MATH-331 Introduction to Real Analysis — Ian Oga
Homework 05 — Fall 2021

Due date: November, 22$^{\text{th}}$ 1:20pm — Total: 57/65.

Très bien!

Exercise	1 (10)	2 (10)	3 (5)	4 (5)	5 (5)	6 (10)	7 (5)	8 (5)	9 (5)	10 (5)
Score	10	10	5	5	3	10	2	2	5	5

Table 1: Scores for each exercises

Instructions: You must answer all the questions below and send your solution by email (to `parisepo@hawaii.edu`). If you decide to not use LaTeX to hand out your solutions, please be sure that after you scan your copy, it is clear and readable. Make sure that you attached a copy of the homework assignment to your homework.
If you choose to use LaTeX, you can use the template available on the course website.
No late homework will be accepted. No format other than PDF will be accepted. Name your file as indicated in the syllabus.

1 — Writing problems

For each of the following problems, you will be asked to write a clear and detailed proof. You will have the chance to rewrite your solution in your semester project after receiving feedback from me.

Exercise 1. (10 pts)

a) Fix any $\delta > 0$ and let $[a, b]$ be an interval with $a < b$. Find a tagged partition $\mathcal{P}$ of $[a, b]$ such that $\|\mathcal{P}\| < \delta$.

b) Suppose that f is Riemann integrable. Show that in the definition of the Riemann integral, the number L is unique. [Remark: This is why we gave it the name $\int_a^b f$.]

Solution:

a) We know that for any positive δ, there exists $n \in \mathbb{N}$ such that $\frac{b-a}{n} < \delta$. We then define $\mathcal{P}$ in the following way:
$\mathcal{P} = \{(a + \frac{b-a}{n} i, [a + \frac{b-a}{n} i, a + \frac{b-a}{n}(i+1)]) : 0 \leq i \leq n-1\}$
Note that the length of each partition is $\frac{b-a}{n}$, and thus the norm of $\mathcal{P}$ is $||\mathcal{P}|| = \frac{b-a}{n} < \delta$. We can also see that the first interval is $[a, a + \frac{b-a}{n}]$ and the last interval is $[a + \frac{b-a}{n}(n-1), b]$. Since every $v_i = u_{i+1}$, this means that the intervals are non-overlapping and cover the interval $[a, b]$.

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10/10

b) Suppose towards a contradiction that L is not unique. Then let L be L_1, L_2 where $L_1 \neq L_2$. Without loss of generality, let $L_2 > L_1$. Then there exists δ such that:
$||\mathcal{P}|| < \delta \to |S(f,\mathcal{P}) - L_1| < \frac{L_2-L_1}{2}$
$||\mathcal{P}|| < \delta \to |S(f,\mathcal{P}) - L_2| < \frac{L_2-L_1}{2}$
5/5 We then have the following for all $||\mathcal{P}|| < \delta$
$\frac{-L_2+L_1}{2} < S(f,\mathcal{P}) - L_1 < \frac{L_2-L_1}{2}$
$\frac{-L_2+3L_1}{2} < S(f,\mathcal{P}) < \frac{L_2+L_1}{2}$
$S(f,\mathcal{P}) < \frac{L_2+L_1}{2}$ ✓
$\frac{-L_2+L_1}{2} < S(f,\mathcal{P}) - L_2 < \frac{L_2-L_1}{2}$
$\frac{L_2+L_1}{2} < S(f,\mathcal{P}) < \frac{3L_2-L_1}{2}$
$\frac{L_2+L_1}{2} < S(f,\mathcal{P})$
We now have $S(f,\mathcal{P}) < \frac{L_2+L_1}{2}$ and $\frac{L_2+L_1}{2} < S(f,\mathcal{P})$, which is a contradiction. Therefore $L_1 = L_2$ and the integral L is unique. □

Exercise 2. (10 pts) Suppose that f and g are Riemann integrable on the interval $[a,b]$.

a) Show that $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.

b) Show that if $f(x) \leq g(x)$ for any $x \in [a,b]$, then $\int_a^b f \leq \int_a^b g$.

