MATH-331 Introduction to Real Analysis — Jeselle-Paige Guillermo

Homework ~~05~~ 06 — Fall 2021

Due date: November, ~~22th 1:20pm~~ 24th 5:00pm — Total: 37/65.

Exercise	1 (10)	2 (10)	3 (5)	4 (5)	5 (5)	6 (10)	7 (5)	8 (5)	9 (5)	10 (5)
Score	8	8	5	5	1	2	3	0	0	5

Table 1: Scores for each exercises

Instructions: You must answer all the questions below and send your solution by email (to `parisepo@hawaii.edu`). If you decide to not use LaTeX to hand out your solutions, please be sure that after you scan your copy, it is clear and readable. Make sure that you attached a copy of the homework assignment to your homework.
If you choose to use LaTeX, you can use the template available on the course website.
No late homework will be accepted. No format other than PDF will be accepted. Name your file as indicated in the syllabus.

1 — Writing problems

For each of the following problems, you will be asked to write a clear and detailed proof. You will have the chance to rewrite your solution in your semester project after receiving feedback from me.

Exercise 1. (10 pts)

a) Fix any $\delta > 0$ and let $[a, b]$ be an interval with $a < b$. Find a tagged partition $\mathcal{P}$ of $[a, b]$ such that $\|\mathcal{P}\| < \delta$.

b) Suppose that f is Riemann integrable. Show that in the definition of the Riemann integral, the number L is unique. [Remark: This is why we gave it the name $\int_a^b f$.]

Exercise 2. (10 pts) Suppose that f and g are Riemann integrable on the interval $[a, b]$.

a) Show that $\int_a^b (f + g) = \int_a^b f + \int_a^b g$.

b) Show that if $f(x) \leq g(x)$ for any $x \in [a, b]$, then $\int_a^b f \leq \int_a^b g$.

Exercise 3. (5 pts) Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable on $[a, b]$ and suppose that $|f(x)| \leq M$ for all $x \in [a, b]$. Show that $\int_a^b f \leq M(b - a)$.

Exercise 4. (5 pts) Suppose that f is Riemann integrable on $[a, b]$. Let $(\mathcal{P}_n)_{n=1}^{\infty}$ be a sequence of tagged partitions of $[a, b]$ such that the sequence $\lim_{n\to\infty} \|\mathcal{P}_n\| = 0$. Prove that the sequence $(S(f, \mathcal{P}_n))_{n=1}^{\infty}$ converges to $\int_a^b f$.

Exercise 5. (5 pts) Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Suppose that f is Riemann integrable on $[a, c]$ for any $c \in (a, b)$. Show that f is Riemann integrable on $[a, b]$. [Hint: Use the Cauchy criterion for integrals.]

2

Homework problems

Answer all the questions below. Make sure to show your work. When we are asking to show that a function is Riemann integrable on an interval $[a, b]$, you must use the definition or the properties of the Riemann integral presented in sections 6.1 and 6.2 respectively.

Exercise 6. (10pts)

a) Define the function $f : [a, b] \to \mathbb{R}$ by $f(x) = k$ for every $x \in [a, b]$ where $k \in \mathbb{R}$ is a fixed constant. Show that f is Riemann integrable on $[a, b]$ and that $\int_a^b k\,dx = k(b - a)$.

b) Let $f(x) = \sin^2(x)$ where $x \in [a, b]$ and assume that the function $g(x) := \cos(kx)$ is integrable on $[a, b]$ for any $k \in \mathbb{R}$. Show that f is Riemann integrable on $[a, b]$.

Exercise 7. (5 pts) Show that the function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} 1 & , \text{ if } 0 \leq x < 1/2 \\ 0 & , \text{ if } 1/2 \leq x \leq 1 \end{cases}$$

is Riemann integrable on $[0, 1]$.

Exercise 8. (5 pts) Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = 1$ if $x = 1/n$ where $n \in \mathbb{N}$, and by $f(x) = 0$ if $x \neq 1/n$, $n \in \mathbb{N}$. Show that f is Riemann integrable on $[0, 1]$.

Exercise 9. (5 pts) Show that the function $f : [0, 1] \to \mathbb{R}$ defined by $f(x) = 0$ if $x \neq 0$ and $f(x) = 4$ if $x = 0$ is Riemann integrable on $[0, 1]$.

