MATH-331 Intro. to Real Analysis — Pierre-Olivier Parisé
Team test 01 — Fall 2021, 09/10/2021

Name of the members of the team: Correction.

Team name: Correction.

Question:	1	2	Total
Points:	10	10	20
Score:	—	—	—

Instructions: You must answer all the questions in teams of 3 and hand out one copy per team. You are allowed to use the lecture notes only. No other tools such as a cell-phone, a calculator, or a laptop. Only your pen and eraser. The space between the questions are there to write the final versions of your answers.

Question 1 (10 pts)

Which of the following inequalities are true, for any $x_1, x_2, x_3 \in \mathbb{R}$? Give a proof or a counter-example.

(a) (2 points) $|x_1 - x_2 + x_3| \leq |x_1| - |x_2| - |x_3|$.

(b) (2 points) $|x_1 - x_2 + x_3| \geq |x_1| - |x_2| + |x_3|$.

(c) (2 points) $|x_1 - x_2 + x_3| \geq |x_1| - |x_2| - |x_3|$.

(d) (2 points) $((-1)^n/n)_{n=1}^{\infty}$ converges to 0.

(e) (2 points) $((-1)^n + (-1)^{n+1})_{n=1}^{\infty}$ converges.

(a) False. Take $x_1 = x_2 = 0$ and $x_3 = 2$:

$$\left.\begin{array}{l} |x_1 - x_2 + x_3| = |x_3| = 2 \\ |x_1| - |x_2| - |x_3| = -2 \end{array}\right\} \Rightarrow 2 \not\leq -2.$$

(b) False. Take $x_2 = 0$. Then

$$|x_1 + x_3| \geq |x_1| + |x_3|.$$

Take $x_1 = 2$, $x_3 = -2$ $\Rightarrow$ $0 \geq 4$.

But $0 \not\geq 4$.

(c) True. By the properties of the absolute value we have

$$|x_1 - x_2 + x_3| \geq |x_1 - x_2| - |x_3|$$
$$\geq |x_1| - |x_2| - |x_3|.$$

(d) True. Let $\varepsilon > 0$. then $\left|\frac{(-1)^n}{n}\right| < \varepsilon$ iff.

$1 < n\varepsilon$.

By the AP with $x = \varepsilon$ & $y = 1$, $\exists N \in \mathbb{N}$ s.t.

$N\varepsilon > 1$.

So, if $n \geq N$, we have

$$\frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

So, since ε was arbitrary, $\frac{(-1)^n}{n} \to 0$.

(e) <u>True.</u> We have

$$(-1)^n + (-1)^{n+1} = 0 \qquad \forall n \geq 1.$$

So, $\left((-1)^n + (-1)^{n+1}\right)_{n=1}^{\infty}$ is just the sequence

$(0)_{n=1}^{\infty}$. This sequence converges to 0. $\square$

QUESTION 2 **(10 pts)**

Suppose $(a_n)_{n=1}^{\infty}$ converges to A, and define the new sequence $b_n = \frac{a_n + a_{n-1}}{2}$ for all $n \geq 1$. Prove that the sequence $(b_n)_{n\geq 1}$ converges to A.

By hypothesis, $a_n \longrightarrow A$.

The goal is to prove that $b_n \longrightarrow A$. Let $\varepsilon > 0$.

Then,

$$\left| \frac{a_n + a_{n-1}}{2} - A \right| = \left| \frac{a_n + a_{n-1} - 2A}{2} \right|$$

$$\leq \frac{|a_n - A| + |a_{n-1} - A|}{2}$$

Choose $N \in \mathbb{N}$ s.t.

$$n \geq N+1 \implies |a_n - A| < \varepsilon.$$

Then, if $n \geq N+1$, then $n-1 \geq N$ &

$$\frac{|a_n - A| + |A_{n-1} - A|}{2} < \frac{\varepsilon + \varepsilon}{2} = \varepsilon.$$

So, $|b_n - A| < \varepsilon$ whenever $n \geq N+1$.

Since $\varepsilon > 0$ was arbitrary, we have

$$b_n \longrightarrow A. \qquad \square$$