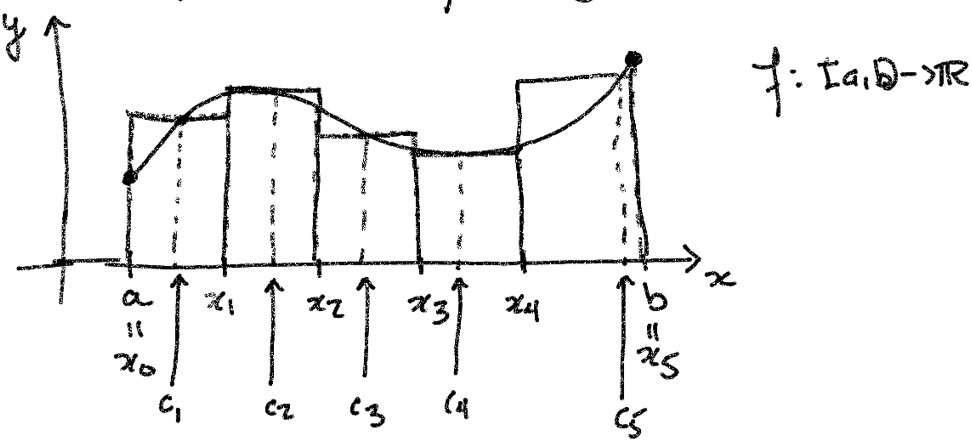
6- Riemann Integral.

Remember, from calculus, that we interpret the integral as the area under the curve $y=f(x)$ of a nonnegative function.

This comes from the following process:



We approximate area as

$$f(c_1)(x_1-x_0)+f(c_2)(x_2-x_1)+\ldots+f(c_5)(x_5-x_4).$$

We can get a better approximation of the area if we increase the numbers of points x_i to divide the interval $[a,b]$ into smaller and smaller pieces. When the length of these pieces goes to zero, so we are taking the limit, and if this limit exists, we get the value of the area.

6.1 Definition.

Rigorously, the Riemann integral is defined with the concepts of tagged partition of an interval and of Riemann Sums.

Def. Let $[a,b]$ with $a<b$.

1) two closed intervals $[s,t]$, $[u,v] \subseteq [a,b]$ are <u>non-overlapping</u> if
$$\mathrm{card}\left([u,v] \cap [a,b]\right) \leq 1 .$$

$\boxed{\text{the number } c \text{ is the tag.}}$

2) A <u>tagged interval</u> in $[a,b]$ is an ordered pair $(c, [u,v])$ s.t. $[u,v] \subseteq [a,b]$ & $c \in [u,v]$.

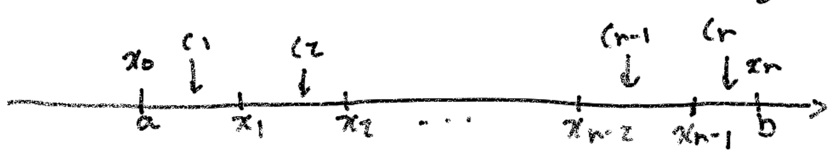
3) A set $\mathcal{P} := \{ (c_i , [x_{i-1}, x_i]) : i = 1,2,\dots,n \}$ is a <u>tagged partition</u> of $[a,b]$ if

a) $[x_{i-1}, x_i]$ are non-overlapping

and (c) $a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$

4) the norm of a tagged partition, denoted by $\|\mathcal{P}\|$, is
$$\|\mathcal{P}\| := \max_i \{ v_i - u_i : i = 1,2,\dots,n \} .$$

<u>Picture</u>



① Let $P_1 = \{(0.25, [0, 0.5]), (0.6, [0.5, 0.75]), (0.8, [0.75, 1])\}$

This is a tagged partition of $[0,1]$ because:

1) each interval are non-overlapping.
2) $\bigcup_{i=1}^{n} [x_{i-1}, x_i] = [0, 0.5] \cup [0.5, 0.75] \cup [0.75, 1] = [0, 1]$.
3) $0 = x_0 < \underset{x_1}{\underset{\parallel}{0.5}} < \underset{x_2}{\underset{\parallel}{0.75}} < 1 = x_3$.

<u>Def</u>. Let $f : [a,b] \to \mathbb{R}$ and $\mathcal{L} = \{(c_i, [u_i, v_i]) : i = 1, \dots, n\}$ be an arbitrary collection of non-overlapping tagged interval. The Rieman sum of f associated to $\mathcal{L}$ is

$$S(f, \mathcal{L}) := \sum_{i=1}^{n} f(c_i)(v_i - u_i).$$

<u>Remark</u>. For a tagged partition P, the riemann sum is

$$S(f, P) = \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} f(c_i)\, \Delta x_i.$$

<u>Example</u> Consider the partition P_1 in the last example. Let $f : [0,1] \to \mathbb{R}$ be $f(x) = x^2$ then

$$S(f, P_1) = (0.25)^2(0.5) + 0.6^2(0.75 - 0.5) + 0.8^2(1 - 0.75)$$
$$= 0.28125.$$

Def. A fct. $f : [a,b] \to \mathbb{R}$ is <u>Riemann integrable</u> on $[a,b]$ if $\exists L \in \mathbb{R}$ s.t. $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. if $\mathcal{P}$ is a tagged partition of $[a,b]$ with $\|\mathcal{P}\| < \delta$, then $|S(f,\mathcal{P}) - L| < \varepsilon$.

The number L is the integral of f on $[a,b]$ we put $L = \int_a^b f(x)dx$ or $L = \int_a^b f$ simply.

$\to$ <u>Notation:</u> $\int_a^b f = \lim_{\|\mathcal{P}\| \to 0} S(f,\mathcal{P})$

<u>Examples</u>

① Let

$$f(x) = \begin{cases} 2, & 0 \le x \le 1/3, \\ 4, & \frac{1}{3} < x \le 2/3, \\ 6, & \frac{2}{3} < x \le 1. \end{cases}$$

We will prove that f is Riemann integrable.

