Chapter 1

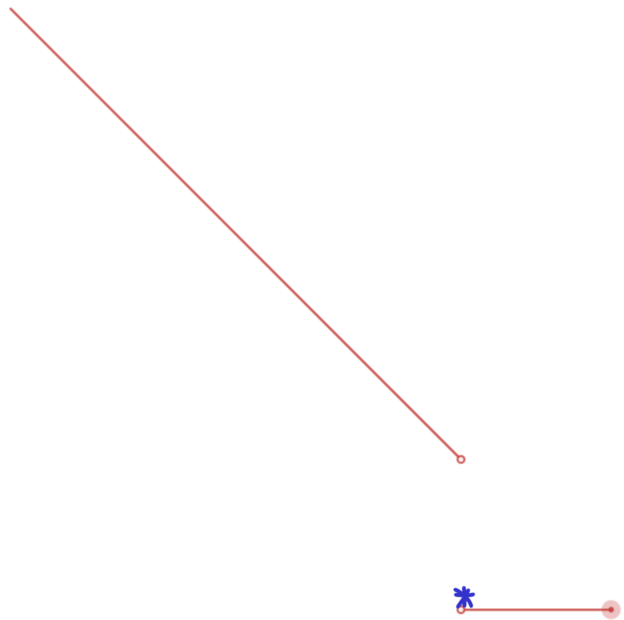
Functions and Limits

1.8 Continuity

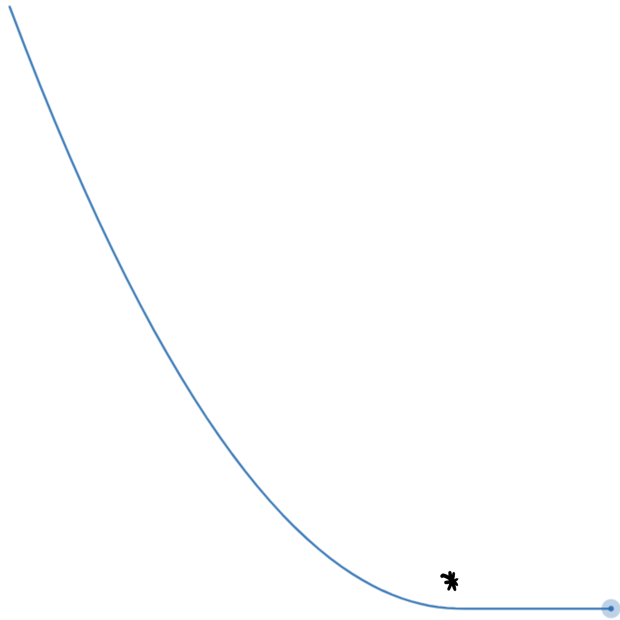
Continuity

Example. What are the main difference(s) between the two following curves?

Illustration: https://www.desmos.com/calculator/hflxgbsemz



(a) Graph of f(x).



(b) Graph of g(x)

(1) red: break point

(2) red: undefined at $*$.

blue: defined at $*$.

(3) red: $\lim_{x \to *} f(x) \nexists$ blue: $\lim_{x \to *} g(x) \exists$

(4) red & blue.

Example. Now, what are the differences between the two following functions?

(a) $f(x) = \begin{cases} 2-x & \text{if } -2 \le x < 1 \\ 0 & \text{if } 1 \le x \le 2 \end{cases}$ (b) $g(x) = \begin{cases} \frac{4}{9}(1-x)^2 & \text{if } -2 \le x < 1 \\ 0 & \text{if } 1 \le x \le 2 \end{cases}$

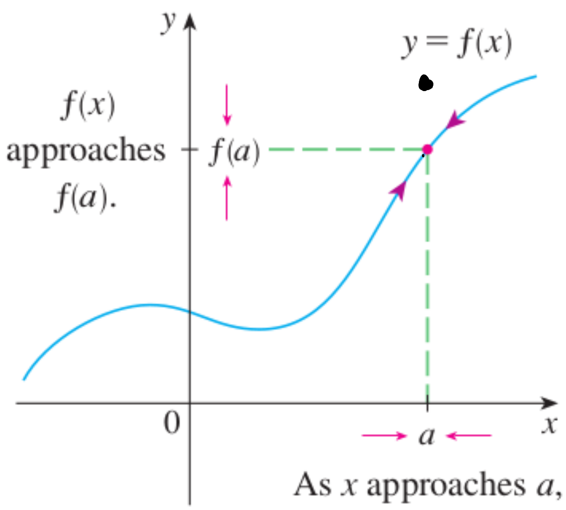
$\downarrow$ red curve

$\downarrow$ blue curve.

need conditions to do calculations.

