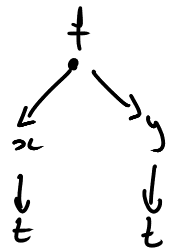
16.3 The Fundamental Theorem for Line Integrals.

Recall. (FTC) $\int_a^b F'(x)\,dx = F(b) - F(a)$

If f is a scalar function, then $\vec{\nabla} f$ is its gradient.



$$\int_a^b \vec{\nabla} f \cdot \vec{r}\,'(t)\,dt = \int_a^b \underbrace{f_x\, x'(t) + f_y\, y'(t)}_{=\frac{d}{dt}(f)}\,dt$$

$$= \int_a^b \frac{d}{dt}(f)\,dt$$

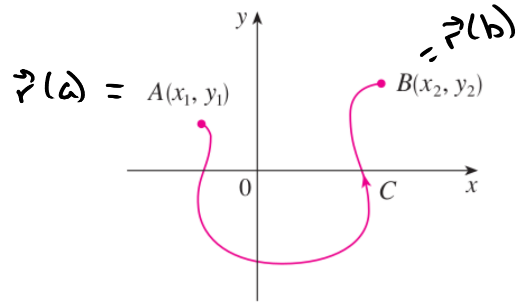
2 Theorem Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \leqslant t \leqslant b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

FTLI

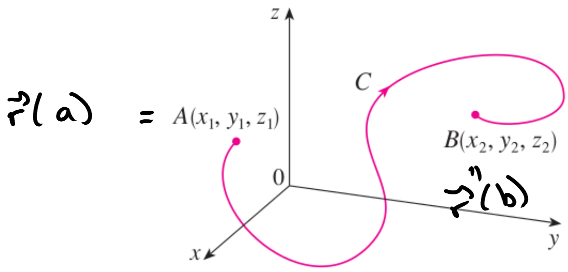
Remarks.

1. In 2D.



$$\int_C \vec{\nabla} f \cdot d\vec{r} = f(x_2, y_2) - f(x_1, y_1).$$

2. In 3D.



$$\int_C \vec{\nabla} f \cdot d\vec{r} = f(x_2, y_2, z_2) - f(x_1, y_1, z_1).$$

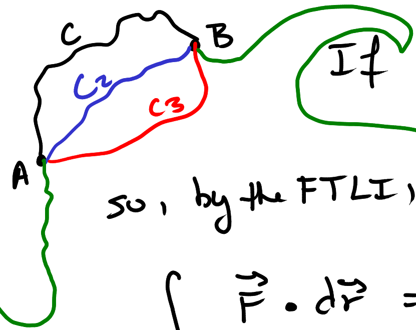
EXAMPLE 1 Find the work done by the gravitational field

$$\boxed{\mathbf{F}(\mathbf{x}) = -\frac{mMG}{|\mathbf{x}|^3}\,\mathbf{x}}$$

in moving a particle with mass m from the point (3, 4, 12) to the point (2, 2, 0) along a piecewise-smooth curve C. (See Example 16.1.4.)

A B

Conservative vector fields: $\vec{F} = \vec{\nabla} f$, some f.



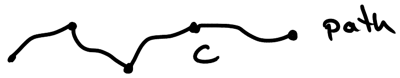
If $f(x,y,z) = \dfrac{mMG}{\sqrt{x^2+y^2+z^2}}$, then $\boxed{\vec{\nabla} f = \vec{F}}$.

So, by the FTLI,

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{\nabla} f \cdot d\vec{r} = f(2,2,0) - f(3,4,12)$$

$$= mMG\left(\frac{1}{2\sqrt{2}} - \frac{1}{13}\right).$$

Independence of Path.



Definition. ① Path: piece-wise smooth curve.

② Independant of Path: $\vec{F}$ is ind. of path if for any two paths C_1 & C_2 starting at A and ending at B, then

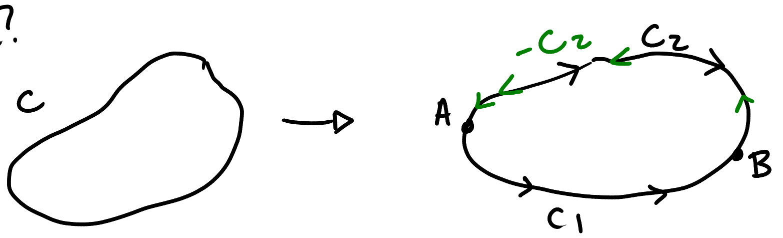
$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}.$$

(Example 4 in 16.2, not true in general)

3 Theorem $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D.

③ Closed path: a path with the same starting & ending points.

