MATH-331 Introduction to Real Analysis — Pierre-Olivier Parisé
Homework 02 — Fall 2021

Due date: 20-09-2021 1:20pm — Total: /70.

| Exercise | 1 (10) | 2 (5) | 3 (5) | 4 (5) | 5 (5) | 6 (10) | 7 (5) | 8 (10) | 9 (5) | 10 (10) |
|---|---|---|---|---|---|---|---|---|---|---|
| Score | | | | | | | | | | |

Table 1: Scores for each exercises

**Instructions:** You must answer all the questions below and send your solution by email (to parisepo@hawaii.edu). If you decide to not use LaTeX to hand out your solutions, please be sure that after you scan your copy, it is clear and readable. Make sure that you attached a copy of the homework assignment to your homework. No late homework will be accepted. No format other than PDF will be accepted. Name your file as indicated in the syllabus.

## 1 Writing problems

For each of the following problems, you will be asked to write a clear and detailed proof. You will have the chance to rewrite your solution in your semester project after receiving feedback from me.

**Exercise 1.** (10 pts)

**a)** Let $\{[a_n, b_n] : n \geq 1\}$ be a family of closed intervals such that $[a_1, b_1] \supset [a_2, b_2] \supset [a_3, b_3] \supset \cdots$. Show that there is a $c \in \mathbb{R}$ such that $c \in [a_n, b_n]$ for all $n \geq \mathbb{N}$. Follow the following steps to prove it:

- **(i)** Prove that for any $n, m \geq 1$, $a_n \leq b_m$. [hint: put $M := \max\{n, m\}$.]
- **(ii)** Show that $\sup\{a_n : n \geq 1\}$ exists.
- **(iii)** Show that $c = \sup\{a_n : n \geq 1\}$ satisfies the requirement.

**b)** Use this last result to prove that the set $\mathbb{R}$ is uncountable. [Hint: Show that any function $f : \mathbb{N} \to \mathbb{R}$ can't be surjective. To do so, construct a sequence of closed intervals such that $f(n) \notin [a_n, b_n]$ with $a_n < b_n$.]

**Solution:** **a)** Let $A := \{a_n : n \geq 1\}$. By the hypothesis, we know that $a_1 \leq a_2 \leq a_3 \leq \cdots$, $b_1 \geq b_2 \geq b_3 \geq \cdots$ and $a_n \leq b_n$ for any $n \geq 1$. In fact, if $n, m \geq 1$ and $M := \max\{n, m\}$, then

$$a_n \leq a_M \leq b_M \leq b_m.$$

So we have $a_n \leq b_m$ for any $n, m \geq 1$. This implies that for any $m \geq 1$, the number $b_m$ is an upper bound for $A$. So, by AC, $\sup A$ exists. Put $c := \sup A$. We will verify that $c$ satisfy all the requirements. Since $c$ is the supremum of $A$, we have $a_n \leq c$ for any $n \geq 1$. Also, since $b_m$ is an upper bound of $A$ for any $m \geq 1$, by the definition of the supremum, we get that $c \leq b_m$ for any $m \geq 1$. Thus, for any $n \geq 1$, we get $a_n \leq c \leq b_n$. In other words, this means $c \in [a_n, b_n]$ for any $n \geq 1$.

**b)** Let $f : \mathbb{N} \to \mathbb{R}$ be a function. Since $f(1) \in \mathbb{R}$, there are real numbers $a_1$ and $b_1$ such that $a_1 < b_1 < f(1)$ (just take $a_1 = f(1) - \varepsilon$ and $b_1 = f(1) - \varepsilon/2$ for $\varepsilon > 0$). Now let $a_2, b_2 \in \mathbb{R}$ such that $a_2 < b_2$ and $[a_2, b_2] \subset [a_1, b_1]$ and $f(2) \notin [a_2, b_2]$. This is possible because

- if $f(2) \notin (a_1, b_1)$, then take $a_2 = a_1/2$ and $b_2 = b_1/2$.
- if $f(2) \in (a_1, b_1)$, then by the density of the rational numbers, there are rational numbers $r, \tilde{r}$ such that $a_1 < r < \tilde{r} < f(2) < b_1$. Set $a_2 = r$ and $b_2 = \tilde{r}$.

Continue in this fashion to construct a sequence of intervals $[a_n, b_n]$ such that

- $[a_1, b_1] \supset [a_2, b_2] \supset [a_3, b_3] \supset \cdots$ and
- $f(n) \notin [a_n, b_n]$ for every $n \geq 1$.

