54/70 good!! :)

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MATH 331 Homework 02

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1 Exercise 1. (10 pts).

a) Let $\{[a_n, b_n] : n \geq 1\}$ be a family of closed intervals such that $[a_1, b_1] \supset [a_2, b_2] \supset [a_3, b_3] \supset \ldots$ Show that there is a $c \in \mathbb{R}$ such that $c \in [a_n, b_n]$ for all $n \geq \mathbb{N}$. Follow the following steps to prove it:

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(i) Prove that for any $n, m \geq 1$, $a_n \leq b_m$. [hint: put $M := \max\{n, m\}$]

(ii) Show that $\sup\{a_n : n \geq 1\}$ exists.

(iii) Show that $c = \sup\{a_n : n \geq 1\}$ satisfies the requirements.

Suppose $\{[a_n, b_n] ; n \geq 1\}$. Then clearly for any $n, m \geq 1$ on the closed interval $[a_n, b_n]$ that $a_n \leq b_n$. The same is true for the closed interval $[a_m, b_m]$ then $a_m \leq b_m$. define $M := \max\{n, m\}$ so that if n is max then $[a_m, b_m] \supset [a_n, b_n]$ so $a_m \leq b_n$ and $a_n \leq b_m$ or if m is max then $[a_n, b_n] \supset [a_m, b_m]$ so $a_n \leq b_m$ and $a_m \leq b_m$. As you can see it is true that $a_n \leq b_m$. (i) Now we must show that the $\sup\{a_n : n \geq 1\}$ exists. It is obvious from the definition $[a_n, b_n] \supset [a_{n+1}, b_{n+1}]$ that $a_{n+1} \geq a_n$ for a $n \in \mathbb{N}$. Then for any $n \geq 1$, a_n will be greater than or equal to all a_i for $1 \leq i \leq n-1$. This means there exists a $M \in \mathbb{R}$ and for all $x \in \{a_n : n \geq 1\}$ then $x \leq M_n$. In particular M_n can be equal to a_n. In addition for all K_n that are upper bounds of $\{a_n : n \geq 1\}$ then $x \leq M_n < K_n$; so $\sup\{a_n : n \geq 1\}$ exist. (ii) We are asked to show that $c = \sup\{a_n : n \geq 1\}$ satisfies the requirements. Let $c = M_n$ then by the same argument above $x \leq c \leq K_n$ where $x \in \{a_n : n \geq 1\}$, $c =$ supremum $\in \mathbb{R}$, $K_n =$ all upper bounds. Therefore it satisfies the requirement of the supremum, since $b_m \geq c = a_n \geq a_{n-1}$, for any n, clearly $c \in [a_n, b_n]$ for all $n \in \mathbb{N}$.

The upper bound can't depend on n (a_n). -1

see solution. -1

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b) Use this last result to prove that the set $\mathbb{R}$ is uncountable. [Hint: show that any function $f: \mathbb{N} \to \mathbb{R}$ can't be surjective. To do so construct a sequence of closed intervals such that $f(n) \notin [a_n, b_n]$ with $a_n < b_n$.]

You have to construct the closed intervals $[a_n, b_n]$.

Suppose $n > m$ such that $a_m \leq a_n$ and $b_n \leq a_m$ with $a_n < b_n$ and $a_m < b_m$. Define $f: \mathbb{N} \to \mathbb{R}$ where $f(n) \notin [a_n, b_n]$ however, $f(n) \in [a_m, a_n] \cup [b_n, b_m]$. $c \in [a_n, b_n]$ but $c \notin [a_m, a_n] \cup [b_n, b_m]$. Then for all $n \geq 1$ $Im(f(n))$ for all $n \geq 1$, $Im(f(n)) = [a_1, b_1]$ or $f(n) \in [a_1, b_1] \neq \mathbb{R}$. Therefore this function is not the set of all real numbers and therefore not surjective. Since a function cannot be found we can say that $\mathbb{R}$ is uncountable. Also $c \in [a_n, b_n]$, but $c \notin [a_m, a_n] \cup [b_n, b_m]$