Let $\varepsilon > 0$ and $\mathcal{P}$ a tagged partition of $[a,b]$ s.t. $\|\mathcal{P}\| < \delta$ (δ is to be determined).

Let $\mathcal{P}_j \subseteq \mathcal{P}$ be the collections

$$\mathcal{P}_1 = \{ (c_i, [x_{i-1}, x_i]) : \; c_i \in [0, 1/3] \}$$

$$\mathcal{P}_2 = \{ (c_i, [x_{i-1}, x_i]) : \; c_i \in (1/3, 2/3] \}$$

& $$\mathcal{P}_3 = \{ (c_i, [x_{i-1}, x_i]) : \; c_i \in (2/3, 1] \}.$$

So, we have

$$S(f, P) = S(f, P_1) + S(f, P_2) + S(f, P_3)$$

Now, each tag c_i of $(c_i, [x_{i-1}, x_i]) \in P_1$ are inside $[0, 1/3)$ and so $f(c_i) = 2$. thus,

$$S(f, P_1) = \sum_{i=1}^{n_1} 2(x_i - x_{i-1}) = 2(x_{n_1} - x_0)$$

$$\Rightarrow \quad S(f, P_1) = 2x_{n_1} \qquad (n_1 = \mathrm{card}(P_1)).$$

Also,

$$S(f, P_2) = \sum_{i=n_1+1}^{n_2} 4(x_i - x_{i-1}) = 4(x_{n_2} - x_{n_1})$$

&

$$S(f, P_3) = \sum_{i=n_2+1}^{n} 6(x_i - x_{i-1}) = 6(x_n - x_{n_2}) = 6(1 - x_{n_2}).$$

Thus,

$$S(f, P) = 6 - 2x_{n_2} - 2x_{n_1}$$

$$\Rightarrow \quad S(f, P) - 4 = 2(1 - (x_{n_2} + x_{n_1})) = 2\left(\frac{1}{3} - x_{n_1} + \frac{2}{3} - x_{n_2}\right)$$

By assumption, $\|P\| < \delta$. So,

$$\frac{1}{3} - \delta < x_{n_1} < \frac{1}{3} + \delta \quad \& \quad \frac{2}{3} - \delta < x_{n_2} < \frac{2}{3} + \delta$$

So, $\left|\frac{1}{3} - x_{n_1}\right| < \delta$ & $\left|\frac{2}{3} - x_{n_2}\right| < \delta$.

Thus,

$$|S(f, \mathcal{P}) - 4| \le 2\left(\left|\frac{1}{3} - x_{n_1}\right| + \left|\frac{2}{3} - x_{n_2}\right|\right)$$
$$< 4\delta.$$

Choose $\delta = \varepsilon/4$. Then we get that if $\|\mathcal{P}\| < \delta$ then

$$|S(f, \mathcal{P}) - 4| < \varepsilon.$$

Thus, $\int_0^1 f(x)\,dx = 4$. □

(2) Define:

$f: [0,1] \to \mathbb{R}$ $\quad f(x) = \begin{cases} 0, & x \notin \mathbb{Q} \\ 1/q, & x = \frac{p}{q}, \ \gcd(p,q) = 1 \end{cases}$

Let $L = 0$ and $\varepsilon > 0$. Suppose $\mathcal{P}$ is a tagged partition s.t. $\|\mathcal{P}\| < \delta$.

Let $E := \{x \in [0,1] : x = \frac{p}{q}$ & $\frac{1}{q} \ge \frac{\varepsilon}{2}\}$.

Then $\exists q_0 \in \mathbb{N}$ s.t.

$$E = \left\{\frac{p}{q} : 0 \le p \le q,\ 1 \le q \le q_0,\ p, q \in \mathbb{N}\right\}.$$

So E is a finite set.

Let $\mathcal{P}_0 \subseteq \mathcal{P}$ be the tagged partitions where each tag are s.t. $f(c_i) \ge \frac{\varepsilon}{2}$. So,

$$S(f, \mathcal{P}) = S(f, \mathcal{P} \setminus \mathcal{P}_0) + S(f, \mathcal{P}_0).$$

So, $\mathcal{P}_0 = \{ (c_{n_{ik}}, [x_{n_{ik}-1}, x_{n_k}]) : k = 1, 2, \ldots, N \}$
is finite & so

$$S(f, \mathcal{P}_0) = \sum_{k=1}^{N} f(c_{n_{ik}}) (x_{n_k} - x_{n_k - 1})$$

$$\leq \sum_{k=1}^{N} 1 \cdot \delta = N\delta .$$

Also,

$$S(f, \mathcal{P} \setminus \mathcal{P}_0) = \sum_{\substack{i=1 \\ i \neq n_k}}^{n} f(c_i) (x_i - x_{i-1})$$

$$< (\varepsilon/2) \sum_{\substack{i=1 \\ i \neq n_k}}^{n} (x_i - x_{i-1})$$

$$\leq (\varepsilon/2) \sum_{i=1}^{n} (x_i - x_{i-1})$$

$$= (\varepsilon/2)(1-0) = (\varepsilon/2).$$

Select δ s.t. $\delta = \frac{(\varepsilon/2)}{N}$. Then, if $\|\mathcal{P}\| < \delta$ with $\mathcal{P}$ a partition, then

$$|S(f, \mathcal{P}) - 0| = S(f, \mathcal{P}) = S(f, \mathcal{P}_0) + S(f, \mathcal{P} \setminus \mathcal{P}_0)$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Thus, $\int_0^1 f(x)\,dx = 0$. $\square$

6.2. Properties.

Thm. Suppose f and g are Riemann integrable on $[a,b]$. Then

(a) The function kf is integrable on $[a,b]$ $\forall k \in \mathbb{R}$ &
$$\int_a^b kf = k\int_a^b f.$$

(b) The function $f+g$ is integ. on $[a,b]$ &
$$\int_a^b (f+g) = \int_a^b f + \int_a^b g.$$

(c) if $f(x) \leq g(x)$, then $\int_a^b f \leq \int_a^b g$.

