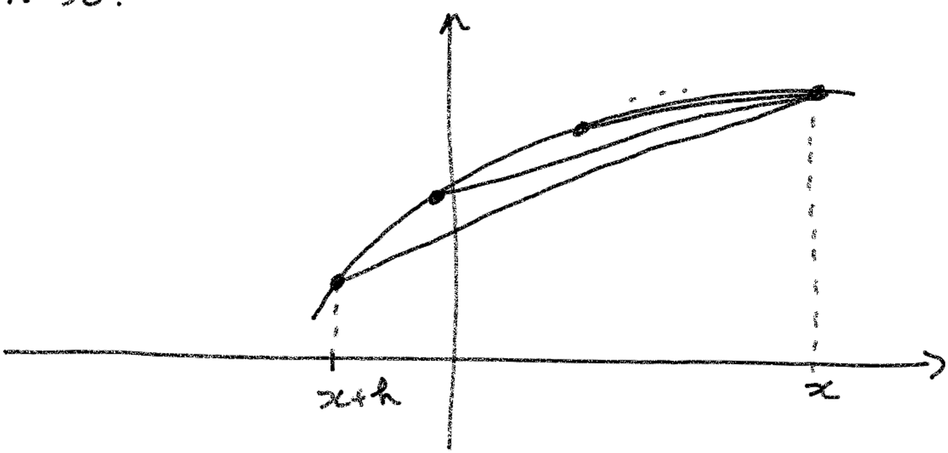
Differentiation

5 - Differentiation

5.1 Definition

From calculus courses, we know that $f'(x)$ represent the slope of the tangent line at x or it is the limit of secant lines passing to $(x+h, f(x+h))$ & $(x, f(x))$ as $h \to 0$.



The slope of the secant lines is

$$\frac{f(x+h) - f(x)}{x+h-x} = \frac{f(x+h) - f(x)}{h}.$$

Def. $f : D \to \mathbb{R}$ & $x_0 \in acc(D)$ & $x_0 \in D$. f is differentiable at x_0 iff

$$\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} \text{ exists.}$$

Remark. • We can put $x = x_0 + h$ & so as $h \to 0$, $x \to x_0$ &

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

• The limit, if it exists, is denoted by $f'(x_0)$.

• $f : D \to \mathbb{R}$, $x_0 \in acc(D)$ and $x_0 \in D$. Then f is differentiable at x_0 iff. for any sequence (x_n), $x_n \in D$, $x_n \neq x_0$ & $x_n \to x_0$,

$$\lim_{n \to \infty} \frac{f(x_n) - f(x_0)}{x_n - x_0} \quad \text{exists}.$$

Examples.

① Let $f(x) = |x|$. Then, if $x_n = 1/n$,

$$\frac{f(x_n) - f(0)}{1/n} = 1$$

but if $x_n = -1/n$, then

$$\frac{f(x_n) - f(0)}{-1/n} = -1.$$

So $f(x) = |x|$ is not differentiable at $x_0 = 0$ even though it is continuous at $x_0 = 0$.

② Take $f(x) = x|x|$. then,

$$\frac{f(x)-f(0)}{x-0} = \frac{x|x|}{x} = |x|$$

and so $\lim_{x\to 0} |x| = 0$ ($|x|$ is continuous).

So, f is differentiable at $x_0 = 0$. We can show that

$$f'(x) = 2|x|.$$

So, f' is continuous.

<u>Thm.</u> $f: D \to \mathbb{R}$ & $x_0 \in acc(D)$ & $x_0 \in D$. If f is differentiable at x_0, then it is continuous at x_0.

<u>Proof.</u> By the limit characterization, we have

$$\lim_{x\to x_0} |f(x)-f(x_0)| = \lim_{x\to x_0} \left|\frac{f(x)-f(x_0)}{x-x_0}\right| |x-x_0|.$$

By the product rule for limits and by the continuity of the abs. value fct:

$$\lim_{x\to x_0} \left|\frac{f(x)-f(x_0)}{x-x_0}\right| |x-x_0| = |f'(x_0)| \cdot 0 = 0.$$

So, $\lim_{x\to x_0} f(x) = f(x_0)$.

