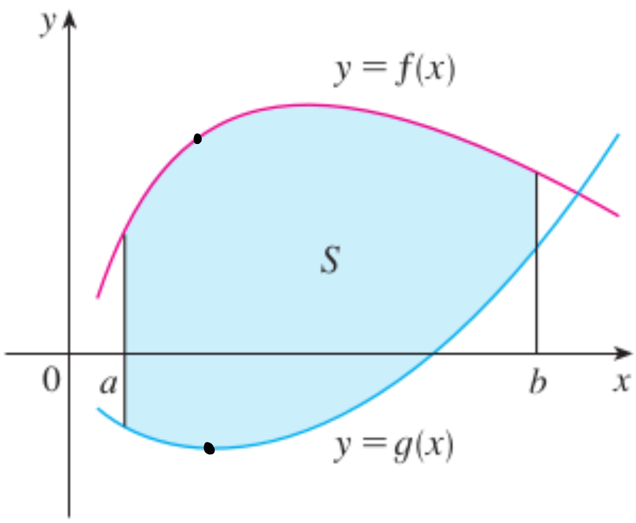
Chapter 5
Applications in integration

5.1 Areas between Curves



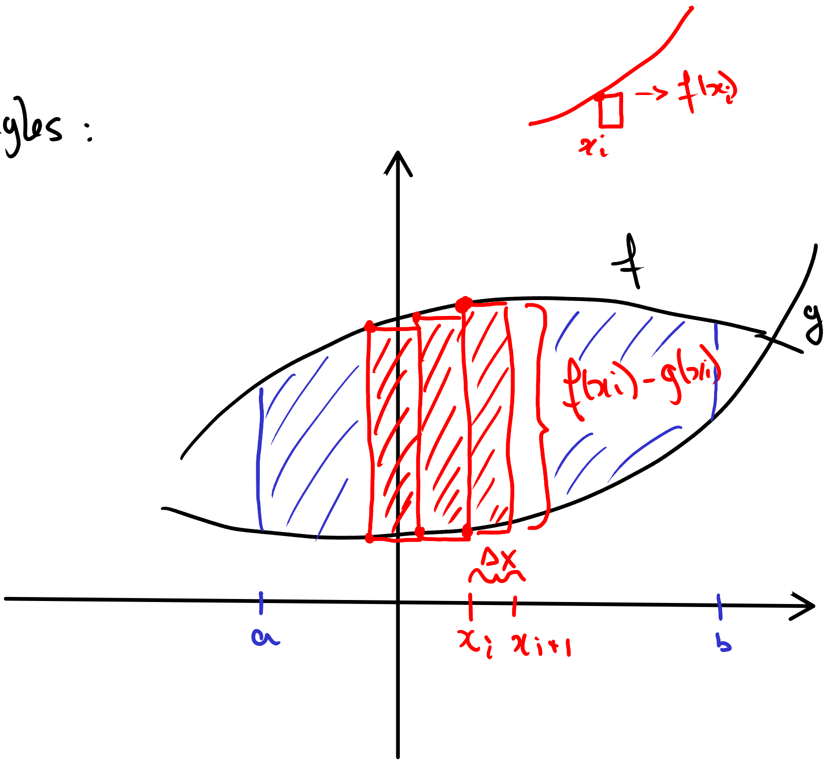
S: area between f & g.
f, g are two functions.

Assumptions: $f(x) \geq g(x)$ for any $a \leq x \leq b$

$\to$ $f(x) - g(x) \geq 0$.

Approximate A(S) by rectangles:

$$A(\underset{R_i}{\square}) = (f(x_i) - g(x_i))\Delta x$$



So,

$$A(S) \approx \sum_{i=1}^{n} A(R_i)$$
$$= \sum_{i=1}^{n} (f(x_i) - g(x_i))\Delta x$$

So, as $n \to \infty$

$$\lim_{n\to\infty} \sum_{i=1}^{n} (f(x_i) - g(x_i))\Delta x = \int_a^b f(x) - g(x) \, dx$$

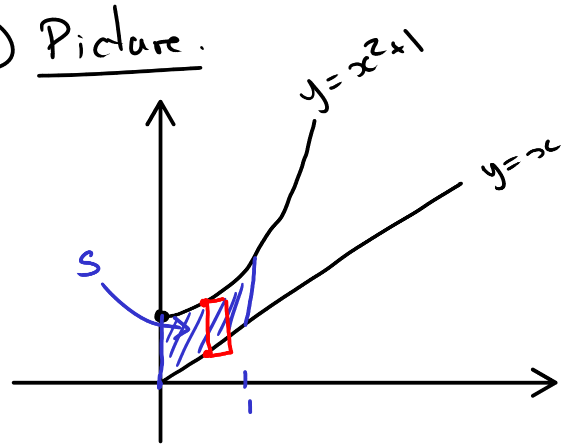
2 The area A of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, $x = b$, where f and g are continuous and $f(x) \geqslant g(x)$ for all x in $[a, b]$, is

$$A = \int_a^b [f(x) - g(x)]\, dx$$

EXAMPLE 1 Find the area of the region bounded above by $y = x^2 + 1$, bounded below by $y = x$, and bounded on the sides by $x = 0$ and $x = 1$.

