Worksheet: Chapter 1

Math 307 — Linear Algebra and Differential equations — Summer 2022

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1. Determine if the following systems have a unique solution. In addition, if they possess a solution, find one using the method you want.

   (a)
   $$\begin{cases} 2x + y = 0 \\ x - 2y = 0 \end{cases}$$

   Answer: Unique. Trivial solution ($x = 0$, $y = 0$)

   (b)
   $$\begin{cases} 2x + y = 1 \\ -4x - 2y = -2 \end{cases}$$

   Answer: Non-unique. The solutions are $y = 1 - 2x$, for any x.

   (c)
   $$\begin{cases} 2x + y = 1 \\ x - 3y = -2 \end{cases}$$

   Answer: Unique. The solution is $x = 1/7$ and $y = 5/7$.

   (d)
   $$\begin{cases} x + 3y = -4 \\ 2x - y = 7 \\ x - y = 5 \end{cases}$$

   Answer: No solutions.

   (e)
   $$\begin{cases} x + 3y = 6 \\ 2x - y = -2 \\ x - 4y = -8 \end{cases}$$

   Answer: Unique. The solution is $x = 0$ and $y = 2$.

   (f)
   $$\begin{cases} x + 3y + z = -4 \\ 2x - y - z = 3 \end{cases}$$

   Answer: Not unique. The solutions are $x = (2z + 5)/7$ and $y = -(3z + 11)/7$, for any z.

2. Let
   $$A = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -1 & 4 \\ 5 & -4 & -4 \\ -2 & 2 & 5 \end{bmatrix}.$$
   Show that
   $$\det(AB) = \det(A)\det(B).$$

3. Let us consider two generic 3×3 upper triangular matrices A and B.

   (a) Show that the product AB will also be upper triangular.

   (b) Show that the product $A^T A$ is a symmetric matrix.

4. Can two non-square matrices commute for the matrix multiplication?
Answer: No, the operation(s) would not be well-defined.

5. Use Cramer's rule to solve the following system of linear equations:

$$\begin{cases} y + 2z = -1, \\ 2x + z = 15, \\ x + 2y = 1. \end{cases} \qquad \textit{Answer}: \qquad \begin{array}{l} x = 7, \\ y = -3, \\ z = 1. \end{array}$$

6. Use the Gaussian elimination method to solve the following system of linear equations:

$$\begin{cases} x + y + z = -3, \\ 2x + y + z = -4, \\ x - 4y + z = 2. \end{cases} \qquad \textit{Answer}: \qquad \begin{array}{l} x = -1, \\ y = -1, \\ z = -1. \end{array}$$

7. Use the Gauss–Jordan elimination method to solve the following system of linear equations:

$$\begin{cases} x = -3, \\ 2x + y = 0, \\ x - 4y + z = -24. \end{cases} \qquad \textit{Answer}: \qquad \begin{array}{l} x = -3, \\ y = 6, \\ z = 3. \end{array}$$

8. Use the elementary operations to find the inverse of the matrix

$$A = \begin{bmatrix} 3 & 2 & 5 \\ 1 & 2 & 0 \\ -2 & -1 & 1 \end{bmatrix}. \qquad \textit{Answer}: \qquad A^{-1} = \frac{1}{19} \begin{bmatrix} 2 & -7 & -10 \\ -1 & 13 & 5 \\ 3 & -1 & 4 \end{bmatrix}.$$

9. Use the adjoint to find the inverse of the matrices

$$A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}.$$

Answer:

$$A^{-1} = \frac{1}{\alpha\delta - \beta\gamma} \begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix} \quad \text{and} \quad B^{-1} = \frac{1}{9} \begin{bmatrix} -2 & 4 & 1 \\ 1 & -2 & 4 \\ 4 & 1 & -2 \end{bmatrix}.$$

10. Use the inverse to solve the following system of linear equations:

$$\begin{cases} 2x - y + 4z = 15/2, \\ 5x - 4y - 4z = -23/2, \\ -2x + 2y + 5z = 12. \end{cases} \qquad \textit{Answer}: \qquad \begin{array}{l} x = 1/2, \\ y = 3/2, \\ z = 2. \end{array}$$