Chapter 2

Derivatives

2.2 The Derivatives as a Function

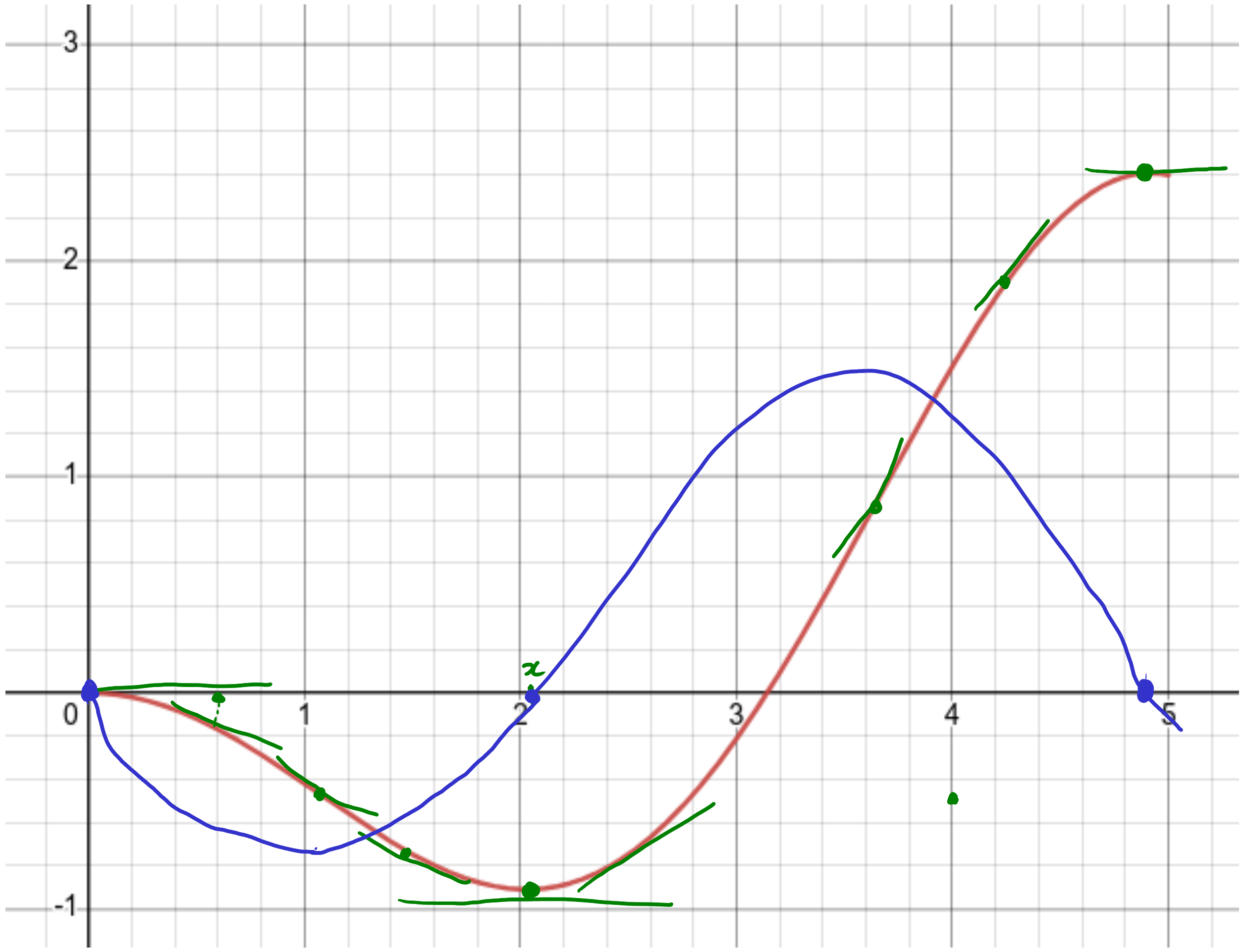
The derivative as a function.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Dom of f': all x such that $f'(x)$ exists.

EXAMPLE 1 The graph of a function f is given . Use it to sketch the graph of the derivative f'.