By a), there is some $c \in \mathbb{R}$ such that $c \in [a_n, b_n]$ for every $n \geq 1$. This implies that $c \neq f(n)$ for every $n \geq 1$ and so $f$ is not surjective. □

**Exercise 2.** (5 pts) Prove that if $a_n \to A$, then $|a_n| \to |A|$.

**Solution:** Let $a_n \to A$. This means that for any $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ such that if $n \geq N$, then $|a_n - A| < \varepsilon$. Let $\varepsilon > 0$ be arbitrary. We know, from the definition of convergence, that there is a $N \in \mathbb{N}$ such that if $n \geq N$, then $|a_n - A| < \varepsilon$. Let $n \geq N$, then, by the properties of the absolute value, we have

$$||a_n| - |A|| \leq |a_n - A| < \varepsilon.$$

So, for any $\varepsilon > 0$, there is a $N \in \mathbb{N}$ such that if $n \geq N$, then $||a_n| - |A|| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we conclude that $|a_n| \to |A|$. □

**Exercise 3.** (5 pts) Let $(a_n)$, $(b_n)$, and $(c_n)$ be sequences of real numbers. Prove that if $a_n \to L$, $b_n \to L$, and $a_n \leq c_n \leq b_n$, then $c_n \to L$.

**Solution:** Let $(a_n)$, $(b_n)$ and $(c_n)$ be sequences such that $a_n \to L$ and $b_n \to L$. Suppose also that $a_n \leq c_n \leq b_n$ for any $n \geq 1$. We want to prove that $c_n \to L$. Let $\varepsilon > 0$. Then, from the definition of convergence, there are $N_A, N_B \in \mathbb{N}$ such that

- if $n \geq N_A$, then $|a_n - L| < \varepsilon$ and;
- if $n \geq N_B$, then $|b_n - L| < \varepsilon$.

These last inequalities are equivalent to $-\varepsilon < a_n - L < \varepsilon$ for $n \geq N_A$ and $-\varepsilon < b_n - L < \varepsilon$ for $n \geq N_B$. The goal is to prove that $|c_n - L| < \varepsilon$. Now, from the hypothesis, we know that $a_n \leq c_n \leq b_n$ for any $n \geq 1$. So, for such $n$, we have

$$a_n - L \leq c_n - L \leq b_n - L$$

Take $N := \max\{N_A, N_B\}$. Then, if $n \geq N \geq N_A$, we get

$$a_n - L > -\varepsilon \quad \Rightarrow \quad c_n - L > -\varepsilon.$$

Also, if $n \geq N \geq N_B$, we get

$$b_n - L < \varepsilon \quad \Rightarrow \quad c_n - L < \varepsilon.$$

So, combining these last two inequalities, if $n \geq N$, then $-\varepsilon < c_n - L < \varepsilon$. This is the same thing as $|c_n - L| < \varepsilon$. Thus, we have just shown that if $\varepsilon > 0$, then there is a $N \in \mathbb{N}$ such that if $n \geq N$, then $|c_n - L| < \varepsilon$. Since $\varepsilon$ was arbitrary, we conclude that $c_n \to L$. $\square$

**Exercise 4.** (5 pts) Prove that if $a_n \to A$ and $a_n \geq 0$ for all $n \geq 1$, then $\sqrt{a_n} \to \sqrt{A}$. Follow the following steps to prove it:

1. Consider the case $A = 0$.
2. Suppose that $A \neq 0$. Show that there is a $N_1 \in \mathbb{N}$ such that if $n \geq N_1$, then $\sqrt{a_n} \geq \sqrt{|A|/2}$. [Hint: use the definition of convergence of $(a_n)_{n\geq 0}$ with a clever choice of $\varepsilon$ and use the properties of the absolute value.]
3. Use the convergence of $(a_n)$ again to find a $N_2$ such that $|a_n - A| < \frac{3}{4}\frac{\varepsilon}{\sqrt{|A|}}$.
4. Express $\sqrt{a_n} - A$ as $\frac{a_n - A}{\sqrt{a_n} + \sqrt{A}}$ and put $N = \max\{N_1, N_2\}$. Conclude.

**Solution:** Let $A = 0$ and $\varepsilon > 0$. Then, $a_n \to 0$ and this implies that there is a $N \in \mathbb{N}$ such that if $n \geq N$, then $|a_n| < \varepsilon^2$. We know that $a_n \geq 0$, then $|a_n| = a_n$. Taking the square root in the last expression gives $\sqrt{a_n} < \varepsilon$ if $n \geq N$. So, for $\varepsilon > 0$, there is a $N \in \mathbb{N}$ such that if $n \geq N$, then $|\sqrt{a_n}| < \varepsilon$. In other words, $\sqrt{a_n} \to \sqrt{0} = 0$.

