

Chapter 1

Functions and Limits

1.6 Calculating Limits Using the Limit Laws

Operations With Limits

EXAMPLE 1

Use the graphs of f and g in Figure 1 to evaluate the following limits, if they exist.

(a) $\lim_{x \rightarrow -2} [f(x) + 5g(x)]$ (b) $\lim_{x \rightarrow 2} [f(x)g(x)]$ (c) $\lim_{x \rightarrow 2} \frac{f(x)}{g(x)}$

(d) $\lim_{x \rightarrow -2} [2f(x)]$ (e) $\lim_{x \rightarrow -2} [f(x) - g(x)]$

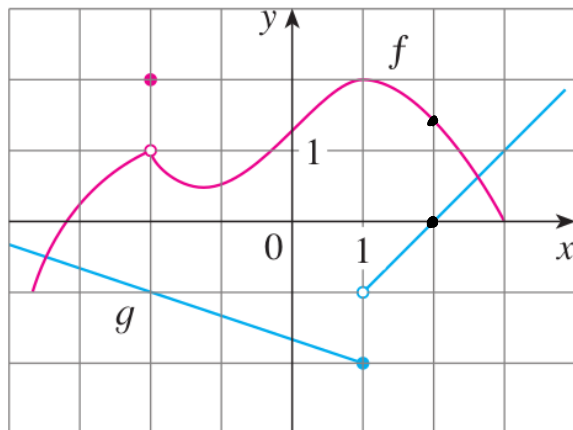


FIGURE 1

Use Desmos

<https://www.desmos.com/calculator/7fy0x0ghia>

$$(a) \lim_{x \rightarrow -2} [f(x) + 5g(x)] = -4$$

$$= 1 + (-5)$$

$$= 1 + 5(-1)$$

$$= \lim_{x \rightarrow -2} f(x) + 5 \lim_{x \rightarrow -2} g(x)$$

$$(b) \lim_{x \rightarrow 2} [f(x)g(x)] = 0 = 1.4 \cdot 0$$

$$= \lim_{x \rightarrow 2} f(x) \lim_{x \rightarrow 2} g(x)$$

$$(c) \lim_{x \rightarrow -2} \frac{f(x)}{g(x)} = -1 = \frac{1}{-1} = \frac{\lim_{x \rightarrow -2} f(x)}{\lim_{x \rightarrow -2} g(x)}$$

$$(d) \lim_{x \rightarrow -2} 2f(x) = 2 \lim_{x \rightarrow -2} f(x) = 2 \cdot 1 = 2$$

$$(e) \lim_{x \rightarrow -2} [f(x) - g(x)] = \lim_{x \rightarrow -2} [f(x) + (-g(x))]$$

$$= \lim_{x \rightarrow -2} f(x) + \lim_{x \rightarrow -2} [-g(x)]$$

$$= \lim_{x \rightarrow -2} f(x) - \lim_{x \rightarrow -2} g(x) = 1 - (-1) = 2$$

Limit Laws Suppose that c is a constant and the limits

$$\lim_{x \rightarrow a} f(x) \quad \text{and} \quad \lim_{x \rightarrow a} g(x)$$

exist. Then

$$1. \lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$$

$$2. \lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$$

$$3. \lim_{x \rightarrow a} [cf(x)] = c \lim_{x \rightarrow a} f(x)$$

$$4. \lim_{x \rightarrow a} [f(x)g(x)] = \lim_{x \rightarrow a} f(x) \lim_{x \rightarrow a} g(x)$$

$$5. \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} \quad \text{if } \lim_{x \rightarrow a} g(x) \neq 0$$

EXAMPLE. Think of three ways of computing the following limit:

$$\lim_{x \rightarrow 2} (1+x)^3$$

1) Graph: $\lim_{x \rightarrow 2} (1+x)^3 = 27$

2) Product: $\lim_{x \rightarrow 2} (1+x)^3 = \lim_{x \rightarrow 2} (1+x)^2 (1+x)$
 $= \lim_{x \rightarrow 2} (1+x)^2 \lim_{x \rightarrow 2} (1+x)$
 $= \lim_{x \rightarrow 2} (1+x) \lim_{x \rightarrow 2} (1+x) \lim_{x \rightarrow 2} (1+x)$
 $= \left(\lim_{x \rightarrow 2} 1+x \right)^3$
 $= \left(\lim_{x \rightarrow 2} 1 + \lim_{x \rightarrow 2} x \right)^3 = (1+2)^3$
 $= 3^3 = \boxed{27}$

