**Math 241**
**Spring 2022**
**Midterm 1**
**February 23rd, 2022**

**Name (Print):** Solutions
**Section Number:** —

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This exam contains 10 pages (including this cover page) and 9 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

**Instructions:**

- You have 75 minutes for the exam.
- You are required to **show your work** and justify your answers for all questions *except where explicitly stated.*
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit, if any.
- No notes or books are allowed.
- No electronic devices are allowed.
- No calculators are allowed.

Academic integrity is expected of all University of Hawaii students at all times, whether in the presence or absence of members of the faculty. Understanding this, I declare I shall not give, use, or receive unauthorized aid in this examination.
**Please sign below to indicate that you have read and agree to these instructions.**

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Signature of student

*For official use only:*

| Question: | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Total |
|---|---|---|---|---|---|---|---|---|---|---|
| Points: | 9 | 15 | 8 | 10 | 12 | 12 | 10 | 10 | 14 | 100 |
| Score: | – | – | – | – | – | – | – | – | – | – |

**Question 1.** (9 points)

The table shows the distance travelled by a bicyclist on a straight line after accelerating from rest.

| Time in seconds | Total distance in feet |
|---|---|
| 0 | 0 |
| 1 | 2 |
| 2 | 4 |
| 3 | 8 |
| 4 | 15 |
| 5 | 30 |
| 6 | 52 |
| 7 | 76 |
| 8 | 101 |

(a) (3 points) Calculate the average speed between 2 and 6 seconds.

$$\frac{52-4}{6-2} = \frac{48}{4} = \boxed{12 \text{ m/s}}$$

(b) (3 points) Compare the average speed of the interval between 0 second and 1 second, and the interval between 1 second and 2 seconds. Between these two intervals, which one has the highest average speed?

$\frac{2-0}{1-0} = 2$ m/s & $\frac{4-2}{2-1} = 2$ None, same.

(c) (3 points) Estimate the average acceleration of the bicyclist at 7 seconds.
(Hint: The average acceleration can be calculated using two average speeds.)

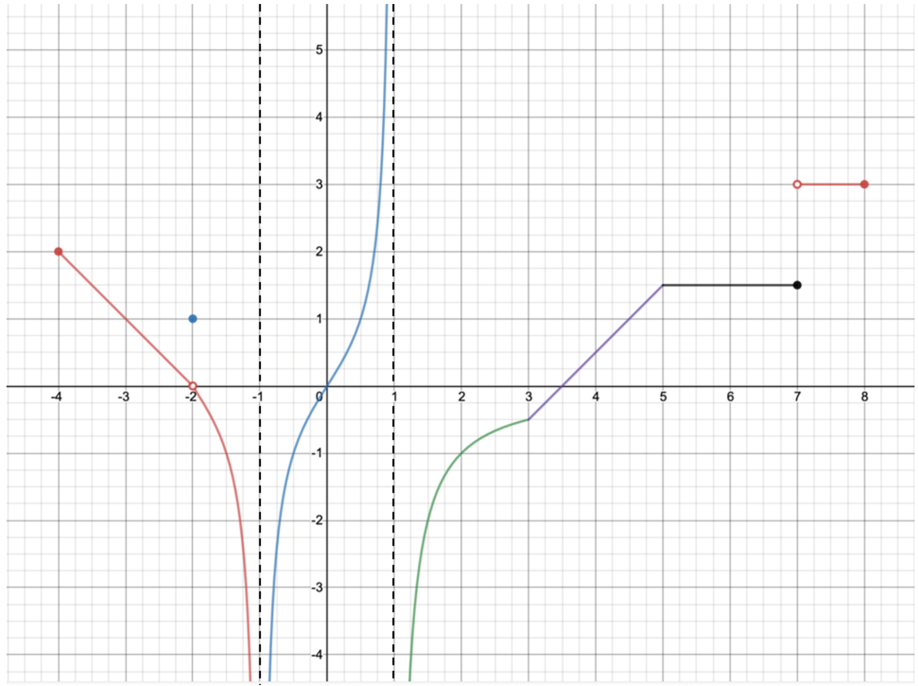
$\frac{76-52}{7-6} = 14$ m/s ( velo. at $t=6$)

$\frac{101-76}{7-6} = 25$ m/s ( velo. at $t=7$)

$\Rightarrow \frac{25-14}{7-6} = \boxed{9 \text{ m/s}^2}$

**Question 2.** (15 points)

The graph of a function $f$ is given below. Assume $f$ has vertical asymptotes at $x = -1$ and $x = 1$. No justifications needed for this problem.




