2 $[a,u],[u,v],[v,b] \subseteq [a,b]$ so
$\int_a^u f + \int_u^v f + \int_v^b f = \int_a^b f$
f is nonnegative so $\int_a^x f ; \int_x^b f \geq 0$
$\int_u^v f = \int_a^b f - \int_a^u f - \int_v^b f \quad \int_a^u f, \int_v^b f \geq 0$
So $\int_a^b f \geq \int_a^b f - \int_a^u f - \int_v^b f$
$\int_u^v f = \int_a^b f - \int_a^u f - \int_v^b f \leq \int_a^b f$
$\int_u^v f \leq \int_a^b f$

1 a) Let P be t.p. of $[u,v]$. $S(\phi,P) = c(v-u)$
If $\|P\| < \delta \to |S(\phi,P) - \int_u^v \phi| < \varepsilon$, if $\int_u^v \phi = c\,l(I)$ then
$|c(v-u) - c\,l(I)| = |c(v-u) - c(v-u)| = 0 < \varepsilon$.
Having $(u,v]$ changes one points value which doesn't change the integrals value.
$[u,u]$ has a single point u, which integrated over anything is 0 because $g(x) = \int_a^x f(t)dt \quad g(a) = 0$.
b) $\int_a^b f_1 + f_2 = \int_a^b f_1 + \int_a^b f_2$ by sum rules of integration.
Using induction, $\int_a^b f_1 + f_2 \dots + f_n = \int_a^b f_1 + \int_a^b f_2 + \dots + \int_a^b f_n$ by sum rules of integration, with $\int_a^b \sum_{k=1}^n f_k = \int_a^b f_1 + \int_a^b f_2 + \dots \int_a^b f_n$
c) Using part a and b, with $\phi = c\chi_I$ on I and 0 everywhere else, $\int_a^b \sum_{k=1}^n c_k \chi_{I_k} = \int_a^b \sum_{k=1}^n c_k\, l(I) = \int_a^b c_1 l(I) + \dots + \int_a^b c_n l(I)$
The integral of ϕ exists because each $\int_a^b c_n l(I_n)$ exists and by sum rule exists.

4 f is continuous on $[a,b]$ it must be bounded. Let $f(d)$ be $\sup(f)$ $[a,b]$ and $f(e)$ be $\inf(f)$. $f(d)(b-a)$ and $f(e)(b-a)$ are riemann sums of f on $[a,b]$ which coorespond to the max and min value of the riemann sum. $\int_a^b f$ is in between these values, So there must be a a value $f(c)$ in between $f(d)$ and $f(e)$ that has $f(c)(b-a) = \int_a^b f$.

3 a) $g = \int_a^x f(t)dt$ $g(b) = 0$, then $g(x) = \int_a^x f(t)dt$ and $g(b) = \int_a^x f(t)dt + \int_x^b f(t)dt$. If f is nonnegative than area cannot cancel out, so $0 = \int_a^x f(t)dt + \int_x^b f(t)dt$ means both integrals must be 0, so $f(x)$ $x \in [a,b]$ must be 0 to ensure area under the curve to be 0

b.) $h(x) = f(x) - g(x)$ $\int_a^b h(x) = \int_a^b f(x) - \int_a^b g(x) = 0$
So $\int_a^b h(x)$ must have $h(x) = 0$ $\forall x \in [a,b]$ meaning $f = g$, or the positive area under the curve equals negative area, meaning $h(x_1) > 0$ & $h(x_2) < 0$. Using IVT, $\exists c \in [a,b)$ s.t. $h(c) = 0$. $f(c) - g(c) = 0$ $f(c) = g(c)$.
Both scenarios have a $f(c) = g(c)$.

