Math 241
Spring 2021
Final
12 May 2021

Name (Print): Pratice
Section Number: 01 & 02 .

Exam length: 120 minutes

This exam contains 17 pages (including this cover page) and 16 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

Instructions:

- You have 120 minutes for the exam, plus 15 minutes before and after to download the exam and upload your solutions. The exam will be available at 11:45am. **The deadline for uploading is the earlier of either 2:15pm or 120 minutes after you start the exam.**
  - Example 1. If you start the assignment at 11:49am, then you must upload it before 1:49pm.
  - Example 2: If you start the assignment at 12:05pm, then you must upload it before 2:15pm.
- You are required to **show your working** and justify your answers for all questions *except where explicitly stated.*
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit, if any.
- All work must be your own. **You are not permitted to discuss the test with anyone else.**
- You may use your personal notes and your textbook, but no other notes.
- You cannot use electronic devices other than to access the textbook, write the exam, to upload or download the exam, or to access the Zoom room to ask questions.
- **To print or not to print:** You may print out the exam and write on it, or you may write your answers on blank paper or on a tablet without printing the exam. If you do this, please clearly indicate which question is being answered, and use a new sheet of paper for a new question (if a question has multiple parts, they may be on the same sheet). Also please write your name and page number on each page.

Academic integrity is expected of all University of Hawaii students at all times, whether in the presence or absence of members of the faculty. Understanding this, I declare I shall not give, use, or receive unauthorized aid in this examination.
Please sign below or on the first page of your answers to indicate that you have read and agree to these instructions.

SIGNATURE OF STUDENT

For official use only:

Question:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	Total
Points:	7	6	9	6	12	8	6	9	9	6	6	8	5	9	6	8	120
Score:																	

Question 1. (7 points)

The movement of two pōpoki are tracked. Pōpoki Alika has a velocimeter strapped to her, while the distance travelled by pōpoki Kapono is tracked via GPS.

Alika

Kapono

Both travel in a straight line from the same location. Their respective speed and distance travelled are recorded below.

Time in seconds	Speed of Alika in meters per second		Distance traveled by Kapono in meters
0	0	0	0
2	2	4	2
4	2	8	10
6	6	20	22
8	12	44	28
10	13	70	32
12	10	90	40
14	4	98	44
		(*)	

(a) (2 points) What is the average acceleration of Alika from time 4 seconds and time 10 seconds?

$$a_{ave.} = \frac{v(b) - v(a)}{b-a} \qquad a=4, \; b=10 \qquad v(4)=2, \; v(10)=13 \qquad a_{ave} = \frac{13-2}{10-4} = \frac{11}{6} \; m/s^2$$

(b) (3 points) Estimate the distance traveled by Alika over the 14 second period.

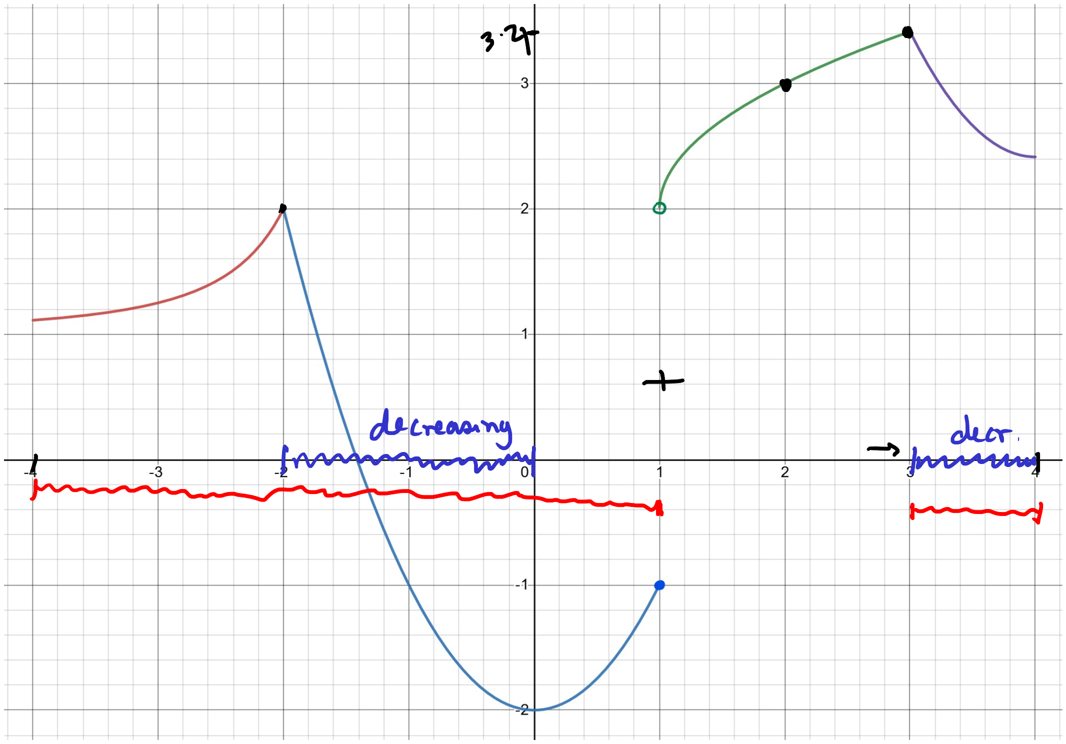
$$\begin{aligned} \text{distance} &= 2\cdot 2 + 2\cdot 2 + 2\cdot 6 + 2\cdot 12 + 2\cdot 13 + 2\cdot 10 + 2\cdot 4 \\ &= 4 + 4 + 12 + 24 + 26 + 20 + 8 \\ &= 98 \text{ meters.} \end{aligned}$$

