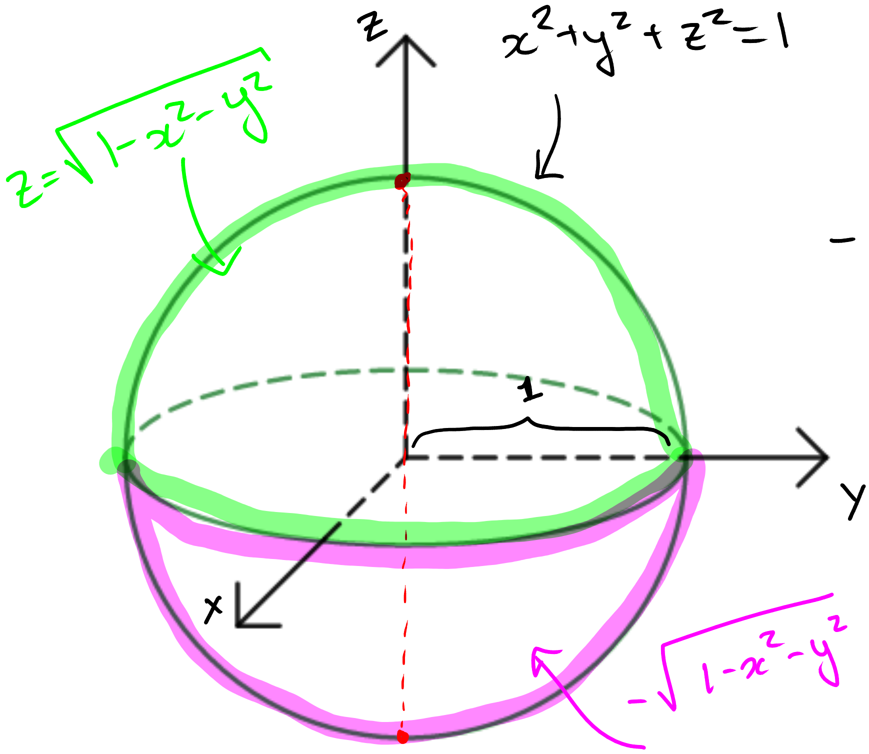
Chapter 15

Multiple Integrals

15.8 Triple integrals in spherical coordinates

Spherical coordinates

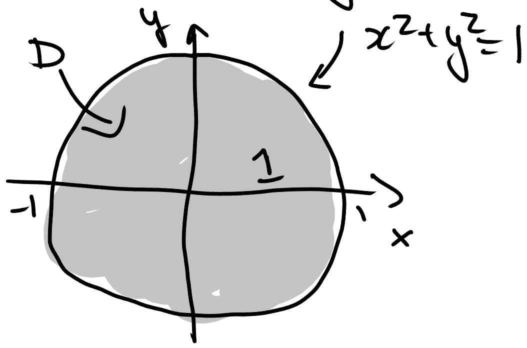
EXAMPLE. Describe the solid bounded by the sphere (picture below).



As a type 1:

$$-\sqrt{1-x^2-y^2} \le z \le \sqrt{1-x^2-y^2}$$

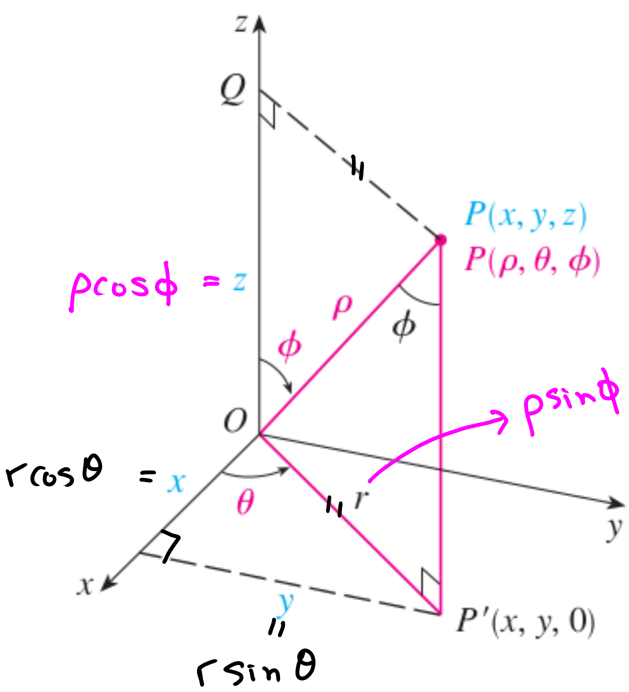
Shadow on xy-plane:



$$D=\{(x,y): -1 \le x \le 1, \; -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}\}$$

$$E=\{(x,y,z): -1 \le x \le 1, \; -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}, \; -\sqrt{1-x^2-y^2} \le z \le \sqrt{1-x^2-y^2}\}.$$

