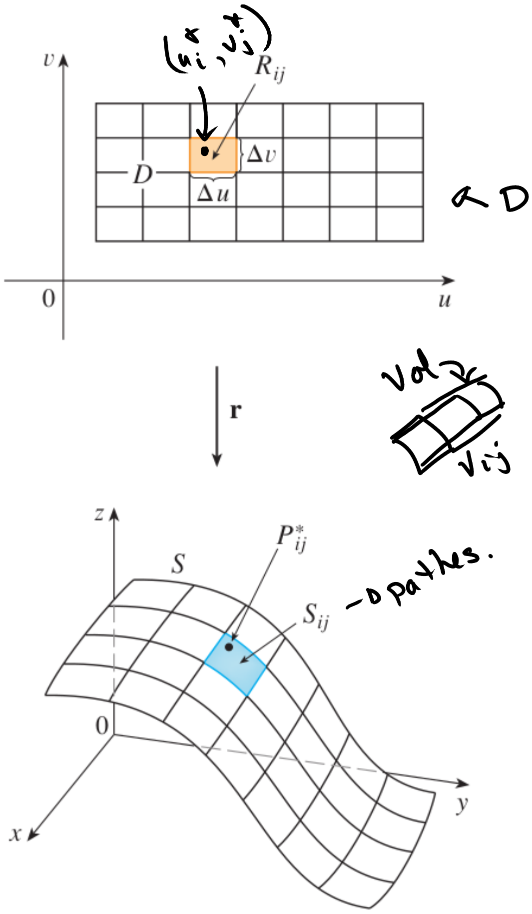
Parametric surfaces.



f: function in 3 variables
S: surface with $\vec{r}(u,v)$ & domain D.

$$\mathrm{Vol}(V_{ij}) \cong A(S_{ij}) \cdot f(P_{ij}^*)$$
$$= \Delta S_{ij} \cdot f(P_{ij}^*)$$

So now, take lim on the number of divisions (number of patches)

$$\boxed{\iint_S f(x,y,z)\,dS = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij}^*)\,\Delta S_{ij}}$$

But,
$$\Delta S_{ij} \cong |\vec{r}_u \times \vec{r}_v|\,\Delta u\,\Delta v.$$

So,
$$\Rightarrow \sum_i \sum_j f(P_{ij}^*)\,\Delta S_{ij} \cong \sum_i \sum_j \underbrace{f(\vec{r}(u_i^*, v_j^*))\,|\vec{r}_u \times \vec{r}_v|}_{g(u,v)}\,\Delta u\,\Delta v.$$

limit on number of patches
$$\to \iint_S f\,dS = \iint_D g(u,v)\,du\,dv.$$

$$\Longrightarrow \boxed{\iint_S f(x,y,z)\,dS = \iint_D \underbrace{f(\mathbf{r}(u,v))\,|\mathbf{r}_u \times \mathbf{r}_v|}_{g(u,v)}\,dA} \to \text{doesn't care about the orientation of } \vec{r}_u \times \vec{r}_v.$$

Mass and center of mass. An aluminum foil S with density $\rho(x,y,z)$.

$$m = \iint_S \rho(x,y,z)\,dS$$

$$\bar{x} = \frac{1}{m}\iint_S x\,\rho(x,y,z)\,dS$$

$$\bar{y} = \frac{1}{m}\iint_S y\,\rho(x,y,z)\,dS$$

$$\bar{z} = \frac{1}{m}\iint_S z\,\rho(x,y,z)\,dS.$$

center of mass:
$(\bar{x}, \bar{y}, \bar{z})$.

EXAMPLE 1 Compute the surface integral $\iint_S x^2\, dS$, where S is the unit sphere $x^2 + y^2 + z^2 = 1$.

① Parametrization & dS.

$$\vec{r}(\theta, \phi) = \langle \underbrace{\cos\theta \sin\phi}_{x},\ \sin\theta \sin\phi,\ \cos\phi \rangle \qquad \begin{array}{l} 0 \le \theta \le 2\pi \\ 0 \le \phi \le \pi. \end{array}$$

$$\vec{r}_\theta = \langle -\sin\theta \sin\phi,\ \cos\theta \sin\phi,\ 0 \rangle$$

$$\vec{r}_\phi = \langle \cos\theta \cos\phi,\ \sin\theta \cos\phi,\ -\sin\phi \rangle$$

$$\Rightarrow \vec{r}_\theta \times \vec{r}_\phi = \langle -\cos\theta \sin^2\phi,\ -\sin\theta \sin^2\phi,\ -\sin\phi \cos\phi \rangle$$

$$\Rightarrow |\vec{r}_\theta \times \vec{r}_\phi| = \sin\phi \ \rightarrow \ dS = \sin\phi\, d\phi\, d\theta$$