Solution:

10/10

a) Since f and g are Riemann integrable, we have L_f and L_g such that for arbitrary ε there exists δ_f, δ_g where:
$||\mathcal{P}|| < \delta_f \to |S(f,\mathcal{P}) - L_f| < \frac{\varepsilon}{2}$
$||\mathcal{P}|| < \delta_g \to |S(g,\mathcal{P}) - L_g| < \frac{\varepsilon}{2}$
Define $\delta = \max(\delta_f, \delta_g)$. Then for any $||\mathcal{P}|| < \delta$:
$|S(f,\mathcal{P}) - L_f| < \frac{\varepsilon}{2}$
$|S(g,\mathcal{P}) - L_g| < \frac{\varepsilon}{2}$
Adding these inequalities:
$|S(f,\mathcal{P}) - L_f| + |S(g,\mathcal{P}) - L_g| < \varepsilon$
$|S(f,\mathcal{P}) - L_f + S(g,\mathcal{P}) - L_g| < \varepsilon$
$|(S(f,\mathcal{P}) + S(g,\mathcal{P})) - (L_f + L_g)| < \varepsilon \qquad (*)$
Let's look at $S(f,\mathcal{P}) + S(g,\mathcal{P})$:
$S(f,\mathcal{P}) + S(g,\mathcal{P}) = \sum_i f(c_i)(v_i - u_i) + \sum_i g(c_i)(v_i - u_i)$
$S(f,\mathcal{P}) + S(g,\mathcal{P}) = [f(c_1)(v_1-u_1) + f(c_2)(v_2-u_2) + f(c_3)(v_3-u_3)\cdots] + [g(c_1)(v_1-u_1) + g(c_2)(v_2-u_2) + g(c_3)(v_3-u_3)\cdots]$
$S(f,\mathcal{P}) + S(g,\mathcal{P}) = [f(c_1)(v_1-u_1) + g(c_1)(v_1-u_1)] + [f(c_2)(v_2-u_2) + g(c_2)(v_2-u_2)] + [f(c_3)(v_3-u_3) + g(c_3)(v_3-u_3)] + \cdots$
$S(f,\mathcal{P}) + S(g,\mathcal{P}) = (f(c_1)+g(c_1))(v_1-u_1) + (f(c_2)+g(c_2))(v_2-u_2) + (f(c_3)+g(c_3))(v_3-u_3) + \cdots$
$S(f,\mathcal{P}) + S(g,\mathcal{P}) = \sum_i (f(c_i)+g(c_i))(v_i-u_i)$
$S(f,\mathcal{P}) + S(g,\mathcal{P}) = \sum_i (f+g)(c_i)(v_i-u_i)$
5/5 $S(f,\mathcal{P}) + S(g,\mathcal{P}) = S(f+g,\mathcal{P})$
Returning to (*) and substituting, we have that for any $||\mathcal{P}|| < \delta$:
$|S(f+g,\mathcal{P}) - (L_f+L_g)| < \varepsilon$
This proves that $f+g$ is Riemann integrable and $\int_a^b (f+g) = L_f + L_g = \int_a^b f + \int_a^b g$.

b) Since $g(x) \geq f(x)$, we can write $g(x) = f(x) + h(x)$ where $h(x) \geq 0$ for all $x \in [a, b]$. We will now prove that $L = \int_a^b h$ is non-negative. Assume towards a contradiction that L is negative. Since f and g are Riemann integrable, h is also Riemann integrable. Then there exists δ such that for any partition where $||\mathcal{P}|| < \delta$, $|S(h, \mathcal{P}) - L| < \frac{-L}{2}$. We now have:
$S(h, \mathcal{P}) - L < \frac{-L}{2}$
5|5 $S(h, \mathcal{P}) < \frac{L}{2}$
$S(h, \mathcal{P}) < 0$ ✓
However, we have that:
$S(h, \mathcal{P}) = \sum_i h(c_i)(v_i - u_i)$
Since $h(x) \geq 0$ for all $x \in [a, b]$, the summation is non-negative and $S(h, \mathcal{P}) \geq 0$. This is a contradiction. Therefore $L = \int_a^b h \geq 0$. Since we defined $g(x) = f(x) + h(x)$, we know that $\int_a^b g = \int_a^b f + \int_a^b h$ from part a. As $\int_a^b h \geq 0$, $\int_a^b g \geq \int_a^b f$, which is what we wanted to prove.
$\square$

Exercise 3. (5 pts) Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable on $[a, b]$ and suppose that $|f(x)| \leq M$ for all $x \in [a, b]$. Show that $\int_a^b f \leq M(b - a)$.

