# M444 – Complex Analysis

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Chapter 4

## Section 4.5: Zeros and Singularities

## Definition 4.5.1

Let $\Omega$ be a region, $f$ be analytic on $\Omega$, and $z_0 \in \Omega$.

① $z_0$ is a **zero of order** $m$ of $f$ if $f(z_0) = 0$ and if there is an analytic function $g$ in a neighborhood $B_r(z_0)$ of $z_0$ such that $g(z_0) \neq 0$ and

$$f(z) = (z - z_0)^m g(z) \quad (z \in B_r(z_0)).$$

② $z_0$ is a **simple zero** of $f$ if it is a zero of order 1 ($m = 1$).

③ $z_0$ is an **isolated zero** if there is a neighborhood $B_r(z_0)$ such that $f(z) \neq 0$ for any $B_r'(z_0)$.

**Example.** Consider $f(z) = z^2 - 2z + 1$.
The function $f$ is analytic on $\Omega = \mathbb{C}$ and $f(1) = 0$.
We see that

$$f(z) = (z-1)^2 = (z-1)^2 g(z)$$

with $g(z) = 1$ is such that $g$ is analytic in $\mathbb{C}$ and $g(1) \neq 0$. Therefore, $z_0 = 1$ is a zero of order 2.
Notice that $f(z) \neq 0$ for any $z \neq 1$. So $z_0 = 1$ is an isolated zero.

**Example.** Consider the function $f(z) = z^3(e^z - 1)$.

The function $f$ is analytic on $\Omega = \mathbb{C}$ and $f(0) = 0$.

To find the order of the zero, we write

$$z^3(e^z - 1) = z^3\Big(\sum_{n=0} \frac{z^n}{n!} - 1\Big) = z^3 \sum_{n=1}^{\infty} \frac{z^n}{n!} = \sum_{m=0}^{\infty} \frac{z^{m+4}}{(m+1)!}.$$

Then

$$z^3(e^z - 1) = z^4 \sum_{m=0}^{\infty} \frac{z^m}{(m+1)!} = z^4 g(z)$$

where $g(z) = \sum_{m=0}^{\infty} \frac{z^m}{(m+1)!}$.

Notice here that, for $|z| \leq R$,

- we have $|z^m|/(m+1)! \leq R^m/(m+1)!$.
- The series $\sum_{m=0}^{\infty} c_m = \sum_{m=0}^{\infty} R^m/(m+1)!$ is convergent from the ratio test:
$$\lim_{m\to\infty} \frac{|c_m|}{|c_{m+1}|} = \frac{1}{R} \lim_{m\to\infty} \frac{1}{m+2} = 0 < 1.$$
- Every function $z^m/(m+1)!$ is analytic on $B_R(0)$.

Therefore $g(z)$ is analytic in any disk $B_R(0)$ and

$$g(0) = 1 + 0 + 0 + \cdots = 1 \neq 0.$$

Hence, the zero $z_0 = 0$ is a zero of order $m = 4$.

Also, we can get that $f(z) \neq 0$ in any neighborhood $B_r(0)$. This means $z_0 = 0$ is an isolated zero.

### Theorem 4.5.2

Let $f$ be an analytic function on a region $\Omega$. Let $z_0 \in \Omega$ such that $f(z_0) = 0$. Then exactly one of the following two assertions holds:

(i) $f$ is identically zero in a neighborhood of $z_0$.

(ii) $z_0$ is an isolated zero of $f$.

**Proof.** Let $B_R(z_0) \subset \Omega$ be an open disk. Then, since $f$ is analytic on $\Omega$, it is also analytic on $B_R(z_0)$. We can therefore write

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad |z - z_0| < R,$$

where $a_n = \frac{f^{(n)}(z_0)}{n!}$.

(1) Assume $a_n = 0$ for any $n$. Then $f(z) = 0$ for any $z \in B_R(z_0)$ and the case (i) is true.

② Assume that case (i) is false, and let $a_n \neq 0$ for some $n \geq 0$.

