Example 1

Consider the function

$$f(x) = \frac{x^2 - 1}{x^2 + 1}.$$

- What is the graph of this function?
- What happens to the numerator if x becomes larger and larger?
- What happens to the denominator if x becomes larger and larger?
- What happens if x becomes larger and larger in the negative values?

Example 5

The function $f(x) = \frac{x^2-1}{x^2+1}$ has $y = 1$ as a HA. $\leftarrow$ $\lim_{x\to\infty}$ or $\lim_{x\to-\infty}$??

$$\lim_{x\to\infty} \frac{x^2-1}{x^2+1} = \frac{\infty}{\infty} \text{ (not defined).}$$

$$\frac{x^2-1}{x^2+1} = \frac{\cancel{x^2}(1-1/x^2)}{\cancel{x^2}(1+1/x^2)} = \frac{1-1/x^2}{1+1/x^2}$$

x	$1/x^2$
1	1
2	1/4
10	1/100
100	1/10000
$\downarrow$	$\downarrow$
∞	0

x	$-1/x^2$
1	-1
2	$-1/4$
10	$-1/100$
100	$-1/10000$
$\downarrow$	$\downarrow$
∞	0

So, $\lim_{x\to\infty} \frac{1}{x^2} = 0$ & $\lim_{x\to\infty} \frac{-1}{x^2} = 0$.

By the sum rule

$$\lim_{x\to\infty} (1 - 1/x^2) = 1 - 0 = 1$$

& $\lim_{x\to\infty} (1 + 1/x^2) = 1 + 0 = 1 \neq 0$

So, by the quotient rule

$$\lim_{x\to\infty} \frac{1-1/x^2}{1+1/x^2} = \frac{\lim 1-1/x^2}{\lim 1+1/x^2} = \frac{1}{1} = 1.$$

Example 8

Using the preceding rule, compute

$$\lim_{x\to\infty} \frac{3\boxed{x^2} - x - 2}{5x^2 + 4x + 1}. \qquad \frac{\infty}{\infty}$$

$$\frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \frac{(3x+2)(x-1)}{(\quad)(\quad)} \quad \times$$

$$\frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \frac{x^2(3 - 1/x - 2/x^2)}{x^2(5 + 4/x + 1/x^2)}$$

$$= \frac{3 - 1/x - 2/x^2}{5 + 4/x + 1/x^2}$$

$$\lim_{x\to\infty} (3 - 1/x - 2/x^2) = 3 - 0 - 2\cdot 0$$
$$= 3$$

$$\lim_{x\to\infty} (5 + 4/x + 1/x^2) = 5 + 4\cdot 0 + 0$$
$$= 5$$

So, (quotient rule)

$$\lim_{x\to\infty} \frac{3 - 1/x - 2/x^2}{5 + 4/x + 1/x^2} = \frac{3}{5}.$$

Example 9

Find the horizontal and vertical asymptotes of

$$f(x) = \frac{\sqrt{2x^2+1}}{3x-5}.$$

$\lim_{x\to\infty}$ $\lim_{x\to-\infty}$.

VA. Denom. is zero if $3x-5=0$

if $x = 5/3$

Replace $x = 5/3$ in $f(x)$

$$\Rightarrow f(5/3) = \frac{\sqrt{2\cdot 25/9 + 1}}{0} = \frac{\sqrt{59/3}}{0} \approx 7.7/0 = \pm\infty$$

Here, we have a V.A. at $x = 5/3$.

H.A. limit at ∞.

$$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to\infty} \frac{\sqrt{x^2(2+1/x^2)}}{x(3-5/x)}$$

$$= \lim_{x\to\infty} \frac{\cancel{x}\sqrt{2+1/x^2}}{\cancel{x}(3-5/x)}$$

$\boxed{x\to\infty \text{, so } x>0,\ \sqrt{x^2}=x}$

$$= \lim_{x\to\infty} \frac{\sqrt{2+1/x^2}}{3-5/x}$$

$$= \frac{\sqrt{2+0}}{3-0} = \frac{\sqrt{2}}{3}$$

so,

$y = \frac{\sqrt{2}}{3}$ is a HA.

