Chapter 1

Functions and Limits

1.5 The Limit of a Function

Intuitive definition of a limit.

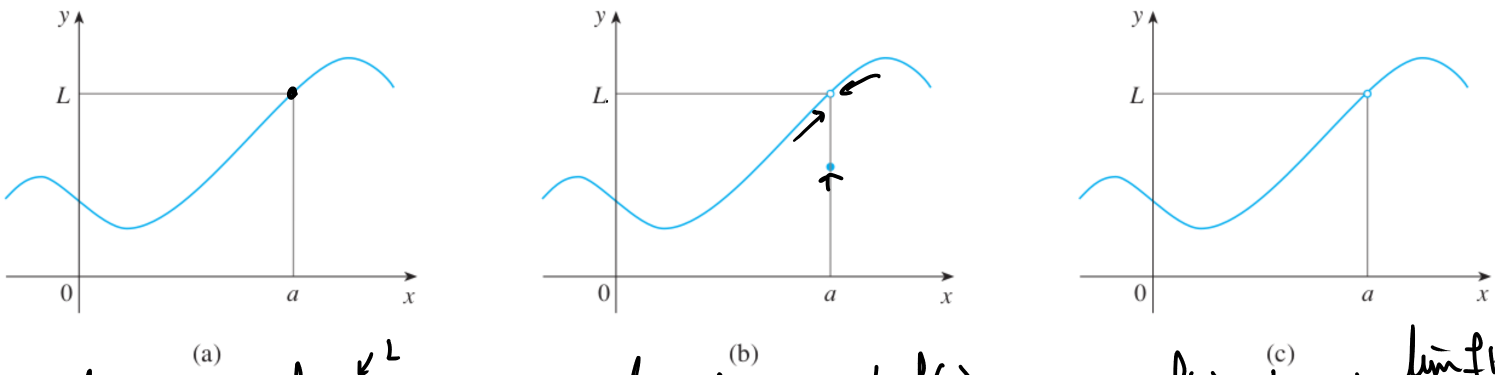
1 Intuitive Definition of a Limit Suppose $f(x)$ is defined when x is near the number a. (This means that f is defined on some open interval that contains a, except possibly at a itself.) Then we write

$$\lim_{x \to a} f(x) = L$$

and say "the limit of $f(x)$, as x approaches a, equals L"

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by restricting x to be sufficiently close to a (on either side of a) but not equal to a.

Three cases:



$\lim_{x \to a} f(x) = f(a) \swarrow L$ $\qquad$ $\lim_{x \to a} f(x) = L \neq f(a)$ $\qquad$ $f(a) \nexists$ but $\lim_{x \to a} f(x) = L$

EXAMPLE 1 Guess the value of $\lim_{x \to 1} \dfrac{x-1}{x^2-1}$.

$f(x) = \dfrac{x-1}{x^2-1}$ $\to$ f is not defined at $x=1$
but f is defined around $x=1$

x	$f(x)$
0.8	0.555556
0.9	0.52632
0.99	0.50251
0.999	0.50025

$\downarrow$ 0.5

x	$f(x)$
1.2	0.45454
1.1	0.47619
1.01	0.49751
1.001	0.49975

$\downarrow$ 0.5

$$\Rightarrow \lim_{x \to 1} \frac{x-1}{x^2-1} = 0.5 .$$

$$\frac{x-1}{x^2-1} = \frac{\cancel{x-1}}{(x+1)\cancel{(x-1)}} = \frac{1}{x+1} \quad (x \neq 1).$$

$\to (x \neq -1)$

EXAMPLE 3 Guess the value of $\lim_{x \to 0} \frac{\sin x}{x}$.

EXAMPLE 4 Investigate $\lim_{x \to 0} \sin \frac{\pi}{x}$.

One-sided Limits.

EXAMPLE 6 The Heaviside function H is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geqslant 0 \end{cases}$$

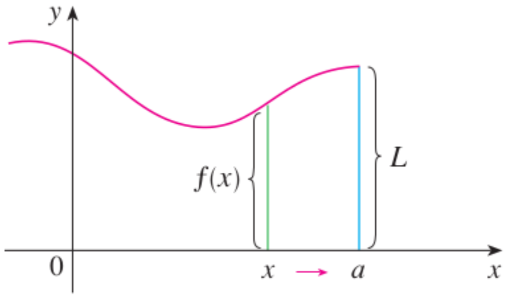
What is the limit when t approached 0 from the right and when t approaches 0 from the left.

Left-hand limits.

2 Definition of One-Sided Limits We write

$$\lim_{x \to a^-} f(x) = L$$

and say the **left-hand limit of $f(x)$ as x approaches a** [or the **limit of $f(x)$ as x approaches a from the left**] is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a with x *less than* a.



(a) $\lim_{x \to a^-} f(x) = L$

Right-hand limits.

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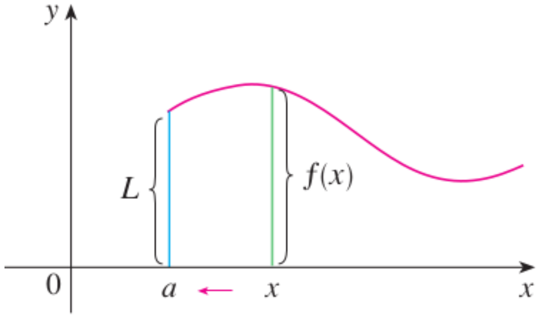


(b) $\lim_{x \to a^+} f(x) = L$

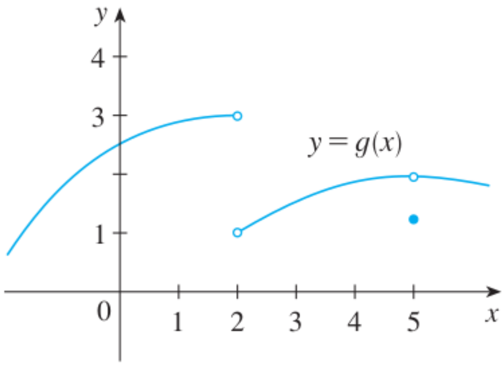
Fundamental Property:

3 $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$

EXAMPLE 7 The graph of a function g is shown in Figure 10. Use it to state the values (if they exist) of the following:

(a) $\lim_{x \to 2^-} g(x)$ (b) $\lim_{x \to 2^+} g(x)$ (c) $\lim_{x \to 2} g(x)$

(d) $\lim_{x \to 5^-} g(x)$ (e) $\lim_{x \to 5^+} g(x)$ (f) $\lim_{x \to 5} g(x)$



Infinite limits.

