

MATH 644

PROBLEM SET

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PROBLEM 1. Prove the parallelogram equality:

$$|z + w|^2 + |z - w|^2 = 2(|z|^2 + |w|^2).$$

PROBLEM 2. Let w be a non-zero complex number and let $n \geq 1$ be a positive integer. Using the polar coordinates, find n solutions to $z^n = w$.

PROBLEM 3. Let z be a non-zero complex number. Show that $0, z, iz$, and $iz + z$ are the vertices of a square.

PROBLEM 4. Prove that there is no complex number z so that

$$|z| - z = i.$$

PROBLEM 5. Find all complex numbers z satisfying the equation

$$4z - 3\bar{z} = \frac{1 - 18i}{2 - i}.$$

PROBLEM 6. Suppose that f is a continuous complex-valued function on a real interval $[a, b]$. Let

$$A = \frac{1}{b - a} \int_a^b f(x) dx.$$

- a) Show that if $|f(x)| \leq |A|$ for all $x \in [a, b]$, then $f \equiv A$.
- b) Show that if $|A| = \frac{1}{b-a} \int_a^b |f(x)| dx$, then $\arg f$ is constant modulo 2π on $\{z : f(z) \neq 0\}$.

PROBLEM 7. Describe geometrically the following subsets:

- a) $\operatorname{Re} z = \operatorname{Im} z$.
- b) $\operatorname{Re} z > 0$.
- c) $\operatorname{Im} z > 0$.
- d) $\frac{\pi}{6} < \arg z < \frac{\pi}{4}$.

PROBLEM 8. Let $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. Prove that \mathbb{T} equipped with the complex multiplication is a commutative group.

PROBLEM 9. Suppose that $\lim_{n \rightarrow \infty} w_n = w$. Is it true then that also

$$\lim_{n \rightarrow \infty} \arg w_n = \arg w?$$

PROBLEM 10. Let $\{z_n\}$ be a sequence of complex numbers such that $\sum_{n=0}^{\infty} z_n$ converges and there is a ϕ such that $|\arg z_n| \leq \phi < \frac{\pi}{2}$ for any $n \geq 0$. Show that the series $\sum_{n=0}^{\infty} z_n$ is absolutely convergent.

PROBLEM 11. Let \mathbb{C}^* be the extended plane, let \mathbb{S}^2 be the sphere $\{(X, Y, Z) : X^2 + Y^2 + Z^2 = 1\}$ and let $\pi : \mathbb{C}^* \rightarrow \mathbb{S}^2$ be the stereographic projection with $\pi(\infty) = (0, 0, 1)$.

- a) Show that straight lines in \mathbb{C} correspond exactly to circles on \mathbb{S}^2 passing through $(0, 0, 1)$.
- b) Show that if $z \neq \infty$, then

$$\chi(z, \infty) = \frac{2}{\sqrt{1 + |z|^2}}.$$

- c) Using the explicit formula of χ in terms of z and w , show that, for any $z, w \in \mathbb{C}^*$,

$$0 \leq \chi(z, w) \leq 2.$$

PROBLEM 12. For what values of z is

$$\sum_{n=0}^{\infty} \left(\frac{z}{1+z} \right)^n$$

convergent? Draw a picture of the region.

PROBLEM 13. Suppose that $\sum_{n \geq 0} a_n (z - z_0)^n$ is a formal power series. Suppose that

$$R := \frac{1}{\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|}$$

exists and is finite.

- a) Show that the power series converges in $\{z : |z - z_0| < R\}$.
- b) Show that the power series diverges in $\{z : |z - z_0| > R\}$.

PROBLEM 14. Define $e^z = \exp(z) := \sum_{n \geq 0} \frac{z^n}{n!}$.

- a) Show that $e^z e^w = e^{z+w}$ (using the power series definition).
- b) Show that $|e^z| = e^{\operatorname{Re} z}$ and $\arg e^z = \operatorname{Im} z$.
- c) Show that $\frac{d}{dz} e^z = e^z$.
- d) Show that, for any non-zero integer n ,

$$\int_0^{2\pi} e^{int} dt = 0.$$

[Hint: Use Fundamental Theorem of Calculus.]

- e) Compute the integral

$$\int e^{nt} \cos(mt) dt.$$

[Hint: Rewrite $\cos(mt)$ as a complex exponential.]

PROBLEM 15. Prove the following assertions.

- a) If f and g are analytic at z_0 , then $(f + g)'(z_0) = f'(z_0) + g'(z_0)$ (Sum rule of differentiation for analytic functions).
- b) If f and g are analytic at z_0 , then $(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$ (Product Rule of differentiation for analytic functions).
- c) If f and g are analytic at z_0 with $g(z_0) \neq 0$, then $(f/g)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{(g(z_0))^2}$ (Quotient rule of differentiation for analytic functions).

- d) If f is analytic at z_0 and g is analytic at $f(z_0)$, then $(g \circ f)'(z_0) = g'(f(z_0))f'(z_0)$ (Chain Rule of differentiation for analytic functions).

Find the derivative of $(z - a)^{-n}$, where n is a positive integer and $a \in \mathbb{C}$.

PROBLEM 16. Let $\Omega \subset \mathbb{C}$. Show that Ω is connected if and only if Ω and \emptyset are the only open and closed subsets of Ω .

PROBLEM 17. Suppose that f and g are two analytic functions on a region (open and connected) Ω . Suppose there is a sequence $(z_n)_{n=1}^{\infty}$ with $z_n \in \Omega$ ($\forall n \geq 1$) such that $f(z_n) = g(z_n)$ ($\forall n \geq 1$). If (z_n) has an accumulation point $z_0 \in \Omega$, then show that $f \equiv g$ on Ω .

