MATH 471 — PROBLEMS SET — FALL 2023

Chapter A

A.1 Sample Space

PROBLEM 1. Write down the sample space for each of the following experiments and write if it is finite, discrete, or continuous.

a) Number of committees of 2 people taken from a group of 3 people.

b) Number of people at the beach every day.

c) The magnitude of the wind speed on a given day.

A.2 Event Space

PROBLEM 2. Suppose three 2-sided fair coins are flipped.

a) Describe the sample space S of this experiment.

b) Let $A = \{\text{the first two tosses are head}\}$. Express A in terms of atomic events.

c) Let $B = \{\text{ the last two tosses are head}\}$. Express A in terms of atomic events.

d) Find $A \cap B$ and interpret it.

PROBLEM 3. Let $S = \{$⚀, ⚁, ⚂, ⚃, ⚄, ⚅$\}$ be the sample space from the experiment of tossing a 6-faced die. Construct an event space with exactly 4 events. Can you construct an event space with 6 events?

PROBLEM 4. Let S be the sample space and let $\mathcal{A}$ be an event space for S.

a) If A, B and C are events, show that $A \cup B \cup C$ is an event.

b) If A and B are events, then show that $A \cap B$ is an event. *[Hint: Use de Morgan's laws to rewrite $A \cap B$.]*

A.3 Axioms of Probability

PROBLEM 5. An unfair coin is tossed two times. So $S = \{(h,h),(h,t),(t,t),(t,h)\}$ and assume $\mathcal{A}$ is all the subsets of S. Assume that a probability measure P is defined by

$$P(\{(h,h)\}) = \frac{1}{9}, P(\{(h,t)\}) = P(\{(t,h)\} = \frac{2}{9}, P(\{(t,t)\}) = \frac{4}{9}.$$

a) Let $A = \{\text{The result of the first toss is tail}\}$. Find $P(A)$.

b) Let $A = \{\text{At least one of the tosses is tail}\}$. Find $P(A)$.

PROBLEM 6. Let $(S, \mathcal{A}, P)$ be a probability space. Show the following assertions:

a) If A, B, and C are events such that $A \cap B = A \cap C = B \cap C = \varnothing$, then $P(A \cup B \cup C) = P(A) + P(B) + P(C)$.

b) If A and B are events with $A \subset B$, then $P(A) \leq P(B)$.

Problem 7. Let $(S, \mathcal{A}, P)$ be a probability space. If A and B are two events, then show that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Problem 8. Let S be a non-empty set and let A be a non-empty subset of S such that A is not all of S. If $\mathcal{A} = \{\varnothing, A, \overline{A}, S\}$, then show that all probability measures on $\mathcal{A}$ have the form

$$\begin{aligned} P(\varnothing) &= 0 & \qquad P(A) &= p \\ P(\overline{A}) &= 1 - p & \qquad P(S) &= 1 \end{aligned}$$

for some number p satisfying $0 \leq p \leq 1$.

Problem 9. Let $S = \{s_1, s_2, \ldots, s_N\}$ be a sample space with exactly N outcomes, and let $\mathcal{A}$ be the family of all subsets of S. Show that the function $P : \mathcal{A} \to \mathbb{R}$ defined by

$$P(A) = \frac{|A|}{N} \quad (A \text{ is an event})$$

is a probability measure.

A.4 Computing Probabilities in the Finite Case

Problem 10. If two dice are rolled, what is the probability that the sum of the upturned faces will equal 7?

Problem 11. If 3 balls are "randomly drawn" from a bowl containing 6 orange balls and 5 blue balls, what is the probability that one of the drawn balls is orange and the other two blue?

Problem 12. A boxcar contains six complex electronic systems. Two of the six are to be randomly selected for thorough testing and then classified as defective or not defective. Two of the six systems are defective. Find the probability that one of the two systems selected will be defective.

A.5 Probability Space For Infinite Sample Spaces

Problem 13. Suppose we toss a fair coin infinitely many times. Let B_i denote the event "the i-th toss lands heads". Interpret the event $B = \cup_{i=1}^{\infty} B_i$ and find $P(B)$. *[Hint: Use the continuity of probability measures.]*

Problem 14. Let B_1, B_2, ... be the list of events defined in the proof of Theorem 1. Show that

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i.$$