Math 644

Chapter 1

Section 1.3: Stereographic Projection

Contents

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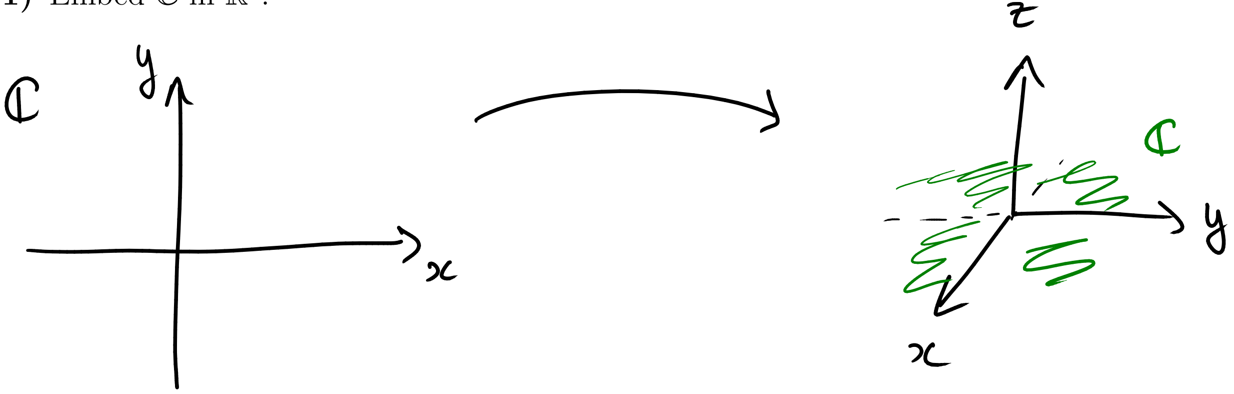
We would like to treat ∞ as any other complex numbers. To do that, we will construct a model using the stereographic projection.

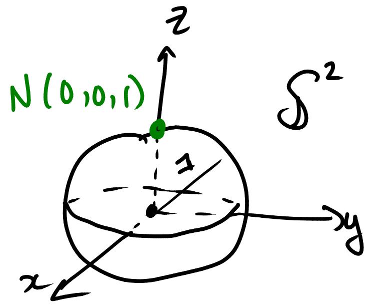
Method

1) Embed $\mathbb{C}$ in $\mathbb{R}^3$.



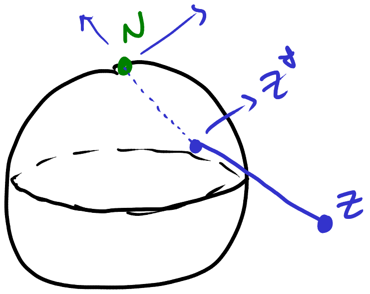
$$x+iy \longmapsto (x,y,0)$$

2) Draw a sphere $\mathbb{S}^2$ with the following characteristics:



- $\mathbb{S}^2 := \{(x,y,z) \in \mathbb{R}^3 \,:\, x^2+y^2+z^2 = 1\}$;
- Denote by $N := (0,0,1)$ the north pole.

3) The stereographic projection:



$$z^* = (x_1, x_2, x_3) \in \mathbb{S}^2$$

$$z = x+iy \in \mathbb{C}.$$

Line: $L(t) = (0,0,1) + [(x,y,0) - (0,0,1)]\,\underset{\in \mathbb{R}}{t}, \quad t \neq 0.$

We know $z^* \in L$ & $z^* \in \mathbb{S}^2$

there is some $t \in \mathbb{R}\setminus\{0\}$ s.t. $z^* = L(t)$.

Point of intersection:

$$t = \frac{2}{x^2+y^2+1} \Longrightarrow \boxed{z^* = \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1} \right)}$$

$\uparrow \pi(z)$

4) Inverse of the stereographic projection:

Given $z^* = (x_1, x_2, x_3)$, find $z \approx (x, y, 0)$

Equation of the line.

$$L(t) = (0,0,1) + [(x_1, x_2, x_3) - (0,0,1)]\, t \quad (t \neq 0)$$

third coord. should be zero:

$$t = 1 - x_3$$

$$x = \frac{x_1}{1 - x_3} \qquad (t \neq 0 \ \& \ x_3 \neq 1)$$

$$y = \frac{x_2}{1 - x_3}$$

So, $z = \pi^{-1}(z^*) = \dfrac{x_1 + x_2 i}{1 - x_3}$.

