## Example 1

Find the general indefinite integral of

$$\int (10x^4 - 2\sec^2 x)\,dx.$$

$$\int 10x^4 - 2\sec^2 x \,dx = \int 10x^4\,dx + \int (-2\sec^2 x)\,dx$$

$$= 10\int x^4\,dx - 2\int \sec^2 x\,dx$$

$$= 10\frac{x^5}{5} + C_1 - 2\tan x + C_2$$

$$= 10\frac{x^5}{5} - 2\tan x + C$$

where $C = C_1 + C_2$.

$$\boxed{\int 10x^4 - 2\sec^2 x\,dx = 2x^5 - 2\tan x + C}$$

## Example 2

Evaluate the indefinite integral $\displaystyle\int \frac{\cos\theta}{\sin^2\theta}\, d\theta$.

$\underset{\frac{\cos\theta}{\sin\theta}}{\cot an\,\theta} \cdot \underset{\frac{1}{\sin\theta}}{\csc\theta}$

$$\int \frac{\cos\theta}{\sin^2\theta}\, d\theta = \int \frac{\cos\theta}{\sin\theta} \cdot \frac{1}{\sin\theta}\, d\theta = \int \cot\theta \cdot \csc\theta \, d\theta$$

$$\boxed{= \csc\theta + C}$$

## Example 3

$t^2 \cdot t^{1/2} = t^{2+1/2} = t^{5/2}$ $\qquad \frac{t^{5/2}}{t^2} = t^{5/2-2}$

Evaluate $\displaystyle\int \frac{2t^2 + \overbrace{t^2\sqrt{t}} - 1}{t^2}\,dt.$

$$\int \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\,dt = \int \frac{2\cancel{t^2}}{\cancel{t^2}} + \frac{\cancel{t^2}\sqrt{t}}{\cancel{t^2}} - \frac{1}{t^2}\,dt$$

$$= \int 2 + \underline{\sqrt{t}} - \frac{1}{t^2}\,dt$$

$$= 2\int 1\,dt + \int \underbrace{\sqrt{t}}_{t^{1/2}}\,dt - \int \frac{1}{t^2}\,dt \quad \to t^{-2}$$

$$= 2t + C_1 + \frac{t^{3/2}}{3/2} + C_2 - \frac{t^{-1}}{-1} + C_3$$

$$= 2t + \frac{2}{3}t^{3/2} + \frac{1}{t} + \underbrace{C_1 + C_2 + C_3}_{=C}$$

$$= 2t + \frac{2}{3}t^{3/2} + \frac{1}{t} + C$$

## Example 4

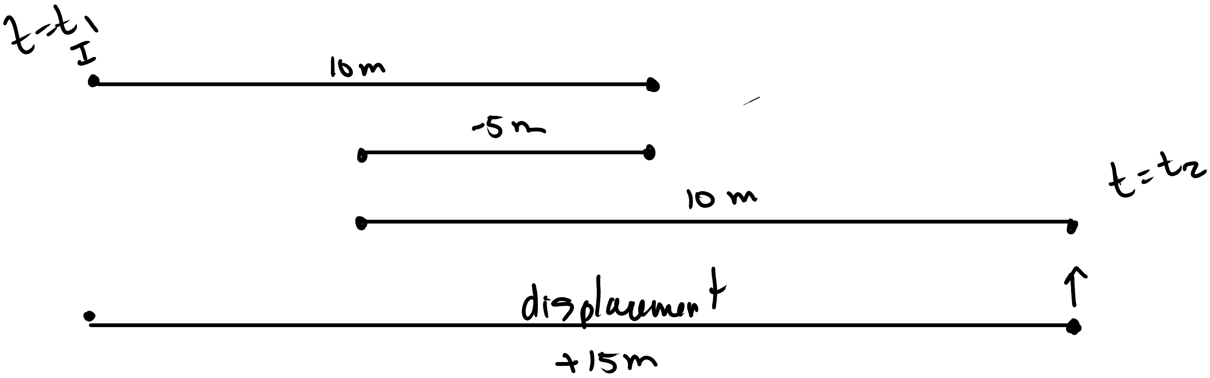
A particle moves along a line so that its velocity at time $t$ is $v(t) = t^2 - t - 6$ (measured in meters per second).

a) Find the displacement of the particle during the time period $1 \le t \le 4$.

