Math 331: Homework 5

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1) The problem states that $|f(x) - f(y)| \leq M|x - y|$. To prove uniform continuity of f, let $\varepsilon > 0$. Then there exists a $\delta > 0$ s.t. $\delta > |x - y|$ then $|f(x) - f(y)| < \varepsilon$.
Choose $\delta = \frac{\varepsilon}{M}$. Since M is a universal constant and does not depend on x and y, we can write δ in terms of M and ε.
Then we see that if $|x - y| < \delta = \frac{\varepsilon}{M}$, then $|f(x) - f(y)| < M \cdot \delta = \frac{\varepsilon}{M} \implies |f(x) - f(y)| < \varepsilon$.

2) The problem states that $\lim_{x\to\infty} f(x) = 0$, or, that $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $|f(x) - 0| < \varepsilon$.
Let $\varepsilon = 1$. Then by the extreme value theorem, $\exists M > 0$ s.t. $\forall x \in (M, \infty), f(x) < \varepsilon$. Since $f(x)$ is positive for these values, we drop the absolute value.
Then $\forall x \in (M, \infty), f(x) < 1$ so the function $f(x)$ is bounded.
We then deal with the closed interval case of $[0, M]$ since we understand the behavior at ∞. Since we now have a closed interval, there exists a supremum and infimum, but we must choose a ε s.t. the supremum is inside the interval. Or, that the sup $> \varepsilon$ for at least one value in $[0, M]$.
We know by EVT that $f(c) \geq f(M)$ and $\varepsilon \geq f((M, \infty))$. Then $f(c) \geq f(x), \forall x \in [0, M]$. So the maximum is attained. 3) We will deal with this in several cases. First, if f maps $a \to a$ or f maps $b \to b$, then we can let $c = a$ or $c = b$, and then $f(c) = c$, for $c \in [a, b]$.
If $f(a) \neq a$ and $f(b) \neq b$, then $f(a) > a$ and $f(b) > b$. Now, suppose we have $f(a) - a$. Since f is continuous and a is continuous, the function is continuous. Similarly, $f(b) - b$ is also continuous. Since $f(a) \neq a$ with $f(a) > a$, $f(a) - a > 0$. Further since $f(b) \neq b$ with $f(b) < b$, $f(b) - b < 0$.
Then, IVT shows that there must be a $c \in [a, b]$ s.t. $f(c) - c = 0$, and in conclusion, $f(c) = c$.

4) Since f is twice differentiable on (a,b) we know that for some number n in (a, b), $f'(n)$ exists. Define $g := f'(n)$, and since f maps from an interval, its range is an interval, so the domain of g is the interval $[c, d]$.
We know g is differentiable on $[c, d]$ because $[c, d] \in [a, b]$. So we have $c < d$ and $g(c) = g(d)$ (from problem statement). Then, by Rolle's thm. there must exist an $x \in (c, d)$ s.t. $g'(x) = 0$. Since $g = f'$, we know $g' = f''$. So $f''(x) = 0$ for some x.

5) a) We will prove that $\lim_{h\to 0} \frac{f(x_0+h)-f(x_0-h)}{2h} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$.

$$
\begin{aligned}
\lim_{h\to 0} \frac{f(x_0 + h) - f(x_0 - h)}{2h} &= \lim_{h\to 0} \frac{f(x + h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0 - h) + f(x_0) - f(x_0)}{2h} \\
&= \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0) - f(x_0 - h) + f(x_0)}{2h} \\
&= \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{2h} + \lim_{h\to 0} \frac{f(x_0) - f(x_0 - h)}{2h}
\end{aligned}
$$

Notice that both $\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$ and $\lim_{h\to 0} \frac{f(x_0)-f(x_0-h)}{h}$ are equal respectively to

$$\lim_{h\to 0} \frac{f(x + h) - f(x)}{h}$$

(with the latter just a negation of the former). Therefore

$$\lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h} + \lim_{h\to 0} \frac{f(x_0) - f(x_0 - h)}{h} = 2 \cdot \lim_{h\to 0} \frac{f(x + h) - f(x)}{h}$$

so then

$$\lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{2h} + \lim_{h\to 0} \frac{f(x_0) - f(x_0 - h)}{2h} = \lim_{h\to 0} \frac{f(x + h) - f(x)}{h}$$

and finally,

$$\lim_{h\to 0} \frac{f(x_0 + h) - f(x_0 - h)}{2h} = \lim_{h\to 0} \frac{f(x + h) - f(x)}{h}.$$

b) The absolute value function, $|x|$ is such a function.

