Math 331 Homework 1

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Exercise 1. Prove that for any $n \in \mathbb{N}, 1 + 2 + ... + n = \frac{n(n+1)}{2}$.

Proof. We will prove this via induction on n. First let's prove the $n = 1$ case. If $n = 1$, plugging into the equation we get

$$\begin{aligned} &\frac{1(1+1)}{2} \\ &= \frac{2}{2} \\ &= 1 \end{aligned}$$

Trivially we know this statement to be true since $1 = 1$.
Now that we know this to be true for n, we will substitute it with $n + 1$.

$$\begin{aligned} 1 + 2 + ... + n + (n+1) &= \frac{(n+1)(n+1+1)}{2} \\ \frac{n(n+1)}{2} + (n+1) &= \frac{(n+1)(n+2)}{2} \end{aligned}$$

We are able to substitute the first n terms on the RHS due to the Induction Hypothesis.

$$\begin{aligned} \frac{n^2+n}{2} + \frac{2(n+1)}{2} &= \frac{n^2+3n+2}{2} \\ \frac{n^2+n}{2} + \frac{2n+2}{2} &= \frac{n^2+3n+2}{2} \\ \frac{n^2+3n+2}{2} &= \frac{n^2+3n+2}{2} \end{aligned}$$

Therefore, this statement is true by induction. $\square$

Exercise 2. Define $f : \mathbb{N} \to \mathbb{N}$ by $f(1) = 1, f(2) = 2, f(3) = 3$ and
$f(n) := f(n-1) + f(n-2) + f(n-3) \quad n \geq 4$

Prove that $f(n) \leq 2^{n-1}$ for all $n \in \mathbb{N}$

Proof. Let's prove this statement by induction on n. For our base case, we'll start with $n = 4$ as defined by the exercise.

$$\begin{aligned} f(4-1) + f(4-2) + f(4-3) &\leq 2^{4-1} \\ f(3) + f(2) + f(1) &\leq 2^3 \\ 3 + 2 + 1 &\leq 8 \\ 6 &\leq 8 \end{aligned}$$

Now that we've got our base case, we'll assume our case holds true for all n. Then plug in $n + 1$.

$$\begin{aligned} f(n+1-1) + f(n+1-2) + f(n+1-3) &\leq 2^{n+1-1} \\ f(n) + f(n-1) + f(n-2) &\leq 2^{n-1} \cdot 2 \end{aligned}$$

From here we can deduce from the induction hypothesis,

$$f(n) \leq 2^{n-1}$$

Because of this, we can focus our attention on the other terms. The RHS is multiplied by two and so because of the IH, we just need to make sure the other two terms are less than 2^{n-1}

$$f(n-1) + f(n-2) \leq 2^{n-1}$$

From here, we know from the original definition of $f(n)$ that $f(n)$ is strictly greater than $f(n-1) + f(n-2)$

$$f(n-1) + f(n-2) \leq f(n) \leq 2^{n-1}$$

So therefore, this statement is true by induction on n. $\square$

Exercise 3. Prove that if A, B, and C are sets, then
(1) $A \sim A$
(2) If $A \sim B$, then $B \sim A$
(3) If $A \sim B$ and $B \sim C$, then $A \sim C$

(1) We need to find a bijective function $f : A \to A$. First, to show that it is injective, we can simply define the function $f(a) = a \;\; \forall a \in A$. This function is injective because every element in A trivially has its own unique output. We

also know this function must be surjective because every element in the range of f is an element of A.

(2) If we know that $A \sim B$, then there must exist a bijective function $f : A \to B$. Now, we must find a function $g : B \to A$. We can simply define g as the inverse of f.

(3) We have two bijective functions $f : A \to B$ and $g : B \to C$. In order to prove that $A \sim C$, we must show that there is a bijective function $y : A \to C$. We can define y as the composition of functions f and g, such that $y : f \circ g$. Therefore, since the composition of two bijective functions results in a bijective function, $A \sim C$.

Exercise 4. Show that any subset of a countable set is countable.

Proof. Let's define some variables. We have $A \subset B$ where B is countable, and we're trying to prove that A must also be countable. We know, by the definition of being countable, that there is an injection $f : B \to \mathbb{N}$. Whether B is countably infinite or has a finite number of variables, let's say A has k elements. Because it's a subset, each of these k elements is also in B. For each of the k elements in A, we can map it to the natural numbers. We set $A_1 \to 1, A_2 \to 2, ..., A_k \to k$. This mapping is trivially an injection because each input from A has a unique output in $\mathbb{N}$.
Now if $A := \varnothing$, then $\varnothing \subset \mathbb{N}$ so it is countable. And if $A \sim B$, then because B is countable, we use the same function that mapped $B \to \mathbb{N}$ to map A to the natural numbers, so A would still be countable. $\square$

Exercise 5. Let $0 < a < b$ be positive real numbers. Prove that
a) $a^2 < b^2$
b) $\sqrt{a} < \sqrt{b}$
a) **Proof.** Let's start with the actual inequality. Then with some algebra, we will show this property.

