Casey Uandason

MATH-331 Introduction to Real Analysis — Created by P.-O. Parisé
Homework 04 — Fall 2021

Due date: October 25^{th} 1:20pm — Total: /70.

Exercise	1 (5)	2 (5)	3 (5)	4 (5)	5 (10)	6 (10)	7 (5)	8 (5)	9 (5)	10 (10)
Score										

Table 1: Scores for each exercises

Instructions: You must answer all the questions below and send your solution by email (to parisepo@hawaii.edu). If you decide to not use LaTeX to hand out your solutions, please be sure that after you scan your copy, it is clear and readable. Make sure that you attached a copy of the homework assignment to your homework.
If you choose to use LaTeX, you can use the template available on the course website.
No late homework will be accepted. No format other than PDF will be accepted. Name your file as indicated in the syllabus.

1 — Writing problems

For each of the following problems, you will be asked to write a clear and detailed proof. You will have the chance to rewrite your solution in your semester project after receiving feedback from me.

Exercise 1. (5 pts) Prove that, if $0 < x < \pi/2$, then $0 \leq \sin x \leq x$ with a geometric argument. [Hint: View $\sin x$ as a point on the unit circle in the first quadrant.]

Exercise 2. (5 pts) Let $f : A \to \mathbb{R}$ and $g : B \to A$ be two functions where $A, B \subset \mathbb{R}$. Let a be an accumulation point of A and b be an accumulation point of B. Suppose that

- $\lim_{t\to b} g(t) = a$.
- there is a $\eta > 0$ such that for any $t \in B \cap (b-\eta, b+\eta)$, $g(t) \neq a$.
- f has a limit at a.

Prove that $f \circ g$ has a limit at b and $\lim_{x\to a} f(x) = \lim_{t\to b} f(g(t))$. [This is the change of variable rule for limits.]

Exercise 3. (5 pts) Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and suppose that $f(x) = 0$ for each rational number x in $[a, b]$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

Exercise 4. (5 pts) Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and suppose that $f(c) > 0$ for some $c \in [a, b]$. Prove that there exist a number η and an interval $[u, v] \subset [a, b]$ such that $f(x) \geq \eta$ for all $x \in [u, v]$.

Exercise 5. (10 pts) Let $f : \mathbb{R} \to \mathbb{R}$ be a function that satisfies $f(x + y) = f(x) + f(y)$ for any real number x and y.

a) Suppose that f is continuous at some point c. Prove that f is continuous on $\mathbb{R}$.

b) Suppose that f is continuous on $\mathbb{R}$ and that $f(1) = k$. Prove that $f(x) = kx$ for all $x \in \mathbb{R}$. [Hint: start with x integer, then x rational, and finally use Exercise 3.]

-2-

HOMEWORK PROBLEMS

Answer all the questions below. Make sure to show your work.

Exercise 6. (10pts) For each of the functions below, say if the limit exists or doesn't exist at the given point. Justify your answer (in other words, prove it!)

a) $f(x) = \sin(1/x)$ if $x \neq 0$ and $x_0 = 0$.

b) $f(x) = x \sin(1/x)$ and $x_0 = 0$.

Exercise 7. (5 pts) Let $c \in (a, b)$ and let f be a function defined on (a, b) except at c. Suppose that $f(x) > 0$ for any $x \in (a, b)\backslash\{c\}$, that $\lim_{x\to c} f(x)$ exists, and that

$$\lim_{x\to c} f(x) = \lim_{x\to c} \left[(f(x))^2 - f(x) - 3\right].$$

Find the value of $\lim_{x\to c} f(x)$. Explain each step carefully.

Exercise 8. (5 pts) Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} x & , x \in \mathbb{Q} \\ -x & , x \notin \mathbb{Q}. \end{cases}$$

is discontinuous at any point of $\mathbb{R}\backslash\{0\}$ and continuous at 0.

Exercise 9. (5 pts) Let $p(x) = x^2 + 2$. Find an interval where p is strictly decreasing and find a formula for its inverse.

