MATH-331 Introduction to Real Analysis | YOUR FULL NAME
Homework 04 | Fall 2021

Due date: October 25^{th} 1:20pm | Total: /70.

Exercise	1 (5)	2 (5)	3 (5)	4 (5)	5 (10)	6 (10)	7 (5)	8 (5)	9 (5)	10 (10)
Score										

Table 1: Scores for each exercises

Instructions: You must answer all the questions below and send your solution by email (to `parisepo@hawaii.edu`). If you decide to not use LaTeX to hand out your solutions, please be sure that after you scan your copy, it is clear and readable. Make sure that you attached a copy of the homework assignment to your homework.
If you choose to use LaTeX, you can use the template available on the course website.
No late homework will be accepted. No format other than PDF will be accepted. Name your file as indicated in the syllabus.

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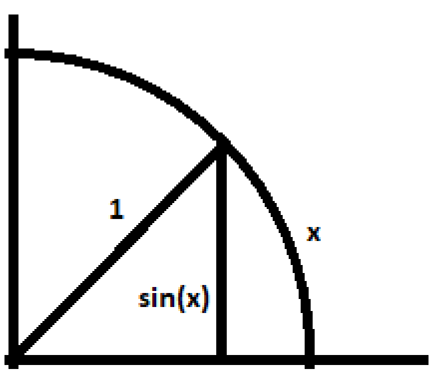
Writing problems

For each of the following problems, you will be asked to write a clear and detailed proof. You will have the chance to rewrite your solution in your semester project after receiving feedback from me.

Exercise 1. (5 pts) Prove that, if $0 < x < \pi/2$, then $0 \leq \sin x \leq x$ with a geometric argument. [Hint: View $\sin x$ as a point on the unit circle in the first quadrant.]



Solution:
With this diagram that I drew of the first quadrant, we can see that our angle x will always be longer than $\sin(x)$. Another way to think of this is to use our basic knowledge that the shortest distance between two points is straight across. So if we take x to the point on the unit circle, then our $\sin(x)$ will be the distance straight down, which is the shortest distance to the x axis.

However, our angle x is also the length of the arc on the circle. On the picture, we can see that it does not take the shortest route to the x-axis. Any deviation from a straight path ($\sin(x)$) will be longer. $\square$

Exercise 2. (5 pts) Let $f : A \to \mathbb{R}$ and $g : B \to A$ be two functions where $A, B \subset \mathbb{R}$. Let a be an accumulation point of A and b be an accumulation point of B. Suppose that

- $\lim_{t \to b} g(t) = a$.
- there is a $\eta > 0$ such that for any $t \in B \cap (b - \eta, b + \eta)$, $g(t) \neq a$.
- f has a limit at a.

Prove that $f \circ g$ has a limit at b and $\lim_{x \to a} f(x) = \lim_{t \to b} f(g(t))$. [This is the change of variable rule for limits.]

Solution:

Exercise 3. (5 pts) Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and suppose that $f(x) = 0$ for each rational number x in $[a, b]$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

Solution: Let $x \in [a, b]$. We have two cases. x can either be rational or irrational. If it is rational, we already know by the assumption that $f(x) = 0$. Now if x is irrational, we can define a sequence r_n such that it converges to x, and each r_n is rational. Now in the assumption, we know that f is continuous on $[a, b]$, so there must exist a limit at x. By a theorem in the textbook, we know that

$$\lim_{n \to \infty} f(r_n) = f(x)$$

However, we defined earlier that each r_n is a rational number, and therefore $f(r_n) = 0 \ \forall \ n > 1$. Therefore, we have that

$$f(x) = 0$$

when x is irrational. We've proven that $f(x) = 0$ when x is rational, and $f(x) = 0$ when x is irrational. Since all numbers in $\mathbb{R}$ can only either be rational or irrational, $f(x) = 0$ for all $x \in [a, b]$. $\square$

Exercise 4. (5 pts) Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and suppose that $f(c) > 0$ for some $c \in [a, b]$. Prove that there exist a number η and an interval $[u, v] \subset [a, b]$ such that $f(x) \geq \eta$ for all $x \in [u, v]$.

