MATH-331 Introduction to Real Analysis — Ian Oga
Homework 04 — Fall 2021

Due date: October 25th 1:20pm — Total: 63/70.

Très bien!

Exercise	1 (5)	2 (5)	3 (5)	4 (5)	5 (10)	6 (10)	7 (5)	8 (5)	9 (5)	10 (10)
Score	5	5	5	5	6	10	5	5	5	7

Table 1: Scores for each exercises

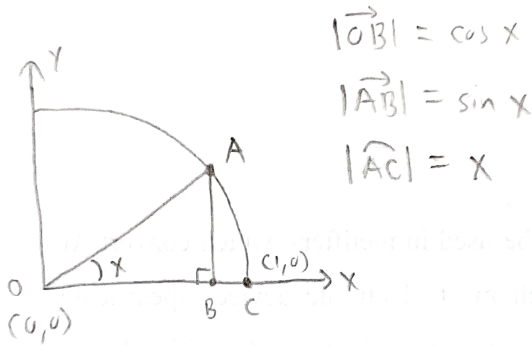
Instructions: You must answer all the questions below and send your solution by email (to `parisepo@hawaii.edu`). If you decide to not use LaTeX to hand out your solutions, please be sure that after you scan your copy, it is clear and readable. Make sure that you attached a copy of the homework assignment to your homework.
If you choose to use LaTeX, you can use the template available on the course website.
No late homework will be accepted. No format other than PDF will be accepted. Name your file as indicated in the syllabus.

1 Writing problems

For each of the following problems, you will be asked to write a clear and detailed proof. You will have the chance to rewrite your solution in your semester project after receiving feedback from me.

Exercise 1. (5 pts) Prove that, if $0 < x < \pi/2$, then $0 \leq \sin x \leq x$ with a geometric argument. [Hint: View $\sin x$ as a point on the unit circle in the first quadrant.]

Solution: Consider the picture below with a quarter circle of radius 1:



The length of $\overline{AB}$ is $\sin(x)$ and the length of arc $\widehat{AC}$ is x. As the shortest distance between two

points is a straight line, $|\overline{AC}| \leq |\overset{\frown}{AC}|$. We now have the following for $0 < x < \frac{\pi}{2}$:
$|\overline{AB}| = \sin(x)$
$|\overline{AB}|^2 = \sin(x)^2$
$|\overline{AB}|^2 \leq \sin(x)^2 + (1 - \cos(x))^2$
$|\overline{AB}| \leq \sqrt{\sin(x)^2 + (1 - \cos(x))^2}$
$|\overline{AB}| \leq \sqrt{|\overline{AB}|^2 + |\overline{BC}|^2}$
$|\overline{AB}| \leq |\overline{AC}|$
$\sin(x) = |\overline{AB}| \leq |\overline{AC}| \leq \overset{\frown}{AC} = x$
$\sin(x) \leq x$
As point A will always be above the x-axis for $0 < x < \frac{\pi}{2}$, we also know that $\sin(x)$ will be positive.
In total, for all $0 < x < \frac{\pi}{2}$, $0 < \sin(x) \leq x$ $\square$

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Exercise 2. (5 pts) Let $f : A \to \mathbb{R}$ and $g : B \to A$ be two functions where $A, B \subset \mathbb{R}$. Let a be an accumulation point of A and b be an accumulation point of B. Suppose that

- $\lim_{t\to b} g(t) = a$.
- there is a $\eta > 0$ such that for any $t \in B \cap (b - \eta, b + \eta)$, $g(t) \neq a$.
- f has a limit at a.

Prove that $f \circ g$ has a limit at b and $\lim_{x\to a} f(x) = \lim_{t\to b} f(g(t))$. [This is the change of variable rule for limits.]

