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9/10/21 (1)

MATH 331 Homework 02

1 Exercise 1. (10pts)

a) Let $\{[a_n, b_n] : n \geq 1\}$ be a family of closed intervals such that $[a_1, b_1] \supset [a_2, b_2] \supset [a_3, b_3] \supset \dots$. Show that there is a $c \in \mathbb{R}$ such that $c \in [a_n, b_n]$ for all $n \geq \mathbb{N}$. Follow the following steps to prove it:

(i) Prove that for any $n, m \geq 1$, $a_n \leq b_m$. [hint: put $M := \max\{n, m\}$]

(ii) Show that $\sup\{a_n : n \geq 1\}$ exists.

(iii) Show that $c = \sup\{a_n : n \geq 1\}$ satisfies the requirements.

Suppose $\{[a_n, b_n] ; n \geq 1\}$. Then clearly for any $n, m \geq 1$ on the closed interval $[a_n, b_n]$ that $a_n \leq b_n$. The same is true for the closed interval $[a_m, b_m]$ then $a_m \leq b_m$. define $M := \max\{n, m\}$ so that if n is max then $[a_m, b_m] \supset [a_n, b_n]$ so $a_m \leq b_n$ and $a_n \leq b_m$ or if m is max then $[a_n, b_n] \supset [a_m, b_m]$ so $a_n \leq b_m$ and $a_m \leq b_m$. As you can see it is true that $a_n \leq b_m$.(i) Now we must show that the $\sup\{a_n : n \geq 1\}$ exists. It is obvious from the definition $[a_n, b_n] \supset [a_{n+1}, b_{n+1}]$ that $a_{n+1} \geq a_n$ for a $n \in \mathbb{N}$. Then for any $n \geq 1$, a_n will be greater than or equal to all a_i for $1 \leq i \leq n-1$. This means there exists a $M \in \mathbb{R}$ and for all $x \in \{a_n : n \geq 1\}$ then $x \leq M_n$. In particular M_n can be equal to a_n. In addition for all K_n that are upper bounds of $\{a_n : n \geq 1\}$ then $x \leq M_n < K_n$; so $\sup\{a_n : n \geq 1\}$ exist.(ii) We are asked to show that $c = \sup\{a_n ; n > 1\}$ satisfies the requirements. Let $c = M_n$ then by the same argument above $x \leq c \leq K_n$ where $x \in \{a_n ; n > 1\}$, $c =$ supremum $\in \mathbb{R}$, $K_n =$ all upper bounds. Therefore it satisfies the requirement of the supremum, since $b_m \geq c = a_n \geq a_{n-1}$, for any n, clearly $c \in [a_n, b_n]$ for all $n \in \mathbb{N}$.

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b) use this last result to prove that the set $\mathbb{R}$ is uncountable. (Hint: show that any function $f: \mathbb{N} \to \mathbb{R}$ can't be surjective. To do so construct a sequence of closed intervals such that $f(n) \notin [a_n, b_n]$ with $a_n < b_n$.)

suppose $n > m$ such that $a_m \leq a_n$ and $b_n \leq a_m$ with $a_n < b_n$ and $a_m < b_m$. Define $f: \mathbb{N} \to \mathbb{R}$ where $f(n) \notin [a_n, b_n]$ however, $f(n) \in [a_m, a_n] \cup [b_n, b_m]$. $c \in [a_n, b_n]$ but $c \notin [a_m, a_n] \cup [b_n, b_m]$. Then for all $n \geq 1$ $Im(f(n))$ for all $n \geq 1$, $Im(f(n)) = [a_1, b_1]$ or $f(n) \in [a_1, b_1] \neq \mathbb{R}$. Therefore this function is not the set of all real numbers and therefore not surjective. Since a function cannot be found we can say that $\mathbb{R}$ is uncountable. Also $c \in [a_n, b_n]$, but $c \notin [a_m, a_n] \cup [b_n, b_m]$

