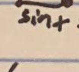
5 a.) If f is continuous at c, then there is an associated limit and sequence of f. This means that $\lim_{x\to c} f(x) = f(c)$. We can then write c as a combination of $\mathbb{R}$ numbers. This will be ~~$f(n_1)+f(n_2)+\ldots+f(n_k)$~~ $= n_1+n_2+\ldots+n_k = c$. This means that $f(n_1)+f(n_2)+\ldots+f(n_k) = f(c)$. With limit rules, we can say the limit exists at all of those points and they add up to $\lim_{x\to c} f(x) = f(c)$. And since they all have associated limits, its continuous at all the points.

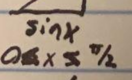
b.) $f(2) = f(1+1) = f(1)+f(1) = k+k = 2k$

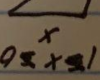
$\mathbb{Z}$ is able to be written in terms of $f(1)$, so kx $(x\in\mathbb{Z})$

$f(q(1)) = (\underbrace{f(1)+f(1)+\ldots+f(1)}_{q}) = qk$. We can write $q \in \mathbb{Q}$ as $\frac{a}{b}$ and we can use $\frac{a}{b}$ as two seperate $\mathbb{Z}$ to get that even with fractions, $f(x) = kx$ $(x\in\mathbb{Q})$

And then in Q3 we saw that if f is continuous at $\mathbb{Q}$ with $Q = kX$, so f is continuous on $\mathbb{R}$ and $f(\mathbb{R}) = kX$.

1  $\sin x = x$ value

 1, $\cos x$, $\sin x$, $0 \le x \le \pi/2$

 1, $\sqrt{1-x^2}$, x, $0 \le x \le 1$

Same effect, x ends early because larger values.

$\frac{d}{dx}\sin x = \cos x$ $\quad \frac{d}{dx} x = 1$ $\quad 0 \le \cos x \le 1$, so x increase at or at greater rate than $\sin x$. Since at $x=0$, $x=0$ & $\sin x = 0$, $x \ge \sin x$

3 Continuous so $|x-x_0|<\delta$ $|f(x)-f(x_0)|<\varepsilon$ $\forall x\in[a,b]$
So if $\mathbb{Q}=x$ $f(x)=0$. $\mathbb{Q}$ is dense in $\mathbb{R}$, so between each number in $\mathbb{R}$ is an infinite amount of $\mathbb{Q}$. Let $x_0\in[a,b]/\mathbb{Q}$, so f is continuous so $|x-x_0|<\delta$ $|f(x)-f(x_0)|<\varepsilon$
If $x_0\in\mathbb{Q}$, then $|x-\mathbb{Q}|<\delta \rightarrow |f(x)|<\varepsilon$
$\mathbb{Q}-\partial<x<\partial+\mathbb{Q}$ $f(\mathbb{Q}-\partial)<f(x)<f(\delta+\mathbb{Q})$
If $\partial\in\mathbb{Q}$, then $0<f(x)<0$, $f(x)=0$. This must be true for all $x\in[a,b]$ to ensure it is continuous if $\delta\in\mathbb{Q}$. With this $|f(x)|=0<\varepsilon$, so ε is arbitrary.

~~4~~ For $f(x)\geq\eta$ $\forall x\in[u,v]$, then ~~[f(u),f(v)]~~ must be strictly increasing or $f(v)\geq\eta$. If η is in $[f(a),f(b)]$, then if $f(a)>\eta$, there $\exists d\in[a,b]$ s.t. $\eta=f(d)<f(a)$. Let $d=u$ and $a=v$. If $f(b)>\eta$, same argument.
Since d and $a\in[a,b]$, $[u,v]\subset[a,b]$

4 If $f(c)>0$ for $c\in[a,b]$, then if $f(a),f(b)>f(c)$, then if $\eta=0$, then $[u,v]$ can be any interval in $[a,b]$.
If either $f(a,b)<0$ or $f(c)$, then we can pick the minimum value in $[u,v]$, and the max in $[u,v]$, and we know that there are values in between them w/ IVT and we can set η to be the minimum so that all numbers in $[u,v]$ are $\geq\eta$

2 $\lim_{t\to b} g(t)=a$ so $|t-b|<\delta_1$ $|g(t)-a|<\varepsilon_1$

$f\circ g = f(g(t))$. a is acc A, so $(a-\delta, a+\delta)\cap A$ is ∞.

So $|t-b|<\delta$ $|f(g(t))-L|<\varepsilon$

As $t\to b$, $g(t)\to a$, so $|t-b|<\delta$ $|f(a)-L|<\varepsilon$

And $\lim_{x\to a} f(x)=L$ so $|x-a|<\delta_2$ $|f(a)-L|<\varepsilon$

so, pick $\delta =: \min\{\delta_1, \delta_2\}$ then $|t-b|<\delta$ $|f(a)-L|<\varepsilon$

$\lim_{x\to a} f(x) = f(a) = f(g(t))$ as $g(t)=a$ $g(t)=a$ at $t\to b$

so $\lim_{t\to b} f(g(t)) = f(a) = \lim_{x\to a} f(x)$

8 $\delta>0$, $\varepsilon>0$, $|x-x_0|<\delta$ $|f(x)-f(x_0)|<\varepsilon$

$|x-0|<\delta$ $|x-0|<\varepsilon$ $|x|<\delta$ & $|x|<\varepsilon$

$\varepsilon=\delta$, $f(x)$ is continuous at 0.

$\mathbb{Q}$ is dense in $\mathbb{R}$, so between every $x\in\mathbb{R}$ are an infinite amount of $\mathbb{Q}$.

