Chapter 3
Applications of Derivatives

3.4 Limits at Infinity; Horizontal Asymptotes

Example. What is the limit of $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$ when x becomes large?

x	$f(x)$
10	$\approx$ 0.99
100	$\approx$ 0.9998
1000	$\approx$ 0.999998
10 000	$\approx$ 0.99999998

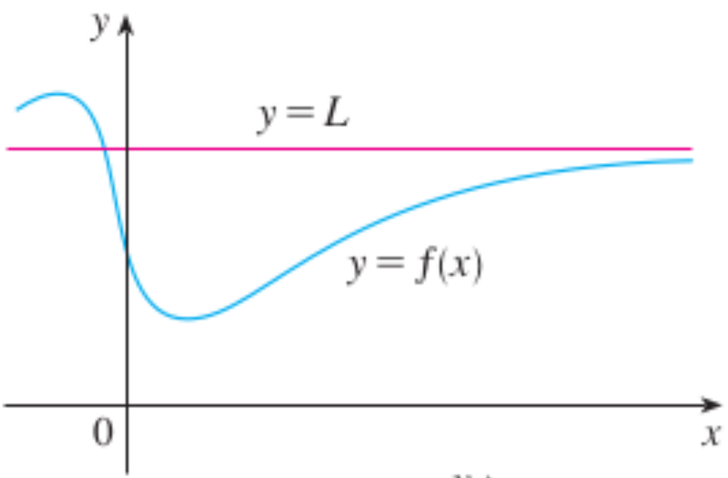
x	$f(x)$
100 000	$\approx$ 0.9999999998
$\vdots$	$\vdots$

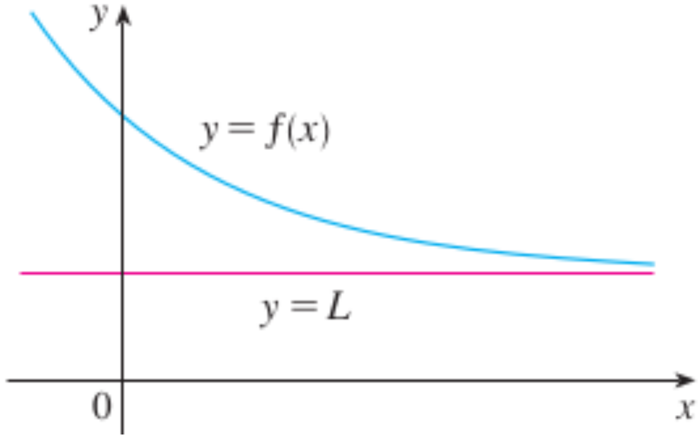
$$\frac{x^2-1}{x^2+1} \longrightarrow 1 \quad \Rightarrow \quad \lim_{x \to +\infty} \frac{x^2-1}{x^2+1} = 1 .$$

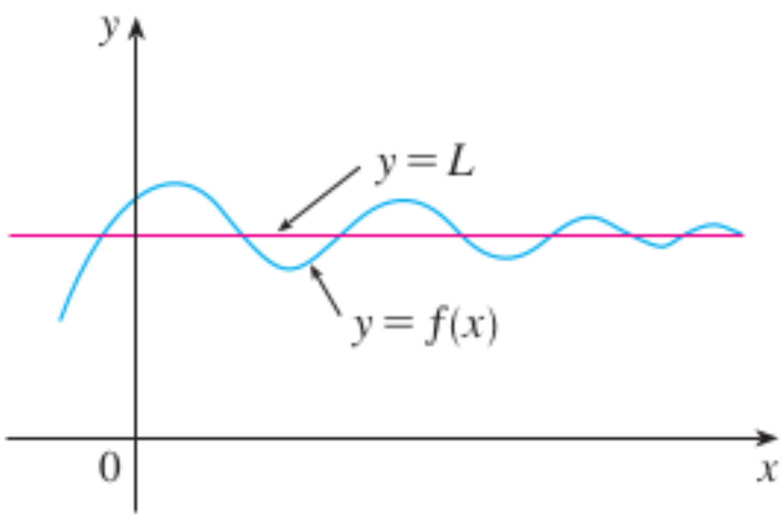
1 Intuitive Definition of a Limit at Infinity Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large.







