Math 644

Problem Sets

Contents

Created by: Pierre-Olivier Parisé

Spring 2023

Chapter 1

Problem 1. Prove the parallelogram equality:

$$|z+w|^2 + |z-w|^2 = 2(|z|^2 + |w|^2).$$

Problem 2. Let w be a non-zero complex number and let $n \geq 1$ be a positive integrer. Using the polar coordinates, find n solutions to $z^n = w$.

Problem 3. Let z be a non-zero complex number. Show that 0, z, iz, and $iz+z$ are the vertices of a square.

Problem 4. Prove that there is no complex number z so that

$$|z| - z = i.$$

Problem 5. Find all complex numbers z satisfying the equation

$$4z - 3\overline{z} = \frac{1-18i}{2-i}.$$

Problem 6. Suppose that f is a continuous complex-valued function on a real interval $[a,b]$. Let

$$A = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

a) Show that if $|f(x)| \leq |A|$ for all $x \in [a,b]$, then $f \equiv A$.

b) Show that if $|A| = \frac{1}{b-a}\int_a^b |f(x)|\,dx$, then $\arg f$ is constant modulo 2π on $\{z \,:\, f(z) \neq 0\}$.

Problem 7. Describe geometrically the following subsets:

a) $\operatorname{Re} z = \operatorname{Im} z$.

b) $\operatorname{Re} z > 0$.

c) $\operatorname{Im} z > 0$.

d) $\frac{\pi}{6} < \arg z < \frac{\pi}{4}$.

Problem 8. Let $\mathbb{T} := \{z \in \mathbb{C} \,:\, |z| = 1\}$. Prove that $\mathbb{T}$ equipped with the complex multiplication is a commutative group.

Problem 9. Suppose that $\lim_{n\to\infty} w_n = w$. Is it true then that also

$$\lim_{n\to\infty} \arg w_n = \arg w?$$

Problem 10. Let $\{z_n\}$ be a sequence of complex numbers such that $\sum_{n=0}^{\infty} z_n$ converges and there is a ϕ such that $|\arg z_n| \leq \phi < \frac{\pi}{2}$ for any $n \geq 0$. Show that the series $\sum_{n=0}^{\infty} z_n$ is absolutely convergent.

Problem 11. Let $\mathbb{C}^*$ be the extended plane, let $\mathbb{S}^2$ be the sphere $\{(X, Y, Z) : X^2 + Y^2 + Z^2 = 1\}$ and let $\pi : \mathbb{C}^* \to \mathbb{S}^2$ be the stereographic projection with $\pi(\infty) = (0, 0, 1)$.

a) Show that straight lines in $\mathbb{C}$ correspond exactly to circles on $\mathbb{S}^2$ passing through $(0, 0, 1)$.

b) Show that if $z \neq \infty$, then

$$\chi(z, \infty) = \frac{2}{\sqrt{1 + |z|^2}}.$$

c) Using the explicit formula of χ in terms of z and w, show that, for any $z, w \in \mathbb{C}^*$,

$$0 \leq \chi(z, w) \leq 2.$$

Chapter 2

Problem 12. For what values of z is

$$\sum_{n=0}^{\infty} \left(\frac{z}{1+z} \right)^n$$

convergent? Draw a picture of the region.

Problem 13. Suppose that $\sum_{n\geq 0} a_n(z-z_0)^n$ is a formal power series. Suppose that

$$R := \frac{1}{\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|}$$

exists and is finite.

a) Show that the power series converges in $\{z \,:\, |z-z_0| < R\}$.

b) Show that the power series diverges in $\{z \,:\, |z-z_0| > R\}$.

Problem 14. Define $e^z = \exp(z) := \sum_{n\geq 0} \frac{z^n}{n!}$.

a) Show that $e^z e^w = e^{z+w}$ (using the power series definition).

b) Show that $|e^z| = e^{\operatorname{Re} z}$ and $\arg e^z = \operatorname{Im} z$.

c) Show that $\frac{d}{dz} e^z = e^z$.

d) Show that, for any non-zero integer n,

$$\int_0^{2\pi} e^{int}\, dt = 0.$$

[Hint: Use Foundamental Theorem of Calculus.]

e) Compute the integral

$$\int e^{nt} \cos(mt)\, dt.$$

[Hint: Rewrite $\cos(mt)$ as a complex exponential.]

