MATH 307

CHAPTER 1

SECTION 1.1: SYSTEMS OF LINEAR EQUATIONS

CONTENTS

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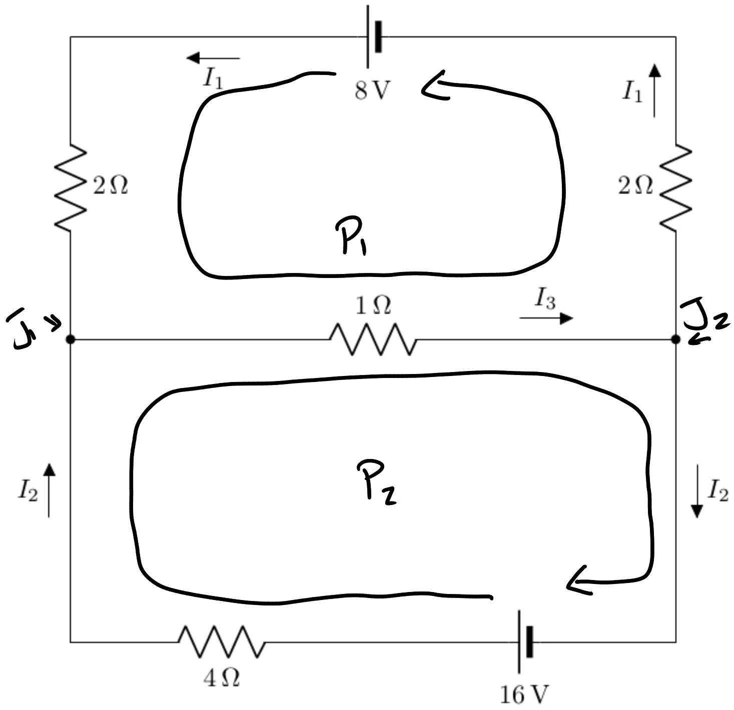

Ohm's Law

- Voltage drop at a resistor is given by $V = IR$.

Kirchhorff's Laws

- Junction: Current flowing into a junction must flow out of it.
- Path: Sum of IR terms in any direction around a closed path is equal to the total voltage in the path in that direction.

<u>Goal:</u> Find the values of I_1, I_2, I_3

J1) $I_1 + I_2 = I_3$

J2) $I_3 = I_1 + I_2$ $\longrightarrow$ $I_1 + I_2 - I_3 = 0$

<u>PATH</u>. P1) $2I_1 + I_3 + 2I_1 = 8 \rightarrow 4I_1 + I_3 = 8$

P2) $4I_2 + I_3 = 16$

To find I_1, I_2, I_3, we must solve the system of lin. eqs.:

$$\begin{cases} I_1 + I_2 - I_3 = 0 \\ 4I_1 \qquad + I_3 = 8 \\ \qquad 4I_2 + I_3 = 16 \end{cases}$$

Linear Equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where

- a_1, a_2, ..., a_n are constants.
- n is the number of variables.
- x_1, x_2, ..., x_n are the variables (unknowns).
- b is the right-hand side constant term.

Systems of Linear Equations

last example
n=3
m=3

$$\begin{array}{ll}
\text{no 1)} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
\text{no 2)} & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
 & \vdots \\
\text{no m)} & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m
\end{array}$$

where

- m is the number of linear equations.
- n is the number of variables.
- a_{11}, ..., a_{mn} are constants.
- b_1, b_2, ..., b_m are the right-hand side constant terms.
- x_1, ..., x_n are the variables (unknowns).

Solution of a System of Linear Equations

A list $(x_1^*, x_2^*, \ldots, x_n^*)$ is a solution to a system of linear equations if it satisfies each equation of the system.

Going back to our previous example

(1, 3, 4) is a solution to our system in the last example.
$\underset{x_1}{\uparrow}\ \underset{x_2}{\uparrow}\ \underset{x_3}{\uparrow}$

$1 + 3 - 4 = 0$ ✓

$4 \cdot 1 + 4 = 8$ ✓

$4 \cdot 3 + 4 = 16$ ✓

Systems of two linear equations with two variables

$$\begin{aligned} x_1 + x_2 &= 0 \\ 2x_1 + x_2 &= 1. \end{aligned}$$

<u>Method 1</u> (Isolate)

