Math 241

Chapter 4

Section 4.2: Definite Integral

Contents

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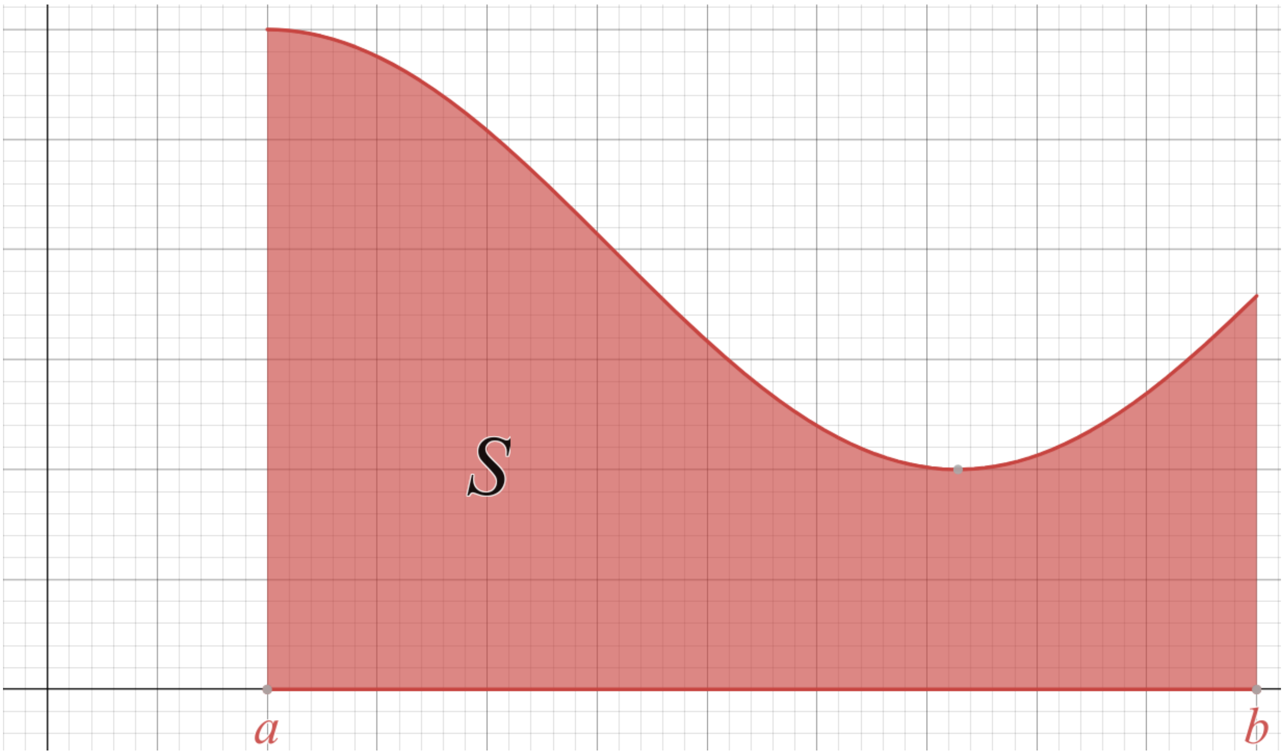
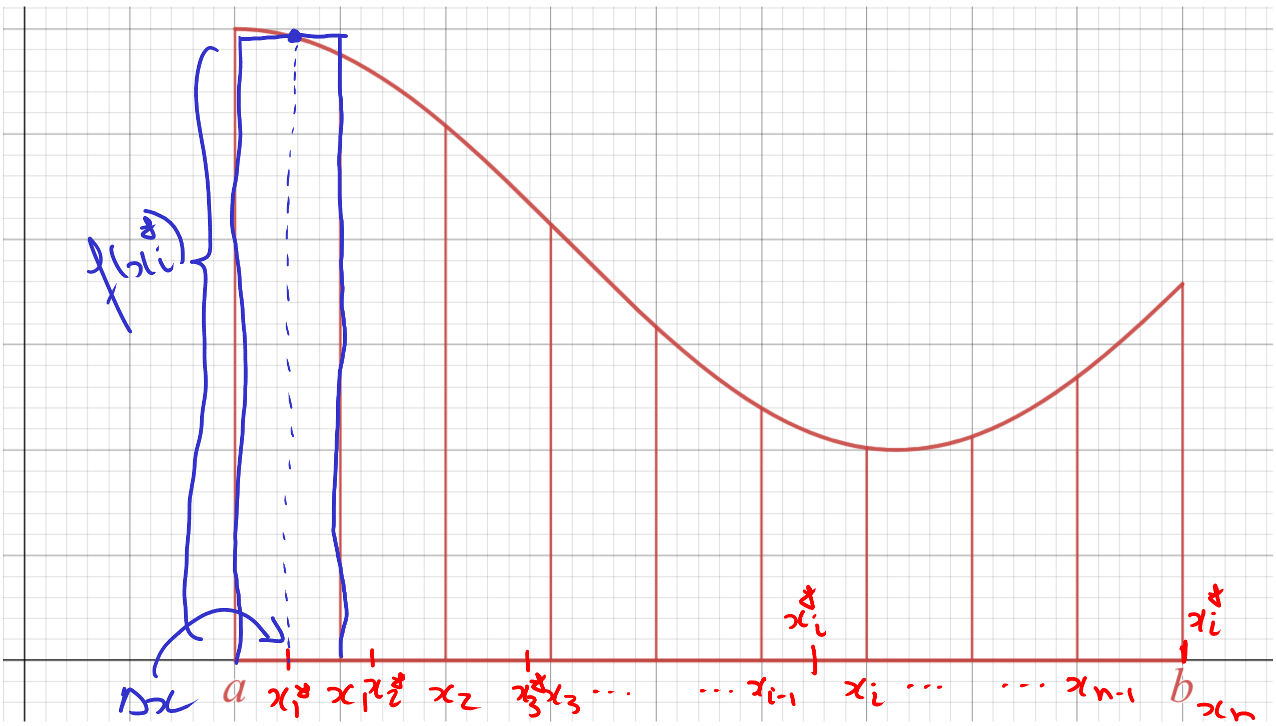
Suppose we have a region S under the graph of a function $y = f(x)$ from $x = a$ to $x = b$.