(a) (6 points) Evaluate each of the following limits, or say the limit does not exist. If the limit is either $\infty$ or $-\infty$, specify which (rather than just saying 'does not exist').

1. $\lim_{x\to -2} f(x)$ $= 0$
2. $\lim_{x\to -1^-} f(x)$ $= -\infty$
3. $\lim_{x\to 1} f(x)$ DNE (or $\nexists$)
4. $\lim_{x\to 7^-} f(x)$ $= 1.5$
5. $\lim_{x\to 7^+} f(x)$ $= 3$
6. $\lim_{x\to 7} f(x)$ DNE

(b) (3 points) For which (if any) values in the interval $[-4, 8]$ is the function $f$ not continuous?

$-2, -1, 1, 7$

(c) (3 points) For which (if any) values in the interval $[-4, 8]$ is $f$ differentiable but not continuous?

$-2, -1, 1, 7$, 3 & 5

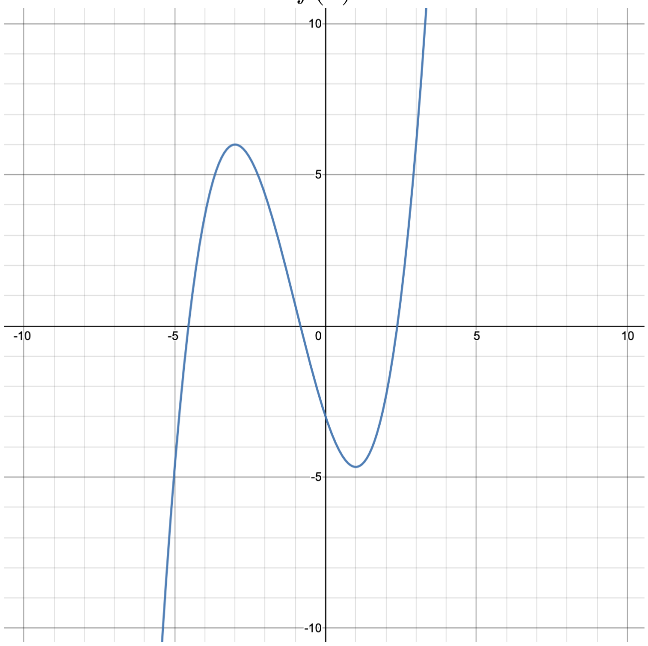
(d) (3 points)For which (if any) values in the interval $[-4, 8]$ is $f$ continuous but not differentiable?

3 & 5

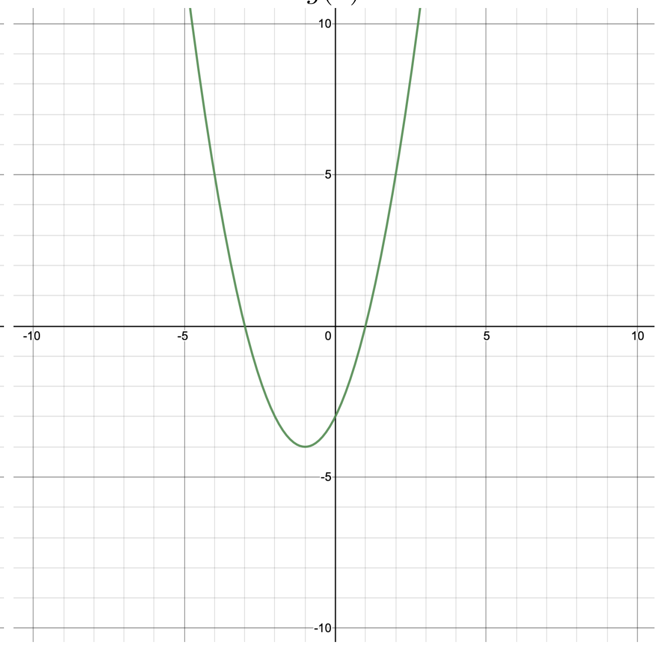
**Question 3.** (8 points)

Given the two graphs below, **roughly** sketch the graphs of their derivative on the blank axes. (4 points for each graph.)

