Math 644

Chapter 2

Section 2.2: Fundamental Theorem of Algebra and Partial Fractions

Contents

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The local behavior of a polynomial (Walking a Dog picture) is really helpful to give a proof of the FTA.

THEOREM 1. Every non-constant polynomial has a zero.

Some precision:

- A function $f : \Omega \subseteq \mathbb{C} \to \mathbb{C}$ has a zero at $a \in \Omega$ if $f(a) = 0$.

LEMMA 2. If $n := \deg p \geq 1$, then $|p(z)| \to \infty$, as $|z| \to \infty$.

Proof.

Let $p(z) = \sum_{k=0}^{n} a_k z^k$, $a_n \neq 0$.

If $|z| \neq 0$, then

$$p(z) = z^n \left(\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \cdots + a_n \right)$$

Since $\frac{1}{|z|^k} \to 0$ as $|z| \to \infty$ $(\forall k)$

then, for $M > 0$ fix, $\exists R_1 > 0$ s.t.

$$\frac{1}{|z|^k} < \frac{|a_n|}{2n\left(\max_{0 \leq k \leq n-1} \{|a_k|\} + 1\right)}, \quad \forall k \in \{1, \ldots, n\}$$

So, if $|z| > R_1$, then $\qquad |z-w| \geq |z| - |w|$

$$\left| \sum_{k=1}^{n} \frac{a_k}{z^k} \right| \leq \sum_{k=1}^{n} \frac{|a_k|}{|z|^k} < \frac{|a_n|}{2}.$$

Now, if $|z| > R_1$, then

$$|p(z)| \geq |z|^n |a_n| - |z|^n \frac{|a_n|}{2} \geq \frac{|z|^n}{2} \to \infty$$

Lemma 3. If $p(z)$ is a polynomial with no zero, then

$$M := \inf\{|p(z)| \,:\, z \in \mathbb{C}\} \in (0, \infty).$$

<u>Proof.</u> First, $p(0) = a_0 \in \mathbb{C} \Rightarrow M \leq |a_0| < \infty$.

Let $(R_n)_{n=1}^{\infty} \subseteq (0, \infty)$ s.t. $R_n \nearrow \infty$.

Let $M_n := \inf\{|p(z)| : |z| \leq R_n\}$. So, the sequence (M_n) is decreasing and bounded below by 0. So, there is M s.t.

$$\lim_{n \to \infty} M_n = M.$$

Since $|p|$ is continuous on $\{z : |z| \leq R_n\}$, then $\exists z_n \in \{z : |z| \leq R_n\}$ s.t. $|p(z_n)| = M_n$.

Suppose that $|z_n| \to \infty$, $|p(z_n)| \to \infty$ $(n \to \infty)$.

So, since $|p(z_n)| = M_n \Rightarrow M_n \longrightarrow \infty$

$$\Rightarrow M = \infty. \quad \#$$

So, there is a $R > 0$ s.t. $|z_n| \leq R$.

So, there is $(z_{n_k})_{k=1}^{\infty}$ s.t. $z_{n_k} \longrightarrow z_0$ for some $z_0 \in \mathbb{C}$.

Continuity $\Rightarrow$ $M = |p(z_0)| > 0$. $\square$

Proof of the FTA.

COROLLARY 4. If p is a polynomial of degree $n \geq 1$, then there are complex numbers z_1, z_2, ..., z_n and a compact constant c such that

$$p(z) = c \prod_{k=1}^{n} (z - z_k).$$

Proof.

Example 5. Find the zeros of $p(z) = z^n - 1$, $n \geq 1$.

Rational Functions

A **rational function** is a quotient of two polynomials. From the FTA, we can write

$$r(z) = \frac{p(z)}{\prod_{j=1}^{N}(z - z_j)^{n_j}}$$

for some $N, n_j \in \mathbb{C}$ and $z_1, z_2, \ldots, z_N \in \mathbb{C}$.

Corollary 6. Let p be a polynomial. Then there is a polynomial $q(z)$ and complex constants $c_{k,j}$ such that

$$\frac{p(z)}{\prod_{j=1}^{N}(z - z_j)^{n_j}} = q(z) + \sum_{j=1}^{N}\sum_{k=1}^{n_j} \frac{c_{k,j}}{(z - z_j)^k}.$$

<u>A simple case:</u>