MATH-331 Introduction to Real Analysis | KaiWei Tang
Homework 05 | Fall 2021

Due date: November, 22^{th} 1:20pm | Total: /65.

Exercise	1 (10)	2 (10)	3 (5)	4 (5)	5 (5)	6 (10)	7 (5)	8 (5)	9 (5)	10 (5)
Score										

Table 1: Scores for each exercises

Instructions: You must answer all the questions below and send your solution by email (to `parisepo@hawaii.edu`). If you decide to not use LaTeX to hand out your solutions, please be sure that after you scan your copy, it is clear and readable. Make sure that you attached a copy of the homework assignment to your homework.
If you choose to use LaTeX, you can use the template available on the course website.
No late homework will be accepted. No format other than PDF will be accepted. Name your file as indicated in the syllabus.

1
Writing problems

For each of the following problems, you will be asked to write a clear and detailed proof. You will have the chance to rewrite your solution in your semester project after receiving feedback from me.

Exercise 1. (10 pts)

a) Fix any $\delta > 0$ and let $[a, b]$ be an interval with $a < b$. Find a tagged partition $\mathcal{P}$ of $[a, b]$ such that $\|\mathcal{P}\| < \delta$.

b) Suppose that f is Riemann integrable. Show that in the definition of the Riemann integral, the number L is unique. [Remark: This is why we gave it the name $\int_a^b f$.]

Solution: a) Let $\mathcal{P}$ be defined as a tp. of $[a, b]$ st. $\mathcal{P} = \{c_i, [x_i, x_{i-1}]\}$ for $x_i = a + i\frac{b-a}{n}$ for n is the number of partitions between $[a, b]$. Thus, we see that by AP, $\forall x > 0$ and $\forall y > 0$, $\exists N \in \mathbb{N}$ st, $xN > y$. We see from our case that our $x = \delta$, $N = n$ and $y = b - a$ for $n\delta > b - a$. From here, we see that in the case of $x_i - x_{i-1} = a + i\frac{b-a}{n} - (a + (i-1)\frac{b-a}{n} = \frac{b-a}{n}$. Thus, each of our partitions are equal length. Thus, by how we defined our $\mathcal{P}$, let us denote $c_i = x_i$ for now we cover all cases under the interval $[a, b]$, thus, $\exists \mathcal{P} \in [a, b]$ st. $\| \mathcal{P} \| < \delta$.

b) Supppose that f is RI, then let us prove by contradiction. Suppose that $\int_a^b f = L_1$ and $\int_a^b f = L_2$. Then by definition, we know that $\int_a^b f = \lim_{\|\mathcal{P}\| \to 0} \mathcal{S}(f, \mathcal{P}) = L_1, L_2$. Then $\forall \epsilon > 0$,

$\exists \delta_1 > 0$ st. $|\mathcal{S}(f,\mathcal{P}) - L_1| < \frac{\epsilon}{2}$ and $\forall \epsilon > 0$, $\exists \delta_2 > 0$ st. $|\mathcal{S}(f,\mathcal{P}) - L_2| < \frac{\epsilon}{2}$. Then we let $\delta = \{\delta_1, \delta_2\}$ and since by assumption L_{12}, then $|L_1 - L_2| > 0$. Thus we can use ghosting and rewrite as,

$$0 < |L_1 - \mathcal{S}(f,\mathcal{P}) + \mathcal{S}(f,\mathcal{P}) - L_2| \leq |L_1 - \mathcal{S}(f,\mathcal{P})| + |\mathcal{S}(f,\mathcal{P}) - L_2| < \epsilon$$

From here, we can let $\epsilon = x$ and take the limit as $x \to 0$ of $\lim_{\|\mathcal{P}\|\to 0} 0 \leq \lim_{\|\mathcal{P}\|\to 0} L_1 - L_2 \leq \lim_{\|\mathcal{P}\|\to 0} x$, thus when we apply the limit we see that $0 \leq L_1 - L_2 \leq 0$, then we see that by squeeze theorem, $L_1 - L_2 = 0$ for $L_1 = L_2$, thus proving uniquness. $\square$

Exercise 2. (10 pts) Suppose that f and g are Riemann integrable on the interval $[a,b]$.

a) Show that $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.

b) Show that if $f(x) \leq g(x)$ for any $x \in [a,b]$, then $\int_a^b f \leq \int_a^b g$.

