Math 241
Spring 2022
Midterm 1
February 23rd, 2022

Name (Print): ____________________
Section Number: ____________________

This exam contains 10 pages (including this cover page) and 9 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

Instructions:

- You have 75 minutes for the exam.
- You are required to **show your work** and justify your answers for all questions *except where explicitly stated*.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit, if any.
- No notes or books are allowed.
- No electronic devices are allowed.
- No calculators are allowed.

Academic integrity is expected of all University of Hawaii students at all times, whether in the presence or absence of members of the faculty. Understanding this, I declare I shall not give, use, or receive unauthorized aid in this examination.
Please sign below to indicate that you have read and agree to these instructions.

SIGNATURE OF STUDENT

For official use only:

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	9	15	8	10	12	12	10	10	14	100
Score:										

Question 1. (9 points)
The table shows the distance travelled by a bicyclist on a straight line after accelerating from rest.

Time in seconds	Total distance in feet
0	0
1	2
→ 2	4
3	8
4	15
5	30
→6	52 ↕
7	76 ↕
8	101

(a) (3 points) Calculate the average speed between 2 and 6 seconds.

$$V_{are} = \frac{52-4}{6-2} = \frac{48}{4} = 12 \text{ feet/s}.$$

(b) (3 points) Compare the average speed of the interval between 0 second and 1 second, and the interval between 1 second and 2 seconds. Between these two intervals, which one has the highest average speed?

$$V^{1}_{ave} = \frac{2-0}{1-0} = 2 \text{ feet/s} \qquad v^{2}_{ave} = \frac{4-2}{2-1} = 2 \text{ feet/s}.$$

$$\boxed{\text{None}}$$

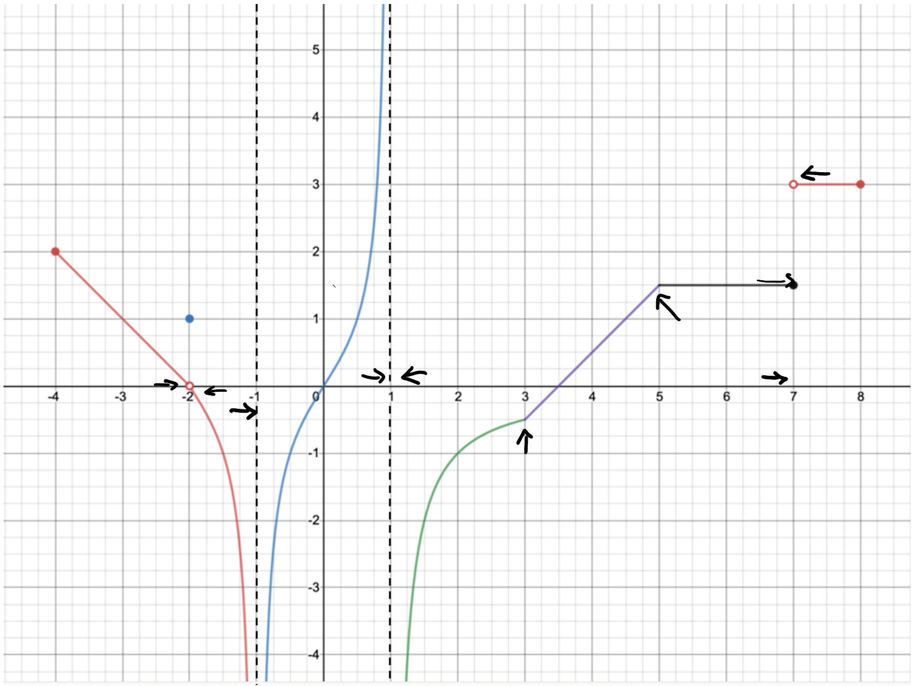
(c) (3 points) Estimate the average acceleration of the bicyclist at 7 seconds.
(Hint: The average acceleration can be calculated using two average speeds.)

$$\rightarrow V^{1}_{ave} = \frac{76-52}{7-6} = 24 \text{ feet/s}$$

$$\rightarrow v^{2}_{ave} = \frac{101-76}{8-7} = 25 \text{ feet/s}$$

$$a_{ave} = \frac{25-24}{8-6} = \boxed{\frac{1}{2}} \text{ feet/s}^2.$$

Question 2. (15 points)
The graph of a function f is given below. Assume f has vertical asymptotes at $x = -1$ and $x = 1$. No justifications needed for this problem.



(a) (6 points) Evaluate each of the following limits, or say the limit does not exist. If the limit is either ∞ or $-\infty$, specify which (rather than just saying 'does not exist').

