MATH-331 Introduction to Real Analysis — YOUR FULL NAME
Homework 03 — Fall 2021

Due date: October 11$^{\text{th}}$ 1:20pm — Total: /70.

Exercise	1 (5)	2 (5)	3 (5)	4 (5)	5 (10)	6 (10)	7 (5)	8 (5)	9 (5)	10 (10)
Score										

Table 1: Scores for each exercises

Instructions: You must answer all the questions below and send your solution by email (to `parisepo@hawaii.edu`). If you decide to not use LaTeX to hand out your solutions, please be sure that after you scan your copy, it is clear and readable. Make sure that you attached a copy of the homework assignment to your homework.
If you choose to use LaTeX, you can use the template available on the course website.
No late homework will be accepted. No format other than PDF will be accepted. Name your file as indicated in the syllabus.

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Writing problems

For each of the following problems, you will be asked to write a clear and detailed proof. You will have the chance to rewrite your solution in your semester project after receiving feedback from me.

Exercise 1. (5 pts) Let $(a_n)_{n=1}^{\infty}$ be an increasing sequence and $(b_n)_{n=1}^{\infty}$ be a decreasing sequence. Let $(c_n)_{n=1}^{\infty}$ be the sequence defined by $c_n = b_n - a_n$. Show that if $\lim_{n\to\infty} c_n = 0$, then the sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ converges and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

Solution: First we need to show that $\forall n, m \in \mathbb{N}$, $b_n \geq a_m$.
If we set $M := max\{n, m\}$, then we have

$$a_m \leq a_M$$

because a_n is increasing

$$b_n \geq b_M$$

because b_n is decreasing

Now we know $b_M \geq a_M$ because $\forall k \in \mathbb{N}$, $b_k \geq a_k$. Lets prove this fact now.
Assume toward a contradiction that $b_n < a_n$. Since

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} b_n - a_n$$
$$\lim_{n\to\infty} b_n - a_n = 0$$

Definition of convergence, $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. if $n \geq N$, then

$$|b_n - a_n| < \epsilon$$
$$\Rightarrow a_n - b_n < \epsilon$$

The LHS is an increasing sequence, meaning for every ϵ, we cannot find an $N \in \mathbb{N}$ such that the previous statement holds. No matter what N we start at, there will alawys be some $N_1 > N$ such that if $n \geq N_1$ then

$$|a_n - b_n| > \epsilon$$

This is a contradiction because we know $(b_n - a_n)$ must converge, but if $b_n < a_n$, then c_n doesnt converge at all. Therefore $b_n \geq a_n$ for all $n > 0$. Now we can use this fact to tie the previous inequalities together.

$$a_n \leq a_M \leq b_M \leq b_m$$

So we have that $a_n \leq b_m$ for all $n, m \geq 1$. This means that for any $n \geq 1$, a_n is bounded above by the number b_m. It also means for any $m \geq 1$, b_m is bounded below by the number a_n. Therefore,
- b_n is bounded below and decreasing meaning it converges
- a_n is bounded above and increasing meaning it also converges

Now that we know a_n and b_n converges, their limit exists. So now,

$$c_n = b_n - a_n$$
$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} (b_n - a_n)$$
$$0 = \lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n$$

by the sum rule of limits

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$$

Exercise 2. (5 pts) Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$, and suppose that x_0 is an accumulation point of D. Suppose that for each sequence $(x_n)_{n=1}^{\infty}$ converging to x_0 with $x_n \in D\backslash\{x_0\}$ for each $n \geq 1$, then the sequence $(f(x_n))_{n=1}^{\infty}$ is Cauchy. Show that f has a limit at x_0.

[Hint: For two sequences (x_n) and (y_n) that satisfy the assumption, define the sequence (z_n) to be $z_{2n} = x_n$ and $z_{2n-1} = y_n$. Show that $(f(z_n))$ converges and the sequence $(f(x_n))$ and $(f(y_n))$ converges to the same limit as $(f(z_n))$. Conclude by a theorem in the lecture notes.]

Exercise 3. (5 pts) Prove that if $f : D \subseteq \mathbb{R} \to \mathbb{R}$ has a limit at $x_0 \in \operatorname{acc} D$, then the limit is unique.

Solution: Define two sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ such that they both converge at x_0. Then the sequences

$$(f(x_n)) \to L_1 \in \mathbb{R}$$
$$(f(y_n)) \to L_2 \in \mathbb{R}$$

Now define a new sequence z_n s.t.

$$z_{2n} = x_n$$
$$z_{2n-1} = y_n$$

z_n converges to x_0 since each of its subsequences converge to x_0. Then we have that the series $(f(z_n))_{n=1}^{\infty}$ converges, and $(f(x_n))$ and $(f(y_n))$ are subsequences of $(z_n(x_n))$, they all must converge to the same point. Therefore,

$$L_1 = L_2$$

$$(f(x_n)) \to L$$
$$(f(y_n)) \to L$$
$$(f(z_n)) \to L$$

Then by a theorem in the textbook, L must be the limit of f at x_0. Now we know L is unique because from our approach, we can have any sequence $(x_n)_{n=1}^{\infty}$ converging to x_0 and $x_n \in D\backslash\{x_0\}$ and still end up with the L being the limit. $\square$

Exercise 4. (5 pts) Suppose $f : D \subseteq \mathbb{R} \to \mathbb{R}$, $g : D \subseteq \mathbb{R} \to \mathbb{R}$ and $h : D \subseteq \mathbb{R} \to \mathbb{R}$ are three functions such that

$$f(x) \leq h(x) \leq g(x) \quad (\forall x \in D).$$

Suppose that f and g have limits at x_0 with $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x)$. Prove that h has a limit at x_0 and

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} h(x) = \lim_{x\to x_0} g(x).$$

Solution: Lets define a sequence $(x_n)_{n-1}^{\infty}$ that converges to x_0 and $x_n \in D\backslash\{x_0\}$.
Then $(f(x_n))_{n=1}^{\infty}$ and $(g(x_n))_{n=1}^{\infty}$ converge at x_0 to L by a theorem in the lecture notes and textbook.
We also know they both converge to L because of our assumption.
Using the same sequence $(x_n)_{n=1}^{\infty}$, we have the sequence $(h(x_n))_{n=1}^{\infty}$ which also converges at x_0. To find out where it converges to, we will use the Squeeze Theorem from sequences.

$$(f(x_n)) \leq (h(x_n)) \leq (g(x_n))$$

By the Squeeze theorem, since $(f(x_n))$ and $(g(x_n))$ both converge to L, $(h(x_n)) \to L$.
This means, by a theorem from the lecture notes, that $\lim_{x\to x_0} h(x)$ exists and that $\lim_{x\to x_0} h(x) = L$ at x_0. $\square$

Exercise 5. (10 pts) Let $f : (0, \infty) \to \mathbb{R}$ be a function. We say that f has a limit at ∞ if there exists a $L \in \mathbb{R}$ such that for any $\varepsilon > 0$, there is a real number $M > 0$ such that if $x > M$, then $|f(x) - L| < \varepsilon$.

a) Show that if $g : (0, \infty) \to \mathbb{R}$ is bounded and $\lim_{x\to\infty} f(x) = 0$, then $\lim_{x\to\infty} f(x)g(x) = 0$.

b) Let $a > 0$ and suppose that $f : (a, \infty) \to \mathbb{R}$ and define $g : (0, 1/a) \to \mathbb{R}$ by $g(x) = f(1/x)$. Show that f has a limit at ∞ if and only if g has a limit at 0.

Solution: a)
We want to show that if

$$h(x) = f(x)g(x)$$

Then there exists a $L = 0 \in \mathbb{R}$ such that for any $\epsilon_1 > 0$, there is a real number $M_1 > 0$ such that if $x > M_1$,

$$|h(x) - 0| < \epsilon_1$$
$$|f(x)g(x)| < \epsilon_1$$

From here, we know that $g(x)$ is bounded, meaning $\exists G > 0$ s.t. $|g(x)| \leq G \ \forall x \geq 1$.

$$|f(x)g(x)| \leq |f(x)G| = G|f(x)|$$

By our assumption, we know $\lim_{x\to\infty} f(x) = 0$, then $\exists L = 0$ and $M_2 \in \mathbb{R}$ s.t. for any ϵ_2, if $x > M_2$,

$$|f(x)| < \epsilon_2$$

We can set $M_1 = M_2$, so that if $x > M_1$,

$$|f(x)g(x)| \leq G|f(x)| < G\epsilon_2$$

From here, we can just set $\epsilon_1 = G\epsilon_2$
Now putting everything together, we have that if $x > M_1$,

$$|f(x)g(x)| < \epsilon_1$$

Therefore, $\lim_{x\to\infty} f(x)g(x) = 0$. □

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Homework problems

Answer all the questions below. Make sure to show your work.

Exercise 6. (10pts) For each of the sequences below, determine its nature (converges or diverges)[1]:

a) (a_n) where $a_n = \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n}$.

[1] You don't need to compute the limit.

b) (a_n) where $a_n = \frac{1+2+\cdots+n}{n^2}$.

Solution: a)
To show that this sequence is convergent, we will show that it is bounded from below and decreasing. First lets show it is decreasing.

$$a_n > a_{n+1}$$
$$\frac{1}{n} + \frac{1}{n+1} + ... + \frac{1}{2n} > \frac{1}{n+1} + ... + \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2}$$

Note that everything except the first term on LHS and the last two terms on RHS will cancel

$$\frac{1}{n} > \frac{1}{2n+1} + \frac{1}{2n+2}$$
$$\frac{1}{n} > \frac{4n+3}{(2n+1)(2n+2)}$$

Then by a property from the order axioms...

$$n < \frac{4n^2+6n+2}{4n+3} \leq 4n^2+6n+2$$
$$n < 4n^2+6n+2$$

The last statement is true for all $n > 0$, so therefore a_n is decreasing.
Now we will prove that it is bounded from below.

$$a_n = \frac{1}{n} + \frac{1}{n+1} + ... + \frac{1}{2n}$$
$$\frac{1}{n} + \frac{1}{n+1} + ... + \frac{1}{2n} > \frac{1}{2n} + ... + \frac{1}{2n}$$
$$= \frac{1}{2n} \cdot (n+1)$$
$$= \frac{1}{2} + \frac{1}{2n}$$

From this, we know that a_n is bounded below by this sequence. By taking the limit of this sequence, we shall know what value a_n is bounded below by.

$$\lim_{n\to\infty} \frac{1}{2} + \frac{1}{2n} = \lim_{n\to\infty} \frac{1}{2} + \lim_{n\to\infty} \frac{1}{2n}$$

By the sum rule of limits

$$= \frac{1}{2}$$

Therefore, $a_n > \frac{1}{2}$. It is bounded below by $\frac{1}{2}$ and is decreasing, so a_n is convergent. $\square$

Solution: b)

$$\begin{aligned} a_n &= \frac{1+2+...+n}{n^2} \\ 1+2+...+n &< n+n+...+n = n(n) = n^2 \\ 1+2+...+n &< n^2 \end{aligned}$$

The denominator grows faster than the numerator, and thus as n approaches infinity, the sequence will converge at 0. $\square$

Exercise 7. (5 pts) Define $g : (0, 1) \to \mathbb{R}$ by $f(x) = \frac{\sqrt{1+x}-1}{x}$. Prove that g has a limit at 0 and find it.

Solution: To start, note that we are looking for the R-H limit at 0 of g. So for our limit definition, we are trying to show that

$$0 < x < \delta$$
$$\exists \epsilon \text{ such that}$$
$$\left| \frac{\sqrt{1+x}-1}{x} - L \right| < \epsilon$$

$g(x)$ is a decreasing function from (0,1) so we can rewrite

$$L - \frac{\sqrt{1+x}-1}{x} < \epsilon$$

Now with some algebra, we find that

$$x < \frac{2\epsilon - 2L + 1}{L^2 - 2L\epsilon + \epsilon^2} = \delta$$

We set the RHS equal to δ so that indeed, for any ϵ, we can find δ such that the inequality holds true.
Now lets find the limit.

