Chapter 4: Integrals
Week 11

Pierre-Olivier Parisé
Calculus I (MATH-241 01/02)

University of Hawai'i
Fall 2021

Upcoming this week

1. 4.2 The Definite Integral
2. 3.9 Antiderivatives
3. 4.3 The Fundamental of Calculus

For a given sample of equidistributed points $x_i^* \in [a, b]$, we create the Riemann sums

$$S_n(f) := \sum_{i=1}^{n} f(x_i^*)\Delta x_i.$$

We also call the points x_i^* a partition of the interval $[a, b]$. The numbers $\Delta x_i = x_i^* - x_{i-1}^*$.

Definition 1

If f is a function on $[a, b]$. The <u>definite integral</u> of f from a to b is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

provided the limit exists and is the same value for any sample of points x_i^* of $[a, b]$.

Remarks:

- The integral $\int_a^b f(x)\,dx$ represent the net area between the curve $y = f(x)$ and the x-axis when $f(x)$ is non-negative.
- Sometimes, it is useful to work with a partition that subdivides the interval $[a, b]$ into subintervals of different lengths. In this case, if Δx_i represents the length of the intervals $[x_{i-1}, x_i]$, then the definite integral can be expressed as

$$\int_a^b f(x)\,dx = \lim_{\Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i.$$

Example 2

Show that $f(x) = 1$ is integrable over the interval $[0, 1]$.

Which functions are integrable.

Theorem 3

If f is continuous on $[a, b]$, or has a finite number of jump discontinuities, then f is integrable on $[a, b]$.

So, we know that

- Any polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ is integrable on $[a, b]$.
- The function $|x|$ is integrable on $[a, b]$.
- Any trigonometric function is integrable on $[a, b]$.

Example 4

Express the following limit in term of an integral:

$$\lim_{n\to\infty} \sum_{i=1}^{n} (x_i^3 + x_i \sin x_i)\Delta x.$$

How so we compute the integral of a function?

Theorem 5

If f is integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$.

Example 6

Using the last Theorem, compute the integral $\displaystyle\int_0^3 (x^2 - 6x)\,dx$.

Theorem 7

If f is integrable on $[a, b]$, then

- $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$
- $\displaystyle\int_a^a f(x)\,dx = 0.$
- $\displaystyle\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx.$

Example 8

Suppose $\int_0^1 f(x)\,dx = 10$ and $\int_2^1 f(x)\,dx = -5$, compute the value of $\int_0^2 f(x)\,dx$.

Theorem 9

If f and g are integrable functions on $[a, b]$ and c is a real number, then

- $\displaystyle\int_a^b c\,dx = c(b-a).$
- $\displaystyle\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$
- $\displaystyle cf(x)\,dx = c\int_a^b f(x)\,dx.$

Example 10

Compute the value of the definite integral $\displaystyle\int_0^1 (4+3x^2)\,dx$.

Theorem 11

If f is integrable and $m \leq f(x) \leq M$ for $x \in [a, b]$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

Example 12

Estimate the integral $\int_1^4 \sqrt{x}\,dx$.

Some other important properties of the integral are

- if $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq 0$.
- if $f(x) \leq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$.

Exercises: 3, 7, 17-20, 21-25 (Use the Theorem 18 instead of the definition), 29, 30, 34, 35-40, 59-64.

Question 13

Can you find a function $F(x)$ such that $F'(x) = 3x^2$?

Definition 14

A function F is called an antiderivative of f on an interval if $F'(x) = f(x)$ for all x in I.

Remark: When you find an antiderivative F, the function $F(x) + C$ where C is a constant is also an antiderivative.

Example 15

Find all the antiderivative of each of the following functions.

a) $f(x) = \sin x$.

b) $f(x) = x^3$.

c) $f(x) = x^{-3}$.

Function	Antiderivative
$cf(x)$	$cF(x) + C$
$f(x) + g(x)$	$F(x) + G(x) + C$
x^n $(n \neq 1)$	$\frac{x^{n+1}}{n+1} + C$
$\cos x$	$\sin x + C$
$\sin x$	$-\cos x + C$
$\sec^2 x$	$\tan x + C$
$\sec x \tan x$	$\sec x$

Table: Table of some functions and their antiderivatives

Example 16

A particle moves in a straight line and has acceleration given by $a(t) = 6t + 4$. Its initial velocity is $v(0) = -6cm/s$ and its initial displacement is $s(0) = 9cm$. Find its position function $s(t)$.

Exercises: 1-20, 21-22, 33-36, 46, 53-58.

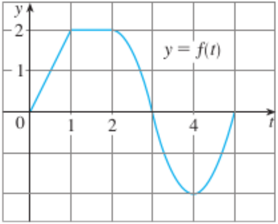
Remember that the derivative can be viewed as a function, that is $f'(x)$.

We can do the same thing for the integral by setting the upper limit in the integral to be x:

$$F(x) := \int_a^x f(t)\,dt.$$

Example 17

Suppose that f is the function given by the graph in the following figure:



If $F(x) := \int_0^x f(t)\,dt$, find the value of $g(0)$, $g(1)$, $g(2)$.

We can prove that

$$\int_0^x x\,dx = \frac{x^2}{2}.$$

As you can see, the integrand is exactly the derivative of $x^2/2$. In fact, this is true in general.

Fondamental Theorem of Calculus (Part 1)

If f is conitnuous on $[a, b]$, then the function F defined by

$$F(x) := \int_a^x f(t)\,dt \quad a \leq x \leq b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $F'(x) = f(x)$.

Example 18

Find the derivative of the function $F(x) = \int_0^x \sqrt{1+t^2}\,dt$.

Remember: a function F is an antiderivative of f if $F'(x) = f(x)$.

Fundamental Theorem of Calculus (Part 2)

If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function F such that $F'(x) = f(x)$.

Example 19

Evaluate the integral $\int_{-2}^{1} x^3\,dx$.

Example 20

Find the area under the cosine curve from 0 to $\pi/2$.

The Fundamental Theorem of Calculus (FTC)

Suppose f is continuous on $[a, b]$.

- If $F(x) = \int_a^x f(x)\,dx$, then $F'(x) = f(x)$.
- If F is an antiderivative of f, then $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$.

The FTC tells us that

- differentition undoes what the integration does.
- integration undoes what the integration does.
- in other words, integration and derivation are inverse processes.

Exercises: 7-18, 19-38, 39-42, 43-46 (only find the exact value).