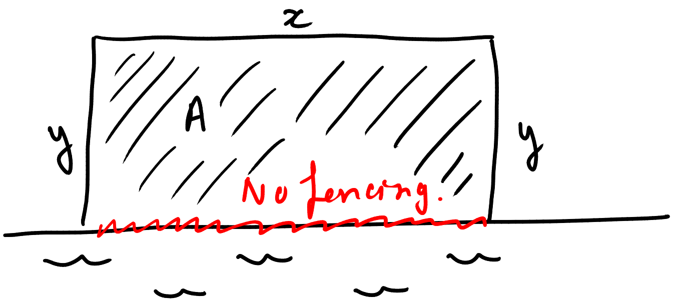
Chapter 3
Applications of Derivatives

3.7 Optimization Problems

EXAMPLE 1 A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

2. Draw a diagram.



3. Notation.

x: width of the field

y: higth of the field.

A: area of the field.

Goal: maximize A.

4. Rule/Equation.

$$A = xy \quad (\text{relation between } x \ \& \ y \ \& \ A).$$

5. Elimination of a variable.

→ Need 3 sides of the rectangle.

→ the total fencing is 2400 ft.

$$2400 = 2y + x$$

$$\Rightarrow \quad x = 2400 - 2y.$$

So, $A = (2400 - 2y)y = 2400y - 2y^2$.

6. Derivative.

$$A'(y) = 2400 - 4y = 0 \iff 2400 = 4y \iff 600 = y$$

$A''(y) = -4 < 0$ for any value of y.

$\Rightarrow$ by the 2^{nd} derivative test, $y = 600$ corresponds to an absolute maximum.

Answer.

$$\boxed{\begin{aligned} x &= 2400 - 2y = 2400 - 1200 = 1200 \text{ ft}. \\ y &= 600 \text{ ft} \\ A &= 720\,000 \text{ ft}^2 \end{aligned}}$$

Recall. c critical number of f &

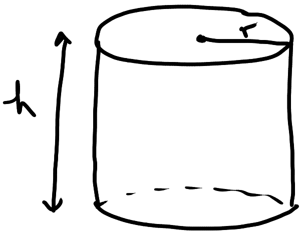
(a) $f''(x) < 0$ for any x, then c is an an abs. max.

(b) $f''(x) > 0$ for any x, then c is an abs. min.

EXAMPLE 2 A cylindrical can is to be made to hold 1 L ($\boxed{1000\ cm^3}$) of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

$\to$ minimize the area.

① Sketch.



Volume cylindre $= \pi r^2 h$

r: radius of the cylindre.

h: high of the cylindre.

A: surface area of the cylindre.

Goal: minimize A.

② Equations.

$$\text{Surface Area} = 2 \times A(\text{circle of radius } r) + 1 \times A(\text{rectangle } h \times 2\pi r)$$

$$\Rightarrow \quad A = 2\pi r^2 + 2\pi r h.$$

We have $V = 1000 \Rightarrow \pi r^2 h = 1000$

$\Rightarrow h = 1000/\pi r^2.$

So, $A(r) = 2\pi r^2 + \dfrac{2000}{r}$. $\to$ $r > 0$ $\leftarrow$ domain of A.

$\to 2000 r^{-1}$ $\quad -2000 r^{-2} = -\dfrac{2000}{r^2}$

③ Optimize!

$$A'(r) = 4\pi r - \frac{2000}{r^2} = 0 \Leftrightarrow 4\pi r = \frac{2000}{r^2}$$

$$\Leftrightarrow r^3 = 500/\pi$$

$$\Leftrightarrow r = \sqrt[3]{500/\pi} \approx 5.419$$

$\to$ If $r < \sqrt[3]{500/\pi}$, then $r^3 < 500/\pi$

$$\Leftrightarrow 4\pi r < \frac{2000}{r^2}$$

$$\Leftrightarrow 4\pi r - \frac{2000}{r^2} < 0 \Leftrightarrow A'(r) < 0.$$

$\to$ If $r > \sqrt[3]{500/\pi}$, then $4\pi r - \dfrac{2000}{r^2} > 0 \Leftrightarrow A'(r) > 0$

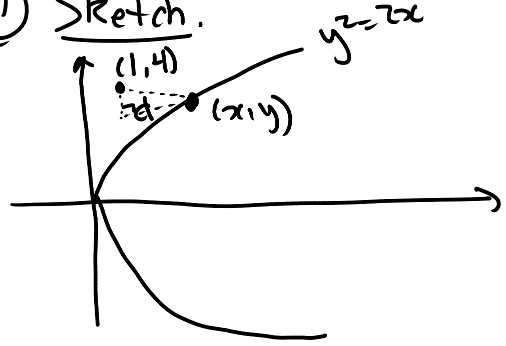
By the 1st derivative test, $r = \sqrt[3]{500/\pi}$ is an absolute mini.