EXAMPLE. Think of three ways of computing the following limit:

$$\lim_{x \rightarrow \pi/4} \cos^2(x)$$

Graph: $\lim_{x \rightarrow \pi/4} \cos^2(x) = 0.5054$

Product rule: $\lim_{x \rightarrow \pi/4} \cos^2(x) = \lim_{x \rightarrow \pi/4} \cos x \lim_{x \rightarrow \pi/4} \cos x$
 $= \left(\lim_{x \rightarrow \pi/4} \cos x \right)^2 = \left(\frac{\sqrt{2}}{2} \right)^2 = \frac{2}{4} = \boxed{\frac{1}{2}}$

General Formula:

$$6. \lim_{x \rightarrow a} [f(x)]^n = \left[\lim_{x \rightarrow a} f(x) \right]^n \quad \text{where } n \text{ is a positive integer}$$

Special cases:

$$\lim_{x \rightarrow a} c = c, \quad \lim_{x \rightarrow a} x = a, \quad \lim_{x \rightarrow a} x^n = a^n.$$

EXAMPLE 2 Evaluate the following limits and justify each step.

(a) $\lim_{x \rightarrow 5} (2x^2 - 3x + 4)$

(b) $\lim_{x \rightarrow -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}$

$$\begin{aligned}
 \text{(a)} \quad \lim_{x \rightarrow 5} (2x^2 - 3x + 4) &= \lim_{x \rightarrow 5} 2x^2 - \lim_{x \rightarrow 5} 3x + \lim_{x \rightarrow 5} 4 \quad [\text{Sum}] \\
 &= 2 \lim_{x \rightarrow 5} x^2 - 3 \lim_{x \rightarrow 5} x + 4 \quad [\text{Constant}] \\
 &= 2 \left(\lim_{x \rightarrow 5} x \right)^2 - 3 \cdot 5 + 4 \quad [\text{Power}] \\
 &= 2 \cdot 5^2 - 3 \cdot 5 + 4 = \boxed{39}
 \end{aligned}$$

(b) Want quotient rule $\rightarrow \lim_{x \rightarrow -2} 5 - 3x = 5 - 3(-2) = 11 \neq 0 \quad \checkmark$

$$\begin{aligned}
 \lim_{x \rightarrow -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} &= \frac{\lim_{x \rightarrow -2} x^3 + 2 \lim_{x \rightarrow -2} x^2 - \lim_{x \rightarrow -2} 1}{\lim_{x \rightarrow -2} 5 - 3x} \quad [\text{Quotient}] \\
 &= \frac{\lim_{x \rightarrow -2} x^3 + 2 \lim_{x \rightarrow -2} x^2 - \lim_{x \rightarrow -2} 1}{11} \quad \begin{array}{l} [\text{Sum} \\ \text{Constant} \\ \text{Diff.}] \end{array} \\
 &= \frac{(-2)^3 + 2(-2)^2 - 1}{11} \quad [\text{Power rule}] \\
 &= \frac{-8 + 8 - 1}{11} \\
 &= \boxed{\frac{-1}{11}}
 \end{aligned}$$

Remark:

Direct Substitution Property If f is a polynomial or a rational function and a is in the domain of f , then

$$\lim_{x \rightarrow a} f(x) = f(a)$$

$$11. \lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)} \quad \text{where } n \text{ is a positive integer}$$

[If n is even, we assume that $\lim_{x \rightarrow a} f(x) > 0$.]

Example. Compute $\lim_{u \rightarrow -2} \sqrt{u^4 + 3u + 6}$.

$$\lim_{u \rightarrow -2} (u^4 + 3u + 6) = (-2)^4 + 3(-2) + 6 = 16 > 0$$

$$\begin{aligned} \lim_{u \rightarrow -2} \sqrt{u^4 + 3u + 6} &= \sqrt{\lim_{u \rightarrow -2} u^4 + 3u + 6} \\ &= \sqrt{16} \\ &= \boxed{4} \end{aligned}$$

EXAMPLE 3 Find $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$.

Quotient:
or subst. $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = \frac{1 - 1}{1 - 1} = \frac{0}{0} \rightarrow$ For now, undefined.

