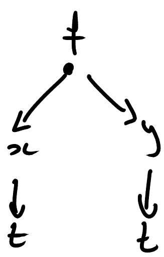
16.3 The Fundamental Theorem for Line Integrals.

Recall. (FTC) $\displaystyle\int_a^b F'(x)\,dx = F(b) - F(a)$

If f is a scalar function, then $\vec{\nabla} f$ is its gradient.



$$\int_a^b \vec{\nabla} f \cdot \vec{r}\,'(t)\,dt = \int_a^b \underbrace{f_x\, x'(t) + f_y\, y'(t)}_{= \frac{d}{dt}(f)}\,dt$$

$$= \int_a^b \frac{d}{dt}(f)\,dt$$

2 Theorem Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \leqslant t \leqslant b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

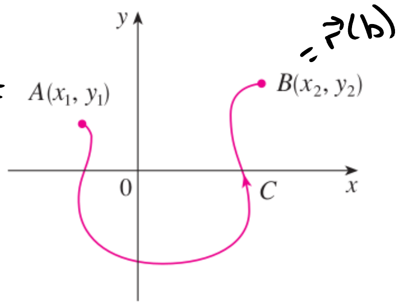
FTLI

Remarks.

1. In 2D.

$\vec{r}(a) =$

$= \vec{r}(b)$

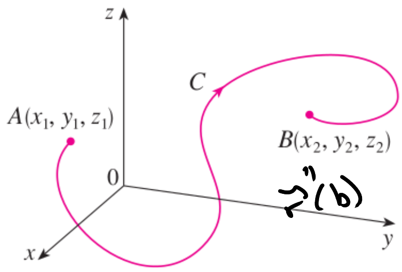


$$\int_C \vec{\nabla} f \cdot d\vec{r} = f(x_2, y_2) - f(x_1, y_1).$$

2. In 3D.

$\vec{r}(a) =$

$= \vec{r}(b)$



$$\int_C \vec{\nabla} f \cdot d\vec{r} = f(x_2, y_2, z_2) - f(x_1, y_1, z_1).$$

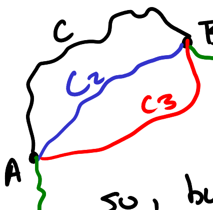
EXAMPLE 1 Find the work done by the gravitational field

$$\boxed{\mathbf{F}(\mathbf{x}) = -\frac{mMG}{|\mathbf{x}|^3}\,\mathbf{x}}$$

in moving a particle with mass m from the point $(3, 4, 12)$ to the point $(2, 2, 0)$ along a piecewise-smooth curve C. (See Example 16.1.4.)

A B

Conservative vector fields: $\vec{F} = \vec{\nabla} f$, some f.



If $f(x,y,z) = \dfrac{mMG}{\sqrt{x^2+y^2+z^2}}$, then $\boxed{\vec{\nabla} f = \vec{F}}$.

so, by the FTLI,

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{\nabla} f \cdot d\vec{r} = f(2,2,0) - f(3,4,12)$$

$$= mMG\left(\frac{1}{2\sqrt{2}} - \frac{1}{13}\right).$$

Independence of Path.

Definition. ① Path: piece-wise smooth curve.

② Independant of Path: $\vec{F}$ is ind. of path if for any two paths C_1 & C_2 starting at A and ending at B, then

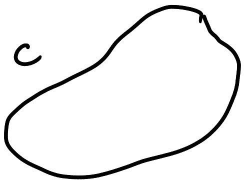
$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}.$$

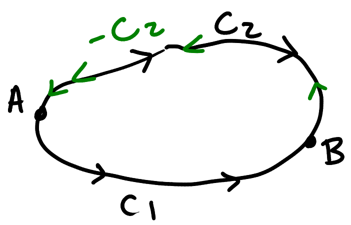
(Example 4 in 16.2, not true in general)

3 Theorem $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D.

③ Closed path: a path with the same starting & ending points.

why?





$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$$

$$\Rightarrow \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r} = 0$$

$$\Rightarrow \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{-C_2} \vec{F} \cdot d\vec{r} = 0$$

$$\Rightarrow \int_{C} \vec{F} \cdot d\vec{r} = 0$$

4 Theorem Suppose $\mathbf{F}$ is a vector field that is continuous on an open connected region D. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D, then $\mathbf{F}$ is a conservative vector field on D; that is, there exists a function f such that $\nabla f = \mathbf{F}$.

④ Open:

⑤ : Open connected:

not separated.

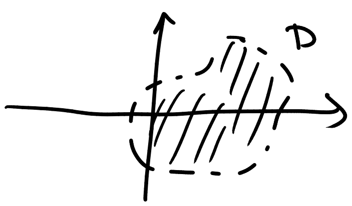
X not connected.

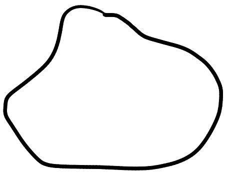
6 Theorem Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order partial derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \text{throughout D}$$

Then $\mathbf{F}$ is conservative. The converse also holds.

⑥ Simply-connected: No holes!





EXAMPLE 2 Determine whether or not the vector field

$$\mathbf{F}(x, y) = (x - y)\,\mathbf{i} + (x - 2)\,\mathbf{j}$$

is conservative.

EXAMPLE 3 Determine whether or not the vector field

$$\mathbf{F}(x, y) = (3 + 2xy)\,\mathbf{i} + (x^2 - 3y^2)\,\mathbf{j}$$

is conservative.

EXAMPLE 4

(a) If $\mathbf{F}(x, y) = (3 + 2xy)\,\mathbf{i} + (x^2 - 3y^2)\,\mathbf{j}$, find a function f such that $\mathbf{F} = \nabla f$.

(b) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve given by

$$\mathbf{r}(t) = e^t \sin t\,\mathbf{i} + e^t \cos t\,\mathbf{j} \qquad 0 \leqslant t \leqslant \pi$$

EXAMPLE 5 If $\mathbf{F}(x, y, z) = y^2\,\mathbf{i} + (2xy + e^{3z})\,\mathbf{j} + 3ye^{3z}\,\mathbf{k}$, find a function f such that $\nabla f = \mathbf{F}$.

Conservation of Energy.