MATH 244

CHAPTER 16

SECTION 16.8: STOKES' THEOREM

CONTENTS

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DEFINITION 1. If $\vec{F} = \langle P, Q, R \rangle$ is a vector field in 3D, then

$$\text{curl } \vec{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle.$$

Another way to write curl $\vec{F}$ is as followed. Define

$$\vec{\nabla} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \implies \text{curl } \vec{F} = \vec{\nabla} \times \vec{F}.$$

EXAMPLE 1. Find the curl of $\vec{F} = \langle xz, xyz, -y^2 \rangle$.

SOLUTION.

$$\text{curl } \vec{F} = \vec{\nabla} \times \vec{F}$$

$$= \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xz & xyz & -y^2 \end{vmatrix}$$

$$= \left\langle \frac{\partial}{\partial y}(-y^2) - \frac{\partial}{\partial z}(xyz), -\left(\frac{\partial}{\partial x}(-y^2) - \frac{\partial}{\partial z}(xz)\right), \frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial y}(xz) \right\rangle$$

$$= \langle -2y - xy, x, yz \rangle$$

THEOREM 1. Let $\vec{F} = \langle P, Q, R \rangle$. If

- P, Q, R have continuous partial derivatives.
- $\text{curl}\,\vec{F} = \vec{0}$.

Then $\vec{F}$ is conservative.

EXAMPLE 2. Let $\vec{F}(x, y, z) = \langle y^2 z^3, 2xyz^3, 3xy^2 z^2 \rangle$.

a) Show that $\vec{F}$ is conservative.

b) Find a function f such that $\vec{F} = \vec{\nabla} f$.

SOLUTION.

a) $\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 z^3 & 2xyz^3 & 3xy^2 z^2 \end{vmatrix}$

$= \langle 6xyz^2 - 6xyz^2, -3y^2 z^2 + 3y^2 z^2,$

$\qquad 2yz^3 - 2yz^3 \rangle$

$= \langle 0, 0, 0 \rangle = \vec{0}.$

$\Rightarrow \vec{F}$ is conservative.

b) $\vec{\nabla} f = \vec{F} \Rightarrow \begin{cases} f_x = y^2 z^3 \\ f_y = 2xyz^3 \\ f_z = 3xy^2 z^2 \end{cases}$

Integrate all the equations:

$$\begin{cases} f(x,y,z) = xy^2z^3 \\ f(x,y,z) = xy^2z^3 \\ f(x,y,z) = xy^2z^3 \end{cases}$$

Final expression:

$$f(x,y,z) = xy^2z^3 + c$$

Recall Green's Theorem:

$$\int_C \vec{F} \cdot d\vec{r} = \iint_D Q_x - P_y \, dA,$$

where C is orientated positively. Writing $\vec{F} = \langle P, Q, 0 \rangle$:

$$Q_x - P_y = \langle 0, 0, Q_x - P_y \rangle \cdot \langle 0, 0, 1 \rangle = \text{curl} \, \vec{F} \cdot \vec{k}$$

so that

$$\int_C \vec{F} \cdot d\vec{r} = \iint_D \vec{F} \cdot \vec{k} \, dA.$$

A particular case of Stokes' Theorem.

THEOREM 2. Assume

- S be an oriented surface bounded by a loop C with orientation induced by the surface.

- $\vec{F} = \langle P, Q, R \rangle$ with P, Q, R having continuous partial derivatives.

Then,

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl} \, \vec{F} \cdot d\vec{S}.$$

Rule of Thumb: What we mean by the orientation induced by the surface is: we apply the right-hand rule with the thumb pointing in the direction of the normal vector.

EXAMPLE 3. Let $\vec{F}(x, y, z) = \langle x + y^2, y + z^2, z + x^2 \rangle$ and C is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. Compute

$$\int_C \vec{F} \cdot d\vec{r}.$$

SOLUTION.

$\textsc{Example 4.}$ Let $\vec{F}(x, y, z) = \langle ze^y, x \cos y, xz \sin y \rangle$ and S be the hemisphere $x^2 + y^2 + z^2 = 16$, $y \geq 0$ oriented in the direction of the positive y-axis. Compute

$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}.$$

$\textsc{Solution.}$