Math 302

Chapter 1

Section 1.2: Basic Concepts

Contents

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What's a DE?

- A **differential equation** (abbreviated by DE) is an equation that contains one or more derivatives of an unknown function. $y = y(x)$.
  - Examples: $T' = -k(T - T_m)$, $y' = x^2$, $x^2y'' + xy' + 2 = 0$.
- The **order** of a DE is the order of the highest derivatives that it contains.
  - Example: $y' = x^2$ is of order 1.
  - Example: $x^2y'' + xy' + 2 = 0$ is of order 2.
- An **Ordinary Differential Equation** (abbreviated ODE) is a DE involving an unknown function of only one variable.
- An **Partial Differential Equation** (abbreviated PDE) is a DE involving an unknown function of more than one variable.

The simplest ODE is of the form

$$y' = f(x) \quad \text{or} \quad y^{(n)} = f(x)$$

where f is a known function of x.

EXAMPLE 1. Find functions $y = y(x)$ satisfying

1. $y' = x^2$.
2. $y'' = \cos(x)$.

1.) $\int y' \, dx = \int x^2 \, dx + C \rightarrow y(x) = \frac{x^3}{3} + C$

2) $g = f' \Rightarrow g' = \cos(x) \Rightarrow g(x) = \sin(x) + C_1$

$\Rightarrow f'(x) = \sin(x) + C_1$

$\Rightarrow f(x) = -\cos(x) + C_1x + C_2$

Our goal is to study general ODEs of the form

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}).$$

What Is a Solution to an ODE?

A **solution** to the ODE

$$y^{(n)}(x) = f(x, y(x), y'(x), \ldots, y^{(n-1)}(x))$$

is a function $y = y(x)$ that verifies the ODE for any x in some open interval (a, b).

Remark:

- Functions that satisfy an ODE at isolated points are not considered solutions.

EXAMPLE 2. Verify that

$$y = \frac{x^2}{3} + \frac{1}{x}$$

is a solution of

$$xy' + y = x^2$$

on $(-\infty, 0)$ and $(0, \infty)$.

Solution and Integral Curves

- The graph of a solution of an ODE is a **solution curve**.
- More generally, a curve C in the plane is said to be an **integral curve** of an ODE if every function $y = y(x)$ whose graph is a segment of C is a solution of the ODE.

EXAMPLE 3. Plot the solutions obtained in Example 2. Are they solution curves of the ODE?

EXAMPLE 4. If a is any positive constant, check that the circle

$$x^2 + y^2 = a^2$$

is an integral curve of $y' = -x/y$.

Example 5. Find a solution of

$$y' = x^3$$

satisfying the additional condition $y(1) = 2$.

Example 6. All the solutions to

$$y'' - 2y' + 3y = 0$$

are the functions

$$y(x) = c_1 e^x + c_2 e^{-3x}$$

where c_1, c_2 are arbitrary constants. Find the solution that satisfies $y(0) = 1$ and $y'(0) = 0$.

An **Initial Value Problem** (abbreviated by IVP) is an ODE with additional **Initial conditions**. The general form of an IVP is

$$y^{(n)}(x) = f(x, y(x), y'(x), \ldots, y^{(n-1)}(x)), \quad y(x_0) = k_0,\, y'(x_0) = k_1, \ldots,\, y^{(n-1)}(x_0) = k_{n-1}.$$

- The largest open interval that contains x_0 on which $y(x)$ is defined and satisfies the ODE is called the **interval of validity** of y.

EXAMPLE 7. Find the interval of validity of the solution to

$$y' = x^3,\, y(1) = 2.$$

Example 8. Find the interval of validity of the solution to the following IVPs:

1. $xy' + y = x^2$, $y(1) = 4/3$.
2. $xy' + y = x^2$, $y(-1) = -2/3$.