Solution: a) Suppose that f, g are RI., then by definition, we know that $\forall \epsilon > 0$, $\exists \delta > 0$ st. for any tp. $\mathcal{P}$ of $[a,b]$, of $\| \mathcal{P} \| < \delta$, then $|\mathcal{S}(f+g,\mathcal{P}) - \int_a^b f + g| < \epsilon$.

$$\begin{aligned}
\mathcal{S}(f+g,\mathcal{P}) &= \sum_{i=1}^{N} (f+g)(c_i)(x_i - x_{i-1}) \\
&= \sum_{i=1}^{N} (f)(c_i)(x_i - x_{i-1}) + (g)(c_i)(x_i - x_{i-1}) \\
&= \mathcal{S}(f,\mathcal{P}) + \mathcal{S}(g,\mathcal{P})
\end{aligned}$$

Thus, we see that $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.

b) By assumption, we know that $f \leq g$. Thus, by defintion, we know that if $\mathcal{S}(f,\mathcal{P}) = \sum_{i=1}^{N} (f)(c_i)(x_i - x_{i-1}$ and $\mathcal{S}(g,\mathcal{P}) = \sum_{i=1}^{N} (g)(c_i)(x_i - x_{i-1})$ then $\mathcal{S}(f,\mathcal{P}) \leq \mathcal{S}(g,\mathcal{P})$. From here, since we know that f, g are RI, we know that $\exists \delta_f > \| \mathcal{P} \|$ for $|\mathcal{S}(f,\mathcal{P}) - \int_a^b f| < \epsilon$ and $\exists \delta_g > \| \mathcal{P} \|$ for $|\mathcal{S}(g,\mathcal{P}) - \int_a^b g| < \epsilon$. Then we allow $\delta = min\{\delta_f, \delta_g\}$. If $\| \mathcal{P} \| < \delta$, then,

$$\begin{aligned}
-\epsilon < \mathcal{S}(f,\mathcal{P}) - \int_a^b f < \epsilon \\
-\epsilon < \mathcal{S}(g,\mathcal{P}) - \int_a^b g < \epsilon
\end{aligned}$$

From assumption, we supposed that $f \leq g$, then we know that by that case, $\mathcal{S}(f,\mathcal{P}) \leq \mathcal{S}(g,\mathcal{P})$. Thus we can solve for $\mathcal{S}(f,\mathcal{P})$ and obtain,

$$\begin{aligned}
\int_a^b f - \epsilon < \mathcal{S}(f,\mathcal{P}) < \epsilon + \int_a^b f \\
\int_a^b g - \epsilon < \mathcal{S}(g,\mathcal{P}) < \epsilon + \int_a^b g
\end{aligned}$$

Then by assumption we know that $\mathcal{S}(f,\mathcal{P}) \leq \mathcal{S}(g,\mathcal{P})$. This means $\int_a^b f - \epsilon \leq \mathcal{S}(g,\mathcal{P})$ and $\mathcal{S}(f,\mathcal{P}) \leq \int_a^b g + \epsilon$. This means $\int_a^b f - \epsilon \leq \int_a^b g + \epsilon$. We add the ϵ to one side and we obtain $\int_a^b f \leq \int_a^b g + 2\epsilon$. We let $\epsilon = x$ and take the limit of the entire inequality and obtain $\lim_{x\to 0} \int_a^b f \leq \lim_{x\to 0} \int_a^b g + \lim_{x\to 0} 2x$ and we see that since f, g are constants, they will remain the same while $2x = 0$ as $x \to 0$. Thus we will obtain $\int_a^b f \leq \int_a^b g$. $\square$

Exercise 3. (5 pts) Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable on $[a,b]$ and suppose that $|f(x)| \leq M$ for all $x \in [a,b]$. Show that $\int_a^b f \leq M(b-a)$.

