$$\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta = \frac{f(z)}{2\pi i} \int_\gamma \frac{1}{\zeta - z}\, d\zeta$$

# Math 644

# Chapter 5

## Section 5.2: Winding Number

Contents




Created by: Pierre-Olivier Parisé


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**LEMMA 1.** If $\gamma$ is a cycle and $a \notin \gamma$, then

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{\zeta - a}\,d\zeta$$

is an integer.

**Proof.** Write $\gamma = \sum_{j=1}^{N} \gamma_j$, $\gamma_j$ are closed curve.

and $$\int_\gamma f(z)\,dz = \sum_{j=1}^{N} \int_{\gamma_j} f(z)\,dz.$$

We may suppose that each $\gamma_j$ is cont. different. (piecewise). We can deal with only one of them. From now on, let $\gamma$ be a closed piecewise cont. diff. curve. $\gamma : [0,1] \to \mathbb{C}$.

Define $$h(x) = \int_0^x \frac{\gamma'(t)}{\gamma(t) - a}\,dt.$$

Then, $h'(x)$ exists & $h'(x) = \dfrac{\gamma'(x)}{\gamma(x) - a}$, except at finitely many $x$.

$$\begin{aligned}\frac{d}{dx}\left[e^{-h(x)}(\gamma(x) - a)\right] &= -h'(x)e^{-h(x)}(\gamma(x) - a) \\ &\quad + e^{-h(x)}\gamma'(x) \\ &= -\gamma'(x)e^{-h(x)} + \gamma'(x)e^{-h(x)} \\ &= 0 \qquad \text{(except at finitely many } x\text{)}.\end{aligned}$$

Since $e^{-h(x)}(\gamma(x)-a)$ is continuous, it must be constant in $[0,1]$

$$\Rightarrow \quad e^{-h(1)}(\gamma(1)-a) = e^{-h(0)}(\gamma(0)-a)$$

$$= e^{0}(\gamma(0)-a)$$

$$= \gamma(1)-a \quad (\gamma \text{ closed curve})$$

Since $a \notin \gamma$,

$$e^{-h(1)} = 1$$

$$\Rightarrow \quad h(1) = 2k\pi i \quad , k \in \mathbb{Z}$$

So,

$$\frac{1}{2\pi i}\int_0^1 \frac{\gamma'(t)}{\gamma(t)-a}\,dt = \frac{h(1)}{2\pi i} = k. \qquad \square$$

**Definition 2.** If $\gamma$ is a cycle, then the **index** or **winding number** of $\gamma$ about $a$ is

$$n(\gamma, a) := \frac{1}{2\pi i} \int_\gamma \frac{1}{\zeta - a}\, d\zeta \quad (a \notin \gamma).$$

**Proposition 3.** Let $\gamma$ be a cycle.

**(a)** $n(\gamma, a)$ is an analytic function of $a$, for $a \notin \gamma$.

**(b)** $n(\gamma, a)$ is constant in each component of $\mathbb{C}\backslash\gamma$.

**(c)** $n(\gamma, a) \to 0$ as $a \to \infty$. In particular, $n(\gamma, a) = 0$ for any $a$ in the unbounded component of $\mathbb{C}\backslash\gamma$.

**(d)** $n(-\gamma, a) = -n(\gamma, a)$.

**(e)** $n(\gamma_1 + \gamma_2, a) = n(\gamma_1, a) + n(\gamma_2, a)$.

**Proof.**

(a) Since the fct. $\zeta \mapsto 1$ is continuous on $\gamma$, then from Lemma 2 in §4.4,

$$n(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{1}{\zeta - a}\, d\zeta$$

is analytic on $\mathbb{C}\backslash\gamma$.

(b) Let $\Omega$ be a component of $\mathbb{C}\backslash\gamma$. Suppose $\Omega$ is bounded, so that $\Omega$ is bounded by some $\gamma_j$ (*), where $\gamma = \sum_j \gamma_j$. From Lemma 1, $n(\gamma_j, a)$ is constant. In other words, $a \in \Omega$, $n(\gamma_j, a) = n(\gamma, a)$.

