Worksheet: Chapter 6

Math 307 — Linear Algebra and Differential equations — Spring 2022 section 3

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1. Solve the following homogeneous systems of differential equations.

   (a)
   $$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ -12 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

   (b)
   $$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11/3 & -1/3 \\ 4/3 & 7/3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

   (c)
   $$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

   (d)
   $$\frac{d}{dt}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & -4 & 3 \\ -2 & 2 & 2 \\ -1 & -4 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

   (e)
   $$\frac{d}{dt}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1/2 & -2 & 3/2 \\ 0 & -2 & 2 \\ 1/2 & -4 & 7/2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

2. Solve the following non-homogeneous system of differential equations.
   The homogeneous versions were solved in problem 1.

   (a)
   $$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ -12 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3e^{3t} \\ 12t \end{bmatrix}$$

   (b)
   $$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11/3 & -1/3 \\ 4/3 & 7/3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3e^{3t} \\ 0 \end{bmatrix}$$

   (c)
   $$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 4\sec(2t) \end{bmatrix}$$

3. Use the Wronskian to determine whether the following solutions Y are made of linearly independent solutions.

   (a)
$$Y = c_1 \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix} + c_2 \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix}$$

   (b)
$$Y = c_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

   (c)
$$Y = c_1 e^t \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + c_2 e^t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_3 e^t \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$$

4. Let us consider the following system of differential equations:
$$\frac{d}{dt} Y = AY. \tag{1}$$

   (a) If $\vec{v}$ is an eigenvector of A associated to the eigenvalue λ, show that $\vec{\psi} = e^{\lambda t}\vec{v}$ is a solution to equation (1).

   (b) If $A\vec{v} = \lambda\vec{v}$ and $A\vec{w} = \lambda\vec{w} + \vec{v}$, show that $\vec{\phi} = te^{\lambda t}\vec{v} + e^{\lambda t}\vec{w}$ is a solution to equation (1).

   (c) If A is a 5×5 matrix, how many linearly independent solutions do we need to construct the general solution?

5. An object of mass $m > 0$ is connected to a wall by a spring of strenght $k > 0$. Its equation of motion is given by the second-order differential equation
$$m\frac{d^2x}{dt^2} + kx = 0, \tag{2}$$
   where x is the distance of the object from its resting position.

   (a) Write equation (2) as a system of first-order differential equation using the position x and its velocity $v = \dfrac{dx}{dt}$.

   (b) Using eigenvalues/vectors, find the general solution of the system found in (a).
   (Note that m and k are both positive real numbers.)

   (c) The initial position is given by $x(0) = 3$ and the initial velocity is given by $v(0) = 0$. Find the position x when $t = \pi\sqrt{\dfrac{m}{k}}$.

Worksheet: Chapter 6 — Answers

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1. (a)
$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{6t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{-6t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

   (b)
$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} t \\ 2t-3 \end{bmatrix}$$

   (c)
$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} \sin(2t) \\ -2\cos(2t) \end{bmatrix} + c_2 \begin{bmatrix} \cos(2t) \\ 2\sin(2t) \end{bmatrix}$$

   (d)
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_3 e^{6t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

   (e)
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_3 e^{t} \begin{bmatrix} t+1 \\ 2t \\ 3t+1 \end{bmatrix}$$

2. (a)
$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{6t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{-6t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} t - \frac{1}{3}e^{3t} \\ \frac{4}{3}e^{3t} - \frac{1}{3} \end{bmatrix}$$

   (b)
$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} t \\ 2t-3 \end{bmatrix} + \begin{bmatrix} (t^2+3t)e^{3t} \\ 2t^2 e^{3t} \end{bmatrix}$$

   (c)
$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} \sin(2t) \\ -2\cos(2t) \end{bmatrix} + c_2 \begin{bmatrix} \cos(2t) \\ 2\sin(2t) \end{bmatrix} + \begin{bmatrix} -2t\sin(2t) + \cos(2t)\ln(\sec(2t)) \\ 4t\cos(2t) + 2\sin(2t)\ln(\sec(2t)) \end{bmatrix}$$

3. (a)
$$W = \begin{vmatrix} \sin(t) & \cos(t) \\ \cos(t) & -\sin(t) \end{vmatrix} \neq 0 \Leftrightarrow \text{Linearly independent}$$

   (b)
$$W = \begin{vmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{vmatrix} \neq 0 \Leftrightarrow \text{Linearly independent}$$

   (c)
$$W = \begin{vmatrix} e^t & e^t & 2e^t \\ e^t & -e^t & 0 \\ 2e^t & e^t & 3e^t \end{vmatrix} = 0 \Leftrightarrow \text{Linearly dependent}$$

4. (a) Partial answer: Substitute Y by $\vec{\psi}$ in equation (1)

   (b) Partial answer: Substitute Y by $\vec{\phi}$ in equation (1)

   (c) 5

5. (a)

$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix}$$

   (b)

$$\begin{bmatrix} x \\ v \end{bmatrix} = c_1 \begin{bmatrix} \cos\left(\sqrt{\frac{k}{m}}t\right) \\ -\sin\left(\sqrt{\frac{k}{m}}t\right) \end{bmatrix} + c_2 \begin{bmatrix} \sin\left(\sqrt{\frac{k}{m}}t\right) \\ \cos\left(\sqrt{\frac{k}{m}}t\right) \end{bmatrix}$$

   (c) $x = -3$