MATH-331 Introduction to Real Analysis — Jeselle-Paige Guillermo
Homework 04 — Fall 2021

Due date: October 25th 1:20pm — Total: /70.

Exercise	1 (5)	2 (5)	3 (5)	4 (5)	5 (10)	6 (10)	7 (5)	8 (5)	9 (5)	10 (10)
Score										

Table 1: Scores for each exercises

Instructions: You must answer all the questions below and send your solution by email (to parisepo@hawaii.edu). If you decide to not use LaTeX to hand out your solutions, please be sure that after you scan your copy, it is clear and readable. Make sure that you attached a copy of the homework assignment to your homework.
If you choose to use LaTeX, you can use the template available on the course website.
No late homework will be accepted. No format other than PDF will be accepted. Name your file as indicated in the syllabus.

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Writing problems

For each of the following problems, you will be asked to write a clear and detailed proof. You will have the chance to rewrite your solution in your semester project after receiving feedback from me.

Exercise 1. (5 pts) Prove that, if $0 < x < \pi/2$, then $0 \leq \sin x \leq x$ with a geometric argument. [Hint: View $\sin x$ as a point on the unit circle in the first quadrant.]

Solution:

Exercise 2. (5 pts) Let $f : A \to \mathbb{R}$ and $g : B \to A$ be two functions where $A, B \subset \mathbb{R}$. Let a be an accumulation point of A and b be an accumulation point of B. Suppose that

- $\lim_{t\to b} g(t) = a$.
- there is a $\eta > 0$ such that for any $t \in B \cap (b - \eta, b + \eta)$, $g(t) \neq a$.
- f has a limit at a.

Prove that $f \circ g$ has a limit at b and $\lim_{x\to a} f(x) = \lim_{t\to b} f(g(t))$. [This is the change of variable rule for limits.]

Solution:

Exercise 3. (5 pts) Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and suppose that $f(x) = 0$ for each rational number x in $[a, b]$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

Solution:

Exercise 4. (5 pts) Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and suppose that $f(c) > 0$ for some $c \in [a, b]$. Prove that there exist a number η and an interval $[u, v] \subset [a, b]$ such that $f(x) \geq \eta$ for all $x \in [u, v]$.

Solution:

Exercise 5. (10 pts) Let $f : \mathbb{R} \to \mathbb{R}$ be a function that satisfies $f(x + y) = f(x) + f(y)$ for any real number x and y.

a) Suppose that f is continuous at some point c. Prove that f is continuous on $\mathbb{R}$.

b) Suppose that f is continuous on $\mathbb{R}$ and that $f(1) = k$. Prove that $f(x) = kx$ for all $x \in \mathbb{R}$. [Hint: start with x integer, then x rational, and finally use Exercise 3.]

Solution: a)

b)

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Homework problems

Answer all the questions below. Make sure to show your work.

Exercise 6. (10pts) For each of the functions below, say if the limit exists or doesn't exist at the given point. Justify your answer (in other words, prove it!)

a) $f(x) = \sin(1/x)$ and $x_0 = 0$.

b) $f(x) = x\sin(1/x)$ adn $x_0 = 0$.

Solution: a)

b)

Exercise 7. (5 pts) Let $c \in (a, b)$ and let f be a function defined on (a, b) except at c. Suppose that $f(x) > 0$ for any $x \in (a, b)\backslash\{c\}$, that $\lim_{x\to c} f(x)$ exists, and that

$$\lim_{x\to c} f(x) = \lim_{x\to c} \left[(f(x))^2 - f(x) - 3\right].$$

Find the value of $\lim_{x\to c} f(x)$. Explain each step carefully.

Solution:

Exercise 8. (5 pts) Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} x & , x \in \mathbb{Q} \\ -x & , x \notin \mathbb{Q}. \end{cases}$$

is discontinuous at any point of $\mathbb{R}\backslash\{0\}$ and continuous at 0.

Solution:

Exercise 9. (5 pts) Let $p(x) = x^2 + 2$. Find an interval where p is strictly decreasing and find a formula for its inverse.

Solution:

Exercise 10. (10 pts) Let $p(x) = ax^3 + bx^2 + cx + d$ be a polynomial of degree 3 and $a > 0$. Prove that p has at least one real root by following these steps:

a) Prove that $\lim_{x\to\infty} p(x) = \infty$.

b) Prove that $\lim_{x\to-\infty} p(x) = -\infty$.

c) Conclude.

