## Problem 1

The function $z-1$ is analytic on $\mathbb{C}$ because it is a polynomial. Now, $(z-1)^2$ is a composition of $z-1$ and $z^2$ which are analytic on $\mathbb{C}$. Therefore, $(z-1)^2$ is analytic. Hence, the function

$$3(z-1)^2 + 2(z-1)$$

is analytic on $\mathbb{C}$ by the sum and product rules.
Now, using the rules for derivatives, we find that

$$\begin{aligned} \big[3(z-1)^2 + 2(z-1)\big]' &= \big[3(z-1)^2\big]' + \big[2(z-1)\big]' \\ &= 3((z-1)^2)' + 2(z-1)' \\ &= 3(2(z-1)(z-1)') + 2(1) \\ &= 6(z-1) + 2. \end{aligned}$$

## Problem 3

We will show that $\operatorname{Im} z$ does not have a complex derivative at every point $z_0 \in \mathbb{C}$.
Fix $z_0 \in \mathbb{C}$.

1. Let $z = x_0 + iy$, with $y \to y_0$. Then
$$\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{y\to y_0} \frac{iy - iy_0}{iy - iy_0} = \lim_{y\to y_0} 1 = 1.$$
2. But, if $z = x + iy_0$, with $x \to x_0$, then
$$\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{x\to x_0} \frac{0}{x-x_0} = 0.$$

We have two possible limits for the difference quotient. Therefore, the complex derivative of $\operatorname{Im} z$ does not exists at any point $z_0 \in \mathbb{C}$.
Here is another way to show that $\operatorname{Im} z$ is not analytic at any $z_0$. Assume that it was (a proof by contradiction). We know that

$$\operatorname{Im} z = \frac{z - \overline{z}}{2i} \quad \Rightarrow \quad \overline{z} = z - 2i \operatorname{Im} z.$$

Since $z$ is analytic and $\operatorname{Im} z$ is assumed to be analytic at $z_0$, then we conclude that $\overline{z}$ is analytic at $z_0$. But we see in the lecture notes that $\overline{z}$ is nowhere analytic. Hence, a contradiction. Therefore, we must have that $\operatorname{Im} z$ is not analytic at any $z_0$.

## Problem 10

The principal branch of the square root $\sqrt{z}$ is analytic on $\mathbb{C}\backslash(-\infty, 0]$, by Example 2.3.13. Since $z-1$ is also analytic, the composition $\sqrt{z-1}$ is analytic on

$$\mathbb{C}\backslash\{z : z-1 \in (-\infty,0]\} = \mathbb{C}\backslash\{z \in (-\infty, 1]\} = \mathbb{C}\backslash(-\infty, 1].$$

The derivative is

$$(\sqrt{z-1})' = \tfrac{1}{2}(z-1)^{(1-2)/2} = \tfrac{1}{2}(z-1)^{-1/2} = \frac{1}{2\sqrt{z-1}}.$$

## Problem 13

We let $z_0 = 1$ and $f(z) = z^{100}$. Then we get

$$\lim_{z\to 1} \frac{z^{100}-1}{z-1} = f'(1) = 100(1)^{99} = 100.$$

## Problem 15

Notice that

$$\frac{1}{z\sqrt{1+z}} - \frac{1}{z} = \frac{1}{z}\Big(\frac{1}{\sqrt{1+z}} - 1\Big) = \frac{\frac{1}{\sqrt{1+z}} - 1}{z}$$

Let $f(z) = \frac{1}{\sqrt{1+z}}$ and $z_0 = 1$. Then

$$\lim_{z\to 0}\Big(\frac{1}{z\sqrt{1+z}} - \frac{1}{z}\Big) = \lim_{z\to 0} \frac{\frac{1}{\sqrt{1+z}} - 1}{z} = f'(0).$$

Now, we have

$$f'(z) = \frac{d}{dz}\Big(\frac{1}{\sqrt{z+1}}\Big) = \frac{(1)'(\sqrt{1+z}) - (1)(\sqrt{1+z})'}{1+z} = -\frac{\frac{1}{2}(1+z)^{-1/2}}{1+z}.$$

If we want to write the derivative with a rational exponent, we use the exponential:

$$-\tfrac{1}{2}\frac{e^{-\frac{1}{2}\operatorname{Log}(1+z)}}{e^{\operatorname{Log}(1+z)}} = -\tfrac{1}{2}e^{-\frac{3}{2}\operatorname{Log}(1+z)} = -\tfrac{1}{2}(1+z)^{-3/2}.$$

Hence,

$$\lim_{z\to 0}\Big(\frac{1}{z\sqrt{1+z}} - \frac{1}{z}\Big) = -\tfrac{1}{2}(1+0)^{-3/2} = -\frac{1}{2}.$$