Proof. We only prove (a). The two other will be given as an exercise. We want to prove that
$$\int_a^b kf \text{ exists } \& \int_a^b kf = k\int_a^b f.$$

Take $L = k\int_a^b f$. Let $\varepsilon > 0$ & $\mathcal{P}$ be a tagged partition of $[a,b]$. Then
$$S(kf, \mathcal{P}) = \sum_{k=1}^{n} kf(c_k)(x_k - x_{k-1})$$
$$= k\left(\sum_{k=1}^{n} f(c_k)(x_k - x_{k-1})\right)$$

We know that f is Riemann integrable. So, $\exists \delta > 0$ s.t. $\forall \mathcal{P}$ tagged partition,

$$\|\mathcal{P}\| < \delta \implies \left| S(f, \mathcal{P}) - \int_a^b f \right| < \frac{\varepsilon}{|k|+1}.$$

So, if $\|\mathcal{P}\| < \delta$, then

$$\left| S(kf, \mathcal{P}) - k\int_a^b f \right| = \left| k\ S(f, \mathcal{P}) - k\int_a^b f \right|$$

$$= |k| \left| S(f, \mathcal{P}) - \int_a^b f \right|$$

$$< |k| \frac{\varepsilon}{|k|+1}$$

$$< \varepsilon.$$

Then, whenever $\|\mathcal{P}\| < \delta$, then

$$\left| S(kf, \mathcal{P}) - k\int_a^b f \right| < \varepsilon.$$

So, $\int_a^b kf$ exists & $\int_a^b kf = k\int_a^b f$. $\square$

<u>Thm</u>. If f is Riemann integrable on $[a,b]$, then f is bounded.

<u>Proof</u>. By def. $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall$ tagged partition $\mathcal{P}$, if $\|\mathcal{P}\| < \delta$ then $\left| S(f, \mathcal{P}) - \int_a^b f \right| < \varepsilon$.

Take $\varepsilon = \frac{1}{2}$ then $\exists \delta > 0$ s.t.

$\forall \mathcal{P}$ tagged partition, $\|\mathcal{P}\| < \delta \implies \left| S(f, \mathcal{P}) - \int_a^b f \right| < \frac{1}{2}$.

Now, if $\mathcal{P}_1$ & $\mathcal{P}_2$ are two tagged partition s.t.

$\|\mathcal{P}_1\| < \delta$ & $\|\mathcal{P}_2\| < \delta$, then

$$|S(f, \mathcal{P}_1) - S(f, \mathcal{P}_2)| \le |S(f, \mathcal{P}_1) - \int_a^b f| + |\int_a^b f - S(f, \mathcal{P}_2)|$$
$$< 1$$

By the AP, $\exists N \in \mathbb{N}$ s.t.

$$\beta = \frac{b-a}{N} < \delta.$$

Define $\mathcal{P}_0 = \{(c_j, [x_{j-1}, x_j])\}$ as

$$x_j = a + j\beta \qquad (j = 0, 1, 2, \dots, N)$$
$$c_j = x_j \qquad (j = 1, 2, \dots, N).$$

Now, we have $\|\mathcal{P}\| < \delta$.

Let $M := \frac{1}{\beta} + \max\{|f(x_j)| : j = 0, 1, \dots, N\}$.

Let $x \in [a, b]$. We will show that $|f(x)| \le M$.

Let $1 \le j \le N$ s.t. $x_{j-1} \le x \le x_j$. Define

$$\mathcal{P}_{00} := \mathcal{P}_0 \setminus \{(x_j\ [x_{j-1}, x_j])\} \cup \{(x, [x_{j-1}, x_j])\}.$$

So, we have

$$S(f, \mathcal{P}_{00}) - S(f, \mathcal{P}_0) = f(x)\beta - f(x_j)\beta$$

$$\Rightarrow f(x) = f(x_j) + \frac{S(f, \mathcal{P}_{00}) - S(f, \mathcal{P}_0)}{\beta}$$

Finally,

$$|f(x)| \le |f(x_j)| + \frac{|S(f, P_{\delta 0}) - S(f, P_\delta)|}{\beta}$$

$$\le \max\{ |f(x_j)| : j = 0, \dots, N\} + \frac{1}{\beta}.$$

$$= M.$$

Thus, for any x, $|f(x)| \le M$, so f is boun. □

Later, we will develop methods for computing $\int_a^b f$. For now, we are only interested in whether or not a function is integrable.

<u>Thm</u> (Cauchy criterion)

A function f is R. int. on $[a,b]$ iff. $\forall \varepsilon > 0, \exists \delta > 0$

$$|S(f, P_1) - S(f, P_2)| < \varepsilon$$

for all tagged partitions P_1, P_2 of $[a,b]$ with $\|P_1\|, \|P_2\| < \delta$.

<u>Proof</u>. The direct implication is obtained by applying this inequality

$$|S(f, P_1) - S(f, P_2)| \le |S(f, P_1) - \int_a^b f| + |\int_a^b f - S(f, P_3)|.$$

Let's tackle the reverse, and suppose that the criterion is satisfied.

For $\varepsilon = 1$, choose $\delta_1 > 0$ s.t.

$$\|\mathcal{P}_1\| < \delta_1 \ \& \ \|\mathcal{P}_2\| < \delta_2 \Rightarrow |S(f, \mathcal{P}_1) - S(f, \mathcal{P}_2)| < 1.$$

For $\varepsilon = 1/2$, choose $0 < \delta_2 < \delta_1$ s.t.

$$\|\mathcal{P}_1\| < \delta_2 \ \& \ \|\mathcal{P}_2\| < \delta_2 \Rightarrow |S(f, \mathcal{P}_1) - S(f, \mathcal{P}_2)| < \tfrac{1}{2}.$$

Continue this process to get a sequence of numbers (δ_n) s.t.

