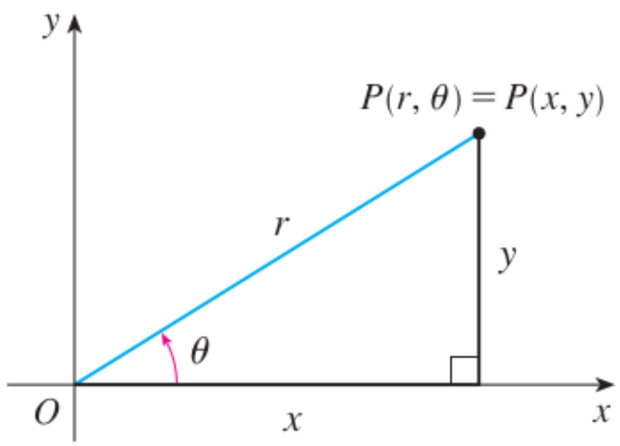
15.7 Triple integrals cylindrical coordinates.

Polar coordinates.



$\underline{\text{Polar} \to \text{Cart.}}$

$x = r\cos\theta \quad , \quad y = r\sin\theta$

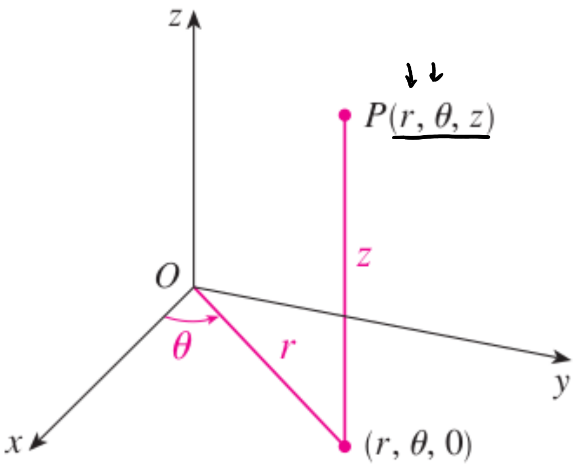
$0 \le \theta \le 2\pi \quad , \quad r \ge 0 .$

$\underline{\text{Cart.} \to \text{Polar}}$

$r = \sqrt{x^2+y^2}$

$\theta = \arctan\left(\frac{y}{x}\right) .$

Cylindrical coordinates.



$\underline{\text{Cyl.} \to \text{Cart (3D)}}$

$x = r\cos\theta \quad y = r\sin\theta \quad z = z$

$\underline{\text{Cart}^{(3D)} \to \text{Cyl.}}$

$r = \sqrt{x^2+y^2}$

$\theta = \arctan\left(\frac{y}{x}\right)$

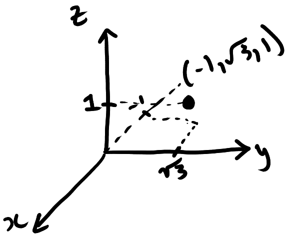
$z = z .$

EXAMPLE 1
(a) Plot the point with cylindrical coordinates $(2, 2\pi/3, 1)$ and find its rectangular coordinates.
(b) Find cylindrical coordinates of the point with rectangular coordinates $(3, -3, -7)$.

(a) $x = r\cos\theta = 2\cos\left(\frac{2\pi}{3}\right) = 2\left(-\frac{1}{2}\right) = -1$

$y = r\sin\theta = 2\sin\left(\frac{2\pi}{3}\right) = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$

$z = z = 1$



(b) $x = 3, \; y = -3 , \; z = -7 .$

$r = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2} .$

$\theta = \arctan\left(-\frac{3}{3}\right) = \arctan(-1)$

$\to \theta = \frac{7\pi}{4} + 2n\pi$

$\to \theta = \frac{7\pi}{4}$. So, $(3\sqrt{2}, 7\pi/4, -7)$

EXAMPLE 2 Describe the surface whose equation in cylindrical coordinates is $\boxed{z = r}$.

