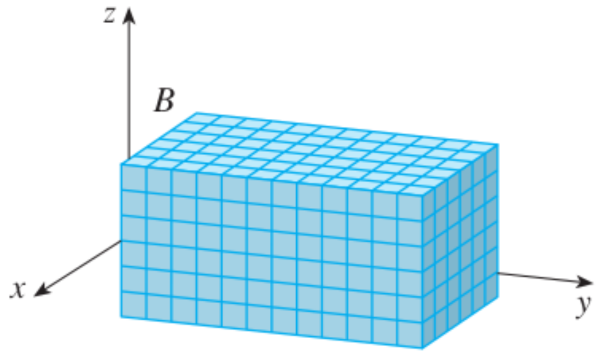
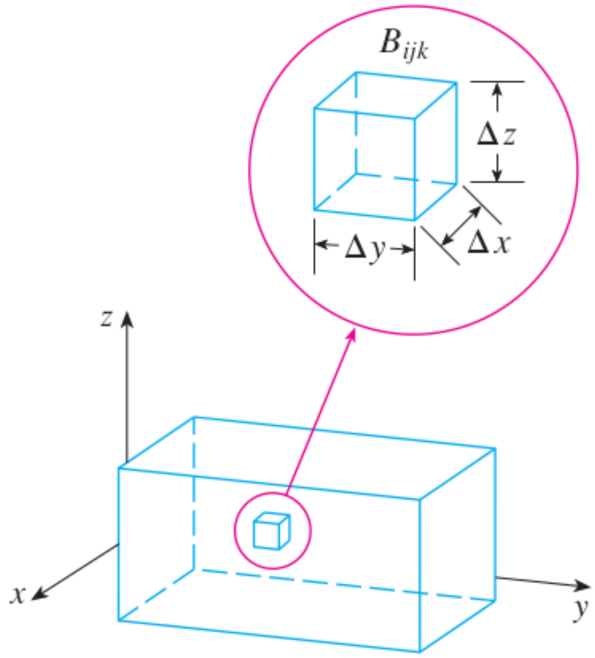
15.6 Triple Integrals.





3 Definition The **triple integral** of f over the box B is

$$\iiint_B f(x, y, z)\, dV = \lim_{l, m, n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\, \Delta V$$

if this limit exists.

4 Fubini's Theorem for Triple Integrals If f is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint_B f(x, y, z)\, dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz$$

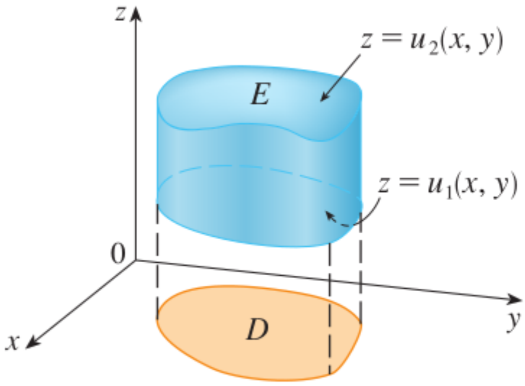
EXAMPLE 1 Evaluate the triple integral $\iiint_B xyz^2\, dV$, where B is the rectangular box given by

$$B = \{(x, y, z) \mid 0 \leqslant x \leqslant 1,\ -1 \leqslant y \leqslant 2,\ 0 \leqslant z \leqslant 3\}$$

General Domains.

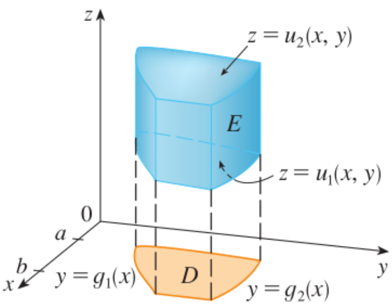
$$\iiint_E f(x, y, z)\, dV = \iiint_B F(x, y, z)\, dV$$

Domain of type 1.



