Math 644

Problem Set

Contents

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Chapter 1

Problem 1. Prove the parallelogram equality:

$$|z+w|^2 + |z-w|^2 = 2(|z|^2 + |w|^2).$$

Problem 2. Let w be a non-zero complex number and let $n \geq 1$ be a positive integrer. Using the polar coordinates, find n solutions to $z^n = w$.

Problem 3. Let z be a non-zero complex number. Show that 0, z, iz, and $iz+z$ are the vertices of a square.

Problem 4. Prove that there is no complex number z so that

$$|z| - z = i.$$

Problem 5. Find all complex numbers z satisfying the equation

$$4z - 3\overline{z} = \frac{1-18i}{2-i}.$$

Problem 6. Suppose that f is a continuous complex-valued function on a real interval $[a,b]$. Let

$$A = \frac{1}{b-a} \int_a^b f(x)\,dx.$$

a) Show that if $|f(x)| \leq |A|$ for all $x \in [a,b]$, then $f \equiv A$.

b) Show that if $|A| = \frac{1}{b-a} \int_a^b |f(x)|\,dx$, then $\arg f$ is constant modulo 2π on $\{z \,:\, f(z) \neq 0\}$.

Problem 7. Describe geometrically the following subsets:

a) $\operatorname{Re} z = \operatorname{Im} z$.

b) $\operatorname{Re} z > 0$.

c) $\operatorname{Im} z > 0$.

d) $\frac{\pi}{6} < \arg z < \frac{\pi}{4}$.

Problem 8. Let $\mathbb{T} := \{z \in \mathbb{C} \,:\, |z| = 1\}$. Prove that $\mathbb{T}$ equipped with the complex multiplication is a commutative group.

Problem 9. Suppose that $\lim_{n\to\infty} w_n = w$. Is it true then that also

$$\lim_{n\to\infty} \arg w_n = \arg w?$$

PROBLEM 10. Let $\{z_n\}$ be a sequence of complex numbers such that $\sum_{n=0}^{\infty} z_n$ converges and there is a ϕ such that $|\arg z_n| \leq \phi < \frac{\pi}{2}$ for any $n \geq 0$. Show that the series $\sum_{n=0}^{\infty} z_n$ is absolutely convergent.

PROBLEM 11. Let $\mathbb{C}^*$ be the extended plane, let $\mathbb{S}^2$ be the sphere $\{(X, Y, Z) : X^2+Y^2+Z^2 = 1\}$ and let $\pi : \mathbb{C}^* \to \mathbb{S}^2$ be the stereographic projection with $\pi(\infty) = (0, 0, 1)$.

a) Show that straight lines in $\mathbb{C}$ correspond exactly to circles on $\mathbb{S}^2$ passing through $(0, 0, 1)$.

b) Show that if $z \neq \infty$, then

$$\chi(z, \infty) = \frac{2}{\sqrt{1 + |z|^2}}.$$

c) Using the explicit formula of χ in terms of z and w, show that, for any $z, w \in \mathbb{C}^*$,

$$0 \leq \chi(z, w) \leq 2.$$

Chapter 2

Problem 12. For what values of z is

$$\sum_{n=0}^{\infty} \left(\frac{z}{1+z}\right)^n$$

convergent? Draw a picture of the region.

Problem 13. Suppose that $\sum_{n\geq 0} a_n(z-z_0)^n$ is a formal power series. Suppose that

$$R := \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

exists and is finite.

a) Show that the power series converges in $\{z : |z-z_0| < R\}$.

b) Show that the power series diverges in $\{z : |z-z_0| > R\}$.

Problem 14. Prove the following assertions.

a) If f and g are analytic at z_0, then $(f+g)'(z_0) = f'(z_0) + g'(z_0)$ (Sum rule of differentiation for analytic functions).

b) If f and g are analytic at z_0, then $(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$ (Product Rule of differentiation for analytic functions).

c) If f and g are analytic at z_0 with $g(z_0) \neq 0$, then $(fg)'(z_0) = \frac{f'(z_0)g(z_0)-f(z_0)g'(z_0)}{(g(z_0))^2}$ (Quotient rule of differentiation for analytic functions).

d) If f is analytic at z_0 and g is analytic at $f(z_0)$, then $(g\circ f)'(z_0) = g'(f(z_0))f'(z_0)$ (Chain Rule of differentiation for analytic functions).

