

MATH 644

PROBLEM SETS

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PROBLEM 1. Prove the parallelogram equality:

$$|z + w|^2 + |z - w|^2 = 2(|z|^2 + |w|^2).$$

PROBLEM 2. Let w be a non-zero complex number and let $n \geq 1$ be a positive integer. Using the polar coordinates, find n solutions to $z^n = w$.

PROBLEM 3. Let z be a non-zero complex number. Show that $0, z, iz$, and $iz + z$ are the vertices of a square.

PROBLEM 4. Prove that there is no complex number z so that

$$|z| - z = i.$$

PROBLEM 5. Find all complex numbers z satisfying the equation

$$4z - 3\bar{z} = \frac{1 - 18i}{2 - i}.$$

PROBLEM 6. Suppose that f is a continuous complex-valued function on a real interval $[a, b]$. Let

$$A = \frac{1}{b - a} \int_a^b f(x) dx.$$

- a) Show that if $|f(x)| \leq |A|$ for all $x \in [a, b]$, then $f \equiv A$.
- b) Show that if $|A| = \frac{1}{b-a} \int_a^b |f(x)| dx$, then $\arg f$ is constant modulo 2π on $\{z : f(z) \neq 0\}$.

PROBLEM 7. Describe geometrically the following subsets:

- a) $\operatorname{Re} z = \operatorname{Im} z$.
- b) $\operatorname{Re} z > 0$.
- c) $\operatorname{Im} z > 0$.
- d) $\frac{\pi}{6} < \arg z < \frac{\pi}{4}$.

PROBLEM 8. Let $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. Prove that \mathbb{T} equipped with the complex multiplication is a commutative group.

PROBLEM 9. Suppose that $\lim_{n \rightarrow \infty} w_n = w$. Is it true then that also

$$\lim_{n \rightarrow \infty} \arg w_n = \arg w?$$

PROBLEM 10. Let $\{z_n\}$ be a sequence of complex numbers such that $\sum_{n=0}^{\infty} z_n$ converges and there is a ϕ such that $|\arg z_n| \leq \phi < \frac{\pi}{2}$ for any $n \geq 0$. Show that the series $\sum_{n=0}^{\infty} z_n$ is absolutely convergent.

PROBLEM 11. Let \mathbb{C}^* be the extended plane, let \mathbb{S}^2 be the sphere $\{(X, Y, Z) : X^2 + Y^2 + Z^2 = 1\}$ and let $\pi : \mathbb{C}^* \rightarrow \mathbb{S}^2$ be the stereographic projection with $\pi(\infty) = (0, 0, 1)$.

- a) Show that straight lines in \mathbb{C} correspond exactly to circles on \mathbb{S}^2 passing through $(0, 0, 1)$.
- b) Show that if $z \neq \infty$, then

$$\chi(z, \infty) = \frac{2}{\sqrt{1 + |z|^2}}.$$

- c) Using the explicit formula of χ in terms of z and w , show that, for any $z, w \in \mathbb{C}^*$,

$$0 \leq \chi(z, w) \leq 2.$$

PROBLEM 12. For what values of z is

$$\sum_{n=0}^{\infty} \left(\frac{z}{1+z} \right)^n$$

convergent? Draw a picture of the region.

PROBLEM 13. Suppose that $\sum_{n \geq 0} a_n (z - z_0)^n$ is a formal power series. Suppose that

$$R := \frac{1}{\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|}$$

exists and is finite.

- a) Show that the power series converges in $\{z : |z - z_0| < R\}$.
- b) Show that the power series diverges in $\{z : |z - z_0| > R\}$.

PROBLEM 14. Define $e^z = \exp(z) := \sum_{n \geq 0} \frac{z^n}{n!}$.

- a) Show that $e^z e^w = e^{z+w}$ (using the power series definition).
- b) Show that $|e^z| = e^{\operatorname{Re} z}$ and $\arg e^z = \operatorname{Im} z$.
- c) Show that $\frac{d}{dz} e^z = e^z$.
- d) Show that, for any non-zero integer n ,

$$\int_0^{2\pi} e^{int} dt = 0.$$

[Hint: Use Fundamental Theorem of Calculus.]

- e) Compute the integral

$$\int e^{nt} \cos(mt) dt.$$

[Hint: Rewrite $\cos(mt)$ as a complex exponential.]

PROBLEM 15. Prove the following assertions.

