MATH-331 Introduction to Real Analysis — Pierre-Olivier Parisé
Homework 01 — Fall 2021

Due date: 09/06/2021 1:20pm — Total: /65.

Instructions: You must answer all the questions below and send your solution by email (to `parisepo@hawaii.edu`). If you decide to not use LaTeX to hand out your solutions, please be sure that after you scan your copy, it is clear and readable. No late homework will be accepted. No format other than PDF will be accepted. Name your file as indicated in the syllabus.

1 Homework problems

Exercise 1. (5 points) Prove that for any $n \in \mathbb{N}$, $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

Solution: We use induction on n. Let $P(n)$ be the proposition

$$\text{“ } 1 + 2 + \cdots + n = \frac{n(n+1)}{2}. \text{ ”}$$

- For $n = 1$, we have $1 = 1 \cdot 2/2$ and so $P(1)$ is true.
- Let $P(n)$ be true. Then, we have

  $$\sum_{k=1}^{n+1} k = \sum_{k=1}^{n} k + (n+1)$$

  and from the induction hypothesis,

  $$\sum_{k=1}^{n+1} k = \frac{n(n+1)}{2} + (n+1) = (n+1)\Big(\frac{n}{2} + 1\Big) = \frac{(n+1)(n+2)}{2}.$$

  So $P(n+1)$ is true.

By the PMI, $P(n)$ is true for any $n \in \mathbb{N}$. □

Exercise 2. (5 points) Define $f : \mathbb{N} \to \mathbb{N}$ by $f(1) = 1$, $f(2) = 2$ and $f(3) = 3$ and

$$f(n) := f(n-1) + f(n-2) + f(n-3) \quad (n \geq 4).$$

Prove that $f(n) \leq 2^{n-1}$ for all $n \in \mathbb{N}$.

Solution: We will use second principle of mathematical induction on n. Let $P(n)$ be the proposition

$$\text{“ } f(n) \leq 2^{n-1}. \text{ ”}$$

- For $n = 1$, $n = 2$ and $n = 3$, $P(n)$ is true. Indeed, we have $f(1) = 1 \leq 1$, $f(2) = 2 \leq 2$ and $f(3) = 3 \leq 4$.
- Now take $n_0 = 3$ and suppose that $P(i)$ is true for any $1 \leq i \leq n$. We have

$$f(n+1) = f(n) + f(n-1) + f(n-2)$$

and using the induction hypothesis, we get

$$f(n+1) \leq 2^{n-1} + 2^{n-2} + 2^{n-3} = 2^{n-3}(4+2+1) = 2^{n-3} \cdot 6.$$

Now, we know that $6 \leq 8$ (relative to the order $\leq$ on $\mathbb{N}$) and so $f(n+1) \leq 2^n$. This means that $P(n+1)$ is true.

By the PMI2, $P(n)$ is true for any $n \in \mathbb{N}$. $\square$

Exercise 3. (5 points) Prove that if A, B and C are sets, then

a) $A \sim A$.

b) If $A \sim B$, then $B \sim A$.

c) If $A \sim B$ and $B \sim C$, then $A \sim C$.

Solution: 1. We can take $f(x) = 1_A(x) := x$, the identity function on the set A. This is a bijection.

2. Let $A\tilde{B}$. Then, by definition, there is a bijection $f : A \to B$. Now, $f^-1 : B \to A$ exists (see your MATH 321 course) and is a bijection to. So, by definition, we get $B\tilde{A}$.

3. Suppose that $A\tilde{B}$ and $B\tilde{C}$. Then, by definition again, there are bijections $f : A \to B$ and $g : B \to C$. Define $h := g \circ f$ (the composition of g with f). We have to show that h is a bijection.

   - Injectivity: Let $h(x) = h(y)$. Then, $g(f(x)) = g(f(y))$. Since g is injective, we infer that $f(x) = f(y)$ and using the injectivity of f, we get that $x = y$.
   - Surjectivity: Let $z \in C$. Since g is surjective, there is a $y \in B$ such that $g(y) = z$. Also, f is surjective. There is a $x \in A$ such that $f(x) = y$. Thus, we get $z = g(f(x)) = h(x)$.

   The function h is the bijection that we were looking for. $\square$

Exercise 4. (5 points) Show that any subset of a countable set is countable.

