Math 331: Homework 1

1\. a) Let I be a subinterval of $[a,b]$ and put $\varphi = c\chi_I$

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Case ①, when $I = [u,v]$, $a \le u < v \le b$, then

$$\int_a^b \phi(x)\,dx = \int_a^u \phi(x)\,dx + \int_u^v \phi(x)\,dx + \int_v^b \phi(x)\,dx$$
$$= \int_a^u 0\,dx + \int_u^v c\,dx + \int_v^b 0\,dx$$
$$= c(v-u) = c\,\ell(I)$$

You didn't show that ϕ is R.I...

Case ②, when $I = (u,v]$ then

$$\int_a^b \phi(x)\,dx = \int_a^u \phi(x)\,dx + \int_u^v \phi(x)\,dx + \int_v^b \phi(x)\,dx$$
$$= \int_u^v c\,dx = c(v-u) = c\,\ell(I)$$

Case ③, when $I = [u,u] = \{u\}$

$$\int_a^b \phi(x)\,dx = \int_a^u \phi(x)\,dx + \int_u^b \phi(x)\,dx$$
$$= \int_a^u 0\,dx + \int_u^b 0\,dx = 0$$

almost.

Since in each case the number of discontinuities of ϕ in $[a,b]$ is finite, ϕ is R.I ∎

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b) We know that, if f_1 and f_2 are R.I. then f_1+f_2 is also R.I and

$$\int_a^b (f_1+f_2)(x)\,dx = \int_a^b f_1(x)\,dx + \int_a^b f_2(x)\,dx \quad (*)$$

Now, let the result be true for $n=k-1$, so $f_1+\dots+f_{k-1}$ is R.I. and

$$\int_a^b (f_1+\dots+f_{k-1})(x)\,dx = \int_a^b f_1(x)\,dx + \dots + \int_a^b f_{k-1}(x)\,dx \quad (**)$$

Now take $n=k$, then let $f_1,\dots,f_k$ be R.I and $f_1+\dots+f_k$ is R.I., and → why?? 2/3

$$\int_a^b (f_1+\dots+f_{k-1}+f_k)(x)\,dx = \int_a^b ((f_1+\dots+f_{k-1})(x) + f_k(x))\,dx$$
$$= \int_a^b (f_1+\dots+f_{k-1})(x)\,dx + \int_a^b f_k(x)\,dx \quad (\text{by } *)$$
$$= \int_a^b f_1(x)\,dx + \dots + \int_a^b f_{k-1}(x)\,dx + \int_a^b f_k(x)\,dx \quad (\text{by } **)$$

By Principle of Mathematical Induction the result is true for all $n \in \mathbb{N}$ ∎

Total: 57/65

	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10
Scores	8/10	3/5	10/10	5/5	5/5	9/10	5/5	5/5	4/5	3/5

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1. c) Given $\phi = \sum_{k=1}^{n} c_k \chi_{I_k}$
from ⓐ $\Rightarrow c_k \chi_{I_k}$ is R.I $\forall k = 1, \dots, n$ 5/5
from ⓑ $\Rightarrow \sum_{k=1}^{n} c_k \chi_{I_k}$ is R.I.
and also
$$\int_a^b \phi(x)\,dx = \int_a^b \sum_{k=1}^{n} c_k \chi_{I_k}(x)\,dx$$
$$= \underbrace{\sum_{k=1}^{n} \int_a^b c_k \chi_{I_k}(x)\,dx}$$ ■ The value?

2. Let $x_i = a + i\Delta x$ and $\Delta x = \frac{b-a}{n}$, where $x_i \in [a,b]$ and n is a natural number.
We are given $0 \le f(x)$ for $x \in [a,b]$, so
$0 \le f(x_i)$
from Homework 6.
$\Rightarrow 0 \le f(x_i)\Delta x$
$\Rightarrow 0 \le \sum_{i=0}^{n} f(x_i)\Delta x$
$\Rightarrow 0 \le \lim_{n\to\infty} \sum_{i=0}^{n} f(x_i)\Delta x$
Okay $\Rightarrow 0 \le \int_a^b f(x)$

You can use: $f(x) \le g(x)\ \forall x \in [a,b] \Rightarrow \int_a^b f \le \int_a^b g \dots$

