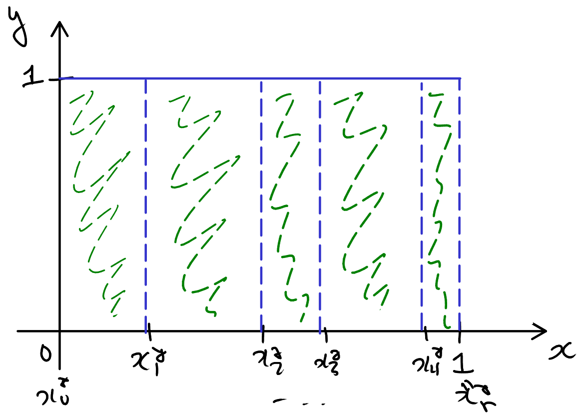
Example 2

Show that $f(x) = 1$ is integrable over the interval $[0, 1]$.



$x_0^*, x_1^*, x_2^*, \ldots, x_n^*$ are pts. in $[0,1]$.

$$\Delta x_i = x_i^* - x_{i-1}^*$$

$$f(x_i^*) = 1$$

$$S_n = \sum_{k=1}^{n} f(x_k^*)\,\Delta x_k = \sum_{k=1}^{n} 1 \cdot (x_k^* - x_{k-1}^*)$$

$$= \sum_{k=1}^{n} (x_k^* - x_{k-1}^*)$$

$\underline{n=2}$ $$\sum_{k=1}^{\boxed{2}} (x_k^* - x_{k-1}^*) = \cancel{x_1^*} - x_0^* + x_2^* - \cancel{x_1^*}$$

$$= x_2^* - x_0^* = 1 - 0 = 1$$

$\underline{n=3}$ $$\sum_{k=1}^{3} (x_k^* - x_{k-1}^*) = x_3^* - x_0^* = 1 - 0 = 1$$

So, $S_n = x_n^* - x_0^* = 1 - 0 = 1$

So, $\displaystyle\lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 = 1.$

$$\boxed{\int_0^1 1\,dx = 1}$$

$g(x) = x$

$g'(x) = 1$

$$\int_0^1 1\,dx = g(1) - g(0) = 1$$

Example 4

Express the following limit in term of an integral: on $[2, 5]$

$$\lim_{n\to\infty} \sum_{i=1}^{n} (x_i^3 + x_i \sin x_i)\Delta x.$$

Goal

$$\lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{(x_i^3 + x_i \sin x_i)} \Delta x = \int_2^5 f(x)\,dx$$

By definition:

$$\int_2^5 f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \widetilde{f(x_i)}\,\Delta x$$

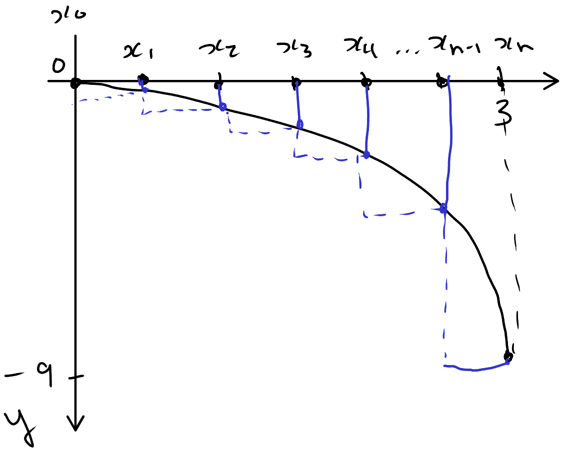
$$f(x) = x^3 + x\sin x .$$

so,

$$\lim_{n\to\infty} \sum_{i=1}^{n} (x_i^3 + x_i \sin x_i)\Delta x = \int_2^5 (x^3 + x\sin x)\,dx$$

Example 6

Using the last Theorem, compute the integral $\int_0^3 (x^2 - 6x)\,dx$. $\to f(x)$



$$\Delta x = \frac{3-0}{n} = \frac{3}{n}$$

$$x_i = a + i\Delta x = 0 + i\frac{3}{n} = i\frac{3}{n}$$

$$f(x_i) = f\left(i\frac{3}{n}\right) = \frac{9i^2}{n^2} - \frac{6\cdot 3i}{n} = \frac{9i^2}{n^2} - \frac{18i}{n}$$

$$\int_0^3 x^2 - 6x\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{9i^2}{n^2} - \frac{18i}{n}\right)\cdot\frac{3}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{27i^2}{n^3} - \frac{54i}{n^2}\right)$$

