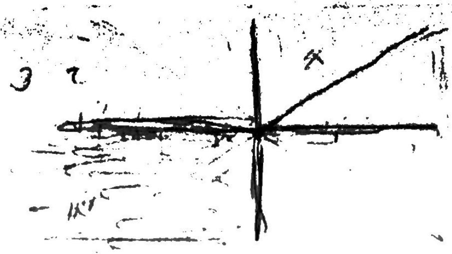


HW #1 Math 331

Ex: 1 Prove that for any $n \in \mathbb{N}$, $1+2+\ldots+n = \frac{n(n+1)}{2}$

- For the base case 1, we observe this statement to be true: $f(1) = \frac{1(1+1)}{2} = \frac{1(2)}{2} = 1$ ✓

So, we can assume this is true for all k such that

$$f(k) = \frac{k(k+1)}{2}$$

Thus we can use induction to prove this is also true for every k+1.

$$f(k+1) = \underbrace{1+2+\ldots+k}+(k+1)$$
$$= \frac{k(k+1)}{2} + k+1 = \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$$
$$= \frac{(k+1)(k+2)}{2} = \frac{(k+1)((k+1)+1)}{2}$$

Substituting an integer 'n' for 'k+1' in the above equation yields $\frac{n(n+1)}{2}$, thus proving the case is true for any $n \in \mathbb{N}$.

Ex 3. Prove that if A, B, and C are sets then:

a.) $A \sim A$ for all elements a_1 in set A_1, there exists an equivalent element in A_2 such that for $a_1 \in A$ and $a_2 \in A_2$ $a_1 - a_2 = 0$
Thus, A is reflexive.

b.) if $A \sim B$, then $B \sim A$
For any $x \in A$ there is also an x so that $x \in B$ because $A \sim B$. So, $x \sim a \in A$ and $a \sim b \in B$, therefore $x \sim b \in B$.
Using the same reasoning, for any $x \in B$ then $x \in A$.
So, $x \sim b \in B$ and $x \sim a \in A$. Therefore $b \sim a$ and $B \sim A$.

c.) If $A \sim B$ and $B \sim C$, then $A \sim C$
Let $a \in A$. Then $a \in B$ because $A \sim B$. Since $b \in B$, $b \in C$, and $a \in B$, this implies that $a \in C$. Therefore every element of

A is also an element of C. Thus, $A \sim C$.

Ex4 Show any subset of a countable set is countable

Suppose $a_1, a_2, a_3 \ldots a_n$ are elements of set A, then because B is a subset of set A $a_n \in B$. Assigning the natural number n to a_n, we say $f(n)$ denotes the number of elements in A belonging to set B. Then, $0 \le f(n) \le n$. Therefore the number of elements in set B is less than or equal to the number of elements in A. Thus, B is countable by the Countability Lemma.

Ex 5. Let $0 < a < b$. Prove that $a^2 < b^2$

a) 1) $a = 0$ Assume b is a positive number greater than a

$a = 0 < b \Rightarrow 0 \cdot b < b \cdot b \Rightarrow 0 < b^2$ therefore $a^2 < b^2$ when $a = 0$.

2.) $a > 0$

$$\begin{aligned} a < b &\Rightarrow a + x = b \\ a^2 < b^2 &\Rightarrow a^2 < (a+x)^2 \\ &\Rightarrow a^2 < a^2 + 2ax + x^2 \\ &\Rightarrow 0 < 2ax + x^2 \checkmark \end{aligned}$$

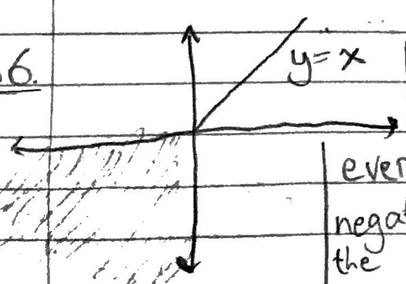
Therefore, when a is greater than 0 $a^2 < b^2$. Thus, proving that $a^2 < b^2$ when $0 < a < b$.

b.) Prove that $\sqrt{a} < \sqrt{b}$

$$\begin{aligned} \text{Let } \quad & 0 < a < b \\ \Rightarrow \quad & 0 < b - a \\ & \sqrt{0} < \sqrt{b} - \sqrt{a} \\ & \sqrt{a} < \sqrt{b} \checkmark \end{aligned}$$

Therefore, when b is greater than a, $\sqrt{b}$ is also greater than $\sqrt{a}$.

Ex6.



The shaded region represents when the points (x,y) satisfy the equation $x + |x| = y + |y|$. This includes every point in the negative region because when (x,y) are negative, both sides of the equation are 0, thus they are equivalent. the line $y = x$ is also included because the equation will be equal when $x + |x|$, or $2x = 2y$. This occurs whenever $y = x$.

