| MATH-331 Introduction to Real Analysis | KaiWei Tang |
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| Homework 03 | Fall 2021 |

Due date: October 11$^{\text{th}}$ 1:20pm  Total: /70.

| Exercise | 1 (5) | 2 (5) | 3 (5) | 4 (5) | 5 (10) | 6 (10) | 7 (5) | 8 (5) | 9 (5) | 10 (10) |
|---|---|---|---|---|---|---|---|---|---|---|
| Score | | | | | | | | | | |

Table 1: Scores for each exercises

**Instructions:** You must answer all the questions below and send your solution by email (to `parisepo@hawaii.edu`). If you decide to not use LaTeX to hand out your solutions, please be sure that after you scan your copy, it is clear and readable. Make sure that you attached a copy of the homework assignment to your homework.
If you choose to use LaTeX, you can use the template available on the course website.
No late homework will be accepted. No format other than PDF will be accepted. Name your file as indicated in the syllabus.

## 1
## Writing problems

For each of the following problems, you will be asked to write a clear and detailed proof. You will have the chance to rewrite your solution in your semester project after receiving feedback from me.

**Exercise 1.** (5 pts) Let $(a_n)_{n=1}^{\infty}$ be an increasing sequence and $(b_n)_{n=1}^{\infty}$ be a decreasing sequence. Let $(c_n)_{n=1}^{\infty}$ be the sequence defined by $c_n = b_n - a_n$. Show that if $\lim_{n\to\infty} c_n = 0$, then the sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ converges and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

**Solution:** before we begin, let us denote what we know thus far from the problem. We know that the sequence $(a_n)$ an increasing sequence. Therefore, $\exists a_{n+1}$ st. $a_n \leq a_{n+1}$. We also know that the sequence $(b_n)$ is decreasing. Therefore, $\exists b_{n+1}$ st. $b_{n+1} \leq b_n$. We also know that the sequence $c_n$ is defined as $c_n = b_n - a_n$ where $\lim_{n\to\infty} c_n = 0$. We can deduce from here since $c_n \to 0$, that means $b_n \to 0$ and $a_n \to 0$ as well for both $(a_n), (b_n) \in (c_n)$ and we know from our theorem in class that for any limit of the sub sequence must converge to the same limit as the parent sequence, thus $(a_n), (b_n)$ converges. We can also see this by using the same method from our midterm, let us take the limit for both sides,

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n$$

Since we know that $\lim_{n\to\infty} c_n = 0$, replace 0 for $\lim_{n\to\infty} c_n$,

$$0 = \lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n$$
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$$

Therefore, we see that if $\lim_{n\to\infty} c_n = 0$, then the sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ converges and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$. $\square$

**Exercise 2.** (5 pts) Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$, and suppose that $x_0$ is an accumulation point of $D$. Suppose that for each sequence $(x_n)_{n=1}^{\infty}$ converging to $x_0$ with $x_n \in D\backslash\{x_0\}$ for each $n \geq 1$, then the sequence $(f(x_n))_{n=1}^{\infty}$ is Cauchy. Show that $f$ has a limit at $x_0$.

[Hint: For two sequences $(x_n)$ and $(y_n)$ that satisfy the assumption, define the sequence $(z_n)$ to be $z_{2n} = x_n$ and $z_{2n-1} = y_n$. Show that $(f(z_n))$ converges and the sequence $(f(x_n))$ and $(f(y_n))$ converges to the same limit as $(f(z_n))$. Conclude by a theorem in the lecture notes.]