Let $A \neq 0$ and $\varepsilon > 0$. For $n \geq 1$, we have

$$\sqrt{a_n} - \sqrt{A} = \frac{a_n - A}{\sqrt{a_n} + \sqrt{A}}. \tag{1}$$

Since $a_n \to A$ with $A \neq 0$, there is a $N_1 \in \mathbb{N}$ such that if $n \geq N$, then $|a_n - A| < \frac{|A|}{2}$ (take $\varepsilon = |A|/2 > 0$ in the definition of convergence). Now, by the properties of the absolute value, we have $|A| - |a_n| \leq ||a_n| - |A|| \leq |a_n - A|$. So, if $n \geq N_1$, then

$$|A| - |a_n| < \frac{|A|}{2} \quad \Rightarrow \quad \frac{|A|}{2} < |a_n|.$$

Taking the square root on each side of the inequality, we obtain $\sqrt{a_n} > \sqrt{\frac{|A|}{2}}$ if $n \geq N_1$. Since $\sqrt{2} < 2$, we also see that $\sqrt{|A|/2} \geq \sqrt{|A|}2$. So,

$$n \geq N_1 \quad \Rightarrow \quad \sqrt{a_n} \geq \frac{\sqrt{|A|}}{2}. \tag{2}$$

By the definition of the convergence of the sequence $(a_n)$, there is a $N_2 \in \mathbb{N}$ such that if $n \geq N_2$, then $|a_n - A| < \frac{3}{4}\sqrt{|A|}\varepsilon$. Put $N := \max\{N_1, N_2\}$. Then, using (**??**), if $n \geq N$, then

$$\left|\sqrt{a_n} - \sqrt{A}\right| = \left|\frac{a_n - A}{\sqrt{a_n} + \sqrt{A}}\right| = \frac{|a_n - A|}{\sqrt{a_n} + \sqrt{A}}.$$

Now, since $n \geq N \geq N_1$, we know that $\sqrt{a_n} \geq \sqrt{|A|}/2$ and so $\sqrt{a_n} + \sqrt{A} \geq \frac{3}{4}\sqrt{|A|}$. Using this, we can bound $|\sqrt{a_n} - \sqrt{A}|$ by

$$\frac{|a_n - A|}{\frac{3}{4}\sqrt{|A|}}$$

and since $n \geq N \geq N_2$, we also have

$$\frac{|a_n - A|}{\frac{3}{4}\sqrt{|A|}} < \frac{\frac{3}{4}|A|\varepsilon}{\frac{3}{4}|A|} = \varepsilon.$$

Thus, we have just shown that if $\varepsilon > 0$ is arbitrary, then there exists a $N \in \mathbb{N}$ such that $|\sqrt{a_n} - \sqrt{A}| < \varepsilon$. We then conclude that $\sqrt{a_n} \to \sqrt{A}$. $\square$

**Exercise 5.** (5 pts) For each sequence $(a_n)_{n=1}^{\infty}$, define the sequence $(\sigma_n)_{n=1}^{\infty}$ by

$$\sigma_n := \frac{a_1 + a_2 + \cdots + a_n}{n} \quad (n \geq 1).$$

Prove that if $a_n \to A$, then $\sigma_n \to A$. Find an example of a divergent sequence $(a_n)$ such that $(\sigma_n)_{n=1}^{\infty}$ converges.

**Solution:** Let $a_n \to A$. We want to prove that $\sigma_n \to A$. This means that for any $\varepsilon > 0$, there is a $N \in \mathbb{N}$ such that if $n \geq N$, then $|\sigma_n - A| < \varepsilon$. Let $\varepsilon > 0$ be arbitrary. From the definition of the convergence of $(a_n)$, there exists a $N_1 \in \mathbb{N}$ such that if $n \geq N_1$, then $|a_n - A| < \varepsilon/2$. So, we get

$$|\sigma_n - A| = \left|\frac{\sum_{k=1}^{n} a_k}{n} - A\right| = \left|\frac{\sum_{k=1}^{n} a_k - nA}{n}\right|$$

Separate the sum from $k = 1$ to $k = N_1 - 1$ and from $k = N_1$ to $k = n$ and use triangle inequality twice to obtain

$$\left|\frac{\sum_{k=1}^{n} a_k - nA}{n}\right| \leq \frac{\sum_{k=1}^{N_1-1} |a_k - A|}{n} + \frac{\sum_{k=N_1}^{n} |a_k - A|}{n}.$$

We know that a convergence sequence is bounded. So, there is a $M > 0$ such that $|a_k| \leq M$ for any $k \geq 1$. Then

$$|a_k - A| \leq |a_k| + |A| \leq M + |A| \quad \forall k \geq 1.$$

Also, when $k \geq N_1$, then $|a_k - A| < \varepsilon/2$. Also, by the AP, there is a natural number $N_2 \in \mathbb{N}$ such that $N_2(\varepsilon/2) > (N_1 - 1)(M + |A|)$.