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2 Exercise 2. (5 pts) Prove that if $a_n \to A$, then $|a_n| \to |A|$

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Proof: suppose a_n converges to A, then by the definition of convergence for all $\varepsilon > 0$, there exist an $N_1 \in \mathbb{N}$, such that for all $n \geq N_1$ we have $|a_n - A| < \varepsilon$. Now suppose $|a_n|$ and take the limit $L = |A|$ such that for all $\varepsilon > 0$, there is an $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$ then $||a_n| - |A|| < \varepsilon$. we have to show this. Let $\varepsilon > 0$ and arbitrary then by the properties of absolute value, 0.2.5 Theorem iv (pg 26) that $||a| - |b|| \leq |a - b|$, so take $|a| = |a_n|$ and $|b| = |A|$ then by this inequality $||a_n| - |A|| \leq |a_n - A|$ but since a_n converges to A it must be true that $|a_n - A| < \varepsilon$ so by transitivity $||a_n| - |A|| \leq |a_n - A| < \varepsilon$. Note to resolve that N_1 and N_2 are not necessarily equal set $N := \max\{N_1, N_2\}$. Thus we have shown that $\varepsilon > 0$ that $||a_n| - |A|| \leq |a_n - A| < \varepsilon$ for all $n \geq N$. Since $\varepsilon > 0$ was arbitrary we have shown that if $a_n \to A$ then $|a_n| \to |A|$. ∎

The N that will work is exactly N_1

3 Exercise 3. (5 pts) Let (a_n), (b_n), and (c_n) be sequences of real numbers. Prove that if $a_n \to L$, $b_n \to L$ and $a_n \leq c_n \leq b_n$, then $c_n \to L$.

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Suppose $a_n \leq c_n \leq b_n$.

Suppose a_n converges to L such that for all $\varepsilon > 0$, there exists an $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$ we have $|a_n - L| < \varepsilon$. Suppose b_n converges to L such that for all $\varepsilon > 0$ there exists an $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$ we have $|b_n - L| < \varepsilon$. Now suppose c_n converges to G where for all $\varepsilon > 0$ there exists an $N_3 \in \mathbb{N}$ such that for all $n \geq N_3$ we have $|c_n - G| < \varepsilon$. We will prove than G must equal L. Now suppose we know that $\{a_n\}_{n=1}^{\infty}$ converges to L and $\{c_n\}_{n=1}^{\infty}$ converges G with $a_n \leq c_n$ for all $n \in \mathbb{N}$ then by 1.2 theorem we know $L \leq G$. Now suppose $\{c_n\}_{n=1}^{\infty}$ converges to G and $\{b_n\}_{n=1}^{\infty}$

You can't suppose that $c_n \to c$. This is what you want to show.

What about (c_n) is divergent.

converges to L with $c_n \leq b_n$ for all $n \in \mathbb{N}$ then we know $G \leq L$ by 1.12 Theorem (pg 48). By using transitivity of $L \leq G$ and $G \leq L$ then $L \leq G \leq L$. The only case this is true is when $G = L$. Therefore we shown that given $\{a_n\}_{n=1}^{\infty}$ converges to L and $\{b_n\}_{n=1}^{\infty}$ converges to L and $a_n \leq c_n \leq b_n$ then $\{c_n\}_{n=1}^{\infty}$ must converge to L. ■

4 Exercise 4 (5pts) prove that if $a_n \to A$ and $a_n \geq 0$ for all $n \geq 1$, then $\sqrt{a_n} \to \sqrt{A}$. Follow the following steps to prove it:

1. consider the case $A = 0$
2. suppose that $A \neq 0$. Show that there is a $N_1 \in \mathbb{N}$ such that if $n \geq N_1$ then $\sqrt{a_n} \geq \sqrt{|A|/2}$.
   [Hint: use the definition of convergence of $(a_n)_{n \geq 0}$ with a clever choice of ε and use the properties of absolute value.]
3. Use the convergence of (a_n) again to find a N_2 such that $|a_n - A| < \frac{3}{4}\frac{\varepsilon}{\sqrt{|A|}}$
4. Express $\sqrt{a_n} - A$ as $\frac{a_n - A}{\sqrt{a_n} + \sqrt{A}}$ and put $N = \max\{N_1, N_2\}$. Conclude.

