1	2	3	4	5	6	7	8	9	10	Total
0/5	3/5	1/5	5/5	5/10	6/10	4/5	1/5	4/5	9	38/70

Not the good way. Write $f(x) = f(t-c+a)$ make $t \to c$ so $x \to a$.

5 a.) If f is continuous at c, then there is an associated limit and sequence of f. This means that $\lim_{x \to c} f(x) = f(c)$. We can then write c as a combination of $\mathbb{R}$ numbers. This will be $n_1 + n_2 + \dots + n_k = c$. This means that $f(n_1)+f(n_2)+\dots+f(n_k) = f(c)$. With limit rules, we can say the limit exists at all of those points and they add up to $\lim_{x \to c} f(x) = f(c)$. And since they all have associated limits, it's continuous at all the points.

5/10 1/5

b.) $f(2) = f(1+1) = f(1)+f(1) = k+k = 2k$

$\mathbb{Z}$ is able to be written in terms of $f(1)$, so kx $(x \in \mathbb{Z})$

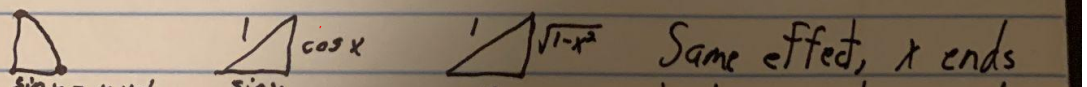
$f(q(1)) = f\left(\frac{a}{b}(1)\right) = f\left[(1)+f(1)+\dots+f(1)\right] = qk$. We can write $q \in \mathbb{Q}$ as $\frac{a}{b}$ and we can use $\frac{a}{b}$ as two seperate $\mathbb{Z}$ to get that even with fractions, $f(x) = kx$ $(x \in \mathbb{Q})$

4/5

More rigor: Induction.

And then in Q3 we saw that if f is continuous at $\mathbb{Q}$ with $\mathbb{Q} = kx$, so f is continuous on $\mathbb{R}$ and $f(\mathbb{R}) = kx$.

1



Same effect, x ends early because larger values.

0/5

$\frac{d}{dx} \sin x = \cos x$ $\frac{d}{dx} x = 1$ $0 \le \cos x \le 1$, so x increase at or at greater rate than $\sin x$. Since at $x=0$, $x=0$ & $\sin x = 0$, $x = \sin x$

We didn't cover the derivative yet.. See the solutions on the course website to see the geometric argument.

Use sequences. Take $x_n \to x_0$ with $x_n \in \mathbb{Q}$

By continuity, $f(x_n) = 0 \to f(x_0)$

$\Rightarrow f(x_0) = 0$

3 Continuous so $|x - x_0| < \delta$ $|f(x) - f(x_0)| < \varepsilon$ $\forall x \in [a,b]$

So if $\mathbb{Q} = x$ $f(x) = 0$. $\mathbb{Q}$ is dense in $\mathbb{R}$, so between each number in $\mathbb{R}$ is an infinite amount of $\mathbb{Q}$. Let $x_0 \in [a,b] / \mathbb{Q}$, so f is continuous so $|x - x_0| < \delta$ $|f(x) - f(x_0)| < \varepsilon$

If $x_0 \in \mathbb{Q}$, then $|x - \mathbb{Q}| < \delta \to |f(x)| < \varepsilon$

$\mathbb{Q} - \delta < x < \delta + \mathbb{Q}$ $f(\mathbb{Q} - \delta) < f(x) < f(\delta + \mathbb{Q})$

If $\delta \in \mathbb{Q}$, then $0 < f(x) < 0$, $f(x) = 0$. This must be true for all $x \in [a,b]$ to ensure it is continuous if $\delta \in \mathbb{Q}$. With this $|f(x)| < 0 < \varepsilon$, so ε is arbitrary.

15

~~For $f(x) \neq \eta$ $\forall x \in [u,v]$, then $[f(u), f(v)]$ must be strictly increasing or $f(v) \neq \eta$. If η is in $[f(a), f(b)]$, then if $f(a) > \eta$, there $\exists d \in [a,b]$ s.t. $\eta > f(d) < f(a)$. Let $d = u$ and $a = v$. If $f(b) > \eta$, same argument. Since d and $a \in [a,b]$, $[u,v] \subset [a,b]$~~

4 If $f(c) > 0$ for $c \in [a,b]$, then if $f(a), f(b) > f(c)$, then if $\eta = 0$, then $[u,v]$ can be any interval in $[a,b]$.

