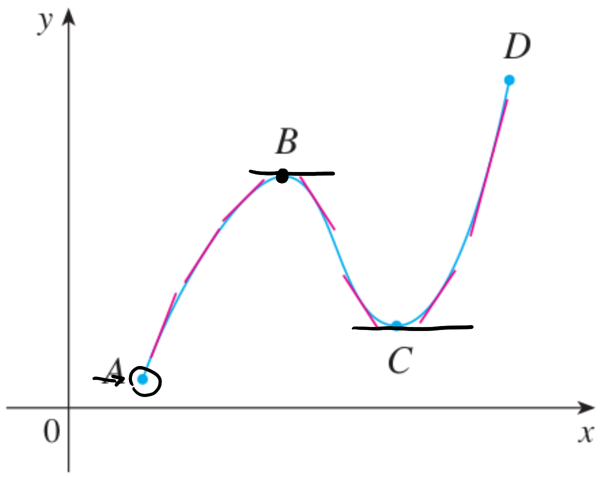
Chapter 3
Applications of Derivatives

3.3 How Derivatives affect the Shape of a Graph

What does f' tells us about f.



	A		B		C		D
$f'(x)$	$\nexists$	+	0	−	0	+	
$f(x)$		↗		↘		↗	

Increasing/Decreasing Test

(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

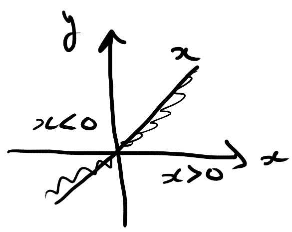
EXAMPLE 1 Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing.

1 Derivative.

$$f'(x) = 12x^3 - 12x^2 - 24x$$
$$= 12x(x^2 - x - 2)$$
$$= 12x(x-2)(x+1)$$

2 Table

factors		-1		0		2	
12	+		+		+		+
x	−		−	0	+		+
$x-2$	−		−		−	0	+
$x+1$	−	0	+		+		+
$f'(x)$	−	0	+	0	−	0	+
$f(x)$	↘		↗		↘		↗



$x - 2 > 0$
$\Leftrightarrow x > 2$

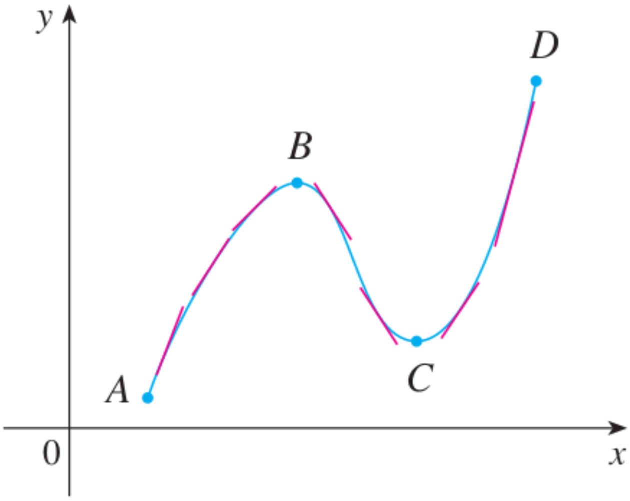
$x + 1 > 0$
$\Leftrightarrow x > -1$

f is decreasing for $-\infty < x < -1$ & $0 < x < 2$.

f is increasing for $-1 < x < 0$ & $x > 2$ $(2 < x < \infty)$

Local Extreme Values.

CRITICAL POINTS $\not\Longrightarrow$ EXTREME VALUES (MAX OR MIN)



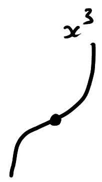
	A		B		C		D
$f'(x)$	$\nexists$	+	0	-	0	+	$\nexists$
$f(x)$	Abs Min	↗	loc MAX	↘	loc Min	↗	Abs Max.

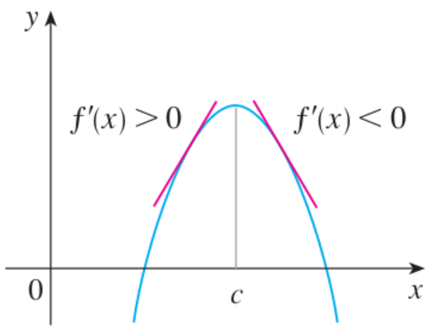
The First Derivative Test Suppose that c is a critical number of a continuous function f.

(a) If f' changes from positive to negative at c, then f has a local maximum at c.

