Math 311

Chapter 1

Section 1.2: Gaussian Elimination

Contents

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Matrix and Augmented Matrix

The system

$$\begin{aligned} x + y + z &= 1 \\ 2x + 2y + z &= 3 \end{aligned}$$

can be put in **matrix form** (array of numbers):

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 3 \end{array}\right]$$

Definition 1.

- The matrix $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix}$ is called the **coefficient matrix** of the system.
- The matrix $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is called the **constant matrix** of the system.

Solving a system of linear equations requires to transform the coefficient matrix in the following form:

$$\begin{bmatrix} 1 & * & * \\ 0 & 1 & * \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & * & * \\ 0 & 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \end{bmatrix}$$

where $*$ denotes any real number.

Definition 2. A matrix is said to be in **row-echelon form** (REF for short) if it satisfies the following three conditions:

1. All **zero rows** are at the bottom.
2. The first nonzero entry from the left in each nonzero row is 1, called the **leading 1** for that row.
3. Each leading 1 is to the right of all leading 1s in the rows above it

Definition 3. A row-echelon matrix is said to be in **reduced row-echelon form** (RREF for short) if, in addition, it satisfies the following condition:

4. Each leading 1 is the only nonzero entry in its column.

Example 1. Circle the matrices in REF. Draw a square around the matrices in RREF.

$$\begin{bmatrix} 1 & * & * \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 0 & 3 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Theorem 1. Every matrix can be transformed, with row operations, to a REF (or RREF).

Algorithm 4. **Gaussian Elinination.** To solve a system of linear equations proceed as follows.

1. Carry the augmented matrix to a reduced row-echelon form using elementary row operations.
2. If a row $\begin{bmatrix} 0 & 0 & \dots & 0 & 1 \end{bmatrix}$ occurs, the system is inconsistent.
3. Otherwise, find the parametric form of the solution set to the system of equations.

Example 2. Solve the following system:

$$\begin{aligned} 3x + y - 4z &= -1 \\ x + 10z &= 5 \\ 4x + y + 6z &= 1 \end{aligned}$$

Solution.

Example 3. Solve the following equation:

$$\begin{aligned} x_1 - 2x_2 - x_3 + 3x_4 &= 1 \\ 2x_1 - 4x_2 + x_3 \phantom{{}+3x_4} &= 5 \\ x_1 - 2x_2 + 2x_3 - 3x_4 &= 4 \end{aligned}$$

Solution.

Note: The variable corresponding to the leading ones are called **leading variables**.

It can be proved that the number r of leading 1s must be the same in each of the row-echelon matrices.

DEFINITION 5. The **rank** of a matrix A is the number of leading 1s in any REF to which A can be carried by row operations.

EXAMPLE 4. Compute the rank of

$$A = \begin{bmatrix} 1 & 1 & -1 & 4 \\ 2 & 1 & 3 & 0 \\ 0 & 1 & -5 & 8 \end{bmatrix}$$

SOLUTION.

Theorem 2. Suppose a system of m equations in n variables is consistent, and that the rank of the coefficient matrix is r.

1. The set of solutions involves $n - r$ parameters.
2. If $r < n$, the system has infinitely many solutions.
3. If $r = n$, the system has a unique solution.

Three situations occur:

- No solution.
- Unique solution.
- Infinitely many solutions.

Example 5. A system of equation with $m = 4$ linear equations and $n = 5$ variables has beed carried to the following REF by row operations:

$$\left[\begin{array}{ccccc|c} 1 & 2 & 1 & 3 & 1 & 1 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(a) Find the rank of the coefficient matrix. (b) Is there no solution, unique solution, or infinitely many solutions?

Solution.