Example 13

Suppose $f(x) = x^3 - x$.

a) Find a formula for $f'(x)$.

b) Sketch the graph of the curve $y = f'(x)$.

a) $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ $\qquad (x+h)^2(x+h)$

$= \lim_{h \to 0} \frac{(x+h)^3 - (x+h) - (x^3 - x)}{h}$

$= \lim_{h \to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x} - h - \cancel{x^3} + \cancel{x}}{h}$

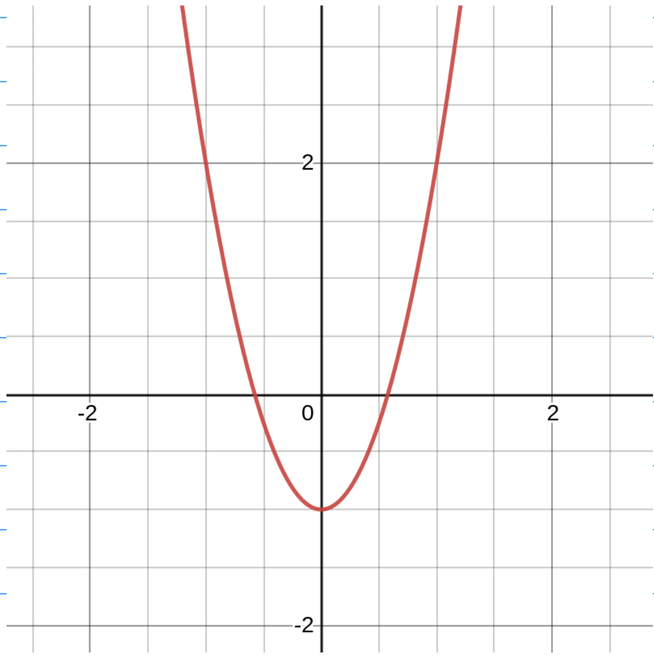
$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 - h}{h}$

$= \lim_{h \to 0} (3x^2 + 3xh + h^2 - 1)$

$= 3x^2 - 1$

So, $f'(x) = 3x^2 - 1$.

b) We have $f'(x) = 3x^2 - 1$



Example 14

If $f(x) = \sqrt{x}$, find the derivative of f and find the domain of f'.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$(a+b)(a-b) = a^2 - b^2$

$$= \lim_{h\to 0} \frac{(\sqrt{x+h} - \sqrt{x})}{h} \cdot \frac{(\sqrt{x+h} + \sqrt{x})}{(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h\to 0} \frac{\cancel{x} + h - \cancel{x}}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$

so, $f'(x) = \dfrac{1}{2\sqrt{x}}$

sqrt $\to$ $x \geq 0$

division by 0 $\to$ $x \neq 0$

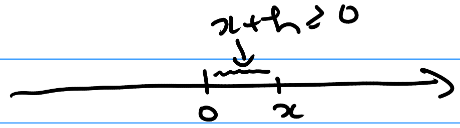
dom $f' = (0, \infty)$

Example 16

Where is the function $f(x) = |x|$ differentiable?

$$|x| = \begin{cases} x & , x \geq 0 \\ -x & , x < 0 . \end{cases}$$

1) When $x > 0$.

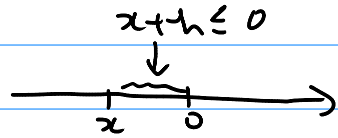


$$f'(x) = \lim_{h \to 0} \frac{|x+h| - |x|}{h} = \lim_{h \to 0} \frac{\cancel{x} + h - \cancel{x}}{h}$$

$$= \lim_{h \to 0} \frac{h}{h} = 1$$

for $x > 0$, $f'(x) = 1$.

2) $x < 0$



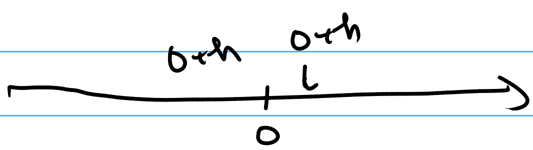
$$f'(x) = \lim_{h \to 0} \frac{|x+h| - |x|}{h}$$

$$= \lim_{h \to 0} \frac{-(x+h) - (-x)}{h}$$

$$= \lim_{h \to 0} \frac{-x - h + x}{h} = \lim_{h \to 0} \frac{-h}{h} = -1$$

for $x < 0$, $f'(x) = -1$

3) $x = 0$



$$\lim_{h \to 0^+} \frac{|0+h| - |0|}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1$$

$$\lim_{h \to 0^-} \frac{|0+h| - |0|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = -1$$

So, $f'(0)$ doesn't exist (undefined). $\square$

Example 19

Find $f''(x)$ of $f(x) = x^3 - x$.

$$(x^3)' = 3x^2 \qquad x' = 1$$

So,

$$f'(x) = 3x^2 - 1$$
$$f''(x) = 6x - 0 = 6x.$$

Doing it with the definition. From Example 13, we know that $f'(x) = 3x^2 - 1$.

