E.I Bivariate Distributions

PROBLEM 1. Let (X, Y) be a random vector with joint distribution $F_{X,Y}$. Prove that, for any $a < c$ and $b < d$,

$$P(a < X \leq b, c < Y \leq d) = F(b, d) + F(a, c) - F(a, d) - F(b, c).$$

E.II Continuous Random Vectors

PROBLEM 2. If (X, Y) are continuous random vector with joint probability density function $f_{X,Y)}$. Prove that

$$P(a \leq X \leq b, c \leq Y \leq d) = P(a < X < b, c < Y < d).$$

PROBLEM 3. If a radioactive particle is randomly located in a square of unit length, a reasonable model for the joint density function for X and Y (the coordinates of the location of the radioactive particle) is

$$f_{X,Y}(x, y) = \begin{cases} kxy & \text{if } (x, y) \in [0, 1] \times [0, 1] \\ 0 & \text{elsewhere} \end{cases}$$

a) Find the value k that makes this a probability density function.

b) Find the joint distribution function for X and Y.

c) Find $P(X \leq 0.5, Y \leq 0.75)$.

PROBLEM 4. Let (X, Y) denote the coordinates of a point chosen at random inside a unit circle whose center is at the origin. Their joint probability density function is

$$f_{X,Y}(x, y) = \begin{cases} 1/\pi & \text{if } x^2 + y^2 \leq 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Find $P(X \leq Y)$.

E.III Marginals and Independence

PROBLEM 5. Let (X, Y) be a continuous random vector with joint probability differentiable density function $f_{X,Y}$. Show that

a) $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy$.

b) $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx$.

PROBLEM 6. Let X and Y be two random variable with joint probability density function

$$f_{X,Y}(x, y) = \begin{cases} 1 & 0 \leq y \leq x \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

a) Sketch $f_{X,Y}$.

b) Are X, Y independent?

Problem 7. Let X and Y be two random variable with joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} 2y+1 & (x,y) \in [0,1]\times[0,1] \\ 0 & \text{elsewhere} \end{cases}$$

a) Sketch $f_{X,Y}$.

b) Are X, Y independent?

Problem 8. A bus arrives at a bus stop at a uniformly distributed time over the interval 0 to 1 hour. A passenger also arrives at the bus stop at a uniformly distributed time over the interval 0 to 1 hour. Assume that the arrival times of the bus and passenger are independent of one another and that the passenger will wait for up to 1/4 hour for the bus to arrive. What is the probability that the passenger will catch the bus?

E.IV Important Measurements

Problem 9. Prove that if X and Y are two independent random variables with average μ_X and μ_Y, then $\mathrm{Var}(X,Y) = 0$.

Problem 10. Prove that if X and Y are two random variables with averages μ_X and μ_Y and standard deviation σ_X and σ_Y, then $\rho(X,Y) \in [-1,1]$.

Problem 11. Let X and Y be random variables with means μ_X and μ_Y and with variance σ_X^2 and σ_Y^2. Use the definition of the covariance to show that

a) $\mathrm{Cov}(X,Y) = \mathrm{Cov}(Y,X)$.

b) $\mathrm{Var}(aX+bY) = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\mathrm{Cov}(X,Y)$.

c) $\mathrm{Cov}(X,X) = \sigma_X^2$.

Problem 12. The random variables X and Y are such that $\mathrm{Exp}(X) = 4$, $\mathrm{Exp}(Y) = -1$, $\sigma_X^2 = 2$ and $\sigma_Y^2 = 8$.

a) What is $\mathrm{Cov}(X,X)$?

b) What is the largest possible value for $\mathrm{Cov}(X,Y)$?