

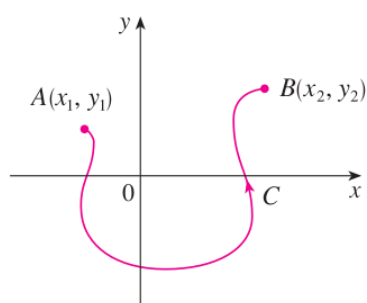
16.3 The Fundamental Theorem for Line Integrals.

2 Theorem Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \leq t \leq b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C . Then

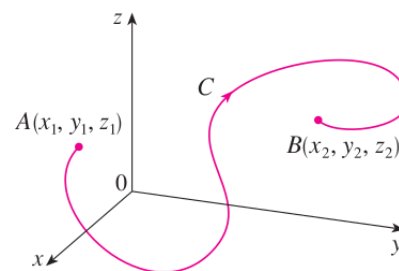
$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

Remarks.

1. In 2D.



2. In 3D.



EXAMPLE 1 Find the work done by the gravitational field

$$\mathbf{F}(\mathbf{x}) = -\frac{mMG}{|\mathbf{x}|^3} \mathbf{x}$$

in moving a particle with mass m from the point $(3, 4, 12)$ to the point $(2, 2, 0)$ along a piecewise-smooth curve C . (See Example 16.1.4.)

Definition.

3 Theorem $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D .

4 Theorem Suppose \mathbf{F} is a vector field that is continuous on an open connected region D . If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D , then \mathbf{F} is a conservative vector field on D ; that is, there exists a function f such that $\nabla f = \mathbf{F}$.

6 Theorem Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ be a vector field on an open simply-connected region D . Suppose that P and Q have continuous first-order partial derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{throughout } D$$

Then \mathbf{F} is conservative. The converse also holds.

EXAMPLE 2 Determine whether or not the vector field

$$\mathbf{F}(x, y) = (x - y) \mathbf{i} + (x - 2) \mathbf{j}$$

is conservative.

EXAMPLE 3 Determine whether or not the vector field

$$\mathbf{F}(x, y) = (3 + 2xy) \mathbf{i} + (x^2 - 3y^2) \mathbf{j}$$

is conservative.

EXAMPLE 4

- (a) If $\mathbf{F}(x, y) = (3 + 2xy) \mathbf{i} + (x^2 - 3y^2) \mathbf{j}$, find a function f such that $\mathbf{F} = \nabla f$.
(b) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve given by

$$\mathbf{r}(t) = e^t \sin t \mathbf{i} + e^t \cos t \mathbf{j} \quad 0 \leq t \leq \pi$$

EXAMPLE 5 If $\mathbf{F}(x, y, z) = y^2 \mathbf{i} + (2xy + e^{3z}) \mathbf{j} + 3ye^{3z} \mathbf{k}$, find a function f such that $\nabla f = \mathbf{F}$.

Conservation of Energy.