Math 644

Chapter 5

Section 5.1: Cauchy's Theorem

Contents

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Proposition 1. If r is a rational function with poles p_1, p_2, ..., p_N and if γ is a closed curve for which $p_k \notin \gamma$, for any $k = 1, 2, \ldots, N$, then

$$\int_\gamma r(\zeta)\, d\zeta = \sum_{k=1}^{N} c_{k,1} \int_\gamma \frac{1}{\zeta - p_k}\, d\zeta,$$

where $c_{k,1} \in \mathbb{C}$, for $k = 1, 2, \ldots, N$.

Proof.

Write $$r(z) = \sum_{k=1}^{N} \sum_{j=1}^{n_k} \frac{c_{k,j}}{(z-p_k)^j} + q(z)$$

From Thm. 4 from sec. 4.4 & from Cauchy's theorem

$$\int_\gamma q(z)\, dz = 0.$$

and $$\int_\gamma \frac{1}{(z-p_k)^j}\, dz = 0, \quad j = 2, 3, \ldots, n_k.$$

So, integrating

$$\Rightarrow \int_\gamma r(z)\, dz = \sum_{k=1}^{N} c_{k,1} \int_\gamma \frac{1}{z-p_k}\, dz. \qquad \square$$

Theorem 2. Suppose γ is a cycle contained in a region Ω, and suppose

$$\int_\gamma \frac{d\zeta}{\zeta - a} = 0 \quad (\forall a \notin \Omega).$$

If f is analytic on Ω, then

$$\int_\gamma f(\zeta)\, d\zeta = 0.$$

Proof.

By Runge's Theorem, we can find a sequence (r_n) of rational functions with poles in $\mathbb{C} \setminus \Omega$ s.t.

$$r_n \longrightarrow f \quad \text{uniformly on } \gamma.$$

By thm. 4 in §4.4, we may choose γ to be piecewise cont. diff. & of finite length.

Therefore:

$$\int_\gamma f(z)\, dz = \lim_{n\to\infty} \int_\gamma r_n(z)\, dz$$

$$= \lim_{n\to\infty} \sum_{k=1}^{M_n} c_{k,1}^n \int_\gamma \frac{1}{z - p_k^n}\, dz$$

$$= 0. \qquad \square$$

THEOREM 3. Suppose γ is a cycle contained in a region Ω, and suppose

$$\int_\gamma \frac{d\zeta}{\zeta - a} = 0 \quad (\forall a \notin \Omega).$$

If f is analytic on Ω and $z \in \mathbb{C}\backslash\gamma$, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta = f(z) \cdot \frac{1}{2\pi i}\int_\gamma \frac{1}{\zeta - z}\, d\zeta.$$

Proof.

Note that the function, for a fixed $z \in \Omega$,

$$g(\zeta) = \begin{cases} \dfrac{f(\zeta) - f(z)}{\zeta - z} & (\zeta \in \Omega \ \& \ \zeta \neq z) \\ f'(z), & \zeta = z \end{cases}.$$

is analytic in Ω.

By Cauchy's Theorem (Thm. 2),

$$\int_\gamma g(\zeta)\, d\zeta = 0.$$

$$\Rightarrow \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{f(z)}{2\pi i}\int_\gamma \frac{1}{\zeta - z} = 0$$

$$\Rightarrow \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta = \frac{f(z)}{2\pi i}\int_\gamma \frac{d\zeta}{\zeta - z} \quad (z \in \Omega).$$

When $z \notin \Omega$, then $\displaystyle\int_\gamma \frac{1}{\zeta - z}\, d\zeta = 0$.

Also, $\zeta \mapsto \frac{f(\zeta)}{\zeta - z}$ is analytic in Ω,

so by Cauchy's Theorem (Thm 2):

$$\int_\gamma \frac{f(\zeta)}{\zeta - z} \, d\zeta = 0 .$$

So,

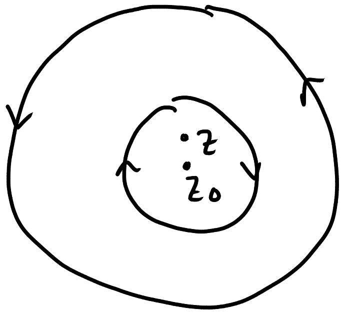
$$0 = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

$$= \frac{f(z)}{2\pi i} \underbrace{\int_\gamma \frac{d\zeta}{\zeta - z}}_{\text{because } = 0} . \qquad \square$$

Example 4. Let $\gamma = \gamma_1 + \gamma_2$ be the cycle formed by $\gamma_1(t) = z_0 + re^{it}$ (clockwise direction) and $\gamma_2(t) = z_0 + Re^{it}$ (counter-clockwise direction), where $0 \le t \le 2\pi$ and $r < R$. Let Ω be the region bounded by γ. If f is analytic on the closure of Ω, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta = \begin{cases} 0 & \text{if } |z - z_0| < r, \\ f(z) & \text{if } r < |z - z_0| < R, \\ 0 & \text{if } |z - z_0| > R. \end{cases}$$

① Suppose $|z - z_0| < r$.



From Cauchy's integral formula,

$$(*) \quad \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta = \frac{f(z)}{2\pi i}\int_\gamma \frac{1}{\zeta - z}\, d\zeta$$

But,

$$\int_\gamma \frac{1}{\zeta - z}\, d\zeta = \int_{\gamma_2} \frac{1}{\zeta - z}\, d\zeta + \int_{\gamma_1} \frac{1}{\zeta - z}\, d\zeta$$

$$= 2\pi i - 2\pi i = 0$$

So, $\displaystyle \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta = 0$.

② $r < |z - z_0| < R$.

From $(*)$,

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta = \frac{f(z)}{2\pi i}\int_\gamma \frac{1}{\zeta - z}\, d\zeta$$

$$= \frac{f(z)}{2\pi i}\Big(\underbrace{\cancel{\int_{\gamma_1} \frac{1}{\zeta - z}\, d\zeta}}_{=0 \quad z \notin \operatorname{int}(\gamma_1)} + \underbrace{\int_{\gamma_2} \frac{1}{\zeta - z}\, d\zeta}_{= 2\pi i}\Big)$$

$$= \frac{f(z)}{2\pi i} \cdot 2\pi i$$

$$= f(z).$$

③ $\underline{|z-z_0| > R.}$

Since $z \notin \mathrm{int}(\gamma_2)$, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta = \frac{f(z)}{2\pi i}\Bigg(\underbrace{\cancel{\int_{\gamma_1} \frac{1}{\zeta - z}\,d\zeta}}_{=0} + \underbrace{\cancel{\int_{\gamma_2} \frac{1}{\zeta - z}\,d\zeta}}_{=0}\Bigg)$$

$$= 0. \qquad \square$$