MATH-331 Introduction to Real Analysis — KaiWei Tang
Homework 07 — Fall 2021

Due date: December, 6^{th} 1:20pm — Total: /65.

Exercise	1 (10)	2 (5)	3 (10)	4 (5)	5 (5)	6 (10)	7 (5)	8 (5)	9 (5)	10 (5)
Score										

Table 1: Scores for each exercises

Instructions: You must answer all the questions below and send your solution by email (to `parisepo@hawaii.edu`). If you decide to not use LaTeX to hand out your solutions, please be sure that after you scan your copy, it is clear and readable. Make sure that you attached a copy of the homework assignment to your homework.
If you choose to use LaTeX, you can use the template available on the course website.
No late homework will be accepted. No format other than PDF will be accepted. Name your file as indicated in the syllabus.

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Writing problems

For each of the following problems, you will be asked to write a clear and detailed proof. All the exercises below can be solve without using the definition with partitions. Try to go back to homework 6 and use some of the exercises there to solve the following problems.

You will have the chance to rewrite your solution in your semester project after receiving feedback from me.

Exercise 1. (10 pts) Prove that a step function is Riemann integrable on $[a, b]$. Follow the steps below.

a) Let I be a subinterval of $[a, b]$ and put $\phi = c\chi_I$. Prove that ϕ is Riemann integrable and that $\int_a^b \phi = c\ell(I)$. [There are three cases to consider: $I = [u, v]$, $I = (u, v]$, and $I = \{u\} = [u, u]$.]

b) Prove by induction that if f_1, f_2, …, f_n are Riemann integrable functions on $[a, b]$, then $f_1 + f_2 + \cdots + f_n$ is Riemann integrable and

$$\int_a^b (f_1 + f_2 + \cdots + f_n) = \int_a^b f_1 + \int_a^b f_2 + \cdots + \int_a^b f_n.$$

c) Write $\phi = \sum_{k=1}^{n} c_k \chi_{I_k}$. Use the second part of this exercise to show that ϕ is Riemann integrable.

Solution: a) Case 1: First, let us turn our c_{χ_I} into a function of $c_{\chi_I}(x)$ for,

$$f(x) = \begin{cases} 0 & , a \leq x \leq u \\ c & , u \leq x \leq v \\ 0 & , v \leq x \leq b \end{cases}$$

By HW6 5, we know that if $\exists c \in [a, b]$ st. if $[a, c]$ is Ri and $[c, b]$ is RI., then $[a, b]$ is RI. From our step function, ler us denote our $c = [u, v]$, then we see that $x = 0$ for $x \in [a, u]$ and since 0 is a constant, we know that any constant is RI. We see the same thing that $x = 0$ $\forall x \in [v, b]$. Thus, we see that its is also RI on the interval $[v, b]$. Then, by our step function we see that $\forall x \in [u, v]$, $\int_u^v x dx = x(u - v) = c$, which we see it is RI. Then by HW6 5, we see that since $[a, c]$ is Ri and $[c, b]$ is RI, then $[a, b]$ is RI, proving case 1.
Case 2: For this problem, we can also use HW6 5.

$$f(x) = \begin{cases} 0 & , a \leq x < u \\ c & , u < x \leq v \\ 0 & , v \leq x \leq b \end{cases}$$

