MATH 311

CHAPTER 2

SECTION 2.3: MATRIX MULTIPLICATION

CONTENTS

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Example 1. Let $f(x) = \sin(x)$, $g(x) = x^2$, and $k(x) = \sqrt{x}$.

a) Find $h = f \circ g$.

b) Find $h = g \circ f$.

c) Is $h = k \circ f$ well-defined?

Solution.

(a) $h(x) = f(g(x)) = f(x^2) = \sin(x^2)$.

(b) $h(x) = g(f(x)) = g(\sin(x)) = \sin^2(x)$

(c) $h(x) =$ undefined for certain $x \in \mathbb{R}$.

Definition 1. Let A be an $m \times n$ matrix and B be an $n \times k$ matrix. We define the composition of $T_A : \mathbb{R}^n \to \mathbb{R}^m$ with $T_B : \mathbb{R}^k \to \mathbb{R}^n$ as the function $T : \mathbb{R}^k \to \mathbb{R}^m$ defined by

$$T(\mathbf{x}) = (T_A \circ T_B)(\mathbf{x}) := T_A(T_B(\mathbf{x}))$$

for every $\mathbf{x} \in \mathbb{R}^k$.

<u>Note:</u> The order is very important! If $k \neq m$, then $T_B \circ T_A$ is not even defined!

Composing Two Matrix Transformation

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 1 & -1 \\ -2 & 1 \end{bmatrix}$. Then, for $\mathbf{x} \in \mathbb{R}^2$,

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{aligned}
(T_A \circ T_B)(\mathbf{x}) &= T_A(T_B(\vec{x})) \\
&= A(B\vec{x}) \\
&= A\left(x_1 \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}\right) \\
&= A\left(x_1 \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}\right) + A\left(x_2 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}\right) \\
&= x_1 A\vec{b}_1 + x_2 A\vec{b}_2 \\
&= \boxed{\begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\end{aligned}$$

$$\boxed{\begin{aligned}
A\vec{b}_1 &= \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix} \\
A\vec{b}_2 &= \begin{bmatrix} 3 \\ 2 \end{bmatrix} \\
\text{so } &\begin{bmatrix} -3 & 3 \\ 2 & 2 \end{bmatrix}
\end{aligned}}$$

In general:

$$\begin{aligned}
(T_A \circ T_B)(\mathbf{x}) &= T_A(T_B(\mathbf{x})) \\
&= A(B\mathbf{x}) \\
&= A(x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_k\mathbf{b}_k) \\
&= A(x_1\mathbf{b}_1) + A(x_2\mathbf{b}_2) + \cdots + A(x_k\mathbf{b}_k) \\
&= x_1(A\mathbf{b}_1) + x_2(A\mathbf{b}_2) + \cdots + x_k(A\mathbf{b}_k) \\
&= [A\mathbf{b}_1 \; A\mathbf{b}_2 \; \cdots \; A\mathbf{b}_k]\mathbf{x}.
\end{aligned}$$

DEFINITION 2. Let A be an $m \times n$ matrix and B be an $n \times k$ matrix with $B = [\mathbf{b}_1\ \mathbf{b}_2\ \cdots\ \mathbf{b}_k]$, where $\mathbf{b}_j$ is the column j of B. The **product matrix** AB is the $m \times k$ matrix defined as follows:

$$AB = A[\mathbf{b}_1\ \mathbf{b}_2\ \cdots\ \mathbf{b}_k] = [A\mathbf{b}_1\ A\mathbf{b}_2\ \cdots\ A\mathbf{b}_k]$$

Notes: The composite transformation $T_A \circ T_B$ is a matrix transformation induced by the matrix AB.

EXAMPLE 2. Compute the product $\underbrace{\begin{bmatrix} 5 & 0 & -7 \\ 1 & 5 & 9 \end{bmatrix}}_{=A} \underbrace{\begin{bmatrix} 3 & 2 \\ 1 & 0 \\ -1 & 3 \end{bmatrix}}_{=B}$.

SOLUTION.

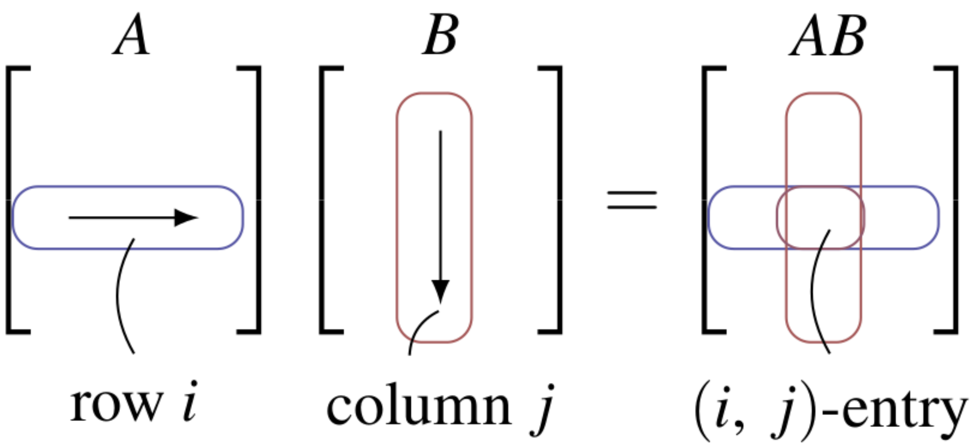
$$AB = [\,A\vec{b}_1 \quad A\vec{b}_2\,]$$

