# SECTION 1.5: Sequences & Series of $\mathbb{C}$-numbers.

A sequence of complex numbers is an ordered list $a_1, a_2, a_3, \ldots, a_n, \ldots$ where $a_n \in \mathbb{C}$ $(a : \mathbb{N} \to \mathbb{C})$.

Notations: $\{a_n\}_{n=1}^{\infty}$ and $(a_n)_{n=1}^{\infty}$.

Examples

- $a_n = \frac{1}{n}$, $n \in \mathbb{N}$. So

$$\{a_n\}_{n=1}^{\infty} = \{1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots\}.$$

$$(a_n)_{n=1}^{\infty} = (1, 1/2, 1/3, \ldots).$$

- $a_n = i^n$, $n \in \mathbb{N}$. So

$$\{a_n\}_{n=1}^{\infty} = \{i, -1, -i, 1, i, -1, -i, 1, \ldots\}$$

## Convergence of sequences

Example: For $\{(1+i)/n\}_{n=1}^{\infty}$, as $n$ gets bigger and bigger, $\frac{1+i}{n}$ gets closer and closer to $0$. How big $n$ should be to get $|a_n| < 0.001$?

$$\Rightarrow \quad \frac{\sqrt{2}}{n} < \frac{1}{1000} \iff 1000\sqrt{2} < n.$$

We would require $n \geq \lfloor 1000\sqrt{2} \rfloor + 1 = 1414 + 1$

$$\iff n \geq 1415.$$

<u>Def.</u> A sequence $\{a_n\}_{n=1}^{\infty}$ converges to a $a \in \mathbb{C}$ if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$\text{if } n \geq N, \text{ then } |a_n - a| < \varepsilon.$$

If $\{a_n\}_{n=1}^{\infty}$ does not converge, we say it diverges.

<u>Remarks</u>:

① Notation: $a_n \to a$ or $\lim_{n\to\infty} a_n = a$.

② Divergent: $a_n \not\to a$.

Negation: $\exists \varepsilon > 0, \forall N \in \mathbb{N}, \exists n \geq N$ s.t. $|a_n - a| \geq \varepsilon$.

<u>THM</u> (thm 1.5.8) Let $a_n = x_n + iy_n$.

$$a_n \to x + iy \iff x_n \to x \ \& \ y_n \to y.$$

Proof.

($\Rightarrow$) Assume that $a_n \longrightarrow x+iy$. Let $\varepsilon>0$.

Notice that

$$|x_n - x| \leq |a_n - (x+iy)| \quad \forall n \in \mathbb{N}.$$

From the def. of $a_n \longrightarrow x+iy$, there's an $N \in \mathbb{N}$ s.t. $|a_n - (x+iy)| < \varepsilon$, $\forall n \geq N$.

So, if $n \geq N$, then

$$|x_n - x| \leq |a_n - (x+iy)| < \varepsilon.$$

So, $x_n \longrightarrow x$ by def. Similarly, you get $y_n \longrightarrow y$.

($\Leftarrow$) Assume $x_n \longrightarrow x$ and $y_n \longrightarrow y$. Recall:

$$|z| \leq |\operatorname{Re} z| + |\operatorname{Im} z|, \quad \forall z \in \mathbb{C}.$$

Let $\varepsilon>0$. then

$$|a_n - (x+iy)| \leq |x_n - x| + |y_n - y|$$

Let $N_1 \in \mathbb{N}$ s.t. if $n \geq N_1$, then

$$|x_n - x| < \varepsilon/2$$

Let $N_2 \in \mathbb{N}$ s.t. if $n \geq N_2$, then

$$|y_n - y| < \varepsilon/2.$$

Let $N = \max\{N_1, N_2\}$. If $n \geq N$, then

$$|a_n - (x+iy)| \leq |x_n - x| + |y_n - y| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

