SAMPLE MIDTERM FALL 2022

1. (15 points) Let $f(x) = x^3 - x$.

   (a) List the intervals where the graph of $f(x)$ is increasing and decreasing.

   $f'(x) = 3x^2 - 1 \implies 3x^2 - 1 = 0 \iff x = \sqrt{1/3}$ or

   $\quad = 3(x^2 - 1/3) \qquad\qquad\qquad x = -\sqrt{1/3}$

   $\implies f'(x) = 3\underbrace{(x + \sqrt{1/3})}_{①}\underbrace{(x - \sqrt{1/3})}_{②}$

factors	$x < $	$-\sqrt{1/3}$	$< x < $	$\sqrt{1/3}$	$< x$
①	$-$		$+$		$+$
②	$-$		$-$		$+$
$f'(x)$	$+$		$-$		$+$

   • $f \nearrow$ on $(-\infty, -\sqrt{1/3}) \cup (\sqrt{1/3}, \infty)$

   • $f \searrow$ on $(-\sqrt{1/3}, \sqrt{1/3})$.

   (b) Find the local maximum and minimum values of $f(x)$.

   • $f \nearrow$ on left & $f \searrow$ on the right
   $\implies x = -\sqrt{1/3}$ is a local max.
   local max. value is: $f(-\sqrt{1/3}) = -\left(\frac{1}{\sqrt{3}}\right)^3 + \frac{1}{\sqrt{3}}$.

   • $f \searrow$ on left & $f \nearrow$ right
   $\implies x = \sqrt{1/3}$ is a local min.
   $\implies$ loc. min. value is $f(\sqrt{1/3})$.

   (c) List the intervals where the graph is concave up and concave down.

   $f''(x) = 6x \implies 6x = 0 \implies x = 0$. (I.P.)

   • $x < 0 \implies 6x < 0 \implies f''(x) < 0 \implies f \frown$ on $(-\infty, 0)$

   • $x > 0 \implies 6x > 0 \implies f''(x) > 0 \implies f \smile$ on $(-\infty, 0)$

   (d) Sketch the graph of the function.

   $x^3 - x = 0$
   $\implies x(x^2 - 1) = 0$
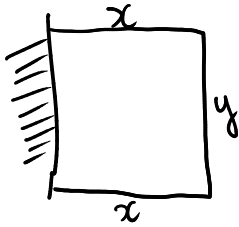 $\implies x = 0$
   $\quad\; x = 1$
   $\quad\; x = -1$
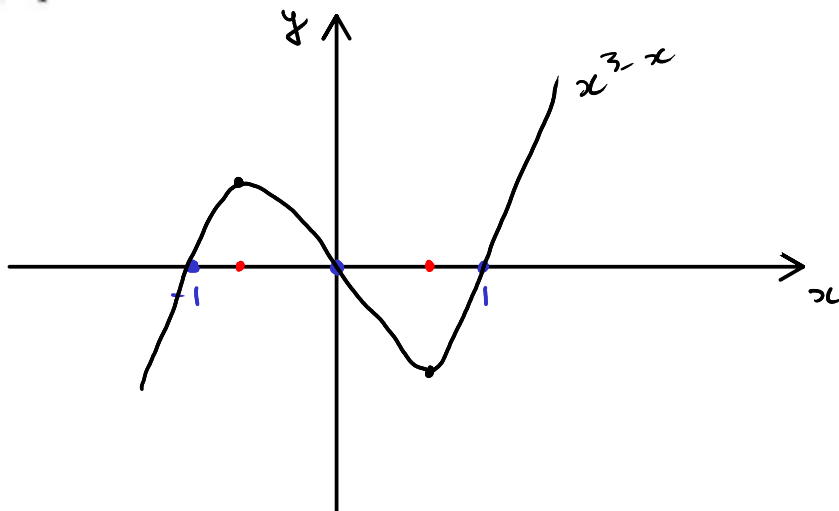
2. (10 points) Emma wants to enclose a rectangular field with total area $200m^2$. Along one side of the field, she will use a pre-existing straight wall, but on the other three sides, she needs to buy fence.

