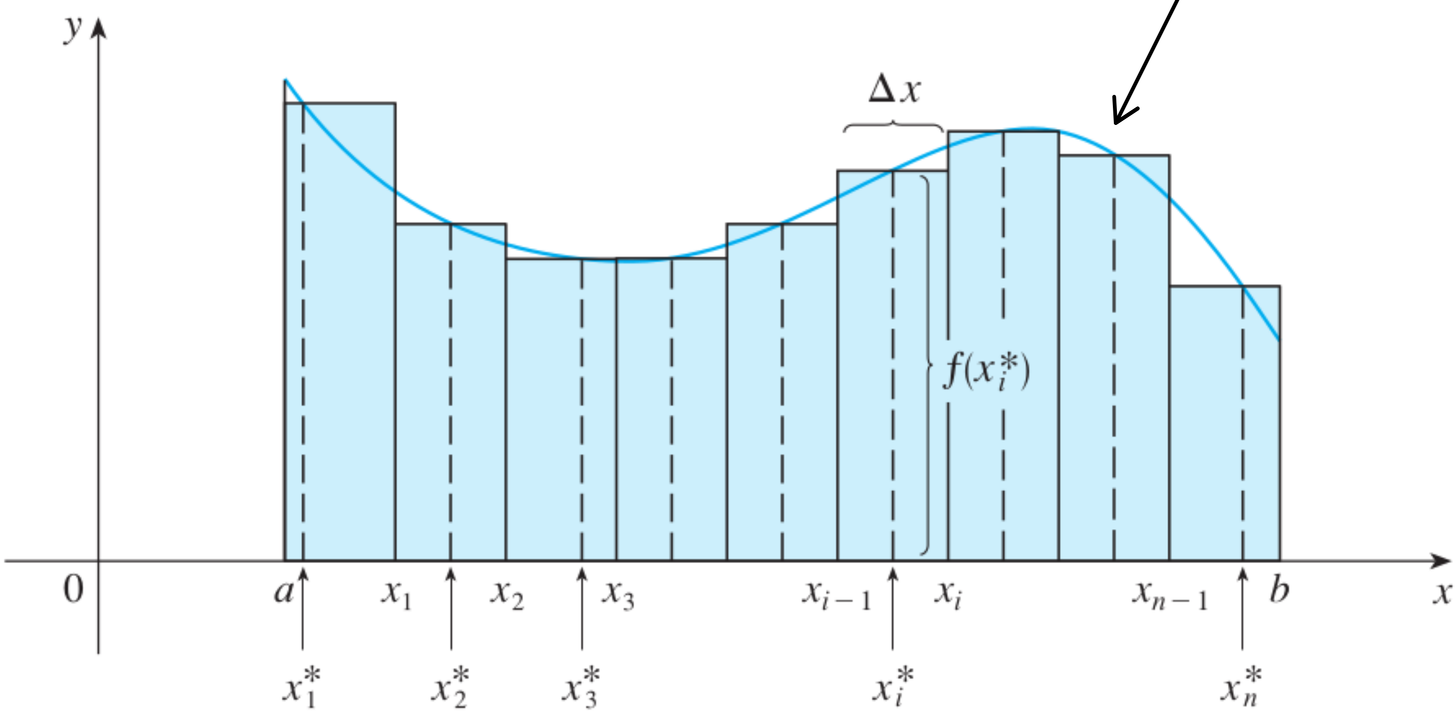
Chapter 15

Multiple Integrals

15.1 Double Integrals over a rectangle

Definite integrals over an interval.

