

# Chapter 4

# Integrals

4.1 Areas and Distances

## The summation symbol.

$$\sum_{i=1}^6 i \leftrightarrow f(a_i) = a_i$$
  

$$\sum_{i=1}^6 \frac{1}{i} \leftrightarrow f(a_i) = \frac{1}{a_i}$$
  

$$\sum_{i=1}^4 \frac{1}{i(i+1)} \leftrightarrow f(a_i) = \frac{1}{a_i}$$

### DEFINITION (Sigma Notation)

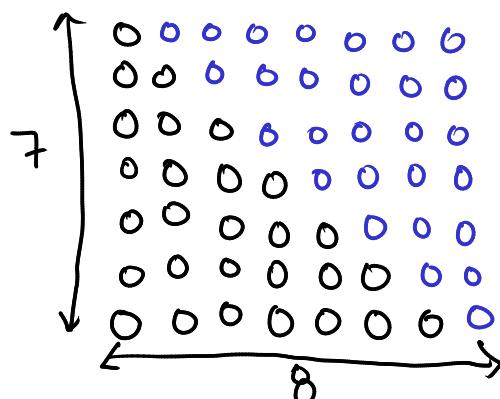
If  $a_1, a_2, a_3, \dots, a_n$  are n numbers, then we can write their ordered sum with the Sigma notation:

$$\begin{aligned}
 ① a_i &= i \\
 ② a_i &= \frac{1}{i} \\
 ③ a_i &= \frac{1}{i(i+1)}.
 \end{aligned}
 \quad a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^n a_i$$

Problem: What is the sum of  $1 + 2 + 3 + 4 + 5 + 6 + 7$ ? Can you compute it in an efficient way?

$$\begin{aligned}
 S &= 1 + 2 + 3 + 4 + 5 + 6 + 7 \\
 S &= 7 + 6 + 5 + 4 + 3 + 2 + 1 \\
 \hline
 &\quad 8 + 8 + 8 + 8 + 8 + 8 + 8
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow 2S &= 7 \times 8 \\
 \Rightarrow S &= \frac{7 \times 8}{2}
 \end{aligned}$$



$$\begin{aligned}
 \text{Total of balls} &= 7 \times 8 = 56 \\
 \text{So, real Total of balls} &= \frac{56}{2} \\
 &= \frac{7 \times 8}{2}
 \end{aligned}$$



Carl Frederich Gauss (1777-1855). Also called the Prince of Mathematics. He discovered the tricks to compute the sum  $1 + 2 + 3 + 4 + 5 + \dots + 100$  when he was around... 7 years old!

General formulas:

$$1) \quad 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

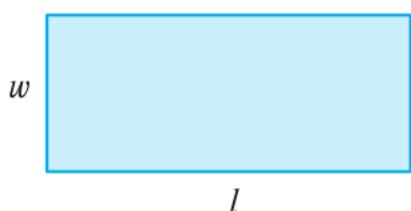
$$2) \quad 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$3) \quad 1^3 + 2^3 + 3^3 + \dots + n^3 = \left( \frac{n(n+1)}{2} \right)^2$$

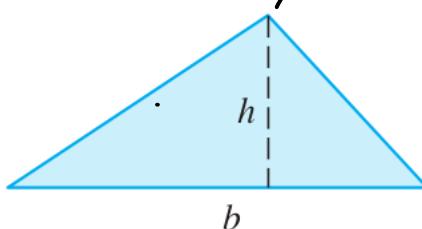
## The Area Problem

What is the area of the following shapes?

