MATH-331 Introduction to Real Analysis | BENDER, LEE'AN
Homework 05 | Fall 2021

Due date: November, 22^{th} 1:20pm | Total: /65.

Exercise	1 (10)	2 (10)	3 (5)	4 (5)	5 (5)	6 (10)	7 (5)	8 (5)	9 (5)	10 (5)
Score										

Table 1: Scores for each exercises

Instructions: You must answer all the questions below and send your solution by email (to `parisepo@hawaii.edu`). If you decide to not use LaTeX to hand out your solutions, please be sure that after you scan your copy, it is clear and readable. Make sure that you attached a copy of the homework assignment to your homework.
If you choose to use LaTeX, you can use the template available on the course website.
No late homework will be accepted. No format other than PDF will be accepted. Name your file as indicated in the syllabus.

1 — Writing problems

For each of the following problems, you will be asked to write a clear and detailed proof. You will have the chance to rewrite your solution in your semester project after receiving feedback from me.

Exercise 1. (10 pts)

a) Fix any $\delta > 0$ and let $[a, b]$ be an interval with $a < b$. Find a tagged partition $\mathcal{P}$ of $[a, b]$ such that $\|\mathcal{P}\| < \delta$.

b) Suppose that f is Riemann integrable. Show that in the definition of the Riemann integral, the number L is unique. [Remark: This is why we gave it the name $\int_a^b f$.]

Solution: :

a) Let $\mathcal{P} := \{(c_i, [x_i, x_{i-1}] : i = 1, 2, ..., n\}$ and let x_i be defined by $x_i = a + i(\frac{b-a}{n})$. To obtain $||\mathcal{P}|| < \delta$, we want

$$|x_i - x_{i-1}| < \delta$$

so that

$$|(a + i(\tfrac{b-a}{n})) - (a + (i-1)(\tfrac{b-a}{n}))| < \delta$$

This simplifies to

$$|\frac{b-a}{n}| < \delta$$

Multiplying n on both sides gives

$$|b-a| < n\delta$$

By AP, there must exist some n that fulfills that inequality, therefore a tagged partition $\mathcal{P} := \{(c_i, [x_i, x_{i-1}] : i = 1, 2, ..., n\}$ must exist such that $||\mathcal{P}|| < \delta$.

b) Suppose toward a contradiction that L is not unique and that $\int_a^b f = L_1$ and $\int_a^b f = L_2$. By the definition of a Riemann Integral, $\int_a^b f = \lim_{||\mathcal{P}|| \to 0} S(f, \mathcal{P}) = L_1$ and L_2 with $L_1 \neq L_2$. So $\forall \epsilon > 0, \exists \delta_1 > 0$ and $\exists \delta_2 > 0$ such that

$$|S(f, \mathcal{P}) - L_1| < \epsilon/2$$
$$|S(f, \mathcal{P}) - L_2| < \epsilon/2$$

Since $L_1 \neq L_2$, $|L_2 - L_1| > 0$. Now, let $\delta := \min\{\delta_1, \delta_2\}$. Now we can expand $|L_2 - L_1| > 0$:

$$0 < |L_2 - L_1|$$
$$0 < |L_2 - S(f, \mathcal{P}) + S(f, \mathcal{P}) - L_1|$$
$$0 < |L_2 - S(f, \mathcal{P})| + |S(f, \mathcal{P}) - L_1|$$
$$0 < |S(f, \mathcal{P}) - L_2| + |S(f, \mathcal{P}) - L_1|$$

Since $|S(f, \mathcal{P}) - L_1| < \epsilon/2$ and $|S(f, \mathcal{P}) - L_2| < \epsilon/2$, we can simplify the inequality to

$$0 < \frac{\epsilon}{2} + \frac{\epsilon}{2}$$
$$0 < \epsilon$$

Since $\epsilon > 0$ was arbitrary,

$$L_2 - L_2 = 0$$
$$L_1 = L_2$$

This contradicts our original assumption, which means that $\int_a^b f = L$ must be unique. $\square$

Exercise 2. (10 pts) Suppose that f and g are Riemann integrable on the interval $[a, b]$.

a) Show that $\int_a^b (f + g) = \int_a^b f + \int_a^b g$.

b) Show that if $f(x) \leq g(x)$ for any $x \in [a, b]$, then $\int_a^b f \leq \int_a^b g$.

