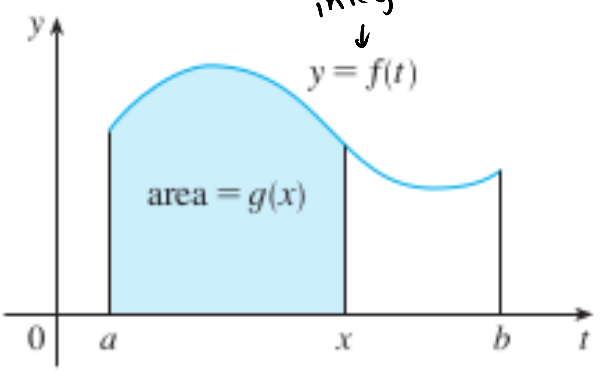
Chapter 4
Integrals

4.3 The Fundamental Theorem of Calculus

Area Function.



$$\int_a^b \underbrace{f(t)}_{\text{integrand}}\,dt.$$

$$g(x) = \int_a^x f(t)\,dt$$

EXAMPLE 1 If f is the function whose graph is shown in Figure 2 and $g(x) = \int_0^x f(t)\,dt$, find the values of $g(0)$, $g(1)$, $g(2)$, $g(3)$, $g(4)$, and $g(5)$. Then sketch a rough graph of g.

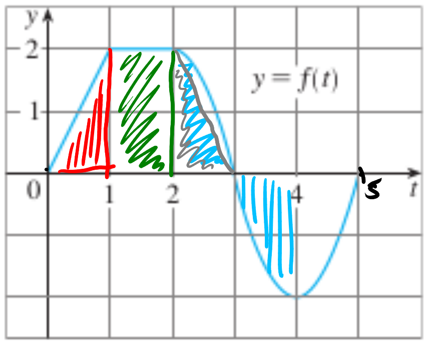


FIGURE 2

$\underline{g(0)}$ $\quad g(0) = \int_0^0 f(t)\,dt = \boxed{0}$.

$\underline{g(1)}$ $\quad g(1) = \int_0^1 f(t)\,dt = A(\triangle) = \frac{2\cdot 1}{2} = \boxed{1}$

$\underline{g(2)}$ $\quad g(2) = \int_0^2 f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt$

$$= A(\triangle) + A(\square)$$

$$= \frac{2\cdot 1}{2} + 2\cdot 1 = \boxed{3}$$

$\underline{g(3)}$ $\quad g(3) = \int_0^3 f(t)\,dt = \int_0^2 f(t)\,dt + \int_2^3 f(t)\,dt$

$$3 + A(D) \approx 1.5$$

$$= \boxed{4.5}$$

$\underline{g(4)}$ $\quad g(4) = \int_0^4 f(t)\,dt = \int_0^2 f(t)\,dt + \int_2^3 f(t)\,dt + \int_3^4 f(t)\,dt$

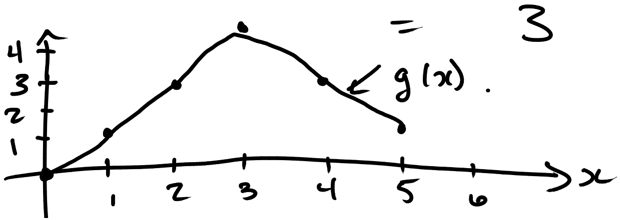
$$= 3 + \cancel{A(D)} - \cancel{A(D)}$$

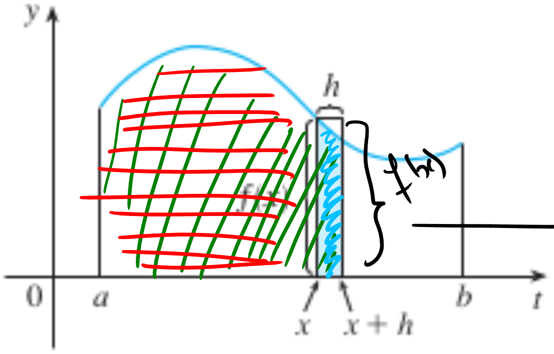
$$= 3$$

$\underline{g(5)}$ $\quad g(5) = \int_0^5 f(t)\,dt = \int_0^4 f(t)\,dt + \int_4^5 f(t)\,dt$

$$= 3 - 1.5 = 1.5$$

Graph.

