Math 311

Chapter 3

Section 3.1: The Cofactor Expansion

Contents

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Goal

Recall that if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det A = ad - bc$ and A is invertible if and only if $\det A \neq 0$.

Goal: To Generalize the determinant to $n \times n$ matrix.

A Basic Example

If A is a 3×3 square matrix and if A is invertible, then we know A can be carried to the identity matrix I.

Following the process of $A \to I$:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \to \begin{bmatrix} a & b & c \\ ad & ae & af \\ ag & ah & ai \end{bmatrix} \to \begin{bmatrix} a & b & c \\ 0 & ae - bd & af - cd \\ 0 & ah - bg & ai - cg \end{bmatrix}$$

Set $u = ae - bd$ and $v = ah - bg$. Then

$$\begin{bmatrix} a & b & c \\ 0 & u & af - cd \\ 0 & v & ai - cg \end{bmatrix} \to \begin{bmatrix} a & b & c \\ 0 & u & af - cd \\ 0 & vu & u(ai - cg) \end{bmatrix} \to \begin{bmatrix} a & b & c \\ 0 & vu & v(af - cd) \\ 0 & 0 & w \end{bmatrix}$$

$w = u(ai - cg) - v(af - cd)$. Hence, if we want to carry on the algorithm, we need that

$$w \neq 0$$

DEFINITION 1. If A is a 3×3 matrix, then

$$\det A := w = aei + bfg + cdh - ceg - afh - bdi.$$

Remark:

- Notice that A is invertible if and only if $\det A \neq 0$.
- Notice that

$$\begin{aligned}\det A &= a(ei - fh) - b(di - fg) + c(dh - eg) \\ &= a \det \begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \det \begin{bmatrix} d & f \\ g & i \end{bmatrix} + c \det \begin{bmatrix} d & e \\ g & h \end{bmatrix}.\end{aligned}$$

- The terms $+a \det \begin{bmatrix} e & f \\ h & i \end{bmatrix}$, $-b \det \begin{bmatrix} d & f \\ g & i \end{bmatrix}$ and $+c \det \begin{bmatrix} d & e \\ g & h \end{bmatrix}$ are called **cofactors** of A and are denoted by $c_{11}(A)$, $c_{12}(A)$ and $c_{13}(A)$ respectively.

EXAMPLE 1. Compute the determinant of $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$.

SOLUTION.

$$\det A = 2 \det \begin{bmatrix} 2 & 3 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} - 3 \det \begin{bmatrix} 2 & 3 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} + 1 \det \begin{bmatrix} 2 & 3 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

$$= 2 \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - 3 \det \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} + \det \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = 5$$

Notice that

$$c_{12}(A) = (-1)^{1+2} \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = (-1)^{1+2} \det \begin{bmatrix} d & f \\ g & i \end{bmatrix}.$$

We denote by A_{ij} the $(n-1) \times (n-1)$ matrix obtained from A by deleting row i and column j.

DEFINITION 2. Let A be an $n \times n$ matrix. The $(\mathbf{i}, \mathbf{j})$**-cofactor** $c_{ij}(A)$ is the scalar defined by

$$c_{ij}(A) = (-1)^{i+j} \det(A_{ij}).$$

Here, $(-1)^{i+j}$ is called the **sign** of the (i, j)-position.

EXAMPLE 2. Find the cofactors of positions $(3, 2)$ of

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}.$$

$$c_{32}(A) = (-1)^{2+3} \det \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= (-1)^5 \big((2)(1) - (1)(1)\big) = \boxed{-1}$$

DEFINITION 3. Let $A = [a_{ij}]$ be an $n \times n$ matrix. The **determinant** of A is defined by

$$\det A = a_{11}c_{11}(A) + a_{12}c_{12}(A) + \cdots + a_{1n}c_{1n}(A).$$

Remark: This is called the **cofactor expansion** of $\det A$ along row 1.

EXAMPLE 3. compute the determinant of $A = \begin{bmatrix} 3 & 4 & 5 & 0 \\ 1 & 7 & 2 & 0 \\ 9 & 8 & -6 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

SOLUTION.

$$\det(A) = 3C_{11}(A) + 4C_{12}(A) + 5C_{13}(A) + 0 \cdot C_{14}(A)$$

$$= 3(-1)^{1+1} \det \begin{bmatrix} 7 & 2 & 0 \\ 8 & -6 & 3 \\ 1 & 1 & 1 \end{bmatrix} + 4(-1)^{1+2} \det \begin{bmatrix} 1 & 2 & 0 \\ 9 & -6 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

$$+ 5(-1)^{1+3} \det \begin{bmatrix} 1 & 7 & 0 \\ 9 & 8 & 3 \\ 1 & 1 & 1 \end{bmatrix} + 0 \cdot (-1)^{1+4} \det \begin{bmatrix} 1 & 7 & 2 \\ 9 & 8 & -6 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= 3(-73) - 4(-21) + 5(-37) - 0$$

$$= \boxed{-320}$$

THEOREM 1. [Proved by Pierre-Simon de Laplace (1749-1827)] The determinant of an $n \times n$ matrix A can be computed by using the cofactor expansion along any row or column of A.

EXAMPLE 4. Compute $\det A$ if $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 2 & 3 & 1 \end{bmatrix}$.

