MATH-244 Calculus IV
Midterm 02

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Last name: Solutions
First name:

Question:	1	2	3	4	5	Total
Points:	20	20	20	20	20	100
Score:	–	–	–	–	–	–

Instructions: Make sure to write your complete name on your copy. You must answer all the questions below and write your answers directly on the questionnaire. At the end of the 75 minutes, hand out your copy.

No devices such as a smart phone, cell phone, laptop, or tablet can be used during the exam. You are not allowed to use the lecture notes, the textbook. You may bring one 2-sided cheat sheet of handwriting notes. You may use a digital calculator (no graphical calculator or symbolic calculator will be allowed).

You must show ALL your work to have full credit. An answer without justification worth no point.

Good luck!

Pierre-Olivier Parisé

Your Signature: –

University of Hawai'i

Question 1 **(20 pts)**

Evaluate[1] the integral

$$\iiint_E (x^2 + y^2)\, dV,$$

where E is the solid between the spheres

$$x^2 + y^2 + z^2 = 4 \quad \text{and} \quad x^2 + y^2 + z^2 = 9.$$

In spherical coordinates:

change spherical /2

$x = \rho\cos\theta\sin\phi$

$y = \rho\sin\theta\sin\phi$

$z = \rho\cos\phi$

Description /6

$\longrightarrow E = \{(\rho, \theta, \phi) : \underset{2pt.}{2 \le \rho \le 3}, \ \underset{2pt.}{0 \le \theta \le 2\pi}, \ \underset{2pt.}{0 \le \phi \le \pi}\}.$

So,

set up integral /10

2pt. 4pt replace x & y

$$\iiint_E x^2+y^2\, dV = \int_0^{\pi}\int_0^{2\pi}\int_2^3 \rho^2\sin^2\phi \underbrace{\rho^2\sin\phi\, d\rho\, d\theta\, d\phi}_{\text{4pt. Jacobian.}}$$

$$= \left(\int_0^{\pi} \sin^3\phi\, d\phi\right)\left(\int_0^{2\pi} d\theta\right)\left(\int_2^3 \rho^4\, d\rho\right)$$

$$= \left(\frac{4}{3}\right) 2\pi \left.\frac{\rho^5}{5}\right|_2^3$$

$$= \frac{4}{3} \cdot 2\pi \cdot \frac{3^5 - 2^5}{5}$$

$$= \frac{1688\pi}{15}$$

calculations. /2

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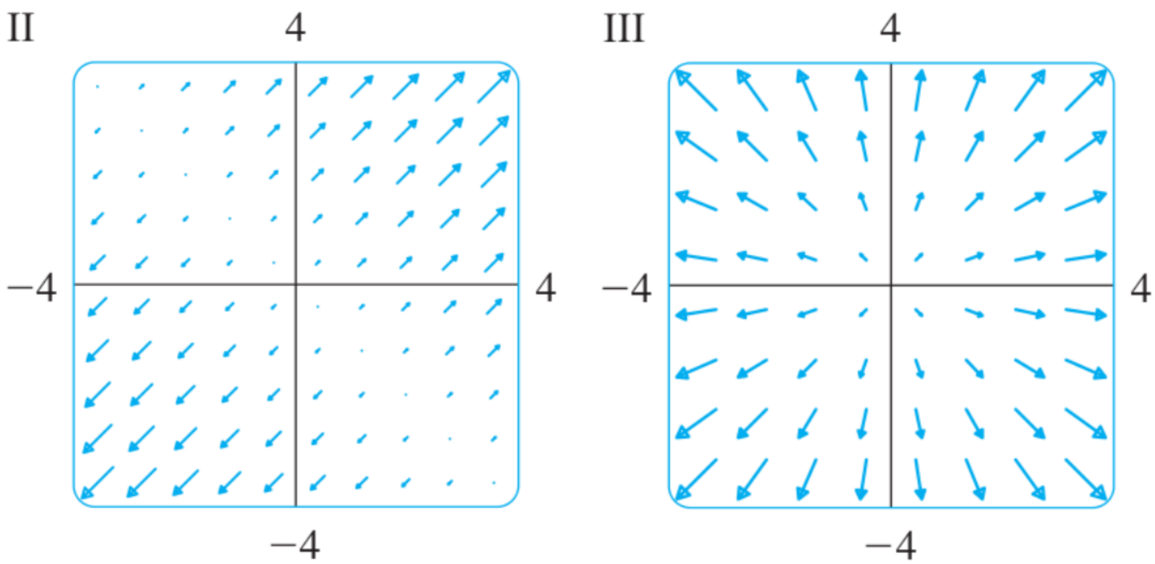
[1] You can take for granted that $\int_0^{\pi} \sin^3(t)\, dt = \frac{4}{3}$.

Question 2 **(20 pts)**

Match the gradient fields $\vec{F}$ with the right vector field plot. Explain your choices (you can use the back of this sheet if you need more space).

