Introduction

We saw in class that the area of a surface S given by the parametric equation $\vec{r}(u,v)$, where $(u,v) \in D$, is given by the following formula

$$\text{Area}(S) = \iint_S dS = \iint_D |\vec{r}_u \times \vec{r}_v| \, dA,$$

where dA is the area differential adapted to D and

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix}.$$

In this worksheet, we will derive formulas for the area of specific types of surfaces.

Area of Graphs

The first type of surface we will work with is the graph of a function in two variables. This is usually given by an expression of the following form:

$$z = f(x,y)$$

with (x,y) restricted to some region D in the xy-plane. A parametrization of a surface S given by the equation $z = f(x,y)$ is

$$\vec{r}(x,y) = \langle x, y, f(x,y) \rangle,$$

where $(x,y) \in D$. Using this parametrization, we find that

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = \langle -f_x, -f_y, 1 \rangle.$$

Hence, the surface area differential takes the following form:

$$dS = |\vec{r}_x \times \vec{r}_y| \, dA = \sqrt{1 + f_x^2 + f_y^2} \, dA.$$

The formula for area of the surface S given by the equation $z = f(x,y)$, where f has continuous partial derivatives, is then

$$\text{Area}(S) = \iint_D \sqrt{1 + f_x^2 + f_y^2} \, dA. \tag{1}$$

Problem 1. Use Formula (1) to compute the area of the following surfaces.

a) $z = x^2 + y^2$, with $0 \leq x \leq 1$, $0 \leq y \leq 1$.

b) $z = \cos(x)\cos(y)$, with $0 \leq x \leq y \leq \pi$.

Area of Surfaces of Revolution

In Calculus I, you probably encountered surfaces of revolution about the x-axis, about the y-axis or maybe about any horizontal/vertical lines. The second type of surfaces we will study are the surfaces of revolution. Our focus here will be on surfaces of revolution obtained by rotating the graph $y = f(x)$ about the x-axis.

PROBLEM 2. Suppose a graph of a function is given by the equation $y = x$, where $0 \leq x \leq 1$. Write or copy/paste the following link `https://www.desmos.com/3d/689fbd2154` in your favorite web browser. Then answer the following questions in order.

a) Move the cursor controlling the parameter a. What is it doing?

b) With the parameter a set to 1, move the cursor controlling the parameter b. What is the orange circle doing?

c) The radius of the orange circle is $f(x)$. Use that to deduce the coordinates y and z of the points (x, y, z) on the surface you see in Desmos.

In general, a parametrization of the surface of revolution generated by the graph of $y = f(x)$ is

$$\vec{r}(x, \theta) = \langle x, f(x)\cos(\theta), f(x)\sin(\theta)\rangle ,$$

where $a \leq x \leq b$ and $0 \leq \theta \leq 2\pi$. From this parametrization, we find that

$$\vec{r}_x \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & f'(x)\cos(\theta) & f'(x)\sin(\theta) \\ 0 & -f'(x)\sin(\theta) & f'(x)\cos(\theta) \end{vmatrix} = \left\langle (f'(x))^2, -(f'(x))\cos(\theta), -f'(x)\sin(\theta) \right\rangle .$$

Hence,

$$dS = |\vec{r}_u \times \vec{r}_\theta|\, dA = |f'(x)|\sqrt{1 + (f'(x))^2}$$

and so

$$\text{Area}(S) = \iint_S dS = \iint_D |f'(x)|\sqrt{1 + (f'(x))^2}\, dA. \tag{2}$$

PROBLEM 3. Compute the area of the following surfaces of revolution.

a) The surface obtained by rotating the graph of $y = x$, $0 \leq x \leq 1$.

b) The surface obtained by rotating the graph of $y = \sin(x)$, $0 \leq x \leq \pi$.

c) The surface obtained by rotating the graph of $y = 1 + \cos(x)$, $0 \leq x \leq \pi$.

Here is a little challenge for you. Assume that a 2D curve C lies entirely above the x-axis and does not intersect itself. Assume the curve has a parametrization $\vec{r}(t) = \langle x(t), y(t)\rangle$, for $a \leq t \leq b$.

PROBLEM 4. Answer the following questions.

a) Find a parametrization of the surface of revolution obtained by rotating the curve C about the x-axis.

b) Modify the code from Desmos to draw the surface obtained by rotating the circle with equation $(0.5\cos(2\pi t), 1 + 0.5\sin(2\pi t))$, $0 \leq t \leq 1$. What is the surface?

c) Find an expression of the area of the surface generated.