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suppose $\{a_n\}_{n=1}^{\infty}$ converges to A and $a_n \geq 0$. In the case where $A = 0$ we have for all $\varepsilon_0 > 0$ there exist an $N \in \mathbb{N}$ such that for all $n \geq N$ we have $|a_n - 0| = |a_n| < \varepsilon_0$. By the definition of convergence take $N = 1$ such that for all $n \geq N = 1$ $a_n \geq 0$ so that $-\varepsilon_0 < a_n < \varepsilon_0$. Therefore $0 \leq a_n < \varepsilon_0$ for all $n \geq 1$. Lets show that if $\varepsilon_1 > 0$ then there exists an $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$ then $|\sqrt{a_n} - \sqrt{A}| = |\sqrt{a_n} - 0| = \sqrt{a_n} < \varepsilon_1$. To show this take $\varepsilon_1 = \sqrt{\varepsilon_0}$ then since we know $0 \leq a_n \leq \varepsilon_0$ then (by HW1 $0 \leq a \leq b$ then $\sqrt{a} < \sqrt{b}$) then $|\sqrt{a_n} - \sqrt{A}| = |\sqrt{a_n} - 0| = \sqrt{a_n}$

not true. You will find a $N \in \mathbb{N}$ $N \geq 1$

$\sqrt{a_n} < \sqrt{\varepsilon_0} = \varepsilon_1$ Thus we have shown that since for $n \geq 1 = N$ that the case where $A = 0$ it is true that $\sqrt{a_n}$ converges to the $\sqrt{A}$.

Now consider that $A \neq 0$ then we must show that

I didnt finish

5 Exercise 5. (5pts) For each sequence $(a_n)_{n=1}^{\infty}$ define the sequence $(\sigma_n)_{n=1}^{\infty}$ by

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$$\sigma_n := \frac{a_1 + a_2 + \dots + a_n}{n} \qquad (n \geq 1)$$

Prove that if $a_n \to A$, then $\sigma_n \to A$. Find an example of a divergent sequence (a_n) such that $(\sigma_n)_{n=1}^{\infty}$ converges.

Proof: suppose that $a_n \to A$ then it must be true that for all $\varepsilon > 0$ there is an $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$ we have $|a_n - A| < \varepsilon$. We are trying to show that for $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$ that $|\sigma_n - A| < \varepsilon$.

Let $\varepsilon > 0$ be arbitrary. Since we know $a_n \to A$ and that $|a_n - A| < \frac{\varepsilon}{n}$ then

$$\left| \frac{a_1 + a_2 + \dots + a_n - A_1 - A_2 - A_3 \dots - A_n}{n} \right|$$

where $A_1 = A_2 = \dots = A_n$

then by the triangle inequality we have

$$\leq |a_1 - A_1| + |a_2 - A_2| + |a_3 - A_3| + \dots + |a_n - A_n|$$

$$\leq \left(\frac{\varepsilon}{n}\right) n = \varepsilon$$

We have shown that if $\varepsilon > 0$ $\therefore$ there is an $N \in N_1$ such that for all $n \geq N_1$ then $\left| \frac{a_1 + a_2 + \dots + a_n}{n} - A \right| < \varepsilon$. Since ε was arbitrary it implies $\{\sigma_n\}_{n=1}^{\infty}$ converges to A.

Here, you have to first suppose that $\exists N \in \mathbb{N}$ s.t. $n \geq N$ $|a_n - A| < \varepsilon$.

I didn't finish

Then, you split the sum from $k=1$ to $k=N-1$ & from $k=N$ to $k=n$.