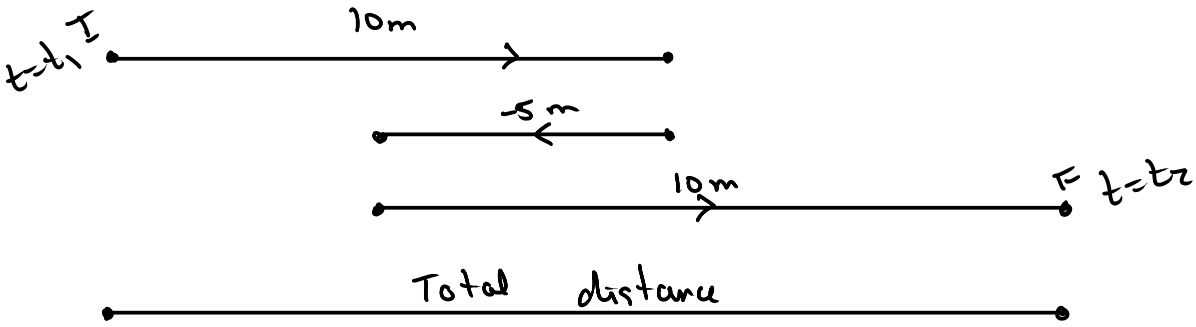
b) Find the distance traveled during this time period.

two concepts.

① Displacement $= \int_{t_1}^{t_2} v(t)\, dt$




② Distance $= \int_{t_1}^{t_2} |v(t)|\, dt$




$10 + 5 + 10 = 25$ m.

a) Displacement $= \int_1^4 t^2 - t - 6\, dt = \left(\frac{t^3}{3} - \frac{t^2}{2} - 6t\right)\Big|_1^4$

$$= \frac{4^3}{3} - \frac{4^2}{2} - 6\cdot 4 - \left(\frac{1}{3} - \frac{1}{2} - 6\right)$$

$$= -4.5 \text{ m.}$$

b) Total distance $= \int_1^4 |t^2 - t - 6t|\, dt$

We have to find an explicit expression for $|t^2 - t - 6|$:

$$|t^2 - t - 6| = \begin{cases} t^2 - t - 6 & \text{if } t^2 - t - 6 \ge 0 \\ -(t^2 - t - 6) & \text{if } t^2 - t - 6 < 0. \end{cases}$$

When is $v(t) \geq 0$

$t^2 - t - 6 \geq 0$ if $(t-6)(t+1) \geq 0$

if $t-6 \geq 0$ and $t+1 \geq 0$

or $t-6 \leq 0$ and $t+1 \leq 0$

if $t \geq 6$ and $t \geq -1$

or

$t \leq 6$ and $t \leq -1$

if $t \geq 6$ or $t \leq -1$

Since $1 \leq t \leq 4$, then $v(t) \leq 0$

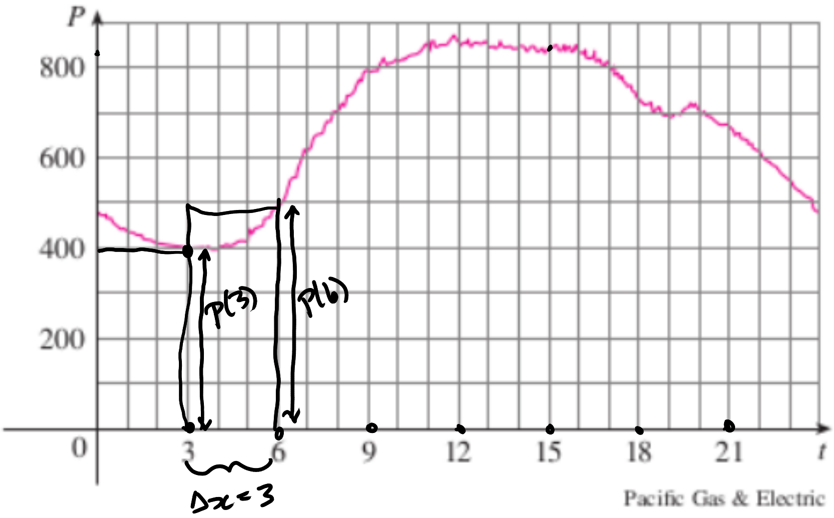
So, $\displaystyle\int_1^4 |v(t)|\,dt = \int_1^4 |t^2 - t - 6|\,dt = \int_1^4 -(t^2 - t - 6)\,dt$

$\displaystyle = -\int_1^4 t^2 - t - 6\,dt$

$= -(-4.5)$ $\boxed{= 4.5\text{ m.}}$

## Example 5

The figure shows the power consumption in the city of San Francisco for a day in September ($P$ is measured in megawatts; $t$ is measured in hours starting at midnight). Estimate the energy used on that day.




Power : $P = \frac{E_f - E_i}{\Delta t}$ (average). $\xrightarrow{\Delta t \to 0}$ $P = \frac{dE}{dt}$.

So, to obtain the net energy used in a day,

$$\int_0^{24} \frac{dE}{dt}\, dt = E(24) - E(0)$$

$$\Rightarrow \int_0^{24} P(t)\, dt = E(24) - E(0).$$

Here, we don't have an expression for E or P, but we have a table (graph) of the values of P.

