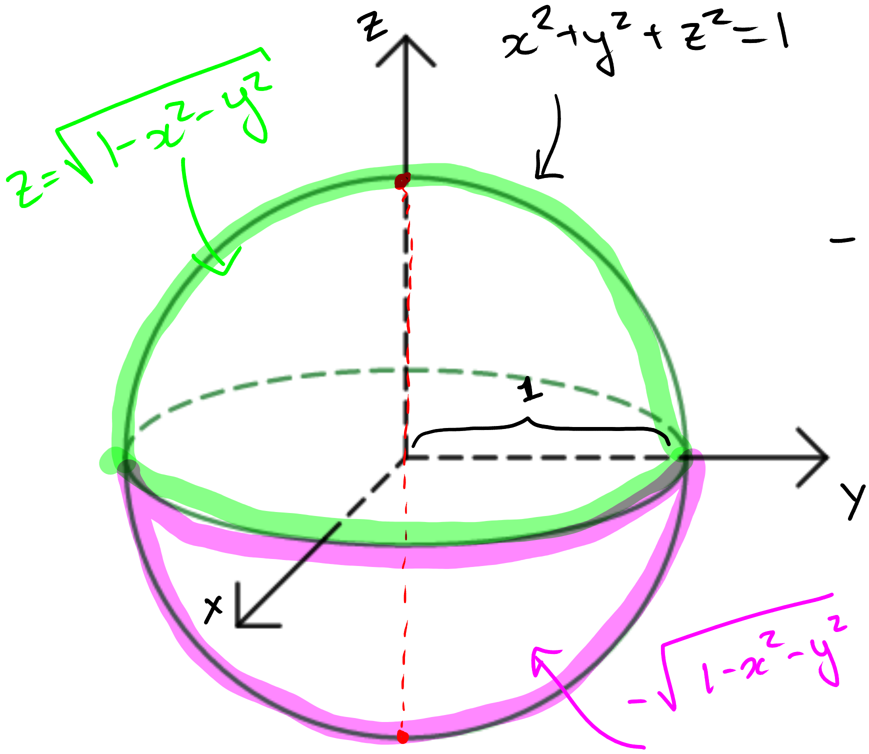
Chapter 15

Multiple Integrals

15.8 Triple integrals in spherical coordinates

Spherical coordinates

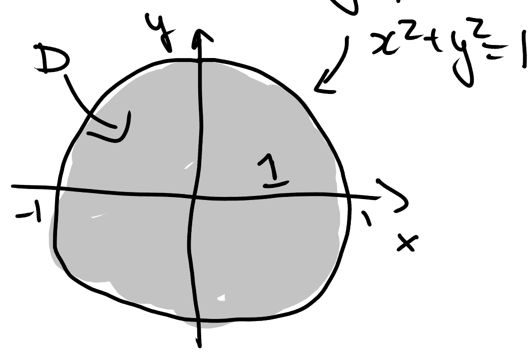
EXAMPLE. Describe the solid bounded by the sphere (picture below).



As a type 1:

$$-\sqrt{1-x^2-y^2} \le z \le \sqrt{1-x^2-y^2}$$

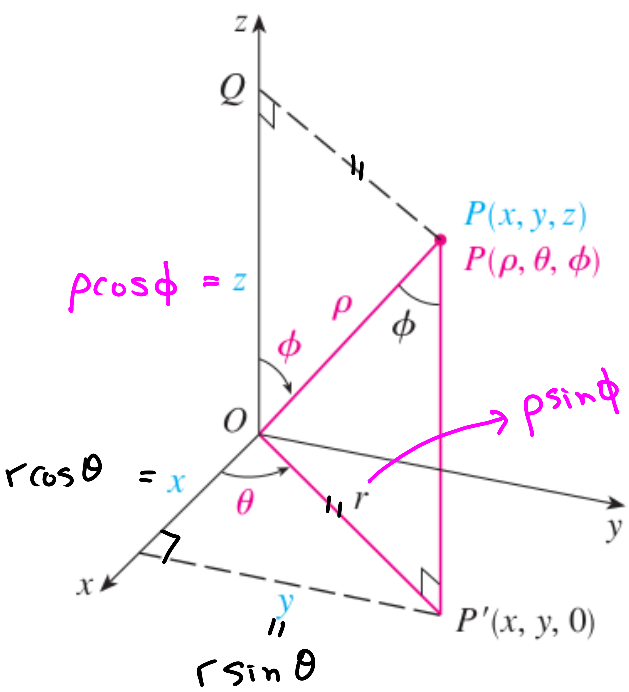
Shadow on xy-plane:



