Math 644

Chapter 2

Section 2.3: Power Series

Contents

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More complicated functions are found by taking limits of polynomials.

EXAMPLE 1. Study the series $\sum_{n=0}^{\infty} z^n$ for a fixed $z \in \mathbb{C}$.

Fix $z \in \mathbb{C}$.

1) If $|z| \geq 1$, then $z^n \not\to 0$ and therefore $\sum_{n=0}^{\infty} z^n$ diverges.

2) If $|z| < 1$. We have

$$S_n = \sum_{k=0}^{n} z^k = \frac{1 - z^{n+1}}{1-z}$$

$$\Rightarrow \left| S_n - \frac{1}{1-z} \right| = \frac{|z|^{n+1}}{|1-z|} \longrightarrow 0 \quad (n \to \infty)$$

So,

$$\sum z^n = \frac{1}{1-z} \qquad (|z| < 1)$$

Power series $(|z|<1)$

function defined on $\mathbb{C} \setminus \{1\}$.

Definition

A formal power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$$

is called a **convergent power series centered (or based) at z_0** if

$\exists\, r > 0$ s.t. $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ conv. in $\{z : |z-z_0| < r\}$.

Convention: $f(z_0) = a_0$ (to avoid 0^0).

Example 2. Find a convergent power series centered at $z_0 \neq a$ representing $\frac{1}{z-a}$.

Write

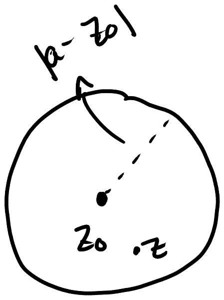
$$\frac{1}{z-a} = \frac{1}{z-z_0+z_0-a} = \frac{1}{-(a-z_0)} \cdot \frac{1}{1-\frac{z-z_0}{a-z_0}} = f(z)$$

If $\left|\frac{z-z_0}{a-z_0}\right| < 1$, $\longrightarrow$ $|z-z_0| < |a-z_0|$

$$\frac{1}{1-\frac{z-z_0}{a-z_0}} = \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(a-z_0)^n}$$

$$\Rightarrow \quad f(z) = \sum_{n=0}^{\infty} \frac{-(z-z_0)^n}{(a-z_0)^{n+1}} \qquad (*)$$

Note



- $(*)$ conv. in $|z-z_0| < |a-z_0|$

  Biggest disk in domain of $\frac{1}{z-a}$

- $(*)$ div. in $|z-z_0| \geq |a-z_0|$.

THEOREM 3. Let $r > 0$ and suppose that $\to$ $n \geq n_0$

a) $|a_n(z-z_0)^n| \leq M_n$ for every z such that $|z-z_0| \leq r$;

b) $\sum_{n=0}^{\infty} M_n < \infty$.

Then $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges uniformly and absolutely in the region

$$D_r := \{z : |z-z_0| \leq r\}.$$

Proof.

Abs conv. Let $z \in D_r$. & $A_N = \sum_{k=0}^{N} |a_n| |z-z_0|^n$.

Then,

$$A_N \leq \sum_{k=0}^{N} M_k \leq \sum_{k=0}^{\infty} M_k < \infty$$

$$\Rightarrow \quad (A_N)_{N=0}^{\infty} \text{ converges.}$$

Unif conv. Let $S_N = \sum_{k=0}^{N} a_n(z-z_0)^k$

We will show that (S_N) is unif. Cauchy.

For $z \in D_r$, & $N > M$,

$$|S_N - S_M| = \left| \sum_{k=M+1}^{N} a_k (z-z_0)^k \right| \leq \sum_{k=M+1}^{N} M_k.$$

$\exists K = K(\varepsilon)$ s.t. if $N > M \geq K$, $\sum_{j=M+1}^{N} M_j < \varepsilon$. So, (S_N) is uniformly Cauchy in D_r. $\square$

Note:

Convergence depends only on the tail of the series. So it is sufficient to satisfy the hypothesis only for $n \geq n_0$, for some non-negative integer n_0.

THEOREM 4. Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a formal power series. Let

$$R := \liminf_{n\to\infty} |a_n|^{-1/n} = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}} \in [0, \infty].$$

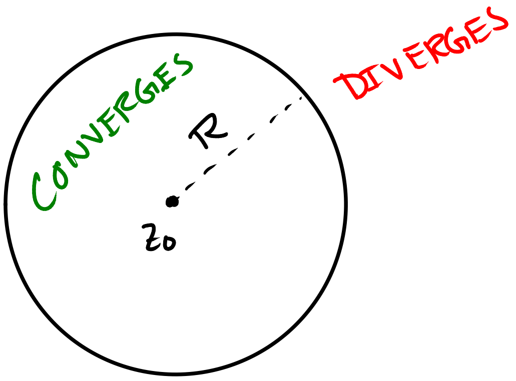
Then, the power series

a) converges abs. in $\{z : |z-z_0| < R\}$;

b) converges uniformly in $\{z : |z-z_0| \leq r\}$, for any $r < R$;

c) diverges in $\{z : |z-z_0| > R\}$.

Notes:



- R is called the radius of convergence of the power series.
- Biggest open disk where the power series converges is $\{z : |z-z_0| < R\}$.
- Information on the decay rate of a_n: for any $S < R$, there is an $n_0 \geq 0$ such that $|a_n| \leq S^{-n}$.

Proof.

Note: Root test does not give any information on the convergence of the power series on the circle

$$\{z : |z - z_0| = R\}.$$

Example 5. Find the Radius of convergence R of the following power series and study their behavior on the boundary of the disk of radius R.

A) $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n}$;

B) $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n^2}$;

C) $\displaystyle\sum_{n=1}^{\infty} n z^n$;

D) $\displaystyle\sum_{n=1}^{\infty} 2^{n^2} z^n$.

Example 6. Let $(a_n)_{n=0}^{\infty}$ be defined by

$$a_n = \begin{cases} 3^{-n} & \text{, if } n \text{ is even} \\ 4^n & \text{, if } n \text{ is odd.} \end{cases}$$

What is the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$.