Chapter 1

Functions and Limits

1.6 Calculating Limits Using the Limit Laws

Limit Laws Suppose that c is a constant and the limits

$$\lim_{x \to a} f(x) = L \quad \text{and} \quad \lim_{x \to a} g(x) = M$$

exist. Then

1. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$ $= L + M$ [Sum rule]

2. $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$ $= L - M$ [Difference rule]

3. $\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$ $= cL$ [Constant Rule]

4. $\lim_{x \to a} [f(x)\, g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$ $= L \cdot M$ [Product Rule]

5. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \dfrac{L}{M}$ if $\lim_{x \to a} g(x) \neq 0$ ($M \neq 0$) [Quotient Rule]

Also apply to $\lim_{x \to a^+}$ or $\lim_{x \to a^-}$

EXAMPLE 1 Use the Limit Laws and the graphs of f and g in Figure 1 to evaluate the following limits, if they exist.

(a) $\lim_{x \to -2} [f(x) + 5g(x)]$ (b) $\lim_{x \to 1} [f(x)g(x)]$ (c) $\lim_{x \to 2} \dfrac{f(x)}{g(x)}$

(a) $\lim_{x \to -2} f(x) = 1$ & $\lim_{x \to -2} g(x) = -1$

$$\lim_{x \to -2} [f(x) + 5g(x)] = \lim_{x \to -2} f(x) + \lim_{x \to -2} 5g(x)$$
$$= 1 + 5 \lim_{x \to -2} g(x)$$
$$= 1 - 5 = \boxed{-4}$$

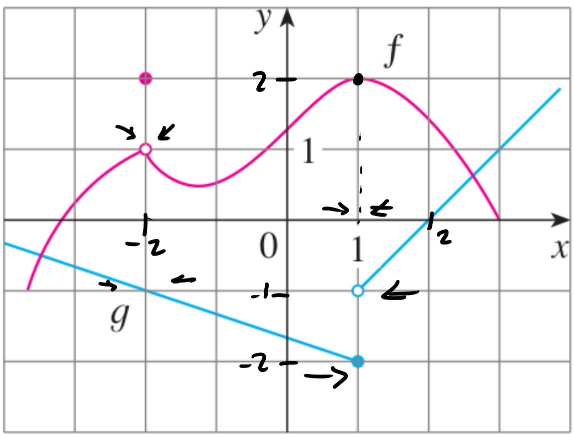


FIGURE 1

(b) $\lim_{x \to 1} f(x) = 2$ & $\lim_{x \to 1} g(x) \nexists$

$\rightarrow$ can't apply the product Rule.

(c) $\lim_{x \to 2} g(x) = 0$ $\rightarrow$ can't apply quotient Rule & $\lim_{x \to 2} \dfrac{f(x)}{g(x)} \nexists$

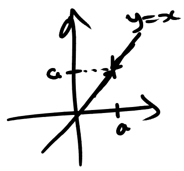
Power Law. $\lim_{x\to 1}(x+1)^2 = \left(\lim_{x\to 1} x+1\right)^2$ $\quad n=1,2,3,4,\ldots \quad (n=0)$

6. $\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$ where n is a positive integer

Three particular cases:

a) $\lim_{x\to a} c = c$

b) $\lim_{x\to a} x = a$



c) $\lim_{x\to a} x^n = \left(\lim_{x\to a} x\right)^n = a^n$

EXAMPLE 2 Evaluate the following limits and justify each step.

(a) $\lim_{x\to 5}(2x^2 - 3x + 4)$ (b) $\lim_{x\to -2}\dfrac{x^3+2x^2-1}{5-3x}$ ← poly. ← poly

(a) $\lim_{x\to 5}(2x^2-3x+4) = \lim_{x\to 5} 2x^2 + \lim_{x\to 5}(-3x) + \lim_{x\to 5} 4$ (Sum Rule).

$= 2\lim_{x\to 5} x^2 - 3\lim_{x\to 5} x + \lim_{x\to 5} 4$ (Constant Rule)

$= 2\cdot 5^2 - 3\cdot 5 + 4$ (Power Rule)

$= \boxed{39}$

(b) $\lim_{x\to -2} 5-3x = \lim_{x\to -2} 5 - 3\lim_{x\to -2} x$ (Diff. Rule + Constant Rule)

$= 5 - 3\cdot(-2)$ (Power Law).

