Math 644

Chapter 4

Section 4.2: Equivalence of Analytic and Holomorphic

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Spring 2023

Definition 1. Let U be an open set and $f : U \to \mathbb{C}$. The function f is holomorphic on U if

- $f'(z) := \lim_{w \to z} \frac{f(w) - f(z)}{w - z}$ exists for all $z \in U$ and;
- $z \mapsto f'(z)$ is continuous on U.

Notes:

- f is holomorphic on U, then f is continuous on U;
- A complex-valued function f is holomorphic on a (generic) set S if it is holomorphic on an open set $U \supset S$.
- There are weaker definitions of a holomorphic functions: For example, one definition does not require that $z \mapsto f'(z)$ is continuous.

Example 2.

a) Any polynomial is a holomorphic function on $\mathbb{C}$.

b) Any rational function is a holomorphic function on their domain.

c) Any power series is a holomorphic function on its disk of convergence.

d) Any analytic function $f : \Omega \to \mathbb{C}$ is a holomorphic function on Ω.

Particular Derivatives:

(*) $f(z) = (z-a)^n$, $n \in \mathbb{Z}$ $(n \neq 0)$ $\to$ $f'(z) = n(z-a)^{n-1}$

$(z \neq a,\ n-1 < 0)$.

(**) $f(z) = a_n z^n + \dots + a_1 z + a_0$

$\Rightarrow$ $f'(z) = n a_n z^{n-1} + \dots + a_1$

(***) Usual rules for derivatives.

THEOREM 3. If f is holomorphic in $\{z : |z - z_0| \leq r\}$, then, for $|z - z_0| < r$,

$$f(z) = \frac{1}{2\pi i} \int_{C_r} \frac{f(\zeta)}{\zeta - z} \, d\zeta,$$

where C_r is the circle of radius r centered at z_0, parameterized in the counter-clockwise direction.

LEMMA 4. Let f be a holomorphic function in a neighborhood of γ and $\gamma : [a, b] \to \mathbb{C}$ be a piecewise continuously differentiable curve, then

$$\int_\gamma f'(z) \, dz = f(\gamma(b)) - f(\gamma(a)).$$

Proof: In particular, $t \mapsto f(\gamma(t))$ is piecewise cont. differentiable. & $\frac{d}{dt}(f(\gamma(t))) = f'(\gamma(t))\,\gamma'(t)$.

Therefore,

$$\int_\gamma f'(z)\,dz = \int_a^b f'(\gamma(t))\,\gamma'(t)\,dt = \int_a^b \frac{d}{dt}(f(\gamma(t)))\,dt$$

So, from FTC, $\int_\gamma f'(z)\,dz = f(\gamma(b)) - f(\gamma(a))$. □

COROLLARY 5. If $\gamma : [a, b] \to \mathbb{C}$ is a closed, piecewise continuously differentiable curve, and if f is holomorphic in a neighborhood of γ, then

$$\int_\gamma f'(z) \, dz = 0.$$

Proof: Comes from Lemma 4 &

$$\gamma(b) = \gamma(a)$$

$$\Rightarrow \quad f(\gamma(b)) - f(\gamma(a)) = 0.$$ □

Corollary 6. If $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges in $B = \{z : |z - z_0| < r\}$, and if $\gamma \subset B$ is a closed, piecewise continuously differentiable curve, then

$$\int_\gamma f(z)\, dz = 0.$$

Proof. Recall that $F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z-z_0)^{n+1}$, then $F'(z) = f(z)$ & F converges in B. So, from Corollary 5,

$$\int_\gamma F'(z) = 0 \Rightarrow \int_\gamma f(z)\, dz = 0. \qquad \square$$

Theorem 7. Let $n \in \mathbb{Z}$, let γ be a piecewise continuously differentiable curve and let $a \notin \gamma$.

a) If $n \neq 1$, then

$$\int_\gamma \frac{1}{(z-a)^n}\, dz = 0.$$

b) If $\gamma = C_r = \{z : |z - z_0| = r\}$, then

$$\frac{1}{2\pi i} \int_{C_r} \frac{1}{z-a}\, dz = \begin{cases} 1 & \text{if } |a - z_0| < r \\ 0 & \text{if } |a - z_0| > r. \end{cases}$$

Proof.

a) For $n \neq 1$, $\frac{d}{dz}\left(\frac{1}{-(n-1)(z-a)^{n-1}} \right) = \frac{1}{(z-a)^n}$ $(z \neq a)$

$$\Rightarrow \int_\gamma \frac{1}{(z-a)^n}\, dz = 0 \text{ by Cor. 5.}$$

b) Let $C_r(t) = z_0 + re^{it}$, $0 \leq t \leq 2\pi$.

