MATH-331 Introduction to Real Analysis
Homework 07

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Due date: December, 6^{th} 1:20pm

Total: 30/65.

Exercise	1 (10)	2 (5)	3 (10)	4 (5)	5 (5)	6 (10)	7 (5)	8 (5)	9 (5)	10 (5)
Score	5	1	4	5	0	0	1	5	4	5

Table 1: Scores for each exercises

Instructions: You must answer all the questions below and send your solution by email (to parisepo@hawaii.edu). If you decide to not use LaTeX to hand out your solutions, please be sure that after you scan your copy, it is clear and readable. Make sure that you attached a copy of the homework assignment to your homework.
If you choose to use LaTeX, you can use the template available on the course website.
No late homework will be accepted. No format other than PDF will be accepted. Name your file as indicated in the syllabus.

1
Writing problems

For each of the following problems, you will be asked to write a clear and detailed proof. All the exercises below can be solve without using the definition with partitions. Try to go back to homework 6 and use some of the exercises there to solve the following problems.

You will have the chance to rewrite your solution in your semester project after receiving feedback from me.

Exercise 1. (10 pts) Prove that a step function is Riemann integrable on $[a, b]$. Follow the steps below.

a) Let I be a subinterval of $[a, b]$ and put $\phi = c\chi_I$. Prove that ϕ is Riemann integrable and that $\int_a^b \phi = c\ell(I)$. [There are three cases to consider: $I = [u, v]$, $I = (u, v]$, and $I = \{u\} = [u, u]$.]

b) Prove by induction that if f_1, f_2, $\ldots$, f_n are Riemann integrable functions on $[a, b]$, then $f_1 + f_2 + \cdots + f_n$ is Riemann integrable and

$$\int_a^b (f_1 + f_2 + \cdots + f_n) = \int_a^b f_1 + \int_a^b f_2 + \cdots + \int_a^b f_n.$$

c) Write $\phi = \sum_{k=1}^n c_k \chi_{I_k}$. Use the second part of this exercise to show that ϕ is Riemann integrable.

Exercise 2. (5 pts) Suppose that f is Riemann integrable on $[a, b]$ and that f is nonnegative (means that $f(x) \geq 0$ for $x \in [a, b]$). Let $u, v \in \mathbb{R}$. Show that if $a \leq u < v \leq b$, then

$$\int_u^v f \leq \int_a^b f.$$

[Hint: Use the following property of the Riemann Integral multiple times: $\int_a^b f = \int_a^c f + \int_c^b f$.]

Exercise 3. (10 pts) Use the Fundamental Theorem of Calculus to solve the following problems:

a) Suppose that f is continuous on $[a, b]$ and that f is nonnegative on $[a, b]$. Show that if $\int_a^b f = 0$, then $f(x) = 0$ for any $x \in [a, b]$.

b) Suppose that f and g are continuous on $[a, b]$ such that $\int_a^b f = \int_a^b g$. Show that there exists a point $c \in (a, b)$ such that $f(c) = g(c)$.

Exercise 4. (5 pts) Let f be a continuous function on $[a, b]$. Prove that there exists a number $c \in [a, b]$ such that $f(c)(b - a) = \int_a^b f$.

Exercise 5. (5 pts) Suppose that f is Riemann integrable on $[a, b]$ and is strictly increasing there. Prove that there exists a point $c \in (a, b)$ such that

$$\int_a^b f = f(a)(c - a) + f(b)(b - c).$$

[Hint: Define the function $g(x) = f(a)(x - a) + f(b)(b - x)$. Show that $\int_a^b f$ is between the numbers $f(a)(b - a)$ and $f(b)(b - a)$ and use the Intermediate Value Theorem.]

2 Homework problems

Answer all the questions below. Make sure to show your work.

Exercise 6. (10pts)

a) Show that the function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & , x \in \mathbb{Q} \\ 0 & , x \notin \mathbb{Q} \end{cases}$$

is not Riemann integrable on $[0, 1]$. [Hint: Use exercise 4 from Homework 6.]

b) Define the two functions $g : [0, 1] \to \mathbb{R}$ and $h : [0, 1] \to \mathbb{R}$ by $g = \chi_{(0,1]}$ and

$$h(x) = \begin{cases} 0 & , x \notin \mathbb{Q} \\ \frac{1}{q} & , x = p/q \in \mathbb{Q}. \end{cases}$$

Use the first part to show that $g \circ h$ is not Riemann integrable on $[0, 1]$. What can you say about the composition of two Riemann integrable functions in light of this last examples?

Exercise 7. (5 pts) Show that if f is continuous on $[a, b]$, then $|f|$ is Riemann integrable on $[a, b]$ and

$$\left| \int_a^b f \right| \leq \int_a^b |f|.$$

[Hint: There is a clever way to show that $|f|$ is Riemann integrable without using the definition with the partitions.]

