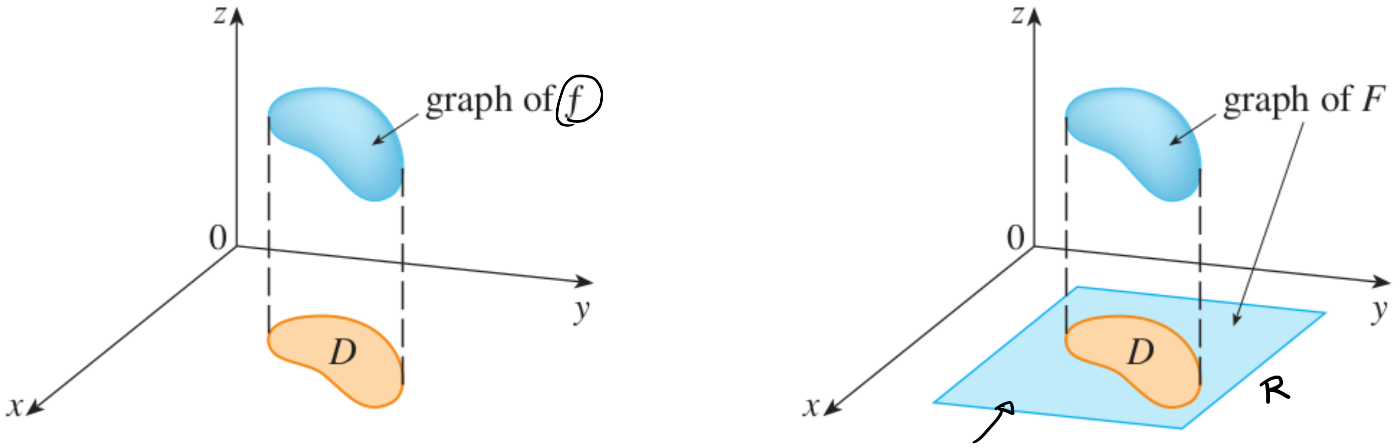
15.2 Double integrals over General Regions.

Definition.

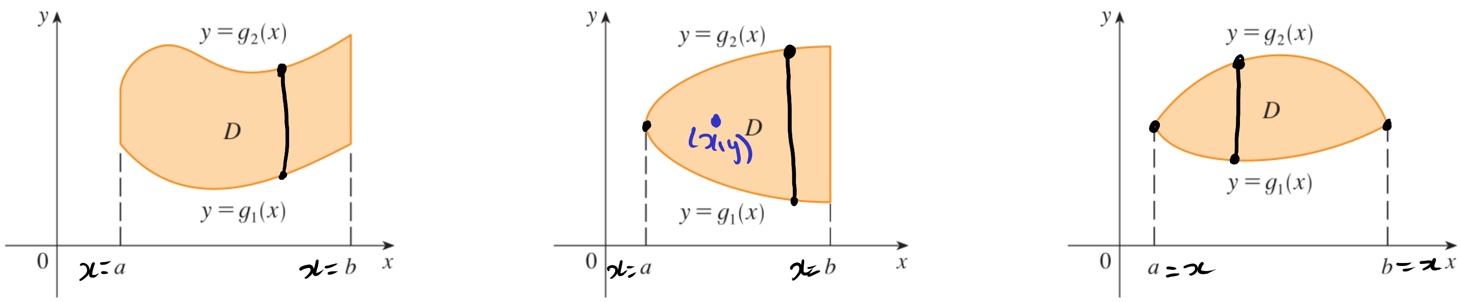
$$\boxed{\iint_D f(x, y)\, dA = \iint_R F(x, y)\, dA}$$

$$F(x,y) = \begin{cases} f(x,y), & (x,y) \text{ in } D \\ 0, & (x,y) \text{ outside of } D. \end{cases}$$

f: function defined on D.