$a = 0$, $b = 1$

① Picture.



x^2+1-x

$\Delta x \to dx$

$f(x) = x^2+1$.

$g(x) = x$.

$(x^2+1-x)\,\Delta x$

$\downarrow$ limit

$(x^2+1-x)\,dx$

② Integrate.

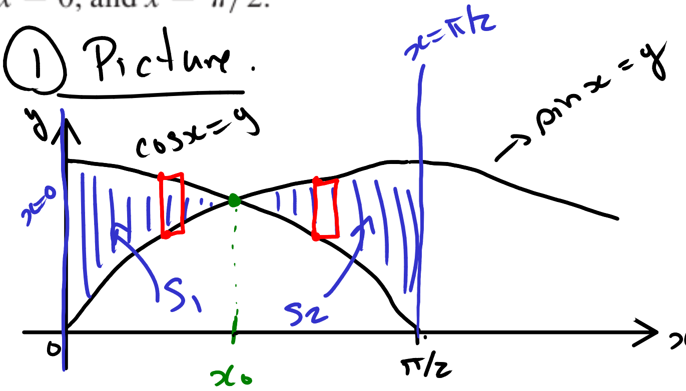
$$A(S) = \int_0^1 x^2+1-x \, dx = \int_0^1 x^2 dx + \int_0^1 1 \, dx - \int_0^1 x \, dx$$

$$= \frac{x^3}{3}\Big|_0^1 + x\Big|_0^1 - \frac{x^2}{2}\Big|_0^1$$

$$= \frac{1}{3} + 1 - \frac{1}{2} = \boxed{\frac{5}{6}}$$

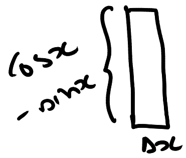
EXAMPLE 6 Find the area of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \pi/2$.

① Picture.

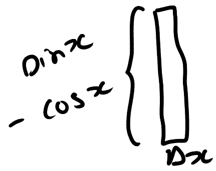


$A(S) = A(S_1) + A(S_2)$

A_1 $\quad 0 \le x \le x_0$



A_2 $\quad x_0 \le x \le \pi/2$



② Find the intersection. $\quad 0 \le x \le \pi/2$

$\cos x = \sin x \longrightarrow x_0 = \pi/4 \qquad \left(\begin{array}{l} \cos \pi/4 = \frac{\sqrt{2}}{2} \\ \sin \pi/4 = \frac{\sqrt{2}}{2} \end{array} \right)$

③ Area of S_1

$$A(S_1) = \int_0^{\pi/4} \cos x - \sin x \, dx = \int_0^{\pi/4} \cos x \, dx - \int_0^{\pi/4} \sin x \, dx$$

$$= \sin x \Big|_0^{\pi/4} - (-\cos x)\Big|_0^{\pi/4}$$

$$= \sin \pi/4 - \cancel{\sin 0}^{\,0} - (-\cos \pi/4 + 1)$$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 1$$

$$= \frac{2\sqrt{2} - 2}{2}$$

④ Area of S_2.

$$A(S_2) = \int_{\pi/4}^{\pi/2} \sin x - \cos x \, dx = \int_{\pi/4}^{\pi/2} \sin x - \int_{\pi/4}^{\pi/2} \cos x \, dx$$

