Math 644

Chapter 2

Section 2.4: Analytic Functions

Contents

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We consider Ω to be an open subset of $\mathbb{C}$, meaning that

$$\forall z \in \Omega, \text{ there is an } r > 0 \text{ such that } \{w : |w - z| < r\} \subset \Omega.$$

Definition 1. Let $f : \Omega \to \mathbb{C}$.

- f is **analytic** at $z_0 \in \Omega$ if there is an $r > 0$ and a power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converging in $B = \{z : |z - z_0| < r\}$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad (\forall z \in B).$$

- f is analytic on Ω if f is analytic at each $z_0 \in \Omega$.

Notes:

- The power series is not necessarily the same for each $z_0 \in \Omega$.
- A function f is analytic on a set E (not necessarily open), if there is an open set $\Omega \supset E$ and an analytic function g on Ω such that $f = g$.

Theorem 2. If f is analytic in Ω, then f is continuous on Ω.

Proof.

Fact: (f_n) conv. unif. to some f & f_n is cont.

$\Downarrow$

f is cont.

Let $z_0 \in \Omega$. $\exists r > 0$ s.t.

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \forall z \in B.$$

From the root test, the power series conv. uniformly in a small enough disk centered at z_0. So, partial sums converge uniformly to f, so f is continuous on $\{z : |z - z_0| \leq r/2\}$. $\square$

Theorem 3. If $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges on $\{z : |z-z_0| < r\}$, then f is analytic on $\{z : |z-z_0| < r\}$.

Proof. Fix z_1 st. $|z_1 - z_0| < r$. By the Binomial Theorem:

$$(z - z_1 + z_1 - z_0)^n = \sum_{k=0}^{n} \binom{n}{k} (z-z_1)^k (z_1 - z_0)^{n-k}.$$

So,

$$f(z) = \sum_{n=0}^{\infty} a_n \left[\sum_{k=0}^{n} \binom{n}{k} (z-z_1)^k (z_1-z_0)^{n-k} \right].$$

For now, suppose we can interchange the order of Σ. Then

$$f(z) = \sum_{k=0}^{\infty} \left[\sum_{n=k}^{\infty} a_n \binom{n}{k} (z_1 - z_0)^{n-k} \right] (z-z_1)^k.$$

From the root test, we know that

$$\sum |a_n| \, |w - z_0|^n \quad \text{conv. if} \quad |w - z_0| < r$$

Set $w := |z - z_1| + |z_1 - z_0| + z_0$

if $|z - z_0| < r - |z_1 - z_0|$, then

$$|w - z_0| = |z - z_1| + |z_1 - z_0| < r$$

&

$$\sum_{n=0}^{\infty} |a_n| |w - z_0|^n = \sum_{n=0}^{\infty} |a_n| \left(|z - z_1| + |z_1 - z_0| \right)^n$$

$$= \sum_{n=0}^{\infty} |a_n| \left(\sum_{k=0}^{n} \binom{n}{k} |z - z_1|^k |z_1 - z_0|^{n-k} \right)$$

Since the LHS is conv., then

$$\sum_{n=0}^{\infty} a_n \left(\sum_{k=0}^{n} \binom{n}{k} (z - z_1)^k (z_1 - z_0)^{n-k} \right)$$ is abs. convergent. $\square$

Theorem 4. Suppose

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n = \sum_{n=0}^{\infty} b_n(z-z_0)^n$$

for all complex numbers in $\{z : |z-z_0| < r\}$. Then $a_n = b_n$ for all $n \geq 0$.

Proof.

Write $c_n = b_n - a_n$. Then

$$\sum_{n=0}^{\infty} c_n (z-z_0)^n = 0 \quad (|z-z_0|<r)$$

Let c_m be the first non-zero coef $(m \geq 1)$.