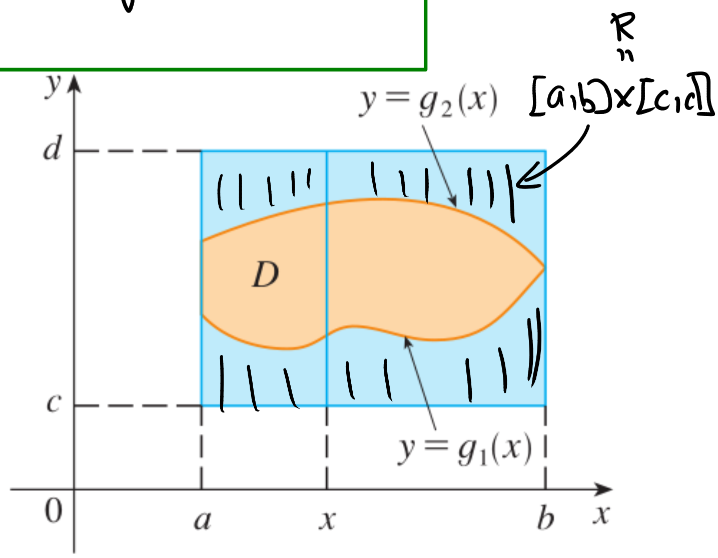
Region of type I.



$$D = \{(x, y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$$

$$\boxed{\iint_D f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx}$$

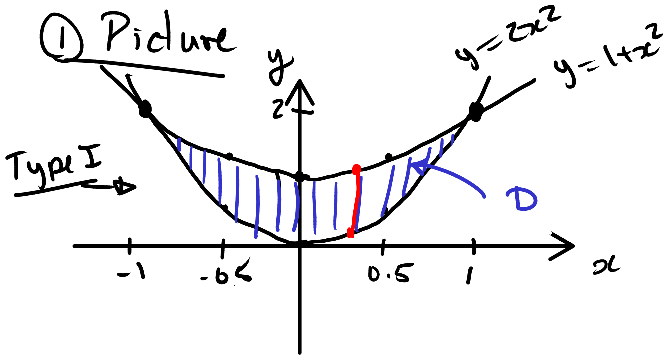
$$\begin{aligned} \iint_D f(x,y)\, dA &= \iint_R F(x,y)\, dA \\ &= \int_a^b \int_c^d F(x,y)\, dy\, dx \\ &= \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx \end{aligned}$$



EXAMPLE 1 Evaluate $\iint_D (x + 2y)\, dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

$z = f(x,y)$

1 Picture



$2x^2 = 1 + x^2 \iff x = \pm 1$

$1 + x^2 \leq y \leq 2x^2$

$D = \{(x,y) : -1 \leq x \leq 1,\ 1+x^2 \leq y \leq 2x^2\}$

2 Integrate.

$$\iint_D (x+2y)\, dA = \int_{-1}^{1} \int_{1+x^2}^{2x^2} (x+2y)\, dy\, dx$$

$$= \int_{-1}^{1} (xy + y^2)\Big|_{y=1+x^2}^{y=2x^2} dx$$

$$= \int_{-1}^{1} \left(2x^3 + 4x^4 - (x + x^3) - \underbrace{(1+x^2)^2}_{1+2x^2+x^4}\right) dx \quad \leftarrow \mathrm{I}$$

$$= \boxed{-\frac{32}{15}}$$

$$\mathrm{I} = \int_{-1}^{1} \cancel{2x^3} + \cancel{4x^4} - \cancel{x} - \cancel{x^3} - \cancel{1} - \cancel{2x^2} - \cancel{x^4}\, dx$$

$$= \int_{-1}^{1} x^3 + 3x^4 - 2x^2 - x - 1\, dx$$

$$= \left(\frac{x^4}{4} + \frac{3x^5}{5} - \frac{2x^3}{3} - \frac{x^2}{2} - x\right)\Big|_{-1}^{1}$$

