# Chapter 1

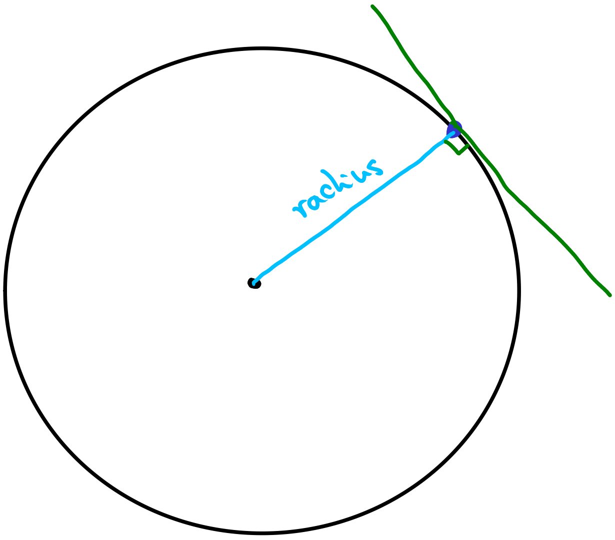
# Functions and Limits

## 1.4 The Tangent and Velocity Problems

# The Tangent problem.

Example. What is the tangent to a circle?

Illustration: https://www.desmos.com/calculator/7qflpgcuay

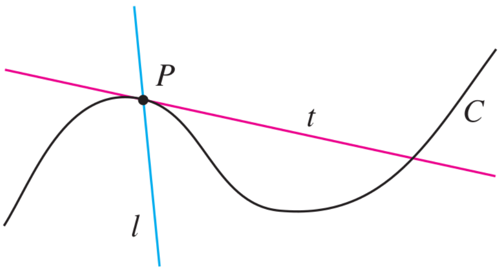



In Geometry, a TANGENT LINE at a given point on a a circle is a line that touches the circle only at that point.

Problems with this definition:

1) Not all curves are circle!

2) For other curves, the tangent line may intersect at several points!



**EXAMPLE 1** Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

Go play around with this problem: https://www.desmos.com/calculator/kbfn4ptdop


① Find slopes of secant lines

$Q(0,0)$ → $m_{PQ} = \dfrac{y_P - y_Q}{x_P - x_Q} = \dfrac{1-0}{1-0} = 1$

$\uparrow x_Q \quad \uparrow y_Q$

$Q(0.5, 0.25)$ → $m_{PQ} = \dfrac{1 - 0.25}{1 - 0.5} = 1.5$

⋮

$Q(a, a^2)$ → $m_{PQ} = \dfrac{1 - a^2}{1 - a}$ make a approaches 1

| $a$ | $m_{PQ}$ |
| --- | --- |
| 0 | 1 |
| 0.5 | 1.5 |
| 0.75 | 1.75 |
| 0.9 | 1.9 |
| 0.99 | 1.99 |
| ↓ | ↓ |
| 1 | 2 |

$$\boxed{\lim_{a \to 1} \frac{1 - a^2}{1 - a} = 2}$$

↑ slope of the tangent line. → $m = 2$

② Find equation of the tangent line

$y - y_0 = m(x - x_0)$ or $y = mx + b$.

$P(1,1)$ belongs to the line & $m = 2$

$\swarrow x_0 \ \downarrow y_0$

$\Rightarrow \boxed{y = 2(x-1) + 1}$ → $\boxed{y = 2x - 1}$

Main concept: The SLOPE of the tangent line is the LIMIT of the slopes of the secant lines.

**EXAMPLE 3** Suppose that a ball is dropped from the upper observation deck of the CN Tower in Toronto, 450 m above the ground. Find the velocity of the ball after 5 seconds.

Galileo: $s(t) = \boxed{4.9t^2}$

distance function

Average velocity: $v_{av} = \dfrac{\text{variation in distance}}{\text{variation in time}}$

$$= \frac{s(5) - s(t)}{5 - t} \quad (t < 5)$$

- Use values of $t$ close to 5:

$\rightarrow$ $t = 0$ : $\dfrac{s(5) - s(0)}{5 - 0} = \dfrac{4.9 \cdot 5^2 - 0}{5 - 0} = 24.5$ m/s

$\rightarrow$ $t = 2.5$ : $\dfrac{s(5) - s(2.5)}{5 - 2.5} = 36.75$ m/s.

Make $t$ approaches 5:

| $t$ | $v_{av}$ |
|---|---|
| 0 | 24.5 |
| 2.5 | 36.75 |
| 4.5 | 46.5 |
| 4.9 | 48.61 |
| 4.99 | 48.951 |
| 4.999 | 48.9951 |
| $\downarrow$ | $\downarrow$ |
| 5 | 49 |

As $t$ approaches 5

$v_{av}$ approaches 49m/s.

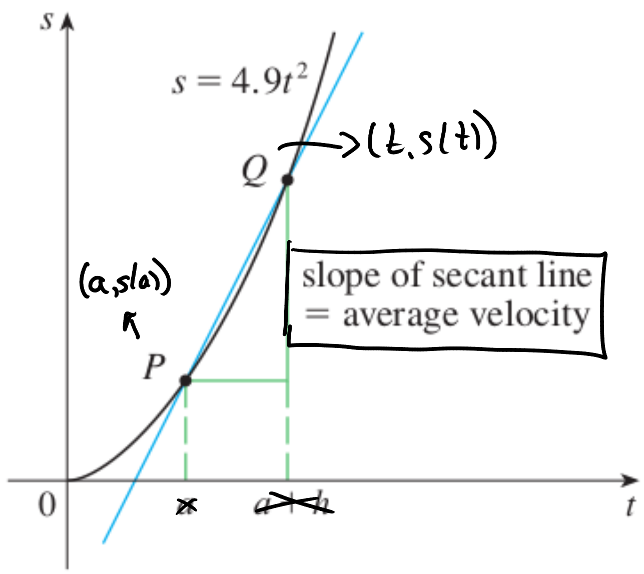
Illustration on Desmos: https://www.desmos.com/calculator/gxry2fylvw

Average velocity.

$$v_{av} = \frac{s(t)-s(a)}{t-a}$$

$$= \frac{-(s(a)-s(t))}{-(a-t)} = \frac{s(a)-s(t)}{a-t}$$

Relation to the secant line.




Instantaneous Velocity.

Make $t$ approaches $a$,

$$v_{av} \longrightarrow v_{inst}$$

Relation to the tangent line.