MATH-244 Calculus IV
Practice Final

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Spring 2022

Last name: Solutions
First name: —

Question:	1	2	3	4	5	Total
Points:	20	20	20	20	20	100
Score:	—	—	—	—	—	—

Instructions: Make sure to write your complete name on your copy. You must answer all the questions below and write your answers directly on the questionnaire. At the end of the 75 minutes, hand out your copy.

No devices such as a smart phone, cell phone, laptop, or tablet can be used during the exam. You are not allowed to use the lecture notes, the textbook. You may bring one 2-sided cheat sheet of handwriting notes. You may use a digital calculator (no graphical calculator or symbolic calculator will be allowed).

You must show ALL your work to have full credit. An answer without justification worth no point.

Good luck!

Pierre-Olivier Parisé

Your Signature:

University of Hawai'i

Question 1 **(20 pts)**

Consider the vector field

$$\vec{F}(x,y) = \left\langle e^{-x} + y^2, e^{-y} + x^2 \right\rangle.$$

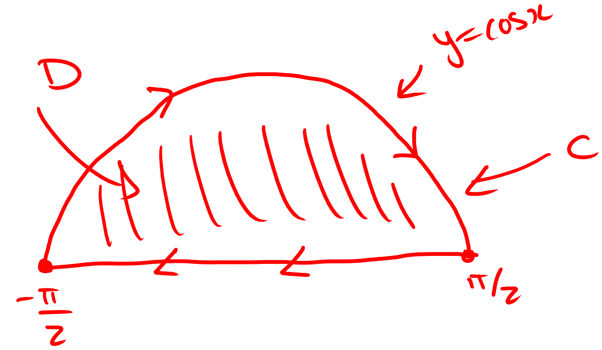
(a) (5 points) Compute the derivatives $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$.

(b) (15 points) If C consists of the arc of the curve $y = \cos x$ from $(-\pi/2, 0)$ to $(\pi/2, 0)$ and the line segment from $(\pi/2, 0)$ to $(-\pi/2, 0)$, then compute[1]

$$\int_C \vec{F} \cdot d\vec{r}.$$

(a) $\frac{\partial P}{\partial y} = \boxed{2y}$ & $\frac{\partial Q}{\partial x} = \boxed{2x}$

(b) Picture



Green's Thm:

$$\int_C \vec{F} \cdot d\vec{r} = \iint_D (Q_x - P_x)\, dA$$

Type I for D $\longrightarrow$ $D = \{(x,y) : -\frac{\pi}{2} \le x \le \frac{\pi}{2},\ 0 \le y \le \cos x\}$

so, $$\iint_D (Q_x - P_y)\, dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\cos x} 2x - 2y \, dy\, dx$$

$$= \int_{-\pi/2}^{\pi/2} 2x\cos x - \cos^2 x \, dx = -\frac{\pi}{2}$$

so, $\boxed{\int_C \vec{F} \cdot d\vec{r} = -\frac{\pi}{2}}$.

[1] You may take for granted that $\int_{-\pi/2}^{\pi/2} 2x\cos x\, dx = 0$ and $\int_{-\pi/2}^{\pi/2} \cos^2 x\, dx = \pi/2$.

Question 2 (20 pts)

Consider the following vector fields:

$$\vec{F}_1(x,y,z) = \left\langle y^2 z, x\cos z, xy^2 \right\rangle, \qquad \vec{F}_2(x,y,z) = \left\langle y^2 z^3, 2xyz^3, 3xy^2 z^2 \right\rangle.$$

(a) (10 points) Which one is conservative? (Explain your answer with a calculation.)

(b) (10 points) Which one is incompressible? (Explain you answer with a calculation.)

(a) Conservative if $\vec{\text{rot}}\ \vec{F} = \vec{0}$.

$$\text{rot}\ \vec{F}_1 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 z & x\cos z & xy^2 \end{vmatrix} = \langle 2xy + x\sin z, -y^2 + y^2, \cos z - 2yz \rangle$$

$$= \langle 2xy + x\sin z, 0, \cos z - 2yz \rangle \neq \vec{0} \quad \times$$

$$\text{rot}\ \vec{F}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 z^3 & 2xyz^3 & 3xy^2 z^2 \end{vmatrix} = \langle 6xyz^2 - 6xyz^2, -3y^2z^2 + 3y^2z^2, 2yz^3 - 2yz^3 \rangle$$

$$= \langle 0, 0, 0 \rangle \quad \checkmark$$

So $\vec{F}_2$ is conservative.

