Section 5.5 — Problem 1 — 10 points

Find the general solutions to the complementary equation.

The complementary equation is

$$y'' + 3y' + 2y = 0$$

The characteristic equation associated to the complementary equation is $r^2+3r'+2 = 0$. Therefore, the roots are $r_1 = -1$ and $r_2 = -2$. The general solution is

$$y_h(x) = c_1e^{-x} + c_2e^{-2x}.$$

Find a particular solution.

The right-hand side contains $\cos x$ and $\sin x$. Those functions are not contained in the complementary equation. We therefore suggest

$$y_{par}(x) = A\cos x + B\sin x.$$

We have

$$\begin{aligned} y' &= -A\sin x + B\cos x \\ y'' &= -A\cos x - B\sin x. \end{aligned}$$

Replacing in the ODE, we get

$$\begin{aligned} y'' + 3y' + 2y &= -A\cos x - B\sin x - 3A\sin x + 3B\cos x + 2A\cos x + 2B\sin x \\ &= (A + 3B)\cos x + (-3A + B)\sin x. \end{aligned}$$

This last expression should be equal to the right-hand side

$$(A + 3B)\cos x + (-3A + B)\sin x = 7\cos x - \sin x.$$

We therefore have to find A, B satisfying

$$\begin{cases} A + 3B = 7 \\ -3A + B = -1 \end{cases}$$

The solution is $A = 9/7$ and $B = 20/7$. The expression of the particular solution is

$$y_{par}(x) = (9/7)\cos x + (20/7)\sin x.$$

General solution.

Combining y_h and y_{par}, we get

$$y(x) = y_h(x) + y_{par}(x) = c_1e^{-x} + c_2e^{-2x} + \frac{9}{7}\cos x + \frac{20}{7}\sin x.$$

Section 5.5 — Problem 3 — 10 points

Find the general solution to the complementary equation.
The complementary equation is

$$y'' + 2y' + y = 0.$$

The characteristic equation associated to the complementary equation is $r^2 + 2r + 1 = 0$. There is only one root: $r_1 = -1$. The solution to the complementary equation is therefore

$$y_h(x) = c_1 e^{-x} + c_2 x e^{-x}.$$

Find a particular solution.
The right-hand side contains the functions $e^x \cos x$ and $e^x \sin x$. Those are not contained in the solution to the complementary equation. Therefore, we suggest

$$y_{par}(x) = e^x (A \cos x + B \sin x).$$

We have

$$\begin{aligned} y'(x) &= e^x(A\cos x + B\sin x) + e^x(-A\sin x + B\cos x) \\ y''(x) &= e^x(A\cos x + B\sin x) + 2e^x(-A\sin x + B\cos x) + e^x(-A\cos x - B\sin x). \end{aligned}$$

Replacing in the ODE, we get

$$\begin{aligned} y'' + 2y' + y &= e^x(A\cos x + B\sin x) + 2e^x(-A\sin x + B\cos x) + e^x(-A\cos x - B\sin x) \\ &\quad + 2e^x(A\cos x + B\sin x) + 2e^x(-A\sin x + B\cos x) + e^x(A\cos x + B\sin x) \\ &= e^x\Big((3A + 3B)\cos x + (3B - 4A)\sin x\Big) \end{aligned}$$

The right-hand side is $e^x(6\cos x + 17\sin x)$. Therefore, we must have

$$3A + 3B = 6 \quad \text{and} \quad 3B - 4A = 17.$$

The solution is $A = -11/7$ and $B = 25/7$. The particular solution is therefore

$$y_{par}(x) = -\frac{11e^x}{7}\cos x + \frac{25e^x}{7}\sin x.$$

General solution.
The general solution is therefore

$$y(x) = y_h(x) + y_{par}(x) = c_1 e^{-x} + c_2 x e^{-x} - \frac{11e^x}{7}\cos x + \frac{25e^x}{7}\sin x.$$

Section 5.5 — Problem 11 — 10 points

Complementary Equation.

The complementary equation is

$$y'' - 2y' + 5y = 0.$$

The characteristic polynomial associated to the complementary equation is $r^2 - 2r + 5 = 0$. The roots are complex numbers and they are

$$r_1 = 1 + 2i \quad \text{and} \quad r_2 = 1 - 2i.$$

Therefore, the solution is

$$y_h(x) = c_1 e^x \cos(2x) + c_2 e^x \sin(2x).$$

Find a particular solution.

The right-hand side unfortunately contains $e^x \cos(2x)$ and $e^x \sin(2x)$. These functions are also multiplied by a polynomial of degree 1. We therefore suggest

$$y_{par}(x) = xe^x \Big((Ax + B)\cos(2x) + (Cx + D)\sin(2x) \Big).$$

Following the hint, we have

$$A = 1,\ B = -1\,, C = 1 \text{ and } D = 1.$$

Therefore, the solution is

$$y_{par}(x) = xe^x \Big((x - 1)\cos(2x) + (x + 1)\sin(2x) \Big).$$

General solution.