Solution: :

a) By the definition of a Riemann Integral, $\int_a^b f = L$. Let $\int_a^b (f + g) = L$ and $\int_a^b f = L_1$ and $\int_a^b g = L_2$. We want to show that $L = L_1 + L_2$. From the limit definition of integrals, we know that $\int_a^b (f+g) = \lim_{||\mathcal{P}|| \to 0||} S((f+g), \mathcal{P})$. From the definition of $S(f, \mathcal{P})$, we know that

$$S((f+g),\mathcal{P}) = \textstyle\sum_{i=1}^{N}(f+g)(c_i)(x_i - x_{i-1})$$

This can be expanded to

$$\begin{gathered}\textstyle\sum_{i=1}^{N}(f(c_i)+g(c_i))(x_i - x_{i-1}) \\ = \textstyle\sum_{i=1}^{N}(f(c_i)(x_i - x_{i-1})) + (g(c_i)(x_i - x_{i-1})) \\ = \textstyle\sum_{i=1}^{N} f(c_i)(x_i - x_{i-1}) + \sum_{i=1}^{N} g(c_i)(x_i - x_{i-1})\end{gathered}$$

From the algebra rules of limits, we know that

$$\lim_{||\mathcal{P}||\to 0}(\textstyle\sum_{i=1}^{N} f(c_i)(x_i - x_{i-1}) + \sum_{i=1}^{N} g(c_i)(x_i - x_{i-1}))$$

is equal to

$$\lim_{||\mathcal{P}||\to 0} \textstyle\sum_{i=1}^{N} f(c_i)(x_i - x_{i-1}) + \lim_{||\mathcal{P}||\to 0} \sum_{i=1}^{N} g(c_i)(x_i - x_{i-1})$$

We can substitute $S(f,\mathcal{P})$ and $S(g,\mathcal{P})$ for their respective limits, so we have

$$\lim_{||\mathcal{P}||\to 0} S(f,\mathcal{P}) + \lim_{||\mathcal{P}||\to 0} S(g,\mathcal{P})$$

These are the values of $\int_a^b f$ and $\int_a^b g$ respectively. So, putting everything together, we have

$$\textstyle\int_a^b (f+g) = \int_a^b f + \int_a^b g$$

Since $\int_a^b (f+g) = L$ and $\int_a^b f = L_1$ and $\int_a^b g = L_2$, we finally obtain

$$L = L_1 + L_2$$

Which proves our assumption.

b) By the definition of Riemann Integral, we know that $\forall \epsilon > 0$, $\exists \delta > 0$ such that if $||\mathcal{P}|| < \delta$, then

$$\begin{gathered}|S(f,\mathcal{P}) - \textstyle\int_a^b f| < \epsilon \\ \text{and } |S(g,\mathcal{P}) - \textstyle\int_a^b g| < \epsilon\end{gathered}$$

We can modify both of these expressions to obtain

$$\begin{gathered}-\epsilon < S(f,\mathcal{P}) - \textstyle\int_a^b f < \epsilon \\ \textstyle\int_a^b f - \epsilon < S(f,\mathcal{P}) < \epsilon + \int_a^b f \\ \text{and} \\ -\epsilon < S(g,\mathcal{P}) - \textstyle\int_a^b g < \epsilon \\ \textstyle\int_a^b g - \epsilon < S(g,\mathcal{P}) < \epsilon + \int_a^b g\end{gathered}$$

From the assumption, we know that $S(f,\mathcal{P}) \leq S(g,\mathcal{P})$. This means, from our expression, that

$$\int_a^b f - \epsilon < S(f,\mathcal{P}) \le S(g,\mathcal{P}) < \int_a^b g + \epsilon$$

So therefore,

$$\int_a^b f - \epsilon \le \int_a^b g + \epsilon$$

We can change this to be

$$\int_a^b f \le \int_a^b g + 2\epsilon$$

We then can set $\epsilon = x$ and take the limit as x goes to ∞ of both sides of the inequality, leaving us with

$$\lim_{x\to\infty} \int_a^b f \le \lim_{x\to\infty}(\int_a^b g + 2\epsilon)$$

Since f and g are constants in these limits, we are left with $\int_a^b f \le \int_a^b g$, which proves our assumption. □

Exercise 3. (5 pts) Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable on $[a,b]$ and suppose that $|f(x)| \le M$ for all $x \in [a,b]$. Show that $\int_a^b f \le M(b-a)$.