$$D=\{(x,y): -1 \le x \le 1, \; -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}\}$$

$$E=\{(x,y,z): -1 \le x \le 1, \; -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}, \; -\sqrt{1-x^2-y^2} \le z \le \sqrt{1-x^2-y^2}\}.$$

Definition



Cartesian $\longrightarrow$ Spherical

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$0 \le \theta \le 2\pi$$

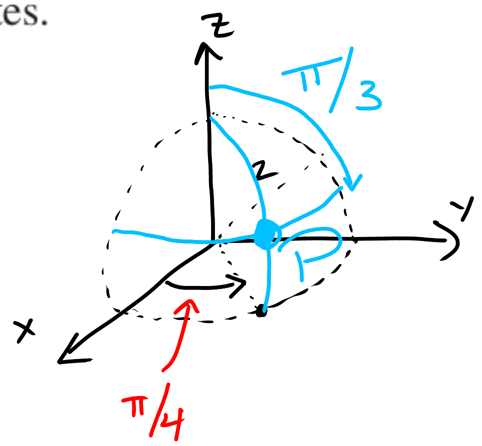
$$z = \rho\cos\phi$$
$$0 \le \phi \le \pi, \quad \rho \ge 0.$$

EXAMPLE 1 The point $(\overset{\rho}{2}, \overset{\theta}{\pi/4}, \overset{\phi}{\pi/3})$ is given in spherical coordinates. Plot the point and find its rectangular coordinates.

$\rho = 2$

$\theta = \pi/4$

$\phi = \pi/3$



$$x = \rho \sin\phi \cos\theta = 2 \sin(\pi/3)\cos(\pi/4)$$
$$= 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \boxed{\frac{\sqrt{6}}{2}}$$

$$y = \rho \sin\phi \sin\theta = 2 \sin(\pi/3)\sin(\pi/4) = \boxed{\frac{\sqrt{6}}{2}}$$

$$z = \rho\cos\phi = 2\cos(\pi/3) = 2\left(\frac{1}{2}\right) = \boxed{1}$$

EXAMPLE 2 The point $(0, 2\sqrt{3}, -2)$ is given in rectangular coordinates. Find spherical coordinates for this point.

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{0^2 + (2\sqrt{3})^2 + (-2)^2}$$
$$= \sqrt{12 + 4} = \boxed{4}.$$

We have $z = \rho\cos\phi \Rightarrow -2 = 4\cos\phi$

$$\Rightarrow -\frac{1}{2} = \cos\phi$$
$$\Rightarrow \boxed{\phi = \frac{2\pi}{3}} \text{ (between 0 and } \pi)$$

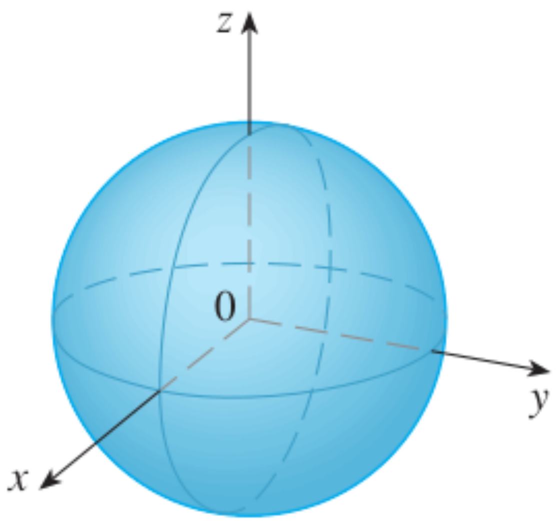
$$0 = x = 2\sin\left(\tfrac{2\pi}{3}\right)\cos(\theta) = \sqrt{3}\cos(\theta)$$
$$\Rightarrow 0 = \sqrt{3}\cos(\theta) \Rightarrow \cos\theta = 0 \Rightarrow \theta = \pi/2 \text{ or } \theta = 3\pi/2.$$

Here $y = 2\sqrt{3} \geq 0 \Rightarrow \boxed{\theta = \pi/2}$

Equations of important solids: $\to$ surface.

