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Last name: Solutions
First name: —
Section: —

Question:	1	2	3	4	5	6	7	8	Total
Points:	10	6	9	20	10	10	15	20	100
Score:	—	—	—	—	—	—	—	—	—

Instructions: Make sure to write your complete name on your copy. You must answer all the questions below and write your answers directly on the questionnaire. At the end of the 75 minutes, hand out your copy.

No devices such as a smart phone, cell phone, laptop, or tablet can be used during the exam. You are not allowed to use the lecture notes, the textbook, or any other notes. You may use a digital calculator (no graphical calculator or symbolic calculator will be allowed).

You must show ALL your work to have full credit. An answer without justification worth no point.

Good luck!

Pierre-Olivier Parisé

Question 1 (10 pts)

The movement of two pōpoki[1] are tracked. Pōpoki Alika has a velocimeter strapped to her, while the distance travelled by pōpoki Kapono is tracked via a GPS. Both travel in a **straight line** from the same location. Their respective speed and distance travelled are recorded below.

Time in seconds	Speed of Alika in meters/s	Distance traveled by Kapono in meters
0	0	0
2	2	2
4	2	10
6	6	22
8	12	28
10	13	32
12	10	40
14	4	44

(a) (3 points) What is the average acceleration of Alika from time 4 seconds to time 10 seconds?

$$a_{ave} = \frac{v(10) - v(4)}{10-4} = \frac{13-2}{10-4} = \frac{11}{6} \approx 1.833 \ m/s^2$$

(b) (4 points) Estimate the velocity of Kapono at time 4 seconds.

$$v(4) \approx \frac{s(4)-s(2)}{4-2} = \frac{10-2}{2} = 4 \ m/s$$

Better $v(4) \approx \dfrac{s(4)-s(6)}{4-6} = \dfrac{10-22}{-2} = 6 \ m/s.$

Take the average:

$$v(4) \approx \frac{4+6}{2} = 5 \ m/s.$$

(c) (3 points) Who, between Alika and Kapono, was faster at time 4 seconds?

Alika $\to$ $v(4) = 2 \ m/s$

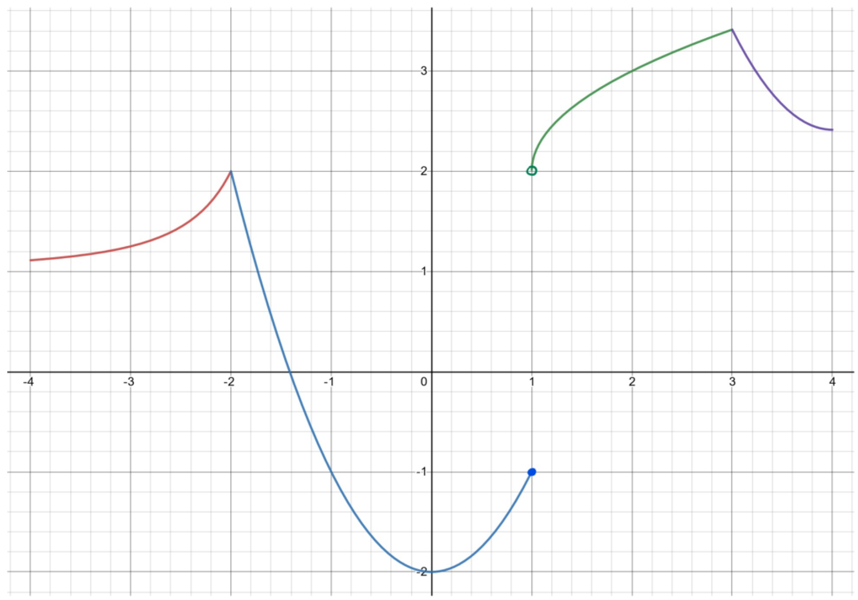
Kapono $\to$ $v(4) \approx 5 \ m/s$

So Kapono was going faster.

[1]This word means 'cat' in hawaiian.

QUESTION 2 **(6 pts)**

Consider the function $f(x)$ with the graph $y = f(x)$ pictured below. The domain of f is $[-4, 4]$.



(a) (1 point) On which interval(s) (if any) is the function decreasing? (no justification needed)

On $[-2, 0]$, $[3, 4]$

(b) (1 point) On which intervals(s) (if any) is the function concave up? (no justification needed)

$[-4, 1]$ & $[3, 4]$

(c) (1 point) Where (if anywhere) is the function not continuous?

