Chapter 1

Functions and Limits

1.5 The Limit of a Function

Intuitive definition of a limit.

1 Intuitive Definition of a Limit Suppose $f(x)$ is defined when x is near the number a. (This means that f is defined on some open interval that contains a, except possibly at a itself.) Then we write

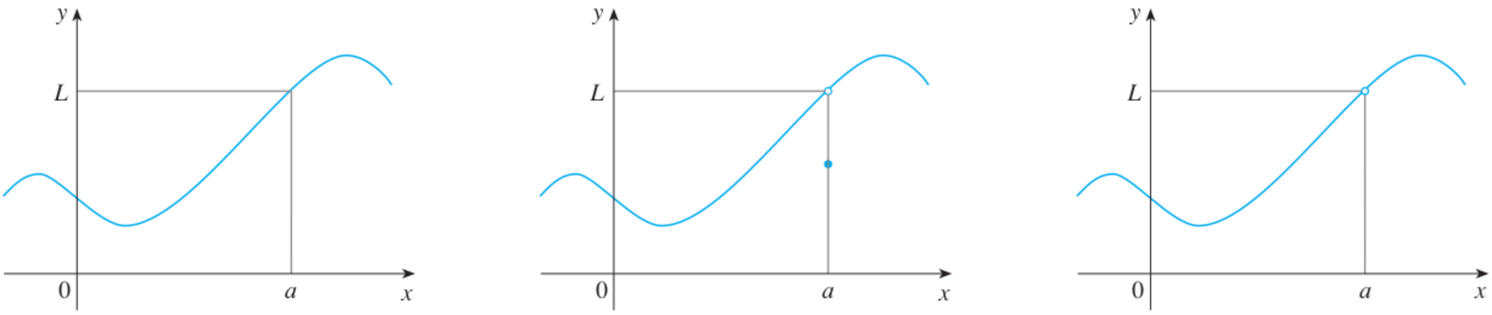
$$\lim_{x \to a} f(x) = L$$

and say "the limit of $f(x)$, as x approaches a, equals L"

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by restricting x to be sufficiently close to a (on either side of a) but not equal to a.

Notations:

Three cases:



(a) (b) (c)

EXAMPLE 1 Guess the value of $\lim_{x \to 1} \dfrac{x - 1}{x^2 - 1}$.

EXAMPLE 2 Estimate the value of $\displaystyle\lim_{t \to 0} \frac{\sqrt{t^2 + 9} - 3}{t^2}$.

EXAMPLE 3 Guess the value of $\displaystyle\lim_{x \to 0} \frac{\sin x}{x}$.

EXAMPLE 4 Investigate $\lim_{x \to 0} \sin \frac{\pi}{x}$.

One-sided Limits.

EXAMPLE 6 The Heaviside function H is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geqslant 0 \end{cases}$$

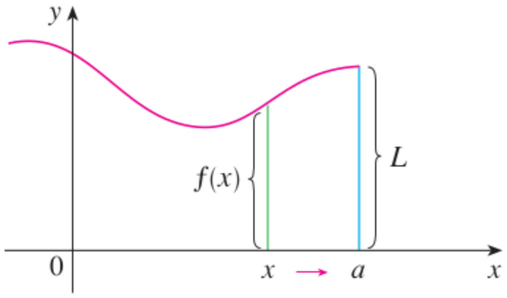
What is the limit when t approached 0 from the right and when t approaches 0 from the left.

Left-hand limits.

2 Definition of One-Sided Limits We write

$$\lim_{x \to a^-} f(x) = L$$

and say the **left-hand limit of $f(x)$ as x approaches a** [or the **limit of $f(x)$ as x approaches a from the left**] is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a with x *less than* a.



(a) $\lim_{x \to a^-} f(x) = L$

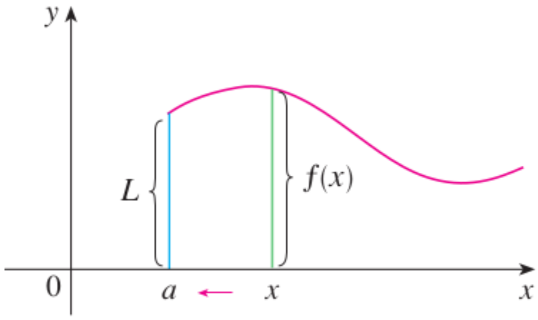
Right-hand limits.

2 Definition of One-Sided Limits We write

$$\lim_{x \to a} f(x) = L$$

and say the ____

__________________________ is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a with x ______________.



(b) $\lim_{x \to a^+} f(x) = L$

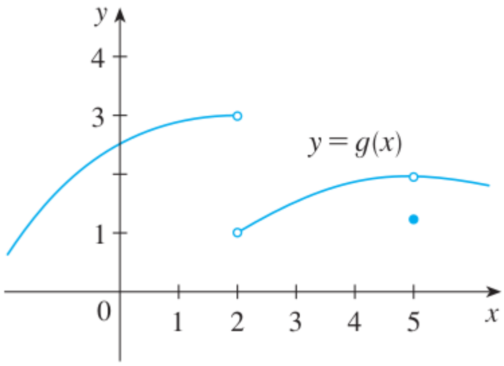
Fundamental Property:

3 $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$

EXAMPLE 7 The graph of a function g is shown in Figure 10. Use it to state the values (if they exist) of the following:

(a) $\lim_{x \to 2^-} g(x)$ (b) $\lim_{x \to 2^+} g(x)$ (c) $\lim_{x \to 2} g(x)$

(d) $\lim_{x \to 5^-} g(x)$ (e) $\lim_{x \to 5^+} g(x)$ (f) $\lim_{x \to 5} g(x)$



Infinite limits.