<u>Conclusion</u>: $\pi : \mathbb{C} \longrightarrow S^2 \setminus \{(0,0,1)\}$

π is a bijection.

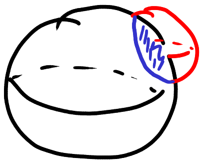
Extend π: $\pi(\infty) := (0,0,1)$

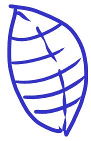
Definition 1. The **extended complex plane** is the set $\mathbb{C}^* := \mathbb{C} \cup \{\infty\}$, where

$$\infty := \pi^{-1}(0,0,1).$$

The Riemann sphere $\mathbb{S}^2$ inherits a topology from the usual topology of $\mathbb{R}^3$ generated by the balls in $\mathbb{R}^3$. In more details:

- A basis for the topology are of the form $B \cap \mathbb{S}^2$, where B is a ball in $\mathbb{R}^3$.

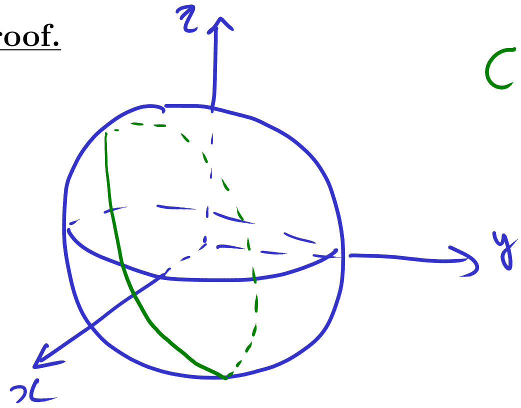




Before describing the topology of $\mathbb{C}^*$, we first show the following.

Theorem 2. Circles in $\mathbb{C}$ correspond precisely to circles on $\mathbb{S}^2 \backslash \{(0,0,1)\}$.

Proof.



C: circle on $S^2 \backslash \{(0,0,1)\}$.

Fact: $C = S^2 \cap P$

for some plane P:

$AX + B \cdot Y + CZ = D$.

Let $z^* = (x_1, x_2, x_3) \in S^2$. Then

$$z^* \in C \iff Ax_1 + Bx_2 + Cx_3 = D$$

Use $\pi(z) = z^*$ to rewrite as

$$A\left(\frac{2x}{x^2+y^2+1}\right) + B\left(\frac{2y}{x^2+y^2+1}\right) + C\left(\frac{x^2+y^2-1}{x^2+y^2+1}\right) = D$$

$$\iff 2Ax + 2By + (C-D)(x^2+y^2) = D+C$$

If $D = C$, then

$$2Ax + 2By = 2C$$

& using π^{-1}, then

$$2A\frac{x_1}{1-x_3} + 2B\frac{x_2}{1-x_3} = 2C$$

$$\Rightarrow \quad Ax_1 + Bx_2 + Cx_3 = C = D$$

$(0,0,1)$ lies on the plane #.

So, $C \neq D$. &

$$\left(x - \frac{A}{C-D}\right)^2 + \left(y - \frac{B}{C-D}\right)^2 = \frac{D + C + A^2 + B^2}{C-D} \quad (*)$$

We have an eq. of a circle in $\mathbb{C}$.

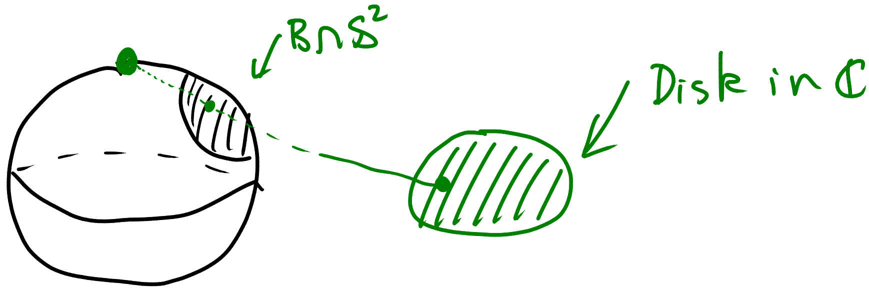
For the other way around, any circle in $\mathbb{C}$ can be put in the form $(*)$. Go backward with π & π^{-1}.