Simplifying: $\frac{x^2 - 1}{x - 1} = \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} = x+1 \quad (x \neq 1)$
 $= g(x) \quad (x \neq 1)$

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} &= \lim_{x \rightarrow 1} \overset{g(x)}{\downarrow} x+1 = \lim_{x \rightarrow 1} x + \lim_{x \rightarrow 1} 1 \\ &= 1 + 1 = \boxed{2} \end{aligned}$$

We have to use the following new substitution rule:

If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$, provided the limits exist.

EXAMPLE 5 Evaluate $\lim_{h \rightarrow 0} \frac{(3+h)^2 - 9}{h}$.

Quotient: $\lim_{h \rightarrow 0} \frac{(3+h)^2 - 9}{h} = \frac{(3+0)^2 - 9}{0} = \frac{0}{0} \rightarrow \text{undefined yet } \frac{0}{0}.$

Simplify: $\frac{(3+h)^2 - 9}{h} = \frac{\cancel{9} + 6h + h^2 - \cancel{9}}{h}$

$$= \frac{6h + h^2}{h} \quad h \neq 0.$$

$$= \frac{(6+h)\cancel{h}}{\cancel{h}} = 6+h = g(h)$$

Now,

$$\lim_{h \rightarrow 0} \frac{(3+h)^2 - 9}{h} = \lim_{h \rightarrow 0} 6+h = 6+0 = \boxed{6}$$

EXAMPLE 6 Find $\lim_{t \rightarrow 0} \frac{\sqrt{t^2 + 9} - 3}{t^2}$.

Subst.: $\frac{0}{0}$ undefined yet
s

Simplify: $\frac{\sqrt{t^2 + 9} - 3}{t^2} \cdot 1 = \frac{\sqrt{t^2 + 9} - 3}{t^2} \cdot \frac{\sqrt{t^2 + 9} + 3}{\sqrt{t^2 + 9} + 3}$

$$= \frac{t^2 + 9 - 9}{t^2 \cdot (\sqrt{t^2 + 9} + 3)}$$

$$= \frac{\cancel{t^2}}{\cancel{t^2}(\sqrt{t^2 + 9} + 3)} = \frac{1}{\sqrt{t^2 + 9} + 3} = g(t)$$

So,

$$\lim_{t \rightarrow 0} \frac{\sqrt{t^2 + 9} - 3}{t^2} = \lim_{t \rightarrow 0} \frac{1}{\sqrt{t^2 + 9} + 3}$$
$$= \frac{\lim_{t \rightarrow 0} 1}{\lim_{t \rightarrow 0} (\sqrt{t^2 + 9} + 3)} = \frac{1}{6} \quad \left[\begin{array}{l} \text{root law} \\ \& \\ \text{sum rule} \\ \text{or} \\ \text{[subst. rule]} \end{array} \right]$$

EXAMPLE 8 Prove that $\lim_{x \rightarrow 0} \frac{|x|}{x}$ does not exist. $\rightarrow \frac{0}{0}$:

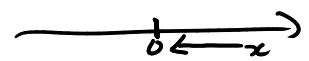
REMARK: ALL THE LIMIT RULES WORK ALSO FOR THE LIMITS FROM THE LEFT AND FROM THE RIGHT.

To prove $\lim_{x \rightarrow 0} \frac{|x|}{x}$ doesn't exist, we will show that

$$\lim_{x \rightarrow 0^+} \frac{|x|}{x} \neq \lim_{x \rightarrow 0^-} \frac{|x|}{x}.$$

$$|x| = x$$

$$1) \lim_{x \rightarrow 0^+} \frac{|x|}{x} = \lim_{x \rightarrow 0^+} \frac{x}{x} = \lim_{x \rightarrow 0^+} 1 = 1$$



$$|-2| = -(-2)$$

$$2) \lim_{x \rightarrow 0^-} \frac{|x|}{x} = \lim_{x \rightarrow 0^-} \frac{-x}{x} = \lim_{x \rightarrow 0^-} -1 = -1$$

$$1) \neq 2) \Rightarrow \lim_{x \rightarrow 0} \frac{|x|}{x} \nexists$$

EXAMPLE 9 If

$$f(x) = \begin{cases} \sqrt{x-4} & \text{if } x > 4 \\ 8 - 2x & \text{if } x < 4 \end{cases}$$

determine whether $\lim_{x \rightarrow 4} f(x)$ exists.

EXAMPLE 11 Show that $\lim_{x \rightarrow 0} x^2 \sin \frac{1}{x} = 0$.

3 The Squeeze Theorem If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$$

then

$$\lim_{x \rightarrow a} g(x) = L$$

