M444 – Complex Analysis

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Chapter 4

Section 4.4: Laurent Series

Open and closed Annular Regions

Let $0 \leq R_1 < R_2 \leq \infty$ and $z_0 \in \mathbb{C}$.

1. The open annular region is the set

$$A_{R_1,R_2}(z_0) := \{z \, : \, R_1 < |z - z_0| < R_2\}.$$

2. The closed annular region is the set

$$\overline{A}_{R_1,R_2}(z_0) := \{z \, : \, R_1 \leq |z - z_0| \leq R_2\}.$$

(a) Open Annular Region **(b) Closed Annular Region**

Theorem 4.4.1

Suppose that f is analytic on the annular region $A_{R_1,R_2}(z_0)$ with $0 \leq R_1 < R_2 \leq \infty$. Then f has a unique **Laurent series representation**:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{a_{-n}}{(z - z_0)^n}, \quad R_1 < |z - z_0| < R_2$$

where

1. the series converges absolutely $\forall z \in A_{R_1,R_2}(z_0)$.
2. the series converges uniformly on every $\overline{A}_{\rho_1,\rho_2}(z_0)$ for $R_1 < \rho_1 < \rho_2 < R_2$.
3. the **Laurent coefficients** a_n, $n \in \mathbb{Z}$, are given by the following formula:
$$a_n = \frac{1}{2\pi i} \int_{C_R(z_0)} \frac{f(w)}{(w - z_0)^{n+1}} \, dw$$
where $R_1 < R < R_2$.

Let $R_1 < \rho_1 < \rho_2 < R_2$ and a $z \in \mathbb{C}$ such that $\rho_1 \leq |z - z_0| \leq \rho_2$. We will show that the Laurent series converges absolutely and uniformly on $\overline{A}_{\rho_1,\rho_2}(z_0)$ using the M-test.

Let r_1, r_2 be chosen so that $R_1 < r_1 < \rho_1 < \rho_2 < r_2 < R_2$.

Let $\rho > 0$ such that $\overline{B_\rho(z)} \subset A_{\rho_1,\rho_2}(z_0)$.

The function $w \mapsto \frac{f(w)}{w-z}$ is analytic on the region $A_{r_1,r_2}(z_0) \backslash \overline{B_\rho(z)}$. Then, according to Cauchy's Formula, we get

$$\frac{1}{2\pi i} \int_{C_{r_2}(z_0)} \frac{f(w)}{w-z}\, dw = \frac{1}{2\pi i} \int_{C_\rho(z)} \frac{f(w)}{w-z}\, dw + \frac{1}{2\pi i} \int_{C_{r_1}(z_0)} \frac{f(w)}{w-z}\, dw.$$

Because f is analytic on $\overline{B_\rho(z)}$, the path integral on $C_\rho(z)$ is equal to $f(z)$, so that

$$\frac{1}{2\pi i}\int_{C_{r_2}(z_0)} \frac{f(w)}{w-z}\,dw = f(z) + \frac{1}{2\pi i}\int_{C_{r_1}(z_0)} \frac{f(w)}{w-z}\,dw$$

so that

$$f(z) = \frac{1}{2\pi i}\int_{C_{r_2}(z_0)} \frac{f(w)}{w-z}\,dw - \frac{1}{2\pi i}\int_{C_{r_1}(z_0)} \frac{f(w)}{w-z}\,dw.$$

Notice that for $w \in C_{r_2}(z_0)$ and $\rho_1 \leq |z - z_0| \leq \rho_2$, we have

$$\frac{|z - z_0|}{|w - z_0|} \leq \frac{\rho_2}{r_2} < 1$$

and

$$\frac{1}{w - z} = \frac{1}{w - z_0 - (z - z_0)} = \frac{\frac{1}{w - z_0}}{1 - \frac{z - z_0}{w - z_0}} = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(w - z_0)^{n+1}}.$$

