Math 331: Homework 6

1. a) Fix $\delta > 0$ and let $[a,b]$ be an interval with $a<b$. It is given that $a<b \Rightarrow b-a>0$, so, $\exists\, n \in \mathbb{N}$ s.t.

$$\frac{b-a}{n} < \delta$$

Consider the partition P of $[a,b]$ defined as $\{a, P_1, P_2, \dots, P_n = b\}$ where

$$\Rightarrow P_i = a + \frac{i(b-a)}{n}$$

$$\Rightarrow P_1 - a = \frac{b-a}{n} < \delta$$

$$\Rightarrow P_k - P_{k-1} = a + \frac{k(b-a)}{n} - a - \frac{(k-1)(b-a)}{n}; \text{ for } k \geq 2$$

$$= \frac{k(b-a)}{n} - \frac{(k-1)(b-a)}{n}$$

$$= \frac{(b-a)}{n} < \delta$$

$$\|P\| < \delta \quad \blacksquare$$

b) Let f be a real-valued function over $[a,b]$ & $L \in \mathbb{R}$. Then f is integrable in $[a,b]$ iff there is a $\delta > 0$ for each $\varepsilon > 0$ s.t for each partition having $\|P\| < \delta$. We can have:

$$|S(f,P) - L| < \varepsilon$$

where L is known and $L = \int_a^b f(x)\,dx$ over $[a,b]$.
Now assume towards a contradiction that L_1 & L_2 are Riemann Integrals of f over $[a,b]$. Let $\varepsilon > 0$. Then for each $i = 1, 2$, $\exists\, \delta_i > 0$ s.t. $\|P\| < \delta_i \Rightarrow |\sigma - L_i| < \varepsilon/2$ whenever P is a partition of $[a,b]$. Take $\delta = \min\{\delta_1, \delta_2\}$. Fix a partition P of $[a,b]$ & suppose $\|P\| < \delta$. Hence

$$0 \leq |L_1 - L_2| \leq |\sigma - L_1| + |\sigma - L_2| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Since $\varepsilon > 0$ was arbitrary

$$0 \leq |L_1 - L_2| < \varepsilon \text{ holds for all } \varepsilon > 0.$$

This forces us to conclude that $|L_1 - L_2| = 0$.
Hence $L_1 = L_2$ and L is unique. $\blacksquare$

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2. a) Suppose that f and g are integrable functions on $[a,b]$. Write $I(f) := \int_a^b f(x)\,dx$ and $I(g) := \int_a^b g(x)\,dx$. Let $\varepsilon > 0$ be arbitrary. We have some $\delta > 0$ such that,

$$\left|\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) - I(f)\right| \le \varepsilon$$

$$\& \left|\sum_{i=1}^{n} g(c_i)(x_i - x_{i-1}) - I(g)\right| \le \varepsilon$$

whenever $P = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a,b]$ with $\|P\| < \delta$ and $c_i \in [x_{i-1}, x_i]$ for $i = 1, 2, \ldots, n$. It follows that

$$\left|\sum_{i=1}^{n} (f+g)(c_i)(x_i - x_{i-1}) - [I(f) + I(g)]\right|$$
$$\le \left|\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) - I(f)\right| + \left|\sum_{i=1}^{n} g(c_i)(x_i - x_{i-1}) - I(g)\right| \le 2\varepsilon.$$

Therefore $f+g$ is integrable on $[a,b]$ and $\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$. ∎

b) By part (a), $h := g - f$ is integrable on $[a,b]$. Since $h(x) \ge 0$ for all $x \in [a,b]$, it is clear that $L(h,P) \ge 0$ for any partition P of $[a,b]$. Hence, $\int_a^b h(x)\,dx = L(h) \ge 0$. Applying part (a) again, we see that

$$\int_a^b g(x)\,dx - \int_a^b f(x)\,dx = \int_a^b h(x)\,dx.$$ ∎

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3. Let $P \in P_{[a,b]}$, i.e., a partition of the interval $[a,b]$. We know that, $f(x) \le M$. Take a lower sum. So we have the infimum of called m_i, so we have,

$$m_i \le M$$

Then,

$$m_i(x_i - x_{i-1}) \le (x_i - x_{i-1})M$$

Taking the sum we get,

$$\sum_{i=1}^{n} m_i(x_i - x_{i-1}) \le \sum_{i=1}^{n} M(x_i - x_{i-1}) = M(b-a)$$

because $x_0 = a$ and $x_n = b$.
Taking the limit when $n \to \infty$ then, because f is R.I.,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \le \lim_{n\to\infty} M(b-a) = M(b-a)$$

