Math 331 HW1

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1 Prove that for any $n \in \mathbb{Z}$, $1 + 2 + ... + n = \frac{n(n+1)}{2}$

Let us use induction to prove the statement. Assume the statement is true and check for base case.

1.1 Base case: $n = 3$

$$\begin{aligned} 1 + 2 + &= \frac{3(3+1)}{2} \\ 6 &= \frac{3(4)}{2} \\ 6 &= \frac{12}{2} \\ 6 &= 6 \end{aligned}$$

Thus we see that our base case is proven, let us prove our induction case of $n + 1$

1.2 Induction: $n + 1$

$$1 + 2 + ... + n + (n + 1) = \frac{n + 1(n + 2)}{2}$$

From here, we know that the case of $1 + 2 + ... + n = \frac{n(n+1)}{2}$, thus let us substitute as such;

$$
\begin{aligned}
\frac{n(n+1)}{2}+n+1 &= \frac{n+1(n+2)}{2} \\
\frac{n(n+1)+2n+2}{2} &= \frac{n+1(n+2)}{2} \\
\frac{n^2+3n+2}{2} &= \frac{n^2+3n+2}{2}
\end{aligned}
$$

Therefore we see that our induction case is true, thus proving the above equation is true for all cases $n \in \mathbb{Z}$

2 Define $f : \mathbb{N} \to \mathbb{N}$ by $f(a) = 1, f(2) = 2, f(3) = 3$ and $f(n) := f(n-1) + f(n-2) + f(n-3)$ for $n \geq 4$. Prove that $f(n) \leq 2^{n-1}$.

For this question, let us prove by induction. First let us establish our base case of $n = 4$.

2.1 Base case: $n = 4$

$$
\begin{aligned}
f(4) :=& f(4-1) + f(4-2) + f(4-3) \\
f(4) :=& f(3) + f(2) + f(1) \\
f(4) :=& 3 + 2 + 1 \\
f(4) :=& 6 \\
f(4) \leq& 2^{4-1} \\
6 \leq& 2^3 \\
6 \leq& 8
\end{aligned}
$$

We see that our base case proves to be true. Let us now prove all cases for $(n-k)$ is true, and therefore all cases for $f(n+1)$ must also be true.

2.2 Induction: $n - k$

Assume that for all cases less than n will work for $n - k$ for $k \geq 1$. Let us prove our hypothesis.

$$
\begin{aligned}
f(n-k) \leq& 2^{n-k-1} \\
f(n-k) \leq& \frac{2^{n-k}}{2} \\
2f(n-k) \leq& 2^{n-k}
\end{aligned}
$$

From here we see that both the LHS and RHS will both be even since the LHS will be multiplied by two and the RHS is the power of two. Thus whatever k might be for $k \geq 1$, our answer for both side of the inequality can either be factored by two or any multiples of two. Thus, we can actually reduce it down to two cases. The first case is when we reduce it down, both LHS and RHS are equal to each other since both can be expressed by a factor or multiple of two. Then let the LHS be p and and RHS be q for $p = q$;

$$\begin{aligned} 2f(n-k) &\leq 2^{n-k} \\ p &\leq q \\ \frac{p}{q} &\leq 1 \end{aligned}$$

We see that this is true. We also see that this is also true for if in our second case, the LHS and RHS, when reduced down are relatively prime, meaning $p \neq q$, thus we can follow the same logic and obtain;

$$\frac{p}{q} \leq 1$$

We know that this is also true since $p \neq q$, we know they cannot reduce down to one since they are relatively prime, thus $1 > \frac{p}{q}$. Therefore proving our induction case, thus proving that all cases that come after n for $n + k$ for $k \geq 1$ must be true as well. QED.

3 Prove that if A, B and C are sets, then;

3.1 $A \sim A$

We can see that if $A \sim A$, it means that they are reflexive of each other. So let us denote an $a \in A$ and if we let $A|A$, then we see that $a|a = 1$ since they are both the same element in the same set, they are divisible of each other, thus proving reflexive.

3.2 $A \sim B$ then $B \sim A$

We can also prove that A is symmetric to B by creating a function $f : A \to B$ for every domain $a \in A$ maps to a codomian $b \in B$, this means there will exists a $f(a) = b$. Since the function f can map from $A \to B$, there must also exists an inverse function of f for $f^{-1}(b) = a$ for we just switched our domain and codomain, thus proving symmetric.

