A WAY MORE SIMPLER PROOF OF THEOREM 5, SECTION 3.1

Theorem 5 If f is analytic in a region Ω, then

$$\limsup_{z \to \partial\Omega} |f(z)| = \sup_{\Omega} f(z).$$

Notice the assumption "Ω is bounded" has been removed.

Proof. If $f \equiv$ const., then the result is immediate.

Suppose $f \not\equiv$ const.

If $z_n \to \partial\Omega$, then $|f(z_n)| \le \sup_{\Omega} |f(z)|$

$$\Rightarrow \limsup_{n \to \infty} |f(z_n)| \le \sup_{\Omega} |f(z)|.$$

Taking the supremum over all sequences $z_n \to \partial\Omega$:

$$\limsup_{z \to \partial\Omega} |f(z)| \le \sup_{\Omega} |f(z)|.$$

So we have to show the other inequality:

$$\sup_{\Omega} |f(z)| \le \limsup_{z \to \partial\Omega} |f(z)|.$$

By definition, there is a sequence $(z_n) \subseteq \Omega$ s.t. $|f(z_n)| \longrightarrow \sup_{\Omega} |f(z)|$.

Now, if (z_{n_k}) s.t. $z_{n_k} \longrightarrow z_0 \in \Omega$, then

$$|f(z_0)| = \sup_{\Omega} |f(z)| \quad \text{(By continuity)}$$

Therefore, taking a disk $B \subseteq \Omega$ containing z_0

$$\Rightarrow |f(z_0)| = \sup_{B} |f(z)|.$$

By the first maximum principle, $f \equiv$ const in B.

By the identity principle $f \equiv$ const. in Ω #

So, no subsequence of (z_n) converges to some $z_0 \in \Omega$.

This forces $z_n \longrightarrow \partial\Omega$, $n \to \infty$.

By definition of $\limsup\limits_{z \to \partial\Omega}$, we see that

$$\lim_{n\to\infty} |f(z_n)| \leq \limsup_{z \to \partial\Omega} |f(z)|$$

$$\Rightarrow \quad \sup_{\Omega} |f(z)| \leq \limsup_{z \to \partial\Omega} |f(z)| . \qquad \square$$