At this step, you have to treat each part individually.

$$|\sigma_n - A| = \left| \frac{\sum_{k=1}^{n} a_k}{n} - A \right| = \left| \frac{\sum_{k=1}^{n} a_k - nA}{n} \right|$$

check the solution

$$\leq \frac{\sum_{k=1}^{n} |a_k - A|}{n} = \frac{\sum_{k=1}^{N} |a_k - A| + \sum_{k=N}^{n} |a_k - A|}{n}$$

6. Exercise 6. (10 pts) Use the definition of convergence to prove that each of the following sequences converges.

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a) $(a_n)_{n=1}^{\infty}$ given by $a_n = 5 + 1/n$ for $n \geq 1$.

Proof: By the definition of convergence we will take $A = 5$ and show that $(5 + 1/n)_{n=1}^{\infty}$ converges to 5. Let $\varepsilon > 0$ then there exists an N such that for all $n \geq N$ we have $|5 + \frac{1}{n} - 5| = |\frac{1}{n}| = \frac{1}{n} < \varepsilon$. By the Archimedean property $[(x, y \in \mathbb{R}, x > 0\ \exists n \in \mathbb{N}$ s.t $(nx > y)]$ we will take $n = N_0$, $x = \varepsilon$, and $y = 1$ then $N_0 \varepsilon > 1$. Here we take $N = N_0$, so if $n \geq N_0$ we have $n\varepsilon \geq N_0 \varepsilon$ (by axiom O4) and by transitivity (by axiom O2) we have $n\varepsilon \geq N_0 \varepsilon > 1$ or $n\varepsilon > 1$ for all $n \geq N_0$. Then this implies the following

$$|5 + \tfrac{1}{n} - 5| = |\tfrac{1}{n}| = \tfrac{1}{n} < \varepsilon \quad \text{for all } n \geq N_0.$$

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We have just shown that if $\varepsilon > 0$, there exist an $N = N_0$ such that, for all $n \geq N_0$ then $|5 + \frac{1}{n} - 5| = \frac{1}{n} < \varepsilon$, since $\varepsilon > 0$ was arbitrary $(5 + \frac{1}{n})_{n=1}^{\infty}$ converges to 5. ∎

b) $(a_n)_{n=1}^{\infty}$ given by $a_n = \frac{3n}{2n+1}$ for $n \geq 1$

Proof: By the definition of convergence we will take $A = \frac{3}{2}$ and show that $(\frac{3n}{2n+1})_{n=1}^{\infty}$ converges to $3/2$. Let $\varepsilon > 0$ then there exists an N such that for all $n \geq N$ we have $|\frac{3n}{2n+1} - \frac{3}{2}| = |\frac{-3}{4n+2}| = \frac{3}{4n+2} < \varepsilon$. By the Archimedean property we will take $n = N_0$, $x = \varepsilon$, and $y = \frac{3}{4}$, then $N_0 \varepsilon > \frac{3}{4}$. Now take $N = N_0$, so if $n \geq N_0$ then we have $n\varepsilon \geq N_0 \varepsilon > \frac{3}{4}$ (by axiom O4 & transitivity axiom O2). Now we have $n\varepsilon > 3/4$ for all $n \geq N_0$. This implies the following:

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↓ on back

$$\left|\frac{3n}{2n+1}-\frac{3}{2}\right| = \left|\frac{-3}{4n+2}\right| = \frac{3}{4n+2} < \frac{3}{4n} < \varepsilon \quad \text{for all } n \geq N_0$$

We have shown that for $\varepsilon > 0$ there exist an $N = N_0$ such that for all $n \geq N_0$ then $\left|\frac{3n}{2n+1}-\frac{3}{2}\right| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary $\left(\frac{3n}{2n+1}\right)_{n=1}^{\infty}$ converges to $3/2$. ∎

7. Exercise 7. (5pts) Prove that the sequence $(a_n)_{n=1}^{\infty} = \left(\frac{2n+1}{n}\right)_{n=1}^{\infty}$ is a Cauchy sequence.