Why?



$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$$

$$\Rightarrow \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r} = 0$$

$$\Rightarrow \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{-C_2} \vec{F} \cdot d\vec{r} = 0$$

$$\Rightarrow \int_{C} \vec{F} \cdot d\vec{r} = 0$$

4 Theorem Suppose $\mathbf{F}$ is a vector field that is continuous on an open connected region D. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D, then $\mathbf{F}$ is a conservative vector field on D; that is, there exists a function f such that $\nabla f = \mathbf{F}$.

④ Open :

⑤ : Open connected:

not separated.

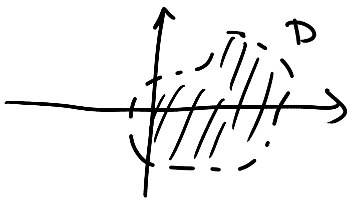
X not connected.

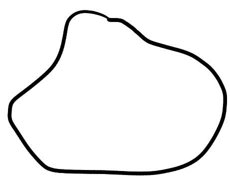
6 Theorem Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order partial derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{throughout D}$$

Then $\mathbf{F}$ is conservative. The converse also holds.

⑥ Simply-connected: No holes!





EXAMPLE 2 Determine whether or not the vector field

$$\mathbf{F}(x, y) = (x - y)\,\mathbf{i} + (x - 2)\,\mathbf{j}$$

is conservative.

$P(x,y) = x - y$
$Q(x,y) = x - 2$

$P_y = -1$
$Q_x = 1$ $\rightarrow$ $P_y = -1 \neq 1 = Q_x \rightarrow$ not conservative!

EXAMPLE 3 Determine whether or not the vector field

$$\mathbf{F}(x, y) = (3 + 2xy)\,\mathbf{i} + (x^2 - 3y^2)\,\mathbf{j}$$

is conservative.

$P(x,y) = 3 + 2xy$
$Q(x,y) = x^2 - 3y^2$.

$P_y = 2x$
$Q_x = 2x$ $\rightarrow$ $P_y = 2x = Q_x \rightarrow$ conservative!! :)

EXAMPLE 4

(a) If $\mathbf{F}(x, y) = (3 + 2xy)\,\mathbf{i} + (x^2 - 3y^2)\,\mathbf{j}$, find a function f such that $\mathbf{F} = \nabla f$.

(b) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve given by

$$\mathbf{r}(t) = e^t \sin t\,\mathbf{i} + e^t \cos t\,\mathbf{j} \qquad 0 \leqslant t \leqslant \pi$$

(a) f s.t. $\vec{\nabla} f = \vec{F}$ $\rightarrow \langle f_x, f_y \rangle = \langle 3 + 2xy, x^2 - 3y^2 \rangle$
$\rightarrow$ ① $f_x = 3 + 2xy$ & ② $f_y = x^2 - 3y^2$.

1st Rig. One.

Int. w.r.t. $x \rightarrow \int f_x\,dx = \int 3 + 2xy\,dx = \underbrace{3x + \frac{2x^2 y}{2} + g(y)}_{= f(x,y)}$

Der. w.r.t. $y \rightarrow f_y = \cancel{x^2} + g'(y) = \cancel{x^2} - 3y^2$
$\Rightarrow \int g'(y) = \int -3y^2 \Rightarrow g(y) = -y^3 + C$

So, $\boxed{f(x,y) = 3x + x^2 y - y^3 + C}$.

2nd method

$f(x,y) = \int f_x\,dx = \int 3 + 2xy\,dx = 3x + x^2 y$
$f(x,y) = \int f_y\,dy = \int x^2 - 3y^2\,dy = yx^2 - y^3$ $\rightarrow f(x,y) = x^2 y + 3x - y^3 + C$

(b) FTLI $\rightarrow \int_C \vec{F} \cdot d\vec{r} = \int_C \vec{\nabla} f \cdot d\vec{r} = f(\vec{r}(\pi)) - f(\vec{r}(0))$.

