Chapter 3: Applications of differentiation
Week 8

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Calculus I (MATH-241 01/02)

University of Hawai'i
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Upcoming this week

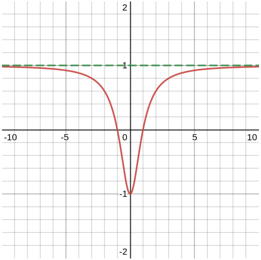
1. 3.4 Limits at infinity; Horizontal Asymptotes
2. 3.5 Summary of Curve Sketching
3. 3.6 Optimization Problems (Part I)

Example 1

Consider the function

$$f(x) = \frac{x^2 - 1}{x^2 + 1}.$$

- What is the graph of this function?
- What happens to the numerator if x becomes larger and larger?
- What happens to the denominator if x becomes larger and larger?
- What happens if x becomes larger and larger in the negative values?

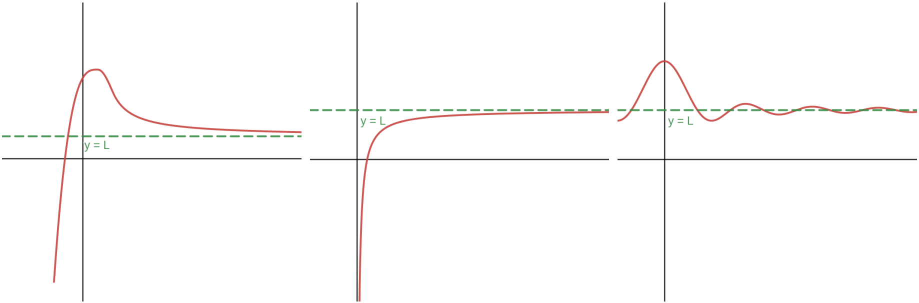


Definition 2 (Limit at infinity)

Let f be a function defined in some interval (a, ∞). Then,

$$\lim_{x \to \infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrary close to L by requiring x to be sufficiently large.



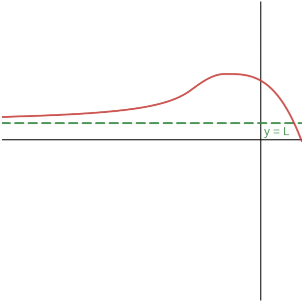
(a) First type of HA (b) Second type of HA (c) Third type of HA

Definition 3 (Limit at infinity)

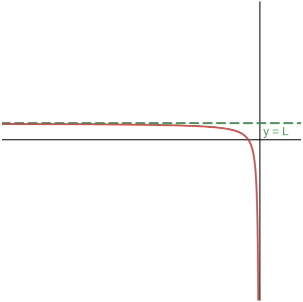
Let f be a function defined in some interval $(-\infty, a)$. Then,

$$\lim_{x \to -\infty} f(x) = L$$

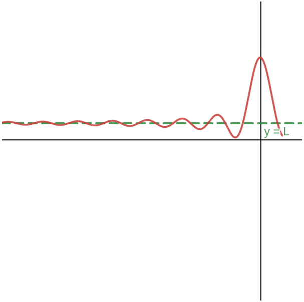
means that the values of $f(x)$ can be made arbitrary close to L by requiring x to be sufficiently large negative values.



(d) Fourth type of HA



(e) Fifth type of HA



(f) Sixth type of HA

Definition 4 (HA)

The line $y = L$ is called an horizontal asymptote (HA) of the curve $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L.$$

Example 5

The function $f(x) = \frac{x^2-1}{x^2+1}$ has $y = 1$ as a HA.

Example 6

Where do you think $1/x$ converges when $x \to \infty$.

Theorem 7

If $r > 0$ is a rational number, then

$$\lim_{x \to \infty} \frac{1}{x^r} = 0.$$

If $r > 0$ is a rational number such that x^r is defined for all $x \in \mathbb{R}$, then

$$\lim_{x \to -\infty} \frac{1}{x^r} = 0.$$

Example 8

Using the preceding rule, compute

$$\lim_{x \to \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}.$$

Example 9

Find the horizontal and vertical asymptotes of

$$f(x) = \frac{\sqrt{2x^2+1}}{3x-5}.$$

Example 10

Compute $\lim_{x\to\infty}(\sqrt{x^2+1} - x)$.

As x becomes larger and larger, we may say that x^r, when $r > 0$ becomes larger and larger. In other words, we have

$$\lim_{x \to \infty} x^r = \infty.$$

This is what we call an infinite limit at infinity.

