MATH 302

CHAPTER 8

SECTION 8.2: LAPLACE TRANSFORMS

CONTENTS

CREATED BY: PIERRE-OLIVIER PARISÉ

FALL 2022

The Laplace transform is REALLY$^{\text{REALLY}}$ useful to solve ODE.

Example 1. Consider the ODE

$$2y''(t) + 3y'(t) + y(t) = 8e^{-2t}$$

with $y(0) = -4$ and $y'(0) = 2$.

Recall: $\begin{cases} Y = L(y) \\ L(y') = sY - y(0) \\ L(y'') = s^2 Y - y(0)s - y'(0) \end{cases}$

Apply L to the ODE.

$$L(2y'') + L(3y') + L(y) = L(8e^{-2t}).$$

$$\to 2L(y'') + 3L(y') + L(y) = 8\,L(e^{-2t})$$

$$\to 2(s^2Y - y(0)s - y'(0)) + 3(sY - y(0)) + Y = \frac{8}{s+2}$$

$$\to 2(s^2 Y + 4s - 2) + 3(sY + 4) + Y = \frac{8}{s+2}$$

$$\to (2s^2 + 3s + 1)Y + 8s + 8 = \frac{8}{s+2}$$

$$\to (2s^2 + 3s + 1)Y = -8s - 8 + \frac{8}{s+2}$$

$$\to Y(s) = \frac{-8s-8}{2s^2+3s+1} + \frac{8}{(s+2)(2s^2+3s+1)}.$$

General Procedure:

1. Apply the Laplace transform to your ODE $ay'' + by' + cy = f(t)$.
2. Apply the properties of the Laplace transform to get

$$a(s^2Y - sf(0) - f'(0)) + b(sY - f(0)) + cY = F.$$

3. Isolate Y:

$$Y = \frac{F + (as + b)f(0) + af'(0)}{as^2 + bs + c}.$$

The last step:

- Take the inverse Laplace transform!

$L^{-1}(L(f)) = f$

Inverse Laplace Transform

Given a Laplace transform $F(s)$ of an unknown function f, we can go backward to find f.

- We denote the **inverse Laplace transform** by L^{-1}.
- We therefore have $f = L^{-1}(F)$.
- How do we find $L^{-1}(F)$?

$L^{-1} \neq \frac{1}{L}$

$f(t)$	$F(s)$
1	$1/s$
t	

Trick: Use the table in the opposite direction!

EXAMPLE 2. Find the inverse Laplace transform of the following functions:

(a) $\dfrac{1}{s^2-1}$.

(b) $\dfrac{s}{s^2+9}$.

(a) $\dfrac{1}{s^2-1} \overset{\text{table}}{\longleftrightarrow} \sinh(t)$

So, $L^{-1}\left(\dfrac{1}{s^2-1}\right) = \sinh(t)$.

(b) $\dfrac{s}{s^2+9} \overset{\text{table}}{\longleftrightarrow} \cos(3t)$

So, $L^{-1}\left(\dfrac{s}{s^2+9}\right) = \cos(3t)$

Linearity of Inverse Transform

If F and G are Laplace transforms of two unknown functions f and g, then

$$L^{-1}(aF + bG) = aL^{-1}(F) + bL^{-1}(G).$$

Example 3. Find

$$L^{-1}\Big(\underbrace{\frac{8}{s+5} + \frac{7}{s^2+3}}_{H(s)}\Big).$$

$$L^{-1}(H(s)) = 8\, L^{-1}\left(\frac{1}{s+5}\right) + 7\, L^{-1}\left(\frac{1}{s^2+3}\right)$$

- $\dfrac{1}{s+5} \xleftrightarrow{\text{table}} e^{-5t}$

$$L^{-1}\left(\frac{1}{s^2+3}\right) = L^{-1}\left(\frac{\sqrt{3}/\sqrt{3}}{s^2+(\underbrace{\sqrt{3}}_{\omega})^2}\right)$$

$$= \frac{1}{\sqrt{3}} L^{-1}\left(\frac{\sqrt{3}}{s^2+(\sqrt{3})^2}\right)$$

- $\dfrac{\sqrt{3}}{s^2+(\sqrt{3})^2} \xleftrightarrow{\text{table}} \sin(\sqrt{3}\,t).$

$$L^{-1}(H(s)) = 8e^{-5t} + \frac{7}{\sqrt{3}}\sin(\sqrt{3}\,t).$$

$$\parallel$$

$$h(t)$$

Example 4. Find

$$\mathcal{L}^{-1}\Big(\overbrace{\frac{3s+8}{s^2+2s+5}}^{H(s)}\Big).$$

Notice:

$$s^2+2s+5 = s^2+2s+1-1+5 = (s+1)^2+4$$

$$\to \quad \frac{3s+8}{s^2+2s+5} = \frac{3s+8}{(s+1)^2+4}$$

$$= \frac{3s+3-3+8}{(s+1)^2+4}$$

$$= \frac{3(s+1)+5}{(s+1)^2+4}$$

$$= \frac{3(s+1)}{(s+1)^2+4} + \frac{5}{(s+1)^2+4}$$

$$\frac{s+1}{(s+1)^2+4} \xleftrightarrow{\text{table}} e^{-t}\cos(2t)$$

$$\frac{2}{(s+1)^2+2^2} \xleftrightarrow{\text{table}} e^{-t}\sin(2t)$$

$$\to \mathcal{L}^{-1}(H(s)) = 3\,\mathcal{L}^{-1}\Big(\frac{s+1}{(s+1)^2+4}\Big) + 5\,\mathcal{L}^{-1}\Big(\frac{2/2}{(s+1)^2+4}\Big)$$

