Chapter 1: Functions and Limits
Week 3

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Upcoming this week

1. 1.7. The precise definition of limit
2. 1.8. Continuity

If we want to tackle detailed proof in mathematics, we have to transfer vague phrases into precise statements with the mathematical language.

> *If $f(x)$ gets closer and closer to a number L as x gets closer and closer to a number a, then $f(x)$ has a limit L.*

We will need the following interpretation of $|\square|$.

Proposition 1

If $\delta > 0$ is a positive real number, then

$$|\square| < \delta \iff -\delta < \square < \delta.$$

Here, $\square$ is a box that contains something.

Example 1

Take $\square = x - 3$ and $\delta = 3$. Then, $|x-3| < 3$ is the same thing as saying that

$$-3 < x - 3 < 3.$$

So, in this situation, $0 < x < 6$.

Example 2

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 2x - 1 & x = 3 \\ 6 & x = 3. \end{cases}$$

Analyze the function near $x = 3$. Limit

In the last example, as we use smaller and smaller values (e.g. 0.1, 0.01, 0.001, ...), it is always possible to find a number δ such that $|x - 3| < \delta$.

Definition 3 (Formal definition of Limit)

Let f be a function defined around a point a. We say that f has a limit L at the point a if whenever $\varepsilon > 0$, there is a number $\delta > 0$ such that

$$0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

Limit

Definition 4 (Right-Hand limit)

Let f be a function defined on the right of a point a. We say that f has a Right-Hand limit L at the point a if whenever $\varepsilon > 0$, there is a number $\delta > 0$ such that

$$a < x < a + \delta \Rightarrow |f(x) - L| < \varepsilon.$$

Definition 5 (Left-Hand limit)

Let f be a function defined on the left of a point a. We say that f has a Left-Hand limit L at the point a if whenever $\varepsilon > 0$, there is a number $\delta > 0$ such that

$$a - \delta < x < a \Rightarrow |f(x) - L| < \varepsilon.$$

▸ R/L-Hand Limits

Definition 6 (Infinity limits)

Let f be a function be defined around a point a. Then $f(x)$ has an infinite limit if whenever $M > 0$ there is a $\delta > 0$ such that

$$0 < |x - a| < \delta \Rightarrow f(x) > M.$$

Example 7

Take $f(x) = 1/x^2$. We know that $\lim_{x\to 0} f(x) = \infty$. Infinite Limits

Definition 8

Let f be a function be defined around a point a. Then $f(x)$ has an negative infinite limit if whenever $M < 0$ there is a $\delta > 0$ such that

$$0 < |x - a| < \delta \Rightarrow f(x) < M.$$

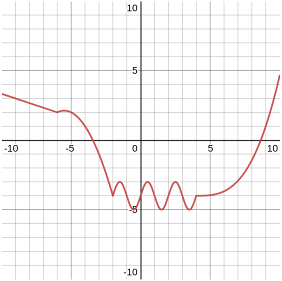
Exercises: 1,2,4,14,16.

We can have a intuitive definition of a continuous function:

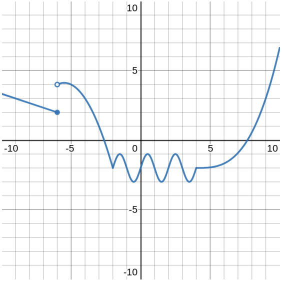
A continuous function is a function such that its graph can be drawn without lifting your pen from your sheet of paper (or your tablet).

Example 9

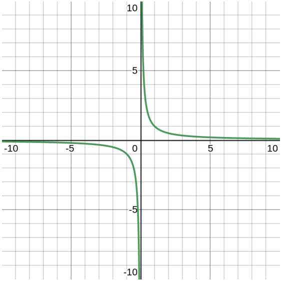
In the following, which function is continuous?



(a) First function



(b) Second function



(c) Third function

Figure: Which one is continuous?

Definition 10

A function f is continuous at a point a if

$$\lim_{x \to a} f(x) = f(a).$$

Remarks. The definition has two implicit statements:

- a belongs to the domain of f ($f(a)$ is well-defined).
- $\lim_{x \to a} f(x)$ exists.
- $\lim_{x \to a} f(x) = f(a)$.

Using the mathematical language: f is continuous at a point a if $a \in \operatorname{dom} f$ and whenever $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

Using the commun language: f is continuous at a point a if $a \in \operatorname{dom} f$ and as x becomes closer and closer to a, so does $f(x)$ with $f(a)$.