Solution: By statement, we know that f is RI. Then $\forall \epsilon > 0$, $\exists \delta > 0$ st. for any tp. $\mathcal{P}_\backslash$ of $[a,b]$, if $\| \mathcal{P}_\backslash \| < \delta$, then $|\mathcal{S}(f,\mathcal{P}_\backslash) - \int_a^b f| < \epsilon$. Thus, we know that $\lim_{n\to\infty} ||\mathcal{P}_n|| = 0$. By definition of limits, we know that $forall \epsilon > 0$, $\exists N > 0$ st. if $n \geq N$, then $|||\mathcal{P}_n|| - L| < \epsilon$. Thus, since $L = 0$ and $||\mathcal{P}_n|| > 0$ we let $P_n < \epsilon$. From here, let $\epsilon = \delta$ for $P_n < \delta$. Then, since we know that $\lim_{n\to\infty} ||\mathcal{P}_n|| = 0$ and by the deifnition of RI, we see that there exists a $\delta > 0$ for $\lim_{||\mathcal{P}_n||\to 0} \mathcal{S}(f,\mathcal{P}_n) = \int_a^b f$. $\square$

Exercise 4. (5 pts) Suppose that f is Riemann integrable on $[a,b]$. Let $(\mathcal{P}_n)_{n=1}^\infty$ be a sequence of tagged partitions of $[a,b]$ such that the sequence $\lim_{n\to\infty} \|\mathcal{P}_n\| = 0$. Prove that the sequence $(S(f,\mathcal{P}_n))_{n=1}^\infty$ converges to $\int_a^b f$.

Solution: By assumption, we know that f is RI. This means that $\forall \epsilon > 0$, $\exists \delta > 0$ st. for any tp. $\mathcal{P}$ of $[a,b]$, if $\| \mathcal{P} \| < \delta$, then $|\mathcal{S}(f,\mathcal{P}) - \int_a^b f| < \epsilon$. Since we rewrote the tp. $\mathcal{P}$ as a sequence of $(\mathcal{P}_n)_{n=1}^\infty$, we know that $\lim_{n\to\infty} \| (\mathcal{P})_n \| = 0$. Thus, our goal is to prove that $(S(f,\mathcal{P}_n))_{n=1}^\infty \to L$ for $L = \int_a^b f$. We can use the same logic as using limits in terms of sequences. By definition, we know that f is RI, then when we apply the definition of sequences to our problem, we would obtain, if $(\| \mathcal{P} \|)_{n=1}^\infty < \delta$ then $|(S(f,\mathcal{P}_n))_{n=1}^\infty - \int_a^b f| < \epsilon$. From here, we see that if we take the limit of $(\| \mathcal{P} \|)_{n=1}^\infty$, we would obtain, $0 < \delta$, which we know it is true because δ has to be bigger than 0, and we also see that if we rewrite in terms of $-\epsilon < (S(f,\mathcal{P}_n))_{n=1}^\infty - \int_a^b f < \epsilon$, we take limit and obtain $-\epsilon < \lim_{n\to\infty} (S(f,\mathcal{P}_n))_{n=1}^\infty - \int_a^b f < \epsilon$, which proves that $(S(f,\mathcal{P}_n))_{n=1}^\infty$ converges to $\int_a^b f$ by the definition of rewriting limits in terms of sequences. $\square$

Exercise 5. (5 pts) Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Suppose that f is Riemann integrable on $[a,c]$ for any $c \in (a,b)$. Show that f is Riemann integrable on $[a,b]$. [Hint: Use the Cauchy criterion for integrals.]