Find the derivative of $(z-a)^{-n}$, where n is a positive integer and $a \in \mathbb{C}$.

Problem 15. Define $e^z = \exp(z) := \sum_{n\geq 0} \frac{z^n}{n!}$.

a) Show that this series converges for all $z \in \mathbb{C}$.

b) Show that $e^z e^w = e^{z+w}$ (using the power series definition).

c) Show that $|e^z| = e^{\mathrm{Re}\, z}$ and $\arg e^z = \mathrm{Im}\, z$.

d) Show that $\frac{d}{dz}e^z = e^z$.

e) Compute the integral

$$\int e^{nt}\cos(mt)\, dt.$$

[Hint: Rewrite $\cos(mt)$ as a complex exponential.]

f) Show that, for any non-zero integer n,

$$\int_0^{2\pi} e^{int}\, dt = 0.$$

PROBLEM 16. Suppose $\sum_{j=0}^{\infty} |a_j|^2 < \infty$.

a) Show that $f(z) = \sum_{j=0}^{\infty} a_j z^j$ is analytic in $\{z : |z| < 1\}$.

b) Compute (with a proof) the following quantity:

$$\lim_{r \to 1^-} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi}.$$

PROBLEM 17. Let $\Omega \subset \mathbb{C}$. Show that Ω is connected if and only if Ω and $\varnothing$ are the only open and closed subsets of Ω.

PROBLEM 18. Suppose that f and g are two analytic functions on a region (open and connected) Ω. Suppose there is a sequence $(z_n)_{n=1}^{\infty}$ with $z_n \in \Omega$ ($\forall n \geq 1$) such that $f(z_n) = g(z_n)$ ($\forall n \geq 1$). If (z_n) has an accumulation point $z_0 \in \Omega$, then show that $f \equiv g$ on Ω.

PROBLEM 19. Suppose f has a power series expansion at 0 which converges in all of $\mathbb{C}$. Suppose also that $\int_{\mathbb{C}} |f(x+iy)|\, dxdy < \infty$. Prove that $f \equiv 0$.

PROBLEM 20. Suppose f is analytic in a connected open set Ω such that, for each $z \in U$, there exists an n (depending on z) such that $f^{(n)}(z) = 0$. Prove that f is a polynomial.

PROBLEM 21. Let f be analytic in a region Ω containing the point $z = 0$. Suppose $|f(1/n)| < e^{-n}$ for $n \geq n_0$, for some integer $n_0 \geq 0$. Prove $f \equiv 0$.

PROBLEM 22. Suppose that $f(z) = az^3 + bz^2 + cz + d$. In addition, suppose that for each $z, w \in \mathbb{C}$ there exists a point ζ on the line segment between z and w with

$$\frac{f(z) - f(w)}{z - w} = f'(\zeta).$$

Show that $a = 0$.

PROBLEM 23. [Hard] Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ have radius of convergence 1 and suppose that $a_n \geq 0$ for all n. Prove that $z = 1$ is a singular point of f. That is, there is no function g analytic in a ball B containing $z = 1$ such that $f = g$ on $B \cap D$.

PROBLEM 24. Let f and g be analytic functions in a region Ω.

a) Show that if $f'(z) = 0$ for all z in a neighborhood of some $z_0 \in \Omega$, then f is constant in Ω, meaning there is a constant $c \in \mathbb{C}$ such that $f(z) = c$ for any $z \in \Omega$.

b) Show that if f and g are analytic in a region Ω with $f'(z) = g'(z)$ for every $z \in \Omega$, then $f - g$ is constant.

Problem 25. Suppose that $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges in $B = \{z : |z - z_0| < r\}$. Show that the power series

$$F(z) := \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z - z_0)^{n+1}$$

converges in B and satisfies $F'(z) = f(z)$ for all $z \in B$. Moreover, show that the radius of convergence of F is the same as the radius of convergence of f.

Problem 26. Show that $\cos^2(z) + \sin^2(z) = 1$ for every $z \in \mathbb{C}$.

Chapter 3

Problem 27. If f is analytic in a region Ω and if there is a $z_0 \in \Omega$ such that

$$|f(z_0)| = \inf_{z \in \Omega} |f(z)|,$$

and if $f(z_0) \neq 0$, then f is constant in Ω.

Problem 28. Let Ω be a region in $\mathbb{C}$. Show that if $f : \Omega \to \mathbb{C}$ is an open map, then f satisfies the maximum modulus principle.