

MATH 244 (Calculus IV), Fall 2021
Midterm Exam 1

I understand it is against the rules to cheat or engage in other academic misconduct during this test.

| | | |
|------------|-----|--|
| Question 1 | 20 | |
| Question 2 | 20 | |
| Question 3 | 20 | |
| Question 4 | 20 | |
| Question 5 | 20 | |
| Total | 100 | |

- This is a closed book, closed notes, no calculator exam. You are only allowed one two-sided cheat sheet.
- You must show your work on all problems. The correct answer with no supporting work may result in no credit. **Put a box around your FINAL ANSWER for each problem and cross out any work that you don't want to be graded.**
- Any student found engaging in academic misconduct will receive a score of 0 on this exam.
- You have 75 minutes to complete the exam, then 15 more minutes to scan and upload your solutions on Gradescope.

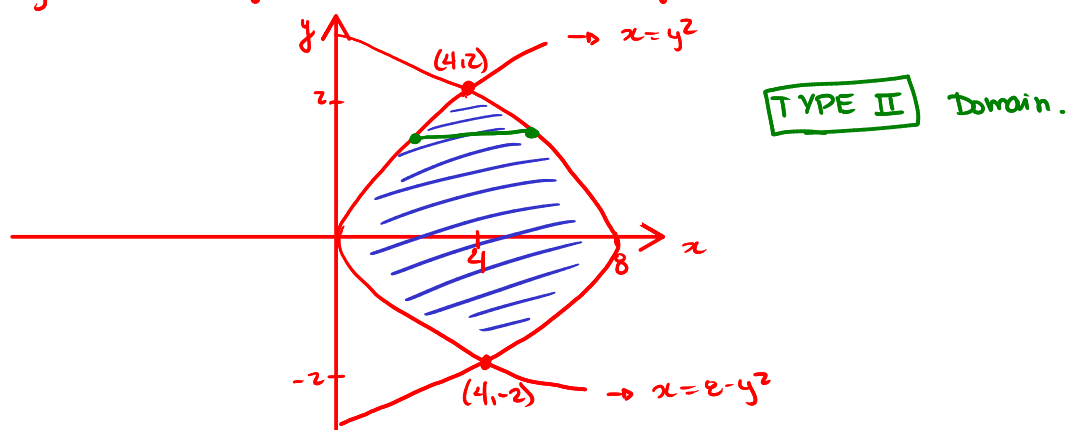
Problem 1.

a. Sketch the region D in the first quadrant bounded by the parabolas $x = y^2$ and $x = 8 - y^2$.

b. Calculate the integral

$$\iint_D y \, dA.$$

(a) $y^2 = 8 - y^2 \Rightarrow y^2 = 4 \Rightarrow y = \pm 2$. So $D = \{(x, y) : y^2 \leq x \leq 8 - y^2, -2 \leq y \leq 2\}$



$$\begin{aligned}
 (b) \quad \iint_D y \, dA &= \int_{-2}^2 \int_{y^2}^{8-y^2} y \, dx \, dy = \int_{-2}^2 y (8 - y^2 - y^2) \, dy \\
 &= \int_{-2}^2 8y - 2y^3 \, dy \\
 &= \left. 4y^2 - \frac{y^4}{2} \right|_{-2}^2 \\
 &= \left(16 - \frac{16}{2} \right) - \left(16 - \frac{16}{2} \right) \\
 &= \boxed{0}.
 \end{aligned}$$

Problem 2.

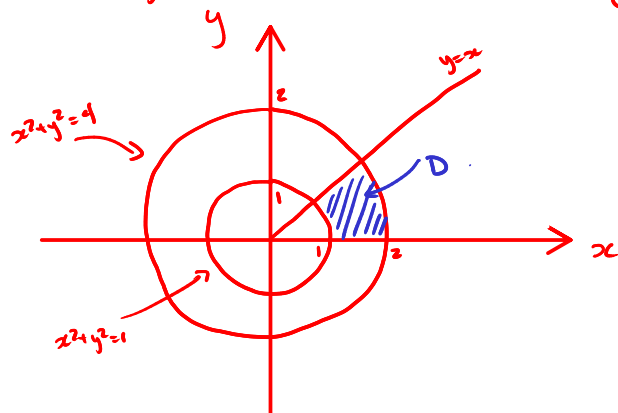
a. Sketch the region D in the xy -plane defined by

$$D := \{(x, y) : 1 \leq x^2 + y^2 \leq 4, 0 \leq y \leq x\}.$$

b. Calculate the integral

$$\iint_D \frac{\arctan(y/x)}{\sqrt{x^2 + y^2}} dA.$$

(a) the region D is bounded by the curves



$x^2 + y^2 = 1$ → circle
 $x^2 + y^2 = 4$ → circle
 $y = 0$ → hor. line
 $y = x$ → line.

(b) This is a sector. Use polar coordinates:

$$x = r \cos \theta \quad \& \quad y = r \sin \theta.$$

$$x^2 + y^2 = 1 \rightarrow r = 1 \quad \& \quad x^2 + y^2 = 4 \rightarrow r = 2$$

Also, $y = x \rightarrow r \cos \theta = r \sin \theta \rightarrow 1 = \tan \theta \rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \dots$
 keep $\theta = \pi/4$.