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Solution: Define function $g(x) = M$ on the interval $[a, b]$. In Exercise 6, we show that g is Riemann integrable and $\int_a^b g = M(b - a)$. Since $|f(x)| \leq M$, $f(x) \leq g(x)$ and we know from Exercise 2b that $\int_a^b f \leq \int_a^b g$. Therefore $\int_a^b f \leq M(b - a)$. $\square$

Exercise 4. (5 pts) Suppose that f is Riemann integrable on $[a, b]$. Let $(\mathcal{P}_n)_{n=1}^{\infty}$ be a sequence of tagged partitions of $[a, b]$ such that the sequence $\lim_{n\to\infty} \|\mathcal{P}_n\| = 0$. Prove that the sequence $(S(f, \mathcal{P}_n))_{n=1}^{\infty}$ converges to $\int_a^b f$.

5|5

Solution: Since f is Riemann integrable, for any ε, there exists a δ such that $||\mathcal{P}|| < \delta \to |S(f, \mathcal{P}) - L| < \varepsilon$. Since $L = \int_a^b f$, $||\mathcal{P}|| < \delta \to |S(f, \mathcal{P}) - \int_a^b f| < \varepsilon$. As $\lim_{n\to\infty} ||\mathcal{P}_n|| = 0$, there exists $m \in \mathbb{N}$ such that $||\mathcal{P}_n|| < \delta$ for all $n > m$. We now have that for an arbitrary ε, there exists $m \in \mathbb{N}$ such that:
$\forall n > m, ||\mathcal{P}_n|| < \delta \to |S(f, \mathcal{P}_n) - \int_a^b f| < \varepsilon$ ✓
$\forall n > m, |S(f, \mathcal{P}_n) - \int_a^b f| < \varepsilon$
This proves that $S(f, \mathcal{P}_n)$ converges to $\int_a^b f$. $\square$

Exercise 5. (5 pts) Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Suppose that f is Riemann integrable on $[a, c]$ for any $c \in (a, b)$. Show that f is Riemann integrable on $[a, b]$. [Hint: Use the Cauchy criterion for integrals.]

Solution: As f is bounded, define M such that $|f(x)| \leq M$. For an arbitrary ε, let $c = b - \frac{\varepsilon}{2M}$. Since f is Riemann integrable on $[a, c]$, there is a δ_1 such that for any partition $\mathcal{P}_1 < \delta_1$, $|S(f, \mathcal{P}_1) - L| < 0.1\varepsilon$. Therefore: → L depends on the number c which change in terms of ε. So L is not a fixed value independant of ε.
$-0.1\varepsilon < S(f, \mathcal{P}_1) - L < 0.1\varepsilon$ (*)
Define $\mathcal{P}_1' = \mathcal{P}_1 \cup (c, [c, b])$. Now $\mathcal{P}_1'$ is a tagged partition over $[a, b]$ where $||\mathcal{P}_1'|| < \max(\delta_1, \frac{\varepsilon}{2M})$ and

$S(f,\mathcal{P}_1') = S(f,\mathcal{P}_1) + f(c)(b-c)$. As $|f(x)| \leq M$ and $c = b - \frac{\varepsilon}{2M}$, we have that:
$|f(c)| \leq M$
$-M \leq f(c) \leq M$
$c = b - \frac{\varepsilon}{2M}$
$b - c = \frac{\varepsilon}{2M}$

$S(f,\mathcal{P}_1') = S(f,\mathcal{P}_1) + f(c)(b-c)$
$\frac{1}{b-c}[S(f,\mathcal{P}_1') - S(f,\mathcal{P}_1)] = f(c)$
$\frac{2M}{\varepsilon}[S(f,\mathcal{P}_1') - S(f,\mathcal{P}_1)] = f(c)$

$-M \leq \frac{2M}{\varepsilon}[S(f,\mathcal{P}_1') - S(f,\mathcal{P}_1)] \leq M$
$-0.5\varepsilon \leq S(f,\mathcal{P}_1') - S(f,\mathcal{P}_1) \leq 0.5\varepsilon$ (**)
Now for a tagged partition $\mathcal{P}$ on $[a,b]$ there exists a δ_2 where if $||\mathcal{P}|| < \delta_2$:
$|S(f,\mathcal{P}) - S(f,\mathcal{P}_1')| < 0.1\varepsilon$
$-0.1\varepsilon < S(f,\mathcal{P}) - S(f,\mathcal{P}_1') < 0.1\varepsilon$ (***)
Now defining $\delta = \max(\delta_1, \delta_2, \frac{\varepsilon}{2M})$, we have that $||\mathcal{P}||, ||\mathcal{P}_1||, ||\mathcal{P}_1'|| < \delta$. Then (*), (**), and (***) are true:
$-0.1\varepsilon < S(f,\mathcal{P}_1) - L < 0.1\varepsilon$
$-0.5\varepsilon \leq S(f,\mathcal{P}_1') - S(f,\mathcal{P}_1) \leq 0.5\varepsilon$
$-0.1\varepsilon < S(f,\mathcal{P}) - S(f,\mathcal{P}_1') < 0.1\varepsilon$
Adding these inequalites:
$-0.7\varepsilon < S(f,\mathcal{P}) - L < 0.7\varepsilon$
$|S(f,\mathcal{P}_1') - L| \leq 0.7\varepsilon$
$|S(f,\mathcal{P}_1') - L| \leq \varepsilon$
This proves that f is Riemann integrable on $[a,b]$. □

L depends on ε, so the value will change when ε change...

