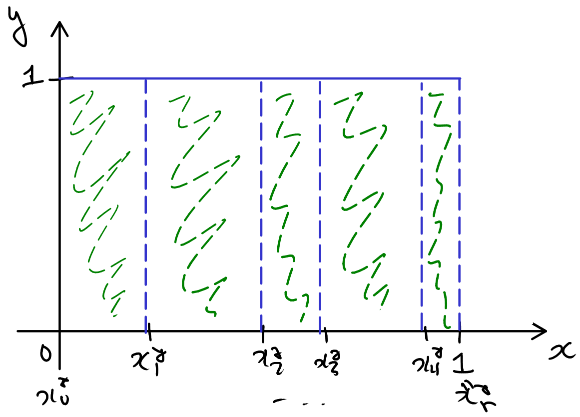
Example 2

Show that $f(x) = 1$ is integrable over the interval $[0, 1]$.



$x_0^*, x_1^*, x_2^*, \ldots, x_n^*$ are pts. in $[0,1]$.

$\Delta x_i = x_i^* - x_{i-1}^*$

$f(x_i^*) = 1$

$$S_n = \sum_{k=1}^{n} f(x_k^*)\,\Delta x_k = \sum_{k=1}^{n} 1 \cdot (x_k^* - x_{k-1}^*)$$

$$= \sum_{k=1}^{n} (x_k^* - x_{k-1}^*)$$

$\underline{n=2}$ $\quad \sum_{k=1}^{\boxed{2}} (x_k^* - x_{k-1}^*) = \cancel{x_1^*} - x_0^* + x_2^* - \cancel{x_1^*}$

$$= x_2^* - x_0^* = 1 - 0 = 1$$

$\underline{n=3}$ $\quad \sum_{k=1}^{3} (x_k^* - x_{k-1}^*) = x_3^* - x_0^* = 1 - 0 = 1$

So, $\quad S_n = x_n^* - x_0^* = 1 - 0 = 1$

So, $\quad \lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 = 1.$

$$\boxed{\int_0^1 1\,dx = 1}$$

$g(x) = x$

$g'(x) = 1$

$\int_0^1 1\,dx = g(1) - g(0) = 1$

Example 4

Express the following limit in term of an integral: on $[2, 5]$

$$\lim_{n\to\infty} \sum_{i=1}^{n} (x_i^3 + x_i \sin x_i)\Delta x.$$

Goal $\lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{(x_i^3 + x_i \sin x_i)} \Delta x = \int_2^5 f(x)\,dx$

By definition:

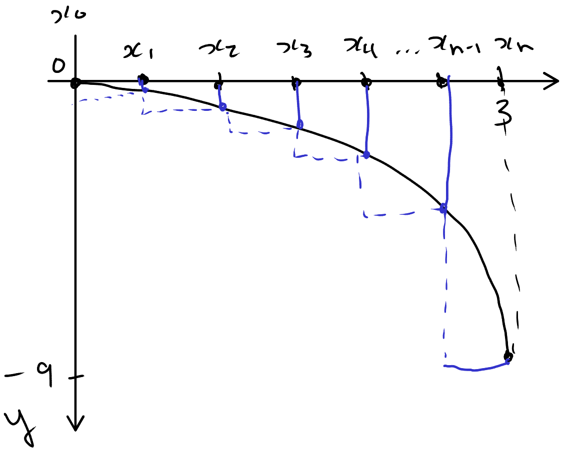
$$\int_2^5 f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \widetilde{f(x_i)}\, \Delta x$$

$$f(x) = x^3 + x \sin x.$$

So, $\lim_{n\to\infty} \sum_{i=1}^{n} (x_i^3 + x_i \sin x_i) \Delta x = \int_2^5 (x^3 + x\sin x)\,dx$

Example 6

Using the last Theorem, compute the integral $\int_0^3 (x^2 - 6x)\,dx$. $\to f(x)$



$$\Delta x = \frac{3-0}{n} = \frac{3}{n}$$

$$x_i = a + i\Delta x = 0 + i\frac{3}{n} = i\frac{3}{n}$$

$$f(x_i) = f\left(i\frac{3}{n}\right) = \frac{9i^2}{n^2} - \frac{6\cdot 3i}{n} = \frac{9i^2}{n^2} - \frac{18i}{n}$$

$$\int_0^3 x^2 - 6x\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{9i^2}{n^2} - \frac{18i}{n}\right)\cdot\frac{3}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{27i^2}{n^3} - \frac{54i}{n^2}\right)$$

$$= \lim_{n\to\infty} \left(27\sum_{i=1}^{n} \frac{i^2}{n^3} - 54\sum_{i=1}^{n} \frac{i}{n^2}\right)$$

$$= \lim_{n\to\infty} \left(\frac{27}{n^3}\sum_{i=1}^{n} i^2 - \frac{54}{n^2}\sum_{i=1}^{n} i\right)$$

$$= \lim_{n\to\infty} \frac{27}{n^3}\,\frac{n\cdot(2n+1)(n+1)}{6} - \frac{54}{n^2}\,\frac{n(n+1)}{2}$$

$$= \lim_{n\to\infty} \frac{27}{n^3}\,\frac{n(2n^2+3n+1)}{6} - \lim_{n\to\infty} \frac{54}{n^2}\,\frac{n^2+n}{2}$$

$$= \lim_{n\to\infty} \frac{27\,(2n^3+3n^2+1)}{6n^3} - \frac{54}{2}$$

$$= \frac{27}{3} - \frac{54}{2} = 9 - 27 = -18$$

$$\boxed{\text{So } \int_0^3 x^2 - 6x\,dx = -18}$$

Example 8

Suppose $\int_0^1 f(x)\,dx = 10$ and $\int_2^1 f(x)\,dx = -5$, compute the value of $\int_0^2 f(x)\,dx$.

Example 10

Compute the value of the definite integral $\displaystyle\int_0^1 (4 + 3x^2)\,dx$.

Example 12

Estimate the integral $\int_1^4 \sqrt{x}\,dx$.