Math 644

Chapter 4

Section 4.3: Approximation by Rational Functions

Contents

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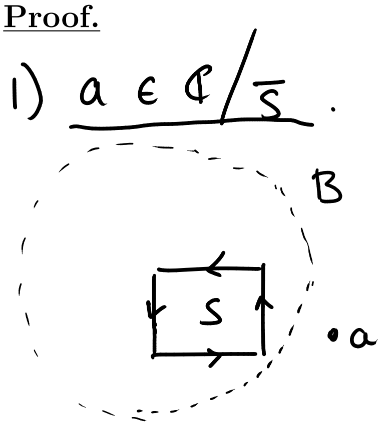
Cauchy Integral Formula For a Square

Theorem 1. If S is an open square with boundary ∂S parameterized in the counter-clockwise direction then

$$\frac{1}{2\pi i}\int_{\partial S}\frac{1}{z-a}\,dz = \begin{cases} 1 & \text{if } a \in S \\ 0 & \text{if } a \in \mathbb{C}\setminus\overline{S}. \end{cases}$$

Proof.

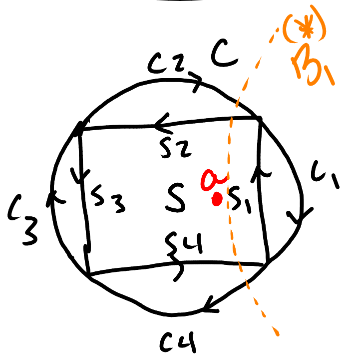
1) $a \in \mathbb{C}\setminus\overline{S}$. There is a disk B s.t. $\overline{S} \subseteq B$ & $a \notin \overline{B}$.



Since $\frac{1}{z-a}$ is analytic in B, from Cor. 11 in sect. 4.2,

$$\int_{\partial S}\frac{1}{z-a}\,dz = 0.$$

2) $a \in S$. Let C be the circle circumscribed to S. Write



$$S = s_1 + s_2 + s_3 + s_4 \quad \& \quad C = c_1 + c_2 + c_3 + c_4$$

For $j = 1,2,3,4$, $\sigma_j := s_j + c_j$ is a closed curve. For each j, we can find a disk B_j s.t. $\sigma_j \subseteq B_j$ & $a \notin B_j$. Therefore, $\int_{\sigma_j}\frac{1}{z-a}\,dz = 0$ by Cor. 4.11 in 4.2.

$$\Rightarrow \int_{S+(-C)}\frac{1}{z-a}\,dz = 0 \Rightarrow \int_S \frac{1}{z-a}\,dz = \int_C \frac{1}{z-a}\,dz = 2\pi i. \quad \square$$

(*) https://www.desmos.com/calculator/65bsuzfc9z

Theorem 2. If f is analytic in a neighborhood of the closure of $\overline{S}$ of an open square S, then, for $z \in S$,

$$f(z) = \frac{1}{2\pi i} \int_{\partial S} \frac{f(\zeta)}{\zeta - z}\, d\zeta,$$

where ∂S is parameterized in the counter-clockwise direction.

Proof. Fix $z \in S$, $\zeta \in \partial S$, $\frac{f(\zeta) - f(z)}{\zeta - z} = \int_0^1 f'(z + t(\zeta - z))\, dt.$

So,

$$\frac{1}{2\pi i} \int_{\partial S} \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta = \lim_{\varepsilon \to 0} \int_\varepsilon^1 \int_{\partial S} \frac{df}{d\zeta}(z + t(\zeta - z))\, d\zeta\, \frac{dt}{t}$$

$$= 0.$$

From thm 1, $f(z) = (2\pi i)^{-1} \int_{\partial S} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$ $\square$

Corollary 3. If f is analytic in a neighborhood of the closure of $\overline{S}$ of an open square S, then

$$\int_{\partial S} f(\zeta)\, d\zeta = 0.$$

Proof.

Define $g(\zeta) = f(\zeta)(\zeta - z)$, $\zeta \in \overline{S}$.

So, g is analytic in $\overline{S}$.

