Math 644

Chapter 4

Section 4.1: Integration on Curves

Contents

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Curves In The Complex Plane

Definition 1. A **curve** is a continuous map $\gamma : I \subset \mathbb{R} \to \mathbb{C}$, where I is (mostly) a closed interval.

Example 2.

a) The circle of radius r and centered at $z_0 \in \mathbb{C}$ is a curve, where $\gamma(t) = z_0 + re^it$, with $t \in [0, 2\pi]$.

b) A straigth line joining a to b is a curve, where $\gamma(t) = (1-t)a + tb$, with $t \in [0, 1]$.

Notes:

- Different curves can have the same image. [Example:
- Use the symbol γ interchangeably to denote the image and the curve itself.
- Arrows on the image γ show how a parametrization $\gamma(t)$ traces the image as $t \in I$ increases.

Definition 3.

- **i)** A curve γ is an **arc** if it is one-to-one.
- **ii)** A curve $\gamma : [a,b] \to \mathbb{C}$ is **closed** if $\gamma(a) = \gamma(b)$.
- **iii)** A curve $\gamma : [a,b] \to \mathbb{C}$ is **simple** if $\gamma : [a,b) \to \mathbb{C}$ is one-to-one.

Example 4.

- **a)** Is $\gamma(t) = a(1-t) + bt$, $0 \leq t \leq 1$ an arc, closed, or simple?
- **b)** Is $\gamma(t) = t^2$, $-1 \leq t \leq 1$ an arc, closed, or simple?
- **c)** Is $\gamma(t) = e^{it}$, $0 \leq t \leq 2\pi$ is an arc, closed, or simple?

Definition 5. A curve $\gamma(t) = x(t) + iy(t)$ is called piecewise continuously differentiable if $\gamma'(t) = x'(t) + iy'(t)$

- **i)** exists and is continuous except for finitely many t;
- **ii)** x' and y' have one-sided limits at the exceptional points.

Notes: If γ is piecewise continuously differentiable, then for $t_1 \neq t_2$,

- we have
$$\gamma(t_2) - \gamma(t_1) = \int_{t_1}^{t_2} x'(t)\,dt + i\int_{t_1}^{t_2} y'(t)\,dt.$$
- $\gamma'(t_1)$ is tangent to γ at $\gamma(t_1)$.

DEFINITION 6. A curve $\psi : [c, d] \to \mathbb{C}$ is called a reparametrization of a curve $\gamma : [a, b] \to \mathbb{C}$ if there is a one-to-one, onto, increasing function $\alpha : [a, b] \to [c, d]$ such that

$$\psi(\alpha(t)) = \gamma(t) \quad \forall t \in [a, b].$$

EXAMPLE 7.

a) Show that $\psi(t) = t^2 + it^4$ $(0 \leq t \leq 1)$ is a reparametrization of $\gamma(t) = t + it^2$ $(0 \leq t \leq 1)$.

b) If $\sigma : [0, 1] \to \mathbb{C}$ is a curve, then show that $\beta : [0, 1] \to \mathbb{C}$ defined by $\beta(t) = \sigma(1 - t)$ is not a reparametrization of σ.

c) Show that any curve $\gamma : [a, b] \to \mathbb{C}$ can be reparametrized to a curve $\psi : [0, 1] \to \mathbb{C}$.

Notes:

- If ψ is a piecewise continuously differentiable reparametrization of a piecewise continuously differentiable curve γ with α also piecewise continuously differentiable, then

$$\psi'(\alpha(t))\alpha'(t) = \gamma'(t).$$

DEFINITION 8. If $\gamma : [a,b] \to \mathbb{C}$ is a piecewise continuously differentiable curve and f is a continuous $\mathbb{C}$-valued function on the image γ, then

$$\int_\gamma f(z)\,dz := \int_a^b f(\gamma(t))\gamma'(t)\,dt.$$

EXAMPLE 9. Compute $\displaystyle\int_\gamma \frac{1}{z}\,dz$, where γ is the circle of radius $1/2$ centered at the origin.

THEOREM 10. The integral of a continuous function over a piecewise continuously differentiable curve does not depend on the parametrization.

Proof.

