# Chapter 4
# Integrals

4.5 The Substitution Rule

Example to start. Find the indefinite integral of $2x\sqrt{1+x^2}$ , that is compute

$$\int 2x\sqrt{1+x^2}\,dx.$$

Find Anti-Derivative of $2x\sqrt{1+x^2}$ ?

$$\left(\sqrt{1+x^2}\right)' = \frac{1}{2\sqrt{1+x^2}}\cdot 2x = \frac{x}{\sqrt{1+x^2}}$$

$$\left((1+x^2)^{3/2}\right)' = \frac{3}{2}(1+x^2)^{1/2}\cdot(1+x^2)' = \frac{3}{2}(1+x^2)^{1/2}\cdot 2x$$

$$= 3x\sqrt{1+x^2}$$

$$\stackrel{?}{=} 2x\sqrt{1+x^2} \;??$$

$$F(x) = \frac{2}{3}(1+x^2)^{3/2}$$

$$\int 2x\sqrt{1+x^2}\,dx = \frac{2}{3}(1+x^2)^{3/2} + C.$$

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Another example. Compute the indefinite integral

$$\int x\sqrt{1+x^2}\,dx.$$

$dx$ : real number

1st. We see that

$$\frac{d}{dx}(1+x^2) = 2x \longrightarrow d(1+x^2) = 2x\,dx$$

2nd. Define $u = 1+x^2 \longrightarrow \frac{du}{dx} = 2x \longrightarrow \frac{1}{2}\frac{du}{dx} = x$

$$\longrightarrow du = 2x\,dx$$

$$\longrightarrow \frac{du}{2} = x\,dx$$

3rd

$$\int \underbrace{x}_{\frac{1}{2}\frac{du}{dx}}\sqrt{1+x^2}\,dx = \int \sqrt{u}\,\frac{du}{2}$$

$$= \frac{1}{2}\frac{u^{3/2}}{3/2} = \frac{1}{3}(1+x^2)^{1/2}$$

$$\int x\sqrt{1+x^2}\,dx = \int \frac{1}{2}\frac{du}{\cancel{dx}}\sqrt{u}\,\cancel{dx}$$

$$= \frac{1}{2}\int \sqrt{u}\,du$$

**4 The Substitution Rule** If $u = g(x)$ is a differentiable function whose range is an interval $I$ and $f$ is continuous on $I$, then

$$\int f'(g(x))\,g'(x)\,dx = \int f(u)\,du$$

$\longrightarrow [f(g(x))]' = f'(g(x)) \cdot g'(x)$

$\int (f(g(x)))'\,dx = f(g(x)) + C$

Relation between du and dx:

$du = \underbrace{g'(x)}_{u'}\,dx$

or $\frac{du}{dx}$

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**EXAMPLE 1** Find $\int x^3 \cos(x^4 + 2)\,dx$.

$u = x^4 + 2 \longrightarrow \frac{du}{dx} = 4x^3 \longrightarrow \frac{1}{4}\frac{du}{dx} = x^3$

$\longrightarrow \frac{1}{4}\,du = x^3\,dx$

$\int x^3 \cos(x^4+2)\,dx = \int \cos(u) \cdot \frac{1}{4}\,du$

$= \frac{1}{4}\int \cos(u)\,du$

$= \frac{1}{4}\sin(u) + C$

$= \boxed{\frac{1}{4}\sin(x^4+2) + C}$

$\int x^3 \cos(x^4+2)\,dx = \int \frac{4}{4} x^3 \cos(x^4+2)\,dx$

$= \frac{1}{4}\int \underbrace{4x^3}_{\frac{du}{dx}} \cos(x^4+2)\,dx$

**EXAMPLE 2** Evaluate $\int \sqrt{2x+1}\,dx$. $\neq \frac{(2x+1)^{3/2}}{3/2} + C$

$u = 2x+1 \to \frac{du}{dx} = 2 \to du = 2\,dx \to \frac{du}{2} = dx$

$$\int \sqrt{2x+1}\,dx = \int \sqrt{u}\,\frac{du}{2}$$
$$= \frac{1}{2}\int u^{1/2}\,du$$
$$= \frac{1}{2}\,\frac{u^{3/2}}{3/2} = \frac{1}{3}u^{3/2}$$
$$= \boxed{\frac{1}{3}(2x+1)^{3/2}}$$

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**EXAMPLE 3** Find $\int \frac{x}{\sqrt{1-4x^2}}\,dx$.

$f(x) = \frac{1}{\sqrt{x}} \to f(1-4x^2) = \frac{1}{\sqrt{1-4x^2}}$.

$u = 1-4x^2 \to \frac{du}{dx} = -8x \to du = -8x\,dx$

$\to -\frac{du}{8} = x\,dx$

$$\int \frac{x}{\sqrt{1-4x^2}}\,dx = \int \frac{1}{\sqrt{u}}\,\left(-\frac{du}{8}\right) = -\frac{1}{8}\int \frac{1}{\sqrt{u}}\,du$$

$\sqrt{u} \to u^{1/2}$

$$= -\frac{1}{8}\int u^{-1/2}\,du$$
$$= -\frac{1}{8}\,\frac{u^{1/2}}{1/2}$$
$$= -\frac{1}{4}\sqrt{u}$$
$$= \boxed{-\frac{1}{4}\sqrt{1-4x^2}}$$

**EXAMPLE 5** Find $\int \sqrt{1+x^2}\, x^5\, dx$.

$$u = 1+x^2 \rightarrow \frac{du}{dx} = 2x\,dx \rightarrow du = 2x\,dx \rightarrow \frac{du}{2} = x\,dx$$

$$\int \sqrt{1+x^2}\, x^5\, dx = \int \sqrt{u}\; x^4\, \underline{x\,dx}$$

$$= \int \sqrt{u}\; x^4\, \frac{du}{2}$$

$$= \int \sqrt{u}\; (x^2)^2\, \frac{du}{2}$$

$$u-1 = x^2 \rightarrow (x^2)^2 = (u-1)^2$$

$$= \int \sqrt{u}\; (u-1)^2\, \frac{du}{2}$$

$$= \frac{1}{2}\int u^{1/2}\,(u^2 - 2u + 1)\, du$$

$$= \frac{1}{2}\int u^{5/2} - 2u^{3/2} + u^{1/2}\, du$$

$$= \frac{1}{2}\left(\frac{u^{7/2}}{7/2} - 2\,\frac{u^{5/2}}{5/2} + \frac{u^{3/2}}{3/2}\right) + C$$

$$= \frac{u^{7/2}}{7} - \frac{2}{5}u^{5/2} + \frac{u^{3/2}}{3} + C$$

$$= \boxed{\frac{(x^2+1)^{7/2}}{7} - \frac{2}{5}(x^2+1)^{5/2} + \frac{(x^2+1)^{3/2}}{3} + C.}$$

When $u = g(x)$, then

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du .$$

**EXAMPLE 7** Evaluate $\displaystyle\int_1^2 \frac{dx}{(3-5x)^2}$.

$u = 3-5x \rightarrow \dfrac{du}{dx} = -5 \rightarrow du = -5dx$

$\rightarrow -\dfrac{du}{5} = dx$

So,

$$\int_1^2 \frac{dx}{(3-5x)^2} = \int_{3-5\cdot(1)}^{3-5\cdot(2)} \frac{1}{u^2} \; -\frac{du}{5}$$

$$= -\frac{1}{5}\int_{-2}^{-7} \frac{1}{u^2}\, du$$

2nd way

$$-\frac{1}{5}\left(\frac{(3-5x)^{-1}}{-1}\right)\Bigg|_1^2$$

$$= \frac{-1}{5}\; \frac{u^{-1}}{-1}\Bigg|_{-2}^{-7}$$

$$= \frac{-1}{5}\left(\frac{1}{-(-7)} - \frac{1}{-(-2)}\right)$$

$$= -\frac{1}{5}\left(\frac{1}{7} - \frac{1}{2}\right)$$

$$= -\frac{1}{5}\left(\frac{-5}{14}\right)$$

$$= \boxed{\frac{1}{14}}$$