Chapter 1: Functions and Limits
Week 1

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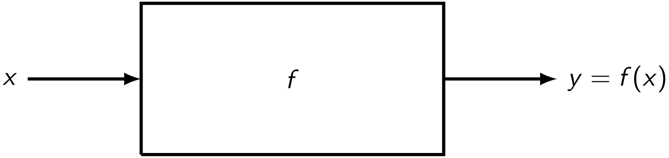
Calculus I (MATH-241 01/02)

University of Hawai'i
Fall 2021

Upcoming this week

1. 1.1 Four ways to represent a function
2. 1.2 Mathematical Models
3. 1.3 New Functions from Old Functions



- x is the independant variable.
- y is the dependant variable.
- all possible inputs x is called the domain. Denoted by $\operatorname{dom} f$.
- all possible outputs $f(x)$ is called the range. Denoted by $\operatorname{rg} f$.

Définition 1

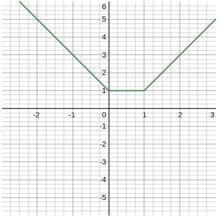
A <u>function</u> is a rule f that assigns to each element $x \in D$, a unique element $f(x) \in E$. We denote this by $f : D \to E$.

There are multiple ways to represent a function :

- verbally : each pineapple costs \$2.
- table of numbers :

Altitude	Temperature
100	70
200	50
300	25

- its graph :



- formula : $f(x) = x^2$.

Exemple 2

Five examples.

Remarks. Two tricks to determine the domain of a function :

- look at possible division by zero.
- look at functions such as $\sqrt{\cdot}$, $\sqrt[4]{\cdot}$, or $\sqrt[n]{\cdot}$ when n is even.

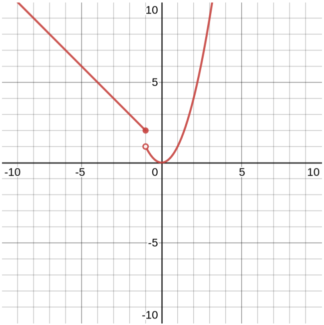
Définition 3

A piece wise function is a function defined by different formulas on different parts of their domain.

Exemple 4

We define

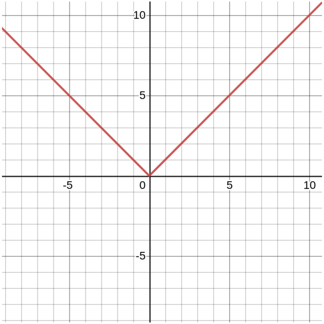
$$f(x) = \begin{cases} 1 - x & x \leq -1 \\ x^2 & x > -1. \end{cases}$$



Exemple 5 (Absolute value)

We define

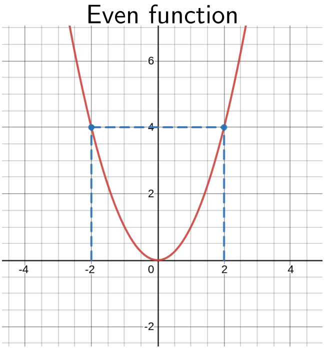
$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0. \end{cases}$$

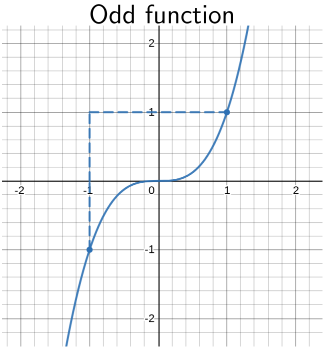


Définition 6

A function $f : D \to E$ is

- even if $f(-x) = f(x)\ \forall x \in D$.
- odd if $f(-x) = -f(x)\ \forall x \in D$.

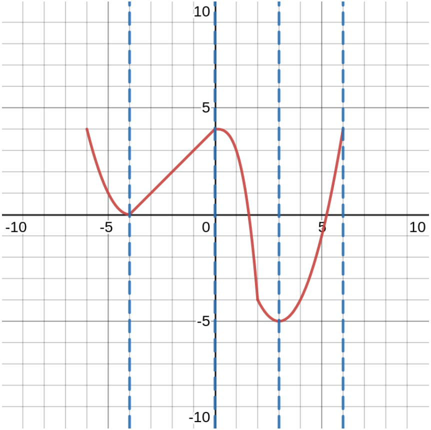




Définition 7

A function f is

- increasing on an interval $[a, b]$ if $x_1 < x_2$ implies that $f(x_1) < f(x_2)$.
- decreasing on an interval $[a, b]$ if $x_1 < x_2$ implies that $f(x_1) > f(x_2)$.



Exemple 8

Let $f(x) = x + 1$ and $g(x) = \frac{x^2-1}{x-1}$.

Définition 9

Two functions f and g are equal if

- $\mathrm{dom}\, f = \mathrm{dom}\, g$ and ;
- $f(x) = g(x)\ \forall x \in \mathrm{dom}\, f$.

Exercises : 1-3, 7-10, 23, 25, 27-32, 38, 41, 47, 51, 54, 57, 64, 70, 71-74, 79.

Functions are essential to built a mathematical model.

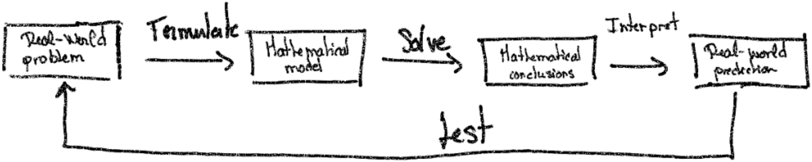


FIGURE – Modeling process

Définition 10

A <u>linear function</u> f has the following form $f(x) = mx + b$ where

- m : slope of the function.
- b : y-intercept.

Remark : The domain of a linear function f is $\operatorname{dom} f := \mathbb{R}$ and $\operatorname{rg} f := \mathbb{R}$.

Exemple 11

Build a linear model based on the following table :

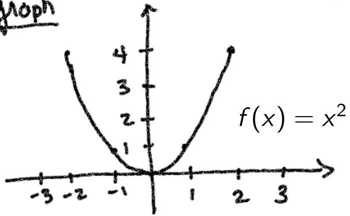
x	$f(x)$
1	1
2	4
3	7
4	10

Remark : When we say that a quantity (expression) T is linear, this means that $T(x) = mx + b$ for some slope m and y-intercept b (x is the independant variable).

Définition 12

A quadratic function f has the following form $f(x) = ax^2 + bx + c$ where a, b, $c \in \mathbb{R}$. The number a is called the leading coefficient.

graph



quadratic formula.

$ax^2 + bx + c = 0$ iff.

$b^2 - 4ac \geq 0$ and

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Parabola summit

$h = -b/2a$

$$k = \frac{4ac - b^2}{4a}$$

Canonical form.

$$y = (x-h)^2 + k.$$

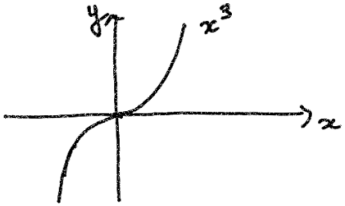
Définition 13

A polynomial is a function P defined as $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ where

- $a_n \neq 0$ is the leading coefficient ;
- n is the degree of P (the highest power of x) ;

Exemple 14

Example. $y = x^3$ polynomial of degree 3.



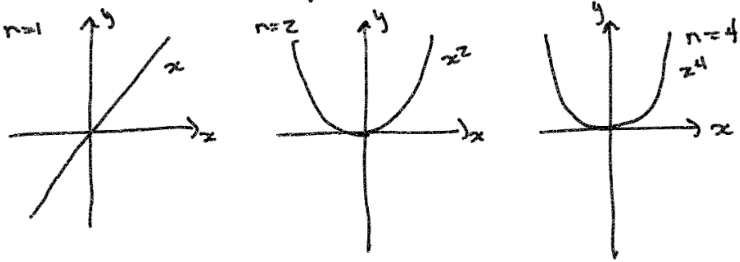
Définition 15

A power function has the form $f(x) = x^a$ where a is a real number (fixed).

Exemple 16

(i) $a = n$, $n \geq 1$ integer.

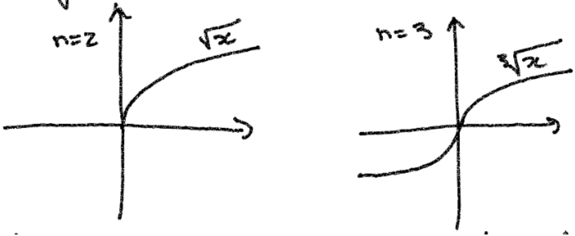
$f(x) = x^n$, called the monomials. They are the building blocks of the polynomials.



Exemple 17

(ii) $a = 1/n$, $n \geq 1$ integer.

$f(x) = x^{1/n} = \sqrt[n]{x}$ root function.



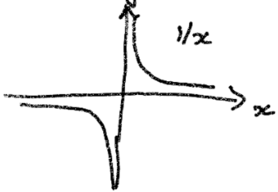
For the root function $f(x) = \sqrt[n]{x}$,

- dom $f = [0, \infty)$ and rg $f = [0, \infty)$ if n is even.
- dom $f = \mathbb{R}$ and rg $f = \mathbb{R}$ if n is odd.

Exemple 18

(iii) $a=-1$. the reciprocal function $f(x) = 1/x$.

