# SECTION 2.2: Limits and Continuity

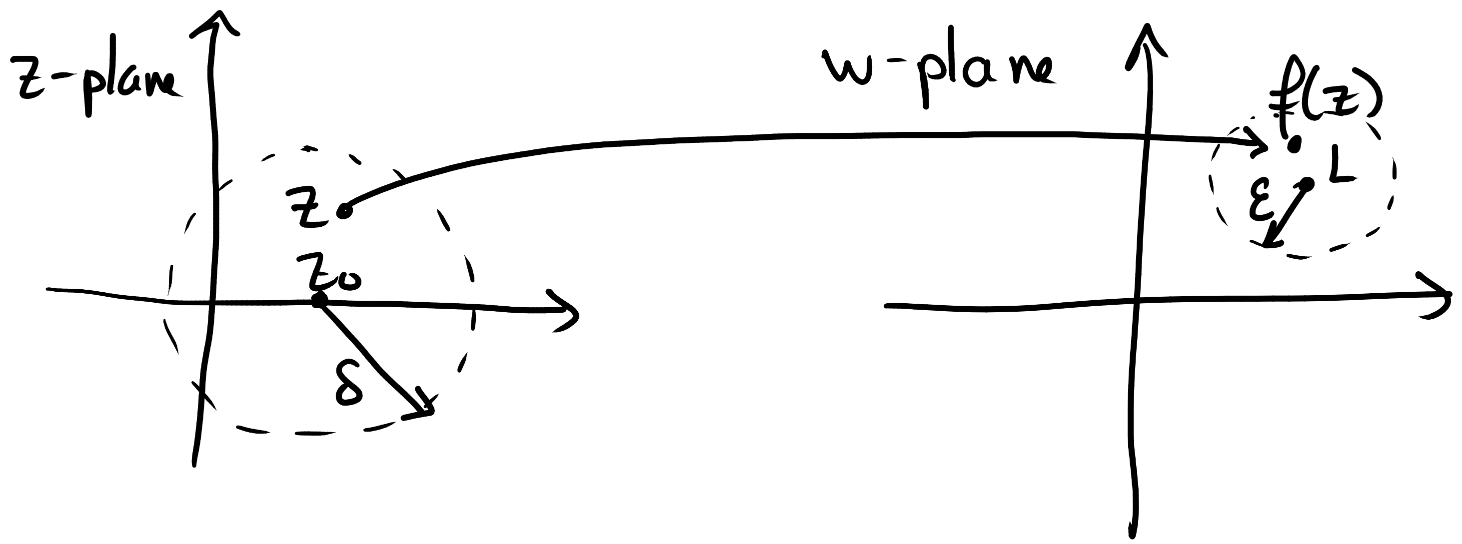
## Limits

**Def.** Let $f: U \to \mathbb{C}$ where $U$ is an open set.

We say $L$ is the limit of $f$ at $z_0 \in U$ if as $z$ approaches $z_0$, $f(z)$ approaches $L$, that is

$\forall \varepsilon > 0, \exists \delta > 0$ s.t.

$$0 < |z - z_0| < \delta \Rightarrow |f(z) - L| < \varepsilon.$$



Prop. 2.2.2 (Uniqueness of limits)
If $f: U \to \mathbb{C}$ is a function defined on an open set $U$, and if $f$ has limit $L$ at $z_0 \in U$, then $L$ is unique.

Proof. Assume that there are two limits $L_1, L_2$ at $z_0$ with $L_1 \neq L_2$. Then

$$|L_1 - L_2| \neq 0$$

Let $\varepsilon = \dfrac{|L_1 - L_2|}{4} > 0$. By def of limits, $\exists \delta_1 > 0$ and $\exists \delta_2 > 0$ such that

$$0 < |z - z_0| < \delta_1 \Rightarrow |f(z) - L_1| < \frac{|L_1 - L_2|}{4}$$

and

$$0<|z-z_0|<\delta_2 \Rightarrow |f(z)-L_2|<\frac{|L_1-L_2|}{4}$$

So, if $|z-z_0|<\min\{\delta_1,\delta_2\}$, then

$$\begin{aligned}|L_1-L_2| &= |L_1-f(z)+f(z)-L_2| \\ &\leq |L_1-f(z)|+|f(z)-L_2| \\ &< \frac{|L_1-L_2|}{4}+\frac{|L_1-L_2|}{4} \\ &= \frac{|L_1-L_2|}{2}\end{aligned}$$

$$\Rightarrow |L_1-L_2|<\frac{|L_1-L_2|}{2}$$

<u>contradiction!</u>

So, $L_1=L_2$. $\square$

<u>Notation</u>:

$$L=\lim_{z\to z_0} f(z) \quad \text{or} \quad f(z)\longrightarrow L \ (z\to z_0).$$

Thm. 2.2.9 Let $U \subset \mathbb{C}$ be an open set. Let $f: U \to \mathbb{C}$ be a function with

$$f(z) = u(z) + i v(z), \quad z \in U.$$

Then

$$\lim_{z \to z_0} f(z) = a + ib \iff \begin{cases} \lim\limits_{z \to z_0} u(z) = a \\ \lim\limits_{z \to z_0} v(z) = b \end{cases}$$

Proof.

