MATH-331 Introduction to Real Analysis | Ian Oga
Homework 07 | Fall 2021

Due date: December, 6th 1:20pm | Total: 60/65.

Exercise	1 (10)	2 (5)	3 (10)	4 (5)	5 (5)	6 (10)	7 (5)	8 (5)	9 (5)	10 (5)
Score	9	5	7	5	5	10	5	5	5	4

Table 1: Scores for each exercises

Instructions: You must answer all the questions below and send your solution by email (to `parisepo@hawaii.edu`). If you decide to not use LaTeX to hand out your solutions, please be sure that after you scan your copy, it is clear and readable. Make sure that you attached a copy of the homework assignment to your homework.
If you choose to use LaTeX, you can use the template available on the course website.
No late homework will be accepted. No format other than PDF will be accepted. Name your file as indicated in the syllabus.

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Writing problems

For each of the following problems, you will be asked to write a clear and detailed proof. All the exercises below can be solve without using the definition with partitions. Try to go back to homework 6 and use some of the exercises there to solve the following problems.

You will have the chance to rewrite your solution in your semester project after receiving feedback from me.

Exercise 1. (10 pts) Prove that a step function is Riemann integrable on $[a, b]$. Follow the steps below.

a) Let I be a subinterval of $[a, b]$ and put $\phi = c\chi_I$. Prove that ϕ is Riemann integrable and that $\int_a^b \phi = c\ell(I)$. [There are three cases to consider: $I = [u, v]$, $I = (u, v]$, and $I = \{u\} = [u, u]$.]

b) Prove by induction that if f_1, f_2, …, f_n are Riemann integrable functions on $[a, b]$, then $f_1 + f_2 + \cdots + f_n$ is Riemann integrable and

$$\int_a^b (f_1 + f_2 + \cdots + f_n) = \int_a^b f_1 + \int_a^b f_2 + \cdots + \int_a^b f_n.$$

c) Write $\phi = \sum_{k=1}^{n} c_k\chi_{I_k}$. Use the second part of this exercise to show that ϕ is Riemann integrable.

Solution: From HW6, exercise 6a, the integral of constant c over interval $[a, b]$ is $c(b - a)$.

a) Consider three cases:
Case 1: $I = [u, v]$ for $v > u$. Then:
$\int_a^b \phi = \int_{[a,u)} \phi + \int_{[u,v]} \phi + \int_{(v,b]} \phi$ 1/2
$\int_a^b \phi = \int_{[a,u)} 0 + \int_{[u,v]} c + \int_{(v,b]} 0$
$\int_a^b \phi = 0 + c(v-u) + 0$
9/10 $\int_a^b \phi = c \cdot \ell(I)$
Case 2: $I = [u, u]$. Then:
$\int_a^b \phi = \int_{[a,u)} \phi + \int_{[u,u]} \phi + \int_{(u,b]} \phi$
$\int_a^b \phi = \int_{[a,u)} 0 + \int_{[u,u]} c + \int_{(u,b]} 0$
$\int_a^b \phi = 0$
$\int_a^b \phi = c \cdot (u-u)$
$\int_a^b \phi = c \cdot \ell(I)$
Case 3: $I = (u, v]$ for $v > u$. Then:
$\int_a^b \phi = \int_{[a,u]} \phi + \int_{(u,v]} \phi + \int_{(v,b]} \phi$
$\int_a^b \phi = \int_{[a,u)} \phi + \int_{[u,v]} \phi - \int_{[u,u]} \phi + \int_{(v,b]} \phi$
$\int_a^b \phi = \int_{[a,u)} 0 + \int_{[u,v]} \phi - \int_{[u,u]} 0 + \int_{(v,b]} 0$
$\int_a^b \phi = c(v-u)$
$\int_a^b \phi = c \cdot \ell(I)$
Therefore $\int_a^b \phi = c \cdot \ell(I)$
.

almost
We have to use Exercise 5 from homework 6.

