Math 241

Chapter 4

Section 4.1: Areas and Distances

Contents

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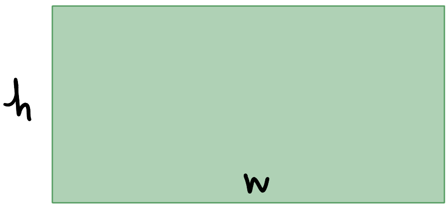
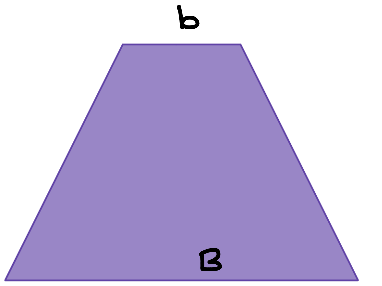
Fall 2022

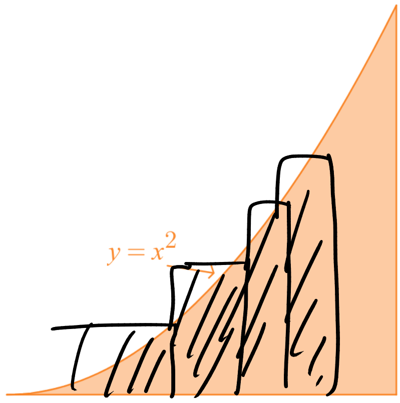
What is the area of the following shapes?



(a) Area = $w \cdot h$



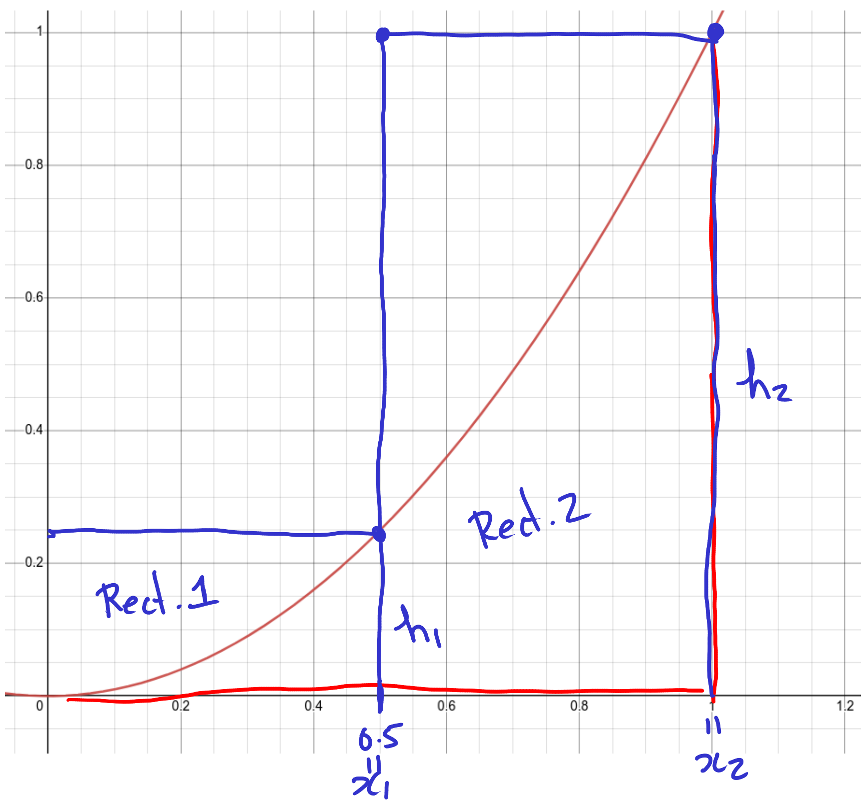
(b) Area = $\frac{b+B}{2}\, h$



(c) Area =

Trick: Use simpler shapes, such as rectangles, to approximate the area.

