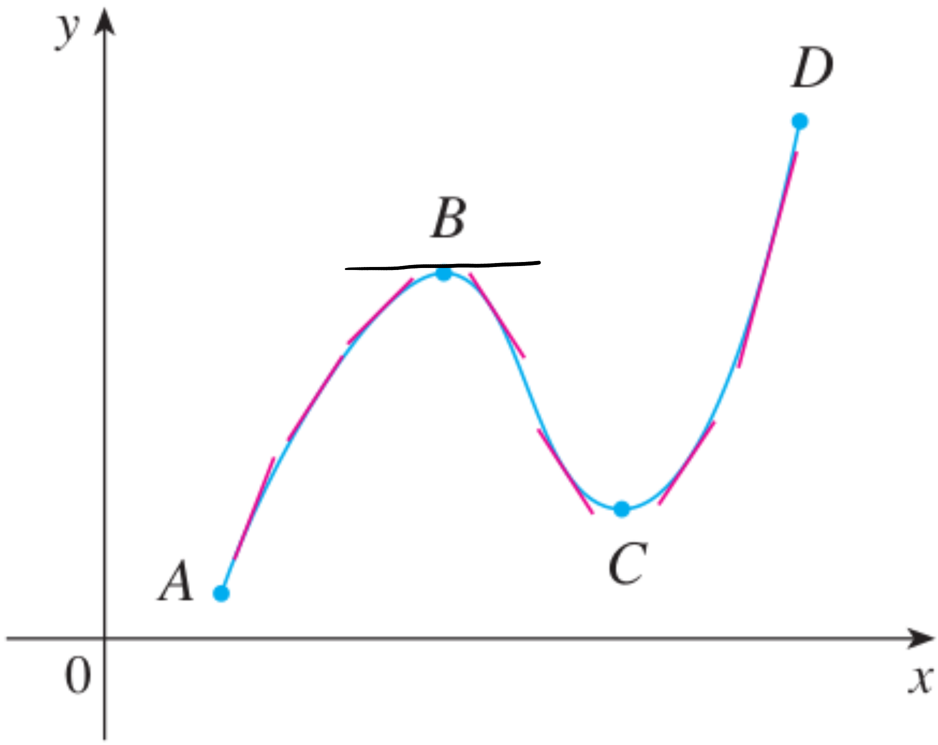
Chapter 3
Applications of Derivatives

3.3 How Derivatives affect the Shape of a Graph

What does f' tells us about f.



	A		B		C		D
$f'(x)$	$\nexists$	$+$	0	$-$	0	$+$	$\nexists$
$f(x)$	Abs min	$\nearrow$	loc max.	$\searrow$	loc. min.	$\nearrow$	Abs. Max.

Conclusion:

Increasing/Decreasing Test

(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

EXAMPLE 1 Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing.

① $\underline{f'(x)}$

$$f'(x) = 12x^3 - 12x^2 - 24x$$
$$= (12x^2 - 12x - 24)x$$
$$= 12(x^2 - x - 2)x$$
$$= 12(x+1)(x-2)x \rightarrow$$

