

# MATH 302

## CHAPTER 8

### SECTION 8.2: LAPLACE TRANSFORMS

CONTENTS
----------

---

<b>From ODE to Algebra</b>	<b>2</b>
<b>Inverse Laplace Transform</b>	<b>4</b>
Linearity of Inverse Transform . . . . .	5
Inverse Laplace Transform of Rational Functions . . . . .	7
Rational Functions with Powers in the Denominator . . . . .	11
<b>The ODE Problem</b>	<b>12</b>

The Laplace transform is REALLY<sup>REALLY</sup> useful to solve ODE.

**EXAMPLE 1.** Consider the ODE

$$2y''(t) + 3y'(t) + y(t) = 8e^{-2t}$$

with  $y(0) = -4$  and  $y'(0) = 2$ .

**General Procedure:**

1. Apply the Laplace transform to your ODE  $ay'' + by' + cy = f(t)$ .
2. Apply the properties of the Laplace transform to get

$$a(s^2Y - sf(0) - f'(0)) + b(sY - f(0)) + cY = F.$$

3. Isolate  $Y$ :

$$Y = \frac{F + (as + b)f(0) + af'(0)}{as^2 + bs + c}.$$

The last step:

- Take the inverse Laplace transform!

Given a Laplace transform  $F(s)$  of an unknown function  $f$ , we can go backward to find  $f$ .

- We denote the **inverse Laplace transform** by  $L^{-1}$ .
- We therefore have  $f = L^{-1}(F)$ .
- How do we find  $L^{-1}(F)$ ?

Trick: Use the table in the opposite direction!

**EXAMPLE 2.** Find the inverse Laplace transform of the following functions:

(a)  $\frac{1}{s^2 - 1}$ .

(b)  $\frac{s}{s^2 + 9}$ .

## Linearity of Inverse Transform

If  $F$  and  $G$  are Laplace transforms of two unknown functions  $f$  and  $g$ , then

$$L^{-1}(aF + bG) = aL^{-1}(F) + bL^{-1}(G).$$

**EXAMPLE 3.** Find

$$L^{-1}\left(\frac{8}{s+5} + \frac{7}{s^2+3}\right).$$

**EXAMPLE 4.** Find

$$L^{-1}\left(\frac{3s+8}{s^2+2s+5}\right).$$

## Inverse Laplace Transform of Rational Functions

**EXAMPLE 5.** Find the inverse Laplace transform of

$$F(s) = \frac{3s + 2}{s^2 - 3s + 2}.$$

**EXAMPLE 6.** Find the inverse transform of

$$F(s) = \frac{6 + (s + 1)(s^2 - 5s + 11)}{s(s - 1)(s - 2)(s + 1)}.$$





General Case:

Suppose your Laplace transform is

$$F(s) = \frac{P(s)}{(s - s_1)(s - s_2) \cdots (s - s_n)}$$

where  $s_1, s_2, \dots, s_n$  are distinct and  $P$  is a polynomial of degree less than  $n$ . Then

$$F(s) = \frac{A_1}{s - s_1} + \frac{A_2}{s - s_2} + \cdots + \frac{A_n}{s - s_n},$$

- $A_1$  is computed by letting  $s = s_1$  in  $G(s) = \frac{P(s)}{(s - s_2) \cdots (s - s_n)}$ .
- $A_2$  is computed by letting  $s = s_2$  in  $G(s) = \frac{P(s)}{(s - s_1)(s - s_3) \cdots (s - s_n)}$ .
- $\vdots$
- $A_n$  is computed by letting  $s = s_n$  in  $G(s) = \frac{P(s)}{(s - s_1)(s - s_2) \cdots (s - s_{n-1})}$ .

**EXAMPLE 7.** Write in partial fractions the following rational function:

$$F(s) = \frac{s^2 + 1}{s^3 - s}.$$

## Rational Functions with Powers in the Denominator

The Heaviside method doesn't work if we encounter powers of monomials in the denominator. What do we do then?

**EXAMPLE 8.** Find the partial fraction expansion of

$$F(s) = \frac{8 - (s + 2)(4s + 10)}{(s + 1)(s + 2)^2}.$$

**EXAMPLE 9.** Using the inverse Laplace transform, complete Example 1.

**EXAMPLE 10.** Use the Laplace transform to solve the initial value problem:

$$y'' - 6y' + 5y = 3e^{2t}, \quad y(0) = 2, \quad y'(0) = 3.$$

