Chapter 2

Derivatives

2.9 Linear Approximations and Differentials.

An observation:

A curve $y = f(x)$ lies very close to its tangent line near the point of tangency. Linearization https://www.desmos.com/calculator/1sp51krlae

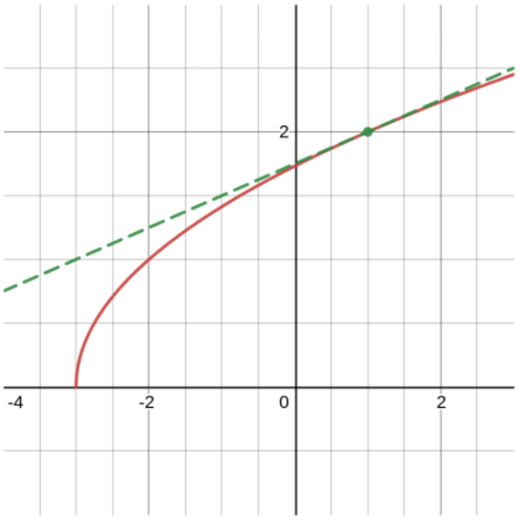


Figure: Linearization near the point of tangency

This suggests to approximate the values of f by the tangent line. This is a really useful procedure because $f(x)$ may be difficult to compute!

$(x_0, y_0) = (a, f(a))$ $\quad y - f(a) = f'(a)(x - a)$

Approximation by the tangent line:

$$f(x) \approx \underbrace{f(a) + f'(a)(x - a)}_{L(x)}$$

So the linearization is

$$L(x) = f(a) + f'(a)(x - a)$$

EXAMPLE 1 Find the linearization of the function $f(x) = \sqrt{x+3}$ at $a = 1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$. Are these approximations overestimates or underestimates?

(1) Linearization:

$$f'(x) = \frac{1}{2}(x+3)^{-1/2} \cdot (1) = \frac{1}{2\cdot\sqrt{x+3}}$$

$$\Rightarrow \quad f'(1) = \frac{1}{2\sqrt{1+3}} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

$$\Rightarrow \quad L(x) = f(1) + \frac{1}{4}(x-1)$$

$$\Rightarrow \quad L(x) = 2 + \frac{1}{4}(x-1).$$

(2) Approx. $\sqrt{3.98}$.

$$\sqrt{x+3} \leftrightarrow \sqrt{3.98} \quad \rightarrow \quad x+3 = 3.98$$

$$\rightarrow \quad x = 0.98$$

$$\Rightarrow \sqrt{3.98} = \sqrt{\underbrace{0.98}_{=x} + 3} \approx L(0.98)$$

$$= 2 + \frac{1}{4}(0.98-1)$$

$$= 2 - \frac{0.02}{4}$$

$$= 2 - 0.005$$

$$= \boxed{1.995}$$

(3) Approx. $\sqrt{4.05}$

$$\sqrt{4.05} \leftrightarrow \sqrt{x+3} \quad \Rightarrow \quad x+3 = 4.05 \Rightarrow x = 1.05$$

$$\Rightarrow \sqrt{4.05} = \sqrt{\underbrace{1.05}_{x} + 3} \approx L(1.05) = 2 + \frac{(1.05-1)}{4}$$

$$= 2 + \frac{0.05}{4} = 2 + 0.0125$$

$$\boxed{2.0125}$$

Differentials.

