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1.

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2. Let $f: D \subseteq \mathbb{R} \to \mathbb{R}$, and suppose that x_0 is an accumulation point of D. Suppose that for each sequence $(x_n)_{n=1}^{\infty}$ converging to x_0 with $x_n \in D \setminus \{x_0\}$ for each $n \geq 1$, then the sequence $(f(x_n))_{n=1}^{\infty}$ is Cauchy. Show that f has a limit at x_0.

Proof: Because the sequence $(f(x_n))_{n=1}^{\infty}$ is cauchy we know that it converges and has a limit L, Suppose that L is not a limit of f at x_0. Thus, there is $\varepsilon > 0$ such that for every $\delta > 0$, there is $x \in D$, with $0 < |x - x_0| < \delta$ and such that $|f(x) - L| \geq \varepsilon$. In particular, for each positive integer n, there is $x_n \in D$ with $0 < |x_n - x_0| < \frac{1}{n}$ such that $|f(x_n) - L| \geq \varepsilon$. The sequence $\{x_n\}_{n=1}^{\infty}$ converges to. x_0 and is a sequence of members of D distinct from x_0; hence, $\{f(x_n)\}_{n=1}^{\infty}$ converges to L, contrary to the fact that $|f(x_n) - L| \geq \varepsilon > 0$ for all n. Thus, L must be the limit of at x_0. ■

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3. To show that a limit is unique we aim to prove that if $\lim_{x \to a} f(x) = X_0$ and $\lim_{x \to a} f(x) = X_1$, then $X_0 = X_1$ and therefore the limit is unique.

Proof: Suppose $\lim_{x \to a} f(x) = X_0$ and $\lim_{x \to a} f(x) = X_1$. Let $\varepsilon > 0$ be arbitrary. There exists δ_1, δ_2 such that if $0 < |x - a| < \delta_1$, then $|f(x) - X_0| < \varepsilon/2$ and if $0 < |x - a| < \delta_2$, then $|f(x) - X_1| < \varepsilon/2$. Pick $\delta = \min\{\delta_1, \delta_2\}$, then if $0 < |x - p| < \delta$, we have:

$$|X_0 - X_1| = |X_0 - X_1 + f(x) - f(x)|$$
$$\leq |f(x) - X_0| + |f(x) - X_1| < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon$$

Because $\varepsilon > 0$ was arbitrary we have $X_0 = X_1$ and therefore f has a limit at $X_0 \in \operatorname{acc} D$ that is unique ∎

4. Denote $L = \lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x)$

Then, $\forall \varepsilon > 0$, $\exists \delta_1 > 0$, when $0 < |x - x_0| < \delta_1$, we have $|f(x) - L| < \varepsilon$, and hence

$$L - \varepsilon < f(x) < L + \varepsilon$$

Similarly, $\forall \varepsilon > 0$, $\exists \delta_2 > 0$, when $0 < |x - x_0| < \delta_1$ we have

$$L - \varepsilon < h(x) < L + \varepsilon$$

Let $\delta = \min(\delta - 1, \delta_2)$. When $0 < |x - x_0| < \delta$, we have

$$L - \varepsilon < f(x) \leq g(x) \leq h(x) < L + \varepsilon$$

namely

$$|g(x) - L| < \varepsilon$$

Thus $\lim_{x \to x_0} g(x) = L$. So $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = \lim_{x \to x_0} g(x)$ ∎

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5. a) If g is bounded, it means that $|g(x)| < c$ for some c, for all x. Now
$$0 \le |f(x)g(x)| = |f(x)||g(x)| \le c|f(x)|$$
Since $\lim_{x\to\infty} f(x) = 0$, we have $\lim_{x\to\infty} |f(x)| = 0$, and so by the squeeze theorem
$$\lim_{x\to\infty} f(x)g(x) = 0$$

b) Let $f:(a,\infty)\to\mathbb{R}$ for some $A>0$, and let $g:(0,\frac{1}{a}]\to\mathbb{R}$ be defined by $g(x) = f(1/x)$

$\Rightarrow$) Suppose f has a limit at ∞
if $x\to\infty \Rightarrow \frac{1}{x} \Rightarrow 0$
$f(\frac{1}{x}) \to 0 \Rightarrow g(x) \to 0$
So $g(x)$ has a limit at 0

