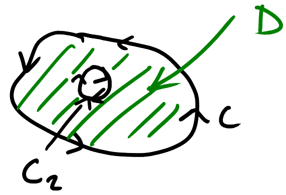
16.9 Divergence Theorem.

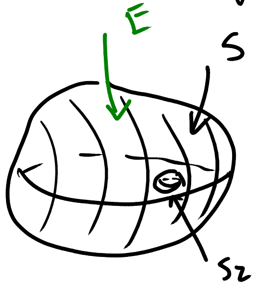


Reminder.

From 16.5:

$$\int_C \vec{F} \cdot \underbrace{d\vec{s}}_{\vec{n}\, ds} = \iint_D \operatorname{div} \vec{F}\, dA$$

There is a generalization to 3D:



E: solid enclosed by S.

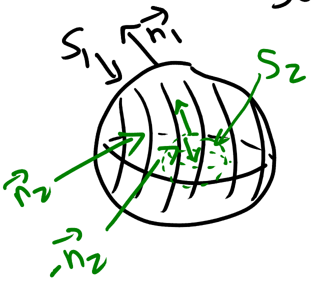
S: boundary of E.

Statement of the Theorem.

The Divergence Theorem Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let **F** be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F}\, dV$$

Remarks. Solid E with a hole inside



S_1 & S_2 enclosed/are bounds for E.

So, boundary of E is $S = S_1 \cup S_2$

$$\Rightarrow \iiint_E \operatorname{div} \vec{F}\, dV = \iint_S \vec{F} \cdot d\vec{S} = \iint_{S_1} \vec{F} \cdot d\vec{S} + \iint_{S_2} \vec{F} \cdot d\vec{S}$$

$$\iint_{S_1} \vec{F} \cdot d\vec{S} = \iint_{S_1} \vec{F} \cdot \vec{n}_1\, dS \quad \& \quad \iint_{S_2} \vec{F} \cdot d\vec{S} = -\iint_{S_2} \vec{F} \cdot \vec{n}_2\, d\vec{S}$$

$$\Rightarrow \iiint_E \operatorname{div} \vec{F}\, dV = \iint_{S_1} \vec{F} \cdot \vec{n}_1\, dS - \iint_{S_2} \vec{F} \cdot \vec{n}_2\, dS.$$

EXAMPLE 1 Find the flux of the vector field $\mathbf{F}(x, y, z) = z\,\mathbf{i} + y\,\mathbf{j} + x\,\mathbf{k}$ over the unit sphere $x^2 + y^2 + z^2 = 1$.

Flux $\rightarrow \iint_S \vec{F} \cdot d\vec{S}$, S: sphere $x^2+y^2+z^2=1$.

Div. Thm. $\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} \, dV$.

$E := \{ (x, y, z) : x^2+y^2+z^2 \leq 1 \}$.

$$\operatorname{div} \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 0 + 1 + 0 = 1$$

so,

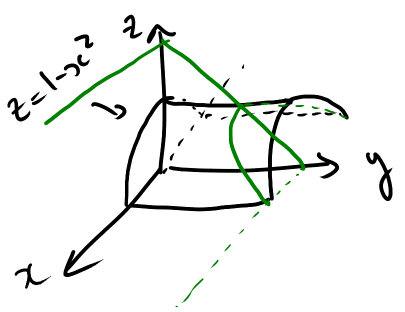
$$\iiint_E \operatorname{div} \vec{F} \, dV = \iiint_E dV = V(E) = \boxed{\frac{4}{3}\pi}$$

