MATH-331 Introduction to Real Analysis | BENDER, LEE'AN
Homework 07 | Fall 2021

Due date: December, 6$^{\text{th}}$ 1:20pm | Total: /65.

Exercise	1 (10)	2 (5)	3 (10)	4 (5)	5 (5)	6 (10)	7 (5)	8 (5)	9 (5)	10 (5)
Score										

Table 1: Scores for each exercises

Instructions: You must answer all the questions below and send your solution by email (to `parisepo@hawaii.edu`). If you decide to not use LaTeX to hand out your solutions, please be sure that after you scan your copy, it is clear and readable. Make sure that you attached a copy of the homework assignment to your homework.
If you choose to use LaTeX, you can use the template available on the course website.
No late homework will be accepted. No format other than PDF will be accepted. Name your file as indicated in the syllabus.

1
Writing problems

For each of the following problems, you will be asked to write a clear and detailed proof. All the exercises below can be solve without using the definition with partitions. Try to go back to homework 6 and use some of the exercises there to solve the following problems.

You will have the chance to rewrite your solution in your semester project after receiving feedback from me.

Exercise 1. (10 pts) Prove that a step function is Riemann integrable on $[a, b]$. Follow the steps below.

a) Let I be a subinterval of $[a, b]$ and put $\phi = c\chi_I$. Prove that ϕ is Riemann integrable and that $\int_a^b \phi = c\ell(I)$. [There are three cases to consider: $I = [u, v]$, $I = (u, v]$, and $I = \{u\} = [u, u]$.]

b) Prove by induction that if f_1, f_2, ..., f_n are Riemann integrable functions on $[a, b]$, then $f_1 + f_2 + \cdots + f_n$ is Riemann integrable and

$$\int_a^b (f_1 + f_2 + \cdots + f_n) = \int_a^b f_1 + \int_a^b f_2 + \cdots + \int_a^b f_n.$$

c) Write $\phi = \sum_{k=1}^n c_k \chi_{I_k}$. Use the second part of this exercise to show that ϕ is Riemann integrable.

Solution: :

a) We treat this as the case where $I = [u, v]$. From the statement, we have

$$f(x) = x\kappa_I(x) = \begin{cases} 0, a \leq x \leq u \\ c, u \leq x \leq v \\ 0, v \leq x \leq b \end{cases}$$

From the previous homework, we know that $\exists c \in [a, b]$ such that $[a, c]$ and $[c, b]$ are Riemann Integrable, then $[a, b]$ is Riemann Integrable. Since $\phi = 0$ on $[a, u]$, and $\phi = 0$ on $[v, b]$, it is R.Int on those intervals because all constants are R. Integrable. Also, since $[u, v]$ is a closed interval, then ϕ is also R. Integrable on $[u, v]$. Therefore , ϕ is R. Integrable on $[a, u]$, $[u, v]$, and $[v, b]$.

b) We treat this as the case where $I = (u, v]$. We know that ϕ is Riemann integrable on $[a, u]$ and $[v, b]$, as they are constant. From the previous homework, we proved using the Cauchy criterion that a function can be integrable from $(a, b]$ or $[a, b)$. Since ϕ is constant from $(u, v]$, then it is integrable from $(u, v]$. (?)

c) We treat this as the case where $I = \{u\} = [u, u]$. Since the interval I is just a point, and we know that all functions are integrable on single points, ϕ must be R. Integrable on I. $\square$

Exercise 2. (5 pts) Suppose that f is Riemann integrable on $[a, b]$ and that f is nonnegative (means that $f(x) \geq 0$ for $x \in [a, b]$). Let $u, v \in \mathbb{R}$. Show that if $a \leq u < v \leq b$, then

$$\int_u^v f \leq \int_a^b f.$$

[Hint: Use the following property of the Riemann Integral multiple times: $\int_a^b f = \int_a^c f + \int_c^b f$.]

