MATH-331 Introduction to Real Analysis — Jeselle-Paige Guillermo

Homework 05 — Fall 2021

Due date: November, 8^{th} 1:20pm — Total: /70.

Exercise	1 (5)	2 (5)	3 (5)	4 (5)	5 (10)	6 (10)	7 (5)	8 (5)	9 (5)	10 (10)
Score										

Table 1: Scores for each exercises

Instructions: You must answer all the questions below and send your solution by email (to `parisepo@hawaii.edu`). If you decide to not use LaTeX to hand out your solutions, please be sure that after you scan your copy, it is clear and readable. Make sure that you attached a copy of the homework assignment to your homework.
If you choose to use LaTeX, you can use the template available on the course website.
No late homework will be accepted. No format other than PDF will be accepted. Name your file as indicated in the syllabus.

1 Writing problems

For each of the following problems, you will be asked to write a clear and detailed proof. You will have the chance to rewrite your solution in your semester project after receiving feedback from me.

Exercise 1. (5 pts) Let $f : \mathbb{R} \to \mathbb{R}$ and suppose that there exists a positive constant M such that $|f(y) - f(x)| \leq M|y - x|$ for all $x, y \in \mathbb{R}$. Prove that f is uniformly continuous on $\mathbb{R}$.

Exercise 2. (5 pts) Let $f : [0, \infty) \to \mathbb{R}$ be nonnegative and continuous such that $\lim_{x\to\infty} f(x) = 0$. Prove that f attained its maximum at some point in $[0, \infty)$.

Exercise 3. Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function such that $f([a, b]) \subseteq [a, b]$. Prove that there is a $c \in [a, b]$ such that $f(c) = c$. [This is one of the many fixed point Theorem.]

Exercise 4. (5 pts) Suppose that $f : (a, b) \to \mathbb{R}$ is twice differentiable on (a, b) and there are two points $c < d$ in (a, b) such that $f'(c) = f'(d)$. Show that there is a point $x \in (c, d)$ such that $f''(x) = 0$.

Exercise 5. (10 pts) Suppose that $f : (a, b) \to \mathbb{R}$ is differentiable at $x_0 \in (a, b)$.

a) Prove that

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0 - h)}{2h} \tag{$\star$}$$

exists and equals $f'(x_0)$.

b) Find a continuous function $f : \mathbb{R} \to \mathbb{R}$ and a point $x_0 \in \mathbb{R}$ such that f is not differentiable at x_0, but the limit $(\star)$ exists.

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Homework problems

Answer all the questions below. Make sure to show your work.

Exercise 6. (10pts)

a) Suppose $r > 0$. Prove that $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = x^r$ is differentiable on $(0, \infty)$ and compute its derivative. [Hint: take for granted that e^x and $\ln x$ are differentiable with $(e^x)' = e^x$ and $(\ln x)' = 1/x$. Rewrite then x^r in terms of a composition of two differentiable functions.]

b) Define $f(x) = \sqrt{x^2 + \sin x + \cos x}$ where $x \in [0, \pi/2]$. Show that f is a differentiable function and find a formula for its derivative.

Exercise 7. (5 pts) Show that $S \subseteq \mathbb{R}$ is closed if and only if $\mathbb{R} \backslash S$ is open.

Exercise 8. (5 pts) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function and define $g(x) = x^2 f(x^3)$. Show that g is differentiable and compute its derivative.

Exercise 9. (5 pts) Prove that $f(x) = \arcsin x$ is differentiable on its domain and find a formula, involving no trigonometric functions, for the derivative of f (justify all your steps!).

Exercise 10. (10 pts) Use the Mean-Value Theorem to show the following inequalities.

a) $ny^{n-1}(x - y) \leq x^n - y^n \leq nx^{n-1}(x - y)$ if $n \in \mathbb{N}$ and $0 \leq y \leq x$.

b) $\sqrt{1 + x} < 1 + \frac{1}{2}x$ for $x > 0$.

