$$(x^2)' = 2x$$

# Math 241

## Chapter 3

### Section 3.9: Antiderivatives

Contents




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Definition

A function $F$ is an **antiderivative** of a function $f$ if $F'(x) = f(x)$.

Example 1. Find an antiderivative of the following functions.

(a) $f(x) = x^2$. (b) $g(x) = 3x^3 + \cos(x)$. (c) $h(x) = x^{2/3} + 4\sec^2(x)$.

(a) $F(x) = x^2 \to F'(x) = 2x$

$F(x) = x^3 \to F'(x) = 3x^2$

$F(x) = \frac{x^3}{3} \to F'(x) = \frac{1}{\cancel{3}}(\cancel{3}x^2) = x^2$

$x^n \to \frac{x^{n+1}}{n+1}$

$n = -1$

(b) $x^3 \to \frac{x^4}{4}$ $\to G(x) = 3\frac{x^4}{4} + \sin(x)$.

$\cos(x) \to \sin(x)$

Check: $G'(x) = \frac{3}{\cancel{4}}(\cancel{4}x^3) + \cos(x)$

$= 3x^3 + \cos(x) = g(x)$.

(c) $x^{2/3} \to \frac{x^{2/3+1}}{2/3+1} = \frac{x^{5/3}}{5/3} = \frac{3}{5}x^{5/3}$

$4\sec^2(x) \to 4\tan x$

$H(x) = \frac{3}{5}x^{5/3} + 4\tan x + C$

Check: $H'(x) = \frac{\cancel{3}}{\cancel{5}}\left(\frac{\cancel{5}}{\cancel{3}}x^{5/3-1}\right) + 4\sec^2 x$

$= x^{2/3} + 4\sec^2 x$. ✓

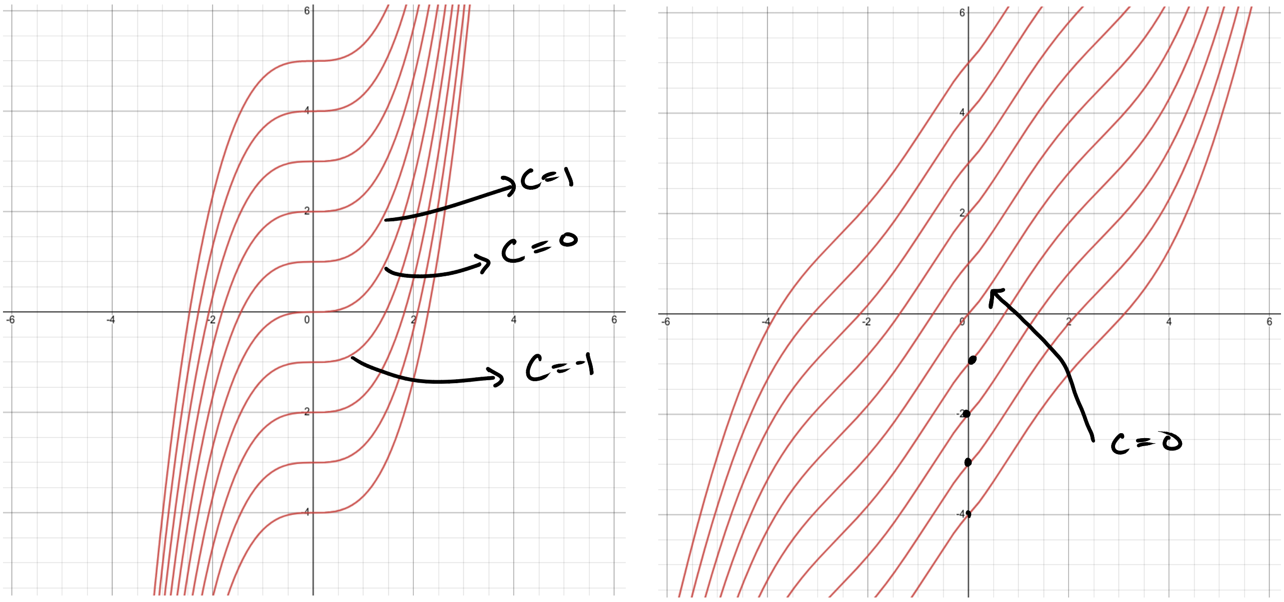
Remarks:

- Recall that $f'(x) = g'(x)$ if and only if $f(x) = g(x) + C$ for some constant $C$.
- There are more than just one antiderivative!

