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2021년 9월 6일

MATH-331 Introduction to Real Analysis — Pierre-Olivier Parisé
Homework 01 — Fall 2021

Due date: 09/06/2021 1:20pm

Instructions: You must answer all the questions below and send your solution by email (to `parisepo@hawaii.edu`). If you decide to not use LaTeX to hand out your solutions, please be sure that after you scan your copy, it is clear and readable. No late homework will be accepted. No format other than PDF will be accepted. Name your file as indicated in the syllabus.

1
HOMEWORK PROBLEMS

Exercise 1. Prove that for any $n \in \mathbb{N}$, $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

Exercise 2. Define $f : \mathbb{N} \to \mathbb{N}$ by $f(1) = 1$, $f(2) = 2$ and $f(3) = 3$ and

$$f(n) := f(n-1) + f(n-2) + f(n-3) \quad (n \geq 4).$$

Prove that $f(n) \leq 2^{n-1}$ for all $n \in \mathbb{N}$.

Exercise 3. Prove that if A, B and C are sets, then

a) $A \sim A$.

b) If $A \sim B$, then $B \sim A$.

c) If $A \sim B$ and $B \sim C$, then $A \sim C$.

Exercise 4. Show that any subset of a countable set is countable.

Exercise 5. Let $0 < a < b$ be positive real numbers. Prove that

a) $a^2 < b^2$.

b) $\sqrt{a} < \sqrt{b}$.

Exercise 6. Sketch the region of the points (x, y) satisfying the following relation: $x + |x| = y + |y|$ (explain your answer).

Exercise 7. If $x \geq 0$ and $y \geq 0$, prove that $\sqrt{xy} \leq \frac{x+y}{\sqrt{2}}$

Exercise 8. Find the infimum and supremum (if they exist) of the following sets. Make sure to justify all your answers:

a) $E := \{x \in \mathbb{R} : x \geq 0 \text{ and } x^2 \leq 9\}$.

b) $E := \{\frac{4n+5}{n+1} : n \in \mathbb{N}\}$.

Writing problems

For each of the following problems, you will be ask to write a clear and detailed proof. You will have the chance to rewrite your solution in your semester project after receiving feedback from me.

Exercise 9. Let A be a non-empty set and $P(A)$ be its power set (the family of all subsets of A). Prove that A is not equivalent to $P(A)$. Deduce that $P(\mathbb{N})$ is not countable. [Hint: Define $C := \{x : x \in A \text{ and } x \notin f(x)\}$.]

Exercise 10. Let $E \subseteq \mathbb{R}$ be bounded from above and $E \neq \emptyset$. For $r \in \mathbb{R}$, let

$$rE := \{rx : x \in E\} \quad \text{and} \quad r + E := \{r + x : x \in E\}.$$

Show that

a) if $r > 0$, then $\sup(rE) = r\sup(E)$.

b) for any $r \in \mathbb{R}$, $\sup(r + E) = r + \sup E$.

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9/5/21

Math 331: Homework 01

1. Exercise 1. Prove that for any $n \in \mathbb{N}$, $1+2+\dots+n = \frac{n(n+1)}{2}$.

Proof by induction

We will use mathematical induction to prove this identity. Let $P(n)$ be the statement

$$1+2+\dots+n = \frac{n(n+1)}{2}$$

Base case: First check $P(1)$ is true. We see that the on the left of the identity is 1 and on the right $\frac{1(1+1)}{2} = 1$ or $1 = \frac{1(1+1)}{2} = 1$. Thus $P(1)$ is true.

Induction step: For $n \in \mathbb{N}$, assume that $P(n)$ is true so that

$$1+2+\dots+n = \frac{n(n+1)}{2}$$

. Now we must show that $P(n+1)$ is true. So we have

$$\begin{aligned} 1+2+\dots+n+n+1 &= \frac{n(n+1)}{2} + n+1 \\ &= \frac{n(n+1)+2(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2} \\ &= \frac{(n+1)((n+1)+1)}{2} \end{aligned}$$

This last equality shows that $P(n+1)$ is true. So from $P(n)$ we showed $P(n+1)$ is true.

