15.9 Change of variables in Multiple integrals.

Change of variable from Calculus I

Put $x = g(u)$, then

$$\int_a^b f(x)\,dx = \int_c^d f(g(u))\,\underbrace{g'(u)\,du}_{\frac{dx}{du}}$$

where $a = g(c)$, $b = g(d)$

$u = g(x)$
$du = g'(x)\,dx$

$x = g(u)$
$dx = g'(u)\,du$

Change of Variable in polar coordinate.

If $x = r\cos\theta$ & $y = r\sin\theta$, then

$$\iint_R f(x,y)\,dA = \iint_S f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

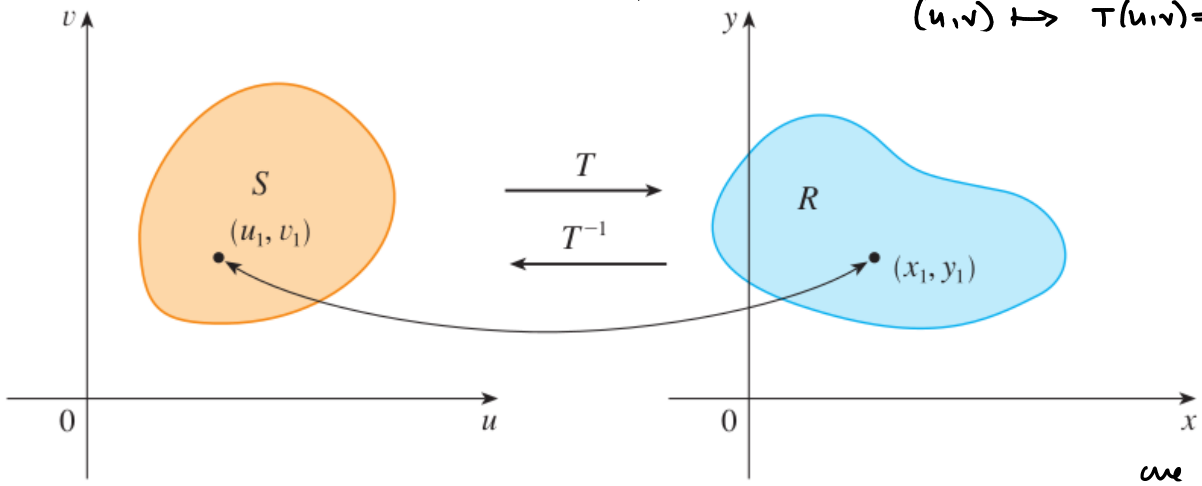
Jacobian of the Polar Transformation.

where R region in xy-plane & S is region in $r\theta$-plane.

Polar coordinates: $T(r,\theta) = (x,y) = (\underbrace{r\cos\theta}_{g}, \underbrace{r\sin\theta}_{h})$.

Transformation in 2D.

Consider transformation $T: S \to R$
$(u,v) \mapsto T(u,v) = (x,y)$



Two equations for x & y. $x = g(u,v)$ & $y = h(u,v)$
or $x = x(u,v)$ or $y = y(u,v)$

Suppose that g_u, g_v, h_u, h_v exists and are continuous. are C^1.

Image: if $(x_1,y_1) = T(u_1,v_1)$, then (x_1,y_1) is the image of (u_1,v_1).

One-to-one: no two points have the same image.