4.2. Algebra of continuous function

Thm. $f: D \to \mathbb{R}$ & $g: D \to \mathbb{R}$ & $x_0 \in acc(D)$ & $x_0 \in D$. If f & g are differentiable at x_0, then

a) $f+g$ is diff. at x_0.

b) fg is diff. at x_0.

c) If $g(x_0) \neq 0$, f/g is differentiable at x_0.

Moreover, in these cases, we have

- $(fg)'(x_0) = f'(x_0)\, g(x_0) + f(x_0)\, g'(x_0)$
- $\left(\frac{f}{g}\right)'(x_0) = \dfrac{f'(x_0)\, g(x_0) - f(x_0)\, g'(x_0)}{(g(x_0))^2}$.

Proof. a) Use the properties of limits.

b) Observe the following

$$(x+\Delta x)(y+\Delta y) - xy$$
$$= \Delta x\, y + x\, \Delta y + \Delta x\, \Delta y$$
$$\approx \underline{\Delta x\, y} + \underline{x\, \Delta y}.$$

So, we have

$$\frac{f(x_0+h)\,g(x_0+h) - f(x_0)\,g(x_0)}{h}$$

$$= \frac{f(x_0+h)\,g(x_0+h) - f(x_0)\,g(x_0+h) + f(x_0)g(x_0+h) - f(x_0)g(x_0)}{h}$$

$$= \left(\frac{f(x_0+h) - f(x_0)}{h}\right) g(x_0+h) + f(x_0)\left(\frac{g(x_0+h) - g(x_0)}{h}\right)$$

Taking the limit as $h \to 0$, we get

$$\lim_{h\to 0} \frac{f(x_0+h)g(x_0+h) - f(x_0)g(x_0)}{h} = f'(x_0)\,g(x_0) + f(x_0)\,g'(x_0)$$

from continuity of g.

c) We use the same trick:

$$\frac{\dfrac{f(x_0+h)}{g(x_0+h)} - \dfrac{f(x_0)}{g(x_0)}}{h} = \frac{f(x_0+h)\,g(x_0) - f(x_0)\,g(x_0+h)}{h\;g(x_0+h)\,g(x_0)}$$

$$= \frac{f(x_0+h)\,g(x_0) - f(x_0)g(x_0) + f(x_0)g(x_0) - f(x_0)g(x_0+h)}{h\;g(x_0+h)\,g(x_0)}$$

$$= \frac{\left(\dfrac{f(x_0+h) - f(x_0)}{h}\right) g(x_0) - f(x_0)\left(\dfrac{g(x_0+h) - g(x_0)}{h}\right)}{g(x_0+h)\,g(x_0)}.$$

Take limit $h \to 0$ & use continuity of g. $\square$

Examples

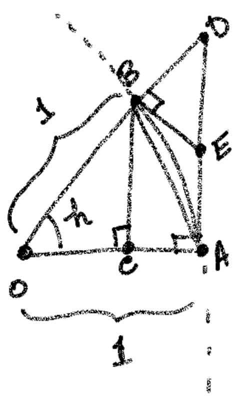
① $f(x) = \sin x$ is differentiable on $\mathbb{R}$.

Proof. $\dfrac{\sin(x+h) - \sin x}{h} = \dfrac{\sin x \cos h + \cos x \sin h - \sin x}{h}$

$$= \sin x \frac{(\cos h - 1)}{h} + \cos x \frac{\sin h}{h}.$$

Now, we know that

$$\sin h \leq h. \quad (h \in (0, \pi/2)).$$



We have arc $AB = h$, so

$$h \leq |\overline{AE}| + |\overline{EB}|$$
$$\leq |\overline{AE}| + |\overline{ED}| = |\overline{AD}|$$
$$= \tan h$$

$$\Rightarrow \quad h \leq \frac{\sin h}{\cos h}$$

$$\Rightarrow \quad \cos h \leq \frac{\sin h}{h}.$$

Thus, by the squeeze theorem for limits:

$$1 = \lim_{h \to 0} \cos h \leq \lim_{h \to 0} \frac{\sin h}{h} \leq \lim_{h \to 0} 1 = 1$$

$$\Rightarrow \quad \lim_{h \to 0} \frac{\sin h}{h} = 1.$$

Also, we have

$$\cos h - 1 = -2\sin^2(h/2)$$

$$\Rightarrow \quad \frac{\cos h - 1}{h} = -\sin(h/2)\cdot\left(\frac{\sin(h/2)}{h/2}\right).$$

$$\Rightarrow \quad \lim_{h\to 0}\frac{\cos h - 1}{h} = \left(-\lim_{h\to 0}\sin(h/2)\right)\left(\lim_{h\to 0}\frac{\sin(h/2)}{h/2}\right)$$

$$= 0\cdot 1 = 0.$$

So, by taking $h\to 0$, we see that

$$\lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = \sin x\lim_{h\to 0}\frac{\cos h - 1}{h} + \cos x\lim_{h\to 0}\frac{\sin h}{h}$$

$$= \cos x.$$

thus, $\sin'(x) = \cos(x)$. □

② If $f(x) = x^n$, then $f'(x) = nx^{n-1}$.

Proof.

$$\frac{x^n - x_0^n}{x - x_0} = \sum_{k=0}^{n-1} x^k x_0^{n-1-k}$$

and so

$$\lim_{x\to x_0}\frac{x^n - x_0^n}{x - x_0} = \sum_{k=0}^{n-1}\lim_{x\to x_0} x^k x_0^{n-1-k} = n x_0^{n-1}.$$ □

(3) $f(x) = x^{-n}$, $n \geq 1$, then $f'(x) = -nx^{-n-1}$ if $x \neq 0$.

<u>Proof</u>. By the quotient rule:

$$f'(x) = \frac{(1)' x^n - (1)(n) x^{n-1}}{x^{2n}} \qquad (x \neq 0)$$

$$= -n \frac{x^{n-1}}{x^{2n}} = -n x^{-n-1}. \qquad \square$$

(4) Any rational f.ts is differentiable where it is defined.

<u>Thm.</u> (Chain rule).

Let $f : D \to \mathbb{R}$, $g : \tilde{D} \to \mathbb{R}$ with $f(D) \subseteq \tilde{D}$

If f is differentiable at x_0 and g is differentiable at $f(x_0)$, then $g \circ f$ is differentiable at x_0 with

$$(g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0).$$

<u>Proof</u>. We have to be careful.

Define

$$F(x) = \begin{cases} \dfrac{g(x) - g(f(x_0))}{x - f(x_0)}, & x \neq f(x_0) \\ g'(f(x_0)), & x = f(x_0). \end{cases}$$

Then, once g is diff. at x_0, then

$\lim_{x \to x_0} F(x) = F(x_0)$ so F is continuous at x_0. Now, for $x \neq x_0$, we have

$$F(x)\left(x - f(x_0)\right) = g(x) - g(f(x_0)).$$

for every $x \in \tilde{D}$. So, replacing x with $f(x)$:

$$F(f(x))\left(f(x) - f(x_0)\right) = g(f(x)) - g(f(x_0)).$$

Now,

$$(g \circ f)'(x_0) = \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0}$$

$$= \lim_{x \to x_0} F(f(x)) \left(\frac{f(x) - f(x_0)}{x - x_0}\right).$$

Since F is continuous & f is diff. at x_0

$$\Rightarrow (g \circ f)'(x_0) = \left(\lim_{x \to x_0} F(f(x))\right)\left(\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}\right)$$

$$= g'(f(x_0))\, f'(x_0)$$

and the proof is completed. $\square$

Thm (Inverse).

Let I be an interval & $f: I \to \mathbb{R}$ be continuous, strictly monotone. Let g be the inverse function of f on I.

Let $c \in I$.

If f is differentiable at c and if $f'(c) \neq 0$, then g is differentiable at $f(c)$ and $g'(f(c)) = 1/f'(c)$.

