MATH-244 Calculus IV
Midterm 01

Created by Pierre-O. Parisé
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Last name: Solution.

First name: ______

Question:	1	2	3	4	5	Total
Points:	20	20	20	20	20	100
Score:	–	–	–	–	–	–

Instructions: Make sure to write your complete name on your copy. You must answer all the questions below and write your answers directly on the questionnaire. At the end of the 75 minutes, hand out your copy.

No devices such as a smart phone, cell phone, laptop, or tablet can be used during the exam. You are not allowed to use the lecture notes, the textbook, or any other notes. You may use a digital calculator (no graphical calculator or symbolic calculator will be allowed).

You must show ALL your work to have full credit. An answer without justification worth no point.

Good luck!

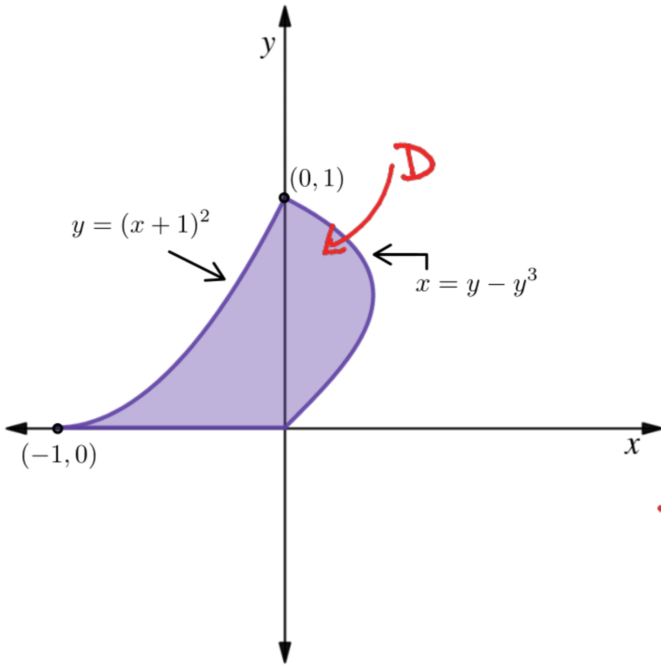
Pierre-Olivier Parisé

Your Signature: –

UNIVERSITY OF HAWAI'I

QUESTION 1 **(20 pts)**

Let D be the region depicted in the figure below. Compute the integral $\iint_D y\,dA$.



① Picture of D.

TYPE II.

$0 \le y \le 1$

$\sqrt{y} - 1 \le x \le y - y^3$

So,

$D = \{(x,y) : \sqrt{y}-1 \le x \le y - y^3,\ 0 \le y \le 1\}.$

12pts.

② Integrate.

$$\iint_D y\,dA = \int_0^1 \int_{\sqrt{y}-1}^{y-y^3} y\,dx\,dy$$

$$= \int_0^1 (y - y^3 - \sqrt{y} + 1)\,y\,dy$$

$$= \int_0^1 y^2 - y^4 - y^{3/2} + y\,dy$$

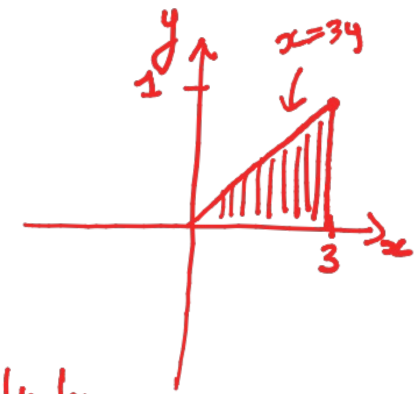
$$= \frac{y^3}{3} - \frac{y^5}{5} - \frac{y^{5/2}}{5/2} + \frac{y^2}{2}\Big|_0^1$$

8pts.

$$= \frac{1}{3} - \frac{1}{5} - \frac{2}{5} + \frac{1}{2} - 0$$

$$= \boxed{7/30}$$

(a) (10 points) Evaluate the iterated integral $\int_0^1 \int_{3y}^3 e^{x^2}\,dx\,dy$.



$D = \{(x,y) :\ 3y \le x \le 3,\ 0 \le y \le 1\}$. 6pt.

order change

$D = \{(x,y) :\ 0 \le x \le 3,\ 0 \le y \le \frac{x}{3}\}$.