2 Exercise 2. (5 pts) Prove that if $a_n \to A$, then $|a_n| \to |A|$

Proof: suppose a_n converges to A, then by the definition of convergence for all $\varepsilon > 0$, there exist an $N_1 \in \mathbb{N}$, such that for all $n \geq N_1$ we have $|a_n - A| < \varepsilon$. Now suppose $|a_n|$ and take the limit $L = |A|$ such that for all $\varepsilon > 0$, there is an $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$ then $||a_n| - |A|| < \varepsilon$. We have to show this. Let $\varepsilon > 0$ and arbitrary then by the properties of absolute value, 0.2.5 Theorem iv (pg 26) that $||a| - |b|| \leq |a - b|$, so take $|a| = |a_n|$ and $|b| = |A|$ then by this inequality $||a_n| - |A|| \leq |a_n - A|$ but since a_n converges to A it must be true that $|a_n - A| < \varepsilon$. so by transitivity $||a_n| - |A|| \leq |a_n - A| < \varepsilon$. Note to resolve that N_1 and N_2 are not necessarily equal set $N := \max\{N_1, N_2\}$. Thus we have shown that $\varepsilon > 0$ that $||a_n| - |A|| \leq |a_n - A| < \varepsilon$ for all $n \geq N$. Since $\varepsilon > 0$ was arbitrary we have shown that if $a_n \to A$ then $|a_n| \to |A|$. ☒

3 Exercise 3. (5 pts) Let (a_n), (b_n), and (c_n) be sequences of real numbers. Prove that if $a_n \to L$, $b_n \to L$ and $a_n \leq c_n \leq b_n$, then $c_n \to L$.

suppose $a_n \leq c_n \leq b_n$.

suppose a_n converges to L such that for all $\varepsilon > 0$, there exists an $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$ we have $|a_n - L| < \varepsilon$. suppose b_n converges to L such that for all $\varepsilon > 0$ there exists an $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$ we have $|b_n - L| < \varepsilon$. Now suppose c_n converges to G where for all $\varepsilon > 0$ there exists an $N_3 \in \mathbb{N}$ such that for all $n \geq N_3$ we have $|c_n - G| < \varepsilon$. We will prove than G must equal L. Now suppose we know that $\{a_n\}_{n=1}^{\infty}$ converges to L and $\{c_n\}_{n=1}^{\infty}$ converges G with $a_n \leq c_n$ for all $n \in \mathbb{N}$ then by 1.2 theorem. we know $L \leq G$. Now suppose $\{c_n\}_{n=1}^{\infty}$ converges to G and $\{b_n\}_{n=1}^{\infty}$

converges to L with $c_n \le b_n$ for all $n \in \mathbb{N}$ then we know $G \le L$ by 1.12 Theorem (pg. 48). By using transitivity of $L \le G$ and $G \le L$ then $L \le G \le L$. The only case this is true is when $G = L$. Therefore we shown that given $\{a_n\}_{n=1}^{\infty}$ converges to L and $\{b_n\}_{n=1}^{\infty}$ converges to L and $a_n \le c_n \le b_n$ then $\{c_n\}_{n=1}^{\infty}$ must converge to L. ∎

4 Exercise 4 (5pts) prove that if $a_n \to A$ and $a_n \ge 0$ for all $n \ge 1$, then $\sqrt{a_n} \to \sqrt{A}$. Follow the following steps to prove it:

1. consider the case $A = 0$
2. suppose that $A \ne 0$. show that there is a $N_1 \in \mathbb{N}$ such that if $n \ge N_1$ then $\sqrt{a_n} \ge \sqrt{|A|/2}$.
   [Hint: use the definition of convergence of $(a_n)_{n \ge 0}$ with a clever choice of ε and use the properties of absolute value.]
3. Use the convergence of (a_n) again to find a N_2 such that $|a_n - A| < \frac{3}{4}\frac{\varepsilon}{\sqrt{|A|}}$
4. Express $\sqrt{a_n} - A$ as $\frac{a_n - A}{\sqrt{a_n} + \sqrt{A}}$ and put $N = \max\{N_1, N_2\}$. Conclude.

suppose $\{a_n\}_{n=1}^{\infty}$ converges to A and $a_n \ge 0$. In the case where $A = 0$ we have for all $\varepsilon_0 > 0$ there exist an $N \in \mathbb{N}$ such that for all $n \ge N$ we have $|a_n - 0| = |a_n| < \varepsilon_0$. By the definition of convergence take "$N = 1$" such that for all $n \ge N = 1$ $a_n \ge 0$ so that $-\varepsilon_0 < a_n < \varepsilon_0$. Therefore $0 \le a_n < \varepsilon_0$ for all $n \ge 1$. Lets show that if $\varepsilon > 0$ then there exists an $N_2 \in \mathbb{N}$ such that for all $n \ge N_2$ then $|\sqrt{a_n} - \sqrt{A}| = |\sqrt{a_n} - 0| = \sqrt{a_n} < \varepsilon_1$. To show this take $\varepsilon_1 = \sqrt{\varepsilon_0}$ then since we know $0 \le a_n \le \varepsilon_0$ then (by HW1 $0 \le a \le b$ then $\sqrt{a} < \sqrt{b}$) then $|\sqrt{a_n} - \sqrt{A}| = |\sqrt{a_n} - 0| = \sqrt{a_n}$