Apply Corollary 2:

$$0 = g(z) = \frac{1}{2\pi} \int_{\partial S} \frac{f(\zeta)(\zeta - z)}{\zeta - z}\, d\zeta = \frac{1}{2\pi} \int_{\partial S} f(\zeta)\, d\zeta. \quad \square$$

Theorem 4. If f is analytic on a compact set K, and if $\varepsilon > 0$, then there is a rational function r so that

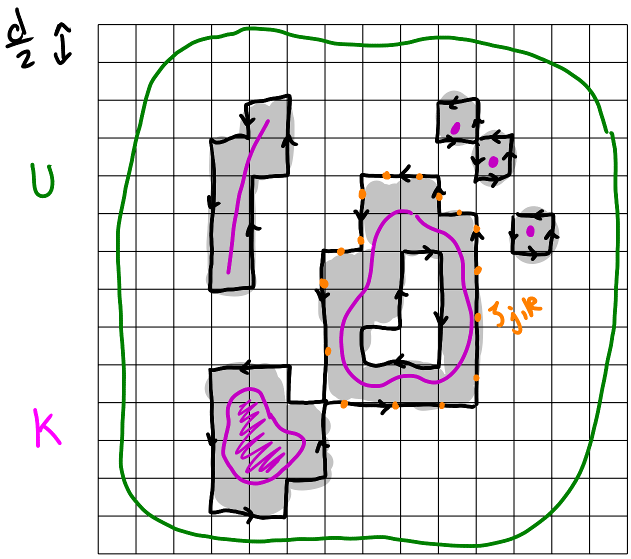
$$\sup_{z \in K} |f(z) - r(z)| < \varepsilon.$$

Proof. Let $U \supseteq K$ be bounded open s.t. f is analytic on U.

Let

$$d := \operatorname{dist}(\partial U, K) := \min\{|z-w| : z \in \partial U, w \in K\}.$$

Construct a grid of closed squares of side length $d/2$



1) Shade the squares intersecting K.

2) Each closed shaded squares $\subseteq U$ (because $\operatorname{diam} = \frac{d}{\sqrt{2}}$).

3) $\{S_k\}$ be the coll. of shaded closed squares

4) $\Gamma := \partial(\cup_j S_j)$, ∂S_j param. counter-clock.

Then Γ is a cycle and $\Gamma \cap K = \emptyset$.

Fix $z \in \operatorname{int}(S_{j_0})$, some j_0 $\Rightarrow$ $\zeta \longmapsto \frac{f(\zeta)}{\zeta - z}$ is analytic in S_j, $\forall j \neq j_0$. From Thm 2 and Cor. 3

$$f(z) = \frac{1}{2\pi i} \int_{\partial S_{j_0}} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

This equality holds $\forall z \in \bigcup S_j \setminus \Gamma$ (continuity).

Fix $z_0 \in K$ and write $\Gamma = \sum_{j=1}^{N} \Gamma_j$, where Γ_j are cont. diff. closed curves. Then

$$f(z_0) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{\zeta - z_0}\, d\zeta = \sum_{j=1}^{N} \frac{1}{2\pi i} \int_{\Gamma_j} \frac{f(\zeta)}{\zeta - z_0}\, d\zeta .$$

Let $0 = t_{j,0} < t_{j,1} < \dots < t_{j,m_j} = 1$ & $\zeta_{j,k} := \zeta(t_{j,k}) \in \Gamma_j$ for $j = 1, \dots, N$, $m_j \in \mathbb{N}$ and $k = 1, \dots, m_j$.

Now, use a Rieman Sum with m_j big enough so that, given $\varepsilon > 0$,

$$(*) \quad \Bigg| f(z_0) - \underbrace{\sum_{j=1}^{N} \sum_{k=0}^{m_j} \frac{f(\zeta_{j,k})\,(\zeta_{j,k+1} - \zeta_{j,k})}{(\zeta_{j,k} - z_0)\, 2\pi i}}_{r_{\mathcal{P}}(z_0).\ (\text{some partition } \mathcal{P})} \Bigg| < \varepsilon$$

From Rie. Int. Theory, $(*)$ holds for any refinements $\mathcal{P}_1$ of the partition $\mathcal{P}_j = \{\zeta_{j,k}\}_{k=1}^{m_j}$, $j = 1, \dots, N$.