Exercise 10. (10 pts) Let $p(x) = ax^3 + bx^2 + cx + d$ be a polynomial of degree 3 and $a > 0$. Prove that p has at least one real root by following these steps:

a) Prove that $\lim_{x\to\infty} p(x) = \infty$.

b) Prove that $\lim_{x\to-\infty} p(x) = -\infty$.

c) Conclude.

[Hint for a): write your polynomial $p(x) = ax^3 + bx^2 + cx + d$ as $x^3(a + b/x + c/x^2 + d/x^3)$ and use the fact that $\lim_{x\to\infty} 1/x^n = 0$ for every $n \geq 1$.]

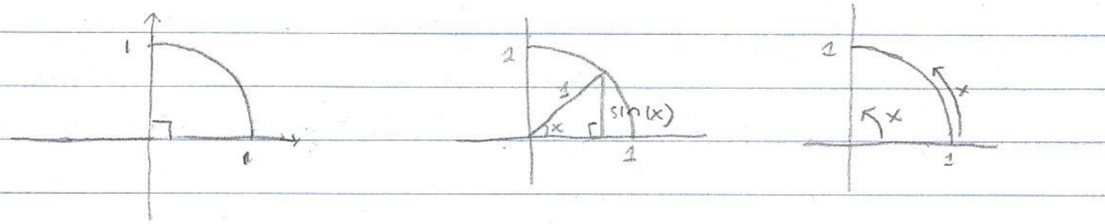
Casey Wardasen

10/22/21

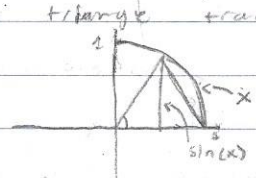
①

Real Analysis MATH 331 - Homework 4

1 Exercise 1. (5 pts) Prove that, if $0 < x < \pi/2$, then $0 \leq \sin x \leq x$ with a geometric argument. [Hint: View $\sin x$ as a point on the unit circle in the first quadrant.



Let x be the angle of the unit circle then the angle in radians is equal to the arclength on the unit circle (r=1) in which x sweeps out. $\sin(x)$ is the vertical length of the right triangle traces on the unit circle. Let $0 < x < \pi/2$. We will prove



that $0 \leq \sin x \leq x$ using area. The area of the slice of the circle is $\frac{x}{2\pi} \cdot \pi r^2 = \frac{x}{2}$. The area of the triangle with base equal to 1 and height $\sin(x)$ is $\frac{1}{2}\text{base} \cdot \text{height} = \frac{1}{2}(1)\sin(x) = \frac{\sin(x)}{2}$, for $0 < x < \frac{\pi}{2}$. As we can see on the diagram, the area of the slice is greater than the area of the triangle so $\frac{\sin(x)}{2} < \frac{x}{2}$ for $0 < x < \pi/2$. If $x = 0$, then the area of both regions are zero, then $0 \leq \frac{\sin(x)}{2} \leq \frac{x}{2}$. This then implies $0 \leq \sin(x) \leq x$. ∎

2 Exercise 2. (5 pts) Let $f : A \to \mathbb{R}$ and $g : B \to A$ be two functions where $A, B \subset \mathbb{R}$. Let a be an accumulation point of A and b an accumulation point of B. Suppose that

- $\lim_{t \to b} g(t) = a$
- there is a $\eta > 0$ such that for any $t \in B \cap (b - \eta, b + \eta)$, $g(t) \neq a$.
- f has a limit at a.

Prove that $f \circ g$ has a limit at b and $\lim_{x \to a} f(x) = \lim_{t \to b} f(g(t))$. [This is the change of variable rule for limits.]

