MATH-331 Introduction to Real Analysis — KaiWei Tang

Homework 05 — Fall 2021

Due date: November 8^{th} 1:20pm — Total: 41/70.

Exercise	1 (5)	2 (5)	3 (5)	4 (5)	5 (10)	6 (10)	7 (5)	8 (5)	9 (5)	10 (10)
Score	5	4	3	4	5	6	0	5	5	4

Table 1: Scores for each exercises

Instructions: You must answer all the questions below and send your solution by email (to `parisepo@hawaii.edu`). If you decide to not use LaTeX to hand out your solutions, please be sure that after you scan your copy, it is clear and readable. Make sure that you attached a copy of the homework assignment to your homework.
If you choose to use LaTeX, you can use the template available on the course website.
No late homework will be accepted. No format other than PDF will be accepted. Name your file as indicated in the syllabus.

1 — Writing problems

For each of the following problems, you will be asked to write a clear and detailed proof. You will have the chance to rewrite your solution in your semester project after receiving feedback from me.

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Exercise 1. (5 pts) Let $f : \mathbb{R} \to \mathbb{R}$ and suppose that there exists a positive constant M such that $|f(y) - f(x)| \leq M|y - x|$ for all $x, y \in \mathbb{R}$. Prove that f is uniformly continuous on $\mathbb{R}$.

Solution: If f is uniformly continuous then $\forall \epsilon > 0$, $\exists \delta > 0$ st. $|y - x| < \delta$ then $|f(y) - f(x)| < \epsilon$. Let us denote δ as $\delta = \frac{\epsilon}{M}$. Thus, we have $|y - x| < \frac{\epsilon}{M}$ for $|y - x|M < \epsilon$. From here, observe from the statement that $|f(y) - f(x)| \leq |y - x|M < M\delta$ where we know $\delta = \frac{\epsilon}{M}$, thus, $|f(y) - f(x)| \leq |y - x|M < M\frac{\epsilon}{M}$, which we see becomes $|f(y) - f(x)| \leq |y - x|M < \epsilon$. Therefore, our function f is continuous. □

Exercise 2. (5 pts) Let $f : [0, \infty) \to \mathbb{R}$ be nonnegative and continuous such that $\lim_{x\to\infty} f(x) = 0$. Prove that f attains its maximum at some point in $[0, \infty)$.

continuity is not the reason...

Solution: By the statement, we already know that $f(x)$ is continuous. Thus, we know that $\forall \epsilon > 0$, $\exists M > 0$ st. $|f(x) - 0| < \epsilon$, $\forall x > M$. From here, let us denote ϵ as some fixed number, then we know that $\forall x > M$, $f(x) < \epsilon$. Therefore, we know that for all x after M is bigger than M is true since $f(x) < \epsilon$ for all $x > M$ and we know our function f is defined at ∞. Thus from here, we check from $[0, M]$. From here, apply the extreme value theorem. Since we know $[0, M]$ is bounded and continuous, then by the extreme value theorem there must exists an max and min, therefore, let C=max$\{\epsilon, f(c)\}$ where $f(c) = sup\{f(x) : x \in [0, M]\}$ and D=min$\{\epsilon, f(c)\}$ where $f(d)$=sup$\{f(x) : x \in [0, m]\}$. Thus, we know that $[0, M]$ is bounded and continuous, and we know that from $[M, \infty]$ is also bounded, therefore f is bounded in $[0, \infty)$. 4/5 (must show that the bound is attained).

Exercise 3. Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function such that $f([a, b]) \subseteq [a, b]$. Prove that there is a $c \in [a, b]$ such that $f(c) = c$. [This one of the many fixed point Theorem.]

You construct g.

Solution: From the hint from email, suppose that a, b are arbitrary and there exists a function g for $g(x) = f(x) - x$. Since we know that f is a continuous function of the interval $[a, b]$, we know that $f(a)$ is defined and $f(b)$ is defined. Therefore, by IVT, we know there must also exists a $C \in (a, b)$ for $fc) = L$ where $c \neq a, b$. From here, we see that if we were to plug into our equation, then $g(c) = f(c) - c$. Thus by IVT, we know that there must exists a $c \in (a, b)$ st. $g(c) = 0$ for $0 = f(c) - c$ st. $f(c) = c$, thus proving our statement that there does exists a $c \in (a, b)$ st. $f(c) = c$. $\square$

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You must show that 0 is between f(a) & f(b).

