HW 8

a) Fix any $\delta > 0$ and let $[a,b]$ be an interval w/ $a<b$. Find a tagged partition $\mathcal{P}$ of $[a,b]$ s.t. $\|\mathcal{P}\| < \delta$.

We want $\frac{b-a}{n} < \delta \Leftrightarrow \underbrace{b-a}_{y} < \underbrace{n\delta}_{x}$

AP. $\forall x > 0, \forall y \in \mathbb{R}, \exists N \in \mathbb{N}$

Let $\mathcal{P} := \{(c_i, [x_{i-1}, x_i]) : i = 1, \ldots, n\}$

$$\tau_i = \begin{cases} x_i = a + i\left(\frac{b-a}{n}\right) \\ x_0 = a \\ x_1 = a + \frac{b-a}{n} \\ x_2 = a + 2\left(\frac{b-a}{n}\right) \end{cases}$$

So, $x_n = a + n\left(\frac{b-a}{n}\right) = b$

and, $x_i = a + i\frac{b-a}{n}$

$\Rightarrow x_i - x_{i-1} = \cancel{\alpha} + i\frac{b-a}{n} - \left(\cancel{\alpha} + (i-1)\frac{b-a}{n}\right)$

$0 < |L_1 - L_2| < \varepsilon$ $\quad = i\frac{b-\cancel{a}}{n} - i\frac{b-\cancel{a}}{n} + \frac{b-a}{n}$

$$\lim_{x\to 0} 0 \leq \lim_{x\to \delta} |L_1 - L_2| \leq \lim_{x\to \delta} x \to 0 \leq |L_1 - L_2| \leq 0 \to L_1 = L_2$$

b) Suppose that f is R.I. Show in the def of RI that the number L is unique

f is R.I. on $[a,b]$ if $\exists L$ st. for every $\varepsilon > 0$, there is a $\delta > 0$ st. $\|\mathcal{P}\| < \delta$ implies $|\sigma - L| < \varepsilon$ where σ is the Riemann Sum of f over the part. $\mathcal{P}$ of $[a,b]$. L is the RI of f over $[a,b]$, $\int_a^b f(x)\,dx = L$

Proof: Assume that L_1 and L_2 are the RI's of f over $[a,b]$.

Goal: show that $L_1 = L_2$.

Let $\varepsilon > 0$. For each $i = 1, 2$ $\exists \delta_i > 0$ s.t. $\|\mathcal{P}\| < \delta_i \Rightarrow |\sigma - L_i| < \frac{\varepsilon}{2}$ when $\mathcal{P}$ is a partition of $[a,b]$

cont'd. →

HW 6 Cont'd.

1b.) cont'd. Take $\delta := \min[\delta_1, \delta_2]$. Fix a partition P of $[a,b]$.
suppose $\|P\| < \delta$. $\delta \leq \delta_i$ for $i = 1, 2$
Thus, $0 \leq |L_1 - L_2| \leq |\sigma - L_1| + |\sigma - L_2| < \varepsilon$
Since $\varepsilon > 0$ is arbitrary, $0 \leq |L_1 - L_2| < \varepsilon$ is true for all $\varepsilon > 0$. Therefore, $|L_1 - L_2| = 0$, and $L_1 = L_2$.
Thus, L is unique.

2.) Suppose f and g are R.I. on $[a,b]$

a.) Show that $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.

since $L(f) = U(f)$ for all RI fct's.

Since f and g are both RI on $[a,b]$, they are both continuous functions. Since f and g are RI, we can write $\int_a^b f$ and $\int_a^b g$ as the lower/upper integrals $L(f) = U(f)$.

so, $\int_a^b f + \int_a^b g = L(f) + L(g) = U(f) + U(g)$

$\Rightarrow U(f+g) \leq U(f) + U(g)$ $\boxed{\text{or } \int_a^b (g+f) \leq \int_a^b f + \int_a^b g}$

$\int_a^b (g+f) \geq \int_a^b f + \int_a^b g =$

and $U(f+g) \geq L(f) + U(g)$ since $L(f) = U(f)$ and $L(g) = U(g)$

$\Rightarrow \int_a^b f + \int_a^b g = \int_a^b (g+f)$

HW 6 Cont'd

3.) Let $f:[a,b]\to\mathbb{R}$ be Riemann Int. on $[a,b]$ and suppose that $|f(x)|\le M$ $\forall x\in[a,b]$ show that $\int_a f \le M(b-a)$

