Math 644

Chapter 3

Section 3.2: Local Behavior

Contents

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DEFINITION 1. A continuous function $f : \Omega \subset \mathbb{C}$, where Ω is open, is an **open map** if $U \subset \Omega$ is open, then $f(U)$ is open.

$\Omega^{\circ} = \Omega$

THEOREM 2. A non-constant analytic function defined on a region is an open map.

Proof.

Let $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ in $\{z : |z-z_0| < R\} \subseteq \Omega$.

Pick $r < R$ and set

$$\delta := \inf \{ |f(z) - f(z_0)| : |z - z_0| = r \}$$

Since the zeros of $f - f(z_0)$ are isolated, we can decrease r s.t. $\delta > 0$.

We want to show

$$\{ w : |w - f(z_0)| < \tfrac{\delta}{2} \} \subseteq f(\{z : |z-z_0| < r\}).$$

Suppose $\exists\, w$ s.t. $|w - f(z_0)| < \frac{\delta}{2}$ but

$w \neq f(z)$ for $|z - z_0| < r$.

So, $\dfrac{1}{f-w}$ is analytic on $\{z : |z-z_0| < r\}$

and if $|z - z_0| = r$, then

$$\frac{1}{|f(z) - w|} \leq \frac{1}{|f(z) - f(z_0)| - |w - f(z_0)|} < \frac{1}{\delta - \frac{\delta}{2}} = \frac{2}{\delta}$$

By the max Principle,

$$\frac{1}{|f(z)-w|} < \frac{2}{\delta} \quad \forall z \in \{z : |z-z_0|<r\}$$

For $z = z_0$

$$\Rightarrow \quad \frac{1}{|f(z_0)-w|} < \frac{2}{\delta} \Rightarrow \frac{\delta}{2} < |f(z_0)-w|$$

A contradiction with $|w-f(z_0)| < \frac{\delta}{2}$.

therefore, $\{w : |w-f(z_0)| < \frac{\delta}{2}\} \subseteq f(\{z : |z-z_0|<r\})$

We can do this for every $z_0 \in \Omega$, so $f(U)$ contains a disk centered at $f(z_0)$ $\forall z_0 \in U$. $\square$

Note:

- An open map always satisfies the maximum modulus principle.

DEFINITION 3. A function f is **one-to-one** if $f(z) = f(w)$ only when $z = w$.

THEOREM 4. If f is analytic at z_0 with $f'(z_0) \neq 0$, then there is an $r > 0$ such that f is one-to-one on $\{z : |z - z_0| < r\}$.

Proof.

Suppose $\forall r > 0$, f is not one-to-one in $\{z : |z - z_0| < r\}$.

Take $r = \frac{1}{n}$. $\exists z_n, w_n \in \{z : |z - z_0| < \frac{1}{n}\}$ s.t.

$z_n \neq w_n$ & $f(z_n) = f(w_n)$. & $z_{n-1} \neq z_n$
& $w_{n-1} \neq w_n$.

Now,

$$f'(z_0) = \lim_{n \to \infty} \frac{f(z_n) - f(w_n)}{z_n - w_n} = 0 \quad \# . \quad \square$$

Note:

- The function $f(z) = e^z$ gives an example of an analytic function which is locally one-to-one, but globally infinite-to-one! The equation $w = e^z$ has infinitely many solutions.
- Theorem 2 and Theorem 4 show that if f is analytic at z_0 with $f'(z_0) \neq 0$, then f is a homeomorphism of a neighborhood of z_0 onto a neighborhood of $f(z_0)$.