Math 331: Homework 1

1.a) Let I be a subinterval of $[a,b]$ and put $\varphi = c\chi_I$

Case ①, when $I = [u,v]$, $a \le u < v \le b$, then

$$\int_a^b \phi(x)\,dx = \int_a^u \phi(x)\,dx + \int_u^v \phi(x)\,dx + \int_v^b \phi(x)\,dx$$
$$= \int_a^u 0\,dx + \int_u^v c\,dx + \int_v^b 0\,dx$$
$$= c(v-u) = c\,\ell(I)$$

Case ②, when $I = (u,v]$ then

$$\int_a^b \phi(x)\,dx = \int_a^u \phi(x)\,dx + \int_u^v \phi(x)\,dx + \int_v^b \phi(x)\,dx$$
$$= \int_u^v c\,dx = c(v-u) = c\,\ell(I)$$

Case ③, when $I = [u,u] = \{u\}$

$$\int_a^b \phi(x)\,dx = \int_a^u \phi(x)\,dx + \int_u^b \phi(x)\,dx$$
$$= \int_a^u 0\,dx + \int_u^b 0\,dx = 0$$

Since in each case the number of discontinuities of ϕ in $[a,b]$ is finite, ϕ is R.I ∎

b) We know that, if f_1 and f_2 are R.I. then $f_1 + f_2$ is also R.I and

$$\int_a^b (f_1+f_2)(x)\,dx = \int_a^b f_1(x)\,dx + \int_a^b f_2(x)\,dx \quad (*)$$

Now, let the result be true for $n = k-1$, so $f_1 + \dots + f_{k-1}$ is R.I. and

$$\int_a^b (f_1+\dots+f_{k-1})(x)\,dx = \int_a^b f_1(x)\,dx + \dots + \int_a^b f_{k-1}(x)\,dx \quad (**)$$

Now take $n=k$, then let $f_1, \dots, f_k$ be R.I and $f_1 + \dots + f_k$ is R.I., and

$$\int_a^b (f_1+\dots+f_{k-1}+f_k)(x)\,dx = \int_a^b \big((f_1+\dots+f_{k-1})(x) + f_k(x)\big)\,dx$$
$$= \int_a^b (f_1+\dots+f_{k-1})(x)\,dx + \int_a^b f_k(x)\,dx \quad (\text{by } *)$$
$$= \int_a^b f_1(x)\,dx + \dots + \int_a^b f_{k-1}(x)\,dx + \int_a^b f_k(x)\,dx \quad (\text{by } **)$$

By Principle of Mathematical Induction the result is true for all $n \in \mathbb{N}$ ∎

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1. c) Given $\phi = \sum_{k=1}^{n} C_k X_{I_k}$
from ⓐ $\Rightarrow C_k X_{I_k}$ is R.I $\forall k = 1, \dots, n$
from ⓑ $\Rightarrow \sum_{k=1}^{n} C_k X_{I_k}$ is R.I.
and also

$$\int_a^b \phi(x)\,dx = \int_a^b \sum_{k=1}^{n} C_k X_{I_k}(x)\,dx$$
$$= \sum_{k=1}^{n} \int_a^b C_k X_{I_k}(x)\,dx \qquad \blacksquare$$

2. Let $x_i = a + i\Delta x$ and $\Delta x = \frac{b-a}{n}$, where $x_i \in [a,b]$ and n is a natural number.
We are given $0 \le f(x)$ for $x \in [a,b]$, so

$$0 \le f(x_i)$$
$$\Rightarrow 0 \le f(x_i)\Delta x$$
$$\Rightarrow 0 \le \sum_{i=0}^{n} f(x_i)\Delta x$$
$$\Rightarrow 0 \le \lim_{n\to\infty} \sum_{i=0}^{n} f(x_i)\Delta x$$
$$\Rightarrow 0 \le \int_a^b f(x)$$

Similarly, since intervals $[a,u], [u,v]$ & $[v,b]$ are in $[a,b]$

$$\Rightarrow 0 \le \int_a^u f(x) \ \&\ 0 \le \int_u^v f(x) \ \&\ 0 \le \int_v^b f(x)$$

