MATH-331 Introduction to Real Analysis — Victor Ho
Homework 02 — Fall 2021

Due date: 20-09-2021 1:20pm — Total: /70.

Exercise	1 (10)	2 (5)	3 (5)	4 (5)	5 (5)	6 (10)	7 (5)	8 (10)	9 (5)	10 (10)
Score										

Table 1: Scores for each exercises

Instructions: You must answer all the questions below and send your solution by email (to `parisepo@hawaii.edu`). If you decide to not use LaTeX to hand out your solutions, please be sure that after you scan your copy, it is clear and readable. Make sure that you attached a copy of the homework assignment to your homework. No late homework will be accepted. No format other than PDF will be accepted. Name your file as indicated in the syllabus.

1 — Writing problems

For each of the following problems, you will be asked to write a clear and detailed proof. You will have the chance to rewrite your solution in your semester project after receiving feedback from me.

Exercise 1. (10 pts)

a) Let $\{[a_n, b_n] : n \geq 1\}$ be a family of closed intervals such that $[a_1, b_1] \supset [a_2, b_2] \supset [a_3, b_3] \supset \cdots$. Show that there is a $c \in \mathbb{R}$ such that $c \in [a_n, b_n]$ for all $n \geq \mathbb{N}$. Follow the following steps to prove it:

- **(i)** Prove that for any $n, m \geq 1$, $a_n \leq b_m$. [hint: put $M := \max\{n, m\}$.]
- **(ii)** Show that $\sup\{a_n : n \geq 1\}$ exists.
- **(iii)** Show that $c = \sup\{a_n : n \geq 1\}$ satisfies the requirement.

b) Use this last result to prove that the set $\mathbb{R}$ is uncountable. [Hint: Show that any function $f : \mathbb{N} \to \mathbb{R}$ can't be surjective. To do so, construct a sequence of closed intervals such that $f(n) \notin [a_n, b_n]$ with $a_n < b_n$.]

Solution: a. Note that a_n will increase and b_n will decrease for larger values of n as $[a_n, b_n]$ is a subset of $[a_{n-1}, b_{n=1}]$, so $[a_n, b_n]$ is between the closed interval of $[a_{n-1}, b_{n=1}]$ so it must be the case that $a_n \geq a_{n-1}$ and $b_n \leq b_{n-1}$. Consider three cases, $n = m, n < m$, and $m > n$. If $n = m$, then $a_n \leq b_n$ because in the closed bound $[a_n, b_n]$, b_n is the upper bound and a_n is the lower bound so $a_n \leq b_n$. If $n < m$, then as mentioned earlier, $a_n < a_m$ so $a_n \leq a_m \leq b_m$, so $a_n \leq b_m$. Now assume towards a contradiction that if $n > m$, $a_n > b_m$. Note that $a_n \leq b_n$ due to our previous conclusion

