Math 644

Chapter 2

Section 2.1: Polynomials

Contents

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What This Class Is About?

We will do calculus with functions

$$f : \Omega \subseteq \mathbb{C} \to \mathbb{C}.$$

- The image of f is $f(\Omega) := \{w = f(z) \,:\, z \in \Omega\}$.
- Since $f(z) \in \mathbb{C}$, for $z \in \Omega$, there are two functions

  $$u : \Omega \subseteq \mathbb{C} \to \mathbb{R} \quad \text{and} \quad v : \Omega \subseteq \mathbb{C} \to \mathbb{R}$$

  such that

  $$f(z) = u(z) + iv(z).$$

- Sometimes, we use $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$.

Definition

The best well-behaved complex-valued functions are polynomials:

$$p(z) = a_0 + a_1 z + \ldots + a_n z^n,$$

where

- $a_0, a_1, \ldots, a_n \in \mathbb{C}$;
- $z \in \mathbb{C}$, so that $\Omega = \mathbb{C}$;
- $a_n \neq 0$, so that $\deg p := n$.

$$z^n - z_0^n = (z - z_0)\left(\sum_{k=0}^{n-1} z^k z_0^{n-k}\right)$$

Theorem 1. A polynomial $p(z)$ is a continuous function.

Note:

- A function $f : \Omega \to \mathbb{C}$ is continuous at z_0 if for any $\varepsilon > 0$, there is a $\delta > 0$ such that if $|z - z_0| < \delta$, then $|f(z) - f(z_0)| < \varepsilon$.
- A function $f : \Omega \to \mathbb{C}$ is continuous on Ω if it is continuous at every $z_0 \in \Omega$.

When the degree of $p(z)$ is 1:

$$p(z) = az + b, \quad a, b \in \mathbb{C} \text{ and } a \neq 0.$$

Some elementary observations:

- $a = 1$. $p(z) = z + b \longrightarrow$ translation.
- $b = 0$. $p(z) = az \longrightarrow$ dilation & rotation

  dilation $\downarrow$ $|a|$, rotation $\downarrow$ $\arg a$.

Consequences: Rewrite as followed:

$$p(z) = a(z + b/a).$$

- translates first by b/a.
- dilates and rotate by $|a|$ and $\arg a$ respectively.

A monomial is

$$p(z) = z^n, \quad n \geq 1.$$

We see that

- $|p(z)| = |z|^n$;
- $\arg p(z) = n \arg z \pmod{2\pi}$.

COROLLARY 2. Each pie slice

$$S_k := \left\{ z : \left| \arg z - \tfrac{2\pi k}{n} \right| < \pi/n \right\} \quad (k = 0, 1, 2, \ldots, n-1)$$

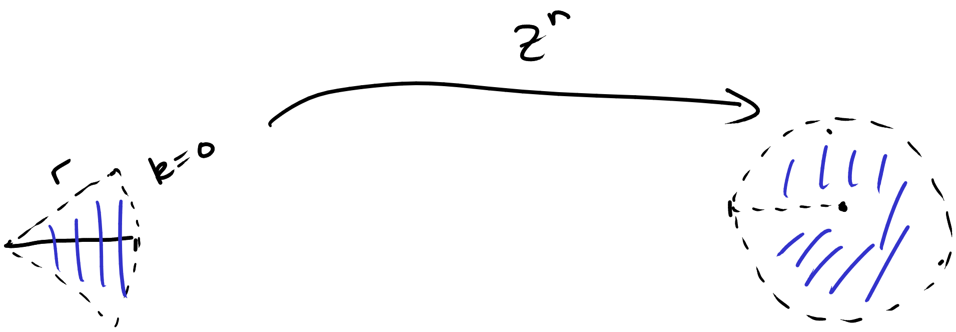
$\cap \{z : |z| < r\}$

is mapped to

$$\{z : |z| < r^n\} \backslash (-r^n, 0)$$

by a monomial $z \mapsto z^n$ $(n \geq 1)$.

Proof.



$$z \in S_k \Rightarrow |z| < r \;\&\; \frac{(2k-1)\pi}{n} < \arg z < (2k+1)\frac{\pi}{n}$$

$$\Rightarrow |z|^n < r^n \;\&\; (2k-1)\pi < \arg(z^n) < (2k+1)\pi.$$

□

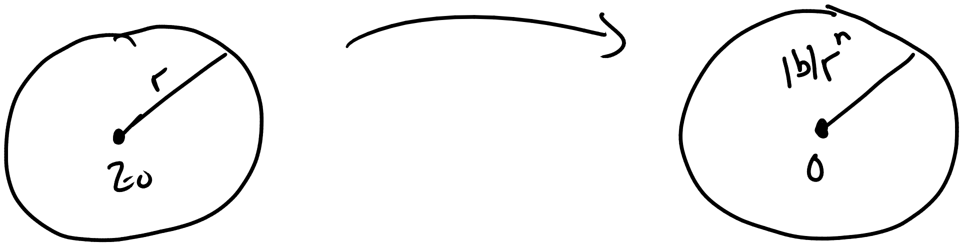
We consider

$$p(z) = b(z - z_0)^n$$

where

- $z_0 \in \mathbb{C}$ is fixed;
- $b \in \mathbb{C}$;
- $n \geq 1$.

