Real Analysis Hw #1

1.) $n \in \mathbb{N}$, $1+2+\ldots+n = \frac{n(n+1)}{2}$

$1+\ldots+n+n+1 = \frac{n(n+1)}{2} + n+1$

$\frac{n(n+1)}{2} + \frac{2n+2}{2}$

$\frac{n^2+n+2n+2}{2}$

$\frac{n^2+3n+2}{2}$

$\frac{(n+1)(n+2)}{2}$ which is the $n+1$ equation

By induction, $1+2+\ldots+n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$

2.) $n=4 = 3+2+1 = 7 \quad 2^{4-1} = 8$

$f(n) \leq 2^{n-1}$

$f(n-1)+f(n-2)+f(n-3) \leq 2^{n-1}$

Lets say the expression holds for all $\mathbb{N}$ n, then $n+1$ would be equal to

$2(f(n-1)+f(n-2)+f(n-3)) \leq 2^n$

$f(n)+f(n-1)+f(n-2)+f(n-3) \leq 2^n$

$f(n+1)+f(n-3) \leq 2^n$

$f(n-3) > 0 \quad f(n-3)+f(n+1) > f(n+1)$

$f(n+1) \leq 2^n$

By strong induction, $f(n) \leq 2^{n-1}$ is true for all $n \in \mathbb{N}$

3.) A) $A \sim A$, there exists a bijection $x=x$ from $A \to A$. Thus, $A \sim A$

B) $A \sim B$, so there exists a bijection $f(x): A \to B$. $f^{-1}(x): B \to A$ thus exists because a bijection has an inverse, and an inverse is a bijection. So $f^{-1}(x)$ is a bijection from $B \to A$, so $B \sim A$.

3c) If $A \sim B$, there exists a bijection $f(x): A \to B$. If $B \sim C$, there exists a bijection $g(x): B \to C$. Let $h(x)$ be $g \circ f$ s.t. $h(x) = g(f(x))$ so that $h(x): A \to C$. Thus a bijection exists between A and C so $A \sim C$.

4.) Let E be a countable set. There exists a bijection $f(x): E \to \mathbb{N}$. If E is finite, its subset will be finite, and thus countable. If E is infinitely countable, then a subset can be finite, or infinite. A finite subset is countable since it is finite. And an infinite subset, can be thought of as a subset A of E. Let B be the subset of everything in ~~E~~, but not in A. $A + B = E$. Since E is countable, and the sum of two countable sets is countable while the sum of an uncountable set is uncountable, A and B must be countable.

5A) $0 < a < b$ $\quad a < b \quad a \cdot a < a \cdot b \quad a^2 < ab \quad a < b \quad a \cdot b < b \cdot b \quad ab < b^2$

$a^2 < ab < b^2 \qquad a^2 < b^2$

B.) $0 < a < b$ so $\sqrt{a} < \sqrt{b}$, $\sqrt{a} = \sqrt{b}$, $\sqrt{a} > \sqrt{b}$. If $\sqrt{a} > \sqrt{b}$, then

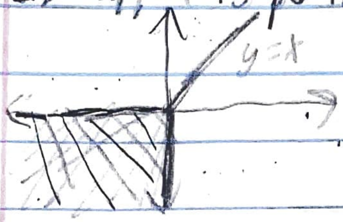
$\sqrt{a} \cdot \sqrt{a} > \sqrt{a}\sqrt{b} \quad a > \sqrt{ab} \quad \sqrt{a} > \sqrt{b} \quad \sqrt{ab} > b \quad a > \sqrt{ab} > b \quad a > b$ #

If $\sqrt{a} = \sqrt{b}$ then $\sqrt{a} \cdot \sqrt{a} = \sqrt{a} \cdot \sqrt{b} \quad a = \sqrt{ab} \quad \sqrt{a} \cdot \sqrt{b} = \sqrt{b} \cdot \sqrt{b} \quad \sqrt{ab} = b$

$a = \sqrt{ab} = b \quad a = b$ #

So $\sqrt{a} < \sqrt{b}$.

6.) In Q_1 (x,y) are positive so $x+x=y+y$ $2x=2y$ $x=y$
In Q_2 x is negative, y is positive so $-x+x=y+y$ $0=2y$ $y=0$
In Q_3, (x,y) are negative so $-x+x=-y+y$ $0=0$ so all (x,y)
In Q_4, x is positive, y is negative $x+x=-y+y$ $2x=0$ $x=0$



7.) $x \geq 0$ $y \geq 0$ $x \geq 0$ $\sqrt{xy} \leq \frac{x+y}{\sqrt{2}}$ $\sqrt{2xy} \leq x+y$

$(\sqrt{2xy})^2 \leq (x+y)^2$ $2xy \leq x^2+2xy+y^2$ $0 \leq x^2+y^2$

$0 \leq x$ $0 \leq x^2$ $0 \leq y$ $0 \leq y^2$ $0+0 \leq x^2+y^2$ $0 \leq x^2+y^2$ ✓
$2xy \leq x^2+2xy+y^2$ $\sqrt{2xy} \leq x+y$ $\sqrt{xy} \leq \frac{x+y}{\sqrt{2}}$

8.) a.) $E:=\{x \in \mathbb{R} : x \geq 0 \text{ and } x^2 \leq 9\}$ $x \geq 0$ $x^2 \leq 9$ $-3 \leq x \leq 3$ $x \leq 3$
The set goes from $x=0$ to $x=\sqrt{3}$. $\inf E = 0$
because x can't be <0 because $x \geq 0$ and $\sup E = 3$
because the case of $x>3$, then $x^2>9$ $x \notin E$
b.) $E:=\{\frac{4n+5}{n+1} : n \in \mathbb{N}\}$ $n=1$ $\frac{4(1)+5}{1+1}=\frac{9}{2}$ $n \to \infty$ $E \to 4$.
$\sup E = \frac{9}{2}$ because 1 is lowest value that can be input to get $9/2$. To get a higher value, you would need to get a smaller number than 1, which is impossible because $n \in \mathbb{N}$

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8 b.) Say $\inf E = x$. $4 \le \frac{4n+5}{n+1} \le 9/2$

We can have $x = 4$, $x < 4$, $x > 4$ minimum is 4

Suppose $x > 4$ $a > 4 - x$

$\frac{4n+5}{n+1} < x$ $4n+5 < xn + x$ $4n - xn < x - 5$ $n(4-x) < x - 5$

In AP, take $x = 4 - x$ and $y = x - 5$, then $n(4-x) > x - 5$. #

9.) A.) A power set can be represented as a binary number with a length of the elements in A. A set with 2 elements Power set can be represented with a 2 digit binary number. With A having n elements, the coorespanding n-digit binary number will have 2^n different values.
If a the xth digit is 1 in the binary number, it means n_x is in the sub-set. If it is 0, n_x is not in the sub-set. Because the size of the power set is 2^n while A is n, they are not equivalent.

B.) Let all $\subseteq \mathbb{N}$ coorespond to an infinite binary number. If 1 is in the subset, the first digit is a one, if two is not in the subset, let the second digit be zero. Let this occur for every $n \in \mathbb{N}$ coorresponding the the x_n digit. If we say it is countable and have a list of every binary representation, we can take the first digit of the first number on the list, the second digit of the second number, and so on, and switch ever 0 to 1 and every 1 to 0. This we every number on the list, Real Analysis Hw #1

9.) and changes a digit. This means that it is a number not on the list, meaning the list is incomplete. This is a contradiction that we could have a list of all of these, and so, $P(\mathbb{N})$ is uncountable.

10.) a.) suppose that $sup(E) = x$. $rE = rx$ $x \in E$
$sup(E) = x$ so $sup(rE) = rx$ with $x = sup(E)$, so rx.
$r sup(E) \to r \cdot (x) = rx$

b.) $sup(E) = x$. Adding $r+E$ shifts the entire function r because $r+E := \{r+x : x \in E\}$. So the $sup(r+E)$ is shifted r, and so if $x = sup(E)$, then the function $r+E$ has $sup(r+E)$ at $r+x$.
And $r + sup E = r + x$.