The **most general antiderivative** of a function $f$ is

$$F(x) + C,$$

where $C$ is a constant.




(a) Several Antiderivatives of $f(x) = x^2$, that is $\frac{x^3}{3} + C$

(b) Several antiderivatives of $f(x) = x^{2/3} + \cos(x)$, that is $\frac{3}{5}x^{5/3} + \sin(x) + C$.

EXAMPLE 2. Find the most general antiderivative of each of the following functions.

**(a)** $f(x) = \sin x$. **(b)** $f(x) = x^n$, $n \geq 0$.

(a) $F(x) = -\cos(x)$ (because $F'(x) = -(-\sin x) = \sin x$).

$\Rightarrow \quad F(x) + C = \boxed{-\cos(x) + C}$

(b) $F(x) = \dfrac{x^{n+1}}{n+1}$

$\Rightarrow \quad F(x) + C = \boxed{\dfrac{x^{n+1}}{n+1} + C}$

| Function | Particular antiderivative | Function | Particular antiderivative |
|---|---|---|---|
| $cf(x)$ | $cF(x)$ | $\cos x$ | $\sin x$ |
| $f(x) + g(x)$ | $F(x) + G(x)$ | $\sin x$ | $-\cos x$ |
| $x^n \ (n \neq -1)$ | $\dfrac{x^{n+1}}{n+1}$ | $\sec^2 x$ | $\tan x$ |
| | | $\sec x \tan x$ | $\sec x$ |

Figure 2: Properties and some Antiderivatives

**EXAMPLE 3.** Find all functions $g$ such that

$$g'(x) = 4\sin x + \frac{2x^5 - \sqrt{x}}{x}.$$

Antiderivatives

① $4\sin x \longrightarrow -4\cos(x)$

② Simplify: $\dfrac{2x^5 - x^{1/2}}{x} = \dfrac{2x^5}{x} - \dfrac{x^{1/2}}{x}$

$= 2x^{5-1} - x^{1/2-1}$

$= 2x^4 - x^{-1/2}$

$2x^4 \longrightarrow \dfrac{2x^5}{5}$

$x^{-1/2} \longrightarrow \dfrac{x^{-1/2+1}}{-\frac{1}{2}+1} = \dfrac{x^{1/2}}{1/2} = 2x^{1/2}$

③ $F(x) = -4\cos(x) + \dfrac{2}{5}x^5 - 2x^{1/2} + C$

or

$F(x) + C$

**EXAMPLE 4.** Find $F$ if $F'(x) = x\sqrt{x}$ and $F(1) = 2$.

$$x\sqrt{x} = x x^{1/2} = x^{1+1/2} = x^{3/2}$$

$$\Rightarrow \quad F(x) = \frac{x^{5/2}}{5/2} + C = \frac{2}{5} x^{5/2} + C.$$

We have $F(1) = 2$

$$\Rightarrow \quad \frac{2}{5}(1)^{5/2} + C = 2$$

$$\Rightarrow \quad C = 2 - \frac{2}{5} = \frac{8}{5}$$

So, $F(x) = \boxed{\frac{2}{5} x^{5/2} + \frac{8}{5}}$

**EXAMPLE 5.** Find $F$ if $F'(x) = \frac{1}{x^2}$ and $F(1) = 2$.

$$\frac{1}{x^2} = x^{-2} \longrightarrow \frac{x^{-2+1}}{-2+1} = \frac{x^{-1}}{-1} = \frac{-1}{x}$$

$$\Rightarrow \quad F(x) = -\frac{1}{x} + C$$

We have $F(1) = 2$

$$\Rightarrow \quad -1 + C = 2$$

$$\Rightarrow \quad C = 3$$

So, $\boxed{F(x) = 3 - \frac{1}{x}}$