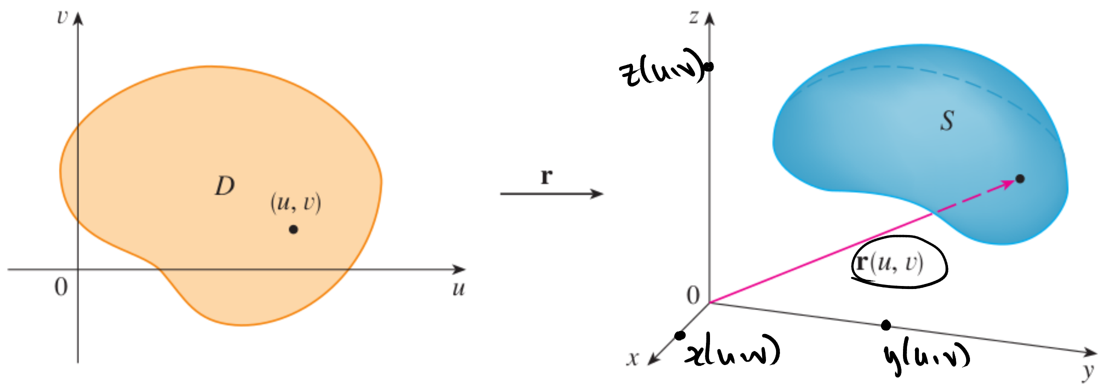
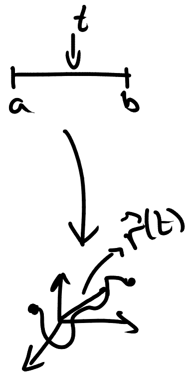
16.6 Parametric surfaces and Their Areas.





Vector expression.

Need three fcts $x, y, z : \underset{\uparrow}{D} \to \mathbb{R}$ 2D region.

$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$

$\vec{r}(u) = \langle x(u), y(u), z(u) \rangle$

Parametric equations.

Given by

$x = x(u,v)$

$y = y(u,v)$

$z = z(u,v)$.

EXAMPLE 1 Identify and sketch the surface with vector equation

$$\mathbf{r}(u, v) = 2\cos u\,\mathbf{i} + v\,\mathbf{j} + 2\sin u\,\mathbf{k}$$

$x(u,v) = 2\cos u$, $y(u,v) = v$, $z(u,v) = 2\sin u$.

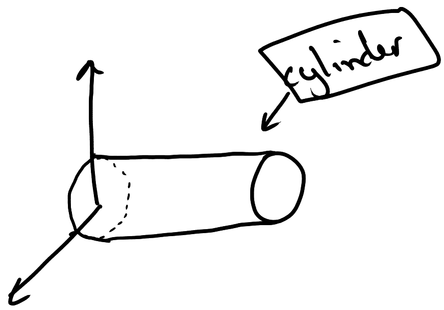
So,

$$x^2 + z^2 = 2^2\cos^2 u + 2^2 \sin^2 u = 4$$

$$\Rightarrow \quad x^2 + z^2 = 4 \quad \to \quad \text{circle.}$$

Now, $y = v$ (no restriction on v)

$\to$ translate the circle $x^2+y^2=4$ along the y-axis.

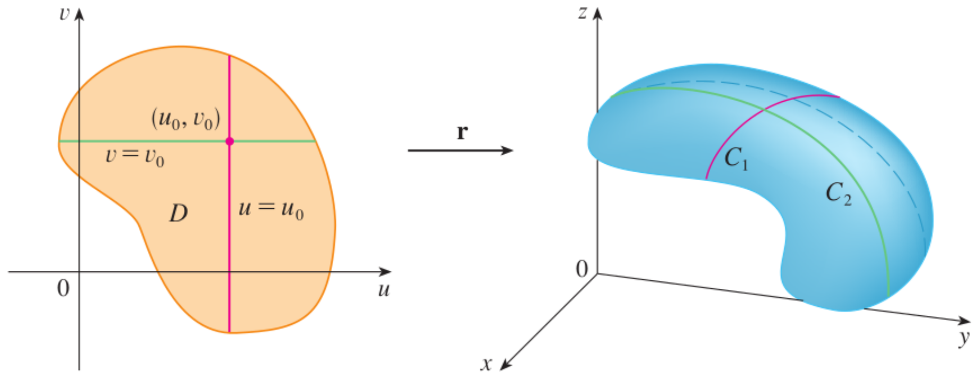


Question: What happen to the surface if we restric one of the parameter?

