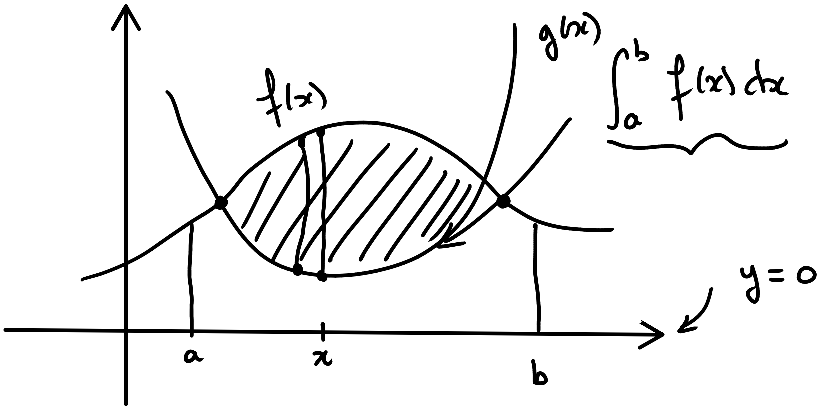
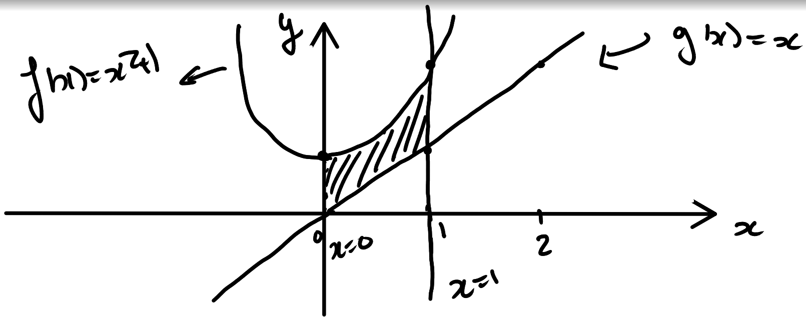


$$\underbrace{\int_a^b f(x)\,dx}\ = \text{Area under the curve } y = f(x)$$

Example 2

Compute the region bounded from above by the curve $f(x) = x^2 + 1$, bounded from below by the curve $g(x) = x$, and bounded on the sides by $x = 0$ and $x = 1$.

① Graphs.



② Area.

$$A = \int_0^1 \underbrace{f(x) - g(x)}_{\geq 0} \, dx = \int_0^1 x^2 + 1 - x \, dx$$

$$= \frac{x^3}{3} + x - \frac{x^2}{2} \Big|_0^1$$

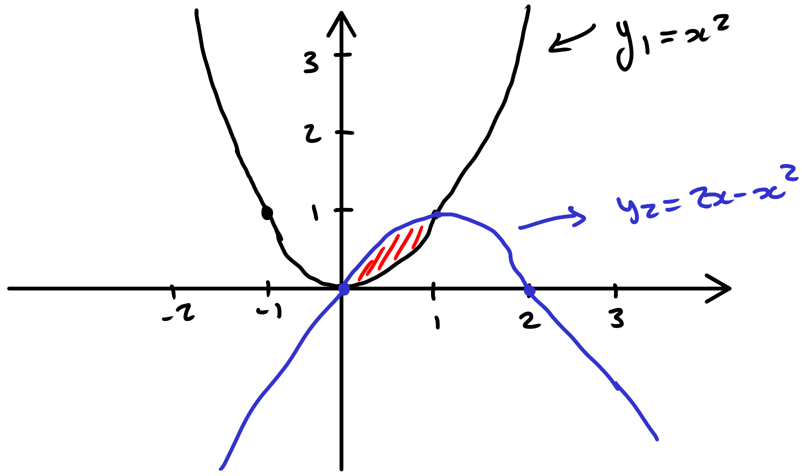
$$\boxed{= \frac{5}{6} u^2}$$

Example 3

Find the area of the region enclosed by the parabola $y = x^2$ and $y = 2x - x^2$.

① Graphs.

$$y_2 = 2x - x^2$$
$$= (2-x)x$$
$$= 0$$
if $x = 2$ & $x = 0$



② Find the intersections between y_1 & y_2

We want to find when $y_1 = y_2$

$y_1 = y_2$ if $x^2 = 2x - x^2$

if $0 = 2x - 2x^2$

if $0 = 2x(1-x)$

if $0 = x$ or $0 = 1 - x$

if $x = 0$ or $x = 1$.