At $x = 1$ because $\lim_{x\to 1} f(x)$ $\nexists$.

(d) (1 point) Where (if anywhere) is the function not differentiable?

$x = -2$
$x = 3$ because there is a corner | $x = 1$ because of the discontinuities.

(e) (1 point) What is $\lim_{x\to 1^+} f(x)$ (a reasonable estimate is OK/ 'does not exist' is a possible answer)?

From the graph, $\boxed{\lim_{x\to 1^+} f(x) = 2}$

$\frac{5}{2} \times 10^{-1} = 0.25$

$\frac{5\times 10^{-2}}{2\times 10^{-1}}$

(f) (1 point) What is $\lim_{x\to 3^-} f'(x)$ (a reasonable estimate is OK/ 'does not exists' is a possible answer)?

$$\lim_{x\to 3^-} f'(x) = f'(3) \approx \frac{f(-3) - f(-3.2)}{-0.2} = \frac{1.25 - 1.2}{-0.2}$$

continuous.

$$= \frac{0.05}{-0.2} = \boxed{0.25}$$

Question 3 (9 pts)

You are given that a function $f(x)$ satisfies the following limits:

$$\lim_{x\to -1} f(x) = 4, \quad \lim_{x\to 3^+} f(x) = -12.$$

Use these to compute the following limits, or say if they are $+\infty$ or $-\infty$. Do not use l'Hôpital's rule.

(a) (3 points) $\displaystyle\lim_{x\to -1} \frac{xf(x)+4x}{x^2+2}$.

Quotient rule. $\displaystyle = \frac{\lim_{x\to -1} xf(x)+4x}{\lim_{x\to -1} x^2+2} = \frac{\left(\lim_{x\to -1} x\right)\left(\lim_{x\to -1} f(x)\right)+4\lim_{x\to -1} x}{\lim_{x\to -1} x^2 + \lim_{x\to -1} 2}$

$\displaystyle = \frac{(-1)(4)+4\cdot(-1)}{1+2}$

$\displaystyle = \boxed{-\frac{8}{3}}$

(b) (3 points) $\displaystyle\lim_{x\to 3^+} \frac{(f(x)+6)^2}{x-3}$.

$\lim_{x\to 3^+} x-3 = 0^+ \longrightarrow$ ~~Quotient rule~~ ~~Substitution~~ Rule

$\displaystyle\lim_{x\to 3^+} (f(x)+6)^2 = \left(\lim_{x\to 3^+} f(x) + \lim_{x\to 3^+} 6\right)^2 = 10^2 = 100 \neq 0$ (a number).

So, $\displaystyle\lim_{x\to 3^+} \frac{(f(x)+6)^2}{x-3} = \frac{100}{0^+} = \boxed{+\infty}$

(c) (3 points) $\displaystyle\lim_{x\to -1}(x^4-3f(x))(x^2+5x+3)$.

Product rule. $\displaystyle = \left(\lim_{x\to -1} x^4 - 3f(x)\right)\left(\lim_{x\to -1} x^2+5x+3\right)$

$\displaystyle = \left(\lim_{x\to -1} x^4 - 3\lim_{x\to -1} f(x)\right)\left((-1)^2-5+3\right)$

$= (1-3\cdot 4)(1-5+3)$

$= (-11)(-1)$

$= \boxed{11}$

QUESTION 4 **(20 pts)**

Find the value of the following limit. You can't use Hôpital's rule to compute the value of the limit.

(a) (10 points) $\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x^2}$.

$$\frac{1-\sqrt{x}}{1-x^2} \to \frac{0}{0} \quad \boxed{\because}$$

But,

$$\frac{1-\sqrt{x}}{1-x^2} = \frac{(1-\sqrt{x})(1+\sqrt{x})}{(1-x)(1+\sqrt{x})}$$

$$= \frac{1-x}{(1-x)(1+\sqrt{x})} = \frac{1}{1+\sqrt{x}} \quad \boxed{\because}$$

So, using $\boxed{\because}$, we have

$$\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x^2} = \lim_{x\to 1} \frac{1}{1+\sqrt{x}} = \boxed{\frac{1}{2}}$$

(b) (10 points) $\lim_{x\to 0} x^4 \cos\left(\frac{4}{x}\right)$.