- $\delta_n > \delta_{n+1}$ $\forall n$ (decreasing)
- $|S(f, \mathcal{P}_1) - S(f, \mathcal{P}_2)| < \frac{1}{n}$ $\forall \|\mathcal{P}_1\| < \delta_n, \|\mathcal{P}_2\| < \delta_n$.

Let $\mathcal{P}_n$ be a tagged partition of $[a,b]$ s.t. $\|\mathcal{P}_n\| < \delta_n$. Then, $\left(S(f, \mathcal{P}_n)\right)_{n=1}^{\infty}$ is a Cauchy sequence. It is then convergent to some $L \in \mathbb{R}$.

We now show that $L = \int_a^b f$.

Let $\varepsilon > 0$ and let $N \in \mathbb{N}$ s.t. $\frac{1}{N} < \frac{\varepsilon}{2}$. There is some $n_0 > N$ s.t. $|S(f, \mathcal{P}_{n_0}) - L| < \frac{\varepsilon}{2}$.

Let $\mathcal{P}$ be a tagged partition s.t.

$$\|\mathcal{P}\| < \delta_{n_0} = \delta.$$

Then, we get

$$|S(f,\mathcal{P}) - L| \leq |S(f,\mathcal{P}) - S(f,\mathcal{P}_{n_0})| + |S(f,\mathcal{P}_{n_0}) - L|$$
$$< \frac{1}{n_0} + \frac{\varepsilon}{2}$$
$$= \frac{1}{N} + \frac{\varepsilon}{2} = \varepsilon .$$

So, whenever $\|\mathcal{P}\| < \delta_{n_0}$, we have
$$|S(f,\mathcal{P}) - L| < \varepsilon .$$

So f is R. int. & $\int_a^b f = L$. $\square$

<u>Remark</u>. Cauchy criterion is not useful to compute or to show that a function is integrable. It is useful when we don't have to actually compute the Rieman sums.

<u>Thm</u>. $f : [a,b] \to \mathbb{R}$ and $c \in (a,b)$.

(a) f is Rie. Integral on $[a,b]$, then f is R. int. on each subinterval $[a,b]$.

(b) If f is R. int. on $[a,c]$ and $[c,b]$, then f is R. int. on $[a,b]$ and $\int_a^b f = \int_a^c f + \int_c^b f$.

<u>Proof</u>.

(a) Let $\varepsilon > 0$. By the Cauchy criterion, $\exists \delta > 0$ s.t.
$$\|\mathcal{P}_1\|, \|\mathcal{P}_2\| < \delta \implies |S(f,\mathcal{P}_1) - S(f,\mathcal{P}_2)| < \varepsilon .$$

Let $[u,v]$ be a subinterval of $[a,b]$ with

$$a < u < v < b.$$

Let P_a be a partition of $[a,u]$ & P_b be a partition of $[v,b]$ s.t. $\|P_a\| < \delta$ & $\|P_b\| < \delta$.
Let P_1 & P_2 be two partitions of $[u,v]$ s.t. $\|P_1\| < \delta$ and $\|P_2\| < \delta$. Put

$$P_1' = P_a \cup P_1 \cup P_b \quad \& \quad P_2' = P_a \cup P_2 \cup P_b$$

and

$$|S(f,P_1) - S(f,P_2)| = |S(f,P_a) + S(f,P_1) + S(f,P_b) - S(f,P_a) - S(f,P_2) - S(f,P_b)|$$

$$= |S(f,P_1') - S(f,P_2')|$$

Now, since $\|P_1'\| < \delta$ & $\|P_2'\| < \delta$ then

$$|S(f,P_1') - S(f,P_2')| < \varepsilon.$$

Thus, $|S(f,P_1) - S(f,P_2)| < \varepsilon$. By the Cauchy criterion on $[u,v]$, f is R. int. on $[u,v]$.

(b) Suppose f is R. int. We want to show that $\int_a^b f$ exists and $\int_a^b f = \int_a^c f + \int_c^b f$.

Since f is R. int on $[a,c]$ and $[c,b]$, it is bounded on $[a,c]$ and $[c,d]$, so it is bounded on $[a,b]$. Let M be this bound.

Let $\delta_a > 0$ & $\delta_b > 0$ s.t.

$$\|P_a\| < \delta_a \Rightarrow \left|S(f, P_a) - \int_a^c f\right| < \frac{\varepsilon}{3}$$

& $$\|P_b\| < \delta_b \Rightarrow \left|S(f, P_b) - \int_c^b f\right| < \frac{\varepsilon}{3}.$$

Let $\delta := \min\{\delta_a, \delta_b, \varepsilon/6M\}$. We will now show that if $\|P\| < \delta$, then

$$\left|S(f,P) - \int_a^b f\right| < \varepsilon.$$

- If c is the endpoint of a tagged interval in P. WLOG, suppose that this tagged interval is

$$(s, [c,v]).$$

Then consider the two subfamilies of P:

$$P_a := \{(c_i, [x_{i-1}, x_i]) : [x_{i-1}, x_i] \subseteq [a,c]\}$$

& $$P_b := \{(c_i, [x_{i-1}, x_i]) : [x_{i-1}, x_i] \subseteq [c,b]\}.$$

Then, since $P_a \cap P_b = \emptyset$,

$$\left|S(f,P) - \int_a^c f - \int_c^b f\right| = \left|S(f, P_a) - \int_a^c f + S(f, P_b) - \int_c^b f\right|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

because $\|P_a\|, \|P_b\| < \delta$.