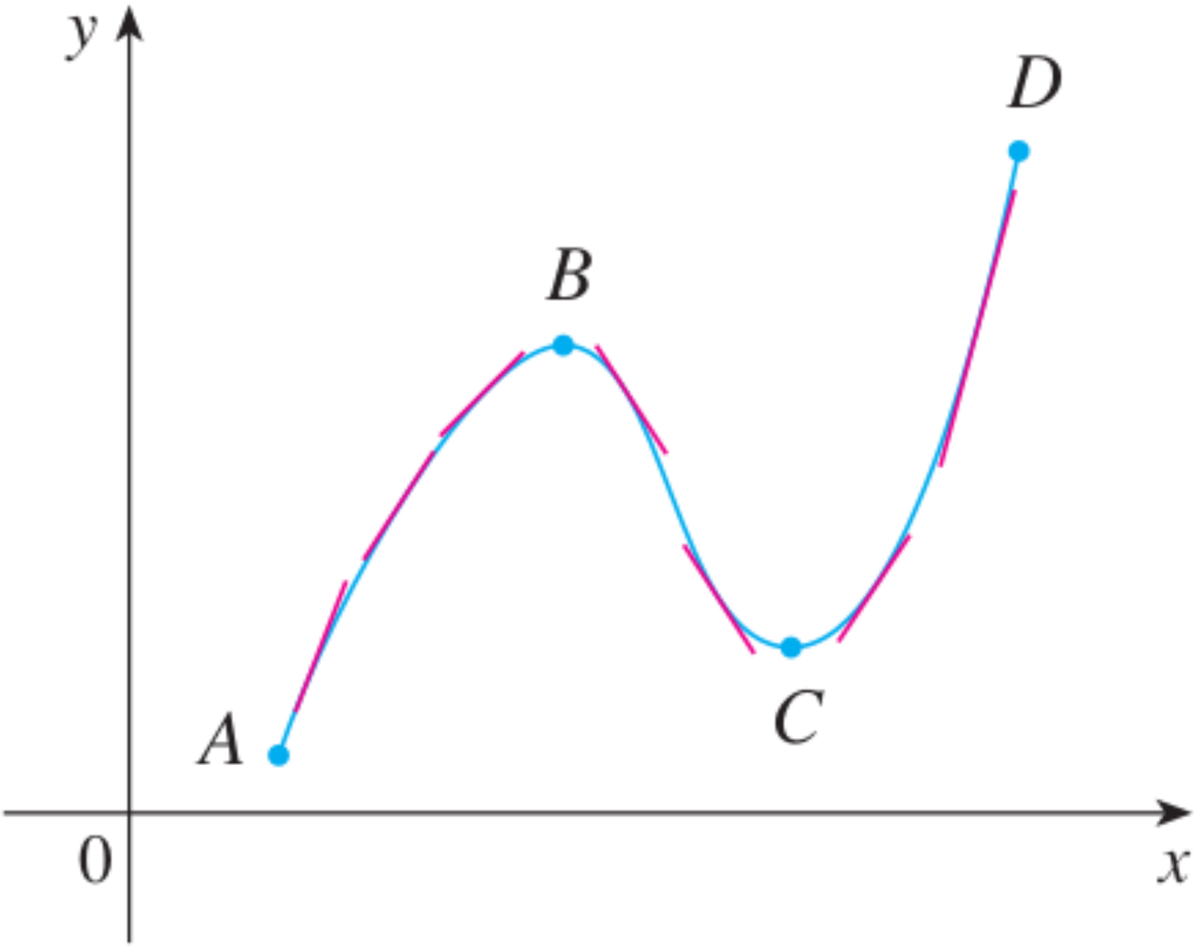
Zeros $x=-1$, $x=0$ & $x=2$.

$x<-1 \rightarrow x+1<0$

Factors	$x<-1$	-1	$-1<x<0$	0	$0<x<2$	2	$x>2$
12	+		+		+		+
$x+1$	−		+		+		+
$x-2$	−		−		−		+
x	−		−		+		+
$f'(x)$	−	0	+	0	−	0	+
$f(x)$	↘	loc. min	↗	loc max	↘	loc. min.	↗

Local Extreme Values.

CRITICAL POINTS $\not\longrightarrow$ EXTREME VALUES (MAX OR MIN)



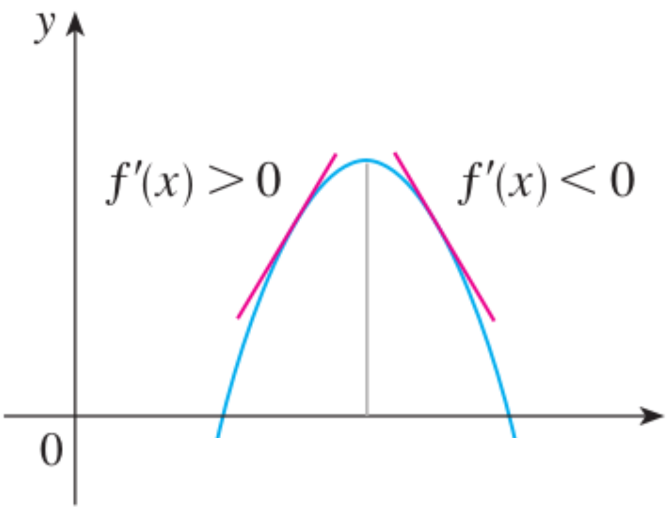
	A		B		C		D
$f'(x)$	$\nexists$	+	0	-	0	+	$\nexists$
$f(x)$	abs. min	$\nearrow$	max	$\searrow$	min	$\nearrow$	abs. max

The First Derivative Test Suppose that c is a critical number of a continuous function f.