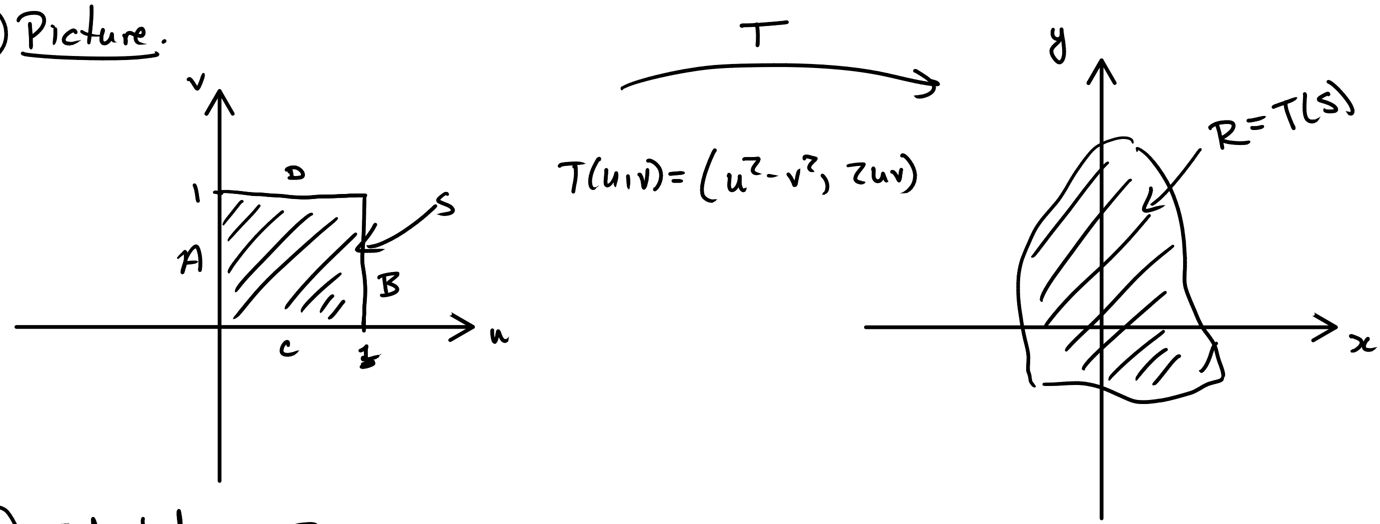
Inverse: T is one-to-one $\to$ T has an inverse $\to$ $T^{-1}: R \to S$
$u = G(x,y)$ & $v = H(x,y)$

EXAMPLE 1 A transformation is defined by the equations

$$x = u^2 - v^2 \qquad y = 2uv$$

Find the image of the square $S = \{(u, v) \mid 0 \leqslant u \leqslant 1,\ 0 \leqslant v \leqslant 1\}$.

① Picture.



② Identifying R.

$x = u^2 - v^2$ & $y = 2uv$.

A) $u = 0$ & $0 \le v \le 1$ $\to$ $x = -v^2$ & $y = 0$

$\to$ $-1 \le x \le 0$ & $y = 0$

B) $u = 1$ & $0 \le v \le 1$ $\to$ $x = 1 - v^2$ & $y = 2v$

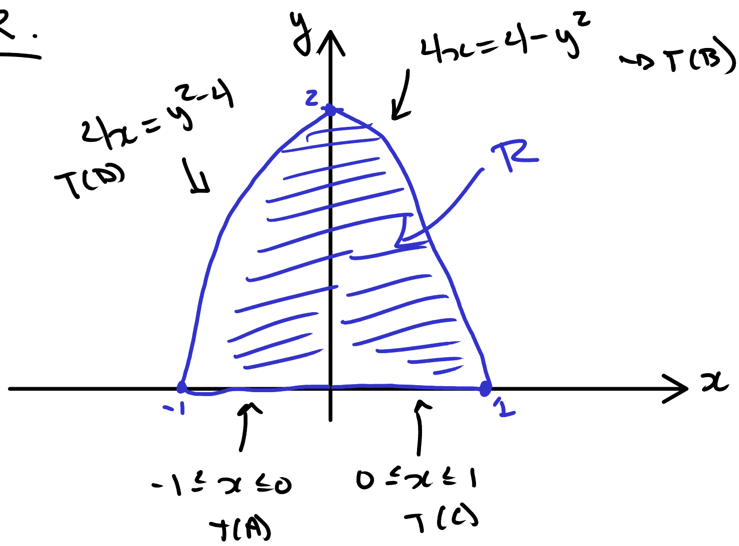
$\to$ $x = 1 - \frac{y^2}{4}$ & $y = 2v$

$\to$ $4x = 4 - y^2$ & $0 \le y \le 2$ $\to$ Parabola.