- If c is not the endpoint of a tagged interval in $\mathcal{P}$. Let $(s, [u, v]) \in \mathcal{P}$ s.t.

$$u < c < v.$$

Consider

$$\mathcal{P}_1 := \{(c_i, [x_{i-1}, x_i]) \in \mathcal{P} : [x_{i-1}, x_i] \subseteq [a, c]\}$$

$$\mathcal{P}_2 := \{(c_i, [x_{i-1}, x_i]) \in \mathcal{P} : [x_{i-1}, x_i] \subseteq [c, b]\}$$

$$\mathcal{P}_a := \mathcal{P}_1 \cup \{(c, [u, c])\}$$

$$\mathcal{P}_b := \mathcal{P}_2 \cup \{(c, [c, v])\}.$$

So, $\|\mathcal{P}_1\| < \delta$, $\|\mathcal{P}_2\| < \delta$ and

$\mathcal{P}_a$, $\mathcal{P}_b$ are tagged partitions of $[a, c]$ & $[c, b]$ respectively with $\|\mathcal{P}_a\| < \delta$ & $\|\mathcal{P}_b\| < \delta$.

Now, we have

$$S(f, \mathcal{P}) = \sum_{\substack{i \\ (c_i, [x_{i-1}, x_i]) \in \mathcal{P}_1}} f(c_i)(x_i - x_{i-1})$$

$$+ f(s)(v - u)$$

$$+ \sum_{\substack{i \\ (c_i, [x_{i-1}, x_i]) \in \mathcal{P}_2}} f(c_i)(x_i - x_{i-1})$$

$$= S(f, \mathcal{P}_a) - f(c)(c - u)$$

$$+ S(f, \mathcal{P}_a) - f(c)(v - c) + f(s)(v - u)$$

$$= S(f, P_a) + S(f, P_b) + (f(s) - f(c))(v-u) .$$

Thus, we get

$$\left| S(f,P) - \int_a^c f - \int_c^b f \right| \le \left| S(f,P_a) - \int_a^c f \right| + \left| S(f,P_b) - \int_c^b f \right|$$

$$+ |f(s) - f(c)| (v-u)$$

$$\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + 2M \cdot \frac{\varepsilon}{6}$$

$$= \varepsilon .$$

Since $\|P_a\| < \delta$, $\|P_b\| < \delta$. Thus, by choosing $\delta = \min\{\delta_a, \delta_b, \varepsilon/6M\}$, then

$\forall P$ tagged partition, $\|P\| < \delta \Rightarrow \left| S(f,P) - \int_a^c f - \int_c^b f \right| < \varepsilon$.

So, $\int_a^b f$ exists (f is R.int.) and

$$\int_a^b f = \int_a^c f + \int_c^b f . \qquad \square$$

<u>Cor</u> If f is R. Int. on $[a,b]$ and g is a function s.t. g differs from f at only one point, then g is R. int. and $\int_a^b f = \int_a^b g$.

<u>Proof.</u> Denote by $c \in [a,b]$ the point where $f(c) \neq g(c)$.

Let $\varepsilon > 0$. By assumption, $\exists \delta_1 > 0$ s.t. $\forall$ t.P. P

$$\|P\| < \delta_1 \Rightarrow \left| S(f,P) - \int_a^b f \right| < \varepsilon .$$

Then, if $\|\mathcal{P}\| < \delta_1$, then two cases:

- $c \neq d_i$ for all $(d_i, [x_{i-1}, x_i]) \in \mathcal{P}$.

So,
$$S(f, \mathcal{P}) = S(g, \mathcal{P}) \quad \text{and so}$$
$$\left| S(g, \mathcal{P}) - \int_a^b f \right| < \varepsilon$$

- $c = d_i$ for some i. Then

$$\begin{aligned} S(f, \mathcal{P}) - S(g, \mathcal{P}) &= f(c)(x_i - x_i) \\ &\quad - g(c)(x_i - x_{i-1}) \\ &= |f(c) - g(c)|\,|x_i - x_{i-1}| \\ &< |f(c) - g(c)|\,\delta_1 . \end{aligned}$$

Let $\delta = \min\left\{ \delta_1, \dfrac{\varepsilon}{|f(c) - g(c)|} \right\}$.

So, if $\|\mathcal{P}\| < \delta$, then

$$\begin{aligned} \left| S(g, \mathcal{P}) - \int_a^b f \right| &\leq | S(g, \mathcal{P}) - S(f, \mathcal{P}) | + \left| S(f, \mathcal{P}) - \int_a^b f \right| \\ &< |f(c) - g(c)|\,\delta + \varepsilon \\ &= 2\varepsilon . \end{aligned}$$

So, g is integrable and $\int_a^b g = \int_a^b f$. $\square$

<u>Remark</u>. If $g: [a,b) \to \mathbb{R}$ and $g(x) = f(x)$ $\forall x \in [a,b)$ where f is R.I. on $[a,b]$, the g is R.I. on $[a,b]$.

6.3 Types of R. int. fcts.

Def. A step function is a function $\phi : [a,b] \to \mathbb{R}$ s.t. there are a finite number of disjoint intervals I_k s.t. ϕ is constant on each I_k.

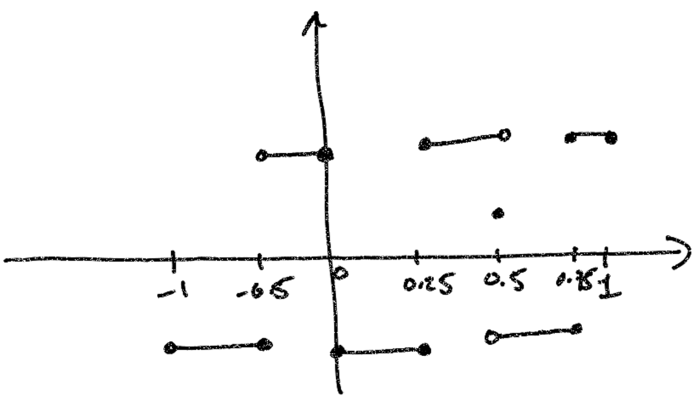
We write

$$\phi = \sum_{k=1}^{N} c_k \chi_{I_k}$$

where χ_{I_k} is the characteristic function of I_k:

$$\chi_{I_k}(x) = \begin{cases} 0, & x \notin I_k \\ 1, & x \in I_k \end{cases}.$$

Picture



thm. If $\phi := \sum_{k=0}^{N} c_k \chi_{I_k}$ is a step function on $[a,b]$, then ϕ is R. int. and

$$\int_a^b \phi = \sum_{k=1}^{N} c_k \, \ell(I_k)$$

where $\ell(I_k)$ is the length of I_k.