Example 1. Using rectangles, approximate the area of the region S under the graph of $y = x^2$ between $x = 0$ and $x = 1$. Go to Desmos: https://www.desmos.com/calculator/gfrgqd4nvx



2 rectangles $(n=2)$

I) Divide the shape in 2 rectangles of

$\Delta x = \frac{1-0}{2} = \frac{1}{2}$

two subintervals $[0, 0.5]$ & $[0.5, 1]$

II) Select the right endpts. of the subintervals:

$x_1 = 0.5 = 0 + 1 \cdot \Delta x$

$x_2 = 1 = 0 + 2 \cdot \Delta x$

$h_1 = f(x_1) = 0.25$

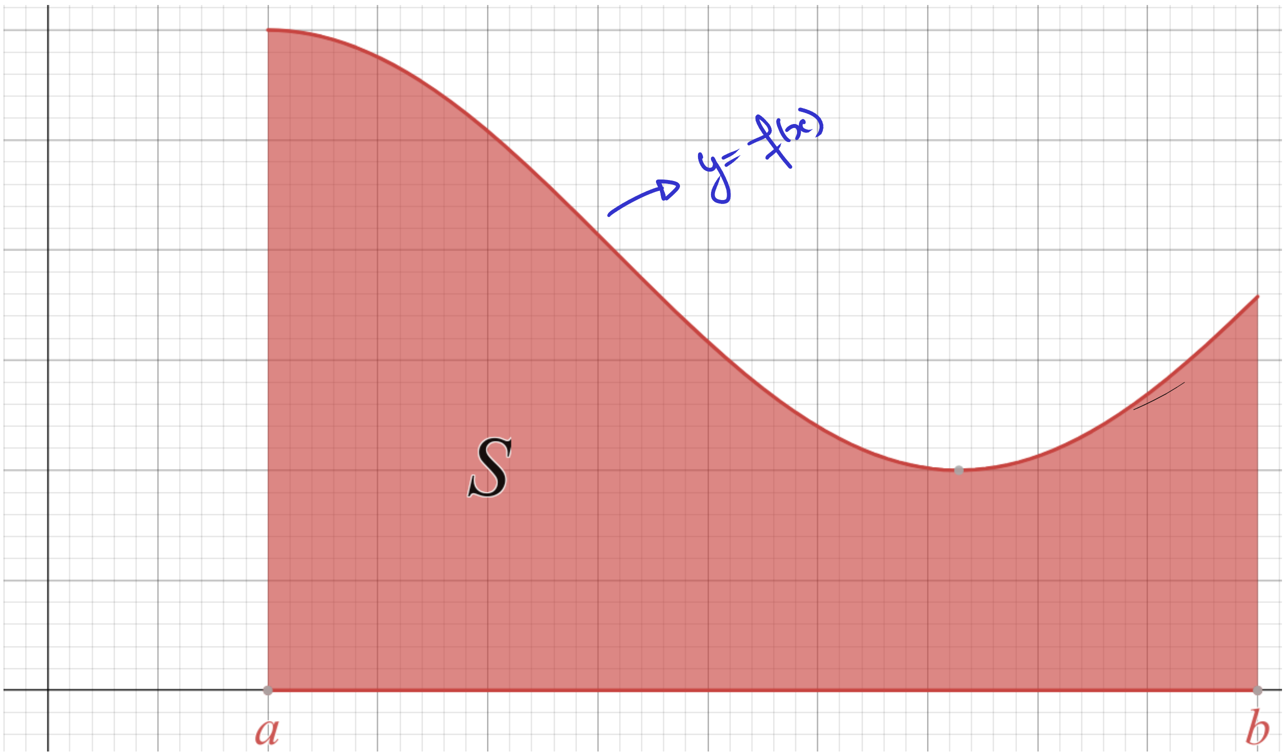
