Math 644

Chapter 3

Section 3.2: Local Behavior

Contents

Created by: Pierre-Olivier Parisé

Spring 2023

Analytic Functions Are Open Maps

Definition 1. A continuous function $f : \Omega \subset \mathbb{C}$, where Ω is open, is an **open map** if $U \subset \Omega$ is open, then $f(U)$ is open.

Theorem 2. A non-constant analytic function defined on a region is an open map.

Proof.

Note:

- An open map always satisfies the maximum modulus principle.

Analytic Functions Are Locally One-To-One

Definition 3. A function f is **one-to-one** if $f(z) = f(w)$ only when $z = w$.

Theorem 4. If f is analytic at z_0 with $f'(z_0) \neq 0$, then there is an $r > 0$ such that f is one-to-one on $\{z : |z - z_0| < r\}$.

Proof.

Note:

- The function $f(z) = e^z$ gives an example of an analytic function which is locally one-to-one, but globally infinite-to-one! The equation $w = e^z$ has infinitely many solutions.
- Theorem 2 and Theorem 4 show that if f is analytic at z_0 with $f'(z_0) \neq 0$, then f is a homeomorphism of a neighborhood of z_0 onto a neighborhood of $f(z_0)$.