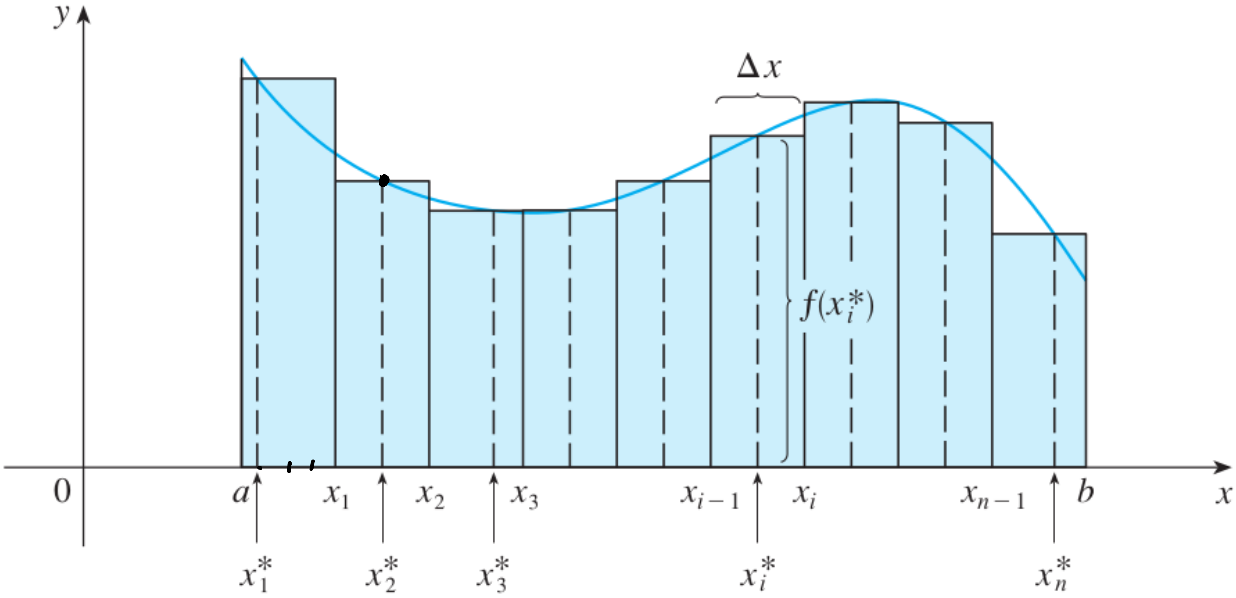
15.1 Double integrals over Rectangles.

Definite integrals over an interval.



f defined on an interval $[a,b]$.

Divide $[a,b]$ into n parts of equal length. Δx

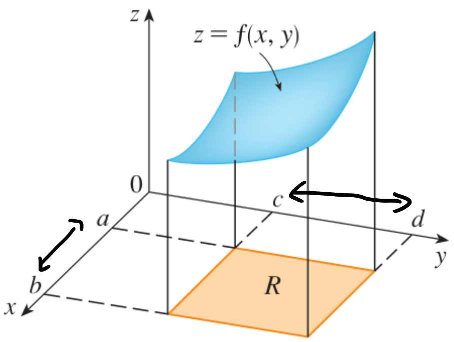
Choose some point $x_i^* \in [x_{i-1}, x_i]$.

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} \underbrace{f(x_i^*)\cdot\Delta x}_{\text{area of each rectangles}}$$

$$= f(x_1^*)\cdot\Delta x + f(x_2^*)\,\Delta x + \cdots + f(x_n^*)\cdot\Delta x$$

So, $$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

Volumes and Double Integrals.



$f(x,y)$: function of x & y.

$R = [a,b] \times [c,d]$

$= \{ (x,y) : a \le x \le b,\ c \le y \le d \}$

Suppose first that

$f(x,y) \ge 0$.

