Chapter 5: Applications of Integration
Week 13

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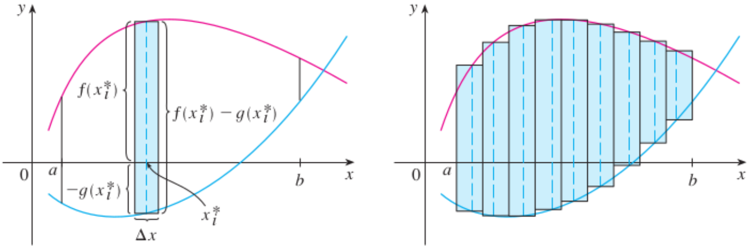
Upcoming this week

1. 5.1 Areas between curves
2. 5.2 Volumes

As we saw last week, the integral is a good tool to compute the area under a curve.

We can also use it to compute the area between two curves $y = f(x)$ and $y = g(x)$ with $f(x) \geq g(x)$.



(a) Typical rectangle (b) Approximating rectangles

Figure: Area between two curves

Definition 1

The area A of a region bounded by the curves $y = f(x)$, $y = g(x)$, $x = a$ and $x = b$, where $f(x) \geq g(x)$ for any $x \in [a, b]$, is

$$A = \int_a^b [f(x) - g(x)]\, dx.$$

Example 2

Compute the region bounded from above by the curve $f(x) = x^2 + 1$, bounded from below by the curve $g(x) = x$, and bounded on the sides by $x = 0$ and $x = 1$.

Example 3

Find the area of the region enclosed by the parabola $y = x^2$ and $y = 2x - x^2$.

Example 4

Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

For general functions (not necessarily $f(x) \geq g(x)$), we compute the total area between two curves as followed.

Definition 5

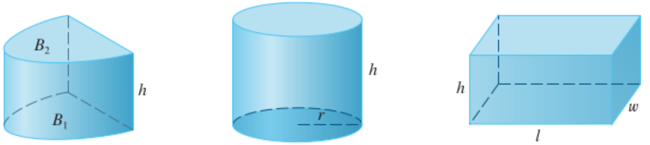
The total area A between the curves $y = f(x)$ and $y = g(x)$ and between $x = a$ and $x = b$ is

$$A = \int_a^b |f(x) - g(x)|\, dx.$$

Example 6

Find the area of the region bounded by the curve $y = \sin x$ and $y = \cos x$ from $x = 0$ and $x = \pi/4$.

Exercises: 5-12, 13-28, 34.



(a) Cylinder $V = Ah$ (b) Circular cylinder $V = \pi r^2 h$ (c) Rectangular box $V = lwh$

Figure: Volume of usual objects

a) If the area of the shapes B_1 and B_2 is A, then the volume is $V = Ah$.
b) The area of the base is πr^2 so the volume is $\pi r^2 h$.
c) The area of the base is wl so the volume is $V = wlh$.

We then see that, usually, the technic to compute the volume of an object is

- Take a horizontal slice parallel to the base.
- Sums up all these slices h times.

We use this technic for arbitrary objects. Take the following object S (think of it as a bread).

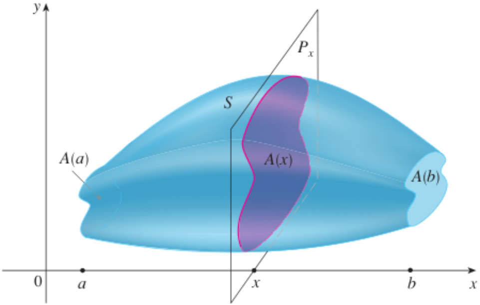


Figure: Slicing an arbitrary object

- Cut the object with a plane P_x at some point x where the plane is perpendicular to the x-axis.
- Call the region of the object $A(x)$.

Now, we cut the bread (object S) in many slices, say n slices.

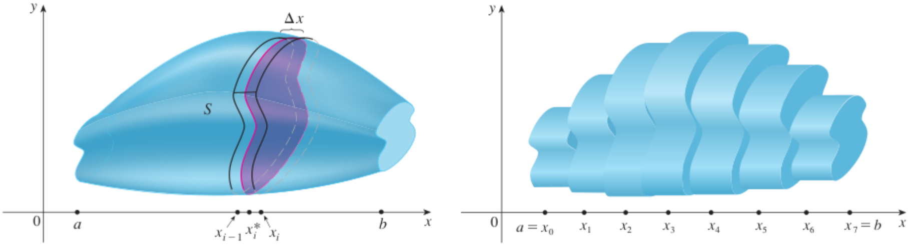


Figure: Slicing in many parts

Now, we will have $V(S) \approx \sum_{i=1}^{n} A(x_i^*)\Delta x$. Take the limit as n goes to ∞.

Definition 7

The volume of an object S is defined as

$$V(S) = \int_a^b A(x)\,dx.$$

Example 8

Show that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.

Example 9

Find the volume of a cylinder of radius r and height h.

We obtain the same answer as we are used to! This is good, it shows that our definition makes sense.

We can rotate functions to obtain 3D objects.

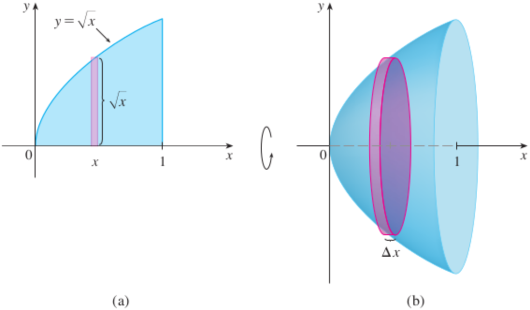


Figure: Rotating the function $f(x) = \sqrt{x}$.

Example 10

Find the volume of the object obtained by rotating the function $f(x) = \sqrt{x}$ $(0 \le x \le 1)$ around the x-axis.

We can also rotate a region about the y-axis.

Example 11

Find the volume of the object obtained by rotating the region enclosed by the curves $y = x^3$, $y = 8$, and $x = 0$ about the y-axis.

We may also rotate about a different line.

Example 12

Find the volume of the object obtained by rotating the region enclosed by the curves $y = x$ and $y = x^2$ about the line $y = 2$.

The previous solid are called <u>solids of revolution</u>. The volume of these solids are either obtained by the formula

$$V = \int_a^b A(x)\,dx \quad \text{or} \quad V = \int_c^d A(y)\,dy.$$

Here is some tips to obtained the formula for A.

- If the cross-section is a disk (as in Examples 10 and 11), we find the radius of the disk (in terms of x or y) and use

$$A = \pi(radius)^2.$$

- If the cross-section is a washer (as in Example 12), we find the inner radius r_{in} and the outer radius r_{out}. We find the area of the washer (in terms of x or y) and we use

$$A = \pi(r_{out})^2 - \pi(r_{in})^2.$$

More examples: Check out examples 7, 8, and 9 in the textbook!
Exercises: 1-18, 19, 29, 31, 39-42, 48, 49, 51, 56, 63,