MATH-241 Calculus I
Midterm 02

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Last name: Solutions.
First name:

Question:	1	2	3	4	5	6	7	8	Total
Points:	10	15	10	10	10	15	20	10	100
Score:	–	–	–	–	–	–	–	–	–

Instructions: Make sure to write your complete name on your copy. You must answer all the questions below and write your answers directly on the questionnaire. At the end of the 75 minutes, hand out your copy.

No devices such as a smart phone, cell phone, laptop, or tablet can be used during the exam. You are not allowed to use the lecture notes, the textbook, or any other notes. You may use a digital calculator (no graphical calculator or symbolic calculator will be allowed).

You must show ALL your work to have full credit. An answer without justification worth no point.

Good luck!

Pierre-Olivier Parisé

Your Signature:

UNIVERSITY OF HAWAI'I

Question 1 **(10 pts)**

Compute the derivative of the following functions. Give all the details of the computations.

(a) (5 points) $f(x) = (2x^3 + 5)^4$.

$f'(x) = 4(2x^3+5)(2x^3+5)'$ [Chain Rule]

$= 4(2x^3+5)\,6x^2$ [Power Rule]

$= 24x^2(2x^3+5)$.

(b) (5 points) $f(x) = \sin(\sqrt{x})$.

$f'(x) = \cos(\sqrt{x})\,(\sqrt{x})'$ [Chain Rule].

$= \cos(\sqrt{x})\,\frac{1}{2\sqrt{x}}$ [Power Rule].

$= \dfrac{\cos\sqrt{x}}{2\sqrt{x}}$.

Question 2 **(15 pts)**

Use implicit differentiation to find the tangent line to the curve at the given point:

$$x^2 - xy - y^2 = 1, \quad \text{at } (2, 1).$$

① Find y'.

$$2x - y - xy' - 2yy' = 0$$

$$\Rightarrow \quad 2x - y = (x+2y)y'$$

$$\Rightarrow \quad \frac{2x-y}{x+2y} = y'$$

② Tangent line.

$$(x_0, y_0) = (2,1)$$
$$m = y'(2).$$

$$y - y_0 = m(x - x_0)$$

$$m = \frac{4-1}{2+2} = 1$$

$$\Rightarrow \quad y - 1 = 1(x-2)$$

$$\Rightarrow \quad y = x - 2 + 1$$

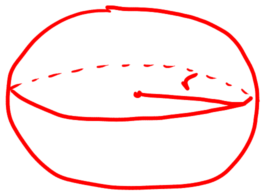
$$\Rightarrow \quad \boxed{y = x - 1}$$

Question 3 **(10 pts)**

If a snowball melts so that its surface area decreases at a rate of $1\text{cm}^2/\text{min}$, find the rate at which the diameter decreases when the diameter is 10cm.

Note: The surface area of a sphere is $A = 4\pi r^2$.

① Sketch.



V: volume
r: radius
d: diameter
A: surface area.

$d = 2r$

$\Rightarrow r = \frac{d}{2}$.

② Equations.

Volume $\rightarrow$ $A = 4\pi r^2 \longrightarrow A = 4\pi\left(\frac{d}{2}\right)^2 = \pi d^2$

③ Rate of Change.

$$\frac{dA}{dt} = \pi 2d \cdot \frac{d(d)}{dt}$$

$$\Rightarrow \quad d' = \frac{A'}{2\pi d}$$

Now $d = 10$ & $A' = -1$

$$\Rightarrow \quad \boxed{d' = \frac{-1}{20\pi} \text{ cm/min}}$$

Question 4 **(10 pts)**

Let $f(x) = \sqrt{1+x}$.

(a) (5 points) Find the linearization of the function f at the point $a = 0$.

We have

$$L(x) = f'(a)(x-a) + f(a).$$

so $f(0) = 1$ & $f'(x) = \frac{1}{2\sqrt{1+x}}$

$$\Rightarrow f'(0) = \frac{1}{2}.$$

So, $L(x) = \frac{x}{2} + 1$

(b) (5 points) Using the linearization, estimate the value of $\sqrt{1.1}$. Explain clearly how you obtained your answers.

$\sqrt{1.1} = \sqrt{1+0.1} \rightarrow x = 0.1$

So, $\sqrt{1.1} = f(0.1) \approx L(0.1) = \frac{0.1}{2} + 1 = 1.05$

$\Rightarrow \sqrt{1.1} \approx 1.05$.

QUESTION 5 **(10 pts)**

The position P of a koholā under the ocean in Hawai'i was tracked by a GPS for 12 hours straight. With the continuous data obtained from the GPS, the graph was plotted, with time, in hours, on the x axis and position, in meters and measured from the surface of the water, on the y axis. Assume the first and second derivatives of the corresponding function exist.

(a) (2 points) Identify, as best as you can, the critical numbers inside the interval $(0, 12)$. (Here 0 and 12 are excluded).

There are 5 CN : $t \approx 1.25$ $t \approx 3.8$ $t \approx 6$ $t \approx 8.6$ $t \approx 11.3$

(b) (4 points) Identify, as best as you can, the local maximums of the function. For each of them, using one of the test, explain why it is a local maximum.

