Assigned date: 11/22/2021 9am
Due date: 11/29/2021 5pm

Last name: Solutions.
First name:
Section:

Question:	1	2	3	4	5	6	Total
Points:	10	10	20	20	20	20	100
Score:							

Instructions: You must answer all the questions below and upload your solutions (in a PDF format) to Gradescope (go to `www.gradescope.com` with the Entry code GEK6Y4). Be sure that after you scan your copy, it is clear and readable. You must name your file like this: `LASTNAME_FIRSTNAME.pdf`. A homework may not be corrected if it's not readable and if it's not given the good name. No other type of files will be accepted (no PNG, no JPG, only PDF) and no late homework will be accepted.

Make sure to show all your work!

Good luck!

Question 1 **(10 points)**

Compute the following definite integrals knowing that $\int_1^4 f(x)\,dx = 12$.

(a) (5 points) $\int_1^2 f(x^2)x\,dx$.

(b) (5 points) $\int_0^{\pi/2} f(3\sin x + 1)\cos x\,dx$.

(a) Put $u = x^2 \Rightarrow du = 2x\,dx \rightarrow x\,dx = \frac{du}{2}$.

$$\int_1^2 f(x^2)\,x\,dx = \int_1^4 f(u)\,\frac{du}{2} = \frac{1}{2}\cdot 12 \boxed{= 16.}$$

(b) Put $u = 3\sin x + 1 \rightarrow du = 3\cos x\,dx$

$\rightarrow \cos x\,dx = \frac{du}{3}$.

So, $$\int_0^{\pi/2} f(3\sin x + 1)\cos x\,dx = \int_1^4 f(u)\,\frac{du}{3}$$

$$= \frac{12}{3} \boxed{= 4}$$

Question 2 **(10 points)**

With the help the concepts of odd and even functions, find the value of the following definite integrals.

(a) (5 points) $\int_{-1}^{1} x \cos x \, dx$

(b) (5 points) $\int_{-1}^{1} \sin x \cos x \, dx$.

(a) $f(x) = x\cos x$ is odd. Indeed:

$$f(-x) = -x\cos(-x) = -x\cos(x) = -f(x).$$

So, $\int_{-1}^{1} x\cos x \, dx = 0.$

(b) $f(x) = \sin x \cos x$ is odd. Indeed:

$$f(-x) = \sin(-x)\cos(-x) = -\sin(x)\cos(x) = -f(x).$$

So, $\int_{-1}^{1} \sin x \cos x \, dx = 0$.

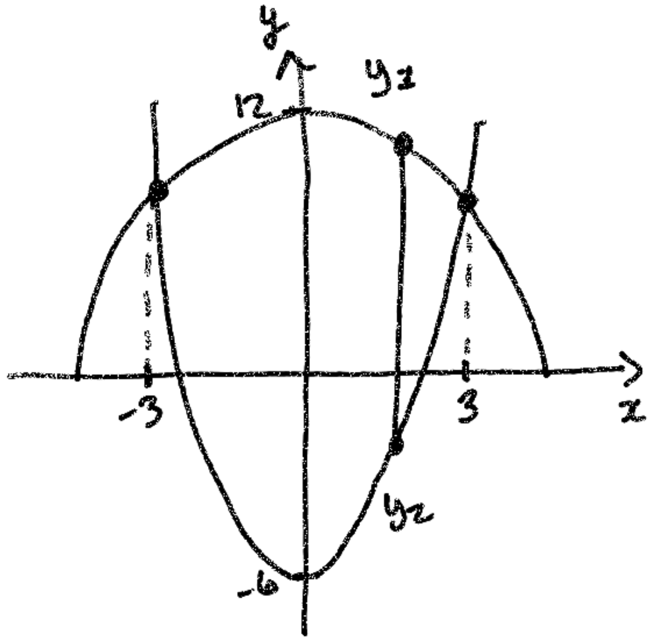
Question 3 (20 points)

Find the area between the given curves.

(a) (10 points) $y = 12 - x^2$ and $y = x^2 - 6$.

(b) (10 points) $y = \cos x$ and $y = 1 - 2x/\pi$. $\left(-\frac{\pi}{2} \le x \le \pi/2\right)$

(a)



① $12 - x^2 = 0 \iff x = \pm\sqrt{12}$

$x^2 - 6 = 0 \iff x = \pm\sqrt{6}$.

