Math 302

Chapter 8

Section 8.3: Unit Step Function

Contents

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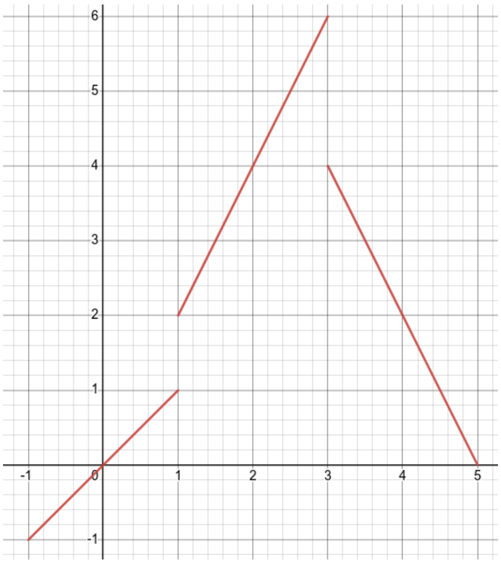
Fall 2022

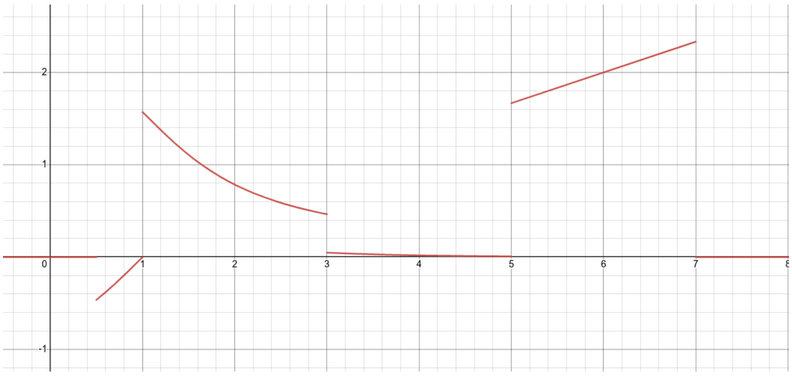
Piecewise Continuous Functions

A piecewise continuous function f is

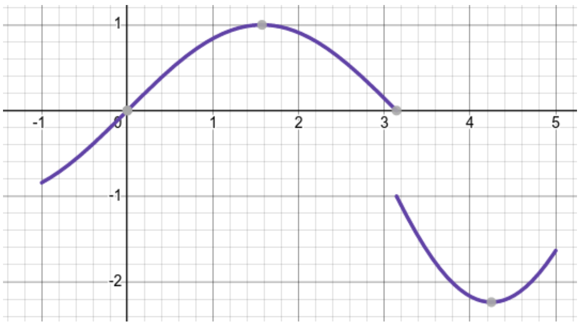
- a function defined on a finite number of intervals $[t_0, t_1]$, $[t_1, t_2]$, ..., $[t_{n-1}, t_n]$;
- such that it is continuous on each interval (t_0, t_1), (t_1, t_2), ..., (t_{n-1}, t_n).



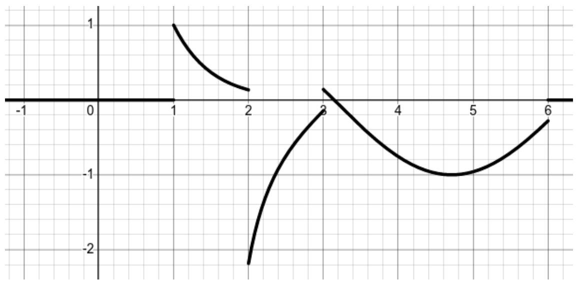
(a) A function $f(x)$



(b) A function $k(x)$



(c) A function $g(x)$



(d) A function $h(x)$

Example 1. Find the Laplace transform of

$$f(t) = \begin{cases} t & 0 < t \leq 1 \\ 2t & 1 < t \leq 3 \\ 10 - 3t & 3 < t \leq 5 \\ 0 & 5 < t. \end{cases}$$

