Math 644

Chapter 4

Section 4.2: Equivalence of Analytic and Holomorphic

Contents

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Holomorphic Functions

Definition 1. Let U be an open set and $f : U \to \mathbb{C}$. The function f is holomorphic on U if

- $f'(z) := \lim_{w \to z} \dfrac{f(w) - f(z)}{w - z}$ exists for all $z \in U$ and;
- $z \mapsto f'(z)$ is continuous on U.

Notes:

- f is holomorphic on U, then f is continuous on U;
- A complex-valued function f is holomorphic on a (generic) set S if it is holomorphic on an open set $U \supset S$.
- There are weaker definitions of a holomorphic functions: For example, one definition does not require that $z \mapsto f'(z)$ is continuous.

Example 2.

a) Any polynomial is a holomorphic function on $\mathbb{C}$.

b) Any rational function is a holomorphic function on their domain.

c) Any power series is a holomorphic function on its disk of convergence.

d) Any analytic function $f : \Omega \to \mathbb{C}$ is a holomorphic function on Ω.

Theorem 3. If f is holomorphic in $\{z : |z - z_0| \leq r\}$, then, for $|z - z_0| < r$,

$$f(z) = \frac{1}{2\pi i} \int_{C_r} \frac{f(\zeta)}{\zeta - z} \, d\zeta,$$

where C_r is the circle of radius r centered at z_0, parameterized in the counter-clockwise direction.

Lemma 4. Let f be a holomorphic function in a neighborhood of γ and $\gamma : [a, b] \to \mathbb{C}$ be a piecewise continuously differentiable curve, then

$$\int_\gamma f'(z) \, dz = f(\gamma(b)) - f(\gamma(a)).$$

Proof:

Corollary 5. If $\gamma : [a, b] \to \mathbb{C}$ is a closed, piecewise continuously differentiable curve, and if f is holomorphic in a neighborhood of γ, then

$$\int_\gamma f'(z) \, dz = 0.$$

Proof:

Corollary 6. If $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges in $B = \{z : |z - z_0| < r\}$, and if $\gamma \subset B$ is a closed, piecewise continuously differentiable curve, then

$$\int_\gamma f(z)\, dz = 0.$$

Proof.

Theorem 7. Let $n \in \mathbb{Z}$, let γ be a piecewise continuously differentiable curve and let $a \notin \gamma$.

a) If $n \neq 1$, then

$$\int_\gamma \frac{1}{(z-a)^n}\, dz = 0.$$

b) If $\gamma = C_r = \{z : |z - z_0| = r\}$, then

$$\frac{1}{2\pi i} \int_{C_r} \frac{1}{z-a}\, dz = \begin{cases} 1 & \text{if } |a - z_0| < r \\ 0 & \text{if } |a - z_0| > r. \end{cases}$$

Proof.

Proof of Cauchy's Integral Formula.

Corollary 8. Let $f : \Omega \to \mathbb{C}$ be a function defined on a region Ω.

a) f is holomorphic in Ω if and only if f is analytic in Ω.

b) Moreover, the series expansion of f based at $z_0 \in \Omega$ converges on the largest open disk centered at z_0 and contained in Ω.

Proof.

Note:

- In particular, if f is analytic in $\mathbb{C}$, then f has a power series expansion which converges in all of $\mathbb{C}$. Such functions are called **entire**.
- From now on, the words "holomorphic" and "analytic" are used interchangably.

Example 9.

a) Show that $f(z) = \frac{z}{e^z - 1}$ is holomorphic in $\mathbb{C}\backslash\{2k\pi i \,:\, k \in \mathbb{Z}, k \neq 0\}$.

b) Use this to show that the radius of the power series based at 0

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} a_n z^n$$

is 2π.

Scholium 10. If f is analytic in $\{z : |z - z_0| \le r\}$ and $C_r = \{z_0 + re^{it} : 0 \le t \le 2\pi\}$, then

a) $\dfrac{f^{(n)}(z_0)}{n!} = \dfrac{1}{2\pi} \displaystyle\int_{C_r} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta$. [Cauchy's Integral Formula for $f^{(n)}$]

b) $\left|\dfrac{f^{(n)}(z_0)}{n!}\right| \le \dfrac{\sup_{C_r} |f(z)|}{r^n}$. [Cauchy's Estimate]

Proof.

Corollary 11. If f is analytic in an open disk B, and if $\gamma \subset B$ is a closed, piecewise continuously differentiable curve, then

$$\int_\gamma f(z)\, dz = 0.$$

Theorem 12. If f is analytic and one-to-one in a region Ω, then the inverse of f, defined on $f(\Omega)$, is analytic.

Lemma 13. If f is an analytic function at z_0 with

$$f(z) - f(z_0) = \sum_{n \geq m} a_n (z - z_0)^n \quad (a_m \neq 0,\ m \geq 2)$$

in some disk B_1 centered at z_0, then there is a $\varepsilon > 0$ and a δ so that $f(z) - w$ has exactly m solutions in $\{z : |z - z_0| < \varepsilon\}$, for any $w \in \{v : |v - f(z_0)| < \delta\}$.

Proof.

Proof of Theorem 12.

Theorem 14. If f is continuous in an open disk B, and if

$$\int_{\partial R} f(\zeta) d\zeta = 0$$

for all closed rectangles $R \subset B$ with sides parallel to the axes, then f is analytic in B.

Proof.