Due date: 09/06/2021 1:20pm

Instructions: You must answer all the questions below and send your solution by email (to `parisepo@hawaii.edu`). If you decide to not use LaTeX to hand out your solutions, please be sure that after you scan your copy, it is clear and readable. No late homework will be accepted. No format other than PDF will be accepted. Name your file as indicated in the syllabus.

1 Homework problems

Exercise 1. Prove that for any $n = \mathbb{N}, 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

Solution: Label the statement $n = \mathbb{N}, 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ as $f(n)$. Note that $f(1)$ is true, as $1 = \frac{1(1+1)}{2}$. Now suppose that $f(k)$ is true. This means that:

$$\begin{aligned}
1 + 2 + \cdots + k &= \frac{k(k+1)}{2} \\
1 + 2 + \cdots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) \\
1 + 2 + \cdots + (k+1) &= \frac{k^2+k}{2} + \frac{2(k+1)}{2} \\
1 + 2 + \cdots + (k+1) &= \frac{k^2+3k+2}{2} \\
1 + 2 + \cdots + (k+1) &= \frac{(k+1)(k+2)}{2} \\
1 + 2 + \cdots + (k+1) &= \frac{(k+1)((k+1)+1))}{2}
\end{aligned}$$

Therefore $f(k) \to f(k+1)$. By induction, $f(n)$ is true for all $n \in \mathbb{N}$ □

Exercise 2. Define $f : \mathbb{N} \to \mathbb{N}$ by $f(1) = 1, f(2) = 2, f(3) = 3$ and

$$f(n) = f(n-1) + f(n-2) + f(n-3) \ (n \geq 4)$$

Prove that $f(n) \leq 2^{n-1}$ for all $n \in \mathbb{N}$

Solution: Note that $f(1) \leq 2^{1-1}$, $f(2) \leq 2^{2-1}$, and $f(3) \leq 2^{3-1}$. Now suppose that $f(x) \leq 2^{x-1}$ for all $0 < x \leq k$. We then have the following for $k \geq 4$:

$$f(k+1) = f(k) + f(k-1) + f(k-2)$$

$$f(k+1) \leq 2^{k-1} + 2^{k-2} + 2^{k-3}$$

$$f(k+1) \leq 2^{k}(2^{-1} + 2^{-2} + 2^{-3})$$

$$f(k+1) \leq \frac{7}{8}(2^{k})$$

$$f(k+1) \leq 2^{k} \text{ (Since } 2^{k} \text{ must be positive)}$$

By strong mathematical induction, we then have that $f(n) \leq 2^{n-1}$ for all $n \in \mathbb{N}$. $\square$

Exercise 3. Prove that if A, B, C are sets, then

a) $A \sim A$.

b) If $A \sim B$, then $B \sim A$.

c) If $A \sim B$ and $B \sim C$, then $A \sim C$.

Solution: $A \sim B$ means that a bijection $f : A \to B$ exists

a) We can create a bijection $f : A \to A$ by defining f as $f(x) = x$. Therefore $A \sim A$.

b) If $A \sim B$, then there exists a bijection $f : A \to B$. Since f is a bijection, $f^{-1} : B \to A$ is also a bijection. Therefore $B \sim A$.

c) If $A \sim B$ and $B \sim C$, then there exist bijections $f : A \to B$ and $g : B \to C$. Then the composite function $g \circ f : A \to C$ is also a bijection. Therefore $A \sim C$. $\square$

Exercise 4. Show that any subset of a countable set is countable.

Solution: Let A be an arbitrary countable set. If A is finite, then any subset of A is also finite and is therefore countable. If A is countably infinite, we must prove that all infinite subsets of A are also countably infinite. Let A' be an arbitrary infinite subset of A. Since A is countably infinite, there exists a bijection $f : A \to \mathbb{N}$. Now define the set $N' = \{f(x) : x \in A'\}$. Since f is a bijection, $A' \sim N'$ and $N' \subset \mathbb{N}$. N' is then countable by Theorem 0.14, and since $A' \sim N'$, A' is also countable, which is what we wanted to prove. $\square$

Exercise 5. Let $0 < a < b$ be positive real numbers. Prove that

a) $a^2 < b^2$.

b) $\sqrt{a} < \sqrt{b}$.

