Problem 6 — 20 points

a) If $A = 0$, then, from the assumption, $f = 0$. Suppose $A \neq 0$ and let $\theta = \arg A$. Then we have $|A| = Ae^{-i\theta}$ and therefore

$$|A| = \frac{1}{b-a} \int_a^b e^{i\theta} f(x)\, dx.$$

Passing to the real part on the right-hand side and since $|e^{i\theta} f(x)| = |f(x)| \leq |A|$, we conclude that $|A| - \mathrm{Re}(e^{-i\theta} f) = 0$ on $[a, b]$. Now, if there is some x such that $b = \mathrm{Im}(e^{-i\theta} f(x)) \neq 0$, then

$$|f(x)| = |e^{-i\theta} f(x)| = \sqrt{|A|^2 + b^2} > |A|$$

which contradicts the assumption. Therefore, $\mathrm{Im}(e^{-i\theta} f) = 0$ on $[a, b]$. We then conclude that $|A| e^{i\theta} = f$ on $[a, b]$.

b) If $|A| = 0$, then $f = 0$ on $[a, b]$. In this case, the argument of f does not make sense. So we let $A \neq 0$. Again, write $|A| = Ae^{-i\theta}$ for some θ. Write $E = \cup_{n=1}^{\infty} (a_n, b_n)$ where $a_n < b_n < a_{n+1}$ for any $n \geq 1$ and $E := \{x \in [a, b] \,:\, f(x) \neq 0\}$. We then have

$$\int_E |f(x)|\, dx = |A| = \int_E |f(x)| e^{i(\arg f(x) - \theta)}\, dx$$

Note that $e^{i \arg f(x)} = \frac{f(x)}{|f(x)|}$ which is continuous on E. Taking the real part on each side, we then find

$$\int_E |f(x)| \Big(1 - \cos(\arg f(x) - \theta) \Big)\, dx = 0.$$

Since $|f(x)|(1 - \cos(\arg f(x) - \theta))$ is non-negative, we can rewrite the equation as

$$\sum_{n=1}^{\infty} \int_{a_n}^{b_n} |f(x)| \Big(1 - \cos(\arg f(x) - \theta) \Big)\, dx = 0.$$

Therefore, $1 - \cos(\arg f(x) - \theta) = 0$ in (a_n, b_n). We then conclude that $\arg f(x) = \theta + 2k\pi$ for some $k \in \mathbb{Z}$ for all $x \in (a_n, b_n)$ and so $\arg f = \theta \,(\mathrm{mod}\, 2\pi)$ on E.

Problem 9 — 5 points

First example. If we do not require that $w \neq 0$, then an easy example is $w_n = \frac{i^n}{n}$.

Second example. If we add the assumption that $w \neq 0$, then we can use $w_n = -1 - \frac{i}{n}$. The argument of w_n is $\arctan(1/n) - \pi$ and $w_n \to -1$. However, $\arg w_n = \arctan(1/n) - \pi \to -\pi \neq \arg(-1)$. According to our convention, $\arg(-1) = \pi$.

Problem 10 — 10 points

First solution: From section 1.2, the series $\sum_{n=1}^{\infty} z_n$ converges absolutely if $\sum_{n=1}^{\infty} \mathrm{Re} z_n$ and $\sum_{n=1}^{\infty} \mathrm{Im} z_n$ converges absolutely. Since $|\arg z_n| < \pi/2$, we observe that $\mathrm{Re} z_n \geq 0$ and therefore $\sum_{n=1}^{\infty} \mathrm{Re} z_n$ converges absolutely, from the assumption. We may suppose, without loss of generality, that $z_n \neq 0$ for any $n \geq 1$. This forces $\mathrm{Re} z_n > 0$. Therefore, from the definition of the argument:

$$\left|\frac{\mathrm{Im} z_n}{\mathrm{Re} z_n}\right| = |\tan(\arg z_n)| \leq |\tan \phi|$$

and we deduce the following inequality $|\mathrm{Im} z_n| \leq |\tan \phi| \mathrm{Re} z_n$. By a comparison test for series, $\sum_{n \geq 1} \mathrm{Im} z_n$ converges absolutely and this concludes the proof.

Second solution: Let $\theta_n := \arg z_n$. We write $|z_n| \cos \theta_n = \mathrm{Re} z_n$. Then, we get $|z_n| = \frac{\mathrm{Re} z_n}{\cos \theta_n}$. Using the fact that $|\theta_n| \leq \phi$ and the fact that $x \mapsto \cos x$ is positive on $[-\phi, \phi]$, we see that

$$0 \leq |z_n| \leq \frac{\mathrm{Re} z_n}{\cos \theta_n} \leq \frac{\mathrm{Re} z_n}{\cos \phi}.$$

Therefore, by a comparison test for series, since $\sum_{n \geq 1} \mathrm{Re} z_n$ converges, we see that $\sum_{n \geq 1} |z_n|$ converges.

Problem 11 — 15 points

a) Let C be a circle on the sphere passing through $(0,0,1)$. Then, we have a plane $P \subset \mathbb{R}^3$ with equation $AX + BY + CZ = D$ such that $C = P \cap \mathbb{S}^2$. Since $(0,0,1) \in P$, we have $C = D$.

Let $z^* := (x_1, x_2, x_3) \in \mathbb{S}^2$. Using the Stereographic projection, the fact that $Ax_1 + Bx_2 + Cx_3 = C$, we must have

$$2Ax + 2Ay = 2C$$

where $\pi(x + iy) = z^*$. This is the equation of a line in $\mathbb{C}$.

Given an equation of a line in $\mathbb{C}$,

$$ax + by = c$$

we then have, using $(x_1, x_2, x_3) = \pi^{-1}(x + iy)$,

$$ax_1 + bx_2 + cx_3 = c.$$

This is an equation of a plane P passing through $(0,0,1)$ and therefore $P \cap \mathbb{S}^2$ traces a circle on $\mathbb{S}^2$ passing through the North Pole.

b) Suppose $w = \infty$. This means that $w^* = (0,0,1)$. Let $z = x = iy$ and let $z^* = \pi(z)$. Then we have

$$\chi(z, \infty) = \|z^* - (0,0,1)\|_{\mathbb{R}^3} = \sqrt{\frac{4|z|^2 + 4}{(1 + |z|^2)^2}} = \frac{2}{\sqrt{1 + |z|^2}}.$$

c) Suppose $z \neq \infty$, $w \neq \infty$ and $z \neq w$ (otherwise it is obvious from the formula). We have

$$|z-w|^2 = |z|^2 + |w|^2 - 2\mathrm{Re}(z\overline{w}) \leq |z|^2 + |w|^2 + 2|z||w|.$$

Therefore,

$$\chi(z,w)^2 \leq 4\frac{(|z|+|w|)^2}{(1+|z|^2)(1+|w|^2)} \leq \frac{(|z|^2+1)(1+|w|^2)}{(1+|z|^2)(1+|w|^2)} = 4,$$

where the last inequality comes from the Cauchy-Schwarz inequality.

Problem 16 — 5 points

Suppose that Ω is connected and let $A \subset \Omega$ satisfies

- $A \neq \varnothing$ and $A \neq \Omega$;
- A is open and closed;
- $\Omega = A \cup (\Omega \backslash A)$.

Since A is closed, $\Omega \backslash A$ is open. We therefore have a contradiction with the definition of connectedness.

