Math 241 Midterm 2, Fall 2020

Name:

Question	Points	Score
1	3	
2	6	
3	7	
4	10	
5	7	
6	7	
7	7	
8	3	
Total:	50	

- The exam is 1 hour long, plus an extra 15 minutes at each end to download the paper, and upload your answers.
- You may not use any electronic devices (other than a tablet to write your answers, if you want) on the test.
- You may use the textbook, and your own personal notes, but no other notes.
- All work must be entirely your own. You cannot discuss the test with anyone else in any way.
- You must show all your work and make clear what your final solution is (for example, by drawing a box around it).
- You will get almost no credit for solutions that are not fully justified.
- You can write your answers on blank paper or on a tablet without printing out the test. If you do this, please make very clear which answer goes with which question, and write your name, and the page number, on each page. You can also print out the test and write on that if you want.
- Good luck!

1. The population of koholā around Hawai'i is graphed, with time on the x axis and population on the y axis. Assume the first and second derivatives of the corresponding function exist.

Answer the first part each of (a), (b), and (c) below with one of the choices (i)-(vii). For each choice, say what property of the first and / or second derivative this corresponds to.

(i) critical

(ii) stable

(iii) going up, and the rate of increase is getting faster

(iv) going up, but the rate of increase is getting slower

(v) going down, and the rate of decrease is getting faster

(vi) going down, but the rate of decrease is getting slower

(vii) none of the above

(a) (1 point) When the graph has a critical point, the population is ... (ii)

This corresponds to the first derivative ... 0

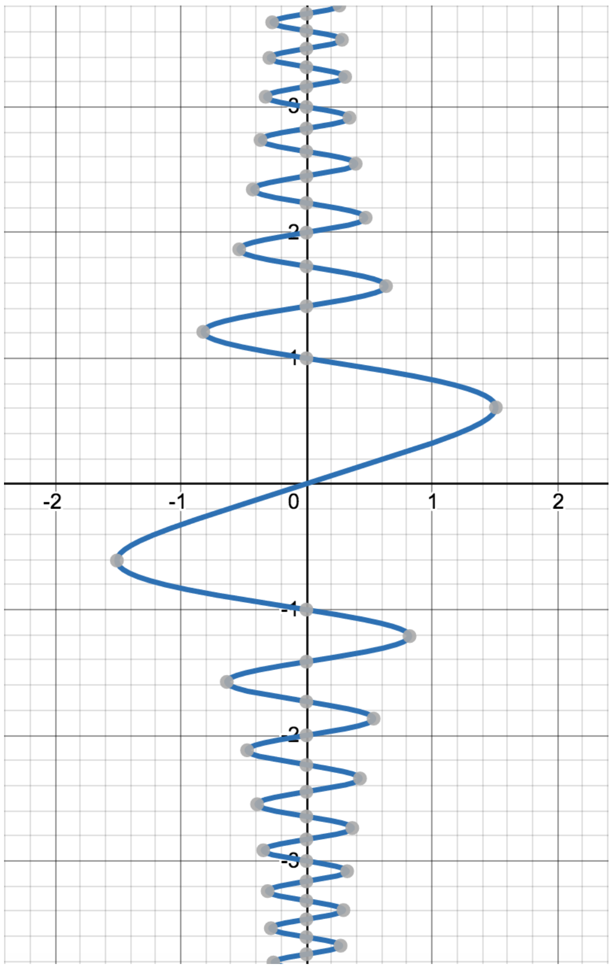
(b) (1 point) When the graph is increasing and concave up, the population is (iii)

This corresponds to the first derivative being ... + and the second derivative being ... +

(c) (1 point) When the graph is decreasing and concave up, the population is (vi)

This corresponds to the first derivative being ... − and the second derivative being ... +

2. (6 points) Compute the tangent line to the curve described by $\sin(\pi y^2) = xy$ at the point $(0, 1)$. Write your answer in the form $y = mx + b$.



① Implicit differentiation.

$$\frac{d}{dx}\left[\sin(\pi y^2)\right] = \frac{d}{dx}(xy)$$

$$\Rightarrow \cos(\pi y^2)\, 2\pi y \frac{dy}{dx} = y + x\frac{dy}{dx}$$

$$\Rightarrow \left(\cos(\pi y^2)\, 2\pi y - x\right)\frac{dy}{dx} = y$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{2\pi y \cos(\pi y^2) - x}$$

② Find the slope.

$$m = \left.\frac{dy}{dx}\right|_{(x,y)=(0,1)} = \frac{1}{-2\pi}$$

$$\text{so } m = \frac{-1}{2\pi}$$

③ Tangent line

$y - y_1 = m(x - x_1)$ $-(x_1, y_1)$ point on the line.

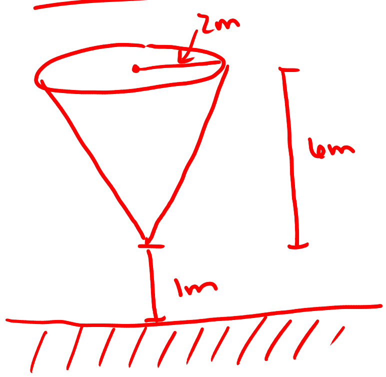
$(x_1, y_1) = (0, 1)$, so

$$y - 1 = \frac{-1}{2\pi}x \Rightarrow \boxed{y = 1 - \frac{x}{2\pi}}$$

3. (7 points) Water is held in an opaque conical tank. The diameter at the top equals 4 meters, the height is 6 meters, and the tank is suspended so the bottom is 1 meter above the ground. You measure that water is coming out of the tank at 0.5 cubic meters per minute, but cannot see the height of the water. How fast is the height of the water changing when the water goes halfway up the height of the cone?

The volume of a cone of radius r and height h is $V = \frac{\pi}{3}r^2h$. Simplify your answer as much as you can (without using a calculator!). Include units in your answer, and make clear whether the rate of change is positive or negative.

① Sketch.



$r = 2m$
$h = 6m$

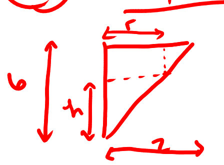
$V = \frac{\pi}{3} r^2 h$

② Goal

$\frac{dV}{dt} = 0.5 \ m^3/min.$

Find $\left.\frac{dh}{dt}\right|_{h=3}$.

③ Expression of r in terms of h.



rules of triangles: $\frac{2}{r} = \frac{6}{h} \Rightarrow \frac{h}{3} = r.$

So, $V = \frac{\pi}{3} \frac{h^3}{9} = \frac{\pi}{27} h^3$

④ Rate of Change.

$$\frac{dV}{dt} = \frac{\pi}{27} 3h^2 \frac{dh}{dt} = \frac{\pi h^2}{9} \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{9}{\pi h^2} \frac{dV}{dt}$$

So, $$\frac{dh}{dt} = \frac{9}{\pi \cdot 9} \cdot 0.5 = \boxed{\frac{1}{2\pi} \ m/min}$$

4. (a) (6 points) Find the absolute minimum and maximum of the function below on the interval $[-1, 4]$ (or say if they don't exist). Make sure you justify your answer, and simplify as much as possible.

$$f(x) = x^{1/3}(x-4)$$

① Critical numbers.

$$f'(x) = \frac{1}{3}x^{-2/3}(x-4) + x^{1/3} = \frac{x-4+3x}{3x^{2/3}} = \frac{4(x-1)}{3x^{2/3}}.$$

- $f'(x) \nexists$ at $x=0$.
- $f'(x) = 0$ at $x = 1$.

② Find max & min.

$$M = \max\{f(-1), f(0), f(1), f(4)\} = \max\{5, 0, -3, 0\} = \boxed{5}$$

$$m = \min\{f(-1), f(0), f(1), f(4)\} = \boxed{-3}$$

(b) (4 points) A function $g(x)$ has *second derivative* $g''(x) = x^{1/3}(x-4)$. Find the interval (or intervals, possibly none) where the original function is concave up, and where it is concave down.

the zeros of $f''(x)$ ← We have change in concavity at the points $x=0$ & $x=4$

① $x < 0 \Rightarrow g''(x) > 0 \Rightarrow g$ concave up.

② $0 < x < 4 \Rightarrow g''(x) < 0 \Rightarrow g$ concave down.

③ $4 < x \Rightarrow g''(x) > 0 \Rightarrow g$ concave up.

5. (a) (4 points) You know that the derivative of a function satisfies the following values.

x	0	5	10	15
$g'(x)$	3	5	8	12

$L(x) = g'(a)(x-a) + g(a)$
here $a=5$.

If you know $g(5) = 24$, use a linearization to estimate $g(6)$.

$$g(6) = g(5+1) \approx g'(5)(1) + g(5)$$
$$= 5 + 24$$
$$= 30$$

so, $g(6) \approx 30$.

(b) (3 points) Compute the horizontal asymptote (or asymptotes, possibly none) of the function defined by

$$f(x) = \frac{4x^2 - 3}{2 - x - x^2}.$$

① At ∞.

$$\lim_{x\to\infty} \frac{4x^2-3}{2-x-x^2} = \lim_{x\to\infty} \frac{4 - 3/x^2}{2/x^2 - 1/x - 1} = \boxed{-4}$$

② At $-\infty$.

$$\lim_{x\to-\infty} \frac{4x^2-3}{2-x-x^2} = \lim_{x\to-\infty} \frac{4 - 3/x^2}{2/x^2 - 1/x - 1} = \boxed{-4}$$

So $y=-4$ is an HA at $\pm\infty$.

6. (7 points) Either using the axes below, or copying these axes[1], draw the graph $y = f(x)$ of a function f that satisfies the following conditions:

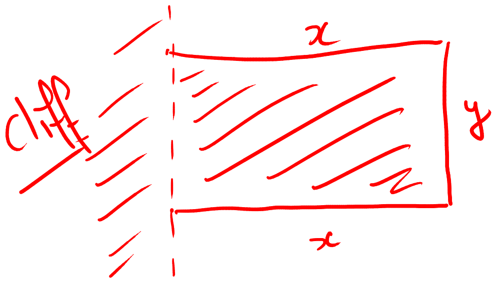
- f has zeros at $x = 0$ and $x = 2$.
- f is defined and continuous everywhere except $x = 3$.
- $f'(x) > 0$ on $(-\infty, 1)$, and $f'(x) < 0$ on $(1, 3)$ and $(3, \infty)$.
- $\lim_{x\to\infty} f(x) = 2$ and $\lim_{x\to-\infty} f(x) = -2$.
- $\lim_{x\to 3^-} f(x) = -\infty$ and $\lim_{x\to 3^+} f(x) = \infty$.
- $f''(x) > 0$ on $(-\infty, 0)$ and $(3, \infty)$, and $f''(x) < 0$ on $(0, 3)$.

Your graph must be careful and accurate, and must unambiguously satisfy the conditions above. Include any asymptotes explicitly in the picture using dashed lines.

[1]If you copy the axes, make sure you label them with numbers as in the picture below.

7. (7 points) You want to enclose off a rectangular or square field using 60 meters of fencing. One side of your field is formed from a (long, straight) cliff, so you only need fencing for three sides of the field. Find the greatest area of field that you can make. Justify your answer.

① Sketch.



x: width square field (meters)

y: length square field (meters)

A: area of the field (m^2).

Goal: max. A.

② Equations.

Relation between x & y.

$2x+y=60$

$\Rightarrow y = 60-2x$

$A = xy$

$\Rightarrow A = x(60-2x)$

$\Rightarrow A = 60x - 2x^2, \quad x \geq 0.$

④ Optimize.

