

# MATH 241

## CHAPTER 4

### SECTION 4.1: AREAS AND DISTANCES

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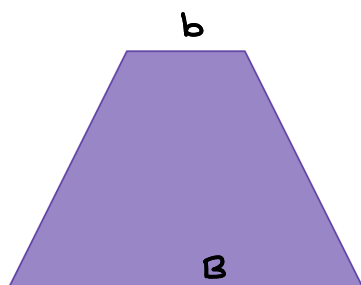
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# AREA PROBLEM

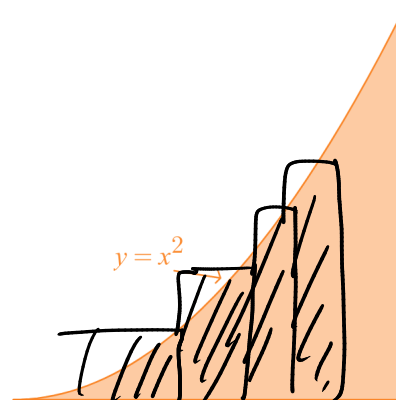
What is the area of the following shapes?



(a) Area =  $w \cdot h$



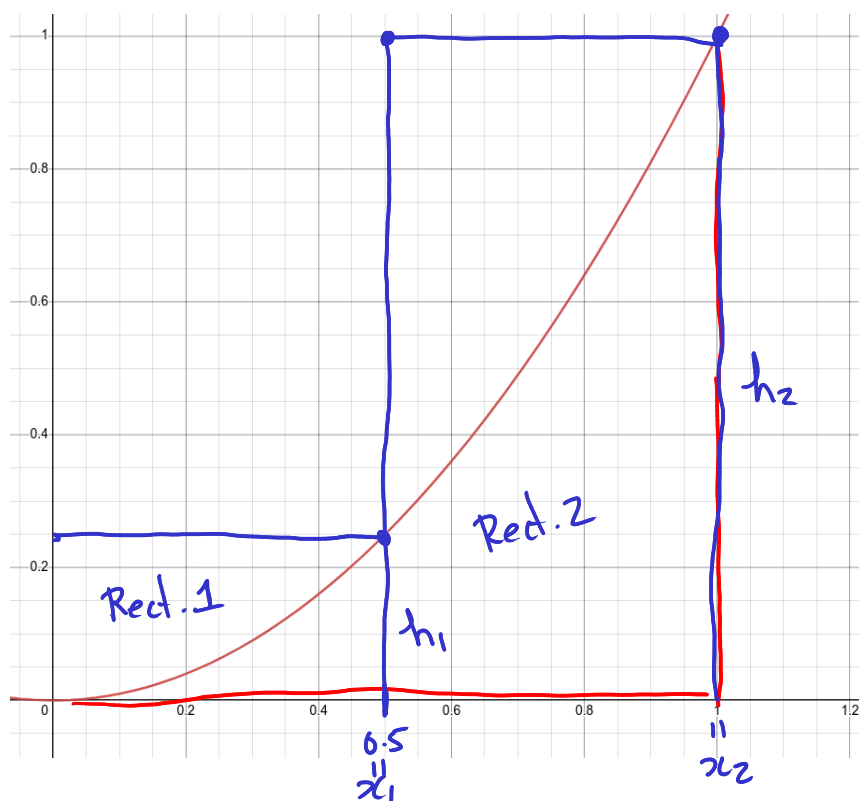
(b) Area =  $\frac{b+B}{2} h$



(c) Area =

Trick: Use simpler shapes, such as rectangles, to approximate the area.

