Chapter 3: Applications of differentiation
Week 7

Pierre-Olivier Parisé
Calculus I (MATH-241 01/02)

University of Hawai'i
Fall 2021

Upcoming this week

1. 3.1 Maximum and Minimum Values
2. 3.2 The Mean Value Theorem
3. 3.3 How Derivatives Affect the Shape of a Graph

Some of the most important applications of differential calculus are optimization problems, in which we are required to find the optimal (best) way of doing something.

Example 1

Examples of optimization problems are

- What is the shape of a can minimizing the manufacturing costs?
- What is the maximum acceleration of a space shuttle?

All of these problems reduced to finding maximum or minimum to functions.

Definition 2

If c is a real number in the domain of f, then $f(c)$ is a

- absolute maximum of f if $f(x) \leq f(c)$ for any x in dom f.
- absolute minimum of f if $f(x) \geq f(c)$ for any x in dom f.
- local maximum of f if $f(x) \leq f(c)$ when x is near c.
- local minimum of f if $f(x) \geq f(c)$ when x is near c.

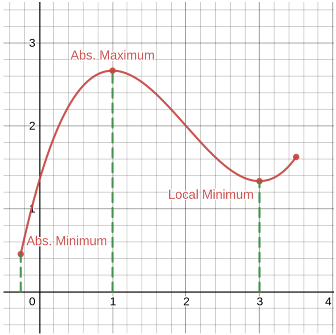


Figure: Illustration of Maxima and Minima

Example 3

Find the local and absolute maxima and minima of the functions

- $f(x) = \cos(x)$.
- $f(x) = x^2$.
- $f(x) = 3x^4 - 16x^3 + 18x^2$

using a graphical tool (like Desmos). Go to Desmos

Remark:

- We also use the terms **global maximum** or **global minimum** to refer to a absolute maximum or absolute minimum.
- Generally, an absolute maximum and absolute minimum are called **extreme values**.

Example 4

Does the function

$$f(x) = \begin{cases} 1/x & 0 < x \leq 1 \\ 0 & x = 0 \end{cases}$$

have a maximum?

Which conditions on the function f will garantee that a maximum and a minimum exist?

Theorem 5 (Extreme Value)

If f is continuous on the closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d of $[a, b]$.

Consider the function

$$f(x) = \cos x.$$

What is happening with the derivative of f? Go to Desmos

Theorem 6 (Fermat's Theorem)

If f has a local maximum or a local minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Example 7

Let $f(x) = x^3$. What is $f'(0)$? Is $f(0)$ a local maximum or local minimum?

Example 8

Let $f(x) = |x|$. Is $f(0)$ a local maximum, global maximum, local minimum, or global minimum?

WARNING: We have to be careful with Fermat's Theorem. It doesn't tell us that every maxima and minima are found by solving the equation

$$f'(x) = 0.$$

Some of them may be found when the derivative doesn't exist.

Definition 9

A critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Example 10

Find the critical numbers of $f(x) = x^{3/5}(4 - x)$.

Here is a systematic method to find extrama values of a function on a closed interval $[a, b]$.

The Closed Interval Method

To find the absolute maximum and absolute minimum of a continuous function f on a closed interval $[a, b]$:

- Find the values of f at the critical numbers of f in (a, b).
- Find the values of f at the endpoints of the interval.
- The largest of the values from the two first steps is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example 11

Find the absolute maximum and minimum values of the function $f(x) = x^3 - 3x^2 + 1$ on the interval $[-1/2, 4]$.

Exercises: 1-6, 9-10, 29-38, 40, 42-46, 48-56, 64, 70, Applied project on Rainbow (if you have time).

We try to find a solution to the equation $x^3 + x - 1 = 0$. How can we be sure that such a solution exist?

Intermediate Value Theorem

If f is a continuous function on an interval $[a, b]$. If $f(a)$ and $f(b)$ have different sign, then there is a number $c \in (a, b)$ such that $f(c) = 0$.

Remark: The number c is not known explicitly. We just know that it exists[1]!

Example 12

Show that the equation $x^3 + x - 1 = 0$ has at least one root.

[1] That's the fantasy of the mathematicians.

Now, we got at least one root to our equation $x^3 + x - 1 = 0$. How can we be sure that there is no other one?

Rolle's Theorem

Let f be a function that satisfies the following three hypotheses:

- f is continuous on the closed interval $[a, b]$.
- f is differentiable on the open interval (a, b).
- $f(a) = f(b)$.

Then there is a number c in (a, b) such that $f'(c) = 0$.

Example 13

Show that the equation $x^3 + x - 1 = 0$ has exactly one root.

Example 14

Consider $f(x) = x^2$.

- Find the slope of the secant line passing through $Q = (0, 0)$ and $P = (2, 4)$.
- Can you find a tangent line to $y = x^2$ with the same slope?