1 Definition A function f is **continuous at a number a** if

$$\lim_{x \to a} f(x) = f(a)$$



Three things to verify to show a function is continuous:

a) The function is defined at x = a.

Find the domain

b) The limit of the function exists at x = a.

$f(x)$ formula → use limit rules

c) The limit of the function at x = a equals the value of the function at x = a.

Discontinuity: $x = a$ is a discontinuity of f if a), or b) or c) is not satisfied

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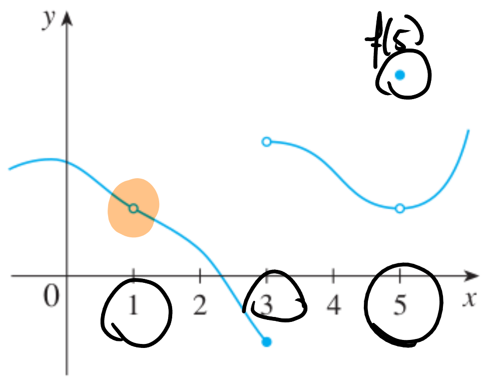
EXAMPLE 1 Figure 2 shows the graph of a function f. At which numbers is f discontinuous? Why?



$x = 1$, because $f(1) \nexists$

$x = 3$, because $\lim_{x \to 3} f(x) \nexists$

$x = 5$, because $\lim_{x \to 5} f(x) \neq f(5)$

Example. Check if the functions in the first example are continuous at x = 1 using the formulas.

(a) a) $f(1)$ exists or not → $f(1) \nexists$

so $\boxed{f \text{ is discontinuous at } x = 1}$

(b) a) $g(1)$? $g(1) = 0$ (from the formula)

b) $\lim_{x \to 1} g(x)$??

$$\lim_{x \to 1^-} g(x) = \lim_{x \to 1^-} \frac{4}{9}(1-x)^2 = 0$$

$$\lim_{x \to 1^+} g(x) = \lim_{x \to 1^+} 0 = 0$$

$$\Rightarrow \lim_{x \to 1} g(x) = 0$$

c) $\lim_{x \to 1} g(x) \stackrel{?}{=} g(1)$ Yes!! ⇒ $\boxed{g \text{ is continuous at } a = 1}$

EXAMPLE 2 Where are each of the following functions discontinuous?

(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2} = \dfrac{\cancel{(x-2)}(x+1)}{\cancel{x-2}}$ (b) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ (c) $f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$

(a) (a) $\mathrm{Dom}(f) = (-\infty, 2) \cup (2, \infty) \Rightarrow$ discontinuous at $x = 2$.

You can verify that

$$\lim_{x \to a} \frac{x^2 - x - 2}{x - 2} = \frac{a^2 - a - 2}{a - 2} \quad (a \neq 2)$$

(b) (a) $\mathrm{Dom}(f) = (-\infty, \infty)$ check at $x = 0$.

(b) $\displaystyle\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{1}{x^2} = +\infty \quad (\nexists)$

$\Rightarrow$ f discontinuous at $a = 0$.

Remark: f is continuous at all other real numbers $(a \neq 0)$.

(c) $\mathrm{Dom}(f) = (-\infty, \infty) \rightarrow$ check at $x = 0$.

(a) $f(0) = 0$ ✓

(b) $\displaystyle\lim_{x \to 0} f(x) \nexists$

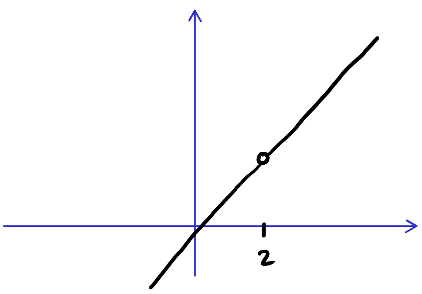
$\displaystyle\lim_{x \to 0^-} f(x) = 0$

$\displaystyle\lim_{x \to 0^+} f(x) = 1$

$\neq$

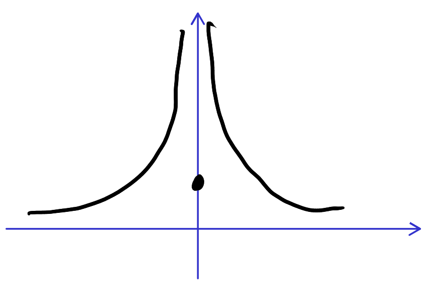
So, f is discontinuous at $a = 0$

3 kinds of discontinuity.



