Chapter 2

Derivatives

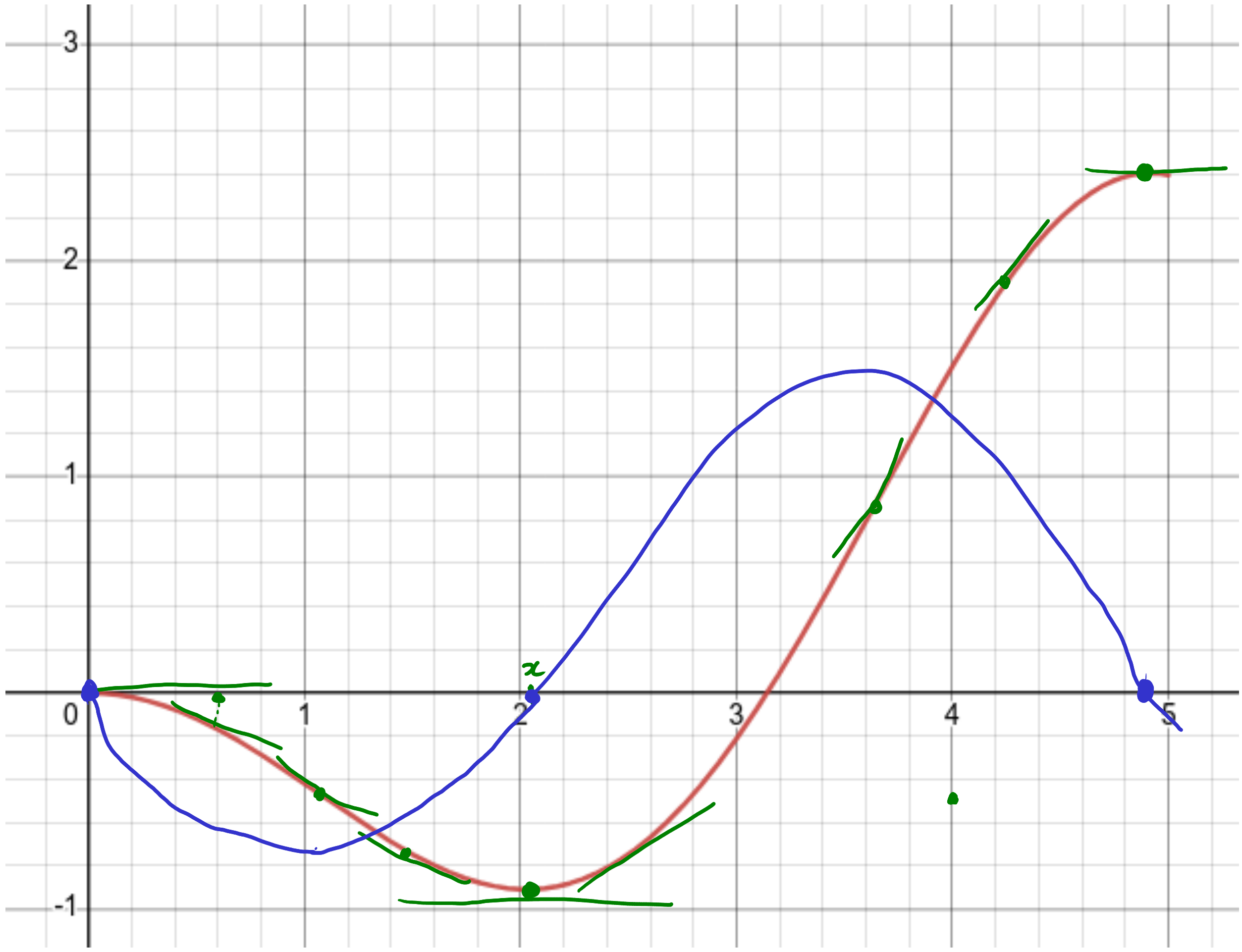
2.2 The Derivatives as a Function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Dom of f': all x such that $f'(x)$ exists.

EXAMPLE 1 The graph of a function f is given . Use it to sketch the graph of the derivative f'.