Solution: We want to show that $|\int_a^b f - M(b-a)| < \epsilon$. Let $g : [a,b] \to \mathbb{R}$ be defined by $g(x) = M$ $\forall x \in [a,b]$. The integral of g from $[a,b]$ is equal to $M(b-a)$ because [see (6a)]. Now we just have two functions, f and g, where $f \le g$ $\forall x \in [a,b]$. Then we can apply the logic from (2b), which implies that if $f \le g$ then $\int_a^b f \le \int_a^b g$. Since $\int_a^b g(x) = M(b-a)$, then $\int_a^b f \le M(b-a)$. □

Exercise 4. (5 pts) Suppose that f is Riemann integrable on $[a,b]$. Let $(\mathcal{P}_n)_{n=1}^{\infty}$ be a sequence of tagged partitions of $[a,b]$ such that the sequence $\lim_{n\to\infty} \|\mathcal{P}_n\| = 0$. Prove that the sequence $(S(f,\mathcal{P}_n))_{n=1}^{\infty}$ converges to $\int_a^b f$.

Solution: From the definition of a Riemann integral, we know that $\forall\epsilon > 0$, $\exists\delta > 0$ such that $||\mathcal{P}|| < \delta$, then, $|S(f,\mathcal{P}) - \int_a^b f| < \epsilon$ $\forall x \in [a,b]$. We are given that $\lim_{n\to\infty} ||\mathcal{P}_n|| = 0$. From the definition of a Riemann Integral we also have that $\int_a^b f = \lim_{||\mathcal{P}_n||\to 0} S(f,\mathcal{P}_n)$. [...] □

Exercise 5. (5 pts) Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Suppose that f is Riemann integrable on $[a,c]$ for any $c \in (a,b)$. Show that f is Riemann integrable on $[a,b]$. [Hint: Use the Cauchy criterion for integrals.]

Solution: We are given that f is Riemann Integrable on $[a,b]$, which means that $\forall\epsilon > 0$, $\exists\delta > 0$ such that if $||\mathcal{P}|| < \delta$ then

$$|S(f,\mathcal{P}) - \int_a^b f| < \epsilon$$

Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two tagged partitions of $[a,b]$ defined by

$$\mathcal{P}_1 := \{(c_i, [x_i, x_{i-1}] \in [a,b]\}$$
$$\mathcal{P}_2 := \{(c_i, [x_i, x_{i-1}]) \in [a,b]\}$$

such that $||\mathcal{P}_1|| < \delta$ and $||\mathcal{P}_2|| < \delta$ so then $|S(f,\mathcal{P}_1) - S(f,\mathcal{P}_2)| < \epsilon$. We let $\epsilon > 0$ and let $c \in [a,b]$ such that $b - c < \epsilon$. We then further divide $\mathcal{P}_1$ and $\mathcal{P}_2$ into 2 disjoint subdivisions:

$$\mathcal{P}_{1a} := \{(c_i, [x_i, x_{i-1}]) \in \mathcal{P}_1 : [x_i, x_{i-1}] \in [a,c]\}$$
$$\mathcal{P}_{1b} := \{(c_i, [x_i, x_{i-1}]) \in \mathcal{P}_1 : [x_i, x_{i-1}] \in [c,b]\}$$

and

$$\mathcal{P}_{2a} := \{(c_i, [x_i, x_{i-1}]) \in \mathcal{P}_2 : [x_i, x_{i-1}] \in [a,c]\}$$
$$\mathcal{P}_{2b} := \{(c_i, [x_i, x_{i-1}]) \in \mathcal{P}_2 : [x_i, x_{i-1}] \in [c,b]\}$$

