Appendix L: Logic and Proofs

L.1 Mathematical Statements

A statement is a sentence (written in words, mathematical symbols, or a combination of the two) that is either true or false.[1]

Example L.1

a) $4 + 11 = 15$.
   This is a statement and it is true.

b) $x > 5$.
   This is not a statement. Grammatically, it is a complete sentence, written in mathematical symbols, with a subject (x) and a predicate (*is greater than* 5). The sentence, however, is neither true or false because the value of x is not specified.

c) If $x = 5$, then $x > 0$.
   This is a statement and it is true.

d) There exists a positive integer n such that $n > 2$.
   This is a statement and it is true.

e) Is the number 20 an even number?
   This is not a statement. A question is neither true or false.

A proof is a piece of writing that demonstrates that a particular statement is true. A statement that we prove to be true is often called a theorem. A statement that we assume without proof is an axiom. A definition is an agreement between the writer (or professor) and the reader (or the student) as to the meaning of a word or phrase. A definition needs no proof.

L.2 Logic and Mathematical Language

In the section on set theory, you will have the chance to practice the methods of proof presented below.

Negation

If P is a statement, then it has a truth value: true or false. The negation of a statement P is defined as *it is not the case that* P. The negation of a statement P will be abbreviated by *not* P or $\neg P$.

[1]Most of the material presented here is from the really good notes retrieved online at https://sites.math.washington.edu/~conroy/m300-general/ConroyTaggartIMR.pdf. Some passages might entirely be copied or modified slightly from this resource.

Example L.2 Consider the statement P: "2 is an even integer". The negation of P is "It is not the case that 2 is an even integer" that may be rewritten as "2 is not an even integer". We may even go further and rewrite $\neg P$ as follows: "2 is an odd integer". Notice that P is true and $\neg P$ is false.

Conjunction and Disjunction

Let's consider two statements P and Q.

- The conjunction of P and Q is the statement "P and Q". It is denoted by $P \wedge Q$ and it is true only when P and Q are true; otherwise it is false.
- The disjunction of P and Q is the statement "P or Q". It is denoted by $P \vee Q$ and it is true when one of the two statements is true.

We can use a truth table to illustrate the conjunction and disjunction of two statements P and Q as shown below.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

(a) Truth table for $P \wedge Q$

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

(b) Truth table for $P \vee Q$

Example L.3

a) Consider the statement P: "2 is a positive integer" and the statement Q: "-4 is a negative integer". The statement $P \wedge Q$ is true because P and Q are true. But $P \wedge (\neg Q)$ is not true because $\neg Q$: "-4 is not a negative integer" is false.

b) Consider the same statements from part a). The statement $P \vee Q$ is true because the integer 2 is a positive integer and only one of the statements P, Q needs to be true. The statement $(\neg P) \vee Q$ is also true because the statement Q is true (-4 is a negative number). But the statement $(\neg P) \vee (\neg Q)$ is not true because $\neg P$ and $\neg Q$ are both false.

We can take the negation of a conjunction and of a disjunction.

Example L.4 A friend tells you the conditions to come to his party. He tells you that you must wear green clothes only AND bring a one-page explanation of why you are at his party. A person that wants to go to your friend's party must satisfies *both* conditions. Anyone who is wearing a non-green piece of clothe will not be allowed at the party. Also, anyone who did not write the one-page essay will not be allowed at the party. Therefore, anyone who does not wear a green outfit or anyone who did not write the one-page essay will not come to the party.

Conclusion: The negation of $P \wedge Q$ is $(\neg P) \vee (\neg Q)$.

Example L.5 Your friend decides to be more welcoming. He tells you the conditions to come to his party remains the same, but only one of them must be meet. In other words, you may wear green clothes only OR bring a one-page explanation of why you are at his party. A person that wants to go to your friend's party must satisfy one of the two conditions. But if the person is not dressed in green clothes and does not bring the one-page essay, then unfortunately, that person will not be allowed to join the party. In other words, if both conditions are not respected by a person, then that person will not be allowed to join the party.

Conclusion: The negation of $P \vee Q$ is $(\neg P) \wedge (\neg Q)$.

