

M444 – Complex Analysis

Pierre-Olivier Parisé

University of Hawai'i at Mānoa
Chapter 3

Section 3.8: Cauchy's Integral Formula

Lemma

Let γ be a positively oriented Jordan curve and let $z_0 \notin \mathbb{C}$. Then

$$\int_{\gamma} \frac{1}{z - z_0} dz = \begin{cases} 0 & z_0 \in \Omega^+ \\ 2\pi i & z_0 \in \Omega^- \end{cases}$$

Proof. Assume $z_0 \in \Omega^+$. Then $f(z) = (z - z_0)^{-1}$ is analytic on $\Omega^- \cup \gamma$ and by Cauchy's Theorem

$$\int_{\gamma} \frac{1}{z - z_0} dz = 0.$$

Assume $z_0 \in \Omega^-$.

- Consider $B_{z_0}(r) \subset \Omega^-$ and let $C := C_{z_0}(r)$ be its boundary.
- Let $[z_1, \dots, z_2]$ and $[z_3, \dots, z_4]$ be two different polygonal curves with starting point on C and terminal point on γ .

Let $f(z) = (z - z_0)^{-1}$. By Cauchy's Theorem, we get

$$\int_{\Gamma_1} \frac{1}{z - z_0} dz = 0 \quad \text{and} \quad \int_{\Gamma_2} \frac{1}{z - z_0} dz = 0.$$

Therefore,

$$\int_{\Gamma_1} f(z) dz + \int_{\Gamma_2} f(z) dz = 0.$$

But

$$\int_{\Gamma_1} f(z) dz = \int_{\gamma_1} f(z) dz + \int_{[z_2, \dots, z_1]} f(z) dz - \int_{C_1} f(z) dz + \int_{[z_3, \dots, z_4]} f(z) dz$$

and

$$\int_{\Gamma_2} f(z) dz = \int_{\gamma_2} f(z) dz + \int_{[z_4, \dots, z_3]} f(z) dz - \int_{C_2} f(z) dz + \int_{[z_1, \dots, z_2]} f(z) dz$$

Adding up :

$$\int_{\gamma_1} f(z) dz + \int_{\gamma_2} f(z) dz - \int_{C_1} f(z) dz - \int_{C_2} f(z) dz = 0$$

and so

$$\int_{\gamma} f(z) dz - \int_C f(z) dz = 0.$$

This implies

$$\int_{\gamma} \frac{1}{z - z_0} dz = \int_C \frac{1}{z - z_0} dz = i \int_0^{2\pi} dt = 2\pi i.$$

This ends the proof. □

Theorem

Assume that

- ① f is an analytic function on a region U .
- ② γ is a Jordan curve such that $\Omega^- \cup \gamma \subset U$.
- ③ $z_0 \in \Omega^-$.

Then, for any $w \in \Omega^-$, we have

$$f(w) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - w} dz.$$

Proof. There is a disk $B_R(w) \subset \Omega^-$ and for any $0 < r \leq R$, we have

$$\int_{\gamma} \frac{f(z)}{z - w} dz = \int_{C_r(w)} \frac{f(z)}{z - w} dz.$$

Goal : To show the right-hand side is $2\pi i f(w)$ if $r \rightarrow 0^+$.

We have

$$\begin{aligned}\int_{C_r(w)} \frac{f(z)}{z-w} dz - 2\pi i f(w) &= \int_{C_r(w)} \frac{f(z) - f(w)}{z-w} dz \\ &= \int_0^{2\pi} \left(\frac{f(w + re^{it}) - f(w)}{re^{it}} \right) ire^{it} dt \\ &= i \int_0^{2\pi} f(w + re^{it}) - f(w) dt.\end{aligned}$$

Therefore,

$$\left| \int_{C_r(w)} \frac{f(z)}{z-w} dz - 2\pi i f(w) \right| \leq \int_0^{2\pi} |f(w + re^{it}) - f(w)| dt.$$

Because f is continuous, $f(w + re^{it}) - f(w) \rightarrow 0$ as $r \rightarrow 0^+$. Hence,

$$\lim_{r \rightarrow 0^+} \int_{C_r(w)} \frac{f(z)}{z-w} dz = 2\pi i f(w).$$

□

Example. Consider the integral

$$\int_{C_2(1)} \frac{z^2 + 3z - 1}{(z + 3)(z - 2)} dz.$$

Notice that -3 is outside of the circle $C_2(1)$ and hence the function

$$f(z) = \frac{z^2 + 3z - 1}{z + 3}$$

is analytic in the interior of and on the circle $C_2(1)$.

Using Cauchy's Integral Formula,

$$\int_{C_2(1)} \frac{z^2 + 3z - 1}{(z + 3)(z - 2)} dz = \int_{C_2(1)} \frac{\frac{z^2 + 3z - 1}{z + 3}}{z - 2} dz = 2\pi i f(2) = \frac{18\pi i}{5}.$$

Differentiation under the integral sign.

- ① γ be a Jordan curve with positively orientation and U be a region.
- ② $\phi : U \times \gamma \rightarrow \mathbb{C}$ be function.
- ③ For any fixed w , $\phi(w, z)$ is continuous in $z \in \gamma$.
- ④ For any z , $\phi(w, z)$ is analytic in $w \in U$.
- ⑤ Complex derivative $\frac{d}{dw}\phi(w, z)$ is continuous in $z \in \gamma$.

Then, the function

$$g(w) = \int_{\gamma} \phi(w, z) dz$$

is analytic in U and its derivative is

$$g'(w) = \int_{\gamma} \frac{d}{dw} \phi(w, z) dz.$$

Theorem

- ① f is analytic on a region U .
- ② γ is a Jordan curve such that $\Omega^- \cup \gamma \subset U$.

Then f has derivatives of **any order** n at all points w in the interior of γ , $f^{(n)}$ are analytic in U , and

$$f^{(n)}(w) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-w)^{n+1}} dz.$$

Proof. For $n = 0$, this is the Cauchy's Integral Formula :

$$f(w) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-w} dz.$$

Notice that $\phi(w, z) = \frac{1}{2\pi i} \frac{f(z)}{z-w}$ is

- continuous in $z \in \gamma$.
- analytic in $w \in \Omega^-$.

Also

$$\frac{d}{dw} \phi(w, z) = \frac{1}{2\pi i} \frac{f(z)}{(w - z)^2}$$

is continuous on γ .

Therefore, differentiating under the integral (DUI), we get

$$f'(w) = \int_{\gamma} \frac{d}{dw} \phi(w, z) dz = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - w)^2} dz.$$

Notice that f' is analytic (apply the first part of DUI).

Reapply differentiating under the integral on the function

$$\begin{aligned} \bullet \quad \phi(w, z) &= \frac{1}{2\pi i} \frac{f(z)}{(z - w)^2} \quad \Rightarrow \quad f''(w) = \frac{2}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - w)^3} dz. \\ \bullet \quad \phi(w, z) &= \frac{2}{2\pi i} \frac{f(z)}{(z - w)^3} \quad \Rightarrow \quad f^{(3)}(w) = \frac{3!}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - w)^4} dz. \end{aligned}$$

To complete the proof, use induction. □