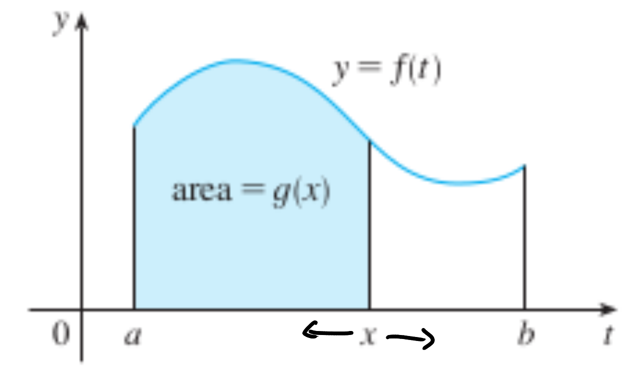
Chapter 4
Integrals

4.3 The Fundamental Theorem of Calculus

Area Function.



$$g(x) = \underbrace{\int_a^x f(t)\,dt}_{\text{Area function.}}$$

$$\cancel{\int_a^x f(x)\,dx}$$

EXAMPLE 1 If f is the function whose graph is shown in Figure 2 and $g(x) = \int_0^x f(t)\,dt$, find the values of $g(0)$, $g(1)$, $g(2)$, $g(3)$, $g(4)$, and $g(5)$. Then sketch a rough graph of g.

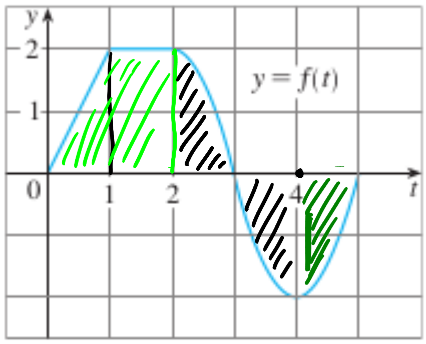


FIGURE 2

(a) $g(0) = \int_0^0 f(t)\,dt = 0$

(b) $g(1) = \int_0^1 f(t)\,dt = \text{Area}(\triangle) = 1$

(c) $g(2) = \int_0^2 f(t)\,dt = \text{Area}(\triangle) + \text{Area}(\square) = 3$

Area($\triangle$)

(d) $g(3) = \int_0^3 f(t)\,dt = \int_0^2 f(t)\,dt + \int_2^3 f(t)\,dt = 3 + \text{Area}(D)$

$\approx 3 + 1 = 4$

(e) $g(4) = \int_0^4 f(t)\,dt = \int_0^2 f(t)\,dt + \int_2^3 f(t)\,dt + \int_3^4 f(t)\,dt$

$= 3 + \text{Area}(\square) - \text{Area}(\square) = 3$

same value

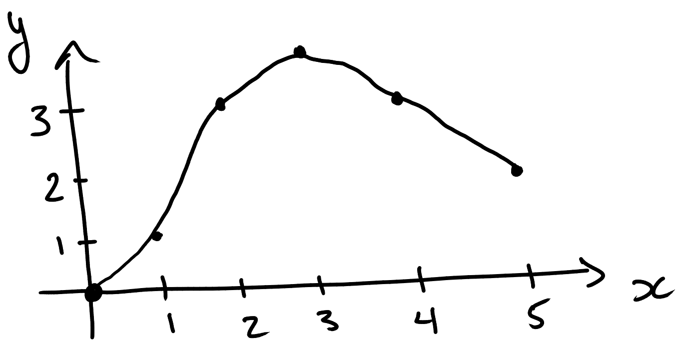
(f) $g(5) = \int_0^5 f(t)\,dt = \int_0^2 f(t)\,dt + \int_2^3 f(t)\,dt + \int_3^4 f(t)\,dt + \int_4^5 f(t)\,dt$

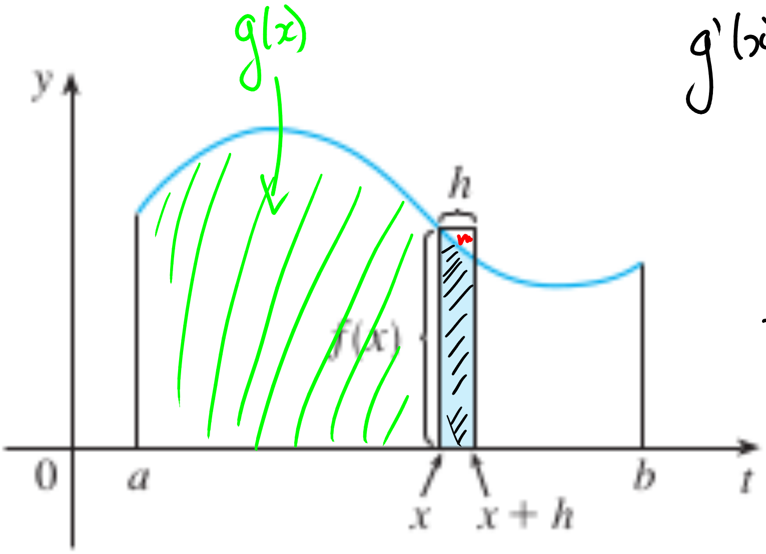
$= 3 + 0 - \text{Area}(D)$

$\approx \text{Area}(\triangle)$

$= 3 - 1 = 2$

(g)





$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}.$$

$$g(x+h) - g(x) \approx f(x) \cdot h$$

$$\Rightarrow \quad \frac{g(x+h) - g(x)}{h} \approx f(x)$$

as $h \to 0$,

$$\frac{g(x+h) - g(x)}{h} \to f(x)$$

The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\,dt \qquad a \leq x \leq b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

EXAMPLE 2 Find the derivative of the function $g(x) = \int_0^x \sqrt{1 + t^2}\,dt$.