By the principal of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$. ■

2. Exercise 2. Define $f: \mathbb{N} \to \mathbb{N}$ by $f(1)=1$, $f(2)=2$ and $f(3)=3$ and

$$f(n) := f(n-1) + f(n-2) + f(n-3) \qquad (n \geq 4)$$

Prove that $f(n) \leq 2^{n-1}$ for all $n \in \mathbb{N}$

Proof by Strong Induction

We want to prove $f(n) \leq 2^{n-1}$ for all $n \in \mathbb{N}$. To do so we will use Theorem 0.9 (Page 14). Let $P(n)$ be the statement $f(n) \leq 2^{n-1}$

Base case: First we must show that $P(1)$, $P(2)$, and $P(3)$ are true

For $n=1$ the inequality gives

$$f(1) = 1 \leq 2^{1-1} = 1$$

Thus for $n=1$, $P(1)$ is true

For $n=2$ the inequality gives

$$f(2) = 2 \leq 2^{2-1} = 2$$

Thus for $n=2$, $P(2)$ is true.

For $n=3$ the inequality gives

$$f(3) = 3 \leq 2^{3-1} = 2^2 = 4$$

Thus for $n=3$, $P(3)$ is true

Induction step: Now assume that $f(i) \leq 2^{i-1}$ is true for $1 \leq i \leq k$. Then

$$f(k+1) = f(k) + f(k-1) + f(k-2) =$$

since $f(k) \leq 2^{k-1}$, $f(k-1) \leq 2^{k-2}$, and $f(k-2) \leq 2^{k-3}$

then

$$\begin{aligned} f(k+1) = f(k) + f(k-1) + f(k-2) &\leq 2^{k-1} + 2^{k-2} + 2^{k-3} \\ &= 2^k 2^{-1} + 2^k 2^{-2} + 2^k 2^{-3} \\ &= 2^k(2^{-1} + 2^{-2} + 2^{-3}) \\ &= 2^k \tfrac{7}{8} \end{aligned}$$

As we can see $f(k+1) \leq \frac{7}{8} 2^k$ and it is obvious that $\frac{7}{8} 2^k \leq 2^k$, so $f(k+1) \leq 2^k$. Thus the formula holds for $n = k+1$.

By the principal of mathematical induction $f(n) \leq 2^{n-1}$ for all $n \in \mathbb{N}$.

3. Exercise 3. Prove that if A, B, and C are sets then

a) $A \sim A$

b) If $A \sim B$ then $B \sim A$

c) If $A \sim B$ and $B \sim C$, then $A \sim C$

Since sets A, B, C are not specified, I will proving these statements generally like the book. This is because we need to show a bijection between both sets which requires a specific function. That function can be determined if A, B, C are specified. Since not a general proof is required.

a) $A \sim A$

Proof: To show $A \sim A$ we need to find a bijective function from A to A. Let's define the function from A onto A as $1_A(a) = a$ for all $a \in A$. To show it is bijective we must prove it is injective. So let $a_1, a_2 \in A$ then suppose $1_A(a_1) = 1_A(a_2)$. By the definition of the 1_A function $a_1 = a_2$. Thus 1_A is an injective function. To show 1_A is surjective we know $im(1_A) = A$. A function that satisfies this quality is said to be surjective. Thus, $A \sim A$.

b) If $A \sim B$, then $B \sim A$

Proof: Suppose $A \sim B$. By the definition of equinumerous sets there must exist a function that is bijective say $f: A \to B$. Then it is true that $f^{-1}: B \to A$ must exist. We must show that f^{-1} is bijective. Since f is 1-1 we know that for $a_1, a_2 \in A$ and $b_1, b_2 \in B$ such that $f(a_1) = b_1$ and $f(a_2) = b_2$, and assuming that $f(a_1) = f(a_2)$ or $b_1 = b_2$ then $a_1 = a_2$. Thus if we assume $f^{-1}(b_1) = f^{-1}(b_2)$ or $a_1 = a_2$ then $b_1 = b_2$. Hence f^{-1} is injective. To show it is surjective know that by 0.6 Theorem (page 10) that $im(f^{-1}) = dom(f) = A$. Hence f^{-1} is surjective. By proving ~~both~~ both we showed that f^{-1} is a bijective function implying that $B \sim A$. ∎

c) If $A \sim B$ and $B \sim C$, then $A \sim C$

Proof: Suppose $A \sim B$ and $B \sim C$. Let $f: A \to B$ and $g: B \to C$ and both f and g bijective functions. To show $A \sim C$ we must define a function, say $g \circ f: A \to C$ and show it's bijective. Suppose $a_1, a_2 \in$ and $(g \circ f)(a_1) = (g \circ f)(a_2)$. By the definition of $g \circ f$ then $g(f(a_1)) = g(f(a_2))$. Since g is 1-1 it implies that $f(a_1) = f(a_2)$ and since f is 1-1 it implies $a_1 = a_2$. Thus $g \circ f$ is injective. To prove that $g \circ f$ is surjective

