Math 307

Chapter 1

Section 1.1: Systems of Linear Equations

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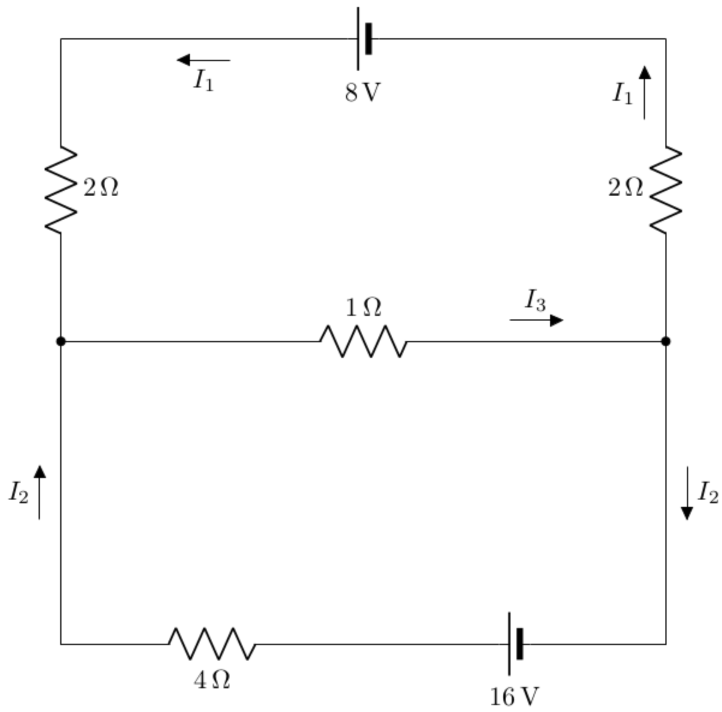

Ohm's Law

- Voltage drop at a resistor is given by $V = IR$.

Kirchhorff's Laws

- Junction: Current flowing into a junction must flow out of it.
- Path: Sum of IR terms in any direction around a closed path is equal to the total voltage in the path in that direction.

Linear Equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where

- a_1, a_2, ..., a_n are constants.
- n is the number of variables.
- x_1, x_2, ..., x_n are the variables (unknowns).
- b is the right-hand side constant term.

Systems of Linear Equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

where

- m is the number of linear equations.
- n is the number of variables.
- a_{11}, ..., a_{mn} are constants.
- b_1, b_2, ..., b_m are the right-hand side constant terms.
- x_1, ..., x_n are the variables (unknowns).

Solution of a System of Linear Equations

A list $(x_1^*, x_2^*, \ldots, x_n^*)$ is a solution to a system of linear equations if it satisfies each equation of the system.

Going back to our previous example

Systems of two linear equations with two variables

$$x_1 + x_2 = 0$$
$$2x_1 + x_2 = 1.$$

Method 1 (Isolate)

Method 2 (Operations)

Gauss-Jordan Elimination

Based on three *elementary operations* on the equations:

- Interchange two equations in the system.
- Replace an equation by a multiple of itself.
- Replace an equation by itself plus a multiple of another equation.

Main GOAL: transform our system into

$$x + 0y + 0z = \tilde{b}_1$$
$$0x + y + 0z = \tilde{b}_2$$
$$0x + 0y + z = \tilde{b}_3.$$

Example 1. Find the solution(s) to the following system of linear equations:

$$\begin{aligned} x - y + z &= 0 \\ 2x - 3y + 4z &= -2 \\ -2x - y + z &= 7. \end{aligned}$$

Augmented Matrix

More efficient way: transform the system in an **augmented matrix**.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \quad \Longrightarrow \quad \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

EXAMPLE 2. Find the augmented matrix of the system of Example 1.

Elementary operations revisited

Elementary operations on linear equations become elementary operations on the rows of the augmented matrix:

- Interchange two rows.
- Replace a row by a multiple of itself.
- Replace a row by itself plus a multiple of another row.

Example 3. Solve the system:

$$\begin{aligned} 2x + 3y - z &= 3 \\ -x - y + 3z &= 0 \\ x + 2y + 2z &= 3 \\ y + 5z &= 3. \end{aligned}$$

Example 4. Solve the system:

$$\begin{aligned} 4x_1 - 8x_2 - x_3 + x_4 + 3x_5 &= 0 \\ 5x_1 - 10x_2 - x_3 + 2x_4 + 3x_5 &= 0 \\ 3x_1 - 6x_2 - x_3 + x_4 + 2x_5 &= 0. \end{aligned}$$

Reduced row-echelon form

Transformed augmented matrix after row operations:

- Any rows of zero (called zero rows) appear at the bottom.
- The first nonzero entry of a nonzero row is 1 (called a leading 1).
- The leading 1 of a nonzero row appears to the right of the leading 1 of any preceding row.
- All the other entries of a column containing a leading 1 are zero.

Consistent Systems vs Inconsistent Systems

- Consistent: means the system of equations has at least one solution.
  - How to recognize that a system is consistent?
    (1) (2)
- Inconsistent: means the system of equations has no solution.
  - How to recognize that a system is inconsistent?
    (1)

Homogeneous System

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

- Trivial solution: $x_1 = x_2 = \cdots = x_n = 0$.

Theorem 5. A homogeneous system of m linear equations in n variables has

- infinitely many nontrivial solutions if $m < n$;
- exactly one (trivial solution) if $m = n$;
- no solution if $m > n$.

Goal. Transform the augmented matrix into an new augmented matrix with the following properties:

- any zero rows appear at the bottom.
- The first nonzero entry of a nonzero row is 1.
- The leading 1 of a nonzero row appears to the right of the leading 1 of any preceding row.

EXAMPLE 6. Determine the values of a, b, and c so that the system

$$\begin{aligned} x - y + 2z &= a \\ 2x + y - z &= b \\ x + 2y - 3z &= c \end{aligned}$$

has solutions.