Math 644

Chapter 2

Section 2.3: Power Series

Contents

Created by: Pierre-Olivier Parisé

Spring 2023

A First Example

More complicated functions are found by taking limits of polynomials.

Example 1. Study the series $\sum_{n=0}^{\infty} z^n$ for a fixed $z \in \mathbb{C}$.

Definition

A formal power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots$$

is called a **convergent power series centered (or based) at $\mathbf{z_0}$** if

Convention: $f(z_0) = a_0$ (to avoid 0^0).

Example 2. Find a convergent power series centered at $z_0 \neq a$ representing $\frac{1}{z-a}$.

Theorem 3. Let $r > 0$ and suppose that

a) $|a_n(z - z_0)^n| \leq M_n$ for every z such that $|z - z_0| < r$;

b) $\sum_{n=0}^{\infty} M_n < \infty$.

Then $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges uniformly and absolutely in the region

$$\{z : |z - z_0| < r\}.$$

Note:

Convergence depends only on the tail of the series. So it is sufficient to satisfy the hypothesis only for $n \geq n_0$, for some non-negative integer n_0.

Theorem 4. Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a formal power series. Let

$$R := \liminf_{n\to\infty} |a_n|^{-1/n} = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}} \in [0,\infty].$$

Then, the power series

a) converges abs. in $\{z : |z-z_0| < R\}$;

b) converges uniformly in $\{z : |z-z_0| \leq r\}$, for any $r < R$;

c) diverges in $\{z : |z-z_0| > R\}$.

Notes:

- R is called the radius of convergence of the power series.
- Biggest open disk where the power series converges is $\{z : |z-z_0| < R\}$.
- Information on the decay rate of a_n: for any $S < R$, there is an $n_0 \geq 0$ such that $|a_n| \leq S^{-n}$.

Proof.

<u>Note:</u> Root test does not give any information on the convergence of the power series on the circle

$$\{z : |z - z_0| = R\}.$$

Example 5. Find the Radius of convergence R of the following power series and study their behavior on the boundary of the disk of radius R.

A) $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n}$;

B) $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n^2}$;

C) $\displaystyle\sum_{n=1}^{\infty} nz^n$;

D) $\displaystyle\sum_{n=1}^{\infty} 2^{n^2} z^n$.

Example 6. Let $(a_n)_{n=0}^{\infty}$ be defined by

$$a_n = \begin{cases} 3^{-n} & \text{, if } n \text{ is even} \\ 4^n & \text{, if } n \text{ is odd.} \end{cases}$$

What is the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$.