SECTION 1.7: TRIG FUNCTIONS AND HYP(E) FUNCTIONS

TRIG FCTS

If $\theta \in \mathbb{R}$, then

$$e^{i\theta} = \cos\theta + i\sin\theta$$

and

$$e^{-i\theta} = \cos\theta - i\sin\theta .$$

So,

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

and

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

DEF 1.7.1 For $z \in \mathbb{C}$, we define

$$\cos(z) := \frac{e^{iz} + e^{-iz}}{2}$$

and

$$\sin(z) := \frac{e^{iz} - e^{-iz}}{2i} .$$

Example 1.7.2 Compute

(a) $\cos(2+i\pi)$ (b) $\sin(i\,5\pi/4)$

Solutions

(a)
$$\cos(2+i\pi) = \frac{1}{2}\left(e^{i(2+i\pi)} + e^{-i(2+i\pi)}\right)$$

$$= \frac{1}{2}\left(e^{-\pi+2i} + e^{\pi-2i}\right)$$

$$= \frac{1}{2}\left(e^{-\pi}(\cos 2 + i\sin 2) + e^{\pi}(\cos 2 - i\sin 2)\right)$$

$$= \frac{1}{2}\left(e^{-\pi}+e^{\pi}\right)\cos 2 + \frac{i}{2}\left(e^{-\pi}-e^{\pi}\right)\sin 2$$

$$= \frac{1}{2}\left(e^{\pi}+e^{-\pi}\right)\cos 2 - i\,\frac{1}{2}\left(e^{\pi}-e^{-\pi}\right)\sin 2$$

$$= \cosh(\pi)\cos(2) - i\sinh(\pi)\sin(2).$$

here, $\cosh(x) = \dfrac{e^{x}+e^{-x}}{2}$ and $\sinh(x) = \dfrac{e^{x}-e^{-x}}{2}$.

(b) $$\sin\left(\frac{i5\pi}{4}\right) = \frac{e^{i(i5\pi/4)} - e^{-i(i5\pi/4)}}{2i}$$

$$= \frac{e^{-5\pi/4} - e^{5\pi/4}}{2i}$$

$$= -\left(\frac{e^{5\pi/4} - e^{-5\pi/4}}{2i}\right)$$

$$= -\frac{1\cdot i}{i\cdot i}\sinh(5\pi/4)$$

$$= i\sinh(5\pi/4).$$

odd: $f(-z) = -f(z)$

even: $f(-z) = f(z)$

<u>Prop. 1.7.3</u>

For any $z \in \mathbb{C}$:

(1) $\cos(-z) = \cos(z)$ & $\sin(-z) = -\sin(z)$

(2) $\cos(z+2\pi) = \cos(z)$ &

$\sin(z+2\pi) = \sin(z)$

Proof. Let $z \in \mathbb{C}$.

(1) $$\cos(-z) = \frac{e^{i(-z)} + e^{-i(-z)}}{2}$$

$$= \frac{e^{-iz} + e^{iz}}{2} = \cos(z)$$

and

$$\sin(-z) = \frac{e^{i(-z)} - e^{-i(-z)}}{2i}$$

$$= \frac{e^{-iz} - e^{iz}}{2i} = -\sin(z).$$

(2) $$\cos(z+2\pi) = \frac{e^{i(z+2\pi)} + e^{-i(z+2\pi)}}{2}$$

$$= \frac{e^{iz+2\pi i} + e^{-iz-2\pi i}}{2}$$

e is $2\pi i$-perid.

$$= \frac{e^{iz} + e^{-iz}}{2} = \cos(z)$$

and

$$\sin(z+2\pi) = \frac{e^{iz+2\pi i} - e^{-iz-2\pi i}}{2i}$$

$$= \frac{e^{iz} - e^{-iz}}{2i} = \sin(z). \qquad \square$$

Prop. the functions $\cos z$ and $\sin z$ are unbounded.

Proof.

For cos : Let $z = iy$. So

$$\cos(z) = \cos(iy) = \frac{e^{-y} + e^{y}}{2} = \cosh(y)$$

$\Rightarrow |\cos(z)| = \cosh(y) \to \infty$ as $y \to \infty$

$\Rightarrow \cos(z)$ is unbounded.

