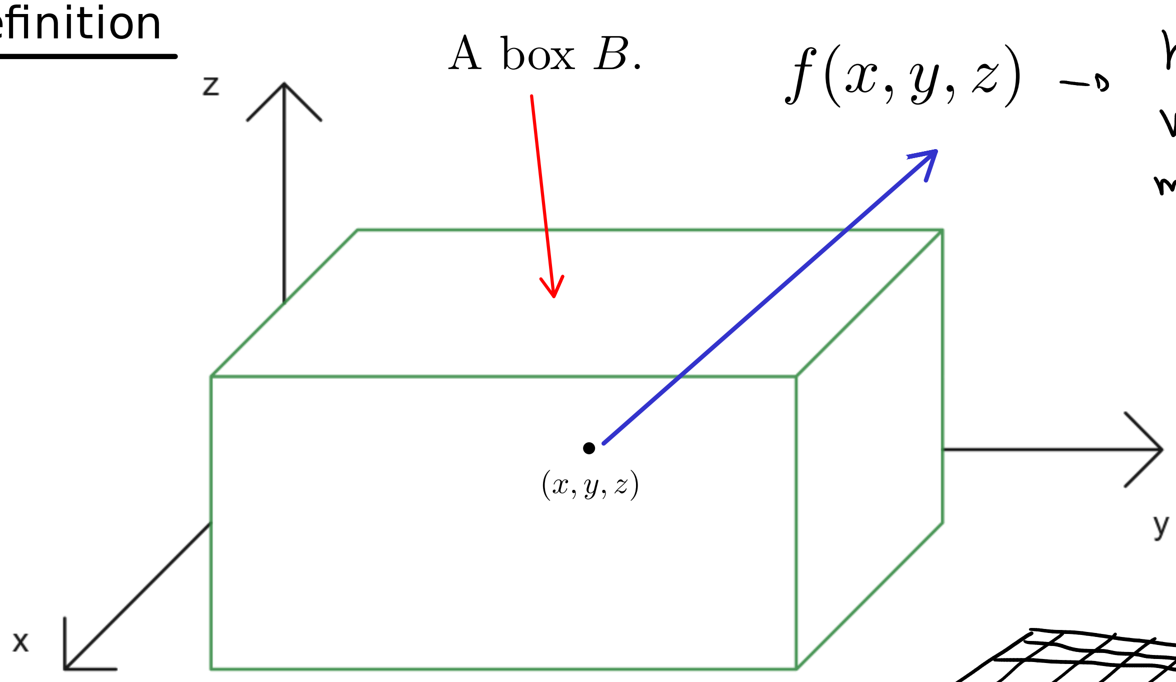
Chapter 15

Multiple Integrals

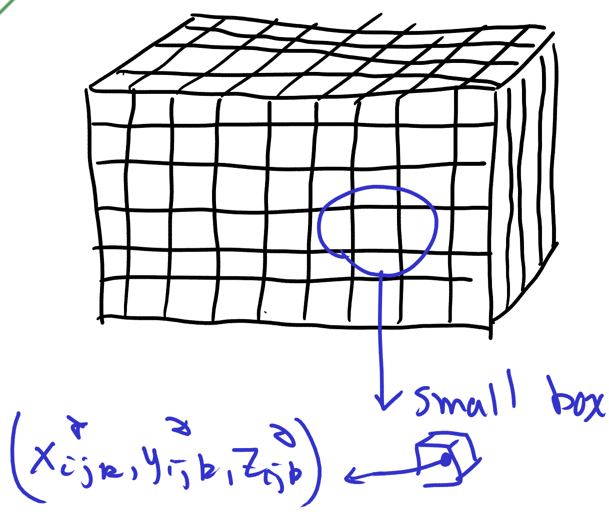
15.6 Triple integrals

Definition



$f(x, y, z) \to$ heat
volume
mass density
$\vdots$

- A box $B = [a, b] \times [c, d] \times [r, s]$
- Divide [a, b] in l parts
- Divide [c, d] in m parts
- Divide [r, s] in n parts



