MATH 644

CHAPTER 2

SECTION 2.1: POLYNOMIALS

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What This Class Is About?

We will do calculus with functions

$$f : \Omega \subseteq \mathbb{C} \to \mathbb{C}.$$

- The image of f is $f(\Omega) := \{w = f(z) \,:\, z \in \Omega\}$.
- Since $f(z) \in \mathbb{C}$, for $z \in \Omega$, there are two functions

  $$u : \Omega \subseteq \mathbb{C} \to \mathbb{R} \quad \text{and} \quad v : \Omega \subseteq \mathbb{C} \to \mathbb{R}$$

  such that

  $$f(z) = u(z) + iv(z).$$

- Sometimes, we use $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$.

Definition

The best well-behaved complex-valued functions are polynomials:

$$p(z) = a_0 + a_1 z + \ldots + a_n z^n,$$

where

- $a_0, a_1, \ldots, a_n \in \mathbb{C}$;
- $z \in \mathbb{C}$, so that $\Omega = \mathbb{C}$;
- $a_n \neq 0$, so that $\deg p := n$.

Theorem 1. A polynomial $p(z)$ is a continuous function.

Note:

- A function $f : \Omega \to \mathbb{C}$ is continuous at z_0 if for any $\varepsilon > 0$, there is a $\delta > 0$ such that if $|z - z_0| < \delta$, then $|f(z) - f(z_0)| < \varepsilon$.
- A function $f : \Omega \to \mathbb{C}$ is continuous on Ω if it is continuous at every $z_0 \in \Omega$.

Linear Polynomials

When the degree of $p(z)$ is 1:

$$p(z) = az + b, \quad a, b \in \mathbb{C} \text{ and } a \neq 0.$$

Some elementary observations:

- $\underline{a = 1.}$

- $\underline{b = 0.}$

Consequences: Rewrite as followed:

$$p(z) = a(z + b/a).$$

- translates first by b/a.
- dilates and rotate by $|a|$ and $\arg a$ respectively.

Monomials

A monomial is

$$p(z) = z^n, \quad n \geq 1.$$

We see that

- $|p(z)| = |z|^n$;
- $\arg p(z) = n \arg z \pmod{2\pi}$.

Corollary 2. Each pie slice

$$S_k := \left\{ z : \left| \arg z - \tfrac{2\pi k}{n} \right| < \pi/n \right\} \quad (k = 0, 1, 2, \ldots, n-1)$$

is mapped to

$$\{z : |z| < r^n\} \backslash (-r^n, 0)$$

by a monomial $z \mapsto z^n$ $(n \geq 1)$.

Proof.

Monomials Translated

We consider

$$p(z) = b(z - z_0)^n$$

where

- $z_0 \in \mathbb{C}$ is fixed;
- $b \in \mathbb{C}$;
- $n \geq 1$.

Picture

Characteristic:

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We first prove the following.

Theorem 3. Any polynomial $p(z) = \sum_{j=0}^{n} a_j z^j$ with $n \geq 1$ can be rewritten as followed:

$$p(z) = \sum_{k=1}^{n} b_k (z - z_0)^k + p(z_0)$$

for $b_1, b_2, \ldots, b_n \in \mathbb{C}$ and $z_0 \in \mathbb{C}$.

Proof.

Corollary 4. If k is the smallest index of all index j such that $b_j \neq 0$ and letting $\zeta := z - z_0$ with $z_0 \in \mathbb{C}$ fixed, then for small ζ, there is a constant C such that

$$\left|p(z_0 + \zeta) - \left(p(z_0) + b_k \zeta^k\right)\right| \leq C|\zeta|^{k+1}.$$

Proof.

Picture of Walking a Dog (WAD)

Explaination of WAD

Corollary 5. We have

$$p(z_0 + \zeta) = p(z_0) + b_k(z - z_0)^k + o((z - z_0)^k), \quad z \to z_0.$$