Chapter 1

Functions and Limits

1.6 Calculating Limits Using the Limit Laws

EXAMPLE 1 Use the graphs of f and g in Figure 1 to evaluate the following limits, if they exist.

(a) $\lim_{x\to -2} [f(x) + 5g(x)]$ (b) $\lim_{x\to 2} [f(x)g(x)]$ (c) $\lim_{x\to 2} \dfrac{f(x)}{g(x)}$

(d) $\lim_{x\to -2} [2f(x)]$ (e) $\lim_{x\to -2} [f(x) - g(x)]$

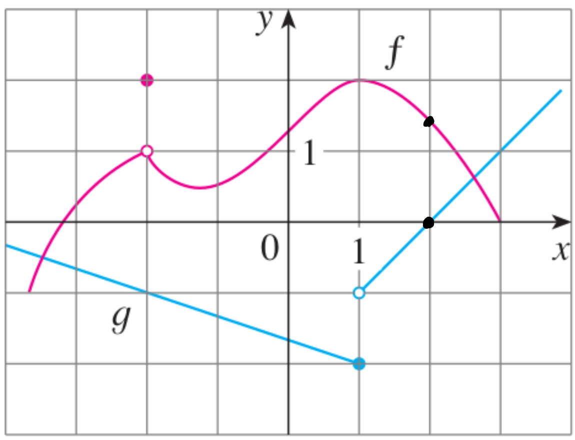


FIGURE 1
Use Desmos
https://www.desmos.com/calculator/7fy0x0ghia

(a) $\lim_{x\to -2} [f(x)+5g(x)] = -4$

$= 1 + (-5)$

$= 1 + 5(-1)$

$= \lim_{x\to -2} f(x) + 5 \lim_{x\to -2} g(x)$

(b) $\lim_{x\to 2} [f(x)g(x)] = 0 = 1.4 \cdot 0$

$= \underbrace{\lim_{x\to 2} f(x)}_{=1.4} \overbrace{\lim_{x\to 2} g(x)}^{0}$

(c) $\lim_{x\to -2} \dfrac{f(x)}{g(x)} = -1 = \dfrac{1}{-1} = \dfrac{\lim_{x\to -2} f(x)}{\lim_{x\to -2} g(x)}$

(d) $\lim_{x\to -2} 2f(x) = 2 \lim_{x\to -2} f(x) = 2 \cdot 1 = 2$

(e) $\lim_{x\to -2} [f(x) - g(x)] = \lim_{x\to -2} [f(x) + (-g(x))]$

$= \lim_{x\to -2} f(x) + \lim_{x\to -2} [-g(x)]$

$= \lim_{x\to -2} f(x) - \lim_{x\to -2} g(x) = 1 - (-1) = 2$

Limit Laws Suppose that c is a constant and the limits

$$\lim_{x\to a} f(x) \quad \text{and} \quad \lim_{x\to a} g(x)$$

exist. Then

1. $\lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x).$
2. $\lim_{x\to a} [f(x) - g(x)] = \lim f(x) - \lim g(x).$
3. $\lim_{x\to a} [cf(x)] = c \lim_{x\to a} f(x).$
4. $\lim_{x\to a} [f(x)g(x)] = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$
5. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$

EXAMPLE. Think of three ways of computing the following limit:

$$\lim_{x\to 2}(1+x)^3$$

1) Graph: $\lim_{x\to 2}(1+x)^3 = 27$

2) Product: $\lim_{x\to 2}(1+x)^3 = \lim_{x\to 2}(1+x)^2(1+x)$

$= \lim_{x\to 2}(1+x)^2 \lim_{x\to 2}(1+x)$

$= \lim_{x\to 2}(1+x) \lim_{x\to 2}(1+x) \lim_{x\to 2}(1+x)$

$= \left(\lim_{x\to 2} 1+x\right)^3$

$= \left(\lim_{x\to 2} 1 + \lim_{x\to 2} x\right)^3 = (1+2)^3$

$= 3^3 = \boxed{27}$

EXAMPLE. Think of three ways of computing the following limit:

$$\lim_{x\to\pi/4}\cos^2(x)$$

Graph: $\lim_{x\to\pi/4}\cos^2(x) = 0.5054$

Product rule: $\lim_{x\to\pi/4}\cos^2(x) = \lim_{x\to\pi/4}\cos x \lim_{x\to\pi/4}\cos x$

$= \left(\lim_{x\to\pi/4}\cos x\right)^2 = \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{2}{4} = \boxed{\frac{1}{2}}$

General Formula:

6. $\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$ where n is a positive integer

Special cases:

$\lim_{x\to a} c = c$, $\lim_{x\to a} x = a$, $\lim_{x\to a} x^n = a^n$.