Suppose that the only subsets of Ω that are open and closed are $\varnothing$ and Ω. If Ω was not connected, then there would be two open sets A and B different from $\varnothing$ and Ω such that $A \cup B = \Omega$ and $A \neq B$. Since $A = \Omega \backslash B$ and B is open, we conclude that A is closed and open. This is a contradiction with our assumption. $\square$

Problem 17 — 10 points

From the hypothesis, there are a $z_0 \in \Omega$ and a subsequence (z_{n_k}) such that $z_{n_k} \to z_0$ and $z_{n_k} \neq z_0$, for any k. Therefore, by continuity of f and g, z_{n_k} and z_0 are zeros of the function $h = f - g$. Since h is analytic in Ω and has a non-isolated zero in Ω, from Corollary 5 in section 2.4, we have $h = 0$ in Ω. Therefore, $f = g$ in Ω. $\square$

Problem 18 — 5 points

We know that $\cos^2 x + \sin^2 x = 1$ for any $x \in \mathbb{R}$. Taking $x_n = 1/n$ which converges to $0 \in \mathbb{R}$ and letting $f(z) = \cos^2(z) + \sin^2(z)$ and $g(z) = 1$ in Problem 17, we conclude $f(z) = g(z)$ for any $z \in \mathbb{C}$. $\square$

Problem 19 — 10 points

Given $n \geq 0$, let $A_n := \{z \in \Omega : f^{(n)}(z) = 0\}$. From the hypothesis, $\Omega = \cup_{n \geq 0} A_n$. Using Baire's Theorem, we can find an N such that $\mathrm{int}(A_N) \neq \varnothing$. Let $z_0 \in \mathrm{int}(A_N)$. There is an $r > 0$ such that $D_r := \{z : |z - z_0| < r\} \subset A_N$. This means that $f^{(N)} = 0$ in D_r. This also means that $f^{(N+k)} = 0$ in D_r, for any $k \geq 0$. If $N = 0$, then $f = 0$ in D_r and by the Identity Principle, $f = 0$ in Ω. Suppose from now on that $N \geq 1$.

The function f is analytic at z_0, so there is a $\rho > 0$ sufficiently small ($\rho < r$) such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad (z \in D_\rho).$$

From Corollary 6 in section 2.5, we have $a_n = \frac{f^{(n)}(z_0)}{n!}$. Therefore, $a_k = 0$ for any $k \geq N$ and the power series of f becomes

$$f(z) = \sum_{n=0}^{N-1} a_n (z - z_0)^n \quad (z \in D_\rho).$$

The polynomial on the right-hand side of the last equality is analytic in $\mathbb{C}$, so in particular in Ω. From the Identity Principle, we conclude that

$$f(z) = \sum_{n=0}^{N-1} a_n (z - z_0)^n \quad (z \in \Omega).$$

This concludes the proof. $\square$

Problem 20 — 10 points

Write $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in some disk $D_r := \{z : |z| < r\}$. By continuity, we have

$$|f(0)| = \lim_{n\to\infty} |f(1/n)| \leq \lim_{n\to\infty} e^{-n} = 0.$$

Therefore, $f(0) = a_0 = 0$. This implies that $f(z) = \sum_{n\geq 1} a_n z^n$.

Now we have

$$|a_1| = |f'(0)| = \lim_{n\to\infty} \left| \frac{f(1/n) - f(0)}{1/n} \right| = \lim_{n\to\infty} n|f(1/n)|$$

and therefore $|a_1| \leq \lim_{n\to\infty} ne^{-n} = 0$. So, $f(z) = \sum_{n\geq 2} a_n z^n$.

We will proceed by induction. Suppose that $a_0 = a_1 = \ldots = a_k = 0$ for some $k \geq 1$. Given z such that $|z| < r$ and $z \neq 0$, we have

$$\frac{f(z)}{z^{k+1}} = \sum_{m=k+1}^{\infty} a_m z^{m-k-1}.$$

By the root test, the power series on the right-hand side is uniformly and absolutely convergent in some disk around the origin. In particular, it is continuous at $z = 0$. Therefore, with $z = 1/n$ and letting $n \to \infty$, we have

$$\lim_{n\to\infty} n^{k+1} f(1/n) = a_{k+1}.$$

Since $\lim_{n\to\infty} n^{k+1} e^{-n} = 0$, we conclude that $a_{k+1} = 0$.

Therefore, $a_n = 0$ for any $n \geq 0$ and $f = 0$ in D_r. By the identity principle, $f = 0$ in Ω. $\square$

Problem 23 — 10 points

Set $b_0 = 0$ and $b_n = a_{n-1}/n$ for $n \geq 1$. Then, $F(z) = \sum_{n\geq 0} b_n (z - z_0)^n$. We use the root test to show that $F(z)$ converges in $\{z : |z - z_0| < R\}$. We have

$$\liminf_{n\to\infty} |b_n|^{-1/n} = \liminf_{n\to\infty} \frac{|a_{n-1}|^{-1/n}}{n^{-1/n}} = \liminf_{n\to\infty} |a_{n-1}|^{-1/n},$$

where the last equality comes from $\lim_{n\to\infty} n^{1/n} = 1$. Let $0 < s < R$. Then we know that $|a_{n-1}| < s^{-(n-1)}$ for some $n \geq n_0$ and therefore

$$\liminf_{n\to\infty} |a_{n-1}|^{-1/n} \geq \lim_{n\to\infty} s^{(n-1)/n} = s.$$

We then have $\liminf |b_n|^{-1/n} \geq s$ for any $0 < s < R$ and therefore the radius of convergence is at least R, the same as the power series $f(z)$. Now, if the radius of convergence of F, say R_F, would be bigger than R, then we could find a $z \in \mathbb{C}$ such that $R < |z| < R_F$ and

$$F(z) = \sum_{n\geq 0} \frac{a_n}{n+1}(z-z_0)^{n+1}.$$

However, we know that, from section 2.5, $F'(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ which would be convergent. By the properties of the radius of convergent, this is impossible (see section 2.4). Therefore, we must have $R_F = R$.

From Theorem 5 in section 2.5, we know that $F'(z)$ exists for any $z \in \mathbb{C}$ such that $|z - z_0| < R$ and

$$F'(z) = \sum_{n=1}^{\infty} n b_n z^{n-1} = \sum_{n=1}^{\infty} a_{n-1} z^{n-1} = f(z).$$

This completes the proof. $\square$

Problem 24 — 10 points

a) For $z \in \mathbb{D}$, we have

$$\sum_{n=0}^{\infty} |a_n| r^n \leq \left(\sum_{n=0}^{\infty} |a_n|^2\right)^{1/2} \left(\sum_{n=0}^{\infty} (|z|^2)^n\right)^{1/2},$$

by the Cauchy-Schwarz inequality for series. The right-hand side of the last inequality is finite from the assumption and from the fact that $|z| < 1$. Therefore the function f has a power series expansion at 0 converging on the whole of $\mathbb{D}$. From Theorem 3 in section 2.4, the function f is analytic in $\mathbb{D}$.

b) In part a), letting $z = re^{i\theta}$ with $0 < r < 1$ and $\theta \in (-\pi, \pi]$, we have

$$f(re^{i\theta}) = \sum_{n\geq 0} a_n r^n e^{in\theta}.$$

Therefore, by the absolute converge of the power series, we have

$$|f(re^{i\theta})|^2 = \sum_{k=0}^{\infty} |a_k|^2 r^{2n} + \sum_{j=0}^{\infty} \sum_{\substack{m=0 \\ m\neq j}}^{\infty} a_j \overline{a_m} r^{j+m} e^{i(j-m)\theta}.$$

Now, by the uniform convergence of the power series on disks $D_r = \{z : |z| \leq r\} \subset \mathbb{D}$, we find

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^2 \, d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n} + \sum_{j=0}^{\infty} \sum_{\substack{m=0 \\ m\neq j}}^{\infty} a_j \overline{a}_m r^{j+m} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(j-m)\theta} \, d\theta\right)$$

The integral of $e^{i(j-m)\theta}$, for $j \neq m$, is 0 and therefore

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^2\, d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}.$$

As $r \to 1^-$, we get

$$\lim_{r\to 1^-} \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^2\, d\theta = \sum_{n=0}^{\infty} |a_n|^2.$$

This completes the proof. $\square$

Problem 27 — 5 points

Since $f(z_0) \neq 0$ and this attains the infimum, then $|f| > 0$ in Ω. Therefore,

$$\frac{1}{|f(z_0)|} = \sup_{z \in \Omega} \frac{1}{|f(z)|}.$$

By the Maximum Principle, $1/f = c$ in Ω, for some $c \in \mathbb{C} \backslash \{0\}$. Therefore, $f = 1/c$ in Ω. $\square$

Problem 30 — 5 points

Notice first, since Ω is a bounded region, each set $\{z \in \Omega \,:\, \mathrm{dist}(z, \partial\Omega) = \delta\}$ is compact (and non-empty if δ is sufficiently small). Also the limit on the RHS exists (can be infinite) by the maximum principle.

