Section 3.2, Problem 20

The function $f(x) = 2x - 1 - \sin x$ is continuous. It is also differentiable at every point. We can apply the IVT and the MVT.

We first use the IVT to show that there is at least one root. We see that $f(0) = -1 < 0$ and $f(\pi) = 2\pi - 1 > 0$. So, letting $N = 0$ in the IVT, we conclude that there is a number c between 0 and π such that $f(c) = 0$.

We secondly use the MVT to show that there is only one root. The derivative of $f(x)$ is $f'(x) = 2 - \cos x$. If there were two roots to the equation $f(x) = 0$, call them c_1 and c_2, then $f(c_1) = f(c_2) = 0$ and from the MVT we conclude that there is a $\tilde{c}$ between c_1 and c_2 such that $f'(\tilde{c}) = 0$. But $f'(x) = 2 - \cos x > 0$ for any number x because $-1 \leq \cos x \leq 1$. This is a contradiction. So, there must be only one root to the equation $f(x) = 0$.

$\boxed{\textbf{Section 3.3, Problem 10}}$

To solve each part of the exercise, we need the first and second derivatives. We have

$$f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x - 2)(x - 1)$$

and

$$f''(x) = 6x - 18 = 6(x - 3).$$

(a) The derivative is defined everywhere. The zeros are $x = 2$ and $x = 1$.

When $x < 1$, then the signs of the factors are $-$ and $-$, so overall $f'(x)$ is positive. The function is then increasing on $(-\infty, 1)$.

When $1 < x < 2$, then the signs of the factors are $-$ and $+$, so overall $f'(x)$ is negative. The function is then decreasing on $(1, 2)$.

When $x > 2$, then the signs of the factors are $+$, and $+$, so overall $f'(x)$ is positive. The function is then increasing on $(2, \infty)$.

(b) From the first derivative test, the value $f(1) = 2$ is local maximum value (from decreasing to increasing).

From the first derivative test, the value $f(2) = 1$ is a local minimum value (from increasing to decreasing).

(c) The second derivative is defined everwhere. The only zero is $x = 3$.

When $x < 3$, then $f''(x)$ is negative. The function is concave down on $(-\infty, 3)$.

When $x > 3$, then $f''(x)$ is positive. The function is concave up on $(3, \infty)$.

Section 3.3, Problem 12

To solve each part of the exercise, we need the derivative and second derivative. We have

$$f'(x) = \frac{x^2 + 1 - 2x^2}{(x^2+1)^2} = \frac{1-x^2}{(1+x^2)^2} = \frac{(1+x)(1-x)}{(1+x^2)^2}$$

and

$$f''(x) = \frac{2x(x^2-3)}{(x^2+1)^3} = -\frac{2x(\sqrt{3}-x)(\sqrt{3}+x)}{(1+x^2)^3}.$$

(a) The derivative is well-defined everywhere. The zeros are $x = \pm 1$. When $x < -1$, the factor $1 + x$ is negative, $1 - x$ is positive and so $f'(x)$ is negative. The function is then decreasing on $(-\infty, -1)$.

When $-1 < x < 1$, then the factor $1+x$ is positive, $1-x$ is positive and so $f'(x)$ is positive. The function is then increasing on $(-1, 1)$.

When $x > 1$, then the factor $1+x$ is positive, $1-x$ is negative and so $f'(x)$ is negative. The function is then decreasing on $(1, \infty)$.

(b) From the first derivative test, the values at $x = -1$ is a local minimum (fron decreasing to increasing). So the local minimum value is $f(-1) = -1/2$.

From the first derivative test, the values at $x = 1$ is a local maximum (from increasing to decreasing). So the local maximum value is $f(1) = 1/2$.

(c) The second derivative exists everywhere. The zeros are $x = 0$, $x = \pm\sqrt{3}$.

When $x < -\sqrt{3}$, the signs of the factors in the numerator are $-$, $-$, $+$, and $-$, so overall $f''(x)$ is negative. The function is then concave downward on $(-\infty, -\sqrt{3})$.

