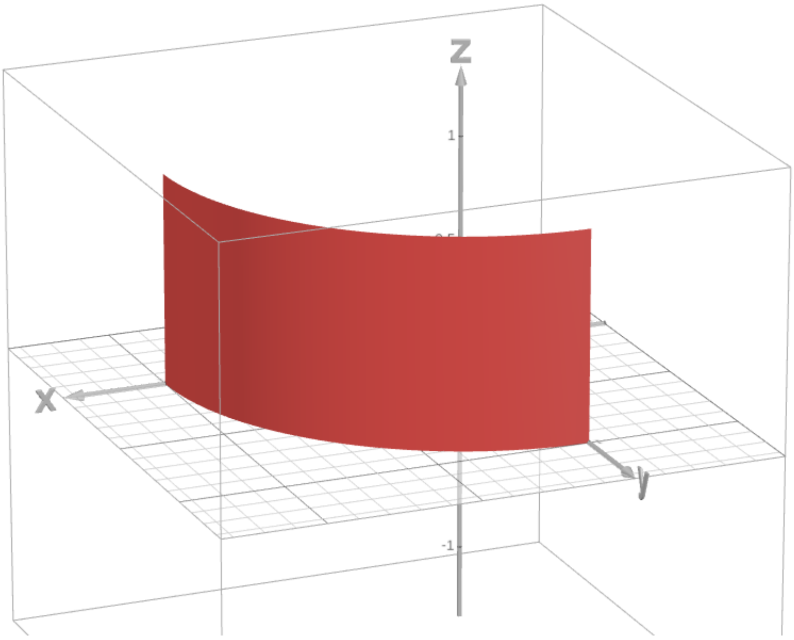
Chapter 16

Vector Calculus

16.7 Surface Integrals

Surface Differential

Example. Find the area of the following parametric surface S:



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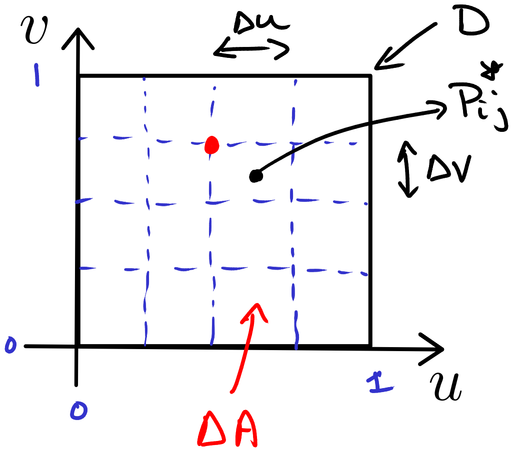
Parametric Equations