Let $z_n \to \partial\Omega$ as $n \to \infty$. Then, z_n is inside some $\Gamma_{\delta_n} := \{z \in \Omega \,:\, d(z, \partial\Omega) = \delta_n\}$. Therefore, from the second version fo the maximum principle, we have

$$|f(z_n)| \leq \sup \Big\{ |f(z)| \,:\, z \in \Gamma_{\delta_n} \Big\}.$$

This implies that

$$\limsup_{n \to \infty} |f(z_n)| \leq \limsup_{n \to \infty} \Big(\sup \Big\{ |f(z)| \,:\, z \in \Gamma_{\delta_n} \Big\} \Big).$$

The limit on the right is simply the RHS of the equation in the statement of the problem. Since this is true for any sequence $z_n \to \partial\Omega$, we conclude that

$$\limsup_{z \to \partial\Omega} |f(z)| \leq \lim_{\delta \to 0} \sup \Big\{ |f(z)| \,:\, d(z, \partial\Omega) = \delta \Big\}.$$

To obtain the other inequality, observe that, since $|f|$ is continuous, then for any sequence $\delta_n \to 0$ $(n \to \infty)$, there is a sequence of distinct points z_n such that $d(z_n, \partial\Omega) = \delta_n$ and

$$\sup \Big\{ |f(z)| \,:\, d(z, \partial\Omega) = \delta_n \Big\} = |f(z_n)|.$$

As $n \to \infty$, we see that $z_n \to \partial\Omega$ and therefore

$$\lim_{n \to \infty} |f(z_n)| = \limsup_{n \to \infty} |f(z_n)| \leq \limsup_{z \to \partial\Omega} |f(z)|.$$

Since this is true for any sequence $\delta_n \to 0$, we then obtain

$$\lim_{\delta \to 0} \sup \Big\{ |f(z)| \,:\, d(z, \partial\Omega) = \delta \Big\} \leq \limsup_{z \to \partial\Omega} |f(z)|$$

and this completes the proof. $\square$

Problem 31 — 5 points

Suppose that $(z_n)_{n\geq 1}$ has a subsequence $(z_{n_k})_{k\geq 1}$ such that $z_{n_k} \to z_0 \in \Omega$. Then $z_{n_k} \in K$ for some compact set $K \subset \Omega$. Therefore, there is a compact set $K \subset \Omega$ containing infinitely many of z_n. Therefore $z_n \not\to \partial\Omega$ as $n \to \infty$.

Now, if $z_n \not\to \partial\Omega$ as $n \to \infty$, then there is a compact set K containing infinitely many z_n. By the Bolzano-Weierstrass Theorem, there is a subsequence z_{n_k} such that $z_{n_k} \to z_0$. Therefore, the sequence (z_n) has a subsequence converging to some point in Ω. $\square$

Problem 32 — 5 points

a) Suppose $|f(z)| = c \in [0, \infty)$ for any $z \in \Omega$. Then, there is at least one $z_0 \in \Omega$ such that $|f(z_0)| = \sup_{z\in\Omega} |f(z)|$. By the Maximum Principle, the function f should be constant in Ω. $\square$

b) **A first solution.** Suppose $\mathrm{Re} f = c \in \mathbb{R}$ in Ω. The function e^f is analytic in Ω because e^z is analytic (from section 2.6) and f is analytic. We know that $|e^{f(z)}| = |e^{\mathrm{Re} f}| = e^c$ for any $z \in \Omega$. Therefore, there is a $z_0 \in \Omega$ such that $|e^{f(z_0)}| = \sup_{z\in\Omega} |e^{f(z)}|$. By the Maximum Principle, we have $e^f = a$ for some $a \in \mathbb{C}\backslash\{0\}$. Since $(e^z)' = e^z \neq 0$, then e^z is locally a homeomorphism. Denote by g its inverse on some $B_r = \{z : |z - z_0| < r\}$ for some $z_0 \in \Omega$ fixed. Then, applying the function g on the equation $e^f = a$, we find that $f = g(a)$ and therefore f is constant in B_r. By the Identity Principle, f is constant in Ω.

A second solution (from one of your classmates). If f would not be constant, then f is an open map. This means that $f(B)$ should be open for any open ball $B \subset \Omega$. But, with $\mathrm{Re} f \equiv c$ for some $c \in \mathbb{C}$, the image of $f(B)$ is the set $\{k + i\mathrm{Im}(f(z)) : z \in B\}$ which is a closed set. This is a contradiction and therefore f is constant. $\square$

Problem 33 — 15 points

a) We will see in Chapter 4 how to obtain this result much more easily. Honestly, I was impressed by all the creativity you've deployed for this problem. Good job everyone.

b) Write $f(z) = \sum_{k\geq 0} a_k z^k$ for any $z \in \mathbb{C}$ such that $|z| < R$ and set $p(z) = \sum_{k=0}^{n} a_k z^k$. Therefore, the function $g(z) = \frac{f(z)-p(z)}{z^n}$ is analytic in $\mathbb{C}$. First, we see that

$$|p(z)| \leq D|z|^n$$

where $|z| > 1$ and $D := n \max_{0\leq k\leq n} |a_k|$. Therefore, from the assumption, we have, for $|z| > L := \max\{M, 1\}$,

$$|g(z)| \leq \frac{C|z|^n + D|z|^n}{|z|^n} = C + D < \infty.$$

From the continuity of g on $\{z : |z| \leq L\}$, the function g is bounded on $\mathbb{C}$. By Liouville's Theorem, we conclude that g is constant, say $g = c$ in $\mathbb{C}$. From the definition of g, we see that

$$f(z) = p(z) + cz^n,$$

which is a polynomial. $\square$

c) If g would be zero, then f is zero and the conclusion is satisfied. If f is zero, then $c = 0$ does the job. So, from now on, suppose that f and g are not identically zero.

Let a be a zero of g. Then, $f(a) = 0$ from the assumption. Therefore, every zero of g is a zero of f and $\mathrm{Ord}_f(a) \geq \mathrm{Ord}_g(a)$, where $\mathrm{Ord}_h(a)$ is the order of the zero a of a function h. For each zero $a \in \mathbb{C}$ of g, there is a disk B_r centered at a of radius $r > 0$ such that $g \neq 0$ on $B_r \backslash \{a\}$. Using power series, we can write

$$f(z) = (z-a)^{n_f} h(z) \quad \text{and} \quad g(z) = (z-a)^{n_g} k(z)$$

where $h(z) \neq 0$, $k(z) \neq 0$ for z is some disk $B_\rho(a)$ and are analytic in $B_\rho(a)$ and n_f, n_g are the order of the zero a of f and g respectively. Therefore, we have

$$f(z)/g(z) = (z-a)^{n_f - n_g} \frac{h(z)}{k(z)} \quad (z \in B_\rho(a) \backslash \{a\}).$$

We then see that f/g is holomorphic in $B_\rho(a)$.