Suppose $\lim_{t \to b} g(t) = a$. Suppose there is a $\eta > 0$ such that for any $t \in B \cap (b - \eta, b + \eta)$, $g(t) \neq a$. Suppose f has a limit at a. Let $\eta = \eta_0 > 0$. Let $f : A \to \mathbb{R}$ and $g : B \to A$ where $A, B \subseteq \mathbb{R}$, with a an accumulation point of A

②

and b an accumulation point at B. We know that since f has a limit at a then, $\lim_{x \to a} f(x) = L_1$. Since $\lim_{t \to b} g(t) = a$ and b an accumulation point then $g(b) \notin Im(g)$; but $Im(g) = A$. Let $x = g(t)$. Define $f \circ g : B \to \mathbb{R}$. Since a is an accumulation point of A and f has a limit at a then if $f: A \to \mathbb{R} = f: Im(g) \to \mathbb{R}$ then $x \in Im(g)$ and $\lim_{x \to a} f(x) = \lim_{x \to a} f(Im(g)) = \lim_{t \to b} f(g(t))$ exists at $f \circ g$ has a limit at b. ☒

3 Exercise 3 (5pts) Let $f: [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and suppose that $f(x) = 0$ for each rational number x in $[a,b]$. Prove that $f(x) = 0$ for all $x \in [a,b]$

Let $f: [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and suppose that $f(x) = 0$ for each rational number x in $[a,b]$. We will prove that even for x that are irrational $f(x) = 0$ for $x \in [a,b]$. Suppose x_1 is rational and $x_1 \in [a,b]$ let $c < x_1$ with $c \in [a,b]$, c is irrational. By theorem 0.22 theorem. Between any two distinct real numbers x_1 and c there is a rational number lets say x_2, so $c < x_2 < x_1$. However between c and x_2 by 0.22 Theorem there is another rational number x_3 so $c < x_3 < x_2 < x_1$. Notice that $f(x_1) = 0$ $f(x_2) = 0$, $f(x_3) = 0$. This recursive definition continues such that for all rational numbers $t \in (c, x_1]$ $f(t) = 0$. Similarly let $y_1 \in [a,b]$ and $y_1 < c$ then by 0.22 theorem $y_1 < y_2 < c$ and $f(y_2) = 0$. Again by 0.22 theorem, $y_2 < y_3 < c$ and $f(y_3) = 0$. Therefore for all rational numbers $t \in [y_2, c)$ then $f(t) = 0$. so $t \in [y_p, c) \cup (c, x_1]$ and $f(c)$ there is a jump if $f(c) \neq 0$. The same argument can be made with every irrational number. By 0.24 Theorem, between any two distinct real numbers say the irrational number J_1 and c ($J_1 \in [a,b]$) then there exist an irrational number between say $J_2 < c$ and between those $J_2 < J_3 < c$ and so on. We can then say $g_1 \in [a,b]$ and $c < g_1$ then by 0.24 theorem $c < g_2 < g_1$. Then for $p \in \{\text{irrational number}\}$ and $p \in [J_1, c) \cup (c, g_1]$. Suppose $f(p) \neq 0$, then since $[a,b]$ is continuous, then $f(p) > 0$ or $f(p) < 0$

for all p. However this is a contradiction. since $f(t) = 0 \neq f(p) > 0$ or $f(p) < 0$ f is no longer continuous on $[a,b]$ since between every rational number there is an irrational number and between every irrational number there is a rational number. Therefore, $f(x) = 0$ for all $[a,b]$ ☒

4 Exercise 4 (5pts) let $f: [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and suppose that $f(c) > 0$ for some $c \in [a,b]$. Prove that there exists a number η and an interval $[u,v] \subset [a,b]$ such that $f(x) \geq \eta$ for all $x \in [u,v]$

Let $f: [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and suppose that $f(c) > 0$ for some $c \in [a,b]$. Suppose $f(c) > 0$ for some $c \in [a,b]$.

5 Exercise 5 (10pts) Let $f: \mathbb{R} \to \mathbb{R}$ be a function that satisfies $f(x+y) = f(x) + f(y)$ for any real number x and y.

a) Suppose that f is continuous at some point c. Prove that f is continuous on $\mathbb{R}$

