Chapter 2: Derivatives
Week 4

Pierre-Olivier Parisé
Calculus I (MATH-241 01/02)

University of Hawai'i
Fall 2021

Upcoming this week

1. 2.1 Derivatives and Rates of change
2. 2.2 Derivatives as a function
3. 2.3 Differentiation formulas

Question 1

Let C be a curve obtained from the equation of a function $y = f(x)$. What is the tangent line at the point $P = (a, f(a))$? ▸ Tangent Line

Answer:

- Take a point $Q = (x, f(x))$ on the graph of f and compute the slope of the secant passing through P and Q:

$$m_{PQ} := \frac{f(x) - f(a)}{x - a}.$$

- Let Q approaches P.

Definition 2

The slope of the tangent line to the curve $y = f(x)$ at the point $P = (a, f(a))$ is the line through P with slope

$$m := \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that the limit exists.

Example 3

Find the tangent line to the curve $y = x^2$ at the point $P = (2, 4)$.

Example 4

Find the slope of the tangent line to the curve $y = 3/x$ at $P = (3, 1)$.

Remarks

- If $h = x - a$, then $x = a + h$ and we can rewrite

$$m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$

- If we zoom in far enough toward the point P, the curve looks almost like a straight line.

Question 5

Say an object moves along a curve $s = f(t)$, where s is the displacement (directed distance) of the object from the origin at time t. Can you compute its instantaneous velocity at time $t = a$?

Answer:

- Take the average velocity $P = (a, s(a))$ to a point $Q = (t, s(t))$:

$$v_{\text{av.}} := \frac{s(t) - s(a)}{t - a} = \frac{s(a+h) - s(a)}{h}.$$

- Let h approaches 0.

Definition 6

The instantaneous velocity of an object moving along a curve $s = f(t)$ at time $t = a$ is

$$v(a) := \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Example 7

Let a ball fall from the CN Tower with position function $s = 4.9t^2$. What is the instantaneous velocity at $t = 5$?

Definition 8

The derivative of a function f at a number a is

$$f'(a) := \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if the limit exists.

Remark: We also have $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$.

Example 9

Find the derivative of the function $f(x) = x^2 - 8x + 9$ at the number $x = a$.

Suppose that y is a quantity that depends on x; so $y = f(x)$.

- When x changes from x_2 to x_1, then the change (the increment of x) is

$$\Delta x := x_2 - x_1.$$

- And the corresponding change (the increment of y) in y is

$$\Delta y := f(x_2) - f(x_1).$$

Definition 10

The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is called the average rate of change of y w.r.t x.

Definition 11

The instantaneous rate of change of y w.r.t. x at $x = x_1$ is the limit of the averaged rates of change of y w.r.t. x as $x_2 \to x_1$; that is

$$\text{Inst. rate of chg.} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Remark: Inst. rate of chg. $= f'(x_1)$.

Exercises: 1, 3, 5, 6, 8, 10, 11, 12, 20, 24, 31-36, 37, 39, 40, 42-44, 50, 59, 60.

In the definition of the derivative, we fixed the number a, but we can make the number a varies. Now, the derivative becomes a function!

Definition 12

The derivative function is the function f' such that

$$f'(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists.

Example 13

Suppose $f(x) = x^3 - x$.

a) Find a formula for $f'(x)$.

b) Sketch the graph of the curve $y = f'(x)$.

Example 14

If $f(x) = \sqrt{x}$, find the derivative of f and find the domain of f'.

There are a lot of ways to denote the derivative of a function:

$$y', f'(x), \frac{dy}{dx}, \frac{df}{dx}, \frac{d}{dx}f(x), Df(x), D_x f(x).$$

- dy/dx, df/dx is a notation invented by Leibniz and is VERY^{VERY} useful when we differentiate composition of functions.
- In Leibniz' notation, we write

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a}$$

to clearly indicate the derivative is evaluated at the point $x = a$.