Desmos: https://www.desmos.com/calculator/o7lfvk2sar



EXAMPLE 3 (a) If $f(x) = \sqrt{x}$, find the derivative of f. State the domain of f'.

(b) Illustrate this formula by comparing the graphs of f and f'. (Do it with Desmos)

(a) $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \to \frac{0}{0}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{\cancel{x} + h + \cancel{\sqrt{x+h}\sqrt{x}} - \cancel{\sqrt{x+h}\sqrt{x}} - \cancel{x}}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \quad (x \neq 0)$$

$$\Rightarrow f'(x) = \frac{1}{2\sqrt{x}} \to \text{Dom}(f') \text{ is } (0, \infty)$$

EXAMPLE 4 Find f' if $f(x) = \dfrac{1-x}{2+x}$.

Dom f is $(-\infty, -2) \cup (-2, \infty)$

Let $x \neq -2$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{1-(x+h)}{2+x+h} - \dfrac{1-x}{2+x}}{h} \to \frac{0}{0}$$

$$= \lim_{h \to 0} \frac{(1-(x+h))(2+x) - (1-x)(2+x+h)}{(2+x+h)(2+x)h}$$

$$= \lim_{h \to 0} \frac{(1-x-h)(2+x) - (1-x)(2+x+h)}{(2+x+h)(2+x)h}$$

$$= \lim_{h \to 0} \frac{\cancel{(1-x)(2+x)} - h(2+x) - \cancel{(1-x)(2+x)} - (1-x)h}{(2+x+h)(2+x)h}$$

$$= \lim_{h \to 0} \frac{-2h - \cancel{hx} - h + \cancel{hx}}{(2+x+h)(2+x)h}$$

$$(\sqrt{x})' = (x^{1/2})'$$

$$\frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-1/2}$$

$$= \lim_{h \to 0} \frac{-2\cancel{h} - \cancel{h}}{(2+x+h)(2+x)\cancel{h}}$$

$$(x^{1/2})' = \frac{1}{2}x^{1/2-1} = \frac{1}{2}x^{-1/2}$$

$$= \lim_{h \to 0} \frac{-3}{(2+x+h)(2+x)} = \boxed{-\frac{3}{(2+x)^2}}$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

Other notations for the derivative.

$y = f(x)$

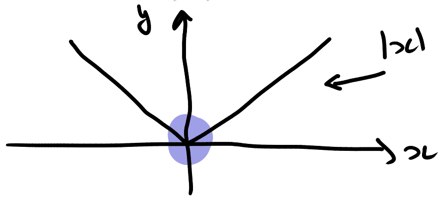
Leibniz notation

$$\boxed{f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)}$$

> **3 Definition** A function f is **differentiable at a** if $f'(a)$ exists. It is **differentiable on an open interval** (a, b) [or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

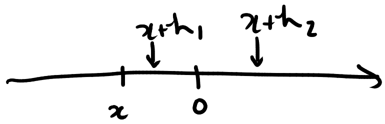
EXAMPLE 5 Where is the function $f(x) = |x|$ differentiable?

Graph:



Problem at $x=0$
(corner)

Case ① $x<0 \Rightarrow |x| = -x$



$x+h$ will be <0 when h approaches 0

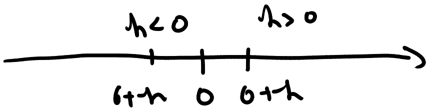
$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{|x+h|-|x|}{h}$$

$$= \lim_{h\to 0} \frac{-(\cancel{x}+h)+\cancel{x}}{h} = \lim_{h\to 0} -1 = -1$$

$\Rightarrow f'(x) = -1.$

Case ② $x>0 \Rightarrow |x| = x. \Rightarrow f'(x) = 1$

Case ③ $x=0.$



$$\lim_{h\to 0^-} \frac{f(\overset{h}{\overset{\shortparallel}{0+h}})-f(0)}{h} = \lim_{h\to 0^-} \frac{|h|-|0|}{h}$$

$$= \lim_{h\to 0^-} \frac{-h-0}{h} = \lim_{h\to 0^-} -1 = -1$$

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{|h|-|0|}{h} = \lim_{h\to 0^+} \frac{h-0}{h} = 1$$

