Problems: 1, 3, 7, 13, 18, 26.

Problem 1

The function $f(z) = \frac{1+z}{z}$ has a pole of order $m = 1$ at $z = 0$. Therefore,

$$\operatorname{Res}(f(z), 0) = \lim_{z \to 0} z f(z) = \lim_{z \to 0} (1 + z) = 1.$$

Problem 3

The function $f(z) = \frac{1+e^z}{z^2} + \frac{2}{z}$ has a singularity at $z_0 = 0$. It is a pole of order $m = 2$ because

$$f(z) = \frac{2}{z^2} + \frac{3}{z} + \frac{1}{2} + \cdots$$

for $|z| > 0$. Using the formula for the residue when we have a pole of order 2, we get

$$\begin{aligned}
\operatorname{Res}(f(z), 0) &= \lim_{z \to 0} \frac{1}{(2-1)!} \frac{d^{2-1}}{dz^{2-1}} (z^2 f(z)) \\
&= \lim_{z \to 0} \frac{d}{dz} (1 + e^z + 2z) \\
&= \lim_{z \to 0} (e^z + 2) \\
&= 3.
\end{aligned}$$

Problem 7

The singularity of $f(z) = 1/z \sin z$ occurs at the zeros of $z \sin z$. We have

$$z \sin z = 0 \iff z = 0 \text{ or} \sin z = 0 \iff z = 0 \text{ or } z = k\pi \iff z = k\pi,\ k \in \mathbb{Z}.$$

For $k \neq 0$ an integer, we have

$$\lim_{z \to k\pi} (z - k\pi) f(z) = \lim_{z \to k\pi} \frac{1}{z \frac{\sin z}{z - k\pi}} = \frac{1}{k\pi \left. \frac{d}{dz} \sin z \right|_{z = k\pi}} = \frac{1}{k\pi \cos(k\pi)} = \frac{(-1)^k}{k\pi}.$$

Therefore, $z = k\pi$ is a pole of order $m = 1$ when $k \neq 0$ and

$$\operatorname{Res}(f(z), k\pi) = \frac{(-1)^k}{k\pi}.$$

For $k = 0$, we have

$$\lim_{z \to 0} |z f(z)| = \lim_{z \to 0} \left| \frac{1}{\sin z} \right| = \infty$$

but
$$\lim_{z\to 0} z^2 f(z) = \lim_{z\to 0} \frac{z^2}{z\sin z} = \lim_{z\to 0} \frac{z}{\sin z} = \frac{1}{\cos(0)} = 1 \neq 0.$$
Therefore $z = 0$ is a pole of order $m = 2$. We use the formula for the residue at a pole of order $m = 2$:
$$\begin{aligned}
\operatorname{Res}(f(z), 0) &= \lim_{z\to 0} \frac{1}{(2-1)!} \frac{d^{2-1}}{dz^{2-1}} \Big(z^2 f(z) \Big) \\
&= \lim_{z\to 0} \frac{d}{dz} \Big(\frac{z}{\sin z} \Big) \\
&= \lim_{z\to 0} \frac{\sin z - z\cos z}{\sin^2 z} \\
&= \lim_{z\to 0} \frac{z \sin z}{2\sin z \cos z} && \text{[Hospital Rule]} \\
&= \lim_{z\to 0} \frac{z}{\cos z} = 0.
\end{aligned}$$

Problem 13

The singularities of $f(z) = \frac{z^2+3z-1}{z(z^2-3)}$ are $z_1 = 0$, and $z_2 = \sqrt{3}$ and $z_3 = -\sqrt{3}$. The only singularity inside $C_1(0)$ is $z = 0$. Therefore
$$\int_{C_1(0)} \frac{z^2+3z-1}{z(z^2-3)}\, dz = 2\pi i \operatorname{Res}(f(z), 0).$$
The singularity $z = 0$ is a pole of order $m = 1$ because
$$\lim_{z\to 0} z f(z) = \lim_{z\to 0} \frac{z^2+3z-1}{z^2-3} = \frac{1}{3} \neq 0.$$
Therefore, from the formula for residue at a pole of order $m = 1$, we get
$$\operatorname{Res}(f(z), 0) = \lim_{z\to 0} z f(z) = \frac{1}{3}.$$
hence
$$\int_{C_1(0)} \frac{z^2+3z-1}{z(z^2-3)}\, dz = \frac{2\pi i}{3}.$$

Problem 18

The function $f(z) = \frac{z^2+1}{(z-1)^2}$ has a singularity at $z = 1$. The only singularity in $C_3(0)$ is $z = 1$.