So,

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$$
$$= \lim_{h \to 0} \frac{3(x+h)^2 - 1 - (3x^2 - 1)}{h}$$
$$= \lim_{h \to 0} \frac{3\cancel{x^2} + 6xh + 3h^2 \cancel{-1} \cancel{-3x^2} \cancel{+1}}{h}$$
$$= \lim_{h \to 0} \frac{6xh + 3h^2}{h}$$
$$= \lim_{h \to 0} (6x + 3h)$$
$$= 6x$$

So,

$$f''(x) = 6x$$

Example 21

Compute the derivatives of the following functions:

a) $f(x) = x^6$

b) $y = t^{1/5}$

c) $y = u^{\pi}$.

d) $u = v^{2/3}$.

$$\frac{(x+h)^6 - x^6}{h}$$

a) $f'(x) = 6x^5$ $(b=6$, power rule$)$

b) $\frac{dy}{dx} = \frac{1}{5} x^{1/5 - 1} = \frac{1}{5} x^{\frac{1}{5} - \frac{5}{5}}$

$= \frac{1}{5} x^{-4/5} = \frac{1}{5x^{4/5}}$.

c) $\frac{dy}{dx} = \pi u^{\pi - 1}$

d) $\frac{du}{dv} = \frac{2}{3} v^{2/3 - 1} = \frac{2}{3} v^{-1/3} = \frac{2}{3v^{1/3}}$.

Example 23

Compute the derivatives of the following functions:

a) $f(x) = x^8 + 12x^5 + 10x^3 - 6x + 5$.

b) $y = (x^2+1)(x^3+2)$.

c) $v = \frac{x^2+x-2}{x^3+6}$.

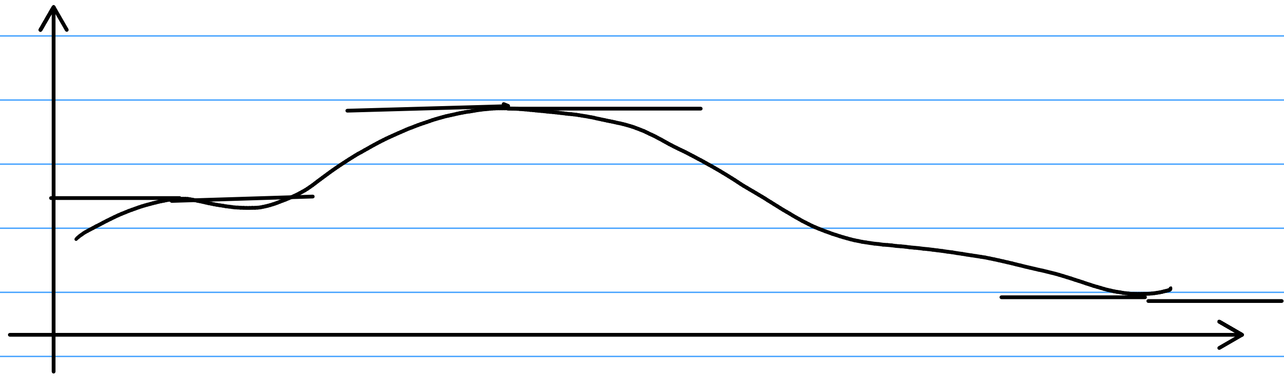
a)
$$f'(x) = (x^8)' + (12x^5)' + (10x^3)' + (-6x)' + 5'$$
$$= 8x^7 + 12(x^5)' + 10(x^3)' - 6(x)' + 5\cdot(1)'$$
$$= 8x^7 + 12\cdot 5x^4 + 10\cdot 3x^2 - 6$$
$$= 8x^7 + 60x^4 + 30x^2 - 6.$$

b)
$$\frac{dy}{dx} = (x^2+1)'(x^3+2) + (x^2+1)(x^3+2)'$$
$$= ((x^2)' + (1)')(x^3+2) + (x^2+1)((x^3)' + (2)')$$
$$= (2x+0)(x^3+2) + (x^2+1)(3x^2+0)$$
$$= 2x(x^3+2) + (x^2+1)3x^2$$
$$= 2x^4 + 4x + 3x^4 + 3x^2$$
$$= 5x^4 + 3x^2 + 4x.$$

c)
$$\frac{dv}{dx} = \frac{(x^2+x-2)'(x^3+6) - (x^2+x-2)(x^3+6)'}{(x^3+6)^2}$$
$$= \frac{(2x+1)(x^3+6) - (x^2+x-2)(3x^2+0)}{(x^3+6)^2}$$
$$= \frac{2x^4 + x^3 + 12x + 6 - (3x^4 + 3x^3 - 6x^2)}{(x^3+6)^2}$$
$$= \frac{-x^4 - 2x^3 + 6x^2 + 12x + 6}{(x^3+6)^2}$$

Example 25

Find the points on the curve $y = x^4 - 6x^2 + 4$ where the tangent line is horizontal.