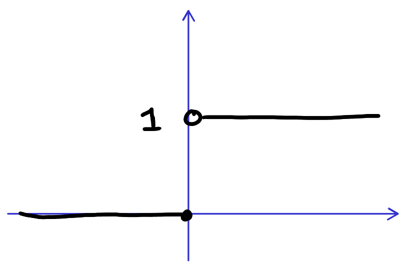
(a) Removable.

$f(a) \nexists$ but $\displaystyle\lim_{x \to a} f(x) \exists$



(b) Infinite discontinuity.

$\displaystyle\lim_{x \to a} f(x) = \pm\infty$



(c) Jump discontinuity.

$\displaystyle\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$

Properties of Continuous Functions.

4 Theorem If f and g are continuous at a and if c is a constant, then the following functions are also continuous at a:

1. $f + g$
2. $f - g$
3. cf
4. fg
5. $\dfrac{f}{g}$ if $g(a) \neq 0$

Consequences:

7 Theorem The following types of functions are continuous at every number in their domains:

- polynomials
- rational functions
- root functions
- trigonometric functions

Substitution Rule Revisited.

$$\lim_{x \to a} f(x) = f(a)$$

EXAMPLE 5 Find $\lim_{x \to -2} \underbrace{\dfrac{x^3 + 2x^2 - 1}{5 - 3x}}_{= f(x)}$.

type of f: rational function $\Rightarrow$ continuous on its domain

Domain: $5 - 3x = 0 \Leftrightarrow \frac{5}{3} = x \to$ Domain is $(-\infty, \frac{5}{3}) \cup (\frac{5}{3}, \infty)$.

$a = -2 \to f$ is cont. at -2 $\Rightarrow \lim_{x \to -2} \dfrac{x^3 + 2x^2 - 1}{5 - 3x} = \dfrac{(-2)^3 + 2(-2)^2 - 1}{5 - 3 \cdot (-2)}$

$= \boxed{-1/11}$

EXAMPLE 7 Evaluate $\lim_{x \to \pi} \underbrace{\dfrac{\sin x}{2 + \cos x}}_{f(x)}$.

$$\lim_{x \to a} f(x) = f(a)$$

$\sin x \to$ continuous
$2 + \cos x \to$ continuous $\to \dfrac{\sin x}{2 + \cos x}$ is continuous on its domain.

Domain: $2 + \cos x = 0 \Leftrightarrow \cos x = -2$ impossible because $-1 \leq \cos x \leq 1$.

No restriction $\to$ Domain is $(-\infty, \infty)$.

f cont. at $\pi \Rightarrow \lim_{x \to \pi} \dfrac{\sin(x)}{2 + \cos(x)} = \dfrac{\sin(\pi)}{2 + \cos(\pi)} = \dfrac{0}{2 - 1} = \boxed{0}$

$\lim_{x \to -\pi} \sin(\sin(x) + \pi + 2x^2 + \tan x)$

Composition of Continuous Functions.

8 Theorem If f is continuous at b and $\lim_{x \to a} g(x) = b$, then $\lim_{x \to a} f(g(x)) = f(b)$. In other words,

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$$

9 Theorem If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a.

Example. Find the value of

$$\lim_{x \to a} h(x) = h(a)$$

$$\lim_{x \to 1/2} \sin(\pi - \pi x^2)$$

$f(x) = \sin(x) \to$ continuous on $(-\infty, \infty)$

$g(x) = \pi - \pi x^2 \to$ continuous on $(-\infty, \infty)$

$\to$ $\sin(\pi - \pi x^2)$ is continuous on its domain.

Domain of $\sin(\pi - \pi x^2)$: $(-\infty, \infty)$.

$\Rightarrow$ $\sin(\pi - \pi x^2)$ is continuous at $1/2$

$$\Rightarrow \lim_{x \to 1/2} \sin(\pi - \pi x^2) = \sin\left(\pi - \pi \left(\tfrac{1}{2}\right)^2\right)$$

$$= \sin\left(\pi - \frac{\pi}{4}\right)$$

$$= \sin\left(\frac{3\pi}{4}\right)$$

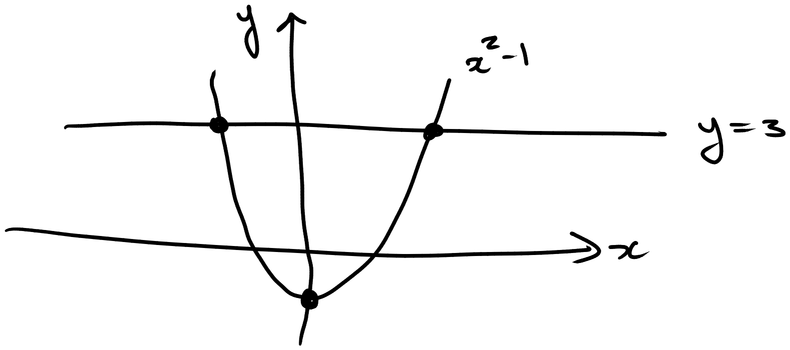
$$= \boxed{\frac{\sqrt{2}}{2}}$$

Example. Suppose we have a function

$$f(x) = x^2 - 1.$$

Does the graph of the function f cross the horizontal line y = 3?

Partial Answer:



Full Answer: x^2-1 is polymonial → continuous on $(-\infty, \infty)$ ✓.

Find two points:

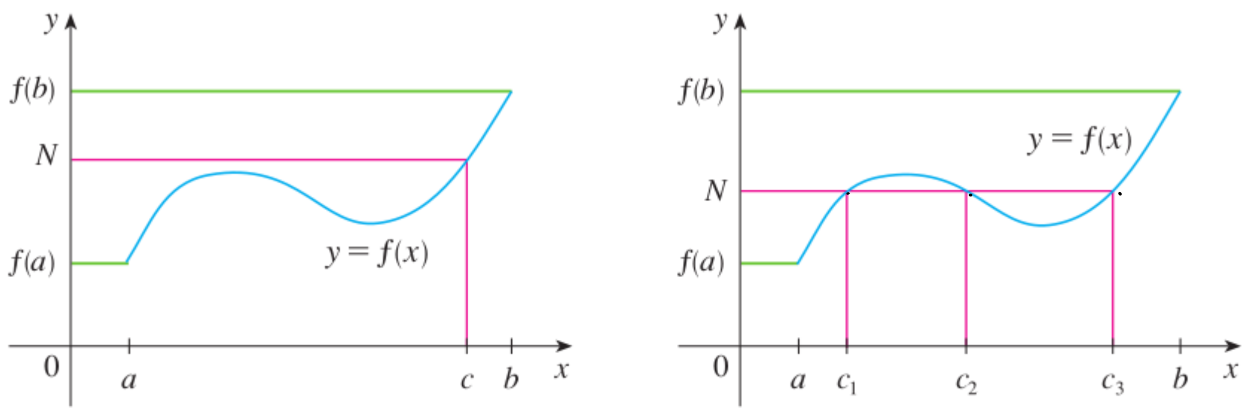
- $x=0 \to f(0) = -1 < 3$ ✓
- $x=3 \to f(3) = 8 > 3$ ✓

so, I must cross the line $y=3$

$\Rightarrow$ there will some r s.t. $\boxed{r^2-1=3}$

(IVP)

10 The Intermediate Value Theorem Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.



(a) one c

(b) multiple c's

EXAMPLE 9 Show that there is a root of the equation

$$\underbrace{4x^3 - 6x^2 + 3x - 2}_{f(x)} = 0$$

between 1 and 2.

$N = 0$

$f(x) = 4x^3 - 6x^2 + 3x - 2$

$a = 1$

$b = 2$

Check the hyp. of IVT

*) f is a polynomial $\Rightarrow$ f is continuous on $(-\infty, \infty)$

**) $f(1) = -1$

$f(2) = 12$

***) $-1 < 0 < 12$

So, there is a c such that $f(c) = 0$

$\Rightarrow 4c^3 - 6c^2 + 3c - 2 = 0$