Solution: From statement, we know that f is RI on $[a,c]$. This means $\forall \epsilon > 0$, $\exists \delta > 0$ st, if $\mathcal{P}$ is a tp. on $[a,c]$ then $|\mathcal{S}(f,\mathcal{P}) - \int_a^b f| < \epsilon$. By apply the hint, the Cauchy critarian states that $\forall \epsilon > 0$, $\exists \delta > 0$ st, if $\mathcal{P}_1$ and $\mathcal{P}_2$ are tp of $[a,c]$ and $||\mathcal{P}_1|| < \delta_c$ and $||\mathcal{P}_2|| < \delta_c$, then $|\mathcal{S}(f,\mathcal{P}_1) - \mathcal{S}(f,\mathcal{P}_2)| < \epsilon$. From here, let $\epsilon > 0$. Then let $c \in (a,b)$ st. $b - c < \epsilon$ for density of rational numbers. We also let $\mathcal{P}_1$ and $\mathcal{P}_2$ be tp of $[a,c]$. Then let us denote two disjoint subfamilies of $\mathcal{P}_1$ and $\mathcal{P}_2$ and call then $\mathcal{P}_{1_a}$, $\mathcal{P}_{1_b}$ and $\mathcal{P}_{2_a}$, $\mathcal{P}_{2_b}$. Let $\mathcal{P}_{1_a} = \{c_i, [x_{i-1}, x_i] \in \mathcal{P}_1 : [x_{i-1}, x_i] \subset [a,c]\}$ and $\mathcal{P}_{1_b} = \{c_i, [x_{i-1}, x_i] \in \mathcal{P}_1 : [x_{i-1}, x_i] \subset [c,b]\}$. We also let $\mathcal{P}_{2_a} = \{c_i, [x_{i-1}, x_i] \in \mathcal{P}_1 : [x_{i-1}, x_i] \subset [a,c]\}$ and

$\mathcal{P}_{2_b} = \{c_i, [x_{i-1}, x_i] \in \mathcal{P}_1 : [x_{i-1}, x_i] \subset [c, b]\}$. From here, we can now joint $\mathcal{P}_1$ and $\mathcal{P}_2$ by introducing $\tilde{\mathcal{P}}_{1_a} = \mathcal{P}_{1_a} \cup \{c, [x_{N_{1a}}, c]\}$, $\tilde{\mathcal{P}}_{1_b} = \mathcal{P}_{1_b} \cup \{c, [c, x_{N_{1a+1}}]\}$. Then by Cauchy criterion we know that $|\mathcal{S}(f, \tilde{\mathcal{P}}_{1_a}) - \mathcal{S}(f, \tilde{\mathcal{P}}_{1_b})| < \epsilon$ for $|\mathcal{S}(f, \tilde{\mathcal{P}}_{i_b})| < M(b-a) < M\epsilon$ since $\mathcal{S}(f, \tilde{\mathcal{P}}_{1_b}) = \sum_{n=1}^{N} f(c_i)(x_{i-1} - x_i) = M \sum_{n=1}^{N}(x_{i-1} - x_i) = M(b-a)$. We do the same for $\mathcal{P}_b$ and we obtain the same results for $\tilde{\mathcal{P}}_{2_a} = \mathcal{P}_{1_a} \cup \{c, [x_{N_{2a}}, c]\}$, $\tilde{\mathcal{P}}_{1_b} = \mathcal{P}_{1_b} \cup \{c, [c, x_{N_{2b+1}}]\}$ and we know that $|\mathcal{S}(f, \tilde{\mathcal{P}}_{i_b})| < M(b-a) < M\epsilon$ since $\mathcal{S}(f, \tilde{\mathcal{P}}_{2_b}) = \sum_{n=1}^{N} f(c_i)(x_{i-1} - x_i) = M \sum_{n=1}^{N}(x_{i-1} - x_i) = M(b-a)$. From here, appply triangle inequality and we see that $|\mathcal{S}(f, \mathcal{P}_1) - |\mathcal{S}(f, \mathcal{P}_1)| = |\mathcal{S}(f, \tilde{\mathcal{P}}_{1_a}) - \mathcal{S}(f, \tilde{\mathcal{P}}_{1_b}) - \mathcal{S}(f, \tilde{\mathcal{P}}_{2_a}) - \mathcal{S}(f, \tilde{\mathcal{P}}_{2_b})| \leq |\mathcal{S}(f, \tilde{\mathcal{P}}_{1_a}) - \mathcal{S}(f, \tilde{\mathcal{P}}_{2_a})| + \mathcal{S}(f, \tilde{\mathcal{P}}_{1_b}) + \mathcal{S}(f, \tilde{\mathcal{P}}_{2_b})|$. We see from the LHS of the addition sign, is RI by definition since it is within the interval $[a, c]$ and the RHS of the addition sign is also true for $|\mathcal{S}(f, \tilde{\mathcal{P}}_{i_b}| < M(b-a) < M\epsilon$. Thus, $|\mathcal{S}(f, \mathcal{P}_1) - |\mathcal{S}(f, \mathcal{P}_1)| < 2M(b-a) < 2M\epsilon$ for $\epsilon + 2M\epsilon = \epsilon(1 + 2M)$ and we let $\epsilon = \frac{\epsilon}{1+2M}$ for $|\mathcal{S}(f, \mathcal{P}_1) - |\mathcal{S}(f, \mathcal{P}_1)| < \epsilon$ for f is RI. ☐

