#2. We see that, if $x \neq -3$

$$f(x) = \frac{2(x^2-9)}{x+3} = \frac{2(x-3)(x+3)}{x+3} = 2(x-3).$$

So, $f(-3) = -12$ and -3 accumulation pt &

$$\lim_{x \to -3} f(x) = \lim_{x \to -3} 2(x-3) = -12.$$

So, $\lim_{x \to -3} f(x) = -12 = f(-3) \Rightarrow f$ continuous at $x = -3$.

#5 We see that, $x \neq 0$,

$$f(x) = \frac{1-\sqrt{x+1}}{\sqrt{x}} = \frac{1-x-1}{\sqrt{x}\,(1+\sqrt{x+1})}$$

$$= \frac{-x}{\sqrt{x}\,(1+\sqrt{x+1})}$$

$$\Rightarrow f(x) = \frac{-\sqrt{x}}{1+\sqrt{x+1}} \qquad (x \neq 0)$$

Just define $f(0) = 0$ so that

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{-\sqrt{x}}{1+\sqrt{x+1}} = 0 = f(0).$$

#7. Since f is continuous, let (x_n) be a sequence of rational numbers such that

$$x_n \longrightarrow x \qquad (x \in \mathbb{R}).$$

Then, $f(x_n) \longrightarrow f(x)$ by continuity

$$\Rightarrow \quad x_n^2 \longrightarrow f(x).$$

But, $x_n^2 \longrightarrow x^2 \Rightarrow f(x) = x^2$ by the uniqueness of limits.

So, $f(\sqrt{2}) = (\sqrt{2})^2 = 2.$

#13 Let $\varepsilon = 1$. Then there is a $\delta > 0$ s.t.

$$\forall x \in D, \quad |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < 1.$$

Take $Q = (x_0 - \delta, x + \delta)$. So, for all $x \in Q \cap D$, $|f(x) - f(x_0)| < 1$

$$\Leftrightarrow \quad |f(x)| - |f(x_0)| < 1$$

$$\Leftrightarrow \quad |f(x)| < 1 + |f(x_0)|$$

Put $M = 1 + |f(x_0)|$.

$\square$

#14 This is a consequence of
$$\big|\,|f(x)| - |f(x_0)|\,\big| \leq |f(x) - f(x_0)|.$$

#15 An elegant solution to this problem is the following. We can prove that

$$\max\{x, y\} = \frac{x+y+|x-y|}{2} \quad (x, y \in \mathbb{R}).$$

- $x \geq y \implies \dfrac{x+y+x-y}{2} = x = \max\{x,y\}.$
- $x < y \implies \dfrac{x+y-(x-y)}{2} = y = \max\{x,y\}.$

So,
$$\max\{f(x), g(x)\} = \frac{f(x)+g(x)+|f(x)-g(x)|}{2}.$$

Since f & g are continuous, so are
$$f+g \quad \& \quad |f+g|.$$

So, by the sum rule, $\max\{f, g\}$ is continuous.

#19. (a) Let $\varepsilon > 0$. There are $\delta_1 > 0$, $\delta_2 > 0$ s.t.

$\forall x, y \in D$,
- $|x-y| < \delta_1 \implies |f(x) - f(y)| < \varepsilon/2$
- $|x-y| < \delta_2 \implies |f(x) - f(y)| < \varepsilon/2$

So, take $\delta := \min\{\delta_1, \delta_2\}$. Then, if $x, y \in D$ s.t. $|x-y| < \delta$, then

$$|f(x)+g(x) - f(y) - g(y)| \leq |f(x)-f(y)| + |g(x)-g(y)|$$
$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So, $f+g$ is uniformly continuous.

(b) No, let $f: \mathbb{R} \to \mathbb{R}$ & $g: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x$ & $g(x) = x$.
f & g are uniformly continuous on $\mathbb{R}$, but $fg(x) = x^2$ is not on $\mathbb{R}$.

