Math 307

Chapter 5

Section 5.5: Similar Matrices, Diagonalization, and Jordan Canonical Form

Contents

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Motivation

Example 1. Let A be the 3×3 matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

Then, (a) compute A^5 (b) find the eigenvalues of A (c) find a basis for each eigenspace.

Remarks

- It is pretty easy to deal with diagonal matrices.
- Our goal is to try to transform a general matrix into a diagonal matrix.

Example 2. Let A be the following 3×3 matrix

$$A = \begin{bmatrix} 6 & -4 & -2 \\ 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Find (a) the eigenvalues of A (b) a basis for each eigenspace (c) compute A^5.

Definition

Diagonalizable Matrices:

An $n \times n$ matrix A is *diagonalizable* if there is a matrix D and an invertible matrix P such that

$$A = P^{-1}DP$$

Facts:

- Let A be an $n \times n$ matrix.
- Let λ_1, λ_2, …, λ_k be the eigenvalues of A.
- Let E_{λ_1}, E_{λ_2}, …, E_{λ_k} be the eigenspaces associated to each eigenvalue.

If $\dim(E_{\lambda_1}) + \dim(E_{\lambda_2}) + \cdots + \dim(E_{\lambda_k}) = n$, then A is diagonalizable.

EXAMPLE 3. Is the matrix from Example 2 diagonalizable?

Example 4. Is the matrix

$$A = \begin{bmatrix} 1 & -2 & -6 \\ -2 & 2 & -5 \\ 2 & 1 & 8 \end{bmatrix}$$

diagonalizable? If so, determine the invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

Example 5. Is the matrix

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

diagonalizable? If so, determine the invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

In general:

An $n \times n$ matrix A is *similar* to an $n \times n$ matrix B if there is an invertible $n \times n$ matrix P such that

$$B = P^{-1}AP.$$

Notation: $A \sim B$ means that A is similar to B.

Facts:

- If A is similar to B and B is similar to C, then A is similar to C.
- If P is the change of bases matrix from α to β and T is a linear transformation, then $[T]_\beta^\beta = P^{-1}[T]_\alpha^\alpha P$. So $[T]_\beta^\beta \sim [T]_\alpha^\alpha$.

Question:

For non-diagonalizable matrices, can we do something?

Answer: Yes! We will replace the diagonal form by the Jordan canonical form.

Jordan blocks

A Jordan block is a square matrix A taking the following shape:

$$A = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{bmatrix}.$$

Why are these type of matrices important?

Example 6. Let A be the matrix

$$A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}.$$

(a) Compute $\det(\lambda I - A)$. (b) Find the dimension of the eigenspaces.

Remark:

- A $n \times n$ Jordan block associated to a number λ has only one eigenvalue.
- The multiplicity of this eigenvalue is necessarily equal to n.
- The eigenspace E_λ is then $\dim(E_\lambda) = n$.
- Jordan blocks are the building blocks for the set of matrices that can't be diagonalizable.

Reduction to Jordan Blocks

EXAMPLE 7. We know that the matrix

$$A = \begin{bmatrix} 1 & -2 & -6 \\ -2 & 2 & -5 \\ 2 & 1 & 8 \end{bmatrix}$$

is not diagonalizable. Find a matrix B, not necessarily a diagonal matrix, such that A is similar to B.

General Procedure: Suppose A is an $n \times n$ matrix.

- Express $\det(\lambda I - A)$ as

$$\det(\lambda I - A) = (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_k)^{m_k}$$

  where m_1 is the multiplicity of λ_1, m_2 is the multiplicity of λ_2, ..., m_k is the multiplicity of λ_k.

- For each λ_j, write

$$A_j = \begin{bmatrix} J_{m_{j-1}+1} & 0 & \cdots & 0 \\ 0 & J_{m_{j-1}+2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_{m_j} \end{bmatrix}.$$

  where each J_p, for $p = m_{j-1} + 1, \dots m_j$, is a Jordan block

$$J_p = \begin{bmatrix} \lambda_j & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_j & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_j & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_j \end{bmatrix}.$$

- Then the Jordan Canonical Form (JCF) is

$$B = \begin{bmatrix} A_1 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ 0 & 0 & A_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_k \end{bmatrix}$$

- The invertible matrix P such that $B = P^{-1}AP$ is more complicated to find. In theory, the method to find P uses the notion of a **generalized eigenvector**. In our situation, we will use Python to find this matrix P.

If you want to know more on the generalized eigenvectors and the Jordan Canonical Form, I suggest to take a look at the following references:

- A more math article: *Down With Determinants!* by Sheldon Axler, `https://www.maa.org/sites/default/files/pdf/awards/Axler-Ford-1996.pdf`.
- A Youtube video: `https://www.youtube.com/watch?v=GVixvieNnyc`.