For each function, find its Laplace transform.

1) $f(t) = \sin(at + b)$, where a and b are constant.
2) $f(t) = \sin t \cos^2(2t)$.
3) $f(t) = (at + b)^2$, where a and b are constants.
4) $f(t) = \cos(at + b)$, where a and b are constants.
5) $f(t) = \sinh(at)$, where a is a constant.
6) $f(t) = \cosh(at)$, where a est une constante.
7) $f(t) = te^{at}$, where a is a constant.
8) $f(t) = t^n e^{at}$, where n is an integer and a is a constant.
9) $f(t) = t \sin at$, where a is a constant.
10) $f(t) = t \cosh(at)$, where a is a constant.
11) $f(t) = t^2 \sinh(at)$, where a is a constant.
12) $f(t) = \sin 3t + \cos 3t$.
13) $f(t) = e^{3t} \cosh(4t) + 20t$.
14) $f(t) = \cos t \sin t$.
15) $f(t) = te^{-t} \sin(2t)$.
16) $f(t) = t^3 \cos t \sin t$.

Answer key

1) By a trigonometric identity, we have that

$$\sin(at+b) = \sin(at)\cos(b) + \cos(at)\sin(b).$$

Therefore from the linearity of the Laplace transform, we obtain

$$L(\sin(at+b)) = \cos(b)L(\sin(at)) + \sin(b)L(\cos(at)) = \frac{a\cos(b)}{s^2+a^2} + \frac{s\sin(b)}{s^2+a^2}$$

and the final answer is:

$$L(\sin(at+b)) = \frac{a\cos b + s\sin b}{s^2+a^2}.$$

2) By a trigonometric identity, we have that

$$\cos^2(2t) = \frac{1+\cos 4t}{2}.$$

Therefore, we get

$$\sin t\cos^2(2t) = \frac{\sin t}{2} + \frac{\sin t\cos 4t}{2}.$$

Using another trigonometric identity, we obtain

$$\sin t\cos^2(2t) = \frac{\sin t}{2} + \frac{\sin(5t)-\sin(3t)}{4}.$$

Now, after using the linearity of the Laplace transform and the table of Laplace transforms, we find that

$$\begin{aligned} L(\sin t\cos^2(2t)) &= \frac{1}{2}L(\sin t) + \frac{1}{4}L(\sin(5t)) - \frac{1}{4}L(\sin(3t)) \\ &= \frac{1}{2(s^2+1)} + \frac{5}{4(s^2+25)} - \frac{3}{4(s^2+9)}. \end{aligned}$$

3) We expand the polynomial:

$$(at+b)^2 = a^2t^2 + 2abt + b^2.$$

Now, we use the linearity of the Laplace transform and the tables:

$$\begin{aligned} L((at+b)^2) = a^2L(t^2) + 2abL(t) + b^2L(1) &= \frac{2a^2}{s^3} + \frac{2ab}{s^2} + \frac{b^2}{s} \\ &= \frac{2a^2 + 2abs + b^2s^2}{s^3}. \end{aligned}$$

4) From a trigonometric identity, we have that

$$\cos(at+b) = \cos(at)\cos(b) - \sin(at)\sin(b).$$

After applying the linearity of the Laplace transform, we end up with

$$\begin{aligned} L(\cos(at+b)) = \cos(b)L(\cos(at)) - \sin(b)L(\sin(at)) &= \frac{s\cos(b)}{s^2+a^2} - \frac{a\sin(b)}{s^2+a^2} \\ &= \frac{s\cos(b) - a\sin(b)}{s^2+a^2}. \end{aligned}$$

5) From the definition of the function sinh, we have

$$L(\sinh(at)) = \frac{1}{2}L(e^{at}) - \frac{1}{2}L(e^{-at}) = \frac{1}{2(s-a)} - \frac{1}{2(s+a)}.$$

Therefore, the final answer is

$$L(\sinh(at)) = \frac{a}{s^2-a^2}.$$

6) By following the same line of reasoning as in the previous question, we obtain

$$L(\cosh(at)) = \frac{s}{s^2-a^2}.$$

7) Using the fact that multiplication by a power of t translates to differentiation of the Laplace transform, we have

$$L(te^{at}) = -\frac{d}{ds}L(e^{at}).$$

We know that

$$L(e^{at}) = \frac{1}{s-a}.$$

Therefore we obtain

$$L(te^{at}) = -\frac{d}{ds}\left(\frac{1}{s-a}\right) = \frac{1}{(s-a)^2}.$$

8) Using the same property as in the previous question, we have

$$L(t^n e^{at}) = (-1)^n \frac{d^n}{ds^n} L(e^{at}).$$

We know that $L(e^{at}) = \frac{1}{s-a}$. Therefore, we get

$$L(t^n e^{at}) = (-1)^n \frac{d^n}{ds^n}\left(\frac{1}{s-a}\right).$$

When $n = 1$, we have $L(te^{at}) = (-1)^{1+1}\left(\frac{1}{(s-a)^2}\right)$. When $n = 2$, we have