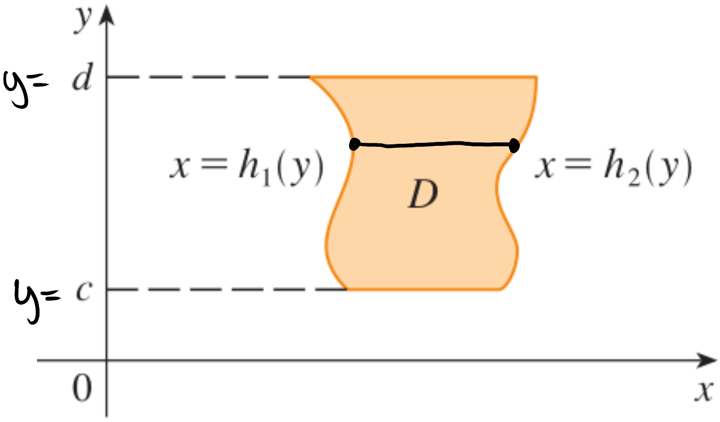
$$= \left(\cancel{\tfrac{1}{4}} + \cancel{\tfrac{3}{5}} - \cancel{\tfrac{2}{3}} - \cancel{\tfrac{1}{2}} - 1\right) - \left(\cancel{\tfrac{1}{4}} - \cancel{\tfrac{3}{5}} + \cancel{\tfrac{2}{3}} - \cancel{\tfrac{1}{2}} + 1\right)$$

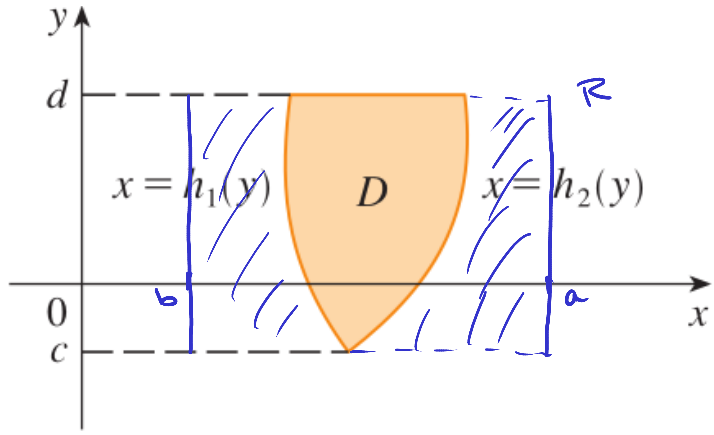
$$= \frac{6}{5} - \frac{4}{3} - 2$$

$$= \frac{18 - 20 - 30}{15} = \frac{-32}{15}$$

Region of Type II.

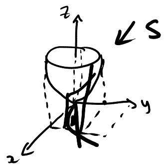
$$D = \{(x, y) \mid c \leq y \leq d,\ h_1(y) \leq x \leq h_2(y)\}$$



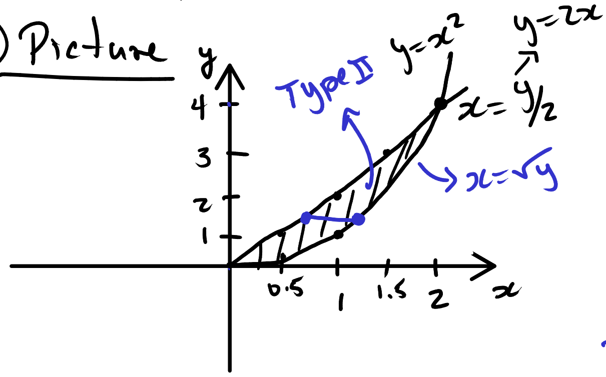


$$\boxed{\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy}$$

EXAMPLE 2 Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the $xy-$plane bounded by the line $x = y/2$ and the parabola $y = x^2$.



① <u>Picture</u>



$x = x^2/2 \to 2x = x^2$

$\to x^2 - 2x = 0 \to x = 0$ or $x = 2$

$x = \sqrt{y}$ $(x \geq 0)$.

$D = \{(x,y) : 0 \leq y \leq 4,\ \frac{y}{2} \leq x \leq \sqrt{y}\}$.

