MATH-331 Introduction to Real Analysis | KaiWei Tang
Homework 04 | Fall 2021

Due date: October 25th 1:20pm | Total: /70.

Exercise	1 (5)	2 (5)	3 (5)	4 (5)	5 (10)	6 (10)	7 (5)	8 (5)	9 (5)	10 (10)
Score										

Table 1: Scores for each exercises

Instructions: You must answer all the questions below and send your solution by email (to `parisepo@hawaii.edu`). If you decide to not use LaTeX to hand out your solutions, please be sure that after you scan your copy, it is clear and readable. Make sure that you attached a copy of the homework assignment to your homework.
If you choose to use LaTeX, you can use the template available on the course website.
No late homework will be accepted. No format other than PDF will be accepted. Name your file as indicated in the syllabus.

1 Writing problems

For each of the following problems, you will be asked to write a clear and detailed proof. You will have the chance to rewrite your solution in your semester project after receiving feedback from me.

Exercise 1. (5 pts) Prove that, if $0 < x < \pi/2$, then $0 \leq \sin x \leq x$ with a geometric argument. [Hint: View $\sin x$ as a point on the unit circle in the first quadrant.]

Solution: We see that x is bounded between 0 and $\frac{\pi}{2}$, and $\sin(x)$ is bounded between 0 and x. Thus $\sin(x)$ is bounded between $0 < \sin(x) \leq x < \frac{\pi}{2}$. We see that by elemetry geometry, the units bounded between 0 and $\frac{\pi}{2}$ are $\frac{\pi}{6}, \frac{\pi}{4}$, and $\frac{\pi}{3}$. From the bounds above, we know that $x \neq 0, \frac{\pi}{2}$, thus, we can check using the other 3 coordinates of $\frac{\pi}{6}, \frac{\pi}{4}$, and $\frac{\pi}{3}$. Let $x = \frac{\pi}{6}$, then our bounds will be $0 < \frac{1}{2} \leq \frac{\pi}{6} < \frac{\pi}{2}$, which we see it is true. We can check for both $\frac{\pi}{6}$ and $\frac{\pi}{4}$ and we obtain $0 < \frac{\sqrt{2}}{2} \leq \frac{\pi}{4} < \frac{\pi}{2}$ and $0 < \frac{\sqrt{3}}{2} \leq \frac{\pi}{3} < \frac{\pi}{2}$ respectivley. Which we see it is true for those coordinates as well. Therefore, we see that it is true for if $0 < x < \frac{\pi}{2}$, then $0 \leq \sin(x) \leq x$. $\square$

Exercise 2. (5 pts) Let $f : A \to \mathbb{R}$ and $g : B \to A$ be two functions where $A, B \subset \mathbb{R}$. Let a be an accumulation point of A and b be an accumulation point of B. Suppose that

- $\lim_{t\to b} g(t) = a$.
- there is a $\eta > 0$ such that for any $t \in B \cap (b - \eta, b + \eta)$, $g(t) \neq a$.

- f has a limit at a.

Prove that $f \circ g$ has a limit at b and $\lim_{x\to a} f(x) = \lim_{t\to b} f(g(t))$. [This is the change of variable rule for limits.]

Solution: From the hints and assumptions given, we know that $\lim_{t\to b} g(t) = a$ and that $\lim_{x\to a} f(x) = L$. By applying the change of variable rule on limits, let us denote $t = f(x)$ for $g(f(x))$. From here, we can understand this as $f \circ g$ for $g : B \to \mathbb{R}$. Thus we can apply the hint and see that our $t \in B \cap (b - \eta, b + \eta)$ st. $g(t) \neq a$. Since we let $t = f(x)$ we know now that $f(x) \in B$ for b is the accumulation point in B, thus if we were to take $\lim_{f(x)\to b} g(f(x)) = b$. From here, observe that $\lim_{x\to a} f(x) = L$, we also see that if we use the change of variable rule once more for $x = g(t)$, we will obtain $f(g(t))$ where we know that $\lim_{t\to b} g(t) = a$ and $f(x)$ is defined at a, thus, we can understand g and f as inverses of each other for g is the inverse of f st. $f(g(x)) = f(x)$ for $\lim_{x\to a} f(x) = \lim_{t\to b} f(g(x))$.