Proof: By the definition of a Cauchy sequence for all $\varepsilon > 0$ there is a positive integer N such that if $m, n \geq N$, then $|a_n - a_m| = \left|\frac{2n+1}{n} - \frac{2m+1}{m}\right| < \varepsilon$. To show this let $\varepsilon > 0$. By the Archimedean property we can choose $N > \frac{2}{\varepsilon}$ $(n = N, x = 1, y = \frac{2}{\varepsilon})$. Then for all $m, n \geq N$ we have the following:

$$\left|\frac{2n+1}{n} - \frac{2m+1}{m}\right| = \left|\frac{m-n}{mn}\right| = \left|\frac{1}{n} - \frac{1}{m}\right|$$

$$\leq \left|\frac{1}{n}\right| + \left|\frac{1}{m}\right| \quad \text{(by the triangle inequality)}$$

$$= \frac{1}{n} + \frac{1}{m}$$

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but we know $m, n \geq N$ and $N > \frac{2}{\varepsilon}$ so by transitivity $m, n > \frac{2}{\varepsilon}$ so, $n > \frac{2}{\varepsilon}$ implies $\frac{1}{n} < \frac{\varepsilon}{2}$ and $m > \frac{2}{\varepsilon}$ implies $\frac{1}{m} < \frac{\varepsilon}{2}$ (by properties (iii) of real numbers), so now,

$$\left|\frac{2n+1}{n} - \frac{2m+1}{m}\right| = \left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{1}{n} + \frac{1}{m} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

We have shown that for $\varepsilon > 0$ there exist an $N > \frac{2}{\varepsilon}$ such that $\left|\frac{2n+1}{n} - \frac{2m+1}{m}\right| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary we have shown that $\left(\frac{2n+1}{n}\right)_{n=1}^{\infty}$ is a Cauchy sequence. ∎

8. Exercise 8. (10pts) Prove that each of the following sequences diverges.

a) $(a_n)_{n=1}^{\infty} = ((-1)^n)_{n=1}^{\infty}$

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Proof: Let's prove that $((-1)^n)_{n=1}^{\infty}$ diverges by showing a contradiction. Let's suppose it converges to A. Set $\varepsilon = 1$ then there must exist an $N \in \mathbb{N}$ such that for all $n \geq N$, then $|(-1)^n - A| < 1$. Note that

$(-1)^n = 1$ for all n that are even. Also, $(-1)^n = -1$ for all n that are odd. Define $(-1)^{2k}$ for $k \in \mathbb{N}$ for the case where $(-1)^n = 1$ and define $(-1)^{2k-1}$ for $k \in \mathbb{N}$ for the case where $(-1)^n = -1$. If $|(-1)^n - A| < 1$ and n is even then $|(-1)^n - A| = |(-1)^{2k} - A| = |1 - A| < 1$ or $-1 < 1 - A < 1$ (by Property of Absolute value) and if we add -1 to each side (axiom O1) then $-2 < -A < 0$. If $|(-1)^n - A| < 1$ and n is odd then $|(-1)^n - A| = |(-1)^{2k-1} - A| = |-1 - A| < 1$ or $-1 < -1 - A < 1$ by the property of absolute value. If you add 1 to the inequality (axiom O1) then $0 < -A < 2$. So we see that $-A \in (-2,0)$ and $-A \in (0,2)$ so $-A \in (-2,0) \cap (0,2)$ $= \emptyset$ which is a contradiction. Therefore, the assumption that $((-1)^n)_{n=1}^{\infty}$ converges is false. Thus, $((-1)^n)_{n=1}^{\infty}$ diverges.

