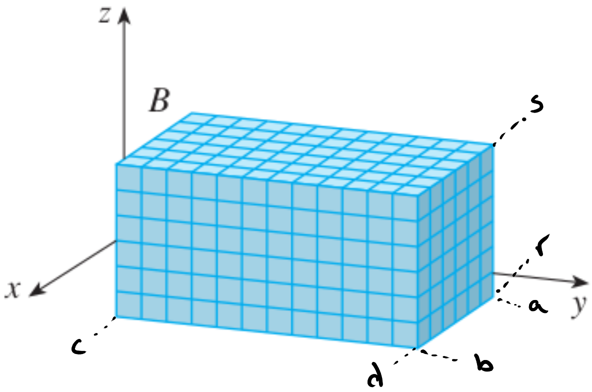
15.6 Triple Integrals.

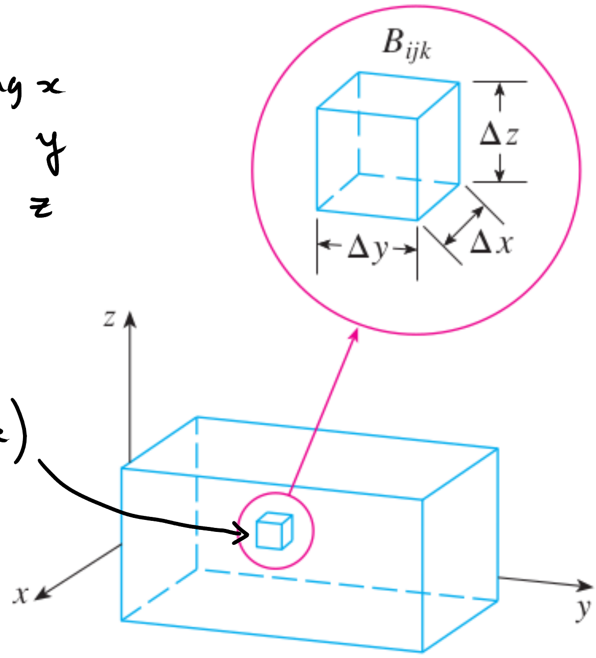
$B = \{(x,y,z) : a \le x \le b,\ c \le y \le d,\ r \le z \le s\}$



function f defined on B.

Separate B into $l \cdot m \cdot n$ boxes
where
- l is number of divisions along x
- m " " " " " y
- n " " " " " z

the width is Δx, Δy & Δz.



Take sample points $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$

We create the Riemann sum

$$\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\ \underbrace{\Delta x\, \Delta y\, \Delta z}_{V(B_{ijk}) \,=\, \Delta V}$$

$$\rightarrow \iiint_B f(x,y,z)\,dV \approx \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\ \Delta V$$

$l \to \infty$
$m \to \infty$
$n \to \infty$

---

3 Definition The **triple integral** of f over the box B is

$$\iiint_B f(x, y, z)\, dV = \lim_{l, m, n \to \infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\, \Delta V$$

if this limit exists.

---

4 Fubini's Theorem for Triple Integrals If f is continuous on the rectangular box $B = \underset{x}{[a, b]} \times \underset{y}{[c, d]} \times \underset{z}{[r, s]}$, then

$$\iiint_B f(x, y, z)\, dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz$$

$\int_r^s\int_a^b\int_c^d f\ dy\,dx\,dz$ ①
$dx\,dz\,dy$ ②
$dz\,dx\,dy$ ③
$dy\,dz\,dx$ ④
$dz\,dy\,dx$ ⑤

EXAMPLE 1 Evaluate the triple integral $\underbrace{\iiint_B xyz^2\,dV}_{=I}$, where B is the rectangular box given by

$$B = \{(x, y, z) \mid 0 \leqslant x \leqslant 1,\ -1 \leqslant y \leqslant 2,\ 0 \leqslant z \leqslant 3\}$$