Corollary 3.

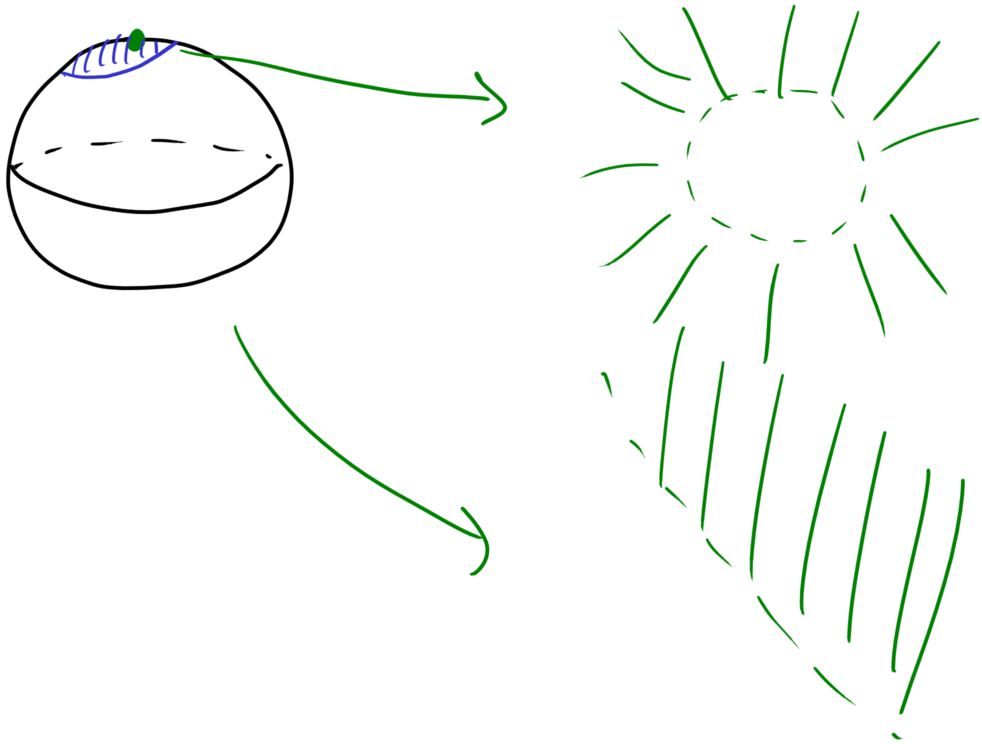
a) Topology of $\mathbb{S}^2$ induces the standard topology on $\mathbb{C}$ under the stereographic projection.

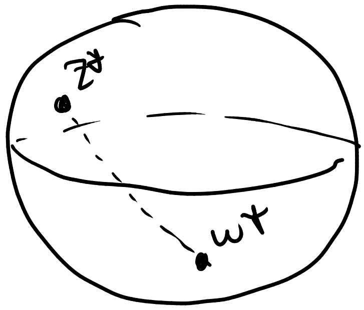
b) Moreover, a basis of neighborhoods for ∞ are of the form $\{z \in \mathbb{C} : |z| > r\} \cup \{\infty\}$, with $r > 0$.

a)



b)





$z^*, w^* \in S^2$

From π, $\exists z, w \in \mathbb{C}^*$

$\pi(z) = z^*$ & $\pi(w) = w^*$

Def. The chordal metric $\chi : \mathbb{C}^* \times \mathbb{C}^* \to [0,\infty)$

is

$$\chi(z,w) := \| z^* - w^* \|_{\mathbb{R}^3}$$

Prop. 1) $(\mathbb{C}^*, \chi)$ is a complete metric space.

2)

$$\chi(z,w) = \begin{cases} \dfrac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}, & z, w \neq \infty \\ \dfrac{2}{\sqrt{1+|z|^2}}, & w = \infty, z \neq \infty \\ \dfrac{2}{\sqrt{1+|w|^2}}, & w \neq \infty, z = \infty. \end{cases}$$

3) $0 \leq \chi(z,w) \leq 2 \quad (\forall z, w \in \mathbb{C}^*)$.