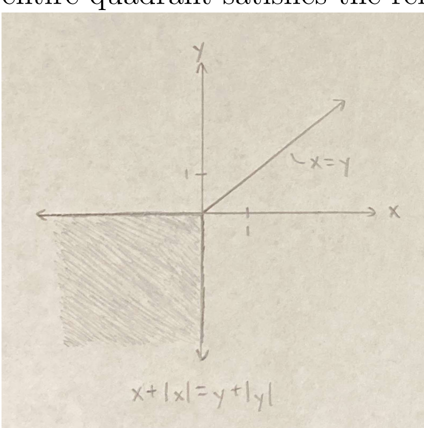
Solution:

a) Consider $a < b$. Since a and b are positive, we can write $a(a) < b(a)$ and $a(b) < b(b)$. Therefore $a^2 < ab$ and $ab < b^2$. Combining these, $a^2 < ab < b^2$ and $a^2 < b^2$.

b) Suppose towards a contradiction that $\sqrt{a} \nless \sqrt{b}$. We will consider two cases.
Case 1: Suppose $\sqrt{a} > \sqrt{b}$. Since $\sqrt{a}$ and $\sqrt{b}$ are positive, we can write $\sqrt{a} \cdot \sqrt{a} > \sqrt{b} \cdot \sqrt{a}$ and $\sqrt{a} \cdot \sqrt{b} > \sqrt{b} \cdot \sqrt{b}$. Then $a > \sqrt{a} \cdot \sqrt{b}$ and $\sqrt{a} \cdot \sqrt{b} > b$. Thus $a > b$. This is a contradiction.
Case 2: Suppose then that $\sqrt{a} = \sqrt{b}$. Squaring both sides, we get $a = b$, which is a contradiction.
Since $\sqrt{a} \ngtr \sqrt{b}$ and $\sqrt{a} \neq \sqrt{b}$, $\sqrt{a} < \sqrt{b}$ by the Order Axiom. □

Exercise 6. Sketch the region of points (x, y) satisfying the following relation: $x + |x| = y + |y|$ (explain your answer).

Solution: Consider this equation in each of the four quadrants.
For $x \geq 0, y \geq 0$: The equation becomes $x + x = y + y$ and then $x = y$
For $x \geq 0, y < 0$: The equation becomes $x + x = y - y$ and then $x = 0$
For $x < 0, y \geq 0$: The equation becomes $x - x = y + y$ and then $y = 0$
For $x < 0, y < 0$: The equation becomes $x - x = y - y$ and then $0 = 0$. As this is always true, the entire quadrant satisfies the relation.



□

Exercise 7. If $x \geq 0$ and $y \geq 0$, prove that $\sqrt{xy} \leq \frac{x+y}{\sqrt{2}}$

Solution: Suppose towards a contradiction that $\sqrt{xy} > \frac{x+y}{\sqrt{2}}$. Then $\sqrt{2xy} > x + y$. From Exercise 5, we know that we can safely square both sides such that:

$$2xy > (x + y)^2$$

$$2xy > x^2 + 2xy + y^2$$

$$0 > x^2 + y^2$$

Since any real number squared is greater than or equal to 0, $x^2 + y^2 \geq 0$. This is a contradiction. Therefore $\sqrt{xy} \ngtr \frac{x+y}{\sqrt{2}}$ and $\sqrt{xy} \leq \frac{x+y}{\sqrt{2}}$. □

Exercise 8. Find the infimum and supremum (if they exist) of the following sets. Make sure to justify all your answers:

a) $E = \{x \in \mathbb{R} : x \geq 0 \text{ and } x^2 \leq 9\}$.

b) $E = \{\frac{4n+5}{n+1} : n \in \mathbb{N}\}$.

Solution:

a) Consider the restriction $x^2 \leq 9$. From Exercise 5, we know that we can safely take the square root of both sides of an inequality. Therefore $\sqrt{x^2} \leq \sqrt{9}$ and $|x| \leq 3$. From Theorem 0.25, this means that $-3 \leq x \leq 3$. In total, the restrictions on x become $x \geq 0$ and $-3 \leq x \leq 3$. Combining these inequalities gives the restriction $0 \leq x \leq 3$. Therefore the infimum of E is 0 and the supremum of E is 3.

b) We will show that the infimum is 4 and the supremum is 4.5. For the infimum, note how $\frac{1}{n+1}$ is positive for all $n \in \mathbb{N}$. Therefore, for all $n \in \mathbb{N}$:

$$\frac{4n+5}{n+1} > \frac{4n+5}{n+1} - \frac{1}{n+1}$$
$$\frac{4n+5}{n+1} > \frac{4n+4}{n+1}$$
$$\frac{4n+5}{n+1} > 4$$

Thus 4 is a lower bound for E. Now suppose $\exists a, a > 4$ and $\forall x \in E, a < x$. Then for all $n \in \mathbb{N}$:

$$a < \frac{4n+5}{n+1}$$
$$a < \frac{4n+4}{n+1} + \frac{1}{n+1}$$
$$a < 4 + \frac{1}{n+1}$$
$$a - 4 < \frac{1}{n+1}$$
$$(n+1)(a-4) < 1$$