b) From Theorem 5.9, we have that:
$\int_a^b (f_1 + f_2) = \int_a^b f_1 + \int_a^b f_2$
Therefore the sum of two Riemann integrable functions is Riemann integrable. Now suppose $\int_a^b \sum_{i=1}^{k} f_i = \sum_{i=1}^{k} \int_a^b f_i$. We then have that:
$\int_a^b \sum_{i=1}^{k+1} f_i = \int_a^b f_{k+1} + \sum_{i=1}^{k} f_i$ ✓
$\int_a^b \sum_{i=1}^{k+1} f_i = \int_a^b f_{k+1} + \int_a^b \sum_{i=1}^{k} f_i$ 5/5
$\int_a^b \sum_{i=1}^{k+1} f_i = \int_a^b f_{k+1} + \sum_{i=1}^{k} \int_a^b f_i$
$\int_a^b \sum_{i=1}^{k+1} f_i = \sum_{i=1}^{k+1} \int_a^b f_i$
Therefore by induction, the sum of arbitrarily many Riemann integrable functions is Riemann integrable. The Riemann integral of the sum of functions is equal to the sum of the Riemann integral of each function.

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c) A step function ϕ is equal to a sum of arbitrarily many step intervals. From part a, we know that each interval is Riemann integrable. Therefore from part b, we know that the sum will also be Riemann integrable. Therefore all step functions ϕ are Riemann integrable. □

Exercise 2. (5 pts) Suppose that f is Riemann integrable on $[a, b]$ and that f is nonnegative (means that $f(x) \geq 0$ for $x \in [a, b]$). Let $u, v \in \mathbb{R}$. Show that if $a \leq u < v \leq b$, then

$$\int_u^v f \leq \int_a^b f.$$

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[Hint: Use the following property of the Riemann Integral multiple times: $\int_a^b f = \int_a^c f + \int_c^b f$.]

Solution: Using the above property for Riemann Integrals:

$\int_a^b f = \int_a^v f + \int_v^b f$

$\int_a^b f = (\int_a^u f + \int_u^v f) + \int_v^b f$

$\int_a^b f - \int_u^v f = \int_a^u f + \int_v^b f$

As $f(x) \geq 0$ for all $x \in [a, b]$, we know from HW6, exercise 2 that $\int_a^u f \geq \int_a^u 0 = 0$ and $\int_v^b f \geq \int_v^b 0 = 0$. Therefore $\int_a^u f + \int_v^b f \geq 0$ and:

$\int_a^b f - \int_u^v f \geq 0$

$\int_a^b f \geq \int_u^v f$

This is what we wanted to prove. $\square$

Exercise 3. (10 pts) Use the Fundamental Theorem of Calculus to solve the following problems:

a) Suppose that f is continuous on $[a, b]$ and that f is nonnegative on $[a, b]$. Show that if $\int_a^b f = 0$, then $f(x) = 0$ for any $x \in [a, b]$.

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b) Suppose that f and g are continuous on $[a, b]$ such that $\int_a^b f = \int_a^b g$. Show that there exists a point $c \in (a, b)$ such that $f(c) = g(c)$.

Solution:

a) $0 = \int_a^b f$. Using Theorem 5.10, we know that for any $x \in [a, b]$:

$0 = \int_a^x f + \int_x^b f$ (*)

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As $f(x) \geq 0$, $\int_c^d f \geq 0$ for any interval [c.d] in [a,b]. Then $\int_a^x f \geq 0$. We now have:

$0 \leq \int_x^b f$

$0 \leq F(b) - F(x)$

Taking the derivative:

$0 \leq -f(x)$

$0 \geq f(x)$

Since $f(x) \geq 0$:

$0 \leq f(x) \leq 0$

Therefore $f(x) = 0$ on $[a, b]$

not exactly, you can't take the derivative and keep the sign of the inequality. You have to use that F is increasing

b) I don't know how to prove this using the Fundamental Theorem of Calculus, but this can be proven using exercise 4. Since $\int_a^b f = \int_a^b g$, $\int_a^b f - \int_a^b g = 0$ and $\int_a^b f - g = 0$. As f and g are continuous, $f - g$ is continuous. Exercise 4 tells us that there then exists a point $c \in [a, b]$ such that $(f(c) - g(c))(b - a) = \int_a^b f - g$. Therefore:

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$(f(c) - g(c))(b - a) = 0$

$f(c) - g(c) = 0$ as $b > a$

$f(c) = g(c)$

okay. Here, you could have used the MVT.