- a) If f and g are analytic at z_0 , then $(f + g)'(z_0) = f'(z_0) + g'(z_0)$ (Sum rule of differentiation for analytic functions).
- b) If f and g are analytic at z_0 , then $(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$ (Product Rule of differentiation for analytic functions).
- c) If f and g are analytic at z_0 with $g(z_0) \neq 0$, then $(f/g)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{(g(z_0))^2}$ (Quotient rule of differentiation for analytic functions).

- d) If f is analytic at z_0 and g is analytic at $f(z_0)$, then $(g \circ f)'(z_0) = g'(f(z_0))f'(z_0)$ (Chain Rule of differentiation for analytic functions).

Find the derivative of $(z - a)^{-n}$, where n is a positive integer and $a \in \mathbb{C}$.

PROBLEM 16. Let $\Omega \subset \mathbb{C}$. Show that Ω is connected if and only if Ω and \emptyset are the only open and closed subsets of Ω .

PROBLEM 17. Suppose that f and g are two analytic functions on a region (open and connected) Ω . Suppose there is a sequence $(z_n)_{n=1}^\infty$ with $z_n \in \Omega$ ($\forall n \geq 1$) such that $f(z_n) = g(z_n)$ ($\forall n \geq 1$). If (z_n) has an accumulation point $z_0 \in \Omega$, then show that $f \equiv g$ on Ω .

PROBLEM 18. Show that $\cos^2(z) + \sin^2(z) = 1$ for every $z \in \mathbb{C}$.

PROBLEM 19. Suppose f is analytic in a connected open set Ω such that, for each $z \in \Omega$, there exists an n (depending on z) such that $f^{(n)}(z) = 0$. Prove that f is a polynomial. [Hint: Use Baire's Theorem.]

PROBLEM 20. Let f be analytic in a region Ω containing the point $z = 0$. Suppose $|f(1/n)| < e^{-n}$ for $n \geq n_0$, for some integer $n_0 \geq 0$. Prove $f \equiv 0$ in Ω .

PROBLEM 21. Let f and g be analytic functions in a region Ω .

- Show that if $f'(z) = 0$ for all z in a neighborhood of some $z_0 \in \Omega$, then f is constant in Ω , meaning there is a constant $c \in \mathbb{C}$ such that $f(z) = c$ for any $z \in \Omega$.
- Show that if f and g are analytic in a region Ω with $f'(z) = g'(z)$ for every $z \in \Omega$, then $f - g$ is constant.

PROBLEM 22. Suppose that $f(z) = az^3 + bz^2 + cz + d$. In addition, suppose that for each $z, w \in \mathbb{C}$ there exists a point ζ on the line segment between z and w with

$$\frac{f(z) - f(w)}{z - w} = f'(\zeta).$$

Show that $a = 0$.

PROBLEM 23. Suppose that $f(z) = \sum_{n=0}^\infty a_n(z - z_0)^n$ converges in $B = \{z : |z - z_0| < r\}$. Show that the power series

$$F(z) := \sum_{n=0}^\infty \frac{a_n}{n+1} (z - z_0)^{n+1}$$

converges in B and satisfies $F'(z) = f(z)$ for all $z \in B$. Moreover, show that the radius of convergence of F is the same as the radius of convergence of f .

PROBLEM 24. Suppose $\sum_{j=0}^\infty |a_j|^2 < \infty$.

- Show that $f(z) = \sum_{j=0}^\infty a_j z^j$ is analytic in $\{z : |z| < 1\}$.

b) Compute (with a proof) the following quantity:

$$\lim_{r \rightarrow 1^-} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi}.$$

PROBLEM 25. Suppose f has a power series expansion at 0 which converges in all of \mathbb{C} . Suppose also that $\int_{\mathbb{C}} |f(x + iy)| dx dy < \infty$. Prove that $f \equiv 0$.

PROBLEM 26. [Hard] Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ have radius of convergence 1 and suppose that $a_n \geq 0$ for all n . Prove that $z = 1$ is a singular point of f . That is, there is no function g analytic in a ball B containing $z = 1$ such that $f = g$ on $B \cap D$.

PROBLEM 27. If f is analytic in a region Ω and if there is a $z_0 \in \Omega$ such that

$$|f(z_0)| = \inf_{z \in \Omega} |f(z)|,$$

and if $f(z_0) \neq 0$, then f is constant in Ω .

PROBLEM 28. Let Ω be a region in \mathbb{C} . Show that if $f : \Omega \rightarrow \mathbb{C}$ is an open map, then f satisfies the maximum modulus principle.