Not define at $x=0$, so $0 \notin \operatorname{dom} f$.



For the reciprocal function $f(x) = 1/x$, $\operatorname{dom} f = \mathbb{R}\backslash\{0\}$ and $\operatorname{rg} f = \mathbb{R}$.

Some basic rules :

- $x^a x^b = x^{a+b}$;
- $(x^a)^b = x^{ab}$;
- $\frac{x^a}{x^b} = x^{a-b}$.

Exemple 19

The reciprocal function has a connection with chemistry and physics. From the Boyle's Law of gazes, when the temperature of a gaz is constant, then its volume is inversely proportional to its pressure P :

$$V(P) = \frac{C}{P}$$

for some constant C.

Définition 20

A rational function f is a quotient of two polynomials :

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polymials.

Remark : For a rational function f, we have

$$\operatorname{dom} f = \{x \in \mathbb{R} \,:\, Q(x) \neq 0\}.$$

Exemple 21

Let $f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4}$. Find the domain of f.

Définition 22

An algebraic function f is a function that can be expressed only in term of the basic operations :

- summation ;
- substraction ;
- multiplication ;
- division ;
- extracting roots (i.e. taking $\sqrt[n]{\cdot}$).

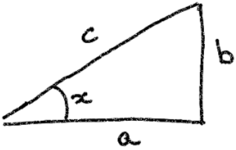
Exemple 23

The function

$$f(x) = \frac{\sqrt{x+1}}{x^2+1} + \frac{x^2(2x-4)}{x-1}$$

is an algebraic function.
The function $\cos(x)$ is not an algebraic function !

Définition 24



$$\cos x = \frac{a}{c} \qquad \sec x = \frac{1}{\cos x}$$

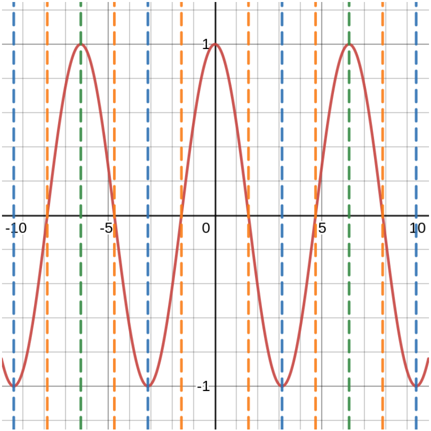
$$\sin x = \frac{b}{c} \qquad \operatorname{cosec} x = \frac{1}{\sin x}$$

$$\tan x = \frac{\sin x}{\cos x} \qquad \operatorname{cotan} x = \frac{1}{\tan x}.$$

Remarks :

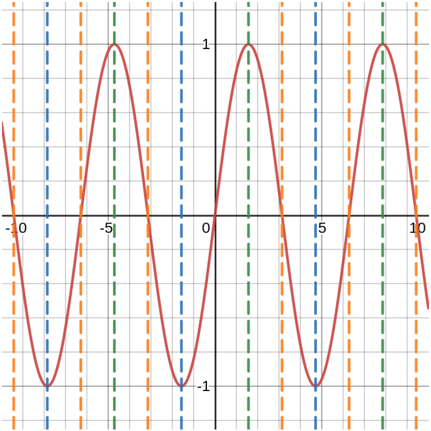
- $\mathrm{dom}(\cos) = \mathrm{dom}(sin) = \mathbb{R}$;
- $\mathrm{rg}(\cos) = \mathrm{rg}(sin) = [-1, 1]$;
- $\mathrm{dom}(\tan) = \{x \,:\, \cos(x) \neq 0\}$;
- $\mathrm{rg}(\tan) = \mathbb{R}$.

Cosine function.



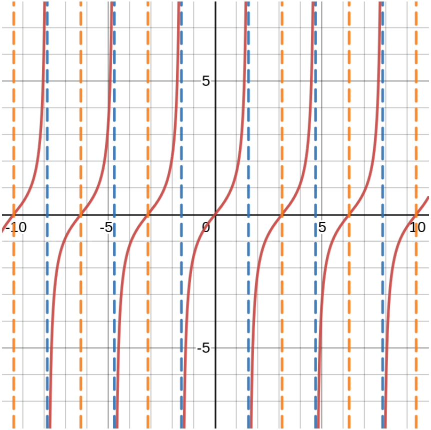
- $\cos x = 0$ if $x = \frac{2n+1}{2}\pi$, $n = \ldots, -2, -1, 0, 1, 2, \ldots$ ($x = -\pi/2$ or $x = 3\pi/2$, etc.)
- $\cos x = 1$ if $x = 2n\pi$, $n = \ldots, -2, -1, 0, 1, 2, \ldots$ ($x = 2\pi$ or $x = 0$, etc.).
- $\cos(x) = -1$ if $x = (2n+1)\pi$, $n = \ldots, -2, -1, 0, 1, 2, \ldots$ ($x = \pi$ or $x = -\pi$, etc.).