Now let $\widetilde{\mathcal{P}}_{1a}$, $\widetilde{\mathcal{P}}_{1b}$, $\widetilde{\mathcal{P}}_{2a}$, $\widetilde{\mathcal{P}}_{2b}$ be defined by

$$\widetilde{\mathcal{P}}_{1a} := \mathcal{P}_{1a} \cup \{c, [x_{N_{1a}}, c]\}$$
$$\widetilde{\mathcal{P}}_{1b} := \mathcal{P}_{1b} \cup \{c, [c, x_{N_{1a}+1}]\}$$
$$\widetilde{\mathcal{P}}_{2a} := \mathcal{P}_{2a} \cup \{c, [x_{N_{2a}}, c]\}$$
$$\widetilde{\mathcal{P}}_{2b} := \mathcal{P}_{2b} \cup \{c, [c, x_{N_{2a}+1}]\}$$

where $c \in [a,b]$. Now we apply the Cauchy Criterion to $\mathcal{P}_1$ and $\mathcal{P}_2$. We split the partitions:

$$|S(f,\mathcal{P}_1) - S(f,\mathcal{P}_2)| < \epsilon$$
$$\Rightarrow |(S(f,\widetilde{\mathcal{P}}_{1a}) + S(f,\widetilde{\mathcal{P}}_{1b})) - (S(f,\widetilde{\mathcal{P}}_{2a}) + S(f,\widetilde{\mathcal{P}}_{2b}))|$$
$$\leq |S(f,\widetilde{\mathcal{P}}_{1a}) - S(f,\widetilde{\mathcal{P}}_{2a})| + |S(f,\widetilde{\mathcal{P}}_{1b}) - S(f,\widetilde{\mathcal{P}}_{2b})| < \epsilon$$

We are given that f is bounded, and we now call the upper bound of f, M. From the original assumption, we can write

$$S(f,\widetilde{\mathcal{P}}_{ia}) \leq M(b-a) \leq M\epsilon$$

Then, applying the Cauchy Criterion,

$$|S(f,\widetilde{\mathcal{P}}_{1a}) - S(f,\widetilde{\mathcal{P}}_{2a})| < 0$$
$$|S(f,\widetilde{\mathcal{P}}_{1b}) - S(f,\widetilde{\mathcal{P}}_{2b})| < \epsilon$$

Thus, we can see that f is integrable from $[a,c]$ and $[c,b]$, proving that if f is Riemann integrable on $[a,c]$ $\forall c \in [a,b]$ then it is integrable on $[a,b]$.
(?) $\square$

2
Homework problems

Answer all the questions below. Make sure to show your work.

Exercise 6. (10pts)

a) Define the function $f : [a,b] \to \mathbb{R}$ by $f(x) = k$ for every $x \in [a,b]$ where $k \in \mathbb{R}$ is a fixed constant. Show that f is Riemann integrable on $[a,b]$ and that $\int_a^b k\,dx = k(b-a)$.

b) Let $f(x) = \sin^2(x)$ where $x \in [a, b]$ and assume that the function $g(x) := \cos(kx)$ is integrable on $[a, b]$ for any $k \in \mathbb{R}$. Show that f is Riemann integrable on $[a, b]$.

Solution: :

a) From the definition of Riemann Integral, $\forall\epsilon > 0$, $\exists\delta > 0$ such that if $||\mathcal{P}|| < \delta$ then

$$|S(f, \mathcal{P}) - k(b - a)| < \epsilon.$$

$S(f, \mathcal{P})$ is defined by $\sum_{i-1}^{N} f(c_i)(x_i - x_{i-1})$. Substituting this in,

$$|\sum_{i-1}^{N} f(c_i)(x_i - x_{i-1}) - k(b - a)| < \epsilon$$

Since f is constant at $f = k$, we can replace $f(c_i)$.

$$\begin{gathered} |\sum_{i-1}^{N} k(x_i - x_{i-1}) - k(b - a)| < \epsilon \\ = |k\sum_{i-1}^{N}(x_i - x_{i-1}) - k(b - a)| < \epsilon \end{gathered}$$

