Math 644

Chapter 5

Section 5.5: The Argument Principle

Contents

Created by: Pierre-Olivier Parisé
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THEOREM 1. Suppose f is meromorphic which not constant in a region Ω with zeros set $\{z_j\}$ and poles set $\{p_k\}$. Suppose γ is a cycle with $\gamma \sim 0$ in Ω, and suppose $\{z_j\} \cap \gamma = \varnothing$ and $\{p_k\} \cap \gamma = \varnothing$. Then

$$n(f(\gamma), 0) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz = \underbrace{\sum_j n(\gamma, z_j) - \sum_k n(\gamma, p_k)}.$$

Notes:

1. The convention is if z is a zero of order k of f, then z appears k times in the list $\{z_j\}$.
2. For the poles, we also have the same convention: if z is a pole of order k of f, then z appears k times in the list $\{p_k\}$.

Proof.

Since zeros & poles of f are isolated & $\gamma \sim 0$ & $\gamma \subset \Omega$,

$\sum_j n(\gamma, z_j)$ & $\sum_k n(\gamma, p_k)$ finite sums.

Middle term $z = f(\gamma(t)) \to dz = f'(\gamma(t))\gamma'(t)\, dt$

$$n(f(\gamma), 0) = \frac{1}{2\pi i} \int_{f(\gamma)} \frac{1}{z}\, dz$$

$$= \frac{1}{2\pi i} \int_a^b \frac{f'(\gamma(t))}{f(\gamma(t))} \gamma'(t)\, dt$$

$$= \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz .$$

Right term.

$$\Omega_1 = \Omega \setminus \left(\{ z_j : n(\gamma, z_j) = 0 \} \cup \{ p_k : n(\gamma, p_k) = 0 \} \right)$$

So, $\gamma \sim 0$ in Ω_1.

Let b be a pole or zero of f of order m:

$$f(z) = (z-b)^{l} g(z)$$

for some $l \in \mathbb{Z}$, g analytic in some disk centered at b & $g(z) \neq 0$ in $B(b,r)$.

Then, $f'(z) = l(z-b)^{l-1} g(z) + (z-b)^{l} g'(z)$

$$\Rightarrow \quad \frac{f'(z)}{f(z)} = \frac{l}{(z-b)} + \frac{g'(z)}{g(z)}, \quad z \in B(b,r)$$

Therefore, $\dfrac{f'(z)}{f(z)} - \dfrac{l}{z-b}$ is analytic in $B(b,r)$.

Repeat this for each pole & zero in $\{ z_j : n(\gamma, z_j) \neq 0 \} \cup \{ p_k : n(\gamma, p_k) \neq 0 \}$,

$$\frac{f'(z)}{f(z)} - \sum_j \frac{l_j}{z-z_j} - \sum_k \frac{l_k}{z-p_k}$$

is analytic in Ω_1.

By Cauchy's Theorem,

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)} - \sum_j \frac{l_j}{z-z_j} - \sum_k \frac{l_k}{z-p_k}\, d\zeta = 0$$

$$\Rightarrow \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\, d\zeta = \sum_j l_j\, n(\gamma, z_j) + \sum_k l_k\, n(\gamma, p_k)$$

$$= \sum_j l_j\, n(\gamma, z_j) - \sum_k (-l_k)\, n(\gamma, p_k).$$

$\square$

THEOREM 2. Suppose γ is a closed curve in a region Ω with $\gamma \sim 0$ in Ω and $n(\gamma, z) = 0$ or $= 1$ for all $z \in \Omega \backslash \gamma$. If f and g are analytic in Ω and satisfy

$$|f(z) + g(z)| < |f(z)| + |g(z)|,$$

for all $z \in \gamma$, then f and g have the same number of zeros enclosed by γ.

Notes:

① Again, the number of zeros of f and g are counted according to their multiplicity.

Proof. By assumption, $f \not\equiv 0$ & $g \not\equiv 0$.

Therefore, $\frac{f}{g}$ is a meromorphic function in Ω. We have

$$(*) \quad \left|\frac{f}{g} + 1\right| < \left|\frac{f}{g}\right| + 1 \quad \text{on } \gamma$$

We have $\quad |w - (-1)| = |w| + 1$

$\iff \quad |w|^2 + 1 + 2\,\mathrm{Re}\, w = |w|^2 + 1 + 2|w|$

$\iff \quad \mathrm{Re}\, w = |w|$

$\iff \quad (\mathrm{Re}\, w)^2 = (\mathrm{Re}\, w)^2 + (\mathrm{Im}\, w)^2, \quad \mathrm{Re}\, w \geq 0$

$\iff \quad \mathrm{Im}(w) = 0$ & $\mathrm{Re}\, w \geq 0$.

$\iff \quad w \in [0, \infty)$.

From $(*)$, $\frac{f}{g}(\gamma)$ omits $[0, \infty)$ and

Since each component of $\mathbb{C} \setminus \frac{f}{g}(\gamma)$ are connected, 0 is in the unbounded component.

So, $n\left(\frac{f}{g}(\gamma), 0\right) = 0$.

$\underset{\text{Principle}}{\overset{\text{Argument}}{\Rightarrow}}$ $\#\{\text{zero of } f\} - \#\{\text{zero of } g\} = 0$.

$\square$

Example 3. Let $f(z) = z^9 - 2z^6 + z^2 - 8z - 2$.

(a) How many zeros does f have in $\{z : |z| < 1\}$?

(b) How many zeros does f have in $\{z : |z| < 2\}$?

(a) $|z^9 - 2z^6 + z^2 - 8z - 2 + 8z|$

$= |z^9 + 2z^6 + z^2 - 2|$

$\leq 1 + 2 + 1 + 2 = 6$ on $|z| = 1$.

Since $|z| = 1$,

$6 < 8 = 8|z| = |8z|$

$\Rightarrow 6 < |z^9 - 2z^6 + z^2 - 8z - 2| + |8z|$

By Rouché's thm:

f & $g(z) = 8z$ have

same # zeros in $\mathbb{D}$.

$\Rightarrow$ f has 1 zeros in $\mathbb{D}$.

(b) z^9 have biggest modulus on $|z| = 2$

$\Rightarrow |f(z) - z^9| \leq 2^7 + 2^2 + 2^4 + 2 \quad (|z| = 2)$

$< 2^8 - 1 < 2^9 = |z|^9$.

By Rouché's Thm, f & z^9 have the same number of zeros in $\{|z| < 2\}$, that is 9.