$a = 0$ $\quad$ $c = -1$ $\quad$ $r = 0$

$b = 1$ $\quad$ $d = 2$ $\quad$ $s = 3$

$$I = \int_0^3 \int_{-1}^2 \int_0^1 xyz^2 \, dx\, dy\, dz$$

$$= \left(\int_0^1 x\, dx\right)\left(\int_{-1}^2 y\, dy\right)\left(\int_0^3 z^2\, dz\right)$$

$$= \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)(9) = \boxed{\frac{27}{4}}$$

$$= \int_0^3 \int_{-1}^2 \int_0^1 xyz^2 \, dx\, dy\, dz$$

$$= \int_0^3 \int_{-1}^2 \left.\frac{x^2}{2}\right|_0^1 yz^2 \, dy\, dz$$

$$= \int_0^3 \int_{-1}^2 \frac{1}{2} yz^2 \, dy\, dz$$

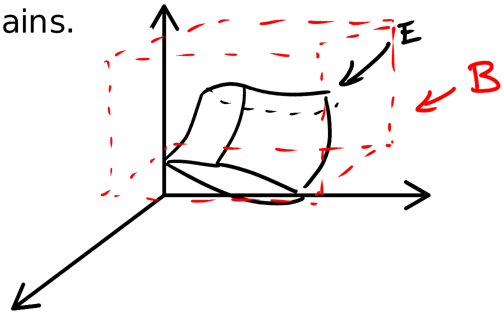
$$= \int_0^3 \left.\frac{y^2}{4}\right|_{-1}^2 z^2 \, dz$$

$$= \int_0^3 \frac{3}{4} z^2 \, dz$$

$$= \frac{3}{4} \left.\frac{z^3}{3}\right|_0^3 = \boxed{\frac{27}{4}}$$

General Domains.

f on E

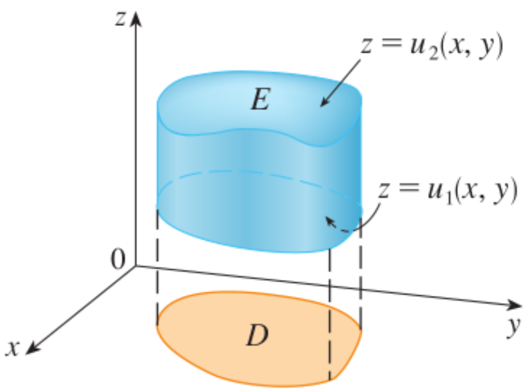


Define

$$F(x,y,z) := \begin{cases} f(x,y,z), & (x,y,z) \in E \\ 0, & (x,y,z) \notin E \end{cases}$$

$$\boxed{\iiint_E f(x, y, z)\, dV = \iiint_B F(x, y, z)\, dV}$$

Domain of type 1.



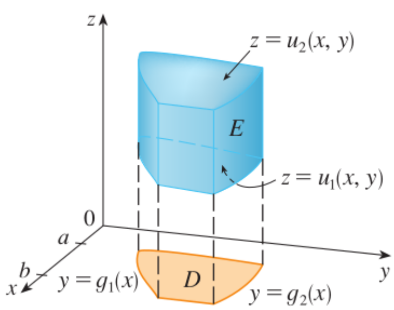
E is bounded in z by $u_2(x,y)$ and $u_1(x,y)$.

$$E := \{(x,y,z) : (x,y) \in D,\ u_1(x,y) \le z \le u_2(x,y)\}$$

D: shadow of E in xy-plane (projection of E over xy-plane).

$$\boxed{\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x, y)=z}^{u_2(x, y)=z} f(x, y, z)\, dz \right] dA}$$

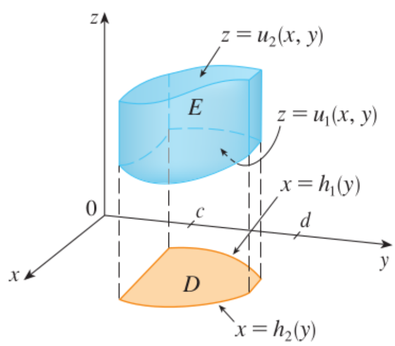
Projection D is of Type I.



$$D = \{(x,y) : a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

$$\boxed{\iiint_E f(x, y, z)\, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz\, dy\, dx}$$

Projection D is of Type II.