If it costs \$2 for each meter of fence, what is the least amount she can spend to enclose her field? (Simplify your answer.)

① Sketch.

② Notations.

x: side length. (width). (meters)
y: side length (heigth). (meters).
P: perimeter (fencing) (meters).
C: total cost.

Goal: minimize C.

③ Formula for cost.

$$C(x,y) = 2(2x) + 2y = 4x + 2y.$$

④ Eliminate a variable.

$$xy = 200 \quad \Rightarrow \quad y = \frac{200}{x}$$

So, $C(x) = 4x + \dfrac{400}{x}$. $(x > 0)$

⑤ Optimize!

$$C'(x) = 4 - \frac{400}{x^2} = 0 \quad \Longleftrightarrow \quad x^2 = 100$$
$$\Longleftrightarrow \quad x = \pm 10$$
$$\Rightarrow \quad x = 10$$
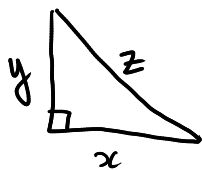
$$C''(x) = 4 + \frac{800}{x^3} > 0 \quad \Rightarrow \quad x = 10 \text{ is an absolute minimum.}$$

⑥ Answer

$$C(10) = 40 + 40 = \boxed{80 \text{ \$}}$$

3. (7 points) A right triangle is changing shape. If the base is 3 meters and expanding at 0.2 meters per minute, and the height is 4 meters and shrinking at 0.1 meters per minute, at what rate is the length of the hypotenuse changing?

① $\underline{Sketch}$.

$x(t) \nearrow$
x: base   (meters).
$y(t) \nwarrow$
y: height   (meters).
$z(t) \nearrow$
z: hypotenuse   (meters).

$\underline{Goal}$: Find $\left. \dfrac{dz}{dt} \right|_{\substack{x=3 \\ y=4}}$ $\overset{\nabla}{\underset{5}{\circ}}$

② $\underline{Link}$.

Pyth. Thm: $\qquad x^2 + y^2 = z^2$

③ $\underline{Differentiate}$

(*) $\qquad 2x \cdot \dfrac{dx}{dt} + 2y \dfrac{dy}{dt} = 2z \dfrac{dz}{dt}$

$\underline{know}$: $\dfrac{dx}{dt} = 0.2 \qquad\qquad x = 3 \qquad\qquad z = \sqrt{9 + 16} = 5$

$\qquad\qquad \dfrac{dy}{dt} = 0.1 \qquad\qquad y = 4$
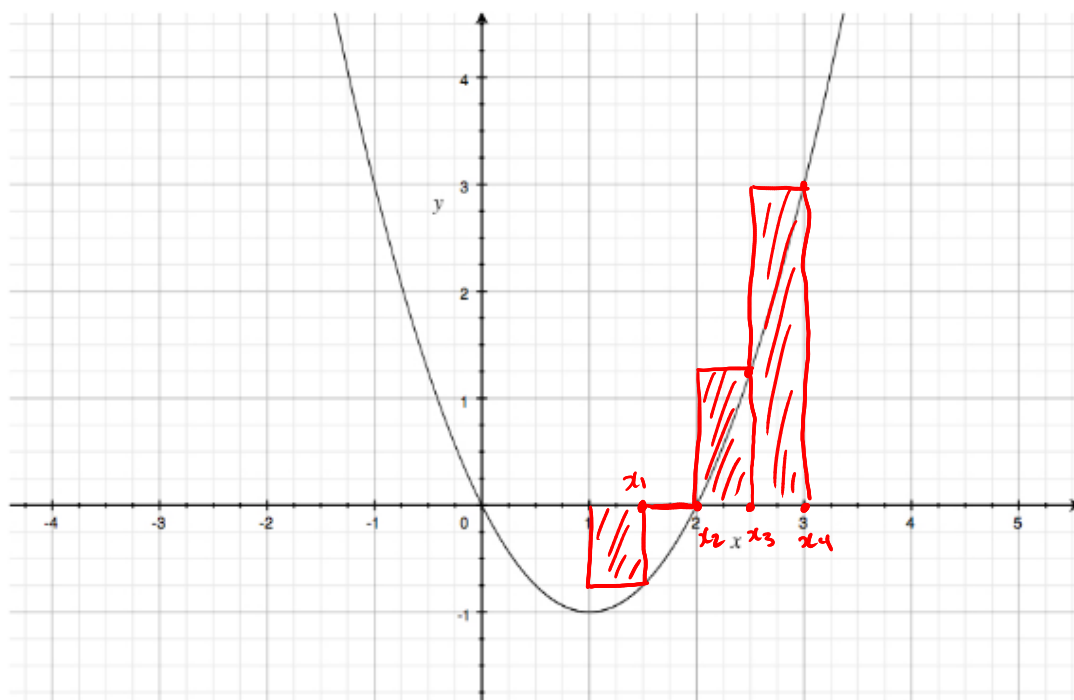
$\Rightarrow \qquad \dfrac{2 \cdot 3 \cdot 2}{10} + \dfrac{2 \cdot 4}{10} = 2 \cdot 5 \cdot \dfrac{dz}{dt}$

