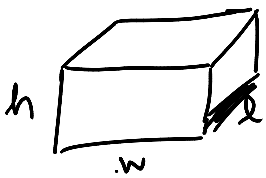
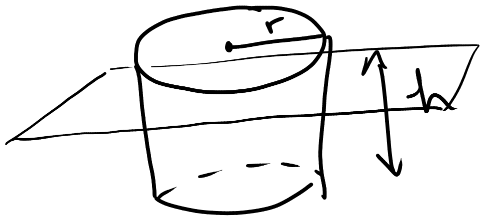
Chapter 5
Applications in integration

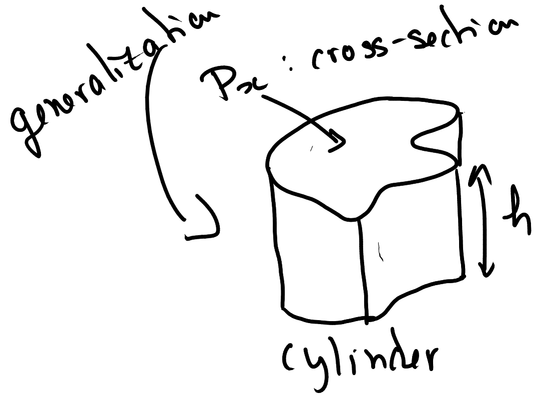
5.2 Volumes



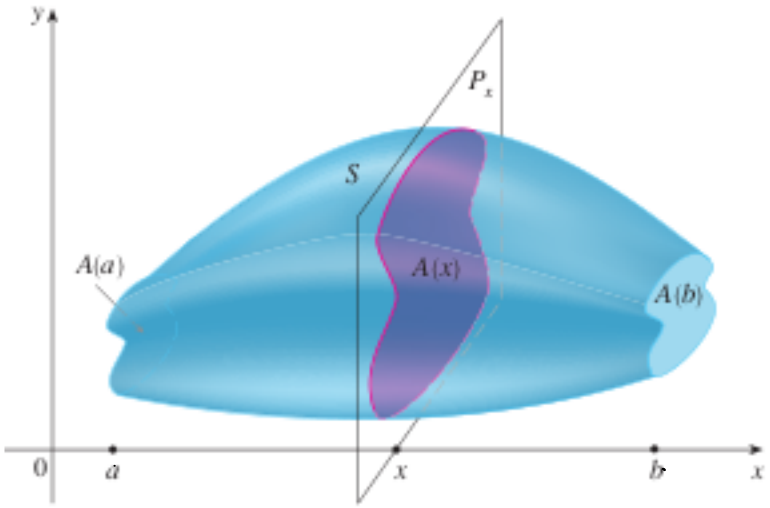
$$\text{Volume} = hwl$$



$$\text{Volume} = \underbrace{\pi r^2}_{\text{Area circle}} \underbrace{h}_{\text{how much I have}}$$



$$\text{Volume} = \text{Area}(P_x) \cdot h$$



S: object (solid)

P_x: cross-section at x

A(x): area of P_x

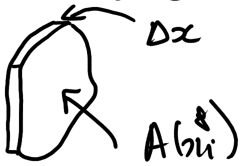
$a \le x \le b$

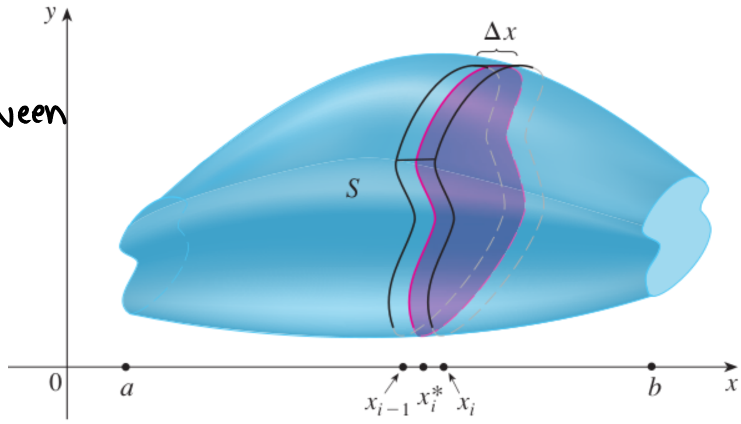
Cut the solid in n slices at $P_{x_1}, P_{x_2}, \dots$