So,

$$\int_0^1 \int_{3y}^3 e^{x^2}\,dx\,dy = \iint_D e^{x^2}\,dA = \int_0^3 \int_0^{\frac{x}{3}} e^{x^2}\,dy\,dx$$

$$= \int_0^3 \frac{x}{3} e^{x^2}\,dx$$

$u = x^2$
$du = 2x\,dx$

4pt.

$$= \frac{1}{6}\int_0^9 e^u\,du$$

$$= \boxed{\frac{1}{6}\left(e^9 - 1\right)}$$

(b) (10 points) Evaluate the iterated integral $\int_0^2 \int_0^{y^2} x^2 y\,dx\,dy$.

We have

$$\int_0^2 \int_0^{y^2} x^2 y\,dx\,dy = \int_0^2 \left.\frac{x^3}{3}\right|_0^{y^2} y\,dy$$ 4pt.

$$= \int_0^2 \frac{y^7}{3}\,dy$$

$$= \left.\frac{y^8}{3\cdot 8}\right|_0^2$$ 4pt.

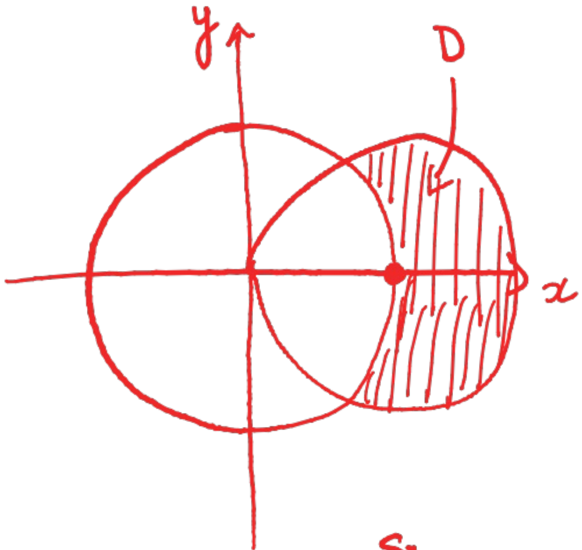
$$= \frac{2^8}{3\cdot 2^3} = \frac{2^5}{3} = \boxed{\frac{32}{3}}$$

2pt.

Question 3 **(20 pts)**

Find the area of the region inside the circle $(x-1)^2+y^2=1$ and outside the circle $x^2+y^2=1$.

① Picture.



4pts

$x = r\cos\theta$ & $y = r\cos\theta$.

$(x-1)^2+y^2=1 \rightarrow r = 2\cos\theta$

$x^2+y^2=1 \rightarrow r=1$

Now,

$1 = 2\cos\theta \Leftrightarrow \theta = \arccos\left(\frac{1}{2}\right)$

$\Leftrightarrow \theta = \pi/3 + 2k\pi$ or

$\theta = -\pi/3 + 2k\pi$

So,

$$D = \left\{(r,\theta): \ 1 \le r \le 2\cos\theta, \ -\frac{\pi}{3} \le \theta \le \frac{\pi}{3}\right\}$$

10pts.

② Area.

4pts. polar.

$$A(D) = \iint_D dA = \int_{-\pi/3}^{\pi/3} \int_1^{2\cos\theta} r \, dr \, d\theta$$

$$= \int_{-\pi/3}^{\pi/3} \left.\frac{r^2}{2}\right|_1^{2\cos\theta} d\theta$$

2pts.

$$= \frac{1}{2}\int_{-\pi/3}^{\pi/3} 4\cos^2\theta - 1 \, d\theta$$

$$= \frac{1}{2}\int_{-\pi/3}^{\pi/3} 4\left(\frac{\cos 2\theta + 1}{2}\right) - 1 \, d\theta$$

$$= \int_{-\pi/3}^{\pi/3} \cos 2\theta + 1/2 \, d\theta$$

$$= \left.\frac{\sin 2\theta}{2}\right|_{-\pi/3}^{\pi/3} + \pi/3 = \boxed{\frac{1}{2} + \frac{\pi}{3}}$$

lower

Question 4 (20 pts)

We would like to construct a fan with rectangular blades of width 1 decimeters and height 2 decimeter (we neglect the thickness). Consider one of the blade where the left corner is positioned at the origin. We know that the metal from the manufacturer to construct one blade has a density of $\rho(x, y) = xy$. What is the moment of inertia about the origin of the blade?

