C.VI Examples of Discrete Random Variables

Let X be a discrete random variable.

Bernouilli distribution

A discrete random variable X has the Bernouilli distribution with parameter $p \in [0,1]$ if $\operatorname{Im} X = \{0,1\}$ and

$$P(X=1) = p \quad \text{and} \quad P(X=0) = 1-p.$$

Used Scenarios: The Bernouilli distribution is usually used to model experiment in which the outcome is "success" or "failure".

Binomial Distribution

Let n be an integer and $q \in [0,1]$. X has the binomial distribution with parameters n and q if $\operatorname{Im} X = \{0,1,2,\ldots,n\}$ and

$$P(X=k) = \frac{n!}{k!(n-k)!} q^k (1-q)^{n-k}, \quad k = 0,1,2,\ldots,n.$$

Used Scenarios: Experiments where the goal is to obtain a certain number of successes in n trials.

Example 8. There are $n=6$ machines to test if they are working properly or not. According to a recent survey, a machine is working properly in 75% of the time. What is the probability that 4 machines are working properly.

Solution. We have $q = 0.75$ and $n = 6$. Let X be the discrete random variable given the number of machines that are working properly. Then $X \sim Bi(6, 0.75)$. Therefore,

$$P(X=4) = \binom{6}{4}(0.75)^4(0.25)^2 = \frac{6!}{4!2!}(0.75)^4(0.25)^2 \approx 0.2966. \qquad \triangle$$

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Poisson Distribution

Let $\lambda > 0$. X has the Poisson distribution if $\operatorname{Im} X = \{0,1,2,\ldots\}$ and

$$p_X(k) = \frac{1}{k!}\lambda^k e^{-\lambda}, \quad k = 0,1,2,\ldots.$$

Used Scenarios: Experiments where the goal is to obtain a certain number of successes in n trials, with n large.

Note: The parameter λ usually refers to the expected number of successes in an experiment (justified later when we introduce expectation of discrete random variables).

Example 9. Consider an experiment that consists of counting the number of α-particles given off in a 1-second interval by 1 gram of radioactive material. If we know from past experience that, on the average, 3.2 such α-particles are given off, what is a good approximation to the probability that no more than 2 α-particles will appear?

Solution. We think of a the surface of the material as a composition of a high number n of particular, that has $3.2/n$ chance of given off. We therefore can approximate the desire probability by a Poisson distribution with parameter $\lambda = nq = 3.2$. Then,

$$P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2) = \frac{e^{-3.2}3.2^0}{0!} + \frac{e^{-3.2}3.2^1}{1!} + \frac{e^{-3.2}3.2^2}{2!} \approx 0.3799.$$

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THEOREM 8. Let X be a discrete random variable which follows a binomial distribution with parameters n and q and let $\lambda = nq$. Then

$$p_X(k) \approx \frac{1}{k!}\lambda^k e^{-\lambda}, \quad k = 0, 1, 2, \ldots,$$

when n is large enough and q is small enough.

Negative Binomial Distribution

Let $q \in (0, 1)$ and $n \geq 0$ be an integer. Then X has the negative binomial distribution with parameters q and n if $\operatorname{Im} X = \{n, n+1, n+2, \ldots\}$ and

$$p_X(k) = \frac{(k-1)!}{(n-1)!(k-n)!} q^n (1-q)^{k-n}, \quad k = n, n+1, n+2, \ldots$$

Used-case Scenarios: Experiments where the goal is to find the probability of having the n-th success after k trials.

EXAMPLE 10. A geological study indicates that an exploratory oil well drilled in a particular region should strike oil with probability 0.2. Find the probability that the third oil strike comes on the fifth well drilled.

Solution. Let X be the number of strikes needed to obtain a third oil strike. In this case, we have $q = 0.2$ and $n = 3$. We are searching for $P(X = 5)$. Then

$$P(X = 5) = \frac{4!}{2!2!}(0.2)^3(0.8)^2 = 0.03072.$$

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Geometric Distribution

Let $q \in (0, 1)$. Then X has the geometric distribution with parameter q if $\operatorname{Im} X = \{1, 2, \ldots\}$ and

$$p_X(k) = (1-q)^{k-1} q, \quad k = 1, 2, 3, \ldots.$$

Used-case Scenarios: Experiments where the goal is to find the probability of the first success to occur within k tries.

EXAMPLE 11. An urn contains 10 red balls and 20 blue balls. Ball are randomly selected, one at a time, until a red one is obtained. If we assume that each selected ball is replaced before the next one is drawn, what is the probability that

a) exactly 3 draws are needed?

b) at least 6 draws are needed.

Solution. Let X be the discrete random variable counting the number of time needed to get a red ball. The random variable X follows a geometric distribution with parameter q, giving the probability of selecting a red ball.

Since the ball is replaced in the urn, the probability of selecting a red ball is always the same, that is $1/3/$. Therefore, $q = 1/3$.

a) Let $k = 3$, so that $P(X = k) = (1 - 1/3)^2(1/3) = 4/27$.

b) What is $P(X \geq 6)$? Using the complement, this is $1 - P(X < 6)$. Therefore,

$$P(X \geq 6) = 1 - P(X = 1) - P(X = 2) - P(X = 3) - P(X = 4) - P(X = 5) \approx 0.8683.$$

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Summary

The table below is a summary of the expected value and variance of each of the examples presented in this section.

Distribution	Expected Value	Variance
$B(q)$	q	$q(1-q)$
$B(n,q)$	nq	$nq(1-q)$
$\mathcal{P}(\lambda)$	λ	λ
$G(q)$	$1/q$	$(1-q)/q^2$
$NB(n,q)$	n/q	$n(1-q)/q^2$

Table 1: Table of Mean and Variance of different distributions