$$\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta = \frac{f(z)}{2\pi i} \int_\gamma \frac{1}{\zeta - z}\, d\zeta$$

Math 644

Chapter 5

Section 5.2: Winding Number

Contents

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Spring 2023

Lemma 1. If γ is a cycle and $a \notin \gamma$, then

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{\zeta - a}\, d\zeta$$

is an integer.

Proof. Write $\gamma = \sum_{j=1}^{N} \gamma_j$, γ_j are closed curve.

and $\int_\gamma f(z)\,dz = \sum_{j=1}^{N} \int_{\gamma_j} f(z)\,dz$.

We may suppose that each γ_j is cont. different. (piecewise). We can deal with only one of them. From now on, let γ be a closed piecewise cont. diff. curve. $\gamma : [0,1] \to \mathbb{C}$.

Define $h(x) = \int_0^x \frac{\gamma'(t)}{\gamma(t) - a}\, dt$.

Then, $h'(x)$ exists & $h'(x) = \frac{\gamma'(x)}{\gamma(x) - a}$, except at finitely many x.

$$\begin{aligned}\frac{d}{dx}\left[e^{-h(x)}(\gamma(x) - a)\right] &= -h'(x)e^{-h(x)}(\gamma(x) - a) \\ &\quad + e^{-h(x)}\gamma'(x) \\ &= -\gamma'(x)e^{-h(x)} + \gamma'(x)e^{-h(x)} \\ &= 0 \qquad \text{(except at finitely many } x\text{)}.\end{aligned}$$

Since $e^{-h(x)}(\gamma(x)-a)$ is continuous, it must be constant in $[0,1]$

$$\Rightarrow \quad e^{-h(1)}(\gamma(1)-a) = e^{-h(0)}(\gamma(0)-a)$$
$$= e^{0}(\gamma(0)-a)$$
$$= \gamma(1)-a \quad (\gamma \text{ closed curve})$$

Since $a \notin \gamma$,

$$e^{-h(1)} = 1$$

$$\Rightarrow \quad h(1) = 2k\pi i \quad , k \in \mathbb{Z}$$

So,

$$\frac{1}{2\pi i}\int_0^1 \frac{\gamma'(t)}{\gamma(t)-a}\,dt = \frac{h(1)}{2\pi i} = k . \qquad \square$$

Definition 2. If γ is a cycle, then the **index** or **winding number** of γ about a is

$$n(\gamma, a) := \frac{1}{2\pi i} \int_\gamma \frac{1}{\zeta - a}\, d\zeta \quad (a \notin \gamma).$$

Proposition 3. Let γ be a cycle.

(a) $n(\gamma, a)$ is an analytic function of a, for $a \notin \gamma$.

(b) $n(\gamma, a)$ is constant in each component of $\mathbb{C}\backslash\gamma$.

(c) $n(\gamma, a) \to 0$ as $a \to \infty$. In particular, $n(\gamma, a) = 0$ for any a in the unbounded component of $\mathbb{C}\backslash\gamma$.

(d) $n(-\gamma, a) = -n(\gamma, a)$.

(e) $n(\gamma_1 + \gamma_2, a) = n(\gamma_1, a) + n(\gamma_2, a)$.

Proof.

(a) Since the fct. $\zeta \mapsto 1$ is continuous on γ, then from Lemma 2 in §4.4,

$$n(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{1}{\zeta - a}\, d\zeta$$

is analytic on $\mathbb{C}\backslash\gamma$.

(b) Let Ω be a component of $\mathbb{C}\backslash\gamma$. Suppose Ω is bounded, so that Ω is bounded by some γ_j (*), where $\gamma = \sum_j \gamma_j$. From Lemma 1, $n(\gamma_j, a)$ is constant. In other words, $a \in \Omega$, $n(\gamma_j, a) = n(\gamma, a)$.

(*) that contributes to the winding number.

