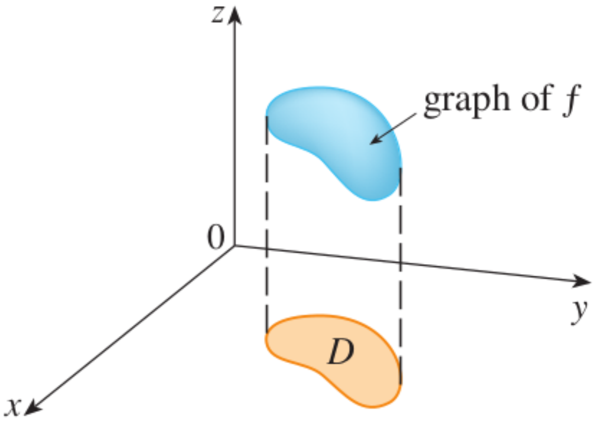
Chapter 15

Multiple Integrals

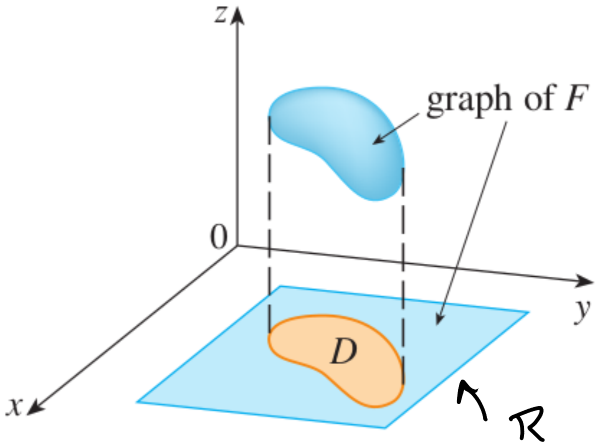
15.2 Double Integrals over genaral regions

Definition.

Given: A function f defined on D



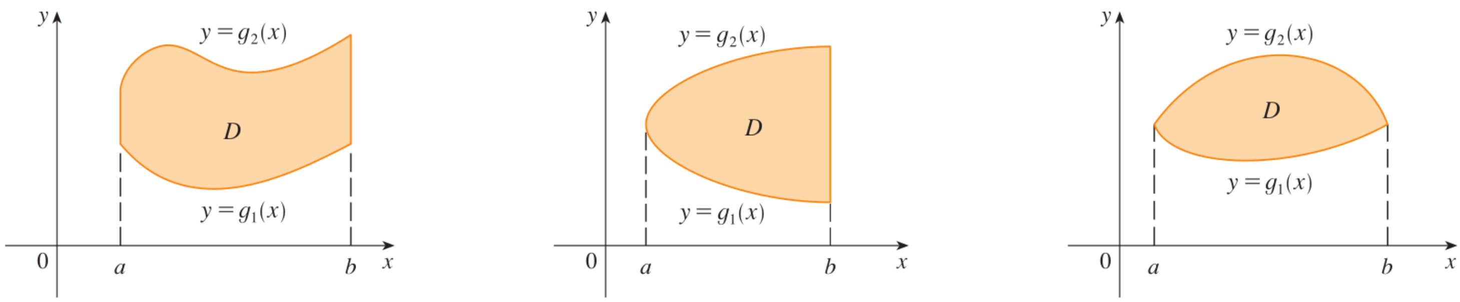
Extend f to a rectangle containing D



$$F(x,y) = \begin{cases} f(x,y), & (x,y) \in D \\ 0, & (x,y) \in R \text{ but } (x,y) \notin D \end{cases}$$

$$\boxed{\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA}$$

Region of type I.