Definition 11. If $\gamma : [a, b] \to \mathbb{C}$ is a curve, then $-\gamma : [-b, -a] \to \mathbb{C}$ is a curve defined by

$$-\gamma(t) := \gamma(-t).$$

Picture

- $-\gamma$ has the same image as γ;
- However, $-\gamma$ traces the image in the opposite direction.

Notes:

- Another way to "inverse" the direction: $\sigma(t) := \gamma(ta + (1 - t)b)$, for $0 \leq t \leq 1$.
- If γ is a piecewise continuously differentiable curve, then

$$\int_{-\gamma} f(z)\, dz = -\int_{\gamma} f(z)\, dz.$$

Definition 12. If γ_1, γ_2, …, γ_n are curves defined on $[0,1]$, then we define their **sum** or **union** $\gamma : [0,n] \to \mathbb{C}$ by setting

$$\gamma(t) := \begin{cases} \gamma_1(t) & 0 \leq t < 1 \\ \gamma_2(t-1) & 1 \leq t < 2 \\ \vdots & \\ \gamma_j(t-j+1) & j-1 \leq t < j \\ \vdots & \\ \gamma_n(t-n+1) & n-1 \leq t \leq n. \end{cases}$$

Corollary 13. If

- f is continuous on each γ_j $(1 \leq j \leq n)$;
- each γ_j is piecewise continuously differentiable and;
- γ is defined as above,

then

$$\int_\gamma f(z)\,dz = \sum_{j=1}^{n} \int_{\gamma_j} f(z)\,dz.$$

Notes:

- From the last Corollary, we will also denote the union of finitely many curves γ_j as $\gamma := \sum_j \gamma_j$.
- If α, β and γ are three curves, then
  - $\alpha + \beta = \beta + \alpha$;
  - $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$.
- In particular, for γ a piecewise continuously differentiable curve, we have

$$\int_{\gamma+(-\gamma)} f(z)\,dz = \int_\gamma f(z)\,dz - \int_\gamma f(z)\,dz = 0.$$

Definition 14. A **cycle** $\gamma = \sum_{j=1}^{n} \gamma_j$ is a finite union of closed curves $\gamma_1, \ldots, \gamma_n$.

Example 15. Let S_1 and S_2 be two (closed) squares such that $S_1 \cap S_2 = \varnothing$. Show that $\partial(S_1 \cup S_2)$ is a cycle.

Corollary 16. If S_1 and S_2 are two (closed) squares sharing exactly one side. Show that, for every continuous function defined on $\partial S_1 \cup \partial S_2$,

$$\int_{\partial S_1} f(z)\, dz + \int_{\partial S_2} dz = \int_{\partial(S_1 \cup S_2)} f(z)\, dz$$

where ∂S_1, ∂S_2 and $\partial(S_1 \cup S_2)$ are parametrized in the counter-clockwise direction.

Definition 17. If $\gamma : [a, b] \to \mathbb{C}$ is a piecewise continuously differentiable curve, and if f is a continuous complex-valued function defined on the image of γ, then we define

$$\int_\gamma f(z)|dz| := \int_a^b f(\gamma(t))|\gamma'(t)|\, dt.$$

Note:

- The length of a piecewise continuously differentiable curve $\gamma : [a, b] \to \mathbb{C}$ is defined by

$$\ell(\gamma) = |\gamma| := \int_\gamma |dz|.$$

Properties:

a) If γ is piecewise continuously differentiable and f is continuous on γ, then

$$\left| \int_\gamma f(z)\, dz \right| \leq \int_\gamma |f(z)|\, |dz|.$$

b) If γ is piecewise continuously differentiable and f is continuous on γ, then

$$\left| \int_\gamma f(z)\, dz \right| \leq \Big(\sup_\gamma |f(z)| \Big) \ell(\gamma).$$

c) If γ is piecewise continuously differentiable and (f_n) is a sequence of continuous function on γ such that $f_n \to f$ uniformly on γ, then

$$\lim_{n\to\infty} \int_\gamma f_n(z)\, dz = \int_\gamma f(z)\, dz.$$

d) Integration on piecewise continuously differentiable curves is linear.