## E.I Bivariate Distributions

Problem 1. See Example 3(c).

## E.II Continuous Random Vectors

Problem 2. Let $R = [a, b] \times [c, d]$. Then, we have

$$P(a \leq X \leq b, c \leq Y \leq d) = \iint_R f_{X,Y}(x, y)\, dA = \int_c^d \int_a^b f_{X,Y}(x, y)\, dxdy$$

Notice that, for any positive integer $n$,

$$P(X = a, c \leq Y \leq d) = \lim_{n\to\infty} P(a \leq X \leq a + \frac{1}{n}, c \leq Y \leq d)$$

by the continuity of probability measure. Therefore,

$$P(X = a, c \leq Y \leq d) = \lim_{n\to\infty} \int_c^d \int_a^{a+\frac{1}{n}} f_{X,Y}(x, y)\, dxdy = \int_c^d \int_a^a f_{X,Y}(x, y)\, dxdy = 0.$$

By performing similar calculations, we have $P(a \leq X \leq b, Y = c) = 0$.

Now, we have

$$R = (\{a\} \times [c, d]) \cup (\{b\} \times [c, d]) \cup ([a, b] \times \{c\}) \cup ([a, b] \times \{d\}) \cup \big((a, b) \times (c, d)\big)$$

and therefore

$$\begin{aligned} P(a \leq X \leq b, c \leq Y \leq d) &= P(X = a, c \leq Y \leq d) + P(X = b, c \leq Y \leq d) + P(a \leq X \leq b, Y = c) \\ &\quad + P(a \leq X \leq b, Y = d)P(a < X < b, c < Y < d) \\ &= 0 + 0 + 0 + 0 + P(a < X < b, c < Y < d) \\ &= P(a < X < b, c < Y < d). \qquad \square \end{aligned}$$

Problem 3.

a) We must have

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dxdy = 1.$$

Therefore, replacing $f_{X,Y}$ by its expression, we find that it must satisfies

$$\int_0^1 \int_0^1 kxy\, dxdy = 1 \quad \iff \quad \frac{k}{4} = 1 \quad \iff \quad k = 4.$$

b) If $x < 0$, then $F_{X,Y}(x, y) = 0$. and if $y < 0$, then $F_{X,Y}(x, y) = 0$ because $f_{X,Y}(x, y) = 0$ there.

So, assume that $x \geq 0$ and $y \geq 0$. There are four cases to consider.

1. Assume $0 \leq x \leq 1$ and $0 \leq y \leq 1$. In that case, we get
$$F_{X,Y}(x,y) = \int_0^y \int_0^x 4uv\, dudv = x^2y^2.$$

2. Assume $0 \leq x \leq 1$ and $y > 1$. In that case, we get
$$F_{X,Y}(x,y) = \int_0^1 \int_0^x 4uv\, dudv = x^2.$$

3. Assume $x > 1$ and $0 \leq y \leq 1$. In that case, we get
$$F_{X,Y}(x,y) = \int_0^y \int_0^1 4uv\, dudv = y^2.$$

4. Assume $x > 1$ and $y > 1$. In that case, we get
$$F_{X,Y}(x,y) = \int_0^1 \int_0^1 4uv\, dudv = 1.$$

Hence, the joint distribution of $X$ and $Y$ is
$$F_{X,Y}(x,y) = \begin{cases} x^2y^2 & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ x^2 & 0 \leq x \leq 1, y > 1 \\ y^2 & x > 1, 0 \leq y \leq 1 \\ 1 & x > 1, y > 1 \\ 0 & \text{elsewhere.} \end{cases}$$

c) Find $P(X \leq 0.5, Y \leq 0.75)$. Since $(0.5, 0.75) \in [0,1] \times [0,1]$, we obtain from (b),
$$P(X \leq 0.5, Y \leq 0.75) = F_{X,Y}(0.5, 0.75) = (0.5)^2(0.75)^2 = \frac{9}{64}. \qquad \square$$

Problem 4. We have $\{X \leq Y\} = \{(X,Y) : X \leq Y\}$. Let $R = \{(x,y) : x \leq y\}$. Therefore,
$$P(X \leq Y) = P((X,Y) \in R) = \iint_R f_{X,Y}(x,y)\, dA = \iint_{D \cap R} \frac{1}{\pi}\, dA = \frac{\text{Area}(D \cap R)}{\pi}.$$
Notice that $D \cap R = \{(r, \theta) : 0 \leq r \leq 1, \pi/4 \leq \theta \leq 5\pi/4\}$ and this is half of the region inside a circle of radius 1. Hence
$$P(X \leq Y) = \frac{\frac{\pi}{2}}{\pi} = \frac{1}{2}. \qquad \square$$