Definition



Cartesian $\longrightarrow$ Spherical

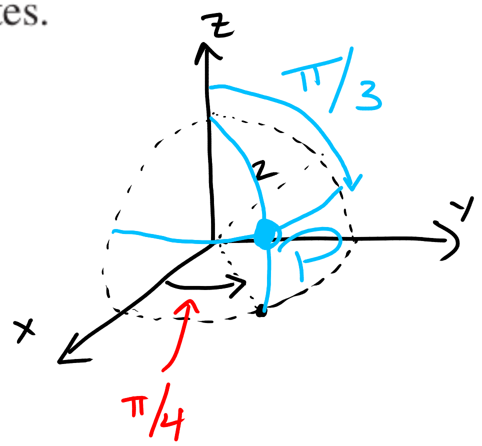
$$x = \rho\sin\phi\cos\theta$$
$$y = \rho\sin\phi\sin\theta$$
$$0 \le \theta \le 2\pi$$

$$z = \rho\cos\phi$$
$$0 \le \phi \le \pi, \quad \rho \ge 0.$$

EXAMPLE 1 The point $(2, \pi/4, \pi/3)$ is given in spherical coordinates. Plot the point and find its rectangular coordinates.

$\rho\ \theta\ \phi$

$\rho = 2$
$\theta = \pi/4$
$\phi = \pi/3$



$$x = \rho \sin\phi \cos\theta = 2 \sin(\pi/3) \cos(\pi/4)$$
$$= 2 \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \boxed{\frac{\sqrt{6}}{2}}$$

$$y = \rho \sin\phi \sin\theta = 2 \sin(\pi/3) \sin(\pi/4) = \boxed{\frac{\sqrt{6}}{2}}$$

$$z = \rho \cos\phi = 2\cos(\pi/3) = 2\left(\frac{1}{2}\right) = \boxed{1}$$

EXAMPLE 2 The point $(0, 2\sqrt{3}, -2)$ is given in rectangular coordinates. Find spherical coordinates for this point.

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{0^2 + (2\sqrt{3})^2 + (-2)^2}$$
$$= \sqrt{12 + 4} = \boxed{4}.$$

We have $z = \rho\cos\phi \Rightarrow -2 = 4\cos\phi$

$$\Rightarrow -\frac{1}{2} = \cos\phi$$
$$\Rightarrow \boxed{\phi = \frac{2\pi}{3}} \text{ (between 0 and } \pi)$$

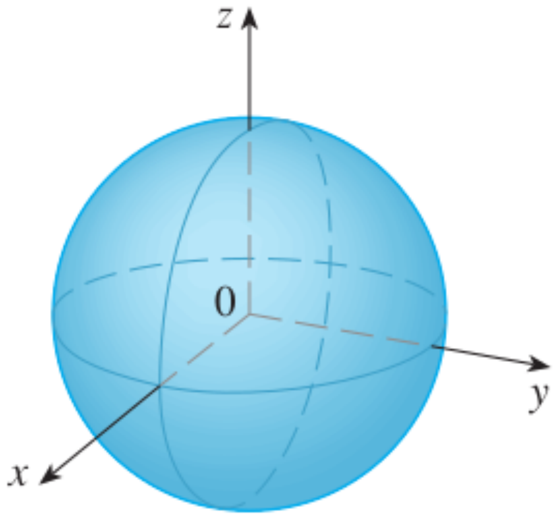
$$0 = x = 2\sin\left(\frac{2\pi}{3}\right)\cos(\theta) = \sqrt{3}\cos(\theta)$$
$$\Rightarrow 0 = \sqrt{3}\cos(\theta) \Rightarrow \cos\theta = 0 \Rightarrow \theta = \pi/2 \text{ or } \theta = 3\pi/2.$$

Here $y = 2\sqrt{3} \geq 0 \Rightarrow \boxed{\theta = \pi/2}$

Equations of important solids.