EXAMPLE 2 Evaluate the following limits and justify each step.

(a) $\lim_{x \to 5} (2x^2 - 3x + 4)$ (b) $\lim_{x \to -2} \dfrac{x^3 + 2x^2 - 1}{5 - 3x}$

(a) $\lim_{x \to 5} (2x^2 - 3x + 4) = \lim_{x \to 5} 2x^2 - \lim_{x \to 5} 3x + \lim_{x \to 5} 4$ [Sum]

$= 2 \lim_{x \to 5} x^2 - 3 \lim_{x \to 5} x + 4$ [Constant]

$= 2 \left(\lim_{x \to 5} x\right)^2 - 3 \cdot 5 + 4$ [Power]

$= 2 \cdot 5^2 - 3 \cdot 5 + 4 = \boxed{39}$

(b) Want quotient rule $\to$ $\lim_{x \to -2} 5 - 3x = 5 - 3(-2) = 11 \neq 0$ ✓

$\lim_{x \to -2} \dfrac{x^3 + 2x^2 - 1}{5 - 3x} = \dfrac{\lim_{x \to -2} x^3 + 2x^2 - 1}{\lim_{x \to -2} 5 - 3x}$ [Quotient]

$= \dfrac{\lim_{x \to -2} x^3 + 2 \lim_{x \to -2} x^2 - \lim_{x \to -2} 1}{11}$ [Sum Constant Diff.]

$= \dfrac{(-2)^3 + 2(-2)^2 - 1}{11}$ [Power rule]

$= \dfrac{-8 + 8 - 1}{11}$

$= \boxed{\dfrac{-1}{11}}$

Remark:

Direct Substitution Property If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a)$$

Root Law.

11. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ where n is a positive integer

[If n is even, we assume that $\lim_{x \to a} f(x) > 0$.]

Example. Compute $\lim_{u \to -2} \sqrt{u^4 + 3u + 6}$.

$$\lim_{u \to -2} (u^4 + 3u + 6) = (-2)^4 + 3(-2) + 6 = 16 > 0$$

$$\lim_{u \to -2} \sqrt{u^4 + 3u + 6} = \sqrt{\lim_{u \to -2} u^4 + 3u + 6}$$

$$= \sqrt{16}$$

$$= \boxed{4}$$

Exception to the Substitution Law

EXAMPLE 3 Find $\lim\limits_{x\to 1} \dfrac{x^2-1}{x-1}$.

Quotient:
or subst.

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \frac{1-1}{1-1} = \frac{0}{0} \to$$ For now, undefined.

Simplify:

$$\frac{x^2-1}{x-1} = \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} = x+1 \quad (x \neq 1)$$

$$= g(x) \quad (x \neq 1)$$

$$\lim_{x\to 1} \frac{x^2-1}{x+1} = \lim_{x\to 1} \overset{g(x)}{\overset{\downarrow}{x+1}} = \lim_{x\to 1} x + \lim_{x\to 1} 1$$

$$= 1 + 1 = \boxed{2}$$

We have to use the following new substitution rule:

> If $f(x) = g(x)$ when $x \neq a$, then $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x)$, provided the limits exist.

EXAMPLE 5 Evaluate $\lim_{h \to 0} \frac{(3+h)^2 - 9}{h}$.

Quotient: $\lim_{h \to 0} \frac{(3+h)^2 - 9}{h} = \frac{(3+0)^2 - 9}{0} = \frac{0}{0} \to$ undefined yet!.

Simplify: $\frac{(3+h)^2 - 9}{h} = \frac{\cancel{9} + 6h + h^2 \cancel{-9}}{h}$

$= \frac{6h + h^2}{h}$ $\qquad h \neq 0.$

$= \frac{(6+h)\cancel{h}}{\cancel{h}} = 6 + h = g(h)$

Now,

$\lim_{h \to 0} \frac{(3+h)^2 - 9}{h} = \lim_{h \to 0} 6 + h = 6 + 0 = \boxed{6}$

EXAMPLE 6 Find $\lim_{t \to 0} \frac{\sqrt{t^2+9}-3}{t^2}$.

