Math 644

Chapter 2

Section 2.4: Analytic Functions

Contents

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We consider Ω to be an open subset of $\mathbb{C}$, meaning that

$$\forall z \in \Omega, \text{ there is an } r > 0 \text{ such that } \{w : |w - z| < r\} \subset \Omega.$$

Definition 1. Let $f : \Omega \to \mathbb{C}$.

- f is **analytic** at $z_0 \in \Omega$ if there is an $r > 0$ and a power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converging in $B = \{z : |z - z_0| < r\}$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n \quad (\forall z \in B).$$

- f is analytic on Ω if f is analytic at each $z_0 \in \Omega$.

Notes:

- The power series is not necessarily the same for each $z_0 \in \Omega$.
- A function f is analytic on a set E (not necessarily open), if there is an open set $\Omega \supset E$ and an analytic function g on Ω such that $f = g$.

Theorem 2. If f is analytic in Ω, then f is continuous on Ω.

Proof.

Where Is A Power Series Analytic?

Theorem 3. If $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges on $\{z : |z - z_0| < r\}$, then f is analytic on $\{z : |z - z_0| < r\}$.

Proof.

Uniqueness Of Power Series Expansion

Theorem 4. Suppose

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = \sum_{n=0}^{\infty} b_n (z - z_0)^n$$

for all complex numbers in $\{z : |z - z_0| < r\}$. Then $a_n = b_n$ for all $n \geq 0$.

Proof.

Note:

- The proof actually shows also that if f is analytic at z_0, then for some $\delta > 0$, either
  - $f(z) \neq 0$ for any $0 < |z - z_0| < \delta$;
  - or $f(z) = 0$ for any $|z - z_0| < \delta$.
- The proof also shows that if f is analytic at z_0, then there is a $r > 0$, an integer $m \geq 1$ and an analytic function g at z_0 such that
  - $g(z) \neq 0$ for any z such that $|z - z_0| < r$;
  - $f(z) - f(z_0) = (z - z_0)^m g(z)$.

Consequences On the Zeros

- A set $\Omega \subset \mathbb{C}$ is called a **region** if it is
  - open;
  - connected, meaning that we can't write $\Omega = U \cup V$, where U and V are open sets in $\mathbb{C}$ such that $V \cap U = \varnothing$.

  Fact: Ω is connected if and only if Ω and $\varnothing$ are the only open and closed subsets of Ω.
- A zero a of a function $f : \Omega \to \mathbb{C}$ is called **isolated** if there is an open disk B centered at a such that $f(z) \neq 0$ for any $z \in B \backslash \{a\}$.

Corollary 5. If f is analytic on a region Ω, then either $f \equiv 0$ or the zeros of f are isolated.

Proof.

Note:

- A consequence of the last Corollary is the **Identity principle**: If f and g are two analytic functions in a region Ω that agree on a set with an accumulation point in Ω, then they must be identical (see Problem 18).
- The last Corollary is not true for continuous functions: $f(x) = x \sin(1/x)$ is an example.