Math 644

Chapter 3

Section 3.3: Growth On $\mathbb{C}$ and $\mathbb{D}$

Contents

Created by: Pierre-Olivier Parisé
Spring 2023

A first consequence of the maximum principle is the famous Liouville's Theorem.

Theorem 1. If f is analytic in $\mathbb{C}$ and bounded, then f is constant.

Proof.

Suppose that $|f| \leq M < \infty$.

Let

$$g(z) = \begin{cases} \dfrac{f(z) - f(0)}{z}, & z \neq 0 \\ f'(0), & z = 0 \end{cases}$$

then g is analytic.

If $|z| = R$, then

$$|g(z)| \leq \frac{2M}{R}$$

By the max principle

$$\sup_{z \in B_R} |g(z)| \leq \frac{2M}{R}$$

where $B_R = \{z : |z| < R\}$. Taking $R \to \infty$

$$\sup_{\mathbb{C}} |g(z)| = 0$$

So, $g \equiv 0$ and so $f(z) = f(0)$. $\square$

A second consequence of the maximum principle is the Schwarz's Lemma.

Theorem 2. Suppose f is analytic in $\mathbb{D}$ and suppose $|f(z)| \leq 1$ and $f(0) = 0$. Then

$$|f(z)| \leq |z|, \tag{1}$$

for all $z \in \mathbb{D}$, and

$$|f'(0)| \leq 1. \tag{2}$$

Moreover, if equality holds in (1) for some $z \neq 0$ or if equality holds in (2), then $f(z) = cz$, where c is a constant with $|c| = 1$.

Proof. The function

$$g(z) = \begin{cases} \frac{f(z)}{z}, & z \neq 0 \\ f'(0), & z = 0 \end{cases}$$

Here, g is analytic in $\mathbb{D}$.

Let $r \in (0,1)$. Then, if $|z| = r$,

$$|g(z)| = \frac{|f(z)|}{r} \leq \frac{1}{r}$$

By the max. principle,

$$|g(z)| \leq \frac{1}{r} \quad \forall z \in \{w : |w| \leq r\}.$$

Let $r \to 1$, so $|g(z)| \leq 1$, $\forall z \in \mathbb{D}$.

Therefore, $|f(z)| \leq |z| \quad \forall z \in \mathbb{D}$

& $|f'(0)| \leq 1$.

If $\exists z \in \mathbb{D}\setminus\{0\}$ s.t. $|f(z)| = |z|$, then

$$|g(z)| = 1 .$$

So, by the max. princ. (1st version), then $g \equiv c$, for some $c \in \mathbb{C}$. Then,

$$f(z) = cz .$$

If $|f'(0)| = 1$, then $|g(0)| = 1$. So by max. princ., $g \equiv c$ & $f(z) = cz$. $\square$

Note:

- A bounded analytic function in $\mathbb{D}$ can't grow too fast in the disk.

Invariant Form of Schwarz's Lemma

Theorem 3. Suppose f is analytic in $\mathbb{D}$ and suppose $|f(z)| < 1$. If $z, a \in \mathbb{D}$, then

$$\left|\frac{f(z) - f(a)}{1 - \overline{f(a)}f(z)}\right| \leq \left|\frac{z-a}{1-\bar{a}z}\right|$$

and

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1-|z|^2}.$$

Proof. For $|c|<1$, the function

$$T_c(z) = \frac{z-c}{1-\bar{c}z} \qquad (z \in \mathbb{C})$$

is analytic on $\mathbb{C} \setminus \{1/\bar{c}\}$. For $|z|=1$,

$$|T_c(z)| = \frac{|z-c|}{|1-\bar{c}z|} = \frac{|z-c|}{|\bar{z}-\bar{c}|} = 1$$

By the max. princ.,

$$|T_c(z)| < 1, \qquad \forall z \in \mathbb{D}.$$

Let $c := f(a) \in \mathbb{D}$. The function $T_c \circ f$ is anatic in $\mathbb{D}$ and

$$g(z) = \begin{cases} \dfrac{(T_c \circ f)(z)}{T_a(z)}, & z \neq a \\[2ex] f'(a) \cdot \dfrac{1-|a|^2}{1-|f(a)|^2}, & z = a. \end{cases}$$

is analytic in $\mathbb{D}$. We have

$$\limsup_{|z|\to 1} \left| \frac{T_c \circ f(z)}{T_a(z)} \right| = \limsup_{|z|\to 1} |T_c \circ f(z)|$$

$$\leq 1$$

because $|T_c(\omega)| \leq 1$ for any $|\omega| \leq 1$.

By the 3rd Version of the Max. princ.,

$$\left| \frac{T_c \circ f(z)}{T_a(z)} \right| \leq 1 \qquad \forall z \in \mathbb{D}.$$

Now,

$$T_c \circ f(z) = \frac{f(z) - f(a)}{1 - \overline{f(a)}\, f(z)}$$

&

$$T_a(z) = \frac{z-a}{1-\bar{a}z}$$

and the statement is true. $\square$

THEOREM 4. If f is analytic in $\mathbb{D}$, $|f| \leq 1$ and $f(z_j) = 0$, for $j = 0, 1, \ldots, n$, then

$$f(z) = \prod_{j=1}^{n} \Big(\frac{z - z_j}{1 - \overline{z}_j z} \Big) g(z),$$

where g is analytic in $\mathbb{D}$ and $|g(z)| \leq 1$ in $\mathbb{D}$.

Proof.

Suppose $f(z_1) = 0$, some $z_1 \in \mathbb{D}$.

The function $g(z) = \dfrac{T_c \circ f(z)}{T_{z_1}(z)}$ is

analytic in $\mathbb{D}$ with $c = f(z_1) = 0$ & $|g(z)| \leq 1$

for any $z \in \mathbb{D}$.

$$g(z) = \frac{f(z)}{\dfrac{z - z_1}{1 - \overline{z}_1 z}}, \quad z \neq z_1$$

$$\Rightarrow \quad f(z) = \frac{z - z_1}{1 - \overline{z}_1 z} g(z), \quad \forall z \in \mathbb{D}.$$

Repeat the argument for $z_2, \ldots, z_n$ to prove the claim. $\square$

$$f(z_1)=0,\ f'(z_1)=0,\ \dots,\ f^{(k)}(0)=0.$$

Growth Rate

COROLLARY 5. If f is non-constant, bounded, and analytic in $\mathbb{D}$, and if z_j $(j \geq 1)$ are the zeros of f (repeated according to their multiplicity), then

$$\sum_{j=1}^{\infty}(1-|z_j|) < \infty.$$

Proof. Suppose, wlog, that $|f| \leq 1$ on $\mathbb{D}$.

<u>Case 1</u> $f(0) \neq 0$, then from thm. 4,

$$f(z) = \prod_{j=1}^{n} \left(\frac{z-z_j}{1-\bar{z}_j z}\right) g(z)$$

and so

$$|f(0)| = \left(\prod_{j=1}^{n} |z_j|\right) |g(0)|$$

$$\leq \prod_{j=1}^{n} |z_j|$$

$$\Rightarrow \log \frac{1}{|f(0)|} \geq \sum_{j=1}^{n} \log \frac{1}{|z_j|}.$$

Using the inequality, $\log\left(\frac{1}{x}\right) \geq 1-x$ valid for $x \in (0,\infty)$.

$$\Rightarrow \log \frac{1}{|f(0)|} \geq \sum_{j=1}^{n} (1-|z_j|)$$

Letting $n \to \infty$, $\displaystyle\sum_{j=1}^{\infty} (1-|z_j|) < \infty.$

Case 2) Assume $f(0)=0$. Write $f(z) = z^k h(z)$.

with $h(0) \neq 0$. & h is analytic in $\mathbb{D}$.

Apply the previous argument to h:

$$\log\left|\frac{1}{h(0)}\right| \geq \sum_{j=k+1}^{n} 1-|z_j|$$

and

$$\sum_{j=1}^{n} 1-|z_j| \leq k + \ln\left|\frac{1}{h(0)}\right|$$

So, let $n \to \infty$,

$$\sum_{j=1}^{\infty} 1-|z_j| < \infty. \qquad \square$$