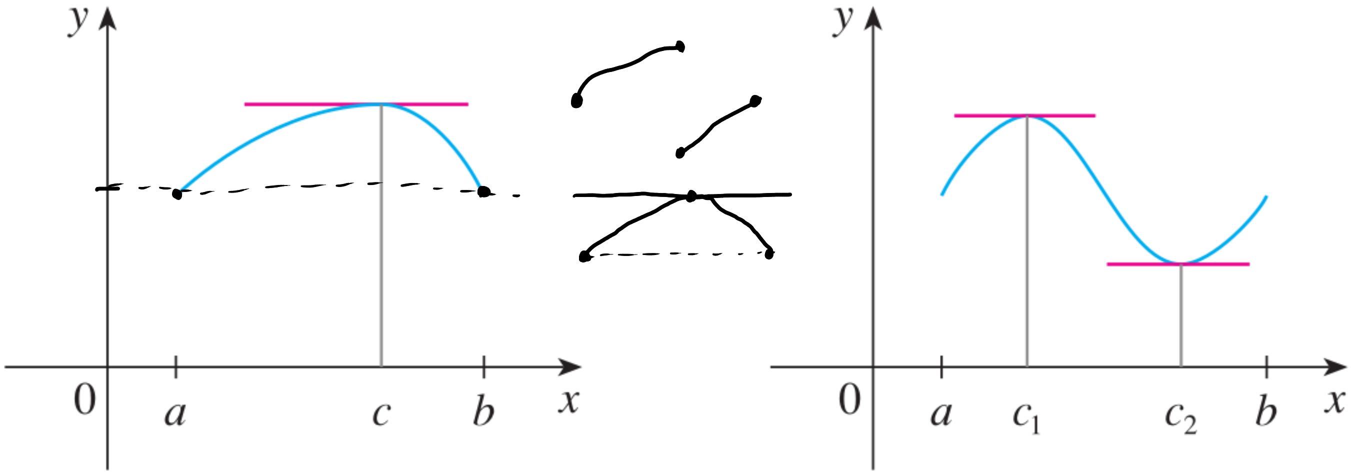
Chapter 3
Applications of Derivatives

3.2 The Mean Value Theorem

The following graphs have a commun geometric property.



(b) (c)

Is there a condition that garantees that a graph of a function has horizontal tangents?

Rolle's Theorem Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
3. $f(a) = f(b)$

Then there is a number c in (a, b) such that $f'(c) = 0$.

EXAMPLE 2 Prove that the equation $x^3 + x - 1 = 0$ has exactly one real root.

① IVT.

$f(x) = x^3 + x - 1$
poly.
continuous.

$f(0) = -1$
$f(1) = 1$

$-1 < 0 < 1$
$f(0)$ $f(1)$

By IVT, there is a $0 < c < 1$ st.
$f(c) = 0$

② Rolle's Thm. (arguing by contradiction).

$\boxed{2.1}$ Suppose that there are two solutions, c_1 & c_2, such that $f(c_1) = 0$ & $f(c_2) = 0$.

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2.2 Apply Rolle's theorem. $[a, b] = [c_1, c_2]$ $(c_1 < c_2)$
or $[c_2, c_1]$ $(c_1 > c_2)$

So, $f'(c) = 0$ at some c.

Assumption 2.1 is not true.

2.3 $f'(x) = 3x^2 + 1$ ($\cancel{x^2 = -\frac{1}{3}}$) so, $f'(x) > 0$ for all x.

The Mean Value Theorem Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

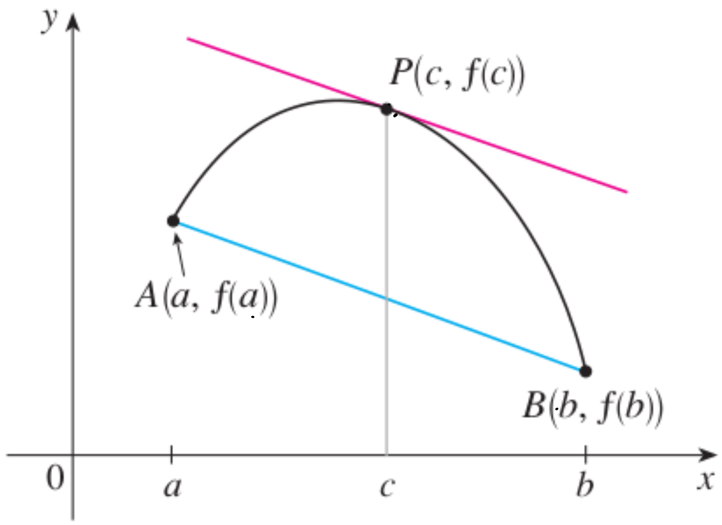
$$f'(c) = \frac{f(b) - f(a)}{b - a} \tag{1}$$

