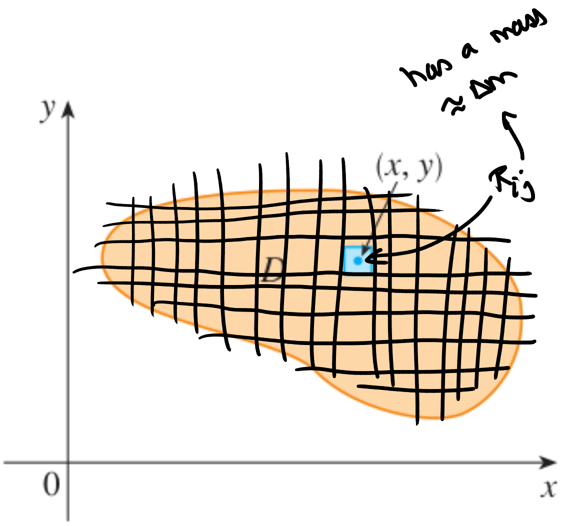
Density, mass and charge.



$\rho(x,y)$: mass density at (x,y)

$$\rho(x,y) = \lim_{\Delta A \to 0} \frac{\Delta m}{\Delta A}$$

$$m \approx \sum_{i=0}^{m} \sum_{j=0}^{n} \underbrace{\rho(x,y) \cdot \Delta A}_{\approx \Delta m}$$

Make $m, n \to \infty$

Total Mass

$$\boxed{m = \iint_D \rho(x,y)\, dA}$$

Total Charge

$$\boxed{Q = \iint_D \sigma(x,y)\, dA}$$

EXAMPLE 1 Charge is distributed over the triangular region D in Figure 3 so that the charge density at (x, y) is $\underline{\sigma(x, y) = xy}$, measured in coulombs per square meter (C/m^2). Find the total charge.

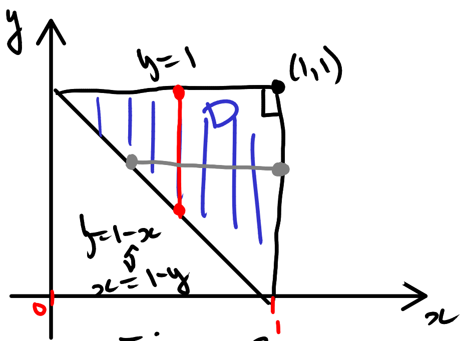


Figure 3

① Describe D

- Type I: $D = \{ (x,y) : 0 \le x \le 1 ,\ 1-x \le y \le 1 \}$
- TYPE II: $D = \{ (x,y) : 1-y \le x \le 1 ,\ 0 \le y \le 1 \}$

② Integrate

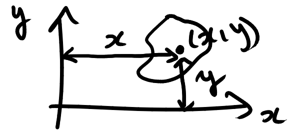
$$Q = \iint_D \sigma(x,y)\, dA = \int_0^1 \int_{1-x}^{1} xy \; dy\, dx$$

$$= \int_0^1 \left. \frac{xy^2}{2} \right|_{1-x}^{1} dx$$

$$= \int_0^1 \frac{x(1)}{2} - \frac{x(1-x)^2}{2} \, dx$$

$$= 5/24 \ C$$

Moments and center of mass.



Moment about the x-axis

$$M_x = \iint_D y\rho(x,y)\,dA$$

Moment about the y-axis

$$M_y = \iint_D x\rho(x,y)\,dA$$

5 The coordinates $(\bar{x}, \bar{y})$ of the center of mass of a lamina occupying the region D and having density function $\rho(x, y)$ are

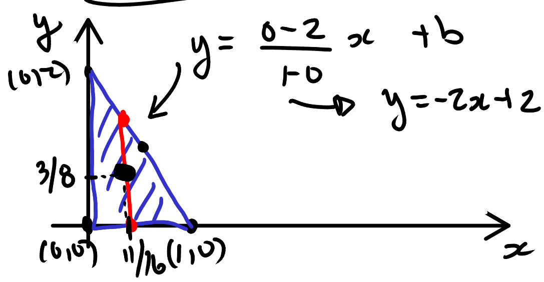
$$\boxed{\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint_D x\rho(x,y)\,dA} \qquad \boxed{\bar{y} = \frac{M_x}{m} = \frac{1}{m}\iint_D y\rho(x,y)\,dA}$$

where the mass m is given by

$$m = \iint_D \rho(x,y)\,dA$$

EXAMPLE 2 Find the mass and center of mass of a triangular lamina with vertices $(0, 0)$, $(1, 0)$, and $(0, 2)$ if the density function is $\rho(x, y) = 1 + 3x + y$.

① Picture



$y = \frac{0-2}{1-0}x + b \to y = -2x+2$

$y = -2x + b$ (1,0) belong to the line.