$$L(t^2e^{at}) = (-1)^{2+2}\left(\frac{2}{(s-a)^3}\right).$$

When $n = 3$, we have

$$L(t^3e^{at}) = (-1)^3\frac{d^3}{ds^3}\left(\frac{1}{s-a}\right) = (-1)^{3+3}\left(\frac{2.3}{(s-a)^4}\right).$$

$$L(t^ne^{at}) = (-1)^{2n}\left(\frac{n!}{(s-a)^{n+1}}\right) = \frac{n!}{(s-a)^{n+1}},$$

where $n! = n.(n-1).(n-2)\cdots 2.1$.

9) We use one of the result from the lecture notes. When $n = 1$, we have that

$$L(t\sin at) = -\frac{d}{ds}L(\sin(at)).$$

We know that $L(\sin(at))$ is equal to $\frac{a}{s^2+a^2}$. Therefore, we conclude that

$$L(t\sin(at)) = -\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right) = \frac{2as}{(s^2+a^2)^2}.$$

10) Using the same result used in the previous question, we have

$$L(t\cosh(at)) = -\frac{d}{ds}L(\cosh(at)).$$

We computed in one of the previous problems that $L(\cosh(at)) = \frac{s}{s^2-a^2}$. Therefore, we get

$$L(t\cosh(at)) = -\frac{s^2-a^2-s(2s)}{(s^2-a^2)^2} = \frac{s^2+a^2}{(s^2-a^2)^2}.$$

11) Again, we have

$$L(t^2\sinh(at)) = (-1)^2\frac{d^2}{ds^2}L(\sinh(at)) = \frac{d^2}{ds^2}\left(\frac{a}{s^2-a^2}\right) = \frac{2a(a^2+3s^2)}{(s^2+a^2)^3}$$

where we used the fact that $L(\sinh(at)) = \frac{a}{s^2-a^2}$.

12) From the linearity of the Laplace transform, we have

$$L(\sin 3t + \cos 3t) = L(\sin 3t) + L(\cos 3t) = \frac{3}{s^2+9} + \frac{s}{s^2+9} = \frac{s+3}{s^2+9}.$$

13) From the linearity of the Laplace transform, we have

$$L(e^{3t}\cosh(4t) + 20t) = L(e^{3t}\cosh(4t)) + 20L(t).$$

Since multiplication by an exponential translates the Laplace transform, we have that

$$L(e^{3t}\cosh(4t)) = \frac{s-3}{(s-3)^2-16}.$$

Also, we have

$$L(t) = \frac{1}{s^2}.$$

Therefore, the final answer is:

$$L(e^{3t}\cosh(4t)) = \frac{s-3}{(s-3)^2-16} + \frac{20}{s^2}.$$

14) From a trigonometric identity, we have

$$\cos t \sin t = \frac{\sin(2t)}{2}.$$

Therefore, from the linearity of the Laplace transform, we get

$$L(\cos t \sin t) = \frac{1}{2}L(\sin(2t)) = \frac{2}{2(s^2+4)} = \frac{1}{s^2+4}.$$

15) First of all, multiplication by t translates to differenting the Laplace transform. Therefore, we obtain

$$L(te^{-t}\sin(2t)) = -\frac{d}{ds}L(e^{-t}\sin(2t)) = -\frac{d}{ds}\left(\frac{2}{(s+1)^2+4}\right) = \frac{4(s+1)}{((s+1)^2+4)^2}.$$

Another way to approach the problem is to first deal with the transform of $t\sin(2t)$. From a result stated in the lecture notes, we have

$$L\left(t\sin(2t)\right) = -\frac{d}{ds}L(\sin(2t)) = -\frac{d}{ds}\left(\frac{2}{s^2+4}\right) = \frac{4s}{(s^2+4)^2}.$$

Now, multiplication by e^{-t} translates the Laplace transform by -1:

$$L\left(e^{-t}t\sin(2t)\right) = \frac{4(s+1)}{((s+1)^2+4)^2}.$$

16) Since there is a multiplication by t^3, we must take the derivative three times of the Laplace transform of $\cos t \sin t$. From the previous problem, we have

$$L(\cos t \sin t) = \frac{1}{s^2+4}.$$

Differentiate three times, we end up with the following final answer:

$$L(t^3\cos t \sin t) = -\frac{24s(s^2-4)}{(s^2+4)^4}.$$