Chapter 4: Integrals
Week 10

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Calculus I (MATH-241 01/02)

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Upcoming this week

1. 4.0 Symbol of summation
2. 4.1 Areas and Distances

Definition 1

The symbole $\sum_{n=k}^{m} a_n$ means to sum the numbers a_k, a_{k+1}, $\cdots$, a_m.

Explicitly, we have

$$\sum_{n=k}^{m} a_n = a_k + a_{k+1} + \cdots + a_m.$$

Example 2

Using the sumbol $\sum$, write the sum $1 + 2 + 3 + 4$.

Example 3

Using the symbol $\sum$, write the sum $1 + 1/2 + 1/3 + 1/4$.

Question 4

What is the sum of the first n positive integers? That is, what is $1+2+3+4+\cdots+n$? Do we have a formula for this?

Values of n	Values of the sum
1	1
2	3
3	6
4	10
$\vdots$	$\vdots$
10	55

Table: Some Values of the Sum of the first positive integers

There is a nice trick to find the formula. Let S be the sum of the positive integers up to 6.

Arrange the sum in this way

$$\begin{aligned} 1+2+3+4+5+6 &= S \\ 6+5+4+3+2+1 &= S. \end{aligned}$$

Sum each equality together:

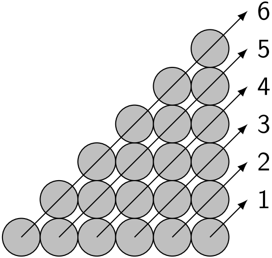
$$\begin{array}{cccccccccccc} 1 & + & 2 & + & 3 & + & 4 & + & 5 & + & 6 = & S \\ 6 & + & 5 & + & 4 & + & 3 & + & 2 & + & 1 = & S \\ \hline 7 & + & 7 & + & 7 & + & 7 & + & 7 & + & 7 = & 2S \end{array}$$

We got six 7, so $6 \cdot 7 = 2S$, so $S = 6 \cdot 7/2 = 21$.

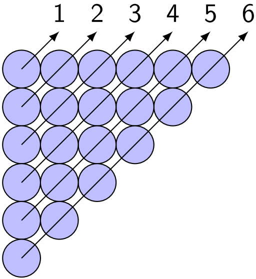
The great Gauss (the prince of mathematics) used this little trick to compute this sum.

Figure: Carl Frederich Gauss

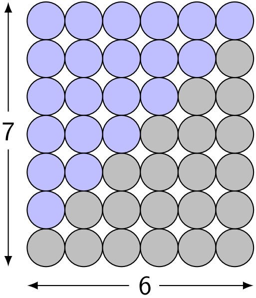
There is another proof of this formula which is really nice.



(a) Initial position of the integers



(b) Another copy of the pyramide



(c) Two copies stacked

$$\text{Numbers} = \frac{6 \times 7}{2}.$$

Theorem 5

In general, we have

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

Theorem 6

In general, we have

$$\sum_{k=1}^{n} k^2 = 1 + 4 + 9 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Theorem 7

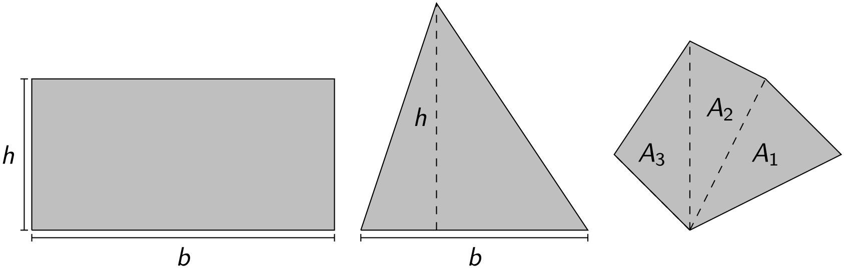
In general, we have

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

Question 8

What is the area of

- a rectangle?
- a triangle?
- a shape delimited by straight lines?



