MATH-241 Calculus 1 | Pierre-Olivier Parisé
Homework 02 | Fall 2021

Assigned date: 09/13/2021 9am
Due date: 09/20/2021 5pm

Last name: Correction.
First name: —
Section: —

Question:	1	2	3	4	5	6	7	Total
Points:	10	10	20	10	15	20	15	100
Score:	—	—	—	—	—	—	—	—

Instructions: You must answer all the questions below and upload your solutions (in a PDF format) to Gradescope (go to `www.gradescope.com` with the Entry code GEK6Y4). Be sure that after you scan your copy, it is clear and readable. You must name your file like this:`LASTNAME_FIRSTNAME.pdf`. A homework may not be corrected if it's not readable and if it's not given the good name. No other type of files will be accepted (no PNG, no JPG, only PDF) and no late homework will be accepted. Good luck!

QUESTION 1 **(10 points)**

Is the following function f continuous at the given point a?

(a) (5 points) $f(x) = \begin{cases} x^2 & x \neq 2 \\ 0 & x = 2. \end{cases}$ at $a = 2$.

(b) (5 points) $f(x) = \frac{x^2+x}{\sin x+1}$ at $a = 0$.

(a) 1. 2 belongs to the domain.

2. We have

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} x^2 = 4$$

3. $f(2) = 0$. So $f(2) \neq 4 = \lim_{x\to 2} f(x)$.

So f is not continuous at $a = 2$.

(b) 1. 0 belongs to the domain because

$\sin 0 + 1 = 1 \neq 0$.

2. We have

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{x^2+x}{\sin x+1} = \frac{\lim_{x\to 0} x^2+x}{\lim_{x\to 0} \sin x+1}$$

$$= \frac{0+0}{\sin 0+1} = \frac{0}{1} = 0.$$

3. $f(0) = 0$. So, $\lim_{x\to 0} f(x) = f(0)$.

Thus, f is continuous at $a = 0$.

Question 2 (10 points)

Where are the following functions continuous?

(a) (5 points) $f(x) = \frac{x^2+x^4}{1+\cos x}$.

(b) (5 points) $f(x) = \frac{x^3}{\sqrt{x^2+x-2}}$.

(a) $f(x) = \frac{g(x)}{h(x)}$ with $g(x) = x^2+x^4$
$h(x) = 1+\cos x$.

According to the quotient rule, f is continuous at all points where $h(x) \neq 0$. We have

$$h(x)=0 \iff 1+\cos x = 0 \iff x = (2n+1)\pi,$$
$$n = \dots,-2,-1,0,1,2,\dots$$

So, f is continuous on $\mathbb{R} \setminus \{(2n+1)\pi : n = \dots,-1,0,1,\dots\}$

(b) $f(x) = \frac{g(x)}{h(x)}$ with $g(x) = x^3$
$h(x) = \sqrt{x^2+x-2}$.

According to root rule, h is continuous where it is defined, so when

$$x^2+x-2 \geq 0 \iff (x+2)(x-1) \geq 0 \iff x \geq 1 \text{ or } x \leq -2.$$

Also, According to the quotient rule, f is continuous where $\sqrt{x^2+x-2} = 0$, so if $x=1$ or $x=-2$.
The function is continuous on

$$(-\infty,-2) \cup (1,\infty).$$

Question 3 (20 points)

Suppose f and g are two continuous function at the point $x = a$. Find the value of following limits. State the appropriate rule that you used to get your answer and show all your work.

(a) (5 points) Find the value of $\lim_{x\to a} f(x)g(x)$ if $f(a) = 2$ and $g(a) = 3$.

(b) (5 points) Find the value of $\lim_{x\to a} f(x)$ if $\sqrt{f(a)} = 2$.

(c) (5 points) Find the value of $\lim_{x\to a} f(x)$ if $\lim_{x\to a}[f(x)g(x) + g(x)] = 2$ and $g(a) = -1$.

(d) (5 points) Find the value of $\lim_{x\to \pi} f(\cos(x))$ if $f(-1) = 0$.

(a) By the product rule, $f(x)g(x)$ is continuous at a.

So, $\lim_{x\to a} f(x)g(x) = f(a)g(a) = 2\cdot 3 = 6$.

(b) By the root rule, $\sqrt{f(x)}$ is continuous at a.

So, $2 = \sqrt{f(a)} = \sqrt{\lim_{x\to a} f(x)} \Rightarrow \lim_{x\to a} f(x) = 2^2 = 4$.

(c) By the product rule, $f(x)g(x)$ is continuous.

By the sum rule, $f(x)g(x) + g(x)$ is continuous.

So, by continuity

$$\lim_{x\to a}[f(x)g(x) + g(x)] = f(a)g(a) + g(a) = 2.$$

$$\Rightarrow \quad -f(a) - 1 = 2 \qquad [g(a) = -1]$$

$$\Rightarrow \quad f(a) = -2-1 = -3.$$

By using continuity again, $\lim_{x\to a} f(x) = -3$.

(d) By the composition rule, $f(\cos(x))$ is continuous.

So, $\lim_{x\to \pi} f(\cos(x)) = f(\cos \pi) = f(-1) = 0$.

Question 4 (10 points)

For what value of the constant c is the function f continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 2 \\ x^3 - cx & \text{if } x \geq 2. \end{cases}$$

For $x \neq 2$, the function is just a polynomial. So it is continuous for all $x \neq 2$. The problem is at $x = 2$.

