MATH-331 Introduction to Real Analysis — BENDER, LEE'AN
Homework 05 — Fall 2021

Due date: November 8^{th} 1:20pm — Total: /70.

Exercise	1 (5)	2 (5)	3 (5)	4 (5)	5 (10)	6 (10)	7 (5)	8 (5)	9 (5)	10 (10)
Score										

Table 1: Scores for each exercises

Instructions: You must answer all the questions below and send your solution by email (to `parisepo@hawaii.edu`). If you decide to not use LaTeX to hand out your solutions, please be sure that after you scan your copy, it is clear and readable. Make sure that you attached a copy of the homework assignment to your homework.
If you choose to use LaTeX, you can use the template available on the course website.
No late homework will be accepted. No format other than PDF will be accepted. Name your file as indicated in the syllabus.

1 Writing problems

For each of the following problems, you will be asked to write a clear and detailed proof. You will have the chance to rewrite your solution in your semester project after receiving feedback from me.

Exercise 1. (5 pts) Let $f : \mathbb{R} \to \mathbb{R}$ and suppose that there exists a positive constant M such that $|f(y) - f(x)| \leq M|y - x|$ for all $x, y \in \mathbb{R}$. Prove that f is uniformly continuous on $\mathbb{R}$.

Solution: The problem posits that if $\exists M \in \mathbb{N}$ such that $\forall x, y \in \mathbb{R}$, $|f(y) - f(x)| < M|y - x|$ then the following is true:
$\forall \epsilon > 0\ \exists \delta > 0$ such that $\forall x, y \in \mathbb{R}$ and $|y - x| < \delta \Rightarrow |f(y) - f(x)| < \epsilon$
from the definition of continuity.
Let $\epsilon > 0$ and $x, y \in \mathbb{R}$ s.t. $|y - x| < \delta$ if $|f(y) - f(x)| < \epsilon$.
Given $|f(y) - f(x)| < M|y - x|$, we can combine this with the definition of continuity to obtain $|f(y) - f(x)| < M|y - x| < M\delta$
Substituting x and y into $f(x)$ we obtain $|f(y) - f(x)| < M\delta = \epsilon$.
Now, we set $\delta = \frac{\epsilon}{M}$. So we have
$\forall x, y \in \mathbb{R}$, $|y - x| < \frac{\epsilon}{M} \Rightarrow |f(y) - f(x)| < \epsilon$.
Since ϵ was arbitrary, f is uniformly continuous on $\mathbb{R}$. $\square$

Exercise 2. (5 pts) Let $f : [0, \infty) \to \mathbb{R}$ be nonnegative and continuous such that $\lim_{x\to\infty} f(x) = 0$. Prove that f attains its maximum at some point in $[0, \infty)$.

Solution: By definition, an absolute maximum occurs when $\forall x \in D\ \exists c \in D$ s.t. $f(x) \leq f(c)$ and $f'(c) = 0$, except where $f(0)$ is the absolute max. We want to show that the function is differentiable $\forall c \in D$, which by definition means that $\lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$ exists. Since f is continuous on $D := [0, \infty)$, by [a theorem], the limit exists $\forall x \in D$ and is therefore differentiable $\forall x \in D$.
Now we want to show that $\forall x \in D$, $\exists c \in D$ s.t. $f(x) \leq f(c)$ and $f'(c) = 0$.
There are 2 possible cases that we need to prove: the case in which $\exists a, b \in D$ where $f(a) = f(b)$ while $a < b$ and the case in which this is not true.
By Rolle's Thm. If you have a function that is continuous and differentiable on (a, b), then $\exists c \in (a, b)$ such that $f'(c) = 0$. By the Axiom of Completeness, any set $(f(a), f(b))$ must contain a supremum, and if there are multiple intervals where $f'(c) = 0$, then one of them has to be the largest, therefore in the case where $f(a) = f(b)$, f must attain its maximum at some point on $[0, \infty)$.
In the case where $f(a) \neq f(b)$, f must be monotone. (If not monotone then there must $\exists a, b \in D$ where $f(a) = f(b)$). If f is monotone increasing, then $f(a) < f(b)\ \forall a < b$. We are given that f is nonnegative, meaning that $f(a), f(b) > 0$. So now we have $f(a) < f(b)$ and $0 < f(b)$. We also have $0 < f(a)$, so $0 < f(a) < f(b)$. We also have $\lim_{x\to\infty} f(x) = 0$ meaning that $f(b) \to 0$. That makes the inequality $0 < f(a) < 0$, which is a contradiction. Therefore, f must be monotone decreasing on D. This means that $\forall a < b$, $f(a) > f(b)$. Since the lower bound of f is 0, then $f(0) > f(x)\ \forall x \in D$. This means that f attains its maximum at $f(0)$. $\square$

Exercise 3. Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function such that $f([a, b]) \subseteq [a, b]$. Prove that there is a $c \in [a, b]$ such that $f(c) = c$. [This one of the many fixed point Theorem.]

Solution: return $\square$

Exercise 4. (5 pts) Suppose that $f : (a, b) \to \mathbb{R}$ is twice differentiable on (a, b) and there are two points $c < d$ in (a, b) such that $f'(c) = f'(d)$. Show that there is a point $x \in (c, d)$ such that $f''(x) = 0$.

