

Due date: October 11<sup>th</sup> 1:20pm

Total: ~~70~~ 70.

Excellent!! 😊

Exercise	1 (5)	2 (5)	3 (5)	4 (5)	5 (10)	6 (10)	7 (5)	8 (5)	9 (5)	10 (10)
Score	5	5	5	5	10	10	5	5	5	10

Table 1: Scores for each exercises

**Instructions:** You must answer all the questions below and send your solution by email (to [parisepo@hawaii.edu](mailto:parisepo@hawaii.edu)). If you decide to not use L<sup>A</sup>T<sub>E</sub>X to hand out your solutions, please be sure that after you scan your copy, it is clear and readable. Make sure that you attached a copy of the homework assignment to your homework.

If you choose to use L<sup>A</sup>T<sub>E</sub>X, you can use the template available on the course website.

No late homework will be accepted. No format other than PDF will be accepted. Name your file as indicated in the syllabus.

—1—  
WRITING PROBLEMS

For each of the following problems, you will be asked to write a clear and detailed proof. You will have the chance to rewrite your solution in your semester project after receiving feedback from me.

**Exercise 1.** (5 pts) Let  $(a_n)_{n=1}^{\infty}$  be an increasing sequence and  $(b_n)_{n=1}^{\infty}$  be a decreasing sequence. Let  $(c_n)_{n=1}^{\infty}$  be the sequence defined by  $c_n = b_n - a_n$ . Show that if  $\lim_{n \rightarrow \infty} c_n = 0$ , then the sequences  $(a_n)_{n=1}^{\infty}$  and  $(b_n)_{n=1}^{\infty}$  converges and  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n$ .

**Solution:** Since  $c_n$  converges,  $c_n$  is bounded. As  $c_n$  is bounded, there exists  $M$  such that  $-M < c_n < M$  for all  $n$ . As  $a_n$  is increasing and  $b_n$  is decreasing, we also know that  $a_n > a_1$  and  $b_n < b_1$  for all  $n$ . Note that:

$$a_n = b_n + (-c_n) < b_1 + M$$

$$b_n = c_n + a_n > -M + a_1$$

Therefore  $a_n$  is bounded from above and  $b_n$  is bounded from below. Since  $a_n$  is increasing and bounded from above and  $b_n$  is decreasing and bounded from below, both  $a_n$  and  $b_n$  must converge.

Now suppose  $\lim_{n \rightarrow \infty} c_n = 0$ . Then:

$$\lim_{n \rightarrow \infty} (b_n - a_n) = 0$$

$$\lim_{n \rightarrow \infty} b_n - \lim_{n \rightarrow \infty} a_n = 0$$

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} a_n$$

Great

5/5

$\lim f(x_n) = L_x$ ,  $\lim f(y_n) = L_y \Rightarrow L_x = L_y$ . The limits are the same okay.

**Exercise 2.** (5 pts) Let  $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ , and suppose that  $x_0$  is an accumulation point of  $D$ . Suppose that for each sequence  $(x_n)_{n=1}^{\infty}$  converging to  $x_0$  with  $x_n \in D \setminus \{x_0\}$  for each  $n \geq 1$ , then the sequence  $(f(x_n))_{n=1}^{\infty}$  is Cauchy. Show that  $f$  has a limit at  $x_0$ .

[Hint: For two sequences  $(x_n)$  and  $(y_n)$  that satisfy the assumption, define the sequence  $(z_n)$  to be  $z_{2n} = x_n$  and  $z_{2n-1} = y_n$ . Show that  $(f(z_n))$  converges and the sequence  $(f(x_n))$  and  $(f(y_n))$  converges to the same limit as  $(f(z_n))$ . Conclude by a theorem in the lecture notes.]

**Solution:** Define  $x_n, y_n$ , and  $z_n$  as given in the hint. As  $z_n$  also satisfies the condition of converging to  $x_0$  with  $z_n \in D \setminus \{x_0\}$  for all  $n \geq 1$ , the sequence  $f(z_n)$  is Cauchy and must converge. Any subsequence of a sequence that converges must also converge to the same limit, so  $f(x_n)$  and  $f(y_n)$  also converge to the same limit as  $f(x_n)$ . By Theorem 2.1, since any sequence converging to  $x_0$  and not containing  $x_0$  implies that the sequence  $f(x_n)$  also converges,  $f$  has a limit at  $x_0$ .  $\square$

**Exercise 3.** (5 pts) Prove that if  $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$  has a limit at  $x_0 \in \text{acc } D$ , then the limit is unique.

**Solution:** Suppose towards a contradiction that the limit at  $x_0$  is not unique. Then there are two limits  $L_1$  and  $L_2$  where  $L_1 \neq L_2$ . Without loss of generality, let  $L_1 < L_2$ . Then:

$$\begin{aligned} \exists \delta_1, \forall x \in D, |x - x_0| < \delta_1 \rightarrow |f(x) - L_1| &< \frac{L_2 - L_1}{2} \\ \exists \delta_2, \forall x \in D, |x - x_0| < \delta_2 \rightarrow |f(x) - L_2| &< \frac{L_2 - L_1}{2} \end{aligned}$$

By defining  $\delta = \min(\delta_1, \delta_2)$ , we get that:

$$\begin{aligned} |x - x_0| < \delta \rightarrow |f(x) - L_1| &< \frac{L_2 - L_1}{2} \\ |x - x_0| < \delta \rightarrow |f(x) - L_2| &< \frac{L_2 - L_1}{2} \end{aligned}$$

We now have the following:

$$\begin{aligned} \frac{L_1 - L_2}{2} &< f(x) - L_1 < \frac{L_2 - L_1}{2} \\ \frac{3L_1 - L_2}{2} &< f(x) < \frac{L_2 + L_1}{2} \\ f(x) &< \frac{L_2 + L_1}{2} \end{aligned}$$

And similarly:

$$\begin{aligned} \frac{L_1 - L_2}{2} &< f(x) - L_2 < \frac{L_2 - L_1}{2} \\ \frac{L_1 + L_2}{2} &< f(x) < \frac{3L_2 - L_1}{2} \\ \frac{L_1 + L_2}{2} &< f(x) \end{aligned}$$

Combining these inequalities,  $f(x) < \frac{L_2 + L_1}{2} < f(x)$ . This is a contradiction, so the limit at  $x_0$  must be unique.  $\square$

**Exercise 4.** (5pts) Suppose  $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ ,  $g : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ , and  $h : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$  are three functions such that

$$f(x) \leq h(x) \leq g(x) \quad (\forall x \in D)$$

Suppose that  $f$  and  $g$  have limits at  $x_0$  with  $\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} g(x)$ . Prove that  $h$  has a limit at  $x_0$  and

$$\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} h(x) = \lim_{x \rightarrow x_0} g(x)$$

**Solution:** Call  $L = \lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} g(x)$ . Since  $f$  and  $g$  have a limit at  $x_0$ , we have the following for an arbitrary  $\varepsilon$ :

$$\begin{aligned}\exists \delta_1, \forall x \in D, |x - x_0| < \delta_1 &\rightarrow |f(x) - L| < \varepsilon \\ \exists \delta_2, \forall x \in D, |x - x_0| < \delta_2 &\rightarrow |g(x) - L| < \varepsilon\end{aligned}$$

By defining  $\delta = \min(\delta_1, \delta_2)$ , we get that:

$$\begin{aligned}\exists \delta, \forall x \in D, |x - x_0| < \delta &\rightarrow |f(x) - L| < \varepsilon \\ \exists \delta, \forall x \in D, |x - x_0| < \delta &\rightarrow |g(x) - L| < \varepsilon\end{aligned}$$

We then have:

$$-\varepsilon < f(x) - L < \varepsilon$$

$$-\varepsilon - L < f(x) < \varepsilon - L$$

and similarly:

$$-\varepsilon - L < g(x) < \varepsilon - L$$

Combining these inequalities:

$$-\varepsilon - L < f(x) \leq h(x) \leq g(x) < \varepsilon - L$$

$$-\varepsilon - L < h(x) < \varepsilon - L$$

$$-\varepsilon < h(x) - L < \varepsilon$$

$$|h(x) - L| < \varepsilon$$

In total,

$$\forall x \in D, |x - x_0| < \delta \rightarrow |h(x) - L| < \varepsilon$$

As  $\varepsilon$  was arbitrary,  $h$  has a limit of  $L = \lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} g(x)$  at  $x_0$ . □

