Math 644

Chapter 6

Section 6.4: Linear Fractional Transformation

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Definition 1. A **linear fractional transformation** (LFT for short) is a non-constant rational function of the form

$$T(z) = \frac{az+b}{cz+d} \quad (a, b, c, d \in \mathbb{C}).$$

Basic Types of LFTs:

① Translation: $T(z) = z + b$;

② Rotation: $T(z) = e^{i\theta} z$;

③ Dilation: $T(z) = az$;

④ Inversion: $T(z) = \frac{1}{z}$.

Facts:

① Any LFT is the composition of basic LFTs.
Proof.

$$\frac{az+b}{cz+d} = \frac{a}{c}\left(1 + \frac{\frac{b}{a} - \frac{d}{c}}{z + \frac{d}{c}}\right)$$

② An LFT $T(z) = \frac{az+b}{cz+d}$ is non-constant if and only if $bc - ad \neq 0$.

Theorem 2. The set of LFTs forms a group under composition.

Proof.

We may only, from fact ①, check if

$$S(z) + b, \quad a\,S(z), \quad e^{i\theta} S(z) \quad \& \quad \frac{1}{S(z)}$$

are LFTs for any given LFT $S(z)$.

$$S(z) = \frac{\alpha z + \beta}{\gamma z + \mu} \rightarrow S(z) + b = \frac{(\alpha + b\gamma) z + (\beta + b\mu)}{\gamma z + \mu}$$

All the other cases are LFT (by calculations).

The inverse: $S^{-1}(z) = \frac{\mu z - \beta}{-\gamma z + \alpha}$. □

THEOREM 3. If f is analytic on $\mathbb{C}\backslash\{z_0\}$ and one-to-one then f is an LFT.

Proof.

We can assume that $z_0 = 0$ (otherwise, $g(z) = f(z+z_0)$ and work with g instead).

In annulus around 0,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n .$$

① 0 is an essential singularity. Let Δ be a punctured disk centered at 0.

In this case, for any open set E, $f(\Delta) \cap E \neq \emptyset$.

In particular, let $\Delta = \{ z : 0 < |z| < \frac{1}{2} \}$ and $E = f(B)$ where $B = \{ z : |z-1| < \frac{1}{2} \}$.

So, E is open because f is an open map.

So, $\exists\, \zeta \in \Delta$ s.t. $f(\zeta) \in f(B)$. Also, $\exists\, z \in B$ such that $f(\zeta) = f(z)$. By assumption,

$$\Rightarrow \zeta = z$$

This is a contradiction because $\Delta \cap B = \emptyset$.

Consequences:

- The automorphisms of the complex plane are the linear functions.
- The automorphisms of the extended complex plane are exactly the set of LFTs.

② 0 is a pole of order n.

Now, $f(z) = \sum_{k=-n}^{\infty} a_k z^k$.

So, $\frac{1}{f}$ has a zero of order n.

Since $\frac{1}{f}$ is one-to-one, from Lemma 13, §4.2,

$n \le 1$.

Now, $f(z) = \sum_{k=-1}^{\infty} a_k z^k$ $\quad (r < |z| < R)$.

③ Change z to $\frac{1}{z}$ and define

$$g(z) = f\left(\frac{1}{z}\right) = a_{-1} z + a_0 + \frac{a_1}{z} + \cdots = \sum_{k=-\infty}^{1} a_{-k} z^k$$

valid $\frac{1}{R} < |z| < \frac{1}{r}$.

Apply ① & ② to the fct. g to see that

$$g(z) = \sum_{k=-m}^{1} a_{-k} z^k, \quad m \le 1.$$

So, From ①-③, $f(z) = \frac{a_{-1}}{z} + a_0 + a_1 z$.

Set $w = f(z)$ and assume $a_{-1} \neq 0$ & $a_1 \neq 0$

$$\Rightarrow \quad w = \frac{a_{-1}}{z} + a_0 + a_1 z$$

$$\Leftrightarrow \quad a_1 z^2 + (a_0 - w) z + a_{-1} = 0$$

There are 2 distinct solutions, contradicting the assumptions on f.

So, $a_{-1} \neq 0$ or $a_1 \neq 0$ (not at the same time)

Also, $a_{-1} = 0$ and $a_1 = 0$ is not possible.

$$\Rightarrow \quad f(z) = a_0 + a_1 z \quad \text{or} \quad f(z) = \frac{a_{-1}}{z} + a_0 .$$

$\square$

Preserving Generalized Circles

Notes:

- A generalized circle is a circle in the complex plane or a line in the complex plane.
- A generalized disk is a a region bounded by a generalized circle (so disks or halfplanes).