$$= \lim_{n\to\infty} \left(27\sum_{i=1}^{n} \frac{i^2}{n^3} - 54\sum_{i=1}^{n} \frac{i}{n^2}\right)$$

$$= \lim_{n\to\infty} \left(\frac{27}{n^3}\sum_{i=1}^{n} i^2 - \frac{54}{n^2}\sum_{i=1}^{n} i\right)$$

$$= \lim_{n\to\infty} \frac{27}{n^3}\,\frac{n\cdot(2n+1)(n+1)}{6} - \frac{54}{n^2}\,\frac{n(n+1)}{2}$$

$$= \lim_{n\to\infty} \frac{27}{n^3}\,\frac{n(2n^2+3n+1)}{6} - \lim_{n\to\infty} \frac{54}{n^2}\,\frac{n^2+n}{2}$$

$$= \lim_{n\to\infty} \frac{27\,(2n^3+3n^2+1)}{6n^3} - \frac{54}{2}$$

$$= \frac{27}{3} - \frac{54}{2} = 9 - 27 = -18$$

$$\boxed{\text{so } \int_0^3 x^2 - 6x\,dx = -18}$$

Example 8

Suppose $\int_0^1 f(x)\,dx = 10$ and $\int_2^1 f(x)\,dx = -5$, compute the value of $\int_0^2 f(x)\,dx$.

Third property:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$a=0$, $b=2$, $c=1$, then

$$\int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx$$

From the 1st property:

$$\int_2^1 f(x)\,dx = -\int_1^2 f(x)\,dx \Rightarrow \int_1^2 f(x)\,dx = -(-5) = 5$$

so, $\int_0^2 f(x)\,dx = 10 + 5 \boxed{= 15}$

Example 10

Compute the value of the definite integral $\int_0^1 (4 + 3x^2)\,dx$.

$f(x) = 4$ $\quad$ $g(x) = 3x^2$

So, by linearity,

$$\int_0^1 4 + 3x^2\,dx = \int_0^1 4\,dx + \int_0^1 3x^2\,dx$$

$$= 4\cdot(1-0) + 3\int_0^1 x^2\,dx \to \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3}$$

$$= 4 + 3\cdot\left(\frac{1}{3}\right)$$

$$\Rightarrow \boxed{\int_0^1 4 + 3x^2\,dx = 5}$$

Example 12

Estimate the integral $\int_1^4 \sqrt{x}\, dx$.

Here, x is between 1 and 4:

$$1 \le x \le 4$$

$$\Rightarrow \quad \sqrt{1} \le \sqrt{x} \le \sqrt{4}$$

$$\Rightarrow \quad \underset{m}{\underset{\shortparallel}{1}} \le \underset{f(x)}{\underset{\shortparallel}{\sqrt{x}}} \le \underset{M}{\underset{\shortparallel}{2}}$$

So, from the last result

$$1 \cdot (4-1) \le \int_1^4 \sqrt{x}\, dx \le 2(4-1)$$

$$\Rightarrow \quad 3 \le \int_1^4 \sqrt{x}\, dx \le 6$$

So,
$$\int_1^4 \sqrt{x}\, dx \approx \frac{6+3}{2} = 4.5$$

close

In fact,
$$\int_1^4 \sqrt{x}\, dx = 4.3333\ldots$$

Example 15

Find all the antiderivative of each of the following functions.

a) $f(x) = \sin x$.

b) $f(x) = x^3$.

c) $f(x) = x^{-3}$.

a) $F(x) = -\cos x + C$ (C is a constant).

b) $F(x) = \dfrac{x^4}{4} + C$

<u>why?</u> $F'(x) = \left(\dfrac{x^4}{4}\right)' + (C)' = \dfrac{4x^3}{4} + 0$

$= x^3$.

c) $F(x) = -\dfrac{x^{-2}}{2} + C$

<u>why?</u> $F'(x) = \left(-\dfrac{x^{-2}}{2}\right)' + (C)' = \dfrac{(-1)(-2)x^{-3}}{2} + 0$

$= x^{-3}$.

Example 16

A particle moves in a straight line and has acceleration given by $a(t) = 6t + 4$. Its initial velocity is $v(0) = -6cm/s$ and its initial displacement is $s(0) = 9cm$. Find its position function $s(t)$.

Reminder: $a(t) = \frac{dv}{dt}(t) = v'(t)$

Also, $v(t) = s'(t)$. $\left(\frac{ds}{dt}(t)\right)$

Antiderivative for $a(t)$ $\quad a(t) = 6t + 4$

$$\Rightarrow \quad v(t) = 3t^2 + 4t + C$$

We have to find the constant C, let $t=0$:

$$-6 = v(0) = 0 + 0 + C \quad \Rightarrow \quad C = -6.$$

So, $v(t) = 3t^2 + 4t - 6$

Antiderivative of $v(t)$ $\quad v(t) = 3t^2 + 4t - 6$

$$s(t) = t^3 + 2t^2 - 6t + D$$

We have to find the constant D: let $t=0$

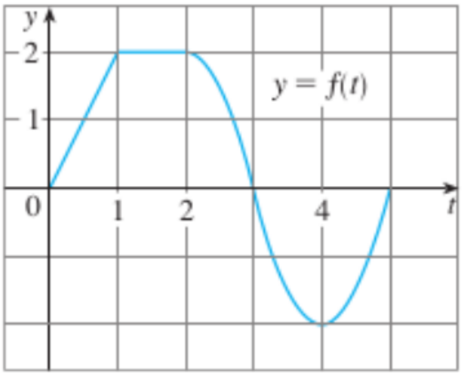
$$9 = s(0) = 0 + 0 - 0 + D$$

$$\Rightarrow D = 9.$$

Thus, $\boxed{s(t) = t^3 + 2t^2 - 6t + 9}$

Example 17

Suppose that f is the function given by the graph in the following figure:



If $F(x) := \int_0^x f(t)\,dt$, find the value of $F(0)$, $F(1)$, $F(2)$.

$$F(0) = \int_0^0 f(t)\,dt = 0$$

$$F(1) = \int_0^1 f(t)\,dt = \text{area of the triangle}$$

$$= 1$$

$$F(2) = \int_0^2 f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt$$

$$= 1 + 2 = 3$$

So, $\boxed{F(0)=0, \quad F(1)=1 \quad \& \quad F(2)=3}$

Example 18

Find the derivative of the function $F(x) = \int_0^x \overbrace{\sqrt{1+t^2}}^{f(t)}\, dt$.

$$F'(x) = \lim_{h \to 0} \frac{\int_0^{x+h} \sqrt{1+t^2}\, dt - \int_0^x \sqrt{1+t^2}\, dt}{h}.$$

In fact, from the FTC,

$$F'(x) = f(x) = \sqrt{1+x^2}.$$

Example 19

Evaluate the integral $\int_{-2}^{1} x^3\, dx$.

$f(x) = x^3 \longrightarrow F(x) = \frac{x^4}{4} \cancel{+ C}$ (take $C=0$)

So,

$$\int_{-2}^{1} x^3\, dx = F(1) - F(-2) = \frac{(1)^4}{4} - \frac{(-2)^4}{4}$$

$$= \frac{1}{4} - 4$$

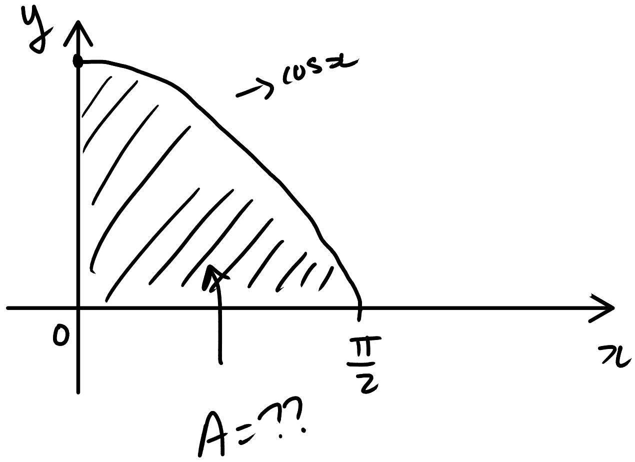
$$= \frac{-15}{4}$$

Thus,

$$\boxed{\int_{-2}^{1} x^3\, dx = -\frac{15}{4}}$$

Example 20

Find the area under the cosine curve from 0 to $\pi/2$.



$$A = \int_0^{\pi/2} \cos x \, dx$$

$$F(x) = \sin x \quad (+C \text{ But } C=0)$$

So, $$\int_0^{\pi/2} \cos x \, dx = F(\pi/2) - F(0) = \sin\frac{\pi}{2} - \sin 0$$
$$= 1$$

Thus, $$\boxed{A = 1 \, u^2}$$