Exercise 3. (5 pts) Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and suppose that $f(x) = 0$ for each rational number x in $[a, b]$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

Solution: Since we know that f is continuous, then f is uniformly continuous on the interval $[a, b]$. Thus, by the definition of continuity, $\forall \epsilon > 0$, $\exists \delta > 0$ st. $\forall x, y \in D$, $|x-y| < \delta$ for $|f(x)-f(y)| < \epsilon$. Let us prove by contradiction. Suppose that there exists a $x_i, y_i \in [a, b]$ st. $f(x_i) \neq 0$ and $f(y_i) \neq 0$. This means that both x_i and y_i are non-rational numbers since by our assumption that if x_i and y_i are rational, then $f(x_i) = 0$ and $f(y_i) = 0$. Thus WLOG, let $y_i < x_i$ then $0 < |x_i - y_i| < \delta$. By the density of rational functions, we know that between two irrational numbers x_i and y_i, there exists a $h \in (x_i, y_i)$ st. h is rational. Since we know that x_i, y_i are irrational numbers we can create a sequence (x_n) st. (x_n) contains all rational numbers in the interval (x_i, y_i) . By the rational density property, we know that $\exists h \in (x_i, y_i)$ st. h is a rational number for $(x_n) = \{h_1, h_2, h_3..., h_n\}$. Thus by our assumption, we have a limit for $\lim_{x_n\to x} f(x_n) = 0$. Thus let us create a subsequence (x_m) st. (x_m) is any non rational number between h_1 and h_n. Therefore, we see that by theorem in class, we know that if the parent sequence tends towards a limit L, then the subsequence also tends towards the same L. Therefore since $\lim_{x_n\to x} f(x_n) = 0$, then $\lim_{x_m\to x} f(x_m) = 0$ for our $L = 0$. Thus since $(x_n), (x_m) \in (x_i, y_i)$ and $(x_i, y_i) \in [a, b]$, we can conclude that $f(x) = 0$ for $x \in [a, b]$. $\square$

Exercise 4. (5 pts) Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and suppose that $f(c) > 0$ for some $c \in [a, b]$. Prove that there exist a number η and an interval $[u, v] \subset [a, b]$ such that $f(x) \geq \eta$ for all $x \in [u, v]$.

Solution: We can prove this by the extreme value theorem given in class. The extreme value theorem states that there exists an f for $f : D \to \mathbb{R}$ and $[a, b] \subset D$. If f is continuous on $[a, b]$, then it is bounded and moreover, $\exists c \in [a, b)$ and $\exists d \in [a, b)$ st. sup$= \{f(x) : x \in [a, b]\} = f(c)$ and inf$= \{f(x) : x \in [a, b]\} = f(d)$. From the statement above, we know that f is continuous, thus, by AP, we know there exists an inf and sup in the interval $[a, b]$. Let us denote the inf as u and sup as v, with some $c \in [u, v]$ for $[u, v] \in [a, b]$ st. $c \in [a, b]$. Thus we have $f(u) = inf$ and $f(v) = sup$ for $f(u) \leq f(c) \leq f(v)$. From here, let us denote $[a, b]$ as the max and min for for the interval $[u, v]$ since we see that $[u, v] \in [a, b]$ for $f(b) \leq f(u) \leq f(v) \leq f(a)$. Then let us denote a η for $\eta \leq f(u)$. Meaning, the number η is smaller than or equal to our inf in the interval $[u, v]$. This mean $\eta \in [u, b]$ from the inf to the min of the interval. As such, we see that we can pick any $x \in [u, v]$ st. $f(x) \geq \eta \ \forall x \in [u, v]$, since $\eta \leq f(u)$, then all elements contained in $[u, v]$ will always be bigger than η. □

Exercise 5. (10 pts) Let $f : \mathbb{R} \to \mathbb{R}$ be a function that satisfies $f(x + y) = f(x) + f(y)$ for any real number x and y.

a) Suppose that f is continuous at some point c. Prove that f is continuous on $\mathbb{R}$.

b) Suppose that f is continuous on $\mathbb{R}$ and that $f(1) = k$. Prove that $f(x) = kx$ for all $x \in \mathbb{R}$. [Hint: start with x integer, then x rational, and finally use Exercise 3.]

