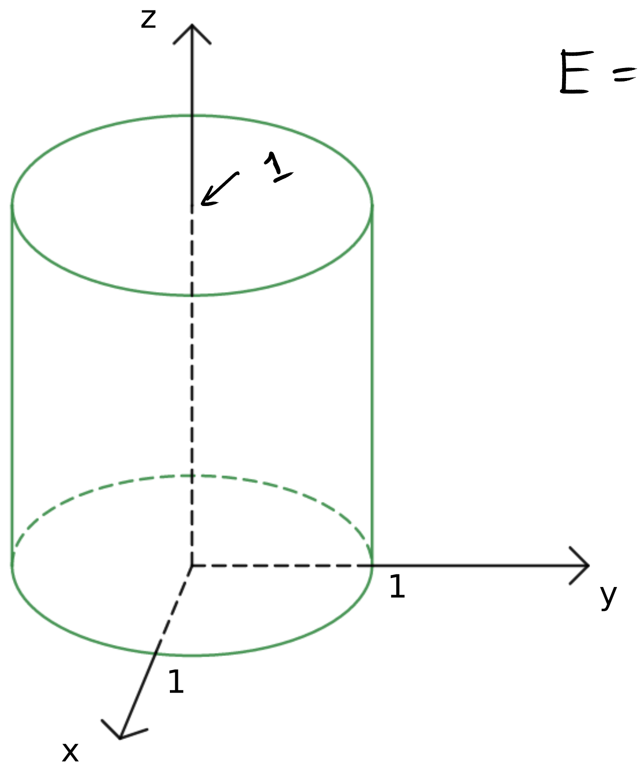
Chapter 15

Multiple Integrals

15.7 Triple integrals in cylindrical coordinates

Cylindrical coordinates

EXAMPLE. Describe the following solid (the interior of a cylinder).



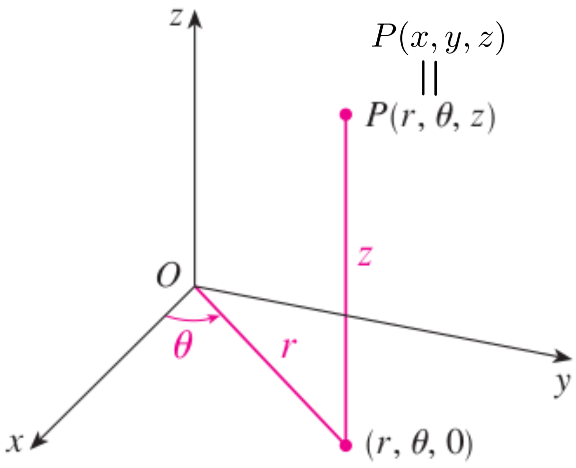
$E = \{ (x, y, z) : \quad 0 \le z \le 1$
$\quad 0 \le x^2 + y^2 \le 1 \}$

Because of the circle, it might be difficult to use this description in a triple integral.

Describe the base of the cylinder using polar coordinates:

$E = \{ (r, \theta, z) : \quad 0 \le r \le 1, \; 0 \le \theta \le 2\pi, \; 0 \le z \le 1 \}$

Definition (when the main axis is the z-axis)



Cylindrical ⟶ Cartesian

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

Cartesian ⟶ Cylindrical

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan(y/x), \quad z = z$$

EXAMPLE 1

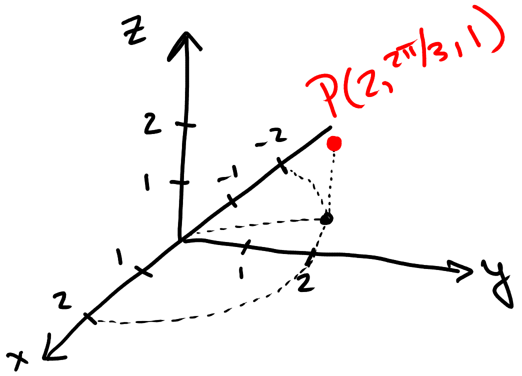
$r \quad \theta \quad z$ $\qquad \frac{\pi}{2} < \frac{2\pi}{3}$

(a) Plot the point with cylindrical coordinates $(2, 2\pi/3, 1)$ and find its rectangular coordinates.

$x \quad y \quad z$

(b) Find cylindrical coordinates of the point with rectangular coordinates $(3, -3, -7)$.