Let $m$ be the least index such that $a_m \neq 0$. This means $a_j = 0$ for $0 \leq j \leq m-1$, but $a_m \neq 0$. Therefore, for $|z - z_0| < R$, we have

$$\begin{aligned} f(z) &= a_m(z-z_0)^m + a_{m+1}(z-z_0)^{m+1} + \cdots \\ &= (z-z_0)^m\big(a_m + a_{m+1}(z-z_0)^{m+1} + \cdots\big) \\ &= (z-z_0)^m g(z) \end{aligned}$$

where $g(z) = a_m + a_{m+1}(z-z_0)^{m+1} + \cdots$ is an analytic function in $B_R(z_0)$ with $g(z_0) = a_m \neq 0$.

Because $|g(z)|$ is a continuous function, we can find a neighborhood $B_r(z_0)$ with $r \leq R$ such that $g(z) \neq 0$ on $B_r(z_0)$. Hence

$$f(z) = (z-z_0)^m g(z) \neq 0$$

for any $z \in B_r'(z_0)$. Hence $z_0$ is an isolated zero, which is case (ii). □

**Consequence.** If $f$ is analytic on a region $\Omega$ and $z_0 \in \Omega$ with $f(z_0) = 0$ is an isolated zero, then there exists

① an integer $m \geq 1$

② a real number $r > 0$

③ an an analytic function $\lambda$ on $B_r(z_0)$ with $\lambda(z) \neq 0$ for any $z \in B_r(z_0)$

such that

$$f(z) = (z - z_0)^m \lambda(z) \quad \forall z \in B_r(z_0).$$

Moreover, it this case, the zero $z_0$ is of order $m$ and

$$f(z_0) = f'(z_0) = \cdots = f^{(m-1)}(z_0) = 0,$$

but $f^{(m)}(z_0) \neq 0$.

**Other consequence.** A nonzero analytic function $f$ on a region $\Omega$ has isolated zeros.

## Theorem 4.5.5 (Identity Principle)

Suppose that

1. $f$ and $g$ are two analytic functions on a region $\Omega$.
2. there is a sequence $(z_n)$ of distinct points of $\Omega$ such that $f(z_n) = g(z_n)$ for all $n$.
3. there is a $z_0 \in \Omega$ such that $z_n \to z_0$.

Then $f(z) = g(z)$ for all $z \in \Omega$!

**Proof.** Notice that $z_n$ is a zero of $h = f - g$. Using continuity, we have

$$h(z_0) = h\left(\lim_{n\to\infty} z_n\right) = \lim_{n\to\infty} h(z_n) = 0.$$

If $h$ is nonzero, then $z_0$ should be an isolated zero. Hence, there is $r > 0$ such that $h(z) \neq 0$ for any $z \in B_r(z_0)$.

However, $z_n \to z_0$ and $h(z_n) = 0$ for all $n$. Therefore, there is some $N$ such that $|z_n - z_0| < r$ $(n \geq N)$ and $h(z_n) = 0$. A contradiction.

Hence, $h(z) = 0$, $\forall z \in \Omega$, showing that $f(z) = g(z)$ $\forall z \in \Omega$. $\square$

**Example.** Let $f(z) = \frac{z^2-1}{z-1}$, for $z \neq 1$.

Then notice that $f$ is analytic in any deleted neighborhood $B_r'(1)$, $r > 0$ but is undefined at $z = 1$. We call $z = 1$ an **isolated singularity** of $f$.

Notice also that

$$\lim_{z\to 1} f(z) = \lim_{z\to 1} \frac{z^2-1}{z-1} = \lim_{z\to 1} \frac{(z+1)(z-1)}{z-1} = 2.$$

Then define $f(1) := 2$. We can now show that $f'(z)$ exists in $B_r(1)$, for $r > 0$. Indeed, $f$ is analytic on $B_r'(1)$ already. Now, at $z = 2$, we have

$$\lim_{z\to 2} \frac{f(z)-2}{z-1} = \lim_{z\to 2} \frac{z^2-z-z+1}{(z-1)(z-1)} = \lim_{z\to 2} \frac{(z-1)^2}{(z-1)^2} = 1.$$

Therefore $z_0 = 1$ is called a **removable singularity**.

### Definition 4.5.8 (Removable Singularity)

An isolated singularity $z_0$ of an analytic function $z_0$ is called **removable** if $f$ can be redefine at $z_0$ so that it is analytic on $B_r(z_0)$.

