MATH-331 Introduction to Real Analysis	Pierre-Olivier Parisé
Homework 02	Fall 2021

Due date: 20-09-2021 1:20pm Total: /70.

Exercise	1 (10)	2 (5)	3 (5)	4 (5)	5 (5)	6 (10)	7 (5)	8 (10)	9 (5)	10 (10)
Score										

Table 1: Scores for each exercises

Instructions: You must answer all the questions below and send your solution by email (to `parisepo@hawaii.edu`). If you decide to not use LaTeX to hand out your solutions, please be sure that after you scan your copy, it is clear and readable. Make sure that you attached a copy of the homework assignment to your homework. No late homework will be accepted. No format other than PDF will be accepted. Name your file as indicated in the syllabus.

1 — Writing problems

For each of the following problems, you will be asked to write a clear and detailed proof. You will have the chance to rewrite your solution in your semester project after receiving feedback from me.

Exercise 1. (10 pts)

a) Let $\{[a_n, b_n] : n \geq 1\}$ be a family of closed intervals such that $[a_1, b_1] \supset [a_2, b_2] \supset [a_3, b_3] \supset \cdots$. Show that there is a $c \in \mathbb{R}$ such that $c \in [a_n, b_n]$ for all $n \geq \mathbb{N}$. Follow the following steps to prove it:

- **(i)** Prove that for any $n, m \geq 1$, $a_n \leq b_m$. [hint: put $M := \max\{n, m\}$.]
- **(ii)** Show that $\sup\{a_n : n \geq 1\}$ exists.
- **(iii)** Show that $c = \sup\{a_n : n \geq 1\}$ satisfies the requirement.

b) Use this last result to prove that the set $\mathbb{R}$ is uncountable. [Hint: Show that any function $f : \mathbb{N} \to \mathbb{R}$ can't be surjective. To do so, construct a sequence of closed intervals such that $f(n) \notin [a_n, b_n]$ with $a_n < b_n$.]

Exercise 2. (5 pts) Prove that if $a_n \to A$, then $|a_n| \to |A|$.

Exercise 3. (5 pts) Let (a_n), (b_n), and (c_n) be sequences of real numbers. Prove that if $a_n \to L$, $b_n \to L$, and $a_n \leq c_n \leq b_n$, then $c_n \to L$.

Exercise 4. (5 pts) Prove that if $a_n \to A$ and $a_n \geq 0$ for all $n \geq 1$, then $\sqrt{a_n} \to \sqrt{A}$. Follow the following steps to prove it:

1. Consider the case $A = 0$.
2. Suppose that $A \neq 0$. Show that there is a $N_1 \in \mathbb{N}$ such that if $n \geq N_1$, then $\sqrt{a_n} \geq \sqrt{|A|/2}$. [Hint: use the definition of convergence of $(a_n)_{n\geq 0}$ with a clever choice of ε and use the properties of the absolute value.]
3. Use the convergence of (a_n) again to find a N_2 such that $|a_n - A| < \frac{3}{4}\frac{\varepsilon}{\sqrt{|A|}}$.
4. Express $\sqrt{a_n} - A$ as $\frac{a_n - A}{\sqrt{a_n}+\sqrt{A}}$ and put $N = \max\{N_1, N_2\}$. Conclude.

Exercise 5. (5 pts) For each sequence $(a_n)_{n=1}^{\infty}$, define the sequence $(\sigma_n)_{n=1}^{\infty}$ by

$$\sigma_n := \frac{a_1 + a_2 + \cdots + a_n}{n} \quad (n \geq 1).$$

Prove that if $a_n \to A$, then $\sigma_n \to A$. Find an example of a divergent sequence (a_n) such that $(\sigma_n)_{n=1}^{\infty}$ converges.

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Homework problems

Exercise 6. (10 pts) Use the definition of convergence to prove that each of the following sequences converges.

a) $(a_n)_{n=1}^{\infty}$ given by $a_n = 5 + 1/n$ for $n \geq 1$.

b) $(a_n)_{n=1}^{\infty}$ given by $a_n = \frac{3n}{2n+1}$ for $n \geq 1$.

Exercise 7. (5 pts) Prove that the sequence $(a_n)_{n=1}^{\infty} = \left(\frac{2n+1}{n}\right)_{n=1}^{\infty}$ is a Cauchy sequence.

Exercise 8. (10 pts) Prove that each of the following sequence diverges.

a) $(a_n)_{n=1}^{\infty} = ((-1)^n)_{n=1}^{\infty}$.

b) $(a_n)_{n=1}^{\infty} = (\sin(\frac{4n+1}{2}\pi))_{n=1}^{\infty}$.

Exercise 9. (5 pts) Give an examples of two sequences (a_n) and (b_n) such that (a_n) and (b_n) don't converge, but $(a_n + b_n)$ converge.

