## Section 16.5, Problem 10

(a) By definition, div $\vec{F} = P_x + Q_y + R_z = P_x + Q_y$ because $R$ is independent of $z$.

If we look only at the $x$ components of $\vec{F}$, that is $P$, we observe that when $y$ varies, there is no change in $P$ (the first coordinate of the vectors in $\vec{F}$), but when $x$ varies, there is a change in $P$. This change is positive because when $x$ increases, the values of $P$ increasin, this means that $P_x > 0$.

If we look only at the $y$-components of the vectors in $\vec{F}$, that is $Q$, we observe that when $x$ varies, there is no change in $Q$ (the second coordinate of the vectors in $\vec{F}$), but when $y$ varies, there is a change in $Q$. This change is positive because when $y$ increases, the values of $Q$ increase, this means that $Q_y > 0$.

Thus, overall, we have $P_x + Q_y > 0$, meaning that div$\vec{F} > 0$.

(b) The fact that $\vec{F}$ doesn't depend on $z$ implies that curl$\vec{F} = \langle 0, 0, Q_x - P_y \rangle$. Thus, depending on the sign of $Q_x - P_y$, the vector curl$\vec{F}$ is orthogonal to the $XY$-plane and points in the direction of the positive $z$-axis if $Q_x - P_y > 0$ and in the direction of the negative $z$-axis if $Q_x - P_y < 0$.

$\boxed{\textbf{Section 16.6, Problem 2 (only Q)}}$

We have to check if there are $u, v$ such that $\vec{r}(u,v) = \langle 2, 3, 3 \rangle$.

For the point $Q$, this means we have to solve the three following equations:

$$1 + u - v = 2, \quad u + v^2 = 3, \quad u^2 - v^2 = 3.$$

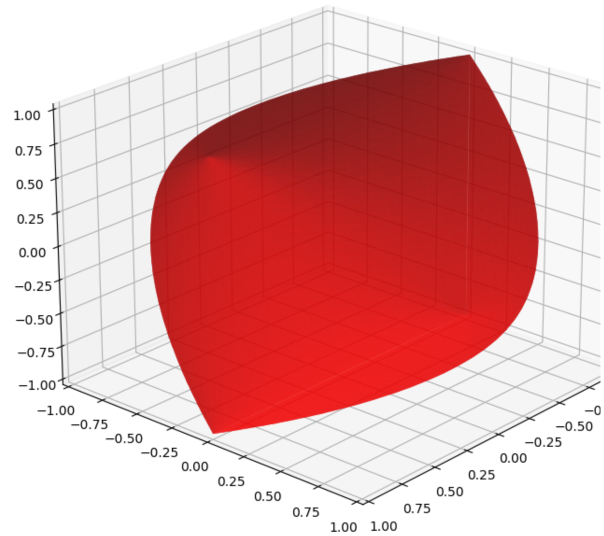
Adding the second equation to the third equation, we obtain

$$u^2 + u = 6 \quad \Rightarrow \quad u^2 + u - 6 = 0 \quad \Rightarrow \quad (u+3)(u-2) = 0.$$

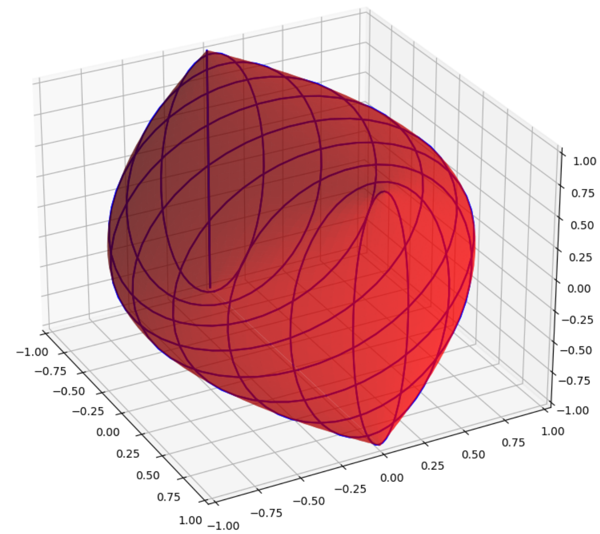
The solutions are $u = -3$ and $u = 2$. we just need one value, say $u = 2$. From the first equation, we see that $u = v$ and so $v = 3$. We just found $(u, v)$ such that $\vec{r}(u,v) = \langle 1, 2, 1 \rangle$ which mean that the point $P$ lies on the surface.