Proof. Let $J := f(I)$ which is an interval. J is the domain of g. Now, since f is strictly increasing, we have

$$\frac{g(y) - g(f(c))}{y - f(c)} = \frac{g(y) - g(f(c))}{x - c} \cdot \frac{x - c}{y - f(c)}$$

and letting $y = f(x)$, $x \neq c$ ($f(x) \neq c$),

$$\frac{g(f(x)) - g(f(c))}{f(x) - f(c)} = \frac{g(f(x)) - g(f(c))}{x - c} \cdot \frac{x - c}{f(x) - f(c)}.$$

But, $g \circ f(x) = x \Rightarrow \dfrac{g(f(x)) - g(f(c))}{x - c} = 1$

So, $\lim_{x \to c} \frac{g(f(x)) - g(f(c))}{f(x) - f(c)} = \lim_{x \to c} \frac{1}{\frac{f(x) - f(c)}{x-c}} = \frac{1}{f'(c)}$. $\square$

Applications.

① the derivative of $f(x) = \cos x$ is $-\sin x$.

Proof. Write $f(x) = \sin\left(\frac{\pi}{2} - x\right)$. Then by the chain rule,

$$f'(x) = \cos\left(\frac{\pi}{2} - x\right) \cdot (-1)$$

$$= -\sin x. \quad \square$$

② the derivative of $f(x) = x^{1/n}$ is

$$f'(x) = \frac{1}{n}\left(\frac{1}{x^{1-1/n}}\right) \quad x > 0.$$

Proof. $x^{1/n}$ is the inverse of x^n. So,

$$g \circ f(x) = x \quad (g = x^n \ \& \ f = x^{1/n})$$

$$\Rightarrow g'(x^{1/n}) \cdot f'(x) = 1$$

$$\Rightarrow n(x^{1/n})^{n-1} \cdot f'(x) = 1$$

$$\Rightarrow f'(x) = \frac{1}{n} \cdot \left(\frac{1}{x^{1-1/n}}\right). \quad \square$$

5.3. Extremums.

Def. Let $f: D \to \mathbb{R}$ be a function & $c \in D$.

- x_0 is a relative maximum (minimum) if there is $\delta > 0$ s.t. $\forall x \in D \cap (c-\delta, c+\delta)$, $f(x) \le f(x_0)$ (resp. $f(x) \ge f(x_0)$).
- x_0 is an absolute maximum (minimum) if $\forall x \in D$, $f(x) \le f(x_0)$.

Remark. • We know that if $f: [a,b] \to \mathbb{R}$ is continuous, then it has an abs. max & min by the Extreme Value Theorem.

- We call rel. max & min relative extremums.
- Abs. extremums are rel. extremums.

Thm (Fermat). If $f: D \to \mathbb{R}$ is differentiable at some $x_0 \in acc(D) \cap D$ & if f has an rel. ext. at x_0, then $f'(x_0) = 0$.

Proof. If x_0 is a rel. max. (WLOG), then there is a $\delta > 0$ s.t. $\forall x \in (x-\delta, x+\delta) \cap D$

$$f(x) \le f(x_0).$$

Let $x_n \to x_0$ with $x_n \ne x_0$ & $x_n \in D$. By starting the sequence further, we may suppose that $x_n \in (x_0, x_0+\delta)$

We know that

$$\lim_{n\to\infty} \left|\frac{f(x_n)-f(x_0)}{x_n-x_0}\right| = \lim_{n\to\infty} \frac{f(x_0)-f(x_n)}{x_n-x_0} = -f'(x_0).$$

Take another sequence $x_n \in D$, $x_n \to x_0$, $x_n \neq x_0$ and $x_n \in (x_0-\delta, x_0)$. Then

$$\lim_{n\to\infty} \left|\frac{f(x_n)-f(x_0)}{x_n-x_0}\right| = \lim_{n\to\infty} \frac{f(x_0)-f(x_n)}{x_0-x_n} = f'(x_0)$$

$$\Rightarrow \quad f'(x_0) + f'(x_0) = 0$$

$$\Rightarrow \quad f(x_0) = 0. \qquad \square$$

5.4 Mean Value Theorem.

We need a first theorem.