Exercise 4. (5 pts) Suppose that $f : (a, b) \to \mathbb{R}$ is twice differentiable on (a, b) and there are two points $c < d$ in (a, b) such that $f'(c) = f'(d)$. Show that there is a point $x \in (c, d)$ such that $f''(x) = 0$.

Solution: Since we know that f is twice differentiable, we know that we can take the limit twice as $\lim_{h\to 0} \frac{f(x-h)-f(x)}{h}$ and then another $\lim_{h\to 0} \frac{f'(x+h)-f'(x)}{h}$. This means that by applying Rolle's theorem, let us denote the first derivative of the function $f'(x) = g(x)$. We know that g is differentiable since g is the first derivative of f and f is twice differentiable. Then we see that $\exists c, d \in (a, b)$ st. $g(c) = g(d)$, then by Rolles theorem, there must also $\exists x \in (c, d)$ st. $g'(c) = 0$. From here, observe that by our assumption of g, we see that there exists $x \in (c, d)$ for $g'(x) = f''(x) = 0$, and since we know that $x \in (c, d) \in (a, b)$, we know that $x \in (a, b)$ for $x \in D$ of f. $\square$

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You must show that g is also continuous on [c,d] to use Rolle's theorem

⊆ (subset).

Exercise 5. (10 pts) Suppose that $f : (a, b) \to \mathbb{R}$ is differentiable at $x_0 \in (a, b)$.

a) Prove that

$$\lim_{h\to 0} \frac{f(x_0 + h) - f(x_0 - h)}{2h} \qquad (\star)$$

exists and equals $f'(x_0)$.

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b) Find a continuous function $f : \mathbb{R} \to \mathbb{R}$ and a point $x_0 \in \mathbb{R}$ such that f is not differentiable at x_0, but the limit $(\star)$ exists.

Solution: By statement, we already know that f is differentiable, now let us check of it is differentiable at some point $x_0 \in D$. We know that he limit above must equal to the limit definition of the derivative, thus apply hint from email and add $f(x_0) - f(x_0)$ to our numerator.

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0-h) + f(x_0) - f(x_0)}{2h}$$
$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0) + f(x_0) - f(x_0-h)}{2h}$$
$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{2h} + \lim_{h\to 0} \frac{f(x_0) - f(x_0-h)}{2h} \quad ✓$$

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We see that both the LHS and RHS are the limit definition of a derivative, thus, we can multiple a 2 in both limits to cancel out the 2 on the bottom and we will obtain,

In fact, you get $\lim_{h\to 0} \frac{f(x_0+h)-f(x_0-h)}{2h} = f'(x_0)$.

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} + \lim_{h\to 0} \frac{f(x_0) - f(x_0-h)}{h}$$

You must change variable: $\to$ $k := -h$

So $\lim_{h\to 0} \frac{f(x_0)-f(x_0-h)}{h} = \lim_{k\to 0} \frac{f(x_0)-f(x_0+k)}{-k}$

for our function f was just the sum of two derivatives of $f = f'(x) + f'(x)$.

b) By class notes, one example of such function is $f(x) = |x|$ for there exists $x_0 \in D$ of f st. $f'(x_0)$ is undefined at 0, but it doesn't mean the limit does not exists as we take the limit from LHS and RHS as we approach 0. $\to$ You must verify this by computing exactly the $\lim_{h\to 0} \frac{|x_0+h| - |x_0-h|}{2h}$. □

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2 HOMEWORK PROBLEMS

Answer all the questions below. Make sure to show your work.

Exercise 6. (10pts)

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a) Suppose $r > 0$. Prove that $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = x^r$ is differentiable on $(0, \infty)$ and compute its derivative. [Hint: take for granted that e^x and $\ln x$ are differentiable with $(e^x)' = e^x$ and $(\ln x)' = 1/x$. Rewrite then x^r in terms of a composition of these two differentiable functions.]

b) Define $f(x) = \sqrt{x^2 + \sin x + \cos x}$ where $x \in [0, \pi/2]$. Show that f is a differentiable function.

Solution: a) We know that from the statement above that $r > 0$, thus for our function $f : D \to \mathbb{R}$ to be defined at $f(x) = x^r$. Suppose f is differentiable, then by the definition of differetiable, we let x_0 be the acc(D) for $x_0 \in D$ and $f : D \to \mathbb{R}$, then f is differentiable at x_0 iff,

$$\exists \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

Use the definition of differentiable and we will attempt our proof.

