Math 644

Chapter 4

Section 4.4: Weierstrass' Theorem

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Theorem 1. Suppose (f_n) is a collection of analytic functions on a region Ω such that $f_n \to f$ uniformly on compact subsets of Ω. Then f is analytic on Ω. Moreover, $f_n' \to f'$ uniformly on compact subsets of Ω.

Lemma 2. If G is integrable on a piecewise continuously differentiable curve γ, then

$$g(z) := \int_\gamma \frac{G(\zeta)}{\zeta - z}\, d\zeta$$

is analytic in $\mathbb{C} \backslash \gamma$ and

$$g'(z) = \int_\gamma \frac{G(\zeta)}{(\zeta - z)^2}\, d\zeta.$$

Proof.

Proof of Weierstrass's Theorem.

Integrating On Continuous Curves

Goal:

- Extend the definition of the integral to continuous maps $\gamma : [a, b] \to \mathbb{C}$.

LEMMA 3. Suppose Ω is a region and suppose $\gamma : [0, 1] \to \Omega$ is continuous. Given $\varepsilon > 0$ with $0 < \varepsilon < \operatorname{dist}(\gamma, \partial\Omega)$, we can find a finite partition $0 = t_0 < t_1 < \cdots < t_n = 1$ so that

a) $\gamma([t_{j-1}, t_j]) \subset B_j := \{z : |z - \gamma(t_j)| < \varepsilon\}$ for every $j = 1, \ldots, n$;

b) $B_j \subset \Omega$ for every $j = 1, \ldots, n$.

Proof.

Construction:

Theorem 4. Suppose Ω is a region and $\gamma : [0,1] \to \mathbb{C}$ is continuous with $\gamma \subset \Omega$. Let σ be the polygonal curve defined in the last page. If f is analytic on Ω, define

$$\int_\gamma f(z)\, dz = \int_\sigma f(z)\, dz.$$

Then this definition of $\int_\gamma f(z)\, dz$ does not depend on the choice of the polygonal curve σ and it agrees with our prior definition if γ is piecewise continuously differentiable.

Proof.