MATH-331 Introduction to Real Analysis — YOUR FULL NAME
Homework 05 — Fall 2021

Due date: November 8^{th} 1:20pm — Total: /70.

Exercise	1 (5)	2 (5)	3 (5)	4 (5)	5 (10)	6 (10)	7 (5)	8 (5)	9 (5)	10 (10)
Score										

Table 1: Scores for each exercises

Instructions: You must answer all the questions below and send your solution by email (to `parisepo@hawaii.edu`). If you decide to not use LaTeX to hand out your solutions, please be sure that after you scan your copy, it is clear and readable. Make sure that you attached a copy of the homework assignment to your homework.
If you choose to use LaTeX, you can use the template available on the course website.
No late homework will be accepted. No format other than PDF will be accepted. Name your file as indicated in the syllabus.

1 Writing problems

For each of the following problems, you will be asked to write a clear and detailed proof. You will have the chance to rewrite your solution in your semester project after receiving feedback from me.

Exercise 1. (5 pts) Let $f : \mathbb{R} \to \mathbb{R}$ and suppose that there exists a positive constant M such that $|f(y) - f(x)| \leq M|y - x|$ for all $x, y \in \mathbb{R}$. Prove that f is uniformly continuous on $\mathbb{R}$.

Solution: To prove that f is uniformly continuous, we must prove that for an arbitrary ε, there exists δ such that for all $x, y \in \mathbb{R}$:

$$|y - x| < \delta \to |f(y) - f(x)| < \varepsilon$$

Let $\delta = \frac{\varepsilon}{M}$. Then if $|y - x| < \delta$:
$|y - x| < \delta$
$|y - x| < \frac{\varepsilon}{M}$
$M|y - x| < \varepsilon$
$|f(y) - f(x)| \leq M|y - x| < \varepsilon$
$|f(y) - f(x)| < \varepsilon$
This is what we wanted to prove. Therefore f is uniformly continuous. □

Exercise 2. (5 pts) Let $f : [0, \infty) \to \mathbb{R}$ be nonnegative and continuous such that $\lim_{x\to\infty} f(x) = 0$. Prove that f attains its maximum at some point in $[0, \infty)$.

Solution: Since $\lim_{x\to\infty} f(x) = 0$, there exists $x_0 \in [0,\infty)$ such that for all $x > x_0$, $|f(x)| < f(0)$. Now consider the interval $[0, x_0]$. By the Extreme Value Theorem, there exists $x_M \in [0, x_0]$ such that for all $x \in [0, x_0]$, $f(x_M) \geq f(x)$. Since $|f(x)| < f(0) \leq f(x_M)$ for all $x > x_0$, we have that $f(x) \leq f(x_M)$ for all $x \in [0,\infty)$. f then attains its maximum at x_M, which is in $[0,\infty)$. $\square$

Exercise 3. Suppose that $f : [a,b] \to \mathbb{R}$ is a continuous function such that $f([a,b]) \subseteq [a,b]$. Prove that there is a $c \in [a,b]$ such that $f(c) = c$. [This one of the many fixed point Theorem.]

Solution: From $f([a,b]) \subseteq [a,b]$ we have that $a \leq f(a)$ and $f(b) \leq b$. Therefore $f(b) - b \leq 0 \leq f(a) - a$. Define function $g : [a,b] \to \mathbb{R}$ such that $g(x) = f(x) - x$. Note that g is continuous since $f(x)$ and x are continuous functions. We now have that $g(b) \leq 0 \leq g(a)$. By the Intermediate Value Theorem, we know that there exists some $c \in [a,b]$ such that $g(c) = 0$. Therefore $f(c) - c = 0$ and $f(c) = c$. This is what we wanted to prove. $\square$

Exercise 4. (5 pts) Suppose that $f : (a,b) \to \mathbb{R}$ is twice differentiable on (a,b) and there are two points $c < d$ in (a,b) such that $f'(c) = f'(d)$. Show that there is a point $x \in (c,d)$ such that $f''(x) = 0$.

