## Problem 2 (b) & (d) (only estimate the derivatives)

(b) It is straigth forward from the graph that $f'(1) \approx 0$.

(d) At $x = 3.5$, $f(3.5) \approx -0.5$ and at $x = 2.5$, $f(2.5) \approx 0.6$. So, we can approximate the derivative, with $h = 0.5$:

$$f'(3) \approx \frac{-0.5 - 0}{0.5} = -1$$

and with $h = -0.5$:

$$f'(3) \approx \frac{0.6 - 0}{-0.5} = -\frac{6}{5}.$$

If we want a better approximation, we can take the average of these values:

$$f'(3) \approx \frac{-1 - 6/5}{2} = -11/10.$$

## Problem 5

The equation of the tangent line at the point $(x_0, y_0) = (2, -4)$ is

$$y + 4 = m(x - 2)$$

where $m = f'(2)$. The derivative is given by the limit of the different quotient:

$$\begin{aligned}\frac{f(2+h) - f(2)}{h} &= \frac{4(2+h) - 3(2+h)^2 + 4}{h} \\ &= \frac{8 + 4h - 3(4 + 4h + h^2) + 4}{h} \\ &= \frac{-4 - 8h - 3h^2 + 4}{h} \\ &= -8 - 3h\end{aligned}$$

and as $h \to 0$, we get $f'(2) = -8$. So, we get

$$y + 2 = -8(x - 2).$$

## Problem 6

The equation of the tangent line at $(2, 3)$ is

$$y - 3 = f'(2)(x - 2).$$

We have to find $f'(2)$. We have $f(x) = x^3 - 3x + 1$, and therefore

$$\begin{aligned} f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} &= \lim_{h\to 0} \frac{(2+h)^3 - 3(2+h) + 1 - 3}{h} \\ &= \lim_{h\to 0} \frac{8 + 12h + 6h^2 + h^3 - 6 - 3h + 1 - 3}{h} \\ &= \lim_{h\to 0} \frac{9h + 2h^2 + h^3}{h} \\ &= \lim_{h\to 0} 9 + 2h + h^2 \\ &= 9. \end{aligned}$$

Therefore, we obtain $f'(2) = 9$. Therefore, the equation of the tangent line is

$$y = 9x - 18 + 3 = 9x - 15.$$

## Problem 25

The domain of the function is $(-\infty, 9]$. The derivative at $x$ is

$$\begin{aligned} f'(x) = \lim_{h\to 0} \frac{\sqrt{9-x-h} - \sqrt{9-x}}{h} &= \lim_{h\to 0} \frac{9 - x - h - 9 + x}{h(\sqrt{9-x-h} + \sqrt{9-x}} \\ &= \lim_{h\to 0} -\frac{1}{\sqrt{9-x-h} + \sqrt{9-x}} \\ &= -\frac{1}{2\sqrt{9-x}}. \end{aligned}$$

So $f'(x) = -1/2\sqrt{9-x}$ and the domain of $f'$ is $(-\infty, 9)$.

## Problem 34

The value of $f'(a)$ is given by

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

Evaluating $f$ at $a+h$ and at $a$ in this expression, we can do some calculations:

$$\begin{aligned}
\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} &= \lim_{h\to 0}\frac{\frac{1}{(a+h)^2}-\frac{1}{a^2}}{h} \\
&= \lim_{h\to 0}\frac{a^2-(a+h)^2}{(a+h)^2a^2h} \\
&= \lim_{h\to 0}\frac{a^2-a^2-2ah-h^2}{(a+h)^2a^2h} \\
&= \lim_{h\to 0}-\frac{2ah+h^2}{(a+h)^2a^2h} \\
&= \lim_{h\to 0}-\frac{2a+h}{(a+h)^2a^2} \\
&= -\frac{2a}{a^4} \\
&= -\frac{2}{a^3}.
\end{aligned}$$

Therefore, we get $f'(a) = -2/a^3$.

## Problem 44

The velocity at $t = 4$ is given by $f'(4)$. This is given by

$$\begin{aligned}
\lim_{h\to 0}\frac{f(4+h)-f(4)}{h} &= \lim_{h\to 0}\frac{10+\frac{45}{5+h}-10-\frac{45}{5}}{h} \\
&= \lim_{h\to 0}\frac{\frac{45}{5+h}-9}{h} \\
&= \lim_{h\to 0}\frac{45-45-9h}{(5+h)h} \\
&= \lim_{h\to 0}-\frac{9h}{(5+h)h} \\
&= \lim_{h\to 0}-\frac{9}{5+h}.
\end{aligned}$$

Evaluating the last limit with the Quotient Rule, we get $f'(4) = -9/5$.

## Problem 60

By definition, we have

$$\begin{aligned}
f'(0) &= \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} \\
&= \lim_{h\to 0}\frac{h^2\sin(1/h)}{h} \\
&= \lim_{h\to 0} h\sin(1/h).
\end{aligned}$$

The last limit exists because

$$-h \leq h\sin(1/h) \leq h$$

for any $h > 0$ and

$$h \leq h\sin(1/h) \leq -h$$

when $h < 0$. We can simplify this by using the absolute value:

$$0 \leq |h\sin(1/h)| \leq |h|$$

because $0 \leq |\sin(1/h)| \leq 1$. Using the Squeeze Theorem, we conclude that

$$\lim_{h \to 0} h\sin(1/h) = 0.$$

Therefore, $f'(0)$ exists and $f'(0) = 0$.