MATH 644

$$\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta = \frac{f(z)}{2\pi i} \int_\gamma \frac{1}{\zeta - z}\, d\zeta$$

CHAPTER 5

SECTION 5.2: WINDING NUMBER

CONTENTS

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LEMMA 1. If γ is a cycle and $a \notin \gamma$, then

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{\zeta - a}\, d\zeta$$

is an integer.

Proof. Write $\gamma = \sum_{j=1}^{N} \gamma_j$, γ_j are closed curve.

and $\int_\gamma f(z)\,dz = \sum_{j=1}^{N} \int_{\gamma_j} f(z)\,dz$.

We may suppose that each γ_j is cont. different. (piecewise). We can deal with only one of them. From now on, let γ be a closed piecewise cont. diff. curve. $\gamma : [0,1] \to \mathbb{C}$.

Define $h(x) = \int_0^x \frac{\gamma'(t)}{\gamma(t) - a}\, dt$.

Then, $h'(x)$ exists & $h'(x) = \frac{\gamma'(x)}{\gamma(x) - a}$, except at finitely many x .

$$\frac{d}{dx}\left[e^{-h(x)}(\gamma(x) - a)\right] = -h'(x)e^{-h(x)}(\gamma(x) - a) + e^{-h(x)}\gamma'(x)$$

$$= -\gamma'(x)e^{-h(x)} + \gamma'(x)\, e^{-h(x)}$$

$$= 0 \qquad \text{(except at finitely many } x\text{)}.$$

Since $e^{-h(x)}(\gamma(x)-a)$ is continuous, it must be constant in $[0,1]$

$$\Rightarrow \quad e^{-h(1)}(\gamma(1)-a) = e^{-h(0)}(\gamma(0)-a)$$

$$= e^{0}(\gamma(0)-a)$$

$$= \gamma(1)-a \quad (\gamma \text{ closed curve})$$

Since $a \notin \gamma$,

$$e^{-h(1)} = 1$$

$$\Rightarrow \quad h(1) = 2k\pi i \quad , k \in \mathbb{Z}$$

So,

$$\frac{1}{2\pi i}\int_0^1 \frac{\gamma'(t)}{\gamma(t)-a}\,dt = \frac{h(1)}{2\pi i} = k. \qquad \square$$

DEFINITION 2. If γ is a cycle, then the **index** or **winding number** of γ about a is

$$n(\gamma, a) := \frac{1}{2\pi i}\int_\gamma \frac{1}{\zeta - a}\,d\zeta \quad (a \notin \gamma).$$

PROPOSITION 3. Let γ be a cycle.

(a) $n(\gamma, a)$ is an analytic function of a, for $a \notin \gamma$.

(b) $n(\gamma, a)$ is constant in each component of $\mathbb{C}\backslash\gamma$.

(c) $n(\gamma, a) \to 0$ as $a \to \infty$. In particular, $n(\gamma, a) = 0$ for any a in the unbounded component of $\mathbb{C}\backslash\gamma$.

(d) $n(-\gamma, a) = -n(\gamma, a)$.

(e) $n(\gamma_1 + \gamma_2, a) = n(\gamma_1, a) + n(\gamma_2, a)$.

Proof.

(a) Since the fct. $\zeta \mapsto 1$ is continuous on γ, then from Lemma 2 in §4.4,

$$n(\gamma, a) = \frac{1}{2\pi i}\int_\gamma \frac{1}{\zeta - a}\,d\zeta$$

is analytic on $\mathbb{C}\backslash\gamma$.

(b) Let Ω be a component of $\mathbb{C}\backslash\gamma$. Suppose Ω is bounded, so that Ω is bounded by some γ_j*, where $\gamma = \sum_j \gamma_j$. From Lemma 1, $n(\gamma_j, a)$ is constant. In other words, $a \in \Omega$, $n(\gamma_j, a) = n(\gamma, a)$.

(*) that contributes to the winding number.

(c) Since $a \notin \gamma$,

$$\frac{1}{|z-a|} \leq \frac{1}{\operatorname{dist}(\gamma, a)} \longrightarrow 0, \quad a \to \infty.$$

therefore, if σ is a polygonal curve as in thm. 4,

$$n(\gamma, a) = n(\sigma, a) \leq \frac{|\sigma|}{2\pi} \frac{1}{\operatorname{dist}(\sigma, a)} \to 0.$$

By lemma 1, & (a) $n(\gamma, a)$ should be constant

$$\Rightarrow \quad n(\gamma, a) = 0 \quad \forall a \in \Omega$$

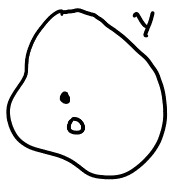
where Ω is the unbounded component of $\mathbb{C} \setminus \gamma$.

