Math 644

Chapter 2

Section 2.3: Power Series

Contents

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A First Example

More complicated functions are found by taking limits of polynomials.

Example 1. Study the series $\sum_{n=0}^{\infty} z^n$ for a fixed $z \in \mathbb{C}$.

Fix $z \in \mathbb{C}$.

1) If $|z| \geq 1$, then $z^n \not\to 0$ and therefore $\sum_{n=0}^{\infty} z^n$ diverges.

2) If $|z| < 1$. We have

$$S_n = \sum_{k=0}^{n} z^k = \frac{1 - z^{n+1}}{1-z}$$

$$\Rightarrow \quad \left| S_n - \frac{1}{1-z} \right| = \frac{|z|^{n+1}}{|1-z|} \longrightarrow 0 \quad (n \to \infty)$$

So,

$$\sum z^n = \frac{1}{1-z} \qquad (|z| < 1)$$

Power series $(|z|<1)$

function defined on $\mathbb{C} \setminus \{1\}$.

Definition

A formal power series

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$$

is called a **convergent power series centered (or based) at z_0** if

$\exists\, r>0$ s.t. $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ conv. in $\{z : |z-z_0|<r\}$.

Convention: $f(z_0) = a_0$ (to avoid 0^0).

Example 2. Find a convergent power series centered at $z_0 \neq a$ representing $\frac{1}{z-a}$.

Write

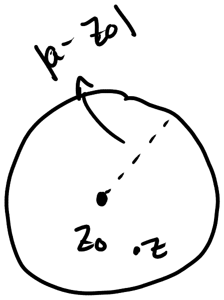
$$\frac{1}{z-a} = \frac{1}{z-z_0+z_0-a} = \frac{1}{-(a-z_0)} \cdot \frac{1}{1-\frac{z-z_0}{a-z_0}} = f(z)$$

If $\left|\frac{z-z_0}{a-z_0}\right| < 1$, $\longrightarrow$ $|z-z_0| < |a-z_0|$

$$\frac{1}{1-\frac{z-z_0}{a-z_0}} = \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(a-z_0)^n}$$

$$\Rightarrow \quad f(z) = \sum_{n=0}^{\infty} \frac{-(z-z_0)^n}{(a-z_0)^{n+1}} \qquad (*)$$

Note



- $(*)$ conv. in $|z-z_0| < |a-z_0|$

  Biggest disk in domain of $\frac{1}{z-a}$

- $(*)$ div. in $|z-z_0| \geq |a-z_0|$.

Theorem 3. Let $r > 0$ and suppose that $\rightarrow$ $n \geq n_0$

a) $|a_n(z - z_0)^n| \leq M_n$ for every z such that $|z - z_0| \leq r$;

b) $\sum_{n=0}^{\infty} M_n < \infty$.

Then $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges uniformly and absolutely in the region

$$D_r := \{z : |z - z_0| \leq r\}.$$

Proof.

Abs conv. Let $z \in D_r$ & $A_N = \sum_{k=0}^{N} |a_n| \, |z - z_0|^n$.

Here,

$$A_N \leq \sum_{k=0}^{N} M_k \leq \sum_{k=0}^{\infty} M_k < \infty$$

$$\Rightarrow \quad (A_N)_{N=0}^{\infty} \quad \text{converges.}$$

Unif conv. Let $S_N = \sum_{k=0}^{N} a_n (z - z_0)^k$

We will show that (S_N) is unif. Cauchy.

For $z \in D_r$, & $N > M$,

$$|S_N - S_M| = \left| \sum_{k=M+1}^{N} a_k (z - z_0)^k \right| \leq \sum_{k=M+1}^{N} M_k.$$

$\exists K = K(\varepsilon)$ s.t. if $N > M \geq K$, $\sum_{j=M+1}^{N} M_j < \varepsilon$. So, (S_N) is uniformly Cauchy in D_r. $\square$

Note:

Convergence depends only on the tail of the series. So it is sufficient to satisfy the hypothesis only for $n \geq n_0$, for some non-negative integer n_0.

Root test.

Theorem 4. Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a formal power series. Let

$$R := \liminf_{n\to\infty} |a_n|^{-1/n} = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}} \in [0,\infty].$$

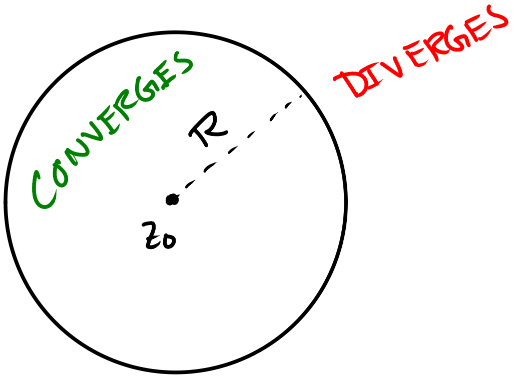
Then, the power series

a) converges abs. in $\{z : |z-z_0| < R\}$;

b) converges uniformly in $\{z : |z-z_0| \le r\}$, for any $r < R$;

c) diverges in $\{z : |z-z_0| > R\}$.

