MATH 644

CHAPTER 6

SECTION 6.2: NORMALITY AND EQUICONTINUITY

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Normality

Definition 1. A collection, or family, $\mathcal{F}$ of continuous functions on a region $\Omega \subset \mathbb{C}$ is said to be **normal on** Ω provided every sequence $(f_n) \subset \mathcal{F}$ contains a subsequence which converges uniformly on compact subsets of Ω.

Example 2. Show if the given family is normal on the given region.

(a) $\mathcal{F}_1 := \{f_n(z) = z^n \,:\, n = 0, 1, \ldots\}$ and $\Omega = \mathbb{D}$.

(b) $\mathcal{F}_2 := \{g_n \,:\, n = 0, 1, \ldots\}$, where $g_n(z) = 1$ if n is even and $g_n(z) = 0$ if n is odd and $\Omega = \mathbb{C}$.

Lemma 3. Suppose Ω

- is a region and;
- $\Omega = \cup_{j=1}^{\infty} \Delta_j$, where $\Delta_j \subset \Omega$ are closed disks.

A family of continuous functions $\mathcal{F}$ is normal on Ω if and only if, for each j, every sequence in $\mathcal{F}$ contains a subsequence which converges uniformly on Δ_j.

Proof.

Theorem 4. A sequence $(f_n) \subset C(\Omega)$ converges uniformly on compact subsets of Ω to $f \in C(\Omega)$ if and only if $\lim_{n\to\infty} \rho(f_n, f) = 0$.

Proof.

Note:

- When $\lim_{n\to\infty} \rho(f_n, f) = 0$, we say that (f_n) **converges locally uniformly** to f on Ω.

Equicontinuous Family of Functions

Definition 5. A family of functions $\mathcal{F}$ defined on a set $E \subset \mathbb{C}$ is

(a) **equicontinuous at $\mathbf{w} \in \mathbf{E}$** if $\forall \varepsilon > 0$, $\exists \delta > 0$ so that

$$z \in E \text{ and } |z - w| < \delta \quad \Longrightarrow \quad |f(z) - f(w)| < \varepsilon, \forall f \in \mathcal{F}.$$

(b) **equicontinuous on $\mathbf{E}$** if it is equicontinuous at each $w \in E$.

(c) **uniformly equicontinuous on $\mathbf{E}$** if $\forall \varepsilon > 0$, $\exists \delta > 0$ so that

$$z, w \in E \text{ and } |z - w| < \delta \quad \Longrightarrow \quad |f(z) - f(w)| < \varepsilon, \forall f \in \mathcal{F}.$$

Example 6. Fix $M > 0$. Show that the family

$$\mathcal{F} := \{f : \mathbb{D} \to \mathbb{C} \; : \; f \text{ analytic and } |f'| \leq M\}$$

is uniformly equicontinuous on $\mathbb{D}$.

Theorem 7. [Arzela-Ascoli] A family of continuous functions $\mathcal{F}$ is normal on a region $\Omega \subset \mathbb{C}$ if and only if

(a) $\mathcal{F}$ is equicontinuous on Ω and;

(b) there is a $z_0 \in \Omega$ so that the collection $\{f(z_0) : f \in \mathcal{F}\}$ is a bounded subset of $\mathbb{C}$.

Proof.

DEFINITION 8. A family $\mathcal{F}$ of continuous functions is said to be **locally bounded** on Ω if

$$\forall w \in \Omega,\ \exists \delta > 0 \text{ and } M < \infty \text{ so that } |z - w| < \delta \Rightarrow |f(z)| \leq M,\ \forall f \in \mathcal{F}.$$

THEOREM 9. Let $\mathcal{F}$ be a family of analytic functions on a region Ω. Then the following are equivalent:

(a) $\mathcal{F}$ is normal on Ω;

(b) $\mathcal{F}$ is locally bounded on Ω;

(c) $\mathcal{F}' := \{f' : f \in \mathcal{F}\}$ is locally bounded on Ω and there is a $z_0 \in \Omega$ so that $\{f(z_0) : f \in \mathcal{F}\}$ is a bounded subset of $\mathbb{C}$.

Proof.