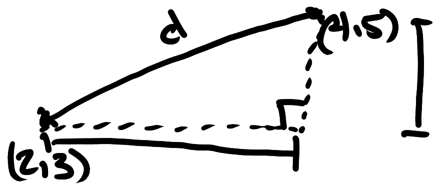
Example 1 $\quad x = y^2/2 \leftarrow y = \pm\sqrt{2x}$

Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$.

distance between two points



$$d = \sqrt{(4-2)^2 + (5-3)^2}$$
$$= \sqrt{2^2 + 2^2}$$
$$= \sqrt{8}.$$

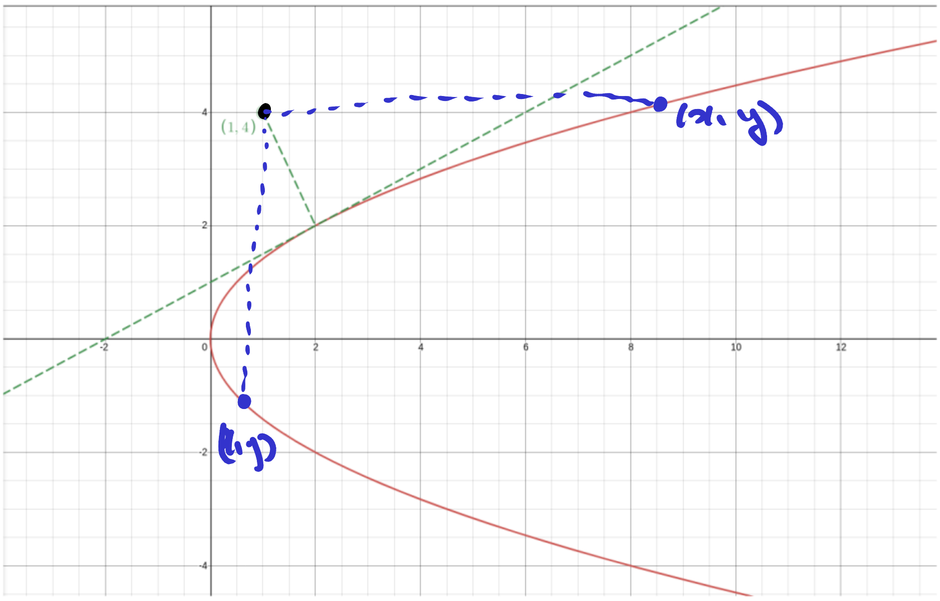
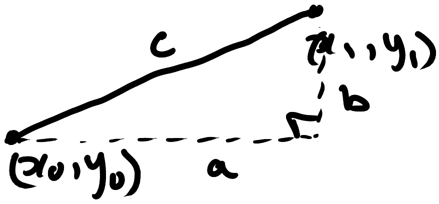


Figure: Drawing of the situation

$$d = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$$



$$c = \sqrt{a^2 + b^2}$$

- (x, y) pt. on the curve.
- coordinates for the pt. not on the curve: $(1, 4)$
- Formula of the curve: $y^2 = 2x$.
- $d = \sqrt{(1-x)^2 + (4-y)^2}$

Goal: find the min of d.

Trick: $D = d^2 = (1-x)^2 + (4-y)^2$

New goal: find the minimum of D.

$$x = y^2/2 \implies D(y) = \left(1 - \frac{y^2}{2}\right)^2 + (4-y)^2$$

$$D'(y) = 2\left(1 - \frac{y^2}{2}\right) \cdot (-y) + 2(4-y) \cdot (-1)$$

$$= -\cancel{2y} + y^3 - 8 + \cancel{2y}$$

$$= y^3 - 8$$

So, $D'(y) = y^3 - 8 = 0 \iff y^3 = 8$

$\iff y = \sqrt[3]{8} = 2$

y		2	
y^3-8	$-$	0	$+$
D	$\searrow$	C.P.	$\nearrow$

So, $y = 2$ is an abs. minimum.

We know that $y = 2 \Rightarrow 2x = 2^2$

$\Rightarrow x = 2$.

$$\boxed{\text{So, the point on the curve } 2x = y^2 \text{ closest to the point } (1,2) \text{ is } (2,2).}$$

Example 2

Find the area of the largest rectangle that can be inscribed in a semicircle of radius 2.

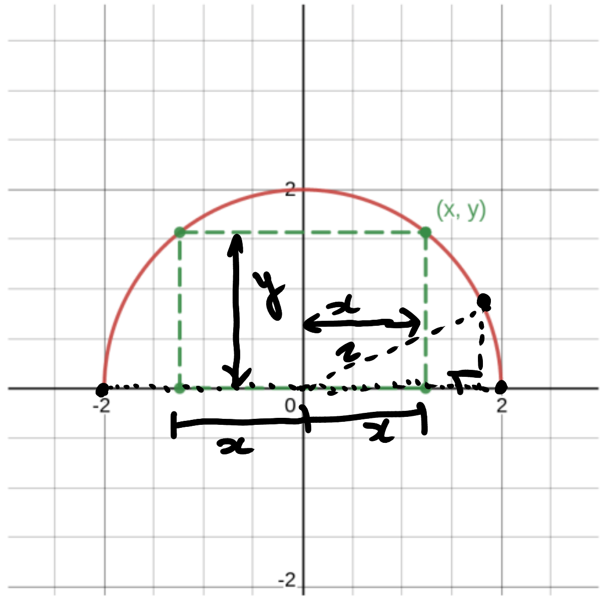


Figure: Drawing of the situation

x : ~~variable for width~~ → distance from the y-axis

y: variable for heigth

A : Area of the rectangle.

<u>Info.</u> • radius of semi-circle : 2

↳ $x^2+y^2 = (\text{radius})^2 = 4.$

• rectangle Area: $A = (\text{width})(\text{heigth}) = 2x \cdot y$

<u>Goal.</u> Find the max of A.

<u>Trick.</u> $-2 \le x \le 2 \to 0 \le x \le 2.$

$$A = 2x \cdot y \overset{y=\sqrt{4-x^2}\ (y\ge 0)}{=} \underbrace{2x}_{f} \underbrace{\sqrt{4-x^2}}_{g}$$

So, we have

$$A'(x) = 2\sqrt{4-x^2} + 2x \frac{-2x}{2\sqrt{4-x^2}}$$

$$= 2\sqrt{4-x^2} - 2x^2 / \sqrt{4-x^2}$$

$$= \frac{2(4-x^2) - 2x^2}{\sqrt{4-x^2}}$$

$$\Rightarrow \quad A'(x) = \frac{8 - 4x^2}{\sqrt{4-x^2}} = \frac{4(2-x^2)}{\sqrt{4-x^2}}$$

So, $A'(x) = 0 \Leftrightarrow 4(2-x^2) = 0 \Leftrightarrow x = \pm\sqrt{2}$

$\Leftrightarrow x = \sqrt{2}$.

x	0		$\sqrt{2}$		2
$2-x^2$		$+$	0	$-$	
$A'(x)$		$+$	0	$-$	
$A(x)$		$\nearrow$	C.P.	$\searrow$	

So, $A(x)$ has a local. max at $x = \sqrt{2}$.

- $A(0) = 0$
- $A(\sqrt{2}) = 4 \longrightarrow$ $\boxed{\text{4 is the biggest area of the rectangle inside the semi-circle}}$
- $A(2) = 0$