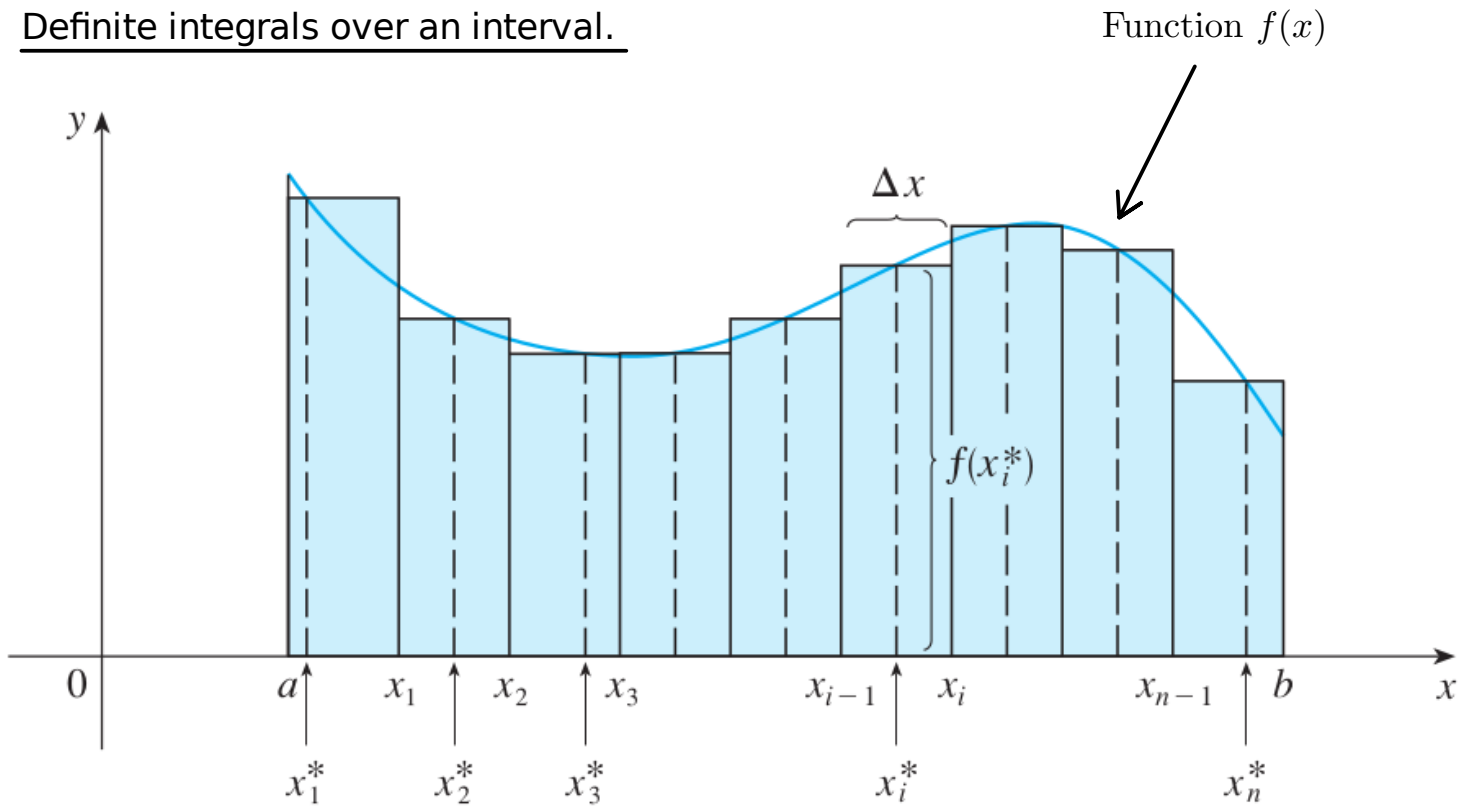


Chapter 15

Multiple Integrals

15.1 Double Integrals over a rectangle

Definite integrals over an interval.



