C.1 Discrete Random Variables

PROBLEM 1. Let $(S, \mathcal{A}, P)$ be a probability space. If X and Y are discrete random variables, show that the mapping $Z : S \to \mathbb{R}$ defined by $Z(s) = X(s)Y(s)$ is a discrete random variable.

PROBLEM 2. Let A be an event from a probability space $(S, \mathcal{A}, P)$. Show that the indicator map 1_A defined by

$$1_A(s) := \begin{cases} 1 & \text{if } s \in A \\ 0 & \text{if } s \notin A, \end{cases}$$

is a discrete random variable.

PROBLEM 3. Let X be a discrete random variable. Show that

a) the set $\{X \leq x\}$ is an event, for every $x \in \mathbb{R}$.

b) the set $\{X < x\}$ is an event, for every $x \in \mathbb{R}$.

c) the set $\{X \geq x\}$ is an event, for every $x \in \mathbb{R}$.

d) the set $\{X > x\}$ is an event, for every $x \in \mathbb{R}$.

PROBLEM 4. Show that X is a discrete random variable if and only if the following conditions hold:

a) $\operatorname{Im} X$ is discrete.

b) $\forall x \in \mathbb{R}$, the set $\{X \leq x\}$ is an event.

C.2 Probability Mass Functions

PROBLEM 5. A card is drawn randomly from a hat containing 5 cards of different colors (say 2 red, 2 blue, and 1 yellow). Let $S = \{r, b, y\}$ be the sample space and $\mathcal{A} = \mathcal{P}(S)$. If the card drawn is red, then the participant looses \$10, if it is blue, the participant wins \$10, and if it is yellow, the participant wins \$20. Let X be the money won by a participant after playing the game. Find the probability mass function of X.

PROBLEM 6. A problem in a test given to small children asks them to match each of three pictures of animals to the word identifying that animal. If a child assigns the three words at random to the three pictures, find the probability distribution (probability mass function) of Y, the number of correct matches.

PROBLEM 7. Five balls, numbered 1, 2, 3, 4, 5, are placed in an urn. Two balls are randomly selected from the five, and their numbers noted. Find the probability distribution for the following random variables: a) X is the *largest* of the two sampled numbers; b) X is the *sum* of the two sampled numbers.

Problem 8. For what value(s) of c is the function p, defined by

$$p(k) = \begin{cases} \frac{c}{k(k+1)} & , k = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

a pmf?

C.3 Functions of Discrete Random Variables

Problem 9. Let X be a discrete random variable with its distribution (probability mass function) p_X given by the following table

x	1	2	3	4
$p_X(x)$	0.4	0.3	0.2	0.1

Find the image of $Y = X^2 - 1$ and its distribution.

Problem 10. Consider the discrete random variable X from the previous problem. Find the image of $Y = \sin(\frac{\pi}{2}X)$ and its distribution.

C.4 Expectation and Variance

Problem 11. A manufacturing company ships its product in two different sizes of truck trailers. Each shipment is made in a trailer with dimensions $8ft \times 10ft \times 30ft$ or $8ft \times 10ft \times 40ft$. If 30% of its shipments are made by using 30-foot trailers and 70% by using 40-foot trailers, find the mean volume shipped per trailer load. (Assume the trailers are always full.)

Problem 12. Two construction contracts are to be randomly assigned to one or more of three firms: A, B, C. Any firm may receive both contracts. If each contract will yield a profit of $\$90,000$ for the firm, find the expected profit for firm A. If firms A and B are actually owned by the same individual, what is the owner's expected total profit?

Problem 13. A single fair die is tossed once. Let X be the number facing up. Find the variance and the standard deviation of X.

Problem 14. If X is a discrete random variable, show that $\text{var}(aX+b) = a^2\text{Var}(X)$ for $a, b \in \mathbb{R}$.

C.5 Conditional Expectation and the Partition Theorem

Problem 15. Let X be a discrete random variable and let $g : \mathbb{R} \to \mathbb{R}$ be a function. Using the definition of the conditional expectation, show that if x is a real number such that $P(X = x) > 0$, then $E(g(X)|X = x) = g(x)$.

C.6 Examples of Discrete Random Variables

Problem 16. A meteorologist in Hawaii recorded X: "the number of days of rain during a 30-day period.". Does X have a binomial distribution?

Problem 17. In 2003, the average combined SAT score (math and verbal) for college-bound students in the United-States was 1026. Suppose that approximately 45% of all high school graduates took this test and that 100 high school graduates are randomly selected from among all high school grads in the United-States. Which of the following discrete random variables has a distribution that can be a binomial distribution? Whenever possible, give the values of n and q.

a) The number of students who took the SAT.

b) The scores of the 100 students in the sample.

c) The number of students in the sample who scored above average on the SAT.

d) The amount of time required by each student to complete the SAT.

PROBLEM 18. If X has a Binomial distribution, show that $E(X) = nq$ and $\mathrm{Var}(X) = nq(1-q)$.