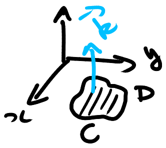
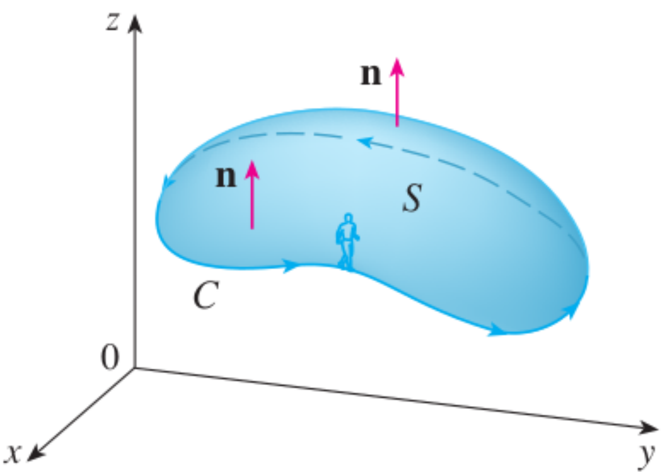
16.8 Stokes' Theorem.

Another story of orientation.



$$\iint_D \operatorname{curl} \vec{F} \cdot \vec{k}\, dA = \int_C \vec{F} \cdot d\vec{r}$$



S: surface with unit normal $\vec{n}$ pointing outward (positive orient.)

C: S induces the positive orientation on C, the boundary of S.

Stokes' Theorem Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\boxed{\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}}$$

<u>Proof</u>. See p. 1175 in textbook. Check wikipedia for a more complete proof.

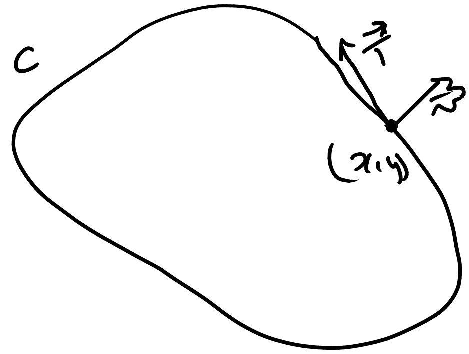
<u>Another Notation</u>.

∂S : C is positive orientation

$$\int_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}.$$

Green's Theorem as a special case of Stokes' Theorem.

Explanation of curl $\vec{F}$.



$\vec{T}$: unit tangent vector at (x,y)

$\vec{v}$: velocity field of a fluid.

C: curve.

$\vec{v} \cdot \vec{T}$ represents the component of $\vec{v}$ in the $\vec{T}$ direction.

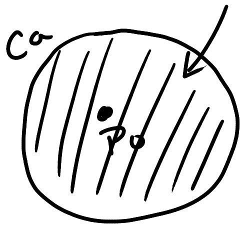
so

- $\vec{v} \cdot \vec{T} > 0 \Rightarrow \vec{v}$ points more in the $\vec{T}$ direction
- $\vec{v} \cdot \vec{T} < 0 \Rightarrow \vec{v}$ points more in the opposite $\vec{T}$ direction.

so, $\int_C \vec{v} \cdot d\vec{r} = \int_C \vec{v} \cdot \vec{T}\, ds$ measures the tendancy for a fluid to move around C.

This quantity is call <u>circulation</u>.

S_a: disk of radius a.



$$\text{curl}\,\vec{v} \approx \text{curl}\,\vec{v}(P_0) \quad \text{on } S_a$$

$$\int_{C_a} \vec{v} \cdot d\vec{r} = \iint_{S_a} \text{curl}\,\vec{F} \cdot \vec{n}\, dS$$

$$\approx \iint_{S_a} \text{curl}\,\vec{F}(P_0) \cdot \vec{n}(P_0) \cdot dS$$

$$= \text{curl}\,\vec{V}(P_0) \cdot \vec{n}(P_0)\, \pi a^2$$

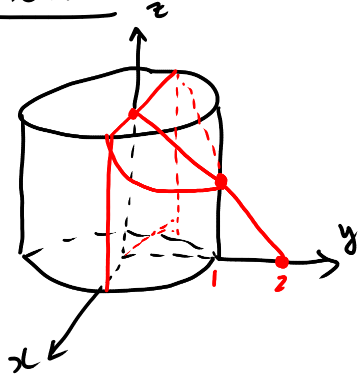
$$\Rightarrow \quad \text{curl}\,\vec{v}(P_0) \cdot \vec{n}(P_0) = \lim_{a \to 0} \frac{\int_{C_a} \vec{v} \cdot d\vec{r}}{\pi a^2}$$