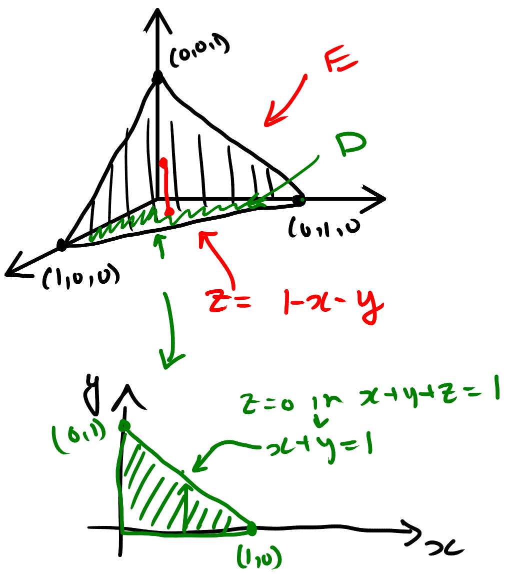
$$D = \{(x,y) : h_1(y) \le x \le h_2(y),\ c \le y \le d\}$$

$$\boxed{\iiint_E f(x, y, z)\, dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz\, dx\, dy}$$

EXAMPLE 2 Evaluate $\overset{=I}{\iiint_E z\, dV}$, where E is the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

yz-p. xz-p. xy-p.

① Picture.



$$E = \{(x,y,z) : (x,y) \in D,\ \underbrace{0}_{u_1} \le z \le \underbrace{1-x-y}_{u_2}\}$$

TYPE I.

$$D = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 1-x\}$$

② Integrate

$$I = \iiint_E z\, dV = \iint_D \left(\int_0^{\overset{z=}{1-x-y}} \overset{f(x,y,z)}{z}\, dz\right) dA$$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\, dz\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} \frac{z^2}{2}\Big|_0^{1-x-y} dy\, dx$$

$$= \int_0^1 \int_0^{1-x} \frac{(1-x-y)^2}{2}\, dy\, dx$$

$(1-x-y)(1-x-y)$
$= 1-x-y$
$\quad -x+x^2+xy$
$\quad -y+xy+y^2$
$= 1-2x-2y+2xy+x^2+y^2$

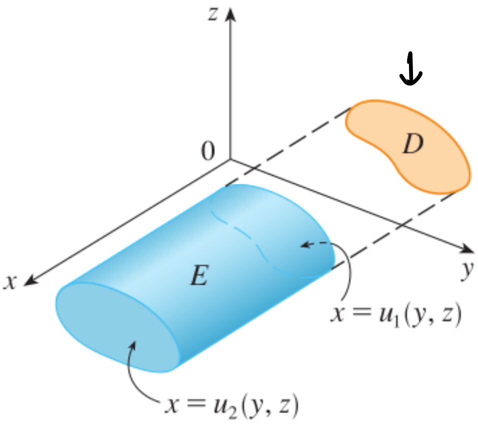
$$= \int_0^1 \int_0^{1-x} \frac{1-2x-2y+2xy+x^2+y^2}{2}\, dy\, dx$$

$$= \int_0^1 \frac{y-2xy-y^2+xy^2+x^2y}{2} + \frac{y^3}{6}\Big|_0^{1-x} dx$$

$$= \int_0^1 \frac{(1-x)-2x(1-x)-(1-x)^2+x(1-x)^2+x^2(1-x)}{2} + \frac{(1-x)^3}{6}\, dx$$

$$= \boxed{1/24}.$$

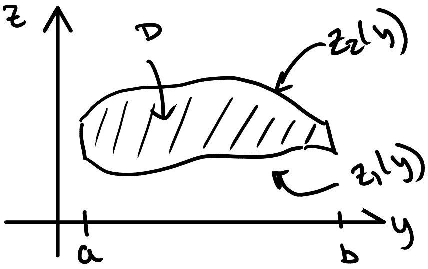
Domains of type 2.



$E = \{(x,y,z) : (y,z) \in D,\ u_1(y,z) \le x \le u_2(y,z)\}$

$$\boxed{\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx\right] dA}$$

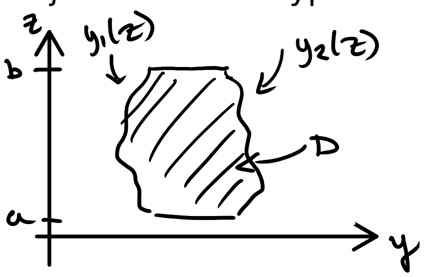
Projection D is of Type I.



