Section 15.6, Problem 4

We have

$$\begin{aligned}
\int_0^1 \int_y^{2y} \int_0^{x+y} 6xy \, dzdxdy &= \int_0^1 \int_y^{2y} 6xy(x+y) \, dxdy \\
&= \int_0^1 \int_y^{2y} 6x^2y + 6xy^2 \, dxdy \\
&= \int_0^1 \left(2x^3y + 3x^2y^2\right)\Big|_{x=y}^{x=2y} \, dy \\
&= \int_0^1 (16y^4 + 12y^4) - (2y^4 + 3y^4) \, dy \\
&= \int_0^1 23y^4 \, dy \\
&= \frac{23}{5}.
\end{aligned}$$

$\boxed{\textbf{Section 15.6, Problem 12}}$

The solid is described in the following way

$$E = \{(x, y, z) \,:\, 0 \le x \le \pi,\, 0 \le y \le \pi - x,\, 0 \le z \le x\}.$$

So,

$$\begin{aligned}\iiint_E \sin y \, dV = \int_0^{\pi} \int_0^{\pi - x} \int_0^{x} \sin y \, dz dy dx &= \int_0^{\pi} x \left. (-\cos y)\right|_{y=0}^{y=\pi - x} \, dx \\ &= \int_0^{\pi} -x(1 + \cos(\pi - x)) \, dx.\end{aligned}$$

After an integration by parts, we get

$$\iiint_E \sin y \, dV = -2 - \pi^2/2 \approx -6.9348.$$

Section 15.6, Problem 20

So we have $x^2 + z^2 \leq y \leq 8 - x^2 - z^2$. We have to intersect the two surfaces to find the domain of integration in the XZ-plane. Equating both equations for the surfaces to y, we get

$$x^2 + z^2 = 8 - x^2 - z^2 \iff x^2 + z^2 = 4.$$

So the domain is a circle of radius 2. Thus, the volume will be given by

$$V = \iiint_E dV = \int_0^{2\pi} \int_0^2 \int_{r^2}^{8-r^2} dyrdrd\theta$$

where we describe the domain in the XZ-plane in polar coordinates. So

$$V = 2\pi \int_0^2 (8 - 2r^2) r\, dr = 2\pi \int_0^4 u\, du = 16\pi.$$

Section 15.6, Problem 34

We have

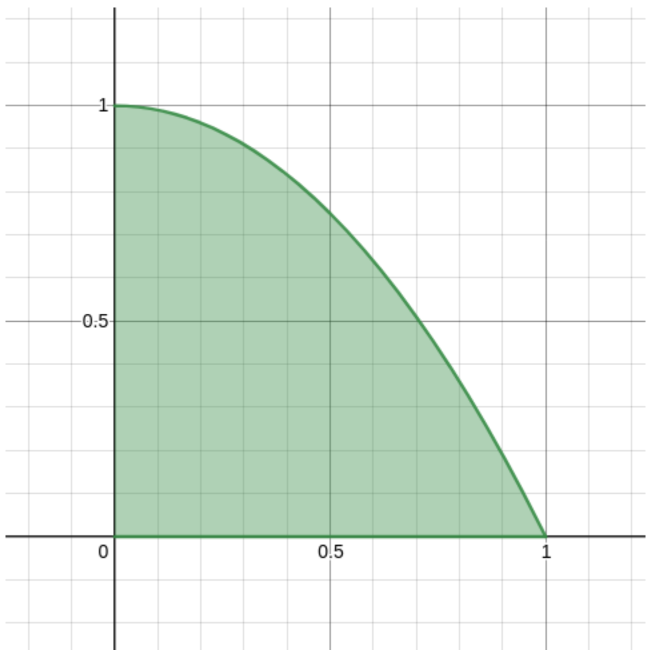
$$E = \{(x, y, z) \,:\, 0 \le x \le 1,\, 0 \le y \le 1 - x,\, 0 \le z \le 1 - x^2\}.$$

The orders we would like are $dzdydx$, $dydxdz$, $dxdydz$, $dzdxdy$, $dxdzdy$.

