Math 307

Chapter 6

Section 6.2: Homogeneous Systems With Constant Coefficients
The Diagonalizable Case

Contents

Created by: Pierre-Olivier Parisé

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Example 1. Determine the general solution to

$$Y' = \underbrace{\begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}}_{A} Y.$$

1) Transform A into a diagonal matrix.

- $\lambda = -1$ & $\lambda = 4$.
- $\dim(E_{-1}) = 1$ & $\dim(E_4) = 1$
- $\dim(E_{-1}) + \dim(E_4) = 2$ ✓ $\to$ A is diagonalizable.

$$D = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} \quad \& \quad P = \begin{bmatrix} 3/2 & -1 \\ 1 & 1 \end{bmatrix}$$

$\Rightarrow \quad A = PDP^{-1}$.

2) Solve the diagonal system.

$(af)' = af'$
$(AY)' = AY'$ (*)

$Y' = AY \to Y' = PDP^{-1}Y \to P^{-1}Y' = DP^{-1}Y$

used (*) $\to$ $\to \underbrace{(P^{-1}Y)}_{Z}' = D\underbrace{(P^{-1}Y)}_{Z}$

Let $Z = P^{-1}Y \Rightarrow Z' = DZ = \begin{bmatrix} \boxed{-1}_{d_1} & 0 \\ 0 & \boxed{4}_{d_2} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$.

$\Rightarrow Z(x) = \begin{bmatrix} c_1 e^{-x} \\ c_2 e^{4x} \end{bmatrix}$.

So, $Y = PZ = \begin{bmatrix} 3/2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{-x} \\ c_2 e^{4x} \end{bmatrix} = \begin{bmatrix} \frac{3c_1}{2} e^{-x} - c_2 e^{4x} \\ c_1 e^{-x} + c_2 e^{4x} \end{bmatrix}$.

Fact: Suppose A and B are $n \times n$ matrices with $B = P^{-1}AP$ for some invertible $n \times n$ matrix P. Then

- If Z is a solution to $Y' = BY$ ($Z' = BZ$), then PZ is a solution to $Y' = AY$.
- If $Z_1, Z_2, \ldots, Z_n$ is a fundamental set of solutions of $Y' = BY$, then $PZ_1, PZ_2, \ldots, PZ_n$ is a fundamental set of solutions to $Y' = AY$.

Example 2. Solve the initial value problem

$$Y' = \underbrace{\begin{bmatrix} 2 & -3 & -3 \\ 2 & -2 & -2 \\ -2 & 1 & 1 \end{bmatrix}}_{A} Y, \quad Y(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

1) Transform into D.

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \& \quad P = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}. \qquad A = PDP^{-1}.$$

2) Solve diagonal system.

$Z = P^{-1}Y \rightarrow Y' = AY$ becomes $Z' = DZ$.

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} Z$$

From 6.1, $$Z(x) = \begin{bmatrix} c_1 e^{2x} \\ c_2 e^{-x} \\ c_3 \end{bmatrix}$$

$$\rightarrow Y_H = PZ = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{2x} \\ c_2 e^{-x} \\ c_3 \end{bmatrix} = \begin{bmatrix} c_1 e^{2x} - c_3 \\ -c_2 e^{-x} - c_3 \\ c_1 e^{2x} + c_2 e^{-x} + c_3 \end{bmatrix}$$

3) Initial conditions.

$$Y(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 - c_3 \\ -c_2 - c_3 \\ c_1 + c_2 + c_3 \end{bmatrix} \rightarrow \begin{cases} c_1 = c_3 + 1 \\ c_2 = -c_3 \\ 0 = c_1 + c_2 + c_3 \end{cases}$$

$$0 = c_3 + 1 - c_3 + c_3 \Rightarrow c_3 = -1$$

$$c_1 = 0$$
$$c_2 = 1$$

So, $$Y(x) = \begin{bmatrix} 1 \\ -e^{-x} + 1 \\ e^{-x} - 1 \end{bmatrix}$$

Complex Exponential Function

For a complex number $z = a + ib$, we define $e^{ix} = \cos(x) + i\sin(x)$.

$$e^z = e^{a+ib} = e^a \cos(b) + ie^a \sin(b).$$

The solution to the differential equation $y' = (a + ib)y$ is

$$y(x) = e^{(a+ib)x}.$$

Finding solutions with complex numbers

Example 3. Find the general solution to

$$Y' = \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}_{A} Y.$$

1) Transform A into D.

$D = \begin{bmatrix} 1-i & 0 \\ 0 & 1+i \end{bmatrix}$ & $P = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}$ $\quad (A = PDP^{-1})$.

2) Solve the diagonal system.

$Z = P^{-1}Y \rightarrow Y' = AY$ becomes $Z' = DZ$.

$$= \begin{bmatrix} 1-i & 0 \\ 0 & 1+i \end{bmatrix} Z.$$

From 6.1, $Z(x) = \begin{bmatrix} c_1 e^{(1-i)x} \\ c_2 e^{(1+i)x} \end{bmatrix}$.

(*) $e^{(1-i)x} = e^{x-ix} = \underbrace{e^x}_{\checkmark} \cdot \underbrace{e^{-ix}}_{??}$

Fact: If $U(x) + iV(x)$ is a solution to $Y' = AY$, then $U(x)$ and $V(x)$ are solutions to $Y' = AY$.

Example 4. Find the general solution to

$$Y' = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} Y.$$