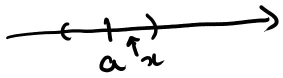
Chapter 1

Functions and Limits

1.5 The Limit of a Function

Intuitive definition of a limit.



1 Intuitive Definition of a Limit Suppose $f(x)$ is defined when x is near the number a. (This means that f is defined on some open interval that contains a, except possibly at a itself.) Then we write

$$\lim_{x\to a} f(x) = \boxed{L}$$

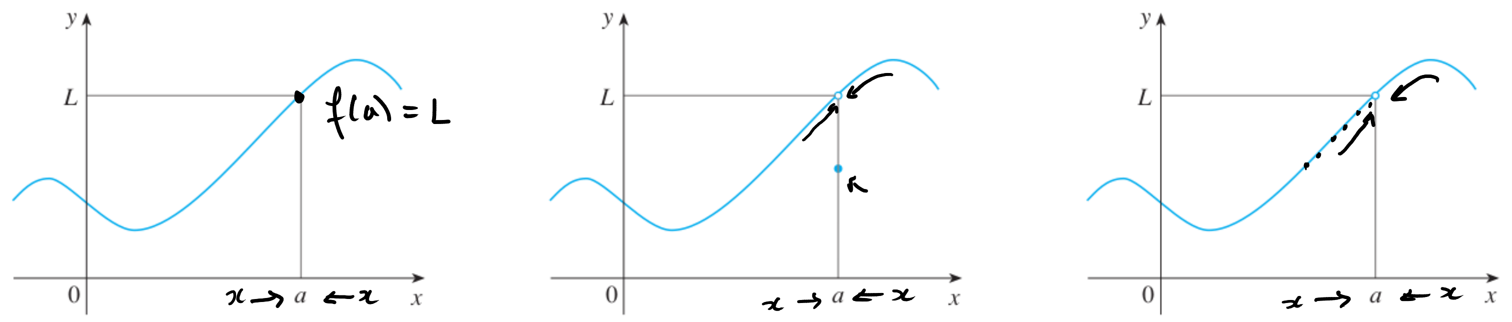
and say "the limit of $f(x)$, as x approaches a, equals L"

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by restricting x to be sufficiently close to a (on either side of a) but not equal to a.

Notations: ① $\lim f(x) = L$

② $f(x) \to L$ as $x \to a$

Three cases:



(a) $\lim_{x\to a} f(x) = f(a) = L$

(b) $\lim_{x\to a} f(x) = L \neq f(a)$

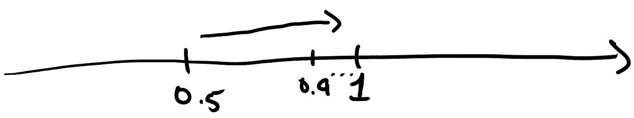
(c) $\lim_{x\to a} f(x) = L$ (exists) but $f(a)$ ~~exists~~

EXAMPLE 1 Guess the value of $\lim_{x\to 1} \dfrac{x-1}{x^2-1}$. $\to$ $f(x) = \dfrac{x-1}{x^2-1}$

$\boxed{L = 0.5}$

$f(0.5) = \dfrac{0.5-1}{0.5^2-1} = 0.666667$

$f(0.9) = \dfrac{0.9-1}{0.9^2-1} = 0.526316$

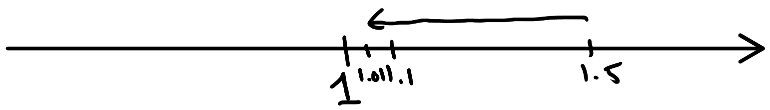


$x<1$	$f(x)$
0.5	0.666667
0.9	0.526316
0.99	0.502513
0.999	0.500250
0.9999	0.500025
↓ 1	↓ 0.5

$f(1.5) = \dfrac{1.5-1}{1.5^2-1} = 0.400000$

$f(1.1) = \dfrac{1.1-1}{1.1^2-1} = 0.476190$

$x>1$	$f(x)$
1.5	0.400000
1.1	0.476190
1.01	0.497512
1.001	0.499750
1.0001	0.499975
↓ 1	↓ 0.5



EXAMPLE 2 Estimate the value of $\lim_{t \to 0} \frac{\sqrt{t^2+9}-3}{t^2}$.

$$f(t) = \frac{\sqrt{t^2+9}-3}{t^2}$$

t	$f(t)$
± 1.0	0.162277...
± 0.1	0.166620...
± 0.05	0.166655...
± 0.01	0.1666666...
$\downarrow$	$\downarrow$
0	$0.1666666... = \frac{1}{6}$