Solution: For the function f, because it is continuous over a closed interval, we can use the EVT to say that $\exists\, d$ such that $sup\{f(x) : x \in [a, b]\} = f(d)$. We can then have $\eta \in [f(a), f(b)]$ and $[u, v] = [d, d]$, the singleton set containing the x-coordinate of the supremum $f(d)$ only. Then by definition of the supremum, we have that $f(x) \geq \eta$ for all $x \in [d, d]$, since $sup\{f(x)\} = f(d) \geq \eta \in [f(a), f(b)]$. Since η and $[u, v]$ were arbitrary, we can do this. $\square$

Exercise 5. (10 pts) Let $f : \mathbb{R} \to \mathbb{R}$ be a function that satisfies $f(x + y) = f(x) + f(y)$ for any real number x and y.

a) Suppose that f is continuous at some point c. Prove that f is continuous on $\mathbb{R}$.

b) Suppose that f is continuous on $\mathbb{R}$ and that $f(1) = k$. Prove that $f(x) = kx$ for all $x \in \mathbb{R}$. [Hint: start with x integer, then x rational, and finally use Exercise 3.]

Solution: **a)**

b)

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Homework problems

Answer all the questions below. Make sure to show your work.

Exercise 6. (10pts) For each of the functions below, say if the limit exists or doesn't exist at the given point. Justify your answer (in other words, prove it!)

a) $f(x) = \sin(1/x)$ and $x_0 = 0$.

b) $f(x) = x\sin(1/x)$ and $x_0 = 0$.

Solution:
a) For this problem, we can show that this series diverges. Say we have two sequences

$$x_n = \frac{1}{n\pi} \to 0, n \to \infty$$
$$y_n = \frac{1}{2\pi n + \frac{\pi}{2}}, n \to \infty$$

Both of these sequences approach 0, so if we plug them into the function, they should both yield the same answer. However, they do not.

$$\sin(\frac{1}{x_n}) = \sin(n\pi) \to 0$$
$$\sin(\frac{1}{y_n}) = \sin(2\pi n + \frac{\pi}{2}) \to 1$$
$$0 \neq 1$$

If this function had a defined limit, then according to a theorem in the lecture notes, inputting two sequences that converge to an identical number should also yield two identical limits. However, we see that it does not. Therefore, this function is divergent.

b) We can split $f(x)$ into the product of two functions.

$$f(x) = h(x)g(x)$$
$$h(x) = x$$
$$g(x) = \sin(\frac{1}{x})$$

Now let's define a sequence $(x_n)_{n=1}^{\infty}$ s.t. (x_n) converges to $x_0 = 0$. Then we know that $h(x) \to 0$ and that $g(x)$ is bounded below by 0 and above by 1. Therefore, by a theorem in the lecture notes, $f(x) = h(x)g(x) \to 0$.

Exercise 7. (5 pts) Let $c \in (a, b)$ and let f be a function defined on (a, b) except at c. Suppose that $f(x) > 0$ for any $x \in (a, b)\backslash\{c\}$, that $\lim_{x\to c} f(x)$ exists, and that

$$\lim_{x\to c} f(x) = \lim_{x\to c} \left[(f(x))^2 - f(x) - 3\right].$$

Find the value of $\lim_{x\to c} f(x)$. Explain each step carefully.

Solution: Since we know that $\lim_{x\to c} f(x)$ exists, let's assign it to $\lim_{x\to c} f(x) = L$. Then we can substitute:

$$L = L^2 - L - 3$$

by the sum rule of limits

$$\begin{aligned} L^2 - 2L - 3 &= 0 \\ (L-3)(L+1) &= 0 \\ L &= 3, -1 \end{aligned}$$

We have two possible values for the limit, but in the assumption we know that $f(x) > 0 \ \forall x \in (a, b)\backslash\{c\}$. Therefore, it cannot converge to a negative value. Therefore, $L = \lim_{x\to c} f(x) = 3$. $\square$

Exercise 8. (5 pts) Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} x & , x \in \mathbb{Q} \\ -x & , x \notin \mathbb{Q}. \end{cases}$$

is discontinuous at any point of $\mathbb{R}\backslash\{0\}$ and continuous at 0.

Solution:

Exercise 9. (5 pts) Let $p(x) = x^2 + 2$. Find an interval where p is strictly decreasing and find a formula for its inverse.