$$x = \cos\left((\pi/2)u\right)$$
$$y = \sin\left((\pi/2)u\right)$$
$$z = v$$

$0 \le u \le 1,\ 0 \le v \le 1.$

1. Divide the uv-region in small rectangles.

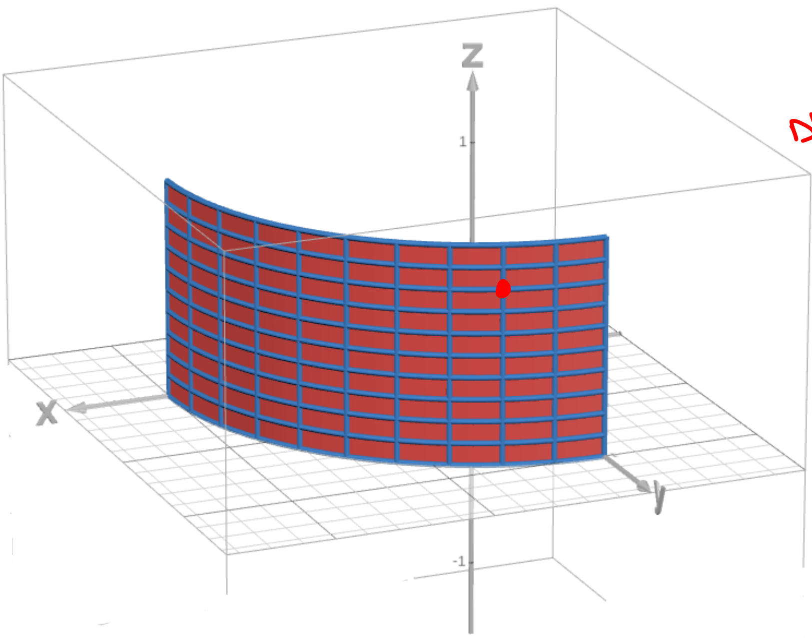


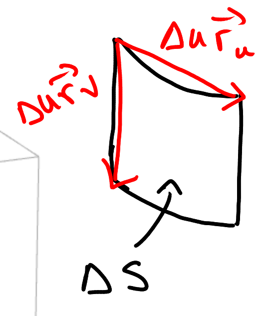
Divide D in small rectangles:

- m parts of length Δu
- n parts of length Δv

Select a sample point P_{ij}^* in each rectangle.

2. Approximate the area of each small piece.





$\Delta S \approx \text{Area}(\square)$

$= |\Delta u \vec{r}_u \times \Delta v \vec{r}_v|$

So,

$$\Delta S \approx |\vec{r}_u \times \vec{r}_v| \underbrace{\Delta u \Delta v}_{\Delta A}$$

3. Sum up.

$$\text{Area}(S) \approx \sum_{i=1}^{m} \sum_{j=1}^{n} |\vec{r}_u \times \vec{r}_v| \, \Delta A$$

Take $m, n \to \infty$

$$\Rightarrow \boxed{\text{Area}(S) = \iint_D |\vec{r}_u \times \vec{r}_v| \, dA}$$

4. Compute the Area.

$$\vec{r}_u = \left\langle -\frac{\pi}{2} \sin\left(\frac{\pi}{2} u\right), \frac{\pi}{2} \cos\left(\frac{\pi}{2} u\right), 0 \right\rangle$$

$$\vec{r}_v = \langle 0, 0, 1 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\frac{\pi}{2} \sin\left(\frac{\pi}{2} u\right) & \frac{\pi}{2} \cos\left(\frac{\pi}{2} u\right) & 0 \\ 0 & 0 & 1 \end{vmatrix} = \left\langle \frac{\pi}{2} \cos\left(\frac{\pi}{2} u\right), \frac{\pi}{2} \sin\left(\frac{\pi}{2} u\right), 0 \right\rangle$$

So,

$$|\vec{r}_u \times \vec{r}_v| = \frac{\pi}{2}.$$

So,

$$\text{Area}(D) = \iint_D \frac{\pi}{2} \, dA = \int_0^1 \int_0^1 \frac{\pi}{2} \, du \, dv$$

$$= \boxed{\frac{\pi}{2}}$$

Surface Area Differential:

$$\boxed{dS = |\vec{r}_u \times \vec{r}_v| dA}$$

Integral of scalar-valued functions.

Data:

- A surface S.
- A parametrization $\vec{r}(u,v)$ of the surface with domain D.
- A scalar-valued function $f(x,y,z)$. → mass density. → temperature.

$$\boxed{\iint_S f(x,y,z)\,dS = \iint_D f(\vec{r}(u,v))|\vec{r}_u \times \vec{r}_v|\,dA}$$

5–20 Evaluate the surface integral.

5. $\iint_S (x+y+z)\,dS$,
S is the parallelogram with parametric equations $x = u + v$, $y = u - v$, $z = 1 + 2u + v$, $0 \leqslant u \leqslant 2$, $0 \leqslant v \leqslant 1$

$f(x,y,z) = x+y+z$, $\quad \vec{r}(u,v) = \langle \underbrace{u+v}_{x}, \underbrace{u-v}_{y}, \underbrace{1+2u+v}_{z} \rangle$.

① Find $\vec{r}_u \times \vec{r}_v$

$\vec{r}_u = \langle 1, 1, 2 \rangle$

$\vec{r}_v = \langle 1, -1, 1 \rangle$

$$\Rightarrow \quad \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 2 \\ 1 & -1 & 1 \end{vmatrix} = \langle 3, -(-1), -2 \rangle$$

