UNIVERSITY
OF HAWAI'I

MATH-331 Intro. to Real Analysis
Final exam

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Last name: Solutions
First name: Final

Question:	1	2	3	4	5	Total
Points:	20	20	20	20	20	100
Score:	–	–	–	–	–	–

Instructions: Make sure to write your complete name on your copy. You must answer all the questions below and write your answers directly on the questionnaire. You have 2 hours to complete the exam. When you are done, hand out your copy and you may leave the classroom.

No devises such as a smart phone, cell phone, laptop, or tablet can be used during the exam. You are not allowed to use the lecture notes and the textbook also. You may use your personal cheat sheet on the exam.

Make sure to show all your work. State clearly any theorem or definition you are using in your proofs or your calculations. Make sure you show clearly that all hypothesis required to use a Theorem are satisfied. No credit will be earned for an answer without explanations.

BE THE BEST VERSION OF YOURSELF!

PIERRE-OLIVIER PARISÉ

Final.

SIGN ↑ TO ACKNOWLEDGE YOU HAD READ AND ACCEPT THE ABOVE RULES.

Question 1 **(20 pts)**

Find the value of the following limits. Write down clearly which properties you are using.

(a) $\lim_{n\to\infty} e^{1/n}$. (/5)

The function $x \mapsto e^x$ is continuous.

Since $\frac{1}{n} \to 0$, by continuity,

$$\lim_{n\to\infty} e^{1/n} = e^{\lim_{n\to\infty} \frac{1}{n}} = e^0 = 1.$$

(b) $\lim_{n\to\infty} x_n$ if $x_1 = 2$ and $x_n = 2 - 1/x_{n-1}$ for $n \geq 2$. (/5)

- $\underline{x_n \leq 2.}$ $x_1 = 2$. If $x_n \leq 2$, then

  $x_{n+1} = 2 - \frac{1}{x_n} \leq 2 - \frac{1}{2} \leq 2$. Induction, $x_n \leq 2$.

- $\underline{x_n \text{ increasing } n \geq 2.}$

  $x_2 = 2 - \frac{1}{2} = \frac{3}{2}$. If $x_n \leq x_{n+1}$, then $-\frac{1}{x_{n+1}} \geq -\frac{1}{x_n}$.

  So, $x_{n+2} = 2 - \frac{1}{x_{n+1}} \geq 2 - \frac{1}{x_n} = x_{n+1}$.

  Induction $\Rightarrow$ $(x_n)_{n=2}^{\infty}$ increasing.

- $\underline{x_n \geq \frac{3}{2}}$ $\underline{\forall n \geq 1}$ (x_n) increasing $\Rightarrow$ $x_n \geq \frac{3}{2}$.

From a Theorem in the lecture notes, $x_n \to A$ (some $A \in \mathbb{R}$)

Thus, $A = 2 - \frac{1}{A} \Rightarrow A^2 - 2A + 1 = 0 \Rightarrow A = 1$.

Thus, $\lim_{n\to\infty} x_n = 1$.

(c) $\lim_{x\to 0} \frac{\sin(x^2)}{2x}$. (You won't be credited if you use l'Hopital's Rule.) (/5)

We see that $\frac{\sin(x^2)}{2x} = \frac{x}{2}\,\frac{\sin(x^2)}{x^2}$.

From the lecture notes, we know that $|\sin(u)| \le |u|$. So,

$$|\sin(x^2)| \le |x^2|$$

$$\Rightarrow \quad -x^2 \le \sin(x^2) \le x^2$$

$$\Rightarrow \quad -1 \le \frac{\sin(x^2)}{x^2} \le 1 \quad (\text{because } x^2 > 0,\ x \neq 0)$$

So, for any $x \in \mathbb{R}$,

$$-\frac{x}{2} \le \frac{x}{2}\,\frac{\sin(x^2)}{x^2} \le \frac{x}{2}$$

Since $\lim_{x\to 0} \frac{x}{2} = 0$ and $\lim_{x\to 0} -\frac{x}{2} = 0$, by the Squeeze theorem for limits, it follows that

$$\lim_{x\to 0} \frac{\sin(x^2)}{2x} = \lim_{x\to 0} \frac{x}{2}\,\frac{\sin(x^2)}{x^2} = 0.$$

(d) $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k \sin(k^2/n^2)}{n^2}$. (Simplify your final answer as much as you can.) (/5)

$x_k = \frac{k}{n}$, $k = 1, 2, \dots, n$. So, $b = 1$ & $a = 0$.

$\Delta x = \frac{b-a}{n} = \frac{1}{n}$.

Thus, $$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k \sin(k^2/n^2)}{n^2} = \int_0^1 x \sin(x^2)\,dx$$

The functions $x \mapsto x$ & $x \mapsto \sin(x^2)$ are cont. so R.I.

We a change of variable

$$\int_0^1 x \sin(x^2)\,dx = \frac{1-\cos(1)}{2}.$$

Thus,

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k \sin(k^2/n^2)}{n^2} = \frac{1-\cos(1)}{2}.$$

QUESTION 2 **(20 pts)**

Answer the following questions. State all the hypothesis of the Theorem you are using and write down clearly which properties you are using.