**EXAMPLE 1.** Using rectangles, approximate the area of the region  $S$  under the graph of  $y = x^2$  between  $x = 0$  and  $x = 1$ . Go to Desmos: <https://www.desmos.com/calculator/gfrgqd4nvx>



2 rectangles ( $n=2$ )

I) Divide the shape  
in 2 rectangles of

$$\Delta x = \frac{1-0}{2} = \frac{1}{2}$$

two subintervals

$[0, 0.5]$  &  $[0.5, 1]$

II) Select the right endpoints  
of the subintervals:

$$x_1 = 0.5 = 0 + 1 \cdot \Delta x$$

$$x_2 = 1 = 0 + 2 \cdot \Delta x$$

$$h_1 = f(x_1) = 0.25$$

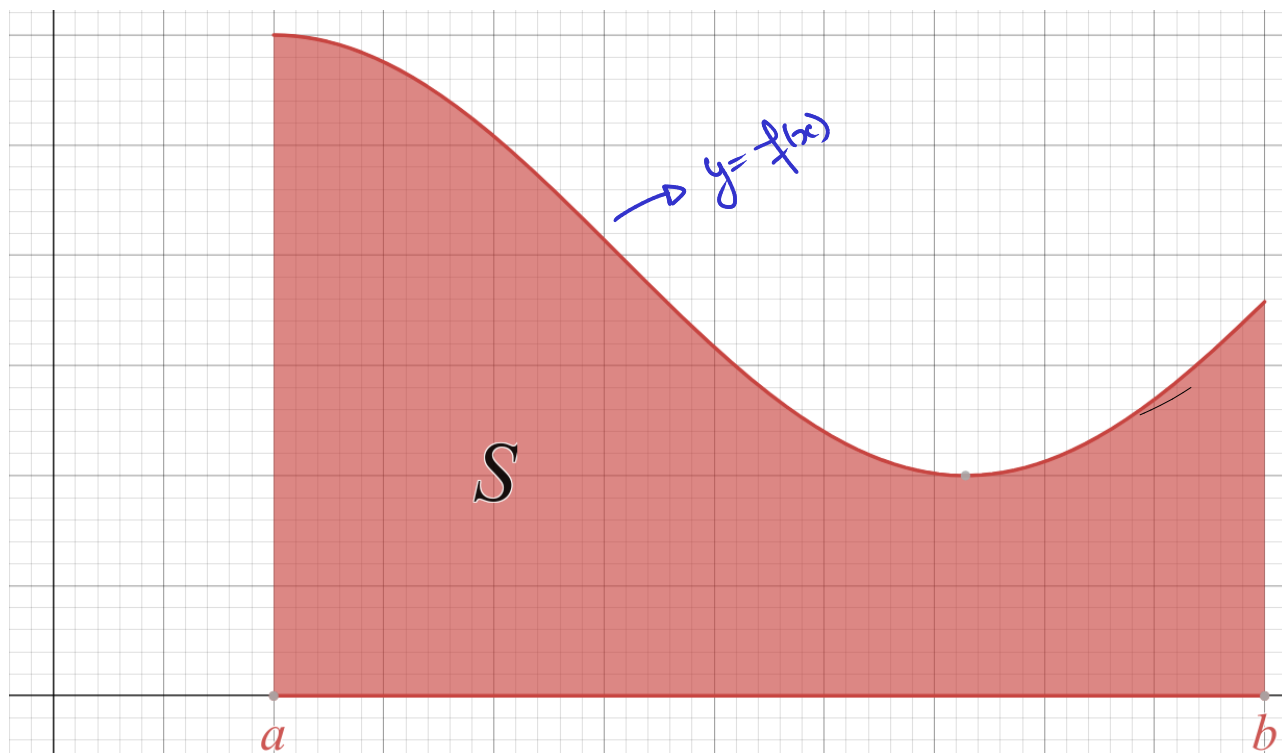