② $12 - x^2 = x^2 - 6$

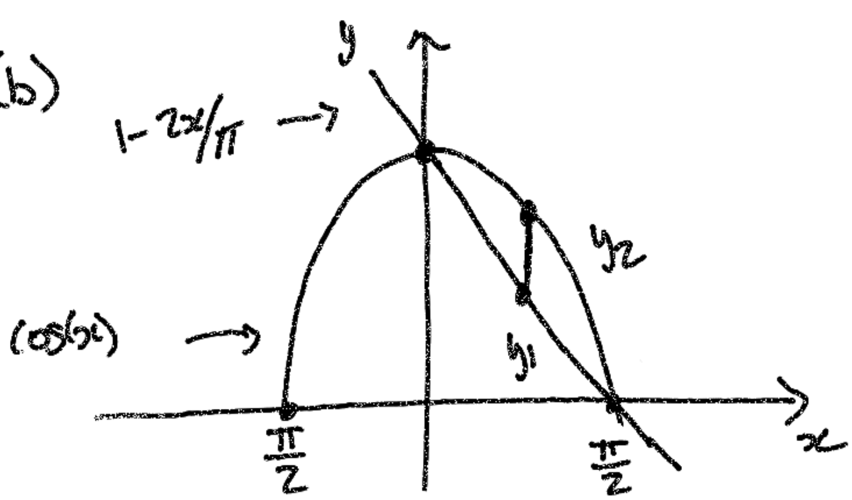
$\iff 18 = 2x^2$

$\iff x = \pm 3$.

So, $\text{Area} = \int_{-3}^{3} y_1 - y_2 \, dx = \int_{-3}^{3} 18 - 2x^2 \, dx$

So, $\boxed{\text{Area} = 108 \, u^2}$

(b)



① $\cos x = 1 - \frac{2x}{\pi}$

if $x = 0$ & $x = \frac{\pi}{2}$.

So, $\text{Area} = \int_0^{\pi/2} y_2 - y_1 \, dx = \int_0^{\pi/2} \cos x - \left(1 - \frac{2x}{\pi}\right) dx$.

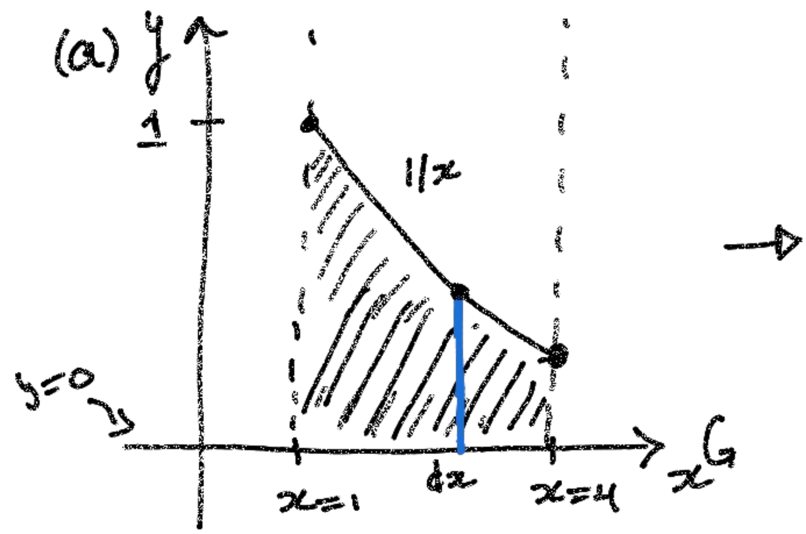
So, $\boxed{\text{Area} = \left(1 + \frac{\pi}{2}\right) u^2}$.

