A.I Sample Space

PROBLEM 1.

a) If the people are labeled a, b and c, then

$$S = \{\{a,b\},\{a,c\},\{b,c\}\}.$$

b) $S = \{0,1,2,\dots\}$ (all non-negative integers).

c) $S = [0,\infty)$ (only the magnitude of the wind speed). △

A.II Event Space

PROBLEM 2.

a) Let h stands for "head" and t stands for "tail". Then

$$S = \{hhh, hht, hth, thh, htt, tht, tth, ttt\}.$$

b) $A = \{hhh\} \cup \{hht\}$.

c) $B = \{hhh\} \cup \{thh\}$.

d) $A \cap B = \{hhh\}$. This means all tosses were head. △

PROBLEM 3. Here is an example of an event space with 8 events:

$$\mathcal{A} = \{\varnothing, \{⚀\}, \{⚁,⚂,⚃,⚄,⚅\}, S\}.$$

The family $\mathcal{A}$ contains $\varnothing$. Also, $\overline{\varnothing} = S$ which is in $\mathcal{A}$, $\overline{\{⚀\}} = \{⚁,⚂,⚃,⚄,⚅\}$ which is in $\mathcal{A}$, $\overline{\{⚁,⚂,⚃,⚄,⚅\}} = \{⚀\}$ which is in $\mathcal{A}$, and $\overline{S} = \varnothing$ which is in S. Therefore, it satisfies property b). Finally, we can see that

- $\varnothing \cup \{⚀\} = ⚀$, $\varnothing \cup \{⚁,⚂,⚃,⚄,⚅\} = \{⚁,⚂,⚃,⚄,⚅\}$ and $\varnothing \cup S = S$ are all in $\mathcal{A}$.
- $\{⚀\} \cup \{⚁,⚂,⚃,⚄,⚅\} = S$ and $\{⚀\} \cup S = S$ are all in $\mathcal{A}$.
- $\{⚁,⚂,⚃,⚄,⚅\} \cup S = S$ is in $\mathcal{S}$.

Therefore, it satisfies property c). Since $\mathcal{A}$ satisfies the requirements in the definition of an event space, it is an event space.

Suppose that $\mathcal{A}$ contains six events, say

$$\mathcal{A} = \{\varnothing, \{⚀\}, \{⚁\}, \{⚁,⚂,⚃,⚄,⚅\}, \{⚀,⚂,⚃,⚄,⚅\}, S\}.$$

The family $\mathcal{A}$ cannot be an event space because it does not satisfy property c) of the definition of an event space. Indeed, if $A = \{\text{⚀}\}$ and $B = \{\text{⚁}\}$, then A, B are events, but $A \cup B = \{\text{⚀}, \text{⚁}\}$ is not an event because it does not belong to $\mathcal{A}$. △

PROBLEM 4.

a) Since A and B are events, then $A \cup B$ is also an event (by b) in the definition). Since $A \cup B$ is an event and C is an event, then $(A \cup B) \cup C$ is also an event (again by b) in the definition).

b) Applying de Morgan's laws, we have $A \cap B = \overline{\overline{A} \cup \overline{B}}$. Since A and B are events, then $\overline{A} \cup \overline{B}$ is also an event (by b) and c) in the definition). Applying b) from the definition, we see that $\overline{\overline{A} \cup \overline{B}}$ is an event. Therefore, $A \cap B$ is an event. △

A.III Axioms of a Probability

PROBLEM 5.

a) We have $A = \{(t,h), (t,t)\} = \{(t,h)\} \cup \{(t,t\}$. Since $\{(t,h)\} \cap \{(t,t)\} = \varnothing$, from the properties of a probability measure, we have

$$P(A) = P(\{(t,h)\}) + P(\{(t,t)\}) = \frac{2}{9} + \frac{4}{9} = \frac{2}{3}.$$

b) We have $A = \{(h,t), (t,h), (t,t)\}$. Using the properties of a probability measure twice (or Problem 7c)), we get

$$P(A) = P(\{(h,t)\}) + P(\{(t,h)\}) + P(\{(t,t)\}) = \frac{2}{9} + \frac{2}{9} + \frac{4}{9} = \frac{8}{9}.$$ △

PROBLEM 6.

a) We have

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C) = \varnothing \cup \varnothing = \varnothing.$$

