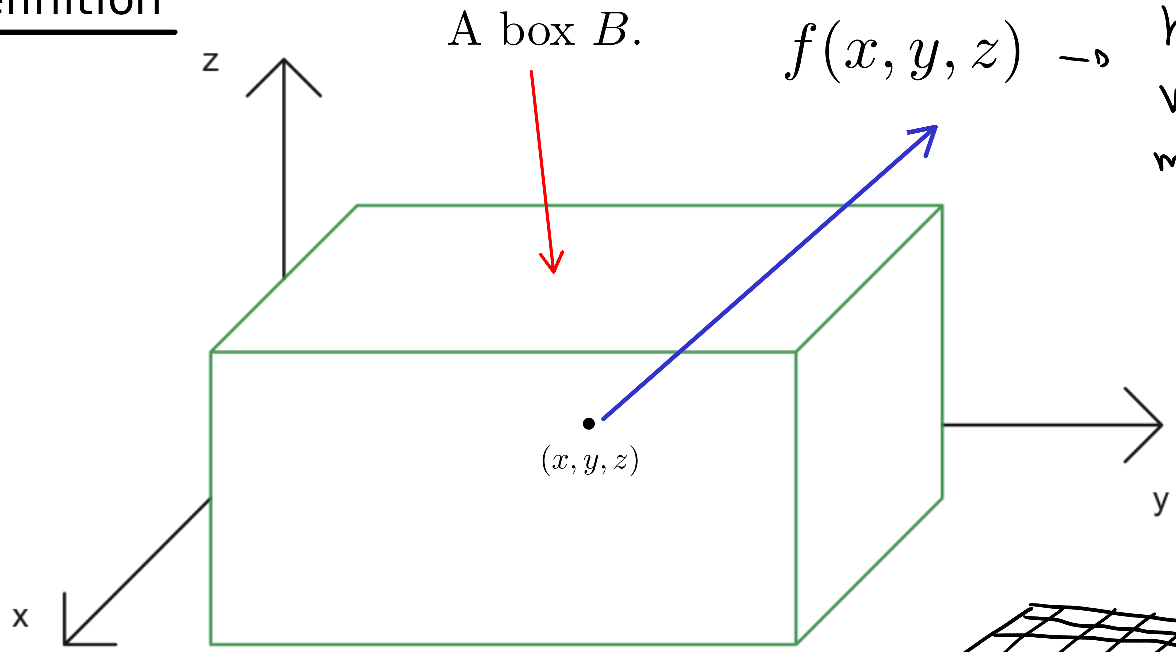
Chapter 15

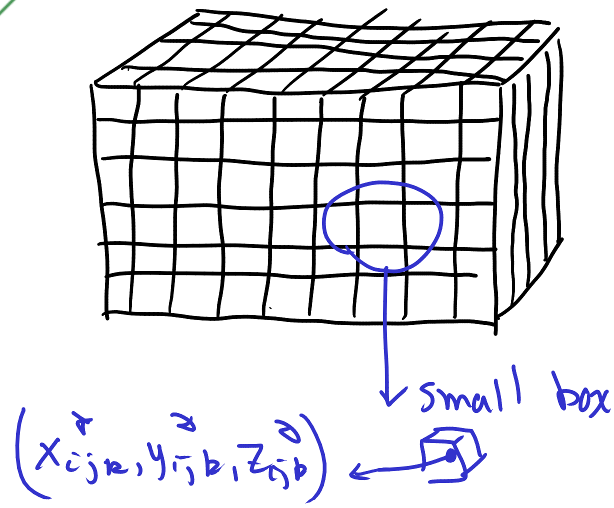
Multiple Integrals

15.6 Triple integrals

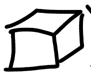
Definition



$\to$ heat
volume
mass density
$\vdots$



- A box $B = [a, b] \times [c, d] \times [r, s]$
- Divide [a, b] in l parts
- Divide [c, d] in m parts
- Divide [r, s] in n parts