**Solution:** For this question, since we assume that the sequence $(x_n)$ is convergent, then $x_n \to x_0$. From the hint given to us, we can define a new sequence $z_n$ for $z_{2n} = x_n$ and $z_{2n-1} = y_n$ where $z_{2n}, z_{2n+1} \in (z_n)$. Since we assumed that $x_n \to x_0$ and $x_n = z_{2n}$ this means $x_n \in z_n$ and $x_n$ is a subsequence of $z_n$. From the theorem in class, we know that for all subsequence of any parents sequence, the limit of the subsequence must equal to the limit of the parent sequence. Then we can redefine a new function $f((z_n))$ for $z_n \to x_0$ then for any subsequence of $(z_n)$ must also go to $x_0$. We can see that this is also true for $z_{2n-1}$ for $y_n = z_{2n-1}$, meaning $y_n \in z_n$ thus $y_n \to x_0$ as well. Therefore we see that for any function of a subsequence, the limit of this function will always converge to the same limit as the function of the parent sequence. As such, we can redefine $L(x)$ as the limit of $(x_n)$, $L(y)$ be the limit of $(y_n)$, and $L(z)$ be the limit of $(z_n)$, for $L(x) = L(y) = L(z) = x_0$. By theorem in class, we know that all convergent sequences are Cauchy, therefore, our sequence $f(x)$ is Cauchy. $\square$

**Exercise 3.** (5 pts) Prove that if $f : D \subseteq \mathbb{R} \to \mathbb{R}$ has a limit at $x_0 \in \operatorname{acc} D$, then the limit is unique.

**Solution:** We can prove this claim similarly to a theorem we proved in class. The theorem states that if $f : D \subseteq \mathbb{R} \to \mathbb{R}$, and $f \to x_0$ where $x_0 \in acc(1)$, then $\exists\delta > 0$ and $\exists M > 0$ st if $x \in (x_0 - \delta, x_0 + \delta) \cap D$, then $|f(x)| \leq M$ for this is locally bounded. We can utilize this and prove by contradiction. Suppose in the function $f$, there exists two different limit at acc$(D)$ for $f(x) \to L$ and $f(x) \to J$. This means there must $\exists\delta_L > 0$ st. $|x - x_0| < \delta_L$ and $x \in D\backslash\{x_0\}$ for $|f(x) - L| < D$ and $\exists\delta_J > 0$ st. $|x - x_0| < \delta_J$ and $x \in D\backslash\{x_0\}$ for $|f(x) - J| < D$. Let us denote a $\delta$ for $\delta :=$min$\{\delta_L, \delta_J\}$. So, if $x \in (x_0 - \delta, x_0 + \delta) \cap D$, then we can rewrite as,

$$|L - J| \leq |L - f(x)| + |f(x) - J| < \epsilon$$

We can let $\epsilon = |L - J|$, thus we see a contradiction where $|L - J| \leq |L - f(x)| + |f(x) - L| < |L - J|$ which is false, therefore we see that our answer is unique. $\square$

**Exercise 4.** (5 pts) Suppose $f : D \subseteq \mathbb{R} \to \mathbb{R}$, $g : D \subseteq \mathbb{R} \to \mathbb{R}$ and $h : D \subseteq \mathbb{R} \to \mathbb{R}$ are three functions such that

$$f(x) \leq h(x) \leq g(x) \quad (\forall x \in D).$$

Suppose that $f$ and $g$ have limits at $x_0$ with $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x)$. Prove that $h$ has a limit at $x_0$ and

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} h(x) = \lim_{x\to x_0} g(x).$$

**Solution:** We know from the theorem given in class that if $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ and where $x_0 \in acc(D)$ st both $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ $\forall x \in D$, then $\lim_{x\to 0} f(x) \leq \lim_{x\to 0} g(x)$. From the statement we can assume that both $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ converges and both have the same limit. Let $x_n$ be a sequence st. $x_n \to x_0$ for $x_n \in D\{x_0\}$, thus $\lim_{x\to 0} f(x) = L$ and $\lim_{x\to 0} g(x) = L$. Since we see that the function $h(x)$ is in between $f(x)$ and $g(x)$, and both functions head towards $L$, we can rewrite as $L \leq h(x) \leq L$ where $h(x)$ is being bounded or squeezed between $L$ and $L$ which we know $h(x)$ cannot be both bigger and smaller than $L$, thus the only way it is true is for $h(x) = L$. So by the squeeze theorem, $\lim_{x\to 0} f(x) = L$ and $\lim_{x\to 0} g(x) = L$ and $f(x) \leq h(x) \leq g(x)$, therefore $\lim_{x\to 0} h(x) = L$ for $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = \lim_{x\to 0} h(x)$.$\square$

**Exercise 5.** (10 pts) Let $f : (0, \infty) \to \mathbb{R}$ be a function. We say that $f$ has a limit at $\infty$ if there exists a $L \in \mathbb{R}$ such that for any $\varepsilon > 0$, there is a real number $M > 0$ such that if $x > M$, then $|f(x) - L| < \varepsilon$.