(c) (2 points) From this data, does it seem likely that Alika and Kapono stayed close together throughout these 14 seconds, or did they get far apart? Give a brief justification.

(*) we cumulate the distances.

Then get far apart most of the time because Alika traveled more distance than Kapono.

Question 2. (6 points)
Consider the function $f(x)$ with the graph $y = f(x)$ pictured below. The domain of f is $[-4, 4]$.



$f'(x) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$

(a) (1 point) On which interval(s) (if any) is the function decreasing?

f is decreasing on $[-2, 0]$ and $[3, 4]$.

(b) (1 point) On which interval(s) (if any) is the function concave up?

f is concave up on $[-4, 1]$ and on $[3, 4]$.

(c) (1 point) Where (if anywhere) is the function **not** continuous?

f is discontinuous at $x = 1$.

(d) (1 point) Where (if anywhere) is the function **not** differentiable?

f is not differentiable at $x = -2$, $x = 1$ & $x = 3$.

(e) (1 point) What is $\lim_{x \to 1^+} f(x)$ (a reasonable estimate is OK / 'does not exist' is a possible answer)?

$\lim_{x \to 1^+} f(x) = 2.$

(f) (1 point) What is $\lim_{x \to 3^-} f'(x)$ (a reasonable estimate is OK / 'does not exist' is a possible answer)?

$\lim_{x \to 3^-} f'(x) \approx f'(3) \approx \frac{f(3) - f(2)}{3 - 2} = \frac{3.24 - 3}{1} = 0.24$

Question 3. (9 points)

You are given that a function $f(x)$ satisfies the following limits:

$$\lim_{x\to -1} f(x) = 4, \quad \lim_{x\to 3+} f(x) = -12, \quad \lim_{x\to\infty} f(x) = 7.$$

Use these to compute the following limits, or say if they are $+\infty$ or $-\infty$. Do **not** use l'Hôpital's rule.

(a) (3 points) $\lim\limits_{x\to -1} \dfrac{xf(x)+4x}{x^2+2}$

$$\lim_{x\to a} \frac{g(x)}{h(x)} = \frac{\lim g(x)}{\underbrace{\lim h(x)}_{\neq 0}}$$

$$\lim_{x\to -1} \frac{xf(x)+4x}{x^2+2} = \frac{\lim\limits_{x\to -1} xf(x)+4x}{\lim\limits_{x\to -1} x^2+2}$$

$$= \frac{\left(\lim\limits_{x\to -1} x\right)\left(\lim\limits_{x\to -1} f(x)\right) + \left(\lim\limits_{x\to -1} 4\right)\left(\lim\limits_{x\to 1} x\right)}{\lim\limits_{x\to -1} x^2 + \lim\limits_{x\to 1} 2}$$

