MATH-331 Introduction to Real Analysis — Kaiwei Tang

Homework 02 — Fall 2021

Due date: 20-09-2021 1:20pm — Total: /70.

Exercise	1 (10)	2 (5)	3 (5)	4 (5)	5 (5)	6 (10)	7 (5)	8 (10)	9 (5)	10 (10)
Score										

Table 1: Scores for each exercises

Instructions: You must answer all the questions below and send your solution by email (to `parisepo@hawaii.edu`). If you decide to not use LaTeX to hand out your solutions, please be sure that after you scan your copy, it is clear and readable. Make sure that you attached a copy of the homework assignment to your homework. No late homework will be accepted. No format other than PDF will be accepted. Name your file as indicated in the syllabus.

1 Writing problems

For each of the following problems, you will be asked to write a clear and detailed proof. You will have the chance to rewrite your solution in your semester project after receiving feedback from me.

Exercise 1. (10 pts)

a) Let $\{[a_n, b_n] : n \geq 1\}$ be a family of closed intervals such that $[a_1, b_1] \supset [a_2, b_2] \supset [a_3, b_3] \supset \cdots$. Show that there is a $c \in \mathbb{R}$ such that $c \in [a_n, b_n]$ for all $n \geq \mathbb{N}$. Follow the following steps to prove it:

- **(i)** Prove that for any $n, m \geq 1$, $a_n \leq b_m$. [hint: put $M := \max\{n, m\}$.]
- **(ii)** Show that $\sup\{a_n : n \geq 1\}$ exists.
- **(iii)** Show that $c = \sup\{a_n : n \geq 1\}$ satisfies the requirement.

b) Use this last result to prove that the set $\mathbb{R}$ is uncountable. [Hint: Show that any function $f : \mathbb{N} \to \mathbb{R}$ can't be surjective. To do so, construct a sequence of closed intervals such that $f(n) \notin [a_n, b_n]$ with $a_n < b_n$.]

Solution:

Exercise 2. (5 pts) Prove that if $a_n \to A$, then $|a_n| \to |A|$.

Solution: Assume that $a_n \to A$, then $\forall \epsilon > 0, \in \mathbb{N}$ st. $n \geq N$. By using the triangle inequality, we see that,

$$\begin{aligned} |a_n| - |A| \leq |a_n - A| &< \epsilon \\ |a_n| - |A| &< \epsilon \end{aligned}$$

Therefore, see that $\epsilon > |a_n| - |A|$, proving that if $a_n \to A$, then $|a_n| \to |A|$. $\square$

Exercise 3. (5 pts) Let (a_n), (b_n), and (c_n) be sequences of real numbers. Prove that if $a_n \to L$, $b_n \to L$, and $a_n \leq c_n \leq b_n$, then $c_n \to L$.

Solution: Assume that $a_n \to$ Ł and b_n. Let N=max$\{N_A, N_B\}$. Since we know that $a_m \leq c_n \leq b_n$, then we know that $a_n b_n$ for both a_n and $b_n \to L$. This means that $\forall \epsilon > 0, \exists N \in \mathbb{N} st. n > N$.

$$\begin{aligned} |a_n - L| \leq& |b_n - L| < \epsilon \\ -\epsilon < a_n - L \leq& b_n - L < \epsilon \\ L - \epsilon < a_n \leq& b_n < \epsilon - L \end{aligned}$$

From here, we know that $a_n \leq c_n \leq b_n$, thus we can rewrite as,

$$L - \epsilon < a_n \leq c_n \leq b_n < \epsilon - L$$
$$L - L - \epsilon < a_n - L \leq c_n - L \leq b_n - L < \epsilon - L - L$$
$$-\epsilon < a_n - L \leq c_n - L \leq b_n - L < \epsilon$$

We know that from here it is just another way for us to rewrite it as,

$$|a_n - L| \leq |c_n - L| \leq |b_n - L| < \epsilon$$

Which proves that if $a_n \leq c_n \leq b_n$ then $c_n L$. $\square$

Exercise 4. (5 pts) Prove that if $a_n \to A$ and $a_n \geq 0$ for all $n \geq 1$, then $\sqrt{a_n} \to \sqrt{A}$. Follow the following steps to prove it:

1. Consider the case $A = 0$.
2. Suppose that $A \neq 0$. Show that there is a $N_1 \in \mathbb{N}$ such that if $n \geq N_1$, then $\sqrt{a_n} \geq \sqrt{|A|/2}$. [Hint: use the definition of convergence of $(a_n)_{n \geq 0}$ with a clever choice of ε and use the properties of the absolute value.]
3. Use the convergence of (a_n) again to find a N_2 such that $|a_n - A| < \frac{3}{4} \frac{\varepsilon}{\sqrt{|A|}}$.
4. Express $\sqrt{a_n} - A$ as $\frac{a_n - A}{\sqrt{a_n} + \sqrt{A}}$ and put $N = \max\{N_1, N_2\}$. Conclude.

Solution: For this problem, let us assume that $A \neq 0$. We can refer back to the lecture notes on the quotient rule and follow the same logic as how we proved our statement in class. Let $\epsilon = \frac{|A|}{2}$.

$$\begin{aligned}
-\frac{|A|}{2} < a_n - A < \frac{|A|}{2} \\
-|A| < 2a_n - 2A < |A| \\
0 < 2a_n - A < 2|A| \\
|A| < 2a_n < 3|A| \\
\frac{|A|}{2} < a_n < \frac{3}{2}|A| \\
\sqrt{\frac{|A|}{2}} < \sqrt{a_n} < \sqrt{\frac{3}{2}|A|}
\end{aligned}$$

From here we see we have our $\sqrt{\frac{|A|}{2}} < \sqrt{a_n}$. □

Exercise 5. (5 pts) For each sequence $(a_n)_{n=1}^{\infty}$, define the sequence $(\sigma_n)_{n=1}^{\infty}$ by

$$\sigma_n := \frac{a_1 + a_2 + \cdots + a_n}{n} \quad (n \geq 1).$$

Prove that if $a_n \to A$, then $\sigma_n \to A$. Find an example of a divergent sequence (a_n) such that $(\sigma_n)_{n=1}^{\infty}$ converges.

Solution: We can begin this proof by the hint given in office hours, and assume that $\sigma_n \to A$.

$$\begin{aligned}
\left|\frac{\sum_{i=1}^{n} a_n}{n} - A\right| < \epsilon \\
\frac{\sum_{i=1}^{n} |a_n - An|}{|n|} < \epsilon \\
\frac{|(a_1 - A) + (a_2 - A) + ...(a_{n-1} - A) + (a_n - A)|}{|n|} < \epsilon
\end{aligned}$$

From here we can rewrite it as the sum from 1 to $n-1$ and 1 to n.

$$\frac{\sum_{i=1}^{n-1} |a_n - A|}{|n|} + \frac{\sum_{i=1}^{n} |a_n - A|}{|n|} < \epsilon$$

From our hypothesis, we assumed that $\sigma_n \to A$, thus at n, can assume that is true, but for $n-1$, we must prove that it also goes to A as well. Let us prove that, for σ_n of $n-1$, goes to A. Assume that σ_{n-1} does not go to A, that means $\exists \epsilon > 0, \forall N \in \mathbb{N}, st.n \geq N$.

$$\begin{aligned}
\frac{|a_{n-1} - A|}{|n|} > \epsilon \\
-n\epsilon > |a_{n-1} - A| > n\epsilon
\end{aligned}$$

We see here that this is a contradiction since $-n\epsilon$ is a negative integer, then we know that $|a_{n-1}-A|$ must also be negative, then it cannot be bigger than positive $n\epsilon$, thus causing a contradiction, therefore proving that $a_{n-1} \to A$. From here, we see that it is true for both n and $n-1$ cases of the sequence a_n, therefore, we see that if $a_n \to A$, then $\sigma_n \to A$ as well. □

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Homework problems

Exercise 6. (10 pts) Use the definition of convergence to prove that each of the following sequences converges.

a) $(a_n)_{n=1}^{\infty}$ given by $a_n = 5 + 1/n$ for $n \geq 1$.

b) $(a_n)_{n=1}^{\infty}$ given by $a_n = \frac{3n}{2n+1}$ for $n \geq 1$.

Solution: By the definition of convergence, $\forall \epsilon > 0, \exists N \in \mathbb{N}, st, n \geq N$. Let us use this definition to prove that it exists. For 6a, we see that the limit goes to 5, therefore, we can rewrite as,

$$\left|5 + \frac{1}{n} - 5\right| < \epsilon$$
$$\left|\frac{1}{n}\right| < \epsilon$$

From here we see this is true from our lecture notes and also from our quiz 1.
For 6b, we use the logic, and see that our absolute sign reduce down to $\frac{3}{4n+2}$ thus,

$$\left|\frac{3}{4n+2}\right| < \epsilon$$

From here we see that the LHS of our inequality if positive since $n \geq 1$ thus we can drop the absolute sign and manipulate it algebraically.

$$\frac{3}{4n+2} < \epsilon$$
$$\left(\frac{3}{2}\right)\frac{1}{2n+1} < \epsilon$$

We see this inequality is very similar to our $\frac{1}{n}$ proof we did in class, so let us follow the same logic.

$$\frac{1}{2n+1} < \frac{2\epsilon}{3}$$
$$2n+1 > \frac{3}{2\epsilon}$$

Since we know that there $\exists N \in \mathbb{N}, st. n \geq N$, then,

$$\begin{aligned}
2n+1 > 2N+1 > \frac{3}{2\epsilon} \\
\frac{1}{2n+1} < \frac{1}{2N+1} < \frac{2\epsilon}{3} \\
\frac{3}{4n+2} < \frac{3}{4N+2} < \epsilon
\end{aligned}$$

We can let $\epsilon = \frac{2\epsilon}{3}$ and we see that it is true for our a_n. $\square$

Exercise 7. (5 pts) Prove that the sequence $(a_n)_{n=1}^{\infty} = \left(\frac{2n+1}{n}\right)_{n=1}^{\infty}$ is a Cauchy sequence.

Solution: For us to prove this statement, we can use the theorem given in class. The Theorem states, "If $a_n \to A$, then a_n is a Cauchy sequence". Let us prove that our $a_n \to A$. We see that for our a_n, our $A = 2$, so let us use this knowldge,

$$\left|\frac{2n+1}{n} - 2\right| < \epsilon$$

We see the LHS of our inequality is positive for $n \geq 1$, thus we can drop our absolute value sign.

$$\begin{aligned}
\frac{2n+1-2n}{n} < \epsilon \\
\frac{1}{n} < \epsilon
\end{aligned}$$

We see that we've already proven this from lecture, thus our $a_n \to A$ and according to our theorem, it is a Cauchy sequence. $\square$

Exercise 8. (10 pts) Prove that each of the following sequence diverges.

a) $(a_n)_{n=1}^{\infty} = ((-1)^n)_{n=1}^{\infty}$.

b) $(a_n)_{n=1}^{\infty} = (\sin(\frac{4n+1}{2}\pi))_{n=1}^{\infty}$.

Solution: For 8a, let us assume that the sequence $(-1)^n$ converges and prove my contradiction. Since we know that the sequence is really within -1 and 1, we can set $\epsilon = 1$.

$$\begin{aligned}
|(-1)^n - A| < \epsilon \\
|(-1)^n - A| < 1 \\
-1 < (-1)^n - A < 1
\end{aligned}$$

We see that if n is even then we would have a positive 1,

$$-1 < 1 - A < 1$$

Which we see it is true. If we let our n be odd, then our 1 would be negative,

$$-1 < -1 - A < 1$$

Which we see causes a contradict, therefore, we see our sequence actually does not converge, thus it diverges.

For 8b, let us do the same and assume contradict, for our $a_n = sin(\frac{2n+1}{2})$. Since we now sine is bounded between -1 and 1, we can again let our ϵ be 1.

$$\left| sin(\frac{4n+1(\pi)}{2}) - A \right| < \epsilon$$
$$-1 < sin(\frac{4n+1(\pi)}{2}) - A < 1$$

From here, we see that if n is odd, then sin will be 1, if our n is even then our sin will be -1. Thus we see the two cases,

$$-1 < 1 - A < 1$$
$$-1 < -1 - A < 1$$

We see that if our n is even our case is true, but if our n is odd, our case is false, thus, contradiction, and our sequence diverges. ☐

Exercise 9. (5 pts) Give an examples of two sequences (a_n) and (b_n) such that (a_n) and (b_n) don't converge, but $(a_n + b_n)$ converge.

Solution: Since we know from class that $\frac{1}{n}$, converges and from problem 8a, we know that $(-1)^n$ does not converge, let us use these two sequences and see if it converges. We can basically solve this like a linear problem and let $(a_n)_{n=1}^{\infty} = (-1)^n$ and $(c_n)_{n=1}^{\infty} = \frac{1}{n}$.

$$a_n + b_n = c_n$$
$$(-1)^n + b_n = \frac{1}{n}$$

Thus we see that our $(b_n)_{n=1}^{\infty} = (-1)^n - \frac{1}{n}$, now let us prove that this sequence diverges. Let us assume that this sequence is bounded and that there exists $M > 0$, $\forall \epsilon > 0, \exists N \in \mathbb{N}, st.n \geq N$.

$$M > (-1)^n - \frac{1}{n}$$

We can use the same trick and let n be odd.

$$M > \frac{1}{2n+1} + 1 > 0$$

We see that this is true. Now check for n be even.

$$M > \frac{1}{2n} - 1 > 0$$

Which we see it is false since $\frac{1}{2n} < 1$, thus it would've been a negative and a negative integer is not greater than 0, therefore, we see it causes a contradiction and our b_n is divergent, thus our divergent sequences a_n and b_n when added together converges. $\square$

Exercise 10. (10 pts) With the limit operations and the writing problems, find the limit of the following sequence with general term

a) $\frac{n^2+4n}{n^2-5}$.

b) $\frac{n}{n^2-3}$.

c) $\frac{\cos n}{n}$. [You can use what you know on the cosine function.]

d) $\left(\sqrt{4-\frac{1}{n}}-2\right)n$.

Solution: 10a) can be done as,

$$\begin{aligned}
&=\frac{n^2+4n}{n^2+5} \\
&=\frac{1+\frac{4}{n}}{1+\frac{5}{n^2}} \\
&=\frac{1+0}{1+0} \\
&=1
\end{aligned}$$

For 10b) we can do something similar.

$$\begin{aligned}
&=\frac{n}{n^2-3} \\
&=\frac{\frac{1}{n}}{1-\frac{3}{n^2}} \\
&=\frac{0}{1-0} \\
&=0
\end{aligned}$$

For 10c) we can be a bit more creative and we know that our cosine functions diverges or it goes towards infinity. So let us split our limits up into two different partitions.

$$\begin{aligned}
&=cos(n)\cdot\frac{1}{n} \\
&=cos(n)\cdot 0 \\
&=0
\end{aligned}$$

For 10d) We can multiple by the conjugate and get our answer,

$$
\begin{aligned}
&= n\left(\sqrt{4-\frac{1}{n}})-2\right)\cdot\left(\frac{1}{n}\cdot\frac{1}{\sqrt{4-\frac{1}{n}}+2}\right)\cdot\left(n\cdot\sqrt{4-\frac{1}{n}}+2\right)\\
&= \frac{n(4-\frac{1}{n}-4}{\sqrt{4-\frac{1}{n}}+2}\\
&= \frac{-1}{\sqrt{4-\frac{1}{n}}+2}
\end{aligned}
$$

From here we know by calculus we can take the limit of top and bottom, and we will get our solution as $\frac{-1}{4}$. $\square$