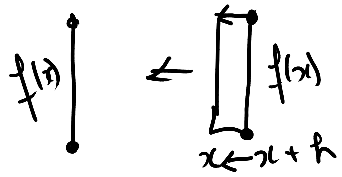




$$\lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = g'(x)$$

$$\to g(x+h) - g(x) \approx h \cdot f(x)$$

$$\to \frac{g(x+h)-g(x)}{h} \approx f(x)$$



$$h \to 0 \quad \to \quad g'(x) = f(x)$$

FTC, I

The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\,dt \qquad a \leqslant x \leqslant b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

Leibniz notation.

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$$

EXAMPLE 2 Find the derivative of the function $g(x) = \int_0^x \overbrace{\sqrt{1+t^2}}^{\text{integrand.}}\,dt$.

$$f(x) = \sqrt{1+x^2}$$

$$g'(x) = \underbrace{\sqrt{1+x^2}}_{f(x)}.$$

EXAMPLE 4 Find $\frac{d}{dx}\underbrace{\int_1^{x^4} \sec t \, dt}_{g(x)}$.

$$g(x) = \int_1^{x^4} \sec t \, dt$$

$$h(x) = x^4$$
$$G(x) = \int_1^{x} \sec t \, dt$$

Now, $g(x) = G(x^4) = \int_1^{x^4} \sec t \, dt = G(h(x))$

By the Chain rule:

$$g'(x) = G'(h(x)) \cdot h'(x)$$

$$G'(x) = \sec(x)$$

$$= \sec(h(x)) \cdot 4x^3$$

$$= \boxed{\sec(x^4) \cdot 4x^3}$$

Example. Find the derivative of the function $f(x) = \int_{\sin x}^{1} \sqrt{1+t^2}\, dt$.

$$\int_{\sin x}^{1} \sqrt{1+t^2}\, dt = -\underbrace{\int_1^{\sin x} \sqrt{1+t^2}\, dt}_{g(x)}$$

$$h(x) = \sin x$$
$$G(x) = \int_0^{x} \sqrt{1+t^2}\, dt$$

$\longrightarrow$ $f(x) = G(\sin x) = G(h(x))$

$$(\underbrace{1+x}_{h(x)})^2 = 2(1+x) \cdot h'(x)$$

By the chain rule,

$$f'(x) = -G'(h(x)) \cdot h'(x)$$

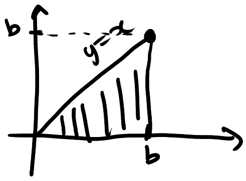
$$G'(x) = \sqrt{1+x^2}$$

$$= \boxed{-\sqrt{1+\sin^2 x} \cdot \cos x}$$

Second part of the Fundamental Theorem of Calculus.

Example. Compute the integral $\int_a^b x\,dx$ where a and b are two numbers such that a < b.

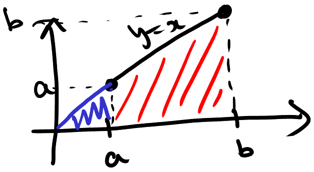
$a=0,\ b=b$



$$\int_0^b x\,dx = A(\triangle) = \frac{b\cdot b}{2} = \frac{b^2}{2} \to F(b)$$

$$F(x) = \frac{x^2}{2} \to F(b)$$

a positive. $b > a$



$$\int_a^b x\,dx = A(\text{trapezoid}) = A(\triangle_b) - \boxed{A(\triangle_a)}$$

$$= \frac{b^2}{2} - \frac{a^2}{2}$$

$$= F(b) - F(a)$$

$F(x) = \frac{x^2}{2}$ & $f(x) = x$ $\to$ $F'(x) = x = f(x)$ $\to$ F is an antiderivative

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Definition A function F is called an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

General Antiderivative:

In the example $F(x) = \frac{x^2}{2}$ is a particular antiderivative

$$GA: \quad \frac{x^2}{2} + C \quad \to \quad GA: \underbrace{F(x)}_{\text{part. Ant.}} + \underbrace{C}_{\text{const.}}$$

Example. Find the general antiderivative of each of the following functions.

(a) $f(x) = x$ (b) $f(x) = \sqrt{x}$ (c) $f(x) = \sin x$ (d) $f(x) = 2x\sin(x^2)$

(a) $\boxed{\frac{x^2}{2} + C}$

(b) $x^{1/2} = f(x)$ $\quad (x^{3/2})' = \frac{3}{2}x^{3/2-1} = \frac{3}{2}x^{1/2}$

$\to \boxed{\frac{2}{3}x^{3/2} + C}$

$\to \left(\frac{2}{3}x^{3/2} + C\right)' = \frac{2}{3}\,\frac{3}{2}x^{3/2-1} + 0$

$= x^{1/2}$

(c) $\boxed{-\cos x + C}$

$\to (-\cos x + C)' = -(-\sin x) + 0 = \sin x$

(d) $x^2 \xrightarrow{'} 2x$

$\sin(x^2) \to \cos(x^2)\cdot 2x$

$\cos(x^2) \rightarrow -\sin(x^2) \cdot 2x$

$-\cos(x^2) \rightarrow -(-\sin(x^2) \cdot 2x) = 2x \sin(x^2)$

$\hookrightarrow \boxed{-\cos(x^2) + C}$

$x^{1/2} \rightarrow \dfrac{x^{1/2+1}}{1/2+1} = \dfrac{x^{3/2}}{3/2} = \dfrac{2}{3} x^{3/2}$

Table of Antiderivaties of some functions.