Vol() = ΔV

Heat in a small box

$$\approx f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \cdot \Delta V$$

Total heat in box

$$\approx \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \cdot \Delta V$$

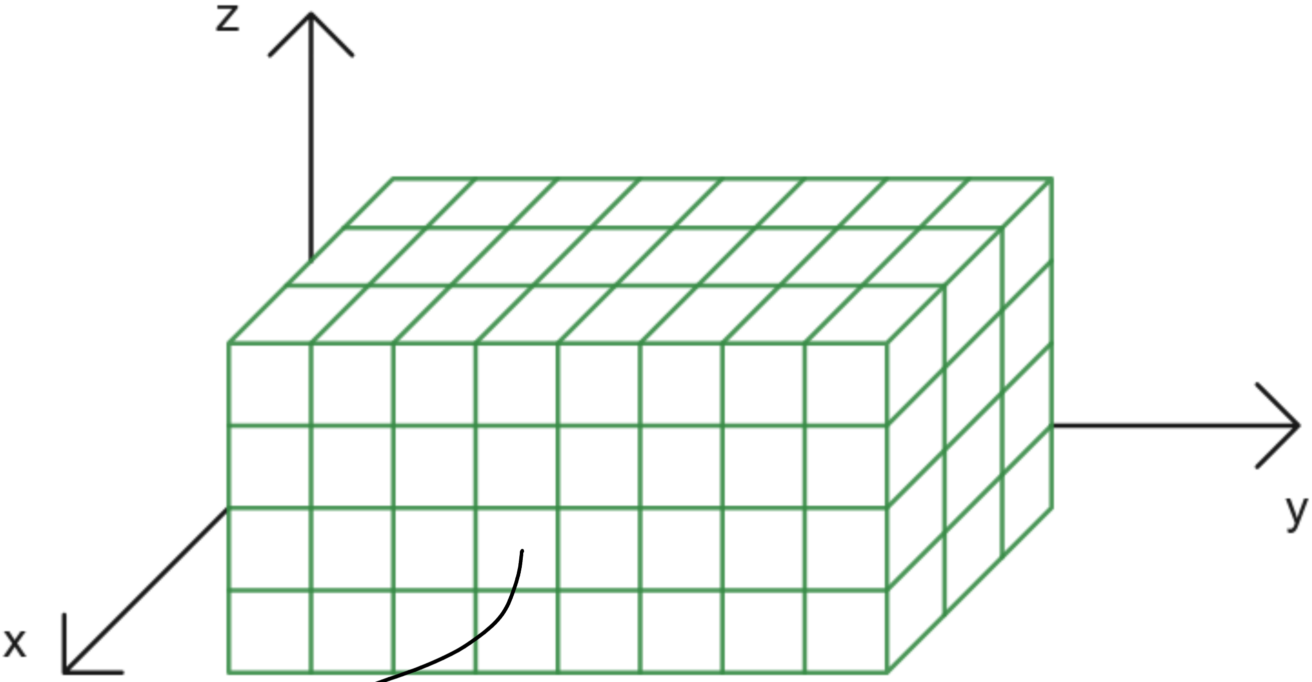
Take the limit as $l, m, n \longrightarrow \infty$

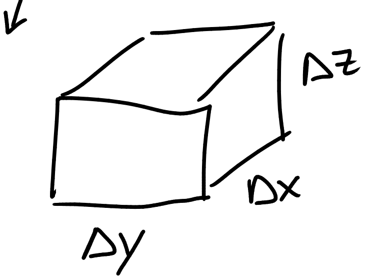
The triple integral of f over the box B is

$$\boxed{\iiint_B f(x, y, z)\, dV = \lim_{l,m,n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \Delta V}$$

Triple integrals in cartesian coordinates

- Write explicitly $B = \{(x,y,z) : a \le x \le b,\ c \le y \le d,\ r \le z \le s\}$
- Divide $[a,b]$ in parts of length Δx.
- Divide $[c,d]$ in parts of length Δy.
- Divide $[r,s]$ in parts of length Δz.