$$= \frac{(-1)(4) + 4\cdot(-1)}{1+2} = \boxed{-\frac{8}{3}}$$

(b) (3 points) $\lim\limits_{x\to 3^+} \dfrac{(f(x)+6)^2}{x-3}$

$$\lim_{x\to 3^+} \frac{(f(x)+6)^2}{x-3}$$

$$\lim_{x\to 3^+} (f(x)+6)^2 = \left(\lim_{x\to 3^+} f(x)+6\right)^2$$

$$= \left(\lim_{x\to 3^+} f(x) + \lim_{x\to 3^+} 6\right)^2$$

$$= (-12+6)^2$$

$$= (-6)^2 = 36$$

$$\lim_{x\to 3^+} \frac{(f(x)+6)^2}{x-3} = \frac{36}{0^+} \boxed{= +\infty}$$

(c) (3 points) $\lim\limits_{x\to\infty} \left(\underbrace{\sqrt{4x^2+x} - 2x}_{\infty - \infty} + f(x)\right) = \dfrac{1}{4} + 7 = \boxed{\dfrac{29}{4}}$

$$\lim_{x\to\infty} \left(\sqrt{4x^2+x} - 2x\right) \cdot \left(\frac{\sqrt{4x^2+x}+2x}{\sqrt{4x^2+x}+2x}\right)$$

$$= \lim_{x\to\infty} \frac{4x^2+x-4x^2}{\sqrt{4x^2+x}+2x} = \lim_{x\to\infty} \frac{x}{\sqrt{4x^2+x}+2x}$$

$$= \lim_{x\to\infty} \frac{\cancel{x}}{\cancel{x}\sqrt{4+\frac{1}{x}} + 2\cancel{x}}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{4+\frac{1}{x}}+2} = \frac{1}{4}.$$

Question 4. (6 points)
Using the limit definition of the derivative, compute $f'(x)$ when $f(x) = \sqrt{6-x}$.

You will get no credit for computing the derivative without using the definition.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{6-x-h} - \sqrt{6-x}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{6-x-h} - \sqrt{6-x}}{h} \cdot \frac{\sqrt{6-x-h} + \sqrt{6-x}}{\sqrt{6-x-h} + \sqrt{6-x}}$$

$$= \lim_{h \to 0} \frac{\cancel{6-x} - h - (\cancel{6-x})}{h\left(\sqrt{6-x-h} + \sqrt{6-x}\right)}$$

$$= \lim_{h \to 0} \frac{-\cancel{h}}{\cancel{h}\left(\sqrt{6-x-h} + \sqrt{6-x}\right)}$$

$$= \lim_{h \to 0} \frac{-1}{\sqrt{6-x-h} + \sqrt{6-x}}$$

$$= \frac{-1}{\sqrt{6-x} + \sqrt{6-x}}$$

$$= \frac{-1}{2\sqrt{6-x}}$$

So,
$$\boxed{f'(x) = \frac{-1}{2\sqrt{6-x}}}$$

Question 5. (12 points)
For this question, let f be a differentiable function such that f and f' take the following values:

x	-1	0	1	2	3
$f(x)$	-4	-3	-2	2	0
$f'(x)$	1	2	-2	4	1

Compute the following derivatives. Simplify your answers as much as you can.

(a) (4 points) Compute $g'(3)$ if $g(x) = f(\sin(\pi x) - \cos(\pi x))$.

$$g'(x) = f'(\sin(\pi x) - \cos(\pi x)) \cdot (\pi \cos(\pi x) + \sin(\pi x)\cdot \pi)$$

$$\Rightarrow g'(3) = f'(\sin(3\pi) - \cos(3\pi)) \cdot (\pi \cos(3\pi) + \pi \sin(3\pi))$$

$$= f'(1) \cdot (-\pi)$$

$$\Rightarrow \boxed{g'(3) = 2\pi}$$

(b) (4 points) Compute $h'(2)$ if $h(x) = \dfrac{f(x)}{x^2+1}$.

$$\left(\frac{a(x)}{b(x)}\right)' = \frac{a'(x)b(x) - a(x)b'(x)}{b(x)^2}$$

$$h'(x) = \frac{f'(x)(x^2+1) - f(x)\cdot(2x)}{(x^2+1)^2}$$

$$\Rightarrow h'(2) = \frac{f'(2)(4+1) - f(2)\cdot 4}{5^2} = \frac{4\cdot 5 - 2\cdot 4}{25}$$

$$= \boxed{\frac{12}{25}}$$

(c) (4 points) Compute $j'(9)$ if $j(x) = f(f(\sqrt{x}))$.

$$j'(x) = f'(f(\sqrt{x})) \cdot (f(\sqrt{x}))'$$

$$= f'(f(\sqrt{x})) \cdot f'(\sqrt{x}) \cdot (\sqrt{x})'$$

$$= f'(f(\sqrt{x})) \cdot f'(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}$$

$$\Rightarrow j'(9) = f'(f(3)) \cdot f'(3) \cdot \frac{1}{2\cdot 3} = f'(0)\cdot 1 \cdot \frac{1}{6}$$

$$\Rightarrow j'(9) = 2\cdot 1 \cdot \frac{1}{6} \Rightarrow \boxed{j'(9) = \frac{1}{3}}$$

Question 6. (8 points)
Let f be a differentiable function. Consider the equation

$$xy^2 - 2f(x)y = -3.$$

(a) (4 points) Compute y'. Your answer should be in terms of x, y, $f(x)$, and $f'(x)$.