Problem 15. Prove the following assertions.

a) If f and g are analytic at z_0, then $(f+g)'(z_0) = f'(z_0) + g'(z_0)$ (Sum rule of differentiation for analytic functions).

b) If f and g are analytic at z_0, then $(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$ (Product Rule of differentiation for analytic functions).

c) If f and g are analytic at z_0 with $g(z_0) \neq 0$, then $(fg)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{(g(z_0))^2}$ (Quotient rule of differentiation for analytic functions).

d) If f is analytic at z_0 and g is analytic at $f(z_0)$, then $(g \circ f)'(z_0) = g'(f(z_0))f'(z_0)$ (Chain Rule of differentiation for analytic functions).

Find the derivative of $(z-a)^{-n}$, where n is a positive integer and $a \in \mathbb{C}$.

Problem 16. Let $\Omega \subset \mathbb{C}$. Show that Ω is connected if and only if Ω and $\varnothing$ are the only open and closed subsets of Ω.

Problem 17. Suppose that f and g are two analytic functions on a region (open and connected) Ω. Suppose there is a sequence $(z_n)_{n=1}^{\infty}$ with $z_n \in \Omega$ ($\forall n \geq 1$) such that $f(z_n) = g(z_n)$ ($\forall n \geq 1$). If (z_n) has an accumulation point $z_0 \in \Omega$, then show that $f \equiv g$ on Ω.

Problem 18. Show that $\cos^2(z) + \sin^2(z) = 1$ for every $z \in \mathbb{C}$.

Problem 19. Suppose f is analytic in a connected open set Ω such that, for each $z \in \Omega$, there exists an n (depending on z) such that $f^{(n)}(z) = 0$. Prove that f is a polynomial. *[Hint: Use Baire's Theorem.]*

Problem 20. Let f be analytic in a region Ω containing the point $z = 0$. Suppose $|f(1/n)| < e^{-n}$ for $n \geq n_0$, for some integer $n_0 \geq 0$. Prove $f \equiv 0$ in Ω.

Problem 21. Let f and g be analytic functions in a region Ω.

a) Show that if $f'(z) = 0$ for all z in a neighborhood of some $z_0 \in \Omega$, then f is constant in Ω, meaning there is a constant $c \in \mathbb{C}$ such that $f(z) = c$ for any $z \in \Omega$.

b) Show that if f and g are analytic in a region Ω with $f'(z) = g'(z)$ for every $z \in \Omega$, then $f - g$ is constant.

Problem 22. Suppose that $f(z) = az^3 + bz^2 + cz + d$. In addition, suppose that for each $z, w \in \mathbb{C}$ there exists a point ζ on the line segment between z and w with

$$\frac{f(z) - f(w)}{z - w} = f'(\zeta).$$

Show that $a = 0$.

Problem 23. Suppose that $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges in $B = \{z : |z - z_0| < r\}$. Show that the power series

$$F(z) := \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z - z_0)^{n+1}$$

converges in B and satisfies $F'(z) = f(z)$ for all $z \in B$. Moreover, show that the radius of convergence of F is the same as the radius of convergence of f.

Problem 24. Suppose $\sum_{j=0}^{\infty} |a_j|^2 < \infty$.

a) Show that $f(z) = \sum_{j=0}^{\infty} a_j z^j$ is analytic in $\{z : |z| < 1\}$.

b) Compute (with a proof) the following quantity:

$$\lim_{r\to 1^-} \int_0^{2\pi} |f(re^{i\theta})|^2 \,\frac{d\theta}{2\pi}.$$

PROBLEM 25. Suppose f has a power series expansion at 0 which converges in all of $\mathbb{C}$. Suppose also that $\int_{\mathbb{C}} |f(x+iy)|\,dxdy < \infty$. Prove that $f \equiv 0$.