6) a) Since x^r is a monomial, we know that it is continuous on $(0,\infty)$. If the derivative of x^r exists for every c in $(0,\infty)$, then it is differentiable on the interval. Or

$$\begin{aligned}
\lim_{h\to 0}\frac{(x+h)^r - x^r}{h} \\
\lim_{h\to 0}\frac{(x)^r + rhx^{r-1} + ... + h^r - x^r}{h} \\
\lim_{h\to 0} rx^{r-1} + ... + h^{r-1} \\
= rx^{r-1}
\end{aligned}$$

So the derivative exists.

b) We will prove this derivative exists with chain rule. Let $f(x) = \sqrt{(x)}$ and $g(x) = x^2 + \sin x + \cos x$. Then we have $(f(g(x)))$ is our function. If we can prove that the two separate functions are differentiable, then their composition is also differentiable.

So,

$$\begin{aligned}
\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h} \\
\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} \\
\lim_{h\to 0}\frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} \\
\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}
\end{aligned}$$

which exists at $h = 0$.

For the inside function, we know that since x^2 is a monomial, it is differentiable and continuous. From class we also proved that the derivatives of $\sin x$ and $\cos x$ exist. So the inside function is also differentiable.

7) Let set S be closed and there exist an x_0 in $\mathbb{R}\setminus S$. Then S will not contain this x_0 since x_0 is not a limit point of S. So, by the definition of an accumulation point, there is a neighborhood for x_0 which is disjoint from S. But if this is true then the elements of the neighborhood must be in $\mathbb{R}\setminus S$. By the definition of an open set, every element in an open set has a neighborhood within the set. So $\mathbb{R}\setminus S$ must be open.

Now assume that $\mathbb{R}\setminus S$ is an open set. Let x_0 be a limit point of S. By the definition of open set, there must then be a neighborhood of x_0 in $\mathbb{R}\setminus S$. But x_0 is a limit point of S so the set of the neighborhood intersected with $\mathbb{R}\setminus S$ cannot be empty. So S must be closed.

8) Since f is a differentiable function, we know the product $x^2f(x^3)$ is also differentiable. Therefore the derivative is

$$\begin{aligned}
(x^2f(x^3))' &= (x^2)'(f(x^3)) + (x^2)(f(x^3))' \\
&= 2x(f(x^3)) + (x^2)(3x^2(f(x^3))') \\
&= 2x(f(x^3)) + 3x^4(f(x^3)').
\end{aligned}$$

9) We know that the inverse of sine is arcsine, so define two functions: $f(x) = \sin x$ and $g(x) = \arcsin x$. We know already that f is differentiable from class.

Since g is the inverse of f, if f maps $D \to R$, then g maps $R \to D$. (Notation note, $R \neq \mathbb{R}$).

The derivative, then, of $g(x_0) = \frac{1}{f'(g(x_0))}$ for some $x_0 \in R$. Since the range of sine is $[-1,1]$, we know then that this is $x_0 \in [-1,1]$. Then, we know that $g'(x_0) = \frac{1}{f'(g(x_0))}$.

From class we know that $f'(x_0) = \cos(x_0)$. So we have $g'(x_0) = \frac{1}{\cos(\arcsin x)}$. We know from an inverse trigonometric identity that $\cos(\arcsin x) = \sqrt{1-x^2}$ so $g'(x_0) = \frac{1}{\sqrt{1-x^2}}$.

10) a) Suppose $n \in \mathbb{N}$ s.t. $n \geq 0$ and $0 \geq y \geq x$. Then

$$\begin{aligned} ny^{n-1}(x-y) \geq x^n - y^n \geq nx^{n-1}(x-y) \\ ny^{n-1} \geq \frac{x^n - y^n}{x-y} \geq nx^{n-1} \end{aligned}$$

Let $f(y) = y^n$ and $f(x) = x^n$. Then

$$f'(y) \leq \frac{f(x) - f(y)}{x-y} \leq f'(x)$$

If $y = x$, the inequalities give $0 \leq 0 \leq 0$ which is true, so we can suppose that $x > y$, which makes our division valid. So this is true.

b) Let $g(x) = \sqrt{1+x}$. Then by MVP, we have that $g'(x) = \frac{g(x)-g(0)}{x=0} = \frac{\sqrt{1+x}-1}{x}$.
We know that $g'(x) = \frac{1}{2}(x+1)^{-\frac{1}{2}} = \frac{1}{2\sqrt{1+x}}$. Since $\frac{1}{2\sqrt{1+x}} = \frac{\sqrt{1+x}-1}{x} \implies \frac{1}{\sqrt{1+x}} = 2\frac{\sqrt{1+x}-1}{x}$. We know also that $\frac{1}{\sqrt{1+x}} < 1$. Then

$$\begin{gathered} 2\frac{\sqrt{1+x}-1}{x} < 1 \\ \frac{\sqrt{1+x}-1}{x} < \frac{1}{2} \\ \sqrt{1+x} < \frac{1}{2}x. \end{gathered}$$

So $\sqrt{1+x} < \frac{1}{2}x + 1$.