$$= \underline{3e^{-t}\cos(2t) + \frac{5}{2}e^{-t}\sin(2t)}$$

Inverse Laplace Transform of Rational Functions

Example 5. Find the inverse Laplace transform of

$$F(s) = \frac{3s+2}{s^2-3s+2}.$$

$\to 1$
$\to 2$

Notice: $s^2 - 3s + 2 = (s - 3/2)^2 + \frac{13}{2}$

Partial Fractions: (Another way)

$$s^2 - 3s + 2 = (s-1)(s-2)$$

Rewrite:
$$\frac{3s+2}{(s-1)(s-2)} = \frac{A_1}{s-1} + \frac{A_2}{s-2}$$
$$= \frac{A_1(s-2) + A_2(s-1)}{(s-1)(s-2)}$$

$$\Rightarrow 3s+2 = A_1(s-2) + A_2(s-1)$$

$s=2 \Rightarrow 3\cdot 2 + 2 = A_1 \cdot 0 + A_2$

$\Rightarrow 8 = A_2$

$s=1 \Rightarrow 3\cdot 1 + 2 = A_1(-1) + A_2 \cdot 0$

$\Rightarrow 5 = -A_1 \Rightarrow A_1 = -5$

So,
$$\frac{3s+2}{s^2-3s+2} = \frac{-5}{s-1} + \frac{8}{s-2}$$

Apply $\mathcal{L}^{-1}$:

$$\mathcal{L}^{-1}(H(s)) = -5\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) + 8\mathcal{L}^{-1}\left(\frac{1}{s-2}\right)$$
$$= -5e^{t} + 8e^{2t}.$$

Example 6. Find the inverse transform of

$$F(s) = \frac{6 + (s+1)(s^2 - 5s + 11)}{s(s-1)(s-2)(s+1)}.$$

$s^2+1 = (s+i)(s-i)$

$$F(s) = \frac{A_1}{s} + \frac{A_2}{s-1} + \frac{A_3}{s-2} + \frac{A_4}{s+1}.$$

$$= \frac{A_1(s-1)(s-2)(s+1) + A_2 s(s-2)(s+1) + A_3 s(s-1)(s+1) + A_4 s(s-1)(s-2)}{s(s-1)(s-2)(s+1)}$$

General Case:
Suppose your Laplace transform is

$$F(s) = \frac{P(s)}{(s-s_1)(s-s_2)\cdots(s-s_n)}$$

where s_1, s_2, …, s_n are distinct and P is a polynomial of degree less than n. Then

$$F(s) = \frac{A_1}{s-s_1} + \frac{A_2}{s-s_2} + \cdots + \frac{A_n}{s-s_n},$$

- A_1 is computed by letting $\boxed{s = s_1}$ in $G(s) = \dfrac{P(s)}{(s-s_2)\cdots(s-s_n)}$.
- A_2 is computed by letting $s = s_2$ in $G(s) = \dfrac{P(s)}{(s-s_1)(s-s_3)\cdots(s-s_n)}$.
- $\vdots$
- A_n is computed by letting $s = s_n$ in $G(s) = \dfrac{P(s)}{(s-s_1)(s-s_2)\cdots(s-s_{n-1})}$.

Rational Functions with Powers in the Denominator

The Heaviside method doesn't work if we encounters powers of monomonials in the denominator. What do we do then?

EXAMPLE 7. Find the partial fraction expansion of

$$F(s) = \frac{8-(s+2)(4s+10)}{(s+1)(s+2)^2}.$$

$$F(s) = \frac{A_1}{s+1} + \frac{A_2}{s+2} + \frac{A_3}{(s+2)^2}$$

$$= \frac{A_1(s+2)^2 + A_2(s+1)(s+2) + A_3(s+1)}{(s+1)(s+2)^2}$$

$$\Rightarrow A_1(s+2)^2 + A_2(s+1)(s+2) + A_3(s+1) = 8-(s+2)(4s+10)$$

$\underline{s=-1}$ $\quad A_1 = 8-6 = 2$

$\underline{s=-2}$ $\quad -A_3 = 8-0 = 8 \Rightarrow A_3 = -8.$

$\underline{s=0}$ $\quad 2(4) + A_2 2 + (-8) = -12$

$$\Rightarrow A_2 = -6$$

$$F(s) = \frac{2}{s+1} - \frac{6}{s+2} - \frac{8}{(s+2)^2}$$

$$L^{-1}(F(s)) = 2L^{-1}\left(\frac{1}{s+1}\right) - 6L^{-1}\left(\frac{1}{s+2}\right) - 8L^{-1}\left(\frac{1}{(s+2)^2}\right)$$

$$= \underline{2e^{-t} - 6e^{-2t} - 8te^{-2t}}.$$

Example 8. Using the inverse Laplace transform, complete Example 1.

$$Y(s) = -\frac{8s-8}{2s^2+3s+1} + \frac{8}{(s+2)(2s^2+3s+1)}.$$

Example 9. Use the Laplace transform to solve the initial value problem:

$$y'' - 6y' + 5y = 3e^{2t}, \quad y(0) = 2,\ y'(0) = 3.$$