Solution: In order to find where $p(x)$ is decreasing, we can find where $p'(x) < 0$.

$$\begin{aligned} p'(x) &= 2x \\ 2x &< 0 \\ x &< 0 \end{aligned}$$

So we know that $p(x)$ is decreasing whenever $x < 0$, or in the interval $(-\infty, 0)$. Now to find the inverse, we can swap $p^{-1}(x)$ with $p(x)$ and x with $p(x)$ then solve for $p^{-1}(x)$.

$$\begin{aligned} x &= p^{-1}(x)^2 + 2 \\ p^{-1}(x) &= \pm\sqrt{x-2} \end{aligned}$$

From here, we know that it must be $p^{-1}(x) = -\sqrt{x-2}$ because we are only looking for when $p(x)$ was decreasing, therefore $p^{-1}(x)$ must also be decreasing. $\square$

Exercise 10. (10 pts) Let $p(x) = ax^3 + bx^2 + cx + d$ be a polynomial of degree 3 and $a > 0$. Prove that p has at least one real root by following these steps:

a) Prove that $\lim_{x\to\infty} p(x) = \infty$.

b) Prove that $\lim_{x\to-\infty} p(x) = -\infty$.

c) Conclude.

[Hint for a): write your polynomial $p(x) = ax^3 + bx^2 + cx + d$ as $x^3(a + b/x + c/x^2 + d/x^3)$ and use the fact that $\lim_{x\to\infty} 1/x^n = 0$ for every $n \geq 1$.]

a) We can first rewrite $p(x)$ as

$$p(x) = x^3\left(a + \frac{b}{x} + \frac{c}{x^2} + \frac{d}{x^3}\right)$$
$$\lim_{x\to\infty} p(x) = \lim_{x\to\infty} x^3 \cdot (\lim_{x\to\infty} a + \lim_{x\to\infty} \frac{b}{x} + \lim_{x\to\infty} \frac{c}{x^2} + \lim_{x\to\infty} \frac{d}{x^3})$$

By the linearity of limits

$$\begin{aligned} &= \lim_{x\to\infty} x^3 \cdot (a + 0 + 0 + 0 + 0) \\ &= \lim_{x\to\infty} ax^3 \\ &\to \infty \end{aligned}$$

b) We must first prove that $\lim_{x\to-\infty} \frac{1}{x} = 0$. We must show that for any $-\delta$, we can find an $\epsilon > 0$ s.t. if $x < -\delta$ then

$$\left|\frac{1}{x} - 0\right| < \epsilon$$
$$-\epsilon < \frac{1}{x} - 0 < \epsilon$$

If we focus on the LHS of the inequality...

$$-\epsilon < \frac{1}{x}$$
$$x < -\frac{1}{\epsilon}$$

We can then set $-\delta = -\frac{1}{\epsilon}$ to satisfy the definition of a limit. Since $-\delta$ and ϵ are arbitrary, $\lim_{x\to-\infty} \frac{1}{x} = 0$.
Now using this, we know that $\lim_{x\to-\infty} 1/x^n = 0$ for every $n \geq 1$ because

$$\begin{aligned} \lim_{x\to-\infty} \frac{1}{x^n} &= \lim_{x\to-\infty} (\frac{1}{x_1} \cdot \frac{1}{x_2} \cdot ... \cdot \frac{1}{x_n}) \\ &= \lim_{x\to-\infty} \frac{1}{x_1} \cdot ... \cdot \lim_{x\to-\infty} \frac{1}{x_n} \end{aligned}$$

by the linearity of limits

$$
\begin{aligned}
&= 0 \cdot 0 \cdot ... \cdot 0 \\
&= 0
\end{aligned}
$$

Knowing this, we can rewrite $p(x)$ the exact same way as for part (a) and go through the same process to show that $\lim_{x \to -\infty} p(x) = -\infty$. Every step will be the same until we get to

$$
\begin{aligned}
\lim_{x \to -\infty} p(x) &= \lim_{x \to -\infty} x^3 \cdot (a + 0 + 0 + 0) \\
&= \lim_{x \to -\infty} ax^3 \\
&\to -\infty
\end{aligned}
$$

c) Using parts (a) and (b), we know that there must be a, b and A, B such that $p(a) = A < 0$ and $p(b) = B > 0$. Let's also define $L = 0$ such that $A < L < B$. Therefore, by the IVT, there must be c such that $a < c < b$ and $p(c) = L$. Therefore we know that $p(x)$ has at least one root where $p(x) = 0$ at $x = c$.