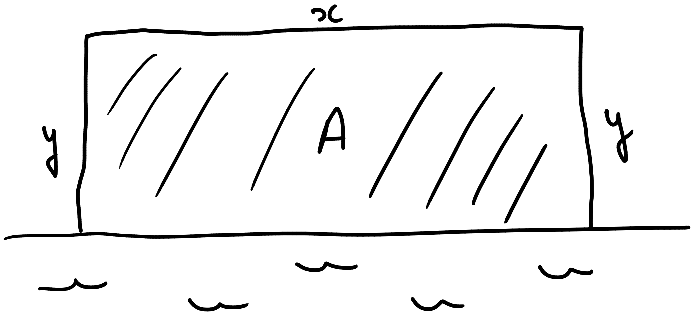
Chapter 3
Applications of Derivatives

3.7 Optimization Problems

EXAMPLE 1 A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

① Sketch



② Notation

x: width of the field (ft)
y: heigth of the field (ft)
A: area of the field. (ft^2)

③ Rule \ Equation.

$$A = xy$$

④ Elimination of one variable.

→ Need only 3 sides.
→ total fencing = 2400 ft

$$2y + x = 2400$$
$$\Rightarrow \quad x = 2400 - 2y$$

So, $A = (2400 - 2y)\, y = 2400y - 2y^2$

⑤ Optimize

$$A' = 2400 - 4y = 0 \quad \Rightarrow \quad 4y = 2400$$
$$\Rightarrow \quad y = 600$$

2nd test: $A''(y) = -4 < 0$ → abs. max at $y = 600$.

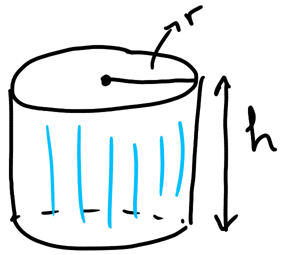
Answer:

$$\boxed{\begin{aligned} x &= 2400 - 2 \cdot 600 = 1200 \text{ ft} \\ y &= 600 \text{ ft} \\ A &= 720\,000 \text{ ft}^2 \end{aligned}}$$

Recall: c critical number
(a) $f''(x) < 0$ (resp. $f''(x) > 0$) for all x, then $f(c)$ is abs. max (resp min).

EXAMPLE 2 A cylindrical can is to be made to hold $\boxed{1\text{ L}}$ (1000 cm^3) of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

① Sketch:



② Notations

r: radius (cm).
h: heigth (cm).
V: volume (cm^3).
A: surface area (cm^2).

Goal: minimize A.

③ Equation

$$A = 2 \cdot A(\bigcirc) + 1 \times A\left(h\, \overset{2\pi r}{\square}\right)$$

$$= 2\pi r^2 + 2\pi r h$$

④ Eliminate one variable.

$$V = 1000 \Rightarrow \pi r^2 h = 1000$$

$$\Rightarrow h = \frac{1000}{\pi r^2}.$$

So, $A = 2\pi r^2 + \dfrac{2 \cdot 1000}{r} = 2\pi r^2 + \dfrac{2000}{r}$, $r > 0$

⑤ Optimize.

$$A'(r) = 4\pi r - \frac{2000}{r^2} = 0$$

$$\Leftrightarrow 4\pi r = \frac{2000}{r^2} \Leftrightarrow 4\pi r^3 = 2000$$

$$\Leftrightarrow r^3 = \frac{500}{\pi}$$

$$\Leftrightarrow r = \sqrt[3]{\frac{500}{\pi}}.$$

$A''(r) = 4\pi + \dfrac{4000}{r^3} > 0 \Rightarrow r = \sqrt[3]{\dfrac{500}{\pi}}$ is an abs. min.

$$h = \frac{1000}{\pi r^2} = \frac{1000}{\pi \left(\frac{500}{\pi}\right)^{2/3}} = \frac{1000}{\pi^{1/3}\, 500^{2/3}}$$

(6) <u>Answer</u>

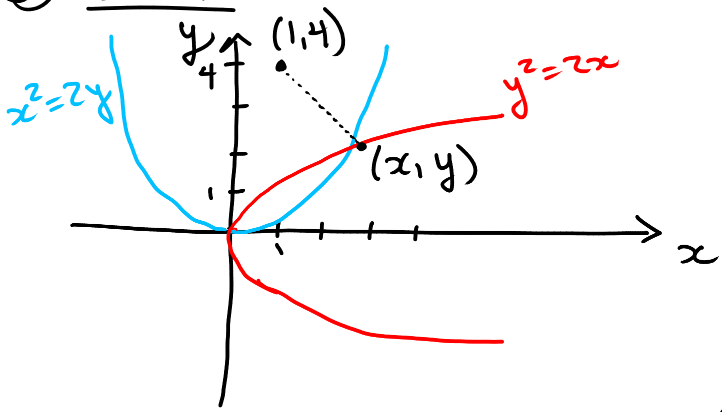
$$\boxed{r = \sqrt[3]{\frac{500}{\pi}}\ \text{cm} \quad \& \quad h = \frac{1000}{\pi^{1/3}\, 500^{2/3}}\ \text{cm}}$$

$\approx 5.419\,\text{cm}$ $\approx 10.839\,\text{cm}.$

EXAMPLE 3 Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$.

$x^2 = 2y$

① Sketch.



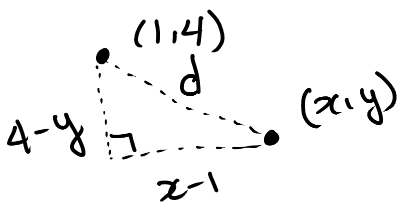
② Notations.

(x,y): point on the parabola.

d: distance between (x,y) & $(1,4)$.

Goal: Minimize d.

③ Equation



$$d = \sqrt{(x-1)^2 + (y-4)^2}$$

④ Eliminate a variable.

We know that $y^2 = 2x \Rightarrow x = \frac{y^2}{2}$

$$\Rightarrow \quad d(y) = \sqrt{\left(\frac{y^2}{2} - 1\right)^2 + (y-4)^2}$$

Trick: $D = d^2 = \left(\frac{y^2}{2} - 1\right)^2 + (y-4)^2$.

⑤ Optimize.

$$D' = 2\left(\frac{y^2}{2} - 1\right) \cdot y + 2(y-4)$$

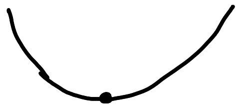
$$= y^3 - 2y + 2y - 8 = y^3 - 8$$

So $D' = 0 \iff y^3 = 8 \iff y = \sqrt[3]{8} = 2$

Also, $D'' = 3y^2$ (can be zero).

1st derivative test: $y < 2 \Rightarrow y^3 < 8 \Rightarrow y^3 - 8 < 0$

$y > 2 \Rightarrow y^3 > 8 \Rightarrow y^3 - 8 > 0$



So, decreasing when $y < 2$

& increasing $y > 2$

$\Rightarrow$ $y = 2$ is a abs. min.

<u>Answer</u>

$$x = \frac{y^2}{2} = 2$$

$$y = 2$$

$$d = \sqrt{5}$$

EXAMPLE 4 A man launches his boat from point A on a bank of a straight river, 3 km wide, and wants to reach point B, 8 km downstream on the opposite bank, as quickly as possible (see Figure 7). He could row his boat directly across the river to point C and then run to B, or he could row directly to B, or he could row to some point D between C and B and then run to B. If he can row 6 km/h and run 8 km/h, where should he land to reach B as soon as possible? (We assume that the speed of the water is negligible compared with the speed at which the man rows.)