Math 644

Chapter 6

Section 6.2: Normality and Equicontinuity

Contents

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DEFINITION 1. A collection, or family, $\mathcal{F}$ of continuous functions on a region $\Omega \subset \mathbb{C}$ is said to be **normal on** Ω provided every sequence $(f_n) \subset \mathcal{F}$ contains a subsequence which converges uniformly on compact subsets of Ω.

EXAMPLE 2. Show if the given family is normal on the given region.

(a) $\mathcal{F}_1 := \{f_n(z) = z^n \,:\, n = 0, 1, \ldots\}$ and $\Omega = \mathbb{D}$.

(b) $\mathcal{F}_2 := \{g_n \,:\, n = 0, 1, \ldots\}$, where $g_n(z) = 1$ if n is even and $g_n(z) = 0$ if n is odd and $\Omega = \mathbb{C}$.

(a) Since $|z|<1 \Rightarrow |z|^n \to 0$, $n \to \infty$ (uniformly) if $z \in K \subseteq \Omega$, K compact

(b) Take sequence $(f_n) \subseteq \mathcal{F}_2$.

$\boxed{\text{Case 1}}$ All index n are odd or all index n are even.

In this case $f_n \equiv 1$ $\forall n$ or $f_n \equiv 0$ $\forall n$.

the whole sequence works for a subsequence.

$\boxed{\text{Case 2}}$ Index are odd or even.

Create $(g_{n_k})_{k=1}^{\infty}$ such that

① $n_k = n$ if n is odd (choose ① finitely many even indexes)

② $n_k = n$ if n is even (choose ② " odd " indexes)

Choose ① if infinitely many odd & even indexes.

therefore $g_{n_k} \equiv 0$ in case 1

or $g_{n_k} \equiv 1$ in case 2

$(g_{n_k})_{k=1}^{\infty}$ converges on compact subsets of $\mathbb{C}$.

$\square$

LEMMA 3. Suppose Ω

- is a region and;
- $\Omega = \cup_{j=1}^{\infty} \Delta_j$, where Δ_j are ~~closed~~ open disks. such that $\overline{\Delta_j} \subset \Omega$

A family of continuous functions $\mathcal{F}$ is normal on Ω if and only if, for each j, every sequence in $\mathcal{F}$ contains a subsequence which converges uniformly on $\overline{\Delta}_j$.

Proof.

($\Rightarrow$). Obvious.

($\Leftarrow$). Suppose $(f_n) \subseteq \mathcal{F}$ be an arbitrary sequence on Ω.

<u>Goal:</u> $\exists (f_{n_k}) \subseteq (f_n)$ s.t. (f_{n_k}) converges unif. on compact subsets of Ω.

- Start with $j=1$.

  $\exists (f_{n_k^1}) \subseteq (f_n)$ s.t. $f_{n_k^1} \to g_1$ unif. on $\overline{\Delta}_1$

- For $j=2$, $\exists (f_{n_k^2}) \subseteq (f_{n_k^1})$ s.t. $f_{n_k^2} \to g_2$ unif. on $\overline{\Delta}_2$.

  $\vdots$

- For $j \geq 2$, $\exists (f_{n_k^j}) \subseteq (f_{n_k^{j-1}})$ s.t. $f_{n_k^j} \to g_j$ unif. on $\overline{\Delta}_j$

<u>Exposition:</u>

$$
\begin{array}{cccccccc}
f_{n_1^1} & f_{n_2^1} & \cdots & f_{n_k^1} & \cdots & \xrightarrow[\overline{\Delta}_1]{\text{unif.}} & g_1 \\
f_{n_1^2} & f_{n_2^2} & \cdots & f_{n_k^2} & \cdots & \xrightarrow[\overline{\Delta}_2]{\text{unif.}} & g_2 \\
\vdots & \vdots & & \vdots & & & \vdots \\
f_{n_1^j} & f_{n_2^j} & \cdots & f_{n_k^j} & \cdots & \xrightarrow[\overline{\Delta}_j]{\text{unif.}} & g_j \\
\vdots & \vdots & & \vdots & & & \vdots
\end{array}
$$

Define the subsequence (h_k) of (f_n) as

$$h_k := f_{n_k^k} \quad , \quad k \geq 1 .$$

Then, $\forall j$, (h_k) converges uniformly on Δ_j.