$\vec{r}(\pi) = (0, -e^{\pi})$
$\vec{r}(0) = (0, 1)$ $\rightarrow f(0, e^{-\pi}) - f(0,1) = e^{3\pi} + C - (-1 + C) = \boxed{e^{3\pi} + 1}$

EXAMPLE 5 If $\mathbf{F}(x, y, z) = y^2\,\mathbf{i} + (2xy \pm e^{3z})\,\mathbf{j} + 3ye^{3z}\,\mathbf{k}$, find a function f such that $\nabla f = \mathbf{F}$.

Goal $\vec{\nabla} f = \vec{F} \leftrightarrow \langle f_x, f_y, f_z \rangle = \vec{F} \leftrightarrow f_x = y^2,\ f_y = 2xy + e^{3z},\ f_z = 3ye^{3z}$

① Integrate w.r.t. x

$$f(x,y,z) \text{``}=\text{''} \int f_x\,dx = \int y^2\,dx = y^2 x$$

② Integrate w.r.t. y.

$$f(x,y,z) \text{``}=\text{''} \int f_y\,dy = \int 2xy + e^{3z}\,dy = xy^2 + ye^{3z}$$

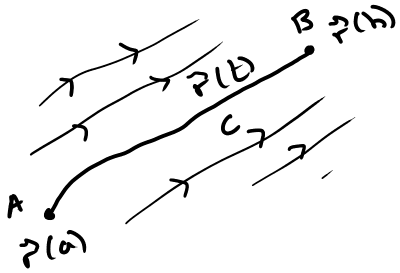
③ Integrate w.r.t. z.

$$f(x,y,z) \text{``}=\text{''} \int f_z\,dz = \int 3ye^{3z}\,dz = ye^{3z}$$

Final expression. $f(x,y,z) = xy^2 + ye^{3z} + C$.

Goal $P(A) + K(A) = P(B) + K(B)$

Conservation of Energy.



Newton's 2nd law:

$$\vec{F}(\vec{r}(t)) = m\,\vec{r}\,''(t) \leftarrow \text{acc.}$$

$$W = \int_C \vec{F}(\vec{r}(t)) \cdot d\vec{r} = \int_a^b m\,\vec{r}\,''(t) \cdot \vec{r}\,'(t)\,dt$$

Recall $\frac{d}{dt}\left(\vec{r}\,'(t) \cdot \vec{r}\,'(t)\right) = \vec{r}\,''(t) \cdot \vec{r}\,'(t) + \vec{r}\,'(t) \cdot \vec{r}\,''(t) = 2\,\vec{r}\,'(t) \cdot \vec{r}\,''(t)$.

$$\Rightarrow W = \frac{m}{2} \int_a^b \frac{d}{dt}\left(\vec{r}\,'(t) \cdot \vec{r}\,'(t)\right) dt = \frac{m}{2}\, \vec{r}\,'(t) \cdot \vec{r}\,'(t) \Big|_a^b$$

$$\Rightarrow W = \frac{m}{2}\, \vec{r}\,'(b) \cdot \vec{r}\,'(b) - \frac{m}{2}\, \vec{r}\,'(a) \cdot \vec{r}\,'(a)$$

$$\Rightarrow W = \frac{m}{2} |\vec{r}\,'(b)|^2 - \frac{m}{2} |\vec{r}\,'(a)|^2 \leftarrow \text{Kinetic energy.}$$

$$\Rightarrow W = \underbrace{\underset{K(B)}{\frac{m}{2} |v(b)|^2} - \underset{K(A)}{\frac{m}{2} |v(a)|^2}}_{\text{net change kinetic energy.}}$$

$\vec{F}$ conservative $\Rightarrow \vec{\nabla} f = \vec{F}$ for some f.

Define $P(x,y,z) = -f(x,y,z) \Rightarrow \vec{\nabla} P = -\vec{\nabla} f$

So, $W = \int_C \vec{F} \cdot d\vec{r} = -\int \vec{\nabla} P \cdot d\vec{r} = -\left(P(B) - P(A)\right) = P(A) - P(B)$

So,

$$K(B) - K(A) = W = P(A) - P(B)$$

$$\Rightarrow \boxed{P(A) + K(A) = P(B) + K(B)}$$

law of
Conservation
of
energy.