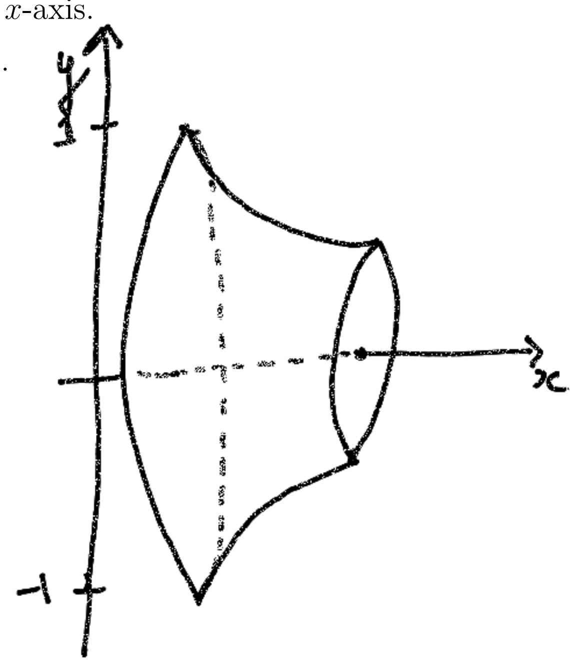
Question 4 (20 points)

Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line.

(a) (10 points) $y = 1/x$, $y = 0$, $x = 1$, $x = 4$, and about the x-axis.

(b) (10 points) $x = 2\sqrt{y}$, $x = 0$, $y = 9$, and about the y-axis.





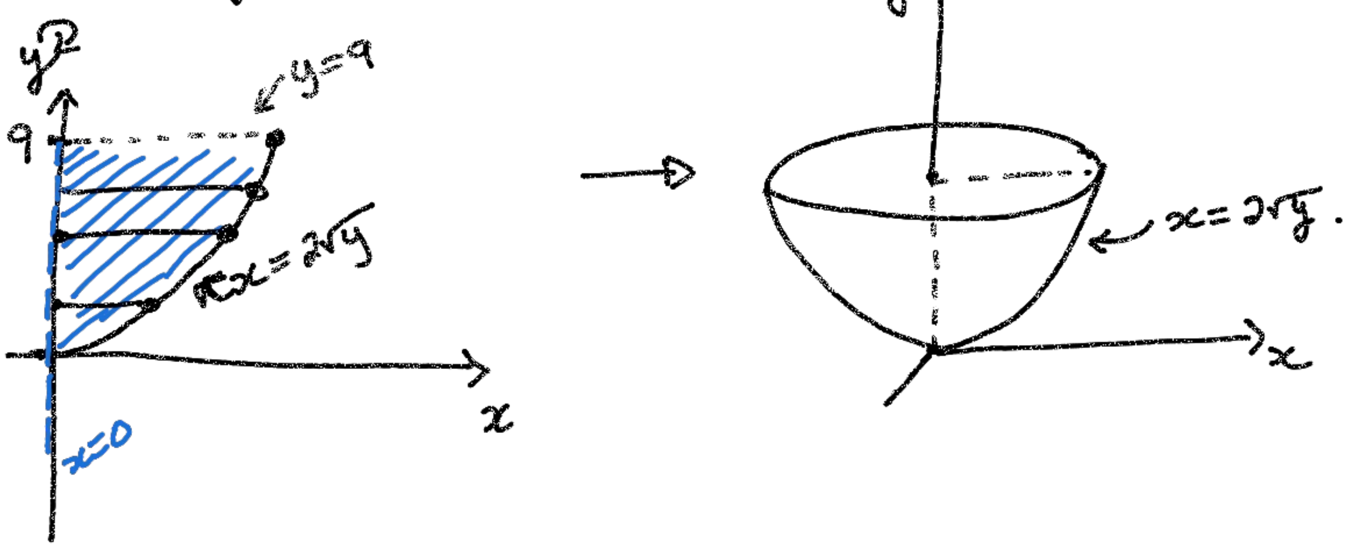
Radius: $y = \frac{1}{x}$ Area disk $A(x) = \pi\left(\frac{1}{x}\right)^2 dx$

So, $V = \int_1^4 \frac{\pi}{x^2}\,dx = \pi\left(\frac{-1}{x}\right)\Big|_{x=1}^{x=4}$