Proof: HW07. $\square$

We will obtain a criterion similar to the squeeze theorem for limits.

<u>Squeeze Thm. for Integrals</u>

$f : [a,b] \to \mathbb{R}$. f is R.I. on $[a,b]$ iff $\forall \varepsilon > 0$ there exists R.I. f'ts g_ε & h_ε on $[a,b]$ s.t.

$$g_\varepsilon(x) \le f(x) \le h_\varepsilon(x) \quad \forall x \in [a,b]$$

&
$$\int_a^b (h_\varepsilon - g_\varepsilon) < \varepsilon$$

<u>Proof.</u>

($\Rightarrow$) This implication is clear because we can choose $g = h = f$ and $\int_a^b h - g = 0$.

($\Leftarrow$) Let $\varepsilon > 0$. Then $\exists\, g_\varepsilon$ & h_ε R.I. on $[a,b]$ s.t.

$$g_\varepsilon(x) \le f(x) \le h_\varepsilon(x) \qquad \forall x \in [a,b]$$

and
$$\int_a^b (h_\varepsilon - g_\varepsilon) < \varepsilon .$$

h_ε is R.I. on $[a,b]$: $\exists\, \delta_1 > 0$ s.t.

$$\|P\| < \delta_1 \Rightarrow \left| S(h_\varepsilon, P) - \int_a^b h_\varepsilon \right| < \varepsilon$$

g_ε is R.I. on $[a,b]$: $\exists\, \delta_2 > 0$ s.t.

$$\|P\| < \delta_2 \Rightarrow \left| S(g_\varepsilon, P) - \int_a^b g_\varepsilon \right| < \varepsilon .$$

Take $\delta := \min\{\delta_1, \delta_2\}$. Then if $\|P\| < \delta$

$$S(g_\varepsilon, P) - \int_a^b g_\varepsilon \le S(f, P) - \int_a^b g_\varepsilon \le S(h_\varepsilon, P) - \int_a^b g_\varepsilon$$

But $\quad S(g_\varepsilon, P) - \int_a^b g_\varepsilon \geq -\varepsilon$

& $\quad \left| S(h_\varepsilon, P) - \int_a^b g_\varepsilon \right| \leq \left| S(h_\varepsilon, P) - \int_a^b h_\varepsilon \right| + \left| \int_a^b h_\varepsilon - g_\varepsilon \right|$

$$< 2\varepsilon.$$

So,

$$-2\varepsilon < -\varepsilon < S(f, P) - \int_a^b g_\varepsilon < 2\varepsilon$$

$$\Rightarrow \quad \left| S(f, P) - \int_a^b g_\varepsilon \right| < 2\varepsilon.$$

Let P_1 & P_2 s.t. $\|P_1\| < \delta$ & $\|P_2\| < \delta$.

Then,

$$\left| S(f, P_1) - S(f, P_2) \right|$$

$$\leq \left| S(f, P_1) - \int_a^b g_\varepsilon \right| + \left| \int_a^b g_\varepsilon - S(f, P_2) \right|$$

$$< 4\varepsilon.$$

Thus, by the Cauchy criterion for integrals, f is R.I. □

Remark. The proof shows that we can construct two sequences g_n of fcts s.t.

$$\lim_{n\to\infty} \int_a^b g_n = \int_a^b f.$$

Thm. If $f : [a,b] \to \mathbb{R}$ is continuous, then f is P.I. on $[a,b]$.

Proof. f cont. on $[a,b]$ $\Rightarrow$ f is uniformly cont.
Let $\varepsilon > 0$. Then $\exists \delta > 0$ s.t.

$$\forall x, y \in [a,b], \quad |x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon .$$

Let $N \in \mathbb{N}$ s.t. $\dfrac{b-a}{N} = \beta < \delta$. Define the points

$$x_j = a + j\beta \qquad j = 0, 1, \dots, N.$$

By the extreme value Theorem, there are $u_j, s_j \in [x_{j-1}, x_j]$ s.t.

$$\sup\{f(x) : x \in [x_{j-1}, x_j]\} = f(s_j)$$
$$\inf\{f(x) : x \in [x_{j-1}, x_j]\} = f(u_j).$$

Define the functions g and h as

$$g = \sum_{j=1}^{N} f(u_j)\, \chi_{[x_{j-1}, x_j]}$$
$$\& \quad h = \sum_{j=1}^{N} f(s_j)\, \chi_{[x_{j-1}, x_j]}.$$

We see that

$$g(x) \le f(x) \le h(x) \quad \forall x \in [a,b]$$

and

$$\int_a^b h-g = \sum_{j=1}^{N} [f(b_j) - f(a_j)] (x_j - x_{j-1})$$
$$\leq \varepsilon \sum_{j=1}^{N} x_j - x_{j-1} = \varepsilon (b-a) .$$

Thus, by the previous thm., the fct. f is R.I. ◻

<u>Remark</u>. The functions h & g are called the upper Riemann sum and the lower Riemann sum of f.

<u>6.4 Fundamental Thm. of Calculus.</u>

If $f: [a,b] \to \mathbb{R}$ is R.I., then it is R.I. on each subinterval of $[a,b]$. So, for a fixed point $c \in [a,b]$, the function

$$F: [a,b] \to \mathbb{R} \quad , \quad F(x) := \int_c^x f(t)\,dt$$

is well-defined. Here, we interprete $\int_c^x f$ as $-\int_x^c$ if $x \leq c$.

<u>Def.</u> A fct. $F: [a,b] \to \mathbb{R}$ is an antiderivative of $f: [a,b] \to \mathbb{R}$ if $F'(x) = f(x) \quad \forall x \in (a,b)$.