If $y = f(x)$, then

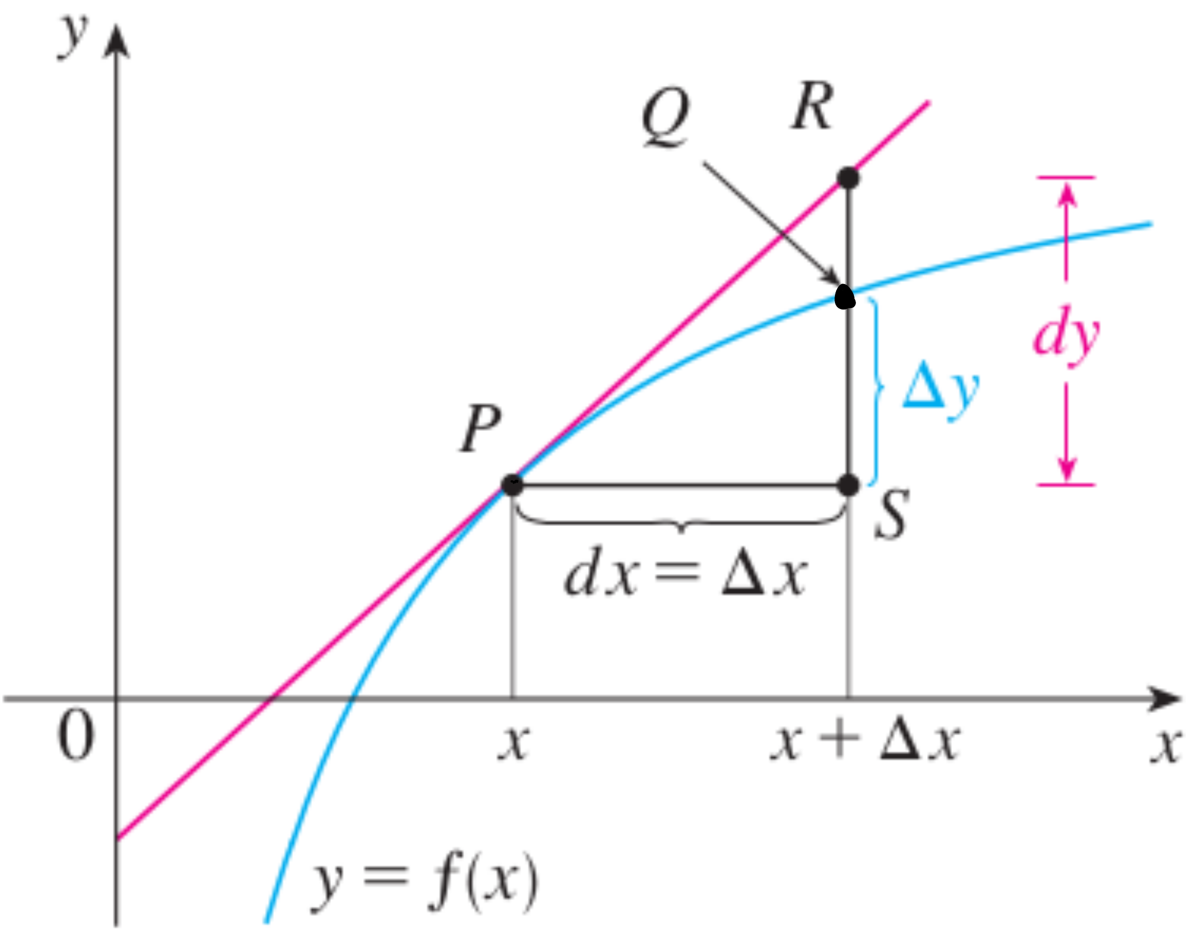
- dx is the differential of x. It's a little increment in the variable x.
- dy is the differential of y and dy is the approximate increment in the variable y given by

$$\boxed{dy = f'(x)dx.} \leftarrow \frac{dy}{dx} = f'(x)$$

Remark:

$$\Delta y \approx f'(x)\,dx = dy \qquad dx = \Delta x$$

Geometric interpretation.



EXAMPLE 3 Compare the values of Δy and dy if $y = f(x) = x^3 + x^2 - 2x + 1$ and x changes (a) from 2 to 2.05 and (b) from 2 to 2.01.

(a) $\Delta y = f(2.05) - f(2)$ $\qquad dx = \Delta x = 0.05$

$dy = f'(2)\, dx$

$\Rightarrow \quad \Delta y = 0.717675$

$dy = (3 \cdot (2)^2 + 2 \cdot 2 - 2)\, 0.05 = 0.7$

(b) $\Delta y = f(2.01) - f(2)$ $\qquad dx = \Delta x = 0.01$

$dy = f'(2)\, dx$

$\Rightarrow \quad \Delta y = 0.140701$.

$dy = 0.140$.