Solution: **a)** From the hint we know that f is continuous at some point c. Thus we know that $\lim_{c \to x_n} f(c) = L$. This means x_n is defined at some point $c \in f$. From here, let us rewrite $x + y = c$ for we know point c is defined, $f(c) = f(x) + f(y)$. From here, we can take the limit of both sides and apply the sum rule, we then obtain

$$\lim_{c \to x_n} f(c) = \lim_{x \to x_n} f(x) + \lim_{y \to x_n} f(y)$$

Observe that $\lim_{c \to x_n} f(c) = L$, thus, rewrite as,

$$L = \lim_{x \to x_n} f(x) + \lim_{y \to x_n} f(y)$$

From here, we know by the way the function f is defined, $\lim_{x \to x_n} f(x) = L$ and $\lim_{y \to x_n} f(y) = L$ for,

$$\begin{aligned} L &= L + L \\ L &= 2L \\ 0 &= 2L - L \\ 0 &= L \end{aligned}$$

Thus we see that for the point c it is continuous at 0. This also means that we can understand the limit for $f(x)$ and $f(y)$ as two functions approaching 0 from the negative and positive side of the graph. Thus, since we know it is continuous at c, it must also be continuous on $\mathbb{R}$ for $f(y) \leq f(c) \leq f(x)$.

b) Assume that f is continuous on $\mathbb{R}$ st. $f(1) = k$. Our goal is to prove that $f(x) = kx \ \forall x \in \mathbb{R}$. By applying our hint, let us first prove that this is true for $x \in \mathbb{Z}$. From the assumption above, we see that $f(1) = k$. We know that our function $f \to 0$ from both the LHS and RHS of the function, thus we know that at the very minimum, it is defined across $\mathbb{Z}$ st. $f(1) = k$ and $f(-1) = -k$ for the way we defined our limit. From here, we know there exists an infinitely many rational numbers in between integers. Thus, apply hint from above and we can let $x \in Z_n$ for Z_n be the parent sequence containing all of the numbers in set $\mathbb{Z}$ that is defined at for f and let Q_n be a subsequence of Z_n that includes all rational numbers between the integers in the set of Z_n that is defined for f. Thus, by the BWT theorem, we know that since all numbers in Z_n is defined or converges to a certain limit L for $\lim_{Z_n \to x_n} f(Z_n) = kx$, then all subsequences of the parent sequence must also converge to the same limit. Therefore, since we know that $\mathbb{R}$ contains all $\mathbb{Z}$ and $\mathbb{Q}$, x is continuous in $\mathbb{R} \ \forall x \in \mathbb{R}$. □

Answer all the questions below. Make sure to show your work.

Exercise 6. (10pts) For each of the functions below, say if the limit exists or doesn't exist at the given point. Justify your answer (in other words, prove it!)

a) $f(x) = \sin(1/x)$ and $x_0 = 0$.

b) $f(x) = x\sin(1/x)$ adn $x_0 = 0$.

Solution: **a)** Since we know that sin$x$ is continuous everywhere, let us assume that $\lim_{x\to x_0} f(x) = L$, thus we are saying there exists a limit L as $sin(\frac{1}{x})$ heads towards $x_0 = 0$. This means $\forall\epsilon > 0$, $\exists\delta > 0$ st. $|x - x_0| < \delta$ for $|f(x) - f(x_0)| < \epsilon$. By this definition of continuity, we see we have a contradiction. We are saying $\forall\epsilon > 0$, $\exists\delta > 0$ st. $|x - x_0| < \delta$ for $|f(x) - f(x_0)| < \epsilon$, thus for $|f(x) - f(x_0)| < \epsilon$, we obtain $|\sin(\frac{1}{x})$-$\sin(\frac{1}{x_0})| < \epsilon$ for $x_0 = 0$, then replace and rewrite as, $|\sin(\frac{1}{x})$-$\sin(\frac{1}{0})| < \epsilon$. We see here that although sin$x$ is continuous, $\nexists\epsilon > 0$ for $|\sin(\frac{1}{x})$-$\sin(\frac{1}{0})| < \epsilon$ st. $f(x_0)$ is undefined when $\sin(\frac{1}{0})$. Thus, there exists no limit at x_0.