Therefore, $A \cup B$ and C are mutually exclusive and

$$P(A \cup B \cup C) = P(A \cup B) + P(C).$$

Since A and B are mutually exclusive, we have $P(A \cup B) = P(A) + P(B)$. Therefore,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C).$$

b) We have $B = A \cup (B \cap \overline{A})$. We have

$$A \cap (B \cap \overline{A}) = A \cap \overline{A} \cap B = \varnothing \cap B = \varnothing$$

so that A and $B \cap \overline{A}$ are mutually exclusive. From the properties of a probability measure, we have

$$P(B) = P(A) + P(B \cap \overline{A}).$$

Since $P(B \cap \overline{A}) \geq 0$, then

$$P(A) \leq P(A) + P(B \cap \overline{A}) = P(B).$$ △

PROBLEM 7. We can rewrite $A \cup B$ as

$$A \cup B = (A \cap \overline{B}) \cup (A \cap B) \cup (\overline{A} \cap B).$$

The three sets on the right hand-side are all mutually exclusive, so from b), we have

$$P(A \cup B) = P(A \cap \overline{B}) + P(A \cap B) + P(\overline{A} \cap B).$$

However, $A = (A \cap \overline{B}) \cup (A \cap B)$ with $A \cap \overline{B} \cap A \cap B = \varnothing$, so that

$$P(A) = P(A \cap \overline{B}) + P(A \cap B)$$

and similarly,

$$P(B) = P(\overline{A} \cap B) + P(A \cap B).$$

So, adding the last two quantities together and subtracting $P(A \cap B)$, we get

$$\begin{aligned} P(A) + P(B) - P(A \cap B) &= P(A \cap \overline{B}) + P(\overline{A} \cap B) + 2P(A \cap B) - P(A \cap B) \\ &= P(A \cap \overline{B}) + P(\overline{A} \cap B) - P(A \cap B). \end{aligned}$$ △

PROBLEM 8. Let $P : \mathcal{A} \to \mathbb{R}$ be a probability measure. In particular, we have $P(S) = 1$. From Problem 7d), we also have $P(\varnothing) = 0$. It remains to show that $P(A) = p$ and $P(\overline{A}) = 1 - p$, for some $0 \leq p \leq 1$.

Set $p = P(A)$ which is a number between 0 and 1 because $P(A)$ is between 0 and 1. Since $A \cap \overline{A} = \varnothing$, we have $P(A) + P(\overline{A}) = 1$ and therefore $P(\overline{A}) = 1 - p$. This completes the proof. △

PROBLEM 9. We will show that P satisfies the three conditions of a probability measure.

a) Let $A \subset S$. Since $|A| \leq |S| = N$, then $P(A) = |A|/N \leq 1$.

b) We have $|S| = N$, so that $P(S) = N/N = 1$.

c) Let $A \subset S$ and $B \subset S$ such that $A \cap B = \varnothing$. Since A and B are disjoint, we have $|A \cup B| = |A| + |B|$. Therefore,

$$P(A \cup B) = \frac{|A \cup B|}{N} = \frac{|A|}{N} + \frac{|B|}{N} = P(A) + P(B).$$

Therefore, the three conditions in the definition of a probability measure are satisfied and P as defined is indeed a probability measure. △

A.IV Computing Probabilities in the Finite Case

PROBLEM 10.

① The sample space S has 36 outcomes. The outcome is a pair of faces from a regular 6-faced die. For example (⚀, ⚁) belongs to S.

② Assuming each outcome are equally likely, we have that each atomic event has probability 1/36 of occuring.

③ Let A denote the event "the sum of the upturned faces equals 7". Then we have

$$A = \{(\epsdice{1}, \epsdice{6}), (\epsdice{2}, \epsdice{5}), (\epsdice{3}, \epsdice{4}), (\epsdice{4}, \epsdice{3}), (\epsdice{5}, \epsdice{2}), (\epsdice{6}, \epsdice{1})\}.$$

We have $|A| = 6$ and therefore $P(A) = \frac{6}{36} = \frac{1}{6}$. $\triangle$

PROBLEM 11.

① If o stands for the color orange and b stands for the color blue, then the outcomes of S are strings formed from the letters o and b. For example, oob is a possible outcome and it means the first and second balls are orange and the third ball is blue. The sample space is then

$$S = \{ooo, oob, obo, boo, obb, bob, bbo, bbb\}$$

which means $|S| = 8$.