PROBLEM 29.

- a) Show geometrically why the maximum principle holds using a “walking the dog” argument. Make it rigorous by following the steps of the proof of the Fundamental Theorem of Algebra.
- b) Use the maximum modulus principle to prove the Fundamental Theorem of Algebra.

PROBLEM 30. Let f be an analytic function defined on some bounded region $\Omega \subset \mathbb{C}$. Show that

$$\limsup_{z \rightarrow \partial\Omega} |f(z)| = \limsup_{\delta \rightarrow 0} \{ |f(z)| : z \in \Omega, \text{dist}(z, \partial\Omega) = \delta \}.$$

PROBLEM 31. Let $\Omega \subset \mathbb{C}$ be a region. Show that $z_n \rightarrow \partial\Omega$ as $n \rightarrow \infty$ if and only if $(z_n)_{n \geq 1}$ has no subsequence converging to a point $z_0 \in \Omega$.

PROBLEM 32. Suppose that f is analytic in a connected open (region) set Ω .

- a) Prove that if $|f(z)|$ is constant on Ω , then f is constant on Ω .
- b) Prove that if $\text{Re } f$ is constant on Ω , then f is constant on Ω .

PROBLEM 33.

- a) Prove that if f is analytic in \mathbb{C} , then $f(z) = \sum_{n \geq 0} a_n z^n$ for any $z \in \mathbb{C}$. In other words, the radius of convergence of the power series $\sum_{n \geq 0} a_n z^n$ representing f at $z = 0$ is $R = \infty$.
- b) Suppose that f is analytic in \mathbb{C} and $|f(z)| \leq C|z|^n$, for some $|z| > M$ and $n \geq 0$. Show that f must be a polynomial.
- c) Suppose that f and g are analytic in \mathbb{C} with $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$. Prove that there exists a constant $c \in \mathbb{C}$ such that $f(z) = cg(z)$ for all $z \in \mathbb{C}$.

PROBLEM 34. Prove that if f is non-constant and analytic on all of \mathbb{C} , then $f(\mathbb{C})$ is dense in \mathbb{C} .

PROBLEM 35. Let f be analytic in \mathbb{D} and suppose $|f(z)| < 1$ on \mathbb{D} . Let $a = f(0)$. Show that f does not vanish in $\{z : |z| < |a|\}$.

PROBLEM 36.

- a) Prove that φ is a one-to-one analytic map of \mathbb{D} onto \mathbb{D} if and only if

$$\varphi(z) = c \left(\frac{z - a}{1 - \bar{a}z} \right) \quad (z \in \mathbb{D}),$$

for some constants c and a , with $|c| = 1$ and $|a| < 1$.

- b) Let f be analytic in \mathbb{D} and satisfy $|f(z)| \rightarrow 1$ as $|z| \rightarrow 1$. Prove that f is rational.

PROBLEM 37.

- a) Suppose p is a non-constant polynomial with all its zeros in the upper half-plane $\mathbb{H} := \{z : \operatorname{Im} z > 0\}$. Prove that all the zeros of p' are contained in \mathbb{H} . [*Hint: Look at the partial fraction expansion of p'/p .*]
- b) Use a) to prove that if p is a polynomial, then the zeros of p' are contained in the (closed) convex hull of the zeros of p . (The closed convex hull is the intersection of all half-planes containing the zeros.)

PROBLEM 38. Suppose f is analytic in \mathbb{D} and $|f(z)| \leq 1$ in \mathbb{D} and $f(0) = 1/2$. Prove that $|f(1/3)| \geq 1/5$.

PROBLEM 39. Suppose f is analytic and non-constant in \mathbb{D} and $|f(z)| \leq M$ on \mathbb{D} . Prove that the number of zeros of f in a disk of radius $1/4$, centered at 0, does not exceed

$$\frac{1}{\ln 4} \ln \left| \frac{M}{f(0)} \right|.$$

PROBLEM 40. Find a parametrization for each set.

- (a) A square of height h and length l .
- (b) The arc of the unit circle starting at 1 and ending at $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$.
- (c) The parabola starting at $(0, 0)$ and ending at $(1, 1)$.
- (d) An ellipse centered at $z_0 \in \mathbb{C}$ with minor axis m and major axis M .

PROBLEM 41. Which of the curves in the previous problem are (i) an arc (ii) closed (ii) simple?