If it was the case, then $\forall \varepsilon > 0, \exists \delta > 0$ s.t.

$$\forall x, y \in \mathbb{R}, \quad |x-y| < \delta \Rightarrow |x^2 - y^2| < \varepsilon.$$

Put $\varepsilon = 1$. So, $\exists \delta > 0$ st.

$$\forall x, y \in \mathbb{R}, \quad |x-y| < \delta \Rightarrow |(x-y)(x+y)| < 1.$$

Take $y = \frac{2N}{\delta}$ and $x = \frac{2N}{\delta}\ \delta/2$ for $N \in \mathbb{N}$. Then
$|x-y| = \frac{2N}{\delta} + \delta/2 - \frac{2N}{\delta} = \delta/2 < \delta$. &
$|(x-y)(x+y)| = |x-y||x+y| = (\delta/2)\left(\frac{4N}{\delta} + \delta/2\right)$

$\Rightarrow \quad |x^2-y^2| = 4N + \frac{\delta^2}{4} > 4N > 1$ #.

So, $f \cdot g$ is not uniformly continuous. □

#16 Homework 5.

#21 First, we have if $x \in [3.4, 5]$,

$0.4 = 3.4 - 3 \le x-3 \le 5-3 = 2$

$$\Rightarrow \frac{1}{2} \le \frac{1}{x-3} \le \frac{1}{0.4} = \frac{5}{2}.$$

Let $x, y \in [3.4, 5]$.

$$|f(x) - f(y)| = \left|\frac{2}{x-3} - \frac{2}{y-3}\right|$$

$$= \left|\frac{2(y-x)}{(x-3)(y-3)}\right| = \frac{2|y-x|}{|x-3||y-3|}$$

Since $x, y \in [3.4, 5]$, then

$$|f(x)-f(y)| = \frac{2|y-x|}{(x-3)(y-3)} \le \frac{25}{4} 2|y-x|.$$

If $\varepsilon > 0$, and $x, y \in D$ s.t. $|x-y| < \frac{2}{25}\varepsilon = \delta$,

then $|f(x) - f(y)| \le \frac{25}{2} \cdot \frac{2}{25} \cdot \varepsilon = \varepsilon$. □

#23. Suppose $\exists\, h \in \mathbb{R}$ s.t. $h \neq 0$ and

$f(x+h) = f(x)$ $\forall x \in \mathbb{R}$ & f is continuous on $\mathbb{R}$.

We may suppose that $h > 0$. Indeed, if $h < 0$, then write $y = x + h$

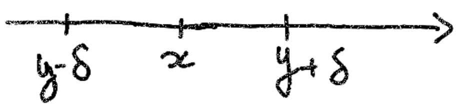
$$f(y) = f(x+h) = f(x) = f(y-h).$$

So, $\forall y \in \mathbb{R}$, $f(y) = f(y-h)$ & $-h > 0$.

Suppose $h > 0$. Let $\delta < h$.

Let $x, y \in \mathbb{R}$ s.t. $|x-y| < \delta$. then

$$|x-y| < h$$



$$\iff \quad y-h < x < y+h$$

This means that there is a natural number s.t.

$$x - Nh = \tilde{x} \in [-h, h] \quad \&$$
$$y - Nh = \tilde{y} \in [-h, h].$$

and
$$|f(x) - f(y)| = |f(\tilde{x} + Nh) - f(\tilde{y} + Nh)|$$
$$= |f(\tilde{x}) - f(\tilde{y})|.$$

Now, f is continuous on $[-h, h]$ and $[-h, h]$ is compact. So, f is uniformly continuous on

$[-h, h]$.

Let $\varepsilon > 0$. Then there is a $\tilde{\delta} > 0$ s.t.

$\forall \tilde{x}, \tilde{y} \in [-h, h]$, $|\tilde{x} - \tilde{y}| < \tilde{\delta} \Rightarrow |f(\tilde{x}) - f(\tilde{y})| < \varepsilon$.