when $n = m$. Therefore we'll have $b_n > a_n > b_m$, which is impossible because if $n > m$, $b_n < b_m$ due to b_n shrinking for higher values of n, therefore if $n < m$ it must be the case that $a_n \leq b_m$. Since we have shown that $a_n \leq b_m, \forall n, m \geq 1$, we can say that a_n is bounded from above by $b_m, \forall n, m \geq 1$. Therefore by the axiom of completeness, a_n has a supremum for $n \geq 1$.
Now I will claim that there is a $c \in \mathbb{R}$ such that $c \in [a_n, b_n]$ where $c = sup\{a_n : n \geq 1\}$. Note that c is an upper bound of a_n as $c \in [a_n, b_n]$. Let $x = sup(a_n)$. For $c \in \mathbb{R}, c \in [a_n, b_n]$, we either have $c < x, c = x, x < c$. If $c < x$, then we have a contradiction as x is supposed to be the least upper, but c will also an upper bound and less than it. If we have $x < c$, $x \neq c$, then it must be outside the interval $[a_n, b_n]$ as c can take any value inside that interval. However being outside the interval means that either $x < a_n$, which is impossible as $x = sup(a_n)$, or $x > b_n$, which is impossible as b_n is also an upper bound of a_n. Therefore we have a contradiction if $x < c$, so it has to be the case that $x = sup(a_n) = c$.
b. Assume towards a contradiction that $\mathbb{R}$ is countable. Therefore there is a bijective function $f : \mathbb{N} \to \mathbb{R}$. Therefore for all $c \in \mathbb{R}$, $n \in \mathbb{N}$, $c = f(n)$. Let $\{[a_n, b_n] : n \geq 1\}$ be a family of closed intervals that contains such that $[a_1, b_1] \supset [a_2, b_2] \supset \cdots \supset [a_n, b_n], a_n < b_n$. Therefore $f(n) \in [a_n, b_n]$ for some $n \in \mathbb{N}$. We can then construct another closed interval, $[a_{n+1}, b_{n+1}]$ such that $[a_n, b_n] \supset [a_{n+1}, b_{n+1}]$, where $f(n) \notin [a_{n+1}, b_{n+1}]$. From the results in 1a, we know that there is a $c \in \mathbb{R}$ where $c \in [a_{n+1}, b_{n+1}]$. However since we have made $[a_{n+1}, b_{n+1}]$ to not include $f(n)$, we have $f(n) \neq c$, which means that there exists a real number that is outside the range of f. Therefore $f : \mathbb{N} \to \mathbb{R}$ is not surjective, so f is not bijective, which contradicts our claim that $\mathbb{R}$ is countable, so $\mathbb{R}$ must be uncountable.

Exercise 2. (5 pts) Prove that if $a_n \to A$, then $|a_n| \to |A|$.

Solution: Since $a_n \to A$, $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that for $n \geq N, |a_n - A| < \epsilon$. Note that $||a_n| - |A|| \leq |a_n - A|$ by the triangle inequality. Therefore we have $||a_n| - |A|| \leq |a_n - A| < \epsilon$ for $n \geq N$, so by order axioms $||a_n| - |A|| < \epsilon$ for $n \geq N$. Therefore by the definition of convergence, $|a_n| \to |A|$. $\square$

Exercise 3. (5 pts) Let (a_n), (b_n), and (c_n) be sequences of real numbers. Prove that if $a_n \to L$, $b_n \to L$, and $a_n \leq c_n \leq b_n$, then $c_n \to L$.

Solution: Since $a_n \to L$ and $b_n \to L$, we have $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that for $n \geq N, |a_n - L| < \epsilon$ and $|b_n - L| < \epsilon$. This is equivalent to $-\epsilon < a_n - L < \epsilon$ and $-\epsilon < b_n - L < \epsilon$. We can then add L to both sides to get $-\epsilon + L < a_n < \epsilon + L$ and $-\epsilon + L < b_n < \epsilon + L$. Since $a_n \leq c_n$, we have $-\epsilon + L < a_n \leq c_n$, which by order axioms implies $-\epsilon + L < c_n$. Also since $c_n \leq b_n$, we have $c_n \leq b_n < \epsilon + L$, which by order axioms imply $c_n < \epsilon + L$. Now we have $-\epsilon + L < c_n < \epsilon + L$, which is equal to $-\epsilon < c_n - L < \epsilon$, which implies $|c_n - L| < \epsilon$. Since $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that for $n \geq N$ implies that $|c_n - L| < \epsilon$, by the definition of convergence we can say that $c_n \to L$. $\square$

Exercise 4. (5 pts) Prove that if $a_n \to A$ and $a_n \geq 0$ for all $n \geq 1$, then $\sqrt{a_n} \to \sqrt{A}$. Follow the following steps to prove it:

1. Consider the case $A = 0$.

2. Suppose that $A \neq 0$. Show that there is a $N_1 \in \mathbb{N}$ such that if $n \geq N_1$, then $\sqrt{a_n} \geq \sqrt{|A|/2}$. [Hint: use the definition of convergence of $(a_n)_{n\geq 0}$ with a clever choice of ε and use the properties of the absolute value.]