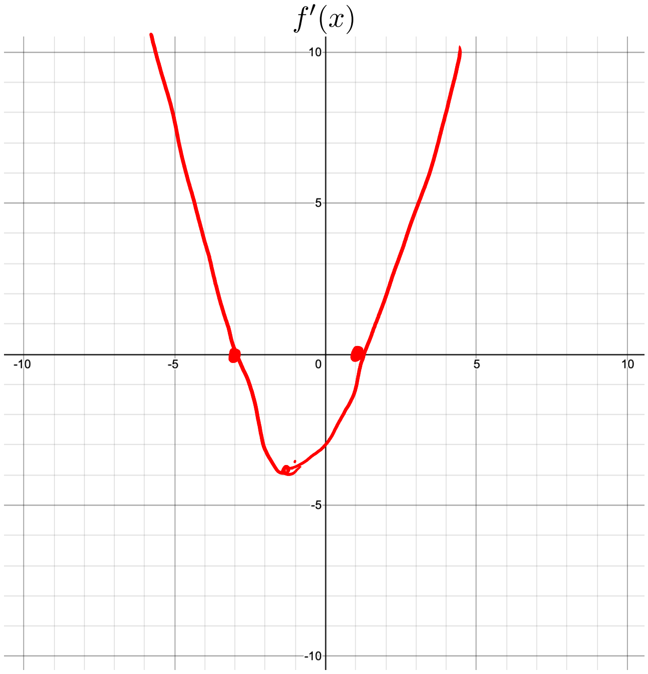
$f(x)$



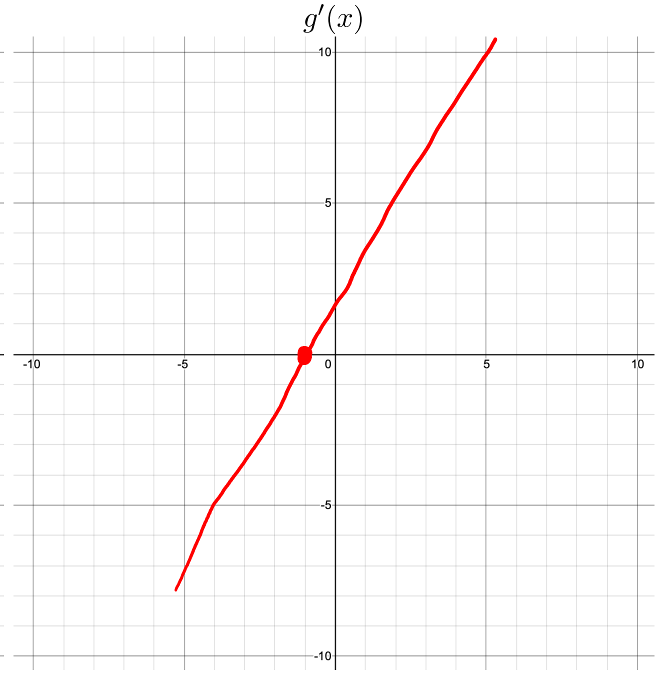

$g(x)$




$f'(x)$




$g'(x)$



**Question 4.** (10 points)
Suppose $f$ is a continuous function that satisfies the following limits:

$$\lim_{x\to -1} f(x) = -2, \qquad \lim_{x\to 0} f(x) = 3$$

Evaluate the following limits. (5 points each.) You may not use L'Hospital's rule, i.e., if you use L'Hospital's rule, you will not get points.

(a) $\lim_{x\to -1} \frac{(x^2-3x-4)}{x+1} f(x)$ $\frac{0}{0}$ ...