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$$\lim_{h\to 0}\frac{(x_0+h)^r-(x_0)^r}{h}$$
$$\lim_{h\to 0}\frac{(x_0)^r+rhx_0^{r-1}+...+h^r-(x_0)^r}{h}$$
$$\lim_{h\to 0}\frac{rhx_0^{r-1}+...+h^r}{h}$$
$$\lim_{h\to 0}\frac{h(rx_0^{r-1}+...+h^{r-1})}{h}$$
$$\lim_{h\to 0}rx_0^{r-1}+...+h^{r-1}$$

This is true if r is a positive integer. When r is irrational, you can't use the Binomial Theorem.

We see from here, all the terms that comes after rx_0^{r-1} has an h coefficients. Observe as we take the limit and apply the sum rule, all limits with the h coefficients will becomes zero for $h\to 0$, thus the only remaining limit is $\lim_{h\to 0} rx_0^{r-1}$. As such, our limit does exists $\forall x_0\in D$, therefore our function is differentiable. We also see that our function f is defined on the interval $(0,\infty)$ since x^r is a monomial and we know from class lecture that all monomials are continuous in their domain. Thus, f is defined in D.

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b) Let use the definition of differentiable once more. For this problem, we can use composition of functions. We know that our $f(x)=\sqrt{x^2+sin(x)+cos(x)}$. Define a $h(x)=\sqrt{x}$ and $g(x)=x^2+sin(x)+cos(x)$. Use composition of functions and obtain $h\circ g$ for $f(x)=h\circ g$. We know by the definition of chain rule that if g is differentiable at x_0 and h is differentiable at $g(x_0)$, then $h\circ g$ is differentiable with $(h\circ g)'(x_0)=g'[(f(x_0)]\cdot f'(x_0)$.

From here, let us check if $g(x)=x^2+sin(x)+cos(x)$ is differentiable at x_0. We know that for $g(x_0)$, x_0^2 is differentiable by our previous problem 6a. Observe that for 6a, we used x^r, and in 6b, we have x^2 and we've proved that x^r is defined on the interval $(0,\infty)$, for our $r=2$ is this case and $2\in(0,\infty)$, therefore x^2 is differentiable. For both $sin(x)$ and $cos(x)$, we've proved $sin(x)$ is differentiable in class, thus by similar methods, $cos(x)$ is also differentiable. Therefore, both $sin(x_0)$ and $cos(x_0)$ are differentiable at x_0. Thus, we see that for $h(g(x_0))$, obeserve $h(g(x_0))=\sqrt{g(x_0)}$ can be rewritten as $h(g(x_0))=(g(x_0))^{\frac{1}{2}}$ and we see this is also similar to our 6a problem for our r in this case is $\frac{1}{2}$ for $\frac{1}{2}\in(0,\infty)$. Thus $h(g(x_0))$ is differentiable at some x_0 for $f'(x)$ exists st. $f'(x)=h'(g(x_0))\cdot g'(x_0)=\frac{2x+cos(x)-sin(x)}{2\sqrt{x^2-sin(x)+cos(x)}}$. □

0/5

Exercise 7. (5 pts) Show that $S\subseteq\mathbb{R}$ is closed if and only if $\mathbb{R}\backslash S$ is open.

What does that mean? How did you infer this?

Solution: a) Foward: Suppose that $\exists S\in\mathbb{R}$ st. the set S is closed. Then that means there must $\exists c\in S$ st. $\forall acc(c)\in S$. This basically means there exists a limit in the closed set S for limit $L=acc(c)$. That means there must also $\exists c_n\in\mathbb{R}\backslash S$ st. $c_n\notin S$ for c_n is not an accumulation point of S. This means there exists some $\delta>0$ st. $(c_n-\delta,c_n+\delta)$ around set S where the interval contains no accumulation points of S. Thus, observe that the interval open is the definition of an open set. Thus, if $S\subset\mathbb{R}$ is closed, then $\mathbb{R}\backslash S$ is open.

not clear...

b) Backwards: Suppose that $\mathbb{R}\backslash S$ is open, that means there exists a $c_n\in\mathbb{R}\backslash S$ for there $\exists\delta$ st. $(c_n-\delta,c_n+\delta)$. What this shows is that c_n is not an accumulation point in the set S since

Where does this set belong?

not clear

The definition of open sets and closed sets are not used properly. Go back in your notes to have a look at these definitions.

we assumed that $\mathbb{R}\backslash S$ is open. Then, there must exists another point $c \in S$ st. the closed set S contains an $acc(c) \in S$ for that is the definition of a closed set. Thus if $\mathbb{R}\backslash S$ is open, then S is closed. $\square$

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Exercise 8. (5 pts) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function and define $g(x) = x^2 f(x^3)$. Show that g is differentiable and compute its derivative.

