MATH-331 Introduction to Real Analysis | YOUR FULL NAME
Homework 07 | Fall 2021

Due date: December, 6th 1:20pm | Total: 48/65.

| Exercise | 1 (10) | 2 (5) | 3 (10) | 4 (5) | 5 (5) | 6 (10) | 7 (5) | 8 (5) | 9 (5) | 10 (5) |
|---|---|---|---|---|---|---|---|---|---|---|
| Score | 0 | 5 | 8 | 5 | 0 | 10 | 5 | 5 | 5 | 5 |

Table 1: Scores for each exercises

**Instructions:** You must answer all the questions below and send your solution by email (to `parisepo@hawaii.edu`). If you decide to not use LaTeX to hand out your solutions, please be sure that after you scan your copy, it is clear and readable. Make sure that you attached a copy of the homework assignment to your homework.
If you choose to use LaTeX, you can use the template available on the course website.
No late homework will be accepted. No format other than PDF will be accepted. Name your file as indicated in the syllabus.

## 1
## WRITING PROBLEMS

For each of the following problems, you will be asked to write a clear and detailed proof. All the exercises below can be solve without using the definition with partitions. Try to go back to homework 6 and use some of the exercises there to solve the following problems.

You will have the chance to rewrite your solution in your semester project after receiving feedback from me.

**Exercise 1.** (10 pts) Prove that a step function is Riemann integrable on $[a, b]$. Follow the steps below.

**a)** Let $I$ be a subinterval of $[a, b]$ and put $\phi = c\chi_I$. Prove that $\phi$ is Riemann integrable and that $\int_a^b \phi = c\ell(I)$. [There are three cases to consider: $I = [u, v]$, $I = (u, v]$, and $I = \{u\} = [u, u]$.]

**b)** Prove by induction that if $f_1, f_2, \ldots, f_n$ are Riemann integrable functions on $[a, b]$, then $f_1 + f_2 + \cdots + f_n$ is Riemann integrable and

$$\int_a^b (f_1 + f_2 + \cdots + f_n) = \int_a^b f_1 + \int_a^b f_2 + \cdots + \int_a^b f_n.$$

**c)** Write $\phi = \sum_{k=1}^n c_k \chi_{I_k}$. Use the second part of this exercise to show that $\phi$ is Riemann integrable.

**Solution:**

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**Exercise 2.** (5 pts) Suppose that $f$ is Riemann integrable on $[a, b]$ and that $f$ is nonnegative (means that $f(x) \geq 0$ for $x \in [a, b]$). Let $u, v \in \mathbb{R}$. Show that if $a \leq u < v \leq b$, then

$$\int_u^v f \leq \int_a^b f.$$

[Hint: Use the following property of the Riemann Integral multiple times: $\int_a^b f = \int_a^c f + \int_c^b f$.]

**Solution:** We have the inequality $a \leq u < v \leq b$. From this, we can rewrite what we're trying to prove as

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$$\int_a^b f = \int_a^u f + \int_u^v f + \int_v^b f$$

Since $f$ is nonnegative, that means each of these terms are greater than or equal to 0. Let's consider the cases. If both $\int_a^u f$ and $\int_v^b f$ both equal 0, then $\int_a^b f = \int_u^v f$. Now if either or both $\int_a^u f$ or $\int_v^b f$ are greater than 0, then $\int_a^b f > \int_u^v f$. Now again, we know that $\int_a^b \not< \int_u^v f$ because either or both $\int_a^u f$ or $\int_v^b f$ must be negative. However, since $f$ is nonnegative, this is impossible. Therefore, if $a \leq u < v \leq b$, then $\int_u^v f \leq \int_a^b f$. $\square$

**Exercise 3.** (10 pts) Use the Fundamental Theorem of Calculus to solve the following problems:

**a)** Suppose that $f$ is continuous on $[a, b]$ and that $f$ is nonnegative on $[a, b]$. Show that if $\int_a^b f = 0$, then $f(x) = 0$ for any $x \in [a, b]$.

**b)** Suppose that $f$ and $g$ are continuous on $[a, b]$ such that $\int_a^b f = \int_a^b g$. Show that there exists a point $c \in (a, b)$ such that $f(c) = g(c)$.