$\sqrt{a_n} < \sqrt{\varepsilon_0} = \varepsilon_1$ Thus we have shown that since for $n \geq 1 = N$ that the case where $A = 0$ it is true that $\sqrt{a_n}$ converges to the $\sqrt{A}$.

Now consider that $A \neq 0$ then we must show that

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5 Exercise 5. (5 pts) For each sequence $(a_n)_{n=1}^{\infty}$ define the sequence $(\sigma_n)_{n=1}^{\infty}$ by

$$\sigma_n := \frac{a_1 + a_2 + \dots + a_n}{n} \quad (n \geq 1)$$

Prove that if $a_n \to A$, then $\sigma_n \to A$. Find an example of a divergent sequence (a_n) such that $(\sigma_n)_{n=1}^{\infty}$ converges

Proof: suppose that $a_n \to A$ then it must be true that for all $\varepsilon > 0$ there is an $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$ we have $|a_n - A| < \varepsilon$. We are trying to show that for $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$ that $|\sigma_n - A| < \varepsilon$

Let $\varepsilon > 0$ be arbitrary. Since we know $a_n \to A$ and that $|a_n - A| < \frac{\varepsilon}{n}$ then

$$\left| \frac{a_1 + a_2 + \dots + a_n - A_1 - A_2 - A_3 \dots - A_n}{n} \right|$$

where $A_1 = A_2 = \dots = A_n$

then by the triangle inequality we have

$$\leq |a_1 - A_1| + |a_2 - A_2| + |a_3 - A_3| + \dots + |a_n - A_n|$$

$$\leq \left(\frac{\varepsilon}{n}\right) n = \varepsilon$$

We have shown that if $\varepsilon > 0$ there is an $N \in N_1$ such that for all $n \geq N_1$ then $\left|\frac{a_1 + a_2 + \dots + a_n}{n} - A\right| < \varepsilon$. Since ε was arbitrary it implies $\{\sigma_n\}_{n=1}^{\infty}$ converges to A.

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6. Exercise 6. (10 pts) Use the definition of convergence to prove that each of the following sequences converges.

a) $(a_n)_{n=1}^{\infty}$ given by $a_n = 5 + 1/n$ for $n \geq 1$.

Proof: By the definition of convergence we will take $A = 5$ and show that $(5 + 1/n)_{n=1}^{\infty}$ converges to 5. Let $\varepsilon > 0$ then there exists an N such that for all $n \geq N$ we have $|5 + \frac{1}{n} - 5| = |\frac{1}{n}| = \frac{1}{n} < \varepsilon$. By the Archimedean property $[(x, y \in \mathbb{R}, x > 0\ \exists n \in \mathbb{N}$ s.t $(nx > y)]$ we will take $n = N_0$, $x = \varepsilon$, and $y = 1$ then $N_0 \varepsilon > 1$. Here we take $N = N_0$, so if $n \geq N_0$ we have $n\varepsilon \geq N_0 \varepsilon$ (by axiom O4) and by transitivity (by axiom O2) we have $n\varepsilon \geq N_0\varepsilon > 1$ or $n\varepsilon > 1$ for all $n \geq N_0$. Then this implies the following

$$|5 + \tfrac{1}{n} - 5| = |\tfrac{1}{n}| = \tfrac{1}{n} < \varepsilon \quad \text{for all } n \geq N_0.$$

