Math 644

Chapter 4

Section 4.4: Weierstrass' Theorem

Contents

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Theorem 1. Suppose (f_n) is a collection of analytic functions on a region Ω such that $f_n \to f$ uniformly on compact subsets of Ω. Then f is analytic on Ω. Moreover, $f_n' \to f'$ uniformly on compact subsets of Ω.

Lemma 2. If G is integrable on a piecewise continuously differentiable curve γ, then

$$g(z) := \int_\gamma \frac{G(\zeta)}{\zeta - z}\, d\zeta$$

is analytic in $\mathbb{C}\backslash\gamma$ and

$$g'(z) = \int_\gamma \frac{G(\zeta)}{(\zeta - z)^2}\, d\zeta.$$

Proof.
Write, for z, $z+h \in \mathbb{C}\backslash\gamma$ $(h \neq 0)$

$$\frac{g(z+h) - g(z)}{h} = \int_\gamma \frac{G(\zeta)}{(\zeta - z - h)(\zeta - z)}\, d\zeta$$

As $h \to 0$, $\dfrac{G(\zeta)}{(\zeta - z - h)(\zeta - z)} \longrightarrow \dfrac{G(\zeta)}{(\zeta - z)^2}$

uniformly on γ. and so $g'(z)$ exists and

$$g'(z) = \lim_{h \to 0} \int_\gamma \frac{G(\zeta)}{(\zeta - z - h)(\zeta - z)}\, d\zeta = \int_\gamma \frac{G(\zeta)}{(\zeta - z)^2}\, d\zeta$$

Moreover, g' is continuous and therefore g is holomorphic on $\mathbb{C}\backslash\gamma$ $\square$

Proof of Weierstrass's Theorem.

1) Let B be a disk with $\overline{B} \subseteq \Omega$.

Then,
$$f_n(z) = \frac{1}{2\pi i} \int_{\partial B} \frac{f_n(\zeta)}{\zeta - z} \, d\zeta, \quad z \in B.$$

Since $\partial B \subseteq \Omega$ is compact, $f_n \to f$ uniformly on ∂B. In particular, f is continuous.

Set
$$F(z) = \frac{1}{2\pi i} \int_{\partial B} \frac{f(\zeta)}{\zeta - z} \, d\zeta, \quad z \in B.$$

By Lemma 2, F is analytic on B. But
$$f(z) = \lim_{n \to \infty} f_n(z) = \frac{1}{2\pi i} \int_{\partial B} \frac{\lim_{n \to \infty} f_n(\zeta)}{\zeta - z} \, d\zeta$$
$$= F(z), \quad \forall z \in B.$$

So, f is analytic on B, so on Ω.

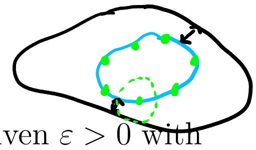
2) From Lemma 2,
$$f_n'(z) = \frac{1}{2\pi i} \int_{\partial B} \frac{f_n(\zeta)}{(\zeta - z)^2} \, d\zeta, \quad z \in B$$
&
$$f'(z) = \frac{1}{2\pi i} \int_{\partial B} \frac{f(\zeta)}{(\zeta - z)^2} \, d\zeta, \quad z \in B.$$

Proof of Weierstrass's Theorem. (continued)

By assumption, we obtain $f_n' \to f'$ unif. on some $\overline{B_0} \subseteq B$.

If $K \subseteq \Omega$ is compact, then cover K by such balls ($\overline{B_0}$) where $f_n' \to f$ unif. $\square$

Goal:



- Extend the definition of the integral to continuous maps $\gamma : [a, b] \to \mathbb{C}$.

Lemma 3. Suppose Ω is a region and suppose $\gamma : [0, 1] \to \Omega$ is continuous. Given $\varepsilon > 0$ with $0 < \varepsilon < \operatorname{dist}(\gamma, \partial\Omega)$, we can find a finite partition $0 = t_0 < t_1 < \cdots < t_n = 1$ so that

a) $\gamma([t_{j-1}, t_j]) \subset B_j := \{z : |z - \gamma(t_j)| < \varepsilon\}$ for every $j = 1, \ldots, n$;

b) $B_j \subset \Omega$ for every $j = 1, \ldots, n$.