$= \pi\left(1 - \frac{1}{4}\right) = \frac{3\pi}{4}$

So, $\boxed{V = \frac{3\pi}{4}\ u^3}$

(b) $x^2 = 4y$



Radius: $x = 2\sqrt{y}$ $\rightarrow$ Area: $A(y) = \pi 4y$

We go from $y=0$ to $y=9$

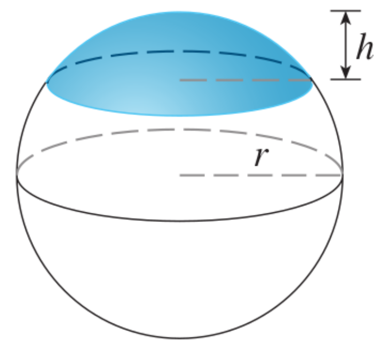
So,

$$V = \int_0^9 \pi 4y \, dy = 4\pi \left. \frac{y^2}{2} \right|_0^9 = 2\pi(81-0)$$

$$\Rightarrow \boxed{V = 162\pi \ u^2}$$

Question 5 $h=1$ **(20 points)**

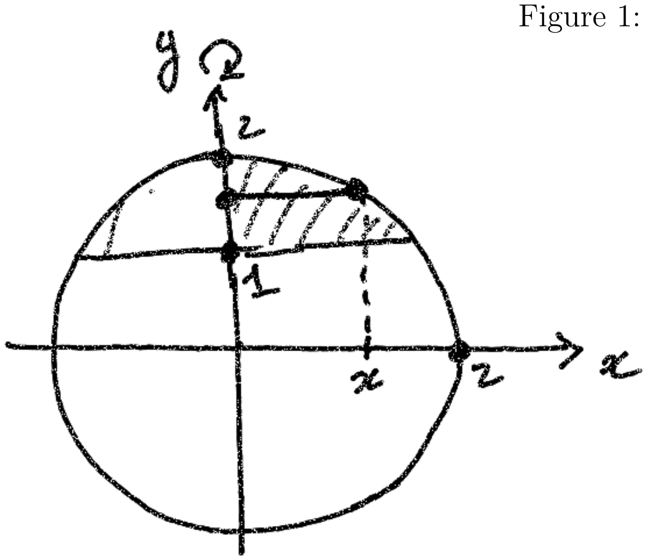
Find the volume of a cap of a sphere with radius $r = 2$ and height [redacted]. See the figure below.



$2-h = 2-1 = 1$

$r = 2.$

Figure 1: Picture of the cap



Equation of a circle:

$x^2 + y^2 = 4$

with $x = \sqrt{4-y^2}$.

from $y = 1$ to $y = 2$

Area $= \pi x^2 = \pi(4-y^2)$

So,

$$V = \int_{2-h}^{2} \pi(4-y^2)\,dy = \pi\left(4y - \frac{y^3}{3}\right)\Big|_1^2$$

$$= \pi\left(\left(8 - \frac{8}{3}\right) - \left(4 - \frac{1}{3}\right)\right)$$

$$= \pi\left(4 - 7/3\right) = \frac{5\pi}{3}.$$

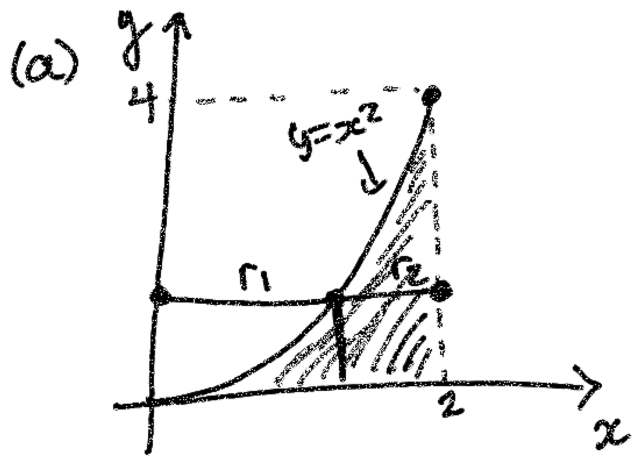
So, $\boxed{V = \frac{5\pi}{3}\ u^3}$

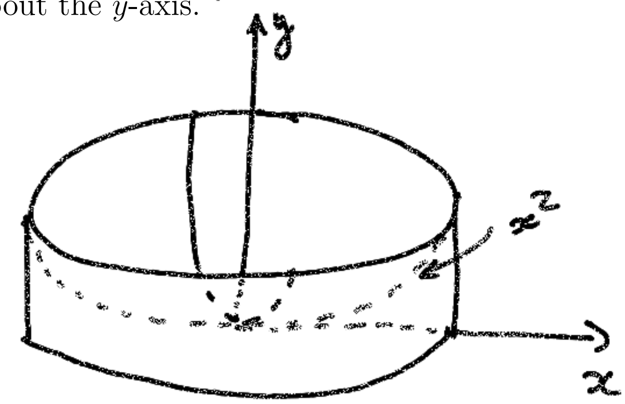
Use the method of cylindrical shells to find the volume of the given solid generated by rotating the region bounded by the given curves about the specified axis.

