Math 644

Chapter 6

Section 6.3: Riemann Mapping Theorem

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Created by: Pierre-Olivier Parisé
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Statement of the Theorem

Theorem 1. Suppose $\Omega \subset \mathbb{C}$ is simply-connected and $\Omega \neq \mathbb{C}$. Then there exists a one-to-one map f of Ω onto $\mathbb{D}$. If $z_0 \in \Omega$, then there is a unique such map with $f(z_0) = 0$ and $f'(z_0) > 0$.

Idea of the proof.

1. Define a family

$$\mathcal{F} = \{f \,:\, f \text{ is one-to-one, analytic, } |f| < 1 \text{ on } \Omega,\, f(z_0) = 0,\, f'(z_0) > 0\}.$$

2. Show $\mathcal{F}$ is normal on Ω.
3. Extract a subsequence $(f_n) \subset \mathcal{F}$ which converges to some f.
4. Show that f has the desire properties.

Lemma 2. The family $\mathcal{F}$ is non-empty and normal in Ω.

Proof.

Theorem 3. [Hurwitz] Suppose $(g_n)_{n=1}^{\infty}$ is a sequence of analytic functions on a region Ω and suppose $g_n(z) \neq 0$ for all $z \in \Omega$ and all n. If g_n converges uniformly to g on compact subsets of Ω, then

- either g is identically zero in Ω or;
- $g(z) \neq 0$ for all $z \in \Omega$.

Proof.

Corollary 4. If $(g_n)_{n=1}^{\infty}$ is a sequence of one-to-one and analytic functions on a region Ω, and if g_n converges to g uniformly on compact subsets of Ω, then

- either g is one-to-one on Ω or;
- g is constant in Ω.

Proof of the Riemann Mapping Theorem.