Math 644

Chapter 5

Section 5.2: Winding Number

Contents

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LEMMA 1. If γ is a cycle and $a \notin \gamma$, then

$$\frac{1}{2\pi i} \int_{\gamma} \frac{1}{\zeta - a} \, d\zeta$$

is an integer.

Proof.

DEFINITION 2. If γ is a cycle, then the **index** or **winding number** of γ about a is

$$n(\gamma, a) := \frac{1}{2\pi i} \int_\gamma \frac{1}{\zeta - a}\, d\zeta \quad (a \notin \gamma).$$

PROPOSITION 3. Let γ be a cycle.

(a) $n(\gamma, a)$ is an analytic function of a, for $a \notin \gamma$.

(b) $n(\gamma, a)$ is constant in each component of $\mathbb{C}\backslash\gamma$.

(c) $n(\gamma, a) \to 0$ as $a \to \infty$. In particular, $n(\gamma, a) = 0$ for any a in the unbounded component of $\mathbb{C}\backslash\gamma$.

(d) $n(-\gamma, a) = -n(\gamma, a)$.

(e) $n(\gamma_1 + \gamma_2, a) = n(\gamma_1, a) + n(\gamma_2, a)$.

Proof.

Some Intuition:

① **Difference in the argument.**

② **Rays and number of connected components.**

Definition 4. Closed curves γ_1 and γ_2 are **homologous** in a region Ω if $n(\gamma_1 - \gamma_2, a) = 0$ for all $a \notin \Omega$ and we write $\gamma_1 \sim \gamma_2$.

Remarks:

- Homology is an equivalence relation on curves in Ω.
- A closed curve is said to be **homologous to 0** if $n(\gamma, a) = 0$ for all $a \notin \Omega$. In this case, we write $\gamma \sim 0$.

Example 5. Show that $\gamma_1(t) = r_1 e^{it}$ and $\gamma_2(t) = r_2 e^{it}$ $(0 \leq t \leq 2\pi)$ are homologous in $\Omega := \{z : |z| < R\}$, where $r_1 < r_2 < R$.

Definition 6. Let Ω be a bounded region in $\mathbb{C}$ bounded by finitely many piecewise continuously differentiable simple closed curves. The **positive orientation** of $\partial\Omega$ is a parametrization that has the following property:

(a) for each $t \in [0, 1]$ where the derivative exists, there is an $\varepsilon(t) > 0$ such that $\gamma(t) + ui\gamma'(t) \in \Omega$, for all $u \in [0, \varepsilon(t)]$.

Notes:

① When the positive orientation is chosen for $\partial\Omega$, then

- $n(\partial\Omega, a) = 0$, for each $a \notin \overline{\Omega}$;
- $n(\partial\Omega, a) = 1$, for each $a \in \Omega$.

Example 7. Find the positive orientation of the boundary of the closed annulus $A := \{z : r_1 \leq |z| \leq r_2\}$.

Definition 8.

(a) A region $\Omega \subset \mathbb{C}^*$ is called **simply-connected** if $\mathbb{C}^* \backslash \Omega$ is connected.

(b) Equivalently, a region Ω is simply-connected if $\mathbb{S}^2 \backslash \pi(\Omega)$ is connected, where π is the stereographic projection.

Example 9. Show that

(a) the unit disk is simply connected;

(b) the vectical strip $\Omega = \{z : 0 < \operatorname{Re} z < 1\}$ is simply connected;

(c) $\mathbb{C} \backslash \{0\}$ is not simply connected.

THEOREM 10.

(a) A region $\Omega \subset \mathbb{C}$ is simply-connected if and only if every cycle in Ω is homologous to 0 in Ω.

(b) If Ω is not simply-connected then we can find a simple closed polygonal curve contained in Ω which is not homologous to 0.

Proof.

Corollary 11. Suppose f is analytic on a simply-connected region Ω. Then

(a) $\displaystyle\int_\gamma f(z)dz = 0$ for all closed curves $\gamma \subset \Omega$;

(b) there exists a function F analytic on Ω such that $F' = f$;

(c) if also $f(z) \neq 0$ for all $z \in \Omega$, then there exists a function g analytic on Ω such that $f = e^g$.

Proof.

① **"Uniqueness" in (b).**

② **"Uniqueness" in (c).**

Definition 12. If g is analytic in a region Ω and if $f = e^g$ then g is called a **logarithm** of f in Ω and is written $g(z) = \log f(z)$. The function g is uniquely determined by its value at one point $z_0 \in \Omega$.

Notes:

① f has countably many logarithms, which differ by $2\pi k i$. To specify $\log f(z)$ uniquely, we have to specify its value at one point $z_0 \in \Omega$.

② We do not claim that we can define a logarithm on $f(\Omega)$ and then composed with f to obtain $\log f(z)$.

Example 13. Consider the function $z \mapsto (z-1)/(z+1)$, for $z \in \mathbb{D}$.