$D = \{(y,z) :$
$a \le y \le b,$
$z_1(y) \le z \le z_2(y)\}$

$$\boxed{\iiint_E f(x,y,z)\,dV = \int_a^b \int_{z_1(y)}^{z_2(y)} \int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx\,dz\,dy}$$

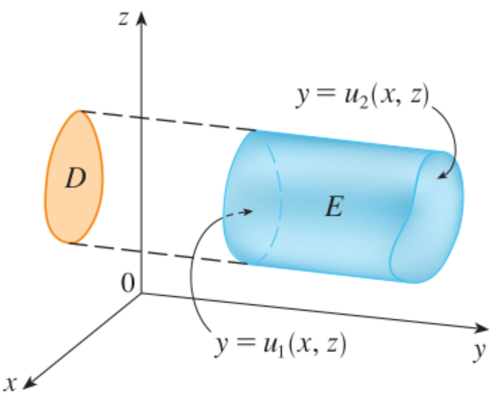
Projection D is of Type II.



$D = \{(y,z) :$
$a \le z \le b,$
$y_1(z) \le y \le y_2(z)\}$

$$\boxed{\iiint_E f(x,y,z)\,dV = \int_a^b \int_{y_1(z)}^{y_2(z)} \int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx\,dy\,dz}$$

Domains of type 3.

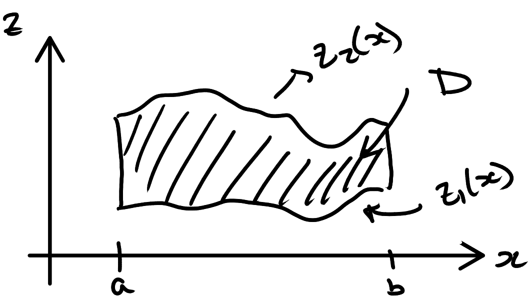


$E = \{(x,y,z) : (x,z) \in D,\ u_1(x,z) \le y \le u_2(x,z)\}$

Polar coordinates $\quad u_2(\overset{x}{r\cos\theta}, \overset{z}{r\sin\theta}) \quad r\,dr\,d\theta$

$$\boxed{\iiint_E f(x,y,z)\,dV = \iint_{\underline{D}} \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,d\underline{y}\right] dA}$$

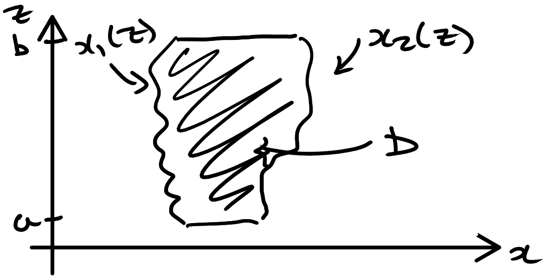
Projection D is of type I.



$D = \{(x,z) : a \le x \le b,\ z_1(x) \le z \le z_2(x)\}.$

$$\boxed{\iiint_E f(x,y,z)\,dV = \int_a^b \int_{z_1(x)}^{z_2(x)} \int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy\,dz\,dx}$$

Projection D is of type II.

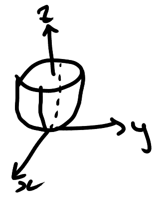


$D = \{(x,z) : x_1(z) \le x \le x_2(z),\ a \le z \le b\}$

$$\boxed{\iiint_E f(x,y,z)\,dV = \int_a^b \int_{x_1(z)}^{x_2(z)} \int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy\,dx\,dz}$$

EXAMPLE 3 Evaluate $\iiint_E \sqrt{x^2 + z^2}\, dV$, where E is the region bounded by the paraboloid $\boxed{y = x^2 + z^2}$ and the plane $\boxed{y = 4.}$

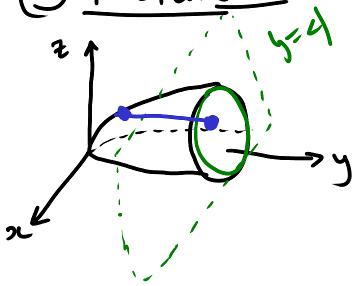
$z = x^2 + y^2$



① Picture E

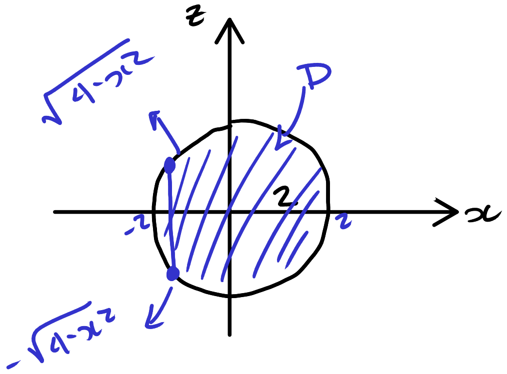
TYPE 3

$E = \{(x,y,z) : x^2 + z^2 \le y \le 4 \ ,\ (x,z) \in D\}$



② Projection D.