Theorem 4. LFTs map generalized circles onto generalized circles

Proof. We may only prove it for the 4 basic LFTs.

① Translation:

circle $\longrightarrow$ circle

line $\longrightarrow$ line

$\text{line} \equiv \operatorname{Re}(z\bar{a}) = 0 \qquad \operatorname{Re}(z\bar{a} + \bar{a}b) = 0$

② Dilation & Rotation

circle $\longrightarrow$ circle (clear)

line $\longrightarrow$ line (clear)

$\operatorname{Re}(z\bar{a}) = 0. \qquad \operatorname{Re}(z\gamma\bar{a}) = 0.$

③ Inversion. $T(z) = \frac{1}{z}$

circle $\equiv |z-c| = r$

Square $|z-c|^2 = r^2$ & set $w = \frac{1}{z}$

$$\Rightarrow \left|\frac{1}{w} - c\right|^2 = r^2 \iff \left(\frac{1}{w} - c\right)\left(\frac{1}{\bar{w}} - \bar{c}\right) = r^2$$

- If $|c|^2 = r^2$, then

$$\frac{1}{|w|^2} - \left(\frac{\bar{c}}{w} - \frac{c}{\bar{w}}\right) + \cancel{|c|^2} = \cancel{|c|^2}$$

$$\iff 2\operatorname{Re}(cw) = 1 \rightarrow \text{line}.$$

• If $|c|^2 \neq r^2$, then by completing the square

$$\left| w + \frac{\bar{c}}{r^2 - |c|^2} \right|^2 = \left(\frac{r}{r^2 - |c|^2} \right)^2 . \rightarrow \text{circle}$$

Line

(A) $\mathrm{Re}(za) = 0 \xrightarrow{\text{invert}} \mathrm{Re}\left(\frac{a\bar{z}}{|z|^2}\right) = 0 \rightarrow \underset{\uparrow \text{ line.}}{\mathrm{Re}(a\bar{z}) = 0.}$

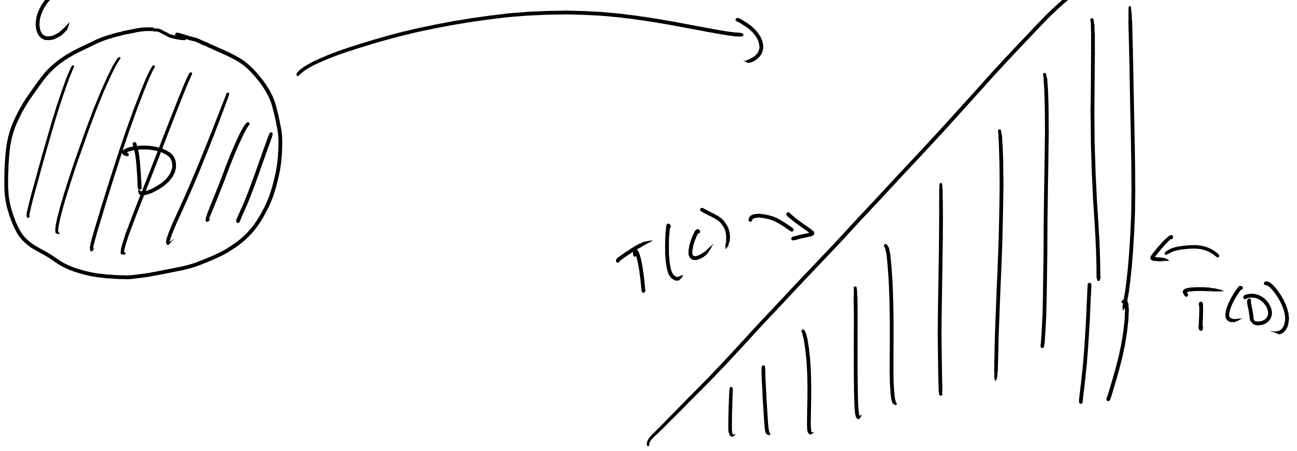
(B) $\mathrm{Re}(za + b) = 0 \xrightarrow[w = \frac{1}{z}]{\text{invert}} \mathrm{Re}\left(\frac{a\bar{w}}{|w|^2}\right) + \mathrm{Re}(b) = 0$

$\mathrm{Re}\, b \neq 0$

$$\rightarrow (2\mathrm{Re}\, b)^2 |w|^2 + a\,\overline{w\, 2\mathrm{Re}(b)} + \bar{a}(w\, 2\mathrm{Re}\, b)$$
$$+ |a|^2 = |a|^2$$

$$\rightarrow \underbrace{|2\mathrm{Re}\, b\, w + a|^2 = |a|^2}_{\text{circle.}}$$

$\square$



THEOREM 5. Given z_1, z_2, z_3 distinct points in $\mathbb{C}^*$, and w_1, w_2, w_3 distinct points in $\mathbb{C}^*$, there is a unique LFTs T such that

$$T(z_i) = w_i, \quad (i = 1, 2, 3).$$

Proof. Assume first $w_1 = 0$, $w_2 = \infty$, $w_3 = 1$.