$\underline{|a - z_0| > r}$ we have

$$\frac{1}{z-a} = \frac{1}{z - z_0 + z_0 - a} = \frac{1}{-(a-z_0)\left(1 - \frac{z-z_0}{a-z_0}\right)}$$

$$= \frac{-1}{a-z_0} \sum_{n=0}^{\infty} \left(\frac{z-z_0}{a-z_0}\right)^n \quad (|z-z_0| \leq r)$$

Therefore,

$$\int_{C_r} \frac{1}{(z-a)}\,dz = -\frac{1}{a-z_0}\sum_{n=0}^{\infty}\frac{i}{(a-z_0)^n}\underbrace{\int_0^{2\pi} r^{n+1}e^{i(n+1)t}\,dt}_{=0\ \forall n\geq 0.}$$

$$= 0.$$

<u>$|a-z_0|<r$</u>

$$\int_{C_r}\frac{1}{z-a}\,dz = \int_0^{2\pi}\frac{ire^{it}}{z_0+re^{it}-a}\,dt$$

$$= i\int_0^{2\pi}\frac{1}{1-\frac{a-z_0}{re^{it}}}\,dt.$$

$$= i\int_0^{2\pi}\sum_{n=0}^{\infty}\frac{(a-z_0)^n}{r^n e^{int}}\,dt$$

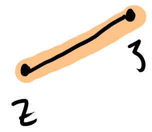
$$= i\sum_{n=0}^{\infty}\frac{(a-z_0)^n}{r^n}\int_0^{2\pi}e^{-int}\,dt$$

$$= 2\pi i.$$

$$\Rightarrow \frac{1}{2\pi i}\int_{C_r}\frac{1}{z-a}\,dz = 1. \qquad \square$$

Proof of Cauchy's Integral Formula.

Suppose $|z-z_0|<r$.



For $\zeta \in C_r$,

$$\frac{f(\zeta)-f(z)}{\zeta - z} = \int_0^1 f'(z+t(\zeta-z))\,dt$$

So,

$$\int_{C_r} \frac{f(\zeta)-f(z)}{\zeta-z}\,d\zeta = \int_{C_r}\int_0^1 f'(z+t(\zeta-z))\,dt\,d\zeta$$

$$\overset{\text{Fubini}}{=} \int_0^1 \int_{C_r} f'(z+t(\zeta-z))\,d\zeta\,dt$$

$$= \lim_{\varepsilon \to 0} \int_\varepsilon^1 \int_{C_r} f'(z+t(\zeta-z))\,t\,d\zeta\,\frac{dt}{t}$$

$$= \lim_{\varepsilon\to 0}\int_\varepsilon^1 \underbrace{\int_{C_r} \frac{d}{d\zeta}\left[f(z+t(\zeta-z))\right] d\zeta}_{=0}\,\frac{dt}{t}$$

$$= 0$$

Therefore,

$$f(z) - \frac{1}{2\pi i}\int_{C_r}\frac{f(\zeta)}{\zeta - z}\,d\zeta = 0.$$

$\square$

Corollary 8. Let $f : \Omega \to \mathbb{C}$ be a function defined on a region Ω.

a) f is holomorphic in Ω if and only if f is analytic in Ω.

b) Moreover, the series expansion of f based at $z_0 \in \Omega$ converges on the largest open disk centered at z_0 and contained in Ω.

Proof.

(a) f analytic in $\Omega \Rightarrow f$ is holomorphic in Ω.

Suppose f is holomorphic in Ω. If $z_0 \in \Omega$, there is a $B_{r_0} = \{ z : |z - z_0| < r_0 \} \subseteq \Omega$. So f is holomorphic on B_{r_0}. Fix $r < r_0$ & from Thm. 3,

$$f(z) = \frac{1}{2\pi i} \int_{C_r} \frac{f(\zeta)}{\zeta - z} \, d\zeta \quad (z \in B_r)$$

$$= \frac{1}{2\pi i} \int_{C_r} f(\zeta) \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}} \, d\zeta$$

$$(*) = \sum_{n=0}^{\infty} \underbrace{\left[\frac{1}{2\pi i} \int_{C_r} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta \right]}_{a_n} (z - z_0)^n .$$

Since $(*)$ converges $\forall z \in B_r$, f is analytic at z_0.

(b) Choose B_{r_0} s.t. $\partial B_{r_0} \cap \partial \Omega \neq \emptyset$. $\square$

Note:

- In particular, if f is analytic in $\mathbb{C}$, then f has a power series expansion which converges in all of $\mathbb{C}$. Such functions are called **entire**.
- From now on, the words "holomorphic" and "analytic" are used interchangably.

Example 9.

