepsilon}$ and our sequence a_n does indeed converge.
b) We hypothesize the sequence converges to $\frac{3}{2}$. By definition of convergence, $\forall \varepsilon, \exists N$ such that $|a_n - A| < \varepsilon$ for $n \leq N$ where $A = \frac{3}{2}$ and $a_n = \frac{3n}{2n+1}$.
So

$$|\frac{3n}{2n+1} - \frac{3}{2}| < \varepsilon$$
$$\frac{3n}{2n+1} - \frac{3}{2} < \varepsilon$$
$$\frac{6n - 3(2n+1)}{2(2n+1)} < \varepsilon$$
$$\frac{6n - 6n - 1}{4n+2} < \varepsilon$$
$$\frac{-1}{4n+2} < \varepsilon$$
$$\frac{1}{\varepsilon} > -(4n+2)$$

We see that the left hand side resembles the sequence $\frac{1}{n}$, which means it converges to 0. Since the sequence converges, the limit is exactly $\frac{3}{2}$.

7. Prove $\frac{2n+1}{n}$ is a Cauchy sequence: By definiton our sequence is Cauchy if $\forall\varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $m, n, \geq N$:

$$|a_n - a_m| < \varepsilon.$$

By theorem 1.3 in the text, every convergent sequence is Cauchy, or a sequence is Cauchy iif it is convergent. Therefore, we will prove convergence:

$$\begin{aligned}\lim_{n\to\infty} \frac{2n+1}{n} &= \lim_{n\to\infty} \frac{1}{n} \cdot \lim_{n\to\infty}(2n+1) \\ &= 0 \cdot \infty \\ &= 0.\end{aligned}$$

The sequence converges to 0, so it is Cauchy.

8. a) We use the definition of Cauchy sequence to prove divergence by contradiction. The definiton states that $\forall\varepsilon > 0, \exists N \in \mathbb{N}, \forall n, m \geq N$,

$$|a_n - a_m| < \varepsilon.$$

We choose an arbitrary ε, say 1. Then $|(-1)^n - (-1)^m| < \varepsilon < 1$.
Choose and arbitrary m, say $m = n + 1$. Then

$$\begin{aligned}|(-1)^n - (-1)^m| &< 1 \\ |(-1)^n - (-1)^{n+1}| &< 1.\end{aligned}$$

We saw in class that $|(-1)^n - (-1)^{n+1}|$ is exactly 2. We then get $2 < 1$ which is a contradiction, so the sequence is divergent.

b)

Assume to a contradiction that $\sin(\frac{2n+1}{2}\pi)$ is a convergent sequence, and let $\varepsilon = 1$. Then

$$\begin{aligned}\sin(\frac{2n+1}{2}\pi) - A &< 1 \\ -1 < \sin(\frac{2n+1}{2}\pi) - A &< 1 \\ A - 1 < \sin(\frac{2n+1}{2}\pi) &< A + 1\end{aligned}$$

If n is even, then we have

$$A - 1 < 1 < A + 1$$

and if n is odd, then we have

$$A - 1 < -1 < A + 1.$$

So $A - 1 < 1 < A + 1 \implies A < 2 < A + 2$ and $A - 1 < -1 < A + 1 \implies A < 0 < A + 2$. Clearly this is a contradiction because A is a limit and cannot be negative.

9. Assume we have two sequences a_n and b_n. The summation of a_n and b_n must be equal to a convergent sequence. Let $a_n = 2n$. We know $2n$ is divergent because it goes to infinity, and we know the sequence $\frac{1}{n}$ is convergent (converges to 0). Then:

$$\begin{aligned}2n + b_n &= \frac{1}{n} \\ b_n &= \frac{1}{n} - 2n \\ b_n &= \frac{1}{n} - \frac{2n^2}{n} \\ b_n &= \frac{1 - 2n^2}{n}\end{aligned}$$

We use the sum rule to evaluate the limit of b_n. $\lim_{n\to\infty} \frac{1}{n} - 2n \implies \lim_{n\to\infty} \frac{1}{n} - \lim_{n\to\infty} 2n = 0 - \infty$. The sequence $b_n = \frac{1-2n^2}{n}$ is divergent since it tends towards $-\infty$.
10. a) $\frac{n^2+4n}{n^2-5}$:

$$\begin{aligned}
\frac{n^2+4n}{n^2-5} &\implies \frac{\frac{n^2}{n^2} - \frac{4n}{n^2}}{\frac{n^2}{n^2} - \frac{5}{n^2}} \\
&\implies \frac{1-\frac{4}{n}}{1-\frac{5}{n^2}} \\
&\implies \frac{1-0}{1-0} \\
&\implies 1.
\end{aligned}$$

b) $\frac{n}{n^2-3}$:

$$\begin{aligned}
\frac{n}{n^2-3} &\implies \frac{\frac{n}{n^2}}{\frac{n^2}{n^2} - \frac{3}{n^2}} \\
\frac{\frac{1}{n}}{1-\frac{3}{n^2}} &\implies \frac{1}{n} \cdot \frac{1}{1-\frac{3}{n^2}} \\
&\implies 0 \cdot \frac{1}{1-0} \\
&\implies 0.
\end{aligned}$$

c) $\frac{cosn}{n}$.

By the product rule, we know $\lim_{n\to\infty} \frac{cosn}{n} = \lim_{n\to\infty} \frac{1}{n} \cdot \lim_{n\to\infty} cosn$. Therefore the limit goes to 0 since we know $\frac{1}{n} \to 0$.

d) Find $\lim_{n\to\infty} n(\sqrt{4-\frac{1}{n}} - 2)$. Multiply by the conjugate:

$$\begin{aligned}
&\lim_{n\to\infty} n(\sqrt{4-\frac{1}{n}} - 2) \\
&\lim_{n\to\infty} n(\sqrt{4-\frac{1}{n}} - 2)\left(\frac{n(\sqrt{4-\frac{1}{n}}+2)}{n(\sqrt{4-\frac{1}{n}}+2)}\right) \\
&\lim_{n\to\infty} \left(\frac{n(4-\frac{1}{n}-4)}{(\sqrt{4-\frac{1}{n}}+2)}\right) \\
&\lim_{n\to\infty} \left(\frac{-1}{\left(\sqrt{4-\frac{1}{n}}+2\right)}\right)
\end{aligned}$$

By quotient rule:

$$\begin{aligned}
\lim_{n\to\infty}\left(\frac{-1}{\left(\sqrt{4-\frac{1}{n}}+2\right)}\right) &= \frac{\lim_{n\to\infty} -1}{\lim_{n\to\infty} \sqrt{4-\frac{1}{n}}+2} \\
&= \frac{-1}{\sqrt{4-0}+2} \\
&= \frac{-1}{4}.
\end{aligned}$$