Thus, our goal is to let v be arbitrarily close to a st. since we know that $[v, b]$ is RI and if we can put v close enough to a, the space between $[a, v]$ can be negligible and we can assume that $[a, v]$ is RI. We know that $[v, b]$ is RI since $x = 0$ for all x in $[v, b]$ and since 0 is a constant and any constant is RI then we know that $[v, b]$ is RI. Then the interval $[a, v]$ will be RI since the integral of a to v will just be $0 + c = c$ and since c is a constant, then c is RI then $[a, v]$ is RI, then by HW5 6, $[a, b]$ is RI.
Case 3: For this case, we see that u is just a point on the function f, meaning u is a constant. B definition, we know that all constants are RI. Thus, f is RI on $[a, b]$
b) From statement, we know that $f_1, f_2..., f_n$ are RI. Thus we first proof base case. By definition, we know that if f is RI, then let us denote a tp $\mathcal{P}$ st. if f_1 and f_2 is RI then $\exists \delta_1, \delta_2$ st. $||\mathcal{P}|| < \delta_1$, then $|\mathcal{S}(f_1, \mathcal{P}) - \int_a^b f_1| < \epsilon$ and $|\mathcal{S}(f_2, \mathcal{P}) - \int_a^b f_2| < \epsilon$. Let $\delta = min\{\delta_1, \delta_2\}$ and if $||\mathcal{P}|| < \delta$, then $|\mathcal{S}(f_1 + f_2, \mathcal{P}) - \int_a^b f_1 + f_2| = |\mathcal{S}(f_1, \mathcal{P}) + \mathcal{S}(f_2, \mathcal{P}) - \int_a^b f_1 - \int_a^b f_2|$. Then by triangle inequality, we see that $|\mathcal{S}(f_1, \mathcal{P}) - \int_a^b f_1| + |\mathcal{S}(f_2, \mathcal{P}) - \int_a^b f_2| = \epsilon + \epsilon$ and we let $\epsilon = 2\epsilon$ and we proved our base case is true.
Now let us prove our $n + 1$ cases. Suppose $\exists \delta_1, \delta_{n+1}$ st. $||\mathcal{P}|| < \delta_1$, then $|\mathcal{S}(f_1, \mathcal{P}) - \int_a^b f_1| < \epsilon$ and $|\mathcal{S}(f_{n+1}, \mathcal{P}) - \int_a^b f_{n+1}| < \epsilon$. Let $\delta = min\{\delta_1, \delta_2 ..., \delta_{n+1}\}$ and if $||\mathcal{P}|| < \delta$, then $|\mathcal{S}(f_1 + f_2, ..., f_{n+1}, \mathcal{P}) - \int_a^b f_1 + f_2 + ... + f_{n+1}| = |\mathcal{S}(f_1, \mathcal{P}) + \mathcal{S}(f_2, \mathcal{P}) + ... + \mathcal{S}(f_{n+1}, \mathcal{P}) - \int_a^b f_1 - \int_a^b f_2| - ... - \int_a^b f_{n+1}$. Then by triangle inequality, we see that $|\mathcal{S}(f_1, \mathcal{P}) - \int_a^b f_1| + |\mathcal{S}(f_2, \mathcal{P}) - \int_a^b f_2| + ... + |\mathcal{S}(f_{n+1}, \mathcal{P}) - \int_a^b f_{n+1}| = n\epsilon$ and we let $n\epsilon = \epsilon$ thus it is true for all $n + 1$ cases as well.
Then we see that if we apply the definition of summation to integrals we will se thatt $\int_a^b f = \sum_{n=1}^N f(c_i)(x_i - x_{i-1})$. Thus we can rewrite our above integral as,

$$\sum_{n=1}^N [f_1(x) + f_2(x) + ...f_n(x)](x_i - x_{i-1}) = \sum_{n=1}^N f_1(x)(x_i - x_{i-1}) + ... + \sum_{n=1}^N f_n(x)(x_i - x_{i-1})$$

Which we see is the LHS of the equal sign my definition of integrals in terms of sums, thus proving they are equal.
c)By definition of step function, we know that $\ell(I_k)$ is the length of I_k. Thus, by using the same

logic as b, we see that $\phi = \sum_{k=1}^{n} C_k \chi I_k$ where $\ell(I_k)$ is just $(x_i - x_{i-1})$, thus we can rewrite as $\phi = \sum_{k=1}^{n} C_k(x_i - x_{i-1})$ and since we know that C_k is just a contains and $(x_i - x_{i-1})$ will also end up as a constant, then this sum is RI for all constants are RI. $\square$

Exercise 2. (5 pts) Suppose that f is Riemann intolerable on $[a, b]$ and that f is non-negative (means that $f(x) \geq 0$ for $x \in [a, b]$). Let $u, v \in \mathbb{R}$. Show that if $a \leq u < v \leq b$, then

$$\int_u^v f \leq \int_a^b f.$$

[Hint: Use the following property of the Riemann Integral multiple times: $\int_a^b f = \int_a^c f + \int_c^b f$.]

