Example 13

Suppose $f(x) = x^3 - x$.

a) Find a formula for $f'(x)$.

b) Sketch the graph of the curve $y = f'(x)$.

a) $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ $\qquad (x+h)^2 (x+h)$

$= \lim_{h \to 0} \dfrac{(x+h)^3 - (x+h) - (x^3 - x)}{h}$

$= \lim_{h \to 0} \dfrac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x} - h - \cancel{x^3} + \cancel{x}}{h}$

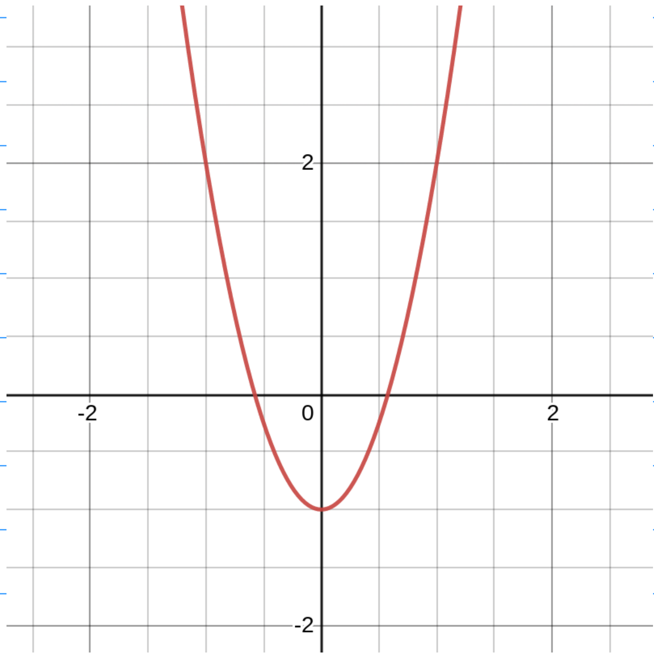
$= \lim_{h \to 0} \dfrac{3x^2h + 3xh^2 + h^3 - h}{h}$

$= \lim_{h \to 0} (3x^2 + 3xh + h^2 - 1)$

$= 3x^2 - 1$

So, $f'(x) = 3x^2 - 1$.

b) We have $f'(x) = 3x^2 - 1$



Example 14

If $f(x) = \sqrt{x}$, find the derivative of f and find the domain of f'.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$(a+b)(a-b) = a^2 - b^2$

$$= \lim_{h \to 0} \frac{(\sqrt{x+h} - \sqrt{x})}{h} \cdot \frac{(\sqrt{x+h} + \sqrt{x})}{(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{\cancel{x} + h - \cancel{x}}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$

So, $f'(x) = \dfrac{1}{2\sqrt{x}}$

sqrt $\to$ $x \geq 0$

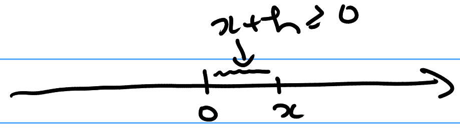
division by 0 $\to$ $x \neq 0$

dom $f' = (0, \infty)$

Example 16

Where is the function $f(x) = |x|$ differentiable?

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0. \end{cases}$$

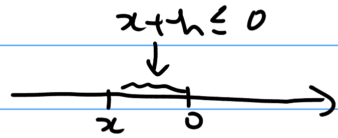


1) When $x > 0$.

$$f'(x) = \lim_{h \to 0} \frac{|x+h| - |x|}{h} = \lim_{h \to 0} \frac{\cancel{x}+h - \cancel{x}}{h}$$

$$= \lim_{h \to 0} \frac{h}{h} = 1$$

for $x > 0$, $f'(x) = 1$.



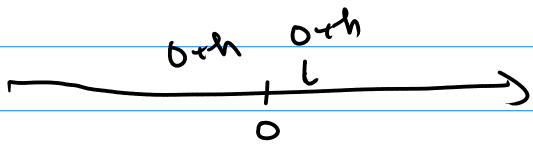
2) $x < 0$

$$f'(x) = \lim_{h \to 0} \frac{|x+h| - |x|}{h}$$

$$= \lim_{h \to 0} \frac{-(x+h) - (-x)}{h}$$

$$= \lim_{h \to 0} \frac{-x - h + x}{h} = \lim_{h \to 0} \frac{-h}{h} = -1$$

for $x < 0$, $f'(x) = -1$



3) $x = 0$

$$\lim_{h \to 0^+} \frac{|0+h| - |0|}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1$$

$$\lim_{h \to 0^-} \frac{|0+h| - |0|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = -1$$

So, $f'(0)$ doesn't exist (undefined). $\square$

Example 19

Find $f''(x)$ of $f(x) = x^3 - x$.

$$(x^3)' = 3x^2 \qquad x' = 1$$

So,

$$f'(x) = 3x^2 - 1$$
$$f''(x) = 6x - 0 = 6x.$$

Doing it with the definition. From Example 13, we know that $f'(x) = 3x^2 - 1$.

So,

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$$
$$= \lim_{h \to 0} \frac{3(x+h)^2 - 1 - (3x^2 - 1)}{h}$$
$$= \lim_{h \to 0} \frac{3\cancel{x^2} + 6xh + 3h^2 - \cancel{1} - \cancel{3x^2} + \cancel{1}}{h}$$
$$= \lim_{h \to 0} \frac{6xh + 3h^2}{h}$$
$$= \lim_{h \to 0} (6x + 3h)$$
$$= 6x$$

So,

$$f''(x) = 6x$$

Example 21

Compute the derivatives of the following functions:

a) $f(x) = x^6$
b) $y = t^{1/5}$
c) $y = u^{\pi}$.
d) $u = v^{2/3}$.