Chapter 1

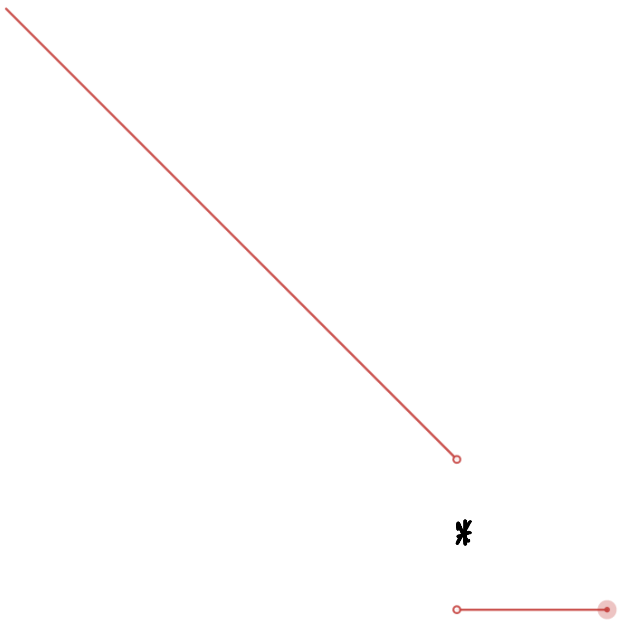
Functions and Limits

1.8 Continuity

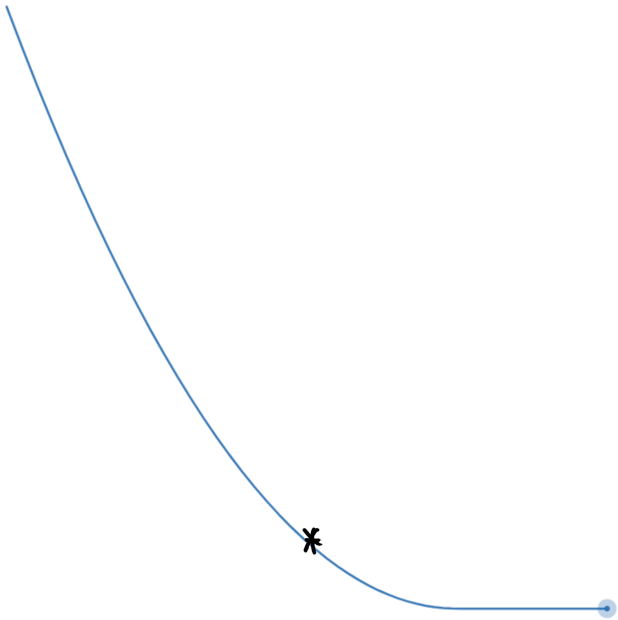
Continuity

Example. What are the main difference(s) between the two following curves?

Illustration: https://www.desmos.com/calculator/hflxgbsemz



(a) Graph of f(x).



(b) Graph of g(x)

(1) red: break point , blue: no break point

(2) red: not defined at (*) , blue: is defined at *

(3) red: $\lim_{x \to *} f(x) \nexists$, blue: $\lim_{x \to *} f(x) \exists$

(4) is red & the other is blue.

Example. Now, what are the differences between the two following functions?

(a) $f(x) = \begin{cases} 2 - x & \text{if } -2 \leq x < 1 \\ 0 & \text{if } 1 \leq x \leq 2 \end{cases}$

↳ red.

(b) $g(x) = \begin{cases} \frac{4}{9}(1-x)^2 & \text{if } -2 \leq x < 1 \\ 0 & \text{if } 1 \leq x \leq 2 \end{cases}$

↳ blue.

1 Definition A function f is **continuous at a number a** if

$$\lim_{x \to a} f(x) = f(a)$$

Three things to verify to show a function is continuous:

a) The function is defined at x = a.

Find the domain.

b) The limit of the function exists at x = a.

Use the limit rules.

c) The limit of the function at x = a equals the value of the function at x = a.

Discontinuity:

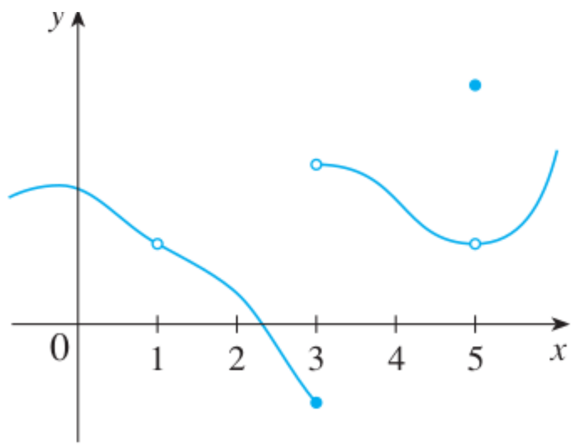
$x=a$ is a discontinuity of $f(x)$ if (a) or (b) or (c) is not satisfied.