Since $a > 4$, $a - 4 > 0$. By the Archimedean Property, we know there must exist some n that will make $(n+1)(a-4) > 1$. This is a contradiction. Therefore there is no lower bound for E that is greater than 4, and 4 is the infimum.
For the supremum, we start with $n \in \mathbb{N}$. Therefore $n \geq 1$ and:

$$0.5n \geq 0.5$$
$$0.5n + (4n + 4.5) \geq 0.5 + (4n + 4.5)$$
$$4.5n + 4.5 \geq 4n + 5$$
$$4.5(n+1) \geq 4n + 5$$
$$4.5 \geq \tfrac{4n+5}{n+1} \text{ (Since } n + 1 > 0)$$

Therefore 4.5 is an upper bound for E. There is no upper bound lower than 4.5 since $\frac{4n+5}{n+1}|_{n=1} = 4.5$. Therefore the supremum of E is 4.5. $\square$

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Writing problems

For each of the following problems, you will be ask to write a clear and detailed proof. You will have the chance to rewrite your solution in your semester project after receiving feedback from me.

Exercise 9. Let A be a non-empty set and $P(A)$ be its power set (the family of all subsets of A). Prove that A is not equivalent to $P(A)$. Deduce that $P(A)$ is not countable. [Hint: Define $C = \{x : x \in A \text{ and } x \notin f(x)\}$.]

Solution: We will consider 3 cases:
Case 1: Suppose A is finite and has n elements. We can define set C as $C = \{\{x\} : x \in A\} \cup \emptyset$. C is a subset of $P(A)$ and has $n+1$ elements. Therefore $P(A)$ must have more than n elements, and cannot be equivalent to A.
Case 2: Suppose A is countably infinite. Then there exists a bijection $f : A \to \mathbb{N}$. This means that the elements of A can be ordered such that the first element of A is $f^{-1}(1)$, the second element is $f^{-1}(2)$, and so on. We can now create a bijection $g : P(A) \to [0,1]$ which is defined as follows:

$$g(X \in P(A)) = \sum_{n\in\mathbb{N}} (2^{-n}x_n) \text{ where } x_n = \{^{1, f^{-1}(n) \in X}_{0, f^{-1}(n) \notin X}$$

Since the real numbers in the interval $[0,1]$ are uncountable and $P(A) \sim \mathbb{R}$, $P(A)$ is also uncountable, and cannot be equivalent to A.
Case 3: Suppose A is uncountable. I do not know how to prove that $A \nsim P(A)$. However, we can define injective function $f : A \to P(A)$ as $f(x) = \{x\}$. This means that $P(A)$ has a cardinality that is greater than or equal to that of A, and $P(A)$ is uncountable. $\square$

Exercise 10. Let $E \subseteq \mathbb{R}$ be bounded from above and $E \neq \emptyset$. For $r \in \mathbb{R}$, let

$$rE = \{rx : x \in E\} \text{ and } r + E = \{r + x : x \in E\}.$$

Show that

a) if $r > 0$, then $\sup(rE) = r\sup(E)$.

b) for any $r \in \mathbb{R}$, $\sup(r + E) = r + \sup(E)$.

Solution: We need to show two things. First is that supremum of a set is greater than or equal to all of set's elements. Second is that no other number less than the supremum is also greater than or equal to all of the set's elements.

a) Suppose $r > 0$. Since $\forall x \in E, x \leq \sup(E)$ and $r > 0$, we must have that $\forall x \in E, rx \leq r\sup(E)$. Therefore $r\sup(E)$ is an upper bound for rE. Now suppose towards a contradiction that $\exists s, s < r\sup(E)$ and $\forall x \in E, rx \leq s$. Since $r > 0$, $\frac{s}{r} < \sup(E)$ and $\forall x \in E, x \leq \frac{s}{r}$. This would mean that $\frac{s}{r}$ is an upper bound for E that is less than $\sup(E)$, which is a contradiction. Therefore $\sup(rE) = r\sup(E)$.

b) Since $\forall x \in E, x \leq$sup(E), we must have that $\forall x \in E, r+x \leq r+sup(E)$. Therefore $r+$sup(E) is an upper bound for $r+E$. Now suppose towards a contradiction that $\exists s, s < r+$sup(E) and $\forall x \in E, r+x \leq s$. Then $s-r <$sup(E) and $\forall x \in E, x \leq s-r$. This would mean that $s-r$ is an upper bound for E that is less than sup(E), which is a contradiction. Therefore sup$(r+E) = r+$sup(E). $\square$