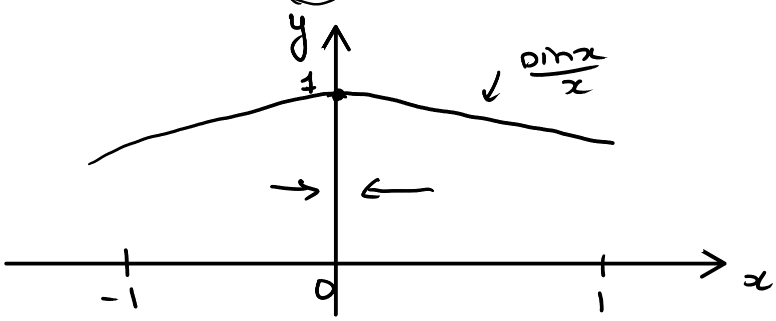
$$\boxed{\lim_{t \to 0} f(t) = \frac{1}{6} = L}$$

t	$f(t)$
1.0	
0.1	
0.01	

t	$f(t)$
-1.0	
-0.1	
-0.01	

EXAMPLE 3 Guess the value of $\lim_{x \to 0} \frac{\sin x}{x}$.

$f(x) = \frac{\sin x}{x}$ (not defined at $x=0$).



$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

EXAMPLE 4 Investigate $\lim_{x\to 0}\sin\left(\frac{\pi}{x}\right)$.

$f(x)=\sin\left(\frac{\pi}{x}\right)$ $\sin\left(\frac{\pi}{x}\right)=1.$

Property. if $\lim_{x\to a} f(x)=L$, then L is unique (one possible value for the limit).

x	$f(x)$
$\frac{1}{2}$	0
$\frac{1}{3}$	0
$\frac{1}{4}$	0
$\vdots$	$\vdots$
$\frac{1}{100}$	0
$\downarrow$	$\downarrow$
0	$0 = L$

$\sin(2\pi)$
$\sin(3\pi)$

x	$f(x)$
0.4	1
0.222	1
0.029	1
0.009	1
$\downarrow$	$\downarrow$
0	$1 = L$

x	$f(x)$
0.286	-1
0.105	-1
0.047	-1
$\downarrow$	$\downarrow$
0	$-1 = L$

3 values is impossible because the limit is unique.

$\lim_{x\to 0}\sin\left(\frac{\pi}{x}\right) \nexists$. $\to$ doesn't exist

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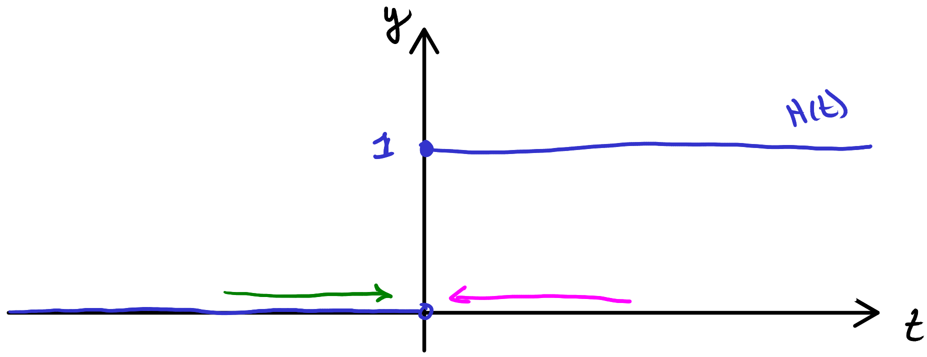
One-sided Limits.

EXAMPLE 6 The Heaviside function H is defined by

$$H(t)=\begin{cases}0 & \text{if } t<0\\ 1 & \text{if } t\geqslant 0\end{cases}$$

What is the limit when t approached 0 from the right and when t approaches 0 from the left.



$\lim_{x\to 0^-} H(t) = \lim_{x\to 0^-} 0 = 0$ | $\lim_{x\to 0^+} H(t) = \lim_{x\to 0^+} 1 = 1$

so $\lim_{x\to 0^-} H(t) \neq \lim_{x\to 0^+} H(t)$

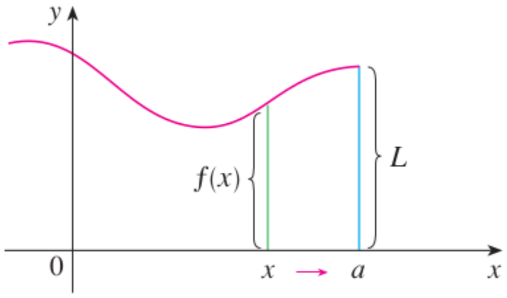
so $\lim_{x\to 0} H(t) \nexists$

Left-hand limits.

2 Definition of One-Sided Limits We write

$$\lim_{x \to a^-} f(x) = L$$

and say the **left-hand limit of $f(x)$ as x approaches a** [or the **limit of $f(x)$ as x approaches a from the left**] is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a with x *less than* a.



