

Conditional Probability, Hypothesis Testing, and the Monty Hall Problem

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On more than one occasion I have heard the comment “Probability does not exist in the real world”, and most recently I heard in the context of plane crashes: “If you ask me what the probability that the next plane I get on will crash, I would say probably 1 in 10,000,000; but if you told me that the pilot was drunk, I might say 1 in 1,000,000. ” The problem here is that there is no “the probability” – by “the probability” we mean “the probability conditioned on the given information, making certain natural assumptions”. Thus, as the information and our assumptions change, so will our probability measure. That is the difficulty in applying probability theory to real-world problems: We often have to infer from the data a natural probability measure to use; however, in the purely theoretical world, that measure is usually just given to us.

In this short note I will work through two examples in depth to show how conditional probability and Bayes’s theorem can be used to solve them: the alcohol testing problem, and the Monty Hall problem.

1 Hypothesis Testing

A prosecutor stands and stares at a jury and holds up a slip of paper containing the results of a lab test: The defendant was high on drugs while flying a commercial airliner, putting 150 people’s lives at risk. The prosecutor says that the test is 98% accurate; that is, if a high person is given the test, the test responds “high” 98% of the time, and if a sober person is given the test, the test responds “clean” 98% of the time. So, the prosecutor concludes that “Beyond a reasonable doubt, the defendant is guilty. You must convict.”

The defense is then allowed to make its case, and it calls a statistician to the stand. The statistician starts by saying, “Wait a minute, the test given to the defendant is only supposed to be used to *eliminate a suspect as being high*, not to determine whether the person was actually high – the error rate of 2% is far too ‘high’, no pun intended.” The jury doesn’t seem to be buying it. The only thing in their mind is that 98%, and “what is that business about *eliminating a suspect* as being different from *determining whether the person is high*”, they wonder.

The statistician continues by saying that if only 2% of the population uses illegal drugs, if you get a test that says “high”, then it would be correct only 50% of the time! This seems to thoroughly confuse the jury – how did they go all the way from 98% down to 50%? The statistician explains “Suppose you have 5000 people, 2% or 100 of whom are illegal users, and so 4900 of whom are clean. If you give these people the test, 2% of those 4900 will result in “high” test results (that is, false positives) while 98 of the 100 drug users will result in a “high” test result. In total you will get

$$(0.02)4900 + 98 = 196$$

test results that read “high”; and, 98 of these will actually be illegal users, and 98 will be innocent. Thus, the test is only 50-50! In the language of conditional probability and Bayes’s theorem,

$$\begin{aligned} P(\text{User}|\text{Tests High}) &= \frac{P(\text{Tests High}|\text{User})P(\text{User})}{P(\text{Tests High}|\text{User})P(\text{User}) + P(\text{Tests High}|\text{Clean})P(\text{Clean})} \\ &= \frac{(0.98)(0.02)}{(0.98)(0.02) + (0.02)(0.98)} \\ &= \frac{1}{2}. \end{aligned}$$

After the rest of the case is presented, the jury adjourns, and they carefully pour over the facts of the case. One of the jurors looks at the numbers and the testimony of the statistician, and becomes flustered that he can’t understand it, and he remarks to the foreman “I was always good at math in high school. I think we should convict.” The foreman rolls his eyes.

In the end, the jury does the right thing, and votes “Not Guilty”. Fortunately for the defendant, the foreman of the jury was an actuary!

1.1 Eliminating Suspects

Just what did the statistician mean by there being a difference between eliminating a suspect and deciding guilt? Well, of course you know what the difference is, but maybe what you don't know is just *how* different they can be. Let me explain: If you want to determine how reliable your test is for eliminating a suspect, you want to compute $P(\text{Clean}|\text{Tests Clean})$, and hope that it is nearly 100%; and if you want your test to be good for deciding if a person is guilty, you want that $P(\text{User}|\text{Tests High})$ is near 100%. We know that the latter is not true if 2% of the population are users, because in that case we computed $P(\text{User}|\text{Tests High}) = 0.5$. But what about the former probability? Again by Bayes's theorem we have that

$$\begin{aligned} P(\text{Clean}|\text{Tests Clean}) &= \frac{P(\text{Tests Clean}|\text{Clean})P(\text{Clean})}{P(\text{Tests Clean}|\text{Clean})P(\text{Clean}) + P(\text{Tests Clean}|\text{User})P(\text{User})} \\ &= \frac{(0.98)(0.98)}{(0.98)(0.98) + (0.02)(0.02)} \\ &= \frac{0.9604}{0.9608} \\ &\approx 99.96\%. \end{aligned}$$

So, the test is *extremely* good at eliminating suspects, but very poor at deciding whether a person is guilty!