$y = y(x)$

$y' = \frac{dy}{dx}$

$$\frac{d}{dx}\left(xy^2 - 2f(x)y\right) = \frac{d}{dx}(-3)$$

$$\to \frac{d}{dx}(xy^2) - 2\frac{d}{dx}(f(x)y) = 0$$

$$\to 1 \cdot y^2 + x \cdot 2y \cdot y' - 2\left(f'(x)y + f(x)y'\right) = 0$$

$$\to y^2 + 2xy \cdot y' - 2f'(x)y - 2f(x)y' = 0$$

$$\to 2xy\, y' - 2f(x)y' = 2f'(x)y - y^2$$

$$\to y' = \boxed{\frac{2f'(x)y - y^2}{2xy - 2f(x)}}$$

(b) (4 points) You are also told that $f(1) = 2$ and $f'(1) = -3$. Find the equation of the tangent line to the curve at the point $(1, 1)$. Write the equation in the form $y = \boxed{m}x + \boxed{b}$

$\overset{\shortparallel}{(x,y)}$

Find m.

$$m = y'(1) = \frac{2\cdot(-3)\cdot 1 - 1^2}{2\cdot 1\cdot 1 - 2\cdot 2} = \frac{-7}{-2} = \frac{7}{2}$$

Find b.

We know that $(1,1)$ is on the tangent

$$\to 1 = \frac{7}{2}\cdot 1 + b \to b = 1 - \frac{7}{2} = \frac{-5}{2}$$

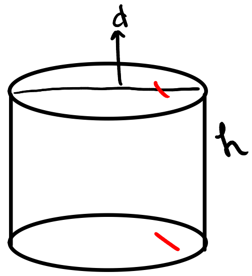
Then, $\boxed{y = \frac{7}{2}x - \frac{5}{2}}$

Question 7. (6 points)

A circular cylinder with a top and bottom is growing in such a way that its height is always three times its diameter.

If the radius of the cylinder is changing at a rate of $\frac{1}{8}$ ft/min, find the rate at which the cylinder's volume is changing when the radius is 4 feet. (Hint: Use the formula $V = \pi r^2 h$.)

↳ Volume of a cylinder.



d : diameter

r : radius

h : height

V : Volume.

Goal. Find $\frac{dV}{dt}$ when $r=4$

We have $h = 3d = 3(2r) = 6r$

so, $V = \pi r^2 (6r) = 6\pi r^3$

$\rightarrow \frac{dV}{dt} = 6\pi \cdot (3r^2) \cdot \frac{dr}{dt} \Rightarrow \frac{dV}{dt} = 6\pi(3 \cdot 4^2) \cdot \frac{1}{8}$

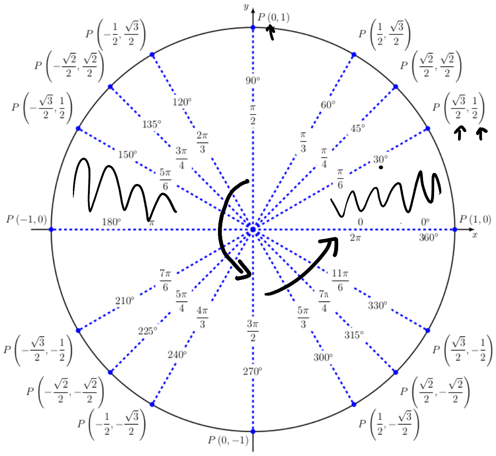
so, $\frac{dV}{dt} = \frac{6 \cdot 3 \cdot 16}{8} \pi = \boxed{36\pi \ ft^2/min}$

Question 8. (9 points)
Consider the function f defined on the interval $[0, 2\pi]$, whose derivative is given below:

$$f'(x) = \frac{2\sin(x) - 1}{2 + \sin(x)}.$$

$-1 \le \sin x \le 1$
$\to 1 \le 2+\sin x \le 3$.

(a) (2 points) On what subintervals (if any) of $[0, 2\pi]$ is f increasing and on which is it decreasing?



