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1. a) We treat this in three cases: $I=[u,v]$, $I=(u,v]$, $I=\{u\}$.

We know from statement that for case one:

$$f(x)=cX_I(x)=\begin{cases} 0, & a\leq x\leq u \\ c, & u\leq x\leq v \\ 0, & v,\leq x\leq b \end{cases}$$

By exo 5 on HW 6, we know that if $\exists c\in[a,b]$ s.t. if $[a,c]$ is R.I. and $[c,b]$ is R.I. then f is R.I. on $[a,b]$.

Since the step function is 0 on $[a,u]$ and $[v,b]$, it is R.I because all constants are Riemann integrable.

Also, by assumption, $\forall x\in[u,v]$, $\int_u^v x\,dx = x(u-v)=c$.

Then, the function f is R.I. on $[a,u]$ $[u,v]$, and $[v,b]$.

Case 2: If f is R.I. and bounded on $[v,b]$ and $[a,v]$ for $v\in[a,b]$, f is R.I on $[a,b]$. Let v get arbitrarily close to a. Then f is R.I. on $[a,v]$ and $[v,b]$, so by HW 6 #5, f is R.I. on $(u,v]$.

Case 3: When $I=\{u\}$, we simply get a constant and all constants are R.I.

b) We know by statement that $f_1, f_2, f_3, \ldots, f_n$ are R.I. functions. Let P be a tagged partition of $[a,b]$.
Let $f_1 + f_2$ be our base case. If both f_1 and f_2 are R.I, then $\exists\, \delta_1, \delta_2 > 0$ s.t

$$\|P\| < \delta_1 \Rightarrow |S(f_1, P) - \int_a^b f_1| < \varepsilon/2$$
$$\|P\| < \delta_2 \Rightarrow |S(f_2, P) - \int_a^b f_2| < \varepsilon/2.$$

Letting $\delta := \min\{\delta_1, \delta_2\}$, we have if $\|P\| < \delta$, then

$$|S(f_1 + f_2, P) - \int_a^b f_1 - \int_a^b f_2| = |S(f_1, P) + S(f_2, P) - \int_a^b f_1 - \int_a^b f_2|$$
$$\leq |S(f_1, P) - \int_a^b f_1| + |S(f_2, P) - \int_a^b f_2|$$
$$< \varepsilon/2 \cdot 2 = \varepsilon$$

So $f_1 + f_2$ is R.I.
Applying this idea to $f_1 + f_2 + \ldots + f_{n+1}$, we have if $f_1, f_2, \ldots, f_{n+1}$ are R.I, then $\exists\, \delta_1, \delta_2, \ldots, \delta_{n+1}$ s.t.

$$\|P\| < \delta_1 \Rightarrow |S(f_1, P) - \int_a^b f_1| < \varepsilon/n+1$$
$$\vdots$$
$$\|P\| < \delta_{n+1} \Rightarrow |S(f_{n+1}, P) - \int_a^b f_{n+1}| < \varepsilon/n+1$$

Let $\delta := \min\{\delta_1, \delta_2, \ldots, \delta_{n+1}\}$. If $\|P\| < \delta$ then

$$|S((f_1 + f_2 + \ldots + f_{n+1}), P) - \int_a^b f_1 - \int_a^b f_2 - \ldots - \int_a^b f_{n+1}| = |S(f_1, P) + S(f_2, P) + \ldots + S(f_{n+1}, P) - \int_a^b f_1 - \int_a^b f_2 - \ldots - \int_a^b f_{n+1}|$$
$$\Rightarrow \leq |S(f_1, P) - \int_a^b f_1| + |S(f_2, P) - \int_a^b f_2| + \ldots |S(f_{n+1}, P) - \int_a^b f_{n+1}|$$
$$< \varepsilon/n+1 + \varepsilon/n+1 + \ldots + \varepsilon/n+1 = \varepsilon.$$

So $f_1 + f_2 + \ldots + f_n$ is R.I.

Since we know $f_1 + f_2 + \ldots + f_n$ is R.I, we now need to prove $\int_a^b (f_1 + f_2 + \ldots + f_n) = \int_a^b f_1 + \int_a^b f_2 + \ldots + \int_a^b f_n$.
Rewrite the left side as a Riemann sum:

$$\sum_{i=1}^n (f_1 + f_2 + f_3 + \ldots + f_n)(c_i)(x_{i-1} - x_i) = \sum_{i=1}^n f_1(c_i) + f_2(c_i) + \ldots + f_n(c_i)(x_{i-1} - x_i)$$

$$\Rightarrow \sum_{i=1}^n f_1(c_i)(x_{i-1} - x_i) + \sum_{i=1}^n (f_2)(c_i)(x_{i-1} - x_i) + \ldots + \sum_{i=1}^n (f_n)(c_i)(x_{i-1} - x_i)$$

Which is exactly equal to $\int_a^b f_1 + \int_a^b f_2 + \ldots \int_a^b f_n$.

c) $\varphi = \sum_{k=1}^{n} c_k \chi I_k$.