We have just shown that if $\varepsilon > 0$, there exist an $N = N_0$ such that, for all $n \geq N_0$ then $|5 + \frac{1}{n} - 5| = \frac{1}{n} < \varepsilon$, since $\varepsilon > 0$ was arbitrary $(5 + \frac{1}{n})_{n=1}^{\infty}$ converges to 5. ∎

b) $(a_n)_{n=1}^{\infty}$ given by $a_n = \frac{3n}{2n+1}$ for $n \geq 1$

Proof: By the definition of convergence we will take $A = \frac{3}{2}$ and show that $(\frac{3n}{2n+1})_{n=1}^{\infty}$ converges to $3/2$. Let $\varepsilon > 0$ then there exists an N such that for all $n \geq N$ we have $|\frac{3n}{2n+1} - \frac{3}{2}| = |\frac{-3}{4n+2}| = \frac{3}{4n+2} < \varepsilon$. By the Archimedean property we will take $n = N_0$, $x = \varepsilon$, and $y = \frac{3}{4}$, then $N_0 \varepsilon > \frac{3}{4}$. Now take $N = N_0$, so if $n \geq N_0$ then we have $n\varepsilon \geq N_0 \varepsilon > \frac{3}{4}$ (by axiom O4 & transitivity axiom O2). now we have $n\varepsilon > 3/4$ for all $n \geq N_0$. This implies the following:

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$$\left|\frac{3n}{2n+1}-\frac{3}{2}\right| = \left|\frac{-3}{4n+2}\right| = \frac{3}{4n+2} < \frac{3}{4n} < \varepsilon \quad \text{for all } n \geq N_0$$

We have shown that for $\varepsilon > 0$ there exist an $N = N_0$ such that for all $n \geq N_0$ then $\left|\frac{3n}{2n+1}-\frac{3}{2}\right| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary $\left(\frac{3n}{2n+1}\right)_{n=1}^{\infty}$ converges to $3/2$. ∎

7. Exercise 7. (5pts) Prove that the sequence $(a_n)_{n=1}^{\infty} = \left(\frac{2n+1}{n}\right)_{n=1}^{\infty}$ is a cauchy sequence.

Proof: By the definition of a cauchy sequence for all $\varepsilon > 0$ there is a positive integer N such that if $m, n \geq N$, then $|a_n - a_m| = \left|\frac{2n+1}{n} - \frac{2m+1}{m}\right| < \varepsilon$. To show this let $\varepsilon > 0$. By the Archimedean property we can choose $N > \frac{2}{\varepsilon}$ $(n = N, x = 1, y = \frac{2}{\varepsilon})$. Then for all $m, n \geq N$ we have the following:

$$\left|\frac{2n+1}{n} - \frac{2m+1}{m}\right| = \left|\frac{m-n}{mn}\right| = \left|\frac{1}{n} - \frac{1}{m}\right|$$
$$\leq \left|\frac{1}{n}\right| + \left|\frac{1}{m}\right| \quad \text{(by the triangle inequality)}$$
$$= \frac{1}{n} + \frac{1}{m}$$

but we know $m, n \geq N$ and $N > \frac{2}{\varepsilon}$ so by transitivity $m, n > \frac{2}{\varepsilon}$ so, $n > \frac{2}{\varepsilon}$ implies $\frac{1}{n} < \frac{\varepsilon}{2}$ and $m > \frac{2}{\varepsilon}$ implies $\frac{1}{m} < \frac{\varepsilon}{2}$ (by properties (iii) of real numbers), so now,

$$\left|\frac{2n+1}{n} - \frac{2m+1}{m}\right| = \left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{1}{n} + \frac{1}{m} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

We have shown that for $\varepsilon > 0$ there exist an $N > \frac{2}{\varepsilon}$ such that $\left|\frac{2n+1}{n} - \frac{2m+1}{m}\right| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary we have shown that $\left(\frac{2n+1}{n}\right)_{n=1}^{\infty}$ is a cauchy sequence. ∎

8. Exercise 8. (10pts) Prove that each of the following sequences diverges.

a) $(a_n)_{n=1}^{\infty} = ((-1)^n)_{n=1}^{\infty}$

Proof: Let's prove that $((-1)^n)_{n=1}^{\infty}$ diverges by showing a contradiction. Let's suppose it converges to A. Set $\varepsilon = 1$ then there must exist an $N \in \mathbb{N}$ such that for all $n \geq N$, then $|(-1)^n - A| < 1$. Note that

