Pierre-Olivier Parisé
Fall 2021, 12/03/2021

Name of the members of the team: ______________________

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Team name (if any): ______________________

Question:	1	2	Total
Points:	10	10	20
Score:			

Instructions: You must answer all the questions in teams of 3 and hand out one copy per team. You are allowed to use the lecture notes only. No other tools such as a cell-phone, a calculator, or a laptop. Only your pen and eraser. The space between the questions are there to write the final versions of your answers.

Question 1 **(10 pts)**

Let $f : (a, b] \to \mathbb{R}$ be a function where $a < b$.

(a) (5 points) How would you define the Riemann integral of f on $(a, b]$? Explain in details your definition.

Solution: First, we have to make sure that the function is Riemann integrable on each $[c, b]$ where $a < c < b$. This is well-defined because the Riemann integral on closed intervals were defined in the lecture notes. Now, we define the integral of f on $(a, b]$ by taking the limit as c goes to a of the integral of f on $[c, b]$. So the definition will be: $f : (a, b] \to \mathbb{R}$ is Riemann integrable on $(a, b]$ if

- f is Riemann integrable on $[c, b]$ for any $c \in (a, b)$.
- the $\lim_{c \to a^+} \int_c^b f$ exists.

We then define the integral of f from a to b by

$$\int_a^b f = \lim_{c \to a^+} \int_c^b .$$

(b) (5 points) Find a function $f : (0, 1] \to \mathbb{R}$ that is Riemann integrable on $(0, 1]$ (with respect to your definition) but is unbounded on $(0, 1]$.

Solution: Take $f(x) = \frac{1}{\sqrt{x}}$ defined on $(0, 1]$. The integral on $[c, 1]$ (for $0 < c < 1$) is

$$\int_c^1 \frac{1}{\sqrt{x}}\, dx = 2x^{1/2}\Big|_c^1 = 2(1 - \sqrt{c}).$$

So, as $c \to 0^+$, we get

$$\int_0^1 \frac{1}{\sqrt{x}}\, dx = \lim_{c \to 0^+} 2(1 - \sqrt{c}) = 2.$$

We see that $f(x) = 1/\sqrt{x}$ is unbounded on $(0, 1]$ because $\lim_{x \to 0^+} f(x) = +\infty$.

Question 2 **(10 pts)**

Find the limit of the sequence $(a_n)_{n=1}^{\infty}$ if

$$a_n = \sum_{k=1}^{n} \frac{k}{k^2 + n^2}.$$

Solution: We have

$$a_n = \sum_{k=1} \frac{1}{n}\left(\frac{k/n}{1 + (k/n)^2}\right).$$

This, as $n \to \infty$, represents the integral from 0 to 1 of the function $f(x) = \frac{x}{1+x^2}$. So,

$$\lim_{n\to\infty} a_n = \int_0^1 \frac{x}{1+x^2}\, dx = (1/2)\log(2).$$