3.3 $A \sim B \sim C$

We have to prove transitive for all sets A, B and C. Let us use the same process we did for $A \sim B$ for us to prove this statement. We already know that the function $f : A \to B$ proves to be true, thus we can create another function $g : B \to C$, for there exists a domain $b \in B$ that maps to a codomain of $c \in C$, thus we have a function for $g(b) = c$ and an inverse function of $g^{-1}(c) = b$. Therefore, we can create a composition of function of $h(g(f))$ or $h(g \circ f)$, thus we will have a function $h : A \to C$ for $h(a) = c$ and an inverse of $h^{-1}(c) = a$, proving transitivity.

4 Show that any subset of a countable set is countable

For this proof let us use $\mathbb{N}$, the set of natrual numbers. In class, we've proved $\mathbb{N}$ is countable. So now let us denote set A for set $A \subseteq \mathbb{N}$. We can prove injective, surjective, and bijective to show that there exists a function $f : A \to \mathbb{N}$.

4.1 Injective

The definition of a injective function is one-to-one, meaning every element in the domain can map to at least one one element in the codomain. Thus if there exists an element $a \in A$ and an element $n \in \mathbb{N}$, there exists a function $f : A \to \mathbb{N}$ for $f(a) = n$. We know that this is true since $\mathbb{N} > A$ in size where A is finite versus $\mathbb{N}$ is infinite. Since $A \subseteq \mathbb{N}$, there will exists at least one element for us to have a mapping of $f(a) = n$ or else $A \nsubseteq \mathbb{N}$. Therefore, proving our injective function.

4.2 Surjective

The definition of surjective states that a surjective function is onto, meaning for every domain that exists it can map to at most one codomain. We see that this is also true following the logic of our previous proof. Since $A \subseteq \mathbb{N}$, for the function $f(a) = n$, there can exists an inverse function of $f^{-1}(n) = a$ for our surjective function. Thus there can map to at most one element for each domain and their respective codomain. Thus proving our surjective function.

4.3 Bijective

For bijective functions, all it needs to be is to be both injective and surjective. We see that we've proven both of them, thus our function is bijective as well, proving that for a subset of any coutable sets, the subject is also countable.

5 Let $0 < a < b$ be positive real numbers. Prove that

5.1 $a^2 < b^2$

From our properties in class, we know that $\forall x \in \mathbb{R}, x^2 \geq 0$. Thus we know that since $a, b \in \mathbb{R}$, $a^2, b^2 \geq 0$. But from our statement above we see that $0 < a < b$, thus actually, $a, b > 0$ for a, b are positive numbers. Therefore all we need to prove is for $a < b$ by the axioms we have in class.

5.1.1 Case 1: $a^2 > b^2$

Since we know that $a, b > 0$ we can square root both sides and a, b will remain positive even when we square root them.

$$\begin{aligned} \sqrt{a} &< \sqrt{b} \\ (\sqrt{a})^2 &< (\sqrt{b})^2 \\ a &< b \end{aligned}$$

By Axiom 1, we know that if $x > y$ then $zx > zy$ for $z > 0$. We can apply such axiom to our proof.

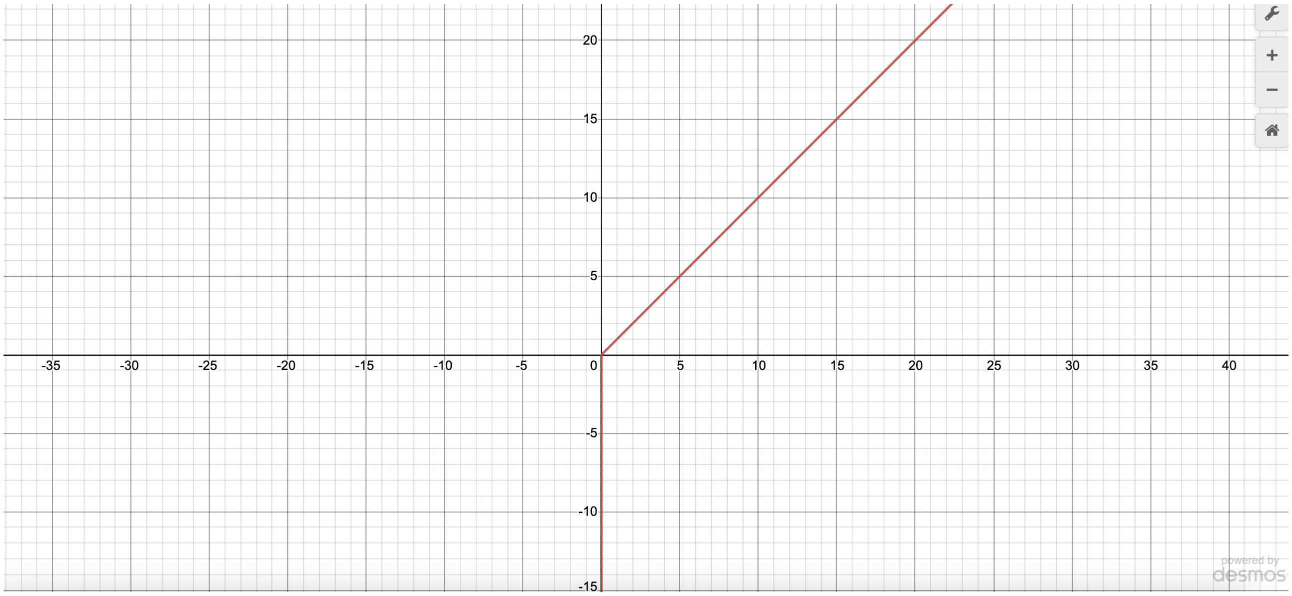
$$\begin{aligned} za &< zb \\ -1(za) &< -(zb) \\ -za &> -zb \\ -za + zb &> 0 \\ z(-a + b) &> 0 \\ -a + b &> 0 \\ b &> a \end{aligned}$$

