Limit of Functions

- Def. of limit
- Limit of fct. & seq.
- Algebra of limits.
- Limit of monotone fct.

3- Limits of functions.

Def. A set $P \subseteq \mathbb{R}$ is a <u>neighborhood</u> of a pt. $x \in \mathbb{R}$ if $\exists \delta > 0$ s.t. $(x-\delta, x+\delta) \subseteq P$.

$(x-\delta, x+\delta)$ fn $\delta > 0$ is a neighborhood of x.

Thm. Let $S \subseteq \mathbb{R}$: Then $x_0 \in acc(E)$ iff. $\exists (x_n)$, $x_n \in S$, $x_n \neq x_0$ s.t. $x_n \longrightarrow x_0$.

Proof. Let $x_0 \in acc(E)$. Choose $\delta := \frac{1}{n}$.
Then, $\forall n \geq 1$, $\exists x_n \in S$ s.t. $|x_n - x_0| < \frac{1}{n}$.
$x_n \neq x_0$ $\forall n \geq 1$.
[$(x_0 - \frac{1}{n}, x_0 + \frac{1}{n}) \cap S$ contain inf. many element].
Let $\varepsilon > 0$. Then $\exists N \in \mathbb{N}$ s.t. $\frac{1}{N} < \varepsilon$ (by AP).

$$n \geq N \Rightarrow |x_n - x_0| < \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

So $x_n \longrightarrow x_0$.

From the other implication, $(x-\varepsilon, x+\varepsilon) \cap S$ will contain infinitely many x_n $(n \geq N)$. $\square$

2.1 Def. of limits.

We want to formalize the notion of

" x close to x_0, the $f(x)$ close to L ".

we will use a similar def. of the sequence.

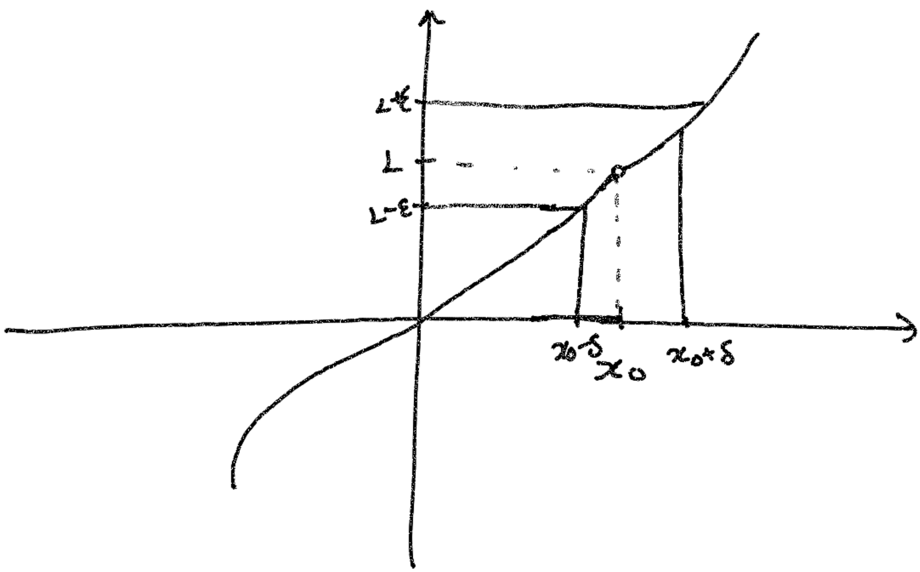
Def. Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ and $x_0 \in \text{acc}(D)$.
f has a limit at x_0 iff $\exists L \in \mathbb{R}$ s.t. $\forall \varepsilon > 0$

$$\exists \delta > 0 \text{ s.t. } \underset{x \in D}{0 < |x - x_0| < \delta} \Rightarrow |f(x) - L| < \varepsilon.$$

Other words. $\exists L \in \mathbb{R}$, $\forall \varepsilon > 0$

$\exists \delta > 0$ s.t. $x \in (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$, $x \in D$

$\Rightarrow f(x) \in (L - \varepsilon, L + \varepsilon)$.



Notation
- $\lim\limits_{x \to x_0} f(x) = L$
- $\lim f(x) = L$ (x_0 is clear)
- $f(x) \to L$ as $x \to x_0$.

Examples ① $f(x) = \dfrac{x^2 - 1}{x - 1}$, compute $\lim\limits_{x \to 1} f(x)$.

② $f(x) = \dfrac{|x|}{x}$, $x \neq 0$. $\lim\limits_{x \to 0} f(x)$?

~~③~~ $f(x) = \sin\left(\dfrac{1}{x}\right)$, $x \in (0,1)$. $\lim\limits_{x \to 0} f(x)$?

③ $f(x) = 1/x$, $x \in (0,1)$. $\lim\limits_{x \to 0} f(x)$.

④ $f : [0,1] \to \mathbb{R}$, $f(x) = \begin{cases} 0, & x \in \mathbb{R} \setminus \mathbb{Q} \\ \dfrac{P}{q}, & x = \dfrac{p}{q} \ \gcd(p,q) = 1. \end{cases}$

[$\varepsilon > 0$, $\exists\, q_0 \in \mathbb{N}$ s.t. $\dfrac{1}{q_0} < \varepsilon$.

There are finite number of $\dfrac{p}{q}$ s.t. $q < q_0$.

Call them $r_1, r_2, \ldots, r_n$.

Assume x_0 is not in the previous list.

Let $\delta := \min \{ |x_0 - r_i| : i = 1, 2, \ldots, n \}$.

Now, if $|x - x_0| < \delta$, then $x \neq r_i$, $i = 1, \ldots, n$.

- $x \in \mathbb{Q} \cap [0,1]$, $x = \dfrac{p}{q}$. But $q \geq q_0$ so

$$|f(x) - 0| = \frac{1}{q} \leq \frac{1}{q_0} < \varepsilon .$$

- $x \in [0,1] \setminus \mathbb{Q}$, then $|f(x) - 0| = 0 < \varepsilon$.

Thus, $f(x) \to 0$, $x \to x_0$.]

2.2. Limits of functions & seq.

<u>Thm.</u> $f : D \subseteq \mathbb{R} \to \mathbb{R}$ and $x_0 \in acc(D)$.
f has a limit at x_0 iff. $\forall (x_n)$, $x_n \in D$, $x_n \to x_0$, $x_n \neq x_0$ $\forall n \geq 1$, $(f(x_n))$ converges.

<u>Proof.</u>

($\Rightarrow$) Suppose $\lim f(x) = L$ and let (x_n) s.t.

$x_n \in D$, $x_n \to x_0$, $x_n \neq x_0$, $\forall n \geq 1$.

Let $\varepsilon > 0$. Then $\exists \delta > 0$ s.t.

$$0 < |x - x_0| < \delta \implies |f(x) - L| < \varepsilon. \quad (*)$$

Now, $x_n \to x_0$. So $\exists N \in \mathbb{N}$ s.t.

$$|x_n - x_0| < \delta.$$

So, by (*), $|f(x_n) - L| < \varepsilon$ if $n \geq N$.

So $f(x_n) \to L$.

($\Leftarrow$) Suppose (*) is true, but f has no limit. Then, $\forall L \in \mathbb{R}$, $\exists \varepsilon > 0$ s.t. $\forall \delta > 0$ $\exists x_L \in (x_0 - \delta, x + \delta)$, $x_L \neq x_0$ s.t.

$$|f(x) - L| \geq \varepsilon$$

Take $\delta = \frac{1}{n}$. Then $\exists x_n \in (x_0 - \frac{1}{n}, x_0 + \frac{1}{n}) \cap D$ s.t. $x_n \neq x_0 \ \forall n \geq 1$ but

$$|f(x_n) - L| \geq \varepsilon.$$

So, by construction, $x_n \longrightarrow x_0$ but

$$|f(x_n) - L| \geq \varepsilon \quad \forall L \in \mathbb{R}.$$

So, $(f(x_n))$ doesn't converge, $\#$. $\blacksquare$

<u>Corollary.</u> $f : D \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ & $x_0 \in \mathrm{acc}(D)$.