$\Rightarrow \qquad\qquad \dfrac{20}{10} = 10 \cdot \dfrac{dz}{dt}$

$\Rightarrow \qquad\qquad \dfrac{2}{10} = \dfrac{dz}{dt} \qquad \Rightarrow \boxed{\dfrac{dz}{dt} = 0.2 \; \text{m}/\text{min}}$

4. Consider the function $f(x) = x^2 - 2x$ as pictured below.

(a) (6 points) Compute a Riemann sum for this function that approximates the integral $\int_1^3 f(x)\,dx$. Use four equal-width intervals for your Riemann sum, and use the right end-point of each interval to determine the height of the corresponding rectangle. You do not have to simplify your answer.

$a = 1$

$b = 3$

$f(x) = x^2 - 2x$

$\Delta x = \dfrac{b-a}{n} = \dfrac{3-1}{4} = \dfrac{2}{4} = \dfrac{1}{2}.$

$x_1 = a + \Delta x = 1 + \dfrac{1}{2} = \dfrac{3}{2} = 1.5$
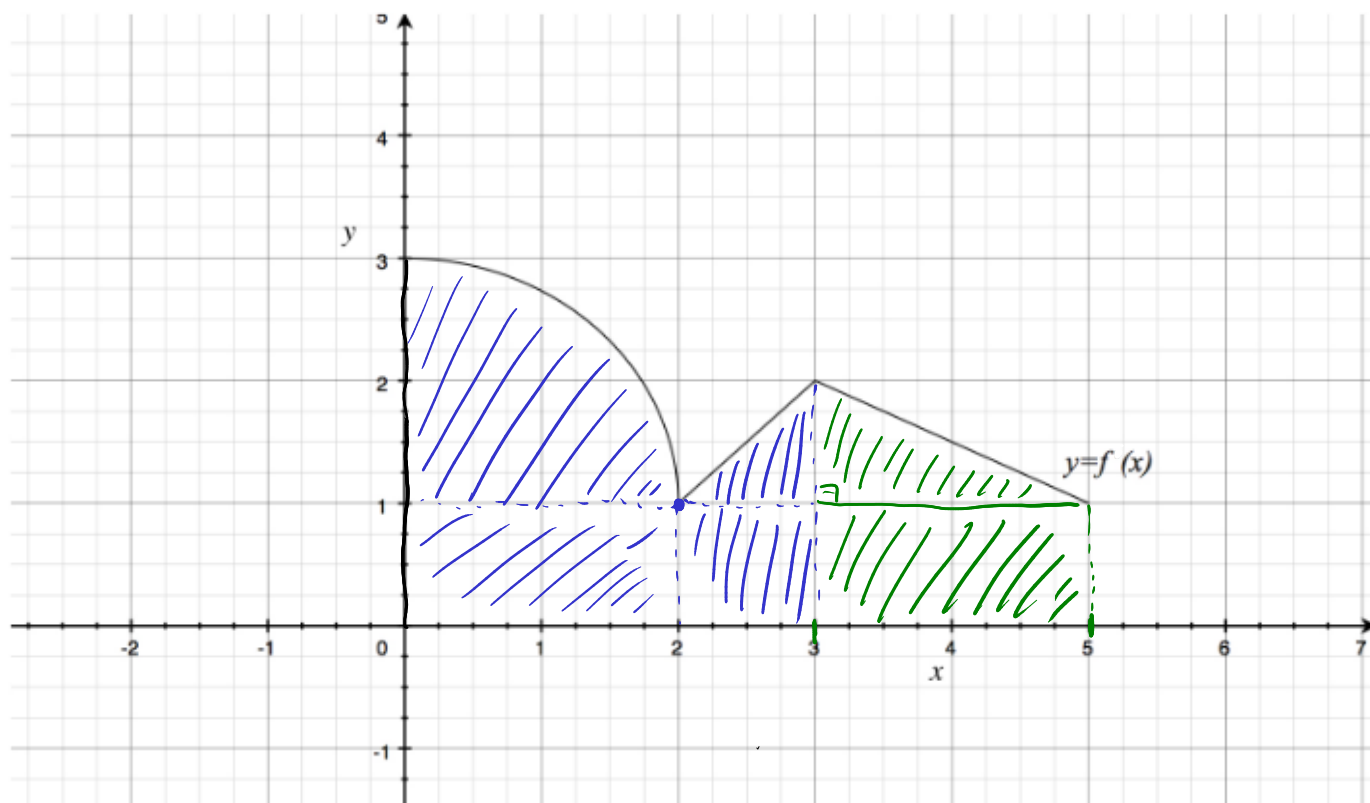
$x_2 = a + 2\Delta x = 1 + 1 = 2$