We have $-1 \le \cos(A) \le 1 \quad \overset{A=\frac{4}{x}}{\Rightarrow} \quad -1 \le \cos\left(\frac{4}{x}\right) \le 1$

$\overset{\cdot x^4}{\Rightarrow} \quad -x^4 \le x^4 \cos\left(\frac{4}{x}\right) \le x^4$.

So, by the Squeeze theorem:

$$-\lim_{x\to 0} x^4 \le \lim_{x\to 0} x^4 \cos\left(\frac{4}{x}\right) \le \lim_{x\to 0} x^4$$

$$\Rightarrow \quad 0 \le \lim_{x\to 0} x^4 \cos\left(\frac{4}{x}\right) \le 0$$

$$\Rightarrow \quad \boxed{\lim_{x\to 0} x^4 \cos\left(\frac{4}{x}\right) = 0}$$

Question 5 **(10 pts)**

Using the definition of the derivative, compute the derivative of the function $f(x) = \sqrt{6-x}$.

By definition

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{6-x-h} - \sqrt{6-x}}{h}$$

$$= \lim_{h \to 0} \frac{(6-x-h) - (6-x)}{h\left(\sqrt{6-x-h} + \sqrt{6-x}\right)}$$

$$= \lim_{x \to 0} \frac{-\cancel{h}}{\cancel{h}\left(\sqrt{6-x-h} + \sqrt{6-x}\right)}$$

$$= \lim_{h \to 0} \frac{-1}{\sqrt{6-x-h} + \sqrt{6-x}}$$

$$= \boxed{\frac{-1}{2\sqrt{6-x}}}$$

Question 6 **(10 pts)**

Consider the function $f(x) = x^3 + x + 1/2$. This function must have at least one zero in the interval $(-1, 0)$. Explain why, making explicit which theorem(s), if any, and which assumtions(s) on f, if any, you are using.

We have f is continuous because it's a polynomial.

We have $f(-1) = (-1)^3 - 1 + 1/2 = -2 + 1/2 = -1.5$

$f(0) = 0 + 0 + 1/2 = 1/2$.

So $N = 0$ is in between -1.5 and $1/2$.

By the IVT, there exist a number c between -1 & 0 st

$f(c) = 0$.

QUESTION 7 **(15 pts)**

Compute the derivatives of the following functions.

(a) (5 points) $f(x) = (x^2+1)^2$.

Product rule. $= \frac{d}{dx}(x^2+1)\,(x^2+1) + \frac{d}{dx}(x^2+1)\,(x^2+1)$

$= 2x(x^2+1) + 2x(x^2+1)$

$= \boxed{4x^3 + 4x}$.

(b) (5 points) $f(x) = \frac{2-\sqrt{x}}{2+\sqrt{x}}$.

Quotient Rule. $= \dfrac{\frac{d}{dx}(2-\sqrt{x})(2+\sqrt{x}) - \frac{d}{dx}(2+\sqrt{x})(2-\sqrt{x})}{(2+\sqrt{x})^2}$

$= \dfrac{-\frac{1}{2\sqrt{x}}(2+\sqrt{x}) - \frac{1}{2\sqrt{x}}(2-\sqrt{x})}{(2+\sqrt{x})^2}$

$= \dfrac{-1/\sqrt{x} - 1 - 1/\sqrt{x} + 1}{(2+\sqrt{x})^2} = \boxed{-\dfrac{2}{\sqrt{x}(2+\sqrt{x})^2}}$

(c) (5 points) $f(x) = \sin x + \tan x$.

Sum rules: $= \frac{d}{dx}(\sin x) + \frac{d}{dx}(\tan x)$

$= \boxed{\cos x + \sec^2 x}$

$= \dfrac{\cos^3 x + 1}{\cos^2 x}$ (also a valid answer).

Question 8 **(20 pts)**

Find the equation of the tangent line to the curve $y = 3 + 4x^2 - 2x^3$ at the point $(2, 3)$.

An equation to the tangent line is

$$y - 3 = m(x-2) \quad , \quad m = y'(2).$$

We have

$$\frac{dy}{dx} = 0 + 8x - 6x^2 = 8x - 6x^2$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{x=2} = 16 - 6\cdot 4 = -8.$$

Thus, $m = -8$ and

$$y - 3 = -8(x-2)$$

$$\Rightarrow y = -8x + 16 + 3$$

$$\Rightarrow \boxed{y = -8x + 19}$$