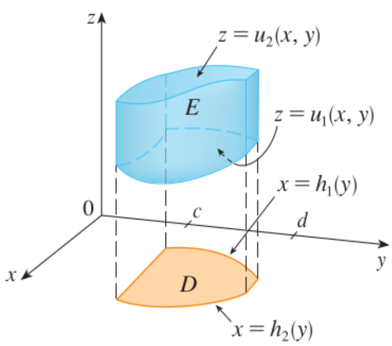
$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz \right] dA$$

Projection D is of Type I.



$$\iiint_E f(x, y, z)\, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz\, dy\, dx$$

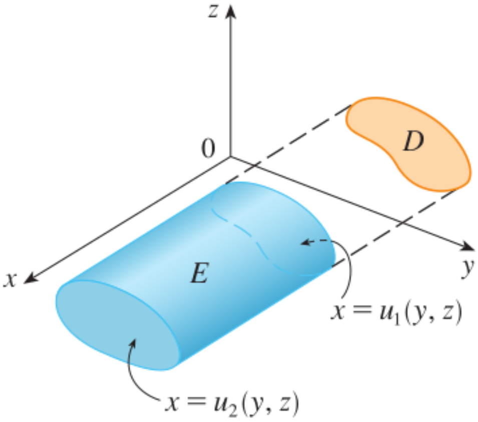
Projection D is of Type II.



$$\iiint_E f(x, y, z)\, dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz\, dx\, dy$$

EXAMPLE 2 Evaluate $\iiint_E z\, dV$, where E is the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

Domains of type 2.



$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z)\, dx \right] dA$$

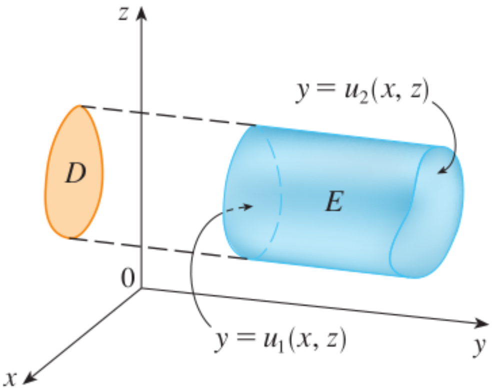
Projection D is of Type I.

$$\iiint_E f(x,y,z)\, dV = \int_a^b \int_{z_1(y)}^{z_2(y)} \int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\, dx\, dz\, dy$$

Projection D is of Type II.

$$\iiint_E f(x,y,z)\, dV = \int_a^b \int_{y_1(z)}^{y_2(z)} \int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\, dx\, dy\, dz$$

Domains of type 3.



$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z)\, dy \right] dA$$

Projection D is of type I.

$$\iiint_E f(x,y,z)\, dV = \int_a^b \int_{z_1(x)}^{z_2(x)} \int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\, dy\, dz\, dx$$

Projection D is of type II.

$$\iiint_E f(x,y,z)\, dV = \int_a^b \int_{x_1(z)}^{x_2(z)} \int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\, dy\, dx\, dz$$

EXAMPLE 3 Evaluate $\iiint_E \sqrt{x^2 + z^2}\, dV$, where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.

EXAMPLE 4 Express the iterated integral $\int_0^1 \int_0^{x^2} \int_0^y f(x, y, z)\, dz\, dy\, dx$ as a triple integral and then rewrite it as an iterated integral in a different order, integrating first with respect to x, then z, and then y.

Application: computing volumes of solids.

$$\boxed{V(E) = \iiint_E dV}$$

EXAMPLE 5 Use a triple integral to find the volume of the tetrahedron T bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.

Other applications.

Mass.

Moments.

Moments of Inertia.

EXAMPLE 6 Find the center of mass of a solid of constant density that is bounded by the parabolic cylinder $x = y^2$ and the planes $x = z$, $z = 0$, and $x = 1$.