Pierre-Olivier Parisé
Spring 2022

Section 16.2, Problem 18

The expression in the line integral of a vector field $\vec{F}$ is $\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)$. This is also the following expression

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = \|\vec{F}\|\|\vec{r}'\| \cos(\theta(t))$$

where $\theta(t)$ is the angle between the two vectors. If $\theta(t)$ is always between 0 and $\pi/2$, then the dot product is always positive because the cosine function is positive there and if $\theta(t)$ is always between $\pi/2$ and π, then the dot product is always negative because the cosine function is negative there.

- C_1. The angle between the tangent vector $\vec{r}'(t)$ (which is always perpendicular to the radius of the circle) and the vector field is always between 0 and $\pi/2$. So the dot product is always positive and so the integral is positive.
- C_2. Along most of the path, the angle between the tangent vector $\vec{r}'(t)$ (which is again perpendicular to the radius of the circle) and the vector field is between $\pi/2$ and π. The length of the vectors in the vector field are also bigger in this area of the curve. At the beginning of the curve C_2, we can see that the angle is less than $\pi/2$, but the modulus of the vectors in the vector field are small in this area. So this makes the dot product also small. So the all the weight goes on the part where the angles are between $\pi/2$ and π and, thus, the line integral is negative.

$\boxed{\textbf{Section 16.2, Problem 22}}$

We have $\vec{r}\,'(t) = (-\sin t)\vec{i} + \cos(t)\vec{j} + \vec{k}$. So,

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,' = -\cos t \sin t + \cos t \sin t + \cos t \sin t = \cos t \sin t.$$

Thus, we obtain

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^\pi \cos t \sin t \, dt = \int_0^\pi (1/2) \sin(2t) \, dt = 0.$$

$\boxed{\textbf{Section 16.2, Problem 41}}$

The line segment is parametrize by

$$\vec{r}(t) = \langle 2t, t, 1-t \rangle \quad (0 \leq t \leq 1).$$

The work done by the vector field over the line segment is given by the line integral of the vector field over the line segment. We have $\vec{r}\,'(t) = \langle 2, 1, -1 \rangle$ and

$$\vec{F}(\vec{r}(t)) = \left\langle 2t - t^2, t - (1-t)^2, 1 - t - 4t^2 \right\rangle.$$

So, the expression of the line integral is

$$\begin{aligned} W &= \int_0^1 \left\langle 2t - t^2, t - (1-t)^2, 1 - t - 4t^2 \right\rangle \cdot \langle 2, 1, -1 \rangle \, dt \\ &= \int_0^1 4t - 2t^2 + t - (1-t)^2 - 1 + t + 4t^2 \, dt \\ &= 7/3 \end{aligned}$$

Section 16.3, Problem 4

We have $P(x,y) = y^2 - 2x$ and $Q(x,y) = 2xy$. Then, we see that

$$P_y = 2y \quad \text{and} \quad Q_x = 2y.$$

So $P_y = Q_x$ and the vector field $\vec{F}$ is conservative.

So, there is a function f such that $\vec{F} = na\vec{b}laf = f_x\vec{i} + f_y\vec{j}$. We must have

$$f_x = y^2 - 2x \quad \text{and} \quad f_y = 2xy.$$

Integrating with respect to x, we find out that $f(x,y) = y^2x - x^2 + g(y)$. Then, differentiating with respect to y, we find $f_y(x,y) = 2xy + g'(y) = 2xy$ and so $g'(y) = 0$. This implies that $g(y) = C$ for a constant C. Thus,

$$f(x,y) = xy^2 - x^2 + C.$$

Section 16.3, Problem 20

If $\vec{r}(t) = (x(t), y(t))$, with $a \leq t \leq b$ is a parametrization of the path C, then the line integral can be expressed as

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt.$$

where $\vec{F}(x, y) = \langle \sin y, x\cos y - \sin y \rangle$.

Now, it is sufficient to show that $\vec{F}$ is conservative. We have $P(x, y) = \sin y$ and $Q(x, y) = x\cos y - \sin y$, so

$$P_y = \cos y \quad \text{ and } \quad Q_x = \cos y.$$

We see that $P_y = Q_x$ and so $\vec{F}$ is conservative.

We now need to find a potential function f. By repeating the procedure from the previous exercise, we obtain

$$f(x, y) = x\sin y + \cos(y).$$

So by the fundamental Theorem for line integrals, we get

$$\int_C \sin y dx + (x\cos y - \sin y)\, dy = f(1, \pi) - f(2, 0) = -1 + 1 = 0.$$