(a) $\lim_{x \to a^-} f(x) = L$

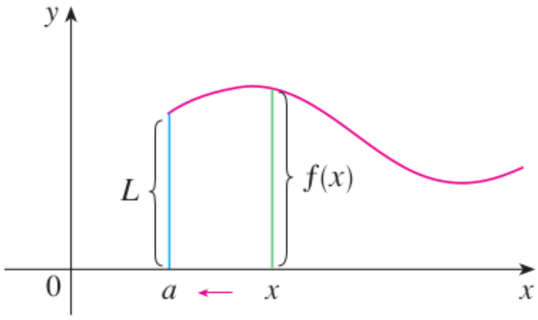
Right-hand limits.

2 Definition of One-Sided Limits We write

$$\lim_{x \to a^+} f(x) = L$$

and say the Right-hand limit of $f(x)$ as x approaches a [or the limit of $f(x)$ as x * is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a with x greater than a.

* approaches a from the right]



(b) $\lim_{x \to a^+} f(x) = L$

Fundamental Property:

3 $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$

EXAMPLE 7 The graph of a function g is shown in Figure 10. Use it to state the values (if they exist) of the following:

(a) $\lim_{x \to 2^-} g(x)$ (b) $\lim_{x \to 2^+} g(x)$ (c) $\lim_{x \to 2} g(x)$

(d) $\lim_{x \to 5^-} g(x)$ (e) $\lim_{x \to 5^+} g(x)$ (f) $\lim_{x \to 5} g(x)$

(a) 3 (b) 1 (c) $\nexists$

(d) 2 (e) 2 (f) $\lim_{x \to 5} g(x) = 2$.

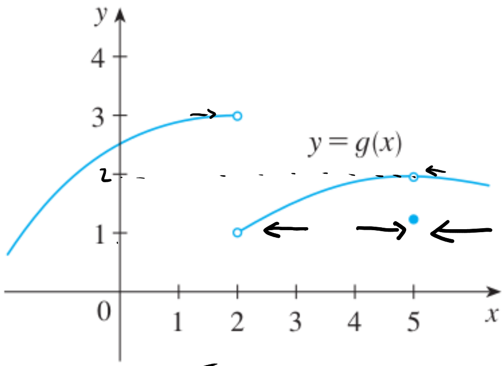


Fig. 10

Infinite limits.

EXAMPLE 8 Find $\lim_{x \to 0} \frac{1}{x^2}$ if it exists. $\to f(x)$

$\frac{1}{0.01} = \frac{1}{1/100}$

x	$f(x)$
± 0.1	100
± 0.01	10000
± 0.001	1000000
± 0.0001	100000000
$\downarrow$	$\downarrow$
0	$+\infty$

$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

~~Example 8 1/2. Find, if it exists, $\lim_{x \to \infty} \left(-\frac{1}{x^2}\right)$.~~

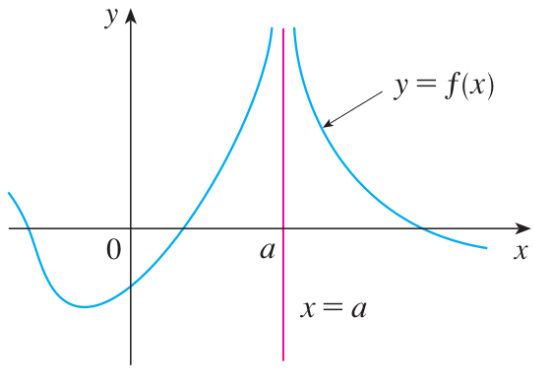
Later (chapter 3)

Positive infinity.

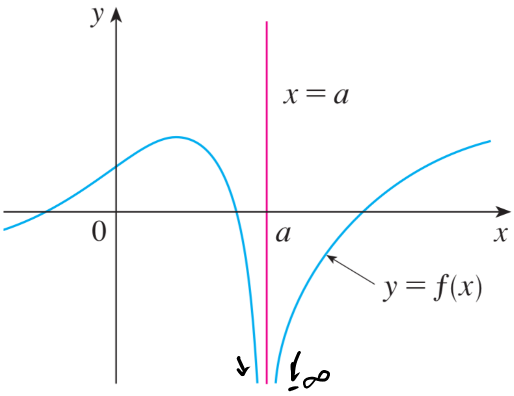
4 Intuitive Definition of an Infinite Limit Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a, but not equal to a.



