Homework 1

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September 2021

Homework Problems

Exercise 1

At $n = 1, 1 = \frac{1(1+1)}{2} = 1$, so the claim holds. Assume that the claim holds at n, so that $1 + 2 + \ldots + n = \frac{n(n+1)}{2}$. We can then add $n+1$ to both sides to get $1+2+\ldots+n+(n+1) = \frac{n(n+1)}{2} + (n+1)$, where we can simplify the right hand side into $\frac{n(n+1)}{2} + \frac{2(n+1)}{2}$. This further simplifies to $\frac{n(n+1)+(2(n+1))}{2}$, which finally simplifies to $\frac{(n+2)(n+1)}{2}$. Now we have $1+2+\ldots+n+(n+1) = \frac{(n+2)(n+1)}{2}$, which is the $n+1$ case, therefore the claim $1+2+\ldots+n = \frac{n(n+1)}{2}$ is true for all $n \in \mathbb{N}$.

Exercise 2

First note that $f(n) = 1 \leq 2^{1-1} = 1, f(2) = 2 \leq 2^{2-1} = 2, f(3) = 3 \leq 2^{3-1} = 4, f(4) = 1 + 2 + 3 = 6 \leq 2^{4-1} = 8$, proving our base case. Then assume that $f(n) \leq 2^{n-1}$ for all $n \in \mathbb{N}$. Note that $f(n+1) = f(n)+f(n-1)+f(n-2)$, which is less than $2f(n)$ as $f(n) + f(n-1) + f(n-2) \leq 2f(n) = f(n) + f(n-1) + f(n-2) + f(n-3)$. Then note that because of the assumption in our inductive step, $f(n) \leq 2^{n-1}$, and if we multiply 2 to both sides, we get $2(n) \leq 2^n$. It follows that $f(n+1) \leq 2f(n) \leq 2^n$, which by the second order axiom implies that $f(n+1) \leq 2^n$, proving that $f(n) \leq 2^{n-1}$ for all $n \in \mathbb{N}$.

Exercise 3

a. A has a bijection with itself as you can simply map every element on A to itself, therefore $A \sim A$.

b. Since $A \sim B$, they have a bijective function $f : A \to B$. Then whenever $a \neq a'$, then $f(a) \neq f(a')$, and the range of f is all of B. We can see that there is also a function $f^{-1} : B \to A$ that is bijective, and it is the inverse of f. This is because when $f(a) \neq f(a')$, then $f^{-1}(f(a)) \neq f^{-1}(f(a'))$ for $f(a) \neq f(a')$, proving that f^{-1} is injective. We can also see that it's surjective as f(a) is injective, meaning that every element in the set had a distinct output in B, and since f^{-1} converts those outputs back to the inputs, the range of f^{-1} will be all of A, proving that it's surjective. Therefore f^{-1} is bijective and $B \sim A$.

c. Since $A \sim B$ they have a bijective function $f : A \to B$, and since $B \sim C$, they also have a bijective function $g : B \to C$. Let $h : A \to C = g(f(n))$. Assume towards a contradiction that h is not injective, so there exists $h(a) = h(a')$ for $a! = a'$. This can be rewritten as $g(f(a)) = g(f(a'))$. Since g is injective, $g(f(a)) = g(f(a'))$ implies $f(a) = f(a')$, and since f is injective, $f(a) = f(a') \to a = a'$, contradicting our claim and proving that h is injective. h will also be injective as since f is surjective, then its range is all of B, and since g is also surjective, the range of g is all of C. It follows that since the range of f is all of B, which makes up the domain of g, and the range of g is all of h, the range of f is also all of C, proving that h is surjective. Therefore since h is injective and surjective, it is bijective, meaning that there is a bijective function $h : A \to C$, meaning that $A \sim C$ if $A \sim B$ and $B \sim C$.