Solution: Let $g(x) = f'(x)$. As f is twice differentiable, g is differentiable and continuous. We now have that for points c and d where $c < d$, $g(c) = g(d)$. By the Mean Value Theorem, there exists $x_0 \in (c,d)$ such that:
$g'(x_0) = \frac{g(c)-g(d)}{c-d}$
As $g(c) = g(d)$ and $c - d < 0$:
$g'(x_0) = \frac{0}{c-d}$
$g'(x_0) = 0$
Substituting $g(x) = f'(x)$:
$(f')'(x_0) = 0$
$f''(x_0) = 0$
We have now found $x \in (c,d)$ where $f''(x) = 0$. $\square$

Exercise 5. (10 pts) Suppose that $f : (a,b) \to \mathbb{R}$ is differentiable at $x_0 \in (a,b)$.

a) Prove that

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0-h)}{2h} \qquad (\star)$$

exists and equals $f'(x_0)$.

b) Find a continuous function $f : \mathbb{R} \to \mathbb{R}$ and a point $x_0 \in \mathbb{R}$ such that f is not differentiable at x_0, but the limit $(\star)$ exists.

Solution:

a) $\lim\limits_{h\to 0}\frac{f(x_0+h)-f(x_0-h)}{2h} = \lim\limits_{h\to 0}\frac{f(x_0+h)-f(x_0)+f(x_0)-f(x_0-h)}{2h}$
$\lim\limits_{h\to 0}\frac{f(x_0+h)-f(x_0-h)}{2h} = \lim\limits_{h\to 0}\frac{f(x_0+h)-f(x_0)}{2h} + \lim\limits_{h\to 0}\frac{f(x_0)-f(x_0-h)}{2h}$
On the right limit, substitute $h = -j$
$\lim\limits_{h\to 0}\frac{f(x_0+h)-f(x_0-h)}{2h} = \lim\limits_{h\to 0}\frac{f(x_0+h)-f(x_0)}{2h} + \lim\limits_{-j\to 0}\frac{f(x_0)-f(x_0+j)}{2(-j)}$
$\lim\limits_{h\to 0}\frac{f(x_0+h)-f(x_0-h)}{2h} = \lim\limits_{h\to 0}\frac{f(x_0+h)-f(x_0)}{2h} + \lim\limits_{-j\to 0}\frac{f(x_0+j)-f(x_0)}{2j}$
$\lim\limits_{h\to 0}\frac{f(x_0+h)-f(x_0-h)}{2h} = 0.5(\lim\limits_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} + \lim\limits_{-j\to 0}\frac{f(x_0+j)-f(x_0)}{j})$
As f is differentiable and $-j \to 0$ implies that $j \to 0$, the right limit is also equal to $f'(x_0)$
$\lim\limits_{h\to 0}\frac{f(x_0+h)-f(x_0-h)}{2h} = 0.5(f'(x_0) + f'(x_0))$
$\lim\limits_{h\to 0}\frac{f(x_0+h)-f(x_0-h)}{2h} = f'(x_0)$

b) Let $f(x) = |x|$ and $x_0 = 0$. f is not differentiable at 0 since $\lim\limits_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} = \lim\limits_{h\to 0}\frac{|h|}{h}$ changes depending on whether h approaches 0 from the negative or positive numbers. However:
$\lim\limits_{h\to 0}\frac{f(x_0+h)-f(x_0-h)}{2h} = \lim\limits_{h\to 0}\frac{|0+h|-|0-h|}{2h}$
$\lim\limits_{h\to 0}\frac{f(x_0+h)-f(x_0-h)}{2h} = \lim\limits_{h\to 0}\frac{|h|-|h|}{2h}$
$\lim\limits_{h\to 0}\frac{f(x_0+h)-f(x_0-h)}{2h} = \lim\limits_{h\to 0}\frac{0}{2h}$
$\lim\limits_{h\to 0}\frac{f(x_0+h)-f(x_0-h)}{2h} = 0$ $\square$

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Homework problems

Answer all the questions below. Make sure to show your work.

