# Chapter 4
# Integrals

4.1 Areas and Distances

The summation symbol.

$$1+2+3+4+5+6 \xrightarrow{\text{Sigma notation}}$$

$$1+1/2+1/3+1/4+1/5+1/6 \xrightarrow{\text{Sigma notation}}$$

$$1/2+1/(2\times 3)+1/(3\times 4)+1/(4\times 5) \xrightarrow{\text{Sigma notation}}$$

**DEFINITION (Sigma Notation)**

If $a_1, a_2, a_3, \ldots, a_n$ are n numbers, then we can write their ordered sum with the Sigma notation:

$$a_1+a_2+a_3+\cdots+a_n=\sum_{i=1}^{n} a_i.$$

Problem: What is the sum of 1 + 2 + 3 + 4 + 5 + 6 + 7? Can you compute it in an efficient way?

Carl Frederich Gauss (1777-1855). Also called the Prince of Mathematics. He discovered the tricks to compute the sum 1 + 2 + 3 + 4 + 5 + ... + 100 when he was around... 7 years old!

General formulas:

1) $$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

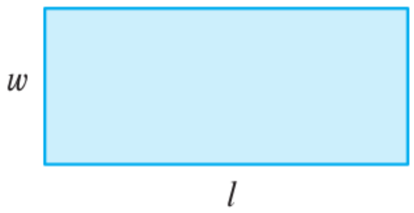
2) $$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

3) $$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

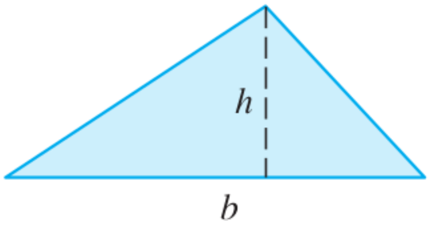
The Area Problem

What is the area of the following shapes?

