Homework 1

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Homework Problems

Exercise 1

Prove that for any $n \in \mathbb{N}, 1+2+...+n = \frac{n(n+1)}{2}$

Proof by Induction:
Let $P(n) = 1+2+...+n = \frac{n(n+1)}{2}$
For $n = 1$,

$$1 = \frac{1(1+1)}{2} = \frac{2}{2} = 1$$

Therefore, $P(n)$ is true for $n = 1$

Assume $P(n)$ is true, we must prove $P(n+1)$. So,

$$\begin{aligned}
P(n+1) = 1+2+...+n+(n+1) &= \frac{(n+1)((n+1)+1)}{2} \\
&= \frac{(n+1)(n+2)}{2} \\
&= \frac{n^2+2n+n+2}{2} \\
&= \frac{n(n+1)+2(n+1)}{2} \\
&= \frac{n(n+1)}{2} + \frac{2(n+1)}{2} \\
&= \frac{n(n+1)}{2} + (n+1)
\end{aligned}$$

Exercise 2

Define $f : \mathbb{N} \to \mathbb{N}$ by $f(1) = 1, f(2) = 2$ and $f(3) = 3$ and
$f(n) := f(n-1) + f(n-2) + f(n-3)$ for $(n \geq 4)$
Prove that $f(n) \leq 2^{n-1}$ for all $n \in \mathbb{N}$

Proof by Induction:
Let $n = 1$, Then

$$\begin{aligned} f(1) &= 1 \leq 2^{1-1} \\ f(1) &= 1 \leq 1 \end{aligned}$$

Therefore, the result is true for $n = 1$

Assume the result is true for $n = k$ and $f(k) \leq 2^{k-1}$
Let $n = k + 1$, then

$$\begin{aligned} f(k+1) &= f(k+1-1) + f(k+1-2) + f(k+1-3) \\ &= f(k) + f(k-1) + f(k-2) \\ &< 2^{k-1} + 2^{k-2} + 2^{k-3} \\ &= 2^k \cdot 2^{-1} + 2^k \cdot 2^{-2} + 2^k \cdot 2^{-3} \\ &= 2^k \cdot \frac{1}{2} + 2^k \cdot \frac{1}{4} + 2^k \cdot \frac{1}{8} \\ &= 2^k(\frac{1}{2} + \frac{1}{4} + \frac{1}{8}) \\ &= 2^k(\frac{7}{8}) \\ f(k+1) &< 2^k \\ &= 2^{(k+1)-1} \end{aligned}$$

Therefore, the results are true for $n = k + 1$
Thus true for all $n \in \mathbb{N}$
Hence, $f(n) \leq 2^{n-1}$ for all $n \in \mathbb{N}$

Exercise 3

Prove that if A, B and C are sets, then
a) $A \sim A$
If $A \sim A$, then there is a $1-1$ function f from A onto A.
So , $1_A(a_1) = 1_A(a_2)$,
then $a_1 = 1_A(a_1) = 1_A(a_2) = a_2$
Since for and $a \in A$, $1_A(a) = a$
Thus, $A \sim A$.

b) If $A \sim B$, then $B \sim A$
If $A \sim B$, then there is a $1-1$ function f from A onto B. To show $B \sim A$, then there is a $1-1$ function g from B onto A.
f^-1 is $1-1$ so domf^-1=im$f$=B, and imf^-1=dom$f$=A. Therefore, $B \sim A$

c) If $A \sim B$ and $B \sim C$, then $A \sim C$
Assuming $A \sim B$ and $B \sim C$, there is a $1-1$ function f from A onto B and there is a $1-1$ function g from B onto C.
If f and g are $1-1$, then $g \circ f$ is $1-1$
The $\text{dom}(g \circ f) = A$ and $\text{im}(g \circ f) = C$ so, there is a $1-1$ function $g \circ f$ from A onto C, and $A \sim C$

Exercise 4

Show that any subset of a countable set is countable.