$\Delta t = 3$

$$\int_0^{24} P(t)\, dt \approx \sum_{i=1}^{8} P(0 + i\Delta t)\, \Delta t$$

$$= P(3)\cdot 3 + P(6)\cdot 3 + P(9)\cdot 3 + P(12)\cdot 3$$

$$+ P(15)\cdot 3 + P(18)\cdot 3 + P(21)\cdot 3 + P(24)\cdot 3$$

$$= 15735 \text{ MW}\cdot\text{h}$$

## Example 6

Find the indefinite integral $\int \underbrace{2x\sqrt{\underbrace{1+x^2}}}\,dx$.

$$\int \sqrt{u}\,du = \frac{u^{3/2}}{3/2} + C = \frac{2}{3}u^{3/2} + C$$

$$u = 1 + x^2 \quad \rightarrow \quad \left(\frac{2}{3}(1+x^2)^{3/2} + C\right)' = \frac{2}{3}\cdot\frac{3}{2}(1+x^2)^{1/2}\cdot(1+x^2)' + \cancel{C'}^{\,0}$$

$$= \underbrace{(1+x^2)^{1/2}\cdot 2x}$$

$$\text{So, } \int 2x\sqrt{1+x^2}\,dx = \frac{2}{3}(1+x^2)^{3/2} + C.$$

## Example 7

Find the indefinite integrals:

a) $\int x^3 \underbrace{\cos(x^4+2)} \, dx$.

b) $\int \sqrt{2x+1} \, dx$.

c) $\int \sqrt{1+x^2} x^5 \, dx$.

a) $f(x) = \cos x$

$u = g(x) = x^4 + 2 \rightarrow \frac{du}{dx} = 4x^3 \rightarrow du = 4x^3 dx$

$\rightarrow \frac{du}{4} = x^3 dx$

So, $\int x^3 \cos(x^4+2) \, dx = \int \underbrace{\cos(x^4+2)} \, x^3 dx$

$= \int \cos(u) \frac{du}{4}$

$= \frac{\sin(u)}{4} + C$

$$\boxed{= \frac{\sin(x^4+2)}{4} + C}$$

b) $f(x) = \sqrt{x}$

$u = g(x) = 2x+1 \rightarrow \frac{du}{dx} = 2 \rightarrow du = 2dx \rightarrow \frac{du}{2} = dx$

$\int \sqrt{2x+1} \, dx = \int \sqrt{u} \frac{du}{2} = \frac{1}{2} \frac{u^{3/2}}{3/2} + C$

$= \frac{u^{3/2}}{3} + C$

$$\boxed{= \frac{(2x+1)^{3/2}}{3} + C}$$

c) $f(x) = \sqrt{x}$

$u = g(x) = \underline{1+x^2} \rightarrow \frac{du}{dx} = 2x \rightarrow du = 2x\,dx \rightarrow \frac{du}{2} = x\,dx$

$$\int \sqrt{1+x^2}\, x^5\, dx = \int \sqrt{1+x^2}\, \underset{(x^2)^2}{\underset{=}{x^4}} \cdot \underbrace{x\,dx}_{du/2}$$

Here, $u = 1+x^2 \rightarrow \underline{u-1} = x^2$

$$= \int \sqrt{u}\,(u-1)^2\, \frac{du}{2}$$

$$= \int u^{1/2}\,(u^2 - 2u + 1)\, \frac{du}{2}$$

$$= \frac{1}{2}\int u^{5/2} - 2u^{3/2} + u^{1/2}\, du$$

$$= \frac{1}{2}\left( \frac{u^{7/2}}{7/2} - \frac{2u^{5/2}}{5/2} + \frac{u^{3/2}}{3/2} \right) + C$$

$$= \frac{u^{7/2}}{7} - \frac{2}{5}u^{5/2} + \frac{u^{3/2}}{3} + C.$$

## Example 9

Compute the value of $\int_0^4 \sqrt{2x+1}\,dx$.

We know that $\int \sqrt{2x+1}\,dx = \dfrac{(2x+1)^{3/2}}{3} + C$

I choose $F(x) = \dfrac{(2x+1)^{3/2}}{3}$

So,

$$\int_0^4 \sqrt{2x+1}\,dx \overset{FTC}{=} \left.\frac{(2x+1)^{3/2}}{3}\right|_0^4$$

$$= \frac{(2\cdot 4+1)^{3/2}}{3} - \frac{(2\cdot 0+1)^{3/2}}{3}$$

$$= \frac{9^{3/2}}{3} - \frac{1^{3/2}}{3}$$

$$= \frac{(\sqrt{9})^3}{3} - \frac{1}{3}$$

$$= \frac{27}{3} - \frac{1}{3} \boxed{= \frac{26}{3}}$$

## Example 10

Compute the value of the definite integrals.

