M444 – Complex Analysis

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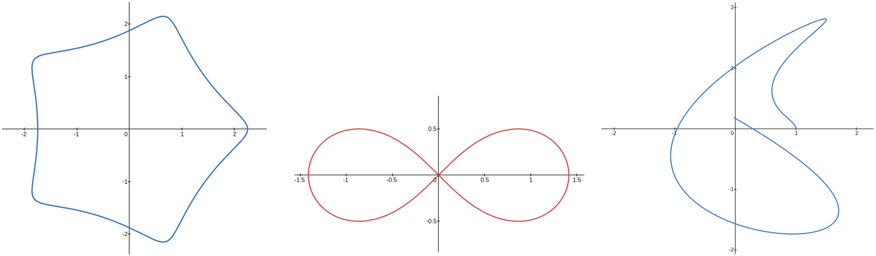
Chapter 3

Section 3.4: Cauchy's Theorem

Definition

A **Jordan curve** γ is a simple, continuous, closed curve. In other words, there is a parametrization $\gamma : [a, b] \to \mathbb{C}$ such that

1. If $a \leq t_1, t_2 < b$, then $\gamma(t_1) \neq \gamma(t_2)$ whenever $t_1 \neq t_2$.
2. $\gamma(a) = \gamma(b)$.
3. γ is continuous.



(a) Jordan curve **(b)** Not Jordan curve **(c)** Not Jordan curve

Figure – Different types of curves

Theorem

Any Jordan curve γ separates $\mathbb{C}$ into two regions :

1. The **interior** of the curve, denoted by Ω^-.
2. The **exterior** of the curve, denoted by Ω^+.

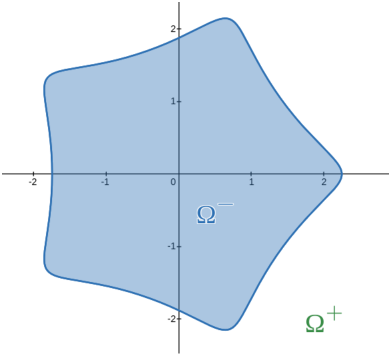


Figure – Interior and Exterior of a Jordan Curve

Definition

Let γ be a Jordan curve with corresponding regions Ω^-, Ω^+. Informally, we say

1. γ has **positive orientation** if traversing the curve, Ω^- is on the left.
2. γ has **negative orientation** if traversing the curve, Ω^- is on the right.

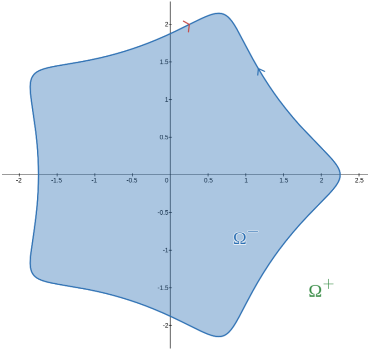


Figure – Two orientations of a Curve

Theorem

1. Assume f is an analytic function on a region U.
2. Assume γ is a Jordan curve such that $\Omega^- \cup \gamma \subset U$.
3. Assume f' is continuous on U.

Then $\displaystyle\int_\gamma f(z)\,dz = 0$.

Proof. Let $f(z) = u(z) + iv(z)$. Let $z(t) = x(t) + iy(t)$, $a \leq t \leq b$ be a parametrization of γ.
By definition,

$$I := \int_\gamma f(z)\,dz = \int_a^b f(z(t))z'(t)\,dt.$$

We have $f(z(t))z'(t) = ux' - vy' + i(vx' + uy')$ and so

$$I = \int_a^b ux' - vy'\,dt + i\int_a^b vx' + uy'\,dt = \int_\gamma udx - vdy + i\int_\gamma vdx + udy.$$

Green's Theorem.

1. Let γ be a Jordan curve with positive orientation.
2. Let Ω^- be the interior of γ.
3. Assume that $P(x,y)$ and $Q(x,y)$ have continuous partial derivatives on $\Omega^- \cup \gamma$.

Then,

$$\int_\gamma P\,dx + Q\,dy = \iint_{\Omega^-} Q_x - P_y\,dA.$$

Since $f' = u_x + iv_x$ and f' is continuous, then u_x and v_x are continuous. Similarly, since $f' = v_y + iu_y$ and f' is continuous, then u_y and v_y are continuous. Therefore, u and v have continuous partial derivatives.

Set $P = u$ and $Q = -v$:

$$\int_\gamma u dx - v dy = \iint_{\Omega^-} -v_x - u_y\,dA = -\iint_{\Omega^-} v_x + u_y\,dA.$$

Set $P = v$ and $Q = u$:

$$\int_{\gamma} vdx + udy = \iint_{\Omega^-} u_x - v_y \, dA.$$

Putting everything together :

$$I = -\iint_{\Omega^-} v_x + u_y \, dA + i \iint_{\Omega^-} u_x - v_y \, dA.$$

Recall the C-R Equations

$$u_x = v_y \quad \text{and} \quad u_y = -v_x.$$

Hence,

$$I = -\iint_{\Omega^-} 0 \, dA + i \iint_{\Omega^-} 0 \, dA = 0.$$

The proof is then completed. $\square$

Note : This is not the general statement of Cauchy's Theorem.

Theorem (General Cauchy's Theorem)

1. Assume f is an analytic function on a region U.
2. Assume γ is a Jordan curve such that $\Omega^- \cup \gamma \subset U$.

Then $\displaystyle\int_\gamma f(z)\,dz = 0$.

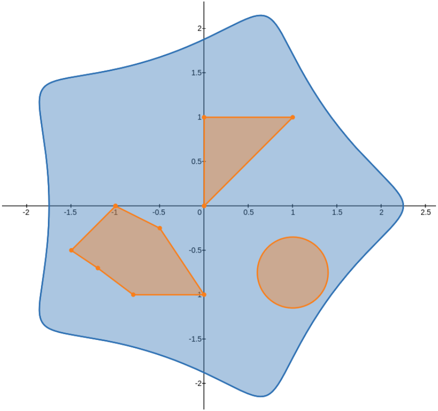
Independence of paths.

1. Let γ_1 and γ_2 be two simple paths in a region U with the same starting and terminal points.
2. Let $\Gamma := \gamma_1 \cup \gamma_2$. Assume that $\Omega^- \subset U$.
3. Let f be analytic on U.

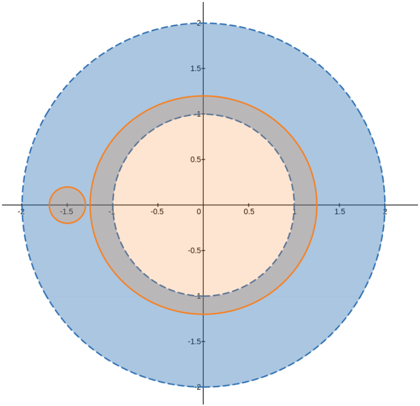
Then $\displaystyle\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$.

Definition

A region Ω is called **simply connected** if the interior of any Jordan curve is contained in Ω.



(a) Simply Connected.



(b) Not Simply Connected.

Corollary

Let f be an analytic function on a simply connected region Ω. If γ is a Jordan curve in Ω, then

$$\int_{\gamma} f(z)\, dz = 0.$$

Consequences.

1. If γ_0 and γ_1 are two paths in Ω with the same initial and terminal points, then

$$\int_{\gamma_0} f(z)\, dz = \int_{\gamma_1} f(z)\, dz.$$

2. There is an analytic function F defined on Ω such that $F' = f$.