Exercise 4

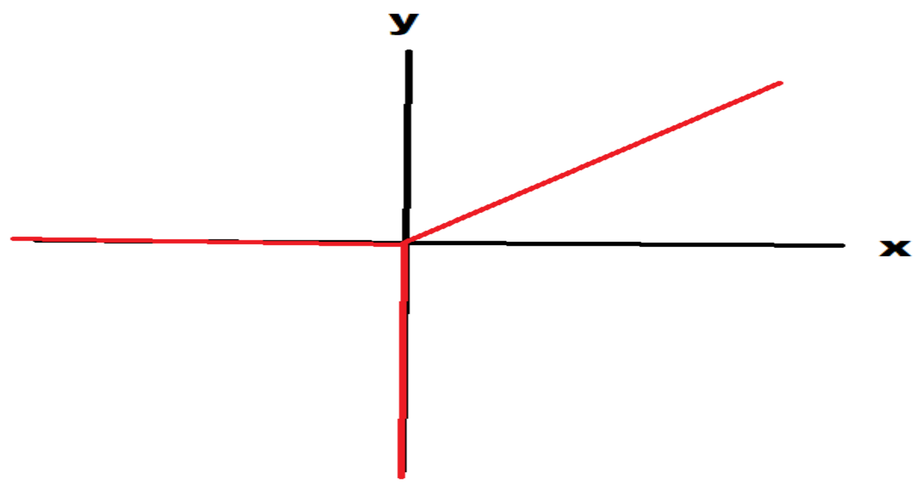
A countable set, A, has a bijective function f with a set of natural numbers. Then let B be a subset of A. Since B is a subset of A, we'll be able to have a function g that maps values of B to values of A, and this function will be injective. Since the composition of injective functions is injective, $f(g(a))$ is injective, meaning that we can map any unique value of B to a unique natural number, which means that B is countable.

Exercise 5

a. Since $a < b$, and a, b are positive real numbers, by order axiom 04, $a^2 < ab$. Similarly, $ab < b^2$. Since $a^2 < ab < b^2$, by order axiom 02, $a^2 < b^2$.

b. Since $a < b$, and a, b are positive real numbers, we can simplify $a < b$ into $a - b < 0$. This is the same as $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) < 0$. We can then divide both sides by $(\sqrt{a}+\sqrt{b})$ to get $\sqrt{a}-\sqrt{b} < 0$, which simplifies to $\sqrt{a} < \sqrt{b}$.

Exercise 6



When $x < 0, y < 0$, $|x| = -x, |y| = -y$, so $x - x = y - y = 0$. Therefore negative x, and negative y have a value of 0. When $x < 0, y \geq 0, |x| = -x, |y| = y$, so $x - x = y + y$ which simplifies to $0 = 2y$. Therefore negative x values have 0 y value. When $x \geq 0, y < 0, |x| = x, |y| = -y$, so $x + x = y - y$, which simplifies to $2x = 0$, so negative y values have an x value of 0. When $x \geq 0, y \geq 0, |x| = x, |y| = y$, so $x + x = y + y$, which simplifies to $2x = 2y$ into $x = y$. So for positive x and y values, the graph has a slope of 1.

Exercise 7

If we square both sides, we get $xy \leq \frac{(x+y)(x+y}{2} = \frac{x^2+y^2+2xy}{2}$. We can then multiply both sides by 2 to get $2xy \leq x^2 + y^2 + 2xy$. Since $x \geq 0$ and $y \geq 0$, it follows that $x^2 \geq 0$ and $y^2 \geq 0$, which means $2xy \leq 2xy + x^2 + y^2$.

Exercise 8

a. First note that since in 5b we have shown $\sqrt{a} < \sqrt{b}$ if $a < b$, we can take x^2 as a, and 9 as b. Therefore since $x^2 \leq 9$, we have $x \leq 3$. Now we have $x \geq 0$

and $x \leq 3$. Since $0, 3 \in \mathbb{R}$, we have $inf(E) = 0$ and $sup(E) = 3$.