③ Area between y_1 & y_2

$$\underset{f(x)}{y_2} \geq \underset{g(x)}{y_1}$$

$$A = \int_0^1 y_2 - y_1 \, dx = \int_0^1 2x - x^2 - x^2 \, dx$$

$$= \int_0^1 2x - 2x^2 \, dx$$

$$= x^2 - \frac{2}{3}x^3 \Big|_0^1$$

$$\boxed{= \frac{1}{3} \text{ unit}^2}$$

Example 4

$y = \pm\sqrt{2x+6}$

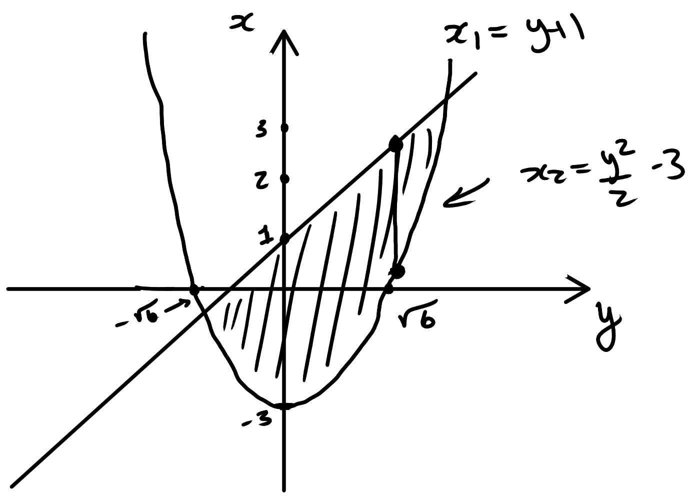
Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

① Graphs.

$x_1 = y+1$

$x_2 = \frac{y^2}{2} - 3$

$x_2 = 0$ if $y = \pm\sqrt{6}$



② Intersections

$x_2 = x_1$ if $\frac{y^2}{2} - 3 = y + 1$ if $y = 4$ & $y = -2$.

$y^2 - 2y - 8 = (y-4)(y+2)$

③ Area

$$\underset{f(x)}{x_1} \geq \underset{g(x)}{x_2}$$

$$A = \int_{-2}^{4} x_1 - x_2 \, dy = \int_{-2}^{4} y + 1 - \left(\frac{y^2}{2} - 3\right) dy$$

$$= \int_{-2}^{4} y - \frac{y^2}{2} + 4 \, dy$$

$$= \frac{y^2}{2} - \frac{y^3}{6} + 4y \Big|_{-2}^{4}$$

$$\boxed{= 18 \text{ units}^2}$$

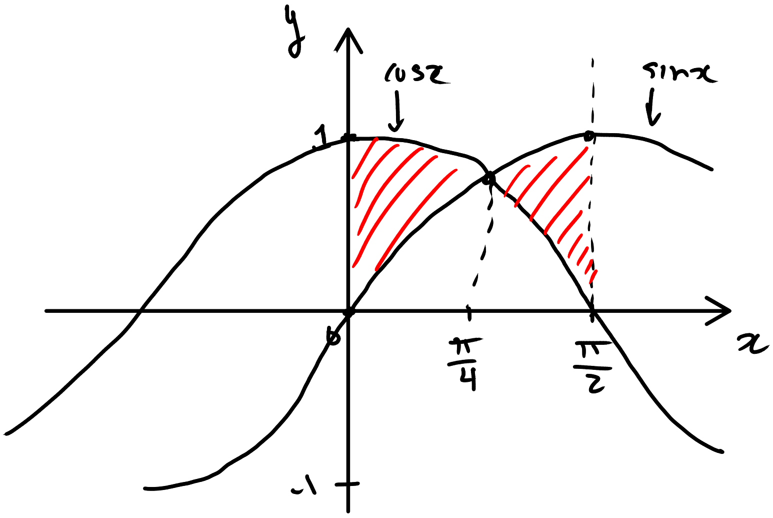
Example 6

Find the area of the region bounded by the curve $y = \sin x$ and $y = \cos x$ from $x = 0$ and $x = \pi/2$.

① Graphs.

$y_1 = \sin x$

$y_2 = \cos x$



Notice that $\cos x \geq \sin x$ on $[0, \pi/4]$

& $\sin x \geq \cos x$ on $[\pi/4, \pi/2]$

② Intersection.

