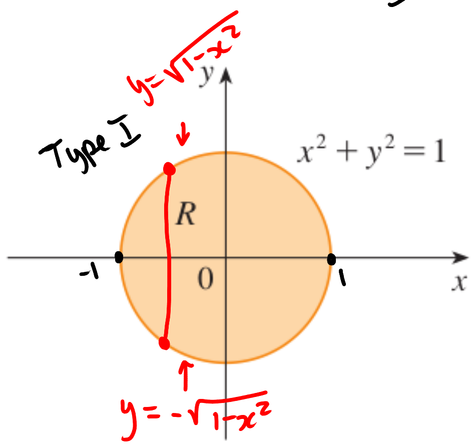
Example. Compute the integral $I = \iint_R x^2 + y^2 \, dA$ where R is the region below.



$y = \pm\sqrt{1-x^2}$

$R = \left\{ (x,y) : \begin{array}{l} -1 \le x \le 1 \\ -\sqrt{1-x^2} \le y \le \sqrt{1-x^2} \end{array} \right\}$

$$I = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^2+y^2 \, dy\, dx$$

$$= \int_{-1}^{1} x^2 y + \frac{y^3}{3} \Big|_{-\sqrt{1-x^2}}^{+\sqrt{1-x^2}} dx$$

$$= \int_{-1}^{1} x^2\sqrt{1-x^2} + \frac{(1-x^2)^{3/2}}{3} + x^2\sqrt{1-x^2} + \frac{(1-x^2)^{3/2}}{3} \, dx$$

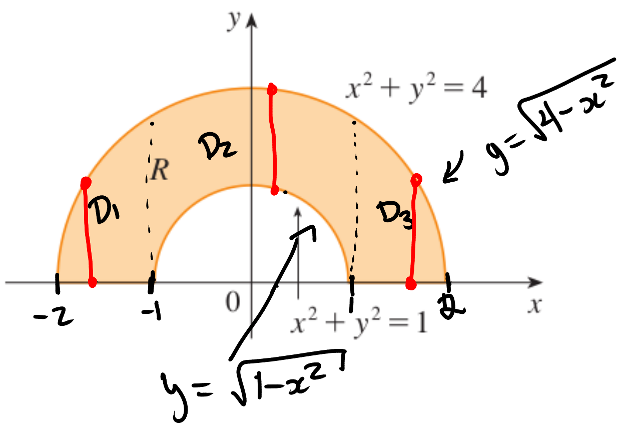
$$= \int_{-1}^{1} 2x^2\sqrt{1-x^2} + \frac{2}{3}(1-x^2)^{3/2} \, dx$$

$$= \frac{\pi}{2}$$

$0 \le r \le 1$

$0 \le \theta \le 2\pi$

Example. Compute the integral $\iint_R x^2 + y^2 \, dA$ where R is the region below.



$D_1 = \left\{ (x,y) : \begin{array}{l} -2 \le x \le -1 \\ 0 \le y \le \sqrt{4-x^2} \end{array} \right\}$

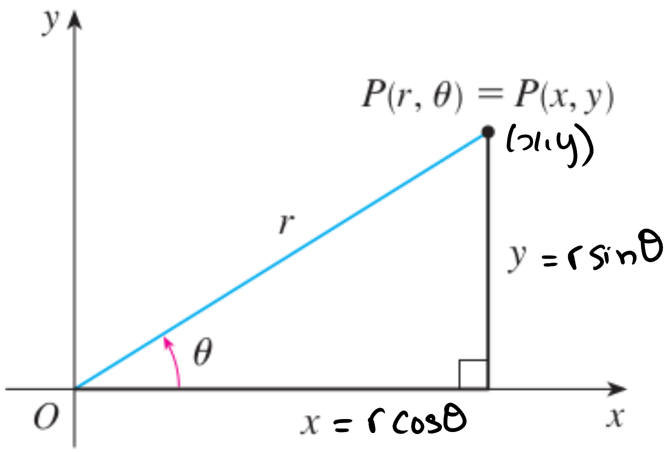
$D_2 = \left\{ (x,y) : \begin{array}{l} -1 \le x \le 1 \\ \sqrt{1-x^2} \le y \le \sqrt{4-x^2} \end{array} \right\}$

