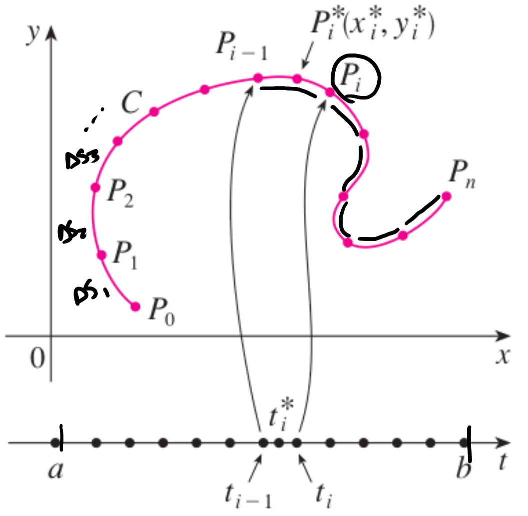
16.2 Line Integrals.

Line integrals in 2D.



C : $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$.
$a \leq t \leq b$

$\vec{r}\,'(t)$ cont.
$\vec{r}\,'(t) \neq 0$
& C is smooth.

f: function in x, y.

Divide [a,b] in n parts, $[t_{i-1}, t_i]$.

We call $P_i = \vec{r}(t_i)$

The corresponding points P_i divide C into n subarcs $\Delta s_1, \Delta s_2, \Delta s_3, \dots, \Delta s_n$

Pick a point $P_i^* = (x_i^*, y_i^*)$ between P_{i-1} & P_i.

We form
$$\sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta s_i$$

push/limit
$n \to \infty$
$\Longrightarrow$

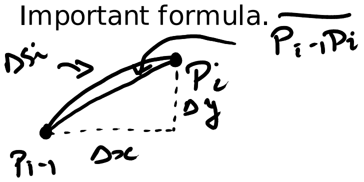
2 Definition If f is defined on a smooth curve C given by Equations 1, then the **line integral of f along C** is

$$\int_C f(x, y)\,\boxed{ds} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta s_i$$

↑ infinitesimal arclength

if this limit exists.

Important formula.



$$\Delta s_i \approx \overline{P_{i-1}P_i} = \sqrt{\Delta x^2 + \Delta y^2}$$

A variation in t induces a variation in Δx & Δy

$$\Rightarrow \Delta s_i \approx \frac{\Delta t}{\Delta t}\sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{\frac{1}{(\Delta t)^2}\left(\Delta x^2 + \Delta y^2\right)}\,\Delta t$$

$$\Rightarrow \Delta s_i \approx \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}\;\Delta t$$

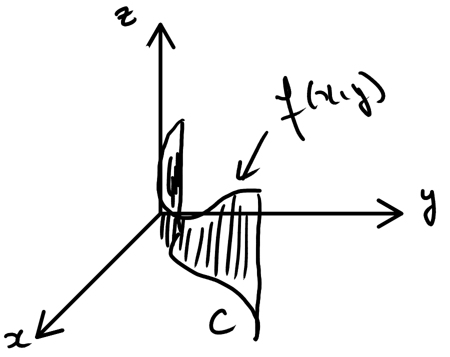
ds

So, $n \to \infty$, then

$$\boxed{\int_C f(x,y)\,ds = \int_a^b f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\;dt}$$

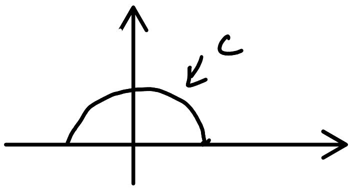
Area.

Integration over a curve of a scalar function represents the area of the fence with base C & heigth f(x,y).



EXAMPLE 1 Evaluate $\int_C (2 + x^2y)\,ds$, where C is the upper half of the $\underbrace{\text{unit}}_{\text{radius}=1}$ circle $x^2 + y^2 = 1$.

