Section 5.4 — Problem 5 — 10 points

Find the general solutions to the complementary equation.
The complementary equation is

$$y'' + 4y = 0.$$

The characteristic equation associated to the complementary equation is $r^2 + 4 = 0$. Therefore, the roots are $r_1 = 2i$ and $r_2 = -2i$. The general solution is

$$y_h(x) = c_1 \cos(2x) + c_2 \sin(2x).$$

Find a particular solution.
We have an exponential times a polynomial of degree two. Also, the number $\alpha = -1$ is not a root of the characteristic polynomial. Therefore, we suggest

$$y_p(x) = e^{-x}(Ax^2 + Bx + C).$$

The respective derivatives are

$$\begin{aligned} y'(x) &= -e^{-x}(Ax^2 + Bx + C) + e^{-x}(2Ax + B) \\ y''(x) &= e^{-x}(Ax^2 + Bx + C) - 2e^{-x}(2Ax + B) + e^{-x}(2A). \end{aligned}$$

Pluging in the original ODE, we get

$$e^{-x}(Ax^2 + Bx + C) - 2e^{-x}(2Ax + B) + 2Ae^{-x} + 4e^{-x}(Ax^2 + Bx + C) = e^{-x}(5x^2 - 4x + 7).$$

Dividing by e^{-x} and collecting similar terms, we get

$$(5A)x^2 + (5B - 4A)x + (5C - 2B + 2A) = 5x^2 - 4x + 7.$$

We see from this equation that $A = 1$. Then, we must have $5B - 4 = -4$ which implies that $B = 0$. Finally, we must also have $5C + 2 = 7$ which implies that $C = 1$. Therefore, we obtain

$$y_p(x) = e^{-x}\left(x^2 + 1\right).$$

General solution.
Combining y_h and y_{par}, we get

$$y(x) = y_h(x) + y_{par}(x) = c_1 \cos(2x) + c_2 \sin(2x) + e^{-x}\left(x^2 + 1\right).$$

Section 5.4 — Problem 11 — 10 points

Find the general solution to the complementary equation.
The complementary equation is

$$y'' + 2y' + y = 0.$$

The characteristic equation associated to the complementary equation is $r^2 + 2r + 1 = 0$. There is only one root: $r_1 = -1$. The solution to the complementary equation is therefore

$$y_h(x) = c_1 e^{-x} + c_2 x e^{-x}.$$

Find a particular solution.
We have an exponential multiplying a polynomial of degree 1. However, the number $\alpha = -1$ is a root of the characteristic polynomial. Moreover, we have e^{-x} and xe^{-x} are solutions to the complementary equation. Therefore, we suggest

$$y_{par}(x) = x^2 e^{-x}(Ax + B) = e^{-x}(Ax^3 + Bx^2).$$

The respective derivatives are

$$\begin{aligned} y'(x) &= -e^{-x}(Ax^3 + Bx^2) + e^{-x}(3Ax^2 + 2Bx) \\ y''(x) &= e^{-x}(Ax^3 + Bx^2) - 2e^{-x}(3Ax^2 + 2Bx) + e^{-x}(6Ax + 2B). \end{aligned}$$

Plugging in the initial ODE, we obtain

$$\begin{aligned} e^{-x}(Ax^3 + Bx^2) - 2e^{-x}(3Ax^2 + 2Bx) + e^{-x}(6Ax + 2B) + 2e^{-x}(3Ax^2 + 2Bx - Ax^3 - Bx^2) \\ + e^{-x}(Ax^3 + Bx^2) = e^{-x}(3x + 2). \end{aligned}$$

Dividing by e^{-x} and collecting similar terms, we obtain

$$6Ax + 2B = 3x + 2$$

We therefore obtain $A = 2$ and $B = 1$. So, the particular solution is

$$y_{par}(x) = e^{-x}(2x^3 + x^2).$$

General solution.
The general solution is therefore

$$y(x) = y_h(x) + y_{par}(x) = c_1 e^{-x} + c_2 x e^{-x} + e^{-x}(2x^3 + x^2).$$

Section 5.4 — Problem 21 — 15 points

Complementary Equation.
The complementary equation is

$$y'' + 3y' - 4y = 0.$$

The characteristic polynomial associated to the complementary equation is $r^2 + 3r - 4 = 0$. The roots are $r = -1$ and $r = 4$. Therefore, the solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^{4x}.$$

Find a particular solution.
We have an exponential times a polynomial of degree two. Also, the number $\alpha = 2$ is a not a root of the characteristic polynomial. We therefore suggest

$$y_{par}(x) = e^{2x}(Ax + B).$$

The respective derivatives are

$$\begin{aligned} y'(x) &= 2e^{2x}(Ax + B) + Ae^{2x} \\ y''(x) &= 4e^{2x}(Ax + B) + 4Ae^{2x} + 2Ae^{2x}. \end{aligned}$$

Plugging in the original ODE, we obtain

$$4e^{2x}(Ax + B) + 4Ae^{2x} + 2Ae^{2x} + 6e^{2x}(Ax + B) + 3Ae^{2x} - 4e^{2x}(Ax + B) = e^{2x}(6x + 7).$$