## E.III Marginals and Independence

Problem 5.

a) The distribution function of $X$ is given by $F_X(x) = \lim_{y \to \infty} F_{X,Y}(x,y)$. Using the fact that $X, Y$ jointly continuous, we get
$$F_X(x) = \int_{-\infty}^x \int_{-\infty}^\infty f_{X,Y}(u,y)\, dydu.$$
Taking the derivative, we get
$$f_X(X) = \int_{-\infty}^\infty f_{X,Y}(x,y)\, dy.$$

b) The distribution function of $Y$ is given by $F_Y(y) = \lim_{x\to\infty} F_{X,Y}(x, y)$. Using the fact again that $X, Y$ are jointly continuous, we get

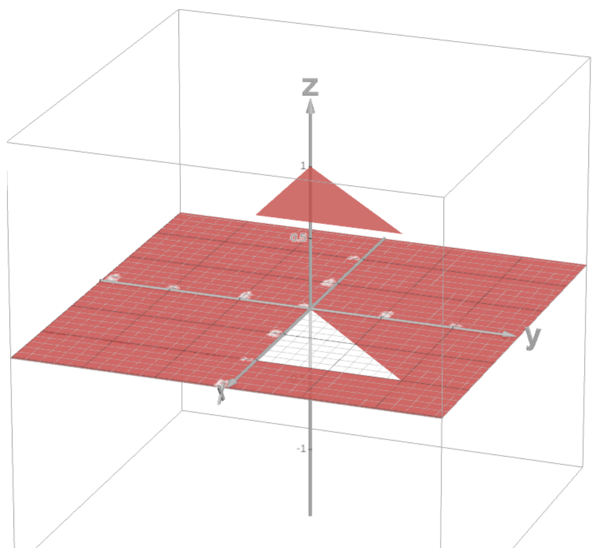
$$F_Y(y) = \int_{-\infty}^{y} \int_{-\infty}^{\infty} f_{X,Y}(x, v)\, dxdv.$$

Taking the derivative, we get

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx. \qquad \square$$

**Problem 6.**

a) Below is a sketch of the density function using Desmos.




b) After some calculations, we get

$$f_X(x) = \begin{cases} x & 0 \le x \le 1 \\ 0 & \text{elsewhere.} \end{cases}$$

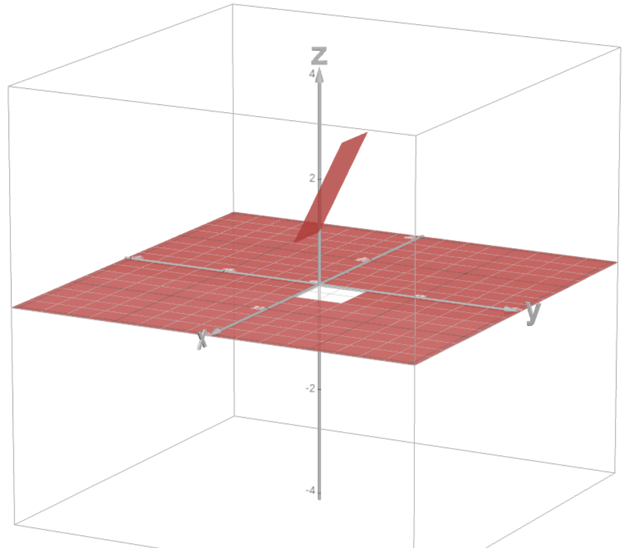
and

$$f_Y(y) = \begin{cases} 1-y & 0 \le y \le 1 \\ 0 & \text{elsewhere.} \end{cases}$$

We see that $f_{X,Y}(x, y) \neq f_X(x)f_Y(y)$ and therefore $X$ and $Y$ are dependent. $\square$

**Problem 7.**

a) Below is a sketch of the density function using Desmos.




b) After some calculations, we get

$$f_X(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{elsewhere.} \end{cases}$$

and

$$f_Y(y) = \begin{cases} y + 1/2 & 0 \leq y \leq 1 \\ 0 & \text{elsewhere.} \end{cases}$$