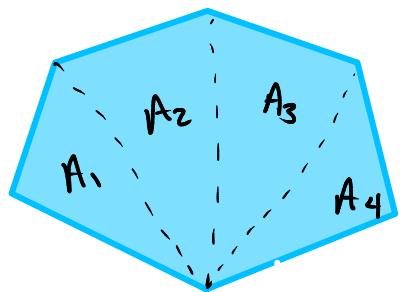
$$A = w \cdot l$$



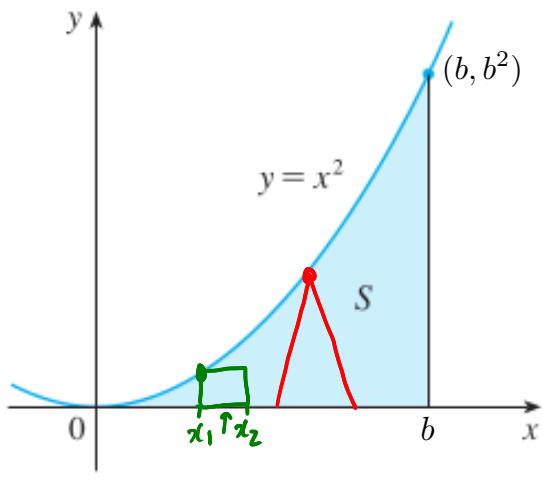
$$A = h \cdot b / 2$$



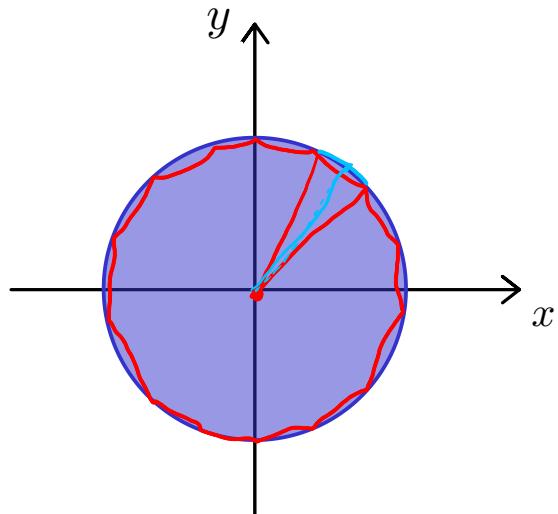
$$A = A_1 + A_2 + A_3 + A_4$$



What about the area of the following shapes



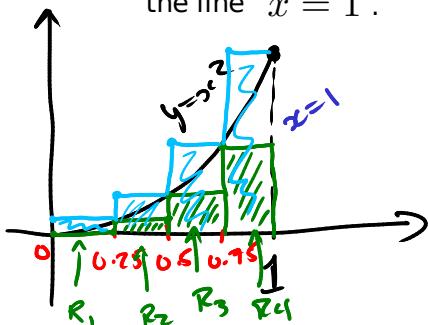
$$A =$$



$$A =$$

Remember the Youtube video that I showed you on the first day?

**Example 1.** Try to estimate the area of the region bounded by the curves  $y = x^2$ , the line  $y = 0$  and the line  $x = 1$ .



3<sup>st</sup>) Compute the area of each rectangle & sum the results.

$$A(R_1) = 0 \cdot 0.25$$

$$A(R_2) = \frac{1}{16} \cdot 0.25 = \frac{1}{64}$$

$$A(R_3) = \frac{1}{4} \cdot 0.25 = \frac{1}{16}$$

$$A(R_4) = \frac{9}{16} \cdot 0.25 = \frac{9}{64}$$

1<sup>st</sup>) Divide in 4 equal parts the interval:



2<sup>nd</sup>) Choose a sample point inside each subinterval:

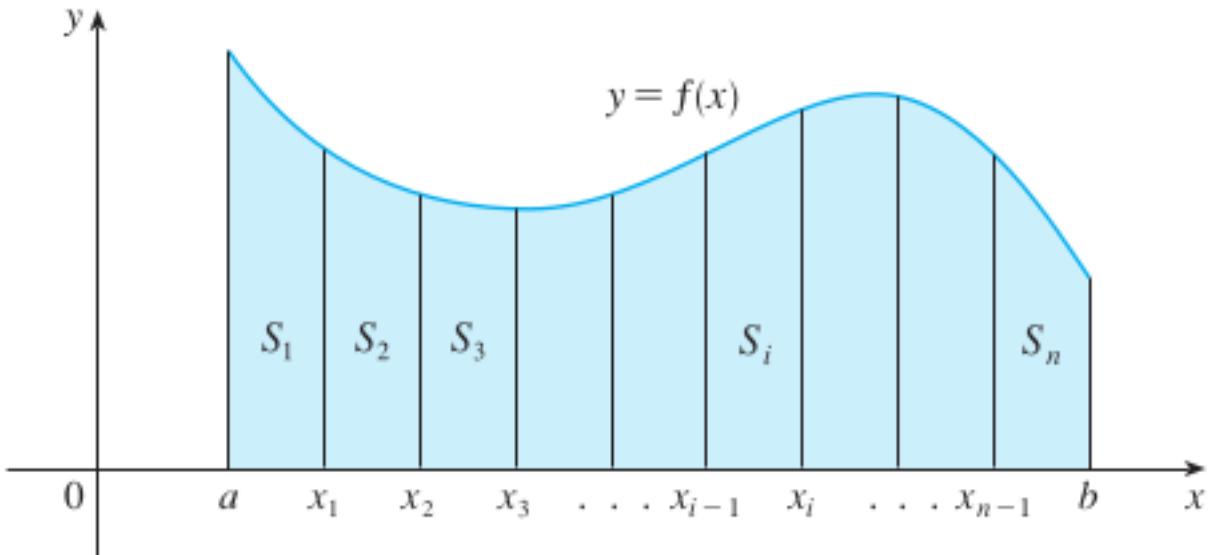
- Left endpoint: 0, 0.25, 0.5, 0.75.