Then we get

$$\int_a^b f(x) - \int_a^u f(x) - \int_v^b f(x) \le \int_a^b f(x)$$

Now use the property of the Riemann Integral we get:

$$\Rightarrow \int_a^v f(x) - \int_a^u f(x) \le \int_a^b f(x)$$
$$\Rightarrow \int_a^u f(x) + \int_u^v f(x) - \int_a^u f(x) \le \int_a^b f(x)$$
$$\Rightarrow \int_u^v f(x) \le \int_a^b f(x) \qquad \blacksquare$$

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3. a) Let us assume towards a contradiction that there exists a point $c \in [a,b]$ s.t $f(c) \neq 0$

$$\Rightarrow f(c) > 0 \qquad (f(x) \geq 0 \ \forall x \in [a,b])$$

f is continuous on $[a,b]$ so $\exists$ an $\varepsilon > 0$ s.t $f(x) > 0 \ \forall x \in (c-\varepsilon, c+\varepsilon)$

$$\Rightarrow \int_{c-\varepsilon}^{c+\varepsilon} f(x) > 0$$

We know that

$$\int_a^b f(x)dx = \int_a^{c-\varepsilon} + \int_{c-\varepsilon}^{c+\varepsilon} f(x)dx + \int_{c+\varepsilon}^{b} f(x)dx$$

and we have

$$\int_a^{c-\varepsilon} f(x)dx \geq 0$$

$$\& \int_{c+\varepsilon}^{b} f(x)dx \geq 0$$

so $\int_a^b f(x)dx > 0 \quad \rightarrow\leftarrow$

so $f(x) = 0 \ \forall x \in [a,b]$ ■

b) Given $f, g: [a,b] \to \mathbb{R}$ is continuous s.t

$$\int_a^b f = \int_a^b g$$

define $h: [a,b] \to \mathbb{R}$ & $h(x) = (f-g)(x)$ with f and g continuous on $[a,b]$ and $h = f - g$ is continuous on $[a,b]$.

By Fundamental Theorem of Calculus there exists

$$H: [a,b] \to \mathbb{R} \quad \text{s.t}$$

$$H'(x) = h(x)$$

and $\int_a^b h(x) = H(b) - H(a)$

so $\int_a^b h = \int_a^b f - g = 0 \Rightarrow H(b) - H(a) = 0$

$$\Rightarrow \exists c \in (a,b) \text{ s.t. } H'(c) = 0$$

$$\Rightarrow h(c) = 0$$

$$\Rightarrow f(c) - g(c) = 0$$

$$\Rightarrow f(c) = g(c)$$ ■

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4. Let $F(x) = \int_a^x f(t)\,dt$. As f is continuous on $[a,b]$, F is differentiable on (a,b) and by the MVT, $\exists c \in (a,b)$ s.t.

$$F'(c) = F(b) - F(a)$$

$$\boxed{\Rightarrow f(c)(b-a) = \int_a^b f}$$

$\blacksquare$

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5. Given f is strictly increasing, define

$g(x) = f(a)(x-a) + f(b)(b-x)$

Since f is strictly increasing $\forall x \in [a,b]$ and $f(a) < f(x) < f(b)$.

Taking the integral

$\Rightarrow \int_a^b f(a)\,dx < \int_a^b f(x)\,dx < \int_a^b f(b)\,dx$

$\Rightarrow f(a)(b-a) < \int_a^b f(x)\,dx < f(b)(b-a)$

Note that $g(a) = f(b)(b-a)$

& $g(b) = f(a)(b-a)$

& since f is continuous $g(x)$ is continuous & since $\int_a^b f(x)dx$ lies between $g(a)$ & $g(b)$. Using the Intermediate Value Theorem $\exists c$ s.t.

$g(c) = \int_a^b f(x)\,dx$

i.e. $f(a)(c-a) + f(b)(b-c) = \int_a^b f(x)\,dx$. ■

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6. a) Let $U(f,P)$ and $L(f,P)$ be the upper and lower sums of f with respect to partition P on $[0,1]$. Let $M_i = \sup\{f(x) \mid x \in I_i\}$ and $m_i = \inf\{f(x) \mid x \in I_i\}$ where I_i is the i^{th} interval of P. Note that $M_i = 1$ for all i, because every interval I_i of P contain rational numbers. On the other hand $m_i = 0$ for all i because every interval I_i of P contain irrational numbers. By definition,

$$U(f,P) = \sum_{i=1}^{n} M_i \mu(I_i) = \sum_{i=1}^{n} 1\mu(I_i) = \sum_{i=1}^{n} \mu(I_i) = 1 - 0 = 1$$

&

$$L(f,P) = \sum_{i=1}^{n} m_i \mu(I_i) = \sum_{i=1}^{n} 0\mu(I_i) = 0$$

Thus f is not R.I. on $[0,1]$ because the upper and lower sums are not equal. ∎

b) Since $g: [0,1] \to \mathbb{R}$ & $h: [0,1] \to \mathbb{R}$ by

$g = \chi_{[0,1]}$, $h(x) = \begin{cases} 0 & , x \in \mathbb{Q} \\ 1/q & , x = p/q \in \mathbb{Q} \end{cases}$

then $g \circ h: [0,1] \to \mathbb{R}$

and $g \circ h(x) = g(h(x)) = g(0)$ if $x \in \mathbb{Q} = 0$

and $g \circ h(x) = g(h(x)) = g(\frac{1}{q})$ if $x = p/q \in \mathbb{Q} = 1$

so $g \circ h(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{Q} \end{cases}$

Hence $g \circ h = f$ and by (a) $g \circ h$ is not integrable on $[0,1]$. ∎

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7. If f is continuous on $[a,b]$ then f is R.I. on $[a,b]$. Then by the same logic we have that $|f|$ is continuous and R.I. on $[a,b]$.
Now we know that

$$\pm x \leq |x| \quad ☆$$

$$\Rightarrow -|x| \leq x \leq |x|$$

Then if we take the integral we have

$$-|f| \leq f \leq |f|$$

$$\Rightarrow -\int |f| \leq \int f \leq \int |f|$$

Then by the logic of ☆

$$\left|\int f\right| \leq \int |f| \quad \blacksquare$$

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8. $f(x) = \int_{\sqrt{x}}^{\sqrt[3]{x}} \frac{1}{1+t^3}\, dt$

Let $g(t) = \frac{1}{1+t^3}$. Since g is continuous $\exists G$ s.t. $g = G'(t)$. Then $G(\sqrt[3]{x}) - G(\sqrt{x})$

$$f'(x) = G(\sqrt[3]{x}) - G(\sqrt{x})$$

$$= \left(\frac{1}{1+\sqrt[3]{x}^3}\right)\frac{1}{3(x^{2/3})} - \frac{1}{1+\sqrt{x}^3}\left(\frac{1}{2\sqrt{x}}\right)$$

$$\boxed{= \frac{1}{3(1+x)(x^{2/3})} - \frac{1}{2(1+x^{3/2})(\sqrt{x})}}$$

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9. $f(1)=0$ and $f'(x)=1+\sin(x^2)\ \forall x>1$

$f'(x)=1+\sin(x^2)$

$\frac{df}{dx}=1+\sin(x^2)$

$\int df=\int 1+\sin(x^2)\,dx$ $\quad u=x^2$

$f(x)=x-\frac{\sqrt{\pi}}{\sqrt{2}}\,S\left(\sqrt{\frac{2}{\pi}}\,x\right)+C$ $\quad du=2x\,dx$

$dx=\frac{1}{2x}\,du$

$$0=1-\sqrt{\frac{\pi}{2}}\,S\left(\sqrt{\frac{2}{\pi}}\right)+C$$

$$C=\sqrt{\frac{\pi}{2}}\,S\left(\sqrt{\frac{2}{\pi}}\right)-1$$

$$\boxed{f(x)=x-\sqrt{\frac{\pi}{2}}\,S\left(\sqrt{\frac{2}{\pi}}\,x\right)+\sqrt{\frac{\pi}{2}}\,S\left(\sqrt{\frac{2}{\pi}}\right)-1}$$

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10. Thinking about this as a Riemann Sum, we factor out an "n^2" in the denominator

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{k^2+n^2}$$
$$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{1+(\frac{k}{n})^2}$$

Which can be rewritten as

$$\lim_{n\to\infty} \int_0^1 \frac{1}{1+x^2}$$
$$\boxed{= \frac{\pi}{4}}$$