$0 = -2\cdot 1 + b \to b = 2$

$D = \{(x,y) : 0 \le x \le 1, \quad 0 \le y \le -2x+2\}$

② Mass.

$$m = \iint_D \rho(x,y)\,dA = \int_0^1\int_0^{-2x+2} 1+3x+y \; dy\,dx$$

$$= \int_0^1 \left. y + 3xy + \frac{y^2}{2} \right|_0^{-2x+2} dx$$

$$= \int_0^1 (-2x+2) + 3x(-2x+2) + \frac{(-2x+2)^2}{2}\,dx$$

$$= \frac{8}{3} \quad (\text{kg})$$

③ Moment in x

$$M_x = \iint_D y\,\rho(x,y)\,dA = \int_0^1 \int_0^{-2x+2} y(1+3x+y)\,dy\,dx$$

$$= \int_0^1 \frac{y^2}{2} + \frac{3xy^2}{2} + \frac{y^3}{3} \Big|_0^{-2x+2} dx$$

$$= \int_0^1 \frac{(-2x+2)^2 + 3x(-2x+2)^2}{2} + \frac{(-2x+2)^3}{3}\,dx$$

$u = -2x+2$
$du = -2dx$

$$= \frac{88}{48}$$

④ Moment y.

$$M_y = \iint_D x\,\rho(x,y)\,dA = \int_0^1 \int_0^{-2x+2} x(1+3x+y)\,dy\,dx$$

$$= \int_0^1 \left(xy + 3x^2 y + \frac{y^2 x}{2}\right)\Big|_0^{-2x+2} dx$$

$$= \int_0^1 x(-2x+2) + 3x^2(-2x+2) + \frac{x(-2x+2)^2}{2}\,dx$$

$$= 1$$

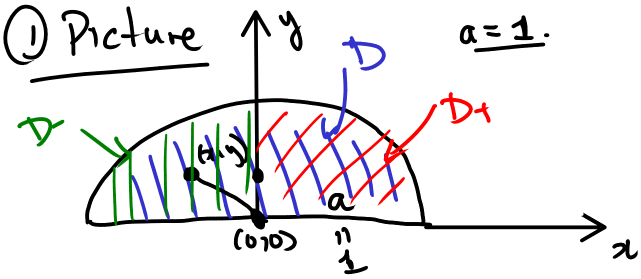
⑤ Center of mass.

$$\bar{y} = \frac{M_x}{m} = \frac{88}{48} \Big/ \frac{8}{3} = \frac{11}{16}$$

$$\bar{x} = \frac{M_y}{m} = 1 \Big/ \frac{8}{3} = \frac{3}{8}$$

EXAMPLE 3 The density at any point on a semicircular lamina is proportional to the distance from the center of the circle. Find the center of mass of the lamina.

① Picture



$a = 1.$

$$D = \{(r,\theta) : 0 \le r \le 1 , \ 0 \le \theta \le \pi\}$$

② Mass.

$$m = \iint_D \underbrace{p(x,y)}_{\overset{\shortparallel}{kr}} \, dA$$

What is $p(x,y)$??

$$p(x,y) = k\sqrt{x^2+y^2} = kr$$

$$x = r\cos\theta , \quad y = r\sin\theta$$

$$m = \int_0^{\pi}\int_0^1 kr \; r\,dr\,d\theta$$

$$= \left(\int_0^{\pi} d\theta\right)\left(\int_0^1 k\,r^2\,dr\right) = \frac{\pi \; k}{3} \qquad \left(\frac{\pi a^3 k}{3}\right)$$

③ Moment in y

$$M_y = \iint_D x\,p(x,y)\,dA \qquad \text{by symmetry.}$$

$$= \iint_{D^-} x\,p(x,y)\,dA + \iint_{D^+} x\,p(x,y)\,dA$$

$$= -\iint_{D^+} x\,p(x,y)\,dA + \iint_{D^+} x\,p(x,y)\,dA = 0 .$$

$$\bar{x} = M_y/m = 0$$

④ Moment in x

$$M_x = \iint_D y \underset{\downarrow \atop kr}{p(x,y)}\, dA = k\int_0^{\pi}\int_0^1 (r\sin\theta)\,\overset{p(x,y) \atop \downarrow}{r}\,r\,dr\,d\theta$$

$$= k\left(\int_0^{\pi} \sin\theta\,d\theta\right)\left(\int_0^1 r^3\,dr\right)$$

$$= k(-\cos\theta)\Big|_0^{\pi}\left(\frac{1}{4}\right)$$

$$= \frac{2}{4}k$$

$$\bar{y} = \frac{M_x}{m} = \frac{2k/4}{\pi k/3} = \frac{3}{2\pi}$$

Moment of Intertia.

