RA

1 a.) i.) Let $n$ be $[a_n, b_n]$. Let $m$ be $[a_m, b_m]$. $n, m \geq 1$. Let $M := \max\{n, m\}$. If $a_n > b_m$, then the beginning of $n$ would be after the end of $m$, meaning $[a_n, b_n] \supset [a_m, b_m]$ is not true, so $a_n \leq b_m$.

ii.) $[a_1, b_1]$ has $a_1 \leq a_2 \leq a_3 \leq \ldots \leq a_n$. So $a_1$ is a lower bound. So $\sup[a_n] \geq a_1$. But if $a_2 \geq \sup[a_n]$, it isn't sup, so $\sup[a_n] \geq a_n$. And $a_n \leq b_m$ so $\sup[a_n] \leq b_m$. $a_n$ is an upper bound. $b_m$ is also an upper bound. Since $a_n \leq b_m$, $b_m$ is the highest upper bound since $a_n$ is the highest a can go. So $\sup[a_n] \leq b_m$.

iii.) If $c = \sup[a_n]$, $c \geq a_n$ for all $n \in \mathbb{N}$ because $c$ is a supremum. And $c \leq b_m$ for $m \in \mathbb{N}$. This means it is between each $[a_n, b_n]$ pair because for every $b_m$, $a_n \leq b_m$ so $c$ lies in every interval.

B.) For any $N$, there is a corresponding $[a_n, b_n]$. If this function starts at $n = 1 \rightarrow [a_1, b_1]$ being the largest interval. Let us exclude $r_1$ from $n = 1$. Then let us have an $n = 2$ s.t. $[a_2, b_2]$ exclude $r_2$. We continue this to $a_n$ excluding $r_n$ numbers, $a_n$ does not include $c$ which is a real number. The list excludes all $r_n$ so it not excluding $c$ means it does not exclude all $r_n$, thus making $r_n$ uncountable because $f: \mathbb{N} \rightarrow \mathbb{R}$ is not surjective going through all $\mathbb{R}$.

RA HW #2

2 $a_n \to A$ then $|a_n - A| < \varepsilon$ $a_n \le |a_n|$ $A \le |A|$

so $|a_n - A| \le ||a_n| - |A|| \le \varepsilon$. $||a_n| - |A|| < \varepsilon$

$\varepsilon$ is arbitrary here, so $|a_n| \to |A|$

3 If $x_n \to X$ $y_n \to Y$ $x_n \le y_n$ $X \le Y$

$a_n \to L$ $c_n \to C$ $a_n \le c_n$ so $L \le C$

$c_n \to C$ $b_n \to L$ $c_n \le b_n$ so $C \le L$

$C \le L \le C$. L cannot be less than L because it needs to be more than L and vice versa so C must be equal to L, so $c_n \to L$

4 1.) $a_n \to 0$ $|a_n| < \varepsilon$ $a_n < \varepsilon$ let $a_n < \varepsilon^2$

$\sqrt{a_n} < \varepsilon$ $|a_n - 0| < \varepsilon$ epsilon arbitrary.

2.) $|a_n - A| < \varepsilon$ let $\varepsilon$ be $|A|/2$

$|a_n - A| \le \frac{|-A|}{2}$ $|a_n| - |A| \le \frac{|-A|}{2}$ $|a_n| < |A| + \frac{|-A|}{2} \ge \frac{|2A - A|}{2}$

$\frac{|A|}{2} \le |a_n|$ $\sqrt{\frac{|A|}{2}} \le \sqrt{|a_n|}$

3.) For $n \ge N_2$, $|a_n - A| < \frac{3}{4}\frac{\varepsilon}{\sqrt{A}}$

4.) $N = \max\{N_1, N_2\}$. $|a_n - A| < \frac{3}{4}\frac{\varepsilon}{\sqrt{A}}$ $\sqrt{a_n} + \sqrt{A} > \sqrt{\frac{A}{2}} + \sqrt{A}$