**Solution:** a) Let $c \in [a, b]$. Now from Exercise 2, we know that

$$\int_a^c f \leq \int_a^b f$$

If we let $u = a, v = c$. Now from the Fundamental Theorem of Calculus, if we have a function $F'(x) = f(x)$. Now we can turn the assumption into the following

$$\begin{aligned} \int_a^b f &= 0 \\ &= F(b) - F(a) = 0 \end{aligned}$$

Now from the inequality, we can turn it into

$$\begin{aligned} F(c) - F(a) &\leq F(b) - F(a) \\ F(c) - F(a) &\leq 0 \\ F(c) &\leq F(a) \end{aligned}$$

not true!

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From the assumption, we know that $f$ is nonnegative on $[a,b]$. This means that the function $F$ is either increasing or constant. Therefore since $c \geq a$, $F(c) \geq F(a)$. It is impossible to have that $F(c) < F(a)$. So continuing from above, $F(c)$ must equal $F(a)$. Then since $F(b) - F(a) = 0$, we also have $F(c) = F(a) = F(b)$. Now if for any $c \in [a,b]$ we have that $F(c) = F(a) = F(b)$, that means we have a constant function $F(x) = k$ for all $x \in [a,b]$ and $k = F(a)$. Now we know that $f(x) = F'(x) = (k)' = 0$. Therefore, if $f$ is continuous and nonnegative on $[a,b]$, and $\int_a^b f = 0$, then $f(x) = 0$ for any $x \in [a,b]$.

b) First, let's just look at a continuous function $h(x)$ on $[a,b]$. Let $H(x) = \int h$. Now by the Mean Value Theorem, we know that there exists $c \in [a,b]$ such that

$$\frac{H(b) - H(a)}{b - a} = H'(c)$$

By the Fundamental Theorem of Calculus, we know that this equals

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$$\frac{\int_a^b h(x)\,dx}{b - a} = h(c)$$

Now going back to this problem, $f, g$ are continuous, and so $f - g$ is also a continuous function. We can apply this previous property and get that there exists a $c \in [a,b]$ such that

$$\begin{aligned} f(c) - g(c) &= \frac{1}{b-a} \int_a^b f(x) - g(x)\,dx \\ &= \frac{1}{b-a} \cdot \left[ \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \right] \end{aligned}$$

Then by the assumption...

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$$\begin{aligned} &= \frac{1}{b-a} \cdot 0 \\ &= 0 \end{aligned}$$

So if $f(c) - g(c) = 0$, then by rearranging, we get that $f(c) = g(c)$. Therefore, there exists a $c \in [a,b]$ such that $f(c) = g(c)$. $\square$

**Exercise 4.** (5 pts) Let $f$ be a continuous function on $[a,b]$. Prove that there exists a number $c \in [a,b]$ such that $f(c)(b-a) = \int_a^b f$.

**Solution:** We know $f$ is continuous. Let $F = \int f$. Since $f$ is continuous, we know that $F$ is also continuous, and we can apply the MVT. By doing this, we know that there exists $c \in [a,b]$ such that

$$\frac{F(b) - F(a)}{b - a} = F'(c)$$

By the Fundamental Theorem of Calculus, we know that $\int_a^b f = F(b) - F(a)$. Plugging this in, we get

$$\frac{\int_a^b f}{b-a} = f(c)$$
$$\Rightarrow \int_a^b \cancel{f} = f(c)(b-a)$$

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Therefore, there exists a number $c \in [a, b]$ such that $f(c)(b-a) = \int_a^b f$. ✓ $\square$

**Exercise 5.** (5 pts) Suppose that $f$ is Riemann integrable on $[a, b]$ and is strictly increasing there. Prove that there exists a point $c \in (a, b)$ such that

$$\int_a^b f = f(a)(c-a) + f(b)(b-c).$$

[Hint: Define the function $g(x) = f(a)(x-a) + f(b)(b-x)$. Show that $\int_a^b f$ is between the numbers $f(a)(b-a)$ and $f(b)(b-a)$ and use the Intermediate Value Theorem.]

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**Solution:**

## 2 Homework problems

Answer all the questions below. Make sure to show your work.

**Exercise 6.** (10pts)

**a)** Show that the function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & , x \in \mathbb{Q} \\ 0 & , x \notin \mathbb{Q} \end{cases}$$

is not Riemann integrable on $[0, 1]$. [Hint: Use exercise 4 from Homework 6.]