Exercise 6. (10pts)

a) Suppose $r > 0$. Prove that $f : (0,\infty) \to \mathbb{R}$ defined by $f(x) = x^r$ is differentiable on $(0,\infty)$ and compute its derivative. [Hint: take for granted that e^x and $\ln x$ are differentiable with $(e^x)' = e^x$ and $(\ln x)' = 1/x$. Rewrite then x^r in terms of a composition of these two differentiable functions.]

b) Define $f(x) = \sqrt{x^2 + \sin x + \cos x}$ where $x \in [0, \pi/2]$. Show that f is a differentiable function.

Solution:

a) e^x and $\ln x$ are defined on all $x \in (0,\infty)$. Therefore the following is true on the domain of f:
$f(x) = x^r$
$f(x) = (e^{\ln x})^r$
$f(x) = e^{r\ln x}$
Define $g(x) = e^x$ and $h(x) = r\ln x$. Then $f(x) = (g\circ h)(x)$ and by the Chain Rule:
$f'(x) = g'(h(x)) \cdot (h'(x))$
$f'(x) = e^{r\ln x} \cdot (\frac{r}{x})$
$f'(x) = e^{\ln x^r} \cdot (\frac{r}{x})$
$f'(x) = x^r \cdot (\frac{r}{x})$
$f'(x) = rx^{r-1}$

b) We know that x^2, $\sin x$, and $\cos x$ are differentiable on $x \in [0, \pi/2]$. Since the sum of differentiable functions is differentiable, $x^2 + \sin x + \cos x$ is differentiable. For $x \in (0, \pi/2]$, we know that $0 < x^2$, $0 \leq \sin x$, and $0 \leq \cos x$. Therefore $0 < x^2 + \sin x + \cos x$. For $x = 0$, $1 = x^2 + \sin x + \cos x$. Therefore on the entire domain $x \in [0, \pi/2]$, $0 < x^2 + \sin x + \cos x$.

We can now define $g : (0, \infty) \to \mathbb{R}$ and $h : [0, \pi/2] \to (0, \infty)$ where $g(x) = x^{0.5}$ and $h(x) = x^2 + \sin x + \cos x$. We know from part a that g is differentiable. Note that $f(x) = (g \circ h)(x)$. Since both g and h are differentiable, f is differentiable. $\square$

Exercise 7. (5 pts) Show that $S \subseteq \mathbb{R}$ is closed if and only if $\mathbb{R} \backslash S$ is open.

Solution: $(\rightarrow)$ Suppose $\mathbb{R} \backslash S$ is open. Suppose towards a contradiction that S is not closed. Then there exists $x_0 \in acc(S)$ where $x_0 \notin S$. Then $x_0 \in \mathbb{R} \backslash S$. As $\mathbb{R} \backslash S$ is open, there exists a δ such that $(x_0 - \delta, x_0 + \delta) \in \mathbb{R} \backslash S$. Since $x_0 \in acc(S)$, there exists infinitely many points in $(x_0 - \delta, x_0 + \delta)$ that are in S. This is a contradiction as $(x_0 - \delta, x_0 + \delta)$ lies completely outside of S. Therefore S is closed.
$(\leftarrow)$ Suppose S is closed. Suppose towards a contradiction that $\mathbb{R} \backslash S$ is not open. Then there exists a $x_0 \in \mathbb{R} \backslash S$ such that for all δ, $(x_0 - \delta, x_0 + \delta) \not\subseteq \mathbb{R} \backslash S$. This means that there exists y such that $y \in S$ and $y \in (x_0 - \delta, x_0 + \delta)$. Now define a sequence $(y_n)_{n=1}^{\infty}$ such that $y_n \in (x_0 - 0.5^n, x_0 + 0.5^n)$ and $y_n \in S$. We have now defined a sequence where $y_n \to x_o$. Therefore $x_0 \in acc(S)$ and $x_0 \notin S$. This contradicts S being closed, and therefore $\mathbb{R} \backslash S$ is open. $\square$

Exercise 8. (5 pts) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function and define $g(x) = x^2 f(x^3)$. Show that g is differentiable and compute its derivative.