$$= -\cos x \Big|_{\pi/4}^{\pi/2} - \sin x \Big|_{\pi/4}^{\pi/2}$$

$$= -\cancel{\cos \pi/2}^{\,0} + \cos \pi/4$$

$$- (\cancel{\sin \pi/2}^{\,1} - \sin \pi/4)$$

$$= \frac{\sqrt{2}}{2} - 1 + \frac{\sqrt{2}}{2}$$

$$= \frac{2\sqrt{2} - 2}{2}$$

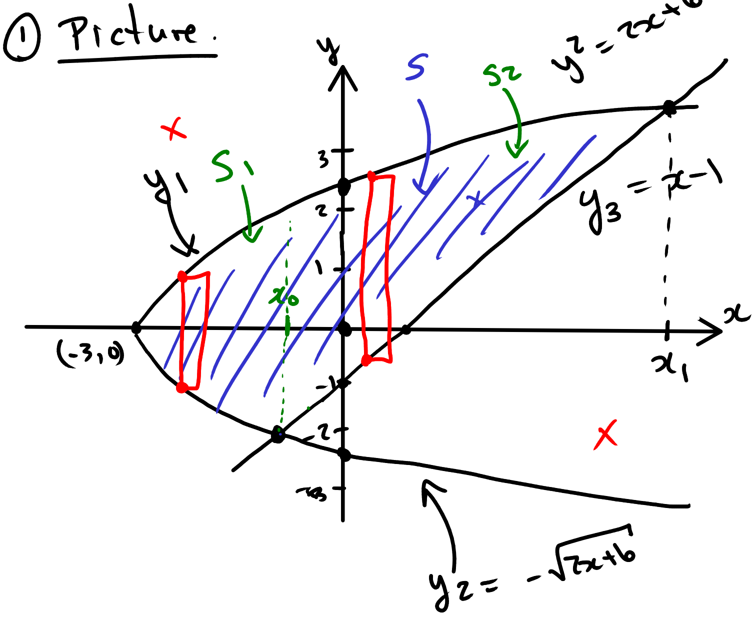
⑤ Answer.

$$A(S) = \frac{2\sqrt{2} - 2}{2} + \frac{2\sqrt{2} - 2}{2} = \boxed{2\sqrt{2} - 2}$$

EXAMPLE 7 Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$. $\to$ $x = \frac{y^2}{2} - 3$

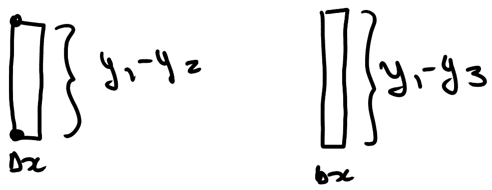
$y^2 = 6 \to y = \pm\sqrt{6}$.

① Picture.



$$y^2 = 2x+6 \to y_1 = \sqrt{2x+6}, \quad y_2 = -\sqrt{2x+6}$$

A_1 $-3 \le x \le x_0$ $\qquad$ A_2 $x_0 \le x \le x_1$



② Intersections.

$y = x-1 \xrightarrow[\text{plug-in}]{}$ $(x-1)^2 = 2x+6 \to x^2 - 2x + 1 = 2x + 6$

$\to x = 5$
$\quad x = -1$

so, $x_0 = -1$ & $x_1 = 5$

③ Area of S_1

$$A(S_1) = \int_{-3}^{-1} y_1 - y_2 \, dx = \int_{-3}^{-1} \sqrt{2x+6} - \left(-\sqrt{2x+6}\right) dx$$

$$= 2\int_{-3}^{-1} \sqrt{2x+6}\, dx \qquad u = 2x+6, \quad du = 2dx$$

$$= \int_0^4 \sqrt{u} \quad du$$

$$= \frac{2}{3} u^{3/2} \Big|_0^4 = \frac{2}{3} 4^{3/2} = \frac{16}{3}$$

④ Area of S_2.

$$A(S_2) = \int_{-1}^{5} y_1 - y_3 \, dx = \int_{-1}^{5} \sqrt{2x+6} - (x-1)\, dx$$

$u = 2x+6$
$du = 2dx$
$\to \frac{du}{2} = dx$

$$= \int_{-1}^{5} \sqrt{2x+6}\, \frac{2dx}{2} - \int_{-1}^{5} x\, dx + \int_{-1}^{5} 1\, dx$$

$$= \frac{1}{2}\int_4^{16} \sqrt{u} \quad du - \frac{x^2}{2}\Big|_{-1}^{5} + x\Big|_{-1}^{5}$$

$$= \frac{1}{2} \quad \frac{u^{3/2}}{3/2}\Big|_4^{16} - \left(\frac{25}{2} - \frac{1}{2}\right) + 5-(-1)$$