$\Delta V = \Delta x \Delta y \Delta z$

As the # of parts $\longrightarrow \infty$

$dV = dx\, dy\, dz$

Fubini's Theorem for triple integrals

$$\boxed{\iiint_B f(x,y,z)\, dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\, dx dy dz}$$

the function must be continuous on the box B.

EXAMPLE 1 Evaluate the triple integral $\iiint_B xyz^2\,dV$, where B is the rectangular box given by

$$B = \{(x, y, z) \mid 0 \leqslant x \leqslant 1,\ -1 \leqslant y \leqslant 2,\ 0 \leqslant z \leqslant 3\}$$

$$\iiint_B xyz^2\,dV = \int_0^3 \int_{-1}^2 \int_0^1 xyz^2\,dx\,dy\,dz\ .$$

$$= \int_0^3 \int_{-1}^2 \left.\frac{x^2}{2}\right|_0^1 yz^2\,dy\,dz$$

$$= \int_0^3 \int_{-1}^2 \frac{1}{2}\,yz^2\,dy\,dz$$

$$= \frac{1}{2} \int_0^3 \int_{-1}^2 yz^2\,dy\,dz$$

$$= \frac{1}{2} \int_0^3 \left.\frac{y^2}{2}\right|_{-1}^2 z^2\,dz$$

$$= \frac{3}{4} \int_0^3 z^2\,dz = \frac{3}{4}\left.\frac{z^3}{3}\right|_0^3 = \boxed{\frac{27}{4}}$$

Comment: $\left(\int_0^1 x\,dx\right)\left(\int_{-1}^2 y\,dy\right)\left(\int_0^3 z^2\,dz\right)$ is also another to calculate the integral.

QUESTION. What are the 5 other configurations of dx, dy, dz in a triple integral?

1. $dV = dz\,dy\,dx$
2. $dV = dz\,dx\,dy$
3. $dV = dy\,dx\,dz$
4. $dV = dy\,dz\,dx$
5. $dV = dx\,dz\,dy$

General Domains.

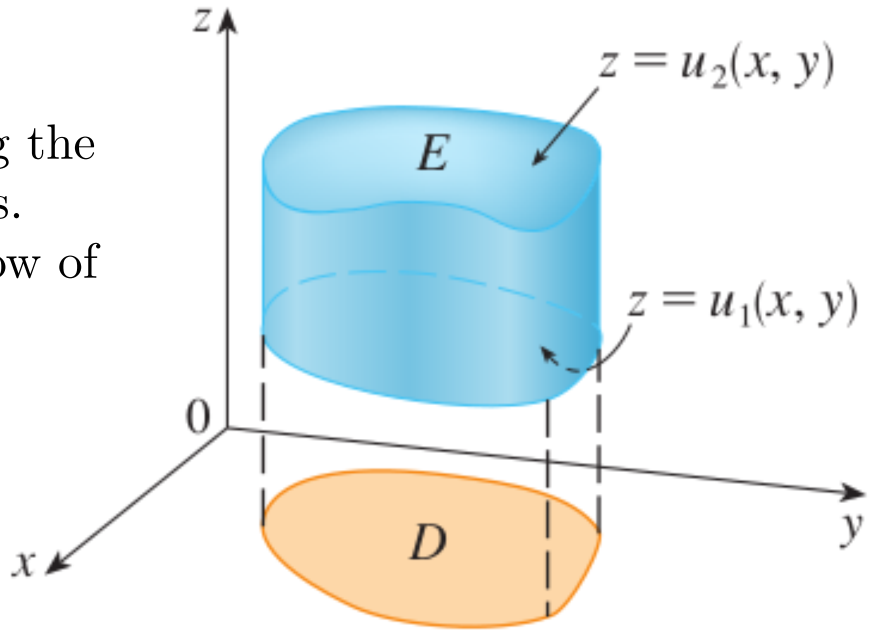
For E a general solid, let B be a box containing E.
Define a function F on B :

