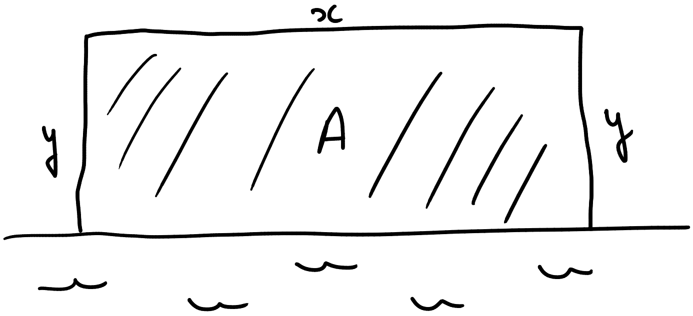
Chapter 3
Applications of Derivatives

3.7 Optimization Problems

EXAMPLE 1 A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

① Sketch



② Notation

x: width of the field (ft)
y: heigth of the field (ft)
A: area of the field. (ft^2)

③ Rule \ Equation.

$$A = xy$$

④ Elimination of one variable.

→ Need only 3 sides.
→ total fencing = 2400 ft

$$2y + x = 2400$$
$$\Rightarrow \quad x = 2400 - 2y$$

So, $A = (2400 - 2y)\,y = 2400y - 2y^2$

⑤ Optimize

$$A' = 2400 - 4y = 0 \quad \Rightarrow \quad 4y = 2400$$
$$\Rightarrow \quad y = 600$$

2nd test: $A''(y) = -4 < 0$ → abs. max at $y = 600$.

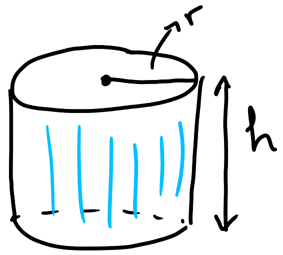
Answer:

$$\boxed{\begin{aligned} x &= 2400 - 2\cdot 600 = 1200 \text{ ft} \\ y &= 600 \text{ ft} \\ A &= 720\,000 \text{ ft}^2 \end{aligned}}$$

Recall: c critical number
(a) $f''(x) < 0$ (resp. $f''(x) > 0$) for all x, then $f(c)$ is abs. max (resp min).

EXAMPLE 2 A cylindrical can is to be made to hold $\overset{1000\ cm^3}{\boxed{1\ \text{L}}}$ of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

① Sketch:



② Notations

r: radius (cm).

h: heigth (cm).

V: volume (cm^3).

A: surface area (cm^2).

Goal: minimize A.

③ Equation

$$A = 2 \cdot A(\bigcirc) + 1 \times A\left(h\,\overset{2\pi r}{\square}\right)$$

$$= 2\pi r^2 + 2\pi r h$$

④ Eliminate one variable.

$$V = 1000 \Rightarrow \pi r^2 h = 1000$$

$$\Rightarrow h = \frac{1000}{\pi r^2}.$$

EXAMPLE 3 Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$.

EXAMPLE 4 A man launches his boat from point A on a bank of a straight river, 3 km wide, and wants to reach point B, 8 km downstream on the opposite bank, as quickly as possible (see Figure 7). He could row his boat directly across the river to point C and then run to B, or he could row directly to B, or he could row to some point D between C and B and then run to B. If he can row 6 km/h and run 8 km/h, where should he land to reach B as soon as possible? (We assume that the speed of the water is negligible compared with the speed at which the man rows.)