EXAMPLE 1 Figure 2 shows the graph of a function f. At which numbers is f discontinuous? Why?

$\underline{x=1}$ $f(1)$ not defined.

$\underline{x=3}$ $\lim_{x \to 3^-} f(x) \neq \lim_{x \to 3^+} f(x)$.

$\underline{x=5}$ $\lim_{x \to 5} f(x) \neq f(5)$.



Example. Check if the functions in the first example are continuous at x = 1 using the formulas.

(1) (a) f should be defined at $x=1$.

but f is not $\longrightarrow$ discontinuous at $x=1$

(2) (a) g should be defined at $x=1$.

Yes & $g(1)=0$.

(b) $\lim_{x\to 1^-} g(x) = \lim_{x\to 1^-} \frac{4}{9}(1-x)^2 = 0$

$\lim_{x\to 1^+} g(x) = \lim_{x\to 1^+} 0 = 0$.

$\Rightarrow \lim_{x\to 1} g(x) = 0$.

(c) $g(1) = 0 = \lim_{x\to 1} g(x)$ ✓

So $g(x)$ is continuous at $x=1$.

EXAMPLE 2 Where are each of the following functions discontinuous?

(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$ (b) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ (c) $f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$

(a) $\text{Dom } f = (-\infty, 2) \cup (2, \infty)$

$f(x) = \dfrac{\cancel{(x-2)}(x+1)}{\cancel{x-2}} = x+1 \quad (x \neq 2).$

Subs. Rule: $\lim\limits_{x \to a} \dfrac{x^2 - x - 2}{x-2} = \dfrac{a^2 - a - 2}{a-2} = f(a) \quad (a \neq 2)$

so f is continuous at every point of $(-\infty, 2) \cup (2, \infty)$.

f is discontinuous at $x = 2$ ($f(2) \nexists$).

(b) $\text{Dom } f = (-\infty, \infty)$.

$\lim\limits_{x \to 0} \dfrac{1}{x^2} = +\infty \neq 1 = f(0) \rightarrow f$ discont. at $x = 0$.

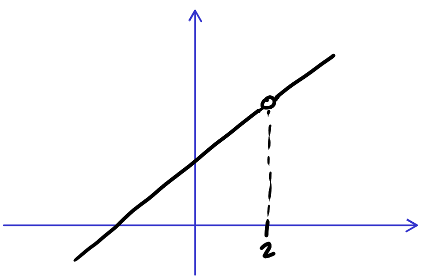
$\boxed{a \neq 0}$

$\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} \dfrac{1}{x^2} = \dfrac{1}{a^2} = f(a)$ ✓ $\rightarrow f$ cont. at every point of $(-\infty, 0) \cup (0, \infty)$

(c) $\lim\limits_{x \to 0^-} f(x) = 0 \neq 1 = \lim\limits_{x \to 0^+} f(x)$. $\rightarrow f$ is discont. at $x = 0$.

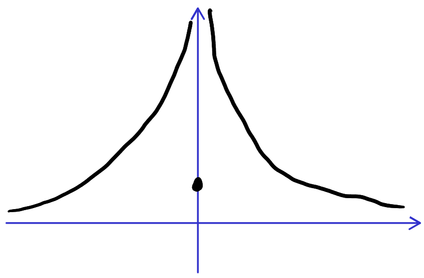
f is cont at all other points in $(-\infty, 0) \cup (0, \infty)$.

3 kinds of discontinuity.



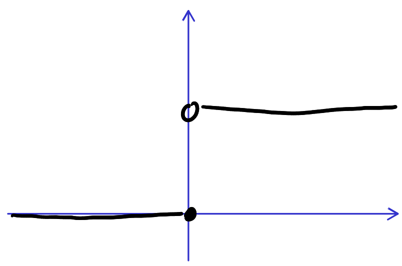
(a) Removable.

$\lim\limits_{x \to 2} f(x) = 3$ ($\exists$) & $f(2) \nexists$



(b) Infinite discontinuity.

$\lim\limits_{x \to 0} f(x) = +\infty$



(c) Jump discontinuity.

$\lim\limits_{x \to 0^-} f(x) \neq \lim\limits_{x \to 0^+} f(x)$

Continuous from left and right.

2 Definition A function f is **continuous from the right at a number a** if

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is **continuous from the left at a** if

$$\lim_{x \to a^-} f(x) = f(a)$$

EXAMPLE. Is the function

$$f(x) = \begin{cases} 1 & , \text{ if } x > 0 \\ 0 & , \text{ if } x \leq 0 \end{cases}$$

(a) continuous from the right at x = 0 (b) continuous from the left at x = 0.

