MATH-331 Introduction to Real Analysis — Pierre-Olivier Parisé
Homework 01 — Fall 2021

Last name: ______________________
First name: ______________________
Section: ______________________

Question:	1	2	3	4	5	6	7	8	Total
Points:	15	15	15	15	10	10	10	10	100
Score:									

Instructions: You must answer all the questions below and upload your solutions to Gradescope (go to `www.gradescope.com` with the Entry code GEK6Y4). If you decide to not use LaTeX to hand out your solutions, please be sure that after you scan your copy, it is clear and readable. No late homework will be accepted.

Question 1 **(15 points)**

Find the domain of the following functions.

(a) (5 points) $f(x) = x^2 + 1$.

(b) (5 points) $f(x) = \sqrt[4]{x-2}$.

(c) (5 points) $f(x) = \frac{1}{x^2-1}$.

Solution:

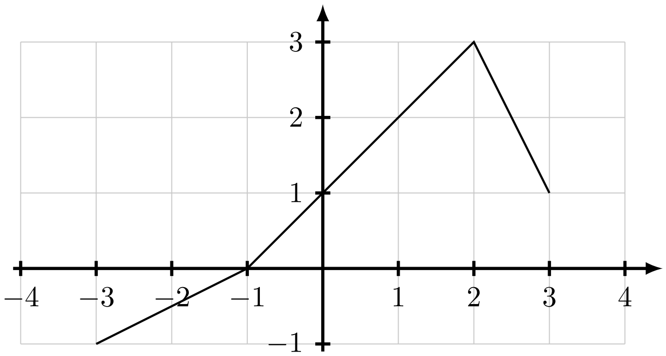
a) The domain is all of $\mathbb{R}$ because there is no restriction.

b) Because the root is an even one, the part inside the root functin must be ≥ 0. So, we must have that $x - 2 \geq 0$ which is equivalent to $x \geq 2$. Thus $\operatorname{dom} f = [2, \infty)$.

c) There is a division by zero if $x^2 - 1 = 0$. This happens when $x^2 = 1$, so when $x = \pm 1$. Thus $\operatorname{dom} f = \mathbb{R} \backslash \{1, -1\}$.

Question 2 **(15 points)**

The function $f(x)$ defined is defined by the following graph:



(a) (3 points) What is $f(-3)$, $f(-1)$ and $f(2)$.

(b) (2 points) Is $f(4)$ and $f(-4)$ defined?

(c) (2 points) Find the value(s) of x for which $f(x) = 1$.

(d) (2 points) Find the domain of the function.

(e) (2 points) Find the range of the function.

(f) (2 points) On what interval is f increasing.

(g) (2 points) Estimate the value of $f(-2)$.

Solution:

a) From the graph, we have $f(-3) = -1$, $f(-1) = 0$, and $f(2) = 3$.

b) The values of $f(4)$ and $f(-4)$ are not defined.

c) We look at the y axis and we see that there are two values of x such that $f(x) = 1$: $x = 0$ and $x = 3$.

d) The domain is $[-3, 3]$.

e) The range is $[-1, 3]$.

f) The function f is increasing on the interval $[-3, 2]$.

g) By looking at the graph, the approximate value of $f(-2)$ is -0.5.

Question 3 **(15 points)**

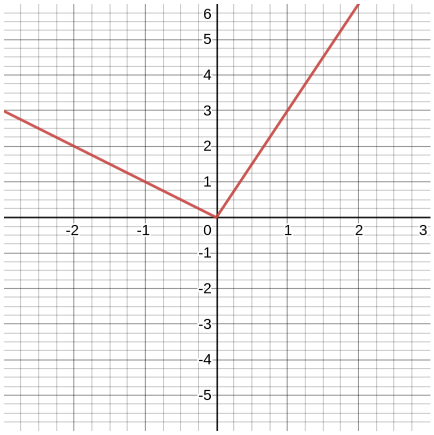
Sketch the graph of the following functions from $x = -3$ to $x = 3$.

(a) (5 points) $f(x) = x + |2x|$.

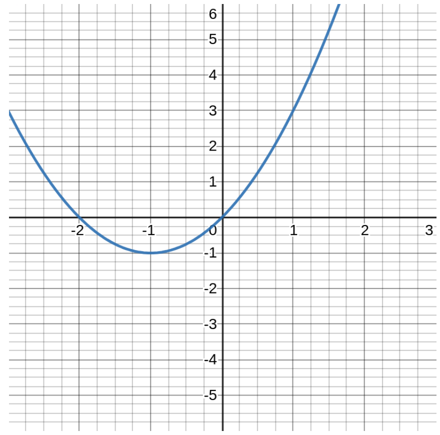
(b) (5 points) $f(x) = 2x + x^2$.

