Math 302

Chapter 2

Section 2.1: Linear First Order Differential Equation

Contents

Created by: Pierre-Olivier Parisé
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What Is A LFODE?

A first order ODE is said to be **linear** (abbreviated LFODE) if it can be written as

$$y' + p(x)y = f(x). \tag{1}$$

- Example: $y' + 3y/x^2 = 1$.
- Example: $xy' - 8x^2y = \sin x$.

More Terminology

- A first order ODE that is not of the form (1), then the ODE is said to be **nonlinear**.
  - Example: $xy' + 3y^2 = 2x$.
  - Example: $yy' + e^y = \tan(xy)$.
- When $f(x) = 0$ for any x, then $y' + p(x)y = 0$ is said to be **homogeneous**.
  - Example: $y' + 3y/x^2 = 0$.
  - Example: $xy' - 8x^2y = 0$.
- When $f(x)$ is not zero, then the LODE is said to be **nonhomogeneous**.

Example 1. Find all the solutions to

$$y' = \frac{1}{x^2}$$

General Solution

We say that a function $y = y(x, c)$ is a **general solution** to (1) if

- For each fixed parameter c, the resulting function $y = y(x, c)$ is a solution to (1) on an an open interval (a, b).
- If $y_1 = y_1(x)$ is a solution to (1) on (a, b), then y_1 can be obtained from the formula $y = y(x, c)$ by choosing c appropriately.

Homogeneous LFODE

We now find the general solution to

$$y' + p(x)y = 0 \tag{2}$$

where p is continuous on an interval (a, b).

Example 2. Let a be a constant (fixed).

1. Find the general solution of $y' - ay = 0$.
2. Solve the initial value problem

$$y' - ay = 0, \quad y(x_0) = y_0.$$

Example 3.

1. Find the general solution of $xy' + y = 0$.
2. Solve the initial value problem

$$xy' + y = 0, \quad y(1) = 3.$$

General facts:

- The general solution to (2) is given by

$$y = ce^{-P(x)}$$

where $P(x) = \int p(x)\, dx$ is any antiderivative of $p(x)$.

- The solution to the IVP

$$y' + p(x)y = 0, \quad y(x_0) = y_0$$

is given by

$$y(x) = y_0 e^{-\int_{x_0}^{x} p(x)\, dx}.$$

Nonhomogeneous LFODE

We now want to find the general solution to

$$y' + p(x)y = f(x)$$

where the functions $p(x)$ and $f(x)$ are continuous on an open interval (a, b).

Remark:

- The homogeneous part $y' + p(x)y = 0$ is called the **complementary equation**.

EXAMPLE 4. Find the general solution of

$$y' + 2y = x^3 e^{-2x}.$$

Summary of The Method

- Find a function y_1 such that $y_1' + p(x)y_1' = 0$
- Write $y = uy_1$ where u is an unknown function.
- Solve $u'y_1 = f(x)$.
- Substitute u in y.

EXAMPLE 5.

1. Find the general solution

$$y' + (\cot x)y = x \csc x.$$

2. Solve the initial value problem

$$y' + (\cot x)y = x \csc x, \quad y(\pi/2) = 1.$$

General Theorem

Suppose

- $p(x)$ and $f(x)$ are continuous on an interval (a, b)
- y_1 is a solution to the complementary equation.

Then the general solution to $y' + p(x)y = f(x)$ is

$$y(x) = y_1(x)\left(c + \int \frac{f(x)}{y_1(x)}\,dx\right)$$

for each x in (a, b).

Existence Theorem

Suppose

- $p(x)$ and $f(x)$ are continuous on an interval (a, b).
- y_1 is a solution to the complementary equation.
- x_0 is an arbitrary number in (a, b) and y_0 is an arbitrary number.

Then the boundary value problem

$$y' + p(x)y + f(x), \quad y(x_0) = y_0$$

has a unique solution which is of the form

$$y(x) = y_1(x)\left(\frac{y_0}{y_1(x_0)} + \int_{x_0}^{x} \frac{f(t)}{y_1(t)}\,dt\right)$$

for each x in (a, b).