MATH-331 Introduction to Real Analysis — Nicholas Lee
Homework 05 — Fall 2021

Due date: November, 22^{th} 1:20pm — Total: /65.

Exercise	1 (10)	2 (10)	3 (5)	4 (5)	5 (5)	6 (10)	7 (5)	8 (5)	9 (5)	10 (5)
Score										

Table 1: Scores for each exercises

Instructions: You must answer all the questions below and send your solution by email (to `parisepo@hawaii.edu`). If you decide to not use LaTeX to hand out your solutions, please be sure that after you scan your copy, it is clear and readable. Make sure that you attached a copy of the homework assignment to your homework.
If you choose to use LaTeX, you can use the template available on the course website.
No late homework will be accepted. No format other than PDF will be accepted. Name your file as indicated in the syllabus.

1
Writing problems

For each of the following problems, you will be asked to write a clear and detailed proof. You will have the chance to rewrite your solution in your semester project after receiving feedback from me.

Exercise 1. (10 pts)

a) Fix any $\delta > 0$ and let $[a, b]$ be an interval with $a < b$. Find a tagged partition $\mathcal{P}$ of $[a, b]$ such that $\|\mathcal{P}\| < \delta$.

b) Suppose that f is Riemann integrable. Show that in the definition of the Riemann integral, the number L is unique. [Remark: This is why we gave it the name $\int_a^b f$.]

Solution: a) Let P be a partition of $[a, b]$, written as

$$P = \{(c_i, [x_{i-1}, x_i]) : i = 1, 2, ..., N\}$$

and let our N be equal to

$$N = \left\lfloor \frac{b-a}{\delta} \right\rfloor + 1$$

Now, let's also construct this partition such that the partitions are identically spaced. So that given this partition, the interval $[a, b]$ is split up into N many section, each with a width of

$$\frac{b-a}{N} = \frac{b-a}{\lfloor \frac{b-a}{\delta} \rfloor + 1}$$

If each section has this width, then if we show that this width is less than δ, we will have shown that $||P|| < \delta$.

$$\frac{b-a}{\lfloor \frac{b-a}{\delta} \rfloor + 1} < \frac{b-a}{\frac{b-a}{\delta}} = \delta$$
$$\frac{b-a}{\lfloor \frac{b-a}{\delta} \rfloor + 1} < \delta$$

Therefore, with δ fixed and this particular partition, we have that $||P|| < \delta$.

b) Let $f : [a, b] \to \mathbb{R}$ be Riemann Integrable on $[a, b]$. That means $\exists L \in \mathbb{R}$ such that $\forall \epsilon > 0$, $\exists \delta > 0$ such that if P is a tagged partition of $[a, b]$, with $||P|| < \delta$, then

$$|S(f, P) - L| < \epsilon$$

Now say for this function f, we have a partition P of $[a, b]$, and two $L_1, L_2 \in \mathbb{R}$ such that

$$|S(f, P) - L_1| < \epsilon$$
$$|S(f, P) - L_2| < \epsilon$$

Now if we can show that $L_1 = L_2$, we will have shown that L is indeed unique. First let's start by combining these two inequalities, such that we get

$$|S(f, P) - L_1| + |S(f, P) - L_2| < 2\epsilon$$

Now by the triangle inequality, we get

$$|S(f, P) - L_1 + S(f, P) - L_2| < 2\epsilon$$
$$|2S(f, P) - L_1 - L_2| < 2\epsilon$$

Now from here, if we plug in $L_1 = L_2$ and get a true statement, then we know that it is a solution, and that $L_1 = L_2$. Now let's just set $L_1 = L_2 = L$.

$$|2S(f, P) - L - L| < 2\epsilon$$
$$|2S(f, P) - 2L| < 2\epsilon$$
$$2|S(f, P) - L| < 2\epsilon$$
$$|S(f, P) - L| < \epsilon$$

This above statement is true because from our own assumption, plugging in both L_1 and L_2 yields

$$|S(f, P) - L_1| < \epsilon$$
$$|S(f, P) - L_2| < \epsilon$$

Both of which we said was true. Therefore, $L_1 = L_2$ is true, and therefore L in the definition of the Riemann Integral is unique. Because no matter what, for the same function, any variable in its place will always be equal to L. $\square$

Exercise 2. (10 pts) Suppose that f and g are Riemann integrable on the interval $[a, b]$.

a) Show that $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.

b) Show that if $f(x) \leq g(x)$ for any $x \in [a, b]$, then $\int_a^b f \leq \int_a^b g$.