Exercise 8. (5 pts) Find $f'(x)$ if $f(x) = \displaystyle\int_{\sqrt{x}}^{\sqrt[3]{x}} \frac{1}{1+t^3}\, dt$ where $x \in [0, 1]$.

Exercise 9. (5 pts) Find a function $f : [1, \infty) \to \mathbb{R}$ such that $f(1) = 0$ and $f'(x) = 1 + \sin(x^2)$ for all $x > 1$.

Exercise 10. (5 pts) By thinking the following sum as a Riemann sum, evaluate

$$\lim_{n \to \infty} \sum_{k=1}^{n} \frac{n}{k^2 + n^2}.$$

① Prove that a step function is Riemann Integral on $[a,b]$. Follow the steps below.

a) Let I be a subinterval of $[a,b]$ and put $\Phi = cX_I$. Prove that Φ is Riemann integrable and that ${}_a\int^b \Phi = c\ell(I)$. [There are three cases to consider: $I=[u,v]$, $I=(u,v]$ and $I=\{u\}=[u,u]$.]

Case 1: $I=[u,v]$ $\quad a \le u < v \le b$

$$\text{Then } {}_a\int^b \Phi(x)dx = {}_a\int^u \Phi(x)dx + {}_u\int^v \Phi(x)dx + {}_v\int^b \Phi(x)dx$$
$$= {}_a\int^b 0dx + {}_u\int^v c\,dx + {}_v\int^b 0dx$$
$$= c(v-u) = c\ell(I)$$

This doesn't prove that ϕ is R. int...

Case 2: $I=(u,v]$

$${}_a\int^b \Phi(x)dx = {}_a\int^u \Phi(x)dx + {}_u\int^v \Phi(x)dx + {}_v\int^b \Phi(x)dx$$
$$= {}_u\int^v c\,dx$$
$$= c(v-u)$$
$$= c\ell(I)$$

Case 3: $I=[u,u]=\{u\}$

$${}_a\int^b \Phi(x)dx = {}_a\int^u \Phi(x)dx + {}_u\int^b \Phi(x)dx$$
$$= {}_a\int^u 0dx + {}_u\int^b 0dx = 0$$

X

5/10

# of discontinuities in each is finite $\Rightarrow \Phi$ is Riemann integrable

b) Prove by induction that if $f_1, f_2, \dots, f_n$ are Riemann integrable functions on $[a,b]$, then $f_1+f_2+\dots+f_n$ is Riemann Integrable and ${}_a\int^b (f_1+f_2+\dots+f_n) = {}_a\int^b f_1 + {}_a\int^b f_2 + \dots + {}_a\int^b f_n$

f_1 and f_2 are RI $\Rightarrow f_1+f_2$ is RI

$${}_a\int^b (f_1+f_2)(x)dx = {}_a\int^b f_1(x)dx + {}_a\int^b f_2(x)dx$$

Let this be true for $n=k-1$

Let $f_1, \dots, f_{k-1}$ be RI

$\Rightarrow f_1+\dots+f_{k-1}$ is RI

$${}_a\int^b (f_1+\dots+f_{k-1})(x)dx = {}_a\int^b f_1(x)dx + \dots + {}_a\int^b f_{k-1}(x)dx$$

Take $n=k$

Let $f_1, \dots, f_k$ be RI

$\Rightarrow f_1+\dots+f_k$ is RI ← this is what you want to prove! x

$\Rightarrow f_1+\dots+f_{k-1}$ and f_k is RI

$\Rightarrow f_1+\dots+f_k$ is RI

$${}_a\int^b (f_1+\dots+f_{k-1}+f_k)(x)dx = {}_a\int^b ((f_1+\dots+f_{k-1})(x)+f_k(x))dx$$
$$= {}_a\int^b (f_1+\dots+f_{k-1})(x)dx + {}_a\int^b f_k(x)dx$$

$\therefore {}_a\int^b (f_1+f_2+\dots+f_n) = {}_a\int^b f_1 + {}_a\int^b f_2 + \dots + {}_a\int^b f_n \quad \forall n \in \mathbb{N}$

c) Write $\Phi = \sum_{k=1}^{n} c_k X_{I_k}$. Use the second part of this exercise to show that Φ is Riemann Integrable

from a) $c_k X_{I_k}$ is RI $\forall k=1,\dots,n$ ✓

from b) $\sum_{k=1}^{n} c_k X_{I_k}$ is RI

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and also ${}_a\int^b \Phi(x)dx = {}_a\int^b \sum_{k=1}^{n} c_k X_{I_k}(x)dx$

$$= \sum_{k=1}^{n} {}_a\int^b c_k X_i(x)dx$$

② Suppose that f is Riemann integrable on [a,b] and that f is non-negative (means that $f(x) \geq 0$ for $x \in [a,b]$) Let $u, v \in \mathbb{R}$. Show that if $a \leq u < v \leq b$, then $\int_u^v f \leq \int_a^b f$