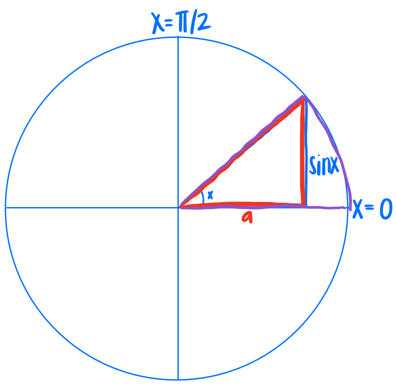
[Hint for a): write your polynomial $p(x) = ax^3 + bx^2 + cx + d$ as $x^3(a + b/x + c/x^2 + d/x^3)$ and use the fact that $\lim_{x\to\infty} 1/x^n = 0$ for every $n \geq 1$.]

Solution: **a)**

b)

c)

① Prove that, if $0<x<\pi/2$, then $0\le \sin x \le x$ with a geometric argument



lets assume $r=1$ for $0<x<\pi/2$

$A_\triangle = \frac{x}{2\pi}\pi r^2 = \frac{x}{2}$

$A_\triangle = \frac{a\sin x}{2}$, let $a=1$, so $A_\triangle = \frac{\sin x}{2}$

$0<\frac{\sin x}{2} < \frac{x}{2} \Rightarrow 0\le \sin x \le x$

$0\le \sin x \le x$
for $x\in[0,\pi/2]$

② Let $f:A\to\mathbb{R}$ and $g:B\to A$ be two functions where $A,B\subset\mathbb{R}$. Let a be an accumulation point of A and b be an accumulation point of B. Suppose that

- $\lim_{t\to b} g(t)=a$
- there is a $\eta>0$ s.t. for any $t\in B\cap(b-\eta,b+\eta)$, $g(t)\ne a$
- f has a limit at a

Prove that $f\circ g$ has a limit at b and $\lim_{x\to a} f(x) = \lim_{t\to b} f(g(t))$

Let $\lim_{x\to a} f(x) = a$

If we make the change of variables

$f(x) = f(g(t))$

Since $\lim_{t\to b} g(t) = a$

$\lim_{x\to a} f(x) = \lim_{g(t)\to a} f(g(t))$

$a = \lim_{x\to a} f(a)$

$a = a$

So, $\lim_{x\to a} f(x) = \lim_{g(t)\to a} f(g(t))$

③ Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and suppose that $f(x)=0$ for each rational number x in $[a,b]$. Prove that $f(x)=0$ for all $x\in[a,b]$

Let x_n be a sequence of rational numbers s.t.

$x\in[a,b]$ and $x_n\in\mathbb{Q}$ then $\exists x_n\to x$

Since f is continuous $\lim_{n\to\infty} f(x_n)\to f(x)$

$f(x_n)=0$ $\forall n\in\mathbb{N}$ since $x_n\in\mathbb{Q}$

so $f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 0 = 0$

$\therefore f(x)=0 \quad \forall x\in[a,b]$

④ Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and suppose that $f(c)>0$ for some $c\in[a,b]$. Prove that there exist a number η and an interval $[u,v]\subset[a,b]$ such that $f(x)\ge\eta$ for all $x\in[u,v]$

⑤ Let $f:\mathbb{R}\to\mathbb{R}$ be a function that satisfies $f(x+y)=f(x)+f(y)$ for any real number x and y

a) Suppose that f is continuous at some point c. Prove that f is continuous on $\mathbb{R}$

If f is continuous at c, $\lim_{x\to c} f(x)=f(c)$ for some $c\in\mathbb{R}$

$$\lim_{x\to a} f(x)=\lim_{x\to c} f(x-c+a)$$

Since $f(x+y)=f(x)+f(y)$ then,

$$\lim_{x\to a} f(x)=\lim_{x\to c} f(x)-f(c)+f(a)$$

$$\lim_{x\to a} f(x)=f(a)$$

So f is continuous at a, where $a\in\mathbb{R}$

b) Suppose that f is continuous on $\mathbb{R}$ and that $f(1)=k$. Prove that $f(x)=kx$ for all $x\in\mathbb{R}$. [Hint: start with x integer, then x rational, and finally use Exercise 3]