Divide $[a,b]$: (n parts)

$a < x_1 < x_2 < \dots < x_i < \dots < x_{n-1} < b$

divisions

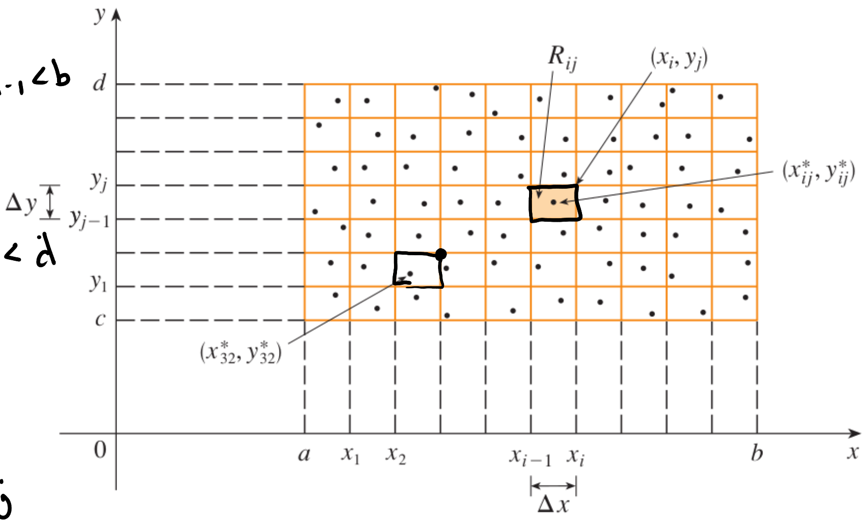
Divide $[c,d]$: (m parts)

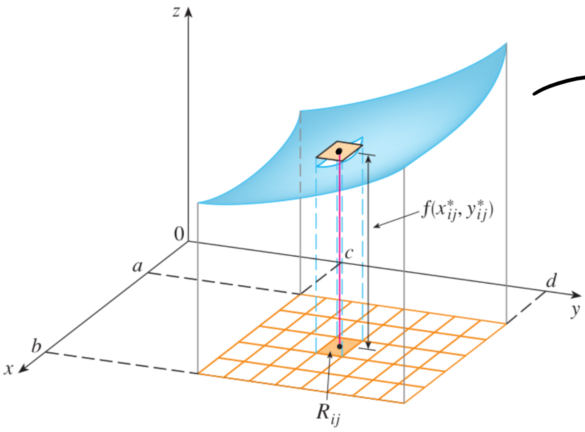
$a < y_1 < y_2 < \dots < y_j < \dots < y_{m-1} < d$

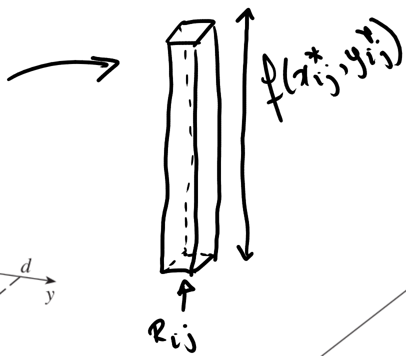
Lengths of each divisions are Δx & Δy.

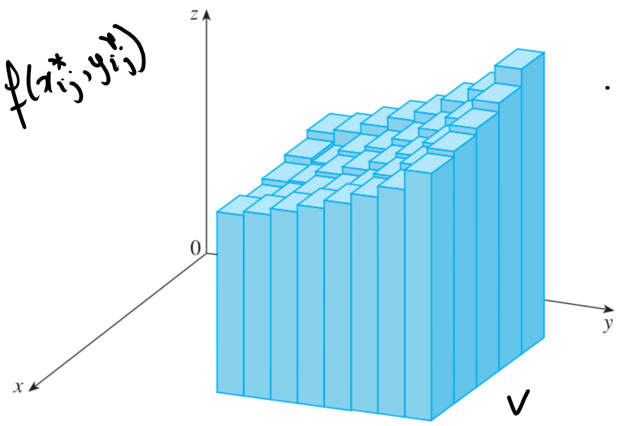
$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$

