Math 644

Chapter 3

Section 3.3: Growth On $\mathbb{C}$ and $\mathbb{D}$

Contents

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A first consequence of the maximum principle is the famous Liouville's Theorem.

Theorem 1. If f is analytic in $\mathbb{C}$ and bounded, then f is constant.

Proof.

Suppose that $|f| \leq M < \infty$.

Let

$$g(z) = \begin{cases} \dfrac{f(z) - f(0)}{z}, & z \neq 0 \\ f'(0), & z = 0 \end{cases}$$

then g is analytic.

If $|z| = R$, then

$$|g(z)| \leq \frac{2M}{R}$$

By the max principle

$$\sup_{z \in B_R} |g(z)| \leq \frac{2M}{R}$$

where $B_R = \{z : |z| < R\}$. Taking $R \to \infty$

$$\sup_{\mathbb{C}} |g(z)| = 0$$

So, $g \equiv 0$ and so $f(z) = f(0)$. $\square$

Schwarz's Lemma

A second consequence of the maximum principle is the Schwarz's Lemma.

Theorem 2. Suppose f is analytic in $\mathbb{D}$ and suppose $|f(z)| \leq 1$ and $f(0) = 0$. Then

$$|f(z)| \leq |z|, \tag{1}$$

for all $z \in \mathbb{D}$, and

$$|f'(0)| \leq 1. \tag{2}$$

Moreover, if equality holds in (1) for some $z \neq 0$ or if equality holds in (2), then $f(z) = cz$, where c is a constant with $|c| = 1$.

Proof.

Note:

- A bounded analytic function in $\mathbb{D}$ can't grow too fast in the disk.

Invariant Form of Schwarz's Lemma

THEOREM 3. Suppose f is analytic in $\mathbb{D}$ and suppose $|f(z)| < 1$. If $z, a \in \mathbb{D}$, then

$$\left|\frac{f(z) - f(a)}{1 - \overline{f(a)}f(z)}\right| \leq \left|\frac{z - a}{1 - \overline{a}z}\right|$$

and

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}.$$

Proof.

Theorem 4. If f is analytic in $\mathbb{D}$, $|f| \leq 1$ and $f(z_j) = 0$, for $j = 0, 1, \ldots, n$, then

$$f(z) = \prod_{j=1}^{n} \left(\frac{z - z_j}{1 - \overline{z}_j z} \right) g(z),$$

where g is analytic in $\mathbb{D}$ and $|g(z)| \leq 1$ in $\mathbb{D}$.

Proof.

Growth Rate

COROLLARY 5. If f is non-constant, bounded, and analytic in $\mathbb{D}$, and if z_j $(j \geq 1)$ are the zeros of f (repeated according to their multiplicity), then

$$\sum_{j=1}^{\infty}(1 - |z_j|) < \infty.$$

Proof.