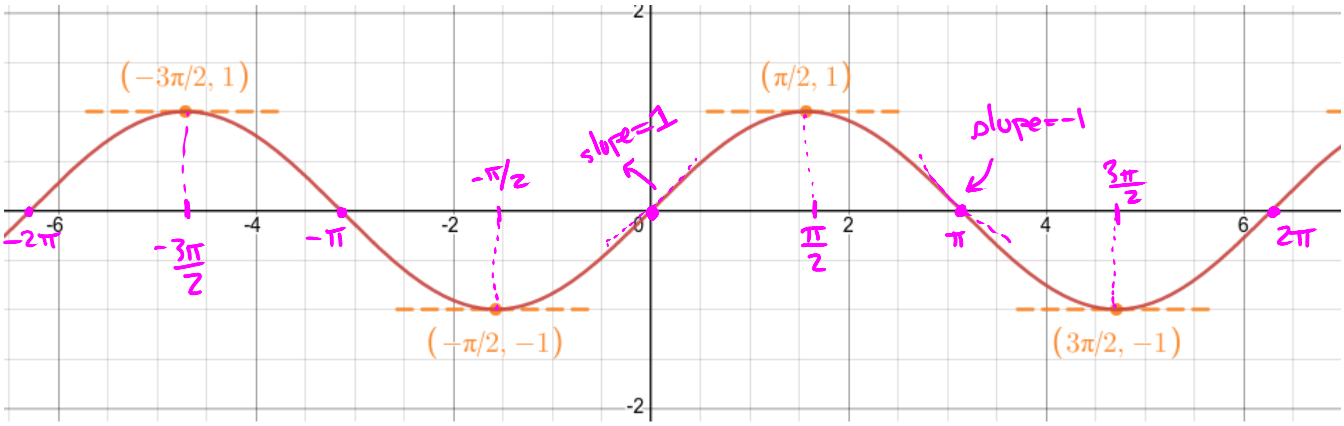
Chapter 2

Derivatives

2.4 Derivatives of Trigonometric Functions

Derivative of the Sine function.



Desmos: https://www.desmos.com/calculator/mhbl7c2hzy

$$\boxed{\frac{d}{dx}(\sin x) = \cos x}$$

Proof.

By def.:
$$\frac{d}{dx}(\sin x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} \to \frac{0}{0}$$

Trig identity:
$$\sin(x+h) = \sin(x)\cos(h) + \sin(h)\cos(x)$$

$$\Rightarrow \frac{\sin(x+h) - \sin(x)}{h} = \frac{\sin(x)\cos(h) + \sin(h)\cos(x) - \sin(x)}{h}$$

$$= \frac{\sin(x)(\cos(h) - 1) + \cos(x)\sin(h)}{h}$$

$$= \sin(x)\,\frac{\cos(h)-1}{h} + \cos(x)\,\frac{\sin(h)}{h}$$

so,

$$\lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} = \lim_{h\to 0} \sin(x)\left(\frac{\cos h - 1}{h}\right) \quad 2$$

$$+ \lim_{h\to 0} \cos(x)\,\frac{\sin h}{h} \quad 1$$

① We can prove that

$$\cos h \leq \frac{\sin h}{h} \leq 1$$

We have $\lim_{h\to 0} \cos h = \cos(0) = 1$

& $\lim_{h\to 0} 1 = 1$

Squeeze Theorem $\Downarrow$

$$\lim_{h\to 0} \frac{\sin h}{h} = 1$$

② $\cos h - 1 = -2\left(\frac{1-\cos h}{2}\right) = -2\sin^2(h/2)$

$$\Rightarrow \lim_{h\to 0} \frac{\cos h - 1}{h} = \lim_{h\to 0} \frac{-2\sin^2(h/2)}{h}$$

$$= \lim_{h\to 0} -2\,\frac{\sin(h/2)}{h}\,\sin(h/2)$$

$$= \lim_{h\to 0} -\frac{\sin(h/2)}{(h/2)}\cdot \sin(h/2)$$

$$= -\lim_{h\to 0} \frac{\sin(h/2)}{(h/2)} \cdot \lim_{h\to 0} \sin(h/2)$$

$$= -\lim_{h\to 0} \frac{\sin(h)}{h} \cdot \sin(0)$$

$$= -1\cdot 0 = 0$$

So

$$\lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h} = \sin(x)\cdot 0 + \cos(x)\cdot 1$$

$$= \boxed{\cos(x)}$$

Trigonometric Functions (reminder).

- $\sec x = \dfrac{1}{\cos x}$
- $\csc x = \dfrac{1}{\sin x}$
- $\tan x = \dfrac{\sin x}{\cos x}$
- $\cot x = \dfrac{1}{\tan x} = \dfrac{\cos x}{\sin x}$

Derivatives of Other Trigonometric Functions.

Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

Proof for the formula for f(x) = tan(x).

EXAMPLE 2 Differentiate $f(x) = \dfrac{\sec x}{1 + \tan x}$. For what values of x does the graph of f have a horizontal tangent?

EXAMPLE 6 Calculate $\lim_{x \to 0} x \cot x$.