Chapter 4
Integrals

4. Indefinite Integrals and the Net Change Theorem

Indefinite Integral.

Previously on Calc I:

Fondamental Theorem of Calculus	$\int_a^b f(x)dx = F(b) - F(a).$

We introduce a notation for the antiderivatives:

indefinite integral

$$\int f(x)\,dx = F(x) \quad \text{means} \quad F'(x) = f(x)$$

$\int f(x)dx$

General A.D.

Example.

a) $\int x^2\,dx = \dfrac{x^3}{3} + C$

b) $\int \cos x\,dx = \sin x + C$

c) $\int \sec^2 x\,dx = \tan x + C$

Table of Indefinite integrals

$\int cf(x)\,dx = c\int f(x)\,dx$	$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$
$\int k\,dx = kx + C$	$\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C \quad (n \neq -1)$
$\int \sin x\,dx = -\cos x + C$	$\int \cos x\,dx = \sin x + C$
$\int \sec^2 x\,dx = \tan x + C$	$\int \csc^2 x\,dx = -\cot x + C$
$\int \sec x \tan x\,dx = \sec x + C$	$\int \csc x \cot x\,dx = -\csc x + C$

Remark: We adopt the convention that the general indefinite integral is valid on a given interval. Thus we write

$$\boxed{\int \frac{1}{x^2}\,dx} = -\frac{1}{x} + C$$

$x^{-2} \to \dfrac{x^{-2+1}}{-2+1}$

with the understanding that it is valid on the interval $(0, \infty)$ or on the interval $(-\infty, 0)$.

EXAMPLE 1 Find the general indefinite integral

$$\int (10x^4 - 2\sec^2 x)\,dx$$

$$\int \underline{10x^4} - \underline{2\sec^2 x}\,dx = \int 10x^4\,dx + \int -2\sec^2 x\,dx$$

$$= 10\underline{\int x^4\,dx} - 2\underline{\int \sec^2 x\,dx}$$

$$= 10\left(x^5/5 + C_1\right) - 2\left(\tan x + C_2\right)$$

$$= 2x^5 + 10C_1 - 2\tan x - 2C_2$$

$$= \boxed{2x^5 - 2\tan x + C}$$

where $C = 10C_1 - 2C_2$.

EXAMPLE 2 Evaluate $\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$.

$$\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \int \frac{\cos\theta}{\sin\theta} \cdot \frac{1}{\sin\theta}\,d\theta$$

$$= \int \cot\theta \cdot \csc\theta\,d\theta$$

$$= \boxed{-\csc\theta + C}$$

because $(\csc\theta)' = -\csc\theta\cot\theta$.

EXAMPLE 4 Find $\int_0^{12} (x - 12 \sin x)\, dx$.

$$\int_0^{12} x - 12\sin x \, dx = \int_0^{12} x \, dx - 12 \int_0^{12} \sin x \, dx$$

$$= \frac{x^2}{2}\Big|_0^{12} - 12\left(-\cos x\right)\Big|_0^{12}$$

$$= \frac{12^2}{2} - \cancel{\frac{0^2}{2}}^{\to 0} - 12\left(-\cos 12 - (-\cancel{\cos 0}^{1})\right)$$

$$= \frac{144}{2} - 12\left(-\cos 12 + 1\right)$$

$$= 72 + 12\cos 12 - 12$$

$$= \boxed{60 + 12\cos 12}$$

EXAMPLE 5 Evaluate $\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\, dt$.

$f(t)$ is continuous on $[1,9]$

$$\frac{2\cancel{t^2} + \cancel{t^2}\sqrt{t} - 1}{\cancel{t^2}} = 2 + \sqrt{t} - \frac{1}{t^2} = 2 + \sqrt{t} - t^{-2}$$

$$\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\, dt = \int_1^9 2 + \sqrt{t} - t^{-2}\, dt$$

t^0

$$= 2\int_1^9 1\, dt + \int_1^9 \sqrt{t}\, dt - \int_1^9 t^{-2}\, dt$$

$$= 2\,\frac{t^{0+1}}{0+1}\Big|_1^9 + \frac{t^{1/2+1}}{1/2+1}\Big|_1^9 - \frac{t^{-2+1}}{-2+1}\Big|_1^9$$

$$= 2t\Big|_1^9 + \frac{2t^{3/2}}{3}\Big|_1^9 + t^{-1}\Big|_1^9$$

$$= 2\cdot 9 - 2\cdot 1 + \frac{2\cdot 9^{3/2}}{3} - \frac{2\cdot 1}{3} + 9^{-1} - 1$$

$$= \boxed{32\frac{4}{9}}$$

Net Change Theorem The integral of a rate of change is the net change:

$F'(x) \to$ rate of change

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

a) Displacement: $\to$ net change in position

$v(t) = s'(t)$ (s: position vector).

$$\text{displ.} = \int_a^b v(t)\,dt = s(b) - s(a)$$

(a: starting point
b: ending point)

b) Total distance traveled:

$$\text{Tot. distance} = \int_a^b |v(t)|\,dt$$

c) Acceleration: net change in velocity $a(t) = v'(t)$.

$$\text{Net chg. in vel.} = \int_a^b \underbrace{v'(t)}_{a(t)}\,dt = v(b) - v(a).$$

EXAMPLE 6 A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ (measured in meters per second).

(a) Find the displacement of the particle during the time period $1 \leqslant t \leqslant 4$. ($\to a$, $\to b$)

(b) Find the distance traveled during this time period.

(a)
$$\text{disp.} = \int_1^4 v(t)\,dt = \int_1^4 t^2 - t - 6\,dt = \frac{t^3}{3} - \frac{t^2}{2} - 6t\Big|_1^4$$

$$= \left(\frac{4^3}{3} - \frac{4^2}{2} - 6\cdot 4\right) - \left(\frac{1}{3} - \frac{1}{2} - 6\right)$$

$$= -\frac{9}{2}\,m = \boxed{-4.5\,m}$$

(b)
$$\text{Total distance} = \int_1^4 |v(t)|\,dt$$

$v(t) = t^2 - t - 6$
$= (t-3)(t+2)$, $t+2 \geq 0$

$t - 3 < 0$: $t < 3$; $t - 3 > 0$: $t > 3$.

$$|v(t)| = \begin{cases} -v(t), & 1 \leq t \leq 3 \\ v(t), & 3 \leq t \leq 4 \end{cases}$$

$$\text{Tot. dist.} = \int_1^3 -v(t)\,dt + \int_3^4 v(t)\,dt$$

$$= \int_1^3 -(t^2 - t - 6)\,dt + \int_3^4 t^2 - t - 6\,dt$$

$$= \frac{61}{6} \approx \boxed{10.17\,m}$$