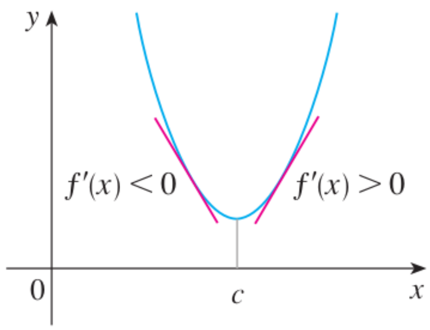
(b) If f' changes from negative to positive at c, then f has a local minimum at c.

(c) If f' is positive to the left and right of c, or negative to the left and right of c, then f has no local maximum or minimum at c.

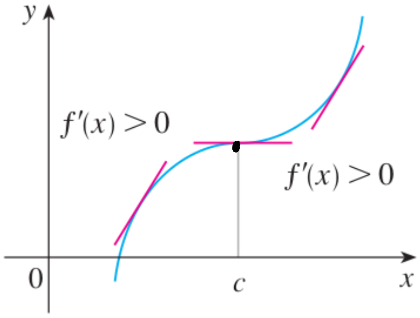




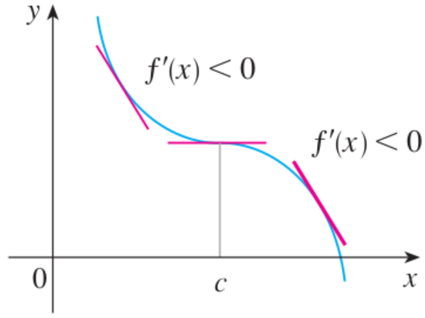
(a) loc max.



(b) loc. min.



(c) no loc. max or min



(d) no local max. or min.

EXAMPLE 3 Find the local maximum and minimum values of the function

$$g(x) = x + 2\sin x \qquad 0 \leqslant x \leqslant 2\pi$$

① Derivative.

$$g'(x) = 1 + 2\cos x$$

Zeros. $g'(x) = 0 \iff 1 + 2\cos x = 0$

$$\iff \cos x = \frac{-1}{2}.$$

$$\iff x = \frac{2\pi}{3} \quad \text{or} \quad x = \frac{4\pi}{3}.$$

② Table.

	0		$2\pi/3$		$4\pi/3$		2π
$1+2\cos x$		$+$	0	$-$	0	$+$	
$x+2\sin x$		$\nearrow$	loc. max.	$\searrow$	loc. min	$\nearrow$	

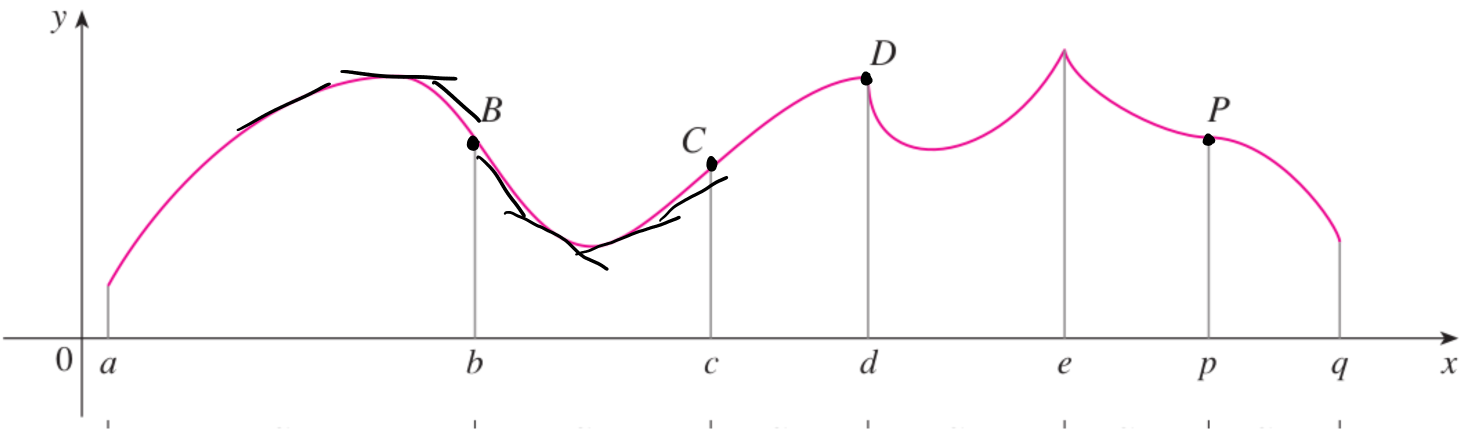
at $x = 2\pi/3 \to f(2\pi/3)$ is a local max. with

$$f(2\pi/3) = \frac{2\pi}{3} + \underbrace{2\sin\frac{2\pi}{3}}_{=\sqrt{3}} \approx \boxed{3.83}$$

at $x = 4\pi/3 \to f(4\pi/3)$ is a local min. with

$$f(4\pi/3) = \frac{4\pi}{3} + \underbrace{2\sin\frac{4\pi}{3}}_{-\sqrt{3}} \approx \boxed{2.46}.$$

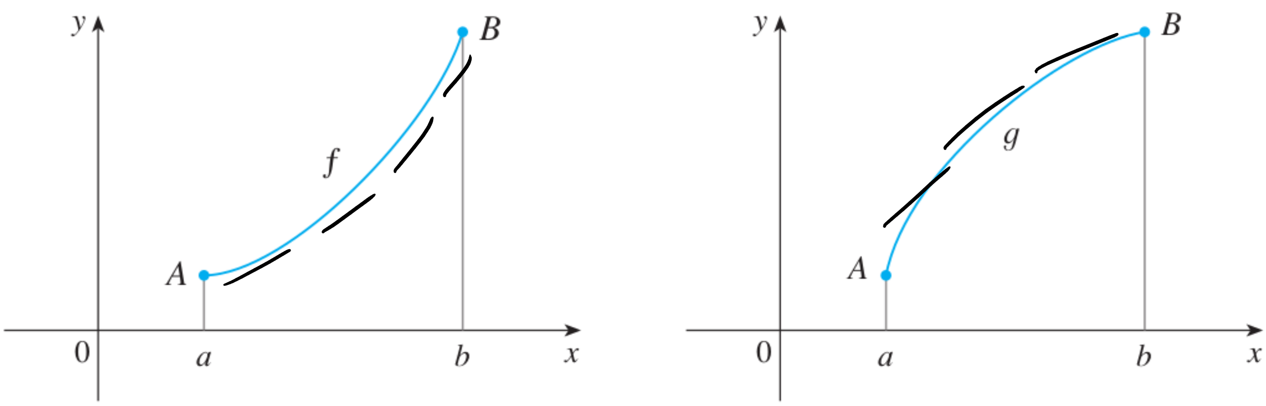
What does f'' tell us about f?



Two important definitions:

1) **Definition** If the graph of f lies above all of its tangents on an interval I, then it is called **concave upward** on I. If the graph of f lies below all of its tangents on I, it is called **concave downward** on I.

2) **Definition** A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.



(a) (b)

Concavity Test

(a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

(b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

Example. Find the interval(s) of concavity of the function $f(x) = x^3 - 3x^2 - 9x + 4$.

① Second derivative.

$f'(x) = 3x^2 - 6x - 9 \to f''(x) = 6x - 6 = 6(x-1)$

Zero. $f''(x) = 0 \iff 6(x-1) = 0$

$\iff x = 1$

② Table.

		1	
$f''(x)$	$-$	0	$+$
$f(x)$	$\cap$		$\cup$

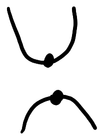
$x - 1 > 0$

$\iff x > 1$.

So, f is concave downward for $x < 1$. $((-\infty, 1))$

& f is concave upward for $x > 1$. $((1, \infty))$.

The Second Derivative Test Suppose f'' is continuous near c.
(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.



REMARK!

- ! $f'(c) = 0$ & $f''(c) = 0$! $\to$ we can't conclude! (example $f(x) = x^3$)
- $f''(x) > 0$ for all x $\Rightarrow$ c is an absolute minimum. (same $f''(x) < 0$ for all x)

EXAMPLE 7 Sketch the graph of the function $f(x) = x^{2/3}(6 - x)^{1/3}$.

① Derivatives

$$f'(x) = \frac{4-x}{x^{1/3}(6-x)^{2/3}} \qquad f''(x) = \frac{-8}{x^{4/3}(6-x)^{5/3}}.$$

Zeros: $f'(x) = 0 \iff 4 - x = 0 \iff x = 4.$
$f''(x) = 0$ impossible $-8 \neq 0$.

existence. f' $\nexists$ if $x = 0$ or $x = 6$.
f'' $\nexists$ if $x = 0$ or $x = 6$.

C.N. 0, 4, 6

② Table.

		0		4		6	
f'	$-$	$\nexists$	$+$	0	$-$	$\nexists$	$-$
f''	$+$	$\nexists$	$-$		$-$	$\nexists$	$+$
f	↘		↗		↘		↘

③ Plot the graph of f.