2 The Monty Hall Problem

Here I will state a variant of the problem, where the math is easier to analyze: There are three doors. Behind two are goats, and behind one is a prize. You begin by picking door 1 as the door you want the host to open. The host looks at you and says "Wait a minute. Let me show you what is behind this door," and you see a goat behind one of doors 2 or 3. He then asks if you want to stay with door 1, or switch to the remaining unopened door.

It turns out that your chances of winning if you switch are 2/3, making certain natural assumptions, such as that the prize was chosen at random to be behind doors 1,2,3, each with equal probability; and, that if the prize was behind door 1, then the host picked door 2 or 3 with equal probability, for the one to reveal.

A quick way to get to this answer of $2/3$ is to realize that if after initially selecting door 1 you shut your ears and eyes, and pay no attention to the host (Monty Hall), your chances of winning with door 1 must be $1/3$. So, the chances that the remaining door was the winner must be $1 - 1/3 = 2/3$. In the next subsections we will give alternative ways to solve the problem.

2.1 Exhausting the Possibilities

One way to solve the problem is to write down the sample space S , and reason for the probability of all the elementary elements; and, then, interpret the event “you win if you switch” as a subset of S .

A natural definition for S is

$$\{(1, 2), (1, 3), (2, 3), (3, 2)\},$$

where the first coordinates are the doors that the prize is behind, and the second coordinate is the door Monty decides to reveal. Since it is equally likely that the prize is behind doors 1, 2, or 3, we know that

$$P(\{(1, 2), (1, 3)\}) = P(\{(2, 3)\}) = P(\{(3, 2)\}) = \frac{1}{3}.$$

Also, we assume that

$$P(\{(1, 2)\}) = P(\{(1, 3)\}),$$

as Monty is not picky about whether to reveal door 2 or door 3 when the prize is behind door 1. With these assumptions, it follows that

$$\begin{aligned} P(\{(1, 2)\}) &= \frac{1}{6} \\ P(\{(1, 3)\}) &= \frac{1}{6} \\ P(\{(2, 3)\}) &= \frac{1}{3} \\ P(\{(3, 2)\}) &= \frac{1}{3}. \end{aligned}$$

Now, the event “You win if you switch to door 2 or 3 that Monty did not reveal” is

$$E = \{(2, 3), (3, 2)\},$$

which has probability

$$P(E) = P(\{(2, 3)\}) + P(\{(3, 2)\}) = \frac{2}{3}.$$

A common mistake here is to assume that the probabilities of all the events of S are equal, which would give $1/2$ for $P(E)$. This problem is a good example of a case where your probability measure on S is not uniform (like it is for fair die or fair coin experiments)!

2.2 Conditional Probability Approach

Although I solved the problem in class using Bayes's theorem, there is actually a much better way to do it using the following fact (which itself is a part of the general Bayes's Theorem): Suppose that E_1, E_2, \dots, E_k are k disjoint events whose union is all of S , the sample space. Then, given an event E we have

$$P(E) = P(E|E_1)P(E_1) + \dots + P(E|E_k)P(E_k).$$

The way we apply this theorem is as follows: As in the previous subsection, let E be the event that you win if you switch to the door 2 or 3 that Monty didn't show you; and, let E_1 be the event that the prize is behind door 1, E_2 is the event the prize is behind door 2, and E_3 is the event the prize is behind door 3. Clearly, E_1, E_2, E_3 are disjoint and their union is S . We now have

$$P(E) = P(E|E_1)P(E_1) + P(E|E_2)P(E_2) + P(E|E_3)P(E_3).$$

Clearly, $P(E|E_1) = 0$, $P(E_1) = P(E_2) = P(E_3) = 1/3$, and $P(E|E_2) = P(E|E_3) = 1$. So,

$$P(E) = 0 + \frac{1}{3} + \frac{1}{3} = \frac{2}{3}.$$

2.3 Two Variations

We consider the following two variations of the problem which look almost identical, but are subtly different (actually, it isn't that subtle if one thinks about it).

Problem 1. Given that the host reveals door 3, what is the probability of winning if you switch?

Problem 2. Given that door 3 has a goat, what is the probability of winning if you switch?

The conditional information in problem 1 is the set $A = \{(1, 3), (2, 3)\}$, and the conditional information in problem 2 is the set $B = \{(1, 3), (1, 2), (2, 3)\}$. So, in problem 2 you are given less information (that is, a larger set of possibilities).

The solution to problem 1 is thus

$$P(E|A) = \frac{P(E \cap A)}{P(A)} = \frac{P(\{(2, 3)\})}{P(A)} = \frac{1/3}{1/6 + 1/3} = \frac{2}{3}.$$

The solution to problem 2 is

$$P(E|B) = \frac{P(E \cap B)}{P(B)} = \frac{1/3}{1/6 + 1/6 + 1/3} = \frac{1}{2}.$$