(c) Since $a \notin \gamma$,

$$\frac{1}{|z-a|} \le \frac{1}{\mathrm{dist}(\gamma,a)} \to 0, \quad a \to \infty.$$

therefore, if σ is a polygonal curve as in thm. 4,

$$n(\gamma,a) = n(\sigma,a) \le \frac{|\sigma|}{2\pi} \frac{1}{\mathrm{dist}(\sigma,a)} \to 0.$$

By lemma 1, & (a) $n(\gamma,a)$ should be constant

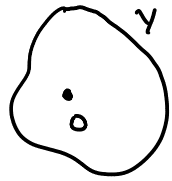
$$\Rightarrow \quad n(\gamma,a) = 0 \qquad \forall a \in \Omega$$

where Ω is the unbounded component of $\mathbb{C}\setminus\gamma$.

(d) Direct calculations.

(e) Direct calculations. $\square$

Some Intuition:



① **Difference in the argument.**

Suppose $\gamma(t) = r(t)e^{i\theta(t)}$ where

- $r(t), \theta(t)$ are piecewise cont. diff.
- $0 \le t \le 1$.
- $\gamma(0) = \gamma(1)$.

Then,

$$n(\gamma, 0) = \mathrm{Re}\left[\frac{1}{2\pi i}\int_\gamma \frac{1}{z}\,dz\right]$$

$$= \mathrm{Re}\left[\frac{1}{2\pi i}\int_0^1 \frac{r'(t)e^{i\theta(t)} + r(t)i\theta'(t)e^{i\theta(t)}}{r(t)e^{i\theta(t)}}\,dt\right]$$

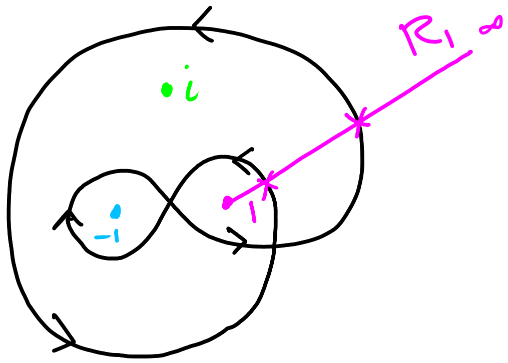
$$= \mathrm{Re}\left[\frac{1}{2\pi i}\int_0^1 \frac{r'(t)}{r(t)} + i\,\theta'(t)\,dt\right]$$

$$= \frac{1}{2\pi}\int_0^1 \theta'(t)\,dt$$

$$= \frac{\theta(1) - \theta(0)}{2\pi}.$$

Net change in the "argument" $\div$ by 2π.

② **Rays and number of connected components.**

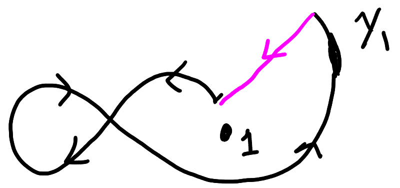


<u>Goal</u>: Find $n(\gamma, 1)$.

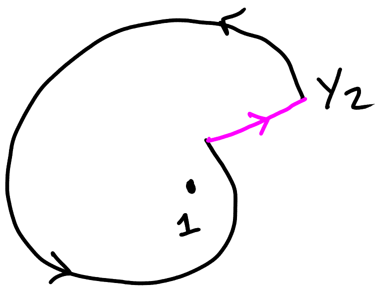
<u>i</u>. Draw ray R_1 from 1 to ∞.

<u>ii</u>. Locate intersections of R_1 with γ.

<u>iii</u> Consider each connected component of $\gamma \setminus R_1$.