$\sqrt{a_n} + \sqrt{A} > \sqrt{\frac{3A}{2}}$ $\frac{1}{\sqrt{a_n} + \sqrt{A}} \ge \sqrt{\frac{2}{3A}}$ $|a_n - A| < \varepsilon$

$\frac{|a_n - A|}{\sqrt{a_n} + \sqrt{A}} < \frac{1}{\sqrt{a_n} + \sqrt{A}} \cdot \varepsilon = \sqrt{\frac{2}{3A}} \cdot \varepsilon$

$|\sqrt{a_n} - \sqrt{A}| < \sqrt{\frac{2}{3A}} \cdot \varepsilon$ If $\varepsilon$ is $\varepsilon \cdot \sqrt{\frac{3A}{2}}$,

then we just have $\varepsilon$

and $|\sqrt{a_n} - A| < \varepsilon$ and $\varepsilon$ is arbitrary so

$\sqrt{a_n} \to \sqrt{A}$

RA #8 and #5

8. $a_n = (-1)^n \quad |a_n - A| < \varepsilon \quad$ diverges $|a_n - A| > \varepsilon$

$|(-1)^n - A| < \varepsilon \quad \varepsilon > |(-1)^n| - |A| \geq |(-1)^n - A|$

$1 - |A| < \varepsilon \quad 1 < \varepsilon + |A| \quad$ Take $\varepsilon = -|A|$

$1 < -|A| + |A| \quad 1 < 0$ #

so $|(-1)^n - A| > -|A|$

b) $\sin\left(\frac{2n+1}{2}\pi\right)_{n=1}^{\infty} \quad \varepsilon < |\sin(\frac{2n+1}{2}\pi) - A| \leq |\sin(\frac{2n+1}{2}\pi)| - |A|$

$\varepsilon + |A| < |\sin(\frac{2n+1}{2}\pi)|$

Let $\varepsilon = -2 - |A|$

$-2 - |A| + |A| < |\sin(\frac{2n+1}{2}\pi)|$

$-2 < |\sin(\frac{2n+1}{2}\pi)| \quad \sin(\frac{2n+1}{2}\pi)$ bounded above by 1 and below by $-1$.

So an $\varepsilon < |a_n - A|$ exists for $\forall N \in \mathbb{N}$, so $\sin(\frac{2n+1}{2}\pi)$ diverges.

5. A divergent sequence where $\sigma_n$ converges would be $((-1)^n)_{n=1}^{\infty}$.

$|a_n - A| < \varepsilon \quad \varepsilon > 0 \quad n = 1 \quad |a_1 - A| < \varepsilon$ all of

$|a_n - A| < \varepsilon \quad$ so $|a_1 - A| < \varepsilon, |a_2 - A| < \varepsilon, \dots |a_n - A| < \varepsilon$

so $|a_1 - A| + |a_2 - A| + |a_3 - A| \dots |a_n - A| < \varepsilon + \varepsilon + \varepsilon \dots \varepsilon = n\varepsilon$

$\left|\frac{a_1 + a_2 + a_3 \dots + a_n - nA}{n}\right| < \varepsilon = |\sigma_n - A| < \varepsilon$

$\varepsilon$ is arbitrary, so $\sigma_n \to A$

Real Analysis HW #2

6 a.) $a_n = 5 + \frac{1}{n}$ $n \geq 1$ $n \to \infty$ $a_n \to 5$

Let $\varepsilon > 0$. $|a_n - A| < \varepsilon$ $|5 + \frac{1}{n} - 5| < \varepsilon$ $|\frac{1}{n}| < \varepsilon$

Let $N \in \mathbb{N}$. Let $n \geq N$. $|\frac{1}{n}| < \varepsilon$ $\frac{1}{n} < \varepsilon$ $1 < n\varepsilon$

$1 \leq N\varepsilon \leq n\varepsilon$ In A.P., Let $y = 1$ and $x = \varepsilon$.

$\exists N_0 \in \mathbb{N}$ s.t. $N_0 \varepsilon > 1$. Take $N = N_0$ If $n \geq N_0$, then

$n\varepsilon \geq N_0 \varepsilon > 1$. so $n\varepsilon > 1$ $(\forall n \geq N_0)$ such that

$n \geq N_0 \to \frac{1}{n} < \varepsilon \to |5 + \frac{1}{n} - 5| < \varepsilon$.

Since $\varepsilon$ was arbitrary, $5 + \frac{1}{n} \to 5$.

b.) $a_n = \frac{3n}{2n+1}$ $n \to \infty$ $a_n \to \frac{3}{2}$ Let $\varepsilon > 0$

$|\frac{3n}{2n+1} - \frac{3}{2}| < \varepsilon$ $|\frac{6n}{4n+2} - \frac{6n+3}{4n+2}| < \varepsilon$ $|\frac{3}{4n+2}| < \varepsilon$

$3 < \varepsilon(4n+2)$ $3 - 2\varepsilon < 4\varepsilon n$. Let $n \geq N$

In A.P. Take $y = 3 - 2\varepsilon$ and $x = 4\varepsilon$

$\exists N_0 \in \mathbb{N}$ such that $4\varepsilon N_0 > 3 - 2\varepsilon$

Take $N = N_0$. If $n \geq N_0$ then $4\varepsilon n > 4\varepsilon N_0 > 3 - 2\varepsilon$

so $4\varepsilon n > 3 - 2\varepsilon$ $(\forall n \geq N_0)$ such that $n \geq N_0$ then

$4\varepsilon n > 3 - 2\varepsilon \to 3 < \varepsilon(4n+2) \to \frac{3}{4n+2} < \varepsilon \to |\frac{3}{4n+2}| < \varepsilon$

$|\frac{6n}{4n+2} - \frac{6n+3}{4n+2}| < \varepsilon \to |\frac{3n}{2n+1} - \frac{3}{2}| < \varepsilon$

Since $\varepsilon$ was arbitrary, $\frac{3n}{2n+1} \to \frac{3}{2}$

7 $\frac{(2n+1)}{n} = a_n$ $|a_n - a_m| < \varepsilon$ $\varepsilon > 0$ $\varepsilon/2 > 0$

$\exists N \in \mathbb{N}$ If $n, m \geq N$. Suppose $\frac{2n+1}{n} \to 2$.

$|a_n - a_m - 2 + 2| \to |a_n - 2 - a_m + 2| \leq |a_n - 2| + |a_m - 2|$

$n > N \to |a_n - 2| < \varepsilon/2$ $m > N \to |a_m - 2| < \varepsilon/2$

$|a_n - 2| + |a_m - 2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$ $|a_n - 2| + |a_m - 2| < \varepsilon$

$\varepsilon$ is arbitrary. Therefore $\frac{2n+1}{n}$ is Cauchy.

9 $a_n = (-1)^n$ this diverges because it alternates between $-1$ and $1$ $\forall n \in \mathbb{N}$.

$b_n = (-1)^{n+1}$ this diverges because it alternates between $1$ and $-1$ $\forall n \in \mathbb{N}$.

$(a_n + b_n)$ converges because $(-1)^n + (-1)^{n+1} = 0$ $\forall n \in \mathbb{N}$ so it converges to $0$.

10 a.) $\frac{n^2+4n}{n^2-5} = \frac{n^2(1+4/n)}{n^2(1-5/n^2)} = \frac{1+4/n}{1-5/n^2}$ $n \to \infty$ $\frac{1+0}{1-0} = \frac{1}{1} = 1$

b.) $\frac{n}{n^2-3} = \frac{n^2(1/n)}{n^2(1-3/n^2)} = \frac{(1/n)}{(1-3/n^2)}$ $n \to \infty = \frac{0}{1-0} = 0$

c.) $\frac{\cos n}{n} = \frac{1}{n}\cos n$ $n \to \infty$ $\frac{1}{n} \to 0$ $0 \cdot \cos n = 0$.

d.) $(\sqrt{4-\frac{1}{n}} - 2)n$ $n \to \infty$ $(\sqrt{4-0} - 2)\infty = (2-2)\infty = 0 \cdot \infty$

$0 \cdot \infty = 0$.