Let P be a partition on [u,v]

Then $Q: a \leq u, P, v \leq b$

Now let $M_1 = \inf_{x \in [a,u]} f(x)$ and $M_2 = \inf_{x \in [v,b]} f(x)$

So $\int_u^v f = \sup\{L(P,f)\}$

$\leq \sup\{L(P,f) + M_1(u-a) + M_2(b-v)\}$

$\leq \sup\{L(Q,f)\}$

$= \int_a^b f$

Thus $\int_u^v f \leq \int_a^b f$

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X

Simply use the fact that

$$\int_a^b f = \int_a^u f + \int_u^v f + \int_v^b f$$

& $f(x) \geq 0$. So,

$\int_a^u f \geq 0$ & $\int_v^b f \geq 0$

$\Rightarrow \int_a^b f \geq \int_u^v f$.

③ Use the fundamental theorem of calculus to solve the following problems:

a) Suppose that f is continuous on [a,b] and that f is nonnegative on [a,b]. Show that if $\int_a^b f = 0$, then $f(x) = 0$ for any $x \in [a,b]$

Suppose that $f(x) > 0$ for some $x \in (a,b)$

$\Rightarrow \exists \delta > 0$ such that for $y \in (x-\delta, x+\delta)$ $|f(y) - f(x)| < \frac{f(x)}{2} = \varepsilon$

$\Rightarrow -\frac{f(x)}{2} < f(y) - f(x) < \frac{f(x)}{2}$

$\Rightarrow f(x) - \frac{f(x)}{2} < f(y)$

$\Rightarrow f(y) > \frac{f(x)}{2} > 0$

So if $y \in (x-\delta, x+\delta)$, $f(y) > 0$

As $\int_a^b f = \int_a^{x-\delta} f + \int_{x-\delta}^{x+\delta} f + \int_{x+\delta}^b f$

$> 0 + \int_{x-\delta}^{x+\delta} f + 0$ [$\because$ f is non-negative]

$> \frac{f(x)}{2}(2\delta)$ X $= f(c)((x+\delta) - (x-\delta))$ [by FTC]

>0 $= f(c) \cdot 2\delta$ for some $c \in (x-\delta, x+\delta)$

> 0

$\Rightarrow \int_a^b f > 0$ #

So $f(x) = 0$ if $x \in (a,b)$ & by continuity

$f(x) = 0$ if $x \in [a,b]$

?? I don't get this part...

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b) Suppose that f and g are continuous on [a,b] such that $\int_a^b f = \int_a^b g$. Show that there exist a point $c \in (a,b)$ such that $f(c) = g(c)$

$\int_a^b f = \int_a^b g$

$\int_a^b f - \int_a^b g = 0$

Use that $F(x) = \int_a^x f$ is increasing and that $F(b) = 0 \Rightarrow F(x) \equiv 0$.

By the FTC,

X $\int_a^b f - g = (f-g)(c)$ for some $c \in (a,b)$ → This is not the FTC...

X

$\Rightarrow (f-g)(c) = 0$

$\Rightarrow f(c) - g(c) = 0$

$\Rightarrow f(c) = g(c)$ for some $c \in (a,b)$

④ Let f be a continuous function on [a,b]. Prove that there exist a number $c\in[a,b]$ such that $f(c)(b-a) = {}_a\int^b f$

consider a function, $F(x) = {}_0\int^x f(x)dx$

Since f is cont on [a,b] & F is also cont. on [a,b] & differentiable on (a,b)

By MVT, $\exists c \in [a,b]$ s.t.

$$\frac{F(b)-F(a)}{b-a} = F'(c)$$

$$\frac{{}_0\int^b f(x)dx - {}_0\int^a f(x)dx}{b-a} = f(c)$$

$${}_0\int^b f(x)dx - {}_0\int^a f(x)dx = f(c)(b-a)$$

$${}_a\int^b f(x)dx = f(c)(b-a)$$

$\therefore \exists c \in (a,b)$ s.t. ${}_a\int^b f(x)dx = f(c)(b-a)$

✓

Nice prove. :)

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⑤ Suppose that f is Riemann integrable on [a,b] and is strictly increasing there. Prove that there exists a point $c\in(a,b)$ s.t.

$${}_a\int^b f = f(a)(c-a) + f(b)(b-c)$$

[Hint: Define the function $g(x) = f(a)(x-a) + f(b)(b-x)$. Show that ${}_a\int^b f$ is btwn the numbers $f(a)(b-a)$ and $f(b)(b-a)$ and use IVT.]