- 

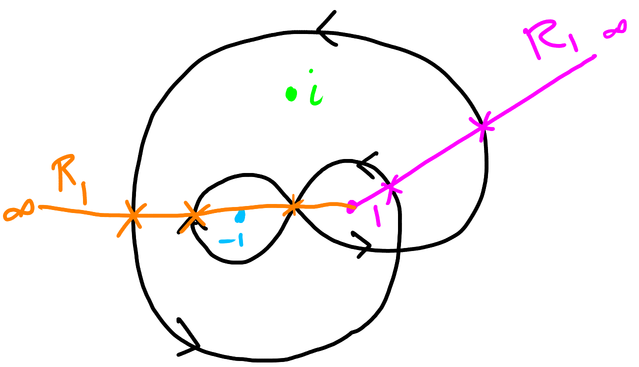
$\longrightarrow$ $n(\gamma_1, 1) = 1$.

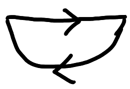
- 

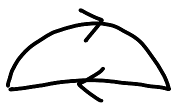
$\longrightarrow$ $n(\gamma_2, 1) = 1$.

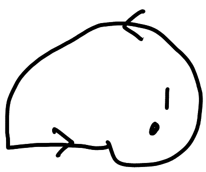
Overall, $n(\gamma, 1) = n(\gamma_1, 1) + n(\gamma_2, 1) = 2$.

<u>Works for any rays:</u>

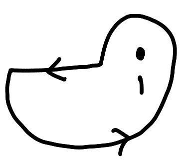


*  $\gamma_1 \longrightarrow n(\gamma_1, 1) = 0$

*  $\gamma_2 \longrightarrow n(\gamma_2, 1) = 0$

*  $\gamma_3 \longrightarrow n(\gamma_3, 1) = 1$

$n(\gamma, 1) = \sum_j n(\gamma_j, 1) = 2$.

*  $\gamma_4 \longrightarrow n(\gamma_4, 1) = 1$

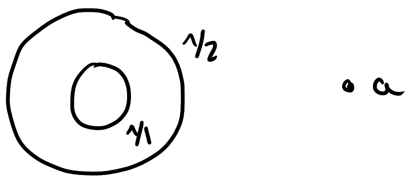
Definition 4. Closed curves γ_1 and γ_2 are **homologous** in a region Ω if $n(\gamma_1 - \gamma_2, a) = 0$ for all $a \notin \Omega$ and we write $\gamma_1 \sim \gamma_2$.

Remarks:

- Homology is an equivalence relation on curves in Ω.
- A closed curve is said to be **homologous to 0** if $n(\gamma, a) = 0$ for all $a \notin \Omega$. In this case, we write $\gamma \sim 0$.

Example 5. Show that $\gamma_1(t) = r_1 e^{it}$ and $\gamma_2(t) = r_2 e^{it}$ $(0 \leq t \leq 2\pi)$ are homologous in $\Omega := \{z : |z| < R\}$, where $r_1 < r_2 < R$.

the curve $\gamma = \gamma_1 - \gamma_2$ is

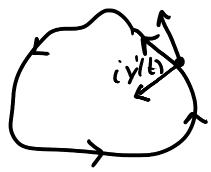


If $a \notin \{z : |z| < R\}$, then

$$n(\gamma_1 - \gamma_2, a) = n(\gamma_1, a) - n(\gamma_2, a) = 0 - 0 = 0 .$$

Definition 6. Let Ω be a bounded region in $\mathbb{C}$ bounded by finitely many piecewise continuously differentiable simple closed curves. The **positive orientation** of $\partial\Omega$ is a parametrization that has the following property:

(a) for each $t \in [0, 1]$ where the derivative exists, there is an $\varepsilon(t) > 0$ such that $\gamma(t) + ui\gamma'(t) \in \Omega$, for all $u \in [0, \varepsilon(t)]$.



