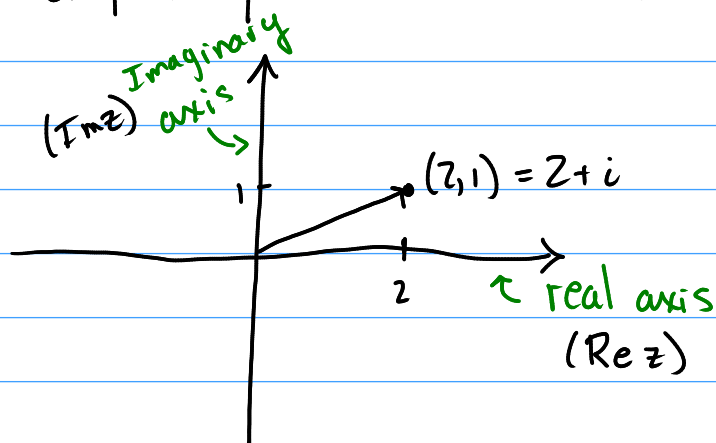


Section 1.2: The Complex Plane

Complex plane \longleftrightarrow Cartesian plane



ABSOLUTE VALUE

DEF if $z = x + iy$, then the absolute value or the norm or the modulus of z is

$$|z| = \sqrt{x^2 + y^2}$$

Note: • if $z = x$, then $|z| = \sqrt{x^2} = |x|$ \leftarrow regular abs. value of \mathbb{R} .

• $|z| = 0 \iff z = 0$.

• $|-z| = |z|$

DEF. If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then the distance between z_1 and z_2 is

$$\begin{aligned} d(z_1, z_2) &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= |z_1 - z_2| (= |z_2 - z_1|) \end{aligned}$$

Ex. 1.2.4 (a) Find the set of complex numbers z s.t.

$$|z + 4 - i| = 2.$$

(b) Find the set of complex numbers z s.t.

$$|z + 4 - i| \leq 2.$$

Sol. (a) Let $z = x + iy$. So

$$|z + 4 - i| = 2 \Leftrightarrow \sqrt{(x+4)^2 + (y-1)^2} = 2$$

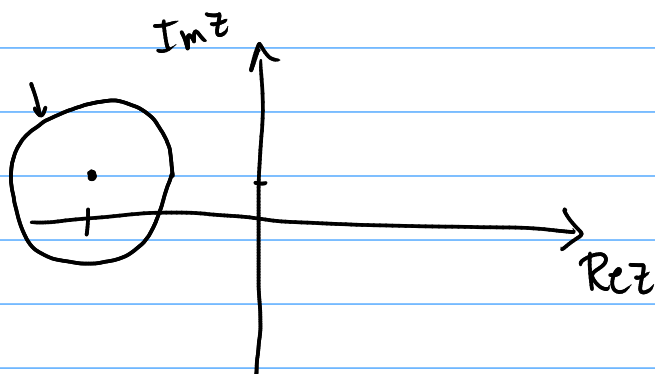
$$\Leftrightarrow (x+4)^2 + (y-1)^2 = 4$$

$$(x-a)^2 + (y-b)^2 = R^2$$

A circle of radius 2 centered at $(-4, 1)$.
The set of z s.t.

$$|z - z_0| = R$$

is a circle centered at z_0 and of radius R

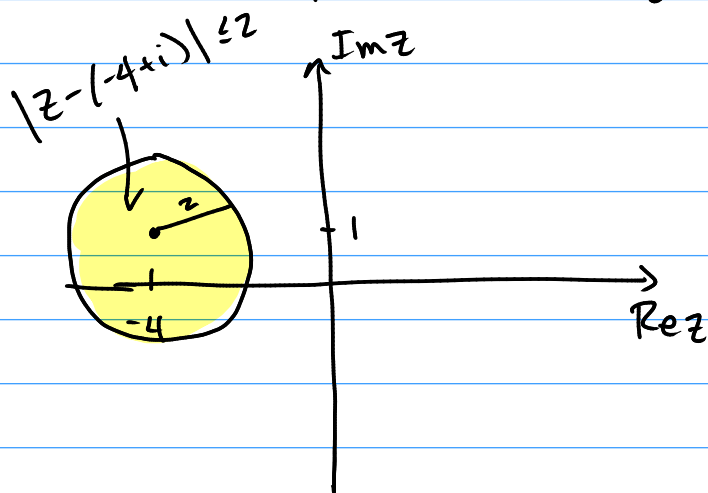


$$(b) |z + 4 - i| \leq 2 \Leftrightarrow |z + 4 - i|^2 \leq 4$$

$$\Leftrightarrow (x+4)^2 + (y-1)^2 \leq 4.$$

A disk of radius 2 and centered at $(-4, 1)$.

In general, $|z - z_0| \leq R$ is a disk of radius R and centered at z_0 .



Prop.
1.2.5

Let z, z_1, z_2, \dots, z_n be complex numbers.

(a) $|z| = \sqrt{z\bar{z}}$ or $|z|^2 = z\bar{z}$

(b) $|\bar{z}| = |z|$

(c) $|z_1 z_2| = |z_1| |z_2|$ (d) $|z_1 z_2 \dots z_n| = |z_1| |z_2| \dots |z_n|$

(e) $|z^n| = |z|^n$, for any $n \geq 0$

(f) $\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$ if $z_2 \neq 0$.

PROOF. Prove (c). Here,

$$\begin{aligned}
 |z_1 z_2|^2 &\stackrel{(a)}{=} (z_1 z_2) \overline{(z_1 z_2)} = z_1 z_2 \bar{z}_1 \bar{z}_2 \\
 &= z_1 \bar{z}_1 z_2 \bar{z}_2 \\
 &\stackrel{(a)}{=} |z_1|^2 |z_2|^2 \quad (\text{take } \sqrt{\cdot})
 \end{aligned}$$