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2 Homework problems

Answer all the questions below. Make sure to show your work.

Exercise 6. (10pts)

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a) Define the function $f : [a,b] \to \mathbb{R}$ by $f(x) = k$ for every $x \in [a,b]$ where $k \in \mathbb{R}$ is a fixed constant. Show that f is Riemann integrable on $[a,b]$ and that $\int_a^b k\,dx = k(b-a)$.

b) Let $f(x) = \sin^2(x)$ where $x \in [a,b]$ and assume that the function $g(x) := \cos(kx)$ is integrable on $[a,b]$ for any $k \in \mathbb{R}$. Show that f is Riemann integrable on $[a,b]$.

Solution:

a) For any tagged partition on f, we have that:
$S(f,\mathcal{P}) = \sum_i f(c_i)(v_i - u_i)$
$S(f,\mathcal{P}) = \sum_i k(v_i - u_i)$
$S(f,\mathcal{P}) = k\sum_i (v_i - u_i)$

5/5 Since $v_i = u_{i+1}$ for all i, the summation becomes the length of the interval.
$S(f, \mathcal{P}) = k(b-a)$
We then have that for $L = k(b-a)$ and any positive ε, δ, norm($\mathcal{P}$)$< \delta \to |S(f,\mathcal{P}) - L| = 0 < \varepsilon$.
This is what we wanted to prove.

b) We know from trigonometry that:
$\cos(2x) = 1 - 2\sin^2(x)$
$\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$
5/5 We know from Theorem 5.9 that a linear sum of bounded Riemann integrable functions in Riemann integrable. Since constants and $\cos(2x)$ are bounded and Riemann integrable, $\sin^2(x)$ must then be Riemann integrable. $\square$

Exercise 7. (5 pts) Show that the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} 1 & \text{, if } 0 \le x < 1/2 \\ 0 & \text{, if } 1/2 \le x \le 1 \end{cases}$$

is Riemann integrable on $[0,1]$.

Solution: We will define a tagged partition $\mathcal{P}_0(\delta)$ for any given δ. For any δ, we know that there exists integer k where $\frac{1}{2k} < \delta$. For such a k, let $\mathcal{P}_0(\delta) = \{(\frac{i-1}{2k}, [\frac{i-1}{2k}, \frac{i}{2k}]) : 1 \le i \le 2k\}$. Note that the norm of this tagged partition is $\frac{1}{2k}$ which is less than δ. We will now find $S(f, \mathcal{P}_0(\delta))$:
$S(f,\mathcal{P}_0(\delta)) = \sum_{i=1}^{2k} f(\frac{i-1}{2k})(\frac{1}{2k})$
$S(f,\mathcal{P}_0(\delta)) = \sum_{i=1}^{k} f(\frac{i-1}{2k})(\frac{1}{2k}) + \sum_{i=k+1}^{2k} f(\frac{i-1}{2k})(\frac{1}{2k})$
$S(f,\mathcal{P}_0(\delta)) = \sum_{i=1}^{k}(\frac{1}{2k}) + \sum_{i=k+1}^{2k} 0 \cdot (\frac{1}{2k})$
$S(f,\mathcal{P}_0(\delta)) = \frac{1}{2}$

how do you know that $|S(f,P) - S(f,P_0)| < \varepsilon$? Essentially, you want to prove that $|S(f,P_1) - S(f,P_2)| < \varepsilon$.

By the Cauchy Criterion, we have that for an arbitrary ε, there exists δ such that:
$||\mathcal{P}|| < \delta, ||\mathcal{P}_0(\delta)|| < \delta \to |S(f,\mathcal{P}) - S(f,\mathcal{P}_0(\delta))| < \varepsilon$
$||\mathcal{P}_0(\delta)|| < \delta$ is guaranteed by our definition of $\mathcal{P}_0(\delta)$.
$||\mathcal{P}|| < \delta \to |S(f,\mathcal{P}) - \frac{1}{2}| < \varepsilon$

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This proves that f is Riemann integrable. $\square$

Exercise 8. (5 pts) Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = 1$ if $x = 1/n$ where $n \in \mathbb{N}$, and by $f(x) = 0$ if $x \neq 1/n$, $n \in \mathbb{N}$. Show that f is Riemann integrable on $[0,1]$.