① Let $f:\mathbb{R}\to\mathbb{R}$ and suppose that there exist a positive constant M such that $|f(y)-f(x)|\le M|y-x|$ for all $x,y\in\mathbb{R}$. Prove that f is uniformly continuous on $\mathbb{R}$.

Let $\varepsilon>0$ & $x,y\in\mathbb{R}$
Let $\delta>0$ s.t. $|y-x|>\delta$
consider $|f(y)-f(x)|<M|y-x|<M\delta$
Choose $\delta=\frac{\varepsilon}{M}$ then $\delta>0$
So, $|f(y)-f(x)|\le M|y-x|<M\delta=M\frac{\varepsilon}{M}=\varepsilon$
and for $\varepsilon>0$ $\exists\delta>0$, $|y-x|<\delta$
$\Rightarrow |f(y)-f(x)|<\varepsilon$ for all x,y in $\mathbb{R}$
Thus f is uniformly cont. on $\mathbb{R}$

② Let $f:[0,\infty)\to\mathbb{R}$ be non-negative and continuous such that $\lim_{x\to\infty}f(x)=0$
Prove that f attained its maximum at some point in $[0,\infty)$.

③ Suppose that $f:[a,b]\to\mathbb{R}$ is a continuous function such that $f([a,b])\subseteq[a,b]$.
Prove that there is a $c\in[a,b]$ such that $f(c)=c$

$f:[a,b]\to\mathbb{R}$ is a continuous function such that $f([a,b])\subseteq[a,b]$
so $f(a)\ge a$, $f(b)\le b$
$\Rightarrow f(a)-a\ge 0$ & $f(b)-b\le 0$
Now let $h(x)=f(x)-x$
$\Rightarrow h(a)=f(a)-a$
$\Rightarrow h(b)=f(b)-b$
so $h(a)\ge 0$ & $h(b)\le 0$
By IVT since h is continuous and $0\in[h(b),h(a)]$
there exists $c\in[a,b]$ s.t. $h(c)=f(c)-c=0$
so $f(c)=c$ for some $c\in[a,b]$

④ Suppose that $f:(a,b)\to\mathbb{R}$ is twice differentiable on (a,b) and there are two points $c<d$ in (a,b) such that $f'(c)=f'(d)$. Show that there is a point $x\in(c,d)$ such that $f''(x)=0$

Given $f:(a,b)\to\mathbb{R}$ is twice differentiable
f & f' is continuous & differentiable
Also $c<d$ in (a,b) s.t. $f'(c)=f'(d)$
so $f'(n)$ is continuous & diff on (a,b)
& $f'(n)$ is continuous & diff on (c,d)
By Rolles Thm $\exists x\in(c,d)$ s.t. $f''(x)=0$

⑤ Suppose that $f:(a,b)\to\mathbb{R}$ is differentiable at $x_0\in(a,b)$

a) Prove that $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0-h)}{2h}$ exists & equals $f'(x_0)$

$$\lim_{x\to 0^-} f(x)=\lim_{h\to 0}\frac{f(x+h)-f(x_0)}{h} \quad (1)$$

$$\lim_{x\to 0^-} f(x)=\lim_{h\to 0}\frac{f(x_0)-f(x-h)}{h} \quad (2)$$

Then $2f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x-h)}{h}$ [by adding (1) & (2)]

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x-h)}{2h}$$

$$f'(x)=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$$

b) Find a continuous function $f:\mathbb{R}\to\mathbb{R}$ and a point $x_0\in\mathbb{R}$ such that f is not differentiable at x_0, but the limit (*) exists.

