1.1. Sequences & convergence.

#4. We have $\frac{3n+7}{n} = 3 + \frac{7}{n}$.

So, $3+7=10$ is an upper bound because

$$\frac{7}{n} \leq 7 \quad \forall n \geq 1$$

$$\Rightarrow \quad 3 + \frac{7}{n} \leq 3+7 = 10 \quad \forall n \geq 1.$$

Also, 3 is a lower bound because

$$\frac{7}{n} \geq 0 \qquad (\forall n \geq 0$$

$$\Rightarrow \quad 3 + \frac{7}{n} \geq 3.$$

#5 An example would be $a_n = (-1)^n$.

#6. (a) Let $\varepsilon > 0$. We have

$$|5 + 1/n - 5| = 1/n.$$

By AP, $\exists N \in \mathbb{N}$ s.t. $N\varepsilon > 1$. So, if $n \geq N$, then

$$1/n \leq \frac{1}{N} < \varepsilon$$

$$\Rightarrow \quad |5 + 1/n - 5| < \varepsilon.$$

Thus, $5 + 1/n \to 5$.

(b) We have $\dfrac{2-2n}{n} = \dfrac{2}{n} - 2$. So choose $A=-2$. Let $\varepsilon>0$. We have that

$$\left|\frac{2}{n} - 2 - (-2)\right| = \frac{2}{n}.$$

By AP, $\exists N \in \mathbb{N}$ s.t. $N\varepsilon > 2$. So, if $n \geq N$, then

$$\frac{2}{n} < \frac{2}{N} < \varepsilon.$$

Thus, $$\frac{2}{n} - 2 \longrightarrow -2.$$

(c) Take $A=0$. Let $\varepsilon>0$. By AP, $\exists N \in \mathbb{N}$ s.t.

$$N\varepsilon > 1 \iff \frac{1}{N} < \varepsilon.$$

By induction, we can prove that

$$n \leq 2^n \quad \forall n \geq 1.$$

So, if $n \geq N$, $2^n \geq n \geq N$ and

$$\frac{1}{2^n} \leq \frac{1}{N} < \varepsilon.$$

Thus, $$|2^{-n} - 0| < \varepsilon \quad \text{whenever } n \geq N.$$

(d) We have $\dfrac{3n}{2n+1} = \dfrac{\frac{3}{2}(2n+1) - \frac{3}{2}}{2n+1} = \dfrac{3}{2} - \dfrac{3/2}{2n+1}$.

Put $A = 3/2$. Let $\varepsilon > 0$. Then

$$\left| \frac{3n}{2n+1} - \frac{3}{2} \right| = \frac{3/2}{2n+1}$$

By AP, select a $N \in \mathbb{N}$ s.t. $N\varepsilon > 3/2$

Let $n \geq N$. Then $2n+1 \geq N$ and

$$\frac{3/2}{2n+1} \leq \frac{3/2}{N} < \varepsilon.$$

Thus, $$\frac{3n}{2n+1} \to \frac{3}{2}.$$

#7. ($\Rightarrow$) Suppose $a_n \to A$. Let $\varepsilon > 0$. Then, $\exists N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow |a_n - A| < \varepsilon$. Take the sequence $b_n = a_n - A$. Then, if $n \geq N$, then

$$|b_n - 0| = |a_n - A| < \varepsilon.$$

So, $b_n \to 0$, in other words, $a_n - A \to 0$.

($\Leftarrow$) Suppose $a_n - A \to 0$. Let $\varepsilon > 0$. Then $\exists N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow |a_n - A - 0| < \varepsilon$. But

$$|a_n - A - 0| = |a_n - A|.$$

So, $n \geq N \Rightarrow |a_n - A| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $a_n \to A$. $\square$

#8. See Question 2 of the Mini-test.

#9. See Exercice 3 of homework 2.

#10 See Exercice 2 of homework 2.

#11 Suppose that $\exists \alpha \in \mathbb{R}$ and $\exists N \in \mathbb{N}$ such that if $n \geq N$, $a_n = \alpha$.

We want to prove that $a_n \to \alpha$, that is

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } n \geq N \Rightarrow |a_n - \alpha| < \varepsilon.$$

Let $\varepsilon > 0$. From the hypothesis, there is a $N \in \mathbb{N}$ s.t. $a_n = \alpha$ if $n \geq N$. Then, for $n \geq N$, $|a_n - \alpha| = |\alpha - \alpha| = 0 < \varepsilon$.

