B.1 Conditional Probabilities

PROBLEM 1. Let $(S, \mathcal{A}, P)$ be a probability space. Suppose two events A and B are given such that $P(A) = 0.5$, $P(B) = 0.3$, and $P(A \cap B) = 0.1$. Find

a) $P(A|B)$.
b) $P(A|A \cup B)$.
c) $P(A \cap B|A \cup B)$.

PROBLEM 2. A balanced die is tossed once. What is the probability the die lands on a 1, given that an odd number was obtained?

PROBLEM 3. Two fair dice are rolled. What is the probability that at least one lands on 6 given that the dice land on different numbers?

PROBLEM 4. Let $(S, \mathcal{A}, P)$ be a probability space. Suppose that two events A and B are given such that $P(A) > 0$, $P(B) > 0$. Prove that if $P(A) < P(A|B)$, then $P(B) < P(B|A)$.

PROBLEM 5. Suppose that $A \subset B$ and that $P(A) > 0$ and $P(B) > 0$. Show that $P(B|A) = 1$ and $P(A|B) = P(A)/P(B)$.

PROBLEM 6. If A and B are mutually exclusive events and $P(B) > 0$, show that

$$P(A|A \cup B) = \frac{P(A)}{P(A) + P(B)}.$$

PROBLEM 7. Let $(S, \mathcal{A}, P)$ be a probability space. If A, B are events with $P(A) > 0$ and $P(B) > 0$, then show that

$$\frac{P(A|B)}{P(\overline{A}|B)} = \frac{P(A)}{P(\overline{A})} \frac{P(B|A)}{P(B|\overline{A})}.$$

B.2 Bayes' Formula

PROBLEM 8. A laboratory blood test is 95% effective in detecting a certain disease when it is, in fact, present. However, the test also yields a "false positive" result for 1% of the healthy people tested[1]. If 0.5% of the population actually have the disease, what is the probability a person has the disease given that the test result is positive?

PROBLEM 9. A total of 46% of the voters in a certain city classify themselves as Independents, whereas 30% classify themselves as Liberals and 24% as Conservative. In a recent local election, 35% of the Independents, 62% of the Liberals, and 58% of the Conservatives voted. A voter is

[1] That is, if a healthy person is tested, then, with probability 0.01, the test result will imply the person has the disease.

chosen at random. Given that this person voted in the local election, what is the probability that the person is a) an Independent? b) a Liberal? c) a Conservative?

PROBLEM 10. When a dice x is tossed it lands on ⚁ with probability 1/2 and all the other outcomes are equally likely to happen. When a dice y is tossed, it lands on ⚂ with probability 1/2 and all the other outcomes are equally likely to happen. Suppose that one of these dice is randomly chosen and then tossed. What is the probability that dice x was tossed, if the die landed on ⚁?

B.3 Independent Events

PROBLEM 11. Three brands of coffee, x, y, and z, are to be ranked according to taste by a judge. Define the following events. A: "Brand x is preferred to y, B: "Brand x is ranked best", C: "Brand x is ranked second best" and D: "Brand x is ranked third best". If the judge actually has no taste preference and randomly assigns ranks to the brands, is event A independent of (a) event B? (b) event C? (c) event D?

PROBLEM 12. Cards are dealt, one at a time, from a standard 52-card deck. If A_i denotes the event "the i-th card dealt is a spade". Are A_1 and A_2 independent?

PROBLEM 13. A system composed of 5 separate components is said to be a parallel system if it functions when at least one of the components functions. For such a system, if component i, independent of other components, functions with probability p_i, $i = 1, 2, \dots, 5$, what is the probability that the system functions?

PROBLEM 14. Let $(S, \mathcal{A}, P)$ be a probability space. Prove that

a) If A and B are independent events with $0 < P(A), P(B) < 1$, then A and $\overline{B}$ are independent.

b) If A and B are independent events with $0 < P(A), P(B) < 1$, then $\overline{A}$ and $\overline{B}$ are independent.