

# Chapter 1

## Functions and Limits

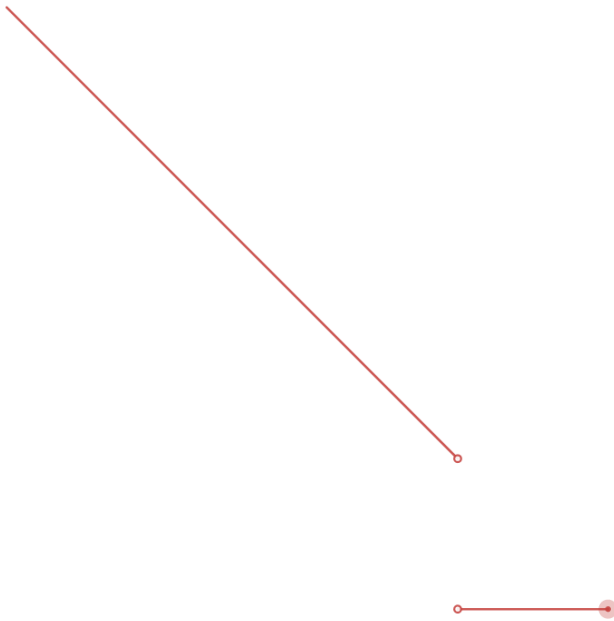
1.8 Continuity

# Continuity

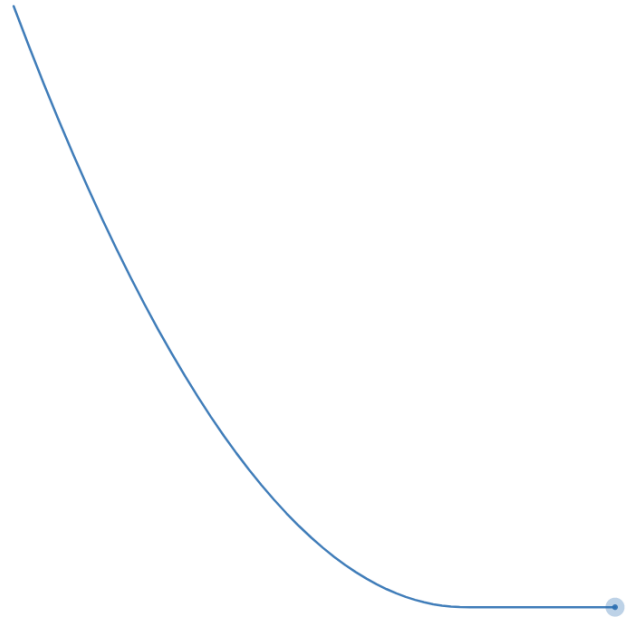
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**Example.** What are the main difference(s) between the two following curves?

Illustration: <https://www.desmos.com/calculator/hflxgbsemz>



(a) Graph of f(x).



(b) Graph of g(x)

**Example.** Now, what are the differences between the two following functions?

$$(a) f(x) = \begin{cases} 2 - x & \text{if } -2 \leq x < 1 \\ 0 & \text{if } 1 \leq x \leq 2 \end{cases} \quad (b) g(x) = \begin{cases} \frac{4}{9}(1 - x)^2 & \text{if } -2 \leq x < 1 \\ 0 & \text{if } 1 \leq x \leq 2 \end{cases}$$

**1 Definition** A function  $f$  is **continuous at a number  $a$**  if

$$\lim_{x \rightarrow a} f(x) = f(a)$$

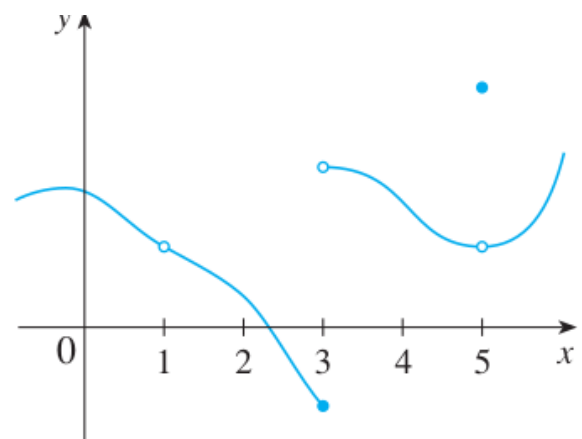
Three things to verify to show a function is continuous:

- a) The function is defined at  $x = a$ .
- b) The limit of the function exists at  $x = a$ .
- c) The limit of the function at  $x = a$  equals the value of the function at  $x = a$ .

Discontinuity:

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**EXAMPLE 1** Figure 2 shows the graph of a function  $f$ . At which numbers is  $f$  discontinuous? Why?



**Example.** Check if the functions in the first example are continuous at  $x = 1$  using the formulas.

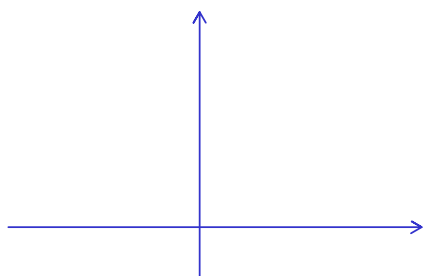
**EXAMPLE 2** Where are each of the following functions discontinuous?

