

Chapter 1

Functions and Limits

1.5 The Limit of a Function

Intuitive definition of a limit.

1 Intuitive Definition of a Limit Suppose $f(x)$ is defined when x is near the number a . (This means that f is defined on some open interval that contains a , except possibly at a itself.) Then we write

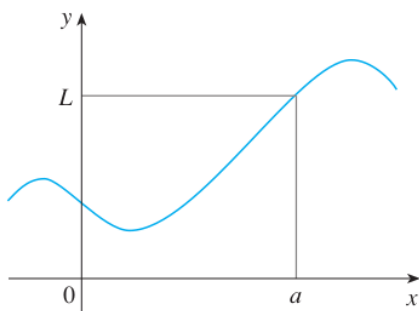
$$\lim_{x \rightarrow a} f(x) = L$$

and say “the limit of $f(x)$, as x approaches a , equals L ”

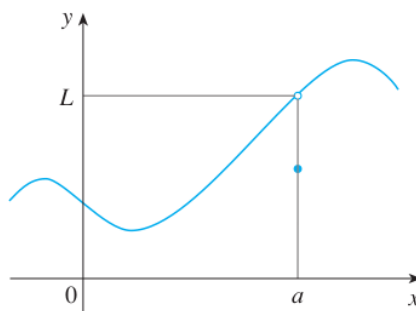
if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by restricting x to be sufficiently close to a (on either side of a) but not equal to a .

Notations:

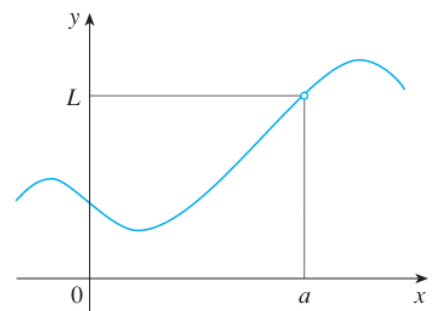
Three cases:



(a)



(b)



(c)

EXAMPLE 1 Guess the value of $\lim_{x \rightarrow 1} \frac{x-1}{x^2-1}$.

EXAMPLE 3 Guess the value of $\lim_{x \rightarrow 0} \frac{\sin x}{x}$.

EXAMPLE 4 Investigate $\lim_{x \rightarrow 0} \sin \frac{\pi}{x}$.

EXAMPLE 6 The Heaviside function H is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

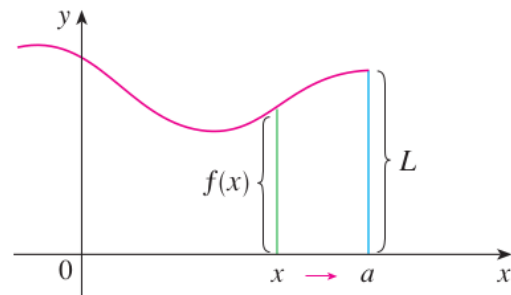
What is the limit when t approached 0 from the right and when t approaches 0 from the left.

Left-hand limits.

2 Definition of One-Sided Limits We write

$$\lim_{x \rightarrow a^-} f(x) = L$$

and say the **left-hand limit of $f(x)$ as x approaches a** [or the **limit of $f(x)$ as x approaches a from the left**] is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a with x less than a .



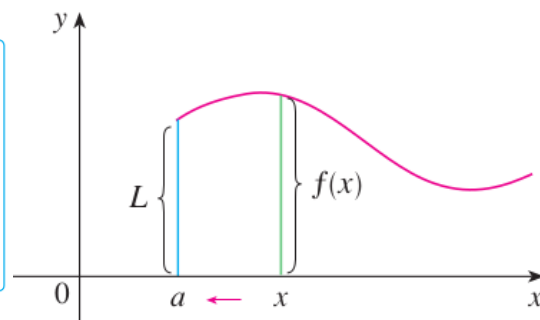
$$(a) \lim_{x \rightarrow a^-} f(x) = L$$

Right-hand limits.

2 Definition of One-Sided Limits We write

$$\lim_{x \rightarrow a^+} f(x) = L$$

and say the **right-hand limit of $f(x)$ as x approaches a** [or the **limit of $f(x)$ as x approaches a from the right**] is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a with x greater than a .



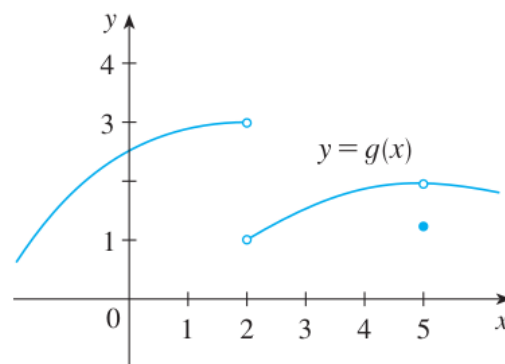
$$(b) \lim_{x \rightarrow a^+} f(x) = L$$

Fundamental Property:

$$\boxed{3} \quad \lim_{x \rightarrow a} f(x) = L \quad \text{if and only if} \quad \lim_{x \rightarrow a^-} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow a^+} f(x) = L$$

EXAMPLE 7 The graph of a function g is shown in Figure 10. Use it to state the values (if they exist) of the following:

- (a) $\lim_{x \rightarrow 2^-} g(x)$ (b) $\lim_{x \rightarrow 2^+} g(x)$ (c) $\lim_{x \rightarrow 2} g(x)$
 (d) $\lim_{x \rightarrow 5^-} g(x)$ (e) $\lim_{x \rightarrow 5^+} g(x)$ (f) $\lim_{x \rightarrow 5} g(x)$



Infinite limits.