Take $\delta := \min\{h, \tilde{\delta}\}$ and let $x, y \in \mathbb{R}$ s.t.

$$|x - y| < \delta.$$

Since $|x-y| < h$, you can find $\tilde{x}, \tilde{y}$ in $[-h, h]$ s.t. $x - Nh = \tilde{x}$ & $y - Nh = \tilde{y}$.

Then, $$|\tilde{x} - \tilde{y}| = |x - y| < \delta \le \tilde{\delta}$$

and so
$$|f(\tilde{x}) - f(\tilde{y})| < \varepsilon.$$

Since $$f(\tilde{x}) = f(\tilde{x} + Nh) = f(x)$$
& $$f(\tilde{y}) = f(\tilde{y} + Nh) = f(y)$$

then
$$|f(x) - f(y)| < \varepsilon.$$

So, f is uniformly continuous on $\mathbb{R}$. $\square$

#26 Suppose that $E \subseteq \mathbb{R}$

Suppose $\forall x_0 \in \mathbb{R}$ s.t. $\exists (x_n) \subseteq \mathbb{R}$ s.t.

- $x_n \in E$.
- $x_n \longrightarrow x_0$.

then $x_0 \in E$.

We want to prove that E is closed, that is $acc(E) \subseteq E$.

Let $x \in acc(E)$. So, $\forall \delta > 0$,

$(x-\delta, x+\delta) \cap E$ has infinitely many elements.

Take $\delta = 1 \Rightarrow (x-1, x+1) \cap E$ has infinitely many elements.

Take $x_1 \in (x-1, x+1) \cap E$.

Take $\delta = \frac{1}{2} \Rightarrow (x-\frac{1}{2}, x+\frac{1}{2}) \cap E$ has infinitely many elements.

So, $[(x-1/2, x+1/2) \cap E] \setminus \{x\}$ has infinitely elements & pick $x_2 \in (x-1/2, x+1/2) \cap E$ with $x_2 \neq x_1$.

Repeat this process so that we get $(x_n)_{n=1}^{\infty}$ s.t.

- $x_n \in E$, • $x_n \neq x_m$, $n \neq m$ • $x_n \longrightarrow x$ $n \to \infty$.

By the assumption, we get that that $x \in E$. Since $x \in acc(E)$ was arbitrary,

$\Rightarrow acc(E) \subseteq E$

$\Rightarrow E$ is closed.

$\square$

#27.

(a) $(a,b) := \{ x \in \mathbb{R} : \ a < x < b \}$.

Let $x \in (a,b)$. Take $\delta := \min\{ b-x, \ x-a \}$

Let $y \in (x-\delta, x+\delta)$. Then $y > x-\delta$ and

so

$$y > x - \delta \quad \&$$

$$\delta < x-a \quad \Rightarrow \quad x-\delta > a$$

$$\Rightarrow \quad y > a.$$

Also,

$$y < x+\delta < x + b - x = b.$$

$$\Rightarrow \quad y < b.$$

So, $a < y < b \Rightarrow y \in (a,b)$.

So, $(x-\delta, x+\delta) \subseteq (a,b) \Rightarrow (a,b)$ is open.

(b) $[a,b] := \{ x \in \mathbb{R} : \ a \le x \le b \}$.

Take $(x_n)_{n=1}^{\infty} \subseteq [a,b]$ s.t. $x_n \to x$. Then

$$b \le x_n \le b \quad \Rightarrow \quad a \le x \le b \Rightarrow x \in [a,b]$$

From exercise 26, $[a,b]$ is closed.