Solution: Similar to Exercise 7, we will define a tagged partition $\mathcal{P}_0(\delta)$ for any given δ. For any δ, we know that there exists integer k where $\frac{1}{k} < \delta$. For such a k, let $\mathcal{P}_0(\delta) = \{(c_i, [\frac{i-1}{k}, \frac{i}{k}]) : 1 \le i \le k\}$. For each c_i, pick an irrational tag. Note that the length of every interval in the partition is $\frac{1}{k}$ which is less than δ. Since the irrationals are dense in the real numbers, this will always be possible. Now consider $S(f, \mathcal{P}_0(\delta))$:
$S(f,\mathcal{P}_0(\delta)) = \sum_i f(c_i)(\frac{1}{k})$

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Since all tags c_i are irrational, $f(c_i)$ is always 0 and the summation is 0. Therefore $S(f, \mathcal{P}_0(\delta)) = 0$.
By the Cauchy Criterion, we have that for an arbitrary ε, there exists δ such that:
$||\mathcal{P}|| < \delta, ||\mathcal{P}_0(\delta)|| < \delta \rightarrow |S(f, \mathcal{P}) - S(f, \mathcal{P}_0(\delta))| < \varepsilon$
$||\mathcal{P}_0(\delta)|| < \delta$ is guaranteed by our definition of $\mathcal{P}_0(\delta)$.
$||\mathcal{P}|| < \delta \rightarrow |S(f, \mathcal{P})| < \varepsilon$
This proves that f is Riemann integrable. □

this doesn't prove that f satisfies the Cauchy criterion.

Exercise 9. (5 pts) Show that the function $f : [0, 1] \rightarrow \mathbb{R}$ defined by $f(x) = 0$ if $x \neq 0$ and $f(x) = 4$ if $x = 0$ is Riemann integrable on $[0, 1]$.

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Solution: Suppose that we have a tagged partition $\mathcal{P}$ where $||\mathcal{P}|| < \delta$. We will prove that $S(f, \mathcal{P}) < 4\delta$.
$S(f, \mathcal{P}) = \sum_{i=1} f(c_i)(v_i - u_i)$
Since f is 0 everywhere except for at $x = 0$, the only way for $S(f, \mathcal{P})$ to be positive is for the interval $[0, v_1]$ to be tagged at 0. As $||\mathcal{P}|| < \delta$, $v_1 < \delta$
$S(f, \mathcal{P}) < f(0)(v_1 - 0)$
$S(f, \mathcal{P}) < 4v_1$
$S(f, \mathcal{P}) < 4\delta$
Since f is non-negative, we also know that $S(f, \mathcal{P}) \geq 0$. Now suppose we have an arbitrary $\varepsilon > 0$. Let $\delta = \frac{\varepsilon}{4}$. Then if $||\mathcal{P}|| < \delta$:
$-\varepsilon < 0 \leq S(f, \mathcal{P}) < 4\delta$
$-\varepsilon < S(f, \mathcal{P}) < \varepsilon$
$|(f, \mathcal{P})| < \varepsilon$
This proves that f is Riemann integrable. □

Exercise 10. (5 pts) Let $\mathcal{P}$ be the following tagged partition of $[-1, 2]$:

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$$\mathcal{P} := \{(-9, [-1, -.8]), (-.7, [-.8, -.3]), (-.1, [-.3, 0]), (.2, [0, 0.2]), (.2, [.2, .4]), (.8, [.4, 1]), \\ (1.42, [1, 1.5]), (1.9, [1.5, 2])\}.$$

Find another partition $\mathcal{P}_0$ such that $\|\mathcal{P}_0\| \leq \|\mathcal{P}\|/3$.

✓ **Solution:** $||\mathcal{P}|| = 0.6$ as the longest interval in $\mathcal{P}$ is $[0.4, 1]$. We must then find a tagged partition $\mathcal{P}_0$ where $||\mathcal{P}_0|| \leq 0.2$. Define the partition as so:
$\mathcal{P}_0 = \{(0.2(i-1), [0.2(i-1), 0.2i]) : -4 \leq i \leq 10\}$ □