Solution: We know that $\forall c \in [a, b]$, then $\int_a^b = \int_a^c + \int_c^b$, meaning that $\int_a^c \leq \int_a^b$ and $\int_c^b \leq \int_a^b$. Similarly, we are given that $[u, v] \subseteq [a, b]$, and $a \leq u < v \leq b$. So,

$$\int_a^u + \int_u^v + \int_v^b = \int_a^b$$

From the previous relationship, we then also know that

$$\begin{gathered} \int_a^u \leq \int_a^b \\ \int_u^v \leq \int_a^b \\ \text{and } \int_v^b \leq \int_a^b \end{gathered}$$

This gives us the relation we were looking for, $\int_u^v \leq \int_a^b$. $\square$

Exercise 3. (10 pts) Use the Fundamental Theorem of Calculus to solve the following problems:

a) Suppose that f is continuous on $[a, b]$ and that f is nonnegative on $[a, b]$. Show that if $\int_a^b f = 0$, then $f(x) = 0$ for any $x \in [a, b]$.

b) Suppose that f and g are continuous on $[a, b]$ such that $\int_a^b f = \int_a^b g$. Show that there exists a point $c \in (a, b)$ such that $f(c) = g(c)$.

Solution: :

a) We are given that $\int_a^b f = 0$. By the FTC, we know that $\frac{d}{dx}\int_a^b f = \frac{d}{dx}0$, so $f(b) - f(a) = 0$, meaning that $f(b) = f(a)$. We now need to show that f is constant and $= 0$ $\forall x \in [a, b]$. By the definition of Riemann Integral, if $\mathcal{P}$ is a tagged partition of $[a, b]$, then $\forall \epsilon > 0$, $\exists \delta > 0$ such that if $||\mathcal{P}|| < \delta$, then $|S(f, \mathcal{P}) - \int_a^b f| < \epsilon$. Substituting in 0 for $\int_a^b f$ gives us $|S(f, \mathcal{P}) - 0| < \epsilon$. So, we get $|S(f, \mathcal{P})| < \epsilon$. Substituting in the value for $S(f, \mathcal{P})$ gives

$$|\textstyle\sum_{i=1}^N f(c_i)(x_i - x_{i-1})| < \epsilon$$

This can be rewritten as

$$-\epsilon < \textstyle\sum_{i=1}^N f(c_i)(x_i - x_{i-1}) < \epsilon$$

We can see that $S(f, \mathcal{P})$ is a reasonable approximation for $\int_a^b f$ (?). If $f \neq 0$ at any $x \in [a, b]$, then we know that $f(c_i) > 0$ at some $c \in [a, b]$, meaning that $\sum_{i=1}^N f(c_i)(x_i - x_{i-1}) > 0$ $\forall x \in [a, b]$. However, this would be contradictory, as $\sum_{i=1}^N f(c_i)(x_i - x_{i-1}) = 0$ as given by the assumption. So we know that f must be 0 for all $x \in [a, b]$.

b) Suppose toward a contradiction that there does *not* exist a $c \in [a, b]$ such that $f(c) = g(c)$. So $f(x) \neq (x)$ $\forall x \in [a, b]$. Then it must be true that either

$$f(x) < g(x)\ \forall x \in [a, b] \text{ OR } f(x) > g(x)\ \forall x \in [a, b]$$

As proven in the previous homework, if $f(x) < g(x)$ $\forall x \in [a, b]$, then $\int_a^b f \leq \int_a^b g$ $\forall x \in [a, b]$. There are now two cases:

$$\textstyle\int_a^b f < \int_a^b g \text{ and } \int_a^b f = \int_a^b g$$

Looking at $\int_a^b f = \int_a^b g$, we can simplify to $\int_a^b f - \int_a^b g = 0$. According to the addition rules of integrals, this can be simplified to $\int_a^b (f - g) = 0$. As proven in part (a), if $\int_a^b f = 0$, then $f(x) = 0$ $\forall x \in [a, b]$. This means we can set $(f - g) = 0$ and obtain $f = g$ as our result. This contradicts our original assumption that either $f < g$ or $g < f$, so that must mean that there must exist $c \in [a, b]$ where $f(c) = g(c)$. $\square$

Exercise 4. (5 pts) Let f be a continuous function on $[a, b]$. Prove that there exists a number $c \in [a, b]$ such that $f(c)(b - a) = \int_a^b f$.