Ω

$\shortparallel$

a) Show that $f(z) = \frac{z}{e^z - 1}$ is holomorphic in $\mathbb{C} \backslash \{2k\pi i : k \in \mathbb{Z}, k \neq 0\}$.

b) Use this to show that the radius of the power series based at 0

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} a_n z^n \qquad a_n = \frac{B_n}{n!} \quad f^{(n)}(0)$$

is 2π.

a) f is continuous on the set $\mathbb{C} \backslash \{2k\pi i : k \in \mathbb{Z}\}$.

At $z=0$, we have, for $h \neq 0$, $|h| < 2\pi$,

$$\frac{h}{e^h - 1} = \frac{h}{\sum_{n=0}^{\infty} \frac{h^n}{n!} - 1} = \frac{h}{\sum_{n=1}^{\infty} \frac{h^n}{n!}} = \frac{1}{\sum_{n=1}^{\infty} \frac{h^{n-1}}{n!}} = 1.$$

So, $f(0) = 1 = \lim_{h \to 0} \frac{h}{e^h - 1}$.

$f'(z)$ exists and is continuous on $\mathbb{C} \backslash \{2\pi k i : k \in \mathbb{Z}\}$.

For $z=0$, we have, for small h:

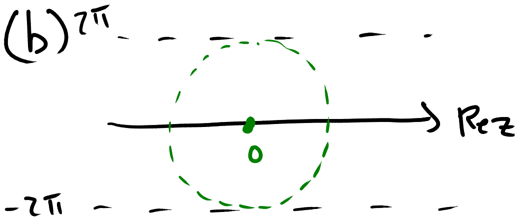
$$\lim_{h \to 0} \frac{\frac{h}{e^h - 1} - 1}{h} = \lim_{h \to 0} \frac{h - e^h + 1}{h(e^h - 1)}$$

$$= \lim_{h \to 0} \frac{-\sum_{n=2}^{\infty} \frac{h^n}{n!}}{h \sum_{n=1}^{\infty} \frac{h^n}{n!}} = \lim_{h \to 0} \frac{-\sum_{n=2}^{\infty} \frac{h^{n-2}}{n!}}{\sum_{n=1}^{\infty} \frac{h^{n-1}}{n!}}$$

$$= -\frac{1}{2}$$

So, f' is continuous on Ω, so holomorphic on Ω.

(b)



The radius of the biggest disk is 2π

$$\Rightarrow R = \liminf_{n \to \infty} |a_n|^{-1/n} = 2\pi.$$

Scholium 10. If f is analytic in $\{z : |z - z_0| \leq r\}$ and $C_r = \{z_0 + re^{it} : 0 \leq t \leq 2\pi\}$, then

a) $\dfrac{f^{(n)}(z_0)}{n!} = \dfrac{1}{2\pi} \displaystyle\int_{C_r} \dfrac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta$. [Cauchy's Integral Formula for $f^{(n)}$]

b) $\left|\dfrac{f^{(n)}(z_0)}{n!}\right| \leq \dfrac{\sup_{C_r} |f(z)|}{r^n}$. [Cauchy's Estimate]

Proof.

a) From the proof of thm. 8 : $|z - z_0| < r$.

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i}\int_{C_r} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta\right)(z - z_0)^n$$

Uniq. of Power series $\Rightarrow$ $\dfrac{f^{(n)}(z_0)}{n!} = \dfrac{1}{2\pi i}\displaystyle\int_{C_r} \dfrac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta$.

b) $$\frac{|f^{(n)}(z_0)|}{n!} = \frac{1}{2\pi}\left|\int_{C_r} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta\right|$$

$$\leq \frac{1}{2\pi}\int_{C_r} \frac{|f(\zeta)|}{r^{n+1}}\, |d\zeta|$$

$$\leq \frac{1}{2\pi r^{n+1}} \sup_{C_r} |f| \; |C_r| \rightarrow 2\pi r$$

$$= \frac{\sup_{C_r} |f|}{r^n}. \qquad \square$$

Corollary 11. If f is analytic in an open disk B, and if $\gamma \subset B$ is a closed, piecewise continuously differentiable curve, then

$$\int_\gamma f(z)\, dz = 0.$$

THEOREM 12. If f is analytic and one-to-one in a region Ω, then the inverse of f, defined on $f(\Omega)$, is analytic.

LEMMA 13. If f is an analytic function at z_0 with

$$f(z) - f(z_0) = \sum_{n \geq m} a_n (z - z_0)^n \quad (a_m \neq 0,\, m \geq 2)$$

in some disk B_1 centered at z_0, then there is a $\varepsilon > 0$ and a δ so that $\overset{\overset{0}{\shortparallel}}{f(z) - w}$ has exactly m solutions in $\{z : |z - z_0| < \varepsilon\}$, for any $w \in \{v : |v - f(z_0)| < \delta\}$.