$f(x,y) = x^2 + y^2$.

② <u>Integrate</u>.

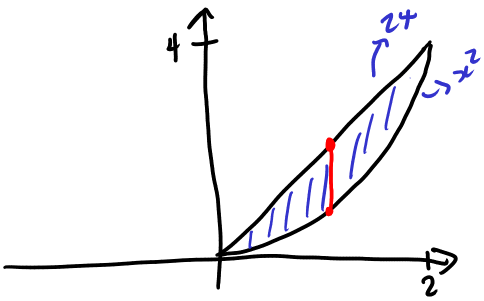
$$V(S) = \iint_D x^2+y^2\, dA = \int_0^4 \int_{\frac{y}{2}}^{\sqrt{y}} (x^2+y^2)\, dx\, dy$$

$$= \int_0^4 \left(\frac{x^3}{3} + xy^2\right)\Big|_{y/2}^{\sqrt{y}} dy$$

$$= \int_0^4 \left(\frac{y^{3/2}}{3} + y^{5/2}\right) - \left(\frac{y^3}{24} + \frac{y^3}{2}\right) dy$$

$$= \boxed{\frac{216}{35}}$$

<u>As a type I Domain.</u>

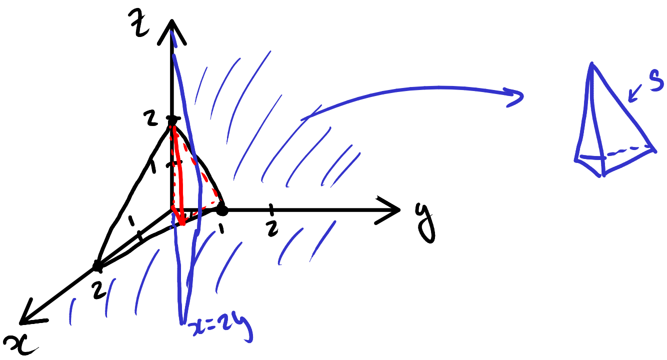


$D = \{(x,y) : 0 \leq x \leq 2,\ x^2 \leq y \leq 2x\}$.

EXAMPLE 4 Find the volume of the tetrahedron bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.

$z = 2 - x - 2y = f(x,y)$.

① Picture



$\underline{z=0}$

$\downarrow$

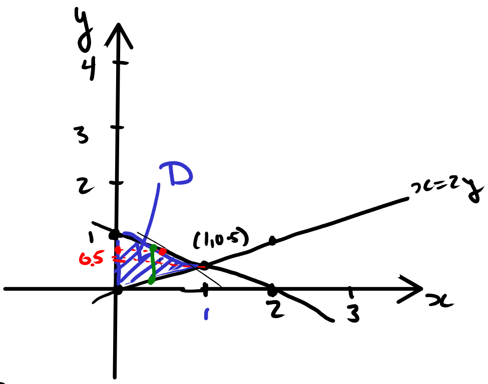
$x + 2y = 2$

$x = 2y$

$\downarrow$

intersection

$4y = 2 \to \boxed{y = \frac{1}{2}}$
$\to \boxed{x = 1}$



what should you do.

~~Is this a domain of~~ type I or type II? Easier to see as a Type I.

$D = \{(x,y) : 0 \le x \le 1,\ \frac{x}{2} \le y \le 1 - \frac{x}{2}\}$.

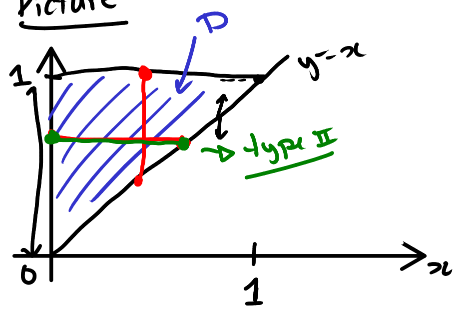
② Integrate

$$V(S) = \iint_D 2 - x - 2y \, dA = \int_0^1 \int_{\frac{x}{2}}^{1-\frac{x}{2}} 2 - x - 2y \, dy \, dx$$

$$= \boxed{1/3}$$

EXAMPLE 5 Evaluate the iterated integral $\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx$.

① Picture



TYPE I

$0 \le x \le 1$
$\underbrace{x \le y}_{y = x} \le 1$

TYPE II.

$0 \le x \le y$
$0 \le y \le 1$

② Integrate

$$\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx = \int_0^1 \int_0^y \sin(y^2)\, dx\, dy$$

$$= \int_0^1 \sin(y^2) \left(\int_0^y dx\right) dy$$

$$= \int_0^1 \sin(y^2)\,(y)\, dy$$

$u = y^2 \to du = 2y\,dy$

$\frac{du}{2} = y\,dy$

$$= \int_0^1 \sin(u) \frac{du}{2}$$

$$= -\frac{\cos(u)}{2}\Big|_0^1 = \boxed{\frac{1 - \cos 1}{2}}$$

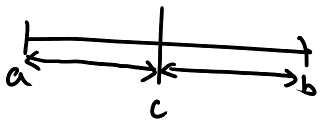
Properties of Double Integrals.

$\boxed{6}$ $\iint_D [f(x,y) + g(x,y)]\,dA = \iint_D f(x,y)\,dA + \iint_D g(x,y)\,dA.$

$\boxed{7}$ $\iint_D c f(x,y)\,dA = c\iint_D f(x,y)\,dA.$

$\boxed{8}$ $f(x,y) \geq g(x,y)$ on D $\to$ $\iint_D f(x,y)\,dA \geq \iint_D g(x,y)\,dA.$

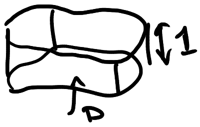
one vari $\to$ $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x,y)\,dx$



$\boxed{9}$ If $D = D_1 \overset{*}{\cup} D_2$ such that $D_1 \cap D_2 = \emptyset$, then

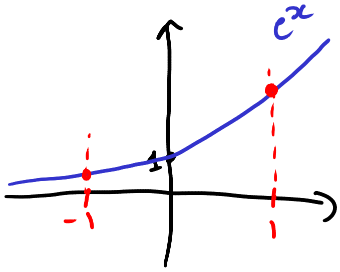
$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA.$$

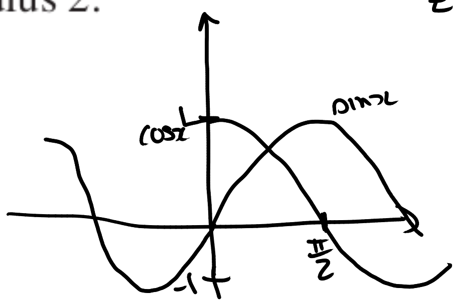
$\boxed{10}$ $A(D) = \iint_D 1\,dA$



$\boxed{11}$ $m \leq f(x,y) \leq M$ $\to$ $m\,A(D) \leq \iint_D f(x,y)\,dA \leq M\,A(D).$

EXAMPLE 6 Use Property 11 to estimate the integral $\iint_D \underbrace{e^{\sin x \cos y}}_{z = f(x,y)}\,dA$, where D is the disk with center the origin and radius 2.





$-1 \leq \sin x \leq 1$

$-1 \leq \cos y \leq 1$

$-1 \leq \sin x \cos y \leq 1$

$$\Rightarrow \quad \underset{m}{\underset{\parallel}{e^{-1}}} \leq e^{\sin x \cos y} \leq \underset{M}{\underset{\parallel}{e^{1}}}$$

$A(D) = A(\text{disque of radius } 2) = \pi 2^2 = 4\pi$

$$\to \quad A(D)e^{-1} \leq \iint_D e^{\sin x \cos(y)}\,dA \leq A(D)\,e$$

$$\to \quad \frac{4\pi}{e} \leq \iint_D e^{\sin x \cos y}\,dA \leq 4\pi e.$$