Math 311

Chapter 1

Section 1.2: Gaussian Elimination

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The system

$$x + \quad y + z = 1$$
$$2x + 2y + z = 3$$

can be put in **matrix form** (array of numbers):

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 3 \end{array}\right]$$

Definition 1.

- The matrix $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix}$ is called the **coefficient matrix** of the system.

- The matrix $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is called the **constant matrix** of the system.

Solving a system of linear equations requires to transform the coefficient matrix in the following form:

$$\begin{bmatrix} 1 & * & * \\ 0 & 1 & * \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & * & * \\ 0 & 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \end{bmatrix}$$

where $*$ denotes any real number.
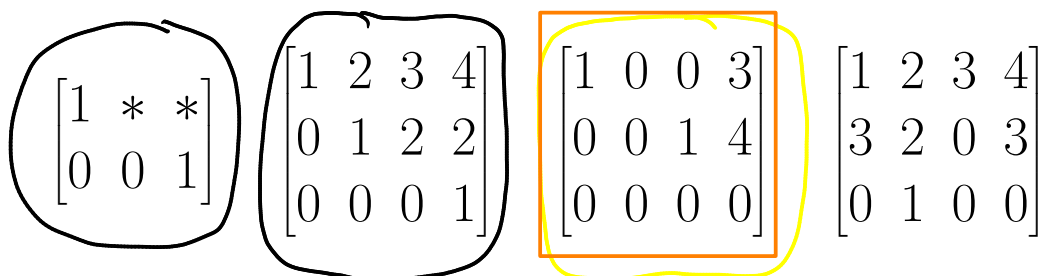
Row-Echelon Form of a matrix

DEFINITION 2. A matrix is said to be in **row-echelon form** (REF for short) if it satisfies the following three conditions:

1. All **zero rows** are at the bottom.

2. The first nonzero entry from the left in each nonzero row is 1, called the **leading 1** for that row.

3. Each leading 1 is to the right of all leading 1s in the rows above it

DEFINITION 3. A row-echelon matrix is said to be in **reduced row-echelon form** (RREF for short) if, in addition, it satisfies the following condition:

4. Each leading 1 is the only nonzero entry in its column.

EXAMPLE 1. Circle the matrices in REF. Draw a square around the matrices in RREF.

$$\begin{bmatrix} 1 & * & * \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 0 & 3 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

THEOREM 1. Every matrix can be transformed, with row operations, to a REF (or RREF).

ALGORITHM 4. **Gaussian Elinination.** To solve a system of linear equations proceed as follows.

1. Carry the augmented matrix to a reduced row-echelon form using elementary row operations.

2. If a row $[0 \ 0 \ \ldots \ 0 \ 1]$ occurs, the system is inconsistent.

3. Otherwise, find the parametric form of the solution set to the system of equations.

EXAMPLE 2. Solve the following system:

$$3x + y - 4z = -1$$
$$x \qquad + 10z = 5$$
$$4x + y + 6z = 1$$

SOLUTION.

1. RREF

$$\begin{bmatrix} 3 & 1 & -4 & | & -1 \\ 1 & 0 & 10 & | & 5 \\ 4 & 1 & 6 & | & 1 \end{bmatrix} \xrightarrow{\cdots} \begin{bmatrix} 1 & 0 & 10 & | & 5 \\ 0 & 1 & -34 & | & -16 \\ 0 & 0 & 0 & | & -3 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & 10 & | & 5 \\ 0 & 1 & -34 & | & -16 \\ 0 & 0 & 0 & | & 1 \end{bmatrix} \quad -\frac{1}{3} R_3 .$$

$$\Rightarrow \quad 0x + 0y + 0z = 1 \quad (\text{third row})$$

$$\Rightarrow \quad 0 = 1$$

2. there is not solution.

EXAMPLE 3. Solve the following equation:

$$x_1 - 2x_2 - x_3 + 3x_4 = 1$$
$$2x_1 - 4x_2 + x_3 = 5$$
$$x_1 - 2x_2 + 2x_3 - 3x_4 = 4$$

SOLUTION.

1. $\begin{bmatrix} 1 & -2 & -1 & 3 & 1 \\ 2 & -4 & 1 & 0 & 5 \\ 1 & -2 & 2 & -3 & 4 \end{bmatrix} \xrightarrow{\cdots} \left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$

2. Write the new system:

① $x_1 - 2x_2 + x_4 = 2$

② $x_3 - 2x_4 = 1$

① $\Rightarrow$ $x_1 = 2 + 2\boxed{x_2} - \boxed{x_4}$

② $\Rightarrow$ $x_3 = 1 + 2\boxed{x_4}$

Set $x_2 = s$ and $x_4 = t$, then

$x_1 = 2 + 2s - t$ $x_3 = 1 + 2t$

$x_2 = s$ $x_4 = t$, $s, t \in \mathbb{R}$

<u>Note:</u> The variable corresponding to the leading ones are called **leading variables**.

It can be proved that the number r of leading 1s must be the same in each of the row-echelon matrices.

DEFINITION 5. The **rank** of a matrix A is the number of leading 1s in any REF to which A can be carried by row operations.

EXAMPLE 4. Compute the rank of

$$A = \begin{bmatrix} 1 & 1 & -1 & 4 \\ 2 & 1 & 3 & 0 \\ 0 & 1 & -5 & 8 \end{bmatrix}$$

SOLUTION.

$$A \longrightarrow \begin{bmatrix} 1 & 1 & -1 & 4 \\ 0 & 1 & -5 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \text{rank } A = 2 .$$

THEOREM 2. Suppose a system of m equations in n variables is consistent, and that the rank of the coefficient matrix is r.

1. The set of solutions involves $n - r$ parameters.

2. If $r < n$, the system has infinitely many solutions.

3. If $r = n$, the system has a unique solution.

Three situations occur:

- No solution.

- Unique solution.

- Infinitely many solutions.

EXAMPLE 5. A system of equation with $m = 4$ linear equations and $n = 5$ variables has been carried to the following REF by row operations:

$$\begin{bmatrix} 1 & 2 & 1 & 3 & 1 & | & 1 \\ 0 & 1 & -1 & 0 & 1 & | & 1 \\ 0 & 0 & 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

(a) Find the rank of the coefficient matrix. (b) Is there no solution, unique solution, or infinitely many solutions?

SOLUTION.