① Parametrization.



$$C : \quad \vec{r}(t) = \overbrace{\cos(t)}^{x(t)}\,\vec{i} + \overbrace{\sin(t)}^{y(t)}\,\vec{j}$$

$$0 \le t \le \pi$$

$$y = \sqrt{1-x^2} \qquad \vec{r}(t) = t\,\vec{i} + \sqrt{1-t^2}\,\vec{j}$$

$$x = t$$

② Integrate.

$$ds = \sqrt{x'(t)^2 + y'(t)^2}\,dt$$

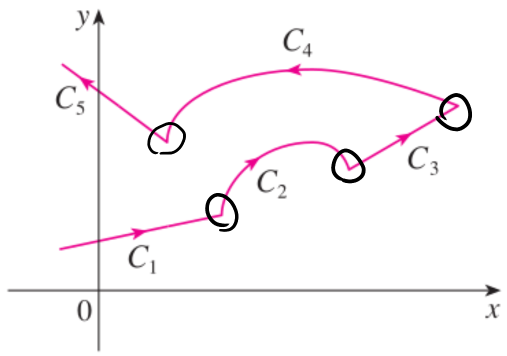
$$x'(t) = -\sin t$$
$$y'(t) = \cos t \quad \Rightarrow \quad ds = \sqrt{\sin^2 t + \cos^2 t}\,dt = dt$$

$$\int_C 2 + x^2y\,ds = \int_0^{\pi} 2 + \cos^2 t \sin t\,dt \qquad u = \cos t$$
$$du = -\sin t\,dt$$

$$= \int_0^{\pi} 2\,dt + \int_0^{\pi} \cos^2 t \sin t\,dt$$

$$= \boxed{2\pi + \frac{2}{3}}$$

Piecewise-smooth curve.



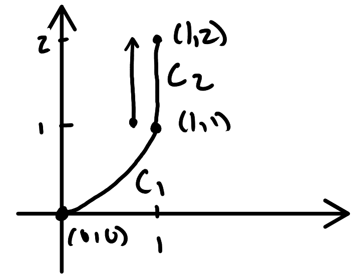
Suppose $C = C_1 \sqcup C_2 \sqcup C_3 \sqcup \dots \sqcup C_n$

then

$$\int_C f(x,y)\,ds = \int_{C_1} f(x,y)\,ds + \int_{C_2} f(x,y)\,ds + \dots + \int_{C_n} f(x,y)\,ds$$

EXAMPLE 2 Evaluate $\int_C 2x\,ds$, where C consists of the arc C_1 of the parabola $y = x^2$ from (0, 0) to (1, 1) followed by the vertical line segment C_2 from (1, 1) to (1, 2).

① Parametrization.



C_1: $\vec{r}_1(t) = x\vec{i} + \overset{y}{x^2}\vec{j}$ $(x=t)$

$0 \le x \le 1$ $\quad = \underline{t}\,\vec{i} + t^2\vec{j}$ $\quad 0 \le t \le 1.$

C_2: $\vec{r}_2(t) = \langle 1,1\rangle + t(\langle 1,2\rangle - \langle 1,1\rangle)$

$= \langle 1, 1+t\rangle$ $\quad 0 \le t \le 1$

$= \underline{\vec{i}} + (1+t)\vec{j}$.

② Integrate.

$C = C_1 \sqcup C_2$

$$\int_C 2x\,ds = \underbrace{\int_{C_1} 2x\,ds}_{I_1} + \underbrace{\int_{C_2} 2x\,ds}_{I_2}$$

$x'(t) = 1$
$y'(t) = 2t$

$$I_1 = \int_{C_1} 2x\,ds = \int_0^1 2t\sqrt{1^2 + (2t)^2}\,dt$$
$$= \frac{1}{6}\left(5\sqrt{5} - 1\right).$$

$u = 1 + (2t)^2$
$du = 4t\,dt$.

$x'(t) = 0$
$y'(t) = 1$

$$I_2 = \int_{C_2} 2x\,ds = \int_0^1 2\cdot 1\sqrt{0^2 + 1^2}\,dt$$
$$= 2\int_0^1 dt = 2$$

So, $\int_C 2s\,ds = \boxed{\frac{1}{6}\left(5\sqrt{5} - 1\right) + 2.}$

Mass and center of mass.