(a) Using the Fundamental Theorem of Calculus, compute the derivative of the function (/10)
$F(x) = \int_{\cos x}^{\sin x} \sqrt{1-t^2}\, dt$ where $x \in [0, \pi/2]$. Simplify your answer as much as you can.

Let $g(x) = \int_0^x \sqrt{1-t^2}\, dt$. then

$$F(x) = g(\sin x) - g(\cos x).$$

By the FTC, we get

$$F'(x) = \sqrt{1-\sin^2 x} \cdot \cos x - \sqrt{1-\cos^2 x} \cdot (-\sin x)$$
$$= \cos(x)\sqrt{1-\sin^2(x)} + \sin(x)\sqrt{1-\cos^2(x)}$$
$$= \cos(x)\sqrt{\cos^2 x} + \sin x \sqrt{\sin^2 x}$$

Since $\cos x \geq 0$ & $\sin x \geq 0$ for $x \in [0, \pi/2]$, then

$$\sqrt{\cos^2 x} = \cos x \quad \& \quad \sqrt{\sin^2 x} = \sin x$$

So,

$$F'(x) = \cos(x) \cdot \cos(x) + \sin(x) \cdot \sin(x)$$
$$= \cos^2(x) + \sin^2(x)$$
$$= \underline{1}.$$

(b) Find $g'(5)$ if g is the inverse of the function $f(x) = x^3 + 2x + 2$. (/5)

The fct. is differentiable because it is a polynomial. So, since $f'(x) = 3x^2 + 2 \neq 0 \ \forall x \in \mathbb{R}$, the inverse g of f exists. This means, from a theorem in the lecture notes that:

$$g'(5) = \frac{1}{f'(g^{-1}(5))}.$$

But $f(1) = 1+2+2 = 5 \Rightarrow g^{-1}(5) = 1$. So

$$g'(5) = \frac{1}{3 \cdot 1 + 2} = \frac{1}{5}.$$

(c) Show that the equation $\cos(2x) = x$ has exactly one solution in the interval $[0, \pi/4]$. (/5)

Let $f(x) = x - \cos(2x)$. f is cont. because the fcts $x \mapsto x$ & $x \mapsto -\cos(2x)$ are continuous.

- We see that $f(0) = -1$ and $f(\pi/4) = \frac{\pi}{4} - 0 = \frac{\pi}{4}$.
  So, $f(0) < 0 < f(\pi/4)$. By the IVT, $\exists c \in (0, \pi/4)$ s.t.
  $$f(c) = 0.$$
- Also, $f'(x) = x + 2\sin(2x)$. The fct. $x \mapsto \sin(2x)$ is strictly increasing on $[0, \pi/4]$. So $f'(x) \neq 0 \ \forall x \in (0, \pi/4)$. If there were $c_1, c_2 \in (0, \pi/4)$ s.t. $f(c_1) = f(c_2) = 0$, then by Rolle's thm., $\exists x \in (0, \pi/4)$ between c_1 & c_2 such that
  $$f'(x) = 0, \quad \#.$$

So, there is only one $c \in (0, \pi/4)$ s.t. $f(c) = 0$

Rewriting this last equation:

$$\exists c \in (0, \pi/4), \quad \cos(2c) = c.$$

bounded

QUESTION 3 **(20 pts)**

Let A and B be two non-empty bounded subsets of $\mathbb{R}$. Give a proof or, if it's false, give a counter-example to the following statements.

(a) If $S \subseteq A$ and S is nonempty, then $\inf A \leq \inf S$. (/10)

(b) If $A \cap B \neq \varnothing$, then $\sup(A \cap B) = \max\{\sup A, \sup B\}$. (/10)

(a) Since A is bounded, $\inf A$ exists (consequence of AC). Since $S \subseteq A$, S is also bounded, so $\inf S$ also exists. Let $s \in S$. Since $S \subseteq A$, by def. of the infimum of A,

$$s \geq \inf A$$

because $a \geq \inf A \ \forall a \in A$ & $s \in A$. So, by the def. of the inf applied to S, this means that

$$\inf S \geq \inf A$$

because $\inf A$ is a lower bound for S and $\inf S \geq l$ for any lower bound l of A.

(b) Not true. Take $A = [0,1]$ & $B = [0,2]$. Then

$\sup A = 1$ & $\sup B = 2$ & $\max\{\sup A, \sup B\} = 2$.

But, $A \cap B = [0,1]$ and $\sup(A \cap B) = 1$.

So, $\sup(A \cap B) = 1 \neq 2 = \max\{\sup A, \sup B\}$.

Question 4 (20 pts)

Let $a > 0$. We say that a function $f : (-a, a) \to \mathbb{R}$ is

- **odd** if $f(-x) = -f(x)$ for any $x \in (-a, a)$;
- **even** if $f(-x) = f(x)$ for any $x \in (-a, a)$.

Suppose $f : (-a, a) \to \mathbb{R}$ is a differentiable function on $(-a, a)$.