1. $\lim_{x\to-2} f(x)$ $= 0$
2. $\lim_{x\to-1^-} f(x)$ $= -\infty$ (DNE)
3. $\lim_{x\to1} f(x)$ $\nexists$ (DNE)
4. $\lim_{x\to7^-} f(x)$ $= 1.5$
5. $\lim_{x\to7^+} f(x)$ $= 3$
6. $\lim_{x\to7} f(x)$ $\nexists$ (DNE).

(b) (3 points) For which (if any) values in the interval $[-4, 8]$ is the function f not continuous?

$-2, -1, 1, 7$

(c) (3 points) For which (if any) values in the interval $[-4, 8]$ is f differentiable but not continuous?

None.

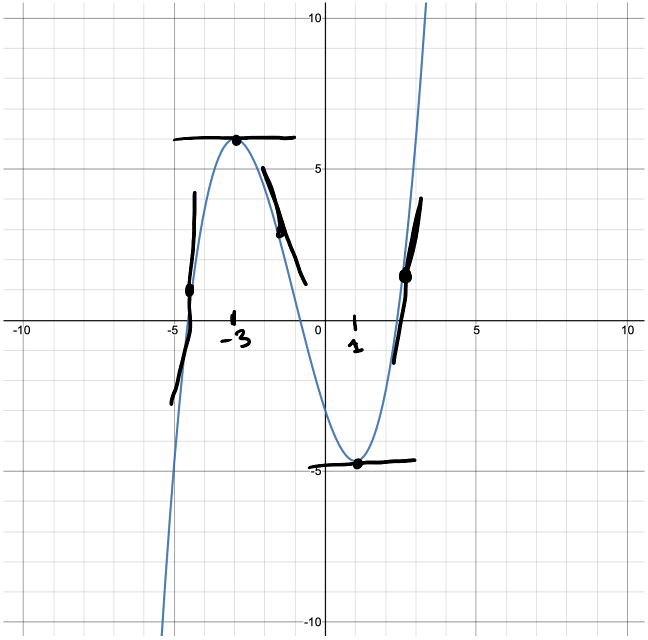
(d) (3 points)For which (if any) values in the interval $[-4, 8]$ is f continuous but not differentiable?

$3, 5$.

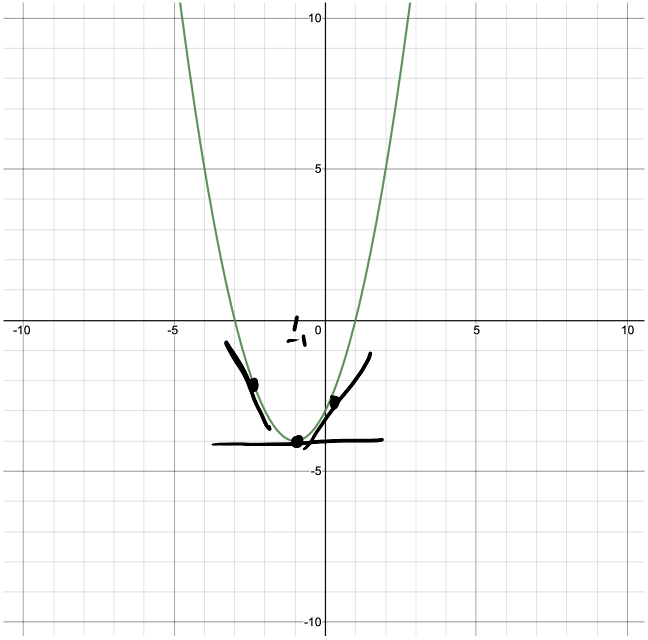
Question 3. (8 points)

Given the two graphs below, **roughly** sketch the graphs of their derivative on the blank axes. (4 points for each graph.)