(*) that contributes to the winding number.

(c) Since $a \notin \gamma$,

$$\frac{1}{|z-a|} \leq \frac{1}{\operatorname{dist}(\gamma, a)} \longrightarrow 0, \quad a \to \infty.$$

therefore, if $\sigma$ is a polygonal curve as in thm. 4,

$$n(\gamma, a) = n(\sigma, a) \leq \frac{|\sigma|}{2\pi} \frac{1}{\operatorname{dist}(\sigma, a)} \to 0.$$

By lemma 1, & (a) $n(\gamma, a)$ should be constant

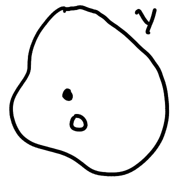
$$\Rightarrow \quad n(\gamma, a) = 0 \qquad \forall a \in \Omega$$

where $\Omega$ is the unbounded component of $\mathbb{C} \setminus \gamma$.

(d) Direct calculations.

(e) Direct calculations. $\square$

Some Intuition:




① **Difference in the argument.**

Suppose $\gamma(t) = r(t)e^{i\theta(t)}$ where

- $r(t)$, $\theta(t)$ are piecewise cont. diff.
- $0 \leq t \leq 1$.
- $\gamma(0) = \gamma(1)$.

Then,

$$n(\gamma, 0) = \mathrm{Re}\left[\frac{1}{2\pi i}\int_\gamma \frac{1}{z}\,dz\right]$$

$$= \mathrm{Re}\left[\frac{1}{2\pi i}\int_0^1 \frac{r'(t)e^{i\theta(t)} + r(t)i\theta'(t)e^{i\theta(t)}}{r(t)e^{i\theta(t)}}\,dt\right]$$

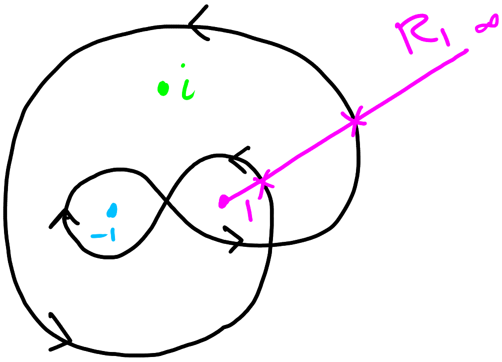
$$= \mathrm{Re}\left[\frac{1}{2\pi i}\int_0^1 \frac{r'(t)}{r(t)} + i\,\theta'(t)\,dt\right]$$

$$= \frac{1}{2\pi}\int_0^1 \theta'(t)\,dt$$

$$= \frac{\theta(1) - \theta(0)}{2\pi}.$$

Net change in the "argument" $\div$ by $2\pi$.

② **Rays and number of connected components.**

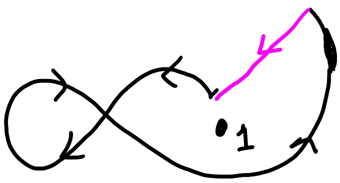



Goal : Find $n(\gamma, 1)$.

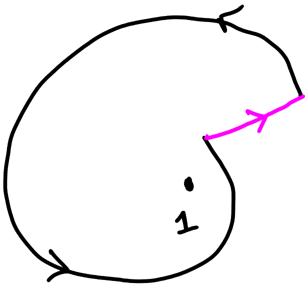
i. Draw ray $R_1$ from 1 to $\infty$.

ii. Locate intersections of $R_1$ with $\gamma$.

iii. Consider each connected component of $\gamma \setminus R_1$.

- 

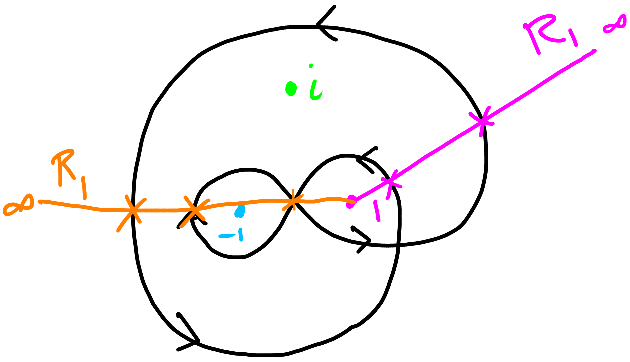
$\gamma_1$ $\longrightarrow$ $n(\gamma_1, 1) = 1$.