Picture



Characteristic:

- $z_0 \longmapsto 0$
- radius to $|b|r^n$
- Disks are mapped to disks.

We first prove the following.

Theorem 3. Any polynomial $p(z) = \sum_{j=0}^{n} a_j z^j$ with $n \geq 1$ can be rewritten as followed:

$$p(z) = \sum_{k=1}^{n} b_k (z - z_0)^k + p(z_0)$$

for $b_1, b_2, \ldots, b_n \in \mathbb{C}$ and $z_0 \in \mathbb{C}$.

Proof.

By induction.

$n=1$ $\quad p(z) = az+b = a(z-z_0+z_0)+b$

$= a(z-z_0) + az_0 + b$

$= a(z-z_0) + p(z_0)$

General: Suppose that it is true for $n=m$.
Let p be of degree $m+1$.

$p(z) - a_{m+1}(z-z_0)^{m+1}$ is of deg $= m$

So,

$$p(z) - a_{m+1}(z-z_0)^{m+1} = \sum_{k=1}^{m} b_k (z-z_0)^k + p(z_0)$$

$$\Rightarrow p(z) = a_{m+1}(z-z_0)^{m+1} + \sum_{k=1}^{m} b_k (z-z_0)^k + p(z_0).$$

$\square$

Corollary 4. If k is the smallest index of all index j such that $b_j \neq 0$ and letting $\zeta := z - z_0$ with $z_0 \in \mathbb{C}$ fixed, then for small ζ, there is a constant C such that

$$\left|p(z_0+\zeta) - \left(p(z_0) + b_k\zeta^k\right)\right| \leq C|\zeta|^{k+1}.$$

Proof.

$$p(z) = \sum_{j=1}^{n} b_j (z-z_0)^j + p(z_0)$$

$$= \sum_{j=k}^{n} b_j (z-z_0)^j + p(z_0)$$

So,

$$\left|p(z_0+\zeta) - \left(p(z_0)+b_k\zeta^k\right)\right| = \left|\sum_{j=k}^{n} b_j \zeta^j - b_k\zeta^k\right|$$

$$= \left|\sum_{j=k+1}^{n} b_j \zeta^j\right|$$

$$= \left|\sum_{j=0}^{n-k-1} b_{j+k+1} \zeta^j\right| |\zeta|^{k+1}$$

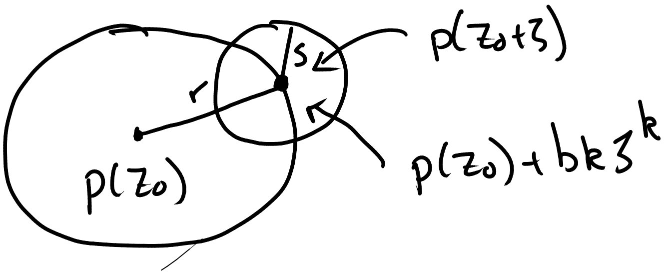
Let $b := \max\{|b_{j+k+1}| : 0 \leq j \leq n-k-1\}$.

So, $$\left|\sum_{j=0}^{n-k-1} b_{j+k+1}\zeta^j\right| \leq |b| \left(\frac{1-|\zeta|^{n-k}}{1-|\zeta|}\right) \quad (|\zeta| \neq 1)$$ $= C$

Let $|\zeta| \leq 1/2$, then

$$\left|p(z_0+\zeta) - \left(p(z_0)+b_k\zeta^k\right)\right| \leq C|\zeta|^{k+1} \quad \square$$

Picture of Walking a Dog (WAD)



Explaination of WAD

Let $s := C|\zeta|^{k+1}$ & $r = |b_k| |\zeta|^k$

- $\zeta \in \{z : |z| < \varepsilon\}$, $p(z_0) + b_k\zeta^k$ wraps k-times around the disk
$$\{z : |z - p(z_0)| < r\}.$$

- $\varepsilon << r$, then $p(z_0+\zeta)$ also traces a path that winds k-times around $p(z_0)$ because
$$p(z_0+\zeta) \in \{z : |z - (p(z_0) + b_k\zeta^k)| < s\}.$$

Corollary 5. We have

$$p(z_0 + \zeta) = p(z_0) + b_k(z - z_0)^k + o((z - z_0)^k), \quad z \to z_0.$$