Thus we get,

$$\int_a^b f(x)\,dx \le M(b-a). \blacksquare$$

4. Given that f is R.I. on $[a,b]$ and $\{P_n\}$ be a sequence of partition of $[a,b]$ s.t.

$$\lim_{n\to\infty} \|P_n\| = 0$$

Since f is integrable on $[a,b]$, for each $\varepsilon > 0$ $\exists$ a $\delta > 0$ s.t for all t.p. P satisfying $\|P\| < \delta$, there exists $|U(f,P_i) - L(f,P_i)| < \varepsilon$. Since $\lim_{n\to\infty}\|P_n\| = 0$, we have $|\|P_n\| - 0| < \delta$ for all $n \ge k \Rightarrow \|P_n\| < \delta$. Therefore $U(f,P_i) - L(f,P_i) < \varepsilon$ for all $n \ge k$. Since f is integrable on $[a,b]$, $L(f,P_n) \le \int_a^b f \le U(f,P_n)$ for $n \in \mathbb{N}$. Also for each P_n, $L(f,P_n) \le S(f,P_n) \le U(f,P_n)$. So we have,

$$|S(f,P_n) - \int_a^b f| \le (U(f,P_n) - L(f,P_n)) < \varepsilon$$

$$\Rightarrow |S(f,P_n) - \int_a^b f| < \varepsilon \quad \text{for all } n \ge k$$

$$\Rightarrow \lim_{n\to\infty} S(f,P_n) = \int_a^b f$$

Hence, $\{S(f,P_n)\}_{n=1}^{\infty}$ converges to $\int_a^b f$. $\blacksquare$

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5. Let the partition P of $[a,b]$ be
$$P := a = x_0 < x_1 < \dots = c < \dots < x_n = b$$
so that for arbitrary Riemann sum
$S(P,f) = \sum_{j=1}^{n} f(c_j)(x_j - x_{j-1})$, we have
$$\left|\int_a^b f(x)\,dx - S(P,f)\right| = \left|\int_a^b f(x)\,dx - \sum_{j=1}^{n} f(c_j)(x_j - x_{j-1})\right| < \varepsilon/3$$
Let $M = \sup_{x\in[a,b]} |f(x)|$ s.t.
$a - c = a - x_1 < \varepsilon/3M$, then
$$\left|\int_a^c f(x)\,dx - S(P',f)\right| = \left|\int_a^c f(x)\,dx - \sum_{j=2}^{n} f(c_j)(x_j - x_{j-1})\right| < \varepsilon/3$$
for each $a < c < a + \varepsilon/3M$,
$$\begin{aligned}\left|\int_a^b f(x)\,dx - \int_a^c f(x)\,dx\right| &\le \left|\int_a^b f(x)\,dx - S(P,f)\right| + |S(P,f) - S(P',f)| + \\ &\qquad \left|\int_a^c f(x)\,dx - S(P',f)\right| \\ &\le 2\varepsilon/3 + |f(c_1)|(a - x_1) \\ &\le 2\varepsilon/3 + M(a - x_1) \\ &< \varepsilon\end{aligned}$$

Since f is Riemann Integrable on $[a,c]$, f is also Riemann Integrable on $[a,b]$. ∎

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6. a) Consider a uniform partition P_n on $[a,b]$ for each n. P_n is a finite sequence of real numbers s.t.

$$a = x_0 < x < \dots < x_n = b$$

Define $I_j = [x_{j-1}, x_j]$ for $1 \le j \le n-1$

$$I_n = [x_{n-1}, x_n]$$

Clearly f is bounded on $[a,b]$. So,

$$m = \inf(f)$$
$$M = \sup(f)$$

Now, fix $j \in \{1, 2, \dots, n\}$

$$x \in I_j$$

$\Rightarrow m_j \le f(t_j) \le M_j$

so $m_j = \inf\{f(x) : x \in I_j\} = k$

& $M_j = \sup\{f(x) : x \in I_j\} = k$

So we have

$$L(f, P_n) = k(b-a) = k(x_n - x_0) = k$$

Then we have

$$\sum_{j=1}^{n} (x_j - x_{j-1}) = \sum_{j=1}^{n} m_j (x_j - x_{j-1})$$
$$\le \sum_{j=1}^{n} f(t_j)(x_j - x_{j-1})$$
$$\le \sum_{j=1}^{n} M_j (x_j - x_{j-1}) = k$$

$$\Rightarrow \sum_{j=1}^{n} (x_j - x_{j-1}) = k(x_n - x_0)$$
$$= k(b-a) = U(f, P_n)$$