Example 11

Let f be the function defined by the following graph:

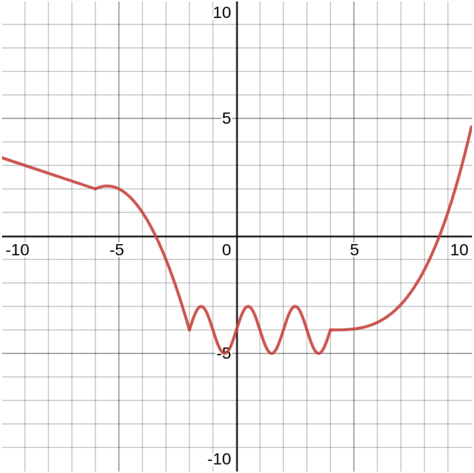


Figure: Is this function continuous at $x = 5$?

Definition 12

Let f be a function defined around a point a.

- f is continuous on the right if $\lim_{x\to a^+} f(x) = f(a)$.
- f is continuous on the left if $\lim_{x\to a^-} f(x) = f(a)$.

Definition 13

A function f is continuous on an interval $[\alpha, \beta]$ if it is continuous at each of the points a in the interval $[\alpha, \beta]$

Remark. We understand continuous at the endpoints α and β to mean continuous from the right and continuous from the left respectively.

Definition 14

A function is discontinuous at a point a if it is not continuous at that point.

Example 15

Are the following functions continuous at the given point a?

a) $f(x) = \frac{x^2-x-2}{x-2}$ at $a = 2$.

b) $f(x) = \begin{cases} \frac{1}{x^2} & x \neq 0 \\ 1 & x = 0. \end{cases}$ at $a = 0$.

c) $f(x) = \begin{cases} \frac{x^2-x-2}{x-2} & x \neq 2 \\ 1 & x = 2 \end{cases}$ at $a = 2$.

Theorem 16

Let f and g be two continuous functions at a point a. Then

- *$f+g$, $f-g$ are continuous functions at a.*
- *fg, cf $(c \in \mathbb{R})$ are continuous functions at a.*
- *f/g is a continuous function at a if $g(a) \neq 0$.*

Example 17

We know that $\lim_{x \to 2} x^2 = 2^2$ and $\lim_{x \to 2} 2x = 2 \cdot 2$. Is the function $f(x) = x^2 + 2x$ continuous at $a = 2$?

We saw in the previous sections that

$$\lim_{x\to a}(a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0) = a_na^n + a_{n-1}a^{n-1} + \cdots + a_0.$$

Theorem 18

- *Any polynomial is a continuous function everywhere; that is on* $(-\infty, \infty)$.
- *Any rational function is continuous whenever it is defined; that is, if* $f(x) = P(x)/Q(x)$, *then it is continuous on the set* $\{x \in \mathbb{R} : Q(x) \neq 0\}$.
- *Any root function is continuous whenever it is defined; that is, if* $f(x) = \sqrt[n]{x}$, *then*
  - *If* n *is even, it is continuous on* $[0, \infty)$.
  - *If* n *is odd, it is continuous on* $(-\infty, \infty)$.

Example 19

Find $\lim_{x\to -2} \frac{x^3+2x^2-1}{5-3x}$.

We have to be careful for the tangent function: it has zero where $\cos(x) = 0$. Elsewhere, there is no problem.

Theorem 20

- *the functions* sin *and* cos *are continuous everywhere; that is on* $(-\infty, \infty)$.
- *the function* tan *is continuous on its domain; that is at every point a except* $a = (2n+1)\pi/2$.

Example 21

On what intervals is each function continuous?

a) $f(x) = x^{100} - 2x^{37} + 75$.

b) $f(x) = \sqrt{x} + \frac{x+1}{x-1} - \frac{x+1}{x^2+1}$.

Example 22

Compute $\lim_{x\to\pi} \frac{\sin x}{2+\cos x}$.

Remember(?): the composition of g with f is defined as

$$(g \circ f)(x) = g(f(x)).$$

Definition 23

Let f be a function continuous at a and g be a function continuous at $b = f(a)$. Then $g \circ f$ is continuous at a and

$$\lim_{x \to a} (g \circ f)(x) = g(f(a)).$$

Example 24

Where are the following functions continuous?

a) $h(x) = \sin(x^2)$.

b) $F(x) = \frac{1}{\sqrt{x^2+7}-4}$.

Exercises: 2, 4-8, 10, 12-18, 23, 24, 35, 37, 46-49, 73.