Combining y_h and y_{par}, we obtain

$$y(x) = y_h(x) + y_{par}(x) = c_1 e^x \cos(2x) + c_2 e^x \sin(2x) + xe^x \Big((x - 1)\cos(2x) + (x + 1)\sin(2x) \Big).$$

Section 5.5 — Problem 23 — 15 points

Complementary Equation.

The complementary equation is

$$y'' - 2y' + 2y = 0.$$

The characteristic polynomial associated to the complementary equation is $r^2 - 2r + 2 = 0$. The roots are complex and they are

$$r_1 = 1 + i \quad \text{and} \quad r_2 = 1 - i.$$

Therefore, the solution is

$$y_h(x) = c_1 e^x \cos(x) + c_2 e^x \sin(x).$$

Find a particular solution.

The right-hand side unfortunately contains $e^x \cos(x)$ and $e^x \sin(x)$. Since these functions are only multiplied by constants, we therefore suggest

$$y_{par}(x) = xe^x \Big(A\cos(x) + B\sin(x) \Big).$$

We find that

$$\begin{aligned} y'(x) &= e^x \Big(A\cos x + B\sin x \Big) + xe^x \Big(A\cos x + B\sin x \Big) + xe^x \Big(-A\sin x + B\cos x \Big) \\ &= e^x \Big(\big((A+B)x + A \big) \cos(x) + \big((B-A)x + B \big) \sin(x) \Big) \end{aligned}$$

and

$$\begin{aligned} y''(x) = e^x &\Big(\big((A+B)x + A \big) \cos(x) + \big((B-A)x + B \big) \sin(x) \Big) \\ &+ e^x \Big((A+B)\cos(x) - \big((A+B)x + A \big) \sin x + (B-A)\sin(x) + \big((B-A)x + B \big) \cos(x) \Big) \\ = e^x &\Big(\big(2Bx + 2(A+B) \big) \cos(x) + \big(-2Ax + 2(B-A) \big) \sin(x) \Big) \end{aligned}$$

We replace in the left-hand side of the ODE to get

$$y'' - 2y' + 2y = e^x \Big(2B\cos(x) - 2A\sin(x) \Big).$$

The right-hand side is $e^x(-6\cos x - 4\sin x)$. Therefore, we must have

$$2B = -6 \quad \text{and} \quad -2A = -4.$$

We conclude that $B = -3$ and $A = 2$ and the solution is

$$y_{par}(x) = xe^x \Big(2\cos x - 3\sin x \Big).$$

General solution.
Combining y_h and y_{par}, we obtain

$$y(x) = y_h(x) + y_{par}(x) = c_1 e^x \cos(x) + c_2 e^x \sin(x) + x e^x \Big(2\cos x - 3\sin x \Big).$$

Initial Value Problem.
We have $y(0) = 1$, so

$$c_1 = 1.$$

The expression of the derivative of the particular solution was computed in the second step. Replacing A and B by their values, we have

$$y'(x) = c_1 e^x \cos x + c_2 e^x \sin x - c_1 e^x \sin x + c_2 e^x \cos x + e^x \bigg(\Big(2 - x \Big) \cos x + \Big(-5x - 3 \Big) \sin x \bigg).$$

Since $y'(0) = 4$, we obtain

$$c_1 + c_2 + 2 = 4 \iff c_1 + c_2 = 2.$$

Solving for c_1 and c_2, we get

$$c_2 = 1.$$

Therefore, the solution to the IVP is

$$y(x) = e^x \cos(x) + e^x \sin(x) + x e^x \Big(2\cos x - 3\sin x \Big).$$

Section 5.6 — Problem 3 — 5 points

We set

$$y = uy_1 = ux.$$

Then the derivative and second derivative are

$$y' = u'x + u$$
$$y'' = u''x + 2u'.$$

Subtituting in the ODE, we obtain

$$x^2(u''x + 2u') - x(u'x + u) + ux = x$$

This simplifies to

$$x^3u'' + x^2u' = x.$$

Dividing through x, we get

$$x^2u'' + xu' = 1.$$

Set $v = u'$ so that $v' = u''$. Therefore, the ODE is reduced to

$$x^2v' + xv = 1.$$

Dividing again by x, we get

$$xv' + v = \frac{1}{x}.$$

The left-hand side can be rewritten as

$$\frac{d}{dx}\Big(xv\Big) = \frac{1}{x}.$$

Integrating with respect to x, we obtain

$$xv = \ln|x| + c_1 \quad \Rightarrow \quad v(x) = \frac{\ln|x|}{x} + \frac{c_1}{x}.$$

Since $v(x) = u'(x)$, integrating again with respect to x the expression of v, we obtain

$$u(x) = \frac{(\ln|x|)^2}{2} + c_1 \ln|x| + c_2.$$

Replacing in $y(x)$, we conclude that the general solution is

$$y(x) = \frac{x(\ln|x|)^2}{2} + c_1 x \ln|x| + c_2 x.$$

From this, we see that a foundamental set of solution for the complementary equation is $\{x, x\ln|x|\}$.

Total (Points): 50.