$\Leftarrow$) Suppose $g(x)$ has a limit at 0
$g(x) \to 0$
$f(1/x) \to 0$
If $1/x \to 0$ as $x\to\infty$, then f has a limit at ∞

6. a) Take $\lim_{n\to\infty} \frac{1}{n} + \frac{1}{n+1} + \dots + \frac{1}{2n}$. By the sum rule, we have $\lim_{n\to\infty} \frac{1}{n} + \lim_{n\to\infty} \frac{1}{n+1} + \dots + \lim_{n\to\infty} \frac{1}{2n}$. Each of the individual limits goes to 0 so $\lim_{n\to\infty} \frac{1}{n} + \frac{1}{n+1} + \dots + \frac{1}{2n} = 0$

b) With some algebra and the sum rule we have $\lim_{n\to\infty} \frac{1+2+\dots+n}{n^2} = \lim_{n\to\infty} \frac{1}{n^2} + \lim_{n\to\infty} \frac{2}{n^2} + \dots + \lim_{n\to\infty} \frac{n}{n^2}$. Each of these limits goes to zero and so $\lim_{n\to\infty} \frac{1+2+\dots+n}{n^2} = 0$

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7. The $\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} = \frac{1}{2}$. To prove this, let $\varepsilon > 0$ be given. We seek $\delta > 0$, so that $|g(x) - 1/2| < \varepsilon$ whenever $0 < |x| < \delta$

Choose $\delta = \varepsilon$. Suppose $0 < |x| < \delta$ with $0 < x < 1$, Then,

$$|g(x) - 1/2| = \left|\frac{\sqrt{1+x}-1}{x} - \frac{1}{2}\right| = \left|\frac{2\sqrt{1+x}-2-x}{2x}\right|$$

$$= \left|\frac{2\sqrt{1+x}-(2+x)}{2x} \cdot \frac{2\sqrt{1+x}+(2+x)}{2\sqrt{1+x}+(2+x)}\right| = \left|\frac{-x^2}{2x(2\sqrt{1+x}+(2+x))}\right|$$

$$< \left|\frac{-x}{8}\right| < |x| < \delta = \varepsilon \quad \blacksquare$$

8. $\lim_{x\to 1} f(x) = 1$

$$\lim_{x\to 1} \frac{f(x)(1-f(x)^2)}{1-f(x)} = \lim_{x\to 1} \frac{(f(x)-f(x)^3)}{1-f(x)} \cdot \frac{(f(x)+f(x^2))}{(f(x)+f(x)^2)}$$

$$= \lim_{x\to 1} \frac{(f(x)-f(x)^3)(f(x)+f(x)^2)}{f(x)-f(x)^2+f(x)^2-f(x)^3}$$

$$= \lim_{x\to 1} f(x) + f(x)^2$$

$$= \lim_{x\to 1} f(x) + \lim_{x\to 1} f(x)^2$$

$$= 1 + 1$$

$$= 2$$

$f(x) = x^n$ then $\lim_{x \to x_0} f(x) = x_0^n$ for any $x_0 \in \mathbb{R}$
If $x_0 \in [0, \infty)$, then $\lim_{x \to x_0} \sqrt{x} = \sqrt{x_0}$

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9. Since f has a limit at x_0, $\forall \varepsilon > 0$, $\exists \delta > 0$, for $x \in D$ with $|x - x_0| < \delta$ we have

$$|f(x) - f(x_0)| < \varepsilon$$

Hence

$$\big||f|(x) - |f|(x_0)\big| \leq |f(x) - f(x_0)| < \varepsilon$$

Thus, $|f|$ has a limit at x_0 ∎

10. a) Assume $f(x)$ is increasing. Let $D = \{x : x \in (\alpha, \beta), f$ does not have a limit at $x\}$. By Lemma 2.7 in the book, $x \in D$ iff $U(x) - L(x) \neq 0$. and, since f is increasing, iff $U(x) - L(x) > 0$. Let