Let A be a countable set and B be a subset of A
Case 1
If A is finite, then B is also finite, because every subset of a finite set is finite.
Thus B is countable.
Case 2
If A is infinite and countable, then $A = \{a_1, a_2, a_3 ...\}$
(i) If B is finite, B is countable.
(ii) If B is infinite, then $B = \{a_{n_1}, a_{n_2}, a_{n_3} ...\}$ where $n_1 < n_2 < n_3 ....$
Meaning $f : \mathbb{N} \to B$ by $f(k) = a_{n_k} \forall k \in \mathbb{N}$ Thus, B is countable.

Exercise 5

Let $0 < a < b$ be positive real numbers. Prove that
a) $a^2 < b^2$
Consider $0 < a < b$,
Then $a^2 < ab$ and $ab < b^2$
Therefore, $0 < a^2 < b^2$

b) $\sqrt{a} < \sqrt{b}$
Proof By Contradiction:
Suppose $\sqrt{a} \geq \sqrt{b}$, consider $0 < a < b$

Case 1: $\sqrt{a} = \sqrt{b}$
Then, $\sqrt{a}\sqrt{a} = \sqrt{a}\sqrt{b}$
and $\sqrt{a}\sqrt{b} = \sqrt{b}\sqrt{b}$
Thus, $\sqrt{a}\sqrt{a} = \sqrt{b}\sqrt{b}$
and $a = b$ which is a contradiction

Case 2: $\sqrt{a} > \sqrt{b}$
Then, $\sqrt{a}\sqrt{a} > \sqrt{a}\sqrt{b}$
and $\sqrt{a}\sqrt{b} > \sqrt{b}\sqrt{b}$
Thus, $\sqrt{a}\sqrt{a} > \sqrt{b}\sqrt{b}$
and $a > b$ which is a contradiction

Therefore, $\sqrt{a} < \sqrt{b}$

Exercise 6

Sketch the region of the points (x, y) satisfying the following relation:
$x + |x| = y + |y|$ (explain your answer). **Last page of PDF**

Exercise 7

If $x \geq 0$ and $y \geq 0$, prove that $\sqrt{xy} \leq \frac{x+y}{\sqrt{2}}$

Since $x \geq 0$ and $y \geq 0$, then $\sqrt{x} \geq 0$ and $\sqrt{y} \geq 0$
Case 1: $\sqrt{x} \geq \sqrt{y}$
$\sqrt{x} - \sqrt{y} \geq 0$
$(\sqrt{x} - \sqrt{y})^2 \geq 0$
$(\sqrt{x})^2 + (\sqrt{y})^2 - 2\sqrt{x}\sqrt{y} \geq 0$
$x + y - 2\sqrt{x}\sqrt{y} \geq 0$
$x + y \geq 2\sqrt{xy}$
$\frac{x+y}{2} \geq \sqrt{xy}$

Case 2: $\sqrt{x} \leq \sqrt{y}$
$\sqrt{x} - \sqrt{y} \leq 0$
$(\sqrt{x} - \sqrt{y})^2 \geq 0$
SAME AS ABOVE. Hence, proved the given inequality.

Exercise 8

Find the infimum and supremum (if they exist) of the following sets. Make sure to justify all your answers:
a) $E := \{x \in \mathbb{R} : x \geq 0 \text{ and } x^2 \leq 9\}$
Then, $x \geq 0$ and $|x| \leq \sqrt{9} \to -3 < x < 3$
$infE = 0$ and $supE = 3$

b) $E := \{\frac{4n+5}{n+1} : n \in \mathbb{N}\}$ The lowest n can be is If $n = 1$ then,

$$\frac{4(1)+5}{(1)+1} = \frac{9}{2}$$

Therefore, $infE = \frac{9}{2}$
Let $x = supE$ we want to show that $x = 4$
There are 3 cases:
(i) $x < 4$
In AP, $(4 - x) > 0$

$$\begin{aligned} n(4-x) &> x-5 \\ 4n - xn &> x-5 \\ 4n+5 &> x+xn \\ 4n+5 &> x(n+1) \\ \frac{4n+5}{n+1} &> x \quad \# \end{aligned}$$