Let $f: \mathbb{R} \to \mathbb{R}$ be a function such that $f(x+y) = f(x) + f(y)$. Suppose that f is continuous at some point c then $f(c)$ is continuous. Suppose $c = x+y$ the $f(c) = f(x+y) = f(x) + f(y)$. Since $f(c)$ is continuous the sum $f(x) + f(y)$ is also continuous. But $c = x+y$ implies $y = c-x$, so $f(y) = f(c-x)$. So $f(c) = f(x) + f(c-x)$. So for all x, $f(c)$ will be continuous. But now we must prove $f(x)$ is continuous on all x. Note that $f(c-x) = f(c) + f(-x)$ then $f(c) = f(x) + f(c) + f(-x)$ so $0 = f(x) + f(-x)$ this sum is continuous as the constant function 0 is continuous. Take $\delta = \varepsilon$ and $x_0 = 0$ then for all $\varepsilon > 0 \ \exists \delta > 0$ then $|x - 0| < \delta$ then $|f(x) - f(0)| = |0 - 0| < \varepsilon$. Since 0 is continuous, so to $f(x) + f(-x)$, now if $f(x)$ is not continuous then since $f(x) = -f(-x)$ then $-f(-x)$ is not continuous

b. Suppose that f is continuous on $\mathbb{R}$ and that $f(1) = k$. Prove that $f(x) = kx$ for all $x \in \mathbb{R}$

6 Exercise 6 (10 pts) For each of the functions below, say if the limit exists or doesn't exist at the given point. Justify your answer (in other words prove it!).

a) $f(x) = \sin(1/x)$ if $x \neq 0$ and $x_0 = 0$

Let $f(x) = \sin(1/x)$ for $x \neq 0$. We will prove the limit does not exist for $x_0 = 0$: Note $x_0 = 0$ is an accumulation point and that $\sin(1/x)$ is bounded by -1 and 1 or $-1 \leq \sin(1/x) \leq 1$ or $|\sin(1/x)| \leq 1$. To prove it doesn't have a limit, for all L, there exists an ε such that for all $\delta > 0$ there is $|x - a| = |x| < \delta$ and $\varepsilon \leq |f(x) - L| = |\sin(1/x) - L| \leq |\sin(1/x)| + |L| \leq 1 + |L|$. So the definition is satisfied for all δ and as long as $\varepsilon \leq 1 + |L|$ for any L, clearly the limit does not exist. ∎

b) $f(x) = x\sin(1/x)$ and $x_0 = 0$

Let $f(x) = x\sin(1/x)$ and $x_0 = 0$. x_0 is an accumulation point. Then it is reasonable that $\lim_{x \to 0} f(x)$ exists, lets say L. Therefore for all $\varepsilon > 0$, then for $|x - 0| < \delta$ then $|f(x) - L| < \varepsilon$. Lets say $L = 0$ since $\sin(1/x)$ is bounded or $-1 \leq \sin(1/x) \leq 1$ and $\lim_{x \to 0} x = 0$. Then $|f(x) - L| = |x\sin(1/x) - 0| < \varepsilon$. So since it is bounded $|\sin(1/x)| \leq 1$. Take $\delta = \varepsilon$ then $|x - 0| = |x| < \varepsilon$. Then $|f(x) - L| = |x\sin(1/x) - 0| = |x\sin(1/x)| = |x| \cdot |\sin(1/x)| \leq \delta \cdot 1 = \varepsilon \cdot 1 = \varepsilon$. So the limit of $f(x)$ exists at $x_0 = 0$. ∎

7 Exercise 7. (5pts) Let $c \in (a,b)$ and let f be a function defined on (a,b) except at c. Suppose that $f(x) > 0$ for any $x \in (a,b)\setminus\{c\}$, that $\lim_{x\to c} f(x)$ exists, and that

$$\lim_{x\to c} f(x) = \lim_{x\to c} [(f(x))^2 - f(x) - 3].$$

Find the value of $\lim_{x\to c} f(x)$. Explain each step carefully.