2 — Homework problems

Answer all the questions below. Make sure to show your work.

Exercise 6. (10pts)

a) Define the function $f : [a, b] \to \mathbb{R}$ by $f(x) = k$ for every $x \in [a, b]$ where $k \in \mathbb{R}$ is a fixed constant. Show that f is Riemann integrable on $[a, b]$ and that $\int_a^b k\, dx = k(b-a)$.

b) Let $f(x) = \sin^2(x)$ where $x \in [a, b]$ and assume that the function $g(x) := \cos(kx)$ is integrable on $[a, b]$ for any $k \in \mathbb{R}$. Show that f is Riemann integrable on $[a, b]$.

Solution: a)We know that by definition, $\int_a^b f = \lim_{\|\mathcal{P}\| \to 0} \mathcal{S}(f, \mathcal{P}) = \sum_{i=1}^{N} f(c_i)(x_i - x_{i-1})$. Thus if we were to replace $f(c_i)$ with our $f(x)$, we see that since $f(x) = k$ for k is a constant, we would obtain, $\int_a^b k = \lim_{\|\mathcal{P}\| \to 0} \sum_{i=1}^{N} k(x_i - x_{i-1})$. Since f is a constant, we can reduce our summation intervals of $x_i - x_{i-1}$ to $b - a$. Then, $\int_a^b k = \lim_{\|\mathcal{P}\| \to 0} k(b-a)$. Since a, b, and k are constants, our limit will equal $k(b-a)$. From here, we know that f is bounded by $M(b-a)$. Therefore let $M = k$ and our area is just $k(b-a)$ for $\int_a^b k = k(b-a)$. Since we know that know $\int_a^b f = L$ and each $\mathcal{S}(f, \mathcal{P})$ has a unique limit, we know that f is RI on $[a, b]$ for our L is unique.

b) By logic of 6a, if we can prove that there exists a unique L for $\int_a^b f = L$ then f is RI for every function has a unique L. Suppose that f is RI, then, $\forall \epsilon > 0$, $\exists \delta > 0$ st. if $\mathcal{P}$ is a tp. and $||\mathcal{P}|| < \delta$ on $[a, b]$, then $|\mathcal{S}(f, \mathcal{P}) - \int_a^b f|$. From here, we apply the definition of an integral and let $\int_a^b f = \lim_{||\mathcal{P}|| \to 0} \mathcal{S}(f, \mathcal{P})$ where,

$$
\begin{aligned}
\mathcal{S}(f,\mathcal{P}) &= \sum_{i=1}^{N} f(c_i)(x_{i-1}-x_i) \\
&= \sum_{i=1}^{N} sin^2(x)(x_{i-1}-x_i) \\
&= \sum_{i=1}^{N} \frac{1-cos(2x)}{2}(x_{i-1}-x_i) \\
&= \frac{1}{2}\sum_{i=1}^{N}(1-cos(2x))(x_{i-1}-x_i) \\
&= \frac{1}{2}\sum_{i=1}^{N}(1-cos(2x))(b-a) \\
&= \frac{1}{2}\sum_{i=1}^{N}(b-a) - \frac{1}{2}\sum_{i=1}^{N} cos(2x)(b-a)
\end{aligned}
$$