Fix $u = 0$, then

$\vec{r}(v) \leftarrow \vec{r}(0,v) = \langle x(0,v), y(0,v), z(0,v) \rangle$

Grid curves.



C_1: $\vec{r}(u_0, v)$
$\rightarrow$ param. of a curve.

C_2: $\vec{r}(u, v_0)$
$\rightarrow$ param. of a curve.

EXAMPLE 2 Use a computer algebra system to graph the surface

$$\mathbf{r}(u, v) = \langle (2 + \sin v)\cos u, (2 + \sin v)\sin u, u + \cos v \rangle$$

Which grid curves have u constant? Which have v constant?

Python: or software

$x = (2+\sin v)\cos u \qquad y = (2+\sin v)\sin u$

$z = u + \cos v$

Grid curves.

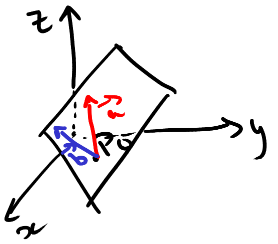
$u = 0 \rightarrow \vec{r}(0,v) = \langle 2+\sin v, 0, \cos v \rangle$
$= \langle 2, 0, 0 \rangle + \langle \sin v, 0, \cos v \rangle$

$v = 0 \rightarrow \vec{r}(u,0) = \langle 2\cos u, 2\sin u, u \rangle$

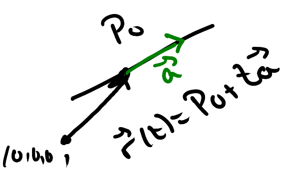
See python's script.

EXAMPLE 3 Find a vector function that represents the plane that passes through the point P_0 with position vector $\mathbf{r}_0$ and that contains two nonparallel vectors $\mathbf{a}$ and $\mathbf{b}$.

$ax + by + cz = d$
$\vec{n} = \langle a, b, c \rangle$
$\vec{n} \cdot (\langle x,y,z \rangle - P_0) = 0$



The points on the plane are obtained by moving along the direction of $\vec{a}$ and $\vec{b}$



1st) Move to P_0.
2nd) Move in the direction $\vec{a}$ &/or $\vec{b}$.

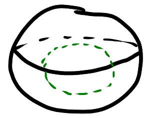
$P_0 = \vec{r}_0$.

$\rightarrow \boxed{\vec{r}(u,v) = \vec{r}_0 + u\vec{a} + v\vec{b}}$

EXAMPLE 4 Find a parametric representation of the sphere

$$x^2 + y^2 + z^2 = a^2$$

$a = 2$
$a = 3$



Recall: $x = \rho\cos\theta\sin\phi \qquad y = \rho\sin\theta\sin\phi \qquad z = \rho\cos\phi$

Fix $\rho = a$, $\quad u = \theta$ & $v = \phi$

$$\Rightarrow \quad \vec{r}(u,v) = \langle a\cos u\sin v,\ a\sin u\sin v,\ a\cos v\rangle$$

$$0 \le u \le 2\pi \qquad 0 \le v \le \pi$$

x, y as parameters

$$\vec{r}_{+}(x,y) = \langle x, y, \sqrt{a^2 - x^2 - y^2}\rangle$$

EXAMPLE 6 Find a vector function that represents the elliptic paraboloid $z = x^2 + 2y^2$.