1) $x_2 = 1 - 2x_1$

2) $x_1 + 1 - 2x_1 = 0$

$\to 1 - x_1 = 0$

$\to 1 = x_1$

3) $x_2 = 1 - 2 \cdot 1 = -1$

solution: $(\underset{x_1}{\underset{\uparrow}{1}}, \underset{x_2}{\underset{\uparrow}{-1}})$

<u>Method 2</u> (Operations)

1) $x_1 + x_2 = 0 \quad E_1$

$2x_1 + x_2 = 1 \quad E_2$

2) $E_1 - E_2 \to E_1$

$$\begin{array}{rl} & x_1 + x_2 = 0 \\ - & 2x_1 + x_2 = 1 \\ \hline & -x_1 + \quad = -1 \end{array}$$

$\to -x_1 = -1$

$\begin{cases} -x_1 = -1 & E_1 \\ 2x_1 + x_2 = 1 & E_2 \end{cases}$

3) $-E_1 \to E_1$

$-(-x_1) = -1 \quad \to x_1 = 1$

$\begin{cases} x_1 = 1 & E_1 \\ 2x_1 + x_2 = 1 & E_2 \end{cases}$

4) $2E_1 - E_2 \to E_2$

$$\begin{array}{rl} & 2x_1 \quad = 2 \\ - & 2x_1 + x_2 = 1 \\ \hline & -x_2 = 1 \end{array}$$

$\begin{cases} x_1 = 1 & E_1 \\ -x_2 = 1 & E_2 \end{cases}$

$\downarrow$

$\boxed{\begin{cases} x_1 = 1 \\ x_2 = -1 \end{cases}}$

Gauss-Jordan Elimination

Based on three *elementary operations* on the equations:

- Interchange two equations in the system.
- Replace an equation by a multiple of itself.
- Replace an equation by itself plus a multiple of another equation.

Main GOAL: transform our system into

$$\begin{aligned} x + 0y + 0z &= \tilde{b}_1 \\ 0x + y + 0z &= \tilde{b}_2 \\ 0x + 0y + z &= \tilde{b}_3. \end{aligned}$$

Example 1. Find the solution(s) to the following system of linear equations:

$$\begin{aligned} 1x - y + z &= 0 & E_1 \\ 2x - 3y + 4z &= -2 & E_2 \\ -2x - y + z &= 7. & E_3 \end{aligned}$$

1) $-2E_1 + E_2 \to E_2$

$$\begin{array}{rl} & -2x + 2y - 2z = 0 \\ + & 2x - 3y + 4z = -2 \\ \hline & -y + 2z = -2 \end{array}$$

$2E_1 + E_3 \to E_3$

$$\begin{array}{rl} & 2x - 2y + 2z = 0 \\ + & -2x - y + z = 7 \\ \hline & -3y + 3z = 7 \end{array}$$

$$\begin{cases} x - y + z = 0 & E_1 \\ -y + 2z = -2 & E_2 \\ -3y + 3z = 7 & E_3 \end{cases}$$

2) $E_1 - E_2 \to E_1$

$$\begin{array}{rl} & x - y + z = 0 \\ - & -y + 2z = -2 \\ \hline & x - z = 2 \end{array}$$

$-3E_2 + E_3 \to E_3$

$$\begin{array}{rl} & 3y - 6z = 6 \\ + & -3y + 3z = 7 \\ \hline & -3z = 13 \end{array}$$

$$\begin{cases} x - z = 2 & E_1 \\ -y + 2z = -2 & E_2 \\ -3z = 13 & E_3 \end{cases}$$

3). $2E_3 + 3E_2 \to E_2$

$$\begin{array}{rl} & -6z = 26 \\ + & -3y + 6z = -6 \\ \hline & -3y = 20 \end{array}$$

$3E_1 - E_3 \to E_1$

$$\begin{array}{rl} & 3x - 3z = 6 \\ - & -3z = 13 \\ \hline & 3x = -7 \end{array}$$

$$\begin{cases} 3x = -7 \\ -3y = 20 \\ -3z = 13 \end{cases}$$

4)

$$\frac{3x}{3} = \frac{-7}{3} \to \qquad \frac{-3y}{-3} = \frac{20}{-3} \to \qquad \frac{-3z}{-3} = \frac{13}{-3} \to$$

$$\boxed{\begin{aligned} x &= -\frac{7}{3} \\ y &= -\frac{20}{3} \\ z &= -\frac{13}{3} \end{aligned}}$$

Augmented Matrix

More efficient way: transform the system in an **augmented matrix**.

