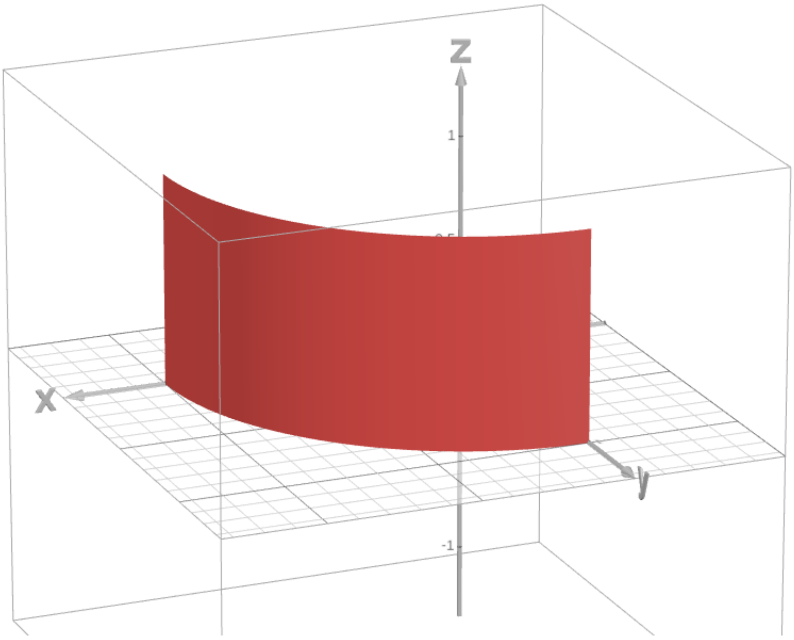
Chapter 16

Vector Calculus

16.7 Surface Integrals

Surface Differential

EXAMPLE. Find the area of the following parametric surface S:



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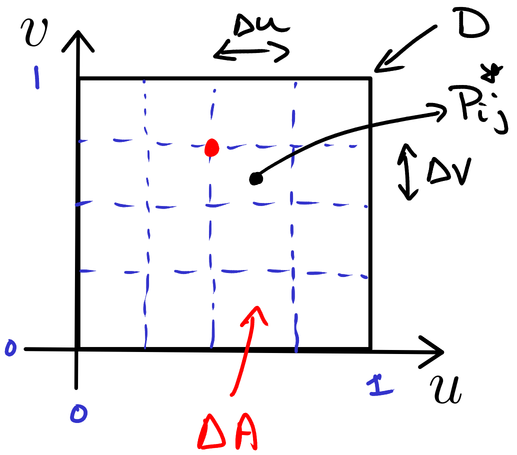
Parametric Equations

$$x = \cos\left((\pi/2)u\right)$$
$$y = \sin\left((\pi/2)u\right)$$
$$z = v$$

$0 \le u \le 1$, $0 \le v \le 1$.

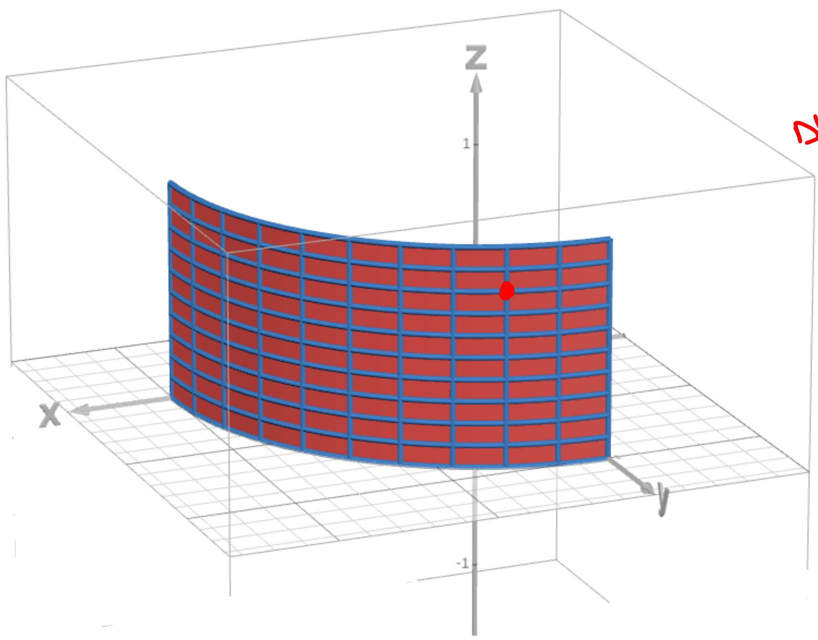
1. Divide the uv-region in small rectangles.