<u>Claim</u>: Moreover $(*)$ holds for any z in a small disk centered at z_0 and any refinements of the partition if ε is replaced by 4ε.

1) Fix j. Consider a segment $I = [a, b]$ in Γ_j.

Fix $\varepsilon > 0$. Since $z_0 \notin \Gamma$, there is a $\delta > 0$ s.t.

$$|\zeta - \eta| < \delta, \ \eta, \zeta \in I \Rightarrow \left| \frac{f(\zeta)}{\zeta - z_0} - \frac{f(\eta)}{\eta - z_0} \right| < \varepsilon.$$

Take a partition $\{I_\ell\}$ of I with $I_\ell = [\mu_{\ell-1}, \mu_\ell]$ s.t. $I = \bigcup_{\ell=1}^{m} I_\ell$ and $|I_\ell| < \delta$. Then, for any $\zeta \in I_\ell$,

$$\left| \frac{f(\zeta)}{\zeta - z_0} - \frac{f(\mu_\ell)}{\mu_\ell - \zeta} \right| < \varepsilon.$$

If $[\eta_{p-1}, \eta_p] \subseteq [\mu_{\ell-1}, \mu_\ell]$ is a refinement, then

$$\zeta \in [\eta_{p-1}, \eta_p] \Rightarrow \left| \frac{f(\zeta)}{\zeta - z_0} - \frac{f(\eta_p)}{\eta_p - z_0} \right| < 2\varepsilon.$$

3) Since $\mathrm{dist}(z_0, \Gamma) > 0$, then for $|z - z_0| \leq \tilde{\delta} < \frac{\mathrm{dist}(z_0, \Gamma)}{2}$

$$\left| \frac{f(\zeta)}{\zeta - z} - \frac{f(\zeta)}{\zeta - z_0} \right| < \varepsilon, \quad \forall \zeta \in I.$$

If $\zeta \in [\eta_{p-1}, \eta_p] \subseteq [\mu_{\ell-1}, \mu_\ell]$ (a refinement):

$$\left| \frac{f(\zeta)}{\zeta - z} - \frac{f(\eta_p)}{\eta_p - z} \right| \leq \left| \frac{f(\zeta)}{\zeta - z} - \frac{f(\zeta)}{\zeta - z_0} \right| + \left| \frac{f(\zeta)}{\zeta - z_0} - \frac{f(\eta_p)}{\eta_p - z} \right|$$

$$< \varepsilon + \left| \frac{f(\zeta)}{\zeta - z_0} - \frac{f(\eta_p)}{\eta_p - z_0} \right| + \left| \frac{f(\eta_p)}{\eta_p - z_0} - \frac{f(\eta_p)}{\eta_p - z} \right|$$

$$< \varepsilon + 2\varepsilon + \varepsilon = 4\varepsilon . \text{ (:)}$$

4) Now,

$$\left| \int_I \frac{f(\zeta)}{\zeta - z} \, d\zeta - \sum_{p=1}^{M} \frac{f(\eta_p)}{\eta_p - z} (\eta_p - \eta_{p-1}) \right|$$

$$\leq \sum_{p=1}^{M} \int_{[\eta_{p-1}, \eta_p]} \left| \frac{f(\zeta)}{\zeta - z} - \frac{f(\eta_p)}{\eta_p - z} \right| d\zeta \leq 4\varepsilon \quad \left(\varepsilon \leftarrow \frac{\varepsilon}{|I|} \right)$$

5) Since Γ_j is a sum of segment I:

$$\left| \int_{\Gamma_j} \frac{f(\zeta)}{\zeta - z} \, d\zeta - \sum_{\substack{I \subseteq \Gamma_j \\ I \text{ segment}}} \sum_{p=1}^{M} \frac{f(\eta_p)}{\eta_{p_1} - z} (\eta_p - \eta_{p-1}) \right| < 4\varepsilon$$

if $\varepsilon \leftarrow \dfrac{\varepsilon}{|\Gamma_j|}$ and $|z - z_0| < \tilde{\delta}$.