Exercise 10. (10 pts) With the limit operations and the writing problems, find the limit of the following sequence with general term

a) $\frac{n^2+4n}{n^2-5}$.

b) $\frac{n}{n^2-3}$.

c) $\frac{\cos n}{n}$. [You can use what you know on the cosine function.]

d) $\left(\sqrt{4 - \frac{1}{n}} - 2\right)n$.

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1.

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2. It is given that $a_n \to A$, so for every $\varepsilon > 0$, there exists $\delta > 0$, such that $|a_n - A| < \varepsilon$, for all $n \in \mathbb{N}$. To prove the sequence $|a_n|$ converges to $|A|$.
That means

$$||a_n| - |A|| < \varepsilon \text{ for all } n \in \mathbb{N}$$
$$||a_n| - |A|| \leq |a_n - A| < \varepsilon \text{ for all } n \in \mathbb{N}$$

Given that the sequence a_n converges to A. Hence, the sequence $|a_n|$ also converges. ■

3. We say that sequence $a_n \to L$ as $n \to \infty$ if given any $\varepsilon > 0$, there exists a positive integer m such that $|a_n - L| < \varepsilon$ $\forall$ $n \geq m$.
Given three sequences a_n, b_n, c_n of reals, $a_n \to L$, $b_n \to L$ as $n \to \infty$ and $a_n \leq c_n \leq b_n$ $\forall n$.
Since $a_n \to L$ and $b_n \to L$ as $n \to \infty$, let $\varepsilon > 0$ be given, then by above definition $\exists$ positive integers m_1, m_2 such that

$$|a_n - L| < \varepsilon \quad \forall n \geq m_1 \to (1)$$
$$|b_n - L| < \varepsilon \quad \forall n \geq m_2 \to (2)$$

Let $m = \max\{m_1, m_2\}$. Now (1), (2) holds for $n \geq m$
Thus

$$|a_n - L| < \varepsilon \quad \forall n \geq m \to (3)$$
$$|b_n - L| < \varepsilon \quad \forall n \geq m \to (4)$$

From (3) and (4) we have

$$L - \varepsilon < a_n < L + \varepsilon \quad \forall n \geq m \to (5)$$
$$L - \varepsilon < b_n < L + \varepsilon \quad \forall n \geq m \to (6)$$

Now given that $a_n \leq c_n \leq b_n$ $\forall n \to (7)$
So for all $n \geq m$, using (5), (6) and (7)

$$L - \varepsilon < a_n \leq c_n \leq b_n \leq L + \varepsilon \quad \forall n \geq m$$
$$L - \varepsilon \leq c_n < L + \varepsilon \quad \forall n \geq m$$
$$\Rightarrow c_n \to L \text{ as } n \to \infty \;■$$

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4. $A_n \to A$ and $a_n \geq 0$ for all $n \geq 1$,
then $\sqrt{a_n} \to \sqrt{A}$.

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5. $\sigma_n := \frac{a_1 + a_2 + \dots + a_n}{n}$ $(n \geq 1)$ $\sigma_n = \frac{a_n}{n}$

Prove that if $a_n \to A$, then $\sigma_n \to A$

Def: A is the limit of $\{a_n\}$ if the following condition is satisfied for every positive number ε, there exists a natural number N, such that if $n \geq N$, then $|a_n - L| < \varepsilon$

Let $\lim_{n\to\infty} a_n \to A$ exists

Given $\sigma_n = \frac{a_1 + a_2 + \dots + a_n}{n} \Rightarrow n\sigma_n = a_1 + a_2 + \dots + a_n$ ③

Replace $n \to (n-1)$ we set $(n-1)\sigma_{n-1} = a_1 + a_2 + \dots + a_{n-1} + a_n$ ④

③ − ④ $\Rightarrow n\sigma_n - (n-1)\sigma_{n-1} = a_n$

$\Rightarrow \lim_{n\to\infty} n\sigma_n - n\sigma_{n-1} + \sigma_{n-1} = \lim_{n\to\infty} a_n = A$

when $n \to \infty$ $\sigma_{n-1} = \sigma_n$

$\Rightarrow \lim_{n\to\infty} n\sigma_n - n\sigma_n + \sigma_n = A$

$\Rightarrow \lim_{n\to\infty} \sigma_n = A = \lim_{n\to\infty} a_n$

Because both the $\lim_{n\to\infty} \sigma_n = A$ and $\lim_{n\to\infty} a_n = A$ then both $a_n \to A$ and $\sigma_n \to A$ ∎

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6. a) $a_n = 5 + \frac{1}{n}$

Claim: $a_n \to 5$

$|a_n - A| = |5 + \frac{1}{n} - 5| = \frac{1}{n} < \varepsilon$ {by Archimedian property}

choose $n' \geq n > \frac{1}{\varepsilon}$

$\Rightarrow a_n \to 5$

b) $a_n = \frac{3n}{2n+1}$

Claim $a_n \to 3/2$

$|a_n - A| = \left|\frac{3n}{2n+1} - \frac{3}{2}\right| = \left|\frac{6n - 6n - 3}{2(2n+1)}\right|$