The Mean Value Theorem

Let f be a function that satisfies the following hypotheses:

- f is continuous on the closed interval $[a, b]$.
- f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) $(a < c < b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently, $f(b) - f(a) = f'(c)(b - a)$. Mean-Value Theorem

Remark: The quantity $\frac{f(b)-f(a)}{b-a}$ can be interpretated as the **average** of the function on the interval $[a, b]$. The Mean Value Theorem tells us that this average can be attained by some value $f'(c)$ where $c \in (a, b)$.

Example 15

Find the numbers $c \in [0, 2]$ such that the average of the function $f(x) = x^3 - x$ on the interval $[0, 2]$ is attained by $f'(c)$.

Important consequences of the Mean Value Theorem are:

- If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).
- If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b); that is, $f(x) = g(x) + c$ where c is a constant. [This will be important when we start to integrate!]

Exercises: 12-18, 19, 20, 29, 32, 34.

Definition 16 (Reminder)

A function f is

- increasing if for any $x_1 < x_2$, then $f(x_1) < f(x_2)$.
- decreasing if for any $x_1 < x_2$, then $f(x_1) > f(x_2)$.

What does f' say about f?

Example 17

Consider $f(x) = x^2$. Where is f increasing? Where is f decreasing? Go to Desmos!

Increasing/Decreasing Test (a.k.a I/D Test)

- If $f'(x) > 0$ on an interval, then f is increasing on that interval.
- If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

Example 18

If $f(x) = x^3 - x$, find where it is increasing and where it is decreasing. Go to Desmos

The First Derivative Test

Suppose c is a critical number of a continuous function f.

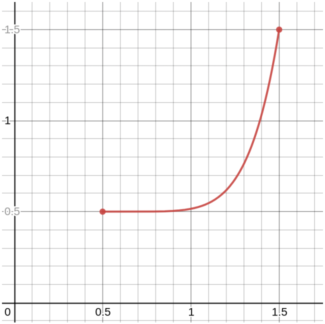
- If f' changes from positive to negative at c, then f has a local maximum at c.
- If f' changes from negative to postive at c, then f has a local minimum at c.
- If f' is positive on the left and right of c, or negative on the left and right of c, then f has no local maximum or minimum at c.

Example 19

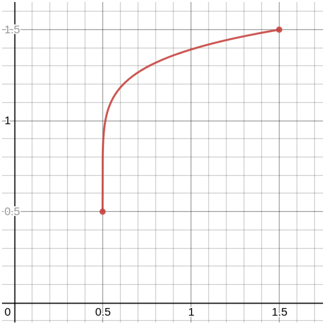
Let $f(x) = x^4 - 2x^3$. Find the local maximum and minimum values of f.

Go to Desmos

Look carefully at those two images:



(a) Concave upward



(b) Concave downward

Definition 20

- If the graph of f lies above all of its tangents on an interval I, then it is called <u>concave upward</u> on I.
- If the graph of f lies below all of its tangents on I, it is called <u>concave downward</u>. Go to Desmos

Concavity Test

- If $f''(x) > 0$ for all x in an interval I, then the graph of f is concave upward on I.
- If $f''(x) < 0$ for all x in an interval I, then the graph of f is concave downward on I.

Remark: So a point where $f''(x) = 0$ is an inflection point.

Example 21

Sketch a possible graph of a function f that satisfies the following conditions:

- $f(0) = 0$, $f(2) = 3$, $f(4) = 6$, and $f'(2) = 0$.
- $f'(x) > 0$ for $0 < x < 2$ and $f'(x) > 0$ for $2 < x < 4$.
- $f''(x) < 0$ for $x < 2$ and $f''(x) > 0$ for $x > 2$.

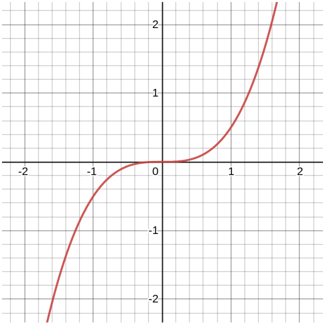


Figure: The graph of $f(x) = x^3$

Definition 22

A point P on a curve $y = f(x)$ is called an <u>inflection point</u> if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward.

There is a second useful test to detect local maxima and local minima.

The Second derivative Test

Suppose that f'' is continuous near c.

- If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
- If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

Warning! The Second Derivative Test is inconclusive when $f''(c) = 0$. This test also fails when $f''(c)$ does not exist. In such a case, the First Derivative Test must be used.

Example 23

Let $f(x) = x^4 - 4x$.

- Find the region where the function is concave upward, concave downward.
- Find the inflection points and the local maxima/minima.
- Use this information to sketch the curve

Exercises: 1-4, 9-10, 13-17, 20-22, 24-27, 33-45, 60.