In total, Exercise 4 tells us that there exists $c \in [a, b]$ where $f(c) = g(c)$. $\square$

Exercise 4. (5 pts) Let f be a continuous function on $[a, b]$. Prove that there exists a number $c \in [a, b]$ such that $f(c)(b-a) = \int_a^b f$.

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Solution: By the Extreme Value Theorem, we know there exists $x_0, x_1 \in [a, b]$ such that f reaches its maximum at x_0 and its minimum at x_1. Consider two cases:
Case 1: Suppose $x_0 = x_1$. Then $f(x_0) = f(x_1)$. Since $f(x_1) \leq f(x) \leq f(x_0)$, f is a constant function. From HW6, exercise 6a, the integral of constant $f(a)$ over $[a, b]$ is $f(a)(b-a)$. As $f(x) = f(a)$, we have $a \in [a, b]$ such that $\int_a^b f = f(a)(b-a)$, which is what we wanted to prove.
Case 2: Suppose $x_0 \neq x_1$. Since $f(x_0) \geq f(x)$ on $[a, b]$, we know from HW6, exercise 2b that $\int_a^b f(x)dx \leq \int_a^b f(x_0)dx$. Then:
$\int_a^b f(x)dx \leq f(x_0)(b-a)$
Similarly, since $f(x_1) \leq f(x)$ on $[a, b]$, $\int_a^b f(x)dx \geq \int_a^b f(x_1)dx$. Then:
$\int_a^b f(x)dx \geq f(x_1)(b-a)$
Combining these equations:
$f(x_1)(b-a) \leq \int_a^b f(x)dx \leq f(x_0)(b-a)$
Now define $g(x) = f(x)(b-a)$. As f is continuous, g is continuous. Then:
$g(x_1) \leq \int_a^b f(x)dx \leq g(x_0)$
By the Intermediate Value Theorem, there then exists c between x_0 and x_1 such that $g(c) = f(c)(b-a) = \int_a^b f(x)dx$. As $x_0, x_1 \in [a, b]$, $c \in [a, b]$. This is what we wanted to prove. □

Exercise 5. (5 pts) Suppose that f is Riemann integrable on $[a, b]$ and is strictly increasing there. Prove that there exists a point $c \in (a, b)$ such that

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$$\int_a^b f = f(a)(c-a) + f(b)(b-c).$$

[Hint: Define the function $g(x) = f(a)(x-a) + f(b)(b-x)$. Show that $\int_a^b f$ is between the numbers $f(a)(b-a)$ and $f(b)(b-a)$ and use the Intermediate Value Theorem.]

Solution: Since $f(x)$ is strictly increasing, $f(x) > f(a)$ on (a, b). Then $\int_a^b f(x)dx > \int_a^b f(a)dx$. Since $f(a)$ is a constant:
$\int_a^b f(x)dx > f(a)(b-a)$
In a similar way, $f(x) < f(b)$ on (a, b). Then $\int_a^b f(x)dx < \int_a^b f(b)dx$ and:
$\int_a^b f(x)dx < f(b)(b-a)$
Putting both equations together:
$f(a)(b-a) < \int_a^b f(x)dx < f(b)(b-a)$
Now define $g(x)$ as given in the hint. Note that $g(a) = f(b)(b-a)$ and $g(b) = f(a)(b-a)$. Therefore:
$g(b) < \int_a^b f(x)dx < g(a)$
As $f(a), f(b)$ are constants, and x is a continuous function, g is a continuous function. Then by the Intermediate Value Theorem, there exists $c \in (a, b)$ such that $g(c) = f(a)(c-a) + f(b)(b-c) = \int_a^b f$. This is what we wanted to prove. □

Answer all the questions below. Make sure to show your work.