From here, we plug it back into the limit and it becomes,

$$
\begin{aligned}
\int_a^b f &= \lim_{||\mathcal{P}||\to 0} \frac{1}{2}\sum_{i=1}^{N}(b-a) - \frac{1}{2}\sum_{i=1}^{N} cos(2x)(b-a) \\
\int_a^b f &= \lim_{||\mathcal{P}||\to 0} \frac{1}{2}\sum_{i=1}^{N}(b-a) - \lim_{||\mathcal{P}||\to 0} \frac{1}{2}\sum_{i=1}^{N} cos(2x)(b-a)
\end{aligned}
$$

Observe that on the LHS of the subtraction sign is RI by 6a for it is just a constant. The RHS of the subtraction sign is also RI from our assumption of $g(x)$. Thus, we see that there exists a unique L for $\int_a^b f = L$ for f is RI. $\square$

Exercise 7. (5 pts) Show that the function $f : [0,1] \to \mathbb{R}$ defined by

$$
f(x) := \begin{cases} 1 & \text{, if } 0 \le x < 1/2 \\ 0 & \text{, if } 1/2 \le x \le 1 \end{cases}
$$

is Riemann integrable on $[0,1]$.

Solution: Suppose that our function f is RI, then, if $\exists L \in \mathbb{R}$ st. $\forall \epsilon > 0$, $\exists \delta > 0$ st. if $\mathcal{P}$ is a tagged partition, of $[a,b]$, with $\parallel \mathcal{P} \parallel < \delta$, then, $|\mathcal{S}(f,\mathcal{P}) - L| < \epsilon$. In our case, we see that by knowldge from calculus one, the value of the integral $\int_0^{\frac{1}{2}} f(x)\,dx = \frac{1}{2}$ for $\int_{\frac{1}{2}}^{1} f(x)\,dx = 0$ for $\frac{1}{2} + 0 = \frac{1}{2}$. Thus, let us rewrite $\mathcal{P} = \{(c_i, [x_{i-1}, x_i]) : i = 1, 2, ..., N\}$ for we know that the Riemman sum is $\mathcal{S}(f,\mathcal{P}) = \sum_{i=1}^{N} f(c_i)(x_i - x_{i-1})$. From here, we let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two collections of tagged partitions such that $\mathcal{P}_1 = \{(c_i, [x_{i-1}, x_i]) : c_i \in [0, \frac{1}{2}]\}$ and $\mathcal{P}_2 = \{(c_i, [x_{i-1}, x_i]) : c_i \in [\frac{1}{2}, 1]\}$. Then we know that $\mathcal{S}(f,\mathcal{P}) = \mathcal{S}(f,\mathcal{P}_1) + \mathcal{S}(f,\mathcal{P}_2)$. Let $N = card(\mathcal{P}_1)$ so that we have,

$$\begin{aligned}
\mathcal{S}(f,\mathcal{P}_1) &= \sum_{i=1}^{N_1} f(c_i)(x_i - x_{i-1}) \\
&= 1(x_i - x_{i-1}) \\
&= 1[x_{N_1} - x_0] \\
&= 1x_{N_1}
\end{aligned}$$

and if $N_2 = card(\mathcal{P}_2)$, then,

$$\begin{aligned}
\mathcal{S}(f,\mathcal{P}_2) &= \sum_{i=1}^{N_2} f(c_i)(x_i - x_{i-1}) \\
&= 0(x_i - x_{i-1}) \\
&= 0[x_{N_2} - x_{N_1}] \\
&= 0
\end{aligned}$$