$h_2 = f(x_2) = 1$

III) Compute the area of the rectangles

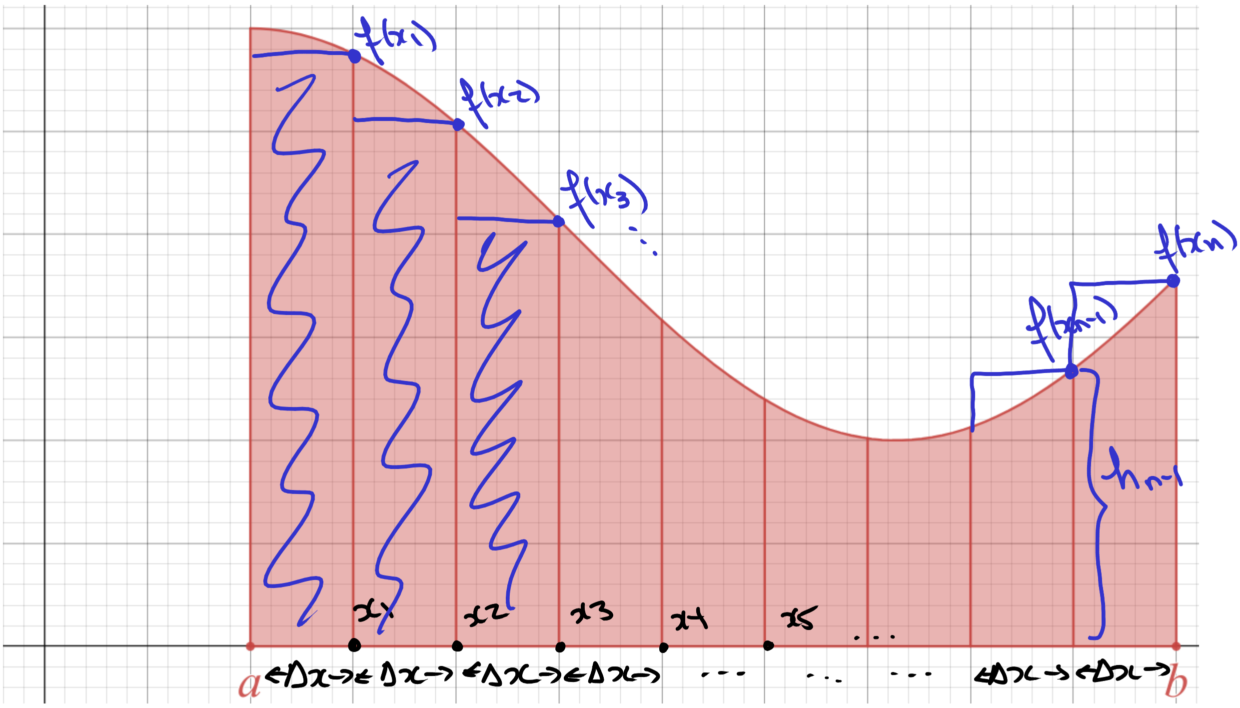
$$\text{Rect. 1} + \text{Rect. 2} = \Delta x \cdot f(x_1) + \Delta x \cdot f(x_2) = \frac{1}{2} \cdot 0.25 + \frac{1}{2} \cdot 1 = \boxed{\frac{5}{8}}$$

Divide and Conquer With the Right Endpoint Rule!

Suppose we want to compute the area of a region S bounded by the graph of some function $y = f(x)$.



STEP I Subdivide the region S into n strips of equal width $\boxed{\Delta x} = (b-a)/n$.