The series converges uniformly for $w \in C_{r_2}(z_0)$ by the M-test because

$$\frac{|z - z_0|^n}{|w - z_0|^{n+1}} \leq \frac{1}{r_2}\left(\frac{\rho_2}{r_2}\right)^n.$$

Therefore, we get

$$\frac{1}{2\pi i} \int_{C_{r_2}(z_0)} \frac{f(w)}{w - z}\, dw = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{C_{r_2}(z_0)} \frac{f(w)}{(w - z_0)^{n+1}}\right)(z - z_0)^n.$$

Notice also that, for $w \in C_{r_1}(z_0)$ and $\rho_1 \leq |z - z_0| \leq \rho_2$, we have

$$\frac{|w - z_0|}{|z - z_0|} \leq \frac{r_1}{\rho_1} < 1$$

and

$$\frac{1}{w - z} = \frac{1}{w - z_0 - (z - z_0)} = -\frac{\frac{1}{z - z_0}}{1 - \frac{w - z_0}{z - z_0}} = -\sum_{n=0}^{\infty} \frac{(w - z_0)^n}{(z - z_0)^{n+1}}.$$

Again, the series converges uniformly for $w \in C_{r_1}(z_0)$ by the M-test:

$$\frac{|w - z_0|^n}{|z - z_0|^{n+1}} \leq \frac{1}{r_1}\left(\frac{r_1}{\rho_1}\right)^n.$$

Therefore, we get

$$\frac{1}{2\pi i} \int_{C_{r_1}(z_0)} \frac{f(w)}{w - z}\, dw = -\sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{C_{r_1}(z_0)} \frac{f(w)}{(w - z_0)^{-n}}\right)(z - z_0)^{-n-1}.$$

Therefore, for $\rho_1 \leq |z - z_0| \leq \rho_2$, we get

$$f(z) = \sum_{n=0}^{\infty} \Big(\frac{1}{2\pi i} \int_{C_{r_2}(z_0)} \frac{f(w)}{(w - z_0)^{n+1}} \Big) (z - z_0)^n$$
$$+ \sum_{n=0}^{\infty} \Big(\frac{1}{2\pi i} \int_{C_{r_1}(z_0)} \frac{f(w)}{(w - z_0)^{-n}} \Big) (z - z_0)^{-n-1}$$

Now, we may choose $R_1 < R < R_2$ and using Cauchy's Theorem, we get

$$a_n := \int_{C_{r_2}(z_0)} \frac{f(w)}{(w - z_0)^{n+1}} \, dw = \int_{C_R(z_0)} \frac{f(w)}{(w - z_0)^{n+1}} \, dw \quad (n \geq 0)$$

and

$$a_{-m} := \int_{C_{r_1}(z_0)} \frac{f(w)}{(w - z_0)^{-m+1}} \, dw = \int_{C_R(z_0)} \frac{f(w)}{(w - z_0)^{-m+1}} \, dw \quad (m \geq 1).$$

$\square$

Consequences: If f is analytic on $B_{R_2}(z_0)$, then

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where

1. the series converges absolutely for any $z \in B_{R_2}(z_0)$.
2. the series converges uniformly on $\overline{B_{\rho_2}}(z_0)$ for any $\rho_2 < R_2$.
3. the coefficient a_n, for $n \geq 0$, can be written as

$$a_n = \frac{1}{2\pi i} \int_{C_R(z_0)} \frac{f(w)}{(w - z_0)^{n+1}} \, dw = \frac{f^{(n)}(z_0)}{n!}$$

where $0 < R < R_2$.

Exponential Function. Recall that

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad z \in \mathbb{C}.$$

① We have $\frac{|z|^n}{n!} \leq \frac{R^n}{n!}$, for any $|z| \leq R$.

② By Weierstrass M-test, the series converges uniformly on any closed disk.

③ We define the **radius of convergence** of the power series as the largest radius R on which the power series converges. In the case above, $R = \infty$.

Geometric Series as a Function. Recall that

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad z \in B_1(0).$$

① We have $|z|^n \leq R^n$, for any $|z| \leq R < 1$.