Desmos: https://www.desmos.com/calculator/o7lfvk2sar



EXAMPLE 3 (a) If $f(x) = \sqrt{x}$, find the derivative of f. State the domain of f'.

(b) Illustrate this formula by comparing the graphs of f and f'. (Do it with Desmos)

(a) $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \to \frac{0}{0}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{\cancel{x} + h + \cancel{\sqrt{x+h}\sqrt{x}} - \cancel{\sqrt{x+h}\sqrt{x}} - \cancel{x}}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \quad (x \neq 0)$$

$$\Rightarrow f'(x) = \frac{1}{2\sqrt{x}} \to \text{Dom}(f') \text{ is } (0, \infty)$$

EXAMPLE 4 Find f' if $f(x) = \dfrac{1-x}{2+x}$.

Dom f is $(-\infty, -2) \cup (-2, \infty)$

Let $x \neq -2$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{1-(x+h)}{2+x+h} - \dfrac{1-x}{2+x}}{h} \to \frac{0}{0}$$

$$= \lim_{h\to 0} \frac{(1-(x+h))(2+x) - (1-x)(2+x+h)}{(2+x+h)(2+x)h}$$

$$= \lim_{h\to 0} \frac{(1-x-h)(2+x) - (1-x)(2+x+h)}{(2+x+h)(2+x)h}$$

$$= \lim_{h\to 0} \frac{\cancel{(1-x)(2+x)} - h(2+x) - \cancel{(1-x)(2+x)} - (1-x)h}{(2+x+h)(2+x)h}$$

$$= \lim_{h\to 0} \frac{-2h - \cancel{hx} - h + \cancel{hx}}{(2+x+h)(2+x)h}$$

$$(\sqrt{x})' = (x^{1/2})'$$

$$\frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-1/2}$$

$$(x^{1/2})' = \frac{1}{2}x^{1/2-1} = \frac{1}{2}x^{-1/2}$$

$$= \lim_{h\to 0} \frac{-2\cancel{h} - \cancel{h}}{(2+x+h)(2+x)\cancel{h}}$$

$$= \lim_{h\to 0} \frac{-3}{(2+x+h)(2+x)} = \boxed{-\frac{3}{(2+x)^2}}$$

$$\lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

Other notations for the derivative.

Leibniz notation

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$

Differentiable Functions.

3 Definition A function f is **differentiable at a** if $f'(a)$ exists. It is **differentiable on an open interval** (a, b) [or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

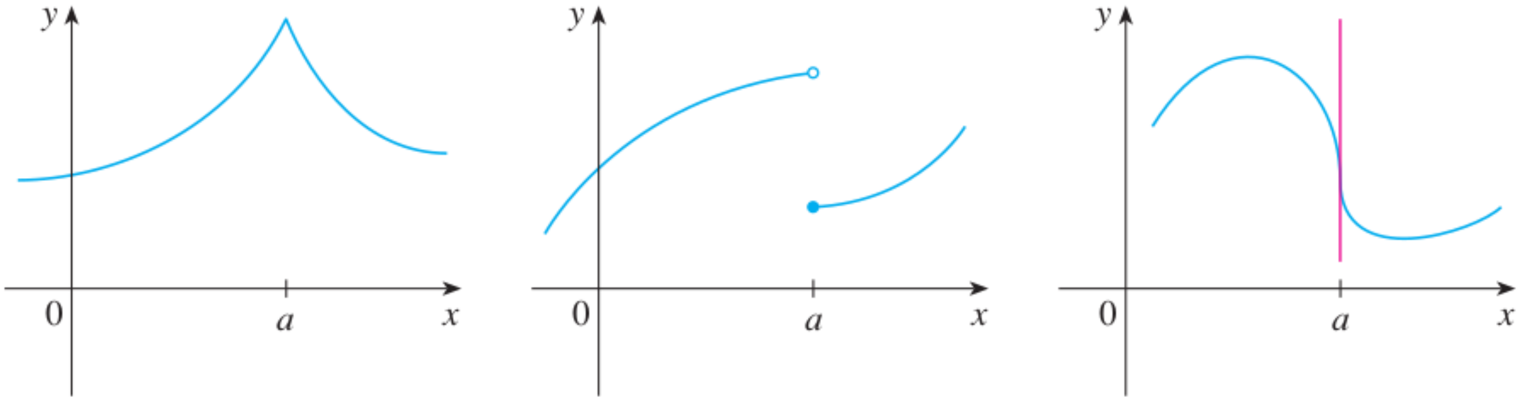
EXAMPLE 5 Where is the function $f(x) = |x|$ differentiable?

