MATH-331 Introduction to Real Analysis — LEE'AN BENDER (25450782)
Homework 02 — Fall 2021

Due date: 20-09-2021 1:20pm — Total: /70.

Exercise	1 (10)	2 (5)	3 (5)	4 (5)	5 (5)	6 (10)	7 (5)	8 (10)	9 (5)	10 (10)
Score										

Table 1: Scores for each exercises

Instructions: You must answer all the questions below and send your solution by email (to `parisepo@hawaii.edu`). If you decide to not use LaTeX to hand out your solutions, please be sure that after you scan your copy, it is clear and readable. Make sure that you attached a copy of the homework assignment to your homework. No late homework will be accepted. No format other than PDF will be accepted. Name your file as indicated in the syllabus.

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Writing problems

For each of the following problems, you will be asked to write a clear and detailed proof. You will have the chance to rewrite your solution in your semester project after receiving feedback from me.

Exercise 1. (10 pts)

a) Let $\{[a_n, b_n] : n \geq 1\}$ be a family of closed intervals such that $[a_1, b_1] \supset [a_2, b_2] \supset [a_3, b_3] \supset \cdots$. Show that there is a $c \in \mathbb{R}$ such that $c \in [a_n, b_n]$ for all $n \geq \mathbb{N}$. Follow the following steps to prove it:

- **(i)** Prove that for any $n, m \geq 1$, $a_n \leq b_m$. [hint: put $M := \max\{n, m\}$.]
- **(ii)** Show that $\sup\{a_n : n \geq 1\}$ exists.
- **(iii)** Show that $c = \sup\{a_n : n \geq 1\}$ satisfies the requirement.

b) Use this last result to prove that the set $\mathbb{R}$ is uncountable. [Hint: Show that any function $f : \mathbb{N} \to \mathbb{R}$ can't be surjective. To do so, construct a sequence of closed intervals such that $f(n) \notin [a_n, b_n]$ with $a_n < b_n$.]

Solution: -

a)

b)

Exercise 2. (5 pts) Prove that if $a_n \to A$, then $|a_n| \to |A|$.

Solution: If $A_n \to A$, then $|a_n| \to |A|$.
We want to prove that $||a_n| - |A|| < \epsilon$.
Assume $\epsilon > 0$ is arbitrary.
$\exists \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow ||a_n| - |A|| < \epsilon$
Let $\epsilon = \epsilon$
$||a_n| - |A|| \leq |a_n - A| < \epsilon$
From the definition of convergence, we know
$\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow |a_n - A| < \epsilon$
Therefore,
$||a_n| - |A|| \leq |a_n - A|$
$||a_n| - |A|| \leq \epsilon$
Since ϵ was arbitrary, $|a_n| \to |A|$ □

Exercise 3. (5 pts) Let (a_n), (b_n), and (c_n) be sequences of real numbers. Prove that if $a_n \to L$, $b_n \to L$, and $a_n \leq c_n \leq b_n$, then $c_n \to L$.

Solution: If $a_n \to L, b_n \to L$, and $a_n \leq c_n \leq b_n \Rightarrow c_n \to L$
Assume that $c_n \to C$
We know that if $a_n \to L, c_n \to C$, and $a_n \leq c_n \Rightarrow L \leq C$
This is also true for $b_n \to L, c_n \to C$, and $c_n \leq b_n \Rightarrow C \leq L$
This means that $L \leq C \leq L$
If $C < L$ there is a contradiction because $C > L$, and if $C > L$ there is a contradiction because $C < L$. Because it is both $\geq$ and $\leq$, assuming $>$ or $<$ contradicts its opposite, which means that $C = L$
Since $c_n \to C = L, c_n \to L$. □

Exercise 4. (5 pts) Prove that if $a_n \to A$ and $a_n \geq 0$ for all $n \geq 1$, then $\sqrt{a_n} \to \sqrt{A}$. Follow the following steps to prove it:

1. Consider the case $A = 0$.
2. Suppose that $A \neq 0$. Show that there is a $N_1 \in \mathbb{N}$ such that if $n \geq N_1$, then $\sqrt{a_n} \geq \sqrt{|A|/2}$. [Hint: use the definition of convergence of $(a_n)_{n \geq 0}$ with a clever choice of ε and use the properties of the absolute value.]
3. Use the convergence of (a_n) again to find a N_2 such that $|a_n - A| < \frac{3}{4} \frac{\varepsilon}{\sqrt{|A|}}$.
4. Express $\sqrt{a_n} - A$ as $\frac{a_n - A}{\sqrt{a_n} + \sqrt{A}}$ and put $N = \max\{N_1, N_2\}$. Conclude.

