B.I Conditional Probabilities

PROBLEM 1. Let $(S, \mathcal{A}, P)$ be a probability space. Suppose two events A and B are given such that $P(A) = 0.5$, $P(B) = 0.3$, and $P(A \cap B) = 0.1$. Find

a) $P(A|B)$. b) $P(A|A \cup B)$. c) $P(A \cap B|A \cup B)$.

PROBLEM 2. A balanced die is tossed once. What is the probability the die lands on a 1, given that an odd number was obtained?

PROBLEM 3. Two fair dice are rolled. What is the probability that at least one lands on 6 given that the dice land on different numbers?

PROBLEM 4. Let $(S, \mathcal{A}, P)$ be a probability space. Suppose that two events A and B are given such that $P(A) > 0$, $P(B) > 0$. Prove that if $P(A) < P(A|B)$, then $P(B) < P(B|A)$.

PROBLEM 5. Suppose that $A \subset B$ and that $P(A) > 0$ and $P(B) > 0$. Show that $P(B|A) = 1$ and $P(A|B) = P(A)/P(B)$.

PROBLEM 6. If A and B are mutually exclusive events and $P(B) > 0$, show that

$$P(A|A \cup B) = \frac{P(A)}{P(A) + P(B)}.$$

B.II Bayes' Formula

PROBLEM 7. A laboratory blood test is 95% effective in detecting a certain disease when it is, in fact, present. However, the test also yields a "false positive" result for 1% of the healthy persons tested[1]. If 0.5% of the population actually has the disease, what is the probability a person has the disease given that the test result is positive?

PROBLEM 8. A total of 46% of the voters in a certain city classify themselves as Independents, whereas 30% classify themselves as Liberals and 24% as Conservative. In a recent local election, 35% of the Independents, 62% of the Liberals, and 58% of the Conservatives voted. A voter is chosen at random. Given that this person voted in the local election, what is the probability that the person is a) an Independent? b) a Liberal? c) a Conservative?

PROBLEM 9. When a dice x is tossed it lands on ⚁ with probability $1/2$ and all the other outcomes are equally likely to happen. When a dice y is tossed, it lands on ⚂ with probability $1/2$ and all the other outcomes are equally likely to happen. Suppose that one of these dice is randomly chosen and then tossed. What is the probability that dice x was tossed, if the die landed on ⚁?

[1] That is, if a healthy person is tested, then, with probability 0.01, the test result will imply the person has the disease.

Problem 10. Let $(S, \mathcal{A}, P)$ be a probability space. If A, B are events, then show that

$$\frac{P(A|B)}{P(\overline{A}|B)} = \frac{P(A)}{P(\overline{A})}\frac{P(B|A)}{P(B|\overline{A})}.$$

B.III Independent Events

Problem 11. Three brands of coffee, x, y, and z, are to be ranked according to taste by a judge. Define the following events. A: "Brand x is preferred to y, B: "Brand x is ranked best", C: "Brand x is ranked second best" and D: "Brand x is ranked third best". If the judge actually has no taste preference and randomly assigns ranks to the brands, is event A independent of (a) event B? (b) event C? (c) event D?

Problem 12. Cards are dealt, one at a time, from a standard 52-card deck. If A_i denotes the event "the i-th card dealt is a spade". Are A_1 and A_2 independent?

Problem 13. A system composed of 5 separate components is said to be a parallel system if it functions when at least one of the components functions. For such a system, if component i, independent of other components, functions with probability p_i, $i = 1, 2, \ldots, 5$, what is the probability that the system functions?

Problem 14. Let $(S, \mathcal{A}, P)$ be a probability space. Prove that if A and B are independent events with $0 < P(A), P(B) < 1$, then so are A and $\overline{B}$. Are $\overline{A}$ and $\overline{B}$ independent?