Math 644

Chapter 6

Section 6.3: Riemann Mapping Theorem

Contents

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Theorem 1. Suppose $\Omega \subset \mathbb{C}$ is simply-connected and $\Omega \neq \mathbb{C}$. Then there exists a one-to-one map f of Ω onto $\mathbb{D}$. If $z_0 \in \Omega$, then there is a unique such map with $f(z_0) = 0$ and $f'(z_0) > 0$.

Idea of the proof.

real & positive.

1. Define a family

$$\mathcal{F} = \{f \,:\, f \text{ is one-to-one, analytic, } |f| < 1 \text{ on } \Omega,\ f(z_0) = 0,\ f'(z_0) > 0\}.$$

2. Show $\mathcal{F}$ is normal on Ω.
3. Extract a subsequence $(f_n) \subset \mathcal{F}$ which converges to some f.
4. Show that f has the desire properties.

Lemma 2. The family $\mathcal{F}$ is non-empty and normal in Ω.

Proof.

Non-empty: Let $w_0 \notin \Omega$ ($\Omega \neq \mathbb{C}$).

Then $f(z) = z - w_0$ is analytic & $\neq 0$ in Ω.

therefore, $\exists\, \varphi : \Omega \to \mathbb{C}$ analytic s.t. $\varphi^2 = f$.

1) $\varphi(z) = \varphi(w) \Rightarrow (\varphi(z))^2 = (\varphi(w))^2$
$\Rightarrow f(z) = f(w)$
$\Rightarrow z = w$. $\to$ φ is one-to-one.

2) $\varphi(z) = -\varphi(w) \Rightarrow (\varphi(z))^2 = (\varphi(w))^2 \Rightarrow z = w$

So, $z \neq w$ then $\varphi(z) \neq -\varphi(w)$

3) φ is an open map, so $\exists\, B(a,r) \subseteq \varphi(\Omega)$.

Notice: $\varphi(w) \in B(a,r) \Leftrightarrow -\varphi(w) \in B(-a,r)$

Therefore, from 2), $\varphi(\Omega)$ avoid $B(-a,r)$.

4) Set $\psi := \dfrac{r}{\varphi + a}$, well defined on Ω because of 3). Then ψ is one-to-one, $|\psi(z)| < 1$ because $|\varphi(z) + a| > r$, for any $z \in \Omega$.

5) Use an automorphism $\varphi_\alpha : \mathbb{D} \to \mathbb{D}$ so that

$$\varphi_\alpha(\psi(z_0)) = 0 \quad \& \quad \frac{d}{dz}\Big(\varphi_\alpha(\psi(z_0))\Big) > 0 .$$

<u>Conclusion:</u> $g(z) = \varphi_\alpha \circ \psi \in \mathcal{F}$.

<u>Normality:</u>

$\mathcal{F}$ locally bounded $\Rightarrow$ $\mathcal{F}$ is normal. $\square$

Theorem 3. [Hurwitz] Suppose $(g_n)_{n=1}^{\infty}$ is a sequence of analytic functions on a region Ω and suppose $g_n(z) \neq 0$ for all $z \in \Omega$ and all n. If g_n converges uniformly to g on compact subsets of Ω, then

- either g is identically zero in Ω or;
- $g(z) \neq 0$ for all $z \in \Omega$.

Proof.

Weierstrass' Theorem $\Rightarrow$ g is analytic on Ω.

Suppose $g \not\equiv 0$, then g has isolated zeros in Ω.

Let $\overline{\Delta} \subseteq \Omega$ be a disk s.t. $g(z) \neq 0$, $\forall z \in \partial\Delta$.

So, $g_n \longrightarrow g$ uniformly on $\partial\Delta$.

Since $|g|$ is continuous on the compact set $\partial\Delta$

$$\min_{z \in \partial\Delta} |g(z)| = |g(z_0)| > 0 \quad (\text{some } z_0 \in \partial\Delta)$$

Let $N \in \mathbb{N}$ be s.t.:

$$\max_{z \in \partial\Delta} |g_N(z) - g(z)| < \frac{|g(z_0)|}{2}$$

then, for any $z \in \partial\Delta$, we have

$$|g_N(z) - g(z)| < \frac{|g(z_0)|}{2} < |g(z_0)| \leq |g(z)|$$

By Rouché's theorem $\Rightarrow$ g_N & g have same # of zeros in Δ, that is none. $\square$

Corollary 4. If $(g_n)_{n=1}^{\infty}$ is a sequence of one-to-one and analytic functions on a region Ω, and if g_n converges to g uniformly on compact subsets of Ω, then

- either g is one-to-one on Ω or;
- g is constant in Ω.