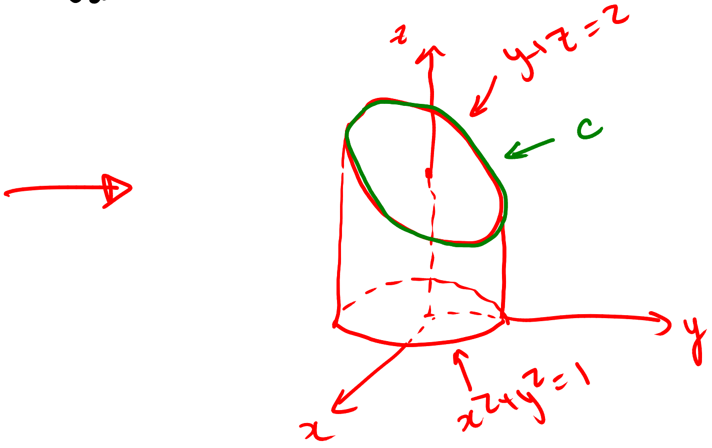
curl $\vec{v}(P_0)$ is tendancy rotation around P_0.

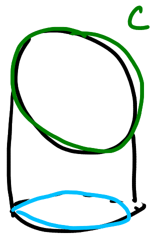
EXAMPLE 1 Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = -y^2\,\mathbf{i} + x\,\mathbf{j} + z^2\,\mathbf{k}$ and C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$. (Orient C to be counterclockwise when viewed from above.)

Recall $\displaystyle\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$

① Picture.





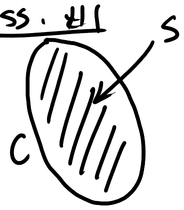


$\vec{r}(\theta) = \langle \cos\theta, \sin\theta, 2 - \sin\theta \rangle$ → Difficult in the integration process.

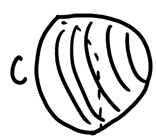
② Stokes' Theorem.

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\, \vec{F} \cdot d\vec{S}.$$

Poss. #1



Poss #2.



half of a sphere of radius 1

We with poss. #1

S surface (Disk).

$\vec{r}(\rho, \theta) = \langle \rho\cos\theta, \rho\sin\theta, 2 - \rho\sin\theta \rangle$

→ $\vec{r}(x, y) = \langle x, y, 2 - y \rangle$

$0 \le \rho \le 1$, $0 \le \theta \le 2\pi$

$0 \le x^2 + y^2 \le 1$

③ Integrate

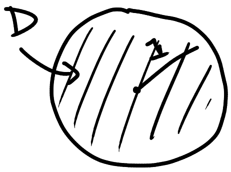
$$\text{curl}\, \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y^2 & x & z^2 \end{vmatrix} = \langle 0, 0, 1 + 2y \rangle$$

$\vec{r}_x = \langle 1, 0, 0 \rangle$ $\qquad \vec{r}_x \times \vec{r}_y = \langle 0, 1, 1 \rangle$

$\vec{r}_y = \langle 0, 1, -1 \rangle$

So, $\displaystyle\iint_S \text{curl}\, \vec{F} \cdot d\vec{S} = \iint_D \langle 0, 0, 1+2y \rangle \cdot \langle 0, 1, 1 \rangle\, dA$

$\displaystyle\qquad = \iint_D 1 + 2y\, dA$



$$= \int_0^1 \int_0^{2\pi} (1 + 2r\sin\theta)\, r\, d\theta\, dr$$

$$= \boxed{\pi}$$

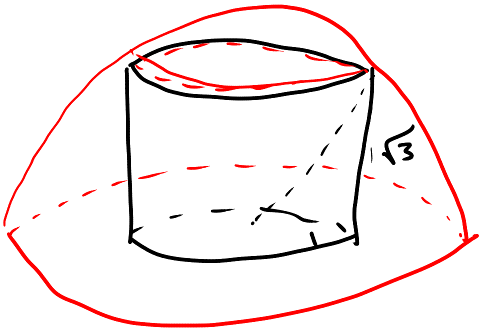
$x = r\cos\theta$
$y = r\sin\theta$

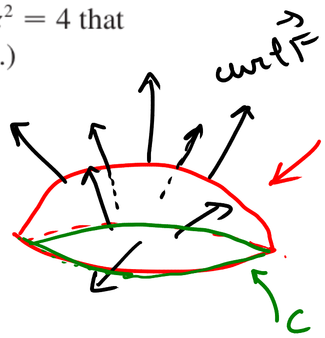
$\vec{r}_\rho = \langle \cos\theta, \sin\theta, -\cos\theta \rangle$

$\vec{r}_\theta = \langle -\rho\sin\theta, \rho\cos\theta, 2 - \rho\cos\theta \rangle$

EXAMPLE 2 Use Stokes' Theorem to compute the integral $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + yz\,\mathbf{j} + xy\,\mathbf{k}$ and S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the xy-plane. (See Figure 4.)

