Math 311

Chapter 6

Section 6.1: Vector Spaces

Contents

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Column Vectors

Recall that

$$\mathbb{R}^n = \{\mathbf{x} \,:\, \mathbf{x} \text{ is an } n \times 1 \text{ vector}\}.$$

① For addition:

$\boxed{\text{A1.}}$ $\vec{x}, \vec{y} \Rightarrow \vec{x} + \vec{y} \in \mathbb{R}^n$.

$\boxed{\text{A2.}}$ $\vec{x} + \vec{y} = \vec{y} + \vec{x}$ (Commutativity).

$\boxed{\text{A3.}}$ $(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$ (Assoc.)

$\boxed{\text{A4.}}$ $\vec{x} + \vec{0} = \vec{x} = \vec{0} + \vec{x}$

$\boxed{\text{A5.}}$ For any $\vec{x}$, there is a $\vec{y}$ s.t.
$\vec{x} + \vec{y} = \vec{y} + \vec{x} = \vec{0}$ (here $\vec{y} = -\vec{x}$)

② For scalar multiplication:

$\boxed{\text{S1.}}$ $\vec{x}$ and $a \in \mathbb{R} \Rightarrow a\vec{x} \in \mathbb{R}^n$.

$\boxed{\text{S2.}}$ $a(\vec{x} + \vec{y}) = a\vec{x} + a\vec{y}$.

$\boxed{\text{S3.}}$ $(a+b)\vec{x} = a\vec{x} + b\vec{x}$

$\boxed{\text{S4.}}$ $a(b\vec{x}) = (ab)\vec{x}$

$\boxed{\text{S5.}}$ $1\vec{x} = \vec{x}$

<u>Conclusion:</u> $\mathbb{R}^n$ is a vector space.

General Definition

Let V be a set of objects called **vectors**. Assume

1. **Vector Addition:** Two vectors $\mathbf{v}$ and $\mathbf{w}$ can be added and denote this operation by $\mathbf{v} + \mathbf{w}$.
2. **Scalar Multiplication:** Any vector $\mathbf{v}$ can be multiplied by any number (scalar) a and denote this operation by $a\mathbf{v}$.

The set V is called a **vector space** if

1. Axioms for the vector addition:

   $\boxed{\text{A1.}}$ <u>Closed:</u> $\mathbf{v}, \mathbf{w} \in V \Rightarrow \mathbf{v} + \mathbf{w} \in V$.

   $\boxed{\text{A2.}}$ <u>Commutativity:</u> $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$.

   $\boxed{\text{A3.}}$ <u>Associativity:</u> $\mathbf{v} + (\mathbf{w} + \mathbf{z}) = (\mathbf{v} + \mathbf{w}) + \mathbf{z}$.

   $\boxed{\text{A4.}}$ <u>Existence of a zero vector:</u> $\mathbf{v} + \mathbf{0} = \mathbf{v} = \mathbf{0} + \mathbf{v}$.

   $\boxed{\text{A5.}}$ <u>Existence of a negative:</u> For each $\mathbf{v}$, there is a $\mathbf{w}$ such that $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v} = \mathbf{0}$.

2. Axioms for the scalar multiplication:

   $\boxed{\text{S1.}}$ $\mathbf{v} \in V \Rightarrow a\mathbf{v} \in V$.

   $\boxed{\text{S2.}}$ $a(\mathbf{v} + \mathbf{w}) = a\mathbf{v} + a\mathbf{w}$.

   $\boxed{\text{S3.}}$ $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.

   $\boxed{\text{S4.}}$ $a(b\mathbf{v}) = (ab)\mathbf{v}$.

   $\boxed{\text{S5.}}$ $1\mathbf{v} = \mathbf{v}$.

Spaces of Matrices

m, n are fixed

EXAMPLE 1. Let $\mathbf{M_{mn}}$ be the set of all $m \times n$ matrices, that is

$$\mathbf{M_{mn}} := \{A : A \text{ is an } m \times n \text{ matrix.}\}$$

Consider the addition and scalar multiplication for matrices. Show that $\mathbf{M_{mn}}$ is a vector space.

SOLUTION.

M_{mn} is a vector space with addition and scalar multiplication as defined in Chapter 2.

Spaces of Polynomials

EXAMPLE 2. Consider the space $\mathbf{P_3}$ of all polynomials of degree at most 3, that is

$$\mathbf{P} := \{a_3x^3 + a_2x^2 + a_1x + a_0 \ : \ a_i \in \mathbb{R}\}.$$

Define 0. $p(x) = q(x)$ iff p, q have same coef.

1. Addition: for two polynomials $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ and $q(x) = b_3x^3 + b_2x^2 + b_1x + b_0$, define $p + q$ as the polynomial

$$\begin{aligned}(p+q)(x) &= p(x) + q(x) \\ &= (a_3+b_3)x^3 + (a_2+b_2)x^2 + (a_1+b_1)x + (a_0+b_0).\end{aligned}$$

2. Scalar multiplicatoin: for a polynomial $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$, define ap as the polynomial

$$(ap)(x) = ap(x) = (aa_3)x^3 + (aa_2)x^2 + (aa_1)x + (aa_0).$$

Show that $\mathbf{P_3}$, with this addition and scalar multiplication, is a vector space.

SOLUTION.

① Addition:

A1 $p+q$ has degree 3 by definition.

So, $p+q \in P_3$.

