

M444 – Complex Analysis

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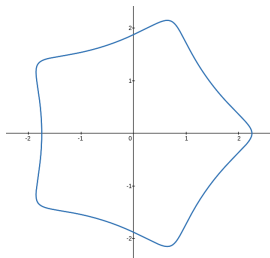
University of Hawai'i at Mānoa
Chapter 3

Section 3.4: Cauchy's Theorem

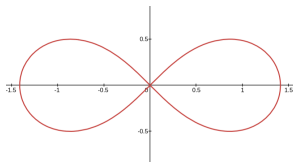
Definition

A **Jordan curve** γ is a simple, continuous, closed curve. In other words, there is a parametrization $\gamma : [a, b] \rightarrow \mathbb{C}$ such that

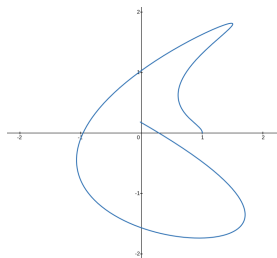
- ① If $a < t_1, t_2 < b$, then $\gamma(t_1) \neq \gamma(t_2)$ whenever $t_1 \neq t_2$.
- ② $\gamma(a) = \gamma(b)$.
- ③ γ is continuous.



(a) Jordan curve



(b) Not Jordan curve



(c) Not Jordan curve

Figure – Different types of curves

Theorem

Any Jordan curve γ separates \mathbb{C} into two regions :

- ① The **interior** of the curve, denoted by Ω^- .
- ② The **exterior** of the curve, denoted by Ω^+ .

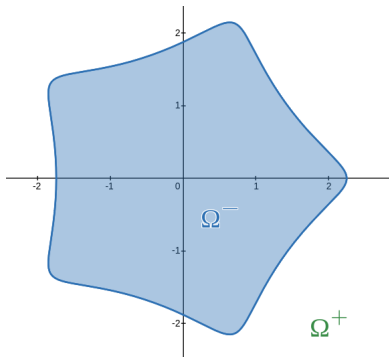


Figure – Interior and Exterior of a Jordan Curve

Definition

Let γ be a Jordan curve with corresponding regions Ω^- , Ω^+ . Informally, we say

- ① γ has **positive orientation** if traversing the curve, Ω^- is on the **left**.
- ② γ has **negative orientation** if traversing the curve, Ω^- is on the **right**.

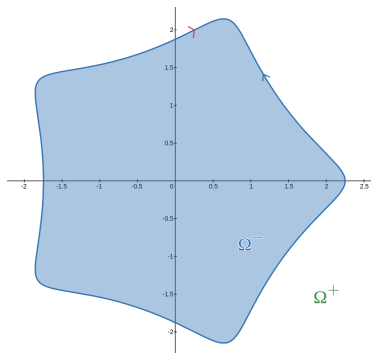


Figure – Two orientations of a Curve

Theorem

- ① Assume f is an analytic function on a region U .
- ② Assume γ is a Jordan curve such that $\Omega^- \cup \gamma \subset U$.
- ③ Assume f' is continuous on U .

Then $\int_{\gamma} f(z) dz = 0$.

Proof. Let $f(z) = u(z) + iv(z)$. Let $z(t) = x(t) + iy(t)$, $a \leq t \leq b$ be a parametrization of γ .

By definition,

$$I := \int_{\gamma} f(z) dz = \int_a^b f(z(t))z'(t) dt.$$

We have $f(z(t))z'(t) = ux' - vy' + i(vx' + uy')$ and so

$$I = \int_a^b ux' - vy' dt + i \int_a^b vx' + uy' dt = \int_{\gamma} udx - vdy + i \int_{\gamma} vdx + udy.$$

Green's Theorem.

- ① Let γ be a Jordan curve with positive orientation.
- ② Let Ω^- be the interior of γ .
- ③ Assume that $P(x, y)$ and $Q(x, y)$ have continuous partial derivatives on $\Omega^- \cup \gamma$.

Then,

$$\int_{\gamma} P dx + Q dy = \iint_{\Omega^-} Q_x - P_y dA.$$

Since $f' = u_x + iv_x$ and f' is continuous, then u_x and v_x are continuous. Similarly, since $f' = v_y + iu_y$ and f' is continuous, then u_y and v_y are continuous. Therefore, u and v have continuous partial derivatives.

Set $P = u$ and $Q = -v$:

$$\int_{\gamma} u dx - v dy = \iint_{\Omega^-} -v_x - u_y dA = - \iint_{\Omega^-} v_x + u_y dA.$$

Set $P = v$ and $Q = u$:

$$\int_{\gamma} v dx + u dy = \iint_{\Omega^-} u_x - v_y dA.$$

Putting everything together :

$$I = - \iint_{\Omega^-} v_x + u_y dA + i \iint_{\Omega^-} u_x - v_y dA.$$

Recall the C-R Equations

$$u_x = v_y \quad \text{and} \quad u_y = -v_x.$$

Hence,

$$I = - \iint_{\Omega^-} 0 dA + i \iint_{\Omega^-} 0 dA = 0.$$

The proof is then completed. □

Note : This is not the general statement of Cauchy's Theorem.

Theorem (General Cauchy's Theorem)

- ① Assume f is an analytic function on a region U .
- ② Assume γ is a Jordan curve such that $\Omega^- \cup \gamma \subset U$.

Then $\int_{\gamma} f(z) dz = 0$.

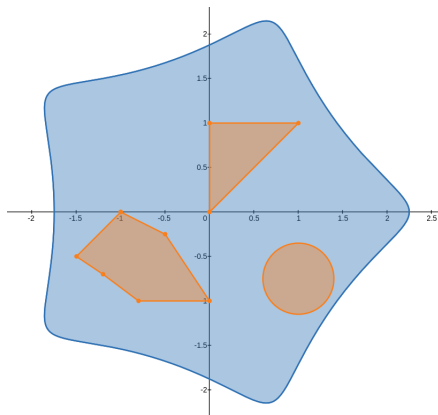
Independence of paths.

- ① Let γ_1 and γ_2 be two simple paths in a region U with the same starting and terminal points.
- ② Let $\Gamma := \gamma_1 \cup \gamma_2$. Assume that $\Omega^- \subset U$.
- ③ Let f be analytic on U .

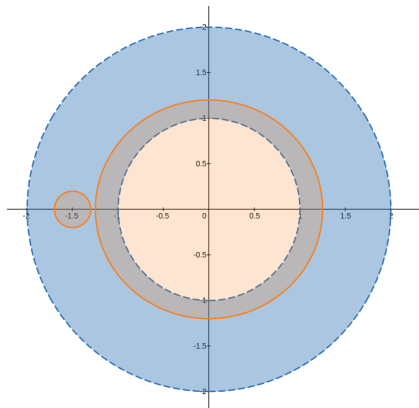
Then $\int_{\gamma_1} f(z) dz = \int_{\gamma_2} f(z) dz$.

Definition

A region Ω is called **simply connected** if the interior of any Jordan curve is contained in Ω .



(a) Simply Connected.



(b) Not Simply Connected.

Corollary

Let f be an analytic function on a simply connected region Ω . If γ is a Jordan curve in Ω , then

$$\int_{\gamma} f(z) dz = 0.$$

Consequences.

- ① If γ_0 and γ_1 are two paths in Ω with the same initial and terminal points, then

$$\int_{\gamma_0} f(z) dz = \int_{\gamma_1} f(z) dz.$$

- ② There is an analytic function F defined on Ω such that $F' = f$.