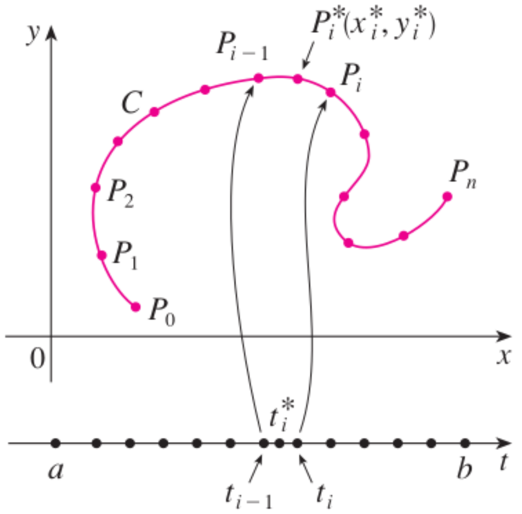
16.2 Line Integrals.

Line integrals in 2D.



2 Definition If f is defined on a smooth curve C given by Equations 1, then the **line integral of f along C** is

$$\int_C f(x, y)\, ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta s_i$$

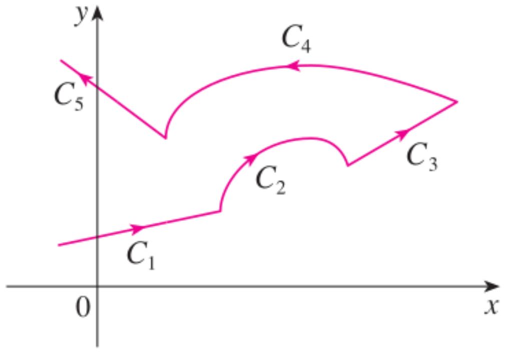
if this limit exists.

Important formula.

Area.

EXAMPLE 1 Evaluate $\int_C (2 + x^2y)\,ds$, where C is the upper half of the unit circle $x^2 + y^2 = 1$.

Piecewise-smooth curve.



EXAMPLE 2 Evaluate $\int_C 2x\,ds$, where C consists of the arc C_1 of the parabola $y = x^2$ from (0, 0) to (1, 1) followed by the vertical line segment C_2 from (1, 1) to (1, 2).

mass.

$$\boxed{m = \int_C \rho(x, y)\, ds}$$

center of mass. $(\overline{x}, \overline{y})$

$$\boxed{\begin{aligned}\overline{x} &= \frac{1}{m}\int_C x\rho(x, y)\, ds \\ \overline{y} &= \frac{1}{m}\int_C y\rho(x, y)\, ds\end{aligned}}$$

EXAMPLE 3 A wire takes the shape of the semicircle $x^2 + y^2 = 1$, $y \geqslant 0$, and is thicker near its base than near the top. Find the center of mass of the wire if the linear density at any point is proportional to its distance from the line $y = 1$.

Two other line integrals.

With respect to x.

$$\boxed{\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))x'(t)\,dt}$$

With respect to y.

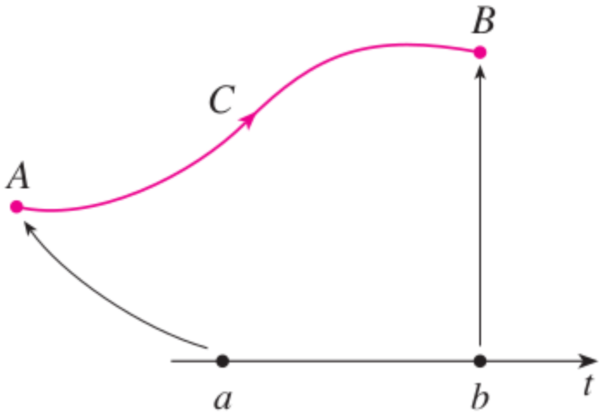
$$\boxed{\int_C f(x,y)\,y = \int_a^b f(x(t),y(t))y'(t)\,dt}$$

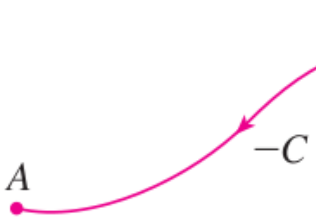
With respect to x and y at the same time.

$$\boxed{\int_C f(x,y)\,dx + \int_C g(x,y)\,dy = \int_C f(x,y)\,dx + g(x,y)\,dy}$$

EXAMPLE 4 Evaluate $\int_C y^2\,dx + x\,dy$, where (a) $C = C_1$ is the line segment from $(-5, -3)$ to $(0, 2)$ and (b) $C = C_2$ is the arc of the parabola $x = 4 - y^2$ from $(-5, -3)$ to $(0, 2)$. (See Figure 7.)

Orientation.





1. $\int_{-C} f(x, y)\, dx = -\int_C f(x, y)\, dx$ 2. $\int_{-C} f(x, y)\, dy = -\int_C f(x, y)\, dy$

3. $\int_{-C} f(x, y)\, ds = \int_C f(x, y)\, ds$

Example. (Take example 4 with -C1).

Line integrals in Space.

In terms of s.

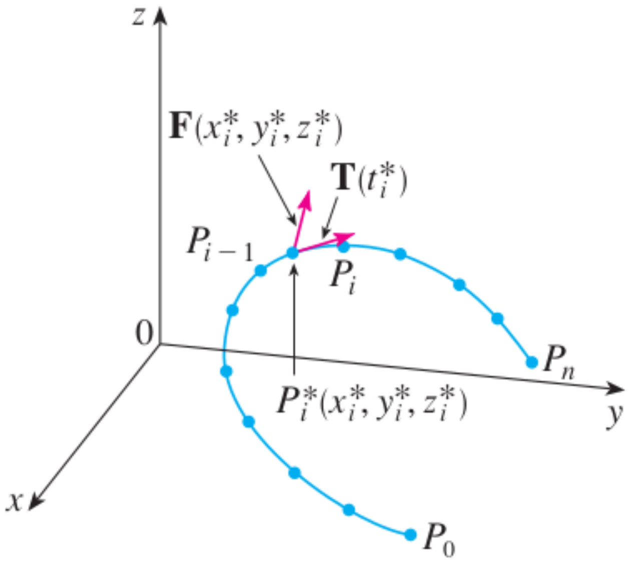
$$\boxed{9} \quad \int_C f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

With respect to x.	With respect to y	With respect to z

EXAMPLE 5 Evaluate $\int_C y \sin z \, ds$, where C is the circular helix given by the equations $x = \cos t$, $y = \sin t$, $z = t$, $0 \leqslant t \leqslant 2\pi$. (See Figure 9.)

EXAMPLE 6 Evaluate $\int_C y\,dx + z\,dy + x\,dz$, where C consists of the line segment C_1 from $(2, 0, 0)$ to $(3, 4, 5)$, followed by the vertical line segment C_2 from $(3, 4, 5)$ to $(3, 4, 0)$.

Line integrals of Vector Fields.



13 Definition Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t)$, $a \leqslant t \leqslant b$. Then the **line integral of F along *C*** is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$$

EXAMPLE 7 Find the work done by the force field $\mathbf{F}(x, y) = x^2\,\mathbf{i} - xy\,\mathbf{j}$ in moving a particle along the quarter-circle $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$, $0 \leqslant t \leqslant \pi/2$.

EXAMPLE 8 Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$ and C is the twisted cubic given by

$$x = t \qquad y = t^2 \qquad z = t^3 \qquad 0 \leqslant t \leqslant 1$$

Line integrals of vector fields and of scalar functions.