$\sum_{i-1}^{N}(x_i - x_{i-1})$ is just equal to the bounds of the function, so we can substitute $(b - a)$:

$$\begin{gathered} |k(b - a) - k(b - a)| < \epsilon \\ 0 < \epsilon \end{gathered}$$

The resulting inequality is true, therefore f is Riemann Integrable on $[a, b]$.

b) $\sin^2(\theta) = \frac{1}{2}(1 - \cos(2\theta))$ (Double angle identity)
So, $\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$. $\cos(kx)$ is Riemann integrable and $\cos(2x) \in \cos(kx)$, so $\cos(2x)$ is Riemann Integrable.
Adding and Multiplying constants does not change the integrability of functions, so $\frac{1}{2} - \frac{1}{2}\cos(2x)$ is integrable and therefore so is $\sin^2(x)$. $\square$

Exercise 7. (5 pts) Show that the function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} 1 & \text{, if } 0 \le x < 1/2 \\ 0 & \text{, if } 1/2 \le x \le 1 \end{cases}$$

is Riemann integrable on $[0, 1]$.

Solution: X $\square$

Exercise 8. (5 pts) Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = 1$ if $x = 1/n$ where $n \in \mathbb{N}$, and by $f(x) = 0$ if $x \neq 1/n$, $n \in \mathbb{N}$. Show that f is Riemann integrable on $[0, 1]$.

Solution: If f is Riemann Integrable on $[0, 1]$, then $\forall\epsilon > 0$, $\exists\delta > 0$ such that if $||\mathcal{P}|| < \delta$ then

$$|S(f, \mathcal{P}) - \int_0^1 f| < \epsilon$$

Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two tagged partitions of $[0,1]$ defined by

$$\mathcal{P}_1 := \{(c_i, [x_i, x_{i-1}] \in [0,1]\}$$
$$\mathcal{P}_2 := \{(c_i, [x_i, x_{i-1}]) \in [0,1]\}$$

such that $||\mathcal{P}_1|| < \delta$ and $||\mathcal{P}_2|| < \delta$ so then $|S(f,\mathcal{P}_1) - S(f,\mathcal{P}_2)| < \epsilon$. We let $\epsilon > 0$ and let $c \in [0,1]$ such that $c - a < \epsilon$. We then further divide $\mathcal{P}_1$ and $\mathcal{P}_2$ into 2 disjoint subdivisions:

$$\mathcal{P}_{1a} := \{(c_i, [x_i, x_{i-1}]) \in \mathcal{P}_1 : [x_i, x_{i-1}] \in [0, c]\}$$
$$\mathcal{P}_{1b} := \{(c_i, [x_i, x_{i-1}]) \in \mathcal{P}_1 : [x_i, x_{i-1}] \in [c, 1]\}$$

and

$$\mathcal{P}_{2a} := \{(c_i, [x_i, x_{i-1}]) \in \mathcal{P}_2 : [x_i, x_{i-1}] \in [0, c]\}$$
$$\mathcal{P}_{2b} := \{(c_i, [x_i, x_{i-1}]) \in \mathcal{P}_2 : [x_i, x_{i-1}] \in [c, 1]\}$$

Now let $\widetilde{\mathcal{P}}_{1a}$, $\widetilde{\mathcal{P}}_{1b}$, $\widetilde{\mathcal{P}}_{2a}$, $\widetilde{\mathcal{P}}_{2b}$ be defined by

$$\widetilde{\mathcal{P}}_{1a} := \mathcal{P}_{1a} \cup \{c, [x_{N_{1a}}, c]\}$$
$$\widetilde{\mathcal{P}}_{1b} := \mathcal{P}_{1b} \cup \{c, [c, x_{N_{1a}+1}]\}$$
$$\widetilde{\mathcal{P}}_{2a} := \mathcal{P}_{2a} \cup \{c, [x_{N_{2a}}, c]\}$$
$$\widetilde{\mathcal{P}}_{2b} := \mathcal{P}_{2b} \cup \{c, [c, x_{N_{2a}+1}]\}$$

