Math 311

Chapter 1

Section 1.3: Homogeneous Equations

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$$x + y = 0$$
$$x + 2y = 0$$

TERMINOLOGY

DEFINITION 1. A system of linear equations in $x_1, \ldots, x_n$ is called **homogeneous** if all the constant terms are zero.

- **Trivial solution:** $x_1 = 0$, $x_2 = 0$, $\ldots$, $x_n = 0$.
- **Non trivial solution:** Any solution in which at least one variable has a nonzero value.

EXAMPLE 1. Show that the following homogeneous system has nontrivial solutions.

$$\begin{aligned} x_1 - x_2 + 2x_3 - x_4 &= 0 \\ 2x_1 + 2x_2 \phantom{{}+ 2x_3} + x_4 &= 0 \\ 3x_1 + x_2 + 2x_3 - x_4 &= 0 \end{aligned}$$

SOLUTION. RREF is $\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$

So, $x_4 = 0$ and

$x_2 - x_3 = 0 \Rightarrow x_2 = x_3$

Also, $x_1 + x_3 = 0$

$\Rightarrow x_1 = -x_3$

$\Rightarrow$ $x_1 = -s$, $x_2 = s$, $x_3 = s$, $x_4 = 0$ $\xRightarrow{s=1}$ $\boxed{x_1 = -1,\ x_2 = 1,\ x_3 = 1,\ x_4 = 0}$

THEOREM 1. If a homogeneous system of linear equations has more variables than equations, then it has a nontrivial solution (in fact, infinitely many).

Linear Combinations

Definition 2.

- An **n-column vector**: $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$.
- Set of all n-column vectors is denoted by $\mathbb{R}^n$.
- **Equality**: $\mathbf{x} = \mathbf{y}$ if $\mathbf{x}$ and $\mathbf{y}$ are of the same size and all entries are the same.
- **Sum** of two n-column vectors $\mathbf{x}$,$\mathbf{y}$ is the new n-column vector $\mathbf{x}+\mathbf{y}$ obtained by adding corresponding entries.
- **Scalar multiplication** $k\mathbf{x}$ of a n-vector $\mathbf{x}$ with a scalar k is obtained by multiplying each entry of $\mathbf{x}$ by k.
- **Linear combination**: A sum of scalar multiples of several column vectors.

Example 2. If $\mathbf{x} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, then

$$\mathbf{x}+\mathbf{y} = \begin{bmatrix} 3-1 \\ -2+1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \text{ and } 2\mathbf{x} = \begin{bmatrix} (2)(3) \\ (2)(-2) \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \end{bmatrix}$$

Example 3. Let

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \mathbf{z} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Determine weither $\mathbf{v}$ and $\mathbf{w}$ are linear combinations of $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$.

Solution.

Basic Solutions

Notation:

- Write n variables x_1, x_2, …, x_n as $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$.

The solution in Example 1 can be written as

$$\mathbf{x} = \begin{bmatrix} -t \\ t \\ t \\ 0 \end{bmatrix} = -t \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

Theorem 2. Any linear combination of solutions to a homogeneous system is again a solution.

Proof. Let $\mathbf{x}$ and $\mathbf{y}$ be two different solutions to a homogeneous system. Let $\mathbf{z} = c\mathbf{x} + d\mathbf{y}$. Then, by definition, each component of $\mathbf{z}$ is $cx_j + dy_j$, for each j. Plugging that in each equation of the system:

$$\begin{aligned} a_{i1}(cx_1 + dy_1) + a_{i2}(cx_2 + dy_2) + \cdots + a_{in}(cx_n + dy_n) \\ &= c(a_{i1}x_1 + \cdots + a_{in}x_n) + d(a_{i1}y_1 + \cdots + a_{in}y_n) \\ &= c(0) + d(0) \\ &= 0 \end{aligned}$$

Therefore, $\mathbf{z}$ is a solution to the homogeneous system.

Example 4. Solve the homogeneous system with coefficient matrix

$$A = \begin{bmatrix} 1 & -2 & 3 & -2 \\ -3 & 6 & 1 & 0 \\ -2 & 4 & 4 & -2 \end{bmatrix}$$

and express the solution as a linear combination of particular solutions.

Solution.

Definition 3. The gaussian algorithm systematically produces solutions to any homogeneous systems of linear equations, called **basic solutions**, one for every parameter.

Hence, the basic solutions in the previous example are

$$\mathbf{x_1} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x_2} = \begin{bmatrix} \frac{1}{5} \\ 0 \\ \frac{3}{5} \\ 1 \end{bmatrix}.$$

Theorem 3. Let A be the coefficient matrix of a homogeneous system of m linear equations in n variables. If A has rank r, then

1. The system has exactly $n - r$ basic solutions, one for each parameter.

2. Every solution is a linear combination of these basic solutions.