Chapter 3
Applications of Derivatives

3.4 Limits at Infinity; Horizontal Asymptotes

Example. What is the limit of $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$ when x becomes large?

x	$f(x)$
10	≈ 0.99
100	≈ 0.9998
1000	≈ 0.999998 .
10000	≈ 0.99999998 !
$\downarrow$	$\downarrow$
∞	1

We say that as

$x \to \infty$

then $f(x) \to 1$

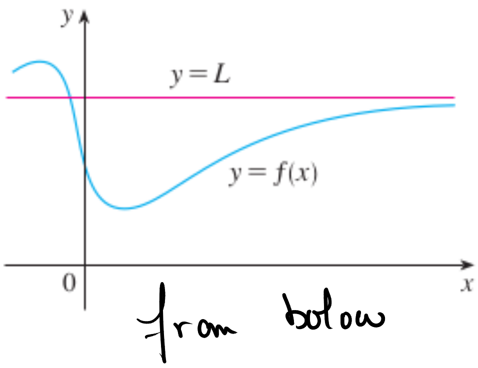
So,

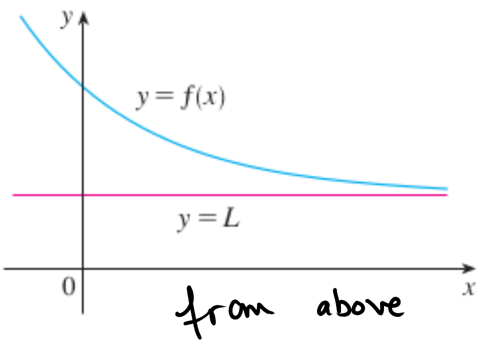
$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2-1}{x^2+1} = 1$$

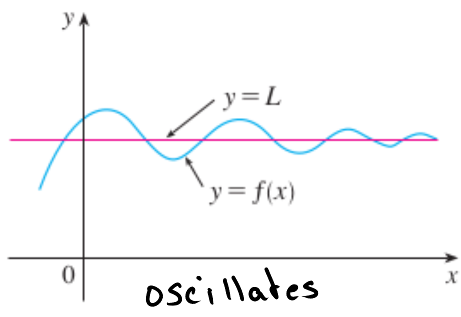
1 Intuitive Definition of a Limit at Infinity Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x\to\infty} f(x) = L \quad \leftarrow L \neq \infty$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large.







Example. What is the limit of $f(x) = \dfrac{x^2-1}{x^2+1}$ when x becomes large? negative large values!

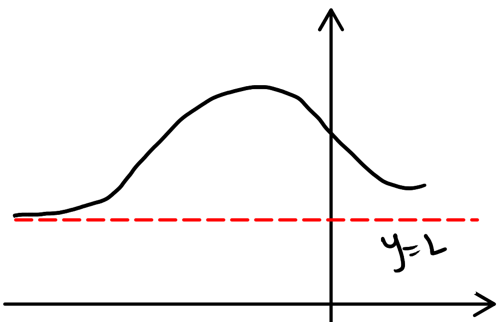
x	$f(x)$
-10	0.98
-100	0.998
-1000	0.9998
$-10\,000$	$0.99999998.$
$\downarrow$	$\downarrow$
$-\infty$	1

We say that as
$$x \longrightarrow -\infty$$
then
$$f(x) \longrightarrow 1$$

2 Definition Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = L \quad \to \text{not } \infty \ (-\infty)$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large negative.



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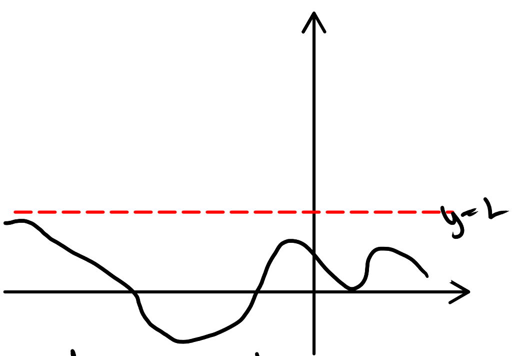

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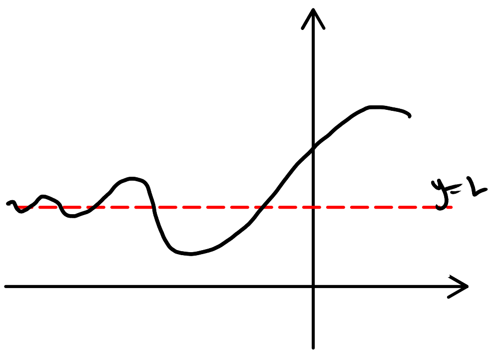

3 Definition The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = L$$

$L=2$

$\lim_{x\to\infty} f(x) = 2$

EXAMPLE 1 Find the infinite limits, limits at infinity, and asymptotes for the function f whose graph is shown in Figure 5.