- **lim at $-\infty$.**

$x \to -\infty$, $x < 0$
$\sqrt{x^2} = -x$

$$\lim_{x \to -\infty} \frac{\sqrt{x^2(2 + 1/x^2)}}{x(3 - 5/x)} = \lim_{x \to -\infty} \frac{(-\cancel{x})\sqrt{2 + 1/x^2}}{\cancel{x}(3 - 5/x)}$$

$$= \lim_{x \to -\infty} \frac{-\sqrt{2 + 1/x^2}}{3 - 5/x}$$

$$= -\frac{\sqrt{2}}{3}.$$

So, $y = -\frac{\sqrt{2}}{3}$ is a HA.

Example 10

Compute $\lim_{x\to\infty}(\sqrt{x^2+1}-x)$. $\infty - \infty$.

$$\sqrt{x^2+1} - x = \frac{(\sqrt{x^2+1}-x)(\sqrt{x^2+1}+x)}{\sqrt{x^2+1}+x}$$

$$= \frac{1}{\sqrt{x^2+1}+x}$$

$$= 1/\sqrt{x^2(1+1/x^2)} + x$$

$$\boxed{\begin{array}{l} x\to\infty \\ x>0 \end{array}}$$

$$= 1/x\sqrt{1+1/x^2} + x$$

$$= 1/x\left(\sqrt{1+1/x^2}+1\right)$$

So,

$$\lim_{x\to\infty} \sqrt{x^2+1} - x = \lim_{x\to\infty} \frac{1}{x\left(\sqrt{1+1/x^2}+1\right)}$$

$$\lim_{x\to\infty} \frac{1}{x} = 0, \quad \lim_{x\to\infty}\left(\sqrt{1+1/x^2}+1\right) = 2$$

So, overall,

$$\lim_{x\to\infty} \sqrt{x^2+1} - x = 0\cdot\left(\frac{1}{2}\right) = 0$$

Example 12

It is wrong to do

$$\lim_{x\to\infty}(x^2 - x) = \lim_{x\to\infty} x^2 - \lim_{x\to\infty} x = \infty - \infty$$

because $\infty - \infty$ is not defined, like $0/0$.

$$x^2 - x = x(x-1)$$

We know that $\lim_{x\to\infty} x = \infty$

Applying the 1st rule, $\lim_{x\to\infty} (x-1) = \infty$

So, from the third rule,

$$\lim_{x\to\infty} x(x-1) = \infty \cdot \infty = \infty$$

Example 15.

$y' = 24x^2 - 42x + 18.$

$y' = 0 \iff x = \frac{3}{4}$ or $x = 1$

$y'' = 48x - 42$

- $f''(3/4) < 0$, local.max.
- $f''(1) > 0$, local.min.

Example 16

With the guideline, sketch the graph of the function

$$f(x) = \frac{2x^2}{x^2-1}.$$

① $\text{Dom} f = \mathbb{R} \setminus \{-1, 1\}.$

② • y-intercept: $f(0) = 0$

• x-intercept: $\dfrac{2x^2}{x^2-1} = 0 \iff x = 0$

③ $f(-x) = \dfrac{2(-x)^2}{(-x)^2-1} = \dfrac{2x^2}{x^2-1} = f(x) \longrightarrow$ even.

④ HA. • $\displaystyle\lim_{x\to\infty} \frac{2x^2}{x^2-1} = \lim_{x\to\infty} \frac{2\cancel{x^2}}{\cancel{x^2}(1-1/x^2)}$

$$= \lim_{x\to\infty} \frac{2}{1-1/x^2}$$

$$= 2$$

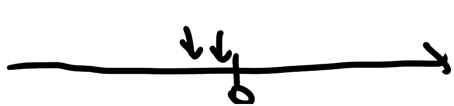
• $\displaystyle\lim_{x\to-\infty} \frac{2x^2}{x^2-1} = 2$

VA. We have $\div$ by 0 when $x = \pm 1$.

• $x \to 1^-$ (x approaches 1 from the left).

x	x^2-1
0.9	-0.19
0.99	-0.0199
	$\downarrow$
$\vdots$	0^-

$$\lim_{x\to 1^-} \frac{2x^2}{x^2-1} = \frac{2}{0^-} = -\infty$$



• $x \to 1^+$

x	x^2-1
1.1	0.21
1.01	0.0201
1.001	0.002001
$\downarrow$	$\downarrow$
1^+	0^+

$$\lim_{x \to 1^+} \frac{2x^2}{x^2-1} = \frac{2}{0+} = +\infty$$

• $x \to -1^-$ $\quad \lim_{x \to -1^-} \frac{2x^2}{x^2-1} = \frac{2}{0^+} = +\infty$

• $x \to -1^+$ $\quad \lim_{x \to -1^+} \frac{2x^2}{x^2-1} = \frac{2}{0^-} = -\infty$