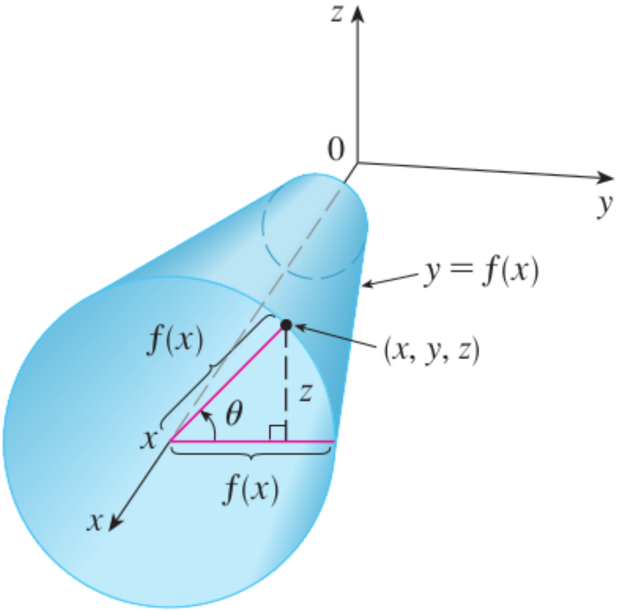
Simple sol.

$u = x$
$v = y$
$$\rightarrow \vec{r}(u,v) = \langle \underbrace{u}_{x}, \underbrace{v}_{y}, \underbrace{u^2 + 2v^2}_{z}\rangle$$

More interesting approach.

EXAMPLE 7 Find a parametric representation for the surface $z = 2\sqrt{x^2 + y^2}$, that is, the top half of the cone $z^2 = 4x^2 + 4y^2$.

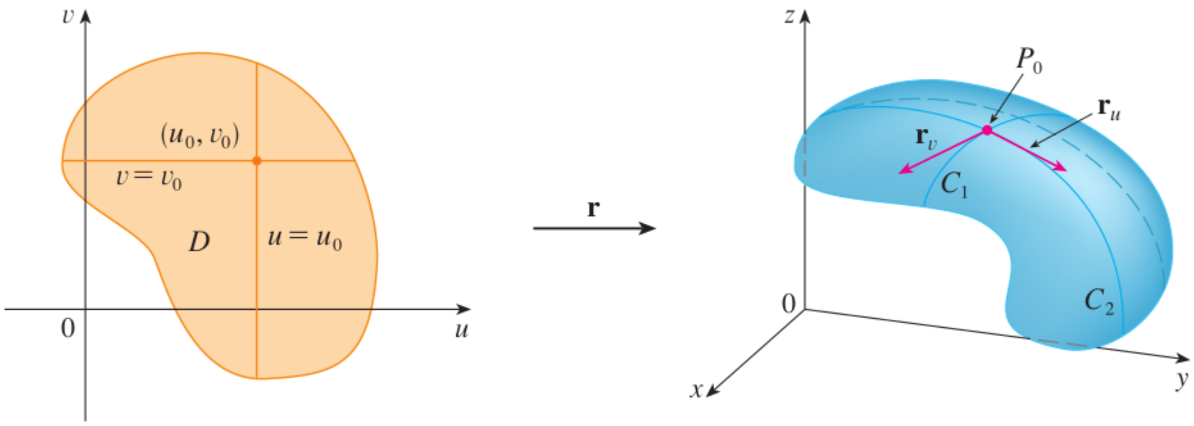
Surfaces of revolution.



Equations.

