Math 644

Chapter 4

Section 4.2: Equivalence of Analytic and Holomorphic

Contents

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DEFINITION 1. Let U be an open set and $f : U \to \mathbb{C}$. The function f is holomorphic on U if

- $f'(z) := \lim_{w \to z} \dfrac{f(w) - f(z)}{w - z}$ exists for all $z \in U$ and;
- $z \mapsto f'(z)$ is continuous on U.

Notes:

- f is holomorphic on U, then f is continuous on U;
- A complex-valued function f is holomorphic on a (generic) set S if it is holomorphic on an open set $U \supset S$.
- There are weaker definitions of a holomorphic functions: For example, one definition does not require that $z \mapsto f'(z)$ is continuous.

EXAMPLE 2.

a) Any polynomial is a holomorphic function on $\mathbb{C}$.

b) Any rational function is a holomorphic function on their domain.

c) Any power series is a holomorphic function on its disk of convergence.

d) Any analytic function $f : \Omega \to \mathbb{C}$ is a holomorphic function on Ω.

Particular Derivatives:

(*) $f(z) = (z-a)^n$, $n \in \mathbb{Z}$ $(n \neq 0)$ $\rightarrow$ $f'(z) = n(z-a)^{n-1}$

$(z \neq a,\ n-1 < 0)$.

(**) $f(z) = a_n z^n + \cdots + a_1 z + a_0$

$\Rightarrow$ $f'(z) = n a_n z^{n-1} + \cdots + a_1$

(***) Usual rules for derivatives.

THEOREM 3. If f is holomorphic in $\{z : |z - z_0| \leq r\}$, then, for $|z - z_0| < r$,

$$f(z) = \frac{1}{2\pi i} \int_{C_r} \frac{f(\zeta)}{\zeta - z} \, d\zeta,$$

where C_r is the circle of radius r centered at z_0, parameterized in the counter-clockwise direction.

LEMMA 4. Let f be a holomorphic function in a neighborhood of γ and $\gamma : [a, b] \to \mathbb{C}$ be a piecewise continuously differentiable curve, then

$$\int_\gamma f'(z) \, dz = f(\gamma(b)) - f(\gamma(a)).$$

Proof: In particular, $t \mapsto f(\gamma(t))$ is piecewise cont. differentiable. &

$$\frac{d}{dt}\left(f(\gamma(t))\right) = f'(\gamma(t))\,\gamma'(t).$$

Therefore,

$$\int_\gamma f'(z)\,dz = \int_a^b f'(\gamma(t))\,\gamma'(t)\,dt = \int_a^b \frac{d}{dt}\left(f(\gamma(t))\right)dt$$

So, from FTC, $\int_\gamma f'(z)\,dz = f(\gamma(b)) - f(\gamma(a))$. □

COROLLARY 5. If $\gamma : [a, b] \to \mathbb{C}$ is a closed, piecewise continuously differentiable curve, and if f is holomorphic in a neighborhood of γ, then

$$\int_\gamma f'(z) \, dz = 0.$$

Proof: Comes from Lemma 4 &

$$\gamma(b) = \gamma(a)$$

$$\Rightarrow \quad f(\gamma(b)) - f(\gamma(a)) = 0.$$ □

Corollary 6. If $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges in $B = \{z : |z-z_0| < r\}$, and if $\gamma \subset B$ is a closed, piecewise continuously differentiable curve, then

$$\int_\gamma f(z)\, dz = 0.$$

Proof. Recall that $F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z-z_0)^{n+1}$, then $F'(z) = f(z)$ & F converges in B. So, from Corollary 5,

$$\int_\gamma F'(z) = 0 \Rightarrow \int_\gamma f(z)\, dz = 0. \quad \square$$

Theorem 7. Let $n \in \mathbb{Z}$, let γ be a piecewise continuously differentiable curve and let $a \notin \gamma$.

a) If $n \neq 1$, then

$$\int_\gamma \frac{1}{(z-a)^n}\, dz = 0.$$

b) If $\gamma = C_r = \{z : |z-z_0| = r\}$, then

$$\frac{1}{2\pi i} \int_{C_r} \frac{1}{z-a}\, dz = \begin{cases} 1 & \text{if } |a-z_0| < r \\ 0 & \text{if } |a-z_0| > r. \end{cases}$$

Proof.

a) For $n \neq 1$, $\frac{d}{dz}\left(\frac{1}{-(n-1)(z-a)^{n-1}} \right) = \frac{1}{(z-a)^n} \quad (z \neq a)$

$$\Rightarrow \int_\gamma \frac{1}{(z-a)^n}\, dz = 0 \text{ by Cor. 5.}$$

b) Let $C_r(t) = z_0 + re^{it}$, $0 \leq t \leq 2\pi$.