we must show that $\operatorname{im}(g \circ f) = C$. suppose $c \in C$.
since $\operatorname{Im}(g) = C$, there is a $b \in B$ such that
$g(b) = c$. Also, since $\operatorname{im}(f) = B$, there is $a \in A$ such
that $f(a) = b$. so we have $(g \circ f)(a) = g(f(a)) = g(b) = c$
Therefore $C \subseteq \operatorname{im}(g \circ f)$. Now let $a \in A$ then $f(a) = b$,
then $(g \circ f)(a) = g(f(a)) \in \operatorname{Im}(g) = C$. so
$\operatorname{im}(g \circ f) \subseteq C$. We have proven $g \circ f$ is bijective,
Therefore $A \sim C$. ∎

4 Exercise 4. show that any subset of a countable set is countable.

proof: suppose C is a countable set, meaning it is finite or
countably infinite. To be finite implies it is either $\emptyset$
or $n \in \mathbb{N}$ such that C has a bijection with the set $\{1, 2, 3, \ldots, n\}$.
suppose $X \subseteq C$. We will prove when $C = \emptyset$ (finite)
$C =$ nonempty set (finite), or countably infinite.

Case 1: If $C = \emptyset$, and suppose $X \subseteq C$ then $X = \emptyset$ since $\emptyset \subseteq \emptyset$
Thus X is finite which is countable.

Case 2: If C is a nonempty (finite) set with bijection
to $\{1, 2, 3, \ldots, n\}$ and if X is any subset of C
then we need to show that X has a bijection
with a subset of $\mathbb{N}$. By definition $f: C \to \{\text{subset of } \mathbb{N}\}$.
Now define $1_X: X \to C$, By definition 1_X function
(page 17) is a bijective function. Since the
composition of bijective functions is bijection
it follows that $f \circ 1_X: X \to \{\text{subset of } \mathbb{N}\} = \{1, 2, \ldots, n\}$
is bijective. Therefore a finite subset of a countable
set is countable

Case 3: If C is a countably infinite set then $C = \{c_1, c_2, c_3, \ldots\}$.
Then if $X \subseteq C = \{c_{n_0}, c_{n_1}, c_{n_2}, \ldots\}$, where $c_{n_0}, c_{n_1}, c_{n_2}$
a so on are elements of C also in X. If the set
$\{n_0, n_1, n_2, \ldots\}$ for $n_i \in \mathbb{N}$ has a largest element then we conclude
it is finite. Otherwise let 1_X have bijection with n_i

then it is obvious from the definition above that 1_x has a bijection with C_{n_i}. Thus X is countably infinite ■

5 Exercise 5: Let $0<a<b$ be positive real numbers. Prove that

a) $a^2<b^2$

b) $\sqrt{a}<\sqrt{b}$

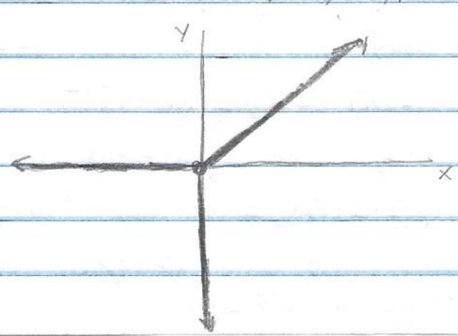
a) $a^2<b^2$

Proof: Suppose $0<a<b$ and $a,b \in \mathbb{R}^+$. Multiply the inequality by a to get $a^2<ab$. Multiply $0<a<b$ by b to get $ab<b^2$. Combining the inequality results in $a^2<b^2$. ■