Repeat this procedure for every zero of g and you obtain $h = f/g$ is analytic in $\mathbb{C}$. From the assumption, we have $|h| \leq 1$ in $\mathbb{C}$. Therefore, the function h is bounded and, by Liouville's Theorme, $h = c$ for some constant $c \in \mathbb{C}$. We then conclude that $f = cg$ in $\mathbb{C}$. $\square$

Problem 34 — 5 points

If $f(\mathbb{C})$ would not be dense, then there is a $w \in \mathbb{C}$ and a disk B_r centered at w of radius $r > 0$ such that $\overline{B}_r \cap \overline{f(\mathbb{C})} = \varnothing$. Therefore, the function $g(z) = \frac{1}{f(z)-w}$ is analytic in $\mathbb{C}$ with $g \neq 0$, because $|f(z) - w| > r$ for every $z \in \mathbb{C}$. However, for any z, we have

$$|g(z)| \leq \frac{1}{r}.$$

By Liouville's Theorem, $g = c$ for some $c \in \mathbb{C}\backslash\{0\}$. We then found that f is constant, contradicting the assumption that f is non-constant.

Problem 36 — 10 points

a) Let $a \in \mathbb{D}$ such that $\varphi(a) = 0$. The number a is the only zero of φ and therefore, from Theorem 4 in Section 3.3, we have

$$\varphi(z) = \frac{z-a}{1-\bar{a}z} g(z),$$

for some analytic function g in $\mathbb{D}$ and $|g| \leq 1$ in $\mathbb{D}$. Now, if $g(a) = 0$, then apply Theorem 4 in Section 3.3 to $g(z)$ again. After n times, where n is the order of the zero a of φ, we can then write

$$\varphi(z) = \left(\frac{z-a}{1-\bar{a}z}\right)^n h(z)$$

If $h(z) = 0$ for some $z \neq a$, then $\varphi(z) = 0$ and so φ would not be one-to-one. Therefore, $h(z) \neq 0$ for every $z \neq a$. Using the power series of φ centered at $z = a$ and dividing through by $(z-a)^n$, we also have that $h(a) \neq 0$. Therefore, $h \neq 0$ in $\mathbb{D}$.

Since $\varphi(\mathbb{D}) = \mathbb{D}$, then $\sup_{z\in\mathbb{D}} |\varphi(z)| = 1$. The same goes for $z \mapsto \frac{z-a}{1-\overline{a}z}$ by the maximum principle (see the proof of Theorem 3 in Section 3.3). Therefore, we have $\sup_{z\in\mathbb{D}} |h(z)| = 1$. We also have that $\frac{1}{h}$ is analytic in $\mathbb{D}$ and

$$\limsup_{|z|\to 1} \frac{1}{|h(z)|} = \limsup_{|z|\to 1} \frac{|\varphi(z)|}{\left(\frac{z-a}{1-\overline{a}z}\right)^n} = 1.$$

From the third version of the maximum modulus principle, $\sup_{z\in\mathbb{D}} 1/|h(z)| = 1$. We then conclude that

$$\inf_{z\in\mathbb{D}} |h(z)| = \sup_{z\in\mathbb{D}} |h(z)| = 1.$$

This implies that $|h(z)| = 1$ for every $z \in \mathbb{D}$. From Problem 30(a), we have $h = c$, for some constant $c \in \mathbb{C}$. Since $|h| = 1$ in $\mathbb{D}$, then $|c| = 1$ and therefore

$$\varphi(z) = c\left(\frac{z-a}{1-\overline{a}z}\right)^n \quad (z \in \mathbb{D}).$$

If $w \in \mathbb{D}$, then there are n distinct complex numbers z such that $\varphi(z) = w$. By assumption, φ should be injective and therefore $n = 1$.

Now suppose you are given the formula of φ. Write φ as

$$\varphi(z) = -c\left(\frac{a-z}{1-\overline{a}z}\right).$$

Notice that, with $\psi(z) = \frac{a-z}{1-\overline{a}z}$, we have $\psi(\psi(z)) = z$. Therefore, ψ is invertible with $\psi^{-1} = \psi$. Moreover, ψ is analytic on $\overline{\mathbb{D}}$ with $|\psi(z)| = 1$ on $\{z : |z| = 1\}$. Therefore, by the Maximum Principle, $\psi : \mathbb{D} \to \mathbb{D}$.

Also, the map $z \mapsto r(z) = -cz$ is a bijective analytic map from $\mathbb{D}$ onto $\mathbb{D}$ with inverse $z \mapsto r^{-1}(z) = -\overline{c}z$. We then see that $\varphi = r \circ \psi(z)$ is the composition of two bijective maps, meaning that φ is also an analytic bijective map. $\square$

b) If the set of zeros of f has an accumulation point in $\mathbb{D}$, then $f = 0$ in $\mathbb{D}$ by the Identity Principle and f is rational in this case.

If there is a sequence (z_n) of zeros of f such that $|z_n| \to 1$, then $|f(z_n)| \to 0$. This contradicts the assumption and therefore f must have a finitely many zeros in $\mathbb{D}$. Let $z_1, \ldots, z_n$ be the zeros of f in $\mathbb{D}$, repeated according to their multiplicity. Then we have, from Theorem 4 in Section 3.3,

$$f(z) = \prod_{k=1}^{n} \frac{z-z_k}{1-\overline{z}_k z} g(z)$$

with g analytic in $\mathbb{D}$ and $|g| \leq 1$ in $\mathbb{D}$. By direct calculations, we see that $\lim_{|z|\to 1} |(z - z_k)/(1-\overline{z_k}z)| = 1$ and therefore $\lim_{|z|\to 1} |g(z)|$ exists and

$$\lim_{|z|\to 1} |g(z)| = \frac{\lim_{|z|\to 1} |f(z)|}{\lim_{|z|\to 1} |\prod_{k=1}^{n} \frac{z-z_k}{1-\overline{z}_k z}|} = 1.$$

If $g(z) = 0$ for some $z \neq z_k$, where $1 \leq k \leq n$, then f has another zero. However, f has exactly n zeros. Therefore, $g(z) \neq 0$ for any z not a zero of f. We also have that $g(z_k) \neq 0$ for any $k = 1, 2, \ldots, n$ (otherwise f would have another zero of higher multiplicity). This means $g(z) \neq 0$ for any $z \in \mathbb{D}$. Therefore, $\lim_{|z| \to 1} 1/|g(z)| = 1$ and by the third Maximum Principle, we conclude that $|g(z)| = 1$ for any $z \in \mathbb{D}$ and $g = c$ from some $c \in \mathbb{T}$. The function f is then a rational function, as required. $\square$

Problem 44 — 5 points

(a) We have $\gamma(t) = 1 + e^{it}$ for $0 \leq t \leq 2\pi$.

(b) Using the parametrization, we have

$$\begin{aligned}\int_\gamma |z|^2\,dz &= \int_0^{2\pi} |1+e^{it}|^2 ie^{it}\,dt \\ &= i\int_0^{2\pi} \Big(2 + e^{it} + e^{-it}\Big)e^{it}\,dt = 2\pi i.\end{aligned}$$

Notice that $z \mapsto |z|^2$ is not analytic in a neighborhood of the circle $\gamma(t) = 1 + e^{it}$, otherwise the integral would be zero. $\square$

Problem 48 — 10 points

For a, b two complex numbers, define the complex interval $[a, b]$ as the image of $\gamma(t) = a+t(b-a)$, for $t \in [0, 1]$. This is simply the straight line from a to b.