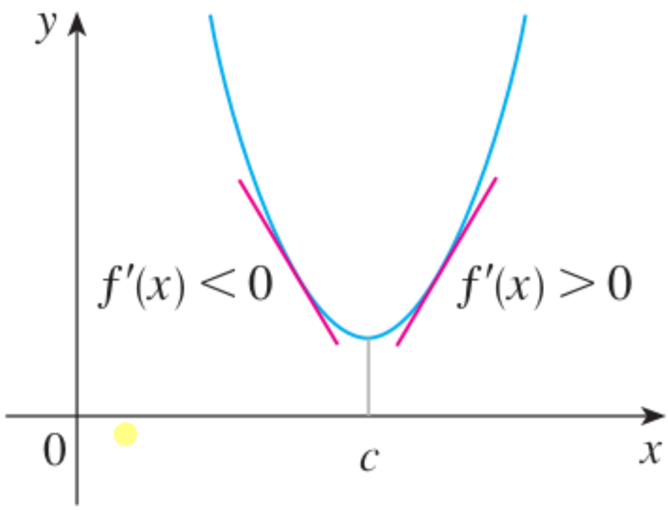
(a) If f' changes from positive to negative at c, then f has a local maximum at c.

(b) If f' changes from negative to positive at c, then f has a local minimum at c.

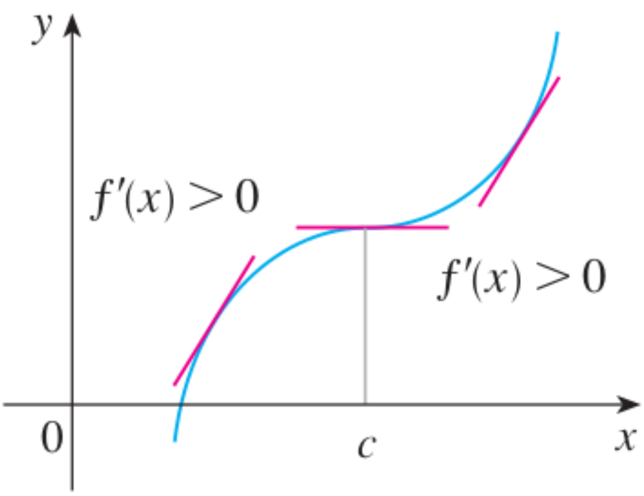
(c) If f' is positive to the left and right of c, or negative to the left and right of c, then f has no local maximum or minimum at c.



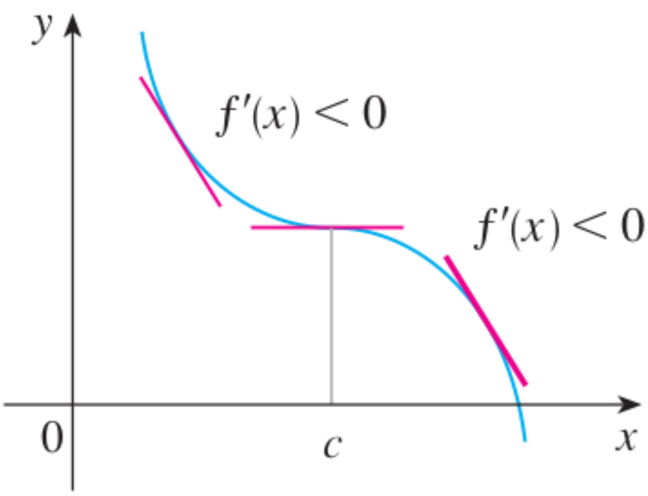
(a) Local maximum



(b) Local minimum



(c) No maximum or minimum



(d) No maximum or minimum

EXAMPLE 3 Find the local maximum and minimum values of the function

$$g(x) = x + 2\sin x \qquad 0 \leqslant x \leqslant 2\pi$$

① Derivative.

$$g'(x) = 1 + 2\cos x$$

Zeros: $g'(x) = 0 \iff 1 + 2\cos x = 0$

$$\iff \cos x = \frac{-1}{2}$$

$$\iff x = \frac{2\pi}{3} \qquad x = \frac{4\pi}{3}$$

② Table.

	0	$< x <$	$\frac{2\pi}{3}$	$< x <$	$\frac{4\pi}{3}$	$< x <$	2π
$1+2\cos x$	$\nexists$	+	0	−	0	+	$\nexists$
$x+2\sin x$		$\nearrow$	loc. max	$\searrow$	loc. min	$\nearrow$	

$\downarrow$

chose $x = \frac{\pi}{2}$

$\Downarrow$

$1 + 2\cos(\frac{\pi}{2}) = 1 > 0$

loc. max value

At $\frac{2\pi}{3}$ $\quad f(\frac{2\pi}{3}) = \frac{2\pi}{3} + 2\sin(\frac{2\pi}{3}) = \frac{2\pi}{3} + \frac{2\sqrt{3}}{2}$

At $\frac{4\pi}{3}$ $\quad f(\frac{4\pi}{3}) = \frac{4\pi}{3} + 2\sin(\frac{4\pi}{3}) = \frac{4\pi}{3} - \frac{2\sqrt{3}}{2}$

loc min value.

