# Chapter 5
# Applications in integration

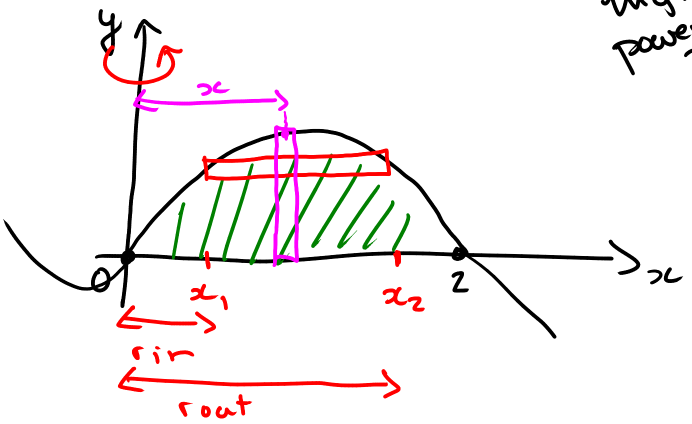
5.3 Volumes by Cylindrical Shells

# Illustrative Example. (Rotation about the y-axis)

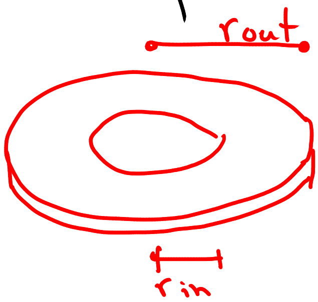
## Example 1.

Find the volume of the solid obtained by rotating about the $y$-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$.

① Picture




coef. front highest power of $x$ $< 0$

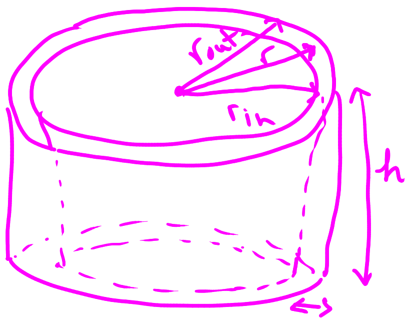

$r_{in} = x_1$

$r_{out} = x_2$

$$\text{Vol(solid)} = \int_a^b \pi (x_2^2 - x_1^2)\, dy$$

$y = x^2$

$\Rightarrow \pm\sqrt{y} = x$




$$
\begin{aligned}
V_1 - V_2 &= \pi r_{out}^2 h - \pi r_{in}^2 h \\
&= \pi h (r_{out}^2 - r_{in}^2) \\
&= \pi h (r_{out} - r_{in})(r_{out} + r_{in}) \\
&= \pi h \, \Delta r \left(\frac{r_{out} + r_{in}}{2}\right) 2 \\
&= 2\pi h r \, \Delta r
\end{aligned}
$$

in our case:

$h = y$

$r = x$

$\Rightarrow \quad V = V_1 - V_2 = 2\pi y x \, \Delta x$

Increase the nb of rectangle:

$$\boxed{V(\text{solid}) = \int_a^b 2\pi \, y \, x \, dx}$$

thickness

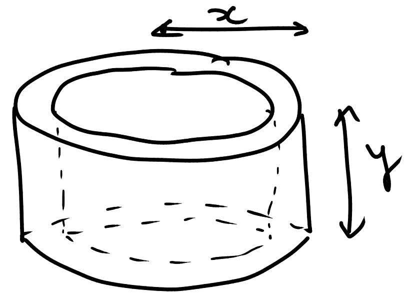
radius

heigth

② Volume:

$a = 0$
$b = 2$

radius $= x$
huigth $= y = 2x^2 - x^3$
thickness $= dx$




$$\text{Vol(solid)} = \int_0^2 2\pi\,(2x^2 - x^3)\,x\,dx$$

$$= 2\pi \int_0^2 2x^3 - x^4\,dx$$

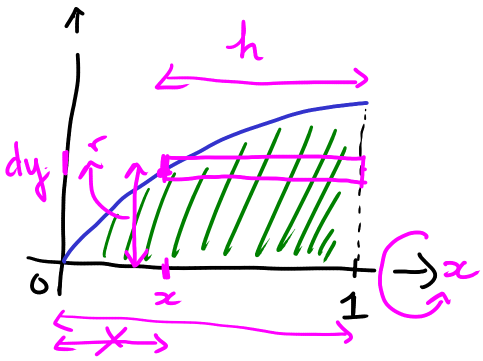
$$= \boxed{16\pi - \frac{64\pi}{5}}$$

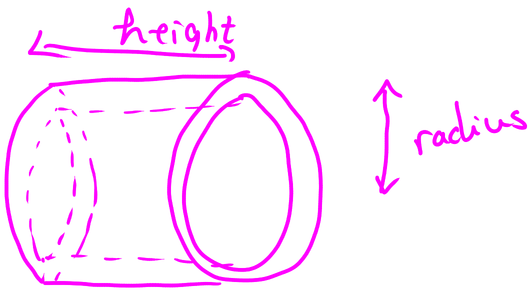
## Example 3.

Use cylindrical shells to find the volume of the solid obtained by rotating about the $x$-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.

$$\boxed{\text{Vol(Solid)} = \int_c^d 2\pi\,(\text{height})(\text{radius})\,dy}$$

① Picture




radius $= \sqrt{x} = y$

height $= 1 - x$

thickness $= dy$

② Volume.

$$\text{Vol(Solid)} = \int_0^1 2\pi\,(1-x)\,y\,dy$$

$$y = \sqrt{x} \Rightarrow y^2 = x$$

$$= \int_0^1 2\pi(1-y^2)\,y\,dy$$

$$= 2\pi\int_0^1 y - y^3\,dy$$

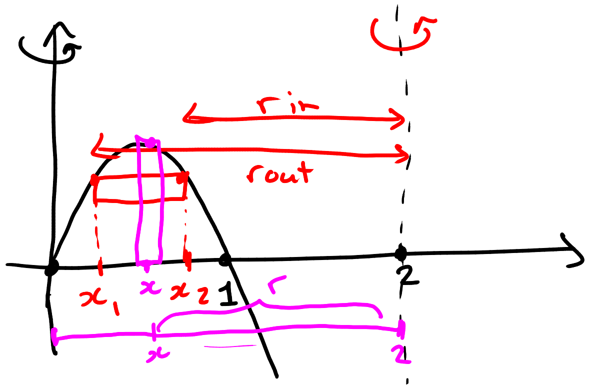
$$= \boxed{\frac{\pi}{2}}$$

## Example 4.

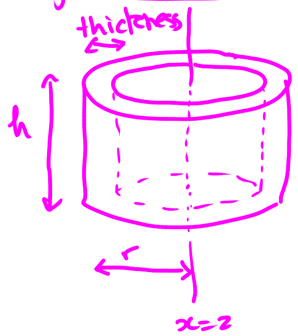
Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $\boxed{x = 2.}$

① Picture

$y = x - x^2 = x(1-x)$




Cylindrical Shells




$r = 2 - x$

$h = y$

thick. $= dx$

② Volume

$$\begin{aligned}
V(\text{Sol}) &= \int_0^1 2\pi(\text{height})(\text{radius})(\text{thickness}) \\
&= \int_0^1 2\pi\, y\, (2-x)\, dx \\
&= \int_0^1 2\pi (x - x^2)(2-x)\, dx \\
&= 2\pi \int_0^1 2x - 2x^2 - x^2 + x^3\, dx \\
&= 2\pi \int_0^1 2x - 3x^2 + x^3\, dx \\
&= \boxed{\frac{\pi}{2}}
\end{aligned}$$