or, equivalently,

$$f(b) - f(a) = f'(c)(b - a) \tag{2}$$

The Meaning: Find a tangent tangent line such that the slope is the as the slope of the secant line joining the points $(a, f(a))$ & $(b, f(b))$.

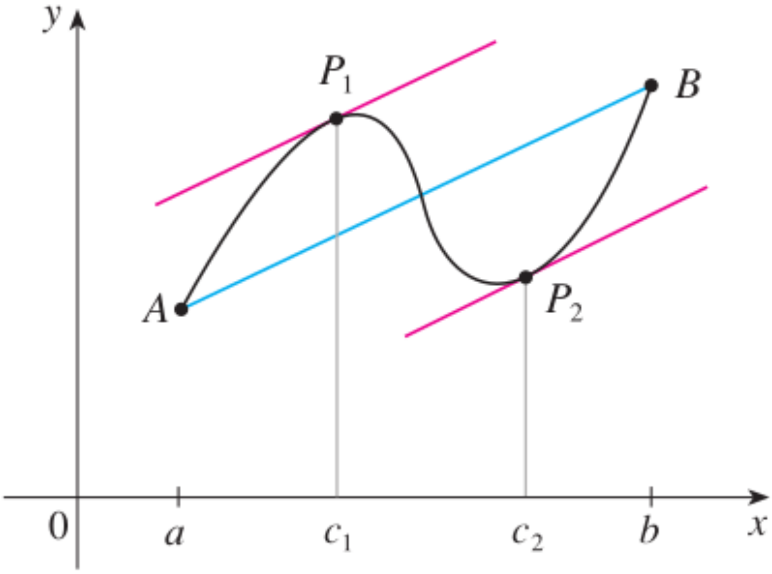
Only one c.



we only have one c s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Multiple c.



We have two numbers c_1 & c_2 s.t.

$$f'(c_1) = \frac{f(b) - f(a)}{b - a}$$

&

$$f'(c_2) = \frac{f(b) - f(a)}{b - a}$$

Example

15–16 Find the number c that satisfies the conclusion of the Mean Value Theorem on the given interval. ~~Graph the function, the secant line through the endpoints, and the tangent line at $(c, f(c))$. Are the secant line and the tangent line parallel?~~

Find c s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

15. $f(x) = \sqrt{x}, \quad [0, 4]$

$b = 4$, $a = 0$, $f'(x) = \dfrac{1}{2\sqrt{x}}$

So, we want to solve

$$\frac{1}{2\sqrt{c}} = \frac{2 - 0}{4 - 0} = \frac{1}{2}$$

$$\Rightarrow \quad \frac{1}{2\sqrt{c}} = \frac{1}{2}$$

$$\Rightarrow \quad 1 = \sqrt{c}$$

$$\Rightarrow \quad \boxed{1 = c}$$

$f(x) = \text{constant}$ $\left(\begin{matrix} 2 = \text{cst.} \\ 3 = \text{cst.} \end{matrix}\right)$

5 Theorem If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).

7 Corollary If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b); that is, $f(x) = g(x) + c$ where c is a constant.

$(x^2)' = 2x$
$(x^2+1)' = 2x$ $\Rightarrow$ $x^2 = (x^2+1) \underbrace{\boxed{-1}}_{\text{cst. } c}$

EXAMPLE 5 Suppose that $f(0) = -3$ and $f'(x) \leqslant 5$ for all values of x. How large can $f(2)$ possibly be?

MVT there is a c between 0 & 2 (a=0 & b=2)

s.t.

$$\frac{f(2) - f(0)}{2} = f'(c) \leq 5$$

So,

$$\frac{f(2) + 3}{2} \leq 5$$

$$\Rightarrow \quad f(2) \leq 10 - 3 = 7$$

So, $\boxed{f(2) \leq 7}$.