PROBLEM 18. Show that $\cos^2(z) + \sin^2(z) = 1$ for every $z \in \mathbb{C}$.

PROBLEM 19. Suppose f is analytic in a connected open set Ω such that, for each $z \in \Omega$, there exists an n (depending on z) such that $f^{(n)}(z) = 0$. Prove that f is a polynomial. [Hint: Use Baire's Theorem.]

PROBLEM 20. Let f be analytic in a region Ω containing the point $z = 0$. Suppose $|f(1/n)| < e^{-n}$ for $n \geq n_0$, for some integer $n_0 \geq 0$. Prove $f \equiv 0$ in Ω .

PROBLEM 21. Let f and g be analytic functions in a region Ω .

- a) Show that if $f'(z) = 0$ for all z in a neighborhood of some $z_0 \in \Omega$, then f is constant in Ω , meaning there is a constant $c \in \mathbb{C}$ such that $f(z) = c$ for any $z \in \Omega$.
- b) Show that if f and g are analytic in a region Ω with $f'(z) = g'(z)$ for every $z \in \Omega$, then $f - g$ is constant.

PROBLEM 22. Suppose that $f(z) = az^3 + bz^2 + cz + d$. In addition, suppose that for each $z, w \in \mathbb{C}$ there exists a point ζ on the line segment between z and w with

$$\frac{f(z) - f(w)}{z - w} = f'(\zeta).$$

Show that $a = 0$.

PROBLEM 23. Suppose that $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges in $B = \{z : |z - z_0| < r\}$. Show that the power series

$$F(z) := \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z - z_0)^{n+1}$$

converges in B and satisfies $F'(z) = f(z)$ for all $z \in B$. Moreover, show that the radius of convergence of F is the same as the radius of convergence of f .

PROBLEM 24. Suppose $\sum_{j=0}^{\infty} |a_j|^2 < \infty$.

- a) Show that $f(z) = \sum_{j=0}^{\infty} a_j z^j$ is analytic in $\{z : |z| < 1\}$.

b) Compute (with a proof) the following quantity:

$$\lim_{r \rightarrow 1^-} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi}.$$

PROBLEM 25. Suppose f has a power series expansion at 0 which converges in all of \mathbb{C} . Suppose also that $\int_{\mathbb{C}} |f(x + iy)| dx dy < \infty$. Prove that $f \equiv 0$.

PROBLEM 26. [Hard] Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ have radius of convergence 1 and suppose that $a_n \geq 0$ for all n . Prove that $z = 1$ is a singular point of f . That is, there is no function g analytic in a ball B containing $z = 1$ such that $f = g$ on $B \cap D$.

PROBLEM 27. If f is analytic in a region Ω and if there is a $z_0 \in \Omega$ such that

$$|f(z_0)| = \inf_{z \in \Omega} |f(z)|,$$

and if $f(z_0) \neq 0$, then f is constant in Ω .

PROBLEM 28. Let Ω be a region in \mathbb{C} . Show that if $f : \Omega \rightarrow \mathbb{C}$ is an open map, then f satisfies the maximum modulus principle.

PROBLEM 29.

- a) Show geometrically why the maximum principle holds using a “walking the dog” argument. Make it rigorous by following the steps of the proof of the Fundamental Theorem of Algebra.
- b) Use the maximum modulus principle to prove the Fundamental Theorem of Algebra.

PROBLEM 30. Suppose that f is analytic in a connected open (region) set Ω .

- a) Prove that if $|f(z)|$ is constant on Ω , then f is constant on Ω .
- b) Prove that if $\operatorname{Re} f$ is constant on Ω , then f is constant on Ω .

PROBLEM 31. Suppose that f and g are analytic in \mathbb{C} with $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$. Prove that there exists a constant $c \in \mathbb{C}$ such that $f(z) = cg(z)$ for all $z \in \mathbb{C}$.

PROBLEM 32. Prove that if f is non-constant and analytic on all of \mathbb{C} , then $f(\mathbb{C})$ is dense in \mathbb{C} .

PROBLEM 33. Let f be analytic in \mathbb{D} and suppose $|f(z)| < 1$ on \mathbb{D} . Let $a = f(0)$. Show that f does not vanish in $\{z : |z| < |a|\}$.

PROBLEM 34.

- a) Prove that φ is a one-to-one analytic map of \mathbb{D} onto \mathbb{D} if and only if

$$\varphi(z) = c \left(\frac{z - a}{1 - \bar{a}z} \right) \quad (z \in \mathbb{D}),$$

for some constants c and a , with $|c| = 1$ and $|a| < 1$.

- b) Let f be analytic in \mathbb{D} and satisfy $|f(z)| \rightarrow 1$ as $|z| \rightarrow 1$. Prove that f is rational.

PROBLEM 35.

- a) Suppose p is a non-constant polynomial with all its zeros in the upper half-plane $\mathbb{H} := \{z : \operatorname{Im} z > 0\}$. Prove that all the zeros of p' are contained in \mathbb{H} . [*Hint: Look at the partial fraction expansion of p'/p .*]

- b) Use **a)** to prove that if p is a polynomial, then the zeros of p' are contained in the (closed) convex hull of the zeros of p . (The closed convex hull is the intersection of all half-planes containing the zeros.)

PROBLEM 36. Suppose f is analytic in \mathbb{D} and $|f(z)| \leq 1$ in \mathbb{D} and $f(0) = 1/2$. Prove that $|f(1/3)| \geq 1/5$.

PROBLEM 37. Suppose f is analytic and non-constant in \mathbb{D} and $|f(z)| \leq M$ on \mathbb{D} . Prove that the number of zeros of f in a disk of radius $1/4$, centered at 0, does not exceed

$$\frac{1}{\ln 4} \ln \left| \frac{M}{f(0)} \right|.$$