Definition 15

Let f be a function and let a be a point in its domain.

- f is differentiable at a if $f'(a)$ exists.
- f is differentiable on an (open) interval (a, b) [or (a, ∞), $(-\infty, a)$, $(-\infty, \infty)$] if it is differentiable at every number of this interval.

Example 16

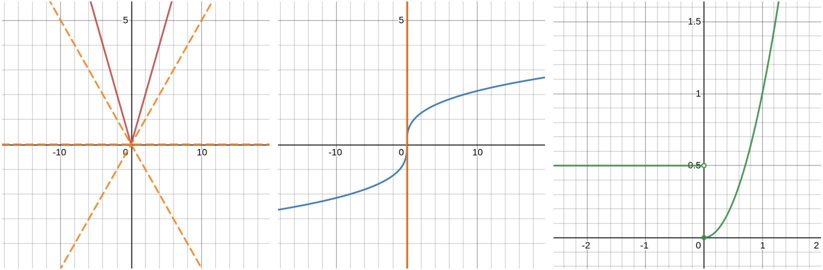
Where is the function $f(x) = |x|$ differentiable?

A nice property of differentiable function is the following.

Theorem 17

If f is differentiable at a, then it is continuous at a.

Warning! The converse is not true. The function $f(x) = |x|$ is continuous at 0 but is not differentiable at 0. Non Differentiable



(a) Spike/corner (b) Vertical line (c) Discontinuities

Figure: When is a function not differentiable?

Now, we can define $f''(x)$ by taking the derivative of the derivative.

Definition 18

The <u>second derivative</u> of f at a point x is

$$f''(x) := \lim_{h\to 0} \frac{f'(x+h) - f(x)}{h}$$

provided this limit exists.

Example 19

Find $f''(x)$ of $f(x) = x^3 - x$.

Remarks:

- Interpretation: f'' is the acceleration of an object with velocity v and position function f.
- You can compute higher and higher derivatives: $f'''(x)$, $f^{(4)}(x)$, ..., $f^{(n)}(x)$ for $n \geq 1$.

Exercises: 3, 13, 19-26, 28, 29, 32, 40-42, 45, 51.

Theorem 20

Let b be any real number.

- $b = 0$, *then* $\frac{d}{dx}(1) = 0$.
- *If* $b \neq 0$, *then* $\frac{d}{dx}(x^b) = bx^{b-1}$.

Example 21

Compute the derivatives of the following functions:

a) $f(x) = x^6$

b) $y = t^{1/5}$

c) $y = u^{\pi}$.

d) $u = v^{2/3}$.

Theorem 22

Let f, g be two differentiable functions and let c be a constant.

- $\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$.
- $\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$.
- $\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$.
- $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$.
- $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$.

Example 23

Compute the derivatives of the following functions:

a) $f(x) = x^8 + 12x^5 + 10x^3 - 6x + 5$.

b) $y = (x^2+1)(x^3+2)$.

c) $v = \frac{x^2+x-2}{x^3+6}$.

Example 24

The equation of motion of a particle is $s = 2t^3 - 5t^2 + 3t + 4$, where s is measured in centimeters and t in seconds.

a) Find the accelation as a function of time.

b) What is the acceleration after 2 seconds.

Example 25

Find the points on the curve $y = x^4 - 6x^2 + 4$ where the tangent line is horizontal.

Example 26

At what points on the hyperbola $xy = 12$ is the tangent line parallel to the line $3x + y = 0$?

Definition 27

The <u>normal line</u> to curve C at a point P is the line through that is perpendicular to the tangent line at P.

Example 28

Find the equations of the tangent and normal lines to the curve $y = \sqrt{x}$ at the point $P = (1, 1)$. Normal line

Exercises: 1-22, 23-26, 28-42, 44, 53, 56-58, 61, 64, 65, 70, 73, 78, 79, 83.