Section 2.5 M444

Problems Set Pierre Parisé

Problem 3

Let $z = x+iy$. then

$$e^{z^2} = e^{x^2-y^2+2xyi} = \underbrace{e^{x^2-y^2}\cos 2xy}_{u} + i\underbrace{e^{x^2-y^2}\sin 2xy}_{v}$$

we have

$$u_x = 2xe^{x^2-y^2}\cos 2xy - 2ye^{x^2-y^2}\sin 2xy$$

$$v_y = -2ye^{x^2-y^2}\sin 2xy + 2xe^{x^2-y^2}\cos 2xy$$

$$\Rightarrow \quad u_x = v_y$$

and

$$u_y = -2ye^{x^2-y^2}\cos 2xy - 2xe^{x^2-y^2}\sin 2xy$$

$$v_x = 2xe^{x^2-y^2}\sin 2xy + 2ye^{x^2-y^2}\cos 2xy$$

$$\Rightarrow \quad u_y = -v_x.$$

Hence, the C-R equation are satisfied and e^{z^2} is analytic everywhere.

Problem 5

Let $z = x+iy \Rightarrow \bar{z} = x - iy$

$$\Rightarrow e^{\bar{z}} = \underbrace{e^x \cos y}_{u} - i\underbrace{e^x \sin y}_{v} .$$

(with the "–" sign)

We have

$$u_x = e^x \cos y \qquad v_y = -e^x \cos y$$

Also,

$$u_y = -e^x \sin y \qquad v_x = -e^x \sin y .$$

The Cauchy-Riemann equations are not satisfied. If they were, then

$$\begin{matrix} u_x = v_y \\ u_y = -v_x \end{matrix} \Rightarrow \begin{matrix} 2e^x \cos y = 0 \\ 2e^x \sin y = 0 \end{matrix} \Rightarrow \begin{matrix} \cos y = 0 \\ \sin y = 0 \end{matrix}$$

Hence, $y = k\pi$ and $y = (2m+1)\frac{\pi}{2}$ for some integers $k, m \in \mathbb{Z}$. But this is impossible. Thus, the C-R equations are not satisfied and $e^{\bar{z}}$ is not analytic.

Problem 6 Assumption: $x \neq 0$ or $y \neq 0$.

We have

$$u = \frac{y}{x^2+y^2}, \quad v = \frac{-x}{x^2+y^2}.$$

Then

$$u_x = \frac{-2xy}{(x^2+y^2)^2}, \quad v_y = \frac{2xy}{(x^2+y^2)^2}$$

Also

$$u_y = \frac{(x^2+y^2) - 2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

and

$$v_x = \frac{-(x^2+y^2) + 2x^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

If the CR equations are satisfied at some x, y, then

$$u_x = v_y \Rightarrow \frac{4xy}{(x^2+y^2)^2} = 0 \Rightarrow x = 0 \text{ or } y = 0. \quad (*)$$

The second set of equations give

$$u_y = -v_x \Rightarrow \frac{2(x^2-y^2)}{(x^2+y^2)^2} = 0$$

Hence, $x^2 = y^2$. (**)

From (*), there are two cases:

1. $x=0$. Then from (**), $y=0$.
   Hence $x=0$ and $y=0$, contradicting the assumption that $x \neq 0$ or $y \neq 0$.

2. $y=0$. Then from (**), $x=0$.
   Hence $x=0$ and $y=0$, contradicting the assumption that $x \neq 0$ or $y \neq 0$.

Hence the CR are not satisfied at any point of $\mathbb{C}$.

Therefore, the function is not analytic.

Problem 12

$$\cosh(z) = \frac{e^{z} + e^{-z}}{2}.$$

Let $z = x + iy$. Hence

$$\cosh(z) = \frac{e^{x} e^{iy} + e^{-x} e^{-iy}}{2}$$

$$= \frac{e^{x} \cos y + i e^{x} \sin y + e^{-x} \cos y - i e^{-x} \sin y}{2}$$

$$= \underbrace{\cosh(x) \cos(y)}_{u} + i \underbrace{\sinh(x) \sin(y)}_{v}$$

we have

$$u_x = \sinh(x) \cos(y) \quad , \quad v_y = \sinh(x) \cos(y)$$

$$\Rightarrow \quad u_x = v_y$$

Also, $u_y = -\cosh(x) \sin(y)$, $v_x = \cosh(x) \sin(y)$

$$\Rightarrow \quad u_y = -v_x.$$

Hence the function is analytic on all $\mathbb{C}$.

Problem 33

Assume that $f = u + iv$ is analytic on a region Ω.

(i) Assume further that $u \equiv c$ on Ω.

Then $u_x = 0$ and $u_y = 0$. Hence, we get $v_x = -u_y = 0$ and $v_y = u_x = 0$.

Since Ω is a region, there is a polygonal path from $z_0 \in \Omega$ to some arbitrary $z \in \Omega$. By the Fundamental Theorem for line integral in $\mathbb{R}^2$ (see Calculus), we have

$$\int_{z_0}^{z} \vec{\nabla} v \cdot d\vec{r} = v(z) - v(z_0).$$

But $\vec{\nabla} v = \vec{0}$ on Ω

$$\Rightarrow \quad 0 = v(z) - v(z_0) \quad \forall z \in \Omega$$

$$\Rightarrow \quad v(z) = v(z_0) \quad \forall z \in \Omega.$$

Hence v is constant.

Conclusion: $f = u + iv = c + i v(z_0)$ is constant on Ω.

② Case $\operatorname{Im} f = c$ is handled in a similar way. $\square$

Problem 34

Assume $f = u + iv$ is analytic in a region Ω.

(a) We know that $f' = u_x + i v_x$.

From the Cauchy-Riemann equations, we have $u_x = v_y$ and $u_y = -v_x$.

Hence,

$$u_y = -v_x \Rightarrow f' = u_x - i u_y.$$

and

$$u_x = v_y \Rightarrow f' = v_y + i v_x.$$

(b) using the formula from (a), we get

$$|f'|^2 = (u_x)^2 + (-u_y)^2 = u_x^2 + u_y^2$$

and

$$|f'|^2 = (v_y)^2 + (v_x)^2 = v_x^2 + v_y^2 .$$

$\square$