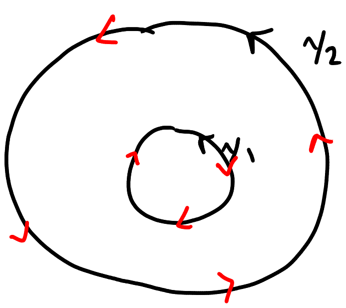
Notes:

① When the positive orientation is chosen for $\partial\Omega$, then

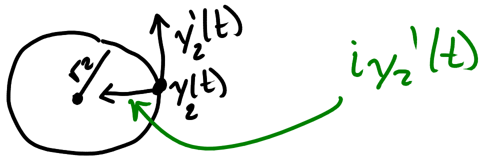
- $n(\partial\Omega, a) = 0$, for each $a \notin \overline{\Omega}$;
- $n(\partial\Omega, a) = 1$, for each $a \in \Omega$.

Example 7. Find the positive orientation of the boundary of the closed annulus $A := \{z : r_1 \leq |z| \leq r_2\}$.

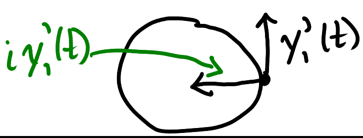
$\gamma_2(t) = r_2 e^{it}$, $\gamma_1(t) = r_1 e^{it}$, $0 \leq t \leq 2\pi$.



• $\gamma_2'(t) = ir_2 e^{it}$



• $\gamma_1'(t) = ir_1 e^{it}$



Change γ_1 into $-\gamma_1$ to obtain positive orientation.

Definition 8.

(a) A region $\Omega \subset \mathbb{C}^*$ is called **simply-connected** if $\mathbb{C}^*\backslash\Omega$ is connected.

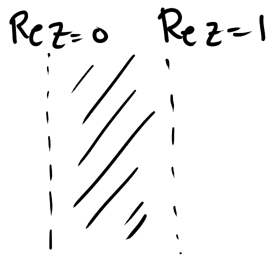
(b) Equivalently, a region Ω is simply-connected if $\mathbb{S}^2\backslash\pi(\Omega)$ is connected, where π is the stereographic projection.

Example 9. Show that

(a) the unit disk is simply connected;

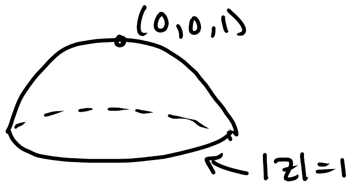
(b) the vectical strip $\Omega = \{z : 0 < \operatorname{Re} z < 1\}$ is simply connected;

(c) $\mathbb{C}\backslash\{0\}$ is not simply connected.



(a) $\mathbb{C}^* \setminus \mathbb{D} = \{ z \in \mathbb{C} : |z| \geq 1 \} \cup \{\infty\}$

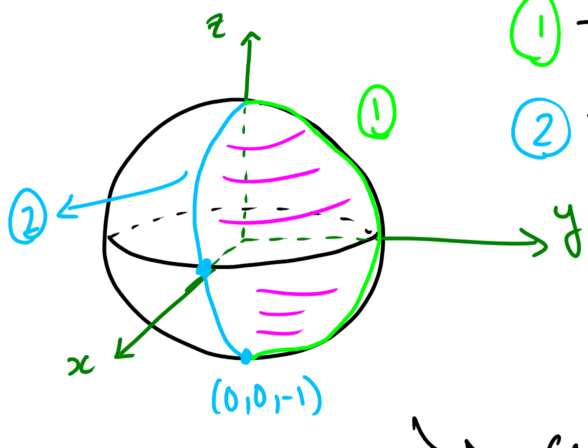
$\overset{\pi}{\cong}$



connected.

(b) $\mathbb{C}^* \setminus \Omega = \{ z \in \mathbb{C} : \operatorname{Re} z \geq 1 \} \cup \{ z \in \mathbb{C} : \operatorname{Re} z \leq 0 \} \cup \{\infty\}$

$\overset{\pi}{\cong}$



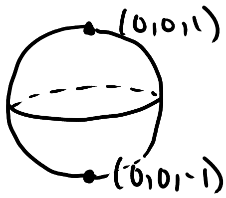
1 $\pi(\operatorname{Re} z = 0) = \left(0, \frac{2y}{y^2+1}, \frac{y^2-1}{y^2+1}\right)$

2 $\pi(\operatorname{Re} z = 1) = \left(\frac{2}{2+y^2}, \frac{2y}{2+y^2}, \frac{y^2}{2+y^2}\right)$

$\to$ connected

(c) $\mathbb{C}^* \setminus (\mathbb{C} \setminus \{0\}) = \{0\} \cup \{\infty\}$

$\overset{\pi}{\cong}$



not simply-connected.