Solution: To show that $f \circ g$ has a limit at b, we must prove that there exists L where for an arbitrary ε, there exists a δ such that for any $t \in B$, $0 < |t - b| < \delta \to |f(g(t)) - L| < \varepsilon$. We will show that L is the limit of f at a

Since f has a limit at a, there exists δ_1 such that for all $x \in A$, $0 < |x - a| < \delta_1 \to |f(x) - L| < \varepsilon$.
Since $\lim\limits_{x\to c} g(t) = a$, there exists δ_2 such that for all $t \in B$, $0 < |t - b| < \delta_2 \to |g(t) - a| < \delta_1$. We then have that:
$t \in (b - \delta_2, b + \delta_2) \to |g(t) - a| < \delta_1$

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Let $\delta = \min(\delta_2, \eta)$. Then for $t \in (b - \delta, b + \delta)$, $g(t) \neq a$ and $0 < |g(t) - a|$. Then:
$t \in (b - \delta, b + \delta) \to 0 < |g(t) - a| < \delta_1$
As $g(t) \in A$:
$0 < |g(t) - a| < \delta_1 \to |f(g(t)) - L| < \varepsilon$
$t \in (b - \delta, b + \delta) \to |f(g(t)) - L| < \varepsilon$ $\square$

Exercise 3. (5 pts) Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and suppose that $f(x) = 0$ for each rational number x in $[a, b]$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

Solution: Suppose towards a contradiction that there exists irrational $x_0 \in [a,b]$ where $f(x_0) = X \neq 0$. Since f is continuous on $[a,b]$, for all ε, there is a δ such that for all $y \in [a,b]$, $|y - x_0| < \delta \to |f(y) - f(x_0)| < \varepsilon$. Let $\varepsilon = |\frac{X}{2}|$. As the rationals are dense in the reals, we can find a rational $y \in [a,b]$ and $y \in (x_0 - \delta, x_0 + \delta)$. Note that $|y - x_0| < \delta$ is satisfied and $f(y) = 0$. We now have the following:
$|f(y) - f(x_0)| < |\frac{X}{2}|$
$|0 - X| < |\frac{X}{2}|$
$|X| < |\frac{X}{2}|$
Since $X \neq 0$, this is a contradiction. Therefore $f(x) = 0$ for all irrational $x \in [a,b]$, and $f(x) = 0$ for all $x \in [a,b]$ $\square$

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Exercise 4. (5 pts) Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and suppose that $f(c) > 0$ for some $c \in [a,b]$. Prove that there exist a number η and an interval $[u,v] \subset [a,b]$ such that $f(x) \geq \eta$ for all $x \in [u,v]$.

Solution: Since f is continuous, there is a $\delta > 0$ such that for all $y \in [a,b]$, $|y - c| < \delta \to |f(y) - f(c)| < \frac{f(c)}{2}$. This is equivalent to the following:
$(-\delta < y - c < \delta) \to (-\frac{f(c)}{2} < f(y) - f(c) < \frac{f(c)}{2})$
$(c - \delta < y < c + \delta) \to (\frac{f(c)}{2} < f(y) < \frac{3f(c)}{2})$
$y \in (c - \delta, c + \delta) \to (\frac{f(c)}{2} < f(y) < \frac{3f(c)}{2})$
$y \in [c - \frac{\delta}{2}, c + \frac{\delta}{2}] \to (\frac{f(c)}{2} < f(y) < \frac{3f(c)}{2})$
In total, we have that for all $y \in [a,b]$ and $y \in [c - \frac{\delta}{2}, c + \frac{\delta}{2}]$, $\frac{f(c)}{2} < f(y) < \frac{3f(c)}{2}$ Since $f(c) > 0$:
$\forall y \in [a,b] \cap [c - \frac{\delta}{2}, c + \frac{\delta}{2}], f(y) > 0$
Defining $\eta = 0$ and $[u,v] = [a,b] \cap [c - \frac{\delta}{2}, c + \frac{\delta}{2}]$ gives us what we need. $\square$

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Exercise 5. (10 pts) Let $f : \mathbb{R} \to \mathbb{R}$ be a function that satisfies $f(x+y) = f(x) + f(y)$ for any real number x and y.

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a) Suppose that f is continuous at some point c. Prove that f is continuous on $\mathbb{R}$.

b) Suppose that f is continuous on $\mathbb{R}$ and that $f(1) = k$. Prove that $f(x) = kx$ for all $x \in \mathbb{R}$. [Hint: start with x integer, then x rational, and finally use Exercise 3.]