Let $y,x\in[a,b]$ and $y\neq x$ \$ $\varepsilon>0$. Since f is R.I. f is bounded on $[a,b]$ by M. Then $|f(y)-f(x)| = |\int_a^y f - \int_a^x f|$

$$= |\int_a^y f + \int_x^a f| = |\int_x^y f|$$

$\Rightarrow |\int_x^y f| \le M|y-x|$

Since x,y are arbitrary pt's in $[a,b]$, it should also follow that

$$\int_a^b f \le M(b-a)$$

4.) Suppose that f is RI on $[a,b]$. Let $(\mathcal{P}_n)_{n=1}^{\infty}$ be a seq. of t.p.'s of $[a,b]$ s.t. the seq. $\lim_{n\to\infty}\|\mathcal{P}_n\|=0$. Prove that the seq. $(S(f,\mathcal{P}_n))_{n=1}^{\infty}$ converges to $\int_a^b f$.

For all $\varepsilon>0$, $\exists\,\delta>0$ such that $\|\mathcal{P}\|<\delta$, then $|S(f,\mathcal{P})-\int_a f|<\varepsilon$.

5.) Let $f:[a,b]\to\mathbb{R}$ be a bounded fct. Suppose that f is Riemann integrable on $[a,c]$ for any $c\in(a,b)$. Show that f is RI on $[a,b]$.

f is RI on $[a,c]$ so $\forall\varepsilon>0$, $\exists\,\delta>0$ s.t. if $\|\mathcal{P}\|<\delta$ in $[a,c]$, then $|S(f,\mathcal{P})-\int_a^c f|<\varepsilon$.

Let $\mathcal{P}_1$ \$ $\mathcal{P}_2$ be tagged part's of $[a,b]$, and let $c\in(a,b)$ st. $b-c<\varepsilon$. Using the cauchy criteria, we have that if $\mathcal{P}_{1a}$ \$ $\mathcal{P}_{2a}$ are t.p's of $[a,c]$, \$ $\|\mathcal{P}_1\|<\delta_\varepsilon$ \$ $\|\mathcal{P}_2\|<\delta_\varepsilon \Rightarrow |S(f,\mathcal{P}_{1a})-S(f,\mathcal{P}_{2a})|<\varepsilon$

? Since $|S(f,\mathcal{P}_{1b})|<M(b-c)<M\cdot\varepsilon$, $|S(f,\mathcal{P}_{2b})|<M(b-c)<M\cdot\varepsilon$

then $|S(f,\mathcal{P}_1)-S(f,\mathcal{P}_2)| = |S(f,\mathcal{P}_{1a})+S(f,\mathcal{P}_{1b})-S(f,\mathcal{P}_{2a})-S(f,\mathcal{P}_{2b})|$.

!Since $|S(f,\mathcal{P}_{ib})|<M(b-c)<M\cdot\varepsilon$ ~~and $c\in[a,b]$~~, this shows that if f is RI on $[a,c]$ where $c\in[a,b]$, it must also be RI on $[a,b]$.

HW 6 Cont'd

6. $f: [a,b] \to \mathbb{R}$ $f(x)=k$ for every $x \in [a,b]$ where $k \in \mathbb{R}$.

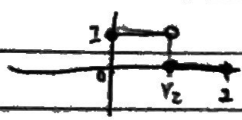
a.) Show that f is RI on $[a,b]$ and that $\int_a^b k\,dx = k(b-a)$

f is RI on $[a,b]$ since it is bounded and continuous.

We know from a th'm in class (FTC) that if G is an antiderivative for f on $[a,b]$, then $\int_a^b f = G(b) - G(a)$. Since $f(x)$ is a constant k, we have that $\int_a^b k\,dx = kx\Big|_a^b = k(b) - k(a) = k(b-a)$.

b.) Let $f(x) = \sin^2 x$ where $x \in [a,b]$ and assume the fct. $g(x) := \cos(kx)$ is integrable on $[a,b]$ for any $k \in \mathbb{R}$. Show that f is RI on $[a,b]$

7.) Show the fct $f: [0,1] \to \mathbb{R}$ defined by $f(x) = \begin{cases} 1, & \text{if } 0 \le x < \frac{1}{2} \\ 0, & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$ is RI on $[0,1]$



steps: show for every partition P with $\|P\| \to 0$

$f(x) = 0$ at discontinuity, so it is ok ✓

bounded + cont. (exception at $x = \frac{1}{2}$, $y = 0$)