Answer

$$\boxed{\begin{aligned} r &\cong 5.419\ cm \\ h &= \frac{1000}{\pi r^2} \cong 10.839\ cm. \end{aligned}}$$

EXAMPLE 3 Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$.

① Sketch.



(x,y) : point on the parabola $y^2=2x$ $(x \geq 0)$.

d: distance between $(1,4)$ & (x,y).

Goal: minimize d!

② Equations.

$$d = \sqrt{(x-1)^2 + (y-4)^2}$$

Key: (x,y) is on the parabola $y^2 = 2x$

$$\Rightarrow x = y^2/2$$

$$\Rightarrow d(y) = \sqrt{(y^2/2 - 1)^2 + (y-4)^2}$$

TRICK: $D = d^2 = (y^2/2 - 1)^2 + (y-4)^2$: optimize D.

③ Optimize!

$$D'(y) = 2(y^2/2 - 1)\cdot y + 2(y-4) = y^3 - 2y + 2y - 8$$
$$= y^3 - 8.$$

$$D'(y) = 0 \iff y^3 - 8 = 0 \iff y = \sqrt[3]{8} = 2.$$

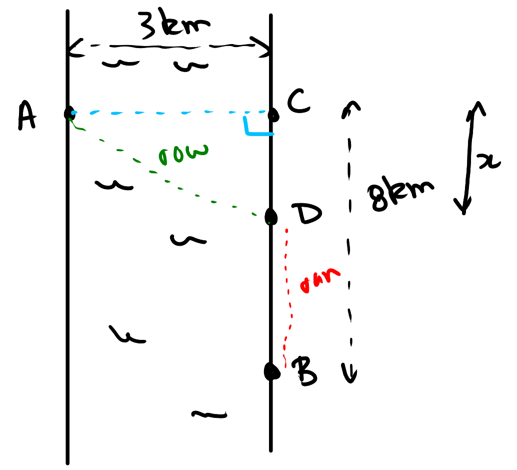
If $y < 2$, then $D'(y) < 0$
If $y > 2$, then $D'(y) > 0$
$\rightarrow$ 1st test $y=2$ is an abs. min.

Answer.

$$\boxed{\begin{aligned} x &= y^2/2 = 4/2 = 2 \\ y &= 2 \\ d &= \sqrt{5} \end{aligned}}$$

EXAMPLE 4 A man launches his boat from point A on a bank of a straight river, 3 km wide, and wants to reach point B, 8 km downstream on the opposite bank, as quickly as possible (see Figure 7). He could row his boat directly across the river to point C and then run to B, or he could row directly to B, or he could row to some point D between C and B and then run to B. If he can row 6 km/h and run 8 km/h, where should he land to reach B as soon as possible? (We assume that the speed of the water is negligible compared with the speed at which the man rows.)

①



row : 6 km/h.

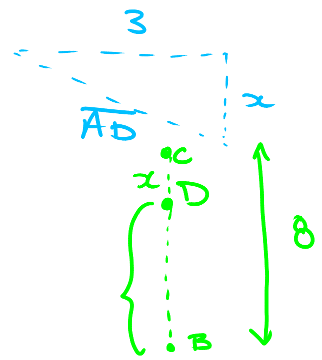
run : 8 km/h.

x : distance from C to D in km
$0 \le x \le 8$

t : time in hour that takes to reach B from A.

Goal : minimize t.