Solution:

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a) We know that for functions f and g continuous at x_0, $f + g$ and fg are continuous at x_0. Since the function $f(x) = -1$ is continuous, we also have that $-f$ and $f - g$ are continuous at x_0. Note that $f(\frac{cx}{a})$ is continuous at $\frac{cx}{a} = c$, or $x = a$. I don't know how to continue from here, but I guess the proof would follow a similar structure to part b. → write

$f(x) = f(x - c + c)$
$= f(x-c) + f(c)$
$\Rightarrow f(x) - f(c) = f(x-c)$.

b) Suppose $f(n) = kn$ for integer n. Then:
$f(n+1) = f(n) + f(1)$
$f(n+1) = kn + k$
$f(n+1) = k(n+1)$ (*)
$f(n) = f(n-1) + f(1)$
$f(n-1) = f(n) - f(1)$

$f(n-1) = kn - k$
$f(n-1) = k(n-1)$ (**)
Using (*), (**), and induction, we have $f(x) = kx$ for any integer x. Now suppose $f(an) = af(n)$ for some integer a. Note that this is clearly true for $a = 1$.
$f((a+1)n) = f(an+n)$
$f((a+1)n) = f(an) + f(n)$
$f((a+1)n) = af(n) + f(n)$
$f((a+1)n) = (a+1)f(n)$ (***)
$f(an) = f(an-n) + f(n)$
$f(an-n) = f(an) - f(n)$
$f((a-1)n) = af(n) - f(n)$
$f((a-1)n) = (a-1)f(n)$ (****)
Using (***), (****), and induction, $f(ax) = af(x)$ for any integer a. Now consider $f(\frac{a}{b})$ for integers a, b.
$f(b \cdot \frac{a}{b}) = bf(\frac{a}{b})$
$f(a) = bf(\frac{a}{b})$
$ka = bf(\frac{a}{b})$
$f(\frac{a}{b}) = \frac{a}{b}k$
Therefore $f(x) = kx$ for any rational number x. We can finish the proof following the same procedure as Exercise 3:
Suppose towards a contradiction that there exists irrational $x_0 \in \mathbb{R}$ where $f(x_0) = X \neq kx_0$. Since f is continuous, for all ε, there is a δ such that for all $y \in \mathbb{R}$, $|y - x_0| < \delta \to |f(y) - f(x_0)| < \varepsilon$. Consider 2 cases:
Case 1: Suppose $X > kx_0$. Let $\varepsilon = X - kx_0$. As the rationals are dense in the reals, we can find a rational $y \in \mathbb{R}$ and $y \in (x_0 - \delta, x_0)$. Note that $|y - x_0| < \delta$ is satisfied and $f(y) = ky$. Also note that $ky < kx_o < X$. We now have the following:
$|f(y) - f(x_0)| < X - kx_0$
$|ky - X| < X - kx_0$
$X - ky < X - kx_0$
$ky > kx_0$
This is a contradiction.
Case 2: Suppose $X < kx_0$. Let $\varepsilon = kx_0 - X$. As the rationals are dense in the reals, we can find a rational $y \in \mathbb{R}$ and $y \in (x_0, x_0 + \delta)$. Note that $|y - x_0| < \delta$ is satisfied and $f(y) = ky$. Also note that $ky > kx_o > X$. We now have the following:
$|f(y) - f(x_0)| < kx_0 - X$ 5|5
$|ky - X| < kx_0 - X$
$ky - X < kx_0 - X$
$ky < kx_0$
This is a contradiction. Therefore $f(x) = kx$ for all irrational x, and $f(x) = kx$ for all $x \in \mathbb{R}$. □

2
Homework problems

Answer all the questions below. Make sure to show your work.

Exercise 6. (10pts) For each of the functions below, say if the limit exists or doesn't exist at the given point. Justify your answer (in other words, prove it!)

a) $f(x) = \sin(1/x)$ and $x_0 = 0$.

b) $f(x) = x\sin(1/x)$ adn $x_0 = 0$.

Solution:

a) The limit does not exist. To show this, suppose towards a contradiction that the limit does exist. Then there exists L and δ where:

$$0 < |x| < \delta \to |\sin(\frac{1}{x}) - L| < 0.5$$

Note that for $x = \frac{1}{2k\pi+\frac{\pi}{2}}$ for any integer k, $\sin(\frac{1}{x}) = 1$. As $\lim\limits_{k\to\infty} \frac{1}{2k\pi+\frac{\pi}{2}} = 0$ we know that there exists some k_1 where $0 < |\frac{1}{2k_1\pi+\frac{\pi}{2}}| < \delta$. Let $x_1 = \frac{1}{2k_1\pi+\frac{\pi}{2}}$. Similarly, for $x = \frac{1}{2k\pi-\frac{\pi}{2}}$ for any integer k, $\sin(\frac{1}{x}) = -1$. There then exists some k_2 where $0 < |\frac{1}{2k_2\pi-\frac{\pi}{2}}| < \delta$. Let $x_2 = \frac{1}{2k_2\pi-\frac{\pi}{2}}$. In total, we have defined x_1, x_2 where $0 < |x_1| < \delta$, $0 < |x_2| < \delta$, $\sin(\frac{1}{x_1}) = 1$, and $\sin(\frac{1}{x_2}) = -1$. We then have the following:

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$|\sin(\frac{1}{x_1}) - L| < 0.5$
$|1 - L| < 0.5$
$-0.5 < 1 - L < 0.5$
$0.5 < L < 1.5$ (*)
$|\sin(\frac{1}{x_2}) - L| < 0.5$
$|-1 - L| < 0.5$
$-0.5 < -1 - L < 0.5$
$-1.5 < L < -0.5$ (**)

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L cannot satisfy both (*) and (**). This is a contradiction. Therefore the limit at 0 does not exist.

b) The limit does exist. To prove this, let $L = 0$. We will then prove that for any ε, there exists δ such that:

$$0 < |x| < \delta \to |x\sin(\frac{1}{x})| < \varepsilon$$

Let $\delta = \varepsilon$. Then $0 < |x| < \varepsilon$. As $|\sin(x)| \leq 1$, we have the following:

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$|x\sin(\frac{1}{x})| = |x| \cdot |\sin(\frac{1}{x})|$
$|x\sin(\frac{1}{x})| \leq |x|$
$|x\sin(\frac{1}{x})| < \delta$
$|x\sin(\frac{1}{x})| < \varepsilon$
This is what we wanted to prove. Therefore the limit of $x\sin(\frac{1}{x})$ at 0 is 0. $\square$

Exercise 7. (5 pts) Let $c \in (a, b)$ and let f be a function defined on (a, b) except at c. Suppose that $f(x) > 0$ for any $x \in (a, b)\backslash\{c\}$, that $\lim_{x\to c} f(x)$ exists, and that

$$\lim_{x\to c} f(x) = \lim_{x\to c} \left[(f(x))^2 - f(x) - 3\right].$$

Find the value of $\lim_{x\to c} f(x)$. Explain each step carefully.

Solution: We have proven several properties of limits:
$\lim\limits_{x\to c}(f(x)+g(x)) = \lim\limits_{x\to c} f(x) + \lim\limits_{x\to c} g(x)$ (*)
$\lim\limits_{x\to c}(a \cdot f(x)) = a \cdot \lim\limits_{x\to c} f(x)$ (**)
$\lim\limits_{x\to c}(f(x)^n) = [\lim\limits_{x\to c} f(x)]^n$ (***)

Call $\lim\limits_{x\to c} f(x) = X$ We then have the following:
$X = \lim\limits_{x\to c}[(f(x))^2 - f(x) - 3]$
$X = \lim\limits_{x\to c}(f(x))^2 + \lim\limits_{x\to c}(-f(x)) + \lim\limits_{x\to c}(-3)$ by (*)
$X = \lim\limits_{x\to c}(f(x))^2 - \lim\limits_{x\to c} f(x) - \lim\limits_{x\to c}(3)$ by (**)
$X = [\lim\limits_{x\to c} f(x)]^2 - \lim\limits_{x\to c} f(x) - \lim\limits_{x\to c}(3)$ by (***)
$X = X^2 - X - 3$
$0 = X^2 - 2X - 3$
$0 = (X-3)(X+1)$
$X = -1$ or $X = 3$
$\lim\limits_{x\to c} f(x) = -1$ or $\lim\limits_{x\to c} f(x) = 3$
However, the limit cannot be -1. To prove this, suppose towards a contradiction that the limit is -1. Then there exists $\delta > 0$ such that for all $x \in (a,b)$, $0 < |x-c| < \delta \to |f(x) - (-1)| < 0.5$. Note that:
$|f(x) - (-1)| < 0.5 \to f(x) + 1 < 0.5 \to f(x) < -0.5$
Since $f(x) > 0$ on (a,b), this condition can never be fulfilled for any δ. This is a contradiction, and the limit cannot be -1. Therefore $\lim\limits_{x\to c} f(x) = 3$ □

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Exercise 8. (5 pts) Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} x & , x \in \mathbb{Q} \\ -x & , x \notin \mathbb{Q}. \end{cases}$$

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is discontinuous at any point of $\mathbb{R}\backslash\{0\}$ and continuous at 0.