Take $N := \max\{N_1, N_2\}$ and $n \geq N$. Putting everything together, we obtain

$$|\sigma_n - A| \leq \frac{(N_1 - 1)(M + |A|)}{n} + \sum_{k=N_1}^{n} \frac{\varepsilon/2}{n} < \frac{(N_1 - 1)(M + |A|)}{N_2} + (n - N_1)(\varepsilon/2)/n < \varepsilon/2 + \varepsilon/2.$$

Then, for any $\varepsilon$, we just proved that there is a $N \in \mathbb{N}$ such that if $n \geq N$, then $|\sigma_n - A| < \varepsilon$. We conclude that $\sigma_n \to A$.

Take $a_n = (-1)^n$. Then $(a_n)$ diverge, but $\sigma_n \to 0$. □

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## Homework problems

**Exercise 6.** (10 pts) Use the definition of convergence to prove that each of the following sequences converges.

**a)** $(a_n)_{n=1}^{\infty}$ given by $a_n = 5 + 1/n$ for $n \geq 1$.

**b)** $(a_n)_{n=1}^{\infty}$ given by $a_n = \frac{3n}{2n+1}$ for $n \geq 1$.

**Solution:** **a)** Take $A = 5$. Let $\varepsilon > 0$ be arbitrary. Then, for any $n \geq 1$, we have

$$|5 - 1/n - 5| = |-1/n| = 1/n.$$

By the AP ($x = \varepsilon$ and $y = 1$), there is a $N_0 \in \mathbb{N}$ such that $N_0\varepsilon > 1$ and so $1/N_0 < \varepsilon$. Take $N = N_0$, so, if $n \geq N_0$, we have

$$|5 - 1/n - 5| = 1/n \leq 1/N_0 < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we just proved that for any $\varepsilon > 0$, there is a $N \in \mathbb{N}$ such that $|a_n - 5| < \varepsilon$. Thus, $a_n \to 5$.

**b)** Take $A = 3/2$. Let $\varepsilon > 0$. We have

$$\left|\frac{3n}{2n+1} - \frac{3}{2}\right| = \left|\frac{6n - 6n - 3}{2(2n+1)}\right| = \frac{3}{2(2n+1)}.$$

By the AP ($x = \varepsilon$ and $y = 1/2$), there is a $N_0 \in \mathbb{N}$ such that $(2N_0 + 1)\varepsilon > \frac{1}{2}$ and so $\frac{1}{2(2N_0+1)} < \varepsilon$. Take $N = N_0$, so if $n \geq N_0$, then we have $2(2n+1) \geq 2(2N_0 + 1)$ and

$$\left|\frac{3n}{2n+1} - \frac{3}{2}\right| = \frac{3}{2(2n+1)} \leq \frac{1}{2(2N_0+1)} < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we just proved that for any $\varepsilon > 0$, there is a $N \in \mathbb{N}$ such that $|a_n - 3/2| < \varepsilon$. Thus $a_n \to 3/2$. □

**Exercise 7.** (5 pts) Prove that the sequence $(a_n)_{n=1}^{\infty} = \left(\frac{2n+1}{n}\right)_{n=1}^{\infty}$ is a Cauchy sequence.

**Solution:** Let $\varepsilon > 0$ be arbitrary. For $n, m \geq 1$, we have

$$\left|\frac{2n+1}{n} - \frac{2m+1}{m}\right| = \left|\frac{2nm + m - 2nm - n}{nm}\right| = \frac{|m-n|}{nm}.$$

By the triangle inequality, $|m-n|/mn \leq (m+n)/mn = \frac{1}{n} + \frac{1}{m}$ and so

$$\left|\frac{2n+1}{n} - \frac{2m+1}{m}\right| \leq \frac{1}{n} + \frac{1}{m}.$$

By the AP ($x = \varepsilon$ and $y = 2$), there is a $N_0 \in \mathbb{N}$ such that $N_0\varepsilon > 2$, so that $\frac{1}{N_0} < \varepsilon/2$. Take $N = N_0$ and let $n, m \geq N_0$. Then, we have

$$\left|\frac{2n+1}{n} - \frac{2m+1}{m}\right| \leq 1/n + 1/m < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we just proved that for any $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ such that if $n, m > N$, then $|a_n - a_m| < \varepsilon$. Thus the sequence is Cauchy. $\square$