0/5

⑥ a) Show that the function $f:[0,1]\to\mathbb{R}$ defined by

$$f(x) = \begin{cases} 1, & x\in\mathbb{Q} \\ 0, & x\notin\mathbb{Q}\end{cases}$$

is not R.I. [Hint: EX 4 from HW6]

0/10

b) Define the two functions $g:[0,1]\to\mathbb{R}$ and $h:[0,1]\to\mathbb{R}$ by $g = \chi_{(0,1]}$ and

$$h(x) = \begin{cases} 0 & x\notin\mathbb{Q} \\ 1/q & x = p/q \in\mathbb{Q}\end{cases}$$

Use the first part to show that $g\circ h$ is not R.I. on [0,1]. What can you say about the composition of two R.I. functions in light of this last examples?

⑦ Show that f is continuous on [a,b] then |f| is Riemann integrable on [a,b] and $|\int_a^b f| \le \int_a^b |f|$

If $\varepsilon > 0$ we set $\varepsilon_0 := \frac{\varepsilon}{b-a}$

Since f is continuous on [a,b] f is uniformly continuous

Hence there is a $\delta > 0$ s.t. $|f(y) - f(x)| < \varepsilon_0$ if $|y - x| < \delta$

Suppose $\|P\| < \delta$ then $|M_i - m_i| \le \varepsilon_0$ Hence

$U(P,f) - L(P,f) = \sum_{i=1}^{n} (M_i - m_i)(c_i) < \varepsilon_0(b-a) = \varepsilon$

$\therefore$ f is riemann integrable $\Rightarrow$ |f| is RI

$\Rightarrow -|f(x)| \le f(x) \le |f(x)|$

$\Rightarrow -\int_a^b |f| \le \int_a^b f \le \int_a^b |f|$

$\Rightarrow |\int_a^b f| \le \int_a^b |f|$

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f cont. $\Rightarrow$ |f| cont.

$\Rightarrow$ |f| is R.int.

Use the fact that

$-|f(x)| \le f(x) \le |f(x)|$

$\Rightarrow -\int_a^b |f(x)| \le \int_a^b f \le \int_a^b |f(x)|$.

⑧ Find $f'(x)$ if $f(x) = \int_{\sqrt{x}}^{\sqrt[3]{x}} \frac{1}{1+t^3}\,dt$ where $x \in [0,1]$

let $g(t) = \frac{1}{1+t^3}$

since g is continuous $\exists G$ st $g = G'(t)$

Then $G(\sqrt[3]{x}) - G(\sqrt{x})$

$f'(x) = G(\sqrt[3]{x}) - G(\sqrt{x})$

$= \left(\frac{1}{1+\sqrt[3]{x}^3}\right)\frac{1}{3(x^{2/3})} - \frac{1}{1+\sqrt{x}^3}\left(\frac{1}{2\sqrt{x}}\right)$

$= \frac{1}{3(1+x)(x^{2/3})} - \frac{1}{2(1+x^{3/2})(\sqrt{x})}$

✓

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⑨ Find a function $f: [1,\infty) \to \mathbb{R}$ such that $f(1) = 0$ and $f'(x) = 1 + \sin(x^2)$ for all $x > 1$

$f(x) = \int 1 + \sin(x^2)\,dx$

$= \int 1\,dx + \int \sin(x^2)\,dx$

$= x + \int \sin(x^2)\,dx$

just need f(x) here

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Let $u = x^2$ $du = 2x\,dx$

so $= x + \cancel{\int \sin(u)\,du \cdot \frac{1}{2x}}$

$= x + \cancel{\frac{1}{2x}\int \sin(u)\,du}$ since $1/2x \in \mathbb{R}$ $\int kf = k\int f$

$= x + \cancel{\frac{1}{2x}(-\cos(u))} + C$

$= x + \cancel{\frac{1}{2x}(-\cos(x^2))} + C$

$f(x) = x + \frac{-\cos(x^2)}{2x} + C$

$f(1) = 0 = 1 + \frac{-\cos(1)}{2} + C$

$\frac{\cos(1)}{2} - 1 = C$

$f(x) = x - \frac{\cos(x^2)}{2x} + \frac{\cos(1)}{2} - 1$

⑩ By thinking the following sum as a Riemann sum evaluate

$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{k^2+n^2}$

$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{\frac{n}{n^2}}{\frac{k^2}{n^2} + \frac{n^2}{n^2}}$

$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{\cancel{1}/n}{(\frac{k}{n})^2 + 1}$

$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{(\frac{k}{n})^2+1} \cancel{1}/n$

$\int_0^1 \frac{1}{1+x^2}$ ✓

$= \tan^{-1} x \,\Big|_0^1$

$= \tan^{-1}(1)$

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