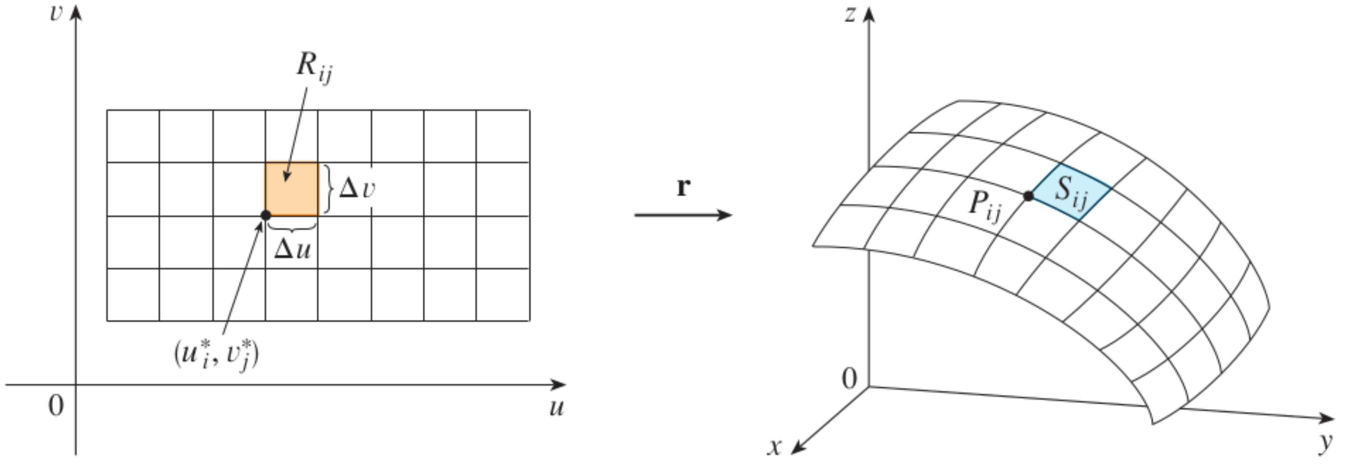
EXAMPLE 8 Find parametric equations for the surface generated by rotating the curve $y = \sin x$, $0 \leqslant x \leqslant 2\pi$, about the x-axis. Use these equations to graph the surface of revolution.

Question: What are the equations of a surface obtained by rotating a function about another axis?

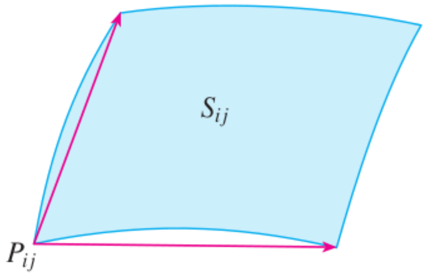


EXAMPLE 9 Find the tangent plane to the surface with parametric equations $x = u^2$, $y = v^2$, $z = u + 2v$ at the point $(1, 1, 3)$.

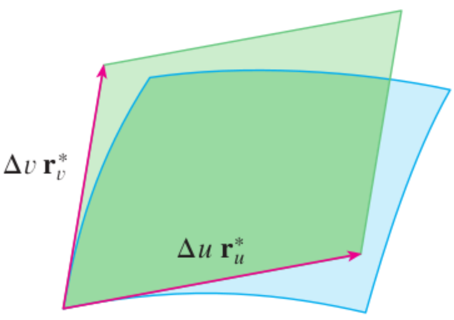
Surface Area.



Closer look.



(a)



(b)

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6 Definition If a smooth parametric surface S is given by the equation

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k} \qquad (u, v) \in D$$

and S is covered just once as (u, v) ranges throughout the parameter domain D, then the **surface area** of S is

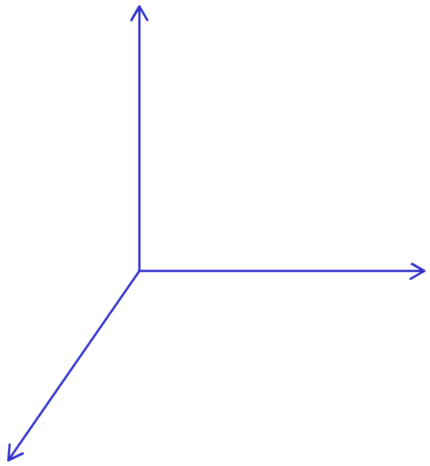
$$\boxed{A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, dA}$$

where $\quad \mathbf{r}_u = \dfrac{\partial x}{\partial u}\mathbf{i} + \dfrac{\partial y}{\partial u}\mathbf{j} + \dfrac{\partial z}{\partial u}\mathbf{k} \qquad \mathbf{r}_v = \dfrac{\partial x}{\partial v}\mathbf{i} + \dfrac{\partial y}{\partial v}\mathbf{j} + \dfrac{\partial z}{\partial v}\mathbf{k}$

41. The part of the plane $x + 2y + 3z = 1$ that lies inside the cylinder $x^2 + y^2 = 3$

(Find the area)

Surface Area of a Graph of a Function.



$$\boxed{A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA}$$

EXAMPLE 11 Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.