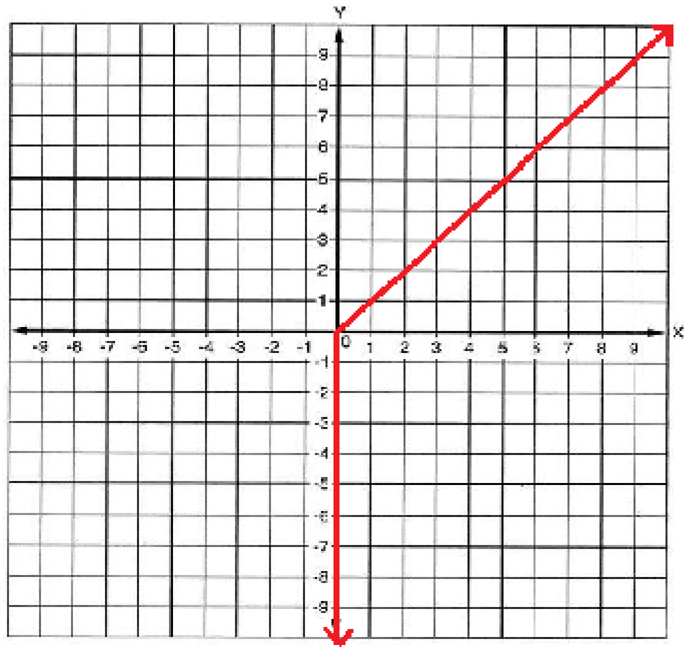
$$\begin{aligned} a &< b \\ a^2 &< ab \\ a^2 &< ab < b^2 \\ a^2 &< b^2 \end{aligned}$$

b) **Proof.** We'll start with the actual inequality again and go from there.

$$
\begin{aligned}
a &< b \\
b - a &> 0 \\
(\sqrt{b} - \sqrt{a})(\sqrt{b} + \sqrt{a}) &> 0 \\
\sqrt{b} - \sqrt{a} &> 0 \\
\sqrt{b} &> \sqrt{a}
\end{aligned}
$$

□

Exercise 6. Sketch the region of the points (x, y) satisfying the following relation: $x + |x| = y + |y|$ (explain your answer)



For positive x and y, it is simply the graph of $y = x$. The reason is because we know that the LHS and RHS will always be positive. So the equation simply becomes $2x = 2y$.
Now on the other hand, we know that x can never equal 0. If y is positive, then x is positive. However, if y is negative, then x will always be 0. Also, if $x = 0$, y can be any negative number and the equation will hold true, so that's why I drew a vertical line going downward from 0 to $-\infty$.

Exercise 7. If $x \geq 0$ and $y \geq 0$, prove that $\sqrt{xy} \leq \frac{x+y}{\sqrt{2}}$

Proof. To prove this, we will expand and simplify this inequality to get an obviously true statement.

$$\begin{aligned}\sqrt{xy} &\leq \frac{x+y}{\sqrt{2}}\\ \sqrt{2xy} &\leq x+y\\ 2xy &\leq x^2+2xy+y^2\end{aligned}$$

Since $x, y \geq 0$, we know this last statement to be true. Therefore this statement is true.$\square$

Exercise 8. Find the infimum and supremum (if they exist) of the following sets. Make sure to justify all your answers:
a) $E := \{x \in \mathbb{R} : x \geq 0 \text{ and } x^2 \leq 9\}$.
b) $E := \{\frac{4n+5}{n+1} : n \in \mathbb{N}\}$.

a) Because $x \geq 0$, the set will never contain a value less than $0^2 = 0$. Therefore, the infimum is 0.
For the supremum, I will show that 3 is the correct supremum.
Proof: Let's suppose $s = sup(E)$. We must show that $x = 3$. There's three possible cases

$$\begin{aligned}&1)s < 3\\ &2)s = 3\\ &3)s > 3\end{aligned}$$

For case 1, we assume $s < 3$. We know $x \geq 0$ from the definition of E. So according to the Archimedes Property, we set $x = x$, and $n = x$, and $y = s$ to get

$$\begin{aligned}x(x) &> s\\ x^2 &> s \quad \#\end{aligned}$$

For case 3, s cannot be greater than 3 we are assuming 3 is the supremum, and showing that x is equal to it.
Therefore, the only option we have left is that $x = 3$ $\square$

b) In this set, the least possible value that n can be is 1. So when we input that into the function, we get a value of

$$\begin{aligned}\frac{4(1)+5}{1+1} &= \frac{4+5}{2}\\ &= \frac{9}{2}\end{aligned}$$

