Chapter 15

Multiple Integrals

15.9 Change of variables in multiple integrals

Change of variable from Calculus I

$u = g(x)$

$du = g'(x)\,dx$

If $x = g(u)$, then

$$\int_a^b f(x)\,dx = \int_c^d f(g(u))g'(u)\,du$$

where $a = g(c)$ and $b = g(d)$.

Change of Variable in polar coordinate.

If $x = r\cos\theta$ and $y = r\sin\theta$, then

$$\iint_D f(x,y)\,dA = \iint_S f(r\cos\theta, r\sin\theta)r\,dr d\theta$$

where R is a region in the xy-plane and
S is a region in the $r\theta$-plane.

Transformation in polar coordinates:

$(x,y) = T(r,\theta) = (r\cos\theta, r\sin\theta)$.

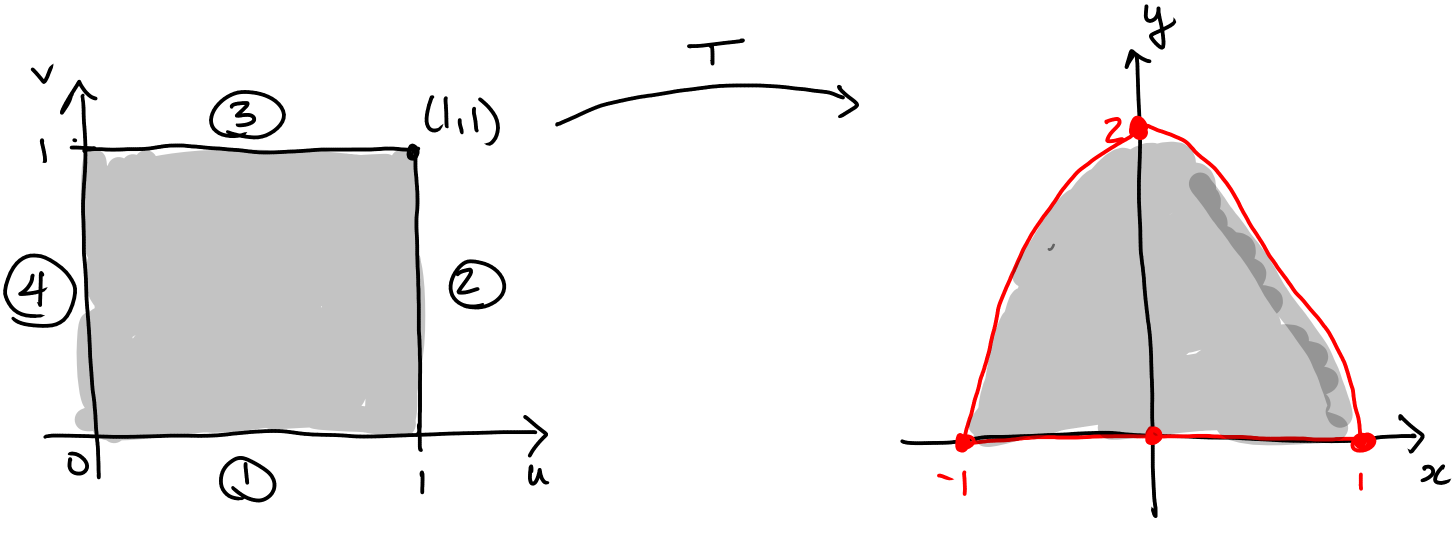
General transformation in 2D.

EXAMPLE 1 A transformation is defined by the equations

$$x = u^2 - v^2 \qquad y = 2uv$$

Find the image of the square $S = \{(u, v) \mid 0 \leqslant u \leqslant 1,\ 0 \leqslant v \leqslant 1\}$.

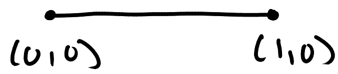
$T(u,v) = (x,y) = (u^2 - v^2, 2uv)$.

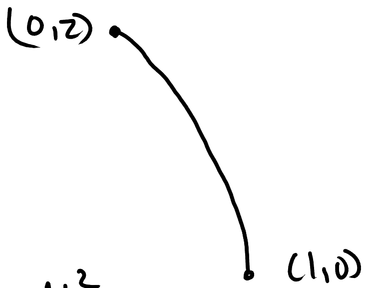


① $\underline{0 \le u \le 1 , \quad v = 0}$

$$x = u^2 - v^2 = u^2 - 0^2 = u^2$$

$$y = 2uv = 2 \cdot u \cdot 0 = 0$$





② $\underline{u = 1 , \quad 0 \le v \le 1}$

$$x = 1 - v^2$$

$$y = 2v \longrightarrow v = \frac{y}{2} \longrightarrow x = 1 - \frac{y^2}{4} , \quad 0 \le y \le 2$$

③ $\underline{0 \le u \le 1 , \quad v = 1}$

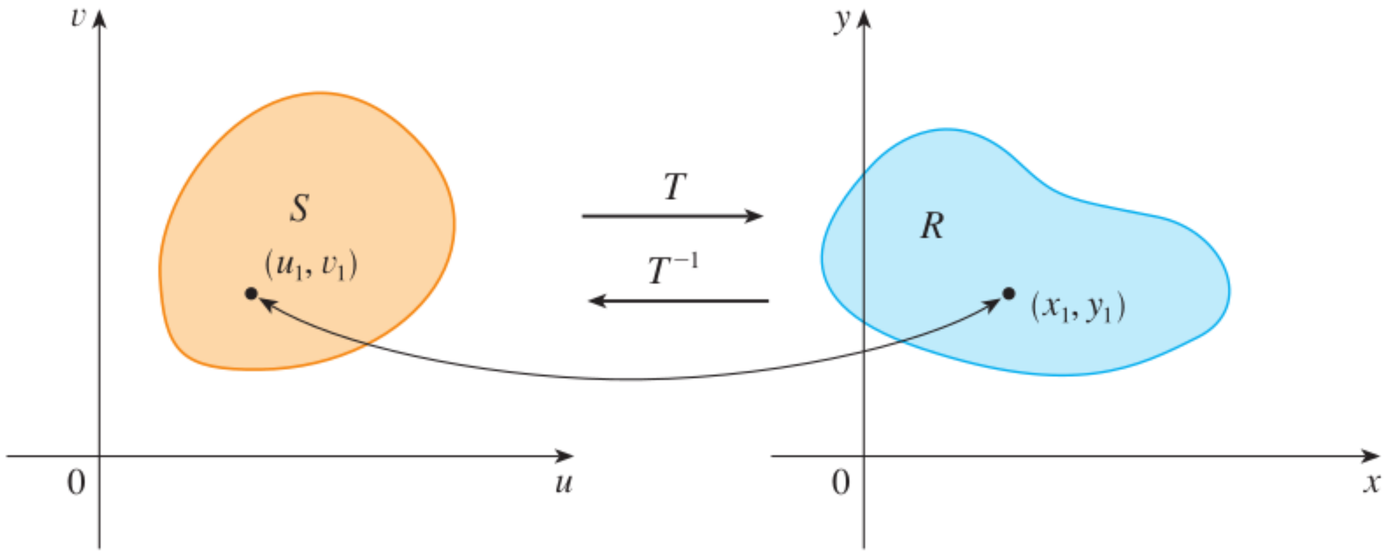
$$x = u^2 - 1 \longrightarrow u = \frac{y}{2} \longrightarrow x = \frac{y^2}{4} - 1 , \quad 0 \le y \le 2$$

$$y = 2u$$

④ $\underline{u = 0 , \quad 0 \le v \le 1}$

$$x = -v^2 \longrightarrow -1 \le x \le 0$$

$$y = 0$$



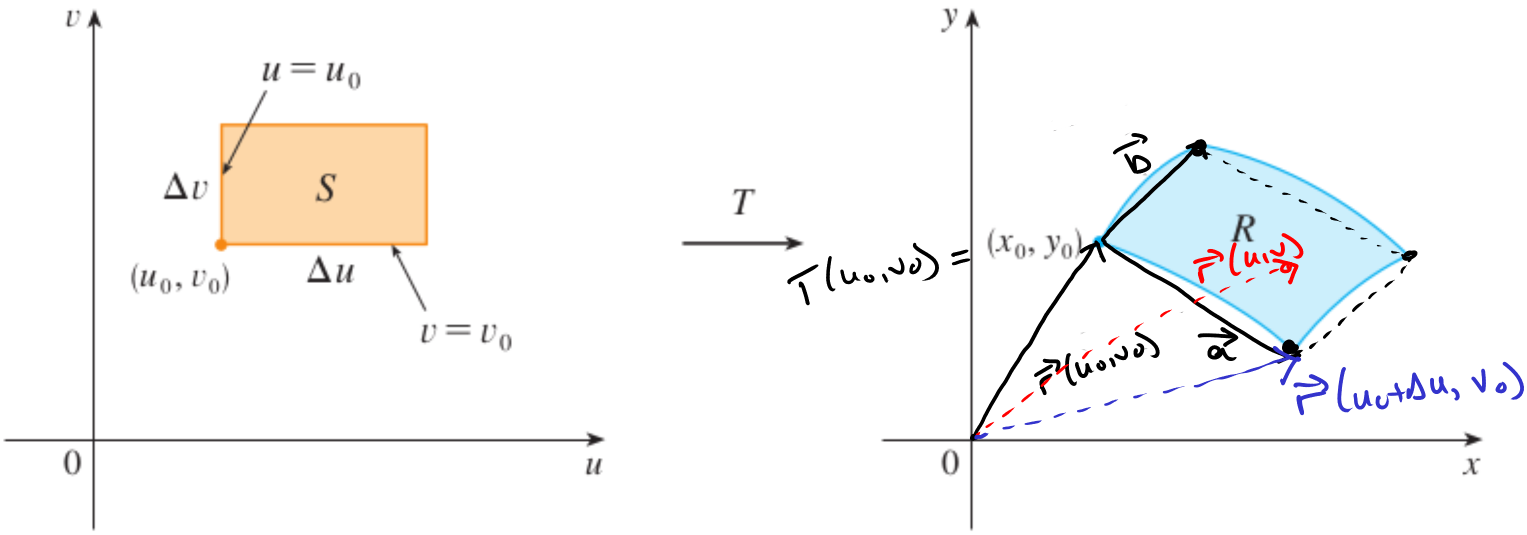
Two equations for x and y:

$$(x, y) = T(u, v) \iff x = x(u, v) \text{ and } y = y(u, v)$$

Image: The region R is the set of possible outputs.

