Preliminaries

- Important sets.
- Principle of mathematical induction.
- Countable sets.

0-Preliminaries.

<u>Example</u> Prove that $\forall x \neq 1$, $\forall n \in \mathbb{N}$

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1-x}.$$

Let $x \neq 1$ be fixed. We will use PIM on n.
Let $P(n)$: $1 + x + \cdots + x^n = \dfrac{1 - x^{n+1}}{1-x}$.

0.1 Important subsets.

- $\mathbb{N} := \{1, 2, 3, 4, \ldots\}$ (or $\mathbb{Z}_+$). $\longrightarrow$ take for granted that $(\mathbb{N}, \leq)$ is well-ordered
- $\mathbb{N}_0 := \{0, 1, 2, 3, \ldots\}$.
- $\mathbb{Z} := \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.
- $\mathbb{Q} := \left\{ x = \frac{a}{b} : \ a, b \in \mathbb{Z} \right\}$.
- $\mathbb{R}$ set of real numbers, $\sqrt{2}, \pi, \frac{4}{3} \in \mathbb{R}$.

0.2 Principle of Mathematical induction

<u>PIM</u> Let $P(n)$ be a proposition on $\mathbb{N}$. If
  1) $P(1)$ is true
  2) $P(n) \Rightarrow P(n+1)$ is true

Then $P(n)$ is true for any $n \in \mathbb{N}$.

The proof of this theorem is base on

$e \leq x \ \forall x \in S$.

<u>WOP</u> Every non empty subset of $\mathbb{N}$ has a minimum.

<u>Example</u> $1 + x + x^2 + \cdots + x^n = \dfrac{1 - x^{n+1}}{1-x}$ $\forall x \neq 1$,

for every $n \in \mathbb{N}$ and,

Let's fixe $x \neq 1$. Let $n \in \mathbb{N}$

$$P(n) := \quad 1 + x + x^2 + \dots + x^n = \frac{1-x^{n+1}}{1-x}.$$

1) We have $(1+x)(1-x) = 1 - x^2$

$$\Rightarrow \quad 1 + x = \frac{1-x^2}{1-x} \quad (x \neq 1).$$

$P(1)$ is true.

2) Suppose $P(n)$ is true. Then

$$1 + x + \dots + x^n = \frac{1-x^{n+1}}{1-x} \quad (IH).$$

We have

$$\sum_{k=0}^{n+1} x^k = \sum_{k=0}^{n} x^k + x^{n+1}$$

$$\overset{IH}{=} \frac{1-x^{n+1}}{1-x} + x^{n+1}$$

$$= \frac{1-x^{n+2}}{1-x}$$

So $P(n+1)$ is true.

By the PMI, $P(n)$ is true $\forall n \in \mathbb{N}$.

<u>PMI 2</u> Let $P(n)$ be a proposition on $\mathbb{N}$. If

1) $P(1), P(2), \dots, P(n_0)$ is true

2) $n \geq n_0$, $P(i)$ true $1 \leq i \leq n \Rightarrow P(n+1)$ true.

Then $P(n)$ is true $\forall n \in \mathbb{N}$.

Example: Let $f: J \to \mathbb{R}$ be defined as

$$f(1) = 3, \quad f(2) = \frac{3}{2} \quad \text{and} \quad f(n) = \frac{f(n-1) + f(n-2)}{2} \quad (n \geq 3).$$

Let $P(n) :=$ "$f(n) = 2 + \left(\frac{-1}{2}\right)^{n-1}$". We will show $P(n)$ true.

1) Choose $n_0 = 2$.

- $f(1) = 3 = 2 + 1 = 2 + \left(\frac{-1}{2}\right)^{n-1} \quad (n=1)$
- $f(2) = \frac{3}{2} = 2 - \frac{1}{2} = 2 + \left(\frac{-1}{2}\right)^{n-1} \quad (n=2).$

So $P(1)$ and $P(2)$ is true.

2) Let s.t. $P(1), P(2), \ldots, P(n)$ is true. We have

$$f(n+1) = \frac{f(n) + f(n-1)}{2} = \frac{2 + \left(\frac{-1}{2}\right)^{n-1} + 2 + \left(\frac{-1}{2}\right)^{n-2}}{2}$$

$$= 2 + \frac{\left(\frac{-1}{2}\right)^{n-2}\left[\frac{-1}{2} + 1\right]}{2}$$

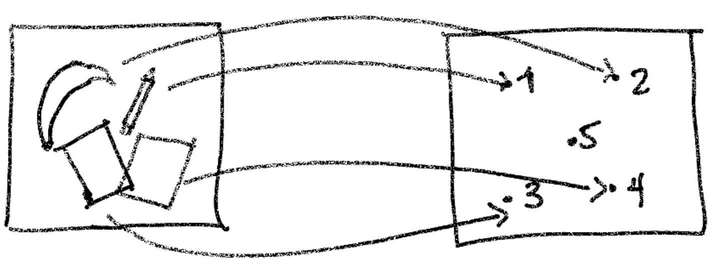
$$= 2 + \left(\frac{-1}{2}\right)^{n-2}\left(\frac{-1}{2}\right)^{2}$$

$$= 2 + \left(\frac{-1}{2}\right)^{n}.$$

So $P(n+1)$ is true.

