Math 644

Chapter 2

Section 2.2: Fundamental Theorem of Algebra and Partial Fractions

Contents

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Spring 2023

The local behavior of a polynomial (Walking a Dog picture) is really helpful to give a proof of the FTA.

Theorem 1. Every non-constant polynomial has a zero.

Some precision:

- A function $f : \Omega \subseteq \mathbb{C} \to \mathbb{C}$ has a zero at $a \in \Omega$ if $f(a) = 0$.

Lemma 2. If $n := \deg p \geq 1$, then $|p(z)| \to \infty$, as $|z| \to \infty$.

Proof.

Let $p(z) = \sum_{k=0}^{n} a_k z^k$, $a_n \neq 0$.

If $|z| \neq 0$, then

$$p(z) = z^n \left(\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \cdots + a_n \right)$$

Since $\frac{1}{|z|^k} \to 0$ as $|z| \to \infty$ $(\forall k)$

then, for $M > 0$ fix, $\exists R_1 > 0$ s.t.

$$\frac{1}{|z|^k} < \frac{|a_n|}{2n\left(\max_{0 \leq k \leq n-1} \{|a_k|\} + 1\right)}, \quad \forall k \in \{1, \ldots, n\}$$

So, if $|z| > R_1$, then $\qquad |z-w| \geq |z| - |w|$

$$\left| \sum_{k=1}^{n} \frac{a_k}{z^k} \right| \leq \sum_{k=1}^{n} \frac{|a_k|}{|z|^k} < \frac{|a_n|}{2}.$$

Now, if $|z| > R_1$, then

$$|p(z)| \geq |z|^n |a_n| - |z|^n \frac{|a_n|}{2} \geq \frac{|z|^n}{2} \to \infty$$

Lemma 3. If $p(z)$ is a polynomial with no zero, then

$$M := \inf\{|p(z)| \,:\, z \in \mathbb{C}\} \in (0, \infty).$$

<u>Proof.</u> First, $p(0) = a_0 \in \mathbb{C} \Rightarrow M \leq |a_0| < \infty$.

Let $(R_n)_{n=1}^{\infty} \subseteq (0,\infty)$ s.t. $R_n \nearrow \infty$.

Let $M_n := \inf\{|p(z)| : |z| \leq R_n\}$. So, the sequence (M_n) is decreasing and bounded below by 0. So, there is M s.t.

$$\lim_{n\to\infty} M_n = M.$$

Since $|p|$ is continuous on $\{z : |z| \leq R_n\}$, then $\exists z_n \in \{z : |z| \leq R_n\}$ s.t. $|p(z_n)| = M_n$.

Suppose that $|z_n| \to \infty$, $|p(z_n)| \to \infty$ $(n \to \infty)$.

So, since $|p(z_n)| = M_n \Rightarrow M_n \longrightarrow \infty$

$$\Rightarrow M = \infty. \quad \#$$

So, there is a $R > 0$ s.t. $|z_n| \leq R$.

So, there is $(z_{n_k})_{k=1}^{\infty}$ s.t. $z_{n_k} \longrightarrow z_0$ for some $z_0 \in \mathbb{C}$.

Continuity $\Rightarrow$ $M = |p(z_0)| > 0$. $\square$

Proof of the FTA.

Suppose that $p(z) \neq 0$, $\forall z \in \mathbb{C}$.

Let $M := \inf \{ |p(z)| : z \in \mathbb{C} \} \in (0, \infty)$

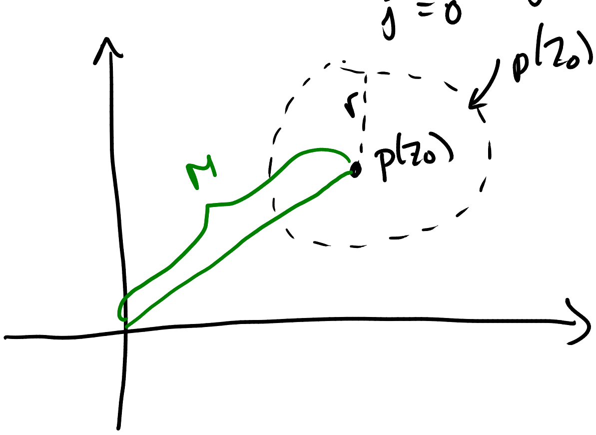
& let $z_0 \in \mathbb{C}$ s.t. $|p(z_0)| = M$.

