

Chapter 5: Applications of Integration

Week 13

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Calculus I (MATH-241 01/02)

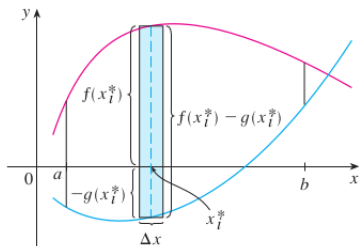
University of Hawai'i
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Upcoming this week

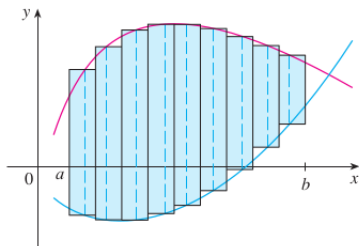
- 1 5.1 Areas between curves
- 2 5.2 Volumes

As we saw last week, the integral is a good tool to compute the area under a curve.

We can also use it to compute the area between two curves $y = f(x)$ and $y = g(x)$ with $f(x) \geq g(x)$.



! (a) Typical rectangle



(b) Approximating rectangles

Figure: Area between two curves

Definition 1

The area A of a region bounded by the curves $y = f(x)$, $y = g(x)$, $x = a$ and $x = b$, where $f(x) \geq g(x)$ for any $x \in [a, b]$, is

$$A = \int_a^b [f(x) - g(x)] dx.$$

Example 2

Compute the region bounded from above by the curve $f(x) = x^2 + 1$, bounded from below by the curve $g(x) = x$, and bounded on the sides by $x = 0$ and $x = 1$.

Example 3

Find the area of the region enclosed by the parabola $y = x^2$ and $y = 2x - x^2$.

Example 4

Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

For general functions (not necessarily $f(x) \geq g(x)$), we compute the total area between two curves as followed.

Definition 5

The total area A between the curves $y = f(x)$ and $y = g(x)$ and between $x = a$ and $x = b$ is

$$A = \int_a^b |f(x) - g(x)| dx.$$

Example 6

Find the area of the region bounded by the curve $y = \sin x$ and $y = \cos x$ from $x = 0$ and $x = \pi/2$.

Exercises: 5-12, 13-28, 34.

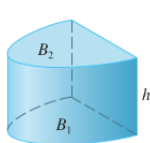
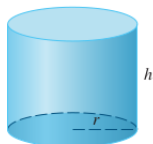
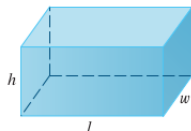
(a) Cylinder $V = Ah$ (b) Circular cylinder $V = \pi r^2 h$ (c) Rectangular box $V = lwh$

Figure: Volume of usual objects

- a) If the area of the shapes B_1 and B_2 is A , then the volume is $V = Ah$.
- b) The area of the base is πr^2 so the volume is $\pi r^2 h$.
- c) The area of the base is wl so the volume is $V = wlh$.

We then see that, usually, the technic to compute the volume of an object is

- Take a horizontal slice parallel to the base.
- Sums up all these slices h times.

We use this technic for arbitrary objects. Take the following object S (think of it as a bread).

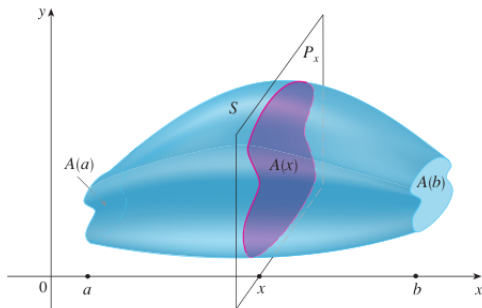


Figure: Slicing an arbitrary object

- Cut the object with a plane P_x at some point x where the plane is perpendicular to the x -axis.
- Call the region of the object $A(x)$.

Now, we cut the bread (object S) in many slices, say n slices.

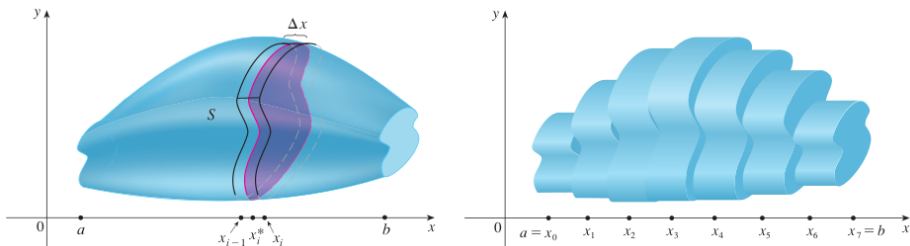


Figure: Slicing in many parts

Now, we will have $V(S) \approx \sum_{i=1}^n A(x_i^*) \Delta x$. Take the limit as n goes to ∞ .

Definition 7

The volume of an object S is defined as

$$V(S) = \int_a^b A(x) dx.$$

Example 8

Show that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.

Example 9

Find the volume of a cylinder of radius r and height h .

We obtain the same answer as we are used to! This is good, it shows that our definition makes sense.

We can rotate functions to obtain 3D objects.

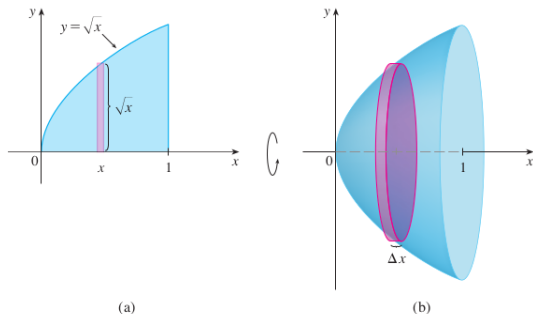


Figure: Rotating the function $f(x) = \sqrt{x}$.

Example 10

Find the volume of the object obtained by rotating the function $f(x) = \sqrt{x}$ ($0 \leq x \leq 1$) around the x -axis.

We can also rotate a region about the y -axis.

Example 11

Find the volume of the object obtained by rotating the region enclosed by the curves $y = x^3$, $y = 8$, and $x = 0$ about the y -axis.

We may also rotate about a different line.

Example 12

Find the volume of the object obtained by rotating the region enclosed by the curves $y = x$ and $y = x^2$ about the line $y = 2$.

The previous solid are called solids of revolution. The volume of these solids are either obtained by the formula

$$V = \int_a^b A(x) dx \quad \text{or} \quad V = \int_c^d A(y) dy.$$

Here is some tips to obtained the formula for A .

- If the cross-section is a disk (as in Examples 10 and 11), we find the radius of the disk (in terms of x or y) and use

$$A = \pi(\text{radius})^2.$$

- If the cross-section is a washer (as in Example 12), we find the inner radius r_{in} and the outer radius r_{out} . We find the area of the washer (in terms of x or y) and we use

$$A = \pi(r_{out})^2 - \pi(r_{in})^2.$$

More examples: Check out examples 7, 8, and 9 in the textbook!

Exercises: 1-18, 19, 29, 31, 39-42, 48, 49, 51, 56, 63,