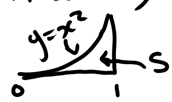
• Divide the interval $[a, b]$ in n subintervals of equal length $\Delta x = (b - a)/n$.



• Select some number x_i^* in each $[x_{i-1}, x_i]$ (can be any number within the subinterval).

• Form the sum: $S_n = \sum_{i=1}^{n} f(x_i^*)\Delta x = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$.

$\text{Area}(S) = \int_0^1 x^2\,dx = \frac{1}{3}$



Definite Integral: For a continuous function f, the definite integral of f is defined by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(x_i^*)\Delta x\right).$$

Important Remarks:

- Description of the terminology:
  - Symbol $\int$: means a "continuous" sum.
  - a: lower bound.
  - b: upper bound.
  - $f(x)$: integrand (what we integrate)
  - dx: variable of integration (similar role as in $\frac{dy}{dx}$).
- The definite integral is a **number!** It does not depend on x! This means that

$$\int_a^b f(x)\,dx = \int_a^b f(r)\,dr = \int_a^b f(t)\,dt = \ldots$$

$\int_a^b f(\square)\,d\square$

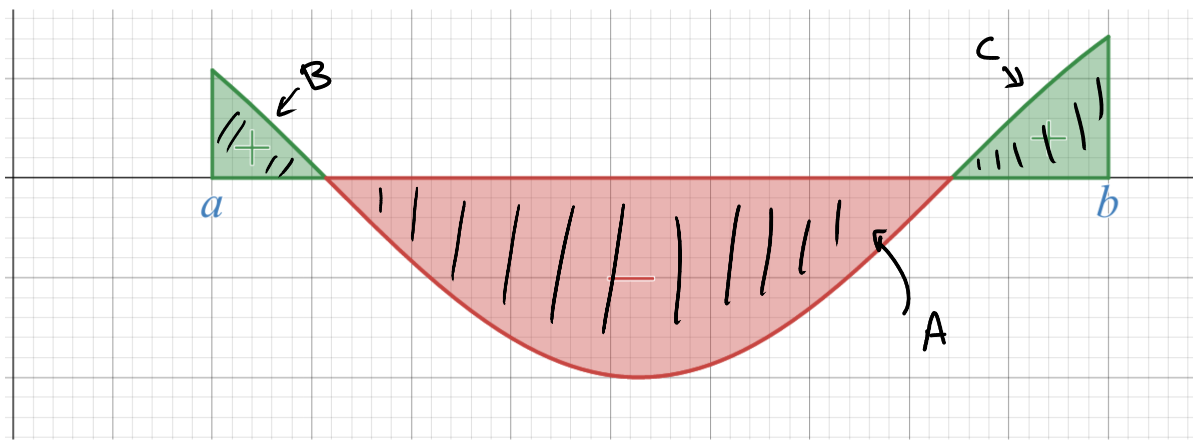
$f(r) = r^2$

$\frac{df}{dr} = 2r$

- The expression S_n are called **Riemann Sums**.
- When $f(x) \geq 0$, then $\int_a^b f(x)\,dx$ is the area of the region S:

$$\text{Area}(S) = \int_a^b f(x)\,dx.$$

- If $f(x)$ is negative somewhere, then $\int_a^b f(x)\,dx$ is the **net area** between the graph of $y = f(x)$ and the horizontal line $y = 0$ (the x-axis).



