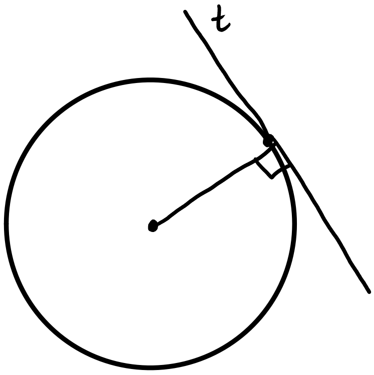
Chapter 1

Functions and Limits

1.4 The Tangent and Velocity Problems

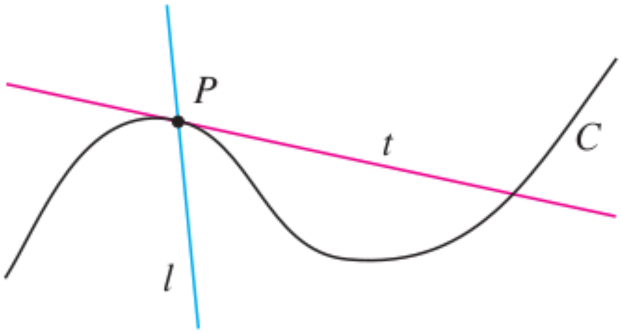
The Tangent problem.

Def. In geometry, a tangent line at a given point on a curve is a line that brushes against the curve.



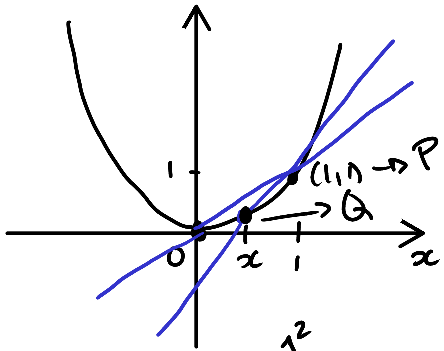
Tangent to a circle

https://www.desmos.com/calculator/itwxbbdwoe



What is the tangent line?

EXAMPLE 1 Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$. https://www.desmos.com/calculator/5eyhh9tfkg



Secant: line that passes through two points of the graph.

slope: $\dfrac{y_2 - y_1}{x_2 - x_1}$: (x_1, y_1), (x_2, y_2) are points on the graph.

m_{PQ} slope of secant passing through P & Q $= \dfrac{\overset{y_2 = 1^2}{1} - \overset{y_1}{x^2}}{\underset{x_2}{1} - \underset{x_1}{x}}$

①

x	m_{PQ}
0	1
0.5	1/2
0.9	1.9
0.999	1.999
↓	↓
1	2

From ①, we see that the slope of the secant line is

$m = 2$

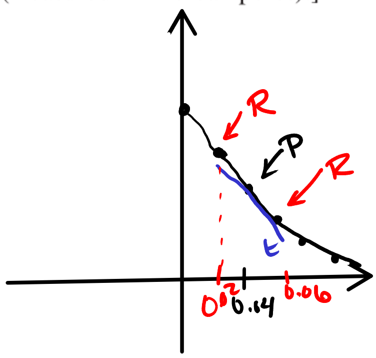
② $y = 2x + b \xrightarrow{(1,1)} 1 = 2 + b \rightarrow b = -1$

so, $\boxed{y = 2x - 1}$

EXAMPLE 2 The flash unit on a camera operates by storing charge on a capacitor and releasing it suddenly when the flash is set off. The data in the table describe the charge Q remaining on the capacitor (measured in microcoulombs) at time t (measured in seconds after the flash goes off). ~~Use the data to draw the graph of this function and~~ estimate the slope of the tangent line at the point where $t = 0.04$. [*Note:* The slope of the tangent line represents the electric current flowing from the capacitor to the flash bulb (measured in microamperes).]

t	Q
0.00	100.00
0.02	81.87
0.04	67.03
0.06	54.88
0.08	44.93
0.10	36.76

m: slope of tangent line.