So, since $\varepsilon > 0$ was arbitrary, $a_n \to \alpha$.

1.2 Cauchy sequences.

#21 Let $S := \{2^n + 1/k : n \,\&\, k \text{ positive integers}\}$.

$x \in acc(S) \iff \forall \delta > 0, (x-\delta, x+\delta) \cap S$ has infinitely many points.

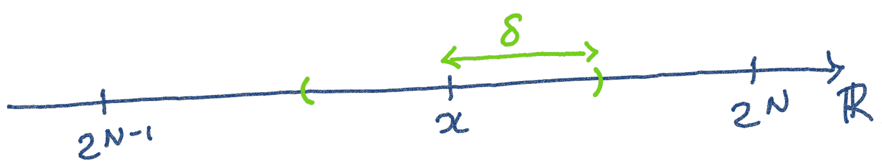
We will show that $acc(S) = \{2^n : n \geq 1\}$.

($\subseteq$) Let $x \notin \{2^n : n \geq 1\}$. The set $\{2^n : n \geq 1\}$

is unbounded. So, there exists a N s.t. $x \leq 2^N$. Since $x \notin \{2^n : n \geq 1\}$ then $x \neq 2^N$ and so $x < 2^N$. By the well-ordering principle of $\mathbb{N}$, select the smallest N s.t. $x < 2^N$. We then have

(*) $$2^{N-1} < x < 2^N.$$

Indeed, if (*) would not be true, then $x < 2^{N-1}$, contradicting the minimality of 2^N. Put $\delta := \min\left\{ \frac{x - 2^{N-1}}{2}, \frac{2^N - x}{2} \right\}$. Here's a picture of the situation:



Then,
$$x + \delta \leq x + \frac{2^N - x}{2} = \frac{x + 2^N}{2} < \frac{2^N + 2^N}{2} = 2^N$$

So, $2^n + \frac{1}{k} \notin (x - \delta, x + \delta)$ $\forall k \geq 1, \forall n \geq N$.

Also, $2^n + 1/k \leq 2^{N-1}$ $\forall n < N-1$ and $\forall k \geq 1$.

So, $\quad 2^n + \frac{1}{k} \notin (x-\delta, x+\delta) \quad \forall n < N-1$
$, \forall k \geq 1$

Finally, by AP, there is a $K \in \mathbb{N}$ s.t.

$$K(x - 2^{N-1} - \delta) > 1$$

So that, after some algebra

$$2^{N-1} + \frac{1}{K} < x - \delta .$$

So, for all $k \geq K$, $\frac{1}{k} \leq \frac{1}{K}$ and so

$$2^{N-1} + \frac{1}{k} < x - \delta .$$

Thus, $(x-\delta, x+\delta) \cap S$ has at most a finite number of elements, which are

$$2^{N-1}+1, \; 2^{N-1}+\frac{1}{2}, \ldots, 2^{N-1}+\frac{1}{K-1} .$$

Thus, x is not an accumulation point of S so $x \notin \mathrm{acc}(S)$. By the contrapositive:

$$x \in \mathrm{acc}(S) \Rightarrow x \in \{2^n : n \geq 1\}.$$

So, $\mathrm{acc}(S) \subseteq \{2^n : n \geq 1\}$.

($\Leftarrow$) Suppose $x \in \{2^n : n \geq 1\}$. Then

$x = 2^N$ for some $N \geq 1$.

Let $\delta > 0$ and by AP, take $K \in \mathbb{N}$ s.t.

$$\frac{1}{K} < \delta.$$

Then, if $k \geq K$,
$$2^N + \frac{1}{k} \leq 2^N + \frac{1}{K}$$
$$< 2^N + \delta$$

and so,

$$2^N + \frac{1}{k} \in (x - \delta, x + \delta) \quad \forall k \geq K.$$

Thus, $(x - \delta, x + \delta)$ contain infinitely many points of S and so $x \in \mathrm{acc}(S)$.

#22. Suppose $S \neq \emptyset$ and S is bounded from above. By the AC, $x := \sup S$ exists.

We want to prove a statement like this:

$$P \to (Q \vee R).$$

To do that, we have to prove that

- $P \wedge (\neg Q) \Rightarrow R$
- $P \wedge (\neg R) \Rightarrow Q$.

First, if S is finite, the result is clearly that $x \in S$. Suppose that S is infinite.

