Example 4

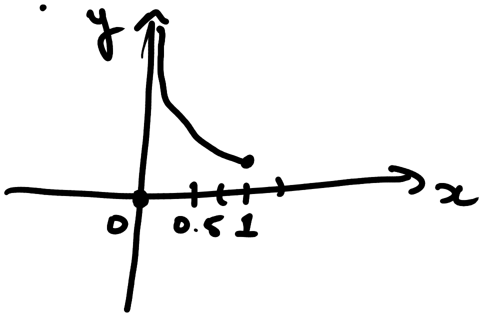
Does the function

$$f(x) = \begin{cases} \overset{\checkmark}{1/x} & 0 < x \le 1 \\ \underset{\nwarrow 2}{0} & x = 0 \end{cases}$$

have a maximum?

x	$f(x)$
0.5	2
0.25	4
0.10	10
0.01	100

x	$f(x)$
1	1
0.5	2
0.25	4
0.1	10
0.01	100



- One at $x = 1$ → $f(x) = 1 \le \frac{1}{x}$ $\forall x \le 1$. $(x \neq 0)$
- Another one at $x = 0$ → $f(0) = 0 \le 1 \le \frac{1}{x}$ $\forall x$.

Example 7

Let $f(x) = x^3$. What is $f'(0)$? Is $f(0)$ a local maximum or local minimum?

$f'(x) = 3x^2$

The solution to $f'(x) = 0$

$\iff 3x^2 = 0$

$\iff x = 0$

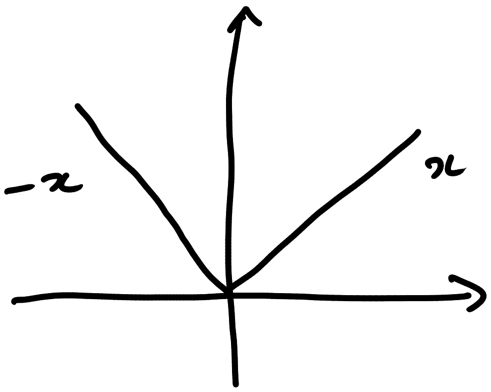
It seems that $x=0$ is a local max or local min.

x	$f(x)$
2	8
1	1
0	0
-1	-1
-2	-8

x is not a maximum and not a minimum!

Example 8

Let $f(x) = |x|$. Is $f(0)$ a local maximum, global maximum, local minimum, or global minimum?



Here $f'(0) \nexists$

But $f(0) = 0$ is an absolute minimum.

x	$f(x)$
-2	2
-1	1
0	0
1	1
2	2

Example 10

Find the critical numbers of $f(x) = x^{3/5}(4-x)$.

1) Find the derivative

$$f'(x) = (x^{3/5})'(4-x) + x^{3/5}\cdot(-1)$$

$$= \frac{3}{5}x^{-2/5}(4-x) - x^{3/5}$$

$$= \frac{3\cdot4}{5}x^{-2/5} - \frac{3}{5}x^{3/5} - x^{3/5}$$

$$= \frac{12}{5}x^{-2/5} - x^{3/5}\left(\frac{8}{5}\right)$$

$$\frac{12}{5x^{2/5}} - \frac{8x^{3/5}}{5} \swarrow$$

$$= \frac{12-8x}{5x^{2/5}}$$

$$= \frac{4(3-2x)}{5x^{2/5}}$$

2) Solve $f'(x)=0$

$$f'(x)=0 \iff \frac{4(3-2x)}{5x^{2/5}} = 0$$

$$\iff x = \frac{3}{2}$$

3) Find $f'(x)$ where $\nexists$

$f'(x)$ $\nexists$ when $x=0$

Answer
critical numbers
are $3/2$ & 0

Example 11

Find the absolute maximum and minimum values of the function $f(x) = \underline{x^3 - 3x^2 + 1}$ on the interval $\underline{[-1/2, 4]}$.

1) Find the critical numbers.

$$f'(x) = 3x^2 - 6x$$

Solve $f'(x) = 0$ (for x).

$$\iff 3x^2 - 6x = 0$$
$$\iff (3x - 6)x = 0$$
$$\iff 3x - 6 = 0 \quad \text{or} \quad x = 0$$
$$\iff x = 2 \quad \text{or} \quad x = 0$$

We have $f(0) = 1$ & $f(2) = -3$.

2) Compute the values of f at endpoints.

$a = \frac{-1}{2}$ & $b = 4$

$f(-1/2) = 1/8$

$f(4) = 17$.