We see that $f_{X,Y}(x, y) = f_X(x) f_Y(y)$ and therefore $X$ and $Y$ are independent. □

**Problem 8.** Let $X$ be the time of arrival of the passenger and let $Y$ be the time of arrival of the bus. We have $0 \leq X \leq 60$ and $0 \leq Y \leq 60$. Also, $X, Y \sim U(0, 60)$. Since there are independent, their joint density function is

$$f_{X,Y}(x, y) = f_X(x) f_Y(y) = \begin{cases} \frac{1}{3600} & (x, y) \in [0, 1] \times [0, 1] \\ 0 & \text{elsewhere.} \end{cases}$$

Let $a$ be the arrival time of the passenger at the bus station. Since the passenger waits 15min for a bus to arrive, the bus must stops at the bus station between $a$min and $(a + 15)$min. Therefore, the probability is

$$P(a \leq X \leq a + 15, a \leq Y \leq a + 15) = \frac{15 \cdot 15}{3600} = \frac{25}{400} = \frac{1}{16} = 0.0625.$$ □

## E.IV Important Measurements

**Problem 9.** Since $X$ and $Y$ are independent, then

$$\mathrm{Cov}(X, Y) = \mathrm{Exp}(XY) - \mathrm{Exp}(X)\mathrm{Exp}(Y) = 0$$

because $\mathrm{Exp}(XY) = \mathrm{Exp}(X)\mathrm{Exp}(Y)$.

Prove that if $X$ and $Y$ are two independent random variables with average $\mu_X$ and $\mu_Y$, then $\mathrm{Cov}(X, Y) = 0$.

**Problem 10.** In this case, we need the Cauchy Schwarz inequality:

$$|\mathrm{Exp}(XY)| \leq \sqrt{\mathrm{Exp}(X^2)}\sqrt{\mathrm{Exp}(Y^2)}.$$

Using that, we see that

$$|\mathrm{Cov}(X,Y)| = \mathrm{Exp}((X-\mu_X)(Y-\mu_Y)) \leq \sqrt{\mathrm{Exp}((X-\mu_X)^2)}\sqrt{\mathrm{Exp}((Y-\mu_Y)^2)} = \sigma_X\sigma_Y.$$

Therefore,

$$|\rho(X,Y)| = \left|\frac{\mathrm{Cov}(X,Y)}{\sigma_X\sigma_Y}\right| \leq 1.. \qquad \square$$

**Problem 11.**

a) By definition,

$$\mathrm{Cov}(X,Y) = \mathrm{Exp}((X-\mu_X)(Y-\mu_Y)) = \mathrm{Exp}((Y-\mu_Y)(X-\mu_X)) = \mathrm{Cov}(Y,X).$$

b) By the formula of the variance,

$$\begin{aligned} \mathrm{Var}(aX+bY) &= \mathrm{Exp}((aX+bY)^2)) - (\mathrm{Exp}(aX+bY))^2 \\ &= a^2\mathrm{Exp}(X^2) + 2ab\mathrm{Exp}(XY) + b^2\mathrm{Exp}(Y^2) - a^2\mu_X^2 - 2ab\mu_X\mu_Y - b^2\mu_Y^2 \\ &= a^2(\mathrm{Exp}(X^2) - \mu_X^2) + b^2(\mathrm{Exp}(Y^2) - \mu_Y^2) + 2ab(\mathrm{Exp}(XY) - \mu_X\mu_Y) \\ &= a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y) + 2ab\mathrm{Cov}(X,Y). \end{aligned}$$

c) By definition,

$$\mathrm{Cov}(X,X) = \mathrm{Exp}((X-\mu_X)(X-\mu_X)) = \mathrm{Var}(X). \qquad \square$$

**Problem 12.**

a) Since $\mathrm{Cov}(X,X) = \mathrm{Var}(X)$, we find that $\mathrm{Cov}(X,X) = 2$.

b) Since $\rho(X,Y) \in [-1,1]$, we see that

$$\frac{\mathrm{Cov}(X,Y)}{\sigma_X\sigma_Y} \leq 1 \quad \Rightarrow \quad \mathrm{Cov}(X,Y) \leq (\sqrt{2})(2\sqrt{2}) = 4. \qquad \square$$