Inertia about the x-axis

$$I_x = \iint_D y^2 \rho(x,y)\,dA$$

Inertia about the y-axis

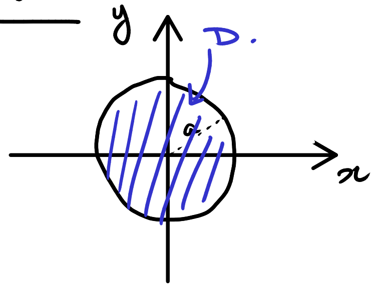
$$I_y = \iint_D x^2 \rho(x,y)\,dA$$

Inertia about the origin

$$I_0 = \iint_D (x^2+y^2)\rho(x,y)\,dA$$

EXAMPLE 4 Find the moments of inertia I_x, I_y, and I_0 of a homogeneous disk D with density $\rho(x, y) = \rho$, center the origin, and radius a.

① Picture



$x = r\cos\theta \qquad y = r\sin\theta$

$D = \{(r,\theta) : 0 \le r \le a,\ 0 \le \theta \le 2\pi\}$

② I_x

$x = r\cos\theta,\ y = r\sin\theta$

$$I_x = \iint_D y^2 \rho(x,y)\,dA = \int_0^{2\pi}\int_0^{a} y^2 \rho\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{a} r^2 \sin^2\theta\, \rho\, r\,dr\,d\theta$$

$$= \rho\left(\int_0^{2\pi} \sin^2\theta\,d\theta\right)\left(\int_0^{a} r^3\,dr\right)$$

$$= \rho\left(\int_0^{2\pi} \frac{1-\cos 2\theta}{2}\,d\theta\right)\left(\frac{r^4}{4}\Big|_0^a\right)$$

$$= \rho\left(\left(\frac{\theta}{2} - \cancel{\frac{\sin 2\theta}{4}}^{\to 0}\right)\Big|_0^{2\pi}\right)\left(\frac{a^4}{4}\right) = \boxed{\rho\pi\frac{a^4}{4}}.$$

③ I_y.

$$I_y = \iint_D x^2 \rho(x,y)\, dA = \int_0^a \int_0^{2\pi} r^2 \cos^2\theta \; \rho r\, dr\, d\theta$$

$$= \rho \left(\int_0^a r^3\, dr\right)\left(\int_0^{2\pi} \cos^2\theta\, d\theta\right)$$

$$= \rho \left(\frac{a^4}{4}\right)\left(\int_0^{2\pi} \frac{1+\cos 2\theta}{2}\, d\theta\right)$$

$$= \rho\, \frac{a^4}{4} \left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right)\Bigg|_0^{2\pi}$$

$$= \boxed{\rho \pi \frac{a^4}{4}.}$$

④ I_0

$$I_0 = \iint_D (x^2+y^2)\rho\, dA = \iint_D x^2\rho + y^2\rho\; dA$$

$$= \iint_D x^2 \rho\, dA + \iint_D y^2 \rho\, dA$$

$$= I_y + I_x$$

$$= \rho\pi \frac{a^4}{4} + \rho\pi\frac{a^4}{4}$$

$$= \boxed{\rho\pi\frac{a^4}{2}.}$$

Challenge.

Compute $I := \int_0^\infty e^{-x^2}\, dx$.

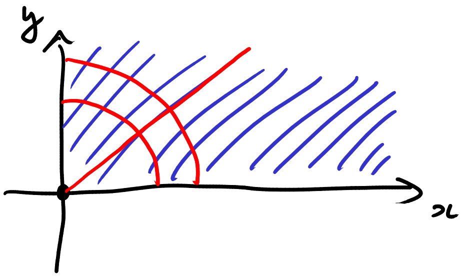
$$I^2 = I \cdot I = \left(\int_0^\infty e^{-x^2} dx\right)\left(\int_0^\infty e^{-x^2} dx\right)$$

$$= \left(\int_0^\infty e^{-x^2} dx\right)\left(\int_0^\infty e^{-y^2} dy\right) \qquad x = y$$

$$= \int_0^\infty \int_0^\infty e^{-x^2} e^{-y^2} \, dx \, dy$$

$$= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \, dx \, dy$$

$D = [0,\infty) \times [0,\infty) = \{ (x,y) : \ 0 \le x < \infty \ , \ 0 \le y < \infty \}.$



$0 \le r < \infty$

$0 \le \theta \le \pi/2$

$$I^2 = \int_0^{\pi/2} \int_0^\infty e^{-r^2} \, r \, dr \, d\theta$$

$$= \left(\int_0^{\pi/2} d\theta \right) \left(\int_0^\infty r e^{-r^2} \, dr \right) \qquad \begin{array}{l} u = r^2 \\ du = 2r\,dr \end{array}$$

$$= \left(\frac{\pi}{2} \right) \left(\int_0^\infty \frac{e^{-u}}{2} \, du \right)$$

$$= \left(\frac{\pi}{2} \right) \left. -\frac{e^{-u}}{2} \right|_0^\infty$$

$$= \left(\frac{\pi}{2} \right) \left(\frac{\overset{0}{-e^{-\infty}} - (\overset{1}{-e^{0}})}{2} \right)$$

$$= \frac{\pi}{4}$$

$$\Rightarrow \quad \boxed{I = \frac{\sqrt{\pi}}{2}}.$$