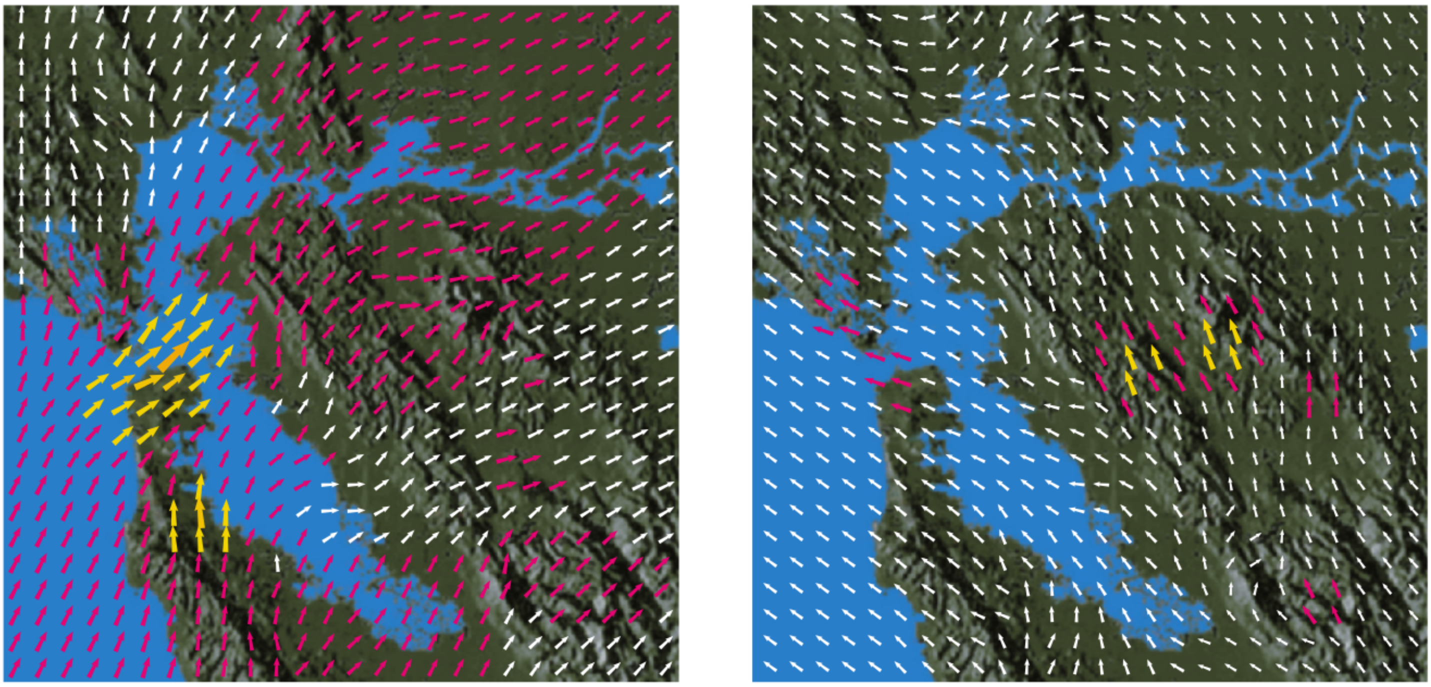
Section 16.1 Vector Fields.

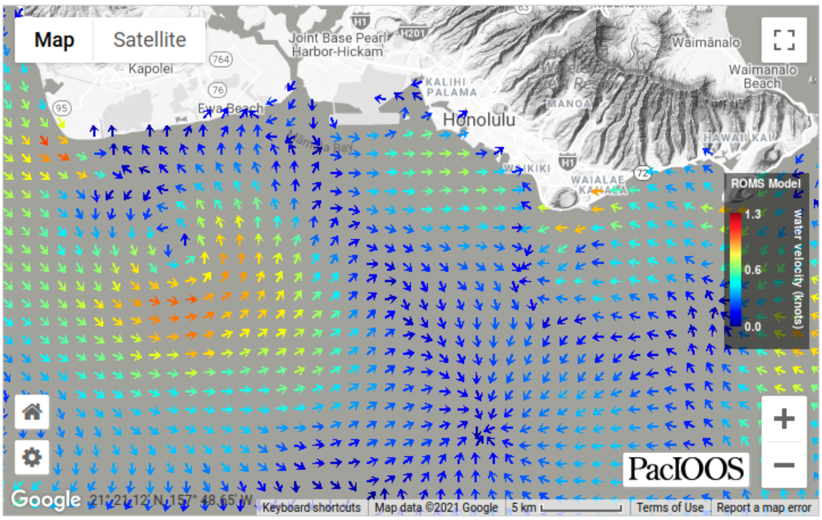
Examples.



