MATH 244 (Calculus IV), Fall 2021
Midterm Exam 1

I understand it is against the rules to cheat or engage in other academic misconduct during this test.

Question 1	20	
Question 2	20	
Question 3	20	
Question 4	20	
Question 5	20	
Total	100	

- This is a closed book, closed notes, no calculator exam. You are only allowed one two-sided cheat sheet.
- You must show your work on all problems. The correct answer with no supporting work may result in no credit. **Put a box around your FINAL ANSWER for each problem and cross out any work that you don't want to be graded.**
- Any student found engaging in academic misconduct will receive a score of 0 on this exam.
- You have 75 minutes to complete the exam, then 15 more minutes to scan and upload your solutions on Gradescope.

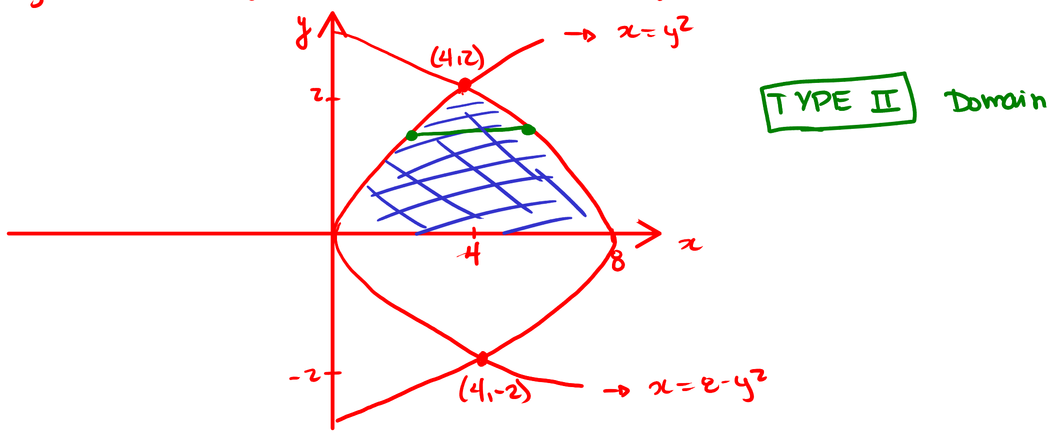
Problem 1.

a. Sketch the region D in the first quadrant bounded by the parabolas $x = y^2$ and $x = 8 - y^2$.

b. Calculate the integral

$$\int\int_D y\, dA.$$

(a) $y^2 = 8 - y^2 \Rightarrow y^2 = 4 \Rightarrow y = \pm 2$. So $D = \{(x,y) : y^2 \le x \le 8 - y^2,\ 0 \le y \le 2\}$



(b) $\iint_D y\, dA = \int_0^2 \int_{y^2}^{8-y^2} y\, dx\, dy = \int_0^2 y\,(8 - y^2 - y^2)\, dy$

$= \int_0^2 8y - 2y^3\, dy$

$= 4y^2 - \frac{y^4}{2}\Big|_0^2$

$= \left(16 - \frac{16}{2}\right)$

$= \boxed{8}$.

Problem 2.

a. Sketch the region D in the xy-plane defined by

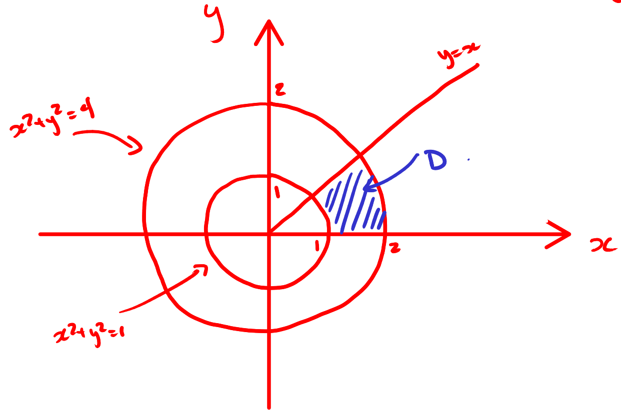
$$D := \{(x, y) : 1 \leq x^2 + y^2 \leq 4, 0 \leq y \leq x\}.$$

b. Calculate the integral

$$\int\int_D \frac{\arctan(y/x)}{\sqrt{x^2+y^2}}\, dA.$$

(a) the region D is bounded by the curves

$x^2+y^2=1$ → circle
$x^2+y^2=4$ → circle
$y=0$ → hor. line
$y=x$ → line.



(b) This is a sector. Use polar coordinates:

$x = r\cos\theta$ & $y = r\sin\theta$.

$x^2+y^2=1 \to r=1$ & $x^2+y^2=4 \to r=2$

Also, $y=x \to r\cos\theta = r\sin\theta \to 1 = \tan\theta \to \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}, \ldots$

keep $\theta = \pi/4$.

So, $y=x \leftrightarrow \theta = \frac{\pi}{4}$.

Also, $y=0 \leftrightarrow \theta = 0$.

then, $D = \{(r,\theta) : 1 \leq r \leq 2, \ 0 \leq \theta \leq \frac{\pi}{4}\}$.

So, $\iint_D \frac{\arctan(y/x)}{\sqrt{x^2+y^2}} = \int_0^{\pi/4}\int_1^2 \frac{\theta}{r}\, r\, dr\, d\theta$

$\theta = \arctan(y/x)$.

$= \left(\int_0^{\pi/4} \theta\, d\theta\right)\left(\int_1^2 dr\right) = \left(\frac{\pi^2/16}{2}\right)(2-1)$

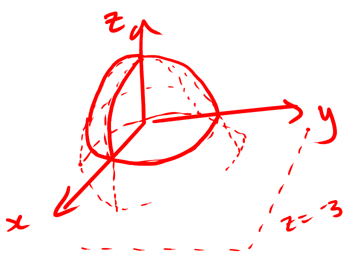
$= \boxed{\frac{\pi^2}{32}}$

Problem 3.