PROBLEM 42. Find the value of

$$\int_{\gamma} x - y + ix^2 dz$$

if γ is

- (a) the straight line joining 0 to $1 + i$;
- (b) the imaginary axis from 0 to i ;
- (c) the line parallel to the real axis from i to $1 + i$.

PROBLEM 43. Compute $\int_{\gamma} \frac{1}{z^3} dz$, where γ is the circle of radius $1/2$ centered at the origin. Compare with Example 9 from Section 4.1.

PROBLEM 44. Let γ denote a circular path with center 1 and radius 1, described once counter-clockwise and starting at the point 2.

- (a) Find a parametrization of γ .
- (b) Find the value of $\int_{\gamma} |z|^2 dz$.

PROBLEM 45. Suppose that f and g are two holomorphic function on a region Ω containing a piecewise continuously differentiable curve $\gamma : [a, b] \rightarrow \mathbb{C}$. Prove the complex analogue of the integration by parts formula:

$$\int_{\gamma} f(z)g'(z) dz = f(z_1)g(z_1) - f(z_0)g(z_0) - \int_{\gamma} f'(z)g(z) dz$$

where $z_1 = \gamma(b)$ and $z_0 = \gamma(a)$.

PROBLEM 46. Let γ be a piecewise continuously differentiable curve. Prove that

$$\int_{-\gamma} f(z) dz = - \int_{\gamma} f(z) dz.$$

PROBLEM 47. Let $\alpha, \beta : [0, 1] \rightarrow \mathbb{C}$ be two curves. Show that $\alpha + \beta = \beta + \alpha$ (in the sense of Definition 12 in Section 4.1).

PROBLEM 48. Prove Corollary 16 in Section 4.1.

PROBLEM 49. Prove all of the properties on page 9 in section 4.1.

PROBLEM 50. Let $B := \{z : |z - 1| < 1\}$.

- (a) Let n be a positive integer. Define a function $g(z) = \sqrt[n]{z}$ on B .
- (b) Show that $g(z) = \sqrt[n]{z}$ is analytic on B .

PROBLEM 51. Let f be analytic in a neighborhood of the closure of a bounded convex set S with piecewise continuously differentiable boundary ∂S . Show that for any $z \in S$,

$$f(z) = \frac{1}{2\pi} \int_{\partial S} \frac{f(\zeta)}{\zeta - z} d\zeta,$$

where ∂S is parameterized in the counter-clockwise direction.

PROBLEM 52. Let f be analytic in a neighborhood of the closure of a bounded convex set S with piecewise continuously differentiable boundary ∂S . Show that

$$\frac{1}{2\pi i} \int_{\partial S} f(\zeta) d\zeta = 0.$$

PROBLEM 53. Show that there is no function f analytic in a neighborhood Ω of $\partial\mathbb{D}$ such that $f'(z) = 1/z$ for any $z \in \Omega$.

PROBLEM 54.

- (a) Use Cauchy's estimate to prove Liouville's Theorem.
- (b) Use Cauchy's estimate to compute a lower bounded on the radius of convergence of the power series representation of a bounded holomorphic function.

PROBLEM 55. Suppose Ω is a region which is symmetric with respect to \mathbb{R} , meaning that $z \in \Omega$ if and only if $\bar{z} \in \Omega$. Set $\Omega := \Omega \cap \mathbb{H}$ and $\Omega^- := \Omega \cap (\mathbb{C} \setminus \overline{\mathbb{H}})$, where $\mathbb{H} := \{z : \operatorname{Im} z > 0\}$. If f is continuous on Ω^+ , continuous on $\Omega^+ \cup (\Omega \cup \mathbb{R})$, and $\operatorname{Im} f = 0$ on $\Omega \cap \mathbb{R}$, then show that the function defined by

$$F(z) := \begin{cases} f(z) & \text{if } z \in \Omega \setminus \Omega^- \\ \overline{f(\bar{z})} & \text{if } z \in \Omega^- \end{cases}$$

is analytic on Ω .

PROBLEM 56. Let $\gamma : [a, b] \rightarrow \Omega$ be a curve, where Ω is a region in \mathbb{C} . Assume that f is analytic in Ω . The goal of this problem is to generalize the integral of f along a curve (not necessarily piecewise continuously differentiable).