Dividing by e^{2x} and collecting similar terms, we obtain

$$6Ax + (7A + 6B) = 6x + 7$$

and therefore $A = 1$ and $B = 0$. The particular solution is therefore

$$y_{par}(x) = xe^{2x}.$$

General solution.
Combining y_h and y_{par}, we obtain

$$y(x) = y_h(x) + y_{par}(x) = c_1 e^{-x} + c_2 e^{4x} + xe^{2x}.$$

Initial Value Problem.
We have $y(0) = 2$, so

$$c_1 + c_2 = 2.$$

We have $y'(x) = -c_1 e^{-x} + 4c_2 e^{4x} + e^{2x} + 2xe^{2x}$ and with $y'(0) = 8$, we obtain

$$-c_1 + 4c_2 = 7.$$

Adding the first equation to the second equation, we obtain

$$5c_2 = 9 \quad \Rightarrow \quad c_2 = 9/5.$$

Replacing c_2 in the first equation by $9/5$, we obtain

$$c_1 = 2 - 9/5 = 1/5.$$

Therefore, the solution to the IVP is

$$y(x) = \frac{e^{-x}}{5} + \frac{9}{5}e^{4x} + xe^{2x}.$$

Section 5.4 — Problem 30(a) — 5 points

Suppose that y is a solution to the constant coefficient equation

$$ay'' + by' + cy = e^{\alpha x} G(x).$$

Dividing by $e^{\alpha x}$, we obtain

$$e^{-\alpha x}(ay'' + by' + cy) = G(x).$$

Define $u = e^{-\alpha x} y(x)$, so that $y(x) = ue^{\alpha x}$. The first and second derivatives of y are therefore

$$\begin{aligned} y' &= \alpha e^{\alpha x} u + e^{\alpha x} u' \\ y'' &= \alpha^2 e^{\alpha x} u + 2\alpha e^{\alpha x} u' + e^{\alpha x} u''. \end{aligned}$$

Replacing those derivatives in the ODE, we have

$$\begin{aligned} e^{-\alpha x}(ay'' + by' + c) &= e^{-\alpha x}\left(a\alpha^2 e^{\alpha x} u + 2a\alpha e^{\alpha x} u' + ae^{\alpha x} u'' + b\alpha e^{\alpha x} u + be^{\alpha x} u' + ce^{\alpha x} u\right) \\ &= a\alpha^2 u + 2a\alpha u' + au'' + b\alpha u + bu' + cu) \\ &= au'' + (2a\alpha + b)u' + (a\alpha^2 + b\alpha + c)u. \end{aligned}$$

Since the left-hand side is equal to $G(x)$, we conclude that u is a solution to the following ODE:

$$au'' + (2a\alpha + b)u' + (a\alpha^2 + b\alpha + c)u = G(x).$$

Denote by $p(r) = ar^2 + br + c$ the characteristic polynomial. Then we can see that

$$2a\alpha + b = p'(\alpha)$$

and

$$a\alpha^2 + b\alpha + c = p(\alpha).$$

Therefore, the function u is a solution to

$$au'' + p'(\alpha)u' + p(\alpha)u = G(x).$$

Now suppose that $y(x) = e^{\alpha x} u$ where u is a solution to

$$au'' + p'(\alpha)u' + p(\alpha)u = G(x).$$

From the calculation above, we see that

$$au'' + p'(\alpha)u' + p(\alpha)u = e^{-\alpha x}(ay'' + by' + c).$$

Therefore, we get

$$e^{-\alpha x}(ay'' + by' + cy) = G(x)$$

and so y is a solution to the ODE

$$ay'' + by' + cy = e^{\alpha x} G(x).$$

Section 5.5 — Problem 7 — 10 points

Complementary Equation.
The complementary equation is

$$y'' + 4y = 0.$$

The characteristic polynomial associated to the complementary equation is $r^2 + 4$. The roots of this polynomial are $r_1 = 2i$ and $r_2 = -2i$. Therefore, the solution is

$$y_h(x) = c_1 \cos(2x) + c_2 \sin(2x).$$

Find a particular solution.
We have a linear combination of $\cos(2x)$ and $\sin(2x)$. However, $\cos(2x)$ and $\sin(2x)$ are in the solutions to the complementary equation. Therefore, we suggest

$$y_{par}(x) = x(A\cos(2x) + B\sin(2x)).$$

The respective derivatives are

$$\begin{aligned} y'(x) &= A\cos(2x) + B\sin(2x) + x(2B\cos(2x) - 2A\sin(2x)) \\ y''(x) &= 4B\cos(2x) - 4A\sin(2x) - x(4A\cos(2x) + 4B\sin(2x)). \end{aligned}$$

Plugging in the original ODE, we get

$$4B\cos(2x) - 4A\sin(2x) - 4Ax\cos(2x) - 4Bx\sin(2x) + 4Ax\cos(2x) + 4Bx\sin(2x) = -12\cos x - 4\sin x$$

which simplifies to

$$4B\cos(2x) - 4A\sin(2x) = -12\cos(2x) - 4\sin(2x).$$

Therefore, we obtain $B = -3$ and $A = 1$. The particular expression is

$$y_{par}(x) = x\cos(2x) - 3x\sin(2x).$$

General solution.
The general solution is

$$y(x) = y_h(x) + y_{par}(x) = c_1\cos(2x) + c_2\sin(2x) + x\cos(2x) - 3x\sin(2x).$$

Total (Points): 50.