Solution: To prove that f is discontinuous at a point $x_0 \neq 0$, suppose towards a contradiction that f is continuous at x_0. Then there exists a δ such that for all $x \in \mathbb{R}$

$$|x - x_0| < \delta \to |f(x) - f(x_0)| < |x_0|$$

Consider two cases:
Case 1: Suppose $x_0 \in \mathbb{Q}$. Then $f(x_0) = x_0$. We know that the irrational numbers are dense in $\mathbb{R}$. There then exists $x_1 \notin \mathbb{Q}$ such that if x_0 is positive $x_1 \in (x_0, x_0 + \delta)$ and if x_0 is negative $x_1 \in (x_0 - \delta, x_0)$. In this way, $|x_1 + x_0| > |2x_0|$ and $|x_1 - x_0| < \delta$. Then $f(x_1) = -x_1$ and we have the following:
$|f(x_1) - f(x_0)| = |-x_1 - x_0|$
$|f(x_1) - f(x_0)| = |x_1 + x_0|$
$|f(x_1) - f(x_0)| > |2x_0|$
This contradicts $|x - x_0| < \delta \to |f(x) - f(x_0)| < |x_0|$. Therefore f is not continuous for $x_0 \in \mathbb{Q}$ and $x \neq 0$.

an easier approach is to use sequences.

Case 2: Suppose $x_0 \notin \mathbb{Q}$. Then $f(x_0) = -x_0$. We know that the rational numbers are dense in $\mathbb{R}$. There then exists $x_1 \in \mathbb{Q}$ such that if x_0 is positive $x_1 \in (x_0, x_0 + \delta)$ and if x_0 is negative $x_1 \in (x_0 - \delta, x_0)$. In this way, $|x_1 + x_0| > |2x_0|$ and $|x_1 - x_0| < \delta$. Then $f(x_1) = x_1$ and we have the following:
$|f(x_1) - f(x_0)| = |x_1 + x_0|$
$|f(x_1) - f(x_0)| > |2x_0|$
This contradicts $|x - x_0| < \delta \to |f(x) - f(x_0)| < |x_0|$. Therefore f is not continuous for $x_0 \notin \mathbb{Q}$ and $x \neq 0$. In total, we have proven that for all $x \neq 0$, f is discontinuous.

To prove f is continuous at 0, we must prove that for all $\varepsilon > 0$, there exists δ such that for all $x \in \mathbb{R}$

$$|x| < \delta \to |f(x) - f(0)| < \varepsilon$$

And thus:

$$|x| < \delta \to |f(x)| < \varepsilon$$

Let $\delta = \varepsilon$. Note that for any $x \in \mathbb{R}$, $|f(x)| = |x|$. We are then proving that for any $\varepsilon > 0$, the following is true for all $x \in \mathbb{R}$:

$$|x| < \varepsilon \to |x| < \varepsilon$$

This is guaranteed. Therefore f is continuous at 0. □

Exercise 9. (5 pts) Let $p(x) = x^2 + 2$. Find an interval where p is strictly decreasing and find a formula for its inverse.

Solution: p is strictly decreasing for $x < 0$, or on the interval $(-\infty, 0)$. To show this, we must prove that $p(x) > p(x + \delta)$ for $x, x + \delta < 0$, $\delta > 0$.
$p(x + \delta) = (x + \delta)^2 + 2$
$p(x + \delta) = x^2 + 2x\delta + \delta^2 + 2$
$p(x + \delta) = x^2 + 2 + \delta(2x + \delta)$
$p(x + \delta) = p(x) + \delta(x + x + \delta)$
$p(x + \delta) - p(x) = \delta(x + x + \delta)$
As $x, x + \delta < 0$, $x + (x + \delta) < 0$. Since $\delta > 0$, $\delta(x + x + \delta) < 0$.
$p(x + \delta) - p(x) < 0$
$p(x + \delta) < p(x)$
Therefore p is decreasing on $(-\infty, 0)$. Note that $p : (-\infty, 0) \to (2, \infty)$. The inverse of p on this interval is $p^{-1}(x) = -\sqrt{x - 2}$. To prove this, we must show that for $x \in (2, \infty)$, $p(p^{-1}(x)) = x$ and for $x \in (-\infty, 0)$, $p^{-1}(p(x)) = x$.
$p(p^{-1}(x)) = (-\sqrt{x - 2})^2 + 2$
$p(p^{-1}(x)) = (x - 2) + 2$ for $x > 2$
$p(p^{-1}(x)) = x$ for $x > 2$
$p^{-1}(p(x)) = -\sqrt{(x^2 + 2) - 2}$
$p^{-1}(p(x)) = -\sqrt{x^2}$
$p^{-1}(p(x)) = -|x|$
$p^{-1}(p(x)) = x$ for $x < 0$