Sphere of radius R.

$$\boxed{\rho = R}$$



$\sqrt{x^2+y^2+z^2} = \sqrt{R^2}$

$\Rightarrow \sqrt{x^2+y^2+z^2} = R$

$\Rightarrow \quad \rho = R$

$0 \le \theta \le 2\pi \ , \ 0 \le \phi \le \pi$

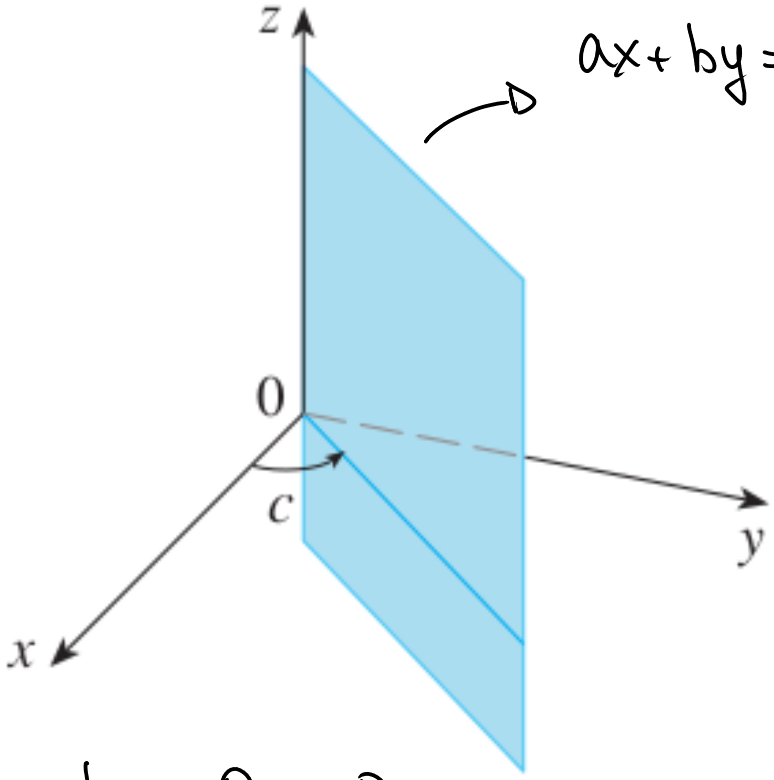
Half planes.

$$\boxed{\theta = c}$$

$0 \le \rho < \infty.$

$0 \le \phi \le \pi.$

$ax + by = 0$



$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

Replace that in

$ax + by = 0$

$\Rightarrow a\rho\sin\phi \cos\theta + b\rho\sin\phi \sin\theta = 0$

$\Rightarrow a\cancel{\rho\sin\phi}\cos\theta = -b\cancel{\rho\sin\phi}\sin\theta$

