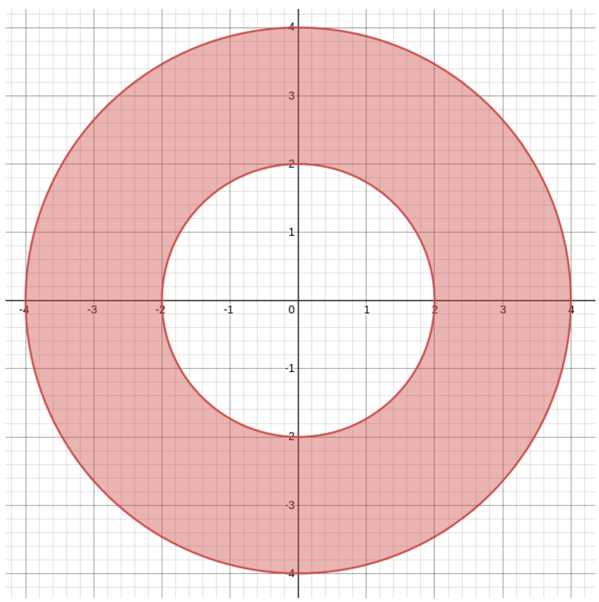
Section 15.3, Problem 22

The surfaces that bounds the z-coordinate are $z = \sqrt{16 - x^2 - y^2}$ and $z = -\sqrt{16 - x^2 - y^2}$. We want the solid outside the cylinder $x^2 + y^2 = 2^2$. Let D be the domain of integration, then the volume of the solid is given by

$$V(S) = \iint_D \sqrt{16 - x^2 - y^2} - (-\sqrt{16 - x^2 - y^2})\, dA = \iint_D 2\sqrt{16 - x^2 - y^2}\, dA.$$

To find D, we will project on the XY-plane. The projections of the cylinder on the XY-plane is a circle of radius 2 and the projection of the sphere on the XY-plane is a circle of radius 4. Thus we want the region between these two circles. We will change in polar coordinates. The equation



of the two circles are $r = 2$ and $r = 4$ and the angle ranges from 0 to 2π. So, in polar coordinates, we have

$$D = \{(r, \theta) \,:\, 2 \leq r \leq 4 \text{ and } 0 \leq \theta \leq 2\pi\}.$$

Thus, using the change of variable formula, we get

$$V(S) = 2\int_0^{2\pi} \int_2^4 (\sqrt{16 - r^2}) r \, dr d\theta = 4\pi \int_2^4 r\sqrt{16 - r^2}\, dr.$$

Setting $u = 16 - r^2$ and completing the calculations for the integral, we get the value

$$V(S) = 32\pi\sqrt{3}.$$

Section 15.3, Problem 32

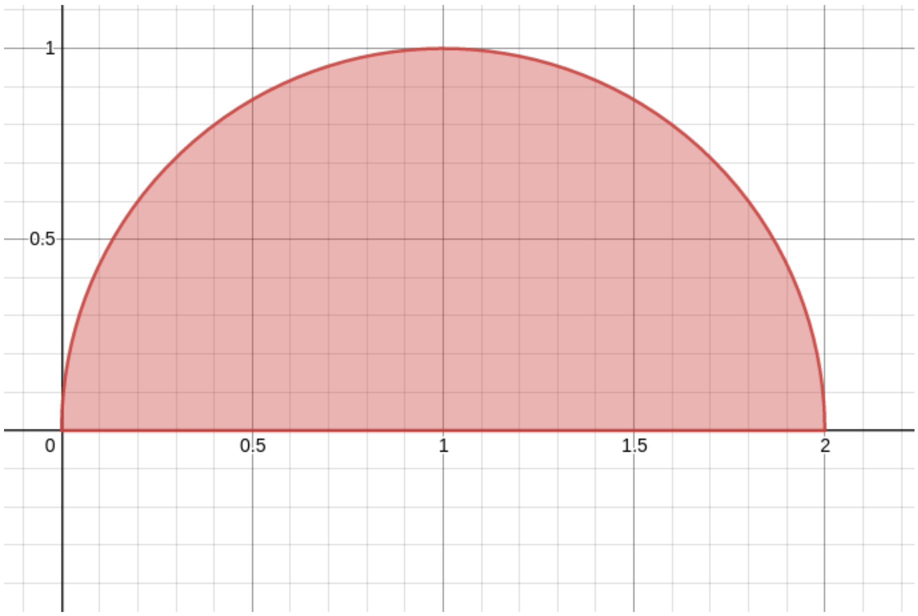
The bounds in the integrals give us

$$D = \{(x, y) : 0 \le x \le 2 \text{ and } 0 \le y \le \sqrt{2x - x^2}\}$$

So, the upper bound of y is half of a circle of radius 1 and center $(1, 0)$ because

$$y = \sqrt{2x - x^2} \iff (x - 1)^2 + y^2 = 1.$$

So the region looks like this



Let's describe the domain D in polar coordinates. Let $x = r\cos\theta$ and $y = r\sin\theta$. The equation of the circle in polar coordinate is $r = 2\cos\theta$ (replace x and y by $r\cos\theta$ and $r\sin\theta$ in the equation $x^2 + y^2 = 2x$). Also, the angle θ will vary from 0 to $\pi/2$ so we cover the upper half of the circle (and its interior). So

$$D = \{(r, \theta) : 0 \le r \le 2\cos\theta \text{ and } 0 \le \theta \le \pi/2\}.$$

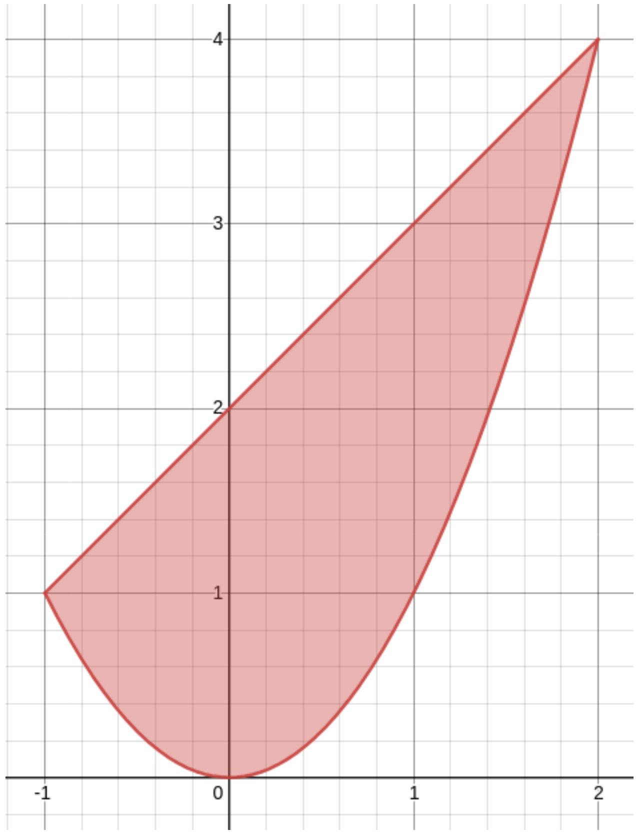
We can now compute the integral, call it I, in polar coordinates:

$$I = \int_0^{\pi/2} \int_0^{2\cos\theta} r^2 \, dr d\theta = \frac{8}{3} \int_0^{\pi/2} \cos^3\theta \, d\theta = \frac{2}{3} \int_0^{\pi/2} 3\cos(\theta) + \cos(3\theta) \, d\theta.$$

After finding the value of the integral, we get $I = 16/9$.

Section 15.4, Problem 8

The shape of the lamina is shown in the picture below. So we get



$$D = \{(x, y) \,:\, -1 \le x \le 2 \text{ and } x^2 \le y \le x + 2\}.$$

The density is $\phi(x) = kx^2$ for some constant k. Then

$$M = \int_{-1}^{2} \int_{x^2}^{x+2} kx^2 \, dydx = 63k/20.$$

The center of mass is given by $(\overline{x}, \overline{y})$. We then compute

$$\overline{x} = \frac{20}{63k} \int_{-1}^{2} \int_{x^2}^{x+2} kx^3 \, dydx = \frac{20}{63} \times \frac{18}{5} = \frac{8}{7}.$$

and

$$\overline{y} = \frac{20}{63} \int_{-1}^{2} \int_{x^2}^{x+2} x^2 y \, dydx = \frac{20}{63} \times \frac{531}{70} = \frac{118}{49}.$$

So the center of mass is $(\overline{x}, \overline{y}) = (8/7, 118/49)$.

Section 15.4, Problem 12 (only the mass)

We have $\rho(x, y) = k(x^2 + y^2)$. In polar coordinate, the disk is described as followed

$$D = \{(r, \theta) \,:\, 0 \leq r \leq 1 \text{ and } 0 \leq \theta \leq \pi/2\}.$$

Setting $x = r\cos\theta$ and $y = r\sin\theta$, we have $dA = rdrd\theta$ and so

$$M = \iint_D \rho(x, y)\, dA = \int_0^{\pi/2} \int_0^1 kr^2 r\, drd\theta = k\pi/8.$$

Section 15.4, Problem 20

The fan is a suare with sides of length 2 with the lower left corner positioned at the origin, so

$$D = [0, 2] \times [0, 2].$$

We have to compare I_x and I_y.

Firstly, we have

$$I_x = \iint_D y^2 \rho(x, y)\, dA = \int_0^2 \int_0^2 y^2(1 + 0.1x)\, dxdy = 88/15.$$

Secondly, we have

$$I_y = \iint_D x^2 \rho(x, y)\, dA = \int_0^2 \int_0^2 x^2(1 + 0.1x)\, dxdy = 92/15.$$

We see that $I_y > I_x$, and so it would be more difficult to rotate the fan blade around the y-axis.