(a)  $f(x) = \frac{x^2 - x - 2}{x - 2}$

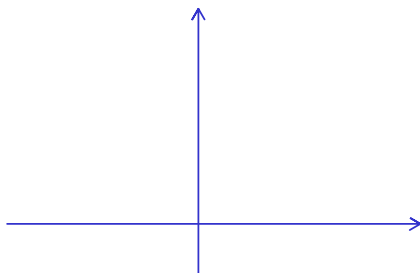
(b)  $f(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

(c)  $f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$

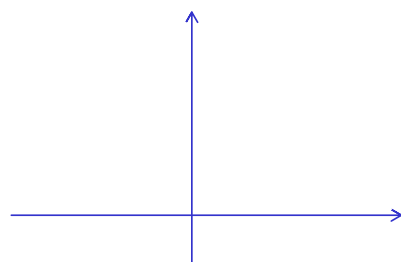
3 kinds of discontinuity.



(a) Removable.



(b) Infinite discontinuity.



(c) Jump discontinuity.

**2 Definition** A function  $f$  is **continuous from the right at a number  $a$**  if

$$\lim_{x \rightarrow a^+} f(x) = f(a)$$

and  $f$  is **continuous from the left at  $a$**  if

$$\lim_{x \rightarrow a^-} f(x) = f(a)$$

**EXAMPLE.** Is the function

$$f(x) = \begin{cases} 1 & , \text{ if } x > 0 \\ 0 & , \text{ if } x \leq 0 \end{cases}$$

(a) continuous from the right at  $x = 0$       (b) continuous from the left at  $x = 0$ .

## Properties of Continuous Functions.

**4 Theorem** If  $f$  and  $g$  are continuous at  $a$  and if  $c$  is a constant, then the following functions are also continuous at  $a$ :

1.  $f + g$

2.  $f - g$

3.  $cf$

4.  $fg$

5.  $\frac{f}{g}$  if  $g(a) \neq 0$

## Consequences:

**7 Theorem** The following types of functions are continuous at every number in their domains:

- polynomials
- rational functions
- root functions
- trigonometric functions

## Substitution Rule Revisited.

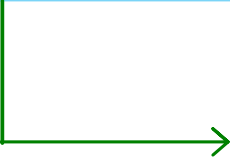
**EXAMPLE 5** Find  $\lim_{x \rightarrow -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}$ .

**EXAMPLE 7** Evaluate  $\lim_{x \rightarrow \pi} \frac{\sin x}{2 + \cos x}$ .



**8 Theorem** If  $f$  is continuous at  $b$  and  $\lim_{x \rightarrow a} g(x) = b$ , then  $\lim_{x \rightarrow a} f(g(x)) = f(b)$ .  
In other words,

$$\lim_{x \rightarrow a} f(g(x)) = f\left(\lim_{x \rightarrow a} g(x)\right)$$



**9 Theorem** If  $g$  is continuous at  $a$  and  $f$  is continuous at  $g(a)$ , then the composition  $f(g(x))$  is continuous at  $a$ .

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**Example.** Find the value of

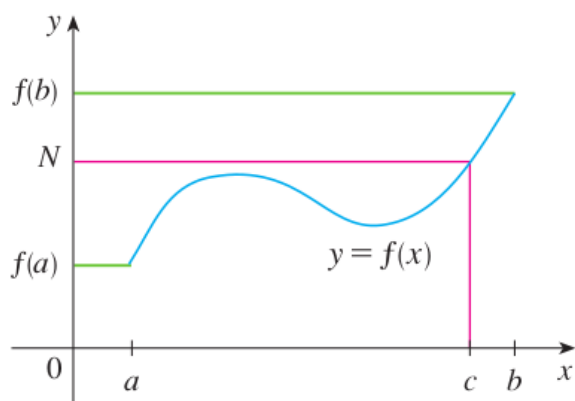
$$\lim_{x \rightarrow 1/2} \sin(\pi - \pi x^2)$$

**EXAMPLE.** Suppose we have a function

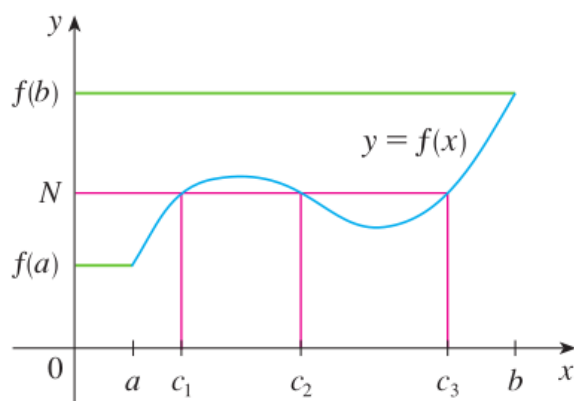
$$f(x) = x^2 - 1.$$

Does the graph of the function  $f$  cross the horizontal line  $y = 3$ ?

**10 The Intermediate Value Theorem** Suppose that  $f$  is continuous on the closed interval  $[a, b]$  and let  $N$  be any number between  $f(a)$  and  $f(b)$ , where  $f(a) \neq f(b)$ . Then there exists a number  $c$  in  $(a, b)$  such that  $f(c) = N$ .



(a) Find one number  $c$



(b) Find multiple numbers  $c$

**EXAMPLE 9** Show that there is a root of the equation

$$4x^3 - 6x^2 + 3x - 2 = 0$$

between 1 and 2.