6) Therefore, from 1) - 5), $(*)$ remains true for any z s.t. $|z - z_0| < \tilde{\delta}$ and any refinement of the partition $\{t_{j,k}\}_{k=1}^{m_j}$, $j = 1, \dots, N$.

This means $$|f(z) - r_p(z)| < 4\varepsilon$$

for any refinement $\mathcal{P}$ and $|z - z_0| < \tilde{\delta}$. $\triangle$

Cover $K \subseteq \bigcup_{\ell=1}^{J} B_\ell$, where $B_\ell = \{ z : |z - z_\ell| < \tilde{\delta}_\ell \}$.

On each B_ℓ, there is a refinement $\mathcal{P}_\ell$ such that

$$\left| f(z) - r_{\mathcal{P}_\ell}(z) \right| < 4\varepsilon, \quad \forall z \in B_\ell .$$

Now, consider $\mathcal{P}$, the union of all refinements $\mathcal{P}_\ell$. $\mathcal{P}$ is a refinement of each $\mathcal{P}_\ell$ and

$$\left| f(z) - r_{\mathcal{P}}(z) \right| < 4\varepsilon \quad \forall z \in \bigcup_{\ell=1}^{J} B_\ell . \quad \square$$

Second Version With Control On the Poles

Definition 5. Let $r(z) = p(z)/q(z)$ be a rational function where p and q are two polynomials with no common zeros. The zeros of q are called the **poles** of the rational function r.

Note:

- If b is a pole of a rational function r, then $|r(z)| \to \infty$ as $z \to b$.

connected & open

Lemma 6. Suppose that U is a region and suppose $b \in U$. Then a rational function with poles in U can be uniformly approximated on $\mathbb{C}\backslash U$ by a rational function with poles only at b.

Proof.

Corollary 7. Suppose U is a region and suppose $\{z : |z| > R\} \subset U$ for some $R < \infty$. Then a rational function with poles only in U can be uniformly approximated on $\mathbb{C} \backslash U$ by a polynomial.

Proof.

Components

DEFINITION 8. Let U be an open set.

a) A polygonal curve in U is a curve consisting of a finite union of line segments.

b) For $a, b \in U$, define $a \sim_U b$ if and only if there is a polynomial curve contained in U with edges parallel to the axis and joining a to b.

THEOREM 9. Let $U \subset \mathbb{C}$ be an open set.

a) Show that the equivalence classes of $\sim_U$ are closed and open (relative to U) and connected.

b) Show that there are countably many equivalent classes.

Note:

- The equivalence classes are called the **components** of U. They are the maximal connected subsets of U.

Closed Components

DEFINITION 10. Suppose $K \subset \mathbb{C}$ is a compact set.

a) For $a, b \in K$, define $a \sim_K b$ if and only if there is a connected subset of K containing a and b.

THEOREM 11. Let $K \subset \mathbb{C}$ be a compact set.

a) Show that the equivalence classes of $\sim_K$ are connected and closed.

b) Show that there might be infinitely many equivalence classes.

Note:

- The equivalence classes are called the **(closed) components** of K.

THEOREM 12. [Runge] Suppose K is a compact set. Choose one point a_n in each bounded component of U_n in $\mathbb{C}\backslash K$. If f is analytic on K and $\varepsilon > 0$, then we can find a rational function r with poles only in the set $\{a_n\}$ such that

$$\sup_{z \in K} |f(z) - r(z)| < \varepsilon.$$

If $\mathbb{C}\backslash K$ has no bounded components, then we may take r to be a polynomial.

Proof.

Third Version of Runge's Theorem: Poles On The Boundary

Theorem 13. If f is analytic on an open set $\Omega \neq \mathbb{C}$, then there is a sequence of rational functions r_n with poles in $\partial\Omega$ so that r_n converges to f uniformly on compact subsets of Ω.

Proof.

Note:

- The improvement of Theorem 9 over Theorem 4 is that the poles of r_n are outside of Ω, not just outside the compact set K on which r_n is close to f.