**Exercise 5.** (10 pts) Let  $f : (0, \infty) \rightarrow \mathbb{R}$  be a function. We say that  $f$  has a limit at  $\infty$  if there exists a  $L \in \mathbb{R}$  such that for any  $\varepsilon > 0$ , there is a real number  $M > 0$  such that if  $x > M$ , then  $|f(x) - L| < \varepsilon$ .

- a) Show that if  $g : (0, \infty) \rightarrow \mathbb{R}$  is bounded and  $\lim_{x \rightarrow \infty} f(x) = 0$ , then  $\lim_{x \rightarrow \infty} f(x)g(x) = 0$ .
- b) Let  $a > 0$  and suppose that  $f : (a, \infty) \rightarrow \mathbb{R}$  and define  $g : (0, 1/a) \rightarrow \mathbb{R}$  by  $g(x) = f(1/x)$ . Show that  $f$  has a limit at  $\infty$  if and only if  $g$  has a limit at 0.

**Solution:**

- a) Since  $g$  is bounded, there exists  $B$  such that  $|g(x)| < B$  for all  $x$ . Since the limit of  $f$  at  $\infty$  is 0, there exists  $M > 0$  such that for all  $x > M$ ,  $|f(x)| < \frac{\varepsilon}{B}$  for an arbitrary  $\varepsilon$ . Then for all  $x > M$ :

$$|f(x)g(x)| = |f(x)||g(x)|$$

$$|f(x)g(x)| < B|f(x)|$$

$$|f(x)g(x)| < \varepsilon$$

Therefore the limit of  $f(x)g(x)$  at  $\infty$  is 0.

5/5

You could also  
used the charact.  
in terms of seq.

10/10

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b) ( $\rightarrow$ ) Suppose  $g$  has a limit at 0. Then for arbitrary  $\varepsilon$ , there exists  $\delta$  and  $L$  such that  $|x| < \delta \rightarrow |g(x) - L| < \varepsilon$ . Substituting  $\frac{1}{y}$  for  $x$ :

$$|\frac{1}{y}| < \delta \rightarrow |g(\frac{1}{y}) - L| < \varepsilon$$

$$\frac{1}{y} < \delta \rightarrow |f(y) - L| < \varepsilon$$

$$y > \frac{1}{\delta} \rightarrow |f(y) - L| < \varepsilon$$

Since  $\delta > 0$ ,  $\frac{1}{\delta} > 0$ . This altogether proves that  $f$  has a limit at  $\infty$

( $\leftarrow$ ) Suppose  $f$  has a limit at  $\infty$ . Then for arbitrary  $\varepsilon$ , there exists  $M > 0$  and  $L$  such that for all  $x > M$ ,  $|f(x) - L| < \varepsilon$ . Substituting  $\frac{1}{y}$  for  $x$ :

$$\frac{1}{y} > M \rightarrow |f(\frac{1}{y}) - L| < \varepsilon$$

$$y < \frac{1}{M} \rightarrow |g(y) - L| < \varepsilon$$

As  $x > 0$ ,  $\frac{1}{y} > 0$  and  $y > 0 > \frac{-1}{M}$ . Therefore:

$$\frac{-1}{M} < y < \frac{1}{M} \rightarrow |g(y) - L| < \varepsilon$$

$$|y| < \frac{1}{M} \rightarrow |g(y) - L| < \varepsilon$$

Since  $M > 0$ ,  $\frac{1}{M} > 0$ . This altogether proves that  $g$  has a limit at 0. S/ □

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## HOMEWORK PROBLEMS

Answer all the questions below. Make sure to show your work.

**Exercise 6.** (10pts) For each of the sequences below, determine its nature (converges or diverges)<sup>1</sup>:

a)  $(a_n)$  where  $a_n = \frac{1}{n} + \frac{1}{n+1} + \dots + \frac{1}{2n}$ .

b)  $(a_n)$  where  $a_n = \frac{1+2+\dots+n}{n^2}$ .