$x_3 = a + 3\Delta x = 1 + \dfrac{3}{2} = \dfrac{5}{2} = 2.5$

$x_4 = a + 4\Delta x = 1 + \dfrac{4}{2} = 1 + 2 = 3$

$\int_1^3 f(x)\,dx \approx R_4 = \Delta x\, f(x_1) + \Delta x\, f(x_2) + \Delta x\, f(x_3) + \Delta x\, f(x_4)$

$= \dfrac{1}{2}\left(1.5^2 - 3\right) + \dfrac{1}{2}\left(2^2 - 4\right)^0 + \dfrac{1}{2}\left(2.5^2 - 5\right)$

$+ \dfrac{1}{2}\left(3^2 - 6\right)$

(b) (2 points) Sketch the rectangles that correspond to part (a) on the graph above.

5. A function f of a single variable x is defined on the interval $[0,5]$
The following picture shows the graph of $f(x)$.

In the picture, the portion of the graph on the interval $(0,2)$ identifies with a quarter-circle of radius 2 and center $(0,1)$; the portions of the graph on the intervals $[2,3]$ and $[3,5]$ are line segments.

(a) (2 points) What is the value of $\int_0^0 f(x)\,dx$?

$$\int_0^0 f(x)\,dx = 0$$

(b) (4 points) What is the value of $\int_0^3 f(x)\,dx$?

$$\int_0^3 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^3 f(x)\,dx$$

$$= \text{Area}(\text{◿}) + \text{Area}(\text{◻}) + \text{Area}(\triangleleft) + \text{Area}(\square)$$

$$= \frac{\pi\,2^2}{4} + 2\cdot 1 + \frac{1\cdot 1}{2} + 1^2$$

$$= \pi + 2 + \frac{1}{2} + 1 = \boxed{\pi + \frac{7}{2}}$$

(c) (4 points) What is the value of $\int_3^5 f(x)\,dx$?

$$\int_3^5 f(x)\,dx = \text{Area}(\square) + \text{Area}(\text{▷})$$

$$= 2\cdot 1 + \frac{1\cdot 2}{2}$$

$$= \boxed{3}$$

6. (6 points) Use linear approximation to estimate the number $(.95)^{10}$.

$$f(x) = x^{10} \qquad\qquad L(x) = f'(a)(x-a) + f(a)$$

1 is close to 0.95 & $1^{10} = 1 \implies a = 1$

$$f'(x) = 10x^9 \implies f'(1) = 10$$

$$f(1) = 1$$

$$\implies L(x) = 10(x-1) + 1$$

Here $x = 0.95$

$$\implies (0.95)^{10} \approx L(0.95) = 10(0.95 - 1) + 1$$

$$= 10(-0.05) + 1$$

$$= \frac{-10 \cdot 5}{100} + 1$$

$$= -\frac{5}{10} + 1$$

$$= -\frac{1}{2} + 1 = \frac{1}{2}$$

So, $\boxed{(0.95)^{10} \approx 0.5}$

7. Compute the following limits.

(a) $\lim\limits_{x\to\infty} \dfrac{4-7x^2}{(x+5)^2}$.

$(x+5)^2 = x^2 + 10x + 25$

$\Rightarrow \quad \lim\limits_{x\to\infty} \quad \dfrac{4-7x^2}{x^2+10x+5} \quad = \quad \dfrac{-7}{1} \quad = \quad \boxed{-7}$

(b) $\lim\limits_{x\to\infty} \dfrac{7-\sqrt{x}}{7+\sqrt{x}}$.

$\lim\limits_{x\to\infty} \quad \dfrac{7-\sqrt{x}}{7+\sqrt{x}} \quad = \quad \lim\limits_{x\to\infty} \dfrac{\sqrt{x}\left(\dfrac{7}{\sqrt{x}}-1\right)}{\sqrt{x}\left(\dfrac{7}{\sqrt{x}}+1\right)}$

$= \lim\limits_{x\to\infty} \dfrac{\dfrac{7}{\sqrt{x}}-1}{\dfrac{7}{\sqrt{x}}+1}$

$= \dfrac{\lim\limits_{x\to\infty} \dfrac{7}{\sqrt{x}}^{\,0}-1}{\lim\limits_{x\to\infty} \dfrac{7}{\sqrt{x}}^{\,0}+1} \quad = \quad \dfrac{-1}{1} \quad = \quad \boxed{-1}$

8. Answer the following.

(a) Given that $\dfrac{1}{2} \le \dfrac{x}{x+1} \le \dfrac{2}{3}$ for any x such that $1 \le x \le 2$,

give an estimate of the following integral:

$$\int_1^2 \frac{x}{x+1}\,dx.$$

From the comparison properties

$$\tfrac{1}{2}(2-1) \le \int_1^2 \frac{x}{x+1}\,dx \le \tfrac{2}{3}(2-1)$$

$$\Rightarrow \qquad \tfrac{1}{2} \le \int_1^2 \frac{x}{x+1}\,dx \le \tfrac{2}{3}$$

take the average:

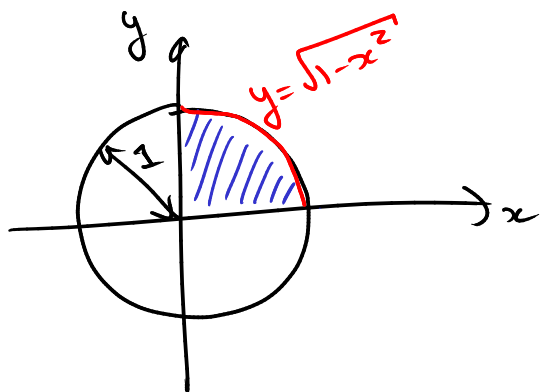$$\int_1^2 \frac{x}{x+1}\,dx \approx \frac{\tfrac{1}{2} + \tfrac{2}{3}}{2} = \boxed{\dfrac{7}{12}}$$

(b) Find the value of the following integral by intepreting it geometrically:

$$\int_0^1 \sqrt{1 - x^2}\,dx.$$

$$y = \sqrt{1-x^2} \ge 0 \qquad \Rightarrow \qquad \text{integral represents area under some curve.}$$

$$\left(\text{square}\right.$$

$$y^2 = 1 - x^2 \qquad \Rightarrow \qquad x^2 + y^2 = 1 \qquad \longrightarrow \qquad \text{circle of radius 1.}$$

$$\Rightarrow \qquad \int_0^1 \sqrt{1-x^2}\,dx = \text{Area}\left(\text{◔}\right)$$

$$= \boxed{\dfrac{\pi}{4}}$$