- Right endpoint: 0.25, 0.5, 0.75, 1.

height of each rectangle is given by

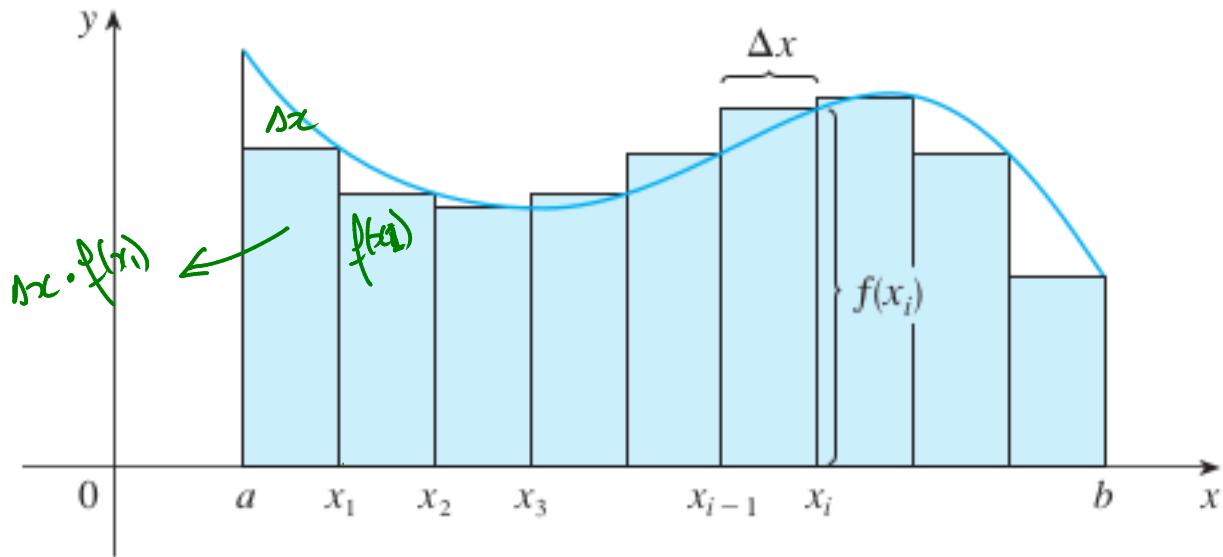
$$\text{height } y(0) = 0^2, y(0.25) = 1/16, y(0.5) = \frac{1}{4}, y(0.75) = \frac{9}{16},$$

$$\text{or height } y(0.25) = 1/16, y(0.5) = 1/4, y(0.75) = 9/16 \text{ & } y(1) = 1.$$



1) Divide the region  $S$  into  $n$  strips.

Choose  $x_1, x_2, \dots, x_{n-1}$  to create  
n equal parts.  
 $\Delta x$ : length of the subintervals (constant)



2) Approximate the area of  $S$  by the sum of the area of each rectangle created.

Right endpoints

If we choose  $x_1, x_2, \dots, x_{n-1}, b$   
then

$$A(S) \approx \sum_{i=1}^n f(x_i) \Delta x$$

$$= f(x_1) \Delta x + f(x_2) \Delta x + \dots + f(x_n) \Delta x$$

here  $x_n = b$ .