$\Rightarrow \quad a \cos\theta = -b \sin\theta$

Note:
$y = \tan(c)\, x$

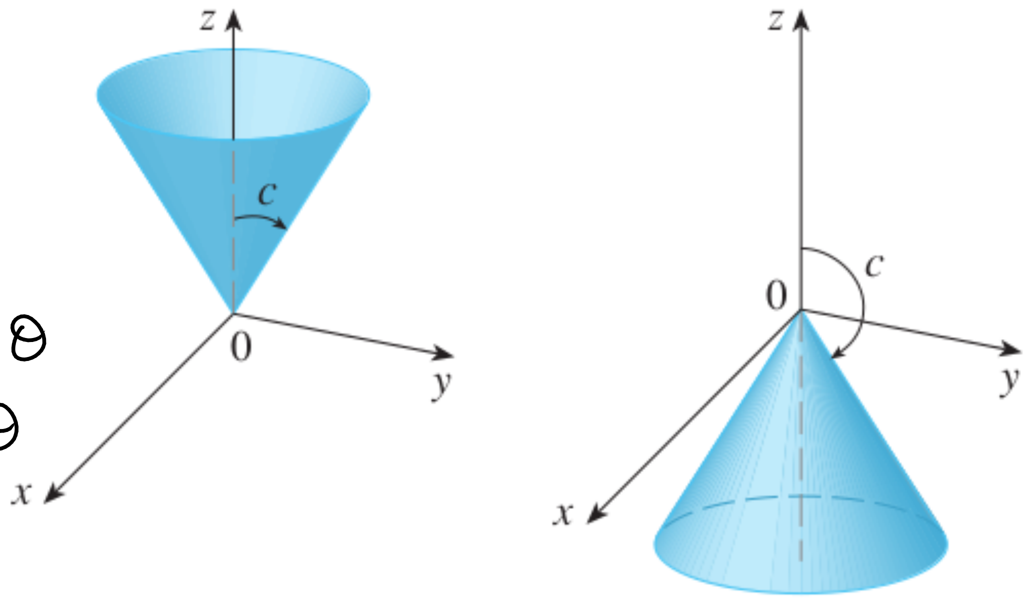
$\Rightarrow \quad -\dfrac{a}{b} = \dfrac{\sin\theta}{\cos\theta}$

$\Rightarrow -\dfrac{a}{b} = \tan\theta \quad \Rightarrow \quad \theta = \overbrace{\arctan\left(\dfrac{-a}{b}\right)}^{=c}$

Cones.

$$\boxed{\phi = c}$$

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$



$0 < c < \pi/2$ $\qquad$ $\pi/2 < c < \pi$

$$z = A\sqrt{x^2+y^2} \Rightarrow \rho\cos\phi = A\sqrt{\rho^2\sin^2\phi(\cos^2\theta+\sin^2\theta)}$$

$$\Rightarrow \rho\cos\phi = A\sqrt{\rho^2\sin^2\phi}$$

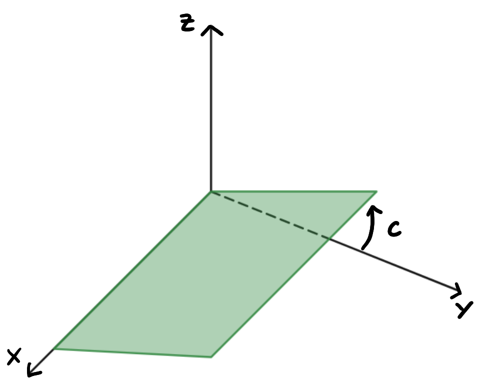
- $A > 0 \Rightarrow z \geq 0 \Rightarrow 0 \leq \phi \leq \frac{\pi}{2}$

$$\Rightarrow \rho\cos\phi = A\rho\sin\phi$$

$$\Rightarrow \frac{1}{A} = \tan\phi \Rightarrow \phi = \underbrace{\arctan\left(\frac{1}{A}\right)}_{c}$$

- $A < 0 \Rightarrow z \leq 0 \Rightarrow \frac{\pi}{2} \leq \phi \leq \pi \Rightarrow \phi = \arctan\left(\frac{1}{A}\right)$

