SECTION 1.6: Exponential Functions

Calculus:

$$x \in \mathbb{R} \longrightarrow e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Complex: replace x by $z \in \mathbb{C}$ and

$$\lim_{n\to\infty} \frac{\left|\frac{z^{n+1}}{(n+1)!}\right|}{\left|\frac{z^n}{n!}\right|} = \lim_{n\to\infty} \frac{|z|}{n+1} = 0$$

By the ratio test:

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad (*)$$

converges absolutely $\forall z \in \mathbb{C}$.

<u>DEF 1.6.1</u> The complex exponential function $\exp(z)$ or e^z is defined as the series $(*)$, that is

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad \forall z \in \mathbb{C}.$$

THM 1.6.2 Let $z, w \in \mathbb{C}$. Then

(1) $e^{z+w} = e^z e^w$.

(2) $e^z \neq 0$ and $e^{-z} = \frac{1}{e^z}$.

(3) $e^{z-w} = \frac{e^z}{e^w}$.

Proof. Assume $z, w \in \mathbb{C}$.

(1) LHS is well-defined.
RHS is also well-defined by the product of two abs. conv. series.

RHS:

$$e^z e^w = \left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right)\left(\sum_{m=0}^{\infty} \frac{w^m}{m!}\right) = \sum_{n=0}^{\infty} c_n$$

where

$$c_n = \sum_{j=0}^{n} \frac{z^j}{j!} \frac{w^{n-j}}{(n-j)!}$$

$$= \frac{1}{n!} \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} z^j w^{n-j}$$

$$= \frac{(z+w)^n}{n!}$$ [Binomial formula]

So,

$$e^z e^w = \sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} = e^{z+w}.$$

② Notice $e^0 = 1$

$$\Rightarrow e^{z-z} = 1$$

$$\Rightarrow e^z e^{-z} = 1.$$

Hence, $e^z \neq 0$ and

$$e^{-z} = 1/e^z.$$

③ We have

$$e^{z-w} = e^z \cdot e^{-w} = e^z/e^w. \quad \square$$

<u>Prop. 1.6.3</u> Let $z = i\theta$, $\theta \in \mathbb{R}$. Then

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Proof. Here,

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}$$

$$= 1 + i\theta + \frac{(-1)\theta^2}{2!} + \frac{(-1)i\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} + \cdots$$

$$= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots + i\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right]$$

$$= \cos\theta + i\sin\theta. \qquad \square$$

Corollary 1.6.4 Let $z = x+iy \in \mathbb{C}$. Then

$$e^z = e^{x+iy} = e^x\cos y + i e^x \sin y.$$

Proof. Write $e^z = e^x e^{iy}$

$$\Rightarrow e^z = e^x \cos y + i e^x \sin y. \qquad \square$$

Example 1.6.5 Compute e^z if

(a) $z = 2 + i\pi$ (b) $3 - i\frac{\pi}{3}$.

Solution.

(a) $$e^{2+i\pi} = e^2\cos(\pi) + i e^2 \sin(\pi)$$
$$= -e^2 + i e^2 (0) = -e^2$$

(b) $$e^{3-i\pi/3} = e^3\cos(-\pi/3) + i e^3 \sin\left(\frac{\pi}{3}\right)$$
$$= e^3\left(1/2\right) - i e^3 \left(\frac{\sqrt{3}}{2}\right)$$
$$= \frac{e^3}{2} - i\frac{\sqrt{3}\, e^3}{2}.$$

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Remark.

① Notice that $x \in \mathbb{R} \Rightarrow e^x > 0$.
But if $z \in \mathbb{C}$, then $e^z \in \mathbb{C}$.

② $$e^{1+i\pi/2} = i e^1$$
$$e^{1+i\pi/2+i2\pi} = e^1\cos(\pi/2 + 2\pi) + e^1 \sin(\pi/2 + 2\pi).$$
$$= e^1\cos(\pi/2) + e^1 \sin(\pi/2)$$
$$= i e^1$$

In general, $\forall z \in \mathbb{C}$, $\forall k \in \mathbb{Z}$.

$$e^{z+i2k\pi} = e^{z}.$$

We say that e^z is $2\pi i$-periodic.

Example 1.6.6 Compute $|e^z|$ and $\text{Arg}(e^z)$ for

(a) $z = 1+i$

Solution. Write $e^z = e^x(\cos y + i\sin y)$ when $z = x+iy$. So

$$|e^z| = e^x = e^1$$

and

$$\text{Arg}(e^z) = 1$$ (because y is between $-\pi$ & π).

Exponential form

If $\theta \in \mathbb{R}$, then

$$\cos\theta + i\sin\theta = e^{i\theta}.$$

therefore, if $r = \sqrt{x^2+y^2}$ and θ is an argument of $z = x+iy$, then

$$z = re^{i\theta}.$$

So, if $z = r_1 e^{i\theta_1}$ and $w = r_2 e^{i\theta_2}$

then

① $z = w \iff r_1 = r_2,\ \theta_1 = \theta_2 + 2k\pi$, $k \in \mathbb{Z}$

② $zw = r_1 r_2 e^{i(\theta_1+\theta_2)}$

All the other arithmetic rules can be computed using the exponential form and the arithmetic rules with the complex exponential fct.

<u>Example 1.6.10</u> Solve $e^z = 1+i$.

<u>Solution.</u> Write $1+i = \sqrt{2}\, e^{i\pi/4}$

and $z = x + iy$

$$\Rightarrow \quad e^x e^{iy} = \sqrt{2}\, e^{i\pi/4}$$

$$\Leftrightarrow \quad e^x = \sqrt{2} \quad \text{and} \quad y = \frac{\pi}{4} + 2k\pi$$

for $k \in \mathbb{Z}$

$\log_e = \log$

$$\Leftrightarrow \quad x = \log(\sqrt{2}) \quad \text{and}$$

$$y = \frac{\pi}{4} + 2k\pi \quad k \in \mathbb{Z}$$

$$\Leftrightarrow \quad z = \log(\sqrt{2}) + i\left(\frac{\pi}{4} + 2k\pi\right), \; k \in \mathbb{Z}. \quad \triangle$$

<u>e^z as a mapping</u> $\left(f(z) = e^z = u + iv\right)$

<u>$x = x_0$</u> then $u = e^{x_0} \cos y, \; v = e^{x_0} \sin y$

$u^2 + v^2 = \left(e^{x_0}\right)^2 \longrightarrow$ circle !

<u>$y = y_0$</u> then $u = e^x \cos y_0, \; v = e^x \sin y_0$

$$v = \left(\sin y_0 / \cos y_0\right) u = \tan(y_0)\, u \; .$$

Desmos: https://www.desmos.com/calculator/exmuixbsdo