Chapter 1: Functions and Limits
Week 2

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Upcoming this week

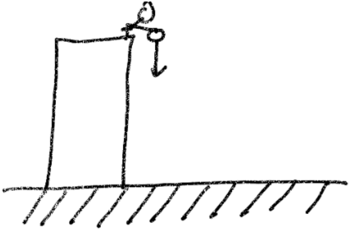
1. 1.4 Introduction to limits
2. 1.5 Limit
3. 1.6 Limit Laws

Definition 1

In geometry, the tangent to a curve at a given point (a, b) of that curve is the straight line that "just touches" the curve at that point. ▸ Circle

Question 2

Can you find the slope of the tangent at the point $(2, 4)$ of the curve generated by the graph of $f(x) = x^2$? ▸ Secant lines



Based on Galileo's work, we can predict the position of the ball:

$$y(t) = 4.9t^2$$

Definition 3

The average velocity between two consecutive times t_1 and t_2 is defined by

$$v_{av.} := \frac{v_2 - v_1}{t_2 - t_1}.$$

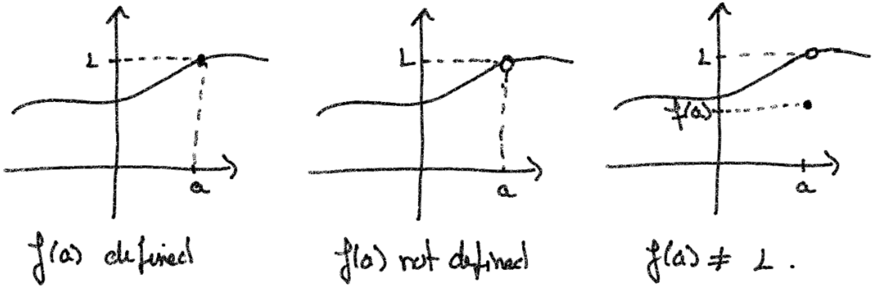
Question 4

Can you find the velocity of the ball after 5 seconds (called the instantaneous velocity)?

Definition 5

Let f be a function defined near (on an interval containing) a point a. If $f(x)$ becomes closer and closer to a number L as x gets closer and closer to the number a (without being equal to a, so $x \neq a$), then L is the limit of f as x approached a and we write

$$\lim_{x \to a} f(x) = L.$$



Example 6

Find intuitivelly the limit of

$$\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2}.$$

t	$f(t)$
1.0	0.162277...
0.5	0.165525...
0.1	0.166620...
0.05	0.166655...
0.01	0.166666...

t	$f(t)$
-1.0	"
-0.5	"
-0.1	"
-0.05	"
-0.01	"

$$0.166666 \approx \frac{1}{6} \Rightarrow \lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2} = \frac{1}{6}$$

Example 7

Can you guess the value of $\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right)$. Graph

x	$f(x)$
1.0	0
0.5	0
1/3	0
0.25	0
0.1	0
0.01	0
⋮	⋮

seems to be zero...

seems to be 1...

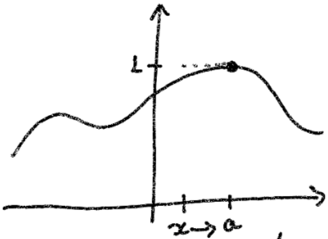
x	$f(x)$
2/3	1.0
2/5	1.0
2/7	1.0
2/9	1.0
2/11	1.0

Here, the limit **can't be two different values**! So the limit does not exists!

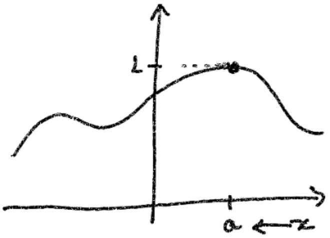
Warning: The examples just presented show that it may be easy or hard to guess the limit. Experimental data can lead to the wrong answer! **We will develop rules to compute the limit in the next section.**

Definition 8

One-sided limit



Left-hand side: $\lim\limits_{x \to a^-} f(x)$



Right-Hand side: $\lim\limits_{x \to a^+} f(x)$

Example 9

Let $f(x) = \frac{1}{x^2}$ and compute it near $x = 0$.