Function $f(x)$



1) Divide the interval in n parts of equal length Δx

2) Name each subinterval $[a, x_1]$, $[x_1, x_2]$, $\dots$, $[x_{n-1}, b]$

3) Choose some point x_1^* in $[a, x_1]$, x_2^* in $[x_1, x_2]$, $\dots$, x_n^* in $[x_{n-1}, b]$

$\Rightarrow$ Total Area of rectangles $= f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$

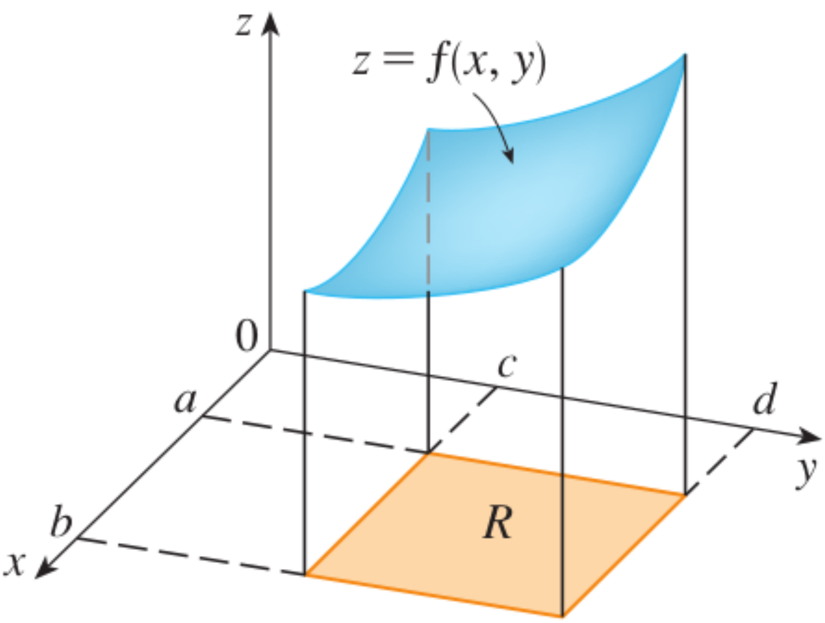
4) Take the limit as $n \to \infty$

$$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x}$$

Useful Fact:

$$\int_a^b f(x)\,dx \cong \sum_{i=1}^{n} f(x_i^*)\,\Delta x \,.$$

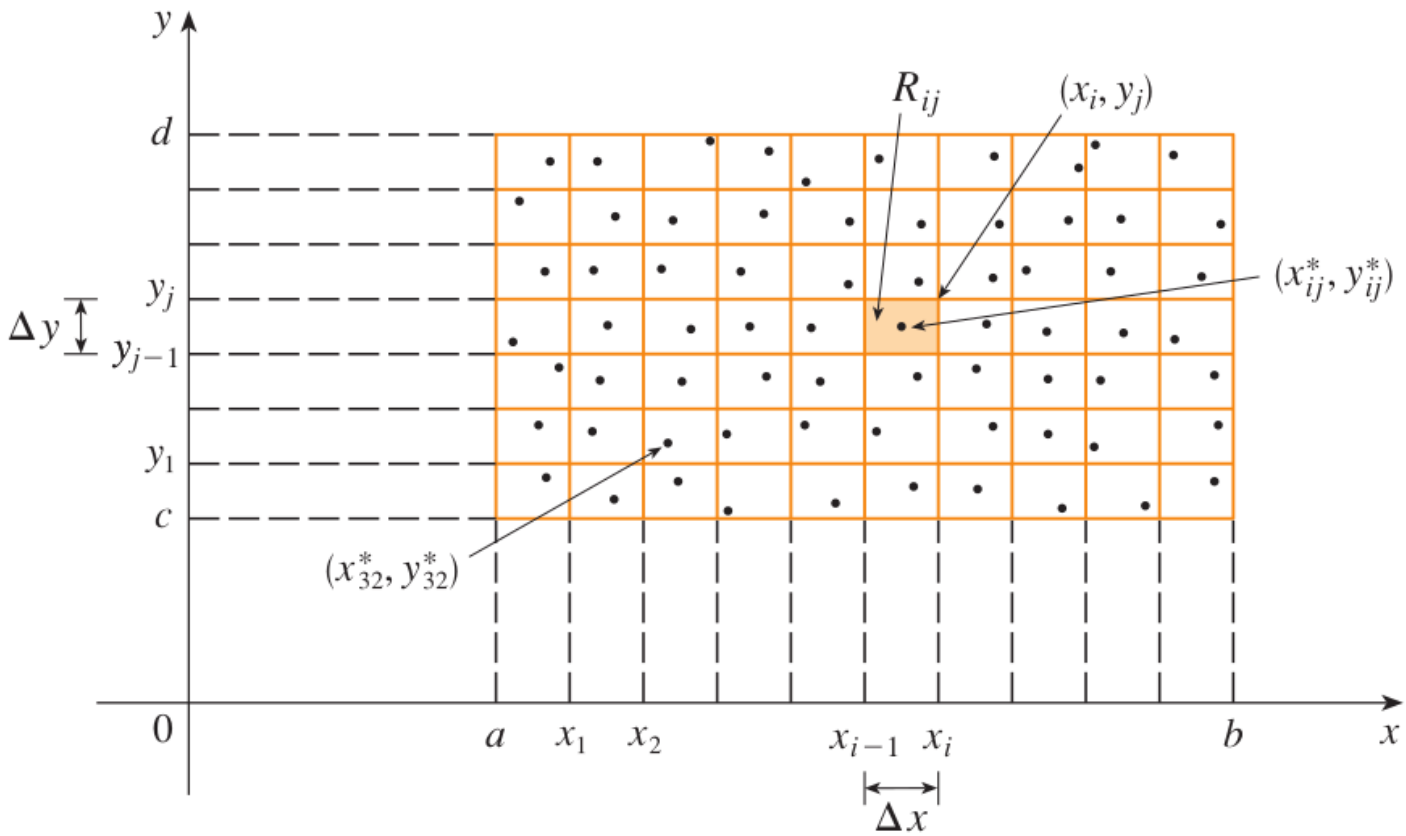
Volumes and Double Integrals.