EXAMPLE 8 Find $\lim_{x \to 0} \frac{1}{x^2}$ if it exists.

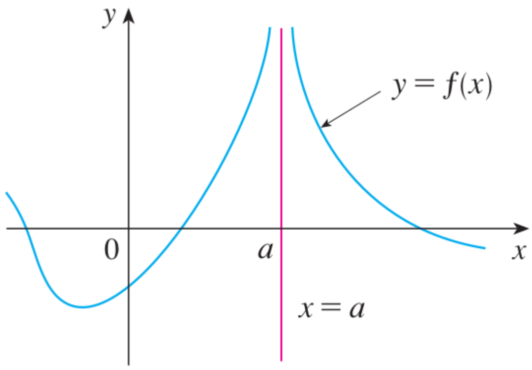
Example 8 1/2. Find, if it exists, $\lim_{x \to \infty} \left(-\frac{1}{x^2}\right)$.

Positive infinity.

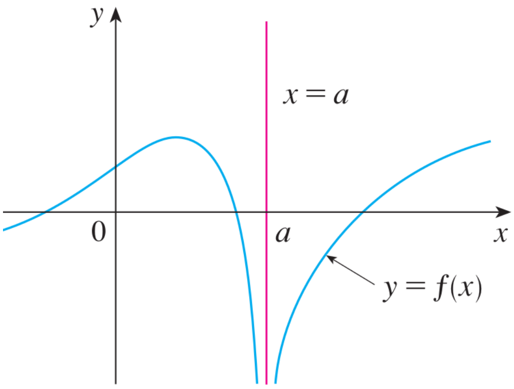
4 Intuitive Definition of an Infinite Limit Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a, but not equal to a.



Negative Infinity



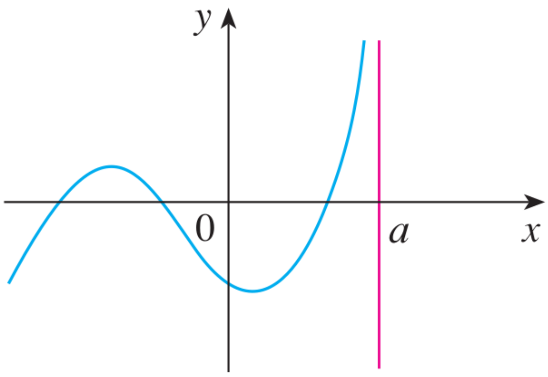
5 Definition Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = -\infty$$

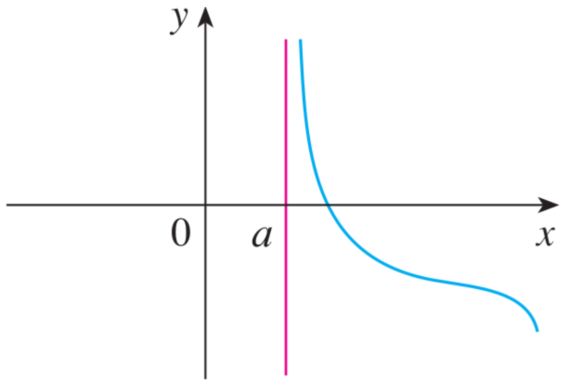
means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a, but not equal to a.

Other types of infinite limits.

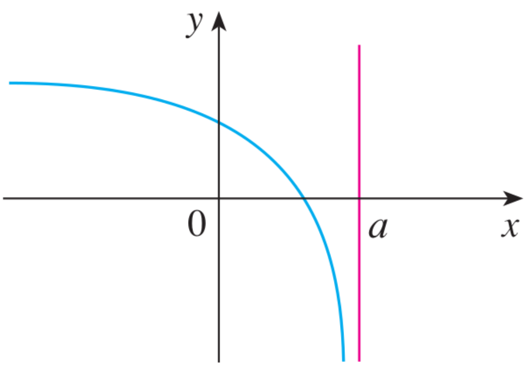
a)



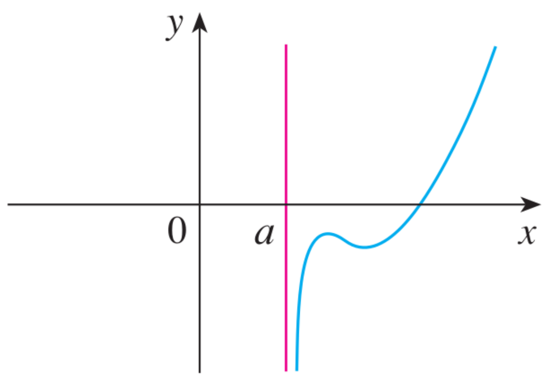
b)



c)



d)



EXAMPLE 9 Find $\lim_{x \to 3^+} \frac{2x}{x - 3}$ and $\lim_{x \to 3^-} \frac{2x}{x - 3}$.

$\boxed{6}$ **Definition** The vertical line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x \to a} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

EXAMPLE 10 Find the vertical asymptotes of $f(x) = \tan x$.