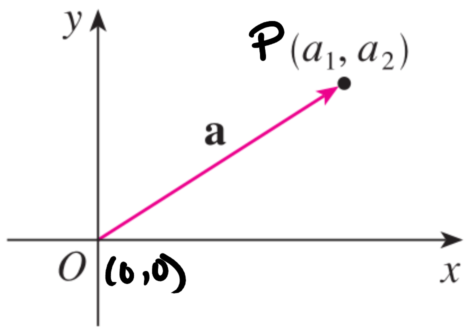
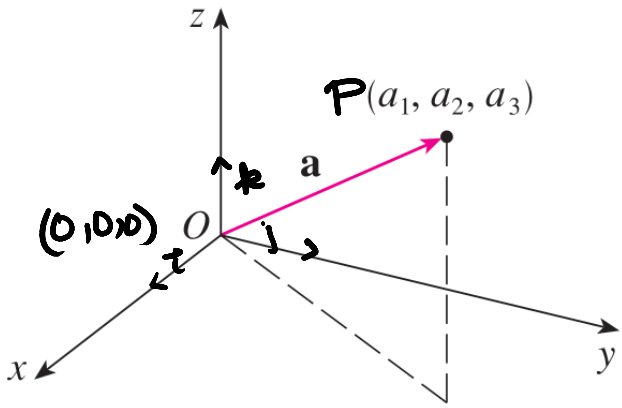
12.5 Equations of Lines and planes.

Vectors.





$\underline{\mathbf{a} = \langle a_1, a_2 \rangle}$

$= P - O$

$\mathbf{a} = \langle a_1, a_2, a_3 \rangle$

$= P - O$

$i = (1,0,0)$, $j = (0,1,0)$, $k = (0,0,1)$.

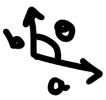
Dot product.

$a = \langle a_1, a_2, a_3 \rangle$

$b = \langle b_1, b_2, b_3 \rangle$

(3D) $a \cdot b = a_1 \cdot b_1 + a_2 \cdot b_2 + a_3 \cdot b_3$

(2D) $a \cdot b = a_1 \cdot b_1 + a_2 \cdot b_2$



Angle: $\cos\theta = \dfrac{a \cdot b}{\|a\| \cdot \|b\|}$

$\|a\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$ $(|a|)$

Cross product (3D)

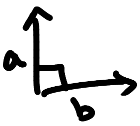
$a = \langle a_1, a_2, a_3 \rangle$

$b = \langle b_1, b_2, b_3 \rangle$

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Direction given by the right-hand rule

Orthogonal vectors.

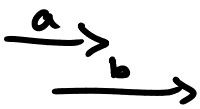


$a \neq 0$, $b \neq 0$

$a \perp b$ if and only if $\cos\theta = 0$

if and only if $a \cdot b = 0$

Parallel vectors.



$a \neq 0$, $b \neq 0$.

$a \parallel b$ if and only if $a \times b = 0$.

$a \parallel b$ if and only if $\cos\theta = 1$

Lines. (3D)

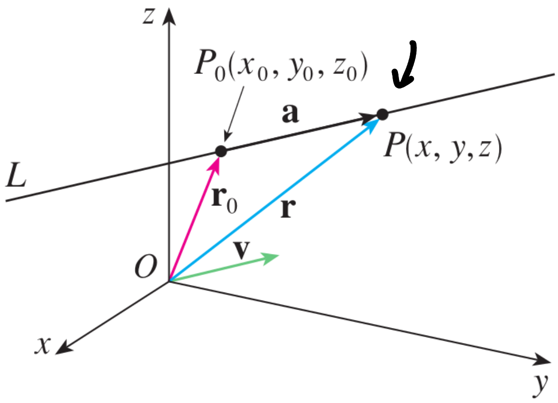
$r = \langle x, y, z \rangle$ (pos. vector)

$r_0 = \langle x_0, y_0, z_0 \rangle$ (Fixed).

$v = \langle a, b, c \rangle$ (director direction line)

$\downarrow$

v parallel to L



Vector equation.

$$\boxed{r = r_0 + tv}$$

$t \in (-\infty, \infty)$

Parametric equation.

$$\boxed{x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct}$$

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle$$

$$= \langle x_0, y_0, z_0 \rangle + \langle ta, tb, tc \rangle$$

$$= \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

Symmetric equations.

$$\boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}}$$

$$t = \frac{x - x_0}{a}$$

$$t = \frac{y - y_0}{b}$$

$$t = \frac{z - z_0}{c}$$

EXAMPLE 1

(a) Find a vector equation and parametric equations for the line that passes through the point (5, 1, 3) and is parallel to the vector $\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$. $\to \langle 1,4,-2\rangle$

(b) Find two other points on the line.