PROBLEM 26. [Hard] Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ have radius of convergence 1 and suppose that $a_n \geq 0$ for all n. Prove that $z = 1$ is a singular point of f. That is, there is no function g analytic in a ball B containing $z = 1$ such that $f = g$ on $B \cap D$.

Chapter 3

Problem 27. If f is analytic in a region Ω and if there is a $z_0 \in \Omega$ such that

$$|f(z_0)| = \inf_{z \in \Omega} |f(z)|,$$

and if $f(z_0) \neq 0$, then f is constant in Ω.

Problem 28. Let Ω be a region in $\mathbb{C}$. Show that if $f : \Omega \to \mathbb{C}$ is an open map, then f satisfies the maximum modulus principle.

Problem 29.

a) Show geometrically why the maximum principle holds using a "walking the dog" argument. Make it rigorous by following the steps of the proof of the Fundamental Theorem of Algebra.

b) Use the maximum modulus principle to prove the Fundamental Theorem of Algebra.

Problem 30. Let f be an analytic function defined on some bounded region $\Omega \subset \mathbb{C}$. Show that

$$\limsup_{z \to \partial \mathbb{D}} |f(z)| = \lim_{\delta \to 0} \sup\{|f(z)| \,:\, z \in \Omega, \mathrm{dist}(z, \partial\Omega) = \delta\}.$$

Problem 31. Let $\Omega \subset \mathbb{C}$ be a region. Show that $z_n \to \partial\Omega$ as $n \to \infty$ if and only if $(z_n)_{n \geq 1}$ has no subsequence converging to a point $z_0 \in \Omega$.

Problem 32. Suppose that f is analytic in a connected open (region) set Ω.

a) Prove that if $|f(z)|$ is constant on Ω, then f is contant on Ω.

b) Prove that if $\mathrm{Re}\, f$ is constant on Ω, then f is constant on Ω.

Problem 33.

a) Prove that if f is analytic in $\mathbb{C}$, then $f(z) = \sum_{n \geq 0} a_n z^n$ for any $z \in \mathbb{C}$. In other words, the radius of convergence of the power series $\sum_{n \geq 0} a_n z^n$ representing f at $z = 0$ is $R = \infty$.

b) Suppose that f is analytic in $\mathbb{C}$ and $|f(z)| \leq C|z|^n$, for some $|z| > M$ and $n \geq 0$. Show that f must be a polynomial.

c) Suppose that f and g are analytic in $\mathbb{C}$ with $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$. Prove that there exists a constant $c \in \mathbb{C}$ such that $f(z) = cg(z)$ for all $z \in \mathbb{C}$.

Problem 34. Prove that if f is non-constant and analytic on all of $\mathbb{C}$, then $f(\mathbb{C})$ is dense in $\mathbb{C}$.

Problem 35. Let f be analytic in $\mathbb{D}$ and suppose $|f(z)| < 1$ on $\mathbb{D}$. Let $a = f(0)$. Show that f does not vanish in $\{z \,:\, |z| < |a|\}$.

Problem 36.

a) Prove that φ is a one-to-one analytic map of $\mathbb{D}$ onto $\mathbb{D}$ if and only if

$$\varphi(z) = c\left(\frac{z-a}{1-\bar{a}z}\right) \quad (z \in \mathbb{D}),$$

for some constants c and a, with $|c| = 1$ and $|a| < 1$.

b) Let f be analytic in $\mathbb{D}$ and satisfy $|f(z)| \to 1$ as $|z| \to 1$. Prove that f is rational.

Problem 37.

a) Suppose p is a non-constant polynomial with all its zeros in the upper half-plane $\mathbb{H} := \{z : \operatorname{Im} z > 0\}$. Prove that all the zeros of p' are contained in $\mathbb{H}$. *[Hint: Look at the partial fraction expansion of p'/p.]*

b) Use **a)** to prove that if p is a polynomial, then the zeros of p' are contained in the (closed) convex hull of the zeros of p. (The closed convex hull is the intersection of all half-planes containing the zeros.)