**Solution:**

a) We will show that this sequence is decreasing. Consider  $a_{n+1}$ :

$$a_{n+1} = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2}$$

$$a_{n+1} = a_n - \frac{1}{n} + \frac{1}{2n+1} + \frac{1}{2n+2}$$

$$a_{n+1} - a_n = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n}$$

Note that  $\frac{1}{2n+1}, \frac{1}{2n+2} < \frac{1}{2n}$ . Therefore  $\frac{1}{2n+1} + \frac{1}{2n+2} < \frac{1}{n}$  and  $\frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n} < 0$

$$a_{n+1} - a_n < 0$$

Therefore the sequence  $a_n$  is decreasing. Since  $n$  is always positive, every term of  $a_n$  is positive and the sequence is bounded from below by 0. Since  $a_n$  is decreasing and bounded from below, the sequence must converge. S/

b)  $1 + 2 + \dots + n$  forms the triangular numbers, and is equal to  $0.5(n^2 + n)$ . We then have that

$$a_n = \frac{0.5(n^2 + n)}{n^2}$$

$$a_n = 0.5 + \frac{0.5}{n}$$

Since  $\frac{0.5}{n}$  will converge,  $a_n$  will converge. ✓ S/S □

<sup>1</sup>You don't need to compute the limit.

**Exercise 7.** (5 pts) Define  $g : (0, 1) \rightarrow \mathbb{R}$  by  $f(x) = \frac{\sqrt{1+x}-1}{x}$ . Prove that  $g$  has a limit at 0 and find it.

5/5

**Solution:** For an arbitrary  $\varepsilon > 0$ , let  $\delta = \varepsilon$ . To show that the limit at 0 exists and is equal to 0.5, we will show that  $|x| < \delta \rightarrow |f(x) - 0.5| < \varepsilon$ .

$$|f(x) - 0.5| = \left| \frac{\sqrt{1+x}-1}{x} - 0.5 \right|$$

$$|f(x) - 0.5| = \left| \frac{(\sqrt{1+x}-1)(\sqrt{1+x}+1)}{x(\sqrt{1+x}+1)} - 0.5 \right|$$

$$|f(x) - 0.5| = \left| \frac{1+x-1}{x(\sqrt{1+x}+1)} - 0.5 \right|$$

$$|f(x) - 0.5| = \left| \frac{1}{\sqrt{1+x}+1} - 0.5 \right|$$

Suppose  $|x| < \delta$ . Then  $-\delta < x < \delta$ . As the domain of  $g$  is only for positive  $x$ ,  $0 < x < \delta$ . Then:

$$2 < \sqrt{1+x}+1 < \sqrt{1+\delta}+1$$

$$0.5 > \frac{1}{\sqrt{1+x}+1} > \frac{1}{\sqrt{1+\delta}+1}$$

$$0 > \frac{1}{\sqrt{1+x}+1} - 0.5 > \frac{1}{\sqrt{1+\delta}+1} - 0.5$$

$$0 < 0.5 - \frac{1}{\sqrt{1+x}+1} < 0.5 - \frac{1}{\sqrt{1+\delta}+1}$$

For  $0 < a < b$ ,  $|a| = a$ . Therefore  $|a| < b$ . Applying that to this inequality:

$$|0.5 - \frac{1}{\sqrt{1+x}+1}| < 0.5 - \frac{1}{\sqrt{1+\delta}+1}$$

$$|f(x) - 0.5| < 0.5 - \frac{1}{\sqrt{1+\delta}+1}$$

$$|f(x) - 0.5| < 0.5 - \frac{1}{\sqrt{1+\varepsilon}+1}$$

We will now show that  $0.5 - \frac{1}{\sqrt{1+\varepsilon}+1} \leq \varepsilon$  for  $\varepsilon > 0$ . Assume towards a contradiction that  $0.5 - \frac{1}{\sqrt{1+\varepsilon}+1} > \varepsilon$ . We now have the following:

$$0 > \varepsilon - 0.5 + \frac{1}{\sqrt{1+\varepsilon}+1}$$

$$0 > (\varepsilon - 0.5)(\sqrt{1+\varepsilon}+1) + 1$$

Since the functions  $\varepsilon - 0.5$  and  $\sqrt{1+\varepsilon}+1$  are increasing, function  $h(\varepsilon) = (\varepsilon - 0.5)(\sqrt{1+\varepsilon}+1) + 1$  is increasing. We then have that  $h(\varepsilon) > h(0) = 0$  for  $\varepsilon > 0$ . Therefore  $0 > h(\varepsilon) > 0$  which is a contradiction. Therefore  $0.5 - \frac{1}{\sqrt{1+\varepsilon}+1} \leq \varepsilon$  for  $\varepsilon > 0$  and  $|f(x) - 0.5| < \varepsilon$ , which proves that  $g$  has a limit at 0.  $\square$

Great!  
I'm impressed.

$h(0) = -0.5 + 1 = 1/2$ . This doesn't break your proof.

**Exercise 8.** (5 pts) Suppose that  $f : (0, 1) \rightarrow \mathbb{R}$  has a limit at  $x_0 = 1$  and  $\lim_{x \rightarrow 1} f(x) = 1$ . Compute the value of the limit

$$\lim_{x \rightarrow 1} \frac{f(x)(1-f(x)^2)}{1-f(x)}.$$

**Solution:**

$$\lim_{x \rightarrow x_0} \frac{f(x)(1-f(x)^2)}{1-f(x)} = \lim_{x \rightarrow 1} \frac{f(x)(1-f(x))(1+f(x))}{1-f(x)}$$

$$\lim_{x \rightarrow x_0} \frac{f(x)(1-f(x)^2)}{1-f(x)} = \lim_{x \rightarrow 1} f(x)(1+f(x))$$

$$\lim_{x \rightarrow x_0} \frac{f(x)(1-f(x)^2)}{1-f(x)} = \lim_{x \rightarrow 1} f(x) \cdot \lim_{x \rightarrow 1} (1+f(x))$$

$$\lim_{x \rightarrow x_0} \frac{f(x)(1-f(x)^2)}{1-f(x)} = \lim_{x \rightarrow 1} f(x) \cdot (1 + \lim_{x \rightarrow 1} f(x))$$

$$\lim_{x \rightarrow x_0} \frac{f(x)(1-f(x)^2)}{1-f(x)} = 1 \cdot (1 + 1)$$

$$\lim_{x \rightarrow x_0} \frac{f(x)(1-f(x)^2)}{1-f(x)} = 2$$

$\square$

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**Exercise 9.** (5 pts) Prove that if  $f : D \rightarrow \mathbb{R}$  has a limit at  $x_0$ , then  $|f|(x) := |f(x)|$  has a limit at  $x_0$ .

**Solution:** If  $f$  has a limit at  $x_0$ , then:

$$\exists L, \forall \varepsilon, \exists \delta, \forall x \in D, |x - x_0| < \delta \rightarrow |f(x) - L| < \varepsilon$$

By the reverse triangle inequality,  $|f(x) - L| < \varepsilon \rightarrow ||f(x)| - |L|| < \varepsilon$ . Therefore

$$\exists L, \forall \varepsilon, \exists \delta, \forall x \in D, |x - x_0| < \delta \rightarrow ||f(x)| - |L|| < \varepsilon$$

This proves that  $|f(x)|$  will have a limit of  $|L|$  at  $x_0$ . □

**Exercise 10.** (10 pts) Using the link between sequences and limits of functions, show the following.

a) If  $f(x) = x^n$  ( $n \geq 0$ ), then  $\lim_{x \rightarrow x_0} f(x) = x_0^n$  for any  $x_0 \in \mathbb{R}$ .

b) If  $x_0 \in [0, \infty)$ , then  $\lim_{x \rightarrow x_0} \sqrt{x} = \sqrt{x_0}$ .