(b) Incompressible if $\text{div}\ \vec{F} = 0$.

$\text{div}\ \vec{F}_1 = 0 + 0 + 0 = 0 \quad \checkmark$

$\text{div}\ \vec{F}_2 = 0 + 2xz^3 + 6xy^2 z = 2xz^3 + 6xy^2 \neq 0 \quad \times$

So $\vec{F}_1$ is incompressible.

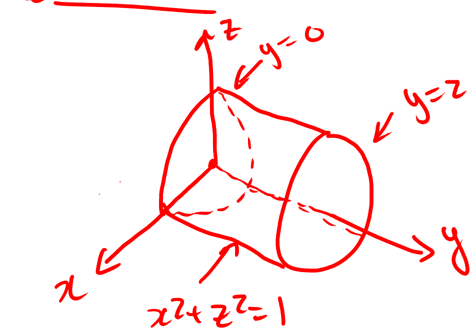
Question 3 **(20 pts)**

Compute the surface integral

$$\iint_S (x + y + z)\, dS$$

where S is the boundary of the solid enclosed by the cylinder $x^2 + z^2 = 1$, $z \geq 0$ and between the planes $y = 0$ and $y = 2$.

① Picture.



Let $x = \cos\theta$ & $z = \sin\theta$

$\Rightarrow \quad \vec{r}(\theta, y) = \langle \cos\theta, \sin\theta, y\rangle$

$0 \leq \theta \leq 2\pi$ & $0 \leq y \leq 2$.

② Integrate.

$$\iint_S (x+y+z)\, dS = \iint_D (\cos\theta + \sin\theta + y)\, |\vec{r}_\theta \times \vec{r}_y|\, dA$$

$$\vec{r}_\theta \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle \cos\theta, \sin\theta, 0\rangle$$

$$\Rightarrow \quad |\vec{r}_\theta \times \vec{r}_y| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

$$\Rightarrow \iint_S (x+y+z)\, dS = \int_0^{2\pi}\int_0^2 \cos\theta + \sin\theta + y \; dy\, d\theta$$

$$= 2\pi \int_0^2 y\, dy = \boxed{4\pi}$$

Question 4 (20 pts)

Consider the force field

$$\vec{F}(x,y,z) = \left\langle z^2, 2xy, 4y^2 \right\rangle .$$

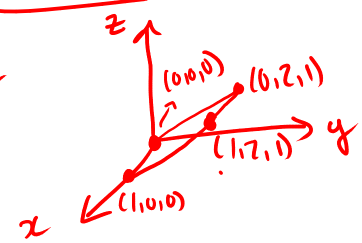
(a) (10 points) Compute rot $\vec{F}$.

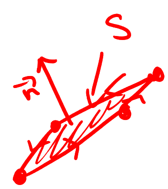
(b) (10 points) Find the work of a particle moving along C in the vector field $\vec{F}$ if C is the polygone with vertices $(0,0,0)$, $(1,0,0)$, $(1,2,1)$, and $(0,2,1)$.

(a) $\operatorname{rot}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z^2 & 2xy & 4y^2 \end{vmatrix} = \langle 8y, 2z, 2y \rangle$

(b) ① Picture

C





Plane

② Stokes' theorem.

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{rot}\vec{F} \cdot d\vec{S} = \iint_D \operatorname{rot}\vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\, dA$$

Equation of S is $\vec{r}(u,v) = \langle 0,0,0\rangle + u\langle 0,2,1\rangle + v\langle 1,0,0\rangle$
$= \langle v, 2u, u\rangle$.

with $0 \le u, v \le 1$.

So, $\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 2 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \langle 0, +1, -2 \rangle$

$$\Rightarrow \iint_D \operatorname{rot}\vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\, dA = \int_0^1\int_0^1 \langle u^2, 4uv, 16u^2\rangle \cdot \langle 0,1,2\rangle\, du\,dv$$
$$= \int_0^1\int_0^1 4uv + 32u^2\, du\,dv$$
$$= \boxed{35/3}$$

Question 5 **(20 pts)**

Consider the vector field

$$\vec{F}(x,y,z) = \left\langle z^2 x, \tfrac{1}{3}y^3 + \tan z, x^2 z + y^2 \right\rangle.$$

Let S be the top half of the sphere $x^2+y^2+z^2=1$.

