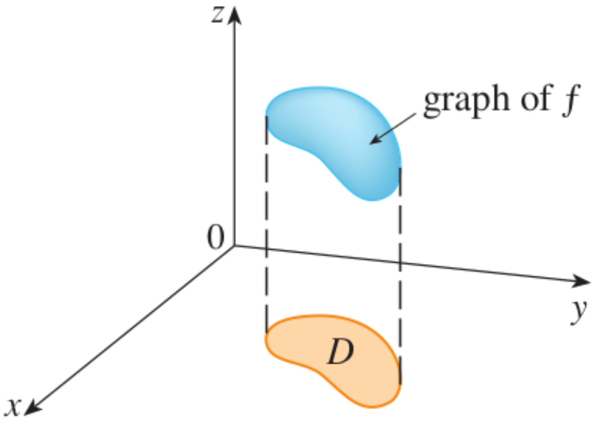
Chapter 15

Multiple Integrals

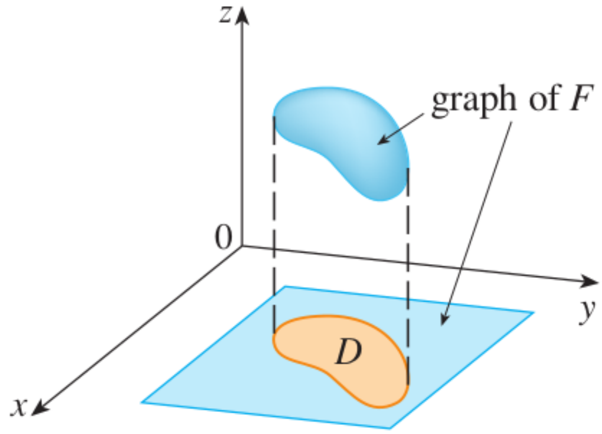
15.2 Double Integrals over genaral regions

Definition.

Given: A function f defined on D

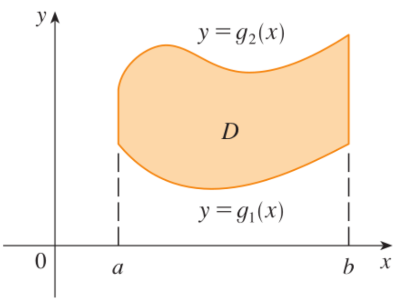


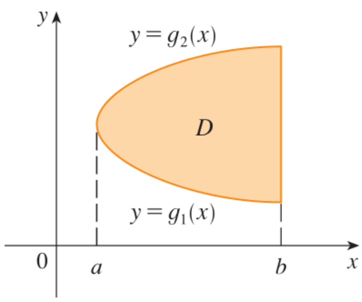
Extend f to a rectangle containing D

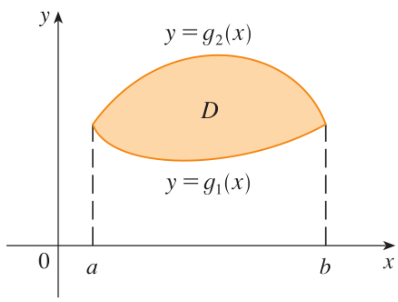


$$\boxed{\iint_D f(x, y)\, dA = \iint_R F(x, y)\, dA}$$

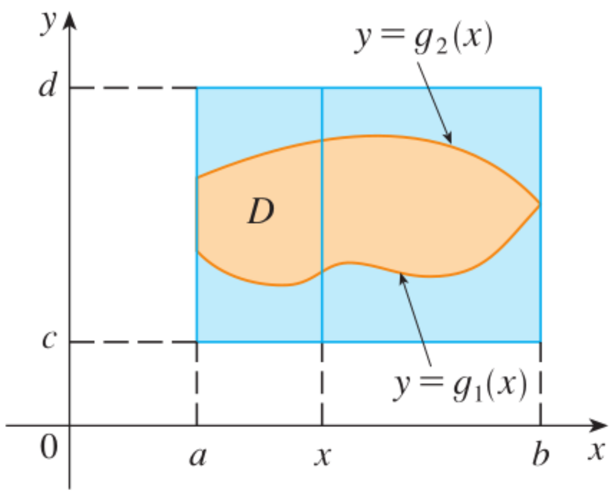
Region of type I.