3. Use the convergence of (a_n) again to find a N_2 such that $|a_n - A| < \frac{3}{4}\frac{\varepsilon}{\sqrt{|A|}}$.

4. Express $\sqrt{a_n} - A$ as $\frac{a_n - A}{\sqrt{a_n}+\sqrt{A}}$ and put $N = \max\{N_1, N_2\}$. Conclude.

Solution: First consider $A = 0$. Note that $\sqrt{A} = \sqrt{0} = 0$. If $A = 0$, then by the definition of convergence, we can let ϵ be arbitrary and take ϵ^2 so that $|a_n - 0| < \epsilon^2$ for $N \in \mathbb{N}$ such that if $n \geq N$. Therefore we can simplify the expression to get $|a_n| < \epsilon^2$, and since $a_n \geq 0$, $a_n < \epsilon^2$. Then we can square root both sides of the inequality to get $\sqrt{a_n} < \sqrt{\epsilon^2} = \epsilon$. Since $\sqrt{a_n} < \epsilon$, and $a_n \geq 0$, we can say that $|\sqrt{a_n}| < \epsilon$, which is equivalent to saying $|\sqrt{a_n} - 0| < \epsilon$. Since ϵ is arbitrary, by the definition of convergence we can say that for $A = 0$, if $a_n \to A$, then $\sqrt{a_n} \to 0 = \sqrt{A}$.
Now if $A \neq 0$ then let $\epsilon = \frac{|A|}{2}$, $\exists N_1 \in \mathbb{N}$ such that if $n \geq N_1$ then $|a_n - A| < \frac{|A|}{2}$. With the properties of absolute values we can then get $|A - a_n| < \frac{|A|}{2}$, where we can use the triangle inequality to get $|A| - |a_n| \leq |A - a_n| < \frac{|A|}{2}$, which implies by the order axioms that $|A| - |a_n| < \frac{\sqrt{|A|}}{2}$. We can then add $|a_n|$ and subtract $\frac{\sqrt{|A|}}{2}$ to both sides to get $|A| - \frac{|A|}{2} < |a_n|$, which can be simplified to $\frac{|A|}{2} < |a_n$. Now we can square root both sides to get $\sqrt{\frac{|A|}{2}} < \sqrt{a_n}$ for $n \geq N_1$.

Now use the definition of convergence for $\epsilon = \frac{\epsilon}{\sqrt{2}}$, $\exists N_2 \in \mathbb{N}$ such that $n \geq N_2$ implies $|a_n - A| < \frac{\epsilon\sqrt{A}}{\sqrt{2}}$.

First note that we can write $|\sqrt{a_n} - \sqrt{A}|$ as $|\frac{a_n - A}{\sqrt{a_n}+\sqrt{A}}|$, which is equal to $\frac{|a_n - A|}{\sqrt{a_n}+\sqrt{A}}$ as $\sqrt{a_n} + \sqrt{A}$ is always positive. Now take $N = max\{N_1, N_2\}$. Therefore for $n \geq N$ we have $\sqrt{a_n} \geq \frac{\sqrt{A}}{\sqrt{2}}$, which implies by the order axioms $\frac{1}{\sqrt{a_n}} \leq \frac{\sqrt{2}}{\sqrt{A}}$. Since $\sqrt{A} > 0$, we have $\frac{1}{\sqrt{a_n}+\sqrt{A}} < \frac{1}{\sqrt{a_n}} \leq \frac{\sqrt{2}}{\sqrt{A}}$, so by order axioms $\frac{1}{\sqrt{a_n}+\sqrt{A}} \leq \frac{\sqrt{2}}{\sqrt{A}}$. We can then multiply both sides by $|a_n - A|$ to get $\frac{|a_n - A|}{\sqrt{a_n}+\sqrt{A}} \leq \frac{\sqrt{2}|a_n - A|}{\sqrt{A}}$. Since $n \geq N \geq N_2$, we know that $|a_n - A| < \frac{\epsilon\sqrt{A}}{\sqrt{2}}$, so $\frac{\sqrt{2}|a_n - A|}{\sqrt{A}} \leq \frac{\sqrt{2}\sqrt{A}\epsilon}{\sqrt{2}\sqrt{A}} = \epsilon$. We can then combine the inequalities to get $\frac{|a_n - A|}{\sqrt{a_n}+\sqrt{A}} \leq \frac{\sqrt{2}|a_n - A|}{\sqrt{A}} < \epsilon$, which by order axioms imply that $\frac{|a_n - A|}{\sqrt{a_n}+\sqrt{A}} < \epsilon$ for all $n \geq N$. Since $\frac{|a_n - A|}{\sqrt{a_n}+\sqrt{A}} = |\sqrt{a_n} - \sqrt{A}|$ we can write $|\sqrt{a_n} - \sqrt{A}| < \epsilon$ for all $n \geq N$, so by the definition of convergence $\sqrt{a_n} \to \sqrt{A}$. □