1. **dzdydx**. Since the bounds depend only on x, we can interchange without problems:

$$\int_0^1 \int_0^{1-x} \int_0^{1-x^2} f(x, y, z)\, dzdydx.$$

2. **dydxdz**. We have to look into the XZ-plane and interchange. The region in this plane are bounded by the curves $x = 0$, $x = 1$, $z = 0$ and $z = 1 - x^2$ and looks like this: So, by seeing



this region as a type two, we get $0 \le x \le \sqrt{1-z}$ and $0 \le z \le 1$. We then obtain

$$\int_0^1 \int_0^{\sqrt{1-z}} \int_0^{1-x} f(x, y, z)\, dydxdz.$$

3. **dxdydz**. We have to look into the XY-plane. We see that $0 \le x \le \sqrt{1-z}$ and $0 \le y \le 1-x$. Here, z is considered as a number which is fixed. If we see this domain as a type II (to interchange the x and the y), we have to deal with two pieces:

- $0 \le y \le 1 - \sqrt{1-z}$, then $0 \le x \le \sqrt{1-z}$.
- $1 - \sqrt{1-z} \le y \le 1$, then $0 \le x \le 1 - y$.

So the integral becomes

$$\int_0^1 \Big(\int_0^{1-\sqrt{1-z}} \int_0^{\sqrt{1-z}} f(x, y, z)\, dxdy + \int_{1-\sqrt{1-z}}^1 \int_0^{1-y} f(x, y, z)\, dxdy \Big)\, dz.$$

4. **dzdxdy**. We look in the XY-plane in the original configuration. From the bounds in the integrals in x and y, the region in the XY-plane is bounded by the curves $x = 0$, $x = 1$, $y = 0$ and $y = 1 - x$. So we interchange easily and get $0 \le x \le 1 - y$ and $0 \le y \le 1$ to get

$$\int_0^1 \int_0^{1-y} \int_0^{1-x^2} f(x, y, z)\, dzdxdy.$$

5. **dxdzdy**. We look at the bounds in x and z. We see these bounds give a region bounded by $x = 0$, $x = 1 - y$, $z = 0$, and $z = 1 - x^2$. Again, we have to split into two cases:

   - $0 \le z \le 1 - (1 - y)^2$, $0 \le x \le 1 - y$;
   - $1 - (1 - y)^2 \le z \le 1$, $0 \le x \le \sqrt{1 - z}$.

   So the integral in this final order looks like

$$\int_0^1 \Big(\int_0^{1-(1-y)^2} \int_0^{1-y} f(x, y, z)\, dxdz + \int_{1-(1-y)^2}^1 \int_0^{\sqrt{1-z}} f(x, y, z)\, dxdz \Big) dy.$$

$\boxed{\textbf{Section 15.6, Problem 42}}$

We have $z = 1 - x - y$ as an upper bound and $z = 0$ as a lower bound. Then, projecting on $z = 0$, we get $x + y = 1$. So the tetahedron is

$$E = \{(x, y, z) \,:\, 0 \le x \le 1,\, 0 \le y \le 1 - x,\, 0 \le z \le 1 - x - y\}.$$

The mass is given by

$$m = \iiint_E y\, dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} y\, dzdydx = 1/24.$$

The center of mass is given by $(\overline{x}, \overline{y}, \overline{z})$ where $\overline{x} = M_{yz}/m$, $\overline{y} = M_{xz}/m$, and $\overline{z} = M_{xy}/m$. We compute

$$\begin{aligned}
M_{yz} &= \iiint_E xy\, dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} xy\, dzdydx = 1/120 \\
M_{xz} &= \iiint_E y^2\, dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} y^2\, dzdydx = 1/60 \\
M_{xy} &= \iiint_E zy\, dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} y^2\, dzdydx = 1/120.
\end{aligned}$$

Thus the center of mass is

$$(\overline{x}, \overline{y}, \overline{z}) = (1/5, 2/5, 1/5).$$