And $\sup_{P_n}(L(f, P_n)) = \lim_{n\to\infty} L(f, P_n) = k(b-a)$

$$\le \lim_{n\to\infty} \sum_{j=1}^{n} f(t_j)(x_j - x_{j-1})$$
$$\le k(b-a) = \lim_{n\to\infty} U(f, P_n) = \inf_{P_n} U(f, P_n)$$

and $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{j=1}^{n} f(t_j)(x_j - x_{j-1}) = \sup L(f, P_n)$

$$= \inf_{P_n} U(f, P_n) = k(b-a) \quad ∎$$

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6. b) First we want to prove $\sin x$ is continuous. Let $\varepsilon > 0$ and $x, y \in \mathbb{R}$. We want $|f(x) - f(y)| < \varepsilon \Rightarrow |\sin x - \sin y| < \varepsilon \Rightarrow |2\cos\frac{x+y}{2}\sin\frac{x-y}{2}|$
Because

$$|2\cos\tfrac{x+y}{2}\sin\tfrac{x-y}{2}| \leq 2|\sin\tfrac{x-y}{2}|$$

it suffices

$$2|\sin\tfrac{x-y}{2}| < \varepsilon$$

when $|x-y| < \delta \Rightarrow |\frac{x-y}{2}| < \delta$
since $|\sin x| \leq |x|$

$$2|\sin\tfrac{x-y}{2}| \leq 2|\tfrac{x-y}{2}| < 2\delta$$

If we choose $\delta = \varepsilon/2$

$$= 2|\sin\tfrac{x-y}{2}| \leq 2|\tfrac{x-y}{2}| < 2(\varepsilon/2) = \varepsilon$$

So we have $\sin x$ is continuous and so is $\sin^2 x$ be continuity rules of multiplication. Because $\sin^2 x$ is continuous it is also R.I. on $[a,b]$. ■

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7. $f(x) = \begin{cases} 1 & 0 \le x < 1/2 \\ 0 & 1/2 \le x \le 1 \end{cases}$

To show $f(x)$ is R.I on $[0,1]$ we have

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$

And we get a sub-interval,

$$[0, \tfrac{1}{n}], [\tfrac{1}{n}, \tfrac{2}{n}], \ldots, [\tfrac{n-2}{2n}, \tfrac{1}{2}], [\tfrac{1}{2}, \tfrac{n+2}{2n}], \ldots, [\tfrac{n-1}{2n}, 1]$$

And

$$L(f,P) = \sum_{i=1}^{n} m_i \Delta x_i$$

So for all $i = 1, 2, \ldots, n$

$$L(f,P) = \sum_{i=1}^{n} m_i (\tfrac{1}{n}) = \sum_{i=1}^{n/2} m_i (\tfrac{1}{n}) + \sum_{i=(n/2+1)}^{n} m_i x_i$$

$$= (m_1 + m_2 + \ldots + m_{n/2} + \ldots + m_n) \cdot \tfrac{1}{n}$$

for sub-interval

$$[0, \tfrac{1}{n}] \cdots [\tfrac{n-2}{2n}, \tfrac{1}{2}], \; m_i = 1$$

$$\& \; [\tfrac{1}{2}, \tfrac{n+2}{2n}] \cdots [\tfrac{n-1}{n}, 1], \; m_i = 0$$

So we have

$$L(f,P) = (1 + 1 + \ldots n/2 \text{ times} + 0 + 0 + \ldots n/2 \text{ times}) \tfrac{1}{n}$$

$$= (1 \times \tfrac{n}{2} + 0) \tfrac{1}{n} = (\tfrac{n}{2})(\tfrac{1}{n}) = \tfrac{1}{2}$$

$$L(f,P) = \tfrac{1}{2}$$

and we have

$$U(f,P) = \sum_{i=1}^{n} M_i \Delta x_i$$

$$= \sum_{i=1}^{n} M_i (\tfrac{1}{n})$$

for sub-interval $[0, \frac{1}{n}] \cdots [\frac{n-2}{2n}, \frac{1}{2}]$, $M_i = 1$

$\& \; [\frac{n-2}{2n}, \frac{1}{2}] \cdots [\frac{n-1}{n}, 1]$, $M_i = 0$

M_i = Value of function of left value of subinterval

m_i = Value of function of right value of subinterval

$$U(f,P) = (1 + 1 + \ldots n/2 \text{ times} + 0) \tfrac{1}{n} = (1 \times \tfrac{n}{2}) \tfrac{1}{n}$$

$$U(f,P) = \tfrac{1}{2}$$

Now $\int_0^1 f(x)\,dx = \lim_{n\to\infty} L(f,P) = \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2}$