Solution: From statement, we know that $f(x) \geq 0$, this mean that $\int_a^b f \geq 0$. From hint, we can break apart our integral by the properties of integral and rewrite as $\int_a^b f = \int_a^u f + \int_u^v f + \int_v^b f$. Then we know that since by assumption that $\int_a^b f \geq 0$, then $\int_a^u f \geq 0$, $\int_u^v f \geq 0$, and $\int_v^b f \geq 0$. From here, we know that since all 3 integrals are greater than or equal to 0, then we know that,

$$\int_a^u f + \int_v^b f \geq 0$$

From here, we can add $\int_u^v f$ tp both sides and obtain,

$$\int_a^u f + \int_v^b f + \int_u^v f \geq \int_u^v f$$

and by properties of integrals, we see that the LHS of the inequality is just $\int_a^b f$, thus $\int_a^b f \geq \int_u^v f$, proving our statement. $\square$

Exercise 3. (10 pts) Use the Fundamental Theorem of Calculus to solve the following problems:

a) Suppose that f is continuous on $[a, b]$ and that f is nonnegative on $[a, b]$. Show that if $\int_a^b f = 0$, then $f(x) = 0$ for any $x \in [a, b]$.

b) Suppose that f and g are continuous on $[a, b]$ such that $\int_a^b f = \int_a^b g$. Show that there exists a point $c \in (a, b)$ such that $f(c) = g(c)$.

Solution: a) By FTC, we know that $F(x) = \int_a^b f$ and $F'(x) = f(x)$. Thus, let $F(x) = \int_a^b f$ for $x \in [a, b]$. By assumption, $F(a) = F(b) = 0$. From here, denote a $y \in [a, b]$ st. $x < y$, then we know the interval $[a, x] \subset [a, y]$. Then we see that $\int_a^x f \leq \int_a^y f$ for f is increasing. Then we know that since $F(a) \leq F(x) \leq F(b)$, then $0 \leq F(x) \leq 0$ for $F(x) = 0$. Then apply FTC and we see that since $F'(x) = f(x)$, and $F(x) = 0$, then $F'(x) = 0 = f(x)$ for any $x \in [a, b]$.
b) We know that by FTC that $F(x) = \int_a^b f$. Then by Rolles theorem, there exists an $F(a) = 0$ and $F(b) = \int_a^b f - g = \int_a^b f - \int_a^b g = 0$. Then there $\exists c \in (a, b)$ st. $F'(c) = f(c) - g(c) = 0$. From here, we see that since $f(c) - g(c) = 0$, $f(c) = g(c)$, proving that there exists an $c \in (a, b)$ st. $g(c) = f(c)$. $\square$

Exercise 4. (5 pts) Let f be a continuous function on $[a, b]$. Prove that there exists a number $c \in [a, b]$ such that $f(c)(b - a) = \int_a^b f$.

Solution: Since f is cont. then it is bounded. From here let us denote a $M = sup(f(x))$ and $m = inf(f(x))$ for $m \leq f(x) \leq M$. From here take integral of the entire inequality and we will see,

$$\begin{aligned} m \leq& f(x) \leq M \\ \int_a^b m \leq& \int_a^b f(x) \leq \int_a^b M \\ m(b-a) \leq& \int_a^b f(x) \leq M(b-a) \\ m \leq& \frac{1}{b-a} \int_a^b f(x) \leq M \end{aligned}$$

From here, apply IVT and we see that let $L = \frac{1}{b-a} \int_a^b f(x)$ for $f(c) = \frac{1}{b-a} \int_a^b f(x)$. Then multiply $b - a$ to $f(c)$ and obtain $f(c)(b - a) = \int_a^b f$, thus proving our statement. □

Exercise 5. (5 pts) Suppose that f is Riemann integrable on $[a, b]$ and is strictly increasing there. Prove that there exists a point $c \in (a, b)$ such that

$$\int_a^b f = f(a)(c - a) + f(b)(b - c).$$

[Hint: Define the function $g(x) = f(a)(x-a)+f(b)(b-x)$. Show that $\int_a^b f$ is between the numbers $f(a)(b - a)$ and $f(b)(b - a)$ and use the Intermediate Value Theorem.]

Solution: Apply hint and define a function $g(x) = f(a)(x - a) + f(b)(b - x)$, we see that since since it is stritly increasing, it is also continuous. Then we also see that $g(a) = f(b)(b - a)$ and $g(b) = (f(a)(b - a)$, then since f is strictyly increasing, we know that $\int_a^b f$ must be between $g(a)$ and $g(b)$. Thus we can apply IVT then there $c \in (a, b)$ st. $g(c) = \int_a^b f$. Then by defintion, we would have $\int_a^b f = \int_a^c f + \int_c^b f = f(a)(c - a) + f(b)(b - c)$. □

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Homework problems

Answer all the questions below. Make sure to show your work.