(a) $r = r_0 + tv$ $\qquad r_0 = \langle 5,1,3\rangle$

$v = \langle 1,2,-2\rangle$

$\Rightarrow r = \langle 5,1,3\rangle + t\langle 1,4,-2\rangle \qquad t \in (-\infty,\infty).$

$r = \langle x,y,z\rangle = \langle 5+t,\ 1+4t,\ 3-2t\rangle$

$\Rightarrow x = 5+t,\ y = 1+4t,\ z = 3-2t.$

$$\boxed{\begin{array}{l} r = \langle 5,1,3\rangle + t\langle 1,4,-2\rangle \quad \& \\ x = 5+t,\ y = 1+4t,\ z = 3-2t \end{array}}$$

(b) $t = 1 \to r_1 = \langle 5,1,3\rangle + \langle 1,4,-2\rangle = \langle 6,5,1\rangle$

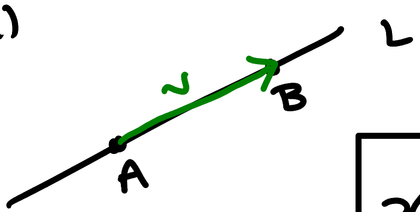
$t = -1 \to r_{-1} = \langle 5,1,3\rangle - \langle 1,2,-2\rangle = \langle 4,-3,5\rangle.$

EXAMPLE 2

(a) Find parametric equations and symmetric equations of the line that passes through the points $A(2, 4, -3)$ and $B(3, -1, 1)$.

(b) At what point does this line intersect the xy-plane?

(a)



$r_0 = \langle 2,4,-3\rangle = \langle x_0,y_0,z_0\rangle$

$v = B - A = \langle 1,-5,4\rangle = \langle a,b,c\rangle$

$$\boxed{\begin{array}{l} x = x_0 + at = 2 + t \\ y = y_0 + bt = 4 - 5t \\ z = z_0 + ct = -3 + 4t \end{array}}$$

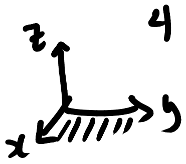
$t = x - 2$

$t = \dfrac{y-4}{-5}$

$t = \dfrac{z+3}{4}$

$\longmapsto \boxed{\dfrac{x-2}{1} = \dfrac{y-4}{-5} = \dfrac{z+3}{4}}$

(b)



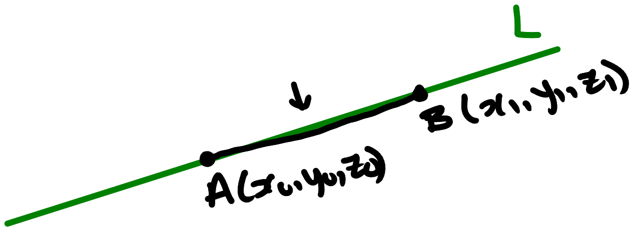
$z = 0$ if $t = \dfrac{3}{4} \to x = 2 + \dfrac{3}{4} = \dfrac{11}{4}$

$y = 4 - \dfrac{15}{4} = \dfrac{1}{4}$

Line segments. $\boxed{r(t) = (1-t)r_0 + tr_1}$

$r_0 = \langle x_0, y_0, z_0 \rangle$

$r_1 = \langle x_1, y_1, z_1 \rangle$



$\forall t \in [0,1]$.

in

$\rightarrow 0 \le t \le 1$

EXAMPLE 3 Show that the lines L_1 and L_2 with parametric equations

$$L_1: \quad x = 1 + t \qquad y = -2 + 3t \qquad z = 4 - t$$
$$L_2: \quad x = 2s \qquad y = 3 + s \qquad z = -3 + 4s$$

are **skew lines**; that is, they do not intersect and are not parallel (and therefore do not lie in the same plane).

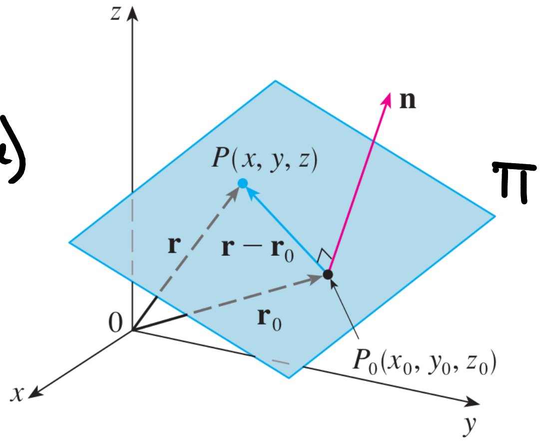
Planes.

$\mathbf{r} = \langle x, y, z \rangle$ (pos. vector)

$\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$ (fixed pt. on the plane)

$\mathbf{r} - \mathbf{r}_0$ is a vector $\parallel$ to π.