$$\text{Vol}(\square) = \Delta V$$

Heat in a small box

$$\approx f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \cdot \Delta V$$

Total heat in box

$$\approx \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \cdot \Delta V$$

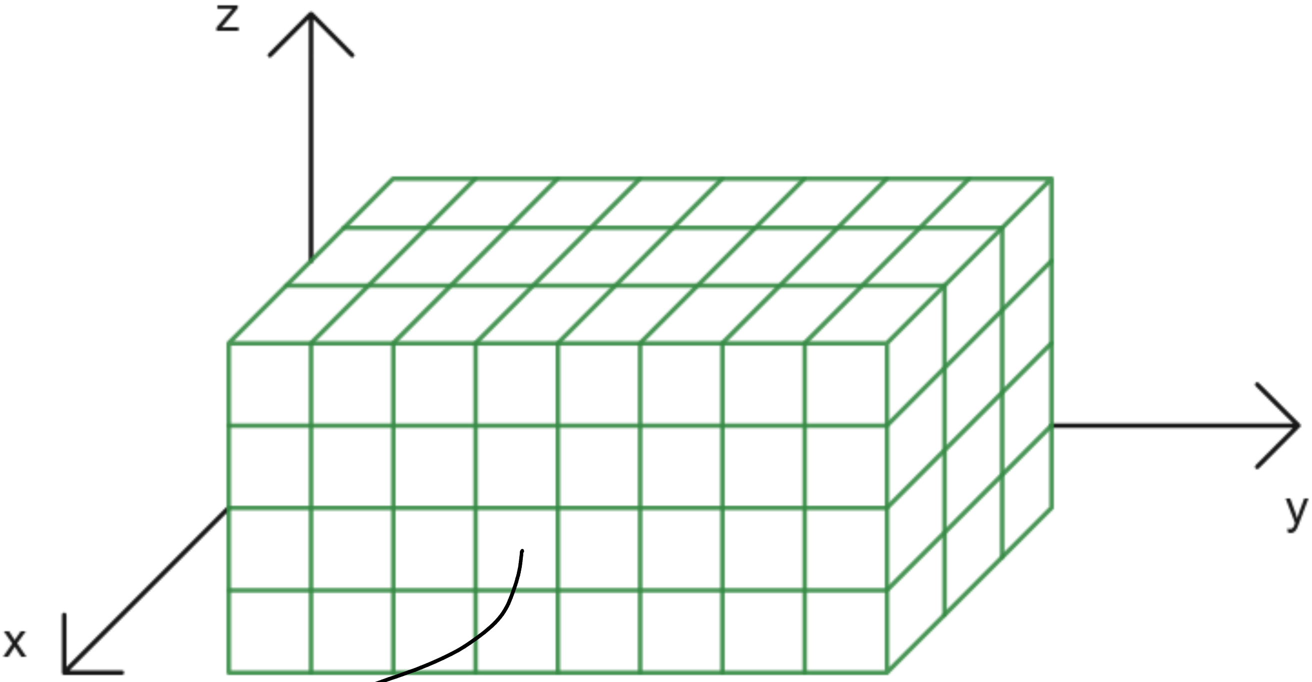
Take the limit as $l, m, n \longrightarrow \infty$

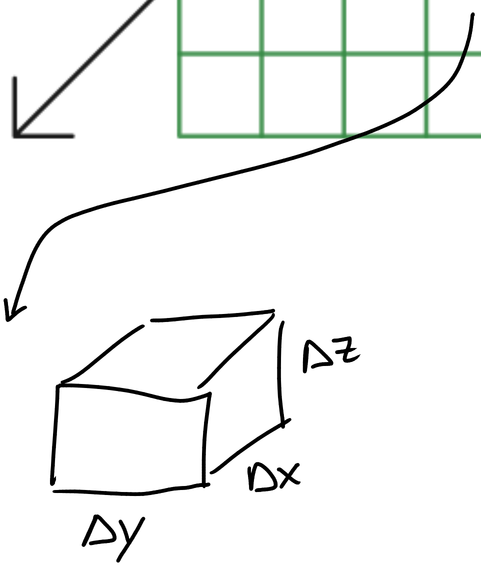
The triple integral of f over the box B is

$$\boxed{\iiint_B f(x, y, z)\, dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \Delta V}$$

Triple integrals in cartesian coordinates

- Write explicitly $B = \{(x, y, z) : a \le x \le b,\, c \le y \le d,\, r \le z \le s\}$
- Divide $[a, b]$ in parts of length Δx.
- Divide $[c, d]$ in parts of length Δy.
- Divide $[r, s]$ in parts of length Δz.





$\Delta V = \Delta x \Delta y \Delta z$

As the # of parts $\to \infty$

$dV = dx\,dy\,dz$

Fubini's Theorem for triple integrals

$$\boxed{\iiint_B f(x, y, z)\, dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\, dx dy dz}$$

the function must be continuous on the box B.

EXAMPLE 1 Evaluate the triple integral $\iiint_B xyz^2\, dV$, where B is the rectangular box given by

$$B = \{(x, y, z) \mid 0 \leqslant x \leqslant 1,\ -1 \leqslant y \leqslant 2,\ 0 \leqslant z \leqslant 3\}$$

$$\iiint_B xyz^2\, dV = \int_0^3 \int_{-1}^2 \int_0^1 xyz^2\, dx\, dy\, dz .$$

$$= \int_0^3 \int_{-1}^2 \frac{x^2}{2}\Big|_0^1 yz^2\, dy\, dz$$

$$= \int_0^3 \int_{-1}^2 \frac{1}{2} yz^2\, dy\, dz$$

$$= \frac{1}{2} \int_0^3 \int_{-1}^2 yz^2\, dy\, dz$$

$$= \frac{1}{2} \int_0^3 \frac{y^2}{2}\Big|_{-1}^2 z^2\, dz$$

$$= \frac{3}{4} \int_0^3 z^2\, dz = \frac{3}{4} \frac{z^3}{3}\Big|_0^3 = \boxed{\frac{27}{4}}$$

Comment: $\left(\int_0^1 x\, dx\right)\left(\int_{-1}^2 y\, dy\right)\left(\int_0^3 z^2\, dz\right)$ is also another to calculate the integral.

QUESTION. What are the 5 other configurations of dx, dy, dz in a triple integral?

1. $dV = dz\, dy\, dx$
2. $dV = dz\, dx\, dy$
3. $dV = dy\, dx\, dz$
4. $dV = dy\, dz\, dx$
5. $dV = dx\, dz\, dy$

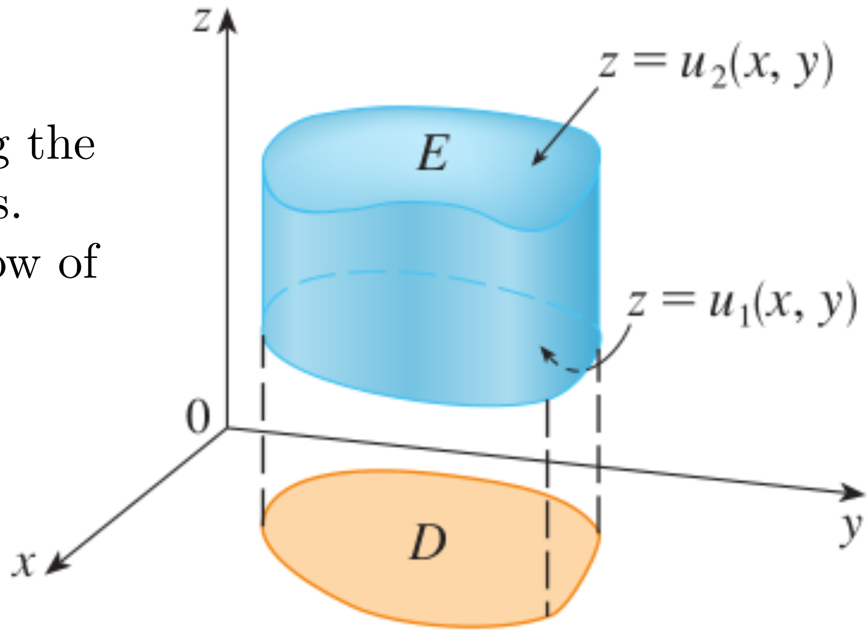
General Domains.

For E a general solid, let B be a box containing E.
Define a function F on B :

$$F(x,y,z) = \begin{cases} f(x,y,z) & \text{if } (x,y,z) \in E \\ 0 & \text{if } (x,y,z) \in B \backslash E. \end{cases}$$

$$\boxed{\iiint_E f(x,y,z)\,dV = \iiint_B F(x,y,z)\,dV}$$

Domain of type 1.

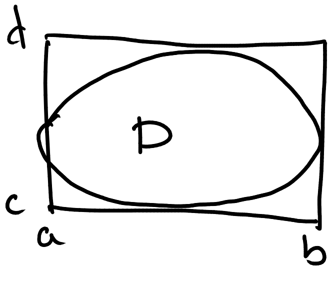


- Solid E is bounded along the $z-$axis by two functions.
- Define D to be the shadow of E in the $xy-$plane.
- The domain D can be of type I or type II.

$$\iiint_E f(x,y,z)\,dV = \iiint_B F(x,y,z)\,dV$$

$$= \int_a^b \int_c^d \int_r^s F(x,y,z)\,dz\,dy\,dx$$

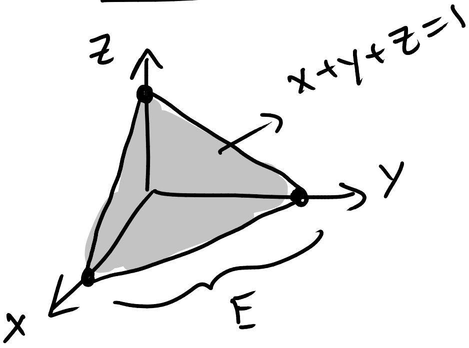
$$= \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right) dA$$



$$\boxed{\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right] dA}$$

EXAMPLE 2 Evaluate $\iiint_E z\, dV$, where E is the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