Solution: We want to show that $\exists x \in (c, d)$ st. $f''(x) = 0$.
Let $g(x) = f'(x)$ where $g := [c, d] \to \mathbb{R}$. Then $g(c) = g(d)$. By Rolle's Theorem, if g is continuous and differentiable on (c,d), then $\exists \kappa \in (c, d)$ s.t. $g'(\kappa) = 0$. Since $g(x) = f'(x)$, then $g'(x) = f''(x)$. Additionally, $\kappa \in (c, d) \subseteq [a, b]$, so $\exists \kappa \in (c, d)$ such that $f''(\kappa) = 0$. (Here I used κ instead of x in the question.) $\square$

Exercise 5. (10 pts) Suppose that $f : (a, b) \to \mathbb{R}$ is differentiable at $x_0 \in (a, b)$.

a) Prove that

$$\lim_{h\to 0} \frac{f(x_0 + h) - f(x_0 - h)}{2h} \tag{$\star$}$$

exists and equals $f'(x_0)$.

b) Find a continuous function $f : \mathbb{R} \to \mathbb{R}$ and a point $x_0 \in \mathbb{R}$ such that f is not differentiable at x_0, but the limit $(\star)$ exists.

Solution: :

a) We are trying to show that

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0 - h)}{2h} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \qquad ()$$

$\frac{f(x_0)-f(x_0)}{0} = \frac{f(x_0)-f(x_0)}{0}$
Well indeed those look the same to me.

b) $f(x) = |x|$ $\square$

Homework problems

Answer all the questions below. Make sure to show your work.

Exercise 6. (10pts)

a) Suppose $r > 0$. Prove that $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = x^r$ is differentiable on $(0, \infty)$. [Hint: take for granted that e^x and $\ln x$ are differentiable with $(e^x)' = e^x$ and $(\ln x)' = 1/x$. Rewrite then x^r in terms of a composition of these two differentiable functions.]

b) Define $f(x) = \sqrt{x^2 + \sin x + \cos x}$ where $x \in [0, \pi/2]$. Show that f is a differentiable function.

Solution: **a)** prove that x^r is differentiable on $(0, \infty)$
$x^r = hog$
$h(x) = e^x$ and $g(x) = \ln(r)$ and $k(x) = y^x$

b) We define $h(x) = \sqrt{x}$ and $g(x) = x^2 + \sin x + \cos x$. Suppose $f(x) = h \circ g$. We know that $h(x)$ is differentiable because $h(x) = x^{1/2}$ and any polynomial is a differentiable function. We also know that $g(x)$ is differentiable because x^2 is differentiable, and $\sin x$ and $\cos x$ are differentiable. Therefore, by addition rules, $g(x)$ is differentiable. Chain rule states that if 2 functions are differentiable on an interval, then their composition is differentiable. Since we defined $f(x) = h \circ g$, and we showed that h and g are differentiable, then f is differentiable. $\square$

Exercise 7. (5 pts) Show that $S \subseteq \mathbb{R}$ is closed if and only if $\mathbb{R} \backslash S$ is open.

Solution: Assume that $\mathbb{R}\backslash S$ is an open set. By the definition of an open set, for each $x \in \mathbb{R}\backslash S$, then there is a $\delta > 0$ such that $(x-\delta, x+\delta) \subseteq \mathbb{R}\backslash S$. Suppose toward a contradiction that S is an open set. Then, by definition, for each $x \in S$, $\exists\delta > 0$ such that $(x-\delta, x+\delta) \subseteq S$. Take a point t such that $t := \sup S$. By the definition of a supremum, Since t is in S, an open set, then $\exists\delta > 0$ such that $(t-\delta, t+\delta) \subseteq S$. However, since t is the supremum of S then $t+\delta$ and $t-\delta$ will be in $\mathbb{R}\backslash S$. But, that means that by the definition, $\mathbb{R}\backslash S$ contains elements that are in S which is contradictory to its nature. Therefore, $S \subseteq \mathbb{R}$ is closed iff $\mathbb{R}\backslash S$ is open. $\square$

Exercise 8. (5 pts) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function and define $g(x) = x^2 f(x^3)$. Show that g is differentiable and compute its derivative.

Solution: We are given that f is a differentiable function and that $g(x) = x^2 f(x^3)$. We know that x^2 is a polynomial, which is differentiable, and f is differentiable, so by multiplication rules, $g(x)$ is differentiable. Chain rule allows us to calculate $g(f(x^3))$.
$g'(x) = 2xf(x^3) + x^2 f'(x^3)3x^2$
$= 2xf(x^3) + 3x^4 f'(x^3)$ $\square$

Exercise 9. (5 pts) Prove that $f(x) = \arcsin x$ is differentiable on its domain and find a formula for the derivative of f (justify all your steps!).

Solution: $\sin^{-1}(x) = y$
$x = \sin y$
$x = \sqrt{1-\cos^2 y}$
derivable via composition $h(x) = \sqrt{x}$ and $g(y) = 1-x^2$ and $k(x) = \cos x$; define $f(x) = h \circ g \circ k$. All are individually derivable; formula for derivative is $f'(x) = h'(g(k(x)))g'(k(x))k'(x)$ $\square$

Exercise 10. (10 pts) Use the Mean-Value Theorem to show the following inequalities.

a) $ny^{n-1}(x-y) \leq x^n - y^n \leq nx^{n-1}(x-y)$ if $n \in \mathbb{N}$ and $0 \leq y \leq x$.

b) $\sqrt{1+x} < 1 + \frac{1}{2}x$ for $x > 0$.

Solution: **a)** $ny^{n-1}(x-y) \leq x^n - y^n \leq nx^{n-1}(x-y)$
$ny^{n-1} \leq \frac{x^n-y^n}{x-y} \leq nx^{n-1}$
Because $0 < y < x$...

b) MIT method $\square$