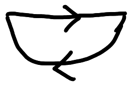
- 

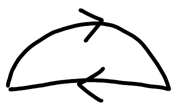
$\gamma_2$ $\longrightarrow$ $n(\gamma_2, 1) = 1$.

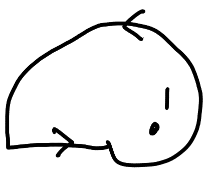
Overall, $n(\gamma, 1) = n(\gamma_1, 1) + n(\gamma_2, 1) = 2$.

Works for any rays:

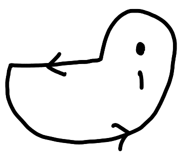



*  $\gamma_1 \longrightarrow n(\gamma_1, 1) = 0$

*  $\gamma_2 \longrightarrow n(\gamma_2, 1) = 0$

*  $\gamma_3 \longrightarrow n(\gamma_3, 1) = 1$

$$n(\gamma, 1) = \sum_j n(\gamma_j, 1) = 2.$$

*  $\gamma_4 \longrightarrow n(\gamma_4, 1) = 1$

# Homologous Curves

**Definition 4.** Closed curves $\gamma_1$ and $\gamma_2$ are **homologous** in a region $\Omega$ if $n(\gamma_1 - \gamma_2, a) = 0$ for all $a \notin \Omega$ and we write $\gamma_1 \sim \gamma_2$.

Remarks:

- Homology is an equivalence relation on curves in $\Omega$.
- A closed curve is said to be **homologous to 0** if $n(\gamma, a) = 0$ for all $a \notin \Omega$. In this case, we write $\gamma \sim 0$.

**Example 5.** Show that $\gamma_1(t) = r_1 e^{it}$ and $\gamma_2(t) = r_2 e^{it}$ $(0 \leq t \leq 2\pi)$ are homologous in $\Omega := \{z : |z| < R\}$, where $r_1 < r_2 < R$.

the curve $\gamma = \gamma_1 - \gamma_2$ is

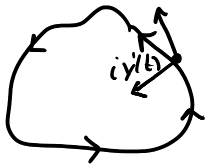

If $a \notin \{z : |z| < R\}$, then

$$n(\gamma_1 - \gamma_2, a) = n(\gamma_1, a) - n(\gamma_2, a) = 0 - 0 = 0.$$

**Definition 6.** Let $\Omega$ be a bounded region in $\mathbb{C}$ bounded by finitely many piecewise continuously differentiable simple closed curves. The **positive orientation** of $\partial\Omega$ is a parametrization that has the following property:

**(a)** for each $t \in [0, 1]$ where the derivative exists, there is an $\varepsilon(t) > 0$ such that $\gamma(t) + ui\gamma'(t) \in \Omega$, for all $u \in [0, \varepsilon(t)]$.

Notes:

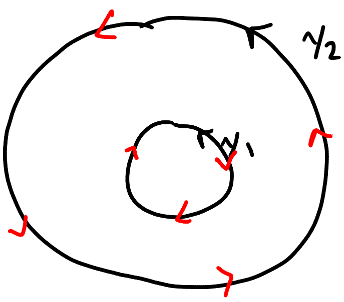



① When the positive orientation is chosen for $\partial\Omega$, then

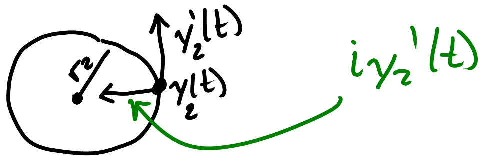
- $n(\partial\Omega, a) = 0$, for each $a \notin \overline{\Omega}$;
- $n(\partial\Omega, a) = 1$, for each $a \in \Omega$.