**Section 16.6, Problem 12**

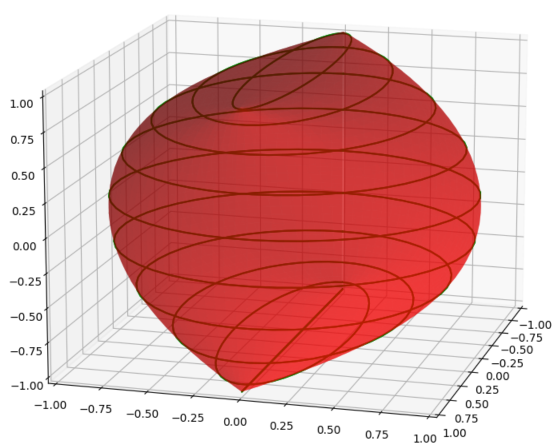
Using either the software on the web, or the python script that I provided you, you obtain the following images.



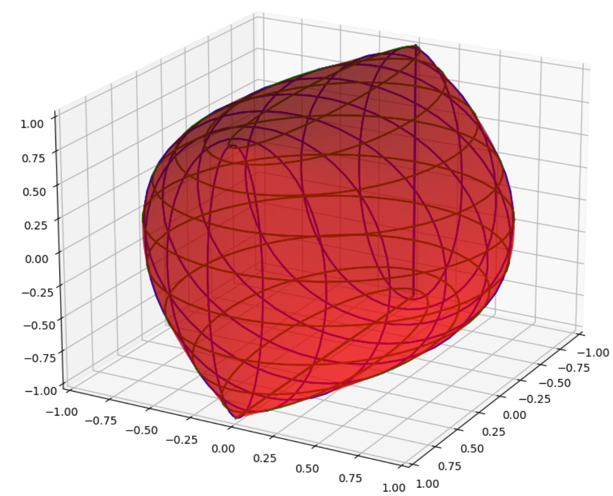

(a) Graph of the surface




(b) Latitudes when $u$ is constant




(c) Longitudes when $v$ is constant




(d) Grid on the surface

This surface is really funny, it looks like a pillow, a really confortable pillow! :)

$\boxed{\textbf{Section 16.6, Problem 20}}$

The parametric equation is

$$\vec{r}(u,v) = \langle 0,-1,5\rangle + u\,\langle 2,1,4\rangle + \langle -3,2,5\rangle = \langle 2u-3v,-1+u+2v,5+4u+5v\rangle\,.$$

**Section 16.6, Problem 26**

The point $(0,0,3)$ lies in the plane. The intersection of a plane and a cylinder is a circle in 3D. So the region will be the interior of a circle (but the circle is not parallel to one of the three planes).

An efficient way of solving this problem is to find two orthogonal vector $\vec{a}$ and $\vec{b}$ parallel to the plane such that they belong to the cylinder and then take a linear combinaison $\langle 0,0,3\rangle + u\vec{a} + v\vec{b}$ where $u^2+v^2 \leq 1$.

A vector parallel to the plane $z = x+3$ is a vector $\vec{a} = \langle a_1, a_2, a_3\rangle$ which is orthogonal to the normal vector of the plane. The normal vector of the plane is $\vec{n} = \langle -1, 0, 1\rangle$. We would also like the tip of the vector $\vec{a}$ belongs to the cylinder, so we also require that $a_1^2 + a_2^2 = 1$. We have to solve

$$\begin{aligned} \vec{a}\cdot\vec{n} &= 0 \\ a_1^2 + a_2^2 &= 1. \end{aligned}$$

This system is explicitly:

$$\begin{aligned} -a_1 + a_3 &= 0 \\ a_1^2 + a_2^2 &= 1. \end{aligned}$$

Since $a_2$ is free, we may put $a_2 = 0$ and so $a_1 = \pm 1$. We keep $a_1 = 1$ and from the first equation, we get $a_3 = 1$. Our vector is then $\vec{a} = \langle 1, 0, 1\rangle$.

We have to find a vector $\vec{b} = \langle b_1, b_2, b_3\rangle$ perpendicular to $\vec{a}$ and lying on the cylinder. These conditions give the following system of equations:

$$\begin{aligned} \vec{b}\cdot\vec{a} &= 0 \\ b_1^2 + b_2^2 &= 1. \end{aligned}$$

Explicitly, it gives the following system of equations:

$$\begin{aligned} b_1 + b_3 &= 0 \\ b_1^2 + b_2^2 &= 1. \end{aligned}$$

A solution to this system is $b_1 = b_3 = 0$ and $b_2 = 1$. So, we obtain $\vec{b} = \langle 0, 1, 0\rangle$.