(a) (10 points) Let S_1 be the disk $x^2+y^2 \le 1$, $z=0$ with the downward orientation. Compute[2]

$$\iint_{S_1} \vec{F}\cdot d\vec{S}.$$

(b) (10 points) Using the Divergence Theorem and (a), deduce the value[3] of

$$\iint_{S} \vec{F}\cdot d\vec{S}.$$

(a) Parametrize S_1 as $\vec{r}(\rho,\theta) = \langle \rho\cos\theta, \rho\sin\theta, 0\rangle$ with $0 \le \rho \le 1$ & $0 \le \theta \le 2\pi$. Choose your normal vector $\vec{n} = \langle 0,0,-1\rangle$ (downward orientatation).

So, $\iint_{S_1} \vec{F}\cdot d\vec{S} = \iint_{S_1} \vec{F}\cdot\vec{n}\, dS = -\iint_{S_1} x^2 z + y^2\, dS$

$$\vec{r}_\rho \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 0 \\ -\rho\sin\theta & \rho\cos\theta & 0 \end{vmatrix} = \langle 0, 0, \rho\rangle$$

$$\Rightarrow \quad dS = |\vec{r}_\rho \times \vec{r}_\theta|\, d\rho\, d\theta = \rho\, d\rho\, d\theta$$

Thus,

$$-\iint_{S_1} x^2 z + y^2\, dS = -\int_0^{2\pi}\int_0^1 (\rho^2\cos^2\theta\cdot 0 + \rho^2\sin^2\theta)\rho\, d\rho\, d\theta$$

$$= -\int_0^{2\pi}\int_0^1 \rho^3 \sin^2\theta\, d\rho\, d\theta$$

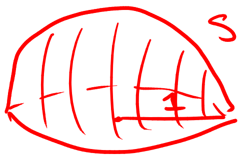
$$= -\left(\int_0^{2\pi}\sin^2\theta\, d\theta\right)\left(\int_0^1 \rho^3\, d\rho\right)$$

$$= \boxed{-\pi/4}$$

[2] You may take for granted that $\int_0^{2\pi}\sin^2\theta\, d\theta = \pi$.

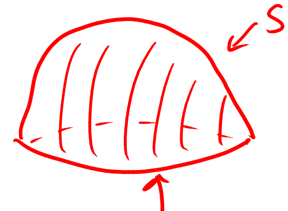
[3] You may take for granted that $\int_0^{\pi/2}\sin^2\phi\, d\phi = \pi/4$.

(b) 1 **Picture**



$x^2+y^2+z^2=1$
$z \geq 0$

close it up!
$\longrightarrow$



S_1: cap $x^2+y^2=1$
$z=0$

Let E be the solid enclosed by S_1 & S.

Thus, $E = \{(x,y,z): \quad x^2+y^2+z^2 \leq 1 \;\&\; z \geq 0\}$.

↳ boundary is $S_1 \cup S$

2 **Divergence thm.**

$$\iint_{S \cup S_1} \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} \, dV$$

$$\operatorname{div} \vec{F} = z^2 + y^2 + x^2 .$$

$$\Rightarrow \iiint_E \operatorname{div} \vec{F} \, dV = \int_0^1 \int_0^{2\pi} \int_0^{\pi/2} (\rho^2) \rho \sin^2\phi \, d\phi \, d\theta \, d\rho$$

$$= \left(\int_0^1 \rho^3 \, d\rho \right) 2\pi \left(\int_0^{\pi/2} \sin^2\phi \, d\phi \right)$$

$$= \boxed{\frac{\pi^2}{8}}$$

3 Find $\iint_S \vec{F} \cdot d\vec{r}$.

$$\iint_{S \cup S_1} \vec{F} \cdot d\vec{r} = \iiint_E \operatorname{div} \vec{F} \, dV = \frac{\pi^2}{4}$$

$$\Rightarrow \iint_S \vec{F} \cdot d\vec{r} + \iint_{S_1} \vec{F} \cdot d\vec{r} = \frac{\pi^2}{4} \Rightarrow \iint_S \vec{F} \cdot d\vec{S} = \boxed{\frac{\pi^2}{8} + \frac{\pi}{4}}$$