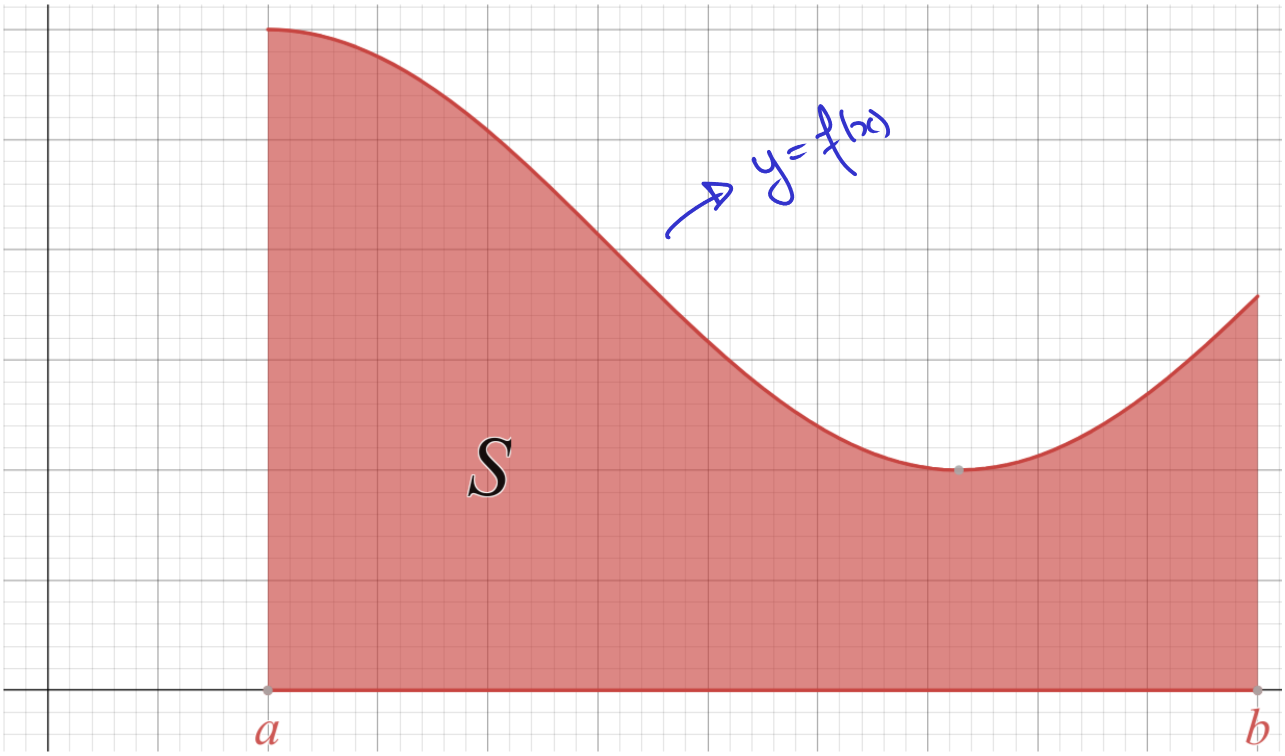
STEP II Choose the right-end point for all subintervals:
$x_1 = a + \Delta x$, $x_2 = a + 2\Delta x$, …, $x_{n-1} = a + (n-1)\Delta x$, $x_n = b$.

STEP III Approximate by adding the area of each rectangle:

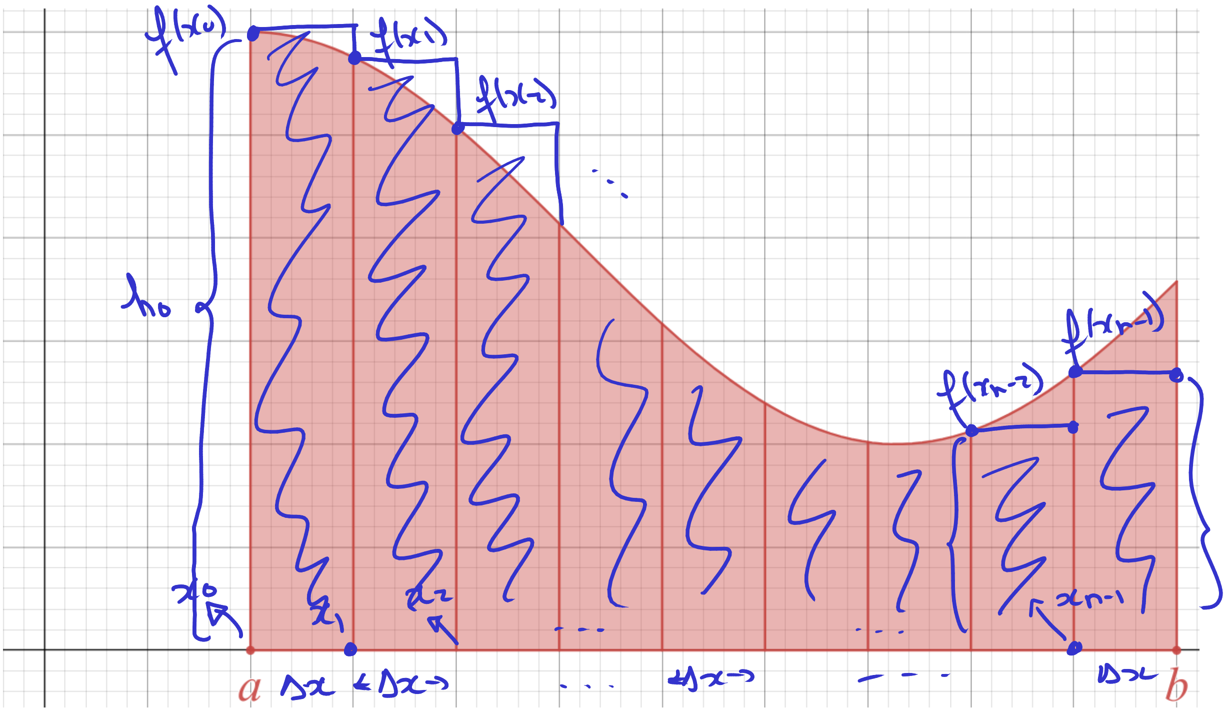
$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x.$$

Divide and Conquer With the Left Endpoint Rule!

Suppose we want to compute the area of a region S bounded by the graph of some function $y = f(x)$ from $x = a$ to $x = b$.



STEP I Subdivide the region S into n strips of equal width $\Delta x = (b-a)/n$.



STEP II Choose the left-end point for all subintervals:
$x_0 = a$, $x_1 = a + \Delta x$, $\ldots$, $x_{n-2} = a + (n-2)\Delta x$, $x_{n-1} = a + (n-1)\Delta x$.

STEP III Approximate by adding the area of each rectangle:

$$L_n = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x.$$

Sigma Notation

We use the symbol $\sum$ to write a summation of numbers compactly:

$a_1 + a_2 + a_3 + \cdots + a_n$

max index $\rightarrow$ n

$$\sum_{i=k}^{n} a_i$$

$a_i \rightarrow$ general term

variable index $\rightarrow$ i

$k \leftarrow$ starting index (min)

Example 2.

a) Expand $\sum_{i=1}^{7} i.$

b) Write $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}$ with the Sigma notation.

c) Write $1 + 3 + 5 + 7 + 9 + 11 + 13$ with the Sigma notation.

$13 = 2i - 1$
$\rightarrow i = 7$

a) min index = 1
max index = 7
general term: i

$$\sum_{i=1}^{7} i = 1 + 2 + 3 + 4 + 5 + 6 + 7$$

b) min index = 1
max index = 7
general term: $\frac{1}{i}$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} = \sum_{i=1}^{7} \frac{1}{i}$$

c) min index = 1
max index = 7
general term = $2i - 1$

$$\sum_{i=1}^{7} (2i-1).$$

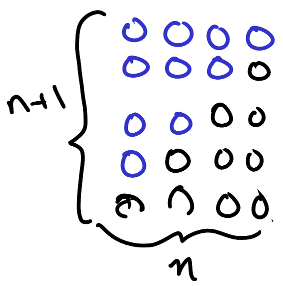
add
$1 = 2\cdot 1 - 1$
$3 = 2\cdot 2 - 1$
$5 = 2\cdot 3 - 1$
$7 = 2\cdot 4 - 1$
$\vdots$
$2i - 1$

even
$2 = 2\cdot 1$
$4 = 2\cdot 2$
$6 = 2\cdot 3$
$8 = 2\cdot 4$
$10 = 2\cdot 5$
$\vdots$
$2i$