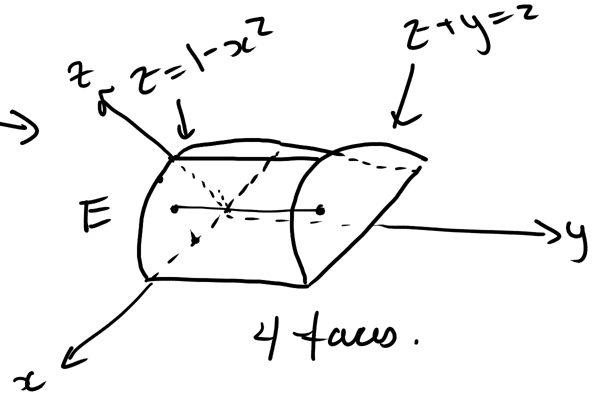
EXAMPLE 2 Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where

$$\mathbf{F}(x, y, z) = xy\,\mathbf{i} + \left(y^2 + e^{xz^2}\right)\mathbf{j} + \sin(xy)\,\mathbf{k}$$

and S is the surface of the region E bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0$, $y = 0$, and $y + z = 2$. (See Figure 2.)

① Picture





TYPE II fn E.

$E = \{(x,y,z) : 0 \le y \le 2-z,\ -1 \le x \le 1,\ 0 \le z \le 1-x^2\}.$

② Div. Thm.

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F}\, dV$$

$$\operatorname{div} \vec{F} = y + 2y + 0 = 3y$$

$$\Rightarrow \iiint_E \operatorname{div} \vec{F}\, dV = \int_{-1}^{1} \int_0^{1-x^2} \int_0^{2-z} 3y\, dy\, dz\, dx$$

$$= \int_{-1}^{1} \int_0^{1-x^2} \frac{3}{2} y^2 \Big|_0^{2-z} dz\, dx$$

$$= \int_{-1}^{1} \int_0^{1-x^2} \frac{3}{2} (2-z)^2\, dz\, dx$$

$$\vdots$$

$$= \boxed{\frac{184}{35}}$$

EXAMPLE 3 In Example 16.1.5 we considered the electric field

$$\mathbf{E}(\mathbf{x}) = \frac{\varepsilon Q}{|\mathbf{x}|^3}\mathbf{x} \qquad \mathbf{x} \to \langle x, y, z\rangle, \quad |\mathbf{x}|^3 \to (x^2+y^2+z^2)^{3/2}$$

(1)

where the electric charge Q is located at the origin and $\mathbf{x} = \langle x, y, z\rangle$ is a position vector. Use the Divergence Theorem to show that the electric flux of $\mathbf{E}$ through any closed surface S_2 that encloses the origin is

$$\iint_{S_2} \mathbf{E}\cdot d\mathbf{S} = 4\pi\varepsilon Q$$

(2) Divergence

$$P(x,y,z) = \frac{x}{(x^2+y^2+z^2)^{3/2}} \to \frac{\partial P}{\partial x} = \frac{-2x^2+y^2+z^2}{(x^2+y^2+z^2)^{5/2}}.$$

$$Q(x,y,z) = \frac{y}{(x^2+y^2+z^2)^{3/2}} \to \frac{\partial Q}{\partial y} = \frac{x^2-2y^2+z^2}{(x^2+y^2+z^2)^{5/2}}$$

$$R(x,y,z) = \frac{z}{(x^2+y^2+z^2)^{3/2}} \to \frac{\partial R}{\partial z} = \frac{x^2+y^2-2z^2}{(x^2+y^2+z^2)^{5/2}}$$

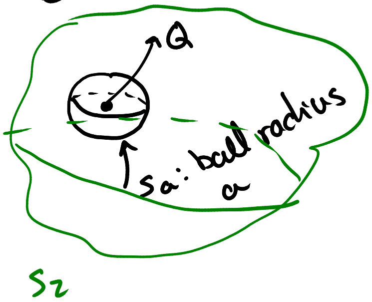
$$\to \operatorname{div}\vec{F} = 0$$

Question

$$\iint_{S_2} \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\, dV = 0\ ?$$

$\to$ problem at $\vec{x} = \vec{0}$ ($\div$ by 0).

(3) The trick.