Solution: a) Since f and g are Riemann integrable, we know that given any ϵ and a tagged partition P of $[a, b]$, there exists a δ such that $||P|| < \delta$, such that

$$|S(f, P) - \int_a^b f| < \epsilon_f$$
$$|S(f, P) - \int_a^b g| < \epsilon_g$$

We must show that the function $f + g$ is also Riemann Integrable, by showing that there exists $L \in \mathbb{R}$ such that for all $\epsilon > 0$, there exists $\delta > 0$ such that if $||P|| < \delta$,

$$|S(f + g, P) - L| < \epsilon$$

Let's set $L = \int_a^b f + \int_a^b g$. So that we must show

$$|S(f + g, P) - (\int_a^b f + \int_a^b g)| < \epsilon$$

If we can do this, we not only show that $f+g$ is Riemann Integrable, but that $\int_a^b f+g = \int_a^b f + \int_a^b g$. From here, let's show that $S(f + g, P) = S(f, P) + S(g, P)$.

$$S(f, P) = \sum_{i=1}^{N} f(c_i)(x_i - x_{i-1})$$
$$S(g, P) = \sum_{i=1}^{N} g(c_i)(x_i - x_{i-1})$$

This is given that $N = \text{card(P)}$

$$\begin{aligned} S(f, P) + S(g, P) &= \sum_{i=1}^{N} f(c_i)(x_i - x_{i-1}) + \sum_{i=1}^{N} g(c_i)(x_i - x_{i-1}) \\ &= \sum_{i=1}^{N} f(c_i)(x_i - x_{i-1}) + g(c_i)(x_i - x_{i-1}) \\ &= \sum_{i=1}^{N} (f(c_i) + g(c_i))(x_i - x_{i-1}) \\ &= S(f + g, P) \end{aligned}$$

Therefore we have that $S(f,P) + S(g,P) = S(f+g,P)$. From here, let's combine the inequalities

$$|S(f,P) - \int_a^b f| < \epsilon_f$$
$$|S(f,P) - \int_a^b g| < \epsilon_g$$

to obtain

$$|S(f,P) - \int_a^b f| + |S(f,P) - \int_a^b g| < \epsilon_f + \epsilon_g$$

By the triangle inequality, we get

$$|S(f,P) - \int_a^b f + S(f,P) - \int_a^b g| < \epsilon_f + \epsilon_g$$
$$|S(f,P) + S(f,P) - \left(\int_a^b f + \int_a^b g\right)| < \epsilon_f + \epsilon_g$$
$$|S(f+g,P) - \left(\int_a^b f + \int_a^b g\right)| < \epsilon_f + \epsilon_g$$

From here, if we set $\epsilon = \epsilon_f + \epsilon_g$, we get

$$|S(f+g,P) - \left(\int_a^b f + \int_a^b g\right)| < \epsilon$$

Which is exactly what we wanted to show. Therefore, since we know that $||P|| < \delta$, and have proven the above statement, we know that $f+g$ is Riemann Integrable and $\int_a^b f + g = \int_a^b f + \int_a^b g$.

b) Since we know f and g are Riemann Integrable, we know that they are bounded. Namely, that there exists $M_f, M_g \in \mathbb{R} > 0$ such that

$$-M_f < f < M_f$$
$$-M_g < g < M_g$$

Now from the assumption that $f(x) \leq g(x)$ for all $x \in [a,b]$, we can assume that

$$M_f \leq M_g$$
$$M_f - M_g \leq 0$$

Now let's look at the functions. From what we're trying to prove, we can rewrite so that we obtain

$$\int_a^b f \leq \int_a^b g$$
$$\int_a^b f - \int_a^b g \leq 0$$

From the previous exercise, we know that

$$\int_a^b f - \int_a^b g = \int_a^b f - g$$

Now using what we've got so far,

$$\int_a^b (f-g) \leq (b-a)(M_f - M_g) \leq 0$$

Therefore, we have that

$$\begin{aligned} \int_a^b (f-g) &\leq 0 \\ \Rightarrow \int_a^b f - \int_a^b g &\leq 0 \\ \Rightarrow \int_a^b f &\leq \int_a^b g \end{aligned}$$

Therefore, if f and g are Riemann Integrable and $f(x) \leq g(x)$ for all $x \in [a,b]$ then

$$\int_a^b f \leq \int_a^b g$$

Exercise 3. (5 pts) Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable on $[a,b]$ and suppose that $|f(x)| \leq M$ for all $x \in [a,b]$. Show that $\int_a^b f \leq M(b-a)$.