3) Take max & min

$$\max\{1, -3, 1/8, 17\} = 17$$
$$\min\{1, -3, 1/8, 17\} = -3$$

Answers: abs maximum is 17

abs minimum is -3

Example 12

Show that the equation $x^3 + x - 1 = 0$ has at least one root.

Here $f(x) = x^3 + x - 1$.

$f(0) = 0 + 0 - 1 = -1$ $a = 0$

$f(1) = 1$ $b = 1$

So, $f(0) < 0$ & $f(1) > 0$.

So, $\exists c \in (0,1)$ s.t. $f(c) = 0$.

In other words, $c^3 + c - 1 = 0$.

Example 13

$f(x) = x^3+x-1$.

Show that the equation $x^3 + x - 1 = 0$ has exactly one root.

Suppose that there are two different roots a & b $(a \neq b)$.

So, $f(a) = 0 = f(b)$.

- f is cont. on $[a,b]$
- f is diff. on (a,b).

So, by Rolle's theorem, $f'(c) = 0$ for some $c \in (a,b)$.

$\updownarrow$ contradiction.

$f'(x) = 3x^2 + 1 \geq 1$, so $f'(x) \neq 0$.

So, we must conclude that there is only one root.

Example 14

Consider $f(x) = x^2$.

- Find the slope of the secant line passing through $Q = (\overset{x^2=0^2}{(0},0)$ and $P = (\overset{x^2=2^2}{2},4)$.
- Can you find a tangent line to $y = x^2$ with the same slope?

- the slope of PQ

$$m_{PQ} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-0}{2-0} = 2$$

- Here $f(x) = x^2$. We have $f'(x) = 2x$

So,

$$f'(x) = 2 \iff 2x = 2$$
$$\iff \boxed{x = 1}$$

Example 15

Find the numbers $c \in [0,2]$ such that the average of the function $f(x) = x^3 - x$ on the interval $[0,2]$ is attained by $f'(c)$.

Here, $a = 0$, $b = 2$.

$f(x) = x^3 - x \quad \rightarrow$

- continuous on $[0,2]$
- differentiable on $(0,2)$.

So, $\exists c \in (0,2)$ s.t.

$$f'(c) = \frac{f(2) - f(0)}{2 - 0}$$

$f'(x) = 3x^2 - 1$

$$\frac{f(2) - f(0)}{2 - 0} = \frac{6 - 0}{2} = 3$$

So,

$$3c^2 - 1 = 3$$

$$\Rightarrow \quad 3c^2 = 4$$

$$\Rightarrow \quad \sqrt{c^2} = \sqrt{4/3}$$

$$\Rightarrow \quad |c| = \sqrt{4/3}$$

$$\Rightarrow \quad c = +\sqrt{4/3} \text{ or } -\sqrt{4/3}$$

The slope of the tangent line is the same as the slope of the secant if

$\boxed{x = 2/\sqrt{3}}$ or $x = \cancel{-2/\sqrt{3}}$ → does not to $[0,2]$.

Example 17

Consider $f(x) = x^2$. Where is f increasing? Where is f decreasing? Go to Desmos!

- f increasing. $x_1 < x_2$ then $f(x_1) < f(x_2)$

$$\begin{tikzcd} f(x_1) \arrow[d] & < & f(x_2) \arrow[d] \\ x_1^2 & < & x_2^2 \end{tikzcd}$$

$x_1 = 2$ $\quad 2 < 3 \Rightarrow$

$x_2 = 3$

$$\begin{tikzcd} 2^2 \arrow[d] & & 3^2 \arrow[d] \\ 4 & < & 9 \end{tikzcd}$$

$x_1 = -2$ $\quad -2 < -1 \Rightarrow$

$x_2 = -1$

$$\begin{tikzcd} (-2)^2 \arrow[d] & & (-1)^2 \\ 4 & > & 1 \end{tikzcd}$$

Suppose $x_1 > 0$, $x_2 > 0$ and $x_1 < x_2$.

$x_1^2 < x_2^2 \Rightarrow f(x_1) < f(x_2)$

So, f is increasing when $x > 0$.

- f is decreasing if $x < 0$.

