MATH-331 Introduction to Real Analysis | Pierre-Olivier Parisé
Homework 01 | Fall 2021

Due date: 09/06/2021 1:20pm | Total: /65.

Instructions: You must answer all the questions below and send your solution by email (to `parisepo@hawaii.edu`). If you decide to not use LaTeX to hand out your solutions, please be sure that after you scan your copy, it is clear and readable. No late homework will be accepted. No format other than PDF will be accepted. Name your file as indicated in the syllabus.

1 Homework problems

Exercise 1. (5 points) Prove that for any $n \in \mathbb{N}$, $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

Exercise 2. (5 points) Define $f : \mathbb{N} \to \mathbb{N}$ by $f(1) = 1$, $f(2) = 2$ and $f(3) = 3$ and

$$f(n) := f(n-1) + f(n-2) + f(n-3) \quad (n \geq 4).$$

Prove that $f(n) \leq 2^{n-1}$ for all $n \in \mathbb{N}$.

Exercise 3. (5 points) Prove that if A, B and C are sets, then

a) $A \sim A$.

b) If $A \sim B$, then $B \sim A$.

c) If $A \sim B$ and $B \sim C$, then $A \sim C$.

Exercise 4. (5 points) Show that any subset of a countable set is countable.

Exercise 5. (10 points) Let $0 < a < b$ be positive real numbers. Prove that

a) $a^2 < b^2$.

b) $\sqrt{a} < \sqrt{b}$.

Exercise 6. (5 points) Sketch the region of the points (x, y) satisfying the following relation: $x + |x| = y + |y|$ (explain your answer).

Exercise 7. (5 points) If $x \geq 0$ and $y \geq 0$, prove that $\sqrt{xy} \leq \frac{x+y}{\sqrt{2}}$

Exercise 8. (10 points) Find the infimum and supremum (if they exist) of the following sets. Make sure to justify all your answers:

a) $E := \{x \in \mathbb{R} \, : \, x \geq 0 \text{ and } x^2 \leq 9\}$.

b) $E := \{\frac{4n+5}{n+1} \, : \, n \in \mathbb{N}\}$.

Writing problems

For each of the following problems, you will be ask to write a clear and detailed proof. You will have the chance to rewrite your solution in your semester project after receiving feedback from me.

Exercise 9. (5 points) Let A be a non-empty set and $P(A)$ be its power set (the family of all subsets of A). Prove that A is not equivalent to $P(A)$. Deduce that $P(\mathbb{N})$ is not countable. [Hint: Define $C := \{x : x \in A \text{ and } x \notin f(x)\}$.]

Exercise 10. (10 points) Let $E \subseteq \mathbb{R}$ be bounded from above and $E \neq \emptyset$. For $r \in \mathbb{R}$, let

$$rE := \{rx : x \in E\} \quad \text{and} \quad r + E := \{r + x : x \in E\}.$$

Show that

a) if $r > 0$, then $\sup(rE) = r\sup(E)$.

b) for any $r \in \mathbb{R}$, $\sup(r + E) = r + \sup E$.