$A'(x) = 60-4x \Rightarrow A'(x)=0 \leftrightarrow x = 15$

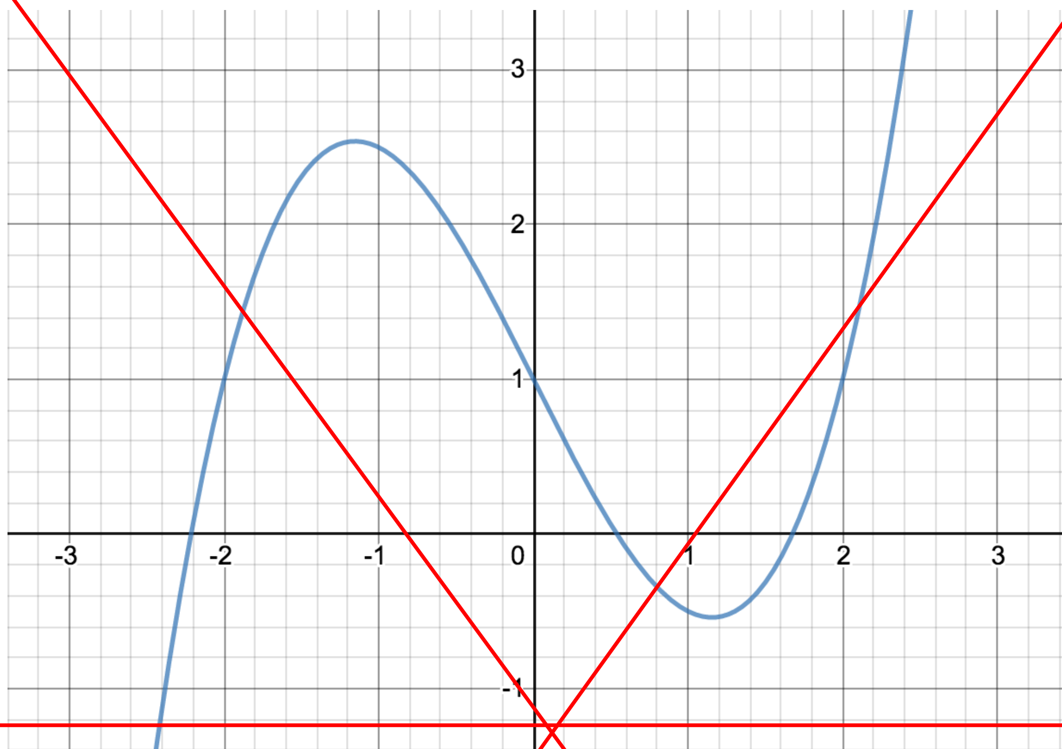
$x<15 \to 4x<60 \to 60-4x>0 \to A'(x)>0$

$x>15 \to 4x>60 \to 60-4x<0 \to A'(x)<0$

$\hookrightarrow$ 1st test $\Rightarrow x = 15$ abs. max.

Answer.: $A(15) = 15 \cdot 30 = \boxed{450 m^2}$

8. You use Newton's method to approximate the zeros of the function pictured.



(a) (1 point) You start with $x_1 = -2$ to approximate the zero near -2. Is x_2 greater than, less than, or exactly equal to the zero?

(b) (1 point) You start with $x_1 = 2$ to approximate the zero near 2. Is x_2 greater than, less than, or exactly equal to the zero?

(c) (1 point) Will you get a better approximation x_2 to the zero near 0.5 by starting with $x_1 = 0$, or with $x_1 = 1$?

Bonus question (5 points):

Batman attempts to pursue the Joker on a moped (the batmobile is in the shop ...). Newton's second law of motion tells him that if a force of F Newtons is applied to his $100kg$ moped, then the acceleration a in meters / second^2 satisfies

$$F = 100a.$$

The moped's unreliable engine provides between 500 and 1200 Newtons of force over a period of 10 seconds. Assuming the moped starts from rest, what is the maximum velocity the moped can attain at the end of 10 seconds, and what is the minimum velocity? Justify your answer using a theorem from class, and explicitly state which theorem.

We know that if $v(t)$ is the velocity, then

$$v'(t) = a(t).$$

By the Mean Value Theorem (MVT), there is a $c \in (0,t)$ such that

$$\frac{v(t) - v(0)}{t - 0} = v'(c) = a(c) = \frac{F(c)}{100}.$$

So,

$$\frac{v(t)}{t} = \frac{F(c)}{100} \qquad 0 < c < t.$$

But, $500 \le F(c) \le 1200$

$$\Rightarrow \quad \frac{v(t)}{t} \ge \frac{500}{100} \quad \& \quad \frac{v(t)}{t} \le \frac{1200}{100}$$

$$\Rightarrow \quad v(t) \ge 5t \quad \& \quad v(t) \le 12t$$

So, for $t = 10$

$$\Rightarrow \quad 50 \le v(10) \le 120$$

So,

minimum speed: 50 meters/s (pretty fast!)

maximum speed: 120 meters/s (really fast!)