- (a) Given $0 < \varepsilon < \text{dist}(\gamma, \partial\Omega)$, show that there is a finite partition $0 = t_0 < t_1 < \cdots < t_n = 1$ so that $\gamma([t_{j-1}, t_j]) \subset B_j := \{z : |z - \gamma(t_j)| < \varepsilon\} \subset \Omega$ for any $j = 1, 2, \dots, n$.
- (b) For a given $0 < \varepsilon < \text{dist}(\gamma, \partial\Omega)$, take a partition from (a). Let $\sigma := \sum_{j=1}^n \sigma_j$ be the polygonal curve, where σ_j is the line segment joining $\gamma(t_{j-1})$ to $\gamma(t_j)$. Define

$$\int_{\gamma} f(\zeta) d\zeta := \int_{\sigma} f(\zeta) d\zeta.$$

Show that this definition does not depend on the choice of the polygonal curve σ .

- (c) Show that the previous definition agrees for a piecewise continuously differentiable curve γ .

PROBLEM 57.

- (a) For $x > 0$, define $x^{-z} := e^{-z \ln(x)}$. Prove that the Riemann Zeta Function

$$\zeta(z) := \sum_{n=1}^{\infty} n^{-z}$$

converges and is analytic in $\{z : \text{Re } z > 1\}$.

- (b) Show that, for $\text{Re } z > 1$,

$$\zeta(z) - \int_1^{\infty} x^{-z} dx = \sum_{n=1}^{\infty} \int_n^{n+1} \int_n^x z t^{-z-1} dt dx.$$

- (c) Use (b) to prove that $(z-1)\zeta(z)$ has a unique analytic extension to $\{z : \text{Re } z > 0\}$.
- (d) Use the fundamental theorem of calculus to give a series for $\zeta(z) - 1/(z-1)$, valid in $\text{Re } z > 0$, which does not involve an integral.

PROBLEM 58. Show that there is a constant $C < \infty$ so that if f is analytic on \mathbb{D} , then

$$|f'(z)| \leq C \int_{\mathbb{D}} |f(x+iy)| dx dy$$

for all $|z| \leq 1/2$. [Hint: Use Cauchy's integral formula.]

PROBLEM 59. Prove that there exists a sequence of polynomials p_k such that

$$\lim_{k \rightarrow \infty} p_k(z) = \begin{cases} 1 & \text{if } \text{Re } z > 0 \\ 0 & \text{if } \text{Re } z = 0 \\ -1 & \text{if } \text{Re } z < 0. \end{cases}$$

PROBLEM 60. Suppose f has a complex derivative at each point of a region Ω . Prove that f is analytic in Ω .

PROBLEM 61. Prove that homology is an equivalence relation on curves in Ω .

PROBLEM 62. If $\sum_{n=-\infty}^{\infty} a_n z^n = \sum_{n=-\infty}^{\infty} b_n z^n$ for $r < |z| < R$ (the series converges in the annulus), then prove that $a_n = b_n$ for any $n \in \mathbb{Z}$.

PROBLEM 63. Find the singularity at ∞ of the following functions. If the singularity is removable, give the value; if the singularity is a zero or pole, give the order.

- | | |
|---------------------------------|-------------------|
| a) $\frac{z^2}{e^z};$ | c) $e^{z/(1-z)};$ |
| b) $\frac{1}{e^{1/z} - 1} - z;$ | d) $z^2 - z.$ |

PROBLEM 64. Find the expansion in powers of z for

- a) $\frac{z^3}{(z^2 + z + 1)(z - 1)}$, which converges in $|z| < 1$.
- b) $\frac{z}{(z^2 + 4)(z - 3)(z - 4)}$, which converges in $3 < |z| < 4$ [*Hint: Use the partial fraction decomposition of the rational function.*]

PROBLEM 65. Let n be an even natural number and α, β be two non-zero real numbers. Prove that the number of roots of the equation

$$z^{2n} + \alpha z^{2n-1} + \beta^2 = 0$$

which have positive real part is equal to n .

PROBLEM 66. How many zeros does $p(z) = 3z^5 + 21z^4 + 5z^3 + 6z + 7$ have in $\overline{\mathbb{D}}$? How many zeros in $\{z : 1 < |z| < 2\}$?

PROBLEM 67. Prove that all of the zeros of the polynomial

$$p(z) = z^n + c_{n-1}z^{n-1} + \cdots + c_1z + c_0$$

lie in the disk centered at 0 with radius

$$R = \sqrt{1 + |c_{n-1}|^2 + \cdots + |c_1|^2 + |c_0|^2}.$$