- Suppose that $x \notin S$. We will prove that $x \in acc(S)$. Let $\delta > 0$. Then, $x - \delta$ is not an upper bound for S (otherwise, we would contradict the def. of x). So, there must must some element $y_1 \in S$ s.t.

$$y_1 \neq x \quad \& \quad x - \delta < y_1 < x .$$

Now, y_1 is not an upper bound for S, so $\exists\, y_2 \in S$, $y_2 \neq y_1$ s.t.

$$y_1 < y_2 < x$$

Here, $y_2 \neq y_1$ because, otherwise y_1 would be an upper bound.

Again, y_2 is not an upper bound of S, so $\exists y_3 \in S$, $y_3 \neq y_2$ s.t. $y_2 < y_3 < x$.

Continuing in this process, we get a sequence $(y_n)_{n=1}^{\infty}$ s.t. $x - \delta < y_n < y_{n+1} < x \;\; \forall n$.

Thus, $(x-\delta, x+\delta) \cap S$ contains infinitely many elements. Thus, $x \in acc(S)$.

- Suppose that $x \notin acc(S)$. Let $\delta = 1$. By def. of the supremum, $\exists\, y_1 \in S$ s.t.

$$x - 1 < y_1 \leq x .$$

So, $(x-1, x+1) \cap S$ is not empty. Also, since $x \notin acc(S)$, $(x-1, x+1) \cap S$ has finitely many elements, say

$$\{y_1, y_2, \dots, y_n\} = (x-1, x+1) \cap S .$$

Let y_k, $k \in \{1, 2, \dots, n\}$ be the max of $\{y_1, \dots, y_n\}$. We will show that $y_k = x$.

Suppose $y_k < x$. Then, y_k would be an upper bound for S! (Why?) So, this would contradict the definition of x. Thus $x = y_k \in S$.

$\square$

1.3 Arithmetic operations on sequences.

#25. Let $a_n \to A$ and $a_n + b_n \to C$. By the product rule, we have that

$$-a_n \to -A.$$

Again by the sum rule, we have

$$b_n = a_n + b_n + (-a_n) \to C - A$$

and so (b_n) converges. □

#26 See homework 2.

#27. Suppose $a_n \to A$ and $a_n b_n \to C$ with $A \neq 0$. By the quotient rule, we have

$$b_n = \frac{a_n b_n}{a_n} \to \frac{C}{A}.$$

So, (b_n) converges. □

#28. See homework 2.

#31 See homework 2.

#32 (c) $|\sin n^2| \le 1$ & $\frac{1}{n} \to 0$ so $\frac{1}{\sqrt{n}} \to 0$ (see homework 2). Thus, by a theorem from the lecture notes:

$$\frac{\sin n^2}{\sqrt{n}} \to 0.$$

(f) We have

$$0 \le \frac{\sqrt{n}}{n+7} \le \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}.$$

Thus, by the squeeze theorem,

$$\frac{\sqrt{n}}{n+7} \to 0.$$

Now, since $(-1)^n$ is bounded and $\frac{\sqrt{n}}{n+7} \to 0$, by a theorem from the lecture notes,

$$\frac{(-1)^n \sqrt{n}}{n+7} \to 0.$$

1.4 Subsequences and monotone sequences.

#36. Let $(a_n)_{n=1}^{\infty}$ be a bounded sequence.

- If the range $\{a_n : n \geq 1\}$ is finite, say

$$\{a_n : n \geq 1\} = \{b_1, b_2, \dots, b_j\}$$

then there must be one b_i that repeats infinitely often. Take $(n_k)_{k=1}^{\infty}$ be the sequence of indexes such that

$$a_{n_k} = b_i \quad \forall k \geq 1.$$

Then $(a_{n_k}) = (b_i)_{k=1}^{\infty}$ is a constant sequence which converges to b_i as $k \to \infty$.

- Suppose that $\{a_n : n \geq 1\}$ is infinite.

By the Bolzano-Weierstrass theorem, $\{a_n : n \geq 1\}$ has an accumulation point, say x.

Let $\delta = 1$. Then $(x-1 \quad x+1)$ contains infinitely many points of $\{a_n : n \geq 1\}$.

So, $\exists\, a_{n_1}$ s.t. $|a_{n_1} - x| < 1$.