Let S_1 be a (closed) square with vertices v_1, v_2, v_3, and v_4 (ordered in counter-clockwise direction). Let S_2 be be a (closed) squares with vertices w_1, w_2, w_3, and w_4 (ordered in counter-clockwise direction) and squaring exactly one side with S_1. Since one side is in common, without loss of generality, suppose that $v_1 = w_2$ and $v_2 = w_1$.

Parametrize ∂S_1, ∂S_2, and $\partial(S_1 \cup S_2)$ in the counter-clockwise direction. Then, since S_1 and S_2 has one side in common and the side common to ∂S_1 and ∂S_2 is traversed in the opposite direction. Therefore, $[v_1, v_2] = [w_2, w_1] = -[v_1, v_2]$. Therefore, we have

$$\begin{aligned}\int_{\partial S_1} f(z)\,dz + \int_{\partial S_2} dz &= \int_{[v_1,v_2]} f(z)\,dz + \int_{[v_2,v_3]} f(z)\,dz + \int_{[v_3,v_4]} f(z)\,dz + \int_{[v_4,v_1]} f(z)\,dz \\ &\quad - \int_{[v_1,v_2]} f(z)\,dz + \int_{[w_2,w_3]} f(z)\,dz + \int_{[w_3,w_4]} f(z)\,dz + \int_{[w_4,w_1]} f(z)\,dz \\ &= \int_{[v_2,v_3]} f(z)\,dz + \int_{[v_3,v_4]} f(z)\,dz + \int_{[v_4,v_1]} f(z)\,dz \\ &\quad + \int_{[w_2,w_3]} f(z)\,dz + \int_{[w_3,w_4]} f(z)\,dz + \int_{[w_4,w_1]} f(z)\,dz \\ &= \int_{\partial(S_1\cup S_2)} f(z)\,dz\end{aligned}$$

where the last equality comes from the fact that

$$\partial(S_1 \cup S_2) = [v_2, v_3] + [v_3, v_4] + [v_4, v_1] + [w_2, w_3] + [w_3, w_4] + [w_4, w_1].$$

This completes the proof. $\square$

Problem 50 — 15 points

(a) Write $z = 1 + re^{it}$ for $r < 1$ and $-\pi < t \leq \pi$. Because $r < 1$, the number $z \neq 0$. Therefore, the $\theta := \arg z$ is defined. We want to solve $w^n = z$. The solutions are given by $w_n := \sqrt[n]{\rho} e^{i(\theta + 2k\pi)/n}$, where $k = 0, 1, \ldots, n-1$ and $z = \rho e^{i\theta}$. Select $k = 0$ and define

$$\sqrt[n]{z} := \sqrt[n]{\rho} e^{i\theta/n},$$

where $z = 1 + re^{it} = \rho e^{i\theta}$ for $-\pi < t \leq \pi$, $0 \leq r < 1$, $-\pi/2 < \theta < \pi/2$, and $0 < \rho < 2\cos(\theta)$. Notice that by definition, $g(z)^n = z$.

(b) Write $z = 1 + w$, where $|w| < 1$. Using the identity

$$(a - b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}) = a^n - b^n,$$

we have, with $a = \sqrt[n]{1 + w + h}$ and $b = \sqrt[n]{1 + w}$:

$$\begin{aligned} \frac{\sqrt[n]{1+w+h} - \sqrt[n]{1+w}}{h} &= \frac{a-b}{h} \frac{a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}}{a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}} \\ &= \frac{a^n - b^n}{h(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})} \\ &= \frac{1}{a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}}. \end{aligned}$$

It remains to show that, as $h \to 0$, $a \to b$. However, if we write $a = \sqrt[n]{\rho_h} e^{i\theta_h/n}$ and $b = \sqrt[n]{\rho} e^{i\theta/n}$, then $a \to b$ if and only if $\sqrt[n]{\rho_h} \to \rho$ and $\theta_h \to \theta$. First, we have $\rho_h = |1 + w + h|$ and $\rho = |1 + w|$. Since $x \mapsto \sqrt[n]{x}$ is continuous for $x > 0$ and the modulus is continuous, we see that $\sqrt[n]{\rho_h} \to \sqrt[n]{\rho}$ as $h \to 0$. Now, $\theta_h = \arctan\big(\mathrm{Im}(1 + w + h)/\mathrm{Re}(1 + w + h)\big)$ and $\theta = \arctan\big(\mathrm{Im}(1+w)/\mathrm{Re}(1+w)\big)$. We know that $v \mapsto \mathrm{Re}(v)$ and $v \mapsto \mathrm{Im}(v)$ are continuous function and therefore $v \mapsto \mathrm{Im}(v)/\mathrm{Re}(v)$ is continuous except for $\mathrm{Re}(v) = 0$. Therefore, for any $|w| < 1$, $w \mapsto \arctan(\mathrm{Im}(1 + w)/\mathrm{Re}(1 + w))$ is continuous. In particular, $\theta_h \to \theta$ as $h \to 0$. In other words, $a \to b$ as $h \to 0$.

Letting $h \to 0$ in the expression of the difference quotient, we find that

$$\lim_{h \to 0} \frac{\sqrt[n]{1+w+h} - \sqrt[n]{1+w}}{h} = \frac{1}{nb^{n-1}} = \frac{1}{n(\sqrt[n]{1+w})^{n-1}} = \tfrac{1}{n}(\sqrt[n]{z})^{1-n}.$$

Therefore, we get $g'(z) = n(\sqrt[n]{z})^{1-n}$ for $z \in B$. We see that g' is continuous in B because $z \mapsto \sqrt[n]{z}$ (we've just shown the derivative exists), $z \mapsto z^{n-1}$ and $z \mapsto z^{-1}$ are continuous functions in B. The function g is then holomorphic and from Corollary 8, in section 4.2, the function g is analytic in B. $\square$

Problem 53 — 5 points

Suppose that this would be possible. Then, we would have

$$\int_{\partial \mathbb{D}} \frac{1}{z}\, dz = \int_{\partial \mathbb{D}} f'(z)\, dz = 0.$$

However, the integral on the left-hand side is $2\pi i$, a contradiction. $\square$

Problem 58 — 10 points

Suppose that $|z| \leq \frac{1}{2}$.

From Lemma 2 in section 4.4, we have

$$f'(z) = \frac{1}{2\pi i} \int_{C_r} \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta,$$

where $\frac{2}{3} \leq r < 1$ and C_r is the circle centered at the origin and of radius r oriented counter clockwise.