If $\forall (x_n)$, $x_n \in D$, $x_n \neq x_0 \ \forall n \geq 1$, $x_n \longrightarrow x_0$, $(f(x_n))_{n=1}^{\infty}$ is a Cauchy seq., then f has a limit at x_0.

<u>Thm.</u> $f : D \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ & $x_0 \in \mathrm{acc}(D)$. If f has a limit at x_0, then $\exists \delta > 0$ & $\exists M > 0$ s.t. $|f(x)| \leq M \quad \forall x \in (x_0 - \delta, x_0 + \delta) \cap D \setminus \{x_0\}$.

<u>Proof.</u> Let $\lim_{x \to x_0} f(x) = L$. For $\varepsilon = 1$, $\exists \delta > 0$ s.t.

$\forall x \in D$, $0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < 1$.

So, for these x, $|f(x)| < \underbrace{1 + |L|}_{M}$. $\square$

This is why the function $f(x) = \frac{1}{x}$ can't have a limit at $x_0 = 0$.

2.3. Algebra of limits

Def. $f, g : D \subseteq \mathbb{R} \to \mathbb{R}$ be two function.

a) $(f+g)(x) := f(x) + g(x) \quad \forall x \in D.$

b) $(fg)(x) := f(g(x)) \quad \forall x \in D.$

c) $\left(\frac{f}{g}\right)(x) := \frac{f(x)}{g(x)} \quad \forall x \in D \left(\begin{array}{c} g(x) \neq 0 \\ \forall x \in D \end{array}\right).$

Thm. Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ & $g : D \subseteq \mathbb{R} \to \mathbb{R}$ & $x_0 \in \text{acc}(D)$. Suppose f & g have a limit at x_0.

(a) $f+g$ has a limit at x_0 &

$$\lim (f+g)(x) = \lim f(x) + \lim g(x).$$

(b) fg has a limit at x_0 &

$$\lim (fg)(x) = \left(\lim f(x)\right)\left(\lim g(x)\right).$$

(c) if $g(x) \neq 0 \ \forall x \in D$, $\lim g(x) \neq 0$, then f/g has a limit at x_0 and

$$\lim \left(\frac{f}{g}\right)(x) = \frac{\lim f(x)}{\lim g(x)}.$$

Proof:

(a) Use the sequence characterization.

(b) Use the δ-ε definition.

(c) Use the sequence characterization. $\square$

Examples of functions

① $f(x) = x$, $x \in \mathbb{R}$. $\lim_{x \to x_0} x = x_0$.

② $f(x) = k$, $x \in \mathbb{R}$, $k \in \mathbb{R}$. $\lim_{x \to x_0} k = k$.

③ $f(x) = x^n$, $x \in \mathbb{R}$, $n \geq 1$. $\lim_{x \to x_0} x^n = x_0^n$.

④ $f(x) = a_n x^n + \dots + a_1 x + a_0$, $x \in \mathbb{R}$, $a_i \in \mathbb{R}$.

$$\lim_{x \to x_0} (a_n x^n + \dots + a_1 x + a_0) = a_n x_0^n + \dots + a_0.$$

⑤ $f(x) = \dfrac{p(x)}{q(x)}$, p, q are polynomials and

$D := \{x \in \mathbb{R} : q(x) = 0\}$.

For $x \in D$, $\displaystyle\lim_{x \to x_0} \frac{p(x)}{q(x)} = \frac{p(x_0)}{q(x_0)}$.