If either $f(a,b) < 0$ or $f(c)$, then we can pick the minimum value in $[u,v]$, and the max in $[u,v]$, and we know that there are values in between them w/ IVT and we can set η to be the minimum so that all numbers in $[u,v]$ are $\geq \eta$

5 | 5

Okay

2 $\lim_{t\to b} g(t) = a$ so $|t-b|<\delta_1$ $|g(t)-a|<\varepsilon$

$f\circ g = f(g(t))$. a is acc A, so $(a-\delta, a+\delta)\cap A$ is ∞.

So $|t-b|<\delta$ $|f(g(t))-L|<\varepsilon$ ~~$|g(t)-a|$~~.

As $t\to b$, $g(t)\to a$, so $|t-b|<\delta$ ~~$|f(a)-L|<\varepsilon$~~

And $\lim_{x\to a} f(x) = L$ so $|x-a|<\delta_2$ $|f(a)-L|<\varepsilon$

so pick $\delta =: \min\{\delta_1, \delta_2\}$ then $|t-b|<\delta$ $|f(a)-L|<\varepsilon$

In fact, you have to take $\delta = \delta_1$ only so that $|g(t)-a|<\delta_2$

3/5

$\lim_{x\to a} f(x) = f(a) = f(g(t))$ as $g(t)\to a$ $g(t)=a$ at $t\to b$

so $\lim_{t\to b} f(g(t)) = f(a) = \lim_{x\to a} f(x)$

Use sequences

8 $\delta>0$, $\varepsilon>0$, $|x-x_0|<\delta$ $|f(x)-f(x_0)|<\varepsilon$ (15)

$|x-0|<\delta$ $|x-0|<\varepsilon$ $|x|<\delta$ & $|x|<\varepsilon$

$\varepsilon=\delta$, $f(x)$ is continuous at 0.

$\mathbb{Q}$ is dense in $\mathbb{R}$, so between every $x\in\mathbb{R}$ are an infinite amount of $\mathbb{Q}$.

$x_0\in\mathbb{Q}$ $|x-\frac{p}{q}|<\delta$ $|f(x)-f(\frac{p}{q})|<\delta \to |f(x)-\frac{p}{q}|<\varepsilon$ $f(x)=-x$

$|x-\frac{p}{q}|<\delta$ $|f(x)-\frac{p}{q}|=|-x-\frac{p}{q}|<\varepsilon$ $\frac{p}{q}-\delta<x<\frac{p}{q}+\delta$ $\delta=\varepsilon$

$\frac{p}{q}-\varepsilon<x<\frac{p}{q}+\varepsilon$ $|-x-\frac{p}{q}|<\varepsilon$ $\frac{p}{q}-\varepsilon<-x<\frac{p}{q}+\varepsilon$ $x\neq 0$ # not continuous

also $\delta=2\varepsilon$ $\frac{p}{q}-2\varepsilon<x<\frac{p}{q}+2\varepsilon$ $x<\frac{p}{2q}+\varepsilon$ $|-\frac{p}{q}-\varepsilon-\frac{p}{q}|\geq\varepsilon$ $|-\frac{2p}{q}-\varepsilon|>\varepsilon$

~~ε not positive~~ $\frac{2p}{q}+\varepsilon>\varepsilon$ $\frac{2p}{q}>0$ ✓

$x_0\in\mathbb{R}$ $|x-x_0|<\delta$ $|f(x)-f(x_0)|\geq\varepsilon$ $x_0-\delta<x<x_0+\delta$ $\delta=\varepsilon$ $x=\mathbb{Q}(\frac{p}{q})$

in between $\mathbb{R}$ $x_0-\delta<\frac{p}{q}<x_0+\delta$ $|f(x)-f(x_0)|\geq\varepsilon$ $|2x|\geq\varepsilon$ $|\frac{p}{q}+x_0|\geq\varepsilon$