Select point (x^*_{ij}, y^*_{ij}) in R_{ij}









Volume of $\square$ $= A(R_{ij}) \cdot f(x^*_{ij}, y^*_{ij})$

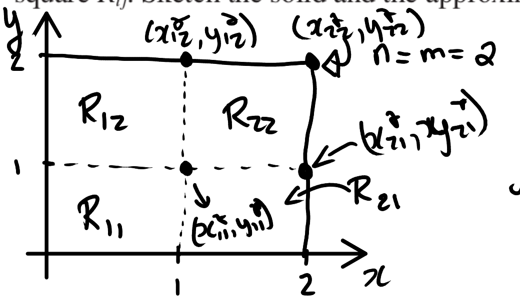
As n, m goes to ∞, the total volume approaches the integral.

$$V = \sum_{i=1}^{n} \sum_{j=1}^{m} f(x^*_{ij}, y^*_{ij}) \cdot A(R_{ij})$$

$$\boxed{\Rightarrow \quad \iint_R f(x,y)\, dA = \lim_{n,m \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x^*_{ij}, y^*_{ij})\, A(R_{ij}).}$$

Here: $\iint_R f(x,y)\, dA \approx \sum_{i=1}^{n} \sum_{j=1}^{m} f(x^*_{ij}, y^*_{ij})\, A(R_{ij})$.

EXAMPLE 1 Estimate the volume of the solid that lies above the square $R = [0, 2] \times [0, 2]$ and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$. Divide R into four equal squares and choose the sample point to be the upper right corner of each square R_{ij}. Sketch the solid and the approximating rectangular boxes. $\longrightarrow f(x,y)$



$$\iint_R 16 - x^2 - 2y^2 \, dA$$

$$\approx f(x_{11}^*, y_{11}^*) \cdot A(R_{11}) + f(x_{12}^*, y_{12}^*) A(R_{12}) + f(x_{21}^*, y_{21}^*) \cdot A(R_{21}) + f(x_{22}^*, y_{22}^*) A(R_{22})$$

$$= f(1,1) \cdot 1 + f(1,2) \cdot 1 + f(2,1) \cdot 1 + f(2,2) \cdot 1$$

$$= 13 + 7 + 10 + 4$$

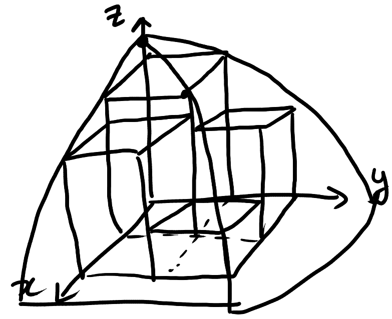
$$\boxed{= 34}$$

$\Delta x = \frac{2}{2} = 1$

$\Delta y = \frac{2}{2} = 1$

$x_1 = 1, \ x_2 = 2$

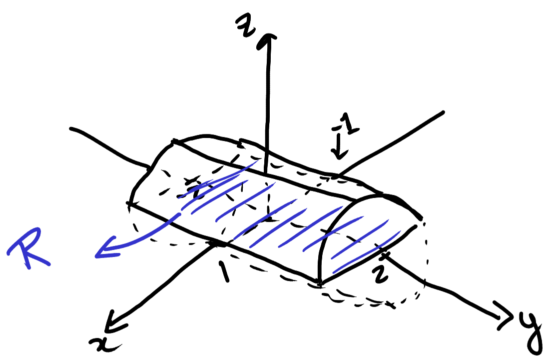
$y_1 = 1, \ y_2 = 2$



EXAMPLE 2 If $R = \{(x, y) \mid -1 \leqslant x \leqslant 1, -2 \leqslant y \leqslant 2\}$, evaluate the integral

$$\iint_R \sqrt{1 - x^2} \, dA$$

Observ.
$z = \sqrt{1 - x^2} \rightarrow x^2 + z^2 = 1 \rightarrow$ cylinder (principle y-axis)

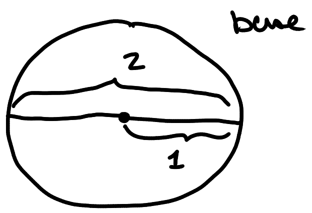


$\iint_R \sqrt{1-x^2} \, dA =$ half of the volume of the cylinder.