b) $\sqrt{a}<\sqrt{b}$

Proof: Suppose $0<a<b$ and $a,b \in \mathbb{R}^+$. Also suppose $\sqrt{a}>\sqrt{b}$. We know from 0.23 Theorem (page 25) that there exist a number namely $\sqrt{a}$ that satisfy $x^2=a$ for $x \in \mathbb{R}$. If we multiply $\sqrt{a}>\sqrt{b}$ by $\sqrt{a}$ then $\sqrt{a}\cdot\sqrt{a}>\sqrt{a}\sqrt{b}$ which simplifies to $a>\sqrt{a}\sqrt{b}$. The same argument from 0.23 Theorem can be used for $x^2=b$ for $x \in \mathbb{R}$, so $\sqrt{b}$ exist. If we multiply $\sqrt{a}>\sqrt{b}$ by $\sqrt{b}$ then $\sqrt{a}\sqrt{b}>\sqrt{b}\sqrt{b}$ or $\sqrt{a}\sqrt{b}>b$. We can combine the inequality to obtain $a>b$ or $b<a$ which is a contradiction. Therefore our assumption that $\sqrt{a}>\sqrt{b}$ was false. Thus it must be true $\sqrt{a}<\sqrt{b}$. ■

6 Exercise 6 Sketch the region of points (x,y) satisfying the following relation: $x+|x|=y+|y|$ (explain your answer)



explanation: we define $|x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$ and

define $|y| = \begin{cases} y & \text{for } y \geq 0 \\ -y & \text{for } y < 0 \end{cases}$. To explain the region

lets make cases.

case 1: Quadrant 1 $x \geq 0$ and $y \geq 0$. Then

$x+|x| = x+x = 2x = 2y = y+y = y+|y|$ or $y=x$.

so $y=x$ for $x \geq 0$ which implies $y \geq 0$. This is valid.

case 2: Quadrant 2 $x<0$ and $y \geq 0$. Then

$x+|x| = x-x = y+y = y+|y|$ or $0 = 2y$

so $y=0$. This value is included in case 1.

case 3: Quadrant 3 $x<0$ and $y<0$. Then

$x+|x| = x-x = y-y = y+|y|$. or $0=0$

This is true but doesn't define a point

case 4: Quadrant 4 $x \geq 0$ and $y<0$. Then

$x+|x| = x+x = y-y = y+|y|$. so $2x=0$

hence $x=0$. This value is included already in prior case

case 5: suppose $y=0$ then $x+|x|=0$ or $-x=|x|$. This is true for $x<0$.

case 6: suppose $x=0$ then $0=y+|y|$ or $-y=|y|$. This is true for $y<0$.

These 6 cases show that $x+|x|=y+|y|$ map to points in quadrant 1 ($x \geq 0$, $y \geq 0$) or points on y-axis ($y<0 \wedge x=0$) and points on x-axis ($x<0 \wedge y=0$).

7. Exercise 7: If $x \geq 0$ and $y \geq 0$, prove that $\sqrt{xy} \leq \frac{x+y}{\sqrt{2}}$.

we will do a proof by contradiction. suppose $x \geq 0$ and $y \geq 0$. Suppose $\sqrt{xy} \geq \frac{x+y}{\sqrt{2}}$. Multiply the inequality by $\sqrt{2}$ which results in $\sqrt{2}\sqrt{xy} \geq x+y$. Then add $-2\sqrt{xy}$ to both sides to get $\sqrt{2}\sqrt{xy} - 2\sqrt{xy} \geq x+y-2\sqrt{xy}$ $= x - 2\sqrt{xy} + y$. Notice that $x - 2\sqrt{xy} + y = (\sqrt{x}-\sqrt{y})^2$ so $\sqrt{2}\sqrt{xy} - 2\sqrt{xy} \geq (\sqrt{x}-\sqrt{y})^2$. The left side of the inequality can be factored to get $(\sqrt{2}-2)\sqrt{xy}$. so $(\sqrt{2}-2)\sqrt{xy} \geq (\sqrt{x}-\sqrt{y})^2$. Notice that $(\sqrt{x}-\sqrt{y})^2 \geq 0$. Now we must show $\sqrt{2}-2 \geq 0$ in order for the inequality to be true as $\sqrt{xy} \geq 0$.

If $\sqrt{2}-2 \geq 0$ then $\sqrt{2} > 2$. Multiply the inequality by $\sqrt{2}$ to get $2 > 2\sqrt{2}$ which can be written as $2-2\sqrt{2} > 0$. By factoring the 2 out $2(1-\sqrt{2}) > 0$ which implies that $1-\sqrt{2} > 0$ or $1 > \sqrt{2}$, By combining the inequality then $1 > 2$ which is a contradiction since by 0.19 Theorem (page 22) $0 < 1$ so by axiom 8 $0+1 < 1+1$ or $1 < 2$. Since $\sqrt{2}-2 < 0$ it is also a contradiction that $(\sqrt{2}-2)\sqrt{xy} \geq (\sqrt{x}-\sqrt{y})^2$ since the right side of the inequality is less than or equal to zero and the right is greater than or equal to zero. Therefore the assumption that $\sqrt{xy} \geq \frac{x+y}{\sqrt{2}}$ was false meaning opposite statement $\sqrt{xy} \leq \frac{x+y}{\sqrt{2}}$ is true. $\blacksquare$