$(-1)^n = 1$ for all n that are even. Also, $(-1)^n = -1$ for all n that are odd. Define $(-1)^{2k}$ for $k \in \mathbb{N}$ for the case where $(-1)^n = 1$ and define $(-1)^{2k-1}$ for $k \in \mathbb{N}$ for the case where $(-1)^n = -1$. If $|(-1)^n - A| < 1$ and n is even then $|(-1)^n - A| = |(-1)^{2k} - A| = |1 - A| < 1$ or $-1 < 1 - A < 1$ (by property of Absolute value) and if we add -1 to each side (axiom O1) then $-2 < -A < 0$. If $|(-1)^n - A| < 1$ and n is odd then $|(-1)^n - A| = |(-1)^{2k-1} - A| = |-1 - A| < 1$ or $-1 < -1 - A < 1$ by the property of absolute value. If you add 1 to the inequality (axiom O1) then $0 < -A < 2$. So we see that $-A \in (-2, 0)$ and $-A \in (0, 2)$ so $-A \in (-2, 0) \cap (0, -2) = \emptyset$ which is a contradiction. Therefore, the assumption than $((-1)^n)_{n=1}^{\infty}$ converges is false. Thus, $((-1)^n)_{n=1}^{\infty}$ diverges.

b) $(a_n)_{n=1}^{\infty} = (\sin(\frac{2n+1}{2}\pi))_{n=1}^{\infty}$

Proof: Let's prove that $(\sin(\frac{2n+1}{2}\pi))_{n=1}^{\infty}$ diverges by showing a contradiction. Suppose $(\sin(\frac{2n+1}{2}\pi))_{n=1}^{\infty}$ converges to A. Set $\varepsilon = 1$ then there must be an $N \in \mathbb{N}$ such that for all $n \geq N$ then $|\sin(\frac{2n+1}{2}\pi) - A| < 1$. Define $\sin(\frac{2n+1}{2}\pi)$ for n is even as $\sin(\frac{2(2k)+1}{2}\pi) = 1$ for $k \in \mathbb{N}$ and define $\sin(\frac{2n+1}{2}\pi)$ for n is odd as $\sin(\frac{2(2k-1)+1}{2}\pi) = -1$ for $k \in \mathbb{N}$. In the case where n is even then $|\sin(\frac{2n+1}{2}\pi) - A| = |\sin(\frac{2(2k)+1}{2}\pi) - A| = |1 - A| < 1$. By property of absolute value $-1 < 1 - A < 1$. and adding -1 to the inequality (axiom O1) then $-2 < -A < 0$. In the case where n is odd then $|\sin(\frac{2n+1}{2}\pi) - A| = |\sin(\frac{2(2k-1)+1}{2}\pi) - A| = |-1 - A| < 1$. By the property of absolute value $-1 < -1 - A < 1$ and by adding 1 to the inequality $0 < -A < 2$. So $-A$ must be an element of both sets or $-A \in (-2, 0) \cap (0, 2) = \emptyset$. This is a contradiction, since we showed that the assumption $(\sin(\frac{2n+1}{2}\pi))_{n=1}^{\infty}$ is false, it must be true. than $((\sin(\frac{2n+1}{2}\pi))_{n=1}^{\infty}$ diverges.

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9 Exercise 9 (5pts) Give an example of two sequences (a_n) and (b_n) such that (a_n) and (b_n) dont converge but (a_n+b_n) converge.

The example is $(a_n)_{n=1}^{\infty}=(n+1)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}=(-n)_{n=1}^{\infty}$.

(i) Lets prove that $(n+1)_{n=1}^{\infty}$ diverges using a contradiction. Suppose $(n+1)_{n=1}^{\infty}$ converges to A. Then by 1.2 Theorem (pg 37) it must be bounded. Based on 1.2 Theorem a_n must be bounded from above $a_n \leq M_1$ where $M \in \mathbb{R}$, so $n+1 \leq M_1$ for all $n \in \mathbb{N}$. However, by the Archimedean principal, (let $x=1$, $n=N_1$, $y=M_1-1$) then $N_1 \geq M_1-1$ for $N_1 \in \mathbb{N}$ and (based on Axiom O1) then $N_1+1 \geq M_1$ which is a contradiction. Therefore $(n+1)_{n=1}^{\infty}$ must diverge.