By the definition of a step function, $I_k = \ell(I_k)$ where ℓ gives the length of the interval I, which we denote as $(x_{i-1} - x_i)$ for $(x_{i-1}, x_i) \in I$.

So we have, $\varphi = \sum_{k=1}^{n} c_k \chi I_k = \sum_{k=1}^{n} c_k (x_{i-1} - x_i)$.
c_k is simply a constant, so this is exactly the Riemann sum formula.

Therefore step functions are R.I.

2. We want to show if $a \le u < v \le b$ then

$$\int_u^v f \le \int_a^b f$$

Since f is nonnegative, then $\int_a^b f \ge 0$.

For this, we will deal with the integrals over three intervals:

$\int_a^u f + \int_u^v f + \int_v^b f = \int_a^b f$ (by properties of the integral).

Again since $f \ge 0$,

$$\int_a^u f \ge 0, \int_u^v f \ge 0, \int_v^b f \ge 0$$

So

$$\int_a^u f + \int_v^b f \ge 0$$

and

$$\int_a^u f + \int_v^b f + \int_u^v f \ge \int_u^v f$$

so

$$\int_a^b f \ge \int_u^v f.$$

3. a) By FTC, $F(x) = \int_a^b f$ since f is continuous and integrable, so $F'(x) = f(x)$. Therefore let $\int_a^x f = F(x)$ with $x \in [a,b]$.
We know $F(a) = F(b) = 0$ by assumption. Let there also be a $y \in [a,b]$ s.t. $x < y$. Then $[a,x] \subseteq [a,y]$.
So, $\int_a^x f \leq \int_a^y f$, so $F(x) \leq F(y)$ meaning that F is an increasing function.
Then, since $F(x) \leq F(b)$, $\forall x \in [a,b]$, then $0 \leq F(x) \leq 0$ so $F(x) = 0$ by the squeeze theorem.
From FTC, $F'(x) = f(x)$, and since the derivative of 0 is 0, $f(x) = 0$.

b) By FTC, $F(x) = \int_a^x f$, for some $x \in [a,b]$. Since $\int_a^b f = \int_a^b g$,
let $F := \int_a^b f - \int_a^b g = 0$. So $F(a) = 0$ and $F(b) = 0$.
We can rewrite $F(b)$, WLOG, as $\int_a^b f - \int_a^b g = 0$, so by Rolle's thm, $\exists c \in (a,b)$ s.t. $F'(c) = 0$. Since $F := \int_a^b f - \int_a^b g$,
$F'(c) = f(c) - g(c) = 0$ so $f = g$.

4. f is continuous and bounded, so let $m := \inf(f)$ and $M := \sup(f)$.
Then $m \leq f(x) \leq M$ and
$$\int_a^b m \leq \int_a^b f(x) \leq \int_a^b M$$
Since m and M are constants:
$$m(b-a) \leq \int_a^b f(x) \leq M(b-a).$$
$b \neq a$ so
$$m \leq \frac{1}{b-a}\int_a^b f(x) \leq M$$
and by IVT:
$$f(c) = \frac{1}{b-a}\int_a^b f(x)$$
for $c \in [a,b]$. So $f(c)(b-a) = \int_a^b f(x)$.

5. Let the function $g(x) := f(a)(x-a) + f(b)(b-x)$.
Notice $g(a) = f(a)(a-a) + f(b)(b-a) = f(b)(b-a)$ and $g(b) = f(a)(b-a) + f(b)(b-b) = f(a)(b-a)$.

Since f is strictly increasing, then $g(a) \le \int_a^b f \le g(b)$, so by IVT, there $\exists c \in (a,b)$ s.t. $g(c) = \int_a^b f$.

By our denoted formula for g then, we have

$$g(c) = \int_a^b f = f(a)(c-a) + f(b)(b-c).$$

6. a) Suppose to a contradiction that f is R.I. Then for any sequence of tagged partitions $(P_n)_{n=1}^{\infty}$ of $[0,1]$, if

$$\lim_{n\to\infty} \|P_n\| = 0$$

then

$$\lim_{n\to\infty} S(f, P_n) = \int_0^1 f.$$

By $\overline{\mathbb{Q}}$, between any two irrational numbers, there must exist a rational number. So, let $(P_n)_{n=1}^{\infty}$ be the sequence of partitions for $x \in \mathbb{Q}$, and $(T_n)_{n=1}^{\infty}$ be the sequence of partitions for $x \notin \mathbb{Q}$. By assumption, $\lim_{n\to\infty} \|P_n\| = \lim_{n\to\infty} \|T_n\| = 0$ and $\int_a^b f = S(f, P_n) = S(f, T_n) = \int_a^b f$.
However, by construction of f, $(P_n)_{n=1}^{\infty} \to 1$ and $(T_n)_{n=1}^{\infty} \to 0$.
So f is not R.I.

b) Taking the same idea as a), assume to a contradiction that $g \circ h$ is R.I.
We know $h(x) \to 0$ for $x \notin \mathbb{Q}$, and $h(x) \to 1/q$ for $x = p/q \in \mathbb{Q}$.
Therefore $g(h(x)) \to x_1$ for $x_1 \in \mathbb{Q}$ and $g(h(x)) \to x_2$ for $x_2 \notin \mathbb{Q}$.