Let $\varepsilon > 0$ small enough s.t.

$$0 < s < r < M$$

where $s = C|\zeta|^{k+1}$ & $r = |b_k||\zeta|^k$

& $p(z) = \sum_{j=0}^{n} b_j z^j$, $k := \inf\{ j : b_j \neq 0 \}$.



$\exists \zeta$ s.t.

$$|p(z_0) + b_k \zeta^k| = M - r$$

Now, we have

$$\begin{aligned} |p(z_0 + \zeta)| &\leq |p(z_0 + \zeta) - p(z_0) - b_k \zeta^k| \\ &\quad + |p(z_0) + b_k \zeta^k| \\ &< s + M - r < M \qquad (s < r) \end{aligned}$$

Contradiction. $\square$

COROLLARY 4. If p is a polynomial of degree $n \geq 1$, then there are complex numbers z_1, z_2, ..., z_n and a compact constant c such that

$$p(z) = c \prod_{k=1}^{n} (z - z_k).$$

Proof. By induction

1) $\underline{n=1}$ $\quad p(z) = az + b = \underset{c}{a}\left(z + \underset{z_1}{\frac{b}{a}}\right)$

2) Suppose the assumption is true for any pol. of degree $m \geq 1$.

Let $q(z) = a_0 + a_1 z + \cdots + a_m z^m + a_{m+1} z^{m+1}$ $(a_{m+1} \neq 0)$.

By FTA, $\exists b \in \mathbb{C}$ s.t. $q(b) = 0$. For $z \neq b$,

$$\frac{q(z)}{z-b} = \frac{q(z) - q(b)}{z-b} = \frac{\sum_{k=1}^{m+1} a_k (z^k - b^k)}{z-b}$$

$$= \frac{\sum_{k=1}^{m+1} a_k \left(\sum_{j=0}^{k-1} z^j b^{k-1-j}\right) \cancel{(z-b)}}{\cancel{z-b}}$$

$$= \sum_{k=1}^{m+1} a_k \left(\sum_{j=0}^{k-1} z^j b^{k-1-j}\right)$$

$$= c \prod_{k=1}^{m} (z - z_k)$$

$$\Rightarrow \quad q(z) = c \left[\prod_{k=1}^{m} (z - z_k)\right](z - b). \qquad \square$$

EXAMPLE 5. Find the zeros of $p(z) = z^n - 1$, $n \geq 1$.

Take $z \in \mathbb{C}$ s.t. $p(z) = 0 \Rightarrow z^n - 1 = 0 \Rightarrow |z| = 1$

So, $\exists \theta \in \mathbb{R}$ s.t. $z = e^{i\theta} = \cos\theta + i\sin\theta$

$$\Rightarrow z^n = 1 \Rightarrow e^{in\theta} = 1$$

$$\Rightarrow \begin{cases} \cos n\theta = 1 \\ \sin n\theta = 0 \end{cases}$$

So, $\theta = \frac{2k\pi}{n}$, $k = 0, 1, 2, \ldots, n-1$.

So, $z_k = e^{i\frac{2k\pi}{n}}$, $k = 0, 1, \ldots, n-1$.

Rational Functions

A **rational function** is a quotient of two polynomials. From the FTA, we can write

$$r(z) = \frac{p(z)}{\prod_{j=1}^{N}(z - z_j)^{n_j}}$$

for some $N, n_j \in \mathbb{C}$ and $z_1, z_2, \ldots, z_N \in \mathbb{C}$.

COROLLARY 6. Let p be a polynomial. Then there is a polynomial $q(z)$ and complex constants $c_{k,j}$ such that

$$\frac{p(z)}{\prod_{j=1}^{N}(z - z_j)^{n_j}} = q(z) + \sum_{j=1}^{N}\sum_{k=1}^{n_j} \frac{c_{k,j}}{(z - z_j)^k}.$$

<u>A simple case:</u> $\deg p < \sum_{j=1}^{N} n_j$.

$$\Rightarrow \frac{p(z)}{\prod_{j=1}^{N}(z - z_j)} = \sum_{j=1}^{N} \frac{c_j}{(z - z_j)} \quad \& \quad c_j = \frac{p(z_j)}{\prod_{k=1, k\neq j}^{N}(z_j - z_k)}.$$