It is also a pole of order $m = 2$ because
$$f(z) = \frac{z^2+1}{(z-1)^2} = \frac{(z-1+1)^2+1}{(z-1)^2} = \frac{(z-1)^2+2(z-1)+2}{(z-1)^2} = 1 + \frac{2}{z-1} + \frac{2}{(z-1)^2}.$$

valid for $|z-1|>0$. By the way, from this expansion, we see immediately that

$$\operatorname{Res}(f(z),1) = a_{-1} = 2.$$

But, using the formula for pole of order $m=2$, we get

$$\begin{aligned}\operatorname{Res}(f(z),1) &= \lim_{z\to 1} \frac{1}{(2-1)!}\frac{d^{2-1}}{dz^{2-1}}\Big((z-1)^2 f(z)\Big)\\ &= \lim_{z\to 1}\frac{d}{dz}(z^2+1)\\ &= \lim_{z\to 1} 2z\\ &= 2.\end{aligned}$$

Therefore, using Cauchy's Residue Theorem, we find that

$$\int_{C_3(0)} \frac{z^2+1}{(z-1)^2}\,dz = 2\pi i \operatorname{Res}(f(z),1) = 4\pi i.$$

Problem 26

The function $f(z) = \frac{1}{z^2(e^z-1)}$ has a singularity at the zeros of $z^2(e^z-1)$. So

$$z^2(e^z-1)=0 \iff z=0 \text{ or } e^z=1 \iff z=0 \text{ or } z=2k\pi i,\ k\in\mathbb{Z} \iff z=2k\pi i,\ z\in\mathbb{Z}.$$

Only the singularity $z=0$ is in the circle $C_{1/2}(0)$. Therefore, from Cauchy's Residue Theorem, we obtain

$$\int_{C_{1/2}(0)} \frac{1}{z^2(e^z-1)}\,dz = 2\pi i \operatorname{Res}(f(z),0).$$

We will find the order of the pole by finding the order of the zero $z=0$ of $z^2(e^z-1)$. Using the Taylor series of e^z centered at $z=0$, we can write

$$z^2(e^z-1) = z^2\Big(\sum_{n=0}^{\infty}\frac{z^n}{n!}-1\Big)z^2\sum_{n=1}^{\infty}\frac{z^n}{n!} = z^3+\frac{z^4}{2}+\frac{z^5}{3!}+\cdots$$

Therefore, the order of the zero $z=0$ of $z^2(e^z-1)$ is of order $m=3$. Hence, the order of the pole at $z=0$ of $f(z)$ is $m=3$ also.

Using the formula for poles of order $m=3$, we get

$$\begin{aligned}\operatorname{Res}(f(z),0) &= \lim_{z\to 0}\frac{1}{(3-1)!}\frac{d^{3-1}}{dz^{3-1}}\Big(z^3 f(z)\Big)\\ &= \frac{1}{2}\lim_{z\to 0}\frac{d^2}{dz^2}\Big(\frac{z}{e^z-1}\Big)\\ &= \frac{1}{2}\lim_{z\to 0}\frac{(z+2)e^z+(z-2)e^{2z}}{(e^z-1)^3}\\ &= \frac{1}{6}\end{aligned}$$

after applying 3 times l'Hospital Rule. Hence,

$$\int_{C_{1/2}(0)} \frac{1}{z^2(e^z-1)}\,dz = \frac{\pi i}{3}.$$