$x_0\in\mathbb{Q}$ $|x-\frac{p}{q}|<\delta$ $|f(x)-f(\frac{p}{q})|<\varepsilon \rightarrow |f(x)-\frac{p}{q}|<\varepsilon$ $f(x)=-x$

$|x-\frac{p}{q}|<\delta$ $|f(x)-\frac{p}{q}|=|-x-\frac{p}{q}|<\varepsilon$ $\frac{p}{q}-\delta<x<\frac{p}{q}+\delta$ $\delta=\varepsilon$

$\frac{p}{q}-\varepsilon<x<\frac{p}{q}+\varepsilon$ $|-x-\frac{p}{q}|<\varepsilon$ $\frac{p}{q}-\varepsilon<-x<\frac{p}{q}+\varepsilon$ $x\neq 0$ # not continuous

Also $\delta=2\varepsilon$ $\frac{p}{q}-2\varepsilon<x<\frac{p}{q}+2\varepsilon$ $x=\frac{p}{2q}+\varepsilon$ $|-\frac{p}{q}-\varepsilon-\frac{p}{q}|\geq\varepsilon$ $|-\frac{3p}{p}-\varepsilon|>\varepsilon$

~~ε is positive~~ $\frac{3p}{p}+\varepsilon>\varepsilon$ $\frac{3p}{q}>0$ ✓

$x_0\in\mathbb{R}$ $|x-x_0|<\delta$ $|f(x)-f(x_0)|\geq\varepsilon$ $x_0-\delta<x<x_0+\delta$ $\delta=\varepsilon$ $x=\mathbb{Q}(\frac{p}{q})$

in between $\mathbb{R}$ $x_0-\delta<\frac{p}{q}<x_0+\delta$ $|f(x)-f(x_0)|\geq\varepsilon$ $\cancel{|2x|\geq\varepsilon}$ $|\frac{p}{q}+x_0|\geq\varepsilon$

9 Decreasing from $(-\infty,0)$ $f^{-1}(f(x))=x$ $f(f^{-1}(x))=x$

$f(x)=x^2+2$ $x=(f^{-1}(x))^2+2$ $x-2=(f^{-1}(x))^2$

$f^{-1}(x)=-\sqrt{x-2}$ $f(f^{-1}(x))=(\sqrt{x-2})^2+2=x-2+2=x$ ✓

HW #5

6 a.) DNE. $\delta>0,\ |x-x_0|<\delta\ \exists\varepsilon\ |f(x)-L|>\varepsilon$

$|x-0|<\delta \quad |x|<\delta \qquad |\sin(\frac{1}{x})-L|\geq\varepsilon$

$-1\leq\sin(a)\leq 1 \qquad \sin(\frac{1}{x})\geq\varepsilon+L\geq-\sin(\frac{1}{x})$

$\frac{1}{x}>\sin^{-1}(\varepsilon+L)>-\frac{1}{x}$

$\frac{1}{\sin^{-1}(\varepsilon+L)}>x>\frac{-1}{\sin^{-1}(\varepsilon+L)} \qquad |x|<\frac{1}{\sin^{-1}(\varepsilon+L)}$

if $\delta=\frac{1}{\sin^{-1}(\varepsilon+L)}$, then there $\exists\varepsilon$ such that $|f(x)-L|\geq\varepsilon$, therefor $\lim_{x\to 0}\sin(\frac{1}{x})=DNE$

b.) $\lim_{x\to 0}x\sin(\frac{1}{x})=0$, Let $\delta>0,\ \varepsilon>0$

$0\leq\sin x\leq 1 \quad 0\leq x\sin x\leq x \qquad |x-0|<\delta \quad |x|<\delta$

$|x\sin(\frac{1}{x})-0|<\varepsilon \quad |x\sin(\frac{1}{x})|<\varepsilon \qquad \delta=\varepsilon$

$|x\sin(\frac{1}{x})|\leq x<\delta=\varepsilon \quad |x\sin(\frac{1}{x})|<\varepsilon$

ε was arbitrary $\lim_{x\to 0}x\sin(\frac{1}{x})=0$

7 $\lim_{x\to c}f(x)=L \qquad \lim_{x\to c}f(x)^2=L^2$ (Limit algebra)

$\lim_{x\to c}-f(x)=-L$ (Limit algebra) $\lim_{x\to c}-3=-3$ (constant limit)

$L=L^2-L-3 \to 0=L^2-2L-3$ solve for L

$\frac{2\pm\sqrt{2^2-4\cdot 1\cdot -3}}{2(1)}=1\pm\sqrt{4+12} \quad 1\pm 4 \quad L=-3,5$

$\lim_{x\to c}f(x)=-3,5$ but $f(x)>0$ so $\lim_{x\to c}f(x)=5$

10 a.) $ax^3+bx^2+cx+d = x^3(a+b/x+c/x^2+d/x^3)$ $\lim_{x\to\infty} = x^3(a+0+0+0)$

$\lim_{x\to\infty} x^3 a$ Let $y=m$ $x=x$ in A.P., $m<nx$ $m^3<n^3x^3$

if $M=m^3$ then $M<nx^3$ so if M is an upper bound,

$\exists x\in\mathbb{R}$ s.t. $x^3>M$ meaning $\lim_{x\to\infty} x^3=\infty$ so

$\lim_{x\to\infty} ax^3+bx^2+cx+d=\infty$

b.) Let M in this case be $-M$, so $-M$ is upper bound. So repeat above steps with $-M$ as upper bound. So $\lim_{x\to-\infty} ax^3+bx^2+cx+d=-\infty$

c.) Since there are positive and negative values, the IVT states there must be $f(c)=0$ $c\in\mathbb{R}$, so the function has at least one real root.