Math 311

Chapter 6

Section 6.1: Vector Spaces

Contents

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DEFINITION

Column Vectors

Recall that

$$\mathbb{R}^n = \{\mathbf{x} : \mathbf{x} \text{ is an } n \times 1 \text{ vector}\}.$$

① For addition:

$\boxed{\text{A1.}}$ $\vec{x}, \vec{y} \Rightarrow \vec{x} + \vec{y} \in \mathbb{R}^n$.

$\boxed{\text{A2.}}$ $\vec{x} + \vec{y} = \vec{y} + \vec{x}$ (Commutativity) .

$\boxed{\text{A3.}}$ $(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$ (Assoc.)

$\boxed{\text{A4.}}$ $\vec{x} + \vec{0} = \vec{x} = \vec{0} + \vec{x}$

$\boxed{\text{A5.}}$ For any $\vec{x}$, there is a $\vec{y}$ s.t.
$\vec{x} + \vec{y} = \vec{y} + \vec{x} = \vec{0}$ (here $\vec{y} = -\vec{x}$)

② For scalar multiplication:

$\boxed{\text{S1.}}$ $\vec{x}$ and $a \in \mathbb{R} \Rightarrow a\vec{x} \in \mathbb{R}^n$.

$\boxed{\text{S2.}}$ $a(\vec{x} + \vec{y}) = a\vec{x} + a\vec{y}$.

$\boxed{\text{S3.}}$ $(a+b)\vec{x} = a\vec{x} + b\vec{x}$

$\boxed{\text{S4.}}$ $a(b\vec{x}) = (ab)\vec{x}$

$\boxed{\text{S5.}}$ $1\vec{x} = \vec{x}$

<u>Conclusion:</u> $\mathbb{R}^n$ is a vector space .

General Definition

Let V be a set of objects called **vectors**. Assume

1. **Vector Addition:** Two vectors $\mathbf{v}$ and $\mathbf{w}$ can be added and denote this operation by $\mathbf{v} + \mathbf{w}$.
2. **Scalar Multiplication:** Any vector $\mathbf{v}$ can be multiplied by any number (scalar) a and denote this operation by $a\mathbf{v}$.

The set V is called a **vector space** if

1. Axioms for the vector addition:

   $\boxed{\text{A1.}}$ Closed: $\mathbf{v}, \mathbf{w} \in V \Rightarrow \mathbf{v} + \mathbf{w} \in V$.

   $\boxed{\text{A2.}}$ Commutativity: $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$.

   $\boxed{\text{A3.}}$ Associativity: $\mathbf{v} + (\mathbf{w} + \mathbf{z}) = (\mathbf{v} + \mathbf{w}) + \mathbf{z}$.

   $\boxed{\text{A4.}}$ Existence of a zero vector: $\mathbf{v} + \mathbf{0} = \mathbf{v} = \mathbf{0} + \mathbf{v}$.

   $\boxed{\text{A5.}}$ Existence of a negative: For each $\mathbf{v}$, there is a $\mathbf{w}$ such that $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v} = \mathbf{0}$.

2. Axioms for the scalar multiplication:

   $\boxed{\text{S1.}}$ $\mathbf{v} \in V \Rightarrow a\mathbf{v} \in V$.

   $\boxed{\text{S2.}}$ $a(\mathbf{v} + \mathbf{w}) = a\mathbf{v} + a\mathbf{w}$.

   $\boxed{\text{S3.}}$ $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.

   $\boxed{\text{S4.}}$ $a(b\mathbf{v}) = (ab)\mathbf{v}$.

   $\boxed{\text{S5.}}$ $1\mathbf{v} = \mathbf{v}$.

Spaces of Matrices

Example 1. Let $\mathbf{M_{mn}}$ be the set of all $m \times n$ matrices, that is

$$\mathbf{M_{mn}} := \{A \,:\, A \text{ is an } m \times n \text{ matrix.}\}$$

Consider the addition and scalar multiplication for matrices. Show that $\mathbf{M_{mn}}$ is a vector space.

Solution.

Spaces of Polynomials

EXAMPLE 2. Consider the space $\mathbf{P_3}$ of all polynomials of degree at most 3, that is

$$\mathbf{P} := \{a_3x^3 + a_2x^2 + a_1x + a_0 \ : \ a_i \in \mathbb{R}\}.$$

Define

1. Addition: for two polynomials $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ and $q(x) = b_3x^3 + b_2x^2 + b_1x + b_0$, define $p + q$ as the polynomial

$$\begin{aligned}(p+q)(x) &= p(x) + q(x)\\ &= (a_3+b_3)x^3 + (a_2+b_2)x^2 + (a_1+b_1)x + (a_0+b_0).\end{aligned}$$

2. Scalar multiplicatoin: for a polynomial $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$, define ap as the polynomial

$$(ap)(x) = ap(x) = (aa_3)x^3 + (aa_2)x^2 + (aa_1)x + (aa_0).$$

Show that $\mathbf{P_3}$, with this addition and scalar multiplication, is a vector space.

SOLUTION.

Note:

① The space of polynomial of degree at most n is denoted by $\mathbf{P_n}$ and is a vector space using the addition and scalar multiplication introduced above.

② The space of all polynomial of any degree is denoted by $\mathbf{P}$ and it is a vector space using the addition and scalar multiplication introduced above.

Weird Example

EXAMPLE 3. Consider the set of all 2×1 vectors $\mathbb{R}^2$. Define the addition and scalar multiplication:

1. $\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 + 1 \end{bmatrix}$.
2. $a\mathbf{x} = a \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_1 \\ ax_2 + a - 1 \end{bmatrix}$.

Show that $\mathbb{R}^2$, with these operations, is a vector space.

SOLUTION.

Non-Example

EXAMPLE 4. Consider the set of all 2×1 vectors $\mathbb{R}^2$. Define the addition and scalar multiplication:

1. $\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 + 1 \end{bmatrix}$.
2. $a\mathbf{x} = a \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_1 \\ ax_2 - 1 \end{bmatrix}$.

Show that $\mathbb{R}^2$, with these operations, is not a vector space.

SOLUTION.

Consider a general vector space V.

① Cancellation: If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, then

$$\mathbf{v} + \mathbf{u} = \mathbf{v} + \mathbf{w} \Longrightarrow \mathbf{u} = \mathbf{w}.$$

② Multiplying by scalar 0:

$$0\mathbf{v} = \mathbf{0}.$$

③ Multiplying by the zero vector:

$$a\mathbf{0} = \mathbf{0}.$$

④ If $a\mathbf{v}\mathbf{0}$, then $a = 0$ or $\mathbf{v} = \mathbf{0}$.

EXAMPLE 5. Simplify the following expression:

$$3\big(2(\mathbf{u} - 2\mathbf{v} - \mathbf{w}) + 3(\mathbf{w} - \mathbf{v}\big) - 7(\mathbf{u} - 3\mathbf{v} - \mathbf{w}).$$

SOLUTION.