and $\int_0^1 f(x)\,dx = \lim_{n\to\infty} U(f,P) = 1/2$

so $\int_0^1 f(x)\,dx = \int_0^1 f(x)\,dx$

so f is R.I. on $[0,1]$ ∎

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8. Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ s.t $N > 2/\varepsilon$. Then, for all $n \geq N$, we have that $1/n < \varepsilon/2$.
Now, let $\delta = \min\{\frac{\varepsilon}{4N}, \frac{1}{4N}\}$ and define the partition

$$P = \{0, \varepsilon/2, \tfrac{1}{N-1} - \delta, \tfrac{1}{N-1} + \delta, \tfrac{1}{N-2} - \delta, \tfrac{1}{N-2} + \delta, \ldots, \tfrac{1}{2} - \delta, \tfrac{1}{2} + \delta, 1 - \delta, 1\}$$

Now, $\sup(f) = 1$ on $[0, \varepsilon/2]$ and on $[\frac{1}{N-k} - \delta, \frac{1}{N-k} + \delta]$ for all $k \leq N-2$, and $\sup(f) = 0$ on all other subintervals determined by P.
So,

$$U(f,P) = 1 \cdot \varepsilon/2 - 0 + 1 \cdot 2\delta + 0 + 1 \cdot 2\delta + \ldots + 0 + 1 \cdot 2\delta + 0 + 1 \cdot \delta$$

Thus,

$$\begin{aligned} U(f,P) &= \varepsilon/2 + \tfrac{\varepsilon}{2N} + \ldots + \varepsilon/2N + \tfrac{\varepsilon}{4N} \\ &= \varepsilon/2 + \sum_{k=1}^{N-2} \varepsilon/2N + \varepsilon/4N \\ &< \varepsilon/2 + \tfrac{N-1}{N-2}\tfrac{\varepsilon}{2} < \varepsilon \end{aligned}$$

On the other hand, $L(f,P) = 0$, since $\inf(f) = 0$ on any sub-interval of $[0,1]$. Therefore,

$$U(f,P) - L(f,P) = U(f,P) - 0 = U(f,P) < \varepsilon$$

Since $\varepsilon > 0$ was arbitrary, f is Riemann Integrable. ∎

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10. $P := \{(-0.9, I_1), (-0.7, I_2), (-0.1, I_3), (0.2, I_4), (0.2, I_5), (0.8, I_6), (1.42, I_7), (1.9, I_8)\}$

Where $I_1 = [-1, -0.8]$
$I_2 = [-0.8, -0.3]$ $\quad I_6 = [0.4, 1]$
$I_3 = [-0.3, 0]$ $\quad I_7 = [1, 1.5]$
$I_4 = [0, 0.2]$ $\quad I_8 = [1.5, 2]$
$I_5 = [0.2, 0.4]$

$|I_1| = -0.8 + 1 = 0.2$ $\quad |I_6| = 0.6$
$|I_2| = 0.5$ $\quad |I_7| = 0.5$
$|I_3| = 0.3$ $\quad |I_8| = 0.5$
$|I_4| = 0.2$
$|I_5| = 0.2$

$$\|P\| = \max\{|I_1|, |I_2|, |I_3|, |I_4|, |I_5|, |I_6|, |I_7|, |I_8|\}$$
$$= 0.6$$

Take $P_0 = \{(-1, [-1, -0.9]), (-0.9, [-0.9, -0.8]), (-0.8, [-0.8, -0.7]), (-0.7, [-0.7, -0.6]), (-0.6, [-0.6, -0.5]), (-0.5, [-0.5, -0.4]), (-0.4, [-0.4, -0.3]), (-0.3, [-0.3, -0.2]), (-0.2, [-0.2, -0.1]), (-0.1, [-0.1, 0]), (0, [0, 0.1]), (0.1, [0.1, 0.2]), (0.2, [0.2, 0.3]), (0.3, [0.3, 0.4]), (0.4, [0.4, 0.5]), (0.5, [0.5, 0.6]), (0.6, [0.6, 0.7]), (0.7, [0.7, 0.8]), (0.8, [0.8, 0.9]), (0.9, [0.9, 1]), (1, [1, 1.1]), (1.1, [1.1, 1.2]), (1.2, [1.2, 1.3]), (1.3, [1.3, 1.4]), (1.4, [1.4, 1.5]), (1.5, [1.5, 1.6]), (1.6, [1.6, 1.7]), (1.7, [1.7, 1.8]), (1.8, [1.8, 1.9]), (1.9, [1.9, 2])\}$

Then P_0 is a partition of $[-1, 2]$, P_0 contains 30 intervals and length of each interval is 0.1.

Then $\|P_0\| = 0.1$ & $\dfrac{\|P\|}{3} = 0.2$

so $\|P_0\| \le \dfrac{\|P\|}{3}$