b. Note that $\frac{4n+5}{n+1}$ for $n \in \mathbb{N}$ is actually a decreasing function. Therefore the smallest value for n in $\mathbb{N}$ will result in $sup(E)$. Since $\frac{4(1)+5}{1+1} = \frac{9}{2}$, we have $sup(E) = \frac{9}{2}$. I will claim that 4 is the infimum. First note that 4 is a lower bound, as $\frac{4n+5}{n+1} < 4$ is a contradiction as $\frac{4n+5}{n+1} < 4$ implies $4n + 5 < 4(n + 1)$ which implies $4n + 5 < 4n + 4$, which is wrong as $n \in \mathbb{N}$. Now let $x = inf(E)$. We know that either $x < 4$, $x = 4$, or $x > 4$. $x \not< 4$, as otherwise it wouldn't be the infimum since 4 is a lower bound. If $x > 4$, then $x - 4 > 0$. We can then use AP, with $x - 4$ for x, and $5 - x$ for y. Therefore by AP $\exists n \in \mathbb{N}$ such that $n(x - 4) > 5 - x$. This can be simplified to $xn - 4n > 5 - x$ to $xn + x > 4n + 5$ to $x(n + 1) > 4n + 5$ to finally $x > \frac{4n+5}{n+1}$. This is a contradiction as we assumed x is the infimum of E. Therefore it has to be the case that $x = 4$, so $inf(E) = 4$.

Writing problems

Exercise 9

Let us assume that there is an bijective function, $f : A \to P(A)$. Let $C := \{x : x \in A, x \notin f(x)\} \subseteq A$. Since $C \in A$ implies $C \in P(A)$, and f is a bijection, there must be some $y \in A$ such that $f(y) = C$. There are then two possibilities, $y \in C$, or $y \notin C$. If $y \in C$, then by the definition of C, $y \notin f(y)$, but since $f(y) = C$ this implies $y \notin C$, so we have a contradiction. If $y \notin C$, then by the definition of C, $y \in f(y)$, but since $f(y) = C$, this implies $y \in C$, so we have a contradiction. Therefore there is no bijection between A and $P(A)$, so $A \neq P(A)$. Since we have shown that a set has no bijection with its power set, it also follows that the set of all natural numbers, $\mathbb{N}$, has no bijection with its power set, $P(\mathbb{N})$, which means that $P(\mathbb{N})$ is not countable.

Exercise 10

a. Let $a = sup(E)$, by the definition of supremum, for all $x \in E$, $a \geq x$. Therefore we have $x \leq a, \forall x \in E$. Since $r > 0$, by order axioms, we have $rx \leq ar, \forall x \in E$, so ar is an upper bound for rE. Let $y = sup(rE)$. Therefore we have $rx \leq y, \forall x \in E$. Since $r > 0$, we can multiply both sides by $\frac{1}{r}$ to get $x \leq \frac{y}{r}, \forall x \in E$, so $\frac{y}{r}$ is an upper bound for E. We want to show that $ar = y$. Since $a, y, r \in \mathbb{R}$, we have either $ar > y, ar < y, ar = y$. $ar < y$ is impossible as y is the supremum, and ar is an upper bound of rE. Assume $ar > y$. Therefore since $r > 0$, $a > \frac{y}{r}$, which is impossible as a is $sup(E)$, and $\frac{y}{r}$ is an upper bound of E. Therefore it has to be the case that $ar = y$, so $rsup(E) = sup(rE)$.

b. Let $a = sup(E)$, by the definition of supremum, for all $x \in E$, $a \geq x$. Therefore we have $x \leq a, \forall x \in E$. Since $r \in \mathbb{R}$, by order axioms, we have

$r+x \leq r+a, \forall x \in E$, so $r+a$ is an upper bound for $r+E$. Let $y = sup(r+E)$. Therefore we have $r+x \leq y, \forall x \in E$. Since $r \in mathbbR$, we can add $-r$ to both sides to get $x \leq y-r, \forall x \in E$, so $y-r$ is an upper bound for E. We want to show that $r+a = y$. Since $a, r, r \in \mathbb{R}$, we have either $r+a > y$, $r+a = y$, or $r+a < y. r+a < y$ is impossible as $y = sup(r+E)$, and $r+a$ is an upper bound of $r+E$. Assume that $r+a > y$. Therefore since $r \in \mathbb{R}$, $r+a-r > y-r$, or $a > y-r$. However this is impossible as $a = sup(E)$, and $y-r$ is an upper bound of E. Therefore it has to be the case that $r+a = y$, so $r + sup(E) = sup(r+E)$.