$\mathbf{n} = \langle a, b, c \rangle$ (normal vector)



Vector equation.

$$\boxed{\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0}$$

$$\langle a, b, c \rangle \cdot (\langle x - x_0, y - y_0, z - z_0 \rangle) = 0$$

$$\Rightarrow \quad a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$\Downarrow$$

Scalar equation.

$$\boxed{\underset{\substack{\uparrow \\ n}}{a}(x - \underset{\substack{\uparrow \\ r_0}}{x_0}) + \underset{\substack{\uparrow \\ n}}{b}(y - \underset{\substack{\uparrow \\ r_0}}{y_0}) + \underset{\substack{\uparrow \\ n}}{c}(z - \underset{\substack{\uparrow \\ r_0}}{z_0}) = 0}$$

$$ax - ax_0 + by - by_0 + cz - cz_0 = 0$$

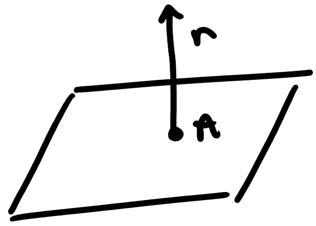
$$* \quad d = -ax_0 - by_0 - cz_0$$

$$\Downarrow$$

Linear equation.

$$\boxed{\underset{\substack{\uparrow \\ a}}{a}x + \underset{\substack{\uparrow \\ b}}{b}y + \underset{\substack{\uparrow \\ c}}{c}z + \underset{\substack{\nwarrow \\ *}}{d} = 0}$$

EXAMPLE 4 Find an equation of the plane through the point $\overbrace{(2, 4, -1)}^{A}$ with normal vector $\mathbf{n} = \langle 2, 3, 4 \rangle$. Find the intercepts and sketch the plane.



① $n = \langle 2, 3, 4 \rangle = \langle a, b, c \rangle$

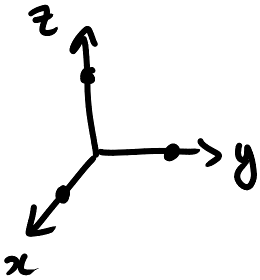
$r_0 = \langle 2, 4, -1 \rangle = \langle x_0, y_0, z_0 \rangle$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$\Rightarrow 2(x-2) + 3(y-4) + 4(z+1) = 0$$

$$\Rightarrow 2x + 3y + 4z - 4 - 12 + 4 = 0$$

$$\Rightarrow 2x + 3y + 4z - 12 = 0.$$



<u>X-axis</u> $(y=0, z=0)$

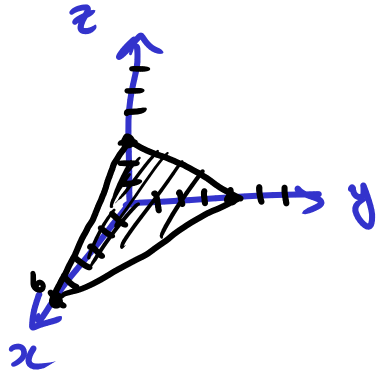
$x = 6$

<u>Y-axis</u> $(x=0, z=0)$

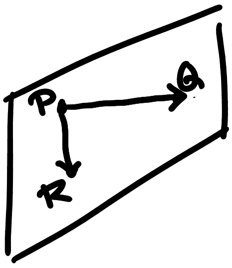
$y = 4$

<u>Z-axis</u> $(x=0, y=0)$

$z = 3$



EXAMPLE 5 Find an equation of the plane that passes through the points $P(1, 3, 2)$, $Q(3, -1, 6)$, and $R(5, 2, 0)$.



$v_1 = Q - P = \langle 2, -4, 4 \rangle$ $\qquad r_0 = (1, 3, 2)$

$v_2 = R - P = \langle 4, -1, -2 \rangle$

$$n = v_1 \times v_2 = \begin{vmatrix} i & j & k \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix}$$

$$= i(12) - j(-20) + k\,14$$

$$= 12i + 20j + 14k$$

$$= \langle 12, 20, 14 \rangle$$

$r = \langle x, y, z \rangle$

$$\Rightarrow n \cdot (r - r_0) = \langle 12, 20, 14 \rangle \cdot (\langle x-1, y-3, z-2 \rangle) = 0.$$

EXAMPLE 6 Find the point at which the line with parametric equations $x = 2 + 3t$, $y = -4t$, $z = 5 + t$ intersects the plane $4x + 5y - 2z = 18$.

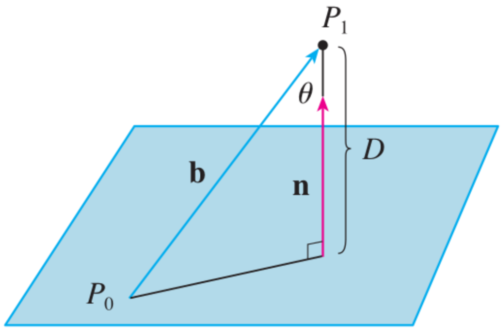
EXAMPLE 7

(a) Find the angle between the planes $x + y + z = 1$ and $x - 2y + 3z = 1$.

(b) Find symmetric equations for the line of intersection L of these two planes.

Distance.

$$\boxed{D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}}$$



EXAMPLE 9 Find the distance between the parallel planes $10x + 2y - 2z = 5$ and $5x + y - z = 1$.