a) $\displaystyle\int_0^4 \sqrt{2x+1}\,dx.$ $\qquad \displaystyle\int_{x=0}^{x=4} \sqrt{2x+1}\,dx$

b) $\displaystyle\int_1^2 \frac{dx}{(3-5x)^2}.$

a) Put $u = 2x+1 \to \dfrac{du}{dx} = 2 \to du = 2\,dx \to \dfrac{du}{2} = dx$

$x=0 \to u = 2\cdot 0+1 = 1$

$x=4 \to u = 2\cdot 4+1 = 9$

So,

$$\int_0^4 \sqrt{2x+1}\,dx = \int_{u=1}^{u=9} \sqrt{u}\,\frac{du}{2} = \frac{1}{2}\frac{u^{3/2}}{3/2}\Bigg|_1^9$$

$$\boxed{= \frac{26}{3}}$$

b) $\displaystyle\int_1^2 \frac{dx}{(3-5x)^2} = \int_1^2 \frac{1}{(3-5x)^2}\,dx.$

① Change of variable. ② Evaluate the integral.

① Let $u = 3-5x \to \dfrac{du}{dx} = -5 \to du = -5\,dx \to \dfrac{du}{-5} = dx$

$x=1 \to u = 3-5\cdot 1 = -2$

$x=2 \to u = 3-5\cdot 2 = -7$

② $\displaystyle\int_1^2 \frac{1}{(3-5x)^2}\,dx = \int_{-2}^{-7} \frac{1}{u^2}\,\frac{du}{-5}$

$$= -\frac{1}{5}\int_{-2}^{-7} u^{-2}\,du$$

$$= \frac{1}{5}\int_{-7}^{-2} u^{-2}\,du$$

$$= \frac{1}{5} \left. \frac{u^{-1}}{-1} \right|_{-7}^{-2}$$

$$= \frac{1}{5} \left( -\left(\frac{1}{-2}\right) - \left(\frac{-1}{-7}\right)\right)$$

$$= \frac{1}{5} \left( \frac{1}{2} - \frac{1}{7} \right)$$

$$= \frac{1}{5} \left( \frac{7-2}{2 \cdot 7} \right)$$

$$\boxed{= \frac{1}{14}}$$

## Example 11

Compute the value of

a) $\int_{-1}^{1} x^2\,dx.$  b) $\int_{-1}^{1} x^3\,dx.$

a) $x^2 = (-x)^2 \rightarrow f(x) = f(-x) \rightarrow f(x) = x^2$ is even.

$$\int_{-1}^{1} x^2\,dx = \underbrace{\int_{-1}^{0} x^2\,dx}_{-1 \le x \le 0} + \int_{0}^{1} x^2\,dx$$

Let $u = -x \rightarrow \frac{du}{dx} = -1 \rightarrow du = -dx \rightarrow -du = dx$

$\underline{-u = x}$

So,

$$\int_{-1=x}^{0=x} x^2\,dx = \int_{1=u}^{0=u} (-u)^2\,(-du)$$

$$= -\int_{1}^{0} u^2\,du$$

$$= \int_{0}^{1} u^2\,du$$

Let $x = t \rightarrow dx = dt$

So, $\int_0^1 x^2\,dx = \int_0^1 t^2\,dt$

Let $u = t \rightarrow du = dt$

So, $\int_0^1 u^2\,dx = \int_0^1 t^2\,dt$

So,

$$\int_{-1}^{1} x^2\,dx = \int_0^1 u^2\,du + \int_0^1 x^2\,dx$$

$$= \int_0^1 t^2\,dt + \int_0^1 t^2\,dt$$

$$= 2\int_0^1 t^2\,dt = 2\int_0^1 x^2\,dx$$

Now, we know $\int_0^1 x^2\,dx = \frac{1}{3}$

$$\Rightarrow \boxed{\int_{-1}^{1} x^2\,dx = \frac{2}{3}}$$

b) $$\int_{-1}^{1} x^3\,dx = \underbrace{\int_{-1}^{0} x^3\,dx}_{-1 \le x \le 0} + \int_0^1 x^3\,dx$$

Let $u = -x \rightarrow \frac{du}{dx} = -1 \rightarrow du = -dx \rightarrow -du = dx$

$\underline{-u = x}$

so, $$\int_{-1=x}^{0=x} x^3\,dx = \int_{1=u}^{0=u} (-u)^3\,(-du)$$

$$= -\int_1^0 -(u)^3\,du$$

$$= -\int_0^1 u^3\,du$$

Now,

$$\int_{-1}^{1} x^3\,dx = -\underbrace{\int_0^1 u^3\,du}_{=\int_0^1 x^3\,dx} + \int_0^1 x^3\,dx$$

$$= -\int_0^1 x^3\,dx + \int_0^1 x^3\,dx \boxed{= 0}$$