Section 2.3 M444

Problems Solutions Pierre Parise

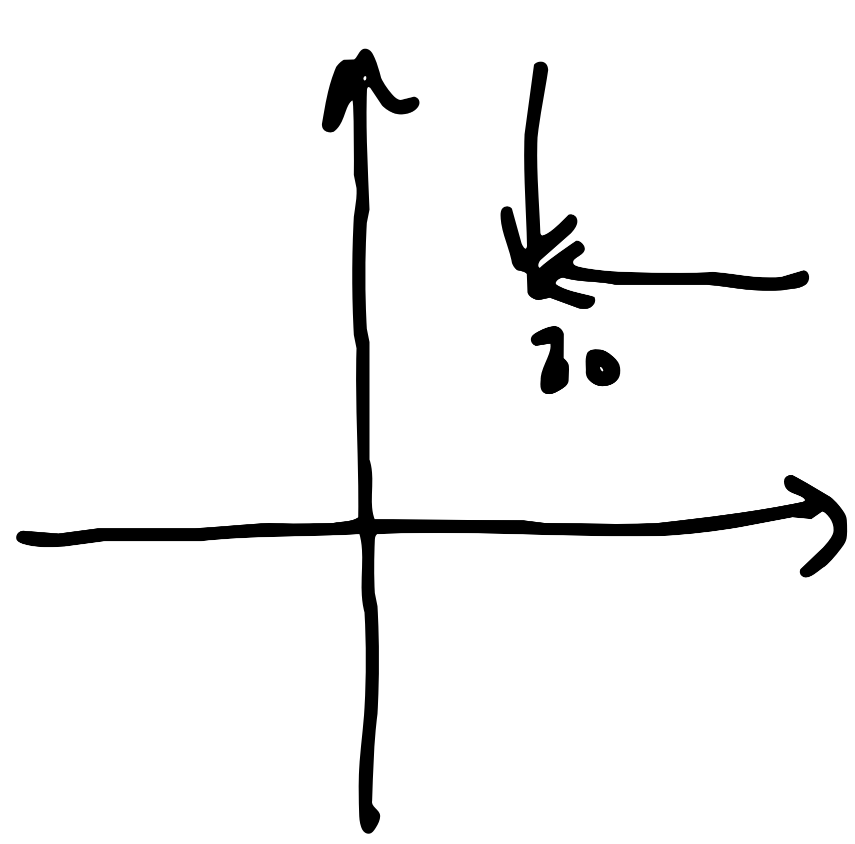
Problem 1

The function is a polynomial. So it is differentiable everywhere.

Problem 3

The function $f(z) = \operatorname{Im} z$ is not analytic anywhere.

Let $z_0 \in \mathbb{C}$ be fixed. Then, we will approach z_0 from the top and from the left as illustrated below:



<u>top</u> $\displaystyle \lim_{z \to z_0} \frac{\operatorname{Im} z - \operatorname{Im} z_0}{z - z_0}$

$$= \lim_{y \to y_0^+} \frac{y - y_0}{x_0 + iy - x_0 - iy_0}$$

$$= \lim_{y \to y_0^+} \frac{y - y_0}{i(y - y_0)} = \frac{1}{i}$$

Right $\displaystyle\lim_{z\to z_0} \frac{\operatorname{Im} z - \operatorname{Im} z_0}{z-z_0} = \lim_{x\to x_0^+} \frac{y_0 - y_0}{x+iy_0-x_0-iy_0}$

$\displaystyle = \lim_{x\to x_0^+} \frac{0}{x-x_0}$

$= 0.$

Hence the limit does not exist and $f'(z_0)$ does not exist $\forall z_0 \in \mathbb{C}$.

Problem 10

We know that the principal branch of the square root $\sqrt{z}$ is analytic on $\mathbb{C} \setminus (-\infty, 0]$, by example 2.3.13.

By Theorem 2.3.12 (the composition), we know that $\sqrt{z-1}$ will be analytic on $\mathbb{C} \setminus \{z : z-1 \in (-\infty, 0]\}$

$= \mathbb{C} \setminus \{z : z \in (-\infty, 1]\}$

$= \mathbb{C} \setminus (-\infty, 1]$.

Problem 13

Let $f(z) = z^{100}$. Then

$$\lim_{z\to 1} \frac{z^{100}-1}{z-1} = \lim_{z\to 1} \frac{f(z)-f(1)}{z-1} = f'(1).$$

We have $f'(z) = 100 z^{99}$.

$$\Rightarrow \lim_{z\to 1} \frac{z^{100}-1}{z-1} = 100.$$

Problem 15

We have, for $z \neq 0$,

$$\frac{1}{z\sqrt{1+z}} - \frac{1}{z} = \frac{1}{z}\left(\frac{1}{\sqrt{1+z}} - 1\right)$$

$$= \frac{\frac{1}{\sqrt{1+z}} - 1}{z}.$$

Let $f(z) = \frac{1}{\sqrt{1+z}}$. Then, $f(0) = \sqrt{1} = 1$

because $\sqrt{\ }$ is the principal branch. Hence

$$\lim_{z\to 0}\left(\frac{1}{z\sqrt{1+z}} - \frac{1}{z}\right) = \lim_{z\to 0} \frac{f(z)-f(0)}{z} = f'(0)$$

we have, by the chain rule

$$f'(z) = g'(h(z)) \cdot h'(z)$$

where $g(z) = \frac{1}{\sqrt{z}}$ and $h(z) = 1+z$.

First:

$$g'(z) = \frac{(1)'\sqrt{z} - (1)(\sqrt{z})'}{(\sqrt{z})^2}$$

$$= -\frac{\frac{1}{2} z^{(1-2)/2}}{z} = -\frac{1}{2} \frac{z^{-1/2}}{z}$$

$$= -\frac{1}{2} \frac{e^{-\frac{1}{2}\operatorname{Log} z}}{e^{\operatorname{Log} z}}$$

$$= -\frac{1}{2} \frac{1}{e^{\operatorname{Log} z + \frac{1}{2}\operatorname{Log} z}}$$

$$= -\frac{1}{2} \frac{1}{e^{\frac{3}{2}\operatorname{Log} z}} = -\frac{1}{2 z^{3/2}}$$

$$\Rightarrow g'(h(z)) = \frac{-1}{2(1+z)^{3/2}}.$$

Since $h'(z) = 1$

$$\Rightarrow \quad f'(z) = \frac{-1}{2(1+z)^{3/2}} \qquad \text{(principal branch)}.$$

So,

$$\lim_{z \to 0} \left(\frac{1}{z\sqrt{1+z}} - \frac{1}{z} \right) = f'(0) = \boxed{-\frac{1}{2}}.$$