Math 644

Chapter 1

Section 1.3: Stereographic Projection

Contents

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How Is The Riemann Sphere Constructed?

We would like to treat ∞ as any other complex numbers. To do that, we will construct a model using the stereographic projection.

Method

1) Embed $\mathbb{C}$ in $\mathbb{R}^3$.

2) Draw a sphere $\mathbb{S}^2$ with the following characteristics:

- $\mathbb{S}^2 := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$;
- Denote by $N := (0, 0, 1)$ the north pole.

3) The stereographic projection:

__**Point of intersection:**__

4) Inverse of the stereographic projection:

<u>**Conclusion**</u>:

DEFINITION 1. The **extended complex plane** is the set $\mathbb{C}^* := \mathbb{C} \cup \{\infty\}$, where

$$\infty := \pi^{-1}(0, 0, 1).$$

The Riemann sphere $\mathbb{S}^2$ inherits a topology from the usual topology of $\mathbb{R}^3$ generated by the balls in $\mathbb{R}^3$. In more details:

- A basis for the topology are of the form $B \cap \mathbb{S}^2$, where B is a ball in $\mathbb{R}^3$.

Before describing the topology of $\mathbb{C}^*$, we first show the following.

Theorem 2. Circles in $\mathbb{C}$ correspond precisely to circles on $\mathbb{S}^2 \backslash \{(0, 0, 1)\}$.

Proof.

Corollary 3.

a) Topology of $\mathbb{S}^2$ induces the standard topology on $\mathbb{C}$ under the stereographic projection.

b) Moreover, a basis of neighborhoods for ∞ are of the form $\{z \in \mathbb{C} : |z| > r\} \cup \{\infty\}$, with $r > 0$.