Example 18

If $f(x) = x^3 - x$, find where it is increasing and where it is decreasing. Go to Desmos

$$f'(x) = 3x^2 - 1 = (\sqrt{3}x - 1)(\sqrt{3}x + 1)$$

$$a^2 - b^2 = (a+b)(a-b).$$

then $f'(x) = 0 \iff \sqrt{3}x - 1 = 0$ or

$\sqrt{3}x + 1 = 0$

$\iff x = 1/\sqrt{3}$ or

$x = -1/\sqrt{3}$

x		$-1/\sqrt{3}$		$1/\sqrt{3}$	
$\sqrt{3}x+1$	$-$	0	$+$	$+$	$+$
$\sqrt{3}x-1$	$-$	$-$	$-$	0	$+$
$f'(x)$	$+$	0	$-$	0	$+$

$x < -1/\sqrt{3} \to \sqrt{3}x < -1 \to \sqrt{3}x + 1 < 0$

$x < 1/\sqrt{3} \to \sqrt{3}x - 1 < 0$

So, $f'(x) > 0$ when x is in $(\infty, -1/\sqrt{3})$ and $(1/\sqrt{3}, \infty) \to f$ is increasing there.

Also, $f'(x) < 0$ when x is in $(-1/\sqrt{3}, 1/\sqrt{3})$ $\to f$ decreases there.

Example 19

Let $f(x) = x^4 - 2x^3$. Find the local maximum and minimum values of f.

Go to Desmos

$$f'(x) = 4x^3 - 6x^2 = 2x^2(2x-3)$$

So, $f'(x) = 0 \iff x^2 = 0$ or $2x - 3 = 0$

$\iff x = 0$ or $x = 3/2$.

$x < \frac{3}{2}$

$2x-3<0$

x		$\downarrow$ 0		$\downarrow$ $3/2$	
2	+	+	+	+	+
x	−	0	+	+	+
x	−	0	+	+	+
$2x-3$	−	−	−	0	+
$f'(x)$	−	0	−	0	+

$x=0$ is not a local minimum or local max.

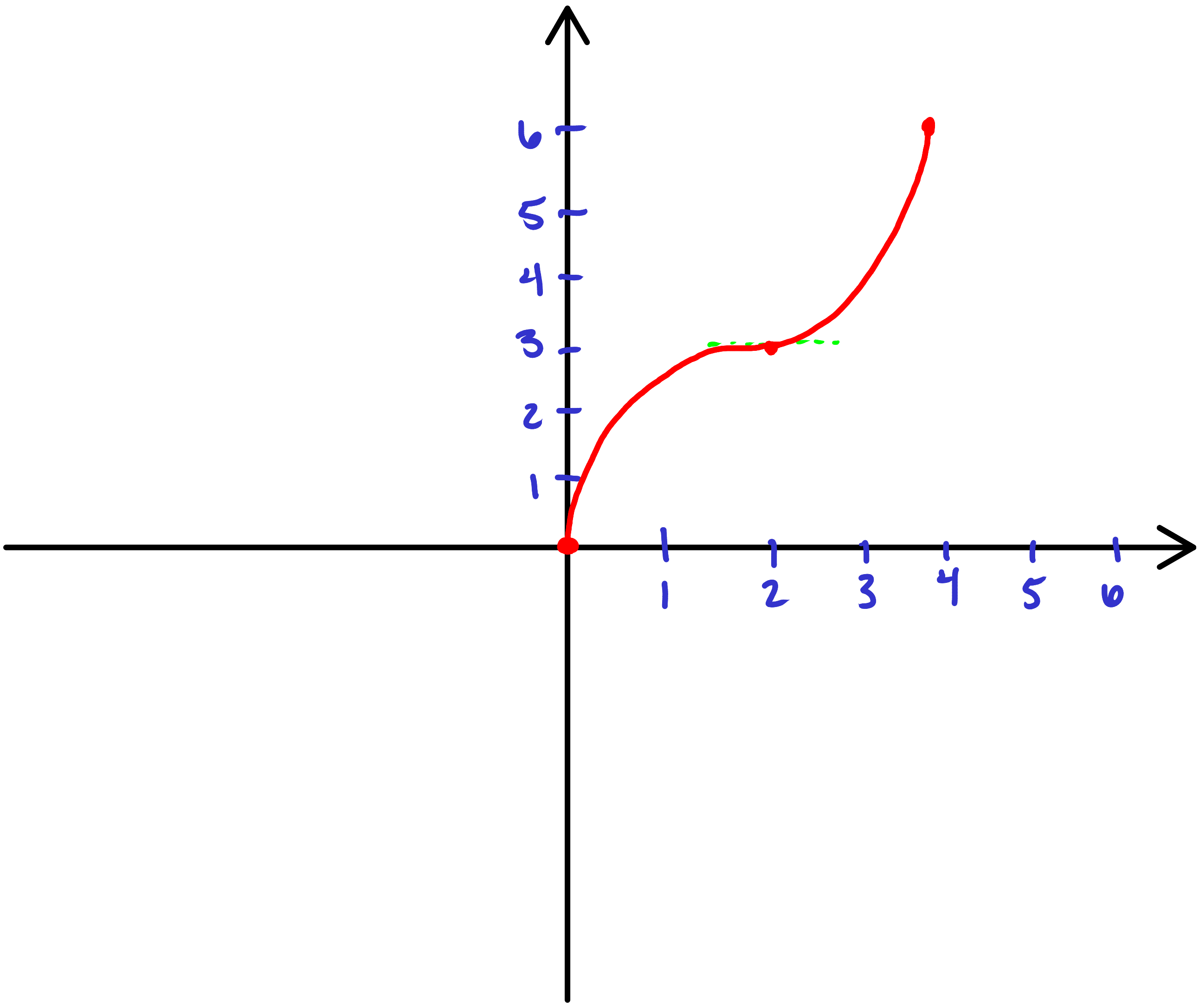
$x = 3/2$ is a local minimum.