C) $0 \le x \le 1$ & $y = 0$

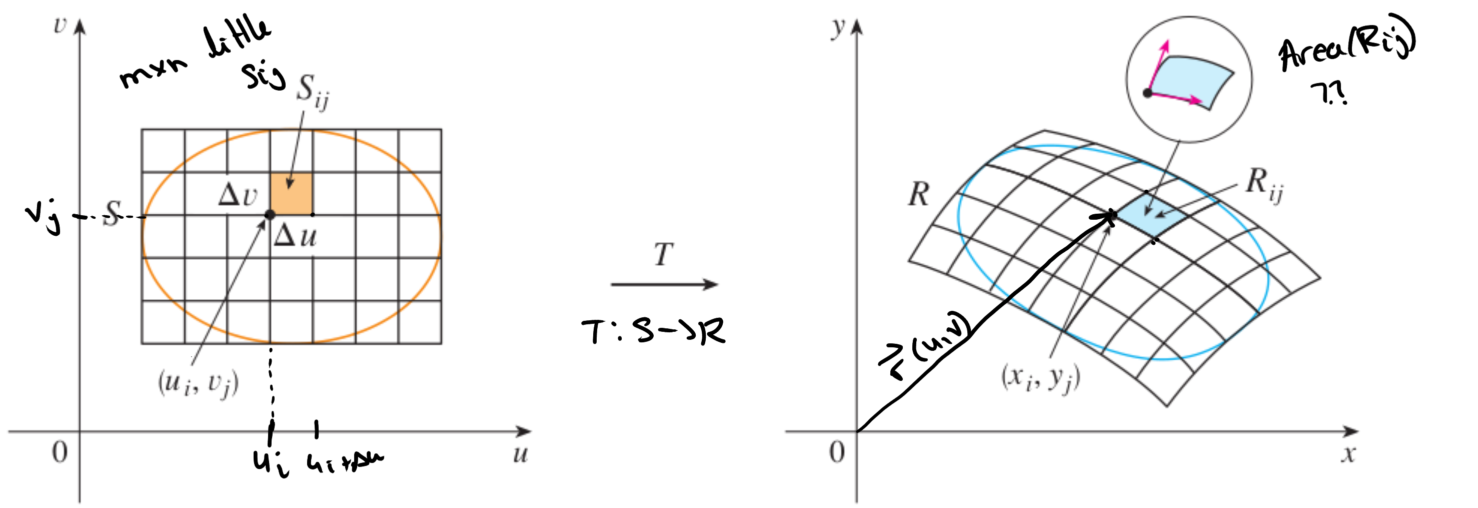
D) $2x = y^2 - 4$ & $0 \le y \le 2$ $\to$ Parabola

③ Picture of R.

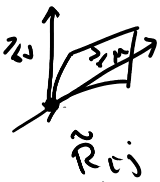


Effect of a change of variables on double integral.

S region in uv-plane $\qquad\qquad$ R region in xy-plane

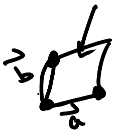


Position vector : $\vec{r}(u,v) = (x,y) = x(u,v)\,\vec{i} + y(u,v)\,\vec{j}$.



Tangent vectors. $\quad \vec{r}_u(u_i,v_j) = \lim\limits_{\Delta u \to 0} \dfrac{\vec{r}(u_i+\Delta u, v_j) - \vec{r}(u_i,v_j)}{\Delta u} = x_u\vec{i} + y_u\vec{j}$

$\vec{r}_v(u_i,v_j) = \lim\limits_{\Delta v \to 0} \dfrac{\vec{r}(u_i, v_j+\Delta v) - \vec{r}(u_i,v_j)}{\Delta v} = x_v\vec{i} + y_v\vec{j}$



Approximate the area by a parallelogram

$\vec{a} = \vec{r}(u_i+\Delta u, v_j) - \vec{r}(u_i,v_j) \approx \Delta u\, \vec{r}_u$ (Δu small)

$\vec{b} = \vec{r}(u_i, v_j+\Delta v) - \vec{r}(u_i,v_j) \approx \Delta v\, \vec{r}_v$ (Δv small)

$$\text{Area}(R_{ij}) \approx \text{Area}(\tilde{R}_{ij}) \approx \|\vec{r}_u \times \vec{r}_v\|\, \Delta u\, \Delta v$$

So,

$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i,y_j)\,\text{Area}(R_{ij})$$

$$\approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x(u_i,v_j), y(u_i,v_j))\,\|\vec{r}_u \times \vec{r}_v\|\,\Delta u\,\Delta v$$

Make $m \to \infty$ & $n \to \infty$

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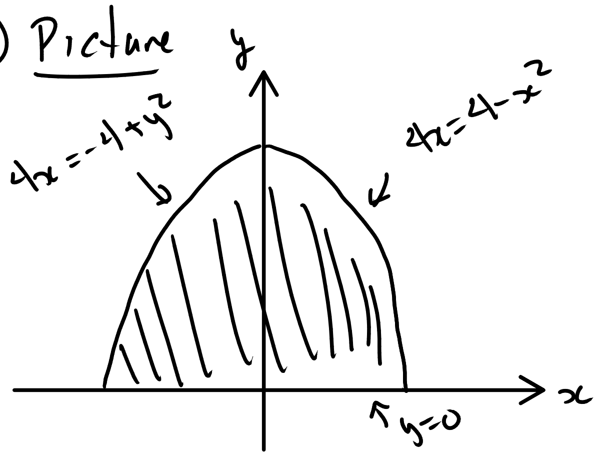
9 Change of Variables in a Double Integral Suppose that T is a C^1 transformation whose Jacobian is nonzero and that T maps a region S in the uv-plane onto a region R in the xy-plane. Suppose that f is continuous on R and that R and S are type I or type II plane regions. Suppose also that T is one-to-one, except perhaps on the boundary of S. Then

$$\boxed{\iint_R f(x,y)\,dA = \iint_S f(x(u,v), y(u,v)) \underbrace{\overset{\|\vec{r}_u \times \vec{r}_v\|}{\overset{\|}{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|}}}_{\text{Jacobian of } T}\, du\, dv}$$

Remark: $\quad \dfrac{\partial(x,y)}{\partial(u,v)} = x_u y_v - x_v y_u = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$

EXAMPLE 2 Use the change of variables $x = u^2 - v^2$, $y = 2uv$ to evaluate the integral $\iint_R y\,dA$, where R is the region bounded by the x-axis and the parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$, $y \geq 0$.

① Picture



$R = \{(x,y) : \frac{y^2}{4} - 1 \leq x \leq 1 - \frac{y^2}{4},\ 0 \leq y \leq 2\}$

From example 1,

$R = T(S)$ where

$S = \{(u,v) : 0 \leq u \leq 1,\ 0 \leq v \leq 1\}$

$T(u,v) = (u^2 - v^2,\ 2uv)$.

② Integrate

$$\iint_R y\,dA = \iint_S 2uv \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = (2u)(2u) - (-2v)(2v)$$
$$= 4(u^2 + v^2) > 0$$

So,

$$\iint_S 2uv \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \int_0^1\int_0^1 2uv\, 4(u^2+v^2)\, du\,dv$$
$$= 8\int_0^1\int_0^1 u^3v + uv^3\, du\,dv$$
$$= \boxed{2}$$

Remark.

If you the inverse transformation $T^{-1} : R \to S$ where
$(u,v) = (u(x,y), v(x,y))$

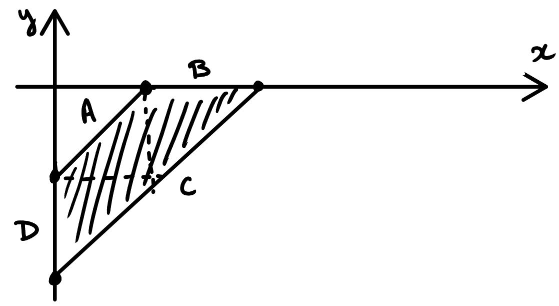
then

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}$$

(Jacobian of T is the inverse of the Jacobian of T^{-1})

EXAMPLE 3 Evaluate the integral $\iint_R e^{(x+y)/(x-y)}\,dA$, where R is the trapezoidal region with vertices $(1, 0)$, $(2, 0)$, $(0, -2)$, and $(0, -1)$.

① Picture



Boundary:

Ⓐ. $y = x - 1$, $0 \le x \le 1$.

Ⓑ. $1 \le x \le 2$ & $y = 0$

Ⓒ. $y = x - 2$, $0 \le x \le 2$.

Ⓓ. $x = 0$, $-2 \le y \le -1$.

② Find the transformation.

From the fct. in the integral, let (xy-plane $\to$ uv-plane)

$$u = x + y \quad \& \quad v = x - y.$$

Solve for x & y:

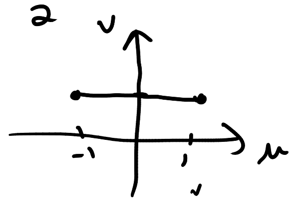
$$u + v = x + y + x - y = 2x \;\to\; x = \frac{u+v}{2}.$$

$$u - v = x + y - x + y = 2y \;\to\; y = \frac{u-v}{2}.$$

(uv-plane $\downarrow$ xy-plane)

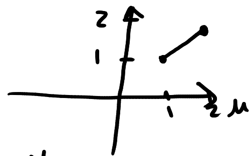
Ⓐ $y = x - 1$, $0 \le x \le 1$ $\to$ $u = 2x - 1$ & $v = 1$, $0 \le x \le 1$

$\to$ $-1 \le u \le 1$ & $v = 1$



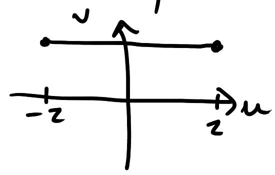
Ⓑ $y = 0$, $1 \le x \le 2$ $\to$ $u = x$ & $v = x$, $1 \le x \le 2$

$\to$ $u = v$, $1 \le u \le 2$



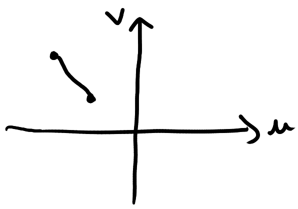
Ⓒ $y = x - 2$, $0 \le x \le 2$ $\to$ $u = 2x - 2$ & $v = 2$

$\to$ $-2 \le u \le 2$ & $v = 2$



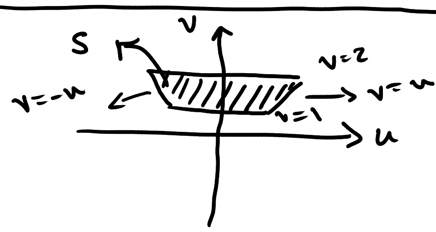
Ⓓ $x = 0$, $-2 \le y \le -1$ $\to$ $u = y$ & $v = -y$, $-2 \le y \le -1$

$\to$ $v = -u$ & $-2 \le u \le -1$



TYPE II.

$$S = \{(u,v) : -v \le u \le v,\; 1 \le v \le 2\}$$



③ Integrate.

$$\iint_R e^{x+y/x-y}\,dA = \iint_S e^{u/v}\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \int_1^2\int_{-v}^{v} e^{u/v}\,|-1/2|\,du\,dv$$

$$= \frac{3}{4}\left(e - e^{-1}\right).$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{1}{2}$$

Effect of change of variable in Triple integrals.

Spherical coordinates. The transformation is

$$T(\rho, \theta, \phi) = (x, y, z) = (\underbrace{\rho\sin\phi\cos\theta}_{x(\rho,\theta,\phi)}, \underbrace{\rho\sin\phi\sin\theta}_{y(\rho,\theta,\phi)}, \underbrace{\rho\cos\phi}_{z(\rho,\theta,\phi)}).$$

So, using T, we get

$$dV = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi .$$

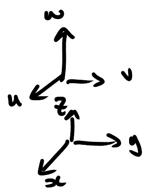
where T maps a region S in the $\rho\theta\phi$-space into a region R in the xyz-space.

Here,

$$\rho^2 \sin\phi = \left| \frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} \right|$$

Jacobian in 3D.

Transformation in 3D.



S region in the uvw-space.
R region in the xyz-space.

We consider a C^1-transformation

$$T: S \to R$$

which is one-to-one ($T^{-1}: R \to S$ exists).

Now, $(x, y, z) = T(u, v, w)$

$x = x(u,v,w)$.
$y = y(u,v,w)$.
$z = z(u,v,w)$.

Jacobian in 3D.

If $T(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w))$

then its Jacobian is

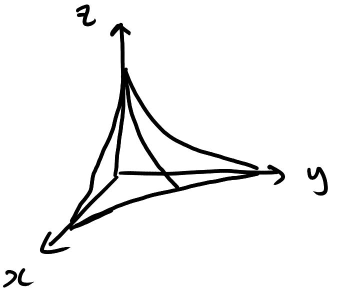
$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix} .$$

$$\iiint_R f(x, y, z)\, dV = \iiint_S f(x(u, v, w), y(u, v, w), z(u, v, w)) \underbrace{\left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right|}\, \underbrace{du\, dv\, dw}$$

Remark: If $T^{-1}: R \to S$, then

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{1}{\left(\frac{\partial(x,y,z)}{\partial(u,v,w)}\right)}$$

56. Use the transformation $x = u^2,\ y = v^2,\ z = w^2$ to find the volume of the region bounded by the surface $\underbrace{\sqrt{x} + \sqrt{y} + \sqrt{z} = 1}$ and the coordinate planes.



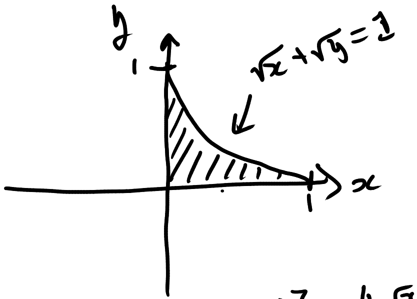
we can deduce that

$0 \le x \le 1$
$0 \le y \le 1$
$0 \le z \le 1$

$\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$
$\sqrt{z} = 1 - \sqrt{x} - \sqrt{y}$

① Picture.

Projection in the xy-plane $(z=0)$ $\to$ $\sqrt{x} + \sqrt{y} = 1$



$R = \{(x,y,z) : 0 \le x \le 1,\ 0 \le y \le (1-\sqrt{x})^2,$
$0 \le z \le (1-\sqrt{x}-\sqrt{y})^2\}$

$$V(R) = \int_0^1 \int_0^{(1-\sqrt{x})^2} \int_0^{(1-\sqrt{x}-\sqrt{y})^2} dz\,dy\,dx .$$

② Change of variable.

$T(u,v,w) = (x,y,z) = (u^2, v^2, w^2)$. $\to$ $x = u^2$, $y = v^2$, $z = w^2$ $\to$ $\sqrt{x} = |u| = u$, $\sqrt{y} = |v| = v$, $\sqrt{z} = |w| = w$

Plug-in the eq. of the surface: $\boxed{\text{where } u,v,w \ge 0}$ Add.

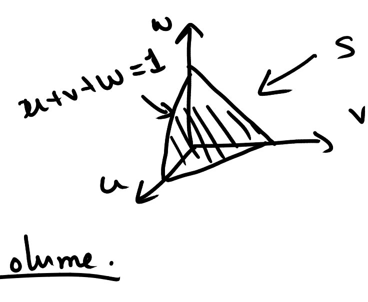
$\to$ $|u| + |v| + |w| = 1$ $\quad (-1 \le u \le 1,\ -1 \le v \le 1,\ -1 \le w \le 1)$.

$\to$ $\pm u \pm v \pm w = 1$ $\quad ($ " , " , " $)$.

ⓐ $u+v+w=1$ ⓑ $-u+v+w=1$ ⓒ $u-v+w=1$

ⓓ $-u-v+w=1$ ⓔ $u+v-w=1$ ⓕ $-u-v-w=1$

ⓖ $u-v-w=1$ ⓗ $-u+v-w=1$ 8 planes.



$S = \{(u,v,w) : 0 \le u \le 1,\ 0 \le v \le 1-u$
$0 \le w \le 1-u-v\}$

③ Volume.

$$V(R) = \iiint_R dV = \int_0^1 \int_0^{1-u} \int_0^{1-u-v} \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| dw\,dv\,du$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 2u & 0 & 0 \\ 0 & 2v & 0 \\ 0 & 0 & 2w \end{vmatrix} = 8uvw$$

So,

$$V(R) = \int_0^1 \int_0^{1-u} \int_0^{1-u-v} 8uvw \; dw\, dv\, du = \boxed{\frac{1}{90} \approx 0.0111}\ldots$$