Unit Step Function

To make the work easier with piecewise continuous function, we introduce the **unit step function**:

$$u(t) := \begin{cases} 0 & t < 0 \\ 1 & t \geq 0. \end{cases}$$

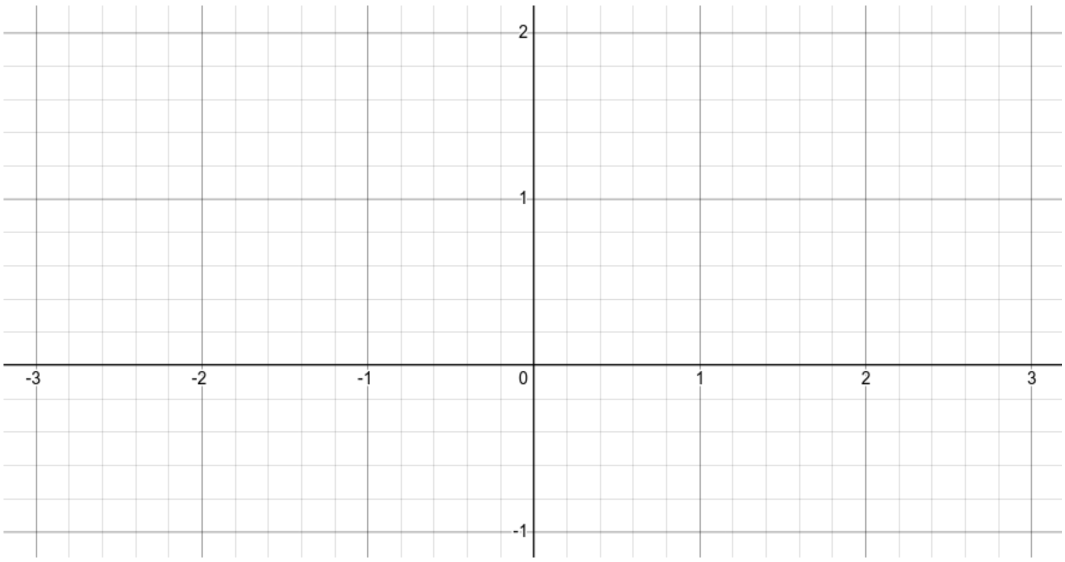


Figure 2: Plot of $u(t)$

Basic Operations

- Translation by a units:

$$u(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a. \end{cases}$$

- Multiplication by c:

$$cu(t) = \begin{cases} 0 & t < 0 \\ c & t \geq 0. \end{cases}$$

- Activation of a function $f(t)$ at time a:

$$f(t)u(t-a) = \begin{cases} 0 & t < a \\ f(t) & t \geq a. \end{cases}$$

- Destruction of a function $f(t)$ at time b and activation of a function $g(t)$ at time b:

$$f(t)u(t-a) + (g(t) - f(t))u(t-b) = \begin{cases} 0 & t < a \\ f(t) & a \leq t < b \\ g(t) & b \leq t. \end{cases}$$

Example 2. Rewrite the function $f(t)$ in Example 1 using the unit step function.

Example 3. A farmer has a field of potatoes of 1 kilometer long. An automated watering system starts at 5:00AM and stops at 8:00AM. The spite of water is 1000 liters per hour. Give an expression of the function $W(t)$ of water used during the day using the unit step function.

Let $a \geq 0$ be a real number and f be a function with a Laplace transform $F(s)$.

- $L(u(t-a)) = \dfrac{e^{-sa}}{s}$.
- $L(u(t-a)f(t)) = e^{-sa}F(s+a)$.
- $L(u(t-a)f(t-a)) = e^{-sa}F(s)$.

Example 4. Find the Laplace transform of

$$f(t) = \begin{cases} \sin(t) & , 0 \leq t < \pi/2 \\ \cos(t) - 3\sin(t) & , \pi/2 \leq t < \pi \\ 3\cos(t) & , t \geq \pi. \end{cases}$$

Example 5. Find

$$L^{-1}\left(\frac{1}{s^2} - e^{-s}\Big(\frac{1}{s^2} + \frac{2}{s}\Big) + e^{-4s}\Big(\frac{4}{s^3} + \frac{1}{s}\Big)\right)$$

We can now allow the forcing function to be a discontinuous function (piecewise continuous).

EXAMPLE 6. Solve the initial value problem

$$y'' - y = f(t), \quad y(0) = -1,\, y'(0) = 2,$$

where

$$f(t) = \begin{cases} t & 0 \leq t < 1 \\ 1 & t \geq 1. \end{cases}$$