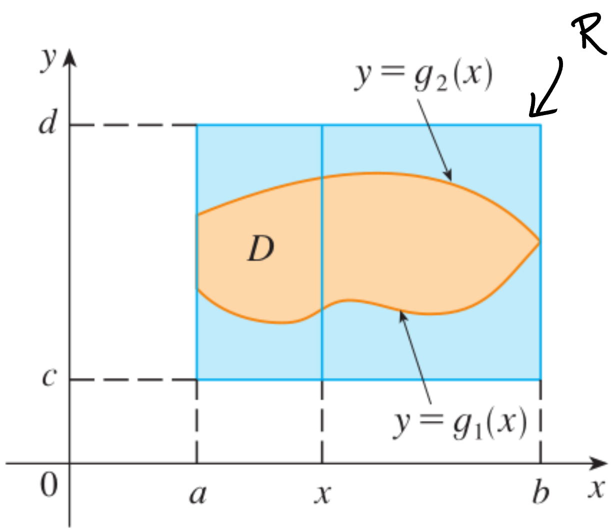


$$D = \{(x,y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$$

$$\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA$$

$$= \int_a^b \int_c^d F(x,y)\,dy\,dx$$

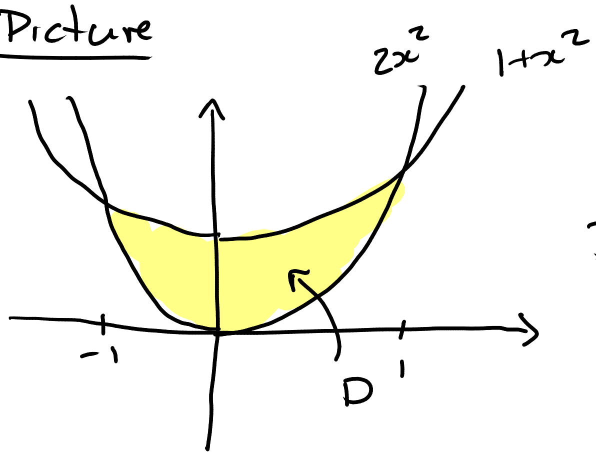
$$= \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$



$$\boxed{\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.}$$

EXAMPLE 1 Evaluate $\iint_D (x + 2y)\, dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

① Picture



$$1 + x^2 = 2x^2$$

$$1 = x^2 \Rightarrow x = \pm 1$$

$$D = \{ (x, y) : -1 \le x \le 1 , \quad 2x^2 \le y \le 1 + x^2 \} .$$

② Integrate

$g_1(x) = 2x^2$ & $g_2(x) = 1 + x^2$

$$\iint_D x + 2y \, dA = \int_{-1}^{1} \int_{2x^2}^{1+x^2} x + 2y \, dy \, dx$$

$$= \int_{-1}^{1} xy \Big|_{2x^2}^{1+x^2} + y^2 \Big|_{2x^2}^{1+x^2} \, dx$$

$$= \int_{-1}^{1} x\left(1 + x^2 - 2x^2\right) + (1+x^2)^2 - (2x^2)^2 \, dx$$

$$= \int_{-1}^{1} x - x^3 + 1 + 2x^2 + x^4 - 4x^4 \, dx$$

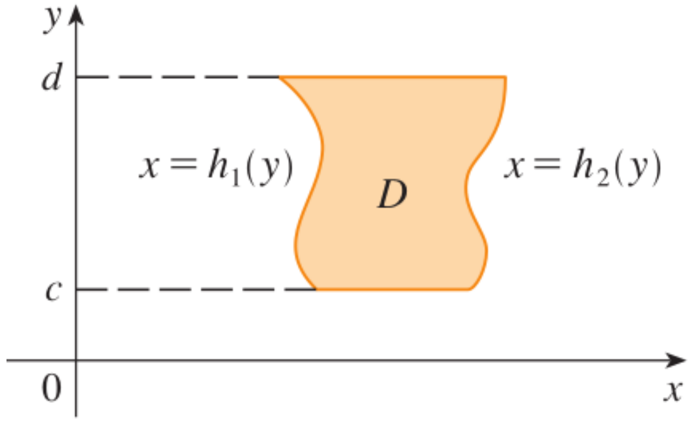
$$= \int_{-1}^{1} 1 + x + 2x^2 - x^3 - 3x^4 \, dx$$