Similarly, since intervals $[a,u], [u,v]$ & $[v,b]$ are in $[a,b]$
$\Rightarrow 0 \le \int_a^u f(x)$ & $0 \le \int_u^v f(x)$ & $0 \le \int_v^b f(x)$
Then we get
$$\int_a^b f(x) - \int_a^u f(x) - \int_v^b f(x) \le \int_a^b f(x)$$
Now use the property of the Riemann Integral we get:
$$\Rightarrow \int_a^v f(x) - \int_a^u f(x) \le \int_a^b f(x)$$
$$\Rightarrow \int_a^u f(x) + \int_u^v f(x) - \int_a^u f(x) \le \int_a^b f(x)$$
$$\Rightarrow \int_u^v f(x) \le \int_a^b f(x)$$ ■

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I'm a bit mixed up in the inequalities... What you should get is
$$\int_a^b f = \int_a^u f + \int_u^v f + \int_v^b f \ge \int_u^v f$$
because $\int_a^u f$ & $\int_v^b f \ge 0$.

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3. a) Let us assume towards a contradiction that there exists a point $c \in [a,b]$ s.t $f(c) \neq 0$

$$\Rightarrow f(c) > 0 \qquad (f(x) \geq 0 \ \forall x \in [a,b])$$

f is continuous on $[a,b]$ so $\exists$ an $\varepsilon > 0$ s.t $f(x) > 0 \ \forall x \in (c-\varepsilon, c+\varepsilon)$

$$\Rightarrow \int_{c-\varepsilon}^{c+\varepsilon} f(x) > 0$$

We know that

$$\int_a^b f(x)dx = \int_a^{c-\varepsilon} + \int_{c-\varepsilon}^{c+\varepsilon} f(x)dx + \int_{c+\varepsilon}^b f(x)dx$$

and we have

$$\int_a^{c-\varepsilon} f(x)dx \geq 0$$

$$\& \int_{c+\varepsilon}^b f(x)dx \geq 0$$

so $\int_a^b f(x)dx > 0 \ \rightarrow\leftarrow$

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✓

so $f(x) = 0 \ \forall x \in [a,b]$ ∎

b) Given $f, g : [a,b] \to \mathbb{R}$ is continuous s.t $\int_a^b f = \int_a^b g$

define $h : [a,b] \to \mathbb{R}$ & $h(x) = (f-g)(x)$ with f and g continuous on $[a,b]$ and $h = f - g$ is continuous on $[a,b]$.

By Fundamental Theorem of Calculus there exists

$$H : [a,b] \to \mathbb{R} \quad \text{s.t}$$

$$H'(x) = h(x)$$

and $\int_a^b h(x) = H(b) - H(a)$

so $\int_a^b h = \int_a^b f - g = 0 \Rightarrow H(b) - H(a) = 0$

$\Rightarrow \exists c \in (a,b)$ s.t. $H'(c) = 0$ → why that? More details...

$\Rightarrow h(c) = 0$

$\Rightarrow f(c) - g(c) = 0$

$\Rightarrow f(c) = g(c)$ ∎

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4. Let $F(x) = \int_a^x f(t)\,dt$. As f is continuous on $[a,b]$, F is differentiable on (a,b) and by the MVT, $\exists c \in (a,b)$ s.t.

$$F'(c) = F(b) - F(a)$$

$$\boxed{\Rightarrow f(c)(b-a) = \int_a^b f}$$

You forgot $(b-a)$:

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$$F'(c) = \frac{F(b) - F(a)}{b-a}.$$

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5. Given f is strictly increasing, define
$$g(x) = f(a)(x-a) + f(b)(b-x)$$
Since f is strictly increasing
$\forall x \in [a,b]$ and $f(a) < f(x) < f(b)$.
Taking the integral
$$\Rightarrow \int_a^b f(a)\,dx < \int_a^b f(x)\,dx < \int_a^b f(b)\,dx$$
$$\Rightarrow f(a)(b-a) < \int_a^b f(x)\,dx < f(b)(b-a)$$
Note that $g(a) = f(b)(b-a)$
& $g(b) = f(a)(b-a)$
& since f is continuous $g(x)$ is continuous & since $\int_a^b f(x)dx$ lies between $g(a)$ & $g(b)$. Using the Intermediate Value Theorem $\exists c$ s.t.
$$g(c) = \int_a^b f(x)\,dx$$
i.e. $f(a)(c-a) + f(b)(b-c) = \int_a^b f(x)\,dx$. ∎

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6. a) Let $U(f,P)$ and $L(f,P)$ be the upper and lower sums of f with respect to partition P on $[0,1]$. Let $M_i = \sup\{f(x) \mid x \in I_i\}$ and $m_i = \inf\{f(x) \mid x \in I_i\}$ where I_i is the i^{th} interval of P. Note that $M_i = 1$ for all i, because every interval I_i of P contain rational numbers. On the other hand $m_i = 0$ for all i because every interval I_i of P contain irrational numbers. By definition,

$$U(f,P) = \sum_{i=1}^{n} M_i \mu(I_i) = \sum_{i=1}^{n} 1\mu(I_i) = \sum_{i=1}^{n} \mu(I_i) = 1 - 0 = 1$$