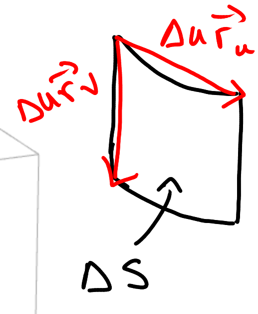


Divide D in small rectangles:
- m parts of length Δu
- n parts of length Δv

Select a sample point P^*_{ij} in each rectangle.

2. Approximate the area of each small piece.





$\Delta S \approx \text{Area}(\ )$

$= |\Delta u \vec{r}_u \times \Delta v \vec{r}_v|$

So,

$$\Delta S \approx |\vec{r}_u \times \vec{r}_v| \underbrace{\Delta u \Delta v}_{\Delta A}$$

3. Sum up.

$$\text{Area}(S) \approx \sum_{i=1}^{m} \sum_{j=1}^{n} |\vec{r}_u \times \vec{r}_v| \, \Delta A$$

Take $m, n \to \infty$

$$\Rightarrow \boxed{\text{Area}(S) = \iint_D |\vec{r}_u \times \vec{r}_v| \, dA}$$

4. Compute the Area.

$$\vec{r}_u = \left\langle -\frac{\pi}{2} \sin\left(\frac{\pi}{2}u\right), \frac{\pi}{2} \cos\left(\frac{\pi}{2}u\right), 0 \right\rangle$$

$$\vec{r}_v = \langle 0, 0, 1 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\frac{\pi}{2} \sin\left(\frac{\pi}{2}u\right) & \frac{\pi}{2} \cos\left(\frac{\pi}{2}u\right) & 0 \\ 0 & 0 & 1 \end{vmatrix} = \left\langle \frac{\pi}{2} \cos\left(\frac{\pi}{2}u\right), \frac{\pi}{2} \sin\left(\frac{\pi}{2}u\right), 0 \right\rangle$$

So,

$$|\vec{r}_u \times \vec{r}_v| = \frac{\pi}{2}.$$

So,

$$\text{Area}(D) = \iint_D \frac{\pi}{2} \, dA = \int_0^1 \int_0^1 \frac{\pi}{2} \, du \, dv$$

$$= \boxed{\frac{\pi}{2}}$$

Surface Area Differential:

$$\boxed{dS = |\vec{r}_u \times \vec{r}_v| dA}$$

Integral of scalar-valued functions.

Data:

- A surface S.
- A parametrization $\vec{r}(u, v)$ of the surface with domain D.
- A scalar-valued function $f(x, y, z)$. → mass density. → temperature.

$$\boxed{\iint_S f(x,y,z)\,dS = \iint_D f(\vec{r}(u,v))|\vec{r}_u \times \vec{r}_v|\,dA}$$

5–20 Evaluate the surface integral.

5. $\iint_S (x + y + z)\,dS$,
S is the parallelogram with parametric equations $x = u + v$, $y = u - v$, $z = 1 + 2u + v$, $0 \leqslant u \leqslant 2$, $0 \leqslant v \leqslant 1$

$f(x,y,z) = x+y+z$, $\quad \vec{r}(u,v) = \langle \underbrace{u+v}_{x}, \underbrace{u-v}_{y}, \underbrace{1+2u+v}_{z} \rangle$.

① Find $\vec{r}_u \times \vec{r}_v$

$\vec{r}_u = \langle 1, 1, 2 \rangle$

$\vec{r}_v = \langle 1, -1, 1 \rangle$

$$\Rightarrow \quad \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 2 \\ 1 & -1 & 1 \end{vmatrix} = \langle 3, -(-1), -2 \rangle$$

② Integral

$$\iint_S x+y+z \; dS = \iint_D (u+v) + (u-v) + (1+2u+v)\,|\vec{r}_u \times \vec{r}_v|\,dA$$