Useful Sum Formulas:

$7(7+1)/2 = 28$

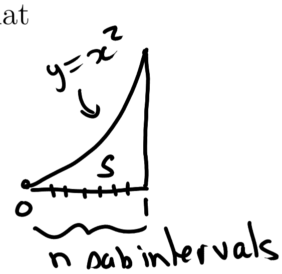
- $\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$;
- $\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$;
- $\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$.



$\frac{(n+1)n}{2}$

Taking the Limit!

EXAMPLE 3. Show that the area of the region S in Example 1 is 1/3. In other words, show that



$$\text{Area}(S) = \lim_{n\to\infty} R_n = 1/3.$$

General: $\Delta x = \frac{b-a}{n} = \frac{1}{n}$

$$x_1 = 0 + 1\cdot\Delta x = \frac{1}{n}$$
$$x_2 = 0 + 2\Delta x = \frac{2}{n}$$
$$\vdots$$
$$x_i = 0 + i\,\Delta x = \frac{i}{n}$$
$$\vdots$$
$$x_n = 0 + n\,\Delta x = \frac{n}{n} = 1$$

$$R_n = \sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\cdot\frac{1}{n}$$
$$= \sum_{i=1}^{n} \frac{i^2}{n^2}\cdot\frac{1}{n}$$
$$= \sum_{i=1}^{n} \frac{i^2}{n^3}$$
$$= \frac{1^2}{n^3} + \frac{2^2}{n^3} + \dots + \frac{i^2}{n^3} + \dots + \frac{n^2}{n^3}$$
$$= \frac{1}{n^3}\left(1^2 + 2^2 + \dots + i^2 + \dots + n^2\right)$$
$$= \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$$

$$\Rightarrow \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) = \lim_{x\to\infty} \frac{1}{x^3}\left(\frac{x(x+1)(2x+1)}{6}\right)$$
$$= \boxed{1/3}$$

General definition of Area: The area of the region S lying under the graph of a function $y = f(x)$ from $x = a$ to $x = b$ is given by

- $\text{Area}(S) = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \Big(f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_n)\Delta x\Big) = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$
- $\text{Area}(S) = \lim_{n\to\infty} L_n = \lim_{n\to\infty} \Big(f(x_0)\Delta x + f(x_1)\Delta x + \dots + f(x_{n-1})\Delta x\Big)$