⑥ $f : D \subseteq \mathbb{R} \to [0, \infty)$, $x_0 \in \operatorname{acc}(D)$. If $\lim f(x) = L$,

then $\displaystyle\lim_{x \to x_0} \sqrt{f(x)} = \sqrt{L}$.

Example Consider $f(x) = \dfrac{\sqrt{4+x} - 2}{x}$, $x \in (0,1)$.

Find $\lim\limits_{x \to 0} f(x)$.

2.4 Limits of monotone functions

Example Let $f(x) = [x]$ where

$[x]$: largest integer less than x.

Here, $\left[\frac{3}{2}\right] = 1$, $[\pi] = 3$, $[-2] = -1$.

For $x_0 \in [n, n+1)$, $f(x) = n$, and so is constant on each interval $(n, n+1)$. So, f will have a limit,

$$\lim_{x \to x_0} f(x) = n.$$

Now, if $x_0 = m \in \mathbb{Z}$, then consider

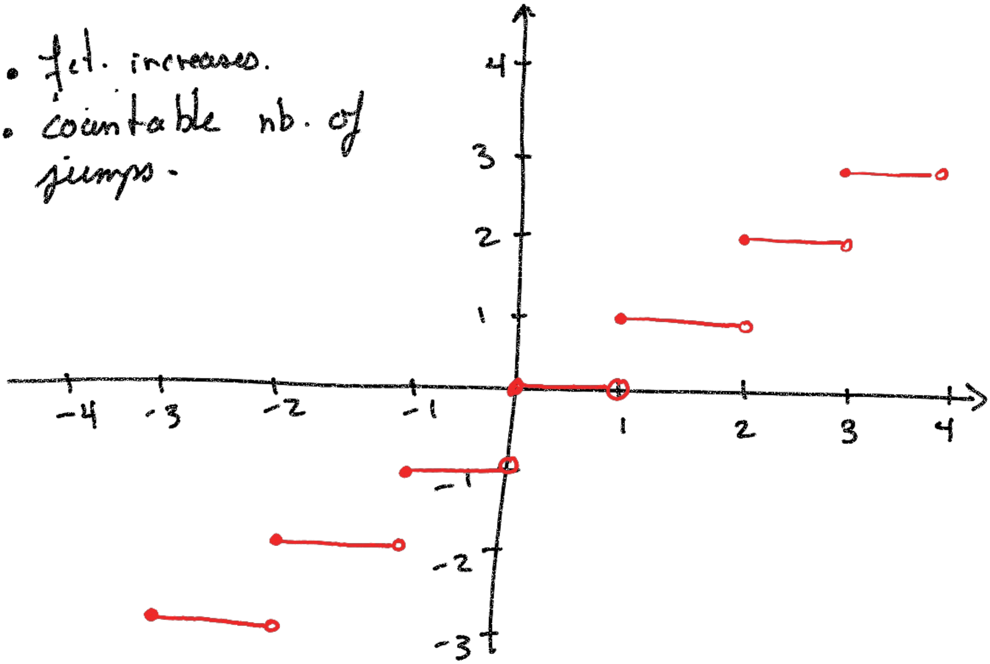
$$x_n = x_0 + \left(\frac{1}{n}\right)(-1)^n \quad , n \geq 1.$$

Then, $x_n \to x_0$, but

- $n = 2k$, $f(x_0 + \frac{1}{n}) = m \xrightarrow[n \to \infty]{} m$
- $n = 2k+1$, $f(x_0 - \frac{1}{n}) = m-1 \longrightarrow m-1$

So, $(f(x_n))_{n=1}^{\infty}$ doesn't converge $\Rightarrow$ f doesn't have a limit at all $x_0 \in \mathbb{Z}$.

- fct. increases.
- countable nb. of jumps.



Def. $f: D \subseteq \mathbb{R} \to \mathbb{R}$. f is said to be

(a) <u>increasing</u> if $x \le y$, then $f(x) \le f(y)$ $\forall x \in D$.

(b) <u>decreasing</u> if $x \le y$, then $f(x) \ge f(y)$ $\forall x \in D$.

(c) <u>monotone</u> if f is increasing or decre.

Consider $f: [\alpha, \beta] \longrightarrow \mathbb{R}$ & f increasing.
Then

$$f(\alpha) \le f(x) \le f(\beta) \quad \forall x \in [\alpha, \beta].$$

So, for $\alpha < x < \beta$, define

$U(x) = \inf\{f(y) : x < y\}$, $L(x) := \sup\{f(y) : y < x\}$

By the AC, $U(x)$ & $L(x)$ are well-define $\forall x$.

$U(x) \geq L(x)$.

Here $U(x) - L(x)$ measures the jump of f at x. [talk about $[x]$]

$y_1 < x < y_2 \Rightarrow f(y_1) < f(y_2)$
$L(x) \leq f(y_2) \leq U(x)$

Lemma. Let $f : [\alpha, \beta] \to \mathbb{R}$ be increasing. Then f has a limit at $x_0 \in (\alpha, \beta)$ iff. $U(x_0) = L(x_0)$ and in this case

$$\lim_{x \to x_0} f(x) = f(x_0) = U(x_0) = L(x_0) .$$

Proof. Suppose f has a limit at x_0, say L. Let $\varepsilon > 0$. Then there is a $\delta > 0$ s.t.

$$|x - x_0| < \delta, \quad x \in [\alpha, \beta] \Rightarrow |f(x) - L| < \varepsilon .$$

Now, $\exists\, x, y \in [\alpha, \beta]$ s.t.

$$x_0 - \delta < x < x_0 < y < x_0 + \delta .$$

By the def. of $U(x_0)$ & $L(x_0)$

$$U(x_0) \leq f(y) < L + \varepsilon$$

and

$$L - \varepsilon < f(x) \leq L(x_0) .$$

So, $L(x_0) - U(x_0) > -2\varepsilon$. Since $L(x_0) - U(x_0) \leq 0$

$$\Rightarrow \quad 0 < U(x_0) - L(x_0) < 2\varepsilon \quad \forall \varepsilon > 0.$$

Thus, $\quad U(x_0) = L(x_0)$. Also,

$$L(x_0) \le f(x_0) \le U(x_0) \quad \Rightarrow \quad f(x_0) = U(x_0) = L(x_0).$$

Finally, from our last calculations with ε & δ

$$\Rightarrow \quad L - \varepsilon < f(x) \le L(x_0) \le U(x_0) \le f(y) < L + \varepsilon$$

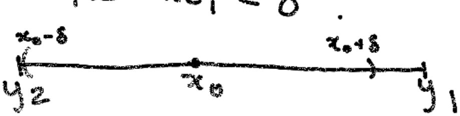
$$\Rightarrow \quad |U(x_0) - L| < \varepsilon \quad \forall \varepsilon > 0$$

$$\Rightarrow \quad U(x_0) = L.$$

Suppose $U(x_0) = L(x_0)$. Then, $f(x_0) = U(x_0) = L(x_0)$.
Let $\varepsilon > 0$.

- $U(x_0) + \varepsilon$, $\exists\, y_1 \in [\alpha, \beta]$ s.t. $y_1 > x_0$
  $$U(x_0) \le f(y_1) < U(x_0) + \varepsilon$$
- $L(x_0) - \varepsilon$, then $\exists\, y_2 \in [\alpha, \beta)$, $y_2 < x_0$
  $$L(x_0) - \varepsilon < f(y_2) \le L(x_0).$$

Take $\delta := \min\{|x_0 - y_1|, |x_0 - y_2|\}$. Let $x \in [\alpha, \beta]$, $|x - x_0| < \delta$.



