UNIVERSITY
OF HAWAI'I

MATH-331 Intro. to Real Analysis
Final exam

Created by Pierre-O. Parisé
Fall 2021, 12/13/2021

Last name: Kershner
First name: Liliana

Question:	1	2	3	4	5	Total
Points:	20	20	20	20	20	100
Score:	7	14	8	5	15	49

Instructions: Make sure to write your complete name on your copy. You must answer all the questions below and write your answers directly on the questionnaire. You have 2 hours to complete the exam. When you are done, hand out your copy and you may leave the classroom.

No devises such as a smart phone, cell phone, laptop, or tablet can be used during the exam. You are not allowed to use the lecture notes and the textbook also. You may use your personal cheat sheet on the exam.

Make sure to show all your work. State clearly any theorem or definition you are using in your proofs or your calculations. Make sure you show clearly that all hypothesis required to use a Theorem are satisfied. No credit will be earned for an answer without explanations.

BE THE BEST VERSION OF YOURSELF!

PIERRE-OLIVIER PARISÉ

Liliana Kershner

SIGN ↑ TO ACKNOWLEDGE YOU HAD READ AND ACCEPT THE ABOVE RULES.

QUESTION 1 (20 pts)

Find the value of the following limits. Write down clearly which properties you are using.

(a) $\lim_{n\to\infty} e^{1/n}$. (3/5)

$\lim_{n\to\infty} e^{1/n} = e^0 = 1$. as $n \to \infty$, $1/n \to 0$.

because e^x is continuous.

(b) $\lim_{n\to\infty} x_n$ if $x_1 = 2$ and $x_n = 2 - 1/x_{n-1}$ (*) for $n \geq 2$. (3/5)

$1\frac{1}{2}, 1\frac{1}{3}, 1\frac{1}{4}$

$x_1 = 2$ then $x_2 = 2 - \frac{1}{x_{n-1}} = 2 - \frac{1}{2} = 3/2$.

$x_5 = 2 - \frac{1}{5/4} = 2 - 4/5 = 1\frac{1}{5}$

$x_3 = 2 - \frac{1}{3/2} = 2 - 2/3 = 4/3$, $x_4 = 2 - \frac{1}{4/3} = 2 - 3/4 = 1\frac{1}{4}$.

x_n is a decreasing sequence, bounded by 2 above. limit exists, call it A. So, from *, $A = 1 - \frac{1}{A}$. Solve for A $\Rightarrow A = 1$

So $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. if $n \geq N$ then $|x_n - A| < \varepsilon$.

Hypothesis: $A = 1$.

Let $\varepsilon > 0$. Then $\exists N \in \mathbb{N}$ s.t. if $n \geq N$, then $|x_n - 1| < \varepsilon$.

$-\varepsilon < x_n - 1 < \varepsilon \Rightarrow 1 - \varepsilon < x_n < \varepsilon + 1$ so $x_n \in (1-\varepsilon, \varepsilon + 1)$

So we can always choose an ε which is greater than x_n, and since ε was arbitrary, $\lim_{n\to\infty} x_n = 1$.

you didn't prove it. You just stated the def.

$N = \frac{1}{\varepsilon + 1}$.

(c) $\lim_{x\to 0} \frac{\sin(x^2)}{2x}$. (You won't be credited if you use l'Hopital's Rule.) goes to 0. (1/5)

$$\lim_{x\to 0} \frac{\sin(x^2)}{2x} = \lim_{x\to 0} \frac{1}{2} \cdot \underbrace{\lim_{x\to 0} \frac{\sin(x^2)}{x}}$$

From calculus identity, $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

$$= \lim_{x\to 0} \frac{1}{2} \cdot 1 = \frac{1}{2}.$$ X

(d) $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k\sin(k^2/n^2)}{n^2}$. (Simplify your final answer as much as you can.) (0/5)

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k\sin\left(\frac{k^2}{n^2}\right)}{n^2} = \lim_{n\to\infty} \sum_{k=1}^{N} \frac{k}{n^2} \cdot \lim_{n\to\infty} \sin\left(\frac{k^2}{n^2}\right)$$

$\lim_{n\to\infty} \sum_{k=1}^{N} \frac{1}{n^2} \cdot k = 0$ since n grows very large.

$\lim_{n\to\infty} \sum_{k=1}^{N} \sin\left(\frac{k^2}{n^2}\right) = \lim_{n\to\infty} \sum_{k=1}^{N} \sin\left(k^2 \cdot \underset{\downarrow 0}{\frac{1}{n^2}}\right) = \sin 0 = 0$ again by previous.