(a) (10 points) $\vec{F} = \langle 2x, 2y \rangle$.

(b) (10 points) $\vec{F} = \langle 2(x+y), 2(x+y) \rangle$.



(a) $\vec{F} = 2\langle x, y \rangle$. So, the vector at (x,y) is in the same direction as $\langle x,y \rangle$ with twice its length.

So, $(x,y) = (1,1) \Rightarrow \vec{F}(1,1) = \langle 2,2 \rangle$ (same direction)
$(x,y) = (-1,1) \Rightarrow \vec{F}(-1,1) = \langle -2,2 \rangle$ (same direction).

Vectors in $\vec{F}$ points outward the origin.

$\Rightarrow$ $\underline{\text{III}}$

(b) $\vec{F} = (x+y)\langle 2,2 \rangle$.

So $\vec{F} = \vec{0}$ when $x+y=0 \Leftrightarrow y=-x$.

We can see in $\underline{\text{II}}$ that along $y=-x$, there is no vector. So

$\underline{\text{II}}$

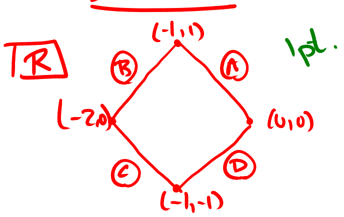
Question 3 **(20 pts)**

Consider the rectangular region R with vertices $(0,0)$, $(-1,1)$, $(-2,0)$, and $(-1,-1)$.

(a) (10 points) Find the image of R under the transformations

$$u = y - x \quad \text{and} \quad v = y + x.$$

Sketch



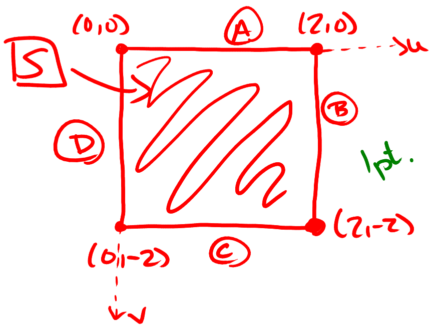
A $y = -x$, $-1 \le x \le 0$. 1pt.

B $y = x + 2$, $-2 \le x \le -1$ 1pt.

C $y = -x - 2$, $-2 \le x \le -1$ 1pt.

D $y = x$, $-1 \le x \le 0$. 1pt.

New Region S



A $u = -x - x = -2x \to 0 \le u \le 2$ 1pt.
$v = -x + x = 0$

B $u = x + 2 - x = 2$ 1pt.
$v = x + 2 + x = 2(x+1) \to -2 \le v \le 0$

C $u = -x - 2 - x = -2(x+1) \to 0 \le u \le 2$ 1pt.
$v = -x - 2 + x = -2$

D $u = x - x = 0$ 1pt.
$v = x + x = 2x \to -2 \le v \le 0$

(b) (10 points) Using the transformation, evaluate the integral

$$\iint_R (x - y)\, dA.$$

Algebra $\Rightarrow$ $x = \dfrac{v-u}{2}$ 2pt. & $y = \dfrac{v+u}{2}$ 2pt.

So, $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{vmatrix} = -1/2$ 2pt. & $S = [0,2] \times [-2,0]$

So, $\displaystyle\iint_R (x-y)\, dA = \int_{-2}^{0}\int_{0}^{2} \frac{v-u}{2} - \left(\frac{v+u}{2}\right) \left|-\frac{1}{2}\right| du\,dv$ 3pt. set-up.

$\displaystyle = \int_{-2}^{0}\int_{0}^{2} -\frac{u}{2}\, du\,dv$

$\displaystyle = -\left(\int_{-2}^{0} dv\right)\left(\int_{0}^{2} \frac{u}{2}\, du\right)$ 1pt calc.

$= \boxed{-2}$.

Question 3 **(20 pts)**

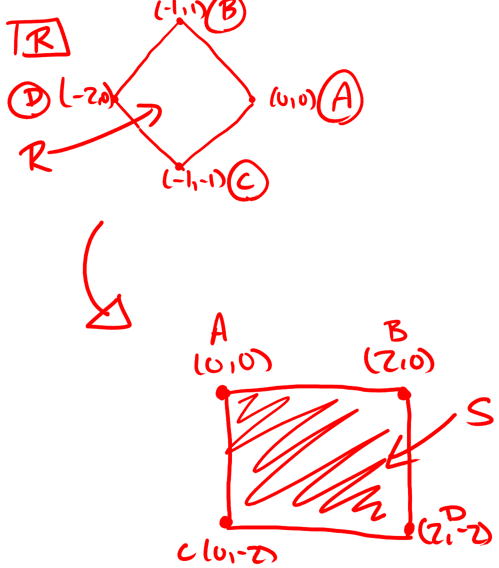
Consider the rectangular region R with vertices $(0,0)$, $(-1,1)$, $(-2,0)$, and $(-1,-1)$.