Exercise 10. (5 pts) Let $\mathcal{P}$ be the following tagged partition of $[-1, 2]$:

$$\begin{aligned}\mathcal{P} := \{&(-9, [-1, -.8]), (-.7, [-.8, -.3]), (-.1, [-.3, 0]), (.2, [0, 0.2]), (.2, [.2, .4]), (.8, [.4, 1]), \\ &(1.42, [1, 1.5]), (1.9, [1.5, 2])\}.\end{aligned}$$

Find another partition $\mathcal{P}_0$ such that $\|\mathcal{P}_0\| \leq \|\mathcal{P}\|/3$.

Exercise 1

a) Fix any $\delta > 0$ and let $[a,b]$ be an interval with $a < b$. Find a tagged partition P of $[a,b]$ such that $\|P\| < \delta$

b) Suppose that f is Riemann integrable. Show that in the definition of the Riemann Integral, the number L is unique. [Remark: this is why we gave it the name ${}_a\int^b f$]

a) Given $a < b \Rightarrow b-a > 0$

$\Rightarrow \exists n \in \mathbb{N}$ st $\frac{b-a}{n} < \delta$

consider P a partition of $[a,b]$ defined $\{a = x_0, x_1, \dots, x_{n-1}, x_n = b\}$ where

$x_i \leftarrow P_i = a + \frac{i(b-a)}{n}$

norm. $\leftarrow P_1 - a = \frac{b-a}{n} < \delta$

$P_k - P_{k-1} = a + \frac{k(b-a)}{n} - a - \frac{(k-1)(b-a)}{n}$

$= \frac{k(b-a)}{n} - \frac{(k-1)(b-a)}{n}$

$= \frac{b-a}{n}$

$< \delta$

$\|P\| < \delta$ ✓

okay.

Can be clarified.

8/10

4/5

b) f is integrable in $[a,b]$ iff $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. for each P where $\|P\| < \delta$ then $|S(f,P) - L| < \varepsilon$

We know $L = {}_a\int^b f(x)$

Goal: show that L is unique

Proof:

Assume L_1 & L_2 are RI of f on $[a,b]$ values for the R.I.

Let $\varepsilon > 0$. Then for each $i = 1,2$ $\exists \delta_i > 0$ s.t.

$\|P\| < \delta_j \Rightarrow |\beta - L_j| < \frac{\varepsilon}{2}$ what is this? Is it the Riemann Sum.

Take $\delta = \min\{\delta_1, \delta_2\}$

$0 \le |L_1 - L_2| \le |\beta - L_1| + |\beta - L_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

Since $\varepsilon > 0$ was arbitrary

$0 \le |L_1 - L_2| < \varepsilon$

$\therefore |L_1 - L_2| = 0 \Rightarrow L_1 = L_2$

4/5

Exercise 2

Suppose that f and g are Riemann Integrable on the interval $[a,b]$.

a) Show that ${}_a\int^b (f+g) = {}_a\int^b f + {}_a\int^b g$

b) Show that if $f(x) \le g(x)$ for any $x \in [a,b]$, then ${}_a\int^b f \le {}_a\int^b g$

a) Let $I(f) := {}_a\int^b f$ and $I(g) := {}_a\int^b g$

Let $\varepsilon > 0$ Then there exist some $\delta > 0$ s.t. if P is s.t. $\|P\| < \delta$, then $|S(f,P) - I(f)| < \varepsilon$.

$|\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) - I(f)| \le \varepsilon$ and $|\sum_{i=1}^{n} g(c_i)(x_i - x_{i-1}) - I(g)| \le \varepsilon$

this should be true

If $P = \{x_0, x_1, \dots, x_n\}$ is a partition of $[a,b]$ and $\|P\| < \delta$ and $c_i \in [x_{i-1}, x_i]$ for $i = 1,2,\dots,n$. Then,

$|\sum_{i=1}^{n} (kf)(c_i)(x_i - x_{i-1}) - kI(f)| = |k| |\sum_{i=1}^{n} (f)(c_i)(x_i - x_{i-1}) - I(f)| \le |c|\varepsilon$

Hence kf is integrable on $[a,b]$ and ${}_a\int^b (kf)(x) = k {}_a\int^b f(x)$

try to use the notations in the lecture notes: $S(f,P)$ Riemann sum of f.