Consider $f(x)=|x|$ at $x=0$

Left Hand derivative

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^-}\frac{|x|}{x}=\lim_{x\to 0^-}\frac{-x}{x}=-1$$

Right Hand derivative

$$\lim_{x\to 0} f(x)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{|x|}{x}=\lim_{x\to 0}\frac{x}{x}=1$$

Thus not differentiable at $x=0$

Left Hand limit

Put $x=0-h$ as $h\to 0$

$$f(x)=\lim_{h\to 0}|0-h|=\lim_{h\to 0}|h|=0$$

Right hand limit

Put $x=0+h$ as $h\to 0$

$$f(x)=\lim_{h\to 0}|0+h|=\lim_{h\to 0}|h|=0$$

Thus the limit exist

⑥ a) Suppose $r>0$. Prove that $f:(0,\infty)\to\mathbb{R}$ defined by $f(x)=x^r$ is differentiable on $(0,\infty)$ and compute its derivative. [Hint: take for granted that e^x and $\ln x$ are differentiable with $(e^x)'=e^x$ and $(\ln x)'=1/x$. Rewrite then x^r in terms of a composition of two differentiable functions.]

Let $g(x)=\ln(x)$, which we know is differentiable.

So $rg(x)$ is also differentiable

$\Rightarrow h(x)=rg(x)=r\ln x=\ln(x^r)$ is differentiable.

Again let $k(x)=e^x$ which we know is differentiable.

Now $k\circ h(x)$ is differentiable

$$k\circ h(x)=k(\ln(x^r))=e^{\ln(x^r)}=x^r=f(x)$$

So $f(x)$ is differentiable

$$f'(x)=(x^r)'=rx^{r-1}$$

b) define $f(x)=\sqrt{x^2+\sin x+\cos x}$ where $x\in[0,\pi/2]$. Show that f is differentiable function & find a formula for its derivative.

$$f'(x)=\left(\sqrt{x^2+\sin x+\cos x}\right)'=\frac{1}{2\sqrt{x^2+\sin x+\cos x}}\cdot(2x+\cos x-\sin x)=\frac{2x+\cos x-\sin x}{2\sqrt{x^2+\sin x+\cos x}}$$

$2\sqrt{x^2+\sin x+\cos x}\neq 0 \quad \forall x\in\mathbb{R}$

$\Rightarrow f'(x)$ is defined $\forall x\in\mathbb{R}$

$\Rightarrow f$ is differential

⑦ Show that $S \subseteq \mathbb{R}$ is closed if and only if $\mathbb{R} \setminus S$ is open

$S \subseteq \mathbb{R}$ is closed if $acc(S) \subseteq S$

$\Rightarrow$ Suppose S is closed and $x_0 \in \mathbb{R} \setminus S$
If S contains all $acc(S)$, x_0 is not an $acc(S)$
so there is a neighborhood, T, of x_0 that does not contain S
Now $T \subset \mathbb{R} \setminus S$ and $\mathbb{R} \setminus S$ is open
$\Leftarrow$ Suppose $\mathbb{R} \setminus S$ is open
To show S is closed we want to show if x_0 is an $acc(S)$, $x_0 \in S$
Let x_0 be an $acc(S)$, $x_0 \in S$ or $x_0 \in \mathbb{R} \setminus S$
$\mathbb{R} \setminus S$ is open, so $x_0 \in \mathbb{R} \setminus S$
so there is a neighborhood, T, of x_0 s.t. $T \subset \mathbb{R} \setminus S$ or $T \cap E$ is empty #
Thus $x_0 \in S$ & S is closed

⑧ Let $f: \mathbb{R} \to \mathbb{R}$ be a differentiable function and define $g(x) = x^2 f(x^3)$.
Show that g is differentiable and compute its derivative.

Given $f: \mathbb{R} \to \mathbb{R}$ is differentiable then,
x^2 and $f(x^3)$ is differentiable

Let $h(x) = x^2$ & $g(x) = h(x) f(x^3)$
since x^2 is differentiable $h(x)$ is too.
Now f & h are differentiable, so g is differentiable.

$$
\begin{aligned}
g'(x) &= (x^2)'(f(x^3)) + x^2 (f(x^3))' \\
&= 2x \cdot f(x^3) + x^2 f'(x^3) \cdot 3x^2 \\
&= 2x \cdot f(x^3) + 3x^4 \cdot f(x^3)
\end{aligned}
$$

⑨ Prove that $f(x) = \arcsin x$ is differentiable on its domain and find a formula involving no trig functions, for the derivative of f.