$\int_a^b f(x)\,dx = B + C - A$

Example 1. Find the value of the following integrals.

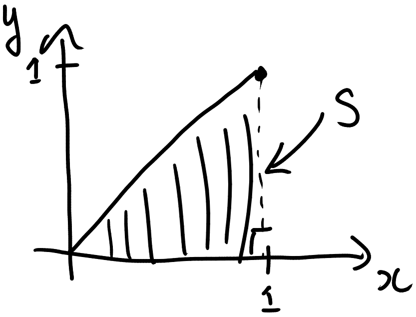
(a) $\displaystyle\int_0^1 x\,dx$. (b) $\displaystyle\int_{-1}^1 x\,dx$. (c) $\displaystyle\int_0^2 |x-1|\,dx$.

(a) $f(x) = x$

var. of int $= x$

$a = 0$

$b = 1$



Since $f(x) \geq 0$

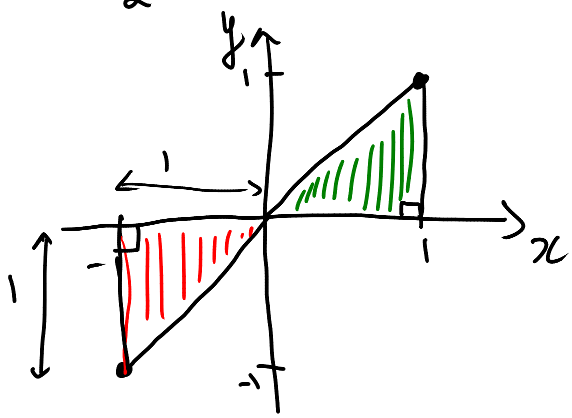
$\int_0^1 x\,dx = \text{Area}(S)$

$= \frac{1 \cdot 1}{2} = \frac{1}{2}$

$\int_a^b x\,dx = \frac{b^2 - a^2}{2}$ (in general).

(b) $a = -1$

$b = 1$



f is negative ...

$\int_{-1}^1 x\,dx = $ Net Area

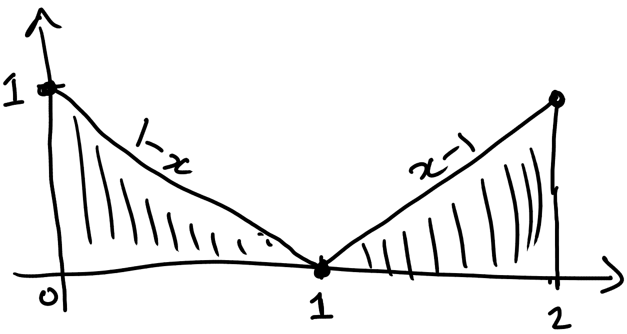
$= A(\triangle) - A(\triangledown)$

$= \frac{1}{2} - \frac{1}{2} = 0$

(c) $f(x) = |x-1|$

$a = 0$

$b = 2$



$f(x) \geq 0$

int

$=$

Area.

$\int_0^2 |x-1|\,dx = \text{Area}(\triangle) + \text{Area}(\triangle)$

$= \frac{1}{2} + \frac{1}{2} = \boxed{1}$

Useful Trick: Try to intepret the integral geometrically!

Playing with Lower and Upper Bounds

- If we change the order of the lower and upper bounds, then

$(a<b)$

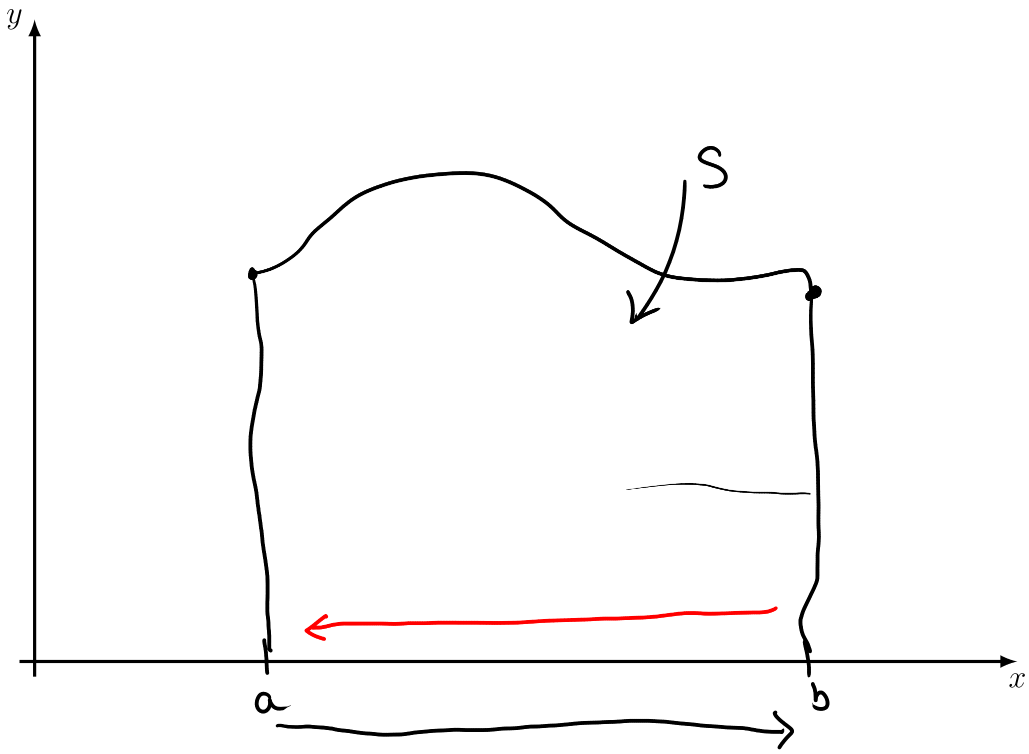
$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

$$\int_2^0 |x-1|\,dx = -\int_0^2 |x-1|\,dx$$

- If the lower and upper bounds are equal, the definite integral is zero, that is

$$\int_a^a f(x)\,dx = 0.$$

Illustration:



Algebraic operations

For two continuous functions $f(x)$ and $g(x)$ on the interval $[a,b]$,

- Addition: $\displaystyle\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$
- Substraction: $\displaystyle\int_a^b (f(x)-g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx.$
- Multiplication by constant: $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx.$

Useful Formulas

Go to Desmos: https://www.desmos.com/calculator/mr9ba23hpz.

x^0

- $\int_a^b 1\,dx = \dfrac{b^1 - a^1}{1}$
- $\int_a^b x^1\,dx = \dfrac{b^2}{2} - \dfrac{a^2}{2} = \dfrac{b^2 - a^2}{2}$
- In general,

$$\int_a^b x^n\,dx = \frac{b^{n+1} - a^{n+1}}{n+1}$$

EXAMPLE 2. Using the properties of the integral and the formulas, find the value of the following integrals.

(a) $\int_0^1 2x^2 - x^4\,dx$.

(b) $\int_{-2}^2 4x^4 - 3x^2\,dx$.

(a)
$$\int_0^1 2x^2 - x^4\,dx = \int_0^1 2x^2\,dx - \int_0^1 x^4\,dx$$
$$= 2\int_0^1 x^2\,dx - \int_0^1 x^4\,dx$$
$$= 2\left(\frac{1^3 - 0^3}{3}\right) - \left(\frac{1^5 - 0^5}{5}\right)$$
$$= \frac{2}{3} - \frac{1}{5} = \boxed{\frac{7}{15}}$$