($\Rightarrow$) Assume $\lim\limits_{z \to z_0} f(z) = L = a + ib$.

WTS: $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$0 < |z - z_0| < \delta \Rightarrow |u(z) - a| < \varepsilon.$$

and

$\forall \varepsilon > 0$, $\exists \delta > 0$, such that

$$0 < |z - z_0| < \delta \Rightarrow |v(z) - b| < \varepsilon.$$

Let $\varepsilon > 0$. By def. of $\lim\limits_{z \to z_0} f(z) = L$,

$\exists \delta > 0$ s.t.

$0 < |z - z_0| < \delta \Rightarrow |f(z) - L| < \varepsilon$. (*)

<u>Recall</u>:

$$|\mathrm{Re}\, w| \leq |w|.$$

Let $z \in U$ such that $0 < |z - z_0| < \delta$.

then

$$|u(z) - a| = |\mathrm{Re}(f(z) - \underbrace{(a+ib)}_{=L})|$$
$$\leq |f(z) - L|$$
$$< \varepsilon.$$

<u>Summary</u>: we found a $\delta > 0$ s.t.

$0 < |z - z_0| < \delta \Rightarrow |u(z) - a| < \varepsilon$.

Repeat same argument for $v(z)$.

($\Leftarrow$) Assume $\lim_{z \to z_0} u(z) = a$ and $\lim_{z \to z_0} v(z) = b$.

<u>WST</u> $\lim_{z \to z_0} f(z) = a+ib$.

i.e. $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$0 < |z - z_0| < \delta \implies |f(z) - (a+ib)| < \varepsilon$.

Let $\varepsilon > 0$. From the definition of limits, $\exists \delta_1 > 0$, $\exists \delta_2 > 0$ such that

$0 < |z - z_0| < \delta_1 \implies |u(z) - a| < \varepsilon/2$ $(\Delta)$

&

$0 < |z - z_0| < \delta_2 \implies |v(z) - b| < \varepsilon/2$. $(\circ)$

<u>Recall</u>:

$$|w| \leq |\operatorname{Re} w| + |\operatorname{Im} w|$$

Let $\delta := \min\{\delta_1, \delta_2\}$.

If $|z-z_0| < \delta$, then

$$|f(z) - (a+ib)| \le |u(z)-a| + |v(z)-b|$$
$$< \varepsilon/2 + \varepsilon/2$$
$$= \varepsilon$$

<u>Summary</u>: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$$0 < |z - z_0| < \delta \Rightarrow |f(z) - (a+ib)| < \varepsilon.$$

So, $\lim_{z \to z_0} f(z) = a+ib$. $\square$

<u>Example</u> Compute

(a) $\lim_{z \to z_0} z$ (b) $\lim_{z \to z_0} z^2$.

<u>Solution</u>.

(a) $z \to z_0 \Leftrightarrow x \to x_0$ and $y \to y_0$.

(b) $z^2 = (x+iy)(x+iy)$

$$= \underbrace{x^2 - y^2}_{u(z)} + \underbrace{(2xy)}_{v(z)} i$$

$$\lim_{z \to z_0} (x^2 - y^2) = \lim_{(x,y) \to (x_0,y_0)} x^2 - y^2$$

$$= x_0^2 - y_0^2 \quad \text{((Calculus)}.$$

$$\lim_{z \to z_0} 2xy = \lim_{(x,y) \to (x_0,y_0)} 2xy$$

$$= 2x_0 y_0$$

Hence:

$$\lim_{z \to z_0} z^2 = \overbrace{(x_0^2 - y_0^2)}^{a} + \overbrace{(2x_0 y_0)}^{b} i$$

$$= z_0^2$$

In fact:

$$\lim_{z \to z_0} z^n = z_0^n \quad .$$

# Properties of limits

① $$\lim_{z \to z_0} \left( f(z) + g(z) \right) = \lim_{z \to z_0} f(z) + \lim_{z \to z_0} g(z)$$

② $$\lim_{z \to z_0} f(z) g(z) = \left( \lim_{z \to z_0} f(z) \right) \left( \lim_{z \to z_0} g(z) \right)$$

③ $$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{\lim_{z \to z_0} f(z)}{\lim_{z \to z_0} g(z)}$$

if $\lim_{z \to z_0} g(z) \neq 0$.

④ $$\lim_{z \to z_0} |z| = |z_0|$$

⑤ $$\lim_{z \to z_0} \overline{z} = \overline{\lim_{z \to z_0} z} \; .$$