Fundamental Thm. of Calculus.

Let $f : [a,b] \to \mathbb{R}$ be R.I. on $[a,b]$. Let $F : [a,b] \to \mathbb{R}$ be defined by $F(x) := \int_a^x f \quad \forall x \in [a,b]$.

a) F is continuous on $[a,b]$.

b) If f is continuous at $x \in [a,b]$, then F is differentiable at x.

c) If G is an anti-derivative of f on $[a,b]$, then $\int_a^b f = G(b) - G(a)$.

Proof.

a) If $x_0 \in [a,b]$ and $x \in [x_0 - \delta, x_0 + \delta] \cap [a,b]$,

$$F(x) - F(x_0) = \int_a^x f - \int_a^{x_0} f$$

$$= \int_{x_0}^x f$$

Since f is R.I., f is bounded, say by M. Then by the properties of the integral, we have

$$-M(x - x_0) \le \int_{x_0}^x f \le M(x - x_0)$$

$$\Rightarrow \left| \int_{x_0}^x f \right| \le M\,|x - x_0|$$

$$\Rightarrow |F(x) - F(x_0)| \le M\,|x - x_0|.$$

Take $\delta = \varepsilon / M$ and $|F(x) - F(x_0)| < \varepsilon$ if $|x - x_0| < \delta$.

b) Let $x \in (a,b)$ and f continuous at x. Let $\varepsilon > 0$. Then there is a $\delta > 0$ s.t. $\forall y \in (x-\delta, x+\delta) \cap [a,b]$

$$|f(y) - f(x)| < \varepsilon .$$

Then, if $|y - x| < \delta$ with $y \in [a,b]$ then

$$\frac{F(y) - F(x)}{y-x} - f(x) = \frac{\int_x^y f - f(x)(y-x)}{y-x}$$

$$= \frac{\int_x^y f - \int_x^y f(x)}{y-x}$$

$$= \frac{\int_x^y f(t) - f(x)\, dt}{y-x}$$

Since $-\varepsilon < f(t) - f(x) < \varepsilon$ because $t \in (x,y)$

$$\Rightarrow \quad -\varepsilon(y-x) < \int_x^y f(t) - f(x)\, dt < \varepsilon(y-x)$$

$$\Rightarrow \quad -\varepsilon < \frac{\int_x^y f(t) - f(x) dt}{y-x} < \varepsilon$$

So,

$$\left| \frac{F(y) - F(x)}{y-x} - f(x) \right| = \left| \frac{\int_x^y f(t) - f(x) dt}{y-x} \right| < \varepsilon .$$

Thus, $F'(x) = f(x)$.

c) Suppose that G is an antiderivative of f. So, G' exists and $G' = f$ on $[a,b]$.

Define $F(x) := \int_a^x f$.

Since $F' = f$ on $[a,b]$, then $F' = G'$ on $[a,b]$.

So, $F' - G' = 0$ on $[a,b]$. and this implies that

$$F = G + k \quad \text{where } k \in \mathbb{R}.$$

So,

$$F(b) - F(a) = G(b) - G(a)$$

&

$$F(b) - F(a) = \int_a^b f$$

$$\Rightarrow \quad G(b) - G(a) = \int_a^b f. \qquad \square$$

6.5. Additional properties

The theorem we want to prove is

Thm. Let $g : [a,b] \to [c,d]$ be R.I. on $[a,b]$. If $f : [c,d] \to \mathbb{R}$ is continuous on $[c,d]$, then $f \circ g$ is R.I. on $[a,b]$.

We won't prove this, but it is nice to see this result and how it applies.

Thm. Suppose f & g are R.I. on $[a,b]$.

(a) The fct. $|f|$ is R.I. and $\left|\int_a^b f\right| \leq \int_a^b |f|$.

(b) The fct. fg is R.I.

(c) If $\beta > 0$ and $g \geq \beta$ on $[a,b]$, then f/g is R.I.

Proof.

(a) The fct. $x \mapsto |x|$ is continuous. So $|f|$ is R.I. from the previous Thm. Also, from the properties, since

$$-|f| \leq f(x) \leq |f|$$

$$\Rightarrow -\int_a^b |f| \leq \int_a^b f \leq \int_a^b |f|$$

$$\Rightarrow \left|\int_a^b f\right| \leq \int_a^b |f|.$$

(b) We see that

$$(f+g)^2 = f^2 + 2fg + g^2$$

$$\Rightarrow fg = \frac{(f+g)^2 - f^2 - g^2}{2}.$$

Since $x \mapsto x^2$ is continuous, the $(f+g)^2$, f^2 & g^2 are R.I. and so is fg.

(c) The function $x \mapsto \frac{1}{x}$ is continuous on any interval that doesn't contain 0. So, $\frac{1}{g(x)}$ is R.I. on $[a,b]$. From b, $f/g = f \cdot \frac{1}{g}$ is R.I. on $[a,b]$.

Remark. We can prove the change of variable formula for integrals also.

6.6 the log (ln) & exp (e^x) fcts.

We now have all the tools to define properly the log function.

Def. For $x > 0$, we define

$$L(x) := \int_1^x \frac{1}{t}\,dt .$$

This is a well-defined fct. since $\frac{1}{x}$ is continuous on every $[1, x]$.

When $x < 1$, the integral in intepreted as

$$-\int_x^1 \frac{1}{t}\,dt .$$

From the FTC, we see that L is continuous and diff. on $(0, \infty)$ with

$$F'(x) = \frac{1}{x} \qquad \forall x \in (0, \infty)$$

and so F is also increasing. We can show the usual properties of the log fct.

Thm. (a) $L(xy) = L(x) + L(y) \quad \forall x, y > 0$.

(b) $L(x^y) = y\,L(x) \quad \forall x > 0$ and $y \in \mathbb{Q}$.

(c) $L(x/y) = L(x) - L(y) \quad \forall x, y > 0$.