$D_3 = \left\{ (x,y) : \begin{array}{l} 1 \le x \le 2 \\ 0 \le y \le \sqrt{4-x^2} \end{array} \right\}$

$$\iint_D x^2+y^2 \, dA = \underbrace{\iint_{D_1} x^2+y^2 \, dA}_{\int_{-2}^{-1}\int_0^{\sqrt{4-x^2}} x^2+y^2 \, dy\, dx} + \iint_{D_2} x^2+y^2 \, dA + \iint_{D_3} x^2+y^2 \, dA.$$

$$= \boxed{\frac{4\pi}{3} - \frac{\sqrt{3}}{2}}$$

Polar coordinate.

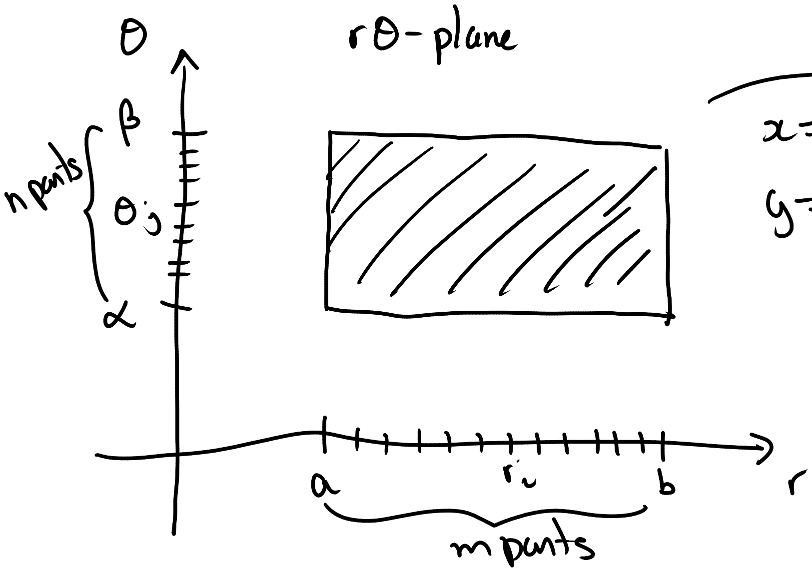


Polar $\longrightarrow$ Cartesian

$x = r\cos\theta \qquad y = r\sin\theta$

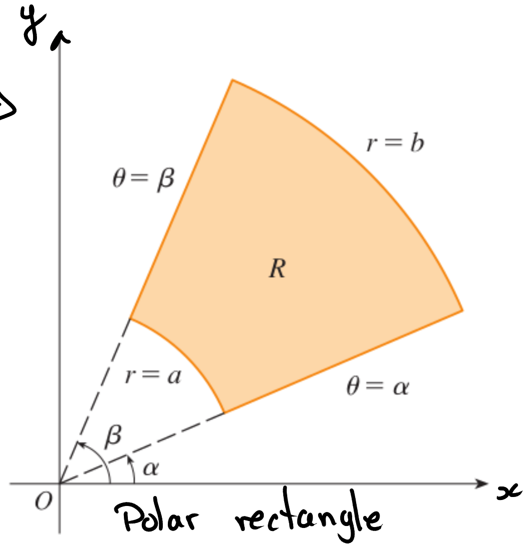
Cartesian $\rightarrow$ Polar

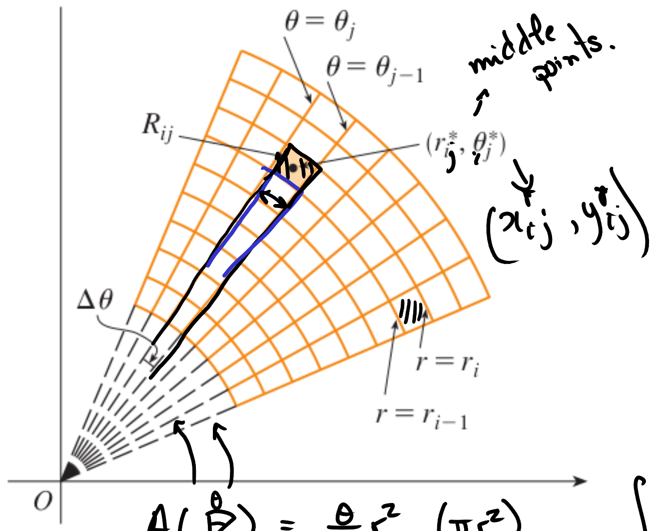
$r = \sqrt{x^2+y^2} \qquad \theta = \arctan\left(\frac{y}{x}\right)$