Example. What is the limit of $f(x) = \dfrac{x^2-1}{x^2+1}$ when x becomes large?

x	$f(x)$
-10	0.98
$\vdots$	
-10000	0.99999998
$\vdots$	$\downarrow$
$\downarrow$	1
$-\infty$	

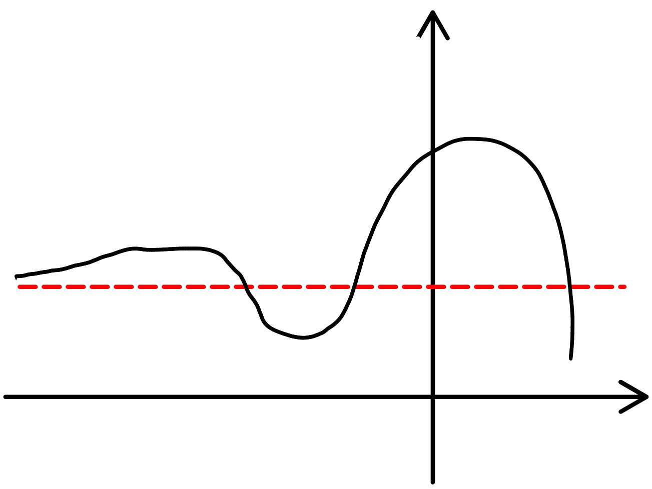
x	$f(x)$

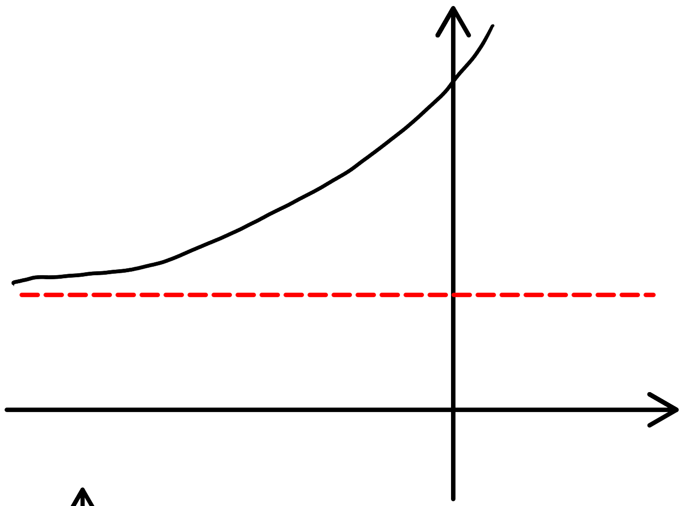
$$\Rightarrow \lim_{x \to -\infty} \frac{x^2-1}{x^2+1} = 1$$

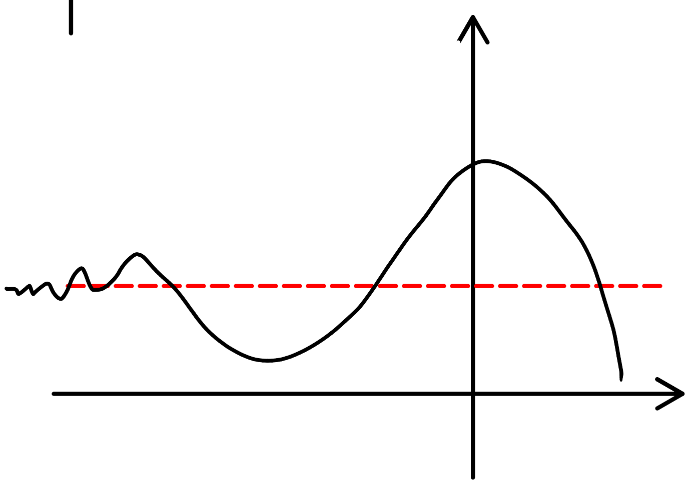
2 Definition Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large negative.







3 Definition The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = L$$

EXAMPLE 1 Find the infinite limits, limits at infinity, and asymptotes for the function f whose graph is shown in Figure 5.