Moreover, $|\sum_{i=1}^{n} (f+g)(c_i)(x_i - x_{i-1}) - [I(f) + I(g)]|$

$\le |\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) - I(f)| + |\sum_{i=1}^{n} g(c_i)(x_i - x_{i-1}) - I(g)|$

$\le 2\varepsilon$ ✓

8/10

Therefore $f+g$ is integrable on $[a,b]$ and ${}_a\int^b (f+g) = {}_a\int^b f + {}_a\int^b g$ ☐

4/5 Can be clarified with an appropriate notation.

b) let $h = g - f$ be integrable on $[a,b]$

Since $h(x) \ge 0$ for all $x \in [a,b]$ then $L(h,P) > 0$ for any partition P of $[a,b]$

Hence ${}_a\int^b h = L(h) \ge 0$ → How do you know that $\int h \ge 0$? You are using the hypothesis for that.

Then, we see ${}_a\int^b g - {}_a\int^b f = {}_a\int^b h \ge 0$

So, ${}_a\int^b g \ge {}_a\int^b f$

okay.

4/5

Exercise 3

Let $f:[a,b]\to\mathbb{R}$ be Riemann integrable on $[a,b]$ and suppose that $|f(x)|\le M$ for all $x\in[a,b]$. Show that ${}_a\int^b f \le M(b-a)$

Let $\mathcal{P}=\{c_i,[x_{i-1},x_i]\}$ be a tagged partition of $[a,b]$

$|S(f,\mathcal{P})| = |\sum_{i=1}^{n} f(c_i)(x_i-x_{i-1})|$

$\le \sum_{i=1}^{n}|f(c_i)||(x_i-x_{i-1})|$

$\le M|(x_i-x_{i-1})|$ → Keep the summation symbol: $\sum_{i=1}^{n}(x_i-x_{i-1})$.

$\le M(b-a)$

5/5

Since f is Riemann integral

$|{}_a\int^b f(x)-S(f,\mathcal{P})| < \varepsilon$ good. be careful. You have to take $\mathcal{P}$ s.t. $\|\mathcal{P}\|<\delta$.

$|{}_a\int^b f(x)| \le |{}_a\int^b f(x)-S(f,\mathcal{P})+S(f,\mathcal{P})|$

$\le |{}_a\int^b f(x)-S(f,\mathcal{P})|+|S(f,\mathcal{P})|$

$\le \varepsilon + M(b-a)$

Since $\varepsilon>0$ is arbitrary

${}_a\int^b f(x) \le M(b-a)$

Exercise 4

Suppose that f is Riemann Integrable on $[a,b]$.
Let $(\mathcal{P}_n)_{n=1}^{\infty}$ be a sequence of tagged partitions of $[a,b]$ such that the sequence $\lim_{n\to\infty}\|\mathcal{P}_n\|=0$. Prove that the sequence $(S(f,\mathcal{P}_n))_{n=1}^{\infty}$ converges to ${}_a\int^b f$

f is Riemann Integrable on $[a,b]$

$\Rightarrow {}_a\int^b f = L$

Consider for each $n\in\mathbb{N}$ $x_n=S(f,\mathcal{P}_n)$

$\Rightarrow$ We have the sequence x_n ✓

Now we show that $x_n\to L$ or $\lim_{n\to\infty} x_n = L$

For $\varepsilon>0$ $\exists N\in\mathbb{N}$ s.t. $|x_n-L|<\varepsilon$ $\forall n\ge N$

Since f is R.I. given $\varepsilon>0$ $\exists\delta>0$ s.t. if $\mathcal{P}$ is a tagged partition of $[a,b]$ with $\|\mathcal{P}\|<\delta$ then

$|S(f,\mathcal{P})-L|<\varepsilon$

5/5

Given $\lim_{n\to\infty}\|\mathcal{P}_n\|=0 \Rightarrow \|\mathcal{P}_n\|\to 0$

there exist an N s.t. $\|\mathcal{P}_n\|<\delta$ $\forall n\ge N$

So,

$|S(f,\mathcal{P}_n)-L|<\varepsilon$

$|x_n-L|<\varepsilon$

$\lim_{n\to\infty} x_n = L$ ✓

$\lim_{n\to\infty} S(f,\mathcal{P}_n) = {}_a\int^b f$

$S(f,\mathcal{P}_n)\to {}_a\int^b f$

Exercise 5

Let $f:[a,b]\to\mathbb{R}$ be a bounded function. Suppose that f is Riemann integrable on $[a,c]$ for any $c\in(a,b)$. Show that f is Riemann integrable on $[a,b]$. [Hint: use the cauchy criterion for integrals]

1/5

let $P:=\{a=x_0<x_1<\ldots<c<\ldots<x_n=b\}$ → this is not a tagged partition. You need to define your tags.