Let $c \in (a,b)$ and Let f be a function defined on (a,b) except c. Suppose $f(x) > 0$ for any $x \in (a,b)\setminus\{c\}$, that $\lim_{x\to c} f(x)$ exists and that $\lim_{x\to c} f(x) = \lim_{x\to c}[(f(x))^2 - f(x) - 3]$. By the algebra of limits $\lim_{x\to c}[(f(x))^2 - f(x) - 3] = \lim_{x\to c}(f(x))^2 - \lim_{x\to c} f(x) - \lim_{x\to c}(3)$. Lets say that $\lim_{x\to c} f(x) = L$ then $\lim_{x\to c}(f(x))^2 - \lim_{x\to c} f(x) - \lim_{x\to c}(3) = L^2 - L - 3 = \lim_{x\to c} f(x) = L$, Thus we are left with the equation $L = L^2 - L - 3$ or $0 = L^2 - 2L - 3$. This can then be factored into $(L+1)(L-3)$ or $0 = L^2 - 2L - 3 = (L+1)(L-3)$. Therefore $L = \lim_{x\to c} f(x) = -1$ or 3. However, since f is defined on $(a,b)\setminus\{c\}$ and $f(x) > 0$ then if $J \in (a,b)\setminus\{c\}$ then $\lim_{x\to J} f(x) > 0$ since by 3.1 theorem f is continuous on J, has limit at J or $\lim_{x\to J} f(x) = f(J)$. The point c is an accumulation point then it has a limit at f if and only if for all $\varepsilon > 0$ there is a $\delta > 0$ such that $0 < |x - c| < \delta$ and $x \in (a,b)$ then $|f(x) - L| < \varepsilon$. So the limit exist and should satisfy $L = \lim_{x\to c} f(x) > 0$. Therefore, the limit $L = 3$. ∎

8 Exercise 8 (5pts) Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} x, & x \in \mathbb{Q} \\ -x, & x \notin \mathbb{Q} \end{cases}$$

is discontinuous at any point $\mathbb{R}\setminus\{0\}$ and continuous at 0.

Let $f : \mathbb{R} \to \mathbb{R}$ and define $f(x) := \begin{cases} x, & x \in \mathbb{Q} \\ -x, & x \notin \mathbb{Q} \end{cases}$. Let $c \in \mathbb{R}\setminus\{0\}$. The definition of discontinuous is If $c \in \mathbb{R}$, then f is discontinuous at c if and only if there exists an $\varepsilon > 0$ for all $\delta > 0$ such that $|x - c| < \delta$, $x \in \mathbb{R}$ and $|f(x) - f(c)| \geq \varepsilon$. Suppose $c_1 \in \mathbb{Q}$ then $|x - c_1| < \delta$ for all δ there exists an $\varepsilon > 0$, take $\varepsilon = \varepsilon_0$ and $|f(x) - f(c)| = |f(x) - c| \geq \varepsilon_0$. If this is the case $|x - c_1| < \delta$ or that $x - c_1 < \delta$ or $c_1 \neq x < \delta$ by definition of absolute value then $c_1 - \delta < x < c_1 + \delta$ which is true for all δ. Also $|f(x) - c_1| \geq \varepsilon_0$ then this implies

$f(x) - c_1 \geq \varepsilon_0$ or $c_1 - f(x) \geq \varepsilon_0$ then $c_1 - \varepsilon_0 \geq f(x) \geq \varepsilon_0 + c_1$. This implies that $\varepsilon_0 < -\varepsilon_0$ which is only true if we take $\varepsilon_0 = 0$. Similarly if $c_2 \notin \mathbb{Q}$ then $|x - c_2| < \delta$ for all $\delta > 0$ then this implies as before $c_2 - \delta < x < c_1 + \delta$ which is always true and also $|f(x) - f(c_2)| = |f(x) + c_2| \geq \varepsilon_1$. This implies $f(x) + c_2 \geq \varepsilon_1$ or $-f(x) - c_2 \geq \varepsilon_1$ then $-c_2 - \varepsilon_1 \geq f(x) \geq \varepsilon_1 - c_2$ so $-\varepsilon_1 \geq \varepsilon_1$ which can only be true if $\varepsilon_1 = 0$. Then $\varepsilon_1 = \varepsilon_0 = 0$. Since for $c_1 \in \mathbb{Q}$ and $c_2 \notin \mathbb{Q}$ satisfy discontinuity, then it is discontinuous on $\mathbb{R} \setminus \{0\}$. If we take $x = 0$ By using the definition of continuous at $x=0$ is true if and only if for all $\varepsilon > 0$, there is a $\delta > 0$ such that $|x - x_0| = |x - 0| < \delta$, $x \in \mathbb{R}$, then $|f(x) - 0| < \varepsilon$, $0 \in \mathbb{Q}$ so $f(x) = x$. Take $\delta = \varepsilon$ then if $|x - 0| = |x| < \delta$ then $|f(x) - 0| = |x - 0| < \delta = \varepsilon$. Thus f is continuous at 0.