Rolle's Theorem.

Let $f: [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). If $f(a)=f(b)$, then $\exists c \in (a,b)$ s.t. $f'(c)=0$.

Proof. By the extreme value Thm, f has a minimum and maximum on $[a,b]$.

Let c_1 & c_2 in $[a,b]$ where

$\max f[a,b] = f(c_1)$ & $\min f[a,b] = f(c_2)$.

If $c_1 = a$ & $c_2 = a$ or $c_1 = a$ & $c_2 = b$

or $c_1 = b$ & $c_2 = a$ or $c_1 = b$ & $c_2 = b$

then f is constant.

If f is constant, then result is clear

because $f'(x) = 0$ $\forall x \in (a,b)$.

Otherwise, c_1 or c_2 are inside (a,b), orthewise

we would fall in one of the four cases

above.

Name c the point inside (a,b). So,

by the previous Theorem on extreme values,

$$f'(c) = 0. \qquad \square$$

<u>Application</u> The polynomial $f(x) = x^3 + 3x + 1$

has exactly 1 root. By the mean value

Theorem, since $f(-1) = -3 < 1 = f(0)$, there

is some $c \in (-1,0)$ s.t. $f(c) = 0$.

Suppose there are two c_1, c_2 s.t. $f(c_1) = 0 = f(c_2)$.

Then, by Rolle's Thm, $\exists\, d \in (c_1, c_2)$ s.t.

$$f'(d) = 0.$$

However, $f'(x) = 3x^2 + 3 > 0 \quad \forall x \in \mathbb{R}$

We have a contradiction.

<u>Mean-Value Thm</u> (MVT)

If $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b), then $\exists c \in (a,b)$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b-a}.$$

<u>Proof</u>. Define $g : [a,b] \longrightarrow \mathbb{R}$ by

$$g(x) := f(x) - \left(\frac{f(b) - f(a)}{b-a} (x-a) + f(a) \right).$$

Then, g is differentiable on (a,b) & cont. on $[a,b]$. We have

$$g(a) = 0 \quad \& \quad g(b) = 0.$$

So, $g(a) = g(b)$ and by Rolle's Thm, $\exists c \in (a,b)$ s.t. $g'(c) = 0$. But

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b-a}$$

$$\Rightarrow \quad f'(c) = \frac{f(b) - f(a)}{b-a}.$$

$\square$

<u>Application</u> We will show that if $p > 1$, then

$$(1+x)^p > px + 1 \quad \text{for any } x > 0$$

Take $f(y) = (1+y)^p$ where $y \in (0, x)$. From Exercise 6a) (HW5) and the Chain rule, f is diff. for any $y \geq -1$. By the MVT, there is a point $t \in (0, x)$ s.t.

$$f'(t) = \frac{f(x) - f(0)}{x} = \frac{(1+x)^p - 1}{x}$$

But, $f'(t) = p(1+x)^{p-1}$ since $p-1>0$, $(1+x)^{p-1} > 1$

$$\Rightarrow \quad (1+x)^p = px(1+x)^{p-1} + 1 > px + 1 \quad . \quad \text{II}$$

<u>Thm.</u> $f : [a,b] \to \mathbb{R}$ be continuous & diff. on (a,b).

a) $f'(x) > 0 \quad \forall x \in (a,b) \Rightarrow f$ increases on $[a,b]$.

b) $f'(x) < 0 \quad \forall x \in (a,b) \Rightarrow f$ decreases on $[a,b]$.

c) $f'(x) = 0$ on $(a,b) \Rightarrow f$ is constant on $[a,b]$.

<u>Proof.</u> We prove only part a). Suppose that $f'(x) > 0 \quad \forall x \in (a,b)$. Then $x < y$ for some $x, y \in [a,b]$. By the MVT, $\exists c \in (x,y)$ s.t.

$$f'(c) = \frac{f(y) - f(x)}{y - x} .$$

So, $f(y) = f(x) + f'(c)(y-x) > f(x)$. $\square$