$S(P,f)=\sum_{i=1}^{n} f(c_i)(x_i-x_{i-1})$

$|{}_a\int^b f-S(P,f)|=|{}_a\int^b f-\sum_{i=1}^{n} f(c_i)(x_i-x_{i-1})|$

$<\varepsilon/3$?? → How can you obtain that?

$|{}_a\int^c f-S(P_2,f)|=|{}_a\int^c f-\sum_{i=1}^{n} f(c_i)(x_i-x_{i-1})|$

$<\varepsilon/3$

let $M=\sup|f(x)|$

$|{}_a\int^c f-{}_a\int^b f|\leq|{}_a\int^b f-S(P,f)|+|S(P,f)-S(P_2,f)|+|{}_a\int^c f-S(P_2,f)|$

$\leq\frac{2\varepsilon}{3}+|f(c_i)|(a-x_i)$

$\leq\frac{2\varepsilon}{3}+M(a-x_i)$

$<\varepsilon$

→ This doesn't prove that f is Riemann integrable. You used in the beginning. that f is R.I.

$\lim_{c\to b}{}_a\int^c f={}_a\int^b f$

$\therefore$ f is R.I on $[a,b]$

Exercise 6

a) define the function $f:[a,b]\to\mathbb{R}$ by $f(x)=k$ for every $x\in[a,b]$ where $k\in\mathbb{R}$ is a fixed constant. show that f is Riemann Integral on $[a,b]$ and that ${}_a\int^b k\,dx=k(b-a)$

2/10

b) Let $f(x)=\sin^2(x)$ where $x\in[a,b]$ and assume that the function $g(x):=\cos(kx)$ is integrable on $[a,b]$ for any $k\in\mathbb{R}$. show that f is Riemann Integrable on $[a,b]$

a) $f:[a,b]\to\mathbb{R}$ be a constant function $\ni f(x)=c\ \forall x\in[a,b]$

let $\{a=x_0,x_1,\ldots,x_{n-1},x_n=b\}$ be a partition of $[a,b]$ → This is not a partition. You need to define the tags!

By the extreme value theorem $\exists u_j,s_j\in[x_{j-1},x_j]$ s.t.

$\sup\{f(x): x\in[x_{j-1},x_j]\}=f(s_j)$

$\inf\{f(x): x\in[x_{j-1},x_j]\}=f(u_j)$

2/5

Now define

$h=\sum_{j=1}^{N} f(s_j)\ x\in[x_{j-1},x_j]$

$h(x_j-x_{j-1})=\sum_{i=1}^{N} k(x_j-x_{j-1})$

$=k(x_n-x_0)$

$=k(b-a)$

$g=\sum_{j=1}^{N} f(u_j)\ x\in[x_{j-1},x_j]$

$g(x_j-x_{j-1})=\sum_{i=1}^{N} k(x_j-x_{j-1})$

$=k(x_n-x_0)$

$=k(b-a)$

So, ${}_a\int^b f=f(s_j)=f(u_j)=k(b-a)$

${}_a\int^b f={}_a\int^b k=k(b-a)$

There is simpler than this.

$S(f,P)=\sum_{i=1}^{n} f(c_i)(x_i-x_{i-1})$

$=k\sum_{i=1}^{n}(x_i-x_{i-1})=k(b-a)$

So, $|S(f,P)-k(b-a)|=0<\varepsilon$.

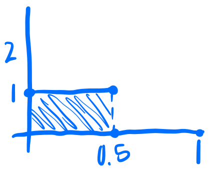
0/5 b) $f(x)=\sin^2 x$ is continuous on $[a,b]$ so f is riemann integrable → You have to use the definition or the properties from the section 6.2.