Now, we can combine the vectors $\vec{a}$ and $\vec{b}$ with the vector $\langle 0,0,3\rangle$ (the points on the plane), to obtain

$$\vec{r}(u,v) = \langle 0,0,3\rangle + u\vec{a} + v\vec{b}.$$

Since $u^2+v^2 \leq 1$, we can use polar coordinates $u = \rho\cos\theta$ and $v = \rho\sin\theta$ with $0 \leq \rho \leq 1$ and $0 \leq \theta \leq 2\pi$. Thus, we get, after collecting all the terms together, the following parametrization of the surface:

$$\vec{r}(u,v) = \langle \rho\cos\theta, \rho\sin\theta, 3+\rho\cos\theta\rangle\,.$$

There is another solution, which is even simpler. We want the portion of the plane inside the cylinder $x^2+y^2 = 1$. This means that we want the region $x^2+y^2 \leq 1$. We can parametrized this region in polar coordinates by setting $x = \rho\cos\theta$ and $y = \rho\sin\theta$ with $0 \leq \rho \leq 1$ and $0 \leq \theta \leq 2\pi$. We can then replace the value of $x$ inside the expression of $z$ to get $z = 3+\rho\cos\theta$. We then get

$$\vec{r}(\rho,\theta) = \langle \rho\cos\theta, \rho\sin\theta, 3+\rho\cos\theta\rangle\,.$$