Definition 10

Let f be function defined near a point a. If $f(x)$ becomes larger and larger as x becomes closer and closer to a, then $f(x)$ is diverging to infinity and we write

$$\lim_{x \to a} f(x) = \infty.$$

Definition 11

Let f be a function defined near a point a. If $f(x)$ takes larger and larger negative values as x becomes closer and closer to a, then $f(x)$ is said to diverge to $-\infty$ and we write

$$\lim_{x \to a} f(x) = -\infty.$$

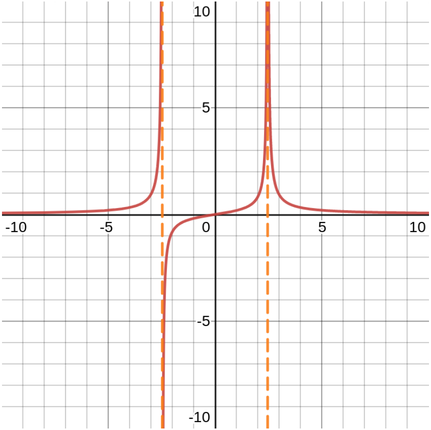
Example 12

Does $f(x) = -\frac{1}{x^2}$ diverges to infinity?

Definition 13

The vertical line $x = a$ is a vertical asymptote for a function f if at least one of the following holds:

- $\lim_{x\to a} f(x) = \infty$;
- $\lim_{x\to a^-} f(x) = \infty$;
- $\lim_{x\to a^+} f(x) = \infty$;
- $\lim_{x\to a} f(x) = -\infty$;
- $\lim_{x\to a^-} f(x) = -\infty$;
- $\lim_{x\to a^+} f(x) = -\infty$.



Example 14

Find

a) $\lim\limits_{x \to 3^+} \frac{2x}{x-3}$.

b) $\lim\limits_{x \to 3^-} \frac{2x}{x-3}$.

Exercises: 4, 6, 8, 10-12, 14(a) (use Desmos), 15, 19, 24, 29, 34, 35 (use Desmos), 40, 46.

Theorem 15

Let f and g be two functions such that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then

a) $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$ *[Limit of sum is sum of limits].*

b) $\lim_{x\to a}[cf(x)] = c \lim_{x\to a} f(x)$ *for any* $c \in \mathbb{R}$.

c) $\lim_{x\to a}[f(x)g(x)] = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right)$ *[Limit of product is product of limits]*

d) $\lim[f(x)/g(x)] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ *if* $\lim_{x\to a} g(x) \neq 0$ *[Limit of quotient is quotient of limits].*

Remark: The rules are also valid for the left-sided and right-sided limits.

Example 16

Say that $\lim_{x\to 2} f(x) = 2$ and $\lim_{x\to 2} g(x) = 5$. Compute

a) $\lim_{x\to 2}[5f(x) - 2g(x)]$;

b) $\lim_{x\to 2}\left[\frac{f(x)g(x)}{f(x)+g(x)}\right]$.

Theorem 17

Let f be a function such that $\lim_{x\to a} f(x)$ exists. If $n \geq 0$ is an integer, then

$$\lim_{x\to a}[f(x)]^n = \left(\lim_{x\to a} f(x)\right)^n.$$

Remarks:

- There are two particular cases: for a constant c

$$\lim_{x\to a} c = c \quad \text{and} \quad \lim_{x\to a} x^n = a^n \,(n \geq 1).$$

- The rule is also valid for the right-sided and left-sided limits.

Example 18

Compute $\lim_{x\to 2}(x^2 - 4x + 3)$.

Theorem 19

Let f be a function such that $\lim_{x\to a} f(x)$ exists. If $n \geq 0$

b) *and if n is odd, then* $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$.

c) *if n is even, and if* $\lim_{x\to a} f(x) > 0$, *then* $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$.

Remark: The rules are also valid for the left-sided and right-sided limits.

Example 20

Compute $\lim_{x\to 5} \frac{2x+4}{\sqrt{x+4}}$.

Example 21

Find the $\lim_{x \to 1} \frac{x^2-1}{x-1}$. ▸ Graph

Definition 22

An indetermination is when we encounter a quotient of the form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$.

Tricks: Rationalize the numerator or the denominator, factorize an expression, or simplify the numerator and denominator by a commun factor.

Example 23

Compute the following limits:

- $\lim_{t \to 0} \frac{\sqrt{t^2+9}-3}{t^2}$.
- $\lim_{h \to 0} \frac{(2+h)^2-4}{h}$.

Theorem 24

Let f be a function. Then

$$\lim_{x\to a} f(x) = L \quad \textit{if and only if} \quad \lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x).$$

Example 25

compute, if it exists, the following limits:

a) $\lim_{x\to 0} |x|$.

b) $\lim_{x\to 0} \frac{|x|}{x}$.

Example 26

Compute, if it exists, the $\lim_{x\to 0} x^2 \sin \frac{1}{x}$.

Theorem 27 (Squeeze theorem)

If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

then $\lim_{x\to a} g(x) = L$.

Exercises: 1-34, 38-40, 42-46