Conditional

A lot of statements we will encounter are of the form "If P, then Q". These statements are called conditional statements. We will use the following notation "$P \Rightarrow Q$" to denote a conditional statement.

The truth value of the statement $P \Rightarrow Q$ depends on the truth values of P and Q.

Example L.6 We think of $P \Rightarrow Q$ as an agreement. Joe makes a deal with his parents. Let P: "Joe did the dishes after dinner" and Q: "Joe got \$5". The agreement is

$$P \Rightarrow Q : \text{ If Joe did the dishes, then he got \$5.}$$

Joe is not required to do the dishes (it is a compulsory act for his love for his family, but also for his love for money). In the case that Joe did the dishes (P is true) and got paid (Q is true), the agreement is met ($P \Rightarrow Q$ is true). In the case that Joe didn't do the dishes (P is false) and didn't get paid (Q is false), the agreement is met ($P \Rightarrow Q$ is true). Since Joe is not required to wash the dishes, his parents may choose to give him \$5 for some other reason. That is, in the case Joe did not do the dishes (P is false) and got \$5 anyway (Q is true), the agreement is still met ($P \Rightarrow Q$ is true). The only instance in which the agreement is not met ($P \Rightarrow Q$ is false) is in the case that Joe did wash the dishes (P is true), but did not get the money from his parents (Q is false).

To summarize, the statement $P \Rightarrow Q$ is true unless P is true and Q is false, like it is shown in the table below.

P	Q	$P \Rightarrow Q$
T	T	T
T	F	T
F	T	F
F	F	T

Truth table for the conditional

To explain the negation of the conditional, we use Joe's story from the previous example. Joe claims that his parents broke their verbal contract, while the parents deny Joe's claim. In other words, Joe's parents say that $P \Rightarrow Q$ is true, while Joe says that $\neg(P \Rightarrow Q)$ is true. If you were Joe's lawyer, what evidence would you have to provide to win the case? You would need to show that Joe washed the dishes and did not get paid. That is, you would need to show that $P \wedge (\neg Q)$ is true.

Conclusion: The negation of the conditional statement $P \Rightarrow Q$ is the statement $P \wedge (\neg Q)$.

Converse and Contrapositive

Definition L.1 The converse of $P \Rightarrow Q$ is the statement $Q \Rightarrow P$. The contrapositive of $P \Rightarrow Q$ is the statement $(\neg Q) \Rightarrow (\neg P)$.

Example L.7 Consider the statements P: "Valérie's cat is hungry" and Q: "Valérie's cat meows".

- The implication $P \Rightarrow Q$ reads as "If Valérie's cat is hungry, then the cat meows".

- The converse $Q \Rightarrow P$ of $P \Rightarrow Q$ reads as "If Valérie's cat meows, then the cat is hungry". Notice that in this contexte, $P \Rightarrow Q$ does not have the same truth value of $Q \Rightarrow P$. For instance, the cat might moews because it wants to be pet.
- The contrapositive of $P \Rightarrow Q$ reads as "If Valérie's cat does not meow, then the cat is not hungry". You can check that $P \Rightarrow Q$ has the same truth value of $(\neg Q) \Rightarrow (\neg P)$.

Equivalent statement

DEFINITION L.2 The statement P if and only if Q, written $P \iff Q$, is readily the statement

$$(P \Rightarrow Q) \wedge (Q \Rightarrow P).$$

Quantifiers

Suppose n is an integer and $P(n)$ is a statement about n.

- If $P(n)$ is true for at least one integer n, then we say "There exists n such that $P(n)$". This type of statement is called an existence statement and the symbol $\exists$ is used as a shortcut for the "there exists" part.
- If $P(n)$ is true no matter what value n takes, then we say "For all n, $P(n)$". This type of statement is called a universal statement and the symbol $\forall$ is used as a shortcut for the "For all" part.

These statements are called quantified statements.

