16.5 Curl and Divergence.

Curl.

Definition.

$\vec{F} = \langle P, Q, R \rangle$ then

$\text{curl}\,\vec{F} = \langle R_y - Q_z,\ P_z - R_x,\ Q_x - P_y \rangle$

In 2D: $\vec{F} = \langle P(x,y), Q(x,y), 0 \rangle$

$\text{curl}\,\vec{F} = \langle 0, 0, Q_x - P_y \rangle$

2D-curl $\vec{F} = Q_x - P_y$.

Cross product formula.

$\vec{\nabla} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & Q \end{vmatrix} = \text{curl}\,\vec{F}$$

$$\boxed{\text{curl}\,\vec{\mathbf{F}} = \vec{\nabla} \times \vec{\mathbf{F}}}$$

EXAMPLE 1 If $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$, find curl $\mathbf{F}$.

$$\text{curl}\,\vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{vmatrix} = (-2y - xy)\vec{i} - (0 - x)\vec{j} + (yz - 0)\vec{k}$$

$$= -y(2 + x)\vec{i} + x\vec{j} + yz\vec{k}.$$

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3 Theorem If f is a function of three variables that has continuous second-order partial derivatives, then

$$\text{curl}(\vec{\nabla} f) = \vec{0}$$

Proof.

$\vec{\nabla} \times (\vec{\nabla} f) = \vec{0}$

$\vec{a} \times c\vec{a} = 0$.

Conservative $\to \vec{\nabla} f = \vec{F}$.

EXAMPLE 2 Show that the vector field $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$ is not conservative.

By contradiction, suppose $\vec{F}$ is conservative.

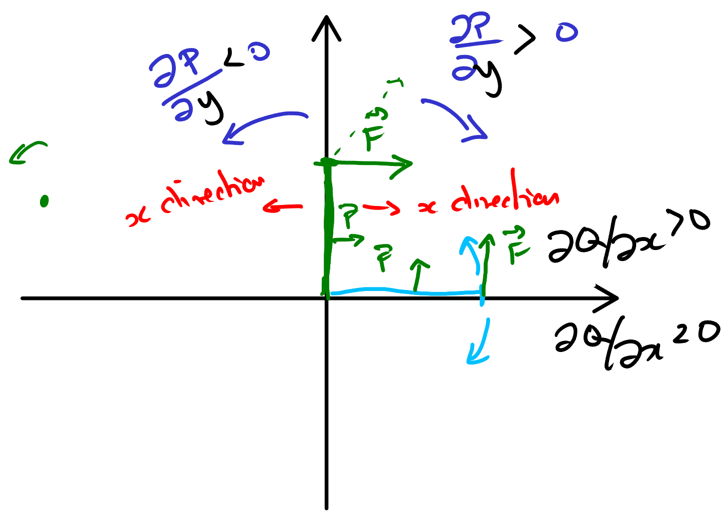
then, $\vec{F} = \vec{\nabla} f$ (from sect. 16.3).

$\Rightarrow$ curl $\vec{F}$ = curl $\vec{\nabla} f = \vec{0}$ (by Thm. 3)

But, in example 1, we saw that curl $\vec{F} \neq \vec{0}$. So, this is a contradiction, and $\vec{F}$ is not conservative.

$\vec{F} = \langle P, Q \rangle$

$\vec{F} = \langle -y, x \rangle$



$\frac{df}{dx} = 0$

$\Rightarrow f(x) = c$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} > 0 \rightarrow \text{clock-wise rotation}$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} < 0 \rightarrow \text{clockwise rotation}$$

4 Theorem If $\mathbf{F}$ is a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and curl $\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is a conservative vector field.

EXAMPLE 3
(a) Show that

$$\mathbf{F}(x, y, z) = y^2z^3\,\mathbf{i} + 2xyz^3\,\mathbf{j} + 3xy^2z^2\,\mathbf{k}$$

is a conservative vector field.
(b) Find a function f such that $\mathbf{F} = \nabla f$.

(a) $\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix} = \langle 6xyz^2 - 6xyz^2, -(3y^2z^2 - 3y^2z^2), 2yz^3 - 2yz^3 \rangle$

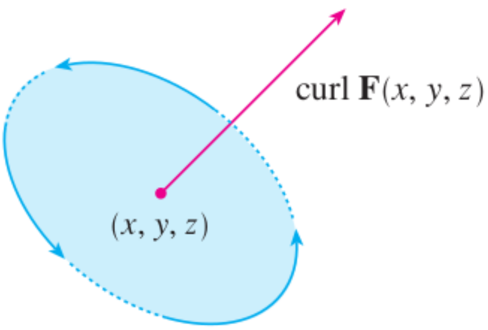
$= \langle 0, 0, 0 \rangle = \vec{0}$.

So, $\vec{F}$ is conservative.

(b) $f_x = y^2z^3 \longrightarrow xy^2z^3$
$f_y = 2xyz^3 \longrightarrow xy^2z^3$
$f_z = 3xy^2z^2 \longrightarrow xy^2z^3$

So, $f(x, y, z) = xy^2z^3 + C$.

Physical interpretation.



- direction represents an axis of rotation.
- length represents how fast the particles are rotating around the axis.

Divergence.

Definition.

$\vec{F} = \langle P, Q, R \rangle$ then

$\operatorname{div} \vec{F} = P_x + Q_y + R_z$

In 2D.