**Example 7.** Find the positive orientation of the boundary of the closed annulus $A := \{z : r_1 \leq |z| \leq r_2\}$.

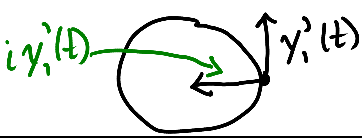
$\gamma_2(t) = r_2 e^{it}$, $\gamma_1(t) = r_1 e^{it}$, $0 \leq t \leq 2\pi$.




- $\gamma_2'(t) = i r_2 e^{it}$




- $\gamma_1'(t) = i r_1 e^{it}$




Change $\gamma_1$ into $-\gamma_1$ to obtain positive orientation.

**Definition 8.**

**(a)** A region $\Omega \subset \mathbb{C}^*$ is called **simply-connected** if $\mathbb{C}^* \setminus \Omega$ is connected.

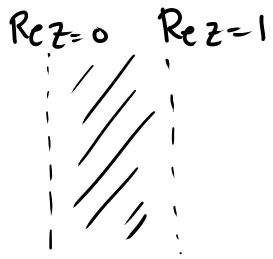
**(b)** Equivalently, a region $\Omega$ is simply-connected if $\mathbb{S}^2 \setminus \pi(\Omega)$ is connected, where $\pi$ is the stereographic projection.

**Example 9.** Show that

**(a)** the unit disk is simply connected;

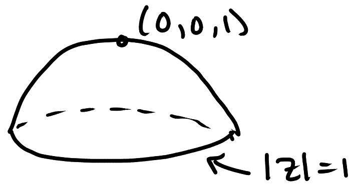
**(b)** the vectical strip $\Omega = \{z : 0 < \operatorname{Re} z < 1\}$ is simply connected;

**(c)** $\mathbb{C} \setminus \{0\}$ is not simply connected.




(a) $\mathbb{C}^* \setminus \mathbb{D} = \{ z \in \mathbb{C} : |z| \geq 1 \} \cup \{\infty\}$

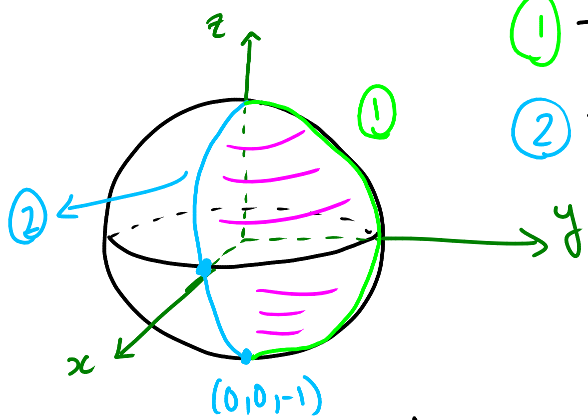
$\overset{\pi}{\cong}$




connected.

(b) $\mathbb{C}^* \setminus \Omega = \{ z \in \mathbb{C} : \operatorname{Re} z \geq 1 \} \cup \{ z \in \mathbb{C} : \operatorname{Re} z \leq 0 \} \cup \{\infty\}$

$\overset{\pi}{\cong}$



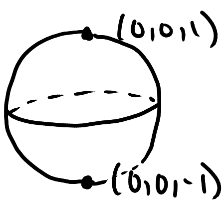

1 $\pi(\operatorname{Re} z = 0) = \left(0, \frac{2y}{y^2+1}, \frac{y^2-1}{y^2+1}\right)$

2 $\pi(\operatorname{Re} z = 1) = \left(\frac{2}{2+y^2}, \frac{2y}{2+y^2}, \frac{y^2}{2+y^2}\right)$

$\rightarrow$ connected

(c) $\mathbb{C}^* \setminus (\mathbb{C} \setminus \{0\}) = \{0\} \cup \{\infty\}$

$\overset{\pi}{\cong}$




not simply-connected.

**Theorem 10.**

(a) A region $\Omega \subset \mathbb{C}$ is simply-connected if and only if every cycle in $\Omega$ is homologous to 0 in $\Omega$.

(b) If $\Omega$ is not simply-connected then we can find a simple closed polygonal curve contained in $\Omega$ which is not homologous to 0.

**Proof.**

(a) $(\Rightarrow)$ $\Omega$ simply-connected.

Let $\gamma$ be a cycle in $\Omega$ and $a \notin \Omega$.

Since $B = \mathbb{C}^* \setminus \Omega$ is connected, $B$ must be in one of the component of $\mathbb{C}^* \setminus \gamma$.

Since $\infty \in B$, $B$ is in the unbounded component of $\mathbb{C}^* \setminus \gamma$ $\Rightarrow$ $n(\gamma, a) = 0$.

$(\Leftarrow)$ Suppose $\mathbb{C}^* \setminus \Omega$ is not connected:

$$\mathbb{C}^* \setminus \Omega = A \cup B,$$

where $A$ & $B$ are closed sets in $\mathbb{C}^*$ and $A \cap B = \emptyset$. WLOG, assume $\infty \in B$.

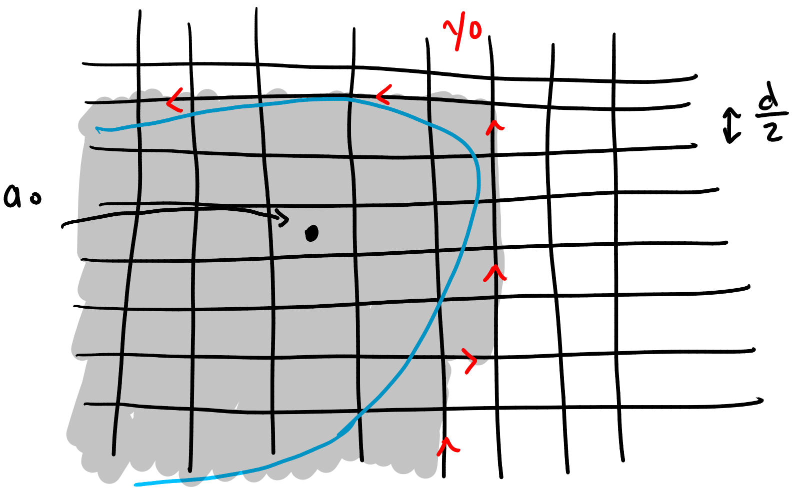
$A$ is closed $\Rightarrow$ $\{z : |z| > R\} \cup \{\infty\} \cap A = \emptyset$

$\Rightarrow$ $A$ is bounded.

Let $a_0 \in A \subseteq \mathbb{C}^* \setminus \Omega$. We construct $\gamma_0 \subseteq \Omega$ s.t.

$$n(\gamma_0, a_0) \neq 0.$$

Let $d = \operatorname{dist}(A, B) = \inf\{|a - b|,\ a \in A, b \in B\} > 0$ & $d = 1$ if $B = \{\infty\}$.




1) Pave plane with squares of side $\frac{d}{2}$, such that $a_0$ center of one square

2) Each square has the positive orientation (counter clockwise)

3) Shade all squares $S_j$ s.t. $\overline{S_j} \cap A \neq \emptyset$

4) Let $\gamma_0$ be the cycle $\cup S_j$ ( after cancelling sides with opposite direction).

5) Now, $\gamma_0 \subseteq \Omega$ because $\gamma_0 \cap (A \cup B) = \emptyset$

6) We have $n(\gamma_0, a_0) = 1$ because $a_0$ is in one bounded component of $\gamma_0$.

7) $\gamma_0 = \sum_{j=1}^{N} \sigma_j$, where $\sigma_j$ is a polygonal closed curve. Then at least one of the $\sigma_j$ is not homologous to $0$. $\square$

↳ This gives you part (b).

**Corollary 11.** Suppose $f$ is analytic on a simply-connected region $\Omega$. Then

**(a)** $\int_\gamma f(z)dz = 0$ for all closed curves $\gamma \subset \Omega$;

**(b)** there exists a function $F$ analytic on $\Omega$ such that $F' = f$;

**(c)** if also $f(z) \neq 0$ for all $z \in \Omega$, then there exists a function $g$ analytic on $\Omega$ such that $f = e^g$.

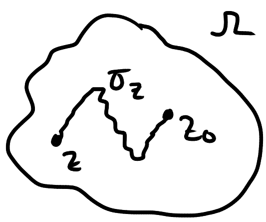
**Proof.**

(a) Since $\gamma \subseteq \Omega$ and $\Omega$ is simply-connected, by Thm. 10, $\gamma \sim 0$. This means that $n(\gamma, a) = 0$ $\forall a \notin \Omega$. By Cauchy's theorem,

$$\int_\gamma f(z)\,dz = 0 .$$

(b) Fix $z_0 \in \Omega$ and define

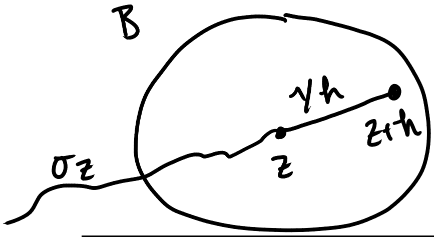
$$F(z) = \int_{\sigma_z} f(\zeta)\,d\zeta$$




If $\sigma_z$ & $\gamma_z$ are two different curves connecting $z$ to $z_0$, then $\sigma_z + (-\gamma_z)$ is a closed path & $\int_{\sigma_z} f(z)\,dz = \int_{\gamma_z} f(z)\,dz$.

$$\frac{F(z+h) - F(z)}{h} = \frac{\int_{\sigma_{z+h}} f(\zeta)\,d\zeta - \int_{\sigma_z} f(z)\,d\zeta}{h}$$

Let $B$ be a disk centered at $z$ s.t. $B \subseteq \Omega$. Then, for $h$ small enough, write




$$\sigma_{z+h} = \sigma_z + \gamma_h$$

$$\Rightarrow \left|\frac{F(z+h)-F(z)}{h}\right| \leq \left|\frac{\int_{\gamma_h} f(\zeta)-f(z)\, d\zeta}{h}\right|$$

$$\leq \sup_{\gamma_h} |f(\zeta)-f(z)| \xrightarrow[h\to 0]{} 0$$

therefore, $F'(z) = f(z)$, $\forall z \in \Omega$.

(c) From the assumption:

$$\frac{f'}{f} \text{ is holomorphic in } \Omega.$$

From (b), $\exists g$ analytic in $\Omega$ s.t.

$$g' = \frac{f'}{f}$$

Set $h = \frac{f}{e^g} = fe^{-g}$ (in $\Omega$).

$$\Rightarrow h' = f'e^{-g} - fe^{-g}\cdot g' = f'e^{-g} - f'e^{-g} = 0$$

$$\Rightarrow h \equiv c \text{ in } \Omega \quad (c \neq 0)$$

Fix $z_0 \in \mathbb{C}$. Set $f(z_0) = e^{a_0 + i\theta_0}$, where

$$e^{a_0} = |f(z_0)| \quad \& \quad \arg f(z_0) = \theta_0 .$$

Let $a = a_0 - \operatorname{Re} g(z_0) + i(\theta_0 - \operatorname{Im} g(z_0))$

$$\Rightarrow \quad (g+a)' = g' = \frac{f'}{f} \qquad \text{and}$$

$$\frac{f(z_0)}{e^{g(z_0)+a}} = 1$$

so that $h \equiv 1$ & $f(z) = e^{\overbrace{g(z)+a}^{\text{new } g(z)}}$, $z \in \Omega$. $\square$

① **"Uniqueness" in (b).**

② **"Uniqueness" in (c).**

Definition 12. If $g$ is analytic in a region $\Omega$ and if $f = e^g$ then $g$ is called a **logarithm** of $f$ in $\Omega$ and is written $g(z) = \log f(z)$. The function $g$ is uniquely determined by its value at one point $z_0 \in \Omega$.

Notes:

① $f$ has countably many logarithms, which differ by $2\pi ki$. To specify $\log f(z)$ uniquely, we have to specify its value at one point $z_0 \in \Omega$.

② We do not claim that we can define a logarithm on $f(\Omega)$ and then composed with $f$ to obtain $\log f(z)$.

Example 13. Consider the function $z \mapsto (z-1)/(z+1)$, for $z \in \mathbb{D}$.