EXAMPLE L.8 Assume throughout this example that n is an integer.

a) "There exists n such that $n > 0$". In this statement, $P(n)$ is "$n > 0$. The statement $P(10)$ is true because $10 > 0$, therefore the statement "$\exists n$ such that $n > 0$" is true because $P(n)$ is true for at least one integer n.

b) "For all n, $n > 0$". The statement $P(-1)$ is false since -1 is not greater than 0. Therefore, the statement "$\forall n$, $P(n)$" is false because $P(n)$ is *not* true for every integer n.

c) "$\exists n$ such that $|n| < 0$". This statement is false because there is no integer n with $|n| < 0$; the absolute value turns every integer into a positive or zero integer.

Here are the ways to negate a quantified statement:

- The negation of "$\exists n$ such that $P(n)$" is "$\forall n$, $\neg(P(n))$".
- The negation of "$\forall n$, $P(n)$" is "$\exists n$ such that $\neg(P(n))$".

L.3 Methods of Proof

We will cover some methods to prove mathematical statements. The two we will cover are direct proofs of a conditional statement and proofs by contradiction.

Direct Proof

There are many ways to proof a conditional statement. The method covered is called "direct proof". If one of the other ways is needed later on in the semester, then I will explain it to you on the spot. This is an agreement between you and me ;).

The "direct proof" method works as followed. We assume the hypothesis (the statement just after the "if") and use definitions, logic, and previously proved results to reach the desired conclusion (the statement after the "then").

Example L.9 Prove the following statement: If a and b are even integers, then $a+b$ is an even integer.

Solution. Suppose that a and b are even integers. In other words, this means a and b are multiples of 2: There exists an integer n such that $a = 2n$ and there exists an integer m such that $b = 2m$. Then

$$a + b = 2n + 2m = 2(n + m).$$

This implies that $a + b$ is a multiple of 2 and therefore it is an even integer. $\triangle$

Proof by Contradiction

In a proof using the method called contradiction, the fact that a statement and its negation have opposite truth values is used. Therefore, to prove that P is true, we suppose instead that the statement $\neg P$ is true and apply logic, definitions, and previous results to arrive at a conclusion known to be false. Then this will imply $\neg P$ must be false and thus P must be true.

Example L.10 No integer is both even and odd.

Solution. Suppose that there is an integer n that is both even and odd (the negation of the statement "$\forall n$, n is neither even or odd", which is equivalent to the statement in the example). Since n is assumed even, $n = 2k$ for some integer k. But n is also assumed odd, so $n = 2l + 1$. Therefore, since $n = n$, we have

$$2k = 2l + 1 \Rightarrow 2k - 2l = 1 \Rightarrow 2(k - l) = 1.$$

Since $k - l$ is an integer, the last equation means that 1 is a multiple of 2 (or that 1 is divisible by 2), which is clearly false! Therefore the assumption that there is an integer n that is both even and odd must be false and it turns out that no integer is both even and odd. $\triangle$

Proof of An Equivalent Statement

To prove the statement "$P \iff Q$", it must be shown that $P \Rightarrow Q$ is true *and* $Q \Rightarrow P$ is true.

Proof of An Existential Statement

To prove a statement of the form "there exists an n such that $P(n)$", the technique used is "construction". This means the object n will be found and be demonstrated that $P(n)$ is true for this choice of n.

Proof of A Universal Statement

The proof of a statement of the form "for all objects n, $P(n)$" is rather more subtle. It is really hard to deal with all objects n at once. Instead, we think of an equivalent way to interpret a universal statement. In fact, the statement "for all objects n, $P(n)$" is equivalent to the statement "If n is such an object, then $P(n)$". For example, the statement "For all integers n, $|n| \geq 0$" has the same meaning as "If n is an integer, then $|n| \geq 0$".

Therefore, to prove a universal statement, we first select a single *arbitrary* object and proved that the conclusion is true for that object. It is really important that the object chosen was arbitrary.

Example L.11 Prove the following statement: For all odd integers n and m, nm is odd.

Solution. Suppose that n is an odd integer and that m is an odd integer. This means there exist integers k and l such that $n = 2k + 1$ and $m = 2l + 1$. Then

$$nm = (2k + 1)(2l + 1) = 4kl + 2k + 2l + 1 = 2(2kl + k + l) + 1.$$

Therefore, the product nm takes the form of an odd integer. $\triangle$