Math 241 Sample Midterm 3, Fall 2020

Name:

Question	Points	Score
1	15	
2	6	
3	8	
4	5	
5	10	
6	6	
Total:	50	

- The exam is 1 hour long, plus an extra 15 minutes at each end to download the paper, and upload your answers.
- You may not use any electronic devices (other than a tablet to write your answers, if you want) on the test.
- You may use the textbook, and your own personal notes, but no other notes.
- All work must be entirely your own. You cannot discuss the test with anyone else in any way.
- You must show all your work and make clear what your final solution is (for example, by drawing a box around it).
- You will get almost no credit for solutions that are not fully justified.
- You can write your answers on blank paper or on a tablet without printing out the test. If you do this, please make very clear which answer goes with which question, and write your name, and the page number, on each page. You can also print out the test and write on that if you want.
- Good luck!

1. Compute the following.

(a) (3 points) $\sum_{i=3}^{7}(i-1)$.

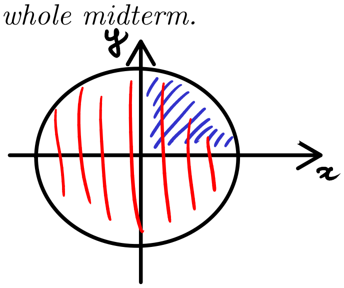
$$\sum_{i=3}^{7}(i-1) = (3-1)+(4-1)+(5-1)+(6-1)+(7-1)$$
$$= 2+3+4+5+6$$
$$= 20$$

(b) (3 points) $\int_0^2 (4-x^2)dx$.

$$\int_0^2 (4-x^2)\,dx = 4x - \frac{x^3}{3}\Big|_0^2 = 8 - \frac{8}{3} - (0)$$
$$= \frac{16}{3}$$

(c) (3 points) $\int_0^2 \sqrt{4-x^2}dx$.

Hint: think geometrically. If you use a technique from Calculus II, you get zero on the whole midterm.



Integral represents $\frac{1}{4}$ of the area of the disk of radius 2 (in red).

Area $= \pi r^2 = \pi \cdot 2^2 = 4\pi$

$$\Rightarrow \int_0^2 \sqrt{4-x^2}\,dx = \frac{4\pi}{4} = \pi$$

(d) (3 points) $\int \csc(4\theta)\cot(4\theta)d\theta$.

Let $u = 4\theta$ $\Rightarrow$ $\int \csc(4\theta)\cot(4\theta)\,d\theta$

$du = 4\,d\theta$

$$= \int \csc(u)\cot(u)\frac{du}{4} = -\frac{\csc(u)}{4} + C$$
$$= -\frac{\csc(4\theta)}{4} + C$$

~~(e)~~ (3 points) $\frac{d}{dx}\int_{\cos(x)}^{3} \cot(t)dt$.

2. Consider the limit

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}\left(\frac{2i}{n}-\left(\frac{2i}{n}\right)^2\right). \tag{1}$$

(a) (1 point) Which of the following integrals equals the limit above? No justification necessary.

$a=0$

$b=2$

$$(i)\ \int_{-2}^{0} x-x^2dx,\quad (ii)\ \int_{-1}^{1} x-x^2dx,\quad (iii)\int_{0}^{2} x-x^2dx.$$

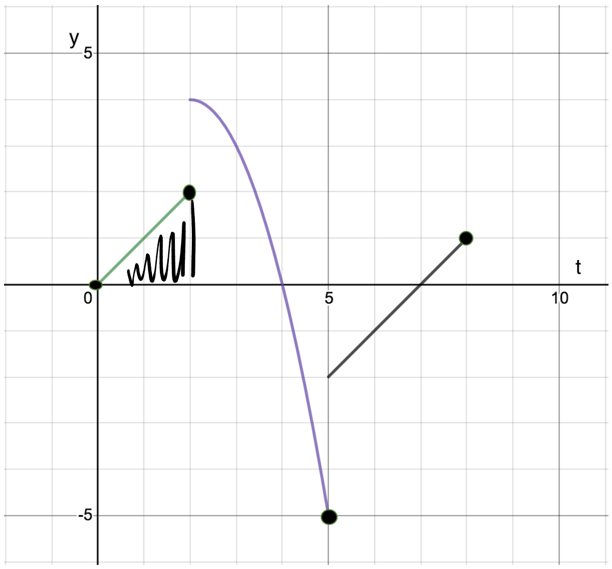
$\Delta x = \frac{b-a}{n} = \frac{2-0}{2}$, $a + i\frac{b-a}{n} = 0 + i\frac{2}{n} = \frac{2}{n}$, $f(x) = x - x^2$

