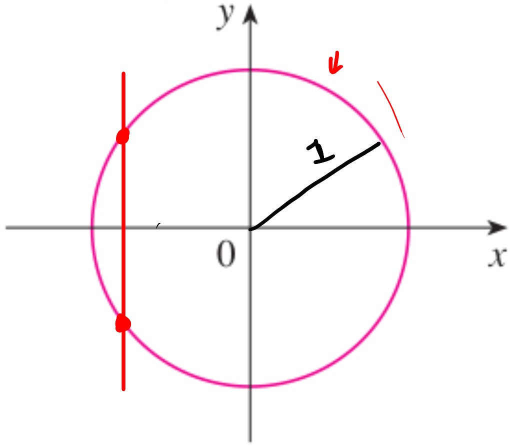
Chapter 2
Derivatives
2.6 Implicit Differentiation

Functions defined implicitly.

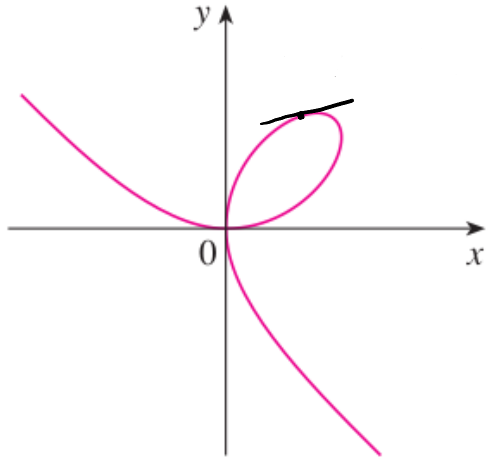
Geometry of curves.

$$x^2 + y^2 = 1$$



Folium of Descartes.

$$x^3 + y^3 = 6xy$$

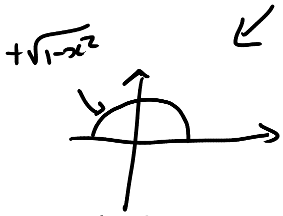


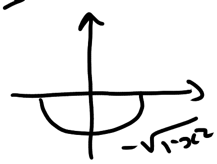
Find $y = f(x)$ s.t.

$$x^2 + (f(x))^2 = 1$$

$$\Rightarrow f(x)^2 = 1 - x^2$$

$$\Rightarrow f(x) = \pm\sqrt{1-x^2}$$





Find $y = f(x)$ s.t.

$$x^3 + (f(x))^3 = 6x\,f(x)$$

$$\Leftrightarrow x^3 - 6x f(x) + (f(x))^3 = 0$$

$$f_1(x) = \sqrt[3]{p^2 - 16q^2} + \sqrt[3]{p^2 - 16q^2}$$

$f_2(x) =$ ... ...

$f_3(x) =$

In Natural Science.

$$\boxed{\left(P + \frac{n^2 a}{V^2}\right)(V - nb) = nRT}$$

- P: Pressure.
- V: Volume.
- T: Temperature.
- R, a, b are constants depending on the gas.

Main steps for implicit differentiation:

1) Take the derivative on each side of the relation.

2) Use the chain rule and other rules to make the computations.

3) Isolate the derivative dy/dx.

EXAMPLE 1

(a) If $x^2 + y^2 = 25$, find $\dfrac{dy}{dx}$.

(b) Find an equation of the tangent to the circle $x^2 + y^2 = 25$ at the point $(3, 4)$.

(a) ① $\dfrac{d}{dx}(x^2+y^2) = \dfrac{d}{dx}(25)$

$\rightarrow \dfrac{d}{dx}(x^2) + \underset{\substack{\uparrow \\ y^2 \rightarrow (f(x))^2}}{\dfrac{d}{dx}(y^2)} = 0 \quad (*) \qquad y = f(x).$

② Chain rule.

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(f(x)^2) = 2f(x)\cdot\frac{d}{dx}f(x)$$

$$\overset{y=f(x)}{=} 2y \cdot \frac{dy}{dx}$$

③ Isolate dy/dx

$(*) \rightarrow 2x + 2y\underbrace{\frac{dy}{dx}}_{A} = 0$

$\rightarrow 2y\dfrac{dy}{dx} = -2x$

$\rightarrow \dfrac{dy}{dx} = \dfrac{-2x}{2y} = \boxed{\dfrac{-x}{y}}$

same!!