By the PMI2, $P(n)$ is true for every $n \in \mathbb{N}$.

0.3 Countable sets.



4 elements
Did a
one-to-one
correspondence.

How to count # elements in a finite set

Def. Two sets A and B are <u>equivalent</u> if $\exists f: A \to B$ s.t. f is a bijection. We write $A \sim B$.

Example. Let $f: \mathbb{N} \to 2\mathbb{N}$ (→ even integers.) be the function define by $f(n) := 2n$.

[Remember: bijection $\Leftrightarrow$ injective & surjective].

1) f is <u>injective</u> $\boxed{1\text{-}1}$: $f(n) = f(m)$, $n, m \in \mathbb{N} \Rightarrow n = m$.

We have $f(n) = f(m) \Rightarrow 2n = 2m$

$\Rightarrow n = m$.

So f is injective.

2) f is <u>surjective</u> $\boxed{\text{onto}}$: $\forall m \in 2\mathbb{N}$, $\exists n \in \mathbb{N}$ s.t. $f(n) = m$.

Since $m \in 2\mathbb{N}$, it is of the form $m = 2n$ $(n \in \mathbb{N})$.

So, taking this $n \Rightarrow m = 2n = f(m)$.

So f is surjective.

Thus, from 1) & 2), f is bijective $\Rightarrow A \sim B$ by def. $\square$

Thm. Let A, B and C be sets. Then

1) $A \sim A$ 2) $A \sim B \Rightarrow B \sim A$ 3) $A \sim B$ and $B \sim C \Rightarrow A \sim C$.

Proof. See exercise 3 of the homework. $\square$

Example. Let $f: \mathbb{N} \to \mathbb{Z}$ be defined by

$$f(n) := \begin{cases} n/2 & , n \text{ even} \\ (-1)\frac{n-1}{2} & , n \text{ odd} \end{cases}$$

We will show that f is a bijection.

1) Let $f(n) = f(m)$. Then n, m are even or n, m odd.

- n, m even $\Rightarrow \frac{n}{2} = \frac{m}{2} \Rightarrow n = m$.
- n, m odd $\Rightarrow (-1)\frac{n-1}{2} = (-1)\frac{m-1}{2} \Rightarrow n = m$.

So f is injective.

2) Let $m \in \mathbb{Z}$.

- $m \geq 1$. Then $\exists n \in 2\mathbb{N}$ s.t. $m = \frac{n}{2}$ (see prev. example)
  $\Rightarrow \exists n \in 2\mathbb{N}$ s.t. $f(n) = \frac{n}{2} = m$.
- $m = 0$. then take $n = 1$.
- $m < 0$. Then $-m \in \mathbb{N}$. $\exists n \in 2\mathbb{N}-1$ s.t. (odd integers.)
  $-m = \frac{n-1}{2}$ (see prev. example for $2\mathbb{N}-1$)
  $\Rightarrow \exists n \in 2\mathbb{N}-1$ s.t. $f(n) = (-1)\frac{n-1}{2} = m$.

So f is a bijection and we get $\mathbb{N} \sim \mathbb{Z}$. $\square$

From last theorem, we see that $\mathbb{Z} \sim 2\mathbb{N} \sim 2\mathbb{N}-1$.

Def. Let A be a set. A is

- countably infinite if $\mathbb{N} \sim A$.
- finite if it is equivalent to $\{1, 2, \dots, n\}$. or is $\emptyset$.
- countable if it is countably infinite or finite.
- uncountable if it is not countable.

The sets $2\mathbb{N}$, $2\mathbb{N}-1$, $\mathbb{Z}$ are countable. Any finite set is countable.

Thm. Any infinite subset of $\mathbb{N}$ is countably infinite.

Proof. Let $S \subseteq \mathbb{N}$ be infinite. Define $f: \mathbb{N} \to S$ recursively:

1) S infinite $\Rightarrow$ S non-empty. By the WOP, it has a smallest element, say $f(1)$.

2) Given $f(1), \dots, f(k)$, $S \setminus \{f(1), \dots, f(k)\}$ is still infinite, so by WOP, it has a smallest element call it $f(k+1)$.

Since $f(k+1) \neq f(i)$ for any $i = 1, 2, \dots, k$, f is 1-1.

