

Chapter 6 : Integral

Suggested Exercises

MATH-331 Introduction to Real Analysis
Pierre-Olivier Parisé

You will find some suggested exercises related to the Riemann integral. Try to solve all of them or, at least, understand the solutions of all the problems in this document.

6.1

DEFINITION

Exercise 1. Let $f(x) = x^2 + x$ defined on the interval $[-1, 2]$. Find the Riemann sum of f corresponding to the following tagged partition of $[-1, 2]$:

$$\mathcal{P} := \{(-9, [-1, -.8]), (-.7, [-.8, -.3]), (-.1, [-.3, 0]), (.2, [0, 0.2]), (.2, [.2, .4]), (.8, [.4, 1]), (1.42, [1, 1.5]), (1.9, [1.5, 2])\}.$$

Exercise 2. Let \mathcal{P} be the tagged partition of the last exercise. Find a tagged partition \mathcal{P}_1 of $[-1, 2]$ such that $\|\mathcal{P}_1\| \leq \|\mathcal{P}\|/3$.

Exercise 3. Prove that the number L in the definition of the Riemann integral is unique.

Exercise 4. Suppose that $f(x) = k$ for all $x \in [a, b]$ where $k \in \mathbb{R}$ is a constant. prove that f is Riemann integrable on $[a, b]$ and $\int_a^b f = k(b - a)$.

Exercise 5. Prove that the function $f : [0, 1] \rightarrow \mathbb{R}$ defined by

$$f(x) := \begin{cases} 0 & , \text{ if } x \neq 0 \\ 4 & , \text{ if } x = 0 \end{cases}$$

is Riemann integrable on $[0, 1]$.

Exercise 6. Prove that the function $h(x) = x$ is Riemann integrable on $[0, 1]$.

6.2

PROPERTIES

Exercise 7. Let $f : [a, b] \rightarrow \mathbb{R}$ be Riemann integrable on $[a, b]$ and suppose that $|f(x)| \leq M$ for any $x \in [a, b]$. Prove that $|\int_a^b f| \leq M(b - a)$.

Exercise 8. Let $f : [a, b] \rightarrow \mathbb{R}$ and $g : [a, b] \rightarrow \mathbb{R}$ be two Riemann integrable functions on $[a, b]$.

a) Show that $f + g$ is Riemann integrable and that $\int_a^b (f + g) = \int_a^b f + \int_a^b g$.

b) Show that if $f(x) \leq g(x)$ for all $x \in [a, b]$, then $\int_a^b f(x) \leq \int_a^b g(x)$.

Exercise 9. Prove that the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 1$ if $x \in \mathbb{Q}$ and $f(x) = 0$ if $x \notin \mathbb{Q}$ is not Riemann integrable on $[0, 1]$.

Exercise 10. Suppose that f is Riemann integrable on $[a, b]$. Let $(\mathcal{P}_n)_{n=1}^\infty$ be a sequence of tagged partitions of $[a, b]$ such that $\|\mathcal{P}_n\| \rightarrow 0$ as $n \rightarrow \infty$. Prove that the sequence $(S(f, \mathcal{P}_n))_{n=1}^\infty$ of Riemann sums converges to $\int_a^b f$.

Exercise 11. Suppose that $f : [a, b] \rightarrow \mathbb{R}$ is a bounded function and that f is Riemann integrable on $[a, c]$ for each $c \in (a, b)$. Prove that f is Riemann integrable on $[a, b]$.

Exercise 12. Prove that if $f : [a, b] \rightarrow \mathbb{R}$ is Riemann integrable on $[a, b]$ and if $g : [a, b] \rightarrow \mathbb{R}$ is a function such that g differed from f only at one point, then g is Riemann integrable and $\int_a^b f = \int_a^b g$.

6.3

TYPES OF RIEMANN INTEGRAL FUNCTIONS

Exercise 13. Consider the step function $\phi : [0, 3] \rightarrow \mathbb{R}$ defined by

$$\phi = 4\chi_{[0,1]} - 3\chi_{(1,1.5)} + 2\chi_{[1.5,2.5]} + \chi_{(2.5,3]}.$$

Exercise 14. Let $\phi : [a, b] \rightarrow \mathbb{R}$ be a step function and suppose that $\psi : [a, b] \rightarrow \mathbb{R}$ differs from ϕ in a finite number of points. Is ψ a step functions? Explain.

