Math 644

Problem Sets

Contents

Created by: Pierre-Olivier Parisé
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Chapter 1

Problem 1. Prove the parallelogram equality:

$$|z+w|^2 + |z-w|^2 = 2(|z|^2 + |w|^2).$$

Problem 2. Let w be a non-zero complex number and let $n \geq 1$ be a positive integrer. Using the polar coordinates, find n solutions to $z^n = w$.

Problem 3. Let z be a non-zero complex number. Show that 0, z, iz, and $iz+z$ are the vertices of a square.

Problem 4. Prove that there is no complex number z so that

$$|z| - z = i.$$

Problem 5. Find all complex numbers z satisfying the equation

$$4z - 3\overline{z} = \frac{1-18i}{2-i}.$$

Problem 6. Suppose that f is a continuous complex-valued function on a real interval $[a,b]$. Let

$$A = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

a) Show that if $|f(x)| \leq |A|$ for all $x \in [a,b]$, then $f \equiv A$.

b) Show that if $|A| = \frac{1}{b-a}\int_a^b |f(x)|\,dx$, then $\arg f$ is constant modulo 2π on $\{z \,:\, f(z) \neq 0\}$.

Problem 7. Describe geometrically the following subsets:

a) $\operatorname{Re} z = \operatorname{Im} z$.

b) $\operatorname{Re} z > 0$.

c) $\operatorname{Im} z > 0$.

d) $\frac{\pi}{6} < \arg z < \frac{\pi}{4}$.

Problem 8. Let $\mathbb{T} := \{z \in \mathbb{C} \,:\, |z| = 1\}$. Prove that $\mathbb{T}$ equipped with the complex multiplication is a commutative group.

Problem 9. Suppose that $\lim_{n\to\infty} w_n = w$. Is it true then that also

$$\lim_{n\to\infty} \arg w_n = \arg w?$$

Problem 10. Let $\{z_n\}$ be a sequence of complex numbers such that $\sum_{n=0}^{\infty} z_n$ converges and there is a ϕ such that $|\arg z_n| \leq \phi < \frac{\pi}{2}$ for any $n \geq 0$. Show that the series $\sum_{n=0}^{\infty} z_n$ is absolutely convergent.

Problem 11. Let $\mathbb{C}^*$ be the extended plane, let $\mathbb{S}^2$ be the sphere $\{(X, Y, Z) : X^2+Y^2+Z^2 = 1\}$ and let $\pi : \mathbb{C}^* \to \mathbb{S}^2$ be the stereographic projection with $\pi(\infty) = (0, 0, 1)$.

a) Show that straight lines in $\mathbb{C}$ correspond exactly to circles on $\mathbb{S}^2$ passing through $(0, 0, 1)$.

b) Show that if $z \neq \infty$, then

$$\chi(z, \infty) = \frac{2}{\sqrt{1 + |z|^2}}.$$

c) Using the explicit formula of χ in terms of z and w, show that, for any $z, w \in \mathbb{C}^*$,

$$0 \leq \chi(z, w) \leq 2.$$

Chapter 2

Problem 12. For what values of z is

$$\sum_{n=0}^{\infty} \left(\frac{z}{1+z} \right)^n$$

convergent? Draw a picture of the region.

Problem 13. Suppose that $\sum_{n\geq 0} a_n(z-z_0)^n$ is a formal power series. Suppose that

$$R := \frac{1}{\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|}$$

exists and is finite.

a) Show that the power series converges in $\{z \,:\, |z-z_0| < R\}$.

b) Show that the power series diverges in $\{z \,:\, |z-z_0| > R\}$.

Problem 14. Define $e^z = \exp(z) := \sum_{n\geq 0} \frac{z^n}{n!}$.

a) Show that $e^z e^w = e^{z+w}$ (using the power series definition).

b) Show that $|e^z| = e^{\operatorname{Re} z}$ and $\arg e^z = \operatorname{Im} z$.

c) Show that $\frac{d}{dz} e^z = e^z$.

d) Show that, for any non-zero integer n,

$$\int_0^{2\pi} e^{int}\, dt = 0.$$

[Hint: Use Foundamental Theorem of Calculus.]

e) Compute the integral

$$\int e^{nt} \cos(mt)\, dt.$$

[Hint: Rewrite $\cos(mt)$ as a complex exponential.]

Problem 15. Prove the following assertions.

a) If f and g are analytic at z_0, then $(f+g)'(z_0) = f'(z_0) + g'(z_0)$ (Sum rule of differentiation for analytic functions).

b) If f and g are analytic at z_0, then $(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$ (Product Rule of differentiation for analytic functions).

c) If f and g are analytic at z_0 with $g(z_0) \neq 0$, then $(fg)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{(g(z_0))^2}$ (Quotient rule of differentiation for analytic functions).

d) If f is analytic at z_0 and g is analytic at $f(z_0)$, then $(g \circ f)'(z_0) = g'(f(z_0))f'(z_0)$ (Chain Rule of differentiation for analytic functions).

Find the derivative of $(z-a)^{-n}$, where n is a positive integer and $a \in \mathbb{C}$.

Problem 16. Let $\Omega \subset \mathbb{C}$. Show that Ω is connected if and only if Ω and $\varnothing$ are the only open and closed subsets of Ω.

Problem 17. Suppose that f and g are two analytic functions on a region (open and connected) Ω. Suppose there is a sequence $(z_n)_{n=1}^{\infty}$ with $z_n \in \Omega$ ($\forall n \geq 1$) such that $f(z_n) = g(z_n)$ ($\forall n \geq 1$). If (z_n) has an accumulation point $z_0 \in \Omega$ with $z_0 \neq z_n$ for all $n \geq 1$, then show that $f \equiv g$ on Ω.

Problem 18. Show that $\cos^2(z) + \sin^2(z) = 1$ for every $z \in \mathbb{C}$.

Problem 19. Suppose f is analytic in a connected open set Ω such that, for each $z \in U$, there exists an n (depending on z) such that $f^{(n)}(z) = 0$. Prove that f is a polynomial. *[Hint: Use Baire's Theorem.]*

Problem 20. Let f be analytic in a region Ω containing the point $z = 0$. Suppose $|f(1/n)| < e^{-n}$ for $n \geq n_0$, for some integer $n_0 \geq 0$. Prove $f \equiv 0$ in Ω.

Problem 21. Let f and g be analytic functions in a region Ω.

a) Show that if $f'(z) = 0$ for all z in a neighborhood of some $z_0 \in \Omega$, then f is constant in Ω, meaning there is a constant $c \in \mathbb{C}$ such that $f(z) = c$ for any $z \in \Omega$.

b) Show that if f and g are analytic in a region Ω with $f'(z) = g'(z)$ for every $z \in \Omega$, then $f - g$ is constant.

Problem 22. Suppose that $f(z) = az^3 + bz^2 + cz + d$. In addition, suppose that for each $z, w \in \mathbb{C}$ there exists a point ζ on the line segment between z and w with

$$\frac{f(z) - f(w)}{z - w} = f'(\zeta).$$

Show that $a = 0$.

Problem 23. Suppose that $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges in $B = \{z : |z - z_0| < r\}$. Show that the power series

$$F(z) := \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z - z_0)^{n+1}$$

converges in B and satisfies $F'(z) = f(z)$ for all $z \in B$. Moreover, show that the radius of convergence of F is the same as the radius of convergence of f.

Problem 24. Suppose $\sum_{j=0}^{\infty} |a_j|^2 < \infty$.

a) Show that $f(z) = \sum_{j=0}^{\infty} a_j z^j$ is analytic in $\{z : |z| < 1\}$.

b) Compute (with a proof) the following quantity:

$$\lim_{r \to 1^-} \int_0^{2\pi} |f(re^{i\theta})|^2 \, \frac{d\theta}{2\pi}.$$

Problem 25. Suppose f has a power series expansion at 0 which converges in all of $\mathbb{C}$. Suppose also that $\int_{\mathbb{C}} |f(x+iy)| \, dxdy < \infty$. Prove that $f \equiv 0$.

Problem 26. [Hard] Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ have radius of convergence 1 and suppose that $a_n \geq 0$ for all n. Prove that $z = 1$ is a singular point of f. That is, there is no function g analytic in a ball B containing $z = 1$ such that $f = g$ on $B \cap D$.

