Chapter 2

Derivatives

2.3 Differentiation Formulas

Constant Function.

$f(x) = c$, $f'(x) = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} \frac{0}{h}$

$= \lim_{h \to 0} 0 = 0$

Derivative of a Constant Function

$$\frac{d}{dx}(c) = 0$$

Power Functions.

n = 1. $f(x) = x$

$$\lim_{h \to 0} \frac{x + h - x}{h} = \lim_{h \to 0} \frac{h}{h} = 1$$

$$\boxed{f'(x) = 1}$$

n = 2. $f(x) = x^2$

$$\boxed{f'(x) = 2x}$$

n = 3. $f(x) = x^3$

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} = \ldots = 3x^2$$

$$\boxed{f'(x) = 3x^2}$$

The Power Rule If n is a positive integer, then

$$\frac{d}{dx}(x^n) = \underline{nx}^{n-1}$$

Multiplication by a constant.

The Constant Multiple Rule If c is a constant and f is a differentiable function, then

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$$

$$\frac{d}{dx}(6x) = 6\frac{d}{dx}(x)$$

Sum.

The Sum Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

Difference.

The Difference Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

EXAMPLE 4 Find the points on the curve $y = x^4 - 6x^2 + 4$ where the tangent line is horizontal.

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^4 - 6x^2 + 4\right) = \frac{d}{dx}(x^4) - \frac{d}{dx}(6x^2) + \frac{d}{dx}(4)$$

$$= 4x^{4-1} - 6\frac{d}{dx}(x^2) + 0$$

$$= 4x^3 - 6\cdot 2x$$

$$= 4x^3 - 12x$$

We want $\frac{dy}{dx} = 0$

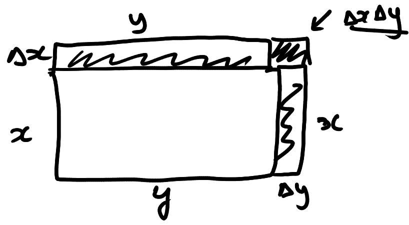
$$\Leftrightarrow \quad 4x^3 - 12x = 0$$

$$\Leftrightarrow \quad 4x(x^2 - 3) = 0$$

$$\Leftrightarrow \quad 4x(x - \sqrt{3})(x + \sqrt{3}) = 0$$

$$\Leftrightarrow \quad \boxed{x = 0, \quad x = \sqrt{3} \ \& \ x = -\sqrt{3}.}$$

Product.



The Product Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

Caution!!!

$$\frac{d}{dx}(fg) \neq \frac{d}{dx}(f)\frac{d}{dx}(g).$$

Example.

$f(x) = x$

$g(x) = x$

$$\frac{d}{dx}(fg) = 2x$$

$$\neq$$

$$\frac{d(f)}{dx} \cdot \frac{d(g)}{dx} = 1$$

Proof.

$$\frac{d}{dx}\Big[\underbrace{f(x)g(x)}_{\text{fct.}}\Big] = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{\big(f(x+h) - f(x)\big)g(x+h)}{h} + f(x)\left(\frac{g(x+h) - g(x)}{h}\right)$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} g(x+h) + \lim_{h\to 0} f(x)\frac{g(x+h) - g(x)}{h}$$

$$= \left(\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\right)\left(\lim_{h\to 0} g(x+h)\right) + \left(\lim_{h\to 0} f(x)\right)\left(\lim_{h\to 0} \frac{g(x+h)-g(x)}{h}\right)$$

$$= f'(x)\,g(x) + f(x)\,g'(x).$$

EXAMPLE 7 If $h(x) = xg(x)$ and it is known that $g(3) = 5$ and $g'(3) = 2$, find $h'(3)$.

① $h'(x) = (x)'g(x) + x\,g'(x) = g(x) + x\,g'(x)$

② $h'(3) = g(3) + 3\,g'(3) = 5 + 3 \cdot 2 = \boxed{11}$

Quotient.

The Quotient Rule If f and g are differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

$\to$ Caution !!

$$\frac{d}{dx}\left(\frac{f}{g}\right) \neq \frac{\frac{d}{dx}(f)}{\frac{d}{dx}(g)}$$

$\longrightarrow$ Example.

$f(x) = x$

$g(x) = 1+x$

$$\bullet\ \frac{d}{dx}\left(\frac{x}{1+x}\right) = \frac{1}{1} = 1$$

$$\not=$$

$$\bullet\ \frac{d}{dx}\left[x/1+x\right] = \frac{1}{(1+x)^2}$$

EXAMPLE 8 Let $y = \dfrac{x^2 + x - 2}{x^3 + 6}$. Compute the derivative.