$t \approx 1.25$ because f is increasing on the left & f is decreasing on the right.

$t \approx 8.6$ because same reason.

(c) (4 points) Identify, as best as you can, the local minimums of the function. For each of them, using one of the test, explain why it is a local minimum.

$t \approx 3.8$ because f is decreasing on the left & f is increasing on the right.

$t \approx 11.3$ because same reason.

Question 6 (15 pts)

Let $f(x) = x^{1/3}(x+4)$. Identify clearly the letter of the part of the question you are answering.

(a) (5 points) The first derivative of f is $f'(x) = \frac{4(x+1)}{3x^{2/3}}$. Find the critical numbers and the interval of increase and decrease.

(b) (5 points) Find the local maximums and local minimums of the function. Using one of the test, explain why there are local maximum or local minimum.

(c) (5 points) The second derivative of f is $f''(x) = \frac{4(x-2)}{9x^{5/3}}$. Find the inflection points and the intervals of concavity.

(a) The CN are
- $f' \nexists \to x=0$
- $f'=0 \to x=-1$

x		-1		0	
$x+1$	$-$	0	$+$	1	$+$
$1/(x^2)^3$	$+$	1	$+$	$\nexists$	$+$
f'	$-$	0	$+$	$\nexists$	$+$

So,

$f'<0$ on $(-\infty,-1) \to f$ is decreasing there

$f'>0$ on $(-1,0)$ & $(0,\infty)$ $\to$ f is increasing there.

(b) f goes from $\searrow$ to $\nearrow$ around $x=-1$, so f has a local minimum there

$\Rightarrow f(-1) = -3$

No change around $x=0$, no max & no min.

(c) I.P. :
- $f'' \nexists \to x=0$
- $f''=0 \to x=2$

x		0		2	
$x-2$	$-$	-2	$-$	0	$+$
$1/x^{5/3}$	$-$	$\nexists$	$+$		$+$
f'	$+$	$\nexists$	$-$	0	$+$

$\longrightarrow$

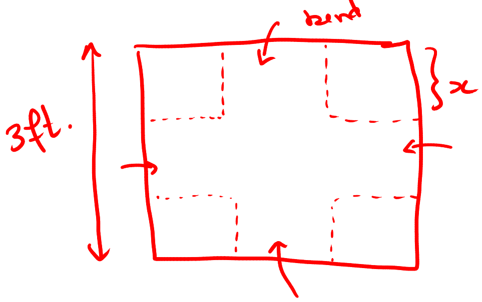
$f''>0$ on $(-\infty,0)$ & $(2,\infty)$, so f is concave up there

$f''<0$ on $(0,2)$ so f is concave down there.

QUESTION 7 **(20 pts)**

A box with an open top is to be constructed from a square piece of cardboard, 3ft wide, by cutting out a square from each of the four corners and bending up the sides. Using the technic learned in class[1], find the largest volume that such a box can have.

① Sketch.



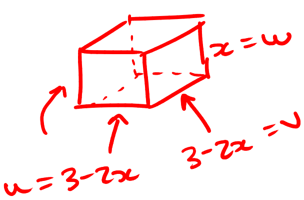
x: length of the sides of the little square

V: volume of the box

Goal

optimize V.

② Equations.



$$V = uvw = (3-2x)^2 x$$

$$0 \leq x \leq \frac{3}{2}$$

③ Optimize.

$$\begin{aligned} V'(x) &= 2(3-2x)x(-2) + (3-2x)^2 \\ &= -12x + 8x^2 + 9 - 12x + 4x^2 \\ &= 9 - 24x + 12x^2 = 3(3 - 8x + 4x^2) \\ &= 3(2x-3)(2x-1) \end{aligned}$$

$$V'(x) = 0 \iff x = \frac{3}{2} \text{ or } x = \frac{1}{2}.$$

$$\begin{aligned} \text{Max} &= \max\{V(0), V(1/2), V(3/2)\} \\ &= \max\{0, 2, 0\} \\ &= \boxed{2 \text{ ft}^3} \end{aligned}$$

[1]No point will be attributed to a solution which doesn't use the derivative.

QUESTION 8 **(10 pts)**

Compute the following limits. Make sure to write the details of your calculations.

(a) (5 points) $\lim_{x\to\infty} \dfrac{3x-2}{2x+1}$.

$$\lim_{x\to\infty} \frac{3-2/x}{2+1/x} = \frac{\lim_{x\to\infty} 3-2/x}{\lim_{x\to\infty} 2+1/x}$$

$$= \boxed{\frac{3}{2}}$$

(b) (5 points) $\lim_{x\to-\infty} \dfrac{\sqrt{2x^2+1}}{3x-5}$.

$x<0 \rightarrow \sqrt{2x^2+1} = \sqrt{x^2}\sqrt{2+1/x^2} = -x\sqrt{2+1/x^2}$

So,

$$\lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to-\infty} -\frac{\sqrt{2+1/x^2}}{3-5/x}$$

$$= -\frac{\sqrt{\lim_{x\to-\infty} 2+1/x^2}}{\lim_{x\to-\infty} 3-5/x}$$

$$= \boxed{-\frac{\sqrt{2}}{3}}$$