So we have $\mathcal{S}(f,\mathcal{P}) = \frac{1}{2}x_{N_1}$ and let us use ghosting and obtain $\mathcal{S}(f,\mathcal{P}) = \frac{1}{2}x_{N_1} - \frac{1}{2} + \frac{1}{2}$ and rewrite as $\mathcal{S}(f,\mathcal{P}) - \frac{1}{2} = \frac{1}{2}x_{N_1} - \frac{1}{2}$. $|\mathcal{S}(f,\mathcal{P}) - \frac{1}{2}| = |x_{N_1} - \frac{1}{2}|$ and since we know that $\parallel \mathcal{P} \parallel < \delta$, then $|\mathcal{S}(f,\mathcal{P}) - \frac{1}{2}| < \delta$. Let $\delta = \varepsilon$ then we get $|\mathcal{S}(f,\mathcal{P}) - \frac{1}{2}| = |x_{N_1} - \frac{1}{2}| < \delta = \varepsilon$. $\square$

Exercise 8. (5 pts) Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = 1$ if $x = 1/n$ where $n \in \mathbb{N}$, and by $f(x) = 0$ if $x \neq 1/n$, $n \in \mathbb{N}$. Show that f is Riemann integrable on $[0,1]$.

Solution: Let us do this problem with the similar logic of number 5. Apply the Cauchy Criterion. By definition, we know that $\forall \epsilon > 0$, $\exists \delta > 0$ st, if $\mathcal{P}_1$ and $\mathcal{P}_2$ are tp of $[a,c]$ and $||\mathcal{P}_1|| < \delta_c$ and $||\mathcal{P}_2|| < \delta_c$, then $|\mathcal{S}(f,\mathcal{P}_1) - \mathcal{S}(f,\mathcal{P}_2)| < \epsilon$. Thus, let us create two partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ for they are tp. of $[a,b]$ and introduce a $c \in [a,b]$ for f is 1 at $\frac{1}{n}$ and we know that $\frac{1}{n}$ is between $0 < \frac{1}{n} \leq 1$ for $\frac{1}{n} > c > \frac{1}{N}$. Thus, we know that from $[0,c]$ is RI. We will show that from $[c,1]$ is RI as well, then by Cauchy criterion, suppose f is RI on $[a,b]$. Denote two disjooint intervals,

$$\begin{aligned}
\mathcal{P}_{1_a} &= \{c_i, [x_{i-1}, x_i] \in \mathcal{P}_1 : [x_{i-1}, x_i] \subset [0,c]\} \\
\mathcal{P}_{1_b} &= \{c_i, [x_{i-1}, x_i] \in \mathcal{P}_1 : [x_{i-1}, x_i] \subset [c,1]\}
\end{aligned}$$

For $\mathcal{P}_1$ and do the same thing for $\mathcal{P}_2$ and denote two disjoint intervals,

$$\begin{aligned}
\mathcal{P}_{2_a} &= \{c_i, [x_{i-1}, x_i] \in \mathcal{P}_2 : [x_{i-1}, x_i] \subset [0,c]\} \\
\mathcal{P}_{2_b} &= \{c_i, [x_{i-1}, x_i] \in \mathcal{P}_2 : [x_{i-1}, x_i] \subset [c,1]\}
\end{aligned}$$

Then introduce,

$$\tilde{\mathcal{P}}_{1_a} = \mathcal{P}_{1_a} \cup \{c, [x_{N_{1a}}, c]\}$$
$$\tilde{\mathcal{P}}_{1_b} = \mathcal{P}_{1_b} \cup \{c, [c, x_{N_{1a}+1}]\}$$

and

$$\tilde{\mathcal{P}}_{2_a} = \mathcal{P}_{2_a} \cup \{c, [x_{N_{2b}}, c]\}$$
$$\tilde{\mathcal{P}}_{2_b} = \mathcal{P}_{2_b} \cup \{c, [c, x_{N_{2b}+1}]\}$$

