Problems: 1, 3, 18, 22.

Problem 1

We have

$$(1-z^2)\sin z = 0 \iff (1-z)(1+z)\sin z = 0 \iff z=1,\ z=-1 \text{ and } z=k\pi$$

where $k \in \mathbb{Z}$.

Order of $\mathbf{z_0 = 1}$. Let $f(z) = (1-z^2)\sin z$. We have $f(1) = 0$. Then

$$f'(z) = -2z\sin z + (1-z^2)\cos z \quad \Rightarrow \quad f'(1) = -2\sin(1) \neq 0.$$

Hence, the order of the zero is $m = 1$.

Order of $\mathbf{z_0 = -1}$. In a similar way, we have $f(-1) = 0$ and

$$f'(-1) = 2\sin(-1) = -2\sin(1) \neq 0.$$

Hence the order of the zero is $m = 1$.

Order of $\mathbf{z_0 = k\pi}$. Let $k \in \mathbb{Z}$ and $z_0 = k\pi$. We have $f(k\pi) = 0$ and

$$f'(k\pi) = (1-k^2\pi^2)\cos(k\pi) = (-1)^k(1-k^2\pi^2) \neq 0.$$

Hence the order of the zero is $m = 1$.

Problem 3

Notice that

$$\frac{z(z-1)^2}{z^2+2z-1} = 0 \iff z(z-1)^2 = 0 \iff z = 0 \text{ or } z = 1.$$

Order of $\mathbf{z_0 = 0}$. Let $f(z) = \frac{z(z-1)^2}{z^2+2z-1}$. Then

$$f'(z) = \frac{z^4+4z^3-8z^2+4z-1}{(z^2+2z-1)^2} \quad \Rightarrow \quad f'(0) = \frac{-1}{(-1)^2} = -1 \neq 0.$$

Hence the order of the zero is $m = 1$.

Order of $\mathbf{z_0 = 1}$. We have

$$f'(1) = \frac{1+4-8+4-1}{(1+2-1)^2} = 0.$$

So we compute

$$f''(z) = \frac{4z(5z^2-6z+3)}{(z^2+2z-1)^3} \quad \Rightarrow \quad f''(1) = \frac{4(2)}{(2)^3} = 1 \neq 0.$$

Hence, $f(1) = f'(1) = 0$ and $f''(1) \neq 0$ and therefore $z_0 = 1$ is a zero of order $m = 2$.

Problem 18

The function $f(z) = \frac{z}{e^z-1}$ is not defined when

$$e^z - 1 = 0 \iff e^z = 1 \iff z = 2k\pi i,\ k \in \mathbb{Z}.$$

- Let $k \neq 0$. We have

$$\lim_{z\to 2k\pi i} |f(z)| = \lim_{z\to 2k\pi i} \frac{|z|}{|e^z - 1|}.$$

Because $e^z = e^z e^{-2k\pi i} = e^{z-2k\pi i}$, we have

$$e^z - 1 = \sum_{n=0}^{\infty} \frac{(z-2k\pi i)^n}{n!} - 1 = \sum_{n=1}^{\infty} \frac{(z-2k\pi i)^n}{n!} = (z-2k\pi i)\sum_{n=0}^{\infty} \frac{(z-2k\pi i)^n}{(n+1)!}$$

so that $f(z) = \frac{z}{(z-2k\pi i)g(z)}$ where $g(z) = \sum_{n=0}^{\infty} \frac{(z-2k\pi i)^n}{(n+1)!}$ is analytic in a neighborhood of $2k\pi i$ and $g(2k\pi i) \neq 0$. Therefore,

$$\lim_{z\to 2k\pi i} \frac{1}{|f(z)|} = \lim_{z\to 2k\pi i} \frac{|(z-2k\pi i)g(z)|}{|z|} = \frac{(0)|g(2k\pi i)|}{2k\pi} = 0.$$

Hence,

$$\lim_{z\to 2k\pi i} |f(z)| = \infty.$$

This means $2k\pi i$ is a pole.

We also have

$$\lim_{z\to 2k\pi i}(z-2k\pi i)f(z) = \lim_{z\to 2k\pi i} \frac{z-2k\pi i}{e^z-1} z = \left(\lim_{z\to 2k\pi i} \frac{z-2k\pi i}{e^z - e^{2k\pi i}}\right)\left(\lim_{z\to 2k\pi i} z\right)$$
$$= \left(\frac{1}{\lim_{z\to 2k\pi i} \frac{e^z - e^{2k\pi i}}{z-2k\pi i}}\right)(2k\pi i).$$

Notice that

$$\lim_{z\to 2k\pi i} \frac{e^z - e^{2k\pi i}}{z-2k\pi i} = \frac{d}{dz}\left(e^z\right)\Big|_{z=2k\pi i} = e^{2k\pi i} = 1.$$

Hence

$$\lim_{z\to 2k\pi i}(z-2k\pi i)f(z) = \left(\frac{1}{1}\right)(2k\pi i) = 2k\pi i \neq 0.$$

Therefore, $z_0 = 2k\pi i$ is a pole of order $m = 1$.

- For $k = 0$, we get

$$\lim_{z\to 0} f(z) = \frac{1}{\frac{d}{dz}(e^z)\big|_{z=0}} = 1 \neq 0.$$

Hence, $z = 0$ is a removable singularity.

Problem 22

The function $f(z) = \frac{1}{(e^z - e^{2z})^2}$ is not defined when

$$e^z - e^{2z} = 0 \iff 1 - e^z = 0 \iff z = 2k\pi i,\ k \in \mathbb{Z}.$$

In the second $\iff$, we used the fact that $e^z \neq 0$ for any $z \in \mathbb{C}$. If

$$g(z) = \frac{1}{f(z)} = (e^z - e^{2z})^2 = e^{2z}(1 - e^z)^2$$

then $z = 2k\pi i$ is an isolated zero of $g(z)$.

We have

$$g'(z) = 2e^{2z}(1 - e^z)^2 + 2e^{2z}(1 - e^z)(-e^z) = 2e^{2z}(1 - e^z)^2 - 2e^{3z}(1 - e^z)$$

and therefore

$$g'(2k\pi i) = 0.$$

We have

$$g''(z) = 2e^{2x}(-9e^x + 8e^{2x} + 2) \quad \Rightarrow \quad g''(2k\pi i) = 2 \neq 0.$$

Hence, we have $g(2k\pi i) = g'(2k\pi i) = 0$ and $g''(2k\pi i) \neq 0$. Therefore, $z = 2k\pi i$ is a zero of order $m = 2$ of $g(z)$. Consequently, $z = 2k\pi i$ is a pole of order $m = 2$ of $f(z)$.