Solution: g is differentiable if the following is defined on all real numbers:
$g'(x) = \lim\limits_{h \to 0} [\frac{g(x+h) - g(x)}{h}]$
$g'(x) = \lim\limits_{h \to 0} [\frac{(x+h)^2 f((x+h)^3) - x^2 f(x^3)}{h}]$
$g'(x) = \lim\limits_{h \to 0} [\frac{x^2 f((x+h)^3) + 2xhf((x+h)^3) + h^2 f((x+h)^3) - x^2 f(x^3)}{h}]$
$g'(x) = \lim\limits_{h \to 0} [\frac{x^2 f((x+h)^3) - x^2 f(x^3)}{h}] + \lim\limits_{h \to 0} [\frac{2xhf((x+h)^3) + h^2 f((x+h)^3)}{h}]$
$g'(x) = x^2 \lim\limits_{h \to 0} [\frac{f((x+h)^3) - f(x^3)}{h}] + \lim\limits_{h \to 0} [2xf((x+h)^3) + hf((x+h)^3)]$
$g'(x) = x^2 f'(x^3) + 2xf(x^3)$
As f, f', and x^n are all defined on the real numbers for integer $n > 0$, $g'(x)$ is defined on all real numbers. $\square$

Exercise 9. (5 pts) Prove that $f(x) = \arcsin x$ is differentiable on its domain and find a formula for the derivative of f (justify all your steps!).

Solution: Note that $f : (-1, 1) \to (\frac{-\pi}{2}, \frac{-\pi}{2})$. $\arcsin x$ is defined as the inverse of $\sin x$ on its domain. Lets then define $f^{-1}(x) : (\frac{-\pi}{2}, \frac{-\pi}{2}) \to (-1, 1)$ where $f^{-1}(x) = \sin x$. We know that $\sin x$ is continuous, differentiable, one-one, and it's derivative is never 0 on the interval $(\frac{-\pi}{2}, \frac{-\pi}{2})$. Therefore $f'(f^{-1}(y)) = \frac{1}{(f^{-1})'(y)}$. Substituting $x = f^{-1}(y)$:

$f'(x) = \frac{1}{(f^{-1})'(f(x))}$
$(f^{-1})'(x) = \frac{d}{dx} \sin x = \cos x$
$f'(x) = \frac{1}{\cos(\arcsin(x))}$
$f'(x) = \sec(\arcsin(x))$ ☐

Exercise 10. (10 pts) Use the Mean-Value Theorem to show the following inequalities.

a) $ny^{n-1}(x-y) \leq x^n - y^n \leq nx^{n-1}(x-y)$ if $n \in \mathbb{N}$ and $0 \leq y \leq x$.

b) $\sqrt{1+x} < 1 + \frac{1}{2}x$ for $x > 0$.

Solution:

a) Let $f : [0, \infty) \to \mathbb{R}$ be $f(x) = x^n$. By the Mean Value Theorem, there exists $y \leq c \leq x$ such that $f'(c) = \frac{f(x)-f(y)}{x-y}$. As $f'(x) = nx^{n-1}$ is an increasing function on $[0, \infty)$, $f'(y) \leq f'(c) \leq f'(x)$ and:
$f'(y) \leq f'(c) \leq f'(x)$
$f'(y) \leq \frac{f(x)-f(y)}{x-y} \leq f'(x)$
$f'(y)(x-y) \leq f(x) - f(y) \leq f'(x)(x-y)$
$ny^{n-1}(x-y) \leq x^n - y^n \leq nx^{n-1}(x-y)$

b) I don't know how to prove this using the Mean Value Theorem, but we can prove this starting from $x > 0$:
$x > 0$
$x^2 > 0$
$\frac{1}{4}x^2 > 0$
$1 + x + \frac{1}{4}x^2 > 1 + x$
$(1 + \frac{1}{2}x)(1 + \frac{1}{2}x) > 1 + x$
$(1 + \frac{1}{2}x)^2 > 1 + x$
$\sqrt{(1 + \frac{1}{2}x)^2} > \sqrt{1+x}$
$|1 + \frac{1}{2}x| > \sqrt{1+x}$
For $x > 0$, $1 + \frac{1}{2}x > 0$. Therefore $|1 + \frac{1}{2}x| = 1 + \frac{1}{2}x$
$1 + \frac{1}{2}x > \sqrt{1+x}$
In total, $\sqrt{1+x} < 1 + \frac{1}{2}x$ for all $x > 0$. ☐