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Exercise 10. (10 pts) Let $p(x) = ax^3 + bx^2 + cx + d$ be a polynomial of degree 3 and $a > 0$. Prove that p has at least one real root by following these steps:

a) Prove that $\lim_{x\to\infty} p(x) = \infty$.

b) Prove that $\lim_{x\to-\infty} p(x) = -\infty$.

c) Conclude.

[Hint for a): write your polynomial $p(x) = ax^3 + bx^2 + cx + d$ as $x^3(a + b/x + c/x^2 + d/x^3)$ and use the fact that $\lim_{x\to\infty} 1/x^n = 0$ for every $n \geq 1$.]

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Solution:

a) Note that $\lim\limits_{x\to\infty} \frac{1}{x^n} = 0$ for all $n \geq 1$ and $\lim\limits_{x\to\infty} x^n = \infty$ for all $n \geq 1$. Therefore:

$$\lim_{x\to\infty} p(x) = \lim_{x\to\infty} [x^3(a + \tfrac{b}{x} + \tfrac{c}{x^2} + \tfrac{d}{x^3})]$$
$$\lim_{x\to\infty} p(x) = \lim_{x\to\infty} (x^3) \cdot [\lim_{x\to\infty} (a) + \lim_{x\to\infty} (\tfrac{b}{x} + \tfrac{c}{x^2} + \tfrac{d}{x^3})]$$
$$\lim_{x\to\infty} p(x) = \infty \cdot (a + 0)$$
$$\lim_{x\to\infty} p(x) = \infty$$

≈ okay Prove the fact that $\lim\limits_{x\to\infty} \frac{1}{x^n} = 0$.

b) To find $\lim\limits_{x\to-\infty} p(x)$, we can find $\lim\limits_{x\to\infty} p(-x)$.

$$\lim_{x\to-\infty} p(x) = \lim_{x\to\infty} p(-x)$$
$$\lim_{x\to-\infty} p(x) = \lim_{x\to\infty} [(-x)^3(a + \tfrac{b}{-x} + \tfrac{c}{(-x)^2} + \tfrac{d}{(-x)^3})]$$
$$\lim_{x\to-\infty} p(x) = \lim_{x\to\infty} [-x^3(a - \tfrac{b}{x} + \tfrac{c}{x^2} - \tfrac{d}{x^3})]$$
$$\lim_{x\to-\infty} p(x) = -\lim_{x\to\infty} (x^3) \cdot [\lim_{x\to\infty} a - \lim_{x\to\infty} \tfrac{b}{x} + \lim_{x\to\infty} \tfrac{c}{x^2} - \lim_{x\to\infty} \tfrac{d}{x^3})]$$
$$\lim_{x\to-\infty} p(x) = -\infty \cdot (a - 0 + 0 - 0)$$
$$\lim_{x\to-\infty} p(x) = -\infty$$

good trick.

c) Since $\lim\limits_{x\to\infty} p(x) = \infty$, there exists M_1 such that for all $x > M_1$, $f(x) > 1$. Since $\lim\limits_{x\to-\infty} p(x) = -\infty$, there exists M_2 such that for all $x < M_2$, $f(x) < -1$. Note that $M_2 \leq M_1$ since if $M_1 < M_2$, there would exist some $M_1 < x < M_2$ such that $1 < f(x) < -1$, which is not possible. Now pick x_1 and x_2 so that $x_2 < M_2 \leq M_1 < x_1$. Therefore $f(x_2) < -1$ and $f(x_1) > 1$. By the Intermediate Value Theorem, there then exists $x_0 \in (x_2, x_1)$ such that $f(x_0) = 0$. x_0 is a real root of $p(x)$. □