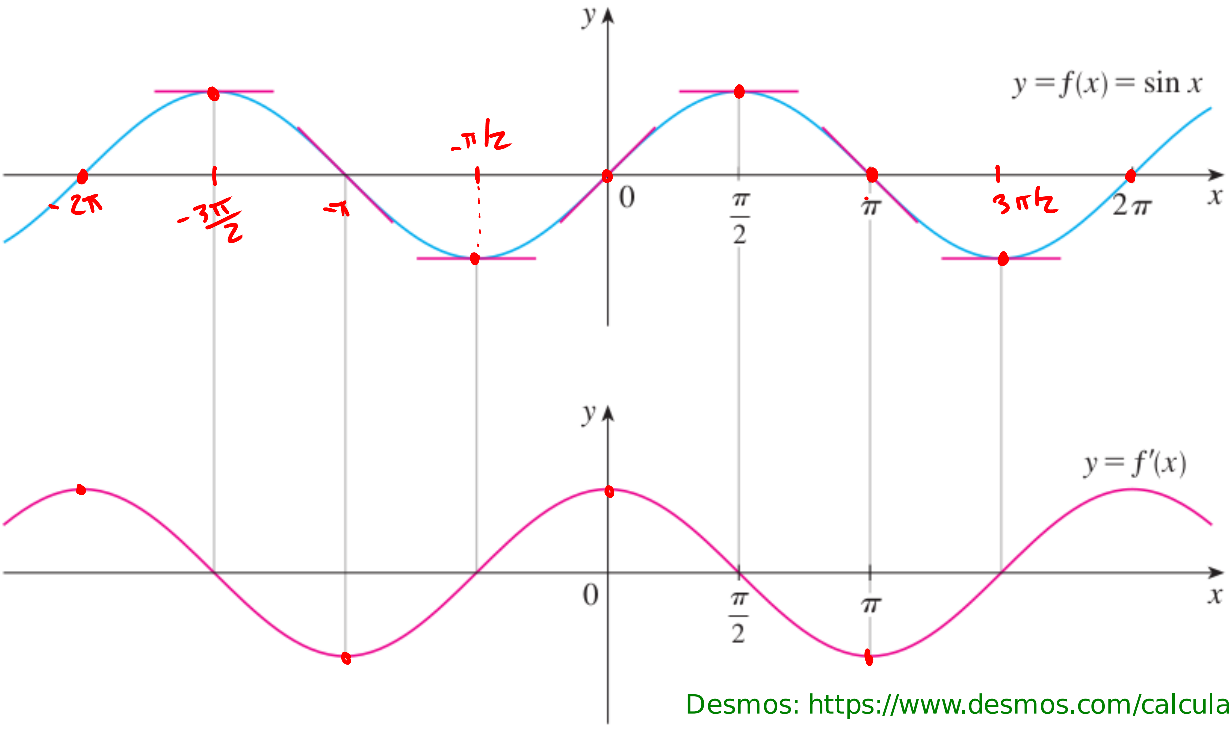
Chapter 2

Derivatives

2.4 Derivatives of Trigonometric Functions

Derivative of the Sine function.



Desmos: https://www.desmos.com/calculator/okfzjutn3q

$$\boxed{\frac{d}{dx}(\sin x) = \cos x}$$

Proof.

$$\lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h} = ?? = \ldots =$$

Trigonometric Functions (reminder).

- $\sec x = \dfrac{1}{\cos x}$
- $\csc x = \dfrac{1}{\sin x}$
- $\tan x = \dfrac{\sin x}{\cos x}$
- $\cot x = \dfrac{1}{\tan x} = \dfrac{\cos x}{\sin x}$

Derivatives of Other Trigonometric Functions.

Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

Proof for the formula for f(x) = tan(x).

$$\tan(x) = \frac{\sin(x)}{\cos(x)} \xrightarrow[\text{rule}]{\text{quotient}} \frac{d}{dx}(\tan x) = \frac{(\sin(x))' \cos x - \sin x (\cos x)'}{(\cos x)^2}$$

$$\to \frac{d}{dx}(\tan x) = \frac{\cos x \cos x - \sin x(-\sin x)}{(\cos x)^2}$$

$$= \frac{(\cos x)^2 + (\sin x)^2}{(\cos x)^2}$$

$\cos^2 x$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \boxed{\sec^2 x}$$

EXAMPLE 2 Differentiate $f(x) = \dfrac{\sec x}{1 + \tan x}$. For what values of x does the graph of f have a horizontal tangent?

① $f'(x) = \dfrac{(\sec x)'(1+\tan x) - \sec x\,(1+\tan x)'}{(1+\tan x)^2}$ Quotient Rule.

$= \dfrac{\sec x \tan x\,(1+\tan x) - \sec x \sec^2 x}{(1+\tan x)^2}$

$\tan^2 x + 1 = \sec^2 x$

$= \dfrac{\sec x\,(\tan x + \tan^2 x - \sec^2 x)}{(1+\tan x)^2}$

$= \dfrac{\sec x\,(\tan x - 1)}{(1+\tan x)^2} = \boxed{\dfrac{\tan x - 1}{\cos x\,(1+\tan x)^2}}$

② Horizontal tangents.

$f'(x) = 0 \iff \tan x - 1 = 0$

$\iff \tan x = 1$

$\iff \boxed{x = n\pi + \pi/4}$, n is integer.

EXAMPLE 6 Calculate $\lim_{x\to 0} x\cot x$.

$$\cot x = \frac{\cos x}{\sin x}$$

$$\lim_{x\to 0} x\cot x = \lim_{x\to 0} \frac{x\cos x}{\sin x} = \lim_{x\to 0} \frac{\cos x}{\frac{\sin x}{x}}$$

$$= \lim_{x\to 0} \frac{\cos x}{\left(\frac{\sin x - \sin 0}{x-0}\right)}$$

Diffrence quotient

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a) \rightarrow a=0 \rightarrow \lim_{x\to 0} \frac{\sin x - \sin 0}{x-0} = \cos(0)$$

$\frac{d}{dx}(\sin x)$

So,

$$\lim_{x\to 0} x\cot x = \frac{\lim_{x\to 0}\cos x}{\underbrace{\lim_{x\to 0}\frac{\sin x - \sin 0}{x-0}}_{\cos 0}} = \frac{\cos(0)}{\cos(0)} = \frac{1}{1} = \boxed{1}$$