Appendix S: Set Theory

S.1 Terminology

A set is presented with a list of its elements surrounded by curly brackets. For instance, $\{1, 2, 3, 4\}$ is a way to present a set. A set may be presented as a family of objects satisfying a certain statement $P(n)$. For instance, $\{n : n \text{ is an odd integer}\}$ would describe the set of odd integers $\{\ldots, -3, -1, 1, 3, \ldots\}$.

Uppercase letters are usually used to denote sets and lowercase letters are used to denote an arbitrary element of a set. For example, $A = \{1, 2, 3, 4\}$. The notation $a \in A$ is used to mean "the element a belongs to the set A". The negation of $a \in A$ will be denoted by $x \notin A$. This means the element a does not belong to the set A.

A reference set or universal set is a set U containing all elements under consideration. For example, a sample space S would be a universal set for an experiment. All the definitions below is based on the assumption of the existence of a universal set U.

Definition S.1 The null set is the set $\varnothing$ containing no element.

Definition S.2 If A and B are two sets, then A is a subset of B if all the elements of the set A belongs to the set B. We denote this by $A \subset B$. The family of all subsets, called the power set, of a set A is denoted by 2^A.

Note: To prove that a set A is a subset of B, the statement "$\forall x \in A, x \in B$" should hold. In other words, we have to prove that the statement "$x \in A \Rightarrow x \in B$.

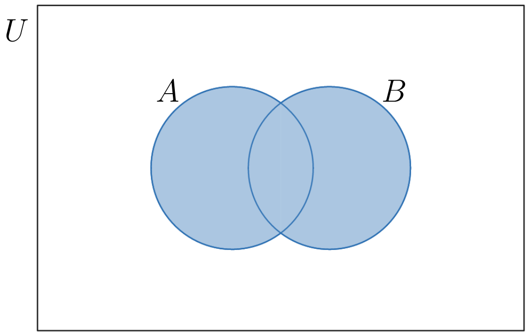
Definition S.3 If A and B are two sets, then we say that A is equal to B, denoted by $A = B$, if $A \subset B$ and $B \subset A$.

In other words, all the elements of A are the same as the elements of B.

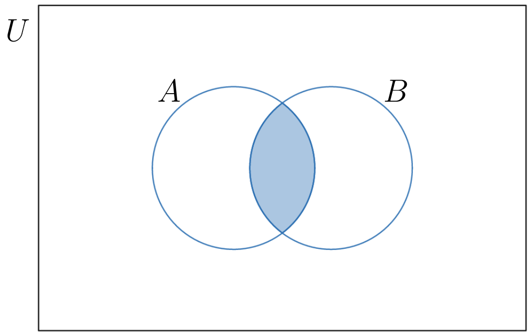
Definition S.4 If a set A has a finite number of elements, then $\#A$ denotes the number of elements in A.

S.2 Operations With Sets

Definition S.5 a) The set $A \cup B$ is the union of A and B. It is the set of elements from A and from B. Note: $A \cup S = S$.
b) The set $A \cap B$ is the set of all elements that are common to A and B. It is called the intersection of A and B. Note: $A \cap S = A$.



(a) Union

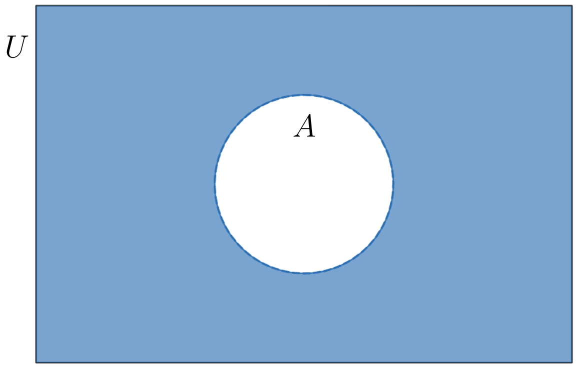


(b) Intersection

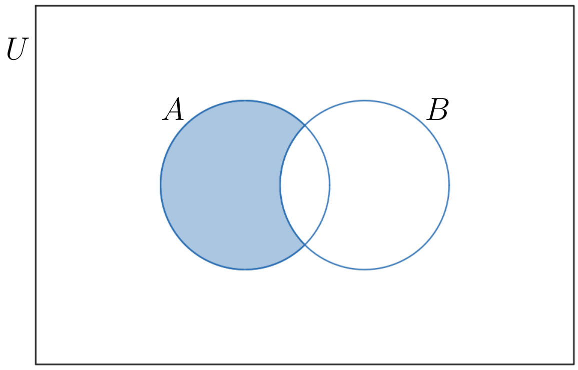
Notes:

- Using the mathematical language from Appendix L, we can rewrite the union of A and B as followed: $A \cup B = \{x \in U \,:\, x \in A \text{ or } x \in B\}$.
- Similarly, we can rewrite the intersection of A and B as followed: $A \cap B = \{x \in U \,:\, x \in A \text{ and } x \in B\}$.