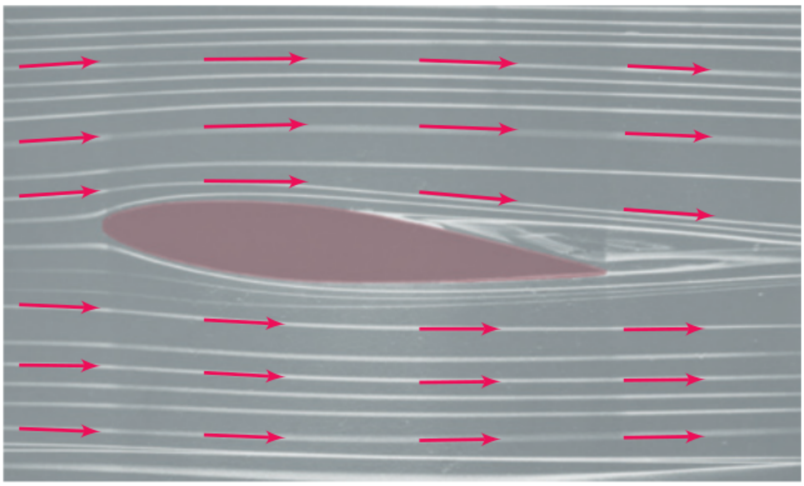
(a) 6:00 PM, March 1, 2010

(b) 6:00 AM, March 1, 2010

FIGURE 1 Velocity vector fields showing San Francisco Bay wind patterns



Map took from
http://www.pacioos.hawaii.edu/currents/model-oahu/



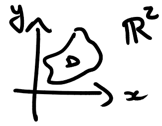
ONERA photograph, Werle, 1974

(b) Airflow past an inclined airfoil

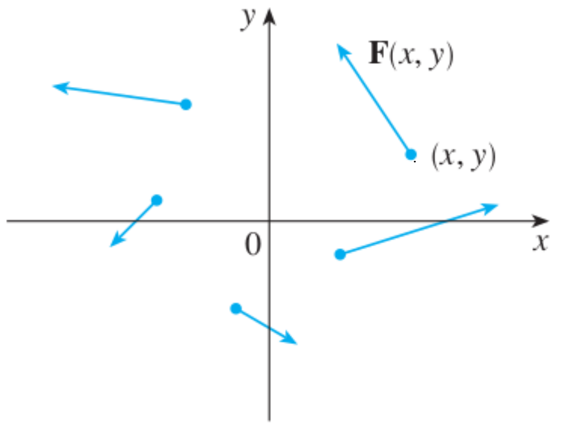
Vector Fields in 2D.

my notation.
↳ $\vec{F}(x,y)$

> **1 Definition** Let D be a set in $\mathbb{R}^2$ (a plane region). A **vector field on** $\mathbb{R}^2$ is a function $\mathbf{F}$ that assigns to each point (x, y) in D a two-dimensional vector $\mathbf{F}(x, y)$.



Representation.



Drawing. Draw a vector representing $\vec{F}(x,y)$ at the point (x,y).

Component Functions.

$\vec{F}(x,y)$ is 2D :

$$\vec{F}(x,y) = P(x,y)\underset{\langle 1,0\rangle}{\vec{i}} + Q(x,y)\underset{\langle 0,1\rangle}{\vec{j}}$$

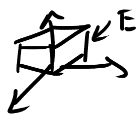
P: x-component of $\vec{F}$

Q: y-component of $\vec{F}$

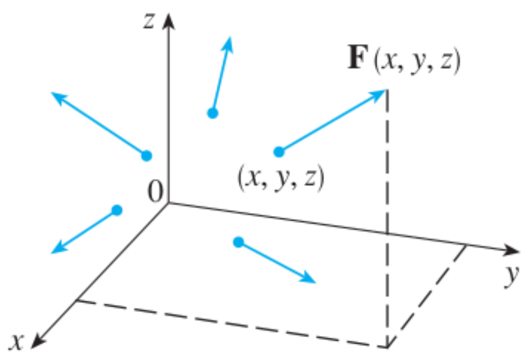
Remark:
P & Q are called scalar fields.
P maps a point to a number.