$x^2 - 3x - 4 = (x-4)(x+1)$

$$\Rightarrow \lim_{x\to -1} \frac{x^2-3x-4}{x+1} f(x) = \lim_{x\to -1} \frac{(x-4)(x+1)}{x+1} f(x)$$

$$= \lim_{x\to -1} (x-4) f(x)$$

$$= (-1-4)(-2) = \boxed{10}$$

(b) $\lim_{x\to 0} \frac{\sqrt{3x^2+16}-4}{x^2 f(x)}$ $\frac{0}{0}$

$$\sqrt{3x^2+16} - 4 = \frac{\left(\sqrt{3x^2+16}-4\right)\left(\sqrt{3x^2+16}+4\right)}{\sqrt{3x^2+16}+4} = \frac{3x^2+16-16}{\sqrt{3x^2+16}+4}$$

$$= \frac{3x^2}{\sqrt{3x^2+16}+4}$$

$$\Rightarrow \lim_{x\to 0} \frac{\sqrt{3x^2+16}-4}{x^2 f(x)} = \lim_{x\to 0} \frac{3\cancel{x^2}}{\cancel{x^2} f(x)\left(\sqrt{3x^2+16}+4\right)}$$

$$= \lim_{x\to 0} \frac{3}{f(x)\left(\sqrt{3x^2+16}+4\right)}$$

$$= \frac{3}{3\cdot 8} = \boxed{\frac{1}{8}}$$

**Question 5.** (12 points)

(a) (8 points) Using *the definition of derivative* (also called the limit process), find the derivative of the function $f(x) = \frac{1}{x+4}$.

You will NOT get any credit unless you use the definition of the derivative!

$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{x+h+4} - \frac{1}{x+4}}{h}$$

$$= \lim_{h \to 0} \frac{x+4-x-h-4}{h(x+h+4)(x+4)}$$

$$= \lim_{h \to 0} \frac{-\cancel{h}}{\cancel{h}(x+h+4)(x+4)}$$

$$= \lim_{h \to 0} \frac{-1}{(x+h+4)(x+4)}$$

$$= \boxed{-\frac{1}{(x+4)^2}}$$

(b) (4 points) Using the function in (a), find the equation of the tangent line to $y = f(x)$ at $(0, \frac{1}{4})$.

Eq. tang. line: $y - \frac{1}{4} = f'(0)\,x$

$$f'(0) = -\frac{1}{16}$$

$$\Rightarrow \boxed{y = \frac{-x}{16} + \frac{1}{4}}$$

**Question 6.** (12 points)
Let $f(x)$ be defined by

$$f(x) = \begin{cases} (x-a)^2 + 2 & \text{if } x < 2 \\ 3 & \text{if } x = 2 \\ a + x & \text{if } x > 2 \end{cases}$$

(a) (8 points) Find all values of $a$ so that $\lim_{x\to 2} f(x)$ exists.

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$$

$$\Leftrightarrow \lim_{x\to 2^-} (x-a)^2 + 2 = \lim_{x\to 2^+} a+x$$

$$\Leftrightarrow (2-a)^2 + 2 = a+2$$

$$\Leftrightarrow (2-a)^2 = a$$

$$\Leftrightarrow 4 - 5a + a^2 = a$$

$$\Leftrightarrow a^2 - 6a + 4 = 0$$

Quadratic formula: $a = \dfrac{6 \pm \sqrt{36-16}}{2} = \dfrac{6 \pm \sqrt{20}}{2}$

$= 3 \pm \sqrt{5}$

So, values are $\boxed{a = 3+\sqrt{5}}$ & $\boxed{a = 3-\sqrt{5}}$

(b) (4 points) Find all possible values of $a$ so that $f(x)$ is continuous at $x = 2$, or show that none exist. Justify your answer.

$f(x)$ continuous if
- $f(2)$ defined ✓
- $\lim_{x\to 2} f(x)$ $\exists$ ✓ for values obtained in (a)
- $f(2) = \lim_{x\to 2} f(x)$ ←

$\underline{a = 3+\sqrt{5}}$

$$\lim_{x\to 2} f(x) = 3+\sqrt{5}+2 = 5+\sqrt{5} \neq 3 = f(2)$$

$\underline{a = 3-\sqrt{5}}$

$$\lim_{x\to 2} f(x) = 3-\sqrt{5}+2 = 5-\sqrt{5} \neq 3 = f(2)$$

$\boxed{\text{None exist}}$

**Question 7.** (10 points)

Suppose $f(x)$ is a function where $f(1) = 1$ and $f'(1) = -1$.