$$h_2 = f(x_2) = 1$$

III) Compute the area of the rectangles

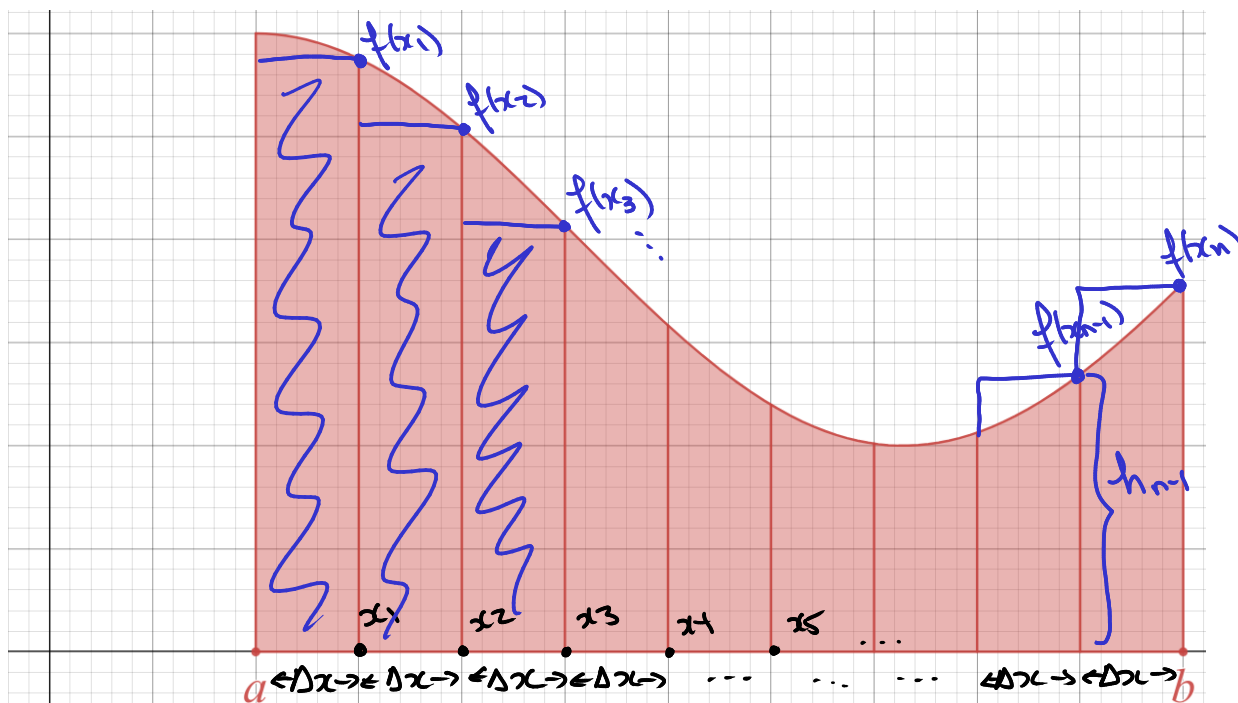
$$\text{Rect. 1} + \text{Rect. 2} = \Delta x \cdot f(x_1) + \Delta x \cdot f(x_2) = \frac{1}{2} \cdot 0.25 + \frac{1}{2} \cdot 1 = \boxed{\frac{5}{8}}$$

## Divide and Conquer With the Right Endpoint Rule!

Suppose we want to compute the area of a region  $S$  bounded by the graph of some function  $y = f(x)$ .



STEP I Subdivide the region  $S$  into  $n$  strips of equal width  $\Delta x = (b - a)/n$ .



STEP II Choose the right-end point for all subintervals:

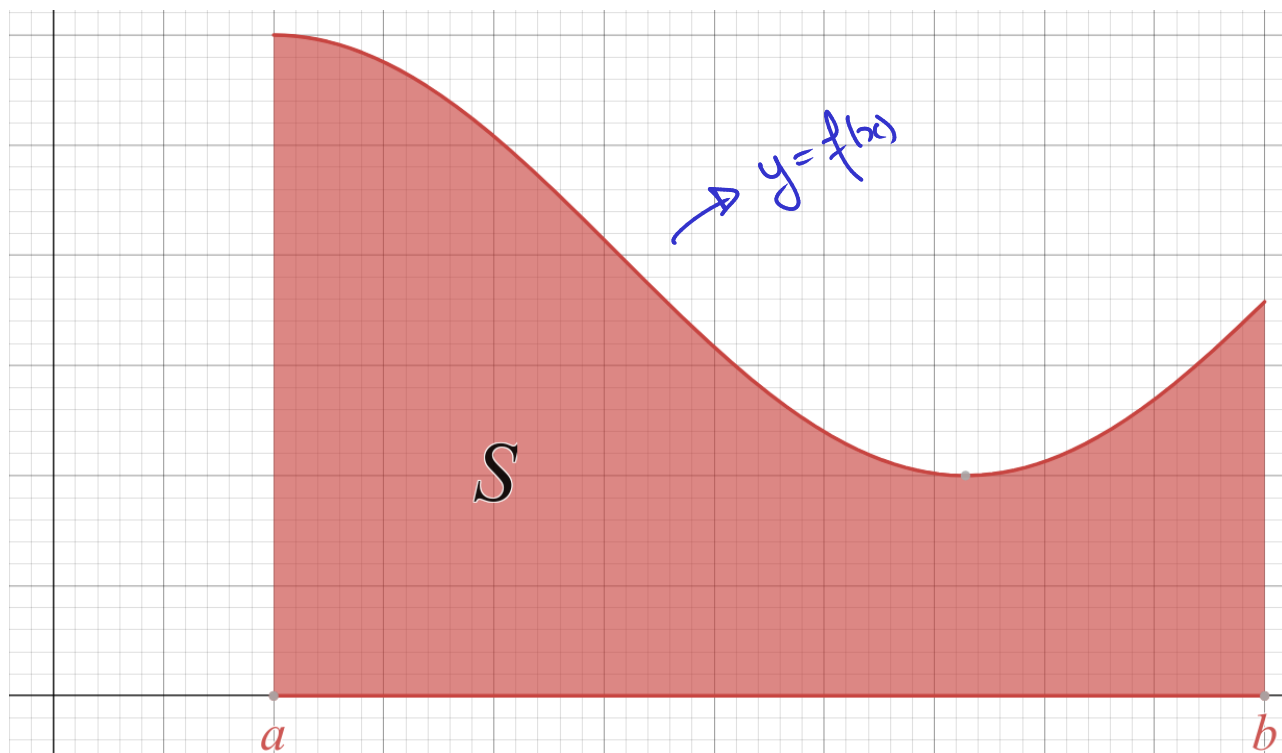
$$x_1 = a + \Delta x, x_2 = a + 2\Delta x, \dots, x_{n-1} = a + (n - 1)\Delta x, x_n = b.$$

STEP III Approximate by adding the area of each rectangle:

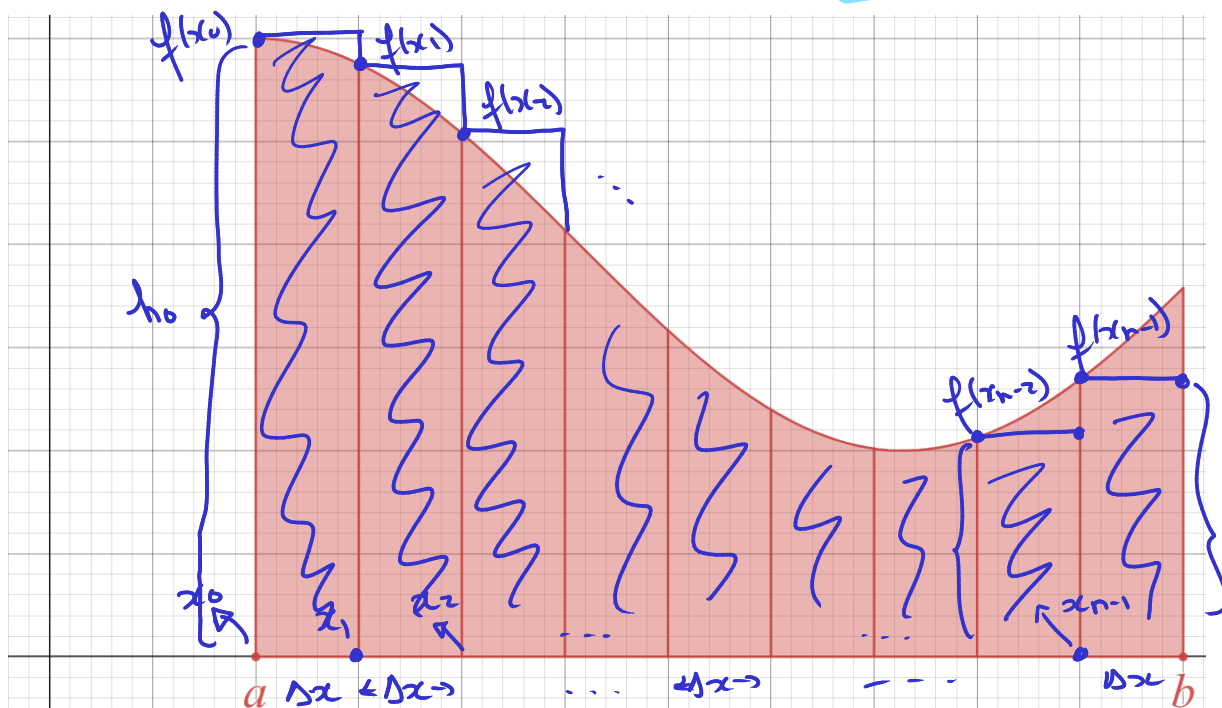
$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_n)\Delta x.$$