(a) $f(0) = 0$, $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 1 = 1$

$\Rightarrow$ $0 \neq \lim_{x \to 0^+} f(x)$

$\Rightarrow$ f is not continuous from the right at $x = 0$.

(b) $f(0) = 0$ & $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 0 = 0$

$\Rightarrow$ $f(0) = \lim_{x \to 0^-} f(x)$

$\Rightarrow$ f is continuous from the left at $x = 0$.

Properties of Continuous Functions.

4 Theorem If f and g are continuous at a and if c is a constant, then the following functions are also continuous at a:

1. $f + g$ **2.** $f - g$ **3.** cf

4. fg **5.** $\dfrac{f}{g}$ if $g(a) \neq 0$

Consequences:

7 Theorem The following types of functions are continuous at every number in their domains:

- polynomials
- rational functions
- root functions
- trigonometric functions

Recall: $\lim_{x \to a} x^n = a^n \to f(x) = x^n$ is continuous

Substitution Rule Revisited.

EXAMPLE 5 Find $\displaystyle\lim_{x \to -2} \underbrace{\frac{x^3 + 2x^2 - 1}{5 - 3x}}_{f(x)}$.

$\text{Dom } f = (-\infty, 5/3) \cup (5/3, \infty).$

f is rational $\Rightarrow$ f is cont. on dom f.

$\Rightarrow$ f is cont. at $x = -2$.

So,

$$\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} = f(-2) = \boxed{-1/11}$$

EXAMPLE 7 Evaluate $\lim_{x\to\pi} \underbrace{\frac{\sin x}{2+\cos x}}_{f(x)}$.

Dom f.

$2+\cos x \overset{??}{\neq} 0$ because of $(*)$

$-1 \le \cos x \le 1$

$\Rightarrow \boxed{1 \le 2+\cos x \le 3 \quad (*)}$

So f is continuous on $(-\infty, \infty) = \text{Dom} f$.

$$\Rightarrow \lim_{x\to\pi} \frac{\sin x}{2+\cos x} = f(\pi) = \frac{\sin(\pi)}{2+\cos(\pi)} = \frac{0}{2-1} = \boxed{0}$$

8 Theorem If f is continuous at b and $\lim_{x \to a} g(x) = b$, then $\lim_{x \to a} f(g(x)) = f(b)$. In other words,

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$$

9 Theorem If g is continuous at a and f is continuous at g(a), then the composition f(g(x)) is continuous at a.

Example. Find the value of

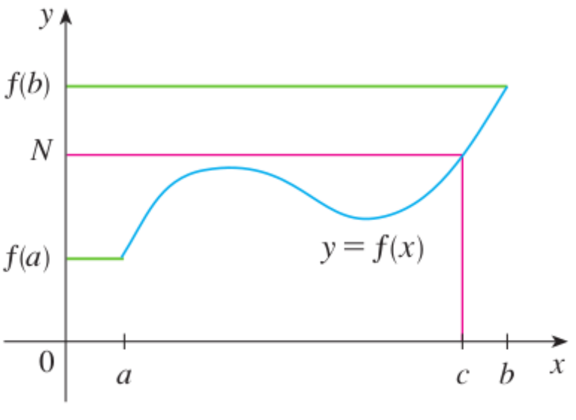
$$\lim_{x \to 1/2} \sin(\pi - \pi x^2)$$

EXAMPLE. Suppose we have a function

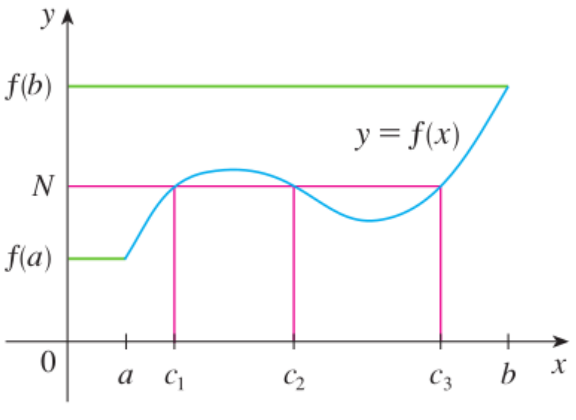
$$f(x) = x^2 - 1.$$

Does the graph of the function f cross the horizontal line y = 3?

10 The Intermediate Value Theorem Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.



(a) Find one number c



(b) Find multiple numbers c

EXAMPLE 9 Show that there is a root of the equation

$$4x^3 - 6x^2 + 3x - 2 = 0$$

between 1 and 2.