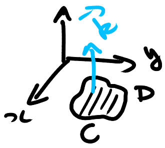
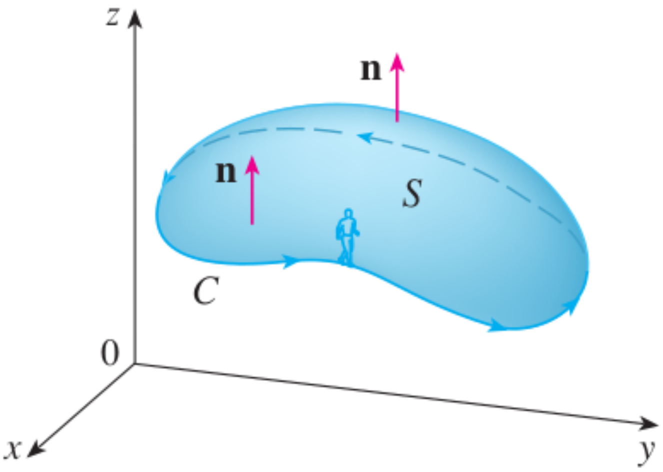
16.8 Stokes' Theorem.

Another story of orientation.



$$\iint_D \operatorname{curl} \vec{F} \cdot \vec{k} \, dA = \int_C \vec{F} \cdot d\vec{r}$$



S: surface with unit normal $\vec{n}$ pointing outward (positive orient.)

C: S induces the positive orientation on C, the boundary of S.

Stokes' Theorem Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\boxed{\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}}$$

Proof. See p.1175 in textbook.
check wikipedia for a more complete proof.

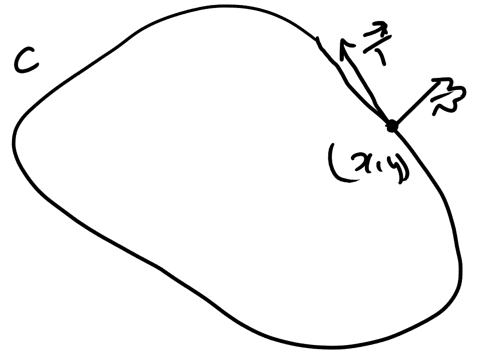
Another Notation.

∂S : C is positive orientation

$$\int_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}.$$

Green's Theorem as a special case of Stokes' Theorem.

Explanation of curl $\vec{F}$.



$\vec{T}$: unit tangent vector at (x,y)

$\vec{v}$: velocity field of a fluid.

C: curve.

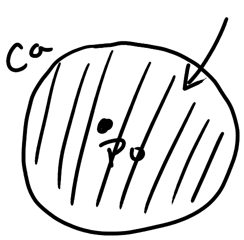
$\vec{v} \cdot \vec{T}$ represents the component of $\vec{v}$ in the $\vec{T}$ direction.

so

- $\vec{v} \cdot \vec{T} > 0 \Rightarrow \vec{v}$ points more in the $\vec{T}$ direction
- $\vec{v} \cdot \vec{T} < 0 \Rightarrow \vec{v}$ points more in the opposite $\vec{T}$ direction.

so, $\int_C \vec{v} \cdot d\vec{r} = \int_C \vec{v} \cdot \vec{T}\, ds$ measures the tendancy for a fluid to move around C.

This quantity is call <u>circulation</u>.



Sa: disk of radius a.

$$\text{curl}\,\vec{v} \approx \text{curl}\,\vec{v}(P_0) \quad \text{on } S_a$$

$$\int_{C_a} \vec{v} \cdot d\vec{r} = \iint_{S_a} \text{curl}\,\vec{F} \cdot \vec{n}\, dS$$

$$\approx \iint_{S_a} \text{curl}\,\vec{F}(P_0) \cdot \vec{n}(P_0) \cdot dS$$

$$= \text{curl}\,\vec{V}(P_0) \cdot \vec{n}(P_0)\, \pi a^2$$

$$\Rightarrow \quad \text{curl}\,\vec{V}(P_0) \cdot \vec{n}(P_0) = \lim_{a \to 0} \frac{\int_{C_a} \vec{v} \cdot d\vec{r}}{\pi a^2}$$

curl $\vec{v}(P_0)$ is tendancy rotation around P_0.

EXAMPLE 1 Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = -y^2\,\mathbf{i} + x\,\mathbf{j} + z^2\,\mathbf{k}$ and C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$. (Orient C to be counterclockwise when viewed from above.)

EXAMPLE 2 Use Stokes' Theorem to compute the integral $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + yz\,\mathbf{j} + xy\,\mathbf{k}$ and S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the xy-plane. (See Figure 4.)

Computing surface integrals when the surface is difficult.

1. A hemisphere H and a portion P of a paraboloid are shown. Suppose $\mathbf{F}$ is a vector field on $\mathbb{R}^3$ whose components have continuous partial derivatives. Explain why

$$\iint_H \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_P \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$