$x = r\cos\theta$

$y = r\sin\theta$





$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, A(R_{ij})$$

$$x_{ij}^* = r_{ij}^*\cos\theta_{ij}^*, \quad y_{ij}^* = r_{ij}^*\sin\theta_{ij}^*$$

$$A(R_{ij}) = A\left(\overset{\Delta\theta}{\triangleright}_{r_i}\right) - A\left(\overset{\Delta\theta}{\triangleright}_{r_{i-1}}\right)$$

$$= \Delta\theta\left(\frac{r_{i-1}+r_i}{2}\right)(r_i - r_{i-1})$$

$$= r_{ij}^*\ \Delta r\ \Delta\theta$$

$$A\left(\overset{\theta}{\triangleright}_r\right) = \frac{\theta}{2}r^2 \quad (\pi r^2)$$

$$\iint_R f(x,y)\,dA \approx \sum_i\sum_j \underbrace{f(r_{ij}^*\cos\theta_{ij}^*, r_{ij}^*\sin\theta_{ij}^*)\ r_{ij}^*\,\Delta r\,\Delta\theta}_{g(r_{ij}^*,\theta_{ij}^*)}$$

$m, n \to \infty$,

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta\int_a^b g(r,\theta)\,dr\,d\theta$$

2 Change to Polar Coodinates in a Double Integral If f is continuous on a polar rectangle R given by $0 \leqslant a \leqslant r \leqslant b$, $\alpha \leqslant \theta \leqslant \beta$, where $0 \leqslant \beta - \alpha \leqslant 2\pi$, then

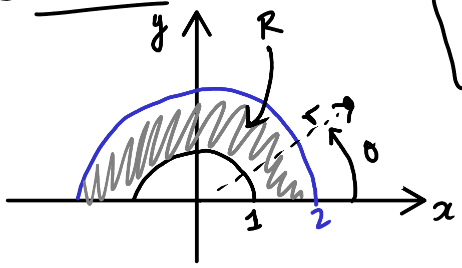
$$\boxed{\iint_R f(x,y)\,\underbrace{dA}_{dxdy} = \int_\alpha^\beta\int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta}$$

$dA = r\,dr\,d\theta$

EXAMPLE 1 Evaluate $\iint_R (3x + 4y^2)\, dA$, where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

$z = f(x,y)$ $\quad 4 = 2^2$

① Picture.



circle polar coordinates. $\quad x = r\cos\theta$, $y = r\sin\theta$

$r^2 = 1 \to r = 1$

$r^2 = 2^2 \to r = 2$

$0 \le \theta \le \pi$

$R = \{(r,\theta) : 1 \le r \le 2$ and $0 \le \theta \le \pi\}$

② Integrating.

$$\iint_R 3x + 4y^2\, dA = \int_0^\pi \int_1^2 (3r\cos\theta + 4r^2\sin^2\theta)\, r\, dr\, d\theta$$

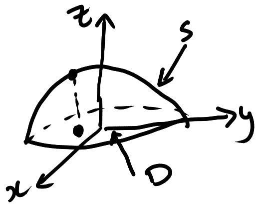
$$= \int_0^\pi \int_1^2 3r^2\cos\theta + 4r^3\sin^2\theta\, dr\, d\theta$$

$$= \int_0^\pi r^3\cos\theta + r^4\sin^2\theta \Big|_1^2 d\theta$$

$$= \int_0^\pi 8\cos\theta + 16\sin^2\theta - (\cos\theta + \sin^2\theta)\, d\theta$$

$$= 15\pi/2$$

EXAMPLE 2 Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$. $= f(x,y)$



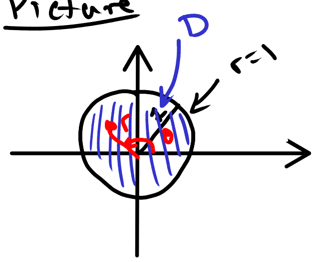
$V(S) = \iint_D 1 - x^2 - y^2\, dA$. What is D??