Solution: From statement above, we know that f is differentiable, and we also know [and the chain rule.] that x^2 is differentiable as well by 6a. Thus, we can just compute thederivative as a product rule for $f'(x) = 2xf(x^3) + f'(x^3)3x^2(x^2) = 2xf(x^3) + f'(x^3)3x^4$. $\square$

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Exercise 9. (5 pts) Prove that $f(x) = \arcsin x$ is differentiable on its domain and find a formula for the derivative of f (justify all your steps!).

Solution: From statement, we know that arcsine is actually the inverse of the sine function. Thus, we know the domain of arcsine is actually the range of sine. From here denote two equations, let $g(x) = sin(x)$ be the inverse of f and $f(x) = arcsin(x)$. Then, observe that we would have two functions $g : I \to J$ and $f : J \to I$. For if f is differentiable then there $\exists x_0 \in D$ for $D = [-1, 1]$, then $f'(x_0) = (g)'(x_0) = \frac{1}{g'(f(x_0))}$. Thus let us check. Since $g(x) = sin(x)$ as g is the inverse of f, we see that $g'(x_0) = cos(x_0)$. Thus, $g'(f(x_0)) = cos(arcsin(x_0))$. By trig id, we see that $cos(arcsin(x_0)) = \sqrt{1 - x_0^2}$, thus we see that $f'(x) = \frac{1}{\sqrt{1-x_0^2}}$. From here, observe that if $x_0 \in [-1, 1]$ then $f'(x) > 0$ for our denominator $\frac{1}{\sqrt{1-x_0^2}}$ is small, thus the smaller the denominator the bigger the answer. Therefore our function is strictly increasing on $[-1, 1]$. Thus we see that our function is differentiable and defined on it's domain of [-1,1] $\square$

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Exercise 10. (10 pts) Use the Mean-Value Theorem to show the following inequalities.

a) $ny^{n-1}(x - y) \leq x^n - y^n \leq nx^{n-1}(x - y)$ if $n \in \mathbb{N}$ and $0 \leq y \leq x$.

b) $\sqrt{1 + x} < 1 + \frac{1}{2}x$ for $x > 0$.

Solution: a) Suppose that $n \in \mathbb{N}$ and $0 \leq y \leq x$. Let us define two functions $f(x) = x^n$ and $f(y) = y^n$ for their derivatives are $f'(x) = nx^{n-1}$ and $f'(y) = ny^{n-1}$. Observe when we manipulate algebraically.

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(*) what you get from the MVP is: $\exists c \in (a,b)$ s.t. $f'(c) = \frac{f(b)-f(a)}{b-a}$. You put $a = x$ & $b = y$.

$$ny^{n-1}(x - y) \leq x^n - y^n \leq nx^{n-1}(x - y)$$

$$ny^{n-1} \leq \frac{x^n - y^n}{x - y} \leq nx^{n-1}$$

$$f'(y) \leq \frac{f(x) - f(y)}{x - y} \leq f'(x)$$

(*) this is not what you get in the MVP!

We see that this is the definition of MVT. Thus, our inequality is true for $ny^{n-1}(x-y) \leq x^n - y^n \leq nx^{n-1}(x - y)$.

I think there is a misunderstanding between a definition and a theorem. Here, you have to verify the hypothesis of the MVP to obtain the inequality.

b) For this question let us denote two functions once again and obtain $f(a) = \sqrt{1+a}$ for $a \in (0, x]$. By definition of MVT, $f'(c) = \frac{f(b)-f(a)}{b-a}$. Observe,

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$$\begin{aligned}
f'(c) &= \frac{1 - \frac{1}{2}x - \sqrt{1+0}}{b-0} \\
f'(c) &= \frac{1 - \frac{1}{2}x - 1}{b} \\
f'(c) &= \frac{\frac{1}{2}x}{b} \\
f'(c) &= \frac{1}{2}xb
\end{aligned}$$

Here $\underbrace{x = c}_{\xi}$

How did the number pass from the denominator and numerator?

??

We know that $\frac{1}{2}xb > 0$ for $x > 0$, thus by MVT, $\sqrt{1+x} < 1 + \frac{1}{2}x$ for $x > 0$. $\square$