$$= \iint_D (4u + v + 1)\,|\langle 3, 1, -2 \rangle|\,dA$$

$$= \int_0^1 \int_0^2 (4u + v + 1) \sqrt{9+1+4} \quad du\,dv$$

$$= \int_0^1 \int_0^2 (4u + v + 1) \sqrt{14} \quad du\,dv$$

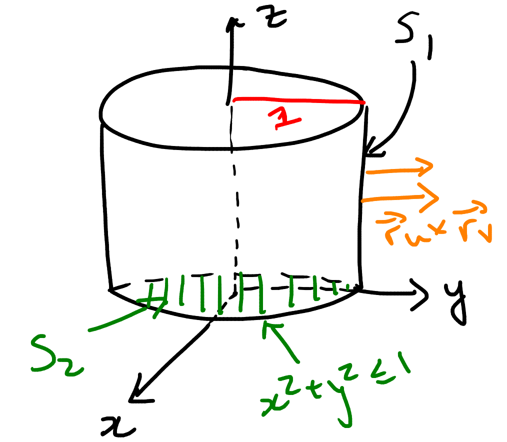
$$= \sqrt{14} \int_0^1 \int_0^2 (4u + v + 1) \, du\,dv$$

$$= \boxed{\sqrt{14} \cdot 11}$$

Example.

Evaluate $\iint_S z\,dS$, where S is the surface whose sides are given by the cylinder $x^2+y^2=1$ from $z=0$ to $z=2$ and whose bottom is the disk $x^2+y^2\le 1$ in the plane $z=0$.

① Picture



$$S = S_1 \sqcup S_2$$

S_1: $\vec{r}(u,v) = \langle \cos u, \sin u, v\rangle$

$0 \le u \le 2\pi, \qquad 0 \le v \le 2.$

S_2: $\vec{r}(u,v) = \langle v\cos u, v\sin u, 0\rangle$

$0 \le u \le 2\pi, \qquad 0 \le v \le 1.$

② Integral $\displaystyle \iint_S z\,dS = \iint_{S_1} z\,dS + \iint_{S_2} z\,dS$

on S_1 $\quad \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin u & \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle \cos u, \sin u, 0\rangle$

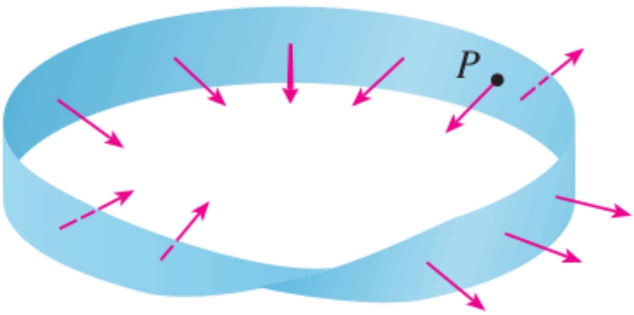
Then, $\displaystyle \iint_{S_1} z\,dS = \int_0^2 \int_0^{2\pi} v\sqrt{1}\,du\,dv = 4\pi$

on S_2 $\quad \displaystyle \iint_{S_2} z\,dS = 0 \qquad$ Why? because $z=0$ in the parametrization of S_2

So: $\displaystyle \iint_S z\,dS = 4\pi + 0 = \boxed{4\pi}$

Surface integral of Vector Fields.

- Non-orientable surfaces.



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- Orientable surface.

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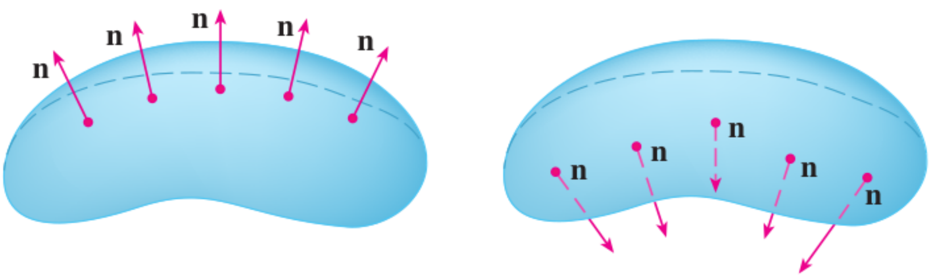


- We assume that the surface is orientable, so that there is a well-defined normal vector.

- For a parametric surface with parametrization $\vec{r}(u, v)$:

$$\boxed{\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}}$$

Example.