E: solid enclosed by S_2 & S_a
$\vec{n}_1$ $\vec{n}_2$

$$\iiint_E \cancelto{0}{\operatorname{div}\vec{F}}\, dV = \iint_{S_1} \vec{F}\cdot\vec{n}_1\, dS - \iint_{S_a} \vec{F}\cdot\vec{n}_2\, dS$$

$$\Rightarrow \iint_{S_1} \vec{E}\cdot\vec{n}_1\, dS = \underbrace{\iint_{S_a} \vec{E}\cdot\vec{n}_2\, dS}.$$

(4) Integrate.

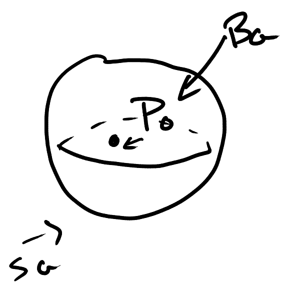
$$\vec{n}_2 = \frac{\vec{x}}{|x|} \to \iint_{S_a} \vec{E}\cdot\vec{n}_2\, dS = \iint_{S_a} \frac{\varepsilon Q}{|x|^3}\vec{x}\cdot\frac{\vec{x}}{|x|}\, dS$$

$$= \frac{\varepsilon Q}{a^2}\iint_{S_a} dS = \boxed{\varepsilon Q\, 4\pi}$$

Application to Fluid Flow. Why $\text{div}\,\vec{F}$ is called the divergence?

$v(x,y,z)$ be a fluid flow & ρ : density of the fluid.

Let $\vec{F} = \rho\vec{v}$.



$\text{div}\,\vec{F} \approx \text{div}\,\vec{F}(P_0)$ (inside B_a)

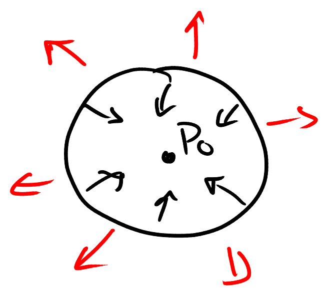
So, by div thm:

$$\iint_{S_a} \vec{F}\cdot d\vec{S} \approx \iiint_{B_a} \text{div}\,\vec{F}(P_0)\,dV$$

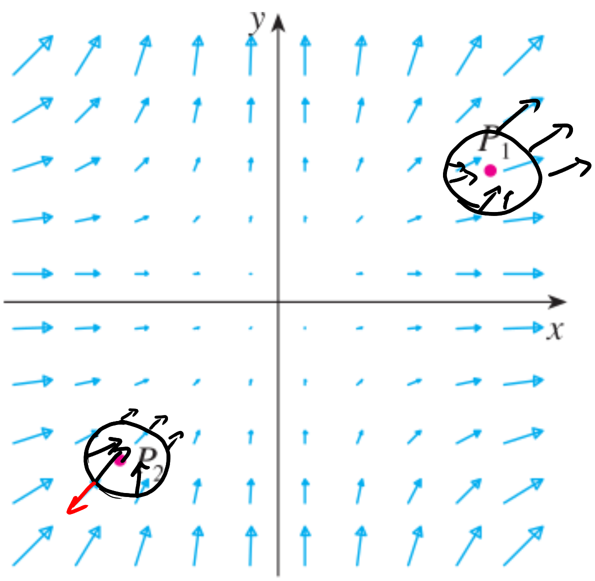
$$= \text{div}\,\vec{F}(P_0)\;V(B_a)$$

Let $a \to 0$

$$\Rightarrow \text{div}\,\vec{F}(P_0) = \lim_{a\to 0} \frac{\iint_{S_a} \vec{F}\cdot d\vec{S}}{V(B_a)}$$



Example of source and sink.



- P_1 is a <u>source</u> because the arrows coming out of the circle are bigger than the arrows coming in.
- P_2 is <u>sink.</u> because the arrows coming out of the circle are smaller than the arrows coming in.