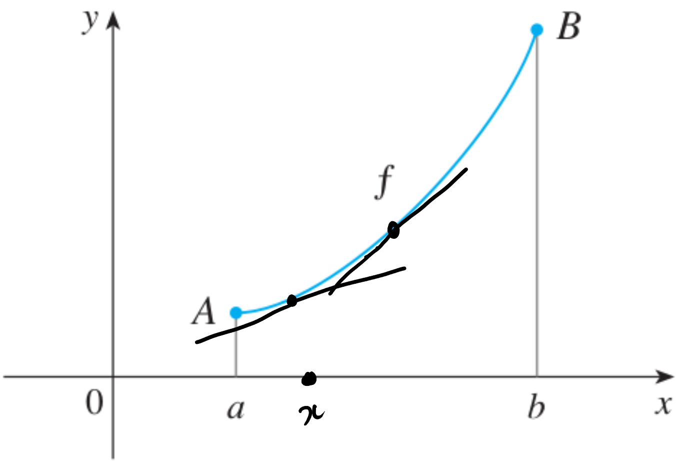
What does f'' tell us about f?

Two important definitions:

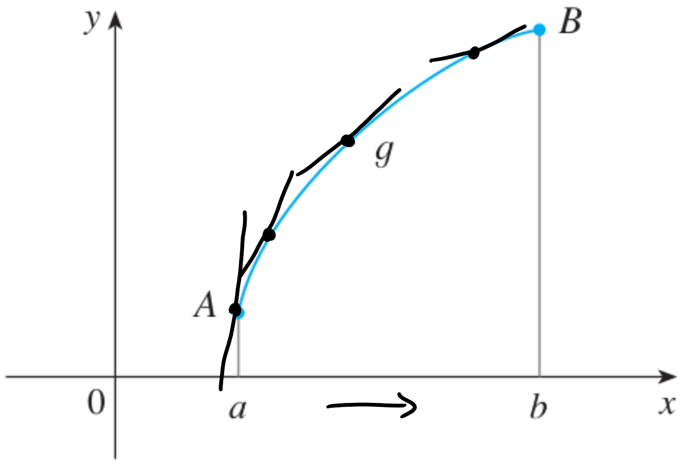
1) **Definition** If the graph of f lies above all of its tangents on an interval I, then it is called **concave upward** on I. If the graph of f lies below all of its tangents on I, it is called **concave downward** on I.

2) **Definition** A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

Find inflection point : when $f''(x) = 0$



(a)



(b)

Concavity Test

(a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

(b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

Example. Find the interval(s) of concavity of the function $f(x) = x^3 - 3x^2 - 9x + 4$.

① Second derivative

$$f'(x) = 3x^2 - 6x - 9$$

$$f''(x) = 6x - 6 = 6(x-1)$$

② Zeros

$$f''(x) = 0 \iff 6(x-1) = 0$$

$$\iff x = 1$$

③ Table.

Factors	$-\infty < x <$	1	$< x < \infty$
6	$+$		$+$
$(x-1)$	$-$		$+$
$f''(x)$	$-$	0	$+$
$f(x)$	$\frown$		$\smile$

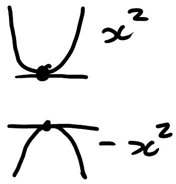
If $x < 1$ $((-\infty, 1))$, then f is concave down.

If $x > 1$ $((1, \infty))$, then f is concave up.

The Second Derivative Test Suppose f'' is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.



REMARK!

- $f''(c) = 0 \Rightarrow$ can't conclude anything.
- $f''(x) > 0$ for any $x \Rightarrow c$ is an absolute minimum
  ($f''(x) < 0$) (absolute maximum).

EXAMPLE. Find the extreme values of the function $f(x) = x^3 + 3x^2$.

① Critical numbers.

$f'(x) = 3x^2 + 6x = 3x(x+2)$

C.N: $f'(x) = 0 \Leftrightarrow 3x(x+2) = 0$

$\Leftrightarrow \underline{x = 0}$ or $\underline{x = -2}$

② 2nd derivative test

$f''(x) = 6x + 6 = 6(x+1)$

$\underline{x=0}$ $\quad f''(0) = 6(0+1) = 6 > 0$

$\Rightarrow x = 0$ is a local min.

$\Rightarrow \underline{f(0) = 0}$

$\underline{x=-2}$ $\quad f''(-2) = 6(-2+1) = -6 < 0$

$\Rightarrow x = -2$ is a local max.

$\Rightarrow \underline{f(-2) = 4}$