$$F(x,y,z) = \begin{cases} f(x,y,z) & \text{if } (x,y,z) \in E \\ 0 & \text{if } (x,y,z) \in B \backslash E. \end{cases}$$

$$\boxed{\iiint_E f(x,y,z)\,dV = \iiint_B F(x,y,z)\,dV}$$

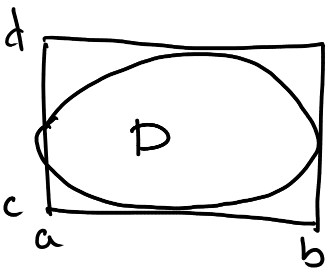
Domain of type 1.

- Solid E is bounded along the $z-$axis by two functions.
- Define D to be the shadow of E in the $xy-$plane.
- The domain D can be of type I or type II.



$$\iiint_E f(x,y,z)\,dV = \iiint_B F(x,y,z)\,dV$$

$$= \int_a^b \int_c^d \int_r^s F(x,y,z)\quad dz\,dy\,dx$$

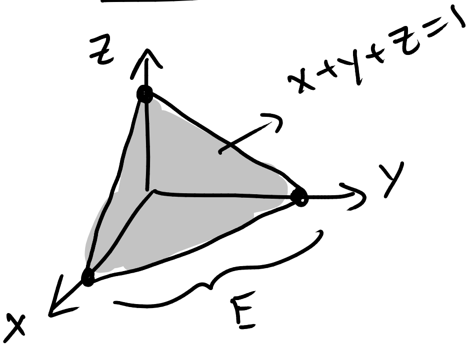


$$= \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right) dA$$

$$\boxed{\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right] dA}$$

EXAMPLE 2 Evaluate $\iiint_E \overset{f(x,y,z)}{z}\, dV$, where E is the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

① Picture



Describe E as a typ 1:

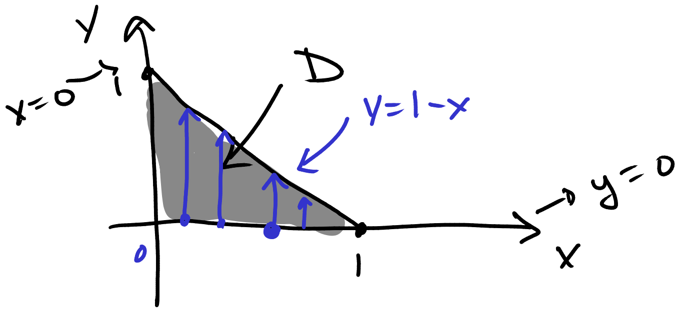
z-values

$z = 0$ (lower surface)

$z = 1 - x - y$

$\Rightarrow \quad \underset{u_1(x,y)}{0} \le z \le \overset{u_2(x,y)}{1 - x - y}$.

Shadow in xy-plane



set $z = 0 \Rightarrow x + y + 0 = 1$

$\Rightarrow x + y = 1$

D as a type I:

$D = \{(x,y) : \ 0 \le x \le 1, \ 0 \le y \le 1 - x\}$

② Integrate

$$\iiint_E z\, dV = \iint_D \left(\int_0^{1-x-y} z\, dz \right) dA$$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\, dz\, dy\, dx \longrightarrow -\frac{(1-x-y)^3}{6} \Big|_0^{1-x}$$