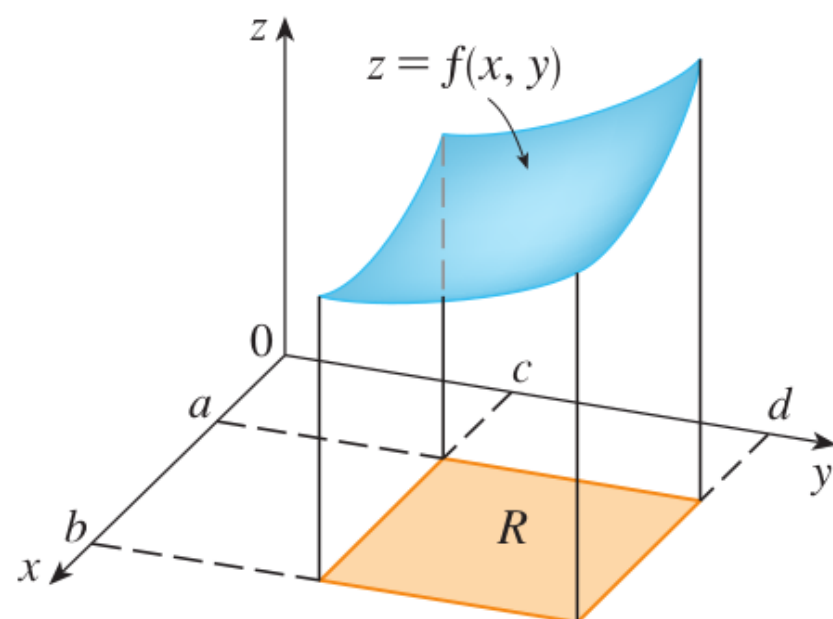
- 1) Divide the interval in n parts of equal length Δx
- 2) Name each subinterval $[a, x_1], [x_1, x_2], \dots, [x_{n-1}, b]$
- 3) Choose some point x_1^* in $[a, x_1], x_2^*$ in $[x_1, x_2], \dots, x_n^*$ in $[x_{n-1}, b]$
 \Rightarrow Total Area of rectangles $= f(x_1^*)\Delta x + f(x_2^*)\Delta x + \dots + f(x_n^*)\Delta x$
- 4) Take the limit as $n \rightarrow \infty$

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x$$

Useful Fact:

$$\int_a^b f(x) dx \approx \sum_{i=1}^n f(x_i^*) \Delta x .$$

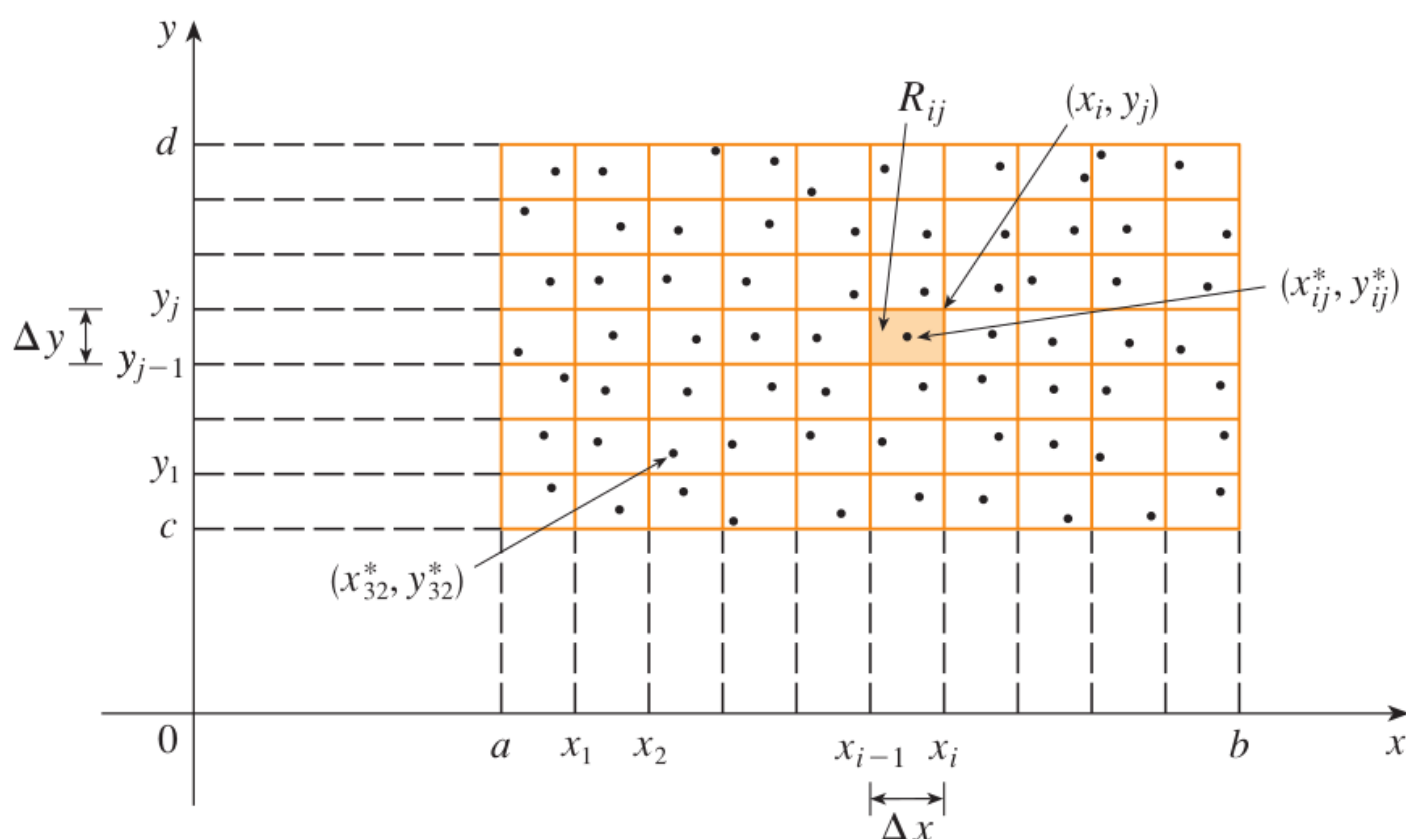
Volumes and Double Integrals.