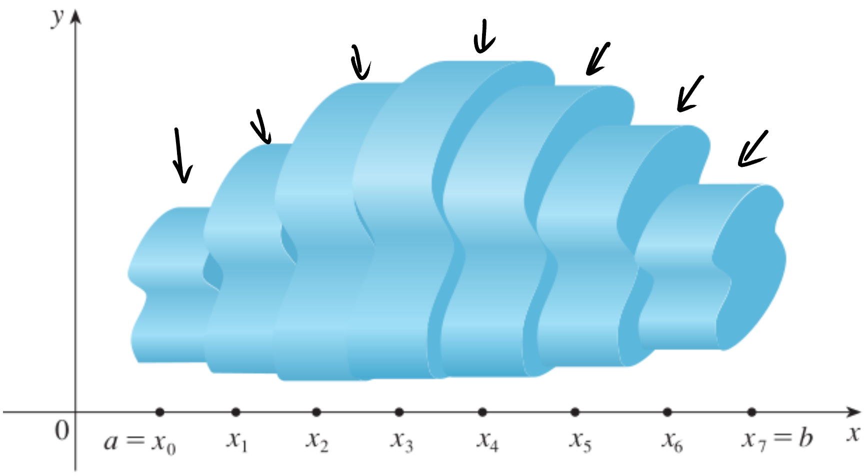
Each slice has width Δx

Take a sample point x_i^* between x_{i-1} & x_i.

Approximate your slice by a cylindre :







$$\mathrm{Vol(slide)} = \boxed{A(x_i^*) \cdot \Delta x}$$

Sum all volume of the slices

$$\Rightarrow \underset{V(S)}{V} \approx \sum_{i=1}^{n} A(x_i^*)\,\Delta x$$

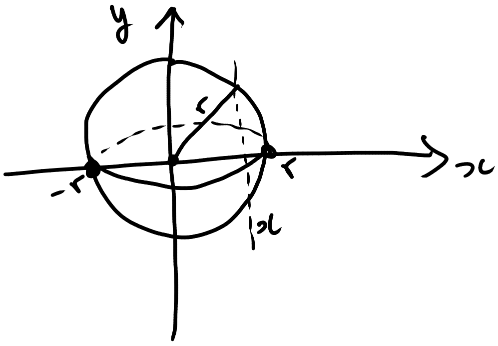
Definition of Volume Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x, through x and perpendicular to the x-axis, is $A(x)$, where A is a continuous function, then the **volume** of S is

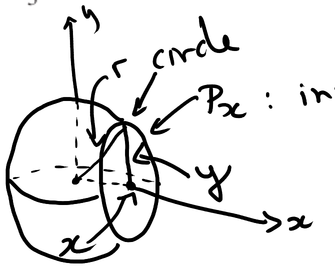
$$V = \lim_{n \to \infty} \sum_{i=1}^{n} A(x_i^*)\,\Delta x = \int_a^b A(x)\,dx$$

Area function of a cross-section P_x.

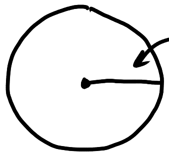
EXAMPLE 1 Show that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.

① Picture.