(a)



$$\frac{2\pi}{3} - \frac{\pi}{2} = \frac{\pi}{6}$$

$$x = 2\cos(2\pi/3) = 2\cdot\left(-\tfrac{1}{2}\right) = -1$$

$$y = 2\sin(2\pi/3) = 2\,\frac{\sqrt{3}}{2} = \sqrt{3}$$

$$z = 1$$

(b) $r = \sqrt{x^2+y^2} = \sqrt{3^2+(-3)^2} = \sqrt{18} = \underline{3\sqrt{2}}$

$\theta = \text{Arctan}\left(\frac{y}{x}\right) = \text{Arctan}\left(\frac{-3}{3}\right) = \underline{\frac{-\pi}{4} \text{ or } \frac{7\pi}{4}}$

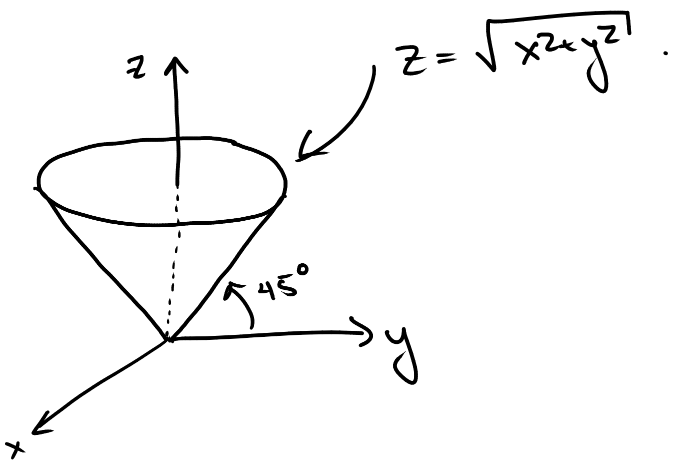
$z = \underline{-7}$

EXAMPLE 2 Describe the surface whose equation in cylindrical coordinates is $z = r$.

$r = \sqrt{x^2+y^2} \longrightarrow z = \sqrt{x^2+y^2}$

$\longrightarrow z^2 = x^2+y^2, \quad z \geq 0.$

<u>Cone</u>:



Note: Principle axis (the z-axis) can be any other axis (x-axis or y-axis) in some applications.

EXAMPLE. Write the equation in cylindrical coordinates and identify the surface.

$$z = x^2 - y^2$$

$x = r\cos\theta$

$y = r\sin\theta$ $\longrightarrow$

$z = z$

$$\begin{aligned} z &= r^2\cos^2\theta - r^2\sin^2\theta \\ &= r^2(\cos^2\theta - \sin^2\theta) \\ &= r^2\cos(2\theta) \end{aligned}$$

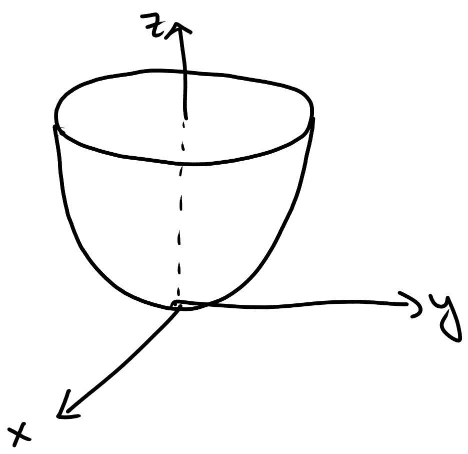
So, $\quad z = r^2\cos(2\theta)$, $\quad r \geq 0$, $\quad 0 \leq \theta \leq 2\pi$

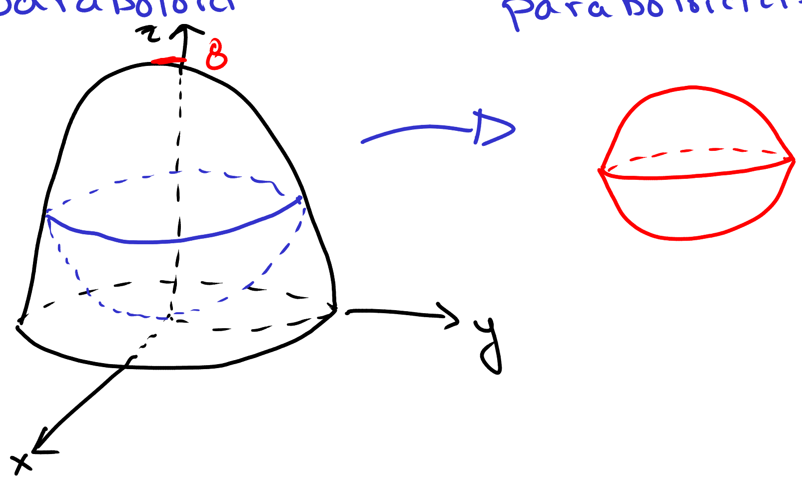
Surface's name: hyperbolic paraboloid.