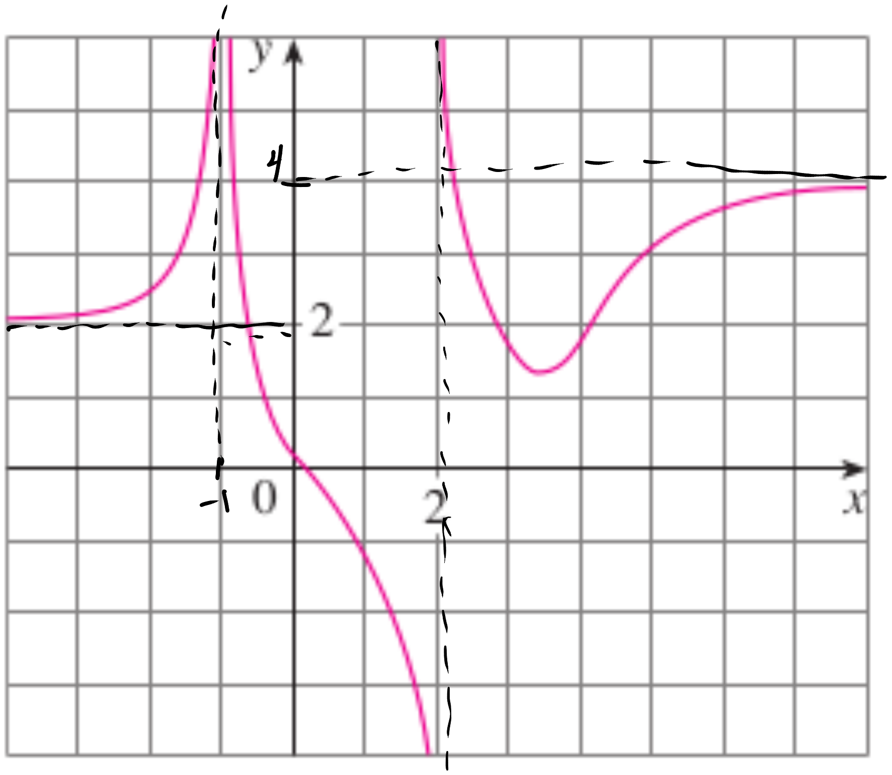


FIGURE 5

A) Infinite limits

$$\lim_{x \to -1} f(x) = +\infty$$

$$\lim_{x \to 2^-} f(x) = -\infty$$

$$\lim_{x \to 2^+} f(x) = +\infty .$$

$x = -1$ is a VA.

$x = 2$ is a VA.

Limits At infinity

$\lim_{x \to +\infty} f(x) = 4 \Rightarrow y = 4$ is a HA.

$\lim_{x \to -\infty} f(x) = 2 \Rightarrow y = 2$ is a HA.

Rules for Limits at infinity.

4 Theorem If $r > 0$ is a rational number, then

$$\lim_{x \to \infty} \frac{1}{x^r} = 0 \qquad \frac{1}{\sqrt{x}}$$

If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x \to -\infty} \frac{1}{x^r} = 0$$

EXAMPLE 3 Evaluate

$$\lim_{x \to \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$$

$$x = \frac{x^2}{x} \qquad -2 = \frac{-2x^2}{x^2}$$

$$\frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \frac{\cancel{x^2}\left(3 - \frac{1}{x} - \frac{2}{x^2}\right)}{\cancel{x^2}\left(5 + \frac{4}{x} + \frac{1}{x^2}\right)}$$

$$= \frac{3 - \frac{1}{x} - \frac{2}{x^2}}{5 + \frac{4}{x} + \frac{1}{x^2}}$$

$$\lim_{x \to \infty}\left(3 - \frac{1}{x} - \frac{2}{x^2}\right) = \lim_{x \to \infty} 3 - \lim_{x \to \infty} \frac{1}{x} - 2 \lim_{x \to \infty} \frac{1}{x^2}$$

$$= 3 - 0 - 2 \cdot 0$$

$$= 3$$

$$\lim_{x \to \infty}\left(5 + \frac{4}{x} + \frac{1}{x^2}\right) = \lim_{x \to \infty} 5 + 4 \lim_{x \to \infty} \frac{1}{x} + \lim_{x \to \infty} \frac{1}{x^2}$$

$$= 5 + 4 \cdot 0 + 0 = 5$$

By quotient rule:

$$\lim_{x \to \infty} \frac{3 - \frac{1}{x} - \frac{2}{x^2}}{5 + \frac{4}{x} + \frac{1}{x^2}} = \boxed{\frac{3}{5}}$$

EXAMPLE 4 Find the horizontal and vertical asymptotes of the graph of the function

$$f(x) = \frac{\sqrt{2x^2 + 1}}{3x - 5}$$

EXAMPLE 5 Compute $\lim_{x \to \infty} \left(\sqrt{x^2 + 1} - x\right)$.

Infinite Limits at Infinity.

The notation

$$\lim_{x \to \infty} f(x) = \infty$$

means that the values of $f(x)$ become larger and larger as the values of x becomes larger and larger. Similar meanings are attached to the following symbols:

$$\lim_{x \to -\infty} f(x) = \infty, \qquad \lim_{x \to \infty} f(x) = -\infty \quad \text{and} \quad \lim_{x \to -\infty} f(x) = -\infty.$$

WARNING!!

EXAMPLE 8 Find $\lim_{x \to \infty} x^3$ and $\lim_{x \to -\infty} x^3$.

EXAMPLE 9 Find $\lim_{x \to \infty} (x^2 - x)$.