Vector Fields in 3D.

> **2 Definition** Let E be a subset of $\mathbb{R}^3$. A **vector field on** $\mathbb{R}^3$ is a function $\mathbf{F}$ that assigns to each point (x, y, z) in E a three-dimensional vector $\mathbf{F}(x, y, z)$.



Representation.



Component Functions.

$\vec{F}(x,y,z)$ is in 3D:

$$\vec{F}(x,y,z) = P(x,y,z)\,\vec{i} \leftarrow \langle 1,0,0\rangle$$
$$+ Q(x,y,z)\,\vec{j} \leftarrow \langle 0,1,0\rangle$$
$$+ R(x,y,z)\,\vec{k} \leftarrow \langle 0,0,1\rangle.$$

P: x-coord. of $\vec{F}$
Q: y-coord. of $\vec{F}$
R: z-coord. of $\vec{F}$.

Remark:
$\vec{F}$ is a continuous vector field if and only if P, Q & R are continuous.

EXAMPLE 1 A vector field on $\mathbb{R}^2$ is defined by $\mathbf{F}(x, y) = -y\,\mathbf{i} + x\,\mathbf{j}$. Describe $\mathbf{F}$ by sketching some of the vectors $\mathbf{F}(x, y)$.

$\langle -y, x\rangle$

Notation. $\vec{x} = \langle x,y,z\rangle$
$\vec{F}(x,y,z) = \vec{F}(\vec{x})$

① Draw a Table.

(x,y)	$\vec{F}(x,y)$
$(1,0)$	$\langle 0,1\rangle$
$(2,2)$	$\langle -2,2\rangle$
$(3,0)$	$\langle 0,3\rangle$
$(0,1)$	$\langle -1,0\rangle$
$(-2,2)$	$\langle -2,-2\rangle$
$(0,3)$	$\langle -3,0\rangle$

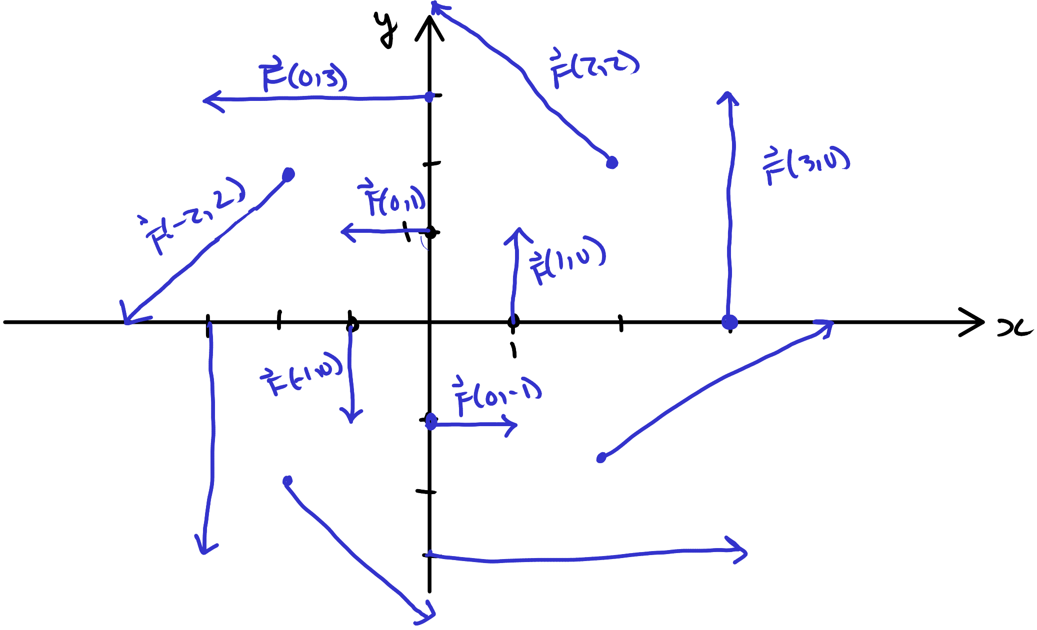
(x,y)	$\vec{F}(x,y)$
$(-1,0)$	$\langle 0,-1\rangle$
$(-2,-2)$	$\langle 2,-2\rangle$
$(-3,0)$	$\langle 0,-3\rangle$
$(0,-1)$	$\langle 1,0\rangle$
$(2,-2)$	$\langle 2,-2\rangle$
$(0,-3)$	$\langle 3,0\rangle$