This implies that

$$|f'(z)| \leq \frac{1}{2\pi} \int_{C_r} \frac{|f(\zeta)|}{|\zeta - z|^2}\, |d\zeta| \quad (\tfrac{2}{3} < r < 1).$$

By integrating for $r = 2/3$ to $r = 1$, we have

$$|f'(z)| \int_{2/3}^{1} dr \leq \int_{2/3}^{1} \int_{0}^{2\pi} \frac{|f(re^{it}|}{|z - re^{it}|^2} r\, dtdr.$$

But, $|z - re^{it}|^2 \geq (r - |z|)^2 \geq (2/3 - 1/2)^2 = 1/36$, so that

$$|f'(z)| \frac{1}{3} \leq 36 \int_{2/3}^{1} \int_{0}^{2\pi} |f(re^{it})|\, r dtdr$$

and this can be simplified to

$$|f'(z)| \leq 54 \int_{A} |f(x + iy)|\, dxdy,$$

where $A := \{z \in \mathbb{C} \,:\, 2/3 < |z| < 1\}$. Since $A \subset \mathbb{D}$, we then get

$$|f'(z)| \leq 54 \int_{\mathbb{D}} |f(x + iy)|\, dxdy.$$

This completes the proof. $\square$

Problem 59 — 5 points

Let $n \geq 1$ be a positive integer and define A_n, B_n and C_n as followed:

$$A_n := \{z \in \mathbb{C} \,:\, \mathrm{Re}z \leq -1/n \text{ and } |\mathrm{Im}z| \leq n\}, \quad B_n := \{z \in \mathbb{C} \,:\, |\mathrm{Re}z| \leq 1/2n \text{ and } |\mathrm{Im}z| \leq n\}$$

and

$$C_n := \{z \in \mathbb{C} \,:\, \mathrm{Re}z \geq 1/n \text{ and } |\mathrm{Im}z| \leq n\}.$$

Let $K_n := A_n \cup B_n \cup C_n$. Define $f_n : K_n \to \mathbb{C}$ to be $f_n(z) := \chi_{A_n \cup B_n}(z) - 2\chi_{A_n}(z) - \chi_{B_n}(z) + \chi_{C_n}(z)$. In other words, we have

$$f_n(z) := \begin{cases} 1 & \text{if } z \in C_n \\ 0 & \text{if } z \in B_n \\ -1 & \text{if } z \in A_n. \end{cases}$$

Notice that f_n is analytic on K_n, because it is analytic on an open set containing K_n.

Because K_n is bounded, from Runge's Theorem, there is a polynomial p_n such that $|p_n - f_n| < \frac{1}{n}$ uniformly on K_n. However, as $n \to \infty$, since C_n increases to $\{z : \mathrm{Re} z > 0\}$, B_n increases to $\{z : \mathrm{Re} z = 0\}$, and A_n increases to $\{z : \mathrm{Re} z < 0\}$, the sequence (f_n) converges pointwise to

$$f(z) := \begin{cases} 1 & \text{if } \mathrm{Re} z > 0 \\ 0 & \text{if } \mathrm{Re} z = 0 \\ -1 & \text{if } \mathrm{Re} < 0. \end{cases}$$

By construction, we therefore have

$$\lim_{n\to\infty} p_n(z) = f(z) \quad (\forall z \in \mathbb{C}).$$

This completes the solution. $\square$

Problem 62 — 5 points

Multiply each side by z^k for $k \in \mathbb{Z}$ to get

$$\sum_{n=-\infty}^{\infty} a_n z^{n+k} = \sum_{n=-\infty}^{\infty} b_n z^{n+k}$$

for $r < |z| < R$. Integrate on a circle γ centered at 0 of radius $(r+R)/2$. Since $n(f_m(\gamma), 0) = 0$ for any $m \neq -1$ and $n(f_m(\gamma), 0) = 1$ for $m = -1$, where $f_m(z) = z^m$, we obtain

$$a_{-k-1} = b_{-k-1}, \ \forall k \in \mathbb{Z}.$$

Problem 63 — 20 points

a) We make the change of variable $w = 1/z$ and therefore

$$\frac{e^{-1/w}}{w^2}$$

and use the expansion of $\exp(z)$ around $z = 0$. We then get, for $w \neq 0$,

$$\frac{e^{-1/w}}{w^2} = \frac{1}{w^2} \sum_{n=0}^{\infty} \frac{1}{w^n n!} = \sum_{n=0}^{\infty} \frac{1}{w^{n+2} n!}.$$

We therefore have a essential singularity at ∞.

b) We have, for $z = 1/w$, $w \neq 0$,

$$\frac{1}{e^w - 1} - \frac{1}{w} = \frac{w + 1 - e^w}{w(e^w - 1)}.$$

Take w such that $0 < |w| < \pi$ so that the denominator is zero only at $w = 0$. Using the power series expansion of e^w around 0, we see that

$$\frac{w + 1 - e^w}{w(e^w - 1)} = -\frac{\sum_{n \geq 2} \frac{w^n}{n!}}{w \sum_{n \geq 1} \frac{w^n}{n!}} = \frac{\sum_{n \geq 0} \frac{w^n}{(n+2)!}}{\sum_{n \geq 0} \frac{w^n}{(n+1)!}}$$

and since the two power series are continuous functions at 0 and the denominator is non-zero at $w = 0$, we get

$$\lim_{w \to 0} \left(\frac{1}{e^{1/z} - 1} - z \right) = -\frac{1/2}{1} = -\frac{1}{2}.$$

Therefore, the point $z = \infty$ is a removable singularity and we have

$$f(\infty) = -\frac{1}{2}.$$

c) Make the change $w = 1/z$, with $w \neq 0$, to get

$$e^{z/(1-z)} = e^{(1/w)/(1-1/w)} = e^{1/(w-1)}.$$

Therefore, as $w \to 0$, we have

$$\lim_{w \to 0} e^{1/(w-1)} = e^{-1}.$$

This means $z = \infty$ is a removable singularity and

$$\lim_{z \to \infty} e^{z/(1-z)} = \frac{1}{e}.$$

d) For $z = 1/w$, with $w \neq 0$, we have

$$z^2 - z = \frac{1}{w^2} - \frac{1}{w}$$

which implies that we have a pole of order 2 at $z = \infty$.

Problem 64 — 10 points

a) Notice that

$$(z^2 + z + 1)(z - 1) = z^3 - 1,$$

so that

$$\frac{z^3}{(z^2 + z + 1)(z - 1)} = \frac{z^3 - 1 + 1}{z^3 - 1} = 1 - \frac{1}{1 - z^3}.$$

For $|z| < 1$, we get

$$\frac{1}{1 - z^3} = \sum_{n \geq 0} z^{3n}$$

and therefore

$$f(z) = 1 - \sum_{n \geq 0} z^{3n} = -\sum_{n \geq 1} z^{3n}.$$

b) First, write $z^2 + 4 = (z - 2i)(z + 2i)$ so that

$$f(z) = \frac{z}{(z - 2i)(z + 2i)(z - 3)(z - 4)}.$$

Using the "little trick" from the section 2.1, we obtain

$$f(z) = \frac{a_1}{z - 2i} + \frac{a_2}{z + 2i} + \frac{a_3}{z - 3} + \frac{a_4}{z - 4},$$

where

$$a_1 = \frac{2i}{4i(2i-3)(2i-4)}, \qquad a_2 = \frac{2i}{4i(2i+3)(2i+4)},$$
$$a_3 = -\frac{3}{(3-2i)(3+2i)}, \qquad a_4 = \frac{4}{(4-2i)(4+2i)}.$$

Let $3 < |z| < 4$, so that $1/4 < 1/|z| < 1/3$. We then have

$$\frac{1}{z-2i} = \frac{1}{z}\left(\frac{1}{1-\frac{2i}{z}}\right) = \frac{1}{z}\sum_{n\geq 0}\left(2i/z\right)^n = \sum_{n\geq 0}(2i)^n/z^{n+1},$$

because $|2i/z| < 2/3 < 1$. Repeat this process for the next two terms in the partial fraction decomposition:

$$\frac{1}{z+2i} = \sum_{n\geq 0}(-2i)^n/(z^{n+1}) \quad \text{and} \quad \frac{1}{z-3} = \sum_{n\geq 0}3^n/z^{n+1}.$$

For the last term, write $z-4$ as $-4(1-z/4)$, so that

$$\frac{1}{z-4} = -\sum_{n\geq 0}z^n/4^{n-1}.$$

Combining everything together, we obtain

$$f(z) = \sum_{n\geq 0}\frac{a_1(2i)^n + a_2(-2i)^n + a_3 3^n}{z^{n+1}} - \sum_{n\geq 0}\frac{a_4}{4^{n-1}}z^n,$$

for $3 < |z| < 4$.