5 $\quad f'(x) = \frac{-4x}{(x^2-1)^2}$ $\qquad f'(x) = 0 \Leftrightarrow x = 0.$

x		-1		0		1	
-4	$-$	$-$	$-$	$-$	$-$	$-$	$-$
x	$-$	$-$	$-$	0	$+$	$+$	$+$
$1/(x^2-1)^2$	$+$	$\nexists$	$+$	$+$	$+$	$\nexists$	$+$
f'	$+$	$\nexists$	$+$	0	$-$	$\nexists$	$-$

• $f' \nearrow$ on $(-\infty, 0)$

• $f' \searrow$ on $(0, \infty)$

6 $\quad f' > 0$ when $-1 \leq x < 0$ &

$f' < 0$ when $0 < x < 1$.

So, $x = 0$ is a local max.

& $f(0) = 0$.

⑦ We have $f''(x) = \dfrac{4(3x^2+1)}{(x-1)^3(x+1)^3}$

$f''(x) = 0 \iff 3x^2 + 1 = 0 \iff x^2 = \frac{-1}{3}$

impossible

So, no zero.

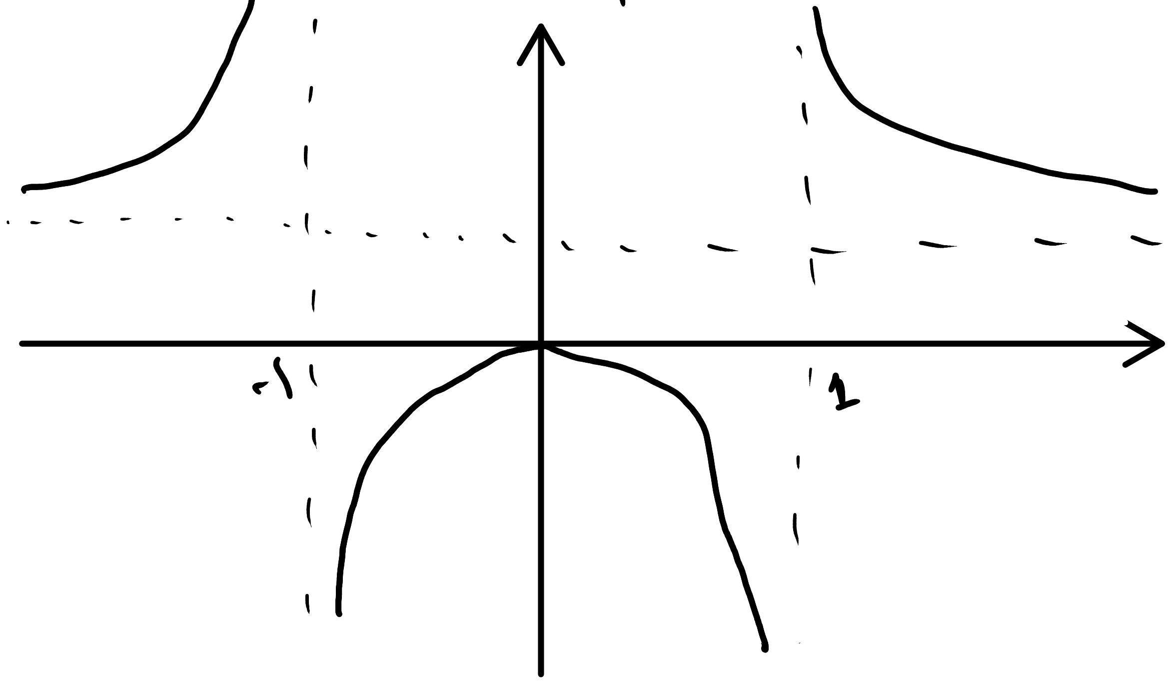
But $f'' \nexists$ if $x = 1$ & $x = -1$.

x		-1		1	
$4(3x^2+1)$	$+$	$+$	$+$	$+$	$+$
$1/(x-1)^3$	$-$	$-$	$-$	$\nexists$	$+$
$1/(x+1)^3$	$-$	$\nexists$	$+$	$+$	$+$
$f''(x)$	$+$	$\nexists$	$-$	$\nexists$	$+$

- f is $\cup$ (upward) on $(-\infty, -1)$ & $(1, \infty)$
- f is $\cap$ (downward) on $(-1, 1)$.