So, $y = x \leftrightarrow \theta = \frac{\pi}{4}$. then,

Also, $y = 0 \leftrightarrow \theta = 0$. $D = \{(r, \theta) : 1 \leq r \leq 2, 0 \leq \theta \leq \frac{\pi}{4}\}.$

So,

$$\iint_D \frac{\arctan(y/x)}{\sqrt{x^2 + y^2}} = \int_0^{\pi/4} \int_1^2 \frac{\theta}{r} r dr d\theta$$

$$= \left(\int_0^{\pi/4} \theta d\theta \right) \left(\int_1^2 dr \right) = \left(\frac{\pi^2/2^2}{2} \right) (2-1)$$

$$\theta = \arctan(y/x).$$

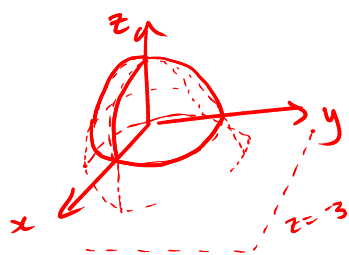
$$= \boxed{\frac{\pi^2}{8}}$$

Problem 3.

Find ~~the surface area~~ ^{the Volume} of the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the plane $z = -3$.

$$V(E) = \iiint_E 1 \, dv.$$

① Description of the solid.



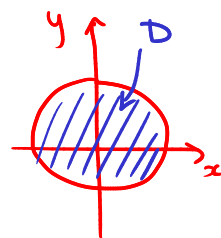
TYPE 1.

$$-3 \leq z \leq 1 - x^2 - y^2$$

Projection



$$\begin{aligned} z &= -3 \\ \downarrow \\ -3 &= 1 - x^2 - y^2 \\ \downarrow \\ x^2 + y^2 &= 4 \\ \downarrow \\ \text{circle radius} &= 2 \end{aligned}$$



Use Polar coordinates: $D = \{ (r, \theta) : 0 \leq r \leq 2 \text{ \& } 0 \leq \theta \leq 2\pi \}$.

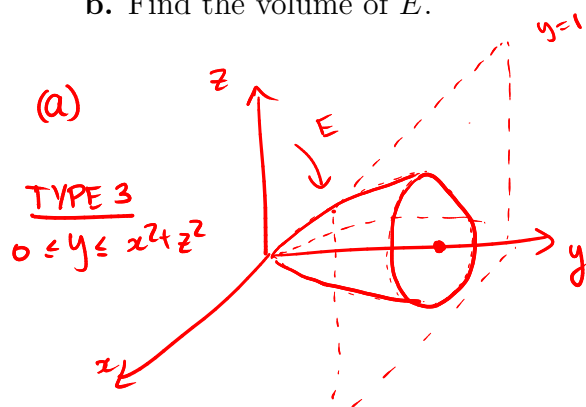
② Find the Volume

$$\begin{aligned} V &= \iiint_E 1 \, dv = \iint_D \left[\int_{-3}^{1-x^2-y^2} dz \right] dA \\ &= \iint_D (4 - x^2 - y^2) \, dA \\ &= \int_0^{2\pi} \int_0^2 (4 - r^2) r \, dr \, d\theta \\ &= 2\pi \left(2r^2 - \frac{r^4}{4} \right) \Big|_0^2 \\ &= 2\pi (8 - 4) \\ &= \boxed{8\pi} \end{aligned}$$

Problem 4.

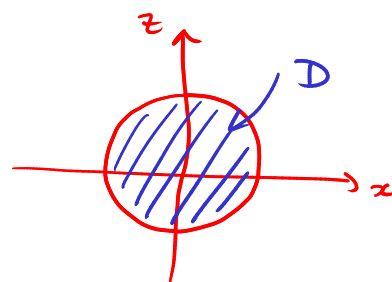
a. Sketch the solid E bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 1$.

b. Find the volume of E .



Project on $y=1$

$1 = x^2 + z^2 \rightarrow$ circle radius 1



$$D = \{(r, \theta) : 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}$$

(b)

$$\begin{aligned}
 V &= \iiint_E 1 \, dV \\
 &= \iint_D \left[\int_0^{x^2+z^2} dy \right] dA \\
 &= \iint_D (x^2 + z^2) \, dA \\
 &= \int_0^{2\pi} \int_0^1 r^2 \, r \, dr \, d\theta \\
 &= 2\pi \left(\frac{1}{4} \right) \\
 &= \boxed{\frac{\pi}{2}}
 \end{aligned}$$

$$\boxed{
 \begin{aligned}
 x &= r \cos \theta \\
 y &= r \sin \theta
 \end{aligned}
 }$$

Problem 5.

~~a. Sketch the surface whose equation in cylindrical coordinates is given by~~

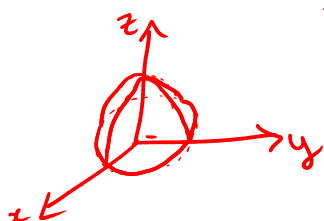
~~$$r^2 + z^2 = 4.$$~~

b. Set up but **do not evaluate** an iterated integral for

$$\int \int \int_E (x + y + z) dV,$$

where E is the solid in the first octant that lies under the paraboloid $z = 4 - x^2 - y^2$.

① Description of E .

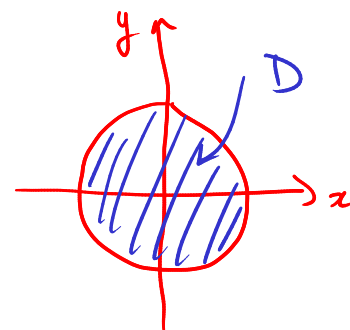


TYPE 1.

$$0 \leq z \leq 4 - x^2 - y^2$$

$$z=0 \rightarrow$$

$$x^2 + y^2 = 4$$



$$D = \{(r, \theta) : 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi\}$$

② Set-up the integral.

$$\begin{aligned} \iiint_E x + y + z dV &= \iint_D \left[\int_0^{4-x^2-y^2} x + y + z dz \right] dA \\ &= \int_0^{2\pi} \int_0^2 \left(\int_0^{4-r^2} r \cos \theta + r \sin \theta + z dz \right) r dr d\theta. \end{aligned}$$