Important Result:

4 Theorem If f is differentiable at a, then f is continuous at a.

Remark:

How can a Function Fail to be diffentiable?



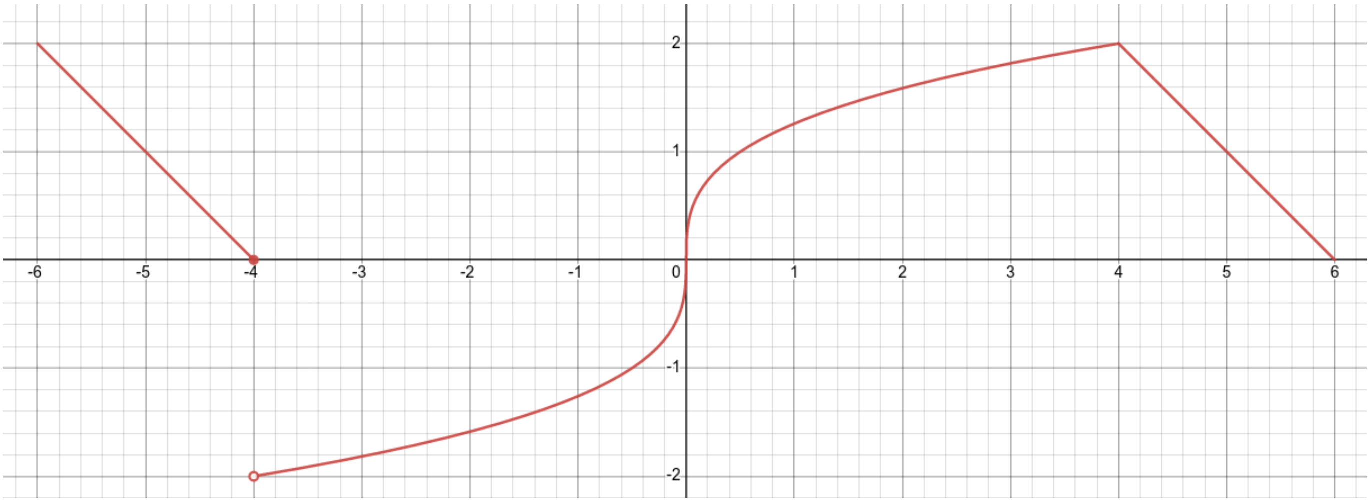
(a) A corner (b) A discontinuity (c) A vertical tangent

(a)

(b)

(c)

Example. The graph of the function is given. State, with reasons, the numbers at which the function is NOT differentiable.
Desmos: https://www.desmos.com/calculator/d0aztxzxta



Higher Derivatives.

Second derivative:

$$\underbrace{\frac{d}{dx}}_{\text{derivative of}}\underbrace{\left(\frac{dy}{dx}\right)}_{\text{first derivative}} = \underbrace{\frac{d^2y}{dx^2}}_{\text{second derivative}}$$

Other notations:

EXAMPLE 6 If $f(x) = x^3 - x$, find and interpret $f''(x)$.

Acceleration:

Third Derivative.

$$y''' = f'''(x) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$$

Jerk: $j = \frac{da}{dt} = \frac{d^3s}{dt^3}$

n-th Derivative.

$$y^{(n)} = f^{(n)}(x) = \frac{d^ny}{dx^n}$$

EXAMPLE 7 If $f(x) = x^3 - x$, find $f'''(x)$ and $f^{(4)}(x)$.