A2

$$(p+q)(x) = (a_3+b_3)x^3 + (a_2+b_2)x^2 + (a_1+b_1)x + (a_0+b_0)$$

$$= a_3x^3 + b_3x^3 + a_2x^2 + b_2x^2 + a_1x + b_1x + a_0 + b_0$$

$$(q+p)(x) = (b_3+a_3)x^3 + (b_2+a_2)x^2 + (b_1+a_1)x + (b_0+a_0)$$

$$= b_3x^3 + a_3x^3 + b_2x^2 + a_2x^2 + b_1x + a_1x + b_0 + a_0$$

The expressions a_ix^i and b_jx^j are $\mathbb{R}$ numbers, so $\mathbb{R}$ number commute

$$\Rightarrow (p+q)(x) = (q+p)(x).$$

A3 Let $r(x) = c_3x^3 + c_2x^2 + c_1x + c_0$.

$$(p+q)+r = \left((a_3+b_3)x^3 + (a_2+b_2)x^2 + (a_1+b_1)x + (a_0+b_0)\right) + r$$

$$= ((a_3+b_3)+c_3)x^3 + ((a_2+b_2)+c_2)x^2$$
$$+ ((a_1+b_1)+c_1)x + ((a_0+b_0)+c_0)$$

$$= (a_3+(b_3+c_3))x^3 + (a_2+(b_2+c_2))x^2$$
$$+ (a_1+(b_1+c_1))x + (a_0+(b_0+c_0))$$

$$= p + (q+r). \quad \checkmark$$

<u>A4</u>. Define: $0(x) = 0x^3 + 0x^2 + 0x + 0 = 0$.

$$\Rightarrow (p+0) = (a_3+0)x^3 + (a_2+0)x^2$$
$$+ (a_1+0)x + (a_0+0)$$

$$= a_3x^3 + a_2x^2 + a_1x + a_0.$$

$$= p. \quad \checkmark$$

<u>A5</u>. Define $q(x) = (-a_3)x^3 + (-a_2)x^2 + (-a_1)x + (-a_0)$

$$\Rightarrow p+q = (a_3-a_3)x^3 + (a_2-a_2)x^2 + (a_1-a_1)x$$
$$+ (a_0-a_0)$$

$$= 0x^3 + 0x^2 + 0x + 0 = 0 .$$

<u>Scalar Multiplication</u>. S3–S5 are verified also.

<u>S1</u>. By definition, ap is a polynomial of degree at most 3.

<u>S2</u>. $$a(p+q) \stackrel{?}{=} ap + aq$$

$$a(p+q) = a\left((a_3+b_3)x^3 + (a_2+b_2)x^2 + (a_1+b_1)x + (a_0+b_0)\right)$$

$$= a(a_3+b_3)x^3 + a(a_2+b_2)x^2 + a(a_1+b_1)x + a(a_0+b_0)$$

$$= a(a_3)x^3 + aa_2x^2 + aa_1x + aa_0 + ab_3x^3 + ab_2x^2 + ab_1x + ab_0 = ap+aq \quad \checkmark$$

<u>Note:</u>

① The space of polynomial of degree at most n is denoted by $\mathbf{P_n}$ and is a vector space using the addition and scalar multiplication introduced above.

② The space of all polynomial of any degree is denoted by $\mathbf{P}$ and it is a vector space using the addition and scalar multiplication introduced above.

Weird Example

EXAMPLE 3. Consider the set of all 2×1 vectors $\mathbb{R}^2$. Define the addition and scalar multiplication:

1. $\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 + 1 \end{bmatrix}$.
2. $a\mathbf{x} = a \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_1 \\ ax_2 + a - 1 \end{bmatrix}$.

Show that $\mathbb{R}^2$, with these operations, is a vector space.

SOLUTION.

① Addition

A1. Since $\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 + 1 \end{bmatrix}$ is a 2×1 vector,

$\vec{x} + \vec{y} \in \mathbb{R}^2$. ✓

A2. $\vec{x} + \vec{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 + 1 \end{bmatrix}$

same commutativity of $\mathbb{R}$ numbers.

$\vec{y} + \vec{x} = \begin{bmatrix} y_1 + x_1 \\ y_2 + x_2 + 1 \end{bmatrix}$

A3. $(\vec{x} + \vec{y}) + \vec{z} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 + 1 \end{bmatrix} + \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$

$$= \begin{bmatrix} x_1 + y_1 + z_1 \\ x_2 + y_2 + 1 + z_2 + 1 \end{bmatrix}$$

Non-Example

EXAMPLE 4. Consider the set of all 2×1 vectors $\mathbb{R}^2$. Define the addition and scalar multiplication:

1. $\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 + 1 \end{bmatrix}$.
2. $a\mathbf{x} = a \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_1 \\ ax_2 - 1 \end{bmatrix}$.

Show that $\mathbb{R}^2$, with these operations, is not a vector space.

SOLUTION.

Properties

Consider a general vector space V.

① Cancellation: If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, then

$$\mathbf{v} + \mathbf{u} = \mathbf{v} + \mathbf{w} \Longrightarrow \mathbf{u} = \mathbf{w}.$$

② Multiplying by scalar 0:

$$0\mathbf{v} = \mathbf{0}.$$

③ Multiplying by the zero vector:

$$a\mathbf{0} = \mathbf{0}.$$

④ If $a\mathbf{v}\mathbf{0}$, then $a = 0$ or $\mathbf{v} = \mathbf{0}$.

Example 5. Simplify the following expression:

$$3\big(2(\mathbf{u} - 2\mathbf{v} - \mathbf{w}) + 3(\mathbf{w} - \mathbf{v}\big) - 7(\mathbf{u} - 3\mathbf{v} - \mathbf{w}).$$

Solution.