

good

58/70

Math 331: Homework 5

1. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function. Suppose \exists a positive constant function $M \ni |f(y) - f(x)| \leq M|y - x|$, for all x, y in \mathbb{R} .
Now we have to show that f is uniformly continuous on \mathbb{R} . Let $\varepsilon > 0$ and let $x, y \in \mathbb{R}$. Let $\delta > 0$ such that $|y - x| < \delta$. Now consider $|f(y) - f(x)| \leq M|y - x| < M\delta$. Choose $\delta = \frac{\varepsilon}{M}$ then $\delta > 0$

(515) $|f(y) - f(x)| \leq M|y - x| < M\delta = M \frac{\varepsilon}{M} = \varepsilon$

i.e. for $\varepsilon > 0 \exists \delta > 0 \ni |y - x| < \delta \Rightarrow |f(y) - f(x)| < \varepsilon$
for all x, y in \mathbb{R} . Hence f is uniformly continuous on \mathbb{R} . ■

Math 331: Homework 5

2. Assume that $f(x)$ is not identically 0. Then there are numbers a, c such that $a \geq 0, c > 0$ such that $f(a) = c$. Now we also know that $\lim_{x \rightarrow \infty} f(x) = 0$. Hence, there is a number $N > 0$ such that $f(x) < c$ for all $x \geq N$. Moreover we can always choose $N > a$. Thus we have $f(x) < c$ for $x \geq N$ and $f(x) = c$ for at least one value $a \in [0, N]$. Let M be the maximum value of $f(x)$ in $[0, N]$. Clearly this is guaranteed because $f(x)$ is continuous in the closed interval $[0, N]$. It is obvious that $M \geq f(a) = c$. And since $f(x) < c \leq M$ for all $x \geq N$. It follows that M is the maximum value of $f(y)$ on $[0, \infty)$. ■

515

Math 331 : Homework 5

3. Suppose that $f: [a,b] \rightarrow \mathbb{R}$ is a continuous function such that $f([a,b]) \subseteq [a,b]$. So we have

$$f(a) \geq a, f(b) \leq b$$

$$\Rightarrow f(a) - a \geq 0, f(b) - b \leq 0 \quad (1)$$

Now we consider $h(x) = f(x) - x$. We get

$$h(a) = f(a) - a$$

$$h(b) = f(b) - b$$

From (1), we get $h(a) \geq 0, h(b) \leq 0$

$$\Rightarrow 0 \in [h(b), h(a)] \quad (2)$$

By the IVT, since h is continuous and $0 \in [h(b), h(a)]$
there exist $c \in [a,b]$, such that

$$h(c) = f(c) - c = 0$$

So $f(c) = c$ for some $c \in [a,b]$ ■

4. Given $f: (a,b) \rightarrow \mathbb{R}$ is twice differentiable we know that f is continuous and differentiable and also f' is continuous and differentiable.

Now given $\exists c < d \in (a,b)$ such that $f'(c) = f'(d)$,
 $f'(x)$ is continuous and differentiable in (a,b)

$\Rightarrow f'(x)$ is continuous and differentiable in (c,d)

Hence we have,

(1) $f'(x)$ is continuous in (c,d)

(2) $f'(x)$ is differentiable in (c,d)

(3) $f'(c) = f'(d)$

Hence by Rolle's Theorem \exists an $x \in (c,d)$ such that $f''(x) = 0$. ■

Math 331 : Homework 5

5. a) We are given,

$$\begin{aligned}
 \lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0-h)}{2h} &= \lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0) + f(x_0) - f(x_0-h)}{2h} \\
 &= \frac{1}{2} \lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h} + \frac{1}{2} \lim_{h \rightarrow 0} \frac{f(x_0-h) - f(x_0)}{-h} \\
 &= \frac{1}{2} f'(x_0) + \frac{1}{2} f'(x_0) = f'(x_0)
 \end{aligned}$$

10/10

5/5

Therefore $\lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0-h)}{2h}$ exists and equals $f'(x_0)$. ■

b) Let $f(x) = |x|$. Then f is not differentiable at $x=0$. However,

$$\begin{aligned}
 \lim_{h \rightarrow 0} \frac{f(h) - f(-h)}{2h} &= \lim_{h \rightarrow 0} \frac{|h| - |-h|}{2h} = 0
 \end{aligned}$$

is convergent. ■

Math 331: Homework 5

7/10

- a) We know $\ln(x) = g(x)$ is differentiable on $(0, \infty)$, and hence, a constant multiple of $g(x)$ is also differentiable, say $h(x) = r g(x) = r \ln(x)$, in $(0, \infty)$
 $\Rightarrow h'(x) = \ln(x)$ is differentiable.