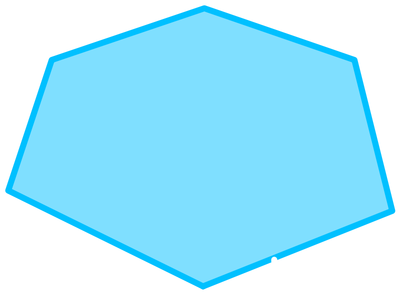
A =



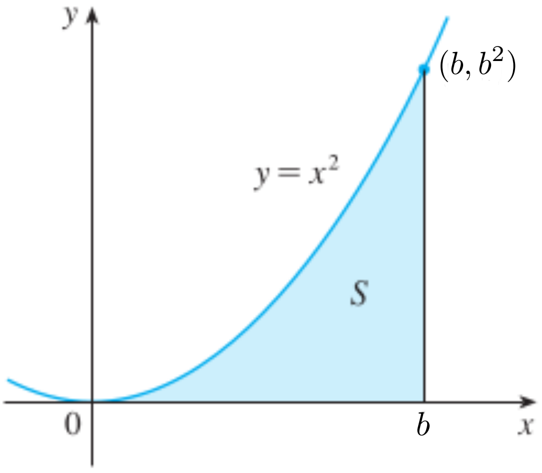

A =



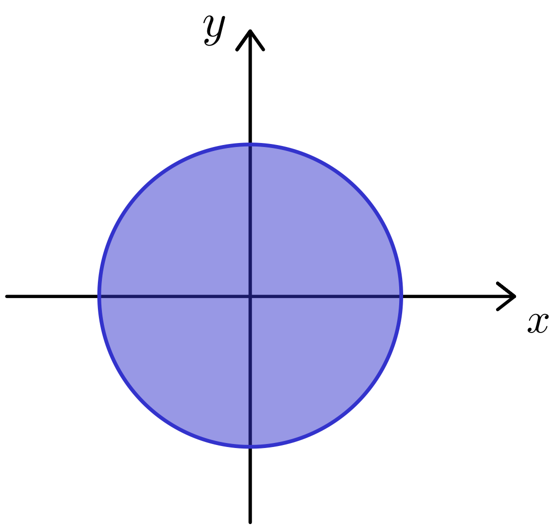

A =



What about the area of the following shapes




A =

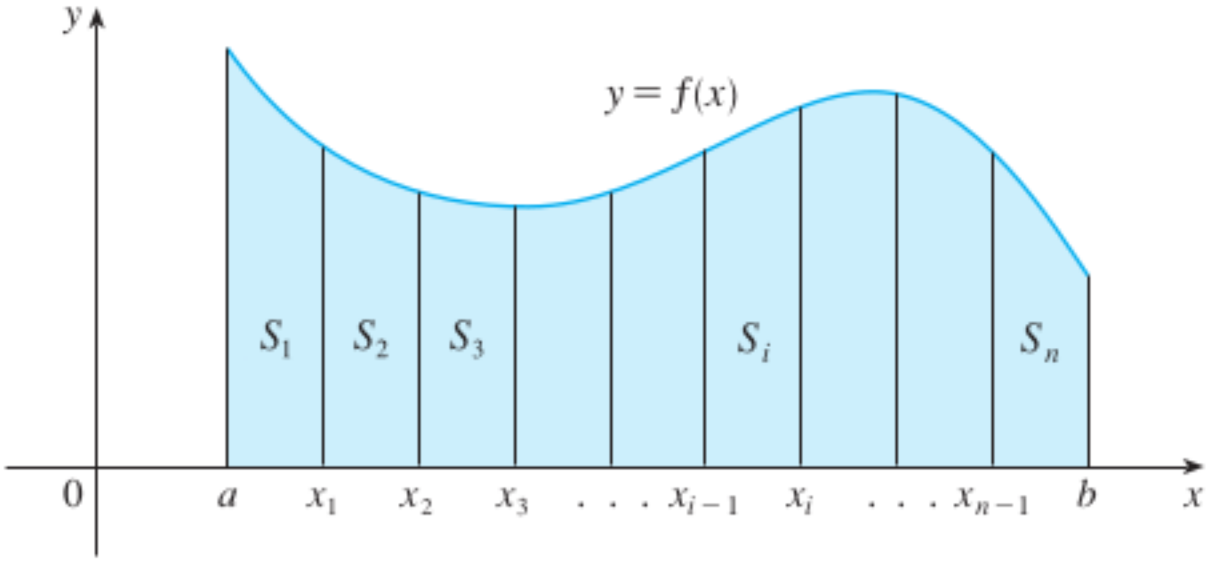



A =

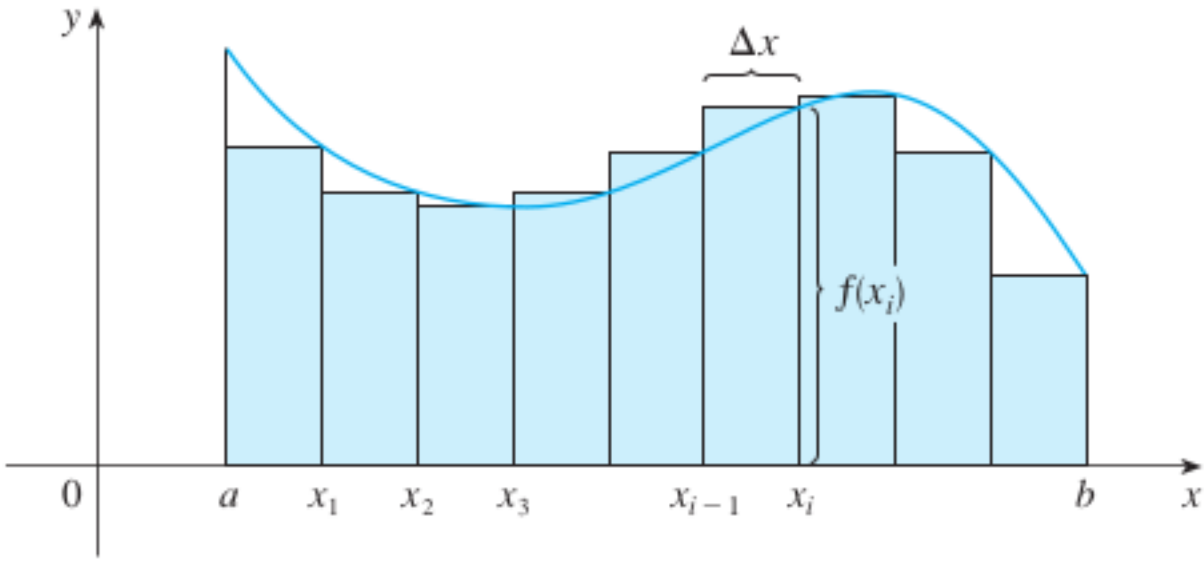
Remember the Youtube video that I showed you on the first day?

Example 1. Try to estimate the area of the region bounded by the curves $y = x^2$ , the line $y = 0$ and the line $x = 1$ .

The key idea: divide and conquer!




1) Divide the region S into n strips.




2) Approximate the area of S by the sum of the area of each rectangle created.

Right endpoints

Left endpoints

2 DEFINITION. The AREA A of the region S that lies under the graph of the continuous function f is the limit of of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x]$$

or
$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} [f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x]$$

Example 2. Show that the area of the region S in example 1 is 1/3. In other words, show that

$$\lim_{n\to\infty} R_n = 1/3.$$

## The distance problem.

If the velocity remains constant, then the distance between the start and the finish line is easy to compute:

$$\text{DISTANCE} = \text{VELOCITY} \times \text{TIME}.$$

What do we do if the velocity varies??

**EXAMPLE 4** Suppose the odometer on our car is broken and we want to estimate the distance driven over a 30-second time interval. We take speedometer readings every five seconds and record them in the following table:

| Time (s) | 0 | 5 | 10 | 15 | 20 | 25 | 30 |
|---|---|---|---|---|---|---|---|
| Velocity (ft/s) | 25 | 31 | 35 | 43 | 47 | 45 | 41 |

velocities were converted from mi/h to ft/s.