Chapter 2

Derivatives

2.9 Linear Approximations and Differentials.

An observation:

A curve $y = f(x)$ lies very close to its tangent line near the point of tangency. Linearization

https://www.desmos.com/calculator/1sp51krlae

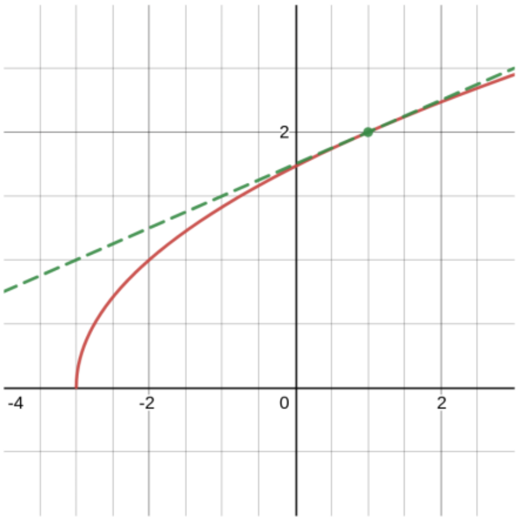


Figure: Linearization near the point of tangency

This suggests to approximate the values of f by the tangent line. This is a really useful procedure because $f(x)$ may be difficult to compute!

Approximation by the tangent line:

$$\boxed{f(x) \approx f(a) + f'(a)(x - a)}$$

↑ approx. at $(a, f(a))$

So the linearization is

$$\boxed{L(x) = f(a) + f'(a)(x - a)}$$

EXAMPLE 1 Find the linearization of the function $f(x) = \sqrt{x + 3}$ at $a = 1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$. ~~Are these approximations over-estimates or underestimates?~~

1) Linearization. $f'(x) = \dfrac{1}{2\sqrt{x+3}} \cdot (1) = \dfrac{1}{2\sqrt{x+3}} \rightarrow f'(1) = \dfrac{1}{4}$

so, since $f(1) = 2 \rightarrow L(x) = 2 + \frac{1}{4}(x - 1) = \frac{1}{4}x + \frac{7}{4}$

2) $\sqrt{3.98} = \sqrt{\underbrace{0.98}_{x} + 3} \approx L(0.98) = \dfrac{0.98}{4} + \dfrac{7}{4} = 1.995.$

3) $\sqrt{4.05} = \sqrt{\underbrace{1.05}_{x} + 3} \approx L(1.05) = \dfrac{1.05}{4} + \dfrac{7}{4} = 2.0125$

Differentials.

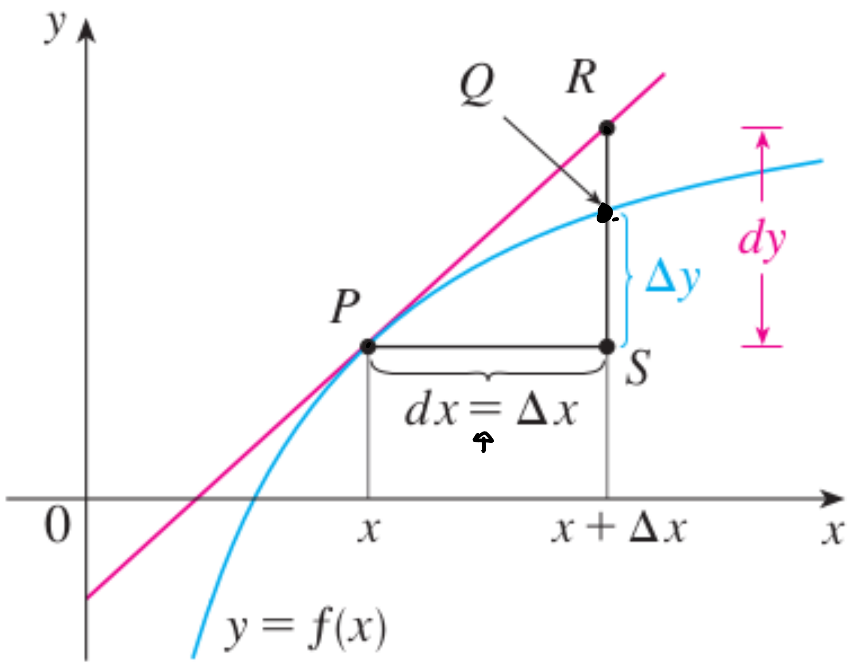
If $y = f(x)$, then

- dx is the differential of x. It's a little increment in the variable x.
- dy is the differential of y and dy is the approximate increment in the variable y given by

$$\Delta y \xrightarrow{\text{approx.}} \boxed{dy = f'(x)dx.}$$

Remark: $\Delta y \approx f'(x)\,dx = dy$ $\qquad dx = \Delta x$

Geometric interpretation.



EXAMPLE 3 Compare the values of Δy and dy if $y = f(x) = x^3 + x^2 - 2x + 1$ and x changes (a) from 2 to 2.05 and (b) from 2 to 2.01.

(a) $f'(x) = 3x^2 + 2x - 2$. so

$$dy = f'(x)\,dx = (3x^2 + 2x - 2)\,dx$$

$x = 2$

$dx = \Delta x = 2.05 - 2 = 0.05$

$$dy = (3(2)^2 + 2\cdot 2 - 2)\,0.05 = 0.7$$

$$\Delta y = f(2.05) - f(2) = 0.717625$$

$$|dy - \Delta y| = 0.017625.$$

(b) $x = 2$

$dx = \Delta x = 0.01$

$$dy = (3\cdot(2)^2 + 2\cdot 2 - 2)\,0.01 = 0.140$$

$$\Delta y = f(2.01) - f(2) = 0.140701$$

$$|dy - \Delta y| = 0.000701$$

EXAMPLE 4 The radius of a sphere was measured and found to be 21 cm with a possible error in measurement of at most 0.05 cm. What is the maximum error in using this value of the radius to compute the volume of the sphere?

$r = 21$ cm
$\Delta r = dr = 0.05$ cm

$$V = \frac{4}{3}\pi r^3$$

$$dV = V'(r)\,dr = (4\pi r^2)\,dr$$

So, $$\Delta V \approx dV = (4\pi(21)^2)(0.05) \approx 277 \text{ cm}^3$$

Relative Error.

$$\frac{|\Delta V|}{|V|} \approx \frac{|dV|}{|V|}$$