② The probability of getting an orange ball is $6/11$ and of getting a blue ball is $5/11$. Therefore,

$$P(\{ooo\}) = \frac{216}{1331}, \quad P(\{oob\}) = P(\{obo\}) = P(\{boo\}) = \frac{180}{1331},$$
$$P(\{obb\}) = P(\{bob\}) = P(\{bbo\}) = \frac{150}{1331}, \quad P(\{bbb\}) = \frac{125}{1331}.$$

③ Let A denote the event "one ball is orange and two balls are blue". Then, we have $A = \{obb, bob, bbo\}$. Therefore, we get

$$P(A) = P(\{obb\}) + P(\{bob\}) + P(\{bbo\}) = 3 \times \frac{150}{1331} = \frac{450}{1331} \approx 0.3381.$$ $\triangle$

PROBLEM 12.

① Let d_1, d_2 be the defective systems and let n_1, n_2, n_3, n_4 be the non defective systems. An example of a possible outcome is $\{d_1, n_2\}$ which means one the systems selected is defective and the other is not. Let S be the sample space. Then there are $\binom{6}{2} = 15$ combinations of two systems out of the six. Therefore, $|S| = 15$.

② Each system are equally likely, so $P(A) = \frac{1}{15}$, where A is an atomic event.

③ Let A be the event "one of the two systems is defective". There are $\binom{2}{1} = 2$ ways of choosing the defective system and then $\binom{5}{1} = 5$ ways of choosing the second system (among the remaining ones). Therefore, $|A| = 2 \times 5 = 10$ and

$$P(A) = \frac{10}{15} = \frac{2}{3}.$$ $\triangle$

A.V Probability Space for Infinite Sample Spaces

PROBLEM 13. The event B can be interpreted in the following way: at least one toss lands heads. Let $A_n := \cup_{i=1}^{n} B_i$. It is easier to compute the probability of the complement $\overline{A}_n$. The event $\overline{A}_n$ can be interpreted as "all tosses lands tails". Since there is n tosses and each of them has a probability 1/2 of landing tails, we see that $P(\overline{A}_n) = (1/2)^n$. Therefore, $P(A_n) = 1 - (1/2)^n$. Since $A_n \subset A_{n+1}$ and $B = \cup_{n=1}^{\infty} A_n$, using the continuity of probability measures, we see that

$$P(B) = \lim_{n\to\infty} P(A_n) = \lim_{n\to\infty} 1 - (1/2)^n = 1. \qquad \triangle$$

PROBLEM 14. By definition $B_1 = A_1$ and $B_i = A_i \cap \overline{B}_{i-1}$, for $i \geq 2$. Using the property that if C and D are two subsets of a bigger set S, then $C \cap D \subset C$, we see that $B_i \subset A_i$ for every $i \geq 1$. If an outcome x is in $\cup_{i=1}^{\infty} B_i$, then it should be in at least one B_i. But $B_i \subset A_i$, so the outcome x should be in A_i. Therefore the outcome should be in $\cup_{i=1}^{\infty} A_i$.

On the other hand, if an outcome x is in $\cup_{i=1}^{\infty} A_i$, then it should be in one A_i for some $i \geq 1$. If $i = 1$, then $A_1 = B_1$ and x belongs to B_1. In this case, x belongs to $\cup_{i=1}^{\infty} B_i$. Assume $i \geq 2$. Then Either x belongs to A_{i-1} or x belongs to $A_i \cap \overline{A}_{i-1}$ because $A_{i-1} \subset A_i$. If x belongs to $A_i \cap \overline{A}_{i=1}$, then x belongs to B_i and so x belongs to $\cup_{i=1}^{\infty} B_i$ in this case. Otherwise, x belongs to A_{i-1}. If $i - 1 = 1$, then we're done because x belongs to B_1. Otherwise, split again in two cases: either x belongs to A_{i-2} or x belongs to $A_{i-1} \cap \overline{A}_{i-2}$. If x belongs to $A_{i-1} \cap \overline{A}_{i-2}$, then x belongs to B_{i-1} and therefore in $\cup_{i=1}^{\infty} B_i$. Otherwise, x belongs to A_{i-2}. If $i - 2 = 1$, then we are done because $A_{i-2} = B_1$. Otherwise, split again in two cases: either x belongs to A_{i-3} or x belongs to $A_{i-2} \cap \overline{A}_{i-3}$. This process will eventually terminate because i is a finite integer. Therefore, the outcome x will be in some B_j, for some $j \geq 1$ and this means it will belong to $\cup_{j=1}^{\infty} B_j$. $\triangle$