Exercise 15. Let $f : [a, b] \rightarrow \mathbb{R}$ be an increasing function and let \mathcal{P} be a tagged partition of $[a, b]$. Suppose that \mathcal{P}_1 is another tagged partition of $[a, b]$ with exactly the same intervals as \mathcal{P} , only tags are different. Prove that

$$|S(f, \mathcal{P}) - S(f, \mathcal{P}_1)| \leq |f(b) - f(a)| \|\mathcal{P}\|.$$

Exercise 16. Consider the function $f : [0, 6] \rightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} \sin x & , \text{ if } 0 \leq x \leq 3 \\ 10 - x & , \text{ if } 3 < x < 4 \\ 1/x & , \text{ if } 4 \leq x \leq 6. \end{cases}$$

Explain carefully why f is Riemann integrable on $[0, 6]$.

Exercise 17. Suppose that f is continuous and nonnegative on $[a, b]$. Show that if $\int_a^b f = 0$, then $f(x) = 0$ for every $x \in [a, b]$. Give an example to show that the nonnegative hypothesis is necessary.

Exercise 18. Suppose that f and g are continuous on $[a, b]$ and that $\int_a^b f = \int_a^b g$. Prove that there exists a point $c \in [a, b]$ such that $f(c) = g(c)$.

Exercise 19. Find the derivative of the function $F(x) = \int_{-x^4}^{x^2} t^2 \sin(t^2) dt$.

Exercise 20. Evaluate $\int_{1/9}^1 \frac{\sin(\pi\sqrt{t}/2)}{\sqrt{t}} dt$.

Exercise 21. Give an example of a function such that the derivative is not Riemann Integrable.

Exercise 22. Suppose that f is continuous and nonnegative on $[a, b]$ and that $\int_a^b f = 0$. Use the Fundamental Theorem of Calculus to prove that $f(x) = 0$ for every $x \in [a, b]$.

Exercise 23. Suppose that f and g are continuous functions on $[a, b]$ and that $\int_a^b f = \int_a^b g$. Use the Fundamental Theorem of Calculus to prove that there exists a point $c \in [a, b]$ such that $f(c) = g(c)$.

Exercise 24. Find a function $f : [0, 1] \rightarrow \mathbb{R}$ that is not Riemann integrable on $[0, 1]$ and a Riemann integrable function $g : [0, 1] \rightarrow \mathbb{R}$ such that $g \circ f$ is Riemann integrable.

Exercise 25. Suppose that f and g are differentiable functions on $[a, b]$ and that f' and g' are Riemann integrable on $[a, b]$. Prove that $f'g$ and fg' are Riemann integrable on $[a, b]$ and that

$$\int_a^b f'g = f(b)g(b) - f(a)g(a) - \int_a^b fg'.$$

What is the name of this integration formula?

Exercise 26. Use the previous exercise to compute $\int_0^{\pi/3} x \sin x dx$.

Exercise 27. Let $g : [a, b] \rightarrow [c, d]$ be differentiable on $[a, b]$ and let $f : [c, d] \rightarrow \mathbb{R}$ be continuous on $[c, d]$. Suppose that g' is Riemann integrable on $[a, b]$. Prove that $(f \circ g)g'$ is Riemann integrable on $[a, b]$ and that

$$\int_a^b (f \circ g)g' = \int_{g(a)}^{g(b)} f.$$

What is the name of this integration formula?

Exercise 28. Use the previous exercise to evaluate $\int_0^3 \frac{6}{\sqrt{x}(1+x)} dx$.

MISCELLANEOUS PROBLEMS

Exercise 29. Let f be a continuous function on $[a, b]$. Prove that there exists a point $c \in [a, b]$ such that $f(c)(b - a) = \int_a^b f$. [This result is often called the Mean Value Theorem for integrals.]

Exercise 30. Suppose that f is a continuous function on $[a, b]$ and that g is a nonnegative Riemann integrable on $[a, b]$. Prove that there exists $c \in [a, b]$ such that $f(c) \int_a^b g = \int_a^b fg$. Show that this result is a generalization of the previous exercise.

Exercise 31. Suppose that $f : [a, b] \rightarrow \mathbb{R}$ is monotone. Prove that there exists a point $c \in (a, b)$ such that

$$\int_a^b f = f(a)(c - a) + f(b)(b - c).$$