Exercise 5. (5 pts) For each sequence $(a_n)_{n=1}^{\infty}$, define the sequence $(\sigma_n)_{n=1}^{\infty}$ by

$$\sigma_n := \frac{a_1 + a_2 + \cdots + a_n}{n} \quad (n \geq 1).$$

Prove that if $a_n \to A$, then $\sigma_n \to A$. Find an example of a divergent sequence (a_n) such that $(\sigma_n)_{n=1}^{\infty}$ converges.

Solution: If $\sigma_n \to A$, then by the definition of convergence, $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that for $n \geq N, |\sigma_n - A| < \epsilon$. Therefore $|\frac{a_1+...+a_n}{n} - A| < \epsilon$. We can simplify the left side of the inequality to get $\frac{a_1+...+a_n-nA}{n} < \epsilon$ to $\frac{a_1-A+...+a_n-A}{n} < \epsilon$. By the triangle inequality, $\frac{a_1-A+...+a_n-A}{n} \leq \frac{|a_1-A|+...|a_n-A|}{n}$. Since we know that $a_n \to A$, by the definition of convergence, $|a_n - A| < \epsilon$. Therefore $|a_1 - A| + \ldots |a_n - A| < n\epsilon$. We can then divide both sides by n to get $\frac{|a_1-A|+...|a_n-A|}{n} < \frac{n\epsilon}{n} = \epsilon$. Therefore we have $|\frac{a_1+...+a_n}{n} - A| \leq \frac{|a_1-A|+...|a_n-A|}{n} < \epsilon$, so by order axioms $|\frac{a_1+...+a_n}{n} - A| < \epsilon$. Since ϵ is arbitrary, by the definition of convergence, $\sigma_n = \frac{a_1+...+a_n}{n} \to A$ if $a_n \to A$.

One example of a divergent series is $a_n = -1^n$. We know that -1^n is divergent from 8a since it'll oscillate between 1 and -1 for even and odd powers of n. However if $\sigma_n = \frac{a_1+...+a_n}{n}$, then $\sigma_n = \frac{-1+1+...-1^n}{n}$. If n is even this will equal to 0 as there will be equal amounts of -1 and 1 that will cancel each other out to get $\frac{0}{n} = 0$. If n is odd, then there will be an extra -1 in the sequence that isn't canceled out, so we're left with $\frac{-1}{n}$, which will converge to 0 as $\frac{1}{n} \to 0$, so $\frac{-1}{n} \to -0 = 0$. Therefore since σ_n will converge to 0 for even and odd values of n, $\sigma_n \to 0$. □

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Homework problems

Exercise 6. (10 pts) Use the definition of convergence to prove that each of the following sequences converges.

a) $(a_n)_{n=1}^{\infty}$ given by $a_n = 5 + 1/n$ for $n \geq 1$.

b) $(a_n)_{n=1}^{\infty}$ given by $a_n = \frac{3n}{2n+1}$ for $n \geq 1$.

