Chapter 2
Derivatives

2.5 Chain Rule

How do you differentiate the function $F(x) = \sqrt{x^2+1}$?

$$F'(x) = \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = \lim_{h\to 0} \frac{\sqrt{(x+h)^2+1} - \sqrt{x^2+1}}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^2+1-(x^2+1)}{h\left(\sqrt{(x+h)^2+1}+\sqrt{x^2+1}\right)}$$

$f(x) = \sqrt{x}$

$f'(x) = \frac{1}{2\sqrt{x}}$

$$= \lim_{h\to 0} \frac{2xh+h^2}{h\left(\sqrt{(x+h)^2+1}+\sqrt{x^2+1}\right)} \quad = 2x+h$$

$g(x) = x^2$

$g'(x) = 2x$

$$= \frac{\boxed{2x}}{\boxed{2\sqrt{x^2+1}}} = \frac{x}{\sqrt{x^2+1}}$$

The Chain Rule If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

outside inside

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

EXAMPLE 1 Find $F'(x)$ if $F(x) = \sqrt{x^2+1}$.

$f(x) = \sqrt{x}$

$g(x) = x^2+1$

$F(x) = f(g(x))$

By the chain rule

$F'(x) = f'(g(x)) \cdot g'(x)$

$$f'(x) = (x^{1/2})' = \frac{1}{2}x^{1/2-1} = \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}}.$$

$$\to f'(g(x)) = f'(x^2+1) = \frac{1}{2\sqrt{x^2+1}}$$

$$g'(x) = (x^2+1)' = (x^2)' + (1)' = 2x+0 = 2x$$

$$\to F'(x) = \frac{1}{\cancel{2}\sqrt{x^2+1}} \cdot \cancel{2}x = \boxed{\frac{x}{\sqrt{x^2+1}}}$$

Main idea:

$$\frac{d}{dx}\underbrace{f}_{\text{outer function}}\underbrace{(g(x))}_{\text{evaluated at inner function}} = \underbrace{f'}_{\text{derivative of outer function}}\underbrace{(g(x))}_{\text{evaluated at inner function}} \cdot \underbrace{g'(x)}_{\text{derivative of inner function}}$$

EXAMPLE 2 Differentiate (a) $y = \sin(x^2)$ and (b) $y = \sin^2 x$.

(a) $F(x) = \sin(x^2)$

$f(x) = \sin x \to f'(x) = \cos x \to f'(x^2) = \cos(x^2)$

$g(x) = x^2 \to g'(x) = 2x$

$$F'(x) = f'(x^2) \cdot (x^2)' = \cos(x^2) \cdot 2x .$$

(b) $F(x) = \sin^2 x = (\sin x)^2$

$f(x) = x^2 \to f'(x) = 2x \to f'(x) = 2\sin x$

$g(x) = \sin x \to g'(x) = \cos x$

$$F'(x) = f'(\sin x) \cdot (\sin x)' = 2\sin x \cos x .$$

$$\frac{dy}{dx} = 2(\sin x)^{2-1} \cdot (\sin x)' = 2 \sin x \cos x .$$

EXAMPLE 4 Find $f'(x)$ if $f(x) = \dfrac{1}{\sqrt[3]{x^2 + x + 1}}$.

$$f'(x) = \frac{d}{dx}\left(\frac{1}{\sqrt[3]{x^2+x+1}}\right) = \frac{d}{dx}\left((x^2+x+1)^{-1/3}\right)$$

$$= -\frac{1}{3}(x^2+x+1)^{-1/3-1} \cdot \frac{d}{dx}(x^2+x+1)$$

$$= -\frac{1}{3}(x^2+x+1)^{-4/3} \cdot (2x+1)$$

$$= -\frac{(2x+1)}{3(x^2+x+1)^{4/3}} .$$

EXAMPLE 6 Differentiate $y = (2x + 1)^5(x^3 - x + 1)^4$.

$$\frac{dy}{dx} = \frac{d}{dx}\left((2x+1)^5 (x^3-x+1)^4\right)$$

$$= \frac{d}{dx}\left[(2x+1)^5\right](x^3-x+1)^4 + (2x+1)^5 \frac{d}{dx}\left[(x^3-x+1)^4\right]$$

$$= 5(2x+1)^4 \frac{d}{dx}(2x+1)\left(x^3-x+1\right)^4 + (2x+1)^5\, 4(x^3-x+1)^3 \frac{d}{dx}(x^3-x+1)$$

$$= 5(2x+1)^4 \cdot 2\,(x^3-x+1)^4 + (2x+1)^5\, 4(x^3-x+1)^3(3x^2-1)$$

$$= (2x+1)^4(x^3-x+1)^3\left[5(x^3-x+1) + 2(2x+1)(3x^2-1)\right]$$

$$= 2(2x+1)^4(x^3-x+1)^3\left[17x^3+6x^2-9x+3\right]$$

So,

$$\boxed{\frac{dy}{dx} = 2(2x+1)^4(x^3-x+1)^3(17x^3+6x^2-9x+3).}$$