$$f(3/2) = \frac{3^4}{2^4} - 2\cdot\frac{3^3}{2^3} = \boxed{\quad\quad}$$

Example 21

Sketch a possible graph of a function f that satisfies the following conditions:

- $f(0) = 0$, $f(2) = 3$, $f(4) = 6$, and $f'(2) = 0$.
- $f'(x) > 0$ for $0 < x < 2$ and $f'(x) > 0$ for $2 < x < 4$.
- $f''(x) < 0$ for $x < 2$ and $f''(x) > 0$ for $x > 2$.



Example 23

Let $f(x) = x^4 - 4x$.

- Find the region where the function is concave upward, concave downward.
- Find the inflection points and the local maxima/minima.
- Use this information to sketch the curve

Zeros. $f(x) = 0 \iff x^4 - 4x = 0$

$\iff x = 0$ or $x = \sqrt[3]{4}$.

Derivative: $f'(x) = 4x^3 - 4 = 4(x^3 - 1)$.

Now, $f'(x) = 0 \iff x^3 - 1 = 0 \iff x = 1$

we also write: $x^3 - 1 = (x-1)(x^2+x+1)$

$\Rightarrow f'(x) = 4(x-1)(x^2+x+1)$.

where x^2+x+1 is never zero and > 0.

2nd derivative: $f''(x) = 12x^2$. So

$f'(x) = 0 \iff x = 0$.

We write $f''(x) = 12 \cdot x \cdot x$.

We remark that $f''(1) = 12 > 0$. So

$x = 1$ is a local minimum. Also,

$f''(x) > 0 \quad \forall x \neq 0 \Rightarrow x = 1$ is an abs. max

Table for f'

x		1	
4	$+$	$+$	$+$
$x-1$	$-$	0	$+$
x^2+x+1	$+$	$+$	$+$
$f'(x)$	$-$	0	$+$

We have $f'(x) < 0 \Rightarrow f$ decreasing on $(-\infty, 1)$

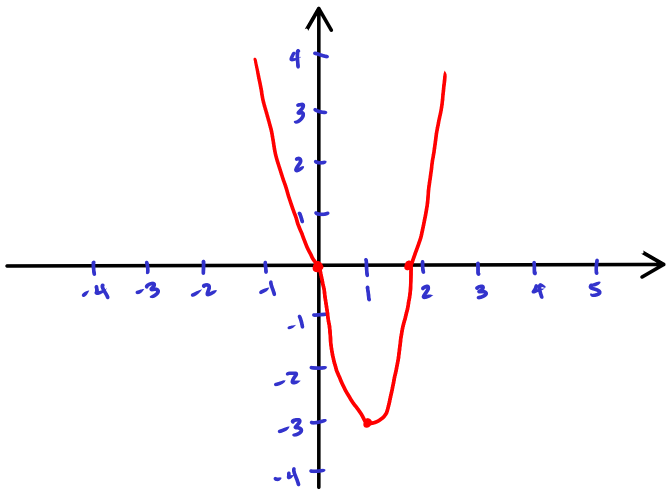
We have $f'(x) > 0 \Rightarrow f$ increasing on $(1, \infty)$

Table for f''

x		0	
12	$+$	$+$	$+$
x	$-$	0	$-$
x	$-$	0	$-$
$f''(x)$	$+$	0	$+$

We have $f''(x) > 0$ on both sides of 0.
So $x=0$ is not an influxion point.
there is no inflection point.

Sketch:



graph of f