**a)** Show that if $g : (0, \infty) \to \mathbb{R}$ is bounded and $\lim_{x\to\infty} f(x) = 0$, then $\lim_{x\to\infty} f(x)g(x) = 0$.

**b)** Let $a > 0$ and suppose that $f : (a, \infty) \to \mathbb{R}$ and define $g : (0, 1/a) \to \mathbb{R}$ by $g(x) = f(1/x)$. Show that $f$ has a limit at $\infty$ if and only if $g$ has a limit at 0.

**Solution:** For 5a, let us recall our class notes on algebra with limits. From our class notes, we can recall our theroem that states for any two functions $f, g$ and suppose the limit of both $f$ and $g$ exists. Then $\lim_{x\to x_0}(fg)(x)$ exists for $\lim_{x\to x_0}(fg)(x) = \lim_{x\to x_0} f(x) \cdot \lim_{x\to x_0} g(x)$ From our statement, we supposed that $g(x)$ is bounded and we know that $\lim_{x\to x_0} f(x) = 0$. Thus, by our theorem, the product of any limits of two function is the product of their limits. We can suppose that $\lim_{x\to x_0} g(x) = L$, thus applying the theorem, $\lim_{x\to x_0}(fg)(x) = \lim_{x\to x_0} f(x) \cdot \lim_{x\to x_0} g(x) = 0 \cdot L = 0$. Thus we see that our limits of the product of $f(x)$ and $g(x)$ is 0. $\square$

**Solution:** For 5b, since it is an iff statement, let us prove it in the forward direction.
Forward: If $f$ has a limit at $\infty$ then $g$ has a limit at 0

If we suppose that $f$ has a limit at $\infty$, then all condition are met for there $\exists L \in \mathbb{R}$ st. for any $\epsilon > 0$, there is a real numver $M > 0$ st. if $x > M$, then $|f(x) - L| < \epsilon$. We can use the example in lecture notes for $f : (0, 1) \to \mathbb{R}$ for $f = (1/x)$ to help us understand this problem. In the example given in class, we see that this problem has no limit at 0 since $1/0$ is undefined and unbounded. Following this logic, we see that since the function $f \in g$, for the function $f(1/x)$ is bounded and defined at for $f(1/\infty) = f(0)$. Meaning when $g \to 0$, $g(0) = f(0)$ for we know $f(0)$ is defined and bounded, thus $g(0)$ has a limit at 0 for $f(1/x)$ has a limit at $\infty$ and since $f \in g$, $g(x)$ is defined as $g \to 0$.

Backwards: If $g$ has a limit at 0 then $f$ has a limit at $\infty$.

Let us use how we proved the theorem from class notes to prove this. Suppose that $g$ has a limit at 0, this means $\exists \delta > 0$ and $\exists M > 0$ st. $x \in (x_0 - \delta, x_0 + \delta) \cap$D, then $|g(x)| \leq M$. Let us define $\epsilon = 1$, then $\exists \delta > 0$ for $|x - x_0| < \delta$ st. $x \in D \backslash \{x_0\}$ which implies that $|g(x) - L| < 1$ where $\lim_{x \to 0} g(x) = L$ for our $x_0 = 0$ since we are trying to prove that $g(x)$ is defined at 0. Applying triangle inequality we get $|g(x)| < 1 + |L|$. Let $M = 1 + |L|$, thus $|g(x)| < M$, proving that $g(x)$ is defined at 0, thus by our assumption, $f(x)$ is also defined at $\infty$. □

## 2

## Homework problems

Answer all the questions below. Make sure to show your work.