$$= \frac{\pi r^2 \cdot h}{2}$$

$$= \frac{\pi \cdot 1^2 \cdot 4}{2}$$

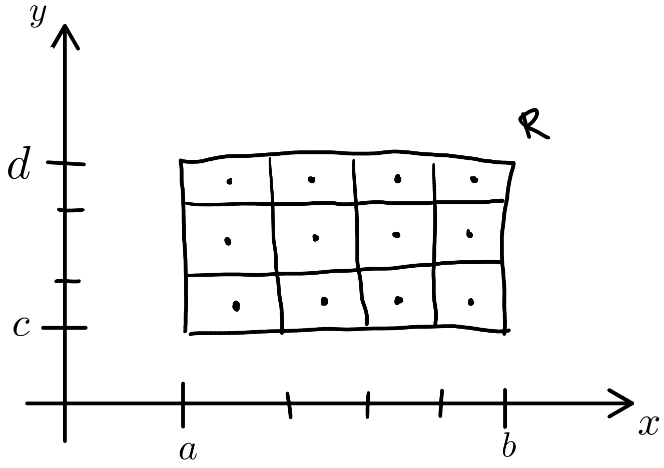
$$= 2\pi .$$



height = 4
radius = 1

So, $\boxed{\iint_R \sqrt{1-x^2} \, dA = 2\pi}$

Midpoint rule.



$R = [a,b] \times [c,d]$

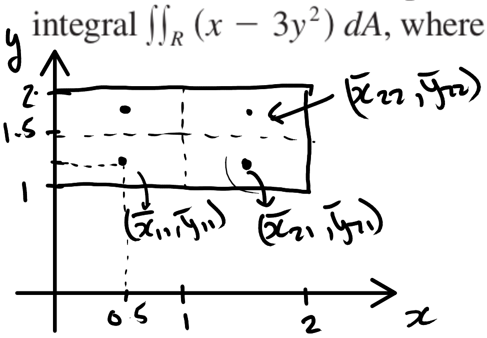
Choose the middle of R_{ij} for the (x_{ij}^*, y_{ij}^*).

$\downarrow$

name this choice $(\bar{x}_{ij}, \bar{y}_{ij})$

$$\boxed{\iint_R f(x,y)\,dA \approx \sum_{i=1}^{n}\sum_{j=1}^{m} f(\bar{x}_{ij}, \bar{y}_{ij})\, A(R_{ij})}$$

EXAMPLE 3 Use the Midpoint Rule with $m = n = 2$ to estimate the value of the integral $\iint_R (x - 3y^2)\, dA$, where $R = \{(x, y) \mid 0 \leqslant x \leqslant 2,\ 1 \leqslant y \leqslant 2\}$.



$\Delta x = \frac{2}{2} = 1$

$\Delta y = \frac{1}{2} = 0.5$

$(\bar{x}_{11}, \bar{y}_{11}) = (0.5, 1.25)$

$(\bar{x}_{11}, \bar{y}_{12}) = (0.5, 1.75)$

$(\bar{x}_{21}, \bar{y}_{21}) = (1.5, 1.25)$

$(\bar{x}_{22}, \bar{y}_{22}) = (1.5, 1.75)$

$$\iint_R (x - 3y^2)\, dA$$

$$\approx f(0.5, 1.25) \cdot \underbrace{A(R_{11})}_{1 \cdot 0.5} + f(0.5, 1.75)\, A(R_{12}) + f(1.5, 1.25) \cdot A(R_{21}) + f(1.5, 1.75) \cdot A(R_{22})$$

$$= -11.875.$$

$$\boxed{\iint_R (x - 3y^2)\, dA \approx -11.875}$$

EXAMPLE 4 Evaluate the iterated integrals.