9 Exercise 9 (5pts) Let $p(x) = x^2 + 2$. Find an interval where p is strictly decreasing and find a formula for its inverse.

Let $p(x) = x^2 + 2$. On the interval a function is strictly decreasing if $x < y$ then $f(x) \geq f(y)$. If $x < 0$ and $y = 0$ then $f(x) = x^2 + 2 \geq f(y) = f(0) = 2$. Therefore $x^2 \geq 0$ which is true. If the function is monotonic and decreasing particularly $f: (-\infty, 0] \to (2, \infty)$ then $f^{-1}: [2, \infty) \to (-\infty, 0]$ is strictly decreasing. $f^{-1}(x) = \pm\sqrt{x-2}$ but the branch that is strictly decreasing is $f^{-1}(x) = -\sqrt{x-2}$.

10 Exercise 10 (10pts) Let $p(x) = ax^3 + bx^2 + cx + d$ be a polynomial of degree 3 and $a > 0$. Prove that p has at least one real root by following these steps.

a) Prove that $\lim_{x \to \infty} p(x) = \infty$

Rewrite $p(x) = ax^3 + bx^2 + cx + d = x^3(a + b/x + c/x^2 + d/x^3)$. By the definition of a limit at infinity of $1/x^n$ then $\lim_{x\to\infty} 1/x^n = 0$ for every $n \geq 1$. By limit algebra $\lim_{x\to\infty} p(x) = \lim_{x\to\infty} ax^3 + bx^2 + cx + d$

$= \left(\lim_{x\to\infty} x^3\right)\left(\lim_{x\to\infty}(a + b/x + c/x^2 + d/x^3)\right) = \left(\lim_{x\to\infty} x^3\right) \cdot a$.

By the formal definition of a limit at infinity, for some $M>0$, we need an N so that if $x>N$, we get $x^3>M$. Let $M>0$. choose $N=\sqrt[3]{M}$ then for all $x>N=\sqrt[3]{M}$ we get $x^3>N^3=(\sqrt[3]{M})^3=M$. Therefore $\lim_{x\to\infty} x^3=\infty$. Thus $\lim_{x\to\infty} P(x)=\lim_{x\to\infty}(x^3)\cdot a = \infty\cdot a=\infty$.

b) Prove that $\lim_{x\to-\infty} P(x)=-\infty$.

Doing the same as in a) $\lim_{x\to-\infty} P(x)=\lim_{x\to-\infty}(x^3(a+b/x+c/x^2+d/x^3))$
$=(\lim_{x\to-\infty} x^3)\cdot a$. To prove $\lim_{x\to-\infty} x^3=-\infty$ we need for some $M<0$ have an $N>0$ such that $x^3<M$ for all $x<N$. choose $N=\sqrt[3]{M}$ then $x<N=\sqrt[3]{M}$ then $x^3<N^3=(\sqrt[3]{M})^3=M$. therefore $\lim_{x\to-\infty} x^3=-\infty$. Then $\lim_{x\to-\infty} P(x)=(\lim_{x\to-\infty} x^3)\cdot a=-\infty\cdot a=-\infty$.

c) conclude.

By the intermediate value theorem suppose $x=a<0$ and $x=b>0$ with $a<b$. $P(a)<y<P(b)$. since a) $\lim_{x\to\infty} P(x)=\infty$ on the interval $[0,\infty)$, $P(x)>0$. thus $P(b)>0$. since b) $\lim_{x\to-\infty} P(x)=-\infty$ on the interval $(-\infty,0]$, $P(x)<0$, thus $P(a)<0$. By IVT then there exist a $c\in(a,b)$ such that $f(c)=y=0$ since $P(x)$ is continuous everywhere on the real line.