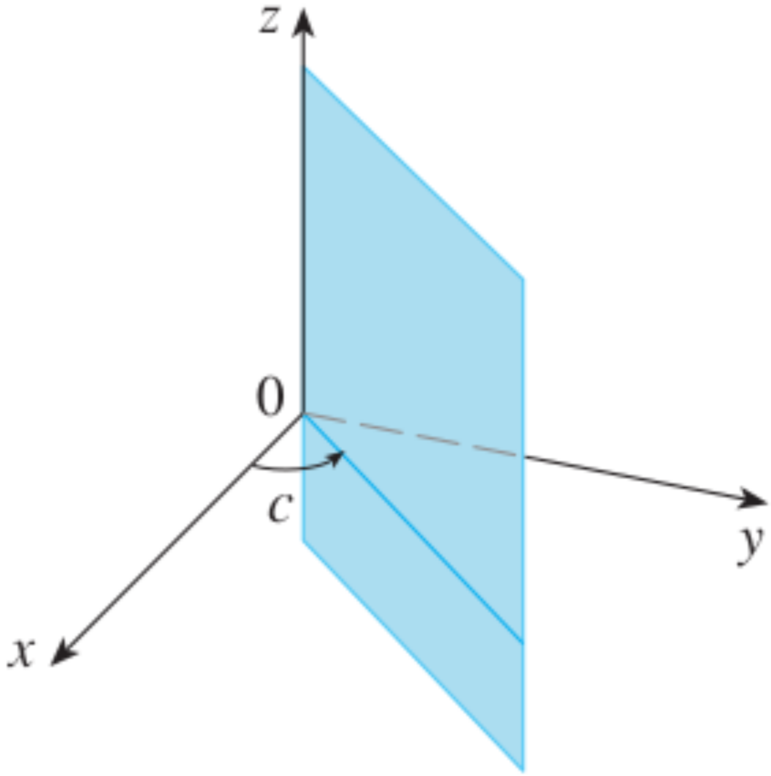
Sphere of radius R.

$$\boxed{\rho = R}$$



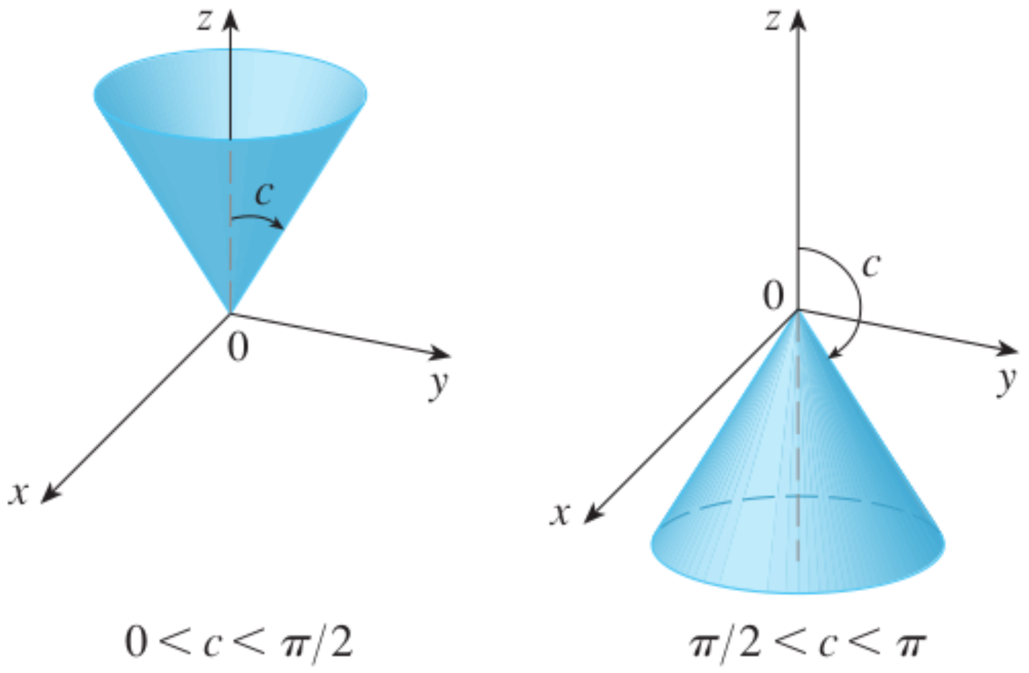
Half planes.