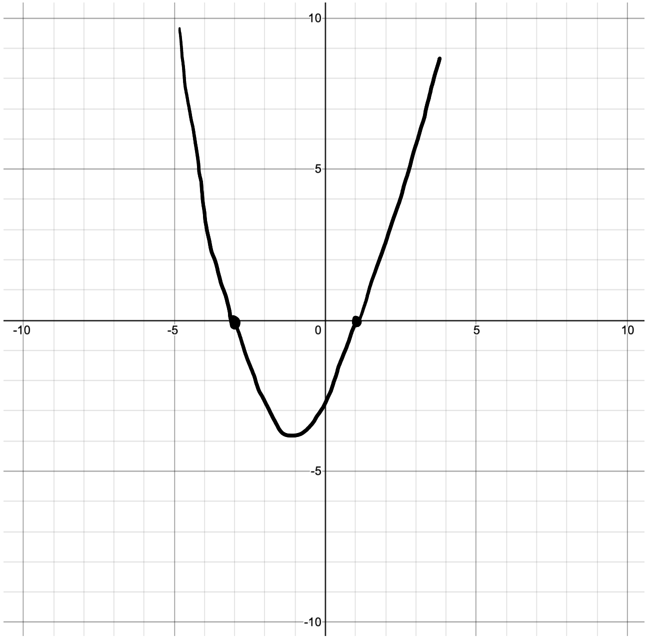
$f(x)$



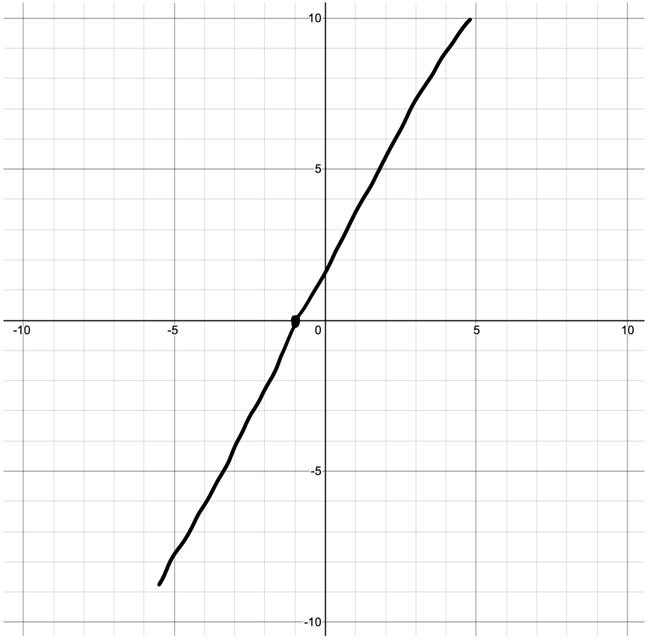
$g(x)$



$f'(x)$



$g'(x)$



Question 4. (10 points)

Suppose f is a continuous function that satisfies the following limits:

$$\lim_{x\to -1} f(x) = -2, \qquad \lim_{x\to 0} f(x) = 3$$

Evaluate the following limits. (5 points each.) You may not use L'Hospital's rule, i.e., if you use L'Hospital's rule, you will not get points.

(a) $\lim_{x\to -1} \frac{(x^2-3x-4)}{x+1} f(x)$ $= \frac{1+3-4}{-1+1}(-2) = \frac{0}{0}!$

$$\frac{x^2-3x-4}{x+1} = \frac{\cancel{(x+1)}(x-4)}{\cancel{x+1}} = x-4.$$

$$\lim_{x\to -1}\frac{x^2-3x-4}{x+1}f(x) = \lim_{x\to -1}(x-4)f(x)$$

$$= \left(\lim_{x\to -1} x-4\right)\left(\lim_{x\to -1} f(x)\right)$$

$$= (-1-4)(-2) = \boxed{10}$$

(b) $\lim_{x\to 0} \frac{\sqrt{3x^2+16}-4}{x^2 f(x)}$ $\frac{0}{0}!$

$$\frac{\sqrt{3x^2+16}-4}{x^2}\cdot\left(\frac{\sqrt{3x^2+16}+4}{\sqrt{3x^2+16}+4}\right) = \frac{3x^2+16-16}{x^2\left(\sqrt{3x^2+16}+4\right)}$$

$$= \frac{3\cancel{x^2}}{\cancel{x^2}\left(\sqrt{3x^2+16}+4\right)}$$

$$= \frac{3}{\sqrt{3x^2+16}+4}$$

$$= \lim_{x\to 0}\frac{3}{\left(\sqrt{3x^2+16}+4\right)f(x)} = \frac{\lim_{x\to 0} 3}{\lim_{x\to 0}\left(\sqrt{3x^2+16}+4\right)\cdot \lim_{x\to 0} f(x)}$$

$$= \frac{\cancel{3}}{8\cdot\cancel{3}} = \boxed{\frac{1}{8}}$$

Question 5. (12 points)

(a) (8 points) Using *the definition of derivative* (also called the limit process), find the derivative of the function $f(x) = \frac{1}{x+4}$.
You will NOT get any credit unless you use the definition of the derivative!

