$u = x^2 \rightarrow \boxed{du = 2x\,dx}$

$$\int_0^1 x\cos(x^2)\,dx = \int_0^1 \cos(u)\,\frac{du}{2}$$

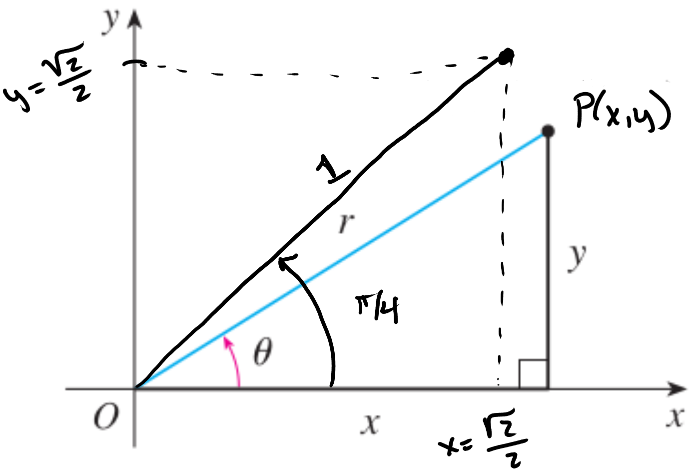
Chapter 15

Multiple Integrals

15.3 Double Integrals in polar coordinates

Polar coordinates

$r = 1$, $\theta = \frac{\pi}{4}$ $\longrightarrow$ $x = 1\cos(\pi/4) = \sqrt{2}/2$, $y = 1\sin(\pi/4) = \sqrt{2}/2$



1) Polar to Cartesian:

$x = r\cos(\theta)$, $y = r\sin\theta$

2) Cartesian to Polar:

$r = \sqrt{x^2 + y^2}$

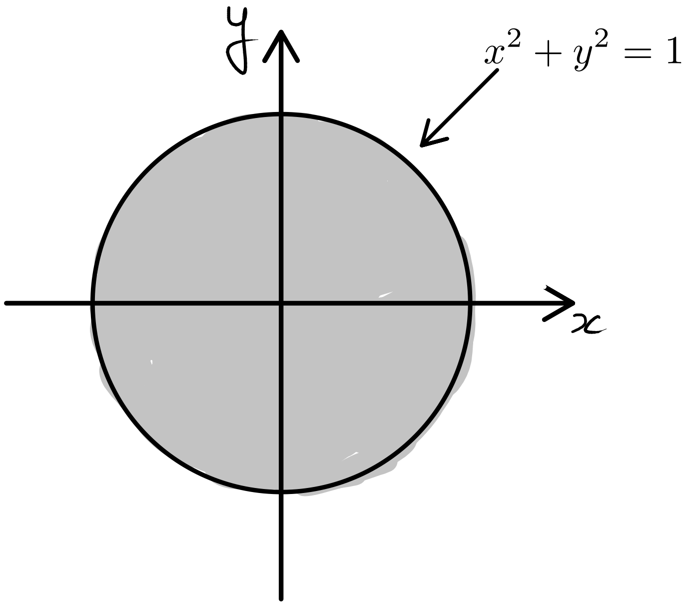
$\tan\theta = \frac{y}{x} \Rightarrow \theta = \arctan\left(\frac{y}{x}\right)$

$\left(\theta = \tan^{-1}\left(\frac{y}{x}\right)\right)$

Why would we use polar coordinates?

$x = 1$, $y = 1$ $\longrightarrow$ $r = \sqrt{2}$, $\theta = \boxed{\pi/4}, \cancel{\frac{5\pi}{4}}, \ldots$

Example. Describe the following region:

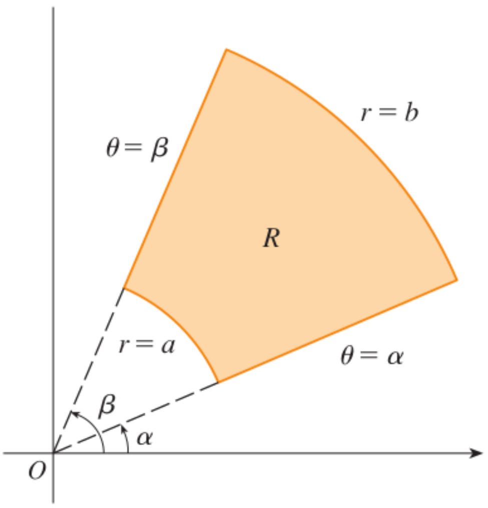


How does it affect the double integral

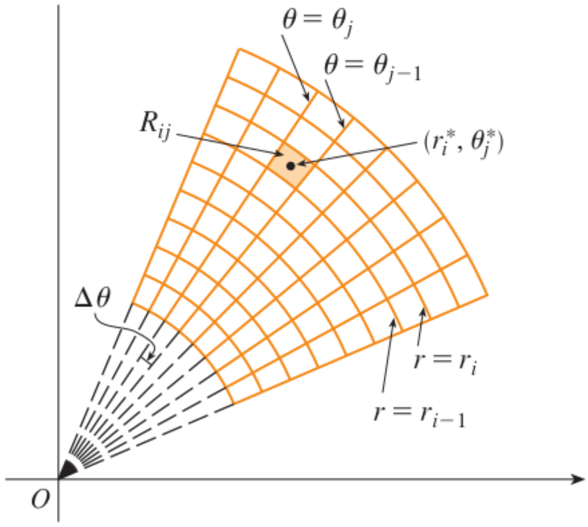
Recall:

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dxdy \longrightarrow \boxed{dA = dxdy}$$
$$= \int_c^d \int_a^b f(x,y)\,dydx \longrightarrow \boxed{dA = dydx}$$

Polar rectangle:



Divide it $\longrightarrow$



Close-up view

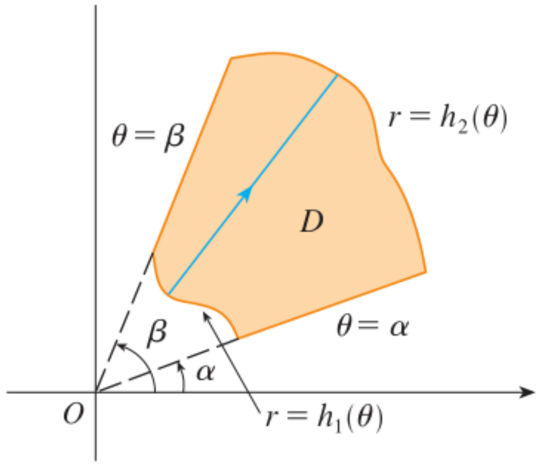
$$\boxed{\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r\,drd\theta}$$

R is a polar rectangle given by $a \le r \le b$ and $\alpha \le \theta \le \beta$, with $\beta - \alpha \le 2\pi$.

EXAMPLE 1 Evaluate $\iint_R (3x + 4y^2)\, dA$, where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

EXAMPLE 2 Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.

More complicated region:



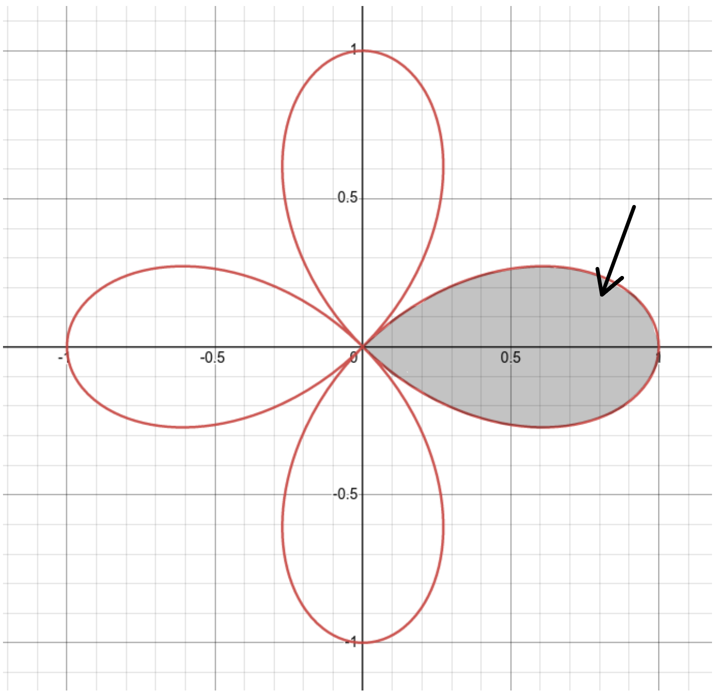
3 If f is continuous on a polar region of the form

$$D = \{(r, \theta) \mid \alpha \leqslant \theta \leqslant \beta,\ h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$$

then
$$\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

EXAMPLE 3 Use a double integral to find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

$\boxed{1}$ PICTURE



EXAMPLE 4 Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.