Question 1

What would be the derivative of the function $f(x) = \sin x$?

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$\sin(A+B) =$
$\sin(A)\cos(B) +$
$\cos(A)\sin(B)$
$\downarrow$
$A = x$
$B = h$

$$= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin h - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x\,(\cos h - 1) + \cos(x)\sin h}{h}$$

$$= \lim_{h\to 0} \left(\sin x \frac{\cos h - 1}{h} + \cos x \frac{\sin h}{h}\right)$$

[Here $\lim_{h\to 0} \frac{\cos h - 1}{h} = 0$, How?

$A = h/2$

$\sin^2 A = \frac{1 - \cos(2A)}{2}$

$$\lim_{h\to 0} \frac{\cos h - 1}{h} = \lim_{h\to 0} \frac{-2\sin^2(h/2)}{h} = \lim_{h\to 0} -\sin(h/2) \cdot \frac{\sin(h/2)}{h/2}$$

$$= (-1) \cdot 0 \cdot 1 = 0]$$

$$= \lim_{h\to 0} \sin x \left(\frac{\cos h - 1}{h}\right) + \lim_{h\to 0} \cos x \frac{\sin h}{h}$$

$$= \sin x \lim_{h\to 0} \cancel{\frac{\cos h - 1}{h}}^{\,0} + \cos x \lim_{h\to 0} \cancel{\frac{\sin h}{h}}^{\,1}$$

$$= \cos x$$

Example 4

Compute the derivative of $f(x) = x^2 \sin(x)$.

$$\frac{d}{dx}(x^2 \sin x) = \frac{d}{dx}(x^2) \sin x + x^2 \frac{d}{dx}(\sin x)$$

$$= 2x \sin x + x^2 \cos x.$$

$$= x(2 \sin x + x \cos x)$$

Example 5

Compute the derivative of

- $f(x) = \frac{1}{\sin x}$. $= \operatorname{cosec} x$
- $f(x) = \frac{1}{\cos x}$. $= \sec x$
- $f(x) = \frac{1}{\tan x}$. $= \operatorname{cotan} x$

- $f'(x) = \dfrac{\cancel{(1)'} \sin x - (1) \cdot (\sin x)'}{\sin^2 x}$

$$= \frac{-\cos x}{\sin^2 x} = \frac{-\cos x}{\sin x} \cdot \frac{1}{\sin x}$$

$$= \frac{-1}{\tan x} \cdot \operatorname{cosec} x$$

$$= -\operatorname{cotan} x \cdot \operatorname{cosec} x$$

- $f'(x) = \dfrac{\cancel{(1)'} \cos x - (1) \cdot (\cos x)'}{\cos^2 x}$

$$= \frac{\sin x}{\cos^2 x} = \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} = \tan x \sec x$$

- $f'(x) = \dfrac{-(1)(\tan x)'}{\tan^2 x}$

$$= \frac{-\sec^2 x}{\tan^2 x}$$

$$= \frac{-1/\cos^2 x}{\sin^2 x/\cos^2 x} = -\frac{1}{\sin^2 x} = -\operatorname{cosec}^2 x$$

Example 7

Compute the derivative of $f(x) = \frac{\sec x}{1+\tan x}$.

- Quotient rule first
- Apply the formulas for the derivative of trig. fcts.

$$f'(x) = \frac{(\sec x)'(1+\tan x) - \sec x(1+\tan x)'}{(1+\tan x)^2}$$

$$= \frac{\sec x \tan x(1+\tan x) - \sec x \sec^2 x}{(1+\tan x)^2}$$

$1+\tan^2 x = \sec^2 x$

$$= \frac{\sec x(\tan x + \tan^2 x - \sec^2 x)}{(1+\tan x)^2}$$

$$= \frac{\sec x(\tan x - 1)}{(\tan x+1)^2}$$

Example 8

Suppose that a the volume of a balloon is given by $V(r) := \frac{4\pi}{3} r^3$ where r is the radius of the balloon. You inflate air in such a way that $r(t) = (t^2 + 1)$ where t is the time (in seconds) after you started to inflate the balloon.

- What is the speed at which the volume increases?

$$V(r) = \frac{4\pi}{3}(t^2+1)^3 \rightarrow V'(r)?$$

Example 10

Find $F'(x)$ if $F(x) = \sqrt{x^2+1}$.

$g(x) = \sqrt{x} = x^{1/2}$, $f(x) = x^2+1$

$[f(x) = \sqrt{x}$, $g(x) = x^2+1]$

$F'(x) = g'(f(x)) \cdot f'(x)$

$= \frac{1}{2}[f(x)]^{1/2-1} \cdot (2x)$

$= \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x$

$= \frac{x}{\sqrt{x^2+1}}$

Example 12

Find y' if $y = (x^3 - 1)^{100}$.

$g(x) = x^3 - 1$

$$y' = 100 (g(x))^{100-1} \cdot g'(x)$$

$$= 100 (x^3 - 1)^{99} \cdot (3x)$$

$$= 300x (x^3 - 1)^{99}$$

Example 15

Let $x^3 + y^3 = 6xy$ be the folium of Descartes.

a) Find y'.

b) Find the equation of the tangent line passing through the point $P = (3, 3)$.

$$x^3 + y^3 = 6xy$$

a) Apply the derivative on each side.

$$(x^3+y^3)' = (6xy)'$$

$$\Rightarrow (x^3)' + (y^3)' = 6[(x)'y + xy']$$

$$\Rightarrow 3x^2 + 3y^2 \cdot y' = 6[y + xy']$$

$$\Rightarrow 3x^2 + 3y^2y' = 6y + 6xy'$$

$$\Rightarrow 3y^2y' - 6xy' = 6y - 3x^2$$

$$\Rightarrow (3y^2 - 6x)y' = 6y - 3x^2$$

$$\Rightarrow y' = \frac{6y - 3x^2}{3y^2 - 6x}.$$

b) $T(x) = mx + b$ $\quad P = (3,3) = (x,y)$.

$$m = y'\Big|_{(x,y)=(3,3)} = \frac{6\cdot 3 - 3\cdot 3^2}{3\cdot 9 - 6\cdot 3} = \frac{-9}{9} = -1$$

$$3 = T(3) = -1\cdot 3 + b \Rightarrow b = 6$$

So,

$$T(x) = -x + 6.$$

Example 16

Reprove that if $y = \tan x$, then $y' = \sec^2 x$.