Let $\delta = 1/2$. Then, also, $\exists\, n_2 > n_1$ s.t.

$$|a_{n_2} - x| < 1/2.$$

Continue this process to construct a subsequence $(a_{n_k})_{k=1}^{\infty}$ s.t.

$$|a_{n_k} - x| < \frac{1}{k} \qquad \forall k \geq 1.$$

Now, we can show that $a_{n_k} \to x$.
Let $\varepsilon > 0$. By AP, $\exists N \in \mathbb{N}$ s.t.

$$\frac{1}{N} < \varepsilon.$$

Take $k \geq N$. Then $\frac{1}{k} \leq \frac{1}{N} < \varepsilon$.
Also, $n_k \geq k \geq N$, so, by construction,

$$|a_{n_k} - x| < \frac{1}{k} \leq \frac{1}{N} < \varepsilon.$$

In other words, $a_{n_k} \to x$. $\square$

<u>#38.</u> Let $c > 1$. By an analogue result for the $\sqrt{\cdot}$, we have $x \leq y \iff \sqrt[n]{x} \leq \sqrt[n]{y}$.
$(n \geq 1)$.

- The sequence $(\sqrt[n]{c})_{n=1}^{\infty}$ is decreasing. Indeed

$$\sqrt[n]{c} > \sqrt[n+1]{c} \iff c^{\frac{n+1}{n}} > c$$

$$\iff \quad c^{1+1/n} > c$$

$$\iff \quad c^{1/n} > 1.$$

$$\iff \quad c > 1^n = 1.$$

The last statement is true, so

$$\sqrt[n]{c} > \sqrt[n+1]{c} \qquad \forall n \geq 1.$$

• The sequence $(\sqrt[n]{c})_{n=1}^{\infty}$ is bounded.

Indeed:

$$\to \sqrt[n]{c} < \sqrt[1]{c} = c \quad \forall n \geq 1$$

$$\to \quad c > 1 \Rightarrow \sqrt[n]{c} > 1 \ \forall n \geq 1.$$

So, it is bounded from below and from above by 1 and c respectively.

Thus, by a theorem from the lecture notes, $\sqrt[n]{c} \longrightarrow A$ for some $A \in \mathbb{R}$.

We know that all subsequence must converge to the same limit, so

$$\sqrt[2n]{c} \longrightarrow A.$$

But $\sqrt[2n]{c} = (\sqrt[n]{c})^{1/2} \longrightarrow \sqrt{A}$.

Then, $A = \sqrt{A} \Rightarrow A^2 = A$

$\Longleftrightarrow A = 1$ or $A = 0$.

But $\sqrt[n]{c} > 1 \quad \forall n \geq 1$, so $A = 0$ is not possible. Thus $A = 1$ and

$$\sqrt[n]{c} \longrightarrow 1 \quad . \quad \square$$

#43. Since $a \leq b$, we have

$$a^n + b^n \leq 2b^n$$

$$\Rightarrow \sqrt[n]{a^n + b^n} \leq \sqrt[n]{2} \cdot b$$

From #38, $\sqrt[n]{2} \leq 2$. So, the sequence

$$\left((a^n + b^n)^{1/n} \right)_{n=1}^{\infty}$$

is bounded (also below by 0).

Now, we have

$$b^n \leq a^n + b^n \leq 2b^n$$

$$\Rightarrow \quad b \leq (a^n + b^n)^{1/n} \leq \sqrt[n]{2}\, b$$

By the squeeze theorem, $(a^n + b^n)^{1/n} \longrightarrow b$. $\square$

#47. Suppose $a_n \to A$. Suppose also that B is an accumulation point of $\{a_n : n \in \mathbb{N}\}$. Suppose that $A \neq B$. Then $|A-B| > 0$.

WLOG, suppose $A > B$ so $|A-B| = A-B$.

By the definition of convergence, $\exists N \in \mathbb{N}$ s.t.

$$n \geq N, \quad |a_n - A| < \frac{A-B}{2}.$$

By definition of an accumulation point, with $\delta = \frac{A-B}{2}$, we have that

$$(B-\delta, B+\delta) \cap \{a_n : n \in \mathbb{N}\}$$

has infinitely many elements.

After some algebra, we have:

$$\frac{A+B}{2} < a_n < \frac{3A-B}{2} \qquad n \geq N.$$

&

$$B+\delta = B + \frac{A-B}{2} = \frac{A+B}{2}.$$

So, the interval would contain only finitely many elements that is $\{a_1, a_2, \dots, a_{N-1}\}$. #.

$\square$