B.1 Conditional Probabilities

Problem 1.

a) $P(A|B) = \frac{P(A\cap B)}{P(B)} = 0.1/0.3 = 1/3$.

b) We have

$$P(A|A\cup B) = \frac{P(A\cap(A\cup B))}{P(A\cup B)} = \frac{P(A)}{P(A\cup B)}.$$

But, we have

$$P(A\cup B) = P(A) + P(B) - P(A\cap B) = 0.5 + 0.3 - 0.1 = 0.7.$$

Therefore, $P(A|A\cup B) = 0.5/0.7 = 5/7$.

c) We have

$$P(A\cap B|A\cup B) = \frac{P((A\cap B)\cap(A\cup B)}{P(A\cup B)}.$$

But, $A\cap B \subset A\cup B$, so $(A\cap B)\cap(A\cup B) = A\cap B$. Therefore,

$$P(A\cap B|A\cup B) = \frac{0.1}{0.7} = \frac{1}{7}. \qquad \triangle$$

Problem 2. The sample space is $S = \{\text{⚀}, \text{⚁}, \text{⚂}, \text{⚃}, \text{⚄}, \text{⚅}\}$ and each single outcome are equally likely. So, $P(A) = 1/6$ for each atomic event A. Let A be the event "dice lands on a 1" and B the event "dice lands on a odd number". Then we have

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P(\{\text{⚀}\})}{P(\{\text{⚀}, \text{⚂}, \text{⚄}\})} = \frac{1/6}{1/2} = \frac{1}{3}. \qquad \triangle$$

Problem 3. The sample space S is all pairs of rolled dice. Every outcome is equally likely, so $P(S) = 1/36$, for every atomic event A.

Let A denote the event "at least one die lands on 6" and let B denote the event "both dice landed on different numbers". We have $P(A) = 11/36$ because $|A| = 11$, $P(B) = 30/36$ because $|B| = 30$ and $P(A\cap B) = 10/36$ because $A\cap B$ is all pairs containing a six except the pair $(\text{⚅}, \text{⚅})$. Therefore,

$$P(A|B) = \frac{10/36}{30/36} = \frac{1}{3}.$$

We see that $P(A|B) > P(A)$, which means knowing B makes A more likely to happen. $\triangle$

Problem 4. Assume that $P(A) < P(A|B)$. By definition of the conditional probability, we have

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$

According to Corollary 1 in the lecture notes, we have $P(B \cap A) = P(A \cap B) = P(B)P(A|B)$. Therefore, we obtain a new expression for $P(B|A)$:

$$P(B|A) = \frac{P(B)P(A|B)}{P(A)} \tag{1}$$

Now, $P(A) < P(A|B)$ implies that

$$\frac{P(B)P(A|B)}{P(A)} > \frac{P(B)P(A)}{P(A)} = P(B). \tag{2}$$

Therefore, we obtain $P(B|A) > P(B)$, or $P(B) < P(B|A)$. △

Problem 5. Assume that $A \subset B$ and $P(A) > 0$, $P(B) > 0$. Since $A \subset B$, we have $A \cap B = A$ and therefore

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(A)}{P(A)} = 1.$$

Also, we have

$$P(A|B) = \frac{P(A \cap B)}{P(A)} = \frac{P(A)}{P(B)}.$$ △

Problem 6. Assume that A and B are mutually exclusive events with $P(B) > 0$. Then we have

$$P(A|A \cup B) = \frac{P(A \cap (A \cup B))}{P(A \cup B)}.$$

We have $A \cap (A \cup B) = (A \cap A) \cup (A \cap B) = A \cup (A \cap B)$. Since A and B are mutually exclusive, we know that $A \cap B = \varnothing$ and therefore

$$A \cap (A \cup B) = A \cup \varnothing = A.$$

Plugging that into the equation for $P(A|A \cup B)$, we obtain

$$P(A|A \cup B) = \frac{P(A)}{P(A \cup B)}. \tag{3}$$

Also, $P(A \cup B) = P(A) + P(B)$ because A and B are mutually exclusive. Replacing in the last equation, we see that

$$P(A|A \cup B) = \frac{P(A)}{P(A) + P(B)}.$$ △

B.2 Bayes' Formula

PROBLEM 7. Let A denotes the event "a person has the desease" and let E be the event "the test detects the desease". From the information in the problem, we have

$$P(E|A) = 0.95, \quad P(E|\overline{A}) = 0.01 \quad \text{and} \quad P(A) = 0.005.$$

We are searching for $P(A|E)$. We have

$$P(A|E) = \frac{P(A \cap E)}{P(E)} = \frac{P(E|A)P(A)}{P(E)}.$$

To find $P(E)$, we use Bayes' formula:

$$P(E) = P(E|A)P(A) + P(E|\overline{A})P(\overline{A}) = 0.95 \cdot 0.005 + 0.01 \cdot 0.995 = 0.0147.$$

Therefore, we get

$$P(A|E) = \frac{0.95 \cdot 0.005}{0.0147} \approx 0.3231. \qquad \triangle$$

PROBLEM 8. Let I denotes the event "A voter is independent", L denotes the event "A voter is liberal", and C denotes the event "A voter is conservative". We have $P(I) = 0.46$, $P(L) = 0.30$, and $P(C) = 0.24$.

a) Let B denotes the event "A voter went voting at the local at the local election". We have

$$P(I|B) = \frac{P(I \cap B)}{P(B)} = \frac{P(I)P(B|I)}{P(B)}.$$

From the information in the problem, we have $P(B|I) = 0.35$, $P(B|L) = 0.62$, $P(B|C) = 0.58$. From Bayes' formula with three events, we have

$$\begin{aligned} P(B) &= P(B|I)P(I) + P(B|L)P(L) + P(B|C)P(C) \\ &= 0.35 \cdot 0.46 + 0.62 \cdot 0.30 + 0.58 \cdot 0.24 \\ &= 0.4862 \end{aligned}$$

and

$$P(I|B) = \frac{0.46 \cdot 0.35}{0.4862} \approx 0.3311.$$

b) We have

$$P(L|B) = \frac{P(L \cap B)}{P(B)} = \frac{P(L)P(B|L)}{P(B)} = \frac{0.30 \cdot 0.62}{0.4862} \approx 0.3826.$$

c) We have

$$P(C|B) = \frac{P(C \cap B)}{P(B)} = \frac{P(B)P(B|C)}{P(B)} = \frac{0.24 \cdot 0.58}{0.4862} \approx 0.2863.$$

[Notice that $P(I|B)+P(L|B)+P(C|B) = 1$ (this comfirms that the mapping $Q(A) = P(A|B)$ is a probability measure.] $\triangle$

Problem 9. Let X be the event "the die x is tossed" and let Y be the event "the die y is tossed". We have $P(X) = P(Y) = 1/2$ because the dice are chosen randomly.

Let A be the event "The die tossed was a ⚁". We have $P(A|X) = 1/2$ from the hypothesis and $P(A|Y) = 1/10$ because there is $1/2$ chance that it lands on ⚂, so $1/10$ chance it lands on any other outcomes. We are looking for $P(X|A)$. We have

$$P(X|A) = \frac{P(X \cap A)}{P(A)} = \frac{P(X)P(A|X)}{P(A)}.$$

We use Bayes' formula to find that

$$P(A) = P(X)P(A|X) + P(Y)P(A|Y) = 0.5 \cdot 0.5 + 0.5 \cdot 0.1 = 0.3.$$

Therefore,

$$P(X|A) = \frac{0.5 \cdot 0.5}{0.3} \approx 0.8333.$$

△

Problem 10. Let $(S, \mathcal{A}, P)$ be a probability space. If A, B are events, then show that

$$\frac{P(A|B)}{P(\overline{A}|B)} = \frac{P(A)}{P(\overline{A})} \frac{P(B|A)}{P(B|\overline{A})}.$$

We have $P(A|B) = \frac{P(A\cap B)}{P(B)}$ and $P(\overline{A}|B) = \frac{P(\overline{A}\cap B)}{P(B)}$. Therefore,

$$\frac{P(A|B)}{P(\overline{A}|B)} = \frac{P(A \cap B)}{P(\overline{A} \cap B)}.$$

We also have $P(A \cap B) = P(A)P(B|A)$ and $P(\overline{A} \cap B) = P(\overline{A})P(B|\overline{A})$. Replacing this into the last equation of the quotient, we obtain

$$\frac{P(A|B)}{P(\overline{A}|B)} = \frac{P(A)P(B|A)}{P(\overline{A})P(B|\overline{A})}.$$

△

B.3 Independent Events

Problem 11. The sample space is given by the different rankings of the brands:

$$S = \{xyz, xzy, yxz, yzx, zxy, zyx\}$$

where, for example, xyz means brand x is the best and brand z is the worst. For atomic event, the probability to occur is $1/6$.

a) We have $A = \{xyz, xzy, zxy\}$, $B = \{xyz, xzy\}$, and $A \cap B = \{xyz, xzy\}$. Therefore,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/3}{1/3} = 1 \neq \frac{1}{2} = P(A).$$

We get $P(A|B) \neq P(A)$ and the events A and B are dependent. Notice that $B \subset A$ and this is why $P(A|B) = 1$.

b) We have $C = \{yxz, zxy\}$ and so $P(C) = 1/3$. We have $A \cap C = \{zxy\}$ and $P(A \cap C) = \frac{1}{6}$. Since $P(A \cap C) = P(A)P(C)$, the event A and C are independent.

c) We have $D = \{yzx, zyx\}$ and so $P(D) = \frac{1}{3}$. We have $A \cap D = \varnothing$ and so $P(A \cap D) = 0$. Since $P(A \cap D) \neq P(A)P(D)$, the event A and D are dependent. △

PROBLEM 12. We have $P(A_1) = 1/4$, because there are four suites in a regular deck of 52 cards. However, given A_1, we have that $P(A_2|A_1) = 12/51 = 4/17$ because there are 12 spades left and 51 cards left in total. Also, we have $P(A_2|\overline{A}_1) = \frac{13}{51}$ because there are 13 spades left if we now that the first card dealt was not a spade and there are 51 cards left in total. Therefore,

$$P(A_2) = P(A_1)P(A_2|A_1) + P(\overline{A}_1)P(A_2|\overline{A}_1) = \left(\frac{1}{4}\right)\left(\frac{4}{17}\right) + \left(\frac{3}{4}\right)\left(\frac{13}{51}\right) = 0.25.$$

Therefore, we see that $P(A_2) \neq P(A_2|A_1)$. This means A_1 and A_2 are not independent. △

PROBLEM 13. Let A_i be the event "The component i is functional". From the assumptions, A_1, A_2, A_3, A_4, and A_5 are independent events. From Problem 14, we also know that $\overline{A}_1$, $\overline{A}_2$, $\overline{A}_3$, $\overline{A}_4$, and $\overline{A}_5$ are independent events. Let A be the event "The system functions". Then $A = \cup_{i=1}^5 A_i$. It is easier to compute $P(\overline{A})$ because of the independence. We have $\overline{A} = \cap_{i=1}^5 \overline{A}_i$ by de Morgan's law. Therefore, by independence, we have

$$\begin{aligned} P(\overline{A}) &= P(\overline{A}_1)P(\overline{A}_2)P(\overline{A}_3)P(\overline{A}_4)P(\overline{A}_5) \\ &= (1-p_1)(1-p_2)(1-p_3)(1-p_4)(1-p_5). \end{aligned}$$

△

PROBLEM 14. Let A and B be independent events.

a) We want to show that $P(A|\overline{B}) = P(A)$, so that A and $\overline{B}$ are independent. From the definition of conditional probabilities, we have

$$P(A|\overline{B}) = \frac{P(A \cap \overline{B})}{P(\overline{B})}.$$

But, we know from Chapter A that $P(A \cap \overline{B}) = P(A) - P(A \cap B)$. Therefore,

$$P(A|\overline{B}) = \frac{P(A) - P(A \cap B)}{P(\overline{B})}.$$

But A and B are independent, which means $P(A \cap B) = P(A)P(B)$ and plugging this in the last equation gives

$$P(A|\overline{B}) = \frac{P(A)(1 - P(B))}{\overline{B}}.$$

Using the fact that $P(\overline{B}) = 1 - P(B)$,

$$P(A)P(\overline{B}) = \frac{P(A)P(\overline{B})}{P(\overline{B})}$$

which simplifies to

$$P(A|\overline{B}) = P(A)$$

since $P(\overline{B}) > 0$. Therefore, A and $\overline{B}$ are independent.

b) We want to show that $\overline{A}$ and $\overline{B}$ are independent. From Part a), we know that A and $\overline{B}$ are independent. Therefore, using Part a) with the event $\overline{B}$ in place of A and the event A in place of B, we deduce $\overline{B}$ and $\overline{A}$ are independent. This is what we wanted to prove. △