Chapter 3
Applications of Derivatives

3.8 Newton's Method

Roots of polynomials.

- for quadratic polynomial $f(x) = ax^2 + bx + c$, the roots are given by:

$ax^2+bx+c = 0$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

ax^3+bx^2+cx+d or $ax^4+bx^3+cx^2+dx+z$

- There are formulas for cubics and quartics (horribly long...).

- For polynomials of degree greater than 4, there is no general formula!

Niels Henrik Abel

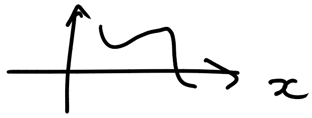
- 1802-1829
- Died from Turberculosis

Evariste Galois

- 1811-1832
- Died in a duel for a mysterious mistress...

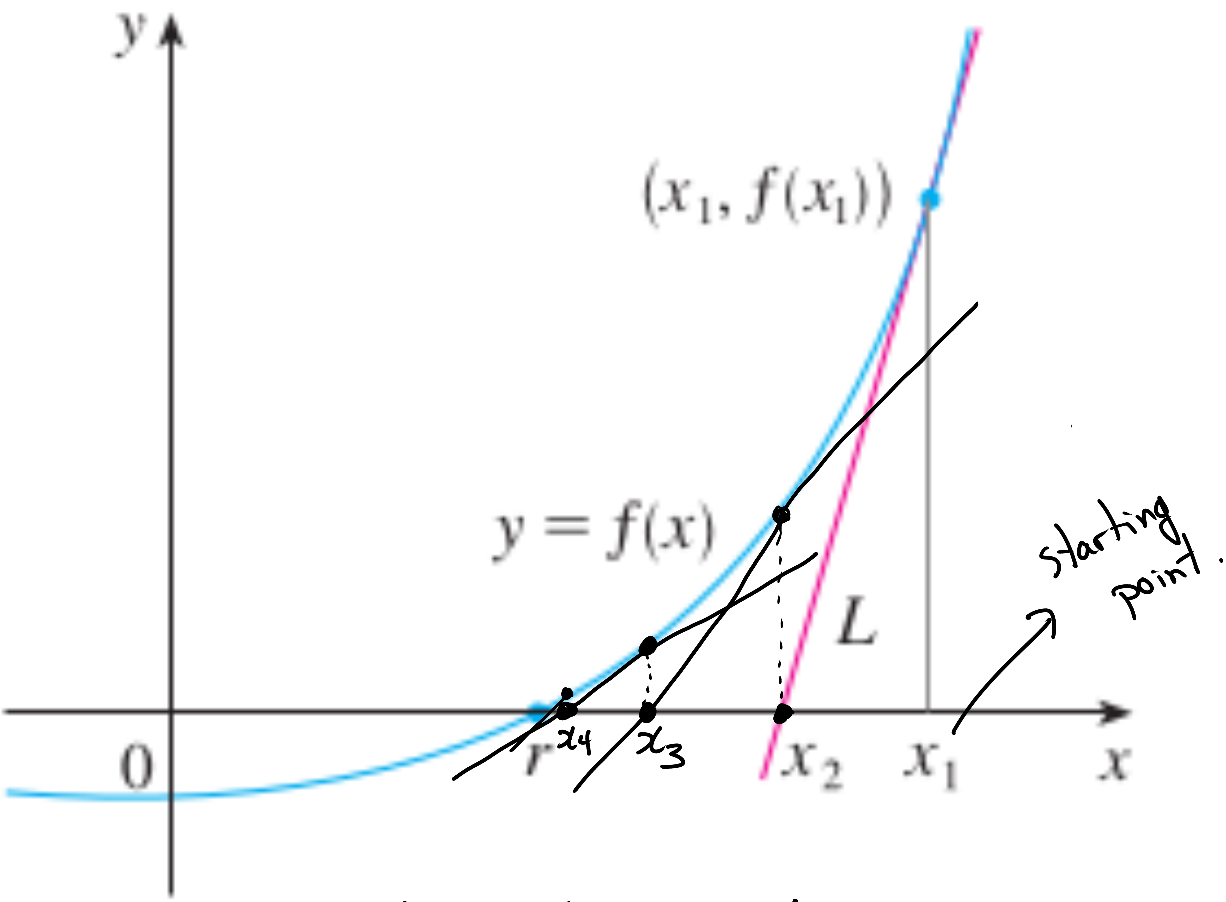
The urgent need of Newton's method!