(a) $\int_0^3 \underbrace{\int_1^2 x^2 y \, dy}_{A(x)} \, dx$ (b) $\int_1^2 \int_0^3 x^2 y \, dx \, dy$

(a) $A(x) = \int_1^2 \underset{\substack{\uparrow \\ \text{constant}}}{x^2} y \, dy = \left.\frac{x^2 y^2}{2}\right|_{1=y}^{2=y} = \frac{x^2}{2}(4-1) = \frac{3x^2}{2}$

$\int_0^3 \underbrace{\int_1^2 x^2 y \, dy}_{A(x)} \, dx = \int_0^3 \frac{3x^2}{2} \, dx = \left.\frac{x^3}{2}\right|_0^3 = \frac{27-0}{2} = \frac{27}{2}$

So, $\int_0^3 \int_1^2 x^2 y \, dy \, dx = \frac{27}{2}$

(b) $\int_1^2 \underbrace{\int_0^3 x^2 y \, dx}\, dy = \int_1^2 \left.\frac{x^3 y}{3}\right|_0^3 dy$

$= \int_1^2 \frac{27-0}{3} \cdot y \, dy$

$= \int_1^2 9y \, dy$

$= \left.\frac{9 \cdot y^2}{2}\right|_1^2$

$= \frac{9 \cdot 4 - 9}{2} = \frac{27}{2}$

$\boxed{10}$ **Fubini's Theorem** If f is continuous on the rectangle $R = \{(x, y) \mid a \leqslant x \leqslant b, c \leqslant y \leqslant d\}$, then

$$\rightarrow \iint_R f(x, y) \, dA = \int_a^b \int_c^d f(x, y) \, dy \, dx = \int_c^d \int_a^b f(x, y) \, dx \, dy$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

EXAMPLE 5 Evaluate the double integral $\iint_R (x - 3y^2)\, dA$, where $R = \{(x, y) \mid 0 \leqslant x \leqslant 2,\ 1 \leqslant y \leqslant 2\}$. (Compare with Example 3.)

a b c d

$$\iint_R (x - 3y^2)\, dA = \int_0^2 \underbrace{\int_1^2 (x - 3y^2)\, dy}\, dx$$

$$= \int_0^2 (xy - y^3)\Big|_{1=y}^{2=y} dx$$

$$= \int_0^2 (x2 - 8) - (x - 1)\, dx$$

$$= \int_0^2 (x - 7)\, dx$$

$$= \left(\frac{x^2}{2} - 7x\right)\Big|_0^2 = 2 - 14 = \boxed{-12}$$

EXAMPLE 6 Evaluate $\iint_R y \sin(xy)\, dA$, where $R = [1, 2] \times [0, \pi]$.

$1 \leq x \leq 2 \quad 0 \leq y \leq \pi$

a b c d

$u = y \qquad dv = \sin(xy)\, dy$

$du = dy \qquad v = -\dfrac{\cos(xy)}{x}$

$$\iint_R y\sin(xy)\, dA = \int_1^2 \int_0^\pi y\sin(xy)\, dy\, dx$$

$u = xy$

$du = y\,dx$

$x = 1 \to u = 1y = y$

$x = 2 \to u = 2 \cdot y = 2y$

$y\sin(xy) = f(x,y)$

$f_x(x,y) = y\cos(xy) \cdot y$

$= y^2 \cos(xy).$

$$= \int_0^\pi \int_{1=x}^{2=x} y\sin(xy)\, dx\, dy$$

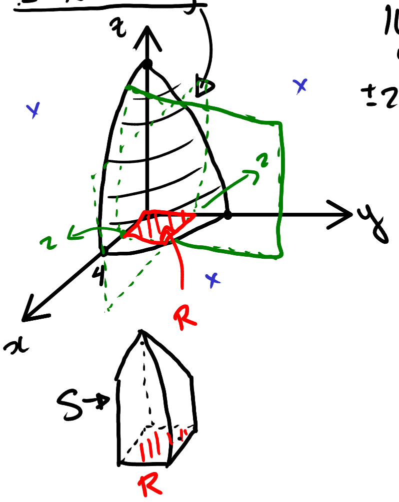
$$= \int_0^\pi \int_y^{2y} \sin(u)\, du\, dy$$

$$= \int_0^\pi (-\cos(u))\Big|_y^{2y} dy$$

$$= \int_0^\pi -\cos(2y) + \cos(y)\, dy = -\frac{\sin(2y)}{2} + \sin(y)\Big|_0^\pi = \boxed{0}$$

EXAMPLE 7 Find the volume of the solid S that is bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$, and the three coordinate planes.