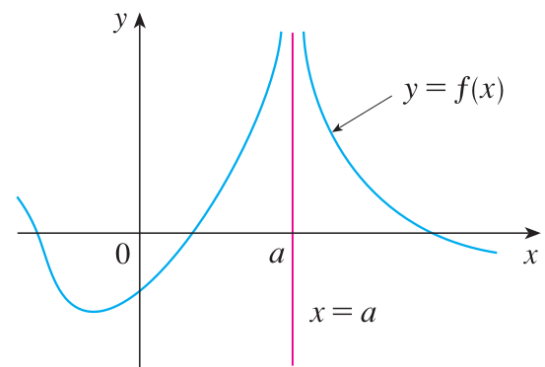
EXAMPLE 8 Find $\lim_{x \rightarrow 0} \frac{1}{x^2}$ if it exists.

Positive infinity.

4 Intuitive Definition of an Infinite Limit Let f be a function defined on both sides of a , except possibly at a itself. Then

$$\lim_{x \rightarrow a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a , but not equal to a .

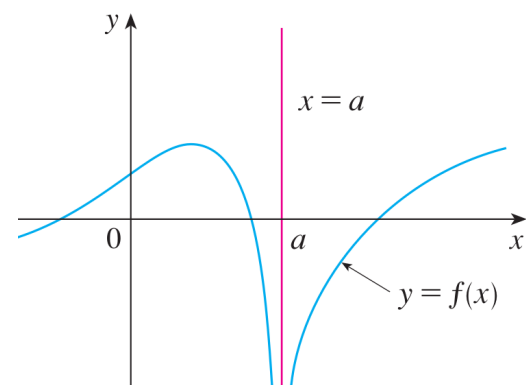


Negative Infinity

5 Definition Let f be a function defined on both sides of a , except possibly at a itself. Then

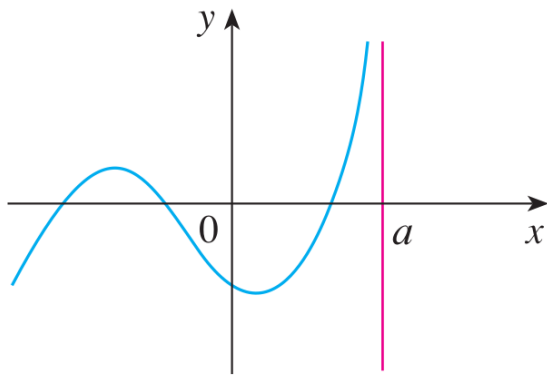
$$\lim_{x \rightarrow a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a , but not equal to a .

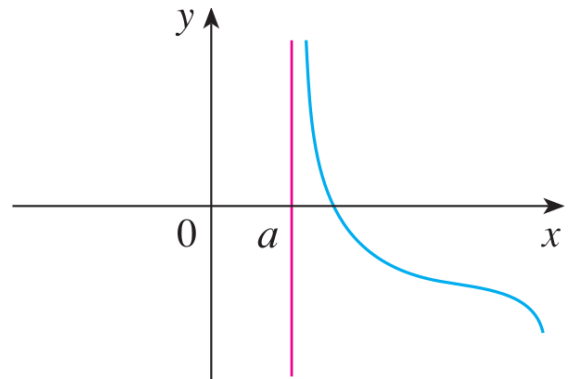


Other types of infinite limits.

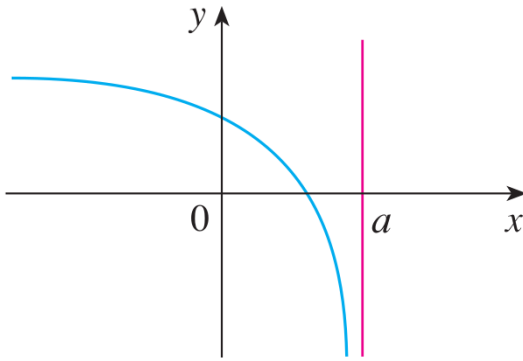
a)



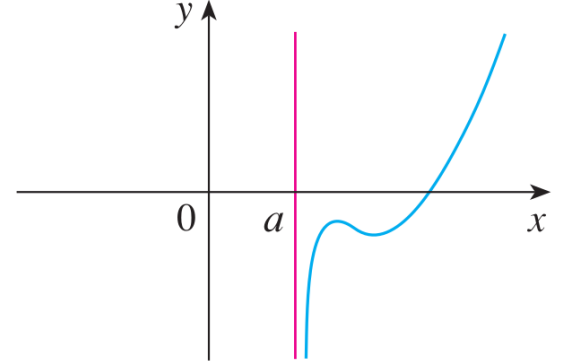
b)



c)



d)



EXAMPLE 9 Find $\lim_{x \rightarrow 3^+} \frac{2x}{x-3}$ and $\lim_{x \rightarrow 3^-} \frac{2x}{x-3}$.

6 Definition The vertical line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x \rightarrow a} f(x) = \infty$$

$$\lim_{x \rightarrow a^-} f(x) = \infty$$

$$\lim_{x \rightarrow a^+} f(x) = \infty$$

$$\lim_{x \rightarrow a} f(x) = -\infty$$

$$\lim_{x \rightarrow a^-} f(x) = -\infty$$

$$\lim_{x \rightarrow a^+} f(x) = -\infty$$

EXAMPLE 10 Find the vertical asymptotes of $f(x) = \tan x$.