② Integrate.

$$\iint_S x^2\, dS = \iint_D (\cos\theta \sin\phi)^2 \sin\phi\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \cos^2\theta \sin^3\phi\, d\phi\, d\theta$$

$$= \Big(\int_0^{2\pi} \underbrace{\cos^2\theta}_{\frac{1+\cos 2\theta}{2}}\, d\theta\Big) \underbrace{\Big(\int_0^{\pi} \sin^3\phi\, d\phi\Big)}_{\text{MT02} \Rightarrow 4/3}$$

$$= \pi \cdot \frac{4}{3}$$

$$= \boxed{\frac{4\pi}{3}}$$

Graphs of functions. $z = g(x,y)$ with $(x,y) \in D$.

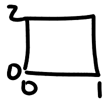
$$\vec{r}(x,y) = \langle x, y, g(x,y) \rangle$$

$$\vec{r}_x = \langle 1, 0, g_x \rangle \quad \to \quad |\vec{r}_x \times \vec{r}_y| = \sqrt{1 + g_x^2 + g_y^2}$$

$$\vec{r}_y = \langle 0, 1, g_y \rangle$$

$$\boxed{\iint_S f(x, y, z)\, dS = \iint_D f(x, y, g(x, y)) \underbrace{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA}_{dS}}$$

EXAMPLE 2 Evaluate $\iint_S y\, dS$, where S is the surface $z = x + y^2$, $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 2$. (See Figure 2.)

$D =$ 

$$\iint_S y\, dS = \iint_D y \sqrt{z_x^2 + z_y^2 + 1}\, dA$$

$$= \int_0^2 \int_0^1 y \sqrt{1 + 4y^2 + 1}\, dx\, dy$$

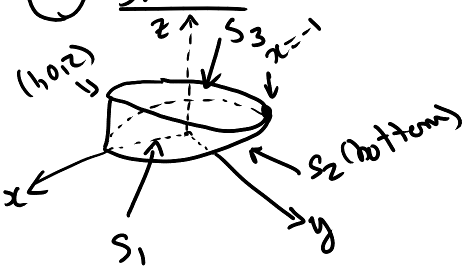
$$= \int_0^2 \int_0^1 y \sqrt{2 + 4y^2}\, dx\, dy$$

$$= \int_0^2 y \sqrt{2 + 4y^2}\, dy \qquad \begin{aligned} u &= 2 + 4y^2 \\ du &= 8y\, dy \end{aligned}$$

$$= \boxed{\frac{13\sqrt{2}}{3}}$$

EXAMPLE 3 Evaluate $\iint_S z\,dS$, where S is the surface whose sides S_1 are given by the cylinder $x^2 + y^2 = 1$, whose bottom S_2 is the disk $x^2 + y^2 \leqslant 1$ in the plane $z = 0$, and whose top S_3 is the part of the plane $z = 1 + x$ that lies above S_2.

① Sketch



Property: $S = S_1 \cup S_2 \cup S_3$

$$\iint_S f\,dS = \iint_{S_1} f\,dS + \iint_{S_2} f\,dS + \iint_{S_3} f\,dS$$

② S_1

$$\vec{r}(\theta, z) = \langle \cos\theta,\ \sin\theta,\ z\rangle$$

$$0 \le \theta \le 2\pi \quad , \quad 0 \le z \le 1+x = 1+\cos\theta$$

$$\vec{r}_\theta = \langle -\sin\theta,\ \cos\theta,\ 0\rangle$$
$$\vec{r}_z = \langle 0,\ 0,\ 1\rangle$$

$$\vec{r}_\theta \times \vec{r}_z = \langle \cos\theta,\ \sin\theta,\ 0\rangle \Rightarrow |\vec{r}_\theta \times \vec{r}_z| = 1$$

$$\Rightarrow \iint_{S_1} z\,dS = \int_0^{2\pi}\int_0^{1+\cos\theta} z\,dz\,d\theta = \frac{3\pi}{2}$$