(c) (5 points) $f(x) = |x| + |x + 1|$

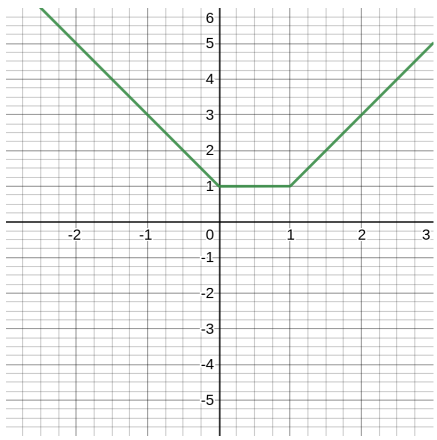
Solution: You should obtained the following graphs.



a)



b)



c)

Question 4 **(15 points)**

For each of the following function, simplify the difference quotient.

(a) (5 points) The function $f(x) = 4 + 3x - x^2$ and the difference quotient $\frac{f(3+h)-f(3)}{h}$.

(b) (5 points) The function $f(x) = \frac{x+5}{x}$ and the difference quotient $\frac{f(x)-f(1)}{x-1}$.

(c) (5 points) The function $f(x) = \frac{x^2-x}{x-1}$ and the difference quotient $\frac{f(w)-f(0)}{w}$.

Solution:

a) Replacing x by $3+h$ and x by 3 in the formula for f, we can simplify the quotient:

$$\begin{aligned}\frac{f(3+h)-f(3)}{h} &= \frac{4+3(3+h)-(3+h)^2-(4)}{h}\\ &= \frac{9+3h-9-6h-h^2}{h}\\ &= \frac{-3h-h^2}{h}\\ &= -3-h\end{aligned}$$

if $h \neq 0$.

b) Replacing x by 1 the formula of f, we can simplify the quotient:

$$\begin{aligned}\frac{f(x)-f(1)}{x-1} &= \frac{\frac{x+5}{x}-6}{x-1}\\ &= \frac{x+5-6x}{x(x-1)}\\ &= \frac{-5(x-1)}{x(x-1)}\\ &= \frac{-5}{x}\end{aligned}$$

if $x \neq 1$.

c) Replacing x by w and x by 0 in the formula for f, we can simplify the quotient:

$$\begin{aligned}\frac{f(w)-f(0)}{w} &= \frac{w^2-w-0}{w}\\ &= \frac{w(w-1)}{w}\\ &= w-1\end{aligned}$$

if $w \neq 1$.

Question 5 **(10 points)**

A Basketball with radius r has a volume of $V(r) = \frac{4}{3}\pi r^3$. Find a function that represents the amount of air required to inflate the ball from a radius of r inches to a radius of $r+1$ inches.

Solution: The basketball has a volume of $V(r)$ when it has a radius r and $V(r+1)$ when it has a radius of $r+1$. So, the amount of air to inflate the ball from $V(r)$ to $V(r+1)$ is $V(r+1)-V(r)$ (in^3). Thus, if we denote by $A(r)$ the amount of air, this function is

$$A(r) := V(r+1)-V(r) = \frac{4\pi}{3}(3r^2+3r+1).$$

Question 6 (10 points)

Express the following functions in the form $f \circ g$. For b), give the domain of the function G.

(a) (5 points) $u(t) = \frac{\tan t}{1+\tan t}$.

(b) (5 points) $G(v) = \sqrt[5]{\frac{x+1}{x}}$.

Solution:

a) Take $f(x) = \frac{x}{1+x}$ and $g(x) = \tan x$. Then we see that $u(t) = f \circ g(t)$.

b) Take $f(x) = \sqrt[5]{x}$ and $g(x) = \frac{x+1}{x}$. Then we see that $G(v) = f \circ g(v)$.

The domain of G is the numbers $x \in \mathbb{R}$ for which $x \neq 0$. Indeed, since the root is given by an odd integer, there is no restriction on the number to put in the radical. However, we must be careful with the division by 0 on the denominator of the function $g(x) = \frac{x+1}{x}$.

Question 7 (10 points)

Use the graph of the function $f(x)$ illustrated in figure 1 to evaluate the limits.

(a) (2 points) $\lim_{x\to -3} f(x)$.

(b) (2 points) $\lim_{x\to -2} f(x)$.

(c) (2 points) $\lim_{x\to 0} f(x)$.

(d) (2 points) $\lim_{x\to 3^-} f(x)$.

(e) (2 points) $\lim_{x\to 3^+} f(x)$.

Solution:

a) The limit is ∞.

b) The limit is $-\infty$.

c) The limit is 0.

d) The limit is 3.

e) The limit is 4.

Question 8 (10 points)

Evaluate the following limits using the Limit Laws.

(a) (5 points) $\lim\limits_{x\to 1}\left[\sqrt[3]{\frac{(2x-11)^2}{3}} + x^2\right]$.

(b) (5 points) $\lim\limits_{y\to -1} \frac{y^2+2y+1}{y+1}$.