Solution:

Exercise 4. (5 pts) Suppose that f is Riemann integrable on $[a,b]$. Let $(\mathcal{P}_n)_{n=1}^{\infty}$ be a sequence of tagged partitions of $[a,b]$ such that the sequence $\lim_{n\to\infty} \|\mathcal{P}_n\| = 0$. Prove that the sequence $(S(f,\mathcal{P}_n))_{n=1}^{\infty}$ converges to $\int_a^b f$.

Solution:

Exercise 5. (5 pts) Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Suppose that f is Riemann integrable on $[a,c]$ for any $c \in (a,b)$. Show that f is Riemann integrable on $[a,b]$. [Hint: Use the Cauchy criterion for integrals.]

Solution:

Homework problems

Answer all the questions below. Make sure to show your work.

Exercise 6. (10pts)

a) Define the function $f : [a, b] \to \mathbb{R}$ by $f(x) = k$ for every $x \in [a, b]$ where $k \in \mathbb{R}$ is a fixed constant. Show that f is Riemann integrable on $[a, b]$ and that $\int_a^b k\,dx = k(b-a)$.

b) Let $f(x) = \sin^2(x)$ where $x \in [a, b]$ and assume that the function $g(x) := \cos(kx)$ is integrable on $[a, b]$ for any $k \in \mathbb{R}$. Show that f is Riemann integrable on $[a, b]$.

Solution: a) We must show that there exists $L \in \mathbb{R}$ such that for all $\epsilon > 0$, there exists $\delta > 0$ such that if P is a tagged partition of $[a, b]$ with $||P|| < \delta$, then

$$|S(f, P) - L| < \epsilon$$

Then if this is true, we also know that $\int_a^b f = L$. Therefore, if we can show that $L = k(b-a)$, then we will have shown f is Riemann Integrable, and that $\int_a^b f = k(b-a)$. Let's set up the inequality.

$$|S(f, P) - k(b-a)| < \epsilon$$

Let's first solve for $S(f, P)$.

$$\begin{aligned} S(f, P) &= \sum_{i=1}^{N} f(c_i)(x_i - x_{i-1}) \\ &= \sum_{i=1}^{N} k(x_i - x_{i-1}) \\ &= k\sum_{i=1}^{N} (x_i - x_{i-1}) \end{aligned}$$

The summation is a telescoping series, such that it equals

$$= k(x_N - x_0)$$

given that $N = \text{card(P)}$. Plugging this back into the inequality, we have

$$\begin{aligned} |k(x_N - x_0) - k(b-a)| &< \epsilon \\ |k||x_N - b + a - x_0| &< \epsilon \\ |x_N - b + a - x_0| &< \frac{\epsilon}{|k|} \end{aligned}$$

By the triangle inequality, we have that

$$|x_N - b + a - x_0| \leq |x_N - b| + |a - x_0|$$

Now from here, because we have that $||P|| < \delta$, we know $|x_N - b| < \delta$ and $|a - x_0| < \delta$. Therefore, we can conclude

$$|x_N - b| + |a - x_0| < 2\delta$$

So far, we have that

$$|x_N - b + a - x_0| \leq |x_N - b| + |a - x_0| < 2\delta$$

If we set $\epsilon = 2\delta|k|$, then we have that

$$|x_N - b + a - x_0| = |S(f, P) - k(b - a)| < \epsilon$$

Therefore, if $||P|| < \delta$, then f is Riemann Integrable and $\int_a^b k = k(b-a)$ because $\int_a^b f = \int_a^b k = k(b - a)$. $\square$

Exercise 7. (5 pts) Show that the function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} 1 & \text{, if } 0 \leq x < 1/2 \\ 0 & \text{, if } 1/2 \leq x \leq 1 \end{cases}$$

is Riemann integrable on $[0, 1]$.