③ S_2

$$\vec{r}(r,\theta) = \langle r\cos\theta,\ r\sin\theta,\ 0\rangle$$

$$\iint_{S_2} z\,dS = \iint_{S_2} 0\,dS = 0 .$$

④ S_3 $\quad z = 1+x$

$$\vec{r}(\rho,\theta) = \langle \overset{x}{\rho\cos\theta},\ \overset{y}{\rho\sin\theta},\ \overset{z}{1+\rho\cos\theta}\rangle$$

$$\iint_{S_3} z\,dS = \iint_{x^2+y^2\le 1} (1+x)\sqrt{1+1^2+0^2}\,dA$$

$$= \sqrt{2}\iint_{x^2+y^2\le 1} (1+x)\,dA$$

$x = r\cos\theta$
$y = r\sin\theta$

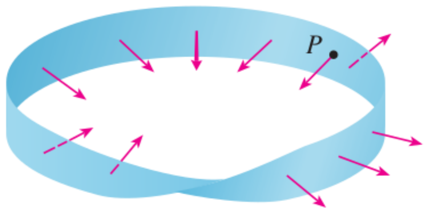
$$= \sqrt{2}\int_0^{2\pi}\int_0^1 (1+r\cos\theta)\,r\,dr\,d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\int_0^1 r + r^2\cos\theta\,dr\,d\theta = \sqrt{2}\pi$$

$$\iint_S z\,dS = \frac{3\pi}{2} + 0 + \sqrt{2}\pi = \boxed{\frac{3+2\sqrt{2}}{2}\pi}$$

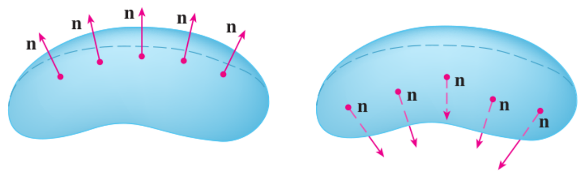
Surface integral of Vector Fields.

Non-orientable surfaces.



Surface has only one side.
This means there is no way of defining a normal properly

Orientable surface.



Surface has two sides.
We can define two normals
- one pointing "outward"
- one pointing "inward".

Special orientations:

1. Graph of a function.

$z = g(x, y)$ then

$$\vec{n} = \frac{\langle -g_x, -g_y, 1\rangle}{\sqrt{1+g_x^2+g_y^2}} \rightarrow \vec{r}_x \times \vec{r}_y$$

$\vec{k}$ - component of $\vec{n} > 0 \rightarrow$ upward
$\vec{k}$ - component of $\vec{n} < 0 \rightarrow$ downward

2. Parametric surface.

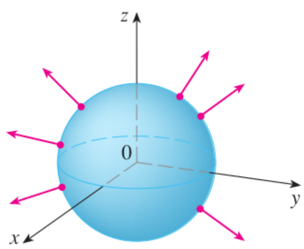
S is given by $\vec{r}(u,v)$, then

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

Example with a sphere.

closed surface: a sphere of radius a.

$$\vec{r}(\theta,\phi) = \langle a\cos\theta\sin\phi,\ a\sin\theta\sin\phi,\ a\sin\phi\rangle$$



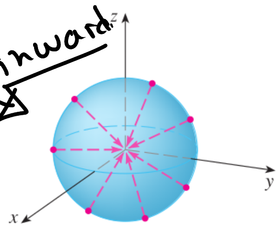
$$\vec{r}_\theta \times \vec{r}_\phi = \langle -a^2\cos\theta\sin^2\phi,\ -a^2\sin\theta\sin^2\phi,\ -a^2\sin\phi\cos\phi\rangle$$

$$|\vec{r}_\theta \times \vec{r}_\phi| = a^2\sin\phi$$

$$\Rightarrow \vec{n} = \langle -\cos\theta\sin\phi,\ -\sin\theta\sin\phi,\ -\cos\phi\rangle$$

$$\rightarrow \vec{n} = \frac{-1}{a}\langle a\cos\theta\sin\phi,\ a\sin\theta\sin\phi,\ a\cos\phi\rangle = \frac{-1}{a}\vec{r}$$

inward

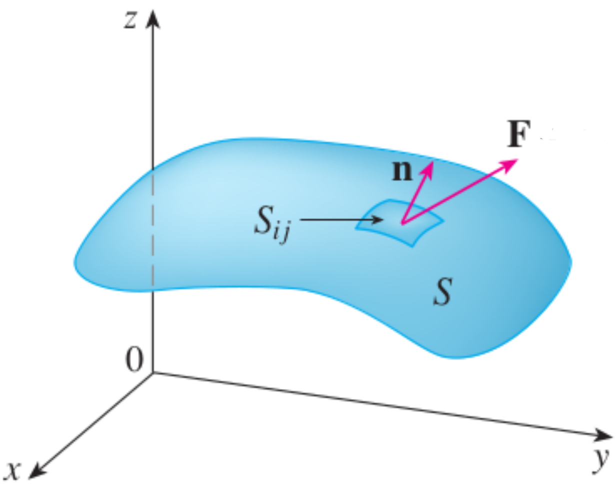


Positive orientation. When $\vec{n}$ points outward.