(d) $\lim_{x \to 0^+} L(x) = -\infty$ and $\lim_{x \to \infty} L(x) = +\infty$

Proof:

(a) $L(xy) = \int_1^{xy} \frac{1}{t}\,dt = \int_1^{x} \frac{1}{t}\,dt + \int_x^{xy} \frac{1}{t}\,dt$

$u = \frac{t}{x} \rightarrow du = \frac{1}{x}\,dt$

$$= \int_1^{x} \frac{1}{t}\,dt + \int_1^{y} \frac{1}{xu}\,x\,du$$

$$= \int_1^{x} \frac{1}{t}\,dt + \int_1^{y} \frac{1}{u}\,du$$

$$= L(x) + L(y).$$

(b) Let $y \in \mathbb{N}$. Then from (a)

$$L(x^y) = L(x^{y-1}x) = L(x^{y-1}) + L(x).$$

Repeat y times to get (induction).

$$L(x^y) = yL(x)$$

Also,

$$L(x) = L((x^{1/y})^y) = yL(x^{1/y})$$

$$\Rightarrow L(x^{1/y}) = \frac{1}{y}L(x).$$

So, for $y = \frac{p}{q}$, we get

$$L(x^{p/q}) = pL(x^{1/q}) = \frac{p}{q}L(x) = yL(x).$$

(c) From (a), $L(x/y) = L(x) + L(1/y)$. It suffices to show $L(1/y) = -L(y)$. We have

$$L(1/y) = \int_1^{1/y} \frac{1}{t}\,dt \overset{u=yt}{\underset{du=ydt}{=}} \int_y^1 \frac{y}{u}\,\frac{du}{y} = \int_y^1 \frac{1}{u}\,du$$

So, $L(1/y) = -\int_1^y \frac{1}{u}\,du = -L(y)$.

(d) We have, for $x<1$

$$L(x) = -\int_x^1 \frac{1}{t}\,dt .$$

<u>Goal</u>: $\forall M>0,\ \exists \delta>0$ s.t.

$0<x<\delta$ then $L(x) < -M$.

Let $\delta := y^N$ for some $N \in \mathbb{N}$. Then

$$L(\delta) = L(y^N) = -N\int_y^1 \frac{1}{t}\,dt .$$

We let N s.t. $-N\int_y^1 \frac{1}{t}\,dt < -M$.

Then, if $0<x<\delta$, then

$$L(x) < L(\delta) = L(y^N) = -N\int_y^1 \frac{1}{t}\,dt < -M .$$

This shows that $\lim_{x\to 0^+} L(x) = -\infty$.

<u>Goal 2</u>: $\forall M>0$, $\exists \delta>0$

$x>\delta$ then $L(x) > M$.

Let $\delta := y^N$ $(y>1)$. Then

$$L(\delta) = L(y^N) = N\int_1^y \frac{1}{t}\,dt$$

We let N s.t. $N\int_1^y \frac{1}{t}\,dt > M$ and let $\delta := y^N$.

Then, if $x>\delta$ then $L(x) > L(\delta) > M$. So $\lim_{x\to\infty} L(x) = \infty$. $\square$

By the IVT, there is a real number e s.t. $L(e) = 1$.

this is the Neperian basis for the usual exponential fct e^x.

Since L is increasing (strictly) on $(0,\infty)$, it has an inverse $E: (-\infty,\infty) \to (0,\infty)$ called the exponential function.

E is continuous, differentiable & strictly increasing on $(-\infty,\infty)$.

<u>Thm.</u> (a) $\lim_{x\to\infty} E(x) = \infty$ and $\lim_{x\to-\infty} E(x) = 0$.

(b) $E'(x) = E(x) \quad \forall x \in \mathbb{R}$

(c) $E(x) = e^x \quad \forall x \in \mathbb{R}$.

<u>Proof.</u>

(a) Since $L(x) \to -\infty$ as $x \to 0^+$, then $E(x) \to 0$ as $x \to -\infty$.

Same: $L(x) \to \infty$ as $x \to \infty$, then $E(x) \to \infty$.

(b) By the inverse formula: if $x = L(y)$, $y > 0$

$$E'(x) = \frac{1}{L'(y)} = y = L^{-1}(x) = E(x)$$

(c) Let $x \in \mathbb{Q}$. Then

$L(E(x)) = x$ and $L(e^x) = x L(e) = x$

$$\Rightarrow \quad L(E(x)) = L(e^x) \overset{L^{-1}}{\Rightarrow} E(x) = e^x.$$

By continuity, $E(x) = e^x$. $\forall x \in \mathbb{R}$.

Homework 7.

<u>Proof</u>. Consider the case where

$$\phi = \phi_0 \chi_I$$

where $I \subseteq [a,b)$ is an interval and $\phi_0 \in \mathbb{R}$.

- $I = [u,v]$, then ϕ is R. int. on $[a,b]$ because
  - * $\phi(x) = \phi_0$ constant on $[u,v]$ is int.
  - * $\phi(x) = 0$ on $[a,u)$ and from the previous thm., is R-I. on $[a,u]$.
  - * $\phi(x) = 0$ on (v,b) and also from previous thm. is R-I. on $[v,b)$.

  Thus, from property of the R.I., ϕ is R-I. on $[a,b]$.
- $I = [u,v)$. Same because now ϕ is R.I. on $[u,v]$.
- $I = (u,v)$. Same because now ϕ is R.I. on $[u,v]$.
- $I = \{u\}$. Same because ϕ differ from 0 at 1 pt.

So ϕ is Rieman integrable. From the properties:

$$\int_a^b \phi = \int_{[a,b)\setminus I} \phi + \int_I \phi = \phi_0 (v-u) = \phi_0 \, \ell(I).$$

Now, if $\phi = \sum_{j=1}^{k} c_j \chi_{I_j}$ with I_j disjoint, then it is Riemann integrable because it it the sum of R.J. fct. Moreover $\int_a^b \phi = \sum_{j=1}^{k} c_j \, \ell(I_j)$. $\square$