$= 11 \neq 0$

We can apply the quotient Rule:

$$\lim_{x\to -2}\frac{x^3+2x^2-1}{5-3x} = \frac{\lim_{x\to -2} x^3+2x^2-1}{\lim_{x\to -2} 5-3x} \quad \text{(Quotient Rule)}$$

$$= \frac{\lim_{x\to -2} x^3 + 2\lim_{x\to -2} x^2 - \lim_{x\to -2} 1}{11} \quad \text{(Sum + Diff + Const.)}$$

$$= \frac{-8+8-1}{11} \quad \text{(Power Rule)}$$

$$= \boxed{-\frac{1}{11}}$$

Root Law.

11. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ where n is a positive integer

[If n is even, we assume that $\lim_{x\to a} f(x) > 0$.]

(a) $\lim_{x\to a} \sqrt[n]{x}$ $(a>0)$

$= \sqrt[n]{\lim_{x\to a} x} = \sqrt[n]{a}$

Example. Compute $\lim_{\boxed{u}\to -2} \underbrace{\sqrt{u^4+3u+6}}_{f(x)}$.

$\lim_{x\to -2} x$ "

$$\lim_{u\to -2} u^4+3u+6 = \lim_{x\to -2} u^4 + 3\lim_{u\to -2} u + \lim_{u\to -2} 6$$

$$= 16 + (-6) + 6$$

$$= 16 > 0$$

So, by Root Law

$$\lim_{u\to -2} \sqrt{u^4+3u+6} = \sqrt{\lim_{u\to -2} u^4+3u+6} = \sqrt{16} = 4.$$

Remark: $\lim_{u\to -2} \underbrace{(u^4+3u+6)}_{f(u)} = (-2)^4 + 3(-2) + 6 = f(-2)$

Direct Substitution Property If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x\to a} f(x) = f(a)$$

Remark: Be careful, not true if a is not in the domain.

EXAMPLE 3 Find $\lim_{x\to 1} \frac{x^2-1}{x-1}$. $\frac{0}{0}$ $x=1$ is not is the domain of $f(x) = \frac{x^2-1}{x-1}$.

~~Quotient Rule~~ ~~Direct subst. Prop.~~

$x \neq 1$

$$\frac{x^2-1}{x-1} = \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} = x+1$$

$$\to \lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} (x+1) = \lim_{x\to 1} x + \lim_{x\to 1} 1 = \boxed{2}$$

Property used:

If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$, provided the limits exist.

EXAMPLE 4 Find $\lim_{x \to 1} g(x)$ where

$$g(x) = \begin{cases} x + 1 & \text{if } x \neq 1 \\ \pi & \text{if } x = 1 \end{cases}$$

$x \neq 1$, $g(x) = x + 1$

$$\lim_{x \to 1} g(x) = \lim_{x \to 1} (x+1) = \boxed{2}$$

EXAMPLE 5 Evaluate $\lim_{h \to 0} \dfrac{(3+h)^2 - 9}{h}$.

$\frac{\sim}{0}$ ~~Quotient rule~~ $h=0$ is not in the domain ~~Subst. Prop.~~

$(3+h)(3+h)$ for $h \neq 0$

$$\frac{(3+h)^2 - 9}{h} = \frac{\cancel{9} + 3h + 3h + h^2 - \cancel{9}}{h} = \frac{6h + h^2}{h}$$

$$= \frac{(6+h)\cancel{h}}{\cancel{h}} = 6 + h$$

$$\lim_{h \to 0} \frac{(3+h)^2 - 9}{h} = \lim_{h \to 0} (6+h) = \boxed{6}$$

EXAMPLE 6 Find $\lim_{t \to 0} \dfrac{\sqrt{t^2+9}-3}{t^2}$.