1) Take the derivative.

$$\frac{dy}{dx} = 4x^3 - 12x$$

2) Find x where $dy/dx = 0$.

$$\frac{dy}{dx} = 0$$

$$\iff 4x^3 - 12x = 0$$

$$\iff (4x^2 - 12)x = 0$$

$$\iff 4x^2 - 12 = 0 \quad \text{or} \quad x = 0$$

$$\iff 4x^2 = 12 \quad \text{or} \quad x = 0$$

$$\iff x^2 = 3 \quad \text{or} \quad x = 0$$

$$\iff x = \pm\sqrt{3}$$

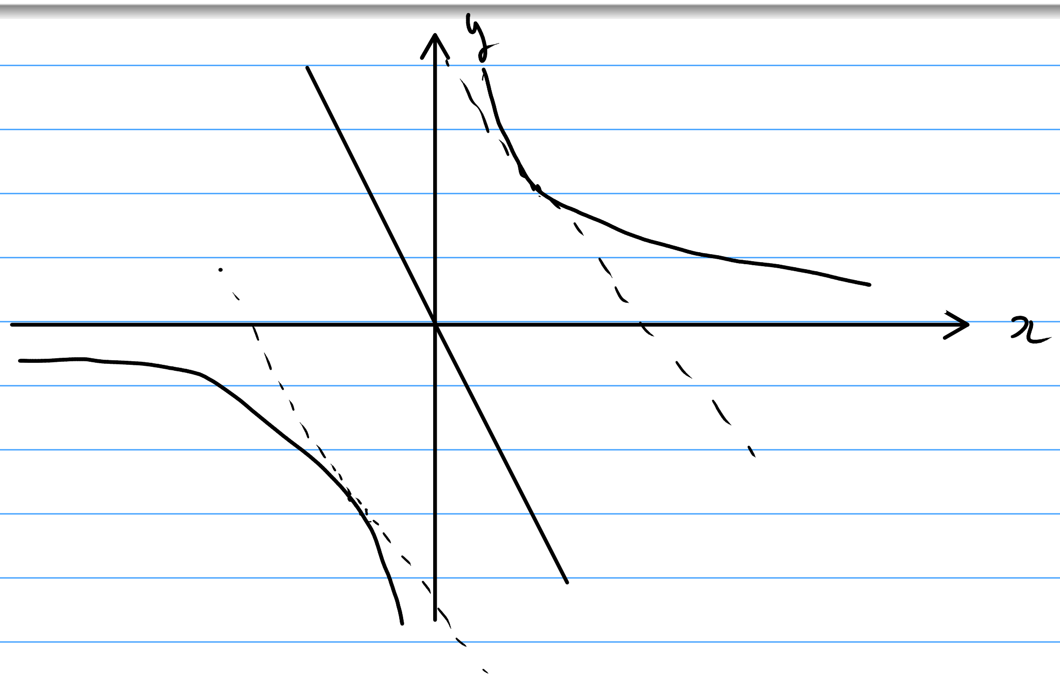
So, the tangent line is horizontal when $x = \sqrt{3}$, $x = -\sqrt{3}$ or $x = 0$.

Example 26

At what points on the hyperbola $xy = 12$ is the tangent line parallel to the line $3x + y = 0$?

$xy = 12$
$\Rightarrow y = 12/x$

$3x + y = 0$
$y = -3x$



<u>Goal</u>: We need to find the slopes of the tangent line to $y = 12/x$ which are equal to the slope of $y = -3x$.

1) $\dfrac{dy}{dx} = 12\left(\dfrac{1}{x}\right)' = 12\,(x^{-1})' = -12x^{-2}$

2) We want

$$-12x^{-2} = -3$$
$$\Leftrightarrow -12\frac{1}{x^2} = -3$$
$$\Leftrightarrow 4 = x^2$$
$$\Leftrightarrow \pm\sqrt{4} = x$$
$$\Leftrightarrow \pm 2 = x$$

The tangent lines are parallel to $3x + y = 0$ at $x = 2$ and $x = -2$. The points are

$y = \dfrac{12}{x} \xrightarrow{x=2} y = 6$ $\qquad (2, 6), (-2, -6)$.

$\xrightarrow[x=-2]{} y = -6$

Example 28

Find the equations of the tangent and normal lines to the curve $y = \sqrt{x}$ at the point $P = (1, 1)$. ▸ Normal line

slope of the normal line

$$m_\perp = -\frac{1}{m}$$ where m is the slope of the line $T(x) = mx + b$.

a) Tangent. $T(x) = mx + b$

$$\frac{dy}{dx} = (\sqrt{x})' = (x^{1/2})' = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}.$$

$$m = \left.\frac{dy}{dx}\right|_{x=1} = \frac{1}{2}$$

Now, $T(1) = 1 \Rightarrow 1 = \frac{1}{2} \cdot 1 + b$

$\Rightarrow b = 1 - 1/2 = 1/2$

So,

$$T(x) = x/2 + 1/2$$

b) Normal line. $N(x) = m_\perp x + b_\perp$

we have $m_\perp = -\frac{1}{m} = -1/{1/2} = -2.$

Also, $N(1) = 1 \Rightarrow 1 = -2 \cdot 1 + b_\perp$

$\Rightarrow b_\perp = 3$

So,

$$N(x) = -2x + 3$$