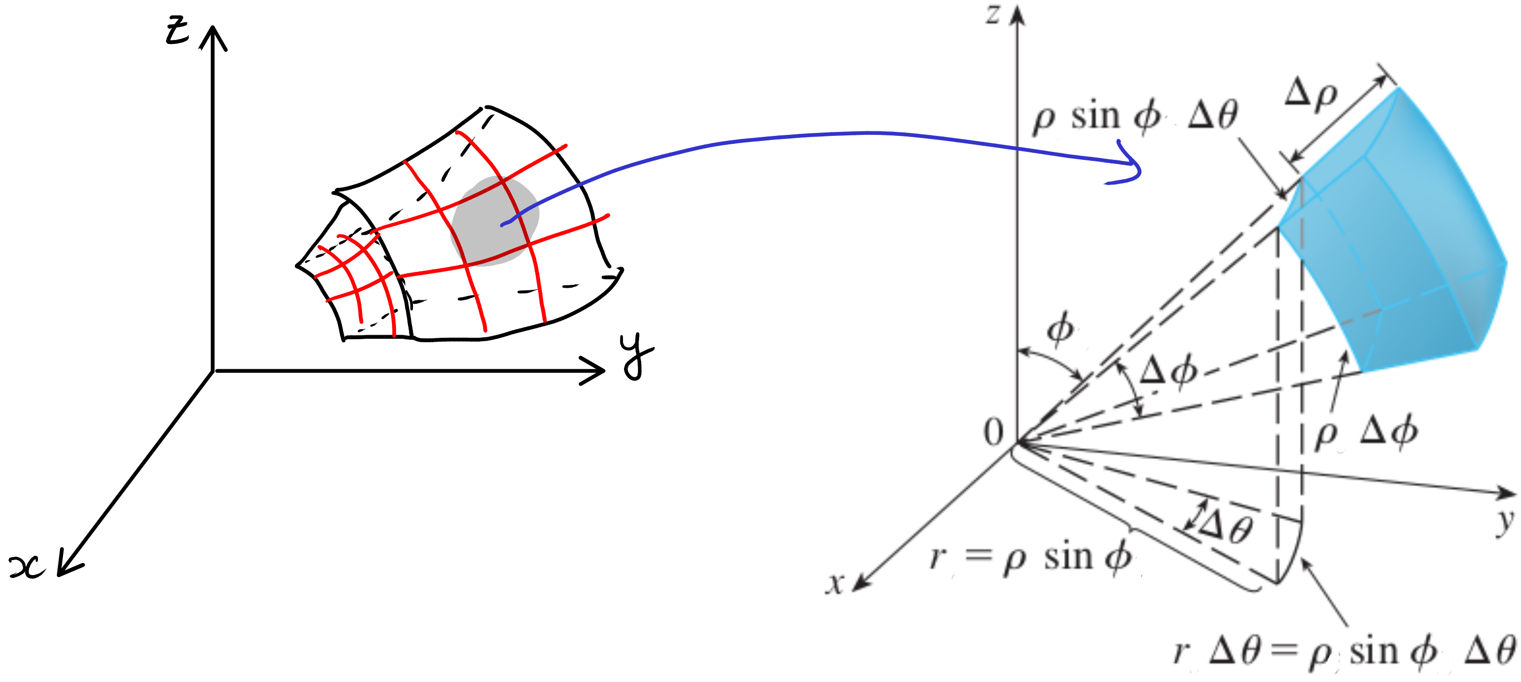
Question. Find the equation of the half-plane in the picture below in spherical coordinates. The plane is making an angle of c with the xy-plane.



Evaluating integrals in sperical coordinates.

Spherical Wedge

$$E = \{(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho\cos\phi) \,:\, a \le \rho \le b,\, \alpha \le \theta \le \beta,\, c \le \phi \le d\}$$



We can show that

$$\Delta V = \bar{\rho}^2 \sin\bar{\phi}\, \Delta\rho\, \Delta\theta\, \Delta\phi$$

$\bar{\rho}$: mid point subdivisions

$\bar{\phi}$: midpoint subdivisions

As the number of subdivisions goes to infinity, we obtain

$$\boxed{dV = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi}$$

Formula for the change of variable (in spherical coordinates).

$$\boxed{\iiint_E f(x,y,z)\, dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi))\, \rho^2 \sin(\phi)\, d\rho\, d\theta\, d\phi}$$

$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

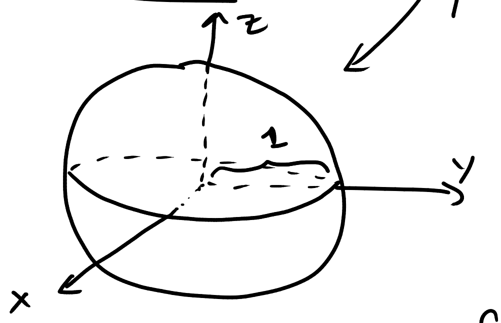
$z = \rho\cos\phi$

EXAMPLE 3 Evaluate $\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\,dV$, where B is the unit ball:

$$B = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$$

① Picture

$\rho = 1$



Radius $0 \leq \rho \leq 1$

Angles.