Let $K \subseteq \Omega$ be a compact subset.

Then K can be cover by finitely many Δ_j, so by finitely many $\overline{\Delta_j}$

Since (h_k) converges uniformly on $\overline{\Delta_j}$, then (h_k) converges uniformly on K. $\square$

Let Ω be a region and write

$$\Omega = \bigcup_{j=1}^{\infty} \Delta_j \quad , \quad \Delta_j \text{ } \cancel{\text{closed}} \text{ open disks. s.t. } \overline{\Delta_j} \subset \Omega .$$

$$C(\Omega) := \{ f : \Omega \to \mathbb{C} : f \text{ continuous on } \Omega \}$$

&

$$\rho(f,g) := \sum_{j=1}^{\infty} \frac{\rho_j(f,g)}{1+\rho_j(f,g)} \cdot \frac{1}{2^j}$$

where
$$\rho_j(f,g) := \sup_{z \in \overline{\Delta_j}} |f(z) - g(z)| .$$

We can show that $(C(\Omega), \rho)$ is a complete metric space.

$(f_n) \subseteq C(\Omega)$ will converge to $f \in C(\Omega)$ iff

$$\rho(f_n, f) \longrightarrow 0 \quad (n \to \infty) .$$

$$\rho(f,g) := \sum_{j=1}^{\infty} \frac{\rho_j(f,g)}{1+\rho_j(f,g)} \cdot \frac{1}{2^j}$$

Theorem 4. A sequence $(f_n) \subset C(\Omega)$ converges uniformly on compact subsets of Ω to $f \in C(\Omega)$ if and only if $\lim_{n\to\infty} \rho(f_n, f) = 0$.

Proof.

($\Rightarrow$) Suppose $(f_n) \subseteq C(\Omega)$ converges uniformly to f on compact subsets of Ω.

Then, in particular, $f_n \to f$ uniformly $\overline{\Delta}_j \subseteq \Omega$, $\forall j \geq 1$.

Choose N_j s.t. $n \geq N_j$

$$\Rightarrow |f_n(z) - f(z)| < \varepsilon, \quad \forall z \in \overline{\Delta}_j.$$

Choose M s.t. $\sum_{j=M}^{\infty} 2^{-j} < \varepsilon$.

Choose $N := \max\{N_1, N_2, \ldots, N_{M-1}\}$, then for $n \geq N$,

$$\rho(f_n, f) \leq \sum_{j=1}^{M-1} \frac{\rho_j(f_n, f)}{1+\rho_j(f_n,f)} 2^{-j} + \sum_{j=M}^{\infty} 2^{-j}$$

$$< \sum_{j=1}^{M-1} \frac{\rho_j(f_n,f)}{1+\rho_j(f_n,f)} 2^{-j} + \varepsilon$$

The function, $x \longmapsto \frac{x}{1+x}$ is increasing on $x > 0$.

Note:

- When $\lim_{n\to\infty} \rho(f_n, f) = 0$, we say that (f_n) **converges locally uniformly** to f on Ω.

Since $\rho_j(f_n, f) < \varepsilon$, then

$$\frac{\rho_j(f_n, f)}{1+\rho_j(f_n, f)} < \frac{\varepsilon}{1+\varepsilon} < \varepsilon$$

So,

$$\rho(f_n, f) < \varepsilon \cdot \left(\sum_{j=1}^{\infty} 2^{-j} \right) + \varepsilon$$

$$\Rightarrow \rho(f_n, f) < \varepsilon \left(\frac{1}{1-\frac{1}{2}} - 1 \right) \quad (n \geq N).$$

$$\Rightarrow \rho(f_n, f) < \varepsilon \qquad (n \geq N).$$

($\Leftarrow$) Suppose $f_n \xrightarrow{\rho} f$, $n \to \infty$. $\forall \varepsilon > 0$, $\exists N$ s.t.

