MATH-241 Calculus I
Midterm 01

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Last name: Solutions.
First name: —

Question:	1	2	3	4	5	6	7	8	Total
Points:	6	15	15	20	10	10	15	9	100
Score:	—	—	—	—	—	—	—	—	—

Instructions: Make sure to write your complete name on your copy. You must answer all the questions below and write your answers directly on the questionnaire. At the end of the 75 minutes, hand out your copy.

No devices such as a smart phone, cell phone, laptop, or tablet can be used during the exam. You are not allowed to use the lecture notes, the textbook, or any other notes. You may use a digital calculator (no graphical calculator or symbolic calculator will be allowed).

You must show ALL your work to have full credit. An answer without justification worth no point.

Good luck!

Pierre-Olivier Parisé

Your Signature: —

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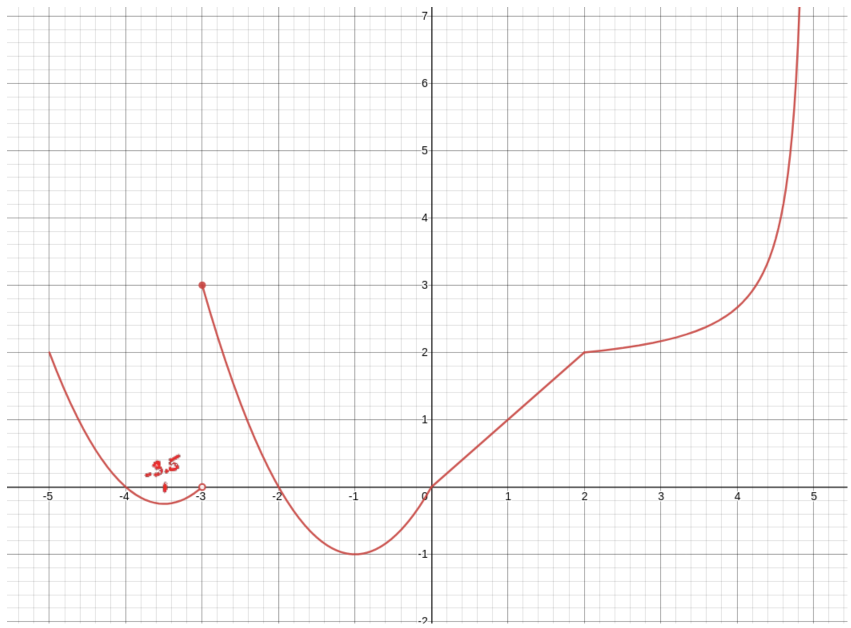
QUESTION 1 **(6 pts)**

Consider the function $f(x)$ with the graph $y = f(x)$ pictured below. The domain of f is $[-5, 5]$.



(a) (1 point) On which interval(s) (if any) is the function decreasing? (no justification needed)

$[-5, -3.5]$, $[-3, -1]$

(b) (1 point) Where (if anywhere) is the function not continuous?

At $x = -3$ because $\lim_{x \to -3^+} f(x) \neq \lim_{x \to -3^-} f(x)$.

(c) (1 point) Where (if anywhere) is the function not differentiable?

(1) $x = -3$ (discontinuity) (2) $x = 0$, corner
(3) $x = 2$, corner

(d) (1 point) What is $\lim_{x \to -3^-} f(x)$?

0

(e) (1 point) What is $\lim_{x \to 5^-} f(x)$?

$+\infty$

(f) (1 point) What is $\lim_{x \to 2^-} \dfrac{f(x) - f(2)}{x - 2}$?

1 (slope of the line).

Question 2 (15 pts)

Find the value of the following limit. No credit will be attributed for using L'Hôpital's rule to find the value of a limit.

(a) (5 points) $\lim_{x\to 0} \frac{x^2-4}{x-2}$.

$$\lim_{x\to 0} \frac{x^2-4}{x-2} = \lim_{x\to 0} \frac{(x+2)(x-2)}{x-2}$$

substitution 5pts.

$$= \lim_{x\to 0} x+2 = \boxed{2}$$

(b) (5 points) $\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9}$.

$$\frac{\sqrt{x}-3}{x-9} = \frac{x-9}{(x-9)(\sqrt{x}+3)} = \frac{1}{\sqrt{x}+3}.$$

3pts

$$\Rightarrow \lim_{x\to 9} \frac{\sqrt{x}-3}{x-9} = \lim_{x\to 9} \frac{1}{\sqrt{x}+3} \overset{2pts.}{=} \frac{1}{3+3} = \boxed{\frac{1}{6}}$$

(c) (5 points) $\lim_{x\to 0} \left(\frac{\cos x - 1}{x}\right)^2$.

3pts.

$$\lim_{x\to 0} \frac{\cos x - 1}{x} = \frac{d}{dx}(\cos x)\Big|_{x=0} = -\sin(0) = 0.$$

So, by the composition rule, we have

$$\lim_{x\to 0} \left(\frac{\cos x-1}{x}\right)^2 = \left(\lim_{x\to 0} \frac{\cos x - 1}{x}\right)^2 = 0^2 = \boxed{0}$$

2pt.

Question 3 (15 pts)

Let $f(x) = 1/x$.

(a) (5 points) State the definition of the derivative of a function at some point a.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$ 3pts.

1pt.

if the limit exists.

1pt.

(b) (10 points) Using the definition of the derivative, find the value of $f'(3)$ if $f(x) = 1/x$. **No credit for a solution using the rules of differentiation.**

$$f'(3) = \lim_{h \to 0} \frac{\frac{1}{3+h} - \frac{1}{3}}{h}$$ 5pts.

$$= \lim_{h \to 0} \frac{3 - (3+h)}{3h(3+h)}$$

$$= \lim_{h \to 0} \frac{-h}{3h(3+h)}$$

5pts for calculation

$$= \lim_{h \to 0} \frac{-1}{(3+h)3}$$

$$= \boxed{-\frac{1}{9}}$$

Question 4 (20 pts)

Find the equation of the tangent line to the curve $y = \frac{x^2+5}{x-2}$ at the point $(1, -6)$.

An equation for the tangent line is

$$\underbrace{y+6}_{1pt.} = \underbrace{m(x-1)}_{1pt} \quad , \quad \underbrace{m}_{2pt.} = y'(1). \qquad \underline{4pts.}$$

The derivative is

$$y'(x) = \frac{(x^2+5)'(x-2) - (x^2+5)(x-2)'}{(x-2)^2}$$

$$= \frac{2x(x-2) - x^2 - 5}{(x-2)^2}$$

$$= \frac{x^2 - 4x - 5}{(x-2)^2}$$

13pts.

10pt. | Quotient 4pts
Sum 2pts
Power 2pts
Cst. 2pts
3pt | algebra.

$$\Rightarrow \quad y'(1) = \frac{1-4-5}{(1-2)^2} = \frac{-8}{(-1)^2} = -8.$$

2pts.

Thus

$$y+6 = -8(x-1)$$

$$\Rightarrow \boxed{y = -8x + 2} \quad 1pt.$$

Question 5 **(10 pts)**

An 'ilio-holo-ika-uaua is fitted with a GPS to track his movement. She gets in the ocean, and swims in a straight line away from the shore. Her position from the shore is recorded every two minutes for ten minutes. The results are recorded in the following table.

Time in minutes	Distance from the shore in meters
0	0
2	20
4	30
6	60
8	80
10	100

(a) (3 points) What is the seal's average velocity between minutes 4 and 8?

$$\text{average vel.} = \frac{s(8) - s(4)}{8-4} = \frac{80-30}{4} = 50/4 = \boxed{12.5 \text{ meters/min}}$$

(b) (4 points) Estimate the seal's velocity at time 6 minutes.

$$v(6) \approx \frac{s(8)-s(6)}{8-6} = \frac{20}{2} = 10$$

$$v(6) \approx \frac{s(6)-s(4)}{6-4} = \frac{30}{2} = 15$$

So, $v(6) \approx \dfrac{10+15}{2} = \boxed{12.5 \text{ meters/min}}$

(c) (3 points) The seal saw a group of fish at time 6 minutes going at 10 m/min. Do you think that the seal could catch the fish?

From (b), $v(6) \approx 12.5$ meters/min.

So, since $v(6) > 10$ meters/min, the seal would be able to catch the fish.

Question 6 (10 pts)

Consider the function $f(x) = 4x^3 - 6x^2 - 6x + 5$. This function must have at least one zero in the interval $(0,1)$. Explain why, making explicit which theorem(s), if any, and which assumtions(s) on f, if any, you are using.

2pts.

f is a polynomial $\rightarrow$ continuous on $[0,1]$.

$f(0) = 5$ 2pts and $f(1) = -3$ 2pts.

So, $f(0) > f(1)$ 2pts. We can use the intermediate value theorem with $N = 0$.

So, we find $c \in (0,1)$ such that

$f(c) = 0$. 2pts.

f has at least one zero in $(0,1)$. ∎

Question 7 (15 pts)

Compute the derivatives of the following functions.

(a) (5 points) $f(x) = \dfrac{\sin x}{x^3 + \cos x}$.

$$f'(x) = \frac{\cos x\,(x^3 + \cos x) - \sin x\,(3x^2 - \sin x)}{(x^3 + \cos x)^2}$$

$$= \frac{x^3 \cos x + \cos^2 x - 3x^2 \sin x + \sin^2 x}{(x^3 + \cos x)^2}$$

$$= \boxed{\frac{x^3 \cos x - 3x^2 \sin x + 1}{(x^3 + \cos x)^2}}$$

2pts quotient rule
1pt. sin rule
1 pt. cos rule.
1 pt. sum+power rule.

(b) (5 points) $f(x) = \dfrac{1 - \sqrt{x}}{1 + \sqrt{x}}$.

$$f'(x) = \frac{-\frac{1}{2\sqrt{x}}(1+\sqrt{x}) - \frac{1}{2\sqrt{x}}(1-\sqrt{x})}{(1+\sqrt{x})^2}$$

$$= \frac{-\frac{1}{2\sqrt{x}}(1+\sqrt{x}+1-\sqrt{x})}{(1+\sqrt{x})^2}$$

$$= \boxed{\frac{-1}{\sqrt{x}(1+\sqrt{x})^2}}$$

2pts quotient Rule
2pts power rule
1pt. Sum rule.

(c) (5 points) $f(x) = x\sqrt{x}$.

$$f(x) = x^{3/2} \to f'(x) = \frac{3}{2} x^{1/2}$$

$$= \boxed{\frac{3}{2}\sqrt{x}}$$

2pts. product rule
2pts. power rule
1pt. answer.
or
1pt. combining powers
3pts. Power rule.
1pt. answer.

Question 8 (9 pts)

Answer each of the following questions. No credit will be attributed for using L'Hôpital's rule to find the value of a limit.

(a) (3 points) Let $f(x) = \begin{cases} Ax & x \le -1 \\ x^2 - 3Ax + 3 & x > -1. \end{cases}$

Find the value of A for which the function f is continuous.

The only problem is at $x=-1$. We must have

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x) \quad \underline{1pt}$$

$$\iff -A = 1+3A+3 \quad \underline{1pt.}$$

$$\iff -4A = 4$$

$$\iff \boxed{A=-1} \quad 1pt.$$

(b) (3 points) If $g(x) = (x+1)f(x)$ and $\lim_{x\to 0} f(x) = 2$, then find $\lim_{x\to 0} g(x)$.

$$\lim_{x\to 0} g(x) = \left(\lim_{x\to 0} x+1\right)\left(\lim_{x\to 0} f(x)\right)$$

$$= 1 \cdot 2$$

$$= \boxed{2}$$

Product 1pt.
Substitution 1pt.
Answer 1pt.

(c) (3 points) Find the value of $\lim_{x\to 3^+} \dfrac{x+4}{x-3}$.

$\lim_{x\to 3^+} x-3 = 0^+$ because $x\to 3$ from the right, so $x>3$.

$$\Rightarrow \lim_{x\to 3^+} \frac{x+4}{x-3} = \frac{3+4}{0^+} = \boxed{+\infty}$$