**Exercise 6.** (10pts) For each of the sequences below, determine its nature (converges or diverges)[1]:

**a)** $(a_n)$ where $a_n = \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n}$.

**b)** $(a_n)$ where $a_n = \frac{1+2+\cdots+n}{n^2}$.

**Solution:** For 6a, we know that from class that a sequence is convergent if it is bounded and monotoned. Thus we see for the above sequence, it is decreasing. Therefore, for the sequence $(a_n)_{n=1}^{\infty}$ $\exists a_{n+1}$ st. $a_{n+1} \leq a_n$ for,

$$
\begin{aligned}
a_{n+1} &\leq a_n \\
0 &\leq a_n - a_{n+1} \\
0 &\leq \left(\frac{1}{n} + \frac{1}{n+2} + \frac{1}{n+3} + ... + \frac{1}{2n}\right) - \left(\frac{1}{n+2} + \frac{1}{n+3} + ... + \frac{1}{2n+1}\right) \\
0 &\leq \frac{1}{n} - \frac{1}{2n+1} \\
0 &\leq \frac{2n+1-n}{n(2n+1)} \\
0 &\leq \frac{n+1}{n(2n+1)} \\
0 &\leq \frac{n}{n(2n+1)} + \frac{1}{n(2n+1)} \\
0 &\leq \frac{1}{2n+1} + \frac{1}{n(2n+1)}
\end{aligned}
$$

Which we see it is true for all $n > 0$, and since our sequence starts at 1 it is true and therefore decreasing. We can also see from here that it is bounded below by $\frac{2}{3}$ since when $n = 1$ our sequence starts at $\frac{2}{3}$, therefore it is bounded below by $\frac{2}{3}$, and it converges. □

[1]You don't need to compute the limit.

**Solution:** For 6b, we see that if we were to multiply top and bottom by the greatest degree, we would have something very similar to 6a, where each of the preceeding term in the numerator will become $\frac{1}{n^2} + \frac{2}{n^2} + ... + \frac{1}{n}$. Again we know that if our sequence is bounded and monotoned then it will converge. We can see here that it is actually increasing for as $\frac{1}{n^2} < \frac{1}{n}$. Thus for this sequence $(a_n)^{n-1}$, $\exists a_{n+1}$ st. $a_n \leq a_{n+1}$ for,

$$a_n \leq a_{n+1}$$
$$0 \leq \left(\frac{1}{n^2+1} + \frac{2}{n^2+1} + \frac{3}{n^2+1} + ... + \frac{1}{n^2+1}\right) - \left(\frac{1}{n^2} + \frac{2}{n^2} + ... + \frac{1}{n}\right)$$
$$0 \leq \left(\frac{1}{n^2+1} - \frac{1}{n^2}\right) + \left(\frac{2}{n^2+1} - \frac{2}{n^2}\right) + ... \left(\frac{1}{n+1} - \frac{1}{n}\right)$$
$$0 \leq \left(\frac{n^2 - n^2 + 1}{n^2(n^2+1)}\right) + \left(\frac{2n^2 - 2n^2 + 1}{n^2(n^2+1)}\right) + ... \left(\frac{n - n + 1}{n(n+1)}\right)$$
$$0 \leq \left(\frac{1}{n^2(n^2+1)}\right) + \left(\frac{2}{n^2(n^2+1)}\right) + ... \left(\frac{1}{n(n+1)}\right)$$

We see that this is definitely true, thus our sequence is increasing. We see that our sequence is actually bounded velow by 0 since each index is closely related to the sequence $\frac{1}{n}$, thus each index gets closer and close to zero as we add on. Therefore our lower bound is zero, thus it converges.□

**Exercise 7.** (5 pts) Define $g : (0, 1) \to \mathbb{R}$ by $f(x) = \frac{\sqrt{1+x}-1}{x}$. Prove that $g$ has a limit at 0 and find it.