Find ~~the surface area~~ the Volume of the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the plane $z = -3$.

$$V(E) = \iiint_E 1 \, dV.$$

① Description of the solid.



TYPE 1.

$-3 \leq z \leq 1 - x^2 - y^2$

Projection

$z = -3$

$\downarrow$

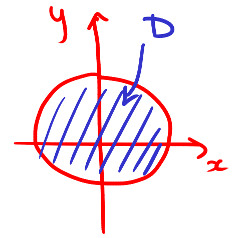
$-3 = 1 - x^2 - y^2$

$\downarrow$

$x^2 + y^2 = 4$

$\downarrow$

circle radius 2



Use Polar coordinates: $D = \{(r, \theta) : 0 \leq r \leq 2 \ \& \ 0 \leq \theta \leq 2\pi\}$.

② Find the volume

$$V = \iiint_E 1 \, dV = \iint_D \left[\int_{-3}^{1-x^2-y^2} dz \right] dA$$

$$= \iint_D 4 - x^2 - y^2 \quad dA$$

$$= \int_0^{2\pi} \int_0^2 (4 - r^2) \, r \, dr \, d\theta$$

$$= 2\pi \left(2r^2 - \frac{r^4}{4} \right) \Big|_0^2$$

$$= 2\pi (8 - 4)$$

$$= \boxed{8\pi}$$

Problem 4.

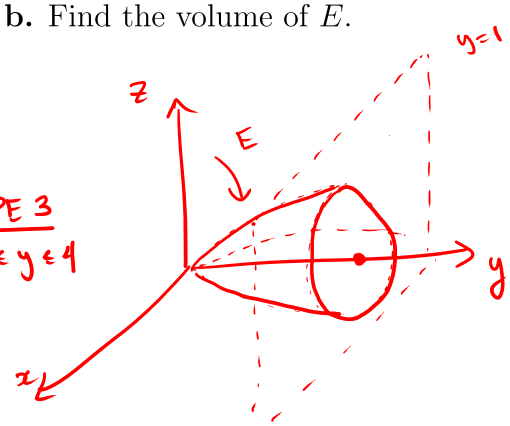
a. Sketch the solid E bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 1$.

b. Find the volume of E.

(a)

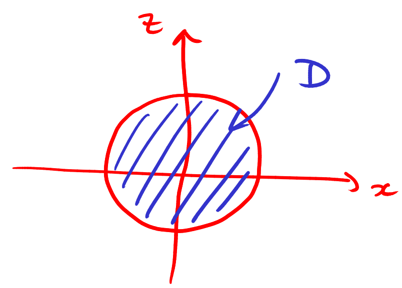
TYPE 3

$x^2+z^2 \le y \le 4$



Project on $y=1$

$1 = x^2+z^2 \to$ circle radius 1



$D = \{(r,\theta) : 0 \le r \le 1, \quad 0 \le \theta \le 2\pi\}$

(b)

$$V = \iiint_E 1 \, dV$$

$$= \iint_D \left[\int_{x^2+z^2}^{4} dy \right] dA$$

$$= \iint_D 4 - x^2 - z^2 \, dA$$

$$\boxed{\begin{aligned} x &= r\cos\theta \\ z &= r\sin\theta \end{aligned}}$$

$$= \int_0^{2\pi} \int_0^1 (4-r^2)\, r \, dr \, d\theta$$

$$= 2\pi \int_0^1 (4-r^2)\, r\,dr$$

$u = 4 - r^2 \to du = -2r\,dr$

$$\int_4^3 u\left(-\frac{du}{2}\right) = \int_3^4 \frac{u}{2}\,du = \frac{u^2}{4}\Big|_3^4 = \frac{16-9}{4} = \frac{7}{4}$$

$$= \boxed{\frac{7\pi}{2}}$$

Problem 5.

~~**a.** Sketch the surface whose equation in cylindrical coordinates is given by~~

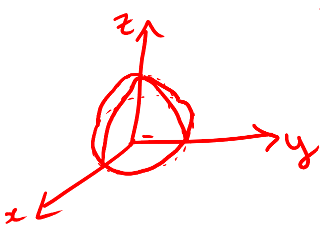
$$\cancel{r^2 + z^2 = 4.}$$

b. Set up but **do not evaluate** an iterated integral for

$$\int\int\int_E (x + y + z)\, dV,$$

where E is the solid in the first octant that lies under the paraboloid $z = 4 - x^2 - y^2$.

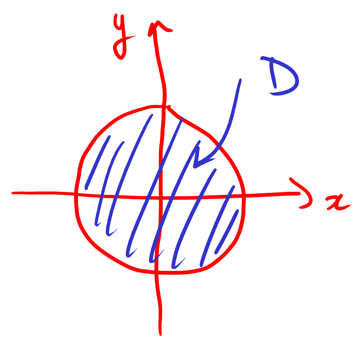
① Description of E.



TYPE 1.

$0 \le z \le 4 - x^2 - y^2$

$z = 0 \longrightarrow x^2 + y^2 = 4$



$D = \{(r, \theta) : 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}$

② Set-up the integral.

$$\iiint_E x + y + z\, dV = \iint_D \left[\int_0^{4 - x^2 - y^2} x + y + z\, dz \right] dA$$

$$= \int_0^{2\pi} \int_0^2 \left(\int_0^{4 - r^2} r\cos\theta + r\sin\theta + z\, dz \right) r\,dr\, d\theta.$$