Note. $\underset{\text{upper half}}{y} = \sqrt{25-x^2} \rightarrow \dfrac{dy}{dx} = \dfrac{-2x}{2\sqrt{25-x^2}} = \dfrac{-x}{\underbrace{\sqrt{25-x^2}}_{=y}} = \dfrac{-x}{y}$

(b) Tangent line at $(\overset{x_0}{3}, \overset{y_0}{4})$

$y - 4 = m(x-3) \qquad m = y'(3) = \left.\dfrac{dy}{dx}\right|_{x=3}$

$m = \dfrac{-3}{4} \rightarrow \boxed{y - 4 = \dfrac{-3}{4}(x-3)}$

EXAMPLE 2

(a) Find y' if $x^3 + y^3 = 6xy$.

(b) Find the tangent to the folium of Descartes $x^3 + y^3 = 6xy$ at the point (3, 3).

(c) At what point in the first quadrant is the tangent line horizontal?

(a)
$$x^3 + y^3 = 6xy$$

$$\Rightarrow \frac{d}{dx}(x^3+y^3) = \frac{d}{dx}(6xy)$$

$$\Rightarrow \frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = 6\left(\frac{d}{dx}(x)\, y + x\frac{d}{dx}(y)\right)$$

$y=f(x)$
$$\Rightarrow 3x^2 + 3y^2 \cdot \frac{d}{dx}(\underbrace{y}_{=f(x)}) = 6\left(y + x\frac{dy}{dx}\right)$$

$$\Rightarrow 3x^2 + 3y^2 \frac{dy}{dx} = 6y + 6x\frac{dy}{dx}$$

$$\Rightarrow 3x^2 - 6y = 6x\frac{dy}{dx} - 3y^2\frac{dy}{dx}$$

$$\Rightarrow 3x^2 - 6y = (6x - 3y^2)\frac{dy}{dx}$$

$$\Rightarrow \boxed{\frac{3x^2-6y}{6x-3y^2} = \frac{dy}{dx}}$$

(b) $\to m = \frac{3\cdot 9 - 6\cdot 3}{6\cdot 3 - 3\cdot 9} = \frac{27-18}{18-27} = \frac{9}{-9} = -1$

$x=3$
$y=3$

$$\to \boxed{y-3 = -(x-3)}.$$

(c) $\frac{dy}{dx} = 0 \to \frac{3x^2-6y}{6x-3y^2} = 0$

$$\to \frac{\boxed{x^2-2y}}{2x-y^2} = 0 \iff x^2-2y = 0$$
$$(3x - y^2 \neq 0)$$

So, $y = \frac{x^2}{2}$. Substitute $y = \frac{x^2}{2}$ in the original equation:

$$x^3 + (x^2/2)^3 = 6x\left(\frac{x^2}{2}\right)$$

$$\Rightarrow x^6 = 16x^3$$

$$\Rightarrow x^3 = \underbrace{16}_{2^3\cdot 2} \Rightarrow \boxed{x = 2\sqrt[3]{2}}.$$

EXAMPLE 3 Find y' if $\sin(x + y) = y^2 \cos x$.

$y = f(x)$

① $\frac{d}{dx}(\sin(x+y)) = \frac{d}{dx}(y^2 \cos x)$

② $\cos(x+y) \frac{d}{dx}(x+y) = \frac{d}{dx}(y^2) \cos x + y^2 \frac{d}{dx}(\cos x)$

$\to [\cos(x+y)] \cdot \left(1 + \frac{dy}{dx}\right) = 2y \cdot \frac{dy}{dx} \cos x + y^2(-\sin x)$

$\to \cos(x+y) + \left(\frac{dy}{dx}\right) \cos(x+y) = 2y \cos x \left(\frac{dy}{dx}\right) - (\sin x) y^2$

③ $\to \cos(x+y) + y^2 \sin x = 2y \cos x \left(\frac{dy}{dx}\right) - \cos(x+y)\left(\frac{dy}{dx}\right)$

$\to \cos(x+y) + y^2 \sin x = [2y \cos x - \cos(x+y)] \left(\frac{dy}{dx}\right)$

$\to \boxed{\frac{\cos(x+y) + y^2 \sin x}{2y \cos x - \cos(x+y)} = \frac{dy}{dx}}$