$\to$ 1st

$\to$ 2nd.

$$y' = \frac{(x^3+6)\left(\overset{①}{\frac{d}{dx}(x^2+x-2)}\right) - (x^2+x-2)\left(\overset{②}{\frac{d}{dx}(x^3+6)}\right)}{(x^3+6)^2}$$

$$\frac{dy}{dx}$$

$$① \quad \frac{d}{dx}(x^2+x-2) = \frac{d}{dx}(x^2) + \frac{d}{dx}(x) - \frac{d}{dx}(2) = 2x+1-0$$

$$= 2x+1$$

$$② \quad \frac{d}{dx}(x^3+6) = \frac{d}{dx}(x^3) + \frac{d}{dx}(6) = 3x^2+0 = 3x^2.$$

$$\to y' = \frac{(x^3+6)(2x+1) - (x^2+x-2)(3x^2)}{(x^3+6)^2}$$

$$= \frac{2x^4 + x^3 + 12x + 6 - (3x^4 + 3x^3 - 6x^2)}{(x^3+6)^2}$$

$$= \boxed{\frac{-x^4 - 2x^3 + 6x^2 + 12x + 6}{(x^3+6)^2}}$$

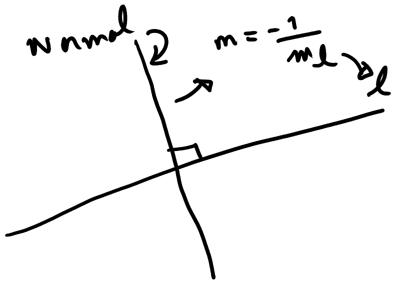
General Power rule.

$x^{1/2} \xrightarrow{d/dx} \frac{1}{2}x^{1/2-1} = \frac{1}{2}x^{-1/2} \leftarrow (\sqrt{x})' = \frac{1}{2\sqrt{x}}$

The Power Rule (General Version) If n is any real number, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

EXAMPLE 12 Find equations of the tangent line and normal line to the curve $y = \sqrt{x}/(1 + x^2)$ at the point $(1, \frac{1}{2})$.
↳ (x_1, y_1)



① Tangent line

$y - y_1 = m(x - x_1)$, $m = y'(x_1)$

$\to y - \frac{1}{2} = m(x - 1)$ $\quad m = y'(1)$,

Here, $y'(x) = \frac{(\sqrt{x})'(1+x^2) - \sqrt{x}(1+x^2)'}{(1+x^2)^2}$

$(x^{1/2})' = \frac{1}{2}x^{-1/2}$
$= \frac{1}{2x^{1/2}}$

$$= \frac{\left(\frac{1}{2\sqrt{x}}\right)(1+x^2) - \sqrt{x}\left((1)' + (x^2)'\right)}{(1+x^2)^2}$$

$$= \frac{1/2\sqrt{x} + x^{3/2}/2 - \sqrt{x}(0 + 2x)}{(1+x^2)^2}$$

$$= \frac{1/2\sqrt{x} + x^{3/2}/2 - 2x^{3/2}}{(1+x^2)^2} = \frac{\frac{1}{2\sqrt{x}} - \frac{3}{2}x^{3/2}}{(1+x^2)^2}$$

$$\to y'(1) = \frac{\frac{1}{2} - \frac{3}{2}}{(1+1)^2} = \frac{-1}{4}$$

So, $y - \frac{1}{2} = -\frac{1}{4}(x-1) \to \boxed{y = -\frac{1}{4}x + \frac{3}{4}}$.

② Normal Line.

$y - y_1 = m_2(x - x_1)$, $m_2 = -\frac{1}{m}$

$\to y - \frac{1}{2} = \frac{-1}{-1/4}(x-1) = 4(x-1)$

$\to \boxed{y = 4x - \frac{15}{4}}$ ← Normal to the tangent.

EXAMPLE 13 At what points on the hyperbola $xy = 12$ is the tangent line parallel to the line $3x + y = 0$?

① Find the curve.

② Find the points.

$$xy = 12 \rightarrow y = \frac{12}{x}.$$

(1.5) Find the derivative.

$$y' = \left(\frac{12}{x}\right)' = (12x^{-1})' = (-1)\,12\,x^{-1-1} = -12x^{-2} = \frac{-12}{x^2}$$

② Find the points.

$$y = \underline{-3}x \rightarrow -3 = y' \leftrightarrow -3 = \frac{-12}{x^2}$$

$$\boxed{\begin{gathered} y = \overset{\downarrow}{m}x + b \\ y - y_1 = \overset{\downarrow}{m}(x - x_1) \end{gathered}}$$

$$\leftrightarrow x^2 = 4$$

$$\leftrightarrow x = \pm 2$$

$$y = \frac{12}{2} = 6$$

$$y = \frac{12}{-2} = -6$$

The points where the tangent line is parallel to $3x+y=0$ are $(2,6)$, $(-2,-6)$.

Summary of Differentiation Formulas.

Table of Differentiation Formulas

$$\frac{d}{dx}(c) = 0 \qquad \frac{d}{dx}(x^n) = nx^{n-1}$$

$$(cf)' = cf' \qquad (f+g)' = f' + g' \qquad (f-g)' = f' - g'$$

$$(fg)' = fg' + gf' \qquad \left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$