#28. Let $D \subseteq \mathbb{R}$ & $D' := acc(D)$. Define

$$\overline{D} := D \cup D'.$$

We want to show that $\overline{D}$ is closed. We will show that $\mathbb{R} \setminus \overline{D}$ is open. We have

$$\mathbb{R} \setminus \overline{D} = \mathbb{R} \cap \overline{D}^c = \mathbb{R} \cap D^c \cap D'^c$$
$$= D^c \cap D'^c.$$

Let $x \in D^c \cap D'^c$. In particular

- $x \in D^c$ $(x \notin D)$
- $x \in D'^c$ $(x \notin acc(D))$.

Since $x \notin acc(D)$, $\exists \delta_0 > 0$ s.t.

$(x-\delta_0, x+\delta_0) \cap D$ contains finitely many elements.

Call them $x_1, x_2, \ldots, x_n$. Each $x_i \neq x$ because $x \notin D$. Put $\delta_i := |x - x_i|$ and

$$\delta := \min\{\delta_i : i = 0, 1, 2, \ldots, n\}.$$

Then, $(x-\delta, x+\delta) \cap D = \emptyset$. This means that $(x-\delta, x+\delta) \subseteq D^c$.

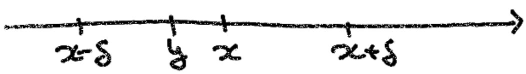
Now we will prove that

$$(x-\delta, x+\delta) \subseteq D'^{c}$$

Suppose, on the contrary, that $\exists y \in (x-\delta, x+\delta)$ s.t. $y \notin D'^{c}$, so $y \in D' = acc(D)$.

This means that $\forall \eta > 0$,

$(y-\eta, y+\eta) \cap D$ has infinitely many elements.



Put $\eta := \min\{x+\delta-y,\ y-x+\delta\}$.

Then, $(y-\eta, y+\eta) \subseteq (x-\delta, x+\delta)$. So,

$$(y-\eta, y+\eta) \cap D \subseteq (x-\delta, x+\delta) \cap D.$$

Since $(y-\eta, y+\eta) \cap D$ contain infinitely many elements, so is $(x-\delta, x+\delta) \cap D$. So, $x \in acc(D)$.
But, $x \notin acc(D)$. A contradiction. So,

$$(x-\delta, x+\delta) \subseteq D'^{c}.$$

Putting everything together,

$$(x-\delta, x+\delta) \subseteq D^{c} \cap D'^{c} = \mathbb{R} \setminus \overline{D}.$$

So, $\mathbb{R}\setminus\overline{D}$ is open & so $\overline{D}$ is closed. $\square$

#29. Suppose that D is bounded, so $\exists M>0$ s.t.

$$|x| \leq M \quad \forall x \in D.$$

Let $x \in \overline{D}$. The we have two cases:

- $x \in D \implies |x| \leq D$.

- $x \in acc(D)$. Then, from #26, we can contruct a sequence $(x_n)_{n=1}^{\infty} \subseteq D$ s.t. $x_n \to x$ (follow the recipe in #26).
  Now, $|x_n| \leq M \iff -M \leq x_n \leq M$
  and taking limit:

$$-M \leq \lim_{n\to\infty} x_n = x \leq M.$$

  So,
$$|x| \leq M.$$

Thus, $\forall x \in \overline{D}$, $|x| \leq M$ & $\overline{D}$ is bounded. $\square$

#30. Let $r_0 \in \mathbb{R}$ and let $A := \{x \in \mathbb{R} : f(x) \neq r_0\}$.
If $A = \emptyset$, then A is open.
Suppose $A \neq \emptyset$. Let $x \in A$. Then $f(x) \neq r_0$.
Consider two cases:

- $\underline{f(x) - r_0 > 0}$. Put $\varepsilon = f(x) - r_0$. Then by the continuity of f, we have $\exists \delta > 0$ s.t.

$\forall t \in \mathbb{R}, \quad |t-x| < \delta \Rightarrow |f(t)-f(x)| < f(x)-r_0$.