So $\lim_{n\to\infty} \sum_{k=1}^{N} \frac{k\sin\left(\frac{k^2}{n^2}\right)}{n^2} = 0 \cdot 0 = 0.$ X

goes to $\int_0^1 x\sin(x^2)\,dx$

QUESTION 2 (20 pts)

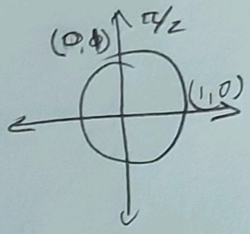
Answer the following questions. State all the hypothesis of the Theorem you are using and write down clearly which properties you are using.

(a) Using the Fundamental Theorem of Calculus, compute the derivative of the function (7/10) $F(x) = \int_{\cos x}^{\sin x} \sqrt{1-t^2}\, dt$ where $x \in [0, \pi/2]$. Simplify your answer as much as you can.

(b written by the upper limit, a by the lower limit)

By FTC: & chain rule.

$$F'(x) = \sqrt{1-(\sin x)^2} \cdot (\sin x)' - \sqrt{1-(\cos x)^2}\,(\cos x)'$$



$$= \sqrt{1-\sin^2 x} \cdot \cos x - \sqrt{1-\cos^2 x} \cdot (-\sin x)$$

by trig identity

$$= \sqrt{\cos^2 x} \cdot \cos x + \sqrt{\sin^2 x}\, \sin x$$

?

$$\Rightarrow x \in [0, \pi/2] \Rightarrow \sqrt{\cos^2 \tfrac{\pi}{2}} \cdot \cos \tfrac{\pi}{2} + \sqrt{\sin^2 \tfrac{\pi}{2}}\, \sin \tfrac{\pi}{2} - (1+0)$$

$$\Rightarrow 0 + 1 - 1 + 0 = 0. \quad \approx$$

$$\cos^2 x + \sin^2 x = 1 \Rightarrow F'(x) = 1.$$

(**2**/5)

(b) Find $g'(5)$ if g is the inverse of the function $f(x) = x^3 + 2x + 2$.

$g'(x) = \frac{1}{f'(x)}$

$\Rightarrow f'(x) = 3x^2 + 2$

$g'(5) = \frac{1}{3(25)+2} = \frac{1}{77}$. ✗

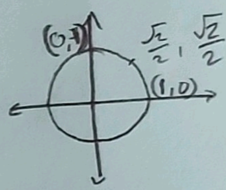
$g'(5) = \frac{1}{f'(g^{-1}(5))} = \frac{1}{f'(1)}$

because $g^{-1}(5) = 1$.

(**5**/5)

(c) Show that the equation $\cos(2x) = x$ has exactly one solution in the interval $[0, \pi/4]$.

Goals: show by IVT $\cos(2x) = x$ has roots on the interval
show by Rolle's there is exactly 1.



Define $f(x) = \cos(2x) - x$.

$f(0) = 1$ positive

$f(\pi/4) = \cos(\pi/2) - \frac{\pi}{4} = 0 - \frac{\pi}{4} =$ negative.

By IVT, f has roots in the interval $[0, \pi/4]$.

Now, let c_1 and c_2 be these roots s.t. $f(c_1) = f(c_2) = 0$.

By Rolle's thm, there $\exists\, c \in (c_1, c_2)$ s.t. $f'(c) = 0$.

$f'(x) = -\sin(2x) \cdot 2 - 1 = -2\sin(2x) - 1$. you're right.

This value will always be negative on $[0, \pi/4]$, so $c_1 = c_2$ and f has only one root. ✓

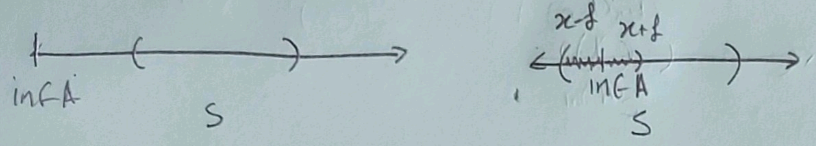
QUESTION 3 (20 pts)

Let A and B be two non-empty subsets of $\mathbb{R}$. Give a proof or, if it's false, give a counter-example to the following statements.

(a) If $S \subseteq A$ and S is nonempty, then $\inf A \leq \inf S$. (8/10)

(b) If $A \cap B \neq \varnothing$, then $\sup(A \cap B) = \max\{\sup A, \sup B\}$. (0/10)

A



a) Assume to a contradiction $\inf A > \inf S$. Define $\inf A := x$, and $\inf S := y$. If $\inf A > \inf S$, there exists a neighborhood Q defined as $(x-\delta, x+\delta)$ around x s.t. $x-\delta$ is outside of A, but contained in S. This would imply there are terms of S not contained in A, but by def. $S \subseteq A$ so this is a contradiction. So $\inf A \leq \inf S$.

justify why more precisely.

b) $A \cap B$ denotes all terms $x \in A, B$ which are shared by both sets.

Let's assume $\sup(A \cap B) = \max\{\sup A, \sup B\}$. WLOG, let $A > B$. ??