Proof. Write $g(z) = \sum_{n=0}^{\infty} \frac{a_{n+m}}{a_m} (z - z_0)^n$, $z \in B_1$ so

$$f(z) - f(z_0) = a_m (z - z_0)^m g(z).$$

Notice, $g(z_0) = 1$ & g is analytic. Take $B_2 \subseteq B_1$ s.t. $g(B_2) \subseteq \{w : |w - 1| < \mu\}$, $\mu < 1$ & g is one-to-one in B_2.

Define $F(z) = \sqrt[n]{g(z)}$ so $F(z)$ is well-defined in B_2 and for $a^m = a_m$:

$$f(z) - f(z_0) = [a(z - z_0) F(z)]^m.$$

Take $B_3 \subseteq B_2$ s.t. F is one-to-one in B_3.

because $F(1) = 1 \neq 0$.

Write $B_3 = \{z : |z - z_0| < \varepsilon\}$, some $\varepsilon > 0$.

Then, for δ small enough s.t. $\{w : |w - f(z_0)| < \delta\}$ included in $f(B_3)$.

$$w - f(z) = 0 \iff w - f(z_0) = [a(z - z_0) F(z)]^m \quad (*)$$

and there are exactly m solutions to $(*)$ because $a(z - z_0) F(z)$ is injective in B_3 $\square$.

Proof of Theorem 12.

We know that f is an open map, so $f^{-1}: f(\Omega) \to \Omega$ is a homeomorphism.

Also, if $f'(z_0) = 0$, then from Lemma 13, f is not injective in some disk B centered at z_0. This contradicts the assumption that f is one-to-one on Ω.

Set $z_0 = f^{-1}(w_0)$, $w_0 \in f(\Omega)$ & $z = f^{-1}(w)$ for $w \in f(\Omega)$.

$$\lim_{w \to w_0} \frac{f^{-1}(w) - f^{-1}(w_0)}{w - w_0} = \lim_{z \to z_0} \frac{z - z_0}{f(z) - f(z_0)}$$

$$= \frac{1}{f'(z_0)}.$$

So, $(f^{-1})'(z_0) = \dfrac{1}{f'(z_0)}$ & is continuous because f' is. $\square$

THEOREM 14. If f is continuous in an open disk B, and if

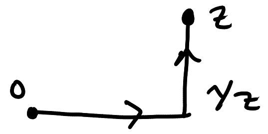
$$\int_{\partial R} f(\zeta)d\zeta = 0$$

for all closed rectangles $R \subset B$ with sides parallel to the axes, then f is analytic in B.

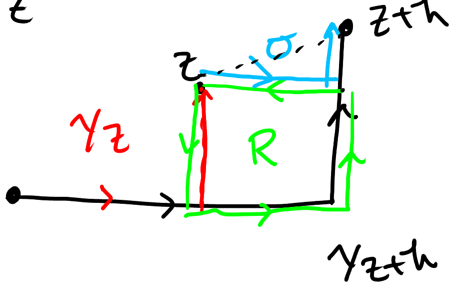
Proof.

We may assume, WLOG, that $B = \mathbb{D}$. $(\varphi(z) = rz + a)$

Define $F(z) = \int_{\gamma_z} f(\zeta)d\zeta$ where $z \in \mathbb{D}$ &



If $|h| < 1 - |z|$, then $\gamma_{z+h} = \gamma_z + \sigma + \partial R$



Now,

$$F(z+h) - F(z) = \int_{\gamma_{z+h}} f(\zeta) - \int_{\gamma_z} f(\zeta)d\zeta$$

$$= \int_\sigma f(\zeta)\, d\zeta .$$

By FTC: $\int_\sigma d\zeta = z + h - z = h$ and so

$$\frac{F(z+h) - F(z)}{h} - f(z) = \frac{1}{h} \int_\sigma f(\zeta) - f(z)\, d\zeta$$

So

$$\left| \frac{F(z+h) - F(z)}{h} - f(z) \right| \leq \sup_\sigma |f(\zeta) - f(z)| \cdot \frac{|\sigma|}{|h|}$$

$$\leq \sup_\sigma |f(\zeta) - f(z)| \cdot \sqrt{2}$$

$|\sigma| \leq \sqrt{2}\, |h|$

As $h \to 0$, $\sup_{\sigma} |f(\zeta) - f(z)| \to 0$ & $F'(z) = f(z)$.

Now, f is continuous, so F' is continuous.

This means F is holomorphic, so analytic in $\mathbb{D}$.

So, $F' = f$ is also analytic. $\square$