EXAMPLE. Sketch the solid described by the given inequalities:

$$r^2 \leq z \leq 8 - r^2$$

$r = \sqrt{x^2+y^2} \longrightarrow \underbrace{x^2+y^2}_{\text{paraboloid}} \leq z \leq \underbrace{8-x^2-y^2}_{\text{paraboloid(ish)}}$

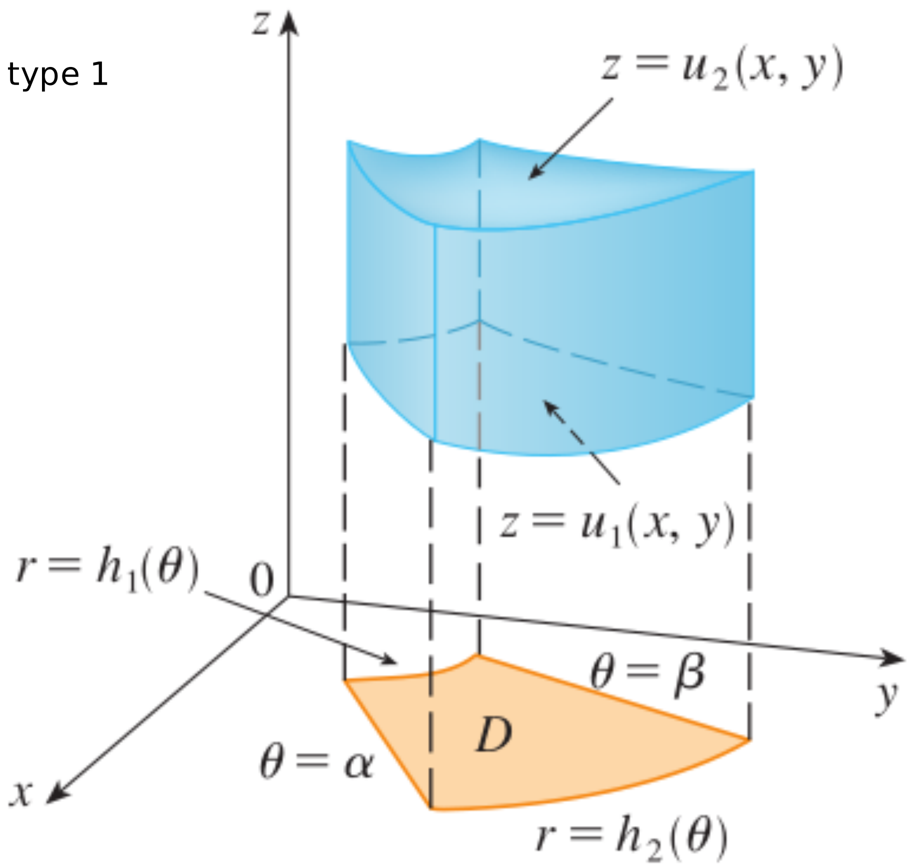




Question. What is the equation of a plane in cylindrical coordinates?

Evaluating triple integrals in cylindrical coordinates.

Solid of type 1



- $E = \{(x, y, z) : (x, y) \in D \text{ and } u_1(x, y) \leq z \leq u_2(x, y)\}$

- Describe D in polar coordinates.

$$\iiint_E f(x, y, z)\, dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \left[\int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, dz \right] r dr d\theta$$

Note: Can be adapted to type 2 and type 3 solids.

EXAMPLE. A solid E lies within the cylinder $x^2 + y^2 = 1$, below the plane $z = 4$, and above the paraboloid $z = 1 - x^2 - y^2$. Find the value of the integral

$$\iiint_E x^2 + y^2 \, dV$$

EXAMPLE 4 Evaluate $\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} (x^2 + y^2)\, dz\, dy\, dx$.