KEY IDEAS:



- The tangent line approximate well the function.
- Replace the fonction with its tangent line.
- Intersect the tangent line with the x-axis.

Data:



Eq. tangent: $y = f'(x_1)(x - x_1) + f(x_1)$

Intersect: $0 = f'(x_1)(x - x_1) + f(x_1)$

$\Rightarrow x = x_1 - \dfrac{f(x_1)}{f'(x_1)}$ $\quad (f'(x_1) \neq 0)$

$\downarrow$

call it x_2

iterate the process:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

Example. Starting with $x_1 = 2$, find the third approximation x_3 to the root of the equation $\underbrace{\frac{x^3}{2} - 3x}_{f(x)} = 0$.

Step1 $\quad x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$$= 2 - \frac{f(2)}{f'(2)}$$

$f(2) = -2$

$f'(x) = \frac{3}{2}x^2 - 3$

$\to f'(2) = 3$

$$\Rightarrow x_2 = 2 - \frac{(-2)}{3} = \frac{8}{3}$$

Step2. $\quad x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

$$= \frac{8}{3} - \frac{f(8/3)}{f'(8/3)}$$

$f(8/3) = \frac{40}{27}$

$f'(8/3) = \frac{69}{9}$

$$\Rightarrow x_3 = \frac{8}{3} - \frac{40/27}{69/9} = \boxed{\frac{512}{207}}$$

$\approx 2.4734.$

n	xn
1	2
2	2.666666667
3	2.473429952
4	2.44983289
5	2.449489815
6	2.449489743
7	2.449489743
8	2.449489743
9	2.449489743
10	2.449489743
11	2.449489743
12	2.449489743
13	2.449489743
14	2.449489743
15	2.449489743

Newton's Method May Fail

Take a look at the formula in Newton's Method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Where do you think this formula might fail?

- Problem when $f'(x_n) \approx 0$.
- x_{n+1} may be outside of the domain.

Example. Redo the last example with $x_1 = -1.14$.
Desmos: https://www.desmos.com/calculator/nm3bpdg95t

Step 1 $\quad x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 1.41$

Step 2 $\quad x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \approx -164.4512\ldots$

Roots

$$\left(\frac{x^2}{2} - 3\right)x = 0 \iff x = 0 \text{ or } x = \pm\sqrt{6}$$

Example.

Starting at $x_1 = 1$, find the second approximation to the root of $\sqrt{x} = 0$.

Desmos: https://www.desmos.com/calculator/nm3bpdg95t

Step 1 $\quad x_2 = 1 - \frac{f(x_1)}{f'(x_1)} \qquad f'(x) = \frac{1}{2\sqrt{x}}$

$= 1 - \frac{\sqrt{1}}{1/2\sqrt{1}} = 1 - 2 = -1$

Step 2. $\quad x_3 = -1 - \frac{f(-1)}{f'(-1)} = -1 - \frac{\sqrt{-1}}{1/2\sqrt{-1}}$.

$x_3 \nexists$

$MANY^{MANY} APPLICATIONS$!!!

- Finding solutions to general equations such as

$$\cos(x) = x$$

- At the core of many numerical methods in ingeneering.

- Gives rise to wonderful fractal pictures:
  Check out 3blue1brown video
  https://www.youtube.com/watch?v=-RdOwhmqP5s