So, $\forall t \in (x-\delta, x+\delta)$, we have

$$f(x)-f(t) \leq |f(t)-f(x)| < f(x)-r_0$$

$$\Rightarrow \quad 0 < f(t)-r_0$$

So, $\forall t \in (x-\delta, x+\delta)$, $f(t) > r_0$.

In other words, $f(t) \neq r_0 \quad \forall t \in (x-\delta, x+\delta)$.

So, $t \in A$, $\forall t \in (x-\delta, x+\delta)$. Thus,

$$(x-\delta, x+\delta) \subseteq A.$$

This implies that A is open.

- $\underline{f(x) < r_0.}$ Repeat the above steps with $\varepsilon := r_0 - f(x)$. $\square$

#31. Put $h(x) = f(x) - g(x)$. Then h is continuous on $[a,b]$. Consider $r_0 = 0$ and the set $A = \{x \in \mathbb{R} : h(x) \neq 0\}$. The the previous exercise tells us that A is open.

So, since $\mathbb{R} \setminus A = \{x \in \mathbb{R} : h(x) = 0\}$
$= \{x \in \mathbb{R} : f(x) = g(x)\} = T$

Then T is closed because A is open.

#33 $\{x : x > 0\} = (0, \infty)$.

Consider the intervals $A_n = (0, n)$ $(n \in \mathbb{N})$.

Then, $(0, \infty) = \bigcup_{n=1}^{\infty} A_n$.

If $(0, \infty)$ was covered by finitely many A's, say $A_{n_1}, A_{n_2}, \ldots, A_{n_k}$, then

$$(0, \infty) \subseteq \bigcup_{i=1}^{k} (0, n_i).$$

So, take $n := \max_{1 \le i \le k} \{ n_i \}$.

Then $\bigcup_{i=1}^{k} (0, n_i) \subseteq (0, n)$ and

so $(0, \infty) \subseteq (0, n) \implies (0, \infty)$ is bounded.

This is a contradiction because $(0, \infty)$ is not bounded.

So $(0, \infty)$ can't be covered by finitely many of the sets A_n. $\square$

No 36, 37, 39

#39 Homework 5.

#42. Put $f(x) = x^3 - 6x^2 + 2.826$.

then, $f(0) = 2.826 > 0$.

$f(1) = 1 - 6 + 2.826 = -3.174 < 0$

By the IVT with $L=0$, there is a $c \in [0,1]$ s.t. $f(c) = 0$.

#44 Homework 5.

#46 Suppose, on the contrary, that it is possible.

Then, there is a continuous function $f: \mathbb{R} \to \mathbb{R}$ s.t. $\forall c \in \mathbb{R}$,

$$f(x) = c$$

has exactly two solutions.

Call these solutions x_c, y_c & suppose $x_c < y_c$

Suppose $c=0$. Then there are $x_0 < y_0$ s.t.

$f(x_0) = f(y_0) = 0$.

Suppose $c=-1$. There there are $x_{-1} < y_{-1}$

$f(x_{-1}) = f(y_{-1}) = -1$.

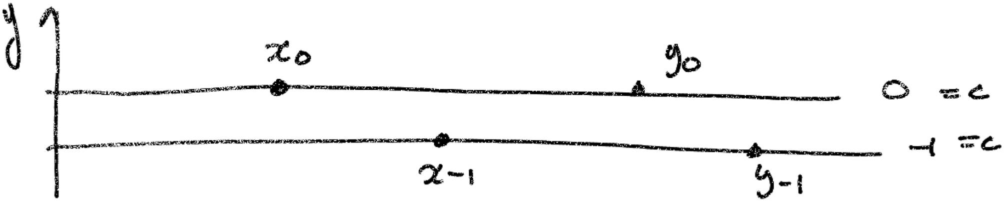
We have $x_{-1} \neq x_0, y_0$ & $y_{-1} \neq x_0, y_0$.

