# Chapter 2: Derivatives
# Week 6

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Calculus I (MATH-241 01/02)

University of Hawai'i
Fall 2021

# Upcoming this week

1. 2.7 Rates of change in the Natural and Social Sciences
2. 2.8 Related Rates
3. 2.9 Linear Approximations and Differentials

It $y = f(x)$, then the derivative $dy/dx$ can be intepretated as the rate of change of $y$ w.r.t. $x$.

This idea has many applications in Science: Physics, Chemistry, Biology, economics, among others.

### Definition 1

The average rate of change of $y$ w.r.t. $x$ over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} := \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

### Definition 2

As $\Delta x \to 0$, we get $f'(x_1)$ which is the instantaneous rate of change of $y$ w.r.t. $x$.

Suppose $s(t)$ represent the position function of an object.

- $v(t) = s'(t)$ represents the instantenous velocity of the object at time $t$.
- $a(t) = v'(t) = s''(t)$ represents the acceleration of the object at time $t$.

Using $v(t)$ and $a(t)$, we can describe oscillatory movement in an equation called differential equation.

**Definition 3**

A differential equation is an equation relating a function $s(t)$ to its derivatives $s'$, $s''$, $s'''$, ...

Suppose you have a pendulum with the following data:

- a massless string of length $L$, a bob of mass $m$, and $g$ is the gravitationnal force.
- $s(t)$ is the distance along the arc from the lowest point to the position of the bob at time $t$.
- $\theta(t)$ is the corresponding angle with the vertical.

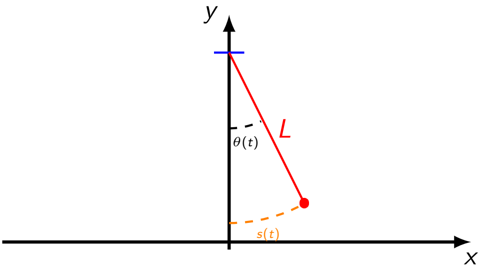


Figure: Ideal pendulum

With some work, we can relate $s$ to $s''$ with the following equation:

$$\theta''(t) = -\frac{g}{L}\theta \quad (|\theta| \lesssim 15^\circ).$$

When you solve for $\theta$ Pendulum:

$$\theta(t) = \sin\left(t\sqrt{L/g}\right).$$

### Example 4

The position of a particle is given by the equation $s = f(t) = t^3 - 6t^2 + 9t$ where $s$ and $t$ are measured in meters and seconds respectively.

a) Find the velocity at time $t$.

b) What is the velocity after $2s$.

c) When is the particle at rest?

f) Find the acceleration at time $t$ and after 4s.

g) Graph the position, velocity, and acceleration functions for $0 \leq t \leq 5$.

h) When is the particle speeding up? When is it slowing down?

Exercises: 1-4, 5-7, 21.

Sometimes, it is hard to measure a quantity when it is changing.

A good example is the radius of a balloon when we are pumping air into it.

However, the volume is much more easier to measure than the radius of the balloon. So if we are able to find an equation related the radius to the volume of the balloon, then we could try to relate their rates of change.

**Example 5**

Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm$^3/s$. How fast is the radius of the balloon increasing when the diamter is 50 cm?

Here's a strategy you can follow to solve related rates problem:

- Read the problem carefully.
- Draw a diagram if possible.
- Introduce notation. Assign symbols to all quantities that are functions of time.
- Express the given information and the required rate in terms of derivatives.

### Example 6

A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall.

### Example 7

A water tank has the shape of an inverted circular cone with base radius 2m and height 4m. If water is being pumped into the tank at a rate of $2\text{m}^3/\text{min}$, find the rate at which the water level is rising when the water is 3m deep.

Exercises: 1-16, 20, 22, 39, 42, 50.

An observation:

*A curve $y = f(x)$ lies very close to its tangent line near the point of tangency.* Linearization

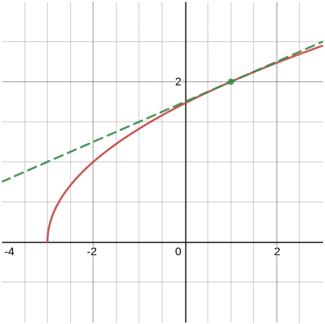


Figure: Linearization near the point of tangency

This suggests to approximate the values of $f$ by the tangent line. This is a really useful procedure because $f(x)$ may be difficult to compute!