## Divide and Conquer With the Left Endpoint Rule!

Suppose we want to compute the area of a region  $S$  bounded by the graph of some function  $y = f(x)$  from  $x = a$  to  $x = b$ .



STEP I Subdivide the region  $S$  into  $n$  strips of equal width  $\Delta x = (b - a)/n$ .



STEP II Choose the left-end point for all subintervals:

$$x_0 = a, x_1 = a + \Delta x, \dots, x_{n-2} = a + (n - 2)\Delta x, x_{n-1} = a + (n - 1)\Delta x.$$

STEP III Approximate by adding the area of each rectangle:

$$L_n = f(x_0)\Delta x + f(x_1)\Delta x + \dots + f(x_{n-1})\Delta x.$$

## Sigma Notation

We use the symbol  $\sum$  to write a summation of numbers compactly:

$$\sum_{i=k}^n a_i$$

### EXAMPLE 2.

- Expand  $\sum_{i=1}^7 i$ .
- Write  $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}$  with the Sigma notation.
- Write  $1 + 3 + 5 + 7 + 9 + 11 + 13$  with the Sigma notation.

### Useful Sum Formulas:

- $\sum_{i=0}^n i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2};$
- $\sum_{i=0}^n i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6};$
- $\sum_{i=0}^n i^3 = 1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2.$

## Taking the Limit!

**EXAMPLE 3.** Show that the area of the region  $S$  in Example 1 is  $1/3$ . In other words, show that

$$\text{Area}(S) = \lim_{n \rightarrow \infty} R_n = 1/3.$$

General definition of Area: The area of the region  $S$  lying under the graph of a function  $y = f(x)$  from  $x = a$  to  $x = b$  is given by

- $\text{Area}(S) = \lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} \left( f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x \right)$
- $\text{Area}(S) = \lim_{n \rightarrow \infty} L_n = \lim_{n \rightarrow \infty} \left( f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x \right)$

## THE DISTANCE PROBLEM

If an object move at constant velocity, then the distance between the start and finish line is easy to compute:

$$\text{DISTANCE} = \text{VELOCITY} \times \Delta\text{TIME} .$$

What do we do if the velocity is not constant?

**EXAMPLE 4.** Suppose the odometer on our car is broken and we want to estimate the distance driven over a 30-second time interval. We take speedometer readings every five seconds and record them in the following table:

Time (s)	0	5	10	15	20	25	30
Velocity (ft/s)	25	31	35	43	47	45	41



Remark:

- The total distance is given by the area under the curve of the velocity function!