Solution: Given this step function, let's show that the integral

$$\int_0^1 f = \frac{1}{2}$$

Let $\epsilon > 0$ and let P be a tagged partition of $[0, 1]$ such that $||P|| < \delta$. We can write P as

$$P = \{(c_i, [x_{i-1}, x_i]) : i = 1, 2, ..., N\}$$

Let P_1 and P_2 be the following:

$$P_1 = \{(c_i, [x_{i-1}, x_i]) : c_i \in [0, 1/2)\}$$
$$P_2 = \{(c_i, [x_{i-1}, x_i]) : c_i \in [1/2, 1]\}$$

We have that P_1 and P_2 are disjoint, and that $P = P_1 + P_2$. So,

$$S(f, P) = S(f, P_1) + S(f, P_2)$$

Let $N_1 = card(P_1)$ and $N_2 = card(P_2)$. Now for each Riemann Sum, we have

$$\begin{aligned} S(f, P_1) &= \sum_{i=1}^{N_1} f(c_i)(x_i - x_{i-1}) \\ &= f(c_i) \sum_{i=1}^{N_1} (x_i - x_{i-1}) \end{aligned}$$

The summation above is a telescoping series, so we have

$$
\begin{aligned}
&= 1 \cdot x_{N_1} \\
&= x_{N_1} \\
S(f, P_2) &= \sum_{i=N_1}^{N_2} f(c_i)(x_i - x_{i-1}) \\
&= 0 \cdot (x_{N_2} - x_{N_1}) \\
&= 0
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
S(f, P) &= x_{N_1} + 0 \\
&= x_{N_1}
\end{aligned}
$$

So retracing back with everything we know, we have that

$$|S(f, P) - 1/2| = |x_{N_1} - 1/2|$$

Now since $||P|| < \delta$

$$|x_{N_1} - 1/2| < \delta$$

Thus,

$$|S(f, P) - 1/2| < \delta$$

We can now just let $\epsilon = \delta$ such that

$$|S(f, P) - 1/2| < \epsilon$$

Now since ϵ can be arbitrary, we have that f is Riemann Integrable on $[0, 1]$. $\square$

Exercise 8. (5 pts) Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = 1$ if $x = 1/n$ where $n \in \mathbb{N}$, and by $f(x) = 0$ if $x \neq 1/n$, $n \in \mathbb{N}$. Show that f is Riemann integrable on $[0, 1]$.

Solution: To show that f is Riemann Integrable, we must show that $\exists L = 0$ such that for all $\epsilon > 0$, there exists a δ such that if P is a tagged partition of $[a, b]$ with $||P|| < \delta$, then

$$|S(f, P) - 0| < \epsilon$$

Now let's define three partitions such that

$$
\begin{aligned}
P_1 &= \{(c_i, [x_i, x_{i-1}]) : c_i \in [0, 1/n)\} \\
P_2 &= \{(c_i, [x_i, x_{i-1}]) : c_i \in [1/n, 1/n]\} \\
P_3 &= \{(c_i, [x_i, x_{i-1}]) : c_i \in (1/n, 1]\}
\end{aligned}
$$

We can see that P_1, P_2, P_3 are disjoint and $P = P_1 + P_2 + P_3$. Therefore,

$$S(f, P) = S(f, P_1) + S(f, P_2) + S(f, P_3)$$

Let's find each Riemann Sum and define N_1, N_2, N_3 as

$$\begin{aligned} N_1 &= \text{card}(P_1) \\ N_2 &= \text{card}(P_2) \\ N_2 &= \text{card}(P_3) \end{aligned}$$

$$\begin{aligned} S(f, P_1) &= \sum_{i=1}^{N_1} f(c_i)(x_i - x_{i-1}) \\ &= \sum_{i=1}^{N_1} 0 \cdot (x_i - x_{i-1}) \\ &= 0 \\ S(f, P_2) &= \sum_{i=N_1}^{N_2} f(c_i)(x_i - x_{i-1}) \\ &= \sum_{i=N_1}^{N_2} 1 \cdot 0 \\ &= 0 \\ S(f, P_3) &= \sum_{i=N_1+N_2}^{N_3} f(c_i)(x_i - x_{i-1}) \\ &= \sum_{i=N_1+N_2}^{N_3} 0 \cdot (x_i - x_{i-1}) \\ &= 0 \end{aligned}$$