Proof. If $0 < \varepsilon < \operatorname{dist}(\gamma, \partial\Omega)$, then by uniform continuity of γ, $\exists\, \delta > 0$ s.t.

$$|\gamma(s) - \gamma(t)| < \varepsilon, \quad \forall\, |s-t| < \delta.$$

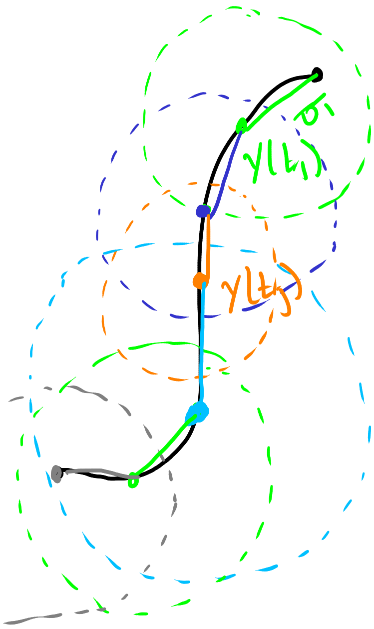
Choose $0 = t_0 < t_1 < \cdots < t_n = 1$ s.t. $\sup_j |t_j - t_{j-1}| < \delta$.

Since $|\gamma(s) - \gamma(t)| < \varepsilon$ if $|s-t| < \delta$, we have

$$|\gamma(t) - \gamma(t_j)| < \varepsilon, \quad \text{if } t \in [t_{j-1}, t_j].$$

$\square$

Construction: $\{t_j\}_{j=0}^{n}$ the partition in Lemma 3.



Let $\sigma = \sum_{j=1}^{n} \sigma_j$ be poly. curve, where

σ_j : straight line from $\gamma(t_{j-1})$ to $\gamma(t_j)$.

γ_j : subarc $\gamma([t_{j-1}, t_j])$, $j = 1, \ldots, n$.

If γ is piecewise cont. diff., then $\beta_j = \gamma_j - \sigma_j$ is closed & f analytic

$$\Rightarrow \int_{\gamma_j} f(z)\,dz = \int_{\sigma_j} f(z)\,dz.$$

THEOREM 4. Suppose Ω is a region and $\gamma : [0,1] \to \mathbb{C}$ is continuous with $\gamma \subset \Omega$. Let σ be the polygonal curve defined in the last page. If f is analytic on Ω, define

$$\int_\gamma f(z)\,dz = \int_\sigma f(z)\,dz.$$

Then this definition of $\int_\gamma f(z)\,dz$ does not depend on the choice of the polygonal curve σ and it agrees with our prior definition if γ is piecewise continuously differentiable.

Proof.

If γ is piecewise diff. (cont.) curve & $\gamma_1, \ldots, \gamma_n$ & $\sigma_1, \ldots, \sigma_n$ are as in the previous page, then

$$\int_{\gamma_j} f(z)\,dz = \int_{\sigma_j} f(z)\,dz$$

$$\Rightarrow \int_\gamma f(z)\,dz = \int_\sigma f(z)\,dz.$$

Now, let γ be continuous.

Let σ be the poly. curve with $\{t_j\}$ const. from Lemma 3.

Select α be another such poly. curve associated with $\{s_j\} \subseteq \{t_j\}$.

Write $\sigma - \alpha = \sum_j \beta_j$ where β_j are closed polygonal curve in each B_j (Lemma 3)

$$\Rightarrow \int_{\sigma-\alpha} f(z)\,dz = \sum_j \int_{\beta_j} f(z)\,dz = 0$$

$$\Rightarrow \int_\sigma f(z)\,dz = \int_\alpha f(z)\,dz.$$

If σ & α are not a refinement of each other, then consider β a poly. curve created from $\{t_j\} \cup \{s_j\}$.

β is a refinement of α and σ

$$\Rightarrow \int_\alpha f(z)\,dz = \int_\beta f(z)\,dz = \int_\sigma f(z)\,dz. \quad \square$$