6a) Let $(P_N)_{n=1}^{\infty}$ be a sequence of tagged partitions of $[a,b]$. Let $(P_{NQ})_{n=1}^{\infty}$ be a subsequence and $(P_{NR})_{n=1}^{\infty}$ be another subsequence. P_{NQ} contains the rational numbers of P and P_{NR} contains everything not rational. Let $\lim_{n \to \infty} ||P_N|| = 0$. Now the sequence $(S(f, P_N))_{n=1}^{\infty}$ converges if all subsequences converge to the same point. However, $(S(f, P_{NQ}))_{n=1}^{\infty} \to 1$ and $(S(f, P_{NR}))_{n=1}^{\infty} \to 0$, so the sequence diverges and $(S(f, P_n))_{n=1}^{\infty} \not\to \int_a^b f$ so $f(x)$ is not R.I.

b.) $g \circ h = \begin{cases} 0, & x \notin \mathbb{Q} \\ x, & x \in \mathbb{Q} \end{cases}$ which is just like part a except replace f with $g \circ h$ and 1 with the value of x. The composition of two functions that are R.I. may not be R.I..

5) Define $g(x) = f(a)(x-a) + f(b)(b-x)$.
$g(a) = f(b)(b-a)$ and $g(b) = f(a)(b-a)$
Since f is strictly increasing, $f(b) > f(a)$.
The area under the curve can then be seen as a riemann sum from a to b. $f(b)(b-a)$ and $f(a)(b-a)$ are R. sums of one partition with a tag either b or a. Because f is strictly increasing $f(a)(b-a)$ is the smallest R. sum and $f(b)(b-a)$ is the largest. So the actual value $\int_a^b f$ is inbetween these. So with $g(x)$, IVT says $g(c) = \int_a^b f = f(a)(c-a) + f(b)(b-c)$

HW #7

8 $f'(x) = ?$ $\quad f(x) = \int_{\sqrt{x}}^{\sqrt[3]{x}} \frac{1}{1+t^3}\,dt$

$f'(x) = \frac{d}{dt}\int_{\sqrt{x}}^{\sqrt[3]{x}} \frac{1}{1+t^3}\,dt = \frac{1}{1+x} \cdot \frac{1}{3(x)^{2/3}} + \frac{1}{1+(x)^{3/2}} \cdot \frac{1}{2\sqrt{x}}$

$f'(x) = \frac{1}{3(x)^{2/3} + 3(x)^{5/3}} + \frac{1}{2\sqrt{x} + 2(x)^{5/3}}$

9 $f(1) = 0 \quad f'(x) = 1 + \sin(x^2) \quad f(x) = \int_1^x 1 + \sin(t^2)\,dt$

10 $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{k^2+n^2} \quad \lim_{n\to\infty} \sum_{k=1}^{n} \Delta x \cdot f(x_k) \quad \Delta x = \frac{b-a}{n} \quad x_k = f(a+\Delta x\cdot k)$

$\frac{n}{k^2+n^2} = \frac{1}{\frac{k^2}{n}+n} = \frac{1/n}{k^2/n^2+1} \quad \Delta x = \frac{1}{n} = \frac{1-0}{n} \quad b = 1, a = 0$

$f(x) = \frac{1}{x^2+1} \quad f(a+\Delta x\cdot k) = \frac{1}{(\frac{k}{n})^2+1} = \frac{1}{\frac{k^2}{n^2}+1}$

$\int_0^1 \frac{1}{x^2+1}\,dx = \int_0^1 (x^2+1)^{-1}\,dx = \arctan(x)\Big|_0^1 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$

7 $|f| = \begin{cases} -f(x), & f(x) < 0 \\ f(x), & f(x) > 0 \end{cases}$ $f(x)$ and $-f(x)$ are R.I. on $[a,b]$ due to being continuous and by continuity rules, so $|f(x)|$ is R.I. on $[a,b]$ due to R.I. rules. So if P is a t.p. of $[a,b]$ and $\|P\| < d$, then $|S(|f|,P) - \int_a^b |f|| < \varepsilon$ $\quad f \le |f|$

$|S(f,P) - \int_a^b f| \le |S(|f|,P) - \int_a^b |f|| < \varepsilon$ $\quad$ Interval > 0

$||S(f,P)| - |\int_a^b f|| \le |S(f,P) - \int_a^b f| \le |S(|f|,P) - \int_a^b |f||$ $\quad |S(f,P)| = S(|f|,P)$

$||S(|f|,P)| - |\int_a^b f|| \le |S(|f|,P) - \int_a^b |f|| \to |\int_a^b f| \le \int_a^b |f|$