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b) $(a_n)_{n=1}^{\infty} = (\sin(\frac{2n+1}{2}\pi))_{n=1}^{\infty}$ ✓

Proof: Let's prove that $(\sin(\frac{2n+1}{2}\pi))_{n=1}^{\infty}$ diverges by showing a contradiction. Suppose $(\sin(\frac{2n+1}{2}\pi))_{n=1}^{\infty}$ converges to A. Set $\varepsilon = 1$ then there must be an $N \in \mathbb{N}$ such that for all $n \geq N$ then $|\sin(\frac{2n+1}{2}\pi) - A| < 1$. Define $\sin(\frac{2n+1}{2}\pi)$ for n is even as $\sin(\frac{2(2k)+1}{2}\pi) = 1$ for $k \in \mathbb{N}$ and define $\sin(\frac{2n+1}{2}\pi)$ for n is odd as $\sin(\frac{2(2k-1)+1}{2}\pi)$ $= -1$ for $k \in \mathbb{N}$. In the case where n is even then $|\sin(\frac{2n+1}{2}\pi) - A| = |\sin(\frac{2(2k)+1}{2}\pi) - A| = |1 - A| < 1$. By property of absolute value $-1 < 1 - A < 1$. and adding -1 to the inequality (axiom O1) then $-2 < -A < 0$. In the case where n is odd then $|\sin(\frac{2n+1}{2}\pi) - A| = |\sin(\frac{2(2k-1)+1}{2}\pi) - A|$ $= |-1 - A| < 1$. By the property of absolute value $-1 < -1 - A < 1$ and by adding 1 to the inequality $0 < -A < 2$. So $-A$ must be an element of both sets or $-A \in (-2,0) \cap (0,2) = \emptyset$. This is a contradiction, since we showed that the assumption $(\sin(\frac{2n+1}{2}\pi))_{n=1}^{\infty}$ is false, it must be true. that $((\sin(\frac{2n+1}{2}\pi))_{n=1}^{\infty}$ diverges.

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9. Exercise 9 (5pts) Give an example of two sequences (a_n) and (b_n) such that (a_n) and (b_n) don't converge but (a_n+b_n) converge.

The example is $(a_n)_{n=1}^{\infty} = (n+1)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty} = (-n)_{n=1}^{\infty}$.

(i) Lets prove that $(n+1)_{n=1}^{\infty}$ diverges using a contradiction. Suppose $(n+1)_{n=1}^{\infty}$ converges to A. Then by 1.2 Theorem (pg 32) it must be bounded. Based on 1.2 Theorem a_n must be bounded from above $a_n \le M_1$ where $M \in \mathbb{R}$, so $n+1 \le M_1$ for all $n \in \mathbb{N}$. However, by the Archimedean principal, (let $x=1$, $n=N_1$, $y=M_1-1$) then $N_1 \ge M_1 - 1$ for $N_1 \in \mathbb{N}$ and (based on Axiom O1) then $N_1+1 \ge M_1$ which is a contradiction. Therefore $(n+1)_{n=1}^{\infty}$ must diverge. ✓

(ii) Lets prove $(b_n)_{n=1}^{\infty} = (-n)_{n=1}^{\infty}$ diverges using a contradiction. Suppose $(-n)_{n=1}^{\infty}$ converges to A. By 1.2 Theorem (pg 32) it must be bounded and say it has a lower bound such that for all n, $S_1 \le a_n$ where $S_1 \in \mathbb{R}$. So $S_1 \le -n$ or $S_1 + n < 0$ (based on axiom O1) for all $n \in \mathbb{N}$. However, based on the Archimedean principal (let $x=1$, $y=-S_1 \in \mathbb{R}$, $n=N_2$) then $-S_1 < N_2$ or $0 < S_1+N_2$ or $-N_2 < S_1$ (based on axiom O1). The existence of N_2 contradicts the assumption that $(-n)_{n=1}^{\infty}$ is bounded, therefore it is unbounded and diverges. ✓