$z = 16 - x^2 - 2y^2$



$R = [0,2] \times [0,2]$

$16 - 2y^2 = 0$

$8 = y^2$

$\pm 2\sqrt{2} = y$

$$V(S) = \iint_R 16 - x^2 - 2y^2\, dA$$

$$= \int_0^2 \int_0^2 16 - x^2 - 2y^2\, dy\, dx$$

$$= \int_0^2 16y - x^2 y - \frac{2y^3}{3}\Big|_0^2 dx$$

$$= \int_0^2 32 - 2x^2 - \frac{16}{3}\, dx$$

$$= \frac{50x}{3} - \frac{2x^3}{3}\Big|_0^2$$

$$= \frac{100}{3} - \frac{16}{3} = \boxed{\frac{84}{3}}$$

EXAMPLE 8 If $R = \underbrace{[0, \pi/2]}_{0 \le x \le \frac{\pi}{2}} \times \underbrace{[0, \pi/2]}_{0 \le x \le \frac{\pi}{2}}$, then compute $\displaystyle\iint_R \sin x \cos y \, dA$.

$$\iint_R \sin x \cos y \, dA = \int_0^{\frac{\pi}{2}} \int_0^{\pi/2} \sin x \cos y \, dy \, dx$$

$$= \int_0^{\pi/2} \sin x \underbrace{\left(\int_0^{\pi/2} \cos y \, dy \right)}_{\text{constant}} dx \longrightarrow + \sin y \Big|_0^{\pi/2} = 1$$

$$= \left(\int_0^{\pi/2} \sin x \, dx \right) \left(\int_0^{\pi/2} \cos y \, dy \right)$$

$$= \left(-\cos x \Big|_0^{\pi/2} \right) \cdot \left(\sin y \Big|_0^{\pi/2} \right)$$

$$\left(-\overset{0}{\cancel{\cos\left(\tfrac{\pi}{2}\right)}} + \overset{1}{\cos(0)} \right)$$

$$= 1 \cdot 1$$

$$= \boxed{1}$$

$$\boxed{11} \quad \iint_R g(x)\, h(y) \, dA = \int_a^b g(x) \, dx \int_c^d h(y) \, dy \qquad \text{where } R = [a, b] \times [c, d]$$

Average Value.

$f(x,y)$ defined on R (rectangle)