Therefore we see that $b > a$, thus our statement is true.

5.2 $\sqrt{a} < \sqrt{b}$

For this case, we've actually proved it in the proof above, if we square both side of the inequality, we will obtain $a < b$, which is true for we've just proved it, thus it hold true for both cases.

6 Sketch the region of the points (x, y) satisfying the following relation: $x + |x| = y + |y|$, explain.



We see this graph as such is due to the properties of absolute value. Since we know that $\pm x \leq |x|$, we know that for both sides of the equation to be equal, we must have the same values for both x and y, in return, we will also have same values for $|x|$ and $|y|$. Therefore, when both x, y are in the negatives, their absolute value would be positive x, y, thus no matter what numbers we put in for $x, y < 0$, our function $x + |x| = y + |y|$ will always equal to zero, thus causing it to essentially look like an $x = 0$ graph until we hit $x, y > 0$ for now both x, y and $|x|, |y|$ are all positive, creating a graph that looks similar to $y = x$. Therefore, by the properties of absolute value, it caused our graph to look as such.

7 If $x \geq 0$ and $y \geq 0$, prove that $\sqrt{xy} \leq \frac{x+y}{\sqrt{2}}$

For this question, let us prove by using the axioms properties we've learned in class for $\sqrt{xy} \leq \frac{x+y}{\sqrt{2}}$.

$$
\begin{aligned}
\sqrt{xy} &\leq \frac{x+y}{\sqrt{2}} \\
\sqrt{x}\cdot\sqrt{y} &\leq \frac{x+y}{\sqrt{2}} \\
(\sqrt{x}\cdot\sqrt{y})^2 &\leq \frac{(x+y)^2}{2} \\
xy &\leq \frac{(x+y)^2}{2} \\
2xy &\leq (x+y)^2 \\
2xy &\leq x^2+2xy+y^2 \\
2xy-2xy &\leq x^2+y^2 \\
0 &\leq x^2+y^2
\end{aligned}
$$

We see from this step that the proof is true, QED.

8 Find the infimum and supremum (if they exist) of the following sets. Make sure you justify.

8.1 $E \coloneqq \{x \in \mathbb{R} : x \geq 0$ and $x^2 \leq 9\}$

For this set E, we see that $x \in \mathbb{R} : x \geq 0$ and $x^2 \leq 9$. This means the set $E \subseteq \mathbb{R}$ for subset E is bounded between $0 \leq x \leq 9$. Which we can see there is clearly a supremum and an infimum. The sefinition of a supremum is the least upper bound, and we see that sup(E)$\in E$ since it is less than or equal to, meaning our x can equal to the LUB of E. We also see there exists an infimum or GLB for our inf(E)$\in E$ as well since it is greater than or equal to 0, meaning 0 is part of the infimum. Therefore set E exists an infimum and supremum for inf(E)=0 and sup(E)=9.

8.2 $E \coloneqq \frac{4n+5}{n+1} : n \in \mathbb{N}$

For this statement, we can see that there exists an supremum or LUB for $n \in \mathbb{N}$, since the set of $\mathbb{N}$ starts at 1, we know that $\frac{9}{2}$ is our LUB or supremum. But we see something really interesting happens as we take the limit for $\frac{4n+5}{n+1}$, the set of E actually goes towards 4. Thus our inf(E)=4. QED.

