Example 1

Consider the function

$$f(x) = \frac{x^2 - 1}{x^2 + 1}.$$

- What is the graph of this function?
- What happens to the numerator if x becomes larger and larger?
- What happens to the denominator if x becomes larger and larger?
- What happens if x becomes larger and larger in the negative values?

Example 5

The function $f(x) = \frac{x^2-1}{x^2+1}$ has $y = 1$ as a HA. ← $\lim_{x\to\infty}$ or $\lim_{x\to-\infty}$??

$$\lim_{x\to\infty} \frac{x^2-1}{x^2+1} = \frac{\infty}{\infty} \quad (\text{not defined}).$$

$$\frac{x^2-1}{x^2+1} = \frac{\cancel{x^2}(1-1/x^2)}{\cancel{x^2}(1+1/x^2)} = \frac{1-1/x^2}{1+1/x^2}$$

x	$1/x^2$
1	1
2	1/4
10	1/100
100	1/10000
↓	↓
∞	0

x	$-1/x^2$
1	-1
2	-1/4
10	-1/100
100	-1/10000
↓	↓
∞	0

So, $\lim_{x\to\infty} \frac{1}{x^2} = 0$ & $\lim_{x\to\infty} \frac{-1}{x^2} = 0$.

By the sum rule

$$\lim_{x\to\infty} (1 - 1/x^2) = 1 - 0 = 1$$

& $$\lim_{x\to\infty} (1 + 1/x^2) = 1 + 0 = 1 \neq 0$$

So, by the quotient rule

$$\lim_{x\to\infty} \frac{1-1/x^2}{1+1/x^2} = \frac{\lim 1-1/x^2}{\lim 1+1/x^2} = \frac{1}{1} = 1.$$

Example 8

Using the preceding rule, compute

$$\lim_{x\to\infty} \frac{3\boxed{x^2} - x - 2}{5x^2+4x+1}. \qquad \frac{\infty}{\infty}$$

$$\frac{3x^2 - x - 2}{5x^2+4x+1} = \frac{(3x+2)(x-1)}{(\quad)(\quad)} \quad \times$$

$$\frac{3x^2 - x - 2}{5x^2+4x+1} = \frac{x^2(3 - 1/x - 2/x^2)}{x^2(5 + 4/x + 1/x^2)}$$

$$= \frac{3 - 1/x - 2/x^2}{5 + 4/x + 1/x^2}$$

$$\lim_{x\to\infty} (3 - 1/x - 2/x^2) = 3 - 0 - 2\cdot 0$$
$$= 3$$

$$\lim_{x\to\infty} (5 + 4/x + 1/x^2) = 5 + 4\cdot 0 + 0$$
$$= 5$$

So, (quotient rule)

$$\lim_{x\to\infty} \frac{3 - 1/x - 2/x^2}{5 + 4/x + 1/x^2} = \frac{3}{5}.$$

Example 9

Find the horizontal and vertical asymptotes of

$$f(x) = \frac{\sqrt{2x^2+1}}{3x-5}.$$

$\lim_{x\to\infty}$ $\lim_{x\to-\infty}$.

VA. Denom. is zero if $3x-5=0$

if $x = 5/3$

Replace $x = 5/3$ in $f(x)$

$$\Rightarrow f(5/3) = \frac{\sqrt{2\cdot 25/9+1}}{0} = \frac{\sqrt{59/3}}{0}$$

$$\approx 7.7/0$$

$$= \pm\infty$$

Here, we have a V.A. at $x = 5/3$.

H.A. limit at ∞.

$$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to\infty} \frac{\sqrt{x^2(2+1/x^2)}}{x(3-5/x)}$$

$$= \lim_{x\to\infty} \frac{\cancel{x}\sqrt{2+1/x^2}}{\cancel{x}(3-5/x)}$$

$\boxed{x\to\infty \text{, so } x>0,\ \sqrt{x^2}=x}$

$$= \lim_{x\to\infty} \frac{\sqrt{2+1/x^2}}{3-5/x}$$

$$= \frac{\sqrt{2+0}}{3-0} = \frac{\sqrt{2}}{3}$$

so,

$y = \frac{\sqrt{2}}{3}$ is a HA.

• <u>lim at $-\infty$.</u>

$x \to -\infty$, $x < \infty$

$\sqrt{x^2} = -x$

$$\lim_{x \to -\infty} \frac{\sqrt{x^2(2 + 1/x^2)}}{x(3 - 5/x)} = \lim_{x \to -\infty} \frac{(-\cancel{x})\sqrt{2 + 1/x^2}}{\cancel{x}(3 - 5/x)}$$

$$= \lim_{x \to -\infty} -\frac{\sqrt{2 + 1/x^2}}{3 - 5/x}$$

$$= -\frac{\sqrt{2}}{3}.$$

So, $y = -\frac{\sqrt{2}}{3}$ is a HA.

Example 10

Compute $\lim_{x\to\infty}(\sqrt{x^2+1}-x)$. $\infty - \infty$.

$$\sqrt{x^2+1}-x = \frac{(\sqrt{x^2+1}-x)(\sqrt{x^2+1}+x)}{\sqrt{x^2+1}+x}$$

$$= \frac{1}{\sqrt{x^2+1}+x}$$

$$= 1 \Big/ \sqrt{x^2(1+1/x^2)} + x$$

$$\boxed{\begin{array}{c} x \to \infty \\ x > 0 \end{array}} \qquad = 1 \Big/ x\sqrt{1+1/x^2} + x$$

$$= 1 \Big/ x\left(\sqrt{1+1/x^2}+1\right)$$

So,

$$\lim_{x\to\infty} \sqrt{x^2+1}-x = \lim_{x\to\infty} \frac{1}{x\left(\sqrt{1+1/x^2}+1\right)}$$

$$\lim_{x\to\infty} \frac{1}{x} = 0, \quad \lim_{x\to\infty}\left(\sqrt{1+1/x^2}+1\right) = 2$$

So, overall,

$$\lim_{x\to\infty} \sqrt{x^2+1}-x = 0\cdot\left(\frac{1}{2}\right) = 0$$

Example 12

It is wrong to do

$$\lim_{x\to\infty}(x^2 - x) = \lim_{x\to\infty} x^2 - \lim_{x\to\infty} x = \infty - \infty$$

because $\infty - \infty$ is not defined, like $0/0$.