Example 4

The position of a particle is given by the equation $s = f(t) = t^3 - 6t^2 + 9t$ where s and t are measured in meters and seconds respectively.

a) Find the velocity at time t.

b) What is the velocity after 2s.

c) When is the particle at rest?

d) ~~f)~~ Find the acceleration at time t and after 4s.

e) ~~g)~~ Graph the position, velocity, and acceleration functions for $0 \le t \le 5$.

f) ~~h)~~ When is the particle speeding up? When is it slowing down?

a) $v(t) = s'(t) = 3t^2 - 12t + 9$.

b) We $v(2) = 3 \cdot 4 - 12 \cdot 2 + 9 = 12 - 24 + 9$
$= -3$

c) This is when $v(t) = 0$

$$\Leftrightarrow 3t^2 - 12t + 9 = 0$$
$$\Leftrightarrow t^2 - 4t + 3 = 0$$
$$\Leftrightarrow (t-3)(t-1) = 0$$
$$\Leftrightarrow \underline{t = 3} \text{ or } \underline{t = 1}$$

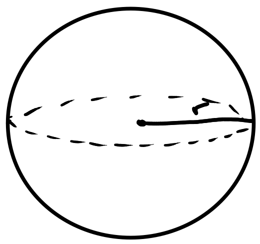
d) $a(t) = v'(t) = 6t - 12$

$a(4) = 6 \cdot 4 - 12 = 12 \text{ m/s}^2$

e) See desmos.

Example 5

Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm^3/s. How fast is the radius of the balloon increasing when the diamter is 50 cm?



V: volume in cm^3

t: time in seconds.

r: radius.

rate of increase of r ?? $\to \dfrac{dr}{dt}$

rate of volume $\to \dfrac{dV}{dt}$.

We have the diameter: 50 cm $\to$ 25 cm radius.

We have

$$V = \frac{4}{3}\pi r^3$$

Here $V = V(t)$ & $r = r(t)$

So,

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$$

$$\Rightarrow \frac{dV}{dt} = \frac{4}{3}\pi \frac{d}{dt}(r^3)$$

$$\Rightarrow \frac{dV}{dt} = \frac{4}{\cancel{3}}\pi\, \cancel{3} r^2 \cdot \frac{dr}{dt}$$

So,

$$\frac{dr}{dt} = \frac{dV/dt}{4\pi r^2}$$

Now, $r = 25$ cm, $dV/dt = 100$ cm/s²

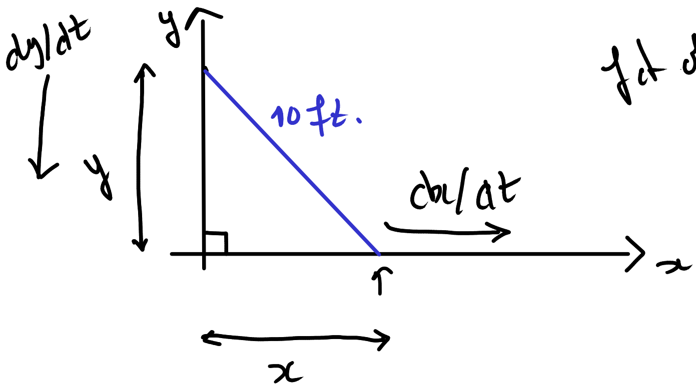
$$\frac{dr}{dt} = \frac{100}{4\pi 25^2} = \frac{100}{100\pi \cdot 25} = \frac{1}{25\pi}$$

So, $\dfrac{dr}{dt} = \dfrac{1}{25\pi}$ cm/s².

Example 6

A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall.

1) Diagram



fct. of t & x: displacement in the x-direction

fct. of t. & y: displacement in the y-direction

$\frac{dx}{dt}$: rate of change of x

$\frac{dy}{dt}$: rate of change of y.

2) Express a relation between x & y.

Goal Find $\frac{dy}{dt}$?

We have, from our friend Pyth,

$$x^2 + y^2 = 100$$

$$\Rightarrow \quad \frac{d}{dt}(x^2+y^2) = \frac{d}{dt}(100)$$

$$\Rightarrow \quad \frac{d}{dt}(x^2) + \frac{d}{dt}(y^2) = 0$$

$$\Rightarrow \quad \cancel{2}x\left(\frac{dx}{dt}\right) + \cancel{2}y \cdot \left(\frac{dy}{dt}\right) = 0$$

$x = \sqrt{100-y^2}$

$$\Rightarrow \quad \frac{dy}{dt} = -\frac{x}{y}\left(\frac{dx}{dt}\right)$$

So,

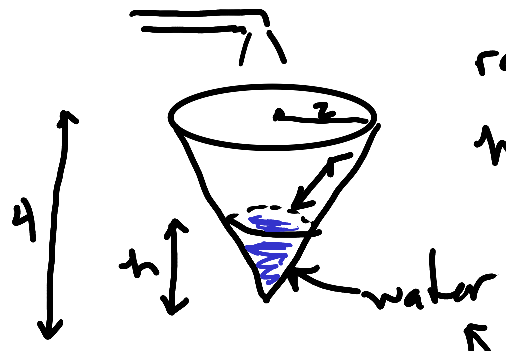
$$\frac{dy}{dt} = -\frac{\sqrt{100-6^2}}{6} \cdot 1 = -\frac{8}{6} = -\frac{4}{3} \text{ ft/s.}$$

$$-\frac{6}{\sqrt{100-6^2}} = -\frac{6}{8} = -\frac{3}{4} \text{ ft/s.}$$

Example 7

A water tank has the shape of an inverted circular cone with base radius 2m and height 4m. If water is being pumped into the tank at a rate of $2\text{m}^3/\text{min}$, find the rate at which the water level is rising when the water is 3m deep.