(ii) Lets prove $(b_n)_{n=1}^{\infty}=(-n)_{n=1}^{\infty}$ diverges using a contradiction. Suppose $(-n)_{n=1}^{\infty}$ converges to A. By 1.2 Theorem (pg 37) it must be bounded and say it has a lower bound such that for all n, $S_1 \leq a_n$ where $S_1 \in \mathbb{R}$. So $S_1 \leq -n$ or $S_1+n<0$ (based on axiom O1) for all $n \in \mathbb{N}$. However, based on the Archimedean principal (let $x=1$, $y=-S_1 \in \mathbb{R}$, $n=N_2$) then $-S_1<N_2$ or $0<S_1+N_2$ or $-N_2<S_1$ (based on axiom O1). The existence of N_2 contradicts the assumption that $(-n)_{n=1}^{\infty}$ is bounded, therefore it is unbounded and diverges.

(iii) in this example, $(a_n)_{n=1}^{\infty}=(n+1)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}=(-n)_{n=1}^{\infty}$ diverges but $(a_n+b_n)_{n=1}^{\infty}$ converges. We will prove that. Suppose $a_n=n+1$, $b_n=-n$ then $(a_n+b_n)_{n=1}^{\infty}=(n+1-n)_{n=1}^{\infty}=(1)_{n=1}^{\infty}$ which is a constant sequence. Let $A=1$. For $\varepsilon>0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have $|1-1|=0<\varepsilon$. Lets consider $N=1$ then if $n \geq 1$ we have $|a_n-A|=|1-1|=0<\varepsilon$ for all $n \geq 1$. We just proved that if $\varepsilon>0$, there exists an $N=1$ such that for all $n \geq N$ $|a_n-1|<\varepsilon$. Since ε was arbitrary we shown that $(a_n+b_n)_{n=1}^{\infty} = (n+1-n)_{n=1}^{\infty} = (1)_{n=1}^{\infty}$ converges.

10 Exercise 10 (10pts) with the limit operations and the writing problems, find the limit of the following sequence with general term.

We are asked to find the limits, not to prove that the sequence converges to A. Also using limit operations as on pg 46 of the book.

a) $\left(\frac{n^2+4n}{n^2-5}\right)_{n=1}^{\infty}$

We can modify the fraction by multiplying by $1/1 = {}^{1}/_{n^2} / {}^{1}/_{n^2}$

so $\left(\frac{n^2+4n}{n^2-5}\right)_{n=1}^{\infty} = \left(\frac{1/n^2}{1/n^2}\left(\frac{n^2+4n}{n^2-5}\right)\right)_{n=1}^{\infty} = \left(\frac{1+4/n}{1-5/n^2}\right)_{n=1}^{\infty}$

by theorem 1.11 states that we can look at the numerator and denominator. $(1+4/n)_{n=1}^{\infty}$ converges to 1 because we proved that $(1)_{n=1}^{\infty}$ converges to 1 and $\left(\frac{1}{n}\right)_{n=1}^{\infty}$ converges to zero proven in the book on pg 34. So $(1+4/n)_{n=1}^{\infty}$ converges to $1+0=1$.

The denominator $(1-5/n^2)_{n=1}^{\infty}$ can be broken up. $(1)_{n=1}^{\infty}$ converges to 1 and $(5/n^2)_{n=1}^{\infty}$ by 1.9 theorem $\left(\frac{1}{n}\right)_{n=1}^{\infty}$ converges to zero so $\left(\frac{1}{n}\cdot\frac{1}{n}\right)_{n=1}^{\infty}$ converges to 0. Therefore $(1-5/n^2)_{n=1}^{\infty}$ converges to $1-0=1$ since $b_n = 1-5/n^2$ doesn't converge to zero or ever equal to zero, we can conclude the limit of the sequence based on 1.11 Theorem is $\left(\frac{n^2+4n}{n^2-5}\right)_{n=1}^{\infty}$ converges to $\frac{1}{1}=1$.

b) $\left(\frac{n}{n^2-3}\right)_{n=1}^{\infty}$

We can modify the fraction by multiplying by $1/1 = {}^{1}/_{n^2} / {}^{1}/_{n^2}$

so $\left(\frac{n}{n^2-3}\right)_{n=1}^{\infty} = \left(\frac{1/n^2}{1/n^2}\,\frac{n}{n^2-3}\right)_{n=1}^{\infty} = \left(\frac{1/n}{1-3/n^2}\right)_{n=1}^{\infty}$.