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mass.

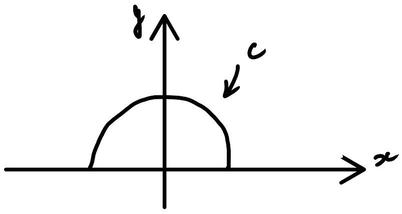
$$\boxed{m = \int_C \rho(x,y)\,ds}$$

center of mass. $(\overline{x}, \overline{y})$

$$\boxed{\begin{aligned} \overline{x} &= \frac{1}{m}\int_C x\rho(x,y)\,ds \\ \overline{y} &= \frac{1}{m}\int_C y\rho(x,y)\,ds \end{aligned}}$$

EXAMPLE 3 A wire takes the shape of the semicircle $x^2 + y^2 = 1$, $y \geqslant 0$, and is thicker near its base than near the top. Find the center of mass of the wire if the linear density at any point is proportional to its distance from the line $y = 1$.

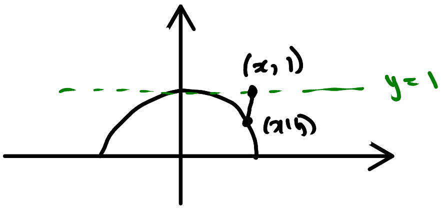
① Parametrization.



$$C: \quad \vec{r}(t) = \underbrace{\cos(t)}_{x}\,\vec{i} + \underbrace{\sin(t)}_{y}\,\vec{j}$$

$$0 \leq t \leq \pi$$

② Density.



$$\rho(x,y) = K(1-y) \qquad \left(\sqrt{(1-y)^2}\right)$$

③ Mass.

$$\begin{aligned} m &= \int_C \rho(x,y)\,ds \\ &= \int_0^\pi K(1-\sin t)\,dt \\ &= K\int_0^\pi 1-\sin t\,dt = (\pi-2)K \end{aligned}$$

$$\begin{aligned} ds &= \sqrt{\sin^2 t + \cos^2 t}\;dt \\ &= dt \end{aligned}$$

④ Center of mass.

$$\begin{aligned} \overline{x} &= \frac{1}{m}\int_C x\rho(x,y)\,ds = \frac{1}{m}\int_0^\pi \cos t\, K(1-\sin t)\,dt \\ &= \frac{1}{(\pi-2)}\int_0^\pi \cos t - \cos t\sin t\,dt = 0 \end{aligned}$$

$$\overline{y} = \frac{1}{m}\int_C y\,\rho(x,y)\,ds = \frac{1}{\cancel{K}(\pi-2)}\int_0^\pi \sin t\,\cancel{K}(1-\sin t)\,dt = \frac{1}{\pi-2}\left(2-\pi/2\right).$$

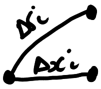
So,
$$\boxed{(\overline{x}, \overline{y}) = \left(0, \frac{4-\pi}{2(\pi-2)}\right)}$$

Two other line integrals.

$\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j}$.

With respect to x.

Replace Δs_i by Δx_i and form



$$\sum_{i=1}^{n} f(x_i, y_i)\,\Delta x_i$$

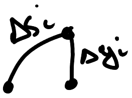
letting $n \to \infty$

$dx = x'(t)\,dt$

$$\boxed{\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))x'(t)\,dt}$$

With respect to y.

Replace Δs_i by Δy_i and form



$$\sum_{i=1}^{n} f(x_i, y_i)\,\Delta y_i$$

Let $n \to \infty$

$dy = g'(t)\,dt$

$$\boxed{\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))y'(t)\,dt}$$

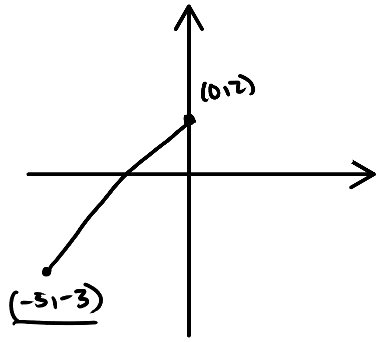
With respect to x and y at the same time.