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&

$$L(f,P) = \sum_{i=1}^{n} m_i \mu(I_i) = \sum_{i=1}^{n} 0\mu(I_i) = 0$$

Thus f is not R.I. on $[0,1]$ because the upper and lower sums are not equal. ∎ okay, using the def. from the book, but this is not what we introduced in the lecture notes...

b) Since $g: [0,1] \to \mathbb{R}$ & $h: [0,1] \to \mathbb{R}$ by $g = \chi_{(0,1]}$, $h(x) = \begin{cases} 0 & , x \notin \mathbb{Q} \\ 1/q & , x = p/q \in \mathbb{Q} \end{cases}$

then $g \circ h: [0,1] \to \mathbb{R}$

and $g \circ h(x) = g(h(x)) = g(0)$ if $x \notin \mathbb{Q} = 0$

and $g \circ h(x) = g(h(x)) = g(\frac{1}{q})$ if $x = p/q \in \mathbb{Q} = 1$

so $g \circ h(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{Q} \end{cases}$

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Hence $g \circ h = f$ and by (a) $g \circ h$ is not integrable on $[0,1]$. ∎

What does that mean? g & h are Riemann integrable but their compo. is not ...

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7. If f is continuous on [a,b] then f is R.I. on [a,b]. Then by the same logic we have that $|f|$ is continuous and R.I. on [a,b].
Now we know that

$$\pm x \leq |x| \quad \star$$

$$\Rightarrow -|x| \leq x \leq |x|$$

Then if we take the integral we have

$$-|f| \leq f \leq |f|$$

$$\Rightarrow -\int |f| \leq \int f \leq \int |f|$$

Then by the logic of $\star$

$$\left|\int f\right| \leq \int |f| \quad \blacksquare$$

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8. $f(x) = \int_{\sqrt{x}}^{\sqrt[3]{x}} \frac{1}{1+t^3}\,dt$

Let $g(t) = \frac{1}{1+t^3}$. Since g is continuous
$\exists G$ s.t. $g = G'(t)$. Then
$G(\sqrt[3]{x}) - G(\sqrt{x}) = f(x)$

$f'(x) = G'(\sqrt[3]{x}) - G'(\sqrt{x})$

$= \left(\frac{1}{1+\sqrt[3]{x}^{\,3}}\right)\frac{1}{3(x^{2/3})} - \frac{1}{1+\sqrt{x}^{\,3}}\left(\frac{1}{2\sqrt{x}}\right)$

$\boxed{= \frac{1}{3(1+x)(x^{2/3})} - \frac{1}{2(1+x^{3/2})(\sqrt{x})}}$

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9. $f(1)=0$ and $f'(x)=1+\sin(x^2)\ \forall x>1$

$f'(x)=1+\sin(x^2)$

$\frac{df}{dx}=1+\sin(x^2)$

What is $S\left(\sqrt{\frac{2}{\pi}}x\right)$??

$\int df=\int 1+\sin(x^2)\,dx$ $\quad u=x^2$

sufficient to take $f(x)=\int_1^x 1+\sin(t^2)dt$

$f(x)=x-\frac{\sqrt{\pi}}{\sqrt{2}}S\left(\sqrt{\frac{2}{\pi}}x\right)+C$ $\quad du=2x\,dx$

$dx=\frac{1}{2x}du$

what are the limits?

$0=1-\sqrt{\frac{\pi}{2}}\,S\left(\sqrt{\frac{2}{\pi}}\right)+C$

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$C=\sqrt{\frac{\pi}{2}}\,S\left(\sqrt{\frac{2}{\pi}}\right)-1$

$$\boxed{f(x)=x-\sqrt{\frac{\pi}{2}}\,S\left(\sqrt{\frac{2}{\pi}}x\right)+\sqrt{\frac{\pi}{2}}\,S\left(\sqrt{\frac{2}{\pi}}\right)-1}$$

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10. Thinking about this as a Riemann Sum, we factor out an "n^2" in the denominator

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{k^2+n^2}$$
$$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{1+(\frac{k}{n})^2}$$

→ You missing the length of the intervals.

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Which can be rewritten as

$$\lim_{n\to\infty} \int_0^1 \frac{1}{1+x^2}$$
$$\boxed{= \frac{\pi}{4}}$$