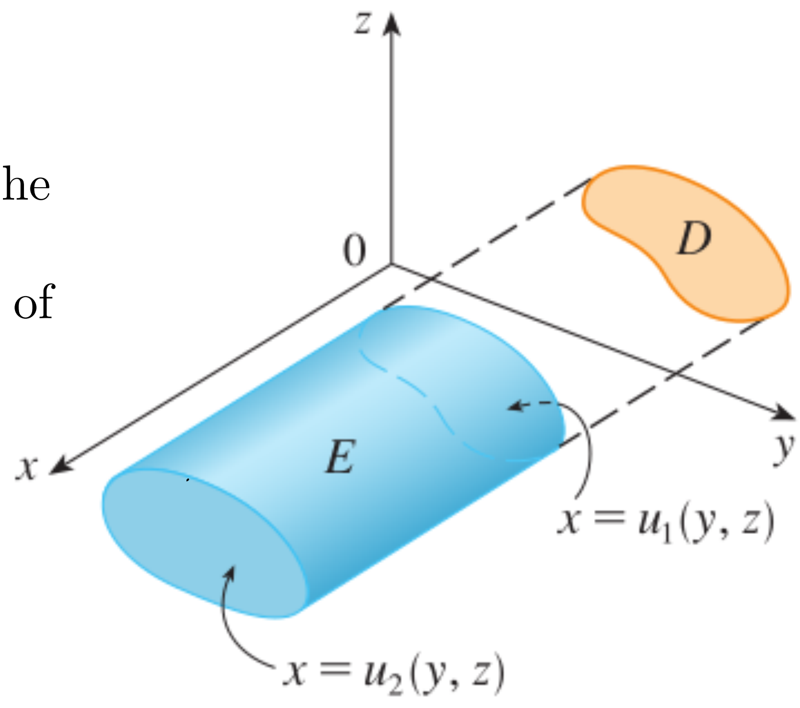
$$= \int_0^1 \int_0^{1-x} \frac{(1-x-y)^2}{2}\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} \frac{1 - 2x - 2y + 2xy + x^2 + y^2}{2}\, dy\, dx$$

$$= \boxed{\frac{1}{24}} \approx 0.0417 .$$

Domains of type 2.

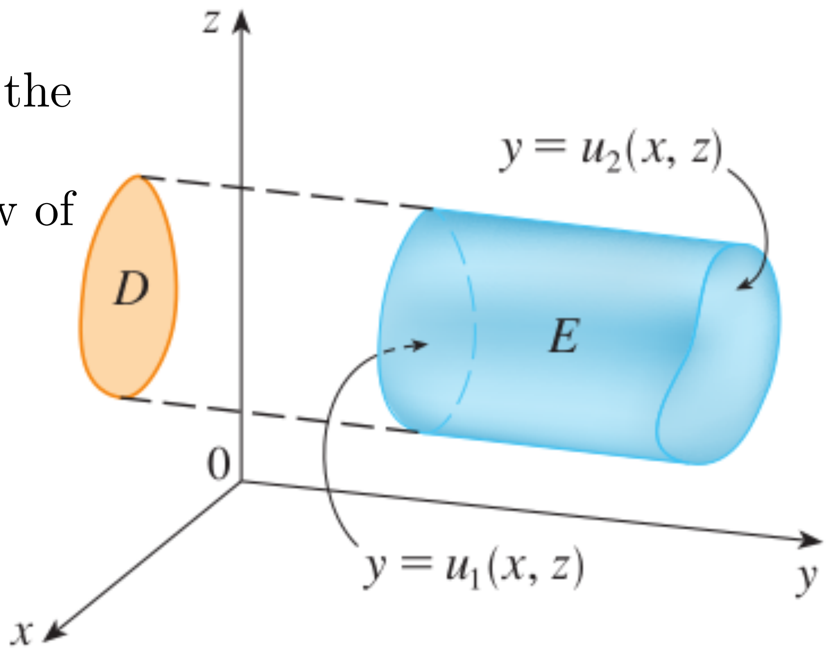
- Solid E is bounded along the $x-$axis by two functions.
- Define D to be the shadow of E in the $yz-$plane.
- The domain D can be of type I or type II.



$$\boxed{\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx\right] dA}$$

Domains of type 3.

