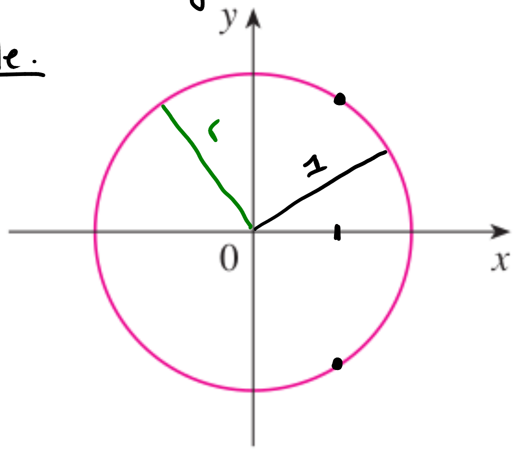
Chapter 2
Derivatives
2.6 Implicit Differentiation

Functions defined implicitly.

Geometry of curves.

$x^2+y^2=1$, $x^2+y^2=r^2$

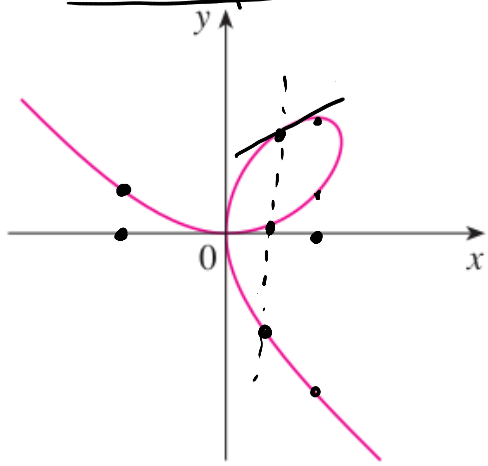
Circle.



$x^2+y^2=1 \Rightarrow y^2=1-x^2$

$\Rightarrow y=\pm\sqrt{1-x^2}$

Folium of Descartes



$x^3+y^3=6xy$

$\Rightarrow y^3-6xy=-x^3$

↳ won't able to isolate y easily.

Key assumption:

We suppose that $y=f(x)$

In Natural Science (Gas' Law).

$$\boxed{\left(P+\frac{n^2a}{V^2}\right)(V-nb)=nRT}$$

- P: Pressure
- V: Volume
- T: Temperature
- R, a, b are constants depending on the gas.

HOW DO WE FIND THE SLOPE/DERIVATIVE OF A FUNCTION $y=f(x)$ IF THE RULE IS GIVEN BY AN IMPLICIT EQUATION?

EXAMPLE 1

(a) If $x^2 + y^2 = 25$, find $\dfrac{dy}{dx}$.

(b) Find an equation of the tangent to the circle $x^2 + y^2 = 25$ at the point (3, 4).

(a) Assumption: $y = f(x)$

① Take derivative of both sides.

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(25).$$

$$\Rightarrow \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$

$$\Rightarrow 2x + 2(y)\cdot\underbrace{\frac{dy}{dx}}_{A} = 0$$

② Isolate $\dfrac{dy}{dx}$

$$2x + 2y\frac{dy}{dx} = 0 \Rightarrow 2y\frac{dy}{dx} = -2x$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\cancel{2}x}{\cancel{2}y}$$

So, $\dfrac{dy}{dx} = -\dfrac{x}{y}$

(b) ③ Find Eq. tangent line.

$(x_0, y_0) = (3, 4)$

$\dfrac{dy}{dx} = -\dfrac{3}{4}$

$$y - y_0 = \left(\frac{dy}{dx}\right)(x - x_0)$$

$$\Rightarrow y - 4 = \left(\frac{dy}{dx}\right)(x-3)$$

$$\Rightarrow \boxed{y = -\frac{3}{4}(x-3) + 4}$$

Main steps for implicit differentiation:

1) Take the derivative on each side of the relation.
2) Use the chain rule and other rules to make the computations.
3) Isolate the derivative dy/dx.

EXAMPLE 2

(a) Find y' if $x^3 + y^3 = 6xy$.

(b) Find the tangent to the folium of Descartes $x^3 + y^3 = 6xy$ at the point (3, 3).

(c) At what point in the first quadrant is the tangent line horizontal?

Desmos: https://www.desmos.com/calculator/efjuccxlrz

(a) Apply $\frac{d}{dx}$ $\quad \frac{d}{dx}(x^3+y^3) = \frac{d}{dx}(6xy)$

$$\Rightarrow \frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = 6\left(\frac{d}{dx}(x)\cdot y + x \cdot \frac{dy}{dx}\right)$$

$$\Rightarrow 3x^2 + 3y^2 \cdot \frac{dy}{dx} = 6\left(y + x\frac{dy}{dx}\right)$$

$$\Rightarrow 3x^2 + 3y^2\frac{dy}{dx} = 6y + 6x\frac{dy}{dx}$$

$$\Rightarrow 3x^2 - 6y = 6x\frac{dy}{dx} - 3y^2\frac{dy}{dx}$$

$$\Rightarrow 3x^2 - 6y = (6x - 3y^2)\frac{dy}{dx}$$

$$\Rightarrow \frac{3x^2-6y}{6x-3y^2} = \frac{dy}{dx}.$$

(b) tangent line $\quad y - y_0 = \frac{dy}{dx}(x - x_0)$

$(x_0, y_0) = (3,3)$

$\Rightarrow x_0 = 3$

$y_0 = 3$

$$\Rightarrow y - 3 = \frac{dy}{dx}(x-3)$$

$$\frac{dy}{dx} = \frac{3\cdot 9 - 6\cdot 3}{6\cdot 3 - 3\cdot 9} = \frac{\cancel{3\cdot 9 - 6\cdot 3}}{-(\cancel{3\cdot 9 - 6\cdot 3})} = \frac{1}{-1} = -1$$

So, $y - 3 = -(x-3) \Rightarrow y = -x + 3 + 3 = 6 - x$

So, $\boxed{y = 6 - x}$

EXAMPLE 3 Find y' if $\sin(x+y) = y^2 \cos x$.

Apply $\frac{d}{dx}$

$$\Rightarrow \frac{d}{dx}\left[\sin(x+y)\right] = \frac{d}{dx}\left[y^2 \cos x\right]$$

$$\Rightarrow \cos(x+y)\cdot\left(1+\frac{dy}{dx}\right) = 2y\left(\frac{dy}{dx}\right)\cos(x) + y^2(-\sin x)$$

$$\Rightarrow \cos(x+y) + \left[\cos(x+y)\right]\frac{dy}{dx} = \left[2y\cos(x)\right]\frac{dy}{dx} - y^2\sin x$$

$$\Rightarrow \cos(x+y) + y^2\sin(x) = 2y\cos(x)\frac{dy}{dx} - \cos(x+y)\frac{dy}{dx}$$

$$\Rightarrow \cos(x+y) + y^2\sin(x) = \left[2y\cos(x) - \cos(x+y)\right]\frac{dy}{dx}$$

$$\Rightarrow \boxed{\frac{\cos(x+y)+y^2\sin(x)}{2y\cos(x)-\cos(x+y)} = \frac{dy}{dx}}.$$