So, $a_n \longrightarrow x+iy$. $\square$

## Properties:

1. Prop. 1.5.2: $a_n \to a \Rightarrow a$ is unique.
2. Prop. 1.5.4: $a_n \to a \Rightarrow (a_n)$ is bounded.
   (bounded: $\exists M > 0$ s.t. $|a_n| \leq M, \forall n$).
3. Prop. 1.5.6: Let $(a_n), (b_n)$ be two seq.
   (i) $a_n \to 0$ and $|b_n| \leq |a_n| \Rightarrow b_n \to 0$.
   (ii) $a_n \to 0$ and $(b_n)$ bounded $\Rightarrow a_n b_n \to 0$.
4. Prop. 1.5.7: Assume $a_n \to a$ and $b_n \to b$.
   (i) $\alpha a_n + \beta b_n \longrightarrow \alpha a + \beta b \qquad \alpha, \beta \in \mathbb{C}$.
   (ii) $a_n b_n \longrightarrow ab$.
   (iii) If $b \neq 0$, $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\frac{a}{b}}{\phantom{x}}$
   (iv) $\bar{a}_n \longrightarrow \bar{a}$ $\left( \displaystyle\lim_{n\to\infty} \bar{a}_n = \overline{\lim_{n\to\infty} a_n} \right)$.
   (v) $|a_n| \longrightarrow |a|$ $\left( \displaystyle\lim_{n\to\infty} |a_n| = \left|\lim_{n\to\infty} a_n\right| \right)$.

**Example** Compute $\lim\limits_{n\to\infty}\left(\dfrac{(3+2i)^2}{n+1}+\dfrac{n+n^2 i}{n^3 i}\right)$.

From the properties:

$$= (3+2i)^2 \lim_{n\to\infty}\frac{1}{n+1} + \lim_{n\to\infty}\frac{n+n^2 i}{n^3 i}$$

$$= 0 + \lim_{n\to\infty}\frac{n+n^2 i}{n^3 i}$$

$$= \lim_{n\to\infty}\left(\frac{1}{n^2 i}+\frac{1}{n}\right)$$

$$= \frac{1}{i}\lim_{n\to\infty}\frac{1}{n^2} + \lim_{n\to\infty}\frac{1}{n} = 0.$$

**Example 1.5.9**

(a) If $|z|<1$, then compute $\lim\limits_{n\to\infty} z^n$.

(b) If $z\neq 1$ and $|z|\geq 1$, then show that $\lim\limits_{n\to\infty} z^n$ does not exist.

**Sol.**

(a) If we want to show that $z^n\to 0$, then we have to consider:

$$|z^n-0| = |z|^n.$$

From Calculus, $\lim_{n\to\infty} r^n = 0$, $0 \le r < 1$.

Put $r = |z|$ $\Rightarrow$ $\lim_{n\to\infty} |z|^n = 0$.

(b) Assume $z \neq 1$ and $|z| \ge 1$.

For a proof by contradiction, assume $\lim_{n\to\infty} z^n = L$, for some $L \in \mathbb{C}$.

We have

$$L = \lim_{n\to\infty} z^n = z \lim_{n\to\infty} z^{n-1} = zL.$$

$$\Rightarrow \quad L = zL$$

Since $|z| \ge 1$ $\Rightarrow$ $|z|^n \ge 1 \overset{n\to\infty}{\Rightarrow} |L| \ge 1$

So, $L \neq 0$ $\Rightarrow$ $\dfrac{L}{L} = \dfrac{zL}{L}$

$$\Rightarrow \quad 1 = z \quad \#.$$

So, $\lim_{n\to\infty} z^n \nexists$.

1.5.10
**DEF.** A sequence $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that if $n, m \ge N$, then $|a_n - a_m| < \varepsilon$.

**THM 1.5.11** Let $\{a_n\}_{n=1}^{\infty}$ be a sequence, $a_n \in \mathbb{C}$.

(i) If $\{a_n\}$ is convergent, then it is Cauchy.

(ii) If $\{a_n\}$ is Cauchy, then $\{a_n\}$ converges.

## Series of complex numbers

An infinite series is an expression of the form

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots$$

$a_n$ → nth term.

Partial sums: $s_n = \sum_{j=0}^{n} a_j = a_0 + a_1 + \cdots + a_n$

**DEF. 1.5.12** $\sum_{n=0}^{\infty} a_n$ converges to some $A \in \mathbb{C}$ if $\lim_{n\to\infty} s_n$ exists and $\lim_{n\to\infty} s_n = A$.

$\boxed{\text{DIGRESSION}}$ Summable theory.

Example $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ (converges)

Example $\sum_{n=1}^{\infty} \frac{1}{n} = +\infty$ (DNC)

Example $\sum_{n=1}^{\infty} (-1)^n$

$s_1 = -1$, $s_2 = 0$, $s_3 = -1$, ...

$$\Rightarrow \quad s_n = \begin{cases} 0, & n \text{ even} \\ -1, & n \text{ odd.} \end{cases}$$

$s_n$ is not Cauchy $\Rightarrow$ $\lim_{n\to\infty} s_n$ DNE.

$\sigma_n$ = average of the first $n$ $s_n$

So, $\sigma_1 = \frac{-1}{1} = -1$

$$\sigma_2 = \frac{-1+0}{2} = -\frac{1}{2}$$

$$\sigma_3 = \frac{-1+0+(-1)}{3} = -\frac{2}{3}$$

$$\sigma_4 = \frac{-1+0+(-1)+0}{4} = -\frac{1}{2}$$

$\vdots$

We can show that $\sigma_n \to -\frac{1}{2}$.

**DEF.** We say that $\sum_{n=1}^{\infty} a_n$ is Cesàro convergent if

- $\lim_{n\to\infty} \sigma_n = \lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n}$

exist

In this, $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sigma_n$.

$\boxed{\text{COME BACK TO OUR SHEEPS}}$

**Example 1.5.13**

If $|z| < 1$, then $\sum_{n=0}^{\infty} z^n$ converges

and $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$.

**Proof:**

We have $S_n = \frac{1 - z^{n+1}}{1-z}$ $(n \geq 1)$

take $n \to \infty$, then

$$\lim_{n\to\infty} S_n = \frac{1}{1-z} - 0 = \frac{1}{1-z}. \quad \square$$

Properties of series and tests:

**Prop. 1.5.15** $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ convergent series, then

① $\sum_{n=0}^{\infty} (\alpha a_n + \beta b_n) = \alpha \sum_{n=0}^{\infty} a_n + \beta \sum_{n=0}^{\infty} b_n$.

② $\sum_{n=0}^{\infty} \overline{a_n} = \overline{\sum_{n=0}^{\infty} a_n}$

③ $\sum_{n=0}^{\infty} \mathrm{Re}(a_n) = \mathrm{Re}\left(\sum_{n=0} a_n\right)$

and $\sum_{n=0}^{\infty} \mathrm{Im}(a_n) = \mathrm{Im}\left(\sum_{n=0}^{\infty} a_n\right)$.

**Prop 1.5.17**

$\sum_{n=0}^{\infty} a_n$ converges $\Rightarrow \lim_{n\to\infty} a_n = 0$.

In other words,

$\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum_{n=0}^{\infty} a_n$ DIV.

**Prop. 1.5.18** (Tail goes to 0)

$$\sum_{n=0}^{\infty} a_n \text{ converges} \Rightarrow \lim_{m\to\infty} \sum_{n=m+1}^{\infty} a_n = 0$$

**Proof.**

Write $s = \sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$

Consider $N \in \mathbb{N}$ such that $N \geq m+1$

$$\sum_{n=m+1}^{N} a_n = s_N - s_m$$

$$\Rightarrow \lim_{N\to\infty} \sum_{n=m+1}^{N} a_n = \sum_{n=m+1}^{\infty} a_n = s - s_m$$

Now

$$\lim_{m\to\infty} \sum_{n=m+1}^{\infty} a_n = s - \lim_{m\to\infty} s_m$$

$$= s - s = 0. \quad \square$$

**DEF 1.5.19**

$\sum_{n=0}^{\infty} a_n$ is absolutely convergent if $\sum_{n=0}^{\infty} |a_n|$ converges.

THM 1.5.21 (Comparison test)

$\sum_{n=1}^{\infty} a_n$ series with $a_n \in \mathbb{C}$ and

$\sum_{n=1}^{\infty} b_n$ is convergent with $b_n \in \mathbb{R}^+$

and $|a_n| \leq b_n$, then

$\sum_{n=0}^{\infty} a_n$ is absolutely convergent.

THM 1.5.23 (Ratio Test)

Let $a_n \neq 0$ be complex numbers.

Define $\rho := \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

and assume the limit exists or is infinite.

(1) If $\rho < 1$, then $\sum_{n=0}^{\infty} a_n$ converges absolutely.

(2) If $\rho > 1$, then $\sum_{n=0}^{\infty} a_n$ DIV.

(3) If $\rho = 1$, then the test is

inconclusive.

**THM 1.5.25** (Root test)

Let $a_n \in \mathbb{C}$ and assume

$$\rho = \lim_{n \to \infty} \sqrt[n]{|a_n|}$$

exists.

(1) $\rho < 1 \Rightarrow \sum_{n=0}^{\infty} a_n$ absolutely converges.

(2) $\rho > 1 \Rightarrow \sum_{n=0}^{\infty} a_n$ DIV.

(3) $\rho = 1 \Rightarrow$ test is inconclusive.