$$= \frac{1}{2}\left(\frac{64}{3/2} - \frac{8}{3/2}\right) - 12 + 6$$

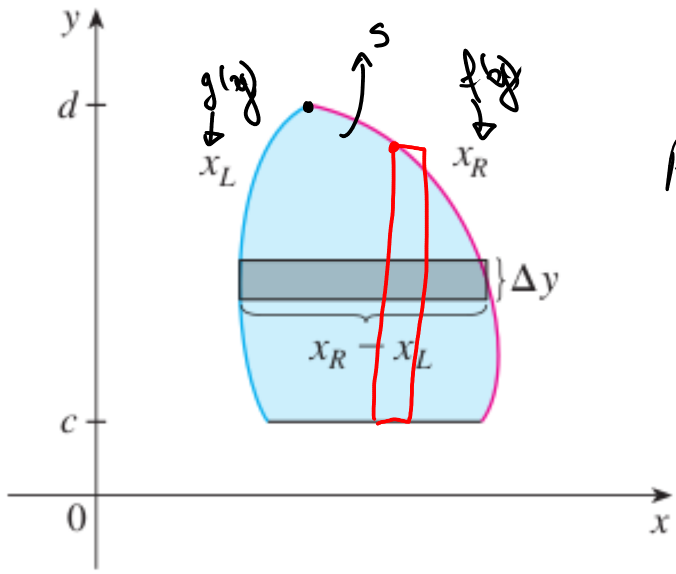
$$= \frac{64}{3} - \frac{8}{3} - 6 = \frac{56}{3} - 6$$

⑤ Answer

$$A(5) = \frac{16}{3} + \frac{56}{3} - 6 = \boxed{\frac{72}{3} - 6} \rightarrow \frac{72}{3} = 24$$

$$\downarrow$$

$$24 - 6 = \boxed{18}$$



bounded by
$x = f(y)$, $x = g(y)$, $y = c$ & $y = d$

Approximate by horizontal rectangles R_i :

$$A(R_i) = (x_R - x_L)\,\Delta y$$

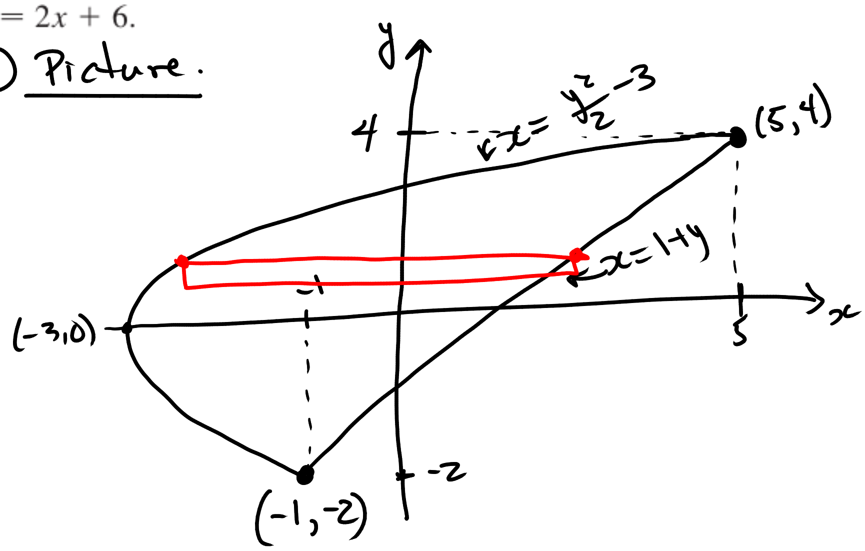
$$\rightarrow \quad A(S) \approx \sum_{i=1}^{n} \underbrace{(x_R - x_L)\,\Delta y}_{A(R_i)}$$

taking $n \to \infty$

$$\Rightarrow \quad A(S) = \boxed{\int_c^b x_R - x_L \, dy} = \int_c^d f(y) - g(y)\, dy$$

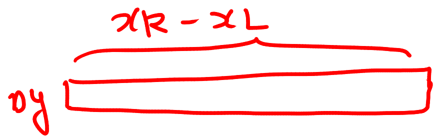
EXAMPLE 7 Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

① Picture.



$x = 1 + y$

$x = \frac{y^2}{2} - 3$



$x_R = 1 + y$

$x_L = \frac{y^2}{2} - 3$

② Integrate.

$$A(S) = \int_{-2}^{4} 1+y - \left(y^2/2 - 3\right) dy$$

$$= \int_{-2}^{4} 4 + y - y^2/2 \, dy$$

$$= 4y\Big|_{-2}^{4} + \frac{y^2}{2}\Big|_{-2}^{4} - \frac{y^3}{6}\Big|_{-2}^{4}$$

$$= (16 + 8) + (8 - 2) - \left(\frac{64}{6} + \frac{8}{6}\right)$$

$$= 24 + 6 - \frac{72}{6} = 30 - \frac{72}{6} = 30 - 12 = \boxed{18}$$