$\Rightarrow \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} \nexists \Rightarrow f'(0) \nexists$

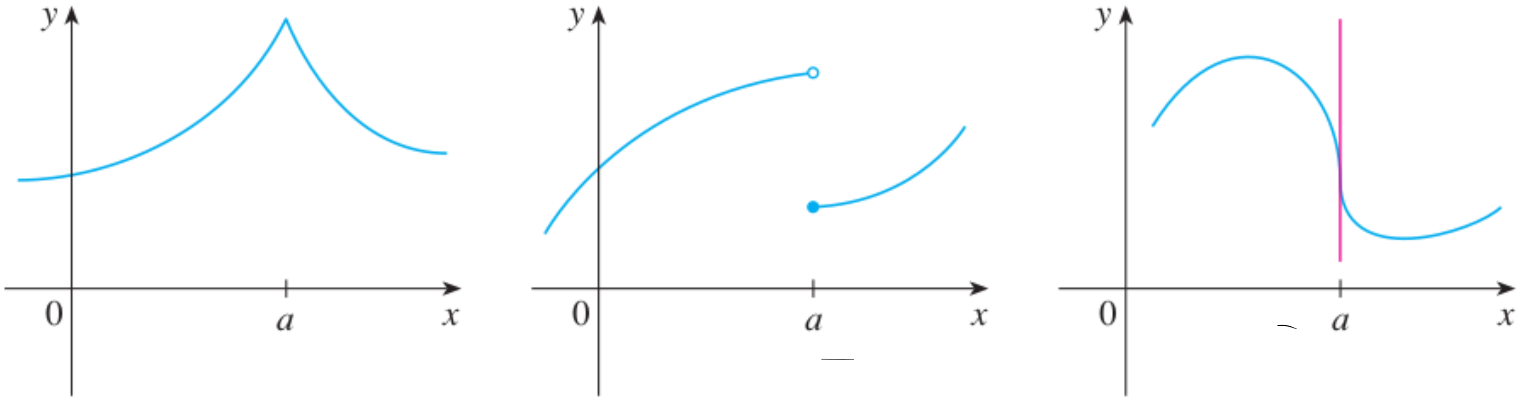
$\Rightarrow f$ not diff. at $x=0.$

Important Result:

> **4 Theorem** If f is differentiable at a, then f is continuous at a.

Remark: f continuous at a does not imply that f is differentiable at a. (Example: $f(x)=|x|$).

How can a Function Fail to be diffentiable?



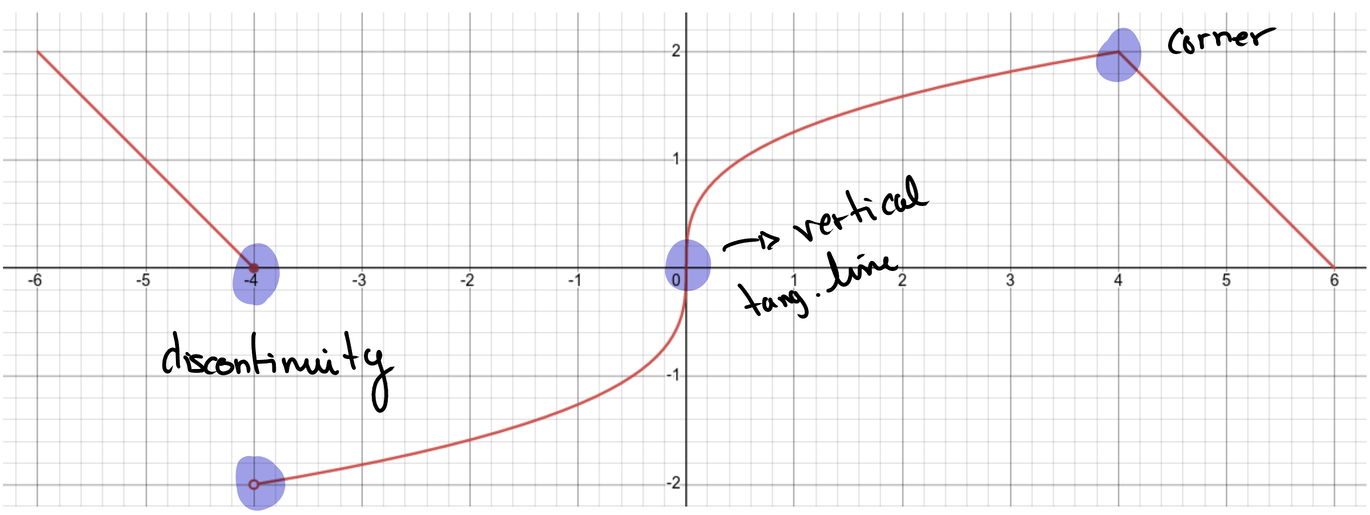
(a) A corner (b) A discontinuity (c) A vertical tangent