b) For this problem, let us apply the squeeze theorem. Suppose we have two functions $g(x)$ and $h(x)$ for $g(x) \le f(x) \le h(x)$. We let $g(x) = \lim_{x\to x_0^-} f(x)$ and $h(x) = \lim_{x_0^+} f(x)$. From here, again since we know that sin$x$ is continuous, we can use the definition of continuity and suppose that $f(x) \to L$ $\forall\epsilon > 0$, $\exists\delta > 0$ st. $|x - x_0| < \delta$ for $|f(x) - f(x_0)| < \epsilon$. We see that if we were to apply this definition for $g(x)$ and $h(x)$ as well, we would obtain $|g(x) - g(x_0)| < \epsilon$ and $|h(x) - h(x_0)| < \epsilon$. Since we supposed that $g(x) = \lim_{x\to x_0^-} f(x)$ and $h(x) = \lim_{x_0^+} f(x)$, we see that $g(x_0) \to 0$ as our x gets bigger and bigger. We also see that $h(x_0) \to 0$ as our x becomes smaller and smaller. Thus by definition of continuity, we see that $g(x_0) = h(x_0) = 0$ for our $L = 0$. Therefore by the squeeze theorem, if $h(x) \to 0$ and $g(x) \to 0$ for $g(x) \le f(x) \le h(x)$, then $f(x) \to 0$ for there exists a limit L at x_0. □

Exercise 7. (5 pts) Let $c \in (a, b)$ and let f be a function defined on (a, b) except at c. Suppose that $f(x) > 0$ for any $x \in (a, b)\backslash\{c\}$, that $\lim_{x\to c} f(x)$ exists, and that

$$\lim_{x\to c} f(x) = \lim_{x\to c} \left[(f(x))^2 - f(x) - 3\right].$$

Find the value of $\lim_{x\to c} f(x)$. Explain each step carefully.

Solution: For this problem, we know that f is a function defined on (a, b) for $\exists c \in (a, b)$ st f is not defined at c. We know that $\exists L$ at c for the $\lim_{x\to c} f(x) = \lim_{x\to c} [(f(x))^2 - f(x) - 3]$. From here, apply sum rule and distribute the limit. Observe,

$$\begin{aligned}
\lim_{x\to c} f(x) &= \lim_{x\to c} \left[(f(x))^2 - f(x) - 3\right] \\
\lim_{x\to c} f(x) &= \lim_{x\to c}(f(x))^2 - \lim_{x\to c} f(x) - \lim_{x\to c} -3 \\
0 &= \lim_{x\to c}(f(x))^2 - 2\lim_{x\to c} f(x) - \lim_{x\to c} -3
\end{aligned}$$

From here, we can solve this like a linear equation and see that our roots are -1 and 3. Since we know that $f(x) > 0 \, \forall x \in (a, b)$, then our L at c must be 3 for $L = 3$ since $f(x) > 0$ thus our $\lim f(x)$ cannot be less than 0. Therefore, the only root that is compatible is 3 for our $\lim_{x \to c} f(x) = 3$. $\square$

Exercise 8. (5 pts) Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} x & , x \in \mathbb{Q} \\ -x & , x \notin \mathbb{Q}. \end{cases}$$

is discontinuous at any point of $\mathbb{R}\backslash\{0\}$ and continuous at 0.

Solution: We see that by the way f is defined, the limit of f approaching from the left of x will be different from the limit as x approaches from the right. Observe, by the definiton of f given, $\lim_{x \to x_n^+} f(x) = x$ and $\lim_{x \to x_n^-} f(x) = -x$. Thus we see the LHS and RHS limits are different, other than at the point 0.We see that although the function f approach x_n at different L $\lim_{0 \to x_n^+} f(0) = 0$ and $\lim_{0 \to x_n^-} f(0) = 0$ for 0=0, thus it is only continuous at 0 by the way f is defined. $\square$

Exercise 9. (5 pts) Let $p(x) = x^2 + 2$. Find an interval where p is strictly decreasing and find a formula for its inverse.