$$
\begin{matrix}
a_{11}\cancel{x_1} + a_{12}\cancel{x_2} + \cdots + a_{1n}\cancel{x_n} = b_1 \\
a_{21}\cancel{x_1} + a_{22}\cancel{x_2} + \cdots + a_{2n}\cancel{x_n} = b_2 \\
\vdots \\
a_{m1}\cancel{x_1} + a_{m2}\cancel{x_2} + \cdots + a_{mn}\cancel{x_n} = b_m
\end{matrix}
\quad \Longrightarrow \quad
\begin{array}{c}
\begin{matrix} x_1 & x_2 & & x_n & b \end{matrix} \\
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn} & b_m
\end{bmatrix}
\end{array}
$$

EXAMPLE 2. Find the augmented matrix of the system of Example 1.

$$
\begin{aligned}
1x - y + z &= 0 \\
2x - 3y + 4z &= -2 \\
-2x - y + z &= 7
\end{aligned}
\quad \Longrightarrow \quad
\begin{array}{c}
\begin{matrix} x & y & z & b \end{matrix} \\
\begin{bmatrix}
1 & -1 & 1 & 0 \\
2 & -3 & 4 & -2 \\
-2 & -1 & 1 & 7
\end{bmatrix}
\end{array}
$$

Elementary operations revisited

Elementary operations on linear equations become elementary operations on the rows of the augmented matrix:

- Interchange two rows.
- Replace a row by a multiple of itself.
- Replace a row by itself plus a multiple of another row.

Example 3. Solve the system:

$$
\begin{aligned}
2x + 3y - z &= 3 \\
-x - y + 3z &= 0 \\
x + 2y + 2z &= 3 \\
y + 5z &= 3.
\end{aligned}
$$

$$
\begin{bmatrix} 2 & 3 & -1 & 3 \\ -1 & -1 & 3 & 0 \\ 1 & 2 & 2 & 3 \\ 0 & 1 & 5 & 3 \end{bmatrix}
\sim
\begin{bmatrix} 2 & 3 & -1 & 3 \\ \boxed{0} & 1 & 5 & 3 \\ 0 & -1 & -5 & -3 \\ 0 & 1 & 5 & 3 \end{bmatrix}
\begin{array}{l} \\ R_1 + 2R_2 \to R_2 \\ R_1 - 2R_3 \to R_3 \\ \\ \end{array}
$$

$$
\sim
\begin{bmatrix} 2 & 0 & -16 & -6 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\begin{array}{l} R_1 - 3R_2 \to R_1 \\ \\ R_2 + R_3 \to R_3 \\ R_2 - R_4 \to R_4 \end{array}
$$

Example 4. Solve the system:

$$\begin{aligned} 4x_1 - 8x_2 - x_3 + x_4 + 3x_5 &= 0 \\ 5x_1 - 10x_2 - x_3 + 2x_4 + 3x_5 &= 0 \\ 3x_1 - 6x_2 - x_3 + x_4 + 2x_5 &= 0. \end{aligned}$$

Reduced row-echelon form

Transformed augmented matrix after row operations:

- Any rows of zero (called zero rows) appear at the bottom.
- The first nonzero entry of a nonzero row is 1 (called a leading 1).
- The leading 1 of a nonzero row appears to the right of the leading 1 of any preceding row.
- All the other entries of a column containing a leading 1 are zero.

Consistent Systems vs Inconsistent Systems

- Consistent: means the system of equations has at least one solution.
  - How to recognize that a system is consistent?
    (1) (2)

- Inconsistent: means the system of equations has no solution.
  - How to recognize that a system is inconsistent?
    (1)

Homogeneous System

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

- Trivial solution: $x_1 = x_2 = \cdots = x_n = 0$.

THEOREM 5. A homogeneous system of m linear equations in n variables has

- infinitely many nontrivial solutions if $m < n$;
- exactly one (trivial solution) if $m = n$;
- no solution if $m > n$.

Goal. Transform the augmented matrix into an new augmented matrix with the following properties:

- any zero rows appear at the bottom.
- The first nonzero entry of a nonzero row is 1.
- The leading 1 of a nonzero row appears to the right of the leading 1 of any preceding row.

Example 6. Determine the values of a, b, and c so that the system

$$\begin{aligned} x - y + 2z &= a \\ 2x + y - z &= b \\ x + 2y - 3z &= c \end{aligned}$$

has solutions.