For sin: Let $z = iy$. Then

$$\sin(z) = \sin(iy) = \frac{e^{-y} - e^{y}}{2i} = \sinh(y)$$

$\Rightarrow |\sin(z)| = |\sinh(y)| = \left|\frac{e^{y} - e^{-y}}{2}\right|$

$\Rightarrow$ as $y \to +\infty$, then $\sinh(y) \to +\infty$

$\Rightarrow |\sin(z)| \to +\infty$ as $y \to +\infty$

$\Rightarrow \sin(z)$ is unbounded. $\square$

<u>Prop. 1.7.6</u> Let $z = x+iy \in \mathbb{C}$. Then

(1) $\cos(z) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$

(2) $\sin(z) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$

<u>Proof</u>. We prove only (1). Let $z = x+iy$.

$$\cos(z) = \frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2}$$

$$= \frac{e^{ix-y} + e^{-ix+y}}{2}$$

$$= \frac{e^{-y}\cos x + i e^{-y}\sin x + e^{y}\cos x - i e^{y}\sin x}{2}$$

$$= \frac{(e^{y}+e^{-y})}{2}\cos x + i\frac{(e^{-y}-e^{y})}{2}\sin x$$

$$= \cosh(y)\cos x - i\sinh(y)\sin(x)$$

$\square$

<u>Prop.</u>

(1) $\sin z = 0 \Leftrightarrow z = k\pi,\ k \in \mathbb{Z}$.

(2) $\cos z = 0 \Leftrightarrow z = \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$.

Proof. Prove (1) only.

$(\Leftarrow)$ $z = k\pi \Rightarrow \sin(k\pi) = \dfrac{e^{ik\pi} - e^{-ik\pi}}{2i}$

$$= \frac{\cos(k\pi) + i\cancel{\sin(k\pi)}^{\to 0} - (\cos k\pi - i\cancel{\sin(k\pi)}^{\to 0})}{2i}$$

$$= 0$$

$(\Rightarrow)$ $\sin(z) = 0 \Rightarrow \dfrac{e^{iz} - e^{-iz}}{2i} = 0$

$$\Rightarrow e^{iz} = e^{-iz}$$

$$\overset{z=x+iy}{\Rightarrow} e^{-y} e^{ix} = e^{y} e^{-ix}$$

$$\Rightarrow e^{-y} = e^{y} \text{ and } x = -x + 2k\pi$$

$$\Rightarrow y = 0 \text{ and } x = k\pi. \quad \square$$

<u>sin & cos as a map</u>

$x = x_0 \to \sin(z) = u + iv \to \left(\dfrac{u}{\cos x_0}\right)^2 - \left(\dfrac{v}{\sin x_0}\right)^2 = 1$

$y = y_0 \to \sin(z) = u + iv \to \left(\dfrac{u}{\cosh y_0}\right)^2 + \left(\dfrac{v}{\sinh y_0}\right)^2 = 1$

DEF 1.7.9

$$\tan z = \frac{\cos z}{\sin z}, \quad \sin z \neq 0$$

$$\cot z = \frac{1}{\tan z}, \quad \cos z \neq 0$$

$$\sec z = \frac{1}{\cos z}, \quad \cos z \neq 0$$

$$\csc z = \frac{1}{\sin z}, \quad \sin z \neq 0.$$

Hyperbolic Functions

DEF 1.7.11 For $z \in \mathbb{C}$

$$\cosh(z) = \frac{e^{z} + e^{-z}}{2}.$$

and

$$\sinh(z) = \frac{e^{z} - e^{-z}}{2}$$

We define

(1) $$\tanh(z) = \frac{\sinh(z)}{\cosh(z)}, \quad \cosh(z) \neq 0$$

(2) $$\operatorname{sech}(z) = \frac{1}{\cosh(z)}, \quad \cosh(z) \neq 0.$$

(3) $\operatorname{csch}(z) = \dfrac{1}{\sinh(z)}$, $\sinh(z) \neq 0$

(4) $\coth(z) = \dfrac{\cosh(z)}{\sinh(z)}$, $\sinh(z) \neq 0$.

<u>Prop. 1.7.12</u> for any $z \in \mathbb{C}$,

(1) $\cosh(iz) = \cos(z)$ & $\sinh(iz) = \sin(z)$.

(2) $\cosh^2(z) - \sinh^2(z) = 1$

<u>Proof.</u>

(2) We have

$$\left(\frac{e^z + e^{-z}}{2}\right)^2 - \left(\frac{e^z - e^{-z}}{2}\right)^2$$

$$= \frac{2+2}{4} = 1. \quad \square$$