There are three requirements:

To satisfy the second requirement, we need

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} cx^2 + 2x$$

equals $\displaystyle\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} x^3 - cx$.

$$\Leftrightarrow \quad c \cdot 4 + 2 \cdot 2 = 2^3 - c \cdot 2$$

$$\Leftrightarrow \quad 4c + 4 = 8 - 2c$$

$$\Leftrightarrow \quad c = \frac{4}{6} = \frac{2}{3}.$$

To satisfy the third requirement, we need that

$$\lim_{x \to 2} f(x) = f(2) = \frac{20}{3} \quad \left(c = \frac{2}{3}\right).$$

But we have that

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = \frac{20}{3}.$$

So, the value $c = \frac{2}{3}$ makes the function cont. $\square$

Question 5 (15 points)

Let $f(x) = 3 + 4x^2 - 2x$.

(a) (5 points) Use the definition of the derivative with the limit (see section 2.1 of the lecture notes) to find the slope of the tangent to the curve $y = f(x)$ at the point where $x = a$. No solution will be credited for using the derivative rules from section 2.3.

(b) (5 points) Find the general equation of the tangent line at a point $(a, f(a))$.

(c) (5 points) Graph the curve and the tangent line at the point $(1, 5)$ on a commun picture.

(a) $$f'(a) = \lim_{h \to 0} \frac{3 + 4(a+h)^2 - 2(a+h) - 3 - 4a^2 + 2a}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{3} + \cancel{4a^2} + 8ah + 4h^2 - \cancel{2a} - 2h \cancel{-3} - \cancel{4a^2} + \cancel{2a}}{h}$$

$$= \lim_{h \to 0} \frac{8ah + 4h^2 - 2h}{h} = \lim_{h \to 0} 8a + 4h - 2$$

$$\Rightarrow f'(a) = 8a - 2.$$

(b) We have $T(x) = mx + b = f'(a)x + b = (8a-2)x + b$

So, since $T(a) = f(a)$

$$\Rightarrow f(a) = (8a-2) \cdot a + b$$

$$\Rightarrow b = f(a) - 8a^2 - 2a.$$

So, $T(x) = (8a-2)x + (f(a) - 8a^2 - 2a)$.

(c) $a = 1,\ f(a) = 5 \Rightarrow T(x) = 6x - 5$.

Question 6 (20 points)

Find the equation of the tangent line and the normal line to the curve $y = x + \sqrt{x}$ at the point $P = (1, 2)$. (You may use the derivative rules for this question).

1) Find tangent line $T(x) = mx + b$.

1.1) Slope m.

$$f'(x) = 1 + \frac{1}{2\sqrt{x}} \Rightarrow m = f'(1) = \frac{3}{2}.$$

1.2) y-intercept.

$$T(1) = 2 = \frac{3}{2} \cdot 1 + b \Rightarrow b = \frac{1}{2}.$$

Thus, $T(x) = \frac{3}{2}x + \frac{1}{2}$.

2) Find Normal line $N(x) = m_\perp x + b_\perp$.

1.1) Slope $m_\perp$

We know that $m \cdot m_\perp = -1 \Rightarrow m_\perp = -\frac{2}{3}$.

1.2) y-intercept.

$$N(1) = 2 \ \& \ N(1) = -\frac{2}{3} \cdot 1 + b_\perp$$

$$\Rightarrow b_\perp = \frac{8}{3}.$$

Thus, $N(x) = -\frac{2}{3}x + \frac{8}{3}$.

Question 7 (15 points)

You may use the derivative rules for this question.

(a) (5 points) Find $h'(0)$ if $h(x) = f(x)g(x)$ and if $f(0) = 1$, $g(0) = 2$, $f'(0) = -1$, and $g'(0) = -2$.

(b) (5 points) Find $f'(x)$ if $f(x) = \frac{x}{x+\frac{1}{x}}$.

(c) (5 points) $G''(r)$ if $G(r) = \sqrt{r} + \sqrt[3]{r}$.

(a) By the product rule, $h'(x) = f'(x)g(x) + f(x)g'(x)$

$$\Rightarrow h'(0) = f'(0)g(0) + f(0)g'(0)$$
$$= (-1)\cdot(2) + (1)\cdot(-2)$$
$$\Rightarrow h'(0) = -4.$$

(b) By the rules: $f'(x) = \dfrac{1\cdot\left(x+\frac{1}{x}\right) - x\left(1 - \frac{1}{x^2}\right)}{(x+1/x)^2}$

$$= \frac{x + \frac{1}{x} - x + \frac{1}{x}}{(x+1/x)^2}$$

$$= \frac{2/x}{(x+1/x)^2} = \frac{2}{x(x+1/x)^2}$$

(c) First, $G'(r) = \dfrac{1}{2\sqrt{r}} + \dfrac{1}{3r^{2/3}}$.

Second,

$$G''(r) = \left(\frac{1}{2\sqrt{r}} + \frac{1}{3r^{2/3}}\right)' = \frac{-1}{4r^{3/2}} - \frac{2}{9r^{5/3}}.$$

Thus,

$$G''(x) = -\left(\frac{1}{4r^{3/2}} + \frac{2}{9r^{5/3}}\right).$$