Function	Particular antiderivative	Function	Particular antiderivative
$cf(x)$	$cF(x)$	$\cos x$	$\sin x$
$f(x) + g(x)$	$F(x) + G(x)$	$\sin x$	$-\cos x$
$x^n \ (n \neq -1)$	$\dfrac{x^{n+1}}{n+1}$	$\sec^2 x$	$\tan x$
		$\sec x \tan x$	$\sec x$

EXAMPLE Find f if $f'(x) = x\sqrt{x}$ and $f(1) = 2$.

1. Find G Anti of $f'(x) = x\sqrt{x}$

$x\sqrt{x} = x\, x^{1/2} = x^{3/2}$

$n = 3/2$

$\hookrightarrow \dfrac{x^{3/2+1}}{3/2+1}$

So, $F(x) = \dfrac{x^{5/2}}{5/2} + C$

$= \dfrac{2}{5} x^{5/2} + C$

2. Find C.

$2 = F(1) = \frac{2}{5} \cdot 1 + C \Rightarrow C = 2 - \frac{2}{5} = \frac{8}{5}$

So, $f(x) = \frac{2}{5} x^{5/2} + \frac{8}{5}$

The Fundamental Theorem of Calculus, Part 2 If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_{x=a}^{x=b}$$

where F is any antiderivative of f, that is, a function F such that $F' = f$.

Consequence on the distance problem:

EXAMPLE 5 Evaluate the integral $\int_{-2}^{1} x^3\,dx$.

$x^3 \to$ continuous on $[2,1]$. ✓

$$F(x) = \frac{x^4}{4} + \boxed{C} \quad (\text{G Anti})$$

$$\int_{-2}^{1} x^3\,dx = F(1) - F(-2) \quad (\text{FTC, part 2})$$

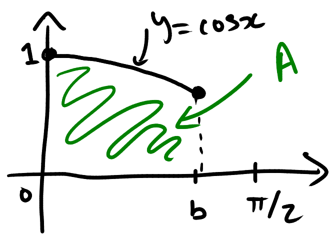
$$= \left(\frac{1}{4} + C\right) - \left(\frac{(-2)^4}{4} + C\right)$$

$$= \frac{1}{4} + \cancel{C} - \frac{16}{4} - \cancel{C}$$

$$= \boxed{-\frac{15}{4}}$$

$$\int_{-2}^{1} x^3\,dx = \frac{x^4}{4}\Big|_{x=-2}^{x=1} = \frac{1}{4} - \frac{16}{4}$$

EXAMPLE 7 Find the area under the cosine curve from 0 to b, where $0 \leqslant b \leqslant \pi/2$.



$$A = \int_0^b \underbrace{\cos x}_{f(x)}\,dx \leftarrow \text{Goal}$$

Anti-Der. $\quad F(x) = \sin x + \boxed{C}$

So,

$$\int_0^b \cos x\,dx \overset{\text{FTC 2}}{=} F(b) - F(0)$$

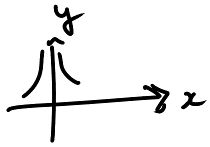
$$= \sin b + \cancel{C} - (\cancel{\sin 0}^{\,0} + \cancel{C})$$

$$= \boxed{\sin b}.$$

EXAMPLE 8 What is wrong with the following calculation?

$$\int_{-1}^{3} \frac{1}{x^2}\,dx = \frac{x^{-1}}{-1}\Bigg]_{-1}^{3} = -\frac{1}{3} - 1 = \boxed{-\frac{4}{3}}$$

① $\frac{1}{x^2}$ is not CONTINUOUS at $x=0$



② $\frac{1}{x^2} \geq 0 \rightarrow \int_{-1}^{3} \frac{1}{x^2}\,dx \geq 0$

but $\frac{-4}{3} < 0$.

<u>Differentiation and Integration as Inverse Processes.</u>

The Fundamental Theorem of Calculus Suppose f is continuous on $[a, b]$.

1. If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$.

2. $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f, that is, $F' = f$.