1 $\lim_{n\to\infty} c_n = \lim_{n\to\infty} (b_n - a_n)$

Using limit rules, we can write this out as

$\lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n$

And since $\lim_{n\to\infty} c_n$ is defined, both of those $\lim_{n\to\infty}$s exist.

We can then use limit sequence relationship to have $\lim_{n\to\infty} b_n = B$ has sequence related with $b_n \to B$ and $\lim_{n\to\infty} a_n = A$ has sequence related with $a_n \to A$ so both a_n and b_n converge.

$\lim_{n\to\infty} c_n = 0 \quad \lim_{n\to\infty} c_n = \lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n$

$0 = \lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$

We say x_n and y_n satisfy the condition.
Let z_n be a sequence with $z_{2n} = x_n$ and $z_{2n-1} = y_n$.
Two subsequences of z_n converge to the same point. These subsequence fill up the real numbers, so any other subsequence also converges. This mean z_n converges and all subsequences converge to x_0,
$z_n \to x_0$. $f(z_n)$ converges because the assumption is that it is cauchy because z_n satisfies all assumptions. Because x_n, y_n, z_n all converge to x_0, the sequences $f(x_n), f(y_n), f(z_n)$ would all converge to $f(x_0)$ using sequence rules. We can then say that f converges to $f(x_0)$ for all subsequences meaning $f \to f(x_0)$. We can then say that f has a limit at $f(x_0)$ so the $\lim_{x \to x_0} f = f(x_0)$ meaning f has a limit at x_0.

3 Say $\lim_{x \to x_0} f = \lim_{x \to x_1} f$ so $|f(x) - L| < \varepsilon$ with $|x - x_1| < \delta$
so $x_1 - \delta < x < x_1 + \delta$ meaning $x \in (x_1 - \delta, x_1 + \delta)$. This is the definition of accumulation point, so $x_1 = \text{acc} D = x_0$.
So $\lim_{x \to x_0} f$ with $x_0 = \text{acc} D$ must be unique because
$|f(x) - L| < \varepsilon$ $|x - x_0| < \delta \to x \in (x_0 - \delta, x_0 + \delta)$

4 If we take the lim of $f(x) \leq h(x)$, then $\lim_{x \to x_0} f(x) \leq \lim_{x \to x_0} h(x)$

And with $h(x) \leq g(x)$, we get $\lim_{x \to x_0} h(x) \leq \lim_{x \to x_0} g(x)$

And $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x)$ so $\lim_{x \to x_0} h(x) \leq \lim_{x \to x_0} f(x)$

$\lim_{x \to x_0} f(x) \leq \lim_{x \to x_0} h(x) \leq \lim_{x \to x_0} f(x)$ so by sandwich theorem, $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = \lim_{x \to x_0} g(x)$

If $f(x)$ and $g(x)$ have limits at x_0, there is a sequence $x_n \to x_0$ $x_0 \in D/\{x_0\}$ so $f(x_n) \to A$ and $g(x_n) \to A$

Since $f(x) \leq h(x) \leq g(x)$, then $f(x_n) \leq h(x_n) \leq g(x_n)$,

And if we take this to infinity, we get that $A \leq h(\infty) \leq A$, so sandwich Theorems says that $h(x_n) \to A$ too, and since this converges, there is an associated $\lim_{x \to x_0} h(x)$.

5 a) If $g:(0, \infty) \to \mathbb{R}$ is bounded, then $g(x) < M$ $\forall x \in (0, \infty)$. Then $\lim_{x \to \infty} f(x) = 0$ so with limit rules we can write $\lim_{x \to \infty} g(x) f(x)$ as $\lim_{x \to \infty} g(x) \cdot 0$

since $g(x) < M$ $\forall x \in (0, \infty)$, $\lim_{x \to \infty} g(x) < M$, so $\lim_{x \to \infty} g(x) \cdot 0 \leq M \cdot 0 = 0$ so $\lim_{x \to \infty} f(x) g(x) \leq 0$.

We can then do same thing with lower bound $-M$ to reach $\lim_{x \to \infty} f(x) g(x) \geq 0$, so by sandwich theorem $\lim_{x \to \infty} f(x) g(x) = 0$.