(d) Direct calculations.

(e) Direct calculations. $\square$

Some Intuition:

① **Difference in the argument.**



Suppose $\gamma(t) = r(t)\, e^{i\theta(t)}$ where

- $r(t)$, $\theta(t)$ are piecewise cont. diff.
- $0 \leq t \leq 1$.
- $\gamma(0) = \gamma(1)$.

Then,

$$n(\gamma, 0) = \mathrm{Re}\left[\frac{1}{2\pi i} \int_\gamma \frac{1}{z}\, dz\right]$$

$$= \mathrm{Re}\left[\frac{1}{2\pi i} \int_0^1 \frac{r'(t)\, e^{i\theta(t)} + r(t)\, i\theta'(t)\, e^{i\theta(t)}}{r(t)\, e^{i\theta(t)}}\, dt\right]$$

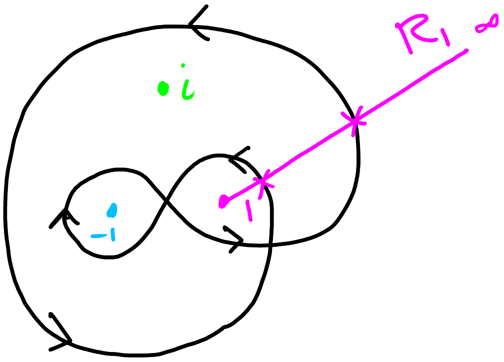
$$= \mathrm{Re}\left[\frac{1}{2\pi i} \int_0^1 \frac{r'(t)}{r(t)} + i\, \theta'(t)\, dt\right]$$

$$= \frac{1}{2\pi} \int_0^1 \theta'(t)\, dt$$

$$= \frac{\theta(1) - \theta(0)}{2\pi}.$$

Net change in the "argument" ÷ by 2π.

② **Rays and number of connected components.**

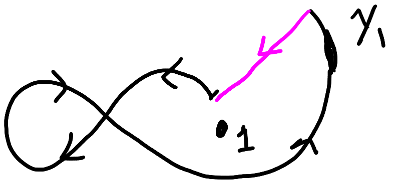


Goal: Find $n(\gamma, 1)$.

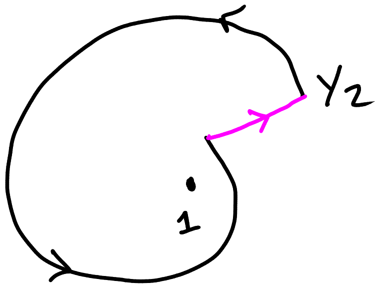
i. Draw ray R_1 from 1 to ∞.

ii. Locate intersections of R_1 with γ.

iii Consider each connected component of $\gamma \setminus R_1$.

- 

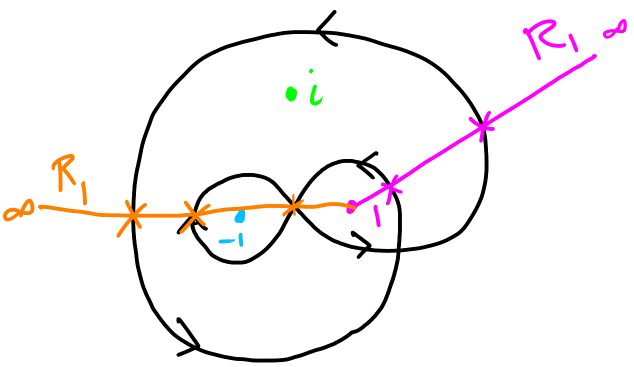
$\longrightarrow$ $n(\gamma_1, 1) = 1$.

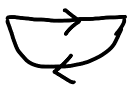
- 

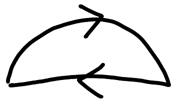
$\longrightarrow$ $n(\gamma_2, 1) = 1$.

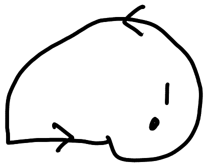
Overall, $n(\gamma, 1) = n(\gamma_1, 1) + n(\gamma_2, 1) = 2$.