4/5

- Again, $k(x) = e^x$ is differentiable, so we will use the fact that composition of differentiable is differentiable, so we have
 $k \circ h(x)$ is differentiable in $(0, \infty)$
 $\Rightarrow k \circ h(x) = k(\ln(x)) = e^{\ln(x)} = x^r = f(x)$

So, $f(x) = x^r$ is also differentiable in $(0, \infty)$ and its derivative is,

$f'(x) = r x^{r-1} \rightarrow$ explain by computing ~~exactly~~ by computing ~~exactly~~ by
from the general formula. █ value of the derivative
with the chain rule.

b) $f(x) = \sqrt{x^2 + \sin x + \cos x}$

We can find $f'(x)$ using the chain rule and show that $f'(x)$ exists for all x

i.e. f is differentiable. By the chain rule,

$$f'(x) = \frac{d}{dx} (\sqrt{x^2 + \sin x + \cos x}) = \frac{1}{2\sqrt{x^2 + \sin x + \cos x}} \times (2x + \cos x - \sin x)$$

$$\Rightarrow f'(x) = \frac{2x + \cos x - \sin x}{2\sqrt{x^2 + \sin x + \cos x}}$$

Clearly, $2\sqrt{x^2 + \sin x + \cos x} \neq 0$, $\forall x \in \mathbb{R}$. So $f'(x)$ is defined for all $x \in \mathbb{R}$ and therefore f is differentiable on \mathbb{R} . █

You have to show first that f is differentiable.

Math. 331: Homework 5

3/5

7. Suppose that S is closed and $x \in \mathbb{R}/S$. Note that x is not a cluster point of S . Thus every neighborhood of x contains no point of S , & there is a neighborhood of x . Say $(x, s) \subset \mathbb{R}/S$. Hence \mathbb{R}/S is open.
- Conversely, suppose that \mathbb{R}/S is open. We shall show that S is closed. Suppose that x is a cluster point of S . Then we claim that $x \in S$. If $x \notin S$ then x is a cluster point of S . Then we claim x is in \mathbb{R}/S , thus \mathbb{R}/S contains a neighborhood of x . Therefore x is not a cluster point of S , and we have a contradiction. Hence S is closed.

thus is
one
neighbor-
hood
It's not
all nigh

I am confused
You suppose
that $x \notin S$.
So, since \mathbb{R}/S
is open, then
 $(x, s) \subset \mathbb{R}/S$.
 $S \cap (x, s) = \emptyset$

Math 331: Homework 5

8. Since x^3 is differentiable, because every polynomial is differentiable, by the chain, $f(x^3)$ is differentiable and,

$$(f(x^3))' = f'(x^3)(x^3)' = 3x^2 f'(x^3)$$

By the multiplication rule, g is differentiable and

$$\begin{aligned} g'(x) &= 2x f(x^3) + x^2 3x^2 f'(x^3) \\ &= 2x f(x^3) + 3x^4 f'(x^3) \end{aligned}$$

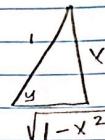
515

9. Let $y = \arcsin(x)$, then it follows that

$$\text{① } \sin(y) = x \text{ and } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

Notice from the triangle if

$$\begin{aligned} y &= \arcsin(x) \text{ the } \sin(y) = \frac{x}{1} = x \text{ &} \\ \cos(y) &= \frac{1}{\sqrt{1-x^2}} = \sqrt{1-x^2} \end{aligned}$$



515

Taking the derivative of ① with respect to x , we have

$$\cos(y) \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos(y)} = \sec(y) = \frac{1}{\sqrt{1-x^2}}$$

You didn't prove
that arcsin is
differentiable.

so the derivative $y = \arcsin(x) = \frac{1}{\sqrt{1-x^2}}$

Math 331 : Homework 5

10. a) Consider a function $f(x) = x^n$, $n \geq 1$. Since $f(x)$ is a polynomial, it is continuous and differentiable everywhere. In particular $f(x)$ is continuous on $[y, x]$ and differentiable on (y, x) .
 By the Mean Value Theorem there exists $c \in (y, x)$ such that

$$f'(c) = \frac{f(x) - f(y)}{x - y}$$

$$\Rightarrow n c^{n-1} = \frac{x^n - y^n}{x - y} \quad (1)$$

515

$$\Rightarrow y^{n-1} < c^{n-1} < x^{n-1}$$

$$\Rightarrow ny^{n-1} < nc^{n-1} < nx^{n-1}$$

$$\Rightarrow ny^{n-1} < \frac{x^n - y^n}{x - y} < nx^{n-1} \quad (\text{from 1})$$



$$\Rightarrow ny^{n-1}(x - y) < x^n - y^n < nx^{n-1}(x - y)$$

- b) If $x > 0$, apply the Mean Value Theorem to $f(x) = \sqrt{1+x}$ on the interval $[0, x]$. There exist $c \in [0, x]$ such that

$$\frac{\sqrt{1+x} - 1}{x - 0} = f(x) - f(0) = f'(c) = \frac{1}{2\sqrt{1+c}} < 1$$

The last inequality holds because $c > 0$. Multiplying by the positive number x and transposing the -1 gives

$$\sqrt{1+x} < 1 + \frac{1}{2} \quad \leftarrow \text{missing } x$$

for $x > 0$.