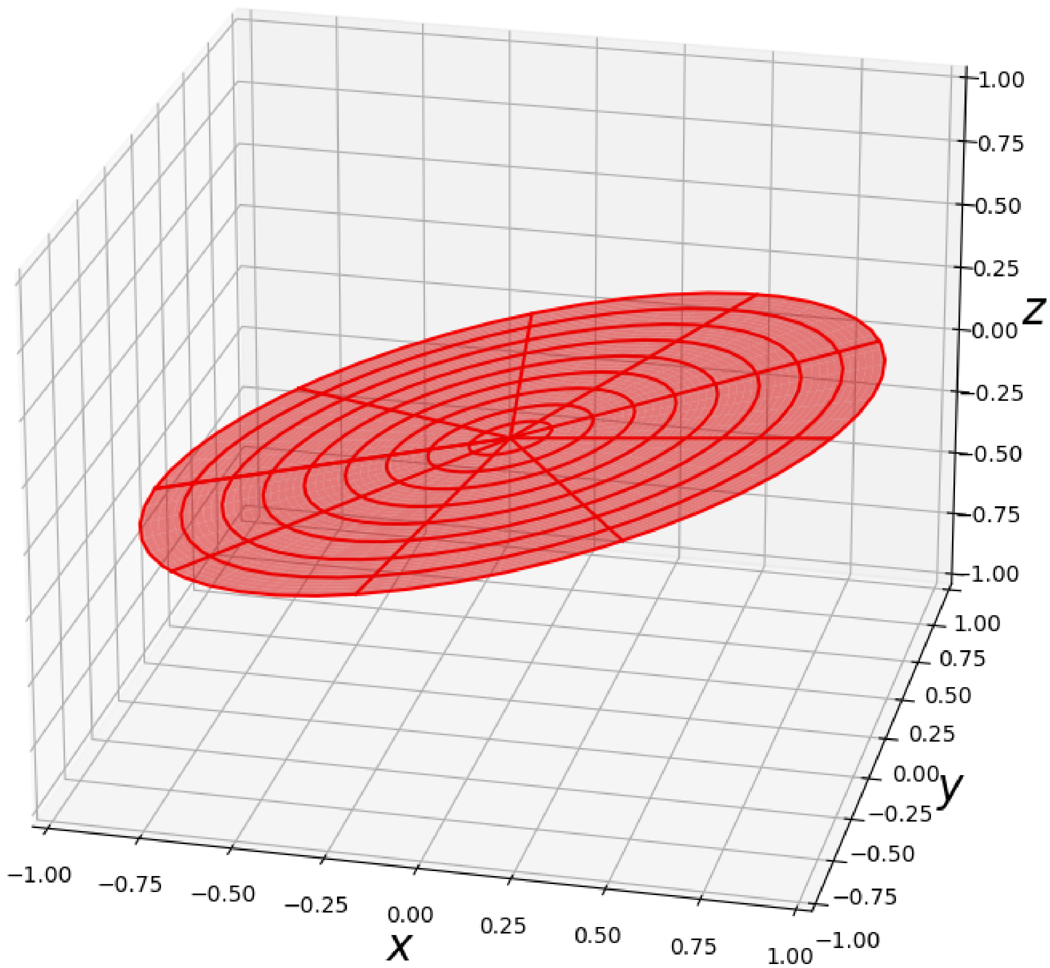



Figure 2: Surface obtained from the parametrization

Section 16.6, Problem 38

We have $\vec{r}_u = \langle -2u, 0, -1\rangle$ and $\vec{r}_v = \langle -2v, -1, 0\rangle$.

We have to find the point $(u_0, v_0)$ such that $\vec{r}(u_0, v_0) = (-1, -1, -1)$. Analyzing the second and third components, we see that $v = 1$ and $u = 1$. Thus, the tangent vectors at $(-1, -1, -1)$ are

$$\vec{r}_u(1,1) = \langle -2, 0, -1\rangle \quad \text{and} \quad \vec{r}_v(1,1) = \langle -2, -1, 0\rangle\,.$$

Thus, the parametric equation of the tangent plane is

$$\vec{r}_\Pi(u,v) = \langle -1,-1,-1\rangle + u\vec{r}_u(1,1) + v\vec{r}_v(1,1) = \langle -1-2u-2v, -1-v, -1-u\rangle$$

where $\Pi$ is the name of the plane (the symbol $\Pi$ is the capital $p$ in greek).

Using python, we obtain the following picture of the tangent plane and the surface.

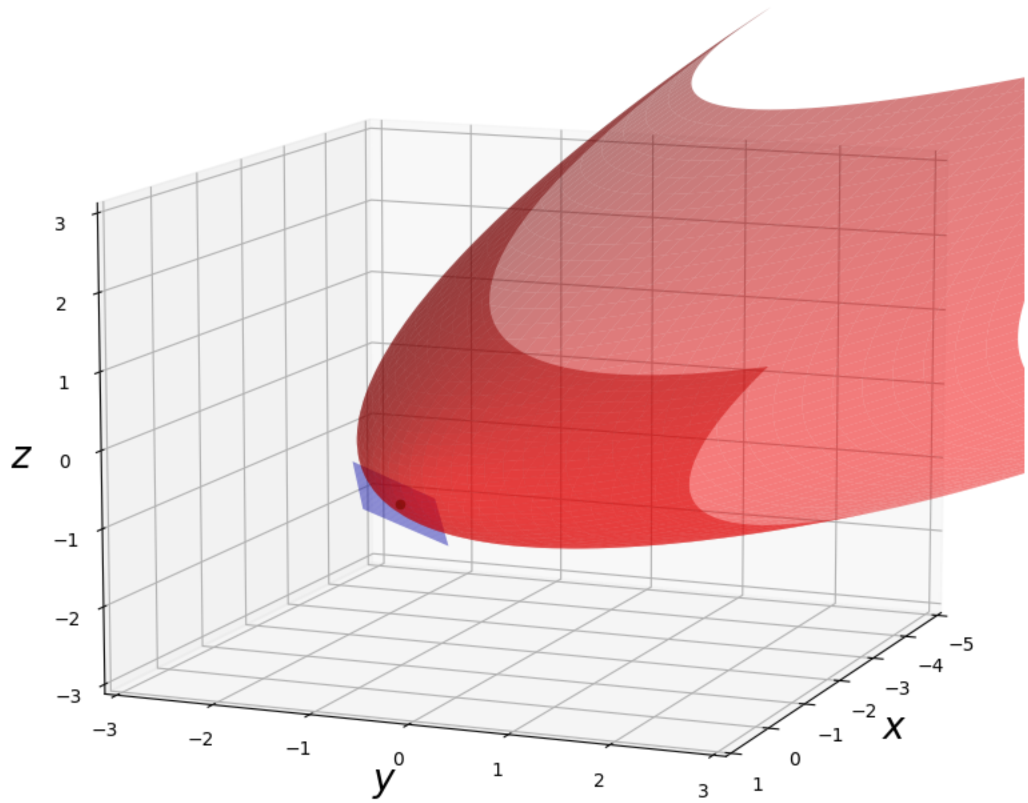



Figure 3: Graph of the surface and its tangent vector