Math 302

Chapter 5

Section 5.5: The Method of Undetermined Coefficient II

Contents

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When The Force Function Is A Trig. Function

We consider the following first basic case:

$$ay'' + by' + cy = F\cos\omega x + G\sin\omega x$$

where F, G and $\overset{\omega}{\cancel{\alpha}}$ are fixed real numbers.

Case I

When $\cos\omega x$ and $\sin\omega x$ are not solution to the complementary equation $ay'' + by' + cy = 0$.

Example 1. Find the general solution to

$$y'' - 2y' + y = 5\cos 2x + 10\sin 2x.$$

① Comple. Eq.

$$y'' - 2y' + y = 0 \to r^2 - 2r + 1 = 0$$

$$\to (r-1)^2 = 0$$

$$\to r = 1 \quad \text{(repeated root).}$$

$$\Rightarrow y_h(x) = c_1 e^x + c_1 x e^x$$

cos(2x) & sin(2x) don't appear

② Part. Sol.

$$y_p(x) = A\cos(2x) + B\sin(2x)$$

$$\Rightarrow y'(x) = -2A\sin(2x) + 2B\cos(2x)$$

$$y''(x) = -4A\cos(2x) - 4B\sin(2x).$$

Replace in the ODE:

$$-4A\cos(2x) - 4B\sin(2x) + 4A\sin(2A) - 4B\cos(2x)$$
$$+ A\cos(2x) + B\sin(2x) = 5\cos(2x) + 10\sin(2x)$$

$$\Rightarrow (-3A - 4B)\cos(2x) + (4A - 3B)\sin(2x)$$
$$= 5\cos(2x) + 10\sin(2x)$$

$$\Rightarrow \begin{cases} -3A - 4B = 5 & (1) \\ 4A - 3B = 10 & (2) \end{cases} \rightarrow \quad 4A = 3B + 10$$
$$\rightarrow \quad A = \frac{3}{4}B + \frac{10}{4}$$

Replace in (1) $\Rightarrow \quad -3\left(\frac{3}{4}B + \frac{10}{4}\right) - 4B = 5$

$$\Rightarrow \left(-\frac{9}{4} - 4\right)B - \frac{30}{4} = 5$$

$$\Rightarrow \frac{-25}{4}B = \frac{50}{4}$$

$$\Rightarrow B = -2$$

From (2) $\Rightarrow \quad 4A + 6 = 10 \Rightarrow A = 1$

So, $\quad y_{par}(x) = \cos(2x) - 2\sin(2x).$

(3) General Solution

$$y(x) = y_h(x) + y_{par}(x) = c_1 e^x + c_2 x e^x + \cos(2x) - 2\sin(2x).$$

Case II

When $\cos \omega x$ or $\sin \omega x$ are solutions to the complementary equation.

EXAMPLE 2. Find the general solution to

$$y'' + 4y = 8 \cos 2x + 12 \sin 2x.$$

① Comple. Equation.

$$y'' + 4y = 0 \longrightarrow r^2 + 4 = 0$$

$$\longrightarrow r^2 = -4$$

$$\longrightarrow r = \pm \sqrt{-4}$$

$$\longrightarrow r = \pm \sqrt{4} \sqrt{-1}$$

$$\longrightarrow r = \pm 2i$$

So, $y_h(x) = c_1 \cos(2x) + c_2 \sin(2x)$.

② Part. solution.

$$y_{par}(x) = x\left[A \cos(2x) + B \sin(2x)\right]$$

$$\Rightarrow y'(x) = A\cos(2x) + B\sin(2x) + x\left[-2A\sin(2x) + 2B\cos(2x)\right]$$

&

$$y''(x) = -2A\sin(2x) + 2B\cos(2x)$$

$$-2A\sin(2x) + 2B\cos(2x)$$

$$- x\left(4A\cos(2x) + 4B\sin(2x)\right).$$

$$= -4A\sin(2x) + 4B\cos(2x) - x\left(4A\cos(2x) + 4B\sin 2x\right)$$

Replace in the ODE:

$$\Rightarrow \quad 4B\cos(2x) - 4A\sin(2x)$$

$$= 8\cos(2x) + 12\sin(2x)$$

$$\Rightarrow \quad 4B = 8 \quad \& \quad -4A = 12$$

$$\Rightarrow \quad B = 2 \quad \& \quad A = -3$$

So, $\quad y_{par}(x) = x\left(-3\cos 2x + 2\sin 2x\right)$

(3) General Solution:

$$y(x) = y_h + y_{par}$$

$$= c_1 \cos 2x + c_2 \sin 2x$$

$$+ x\left(2\sin 2x - 3\cos 2x\right).$$

We consider the following second basic case:

$$ay'' + by' + cy = F(x)\cos \omega x + G(x)\sin \omega x$$

where ω is a fixed real number and F, G are two polynomials.

There are still two cases: weither $\cos \omega x$ and $\sin \omega x$ are or are not solutions to the complementary equation.

Example 3. Find the general solution to

$$y'' + 3y' + 2y = (16 + 20x)\cos x + 10\sin x.$$

① Compl. Eq.

$$y'' + 3y' + 2y = 0 \Rightarrow r^2 + 3r + 2 = 0$$

$$\Rightarrow (r+2)(r+1) = 0$$

$$\Rightarrow r = -1 \text{ \& } r = -2.$$

So, $y_h(x) = c_1 e^{-x} + c_2 e^{-2x}$

② Particular solutions.

$$y_{par}(x) = (Ax+B)\cos(x) + (Cx+D)\sin(x).$$

$$y' = A\cos x - (Ax+B)\sin(x) + C\sin x + (Cx+D)\cos(x)$$

$$y'' = -A\sin x - A\sin x - (Ax+B)\cos x + C\cos x + C\cos x - (Cx+D)\sin x$$

$$= -2A\sin x + 2C\cos x - (Ax+B)\cos x - (Cx+D)\sin x$$

Replace in the ODE:

$$-2A\sin x - 2C\cos x - (Ax+B)\cos x - (Cx+D)\sin x$$

$$-3A\cos x + 3(Ax+B)\sin(x) - 3C\sin x$$
$$- 3((Cx+D)\cos(x)$$

$$+2(Ax+B)\cos(x) + 2((Cx+D))\sin(x)$$

$$= [B+2A+3D+2C + (A+3C)x]\cos(x)$$

$$+ [D+3C-3B-2A + (C-3A)x]\sin x.$$

$$= (16+20x)\cos(x) + 10\sin x.$$

$$\Rightarrow \quad B+2A+3D+2C = 16 \quad \& \quad A+3C = 20$$

$$D+3C-3B-2A = 10 \quad \& \quad C-3A = 0$$

$$\Rightarrow \quad A = 2, \quad B = 1, \quad C = 6 \quad \& \quad D = -1.$$

Therefore:

$$y_{par}(x) = (2x+1)\cos x + (6x-1)\sin x.$$

③ General Solution:

$$y(x) = y_h(x) + y_{par}(x)$$

$$= c_1 e^{-x} + c_2 e^{-2x} + (2x+1)\cos(x) + (6x-1)\sin x.$$

We now consider the more general case

$$ay'' + by' + c = e^{\alpha x} \left(F(x) \cos \omega x + G(x) \sin \omega x\right)$$

where α, ω are real numbers with $\omega \neq 0$ and F, G are polynomials.

There are also two cases: weither $e^{\alpha x} \cos \omega x$ and/or $e^{\alpha x} \sin \omega x$ are or are not solutions to the complementary equation.

Example 4. Find the general solution of

$$y'' + 2y' + 5y = e^{-x} \left((6 - 16x) \cos 2x - (8 + 8x) \sin 2x\right).$$

Recap

$ar^2 + br = 0$
$(ar+b)r = 0$

A particular solution of

$c = 0$

$$ay'' + by' + cy = \cancel{e^{\alpha x}} \left(F(x)\cos\omega x + G(x)\sin\omega x\right)$$

where $\omega \neq 0$ has the form

- when $\cancel{e^{\alpha x}}\cos\omega x$ and $\cancel{e^{\alpha x}}\sin\omega x$ are not solutions to the complementary equation,
$$y_{par}(x) = \cancel{e^{\alpha x}} \left(A(x)\cos\omega x + B(x)\sin\omega x\right),$$
with $A(x)$ and $B(x)$ are polynomials of the same degree as the biggest degree between $F(x)$ and $G(x)$
- When $e^{\alpha x}\cos\omega x$ and $e^{\alpha x}\sin\omega x$ are solutions to the complementary equation,
$$y_{par}(x) = xe^{\alpha x} \left(A(x)\cos\omega x + B(x)\sin\omega x\right),$$
with $A(x)$ and $B(x)$ are polynomials of the same degree as the highest degree between the polynomials $F(x)$ and $G(x)$.