$f(x) = \sqrt{1+x^2}$, $\quad a = 0$.

$$\Rightarrow \quad g'(x) = \sqrt{1+x^2} \qquad (\text{FTC, Part I}).$$

Example. Find $\frac{d}{dx}\left(\int_{\underset{a}{1}}^{x^4} \sec(t)\,dt\right) \longrightarrow G(x)$.

$\underline{a=1}$ Since x^4 is in place of x, ~~FTC~~

$$g(x) = \int_1^x \sec(t)\,dt.$$

So

$$G(x) = g(x^4) = \int_1^{x^4} \sec(t)\,dt$$

By the Chain Rule

$$\frac{d}{dx}(G)(x) = g'(x^4) \cdot \frac{d}{dx}(x^4)$$

FTC: $g'(x) = \sec(x) \Rightarrow g'(x^4) = \sec(x^4)$

So, $$\frac{d}{dx} G(x) = \boxed{\sec(x^4)\,(4x^3)}$$

Example. Find the derivative of the function $f(x) = \int_{\sin x}^{1} \sqrt{1+t^2}\, dt$

$$g(x) = \int_a^x f(t)\,dt.$$

Prop. of the integral: $\int_{\sin x}^{1} \sqrt{1+t^2}\,dt = -\int_1^{\sin x} \sqrt{1+t^2}\,dt$

$$f(x) = -\,g(\sin x) = -\int_1^{\sin x} \sqrt{1+t^2}\,dt$$

Chain Rule: $f'(x) = -\,g'(\sin x) \cdot \frac{d}{dx}(\sin x)$

FTC: $g'(x) = \sqrt{1+x^2} \to g'(\sin x) = \sqrt{1+\sin^2 x}$.

Finally:

$$\boxed{f'(x) = -\sqrt{1+\sin^2 x} \cdot \cos(x)}$$

Second part of the Fundamental Theorem of Calculus.

Example. Compute the integral $\int_a^b x\,dx$ where a and b are two numbers such that a < b.

Formula: $\int_a^b x\,dx = \frac{b^2 - a^2}{2}$

Anti derivate

$$\frac{x^2}{2} \xrightarrow{x=b} \frac{b^2}{2}$$

$$\frac{x^2}{2} \xrightarrow{x=a} \frac{a^2}{2}$$

$$\Rightarrow \int_a^b x\,dx = \left.\frac{x^2}{2}\right|_a^b = \frac{b^2}{2} - \frac{a^2}{2}$$

$x=b$

$x=a$

The Fundamental Theorem of Calculus, Part 2 If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$$\int_a^b f(x)\,dx = \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1}$$

$f(x) = x^n$

$F(x) = \frac{x^{n+1}}{n+1}$

where F is any antiderivative of f, that is, a function F such that $F' = f$.

Example. Evaluate the integral $\int_{-2}^{1} x^3\,dx$.

Anti-derivative of x^3: $\frac{x^4}{4} + \cancel{C}^{=0}$

$$\int_{-2}^{1} x^3\,dx = \left.\frac{x^4}{4}\right|_{-2}^{1} = \frac{1^4}{4} - \frac{(-2)^4}{4}$$

$$= \boxed{\frac{-15}{4}}$$

Example. Find the value of the integral $\int_0^{\pi} \underbrace{(3x^2 - \sin(\cancel{x}x) + \cos(x) - x\sin(x))}_{f(x)}\, dx.$

$$\int_0^{\pi} f(x)\,dx = \int_0^{\pi} 3x^2\,dx - \int_0^{\pi} \sin x + \int_0^{\pi} \cos x\,dx$$
$$- \int_0^{\pi} \cancel{x\sin x\,dx}$$

$$3x^2 \longrightarrow 3\frac{x^3}{3} = x^3$$

$$\sin x \longrightarrow -\cos x$$

$$\cos x \longrightarrow \sin x \qquad (-x\cos x)'$$

$$\cancel{x\sin x} \longrightarrow \qquad = -\cos x + x\sin x$$

$$\int_0^{\pi} 3x^2\,dx = x^3\Big|_0^{\pi} = \pi^3 - 0^3 = \pi^3$$

$$\int_0^{\pi} \sin x\,dx = -\cos x\Big|_0^{\pi} = -\cos\pi - (-\cos 0)$$
$$= 1 + 1 = 2$$

$$\int_0^{\pi} \cos x\,dx = \sin x\Big|_0^{\pi} = \sin\pi - \sin 0 = 0$$

So,

$$\int_0^{\pi} f(x)\,dx = \pi^3 - 2 + 0 = \boxed{\pi^3 - 2}$$

EXAMPLE 8 What is wrong with the following calculation?

$$\int_{-1}^{3} \frac{1}{x^2}\,dx = \frac{x^{-1}}{-1}\Bigg]_{-1}^{3} = -\frac{1}{3} - 1 = -\frac{4}{3}$$

Differentiation and Integration as Inverse Processes.

The Fundamental Theorem of Calculus Suppose f is continuous on $[a, b]$.

1. If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$.

2. $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f, that is, $F' = f$.