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{x+h+4} - \frac{1}{x+4}}{h}$$

$$= \lim_{h \to 0} \frac{\left(\frac{\cancel{x}+\cancel{4} - (\cancel{x}+h+\cancel{4})}{(x+h+4)(x+4)} \right)}{h}$$

$$= \lim_{h \to 0} \frac{-\cancel{h}}{(x+h+4)(x+4)\cancel{h}}$$

$$= \lim_{h \to 0} \frac{-1}{(x+h+4)(x+4)} = \frac{-1}{(x+4)(x+4)} = \boxed{\frac{-1}{(x+4)^2}}$$

$f'(x)$

(b) (4 points) Using the function in (a), find the equation of the tangent line to $y = f(x)$ at $(0, \frac{1}{4})$.

x_0 y_0

$$y - \frac{1}{4} = f'(0)(x - 0)$$

$$\Rightarrow f'(0) = \frac{-1}{(0+4)^2} = -\frac{1}{16}$$

$$\Rightarrow y - \frac{1}{4} = \frac{-1}{16}x \Rightarrow \boxed{y = \frac{1}{4} - \frac{x}{16}}$$

Question 6. (12 points)
Let $f(x)$ be defined by

$$f(x) = \begin{cases} (x-a)^2 + 2 & \text{if } x < 2 \\ 3 & \text{if } x = 2 \\ a + x & \text{if } x > 2 \end{cases}$$

(a) (8 points) Find all values of a so that $\lim_{x \to 2} f(x)$ exists.

$\lim_{x\to 2} f(x)$ exists if ① $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$ ② .

① $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x-a)^2 + 2 = (2-a)^2 + 2$

② $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} a + x = a + 2$

$\Rightarrow$ ① = ② $\Rightarrow (2-a)^2 + 2 = a + 2$

$\Rightarrow (2-a)^2 + \cancel{2} - a - \cancel{2} = 0$

$\Rightarrow 4 - 4a + a^2 - a = 0$

$\Rightarrow 4 - 5a + a^2 = 0$.

$\Rightarrow (a-4)(a-1) = 0$

$\Rightarrow \boxed{a = 4}$ or $\boxed{a = 1}$

(b) (4 points) Find all possible values of a so that $f(x)$ is continuous at $x = 2$, or show that none exist. Justify your answer.

f cont. at $x = 2$ if

- $f(2)$ exists. ✓
- $\lim_{x\to 2} f(x)$ exists ✓
- $f(2) = \lim_{x\to 2} f(x)$. ←

$a = 4$
$a = 1$

$\underline{a=4}$. $\lim_{x\to 2} f(x) = \lim_{x\to 2^+} f(x) = a + 2 = 6 \neq 3 = f(2)$

$\underline{a=1}$ $\lim_{x\to 2} f(x) = \lim_{x\to 2^+} f(x) = a + 2 = 3 = f(2)$ ✓

[Answer: f is continuous at $x=2$ if $a=1$]

Question 7. (10 points)

Suppose $f(x)$ is a function where $f(1) = 1$ and $f'(1) = -1$.

(a) (5 points) Let $g(x) = x^3 f(x) + 2$. Find $g'(1)$.

$$g'(x) = \left(x^3 f(x) + 2\right)' = \left(x^3 f(x)\right)' + (3)'$$

$$= 3x^2 \, f(x) + x^3 \, f'(x) + 0$$

$$= 3x^2 f(x) + x^3 f'(x)$$

$$\Rightarrow g'(1) = 3 \cdot 1^2 \cdot \underset{\downarrow}{f(1)} + 1^3 \cdot \underset{\downarrow}{f'(1)}$$

$$= 3 \cdot 1 + 1 \cdot (-1)$$

$$= 3 - 1 = \boxed{2}$$

(b) (5 points) Let $h(x) = \sqrt{4\sin(\pi x) + 3f(x)}$. Find $h'(1)$.

$$h'(x) = \left(\sqrt{4\sin(\pi x) + 3f(x)}\right)' = \left[\left(4\sin(\pi x) + 3f(x)\right)^{1/2}\right]'$$

$$= \frac{1}{2}\left(4\sin(\pi x) + 3f(x)\right)^{-1/2} \cdot \left(4\sin(\pi x) + 3f(x)\right)'$$

$$🟦' = \left(4\sin(\pi x)\right)' + \left(3f(x)\right)' = 4\left(\sin(\pi x)\right)' + 3\left(f(x)\right)'$$

$$= 4\left[\cos(\pi x)\right] \pi + 3f'(x)$$

$$h'(1) = \frac{1}{2}\left[4 \overset{0}{\cancel{\sin(\pi)}} + 3 \overset{1}{f(1)}\right]^{-1/2} \cdot \left[4\pi \overset{-1}{\cos(\pi)} + 3 \overset{-1}{f'(1)}\right]$$

$$= \frac{1}{2} \; 3^{-1/2} \cdot (-4\pi - 3) = \boxed{\frac{-4\pi - 3}{2\sqrt{3}}}.$$