Therefore the supremum is $\frac{9}{2}$. The reason why it's the supremum is because this function is a decreasing function as n increases, so the least possible n will actually be the max value.
Now for the infimum, I will prove that $inf(E) = 4$.
Proof. Let's suppose that $x = inf(E)$. Now we need to show that $x = 4$. There are three possible cases

$$1)x < 4$$
$$2)x = 4$$
$$3)x > 4$$

For case 1, this scenario is impossible. We are assuming that 4 is the infimum, so we cannot have x be less than it.
For case 3, suppose toward a contradiction that $x > 4$. Now putting this into the form of the Archimedes Property, we can set $x = (x - 4)$ and $y = 5 - x$. To be precise, $(x - 4) > 0$ because we are assuming that $x > 4$. From here, we can do some algebra to reach a contradiction.

$$\begin{aligned}
n(x-4) &< 5 - x \\
nx - 4n &< 5 - x \\
-4n - 5 &< -nx - x \\
-4n - 5 &< x(-n-1) \\
\frac{-4n-5}{-n-1} &< x \\
\frac{(-1)(4n+5)}{(-1)(n+1)} &< x \\
\frac{4n+5}{n+1} &< x\#
\end{aligned}$$

Therefore, since we eliminated two of the three cases, it must be that $x = 4 = inf(E)$. $\square$

Exercise 9. Let A be a non-empty set and $P(A)$ be its power set. Prove that A is not equivalent to $P(A)$. Deduce that $P(\mathbb{N})$ is not countable. [Hint: Define $C := \{x : x \in A \text{ and } x \notin f(x)\}$.]

Proof. We must first prove that $A \neq P(A)$. This can be done trivially. Say we have a set A with $|A| = k$. Then we know the $|P(A)| = 2^k$. This is because for every element in A, we have a "choice" whether to include that element in a set or not. By the multiplication rule of counting, the total possible number of sets (the power set) is equal to 2^k. Now, since $|A| \neq |P(A)|$, these two sets cannot be equal. $\square$

Now to deduce that $P(\mathbb{N})$ is not countable, we can use a method called diagonalization.

Proof. Let's assume toward a contradiction that $P(\mathbb{N})$ is countable. This means that there is an injection $f : P(\mathbb{N}) \to \mathbb{N}$. Now like before, we know that each set in $P(\mathbb{N})$ is a combination of "yes" and "no" for each element in $\mathbb{N}$. For example, in the power set, the set $\{1, 2, 3\}$ is a "yes", "yes", "yes", "no", "no",...,"no". We say "yes" to the numbers $1, 2, 3$ while saying "no" to the rest of the natural numbers. So let's assume we have the entirety of $P(\mathbb{N})$ mapped this way, as a combination of "yes" and "no". Now from here, we can always find an element in the set $C := \{x : x \in P(\mathbb{N}) \text{ and } x \notin f(x)\}$, $f(x)$ being the natural numbers. For each element $x_i \in P(\mathbb{N})$, we take the $i'th$ number and flip the "yes" and "no". For example, say the first element we map ($i = 1$) from the power set is $\varnothing$ (all "no"). Then the second is the set of even numbers ($i = 2$, only "yes" on the even numbers). And so on for every single set possible. If we take the $i'th$ number and flip "yes" to "no" and vice versa, we will always find a new set with a new combination of numbers, since it'll differ from every prior set by the $i'th$ number. Therefore, even with "every" single element mapped to the natural numbers, we can always find one more set in the power set that the natural numbers cannot account for. Therefore, it is uncountable.$\square$

Exercise 10. Let $E \subseteq \mathbb{R}$ be bounded from above and $E \neq \varnothing$. For $r \in \mathbb{R}$, let

$$rE := \{rx : x \in E\}$$
$$r + E := \{r + x : x \in E\}$$

Show that
a) if $r > 0$, then $sup(rE) = rsup(E)$
b) for any $r \in \mathbb{R}$, $sup(r + E) = r + sup(E)$

a) **Proof.** In the set E, let each element be denoted e_i such that $E := \{e_1, e_2, ..., e_i\}$ and have them ordered $e_1 < e_2 < ... < e_i$. From here, we can deduce that $e_i \leq sup(E)$. We can then multiply both sides by r to get $re_i \leq rsup(E)$. Therefore, $rsup(E)$ is a valid supremum for the set rE since $re_i \in rE$ and is its greatest element. It is itself an upper bound, and there is nothing less than it that is still a valid upper bound. We know that because e_i was the "last" element before surpassing the supremum. Therefore, by multiplying both sides of the inequality earlier by r, we know that re_i is the "last" element before surpassing $rsup(E)$.$\square$

b) **Proof.** We'll approach this similarly to the previous problem, but instead of multiplying both sides by r, we add. Following the same reasoning, e_i is the last and greatest element of the set E before surpassing $sup(E)$. Therefore, we have the inequality $e_i \leq sup(E)$. Now we add r to both sides to get $r + e_i \leq sup(E) + r$. Since $(r + e_i) \in r + E$, and $r + e_i$ is the greatest element $\in (r + E)$, $sup(E) + r$ is a valid supremum. Every element is less than it by the inequality, and there is no element less than it that is still a valid supremum because $r + e_i$ was itself the greatest element before surpassing $sup(E + r)$.$\square$