So we construct two sequences, $(P_n)_{n=1}^{\infty}$ and $(T_n)_{n=1}^{\infty}$ such that $(P_n)_{n=1}^{\infty} \to x_1$ and $(T_n)_{n=1}^{\infty} \to x_2$. If $g(h(x))$ is R.I., then by definition, $\lim_{n\to\infty} S(f, P_n) = \lim_{n\to\infty} S(f, T_n) = \int_a^b f$. However, by nature of g as a step function, $x_1 \neq x_2$, so $g \circ h$ is not R.I.
We can say that the composition of two R.I. functions will be R.I.

7. From previous work, we know that if f is continuous, $|f|$ is also continuous, so it is R.I. on $[a,b]$.

Then,

$$\int_a^b |f| = \lim_{\|P\|\to 0} \sum_{n=1}^{N} |f(c_i)|(x_{i-1} - x_i)$$

$$= \lim_{\|P\|\to 0} \left[|f(c_1)| + |f(c_2)| + \ldots |f(c_n)|\right](x_{i-1} - x_i)$$

While

$$\left|\int_a^b f\right| = \left|\lim_{\|P\|\to 0} \sum_{n=1}^{N} f(c_i)(x_{i-1} - x_i)\right|$$

$$= \lim_{\|P\|\to 0} \left|\left(f(c_1) + f(c_2) + \ldots f(c_n)\right)\right| \left|(x_{i-1} - x_i)\right|$$

The triangle inequality states $|a+b| \leq |a| + |b|$, so

$$|f(c_1) + f(c_2) + \ldots + f(c_n)| \leq |f(c_1)| + \ldots + |f(c_n)|$$

so

$$\left|\int_a^b f\right| \leq \int_a^b |f| .$$

8. By FTC, we know $\frac{d}{dx}\int_{g(x)}^{f(x)} h(t)\,dt = h(f(x))\cdot f'(x) - h(g(x))\cdot g'(x)$.

So:

$$\int_{\sqrt{x}}^{\sqrt[3]{x}} \frac{1}{1+t^3}\,dt$$

$$h(f(x))\cdot f'(x) = \frac{1}{1+(\sqrt[3]{x})^3}\cdot(\sqrt[3]{x})' = \frac{1}{1+x}\left(\frac{1}{3}\cdot x^{-2/3}\right)$$
$$= \frac{1}{1+x}\cdot\frac{1}{3\sqrt[3]{x^2}}$$

$$h(g(x))\cdot g'(x) = \frac{1}{1+(\sqrt{x})^3}\cdot(\sqrt{x})' = \frac{1}{1+\sqrt{x^3}}\cdot\frac{1}{2\sqrt{x}}$$

$$\int_{\sqrt{x}}^{\sqrt[3]{x}} \frac{1}{1+t^3}\,dt = \left(\frac{1}{1+x}\cdot\frac{1}{3\sqrt[3]{x^2}}\right) - \left(\frac{1}{1+\sqrt{x^3}}\cdot\frac{1}{2\sqrt{x}}\right)$$

9. By the theorem that shows continuous functions are Riemann integrable, we will prove f is R.I.

We know from previous work that x^2 is continuous everywhere and $\sin x$ is continuous everywhere, so because the composition of two continuous functions is also continuous, $\sin(x^2)$ is continuous. Also 1 is just a constant so $f'(x) = 1 + \sin(x^2)$ is continuous.

Therefore, $f'(x)$ is also continuous and integrable. So, by FTC, $F'(x) = f(x)$, so $f(x)$ exists, and is equal to $\int_a^b 1+\sin(x^2)\,dx$.

10. If our expression is a Riemann sum, then we want to find Δx.
So,

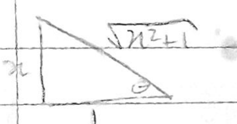
$$\sum_{k=1}^{n} \frac{n}{k^2+n^2} = \sum_{k=1}^{n} \frac{n}{\frac{k^2}{n}+n} \cdot \frac{1}{n}$$

$$= \sum_{k=1}^{n} \frac{1}{\frac{k^2}{n^2}+1} \cdot \frac{1}{n}$$

So $\Delta x = 1/n$ and $f(x) = \frac{1}{x^2+1}$.
Since $\lim_{n\to\infty} \frac{1}{n} = 0$ and $\lim_{n\to\infty} \frac{k}{n} = 1$, we have

$$\int_0^1 \frac{1}{x^2+1}\,dx$$

With a trig sub, we have



$$\Rightarrow \int \frac{1}{1+\tan^2\theta} \sec^2\theta \, d\theta = \int \frac{1}{\sec^2\theta} \sec^2\theta \, d\theta$$

$$= \int d\theta = \theta + C$$

$$= \arctan 1 - \arctan 0$$

$$= \arctan 1 = \frac{\pi}{4}.$$