

M444 – Complex Analysis

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Chapter 2

Section 2.5: The Cauchy-Riemann Equations

Recall the following :

- ① $z = (x, y)$, a point in \mathbb{R}^2 .
- ② $f(z) = f(x, y) = u(x, y) + iv(x, y)$.
- ③ If $\phi = \phi(x, y)$, then

$$\frac{\partial \phi}{\partial x}(x_0, y_0) = \phi_x(x_0, y_0) = \lim_{\Delta x \rightarrow 0} \frac{\phi(x_0 + \Delta x, y_0) - \phi(x_0, y_0)}{\Delta x}$$

and

$$\frac{\partial \phi}{\partial y}(x_0, y_0) = \phi_y(x_0, y_0) = \lim_{\Delta y \rightarrow 0} \frac{\phi(x_0, y_0 + \Delta y) - \phi(x_0, y_0)}{\Delta y}.$$

Notice that

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} \iff f'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z_0)}{\Delta z}$$

Set $\Delta z = \Delta x$, for $\Delta x \in \mathbb{R}$. Then

$$\begin{aligned} f'(z_0) &= \lim_{\Delta x \rightarrow 0} \frac{f(z_0 + \Delta x) - f(z_0)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} \\ &\quad + i \lim_{\Delta x \rightarrow 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x} \\ &= u_x(x_0, y_0) + i v_x(x_0, y_0). \end{aligned}$$

Conclusion : $f'(z_0) = u_x(x_0, y_0) + i v_x(x_0, y_0)$.

Set $\Delta z = i\Delta y$, for $\Delta y \in \mathbb{R}$. Then

$$\begin{aligned} f'(z_0) &= \lim_{\Delta y \rightarrow 0} \frac{f(z_0 + \Delta y) - f(z_0)}{i\Delta y} \\ &= \lim_{\Delta y \rightarrow 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{i\Delta y} \\ &= \lim_{\Delta y \rightarrow 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{i\Delta y} + i \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{i\Delta y} \\ &= \frac{u_y(x_0, y_0)}{i} + v_y(x_0, y_0) \\ &= v_y(x_0, y_0) - iu_y(x_0, y_0). \end{aligned}$$

Conclusion : $f'(z_0) = v_y(x_0, y_0) - iu_y(x_0, y_0)$.

Theorem (Cauchy-Riemann Equations, Necessary conditions)

If $f = u + iv$ is analytic in an open set U , then

① u_x, u_y, v_x, v_y exist.

② $u_x = v_y$ and $u_y = -v_x$ (C-R equations).

Example : $f(z) = \bar{z}$ is not analytic.

Indeed, $u(x, y) = x$ and $v(x, y) = -y$. But

$$u_x(x, y) = 1 \neq -1 = v_y(x, y).$$

Hence, the C-R equations are not satisfied. So \bar{z} is not analytic.

Theorem (Cauchy-Riemann Equations; Corollary 2.5.2)

Let $f = u + iv$ be a function defined on an open set U . Assume that

① u_x, u_y, v_x, v_y exist and are continuous on U .

② $u_x = v_y$ and $u_y = -v_x$ on U .

Then f is analytic on U and

$$f' = u_x + iv_x = v_y - iu_y.$$

Example : We have $e^z = e^x \cos y + ie^x \sin y$.

① $u_x(x, y) = e^x \cos y$ and $v_y = e^x \cos y$, and so $u_x = v_y$.

② $u_y(x, y) = -e^x \sin y$ and $v_x = e^x \sin y$, and so $u_y = -v_x$.

Hence, e^z is analytic on \mathbb{C} and

$$(e^z)' = u_x + iv_x = e^x \cos y + ie^x \sin y = e^z.$$

Consequences :

- ① $\frac{d}{dz} \operatorname{Log}(z) = \frac{1}{z}$, for $z \in \mathbb{C} \setminus (-\infty, 0]$.
- ② $\frac{d}{dz} z^\alpha = \alpha z^{\alpha-1}$, for¹ $z \in \mathbb{C} \setminus (-\infty, 0]$. [Reason : $z^\alpha = e^{\alpha \operatorname{Log} z}$.]
- ③ $\frac{d}{dz} \sin(z) = \cos(z)$ and $\frac{d}{dz} \cos(z) = -\sin(z)$.
- ④ $\frac{d}{dz} \sinh(z) = \cosh(z)$ and $\frac{d}{dz} \cosh(z) = \sinh(z)$.

Proof of ①.

We have $z = e^{\operatorname{Log} z}$. Therefore

$$(z)' = (e^{\operatorname{Log} z})' \Rightarrow 1 = e^{\operatorname{Log} z} (\operatorname{Log} z)' \Rightarrow \frac{1}{e^{\operatorname{Log} z}} = (\operatorname{Log} z)'.$$

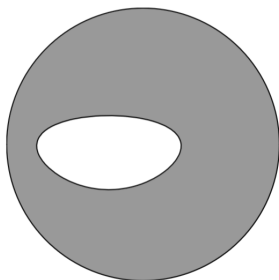
$$\text{Hence } (\operatorname{Log} z)' = \frac{1}{z}.$$



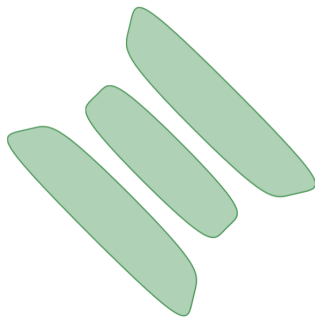
1. Also, $\alpha \neq 0$.

A **region** is a set $\Omega \subset \mathbb{C}$ such that

- ① Ω is open.
- ② Any two points $z, w \in \Omega$ can be connected by a polygonal curve.



(a) Connected Set



(b) Disconnected Set

Figure – Examples of connected and disconnected sets

Theorem

If f is analytic on a region Ω and $f'(z) = 0$ for every $z \in \Omega$, then there is a $c \in \mathbb{C}$ such that $f(z) = c$ for any $z \in \Omega$.

Proof.

- ① Fix $w \in \Omega$ and let $z \in \Omega$ with $z \neq w$. Let C be a polygonal curve joining w to z in Ω .
- ② Recall that $f'(z) = u_x + iv_x = v_y - iu_y \Rightarrow u_x = u_y = v_x = v_y = 0$.
- ③ Therefore, $\nabla u = \vec{0}$ and $\nabla v = \vec{0}$.
- ④ From the Fundamental Theorem for line integrals, we get

$$u(z) - u(w) = \int_C \nabla u \cdot d\vec{r} = 0 \quad \Rightarrow \quad u(z) = u(w).$$

Similarly, $v(z) = v(w)$.

- ⑤ Hence $f(z) = u(w) + iv(w) = f(w)$, a constant. □

WARNING !

In the last result, Ω must be a region (open and connected).

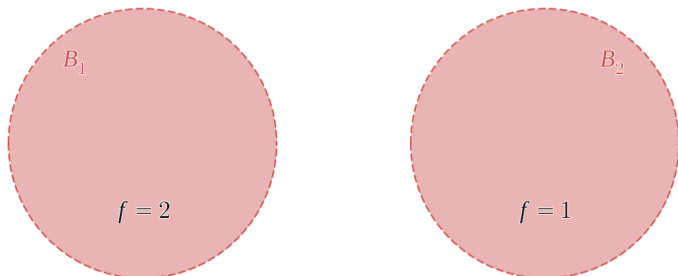


Figure – An analytic function f such that

- ① $f'(z) = 0$ on $\Omega = B_1 \cup B_2$;
- ② but f is not constant on Ω .