<u>$|a-z_0| > r$</u> we have

$$\frac{1}{z-a} = \frac{1}{z - z_0 + z_0 - a} = \frac{1}{-(a-z_0)\left(1 - \frac{z-z_0}{a-z_0}\right)}$$

$$= \frac{-1}{a-z_0} \sum_{n=0}^{\infty} \left(\frac{z-z_0}{a-z_0}\right)^n \quad (|z-z_0| \leq r)$$

Therefore,

$$\int_{C_r} \frac{1}{(z-a)}\,dz = -\frac{1}{a-z_0} \sum_{n=0}^{\infty} \frac{i}{(a-z_0)^n} \underbrace{\int_0^{2\pi} r^{n+1} e^{i(n+1)t}\,dt}_{=0 \ \forall n \geq 0.}$$

$$= 0.$$

<u>$|a-z_0|<r$</u>

$$\int_{C_r} \frac{1}{z-a}\,dz = \int_0^{2\pi} \frac{ire^{it}}{z_0+re^{it}-a}\,dt$$

$$= i\int_0^{2\pi} \frac{1}{1-\frac{a-z_0}{re^{it}}}\,dt.$$

$$= i\int_0^{2\pi} \sum_{n=0}^{\infty} \frac{(a-z_0)^n}{r^n e^{int}}\,dt$$

$$= i \sum_{n=0}^{\infty} \frac{(a-z_0)^n}{r^n} \int_0^{2\pi} e^{-int}\,dt$$

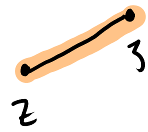
$$= 2\pi i.$$

$$\Rightarrow \frac{1}{2\pi i}\int_{C_r} \frac{1}{z-a}\,dz = 1.$$

$\square$

Proof of Cauchy's Integral Formula.

Suppose $|z - z_0| < r$.



For $\zeta \in C_r$,

$$\frac{f(\zeta) - f(z)}{\zeta - z} = \int_0^1 f'(z + t(\zeta - z))\,dt$$

So,

$$\int_{C_r} \frac{f(\zeta) - f(z)}{\zeta - z}\,d\zeta = \int_{C_r} \int_0^1 f'(z + t(\zeta - z))\,dt\,d\zeta$$

$$\overset{\text{Fubini}}{=} \int_0^1 \int_{C_r} f'(z + t(\zeta - z))\,d\zeta\,dt$$

$$= \lim_{\varepsilon \to 0} \int_\varepsilon^1 \int_{C_r} f'(z + t(\zeta - z))\,t\,d\zeta\,\frac{dt}{t}$$

$$= \lim_{\varepsilon \to 0} \int_\varepsilon^1 \underbrace{\int_{C_r} \frac{d}{d\zeta}\left[f(z + t(\zeta - z))\right] d\zeta}_{=0}\,\frac{dt}{t}$$

$$= 0$$

Therefore,

$$f(z) - \frac{1}{2\pi i} \int_{C_r} \frac{f(\zeta)}{\zeta - z}\,d\zeta = 0. \qquad \square$$

Corollary 8. Let $f : \Omega \to \mathbb{C}$ be a function defined on a region Ω.

a) f is holomorphic in Ω if and only if f is analytic in Ω.

b) Moreover, the series expansion of f based at $z_0 \in \Omega$ converges on the largest open disk centered at z_0 and contained in Ω.

Proof.

(a) f analytic in Ω $\Rightarrow$ f is holomorphic in Ω.

Suppose f is holomorphic in Ω. If $z_0 \in \Omega$, there is a $B_{r_0} = \{z : |z-z_0| < r_0\} \subseteq \Omega$. So f is holomorphic on B_{r_0}. Fix $r < r_0$ & from Thm. 3,

$$f(z) = \frac{1}{2\pi i}\int_{C_r} \frac{f(\zeta)}{\zeta - z}\, d\zeta \quad (z \in B_r)$$

$$= \frac{1}{2\pi i}\int_{C_r} f(\zeta) \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(\zeta-z_0)^{n+1}}\, d\zeta$$

$$(*) = \sum_{n=0}^{\infty} \underbrace{\left[\frac{1}{2\pi i}\int_{C_r} \frac{f(\zeta)}{(\zeta-z_0)^{n+1}}\, d\zeta\right]}_{a_n} (z-z_0)^n .$$

Since $(*)$ converges $\forall z \in B_r$, f is analytic at z_0.

(b) Choose B_{r_0} s.t. $\partial B_{r_0} \cap \partial\Omega \neq \emptyset$. $\square$

Note:

- In particular, if f is analytic in $\mathbb{C}$, then f has a power series expansion which converges in all of $\mathbb{C}$. Such functions are called **entire**.
- From now on, the words "holomorphic" and "analytic" are used interchangably.

Example 9.

a) Show that $f(z) = \frac{z}{e^z - 1}$ is holomorphic in $\underbrace{\mathbb{C}\backslash\{2k\pi i : k \in \mathbb{Z}, k \neq 0\}}_{}$. $\;\Omega$

b) Use this to show that the radius of the power series based at 0

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} a_n z^n$$

$f^{(n)}(0)$

$$a_n = \frac{B_n}{n!}$$

is 2π.

a) f is continuous on the set $\mathbb{C} \backslash \{2k\pi i : k \in \mathbb{Z}\}$.