Solution: For this polynomial, we see that the interval which is strictly decreasing would be from $(-\infty, 0]$ for the parent function of $y- = x^2$ is a parabola and it is only decreases on it's domain from $(-\infty, 0]$. For us to find the inverse of this particular function, use elementary algebraic skills and solve,

$$\begin{aligned} p(x) &= x^2 + 2 \\ y &= x^2 + 2 \\ x &= y^2 + 2 \\ y^2 &= x - 2 \\ y &= \sqrt{x - 2} \end{aligned}$$

Exercise 10. (10 pts) Let $p(x) = ax^3 + bx^2 + cx + d$ be a polynomial of degree 3 and $a > 0$. Prove that p has at least one real root by following these steps:

a) Prove that $\lim_{x \to \infty} p(x) = \infty$.

b) Prove that $\lim_{x \to -\infty} p(x) = -\infty$.

c) Conclude.

[Hint for a): write your polynomial $p(x) = ax^3 + bx^2 + cx + d$ as $x^3(a + b/x + c/x^2 + d/x^3)$ and use the fact that $\lim_{x \to \infty} 1/x^n = 0$ for every $n \geq 1$.]

Solution: **a)** Apply hint and rewrite polynomial,

$$p(x) = x^3\left(a + \frac{b}{x} + \frac{c}{x^2} + \frac{d}{x^3}\right)$$

Apply product rule and rewrite as,

$$\lim_{x\to\infty} p(x) = \lim_{x\to\infty} x^3 \cdot \lim_{x\to\infty}\left(a + \frac{b}{x} + \frac{c}{x^2} + \frac{d}{x^3}\right)$$

From here apply sum rule to RHS of the multiplication,

$$\lim_{x\to\infty} p(x) = \lim_{x\to\infty} x^3 \cdot \left(\lim_{x\to\infty} a + \lim_{x\to\infty}\frac{b}{x} + \lim_{x\to\infty}\frac{c}{x^2} + \lim_{x\to\infty}\frac{d}{x^3}\right)$$

Apply limit to each term and second part of the hint,

$$\lim_{x\to\infty} p(x) = \infty \cdot (a + 0 + 0 + 0)$$
$$\lim_{x\to\infty} p(x) = \infty \cdot a$$
$$\lim_{x\to\infty} p(x) = \infty$$

b) Apply hint and rewrite polynomial,

$$p(x) = x^3\left(a + \frac{b}{x} + \frac{c}{x^2} + \frac{d}{x^3}\right)$$

Apply product rule and rewrite as,

$$\lim_{x\to-\infty} p(x) = \lim_{x\to-\infty} x^3 \cdot \lim_{x\to-\infty}\left(a + \frac{b}{x} + \frac{c}{x^2} + \frac{d}{x^3}\right)$$

From here apply sum rule to RHS of the multiplication,

$$\lim_{x\to-\infty} p(x) = \lim_{x\to-\infty} x^3 \cdot \left(\lim_{x\to-\infty} a + \lim_{x\to-\infty}\frac{b}{x} + \lim_{x\to-\infty}\frac{c}{x^2} + \lim_{x\to-\infty}\frac{d}{x^3}\right)$$

Apply limit to each term and second part of the hint,

$$\lim_{x\to-\infty} p(x) = -\infty \cdot (a + 0 + 0 + 0)$$
$$\lim_{x\to-\infty} p(x) = -\infty \cdot a$$
$$\lim_{x\to-\infty} p(x) = -\infty$$

c) Since we know that the polynomial $p(x) = ax^3 + bx^3 + cx + d$ for $a > 0$ and it is continuous everywhere, assume that all other constants $b, c, d = 0$, then all that is remained is $p(x) = ax^3$. The parent function $y = ax^3$ is an odd function and it crosses over the x axis for so long as $a > 0$. Since we know that $a > 0$, therefore there exists at least 1 root for $p(x) = ax^3 + bx^2 + cx + d$. $\square$