(a) when $\displaystyle\lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h} \neq \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}$.

(b) when f is not continuous at $x=a$.

(c) $\displaystyle\lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h} = \pm\infty$ or $\displaystyle\lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h} = \pm\infty$.

Example. The graph of the function is given. State, with reasons, the numbers at which the function is NOT differentiable.
Desmos: https://www.desmos.com/calculator/d0aztxzxta



f not diff. at $x=-4$, $x=0$ & $x=4$.

Higher Derivatives.

Second derivative:

$$\underbrace{\frac{d}{dx}}_{\text{derivative of}}\underbrace{\left(\frac{dy}{dx}\right)}_{\text{first derivative}} = \underbrace{\frac{d^2y}{dx^2}}_{\text{second derivative}}$$

Other notations:

$f''(x)$

or

y''

EXAMPLE 6 If $f(x) = x^3 - x$, find and ~~interpret~~ $f''(x)$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^3 - (\cancel{x}+h) - x^3 + \cancel{x}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x^3} - h}{h}$$

$(x+h)^2 = x^2 + 2xh + h^2$

$$= \lim_{h\to 0} \frac{3x^2\cancel{h} + 3xh^{\cancel{2}\,1} + h^{\cancel{3}\,2} - h}{h}$$

$$= \lim_{h\to 0} \left(3x^2 + 3xh + h^2 - 1\right) = \boxed{3x^2 - 1}$$

$$f''(x) = \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h\to 0} \frac{3(x+h)^2 \cancel{-1} - 3x^2 \cancel{+1}}{h} \quad \frac{0}{0}$$

$$= \lim_{h\to 0} \frac{3(x^2 + 2xh + h^2) - 3x^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{3x^2} + 6xh + 3h^2 - \cancel{3x^2}}{h}$$

$$= \lim_{h\to 0} \frac{6x\cancel{h} + 3h^{\cancel{2}\,1}}{\cancel{h}} = \lim_{h\to 0} 6x + 3h$$

$$= \boxed{6x}$$

Acceleration: derivative of velocity $\to v(t)$

"

second derivative of position function. $\to s(t)$

$$a(t) = v'(t) = \lim_{h\to 0} \frac{v(t+h) - v(t)}{h}$$

$$= \lim_{h\to 0} \frac{s'(t+h) - s'(t)}{h} = s''(t).$$

Third Derivative.

$$y''' = f'''(x) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$$

Jerk: $j = \dfrac{da}{dt} = \dfrac{d^3s}{dt^3}$

n-th Derivative.

$$y^{(n)} = f^{(n)}(x) = \frac{d^n y}{dx^n}$$

EXAMPLE 7 If $f(x) = x^3 - x$, find $f'''(x)$ and $f^{(4)}(x)$.

$f'(x) = 3x^2 - 1$ & $f''(x) = 6x$

$$f'''(x) = \lim_{h \to 0} \frac{f''(x+h) - f''(x)}{h}$$

$$= \lim_{h \to 0} \frac{6(x+h) - 6x}{h} = \lim_{h \to 0} \frac{6\cancel{h}}{\cancel{h}} = \boxed{6}$$

$$f''''(x) = \lim_{h \to 0} \frac{f'''(x+h) - f'''(x)}{h}$$

$$= \lim_{h \to 0} \frac{6 - 6}{h} = \lim_{h \to 0} 0 = \boxed{0}$$