Solution: a. I will claim that $a_n \to 5$. Therefore by the definition of convergence, $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $n \geq N$ implies $|5 + \frac{1}{n} - 5| < \epsilon$. This can be simplified to $|\frac{1}{n}| < \epsilon$. Note that $\frac{1}{n} \to 0$. This is because if $\frac{1}{n} \to 0$, then $|\frac{1}{n} - 0| = \frac{1}{n} < \epsilon$ for all $n \geq N$. By AP with $x = \epsilon, y = 1$, $\exists N \in \mathbb{N}$ such that $N\epsilon > 1$, which simplifies to $N > \frac{1}{\epsilon}$. Therefore by order axioms $\frac{1}{N} < \epsilon$. If we have $n \geq N$, we'll then have $\frac{1}{n} \leq \frac{1}{N} < \epsilon$. Therefore we have $\frac{1}{n} < \epsilon$ for all $n \geq N$, so by the definition of convergence, $\frac{1}{n} \to 0$. Therefore $a_n = 5 + \frac{1}{n} \to 5$, so $a_n \to 5$.
b. First we let $\epsilon > 0$. I will now claim that $a_n \to \frac{3}{2}$. Therefore by the definition of convergence, $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $n \geq N$ implies $|\frac{3n}{2n+1} - \frac{3}{2}| < \epsilon$. We can simplify the inside of the absolute value function to $|\frac{6n}{2(2n+1)} - \frac{3(2n+1)}{2(2n+1)}|$ into $|\frac{6n-6n-3}{2(2n+1)}|$ into $|\frac{3}{4n+2}|$. Since $n \in \mathbb{N}, |\frac{3}{4n+2}| = \frac{3}{4n+2}$. We can then use AP with $x = \epsilon, y = 1$, $\exists N \in \mathbb{N}$ such that $N\epsilon > 1$, which simplifies to $N > \frac{1}{\epsilon}$. Therefore by order axioms $\frac{1}{N} < \epsilon$. If we have $n \geq N$, we'll then have $\frac{1}{n} \leq \frac{1}{N} < \epsilon$. We can then multiply all elements of the inequality by $\frac{3}{4}$ to get $\frac{3}{4n} \leq \frac{3}{4N} < \frac{3\epsilon}{4}$. Note that $\frac{3}{4n+2} < \frac{3}{4n}$ and $\frac{3\epsilon}{4} < \epsilon$. Therefore we have $\frac{3}{4n+2} < \frac{3}{4n} \leq \frac{3}{4N} < \frac{3\epsilon}{4} < \epsilon$, and therefore $\frac{3}{4n+2} < \epsilon$. Since ϵ is arbitrary, $a_n \to \frac{3}{2}$. □

Exercise 7. (5 pts) Prove that the sequence $(a_n)_{n=1}^{\infty} = \left(\frac{2n+1}{n}\right)_{n=1}^{\infty}$ is a Cauchy sequence.

Solution: Note that $a_n \to 2$. This is because $a_n = \frac{2n+1}{n} = \frac{2n}{n} + \frac{1}{n} = 2 + \frac{1}{n}$.We know that if $a_n \to A$ and $b_n \to B$ then $a_n + b_n \to A + B$. Therefore we can take $b_n = 2$ and $c_n = \frac{1}{n}$ so that $a_n = b_n + c_n$. Since 2 is a constant, $b_n \to 2$, and from my work done in 6a, we know that $\frac{1}{n} \to 0$. Therefore $a_n = b_n + c_n \to 2 + 0$, so $a_n \to 2$. Finally, from theorem 1.3 in the textbook, we know that any convergent sequence is a Cauchy sequence, and since $a_n \to 2$, a_n is a Cauchy sequence. □

Exercise 8. (10 pts) Prove that each of the following sequence diverges.

a) $(a_n)_{n=1}^{\infty} = ((-1)^n)_{n=1}^{\infty}$.

b) $(a_n)_{n=1}^{\infty} = (\sin(\frac{2n+1}{2}\pi))_{n=1}^{\infty}$.

