SECTION 1.8: Logarithms and powers

Log function(s)

Let $z \in \mathbb{C}$ and $w \in \mathbb{C}$.

$$w = \log(z) \iff e^{w} = z$$

Let $w = u + iv$ and $z = re^{i\theta}$ with $z \neq 0$. Then

$$e^{w} = z \iff e^{u} e^{iv} = r e^{i\theta}$$

$$\iff e^{u} = r \quad \text{and} \quad v = \theta + 2k\pi, \; k \in \mathbb{Z}$$

$$\iff u = \log(r) \quad \text{and} \quad v = \theta + 2k\pi, \; k \in \mathbb{Z}$$

Thus, the complex logarithm of $z \in \mathbb{C} \setminus \{0\}$ with $z = re^{i\theta}$ is

$$\log(z) = \log(r) + i(\theta + 2k\pi)$$

with $k \in \mathbb{Z}$.

Another notation:

$$\log(z) = \log|z| + i \arg(z)$$

$$= \{ \log|z| + (\operatorname{Arg}(z) + 2k\pi) i : k \in \mathbb{Z} \}$$

<u>Example 1.8.1</u>

(a) $\log(i) = \log|i| + (\operatorname{Arg}(i) + 2k\pi) i$

Here, $|i| = 1$

and $\operatorname{Arg}(i) = \pi/2$

$$\Rightarrow \log(i) = \log(1) + \left(\frac{\pi}{2} + 2k\pi\right) i$$

with $k \in \mathbb{Z}$.

(b) $\log(1+i) = \log\sqrt{2} + \left(\frac{\pi}{4} + 2k\pi\right) i$

with $k \in \mathbb{Z}$.

(c) $\log(-2) = \log(2) + (\pi + 2k\pi)i$

with $k \in \mathbb{Z}$.

$$\Rightarrow \log(-z) = \{\ldots, \log z - 3\pi i, \log z - \pi i, \log z + \pi i, \ldots\}.$$

DEF 1.8.2 The principal value or principal branch of the complex logarithm is defined by

$$\mathrm{Log}(z) = \overset{\ln}{\log}|z| + i\,\mathrm{Arg}(z)$$

for $z \neq 0$.

Example 1.8.3

(a) $\mathrm{Log}(i) = \log(1) + \frac{\pi}{2} i = i\frac{\pi}{2}$.

(b) $\mathrm{Log}(5) = \log(5)$

(c) $\text{Log}\big(\underbrace{e^{6\pi i}}_{=1}\big) = \log(1) + i\,0 = 0$

Remarks

(1) $x \in \mathbb{R}$ and $x > 0 \Rightarrow \text{Log}(x) = \log(x)$.

(2) $x \in \mathbb{R}$ and $x < 0 \Rightarrow \text{Log}(x) = \log|x| + i\pi$

$\log|z| + i\,\text{Arg}(z)$

(3) $\forall z \in \mathbb{C}\setminus\{0\}, \quad e^{\text{Log}\, z} = z$.

But, $\text{Log}(e^z)$ is not necessarily equal to z! In fact,

$$\text{Log}(e^z) = z \Leftrightarrow -\pi < \text{Im}\, z \leq \pi.$$

(4) $x_1, x_2 \in \mathbb{R}$ and $x_1 > 0$, $x_2 > 0$

$$\Rightarrow \log(x_1 x_2) = \log(x_1) + \log(x_2).$$

But,

$$\text{Log}((-1)(-1)) = \text{Log}(1) = 0$$

and

$$\text{Log}(-1) = i\pi \quad , \text{ so that}$$

$$\text{Log}(-1) + \text{Log}(-1) = 2\pi i \neq 0 = \text{Log}\big((-1)(-1)\big).$$

<u>Powers of z</u>

For $x > 0$, and $a > 0$, then

$$x^a = e^{a \ln x}$$

For $z \in \mathbb{C}\setminus\{0\}$, and $a \in \mathbb{C}\setminus\{0\}$,

we define

$$z^a = e^{a \log z} \quad .$$