4 If we take the lim of $f(x) \leq h(x)$, then $\lim_{x \to x_0} f(x) \leq \lim_{x \to x_0} h(x)$

And with $h(x) \leq g(x)$, we get $\lim_{x \to x_0} h(x) \leq \lim_{x \to x_0} g(x)$

And $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x)$ so $\lim_{x \to x_0} h(x) \leq \lim_{x \to x_0} f(x)$

$\lim_{x \to x_0} f(x) \leq \lim_{x \to x_0} h(x) \leq \lim_{x \to x_0} f(x)$ so by sandwich theorem,

$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = \lim_{x \to x_0} g(x)$

If $f(x)$ and $g(x)$ have limits at x_0, there is a sequence $x_n \to x_0$ $x_n \in D/x_0$, so $f(x_n) \to A$ and $g(x_n) \to A$

Since $f(x) \leq h(x) \leq g(x)$, then $f(x_n) \leq h(x_n) \leq g(x_n)$,

And if we take this to infinity, we get that $A \leq h(\infty) \leq A$, so sandwich Theorems says that $h(x_n) \to A$ too, and since this converges, there is an associated $\lim_{x \to x_0} h(x)$.

5 a) If $g:(0,\infty) \to \mathbb{R}$ is bounded, then $g(x) < M$ $\forall x \in (0,\infty)$. Then, $\lim_{x \to \infty} f(x) = 0$ so with limit rules we can write $\lim_{x \to \infty} g(x) f(x)$ as $\lim_{x \to \infty} g(x) \cdot 0$

since $g(x) < M$ $\forall x \in (0,\infty)$, $\lim_{x \to \infty} g(x) < M$, so

$\lim_{x \to \infty} g(x) \cdot 0 \leq M \cdot 0 = 0$ so $\lim_{x \to \infty} f(x) g(x) \leq 0$.

We can then do same thing with lower bound $-M$ to reach $\lim_{x \to \infty} f(x) g(x) \geq 0$, so by sandwich theorem $\lim_{x \to \infty} f(x) g(x) = 0$.

5 b.) If $\lim_{x\to 0} g(x)$ exists, then $|g(0)-L|<\varepsilon$, $|x|<\delta$

So $g(0)$ approachs L, so plugging this into $f(x)$, we get $g(0)=f(1/0)=f(\infty)$. So as $x\to\infty$ $f(x)\to\frac{1}{L}$. So $g(x)=f(\frac{1}{x})$ so $|f(\infty)-\frac{1}{L}|<\varepsilon$

$|x|<\delta$ for g has a coorespording $|\frac{1}{x}|<\delta \to x>\frac{1}{\delta}$ for f.

We can use these to have $M=\frac{1}{\delta}$ where $\forall\varepsilon>0$, $\exists M=\frac{1}{\delta}$ that has $|f(\infty)-\frac{1}{L}|<\varepsilon$ and $x>M$.

If the $\lim_{x\to 0} g(x)=DNE$ then there is no δ to satisfy $M=\frac{1}{\delta}$.

Real Analysis HW #3

6.) a.) The sequence $\frac{1}{n}$ converges. We can rewrite this sequence as a bunch of subsequences as $\frac{1}{n}, \frac{1}{n+1}, \dots, \frac{1}{2n}$. All of these have n in the denominator, meaning all of them converge to 0. The sequence is then made up of sequences that converge, sequence addition says it converges to 0.

6.) $a_n = \frac{1+2+\dots+n}{n^2}$ $\quad n=1 \; a_n = \frac{1}{1^2} = \frac{1}{1} = 1 \quad n=2 \; a_n = \frac{1+2}{2^2} = \frac{3}{4}$

$a_{n+1} = \frac{1+2+\dots+n+n+1}{(n+1)^2}$

bounded? $n \to \infty$, $\frac{1+2+\dots n}{n^2} = \frac{n(\frac{1}{n}+\frac{2}{n}+\dots+1)}{n^2} = \frac{(\frac{1}{n}+\frac{2}{n}+\dots+1)}{n}$ $\quad n \to \infty \; a_n \to 0$

Bounded below by 0. Bounded above by $n=1 \; a_n = 1$

$a_n \geq a_{n+1}$ $\quad \frac{1+2+\dots+n}{n^2} \geq \frac{1+2+\dots+n+1}{(n+1)^2}$

$(1+2+\dots+n)(n+1)^2 \geq (1+2+\dots+n+1)n^2$

$(1+\dots+n)(n^2+2n+1) \geq (1\dots+n+1)n^2$

$(n^2+\dots+n^3)(2n+\dots+2n^2)(1+\dots+n) \geq (n^2+\dots+n^3+n^2)$

$(2n+\dots 2n^2)(1+\dots+n) \geq n^2$

have $2n^2$ so $-n^2$

$(2n+\dots) \geq 0$ $\quad n$ is positive so decreasing

Decreasing and bounded means (a_n) converges.