2, 3c, 8a,b, 9, 10

$\sqrt{2} + \cancel{z} = 2$

$z = 2 - \sqrt{2}$

$2xy - \sqrt{2}xy$

$3 > x \Leftrightarrow 3 - x > 0$

$n(3-x) > 0$

HW 1 Cont'd

Ex 7. If $x \geq 0$ and $y \geq 0$ prove that $\sqrt{xy} \leq \frac{x+y}{\sqrt{2}}$

since x and $y > 0$, we can substitute $x = u^2$ and $y = v^2$

$\Rightarrow \sqrt{u^2v^2} \leq \frac{u^2+v^2)}{\sqrt{2}} \Rightarrow uv \leq \frac{u^2+v^2}{\sqrt{2}}$

$\sqrt{2}uv \leq u^2 + v^2$

$0 \leq u^2 - \sqrt{2}uv + v^2$ * Substitute x and y back into the equation

$0 \leq x^2 - \sqrt{2}xy + y^2$

$2 - \sqrt{2}xy \leq x^2 + 2xy + y^2$ * add $2-\sqrt{2}(x,y)$ both sides

$xy \leq \frac{x^2+2xy+y^2}{2-\sqrt{2}}$ * Divide both sides by $2-\sqrt{2}$

$xy \leq \frac{(x+y)^2}{2-\sqrt{2}} \Rightarrow \sqrt{xy} \leq \frac{x+y}{\sqrt{2} \frac{4}{\sqrt{2}}}$

8.) Find the infimum and supremum of the following sets

a) $E := \{ x \in \mathbb{R} : x \geq 0$ and $x^2 \leq 9$

$x \geq 0 \quad x^2 \leq 9 \Rightarrow x \leq 3$

$E \neq \emptyset$ because $3 \in E$

If $x > 3$ then $\left.\begin{array}{l} x \cdot x > 3 \cdot x \\ x \cdot 3 > 3 \cdot 3 \end{array}\right\} \Rightarrow x^2 > 3^2 \Rightarrow x^2 > 9$

so, $x^2 > 3 \Rightarrow x \notin E$

so, $x \in E \Rightarrow x \leq 3$ Therefore, 3 is an upper bound for E. By AC the supremum exists.

Suppose $x^2 > 9 \Rightarrow x > 3$ Let $\delta := \min\{3, \frac{p-x^2}{2x+1}$ * $\delta \leq \Leftrightarrow \delta^2 \leq \delta \leq 3$

$(x+\delta)^2 = x^2 + 2x\delta + \delta^2 \geq \underbrace{x^2 - 2x\delta} = x^2 + p - x^2 = p$

So, $(x-\delta)^2 \geq p \geq y^2 \quad \forall y \in E$

$x - \delta$ is an upper bound of E and $x - \delta < x$ ※

This is a contradiction to the proof above which shows that $x-\delta$ is a an upper bound of E. Thus $3 != \sup E$

8a.) Cont'd. Using the previous arguement, we know set E is nonempty and we can prove the existence of an infimum.

$0 \leq x \qquad x^2 \leq 9 \Rightarrow x \leq 3$

$\Rightarrow 0 \leq x \leq 3$

Suppose $0 \leq p \leq 3$ then

$\frac{1}{p} \geq 3$ s.t. $x^2 = \frac{1}{p}$ So, $p = \frac{1}{x^2} = \left(\frac{1}{x}\right)^2$ thus $\frac{1}{x} \geq 0$

Therefore the infimum of E exists and $0 := \inf E$

8 b.) $E := \left\{ \frac{4n+5}{n+1} : n \in \mathbb{N} \right\} \qquad \overset{\to \infty}{4} \leq \overset{\to 1}{\frac{4n+5}{n+1}} \leq 4.5/\frac{9}{2} \quad \forall n \geq 1$ (assuming $0 \notin \mathbb{N}$)

1) if $n = 1$, then $\frac{4n+5}{n+1} = \frac{9}{2} \Rightarrow \frac{9}{2} \in E \Rightarrow \frac{9}{2} \in E \Rightarrow \sup E = \frac{9}{2}$

Suppose $x < \frac{9}{2} \Rightarrow 9 - 2x > 0$

Using AP use $9-2x$ (for x) ; use $x-5$ (for y)

$$\boxed{\begin{aligned} \frac{4n+5}{n+1} > x &\Leftrightarrow 4n+5 > nx + x \\ &\Leftrightarrow 4n - nx > x - 5 \\ &\quad n(4-x) > \boxed{x-5} \end{aligned}}$$

$\exists n \in \mathbb{N}$ s.t. $n(9-2x) > x - 5 \Rightarrow \frac{4n+5}{x-5} > x$

This is a contradiction because $\frac{9}{2}$ is the supremum.

9.) $A \notin \emptyset$; $P(A) :=$ "powerset A"

pt. 1 * The set A is defined as a "nonempty" set. The powerset A contains the empty set along with all subsets of A, therefore $A \notin \emptyset \neq P(A) := \{\emptyset, a_1, a_2 \ldots a_n : a_n \in P(A)\}$.

Additionally, suppose we define $C := \{x : x \in A \text{ and } x \notin f(x)\}$

To show that C (which represents $P(A)$) is not equal to set A, we can consider the elements $a \in A$. Based on the definition of C, $a \in C$ iff $a \notin f(a)$. Therefore $C \cap \{a\} \neq a \cap \{a\}$, so, C is not equal to the set A.

2 * The set of all natural numbers $\mathbb{N}$ is countalbly infinite, so, there is a one-one correspondence from $\mathbb{N}$ to every element in the set $\mathbb{N}$. Assuming $P(\mathbb{N})$ is also countalble, we can assign every subset $x \in \mathbb{N}$ to a natural number $n \in \mathbb{N}$.

$\to$

To show these two sets $\mathbb{N}$ and $P(\mathbb{N})$ are equinumerous an attempt to pair their elements is shown below.

$$\mathbb{N}\left\{\begin{array}{ccc} 1 & \leftrightarrow & \{1,7\} \\ 2 & \leftrightarrow & \{5,6\} \\ 3 & \leftrightarrow & \{4,2\} \\ 4 & \leftrightarrow & \{1,3\} \\ \vdots & & \vdots \end{array}\right\} P(\mathbb{N})$$

So, some elements in $\mathbb{N}$ are paired with subsets "selfish" containg the same number. (Ex 1) The set $P(\mathbb{N})$ contains this set of "selfish" sets as an element as well. Thus, pairing this "selfish" element to a natural number is a contradiction. If the set of selfish elements "A" is paired with some natural number "a" where $a \in A$, the definition of A is contradicted. Therefore, no element of $P(\mathbb{N})$ which maps to $\mathbb{N}$ can exist. Thus $P(\mathbb{N})$ is uncountable.

10.) Let $E \subseteq \mathbb{R}$ be bounded from above and $E \neq \emptyset$ for $r \in \mathbb{R}$, let $rE := \{rx : x \in E\}$ and $r+E := \{r+x : x \in E$

a.) Show if $r > 0$, then $\sup(rE) = r\sup(E)$

Let $E := \{x : x \in E\}$, so $rE := \{rx : x \in E\}$ is the image of $\{E\}$ after multiplying every image by r.

So, $r(x) : x \in E = rx : x \in E$. Therefore $A = \sup(E)$ so, $B = rA = \sup(rE)$. The set rE contains an integer r, such that $E \neq \emptyset$, and has the upper bound B so, $\sup(E)$ exists. Suppose by contradiction that $\sup(rE) < B$. For an element $y \in rE$ there exists an element $x \in E$ such that $rx = y < A$ and $x = \frac{y}{r} < A$. So, there exists another value w such that $\frac{y}{r} < w < A$ which maps to rE, but is greater than B. This contradicts that $B = \sup(rE)$, so, $B = rA = \sup(rE)$.

b. $\rightarrow$

10 b.) For any $r \in \mathbb{R}$, $\sup(r+E) = r + \sup E$

* Let $x = r+E$ then $x = r+y$ $\forall y \in E$. By the definition of a supremum, $y \leq \sup E$, so $x \leq r + \sup E$. Since $x \in r+E$ was arbitrary, $r + \sup E$ is an upper bound for $r+E$. So, $\sup(r+E) \leq r + \sup E$.
* Let $y \in E$, then $r+y \in r+E$. So, since a supremum is an upper bound, $r+y \leq \sup(r+E)$ and therefore $y \leq \sup(r+E) - r$. So, $\sup(r+E) - r$ is an upper bound on E, and $\sup E \leq \sup(r+E) - r$ therefore $\sup(r+E) \geq r + \sup E$.

Using this in addition to the above arguement shows definitively that $\sup(r+E) = r + \sup(E)$

2.) $\mathbb{N} \to \mathbb{N}$ $\quad f(1)=1,\ f(2)=2,\ f(3)=3$ $\quad$ 1, 2, 3, 6,

$f(n) := f(n-1) + f(n-2) + f(n-3)$

<u>induction</u>

$f(1) = 1 \leq 2^{1-1} = 1$ ✓

$f(n+1) := f(n) + f(n-1) + f(n-2) \leq 2^n$