THEOREM 10.

(a) A region $\Omega \subset \mathbb{C}$ is simply-connected if and only if every cycle in Ω is homologous to 0 in Ω.

(b) If Ω is not simply-connected then we can find a simple closed polygonal curve contained in Ω which is not homologous to 0.

Proof.

(a) ($\Rightarrow$) Ω simply-connected.

Let γ be a cycle in Ω and $a \notin \Omega$.

Since $B = \mathbb{C}^* \setminus \Omega$ is connected, B must be in one of the component of $\mathbb{C}^* \setminus \gamma$.

Since $\infty \in B$, B is in the unbounded component of $\mathbb{C}^* \setminus \gamma \Rightarrow n(\gamma, a) = 0$.

($\Leftarrow$) Suppose $\mathbb{C}^* \setminus \Omega$ is not connected:

$$\mathbb{C}^* \setminus \Omega = A \cup B,$$

where A & B are closed sets in $\mathbb{C}^*$ and $A \cap B = \emptyset$. WLOG, assume $\infty \in B$.

A is closed $\Rightarrow \{z : |z| > R\} \cup \{\infty\} \cap A = \emptyset$

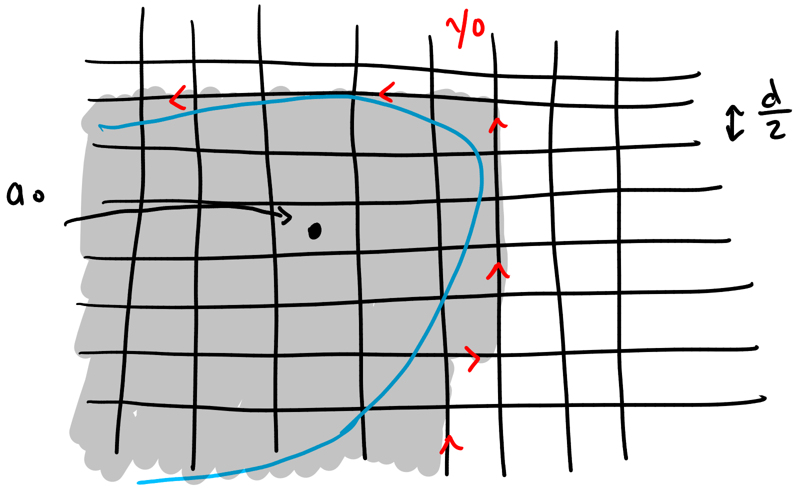
$\Rightarrow$ A is bounded.

Let $a_0 \in A \subseteq \mathbb{C}^* \setminus \Omega$. We construct $\gamma_0 \subseteq \Omega$ s.t.

$$n(\gamma_0, a_0) \neq 0.$$

Let $d = \mathrm{dist}(A, B) = \inf \{ |a - b|, a \in A, b \in B\} > 0$

& $d = 1$ if $B = \{\infty\}$.



1) Pave plane with squares of side $\frac{d}{2}$, such that a_0 center of one square

2) Each square has the positive orientation (counter clockwise)

3) Shade all squares S_j s.t. $\overline{S_j} \cap A \neq \emptyset$

4) Let γ_0 be the cycle $\cup S_j$ (after cancelling sides with opposite direction).

5) Now, $\gamma_0 \subseteq \Omega$ because $\gamma_0 \cap (A \cup B) = \emptyset$

6) We have $n(\gamma_0, a_0) = 1$ because a_0 is in one bounded component of γ_0.