7.) What is limit? Has $\frac{0}{0}$, $N' = \frac{1}{2\sqrt{1+x}}$ $D' = 1$

$$\lim_{x \to 0} \frac{1}{2\sqrt{1+x}} = \frac{1}{2} \qquad \frac{\sqrt{1+x} - 1}{x} = \frac{1+x-1}{x(\sqrt{1+x}+1)} = \frac{1}{\sqrt{1+x}+1} \quad \lim_{x \to 0} = \frac{1}{2}$$

Let $\varepsilon > 0$, $\exists \delta$ s.t. $|f(x) - L| < \varepsilon$ $|x - x_0| < \delta$

$$\left| \frac{\sqrt{1+x} - 1}{x} - \frac{1}{2} \right| < \varepsilon \to \left| \frac{1}{\sqrt{1+x}+1} - \frac{1}{2} \right| < \varepsilon$$

$$\left(\frac{1}{2} - \varepsilon < \frac{1}{\sqrt{1+x}+1} < \frac{1}{2} + \varepsilon \right)^{-1} \to \frac{1}{\frac{1}{2} - \varepsilon} \geq \sqrt{1+x} + 1 \geq \frac{1}{\frac{1}{2} + \varepsilon}$$

$$\frac{1}{\frac{1}{2} - \varepsilon} - 1 \geq \sqrt{1+x} \geq -1 + \frac{1}{\frac{1}{2} + \varepsilon}$$

$$1 - \frac{2}{\frac{1}{2} - \varepsilon} + \left(\frac{1}{\frac{1}{2} - \varepsilon} \right)^2 \geq 1 + x \geq 1 - \frac{2}{\frac{1}{2} + \varepsilon} + \left(\frac{1}{\frac{1}{2} + \varepsilon} \right)^2$$

$$-\frac{2}{\frac{1}{2} - \varepsilon} + \left(\frac{1}{\frac{1}{2} - \varepsilon} \right)^2 \geq x \geq \frac{-2}{\frac{1}{2} + \varepsilon} + \left(\frac{1}{\frac{1}{2} + \varepsilon} \right)^2 \to \frac{-2}{\frac{1}{2} - \varepsilon} + \frac{1}{\frac{1}{4} - \varepsilon + \varepsilon^2} > x > \frac{-2}{\frac{1}{2} + \varepsilon} + \frac{1}{\frac{1}{4} + \varepsilon + \varepsilon^2}$$

$$\frac{2\varepsilon}{\frac{1}{4} - \varepsilon + \varepsilon^2} > x > \frac{-2\varepsilon}{\frac{1}{4} + \varepsilon + \varepsilon^2}$$

Let $\delta = \max\left(\left| \frac{2\varepsilon}{\frac{1}{4} - \varepsilon + \varepsilon^2} \right|, \left| \frac{-2\varepsilon}{\frac{1}{4} + \varepsilon + \varepsilon^2} \right| \right)$ then $|x - 0| < \delta$

$\varepsilon > 0$ was arbitrary, so $\lim_{x \to 0} \frac{\sqrt{1+x} - 1}{x} = \frac{1}{2}$

8.) $\lim_{x\to 1} f(x) = 1$ $\quad x_0 = 1$

$\lim_{x\to 1} \left(\frac{f(x)(1-f(x)^2)}{1-f(x)} \right)$ $\quad \frac{f(x)(1-f(x)^2)}{1-f(x)} \cdot \frac{1+f(x)}{1+f(x)}$

$\frac{f(x)(1-\cancel{f(x)^2})(1+f(x))}{(1-\cancel{f(x)^2})} = f(x)(1+f(x)) = f(x) + f(x)^2$

$\lim_{x\to 1} f(x)^2 = (\lim_{x\to 1} f(x))^2 = (1)^2 = 1$

$\lim_{x\to 1} f(x) + f(x)^2 = 1 + 1^2 = 2.$

9 $\varepsilon > 0$ $\delta > 0$ $|f(x) - f(x_0)| < \varepsilon$ $|x - x_0| < \delta$

So if $\varepsilon > 0$ and $\delta > 0$, we want to show that

$||f|(x) - |f|(x_0)| < \varepsilon$ & $|x - x_0| < \delta$

$|f|(x) = |f(x)|$ $\quad |f|(x_0) = |f(x_0)|$

$||f(x)| - |f(x_0)|| \le |f(x) - f(x_0)| < \varepsilon$

$||f|(x) - |f|(x_0)| < \varepsilon$ and $|x - x_0| < \delta$

ε was arbitrary, so $\lim_{x\to x_0} |f|(x)$ exists.

10 a.) Let (x_n) be a sequence such that $x_k \to x_0$, $(x_k) \neq x_0$, $x_k \in \mathbb{R}$

We then have $f(x_k) \to x_0$. With sequence rules, we have that $f(x_k)^n \to x_0^n$. Then $f(x_k)^n \to x_0^n$ is related to the $\lim_{x\to x_0} x^n \to x_0^n$.