Exercise 6. (10pts)

a) Show that the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & , x \in \mathbb{Q} \\ 0 & , x \notin \mathbb{Q} \end{cases}$$

is not Riemann integrable on $[0,1]$. [Hint: Use exercise 4 from Homework 6.]

b) Define the two functions $g : [0,1] \to \mathbb{R}$ and $h : [0,1] \to \mathbb{R}$ by $g = \chi_{(0,1]}$ and

p. 10

$$h(x) = \begin{cases} 0 & , x \notin \mathbb{Q} \\ \frac{1}{q} & , x = p/q \in \mathbb{Q}. \end{cases}$$

Use the first part to show that $g \circ h$ is not Riemann integrable on $[0,1]$. What can you say about the composition of two Riemann integrable functions in light of this last examples?

Solution:

a) Suppose towards a contradiction that f is Riemann integrable on [0,1]. Then there exists L, δ such that for any partition $\mathcal{P}$ where $||\mathcal{P}|| < \delta$, $|S(f,\mathcal{P}) - L| < 0.5$. Consider partition $\mathcal{P} = \{(c_i, [u_i, v_i])\}$ where $||\mathcal{P}|| < \delta$. Since both the rationals and irrationals are dense in the real numbers, we can always find an rational and irrational number in each interval $[u_i, v_i]$. Define partition $\mathcal{P}_0 = \{(a_i, [u_i, v_i])\}$ so that every a_i is irrational. Then define partition $\mathcal{P}_1 = \{(b_i, [u_i, v_i])\}$ so that every b_i is rational. Note that $||\mathcal{P}_0||, ||\mathcal{P}_1|| < \delta$. Since all a_i are irrational and all b_i are rational, $S(f, \mathcal{P}_0) = 0$ and $S(f, \mathcal{P}_1) = 1$. We now have the following:
$|S(f, \mathcal{P}_0) - L| < 0.5$
$|L| < 0.5$
$-0.5 < L < 0.5$ (*) ✓
$|S(f, \mathcal{P}_1) - L| < 0.5$
$|1 - L| < 0.5$
$-0.5 < 1 - L < 0.5$
$-1.5 < -L < -0.5$
$0.5 < L < 1.5$ (**)
L cannot satisfy both (*) and (**). This is a contradiction. Therefore f is not Riemann integrable.

b) Note that for $x \notin \mathbb{Q}$, $h(x) = 0 \notin (0,1]$. Therefore $g(h(x)) = 0$ for $x \notin \mathbb{Q}$. For $x \in \mathbb{Q}$, $h = \frac{1}{q}$ for natural number q. Therefore $0 < h(x) \leq 1$ and $h(x) \in (0,1]$. Thus $g(h(x)) = 1$ for $x \in \mathbb{Q}$. This means that $g \circ h$ is equal to f from part a, and is not Riemann integrable. This example shows that the composition of two Riemann integrable functions is not always Riemann integrable. ☐

Exercise 7. (5 pts) Show that if f is continuous on $[a,b]$, then $|f|$ is Riemann integrable on $[a,b]$ and

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$$\Big|\int_a^b f\Big| \le \int_a^b |f|.$$

[Hint: There is a clever way to show that $|f|$ is Riemann integrable without using the definition with the partitions.]

Solution: From Theorem 3.4, if function f is continuous at x_0 and function g is continuous at $f(x_0)$, the composition $g \circ f$ is continuous at x_0. By extension, for functions f and g continuous on all real numbers, their composition $g \circ f$ is continuous on all real numbers. As $|x|$ is continuous, the composition $|f(x)|$ is continuous. All continuous functions are Riemann integrable, so $|f(x)|$ is Riemann integrable.
Note that $-|f| \le f \le |f|$. From HW6, exercise 2b we then have that:
$\int_a^b -|f| \le \int_a^b f \le \int_a^b |f|$
$-\int_a^b |f| \le \int_a^b f \le \int_a^b |f|$ ✓
$|\int_a^b f| \le \int_a^b |f|$
This is what we wanted to prove. □

Exercise 8. (5 pts) Find $f'(x)$ if $f(x) = \displaystyle\int_{\sqrt{x}}^{\sqrt[3]{x}} \frac{1}{1+t^3}\,dt$ where $x \in [0,1]$.