$= \frac{3}{2(2n+1)} < \varepsilon$

$\Rightarrow \frac{3}{2\varepsilon} > 2n+1$

$\Rightarrow \frac{3}{2\varepsilon} - 1 < n$

choose

$n' \geq n > \frac{3 - 2\varepsilon}{6}$

$\Rightarrow a_n \to 3/2$

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7. $(a_n)_{n=1}^{\infty} = \left(\frac{2n+1}{n}\right)_{n=1}^{\infty}$

Let $\varepsilon > 0$ be given and if $n > m$; Consider

$$|a_n - a_m| = \left|\frac{2n+1}{n} - \frac{2m+1}{m}\right|$$
$$= \left|\frac{m(2n+1) - n(2m+1)}{n \cdot m}\right|$$
$$= \left|\frac{2mn + m - 2mn - n}{n \cdot m}\right|$$
$$= \left|\frac{m-n}{n \cdot m}\right|$$
$$= -1\left(\frac{n-m}{n \cdot m}\right) < \frac{-1}{m}$$
$$\Rightarrow |a_n - a_m| < \frac{-1}{m} < \varepsilon; \text{ provided } m > \frac{-1}{\varepsilon}$$

Let m be a positive integer greater than $-1/\varepsilon$. Then $|a_n - a_m| < \varepsilon \;\; \forall\, n \geq m$

Hence $(a_n)_{n=1}^{\infty} = \left(\frac{2n+1}{n}\right)_{n=1}^{\infty}$ is a cauchy sequence ∎

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8. a) $(a_n)_{n=1}^{\infty} = ((-1)^n)_{n=1}^{\infty}$

Therefore

$(a_n)_{n=1}^{\infty} = (a_1, a_2, a_3, \dots)$
$= (-1, 1, -1, 1, \dots)$

Therefore the sequence $(a_n)_{n=1}^{\infty}$ oscillates infinitely between -1 and 1, so $\lim_{n\to\infty} f(n)$ does not exist

Therefore, $(a_n)_{n=1}^{\infty}$ is a divergent sequence.

b) $(a_n)_{n=1}^{\infty} = \left(\sin\left(\frac{4n+1}{2}\pi\right)\right)_{n=1}^{\infty}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sin\left(\frac{4n+1}{2}\pi\right)$

Now the limit depends on an n value so the limit does not exist and the sequence diverges

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9. take $(a_n) = n$
and $(b_n) = -n$
both of these diverge

$(a_n + b_n) = (n - n) = (0)$ is convergent

10. a) $\frac{n^2+4n}{n^2-5} \Rightarrow \lim_{n\to\infty}\left(\frac{n^2+4n}{n^2-5}\right) = \lim_{n\to\infty}\left(\frac{n^2\left(1+\frac{4n}{n^2}\right)}{n^2-5}\right)$

$= \lim_{n\to\infty}\left(\frac{n(n+4)}{n^2-5}\right) = \lim_{n\to\infty}\left(\frac{\frac{n+4}{n}}{1-\frac{5}{n^2}}\right) = \frac{\lim_{n\to\infty}\left(\frac{n+4}{n}\right)}{\lim_{n\to\infty}\left(1-\frac{5}{n^2}\right)} = \frac{1}{1} = 1$

so $\boxed{\lim_{n\to\infty} = 1}$

b) $\frac{n}{n^2-3} = \lim_{n\to\infty}\left(\frac{n}{n^2-3}\right) = \lim_{n\to\infty}\left(\frac{\frac{1}{n}}{1-\frac{3}{n^2}}\right) = \frac{\lim_{n\to\infty}\left(\frac{1}{n}\right)}{\lim_{n\to\infty}\left(1-\frac{3}{n^2}\right)}$

$= \frac{0}{1} = 0$ So $\boxed{\lim_{n\to\infty}\left(\frac{n}{n^2-3}\right) = 0}$

c) $\frac{\cos n}{n} = \lim_{n\to\infty}\left(\frac{\cos n}{n}\right)$

apply the squeeze theorem: so $\boxed{\lim_{n\to\infty}\frac{\cos n}{n} = 0}$

d) $\left(\sqrt{4-\frac{1}{n}} - 2\right)n = \lim_{n\to\infty}\left(\sqrt{4-\frac{1}{n}} - 2\right) = \lim_{n\to\infty}\left(\frac{-\frac{1}{n}}{\sqrt{4-\frac{1}{n}}+2}\, n\right)$

$= \lim_{n\to\infty}\left(-\frac{1}{\sqrt{\frac{4n-1}{n}}+2}\right) = -\frac{\lim_{n\to\infty}(1)}{\lim_{n\to\infty}\left(\sqrt{\frac{4n-1}{n}}+2\right)} = \frac{-1}{4}$

so $\boxed{\lim_{n\to\infty}\left(\sqrt{4-\frac{1}{n}} - 2\right)n = \frac{-1}{4}}$