Solution: Let $M := \max\{f(x) : x \in [a, b]\}$ and $m := \min\{f(x) : x \in [a : b]\}$. On the previous homework, we proved that if $f(x) < M$ $\forall x \in [a, b]$, then $\int_a^b f < M(b - a)$. This is also true the other way, in which if $f(x) > m$ $\forall x \in [a, b]$, then $\int_a^b f > m(b - a)$. Combining these, we have

$$m(b - a) < \textstyle\int_a^b f < M(b - a)$$

We can then simplify to

$$m < \frac{\int_a^b f}{b-a} < M$$

So $\frac{\int_a^b f}{b-a}$ must lie between M and m, both of which lie on $f(x)$. Therefore, by IVT, there must exist some value $c \in [a,b]$ where

$$f(c) = \frac{\int_a^b f}{b-a}$$

We can multiply $b-a$ to both sides to obtain

$$f(c)(b-a) = \int_a^b f$$

which was our goal, proving the original statement. $\square$

Exercise 5. (5 pts) Suppose that f is Riemann integrable on $[a,b]$ and is strictly increasing there. Prove that there exists a point $c \in (a,b)$ such that

$$\int_a^b f = f(a)(c-a) + f(b)(b-c).$$

[Hint: Define the function $g(x) = f(a)(x-a) + f(b)(b-x)$. Show that $\int_a^b f$ is between the numbers $f(a)(b-a)$ and $f(b)(b-a)$ and use the Intermediate Value Theorem.]

Solution: Since this function is strictly increasing on $[a,b]$, we know that $f(b) := \max\{f\}$ and $f(a) := \min\{f\}$. By problem 4, we know that $f(a)(b-a) < \int_a^b f < f(b)(b-a)$. By the IVT, we know that $\exists c \in [a,b]$ such that $f(b)(b-c) \leq \int_a^b f$. Ultimately, we want to have

$$f(a)(c-a) + f(b)(b-c) = \int_a^b f$$

Let $f(a)(c-a) > 0$. Since both $f(a)(c-a)$ and $f(b)(b-c)$ are $< \int_a^b$, $f(b)(b-c) \neq \int_a^b$, since otherwise $f(a)(c-a) = 0$, so $f(b)(b-c) = f(b)(b-a)$ which we know to be $> \int_a^b f$. So, by the IVT, we know since

$$f(a)(b-a) < f(a)(c-a) + f(b)(b-c) < f(b-a)$$

there must exist $c \in [a,b]$ such that $f(a)(c-a) + f(b)(b-c) = \int_a^b f$. $\square$

2

Homework problems

Answer all the questions below. Make sure to show your work.

Exercise 6. (10pts)

a) Show that the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & , x \in \mathbb{Q} \\ 0 & , x \notin \mathbb{Q} \end{cases}$$

is not Riemann integrable on $[0,1]$. [Hint: Use exercise 4 from Homework 6.]

b) Define the two functions $g : [0,1] \to \mathbb{R}$ and $h : [0,1] \to \mathbb{R}$ by $g = \chi_{(0,1]}$ and

$$h(x) = \begin{cases} 0 & , x \notin \mathbb{Q} \\ \frac{1}{q} & , x = p/q \in \mathbb{Q}. \end{cases}$$

Use the first part to show that $g \circ h$ is not Riemann integrable on $[0,1]$. What can you say about the composition of two Riemann integrable functions in light of this last examples?

Solution: :

a) We want to show that because the rational and irrational numbers are too dense, the function is not continuous and therefore not integrable. By the definition of continuity, $\forall \epsilon > 0$, $\exists \delta > 0$ such that $|x - x_0| < \delta$ if $|f(x) - f(x_0)| < \epsilon$. Since rational and irrational numbers are dense, we can choose an $x \in \mathbb{Q}$ and $x_0 \notin \mathbb{Q}$ such that $|x - x_0| < \delta$. Since x is rational and x_0 is irrational, then $f(x) = 1$ and $f(x_0) = 0$. Then we have

$$|f(x) - f(x_0)| = |1 - 0| = |1| < \epsilon$$

Since ϵ is arbitrary, we can set $\epsilon < 1$, which would cause a contradiction. This would mean that f is not continuous on $[0,1]$, so it is also not integrable on $[0,1]$.

b) By the same logic as above, rational and irrational numbers are dense, so choosing for $h(x)$ an $x \in \mathbb{Q}$ and $x_0 \notin \mathbb{Q}$ where $|x - x_0| < \delta$ would cause $|f(x) - f(x_0)| = |\frac{1}{q} - 0| = |\frac{1}{q}| < \epsilon$. Since ϵ is arbitrary, we can set $\epsilon < \frac{1}{q}$, causing a contradiction. So h is not continuous. $g \circ h$ can be defined by $g(h(x)) = h_{[0,1]}$. From $[0,1]$, h we proved to be not continuous, so that means that g must not be continuous either. Therefore neither g nor h are differentiable. $\square$

Exercise 7. (5 pts) Show that if f is continuous on $[a,b]$, then $|f|$ is Riemann integrable on $[a,b]$ and

$$\Big| \int_a^b f \Big| \leq \int_a^b |f|.$$

[Hint: There is a clever way to show that $|f|$ is Riemann integrable without using the definition with the partitions.]