② Equations. $v = \dfrac{\text{distance}}{\Delta t}$



length of $\overline{AD}$ = $\sqrt{3^2 + x^2}$ km.

length of $\overline{DB}$ = $8 - x$ km

$$t = \frac{\text{length of } \overline{AD}}{\text{speed rowing}} + \frac{\text{length } \overline{DB}}{\text{speed running}}$$

$$= \frac{\sqrt{9+x^2}}{6} + \frac{8-x}{8}$$

$$\Rightarrow t(x) = \frac{\sqrt{9+x^2}}{6} + 1 - \frac{x}{8}$$

$\boxed{0 \le x \le 8}$

③ Optimize.

$$t'(x) = \frac{1}{\cancel{2}} \cdot \frac{1}{\sqrt{9+x^2}\,6} \cdot \cancel{2}x - \frac{1}{8}$$

$$= \frac{x}{6\sqrt{9+x^2}} - \frac{1}{8}.$$

$$t'(x) = 0 \iff \frac{x}{6\sqrt{9+x^2}} - \frac{1}{8} = 0$$
$$\iff \frac{4}{3}x = \sqrt{9+x^2}$$
$$\iff \frac{16}{9}x^2 = 9 + x^2$$
$$\iff \frac{7}{9}x^2 = 9$$
$$\iff x^2 = \frac{81}{7} \iff x = \pm\frac{9}{\sqrt{7}}$$

Discard $-\frac{9}{\sqrt{7}} \rightarrow x = \frac{9}{\sqrt{7}} \approx 3.401$ km.

Use the closed interval method:

$$\text{minimum} = \min\{t(0),\ t(9/\sqrt{7}),\ t(8)\}$$
$$= \min\{1.5,\ 1.33,\ 1.42\} = 1.33.$$

So, the distance he should land from C is $\boxed{x \approx 3.401 \text{ km}}$.

<u>Remark</u>. First derivative test for abs max & min.

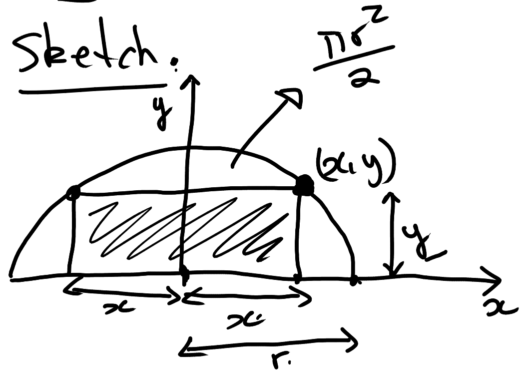
Suppose c is a critical number for f.

(a) If $f'(x) < 0$ for all $x < c$ (on the left of c) & $f'(x) > 0$ for all $x > c$ (on the right of c), then c corresponds to an absolute minimum.

(b) If $f'(x) > 0$ for all $x < c$ (on the left of c) & $f'(x) < 0$ for all $x > c$ (on the right of c), then c corresponds to an absolute maximum.

EXAMPLE 5 Find the area of the largest rectangle that can be inscribed in a semicircle of radius r.

① Sketch.



x: abscisse $\quad 0 \le x \le r$

y: y-value of (x,y) $\quad 0 \le y \le r$

(x,y): point on the circle

A: area of the rectangle.

Goal: maximize A.

② Equations.

$A = 2xy$

Eq. circle: $x^2 + y^2 = r^2$

$\Rightarrow y = \sqrt{r^2 - x^2}$

$$\Rightarrow A(x) = 2\underset{\substack{\uparrow \\ 1^{st}}}{x}\underset{\substack{\uparrow \\ 2^{nd}}}{\sqrt{r^2-x^2}} \quad \rightarrow \quad \sqrt{r^2 - \frac{r^2}{2}} = \frac{r}{\sqrt{2}}$$

③ Optimize.

$$A'(x) = 2\sqrt{r^2-x^2} + \cancel{2}x \frac{1}{\cancel{2}\sqrt{r^2-x^2}} \cdot (-2x)$$

$$= 2\sqrt{r^2-x^2} - \frac{2x^2}{\sqrt{r^2-x^2}}$$

$A'(x) \nexists$ for $x = \pm r$

$\quad \hookrightarrow x = r$

$$A'(x) = 0 \iff 2\sqrt{r^2-x^2} - \frac{2x^2}{\sqrt{r^2-x^2}} = 0$$

$$\iff 2\sqrt{r^2-x^2} = \frac{2x^2}{\sqrt{r^2-x^2}}$$

$$\iff 2(r^2-x^2) = 2x^2$$

$$\iff 2r^2 = 4x^2$$

$$\iff x = \sqrt{1/2}\, r = \frac{r}{\sqrt{2}}.$$

$$\text{max. Area} = \max\{A(0),\ A(r/\sqrt{2}),\ A(r)\}$$

$$= \max\{0,\ r^2,\ 0\} = \boxed{r^2}$$