line integral.

$$\boxed{\int_C f(x,y)\,dx + \int_C g(x,y)\,dy = \underline{\int_C f(x,y)\,dx + g(x,y)\,dy}}$$

EXAMPLE 4 Evaluate $\int_C y^2\,dx + x\,dy$, where (a) $C = C_1$ is the line segment from $(-5, -3)$ to $(0, 2)$ and (b) $C = C_2$ is the arc of the parabola $x = 4 - y^2$ from $(-5, -3)$ to $(0, 2)$. (See Figure 7.)

(a) ① Picture & Parametrize



$$\vec{r}(t) = \langle -5,-3\rangle + t\left(\langle 0,2\rangle - \langle -5,-3\rangle\right)$$

$$= \langle \underbrace{-5+5t}_{x(t)}, \underbrace{-3+5t}_{y(t)} \rangle \qquad 0 \le t \le 1$$

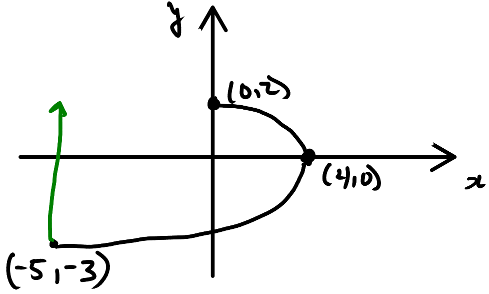
② Integrate. $x'(t) = 5$, $y'(t) = 5$

$$\int_C y^2\,dx + x\,dy = \int_C y^2\,dx + \int_C x\,dy$$

$$= \int_0^1 (-3+5t)^2\, 5\,dt + \int_0^1 (-5+5t)\, 5\,dt$$

$$= -5/6 \approx \boxed{-0.833\ldots}$$

(b) ① Picture & parametrize.



$$\vec{r}(t) = \langle \overset{x}{4-y^2}, \overset{y}{y} \rangle \quad \boxed{y=t} \quad = \langle \overset{x(t)}{4-t^2}, \overset{y(t)}{t} \rangle$$

$$-3 \le t \le 2$$

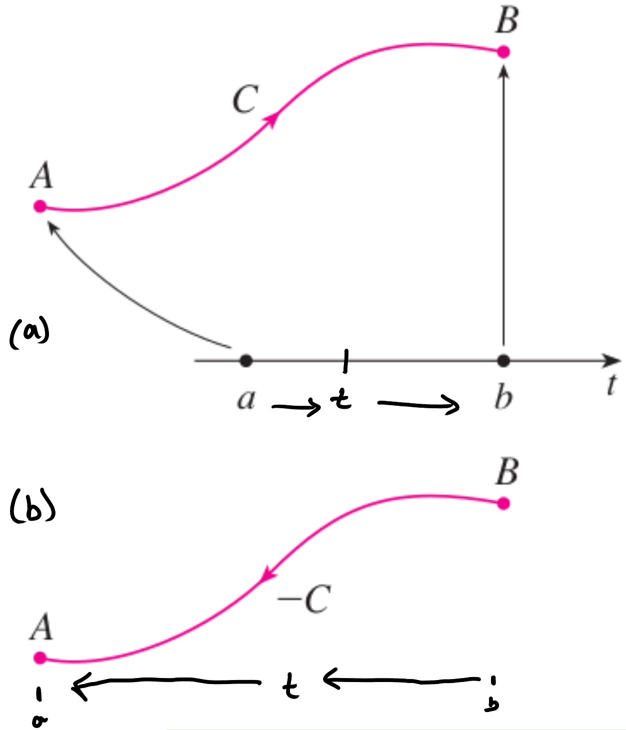
② Integrate. $x'(t) = -2t$, $y'(t) = 1$.

$$\int_C y^2\,dx + x\,dy = \int_C y^2\,dx + \int_C x\,dy$$

$$= \int_{-3}^{2} t^2(-2t)\,dt + \int_{-3}^{2} 4 - t^2\,dt$$

$$= \int_{-3}^{2} -2t^3 + 4 - t^2\,dt = \frac{245}{6} \approx \boxed{40.833\ldots}$$

Orientation.