Find a normal vector at every point of a sphere of equation

$$x^2+y^2+z^2=1$$

$$\vec{r}(\theta,\phi)=\langle \cos\theta\sin\phi,\ \sin\theta\sin\phi,\ \cos\phi\rangle$$

$$0\le\theta\le 2\pi,\qquad 0\le\phi\le\pi$$

$$\vec{r}_\theta=\langle -\sin\theta\sin\phi,\ \cos\theta\sin\phi,\ 0\rangle$$

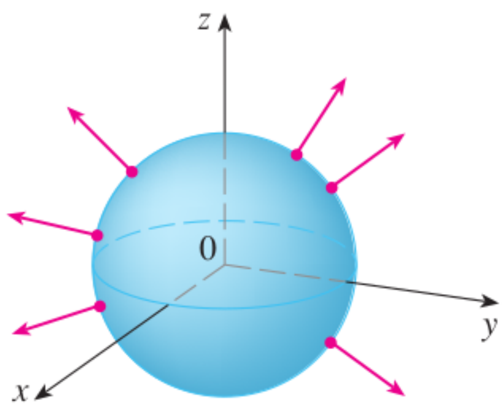
$$\vec{r}_\phi=\langle \cos\theta\cos\phi,\ \sin\theta\cos\phi,\ -\sin\phi\rangle$$

$$\Rightarrow\quad \vec{r}_\theta\times\vec{r}_\phi=\langle -\cos\theta\sin^2\phi,\ -\sin\theta\sin^2\phi,\ -\sin\phi\cos\phi\rangle$$

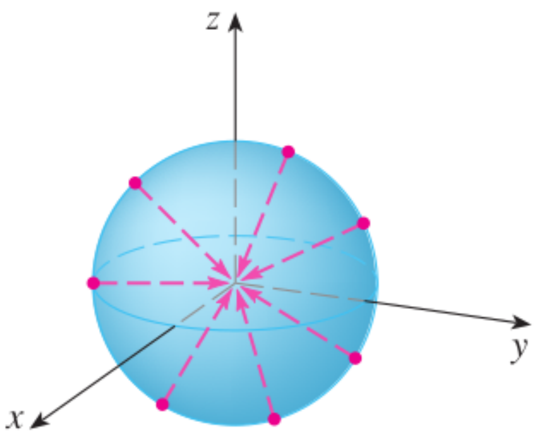
$$\Rightarrow\quad |\vec{r}_\theta\times\vec{r}_\phi|=\sin\phi$$

Thus, $$\vec{n}=\frac{\vec{r}_\theta\times\vec{r}_\phi}{|\vec{r}_\theta\times\vec{r}_\phi|}=\langle -\cos\theta\sin\phi,\ -\sin\theta\sin\phi,\ -\cos\phi\rangle=-\vec{r}(\theta,\phi).$$

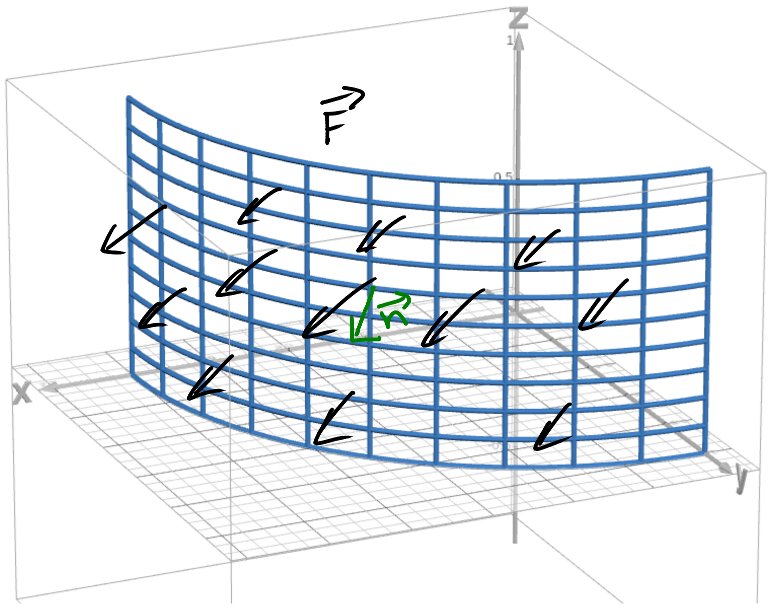
Positive orientation of a closed surface.



Negative orientation of a closed surface.



Flux integral (or Surface integral).



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Data:

- An orientable surface S.
- A parametrization $\vec{r}(u, v)$ of the surface.
- A vector field $\vec{F}(x, y, z)$.