Definition 11

The notation

$$\lim_{x \to \infty} f(x) = \infty$$

means that the values of $f(x)$ become larger and larger as the values of x becomes larger and larger. Similar meanings are attached to the following symbols:

$$\lim_{x \to -\infty} f(x) = \infty, \quad \lim_{x \to \infty} f(x) = -\infty \quad \text{and} \quad \lim_{x \to -\infty} f(x) = -\infty.$$

Example 12

It is wrong to do

$$\lim_{x\to\infty}(x^2 - x) = \lim_{x\to\infty} x^2 - \lim_{x\to\infty} x = \infty - \infty$$

because $\infty - \infty$ is not defined, like $0/0$.

Theorem 13

If $\lim_{x\to\infty} f(x) = \infty$ *and if* $\lim_{x\to\infty} g(x) = \infty$, *then*

- $\lim_{x\to\infty}(f(x) + c) = \infty$ *for any number* c.
- $\lim_{x\to\infty}(f(x) + g(x)) = \infty$.
- $\lim_{x\to\infty} f(x)g(x) = \infty$.

Example 14

Redo example 12 with these rules at hand.

Exercises: 1, 3, 7-12, 15-20, 22-32, 35, 36, 38-40, 48-51.

Nowadays, technology is really useful to sketch the graph of a function. For example, we use Desmos in the course to illustrate concepts and to draw the graph of functions.

Question

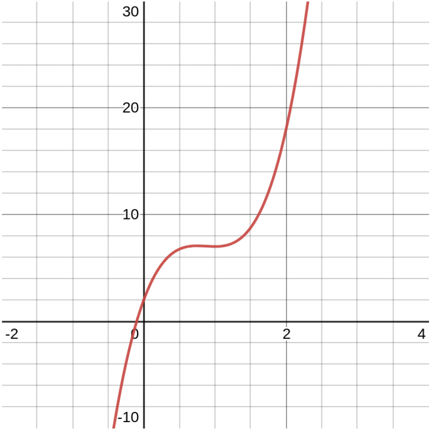
So, why do we have to do it by hand, if computers can do it in our place?

There are many reasons, but here are a few:

- It gives you a good opportunity to pratice and to see if you understand the concepts.
- Sometimes, a graph drawn by a computer can be misleading. Calculus is here for us to make sure that we don't make assumptions on anything.

Example 15

Let $y = 8x^3 - 21x^2 + 18x + 2$. The graph of this function is presented below.



It doesn't seem to have a maximum. But if you compute the derivative, you will see that this function has a minimum and a maximum!

Calculus is a good tool to help us to sketch the graph of a function. Here are the steps to follow to have all the information to sketch a curve $y = f(x)$.

1. Find the domain of the function.
2. Find the y-intercept, that is $f(0)$.
3. Search for symmetries in the function (facultative)
   - If $f(x) = f(-x)$, then the function is even.
   - If $-f(x) = f(-x)$, then the function is odd.
   - If $f(x+p) = f(x)$, then the function repeats itself after a period p (it is periodic).
4. Find the asymptotes of the function:
   - The Horizontal asymptotes.
   - The Vertical asymptotes.
5. Find the intervals of increase and decrease.
6. Find the local maximum and minimum values.
7. Find the concavity and the points of inflections.

Example 16

With the guideline, sketch the graph of the function

$$f(x) = \frac{2x^2}{x^2 - 1}.$$

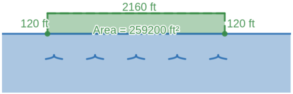
Example 17

With the guideline, sketch the graph of the function

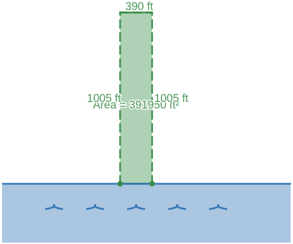
$$f(x) = \frac{\cos x}{2 + \sin x}.$$

Example 18

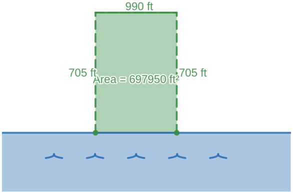
A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area? Field problem



(g) Small Height



(h) Big Height



(i) Good Height

Figure: Some examples

Example 19

Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$.

Parabola distance problem

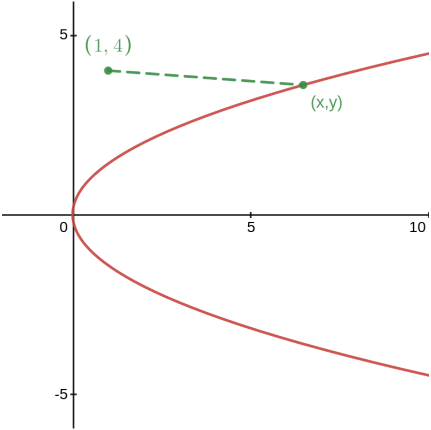


Figure: Minimum Distance Problem