Define y as the supremum of A. Then $\sup A > \sup B$, $y \notin B$.

However, $\max\{\sup A, \sup B\} = y$, yet $y \notin A \cap B$ so $y \neq \sup(A \cap B)$.

So this can't be true for all sets A and B.

X

not true

QUESTION 4 (20 pts)

Let $a > 0$. We say that a function $f : (-a, a) \to \mathbb{R}$ is

- **odd** if $f(-x) = -f(x)$ for any $x \in (-a, a)$;
- **even** if $f(-x) = f(x)$ for any $x \in (-a, a)$.

Suppose $f : (-a, a) \to \mathbb{R}$ is a differentiable function on $(-a, a)$.

(a) Show that the function f is even if, and only if, f' is odd. (2.5/10)

(b) Show that the function f is odd if, and only if, f' is even and $f(0) = 0$. (2.5/10)

being not odd $\not\Rightarrow$ being even.

a) Assume WLOG, f is even and f' is even by contraction.
Then $f(-x) = f(x)$ and $f'(-x) = f'(x)$.
Then by MVP, $\exists c \in (-a, a)$ s.t

Ghosting

$$f'(-c) = \frac{f(a) - f(-a)}{a-(-a)} = f'(c) \Rightarrow \frac{f(a) - f(a) + f(a) - f(-a)}{2a} \Rightarrow$$

$$\frac{f(a) - f(a)}{2a} + \frac{f(a) + f(-a)}{2a}$$

Since $f(-a) = f(a)$:

what is the contradiction?

half of the points of $\Rightarrow$

$$\frac{0}{2a} + \frac{0}{2a} = 0. \quad f'(-c) = 0 = f'(c).$$

This implies that there is no c which gives a value different from 0 which can't be true.

X

b) Assume to a contradiction again that f is odd and f' is odd. 2.5
Then $f(-x) = -f(x)$ and $f'(x) = -f'(x)$.
Since f' is cont and $-a < a$, and $f'(a) > f'(-a)$, by IVT, there exists a $c_1 \in (a,a)$ s.t. $f'(c_1) = 0$.
$-f'(a)$

$\approx$

By the exact same logic, $\exists c_2 \in (-a, a)$ s.t. $f(c_2) = 0$.
However, $c_1 \neq c_2$, so this can't work. So f' must be even and $f(0) = 0$ for f to be odd.

what is the contradiction?

QUESTION 5 (20 pts)

Answer the following questions with **True** or **False**. Write down you answers on the line at the end of each question. Justify briefly your answer in the space after the statement of the problem.

(a) Any subset of the real numbers has a supremum. (0/ 4)

This would be true only if every subset were bounded.
Is every subset of real numbers bounded? ✗
Yes, let $S \subseteq \mathbb{R}$, then S will be bounded.

What about $\mathbb{N}$?

(a) True ✗

(b) If $f(x) = 2x$ when $x \in \mathbb{Q}$ and $f(x) = -x$ if $x \notin \mathbb{Q}$, then f has a limit at $x = 1$. (4/ 4)

Define $x_n = x \in \mathbb{Q} \Rightarrow 2x_n$
$y_n = x \notin \mathbb{Q} = -y_n$ } $x_n, y_n \to x_0$

f has a limit when $2x_0 = -x_0$

for $x_0 = 1$, $2 \cdot 1 = -1 \Rightarrow 2 = -1$ contradiction

(b) False ✓

(c) The sequence $(x_n)_{n=1}^{\infty}$ defined by $x_n = (-1)^n$ has a convergent subsequence. (4/ 4)

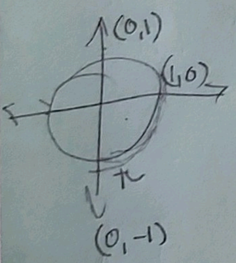
Since $(-1)^n$ oscillates between 1 and -1, there are two convergent subsequences,

$\{1, 1, 1, \ldots\}$

$\{-1, -1, -1, \ldots\}$ ✓

(c) True

(d) If f is differentiable on $(0, 2)$, if $f(1) = 1$, $f'(1) = 2$, and if $g(x) = f(x^2)\cos(\pi x)$, then $g'(1) = -2$. (3/ 4)



$g' = (f(x^2))' \cos(\pi x) + f(x^2)(-\sin(\pi x)\pi)$

$g'(1) = 2\cos(\pi) - \pi f \sin \pi)$ chain rule

$= 0 - 0$

(d) False ✓

(e) If $f : [a, b] \to \mathbb{R}$ and $g : [c, d] \to [a, b]$ are two continuous functions, then $f \circ g$ is Riemann integrable on $[a, b]$. (4/ 4)

$f(g(x))$ since f is defined on the image of g and both are continuous, then true.

$\int_a^b f(g(x))dx$

(e) True ✓