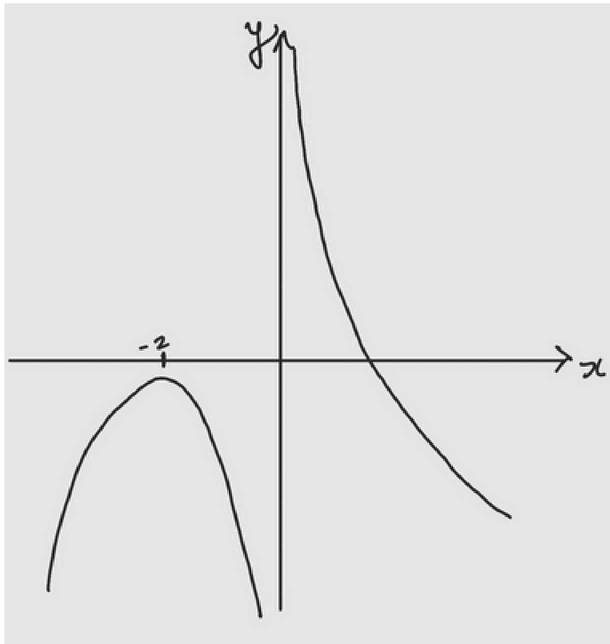
When $-\sqrt{3} < x < 0$, the signs of the factors in the numerator are $-$, $-$, $+$, and $+$, so overall $f''(x)$ is positive. The function is then concave upward on $(-\sqrt{3}, 0)$.

When $0 < x < \sqrt{3}$, the signs of the factors in the numerator are $-$, $+$, $+$, and $+$, so overall $f''(x)$ is negative. The function is then concave downward on $(0, \sqrt{3})$.

When $x > \sqrt{3}$, the signs of the factors in the numerator are $-$, $+$, $-$, and $+$, so overall $f''(x)$ is positive. The function is then concave upward on $(\sqrt{3}, \infty)$.

Section 3.3, Problem 22

Here is a possible graph of a function with the desire properties.



Section 3.4, Problem 8

We factor the greatest power of x:

$$\frac{9x^3+8x-4}{3-5x+x^3} = \frac{x^3(9+8/x^2-4/x^3)}{x^3(3/x^3-5/x^2+1)} = \frac{9+8/x^2-4/x^3}{3/x^3-5/x^2+1}.$$

We have

$$\lim_{x\to\infty} 9+8/x^2-4/x^3 = \lim_{x\to\infty} 9+8\lim_{x\to\infty} 1/x^2 - 4\lim_{x\to\infty} 1/x^3 = 9+8\times 0-4\times 0 = 9$$

and

$$\lim_{x\to\infty} 3/x^3-5/x^2+1 = 3\lim_{x\to\infty} 1/x^3 - 5\lim_{x\to\infty} 1/x^2 + \lim_{x\to\infty} 1 = 3\times 0-5\times 0+1 = 1.$$

So, we obtain

$$\lim_{x\to\infty} \frac{9x^3+8x-4}{3-5x+x^3} = \lim_{x\to\infty} \frac{9+8/x^2-4/x^3}{3/x^3-5/x^2+1} = \frac{\lim_{x\to\infty} 9+8/x^2-4/x^3}{\lim_{x\to\infty} 3/x^3-5/x^2+1} = \frac{9}{1} = 9$$

and then

$$\lim_{x\to\infty} \sqrt{\frac{9x^3+8x-4}{3-5x+x^3}} = \sqrt{\lim_{x\to\infty} \frac{9x^3+8x-4}{3-5x+x^3}} = \sqrt{9} = 3.$$

Section 3.4, Problem 18

We have

$$\sqrt{1+4x^6} = \sqrt{x^6(1/x^6+4)} = |x|^3\sqrt{1/x^6+4}.$$

Now, since $x<0$, we have $|x| = -x$ and so

$$\sqrt{1+4x^6} = -x^3\sqrt{1/x^6+4}.$$

Then, we can rewrite the limit and compute it:

$$\lim_{x\to-\infty}\frac{\sqrt{1+4x^6}}{2-x^3} = \lim_{x\to-\infty}\frac{-x^3\sqrt{1/x^6+4}}{x^3(2/x^3-1)} = \lim_{x\to-\infty}-\frac{\sqrt{1/x^6+4}}{2/x^3-1} = -\frac{\sqrt{\lim_{x\to-\infty}1/x^6+4}}{\lim_{x\to-\infty}2/x^3-1} = 2.$$