8. Exercise 8: Find the infimum and supremum (if they exist) of the following sets. Make sure to justify all your answers:

a) $E := \{x \in \mathbb{R} : x \geq 0 \text{ and } x^2 \leq 9\}$

b) $E := \{\frac{4n+5}{n+1} : n \in \mathbb{N}\}$

a) $E := \{x \in \mathbb{R} : x \geq 0 \text{ and } x^2 \leq 9\}$

Before we find $\inf(E)$ and $\sup(E)$ if they exist, lets define $x^2 \leq 9$ since it hasn't been proven.

By the definition of an even function $x^2 = x^2$ for $x \geq 0$. Likewise $x^2 = (-x)^2$ for $x < 0$. Based on exercise 5a if $0 < a < b$ then $a^2 < b^2$. Therefore, for $x > 0$ $x^2 \leq 3^2$. Although in exercise, I proved that $0 < a < b$ implies $a^2 < b^2$ it is true that if $a^2 < b^2$ implies $a < b$ by simply undoing the steps. Thus $x^2 \leq 3^2$ implies $x \leq 3$ for $x > 0$ but for $x < 0$ then $x^2 = (-x)^2 \leq 3^2$ implies $-x \leq 3$ or $-3 \leq x$. Combining the inequality we obtain $-3 \leq x \leq 3$. Therefore we proved that $x^2 \leq 9$ implies $-3 \leq x \leq 3$.

Suppose $E := \{x \in \mathbb{R} : x \geq 0 \text{ and } x^2 \leq 9\}$. Then, by the definition above $E := \{x \in \mathbb{R} : x \geq 0 \text{ and } -3 \leq x \leq 3\}$. This is simply the intersection of two sets $\{x \in \mathbb{R} : x \geq 0\} \cap \{x \in \mathbb{R} : -3 < x \leq 3\}$ which equals $\{x \in \mathbb{R} : 0 \leq x \leq 3\} = [0,3]$. The $\inf(E)$ is an lower bound $M \in \mathbb{R}$ such that $M \leq x \in [0,3]$. M is also the greatest lower bound such that for any b that are lower bound $b \leq M$. Therefore $b \leq M \leq x$ and the value M must be 0. $\inf(E) = 0$ as it satisfies being a lower bound and greatest lower bound. The $\sup(E) = 3$ as it is an upper bound $x \in [0,3]$ and $x \leq M = 3$. Also it is the least upper bound so for all b that are upper bound $M = 3 \leq b$. so $\inf(E) = 0$, $\sup(E) = 3$.

b: $E := \left\{\frac{4n+5}{n+1} : n \in \mathbb{N}\right\}$

Assume $\mathbb{N} = \{1, 2, 3, \ldots\}$. I will assume that the $\inf(E)$ and $\sup(E)$ if they exist will be an element of $\mathbb{R}$ as in the book than defined for integers. The motivation of the prior assumption is that $\frac{4n+5}{n+1}$ produces rational numbers not all integers.

Lets produce values for $n = 1, 2, 3, 4$. For $n=1$ $\frac{4n+5}{n+1} = \frac{4(1)+5}{(1)+1} = \frac{9}{2} = 4.5$. For $n = 2$, $\frac{4(2)+5}{2+1} = \frac{13}{3} = 4.\bar{3}$. For $n=3$, $\frac{4(3)+5}{3+1} = \frac{17}{4} = 4.25$. For $n=4$, $\frac{4(4)+5}{4+1} = \frac{21}{5} = 4.2$. As n increases, $\frac{4n+5}{n+1}$ appears to approach 4. so $\inf(E) = 4$. It is an lower bound as for all $x \in E$, $4 \leq x$. Also its a greatest lower bound as for all $b \in$ lower bound $b \leq 4$. The $\sup(E) = \frac{9}{2} = 4.5$. 4.5 is an upper bound as for all $x \in E$ $x \leq \frac{9}{2}$. In addition, its a lowest upper bound so for all $b \in$ upper bound $\frac{9}{2} \leq b$. so $\inf(E) = 4$ and $\sup(E) = \frac{9}{2} = 4.5$.