Works for any rays:



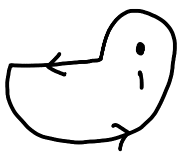
*  $\gamma_1 \longrightarrow n(\gamma_1, 1) = 0$

*  $\gamma_2 \longrightarrow n(\gamma_2, 1) = 0$

* 

$\gamma_3 \longrightarrow n(\gamma_3, 1) = 1$

$n(\gamma, 1) = \sum_j n(\gamma_j, 1) = 2$.

* 

$\gamma_4 \longrightarrow n(\gamma_4, 1) = 1$

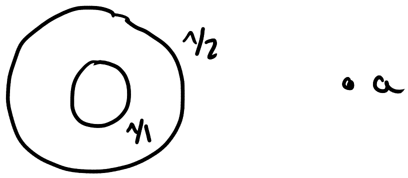
DEFINITION 4. Closed curves γ_1 and γ_2 are **homologous** in a region Ω if $n(\gamma_1 - \gamma_2, a) = 0$ for all $a \notin \Omega$ and we write $\gamma_1 \sim \gamma_2$.

Remarks:

- Homology is an equivalence relation on curves in Ω.
- A closed curve is said to be **homologous to 0** if $n(\gamma, a) = 0$ for all $a \notin \Omega$. In this case, we write $\gamma \sim 0$.

EXAMPLE 5. Show that $\gamma_1(t) = r_1 e^{it}$ and $\gamma_2(t) = r_2 e^{it}$ $(0 \leq t \leq 2\pi)$ are homologous in $\Omega := \{z : |z| < R\}$, where $r_1 < r_2 < R$.

the curve $\gamma = \gamma_1 - \gamma_2$ is



If $a \notin \{z : |z| < R\}$, then

$$n(\gamma_1 - \gamma_2, a) = n(\gamma_1, a) - n(\gamma_2, a) = 0 - 0 = 0 .$$

DEFINITION 6. Let Ω be a bounded region in $\mathbb{C}$ bounded by finitely many piecewise continuously differentiable simple closed curves. The **positive orientation** of $\partial\Omega$ is a parametrization that has the following property:

(a) for each $t \in [0, 1]$ where the derivative exists, there is an $\varepsilon(t) > 0$ such that $\gamma(t) + ui\gamma'(t) \in \Omega$, for all $u \in [0, \varepsilon(t)]$.

Notes:

① When the positive orientation is chosen for $\partial\Omega$, then

- $n(\partial\Omega, a) = 0$, for each $a \notin \overline{\Omega}$;
- $n(\partial\Omega, a) = 1$, for each $a \in \Omega$.

EXAMPLE 7. Find the positive orientation of the boundary of the closed annulus $A := \{z : r_1 \leq |z| \leq r_2\}$.

DEFINITION 8.

(a) A region $\Omega \subset \mathbb{C}^*$ is called **simply-connected** if $\mathbb{C}^* \backslash \Omega$ is connected.

(b) Equivalently, a region Ω is simply-connected if $\mathbb{S}^2 \backslash \pi(\Omega)$ is connected, where π is the stereographic projection.

EXAMPLE 9. Show that

(a) the unit disk is simply connected;

(b) the vectical strip $\Omega = \{z : 0 < \operatorname{Re} z < 1\}$ is simply connected;

(c) $\mathbb{C} \backslash \{0\}$ is not simply connected.

THEOREM 10.

(a) A region $\Omega \subset \mathbb{C}$ is simply-connected if and only if every cycle in Ω is homologous to 0 in Ω.

(b) If Ω is not simply-connected then we can find a simple closed polygonal curve contained in Ω which is not homologous to 0.

Proof.

Corollary 11. Suppose f is analytic on a simply-connected region Ω. Then

(a) $\displaystyle\int_\gamma f(z)dz = 0$ for all closed curves $\gamma \subset \Omega$;

(b) there exists a function F analytic on Ω such that $F' = f$;

(c) if also $f(z) \neq 0$ for all $z \in \Omega$, then there exists a function g analytic on Ω such that $f = e^g$.

Proof.

① **“Uniqueness” in (b).**

② **“Uniqueness” in (c).**

Definition 12. If g is analytic in a region Ω and if $f = e^g$ then g is called a **logarithm** of f in Ω and is written $g(z) = \log f(z)$. The function g is uniquely determined by its value at one point $z_0 \in \Omega$.

Notes:

① f has countably many logarithms, which differ by $2\pi k i$. To specify $\log f(z)$ uniquely, we have to specify its value at one point $z_0 \in \Omega$.

② We do not claim that we can define a logarithm on $f(\Omega)$ and then composed with f to obtain $\log f(z)$.

Example 13. Consider the function $z \mapsto (z-1)/(z+1)$, for $z \in \mathbb{D}$.