

Question 1

What would be the derivative of the function $f(x) = \sin x$?

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} & \sin(A+B) &= \sin(A)\cos(B) + \cos(A)\sin(B) \\
 &= \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h} & & \downarrow \\
 &= \lim_{h \rightarrow 0} \frac{\sin(x)\cos(h) + \cos(x)\sin h - \sin x}{h} & A=x & \\
 & & B=h & \\
 &= \lim_{h \rightarrow 0} \frac{\sin x (\cos h - 1) + \cos x \sin h}{h} \\
 &= \lim_{h \rightarrow 0} \left(\sin x \frac{\cos h - 1}{h} + \cos x \frac{\sin h}{h} \right)
 \end{aligned}$$

There $\lim_{h \rightarrow 0} \frac{\cos h - 1}{h} = 0$, How?

$A = h/2$
 $\sin^2 A = 1 - \frac{\cos(2A)}{2}$

$$\begin{aligned}
 \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} &= \lim_{h \rightarrow 0} \frac{-2 \sin^2(h/2)}{h} = \lim_{h \rightarrow 0} \frac{-\sin(h/2) \cdot \sin(h/2)}{h/2} \\
 &= (-1) \cdot 0 \cdot 1 = 0
 \end{aligned}$$

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \sin x \left(\frac{\cos h - 1}{h} \right) + \lim_{h \rightarrow 0} \cos x \frac{\sin h}{h} \\
 &= \sin x \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} + \cos x \lim_{h \rightarrow 0} \frac{\sin h}{h} \\
 &= \cos x
 \end{aligned}$$

Example 4

Compute the derivative of $f(x) = x^2 \sin(x)$.

$$\begin{aligned}\frac{d}{dx} (x^2 \sin x) &= \frac{d}{dx} (x^2) \sin x + x^2 \frac{d}{dx} (\sin x) \\ &= 2x \sin x + x^2 \cos x. \\ &= x(2 \sin x + x \cos x)\end{aligned}$$