Two possibilities can occur:

1) $x_{-1} < x_0 < y_0 < y_{-1}$ or;

2) $x_0 < x_{-1} < y_{1} < y_0$.

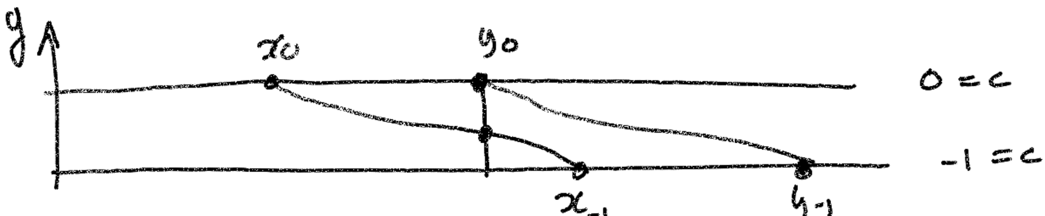
Indeed, consider the situation



$x_0 < x_{-1} < y_0 < y_{-1}$.

Then, for $n = -1/2$, $\exists\, c_1 \in (x_0, x_{-1})$, $\exists\, c_2 \in (x_{-1}, y_0)$ $\exists\, c_3 \in (y_0, y_{-1})$ s.t. $f(c_1) = f(c_2) = f(c_3) = \frac{-1}{2}$.
This contradicts the fact that $f(x) = -\frac{1}{2}$ has only two distinct solution.

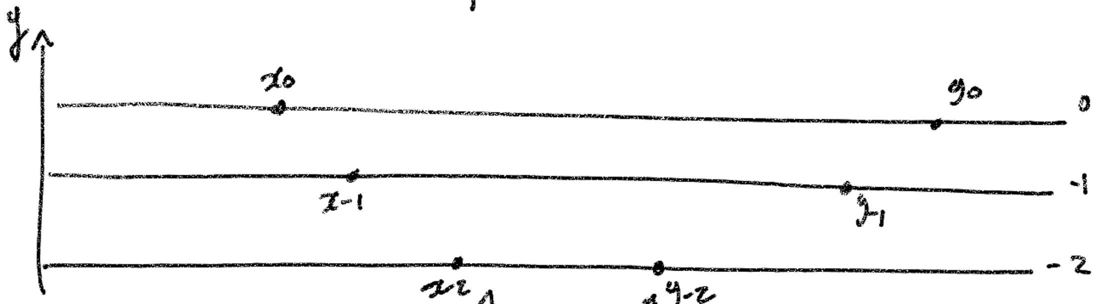
Consider the other situation



$x_0 < y_0 < x_{-1} < y_{-1}$ $\to$ f is not a function.

All other situations are treated in the same way. WLOG, suppose that we are in situation 2) : $x_0 < x_{-1} < y_{-1} < y_0$.
For the situation 1), consider instead $n = 1$.

Consider $\eta = -2$. Then there are $x_{-2} < y_{-2}$ s.t. $f(x_{-2}) = f(y_{-2}) = -2$.



the only situation that can occur is that

$$x_0 < x_{-1} < x_{-2} < y_{-2} < y_{-1} < y_0 .$$

Continue this process and create two sequences $(x_{-n})_{n=1}^{\infty}$ & $(y_{-n})_{n=1}^{\infty}$ s.t.

- $x_0 < x_{-1} < x_{-2} < \dots < x_{-n} < y_{-n} < \dots < y_0$.
- $f(x_{-n}) = -n \quad \forall n \geq 1$.

Since $(x_{-n})_{n=1}^{\infty}$ is increasing and bounded by y_0, then it converges to some $x \in \mathbb{R}$.

By continuity, $\lim\limits_{n\to\infty} f(x_{-n}) = f(x) \in \mathbb{R}$.

But, $\lim\limits_{n\to\infty} f(x_n) = \lim\limits_{n\to\infty} -n = -\infty$.

This is a contradiction !

TJ