⑥ a) $f(x)=\sin(1/x)$ and $x_0=0$

Let $x_n=\frac{1}{n\pi}\to 0$

$y_n=\frac{1}{2n\pi+\pi/2}\to 0$

$\sin(\frac{1}{x_n})=\sin(n\pi)\to 0$

$\sin(\frac{1}{\frac{1}{2n\pi+\pi/2}})=\sin(2n\pi+\pi/2)\to 1$

$\lim(f(x_n))\neq\lim(f(y_n))$, so $\lim f(x)$ does not exist at $x_0=0$

b) $f(x)=x\sin(1/x)$ and $x_0=0$

We found the limit on $\sin\frac{1}{x}$ does not exist at $x_0=0$.
$\sin\frac{1}{x}$ is bounded above by 1 and bounded below by -1.

$$-1\leq\sin\frac{1}{x}\leq 1$$

$$\lim_{x\to 0} x(-1)\leq\lim_{x\to 0} x\sin(\tfrac{1}{x})\leq\lim_{x\to 0} x(1)$$

Let $x=0$, $\quad 0\leq 0\leq 0$

By squeeze thm, $\lim_{x\to 0} x\sin(\frac{1}{x})=0$ at $x_0=0$

⑦ Let $c\in(a,b)$ and let f be a function defined on (a,b) except at c. Suppose that $f(x)>0$ for any $x\in(a,b)\setminus\{c\}$, that $\lim_{x\to c} f(x)$ exists, and that

$$\lim_{x\to c} f(x)=\lim_{x\to c}[(f(x))^2-f(x)-3]$$

Find the value of $\lim_{x\to c} f(x)$. Explain each step carefully.

$$\lim_{x\to c} f(x)=\lim_{x\to c}[(f(x))^2-f(x)-3]$$

$$=\lim_{x\to c}[f(x)^2]-\lim_{x\to c} f(x)-\lim_{x\to c} 3$$

$$=\lim_{x\to c} f(x)\lim_{x\to c} f(x)-\lim_{x\to c} f(x)-3$$

⑧ Prove that the function $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x)\begin{cases} x, & x\in\mathbb{Q} \\ -x, & x\notin\mathbb{Q}\end{cases}$$

is discontinuous at any point of $\mathbb{R}\setminus\{0\}$ and continuous at 0

If $x\in\mathbb{R}$ then $\exists (x_n)\subseteq\mathbb{Q}$ s.t $\lim_{n\to\infty} x_n = x$

Claim: for all $a\in\mathbb{R}$, f is discontinuous at a

Proof: Suppose $a\in\mathbb{R}\setminus\mathbb{Q}$

Take $(x_n)\subseteq\mathbb{Q}$ st $x_n\to a$ as $n\to\infty$

$f(x_n)=x \;\; \forall n\in\mathbb{N}$

$\Rightarrow \lim_{n\to\infty} f(x_n)=x$

$f(a)=-x$

$\lim_{n\to\infty} f(x_n)\neq f(a)$

f is discontinuous at a

We want to find if f is continuous at 0

let $\varepsilon>0$. $\delta=\varepsilon$

$$|f(x)-f(0)|=|f(x)-0|=\begin{cases}|x| & \text{if } x\in\mathbb{Q}\\ |-x| & \text{if } x\notin\mathbb{Q}\end{cases}$$

$= x$

$\leq \varepsilon$

So f is continuous on 0

⑨ Let $P(x)=x^2+2$. Find an interval where P is strictly decreasing and find a formula for its inverse

$P(x)=x^2+2$ is strictly decreasing

$P'(x)<0$

$P'(x)=2x\leq 0$ when $x\leq 0$

$P(x)$ is strictly decreasing when $x\leq 0$

$P^{-1}(x)=\sqrt{x-2}$

⑩ Let $P(x)=ax^3+bx^2+cx+d$ be a polynomial of degree 3 and $a>0$. Prove that P has at least one real root by following these steps:

a) Prove that $\lim_{x\to\infty} P(x)=\infty$

b) Prove that $\lim_{x\to\infty^-} P(x)=-\infty$

c) Conclude

a) $\lim_{x\to\infty} f(x) = \lim_{x\to\infty}(ax^3+bx^2+cx+d)$

$= \lim_{x\to\infty}\left[x^3\left(a+\frac{b}{x}+\frac{c}{x^2}+\frac{d}{x^3}\right)\right]$

$= \infty$

b) Similarly $\lim_{x\to\infty^-} f(x)=-\infty$

c) So $\exists x_1$ s.t. $x_1>0$ and $f(x_1)>0$

$\exists x_2$ s.t $x_2<0$ and $f(x_2)<0$

$\therefore$ $f(x_1)\cdot f(x_2)<0$ and P has at least one real root