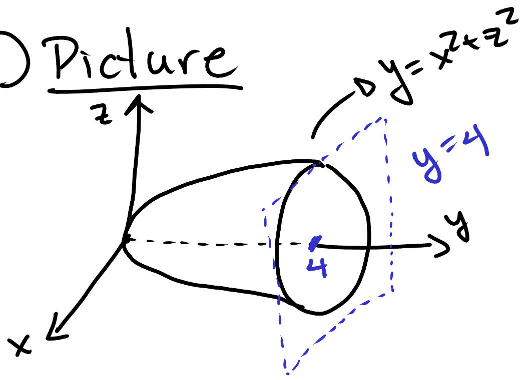
- Solid E is bounded along the $y-$axis by two functions.
- Define D to be the shadow of E in the $xz-$plane.
- The domain D can be of type I or type II.



$$\boxed{\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy\right] dA}$$

EXAMPLE 3 Evaluate $\iiint_E \sqrt{x^2+z^2}\, dV$, where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.

① Picture



Type 3: y-values are bounded by two functions.

y-values: $x^2+z^2 \le y \le 4$.

Shadow in xz-plane

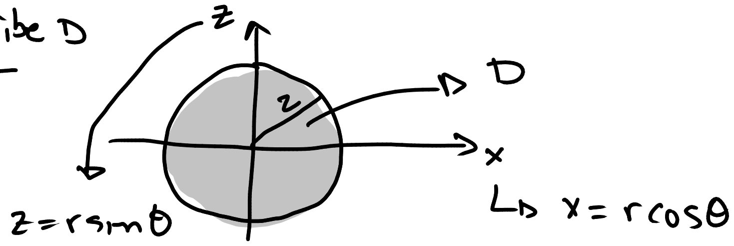
Set $y=4 \Rightarrow 4 = x^2+z^2$

↳ circle radius = 2.

$x = r\cos\theta,\ z = r\sin\theta$

Describe D

$D = \{(r,\theta) : 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}$



② Integrate

$$\iiint_E (x^2+z^2)^{1/2}\, dV = \iint_D \left(\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\, dy\right) dA$$

$$= \iint_D \sqrt{x^2+z^2}\,\left(4 - (x^2+z^2)\right) dA$$

$$= \int_0^{2\pi}\int_0^2 r\,(4-r^2)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 4r^2 - r^4\, dr\, d\theta$$

$$= \left(\int_0^2 4r^2 - r^4\, dr\right)\left(\int_0^{2\pi} 1\, d\theta\right)$$

$$= \boxed{\frac{128\pi}{15}} \approx 26.808.$$

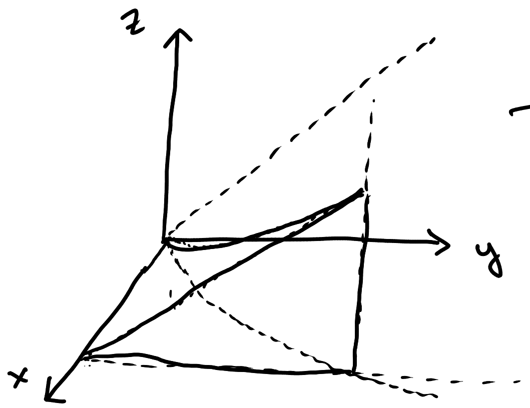
EXAMPLE 4 Express the iterated integral $\int_0^1 \int_0^{x^2} \int_0^y f(x, y, z)\, dz\, dy\, dx$ as a triple integral and then rewrite it as an iterated integral in a different order, integrating first with respect to x, then z, and then y.

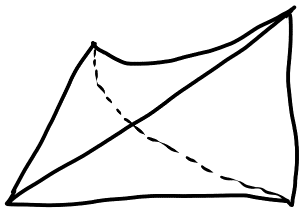
<u>Goal:</u> $dx\,dz\,dy$.

① <u>Picture</u>

Type 1

$$E = \{ (x, y, z) : 0 \le x \le 1,\ 0 \le y \le x^2,\ 0 \le z \le y \}$$





Application: computing volumes of solids.

$$\boxed{\mathrm{Vol}(E) = \iiint_E dV}$$

EXAMPLE. Use a triple integral to find the volume of the tetrahedron T bounded by the planes $x + 2y + z = 2, x = 2y, x = 0,$ and $z = 0$.