$P(x,y) = -y$
$Q(x,y) = x$

$\vec{F}(1,0) = 0\vec{i} + \vec{j}$

$\vec{F}(2,2) = -2\vec{i} + 2\vec{j}$

$\vec{F}(3,0) = 0\vec{i} + 3\vec{j}$

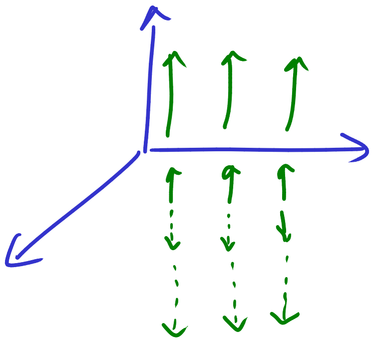


EXAMPLE 2 Sketch the vector field on $\mathbb{R}^3$ given by $\mathbf{F}(x, y, z) = z\,\mathbf{k}$.

$P(x,y,z) = 0$

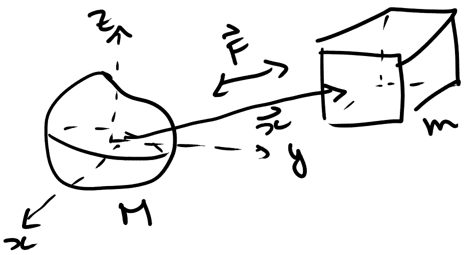
$Q(x,y,z) = 0$

$R(x,y,z) = z$



See python script.

EXAMPLE 4 Newton's Law of Gravitation.



Law tells your

$$\|\vec{F}\| = \frac{mMG}{r^2}$$

r: distance between two objects.

G: gravitational constant.

Suppose that M is at the origin.

Let $\vec{x} = \langle x, y, z \rangle$ be the position vector of the mass m.

Then $r = \|\vec{x}\| \Rightarrow r^2 = \|\vec{x}\|^2$.

Since $M > m$, the mass m will be attracted to M.

The direction of the force is

$$-\frac{\vec{x}}{\|\vec{x}\|}$$

So,

$$\vec{F}(x, y, \vec{z}) = \|\vec{F}\| \cdot \left(-\frac{\vec{x}}{\|\vec{x}\|}\right)$$

$$\Rightarrow \boxed{\vec{F}(x, y, z) = -\frac{mMG}{\|\vec{x}\|^3}\vec{x}} \to \text{Gravitational Vec. Field}$$

<u>More examples.</u>

- Force fied around an electric change Q

$$\vec{F}(\vec{x}) = \frac{\varepsilon_0 q Q}{\|\vec{x}\|^3}\vec{x}.$$

- Electric fied around Q

$$\vec{E}(\vec{x}) = \frac{\vec{F}(\vec{x})}{q} = \frac{\varepsilon_0 Q}{\|\vec{x}\|^3}\vec{x}.$$

Gradient Fields.

Gradient.

If $f: \mathbb{R}^2 \to \mathbb{R}$, then $\vec{\nabla} f(x,y) = \langle f_x, f_y \rangle = f_x \vec{i} + f_y \vec{j}$

If $f: \mathbb{R}^3 \to \mathbb{R}$, then $\vec{\nabla} f(x,y,z) = \langle f_x, f_y, f_z \rangle = f_x \vec{i} + f_y \vec{j} + f_z \vec{k}$

Called Gradient Vector Fields.

EXAMPLE 6 Find the gradient vector field of $f(x, y) = x^2y - y^3$. Plot the gradient vector field together with a contour map of f. How are they related?

1. Gradient.

$f_x = 2xy \qquad f_y = x^2 - 3y^2$

$$\vec{\nabla} f(x,y) = 2xy\,\vec{i} + (x^2 - 3y^2)\vec{j}.$$

$P(x,y) = 2xy \qquad Q(x,y) = x^2 - 3y^2$

2. Plot the gradient field.

Python. :)

Conservative Vector Fields.

- Vector field $\vec{F}$ is conservative if there is a scalar function f such that $\vec{F}$ is the gradient of f, that is

$$\vec{F} = \vec{\nabla} f.$$

- The function f is called the potential function of $\vec{F}$.

For example, if

$$f(x,y,z) = \frac{mMG}{\sqrt{x^2+y^2+z^2}}$$

then f is a potential function for the gravitational field

$$\vec{F} = -\frac{mMG}{\|\vec{x}\|^3}\vec{x}$$

$(x^2+y^2+z^2)^{3/2}$