Math 644

Chapter 2

Section 2.6: Examples Of Analytic Maps

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Theorem 1. The power series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}$$

has radius of convergence $R = \infty$.

Proof.

$$R = \liminf_{n\to\infty} \left(\frac{1}{n!}\right)^{-1/n} = \infty .$$

or

$$R^{-1} = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0 \Rightarrow R = \infty .$$

Definition 2. We define the exponential function to be

$$\exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad (z \in \mathbb{C}).$$

Facts:

- $\exp(z)$ is analytic in $\mathbb{C}$.
- $\exp(z+w) = \exp(z)\exp(w)$.
- $\exp(it) = \cos(t) + i\sin(t)$ for $t \in \mathbb{R}$.
- $|e^z| = e^{\operatorname{Re} z}$ and $\arg e^z = \operatorname{Im} z$.
- $\frac{d}{dz}e^z = e^z$.
- $\exp(z + 2k\pi i) = \exp(z)$, for any $k \in \mathbb{Z}$.

Theorem 3. The exponential function has the following additional properties:

a) $\exp(z) \neq 0$ for any $z \in \mathbb{C}$.

b) $\exp(z)$ is locally one-to-one.

Proof.

(a) $\exp(z) = 0 \Rightarrow e^{\operatorname{Re} z} = 0$ #.

(b) From next chapter,
$f'(z_0) \neq 0$ & f analytic $\Rightarrow$ $\exists r > 0$ s.t. f is one-to-one in $\{|z - z_0| < r\}$.

Use part (a) & $(e^z)' = e^z$. $\square$

DEFINITION 4. We define, for $z \in \mathbb{C}$,

$$\cos(z) := \frac{\exp(iz) + \exp(-iz)}{2} \quad \text{and} \quad \sin(z) := \frac{\exp(iz) - \exp(-iz)}{2i}.$$

THEOREM 5. The functions cos and sin are analytic on $\mathbb{C}$. Moreover, we have

$$\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \quad \text{and} \quad \sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \quad (z \in \mathbb{C}).$$

Facts:

- We have $\exp(z) = \cos(z) + i\sin(z)$, for any $z \in \mathbb{C}$
- $\cos^2(z) + \sin^2(z) = 1$ for any $z \in \mathbb{C}$.
- $\cos z = 0$ if and only if $z = \frac{\pi}{2} + k\pi$ $(k \in \mathbb{Z})$.
- $\sin z = 0$ if and only if $z = k\pi$ $(k \in \mathbb{Z})$
- $\frac{d}{dz}\cos z = -\sin z$ and $\frac{d}{dz}\sin z = \cos z$.

Other trigonometric functions:

- $\tan z := \frac{\sin z}{\cos z}$ (analytic for z such that $\cos z \neq 0$).
- $\cot z := \frac{\cos z}{\sin z}$ (analytic for z such that $\sin z \neq 0$).
- $\sec z := \frac{1}{\cos z}$ (analytic for z such that $\cos z \neq 0$).
- $\csc z := \frac{1}{\sin z}$ (analytic for z such that $\sin z \neq 0$).

Hyperbolic Functions

Definition 6. We define, for $z \in \mathbb{C}$,

$$\cosh(z) := \frac{\exp(z) + \exp(-z)}{2} \quad \text{and} \quad \sinh(z) := \frac{\exp(z) - \exp(-z)}{2}.$$

Theorem 7. The functions cosh and sinh are analytic on $\mathbb{C}$. Moreover, we have

$$\cosh(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \quad \text{and} \quad \sinh(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \quad (z \in \mathbb{C}).$$

Facts:

- $\frac{d}{dz} \cosh z = \sinh z$ and $\frac{d}{dz} \sinh z = \cosh z$.
- $\cosh^2 z - \sin^2 z = 1$, for every $z \in \mathbb{C}$.

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