Approx flux through $\Delta\vec{S}$:

$$\vec{F} \cdot (\vec{r}_u \times \vec{r}_v) \underbrace{\Delta u \, \Delta v}_{\Delta A}$$

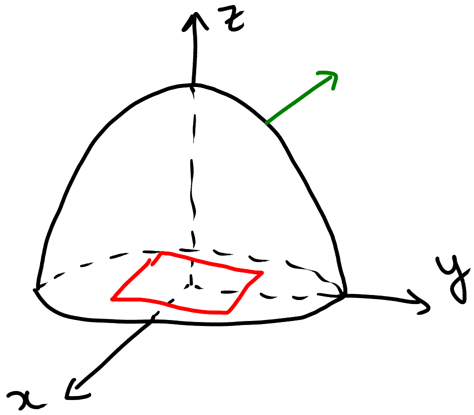
$$\boxed{\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\, dA}$$

D: region of the parameters u and v.

EXAMPLE.

Find the flux integral of $\vec{F}(x, y, z) = \langle xy, yz, zx \rangle$ through the part of the paraboloid $z = 4 - x^2 - y^2$ lying above the square $[0, 1] \times [0, 1]$ and with upward orientation.

Parametrization



$$\vec{r}(x, y) = \langle x, y, 4 - x^2 - y^2 \rangle$$

for $0 \le x \le 1, \quad 0 \le y \le 1.$

$\underline{\vec{r}_x \times \vec{r}_y}$

> 0 upward ✓

$$\vec{r}_x = \langle 1, 0, -2x \rangle$$
$$\vec{r}_y = \langle 0, 1, -2y \rangle$$
$$\rightarrow \quad \vec{r}_x \times \vec{r}_y = \langle 2x, 2y, 1 \rangle$$

<u>Integral</u>

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \langle xy, y(4-x^2-y^2), x(4-x^2-y^2) \rangle \cdot \langle 2x, 2y, 1 \rangle \, dA$$

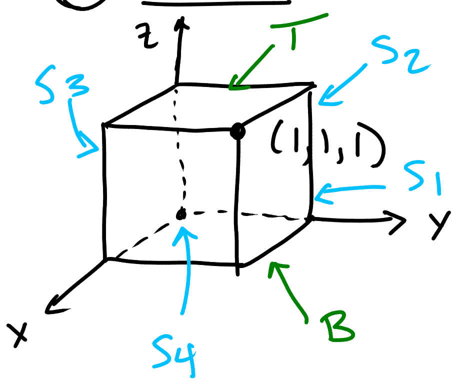
$$= \int_0^1 \int_0^1 2x^2y + 2y^2(4-x^2-y^2) + x(4-x^2-y^2) \, dx\,dy$$

$$= \boxed{\frac{713}{180}} \approx 3.9611$$

EXAMPLE.

Find the flux integral of $\vec{F}(x,y,z) = \langle x, 2y, 3z\rangle$ if S is a cube with diagonal $(0,0,0)$ to $(1,1,1)$ and S has the positive orientation.

① Picture



$$S = T \cup B \cup S_1 \cup S_2 \cup S_3 \cup S_4$$

$T: \ \vec{r}(u,v) = \langle u, v, 1\rangle \quad (0 \le u,v \le 1)$

$B: \ \vec{r}(u,v) = \langle u, v, 0\rangle \quad$ " "

$S_1: \langle u, 1, v\rangle \qquad S_3: \langle u, 0, v\rangle$

$S_2: \langle 0, u, v\rangle \qquad S_4: \langle 1, u, v\rangle$

② Integrate

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_T \vec{F}\cdot d\vec{S} + \iint_B \vec{F}\cdot d\vec{S} + \iint_{S_1} \vec{F}\cdot d\vec{S} + \iint_{S_2} \vec{F}\cdot d\vec{S} + \iint_{S_3} \vec{F}\cdot d\vec{S} + \iint_{S_4} \vec{F}\cdot d\vec{S}$$

$$= \iint_D \langle u, 2v, 3\rangle \cdot \langle 0,0,1\rangle\, dA + \iint_D \langle u, 2v, 0\rangle \cdot \langle 0,0,-1\rangle\, dA$$

$$+ \iint_D \langle u, 2, 3v\rangle \cdot \langle 0,1,0\rangle\, dA + \iint_D \langle u, 0, 3v\rangle \cdot \langle 0,-1,0\rangle\, dA$$

$$+ \iint_D \langle 0, 2u, 3v\rangle \cdot \langle -1,0,0\rangle\, dA + \iint_D \langle 1, 2u, 3v\rangle \cdot \langle 1,0,0\rangle\, dA$$

$$= \boxed{6}$$

Gauss' Law

The net charge enclosed by a closed surface S is

$$Q = \varepsilon_0 \iint_S \vec{E} \cdot d\vec{S}$$

where $\vec{E}$ is the electric field and ε_0 is the permittivity of free space.