$$A\vec{b}_1 = \begin{bmatrix} 5 & 0 & -7 \\ 1 & 5 & 9 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 22 \\ -1 \end{bmatrix}$$

$$A\vec{b}_2 = \begin{bmatrix} 5 & 0 & -7 \\ 1 & 5 & 9 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -11 \\ 29 \end{bmatrix}$$

$$\rightarrow \boxed{AB = \begin{bmatrix} 22 & -11 \\ -1 & 29 \end{bmatrix}}$$

Dot Product Rule



EXAMPLE 3. If $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ -2 & 1 \\ 0 & 6 \end{bmatrix}$, find AB.

SOLUTION.

Compability Rule: The product of matrices A and B is only defined when the number of columns of A is equal to the number of rows of B.

Example 4. (a) Compute the $(2,4)$-entry of AB if

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 & 6 & 0 \\ 0 & 2 & 3 & 4 \\ -1 & 0 & 5 & 8 \end{bmatrix}.$$

(b) Is BA well defined?

Solution.

EXAMPLE 5. Let $A = \begin{bmatrix} 6 & 9 \\ -4 & -6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$. Compute A^2, AB, BA, $(AB)^\top$ and $B^\top A^\top$.

SOLUTION.

Note: In general, $AB \neq BA$. If $AB = BA$, then we say that A and B **commute**.

THEOREM 1. Let a be a real number, and A, B, C are matrices of sizes such that the indicated matrix products are defined. Then:

1) $IA = A$ and $AI = A$, where I denotes the identity matrix of proper size.
2) $A(BC) = (AB)C$.
3) $A(B + C) = AB + AC$.
4) $(B + C)A = BA + CA$.
5) $a(AB) = (aA)B = A(aB)$.
6) $(AB)^\top = B^\top A^\top$.

PROOF.

1) Assume that $A = [\mathbf{a}_1\ \mathbf{a}_2\ \cdots\ \mathbf{a}_n]$ is of dimension $m \times n$ and I is the $m \times m$ identity matrix. Then

$$\begin{aligned} IA &= [I\mathbf{a}_1\ I\mathbf{a}_2\ \cdots\ I\mathbf{a}_k] \\ &= [\mathbf{a}_1\ \mathbf{a}_2\ \cdots\ \mathbf{a}_k] \end{aligned}$$

where we used that $I\mathbf{x} = \mathbf{x}$ from Example 4 in Section 2.2.

2) If we write A in terms of its columns:

$$\begin{aligned} (B + C)A &= [(B + C)\mathbf{a}_1\ \cdots\ (B + C)\mathbf{a}_n] \\ &= [B\mathbf{a}_1 + C\mathbf{a}_1\ \cdots\ B\mathbf{a}_n + C\mathbf{a}_n] \\ &= [B\mathbf{a}_1\ \cdots\ B\mathbf{a}_n] + [C\mathbf{a}_1\ \cdots\ C\mathbf{a}_n] \\ &= BA + CA. \end{aligned}$$

$\square$

Example 6. Simplify the following expression:

$$A(3B - C) + (A - 2B)C + 2B(C + 2A)$$

where A, B, C represent matrices.

Solution.

Example 7. Show that $AB = BA$ if and only if $(A - B)(A + B) = A^2 - B^2$.

Solution.

Definition 3. A matrix is said to be **partitioned into blocks** if the entries of the matrix are themselves matrices.

Example 8. Writing $A = [\mathbf{a}_1 \; \mathbf{a}_2 \; \cdots \; \mathbf{a}_n]$ in terms of its columns.

Matrix Product with Blocks

Example 9. (a) Find a "nice" partition into blocks for the following matrices

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 2 & -1 & 4 & 2 & 1 \\ 3 & 1 & -1 & 7 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & -2 \\ 5 & 6 \\ 7 & 3 \\ -1 & 0 \\ 1 & 6 \end{bmatrix}.$$

(b) Use that to compute AB.

Solution.

Example 10. Obtain a formula for A^5 where $A = \begin{bmatrix} I & X \\ 0 & 0 \end{bmatrix}$ is a square matrix and I is an identity matrix.

Solution.

Notes:

- Block Multiplication is useful in theory.
- It is also usuful in computing products of large matrices in a computer with limited memory capacity.