7) $\gamma_0 = \sum_{j=1}^{N} \sigma_j$, where σ_j is a polygonal closed curve. Then at least one of the σ_j is not homologous to 0. $\square$

$\hookrightarrow$ This gives you part (b).

Corollary 11. Suppose f is analytic on a simply-connected region Ω. Then

(a) $\int_\gamma f(z)dz = 0$ for all closed curves $\gamma \subset \Omega$;

(b) there exists a function F analytic on Ω such that $F' = f$;

(c) if also $f(z) \neq 0$ for all $z \in \Omega$, then there exists a function g analytic on Ω such that $f = e^g$.

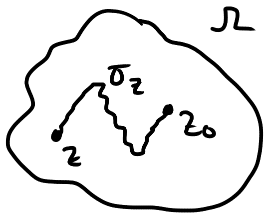
Proof.

(a) Since $\gamma \subseteq \Omega$ and Ω is simply-connected, by Thm. 10, $\gamma \sim 0$. This means that $n(\gamma, a) = 0$ $\forall a \notin \Omega$. By Cauchy's theorem,

$$\int_\gamma f(z)\,dz = 0.$$

(b) Fix $z_0 \in \Omega$ and define

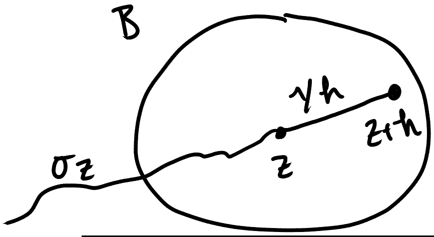
$$F(z) = \int_{\sigma_z} f(\zeta)\,d\zeta$$



If σ_z & γ_z are two different curves connecting z to z_0, then $\sigma_z + (-\gamma_z)$ is a closed path & $\int_{\sigma_z} f(z)\,dz = \int_{\gamma_z} f(z)\,dz$.

$$\frac{F(z+h) - F(z)}{h} = \frac{\int_{\sigma_{z+h}} f(\zeta)\,d\zeta - \int_{\sigma_z} f(\zeta)\,d\zeta}{h}$$

Let B be a disk centered at z s.t. $B \subseteq \Omega$. Then, for h small enough, write



$$\sigma_{z+h} = \sigma_z + \gamma_h$$

$$\Rightarrow \left|\frac{F(z+h)-F(z)}{h}\right| \leq \left|\frac{\int_{\gamma_h} f(\zeta)-f(z)\,d\zeta}{h}\right|$$

$$\leq \sup_{\gamma_h} |f(\zeta)-f(z)| \xrightarrow[h\to 0]{} 0$$

therefore, $F'(z) = f(z)$, $\forall z \in \Omega$.

(c) From the assumption:

$$\frac{f'}{f} \text{ is holomorphic in } \Omega.$$

From (b), $\exists g$ analytic in Ω s.t.

$$g' = \frac{f'}{f}$$

Set $h = \dfrac{f}{e^g} = fe^{-g}$ (in Ω).

$$\Rightarrow h' = f'e^{-g} - fe^{-g}\cdot g' = f'e^{-g} - f'e^{-g} = 0$$

$$\Rightarrow h \equiv c \text{ in } \Omega \quad (c \neq 0)$$

Fix $z_0 \in \mathbb{C}$. Set $f(z_0) = e^{a_0 + i\theta_0}$, where

$$e^{a_0} = |f(z_0)| \;\&\; \arg f(z_0) = \theta_0 .$$

Let $a = a_0 - \operatorname{Re} g(z_0) + i(\theta_0 - \operatorname{Im} g(z_0))$

$$\Rightarrow \quad (g+a)' = g' = \frac{f'}{f} \quad \text{and}$$

$$\frac{f(z_0)}{e^{g(z_0)+a}} = 1$$

so that $h \equiv 1$ & $f(z) = e^{\overbrace{g(z)+a}^{\text{new } g(z)}}$, $z \in \Omega$. $\square$

① **"Uniqueness" in (b).**

If F & G exist with $F' = G' = f$ in Ω

then $(F-G)' = 0 \Rightarrow F = G + C$, $C \in \mathbb{C}$.