**b)** Define the two functions $g : [0, 1] \to \mathbb{R}$ and $h : [0, 1] \to \mathbb{R}$ by $g = \chi_{(0,1]}$ and

$$h(x) = \begin{cases} 0 & , x \notin \mathbb{Q} \\ \frac{1}{q} & , x = p/q \in \mathbb{Q}. \end{cases}$$

Use the first part to show that $g \circ h$ is not Riemann integrable on $[0, 1]$. What can you say about the composition of two Riemann integrable functions in light of this last examples?

**Solution:** a) First, let's define two sequences of tagged partitions such that

$$\lim_{n\to\infty} ||P_{1_n}|| = 0$$
$$\lim_{n\to\infty} ||P_{2_n}|| = 0$$

And getting more specific, let's define each partition to be

$$P_1 = \{(c_i, [x_{i-1}, x_i]) : c_i \in \mathbb{Q}\}$$
$$P_2 = \{(c_i, [x_{i-1}, x_i]) : c_i \notin \mathbb{Q}\}$$

Now from Exercise 4 from Homework 6, we know that if $f$ is Riemann Integrable, then

$$(S(f, P_{1_n}))_{n=1}^{\infty} \to \int_a^b f$$
$$(S(f, P_{2_n}))_{n=1}^{\infty} \to \int_a^b f$$

This property holds true if $f$ is Riemann Integrable. Therefore, it is enough to show that the property above is false, to show that $f$ is indeed not Riemann Integrable. First, let's start with $P_1$, and let $n = \text{card}(P_1)$.

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$$\begin{aligned}(S(f, P_{1_n}))_{n=1}^{\infty} &= \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}\right) \\ &= \lim_{n\to\infty}\left(\sum_{i=1}^{n}(x_i - x_{i-1})\right) \\ &= \lim_{n\to\infty}(x_n - x_0) \\ &= 1 - 0 \\ &= 1\end{aligned}$$

Now for $P_2$, letting $n = \text{card}(P_2)$, we have the same setup.

$$\begin{aligned}(S(f, P_{2_n}))_{n=1}^{\infty} &= \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1})\right) \\ &= \lim_{n\to\infty}\left(\sum_{i=1}^{n} 0 \cdot (x_i - x_{i-1})\right) \\ &= \lim_{n\to\infty}(0) \\ &= 0\end{aligned}$$

So we have that ✓

$$(S(f, P_{1_n}))_{n=1}^{\infty} \to 1$$
$$(S(f, P_{2_n}))_{n=1}^{\infty} \to 0$$

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$\int_0^1 f$ cannot equal both 1 and 0, therefore by contradiction, $f$ must not be Riemann Integrable.

b) Let's dissect the function $g \circ h$. If $x \in \mathbb{Q}$, we have that $h(x) = 1/q$, and we know that $1/q \in (0,1]$ so $g(1/q) = 1$. If $x \notin \mathbb{Q}$, then $h(x) = 0$. Then since $0 \notin (0,1]$, we have that $g(0) = 0$. We also know $x$ must either be in $\mathbb{Q}$ or not, so these are the only two possibilities for $x$. So summarizing these properties, let's define a function $f$ such that $f = g \circ h$. Then we have that

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$$f(x) = \begin{cases} 1 & , x \in \mathbb{Q} \\ 0 & , x \notin \mathbb{Q} \end{cases}$$

This is the same function as the last exercise, which we've already proven is not Riemann Integrable. Therefore the composition $g \circ h$ is not Riemann Integrable. This shows that even if we have two Riemann Integrable functions, that doesn't necessarily mean that their composition is also Riemann Integrable. $\square$

**Exercise 7.** (5 pts) Show that if $f$ is continuous on $[a,b]$, then $|f|$ is Riemann integrable on $[a,b]$ and

$$\left| \int_a^b f \right| \leq \int_a^b |f|.$$

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[Hint: There is a clever way to show that $|f|$ is Riemann integrable without using the definition with the partitions.]

**Solution:** If $f$ is continuous on $[a,b]$, we know that it is also Riemann Integrable on $[a,b]$. Then from here, say we have this setup:

✓

$$-|f| \leq f \leq |f|$$

Then from a previous homework, we can take the integral of each part and know that it remains true.

$$-\int_a^b |f| \leq \int_a^b f \leq \int_a^b |f|$$
$$\left| \int_a^b f \right| \leq \int_a^b |f|$$

**Exercise 8.** (5 pts) Find $f'(x)$ if $f(x) = \displaystyle\int_{\sqrt{x}}^{\sqrt[3]{x}} \frac{1}{1+t^3}\, dt$ where $x \in [0,1]$.