Therefore $S(f, P) = S(f, P_1) + S(f, P_2) + S(f, P_3) = 3(0) = 0$. Plugging this back into the definition of a Riemann Integrable function, we get

$$\begin{aligned} |S(f, P)| &< \epsilon \\ 0 &< \epsilon \end{aligned}$$

Since we defined $\epsilon > 0$, we know this equation to be true. Therefore, given this function and $||P|| < \delta$, f is Riemann Integrable. $\square$

Exercise 9. (5 pts) Show that the function $f : [0, 1] \to \mathbb{R}$ defined by $f(x) = 0$ if $x \neq 0$ and $f(x) = 4$ if $x = 0$ is Riemann integrable on $[0, 1]$.

Solution: Let's first guess that $\int_0^1 f = 0$. Now to show that f is Riemann Integrable, we must show that $\exists L = 0$ such that for all $\epsilon > 0$, there exists a δ such that if P is a tagged partition of $[a, b]$ with $||P|| < \delta$, then

$$|S(f, P) - 0| < \epsilon$$

Let's first define two partitions P_1, P_2 as

$$\begin{aligned} P_1 &= \{(c_i, [x_i, x_{i-1}]) : c_i \in [0, 0]\} \\ P_2 &= \{(c_i, [x_i, x_{i-1}]) : c_i \in (0, 1]\} \end{aligned}$$

We can see that P_1 and P_2 are disjoint and that $P = P_1 + P_2$. Then we know that

$$S(f, P) = S(f, P_1) + S(f, P_2)$$

Now let's have $N_1 = \text{card}(P)$ and $N_2 = \text{card}(P)$.

$$\begin{aligned} S(f, P_1) &= \sum_{i=1}^{N_1} f(c_i)(x_i - x_{i-1}) \\ &= \sum_{i=1}^{N_1} 4(0 - 0) \\ &= 0 \\ S(f, P_2) &= \sum_{i=N_1}^{N_2} f(c_i)(x_i - x_{i-1}) \\ &= \sum_{i=N_1}^{N_2} 0(x_i - x_{i-1}) \\ &= 0 \end{aligned}$$

Therefore,

$$S(f, P) = S(f, P_1) + S(f, P_2) = 0$$

Plugging this back into the inequality from the definition of a Riemann Integrable function, we have

$$\begin{aligned} |S(f, P) - 0| &< \epsilon \\ 0 &< \epsilon \end{aligned}$$

Since we defined $\epsilon > 0$, we know that this inequality is true. Therefore, given this function f, it is Riemann Integrable. $\square$

Exercise 10. (5 pts) Let $\mathcal{P}$ be the following tagged partition of $[-1,2]$:

$$\mathcal{P} := \{(-9,[-1,-.8]),(-.7,[-.8,-.3]),(-.1,[-.3,0]),(.2,[0,0.2]),(.2,[.2,.4]),(.8,[.4,1]),\\ (1.42,[1,1.5]),(1.9,[1.5,2])\}.$$

Find another partition $\mathcal{P}_0$ such that $\|\mathcal{P}_0\| \leq \|\mathcal{P}\|/3$.

Solution: To satisfy this requirement, we only need to find the largest interval(s) in P and divide it into 3. Then if any other interval is larger than this new interval, then split it up until the intervals are less.
The largest interval in P is $(.8,[0.4,1])$ with a width of 0.6. Let's divide this into three sections.

$$[0.4,1] = [0.4,0.6] \cup [0.6,0.8] \cup [0.8,1]$$

Now we must make sure every other interval is less than or equal to a width of 0.2. After doing this, we end up with the partition

$$\mathcal{P}_0 := \{(-9,[-1,-.8]),(-.7,[-.8,-.7]),(-.6,[-.7,-.6]),(-.4,[-.6,-.4]),(-.2,[-.4,-.3]),\\ (-.1,[-.3,-.1]),(0,[-.1,0]),(.2,[0,0.2]),(.2,[.2,.4]),(.4,[.4,.6]),(.6,[.6,.8]),(.8,[.8,1]),\\ (1.42,[1,1.5]),(1.5,[1.5,1.7]),(1.7,[1.7,1.9]),(1.9,[1.9,2])\}$$

The widest interval in P_0 is 0.2 Whereas the widest interval in P was 0.6. Therefore,

$$||P_0|| \leq 0.2 \leq ||P||/3$$