Math 644

Chapter 5

Section 5.4: Laurent Series

Contents

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Laurent Series

Theorem 1. Suppose f is analytic on $A = \{z : r < |z-a| < R\}$. Then there is a unique sequence $(a_n) \subset \mathbb{C}$ so that

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n,$$

where the series converges uniformly and absolutely on compact subsets of A. Moreover,

$$a_n = \frac{1}{2\pi i} \int_{C_s} \frac{f(\zeta)}{(\zeta - a)^{n+1}}\, d\zeta,$$

where C_s is the circle centered at a with radius s, $r < s < R$, oriented counter-clockwise.

Proof.

Types of Singularities

Definition 2. A function f has an **isolated singularity** at b if f is analytic in $\{z : 0 < |z-b| < \varepsilon\}$ for some $\varepsilon > 0$ and $f(b)$ is not defined.

Let $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-b)^n$.

① **Removable singularity.**

② **Zero of order $\mathbf{n_0}$.**

③ **Pole of order $\mathbf{n_0}$.**

④ **Essential singularity.**

Definition 3. A zero or pole is called **simple** if the order is 1.

Example 4. Find the singularities of the following functions. If it is a zero or a pole, give the order.

(a) $f(z) = e^{-1/z}$.

(b) $f(z) = \frac{e^z}{z^2}$.

Definition 5. If f is analytic in $\{z : |z| > R\}$, then $f(1/z)$ has an isolated singularity at 0 and we say that f has an **isolated singularity at** ∞.

Notes:

① The type of singularities at ∞ are based on the Laurent expansion of $f(1/z)$ at 0.

② Given the Laurent expansion of $f(1/z) = \sum_{n=-\infty}^{\infty} b_n z^n$ around $z = 0$, the Laurent expansion of $f(z)$ at ∞ is given by

$$\sum_{n=-\infty}^{\infty} a_n z^n$$

with $a_n = b_{-n}$, $n \in \mathbb{Z}$.

③ An essential singularity at ∞ is therefore characterized by $a_n \neq 0$ for infinitely many positive integers n.

Definition 6. If f is analytic in a region Ω except for isolated poles in Ω then we say that f is **meromorphic in Ω**. A meromorphic function in $\mathbb{C}$ is sometimes just called meromorphic.

Facts:

① If f is meromorphic in Ω and not identically 0, then $1/f$ is meromorphic in Ω.

② A complex number $b \in \Omega$ is a zero of order k of a meromorphic function $f \not\equiv 0$ in Ω if and only if $b \in \Omega$ is a pole of order k of the meromorphic function $1/f$.

③ If f and g are two meromorphic function in Ω with $g \not\equiv 0$ and if b is a zero of order k and a zero of order m for f and g respectively, then the order of the zero/pole of f/g is $|k - m|$.

Theorem 7. If f is analytic in $U = \{z : 0 < |z - b| < \delta\}$ for some $b \in \mathbb{C}$ and $\delta > 0$, then if b is an essential singularity for f, then $f(U)$ is dense in $\mathbb{C}$.

Proof.