(a) (10 points) $y = x^2$, [redacted], and about the y-axis. $x = 2$, $y = 0$,

(b) (10 points) $y = x^3$, $y = 8$, $x = 0$, and about $x = 3$.

(a)





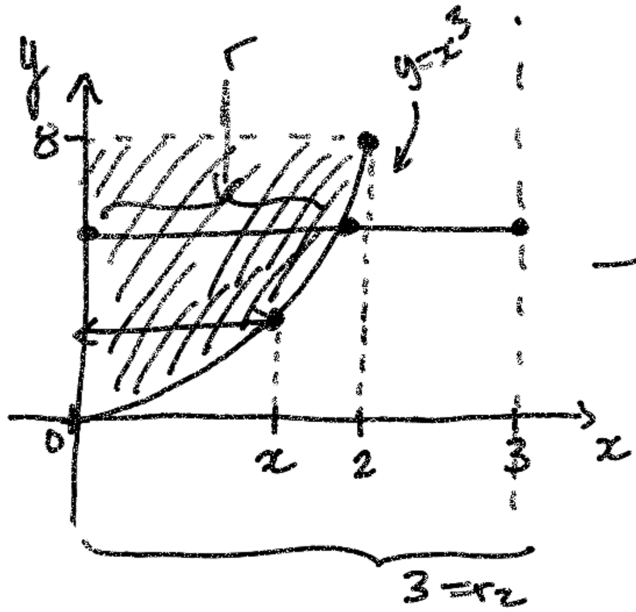
the heigh is $f(x) = y = x^2$. x goes from 0 to 2 , so

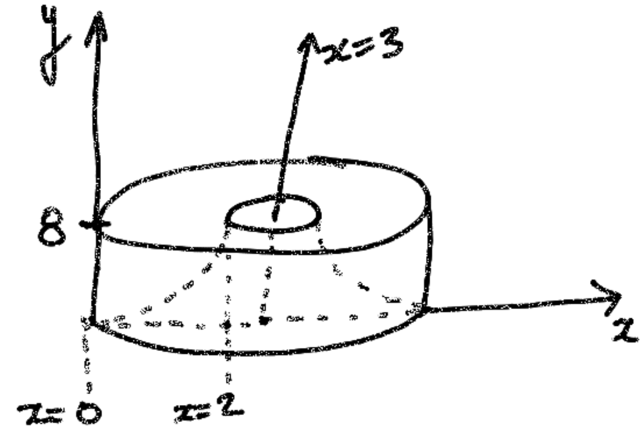
$$V = \int_0^2 2\pi x \, f(x) \, dx$$

$$= 2\pi \int_0^2 x \cdot x^2 \, dx = 2\pi \left(\frac{x^4}{4} \right) \Big|_0^2 = 8\pi$$

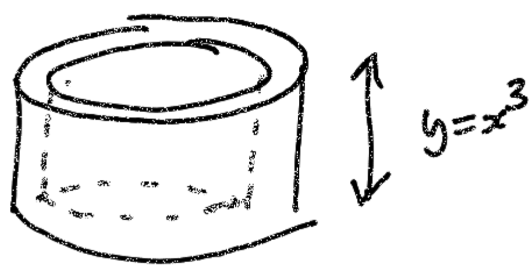
So, $\boxed{V = 8\pi \ u^2}$

(b)





Typical cylinder:



Radius will be $3-x$. So

$$V = \int_0^2 2\pi(3-x)f(x)\,dx = \int_0^2 2\pi(3-x)x^3\,dx$$

$$= 2\pi\left(\frac{3x^4}{4} - \frac{x^5}{5}\right)\Big|_0^2$$

$$= 2\pi\left(3\cdot 4 - \frac{32}{5}\right)$$

$$= 2\pi\left(\frac{60-32}{5}\right)$$

So,

$$\boxed{V = \frac{56}{5}\pi \ u^3}$$