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Math 331 Midterm 1

1a) Let $S \subseteq \mathbb{R}$ be bounded from below
Prove S has an infimum
Since $S \subseteq \mathbb{R}$ has a lower bound, there is some $x \in S$ that is the least value contained in S. By the definition of a lower bound, (the least number contained in $S \subseteq \mathbb{R}$) we know there exists some $x \in S$ that is the greatest lower bound. Since $S \subseteq \mathbb{R}$, this number x must be contained in the set of real numbers $\mathbb{R}$. This meets the criteria for the definition of an infimum (the greatest lower bound $x \in \mathbb{R}$) so, $\inf(S)$ exists.

b) Let $x := \inf(S)$.
Since x is the greatest lower bound of S, there exists no value $s \in S$ st. $s < x$. Setting $\varepsilon > 0$, we have that $x < x + \varepsilon$. $\forall s \in S$ we also have that $x \leq s$ by def. of an infimum. Since $\varepsilon > 0$ and $S \subseteq \mathbb{R}$ we can combine the inequality to show there must be some $s \in S$ st. $x \leq s < x + \varepsilon$.

2. Show if $(a_n)_{n=0}^{\infty}$ is Cauchy, $(a_n^2)_{n=1}^{\infty}$ is Cauchy
Let $(a_n)_{n=0}^{\infty}$ be a Cauchy seq. st. if $n \geq N_1$ $|a_n - L| < \varepsilon$. $(\forall \varepsilon > 0)$
Since by def. of a Cauchy seq. we know a_n converges, we can reference the product rule of convergent sequences. If a_n converges, $a_n \cdot a_n$ also converges. So, $a_n \cdot a_n = a_n^2$ is a convergent sequence. Thus, $\forall n \geq N_1$ $|a_n^2 - L| < \varepsilon$. Since convergence shows a sequence is Cauchy, and the value $n=1$, does not impact convergence, $(a_n^2)_{n=1}^{\infty}$ is a Cauchy sequence.

3. Let $(a_n)_{n \geq 1}^{\infty}$ be a seq. of non-neg $\mathbb{R}$ ($a_n \geq$ for any $n \geq 1$)
Define $b_n = a_1 + a_2 + \ldots a_n$ Show if $(b_n)_{n=1}^{\infty}$ bounded from above, then (b_n) converges.
Let b_n be bounded from above and $\varepsilon > 0$. So, if b_n is bounded

from above, and $a_n \in \mathbb{R}$, b_n is the sum of finitely many real numbers and converges to some limit $L \in \mathbb{R}$
So, $|b_n - L| < \varepsilon$. Thus, by the def. of convergence, b_n converges.

4. Let $(a_n)_{n=1}^{\infty}$ be the seq. non-neg $\mathbb{R}$ defined by: $a_n = \sqrt{a_{n-1}} + \sqrt{a_{n-2}}$ $(n \geq 3)$
Since a_n is defined recursively, and $n \geq 3$
sorry, idk :(
we know that the limit of a_n will be some value $A \in a_n$ s.t. $A = \sqrt{a_{n-1}} + \sqrt{a_{n-2}}$.
So, by PMI, $a_{n+1} = \sqrt{a_n} + \sqrt{a_{n-1}}$, $\forall n \geq 3$.
Thus, $a_{n+1}^2 = a_n + a_{n-1} \Rightarrow a_{n+1}^2 - a_{n-1} = a_n$
So, since this is true for a_{n+1}, it also follows for a_n
and $\lim(a_n) = a_n^2 - a_n = A$