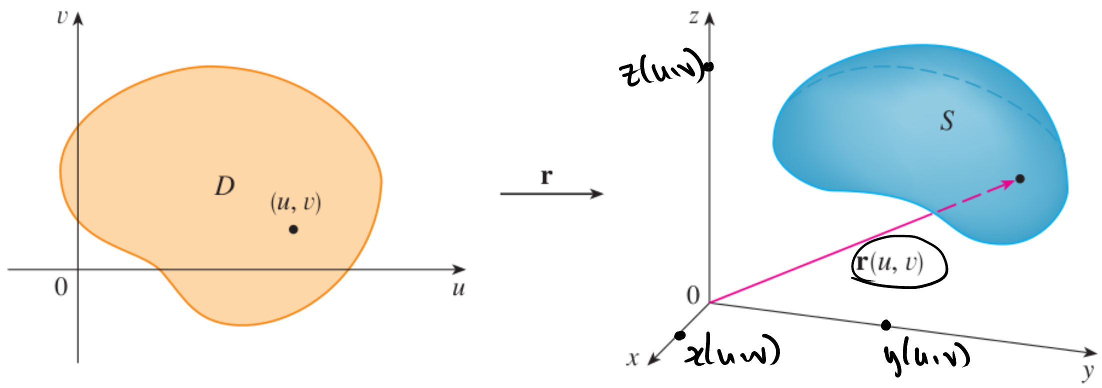
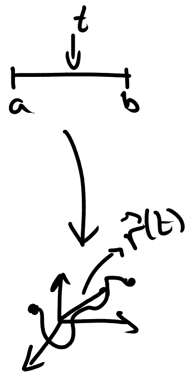
16.6 Parametric surfaces and Their Areas.





Vector expression.

Need three fcts $x, y, z : D \to \mathbb{R}$ (2D region)

$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$

$\vec{r}(u) = \langle x(u), y(u), z(u) \rangle$

Parametric equations.

Given by

$x = x(u,v)$

$y = y(u,v)$

$z = z(u,v)$.

EXAMPLE 1 Identify and sketch the surface with vector equation

$$\mathbf{r}(u, v) = 2\cos u\,\mathbf{i} + v\,\mathbf{j} + 2\sin u\,\mathbf{k}$$

$x(u,v) = 2\cos u$, $y(u,v) = v$, $z(u,v) = 2\sin u$.

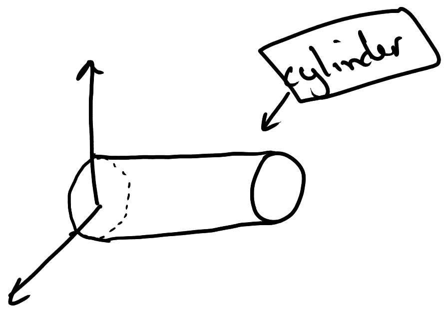
So,

$$x^2 + z^2 = 2^2\cos^2 u + 2^2 \sin^2 u = 4$$

$\Rightarrow$ $x^2 + z^2 = 4$ $\to$ circle.

Now, $y = v$ (no restriction on v)

$\to$ translate the circle $x^2 + y^2 = 4$ along the y-axis.

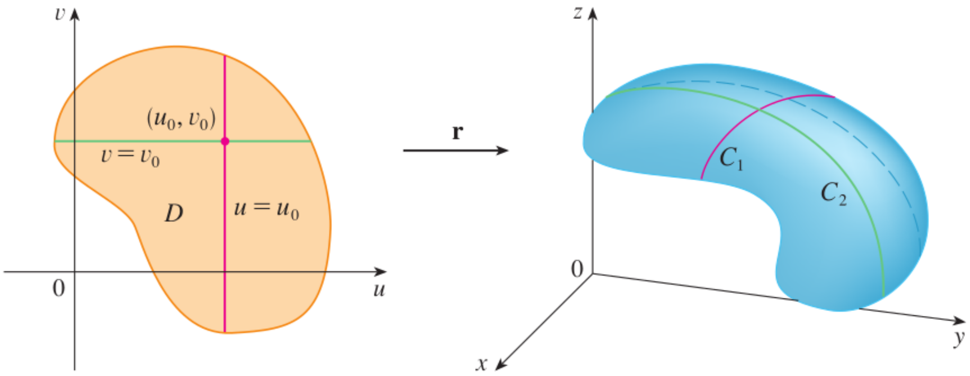


Question: What happen to the surface if we restric one of the parameter?