(a) Show that the function f is even if, and only if, f' is odd. (/10)

(b) Show that the function f is odd if, and only if, f' is even and $f(0) = 0$. (/10)

(a) f is diff. on $(-a,a)$. Define

$$h(x) = f(x) - f(-x).$$

the fcts, $x \mapsto f(x)$, $x \mapsto -f(-x)$ are differentiable by assumptions & by the chain rule & product rule respectively.

($\Rightarrow$) Suppose f is even. Then $h(x) = f(x) - f(-x) = 0$. for all $x \in (-a,a)$. So, $h'(x) = 0 \quad \forall x \in (-a,a)$

$$\Rightarrow \quad f'(x) + f'(-x) = 0 \quad \forall x \in (-a,a)$$
$$\Rightarrow \quad f'(-x) = -f(x) \quad \forall x \in (-a,a)$$
$$\Rightarrow \quad f' \text{ is odd.}$$

($\Leftarrow$) Suppose f' is odd. then $h'(x) = 0 \quad \forall x \in (-a,a)$.

this means that $h(x) = k \quad \forall x \in (-a,a)$ ($k \in \mathbb{R}$ cst.)

But $h(0) = f(0) - f(-0) = 0 \Rightarrow k = 0$. thus,

$$h(x) = 0 \quad \forall x \in (-a,a)$$
$$\Rightarrow \quad f(-x) = f(x) \quad \forall x \in (-a,a)$$
$$\Rightarrow \quad f \text{ is even.}$$

(b) Define $g(x) = f(x) + f(-x)$. g is diff. on $(-a,a)$.

($\Rightarrow$) Suppose f is odd. Then, $g(x) = f(x) + f(-x) = 0$ for any $x \in (-a,a)$. So,

$$g'(x) = 0 \quad \forall x \in (-a,a)$$
$$\Rightarrow f'(x) - f'(-x) = 0 \quad \forall x \in (-a,a)$$
$$\Rightarrow f'(x) = f'(-x) \quad \forall x \in (-a,a).$$
$$\Rightarrow f' \text{ is even}.$$

Also, $f(0) = 0$ because f is odd. So,

$$f(0) = -f(0) \Rightarrow 2f(0) = 0$$
$$\Rightarrow f(0) = 0.$$

($\Leftarrow$) Suppose f' is even & $f(0)=0$. Then

$$h'(x) = f'(x) - f'(-x) = 0 \quad \forall x \in (-a,a)$$

So, $h'(x) = 0 \ \forall x \in (-a,a) \Rightarrow h(x) = k \quad \forall x \in (-a,a)$
$(k \in \mathbb{R} \text{ cst.})$.

Since $f(0) = 0 \Rightarrow h(0) = 0 \Rightarrow k = 0.$

Thus, $f(x) + f(-x) = 0 \quad \forall x \in (-a,a)$

$$\Rightarrow f(-x) = -f(-x) \quad \forall x \in (-a,a)$$
$$\Rightarrow f \text{ is odd}.$$

QUESTION 5 **(20 pts)**

Answer the following questions with **True** or **False**. Write down you answers on the line at the end of each question. Justify briefly your answer in the space after the statement of the problem.

(a) Any subset of the real numbers has a supremum. (/ 4)

$A = \mathbb{R}$ is a counter-example.

(a) False.

(b) If $f(x) = 2x$ when $x \in \mathbb{Q}$ and $f(x) = -x$ if $x \notin \mathbb{Q}$, then f has a limit at $x = 1$. (/ 4)

- $(x_n)_{n=1}^{\infty} \subseteq \mathbb{Q}$ with $x_n \to 1$ $\Rightarrow \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 2x_n = 2.$
$\neq$
- $(x_n)_{n=1}^{\infty} \subseteq \mathbb{R}\setminus\mathbb{Q}$ with $x_n \to 1$ $\Rightarrow \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} -x_n = -1$

(b) False.

(c) The sequence $(x_n)_{n=1}^{\infty}$ defined by $x_n = (-1)^n$ has a convergent subsequence. (/ 4)

Define $(x_{n_k})_{k=1}^{\infty}$ by $x_{n_k} = (-1)^{2k} = 1.$

$x_{n_k} \xrightarrow[k\to\infty]{} 1$.

(c) True.

(d) If f is differentiable on $(0, 2)$, if $f(1) = 1$, $f'(1) = 2$, and if $g(x) = f(x^2)\cos(\pi x)$, then $g'(1) = -2$. (/ 4)

$$g'(x) = 2x f'(x^2)\cos(\pi x) + f(x^2)(-\sin(\pi x))\,\pi$$

$$\Rightarrow g'(1) = 2 \cdot f'(1) \cdot \cos(\pi) + f(1)(-\sin(\pi)) \cdot \pi$$
$$= -2$$

(d) True.

(e) If $f : [a, b] \to \mathbb{R}$ and $g : [c, d] \to [a, b]$ are two continuous functions, then $f \circ g$ is Riemann integrable on $[a, b]$. (/ 4)

f, g cont. $\Rightarrow f\circ g$ is cont.

$\Rightarrow f\circ g$ is R.I.

(e) True.