Negative orientation: $\vec{n}$ points inward.

The convention is to take the positive orientation.

Flux integral (or Surface integral).



Suppose S is a surface & $\vec{F} = \rho \vec{v}$ ($\vec{v}$ speed & ρ: density).

S_{ij} : patches of the surface S.

Approximate the mass of fluid passing through S_{ij} by

$$(\vec{n} \cdot \vec{F})\, A(S_{ij}) \to (\text{kg}/\text{s}).$$

Add all the contributions & take the limit on the number of patches then

8 Definition If $\mathbf{F}$ is a continuous vector field defined on an oriented surface S with unit normal vector $\mathbf{n}$, then the **surface integral of F over** S is

$$\boxed{\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS}$$

This integral is also called the **flux** of $\mathbf{F}$ across S.

- Parametric surface: Integral formula.

If S is given by $\vec{r}(u,v)$, then

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

&

$$dS = |\vec{r}_u \times \vec{r}_v|\, dA$$

Then,

$$d\vec{S} = (\vec{r}_u \times \vec{r}_v)\, dA$$

$$\boxed{\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA}$$

x, y, z by the coordinates of $\vec{r}(u,v)$.

- Graph of a function: Integral formula.

$z = g(x,y)$, then $\vec{r}_x \times \vec{r}_y = \langle -g_x, -g_y, 1 \rangle$

Also, if $\vec{F} = \langle P, Q, R \rangle$, then

$$\vec{F} \cdot (\vec{r}_x \times \vec{r}_y) = -P g_x - Q g_y + R$$

$$\boxed{\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(-P \frac{\partial g}{\partial x} - Q \frac{\partial g}{\partial y} + R \right) dA}$$

EXAMPLE 4 Find the flux of the vector field $\mathbf{F}(x, y, z) = z\,\mathbf{i} + y\,\mathbf{j} + x\,\mathbf{k}$ across the unit sphere $x^2 + y^2 + z^2 = 1$.

① Parametrization.

$$\vec{r}(\theta, \phi) = \langle \cos\theta \sin\phi,\ \sin\theta \sin\phi,\ \cos\phi \rangle \qquad \begin{array}{l} 0 \le \theta \le 2\pi \\ 0 \le \phi \le \pi \end{array}$$

so,

$$\vec{r}_\theta \times \vec{r}_\phi = \langle -\cos\theta \sin^2\phi,\ -\sin\theta \sin^2\phi,\ -\cos\phi \sin\phi \rangle.$$

② Integrate.

$$F(\vec{r}(\theta,\phi)) = \vec{F}(\cos\theta \sin\phi,\ \sin\theta \sin\phi,\ \cos\phi)$$
$$= \langle \cos\phi,\ \sin\theta \sin\phi,\ \cos\theta \sin\phi \rangle$$

$$\vec{F}(\vec{r}(\theta,\phi)) \cdot (\vec{r}_\theta \times \vec{r}_\phi) = -\cos\theta\cos\phi\sin^2\phi - \sin^2\theta\sin^3\phi - \cos\theta\cos\phi\sin^2\phi$$
$$= -2\cos\theta\cos\phi\sin^2\phi - \sin^2\theta\sin^3\phi.$$

so,

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot (\vec{r}_\theta \times \vec{r}_\phi)\, dA$$

$$= \int_0^\pi \int_0^{2\pi} -2\cos\theta\cos\phi\sin^2\phi - \sin^2\theta\sin^3\phi \, d\theta\, d\phi$$

$$= -2\left(\int_0^{2\pi} \cos\theta\, d\theta\right)\left(\int_0^\pi \cos\phi \sin^2\phi\, d\phi\right)$$

$$- \left(\int_0^{2\pi} \sin^2\theta\, d\theta\right)\left(\int_0^\pi \sin^3\phi\, d\phi\right)$$

$$= 0 - \frac{4\pi}{3}$$

$$= \boxed{-\frac{4\pi}{3}}$$

EXAMPLE 5 Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = y\,\mathbf{i} + x\,\mathbf{j} + z\,\mathbf{k}$ and S is the boundary of the solid region E enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.

Applications to Physics.

Electric Flux.

$$\boxed{\iint_S \mathbf{E} \cdot d\mathbf{S}}$$

Gauss' Law.

$$\boxed{Q = \varepsilon_0 \iint_S \mathbf{E} \cdot d\mathbf{S}}$$

Heat flow.

$$\boxed{-K \iint_S \nabla u \cdot d\mathbf{S}}$$

EXAMPLE 6 The temperature u in a metal ball is proportional to the square of the distance from the center of the ball. Find the rate of heat flow across a sphere S of radius a with center at the center of the ball.