If $0 < |z-z_0| < r$, then

$$(z-z_0)^{-m} \sum_{n=m}^{\infty} c_n (z-z_0)^n = \sum_{n=0}^{\infty} c_{n+m}(z-z_0)^n$$
$$\equiv F(z).$$

The series defining F is convergent on $\{z : 0 < |z-z_0| < r\}$. By the root test, the series for F converges in a disk containing z_0.

$$\left[0<\rho<r \Rightarrow |c_n| \leq \rho^{-n} \Rightarrow \rho^{1+\frac{m}{j}} \leq |c_{j+m}|^{-1/j} \right.$$
$$\left. \Rightarrow \rho \leq \liminf_{j\to\infty} |c_{j+m}|^{-1/j} \right]$$

F is continuous, so there is a $\delta > 0$ s.t.

$$|z-z_0| < \delta \Longrightarrow |F(z) - F(z_0)| < \frac{|c_m|}{2}$$

$$\Rightarrow \quad |F(z) - c_m| < \frac{|c_m|}{2}.$$

If $F(z) = 0$ for some $|z - z_0| < \delta$, then

$$|0 - c_m| < \frac{|c_m|}{2} \Rightarrow |c_m| < \frac{|c_m|}{2}$$

$$\Rightarrow \#$$

So, $F \neq 0$ on $\{z : |z - z_0| < \delta\}$.

Then,

$$\sum_{n=m}^{\infty} c_n (z - z_0)^n = (z - z_0)^m F(z) \neq 0$$

on $\{z : 0 < |z - z_0| < \delta\}$. We have a contradiction with the assumption

$$\sum c_n (z - z_0)^n = 0 \quad \text{on} \quad \{z : |z - z_0| < r\}. \square$$

$f(z_0) = 0$

Note:

- The proof actually shows also that if f is analytic at z_0, then for some $\delta > 0$, either
  - $f(z) \neq 0$ for any $0 < |z - z_0| < \delta$;
  - or $f(z) = 0$ for any $|z - z_0| < \delta$.
- The proof also shows that if f is analytic at z_0, then there is a $r > 0$, an integer $m \geq 1$ and an analytic function g at z_0 such that
  - $g(z) \neq 0$ for any z such that $|z - z_0| < r$;
  - $f(z) - f(z_0) = (z - z_0)^m g(z)$.

Consequences On the Zeros

- A set $\Omega \subset \mathbb{C}$ is called a **region** if it is
  - open;
  - connected, meaning that we can't write $\Omega = U \cup V$, where U and V are open sets in $\mathbb{C}$ such that $V \cap U = \varnothing$.

  Fact: Ω is connected if and only if Ω and $\varnothing$ are the only open and closed subsets of Ω.
- A zero a of a function $f : \Omega \to \mathbb{C}$ is called **isolated** if there is an open disk B centered at a such that $f(z) \neq 0$ for any $z \in B \backslash \{a\}$.

Corollary 5. If f is analytic on a region Ω, then either $f \equiv 0$ or the zeros of f are isolated.

Proof.

Let E be the set of non-isolated zero.

1) E is closed.

$z_n \to z$ with $(z_n) \subseteq E$, $z_n \neq z$.

then, by cont., $f(z_n) = 0 \Rightarrow f(z) = 0$ & $z \in E$.

2) E is open

$z \in E$, then $\begin{cases} f \equiv 0 \text{ in } \{w : |w - z_0| < \delta\} \\ \text{or} \\ f(w) \neq 0 \text{ in } \{w : 0 < |w - z_0| < \delta\}. \end{cases}$

the second alternative can't happen. this means $\{w : |w - z_0| < \delta\} \subseteq E$.

Therefore E is open & closed $\Rightarrow$ $E = \phi$ or $E = \Omega$. ◻

Note:

- A consequence of the last Corollary is the **Identity principle**: If f and g are two analytic functions in a region Ω that agree on a set with an accumulation point in Ω, then they must be identical (see Problem 18).
- The last Corollary is not true for continuous functions: $f(x) = x \sin(1/x)$ is an example.