Left endpoints

If we choose the left-end  
points  $a, x_1, x_2, \dots, x_{n-1}$  then

$$A(S) \approx \sum_{i=1}^n f(x_{i-1}) \Delta x$$

$$= f(x_0) \Delta x + f(x_1) \Delta x + \dots + f(x_{n-1}) \Delta x$$

here  $x_0 = a$ .

2 DEFINITION. The AREA A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} [f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x]$$

$$\text{or } A = \lim_{n \rightarrow \infty} L_n = \lim_{n \rightarrow \infty} [f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x]$$

**Example 2.** Show that the area of the region S in example 1 is  $1/3$ . In other words, show that

$$\lim_{n \rightarrow \infty} R_n = 1/3.$$

Desmos: see the illustration.

$$[a, b] = [0, 1], \quad \Delta x = \frac{1-0}{n} = \frac{1}{n}, \quad \text{and}$$

$$\begin{aligned} x_1 &= 0 + 1\Delta x = 0 + \frac{1}{n} = \frac{1}{n} & x_i &= 0 + i \Delta x = \frac{i}{n} \\ x_2 &= 0 + 2\Delta x = 0 + \frac{2}{n} = \frac{2}{n} & x_n &= n \cdot \Delta x = \frac{n}{n} = 1 = b \end{aligned}$$

So,

$$\begin{aligned} R_n &= f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x + \cdots + f(x_n)\Delta x \\ f(x) = x^2 &= x_1^2 \Delta x + x_2^2 \Delta x + \cdots + x_i^2 \Delta x + \cdots + x_n^2 \Delta x \\ &= \left(\frac{1}{n}\right)^2 \left(\frac{1}{n}\right) + \left(\frac{2}{n}\right)^2 \frac{1}{n} + \cdots + \left(\frac{i}{n}\right)^2 \frac{1}{n} + \cdots + \left(\frac{n}{n}\right)^2 \left(\frac{1}{n}\right) \\ &= \frac{1}{n^3} + \frac{2^2}{n^3} + \cdots + \frac{i^2}{n^3} + \cdots + \frac{n^2}{n^3} \\ &= \frac{1}{n^3} \left( 1^2 + 2^2 + \cdots + i^2 + \cdots + n^2 \right) \end{aligned}$$

$$= \frac{1}{n^3} \sum_{i=1}^n i^2 = \frac{1}{n^3} \frac{n(n+1)(2n+1)}{6}$$

$$\begin{aligned} \text{So, } A &= \lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} \frac{(n^2+n)(2n+1)}{6n^3} = \lim_{n \rightarrow \infty} \frac{2n^3 + 3n^2 + n}{6n^3} \\ &= \lim_{n \rightarrow \infty} \frac{2 + 3/n + 1/n^2}{6} = \boxed{\frac{1}{3}} \end{aligned}$$

## The distance problem.

If the velocity remains constant, then the distance between the start and the finish line is easy to compute:

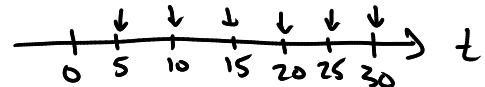
$$\text{DISTANCE} = \text{VELOCITY} \times \text{TIME}.$$

What do we do if the velocity varies??

**EXAMPLE 4** Suppose the odometer on our car is broken and we want to estimate the distance driven over a 30-second time interval. We take speedometer readings every five seconds and record them in the following table:

Time (s)	0	→ 5	→ 10	15	20	25	30
Velocity (ft/s)	25	31	35	43	47	45	41

velocities were converted from mi/h to ft/s.



Right-end points

$$\Delta t = 5$$

$$t_1 = 5$$

$$t_2 = 10$$

$$t_3 = 15$$

$$t_4 = 20$$

$$t_5 = 25$$

$$t_6 = 30$$

velocity at 5s.

$$\begin{aligned} \text{distance} \approx & \Delta t \cdot v(5) + \Delta t \cdot v(10) + \Delta t \cdot v(15) \\ & + \Delta t \cdot v(20) + \Delta t \cdot v(25) + \Delta t \cdot v(30) \end{aligned}$$

$$= 5(31 + 35 + 43 + 47 + 45 + 41)$$

$$= 5 \cdot 242$$

$$= \boxed{1210 \text{ ft}}$$