Solution: We know from (?) that $\pm x \leq |x|$. This can be further expanded to $-|x| < x < |x|$. We can use f in place of x giving $\pm f \leq |f|$. If we integrate both sides we end up with $\int_a^b f \leq \int_a^b |f|$. We can also use this relation with $\int_a^b f$ giving $\int_a^b f \leq |\int_a^b f|$. This ultimately leaves us with 2 inequalities:

$$-\int_a^b |f| \leq \int_a^b f \leq \int_a^b f$$
and
$$-|\int_a^b f| \leq \int_a^b f \leq |\int_a^b f$$

By the definition of Rieman integrals, we can use the following substitutions:

$$\int_a^b |f| = \lim_{||\mathcal{P}|| \to 0} \sum_{i=1}^N |f(c_i)(x_i - x_{i-1})|$$
and
$$|\int_a^b f| = |\lim_{||\mathcal{P}|| \to 0} \sum_{i=1}^N f(c_i)(x_i - x_{i-1})|$$

These can then be further expanded to

$$\int_a^b |f| = \lim_{||\mathcal{P}|| \to 0}(|f(c_1)(x_1 - x_0)| + |f(c_2)(x_2 - x_1)| + ... + |f(c_N)(x_N - x_{N-1})|)$$
and
$$|\int_a^b f| = \lim_{||\mathcal{P}|| \to 0} |f(c_1)(x_1 - x_0) + f(c_2)(x_2 - x_1) + ... + f(c_N)(x_N - x_{N-1})|$$

We know from the triangle inequality that $|a + b| \leq |a| + |b|$, so by that logic, $|\int_a^b f| \leq \int_a^b |f|$. □

Exercise 8. (5 pts) Find $f'(x)$ if $f(x) = \displaystyle\int_{\sqrt{x}}^{\sqrt[3]{x}} \frac{1}{1+t^3}\, dt$ where $x \in [0, 1]$.

Solution: $f(x) = \int_{\sqrt{x}} {}^{\sqrt[3]{x}} \frac{1}{1+t^3}\, dt$.
$\frac{d}{dx} f(x) = \frac{d}{dx} \int_{\sqrt{x}} {}^{\sqrt[3]{x}} \frac{1}{1+t^3}$
$f'(x) = \frac{1}{1+x} - \frac{1}{1+x^{3/2}}$

simplify

$f'(x) = \frac{x^{3/2} - x}{(1+x)(1+x^{3/2})}$. □

Exercise 9. (5 pts) Find a function $f : [1, \infty) \to \mathbb{R}$ such that $f(1) = 0$ and $f'(x) = 1 + \sin(x^2)$ for all $x > 1$.

Solution: We want to find $F(x) = \int 1 + \sin x^2$. So $F'(x) = f(x)$. By FTC, $f(x) = \int_0^x F'(t)dt$.
Since we are given $f(1) = 0$, we know that f is differentiable, and therefore it is also integrable, and likewise for $F(x)$. Therefore f must exist and is defined by $f(x) = \int_0^x F'(t)dt$. □

Exercise 10. (5 pts) By thinking the following sum as a Riemann sum, evaluate

$$\lim_{n \to \infty} \sum_{k=1}^n \frac{n}{k^2 + n^2}.$$

Solution: $\lim_{n \to \infty} \sum_{k=1}^n \frac{n}{n^2} \frac{1}{\frac{k^2}{n^2} + 1}$
$= \lim_{n \to \infty} \sum_{k=1}^n \frac{1}{n} \frac{1}{\frac{k^2}{n^2} + 1}$
$= \lim_{n \to \infty} \sum_{k=1}^n \frac{1}{n} \frac{1}{\frac{k^2}{n^2} + \frac{n^2}{n^2}}$
$= \lim_{n \to \infty} \sum_{k=1}^n \frac{1}{n} \frac{1}{\frac{k^2 + n^2}{n^2}}$
$= \lim_{n \to \infty} \sum_{k=1}^n \frac{1}{n} \frac{n^2}{k^2 + n^2}$ [...] □