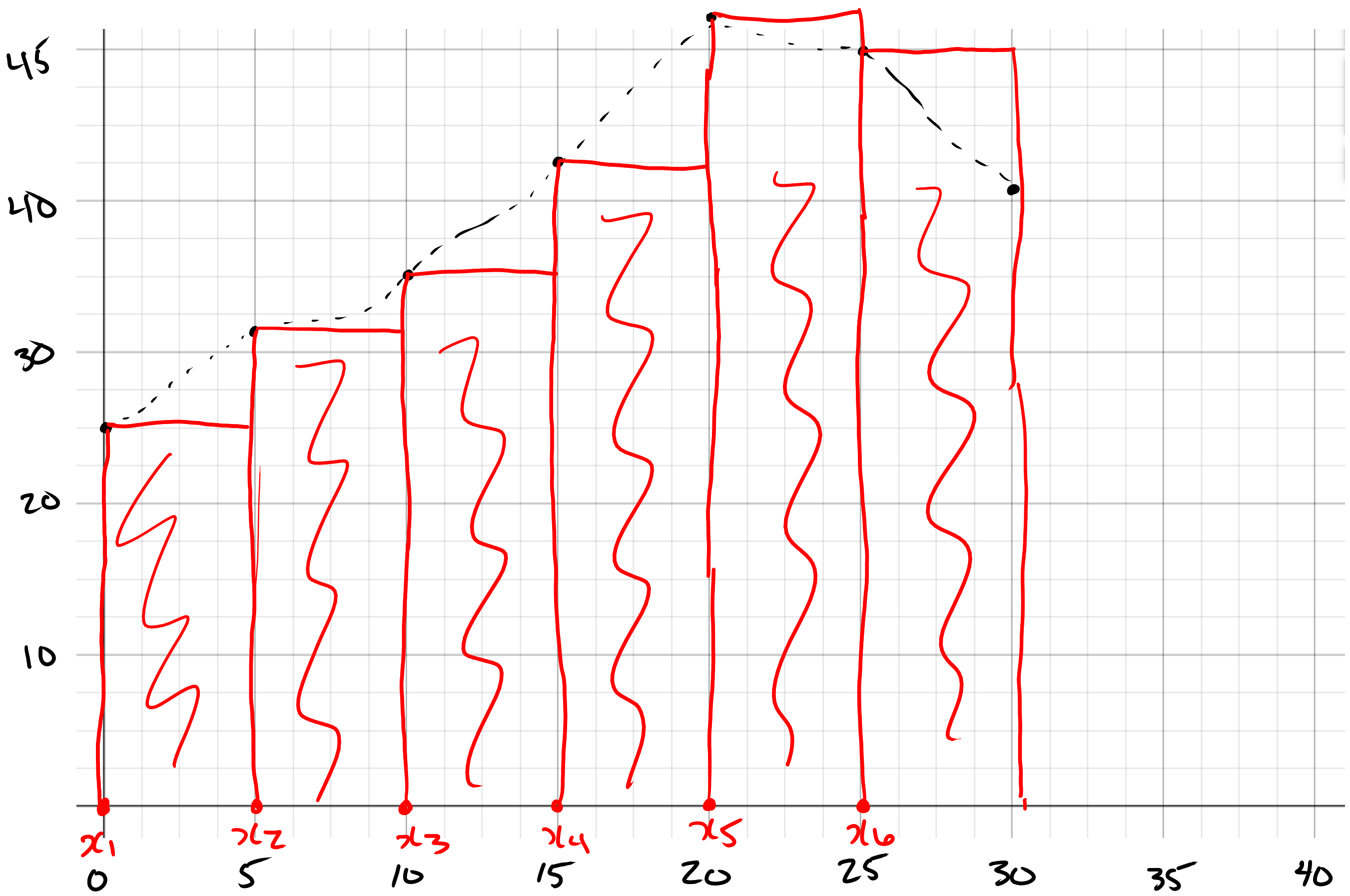
If an object move at constant velocity, then the distance between the start and finish line is easy to compute:

$$\text{Distance} = \text{Velocity} \times \Delta\text{Time}\ .$$

What do we do if the velocity is not constant?

Example 4. Suppose the odometer on our car is broken and we want to estimate the distance driven over a 30-second time interval. We take speedometer readings every five seconds and record them in the following table:

Time (s)	0	5	10	15	20	25	30
Velocity (ft/s)	25	31	35	43	47	45	41



$\Delta x = 5$

$x_1 = 0$ $\quad x_4 = 15$

$x_2 = 5$ $\quad x_5 = 20$

$x_3 = 10$ $\quad x_6 = 25$

Total Distance

$= 5 \cdot 25 + 5 \cdot 31$

$+ 5 \cdot 35 + 5 \cdot 43$

$+ 5 \cdot 47 + 5 \cdot 45$

$\Rightarrow$ Total Distance $\approx \boxed{1130 \text{ ft}}$.

Remark:

- The total distance is given by the area under the curve of the velocity function!