Notes:



- R is called the radius of convergence of the power series.
- Biggest open disk where the power series converges is $\{z : |z-z_0| < R\}$.
- Information on the decay rate of a_n: for any $S < R$, there is an $n_0 \ge 0$ such that $|a_n| \le S^{-n}$.

Proof. Suppose $R > 0$.

Let $|z-z_0| \le r < \rho < R$. By def.:

$$\liminf_{n\to\infty} |a_n|^{-1/n} = \lim_{n\to\infty} \inf_{k\ge n} |a_k|^{-1/k} = R$$

$\exists N \in \mathbb{N}$ s.t.

$$\rho < \inf_{k\ge N} |a_k|^{-1/k} \le R$$

So, $\rho < |a_k|^{-1/k} \quad \forall k \ge N$.

$\Rightarrow |a_k| < \rho^{-k} \quad \forall k \ge N$

So, $\forall k \ge N$

$$|a_k|\,|z-z_0|^k \le \left(\frac{r}{\rho}\right)^k.$$

Let $M_k = \left(\frac{r}{\rho}\right)^k$ & since $r < \rho$,

$$\sum_{k=N}^{\infty} M_k \quad \text{converges.}$$

So, by the M-test, power series converges uniformly & absolutely in $\{z : |z-z_0| \leq r\}$.

This proves (a) & (b).

Now, let $|z-z_0| > R$. Let ρ s.t.

$$R < \rho < |z-z_0|.$$

From def. of liminf, there are $(n_k)_{k=1}^{\infty}$ $n_k < n_{k+1}$ s.t.

$$|a_{n_k}|^{-1/n_k} < \rho.$$

So,

$$|a_{n_k}(z-z_0)^{n_k}| > \frac{|z-z_0|^{n_k}}{\rho^{n_k}} \longrightarrow \infty \quad (k \to \infty)$$

So, $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ diverges. $\square$

Note: Root test does not give any information on the convergence of the power series on the circle

$$\{z : |z-z_0| = R\}.$$

Example 5. Find the Radius of convergence R of the following power series and study their behavior on the boundary of the disk of radius R.

A) $\sum_{n=1}^{\infty} \frac{z^n}{n}$;

B) $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$;

C) $\sum_{n=1}^{\infty} n z^n$;

D) $\sum_{n=1}^{\infty} 2^{n^2} z^n$.

A) $R = \liminf \frac{1}{|1/n|^{-1/n}} = \lim_{n\to\infty} \frac{1}{n^{1/n}} = 1$.

$z=1 \to$ power series diverges.

$z=-1 \to$ power series converges.

B) $R = \liminf \frac{1}{|1/n^2|^{-1/n}} = \left(\lim_{n\to\infty} \frac{1}{n^{1/n}}\right)^2 = 1$.

If $|z|=1 \Rightarrow \sum \frac{|z|^n}{n^2} = \sum \frac{1}{n^2} \to$ conv. on $\mathbb{T}$.

C) $R = \liminf \frac{1}{n^{-1/n}} = \lim n^{1/n} = 1$

If $|z|=1 \Rightarrow n|z|^n = n \longrightarrow \infty$

$\Rightarrow$ power series diverges on all of $\mathbb{T}$

D) $R = \liminf \frac{1}{(2^{n^2})^{-1/n}} = \liminf 2^{-n} = 0$

$\sum_{n=0}^{\infty} 2^{n^2} z^n$ is not a convergent power series.

EXAMPLE 6. Let $(a_n)_{n=0}^{\infty}$ be defined by

$$a_n = \begin{cases} 3^{-n} & \text{, if } n \text{ is even} \\ 4^{n} & \text{, if } n \text{ is odd.} \end{cases}$$

What is the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$.

$$\liminf_{n\to\infty} |a_n|^{-1/n} = \inf \left[\operatorname{acc} \left(|a_n|^{-1/n} \right)_{n=0}^{\infty} \right]$$

Here $\operatorname{acc}\left(|a_n|^{-1/n}\right) = \left\{ \frac{1}{4}, 3 \right\}$

$$\Rightarrow \liminf_{n\to\infty} |a_n|^{-1/n} = \frac{1}{4} = R.$$

Example where $\lim \frac{a_{n+1}}{a_n}$ doesn't exist and therefore can't give any information on converge of the power series.

(See problem 13, D'Alembert ratio test).