(a) (5 points) Let $g(x) = x^3 f(x) + 2$. Find $g'(1)$.

$$g'(x) = 3x^2 f(x) + x^3 f'(x)$$

$$\Rightarrow g'(1) = 3 \cdot 1^2 \cdot f(1) + 1^3 f'(1)$$

$$= 3 \cdot 1 + 1 \cdot (-1) = \boxed{2}$$

(b) (5 points) Let $h(x) = \sqrt{4\sin(\pi x) + 3f(x)}$. Find $h'(1)$.

$$h'(x) = \frac{1}{2}\left[4\sin(\pi x) + 3f(x)\right]^{-1/2} \cdot \left(4\pi\cos(\pi x) + 3f'(x)\right)$$

$$\Rightarrow h'(1) = \frac{1}{2}\left[4\sin(\pi) + 3\right]^{-1/2} \cdot \left(4\pi\cos(\pi) - 3\right)$$

$$= \boxed{\frac{1}{2}(3)^{-1/2}(4\pi - 3)} \quad \text{or} \quad \frac{4\pi - 3}{2\sqrt{3}}$$

**Question 8.** (10 points)

Use implicit differentiation to find an equation of the tangent line to the following cardioid

$$x^2 + y^2 = (2x^2 + 2y^2 - x)^2 \text{ at the point } \left(0, -\frac{1}{2}\right)$$

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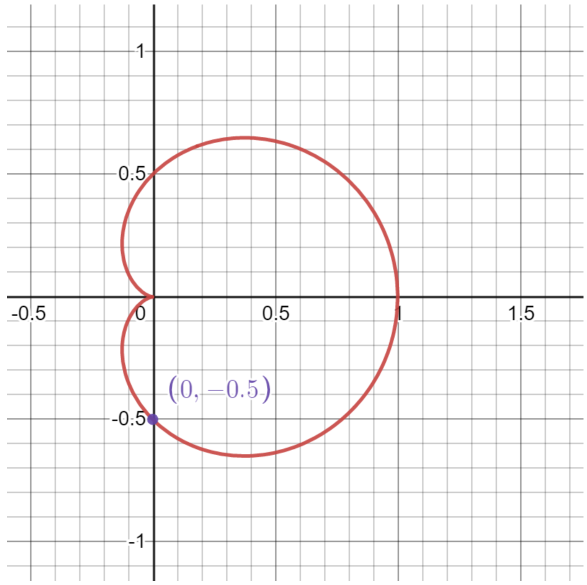


① Find $\frac{dy}{dx}$

$$2x + 2yy' = 2(2x^2+2y^2-x)(4x+4yy'-1)$$

$$\Rightarrow 2x + 2yy' = 2(2x^2+2y^2-x)(4x-1) + 8yy'(2x^2+2y^2-x)$$

$$\Rightarrow 2x - 2(2x^2+2y^2-x)(4x-1) = 8yy'(2x^2+2y^2-x) - 2yy'$$

$$\Rightarrow \frac{2x-2(2x^2+2y^2-x)(4x-1)}{8y(2x^2+2y^2-x)-2y} = y'$$

② Tangent line

Replace $x$ & $y$ by $(0, -\frac{1}{2}) \Rightarrow y' = -1$

So, $y + \frac{1}{2} = (-1)(x-0) \Rightarrow \boxed{y = -x - \frac{1}{2}}$

**Question 9.** (14 points)
Suppose that an object moves along a line over time. Its position is given by

$$x(t) = -0.02t^2 + 50t + 100.$$

(a) (4 points) What is the average speed of the object between the time $t = 0$ and $t = 1000$?

$$\frac{x(1000) - x(0)}{1000 - 0} = \frac{-0.02 \cdot 1000\,000 + 50\,000 + 100 - 100}{1000}$$

$$= \frac{-20\,000 + 50000}{1000}$$

$$= \frac{30\,000}{1000} = \boxed{30}$$

(b) (5 points) What is the velocity of the object when $t = 500$?

$$x'(t) = -0.04t + 50 \Rightarrow x'(500) = -0.04 \cdot 500 + 50$$

$$= -20 + 50$$

$$= \boxed{30}$$

(c) (5 points) What is the acceleration of the object when $t = 10$?

$$x''(t) = -0.04 \Rightarrow x''(10) = \boxed{-0.04}.$$