## Theorem 4.5.12

Assume that $f$ is analytic on $0 < |z - z_0| < R$. The following are equivalent:

1. $f$ has a removable singularity at $z_0$.
2. $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ for $0 < |z - z_0| < R$.
3. $\lim_{z \to z_0} f(z)$ exists.
4. $\lim_{z \to z_0} |f(z)|$ exists and is <u>finite</u>.
5. $f$ is bounded in a neighborhood of $z_0$.
6. $\lim_{z \to z_0} (z - z_0) f(z) = 0$.

**Note.** If $z_0$ is a removable singularity, then we get

$$f(z_0) = \lim_{z \to z_0} f(z) = a_0.$$

**Example.** Consider $f(z) = \frac{\cos z}{z}$, for $z \neq 0$.

Recall that for a singularity to be removable, we need to verify that

$$\lim_{z \to 0} |f(z)|$$

exists and is finite.

We have

$$\lim_{z \to 0} \frac{1}{|f(z)|} = \lim_{z \to 0} \frac{|z|}{|\cos z|} = \frac{0}{1} = 0$$

and hence

$$\lim_{z \to 0} |f(z)| = \infty.$$

The singularity $z = 0$ is not removable and we will call it a pole.

Definition 4.5.8 (Poles)

A isolated singularity $z_0$ of an analytic function is called a **pole** if

$$\lim_{z \to z_0} |f(z)| = \infty.$$

Expanding $\cos z$ in its Taylor series around $z_0 = 0$, we get

$$f(z) = \frac{1}{z} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} = \frac{1}{z} - \frac{z}{2} + \frac{z^3}{24} - \frac{z^5}{720} + \cdots .$$

We notice that $a_{-1} = 1$ and $a_{-n} = 0$ for any $n \geq 2$.

The highest index $m$ such that $a_{-m} \neq 0$ and $a_{-n} = 0$ for any $n \geq m$ is called the **order of the pole**.

Equivalently, we can define the order of a pole $z_0$ of a function $f$ as the order of the zero $z_0$ of the function $g(z) = \frac{1}{f(z)}$ for $z \neq z_0$ and $g(z_0) = 0$.

## Theorem 4.5.15

Let $m \geq 1$ be an integer and $R > 0$. Assume that $f$ is analytic on $A_{0,R}(z_0)$. Then the following are equivalent.

1. $f$ has a pole of order $m$ at $z_0$.
2. There is an $r > 0$ and a non-vanishing analytic function $\phi$ on $B_r(z)$ such that
$$f(z) = \frac{\phi(z)}{(z - z_0)^m}, \quad 0 < |z - z_0| < \min\{r, R\}.$$
3. There exists a complex number $\alpha \neq 0$ such that
$$\lim_{z \to z_0} (z - z_0)^m f(z) = \alpha.$$
4. The Laurent series expansion of $f$ has the form
$$f(z) = \frac{a_{-m}}{(z - z_0)^m} + \frac{a_{-m+1}}{(z - z_0)^{m-1}} + \cdots + \frac{a_{-1}}{z - z_0} + a_0 + \cdots .$$

**Example.** Consider $f(z) = e^{1/z}$, for $z \neq 0$.

The point $z = 0$ is a pole or a removable singularity if either

- $\lim_{z\to 0} |f(z)|$ exists and is finite.
- $\lim_{z\to 0} |f(z)| = \infty$.

However, if $z = iy$ with $y \to 0$, then

$$\lim_{z\to 0} |f(z)| = \lim_{y\to 0} |e^{-i/y}| = 1;$$

and if $z = x$ with $x \to 0^+$, then

$$\lim_{z\to 0} |f(z)| = \lim_{x\to 0^+} e^{1/x} = \infty.$$

So $\lim_{z\to 0} |f(z)|$ does not exist!

Definition 4.5.8 (Essential Singularities)

An isolated singularity $z_0$ of an analytic function is called an **essential singularity** if

$$\lim_{z\to z_0} |f(z)| \text{ does not exist.}$$

Notice that

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{(1/z)^n}{n!} = 1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \cdots .$$

We have $a_{-m} \neq 0$ for infinitely many integer $m > 0$.

**Theorem 4.5.17**

Suppose that $f$ is analytic in a region $\Omega \backslash \{z_0\}$. Let

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{a_{-m}}{(z - z_0)^m}$$

be the Laurent expansion of $f$ in some $A_{0,R}(z_0)$.
Then, $z_0$ is an essential singularity if and only if $a_{-n} \neq 0$ for infinitely many $n > 0$.