MATH 644

CHAPTER 4

SECTION 4.3: APPROXIMATION BY RATIONAL FUNCTIONS

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Cauchy Integral Formula For a Square

Theorem 1. If S is an open square with boundary ∂S parameterized in the counter-clockwise direction then

$$\frac{1}{2\pi i}\int_{\partial S}\frac{1}{z-a}\,dz=\begin{cases}1 & \text{if } a\in S\\ 0 & \text{if } a\in\mathbb{C}\backslash\overline{S}.\end{cases}$$

Proof.

Theorem 2. If f is analytic in a neighborhood of the closure of $\overline{S}$ of an open square S, then, for $z \in S$,

$$f(z) = \frac{1}{2\pi i} \int_{\partial S} \frac{f(\zeta)}{\zeta - z} \, d\zeta,$$

where ∂S is parameterized in the counter-clockwise direction.

Proof.

Corollary 3. If f is analytic in a neighborhood of the closure of $\overline{S}$ of an open square S, then

$$\int_{\partial S} f(\zeta) \, d\zeta = 0.$$

Proof.

THEOREM 4. If f is analytic on a compact set K, and if $\varepsilon > 0$, then there is a rational function r so that

$$\sup_{z \in K} |f(z) - r(z)| < \varepsilon.$$

Proof.

Definition 5. Let $r(z) = p(z)/q(z)$ be a rational function where p and q are two polynomials with no common zeros. The zeros of q are called the **poles** of the rational function r.

Note:

- If b is a pole of a rational function r, then $|r(z)| \to \infty$ as $|z| \to \infty$.

Lemma 6. Suppose that U is a region and suppose $b \in U$. Then a rational function with poles in U can be uniformly approximated on $\mathbb{C} \backslash U$ by a rational function with poles only at b.

Proof.

Corollary 7. Suppose U is a region and suppose $\{z : |z| > R\} \subset U$ for some $R < \infty$. Then a rational function with poles only in U can be uniformly approximated on $\mathbb{C} \backslash U$ by a polynomial.

Proof.

Components

DEFINITION 8. Let U be an open set.

a) A polygonal curve in U is a curve consisting of a finite union of line segments.

b) For $a, b \in U$, define $a \sim_U b$ if and only if there is a polynomial curve contained in U with edges parallel to the axis and joining a to b.

THEOREM 9. Let $U \subset \mathbb{C}$ be an open set.

a) Show that the equivalence classes of $\sim_U$ are closed and open (relative to U) and connected.

b) Show that there are countably many equivalent classes.

Note:

- The equivalence classes are called the **components** of U. They are the maximal connected subsets of U.

Closed Components

DEFINITION 10. Suppose $K \subset \mathbb{C}$ is a compact set.

a) For $a, b \in K$, define $a \sim_K b$ if and only if there is a connected subset of K containing a and b.

THEOREM 11. Let $K \subset \mathbb{C}$ be a compact set.

a) Show that the equivalence classes of $\sim_K$ are connected and closed.

b) Show that there might be infinitely many equivalence classes.

Note:

- The equivalence classes are called the **(closed) components** of K.

THEOREM 12. [Runge] Suppose K is a compact set. Choose one point a_n in each bounded component of U_n in $\mathbb{C}\backslash K$. If f is analytic on K and $\varepsilon > 0$, then we can find a rational function r with poles only in the set $\{a_n\}$ such that

$$\sup_{z \in K} |f(z) - r(z)| < \varepsilon.$$

If $\mathbb{C}\backslash K$ has no bounded components, then we may take r to be a polynomial.

<u>**Proof.**</u>

Theorem 13. If f is analytic on an open set $\Omega \neq \mathbb{C}$, then there is a sequence of rational functions r_n with poles in $\partial\Omega$ so that r_n converges to f uniformly on compact subsets of Ω.

Proof.

Note:

- The improvement of Theorem 9 over Theorem 4 is that the poles of r_n are outside of Ω, not just outside the compact set K on which r_n is close to f.