$f(x) = \sin^{-1} x$
$\Rightarrow \sin(f(x)) = x$
$(\sin(f(x)))' = x'$
$\cos(f(x)) \cdot f(x) = 1$

$$
\begin{aligned}
f'(x) &= \frac{1}{\cos(f(x))} \\
&= \frac{1}{\sqrt{1 - \sin^2(f(x))}} \quad \text{from trig iden: } \sin^2 y + \cos^2 y = 1 \\
&= \frac{1}{\sqrt{1 - x^2}} \quad \text{since } \sin(f(x)) = x
\end{aligned}
$$

⑩ Use the mean-value theorem to show the following inequalities

<u>MVT</u>: If $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b), then $\exists c\in(a,b)$ s.t. $f'(c)=\frac{f(b)-f(a)}{b-a}$

a) $ny^{n-1}(x-y)\le x^n-y^n\le nx^{n-1}(x-y)$ if $n\in\mathbb{N}$ and $0\le y\le x$

$f(x)=x^n-y^n-ny^{n-1}(x-y)$

Now $f'(x)=nx^{n-1}-ny^{n-1}\frac{dy}{dx}-n(n-1)y^{n-2}\frac{dy}{dx}(x-y)-ny^{n-1}(1-\frac{dy}{dx})$

$=nx^{n-1}\cancel{-ny^{n-1}\frac{dy}{dx}}-n(n-1)y^{n-2}\frac{dy}{dx}(x-y)-ny^{n-1}\cancel{+ny^{n-1}\frac{dy}{dx}}$

$=nx^{n-1}-n(n-1)y^{n-2}\frac{dy}{dx}(x-y)-ny^{n-1}$

$=n(x^{n-1}-y^{n-1})-n(n-1)y^{n-2}\frac{dy}{dx}(x-n)$

since $f(x)$ is increasing for $x>0$ $f(x)>0$ for $x>0$, so

$f(x)-f(0)>0$

$\frac{f(x)-f(0)}{x-0}>0$

$f'(x)>0$

$n(x^{n-1}-y^{n-1})-n(n-1)y^{n-2}\frac{dy}{dx}(x-n)>0$

Thus $f(x)>0$ and $ny^{n-1}(x-y)<x^n-y^n$ if $x>0$

$g(x)=nx^{n-1}(x-y)$

$g'(x)=n(n-1)x^{n-1}(x-y)+nx^{n-1}(1-\frac{dy}{dx})$

$g'(x)>0$ for $y>0$, $x>1$. Thus

$g(x)>g(0)$ for $x>0$

$g(x)>0$

$x^n-y^n<nx^{n-1}(x-y)$ if $y>0$ & $n\ge1$

So, $ny^{n-1}(x-y)\le x^n-y^n\le nx^{n-1}(x-y)$

b) $\sqrt{1+x}<1+\frac{1}{2}x$ for $x>0$

$f(x)=\sqrt{1+x}$ for $x>0$

By MVT, $f(x)$ is differentiable for $x>0$ so therefore $\exists c\in[0,x]$ such that $f'(x)=\frac{f(x)-f(0)}{x-0}$

$\Rightarrow \frac{1}{2\sqrt{1+x}}=\frac{\sqrt{1+x}-1}{x}$

$\Rightarrow x=2(1+x)-2\sqrt{1+x}$

$x=2+2x-2\sqrt{1+x}$

$2\sqrt{1+x}=2+x$

$\sqrt{1+x}=1+\frac{x}{2}$

Since $x>0$, $\sqrt{1+x}<1+\frac{n}{2}$