Real Analysis HW #6

1 a.) $b-a$ is length of segment, set $l := \min\{N, b-a\}$ where $\frac{b-a}{N} < \delta$ (by A.P.) then a partition with lengths $\frac{b-a}{l}$, the $\|P\|$ will be less than δ

b.) Suppose $\int_a^b f_1 \neq \int_a^b f_2$ are the two RI values of f. f is RI, so $\|P\| < \delta_1$, $|S(P,f) - \int_a^b f_1| < \varepsilon/2$ and $\|P\| < \delta_2$ $|S(P,f) - \int_a^b f_2| < \varepsilon/2$

Set $\delta := \min\{\delta_1, \delta_2\}$ And $|\int_a^b f_1 - \int_a^b f_2| \neq 0$

$\|P\| < \delta$, then $|\int_a^b f_1 - \int_a^b f_2| = |\int_a^b f_1 - S(f,P) + S(f,P) - \int_a^b f_2|$

$\leq |S(f,P) - \int_a^b f_1| + |S(f,P) - \int_a^b f_2| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$

set $\varepsilon = \frac{|\int_a^b f_1 - \int_a^b f_2|}{2}$

$|\int_a^b f_1 - \int_a^b f_2| < \frac{|\int_a^b f_1 - \int_a^b f_2|}{2}$ $\qquad 1 < \frac{2}{2} \qquad 2 < 1$ #

$\int_a^b f$ is unique

2 a.) $\|P\| < \delta_1 \Rightarrow |S(f,P) - \int_a^b f| < \varepsilon/2$ $\quad \|P\| < \delta_2 \Rightarrow |S(g,P) - \int_a^b g| < \varepsilon/2$

set $\delta = \min\{\delta_1, \delta_2\}$ then $\|P\| < \delta$ $|S(f+g,P) - \int_a^b f - \int_a^b g| < \varepsilon$

$S(f+g, P) = \sum_{i=1}^N [f+g](c_i)(x_i - x_{i-1}) = \sum_{i=1}^N f(c_i)(x_i - x_{i-1}) + \sum_{i=1}^N g(c_i)(x_i - x_{i-1})$

number so distribute

$S(f+g, P) = S(f,P) + S(g,P) \rightarrow |S(f,P) - \int_a^b f + S(g,P) - \int_a^b g| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$

2 b) $f(x) \leq g(x)$ $\int_a^b f \leq \int_a^b g$:

$S(f,p) \leq S(g,p) \to \sum_{r=1}^n f(c_i)(x_i - x_{i-1}) \leq \sum_{i=1}^n g(c_i)(x_i - x_{i-1})$ because $f \leq g$

Let $\xi = \frac{\int_a^b f - \int_a^b g}{2}$. $\|P\| < \delta_1 \to |S(f,P) - \int_a^b f| < \frac{\int_a^b f + \int_a^b g}{2}$:b

$\|P\| < \delta_2 \to |S(g,P) - \int_a^b g| < \frac{\int_a^b f + \int_a^b g}{2}$

take $\delta := \min\{\delta_1, \delta_2\}$

then $-\frac{\int_a^b f + \int_a^b g}{2} < S(f,P) - \int_a^b f < \frac{\int_a^b f + \int_a^b g}{2}$

then $-\frac{\int_a^b f + \int_a^b g}{2} < S(g,P) - \int_a^b g < \frac{\int_a^b f + \int_a^b g}{2}$

$\frac{\int_a^b f + \int_a^b g}{2} < S(f,P)$ $S(g,P) < \frac{\int_a^b f + \int_a^b g}{2}$

so $S(g,P) < \frac{\int_a^b f + \int_a^b g}{2} < S(f,P)$

Which is a contradiction of $S(f,P) \leq S(g,P)$ #

3 We know that $\int_a^b k = k(b-a)$ for some constant k. If we set $M = k$ then $\int_a^b M = M(b-a)$. We know that $|f(x)| \leq M$ so, from #2b, we know that $\int_a^b f \leq \int_a^b M = M(b-a)$ so $\int_a^b f \leq M(b-a)$

4 $\lim_{n \to \infty} \|P_n\| \to 0$ $|\|P_n\| - 0| < \varepsilon$ $\forall \varepsilon > 0, \exists N$ s.t. $n \geq N$ then $\|P_n\| < \varepsilon$

$n \geq N$ then $|S(f,P_N) - \int_a^b f| < \varepsilon$ $S(f,P_N) = \sum_{i=1}^N S(c_i)(x_i - x_{i-1})$

$(x_i - x_{i-1}) \to 0$ so 0 $|0 - \int_a^b f| < \varepsilon$ $\int_a^b f < \varepsilon$

6 a.) Let P be tagged partition with $\|P\| < \delta$.