~~(b)~~ (5 points) Use the formulas

$$\sum_{i=1}^{n} i=\frac{n(n+1)}{2}\qquad\text{and}\qquad\sum_{i=1}^{n} i^2=\frac{n(n+1)(2n+1)}{6}$$

to compute the limit from line (1).

3. The picture below is the graph $y = f(x)$ of a function f defined on the interval $[0, 8]$.



Let $g(x) = \displaystyle\int_0^x f(t)dt$ for $0 \le x \le 8$.

(a) (2 points) What are $g'(6)$ and $g(2)$? No justification required.

$g(2) =$ Area triangle $= 2$. / $g'(6) \overset{FTC}{=} f(6) = -1$

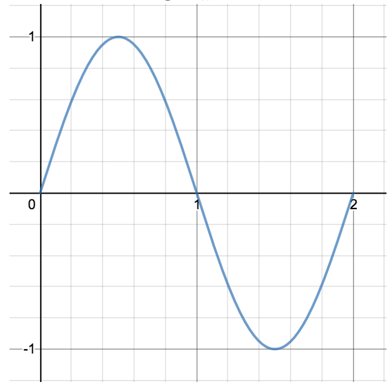
(b) (5 points) Use a Riemann sum with four equal-width intervals and the midpoint rule to estimate $g(8) = \displaystyle\int_0^8 f(x)dx$. Draw the rectangles in the Riemann sum (either on the graph above, or copy it out).

(c) (1 point) Is your answer to (b) less than or greater than the actual value of $g(8)$? No justification needed.

4. A mo'o starts from rest, and accelerates according to the formula

$$a(t) = \sin(\pi t) \quad \text{meters / second}^2, \quad 0 \le t \le 2$$

as pictured below (the first picture is the mo'o, the second is the graph of its acceleration).



(a) (1 point) For which value of t in $[0, 2]$ is the mo'o going fastest? No justification necessary.

At $t = 1$.

(b) (4 points) What is the velocity of the mo'o at the end of the 2 second period? Justify your answer.

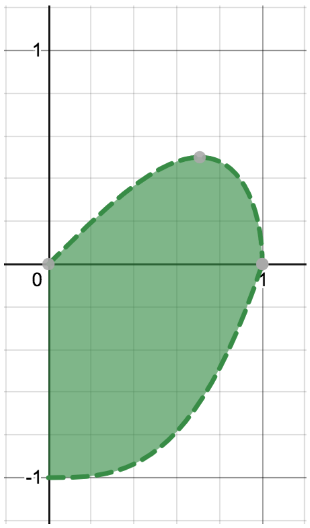
$$v(2) = \int_0^2 a(t)\,dt = \int_0^2 \sin(\pi t)\,dt$$

$$= -\frac{\cos(\pi t)}{\pi}\Big|_0^2$$

$$= -\frac{\cos(\pi 2) + \cos(\pi 0)}{\pi}$$

$$= \frac{-1+1}{\pi} \quad \boxed{= 0 \quad \text{meters/seconds}}$$

5. You are asked to compute the area of the shaded region between $y = x^3 - 1$ and $y = x\sqrt{1-x^2}$ as pictured on a calculus homework.