② Integral

$$\iint_S x+y+z \; dS = \iint_D (u+v)+(u-v)+(1+2u+v)\,|\vec{r}_u \times \vec{r}_v|\,dA$$

$$= \iint_D (4u+v+1)\,|\langle 3,1,-2\rangle|\,dA$$

$$= \int_0^1 \int_0^2 (4u + v + 1)\sqrt{9+1+4}\ du\,dv$$

$$= \int_0^1 \int_0^2 (4u + v + 1)\sqrt{14}\ du\,dv$$

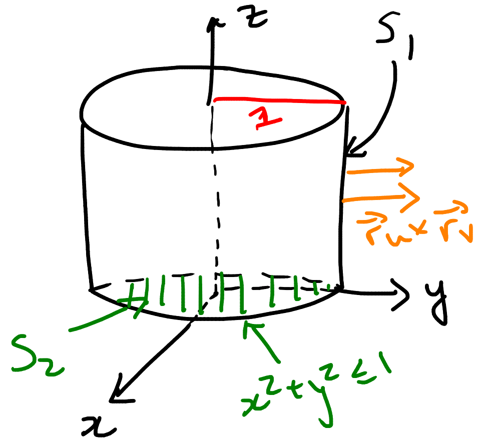
$$= \sqrt{14} \int_0^1 \int_0^2 (4u + v + 1)\ du\,dv$$

$$= \boxed{\sqrt{14} \cdot 11}$$

EXAMPLE.

Evaluate $\iint_S z\,dS$, where S is the surface whose sides are given by the cylinder $x^2+y^2=1$ from $z=0$ to $z=2$ and whose bottom is the disk $x^2+y^2\le 1$ in the plane $z=0$.

① Picture



$$S = S_1 \sqcup S_2$$

S_1: $\vec{r}(u,v) = \langle \cos u, \sin u, v\rangle$

$0 \le u \le 2\pi, \quad 0 \le v \le 2.$

S_2: $\vec{r}(u,v) = \langle v\cos u, v\sin u, 0\rangle$

$0 \le u \le 2\pi, \quad 0 \le v \le 1.$

② Integral $\displaystyle\iint_S z\,dS = \iint_{S_1} z\,dS + \iint_{S_2} z\,dS$

on S_1 $\quad \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin u & \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle \cos u, \sin u, 0\rangle$

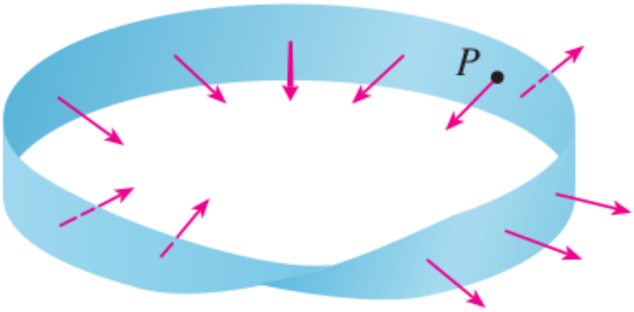
Then, $\displaystyle\iint_{S_1} z\,dS = \int_0^2 \int_0^{2\pi} v\sqrt{1}\,du\,dv = 4\pi$

on S_2 $\quad \displaystyle\iint_{S_2} z\,dS = 0$ $\qquad$ Why? because $z=0$ in the parametrization of S_2

So: $\displaystyle\iint_S z\,dS = 4\pi + 0 = \boxed{4\pi}$

Surface integral of Vector Fields.

- Non-orientable surfaces.



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- Orientable surface.

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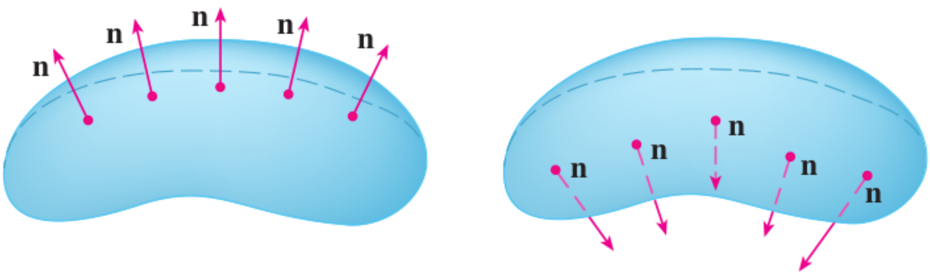


- We assume that the surface is orientable, so that there is a well-defined normal vector.

- For a parametric surface with parametrization $\vec{r}(u, v)$:

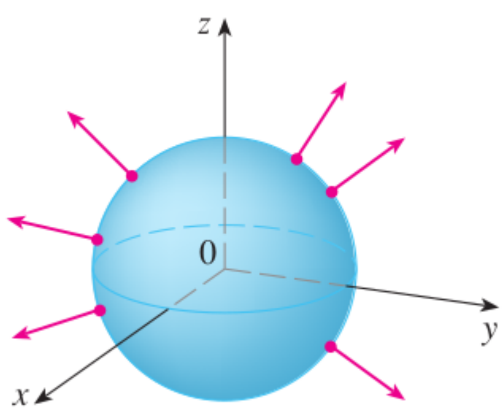
$$\boxed{\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}}$$

Example.

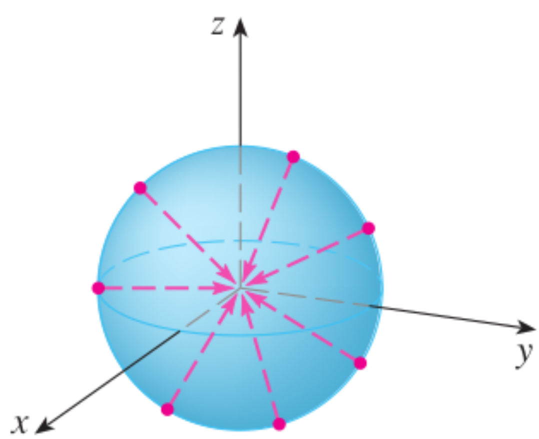
Find a normal vector at every point of a sphere of equation

$$x^2 + y^2 + z^2 = 1$$

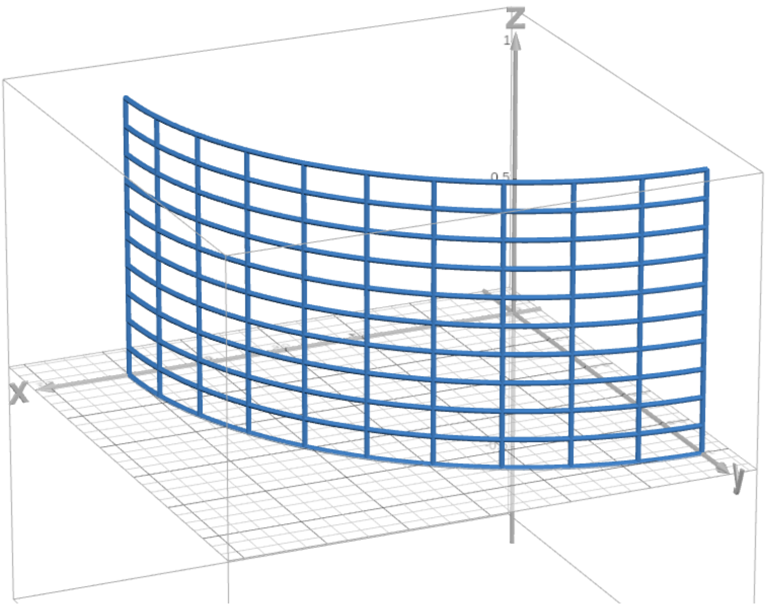
Positive orientation of a closed surface.



Negative orientation of a closed surface.



Flux integral (or Surface integral).



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Data:

- An orientable surface S.
- A parametrization $\vec{r}(u,v)$ of the surface.
- A vector field $\vec{F}(x,y,z)$.

$$\boxed{\int_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot(\vec{r}_u \times \vec{r}_v)\, dA}$$

EXAMPLE.

Find the flux integral of $\vec{F}(x,y,z) = \langle xy, yz, zx\rangle$ through the part of the paraboloid $z = 4 - x^2 - y^2$ lying above the square $[0,1]\times[0,1]$ and with upward orientation.

EXAMPLE.

Find the flux integral of $\vec{F}(x, y, z) = \langle x, 2y, 3z \rangle$ if S is a cube with diagonal $(0, 0, 0)$ to $(1, 1, 1)$ and S has the positive orientation.

Gauss' Law

The net charge enclosed by a closed surface S is

$$Q = \varepsilon_0 \iint_S \vec{E} \cdot d\vec{S}$$

where $\vec{E}$ is the electric field and ε_0 is the permittivity of free space.