A parametrization $\vec{r}(t)$ of a curve C determines an orientation:

- -> Positive orientation for increasing values of t.
- -> Opposite orientation for decreasing values of t.

- In (a), the curve is positively oriented.
- In (b), the curve is in the opposite orientation. We denote it by −C

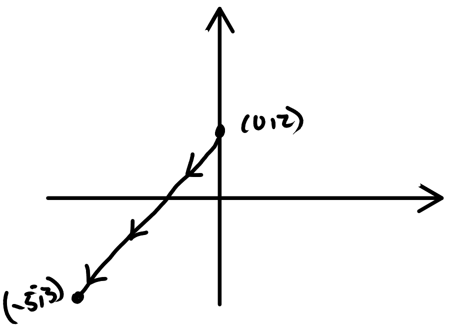
1. $\int_{-C} f(x, y)\, dx = -\int_C f(x, y)\, dx$ 2. $\int_{-C} f(x, y)\, dy = -\int_C f(x, y)\, dy$

3. $\int_{-C} f(x, y)\, \underline{ds} = \int_C f(x, y)\, ds$ $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

Example. (Take example 4 with -C1).

C : line segment joining (−5,−3) to (0,2)

−C : line segment joining (0,2) to (−5,−3).



$$\vec{r}(t) = \langle 0,2\rangle + t(\langle -5,-3\rangle - \langle 0,2\rangle)$$

$$= \langle \underbrace{-5t}_{x(t)}, \underbrace{2-5t}_{y(t)} \rangle \qquad 0 \le t \le 1$$

$$\int_{-C} y^2 dx + x\,dy = \int_0^1 (2-5t)^2 (-5)\, dt + \int_0^1 (-5t)(-5)\, dt$$

$$= 5/6 \approx \boxed{0.833}$$

$$\underbrace{-(-5/6)} \leftarrow = -\int_C y^2 dx + x\,dy$$

Line integrals in Space.

Let $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ $\quad a \le t \le b$

be a parametrization of a curve C in $\mathbb{R}^3$.

Then:

In terms of s.

$\int_{-C} = \int_C$

$$\boxed{9} \quad \int_C f(x, y, z)\, \underset{\text{arclength}}{\underset{\uparrow}{ds}} = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

With respect to x.	With respect to y	With respect to z
Replace ds by dx and so $dx = x'(t)\, dt$	Replace ds by dy and so $dy = y'(t)\, dt$	Replace ds by z and so $dz = z'(t)\, dt$

In terms of x, y, z: sum them together.

EXAMPLE 5 Evaluate $\int_C y \sin z\, ds$, where C is the circular helix given by the equations $x = \cos t$, $y = \sin t$, $\underline{z = t}$, $0 \leqslant t \leqslant 2\pi$. (See Figure 9.)

First,

$$ds = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt = \sqrt{\sin^2 t + \cos^2 t + 1}\, dt = \sqrt{2}\, dt$$

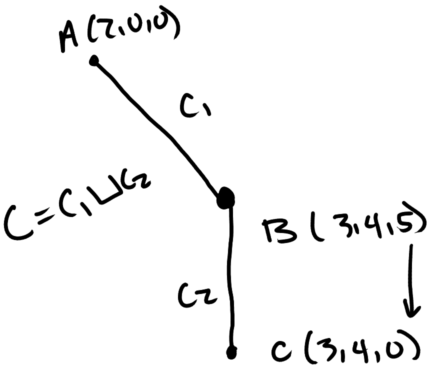
$$\int_C y \sin z\, ds = \int_0^{2\pi} \sin t \sin t \sqrt{2}\, dt = \int_0^{2\pi} \sqrt{2} \sin^2 t\, dt$$

$$= \boxed{\sqrt{2}\, \pi}$$

EXAMPLE 6 Evaluate $\int_C y\,dx + z\,dy + x\,dz$, where C consists of the line segment C_1 from (2, 0, 0) to (3, 4, 5), followed by the vertical line segment C_2 from (3, 4, 5) to (3, 4, 0).