Problem 38. Suppse f is analytic in $\mathbb{D}$ and $|f(z)| \leq 1$ in $\mathbb{D}$ and $f(0) = 1/2$. Prove that $|f(1/3)| \geq 1/5$.

Problem 39. Suppose f is analytic and non-constant in $\mathbb{D}$ and $|f(z)| \leq M$ on $\mathbb{D}$. Prove that the number of zeros of f in a disk of radius $1/4$, centered at 0, does not exceed

$$\frac{1}{\ln 4} \ln \left| \frac{M}{f(0)} \right|.$$

Chapter 4

Problem 40. Find a parametrization for each set.

(a) A square of height h and length l.

(b) The arc of the unit circle starting at 1 and ending at $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$.

(c) The parabola starting at $(0,0)$ and ending at $(1,1)$.

(d) An ellipse centered at $z_0 \in \mathbb{C}$ with minor axis m and major axis M.

Problem 41. Which of the curves in the previous problem are (i) an arc (ii) closed (ii) simple?

Problem 42. Find the value of

$$\int_\gamma x - y + ix^2 \, dz$$

if γ is

(a) the straight line joining 0 to $1+i$;

(b) the imaginary axis from 0 to i;

(c) the line parallel to the real axis from i to $1+i$.

Problem 43. Compute $\displaystyle\int_\gamma \frac{1}{z^3} \, dz$, where γ is the circle of radius $1/2$ centered at the origin. Compare with Example 9 from Section 4.1.

Problem 44. Let γ denote a circular path with center 1 and radius 1, described once counter-clockwise and starting at the point 2.

(a) Find a parametrization of γ.

(b) Find the value of $\displaystyle\int_\gamma |z|^2 \, dz$.

Problem 45. Suppose that f and g are two holomorphic function on a region Ω containing a piecewise continuously differentiable curve $\gamma : [a,b] \to \mathbb{C}$. Prove the complex analogue of the integration by parts formula:

$$\int_\gamma f(z) g'(z) \, dz = f(z_1)g(z_1) - f(z_0)g(z_0) - \int_\gamma f'(z) g(z) \, dz$$

where $z_1 = \gamma(b)$ and $z_0 = \gamma(a)$.

Problem 46. Let γ be a piecewise continuously differentiable curve. Prove that

$$\int_{-\gamma} f(z) \, dz = -\int_\gamma f(z) \, dz.$$

Problem 47. Let $\alpha, \beta : [0,1] \to \mathbb{C}$ be two curves. Show that $\alpha + \beta = \beta + \alpha$ (in the sense of Definition 12 in Section 4.1).

Problem 48. Prove Corollary 16 in Section 4.1.

Problem 49. Prove all of the properties on page 9 in section 4.1.

Problem 50. Let Ω be an open set.

(a) Let n be a positive integer. Define a function $g(z) = \sqrt[n]{z}$ on $B = \{z : |z-1| < 1\}$.

(b) Show that $g(z) = \sqrt[n]{z}$ is holomorphic on B.

Problem 51. Let f be analytic in a neighborhood of the closure of a bounded convex set S with piecewise continuously differentiable boundary ∂S. Show that for any $z \in S$,

$$f(z) = \frac{1}{2\pi} \int_{\partial S} \frac{f(\zeta)}{\zeta - z} \, d\zeta,$$

where ∂S is parameterized in the counter-clockwise direction.

Problem 52. Let f be analytic in a neighborhood of the closure of a bounded convex set S with piecewise continuously differentiable boundary ∂S. Show that

$$\frac{1}{2\pi i} \int_{\partial S} f(\zeta) \, d\zeta = 0.$$

Problem 53. Show that there is no function f defined in a neighborhood Ω of $\partial \mathbb{D}$ such that $f'(z) = 1/z$ for any $z \in \Omega$.

Problem 54.

(a) Use Cauchy's estimate to prove Liouiville's Theorem.