⑧ <u>Sketch</u>.

x		-1		0		1	
f'	$+$	$\nexists$	$+$	0	$-$	$\nexists$	$-$
f''	$+$	$\nexists$	$-$	$-$	$-$	$\nexists$	$+$
f	$\nearrow$	VA	$\nearrow$	loc. max. $f(0)=0$	$\searrow$	VA	$\searrow$



Example 17

With the guideline, sketch the graph of the function
$$f(x) = \frac{\cos x}{2 + \sin x}.$$

① Dom $f = \mathbb{R}$ because $2+\sin x \neq 0$ for any number x.

② y-intercept. $f(0) = \dfrac{\cos(0)}{2+\sin(0)} = \dfrac{1}{2}$

x-intercept. $f(x) = 0 \iff \cos x = 0$

$\iff x = \dfrac{\pi}{2} + \pi k$

k integer.

③ f is not odd and not even.

f is 2π-periodic $\to$ $\cos(x+2\pi) = \cos(x)$

$\sin(x+2\pi) = \sin(x)$.

We can restrict the sketch to $[0, 2\pi]$.

④ HA No horizontal asymptotes

VA No vertical asymptotes.

⑤ $f'(x) = -\dfrac{2\sin x + 1}{(2+\sin x)^2}$.

$$f'(x) = 0 \iff 2\sin x + 1 = 0$$

$$\iff \sin x = -1/2$$

$$\iff x = \frac{11\pi}{6} + 2k\pi \quad k \text{ integer.}$$

or

$$x = \frac{7\pi}{6} + 2k\pi$$

We have $x = 7\pi/6$ and $x = \frac{11\pi}{6}$

because $x \in [0, 2\pi]$.

x	0		$\frac{7\pi}{6}$		$\frac{11\pi}{6}$		2π
-1		$-$	$-$	$-$	$-$	$-$	
$2\sin x+1$		$+$	0	$-$	0	$+$	
$1/(2+\sin x)^2$		$+$	$+$	$+$	$+$	$+$	
$f'(x)$		$-$	0	$+$	0	$-$	

- $f \searrow$ on $(0, 7\pi/6)$ & $(\frac{11\pi}{6}, 2\pi)$.
- $f \nearrow$ on $(\frac{7\pi}{6}, \frac{11\pi}{6})$.

(b) By the 1^{st} derivative test

- $x = \frac{7\pi}{6}$ is a local min.
- $x = \frac{11\pi}{6}$ is a local max.

$f(7\pi/6) = -\frac{\sqrt{3}}{3}$ & $f\left(\frac{11\pi}{6}\right) = \frac{\sqrt{3}}{3}$

7 $f''(x) = -\dfrac{2\cos x\,(1-\sin x)}{(2+\sin x)^2}$

$f''(x) = 0 \iff \left[\begin{array}{ll} x = \frac{\pi}{2} \text{ or } x = \frac{3\pi}{2} & \leftarrow \cos x \\ x = \frac{\pi}{2} & \leftarrow 1-\sin x \end{array}\right.$