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Solution: The Fundamental Theorem of Calculus tells us that:
$\int_a^b f(t)dt = G(b) - G(a)$
where G is the antiderivative of f. This means that $G'(x) = f(x)$. If we substitute functions in for integration bounds a, b we get the following:
$\int_{h_1(x)}^{h_2(x)} f(t)dt = G(h_2(x)) - G(h_1(x))$
Taking the derivative:
$\frac{d}{dx}\int_{h_1(x)}^{h_2(x)} f(t)dt = G'(h_2(x))h_2'(x) - G'(h_1(x))h_1'(x)$
$\frac{d}{dx}\int_{h_1(x)}^{h_2(x)} f(t)dt = f(h_2(x))h_2'(x) - f(h_1(x))h_1'(x)$
Applying this to the original problem:
$f'(x) = \frac{d}{dx}\int_{\sqrt{x}}^{\sqrt[3]{x}} \frac{1}{1+t^3}dt$
$f'(x) = \frac{1}{1+(\sqrt[3]{x})^3} \cdot (\frac{d}{dx}x^{1/3}) - \frac{1}{1+(\sqrt{x})^3} \cdot (\frac{d}{dx}x^{1/2})$
$f'(x) = \frac{1}{1+x} \cdot (\frac{1}{3}x^{-2/3}) - \frac{1}{1+x^{3/2}} \cdot (\frac{1}{2}x^{-1/2})$
$f'(x) = \frac{1}{3x^{2/3}+3x^{5/3}} - \frac{1}{2x^{1/2}+2x^2}$
$f'(x) = \frac{1}{3\sqrt[3]{x^2}+3\sqrt[3]{x^5}} - \frac{1}{2\sqrt{x}+2x^2}$

Exercise 9. (5 pts) Find a function $f : [1,\infty) \to \mathbb{R}$ such that $f(1) = 0$ and $f'(x) = 1 + \sin(x^2)$ for all $x > 1$.

Solution: The Fundamental Theorem of Calculus tells us that:
$\int_a^b f'(t)dt = f(b) - f(a)$
$\int_1^x f'(t)dt = f(x) - f(1)$
$\int_1^x 1 + \sin(t^2)dt = f(x)$
This gives us $f(x) = \int_1^x 1 + \sin(t^2)dt$ which satisfies the conditions. $\square$

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Exercise 10. (5 pts) By thinking the following sum as a Riemann sum, evaluate

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{k^2 + n^2}.$$

Solution: We would like to write this summation in the following form for a Riemann sum:
$\sum_{k=1}^{n} f(kx)(x)$

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Here x represents the length of the intervals. As we would like the length of the intervals to become arbitrarily small, we should choose x so that it goes to 0 as $n \to \infty$. We can then write our summation as follows:
$\sum_{k=1}^{n} \frac{n}{k^2+n^2} = \sum_{k=1}^{n} (\frac{n^2}{k^2+n^2})(\frac{1}{n})$
$\sum_{k=1}^{n} \frac{n}{k^2+n^2} = \sum_{k=1}^{n} (\frac{1}{(\frac{k}{n})^2+1})(\frac{1}{n})$
We have let $x = \frac{1}{x}$ and now have the function $f(x) = \frac{1}{x^2+1}$. We now evaluate:
$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{k^2+n^2} = \lim_{n\to\infty} \sum_{k=1}^{n} f(\frac{k}{n})\frac{1}{n}$
This represents the integral of f. As the first term of the summation is at $x = \frac{1}{n}$ and n approaches infinity, the lower bound of the integral is at 0. Similarly, the last term of the summation is at $x = \frac{n}{n} = 1$. Therefore the upper bound of the integral is at 1.
$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{k^2+n^2} = \int_0^1 f(x)dx$
$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{k^2+n^2} = \int_0^1 \frac{1}{x^2+1}dx$

Arctan(x) $\Big|_0^1$

$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{k^2+n^2} = (\text{Arctan}\,(x+1))_{x=0}^{x=1}$
$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{k^2+n^2} = \text{Arctan}\,(2) - \text{Arctan}\,(1)$