(iii) in this example, $(a_n)_{n=1}^{\infty} = (n+1)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty} = (-n)_{n=1}^{\infty}$ diverges but $(a_n+b_n)_{n=1}^{\infty}$ converges. We will prove that. Suppose $a_n = n+1$, $b_n = -n$ then $(a_n+b_n)_{n=1}^{\infty} = (n+1-n)_{n=1}^{\infty} = (1)_{n=1}^{\infty}$ which is a constant sequence. Let $A=1$. For $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \ge N$ we have $|1-1| = 0 < \varepsilon$. Lets consider $N=1$ then if $n \ge 1$ we have $|a_n - A| = |1-1| = 0 < \varepsilon$ for all $n \ge 1$. We just proved that if $\varepsilon > 0$, there exists an $N=1$ such that for all $n \ge N$ $|a_n - A| < \varepsilon$. Since ε was

Excellent.

arbitrary we shown that $(a_n+b_n)_{n=1}^{\infty} = (n+1-n)_{n=1}^{\infty} = (1)_{n=1}^{\infty}$ converges.

10 Exercise 10 (10 pts) With the limit operations and the writing problems, find the limit of the following sequence with general term.

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We are asked to find the limits, not to prove that the sequence converges to A. Also using limit operations as on pg 46 of the book.

a) $\left(\frac{n^2+4n}{n^2-5}\right)_{n=1}^{\infty}$

We can modify the fraction by multiplying by $1/1 = {}^{1}/_{n^2} / {}^{1}/_{n^2}$

so $\left(\frac{n^2+4n}{n^2-5}\right)_{n=1}^{\infty} = \left(\frac{1/n^2}{1/n^2}\left(\frac{n^2+4n}{n^2-5}\right)\right)_{n=1}^{\infty} = \left(\frac{1+4/n}{1-5/n^2}\right)_{n=1}^{\infty}$

by theorem 1.11 states that we can look at the numerator and denominator. $(1+4/n)_{n=1}^{\infty}$ converges to 1 because we proved that $(1)_{n=1}^{\infty}$ converges to 1 and $(\frac{1}{n})_{n=1}^{\infty}$ converges to zero proven in the book on pg 34. So $(1+4/n)_{n=1}^{\infty}$ converges to $1+0=1$. The denominator $(1-5/n^2)_{n=1}^{\infty}$ can be broken up. $(1)_{n=1}^{\infty}$ converges to 1 and $(5/n^2)_{n=1}^{\infty}$ by 1.9 theorem $(\frac{1}{n})_{n=1}^{\infty}$ converges to zero so $\left(\frac{1}{n}\cdot\frac{1}{n}\right)_{n=1}^{\infty}$ converges to 0. Therefore $(1-5/n^2)_{n=1}^{\infty}$ converges to $1-0=1$ since $b_n = 1-5/n^2$ doesn't converge to zero or ever equal to zero, we can conclude the limit of the sequence based on 1.11 Theorem is $\left(\frac{n^2+4n}{n^2-5}\right)_{n=1}^{\infty}$ converges to $\frac{1}{1}=1$.

b) $\left(\frac{n}{n^2-3}\right)_{n=1}^{\infty}$

We can modify the fraction by multiplying by $1/1 = {}^{1}/_{n^2} / {}^{1}/_{n^2}$

so $\left(\frac{n}{n^2-3}\right)_{n=1}^{\infty} = \left(\frac{1/n^2}{1/n^2}\,\frac{n}{n^2-3}\right)_{n=1}^{\infty} = \left(\frac{1/n}{1-3/n^2}\right)_{n=1}^{\infty}$.

By theorem 1.11, we must look at the numerator and denominator separately. The numerator $(1/n)_{n=1}^{\infty}$ we already prove on pg 34 of the book and it converges to 0. The denominator $(1-3/n^2)_{n=1}^{\infty}$ converges to 1 since $(1)_{n=1}^{\infty}$ converges to 1 and $(1/n)_{n=1}^{\infty}$ converges to zero so by 1.9 theorem $(\frac{1}{n^2})_{n=1}^{\infty}$ converges

to 0. This means $(1-3/n^2)_{n=1}^{\infty}$ converges to $1-3\cdot 0=1$.
so since the denominator does not converge to zero with no steps $b_n = 0$, $b\in\{-2,1,6,\dots\}$ then by theorem 1.11, the limit of a fraction is the limit of numerator divided by denominator or $0/1 = 0$.