Solution:

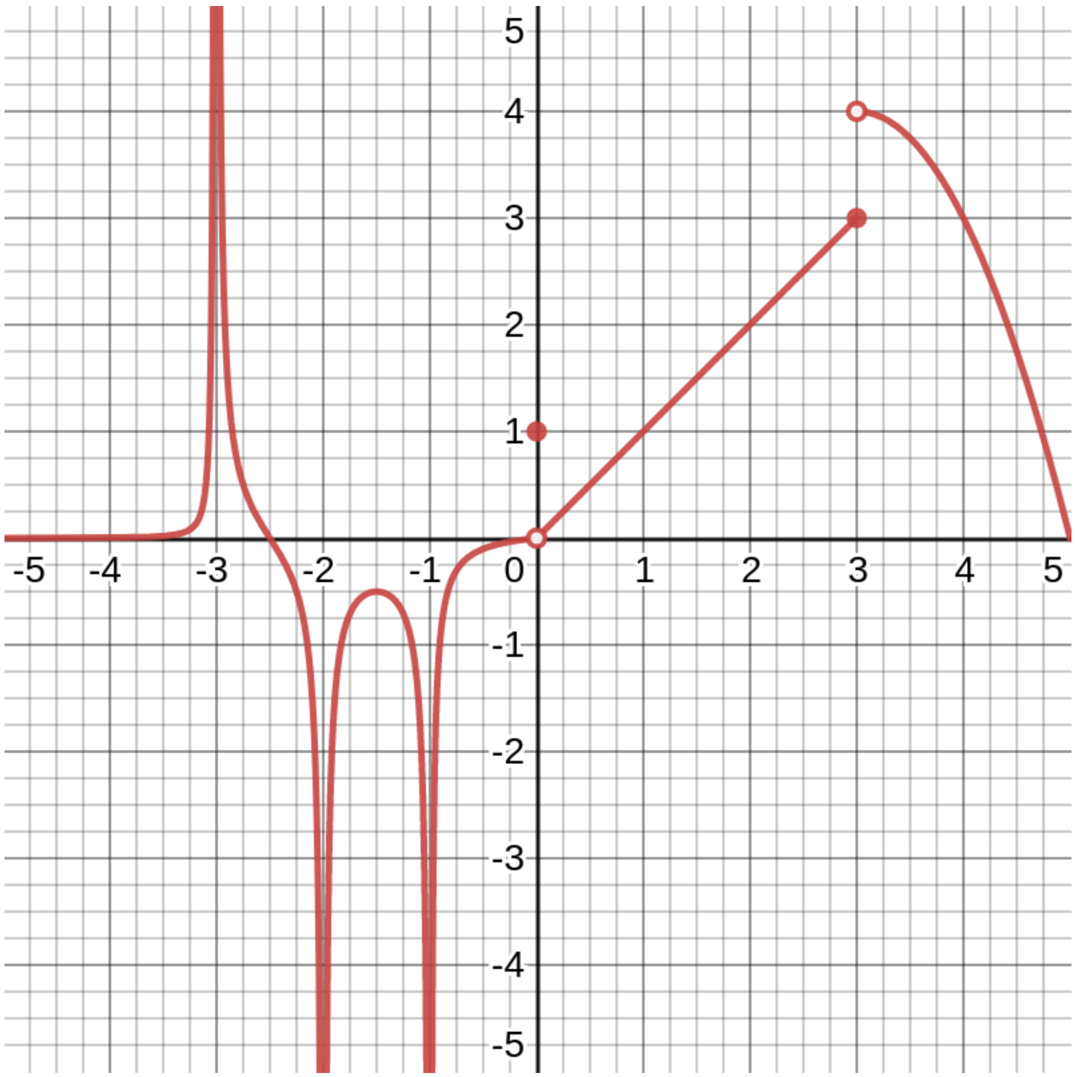


Figure 1: Graph of the function in question 7

a) The Limit laws used are written in the brackets on the right of the equation:

$$\begin{aligned}
\lim_{x\to 1}\left[\sqrt[3]{\frac{(2x-11)^2}{3}}+x^2\right] &= \lim_{x\to 1}\sqrt[3]{\frac{(2x-11)^2}{3}}+\lim_{x\to 1}x^2 \quad \left[\text{Basic law a)}\right]\\
&= \sqrt[3]{\lim_{x\to 1}\frac{(2x-11)^2}{3}}+1\cdot 1 \quad \left[\text{Root law and power law}\right]\\
&= \sqrt[3]{\frac{1}{3}\left(\lim_{x\to 1}(2x-11)\right)^2}+1 \quad \left[\text{Basic law b) and Power law}\right]\\
&= \sqrt[3]{\frac{1}{3}\left(2\lim_{x\to 1}x-\lim_{x\to 1}11)\right)^2}+1 \quad \left[\text{Basic laws a) and b)}\right]\\
&= \sqrt[3]{\frac{1}{3}\left(2-11\right)^2}+1 \quad \left[\text{Power law}\right]\\
&= 3+1\\
&= 4.
\end{aligned}$$

b) The Limit laws used are written in the brackets on the right of the equation:

$$\begin{aligned}
\lim_{y\to -1}\frac{y^2+2y+1}{y+1} &= \lim_{h\to -1}\frac{(y+1)^2}{y+1}\\
&= \lim_{y\to -1}(y+1)\\
&= \lim_{y\to -1}y+\lim_{y\to -1}1 \quad \left[\text{Basic law a)}\right]\\
&= -1+1 \quad \left[\text{Power law}\right]\\
&= 0.
\end{aligned}$$