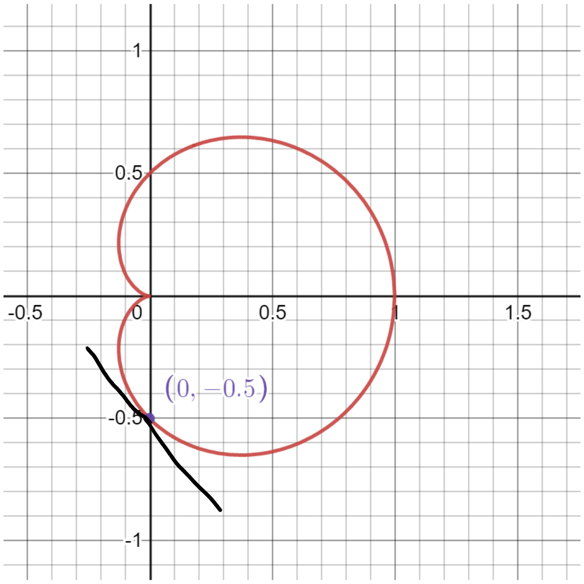
Question 8. (10 points)

Use implicit differentiation to find an equation of the tangent line to the following cardioid

$$x^2 + y^2 = (2x^2 + 2y^2 - x)^2 \text{ at the point } \left(0, -\frac{1}{2}\right)$$

.

$y = f(x)$



① Impl. Diff.

$$\frac{d}{dx}\left(x^2+y^2\right) = \frac{d}{dx}\left(\left(2x^2+2y^2-x\right)^2\right)$$

$$\Rightarrow \quad 2x + 2y\frac{dy}{dx} = 2\left(2x^2+2y^2-x\right)\cdot\left(4x+4y\cdot\frac{dy}{dx}-1\right)$$

$$\Rightarrow \quad 2x + 2y\frac{dy}{dx} = 8y\frac{dy}{dx}\left(2x^2+2y^2-x\right) + 2(4x-1)\left(2x^2+2y^2-x\right)$$

$$\Rightarrow \quad 2x - 2(4x-1)\left(2x^2+2y^2-x\right) = 8y\frac{dy}{dx}\left(2x^2+2y^2-x\right) - 2y\frac{dy}{dx}$$

$$\Rightarrow \quad 2x - 2(4x-1)\left(2x^2+2y^2-x\right) = \left(8y\left(2x^2+2y^2-x\right) - 2y\right)\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{2x-2(4x-1)\left(2x^2+2y^2-x\right)}{8y\left(2x^2+2y^2-x\right)-2y} = \frac{dy}{dx}.$$

② Find Eq. Tangent.

$$y + \frac{1}{2} = \frac{dy}{dx} \cdot (x - 0)$$

Replace $x=0$ & $y=-\frac{1}{2}$.

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(-2)(-1)(\frac{2}{4})}{8(-\frac{1}{2})(\frac{2}{4}) - 2(-\frac{1}{2})} = \frac{1}{-2+1} = \frac{1}{-1} = -1$$

So,
$$y + \frac{1}{2} = -1\,(x)$$

$$\Rightarrow \quad \boxed{y = -x - \frac{1}{2}}$$

Question 9. (14 points)

Suppose that an object moves along a line over time. Its position is given by

$$x(t) = -0.02t^2 + 50t + 100.$$

(a) (4 points) What is the average speed of the object between the time $t = 0$ and $t = 1000$?

$$V_{ave} = \frac{\Delta x}{\Delta t} = \frac{-0.02(1000)^2 + 50 \cdot 1000 + 100 - 100}{1000}$$

$$= \frac{-0.02(1\,000\,000) + 50\,000}{1000}$$

$$= \frac{-20\,000 + 50\,000}{1000} = \frac{30\,\cancel{000}}{1\cancel{000}} = \boxed{30}$$

(b) (5 points) What is the velocity of the object when $t = 500$?

$$x'(t) = -0.04t + 50$$

$$\Rightarrow \quad x'(500) = -0.04(500) + 50$$

$$= -20 + 50 = \boxed{30}$$

(c) (5 points) What is the acceleration of the object when $t = 10$?

$$x''(t) = -0.04 .$$

$$\Rightarrow \quad x''(10) = \boxed{-0.04}$$