Problem 65 — 10 points

Write $p(z) = z^{2n} + \alpha z^{2n-1} + \beta^2$, with $n \geq 2$ an even integer.

Consider the close curve γ defined as followed:

$$\gamma(t) := \begin{cases} it & \text{if } -R < t < R \\ Re^{it} & \text{if } -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}, \end{cases}$$

where $R > \beta$ is to be chosen later. We will use Rouché's Theorem with $p(z)$ and $g(z) = z^{2n} + \beta^2$. We therefore have to verify that $|p(z) + g(z)| < |p(z)| + |g(z)|$ for $z \in \gamma$.

First Proof. First, notice from the inverse triangle inequality that

$$|p(z)| + |g(z)| = |z^{2n} + \alpha z^{2n-1} + \beta^2| + |z^{2n} + \beta^2| \geq |\alpha z^{2n-1}| - |z^{2n} + \beta^2| + |z^{2n} + \beta^2| = |\alpha z^{2n-1}|.$$

Can we have equality when using the inverse triangle inequality? Equality occurs in the inverse triangle inequality if and only if

$$|a+b| = |a| - |b| \iff \mathrm{Re}(a\bar{b}) = -|a\bar{b}| \iff \mathrm{Re}(a\bar{b}) \leq 0 \text{ and } \mathrm{Im}(a\bar{b}) = 0.$$

In our case, we have $a = \alpha z^{2n-1}$ and $b = z^{2n} + \beta^2$, for $z \in \gamma$. So, we have equality if and only if

$$\mathrm{Im}\Big(\alpha z^{2n-1}(\overline{z}^{2n} + \beta^2)\Big) = 0 \quad \text{ and } \quad \mathrm{Re}\Big(\alpha z^{2n-1}(\overline{z}^{2n} + \beta^2)\Big) \leq 0.$$

This is also equivalent to

$$|z|^{4n-2}\mathrm{Im}(\overline{z}) = -\beta^2 \mathrm{Im}(z^{2n-1}) \quad \text{ and } \quad |z|^{4n-2}\mathrm{Re}(\overline{z}) \leq -\beta^2 \mathrm{Re}(z^{2n-1}).$$

If $z = iy$, with $y \neq 0$, then

$$-y^{4n-1} = \beta^2 y^{2n-1} \iff y^{2n} = -\beta^2$$

which is impossible. If $y = 0$, then

$$|p(iy) - (iy)^{2n} - \beta^2| = |\alpha(iy)^{2n-1}| = 0 < \beta^2 < |p(iy)| + |g(iy)|.$$

Therefore, for $z = iy$, equality in the inverse triangle inequality can't occur and we have $|p(z) - (z^{2n} + \beta^2)| < |p(z)| + |z^{2n} + \beta^2|$.

Now, let $z = Re^{i\theta}$, for $-\pi/2 \leq \theta \leq \pi/2$. We then have

$$-R^{4n-1}\sin(\theta) = -\beta^2 R^{2n-1}\sin((2n-1)\theta) \iff R^{2n}\sin(\theta) = \beta^2 \sin((2n-1)\theta).$$

This equation has a solution at $\theta = 0$, but have no other solution if $R^{2n} > (2n-1)\beta^2$ (the slope of the function $\beta^2 \sin((2n-1)\theta)$ at $\theta = 0$). At $\theta = 0$, we have $z = R$ and

$$|p(R) - R^{2n} - \beta^2| = |\alpha|R^{2n-1} < R^{2n} + \beta^2$$

if we choose $R > |\alpha|$. Also, the second condition is now

$$R^{2n}\cos(\theta) \leq \beta^2 \cos((2n-1)\theta)$$

and since $R > (2n-1)\beta^2$, we always have $R^{2n}\cos\theta > \beta^2\cos((2n-1)\theta)$ for $\theta \neq -\pi/2$ and $\theta \neq \pi/2$. When $\theta = \pi/2$ or $\theta = -\pi/2$, we are in the previous situation where $z = iy$, for some $-R \leq y \leq R$. Therefore, for our choice of R, equality in the inverse triangle inequality can't occur and so $|p(z) - (z^{2n} + \beta^2)| < |p(z)| + |z^{2n} + \beta^2|$.

In summary, we just proved that equality is not possible on γ and therefore

$$|p(z) - g(z)| < |p(z)| + |g(z)|, \quad z \in \gamma.$$

By Rouché's Theorem, the functions p and $z^{2n} + \beta^2$ have the same zeros inside the halfdisk bounded by the curve γ.

Second proof. Notice that, for $z \neq 0$,

$$\frac{z^{2n} + \alpha z^{2n-1} + \beta^2}{\alpha z^{2n-1}} = z + 1 + \frac{\beta^2}{z^{2n-1}}.$$

Therefore, we have

$$\lim_{|z|\to\infty} \left| \frac{z^{2n} + \alpha z^{2n-1} + \beta^2}{\alpha z^{2n-1}} \right| = \lim_{|z|\to\infty} \left| z + 1 + \frac{\beta^2}{z^{2n-1}} \right| = \infty.$$

Given $M > 1$, there is an $R_1 > 0$ such that, if $|z| \geq R_1$, then

$$\left| \frac{z^{2n} + \alpha z^{2n-1} + \beta^2}{\alpha z^{2n-1}} \right| \geq M.$$

Therefore, for $|z| = R_1$, we have

$$|z^{2n} + \alpha z^{2n-1} + \beta^2| \geq M|\alpha z^{2n-1}| > |\alpha z^{2n-1}|.$$

Therefore, we have $|f(z) - g(z)| < |f(z)|$ when $z = R_1 e^{it}$, with $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$.

Now, if $z = iy$, for $|y| \leq R_1$, then

$$|z^{2n} + \alpha z^{2n-1} + \beta^2| = |y^{2n} + i\alpha y^{2n-1} + \beta^2| > \sqrt{(\alpha y^{2n-1})^2} = |\alpha y^{2n-1}|.$$

Therefore, we also have $|f(z) - g(z)| < |f(z)|$.

Now, select $R := \max\{R_1, \sqrt[n]{|\beta|}\}$. Then we have $|f(z) - g(z)| < |f(z)| + |g(z)|$, for every $z \in \gamma$.

Conclusion. The zeros of $z^{2n} + \beta^2$ are

$$z_k = \sqrt[n]{|\beta|} e^{i\left(\frac{(2k+1)\pi}{2n}\right)},$$

for $k = 0, 1, \ldots, 2n-1$. Since n is even, there are exactly n roots in the first and fourth quadrant and none on the real or imaginary axis. Since $\sqrt[n]{|\beta|} < R$, these roots are in the half disk bounded by the curve γ. Therefore, the polynomial $p(z)$ has n roots in the half disk bounded by the curve γ. In other words, $p(z)$ has n roots with positive real part.

Problem 67 — 5 points

Let $|z| = R$. By Cauchy-Schwarz, we have

$$|p(z) - z^n| \leq \sum_{k=0}^{n-1} |c_k||z|^k \leq \left(\sum_{k=0}^{n-1} |c_k|^2 \right)^{1/2} \left(\sum_{k=0}^{n-1} (R^2)^k \right)^{1/2} = (R^2 - 1)^{1/2} \left(\frac{R^{2n} - 1}{R^2 - 1} \right)^{1/2}$$

and therefore

$$|p(z) - z^n| \leq (R^{2n} - 1)^{1/2} < R^n = |z|^n$$

Therefore, by Rouché's Theorem, we conclude that p and z^n have the same number of zeros in $\{z : |z| < R\}$.