so, P_x



$x^2+y^2=r^2$

So, radius $= y = \sqrt{r^2-x^2}$

② Volume. $-r \le x \le r$

$$V(S) = \int_{-r}^{r} A(x)\,dx = \int_{-r}^{r} \pi \text{ radius}^2\,dx$$

$$= \int_{-r}^{r} \pi \left(\sqrt{r^2-x^2}\right)^2 dx$$

$$= \int_{-r}^{r} \pi (r^2-x^2)\,dx$$

$$= \pi \int_{-r}^{r} r^2\,dx - \pi \int_{-r}^{r} x^2\,dx$$

$$= \pi r^2 \int_{-r}^{r} dx - \pi \int_{r}^{r} x^2\,dx$$

$$= \pi r^2 \; x\Big|_{-r}^{r} - \pi \left.\frac{x^3}{3}\right|_{-r}^{r}$$

$$= \pi r^2 (r-(-r)) - \pi \left(\frac{r^3}{3} - \frac{(-r)^3}{3}\right)$$

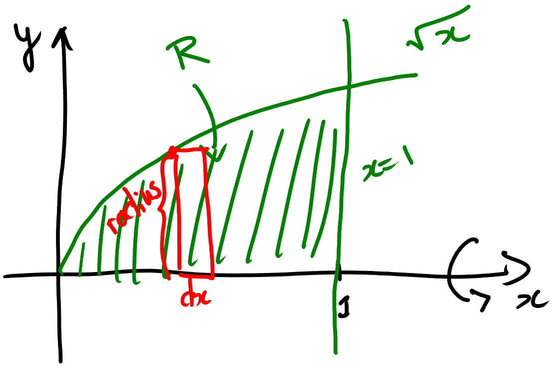
$$= \boxed{\frac{4\pi r^3}{3}}$$

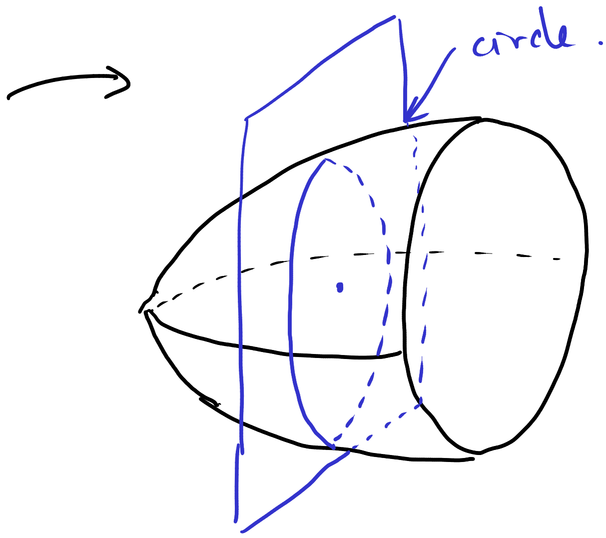
Solid of revolution.

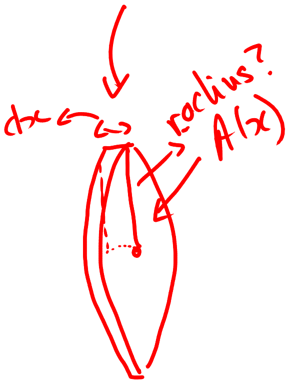
Rotation about the x-axis.

EXAMPLE 2 Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1. ~~Illustrate the definition of volume~~ by ~~sketching a typical approximating cylinder.~~

① Picture region.







radius $= y = \sqrt{x}$

② volume. $0 \le x \le 1$

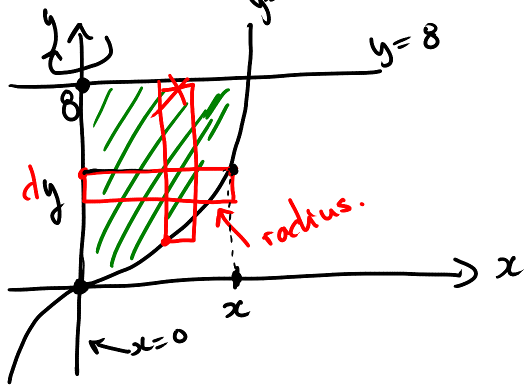
$$V(S) = \int_0^1 A(x)\,dx = \int_0^1 \pi (\text{radius})^2\,dx$$

$$= \int_0^1 \pi (\sqrt{x})^2\,dx$$

$$= \int_0^1 \pi x\,dx = \boxed{\frac{\pi}{2}}$$

EXAMPLE 3 Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$, and $x = 0$ about the y-axis.

① Picture.



$$\text{radius} = x = \sqrt[3]{y}$$

$$\rightarrow A(y) = \pi(\text{radius})^2 = \pi\, y^{2/3}$$

② Volume

$$V(S) = \int_0^8 A(y)\,dy = \int_0^8 \pi y^{2/3}\,dy$$

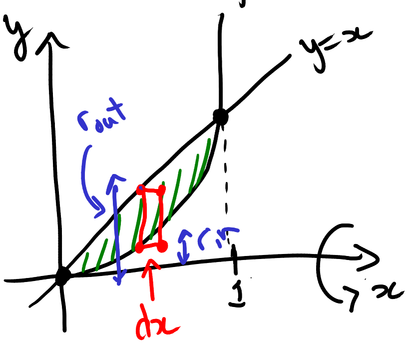
$$= \pi \int_0^8 y^{2/3}\,dy$$

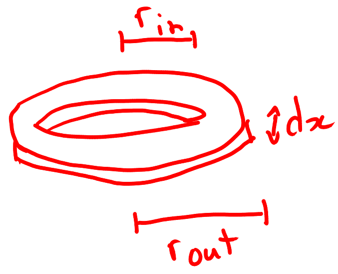
$$= \pi \left.\frac{y^{5/3}}{5/3}\right|_0^8 = \boxed{\frac{96\pi}{5}}$$

Cross-section as a washer.