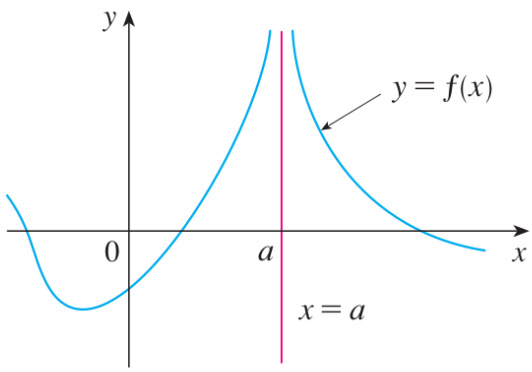
EXAMPLE 8 Find $\lim_{x \to 0} \frac{1}{x^2}$ if it exists.

Positive infinity.

4 Intuitive Definition of an Infinite Limit Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a, but not equal to a.

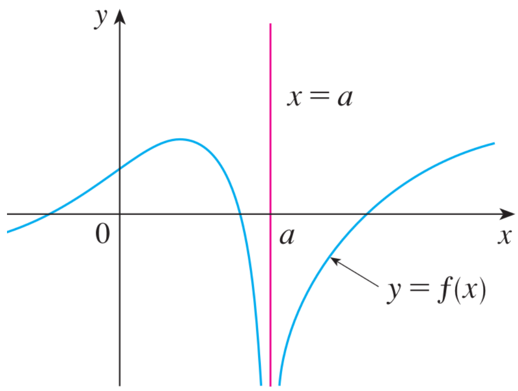


Negative Infinity

5 Definition Let f be a function defined on both sides of a, except possibly at a itself. Then

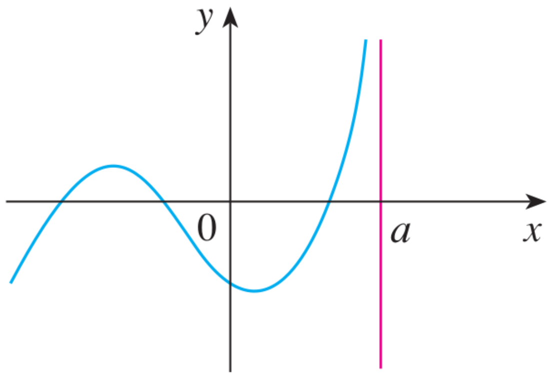
$$\lim_{x \to a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a, but not equal to a.

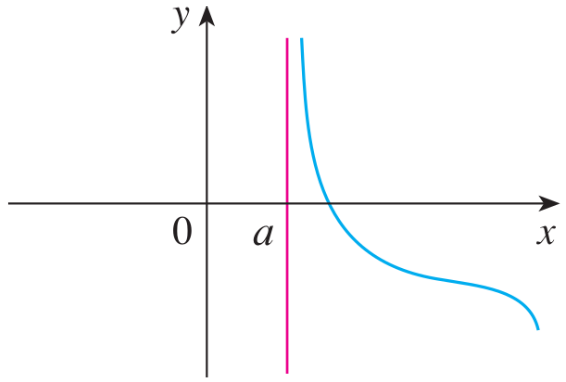


Other types of infinite limits.

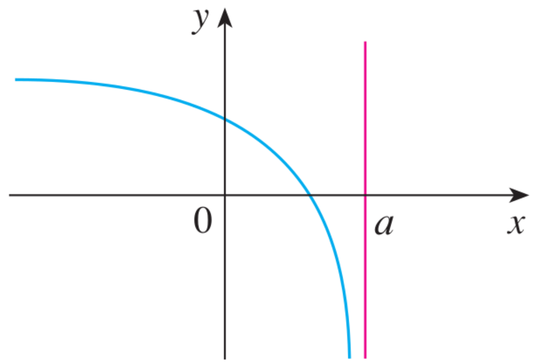
a)



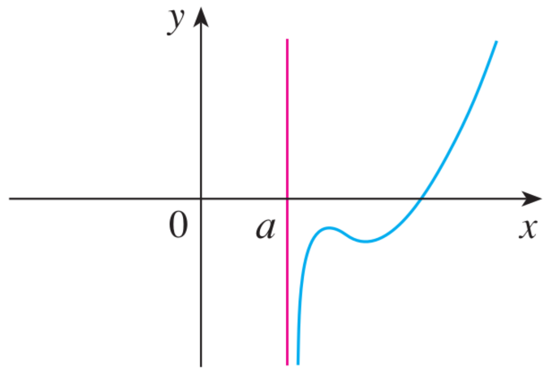
b)



c)



d)



EXAMPLE 9 Find $\lim_{x \to 3^+} \dfrac{2x}{x - 3}$ and $\lim_{x \to 3^-} \dfrac{2x}{x - 3}$.

6 Definition The vertical line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x \to a} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

EXAMPLE 10 Find the vertical asymptotes of $f(x) = \tan x$.