Chapter 2

Derivatives

2.8 Related Rates.

Rate of change: $\frac{dy}{dx}$, $\frac{ds}{dt}$ (velocity), $\frac{dv}{dt}$ (accelaration).

EXAMPLE 1 Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm^3/s. How fast is the radius of the balloon increasing when the diameter is 50 cm?

1. Identify what is given & is unknown
   - Vol. ↑ at a rate of 100 cm^3/s
   - rate of change of the radius is unknown.

2. Give names.

V : volume of air of the balloon | r : radius of the balloon.

- $\frac{dV}{dt} = 100 \ cm^3/s$
- unknown : $\left.\frac{dr}{dt}\right|_{r=25}$ (diameter = 50)

3. Link between V & r

$$V = \frac{4}{3}\pi r^3 \quad \text{(volume of the sphere)}$$

4. Use the chain rule.

$r = r(t)$

$$\frac{dV}{dt} = \frac{d}{dt}\Big(\underbrace{\tfrac{4}{3}\pi}_{\text{cst.}} r^3\Big) = \frac{4}{3}\pi \frac{d}{dt}(r^3)$$

$$\rightarrow \frac{dV}{dt} = \frac{4}{\cancel{3}}\pi \, \cancel{3} r^2 \underbrace{\left(\frac{dr}{dt}\right)}_{\text{unknown rate of change}}$$

$$\rightarrow \frac{dr}{dt} = \frac{(dV/dt)}{4\pi r^2}$$

So, $r = 25$ & $\frac{dV}{dt} = 100 \Rightarrow \left.\frac{dr}{dt}\right|_{r=25} = \frac{100}{4\pi (25)^2} = \frac{1}{25\pi}$

$$\frac{\cancel{4}\cdot\cancel{25}}{\cancel{4}\pi\cdot 25\cdot\cancel{25}}$$

5. Answer $\left.\frac{dr}{dt}\right|_{r=25} = \frac{1}{25\pi}$ cm/s.

Key Steps.

1) Identify the given information and the unknown.

2) Introduce notation.

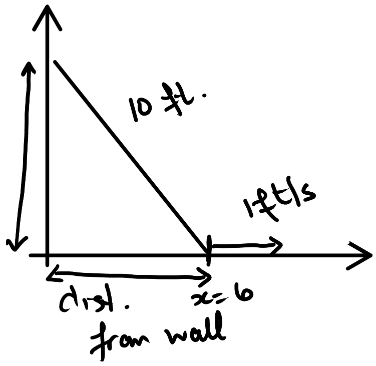
3) Restate the given information and the unknown with the new notation.

4) Connect the variables together with an equation.

5) Apply the chain rule to find the related rates.

EXAMPLE 2 A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?

① Given vs unknown



x : distance of the bottom of the ladder from the wall

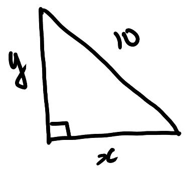
y : distance from the floor to the top of the ladder.

? : $\frac{dy}{dt}$ (rate of change of the positive of the top of the ladder).

given: $x = 6$ (6 ft. from the wall)

$\frac{dx}{dt} = 1$ ft/s.

② Link



Pythagorean thm: $x^2 + y^2 = 10^2 = 100$

$\Rightarrow y = \sqrt{100 - x^2}$

③ Chain rule.

$x = x(t)$
$y = y(t)$

$$\frac{d}{dt}(x^2+y^2) = \frac{d}{dt}(100)$$

$$\Rightarrow \frac{d}{dt}(x^2) + \frac{d}{dt}(y^2) = 0$$

$$\Rightarrow 2x\left(\frac{dx}{dt}\right) + 2y\left(\frac{dy}{dt}\right) = 0$$

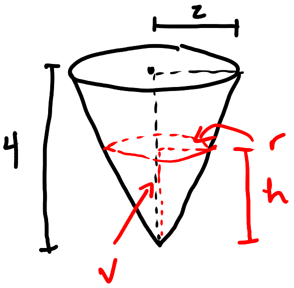
$$\Rightarrow 2x\left(\frac{dx}{dt}\right) + 2\sqrt{100-x^2}\left(\frac{dy}{dt}\right) = 0$$

$$\frac{dy}{dt} = \frac{-x}{\sqrt{100-x^2}}\left(\frac{dx}{dt}\right)$$

$$\Rightarrow \left.\frac{dy}{dt}\right|_{x=6} = \frac{-6}{\sqrt{100-36}}(1) = \boxed{\frac{-3}{4} \text{ ft/s}}$$

EXAMPLE 3 A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of 2 m^3/min, find the rate at which the water level is rising when the water is 3 m deep.

① Given vs unknown.



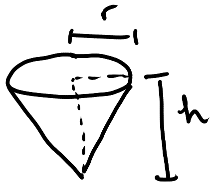
V: volume of water in the tank.

r: radius of the water-cone shape in red.

h: heigth of the water-coned shape in red.

$\cancel{?}$ $\frac{dV}{dt} = 2$ m^3/min. ? $\frac{dh}{dt}$ = unknown.

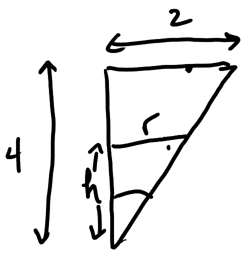
② Link.



$$V = \frac{1}{3}\pi r^2 h$$

Find a way to replace r with an expression in h.

Trick.



$$\frac{2}{r} = \frac{4}{h} \to r = \frac{h}{2}$$

so, $$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3.$$

③ Chain Rule.

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{\pi}{12}h^3\right) = \frac{\pi}{12}\frac{d}{dt}(h^3) \qquad h = h(t)$$

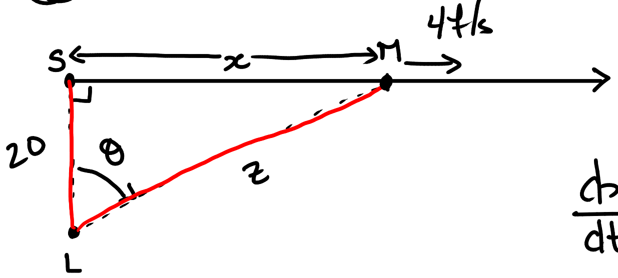
$$\to \frac{dV}{dt} = \frac{\pi}{12}3h^2\cdot\left(\frac{dh}{dt}\right) = \frac{\pi}{4}h^2\left(\frac{dh}{dt}\right)$$

$$\to \frac{dh}{dt} = \frac{4}{\pi h^2}\left(\frac{dV}{dt}\right) \to \frac{4}{\pi(3m)^2}\,2\,m^3/min \quad \to \frac{m^3}{m^2} = m$$

so, $$\left.\frac{dh}{dt}\right|_{h=3} = \frac{4}{\pi 3^2}(2) = \boxed{\frac{8}{9\pi}\ m/min}$$

EXAMPLE 5 A man walks along a straight path at a speed of 4 ft/s. A searchlight is located on the ground 20 ft from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 15 ft from the point on the path closest to the searchlight?

① Given & unknown.



x: distance from M to S.

θ: angle between the start S and the man M

$$\frac{dx}{dt} = 4\,ft/s \qquad \frac{d\theta}{dt} = ?? \text{ unknown.}$$

② Link.

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{x}{20}$$

$$\rightarrow \quad x = 20\tan\theta$$

③ Chain rule. $\qquad \theta = \theta(t)$

$$\frac{dx}{dt} = 20\sec^2(\theta)\frac{d\theta}{dt}$$

$$\rightarrow \quad \frac{dx}{dt} = \frac{20}{\cos^2\theta}\left(\frac{d\theta}{dt}\right)$$

$$\rightarrow \quad \frac{d\theta}{dt} = \frac{\cos^2\theta}{20}\left(\frac{dx}{dt}\right)$$

$$\tan\theta = \frac{x}{20} \rightarrow \tan\theta = \frac{15}{20} = \frac{3}{4} \rightarrow \theta =$$

$$\cos\theta = \frac{15}{z} = \frac{15}{\sqrt{15^2+20^2}} = \frac{4}{5}$$

So,
$$\frac{d\theta}{dt} = \frac{16/25}{20}(4) = \boxed{0.128\ rad./s}$$

1. Read the problem carefully.
2. Draw a diagram if possible.
3. Introduce notation. Assign symbols to all quantities that are functions of time.
4. Express the given information and the required rate in terms of derivatives.
5. Write an equation that relates the various quantities of the problem. If necessary, use the geometry of the situation to eliminate one of the variables by substitution (as in Example 3).
6. Use the Chain Rule to differentiate both sides of the equation with respect to t.
7. Substitute the given information into the resulting equation and solve for the unknown rate.