$$D_n = \{x : U(x) - L(x) > \tfrac{1}{n}\}$$

It is clear that $D = \bigcup_{n=1}^{\infty} D_n$. Now, suppose $\{x_1, \dots, x_r\} \subset D_n$ with $\alpha < x_1 < x_2 < \dots < x_r < \beta$. Choose $z_1, \dots, z_{r+1}$ such that $\alpha < z_1 < x_1$, $x_i < z_{i+1} < x_{i+1}$ for $i = 1, 2, \dots, r-1$, and $x_r < z_{r+1} < \beta$. Now for each i, $f(z_i) \leq L(x_i)$ and $U(x_i) \leq f(z_{i+1})$ hence

$$f(z_{i+1}) - f(z_i) \geq U(x_i) - L(x_i) > \tfrac{1}{n}$$. now

$$f(\beta) - f(\alpha) = f(\beta) - f(z_{r+1}) + \sum_{k=2}^{n+1} [f(z_k) - f(z_{k-1})] + f(z_1) - f(\alpha) \geq r\left(\tfrac{1}{n}\right)$$

$$f(\beta) - f(\alpha) = f(\beta) - f(z_{r+1}) + \sum_{k=2}^{r+1} [f(z_k) - f(z_{k-1})] + f(z_1) - f(\alpha) \geq r\left(\tfrac{1}{n}\right)$$

Since $f(\beta) - f(\alpha) > 0$ is a fixed real number, it is necessary that $r \leq n[f(\beta) - f(\alpha)]$. Therefore, D_n is finite for each n; hence whenever the limit exists $\lim_{x \to x_0} f(x) = x_0^n$

If $x_0 \in [0, \infty)$, then $\lim_{x \to x_0} \sqrt{x} = \sqrt{x_0}$

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10. b) We need to prove that for any point $x_0 \in [0, \infty)$, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|x - x_0| < \delta \Rightarrow |\sqrt{x} - \sqrt{x_0}| < \varepsilon$$

So, to find a δ, we turn to the inequality $|\sqrt{x} - \sqrt{x_0}| < \varepsilon$. Since we want an expression involving $|x - x_0|$, multiply by the conjugate to remove the square roots.

$$|\sqrt{x} - \sqrt{x_0}| < \varepsilon \Rightarrow |\sqrt{x} - \sqrt{x_0}| \cdot |\sqrt{x} + \sqrt{x_0}| < \varepsilon \cdot |\sqrt{x} + \sqrt{x_0}|$$

$$|x - x_0| < \varepsilon \cdot |\sqrt{x} + \sqrt{x_0}| \quad (1)$$

Now if you require that $|x - x_0| < 1$, then it follows that $x - x_0 < 1$, so $x_0 - 1 < x < x_0 + 1$, and therefore that $\sqrt{x} < \sqrt{x_0 + 1}$. Therefore, $\sqrt{x} + \sqrt{x_0} < \sqrt{x_0 + 1} + \sqrt{x_0}$, which combined with (1) tells us that

$$|x - x_0| < \varepsilon (\sqrt{x_0 + 1} + \sqrt{x_0})$$

So let $\delta = \min(1, \varepsilon(\sqrt{x_0 + 1} + \sqrt{x_0}))$. This proves that $\lim_{x \to x_0} \sqrt{x} = \sqrt{x_0}$

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11. a) To prove that f has a limit at every point, I interpret this to prove that $\lim_{x \to a} f(x)$ exists for every a, given that it exist for 0.
We have:
$\lim_{x \to a} f(x) = \lim_{h \to 0} f(a+h) = \lim_{h \to 0} f(a)f(h) = f(a) \cdot \lim_{h \to 0} f(h)$.
Since we know that the expression the right exists, you can go backwards to see that the expression on the left also exists

b) Assume that $\lim_{x \to 0} f(x) \neq 1$, then we must show that f(x) is zero for all x.

$f(x) = f(x-h) \cdot f(h)$, then h goes to zero, and use the explanation above that f is continuous, and part (a) that the limit exists in every point. Then we get $f(x) = \lim_{h \to 0} f(x-h) \cdot \lim_{h \to 0} f(h) = f(x) \cdot \lim_{h \to 0} f(h)$.
And since we know that the last limit is not 1, f(x) must be zero.