**Solution:** Let us find the limit at 0 and prove that it is at 0. We see that for us to find the $\lim_{x\to\infty} f(x)$, we can do some algebraic manipulation.

$$= \frac{\sqrt{1+x} - 1}{x} \cdot \frac{\frac{1}{x}}{\frac{1}{x}}$$
$$= \frac{\sqrt{1+x}}{x} - \frac{1}{x}$$

From here, we know from our lecture notes in class that if $x \to \infty$, then for any number divided by infinity, it will equal zero. Therefore we we actually simplified it to 0+0=0. Therefore, our limit is 0. □

**Exercise 8.** (5 pts) Suppose that $f : (0, 1) \to \mathbb{R}$ has a limit at $x_0 = 1$ and $\lim_{x\to 1} f(x) = 1$. Compute the value of the limit

$$\lim_{x\to 1} \frac{f(x)(1 - f(x)^2)}{1 - f(x)}.$$

**Solution:** We can do this question similarly to how we did the example question of $\frac{\sqrt{4+x}-2}{x}$ in class. Since we know that $\lim_{x\to\infty} f(x) = 1$, let us break the bigger limit apart to see the full picture. Let us multiply top and bottom by the conjugate, $(1 + f(x))$,

$$
\begin{aligned}
&=\frac{f(x)(1-f(x)^2)}{1-f(x)} \cdot \frac{(1+f(x))}{(1+f(x))}\\
&=\frac{f(x)(1-f(x)^2)(1+f(x))}{1-f(x)^2}\\
&=f(x)(1+f(x))\\
&=1(1+1)\\
&=1(2)\\
&=2
\end{aligned}
$$

We see that since we know the $\lim_{x\to\infty} f(x) = 1$, we can assume that for $f(x)(1+f(x))$, $f(x) = 1$, thus we have $1(1 + 1) = 2$. Therefore, our limit is 2. $\square$

**Exercise 9.** (5 pts) Prove that if $f : D \to \mathbb{R}$ has a limit at $x_0$, then $|f|(x) := |f(x)|$ has a limit at $x_0$.

**Solution:** From our notes in lecture, we can assume that if $\exists \lim_{x_n\to x_0} f(x) = L$, then we know that $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$, st. $n \geq N$. Therefore, we know that $|f(x) - L| < \epsilon$, then by triangle inequality, we know that $||f(x)| - |L|| \leq |f(x) - L| < \epsilon$ where $||f(x)| - |L|| < \epsilon$. We know that this is true since by theorem in class, we know that $\lim_{x_n\to x_0} f(x) = L$ for $\lim_{x_n\to x_0} |f(x)| = |L|$, therefore, we see it is true. $\square$

**Exercise 10.** (10 pts) Using the link between sequences and limits of functions, show the following.

**a)** If $f(x) = x^n$ $(n \geq 0)$, then $\lim_{x\to x_0} f(x) = x_0^n$ for any $x_0 \in \mathbb{R}$.

**b)** If $x_0 \in [0, \infty)$, then $\lim_{x\to x_0} \sqrt{x} = \sqrt{x_0}$.

**Solution:** For 10a, we know that from limit arithmetic, that if we were to have $(f - g)(x) = f(x) - g(x)$. Thus, if we assume that $f(x) = x^n$, then $f(x_0) = x_0^n$. We use limit arithmetic and deduce $f(x) - f(x_0) = x^n - x_0^n$ for we see it is the definition of a limit of $|x^n - x_0^n| < \delta$ where we can let $\delta = \epsilon$ for $|x^n - x_0^n| < \epsilon$. $\square$

**Solution:** We can prove this statement from a hint given to us in the lecture notes. From our lecture notes, we see that for any $\lim_{x\to\infty} f(x) = L$, if $\lim_{x\to\infty} \sqrt{f(x)} = \sqrt{L}$. Thus from our question if our $x_0 \in [0, \infty)$, for $x_0 = L$ then let $f(x) = x$, for $\lim_{x\to\infty} \sqrt{x} = \sqrt{x_0}$. $\square$

## 3
## Bonus

**Exercise 11.** Assume that $f : \mathbb{R} \to \mathbb{R}$ such that $f(x + y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$.

**a)** Show that $f$ has a limit at every point of $\mathbb{R}$.

**b)** Show that either $\lim_{x\to 0} f(x) = 1$ or $f(x) = 0$ for any $x \in \mathbb{R}$.