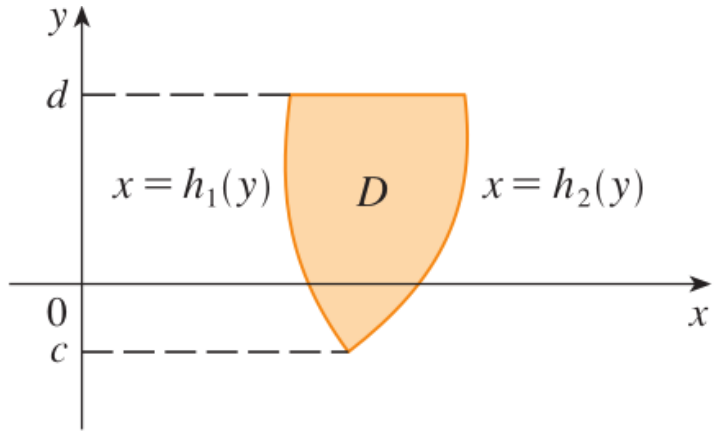
$$= \boxed{\frac{32}{15}} \approx 1.3167 .$$

$$\left(x + \frac{x^2}{2} + \frac{2x^3}{3} - \frac{x^4}{4} - \frac{3x^5}{5} \right) \Bigg|_{-1}^{1}$$

Region of Type II.

$$D = \{(x, y) \mid c \leq y \leq d,\ h_1(y) \leq x \leq h_2(y)\}$$





$$\boxed{\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy}$$

EXAMPLE. Evaluate $\displaystyle\iint_D e^{-y^2}\, dA$, where D is the region bounded by the lines $x = 0, x = 3$ and $x = y$.

EXAMPLE. Find the volume of the tetrahedron bounded by the planes $x + 2y + z = 2$, $x = 2y$, $y = 0$, and $z = 0$.

EXAMPLE 5 Evaluate the iterated integral $\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx$.

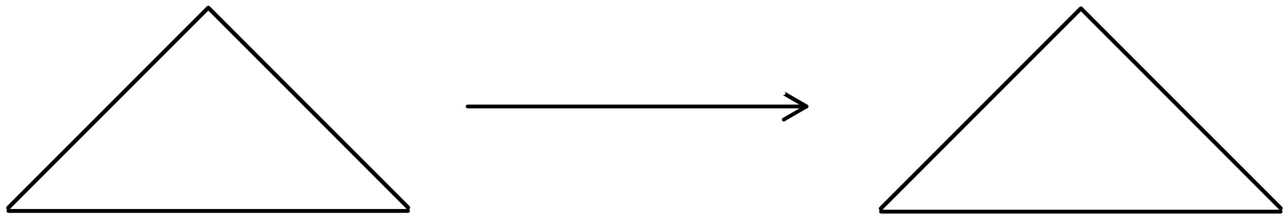
$\boxed{6}$ $$\iint_D \big(f(x,y)+g(x,y)\big)\,dA = \iint_D f(x,y)\,dA + \iint_D g(x,y)\,dA$$

$\boxed{7}$ $$\iint_D cf(x,y)\,dA = c\iint_D f(x,y)\,dA$$

$\boxed{8}$ If $f(x,y) \geq g(x,y)$ on D, then $\displaystyle\iint_D f(x,y)\,dA \geq \iint_D g(x,y)\,dA$

$\boxed{9}$ If $D = D_1 \cup D_2$, with $D_1 \cap D_2 = \emptyset$, then

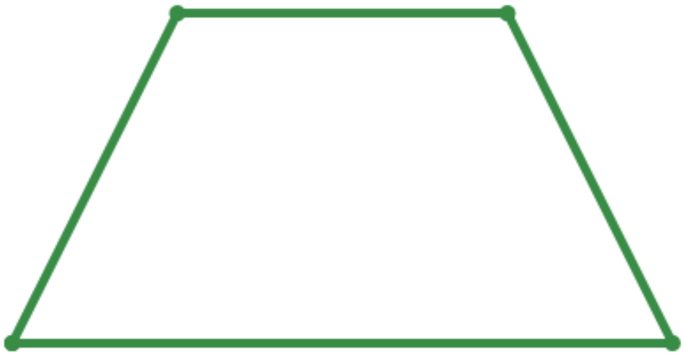
$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$$



$\boxed{10}$ $$Area(D) = \iint_D 1\,dA$$

$\boxed{11}$ If $m \leq f(x,y) \leq M$, then $\displaystyle m \cdot Area(D) \leq \iint_D f(x,y)\,dA \leq M \cdot Area(D)$

Example. Find the area of the trapezoid below:



Challenge. Find the area of the hexagone below using properties 9 and 10: