

Chapter 16

Vector Calculus

16.6 Parametric Surfaces

Generic surfaces in 3D.

EXAMPLE. Using Python, draw the surface described by the equation
$$x^2 + y^2 + z^2 = 1.$$

① Cartesian coordinates

Let x, y be such that $x^2 + y^2 \leq 1$

$$\Rightarrow z = \sqrt{1 - x^2 - y^2} \rightarrow \vec{r}(x, y) = \langle x, y, \sqrt{1 - x^2 - y^2} \rangle$$

$$\text{or } z = -\sqrt{1 - x^2 - y^2} \rightarrow \vec{r}(x, y) = \langle x, y, -\sqrt{1 - x^2 - y^2} \rangle$$

② Spherical coordinates

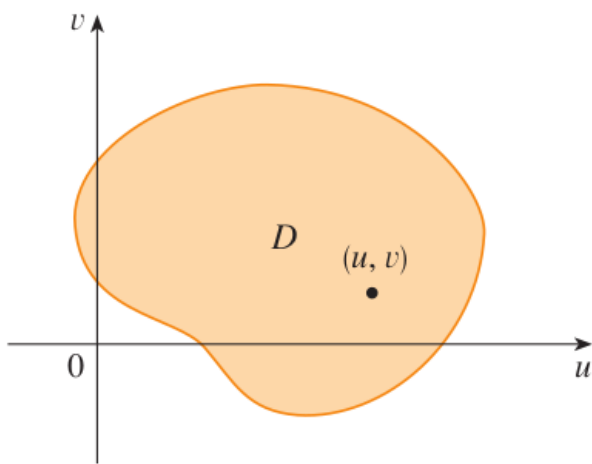
$$\begin{array}{lcl} x = \rho \cos \theta \sin \phi & \overset{\rho=1}{=} & \cos \theta \sin \phi \\ y = \rho \sin \theta \sin \phi & = & \sin \theta \sin \phi \\ z = \rho \cos \phi & = & \cos \phi \end{array} \quad \left\{ \begin{array}{l} 0 \leq \theta \leq 2\pi \\ 0 \leq \phi \leq \pi \end{array} \right.$$

$$\vec{r}(\theta, \phi) = \langle \cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi \rangle$$

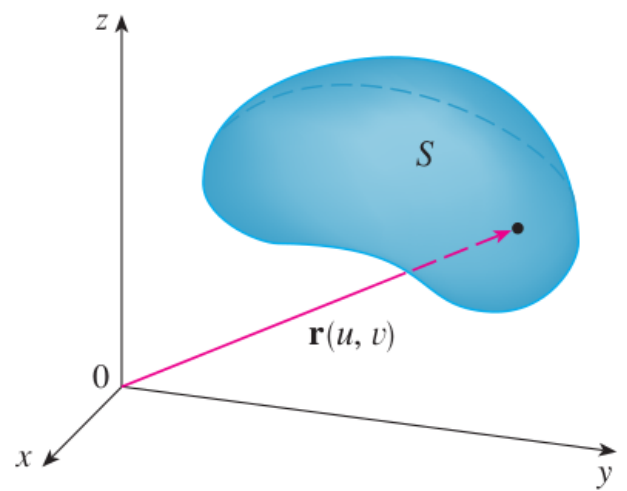
Polar coordinates

$$\vec{r}(\rho, \theta) = \langle \rho \cos \theta, \rho \sin \theta, \sqrt{1 - \rho^2} \rangle$$

$$0 \leq \theta \leq 2\pi, \quad 0 \leq \rho \leq 1.$$



$\xrightarrow{\mathbf{r}}$



$$\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle \quad \left((u, v) \in D \right)$$

Vector function
or
parametrization

3–6 Identify the surface with the given vector equation.

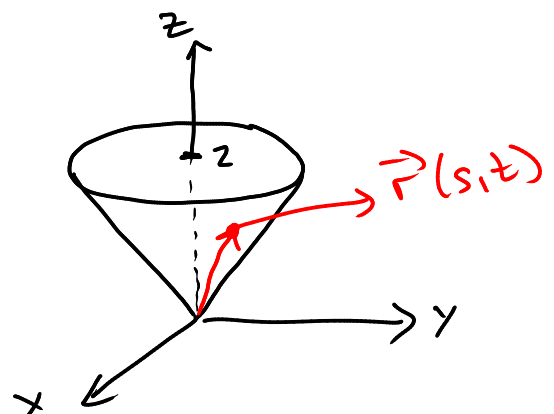
5. $\mathbf{r}(s, t) = \langle \underset{z}{s \cos t}, \underset{y}{s \sin t}, \underset{z}{s} \rangle$ where $0 \leq s \leq 2$, and $0 \leq t \leq 2\pi$.

Notice that

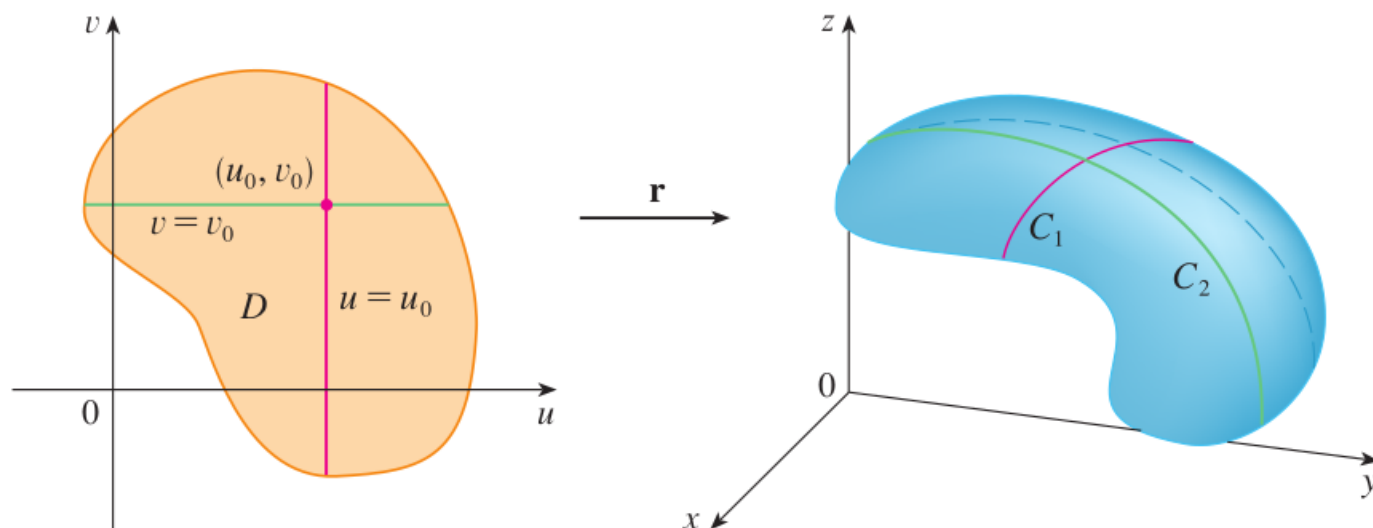
$$\begin{aligned} x^2 + y^2 &= s^2 \cos^2 t + s^2 \sin^2 t \\ &= s^2 = z^2 \end{aligned}$$

So, $z^2 = x^2 + y^2 \xrightarrow{z \geq 0} z = \sqrt{x^2 + y^2}$.
cone
(cône)

That's a cone



Grid curves.



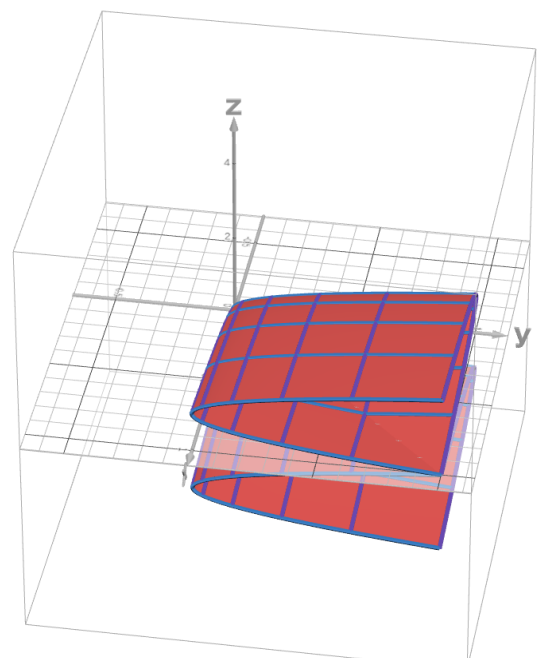
- $C_1 : \vec{r}(v) = \vec{r}(u_0, v)$
- $C_2 : \vec{r}(u) = \vec{r}(u, v_0)$

7-12 Use a computer to graph the parametric surface. Get a printout and indicate on it which grid curves have u constant and which have v constant.

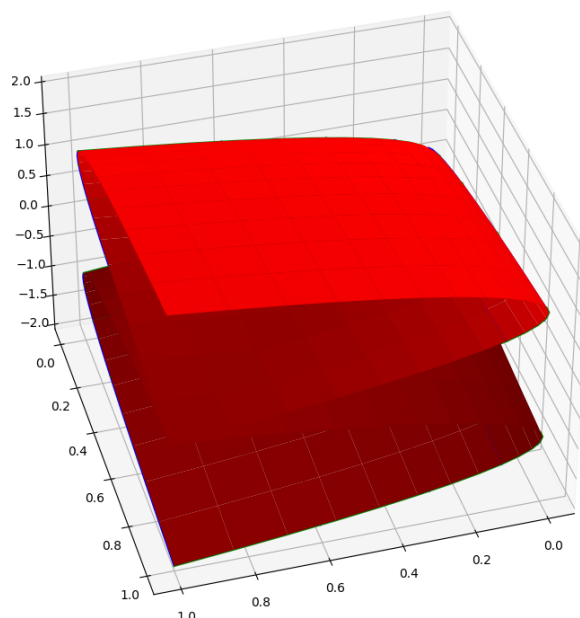
7. $\mathbf{r}(u, v) = \langle u^2, v^2, u + v \rangle$, <https://www.desmos.com/3d/fb472f71c5>
 $-1 \leq u \leq 1, -1 \leq v \leq 1$