Exercise 7

Show that the function $f:[0,1]\to\mathbb{R}$ defined by

$$f(x) := \begin{cases} 1, & \text{if } 0 \le x < 1/2 \\ 0, & \text{if } 1/2 \le x \le 1 \end{cases}$$

is Riemann integrable on $[0,1]$



$\int_0^1 f = 1/2$

let $\varepsilon > 0$

Suppose $\delta > 0$ and $\mathcal{P}$ is a tagged partition st $\|\mathcal{P}\| < \delta$

let $\mathcal{P}_1 = \{(c_i, [x_{i-1}, x_i]) : c_i \in [0, 1/2)$

$\mathcal{P}_2 = \{(c_i, [x_{i-1}, x_i]) : c_i \in [1/2, 1]$

We have $S(f,\mathcal{P}) = S(f,\mathcal{P}_1) + S(f,\mathcal{P}_2)$

Now $N_1 = \mathrm{card}(\mathcal{P}_1)$

$S(f,\mathcal{P}_1) = \sum_{i=1}^{N_1} f(c_i)(x_i - x_{i-1}) = 1 \sum_{i=1}^{N_1} (x_i - x_{i-1}) = x_{N_1}$

and $N_2 = \mathrm{card}(\mathcal{P}_2)$

3/5

$S(f,\mathcal{P}_2) = 0(x_{N_1+N_2} - x_{N_1}) = 0$

so $S(f,\mathcal{P}) = x_{N_1} + 0 = x_{N_1}$

Since $\|\mathcal{P}\| < \delta$

then $x_{N_1} < \delta$ not true! imagine if x_{N_1} close to $\frac{1}{2}$. Then for any $\delta < \frac{1}{2}$, $x_{N_1} > \delta$.

$S(f,\mathcal{P}) = x_{N_1} < \delta$

$S(f,\mathcal{P}) < \delta$ ✗

Choose $\delta = \varepsilon$

Then $|S(f,\mathcal{P}) - 1/2| < \varepsilon$

Exercise 8

Let $f:[0,1]\to\mathbb{R}$ be defined by $f(x)=1$ if $x = 1/n$ where $n\in\mathbb{N}$ and by $f(x)=0$ if $x \ne 1/n$, $n\in\mathbb{N}$. Show that f is Riemann integrable on $[0,1]$

0/5

$$f(x) = \left\{ \begin{matrix} 1 & x = 1/n \\ 0 & x \ne 1/2 \end{matrix} \right\} \text{ in } [0,1]$$

Exercise 9

Show that the function $f:[0,1]\to\mathbb{R}$ defined by $f(x)=0$ if $x\neq 0$ and $f(x)=4$ if $x=0$ is Riemann Integrable on $[0,1]$

0/5

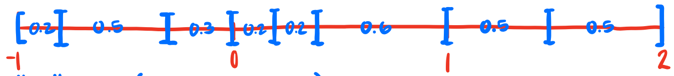
Exercise 10

Let P be the following tagged partition of $[-1,2]$

$$P:=\{(-.9,[-1,-.8]),(-0.7,[-0.8,-0.3]),(-0.1,[-0.3,0]),(0.2,[0,0.2]),(0.2,[0.2,0.4]),(0.8,[0.4,1])(1.42,[1,1.5]),(1.9,[1.5,2])\}$$

Find another partition P_0 s.t. $\|P_0\|\leq\|P\|/3$

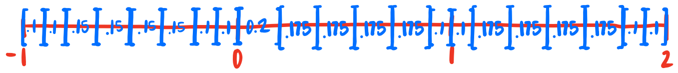
5/5



$\|P\| = \max\{x_i - x_{i-1} : i=1,2,\dots,N\} = 0.6$

$\|P\|/3 = 0.6/3 = 0.2$

$$P_0:=\{(-.95,[-1,-.9]),(-.85[-.9,-.8])(-.75[-.8,-.65])(0.65,[-.65,-.5]),(-.45,[-.5,-.35])(-.25,[-.35,-.2]),(-.1,[-.2,-.1]),(-.05,[-.1,0]),(0.1,[0,0.2]),(0.3,[0.2,0.375])(0.5,[0.375,.55])(0.6,[.55,.725])(0.8,[.725,.9])(.9,[.9,1]),(1.05,[1,1.1])(1.2,[1.1,1.275])(1.3,[1.275,1.45])(1.5,[1.45,1.625])(1.7,[1.625,1.8])(1.85,[1.8,1.9])(1.95,[1.9,2])\}=0.2$$



$\|P_0\| = 0.2$