Then by logic of 5, $|\mathcal{S}(f, \mathcal{P}_1) - |\mathcal{S}(f, \mathcal{P}_1)| < 2M(b-a) < 2M\epsilon$ for $\epsilon + 2M\epsilon = \epsilon(1+2M)$ and we let $\epsilon = \frac{\epsilon}{1+2M}$ for $|\mathcal{S}(f, \mathcal{P}_1) - |\mathcal{S}(f, \mathcal{P}_1)| < \epsilon$ for f is RI. $\square$

Exercise 9. (5 pts) Show that the function $f : [0, 1] \to \mathbb{R}$ defined by $f(x) = 0$ if $x \neq 0$ and $f(x) = 4$ if $x = 0$ is Riemann integrable on $[0, 1]$.

Solution: Suppose that our function is RI. Then we know that if $\forall L \in \mathbb{R}$ st. $\forall\epsilon > 0$, $\exists\delta > 0$ st. if $\mathcal{P}$ is a tagged partition, of $[a, b]$, with $\parallel \mathcal{P} \parallel < \delta$, then, $|\mathcal{S}(f, \mathcal{P}) - L| < \epsilon$. We can see this as a step functions of

$$f(x) := \begin{cases} 0 & \text{, if } x \neq 0 \\ 4 & \text{, if } x = 0 \end{cases}$$

This means we can use the definition of step functions and say that if $\sum_{k=1}^{N} C_k \ell(I_k)$ is a step function, on $[a, b]$, then ϕ is RI on $[a, b]$ and $\int_a^b \phi = \sum_{k=1}^{N} C_k \ell(I_k)$. From here, we see that if we were to compute for our integral, we would obtain that our $L = 0$ by using the sum formula,

$$\begin{aligned} &= \sum_{k=1}^{N} C_k \ell(I_k) \\ &= 0(1-0) \\ &= 0 \end{aligned}$$

We see that our sum is 0 when $C_k = 0$. When $C_k = 4$, we see that same thing for,

$$\begin{aligned} &= \sum_{k=1}^{N} C_k \ell(I_k) \\ &= 4(0-0) \\ &= 0 \end{aligned}$$

Thus, our total sum is 0 for there exists $L = 0$. Then we see that plugging this back into our definition of RI, $\forall L \in \mathbb{R}$ st. $\forall\epsilon > 0$, $\exists\delta > 0$ st. if $\mathcal{P}$ is a tagged partition, of $[a, b]$, with $\parallel \mathcal{P} \parallel < \delta$, then, $|\mathcal{S}(f, \mathcal{P})| < \epsilon$. Therefore, since we know there there exists a unique L, for $L = 0$, then our $\lim_{||\mathcal{P}|| \to 0} = 0$ for our solution is a constant. By 6a, we know that any constant is RI, then f must be RI on $[a, b]$. $\square$

Exercise 10. (5 pts) Let $\mathcal{P}$ be the following tagged partition of $[-1, 2]$:

$$\mathcal{P} := \{(-9, [-1, -.8]), (-.7, [-.8, -.3]), (-.1, [-.3, 0]), (.2, [0, 0.2]), (.2, [.2, .4]), (.8, [.4, 1]), (1.42, [1, 1.5]), (1.9, [1.5, 2])\}.$$

Find another partition $\mathcal{P}_0$ such that $\|\mathcal{P}_0\| \leq \|\mathcal{P}\|/3$.

Solution: $[-0.9, [-1, -0.8]], [-0.7, [-0.8, -0.6]], [-0.5, [-0.6, -0.4]], [-0.3, [-0.4, -0.2]],$ $[-0.1, [-0.2, 0]], [0.1, [0, 0.2]], [0.3, [0.2, 0.4]], [0.5[0.4, 0.6]], [0.7, [0.6, 0.8]], [0.9, [0.8, 1]],$ $[1.1, [1, 1.2]], [1.3,]1.2, 1.4]], [1.5[1.4, 1.6]], [1.7, [1.6, 1.8]], [1.85, [1.8, 1.9]], [1.95, [1.9, 2]].$ $\square$