Subst: $\frac{0}{0}$ undefined yet

Simplify: $\frac{\sqrt{t^2+9}-3}{t^2} \cdot 1 = \frac{\sqrt{t^2+9}-3}{t^2} \cdot \frac{\sqrt{t^2+9}+3}{\sqrt{t^2+9}+3}$

$$= \frac{t^2 + \cancel{9} - \cancel{9}}{t^2 \cdot \left(\sqrt{t^2+9}+3\right)}$$

$$= \frac{\cancel{t^2}}{\cancel{t^2}\left(\sqrt{t^2+9}+3\right)} = \frac{1}{\sqrt{t^2+9}+3} = g(t)$$

So,

$$\lim_{t \to 0} \frac{\sqrt{t^2+9}-3}{t^2} = \lim_{t \to 0} \frac{1}{\sqrt{t^2+9}+3}$$

$$= \frac{\lim_{t \to 0} 1}{\lim_{t \to 0} \left(\sqrt{t^2+9}+3\right)} = \frac{1}{6}$$

[root law & sum rule] or [subst. rule]

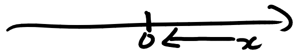
EXAMPLE 8 Prove that $\lim_{x\to 0} \frac{|x|}{x}$ does not exist. $\longrightarrow \frac{0}{0}$:

REMARK: ALL THE LIMIT RULES WORK Also FOR THE LIMITS FROM THE LEFT AND FROM THE RIGHT.

To prove $\lim_{x\to 0} \frac{|x|}{x}$ doesn't exist, we will show that

$$\lim_{x\to 0^+} \frac{|x|}{x} \neq \lim_{x\to 0^-} \frac{|x|}{x}.$$

$|x| = x$



1) $\lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} \frac{x}{x} = \lim_{x\to 0^+} 1$
$= 1$

$|-2| = -(-2)$

2) $\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = \lim_{x\to 0^-} -1$
$= -1$

1) $\neq$ 2) $\Rightarrow \lim_{x\to 0} \frac{|x|}{x} \nexists$

EXAMPLE 9 If

$$f(x) = \begin{cases} \sqrt{x-4} & \text{if } x > 4 \\ 8 - 2x & \text{if } x < 4 \end{cases}$$

$f(4)$

determine whether $\lim_{x\to4} f(x)$ exists.

1) Left-hand limit.

$$\lim_{x\to4^-} f(x) = \lim_{x\to4^-} 8-2x = \lim_{x\to4^-} 8 - 2\lim_{x\to4^-} x$$

$$= 8 - 2\cdot4 = 0$$

2) Right-Hand Limit.

$$\lim_{x\to4^+} f(x) = \lim_{x\to4^+} \sqrt{x-4} = \sqrt{\lim_{x\to4^+} x-4}$$

$$= \sqrt{4-4} = 0$$

3) Conclusion

$$\lim_{x\to4^-} f(x) = 0 = \lim_{x\to4^+} f(x)$$

$$\Rightarrow \boxed{\lim_{x\to4} f(x) = 0}$$

EXAMPLE 11 Show that $\lim_{x \to 0}\left[x^2 \sin\left(\frac{1}{x}\right)\right] = 0$.

Pretend: $\lim_{x \to 0}\left[x^2 \sin\left(\frac{1}{x}\right)\right] \cancel{=} \left(\lim_{x \to 0} x^2\right)\left(\lim_{x \to 0} \sin\left(\frac{1}{x}\right)\right)$.

$\exists$ $\nexists$

Good Way to do it:

$-1 \le \sin(A) \le 1$ $\to$ true for any angle A.

Choose $A = \frac{1}{x}$ $\Rightarrow$ $-1 \le \sin\left(\frac{1}{x}\right) \le 1$ $(x \neq 0)$

mult by x^2 $\Rightarrow$ $-x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2$ (*)

We have $\lim_{x \to 0} -x^2 = 0$ (**)

& $\lim_{x \to 0} x^2 = 0$ (***)

(*) with (**) & (***)

$\Rightarrow$ $\boxed{\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0}$

3 The Squeeze Theorem If $f(x) \le g(x) \le h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then

$$\lim_{x \to a} g(x) = L$$