Solution: Let S be a countable set and let $E \subseteq S$. If E is finite, then it is countable. Suppose that E is infinite. Since S is countable, there is a bijection $f : S \to \mathbb{N}$. Since f is an injection, $f(E)$ is a subset of $\mathbb{N}$ and must be infinite also. So, $f(E)$ is a non-empty infinite subset of $\mathbb{N}$. By a Theorem of the lecture notes (Preliminaries-0), $f(E)$ is also countable. So there is a bijection $g : f(E) \to \mathbb{N}$. Now, the function $h : E \to \mathbb{N}$ defined by $h := g \circ f$ is the bijective function that we were looking for. $\square$

Exercise 5. (10 points) Let $0 < a < b$ be positive real numbers. Prove that

a) $a^2 < b^2$.

b) $\sqrt{a} < \sqrt{b}$.

Solution: **a)** By the Axiom O4, we have that $a \cdot a < b \cdot a$. Similarly, by the Axiom O4, we have $a \cdot b < b \cdot b$. From the commutativity of the multiplication, we have $ba = ab$. By Axiom O2, we get that $a^2 < b^2$.

b) From Axiom O3, $\sqrt{a} < \sqrt{b}$, $\sqrt{a} = \sqrt{b}$, or $\sqrt{a} > \sqrt{b}$.

- Suppose that $\sqrt{a} = \sqrt{b}$. Then, by definition of the positive square-root, we have $(\sqrt{a})^2 = a$ and $(\sqrt{b})^2 = b$. So, $a = b$. This contradicts our hypothesis that $a < b$
- Suppose that $\sqrt{a} > \sqrt{b}$. Then, from a), we get $(\sqrt{b})^2 < (\sqrt{a})^2$ or equivalently, $b < a$. This last assumption contradicts our hypothesis saying that $a < b$.

So, we must conclude that $\sqrt{a} < \sqrt{b}$. $\square$

Exercise 6. (5 points) Sketch the region of the points (x, y) satisfying the following relation: $x + |x| = y + |y|$ (explain your answer).

Solution: There are four cases to consider:

- $\underline{x \geq 0,\ y \geq 0}$. In this case, we have $x + x = y + y$ and so $x = y$. Let $S_1 := \{(x, y)\ :\ y = x\}$.
- $\underline{x \geq 0,\ y < 0}$. In this case, we have $x + x = y - y$ and so $x = 0$. Let $S_2 := \{(0, y)\ :\ y < 0\}$.
- $\underline{x < 0,\ y \geq 0}$. In this case, we have $x - x = y + y$ and so $y = 0$. Let $S_3 := \{(x, 0)\ :\ x < 0\}$.
- $\underline{x < 0,\ y < 0}$. In this case, we have $x - x = y - y$ and so the relation is always satisfied in this case. Let $S_4 := \{(x, y)\ :\ x < 0,\ y < 0\}$.

Figure 1, the region $\cup_{i=1}^4 S_i$ is represented. $\square$

Exercise 7. (5 points) If $x \geq 0$ and $y \geq 0$, prove that $\sqrt{xy} \leq \frac{x+y}{\sqrt{2}}$

Solution: Let $x \geq 0$ and $y \geq 0$. Then, with a little algebra, we get $(x + y)^2 = x^2 + 2xy + y^2$. Now, since $x^2 \geq 0$ and $y^2 \geq 0$, we have

$$(x + y)^2 = x^2 + 2xy + y^2 \geq 2xy \quad \Rightarrow \quad xy \leq \frac{(x + y)^2}{2}.$$

Taking the square-root on each side and using exercise **b)**, we get $\sqrt{xy} \leq \frac{x+y}{\sqrt{2}}$. $\square$

Exercise 8. (10 points) Find the infimum and supremum (if they exist) of the following sets. Make sure to justify all your answers:

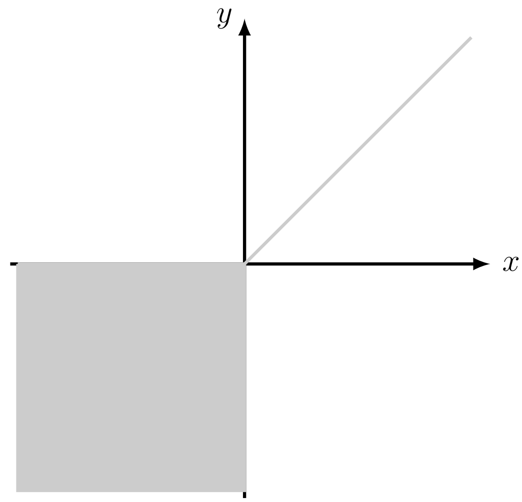

Figure 1: Representation of the set $S_1 \cup S_2 \cup S_3 \cup S_4$

a) $E := \{x \in \mathbb{R} : x \geq 0 \text{ and } x^2 \leq 9\}$.

b) $E := \{\frac{4n+5}{n+1} : n \in \mathbb{N}\}$.