Recall that the equation of the tangent line at the point $P = (a, f(a))$ is

$$L(x) = f(a) + f'(a)(x - a).$$

### Definition 8

The function $L(x)$ is called the <u>linearization</u> of $f$ at the point $a$. It has the following property:

$$f(x) \approx L(x)$$

near $x = a$.

## Example 9

Find a linearization of $f(x) = \sqrt{x+3}$ at $a = 1$. Use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$

| | $x$ | $L(x)$ | Actual value |
|---|---|---|---|
| $\sqrt{3.9}$ | 0.9 | 1.975 | 1.97484176... |
| $\sqrt{3.98}$ | 0.98 | 1.995 | 1.99499373... |
| $\sqrt{4}$ | 1 | 2 | 2.00000000... |
| $\sqrt{4.1}$ | 1.1 | 2.025 | 2.02484567... |
| $\sqrt{5}$ | 2 | 2.25 | 2.23606797... |
| $\sqrt{6}$ | 3 | 2.5 | 2.44948974... |

The Linearization of a function is closely related to another concept: the differentials.

### Definition 10

If $y = f(x)$, then

- $dx$ is the differential of $x$. It's a little increment in the variable $x$.
- $dy$ is the differential of $y$ and $dy$ is the approximate increment in the variable $y$ given by

$$dy = f'(x)dx.$$

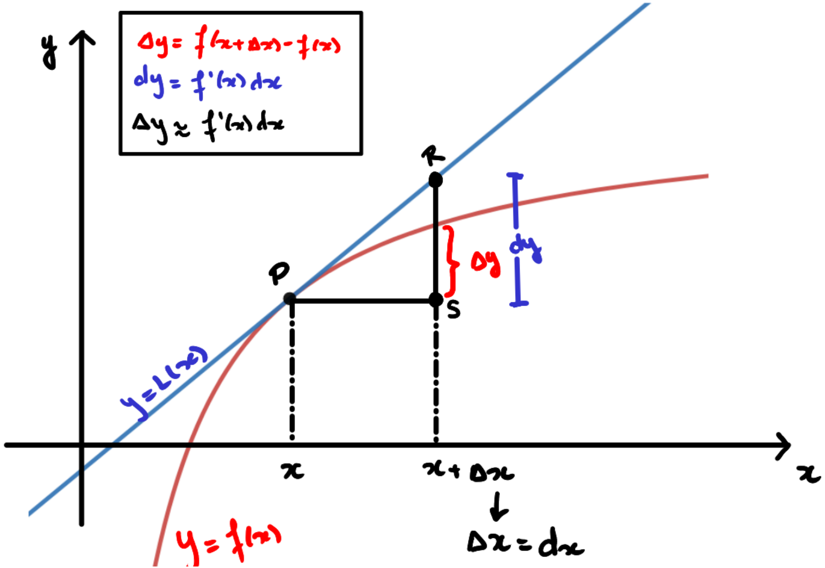


Figure: Geometric meaning of differentials

## Example 11

Compare the values of $\Delta y$ and $dy$ if $y = f(x) = x^3 + x^2 - 2x + 1$ and $x$ changes from

a) 2 to 2.05.

b) 2 to 2.01.

Th linear approximation $f(x) \approx L(x)$ can be rewritten in the following new form:

$$f(x + dx) - f(x) = \Delta y \approx dy \quad \Rightarrow \quad f(x + dx) \approx f(x) + dy.$$

## Example 12

The radius of a sphere was measured and found to be 21cm with a possible error in measurement of at most 0.05cm. What is the maximum error in using this value of the radius to compute the volume of the sphere?

Note: Relative error is $\frac{\Delta V}{V} \approx \frac{dV}{V}$.

Exercises: 1-4, 6-8, 11, 12, 15-22, 23, 32, 39.