Given:

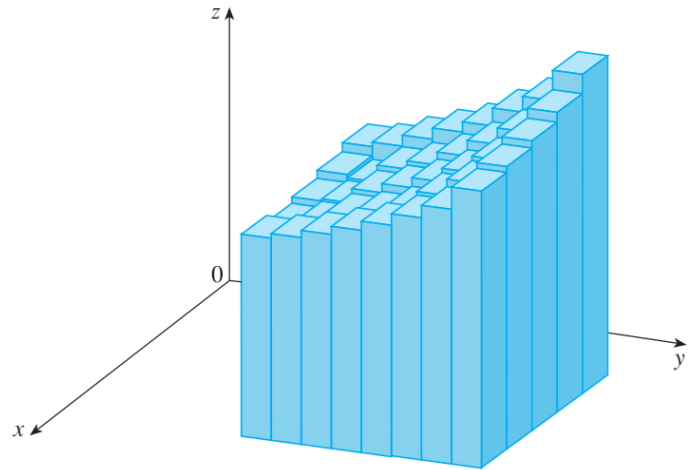
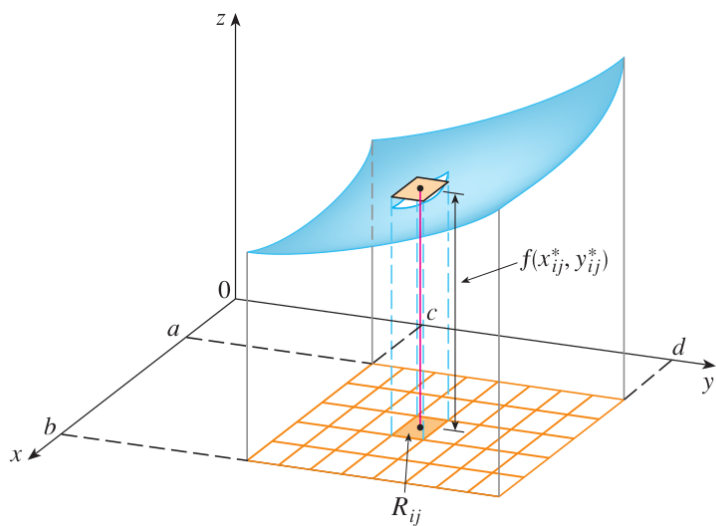
- A function $z = f(x, y)$
- The domain $R = [a, b] \times [c, d]$

1st Step: Divide the domain to create a grid.



- 1) Divide $[a, b]$ in m equal parts Δx
- 2) Divide $[c, d]$ in n equal parts Δy
- 3) Create the rectangle $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$
- 4) Select a point (x_{ij}^*, y_{ij}^*) in R_{ij}

2nd Step: Approximate the volume by "buildings"