(ii) $x > 4$
This is impossible, since we are assuming 4 is the supremum
Therefore, the $supE = 4$

Writing Problems

Exercise 9

Let A be a non-empty set and $P(A)$ be its power set (the family of all subsets of A). Prove that A is not equivalent to $P(A)$. Deduce that $P(\mathbb{N})$ is not countable. [Hint: Define $C := \{x : x \in A \text{ and } x \notin f(x)\}$.]

Goal: Prove that A is not equivalent to $P(A)$.
Consider a fuction $f : A \to P(A)$
Let $C := \{x : x \in A \text{ and } x \notin f(x)\}$ We must prove f is not surjective
Let's assume f is surjective
Then every element in $P(A)$ has a pre-image in A
Meaning for $C \in P(A), \exists a \in A$ so that $f(a) = C$

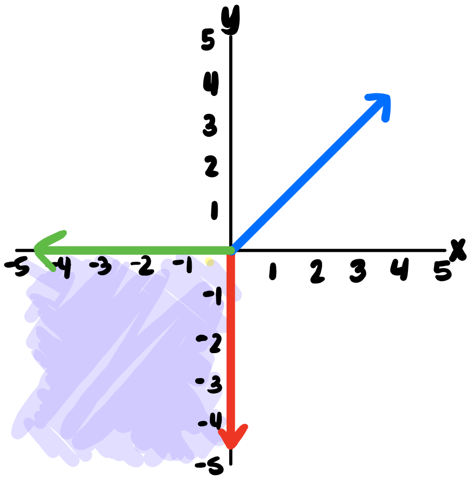
Suppose $a \in C$, so from the def of C, $a \notin f(a)$.
But since $f(a) = C$ then $a \notin C$ #
Suppose $a \notin C$, so from the def of C, $a \in f(a)$.
But since $f(a) = C$ then $a \in C$ #
Therefore, f is not surjective and A is not equivalent to $P(A)$.

Exercise 10

Let $E \subseteq \mathbb{R}$ be bounded from above and $E \neq \emptyset$. For $r \in \mathbb{R}$, let
$rE := \{rx : x \in E\}$ and $r + E := \{r + x : x \in E\}$
Show that
a) if $r > 0$, then $\sup(rE) = r\sup(E)$
Define $rE := \{rx : x \in E\}$ and $r > 0$.
$(rE) = rx_1, rx_2, ..., rx_n \forall x \in E$
$\sup(rE) = r(x_n)$
Lets say set $E = \{x_1, x_2, ..., x_n\}$
If every number in set E is multiplied by r, then $r\sup(E) = r(x_n)$.
This shows $\sup(rE) = r(x_n)$=rsup(E)

b) if $r \in \mathbb{R}$, then $\sup(r + E) = r$+$\sup(E)$
Define $r + E := \{r + x : x \in E\}$ and $r > 0$.
$(r + E) = r + x_1, r + x_2, ..., r + x_n \forall x \in E$
$\sup(r + E) = r + x_n$
Lets say set $E = \{x_1, x_2, ..., x_n\}$
Similarly to 10a, If r is added to every number in set E, then
r+$\sup(E) = r + x_n$.
This shows $\sup(r + E) = r + x_n = r$+$\sup(E)$

Homework #1 : Exercise #6



$x + |x| = y + |y|$

$x, y \geq 0$

$$x + |x| = y + |y|$$
$$2x = 2y$$
$$x = y$$

$x \geq 0$ and $y \leq 0$

$$x + |x| = -y + |-y|$$
$$2x = 0$$
$$x = 0$$

$x \leq 0$ and $y \geq 0$

$$-x + |-x| = y + |y|$$
$$0 = 2y$$
$$0 = y$$

$x, y \leq 0$

$$-x + |-x| = -y + |-y|$$
$$0 = 0$$

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