$$(\sqrt{A} + \underset{B}{\cancel{\sqrt{B}}})(\sqrt{A} - \underset{B}{\cancel{\sqrt{B}}}) = A - B^2$$

$$\frac{\sqrt{t^2+9}-3}{t^2} = \frac{(\sqrt{t^2+9}-3)}{t^2}\frac{(\sqrt{t^2+9}+3)}{(\sqrt{t^2+9}+3)} \to \frac{1}{1}$$

$$= \frac{t^2+9-9}{t^2(\sqrt{t^2+9}+3)} = \frac{\cancel{t^2}}{\cancel{t^2}(\sqrt{t^2+9}+3)} = \frac{1}{\sqrt{t^2+9}+3}$$

$$\lim_{t\to 0}(\sqrt{t^2+9}+3) = \sqrt{\lim_{t\to 0} t^2+9} + \lim_{t\to 0} 3 \qquad \left(\begin{array}{l}① \text{ Sum Rule} \\ ② \text{ Root Rule}\end{array}\right)$$

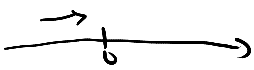
$$= 3+3 = 6 \neq 0$$

$$\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2} = \lim_{t\to 0} \frac{1}{\sqrt{t^2+9}+3} = \frac{\lim_{t\to 0} 1}{\lim_{t\to 0}\sqrt{t^2+9}+3} = \boxed{\frac{1}{6}}$$

EXAMPLE 7 Show that $\lim_{x\to 0} |x| = 0$.

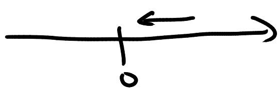
$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Recall $\lim_{x\to 0} |x| = 0$ if, and only if, ① $\lim_{x\to 0^-} |x| = 0$
② $\lim_{x\to 0^+} |x| = 0$



① $\lim_{x\to 0^-} |x| \underset{x<0}{=} \lim_{x\to 0^-} (-x) = 0$ ✓ (Power rule, Constant rule). Power rule for R-H limits and L-H limits.

② $\lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0$ ✓ (Powerrule).



So, $\boxed{\lim_{x\to 0} |x| = 0}$.

EXAMPLE 8 Prove that $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

EXAMPLE 9 If

$$f(x) = \begin{cases} \sqrt{x - 4} & \text{if } x > 4 \\ 8 - 2x & \text{if } x < 4 \end{cases}$$

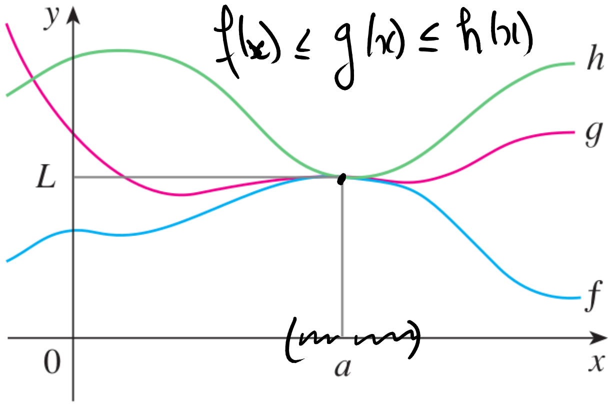
determine whether $\lim_{x \to 4} f(x)$ exists.

The Squeeze Theorem.

3 The Squeeze Theorem If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

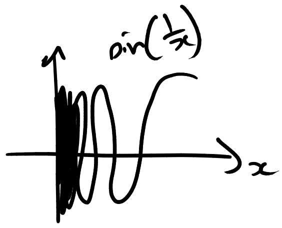
then $$\lim_{x \to a} g(x) = L$$



EXAMPLE 11 Show that $\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$.

$g(x) = x^2 \sin\left(\frac{1}{x}\right)$

$\lim_{x \to 0} \sin\left(\frac{1}{x}\right) \nexists$



$$\boxed{-1 \leq \sin A \leq 1}$$

$A = \frac{1}{x} \quad \rightarrow \quad -1 \leq \sin\left(\frac{1}{x}\right) \leq 1$

$$\rightarrow \quad \underbrace{-x^2}_{= f(x)} \leq \underbrace{x^2 \sin\left(\frac{1}{x}\right)}_{= g(x)} \leq \underbrace{x^2}_{= h(x)}$$

$\lim_{x \to 0} f(x) = \lim_{x \to 0} -x^2 = -0 = 0$

$\lim_{x \to 0} h(x) = \lim_{x \to 0} x^2 = 0$ same

So, by the Squeeze thm.,

$\lim_{x \to 0} g(x) = 0$.