**Exercise 8.** (10 pts) Prove that each of the following sequence diverges.

**a)** $(a_n)_{n=1}^{\infty} = ((-1)^n)_{n=1}^{\infty}$.

**b)** $(a_n)_{n=1}^{\infty} = (\sin(\frac{2n+1}{2}\pi))_{n=1}^{\infty}$.

**Solution:** **a)** Suppose that the sequence converges to $A$. Let $\varepsilon = |A|$, if $A \neq 0$ and $\varepsilon = 1/2$, if $A = 0$, in the definition of convergence of a sequence. Then there exists a $N \in \mathbb{N}$ such that if $n \geq N$, then $|a_n - A| < \varepsilon$.

If $A \neq 0$ and if $n \geq N$, then, by the properties of the absolute value, we have

$$|a_n - A| < \varepsilon \iff -|A| < a_n - A < |A| \iff A - |A| < (-1)^n < A + |A|.$$

- If $A > 0$, then $A - |A| = 0$ and for any $n \geq N$, $0 < (-1)^n$ which is false if $n$ is odd.
- If $A < 0$, then $A + |A| = 0$ and for any $n \geq N$, $(-1)^n < 0$ which is false if $n$ is even.

If $A = 0$, then $|(-1)^n| < 1/2$ and so $1 < 1/2$ a contradiction.

Thus, the sequence $(a_n)_{n=1}^{\infty}$ is not convergent.

**b)** We see that $\sin(4n+1)\pi/2 = (-1)^n$. So, from a), it is not a convergent sequence. $\square$

**Exercise 9.** (5 pts) Give an examples of two sequences $(a_n)$ and $(b_n)$ such that $(a_n)$ and $(b_n)$ don't converge, but $(a_n + b_n)$ converge.

**Solution:** Let $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$, then $a_n + b_n = 0$. The sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ both diverge, but $(a_n + b_n)_{n=1}^{\infty}$ converge. $\square$

**Exercise 10.** (10 pts) With the limit operations and the writing problems, find the limit of the following sequence with general term

**a)** $\frac{n^2+4n}{n^2-5}$.

**b)** $\frac{n}{n^2-3}$.

**c)** $\frac{\cos n}{n}$. [You can use what you know on the cosine function.]

**d)** $\left(\sqrt{4-\frac{1}{n}}-2\right)n$.

**Solution:** **a)** We can't use the limit rules directly. We rearrange the expression:

$$\frac{n^2+4n}{n^2-5} = \frac{1+4/n}{1-5/n^2}.$$

Now, $1/n \to 0$, so $1+4/n \to 1$. Also, $1/n^2 \to 0$ according to the product rule and so $1-5/n^2 \to 1$. Thus, by the quotient rule, we get

$$\lim_{n\to\infty} \frac{n^2+4n}{n^2-5} = \frac{1}{1} = 1.$$

**b)** Again, we can't use the limit rules directly. We rearrange the expression:

$$\frac{n}{n^2-3} = \Big(\frac{1}{n}\Big)\Big(\frac{1}{1-3/n^2}\Big).$$

We know that $1/n \to 0$ and $1/n^2 \to 0$. So, by the limit rules, $1-3/n^2 \to 1$. Thus, by the product rule, we get

$$\lim_{n\to\infty} \frac{n}{n^2-3} = \Big(\lim_{n\to\infty} \frac{1}{n}\Big)\Big(\lim_{n\to\infty} \frac{1}{1-3/n^2}\Big) = 0.$$

**c)** We know that $|\cos(x)| \leq 1$ for any $x \in \mathbb{R}$. So, use a Theorem in the lecture notes with $(a_n) = (1/n)_{n=1}^{\infty}$ and $(b_n) = (\cos n)_{n=1}^{\infty}$, we have that $a_n b_n \to 0$ since $1/n \to 0$. In other words, $\cos(n)/n \to 0$.

**d)** Here we can't apply the limit rules directly. We have to rearrange the expression. We have

$$\left(\sqrt{4-\frac{1}{n}}-2\right)n = \frac{(4-1/n-4)n}{\sqrt{4-1/n}+2} = \frac{-1}{\sqrt{4-1/n}+2}.$$

Now, from the working problems, since $1/n \to 0$ and so $4-1/n \to 4$, we get that $\sqrt{4-1/n} \to \sqrt{4} = 2$. Thus, from the quotient rule,

$$\lim_{n\to\infty} \left(\sqrt{4-\frac{1}{n}}-2\right)n = -\frac{1}{2+2} = -\frac{1}{4}.$$