9 Let A be a non-empty set and $P(A)$ be its power set (the family of all subsets of A). Prove that A is not equivalent to $P(A)$. Deduce that $P(\mathbb{N})$ is not countable. [Hint: Define $C := \{x \in A$ and $x \notin f(x)\}$

Since equivalents functions are functions that must include reflexive, symmetric, and transitive, if we were to prove any one of them is not true, then $A \sim P(A)$ is false. Thus from the hint, let us prove by contradiction and assume that the function $f : A \to P(A)$ is true.

Using our hint, we can see that if $c = f(c)$ for there exists a $c \in A$, then $c \notin f(c)$. Which in our case makes no sense since we agreed that $C \subseteq A$ and since $c \in C$ then $c \in A$ thus if there should exists an symmetric function then $c \in P(A)$. But that is false due to how we defined C for there exists an $x \in A$ for $x \notin f(x)$, thus $c \notin f(c)$. This will cause a contradiction for our statement assuming that our surjective function is true, therefore it is false, and A is not equivalent to $P(A)$, QED.

10 Let $E \subseteq \mathbb{R}$ be bounded from above and $E \neq \emptyset$. For $r \in \mathbb{R}$, let $rE := \{rx : x \in E\}$ and $r + E := \{r + x : x \in E\}$. Show,

For this problem, let us understand what sup(E) means. A supremum is defined as the least upper bound of subset within any set. In our case, our subset is $E \subseteq \mathbb{R}$. The simplest way we can imagine this is on a number line. Let us imagine the line of set $\mathbb{R}$, and within this line, a section of it has been bounded, and that section is the subset of $\mathbb{R}$, which we denote it as subset E. With that in mind, let us prove the two statements below.

10.1 If $r > 0$ then sup(rE)=$r$sup(E)

For this problem we see that $r > 0$ for $r \in \mathbb{R}$ where E is bounded from above, and $rE := \{rx : x \in E\}$. This means for the set of E there exists a $r \in R$ for which when multiplied to E, set E increases by a factor of r. So from here, let us assume the statement sup(rE)=$r$sup(E) is false for sup(rE)$\neq$ $r$sup(E) and prove by contradiction.

So assume that sup(rE)$\neq$ $r$sup(E). By field axiom 1, we know that multiplication does not require a specific order in which the elements are multiplied to. As explained in field axiom 1;

$$(x \cdot y) \cdot z = x \cdot (y \cdot z)$$

We can apply this logic in our problem. On the LHS of our assumption, E has already been increase by a factor of r, meaning, let $x = r$ and $y \cdot z = E$ for the LHS be sup($x \cdot y \cdot z$). Thus, let us apply the same logic on the RHS. Let $x = r$ and $y \cdot z = E$ once again, for the

RHS will now become $x\sup(y \cdot z)$ for;

$$sup(x \cdot y \cdot z) \neq xsup(y \cdot z)$$

By using the properties of field axiom 1, we know that the LHS has factors of x, y, z and the RHS also has factors of x, y, z since it does not matter the order of which we multiplied, our product will all be the same, therefore, we can actually divide both sides by x since we know both sides are factors of the 3 elements and we will obtain;

$$sup(y \cdot z) \neq sup(y \cdot z)$$
$$sup(E) \neq sup(E)$$

Which we see has a contradiction since $E \subseteq \mathbb{R}$, thus the $\sup(E)$ must equal to the $\sup(E)$ of the same set. Thus creating a contradiction, therefore, proving that $\sup(rE)$=$r\sup(E)$. QED.

10.2 For any $r \in \mathbb{R}$, sup($r + E$)=r+sup(E)

For this problem, we can do the same as we did for our previous proof. By field axiom 1 once again, we can see the properties of addition does not require an specfic order. As explained by field axiom 1;

$$(x + y) + z = x + (y + z)$$

By applying this logic to our current statement we can again prove this by contradiction and assume $\sup(r + E) \neq r$+$\sup(E)$. From here, let $r = x$ and $E = y + z$. We would obtain the equation;

$$sup(x + y + z) \neq x + sup(y + z)$$

Similarly from the previous problem, we see that for subset E, the LHS has $\sup(E + r)$, thus we are finding the supremum with the subset E plus r. While the RHS its finding $\sup(E)$ first, and then adding r. As we saw in field axiom 1. the order in which we add does not matter, our sum will still remain the same since addition is commutative where the order does not change the sum. Therefore, we can actually subtract x from both sides and obtain;

$$sup(y + z) \neq sup(y + z)$$

Which we see again, is another contradiction, thus proving that $\sup(r + E)$=r+$\sup(E)$ is true. QED.