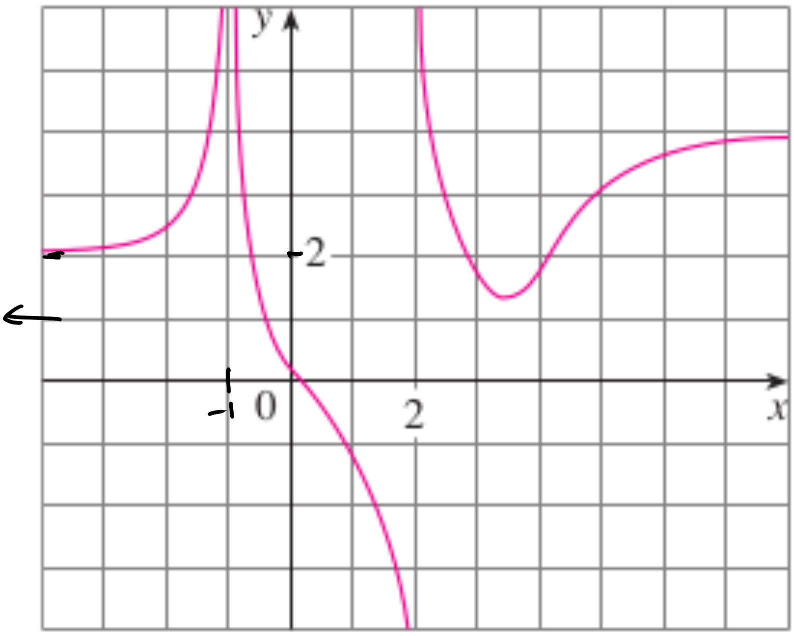

FIGURE 5

A) Infinite limits.

$\lim_{x\to-1^-} f(x) = +\infty$ & $\lim_{x\to-1^+} f(x) = +\infty$

$\to$ a vertical asymptote at $x=-1$
(VA)

$\lim_{x\to2^-} f(x) = -\infty$ & $\lim_{x\to2^+} f(x) = +\infty$

$\to$ a VA at $x=2$.

B) Limits at infinity.

$\lim_{x\to\infty} f(x) = 4 \longrightarrow$ a horizontal asymptote (HA) at $y=4$

$\lim_{x\to-\infty} f(x) = 2 \longrightarrow$ a horizontal asymptote (HA) at $y=2$.

4 Theorem If $r > 0$ is a rational number, then

$$\lim_{x\to\infty} \frac{1}{x^r} = 0$$

If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x\to-\infty} \frac{1}{x^r} = 0$$

EXAMPLE 3 Evaluate

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$$

$\frac{\infty}{\infty}$

① Factor

$$\frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \frac{\cancel{x^2}(3 - 1/x - 2/x^2)}{\cancel{x^2}(5 + 4/x + 1/x^2)} = \frac{3 - 1/x - 2/x^2}{5 + 4/x + 1/x^2}.$$

② Rules (standard) of limits.

$$\lim_{x\to\infty} 3 - 1/x - 2/x^2 = \lim_{x\to\infty} 3 - \underbrace{\lim_{x\to\infty} 1/x}_{=0} - 2\underbrace{\lim_{x\to\infty} \frac{1}{x^2}}_{=0}$$

$$= 3 - 0 - 2\cdot 0 = 3$$

$$\lim_{x\to\infty} 5 + 4/x + 1/x^2 = \lim_{x\to\infty} 5 + 4\underbrace{\lim_{x\to\infty} \frac{1}{x}}_{=0} + \underbrace{\lim_{x\to\infty} \frac{1}{x^2}}_{=0}$$

$$= 5 + 4\cdot 0 + 0 = 5$$

Then

$$\lim_{x\to\infty} \frac{3 - 1/x - 2/x^2}{5 + 4/x + 1/x^2} = \frac{\lim_{x\to\infty} 3 - 1/x - 2/x^2}{\lim_{x\to\infty} 5 + 4/x + 1/x^2} = \boxed{\frac{3}{5}}$$

↓

HA

EXAMPLE 4 Find the horizontal and vertical asymptotes of the graph of the function

$\lim_{x \to +\infty} \frac{\sqrt{2x^2+1}}{3x-5}$.

$$f(x) = \frac{\sqrt{2x^2+1}}{3x-5}$$

① Limits at infinity. (HA).

Take the square $\rightarrow$ $\frac{2x^2+1}{(3x-5)^2} = \frac{2x^2+1}{9x^2-30x+25}$

1) $\lim_{x\to\infty} \frac{2x^2+1}{9x^2-30x+25} = \lim_{x\to\infty} \frac{2+1/x^2}{9-30/x+25/x^2}$

$= \frac{2+0}{9-30\cdot 0+25\cdot 0} = \frac{2}{9}$

$\Rightarrow \lim_{x\to\infty} f(x) = \sqrt{\frac{2}{9}} = \frac{\sqrt{2}}{3}$.