$\cos x = \sin x$ iff $1 = \tan x$ iff $x = \frac{\pi}{4} + k\pi$

$\downarrow$

$x = \frac{\pi}{4}$ because $0 \leq x \leq \frac{\pi}{2}$

③ Tol. Area

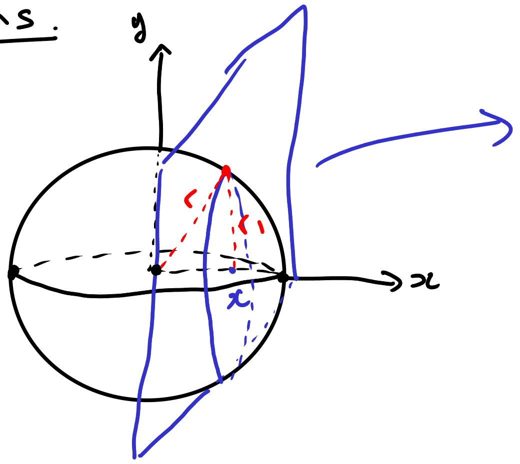
$$A = \int_0^{\pi/2} |\cos x - \sin x| \, dx = \int_0^{\pi/4} \overbrace{\cos x - \sin x}^{\geq 0} \, dx + \int_{\pi/4}^{\pi/2} \overbrace{\sin x - \cos x}^{\geq 0} \, dx$$

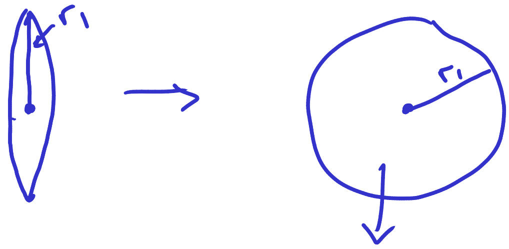
$$\boxed{\approxeq 0.828427 \text{ units}^2}$$

Example 8

Show that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.

(1) Graphs.





$A(x) = \pi r_1^2$

(2) Find $A(x)$

Find the radius r_1 first:

$$r_1^2 + x^2 = r^2 \Rightarrow r_1^2 = r^2 - x^2$$
$$\Rightarrow r_1 = \sqrt{r^2 - x^2}$$

So $\quad A(x) = \pi r_1^2 = \pi\left(\sqrt{r^2-x^2}\right)^2 = \pi\left(r^2 - x^2\right)$

(3) Find the Volume $\quad -r \le x \le r$

$$V = \int_{-r}^{r} \underbrace{A(x)}_{\text{area of slide}} \underbrace{dx}_{\text{thickness}} = \int_{-r}^{r} \pi(r^2 - x^2)\, dx$$

$$= \pi\left(r^2 x - \frac{x^3}{3}\right)\Big|_{-r}^{r}$$

$$= \pi\left(r^3 - \frac{r^3}{3} - \left(-r^3 + \frac{r^3}{3}\right)\right)$$

$$= \pi\left(2r^3 - \frac{2r^3}{3}\right)$$

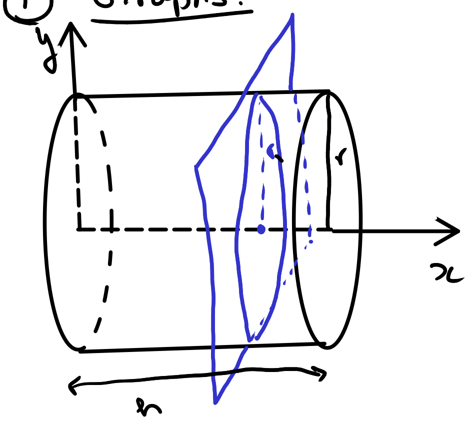
$$= 2\pi\left(r^3 - \frac{r^3}{3}\right) = 2\pi\left(\frac{2r^3}{3}\right)$$

$$\boxed{= \frac{4\pi r^3}{3}}$$

Example 9

Find the volume of a cylinder of radius r and height h.

① Graphs.



The slice is a circle with the same radius as the base of the cylinder.