(b) Use Cauchy's estimate to compute a lower bounded on the radius of convergence of the power series representation of a bounded holomorphic function.

Problem 55. Suppose Ω is a region which is symmetric with respect to $\mathbb{R}$, meaning that $z \in \Omega$ if and only if $\overline{z} \in \Omega$. Set $\Omega := \Omega \cap \mathbb{H}$ and $\Omega^- := \Omega \cap (\mathbb{C} \backslash \overline{\mathbb{H}})$, where $\mathbb{H} := \{z : \operatorname{Im} z > 0\}$. If f is continuous on Ω^+, continuous on $\Omega^+ \cup (\Omega \cup \mathbb{R})$, and $\operatorname{Im} f = 0$ on $\Omega \cap \mathbb{R}$, then show that the function defined by

$$F(z) := \begin{cases} f(z) & \text{if } z \in \Omega \backslash \Omega^- \\ \\ \overline{f(\overline{z})} & \text{if } z \in \Omega^- \end{cases}$$

is analytic on Ω.

PROBLEM 56. Let $\gamma : [a, b] \to \Omega$ be a curve, where Ω is a region in $\mathbb{C}$. Assume that f is analytic in Ω. The goal of this problem is to generalized the integral of f along a curve (not necessarily piecewise continuously differentiable).

(a) Given $0 < \varepsilon < \operatorname{dist}(\gamma, \partial\Omega)$, show that there is a finite partition $0 = t_0 < t_1 < \cdots < t_n = 1$ so that $\gamma([t_{j-1}, t_j]) \subset B_j := \{z : |z - \gamma(t_j)| < \varepsilon\} \subset \Omega$ for any $j = 1, 2, \ldots, n$.

(b) For a given $0 < \varepsilon < \operatorname{dist}(\gamma, \partial\Omega)$, take a partition from (a). Let $\sigma := \sum_{j=1}^{n} \sigma_j$ be the polygonal curve, where σ_j is the line segment joining $\gamma(t_{j-1})$ to $\gamma(t_j)$. Define

$$\int_\gamma f(\zeta)\, d\zeta := \int_\sigma f(\zeta)\, d\zeta.$$

Show that this definition does not depend on the choice of the polygonal curve σ.

(c) Show that the previous definition agrees for a piecewise continuously differentiable curve γ.

PROBLEM 57.

(a) For $x > 0$, define $x^{-z} := e^{-z\ln(x)}$. Prove that the Riemann Zeta Function

$$\zeta(z) := \sum_{n=1}^{\infty} n^{-z}$$

converges and is analytic in $\{z : \operatorname{Re} z > 1\}$.

(b) Show that, for $\operatorname{Re} z > 1$,

$$\zeta(z) - \int_1^\infty x^{-z}\, dx = \sum_{n=1}^{\infty} \int_n^{n+1} \int_n^x z t^{-z-1}\, dt dx.$$

(c) Use (b) to prove that $(z-1)\zeta(z)$ has a unique analytic extension to $\{z : \operatorname{Re} z > 0\}$.

(d) Use the fundamental theorem of calculus to give a series for $\zeta(z) - 1/(z-1)$, valid in $\operatorname{Re} z > 0$, which does not involve an integral.

PROBLEM 58. Show that there is a constant $C < \infty$ so that if f is analytic on $\mathbb{D}$, then

$$|f'(z)| \leq C \int_{\mathbb{D}} |f(x + iy)|\, dxdy$$

for all $|z| \leq 1/2$. [Hint: Use Cauchy's integral formula.]

PROBLEM 59. Prove that there exists a sequence of polynomials p_k such that

$$\lim_{k\to\infty} p_k(z) = \begin{cases} 1 & \text{if } \operatorname{Re} z > 0 \\ 0 & \text{if } \operatorname{Re} z = 0 \\ -1 & \text{if } \operatorname{Re} z < 0. \end{cases}$$

PROBLEM 60. Suppose f has a complex derivative at each point of a region Ω. Prove that f is analytic in Ω.