M444 – Complex Analysis

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Chapter 2

Section 2.5: The Cauchy-Riemann Equations

Recall the following :

1. $z = (x, y)$, a point in $\mathbb{R}^2$.
2. $f(z) = f(x, y) = u(x, y) + iv(x, y)$.
3. If $\phi = \phi(x, y)$, then

$$\frac{\partial \phi}{\partial x}(x_0, y_0) = \phi_x(x_0, y_0) = \lim_{\Delta x \to 0} \frac{\phi(x_0 + \Delta x, y_0) - \phi(x_0, y_0)}{\Delta x}$$

and

$$\frac{\partial \phi}{\partial y}(x_0, y_0) = \phi_y(x_0, y_0) = \lim_{\Delta y \to 0} \frac{\phi(x_0, y_0 + \Delta y) - \phi(x_0, y_0)}{\Delta y}.$$

Notice that

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \iff f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z_0)}{\Delta z}$$

Set $\Delta z = \Delta x$, for $\Delta x \in \mathbb{R}$. Then

$$
\begin{aligned}
f'(z_0) &= \lim_{\Delta x \to 0} \frac{f(z_0 + \Delta x) - f(z_0)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} \\
&\qquad + i \lim_{\Delta x \to 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x} \\
&= u_x(x_0, y_0) + i v_x(x_0, y_0).
\end{aligned}
$$

Conclusion : $f'(z_0) = u_x(x_0, y_0) + i v_x(x_0, y_0)$.

Set $\Delta z = i\Delta y$, for $\Delta y \in \mathbb{R}$. Then

$$\begin{aligned}
f'(z_0) &= \lim_{\Delta y \to 0} \frac{f(z_0 + i\Delta y) - f(z_0)}{i\Delta y} \\
&= \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{i\Delta y} \\
&= \lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{i\Delta y} + i\frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{i\Delta y} \\
&= \frac{u_y(x_0, y_0)}{i} + v_y(x_0, y_0) \\
&= v_y(x_0, y_0) - iu_y(x_0, y_0).
\end{aligned}$$

Conclusion : $f'(z_0) = v_y(x_0, y_0) - iu_y(x_0, y_0)$.

Conclusion :

$$u_x(x_0, y_0) + iv_x(x_0, y_0) = f'(x_0, y_0) = v_y(x_0, y_0) - iu_y(x_0, y_0).$$

Theorem (Cauchy-Riemann Equations, Necessary conditions)

If $f = u + iv$ is analytic in an open set U, then

1. u_x, u_y, v_x, v_y exist.
2. $u_x = v_y$ and $u_y = -v_x$ (C–R equations).

Example : $f(z) = \overline{z}$ is not analytic.

Indeed, $u(x, y) = x$ and $v(x, y) = -y$. But

$$u_x(x, y) = 1 \neq -1 = v_y(x, y).$$

Hence, the C–R equations are not satisfied. So $\overline{z}$ is not analytic.

Theorem (Cauchy-Riemann Equations ; Corollary 2.5.2)

Let $f = u + iv$ be a function defined on an open set U. Assume that

1. u_x, u_y, v_x, v_y exist and are continuous on U.
2. $u_x = v_y$ and $u_y = -v_x$ on U.

Then f is analytic on U and

$$f' = u_x + iv_x = v_y - iu_y.$$

Example : We have $e^z = e^x \cos y + ie^x \sin y$.

1. $u_x(x, y) = e^x \cos y$ and $v_y = e^x \cos y$, and so $u_x = v_y$.
2. $u_y(x, y) = -e^x \sin y$ and $v_x = e^x \sin y$, and so $u_y = -v_x$.