① Parametrize.

$$\int_C (\ldots) = \int_{C_1} (\ldots) + \int_{C_2} (\ldots).$$



$$C_1 : \vec{r}_1(t) = \langle 2,0,0\rangle + t(\langle 3,4,5\rangle - \langle 2,0,0\rangle)$$

$$= \langle \underbrace{2+t}_{x(t)}, \underbrace{4t}_{y(t)}, \underbrace{5t}_{z(t)} \rangle \qquad 0 \le t \le 1$$

$$C_2 : \vec{r}_2(t) = \langle \underbrace{3}_{x}, \underbrace{4}_{y}, \underbrace{5-5t}_{z(t)} \rangle \qquad 0 \le t \le 1$$

② Integrate.

$$I = \int_C y\,dx + z\,dy + x\,dz = \int_C y\,dx + \int_C z\,dy + \int_C x\,dz$$

$$= \underline{\int_{C_1} y\,dx} + \underline{\int_{C_2} y\,dx} + \underline{\int_{C_1} z\,dy} + \underline{\int_{C_2} z\,dy}$$

$$+ \underline{\int_{C_1} x\,dz} + \underline{\int_{C_2} x\,dz}.$$

On C_1 $(\vec{r}_1(t))$

$dx = x'(t)dt = dt \qquad dy = y'(t)dt = 4\,dt \qquad dz = z'(t)dt = 5\,dt$

$$\int_{C_1} y\,dx + \int_{C_1} z\,dy + \int_{C_1} x\,dz = \int_0^1 4t\,dt + \int_0^1 (5t)4\,dt + \int_0^1 (2+t)5\,dt$$

$$= \int_0^1 4t + 20t + 10 + 5t\,dt = 24.5$$

On C_2 $(\vec{r}_2(t))$

$dx = 0 \qquad dy = 0 \qquad dz = -5\,dt$

$$\cancelto{0}{\int_{C_2} y\,dx} + \cancelto{0}{\int_{C_2} z\,dy} + \int_{C_2} x\,dz = \int_0^1 3(-5)\,dt = -15$$

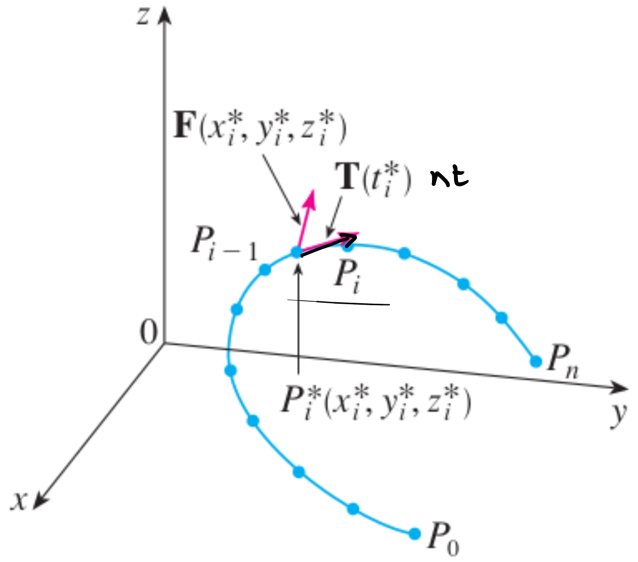
So,

$$I = 24.5 - 15 = \boxed{9.5}$$

Line integrals of Vector Fields.

WORK. = $\vec F \cdot \vec D$ (F · D)

C: curve $\vec r(t)$ (a ≤ t ≤ b)



Divide C in n pieces:

$\Delta s_1, \Delta s_2, \Delta s_3, \dots, \Delta s_n$.

$\vec F = P\vec i + Q\vec j + R\vec k$

Let P_i be the points that divide on the curves.

