Math 241

Chapter 5

Section 5.1: Area Between Curves

Contents

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Non Intersecting Regions

Desmos: https://www.desmos.com/calculator/o7vvfgfwzy

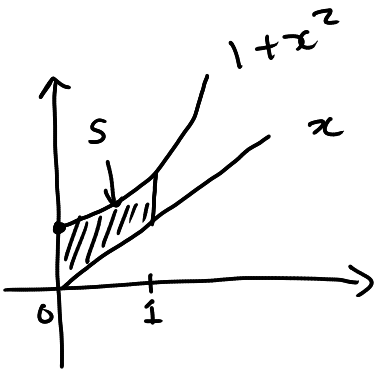
Given two functions $f(x)$ and $g(x)$ such that

$$g(x) \leq f(x) \quad a \leq x \leq b,$$

the area of the region S enclosed by $f(x)$, $g(x)$, $x = a$ and $x = b$ is

$$\text{Area}\,(S) = \int_a^b f(x) - g(x)\,dx.$$

Example 1. Find the area of the region bounded above by $y = x^2 + 1$, bounded below by $y = x$, and bounded on the sides by $x = 0$ and $x = 1$.



a b

$$\text{Area}(S) = \int_a^b f(x) - g(x)\,dx$$

$$= \int_0^1 x^2 + 1 - x\,dx$$

$$= \left(\frac{x^3}{3} + x - \frac{x^2}{2}\right)\Big|_0^1$$

$$= \frac{1}{3} + 1 - \frac{1}{2} - \left(\frac{0^3}{3} + 0 - \frac{0^2}{2}\right)$$

$$= \boxed{\frac{5}{6}}$$

EXAMPLE 2. Find the area of the region enclosed by the functions $y = x^2$ and $y = x + 2$.

