Math 644

Chapter 5

Section 5.4: Laurent Series

Contents

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Theorem 1. Suppose f is analytic on $A = \{z : r < |z-a| < R\}$. Then there is a unique sequence $(a_n) \subset \mathbb{C}$ so that

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n,$$

where the series converges uniformly and absolutely on compact subsets of A. Moreover,

$$a_n = \frac{1}{2\pi i} \int_{C_s} \frac{f(\zeta)}{(\zeta - a)^{n+1}}\, d\zeta,$$

where C_s is the circle centered at a with radius s, $r < s < R$, oriented counter-clockwise.

Proof.

For $r < s < R$, define

$$f_s(z) := \int_{C_s} \frac{f(\zeta)}{\zeta - z}\, d\zeta, \quad z \notin C_s$$

where $C_s := \{ a + se^{it} : 0 \leq t \leq 2\pi \}$. Then f_s is analytic on $\mathbb{C} \setminus C_s$.

Fix z s.t. $r < |z-a| < s_1 < s_2 < R$.

Then,

$$n(C_{s_2} - C_{s_1}, z) = \frac{1}{2\pi i} \int_{C_{s_2}} \frac{1}{\zeta - z}\, d\zeta - \frac{1}{2\pi i} \int_{C_{s_1}} \frac{1}{\zeta - z}\, d\zeta$$

$$= 1 - 1 = 0.$$

By Cauchy's integral Formula,

$$f_{s_2}(z) - f_{s_1}(z) = 0.$$

$\Rightarrow$ $f_s(z)$ does not depend on s, if

$r < |z-a| < s < R$. Expanding $\dfrac{1}{\zeta - z}$ as a power series in $z-a$, for $|z-a| < s < R$:

$$(1) \quad f_s(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{C_s} \frac{f(\zeta)}{(\zeta - a)^{n+1}} \, d\zeta \right) (z-a)^n$$

Likewise, $f_s(z)$ does not depend on s when $r < s < |z-a| < R$. Writing

$$\frac{1}{z-\zeta} = \frac{1}{z-a+a-\zeta} = \frac{1}{z-a} \left(\frac{1}{1 - \frac{\zeta - a}{z-a}} \right)$$

and expand in powers of $\dfrac{\zeta - a}{z-a}$, for $r < s < |z-a| < R$, to get

$$(2) \quad f_s(z) = \sum_{n=0}^{\infty} \left(\int_{C_s} \frac{f(\zeta)}{(\zeta - a)^{-n}} \, d\zeta \right) (z-a)^{-n-1}$$

Now, if $r < s_1 < |z-a| < s_2 < R$, then $n(C_{s_2} - C_{s_1}, z) = 1$ and by Cauchy's integral formula:

$$f(z) = \frac{1}{2\pi i} \int_{C_{s_2} - C_{s_1}} \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

$$= f_{s_2}(z) - f_{s_1}(z).$$

Use (1) & (2) to obtain the series. $\square$

Types of Singularities

Definition 2. A function f has an **isolated singularity** at b if f is analytic in $\{z : 0 < |z - b| < \varepsilon\}$ for some $\varepsilon > 0$ and $f(b)$ is not defined.

Let $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-b)^n$.

① **Removable singularity.**

If $a_n = 0$, $\forall n < 0$.

② **Zero of order $\mathbf{n_0}$.**

If $a_n = 0$, $\forall n < n_0$ for some $n_0 > 0$.

③ **Pole of order $\mathbf{n_0}$.**

If $a_n = 0$, $\forall n < -n_0$, for some $n_0 > 0$

④ **Essential singularity.**

If $a_n \neq 0$ for infinitely many $n < 0$.

Definition 3. A zero or pole is called **simple** if the order is 1.

Example 4. Find the singularities of the following functions. If it is a zero or a pole, give the order.

(a) $f(z) = e^{-1/z}$.

(b) $f(z) = \frac{e^z}{z^2}$.

(a) $z = 0$ is the singularity (well-defined $\forall z \neq 0$).
(analytic $\forall z \neq 0$)

$$f(z) = e^{-1/z} = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{(-1)^n}{z^n} \quad (z \neq 0).$$

<u>Type:</u> essential.

(b) $z = 0$ is the singularity (analytic $\forall z \neq 0$)

$$f(z) = \frac{e^z}{z^2} = \frac{1}{z^2} \sum_{n=0}^{\infty} \frac{z^n}{n!} = \sum_{n=-2}^{\infty} \frac{z^n}{(n+2)!}$$

<u>Type:</u> pole, order 2.

Definition 5. If f is analytic in $\{z : |z| > R\}$, then $f(1/z)$ has an isolated singularity at 0 and we say that f has an **isolated singularity at** ∞.

<u>**Notes:**</u>

① The type of singularities at ∞ are based on the Laurent expansion of $f(1/z)$ at 0.

② Given the Laurent expansion of $f(1/z) = \sum_{n=-\infty}^{\infty} b_n z^n$ around $z = 0$, the Laurent expansion of $f(z)$ at ∞ is given by

$$\sum_{n=-\infty}^{\infty} a_n z^n$$

with $a_n = b_{-n}$, $n \in \mathbb{Z}$.

③ An essential singularity at ∞ is therefore characterized by $a_n \neq 0$ for infinitely many positive integers n.

Definition 6. If f is analytic in a region Ω except for isolated poles in Ω then we say that f is **meromorphic in Ω**. A meromorphic function in $\mathbb{C}$ is sometimes just called meromorphic.

Facts:

① If f is meromorphic in Ω and not identically 0, then $1/f$ is meromorphic in Ω.

② A complex number $b \in \Omega$ is a zero of order k of a meromorphic function $f \not\equiv 0$ in Ω if and only if $b \in \Omega$ is a pole of order k of the meromorphic function $1/f$.

③ If f and g are two meromorphic function in Ω with $g \not\equiv 0$ and if b is a zero of order k and a zero of order m for f and g respectively, then the order of the zero/pole of f/g is $|k-m|$.

Theorem 7. If f is analytic in $U = \{z : 0 < |z-b| < \delta\}$ for some $b \in \mathbb{C}$ and $\delta > 0$, then if b is an essential singularity for f, then $f(U)$ is dense in $\mathbb{C}$.

Proof. Suppose $f(U)$ is not dense in $\mathbb{C}$.

there is a $w \in \mathbb{C}$ s.t. $\overline{B(w,\varepsilon)} \subseteq \mathbb{C} \setminus \overline{f(U)}$.

Then,

$$g(z) = \frac{1}{f(z)-w}, \quad z \in U$$

is analytic in U. We have, $\forall z \in U$

$$|(z-b)g(z)| \leq \frac{|z-b|}{\varepsilon}.$$

If $z \to b$, $(z-b)g(z) \longrightarrow 0$. Then by Riemann's Removability theorem, g is analytic on U.

So, $$f(z) - w = \frac{1}{g(z)} \quad \text{in } U \cup \{b\}$$

Since the zeros of g are isolated, $f(z)-w$ is a meromorphic function in $U \cup \{b\}$. Therefore, f has only poles as singularities, in particular f has a pole at b. But this is a contradiction with the fact that b is an essential sing.