Sine function.



- $\sin x = 0$ if $x = n\pi\pi$, $n = \ldots, -2, -1, 0, 1, 2, \ldots$ ($x = -\pi$ or $x = \pi$, etc.)
- $\sin x = 1$ if $x = \frac{4n+1}{2}\pi$, $n = \ldots, -2, -1, 0, 1, 2, \ldots$ ($x = \pi/2$ or $x = -3\pi/2$, etc.).
- $\sin(x) = -1$ if $x = \frac{4n+3}{2}\pi$, $n = \ldots, -2, -1, 0, 1, 2, \ldots$ ($x = 3\pi/2$ or $x = -\pi/2$, etc.).

Tangent function.



- $\tan x = 0$ if $\sin x = 0$ if $x = n\pi$.
- Vertical asymtotes at $x = \frac{2n+1}{2}\pi$, $n = \ldots, -2, -1, 0, 1, 2, \ldots$ ($x = -\pi/2$ or $x = \pi/2$, etc.)

Properties :

- $\sin(x + 2\pi) = \sin(x)$;
- $\cos(x + 2\pi) = \cos(x)$;
- $\tan(x + \pi) = \tan(x)$.

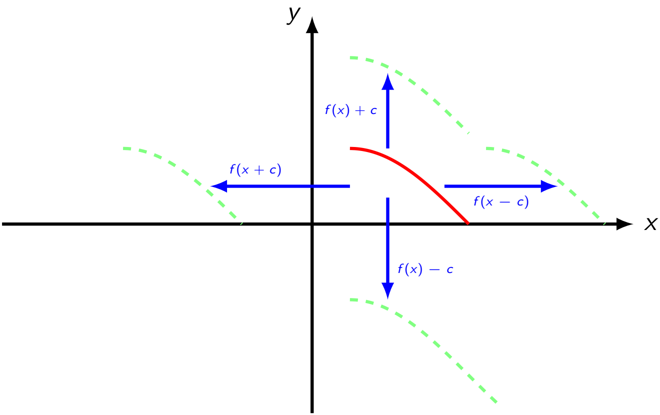
More properties on the trigonometric sheet (see on the course website).

Exercises : 1 (not a) and b)), 3, 5, 6, 8, 12, 16, 20.

Définition 25

If f is a function, we can shift it in 4 ways : for $c > 0$

- $f(x) + c$ (upward shift) ;
- $f(x) - c$ (downward shift) ;
- $f(x - c)$ (shift to the right) ;
- $f(x + c)$ (shift to the left).



Définition 26

If f is a function, we can shrink/stretch it in 4 ways : for $c > 1$

- $cf(x)$ (stretch vertically by factor c) ;
- $(1/c)f(x)$ (shrink vertically by factor c) ;
- $f(cx)$ (shrink horizontally by factor c) ;
- $f(x/c)$ (stretch horizontally by factor c).

Exemple 27

Sketch the graph of the functions

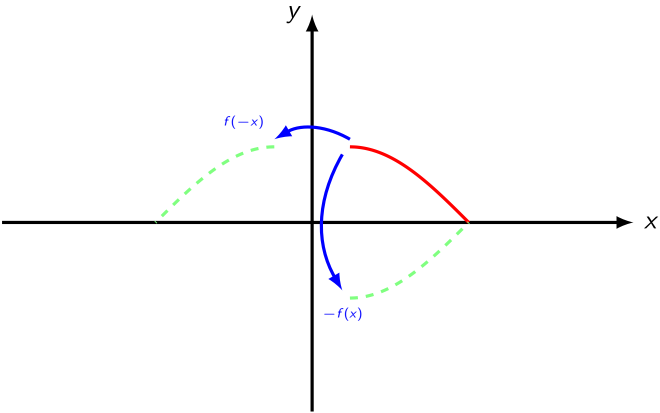
a) $y = \sin 2x$.

b) $y = |x^2 - 1|$.

Définition 28

If f is a function, we can reflect it in 2 ways :

- $-f(x)$ (a reflexion w.r.t. the x-axis) ;
- $f(-x)$ (a reflexion w.r.t. the y-axis).



Définition 29

The composition of a function g with a function f is defined as

$$(g \circ f)(x) := g(f(x)).$$

Remark : g must be defined on the range of f !

Exemple 30

a) Compute $g \circ f$ and $f \circ g$ if $g(x) = x^2$ and $f(x) = x - 1$

b) Let $f(x) = \sqrt{x}$ and $g(x)\sqrt{2 - x}$. Find the expression of $g \circ f$ and its domain.

Exercises : 1-3, 6, 9-18, 31-33, 35, 43, 44, 52.