**Solution:** Let's first rewrite $f(x)$ and solve from there.

$$\begin{aligned} f(x) &= \int_{\sqrt{x}}^{c} \frac{1}{1+t^3}\, dt + \int_c^{\sqrt{x}} \frac{1}{1+t^3}\, dt \\ &= -\int_c^{\sqrt{x}} \frac{1}{1+t^3}\, dt + \int_c^{\sqrt{x}} \frac{1}{1+t^3}\, dt \end{aligned}$$

Now we can take the derivative of each term.

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$$\begin{aligned}
f'(x) &= -\left[\frac{1}{1+x^{3/2}}\right]\cdot(\sqrt{x})' + \left[\frac{1}{1+x}\right]\cdot(\sqrt[3]{x})' \\
&= -\left[\frac{1}{1+x^{3/2}}\right]\cdot\frac{1}{2x^{1/2}} + \left[\frac{1}{1+x}\right]\cdot\frac{1}{3x^{2/3}} \\
&= -\frac{1}{2x^{1/2}+x^2} + \frac{1}{3x^{2/3}+3x^{5/3}}
\end{aligned}$$

**Exercise 9.** (5 pts) Find a function $f : [1,\infty) \to \mathbb{R}$ such that $f(1) = 0$ and $f'(x) = 1 + \sin(x^2)$ for all $x > 1$.

**Solution:** First let's integrate $f'(x)$.

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$$\begin{aligned}
f(x) &= \int 1 - \sin(x^2)\,dx \\
&= \int 1\,dx - \int \sin(x^2)\,dx \\
&= x - \int \sin(x^2)\,dx + C
\end{aligned}$$

Now it is known that

$$\int \sin(x^2)\,dx = \frac{\sqrt{\pi}S\left(\frac{\sqrt{2}}{\sqrt{\pi x}}\right)}{\sqrt{2}}$$

This is a special function. So far we have

$$f(x) = x - \frac{\sqrt{\pi}}{\sqrt{2}}S\left(\sqrt{\frac{2}{\pi}}x\right) + C$$

okay. Thanks.

From here, it's given that $f(1) = 0$, so we can actually solve for $C$.

$$\begin{aligned}
0 &= 1 - \sqrt{\frac{\pi}{2}}S\left(\sqrt{\frac{2}{\pi}}\right) + C \\
C &= \sqrt{\frac{\pi}{2}}S\left(\sqrt{\frac{2}{\pi}}\right) - 1
\end{aligned}$$

Now plugging back in for $C$, we have

$$f(x) = x - \sqrt{\frac{\pi}{2}}S\left(\sqrt{\frac{2}{\pi}}x\right) + \sqrt{\frac{\pi}{2}}S\left(\sqrt{\frac{2}{\pi}}\right) - 1$$

**Exercise 10.** (5 pts) By thinking the following sum as a Riemann sum, evaluate

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{k^2+n^2}.$$

**Solution:** Lets start with the general fact we know, that

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$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x \cdot f(x_i)$$
$$\Delta x = \frac{b-a}{n}$$
$$x_i = a + \Delta x \cdot i$$

Now from here, we can have

$$\frac{n}{n^2+k^2} = \frac{1}{n} \cdot \frac{1}{1+(k/n)^2}$$

Thus we can assign $\Delta x = 1/n$ and $f(x_i) = \frac{1}{1+(k/n)^2}$ and have $x_i = k/n$. This also makes sense because

$$\begin{aligned} x_i &= a + \Delta x \cdot k \\ &= 0 + \frac{1}{n} \cdot k \\ &= \frac{k}{n} \end{aligned}$$

Now applying everything, we have

$$\begin{aligned} \lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{n^2+k^2} &= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n} \cdot \frac{1}{1+(k/n)^2} \\ &= \int_0^1 \frac{1}{1+x^2}\,dx \end{aligned}$$

We know the limits of integration are from 0 to 1 because we know

$$\begin{aligned} \Delta x &= \frac{b-a}{n} \\ \frac{1}{n} &= \frac{b-a}{n} \\ 1 &= b-a \\ a &= 0 \\ b &= 1 \end{aligned}$$

Now going back,

$$\begin{aligned} \int_0^1 \frac{1}{1+x^2}\,dx &= \arctan(1) - \arctan(0) \\ &= \frac{\pi}{4} - 0 \\ &= \frac{\pi}{4} \end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{k^2+n^2} = \frac{\pi}{4}$$