Area $= \int_0^1 (x^3-1)\,dx - \int_0^1 x\sqrt{1-x^2}\,dx$

Set $u = 1-x^2 \Rightarrow du = -2x\,dx$
$\Rightarrow x\,dx = (-\frac{1}{2}du)$

$\Rightarrow$ Area $= \int_0^1 (x^3-1)\,dx - \int_0^1 u^{\frac{1}{2}} \cdot (-\frac{1}{2}du)$

$= 3x^2 \Big|_0^1 + \frac{1}{2}\frac{u^{3/2}}{3/2}\Big|_0^1$

$= 3 - 0 + \frac{1^{3/2}}{3} - \frac{0}{3}$

$= \boxed{3\frac{1}{3}}$

On the right is your friend's solution to the problem. Your friend knows this is wrong but cannot find their own mistakes.

(a) (1 point) Why is the answer 'obviously' wrong (a very brief explanation is ok)?

(b) (5 points) There are **three** mistakes in the work above. What are they (very brief explanations are ok)?

(c) (4 points) Compute the actual area.

$A = \int_0^1 x\cdot\sqrt{1-x^2}\,dx - \int_0^1 x^3-1\,dx$

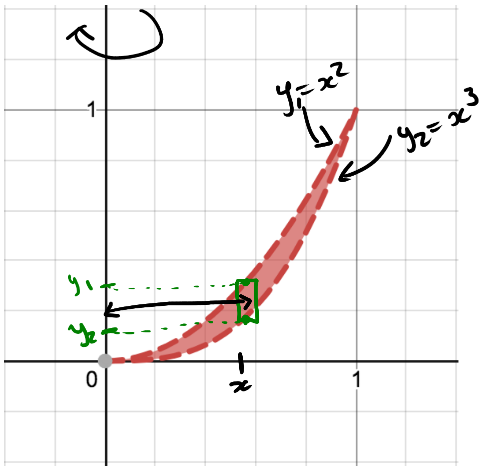
$u = 1-x^2$
$du = -2x\,dx$
$-\frac{du}{2} = dx$

$= \int_1^0 \sqrt{u}\left(\frac{-du}{2}\right) - \left(\frac{x^4}{4} - x\right)\Big|_0^1$

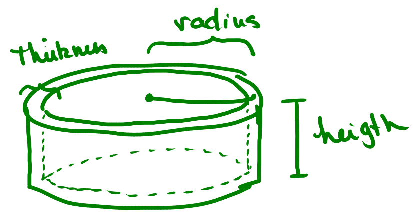
$= \frac{-1}{2}\frac{u^{3/2}}{3/2}\Big|_1^0 - \left(\frac{1}{4} - 1\right) = -\frac{1}{3}(-1) - \left(\frac{-3}{4}\right) = \frac{1}{3} + \frac{3}{4}$

$\boxed{= \frac{13}{12}}$

6. (6 points) Consider the shaded region between the graphs $y = x^2$ and $y = x^3$ as pictured.



Cylindrical shells because y is given explicly in tums of x.



radius $= x$

thickness $= \Delta x = dx$

heigth $= y_1 - y_2$

Compute the volume of the solid you get by rotating this around the y axis.

$$V = \int_0^1 \text{radius} \cdot \text{heigth} \cdot 2\pi \cdot \text{thickness}$$

$$= \int_0^1 x \, (x^2 - x^3) \cdot 2\pi \, dx$$

$$= 2\pi \int_0^1 x^3 \, dx - 2\pi \int_0^1 x^4 \, dx$$

$$= 2\pi \left. \frac{x^4}{4} \right|_0^1 - 2\pi \left. \frac{x^5}{5} \right|_0^1$$

$$= \frac{2\pi}{4} - \frac{2\pi}{5} = \frac{2\pi}{20} \boxed{= \frac{\pi}{10} \text{ units}^3}$$

~~Bonus question (5 points):~~

The marginal cost of producing x toy he'e is given by $m(x)$ dollars.

(a) Write down an integral that equals the cost $d(x)$ of doubling production in terms of $m(x)$, given that x toys have been produced already.

(b) Compute the derivative of $d(x)$ in terms of $m(x)$.