① Increasing? $f'(x) > 0$ if $2\sin x > 1$
if $\sin x > 1/2$
if $\frac{\pi}{6} < x < \frac{5\pi}{6}$.

f is increasing on $[\frac{\pi}{6}, \frac{5\pi}{6}]$.

② Decreasing? $f'(x) < 0$ if $2\sin x < 1$
if $\sin x < \frac{1}{2}$
if $0 \le x < \pi/6$
or $\frac{5\pi}{6} < x \le 2\pi$

f is decreasing on $[0, \pi/6]$ and $[\frac{5\pi}{6}, 2\pi]$.

(b) (2 points) Where (if anywhere) does f have a local minimum or a local maximum?

$f'(x) = 0$ if $2\sin(x) - 1 = 0$ if $\sin x = \frac{1}{2}$ if $x = \frac{\pi}{6}, \frac{5\pi}{6}$.

x		$\pi/6$		$5\pi/6$	
$f'(x)$	↘	0	↗	0	↘
		loc. min		loc. max.	

loc min. at $x = \frac{\pi}{6}$

loc max at $x = \frac{5\pi}{6}$.

(c) (5 points) On what subintervals (if any) of $[0, 2\pi]$ is f concave up or concave down, and where (if anywhere) does f have an inflection point?

$$f''(x) = \left(\frac{2\sin x - 1}{2+\sin x}\right)' = \frac{2\cos x(2+\sin x) - (2\sin x - 1)\cos x}{(2+\sin x)^2}$$

$$= \frac{4\cos x + \cos x}{(2+\sin x)^2} = \frac{5\cos x}{(2+\sin x)^2}$$

x		$\frac{\pi}{2}$		$\frac{3\pi}{2}$	
f''	+	0	−	0	+
	∪	P.I.	∩	P.I.	∪

f ∪ on $[0, \frac{\pi}{2}]$ & $[\frac{3\pi}{2}, 2\pi)$

f ∩ on $[\frac{\pi}{2}, \frac{3\pi}{2}]$.

2 P.I. at $x = \frac{\pi}{2}$
& $x = \frac{3\pi}{2}$.

Question 9. (9 points)
Let $f(x) = x^3 + x + \frac{1}{2}$. $\rightarrow f'(x) = 3x^2 + 1$

(a) (3 points) Use a linear approximation or differential to estimate how much f changes when x changes by 0.1 relative to $x = 0$.

1 Find tangent at $x=0$

$y = mx + b$.

$m = f'(0) = 1$

I know that $f(0) = \frac{1}{2}$. So

$\frac{1}{2} = 1 \cdot 0 + b \rightarrow b = \frac{1}{2}$.

$\Rightarrow y = x + \frac{1}{2}$

2 Approximate.

$f(0.1) \approx 0.1 + \frac{1}{2}$

$= \frac{1}{10} + \frac{5}{10} = \frac{6}{10}$

$\rightarrow \boxed{f(0.1) \approx \frac{3}{5}}$

(b) (3 points) The function f must have at least one zero in the interval $(-1, 0)$. Explain why, making explicit which theorem(s), if any, and which assumption(s) on f, if any, you are using.

Intermediate Value Theorem.

Here, $f(-1) = -1 - 1 + \frac{1}{2} = -\frac{3}{2} < 0$ & $f(0) = 0 + 0 + \frac{1}{2} = \frac{1}{2} > 0$.

So, there be at least one point c in $(-1, 0)$ s.t.
$f(c) = 0$ (so f has at least one zero in $(-1, 0)$).

(c) (3 points) The function f must have at most one zero in the interval $(-1, 0)$. Explain why, making explicit which theorem(s), if any, and which assumption(s) on f, if any, you are using.

Rolle's Theorem. $f'(x) = 3x^2 + 1$

Here, $f'(x) \geq 1 > 0$ (never equal to zero)
for any x in $(-1, 0)$.

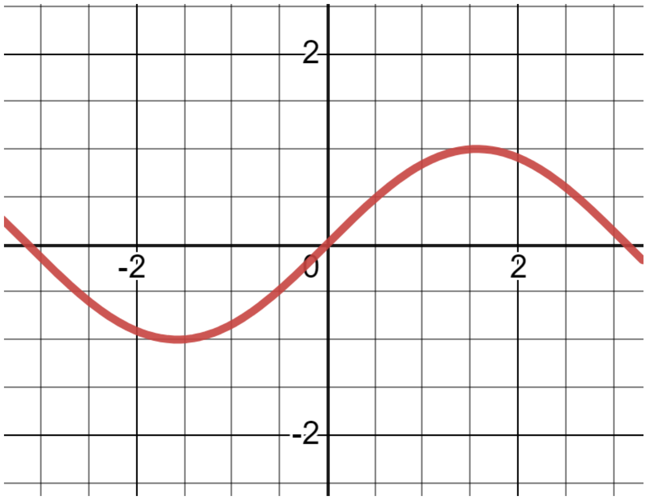
So, there is at most one zero.