Exercise 6. (10pts)

a) Show that the function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & , x \in \mathbb{Q} \\ 0 & , x \notin \mathbb{Q} \end{cases}$$

is not Riemann integrable on $[0, 1]$. [Hint: Use exercise 4 from Homework 6.]

b) Define the two functions $g : [0,1] \to \mathbb{R}$ and $h : [0,1] \to \mathbb{R}$ by $g = \chi_{(0,1]}$ and

$$h(x) = \begin{cases} 0 & , x \notin \mathbb{Q} \\ \frac{1}{q} & , x = p/q \in \mathbb{Q}. \end{cases}$$

Use the first part to show that $g \circ h$ is not Riemann integrable on $[0,1]$. What can you say about the composition of two Riemann integrable functions in light of this last examples?

Solution: a) Let us prove this statement by contradiction. Suppose that f is RI., then for any sequence of tp. $(\mathcal{P})_{n=1}^{\infty}$ st. $\lim_{n\to 0} ||\mathcal{P}|| = 0$, then $\lim_{n\to 0} \mathcal{S}(f, \mathcal{P}_n) = \int_a^b f$. From here, we know that by the density of rational numbers that there exists a irrational number between any rational number. Thus, let us denote $\mathcal{P}_n$ as the sequence of $(\mathcal{P})_{n=1}^{\infty} \to x$ for $x \in \mathbb{Q}$ and a T_n as the sequence of $(T_n)_{n=1}^{\infty} \to x$ for $x \notin \mathbb{Q}$. Then, by our assumption, $\lim_{n\to 0} ||\mathcal{P}|| = 0$ and $\lim_{n\to 0} ||T_n|| = 0$ for $\mathcal{S}(f, \mathcal{P}_n) = \int_a^b f$ and $\mathcal{S}(f, T_n) = \int_a^b f$. But we know that every integral has is unique and by definition of out function we see that our $(\mathcal{P})_{n=1}^{\infty} \to 1$ and $(T_n)_{n=1}^{\infty} \to 0$, meaning, $\lim_{n\to 0} \mathcal{S}(f, \mathcal{P}_n) \neq \lim_{n\to 0} \mathcal{S}(f, T_n)$ which cannot happen since every integral is unqiue. Thus, we see that our assumption fails and our f is not RI.
b) Let us again do a proof my contradiction. Suppose that $g \circ h$ is RI. and since $h \to 0$ for $x \in \mathbb{Q}$ and $h \to \frac{1}{q}$ for $x \notin \mathbb{Q}$, then that means $g(h(x) \to x_1$ for $x \in \mathbb{Q}$ and $g(h(x)) \to x_2$ for $x \notin \mathbb{Q}$. Since we have two different outputs for x, the composition function will also have two outputs. Then define $(\mathcal{P})_{n=1}^{\infty} \to x$ for $x \in \mathbb{Q}$ and a T_n as the sequence of $(T_n)_{n=1}^{\infty} \to x$ for $x \notin \mathbb{Q}$. Then by assumption, $\lim_{n\to 0} ||\mathcal{P}|| = 0$ and $\lim_{n\to 0} ||T_n|| = 0$ for $\mathcal{S}(f, \mathcal{P}_n) = \int_a^b f$ and $\mathcal{S}(f, T_n) = \int_a^b f$. Then we see that since $(\mathcal{P})_{n=1}^{\infty} \to 0$ and $(T_n)_{n=1}^{\infty} \to \frac{1}{q}$, then $x_1 \neq x_2$ for our limit is not unique thus our f is not RI. $\square$

Exercise 7. (5 pts) Show that if f is continuous on $[a,b]$, then $|f|$ is Riemann integrable on $[a,b]$ and

$$\left| \int_a^b f \right| \leq \int_a^b |f|.$$

[Hint: There is a clever way to show that $|f|$ is Riemann integrable without using the definition with the partitions.]

Solution: By theorem in class, we know that if f is cont. then $|f|$ is also cont., then $|f|$ is RI. By using triangle inequality, we know that $|x+y| \geq |x+y|$. Thus we apply this property to our problem and we see that

$$\int_a^b |f| = \lim_{||\mathcal{P}||\to 0} \sum_{n=1}^{N} |f(c_i)|(x_i = x_{i-1}) = [|f(c_1)| + |f(c_2)| + ...|f(c_n)|](x_i - x_{i-1})$$

$$\left| \int_a^b f \right| = \left| \lim_{||\mathcal{P}||\to 0} \sum_{n=1}^{N} f(c_i)(x_i = x_{i-1}) \right| = |f(c_1) + f(c_2) + ...f(c_n)| \, (x_i - x_{i-1})$$