Domain: The region S is the set of all possible inputs.

Effect of a change of variables in double integral.



Goal: Find how dA is transformed after the transformation.

$$\text{Area}(R) \approx \|\vec{a} \times \vec{b}\| .$$

Recall $\quad \vec{r}_u(u_0,v_0) = \lim_{\Delta u \to 0} \dfrac{\vec{r}(u_0+\Delta u, v_0) - \vec{r}(u_0,v_0)}{\Delta u}$.

$$\vec{r}_v(u_0,v_0) = \lim_{\Delta v \to 0} \dfrac{\vec{r}(u_0, v_0+\Delta v) - \vec{r}(u_0,v_0)}{\Delta v}.$$

When Δu and Δv are small :

$$\vec{a} = \vec{r}(u_0+\Delta u, v_0) - \vec{r}(u_0,v_0) \approx \Delta u\, \vec{r}_u(u_0,v_0).$$

$$\vec{b} \approx \Delta v\, \vec{r}_v(u_0,v_0)$$

So,

$$\text{Area}(R) \approx \|\vec{r}_u \times \vec{r}_v\|\, \Delta u\, \Delta v$$

$$\Rightarrow \quad \Delta A \approx \|\vec{r}_u \times \vec{r}_v\|\, \Delta u\, \Delta v \qquad \left(\Delta y \approx f'(a)\, \Delta x \right)$$

Make $\Delta u, \Delta v \to 0$ (or nb. of divisions $\to \infty$):

$$\Rightarrow \quad dA = \underbrace{\|\vec{r}_u \times \vec{r}_v\|}_{\text{more explicit expression please.}}\, du\, dv$$

Here, $\vec{r}(u,v) = T(u,v) = (x(u,v), y(u,v)) \hookrightarrow$ 3D

$\Rightarrow \vec{r}_u = \langle x_u, y_u, 0\rangle, \quad \vec{r}_v = \langle x_v, y_v, 0\rangle$

$$\Rightarrow \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_u & y_u & 0 \\ x_v & y_v & 0 \end{vmatrix} = \underbrace{(x_u y_v - x_v y_u)}_{\text{Jacobian } \frac{\partial(x,y)}{\partial(u,v)}} \vec{k}$$

$$\Rightarrow \quad dA = |x_u y_v - x_v y_u| \, du\, dv.$$

useful notation: $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$

$$\boxed{dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dv\,du}$$

type I

or

$$\boxed{dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv}$$

type II

Remarks:

① $\dfrac{\partial(x,y)}{\partial(u,v)} = x_u y_v - x_v y_u$ is called the <u>Jacobian</u>. of the transform. T.

② If T^{-1} exists, then

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{\left(\frac{\partial(x,y)}{\partial(u,v)}\right)}.$$

③ The formulas for $\dfrac{\partial(x,y)}{\partial(u,v)}$ and dA work when T is a C^1-transformation (this means the derivatives exist and are continuous).

EXAMPLE 2 Use the change of variables $x = u^2 - v^2$, $y = 2uv$ to evaluate the integral $\iint_R y\, dA$, where R is the region bounded by the x-axis and the parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$, $y \geqslant 0$.

EXAMPLE 3 Evaluate the integral $\iint_R e^{(x+y)/(x-y)}\,dA$, where R is the trapezoidal region with vertices $(1, 0)$, $(2, 0)$, $(0, -2)$, and $(0, -1)$.

Effect of change of variable in Triple integrals.

Spherical coordinates.

$$(x,y,z) = T(\rho,\theta,\phi) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$$

This implies that

$$dV = \underbrace{\rho^2 \sin\phi}\, d\rho\, d\theta\, d\phi$$

Jacobien of the transformation.

Transformation in 3D:

- A function T from a region S in the uvw-space into a region R in the xyz-space.
- So
$$(x,y,z) = T(u,v,w)$$
$$\Updownarrow$$
$$x = x(u,v,w),\ y = y(u,v,w) \text{ and } z = z(u,v,w)$$

Jacobian in 3D:

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

$$\boxed{\iiint\limits_R f(x,y,z)\,dV = \iiint\limits_S f(x(u,v,w), y(u,v,w), z(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\,dv\,dw}$$

Important fact: If $T^{-1} : R \to S$ exists, then $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = \dfrac{1}{\frac{\partial(x,y,z)}{\partial(u,v,w)}}$

56. Use the transformation $x = u^2,\ y = v^2,\ z = w^2$ to find the volume of the region bounded by the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$ and the coordinate planes.