$0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \pi$

$$E = \{(\rho, \theta, \phi) : 0 \leq \rho \leq 1,\ 0 \leq \theta \leq 2\pi,\ 0 \leq \phi \leq \pi\}.$$

② Integrate $\longrightarrow$ $\rho^2 = x^2+y^2+z^2 \to$ shortcut.

$$\iiint_E e^{(x^2+y^2+z^2)^{3/2}}\,dV$$

$$= \int_0^{\pi}\int_0^{2\pi}\int_0^1 \left(e^{(\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta + \rho^2\cos^2\phi)^{3/2}}\right) \rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

$$= \int_0^{\pi}\int_0^{2\pi}\int_0^1 e^{(\rho^2\sin^2\phi + \rho^2\cos^2\phi)^{3/2}}\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

$$= \int_0^{\pi}\int_0^{2\pi}\int_0^1 e^{(\rho^2)^{3/2}}\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

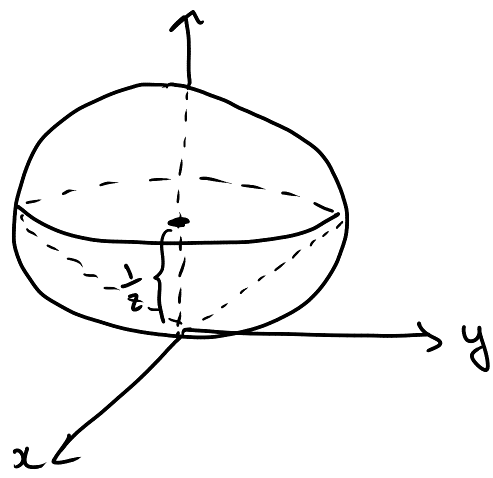
$$= \int_0^{\pi}\int_0^{2\pi}\int_0^1 e^{\rho^3}\cdot\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

$$= \underbrace{\left(\int_0^1 e^{\rho^3}\rho^2\,d\rho\right)}_{u=\rho^3 \to du = 3\rho^2 d\rho}\left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi}\sin\phi\,d\phi\right) = \boxed{4\pi\left(\frac{e-1}{3}\right)}.$$

EXAMPLE 4 Use spherical coordinates to find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$.

① Picture

$$x^2+y^2+z^2 = z$$
$$\Updownarrow$$
$$x^2+y^2+z^2-z+\frac{1}{4} = \frac{1}{4}$$
$$\Updownarrow$$
$$x^2+y^2+\left(z-\frac{1}{2}\right)^2 = \frac{1}{4} = \left(\frac{1}{2}\right)^2$$



Replace eq. cone in eq. sphere:

$$x^2+y^2+x^2+y^2 = \sqrt{x^2+y^2}$$

$$\Rightarrow \quad 2\underbrace{(x^2+y^2)}_{\left(\sqrt{x^2+y^2}\right)^2} = \sqrt{x^2+y^2} \quad \Rightarrow \quad 2\sqrt{x^2+y^2} = 1$$

$$\Rightarrow \quad x^2+y^2 = \frac{1}{4}$$

Cone

$\phi = \pi/4$

Sphere

$$x^2+y^2+z^2 = z$$
$$\Rightarrow \quad \rho^2 = \rho\cos\phi$$
$$\Rightarrow \quad \rho = \cos\phi$$

$$E = \left\{(\rho,\theta,\phi) : 0 \le \rho \le \cos\phi,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \frac{\pi}{4}\right\}$$

② Volume

$$\mathrm{Vol}(E) = \iiint_E 1\, dV = \int_0^{\pi/4}\int_0^{2\pi}\int_0^{\cos\phi} \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$$

$$= \int_0^{\pi/4}\int_0^{2\pi} \sin\phi \, \frac{\cos^3\phi}{3}\, d\theta\, d\phi \qquad \left(\begin{array}{l} u = \cos\phi \\ du = -\sin\phi\, d\phi \end{array}\right)$$

$$= \boxed{\frac{\pi}{24}}$$