Given:

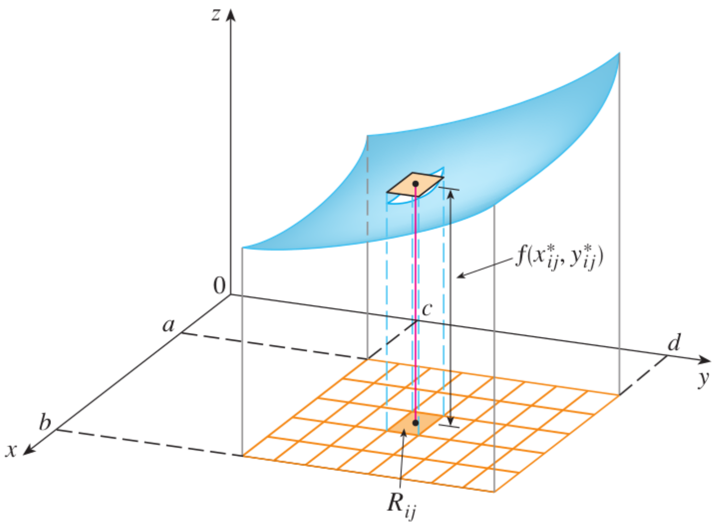
- A function $z = f(x, y)$
- The domain $R = [a, b] \times [c, d]$

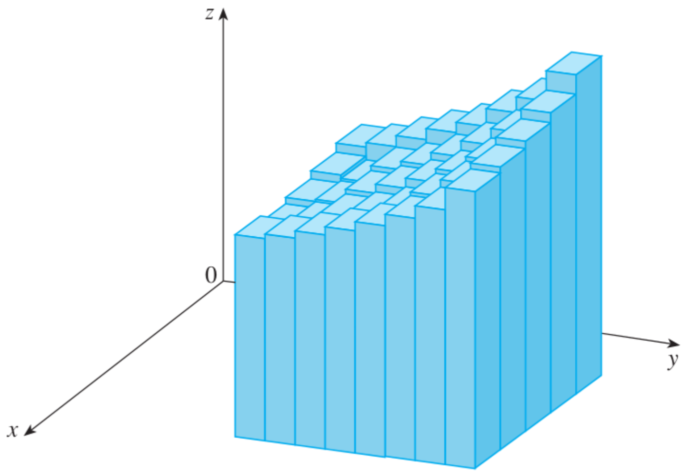
1st Step: Divide the domain to create a grid.