Fix $u = 0$, then

$\vec{r}(v) \leftarrow \vec{r}(0,v) = \langle x(0,v), y(0,v), z(0,v) \rangle$

Grid curves.



C_1: $\vec{r}(u_0, v)$
$\hookrightarrow$ param. of a curve.

C_2: $\vec{r}(u, v_0)$
$\hookrightarrow$ param. of a curve.

EXAMPLE 2 Use a computer algebra system to graph the surface

$$\mathbf{r}(u, v) = \langle (2 + \sin v) \cos u, (2 + \sin v) \sin u, u + \cos v \rangle$$

Which grid curves have u constant? Which have v constant?

Python: or software

$x = (2+\sin v) \cos u \qquad y = (2+\sin v) \sin u$

$z = u + \cos v$

Grid curves.

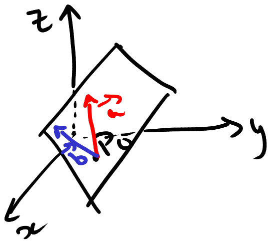
$u = 0 \ \rightarrow \ \vec{r}(0, v) = \langle 2 + \sin v, 0, \cos v \rangle$
$= \langle 2, 0, 0 \rangle + \langle \sin v, 0, \cos v \rangle$

$v = 0 \ \rightarrow \ \vec{r}(u, 0) = \langle 2 \cos u, 2 \sin u, u \rangle$

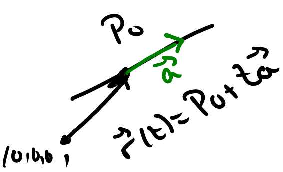
see python's script.

EXAMPLE 3 Find a vector function that represents the plane that passes through the point P_0 with position vector $\mathbf{r}_0$ and that contains two nonparallel vectors $\mathbf{a}$ and $\mathbf{b}$.

$ax + by + cz = d$
$\vec{n} = \langle a, b, c \rangle$
$\vec{n} \cdot (\langle x, y, z \rangle - P_0) = 0$



The points on the plane are obtained by moving along the direction of $\vec{a}$ and $\vec{b}$



1st) Move to P_0.
2nd) Move in the direction $\vec{a}$ &/or $\vec{b}$.

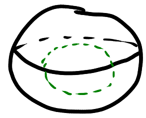
$P_0 = \vec{r}_0$.

$\rightarrow \boxed{\vec{r}(u, v) = \vec{r}_0 + u\vec{a} + v\vec{b}}$

EXAMPLE 4 Find a parametric representation of the sphere

$$x^2 + y^2 + z^2 = a^2$$

$a = 2$
$a = 3$



Recall: $x = \rho \cos\theta \sin\phi \qquad y = \rho \sin\theta \sin\phi \qquad z = \rho \cos\phi$

Fix $\rho = a$, $\quad u = \theta$ & $v = \phi$

$$\Rightarrow \quad \vec{r}(u,v) = \langle a \cos u \sin v,\ a \sin u \sin v,\ a \cos v \rangle$$

$$0 \le u \le 2\pi \qquad 0 \le v \le \pi$$

x, y as parameters

$$\vec{r}_+(x,y) = \langle x, y, \sqrt{a^2 - x^2 - y^2} \rangle$$

EXAMPLE 6 Find a vector function that represents the elliptic paraboloid $z = x^2 + 2y^2$.

Simple sol.

$u = x$
$v = y$ $\quad \to \quad \vec{r}(u,v) = \langle \underset{x}{u}, \underset{y}{v}, \underbrace{u^2 + 2v^2}_{z} \rangle$

More interesting approach.

$u = \rho$ (radius) $\qquad x = \rho\cos\theta$
$v = \theta$ (angle) $\qquad \cancel{y} = \rho \sin\theta \quad \to \quad z = \rho^2 \cos^2\theta + 2\rho^2 \sin^2\theta$.

Instead, $\to y = \frac{\rho}{\sqrt{2}} \sin\theta \quad \to \quad z = \rho^2$

So, $\quad \vec{r}(\rho, \theta) = \langle \rho\cos\theta, \frac{\rho}{\sqrt{2}} \sin\theta, \rho^2 \rangle$. $\qquad \rho \ge 0$, $\quad 0 \le \theta \le 2\pi$

EXAMPLE 7 Find a parametric representation for the surface $z = 2\sqrt{x^2 + y^2}$, that is, the top half of the cone $z^2 = 4x^2 + 4y^2$.

$$x^2 + y^2 - \frac{z^2}{4} = 1$$

$x = \rho\cos\theta \qquad y = \rho\sin\theta \qquad \rho \ge 0$ & $0 \le \theta \le 2\pi$.

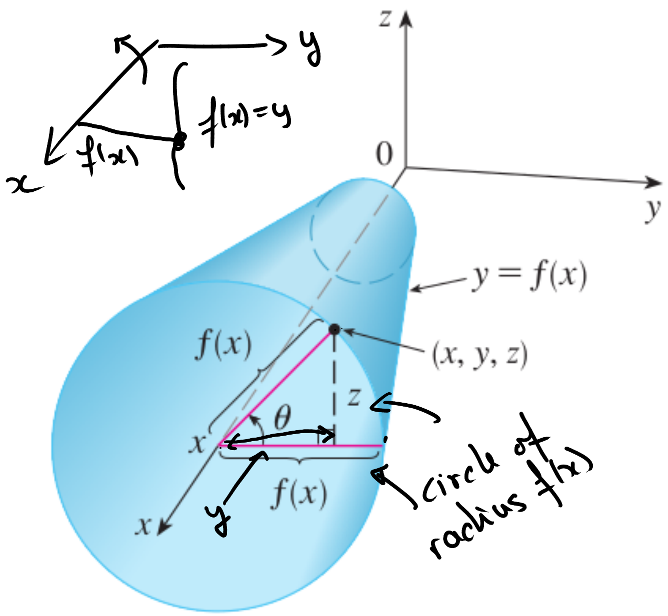
$$\Rightarrow \quad z = 2\sqrt{\rho^2} = 2\rho$$

$$\vec{r}(\underset{u}{\rho}, \underset{v}{\theta}) = \langle \rho\cos\theta, \rho\sin\theta, 2\rho \rangle.$$

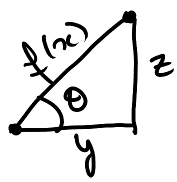
Also,

$$\vec{r}(\underset{u}{x}, \underset{v}{y}) = \langle x, y, 2\sqrt{x^2 + y^2} \rangle.$$

Surfaces of revolution.



Equations. f in one variable. $a \le x \le b$



$$z = f(x)\sin\theta$$
$$y = f(x)\cos\theta$$

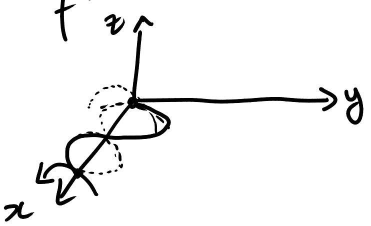
So,

$$\vec{r}(x,\theta) = \langle x,\ f(x)\cos\theta,\ f(x)\sin\theta \rangle$$

$$a \le x \le b \quad \& \quad 0 \le \theta \le 2\pi$$

EXAMPLE 8 Find parametric equations for the surface generated by rotating the curve $y = \sin x$, $0 \leqslant x \leqslant 2\pi$, about the x-axis. Use these equations to graph the surface of revolution.

$f(x) = \sin x \qquad 0 \le x \le 2\pi$

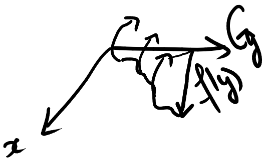


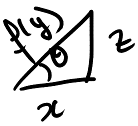
$$\vec{r}(x,\theta) = \langle x,\ \sin x\cos\theta,\ \sin x \sin\theta \rangle$$

(see Python script)

Question: What are the equations of a surface obtained by rotating a function about another axis?

About y-axis. $x = f(y)$

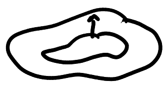




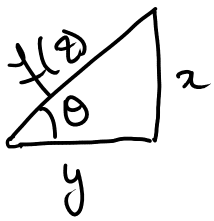
$$z = f(y)\sin\theta$$
$$x = f(y)\cos\theta$$

$$\vec{r}(y,\theta) = \langle f(y)\cos\theta,\ y,\ f(y)\sin\theta \rangle$$

$$a \le y \le b \quad , \quad 0 \le \theta \le 2\pi$$



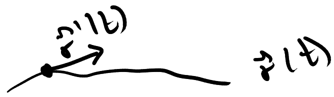
About z-axis. $y = f(z)$.



$$x = f(z)\sin\theta$$
$$y = f(z)\cos\theta$$

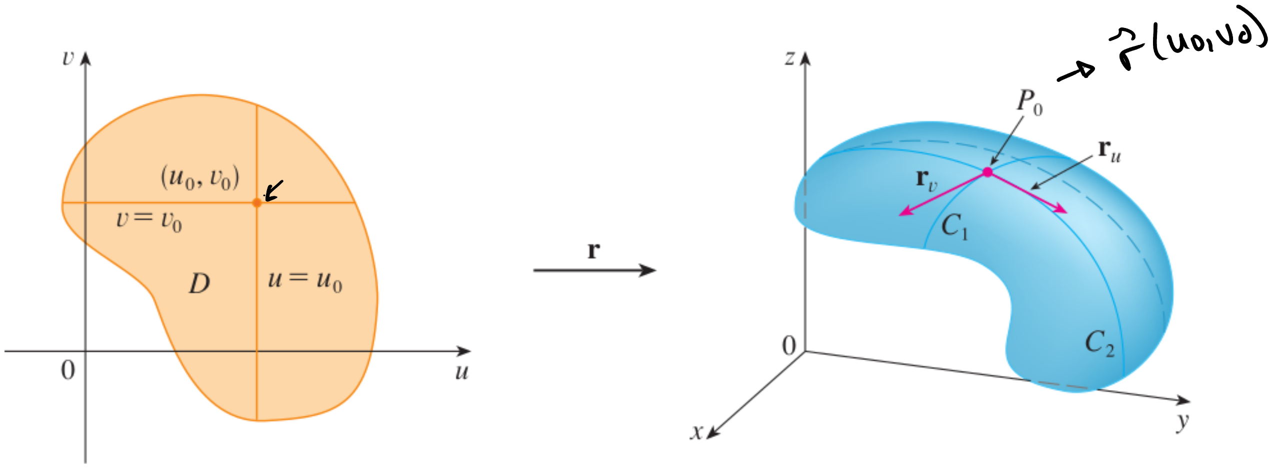
$$\vec{r}(z,\theta) = \langle f(z)\sin\theta,\ f(z)\cos\theta,\ z \rangle$$

$$a \le z \le b \quad \& \quad 0 \le \theta \le 2\pi.$$



Tangent Planes.

Recall. $\vec{r}(u,v) = P_0 + u\vec{a} + v\vec{b}$ (parametrization of a plane).



$\underline{u=u_0}$ $\vec{r}(u_0,v)$ represents a curve C_1. So, $\vec{r}_v(u_0,v_0)$ is the tangent vector at P_0 of C_1.

$\underline{v=v_0}$ $\vec{r}(u,v_0)$ represents a curve C_2. So, $\vec{r}_u(u_0,v_0)$ is the tangent vector at P_0 of C_2.

<u>Eq. tangent plane at P_0</u>: $\vec{r}(u,v) = P_0 + u\,\vec{r}_u(u_0,v_0) + v\,\vec{r}_v(u_0,v_0)$

EXAMPLE 9 Find the tangent plane to the surface with parametric equations $x = u^2$, $y = v^2$, $z = u + 2v$ at the point $\underline{(1, 1, 3)}$. $\to P_0 \to$

① <u>Derivatives.</u>

$$\vec{r}(u,v) = \langle u^2, v^2, u+2v\rangle \longrightarrow \quad \vec{r}_u = \langle 2u, 0, 1\rangle, \quad \vec{r}_v = \langle 0, 2v, 2\rangle$$

② <u>Find u_0, v_0 s.t. $\vec{r}(u_0,v_0) = \langle 1,1,3\rangle$</u>

$$\langle u_0^2, v_0^2, u+2v\rangle = \langle 1,1,3\rangle$$

$$\Rightarrow \quad u_0^2 = 1, \quad v_0^2 = 1, \quad u_0 + 2v_0 = 3$$

$$\Rightarrow \quad u_0 = \pm 1, \quad v_0 = \pm 1, \quad u_0 + 2v_0 = 3$$

$$u_0 = 1 \ \& \ v_0 = 1 \Rightarrow u_0 + 2v_0 = 3 \ \checkmark$$

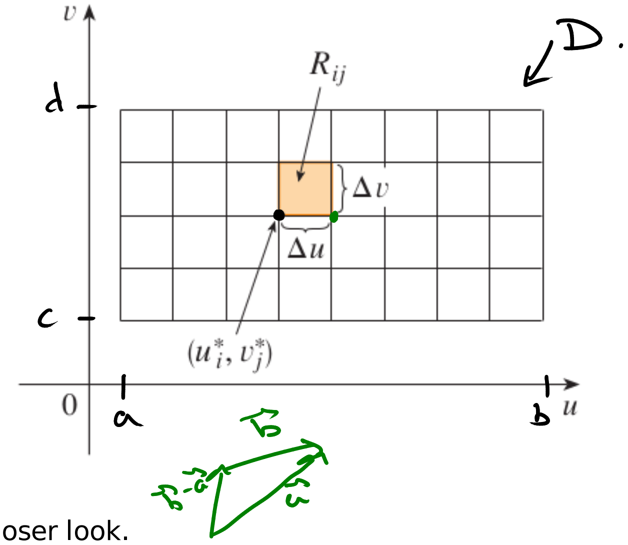
$$\vec{r}_u(u_0,v_0) = \vec{r}_u(1,1) = \langle 2,0,1\rangle$$

$$\vec{r}_v(u_0,v_0) = \vec{r}_v(1,1) = \langle 0,2,2\rangle$$

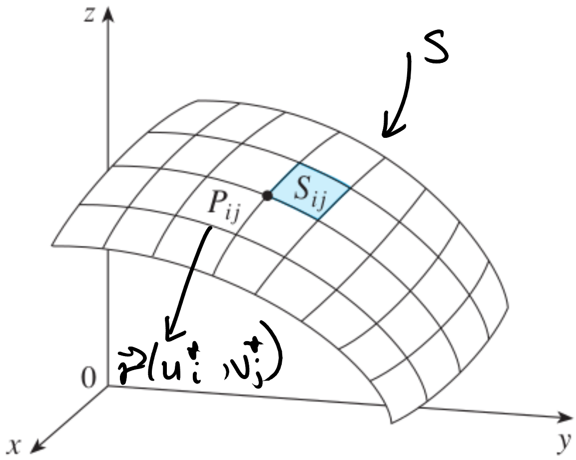
$$\vec{r}_{\pi}(u,v) = P_0 + u\langle 2,0,1\rangle + v\langle 0,2,2\rangle = \boxed{\langle 1+2u,\ 1+2v,\ 3+u+3v\rangle}$$

Surface Area.

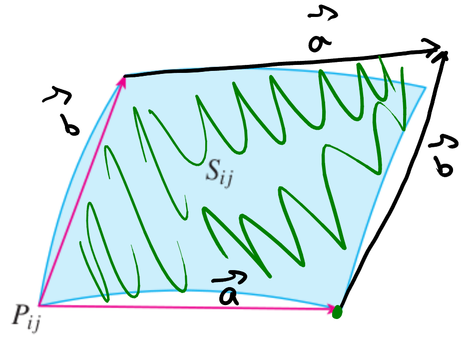
Say S is a surface $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$.
Let D be the domain of the surface.



r $\rightarrow$



Closer look.



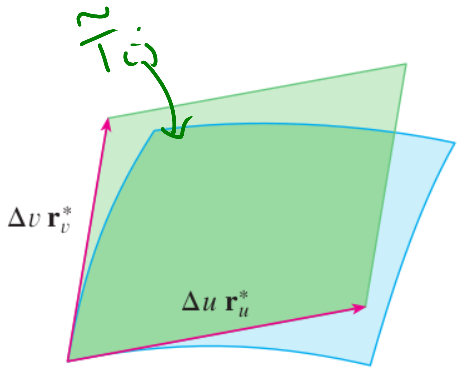
(a)

$S_{ij} \approx$ parallelogram T_{ij} generated by $\vec{a}$ & $\vec{b}$

$\vec{a} = \vec{r}(u_i + \Delta u, v_j) - \vec{r}(u_i, v_j)$

$\vec{b} = \vec{r}(u_i, v_j + \Delta v) - \vec{r}(u_i, v_j)$

$\Rightarrow A(S_{ij}) \approx A(T_{ij})$



(b)

But, $\vec{a} \approx \vec{r}_u(u_0, v_0)\, \Delta u$

$\vec{b} = \vec{r}_v(u_0, v_0)\, \Delta v$

So, $A(T_{ij}) \approx A(\tilde{T}_{ij})$.

But $A(\tilde{T}_{ij}) = |\Delta u\, \vec{r}_u(u_0,v_0) \times \Delta v\, \vec{r}_v(u_0,v_0)|$

$= \Delta u\, \Delta v\, |\vec{r}_u(u_0,v_0) \times \vec{r}_v(u_0,v_0)|$

$A(S) \approx \sum A(S_{ij}) \approx \sum A(\tilde{T}_{ij}) \approx \sum \underbrace{|\vec{r}_u \times \vec{r}_v|}_{f(u,v)}\, \Delta u\, \Delta v$.

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6 Definition If a smooth parametric surface S is given by the equation

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k} \qquad (u, v) \in D$$

$dS = \underbrace{|\vec{r}_u \times \vec{r}_v|}\, dA$

and S is covered just once as (u, v) ranges throughout the parameter domain D, then the **surface area** of S is

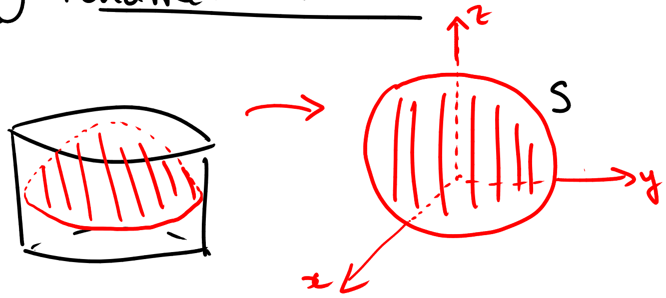
$$\boxed{A(S) = \iint_D \underbrace{|\mathbf{r}_u \times \mathbf{r}_v|}_{f(u,v)}\, dA}$$

where $\quad \mathbf{r}_u = \dfrac{\partial x}{\partial u}\mathbf{i} + \dfrac{\partial y}{\partial u}\mathbf{j} + \dfrac{\partial z}{\partial u}\mathbf{k} \qquad \mathbf{r}_v = \dfrac{\partial x}{\partial v}\mathbf{i} + \dfrac{\partial y}{\partial v}\mathbf{j} + \dfrac{\partial z}{\partial v}\mathbf{k}$

41. The part of the plane $x + 2y + 3z = 1$ that lies inside the cylinder $x^2 + y^2 = 3$

(Find the area)

① Parametrization.



$z = \frac{1}{3} - \frac{x}{3} - \frac{2}{3}y$

w.r.t. $x^2+y^2 \le 3 = (\sqrt{3})^2$

$x = \rho\cos\theta \quad y = \rho\sin\theta$

$\Rightarrow \quad z = \frac{1}{3} - \frac{\rho}{3}\cos\theta - \frac{2}{3}\rho\sin\theta$

with $0 \le \rho \le \sqrt{3}$ & $0 \le \theta \le 2\pi$.

$\vec{r}(\rho,\theta) = \left\langle \rho\cos\theta, \rho\sin\theta, \frac{1}{3} - \frac{\rho}{3}\cos\theta - \frac{2}{3}\rho\sin\theta \right\rangle.$

② Surface Area.

ρ,θ-plane.

ds = jacobian

$A(S) = \iint_D |\vec{r}_u \times \vec{r}_v| \, dA$. $\quad u=\rho$ & $v=\theta$.

