Chapter 4
Integrals

4.5 The Substitution Rule

Example to start. Find the indefinite integral of $2x\sqrt{1+x^2}$, that is compute

$$\int 2x\sqrt{1+x^2}\,dx.$$

Goal: Find $F(x)$ s.t. $F'(x) = 2x\sqrt{1+x^2}$.

- $\sqrt{x} \longrightarrow \dfrac{x^{3/2}}{3/2}$

- $\dfrac{(1+x^2)^{3/2}}{3/2} \xrightarrow{d/dx} \dfrac{\cancel{3/2}\,(1+x^2)^{1/2}}{\cancel{3/2}} \cdot \dfrac{d}{dx}(1+x^2)$

$$\longrightarrow \sqrt{1+x^2}\cdot 2x$$

$$\Rightarrow \int 2x\sqrt{1+x^2}\,dx = \frac{2}{3}(1+x^2)^{3/2} + C$$

Another example. Compute the indefinite integral

$$\int 2x\sqrt{1+x^2}\,dx.$$

1st: Notice that $\dfrac{d}{dx}(1+x^2) = 2x \Rightarrow d(1+x^2) = 2x\,dx$

2st: Let $u = 1+x^2 \rightarrow \dfrac{du}{dx} = 2x \rightarrow du = 2x\,dx$

3rd: $\displaystyle\int 2x\sqrt{1+x^2}\,dx = \int \underbrace{\sqrt{1+x^2}}_{u}\ \underbrace{2x\,dx}_{du}$

$$= \int \sqrt{u}\,du$$

$$= \frac{u^{3/2}}{3/2} + C$$

$$\Rightarrow I = \frac{2}{3}(1+x^2)^{3/2} + C$$

Substitution Rule. If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f'(g(x))g'(x)\,dx = \int f(u)\,du.$$

Relation between du and dx:

$$u = g(x) \Rightarrow \frac{du}{dx} = g'(x) \Rightarrow du = g'(x)\,dx$$

EXAMPLE 1 Find $\int x^3 \cos(x^4 + 2)\,dx$.

$$u = x^4 + 2 \Rightarrow \frac{du}{dx} = 4x^3 \Rightarrow du = 4x^3\,dx$$

$$\Rightarrow \frac{1}{4}\,du = x^3\,dx$$

$$\int x^3 \cos(x^4+2)\,dx = \int \cos(u)\ \frac{1}{4}\,du$$

$$= \frac{1}{4}\int \cos(u)\,du$$

$$= \frac{1}{4}\sin(u) + C$$

$$= \boxed{\frac{1}{4}\sin(x^4+2) + C}$$

EXAMPLE 2 Evaluate $\int \sqrt{2x+1}\,dx$.

$$\times \int \sqrt{2x+1}\,dx = \int (2x+1)^{1/2}\,dx = \frac{(2x+1)^{3/2}}{3/2} + C$$

$$\left(\frac{(2x+1)^{3/2}}{3/2} + C\right)' = (2x+1)^{1/2} \cdot 2$$

$$u = 2x+1 \longrightarrow \frac{du}{dx} = 2 \longrightarrow \frac{1}{2}\,du = dx$$

$$\int \sqrt{2x+1}\,dx = \int \sqrt{u}\,\frac{1}{2}\,du = \frac{1}{2}\,\frac{u^{3/2}}{3/2} + C = \boxed{\frac{(2x+1)^{3/2}}{3} + C}$$

EXAMPLE 3 Find $\int \frac{x}{\sqrt{1-4x^2}}\,dx$.

$$u = 1-4x^2 \longrightarrow \frac{du}{dx} = -8x \longrightarrow \boxed{-\frac{1}{8}\,du = x\,dx}$$

$$\Rightarrow \int \frac{x}{\sqrt{1-4x^2}}\,dx = \int \frac{1}{\sqrt{u}}\,-\frac{1}{8}\,du$$

$$= -\frac{1}{8}\int u^{-1/2}\,du$$

$$= -\frac{1}{8}\,\frac{u^{1/2}}{1/2} + C$$

$$= \boxed{-\frac{1}{4}\,(1-4x^2)^{1/2} + C}$$

EXAMPLE 5 Find $\int \sqrt{1+x^2}\, x^5\, dx$.

$$u = 1 + x^2 \longrightarrow \frac{du}{dx} = 2x\, dx \longrightarrow \frac{1}{2} du = x\, dx$$

$$\int \sqrt{1+x^2}\, x^5\, dx = \int \sqrt{u}\, x^4\, x\, dx$$

$$= \int \sqrt{u}\, x^4\, \frac{1}{2} du$$

$$= \frac{1}{2} \int \sqrt{u}\, (x^2)^2\, du$$

Notice: $u = 1 + x^2 \implies x^2 = u - 1$

$\implies x^4 = (u-1)^2$

$$\sqrt{1+x^2} \neq \sqrt{1} + \sqrt{x^2}$$

So,

$$\frac{1}{2} \int \sqrt{u}\, (x^2)^2\, dx = \frac{1}{2} \int \sqrt{u}\, (u-1)^2\, du$$

$$= \frac{1}{2} \int u^{1/2} (u^2 - 2u + 1)\, du$$

$$= \frac{1}{2} \int u^{5/2} - 2u^{3/2} + u^{1/2}\, du$$

$\frac{4}{5} u^{5/2} \leftarrow$

$$= \frac{1}{2} \left(\frac{u^{7/2}}{7/2} - 2\frac{u^{5/2}}{5/2} + \frac{u^{3/2}}{3/2} \right) + C$$

$$= \boxed{\frac{(1+x^2)^{7/2}}{7} - \frac{2}{5}(1+x^2)^{5/2} + \frac{(1+x^2)^{3/2}}{3} + C}$$

$u = g(x)$

$$\int_{a=x}^{b=x} \boxed{}\, dx = \int_{g(a)}^{g(b)} \boxed{}\, du$$

EXAMPLE 7 Evaluate $\displaystyle\int_1^{2=x} \frac{dx}{(3-5x)^2}$.

Is it $\dfrac{(3-5x)^{-1}}{-1}$??

$\displaystyle\int_{1}^{2} \cancel{\frac{1}{x^2}\,dx}$

$u = 3-5x \longrightarrow \dfrac{du}{dx} = -5 \longrightarrow -\dfrac{1}{5}\,du = dx$

$$\int_1^2 \frac{dx}{(3-5x)^2} = \int_{3-5\cdot(1)}^{3-5\cdot(2)} \frac{1}{u^2}\; -\frac{1}{5}\,du$$

$$= -\frac{1}{5}\int_{-2}^{-7} u^{-2}\,du$$

$$= \frac{1}{5}\int_{-7}^{-2} u^{-2}\,du$$

$$= \frac{1}{5}\;\frac{u^{-1}}{-1}\Bigg|_{-7}^{-2}$$

$-\dfrac{1}{-2} - (-1)\dfrac{1}{-7}$

$\left(\dfrac{1}{2} - \dfrac{1}{7}\right)$

$\dfrac{7-2}{14} = \dfrac{5}{14}$

$$= \boxed{\frac{1}{14}}$$