**Section 3.1 — Problem 30 — 5 points**

The derivative of $f$ is

$$f'(x) = 3x^2 + 12x - 15.$$

The critical numbers of $f$ are the numbers $c$ such that $f'(c) = 0$ or $f'(c)$ does not exist. Since $f'(x)$ is a polynomial, then $f'(x)$ always exists. We therefore need only to find the numbers $c$ such that $f'(c) = 0$. We see that

$$f'(x) = 3(x^2 + 4x - 5) = 3(x - 1)(x + 5).$$

Therefore, $f'(c) = 0$ if and only if $c = 1$ or $c = -5$. The critical numbers are

$$c = 1 \text{ and } c = -5.$$

## Section 3.1 — Problem 34 — 5 points

The function is not differentiable at $c = 4/3$ because $g$ has a corner there. Therefore $c = 4/3$ is a critical point.

Now, for $t < 4/3$, we have that $3t - 4 < 0$ and

$$g(t) = -(3t - 4) = 4 - 3t.$$

We have $g'(t) = -3$ and the number $-3$ is never zero. So no critical point for $t < 4/3$.

Now, for $t > 4/3$, we have that $3t - 4 > 0$ and

$$g(t) = 3t - 4.$$

We have $g'(t) = 3$ and the number 3 is never zero. So, no critical point for $t > 4/3$.

In summary, there is only one critical number at $c = 4/3$.

**Section 3.1 — Problem 38 — 5 points**

The derivative of $g$ is

$$g'(x) = -\frac{2}{3}\frac{x}{(4-x^2)^{2/3}}.$$

The derivative does not exist when the denominator is zero. The denominator is zero if

$$4 - x^2 = 0 \iff x = \pm 2.$$

The derivative is zero if the numerator of $g'$ is zero. The numerator is zero if $x = 0$.

Therefore, the critical points are $c = -2$, $c = 0$, and $c = 2$.

**Section 3.1 — Problem 52 — 5 points**

The derivative of $f$ is

$$f'(x) = \frac{(x^2 - x + 1) - x(2x - 1)}{(x^2 - x + 1)^2} = \frac{-x^2 + 1}{(x^2 - x + 1)^2} = -\frac{1 - x^2}{(x^2 - x + 1)^2}.$$

The critical points of $f$ are

- When $f'(x)$ does not exists. The denominator is zero if

  $$x^2 - x + 1 = 0.$$

  The discrimant of this quadratic is

  $$b^2 - 4ac = 1 - 4 = -3.$$

  Since the discrimant is negative, the expression $x^2 - x + 1$ is never zero.

- When $f'(x)$ is zero. The derivative $f'$ is zero if

  $$1 - x^2 = 0 \iff x = \pm 1.$$

The critical points of $f$ are therefore $c = -1$ and $c = 1$.

Using the closed interval method, one critical point is within the interval $[0, 3]$. Therefore, we have

$$\max f(x) = \max\{f(0), f(1), f(3)\} = \max\{0, 1, 3/7\} = 1.$$

**Section 3.2 — Problem 12 — 5 points**

Since $f$ is a polynomial, then it is continuous and differentiable on $[-2, 2]$. Therefore, the hypothesis of the MVP are satisfied. We want to find all solutions $c$ to

$$f'(c) = \frac{f(2) - f(-2)}{2 - (-2)} = \frac{f(2) - f(-2)}{4}.$$

The derivative of $f$ is

$$f'(x) = 3x^2 - 3.$$

Therefore, we look for numbers $c$ such that

$$3c^2 - 3 = \frac{4 - 0}{4} = 1.$$

So, $c$ should be a solution of

$$3c^2 = 4 \iff c = \pm\frac{2}{\sqrt{3}}.$$

The numbers that satisfy the Mean Value Theorem are $c = -2/\sqrt{3}$ and $c = 2/\sqrt{3}$.

**Section 3.2 — Problem 30 — 5 points**

Fix $b > 0$. An odd function on $[-b, b]$ means that $f(-x) = -f(x)$ for any $x$ in $[-b, b]$.

Since $f$ is differentiable, from the Mean Value Theorem, there exists a $c$ in $(-b, b)$ such that

$$f'(c) = \frac{f(b) - f(-b)}{2b}.$$

However, $f(-b) = -f(b)$ and therefore

$$\frac{f(b) - f(-b)}{2b} = \frac{f(b) + f(b)}{2b} = \frac{f(b)}{b}.$$

So, combining everything together, there exists a $c$ in $(-b, b)$ such that

$$f'(c) = \frac{f(b)}{b}.$$

This completes the proof.

**Section 3.3 — Problem 10 — 10 points**

a) The derivative of $f$ is

$$f'(x) = 6x^2 - 18x + 12.$$

We see that

$$f'(x) = 6(x - 3x + 2) = 6(x-2)(x-1).$$

Therefore, the zeros of $f'$ are $x = 2$ and $x = 1$.