↳ Justify (4/5)

9 Decreasing from $(-\infty,0)$ $f^{-1}(f(x))=x$ $f(f^{-1}(x))=x$

$f(x)=x^2+2$ $x=(f^{-1}(x))^2+2$ $x-2=(f^{-1}(x))^2$

$f^{-1}(x)=-\sqrt{x-2}$ $f(f^{-1}(x))=(-\sqrt{x-2})^2+2=x-2+2=x$ ✓

HW #5

6 a) DNE. $\delta>0$, $|x-x_0|<\delta$ $\exists\varepsilon$ $|f(x)-L|\geq\varepsilon$

consider sequences instead.

$|x-0|<\delta$ $|x|<\delta$ $|\sin(\frac{1}{x})-L|\geq\varepsilon$

$-1\leq\sin(a)\leq 1$ $\sin(\frac{1}{x})\geq\varepsilon+L\geq-\sin(\frac{1}{x})$

$\frac{1}{x}\geq\sin^{-1}(\varepsilon+L)\geq-\frac{1}{x}$ → You need to prove that arcsin is increasing

$\frac{1}{\sin^{-1}(\varepsilon+L)}\geq x\geq\frac{-1}{\sin^{-1}(\varepsilon+L)}$ $|x|<\frac{1}{\sin^{-1}(\varepsilon+L)}$

if $\delta=\frac{1}{\sin^{-1}(\varepsilon+L)}$, then there $\exists\varepsilon$ such that $|f(x)-L|\geq\varepsilon$, therefor $\lim_{x\to 0}\sin(\frac{1}{x})=DNE$

6/10

b.) $\lim_{x\to 0}x\sin(\frac{1}{x})=0$, Let $\delta>0$, $\varepsilon>0$

$0\leq\sin x\leq 1$ $0\leq x\sin x\leq x$ $|x-0|<\delta$ $|x|<\delta$

$|x\sin(\frac{1}{x})-0|<\varepsilon$ $|x\sin(\frac{1}{x})|<\varepsilon$ $\delta=\varepsilon$

$|x\sin(\frac{1}{x})|\leq x<\delta=\varepsilon$ $|x\sin(\frac{1}{x})|<\varepsilon$

ε was arbitrary $\lim_{x\to 0}x\sin(\frac{1}{x})=0$

1/5

5/5

4/5

7 $\lim_{x\to c}f(x)=L$ $\lim_{x\to c}f(x)^2=L^2$ (Limit algebra)

$\lim_{x\to c}-f(x)=-L$ (Limit algebra) $\lim_{x\to c}-3=-3$ (constant limit)

$L=L^2-L-3$ → $0=L^2-2L-3$ solve for L

$\frac{2\pm\sqrt{2^2-4\cdot1\cdot-3}}{2(1)}=1\pm\sqrt{4+12}$ 1 ± 4 $L=-3,5$

$\lim_{x\to c}f(x)=-3,5$ but $f(x)>0$ so $\lim_{x\to c}f(x)=5$

3

9/10

10 a.) $ax^3+bx^2+cx+d = x^3(a+b/x+c/x^2+d/x^3)$ $\lim_{x\to\infty} = x^3(a+0+0+0)$

→ more rigor. Prove that $\frac{1}{x}, \frac{1}{x^2}, \frac{1}{x^3}$ goes to zero

$\lim_{x\to\infty} x^3 a$ Let $y=m$ $x=x$ in A.P., $m<nx$ $m^3<n^3x^3$

if $M=m^3$ then $M<nx^3$ so if M is an upper bound,

$\exists x\in\mathbb{R}$ s.t. $x^3>M$ meaning $\lim_{x\to\infty} x^3=\infty$ so

$\lim_{x\to\infty} ax^3+bx^2+cx+d=\infty$ okay

b.) Let M in this case be $-M$, so $-M$ is upper bound. So repeat above steps with $-M$ as ~~upper~~ lower bound. So $\lim_{x\to-\infty} ax^3+bx^2+cx+d=-\infty$ okay

c.) Since these are positive and negative values, the IVT states there must be $f(c)=0$ $c\in\mathbb{R}$, so the function has at least one real root.

okay.