1) <u>Draw a diagram.</u>



radium 2m

heigth 4m

r = radium of the cone filled with water

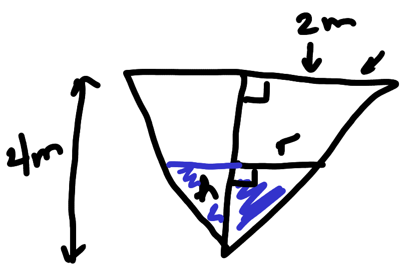
h = heigth " "
" " "
" .

V = Volume of the water

Goal: find $\dfrac{dh}{dt}$ at 3m (when $h=3$).

2) <u>Find a relation between V and h</u>

$$V = \frac{\pi r^2 h}{3} \quad \text{(Volume of a cone).}$$



$$\frac{2}{r} = \frac{4}{h}$$

So, $r = \dfrac{h}{2}$.

Then, $$V = \frac{\pi (h/2)^2 h}{3} = \frac{\pi\, h^3}{12}.$$

Now, $$\frac{dV}{dt} = \frac{\pi}{12}\frac{d}{dt}(h^3) = \frac{\pi}{12} 3h^2 \frac{dh}{dt}$$

$h=3 \Rightarrow$ $$2 = \frac{\pi}{4}(3)^2 \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{8}{9\pi}\ \text{m/min.}$$

Example 9

Find a linearization of $f(x) = \sqrt{\underbrace{x+3}}$ at $a = 1$. Use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$

We have $L(x) = f(a) + f'(a)(x-a)$

Here $a=1 \Rightarrow L(x) = f(1) + \underline{f'(1)}(x-1)$.

1) <u>Derivative of f.</u>

$$f'(x) = \left((x+3)^{1/2}\right)' = \frac{1}{2}(x+3)^{1/2-1} \cdot \frac{d}{dx}(x+3)$$

$$= \frac{1}{2\sqrt{x+3}}$$

So, $f'(1) = \dfrac{1}{2\sqrt{1+3}} = \dfrac{1}{4}$

We also have $f(1) = \sqrt{1+3} = 2$

Then $L(x) = 2 + \dfrac{x-1}{4}$.

2) <u>Compute $\sqrt{3.98}$</u> We have $3.98 = 3 + 0.98$

$$= 3 + x$$

$\Rightarrow x = 0.98$.

So, $\sqrt{3.98} = f(0.98) \approx L(0.98)$

$$= 2 + \frac{(0.98-1)}{4}$$

$$= 1.995$$

3) <u>Compute $\sqrt{4.05}$</u> We have $4.05 = x + 3$

$\Rightarrow x = 1.05$

So, $\sqrt{4.05} = f(1.05) \approx L(1.05)$

$= 2 + \dfrac{(1.05 - 1)}{4}$

$= 2.0125$

Example 11

Compare the values of Δy and dy if $y = f(x) = x^3 + x^2 - 2x + 1$ and x changes from

a) 2 to 2.05.

b) 2 to 2.01. $\to f(2.1) = 9.14 \qquad f(2) = 9$

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a) Here $x = 2$ and $x + \Delta x = 2.05 \to \Delta x = 0.05$

Usually, we put $dx = \Delta x = 0.05$

So,

$$dy = f'(x) \cdot dx = f'(2) \cdot 0.05$$

$f'(x) = 3x^2 + 2x - 2 \to f'(2) = 14.$

So, $dy = 14 \cdot 0.05 = 0.7.$

Also, $\Delta y = f(x + \Delta x) - f(x) = f(2.05) - f(2)$

$= 0.7075$

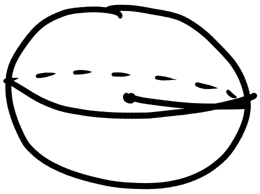
b) $x = 2$, $x + \Delta x = 2.01 \to \Delta x = 0.01$

$$dy = f'(2)\, dx = 14 \times 0.01 = 0.14$$

$$\Delta y = f(2.01) - f(2) = 0.14.$$

Example 12

The radius of a sphere was measured and found to be 21cm with a possible error in measurement of at most 0.05cm. What is the maximum error in using this value of the radius to compute the volume of the sphere?



$$V = \frac{4}{3}\pi r^3$$

$r = 21$ cm

<u>Def.</u> • Δr is error in the measurement

• ΔV is the error in the calculations of the volume

<u>Goal</u> find ΔV.

we gonna put $\Delta V \approx dV$

$$dV = V'(21)\, dr = (4\pi \cdot 21^2) \cdot 0.05 \approx 277.08847$$

so, $\Delta V \approx 277.08847$.