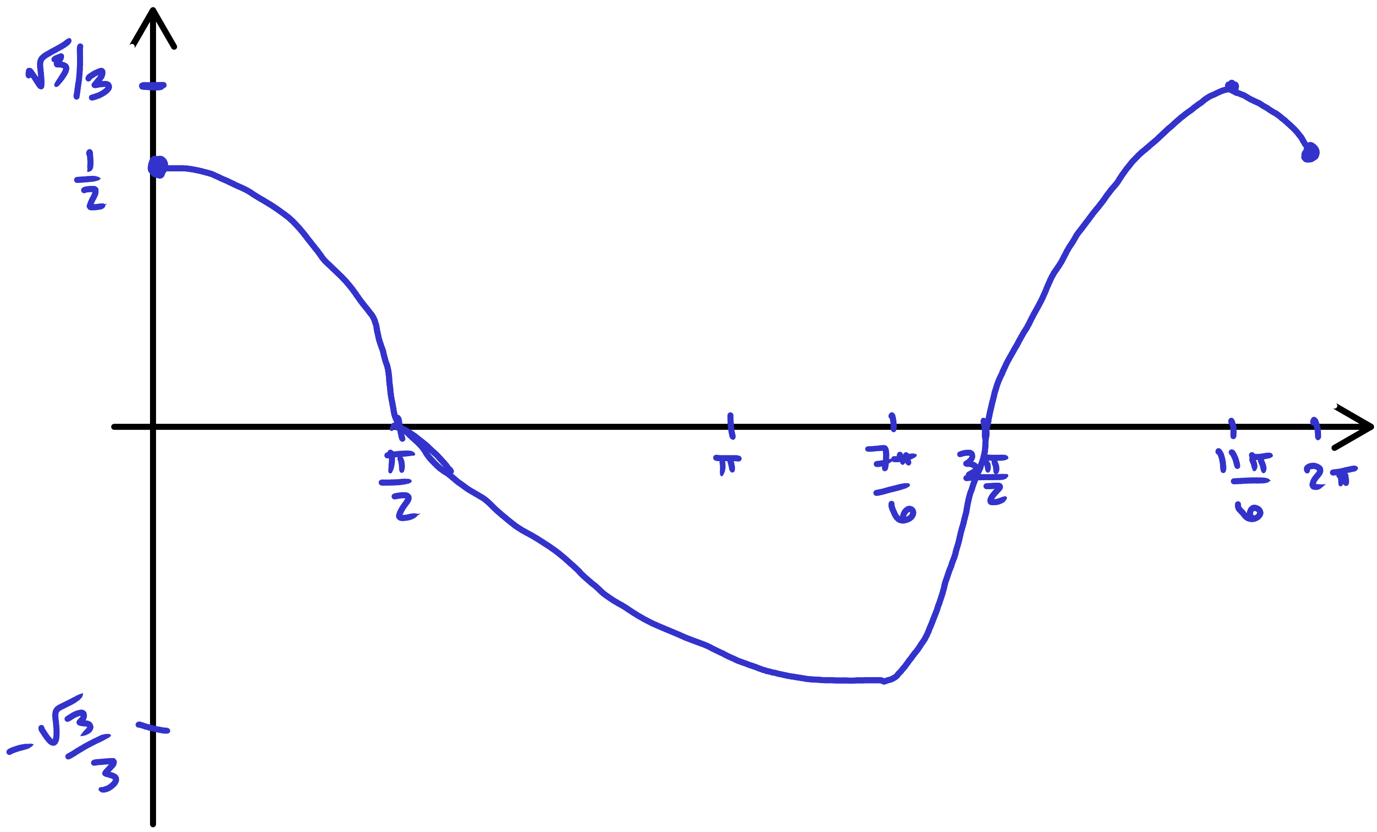
x	0		$\pi/2$		$3\pi/2$		2π
$-2\cos x$		$-$	0	$+$	0	$-$	
$1-\sin x$		$+$	0	$+$	2	$+$	
$(2+\sin x)^2$		$+$	$+$	$+$	$+$	$+$	
$f''(x)$		$-$	0	$+$	0	$-$	

- f is $\cap$ on $(0, \pi/2)$ & $(3\pi/2, 2\pi)$
- f is $\cup$ on $(\pi/2, 3\pi/2)$.

$\pi/2$ & $3\pi/2$ are infl. points.

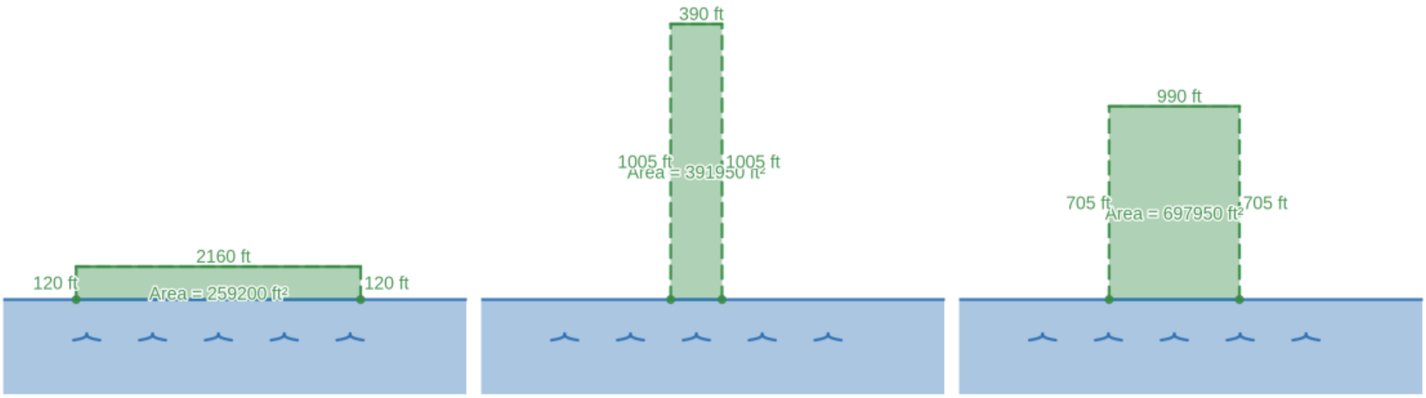
⑧ Sketch.