Solution: a. Note that is n is even, $-1^n = 1$, and if n is odd, $-1^n = -1$. Now assume towards a contradiction that a_n converges. Therefore $\forall \epsilon > 0$, $\exists N$ such that $\forall n, n \geq N, |a_n - A| < \epsilon$. Let $\epsilon = 1$. When n is even, $-1^n = 1$, so $|1 - A| < 1$, so $-1 < 1 - A < 1$. We can subtract one to both sides to get $-2 < -A < 0$. When n is odd, $-1^n = -1$, so $|-1 - A| < 1$, so $-1 < -1 - A < 1$. We can add one to both sides to get $0 < -A < 2$. Therefore we have $-A < 0$ but also $0 < -A$. This contradicts our claim that $\forall \epsilon > 0$, $\exists N$ such that $\forall n, n \geq N, |a_n - A| < \epsilon$, so a_n diverges.
b. Note that $sin(\frac{2n+1}{2}\pi) = sin(\frac{2n\pi}{2} + \frac{\pi}{2} = sin(n\pi + \frac{\pi}{2})$. Also note that $sin(x)$ has a period of 2π. Therefore if n is even, $sin(n\pi + \frac{\pi}{2}) = sin(\frac{\pi}{2})$ as adding $n\pi$ when n is even is the same as adding multiples of 2π, which will not change the sin function as it has a period of 2π. Also note that if n is odd, then $sin(n\pi + \frac{\pi}{2}) = sin(\pi + \frac{\pi}{2}) = sin(\frac{3\pi}{2})$. This if $n\pi$ is odd, we can treat it as $m\pi + \pi$ where m is even, which will mean that we're adding multiples of 2π to $\frac{3\pi}{2})$, which will cause no change to the sin function as it has a period of 2π. Also note that $sin(\frac{\pi}{2}) = 1$, and $sin(\frac{3\pi}{2}) = -1$. We can then use a similar argument to 8a to prove that a_n diverges. ——— Now assume towards a contradiction that a_n converges. Therefore $\forall \epsilon > 0$, $\exists N$ such that $\forall n, n \geq N, |a_n - A| < \epsilon$. Let $\epsilon = 1$. When n is even, $sin(\frac{2n+1}{2}\pi) = 1$, so $|1 - A| < 1$, so $-1 < 1 - A < 1$. We can subtract one to both sides to get $-2 < -A < 0$. When n is odd, $sin(\frac{2n+1}{2}\pi) = -1$, so $|-1 - A| < 1$, so $-1 < -1 - A < 1$. We can add one to both sides to get $0 < -A < 2$. Therefore we have $-A < 0$ but also $0 < -A$. This contradicts our claim that $\forall \epsilon > 0$, $\exists N$ such that $\forall n, n \geq N, |a_n - A| < \epsilon$, so a_n diverges. □

Exercise 9. (5 pts) Give an examples of two sequences (a_n) and (b_n) such that (a_n) and (b_n) don't converge, but $(a_n + b_n)$ converge.

Solution: Let $a_n = -1^n, b_n = -1^{n+1}$. We know from 8a that a_n diverges, and we can use a similar argument to show that b_n will diverge. However $a_n + b_n$ will converge as when n is even, $a_n = 1, b_n = -1$ so $a_n + b_n = 0$. Similarly, when n is odd, $a_n = -1, b_n = 1$ so $a_n + b_n = 0$. Therefore for all values of n $a_n + b_n = 0$. Since $0 \to 0$, $a_n + b_n \to 0$. □

Exercise 10. (10 pts) With the limit operations and the writing problems, find the limit of the following sequence with general term

a) $\frac{n^2+4n}{n^2-5}$.

b) $\frac{n}{n^2-3}$.

c) $\frac{\cos n}{n}$. [You can use what you know on the cosine function.]

d) $\left(\sqrt{4 - \frac{1}{n}} - 2\right)n$.