1) Volume of a building: $\Delta A \cdot f(x_{ij}^*, y_{ij}^*)$

2) Total volume:

$$V = \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \cdot \Delta A$$

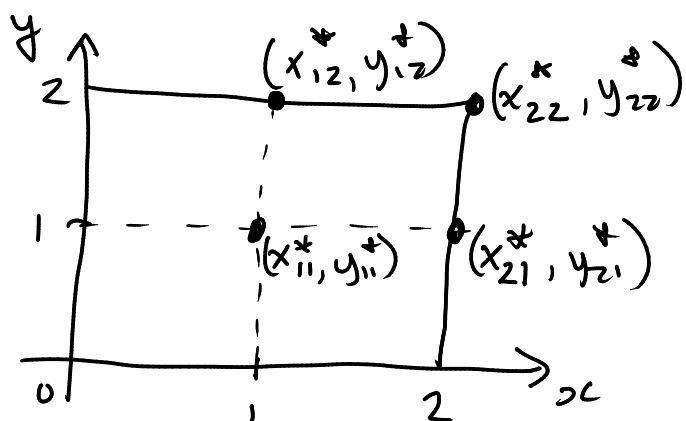
3) Take the limit as $m, n \rightarrow \infty$:

$$\iint_R f(x, y) dA = \lim_{n, m \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A$$

Useful Fact:

$$\iint_R f(x, y) dA \approx \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A$$

EXAMPLE 1 Estimate the volume of the solid that lies above the square $R = [0, 2] \times [0, 2]$ and below the elliptic paraboloid $z = 16 - x^2 - y^2$. Divide R into four equal squares and choose the sample point to be the upper right corner of each square R_{ij} . Sketch the solid and the approximating rectangular boxes.



$$(x_{11}^*, y_{11}^*) = (1, 1) \quad (x_{21}^*, y_{21}^*) = (2, 1)$$

$$(x_{12}^*, y_{12}^*) = (1, 2) \quad (x_{22}^*, y_{22}^*) = (2, 2)$$

$$f(x, y) = 16 - x^2 - y^2$$

$$\Delta A = 1$$

$$\iint_R 16 - x^2 - y^2 \, dA \approx \sum_{i=1}^2 \sum_{j=1}^2 [16 - (x_{ij}^*)^2 - (y_{ij}^*)^2] \Delta A$$

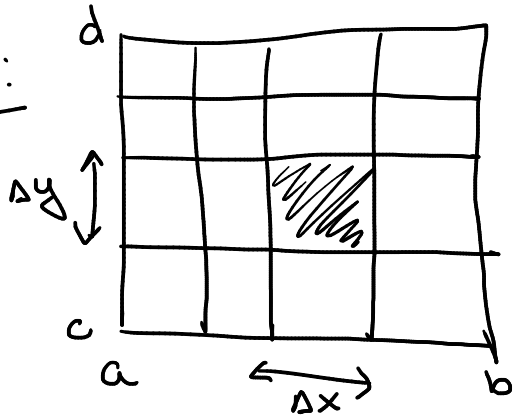
$$= (16 - 1^2 - 1^2) \cdot 1 + (16 - 1^2 - 2^2) \cdot 1$$

$$+ (16 - 2^2 - 1^2) \cdot 1 + (16 - 2^2 - 2^2) \cdot 1$$

$$= \boxed{44}$$

Evaluating Integrals with cartesian coordinates: Iterated Integrals

Grid:



$[a, b]$ in m parts Δx .

$[c, d]$ in n parts by

$$\begin{array}{lcl} \begin{array}{c} \Delta y \\ \square \\ \Delta x \end{array} \rightarrow & \Delta A = \Delta x \Delta y \xrightarrow{\min \rightarrow \infty} & dA = dx dy \quad (1) \\ & \Delta A = \Delta y \Delta x \xrightarrow{\min \rightarrow \infty} & dA = dy dx \quad (2) \end{array}$$

$$\frac{d}{dx}(2) = 0$$

$$\frac{d}{dx}(2x) = 2$$

$$\frac{\partial}{\partial x} (x^2 + y^2) = 2x$$

$$\frac{\partial}{\partial y}(xy^2) = x \frac{\partial}{\partial y}(y^2) = 2xy.$$

Fubini's Theorem:

$$\iint_R f(x, y) dA = \int_c^d \int_a^b f(x, y) dx dy = \int_a^b \int_c^d f(x, y) dy dx$$

Called "iterated integrals"

EXAMPLE 4 Evaluate the iterated integrals.

$$(a) \int_0^3 \left[\int_1^2 x^2 y \, dy \right] dx = I_1 \quad (b) \int_1^2 \left[\int_0^3 x^2 y \, dx \right] dy = I_2$$

$$(a) I_1 = \begin{array}{l} \text{first } \int_1^2 x^2 y \, dy \\ \text{second } \int_0^3 (\dots) \, dx \end{array}$$

$$\int_1^2 x^2 y \, dy = x^2 \int_1^2 y \, dy = x^2 \left. \frac{y^2}{2} \right|_1^2 = x^2 \left(2 - \frac{1}{2} \right) = \frac{3x^2}{2}$$