**Solution:** By Theorem 2.1, we will show that for any sequence  $x_m$  converging to  $x_0$  where  $x_m \in D \setminus x_0$  for all  $m$ , the sequence  $f(x_m)$  converges to the given value.

a) Since  $x_m$  converges to  $x_0$ ,  $x_m$  is bounded. There then exists  $M$  such that  $x_m \leq M$  for all  $m$ . For  $n=0$ ,  $f(x) = 1$  and  $\lim_{x \rightarrow x_0} f(x) = x_0^0 = 1$ . Now suppose that if  $f(x) = x^i$  ( $i \geq 0$ ),

$\lim_{x \rightarrow x_0} f(x) = x_0^i$  for all  $i \leq k$ . We then have the following:

$$\exists N_1, \forall m > N_1, |x_m - x_0| < \frac{\varepsilon}{2M^k}$$

$$\exists N_2, \forall m > N_2, |x_m^k - x_0^k| < \frac{\varepsilon}{2x_0}$$

Now for all  $m > N$  for  $N = \max(N_1, N_2)$ :

$$|x_m^{k+1} - x_0^{k+1}| = |x_m^{k+1} - x_m^k x_0 + x_m^k x_0 - x_0^{k+1}|$$

$$|x_m^{k+1} - x_0^{k+1}| = |x_m^k(x_m - x_0) + x_0(x_m^k - x_0^k)|$$

$$|x_m^{k+1} - x_0^{k+1}| \leq |x_m^k(x_m - x_0)| + |x_0(x_m^k - x_0^k)|$$

$$|x_m^{k+1} - x_0^{k+1}| \leq x_m^k |x_m - x_0| + x_0 |x_m^k - x_0^k|$$

$$|x_m^{k+1} - x_0^{k+1}| < x_m^k \frac{\varepsilon}{2M^k} + x_0 \frac{\varepsilon}{2x_0}$$

$$\text{As } x_m \leq M, x_m^k \leq M^k$$

$$|x_m^{k+1} - x_0^{k+1}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$|x_m^{k+1} - x_0^{k+1}| < \varepsilon$$

Therefore  $\lim_{x \rightarrow x_0} x^{k+1} = x_0^{k+1}$ . By strong induction,  $\lim_{x \rightarrow x_0} x^n = x_0^n$  for any integer  $n \geq 0$ .

b) Since  $x_m$  converges to  $x_0$ , we have:

$$\exists N, \forall m > N, |x_m - x_0| < \varepsilon(\sqrt{x_0})$$

Now for all  $m > N$ :

$$|\sqrt{x_m} - \sqrt{x_0}| = \left| \frac{x_m - x_0}{\sqrt{x_m} + \sqrt{x_0}} \right|$$

$$|\sqrt{x_m} - \sqrt{x_0}| = \frac{|x_m - x_0|}{|\sqrt{x_m} + \sqrt{x_0}|}$$

$$|\sqrt{x_m} - \sqrt{x_0}| < \frac{\sqrt{x_0}}{\sqrt{x_m} + \sqrt{x_0}} \varepsilon$$

$$|\sqrt{x_m} - \sqrt{x_0}| < \varepsilon$$

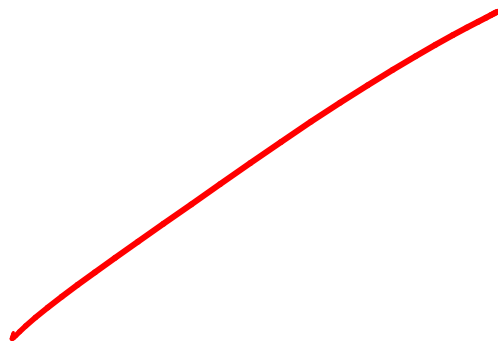
Therefore  $\lim_{x \rightarrow x_0} \sqrt{x} = \sqrt{x_0}$  □

**Exercise 11.** Assume that  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that  $f(x+y) = f(x)f(y)$  for all  $x, y \in \mathbb{R}$ .

- a) Show that  $f$  has a limit at every point of  $\mathbb{R}$ .
- b) Show that either  $\lim_{x \rightarrow 0} f(x) = 1$  or  $f(x) = 0$  for any  $x \in \mathbb{R}$ .

**Solution:**

- a)
- b)



see the solutions.