Contradiction: Suppose c_1 & c_2 s.t. $f(c_1) = f(c_2) = 0$.
Rolle's theorem $\Rightarrow \exists x$ between c_1 & c_2 s.t. $f'(x) = 0$.
But $f'(x) \neq 0$. this is contradiction.

Question 10. (6 points)
A rectangular box has a base that is a square. The perimeter of the base plus three times the height of the box is equal to 3 ft. What is the largest possible volume for such a box, and what are its dimensions? Justify your answer.

Question 11. (6 points)

Below is a graph of the function $y = \sin(x)$:



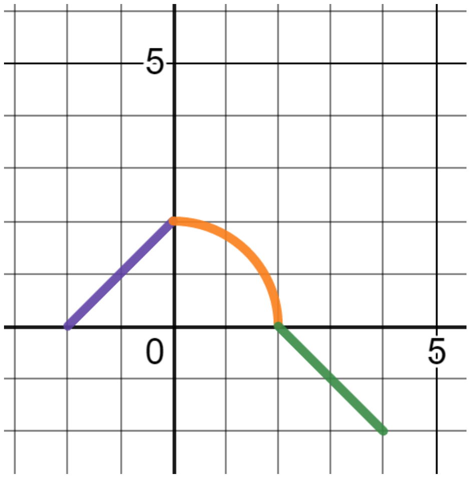
(a) (3 points) Use the right-endpoint rule with $n = 4$ to estimate $\int_0^{\pi/2} \sin x \, dx$. (No simplification is needed).

(b) (1 point) On the figure above, or on a sketched copy of the graph, plot the rectangles that you used for the estimation.

(c) (2 points) Is this estimate over or under the actual answer? Or is it impossible to tell? Explain your answer.

Question 12. (8 points)

Below is a graph of the function g. For x in $[-2, 4]$ we define $f(x) = \int_0^x g(t)\,dt$.



(a) (3 points) Find $f(0)$ and $f(4)$.

(b) (2 points) Find all critical points of the function f on the interval (-2,4).

(c) (3 points) Find the absolute maximum value of $f(x)$ on the interval [-2,4]

Question 13. (5 points)

The Fundamental Theorem of Calculus says the following. If f is a continuous function on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\,dt$$

is continuous on $[a, b]$ and differentiable on (a, b), with $g'(x) = f(x)$.

A function $h(x)$ is defined for $x \in [0, \pi/2]$ by

$$h(x) = \int_{\cos x}^{\sin x} \sqrt{1 - t^2}\,dt.$$

Find $h'(x)$ by using the above version of the Fundamental Theorem of Calculus.

You will get no credit if you do not use this version of the Fundamental Theorem of Calculus.

Question 14. (9 points)

a) (3 points) Solve the indefinite integral $\int (x^2 - 5\sin x + \sec^2 x)\ dx$

b) (3 points) Solve the definite integral $\int_2^{23} \frac{1}{\sqrt{x+2}}\ dx$

c) (3 points) Solve the indefinite integral $\int \cos x\ \sin^2 x\ \sqrt{1-\sin^3 x}\ dx$

Question 15. (6 points)

a) (2 points) Graph and shade the area bounded by the following. Label points of intersection and endpoints of the graph.

$$y = |x|,\ y = x^3,\ -1 \leq x \leq 1$$

b) (4 points) Solve for the area of the shaded region.

Question 16. (8 points)

Setup the integral but **DO NOT EVALUATE** the following volumes using the prescribed method.

a) (4 points) The volume of the solid generated using the area bounded by

$$y = 0, \quad y = \sqrt{x^2 + 1}, \quad 0 \leq x \leq 5$$

rotated about the y-axis. Use the cylindrical shell method (setup but **do not solve** the integral).

b) (4 points) The volume of the solid generated using the area bounded by

$$-y^2 + 2y = x, \quad x = 0$$

rotated about the y-axis. Use the disk method (also known as the washer method; setup but **do not solve** the integral).