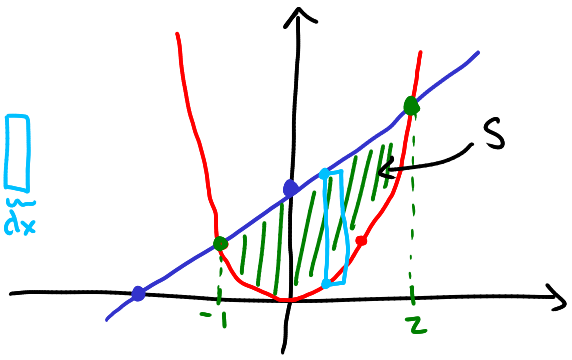
① Sketch & Intersects

$x^2 = x+2$

$\Rightarrow x^2 - x - 2 = 0$

$\Rightarrow (x+1)(x-2) = 0$

$\Rightarrow$ $x = -1$ or $x = 2$



② Integrate

$$\text{Area}(S) = \int_{-1}^{2} x+2-x^2\,dx$$

$$= \left(\frac{x^2}{2} + 2x - \frac{x^3}{3}\right)\Big|_{-1}^{2}$$

$$= \left(\frac{4}{2} + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)$$

$$= 2 + 4 - \frac{8}{3} - \frac{1}{2} + 2 - \frac{1}{3}$$

$$= 8 - 3 - \frac{1}{2}$$

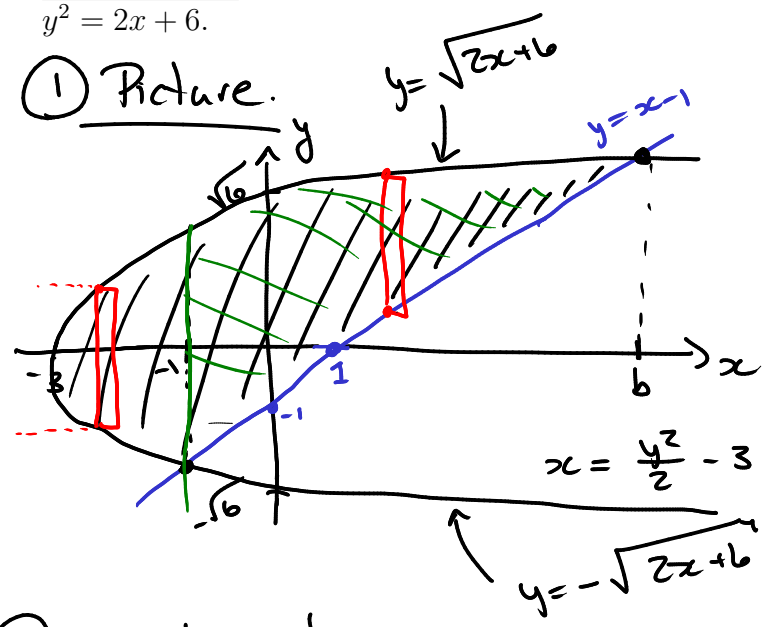
$$= \boxed{\frac{9}{2}}$$

General Steps:

1. Find the intersection between the two curves.
2. Draw a picture.
3. Set up the definite integral.
4. Evaluate the definite integral.

Example 3. Find the area of the region enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

① Picture.



$x = \frac{y^2}{2} - 3$

$(y = \frac{x^2}{2} - 3)$

$0 = \frac{y^2}{2} - 3 \Leftrightarrow y = \pm\sqrt{6}$

② Intersections.

(a) $y+1 = x = \frac{y^2}{2} - 3 \Rightarrow y+1 = \frac{y^2}{2} - 3$

$\Rightarrow 0 = \frac{y^2}{2} - y - 4$

(b) $x = y - 1 \Rightarrow (x-1)^2 = 2x+6$

$\Rightarrow x = 5$ (b) & $x = -1$ (a)

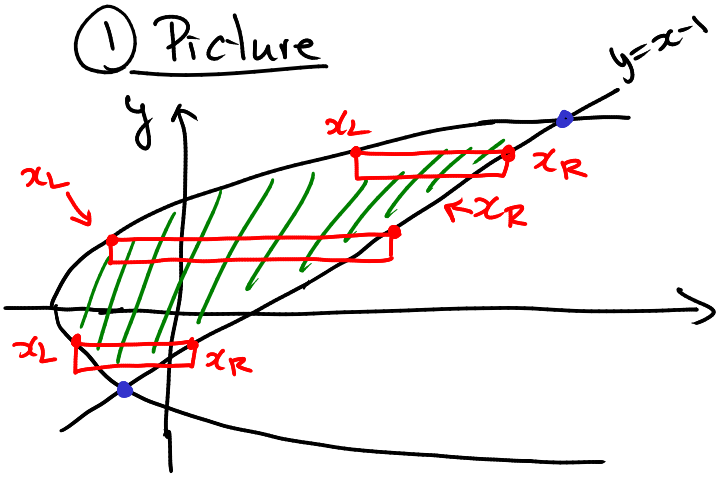
③ Integrate

$\text{Area}(S) = \int_{-1}^{5} \text{upper} - \text{lower}\ dx$

↖ not right.

EXAMPLE 4. Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

① Picture



$$x_R = y + 1$$
$$x_L = \frac{y^2}{2} - 3$$

$$\text{Area}(S) = \int_c^d x_R - x_L \, dy$$

② Intersections

$$y + 1 = \frac{y^2}{2} - 3 \Rightarrow 0 = \frac{y^2}{2} - y - 4$$
$$\Rightarrow 0 = y^2 - 2y - 8$$
$$\Rightarrow 0 = (y+2)(y-4)$$
$$\Rightarrow y = \underbrace{-2}_{c} \text{ or } y = \underbrace{4}_{d}$$

③ Area

$$\text{Area}(S) = \int_{-2}^{4} y + 1 - \left(\frac{y^2}{2} - 3\right) dy$$
$$= \int_{-2}^{4} -\frac{y^2}{2} + y + 4 \, dy$$
$$= \boxed{18}$$