At $z=0$, we have, for $h \neq 0$, $|h| < 2\pi$,

$$\frac{h}{e^h - 1} = \frac{h}{\sum_{n=0}^{\infty} \frac{h^n}{n!} - 1} = \frac{h}{\sum_{n=1}^{\infty} \frac{h^n}{n!}} = \frac{1}{\sum_{n=1}^{\infty} \frac{h^{n-1}}{n!}} = 1.$$

So, $f(0) = 1 = \lim_{h \to 0} \frac{h}{e^h - 1}$.

$f'(z)$ exists and is continuous on $\mathbb{C} \backslash \{2\pi k i : k \in \mathbb{Z}\}$.

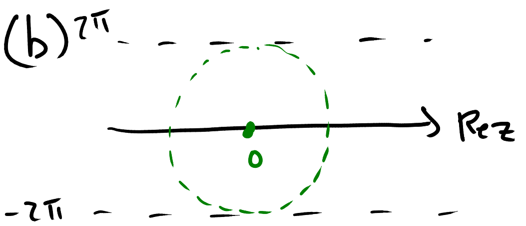
For $z = 0$, we have, for small h:

$$\lim_{h \to 0} \frac{\frac{h}{e^h - 1} - 1}{h} = \lim_{h \to 0} \frac{h - e^h + 1}{h(e^h - 1)}$$

$$= \lim_{h \to 0} \frac{-\sum_{n=2}^{\infty} \frac{h^n}{n!}}{h \sum_{n=1}^{\infty} \frac{h^n}{n!}} = \lim_{h \to 0} \frac{-\sum_{n=2}^{\infty} \frac{h^{n-2}}{n!}}{\sum_{n=1}^{\infty} \frac{h^{n-1}}{n!}}$$

$$= -\frac{1}{2}$$

So, f' is continuous on Ω, so holomorphic on Ω.

(b) 

The radius of the biggest disk is 2π

$$\Rightarrow R = \liminf_{n \to \infty} |a_n|^{-1/n} = 2\pi.$$

Scholium 10. If f is analytic in $\{z : |z - z_0| \leq r\}$ and $C_r = \{z_0 + re^{it} : 0 \leq t \leq 2\pi\}$, then

a) $\dfrac{f^{(n)}(z_0)}{n!} = \dfrac{1}{2\pi} \displaystyle\int_{C_r} \dfrac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta$. [Cauchy's Integral Formula for $f^{(n)}$]

b) $\left| \dfrac{f^{(n)}(z_0)}{n!} \right| \leq \dfrac{\sup_{C_r} |f(z)|}{r^n}$. [Cauchy's Estimate]

Proof.

a) From the proof of thm. 8 : $|z - z_0| < r$.

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{C_r} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta \right) (z - z_0)^n$$

Uniq. of Power series $\Rightarrow$ $\dfrac{f^{(n)}(z_0)}{n!} = \dfrac{1}{2\pi i} \displaystyle\int_{C_r} \dfrac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta$.

b) $$\frac{|f^{(n)}(z_0)|}{n!} = \frac{1}{2\pi} \left| \int_{C_r} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta \right|$$

$$\leq \frac{1}{2\pi} \int_{C_r} \frac{|f(\zeta)|}{r^{n+1}} \, |d\zeta|$$

$$\leq \frac{1}{2\pi r^{n+1}} \sup_{C_r} |f| \; \underbrace{|C_r|}_{2\pi r}$$

$$= \frac{\sup_{C_r} |f|}{r^n}. \qquad \square$$

Corollary 11. If f is analytic in an open disk B, and if $\gamma \subset B$ is a closed, piecewise continuously differentiable curve, then

$$\int_\gamma f(z) \, dz = 0.$$

Theorem 12. If f is analytic and one-to-one in a region Ω, then the inverse of f, defined on $f(\Omega)$, is analytic.

Lemma 13. If f is an analytic function at z_0 with

$$f(z) - f(z_0) = \sum_{n \geq m} a_n (z - z_0)^n \quad (a_m \neq 0,\, m \geq 2)$$

in some disk B_1 centered at z_0, then there is a $\varepsilon > 0$ and a δ so that $f(z) - w$ has exactly m solutions in $\{z : |z - z_0| < \varepsilon\}$, for any $w \in \{v : |v - f(z_0)| < \delta\}$.

Proof.

Proof of Theorem 12.

Morera's Theorem

Theorem 14. If f is continuous in an open disk B, and if

$$\int_{\partial R} f(\zeta) d\zeta = 0$$

for all closed rectangles $R \subset B$ with sides parallel to the axes, then f is analytic in B.

Proof.