SOLUTION. Use the 1st column.

$$\det A = (1)C_{11}(A) + \cancel{(0)C_{21}(A)} + \cancel{(0)C_{31}(A)} + \cancel{(0)C_{41}(A)}$$

$$= (1)(-1)^{1+1}\det\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}$$

$$= (1)\Big[\cancel{(0)(-1)^{2+1}\det\begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}} \to 0$$

$$+ (1)(-1)^{2+2}\det\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

$$+ (1)(-1)^{2+3}\det\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\Big]$$

$$= (1)\big(0 + (1)(1)(-1) + (1)(-1)(1)\big) = \boxed{-2}$$

$$\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0 - (1)\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + 0$$
$$= -1$$

Interchanging two rows

EXAMPLE 5. Show that

$$\det \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{A} = -\det \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{B}.$$

SOLUTION. Notice: $A \to B$ operation was $R_1 \leftrightarrow R_2$.

Matrix:
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

LATER.

$$\Rightarrow B = EA \Rightarrow \det B = \det(EA) = \det E \det A.$$

Here $\det E = -1 \Rightarrow \det B = (-1)\det A$

$$\Rightarrow \det A = (-1)\det B$$

THEOREM 2. If B is an $n \times n$ matrix obtained from interchanging two rows of an $n \times n$ matrix A, then

$$\det(B) = -\det(A).$$

Remark: This fact is still true if we interchange two *columns* (instead of rows).

Scaling a row

EXAMPLE 6. Show that

$$\det \overset{B}{\begin{bmatrix} 2 & 6 & 8 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}} = 2 \det \overset{A}{\begin{bmatrix} 1 & 3 & 4 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}}.$$

SOLUTION. Notice: $A \to B$ operation was $2R_1 \to$ new R_1 of B.

Matrix: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to E = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

So, $B = EA \Rightarrow \det(B) = \det(EA)$

$= \det(E)\det(A).$

$\det E = (2)\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 2 \Rightarrow \det(B) = 2\det(A).$

THEOREM 3. If B is an $n \times n$ matrix for which column j is obtained by multiplying k times the column j of an $n \times n$ matrix A, with $k \neq 0$, then

$$\det(B) = k \det(A).$$

Remark: This fact is still true if a column j of a matrix B is obtained by multiplying the column j of a given matrix A by a nonzero scalar

Subtracting a Multiple of a Row

EXAMPLE 7. Show that

$$\det \overset{\to B}{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}} = \det \overset{\to A}{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 3 \\ 1 & 2 & 3 \end{bmatrix}}.$$

SOLUTION. Notice: $A \to B$ replace R_2 by $R_2 - 2R_1$

Matrix: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E$

$B = EA \Rightarrow \det(B) = \det(E)\det(A)$

So, $\begin{vmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = (1)\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - (0) + (0) = 1.$

So, $\det(B) = (1)\det(A)$.

THEOREM 4. If a the row j of a matrix B is obtained by subtracting a multiple of a row of a matrix A to the row j of A, then

$$\det(B) = \det(A).$$

Remark: This remains true if we replace the row operation by the corresponding column operation.

THEOREM 5. Let A be an $n \times n$ matrix.

1. If A has a row (or column) of zero, then $\det(A) = 0$.
2. If A has two identical rows (or columns), then $\det(A) = 0$.

PROOF.

1. Developing $\det(A)$ along the row of zero, then $\det(A) = 0$.
2. Assume that the two identical rows have index p and q. Let B be the matrix obtained by interchanging rows p and q of A. Then, $A = B$. But, $\det(B) = -\det(A)$, which implies that $2\det(A) = 0$, hence $\det(A) = 0$. $\square$

EXAMPLE 8. Find the values of x for which $\det(A) = 0$, where
$A = \begin{bmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{bmatrix}$. $R_2 - xR_1$
$R_3 - xR_1$

SOLUTION.

$$\det A = \det \begin{bmatrix} 1 & x & x \\ 0 & 1-x^2 & x-x^2 \\ 0 & x-x^2 & 1-x^2 \end{bmatrix}$$

$1-x^2 = (1-x)(1+x)$

$x-x^2 = (1-x)\,x$

$$= (1-x)^2 \det \begin{bmatrix} 1 & x & x \\ 0 & 1+x & x \\ 0 & x & 1+x \end{bmatrix}$$

$$= (1-x)^2 \det \begin{bmatrix} 1 & x & x \\ 0 & 1+x & x \\ 0 & 1+2x & 1+2x \end{bmatrix} \quad R_3 + R_2$$

$$= (1-x)^2 (1+2x) \det \begin{bmatrix} 1 & x & x \\ 0 & 1+x & x \\ 0 & 1 & 1 \end{bmatrix}$$

$$= (1-x)^2 (1+2x) \big((1)(1+x-x) \big)$$

$$= (1-x)^2 (1+2x)$$

Now, $\det A = 0 \iff \boxed{x = 1 \text{ or } x = \frac{-1}{2}}$.

Example 9. Compute $\det(A)$ if $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 3 & 4 & 5 & 0 \\ 4 & 3 & 2 & 10 \end{bmatrix}$.

Solution.

Definition 4. A matrix A is

1. **lower triangle** if all the entries above the main diagonal are zero.
2. **upper triangle** if all the entries below the main diagonal are zero.
3. **triangular** if it is lower triangle or upper triangle.

Theorem 6. If $A = [a_{ij}]$ is an $n \times n$ triangular matrix, then $\det(A) = a_{11}a_{22} \cdots a_{nn}$.