$\operatorname{div} \vec{F} = P_x + Q_y$.

Dot product formula.

$\vec{\nabla} = \langle \partial/\partial x, \partial/\partial y, \partial/\partial z \rangle$

Then

$$\operatorname{div} \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \vec{F} = \vec{\nabla} \cdot \vec{F}.$$

$$\boxed{\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F}}$$

EXAMPLE 4 If $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$, find div $\mathbf{F}$.

$$\operatorname{div} \vec{F} = \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(-y^2)$$

$$= z + xz$$

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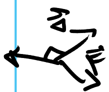
11 Theorem If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and P, Q, and R have continuous second-order partial derivatives, then

$$\operatorname{div} \operatorname{curl} \vec{\mathbf{F}} = 0$$

$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = \vec{0}$.



EXAMPLE 5 Show that the vector field $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$ can't be written as the curl of another vector field, that is, $\mathbf{F} \neq \operatorname{curl} \mathbf{G}$.

By contradiction, suppose that it is possible: there is a $\vec{G}$ st.

$$\vec{F} = \operatorname{curl} \vec{G}.$$

By Thm. 11, $\operatorname{div} \vec{F} = \operatorname{div} \operatorname{curl} \vec{G} = 0$.

By, by example 4, $\operatorname{div} \vec{F} \neq 0$, # ($\rightarrow\!\leftarrow$). So $\vec{F} \neq \operatorname{curl} \vec{G}$, $\forall \vec{G}$.

Incompressible Flow.

$\vec{F}$ velocity of a fluid.

$\operatorname{div} \vec{F}$ measures the net rate of change of the mass of the fluid flowing at a giving point.

- $\operatorname{div} \vec{F} > 0 \to (x, y, z)$ source
- $\operatorname{div} \vec{F} < 0 \to (x, y, z)$ sink
- $\operatorname{div} \vec{F} = 0 \to (x, y, z)$ is incompressible

Laplace's Equation.

$f(x, y, z)$, then

$$\vec{\nabla} \cdot (\vec{\nabla} f) = \operatorname{div} \vec{\nabla} f = \underbrace{f_{xx} + f_{yy} + f_{zz}}_{\text{Laplace's eq. when } = 0}.$$

Notation (Laplacian)

$$\vec{\nabla}^2 f = \Delta f = f_{xx} + f_{yy} + f_{zz}.$$

Vector Forms of Green's Theorem.

I. First Formula with curl.

$\vec{F} = \langle P, Q, 0 \rangle$

$\text{curl}\,\vec{F} = \langle 0, 0, Q_x - P_y \rangle$

$\langle 0, 0, Q_x - P_y \rangle \cdot \langle 0, 0, 1 \rangle$

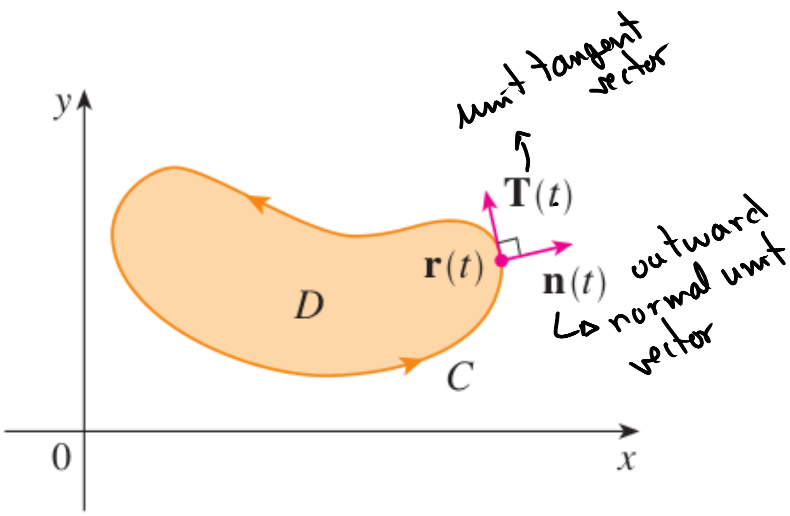
$$\boxed{\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\text{curl}\,\mathbf{F}) \cdot \mathbf{k}\, dA}$$

vector form of Green's thm.

$\iint_D \underline{Q_x - P_y}\, dA$

3rd component of the 2D-curl.

II. Second formula with divergence.



$\vec{T}(t) = \dfrac{\langle x'(t), y'(t) \rangle}{|\vec{r}\,'(t)|}$

$\vec{n}(t) = \dfrac{\langle y'(t), -x'(t) \rangle}{|\vec{r}\,'(t)|}$

Use these vectors for the ds

$$\boxed{\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_D \text{div}\,\mathbf{F}(x, y)\, dA}$$

Stokes' theorem.

Maxwell's Equations.

$\vec{E}$: electric field $\quad \vec{B}$: magnetic field

$$\text{div}\,\vec{E} = 0$$

$$\text{div}\,\vec{B} = 0$$

$$\begin{cases} \text{curl}\,\vec{E} = -\dfrac{\partial \vec{B}}{\partial t} \\ \text{curl}\,\vec{B} = \mu_0 \left(\vec{J} + \varepsilon_0 \dfrac{\partial \vec{E}}{\partial t} \right) \end{cases}$$