$$I_1 = \int_0^3 \underbrace{\int_1^2 x^2 y \, dy}_{= \frac{3x^2}{2}} dx = \int_0^3 \frac{3x^2}{2} dx = \frac{3}{2} \left. \frac{x^3}{3} \right|_0^3 = \boxed{\frac{27}{2}}$$

$$(b) \int_0^3 x^2 y \, dx = y \int_0^3 x^2 \, dx = y \left. \frac{x^3}{3} \right|_0^3 = 9y$$

$$\begin{aligned} I_2 &= \int_1^2 \underbrace{\int_0^3 x^2 y \, dx}_{9y} dy = \int_1^2 9y \, dy = \left. \frac{9y^2}{2} \right|_1^2 \\ &= 18 - \frac{9}{2} \\ &= \boxed{\frac{27}{2}} \end{aligned}$$

Notice: $I_1 = I_2$. The order doesn't matter.

Example. Evaluate the following integral:

$$R = [0, 1] \times [0, 1]$$

$$I = \int_0^1 \left[\int_0^1 v(u^2 + v^2)^4 du \right] dv$$

$$\int_0^1 v(u^2 + v^2)^4 du = v \int_0^1 (u^2 + v^2)^4 du$$

Change the order! (to use a u-sub method).

$$I = \int_0^1 \int_0^1 v(u^2 + v^2)^4 dv du$$

$$\left(\begin{array}{l} w = u^2 + v^2 \rightarrow dw = 2v dv \rightarrow \frac{dw}{2} = v dv \\ \rightarrow \end{array} \right. I = \int_0^1 \int_{u^2}^{u^2+1} w^4 \frac{dw}{2} du$$

$$= \int_0^1 \frac{1}{2} \frac{w^5}{5} \Big|_{u^2}^{u^2+1} du$$

$$= \frac{1}{10} \int_0^1 (u^2+1)^5 - u^{10} du$$

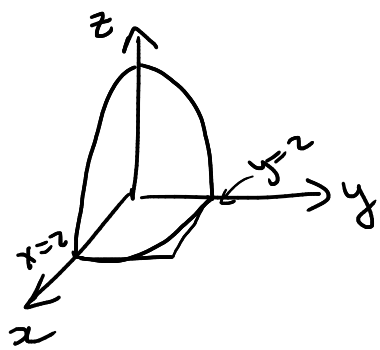
$$= \frac{1}{10} \int_0^1 \cancel{u^{10}} + 5u^8 + 10u^6 + 10u^4 + 5u^2 + 1 - \cancel{u^{10}} du$$

$$= \frac{1}{10} \int_0^1 5u^8 + 10u^6 + 10u^4 + 5u^2 + 1 du$$

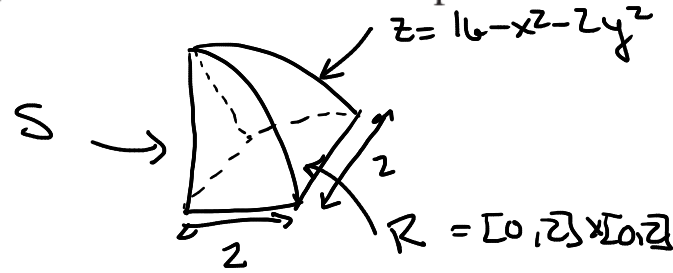
$$= \boxed{\frac{419}{630}}$$

EXAMPLE 7 Find the volume of the solid S that is bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$, and the three coordinate planes.

↳ $z = 16 - x^2 - 2y^2$



solid



$$\text{Vol}(S) = \iint_R (16 - x^2 - 2y^2) \, dA$$

So,

$$\text{Vol}(S) = \int_0^2 \int_0^2 (16 - x^2 - 2y^2) \, dx \, dy$$

$$= \int_0^2 \left[16x - \frac{x^3}{3} - 2y^2x \right]_0^2 \, dy$$

$$= \int_0^2 \left(32 - \frac{8}{3} - 4y^2 \right) \, dy$$

$$= \left[32y - \frac{8}{3}y - \frac{4y^3}{3} \right]_0^2$$

$$= \left[64 - \frac{16}{3} - \frac{32}{3} \right]$$

$$= \frac{192 - 16 - 32}{3} = \frac{144}{3} \approx 48$$

EXAMPLE 8 If $R = [0, \pi/2] \times [0, \pi/2]$, then compute $\iint_R \sin x \cos y \, dA$.

Useful Fact:

$$\iint_R g(x)h(y) \, dA = \left(\int_a^b g(x) \, dx \right) \left(\int_c^d h(y) \, dy \right)$$

where $R = [a, b] \times [c, d]$