

Chapter 1

Functions and Limits

1.6 Calculating Limits Using the Limit Laws

Operations With Limits

EXAMPLE 1

Use the graphs of f and g in Figure 1 to evaluate the following limits, if they exist.

- (a) $\lim_{x \rightarrow -2} [f(x) + 5g(x)]$ (b) $\lim_{x \rightarrow 2} [f(x)g(x)]$ (c) $\lim_{x \rightarrow 2} \frac{f(x)}{g(x)}$
 (d) $\lim_{x \rightarrow -2} [2f(x)]$ (e) $\lim_{x \rightarrow -2} [f(x) - g(x)]$

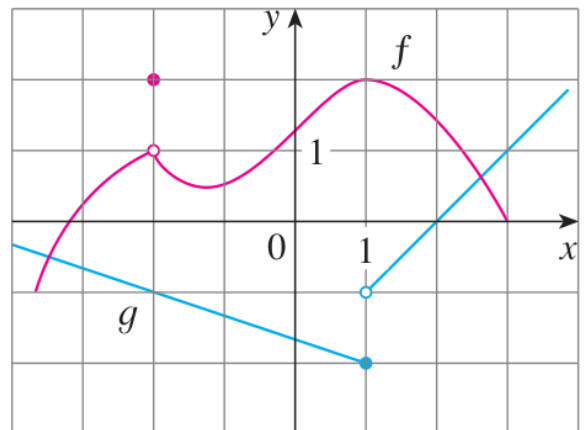


FIGURE 1

Use Desmos

<https://www.desmos.com/calculator/7fy0x0ghia>

Limit Laws Suppose that c is a constant and the limits

$$\lim_{x \rightarrow a} f(x) \quad \text{and} \quad \lim_{x \rightarrow a} g(x)$$

exist. Then

$$1. \lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$$

$$2. \lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$$

$$3. \lim_{x \rightarrow a} [cf(x)] = c \lim_{x \rightarrow a} f(x)$$

$$4. \lim_{x \rightarrow a} [f(x)g(x)] = \lim_{x \rightarrow a} f(x) \lim_{x \rightarrow a} g(x)$$

$$5. \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} \quad \text{if } \lim_{x \rightarrow a} g(x) \neq 0$$

EXAMPLE. Think of three ways of computing the following limit:

$$\lim_{x \rightarrow 2} (1 + x)^3$$

EXAMPLE. Think of three ways of computing the following limit:

$$\lim_{x \rightarrow \pi/4} \cos^2(x)$$

General Formula:

$$\mathbf{6.} \quad \lim_{x \rightarrow a} [f(x)]^n = \left[\lim_{x \rightarrow a} f(x) \right]^n \quad \text{where } n \text{ is a positive integer}$$

EXAMPLE 2 Evaluate the following limits and justify each step.

(a) $\lim_{x \rightarrow 5} (2x^2 - 3x + 4)$

(b) $\lim_{x \rightarrow -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}$

Remark:

Direct Substitution Property If f is a polynomial or a rational function and a is in the domain of f , then

$$\lim_{x \rightarrow a} f(x) = f(a)$$

$$\mathbf{11.} \quad \lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)} \quad \text{where } n \text{ is a positive integer}$$

[If n is even, we assume that $\lim_{x \rightarrow a} f(x) > 0$.]

Example. Compute $\lim_{u \rightarrow -2} \sqrt{u^4 + 3u + 6}$.

EXAMPLE 3 Find $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$.

We have to use the following new substitution rule:

If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$, provided the limits exist.

EXAMPLE 5 Evaluate $\lim_{h \rightarrow 0} \frac{(3 + h)^2 - 9}{h}$.

EXAMPLE 6 Find $\lim_{t \rightarrow 0} \frac{\sqrt{t^2 + 9} - 3}{t^2}$.

EXAMPLE 8 Prove that $\lim_{x \rightarrow 0} \frac{|x|}{x}$ does not exist.

EXAMPLE 9 If

$$f(x) = \begin{cases} \sqrt{x-4} & \text{if } x > 4 \\ 8 - 2x & \text{if } x < 4 \end{cases}$$

determine whether $\lim_{x \rightarrow 4} f(x)$ exists.

EXAMPLE 11 Show that $\lim_{x \rightarrow 0} x^2 \sin \frac{1}{x} = 0$.

3 The Squeeze Theorem If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$$

then

$$\lim_{x \rightarrow a} g(x) = L$$