① Find the D.

$z = 0 \to 0 = 1 - x^2 - y^2 \to x^2 + y^2 = 1$

$\Rightarrow D = \{(x,y) : x^2 + y^2 \le 1\}$

② Picture



$D = \{(r,\theta) : 0 \le r \le 1, \ 0 \le \theta \le 2\pi\}$

(*) $\int_0^{2\pi} 1\, d\theta = \theta \Big|_0^{2\pi} = 2\pi$

(**) $\int_0^1 r - r^3\, dr = \frac{r^2}{2} - \frac{r^4}{4} \Big|_{0=r}^{1=r} = \frac{1}{2} - \frac{1}{4} - 0 = \frac{1}{4}$

③ Integrate. $x = r\cos\theta$, $y = r\sin\theta \to dA = r\, dr\, d\theta$

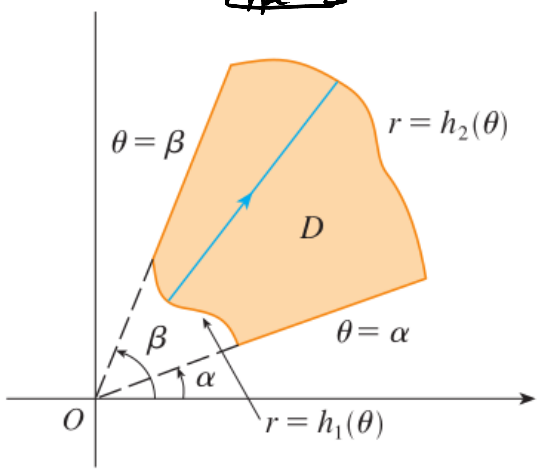
$$\iint_D 1 - x^2 - y^2\, dA = \int_0^{2\pi} \int_0^1 (1 - r^2\cos^2\theta - r^2\sin^2\theta)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 \underline{(1 - r^2) r\, dr}\, d\theta$$

$$\overset{*}{=} \left(\int_0^{2\pi} d\theta\right)\left(\int_0^1 (1 - r^2) r\, dr\right)^{**} = \frac{\pi}{2}$$

More complicated region:

Type I



3 If f is continuous on a polar region of the form

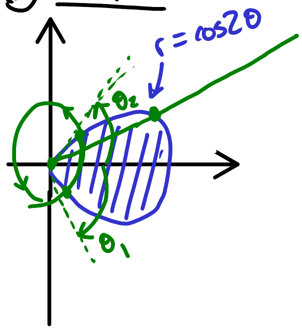
$$D = \{(r, \theta) \mid \alpha \leqslant \theta \leqslant \beta,\ h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$$

then

$$\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

EXAMPLE 3 Use a double integral to find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$. $A = \iint_D 1\, dA$.

① Picture



to find θ, we have to plug in 0:

$$0 = \cos 2\theta \Leftrightarrow 2\theta = \frac{\pi}{2} + k\pi \quad k = \dots, -1, 0, 1, \dots$$

$$\Leftrightarrow \theta = \frac{\pi}{4} + k\frac{\pi}{2}, \quad k = \dots, -1, 0, 1, \dots$$

$k = 0 \to \theta_2 = \pi/4$ $\qquad$ $k = -1 \to \theta_1 = -\frac{\pi}{4}$

$D = \{(r, \theta) : 0 \le r \le \cos 2\theta,\ -\frac{\pi}{4} \le \theta \le \pi/4\}$.

