Math 644

Chapter 3

Section 3.1: The Maximum Principle

Contents

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First Version

Theorem 1. Suppose f is analytic in a region Ω. If there exists a $z_0 \in \Omega$ such that

$$|f(z_0)| = \sup_{z \in \Omega} |f(z)|,$$

then f is constant in Ω.

Lemma 2. If $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ which converges in $\{z : |z - z_0| < r_0\}$ for some $r_0 > 0$, then for $r < r_0$

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})\, dt.$$

Proof.

Proof of the Maximum Modulus Principle.

Note:

- If f is analytic in Ω and there is a $z_0 \in \Omega$ such that $|f(z_0)| = \inf_{z\in\Omega} |f(z)|$ and $|f(z_0)| \neq 0$, then f is constant in Ω.

Second version

Corollary 3. If f is a non-constant analytic function in a bounded region Ω, and if f is continuous on $\overline{\Omega} = \mathrm{clos}(\Omega)$, then

$$\max_{z \in \overline{\Omega}} |f(z)|$$

occurs on $\partial\Omega$, but not in Ω.

Note:

- The requirement that Ω is bounded is necessary: the function $f(z) = e^{-iz}$ is
  - analytic in the upper half-plane $\mathbb{H} := \{z : \operatorname{Im} z > 0\}$;
  - continuous on $\{z : \operatorname{Im} z \geq 0\}$ and;
  - has absolute value 1 on the real line $\mathbb{R}$.

  However, f is not bounded by 1 in $\mathbb{H}$.

Let Ω be a region in $\mathbb{C}$.

- A sequence $(z_n)_{n\geq 1}$ tends to $\partial\Omega$ if for any compact subset $K \subset \Omega$, there exists an $N \in \mathbb{N}$ such that $z_n \notin K$, when $n \geq N$.
- The region Ω can be unbounded. In this case, we consider the region as lying in $\mathbb{C}^*$ and ∞ might be on $\partial\Omega$.
- If $f : \Omega \to \mathbb{C}$ is a continuous function, then

$$\limsup_{z\to\partial\Omega} |f(z)| := \sup\Big\{ \limsup_{n\to\infty} |f(z_n)| \;:\; z_n \to \partial\Omega \Big\}.$$

  We can show that, if Ω is bounded, then

$$\limsup_{z\to\partial\Omega} |f(z)| = \lim_{\delta\to 0} \sup\{|f(z)| \;:\; z \in \Omega, \mathrm{dist}(z, \partial\Omega) = \delta\}$$

EXAMPLE 4.

a) Show that $z_n \to \partial\mathbb{D}$ if and only if $|z_n| \to 1$, as $n \to \infty$.

b) Let $\Omega := \{z \;:\; |z| > 2\}$. Compute $\displaystyle\limsup_{z\to\partial\Omega} \left|\frac{1+z}{1-z}\right|$.

THEOREM 5. If f is analytic on a bounded region Ω, then

$$\limsup_{z \to \partial\Omega} |f(z)| = \sup_{\Omega} |f(z)|.$$

Proof.

Note:

- If f is continuous on $\overline{\Omega}$, then we recover the second version of the Maximum Principle.