② **"Uniqueness" in (c).**

If g & h are such

$$e^g = f = e^h .$$

then $g' e^g = h' e^h \Rightarrow g' f = h' f$

$\Rightarrow g' = h' \quad (f(z) \neq 0)$.

So, $g = h + c$, some $c \in \mathbb{C}$.

But, $e^h = e^g = e^{h+c} \Rightarrow e^c = 1$

So, $c = 2k\pi i$, $k \in \mathbb{Z} \Rightarrow g = h + 2k\pi i$, $k \in \mathbb{Z}$.

Definition 12. If g is analytic in a region Ω and if $f = e^g$ then g is called a **logarithm** of f in Ω and is written $g(z) = \log f(z)$. The function g is uniquely determined by its value at one point $z_0 \in \Omega$.

Notes:

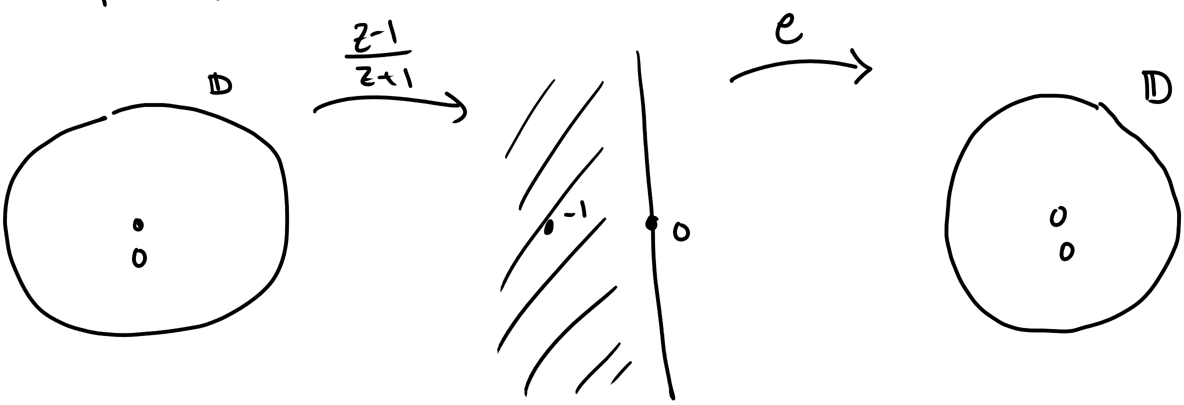
① f has countably many logarithms, which differ by $2\pi k i$. To specify $\log f(z)$ uniquely, we have to specify its value at one point $z_0 \in \Omega$.

② We do not claim that we can define a logarithm on $f(\Omega)$ and then composed with f to obtain $\log f(z)$.

Example 13. Consider the function $z \mapsto (z-1)/(z+1)$, for $z \in \mathbb{D}$.

$z \mapsto \frac{z-1}{z+1}$ maps $\mathbb{D}$ onto $\{z : \operatorname{Re} z < 0\}$.

Therefore, $e^{\frac{z-1}{z+1}}$ maps $\mathbb{D}$ onto $\mathbb{D} \setminus \{0\}$



So, $f(z) = e^{\frac{z-1}{z+1}}$ satisfies the hyp. of Cor. 11. (c)

Say the function $g(z)$ is

$$g(z) = \log f(z) = \frac{z-1}{z+1}, \quad \forall z \in \mathbb{D}.$$

However, we can't define $\log z$ on $f(\mathbb{D})$ as an analytic function, otherwise

$$0 = \int_{C_r} \log z \, dz = -2\pi r, \qquad 0 < r < 1.$$