① Picture



Describe E as a typ 1:

z-values

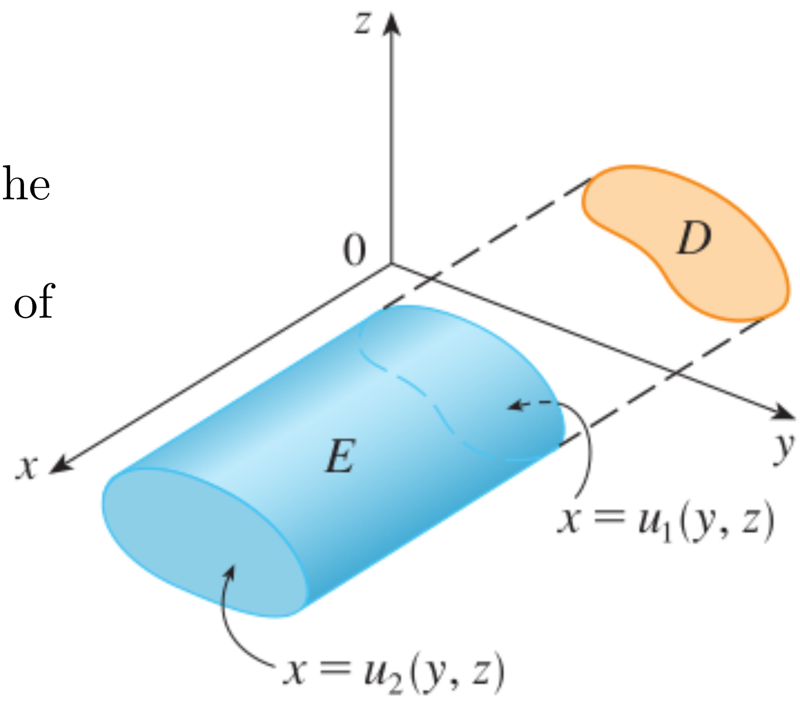
$z = 0$ (lower surface)

$z = 1 - x - y$

$\Rightarrow \quad 0 \le z \le 1 - x - y$.

Domains of type 2.

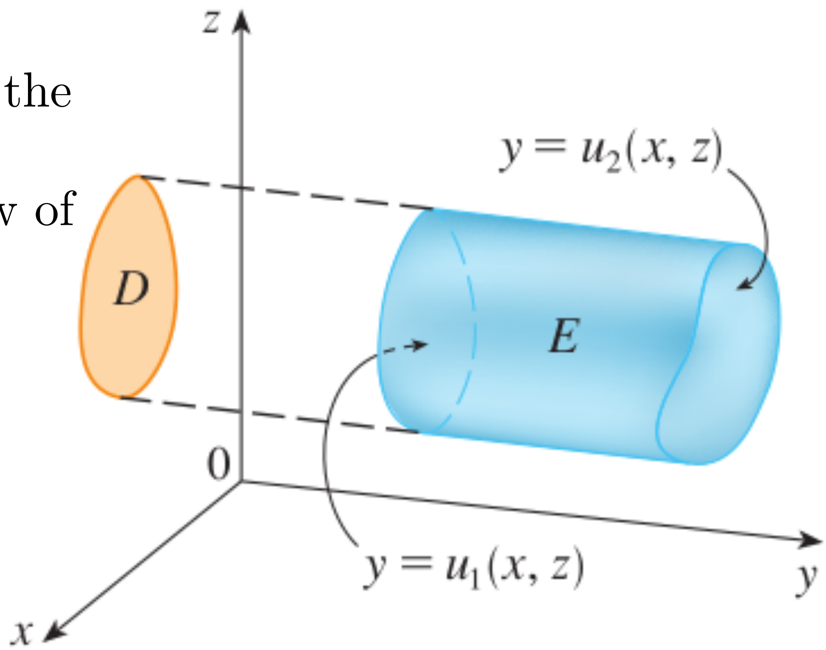
- Solid E is bounded along the $x-$axis by two functions.
- Define D to be the shadow of E in the $yz-$plane.
- The domain D can be of type I or type II.



$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx\right] dA$$

Domains of type 3.

- Solid E is bounded along the $y-$axis by two functions.
- Define D to be the shadow of E in the $xz-$plane.
- The domain D can be of type I or type II.



$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy\right] dA$$

EXAMPLE 3 Evaluate $\iiint_E \sqrt{x^2 + z^2}\, dV$, where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.

EXAMPLE 4 Express the iterated integral $\int_0^1 \int_0^{x^2} \int_0^y f(x, y, z)\, dz\, dy\, dx$ as a triple integral and then rewrite it as an iterated integral in a different order, integrating first with respect to x, then z, and then y.

Application: computing volumes of solids.

$$\boxed{\mathrm{Vol}(E) = \iiint_E dV}$$

EXAMPLE. Use a triple integral to find the volume of the tetrahedron T bounded by the planes $x + 2y + z = 2, x = 2y, x = 0,$ and $z = 0.$