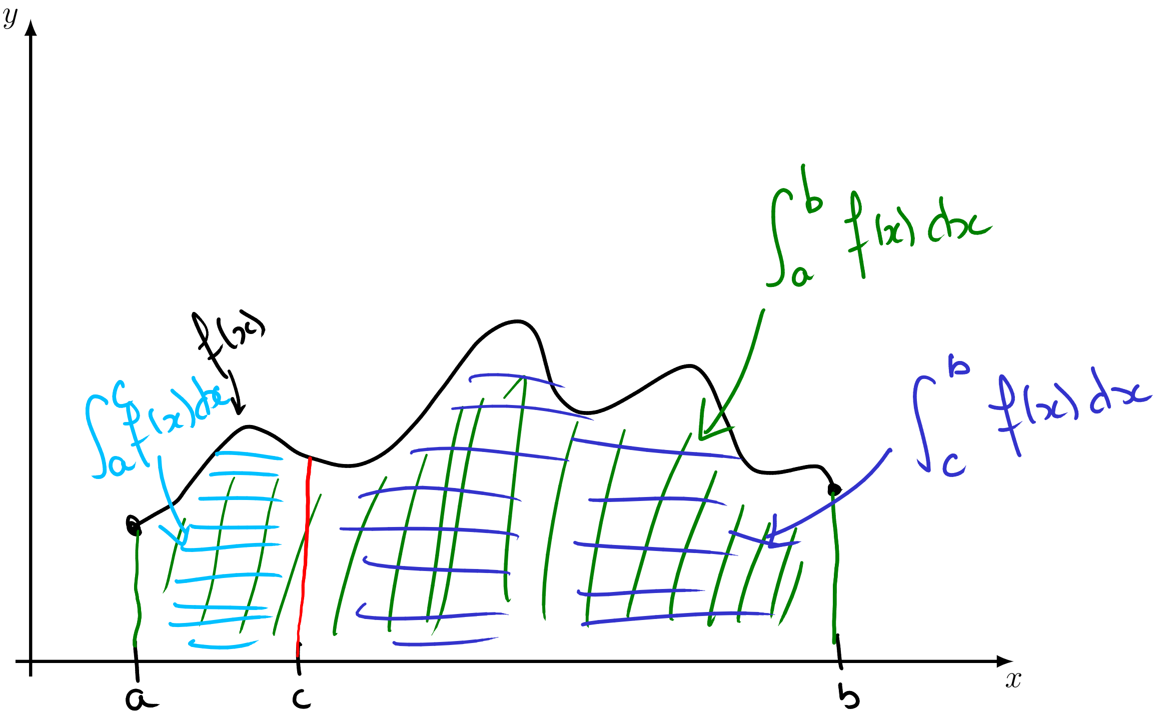
(b)
$$\int_{-2}^2 4x^4 - 3x^2\,dx = \int_{-2}^2 4x^4\,dx - \int_{-2}^2 3x^2\,dx$$
$$= 4\int_{-2}^2 x^4\,dx - 3\int_{-2}^2 x^2\,dx$$
$$= 4\left(\frac{2^5 - (-2)^5}{5}\right) - 3\left(\frac{2^3 - (-2)^3}{3}\right)$$
$$= \boxed{\frac{176}{5}}$$

Cutting the domain

Let $a < c < b$ and $f(x)$ be a continuous function on $[a, b]$. Then

$$\underbrace{\int_a^b f(x)\,dx}_{} = \underbrace{\int_a^c f(x)\,dx}_{} + \underbrace{\int_c^b f(x)\,dx}_{}.$$

Illustration:



Example 3. If it is known that $\int_0^{10} f(x)\,dx = 17$ and $\int_0^{8} f(x)\,dx = 12$, then find $\int_8^{10} f(x)\,dx$.

$$\int_0^{10} f(x)\,dx = \int_0^8 f(x)\,dx + \underbrace{\int_8^{10} f(x)\,dx}_{??}$$

$$\Rightarrow \quad 17 = 12 + \int_8^{10} f(x)\,dx$$

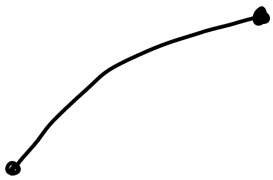
$$\Rightarrow \quad \int_8^{10} f(x)\,dx = \boxed{5}$$

Comparison Properties

- If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq 0$.
- If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.
- If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq \underline{M(b-a)}.$$

EXAMPLE 4. Use the last comparison property to estimate $\int_1^4 \sqrt{x}\,dx$.



We know that

$$\underset{m}{1} \leq \sqrt{x} \leq \overset{M}{2}$$

$$\Rightarrow \quad 1(4-1) \leq \int_1^4 \sqrt{x}\,dx \leq 2(4-1)$$

$$\Rightarrow \quad \underline{\underline{3}} \leq \int_1^4 \sqrt{x}\,dx \leq \underline{\underline{6}}$$

Average:

$$\int_1^4 \sqrt{x}\,dx \approx \frac{6+3}{2} = 4.5$$

<u>Remark:</u>

$$\int_1^4 x^{1/2}\,dx = \frac{4^{1/2+1} - 1^{1/2+1}}{1/2+1} = \frac{4^{3/2}-1}{3/2}$$

$$= \frac{14}{3}$$