So, $\quad x_0 - \delta < x < x_0 \Rightarrow y_1 < x$

$$\Rightarrow L(x_0) - \varepsilon < f(y_2) \leq f(x) \leq f(y_1) < U(x_0) + \varepsilon.$$

Similarly, $\quad x_0 < x < x_0 + \delta \Rightarrow x < y_1$

$$\Rightarrow L(x_0) - \varepsilon < f(y_2) \leq f(x) \leq f(y_1) < U(x_0) + \varepsilon.$$

Thus, in the two cases, we get

$$L(x_0) - \varepsilon < f(x) < U(x_0) + \varepsilon$$

But $\quad L(x_0) = U(x_0) = f(x_0)$

$$\Rightarrow \quad -\varepsilon < f(x) - f(x_0) < \varepsilon$$

So, $\quad |f(x) - f(x_0)| < \varepsilon$ and $\lim\limits_{x \to x_0} f(x) = f(x_0)$ $\square$

We assumed that f was increasing. We can do it when f is decreasing (why? Take $-f$).

<u>Remark</u>. We have neglected the points α & β. U can't be define at β and L can't be define at α. See exercise 24 of the book.

<u>Thm</u> Let $f : [\alpha, \beta] \to \mathbb{R}$ be monotonic. Let
$D := \{ x \in \mathbb{R} : \alpha \leq x \leq \beta \ \& \ \lim_{x \to x_0} f(x) \nexists \}$.

Then D is countable.

<u>Proof</u>. Suppose f is increasing. By the previous lemma,

$$x_0 \in D \iff U(x_0) \neq L(x_0) \iff U(x_0) - L(x_0) > 0.$$

Define

$$D_n := \{ x \in [\alpha, \beta] : U(x) - L(x) \geq \tfrac{1}{n} \}.$$

It is obvious that $D = \bigcup_{n \geq 1} D_n$ (Use AP).

Let $x_1, x_2, \ldots, x_r \in D$ for some $r \in \mathbb{N}$ s.t.

$$\alpha < x_1 < x_2 < \cdots < x_r < \beta.$$

Let $z_1, z_2, \ldots, z_{r+1} \in [\alpha, \beta]$ s.t.

$$\alpha < z_1 < x_1, \quad x_i < z_{i+1} < x_{i+1} \qquad i = 1, 2, \ldots, r-1$$

$$x_r < z_{r+1} < \beta.$$

For each i, $\quad z_i \leq x_i \Rightarrow \quad f(z_i) \leq L(x_i)$

Also, for each i, $x_i \leq z_{i+1} \Rightarrow U(x_i) \leq f(z_{i+1})$.

This implies that

$$f(z_i) - f(z_{i-1}) \geq U(x_i) - L(x_i) \geq \frac{1}{n} \quad (i=2,\dots,r+1).$$

Now, we get

$$f(\beta) - f(\alpha) = \overbrace{f(\beta) - f(z_{r+1})}^{\geq 0} + \sum_{i=2}^{r+1} [f(z_i) - f(z_{i-1})] + \underbrace{f(z_1) - f(\alpha)}_{\geq 0}$$

$$\geq \sum_{i=2}^{r+1} [f(z_i) - f(z_{i-1})] \geq r \cdot \left(\frac{1}{n}\right)$$

$$\Rightarrow \quad r \leq n \, [f(\beta) - f(\alpha)].$$

So, r is bounded! Thus, D_n must be finite. Since $D = \bigcup_{n\geq 1} D_n$, D must be countable. ∎

3.5 Other concepts.

Def Right-hand and left-hand limits.

Def limit at ∞ and $-\infty$.

Good

Suggested problems from the book.

- Section 2.1 : 1-5, 8, 10-12, 14
- Section 2.2 : 16, 18-20, 22
- Section 2.3 : 23, 24
- Miscellaneous : 26.
- Projects : 2.2, 2.3, 2.4.