Solution: **a)** The set E is bounded below by 0 and $0 \in E$. So $\inf E = 0$. The set E is also bounded above by 3 because if $x > 3$, then $x^2 > 9$ and so $x \notin E$. Since $3 \in E$, we have $\sup E = 3$.

b) We will find first the infimum and then the supremum.

- An upper bound for E would be $\frac{9}{2}$. Indeed, we have $\frac{4n+5}{n+1} \leq \frac{9}{2}$ if and only if

$$8n + 10 \leq 9n + 9 \iff 1 \leq n$$

where the condition $n \geq 1$ is always satisfied. So, $\frac{4n+5}{n+1} \leq \frac{9}{2}$ for every $n \in \mathbb{N}$. Also, $\frac{9}{2} \in E$ (just set $n = 1$) and so $\sup E = \frac{9}{2}$.

- A lower bound for E would be 4. Indeed, we have $\frac{4n+5}{n+1} > 4$ if and only if

$$4n + 5 > 4n + 4 \iff 1 > 0$$

which is always true. So, $\frac{4n+5}{n+1} > 4$ for any $n \in \mathbb{N}$ and this implies that $\inf E \geq 4$. We will show that $\inf E = 4$. Since $\inf E \leq 4$, we have $\inf E = 4$ or $\inf E > 4$. Let $x = \inf E$ and suppose that $x > 4$. By the Achimedian property with $x - 4 > 0$ in place of x and $5 - x$ in place of y, there is a $n \in \mathbb{N}$ such that

$$5 - x < n(x - 4)$$

After some algebra, we get $\frac{4n+5}{n+1} < x$ which contradicts the definition of x (it must be a lower bound for E). Then, we must take the other possibility, that is $\inf E = 4$. $\square$

2
Writing problems

For each of the following problems, you will be ask to write a clear and detailed proof. You will have the chance to rewrite your solution in your semester project after receiving feedback from me.

Exercise 9. (5 points) Let A be a non-empty set and $P(A)$ be its power set (the family of all subsets of A). Prove that A is not equivalent to $P(A)$. Deduce that $P(\mathbb{N})$ is not countable. [Hint: Define $C := \{x : x \in A \text{ and } x \notin f(x)\}$.]

Hints: Argue by contradiction.

- Suppose that $A \sim P(A)$.
- Let $f : A \to P(A)$ be a bijection between A and $P(A)$ (why can you say that?).
- Define $C = \{x : x \in A \text{ and } x \notin f(x)\}$.
- There exists a $x \in A$ such that $f(x) = C$ (Why can you find such a x?).
- Can you answer the question "does $x \in f(x)$"? This should lead you to a contradiction. □

Exercise 10. (10 points) Let $E \subseteq \mathbb{R}$ be bounded from above and $E \neq \emptyset$. For $r \in \mathbb{R}$, let

$$rE := \{rx : x \in E\} \quad \text{and} \quad r + E := \{r + x : x \in E\}.$$

Show that

a) if $r > 0$, then $\sup(rE) = r\sup(E)$.

b) for any $r \in \mathbb{R}$, $\sup(r + E) = r + \sup E$.

Hints: First of all, $\sup E$ exists by the Axiom of completude (because E is bounded from above). Set $s := \sup E$.

For a), you need to prove that

- $rx \leq rs$ for any $rx \in rE$, so that rs is an upper bound for rE.
  - For any $x \in E$, we know that $x \leq s$ (Why?).
  - Which axiom can be used to get $rx \leq rs$.
- for any upper bound b of rE, $rs \leq b$.
  - Let b be an upper bound for rE.
  - Then, $rx \leq b$. Why can you conclude that b/r is an upper bound for E?
  - Then, $s \leq b/r$, Why should that be the case?
  - Which axiom can be used to conclude.

For b), you follow the same guideline as before. □