$$\boxed{\theta = c}$$

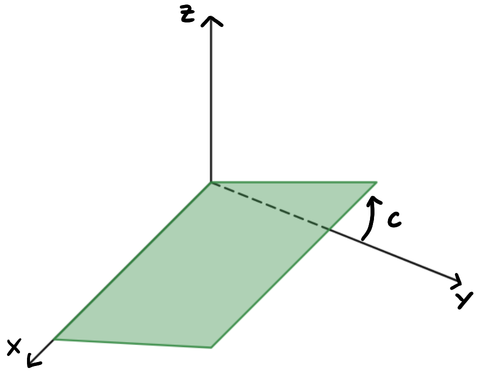


Cones.

$$\boxed{\phi = c}$$



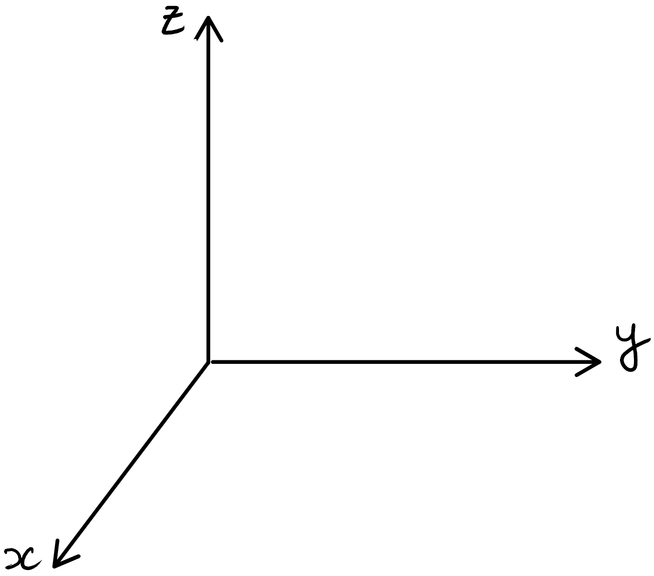
Question. Find the equation of the half-plane in the picture below in spherical coordinates. The plane is making an angle of c with the xy-plane.

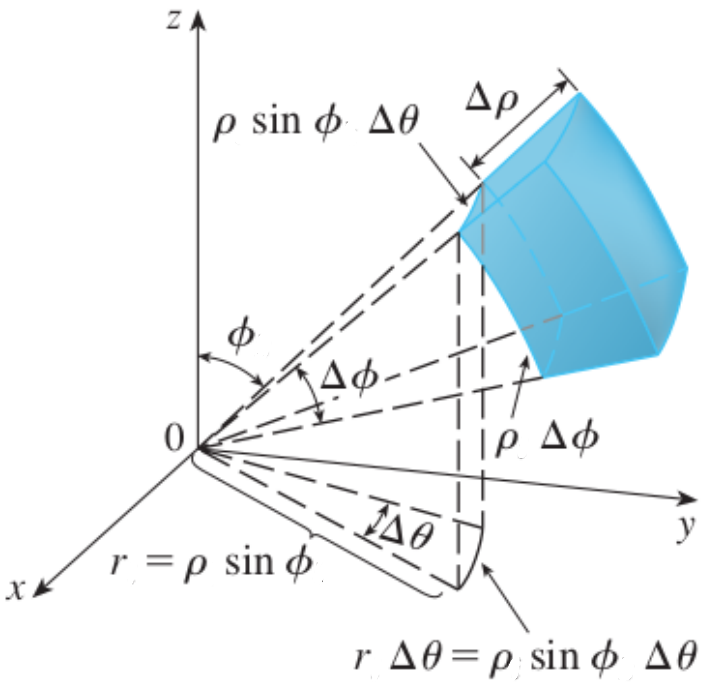


Evaluating integrals in sperical coordinates.

Spherical Wedge

$$E = \{(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi) \,:\, a \le \rho \le b,\, \alpha \le \theta \le \beta,\, c \le \phi \le d\}$$





We can show that

$$\Delta V = \rho^2 \sin\phi \,\Delta\rho \,\Delta\theta \,\Delta\phi$$

As the number of subdivisions goes to infinity, we obtain

$$\boxed{dV = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi}$$

Formula for the change of variable (in spherical coordinates).

$$\boxed{\iiint_E f(x,y,z)\, dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi))\; \rho^2 \sin(\phi)\, d\rho \, d\theta \, d\phi}$$

EXAMPLE 3 Evaluate $\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\, dV$, where B is the unit ball:

$$B = \{(x, y, z) \mid x^2 + y^2 + z^2 \leqslant 1\}$$

EXAMPLE 4 Use spherical coordinates to find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$.