Let $y = 4 \ \rightarrow \ 4 = x^2 + z^2 \ \rightarrow$ circle radius 2.



$D = \{(r,\theta) : 0 \le r \le 2 \ ,\ 0 \le \theta \le 2\pi\}$

$x = r\cos\theta$
$y = r\sin\theta$

③ Integrate

$$\iiint_E \sqrt{x^2+z^2}\, dV = \iint_D \left(\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\, dy \right) dA$$

$$= \iint_D \sqrt{x^2+y^2} \ \ y \Big|_{x^2+z^2}^{4} \ dA$$

$$= \iint_D \sqrt{x^2+y^2} \ (4 - (x^2+z^2))\ dA$$

$x = r\cos\theta$
$z = r\sin\theta$

$$= \int_0^{2\pi}\int_0^{2} r \ (4 - r^2)\ r\, dr\, d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{2} 4r^2 - r^4\, dr\right)$$

$$= (2\pi) \left(\frac{4r^3}{3} - \frac{r^5}{5}\right)\Big|_0^2$$

$$= 2\pi \left(\frac{32}{3} - \frac{32}{5}\right)$$

$$= \boxed{\frac{128\pi}{15}}.$$

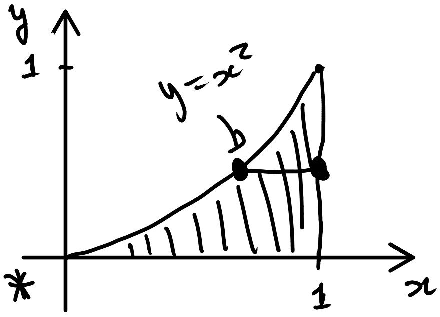
EXAMPLE 4 Express the iterated integral $\int_0^1\int_0^{x^2}\int_0^y f(x, y, z)\,dz\,dy\,dx$ as a triple integral and then rewrite it as an iterated integral in a different order, integrating first with respect to x, then z, and then y.

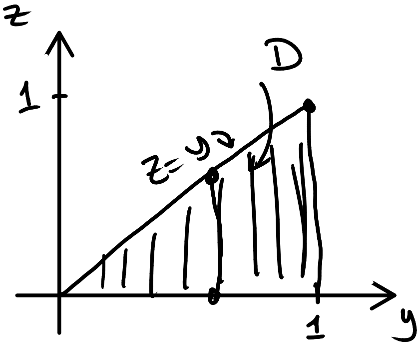
$\rightarrow \iiint_E f(x,y,z)\,dV$

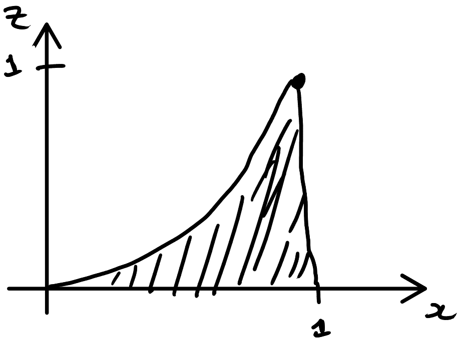
① Identify E.

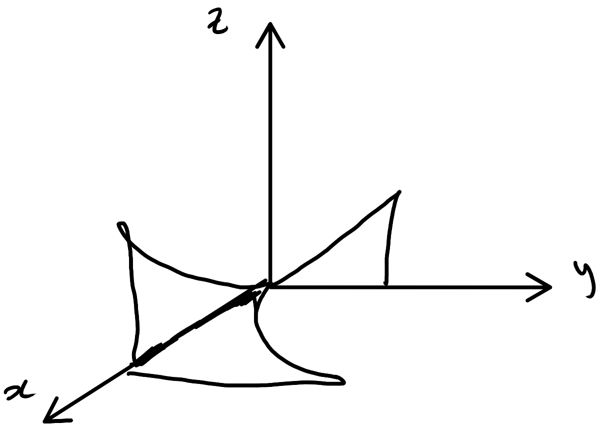
$E = \{ (x,y,z) : \quad 0 \le x \le 1, \quad 0 \le y \le x^2, \quad 0 \le z \le y \}$

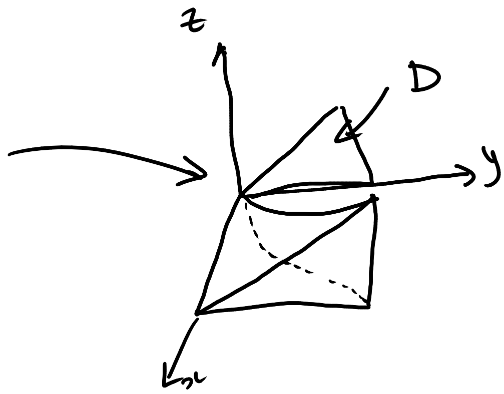
② Projections.