Writing Problems

9 Exercise 9. Let A be a nonempty set and $P(A)$ be its power set (the family of all subsets of A). Prove that A is not equivalent to $P(A)$. Deduce that $P(\mathbb{N})$ is not countable. [Hint: Define $C := \{x : x \in A \text{ and } x \notin f(x)\}$.]

Proof:
To show that A and $P(A)$ are not equivalent, let's assume its possible. Define $f: A \to P(A)$ and f is bijective. Define $C := \{x : x \in A \text{ and } x \notin f(x)\}$. As we can see C must exist in the $\operatorname{im}(f)$ which is a set of sets. By the definition of C there must be an $a \in C$ such that $f(a) = C$. But by the definition of C $a \in C$ means $a \in C$ and $a \notin f(a)$. This is a contradiction. If there is such an element, lets say $b \in A$ and $b \notin f(b)$ then again it contradicts as $b \in C = f(b)$. Therefore, f does not exist so $A \nsim P(A)$. ∎

From the proof above then $\mathbb{N} \sim P(\mathbb{N})$ are not equivalent since there exist no bijective function $f: \mathbb{N} \to P(\mathbb{N})$. Therefore $P(\mathbb{N})$ is not countable despite $\mathbb{N}$ being countable.

10 Exercise 10. Let $E \subseteq \mathbb{R}$ be bounded from above and $E \neq \emptyset$. For $r \in \mathbb{R}$, let
$$rE := \{rx : x \in E\} \quad \text{and} \quad r + E := \{r + x : x \in E\}$$
Show that

a) if $r > 0$, then $\sup(rE) = r\sup(E)$

b) for any $r \in \mathbb{R}$, $\sup(r+E) = r + \sup E$

a) if $r > 0$, then $\sup(rE) = r\sup(E)$

Proof: Suppose $E \subseteq \mathbb{R}$ be bounded from above and $E \neq \emptyset$. For $r \in \mathbb{R}$, let $rE := \{rx : x \in E\}$. By the definition of $\sup(E)$ it must be an upper bound and a least upper bound. For it to be an upper bound, there must exist an M, say M_0 such that $x \le M_0$ for all $x \in E$. For M_0 to be a least upper bound, for all b_0 that are an upper bound of E $M_0 \le b_0$. The combined inequality is $x \le M_0 \le b_0$ when $M_0 = \sup(E)$. The $\sup(rE)$ should be defined similarly with $\sup(rE) = M_1$ and satisfying the inequality $rx \le M_1 \le b_1$ for all $x \in E$ and b_1 that are an upper bound of rE. If we multiply $M_0 = \sup(E)$ and the respective inequality by r we get $rM_0 = r\sup(E)$ and $rx \le rM_0 \le rb_0$ as we can see $M_1 = rM_0$ and $b_1 = rb_0$. Thus for $r > 0$ $\sup(rE) = r\sup(E)$. ∎

(b) for any $r \in \mathbb{R}$, $\sup(r+E) = r + \sup(E)$

Proof: Suppose $E \subseteq \mathbb{R}$ be bounded from above and $E \neq \emptyset$. For $r \in \mathbb{R}$, let $r + E := \{r + x : x \in E\}$. By the definition of $\sup(E)$ it has to be an upper bound and a least upper bound. For it to be an upper bound, there exists an M, say M_0 such that $x \le M_0$ for all $x \in E$. For M_0 to be considered a least upper bound, for all b_0 that are an upper bound of E $M_0 \le b_0$. The combined inequality $x \le M_0 \le b_0$ where $M_0 = \sup(E)$. The $\sup(r+E)$ should be defined similarly as above with $\sup(r+E) = M_1$ and satisfy the inequality $r + E \le M_1 \le b_1$ for all $x \in E$ and b_1 that are an upper bound of $r+E$. Notice if we add r to $M_0 = \sup(E)$ and to the inequality $x \le M_0 \le b_0$ then $r + M_0 = r + \sup(E)$ and $r + x \le r + M_0 \le r + b_0$. As we can see, for $\sup(r+E)$ to be a least upper bound then $M_1 = \sup(r+E) = r + M_0 = r + \sup E$, and $b_1 = r + b_0$. Thuse for any $r \in \mathbb{R}$, $\sup(r+E) = r + \sup(E)$. ∎