$$\begin{aligned} &\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} \\ &t = x+1 \\ \lim_{t\to 1} \frac{\sqrt{t}-1}{t-1} &= \lim_{t\to 1} \frac{\sqrt{t}-1}{(\sqrt{t}+1)(\sqrt{t}-1)} \\ &= \lim_{t\to 1} \frac{1}{\sqrt{t}+1} \\ &= \frac{1}{2} = \lim_{x\to 0} g(x) \end{aligned}$$

Exercise 8. (5 pts) Suppose that $f : (0, 1) \to \mathbb{R}$ has a limit at $x_0 = 1$ and $\lim_{x\to 1} f(x) = 1$. Compute the value of the limit

$$\lim_{x\to 1} \frac{f(x)(1 - f(x)^2)}{1 - f(x)}.$$

Solution:

$$\begin{aligned}
\frac{f(x)(1-f(x)^2}{1-f(x)} &= \frac{f(x)(1+f(x))(1-f(x))}{1-f(x)} \\
&= f(x)(1+f(x)) \\
\lim_{x\to\infty} f(x)(1+f(x)) &= \lim_{x\to\infty} f(x) \cdot (\lim_{x\to\infty} 1 + \lim_{x\to\infty} f(x))
\end{aligned}$$

by the sum and product rule of limits

$$\begin{aligned}
&= 1 \cdot (1+1) \\
&= 2
\end{aligned}$$

Exercise 9. (5 pts) Prove that if $f : D \to \mathbb{R}$ has a limit at x_0, then $|f|(x) := |f(x)|$ has a limit at x_0.

Solution: Let $(x_n)_{n=1}^{\infty}$ be a sequence that converges to x_0, and $x_n \in D\backslash\{x_0\}$. Then by a theorem from the lecture notes and textbook, $(f(x_n))_{n=1}^{\infty}$ converges at x_0.
Then by our previous homework on sequences, $|(f(x_n))|$ also converges at x_0.
Then by the same theorem, $|f(x)|$ must have a limit at x_0 $\square$

Exercise 10. (10 pts) Using the link between sequences and limits of functions, show the following.

a) If $f(x) = x^n$ $(n \geq 0)$, then $\lim_{x\to x_0} f(x) = x_0^n$ for any $x_0 \in \mathbb{R}$.

b) If $x_0 \in [0, \infty)$, then $\lim_{x\to x_0} \sqrt{x} = \sqrt{x_0}$.

Solution: a)
Define a sequence $(x_k)_{k=1}^{\infty}$ that converges to x_0. We know from sequences that if

$$\begin{aligned}
x_k &\to x_0 \\
x_k^n &\to x_0^n
\end{aligned}$$

In the last statement, notice that

$$\begin{aligned}
x_k^n &= (f(x_k)) \\
x_0^n &= f(x_0) \\
(f(x_k))_{k=1}^{\infty} &\to f(x_0) \\
\lim_{x\to x_0} f(x) &= x_0^n
\end{aligned}$$

Solution: b)
Define a sequence $(x_n)_{n=1}^{\infty}$ s.t. it converges to x_0 and $x_n \in [0, \infty)\backslash\{x_0\}$. Lets also define a function $f(x) = \sqrt{x}$.
Then we can substitute

$$\lim_{x \to x_0} f(x) = \sqrt{x_0}$$

We know by the theorem from the textbook that the sequence $(f(x_n))_{n=1}^{\infty}$ converges at x_0. We know by sequences from a previous homework that if we have

$$\begin{aligned} x_n &\to x_0 \\ \sqrt{x_n} &\to \sqrt{x_0} \\ \Rightarrow (f(x_n))_{n=1}^{\infty} &\to f(x_0) \\ \lim_{n \to x_0} f(x) &\to f(x_0) \\ \Rightarrow \lim_{n \to x_0} \sqrt{x} &\to \sqrt{x_0} \end{aligned}$$

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Bonus

Exercise 11. Assume that $f : \mathbb{R} \to \mathbb{R}$ such that $f(x + y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$.

a) Show that f has a limit at every point of $\mathbb{R}$.

b) Show that either $\lim_{x \to 0} f(x) = 1$ or $f(x) = 0$ for any $x \in \mathbb{R}$.