x	0		$\frac{\pi}{2}$		$\frac{7\pi}{6}$		$\frac{3\pi}{2}$		$\frac{11\pi}{6}$		2π
f'		$-$	$-$	$-$	0	$+$	$+$	$+$	0	$-$	
f''		$-$	0	$+$ $+$		$+$	0	$-$	$-$	$-$	
f		$\searrow$	I.P.	$\searrow$	loc. min.	$\nearrow$	I.P.	$\nearrow$	loc max.	$\searrow$	



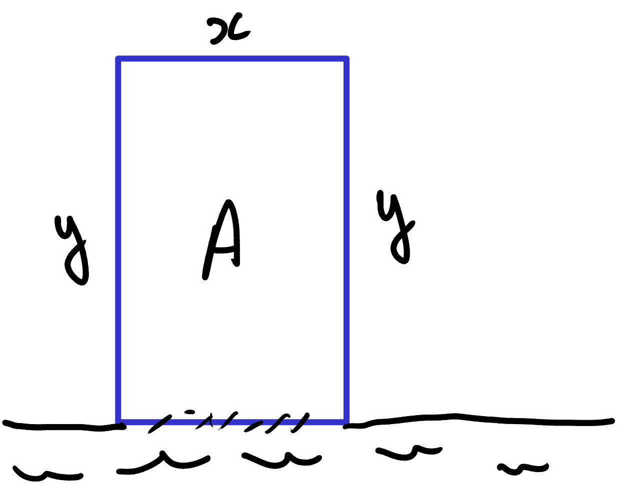
Example 18

A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area? Field problem



(g) Small Height (h) Big Height (i) Good Height

Figure: Some examples



x: width (ft).

y: height (ft)

A: area. (ft^2)

[$A = xy$. to optimize.

Info. $2y + x = 2400$

$$\Rightarrow x = 2400 - 2y$$

So, $A(y) = (2400 - 2y)\,y = 2400y - 2y^2$

$A'(y) = 2400 - 4y$

So, $A'(y) = 0 \iff y = 600$

y		600	
A'	$+$	0	$-$

We have a global max and

$$A = 600(1200) = 720000 \text{ ft}^2$$

Now, $x = 2400 - 2 \cdot 600 = 1200$

So, the dimensions that maximize the area is $x = 1200$ ft &

$y = 600$ ft.

Example 19

Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$.

Parabola distance problem

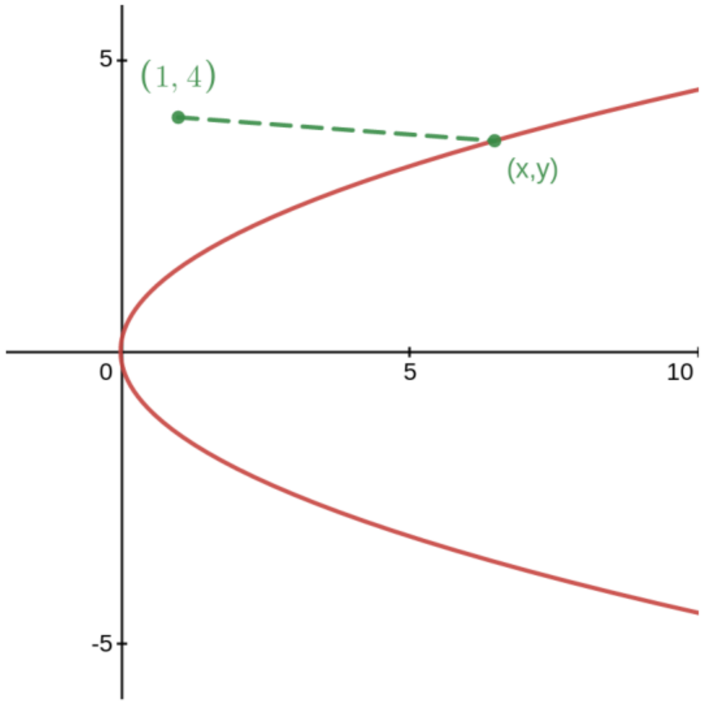


Figure: Minimum Distance Problem