$\vec{r}_\rho = \left\langle \cos\theta, \sin\theta, -\frac{\cos\theta}{3} - \frac{2}{3}\sin\theta \right\rangle$

& $\vec{r}_\theta = \left\langle -\rho\sin\theta, \rho\cos\theta, \frac{\rho}{3}\sin\theta - \frac{2}{3}\rho\cos\theta \right\rangle.$

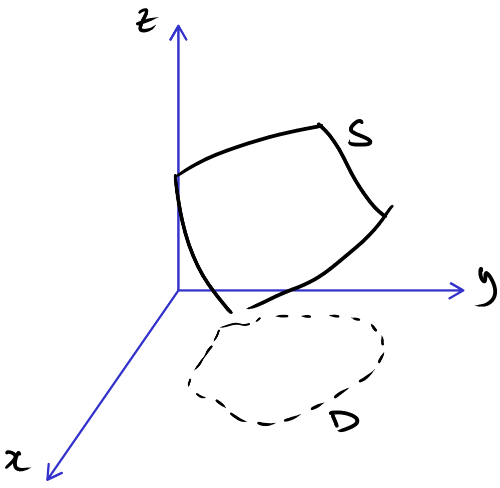
$$\vec{r}_\rho \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & -\frac{\cos\theta}{3} - \frac{2}{3}\sin\theta \\ -\rho\sin\theta & \rho\cos\theta & \frac{\rho}{3}\sin\theta - \frac{2}{3}\rho\cos\theta \end{vmatrix}$$

$$= \left\langle \rho/3, -2/3\rho, \rho \right\rangle$$

$\Rightarrow |\vec{r}_\rho \times \vec{r}_\theta| = \frac{\rho}{3}\sqrt{14}$.

$$A(S) = \int_0^{2\pi}\int_0^{\sqrt{3}} \frac{\rho}{3}\sqrt{14} \, d\rho \, d\theta$$

Surface Area of a Graph of a Function.



$z = g(x,y)$

$\vec{r}(x,y) = \langle x, y, g(x,y) \rangle \qquad (x,y) \in D$

$\vec{r}_x = \langle 1, 0, g_x \rangle$

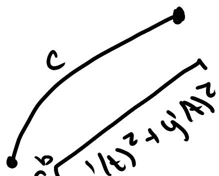
$\vec{r}_y = \langle 0, 1, g_y \rangle$

$\vec{r}_x \times \vec{r}_y = \langle -g_x, -g_y, 1 \rangle$

$\Rightarrow \quad |\vec{r}_x \times \vec{r}_y| = \sqrt{g_x^2 + g_y^2 + 1}$

$g_x = z_x, \quad g_y = z_y$

$\left(\frac{\partial z}{\partial x}\right)^2$

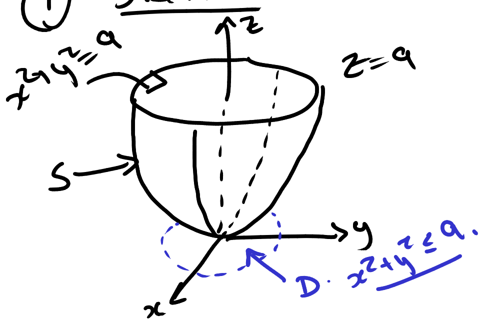


$L(C) = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt$

$$\boxed{A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA}$$

EXAMPLE 11 Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

① Sketch.



② Formula.

$z = x^2 + y^2 \rightarrow \quad z_x = 2x$

$z_y = 2y$

$1 + z_x^2 + z_y^2 = 1 + 4x^2 + 4y^2$

So,

$$A(S) = \iint_{D:\, x^2+y^2 \le 9} \sqrt{1 + 4(x^2+y^2)}\, dA$$

$x = r\cos\theta \qquad 0 \le r \le 3$

$y = r\sin\theta \qquad 0 \le \theta \le 2\pi$

$$= \int_0^{2\pi}\int_0^3 \sqrt{1+4r^2}\, r\, dr\, d\theta$$

$u = 1 + 4r^2$

$du = 8r\, dr$

$$= \frac{\pi}{6}\left(37^{3/2} - 1\right)$$

$$\approx \boxed{117.32}$$