Math 644

Chapter 6

Section 6.4: Linear Fractional Transformation

Contents

Created by: Pierre-Olivier Parisé

Spring 2023

Group of LFTs

Definition 1. A **linear fractional transformation** (LFT for short) is a non-constant rational function of the form

$$T(z) = \frac{az+b}{cz+d} \quad (a, b, c, d \in \mathbb{C}).$$

Basic Types of LFTs:

1. Translation: $T(z) = z + b$;
2. Rotation: $T(z) = e^{i\theta} z$;
3. Dilation: $T(z) = az$;
4. Inversion: $T(z) = \frac{1}{z}$.

Facts:

- Any LFT is the composition of basic LFTs.
  Proof.
- An LFT $T(z) = \frac{az+b}{cz+d}$ is non-constant if and only if $bc - ad \neq 0$.

Theorem 2. The set of LFTs forms a group under composition.

Proof.

Characterization of LFTs

Theorem 3. If f is analytic on $\mathbb{C}\backslash\{z_0\}$ and one-to-one then f is an LFT.

Proof.

Consequences:

- The automorphisms of the complex plane are the linear functions.
- The automorphisms of the extended complex plane are exactly the set of LFTs.

Preserving Generalized Circles

Notes:

- A generalized circle is a circle in the complex plane or a line in the complex plane.
- A generalized disk is a a region bounded by a generalized circle (so disks or halfplanes).

Theorem 4. LFTs map generalized circles onto generalized circles

Proof.

Uniqueness of LFTs

Theorem 5. Given z_1, z_2, z_3 distinct points in $\mathbb{C}^*$, and w_1, w_2, w_3 distinct points in $\mathbb{C}^*$, there is a unique LFTs T such that

$$T(z_i) = w_i, \quad (i = 1, 2, 3).$$

Proof.