(1) $P(0.04, 67.03)$
$R(0.02, 81.87)$

$$m_{PR} = \frac{67.03 - 81.87}{0.04 - 0.02} = -742.00$$

(2) $P(0.04, 67.03)$
$R(0.06, 54.88)$

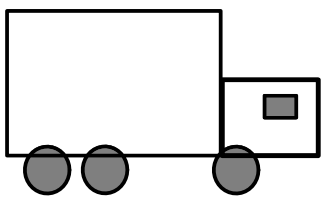
$$m_{PR} = \frac{67.03 - 54.88}{0.04 - 0.06} = -607.50$$

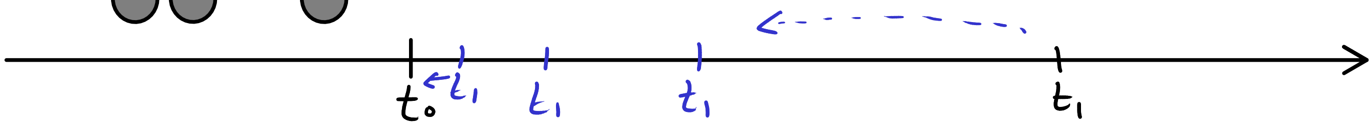
$$m \approx \frac{-742.00 - 607.50}{2} = -674.75 \text{ microC./sec.}$$

The Velocity Problem.

$s(t)$ = position function

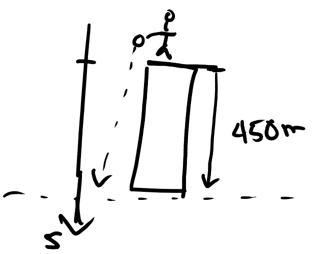
$$V_{ave} = \frac{s(t_1) - s(t_0)}{t_1 - t_0} \quad (\text{average velocity on } [t_0, t_1])$$





EXAMPLE 3 Suppose that a ball is dropped from the upper observation deck of the CN Tower in Toronto, 450 m above the ground. Find the velocity of the ball after 5 seconds.

Galileo: $s(t) = 4.9t^2$



$t_0 = 5$ sec.

and $t_1 > 5$

Then $$v_{ave} = \frac{s(t_1) - s(5)}{t_1 - 5}$$

t_1	v_{ave}
6	53.9
5.1	49.49
5.01	49.049
5.001	49.0049
↓	↓
5	49

average on $[5,6]$ →

→ $\frac{s(6) - s(5)}{6 - 5}$

$\boxed{v = 49 \text{ m/s}}$

↓
inst. velocity

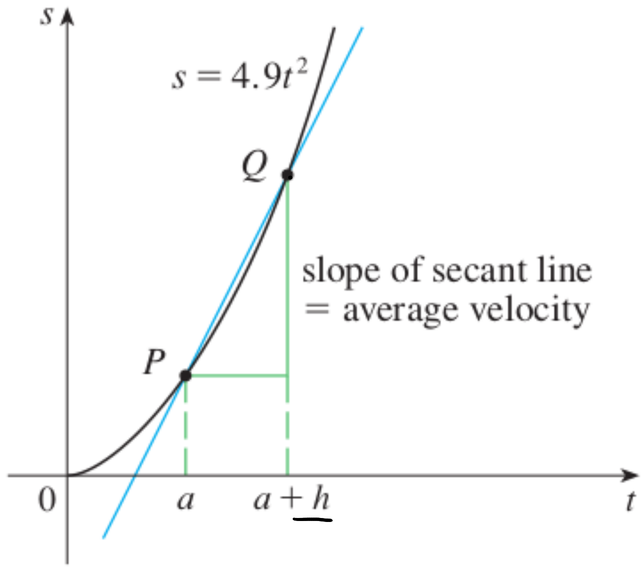
Average velocity.

$P(a, s(a))$
$Q(a+h, s(a+h))$

$$\underline{v_{ave}} = \frac{s(a+h) - s(a)}{a+h-a}$$
$$= \frac{s(a+h) - s(a)}{h}$$

$\underline{Ex}$: $5.001 \rightarrow h = 0.001$
$a = 5$
$a + h = 5.001$

Relation to the secant line.



Instantaneous Velocity.

$$v = \lim_{h \to 0} \underbrace{\frac{s(a+h) - s(a)}{h}}_{v_{ave}.}$$

Relation to the tangent line.