(r, θ, z) where $r = \sqrt{x^2+y^2}$

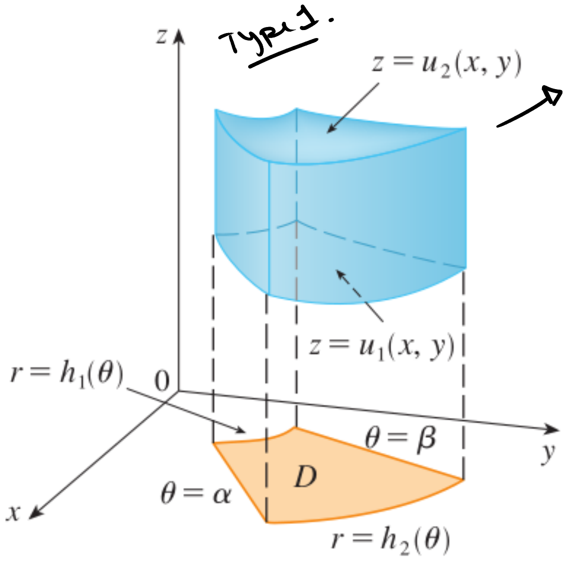
$\tan\theta = y/x .$

$\to z = \sqrt{x^2+y^2}$

$\to z^2 = x^2+y^2$

We get a cone.

Evaluating triple integrals.



$E = \{ (x,y,z) : (x,y) \in D$ and $u_1(x,y) \le z \le u_2(x,y) \}$

$$\iiint_E f(x,y,z)\, dV = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz \right) dA$$

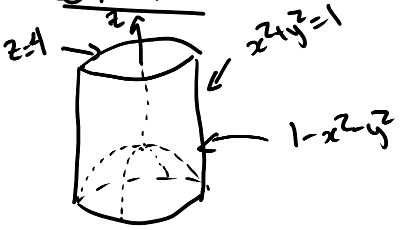
D is described is polar coordinates

$\to dA = r\,dr\,d\theta$.

$$\boxed{\iiint_E f(x, y, z)\, dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta,\, r\sin\theta)}^{u_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta}$$

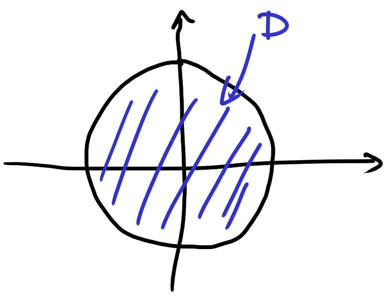
EXAMPLE 3 A solid E lies within the cylinder $x^2 + y^2 = 1$, below the plane $z = 4$, and above the paraboloid $z = 1 - x^2 - y^2$. (See Figure 8.) The density at any point is proportional to its distance from the axis of the cylinder. Find the mass of E.

① Picture.



$E = \{ (x,y,z) : (x,y) \in D \,,\ 1-x^2-y^2 \le z \le 4 \}$.

② Projection.



$D = \{ (x,y) : \ x^2+y^2 \le 1 \}$

$= \{ (r,\theta) : \ 0 \le r \le 1 \,,\ 0 \le \theta \le 2\pi \}$

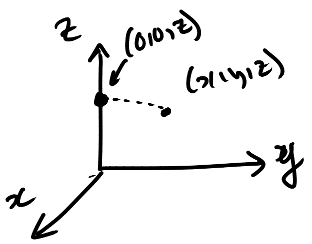
④ Mass. $x = r\cos\theta,\ y = r\sin\theta,\ z = z$

$$m = \iiint_E \rho(x,y,z)\, dV$$

$$= \int_0^{2\pi} \int_0^1 \int_{1-r^2}^4 k\, r\, dz\, r\, dr\, d\theta$$

$$= \frac{12\pi}{5} k$$

③ Density.



$\rho(x,y,z) = k\sqrt{x^2+y^2}$

EXAMPLE 4 Evaluate $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dz\,dy\,dx. = I$

① Description.

$$E = \{(x,y,z) : -2 \le x \le 2, \; -\sqrt{4-x^2} \le y \le \sqrt{4-x^2}, \; \sqrt{x^2+y^2} \le z \le 2\}.$$

② Projection x,y plane.

y is bounded by $y = -\sqrt{4-x^2}$ & $y = \sqrt{4-x^2}$

$\Rightarrow \; y^2 = 4 - x^2$

$\Rightarrow \; x^2+y^2 = 4 \rightarrow$ circle radius 2.

$$D = \{(x,y) : x^2+y^2 \le 4\}$$
$$= \{(r,\theta) : 0 \le r \le 2, \; 0 \le \theta \le 2\pi\}.$$

③ Evaluate.

$$I = \iiint_E (x^2+y^2)\,dV = \int_0^{2\pi}\int_0^2 \left(\int_r^2 r^2\,dz\right) r\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^2 \left(\int_r^2 dz\right) r^3\,dr\,d\theta$$
$$= \boxed{\frac{16}{5}\pi}.$$