$I_0 = I_x + I_y$. 2pts.

① $I_x = \iint_R y^2 \rho(x,y)\, dA = \int_0^1 \int_0^2 y^3 x \, dy\, dx$

$= \left(\frac{x^2}{2}\Big|_0^1\right)\left(\frac{y^4}{4}\Big|_0^2\right)$ 8pts.

$= \left(\frac{1}{2}\right)\left(\frac{2^4}{4}\right) = 2$

② $I_y = \iint_R x^2 \rho(x,y)\, dA = \int_0^1 \int_0^2 x^3 y \, dy\, dx$ 8pts.

$= \left(\frac{x^4}{4}\Big|_0^1\right)\left(\frac{y^2}{2}\right)_0^2 = \frac{1}{2}$

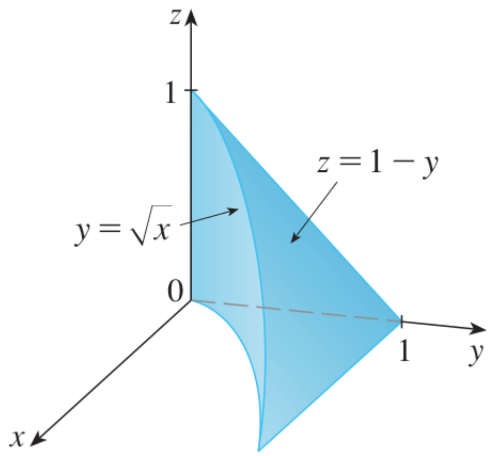
So,

$I_0 = 2 + \frac{1}{2} = \boxed{\frac{5}{2}}$

2pts.

Question 5 **(20 pts)**

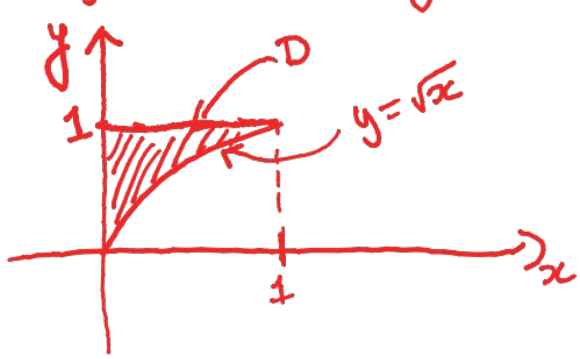
Consider the solid E below which is bounded by $x=0$, $z=0$, $y=\sqrt{x}$, and $z=1-y$. Set up the triple integral $\iiint_E f(x,y,z)\,dV$ as an iterated integral in two ways.



10pts:

TYPE 1. $0 \le z \le 1-y$.

Projection on xy-plane



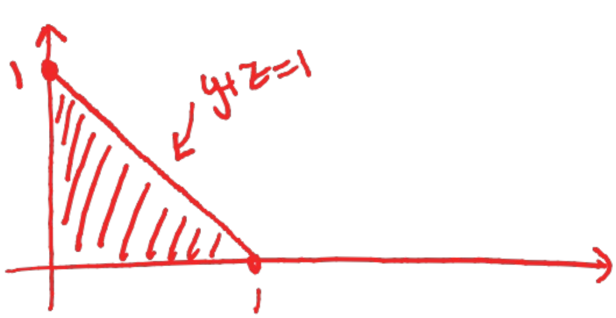
$$\boxed{\iiint_E f(x,y,z)\,dV = \int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)\,dz\,dy\,dx.}$$

$D = \{(x,y) : 0 \le x \le 1,\ \sqrt{x} \le y \le 1\}$

10pts.

TYPE 2. $0 \le x \le y^2$

Projection on yz-plane



$$\boxed{\iiint_E f(x,y,z)\,dV = \int_0^1 \int_0^{1-y} \int_0^{y^2} f(x,y,z)\,dx\,dz\,dy}$$

$D = \{(y,z) : 0 \le y \le 1,\ 0 \le z \le 1-y\}$