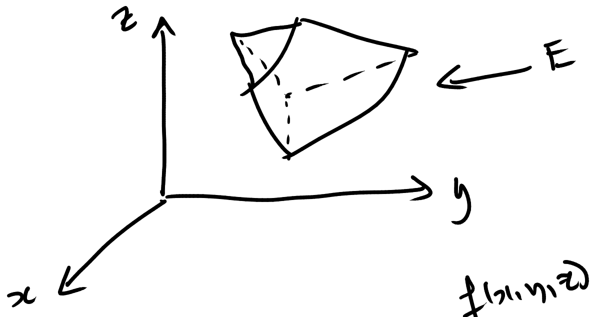
Here, from (*), $\sqrt{y} \le x \le 1$.

④ Rewrite the integral.

$$\iiint_E f(x,y,z)\,dV = \iint_D \left(\int_{\sqrt{y}}^1 f(x,y,z)\,dx \right) dA$$

$$= \int_0^1 \int_0^y \int_{\sqrt{y}}^1 f(x,y,z)\,dx\,dz\,dy$$

Application: computing volumes of solids.

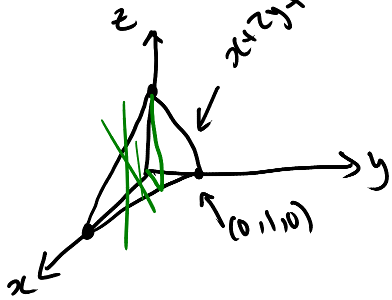


$f(x,y,z) = 1$

$$\boxed{V(E) = \iiint_E dV}$$

EXAMPLE 5 Use a triple integral to find the volume of the tetrahedron T bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.

① Picture.



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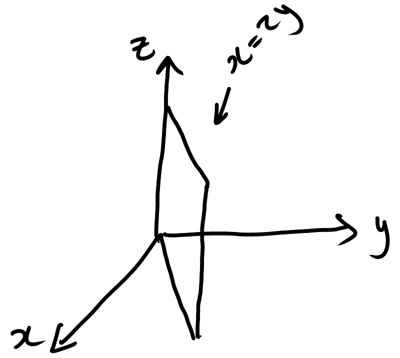
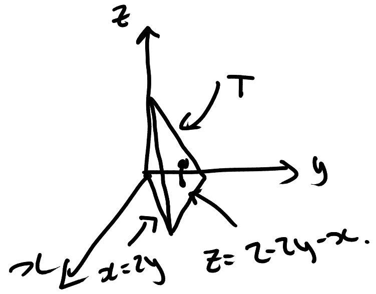
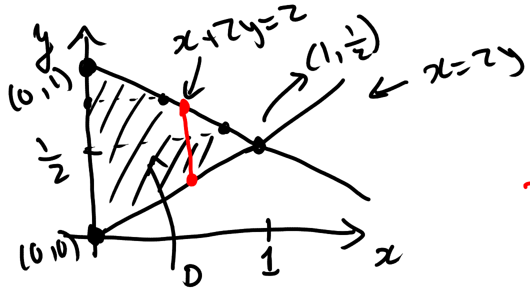


$=$



$$T = \{(x,y,z) : (x,y) \in D, \quad 0 \le z \le 2 - x - 2y\}$$

② Projection.



intersection. $4y = 2 \longrightarrow y = \frac{1}{2}$

TYPE I.

$$D = \{(x,y) : 0 \le x \le 1, \; 1 - \tfrac{x}{2} \le y \le \tfrac{x}{2}\}$$

③ Integrate.

$$V(T) = \iiint_T 1 \, dV = \int_0^1 \int_{1-\frac{x}{2}}^{\frac{x}{2}} \int_0^{2-x-y} dz\, dy\, dx = \boxed{1/3}$$

Other applications.