c) $\left(\frac{\cos n}{n}\right)_{n=1}^{\infty}$

Note $\cos n$ is a bounded sequence particularly $-1 \leq \cos n \leq 1$ meaning $|\cos n| \leq 1$

Based on the axiom O4 we can multiply both sides by $\frac{1}{n}$, so $\frac{|\cos n|}{n} < \frac{1}{n}$. Therefore $0 \leq \left(\frac{|\cos n|}{n}\right)_{n=1}^{\infty} \leq \left(\frac{1}{n}\right)_{n=1}^{\infty}$.
By exercise 3 $a_n \leq c_n \leq b_n$ if $a_n = 0$ converges to 0 and $b_n = (\frac{1}{n})$ converges to zero, therefore $c_n = \frac{|\cos n|}{n}$ converges to 0.

Alternatively, 1.13 theorem (pg 49) Break $\left(\frac{\cos n}{n}\right)_{n=1}^{\infty}$ into $\left(\frac{1}{n}\cdot \cos n\right)_{n=1}^{\infty}$ where $(a_n)_{n=1}^{\infty} = \left(\frac{1}{n}\right)_{n=1}^{\infty}$ converges to zero and $(b_n)_{n=1}^{\infty} = (\cos n)_{n=1}^{\infty}$ is bounded. Therefore $\left(\frac{1}{n}\cdot \cos n\right)_{n=1}^{\infty}$ converges to zero or $0\cdot \cos n = 0$.

d) $(\sqrt{4-\frac{1}{n}}-2)\,n$

Lets manipulate $(\sqrt{4-\frac{1}{n}}-2)\,n$ by multiplying by $1/1 = (\sqrt{4-1/n}+2)/(\sqrt{4-1/n}+2)$ then we have

$$n\left(\sqrt{4-1/n}-2\right)\left(\frac{\sqrt{4-1/n}+2}{\sqrt{4-1/n}+2}\right) = n\left(\frac{\cancel{4}-1/n-\cancel{4}}{\sqrt{4-1/n}+2}\right)$$

$$= \frac{-1}{\sqrt{4-1/n}+2}$$

so $\left[(\sqrt{4-1/n}-2)\,n\right]_{n=1}^{\infty} = \left(\frac{-1}{\sqrt{4-1/n}+2}\right)_{n=1}^{\infty}$. Now based on 1.11 Theorem lets look at numerator and denominator. The numerator $(-1)_{n=1}^{\infty}$ converges to -1. The denominator is $\sqrt{4-1/n}+2$ and at first use Exercise 4. $(4-1/n)_{n=1}^{\infty}$ converges to 4 as $(4)_{n=1}^{\infty}$ converges to 4 and $+(\frac{1}{n})_{n=1}^{\infty}$ converges to 0. Then $(\sqrt{4-1/n})_{n=1}^{\infty}$ must converge to $\sqrt{4}$ or 2.

so the denominator $(\sqrt{4-1/n}+2)_{n=1}^{\infty}$ can be broken into $(\sqrt{4-1/n})_{n=1}^{\infty}$ which converges to 2 and $(2)_{n=1}^{\infty}$ converges to 2 so therefore $(\sqrt{4-1/n}+2)_{n=1}^{\infty}$ converges to $(2+2)_{n=1}^{\infty}$ $= (4)_{n=1}^{\infty}$ which converge to 4. so by 1.11 Theorem the numerator converges to -1 and denominator to 4 so $B \neq 0$ and $|b_n|$ is bounded away from zero $b_n \in [\sqrt{8}+2, 2)$ then the limit of $((\sqrt{4-1/n}-2)n)_{n=1}^{\infty} = \left(\frac{-1}{\sqrt{4-1/n}+2}\right)_{n=1}^{\infty}$ converges to $-1/4$.