② Find $A(x)$

$r_1 = r \Rightarrow A(x) = \pi r^2$

③ Volume

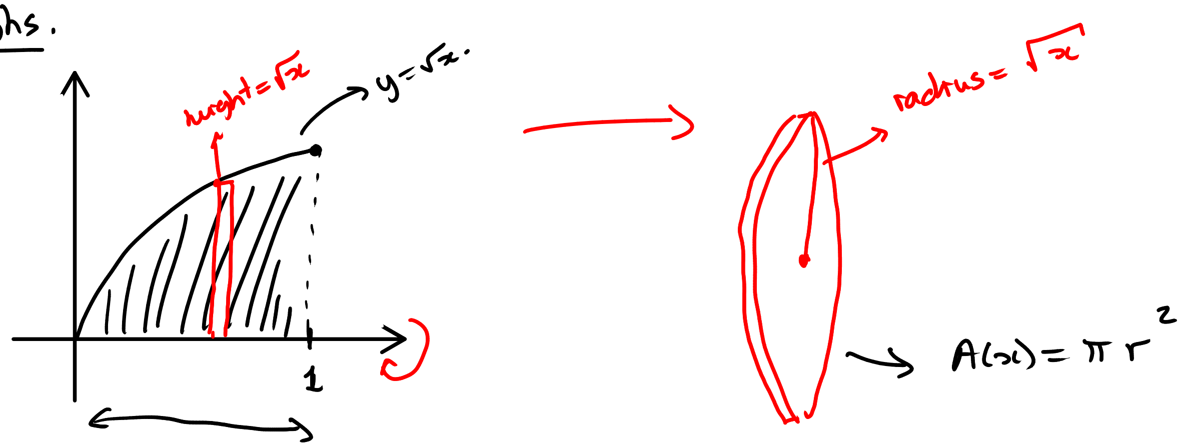
$$V = \int_0^h A(x)\,dx = \int_0^h \pi r^2\,dx = \pi r^2 x \Big|_0^h$$

$$\boxed{= \pi r^2 h}$$

Example 10

Find the volume of the object obtained by rotating the function $f(x) = \sqrt{x}$ $(0 \le x \le 1)$ around the x-axis.

① Graphs.



② Find $A(x)$

So, $A(x) = \pi r^2 = \pi (\sqrt{x})^2 = \pi x$

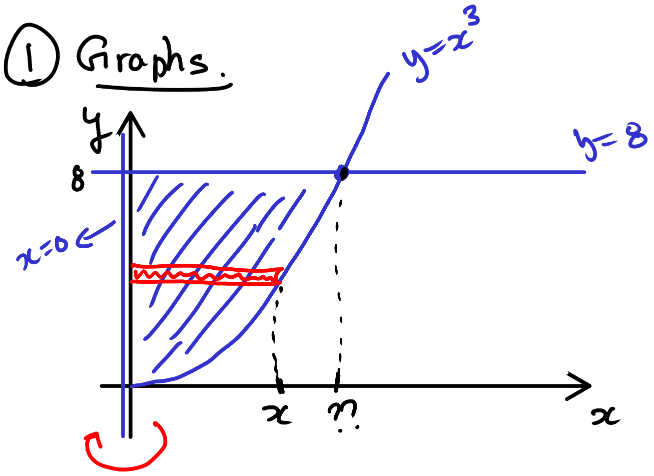
③ Find Volume.

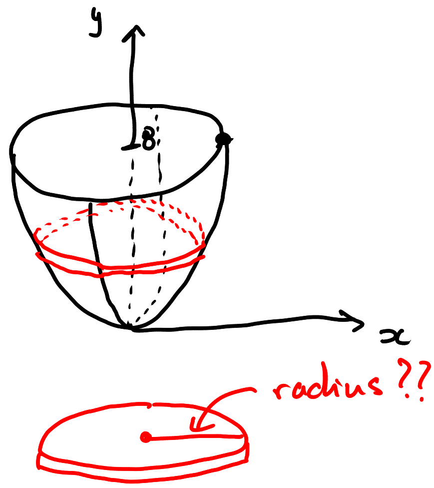
$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi x \, dx = \pi \frac{x^2}{2} = \boxed{\frac{\pi}{2} \text{ units}^2}$$

Example 11

Find the volume of the object obtained by rotating the region enclosed by the curves $y = x^3$, $y = 8$, and $x = 0$ about the $\underline{y\text{-axis}}$.

① Graphs.





② Find $A(x)$

radius $= x = \sqrt[3]{y} = y^{1/3}$

So $A(y) = \pi\,(\text{radius})^2 = \pi\, y^{2/3}$.

③ Volume. Here we see that $y = 8$ iff $x^3 = 8$
iff $x = \sqrt[3]{8} = 2$.