② By Weierstrass M-test, the series converges uniformly on any closed disk.

③ In the case above, $R = 1$.

Logarithmic Function. Recall that uniformly in $z \in \overline{B_R}(0)$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad \Rightarrow \quad \int_{[0,z]} \frac{1}{1-w}\, dw = \sum_{n=0}^{\infty} \int_{[0,z]} w^n \, dw$$

$$\Rightarrow \quad -\operatorname{Log}(1-z) = \sum_{n=0}^{\infty} \frac{z^{n+1}}{n+1}$$

Hence,

$$\operatorname{Log}(1-z) = -\sum_{n=0}^{\infty} \frac{z^{n+1}}{n+1} \quad (|z| < 1).$$

1. We have $\frac{|z|^{n+1}}{n+1} \leq R^n$, for any $|z| \leq R < 1$.
2. By Weierstrass M-test, the series converges uniformly on any closed disk.
3. In the case above, $R = 1$.

Example. Consider

$$f(z) = \frac{3}{(1+z)(2-z)}$$

and $z_0 = 0$.

① The function is analytic on $\mathbb{C}\backslash\{-1, 2\}$. So there are three regions to analyze: $B_1(0)$, $A_{1,2}(0)$, and $A_{1,\infty}(0)$.

② We will use partial fraction decomposition:

$$f(z) = \frac{1}{1+z} + \frac{1}{2-z}.$$

In the unit disk. We have

$$\frac{1}{1+z} = \frac{1}{1-(-z)} = \sum_{n=0}^{\infty} (-1)^n z^n$$

and

$$\frac{1}{2-z} = -\frac{1}{2}\frac{1}{1-\frac{z}{2}} = -\frac{1}{2}\sum_{n=0}^{\infty} \frac{z^n}{2^n} = \sum_{n=0}^{\infty} \Big(-\frac{z^n}{2^{n+1}}\Big).$$

Therefore

$$f(z) = \sum_{n=0}^{\infty} \Big((-1)^n - \frac{1}{2^{n+1}}\Big) z^n.$$

This is actually a power series centered at $z_0 = 0$. We have

- $a_n = (-1)^n - \frac{1}{2^{n+1}}$ for $n \geq 0$.
- $a_n = 0$ for $n < 0$.

In the second annular region. Since $|z| > 1$, we now have

$$\frac{1}{1-z} = -\frac{1}{z}\Big(\frac{1}{1-1/z}\Big) = \sum_{n=0}^{\infty}\Big(-\frac{1}{z^{n+1}}\Big).$$

Since $|z| < 2$, we have

$$\frac{1}{2-z} = \sum_{n=0}^{\infty}\Big(-\frac{z^n}{2^{n+1}}\Big).$$

Therefore,

$$f(z) = \sum_{n=0}^{\infty}\Big(-\frac{z^n}{2^{n+1}}\Big) + \sum_{n=0}^{\infty}\Big(-\frac{1}{z^{n+1}}\Big).$$

This is the Laurent series representation of f in $A_{1,2}(0)$. We have

- $a_n = -\frac{1}{2^{n+1}}$ for $n \geq 0$.
- $a_n = -1$ for $n < 0$.

In the third annular region. Since $|z| > 2$, we have

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} \Big(-\frac{1}{z^{n+1}} \Big)$$

and

$$\frac{1}{2-z} = -\frac{1}{z}\Big(\frac{1}{1-2/z}\Big) = -\frac{1}{z}\sum_{n=0}^{\infty}\frac{2^n}{z^n} = \sum_{n=0}^{\infty}\Big(-\frac{2^n}{z^{n+1}}\Big).$$

Therefore,

$$f(z) = \sum_{n=0}^{\infty}(-1-2^n)\frac{1}{z^{n+1}}.$$

This is the Laurent series representation of f in $A_{2,\infty}(0)$. We have

$a_n = 0$ for any $n \geq 0$.

$a_n = -1 - 2^{-n}$ for any $n < 0$.