Hence, e^z is analytic on $\mathbb{C}$ and

$$(e^z)' = u_x + iv_x = e^x \cos y + ie^x \sin y = e^z.$$

Consequences :

1. $\frac{d}{dz}\operatorname{Log}(z) = \frac{1}{z}$, for $z \in \mathbb{C} \backslash (-\infty, 0]$.
2. $\frac{d}{dz} z^\alpha = \alpha z^{\alpha - 1}$, for [1] $z \in \mathbb{C} \backslash (-\infty, 0]$. [Reason : $z^\alpha = e^{\alpha \operatorname{Log} z}$.]
3. $\frac{d}{dz}\sin(z) = \cos(z)$ and $\frac{d}{dz}\cos(z) = -\sin(z)$.
4. $\frac{d}{dz}\sinh(z) = \cosh(z)$ and $\frac{d}{dz}\cosh(z) = \sinh(z)$.

Proof of 1.
We have $z = e^{\operatorname{Log} z}$. Therefore

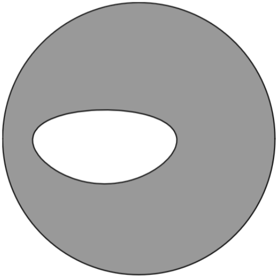
$$(z)' = (e^{\operatorname{Log} z})' \Rightarrow 1 = e^{\operatorname{Log} z}(\operatorname{Log} z)' \Rightarrow \frac{1}{e^{\operatorname{Log} z}} = (\operatorname{Log} z)'.$$

Hence $(\operatorname{Log} z)' = \frac{1}{z}$. □

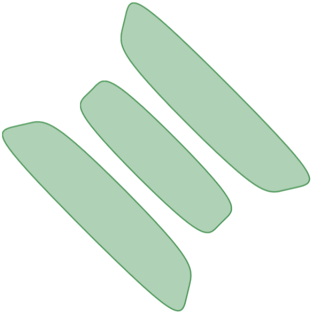
1. Also, $\alpha \neq 0$.

A **region** is a set $\Omega \subset \mathbb{C}$ such that

1. Ω is open.
2. Any two points $z, w \in \Omega$ can be connected by a polygonal curve.



(a) Connected Set



(b) Disconnected Set

Figure – Examples of connected and disconnected sets

Theorem

If f is analytic on a region Ω and $f'(z) = 0$ for every $z \in \Omega$, then there is a $c \in \mathbb{C}$ such that $f(z) = c$ for any $z \in \Omega$.

Proof.

① Fix $w \in \Omega$ and let $z \in \Omega$ with $z \neq w$. Let C be a polygonal curve joining w to z in Ω.

② Recall that $f'(z) = u_x + iv_x = v_y - iu_y \Rightarrow u_x = u_y = v_x = v_y = 0$.

③ Therefore, $\vec{\nabla} u = \vec{0}$ and $\vec{\nabla} v = \vec{0}$.

④ From the Fundamental Theorem for line integrals, we get

$$u(z) - u(w) = \int_C \vec{\nabla} u \cdot d\vec{r} = 0 \quad \Rightarrow \quad u(z) = u(w).$$

Similarly, $v(z) = v(w)$.

⑤ Hence $f(z) = u(w) + iv(w) = f(w)$, a constant. □

WARNING !

In the last result, Ω must be a region (open and connected).

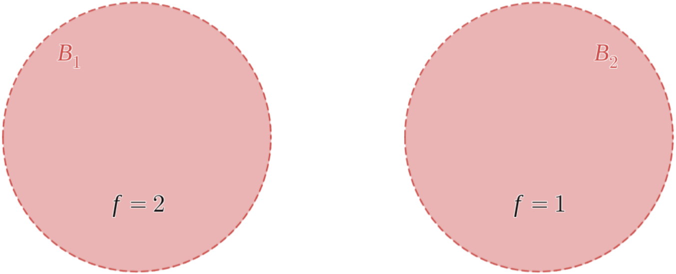


Figure – Definition of an analytic function f on $B_1 \cup B_2$

The function f satisfies :

1. $f'(z) = 0$ on $\Omega = B_1 \cup B_2$;
2. But f is not constant on Ω.