$$n \geq N \Rightarrow \sum_{j=1}^{\infty} 2^{-j} \frac{\rho_j(f_n, f)}{1+\rho_j(f_n, f)} < \varepsilon.$$

In particular, $\forall j$

$$n \geq N \Rightarrow 2^{-j} \frac{\rho_j(f_n, f)}{1+\rho_j(f_n, f)} < \varepsilon$$

Then, $\forall j$

$$n \geq N, \quad \rho_j(f_n, f) < \frac{2^j \varepsilon}{1 - 2^j \varepsilon}.$$

So, $f_n \to f$ uniformly on $\overline{\Delta_j}$.

If $K \subseteq \Omega$ is compact, then $K \subseteq \bigcup_{k=1}^{N} \overline{\Delta_{jk}}$ and $f_n \to f$ uniformly on each of $\overline{\Delta_{jk}}$. $\square$

Equicontinuous Family of Functions

Definition 5. A family of functions $\mathcal{F}$ defined on a set $E \subset \mathbb{C}$ is

(a) **equicontinuous at $\mathbf{w} \in \mathbf{E}$** if $\forall \varepsilon > 0$, $\exists \delta > 0$ so that

$$z \in E \text{ and } |z - w| < \delta \quad \Longrightarrow \quad |f(z) - f(w)| < \varepsilon, \forall f \in \mathcal{F}.$$

(b) **equicontinuous on E** if it is equicontinuous at each $w \in E$.

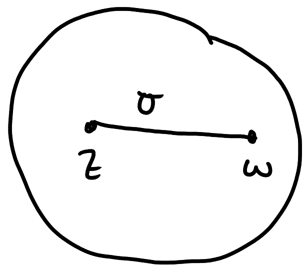
(c) **uniformly equicontinuous on E** if $\forall \varepsilon > 0$, $\exists \delta > 0$ so that

$$z, w \in E \text{ and } |z - w| < \delta \quad \Longrightarrow \quad |f(z) - f(w)| < \varepsilon, \forall f \in \mathcal{F}.$$

Example 6. Fix $M > 0$. Show that the family

$$\mathcal{F} := \{f : \mathbb{D} \to \mathbb{C} \ : \ f \text{ analytic and } |f'| \le M\}$$

is uniformly equicontinuous on $\mathbb{D}$.



$$f(z) - f(\omega) = \int_\sigma f'(z)\, dz$$

$$\Rightarrow \quad |f(z) - f(\omega)| \le \int_\sigma |f'(z)|\, |dz|$$

$$\le |z - \omega|\, M$$

Set $\delta = \dfrac{\varepsilon}{M}$. □

Theorem 7. [Arzela-Ascoli] A family of continuous functions $\mathcal{F}$ is normal on a region $\Omega \subset \mathbb{C}$ if and only if

(a) $\mathcal{F}$ is equicontinuous on Ω and;

(b) there is a $z_0 \in \Omega$ so that the collection $\{f(z_0) \,:\, f \in \mathcal{F}\}$ is a bounded subset of $\mathbb{C}$.

Proof.

Definition 8. A family $\mathcal{F}$ of continuous functions is said to be **locally bounded** on Ω if

$$\forall w \in \Omega,\ \exists \delta > 0 \text{ and } M < \infty \text{ so that } |z - w| < \delta \Rightarrow |f(z)| \leq M,\ \forall f \in \mathcal{F}.$$

Theorem 9. Let $\mathcal{F}$ be a family of analytic functions on a region Ω. Then the following are equivalent:

- **(a)** $\mathcal{F}$ is normal on Ω;
- **(b)** $\mathcal{F}$ is locally bounded on Ω;
- **(c)** $\mathcal{F}' := \{f' : f \in \mathcal{F}\}$ is locally bounded on Ω and there is a $z_0 \in \Omega$ so that $\{f(z_0) : f \in \mathcal{F}\}$ is a bounded subset of $\mathbb{C}$.

Proof.