- if $x < 1$, then $x < 2$ also. Therefore, $x-1 < 0$ and $x-2 < 0$. The product $(x-2)(x-1)$ is positive, being the product of two negative quantities. So $f'(x) > 0$ when $x < 1$. Hence $f$ is increasing for $x < 1$.
- if $x > 1$ and $x < 2$. Therefore, $x - 1 > 0$ and $x - 2 < 0$. The product $(x-2)(x-1)$ is negative, being the product of a negative quantity by a positive quantity. So $f'(x) < 0$ when $x > 1$ and $x < 2$. Hence $f$ is decreasing for $1 < x < 2$.
- If $x > 2$, then $x > 1$ also. Therefore $x-1 > 0$ and $x-2 > 0$. The product $(x-2)(x-1)$ is positive, being the product of two positive quantities. So $f'(x) > 0$ when $x > 2$. Hence $f$ is increasing for $x > 2$.

b) We know that $f'(x) = 6(x-2)(x-1)$. Therefore, the zeros of the derivative are $x = 1$ and $x = 2$. The derivative exists everywhere.

- $x = 1$. In this case, we see that $f$ is increasing for $x < 1$ and $f$ is decreasing for $1 < x < 2$. Therefore, from the first derivative test, $f$ attains a local maximum at $x = 1$.
- $x = 2$. In this case, we see that $f$ is decreasing for $1 < x < 2$ and increasing for $x > 2$. Therefore, from the first derivative test, $f$ attains a local minimum at $x = 2$.

c) The second derivative of $f$ is

$$f''(x) = 12x - 18 = 6(2x - 3).$$

The zeros of $f''$ are $x = 3/2$.

- When $x < 3/2$, then $2x - 3 < 0$. Therefore, $f''(x) < 0$. This means that $f$ is concave downward.
- When $x > 3/2$, then $2x - 3 > 0$. Therefore, $f''(x) > 0$. This means that $f$ is concave up..

**Section 3.3 — Problem 16 — 10 points**

**With the First Derivative Test.** The derivative of $f$ is

$$f'(x) = \frac{x(x-2)}{(x-1)^2}.$$

The critical numbers of $f$ are $c = 0$, $c = 1$, and $c = 2$.

- $c = 0$.
  - When $x < 0$, then $x - 2 < 0$. Since $(x - 1)$ is squared, then $(x - 1) > 0$. Therefore, we see that $f'(x) > 0$ for $x < 0$ because $f'(x)$ is the product of a two negative quantities and a positive quantity. So $f$ is increasing when $x < 0$.
  - When $0 < x < 1$, then $x - 2 < 0$ and $x - 1 < 0$. But $(x - 1)$ is squared, so $(x - 1)^2 > 0$. Therefore, we see that $f'(x) < 0$ because $f'(x)$ is the product of two positive quantities and a negative quantity. Therefore, $f$ is decreasing if $0 < x < 1$.

  Using the First Derivative Test, we conclude that $c = 0$ is a local maximum of $f$.
- $c = 1$.
  - When $0 < x < 1$, then $x - 2 < 0$ and $x > 0$. Since $(x - 1)^2 > 0$, then $f'(x) < 0$ because it is a product of two positive quantities and a negative quantity. So $f$ is decreasing on $0 < x < 1$.
  - When $1 < x < 2$, then $x - 2 < 0$ and $x > 0$. Since $(x - 1)^2 > 0$, then $f'(x) < 0$ because it is a product of two positive quantities and a negative quantity. So $f$ is decreasing on $1 < x < 2$.

  Since there is no change in the sign of $f'(x)$, we conclude that $c = 1$ is not a local maximum, nor a local minimum.
- $c = 2$.
  - When $1 < x < 2$, then $x - 2 < 0$ and $x > 0$. Since $(x - 1)^2 > 0$, then $f'(x) < 0$ because it is a product of two positive quantities and a negative quantity. So $f$ is decreasing on $1 < x < 2$.
  - When $x > 2$, then $x - 2 > 0$ and $x > 0$. Since $(x - 1)^2 > 0$, then $f'(x) > 0$ because it is the product of three positive quantities. So $f$ is increasing on $x > 2$.

  By the First derivative Test, we conclude that $f$ has a local minimum at $c = 2$.

**With the Second Derivative Test.** From the first part, we know that the critical numbers of $f$ are $c = 0$, $c = 1$, and $c = 2$. The second derivative of $f$ is

$$f''(x) = \frac{2}{(x-1)^3}.$$

- $c = 0$. We have $f''(0) = \frac{2}{(-1)^3} = -2$. Since $f''(0) < 0$, by the Second Derivative Test, we conclude that $c = 0$ is a local maximum.

- $c = 1$. We have $f''(1)$ does not exist. So we can't conclude anything from the Second Derivative Test.
- $c = 2$. We have $f''(2) = \frac{2}{(1)^3} = 2$. Since $f''(2) > 0$, by the Second Derivative Test, we conclude that $c = 2$ is a local minimum.

Remark: I prefer the Second Test Derivative.

TOTAL (POINTS): 50.