EXAMPLE 4 The region $\mathscr{R}$ enclosed by the curves $y = x$ and $y = x^2$ is rotated about the x-axis. Find the volume of the resulting solid.

① Picture.





$$r_{out} = x \longrightarrow \pi r_{out}^2 = \pi x^2$$

$$r_{in} = x^2 \longrightarrow \pi r_{in}^2 = \pi x^4$$

$$\Rightarrow A(x) = \pi r_{out}^2 - \pi r_{in}^2 = \pi x^2 - \pi x^4$$

② Volume.

$$V(S) = \int_0^1 A(x)\,dx = \int_0^1 \pi x^2 - \pi x^4\,dx$$

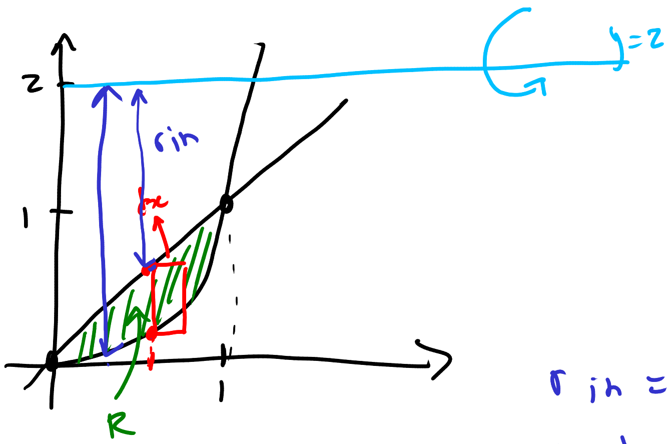
$$= \int_0^1 \pi x^2\,dx - \int_0^1 \pi x^4\,dx$$

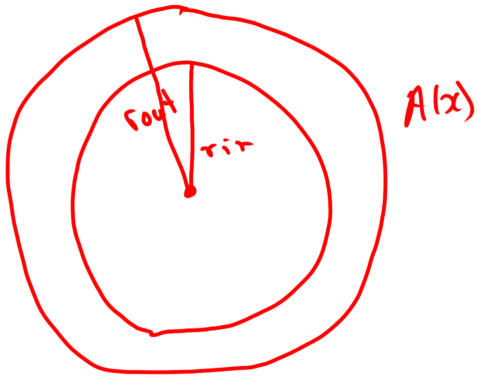
$$= \pi \left.\frac{x^3}{3}\right|_0^1 - \pi \left.\frac{x^5}{5}\right|_0^1$$

$$= \boxed{\frac{2\pi}{15}}$$

EXAMPLE 5 Find the volume of the solid obtained by rotating the region in Example 4 about the line $y = 2$.

① Picture.





$r_{in} = 2 - x \rightarrow \pi r_{in}^2 = \pi(2-x)^2$

$r_{out} = 2 - x^2 \rightarrow \pi r_{out}^2 = \pi(2-x^2)^2$

$$A(x) = \pi(2-x^2)^2 - \pi(2-x)^2$$

② Volume.

$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi(2-x^2)^2 - \pi(2-x)^2\,dx$$

$$= \pi\int_0^1 (2-x^2)^2\,dx - \pi\int_0^1 (2-x)^2\,dx$$

$$= \pi\int_0^1 4 - 4x^2 + x^4\,dx - \pi\int_0^1 4 - 4x + x^2\,dx$$

$$= \pi\int_0^1 4\,dx - \pi 4\int_0^1 x^2\,dx + \pi\int_0^1 x^4\,dx$$

$$- \pi\int_0^1 4\,dx + \pi 4\int_0^1 x\,dx - \pi\int_0^1 x^2\,dx$$

$$= \boxed{\frac{8\pi}{15}}$$

- If the cross-section is a disk (as in Examples 1–3), we find the radius of the disk (in terms of x or y) and use

$$A = \pi(\text{radius})^2$$

- If the cross-section is a washer (as in Examples 4 and 5), we find the inner radius r_{in} and outer radius r_{out} from a sketch (as in Figures 8, 9, and 10) and compute the area of the washer by subtracting the area of the inner disk from the area of the outer disk:

$$A = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2$$