# Math 307

## Chapter 1

### Section 1.3: Inverses of Matrices

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## For Real Numbers

EXAMPLE 1. Find the value of $x$ if

1. $2x - 1 = 0$.
2. $x^2 - x = 0$.

Secretly:

- In the first equation, we multiplied by the inverse of 2, which is $1/2$, because $(1/2)2 = 1$.

- In the second equation, we examined the values of $x$ and made sure we avoid the value 0 because 0 is not "divisible". In other words, it doesn't have an inverse.

## For Matrices

We say that a square matrix $A$ is invertible if there is another matrix $B$ such that

$$AB = BA = I.$$

Remarks:

- Not all non-zero square matrices are invertible.

- Matrices that are invertible are called **nonsingular** and matrices that are not invertible are called **singular**.

- If the inverse exists, then there is only one inverse and we denote it by $A^{-1}$.

EXAMPLE 2. Verify that the matrix $B$ is the inverse of $A$ if

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \quad B = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}.$$

## Properties of Inverses

Find the inverse of the product

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 2 & 1 & 1 \\ 4 & 5 & 1 \end{bmatrix}$$

if you know how to find the inverse with Python.

GENERAL FACTS: Let $A$ and $B$ be matrices of the same size and let $m$ be a positive integer.

- If $A$ and $B$ are invertible, then $AB$ is invertible with $(AB)^{-1} = B^{-1}A^{-1}$.

- If $A$ is invertible, then $A^{-1}$ is also invertible and $(A^{-1})^{-1} = A$.

- If $A$ is invertible, then $A^m$ is also invertible and $(A^m)^{-1} = (A^{-1})^m$.

- Suppose that $A$ and $B$ are $n \times n$ matrices such that $AB = I$ or $BA = I$. Then $A$ has an inverse and $A^{-1} = B$.

For numbers, finding the inverses is quite straightforward, or should we say "we are used to divide with numbers".

## Little Warm-up

For matrices, it is not that obvious.

EXAMPLE 4. Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}.$$

## Systematic method with Augmented Matrices

Given a square matrix $A = [a_{ij}]$, we "augment" $A$ with the identity matrix:

$$\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Now, the goal, if possible, is to perform row operations to change the left-side (the matrix $A$) into the identity matrix, that is:

$$\begin{bmatrix} I & B \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 & b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & 1 & \cdots & 0 & b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}.$$

Remark:

- When it's possible to transform the augmented matrix $\begin{bmatrix} A & I \end{bmatrix}$ into the augmented matrix $\begin{bmatrix} I & B \end{bmatrix}$, then $B$ is the inverse of $A$.

- When it's not possible to transform $\begin{bmatrix} A & I \end{bmatrix}$ into $\begin{bmatrix} I & B \end{bmatrix}$, then $A$ is singular.

EXAMPLE 5. If possible, find the inverse of the following matrix:

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 2 & 1 & 1 \\ 4 & 5 & 1 \end{bmatrix}.$$

EXAMPLE 6. If possible, find the inverse of the following matrix:

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -3 & -1 \\ 1 & -1 & -1 \end{bmatrix}.$$

## Inverses to Solve Systems

If you have a given system of linear equations

$$AX = B$$

where $A$ is a nonsingular matrix, then you can find $X$ (the vector of solutions) by multiplying on the left the whole equation by the inverse $A^{-1}$:

$$A^{-1}AX = A^{-1}B \quad \Rightarrow \quad X = A^{-1}B.$$

EXAMPLE 7. Solve the system

$$
\begin{aligned}
2x + y + 3z &= 6 \\
2x + y + z &= -12 \\
4x + 5y + z &= 3.
\end{aligned}
$$

When we are performing row operations, we are in fact performing matrix multiplication with special matrices that we call elementary matrices.

## Three types

- An elementary matrix obtained by interchanging two rows of $I$.

- An elementary matrix obtained by multiplying a row $I$ by a nonzero number.

- An elementary matrix obtained by replacing a row of $I$ by itself plus a multiple of another row of $I$.

EXAMPLE 8. Here are some examples of dimensions $3 \times 3$:

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad E_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

## Some mysteries Unraveled!

When we were performing row operations on a matrix $A$, we were in fact performing a multiplication of an elementary matrix with $A$. Here are some facts related to this:

- If $E$ is obtained by interchanging rows $i$ and $j$ of $I$, then $EA$ is the matrix obtained from $A$ by interchanging rows $i$ and $j$ of $A$.

- If $E$ is obtained by multiplying row $i$ of $I$ by a scalar $c$, then $EA$ is the matrix obtained from $A$ by multiplying row $i$ of $A$ by $c$.

- If $E$ is obtained by replacing row $i$ of $I$ by itself plus $c$ times the row $j$ of $I$, then $EA$ is the matrix obtained from $A$ by replacing row $i$ of $A$ by itself plus $c$ times row $j$ of $A$.

EXAMPLE 9.    Give the elementary matrices used in Example 5.  At each step, using the elementary matrices, give the expression of the matrix resulting from the row operations.

## Inverses of elementary matrices

Consider the following elementary matrices

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

For each of them, find the inverse.

Remarks: In general, if $E$ is an elementary matrix, then $E$ is invertible and:

- If $E$ is obtained by interchanging two rows of $I$, then $E^{-1} = E$;

- If $E$ is obtained by multiplying row $i$ of $I$ by a nonzero scalar $c$, then $E^{-1}$ is the matrix obtained by multiplying row $i$ of $I$ by $1/c$;

- If $E$ is obtained by replacing row $i$ of $I$ by itself plus $c$ times row $j$ of $I$, then $E^{-1}$ is the matrix obtained by replacing row $i$ of $I$ by itself plus $-c$ times row $j$ of $I$.

Consequences:

- A square matrix $A$ is invertible if and only if $A$ is a product of elementary matrices.