Solution: - □

Exercise 5. (5 pts) For each sequence $(a_n)_{n=1}^{\infty}$, define the sequence $(\sigma_n)_{n=1}^{\infty}$ by

$$\sigma_n := \frac{a_1 + a_2 + \cdots + a_n}{n} \quad (n \geq 1).$$

Prove that if $a_n \to A$, then $\sigma_n \to A$. Find an example of a divergent sequence (a_n) such that $(\sigma_n)_{n=1}^{\infty}$ converges.

Solution: If $a_n \to A$, then $\sigma_n \to A$.
$\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n \geq N, \Rightarrow |a_n - A| < \epsilon$
$\sigma_n := \frac{a_1+a_2+...+a_n}{n}, \forall n \geq 1$
Because $a_n \to A$, $\lim_{n\to\infty} a_n = A$.
This then means that $\lim_{n\to\infty} \sigma_n := \frac{a_1+a_2+...+A+A+...}{n} \Rightarrow \frac{m+\infty(A)}{\infty}$ where m is an arbitrary sum of the first $n, (\forall n \leq N)$ numbers of a_n.
Simplifying, we get
$\frac{m}{\infty} + \frac{\infty(A)}{\infty}$
$= 0 + (A) = A$
Therefore, $\sigma_n \to A$. ☐

HOMEWORK PROBLEMS

Exercise 6. (10 pts) Use the definition of convergence to prove that each of the following sequences converges.

a) $(a_n)_{n=1}^{\infty}$ given by $a_n = 5 + 1/n$ for $n \geq 1$.

b) $(a_n)_{n=1}^{\infty}$ given by $a_n = \frac{3n}{2n+1}$ for $n \geq 1$.

Solution: .

a) $\lim_{n\to\infty}(5 + \frac{1}{n})$
$= 5 + \frac{1}{\infty}$
$= 5 + 0 = 5$
$a_n \to 5$

b) $\lim_{n\to\infty}(\frac{3n}{2n+1})$
$= \frac{3(\infty)}{2(\infty)+1}$
$= \frac{3}{2}$ ☐

Exercise 7. (5 pts) Prove that the sequence $(a_n)_{n=1}^{\infty} = \left(\frac{2n+1}{n}\right)_{n=1}^{\infty}$ is a Cauchy sequence.

Solution: If $a_n \to A$, then the sequence is a Cauchy sequence.
$\lim_{n\to\infty}(\frac{2n+1}{n})$
$= \frac{2(\infty)+1}{\infty}$
$= 2$
Therefore, $a_n \to 2$.
Since a_n converges to a number (2), then that means it is a Cauchy sequence. □

Exercise 8. (10 pts) Prove that each of the following sequence diverges.

a) $(a_n)_{n=1}^{\infty} = ((-1)^n)_{n=1}^{\infty}$.

b) $(a_n)_{n=1}^{\infty} = (\sin(\frac{4n+1}{2}\pi))_{n=1}^{\infty}$.

Solution: **a)** A sequence diverges if $\forall\epsilon > 0, \exists N \in \mathbb{N}$, such that $\forall n \geq N, |a_n - A| > \epsilon$, OR if it does not converge.
Let $\epsilon > 0$. Assume toward a contradiction that $\forall\epsilon > 0, \exists N \in \mathbb{N}$, such that $\forall n \geq N, |a_n - A| < \epsilon$.
$|(-1)^n - A| < \epsilon$
$||(-1)^n| - |A|| \leq |(-1)^n - A| < \epsilon$
$|1 - A| < \epsilon$
...

b) - □

Exercise 9. (5 pts) Give an examples of two sequences (a_n) and (b_n) such that (a_n) and (b_n) don't converge, but $(a_n + b_n)$ converge.

Solution: .
$(a_n)_{n=1}^{\infty} = (-n)_{n=1}^{\infty}$
$(b_n)_{n=1}^{\infty} = (n)_{n=1}^{\infty}$
a_n and b_n both diverge, but $(a_n + b_n) \to 0$

Exercise 10. (10 pts) With the limit operations and the writing problems, find the limit of the following sequence with general term

a) $\frac{n^2+4n}{n^2-5}$.

b) $\frac{n}{n^2-3}$.

c) $\frac{\cos n}{n}$. [You can use what you know on the cosine function.]

d) $\left(\sqrt{4-\frac{1}{n}}-2\right)n$.

Solution: **a)** $\lim_{n\to\infty}(\frac{n^2+4n}{n^2-5}) = 1$

b) $\lim_{n\to\infty}(\frac{n}{n^2-3}) = 0$

c) $\lim_{n\to\infty}(\frac{cos(n)}{n}) = 0$

d) $\lim_{n\to\infty}(\sqrt{4-\frac{1}{n}}-2)n) = 0$ □