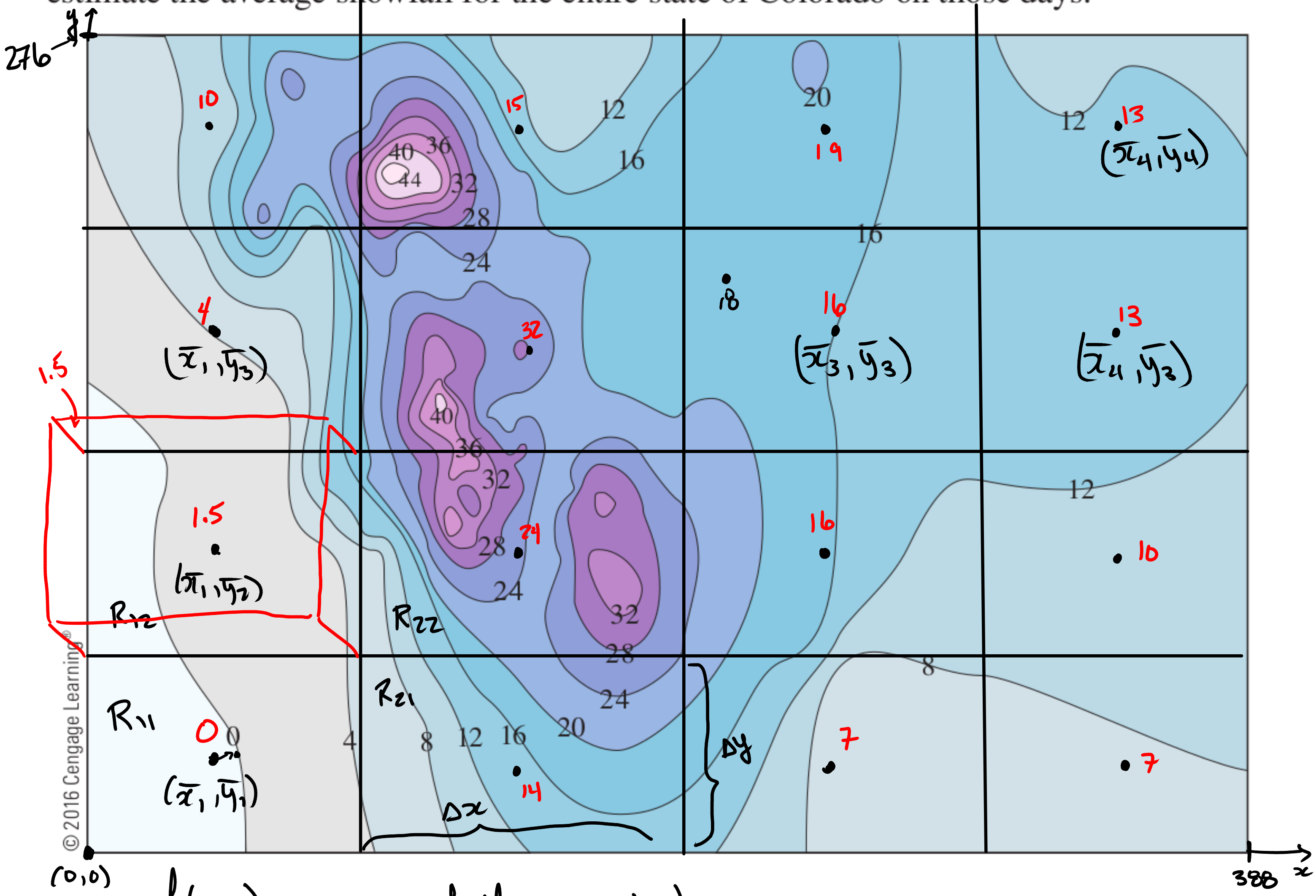
$$\boxed{f_{\text{ave}} = \frac{1}{A(R)} \iint_R f(x,y)\, dA}$$

f_{ave}: average of f. $A(R)$: area of R

$$\underline{f(x_0)} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$f(x_0)$: mean-value of f or average of f.

EXAMPLE 9 The contour map in Figure 18 shows the snowfall, in inches, that fell on the state of Colorado on December 20 and 21, 2006. (The state is in the shape of a rectangle that measures 388 mi west to east and 276 mi south to north.) Use the contour map to estimate the average snowfall for the entire state of Colorado on those days.



$f(x,y) \to$ snowfall in inches.

$f_{\text{ave}} \to$ average snowfall.

$R = [0,388]\times[0,276]$

So,

$$f_{\text{ave}} = \frac{1}{A(R)} \iint_R f(x,y)\,dA$$

① Estimate the integral

Mid-point rule $m=n=4$

$\Delta x = 388/4 = 97$

$\Delta y = 276/4 = 69$

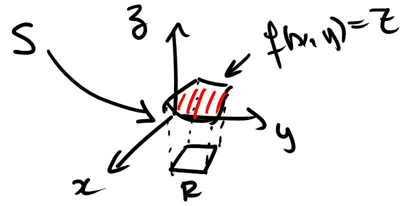
$$\iint_R f(x,y)\,dA = \sum_{i=1}^{4}\sum_{j=1}^{4} f(\bar{x}_i,\bar{y}_j)\,A(R_{ij})$$

$\approx$ 1 348 639.5 inches $\cdot$ miles^2

② $A(R) = 388 \cdot 276 = 107\,088$ miles^2

So, $$f_{ave} = \frac{1\,348\,639.5 \text{ inches} \cdot \cancel{\text{miles}^2}}{107\,088 \ \cancel{\text{miles}^2}}$$

$$= \boxed{12.59375 \text{ inches}}$$



Remark. When $f(x,y) \geq 0$, then

$$\underbrace{\iint_R f(x,y)\, dA}_{\text{Vol of } S} = \underbrace{f_{ave} \cdot A(R)}_{\text{Vol. of}}$$