Apply Hurwitz to $g_n - g_n(w)$ / $g - g(w)$ on $\Omega \setminus \{w\}$, $w \in \Omega$.

From Lemma 2, $\mathcal{F} \neq \emptyset$ & $\mathcal{F}$ is normal.

Let $M = \sup\{ f'(z_0) : f \in \mathcal{F} \} > 0$.

Let $(f_n) \subseteq \mathcal{F}$ s.t. $f_n'(z_0) \xrightarrow[n\to\infty]{} M$.

Replacing (f_n) by one of its subsequence (normality) we may assume that $f_n \to f$ locally uniformly (some f).

By Weierstrass' theorem, f is analytic and

$$f_n' \to f' \text{ locally uniformly.}$$

This implies that $f'(z_0) = M \neq \infty$.

Also, by Hurwitz, f is one-to-one.

Also, $\underbrace{\lim_{n\to\infty} f_n(z_0)}_{= 0} = f(z_0) = 0$.

<u>Conclusion 1</u>: $f \in \mathcal{F}$.

We now have to show that $f(\Omega) = \mathbb{D}$.

Suppose $\exists \zeta \in \mathbb{D}$ s.t. $f(z) \neq \zeta$, $\forall z \in \Omega$.

Let $g_1(z) := \dfrac{f(z)-3}{1-\bar{3}f(z)} =: T_1 \circ f(z) \quad (z \in \Omega)$.

then $g_1(z) \neq 0 \quad \forall z \in \Omega$ & Ω is simply-connected

$\Rightarrow \exists\, g_2 : \Omega \to \mathbb{D}$ s.t. $g_2^2 = g_1$.

Notice that g_2 is also one-to-one.

Set

$$g(z) := \frac{g_2(z) - g_2(z_0)}{1 - \overline{g_2(z_0)}\, g_2(z)} =: T_2 \circ g_2(z) \quad (z \in \Omega).$$

then, by construction, g is one-to-one & $g(z_0) = 0$.

If $\lambda = \dfrac{|g'(z_0)|}{g'(z_0)}$, then $\lambda g \in \mathcal{F}$.

Set $\varphi := T_1^{-1} \circ S \circ T_2^{-1}$, where $S(z) = z^2$.

Since T_1^{-1} & T_2^{-1} are automorphisms of $\mathbb{D}$,

the map φ is a 2-to-1 map of $\mathbb{D}$ onto $\mathbb{D}$

and

$$\begin{aligned} \varphi(0) &= T_1^{-1} \circ S \circ T_2^{-1}(0) \\ &= T_1^{-1} \circ S(g_2(z_0)) \\ &= T_1^{-1}(g_1(z_0)) = f(z_0) = 0 \end{aligned}$$

$\Rightarrow \varphi(0) = 0$.

Similar calculations show that $f(z) = \varphi \circ g(z)$.

By Schwarz's Lemma, $|\varphi'(0)| < 1$ (otherwise, if $|\varphi'(0)| = 1$, then $\varphi(z) = \lambda z$, some $|\lambda| = 1$, $\divideontimes$)

So,

$$f'(z_0) = |f'(z_0)| = |\varphi'(g(z_0))| \cdot |g'(z_0)|$$
$$= |\varphi'(0)| \cdot |g'(z_0)|$$
$$< |g'(z_0)| = (\lambda g)'(z_0)$$

this contradicts the maximality of $f'(z_0)$

$$\Rightarrow \quad f(\Omega) = \mathbb{D}.$$

<u>Uniquess</u>

$\exists f, g \in \mathcal{F}$ with desire properties.

then, $h = f \circ g^{-1}$

s.t. $h(0) = 0$

$$\Rightarrow \quad h(z) = \lambda z.$$

But, $\lambda = 1$ because $h'(0) = \frac{f'(z_0)}{g'(z_0)} \in (0, \infty)$.

$$\Rightarrow \quad f = g.$$

$\square$