Negative Infinity



5 Definition Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\boxed{\lim_{x \to a} f(x) = -\infty}$$

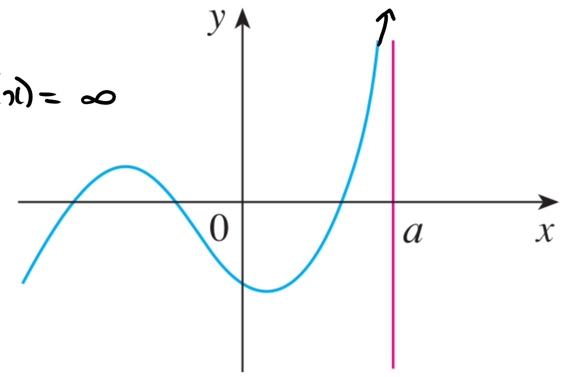
means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a, but not equal to a.

$\infty \cancel{+} \infty$ $\quad \cancel{\frac{\infty}{\infty}}$

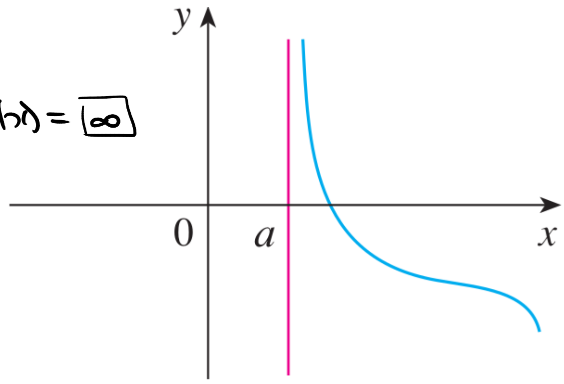
$\infty \cancel{-} \infty$

Other types of infinite limits.

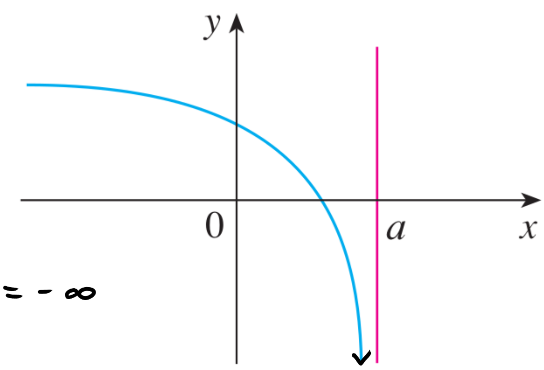
a) $\lim_{x \to a^-} f(x) = \infty$



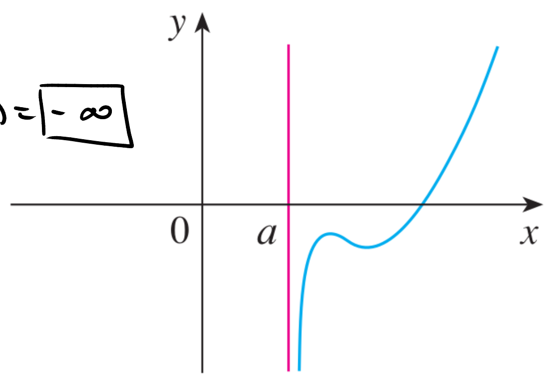
b) $\lim_{x \to a^+} f(x) = \boxed{\infty}$



c) $\lim_{x \to a^-} f(x) = -\infty$



d) $\lim_{x \to a^+} f(x) = \boxed{-\infty}$



EXAMPLE 9 Find $\underbrace{\lim_{x\to 3^+} \frac{2x}{x-3}}_{(a)}$ and $\lim_{x\to 3^-} \frac{2x}{x-3}$.

(a) $\lim_{x\to 3^+} \frac{2x}{x-3}$ From the graph, $\lim_{x\to 3^+} \frac{2x}{x-3} = \infty$

(b) $\lim_{x\to 3^-} \frac{2x}{x-3}$, From the graph, $\lim_{x\to 3^-} \frac{2x}{x-3} = -\infty$.

$\lim_{x\to 3} \frac{2x}{x-3}$?? $\to$ we can't say the value ...

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6 Definition The vertical line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x\to a} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty \qquad \lim_{x\to a^+} f(x) = \infty$$

$$\lim_{x\to a} f(x) = -\infty \qquad \lim_{x\to a^-} f(x) = -\infty \qquad \lim_{x\to a^+} f(x) = -\infty$$

EXAMPLE 10 Find the vertical asymptotes of $f(x) = \tan x$.

$f(x) = \frac{\sin x}{\cos x}$, $\cos x = 0 \iff x = \frac{\pi}{2} + k\pi \quad (k = \ldots, -2, -1, 0, 1, 2)$

Vertical asymptotes are at $x = \frac{\pi}{2} + k\pi$.

Let's take a look at the graph.

$\lim_{x\to \frac{\pi}{2}^-} \tan x = +\infty$ & $\lim_{x\to \frac{\pi}{2}^+} \tan x = -\infty$.

$\vdots$