(d) Rectangle $A = bh$

(e) Triangle $A = bh/2$

(f) Polygonal shape $A = A_1 + A_2 + A_3$

Question 9

Can you find the area of the shape in the picture below? Area under a curve

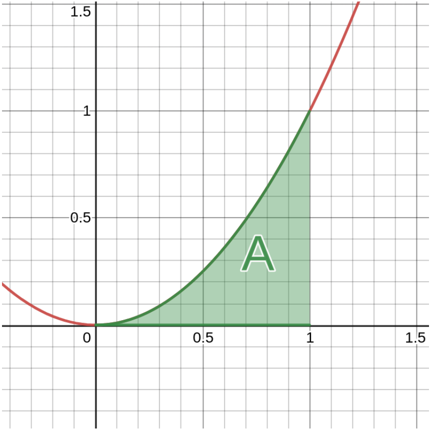


Figure: Area under the curve $y = x^2$ on $[0, 1]$

Example 10

Approximate the area under the curve $y = x^2$ for $x \in [0, 1]$ with 4 rectangles.

Definition 11

Let f be a function on an interval $[a, b]$. Set $\Delta x := \frac{b-a}{n}$.

- The upper sum of triangles is defined as

$$R_n := \sum_{k=1}^{n} f(a + k\Delta x)\Delta x.$$

- The lower sum is defined as

$$L_n := \sum_{k=1}^{n} f(a + (k-1)\Delta x)\Delta x.$$

Remarks

- The points $x_k = a + k\Delta x$ are called the right-end points.
- The points $x_k = a + (k-1)\Delta x$ are called the left-end points.

n	L_n	R_n
10	0.28500000	0.3850000
20	0.3087500	0.3587500
100	0.3283500	0.3383500
1000	0.3328335	0.3338335

So, the area seems to approach $0.33333... = 1/3$.

Example 12

Prove that $\lim_{n\to\infty} R_n = \frac{1}{3}$.

We can use this idea to apply to the general case. Let $[a, b]$ be an interval and let $\Delta x := \frac{b-a}{n}$. Take x_n a list of points such that

$$
\begin{aligned}
x_0 &= a, & x_1 &= a + \Delta x, \\
x_2 &= a + 2\Delta x, & x_3 &= a + 3\Delta x, \\
&\vdots & &\vdots \\
x_{n-1} &= a + (n-1)\Delta x, & x_n &= a + n\Delta x.
\end{aligned}
$$

Put $R_n := \sum_{k=1}^{n} f(x_k)\Delta x$.

Definition 13

Let f be a function defined on the interval $[a, b]$. The area A of the region S lying under the graph of the function f is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \left(f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x \right).$$

Remarks

- We can take any point $x_k^* \in [x_{k-1}, x_k]$ and form the sum $\sum_{k=1}^{n} f(x_k^*)\Delta x$.

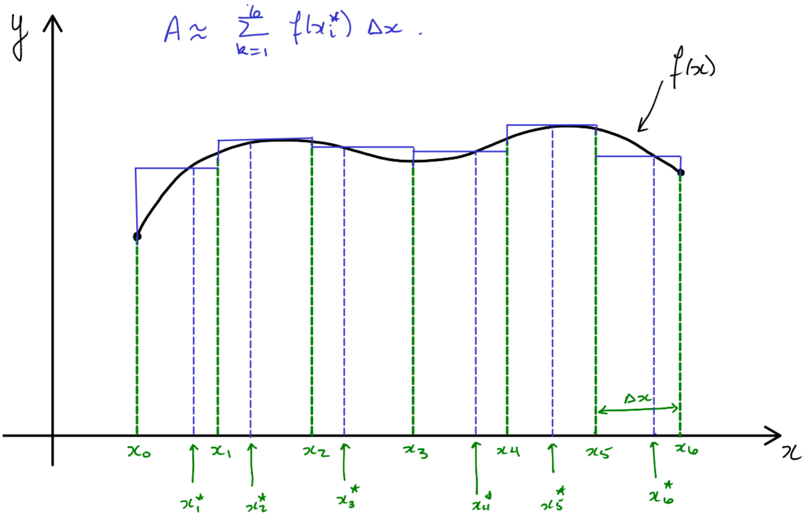


Figure: Sum of rectangles obtained from sample points

Exercises: 3, 4, 14, 21-26. Read section on the distance problem.