DEFINITION S.6 a) The complement of a set is the new set $\overline{A}$ of all elements in U that are not in A. Note that $A \cup \overline{A} = U$.
b) The set difference of two sets A and B is the set $A \cap \overline{B}$. In other words, it is the set of elements that are in A, but not in B.



(a) Complement



(b) Difference

Notes:

- We can rewrite the complement of a set A as followed: $\overline{A} = \{x \in U \,:\, x \notin A\}$.
- We can rewrite the set difference of A and B as followed: $A \cap \overline{B} = \{x \in U \,:\, x \in A \text{ and } x \notin B\}$.

DEFINITION S.7 Two sets, A and B, are disjoint or mutually exclusive if $A \cap B = \varnothing$.

EXAMPLE S.1 Let $U = \{1, 2, 3, 4, 5\}$. The sets $A = \{1, 2\}$ and $B = \{3\}$ are mutually exclusive because they have nothing in common, meaning $A \cap B = \varnothing$.

S.3 Important Laws for Set Algebra

Theorem S.1

a) Commutative laws:

$$A \cup B = B \cup A \quad \text{and} \quad A \cap B = B \cap A.$$

b) The distributive laws:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \tag{S.1}$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C). \tag{S.2}$$

Proof. We will prove one of the commutative laws and one of the distributive laws. The other formulas are left as an exercise.

a) To prove the equation $A \cup B = B \cup A$, we have to show (i)$A \cup B \subset B \cup A$ and; (ii)$B \cup A \subset A \cup B$.

   (i) Assume $x \in A \cup B$. Then, $x \in A$ or $x \in B$ by definition of the union of sets. But, this is the same thing as writing $x \in B$ or $x \in A$ (the order of the presentation is unimportant). Therefore, the element x belongs to $B \cup A$ by definition of the union of B and A.

   (ii) Now, assume $x \in B \cup A$. Then $x \in B$ or $x \in A$ from the definition of $B \cup A$. Since the order of the presentation is unimportant, $x \in A$ or $x \in B$. Therefore, $x \in A \cup B$ by definition of the union of A and B.

   Let's wrap this up. We just proved that $A \cup B \subset B \cup A$ and that $B \cup A \subset A \cup B$. From the definition of equality of sets, this means $A \cup B = B \cup A$.

b) We prove the equation $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

   (i) We start by proving that $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$. Assume $x \in A \cap (B \cup C)$. Then, by definition of the intersection of two sets, this means $x \in A$ and $x \in B \cup C$. By definition of the union of two sets, $x \in B \cup C$ implies that $x \in B$ or $x \in C$. Since x belongs to A in both cases, then if x belongs to B, we conclude that x belongs to $A \cap B$, but if x belongs to C, we conclude that x belongs to $A \cap C$. Therefore, x belongs to $A \cap B$ or to $A \cap C$. By definition of the union, the element x belongs to $(A \cap B) \cup (A \cap C)$.

   (ii) Now we prove that $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$. Assume $x \in (A \cap B) \cup (A \cap C)$. Then, this means that x belongs to $A \cap B$ or to $A \cap C$. The fact $x \in A \cap B$ implies that $x \in A$ and $x \in B$. The second fact that $x \in A \cap C$ implies that $x \in A$ and $x \in C$. Since x belongs to A in both cases, we see that $x \in A$ and x belongs to B or to C. In other words, the element x belongs to A and to $B \cup C$, which means that $x \in A \cap (B \cup C)$.

   From (i) and (ii), we can then conclude that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

□

Theorem S.2 De Morgan's laws:

$$\overline{A \cap B} = \overline{A} \cup \overline{B} \quad \text{and} \quad \overline{A \cup B} = \overline{A} \cap \overline{B}.$$

Proof. We only prove the first equation and leave the proof of the second one as an exercise.

Suppose $x \in \overline{A \cap B}$. From the definition of the complement of a set, $x \notin A \cap B$. this means that x does not belong to $A \cap B$. We have to negate the definition of intersection. We have $x \in A \cap B$ when $x \in A$ *and* $x \in B$. The negation is $x \notin A$ or $x \notin B$. Therefore, $x \in \overline{A}$ or $x \in \overline{B}$. From the definition of the union of two sets, we see that $x \in \overline{A} \cup \overline{B}$.

Now, assume $x \in \overline{A} \cup \overline{B}$. This means that $x \notin A$ or $x \notin B$. From the last paragraph, this last statement is the negation of $x \in A \cap B$. Therefore, $x \notin A \cap B$. In other words, x belongs to $\overline{A \cap B}$.

We can then conclude that $\overline{A \cap B} = \overline{A} \cup \overline{B}$. $\square$