① Picture.



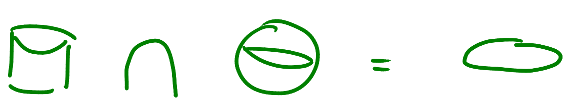


$$\vec{r} = \langle 2\cos\theta\sin\phi,\ 2\sin\theta\sin\phi,\ 2\cos\phi\rangle$$

$$0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi/6$$

② Stokes' Theorem.

$$\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \int_C \vec{F}\cdot d\vec{r}$$

C :  (circle).

$x^2+y^2=1$ $\qquad$ $x^2+y^2+z^2=4$

$$\Rightarrow\ 1+z^2=4 \ \Rightarrow\ z^2=3 \ \Rightarrow\ z=\sqrt{3}$$

So, $C: \vec{r}(\theta) = \langle \cos\theta, \sin\theta, \sqrt{3}\rangle \qquad 0 \le \theta \le 2\pi$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \langle \cos\theta\sqrt{3},\ \sin\theta\sqrt{3},\ \cos\theta\sin\theta\rangle \cdot \langle -\sin\theta,\ \cos\theta,\ 0\rangle\, d\theta$$

$$= \int_0^{2\pi} -\cancel{\sin\theta\cos\theta}\sqrt{3} + \cancel{\sin\theta\cos\theta}\sqrt{3} + 0\, d\theta$$

$$= \boxed{0}$$

$$\Rightarrow \iint_S \operatorname{curl}\vec{F}\, d\vec{S} = 0.$$

Computing surface integrals when the surface is difficult.

Last Example: Computed $\iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$ by knowing only the values of $\vec{F}$ on C.

In math equation: If S_1 & S_2 are two surfaces that share a common boundary C, then

$$\iint_{S_1} \text{curl}\,\vec{F}\cdot d\vec{S} \underset{\text{Stokes}}{=} \int_C \vec{F}\cdot d\vec{r} \underset{\text{Stokes}}{=} \iint_{S_2} \text{curl}\,\vec{F}\cdot d\vec{S}.$$

This means that the integral over a ~~difficult~~ hard surface can be changed to an integral over an easier surface.

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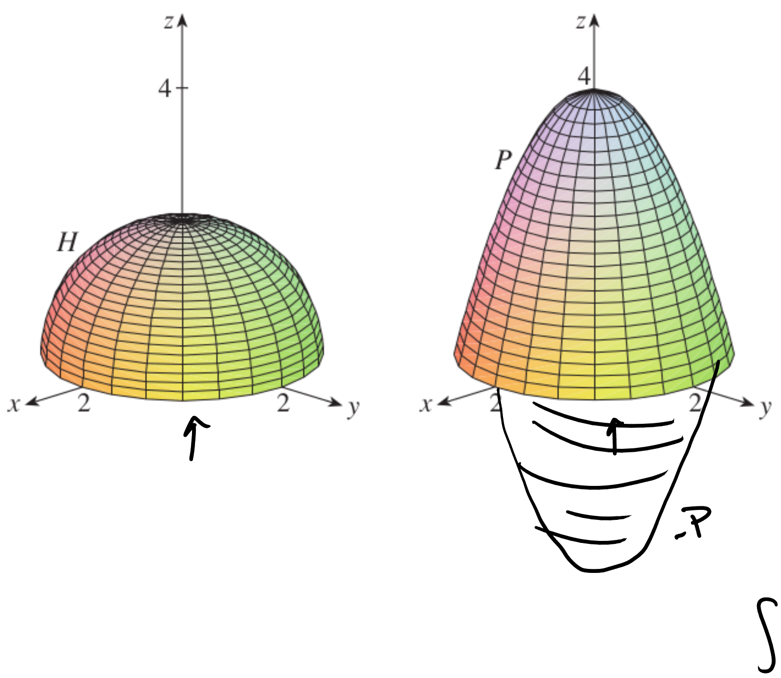
1. A hemisphere H and a portion P of a paraboloid are shown. Suppose $\mathbf{F}$ is a vector field on $\mathbb{R}^3$ whose components have continuous partial derivatives. Explain why

$$\iint_H \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = \iint_P \text{curl}\,\mathbf{F}\cdot d\mathbf{S}$$



boundary of H:

$C: x^2+y^2=4$

boundary of P:

$C: x^2+y^2=4$

H & P share a common boundary

$$\Rightarrow \iint_H \text{curl}\,\vec{F}\cdot d\vec{S} = \iint_P \text{curl}\,\vec{F}\cdot d\vec{S}.$$

$$\iint_{-P} \text{curl}\,\vec{F}\cdot d\vec{S} = -\iint_P \text{curl}\,\vec{F}\cdot d\vec{S}$$