$\sqrt{ab} = \sqrt{a}\cdot\sqrt{b}$

Don't square $\rightarrow$ 2) $\lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to-\infty} \frac{\sqrt{x^2(2+1/x^2)}}{x(3-5/x)}$

$\sqrt{x^2} = |x| = -x$

$= \lim_{x\to\infty} \frac{\sqrt{x^2}\sqrt{2+1/x^2}}{x\,(3-5/x)}$

$= \lim_{x\to-\infty} \frac{-\cancel{x}}{\cancel{x}} \frac{\sqrt{2+1/x^2}}{3-5/x}$

$= -\lim_{x\to-\infty} \frac{\sqrt{2+1/x}}{3-5/x}$

$= -\frac{\sqrt{2+0}}{3-5\cdot 0} = -\frac{\sqrt{2}}{3}$

HA.

$y = \frac{\sqrt{2}}{3}$ $y = -\frac{\sqrt{2}}{3}$

② Vertical Asymptotes (VA)

$3x-5=0 \rightarrow x = \frac{5}{3}$.

$\lim_{x\to(5/3)^-} \frac{\sqrt{2x^2+1}}{3x-5} = \frac{\sqrt{50/9+1}}{0^-} = -\infty$.

$x < 5/3 \rightarrow 3x-5 < 0$

So, VA at $x = \frac{5}{3}$.

EXAMPLE 5 Compute $\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)$.

$x^2 \nearrow \infty$

$x \nearrow \infty$

WRONG WAY. ✗ bad

$$\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)$$

$$= \lim_{x\to\infty}\sqrt{x^2+1} - \lim_{x\to\infty} x$$

$$= \infty - \infty \cancel{=} 0$$

Good WAY. solution: conjugate!

$$\sqrt{x^2+1}-x = \left(\sqrt{x^2+1}-x\right)\cdot\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}$$

$$= \frac{x^2+1-x^2}{\sqrt{x^2+1}+x}$$

$$= \frac{1}{\sqrt{x^2+1}+x}$$

so,

$$\lim_{x\to\infty}\sqrt{x^2+1}-x = \lim_{x\to\infty}\frac{1}{\sqrt{x^2+1}+x}$$

$$= \lim_{x\to\infty}\frac{1}{\sqrt{x^2(1+1/x^2)}+x}$$

$$= \lim_{x\to\infty}\frac{1}{x\sqrt{1+1/x^2}+x} \qquad \left(\sqrt{x^2}=x,\ x>0\right)$$

$$= \lim_{x\to\infty}\frac{1}{x\left(\sqrt{1+1/x^2}+1\right)}$$

$\lim_{x\to\infty}(x^2-x) = \infty - \cancel{\infty}$

$$= \left(\lim_{x\to\infty}\frac{1}{x}\right)\left(\lim_{x\to\infty}\frac{1}{\sqrt{1+1/x^2}+1}\right)$$

$$= 0\cdot\frac{1}{\sqrt{1+0}+1} = \boxed{0}$$

The notation

$$\lim_{x \to \infty} f(x) = \infty$$

means that the values of $f(x)$ become larger and larger as the values of x becomes larger and larger. Similar meanings are attached to the following symbols:

$$\lim_{x \to -\infty} f(x) = \infty, \qquad \lim_{x \to \infty} f(x) = -\infty \quad \text{and} \quad \lim_{x \to -\infty} f(x) = -\infty.$$

WARNING!! Be careful: $\infty - \infty$ is undefined!!

EXAMPLE 8 Find $\lim_{x \to \infty} x^3$ and $\lim_{x \to -\infty} x^3$.

$\lim_{x \to \infty} x^3 = \infty$ (see Desmos).

$\lim_{x \to -\infty} x^3 = -\infty$ (see Desmos).

EXAMPLE 9 Find $\lim_{x\to\infty} (x^2 - x)$. $\infty - \infty$ (indetermination)

$$\lim_{x\to\infty} (x^2 - x) = \lim_{x\to\infty} x(x-1)$$

Product rule for infinite limits. $\lim_{x\to\infty} f(x) = \infty$ & $\lim_{x\to\infty} g(x) = \infty$

$$\Rightarrow \lim_{x\to\infty} f(x)\,g(x) = \infty \cdot \infty = \infty$$

$f(x) = x$ & $g(x) = x-1$

$$\Rightarrow \lim_{x\to\infty} x(x-1) = \left(\lim_{x\to\infty} x\right)\left(\lim_{x\to\infty} x-1\right)$$

Substracting a constant Rule: $\lim_{x\to\infty} g(x) = \infty$, then $\lim_{x\to\infty} (g(x) - c) = \infty$ for any constant c.

$\downarrow$

$g(x) = f(x) = x$

So, $\lim_{x\to\infty} (x-1) = \infty - 1 = \infty$

So, $\lim_{x\to\infty} (x^2 - x) = \infty \cdot \infty = \boxed{\infty}$.