Let $\vec T_i$ be the unit tangent vector at P_i. So,

$$W_i = \vec F \cdot (\vec T_i \Delta s_i) = (\vec F \cdot \vec T_i)\Delta s_i$$

So,

$$W \approx \sum_{i=1}^{n} W_i = \sum_{i=1}^{n} (\vec F \cdot \underset{\frac{\vec r'_i}{\|\vec r'_i\|}}{\underset{\uparrow}{\vec T_i}})\Delta s_i$$

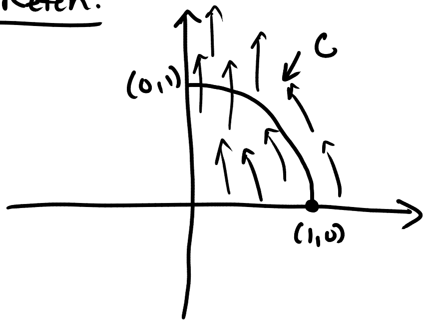
Let n→∞

13 Definition Let **F** be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t)$, $a \leqslant t \leqslant b$. Then the **line integral of F along C** is

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \boxed{\int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds \leftarrow$$

EXAMPLE 7 Find the work done by the force field $\mathbf{F}(x, y) = x^2\,\mathbf{i} - xy\,\mathbf{j}$ in moving a particle along the quarter-circle $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$, $\underset{\overset{\downarrow}{a}}{0} \leqslant t \leqslant \underset{\overset{\downarrow}{b}}{\pi/2}$.

① Sketch.



$\vec r'(t) = -\sin t\,\vec i + \cos t\,\vec j$

$\vec F(\vec r(t)) = \cos^2 t\,\vec i - \cos t \sin t\,\vec j$

$\vec F \cdot \vec r'(t) = -\sin t \cos^2 t - \sin \cos^2 t$

$= -2\sin t \cos^2 t$

② Integrate.

$$W = \int_C \vec F \cdot d\vec r = \int_0^{\pi/2} \underbrace{\vec F(\vec r(t)) \cdot \vec r'(t)}_{??}\, dt$$

$$= \int_0^{\pi/2} -2\sin t \cos^2 t\, dt$$

$u = \cos t$
$du = -\sin t\, dt$

$$= \boxed{-2/3}$$

EXAMPLE 8 Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$ and C is the twisted cubic given by

$$x = t \qquad y = t^2 \qquad z = t^3 \qquad 0 \leqslant t \leqslant 1$$

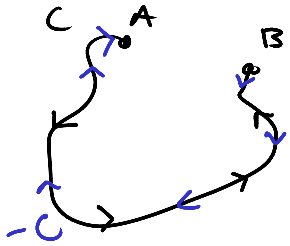
$$\vec{r}(t) = \langle \underset{x}{t}, \underset{y}{t^2}, \underset{z}{t^3} \rangle \qquad 0 \le t \le 1$$

$$\vec{r}\,'(t) = \langle 1, 2t, 3t^2 \rangle \qquad \vec{F}(\vec{r}(t)) = \langle t^3, t^5, t^4 \rangle$$

$$\Rightarrow \quad \vec{r}\,'(t) \cdot \vec{F}(\vec{r}(t)) = t^3 + 2t^6 + 3t^6$$
$$= t^3 + 5t^6.$$

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$
$$= \int_0^1 t^3 + 5t^6\, dt$$
$$= \boxed{\frac{27}{28}}$$

<u>Remark.</u> $\int_{-C} \vec{F} \cdot d\vec{r} = -\int_C \vec{F} \cdot d\vec{r}$



Line integrals of vector fields and of scalar functions.

line integral of vector fields.

$$\vec{F} = \langle P, Q, R \rangle \qquad \Rightarrow \quad \int_C \vec{F} \cdot d\vec{r} = \int_C P x'(t) + Q y'(t) + R z'(t)\, dt$$

$$\vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle$$

$$= \underbrace{\int_C P\, dx + Q\, dy + R\, dz}_{\text{line integral of scalar functions.}}$$

because $dx = x'(t)\, dt$ $\quad dz = z'(t)\, dt$
$dy = y'(t)\, dt$