Worksheet: Chapter 5

Math 307 — Linear Algebra and Differential equations — Spring 2022 section 3

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1. Consider the following matrices:

$$A = \begin{bmatrix} 0 & -1/2 \\ 2 & 0 \end{bmatrix}, \qquad B = \frac{1}{2}\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} -1 & 0 & 1 \\ -3 & 2 & 1 \\ -3 & 0 & 3 \end{bmatrix}$$

For each of these matrice, do the following.

(a) Find its eigenvalues.

(b) Find its eigenvectors.

(c) Determine if the matrix is diagonalizable, and if so, find a matrix P that would diagonalize the matrix.

(d) Write a Jordan canonical form of the matrix.

Answers:

$$A: \quad \lambda = \pm i, \quad v = [\pm i, 2]^T, \quad \text{diagonalizable}, \quad P = \begin{bmatrix} i & -i \\ 2 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

$$B: \quad \lambda = 1, \quad v = [1, 1]^T, \quad \text{non-diagonalizable}, \quad D = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$C: \quad \begin{array}{l} \lambda_1 = 0, \\ \lambda_2 = 2, \\ \end{array} \quad \begin{array}{l} v_1 = [1,1,1]^T, \\ v_2 = [1,0,3]^T, \\ v_3 = [0,1,0]^T, \end{array} \quad \text{diagonalizable}, \quad P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 3 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

2. Let us consider the following matrices:

$$A = \frac{1}{2}\begin{bmatrix} 2 & 1 & -1 \\ -2 & 5 & -1 \\ -2 & 1 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \qquad C = \frac{1}{2}\begin{bmatrix} 0 & 3 & -1 \\ -4 & 7 & -1 \\ 0 & -1 & 3 \end{bmatrix}.$$

(a) Are A and B similar matrices? Justify.

(b) Are B and C similar matrices? Justify.

(c) Are A and C similar matrices? Justify.

Answers:
(a) Yes, A has the same eigenvalues as B and A is diagonalizable.
(b) No, C is not diagonalizable, so it is impossible to get to B.
(c) No, similarity is transitive, so A cannot be similar to C.

3. Prove that a square matrix is invertible if and only if none of its eigenvalues are zero.
Partial answer: Write the matrix in its Jordan canonical form and check the determinant.

4. Consider the following linear transformations:

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -x-2y \\ 2x+3y \end{bmatrix}$$

$$g\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \qquad g\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 2 \end{bmatrix}.$$

In addition, consider that α is the standard basis of $\mathbb{R}^2$ and β is the basis

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

(a) Find the matrix A such that $f([x,\ y]^T) = A[x,\ y]^T$ in the basis α.
(b) Find $[f]_\beta^\beta$.
(c) Find the matrix B such that $g([x,\ y]^T) = B[x,\ y]^T$ in the basis α.
(d) Find $[g]_\beta^\beta$.
(e) Find a matrix C such that $h([x,\ y]^T) = C[x,\ y]^T$ in the basis α, where $h(v) = (f \circ g)(v) = f(g(v))$.
(f) Find the eigenvalues and eigenvectors of h.
(g) Find the dimension of the kernel of h.
(h) Find the dimension of the range of h.

Answers:

$$\text{(a)}\quad A = \begin{bmatrix} -1 & -2 \\ 2 & 3 \end{bmatrix}, \qquad \text{(b)}\quad [f]_\beta^\beta = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix},$$

$$\text{(c)}\quad B = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}, \qquad \text{(d)}\quad [g]_\beta^\beta = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix},$$

$$\text{(e)}\quad C = \begin{bmatrix} -4 & -2 \\ 6 & 4 \end{bmatrix}, \qquad (f)\quad \begin{array}{ll} \lambda_1 = -2, & v_1 = [-1, 1]^T, \\ \lambda_2 = 2, & v_2 = [-1, 3]^T, \end{array}$$

$$(g)\quad \dim(\ker(C)) = 0, \qquad (h)\quad \dim(\text{range}(C)) = 2.$$

5. Determine if the following transformations are linear ones.

(a) $f : \mathbb{R}^3 \to \mathbb{R}^2$, where $f([x,\ y,\ z]^T) = [x+y+z,\ x-z]^T$.
(b) $f : \mathbb{R}^3 \to \mathbb{R}^2$, where $f([x,\ y,\ z]^T) = [x+1,\ y-z]^T$.
(c) $f : \mathbb{R}^3 \to \mathbb{R}^2$, where $f([x,\ y,\ z]^T) = [xyz,\ x-z]^T$.
(d) $f : P^2 \to P^1$, where $f(ax^2+bx+c) = \frac{d}{dx}(ax^2+bx+c)$.

Answer:
(a) Linear, (b) Not linear, (c) Not linear, (d) Linear