From here, we know that $|f(c_1)| + |f(c_2)| + ...|f(c_n)| \geq |f(c_1) + f(c_2) + ...f(c_n)|$ by our property above, and we also know that if $\exists x, y, z \in \mathbb{R}$ st. $x > y$ then $zx > zy$. Then we can let $(x_i - x_{i-1}) = z$ and we obtain $|f(c_1)| + |f(c_2)| + ...|f(c_n)|(x_i - x_{i-1}) \geq (x_i - x_{i-1})|f(c_1) + f(c_2) + ...f(c_n)|$, which we see it is true, therefore proving our statement. $\square$

Exercise 8. (5 pts) Find $f'(x)$ if $f(x) = \displaystyle\int_{\sqrt{x}}^{\sqrt[3]{x}} \frac{1}{1+t^3}\, dt$ where $x \in [0,1]$.

Solution: Split our integral up as $\int_0^{\sqrt{x}} \frac{1}{1+t^3} dt$ and $\int_0^{\sqrt[3]{x}} \frac{1}{1+t^3} dt$. Define a $\phi(x) = \sqrt{x}$, then

$$F(x) = \int_0^{\phi(x)} \frac{1}{1+t^3} dt$$

Define a $G(x) = \int_0^{\sqrt{x}} \frac{1}{1+t^3} dt$, then $G(\phi(x)) = F(x)$. From here, apply chain rule and we obtain $F'(x) = \frac{1}{2\sqrt{x}+2\sqrt[3]{x^2}} = -\frac{1}{2\sqrt{x}+2\sqrt[3]{x^2}}$ since we want our integral to be from $\sqrt{x}$ to 0and by property of integrals, $\int_0^{\sqrt{x}} \frac{1}{1+t^3} dt = -\int_{\sqrt{x}}^{0} \frac{1}{1+t^3} dt$. We do that same for the other integral and define a $\varphi(x) = \sqrt[3]{x}$. Define $F(x) = \int_0^{\varphi(x)} \frac{1}{1+t^3} dt$ and $H(x) = \int_0^{\sqrt[3]{x}} \frac{1}{1+t^3} dt$ for $H(\varphi(x)) = F(x)$, then by chain rule,$F'(x) = \frac{1}{3\sqrt[3]{x^2}+3\sqrt[3]{x^5}}$. Then our final answer will be the sum of these two answers of $F'(x) = \frac{1}{3\sqrt[3]{x^2}+3\sqrt[3]{x^5}} - \frac{1}{2\sqrt{x}+2\sqrt[3]{x^2}}$. □

Exercise 9. (5 pts) Find a function $f : [1, \infty) \to \mathbb{R}$ such that $f(1) = 0$ and $f'(x) = 1 + \sin(x^2)$ for all $x > 1$.

Solution: By FTC, we know that $F'(x) = f(x)$. Thus we see that our $f'(x) = 1 + sin(x^2)$, then by definition of continuity, we know that 1 is continuous and $sin(x^2)$ is continuous for all sine functions are continuous by lecture notes. Then there has to exists a $f(x)$ by FTC since 1 is a constant, we know it's RI. and we also know that $sin(x^2)$ is also RI for it is a composition of functions. We let $f(x) = sin$ and we know sine is RI. and $g(x) = x^2$ is RI by previous hw, then $f(g(x))$ is RI. From here, the sum of two RI functins are also RI. Then by FTC, $f(x) = \int_a^b F'(x) = \int_a^b 1 + sin(x^2)$, thus $f(x)$ exists, by FTC. □

Exercise 10. (5 pts) By thinking the following sum as a Riemann sum, evaluate

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{k^2+n^2}.$$

Solution:

$$\frac{n}{k^2+n^2}$$
$$\frac{n}{\frac{k^2}{n^2}+1} \cdot \frac{1}{n^2}$$
$$\frac{1}{\frac{k^2}{n^2}+1} \cdot \frac{1}{n}$$

From here, let $\Delta x = \frac{1}{n}$ and let $x = \frac{k}{n}$ and we obtain $\frac{1}{x^2+1}$ and we see our upper and lower bound is 0 and 1. Thus take integral and obtain,

$$\begin{aligned} &= \int_0^1 \frac{1}{x^2+1} dx \\ &= arctan(1) - arctan(0) \\ &= \frac{\pi}{4} \end{aligned}$$