where $c \in [0,1]$. Now we apply the Cauchy Criterion to $\mathcal{P}_1$ and $\mathcal{P}_2$. We split the partitions:

$$|S(f,\mathcal{P}_1) - S(f,\mathcal{P}_2)| < \epsilon$$
$$\Rightarrow |(S(f,\widetilde{\mathcal{P}}_{1a}) + S(f,\widetilde{\mathcal{P}}_{1b})) - (S(f,\widetilde{\mathcal{P}}_{2a}) + S(f,\widetilde{\mathcal{P}}_{2b}))|$$
$$\leq |S(f,\widetilde{\mathcal{P}}_{1a}) - S(f,\widetilde{\mathcal{P}}_{2a})| + |S(f,\widetilde{\mathcal{P}}_{1b}) - S(f,\widetilde{\mathcal{P}}_{2b})| < \epsilon$$

We are given that f is bounded, and we now call the upper bound of f, M. From the original assumption, we can write

$$S(f,\widetilde{\mathcal{P}}_{ia}) \leq M(b-a) \leq M\epsilon$$

Then, applying the Cauchy Criterion,

$$|S(f,\widetilde{\mathcal{P}}_{1a}) - S(f,\widetilde{\mathcal{P}}_{2a})|$$
$$|M\epsilon - M\epsilon| = 0$$
$$\text{and } |S(f,\widetilde{\mathcal{P}}_{1b}) - S(f,\widetilde{\mathcal{P}}_{2b})| < \epsilon$$

Thus, we can see that f is integrable from $[0,c]$ and $[c,1]$, proving that if f is Riemann integrable on $[0,c]$ $\forall c \in [0,1]$ then it is integrable on $[0,1]$. □

Exercise 9. (5 pts) Show that the function $f : [0,1] \to \mathbb{R}$ defined by $f(x) = 0$ if $x \neq 0$ and $f(x) = 4$ if $x = 0$ is Riemann integrable on $[0,1]$.

Solution: See above. □

Exercise 10. (5 pts) Let $\mathcal{P}$ be the following tagged partition of $[-1, 2]$:

$$\mathcal{P} := \{(-9, [-1, -.8]), (-.7, [-.8, -.3]), (-.1, [-.3, 0]), (.2, [0, 0.2]), (.2, [.2, .4]), (.8, [.4, 1]), (1.42, [1, 1.5]), (1.9, [1.5, 2])\}.$$

Find another partition $\mathcal{P}_0$ such that $\|\mathcal{P}_0\| \leq \|\mathcal{P}\|/3$.

Solution: :

Let P be the tag/par of $[-1,2]$: $P := \{ \overset{.2}{\underbrace{(-9,[-1,-.8])}}, \overset{.5}{\underbrace{(-.7,[-.8,-.3])}}, \overset{.3}{\underbrace{(-.1,[-.3,0])}}, \overset{.2}{\underbrace{(.2,[0,.2])}}, \overset{.2}{\underbrace{(.2,[.2,.4])}},$
$\overset{.6}{\underbrace{(.8,[.4,1])}}, \overset{.5}{\underbrace{(1.42,[1,1.5])}}, \overset{.5}{\underbrace{(1.9,[1.5,2])}} \}$

Find another partition P_0 s.t. $\|P_0\| \leq \|P\|/3$.

P_0 of $[-1,2]$ $\qquad \|P\| = \frac{.6}{3} = .2 = \|P_0\|$

$$\begin{aligned} P_0 := \{ & (-.9,[-1,-.8]), \\ & (-.7,[-.8,-.6]), \\ & (-.5,[-.6,-.4]), \\ & (-.3,[-.4,-.2]), \\ & (-.1,[-.2,0]), \\ & (.1,[0,.2]), \\ & (.3,[.2,.4]), \\ & (.5,[.4,.6]), \\ & (.7,[.6,.8]) \\ & (.9,[.8,1]), \\ & (1.1,[1,1.2]), \\ & (1.3,[1.2,1.4]), \\ & (1.5,[1.4,1.6]), \\ & (1.7,[1.6,1.8]), \\ & (1.9,[1.8,2])\} \end{aligned}$$

□