Math 644

Chapter 4

Section 4.4: Weierstrass' Theorem

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THEOREM 1. Suppose (f_n) is a collection of analytic functions on a region Ω such that $f_n \to f$ uniformly on compact subsets of Ω. Then f is analytic on Ω. Moreover, $f_n' \to f'$ uniformly on compact subsets of Ω.

LEMMA 2. If G is integrable on a piecewise continuously differentiable curve γ, then

$$g(z) := \int_\gamma \frac{G(\zeta)}{\zeta - z}\, d\zeta$$

is analytic in $\mathbb{C}\backslash\gamma$ and

$$g'(z) = \int_\gamma \frac{G(\zeta)}{(\zeta - z)^2}\, d\zeta.$$

Proof.

Write, for z, $z+h \in \mathbb{C}\backslash\gamma$ $(h \neq 0)$

$$\frac{g(z+h) - g(z)}{h} = \int_\gamma \frac{G(\zeta)}{(\zeta - z - h)(\zeta - z)}\, d\zeta$$

As $h \to 0$, $\dfrac{G(\zeta)}{(\zeta - z - h)(\zeta - z)} \longrightarrow \dfrac{G(\zeta)}{(\zeta - z)^2}$

uniformly on γ. and so $g'(z)$ exists and

$$g'(z) = \lim_{h \to 0} \int_\gamma \frac{G(\zeta)}{(\zeta - z - h)(\zeta - z)}\, d\zeta = \int_\gamma \frac{G(\zeta)}{(\zeta - z)^2}\, d\zeta$$

Moreover, g' is continuous and therefore g is holomorphic on $\mathbb{C}\backslash\gamma$ $\square$

Proof of Weierstrass's Theorem.

1) Let B be a disk with $\overline{B} \subseteq \Omega$.

Then,
$$f_n(z) = \frac{1}{2\pi i} \int_{\partial B} \frac{f_n(\zeta)}{\zeta - z} \, d\zeta, \quad z \in B.$$

Since $\partial B \subseteq \Omega$ is compact, $f_n \to f$ uniformly on ∂B. In particular, f is continuous.

Set
$$F(z) = \frac{1}{2\pi i} \int_{\partial B} \frac{f(\zeta)}{\zeta - z} \, d\zeta, \quad z \in B.$$

By Lemma 2, F is analytic on B. But
$$f(z) = \lim_{n\to\infty} f_n(z) = \frac{1}{2\pi i} \int_{\partial B} \frac{\lim_{n\to\infty} f_n(\zeta)}{\zeta - z} \, d\zeta$$
$$= F(z), \quad \forall z \in B.$$

So, f is analytic on B, so on Ω.

2) From Lemma 2,
$$f_n'(z) = \frac{1}{2\pi i} \int_{\partial B} \frac{f_n(\zeta)}{(\zeta - z)^2} \, d\zeta, \quad z \in B$$
&
$$f'(z) = \frac{1}{2\pi i} \int_{\partial B} \frac{f(\zeta)}{(\zeta - z)^2} \, d\zeta, \quad z \in B.$$

Proof of Weierstrass's Theorem. (continued)

By assumption, we obtain $f_n' \to f'$ unif. on some $\overline{B_0} \subseteq B$.

If $K \subseteq \Omega$ is compact, then cover K by such balls ($\overline{B_0}$) where $f_n' \to f$ unif. $\square$

Integrating On Continuous Curves

Goal:

- Extend the definition of the integral to continuous maps $\gamma : [a, b] \to \mathbb{C}$.

LEMMA 3. Suppose Ω is a region and suppose $\gamma : [0, 1] \to \Omega$ is continuous. Given $\varepsilon > 0$ with $0 < \varepsilon < \text{dist}(\gamma, \partial\Omega)$, we can find a finite partition $0 = t_0 < t_1 < \cdots < t_n = 1$ so that

a) $\gamma([t_{j-1}, t_j]) \subset B_j := \{z : |z - \gamma(t_j)| < \varepsilon\}$ for every $j = 1, \ldots, n$;

b) $B_j \subset \Omega$ for every $j = 1, \ldots, n$.

Proof.

Construction:

Theorem 4. Suppose Ω is a region and $\gamma : [0,1] \to \mathbb{C}$ is continuous with $\gamma \subset \Omega$. Let σ be the polygonal curve defined in the last page. If f is analytic on Ω, define

$$\int_\gamma f(z)\,dz = \int_\sigma f(z)\,dz.$$

Then this definition of $\int_\gamma f(z)\,dz$ does not depend on the choice of the polygonal curve σ and it agrees with our prior definition if γ is piecewise continuously differentiable.

Proof.