Math 471

More Problems Set

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A.III Axioms of Probability

Problem 1. Let $(S, \mathcal{A}, P)$ be a probability space. If A and B are two events, then show that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Problem 2. Let S be a non-empty set and let A be a non-empty subset of S such that A is not all of S. If $\mathcal{A} = \{\varnothing, A, \overline{A}, S\}$, then show that all probability measures on $\mathcal{A}$ have the form

$$\begin{aligned} P(\varnothing) &= 0 & \qquad P(A) &= p \\ P(\overline{A}) &= 1 - p & \qquad P(S) &= 1 \end{aligned}$$

for some number p satisfying $0 \leq p \leq 1$.

Problem 3. Let $S = \{s_1, s_2, \ldots, s_N\}$ be a sample space with exactly N outcomes, and let $\mathcal{A}$ be the family of all subsets of S. Show that the function $P : \mathcal{A} \to \mathbb{R}$ defined by

$$P(A) = \frac{|A|}{N} \quad (A \text{ is an event})$$

is a probability measure.

A.IV Computing Probabilities in the Finite Case

Problem 4. A boxcar contains six complex electronic systems. Two of the six are to be randomly selected for thorough testing and then classified as defective or not defective. Two of the six systems are defective. Find the probability that one of the two systems selected will be defective.

Problem 5. A poker hand consists of 5 cards. If the cards have distinct consecutive numerical values and are not all of the same suit, we say that the hand is straight. For instance, the hands 5♠ 6♠ 7♠ 8♠ 9♥ or 6♥ 7♠ 8♦ 9♣ 10♥ are straight hands. What is the probability that one is dealt a straight hand? Assume that all possible pocker hands are equally likely.

Problem 6. Three teams of three people have to be selected from a group of 5 mathematicians, 2 engineers, 1 astrophysicist and 1 atmospheric scientist. If the selection is made randomly, what is the probability that at least one person in each team is a mathematician?

Problem 7. [Extra] A poker hand consists of 5 cards. What is the probability that a poker hand consists of all cards of the same suite and not consecutive values[1]. A valid example would be 2♥ 4♥ 5♥ 6♥ 7♥. Examples of non-valid poker hands are 2♥ 3♥ 4♥ 5♥ 6♥ and 2♥ 4♠ 5♠ 6♠ 7♠. Also, it is prohibited to create sequences of consecutive numbers by connecting the Ace with the 2. For example, the poker hand K♥ A♥ 2♥ 3♥ 4♥ is not valid. Assume all poker hands are equally likely.

[1]Here, by the "value" of a card, we mean the numerical values from 2 to 10 together with the Ace, King, Queen, Jack.

A.V Probability Space For Infinite Sample Spaces

PROBLEM 8. Let B_1, B_2, ... be the list of events defined in the proof of Theorem 1. Show that

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i.$$

Chapter B

B.I Conditional Probabilities

Problem 1. Two fair dice are rolled. What is the probability that at least one lands on 6 given that the dice land on different numbers?

Problem 2. Suppose that an urn contains 35 red balls and 20 blue balls. We draw 2 balls from the urn without replacement. If we assume that every ball is equally likely to be drawn, what is the probability that both balls drawn are red? Use the formula in Corollary 1.

B.II Bayes' Formula

Problem 3. A total of 46% of the voters in a certain city classify themselves as Independents, whereas 30% classify themselves as Liberals and 24% as Conservative. In a recent local election, 35% of the Independents, 62% of the Liberals, and 58% of the Conservatives voted. A voter is chosen at random. Given that this person voted in the local election, what is the probability that the person is a) an Independent? b) a Liberal? c) a Conservative?

Problem 4. When a dice x is tossed it lands on ⚁ with probability 1/2 and all the other outcomes are equally likely to happen. When a dice y is tossed, it lands on ⚂ with probability 1/2 and all the other outcomes are equally likely to happen. Suppose that one of these dice is randomly chosen and then tossed. What is the probability that dice x was tossed, if the die landed on ⚁?

B.III Independent Events

Problem 5. Let A, B be two events.

a) If $P(A) = 0.2$, $P(B) = 0.3$, and $P(A \cap B) = 0.3$, are A and B independent?

b) If $P(A) = 0.2$, $P(B) = 0.3$, and $P(A \cap B) = 0.06$, are A and B independent?

Problem 6. Let $(S, \mathcal{A}, P)$ be a probability space. Suppose that two events A and B are given such that $P(A) > 0$, $P(B) > 0$. Prove that if $P(A) < P(A|B)$, then $P(B) < P(B|A)$.

Problem 7. Suppose that $A \subset B$ and that $P(A) > 0$ and $P(B) > 0$. Show that $P(B|A) = 1$ and $P(A|B) = P(A)/P(B)$.

Problem 8. If A and B are mutually exclusive events and $P(B) > 0$, show that

$$P(A|A \cup B) = \frac{P(A)}{P(A) + P(B)}.$$

Problem 9. A system composed of 5 separate components is said to be a parallel system if it functions when at least one of the components functions. For such a system, if component i, independent of other components, functions with probability p_i, $i = 1, 2, \ldots, 5$, what is the probability that the system functions?

TOTAL NUMBER OF PROBLEMS: **17**