F.1 Moments

PROBLEM 1. From the definition of the k-th moment, we compute

$$\mathrm{Exp}(X^k) = \int_{-\infty}^{\infty} x^k f_X(x)\, dx = \int_a^b \frac{x^k}{b-a}\, dx.$$

After integrating x^k, we obtain the answer:

$$\mathrm{Exp}(X^k) = \frac{b^{k+1} - a^{k+1}}{(k+1)(b-a)}. \qquad \triangle$$

F.2 Moment Generating Function

PROBLEM 2. The moment generating function of the normal distribution is

$$M_X(t) = e^{\mu_X t + \frac{1}{2}\sigma_X^2 t^2}.$$

Since $\mu_X = 0$ and $\sigma_X^2 = 1$, we then get $M_X(t) = e^{t^2/2}$. From Theorem F.1, we know that

$$\mathrm{Exp}(X^3) = M_X^{(3)}(0).$$

We have

$$M_X^{(3)}(t) = e^{t^2/2} t(t^2+3) \quad \Rightarrow \quad \mathrm{Exp}(X^3) = 0. \qquad \triangle$$

PROBLEM 3. We will identify the moment generating function of $aX + b$. From the definition of the moment generating function, we have

$$M_{aX+b}(t) = \mathrm{Exp}(e^{t(aX+b)}) = e^{tb}\mathrm{Exp}(e^{taX}) = e^{tb} M_X(at).$$

Since $X \sim N(\mu_X, \sigma_X)$, we know that $M_X(t) = e^{\mu_X t + \frac{1}{2}\sigma_X^2 t^2}$. Substituting this into the equation for $M_{aX+b}(t)$, we find that

$$M_{aX+b}(t) = e^{tb} e^{\mu_X at + \frac{1}{2}\sigma_X^2 a^2 t^2} = e^{(b+\mu_X a)t + \frac{1}{2}\sigma_X^2 a^2 t^2}.$$

The moment generating function of $aX + b$ is then the moment generating function of a normal distribution with average $b + \mu_X a$ and variance $\sigma_X^2 a^2$. Hence, $aX + b \sim N(b + \mu_X a, \sigma_X^2 a^2)$. $\triangle$

PROBLEM 4.

a) Since X is an exponential distribution and $\lambda = 1/\theta$, we find that

$$M_X(t) = \frac{1/\theta}{1/\theta - t} = \frac{1}{1 - \theta t},$$

for $t < 1/\theta$.

b) We know that

$$\mathrm{Exp}(X) = M_X'(0) \quad \text{and} \quad \mathrm{Var}(X) = \mathrm{Exp}(X^2) - (\mathrm{Exp}(X))^2 = M_X''(0) - (M_X'(0))^2.$$

We compute

$$M_X'(t) = \frac{\theta}{(1-\theta t)^2} \quad \text{and} \quad M_X''(t) = \frac{2\theta^2}{(1-\theta t)^3}$$

Hence

$$\mathrm{Exp}(X) = \frac{\theta}{(1-\theta(0))^2} = \theta$$

and

$$\mathrm{Var}(X) = \frac{2\theta^2}{(1-\theta(0))^3} - (\theta)^2 = \theta^2.$$

△