- $z_1 = \infty$, $z_2 = 0$, $z_3 \neq 0, \infty$ $\longrightarrow$ $T(z) = \dfrac{z_3}{z}$
- $z_1 \neq \infty$, $z_2 = \infty$, $z_3 \neq \infty$ $\longrightarrow$ $T(z) = \dfrac{z - z_1}{z_3 - z_1}$
- $z_1 \neq \infty$, $z_2 \neq \infty$, $z_3 = \infty$ $\longrightarrow$ $T(z) = \dfrac{z - z_1}{z - z_2}$
- $z_1 \neq \infty$, $z_2 \neq \infty$, $z_3 \neq \infty$ $\longrightarrow$ $T(z) = \dfrac{z - z_1}{z - z_2} \cdot \dfrac{z_3 - z_2}{z_3 - z_1}$.

Fix z_1, z_2, z_3 & w_1, w_2, w_3. Choose LFTs R and S s.t.

$$R(z_1) = S(w_1) = 0$$
$$R(z_2) = S(w_2) = \infty$$
$$R(z_3) = S(w_3) = 1$$

the LFT is $T = S^{-1} \circ R$.

<u>Uniqueness</u> Suppose there is another LFT U s.t. $U(z_i) = w_i$ $(i = 1, 2, 3)$.

If $V = S \circ U \circ R^{-1}$, then V is an LFT

& $V(0) = 0$, $V(1) = 1$ & $V(\infty) = \infty$.

$\Rightarrow$ $V(z) = z$

So, $U = S^{-1} \circ R = T$. $\square$

Cayley Transform

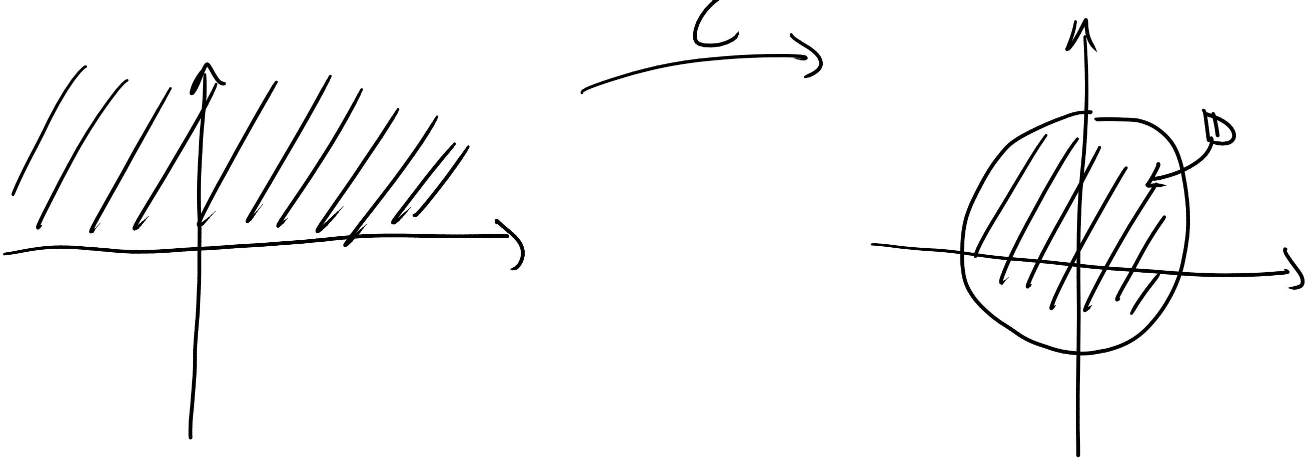
Def.: $\displaystyle C(z) = \frac{z-i}{z+i}.$

Zero: $z = i$

Pole: $z = -i$

Let $\mathbb{H}^+ := \{ z : \operatorname{Im} z > 0 \}$ & $\mathbb{D} := \{ z : |z| < 1 \}$

Then $C(\mathbb{H}^+) = \mathbb{D}$



Also, $C(\mathbb{H}^-) = \mathbb{C} \setminus \overline{\mathbb{D}}$ where $\mathbb{H}^- := \mathbb{C} \setminus \overline{\mathbb{H}^+}$