(a) (10 points) Find the image of R under the transformations

$$u = y - x \quad \text{and} \quad v = y + x.$$

Second way:

Sketch



A $(0,0) \longmapsto$ $u = 0 - 0 = 0$, $v = 0 + 0 = 0$ $(0,0)$

B $(-1,1) \longmapsto$ $u = 1 - (-1) = +2$, $v = 1 - 1 = 0$ $(2,0)$

C $(-1,-1) \longmapsto$ $u = -1 - (-1) = 0$, $v = -1 - 1 = -2$ $(0,-2)$

D $(-2,0) \longmapsto$ $u = 0 - (-2) = 2$, $v = 0 - 2 = -2$ $(2,-2)$

(b) (10 points) Using the transformation, evaluate the integral

$$\iint_R (x - y)\, dA.$$

Question 4 (20 pts)

Evaluate the following line integrals.

(a) (10 points) $\int_C y\,ds$, where C is given by the parametrization $x(t) = t^2$ and $y(t) = 2t$ with $0 \le t \le 3$.

$$ds = \sqrt{x'(t)^2 + y'(t)^2}\,dt = \sqrt{4t^2+4}\,dt = 2\sqrt{t^2+1}\,dt$$ 5pts.

So,

$$\int_C y\,ds = \int_0^3 2t\sqrt{t^2+1}\,dt$$ 3pts. $u = t^2+1$

$$= \int_1^{10} 2\sqrt{u}\,dt$$

$$= 2\frac{u^{3/2}}{3/2}\Big|_1^{10} = \frac{4}{3}\left(\sqrt{10^3} - 1\right)$$ 2pts.

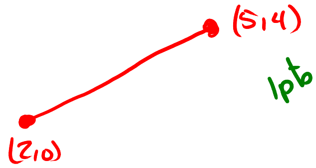
$\sqrt{10\cdot10\cdot10} = \sqrt{1000} = \sqrt{100\cdot10} = 10\sqrt{10}$

$$= \frac{4}{3}\left(10\sqrt{10} - 1\right)$$

$$= \boxed{40.83}$$

(b) (10 points) $\int_C \left(\frac{y}{4} - \frac{x}{3}\right) dy$ where C is the line segment from $(2,0)$ to $(5,4)$.

① Parametrization



1pt

4pts.

$\vec{r}(t) = \langle 3t+2, 4t \rangle \quad 0 \le t \le 1$

$\vec{r}\,'(t) = \langle 3, 4 \rangle$

$\hookrightarrow y'(t)$.

/5

② Integral.

$$\int_C \left(\frac{y}{4} - \frac{x}{3}\right) dy = \int_0^1 \left(t - t - \frac{2}{3}\right) 4\,dt$$ 3pts.

$$= \int_0^1 -\frac{8}{3}\,dt$$

/5

$$= \boxed{-\frac{8}{3}}$$ 2pts.

QUESTION 5 (20 pts)

Consider the following line integral

$$\int_C 2xe^{-y}\,dx + (2y - x^2e^{-y})dy$$

where C is the curve given by $x(t) = t$ and $y(t) = (t-1)^4$ with $1 \le t \le 2$.

(a) (5 points) Is the integral independent of the path? Explain.

(b) (15 points) Find the value of the integral.

(a) $P(x,y) = 2xe^{-y}$ 2pts $\quad \Rightarrow \quad P_y = -2xe^{-y}$

$Q(x,y) = 2y - x^2e^{-y}$ 2pts. $\quad Q_x = 0 - 2xe^{-y}$

$\Rightarrow P_y = Q_x$ 1pt.

$\Rightarrow$ YES IT IS!

(b) ① Potential function.

$f_x = P = 2xe^{-y}$ $\xrightarrow{\text{Integrate}}$ $f(x,y) = \underline{x^2e^{-y}}$ 2pts.

$f_y = Q = 2y - x^2e^{-y}$ 2pts $\quad f(x,y) = y^2 + \underline{x^2e^{-y}}$ 2pts.

so $\rightarrow f(x,y) = x^2e^{-y} + y^2 + C$. 2pts.

② FTLI.

We have $\displaystyle\int_C \vec{F}\cdot d\vec{r} = f(\vec{r}(2)) - f(\vec{r}(1))$ 3pts.

where $\vec{F}(x,y) = \langle 2xe^{-y}, 2y - x^2e^{-y}\rangle$.

$\vec{r}(2) = \langle 2,1\rangle$
$\vec{r}(1) = \langle 1,0\rangle$ $\rightarrow f(2,1) - f(1,0) = 4e^{-1} + 1 - (1) = \dfrac{4}{e}$ 3pts.

so, $\boxed{\displaystyle\int_C \vec{F}\cdot d\vec{r} = \frac{4}{e}}$ 1pt.