Chapter 3

Problem 27. If f is analytic in a region Ω and if there is a $z_0 \in \Omega$ such that

$$|f(z_0)| = \inf_{z \in \Omega} |f(z)|,$$

and if $f(z_0) \neq 0$, then f is constant in Ω.

Problem 28. Let Ω be a region in $\mathbb{C}$. Show that if $f : \Omega \to \mathbb{C}$ is an open map, then f satisfies the maximum modulus principle.

Problem 29.

a) Show geometrically why the maximum principle holds using a "walking the dog" argument. Make it rigorous by following the steps of the proof of the Fundamental Theorem of Algebra.

b) Use the maximum modulus principle to prove the Fundamental Theorem of Algebra.

Problem 30. Let f be an analytic function defined on some bounded region $\Omega \subset \mathbb{C}$. Show that

$$\limsup_{z \to \partial \mathbb{D}} |f(z)| = \lim_{\delta \to 0} \sup\{|f(z)| \,:\, z \in \Omega, \operatorname{dist}(z, \partial\Omega) = \delta\}.$$

Problem 31. Suppose that f is analytic in a connected open (region) set Ω.

a) Prove that if $|f(z)|$ is constant on Ω, then f is contant on Ω.

b) Prove that if $\operatorname{Re} f$ is constant on Ω, then f is constant on Ω.

Problem 32.

a) Prove that if f is analytic in $\mathbb{C}$, then $f(z) = \sum_{n \geq 0} a_n z^n$ for any $z \in \mathbb{C}$. In other words, the radius of convergence of the power series $\sum_{n \geq 0} a_n z^n$ representing f at $z = 0$ is $R = \infty$.

b) Suppose that f is analytic in $\mathbb{C}$ and $|f(z)| \leq C|z|^n$, for some $|z| > M$ and $n \geq 0$. Show that f must be a polynomial.

c) Suppose that f and g are analytic in $\mathbb{C}$ with $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$. Prove that there exists a constant $c \in \mathbb{C}$ such that $f(z) = cg(z)$ for all $z \in \mathbb{C}$.

Problem 33. Prove that if f is non-constant and analytic on all of $\mathbb{C}$, then $f(\mathbb{C})$ is dense in $\mathbb{C}$.

Problem 34. Let f be analytic in $\mathbb{D}$ and suppose $|f(z)| < 1$ on $\mathbb{D}$. Let $a = f(0)$. Show that f does not vanish in $\{z \,:\, |z| < |a|\}$.

Problem 35.

a) Prove that φ is a one-to-one analytic map of $\mathbb{D}$ onto $\mathbb{D}$ if and only if

$$\varphi(z) = c\left(\frac{z-a}{1-\bar{a}z}\right) \quad (z \in \mathbb{D}),$$

for some constants c and a, with $|c| = 1$ and $|a| < 1$.

b) Let f be analytic in $\mathbb{D}$ and satisfy $|f(z)| \to 1$ as $|z| \to 1$. Prove that f is rational.

Problem 36.

a) Suppose p is a non-constant polynomial with all its zeros in the upper half-plane $\mathbb{H} := \{z : \operatorname{Im} z > 0\}$. Prove that all the zeros of p' are contained in $\mathbb{H}$. *[Hint: Look at the partial fraction expansion of p'/p.]*

b) Use **a)** to prove that if p is a polynomial, then the zeros of p' are contained in the (closed) convex hull of the zeros of p. (The closed convex hull is the intersection of all half-planes containing the zeros.)

Problem 37. Suppse f is analytic in $\mathbb{D}$ and $|f(z)| \leq 1$ in $\mathbb{D}$ and $f(0) = 1/2$. Prove that $|f(1/3)| \geq 1/5$.

Problem 38. Suppose f is analytic and non-constant in $\mathbb{D}$ and $|f(z)| \leq M$ on $\mathbb{D}$. Prove that the number of zeros of f in a disk of radius $1/4$, centered at 0, does not exceed

$$\frac{1}{\ln 4} \ln\left|\frac{M}{f(0)}\right|.$$