$S(f,P) = \sum_{i=1}^{N} f(c_i)(x_i - x_{i-1})$ $f(c_i) \; \forall c_i \in c_1, \ldots, c_N = k$

$S(f,P) = \sum_{i=1}^{N} k(x_i - x_{i-1})$ since P is tagged partition,

$\sum_{i=1}^{N} (x_i - x_{i-1}) = (b-a)$ $\sum_{i=1}^{N} k(x_i - x_{i-1}) = k\sum_{i=1}^{N}(x_i - x_{i-1}) = k(b-a)$

So $|S(f,P) - \int_a^b f| < \varepsilon$ for $\varepsilon > 0$ $|S(f,P) - k(b-a)| = |k(b-a) - k(b-a)|$

$= 0 < \varepsilon$ so $|S(f,P) - \int_a^b f| < \varepsilon$ so f is RI on $[a,b]$

and $\int_a^b f = k(b-a)$

b.) $\int_a^b \sin^2(x)$ $\sin^2(x) = \frac{1-\cos(2x)}{2} = \frac{1}{2} - \frac{1}{2}\cos(2x)$

$\int_a^b \frac{1}{2} - \frac{1}{2}\cos(2x) \to \int_a^b \frac{1}{2} + \int_a^b -\frac{1}{2}\cos(2x)$ $2 \in \mathbb{R}$ so RI

$\frac{1}{2}(a,b) - \frac{1}{2}\int_a^b \cos(2x)$ so $\int_a^b \sin^2(2x)$ is RI on $[a,b]$

7 Let P be t.p. of $[0,1]$ s.t. $\|P\| < \delta$.

Say $\int_a^1 f = \frac{1}{2}$ then $|S(f,P) - \frac{1}{2}| < \varepsilon$

$S(f,P) = \sum_{i=1}^{N} f(c_i)(x_i - x_i)$ If $c_i \geq \frac{1}{2}$, then $f(c_i) = 0$ and

if $c_i < \frac{1}{2}$, then $f(c_i) = 1$ so $S(f,P) = \sum_{i=1}^{N}(x_{i-1} - x_i) = \frac{1}{2} - 0$

$= \frac{1}{2}$ so $|\frac{1}{2} - \frac{1}{2}| < \varepsilon$ $0 < \varepsilon$

ε was arbitrary. f is RI on $[0,1]$.

8 Say $\int_0^1 f = 0$ b.c. more $\mathbb{R}$ than $\mathbb{N}$. P be t.p. $[0,1]$ s.t.

$\|P\| < \delta$ then $|S(f,P)| < \varepsilon$ $S(f,P) = \sum_{i=1}^{N} f(c_i)(x_{i-1} - x_i)$

$E_\varepsilon := \{x = \frac{i}{N} : \frac{i}{N} \geq \frac{\varepsilon}{2}\}$ for $x \in E_\varepsilon$ then $N \leq \frac{2}{\varepsilon}$ there is a N_0 s.t. $N_0 = \lfloor \frac{2}{\varepsilon} \rfloor$

Since set $[0,1]$ is finite, E_ε is finite. $P_0 \subseteq P$ s.t. $P_0 = \{(c_i : [x_{i-1}, x_i], f(c_i) \geq \frac{\varepsilon}{2}\}$

$P = P | P_0 \cup P_0$, P_0 is finite bc E_ε is. $S(f,P) = S(f,P | P_0) \cup S(f,P_0)$

$S(f,P) = \sum_{i=1}^{N} f(c_i)(x_{nk} - x_{n(k-1)}) \frac{\delta}{N}$ $S(f,P_0|P_0) = \sum_{i=1}^{N} f(c_i)(x_{i-1} - x_i) < \frac{\varepsilon}{2}$

$\delta = \frac{N\varepsilon}{2}$ $|S(f,P)| = S(f,P_0) + S(f,P|P_0) < \frac{\varepsilon}{2} + \frac{\delta}{N} = \frac{\varepsilon}{2} + \frac{N\varepsilon}{2N} = \varepsilon$ So $\int_0^1 f = 0$

4 Let P be a t.p. of $[0,1]$ s.t. $\|P\| < \delta$. then

say $\int_0^1 f = 0$. Then $|S(f,P) - 0| < \varepsilon \Rightarrow |S(f,P)| < \varepsilon$

$S(f,P) = \sum_{i=1}^{N} f(c_i)(x_{i-1} - x_i)$. if $c_i > 0$, then $f(c_i) = 0$

so $\sum_{i=1}^{N} 4 \ (0-0) = 0$ so $S(f,P) = 0$

$0 < \varepsilon$ ε w.r.t arbitrary so f is RI on $[0,1]$

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10 $\|P\| = 0.2$ So $\|P_0\| \le \frac{0.2}{3}$ Have 18 partitions
from -1 to 2 that are $\frac{0.2}{3}$ units long so that
$\|P_0\| = \frac{0.2}{3}$.

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5 $c = b - \varepsilon$ $\|P_1\| < \delta$ $|S(f,P_1) - \int_a^c f| < \varepsilon$ P_1 t.p. of $[a,c]$

P_2 t.p. of $[a,b] = [a, c+\varepsilon]$ $S(f,P_2) = S(f,P_1) + f(c_b)(c+\varepsilon-c)$

$S(f,P_2) = S(f,P_1) + f(c_b)\varepsilon$

$|S(f,P_2) - \int_a^b f| < \varepsilon$ $|S(f,P_1) + \varepsilon f(c_b) - \int_a^b f| \le |S(f,P_1) - \int_a^b f| - \varepsilon f(c_b)|$

$|S(f,P_1) - \int_a^b f| < \varepsilon(1 + f(c_b))$

Set ε in beginning to $\frac{\varepsilon}{1 + f(c_b)}$

ε arbitrary so f is RI on $[a,b]$.