So,

$$V = \int_0^8 A(y)\,dy = \int_0^8 \pi\, y^{2/3}\,dy = \pi \left.\frac{y^{5/3}}{5/3}\right|_0^8$$

$$= \frac{3\pi}{5}\left(8^{5/3} - 0^{5/3}\right)$$

$$= \frac{3\pi}{5}\left(\left(8^{1/3}\right)^5 - 0\right)$$

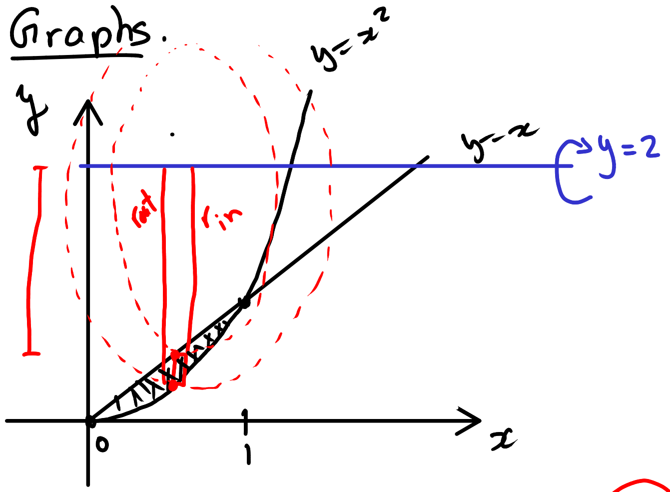
$$= \frac{3\pi}{5}\left(2^5 - 0\right)$$

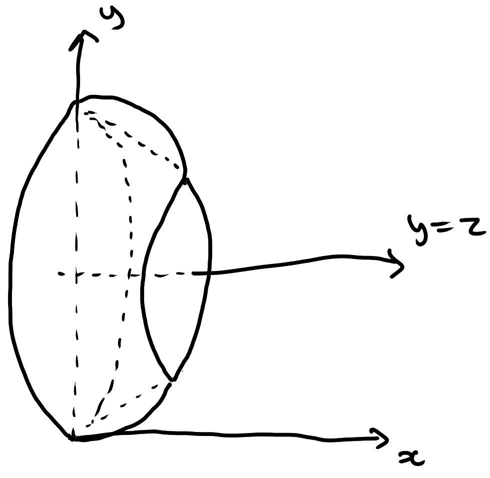
$$= \frac{3 \cdot 32\pi}{5} \boxed{= \frac{96}{5}\pi \text{ units}^3}$$

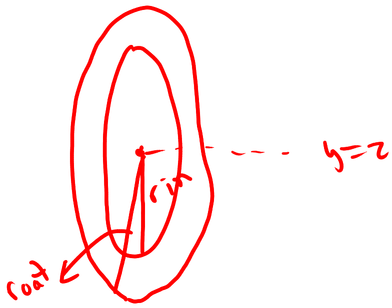
Example 12

Find the volume of the object obtained by rotating the region enclosed by the curves $y = x$ and $y = x^2$ about the line $y = 2$.

① Graphs.







② Find $A(x)$

$r_{in} = 2 - x$ $\qquad$ $r_{out} = 2 - x^2$

$$A(x) = \pi r_{out}^2 - \pi r_{in}^2$$
$$= \pi\left((2-x^2)^2 - (2-x)^2\right)$$
$$= \pi\left(\cancel{4} - 4x^2 + x^4 - \cancel{4} + 4x - x^2\right)$$
$$= \pi\left(x^4 - 5x^2 + 4x\right)$$

③ Find intersections

$y = x = x^2 \Rightarrow x^2 - x = 0$

$\Rightarrow x(x-1) = 0$

$\Rightarrow x = 0$ or $x = 1$.

④ Volume.

$$V = \int_0^1 A(x)\,dx = \int_0^1 (x^4 - 5x^2 + 4x)\pi \, dx$$

$$= \pi \left(\frac{x^5}{5} - \frac{5x^3}{3} + \frac{4x^2}{2}\right)\Big|_0^1$$

$$= \pi \left(\frac{1}{5} - \frac{5}{3} + 2 - 0\right)$$

$$= \pi \left(\frac{1}{5} + \frac{1}{3}\right)$$

$$= \pi \frac{8}{15}$$

So, $\boxed{V = \frac{8\pi}{15} \text{ units}^3}$