Let $s \in S$. Then, $s \in \{f(1), f(2), \dots, f(s)\}$. If not, then $s \in S \setminus \{f(1), \dots, f(i)\}$ for any $1 \leq i \leq s-1$

So, by definition of each $f(i)$, we must have

$$s \geq f(1) \ \& \ s \geq f(i+1) \text{ for } i = 1, 2, \dots, s-1.$$

So, $\#\{x \in S : x \leq s\} > s$. This contradicts the

fact that $\#\{x \in S : x \leq s\} \leq \#\{x \in \mathbb{N} : x \leq s\} = s$.
So, $s \in \{f(1), \ldots, f(s)\}$. then, there is a $n \in \mathbb{N}$ such that $f(n) = s$. So f is surjective. $\square$

<u>Cor.</u> Let S be a set and $f : S \to \mathbb{N}$ be injective. Then S is countable.

<u>Proof.</u> If $f(S)$ is finite, then it is countable. Suppose $f(S)$ is infinite. Then $f(S)$ is a subset of $\mathbb{N}$, so it is countably infinite by the previous theorem. So, $\exists g : f(S) \to \mathbb{N}$ where g is a bijection. So,

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[dr, "g \circ f"'] & f(S) \arrow[d, "g"] \\ & \mathbb{N} \end{tikzcd}$$

is bijection. $\square$

<u>thm</u> If A & B are countable, then $A \times B$ is.

<u>Proof.</u> Define $h : A \times B \to \mathbb{N}$ by

$$h(a, b) := 2^{f(a)} 3^{g(b)}$$

where $f : A \to \mathbb{N}$ & $g : B \to \mathbb{N}$ are bijections.
By the Unique Factorization theorem, h is 1-1.
So, $A \times B$ is countable by the previous Cor. $\square$

Thm If A & B are countable sets, then $A \cup B$ is a countable set. More generally, if each A_n are countable sets, then $\bigcup_{n \geq 1} A_n$ is countable.

Recall: $\bigcup_{n \geq 1} A_n = \{ x : \exists n \in \mathbb{N} \text{ s.t. } x \in A_n \}$.

Proof. We will show the general case. Let $f_n : A_n \to \mathbb{N}$ be a bijection. We will construct a function $f : \bigcup_{n \geq 1} A_n \to \mathbb{N} \times \mathbb{N}$ in the following.

Let $x \in \bigcup_{n \geq 1} A_n$. Then, $x \in A_n$ for some n. By the WOP, there is a smallest such n, call it m. Define $f : \bigcup_{n \geq 1} A_n \to \mathbb{N}$ by

$$f(x) := (m, f_m(x)).$$

Since m is the smallest of all integers n s.t. $x \in A_n$, this means that f is well-defined. Now, if $f(x) = f(y) \Rightarrow (m, f_m(x)) = (\tilde{m}, f_{\tilde{m}}(y))$

$$\Rightarrow \tilde{m} = m \text{ and } f_m(x) = f_{\tilde{m}}(y)$$

$$\Longrightarrow \quad f_m(x) = f_m(y) \implies x = y.$$

So, f is injective and by the corollary, $A \cup B$ is countable, because $\mathbb{N} \times \mathbb{N}$ is countable by the previous theorem. $\square$

Exercises.

#1. Prove that $\forall n \in \mathbb{N}$, $1+2+3+\dots+n = \frac{n(n+1)}{2}$.

#2. Prove that $\forall n \in \mathbb{N}$, $1+3+5+\dots+(2n-1) = n^2$.

#3. Prove that $n^2 < 2^n$ $\forall n \in \mathbb{N}$ s.t. $n \geq 5$.

#4. Define $f: \mathbb{N} \to \mathbb{N}$ by $f(1)=1$, $f(2)=2$, $f(3)=3$ and

$$f(n) = f(n-1) + f(n-2) + f(n-3), \quad n \geq 4.$$

Prove that $f(n) \leq 2^n$ $\forall n \in \mathbb{N}$.

#5. (HW1) Prove that if A & B are sets then

1) $A \sim A$. 2) $A \sim B \Rightarrow B \sim A$.

3) $A \sim B$ and $B \sim C \Rightarrow A \sim C$.

#6. Prove that $\mathbb{Q}$ is countable.

#7. (HW1) Prove that any subset of a countable set is countable.

#8 (HW1) Let $n \in \mathbb{N}$ and $A_1, A_2, \dots, A_n$ be countable sets. Show that $A_1 \times A_2 \times \dots \times A_n$ is countable.

#9 (HW1) (i) Let P_n be the set of all polynomials of degree n with integer coefficients. Prove that P_n is countable. [Hint: Use the last exercise].

(ii) Deduce that the set P of all polynomials with integer coefficients is countable.

#10 For a set A, let $P(A)$ be the family of subsets of A.

Show that A is not equivalent to $P(A)$.
[Hint: suppose $f: A \to P(A)$ and define
$C := \{x : x \in A \text{ and } x \notin f(x)\}$. Show that $C \notin \operatorname{im} f$
where $\operatorname{im} f := \{ f(x) : x \in A \}$.]

From the book :

0.3 : 20, 22,
0.4 : 37, 38.