Solution: a. If we divide the fraction by n^2 we have $\frac{1+\frac{4}{n}}{1-\frac{5}{n^2}}$, which is equal to $\frac{1+\frac{4}{n}}{1-(\frac{5}{n}(\frac{5}{n}))}$. Note that since $4 \to 4$. From 6a we know that $\frac{1}{n} \to 0$, and since $a_n b_n \to AB$, we can take 4 as a_n and $\frac{1}{n}$ as b_n to have $\frac{4}{n} \to 4(0) = 0$. Since $a_n + b_n \to A + B$, and $1 \to 1$, we can take $a_n = 1, b_n = \frac{4}{n}$ to get $1+\frac{4}{n} \to 1+0 = 1$. Similarly $\frac{5}{n} \to 5$ since we can take $a_n = 5, b_n = \frac{1}{n}$ and $5 \to 5$. Since $a_n b_n \to AB$, we can take $a_n = b_n = \frac{5}{n}$ to get $\frac{5}{n} \times \frac{5}{n} \to 0 \times 0$. Therefore since $1 \to 1$, $1 - (\frac{5}{n}(\frac{5}{n})) \to 1 - 0$. Finally we know that $\frac{a_n}{b_n} \to \frac{A}{B}$, so we can take $a_n = 1 + \frac{4}{n}, b_n = 1 - (\frac{5}{n}(\frac{5}{n}))$. Since $a_n \to 1$, and $b_n \to 1$, $\frac{a_n}{b_n} \to 1$. Therefore we can say that $\frac{n^2+4n}{n^2-5} \to 1$.

b. We can divide the fraction by n^2 to get $\frac{\frac{1}{n}}{1-\frac{3}{n^2}}$ which is equal to $\frac{\frac{1}{n}}{1-(\frac{3}{n}(\frac{3}{n}))}$. We know that $\frac{1}{n} \to 0$. Since $a_n b_n \to AB$, we can take 3 as a_n and $\frac{1}{n}$ as b_n, so $\frac{3}{n} \to 3(0) = 0$. We can then take $a_n = b_n = \frac{3}{n}$ so that $a_n b_n = \frac{3}{n}(\frac{3}{n}) \to 0(0) = 0$. Therefore since $1 \to 1$, $1 - (\frac{3}{n}(\frac{3}{n})) \to 1 - 0 = 1$. Finally since $\frac{a_n}{b_n} \to \frac{A}{B}$, we can take $a_n = \frac{1}{n}, b_n = 1 - (\frac{3}{n}(\frac{3}{n}))$ so $\frac{a_n}{b_n} \to \frac{0}{1} = 0$. Therefore $\frac{n}{n^2-3} \to 0$.

c. Note that $\frac{cos(n)}{n} = \frac{1}{n}(cos(n))$. Also note that $cos(n)$ is bounded by -1 and 1. By theorem 1.13 in the textbook, if $a_n \to 0$ and b_n is bounded, then $a_n b_n \to 0$. Therefore we can take $a_n = \frac{1}{n}, b_n = cos(n)$, and since $\frac{1}{n} \to 0$ and $cos(n)$ is bounded, $\frac{cos(n)}{n} = \frac{1}{n}(cos(n)) \to 0$.

d. $(\sqrt{4 - \frac{1}{n}} - 2)n = \frac{(\sqrt{4-\frac{1}{n}}-2)(\sqrt{4-\frac{1}{n}}+2)n}{\sqrt{4-\frac{1}{n}}+2} = \frac{(4-\frac{1}{n}-4)n}{\sqrt{4-\frac{1}{n}}+2} = \frac{4n-\frac{n}{n}-4n}{\sqrt{4-\frac{1}{n}}+2} = \frac{-1}{\sqrt{4-\frac{1}{n}}+2}$. Since $4 \to 4$, $\frac{1}{n} \to 0$, $4 - \frac{1}{n} \to 4 - 0 = 4$. Since $4 - \frac{1}{n} \to 4$, $\sqrt{4 - \frac{1}{n}} \to \sqrt{4} = 2$. Since $2 \to 2$, $\sqrt{4 - \frac{1}{n}} + 2 \to 2 + 2 = 4$. Since $-1 \to -1$, $\frac{-1}{\sqrt{4-\frac{1}{n}}+2} \to \frac{-1}{4}$. Therefore $(\sqrt{4 - \frac{1}{n}} - 2)n \to \frac{-1}{4}$. □