Mass. $\rho(x,y,z)$: density of E . .

$$m = \iiint_E \rho(x,y,z)\, dV$$

Moments.

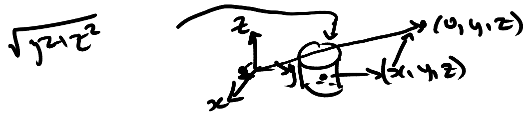
$$M_{yz} = \iiint_E x\rho(x,y,z)\, dV$$

$$M_{xz} = \iiint_E y\rho(x,y,z)\, dV$$

$$M_{xy} = \iiint_E z\rho(x,y,z)\, dV$$

center of mass : $(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m}\right)$

Moments of Inertia.



$$I_x = \iiint_E (y^2+z^2)\,\rho(x,y,z)\, dV$$

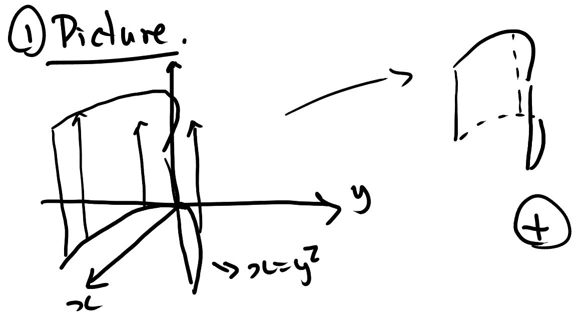
$$I_y = \iiint_E (x^2+z^2)\,\rho(x,y,z)\, dV$$

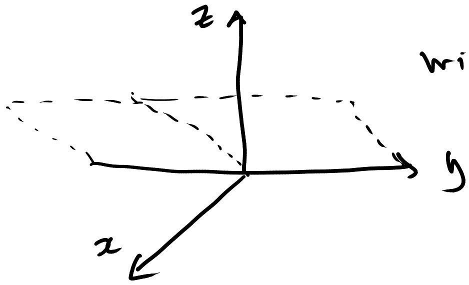
$$I_z = \iiint_E (x^2+y^2)\,\rho(x,y,z)\, dV$$

EXAMPLE 6 Find the center of mass of a solid of constant density that is bounded by the parabolic cylinder $x = y^2$ and the planes $x = z$, $z = 0$, and $x = 1$.

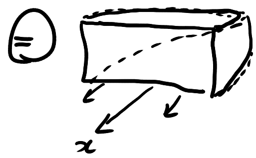
$\rho(x,y,z) = K$

① Picture.



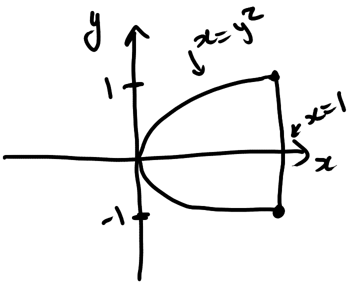


with $z=0$ & $x=1$.



② Description.

TYPE 1. $E = \{(x,y,z) : (x,y) \in D,\ 0 \le z \le x\}$.



$D = \{(x,y) : y^2 \le x \le 1,\ -1 \le y \le 1\}$

③ Mass.

$$m = \iiint_E \rho(x,y,z)\, dV = K\int_{-1}^{1}\int_{y^2}^{1}\int_0^x dz\,dx\,dy = \frac{4K}{5}.$$

④ Moments.

$$M_{yz} = \iiint_E x\rho(x,y,z)\, dV = K\int_{-1}^{1}\int_{y^2}^{1}\int_0^x x\,dz\,dx\,dy = \frac{4K}{7}.$$

$$M_{xy} = \iiint_E z\,\rho(x,y,z)\,dV = K\int_{-1}^{1}\int_{y^2}^{1}\int_0^{x} z\,dz\,dx\,dy = \frac{2K}{7}.$$

$$M_{xz} = \iiint_E y\,\rho(x,y,z)\,dV = K\iiint_E y\,dV$$

$$= K\left(\iiint_{E^+} y\,dV + \iiint_{E^-} y\,dV\right)$$

$y \geq 0$ $\quad$ $y \leq 0$

$$\overset{\text{symmetry}}{=} K\left(\iiint_{E^+} y\,dV - \iiint_{E^+} y\,dV\right)$$

$$= 0.$$

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m}\right) = \left(5/7,\ 0,\ 5/14\right).$$