Worksheet: Chapter 2

Math 307 — Linear Algebra and Differential equations — Spring 2022 section 3

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1. Use the Wronskian to determine whether the following functions are linearly independent or not.
   (a) $\sin(x)$ and $\cos(x)$. *Answer: Linearly independent*

   (b) x^3, x^2, x and 1. *Answer: Linearly independent*

   (c) x^2+1, $x-1$ and $2x^2+2x$ *Answer: Linearly dependent*

   (d) e^{2x} and e^{-x}. *Answer: Linearly independent*

   (e) e^x and xe^x. *Answer: Linearly independent*

   (f) $\sin(2x)$, $\sin(x)\cos(x)$ and $\cos(2x)$. *Answer: Linearly dependent*

2. For the following sets of vectors and vector space, determine if the sets of vectors form a basis for the associated vector space. If they do not form a basis, add/substract vectors to obtain a basis.
   (a) Vector set: $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Vector space: $\mathbb{R}^3$.
   Answer: It is a basis

   (b) Vector set: $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$. Vector space: $\mathbb{R}^3$.
   Answer: Missing one linearly independent vector, e.g. substitute v_3 by $[1,0,1]^T$

   (c) Vector set: $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$. Vector space: $\mathrm{Span}\{v_1, v_2, v_3\}$.
   Answer: The vector v_1, v_2 and v_3 are lin. dep. Taking one out will form a basis.

   (d) Vector set: $v_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Vector space: $M_{2\times 2}$.
   Answer: Add a vector v_4 that is linearly independent, e.g. the identity matrix.

   (e) Vector set: $v_1 = x^3+x$, $v_2 = x^2+1$, $v_3 = x$ and $v_4 = x^3-2x^2+x$. Vector space: P_3.
   Answer: It is a basis.

3. (a) Show that the Hermite polynomials H_n up to order $n = 4$ form a basis for P_4.
   (b) Express the standard basis of P_4 as a linear combination of H_4.

Hermite polynomials	Standard basis
$h_0 = 1$	$e_0 = 1$
$h_1 = 2x$	$e_1 = x$
$h_2 = 4x^2 - 2$	$e_2 = x^2$
$h_3 = 8x^3 - 12x$	$e_3 = x^3$
$h_4 = 16x^4 - 48x^2 + 12$	$e_4 = x^4$

Answer:

$$e_0 = h_0, \qquad e_1 = \frac{1}{2}h_1, \qquad e_2 = \frac{1}{4}h_2 + \frac{1}{2}h_0, \qquad e_3 = \frac{1}{8}h_3 + \frac{3}{4}h_1, \qquad e_4 = \frac{1}{16}h_4 + \frac{3}{4}h_2 + \frac{3}{4}h_0.$$

4. Is $[1, -1, 2]$ in $\text{Span}\{[0, 1, 2], [2, 0, 1], [1, 2, 0]\}$?
   Answer: Yes.

5. (a) Show that the set of $n \times n$ diagonal matrices forms a subspace of $M_{n\times n}$.
   (b) What is the dimension of the vector space formed by the set of $n \times n$ diagonal matrices?
   Answer: n