By theorem 1.11. we must look at the numerator and denominator separately. The numerator $(1/n)_{n=1}^{\infty}$ we already prove on pg 34 of the book and it converges to 0. The denominator $(1-3/n^2)_{n=1}^{\infty}$ converges to 1 since $(1)_{n=1}^{\infty}$ converges to 1 and $(1/n)_{n=1}^{\infty}$ converges to zero so by 1.9 theorem $\left(\frac{1}{n^2}\right)_{n=1}^{\infty}$ converges

to 0. This means $(1-3/n^2)_{n=1}^{\infty}$ converges to $1-3\cdot 0=1$.
so since the denominator does not converge to zero with no steps $b_n = 0$, $b\in\{-2,1,6,...\}$ then by theorem 1.11, the limit of a fraction is the limit of numerator divided by denominator or $0/1=0$.

c) $\left(\frac{\cos n}{n}\right)_{n=1}^{\infty}$

Note $\cos n$ is a bounded sequence particularly $-1\le \cos n \le 1$ meaning $|\cos n| \le 1$
Based on the axiom O4 we can multiply both sides by $\frac{1}{n}$, so $\frac{|\cos n|}{n} < \frac{1}{n}$. Therefore $0 \le \left(\frac{|\cos n|}{n}\right)_{n=1}^{\infty} \le \left(\frac{1}{n}\right)_{n=1}^{\infty}$.
By exercise 3 $a_n \le c_n \le b_n$ if $a_n=0$ converges to 0 and $b_n=(\frac{1}{n})$ converges to zero, therefore $c_n=\frac{|\cos n|}{n}$ converges to 0.
Alternatively, 1.13 theorem (pg 49) Break $\left(\frac{\cos n}{n}\right)_{n=1}^{\infty}$ into $(\frac{1}{n}\cdot \cos n)_{n=1}^{\infty}$ where $(a_n)_{n=1}^{\infty} = (\frac{1}{n})_{n=1}^{\infty}$ converges to zero and $(b_n)_{n=1}^{\infty} = (\cos n)_{n=1}^{\infty}$ is bounded. Therefore $(\frac{1}{n}\cdot\cos n)_{n=1}^{\infty}$ converges to zero or $0\cdot\cos n = 0$.

d) $(\sqrt{4-\frac{1}{n}}-2)n$

Lets manipulate $(\sqrt{4-\frac{1}{n}}-2)n$ by multiplying by $1/1 = (\sqrt{4-1/n}+2)/(\sqrt{4-1/n}+2)$ then we have

$$n(\sqrt{4-1/n}-2)\left(\frac{\sqrt{4-1/n}+2}{\sqrt{4-1/n}+2}\right) = n\left(\frac{\cancel{4}-1/n-\cancel{4}}{\sqrt{4-1/n}+2}\right)$$

$$= \frac{-1}{\sqrt{4-1/n}+2}$$

so $\left((\sqrt{4-1/n}-2)n\right)_{n=1}^{\infty} = \left(\frac{-1}{\sqrt{4-1/n}+2}\right)_{n=1}^{\infty}$. No based on 1.11 Theorem lets look at numerator and denominator. The numerator $(-1)_{n=1}^{\infty}$ converges to -1. The denominator is $\sqrt{4-1/n}+2$ and at first use Exercise 4. $(4-1/n)_{n=1}^{\infty}$ converges to 4 as $(4)_{n=1}^{\infty}$ converges to 4 and $(\frac{1}{n})_{n=1}^{\infty}$ converges to 0. Then $(\sqrt{4-1/n})_{n=1}^{\infty}$ must converge to $\sqrt{4}$ or 2.

so the denominator $\left(\sqrt{4-1/n}+2\right)_{n=1}^{\infty}$ can be broken into $\left(\sqrt{4-1/n}\right)_{n=1}^{\infty}$ which converges to 2 and $(2)_{n=1}^{\infty}$ converges to 2 so therefore $\left(\sqrt{4-1/n}+2\right)_{n=1}^{\infty}$ converges to $(2+2)_{n=1}^{\infty}$ $= (4)_{n=1}^{\infty}$ which converges to 4. so by 1.11 Theorem the numerator converges to -1 and denominator to 4 so $B \neq 0$ and $|b_n|$ is bounded away from zero $b_n \in [\sqrt{3}+2, 2)$ then the limit of $\left(\left(\sqrt{4-1/n}-2\right)n\right)_{n=1}^{\infty} = \left(\frac{-1}{\sqrt{4-1/n}+2}\right)_{n=1}^{\infty}$ converges to $-1/4$.