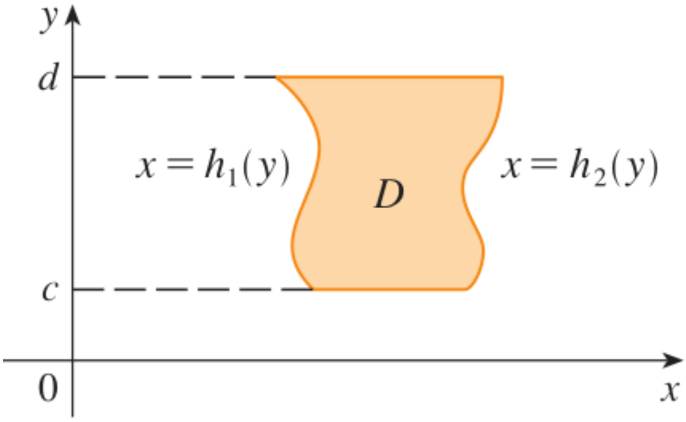
$$D = \{(x, y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$$

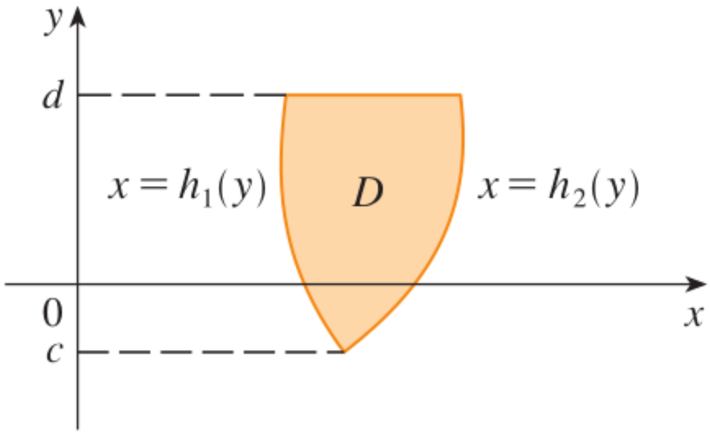


EXAMPLE 1 Evaluate $\iint_D (x + 2y)\, dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

Region of Type II.

$$D = \{(x, y) \mid c \leqslant y \leqslant d,\ h_1(y) \leqslant x \leqslant h_2(y)\}$$





$$\boxed{\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy}$$

EXAMPLE. Evaluate $\displaystyle\iint_D e^{-y^2}\, dA$, where D is the region bounded by the lines $x = 0, x = 3$ and $x = y$.

EXAMPLE. Find the volume of the tetrahedron bounded by the planes $x + 2y + z = 2$, $x = 2y$, $y = 0$, and $z = 0$.

EXAMPLE 5 Evaluate the iterated integral $\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx$.

Properties of Double Integrals.

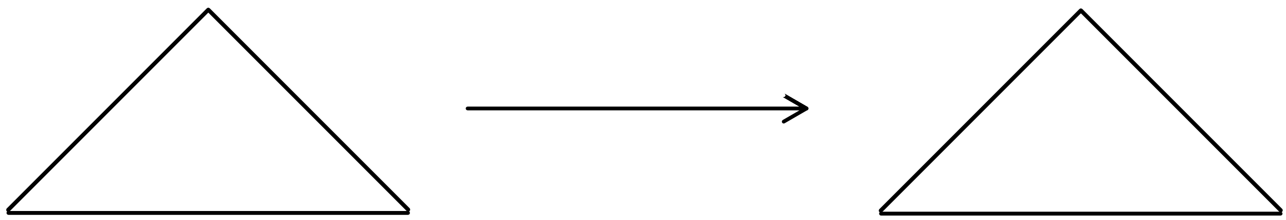
$$\boxed{6} \quad \iint_D \big(f(x,y)+g(x,y)\big)\,dA = \iint_D f(x,y)\,dA + \iint_D g(x,y)\,dA$$

$$\boxed{7} \quad \iint_D cf(x,y)\,dA = c\iint_D f(x,y)\,dA$$

$\boxed{8}$ If $f(x,y) \geq g(x,y)$ on D, then $\displaystyle\iint_D f(x,y)\,dA \geq \iint_D g(x,y)\,dA$

$\boxed{9}$ If $D = D_1 \cup D_2$, with $D_1 \cap D_2 = \emptyset$, then

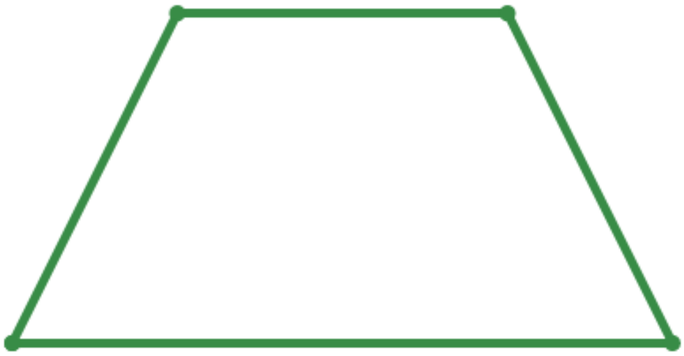
$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$$



$$\boxed{10} \quad Area(D) = \iint_D 1\,dA$$

$\boxed{11}$ If $m \leq f(x,y) \leq M$, then $\displaystyle m \cdot Area(D) \leq \iint_D f(x,y)\,dA \leq M \cdot Area(D)$

Example. Find the area of the trapezoid below:



Challenge. Find the area of the hexagone below using properties 9 and 10: