Math 644

Chapter 1

Section 1.1: Complex Numbers

Contents

Created by: Pierre-Olivier Parisé
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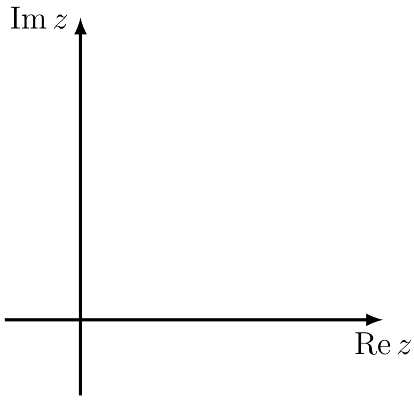
Definitions

- $\mathbb{C} := \{(a,b) \,:\, a,b \in \mathbb{R}\}$.
- $i \sim (0,1)$ and $1 \sim (1,0)$, so that

$$z \in \mathbb{C} \iff z = a + ib.$$

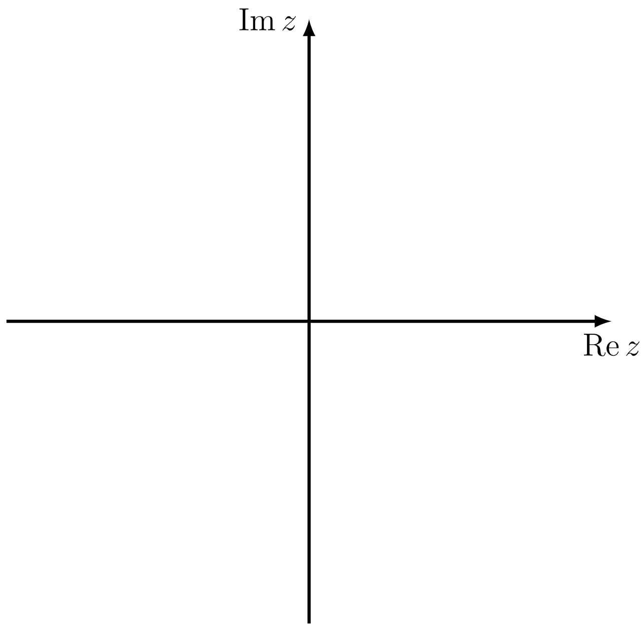


- Polar representation:

$$z = a + ib \iff a = r\cos\theta,\ b = r\sin\theta \iff z \simeq (r,\theta),$$

  where $r = \sqrt{a^2 + b^2}$ and $\tan\theta = b/a$.

  Note: θ is not defined for $z = 0$.



- Exponential form:

$$z = re^{i\theta}$$

  where $e^{i\theta} = \cos\theta + i\sin\theta$ and $r = \sqrt{a^2 + b^2}$.
- $z = a + ib$, then
  - $\operatorname{Re} z := a$;
  - $\operatorname{Im} z := b$.

Basic Arithmetic

Let $z = a + ib \simeq (r, \theta)$ and $w = c + id \simeq (\rho, \psi)$, then

- **Addition:** $z + w := (a + c) + i(b + d)$;
- **Multiplication:** $z \cdot w := (ac - bd) + i(ad + bc)$;
- **Equal:** $z = w \iff a = c$ and $b = d$;
- **Mult. in Polar form:** $z \cdot w \simeq (r\rho, \theta + \psi)$;
- **Equal in Polar form:** $z = w \iff r = \rho$ and $\theta = \psi + 2k\pi, \quad k \in \mathbb{Z}$;
- **Mult. in Exponential form:** $zw = r\rho e^{i(\theta+\psi)}$;
- $(\mathbb{C}, +, \cdot)$ is a commutative field with
  - Additive zero is $z = 0$;
  - Multiplicative identity is $z = 1$.

Example 1.

a) Compute $(2 + 2i)(-1 + i)$.

b) Find r, θ for $2 + 2i$, find ρ, ψ for $-1 + i$, and compute $(2+2i)(-1+i)$ using polar coordinates.

c) Compute $(a + ib)(a - ib)$.

d) Find an expression for the inverse of $a + ib$.

Other Operations

Let $z = a + ib \simeq (r, \theta)$ and $w = c + id \simeq (\rho, \psi)$.

- **Absolute Value or modulus:** $|z| := \sqrt{a^2 + b^2}$.
- **Argument:** $\arg z := \theta$. Common convention is to choose $\arg z \in (-\pi, \pi]$.
- **Complex Conjugate:** $\overline{z} := a - ib$.
- **Conjugate and Modulus:** $|z|^2 = z\overline{z}$.
- **Division revisited:**

  ★ $\displaystyle \frac{1}{w} = \frac{\overline{w}}{w\overline{w}} = \frac{\overline{w}}{|w|^2} = \frac{1}{r}e^{-i\theta};$ ★ $\displaystyle \frac{z}{w} = \frac{z\overline{w}}{|w|^2} = \frac{r}{\rho}e^{i(\theta-\psi)}.$

Theorem 2. For $z, w \in \mathbb{C}$, then

a) $|zw| = |z||w|$;

b) $|z/|z|| = 1$;

c) $|e^{i\theta}| = 1$;

d) $\operatorname{Re} z = \frac{z+\overline{z}}{2}$;

e) $\operatorname{Im} z = \frac{z-\overline{z}}{2i}$;

f) $\overline{z + w} = \overline{z} + \overline{w}$;

g) $\overline{zw} = \overline{z}\overline{w}$;

h) $\overline{\overline{z}} = z$;

i) $|z| = |\overline{z}|$;

j) $\arg(zw) = \arg(z) + \arg(w) \pmod{2\pi}$;

k) $\arg(\overline{z}) = -\arg(z) = 2\pi - \arg z \pmod{2\pi}$.

Proof. Prove some of the above properties.

For $z, w \in \mathbb{C}$, we define a function $d : \mathbb{C} \times \mathbb{C} \to [0, \infty)$ by

$$d(z, w) := |z - w|.$$

THEOREM 3. $(\mathbb{C}, d)$ is a complete metric space.

Proof. Prove this assertion in two lines.

Important Subsets

- **Open disc:** For $a \in \mathbb{C}$ and $r \in [0, \infty)$, an open disc is the set

$$\{z \in \mathbb{C} : |z - a| < r\}.$$

- **Closed disc:** For $a \in \mathbb{C}$ and $r \in [0, \infty)$, a closed disc is the set

$$\{z \in \mathbb{C} : |z - a| \leq r\}.$$

- **Circles:** For $a \in \mathbb{C}$ and $r \in [0, 1)$, a circle is the set

$$\{z \in \mathbb{C} : |z - a| = r\} = \partial\{z \in \mathbb{C} : |z - a| < r\}.$$

- **Open unit disc:** We denote by $\mathbb{D}$ the open unit disc, meaning

$$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}.$$

- **Unit circle:** We denote by $\mathbb{T}$ the unit circle, meaning

$$\mathbb{T} := \partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}.$$

Note: The topology of $\mathbb{C}$ is generated by the family of all open discs.