1) Divide [a, b] in m equal parts Δx
2) Divide [c, d] in n equal parts Δy
3) Create the rectangle $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$
4) Select a point (x_{ij}^*, y_{ij}^*) in R_{ij}

2nd Step: Approximate the volume by "buildings"





1) Volume of a building: $\Delta A \cdot f(x_{ij}^*, y_{ij}^*)$

2) Total volume:

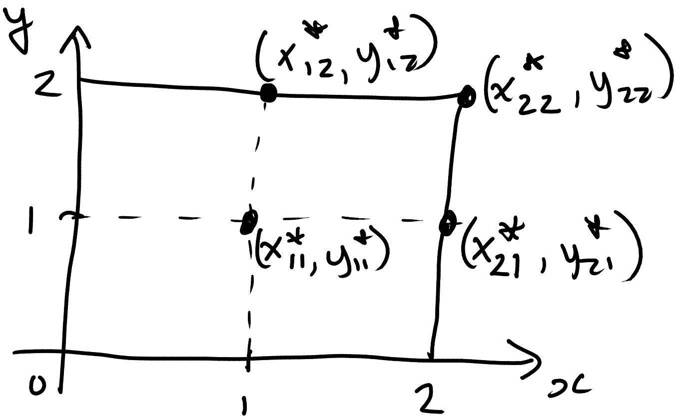
$$V = \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \cdot \Delta A$$

3) Take the limit as $m, n \to \infty$:

$$\boxed{\iint_R f(x,y)\, dA = \lim_{n,m\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A}$$

Useful Fact: $\iint_R f(x,y)\, dA \cong \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A$

EXAMPLE 1 Estimate the volume of the solid that lies above the square $R = [0, 2] \times [0, 2]$ and below the elliptic paraboloid $z = 16 - x^2 - y^2$. Divide R into four equal squares and choose the sample point to be the upper right corner of each square R_{ij}. Sketch the solid and the approximating rectangular boxes.



$(x_{11}^*, y_{11}^*) = (1, 1)$ $(x_{21}^*, y_{21}^*) = (2, 1)$

$(x_{12}^*, y_{12}^*) = (1, 2)$ $(x_{22}^*, y_{22}^*) = (2, 2)$

$f(x, y) = 16 - x^2 - y^2$.

$\Delta A = 1$

$$\iint_R 16 - x^2 - y^2 \, dA \cong \sum_{i=1}^{2} \sum_{j=1}^{2} \left[16 - (x_{ij}^*)^2 - (y_{ij}^*)^2\right] \Delta A$$

$$= (16 - 1^2 - 1^2) \cdot 1 + (16 - 1^2 - 2^2) \cdot 1$$

$$+ (16 - 2^2 - 1^2) \cdot 1 + (16 - 2^2 - 2^2) \cdot 1$$

$$= \boxed{44}$$

Evaluating Integrals with cartesian coordinates: Iterated Integrals

Fubini's Theorem:

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)dxdy = \int_a^b \int_c^d f(x,y)\,dydx$$

EXAMPLE 4 Evaluate the iterated integrals.

(a) $\int_0^3 \int_1^2 x^2 y \, dy \, dx$ (b) $\int_1^2 \int_0^3 x^2 y \, dx \, dy$

Example. Evaluate the following integral:

$$\int_0^1 \int_0^1 v(u^2 + v^2)^4 \, du dv$$

EXAMPLE 7 Find the volume of the solid S that is bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$, and the three coordinate planes.

EXAMPLE 8 If $R = [0, \pi/2] \times [0, \pi/2]$, then compute $\displaystyle\iint_R \sin x \cos y \, dA$.

Useful Fact:

$$\boxed{\iint_R g(x)h(y)\,dA = \left(\int_a^b g(x)\,dx\right)\left(\int_c^d h(y)\,dy\right)}$$

where $R = [a, b] \times [c, d]$