MATH-331 Introduction to Real Analysis — Created by P.-O. Parisé
Homework 03 — Fall 2021

Due date: October 11th 1:20pm — Total: /70.

Exercise	1 (5)	2 (5)	3 (5)	4 (5)	5 (10)	6 (10)	7 (5)	8 (5)	9 (5)	10 (10)
Score										

Table 1: Scores for each exercises

Instructions: You must answer all the questions below and send your solution by email (to parisepo@hawaii.edu). If you decide to not use LaTeX to hand out your solutions, please be sure that after you scan your copy, it is clear and readable. Make sure that you attached a copy of the homework assignment to your homework.
If you choose to use LaTeX, you can use the template available on the course website.
No late homework will be accepted. No format other than PDF will be accepted. Name your file as indicated in the syllabus.

1 Writing problems

For each of the following problems, you will be asked to write a clear and detailed proof. You will have the chance to rewrite your solution in your semester project after receiving feedback from me.

Exercise 1. (5 pts) Let $(a_n)_{n=1}^{\infty}$ be an increasing sequence and $(b_n)_{n=1}^{\infty}$ be a decreasing sequence. Let $(c_n)_{n=1}^{\infty}$ be the sequence defined by $c_n = b_n - a_n$. Show that if $\lim_{n\to\infty} c_n = 0$, then the sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ converges and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

Solution: Suppose that $\lim_{n\to\infty} c_n = 0$. Since (a_n) is increasing, we have that $a_n \leq a_{n+1}$ for any $n \geq 1$, so $-a_n \geq -a_{n+1}$ for any $n \geq 1$. So, $(-a_n)$ is a decreasing sequence. Thus, since (b_n) is decreasing, we get, for any $n \geq 1$,

$$c_n = b_n - a_n \geq b_{n+1} - a_{n+1} = c_{n+1}.$$

Thus, (c_n) is decreasing. This implies that, if $m, n \geq 1$ such that $m \leq n$, then $b_m - a_m \geq b_n - a_n = c_n$ and taking $n \to \infty$, we get that

$$b_m - a_m \geq \lim_{n\to\infty} c_n = 0.$$

Thus, $a_m \leq b_m$ for any $m \geq 1$.

Since the sequence (b_n) is decreasing, we see that $a_m \leq b_m \leq b_1$ for any $m \geq 1$ and so (a_n) is bounded from above. Since $a_m \geq a_1$ for any $m \geq 1$, the sequence (a_n) is bounded. So by a

Theorem from the lecture notes, the sequence (a_n) converges. Also, since (a_n) is increasing, we have $b_m \geq a_m \geq a_1$ for any $m \geq$, so the sequence (b_n) is bounded from below. Since $b_m \leq b_1$ for any $m \geq 1$, we see that the sequence (b_n) is bounded. By a Theorem from the lecture notes, the sequence (b_n) is convergent.

Call A and B the limits of the sequences (a_n) and (b_n). Then, from the sum rules for limits of sequences, we have

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n = B - A.$$

Since, $\lim_{n\to\infty} c_n = 0$, then $A = B$. $\square$

Exercise 2. (5 pts) Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$, and suppose that x_0 is an accumulation point of D. Suppose that for each sequence $(x_n)_{n=1}^{\infty}$ converging to x_0 with $x_n \in D\backslash\{x_0\}$ for each $n \geq 1$, then the sequence $(f(x_n))_{n=1}^{\infty}$ is Cauchy. Show that f has a limit at x_0.

[Hint: For two sequences (x_n) and (y_n) that satisfy the assumption, define the sequence (z_n) to be $z_{2n} = x_n$ and $z_{2n-1} = y_n$. Show that $(f(z_n))$ converges and the sequence $(f(x_n))$ and $(f(y_n))$ converges to the same limit as $(f(z_n))$. Conclude by a theorem in the lecture notes.]

Solution: By the assumptions, since $(f(x_n))_{n\geq 1}$ is Cauchy, it is convergent, say to $A(x)$ where $x = (x_n)$. So the limit depends on the choice of the sequence x. We will show first that in fact, it does not depend on the choice of the sequence and the values are the same. Let $x = (x_n)_{n\geq 1}$ and $y = (y_n)_{n\geq 1}$ be two sequences satisfying the conditions of the assumptions. Then $\lim_{n\to\infty} f(y_n) = A(y)$ and $\lim_{n\to\infty} f(x_n) = A(x)$. Define a new sequence $z = (z_n)$ in the following way

- $z_{2n} = x_n$ for $n \geq 1$.
- $z_{2n-1} = y_n$ for $n \geq 1$.

Then the sequence (z_n) satifies the conditions in the assumptions and so the sequence $(f(z_n))$ is Cauchy. It does converge to some limit $A(z)$. However, the sequence (x_n) is a subsequence of (z_n). So, it must also converge to $A(z)$. Then, we must have $A(x) = A(z)$. Also, the sequence (y_n) is a subsequence of (z_n). So, it must converge to $A(z)$. Then, we must also have $A(y) = A(z)$. Thus, $A(x) = A(y)$.

So, if (x_n) is a sequence that satifies the assumptions, then $(f(x_n))$ converges to some $L \in \mathbb{R}$ which does not depend on the choice of $(x_n)_{n\geq 0}$. From a Theorem from the lecture notes, we conclude that the fonction has a limit at x_0 and $\lim_{x\to x_0} f(x) = L$. $\square$

Exercise 3. (5 pts) Prove that if $f : D \subseteq \mathbb{R} \to \mathbb{R}$ has a limit at $x_0 \in \operatorname{acc} D$, then the limit is unique.

Solution: Suppose that there are two values L_1 and L_2 with $L_1 \neq L_2$ such that $f(x) \to L_1$ and $f(x) \to L_2$ as $x \to x_0$. Take $\varepsilon := |L_1 - L_2|/2$. Then there is a $\delta_1 > 0$ and $\delta_2 > 0$ such that

- if $|x - x_0| < \delta$ and $x \in D\backslash\{x_0\}$, then $|f(x) - L_1| < |L_1 - L_2|/2$.
- if $|x - x_0| < \delta_2$ and $x \in D\backslash\{x_0\}$. then $|f(x) - L_2| < |L_1 - L_2|/2$.

let $\delta := \min\{\delta_1, \delta_2\}$ and pick a $x \in D\backslash\{x_0\}$ such that $|x - x_0| < \delta$. Then, we have

$$|L_1 - L_2| \leq |L_1 - f(x)| + |f(x) - L_2| < |L_1 - L_2|/2 + |L_1 - L_2|/2 = |L_1 - L_2|.$$

So $|L_1 - L_2| < |L_1 - L_2|$, a contradiction. So $L_1 = L_2$. $\square$

Exercise 4. (5 pts) Suppose $f : D \subseteq \mathbb{R} \to \mathbb{R}$, $g : D \subseteq \mathbb{R} \to \mathbb{R}$ and $h : D \subseteq \mathbb{R} \to \mathbb{R}$ are three functions such that

$$f(x) \leq h(x) \leq g(x) \quad (\forall x \in D).$$

Suppose that f and g have limits at x_0 with $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x)$. Prove that h has a limit at x_0 and

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} h(x) = \lim_{x\to x_0} g(x).$$

Solution: Call L the commun limit of f and g. We will use the characterization in terms of sequences of the limits of functions. Suppose (x_n) is a sequence such that $x_n \to x_0$ with $x_n \in D\backslash\{x_0\}$ for any $n \geq 1$. Then, we know that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = L$. Also, we have $f(x_n) \leq h(x_n) \leq g(x_n)$ for any $n \geq 1$. From the squeeze Theorem for sequences (see homework 2), we have that $\lim_{n\to\infty} h(x_n)$ exists and $\lim_{n\to\infty} h(x_n) = L$. Since this is true for any sequences (x_n), we conclude that $\lim_{x\to x_0} h(x) = L$. $\square$

Exercise 5. (10 pts) Let $f : (0, \infty) \to \mathbb{R}$ be a function. We say that f has a limit at ∞ if there exists a $L \in \mathbb{R}$ such that for any $\varepsilon > 0$, there is a real number $M > 0$ such that if $x > M$, then $|f(x) - L| < \varepsilon$.

a) Show that if $g : (0, \infty) \to \mathbb{R}$ is bounded and $\lim_{x\to\infty} f(x) = 0$, then $\lim_{x\to\infty} f(x)g(x) = 0$.

b) Let $a > 0$ and suppose that $f : (a, \infty) \to \mathbb{R}$ and define $g : (0, 1/a) \to \mathbb{R}$ by $g(x) = f(1/x)$. Show that f has a limit at ∞ if and only if g has a limit at 0.

Solution: **a)** Let $g : (0, \infty) \to \mathbb{R}$ be bounded. Then, there is a $N > 0$ such that $|g(x)| \leq N$ for any $x \in (0, \infty)$. Let $\varepsilon > 0$. Since f has a limit at ∞, there is a $M > 0$ such that if $x > M$, then $|f(x)| < \varepsilon/N$. Then, if $x > M$, then

$$|f(x)g(x)| \leq |f(x)|N < \varepsilon.$$

So, $\lim_{x\to\infty} f(x)g(x) = 0$.

b) Suppose f has a limit at ∞, say L. Then, there is a $M > 0$ such that if $x > M$, then $|f(x) - L| < \varepsilon$. Take $\delta := \min\{1/M, 1/a\}$. Then, if $x < \delta$, then $1/x > M$ and so

$$|g(x) - L| = |f(1/x) - L| < \varepsilon.$$

Thus, $\lim_{x\to 0} g(x) = L$.

Suppose now that $\lim_{x\to 0} g(x) = L$. Then there is a $\delta > 0$ such that $|x| < \delta$ and $0 < x < 1/a$, then $|g(x) - L| < \varepsilon$. Take $M := \frac{1}{\delta}$. Then, if $x > M$, then $\frac{1}{x} < 1/M = \delta$. This implies that

$$|f(x) - L| = |g(1/x) - L| < \varepsilon.$$

So $\lim_{x\to\infty} f(x) = L$. $\square$

Answer all the questions below. Make sure to show your work.

Exercise 6. (10pts) For each of the sequences below, determine its nature (converges or diverges)[1]:

a) (a_n) where $a_n = \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n}$.

b) (a_n) where $a_n = \frac{1+2+\cdots+n}{n^2}$.

Solution: 1. The sequence (a_n) is decreasing and bounded.

- Decreasing: We have $a_{n+1} \leq a_n$ iff
$$\sum_{k=n+1}^{2n+2} \frac{1}{k} \leq \sum_{k=n}^{2n} \frac{1}{k}$$
which is iff
$$\frac{1}{2n+1} + \frac{1}{2n+2} \leq \frac{1}{n}.$$
But, $2n+1, 2n+2 \geq 2n$ and so $\frac{1}{2n+1}, \frac{1}{2n+2} \leq \frac{1}{2n}$. This implies that
$$\frac{1}{2n+1} + \frac{1}{2n+1} \leq \frac{2}{2n} = \frac{1}{n}.$$
So, we conclude that $a_{n+1} \leq a_n$.
- Bounded: Since (a_n) is decreasing, $a_n \leq a_1 = \frac{3}{2}$. Also, since for any $k \geq n$, $\frac{1}{k} \leq \frac{1}{n}$, we get
$$\sum_{k=n}^{2n} \frac{1}{k} \geq \sum_{k=n}^{2n} \frac{1}{2n} = \frac{n+1}{n} \geq 1.$$

So, since (a_n) is decreasing and bounded, it must converge to a limit.

2. Since $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$, we get $a_n = \frac{n+1}{2n}$. The sequence (a_n) is bounded and increasing.

- Increasing: For any $n \geq 1$, we have $1/(n+1) \leq 1/n$, so
$$a_n = (n+1)/2n = \frac{1}{2} + \frac{1}{2n} \geq \frac{1}{2} + \frac{1}{2(n+1)} = a_{n+1}.$$
- Bounded: Since (a_n) is decreasing, $a_n \leq a_1 = 1$. Also, since all terms in the sequence of positive, we have $a_n \geq 0$.

[1] You don't need to compute the limit.

Since a_n is decreasing and bounded, it must converge to a limit.

Another way is to use the limit rules. We know that $\frac{n+1}{2n} = \frac{1}{2} + \frac{1}{2n}$. So, this limit must converge to $\frac{1}{2}$. $\square$

Exercise 7. (5 pts) Define $g : (0,1) \to \mathbb{R}$ by $f(x) = \frac{\sqrt{1+x}-1}{x}$. Prove that g has a limit at 0 and find it.

Solution: We have $g(x) = \frac{1}{\sqrt{x+1}+1}$. The function $x \mapsto 1$ has a limit at 0 and $x \mapsto \sqrt{x+1} - 1$ has a limit at 0 which is $\sqrt{1} + 1 \neq 0$. So, by the quotient rule, the limit of g exists at 0 and

$$\lim_{x\to 0} g(x) = \frac{\lim_{x\to 0} 1}{\lim_{x\to 0} \sqrt{1+x}+1} = \frac{1}{2}.$$

Exercise 8. (5 pts) Suppose that $f : (0,1) \to \mathbb{R}$ has a limit at $x_0 = 1$ and $\lim_{x\to 1} f(x) = 1$. Compute the value of the limit

$$\lim_{x\to 1} \frac{f(x)(1-f(x)^2)}{1-f(x)}.$$

Solution: We have $\frac{f(x)(1-f(x)^2)}{1-f(x)} = \frac{f(x)(1+f(x))(1-f(x))}{1-f(x)} = f(x)(1+f(x))$. So, by the product rule, we get

$$\lim_{x\to 1} \frac{f(x)(1-f(x)^2)}{1-f(x)} = \lim_{x\to 1} f(x)(1+f(x)) = 2.$$

Exercise 9. (5 pts) Prove that if $f : D \to \mathbb{R}$ has a limit at x_0, then $|f|(x) := |f(x)|$ has a limit at x_0.

Solution: Let $\varepsilon > 0$. Since f has a limit at x_0, say L, there is a $\delta > 0$ such that if $|x - x_0| < \delta$ and $x \in D\backslash\{x_0\}$, then $|f(x) - L| < \varepsilon$. Using the reverse triangle inequality, we have $||f(x)| - |L|| \leq |f(x) - L|$. So, if $|x - x_0| < \delta$ and $x \in D\backslash\{x_0\}$, then $||f(x)| - |L|| \leq |f(x) - L| < \varepsilon$. So, $\lim_{x\to x_0} |f(x)| = |L|$. $\square$

Exercise 10. (10 pts) Using the link between sequences and limits of functions, show the following.

a) If $f(x) = x^n$ $(n \geq 0)$, then $\lim_{x\to x_0} f(x) = x_0^n$ for any $x_0 \in \mathbb{R}$.

b) If $x_0 \in [0,\infty)$, then $\lim_{x\to x_0} \sqrt{x} = \sqrt{x_0}$.

Solution:

a) Let (x_n) be a sequence such that $x_k \to x_0$ and $x_k \in \mathbb{R}\backslash\{x_0\}$. Using induction (using the product rule), we can prove that $x_k^n \to x_0^n$. So, the limit of $(f(x_k))_{k=1}^{\infty}$ exists and is x_0^k. By a Theorem from the lecture notes, we conclude that the limit exists and must be x_0^n.

b) Let $x_0 \in [0,\infty)$ and (x_n) be a sequence such that $x_n \to x_0$ with $x_n \in [0,\infty)\backslash\{x_0\}$. Then as we see for sequences (see homework 2), we have $\sqrt{x_n} \to \sqrt{x_0}$. Since (x_n) was arbitrary, from a Theorem on characterization of limits in terms of sequences, we conclude that $\lim_{x\to x_0} \sqrt{x}$ exists and must be $\sqrt{x_0}$. $\square$

3 Bonus

Exercise 11. Assume that $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. Suppose that f has a limit at some point.

a) Show that f has a limit at every point of $\mathbb{R}$.

b) Show that either $\lim_{x\to 0} f(x) = 1$ or $f(x) = 0$ for any $x \in \mathbb{R}$.

Solution: **a)** Let L be the limit of f at the point x_0. Remark that the limit of a function f at a point x_0 can be rewritten as

$$\lim_{h\to 0} f(x_0 + h) = L$$

which means that for any $\varepsilon > 0$, there is a number $\delta > 0$ such that if $|h| < \delta$ and $h \neq 0$, then $|f(x_0 + h) - L| < \varepsilon$.

Let $h \in \mathbb{R}$. Then, $f(x_0 + h) = f(x_0)f(h)$. So, if $f(x_0) \neq 0$, we get that $f(h) = \frac{f(x_0+h)}{f(x_0)}$. We know that the limit as $h \to 0$ on the right-hand side exists and by the quotient rule, is equal to $L/f(x_0)$. So, the limit of f at 0 exists and is $L/f(x_0)$. Now, since for any $y \in \mathbb{R}$, we have $f(y+h) = f(y)f(h)$, we conclude that the limit exists at y, and the limit is equal to $Lf(y)/f(x_0)$.

b) We made the assumption that $f(x_0) \neq 0$. If $f(x_0) = 0$, then for any $x \in \mathbb{R}$, we have $f(x) = f(x - x_0)f(x_0) = 0$. Thus, f is identically zero on $\mathbb{R}$. In the case $f(x_0) \neq 0$, then we know that the limit at $x = 0$ exists and is equal to $L/f(x_0)$. Also, since $0 = 0 + 0$, we have $f(0) = f(0)^2$ and so $f(0) = 1$ (otherwise f is identically zero).

Let n be a natural number. Then $f(1) = 1$ if $n = 1$. Also, if $n = 2$, we have $f(1) = f(1/2 + 1/2) = f(1/2)^2$ and so $f(1/2) = \sqrt{f(1)}$. By induction, we can show that $f(1/n) = \sqrt[n]{f(1)}$. For sure, $f(1) > 0$, otherwise, we would have $f(1/2) = \sqrt{f(1)}$ with $f(1) < 0$ (which is impossible for real numbers since the square of any number is positive or zero). This implies that

$$\lim_{n\to\infty} f(1/n) = \lim_{n\to\infty} (f(1))^{1/n} = 1$$

by a computation we did in the lecture notes on the sequences (see last example). Since taking any sequence $h_n \to 0$ results in the same limit, we must have that

$$\lim_{x\to 0} f(x) = 1 = f(0).$$