Problem 72 — 10 points

Let $0 \leq |z| \leq \rho < 1$. By Cauchy's Integral Formula, given $0 < \rho < r < 1$, we have

$$f(z) = \frac{1}{2\pi i} \int_{C_r} \frac{f(\zeta)}{\zeta - z} \, d\zeta,$$

where C_r is the circle of radius r centered at the origin. Writing $\zeta = re^{i\theta}$, for $0 \leq \theta \leq 2\pi$, we get

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(re^{i\theta})}{re^{i\theta} - z} re^{i\theta} \, d\theta.$$

Since f is continuous on $\overline{\mathbb{D}}$, it is uniformly continuous there and, in particular, it is uniformly continuous on $A := \{z \, : \, r_0 \leq |z| \leq 1\}$, for $\rho < r_0 < 1$. Moreover, since $z \notin A$, the function $w \mapsto \frac{f(w)w}{w-z}$ is uniformly continuous on A. Therefore, as $r \to 1^-$, we get

$$f(z) = \lim_{r \to 1^-} \frac{1}{2\pi} \int_0^{2\pi} \frac{f(re^{i\theta})}{re^{i\theta} - z} re^{i\theta} \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{i\theta})e^{i\theta}}{e^{i\theta} - z} \, d\theta.$$

Since $|z| \leq \rho$, then $|e^{i\theta} - z| \geq 1 - \rho$, for any $\theta \in [0, 2\pi]$ and therefore

$$|f(z)| \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(e^{i\theta})}{1 - \rho} \, d\theta \leq \frac{M}{2\pi(1 - \rho)}.$$

This is true for any z such that $|z| \leq \rho$ and any $f \in \mathcal{F}_M$. The family $\mathcal{F}_M$ is locally bounded and therefore is a normal family, by Theorem 9 in section 6.2. □

Problem 73 — 10 points

We will show that $\mathcal{F}$ is locally bounded. Fix $0 < s < r_0 < 1$.

By Cauchy's Integral formula, for $|z| < s$, we have

$$f(z) = \frac{1}{2\pi i} \int_{C_r} \frac{f(\zeta)}{\zeta - z} \, d\zeta,$$

for $s < r < 1$. Then, this implies that

$$|f(z)| \leq \frac{1}{r - s} \int_0^{2\pi} |f(re^{it})| r \, dt \leq \frac{1}{r_0 - s} \int_0^{2\pi} |f(re^{it})| r \, dt$$

Integrate with respect to r from r_0 to 1 to obtain

$$|f(z)|(1 - r_0) \leq \frac{1}{r_0 - s} \int_{r_0}^1 \int_0^{2\pi} |f(re^{it})| r \, dt dr \leq \frac{1}{r_0 - s} \int_0^1 \int_0^{2\pi} |f(re^{it})| r \, dt dr.$$

Changing to cartesian coordinates, we obtain

$$|f(z)| \le C \int_{\mathbb{D}} |f(x+iy)|\, dxdy \le C,$$

for any $|z| < s$ and where $C = \frac{1}{(1-r_0)(r_0-s)}$. Therefore, the family $\mathcal{F}$ is locally bounded on $\mathbb{D}$. From Theorem 9 in section 6.2, the family $\mathcal{F}$ is normal on $\mathbb{D}$.

Problem 75 — 25 points

a) (10pts) We have $f_n'(a) = f'(f_{n-1}(a))f_{n-1}'(a)$. By induction and using the fact that $f(a) = a$, we see that $f_n'(a) = (f'(a))^n$, for any $n \ge 1$.

Now, since Ω is bounded and $f(\Omega) \subset \Omega$, the family $\{f_n : n \ge 1\}$ is a normal family of analytic function on Ω.

From Theorem 9 in section 6.2, we know that $\{f_n' : n \ge 1\}$ is locally bounded. In particular, we have that $\{f_n'(a) : n \ge 1\}$ is bounded. Using the fact that $f_n'(a) = (f'(a))^n$, for any $n \ge 1$, we see that $|f'(a)| \le 1$, otherwise the set $\{f_n'(a) : n \ge 1\}$ would be unbounded.

b) (10pts) Assume that $f'(a) = 1$. Then the power series of f around $z = a$ is of the form

$$f(z) = a + (z-a) + c_m(z-a)^m + \cdots = z + c_m(z-a)^m + \cdots$$

on $B_1 := \{z : |z-a| < r_1\} \subset \Omega$. Then, with $f^2 := f \circ f$, we see that

$$f^2(z) = f(z) + c_m(f(z) - a)^m + \cdots$$

in some $B_2 := \{z : |z-a| < r_2\} \subset B_1$ and $B_2 \subset f(B_1)$. Therefore, we obtain

$$\begin{aligned} f^2(z) &= z + c_m(z-a)^m + c_m(z - a + c_m(z-a)^m + \cdots)^m + \cdots \\ &= z + 2c_m(z-a)^m + \cdots . \end{aligned}$$

Using induction, we then see that

$$f_n(z) = z + nc_m(z-a)^m + \cdots .$$

Now, since $\{f_n : n \ge 1\}$ is a normal family, from Theorem 9 in section 6.2, $\{f_n^{(m)} : n \ge 1\}$ is locally bounded (just successively apply Theorem 9 to $\{f_n' : n \ge 1\}$, ..., $\{f_n^{(m-1)} : n \ge 1\}$).

This means that $\{f_n^{(m)}(a) : n \ge 1\}$ is bounded, say by L. Since $nc_m = \frac{f_n^{(m)}(a)}{m!}$, then we have

$$|c_m| \le \frac{L/m!}{n} \to 0 \quad (\text{ as } n \to \infty).$$

Therefore, $c_m = 0$, for any m and this implies that $f(z) = z$.

c) (5pts) Let $\alpha = f'(a)$, then $|\alpha| = 1$. Since $\{f_n : n \ge 1\}$ is a normal family, then there is a subsequence (f_{n_k}) such that $f_{n_k} \to g$ locally uniformly, where g is analytic on Ω. We may relabel the subsequence, and suppose that $f_n \to g$ locally uniformly.

Choose an increasing subsequence $(n_k)_{k\geq 1}$ of positive integers such that $\alpha^{n_k} \to 1$. Then, since $f(\Omega) \subset \Omega$, we have $g(\Omega) \subset \Omega$. Also, by Weierstrass' Theorem, we have $g'(a) = 1$. From part (b), we see that $g(z) = z$.

If $f(z) = f(w)$, for $z, w \in \Omega$, then $f_{n_k}(z) = f_{n_k}(w)$. Taking $k \to \infty$, we see that $g(z) = g(w)$, which implies that $z = w$. So the function f is one-to-one.

Suppose that there is a $w \in \Omega$ such that $w \notin f(\Omega)$. Since $f_2(\Omega) \subset f(\Omega)$ and therefore, by induction, we have $f_n(\Omega) \subset f(\Omega)$. This means $w \notin f_n(\Omega)$ for any $n \geq 1$. We know that $f_{n_k} \to g$ locally uniformly on Ω, so $f_{n_k} - w \to g - w$ locally uniformly on Ω. Since $f_{n_k}(z) - w \neq 0$ for any $z \in \Omega$, we must have also that $g(z) - w \neq 0$ for any $z \in \Omega$, by Hurwitz's Theorem (Theorem 3 in section 6.3). Since $w \in \Omega$, this is impossible and therefore f much be onto, meaning $f(\Omega) = \Omega$.

Problem 76 — 5 points

Since all imaginary parts are different, we can order the list w_1, w_2, w_3, w_4 by increasing imaginary part. Suppose the list was in such an order.

Let $c := \frac{\mathrm{Im}(w_2)+\mathrm{Im}(w_3)}{2}$. Then, the transformation

$$\phi(z) := \frac{(z - ic) - i}{(z - ic) + i} = C(z - ic)$$

will do the job.