If $u = u_0$, then
 $C_1 : \vec{r}(v) = \langle u_0^2, v^2, u_0 + v \rangle$

If $v = v_0$, then
 $C_2 : \vec{r}(u) = \langle u^2, v_0^2, u + v_0 \rangle$



With Desmos



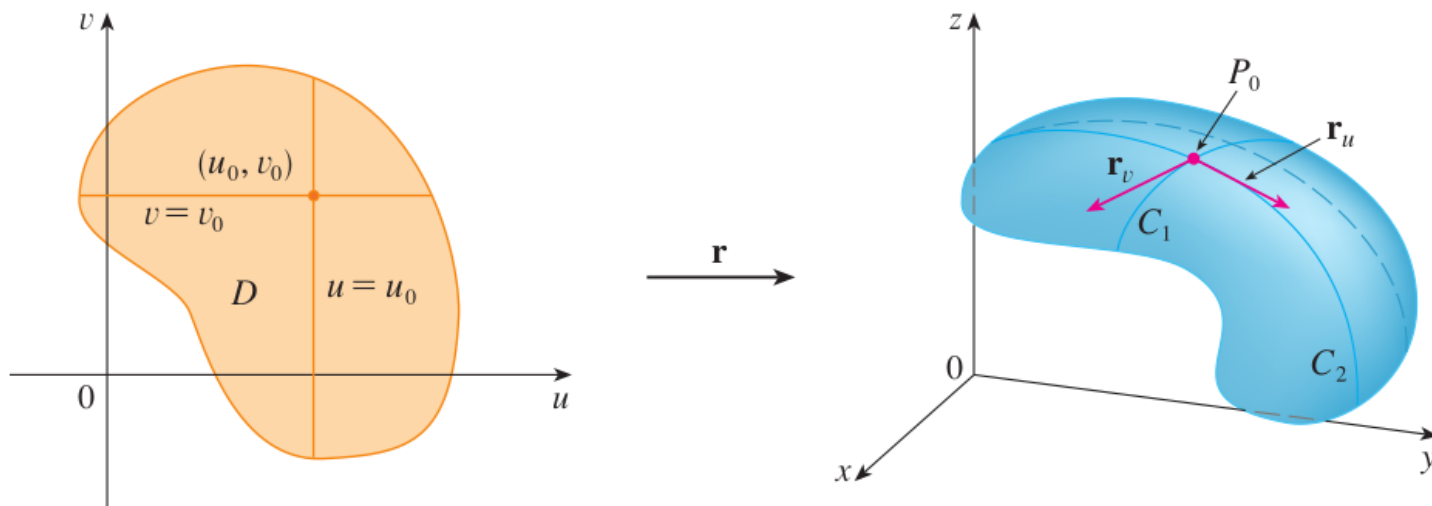
With Python

EXAMPLE 3 Find a vector function that represents the plane that passes through the point P_0 with position vector \mathbf{r}_0 and that contains two nonparallel vectors \mathbf{a} and \mathbf{b} .

19–26 Find a parametric representation for the surface.

- 23.** The part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the cone $z = \sqrt{x^2 + y^2}$

Tangent Planes.



- $u = u_0$ (constant). $\vec{r}(v) = \vec{r}(u_0, v)$ represents the curve C_1 . Hence, \vec{r}_v is the tangent vector to C_1 at P_0 .
- $v = v_0$ (constant). $\vec{r}(u) = \vec{r}(u, v_0)$ represents the curve C_2 . Hence, \vec{r}_u is the tangent vector to C_2 at P_0 .

Equation of the tangent plane at P_0 :

$$\vec{r}(u, v) = \langle x_0, y_0, z_0 \rangle + u\vec{r}_u(u_0, v_0) + v\vec{r}_v(u_0, v_0)$$

where $-\infty < u < \infty, -\infty < v < \infty$.

37–38 Find an equation of the tangent plane to the given parametric surface at the specified point. Graph the surface and the tangent plane. <https://www.desmos.com/3d/dfc50f1356>

37. $\mathbf{r}(u, v) = u^2\mathbf{i} + 2u \sin v\mathbf{j} + u \cos v\mathbf{k}; \quad u = 1, v = 0$