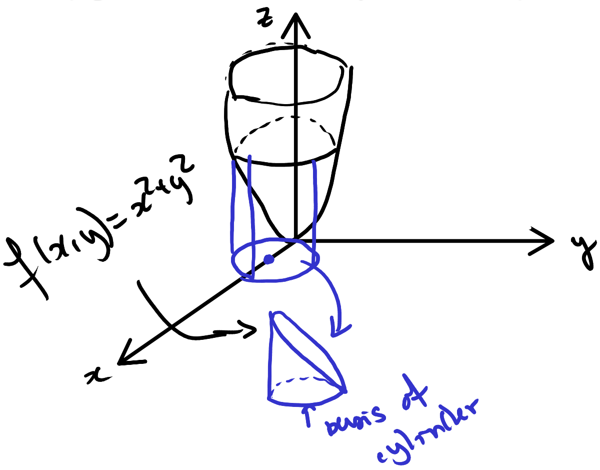
② Integrate. $dA = r\,dr\,d\theta$

$$A = \int_{-\frac{\pi}{4}}^{\pi/4} \int_0^{\cos 2\theta} r\, dr\, d\theta = \int_{-\pi/4}^{\pi/4} \frac{\cos^2 2\theta}{2}\, d\theta = \int_{-\pi/4}^{\pi/4} \frac{1 + \cos(4\theta)}{4}\, d\theta$$

$$= \frac{\theta}{4} + \frac{\sin(4\theta)}{16}\Big|_{-\pi/4}^{\pi/4} = \frac{\pi/4 - (-\pi/4)}{4} + \frac{(0 - 0)}{16} = \boxed{\frac{\pi}{8}}$$

EXAMPLE 4 Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$. →

$$x^2 + y^2 = 2x$$
$$\to x^2 - 2x + y^2 = 0$$
$$\to x^2 - 2x + 1 - 1 + y^2 = 0$$
$$\to (x-1)^2 + y^2 = 1$$



D = base of the cylinder

Goal: decribe boundary of D:

(*) $x^2 + y^2 = 2x$

① Boundary of D.

$x = r\cos\theta$
$y = r\sin\theta$

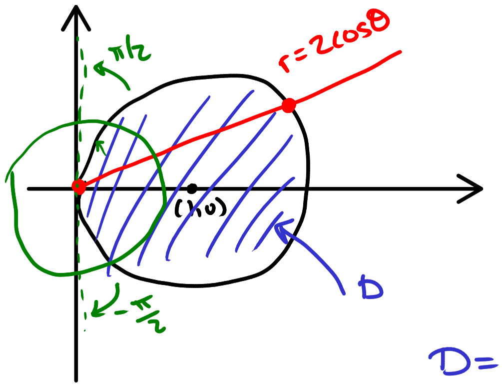
$\xrightarrow{(*)}$ $r^2\cos^2\theta + r^2\sin^2\theta = 2r\cos\theta$

$\to r^2 \underbrace{(\cos^2\theta + \sin^2\theta)}_{=1} = 2r\cos\theta$

$$\to \quad r^2 = 2r\cos\theta$$

$$\to \quad r = 2\cos\theta$$

② Picture of D



$0 \le r \le 2\cos\theta$

$r = 0 \to \cos\theta = 0 \to \theta = \frac{\pi}{2} + k\pi$

$k = 0 \to \theta_2 = \frac{\pi}{2}$

$k = -1 \to \theta_1 = -\frac{\pi}{2}$

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$D = \{ (r,\theta) : \ 0 \le r \le 2\cos\theta \ , \ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \}.$

③ Integrate

$$V(S) = \iint_D x^2 + y^2 \, dA \quad \longrightarrow \quad \int r^3 \, dr = \frac{r^4}{4}$$

$$= \int_{-\frac{\pi}{2}}^{\pi/2} \int_0^{2\cos\theta} r^2 \, r \, dr \, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\pi/2} \frac{16\cos^4\theta}{4} \, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} 4 \, (\cos^2\theta)^2 \, d\theta = 4 \int_{-\pi/2}^{\pi/2} \left(\frac{1 + \cos(2\theta)}{2} \right)^2 d\theta$$

$$= \int_{-\pi/2}^{\pi/2} 1 + 2\cos 2\theta + \cos^2 2\theta \, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} 1 + 2\cos 2\theta + \frac{1 + \cos(4\theta)}{2} \, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{3}{2} + 2\cos 2\theta + \frac{\cos 4\theta}{2} \, d\theta$$

$$= \frac{3}{2}\theta + \sin 2\theta + \frac{\sin 4\theta}{8} \Big|_{-\pi/2}^{\pi/2}$$

$$= \frac{3}{2}(\pi) + (0 - 0) + \left(\frac{0 - 0}{8} \right)$$

$$= \boxed{\frac{3\pi}{2}}$$