Math 302

Chapter 1

Section 1.2: Basic Concepts

Contents

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- A **differential equation** (abbreviated by DE) is an equation that contains one or more derivatives of an unknown function. $y = y(x)$.
  - Examples: $T' = -k(T - T_m)$, $y' = x^2$, $x^2y'' + xy' + 2 = 0$.
- The **order** of a DE is the order of the highest derivatives that it contains.
  - Example: $y' = x^2$ is of order 1.
  - Example: $x^2y'' + xy' + 2 = 0$ is of order 2.
- An **Ordinary Differential Equation** (abbreviated ODE) is a DE involving an unknown function of only one variable.
- An **Partial Differential Equation** (abbreviated PDE) is a DE involving an unknown function of more than one variable.

The simplest ODE is of the form

$$y' = f(x) \quad \text{or} \quad y^{(n)} = f(x)$$

where f is a known function of x.

EXAMPLE 1. Find functions $y = y(x)$ satisfying

1. $y' = x^2$.
2. $y'' = \cos(x)$.

1.) $\int y'\,dx = \int x^2\,dx + c \rightarrow y(x) = \frac{x^3}{3} + c$

2) $g = f' \Rightarrow g' = \cos(x) \Rightarrow g(x) = \sin(x) + c_1$

$\Rightarrow f'(x) = \sin(x) + c_1$

$\Rightarrow f(x) = -\cos(x) + c_1x + c_2$

Our goal is to study general ODEs of the form

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}).$$

$x^2y'' + xy' + 2 = 0$

$\Rightarrow x^2y'' = -xy' - 2$

$\Rightarrow y'' = \underbrace{\frac{-xy' - 2}{x^2}}_{= f(x, y, y')}$

What Is a Solution to an ODE?

A **solution** to the ODE

$$y^{(n)}(x) = f(x, y(x), y'(x), \ldots, y^{(n-1)}(x))$$

is a function $y = y(x)$ that verifies the ODE for any x in some open interval (a, b).

Remark:

- Functions that satisfy an ODE at isolated points are not considered solutions.

EXAMPLE 2. Verify that

$$y = \frac{x^2}{3} + \frac{1}{x}$$

is a solution of

$$xy' + y = x^2 \quad \to \quad y' = \frac{x^2 - y}{x}$$

$f(x,y)$

on $(-\infty, 0)$ and $(0, \infty)$.

$$y' = \frac{2x}{3} - \frac{1}{x^2} \qquad (x \neq 0)$$

$$\Rightarrow \quad xy' + y = x^2 \quad \Rightarrow \quad x\left(\frac{2x}{3} - \frac{1}{x^2}\right) + \frac{x^2}{3} + \frac{1}{x} = x^2$$

$$\Rightarrow \quad \frac{2x^2}{3} - \cancel{\frac{1}{x}} + \frac{x^2}{3} + \cancel{\frac{1}{x}} = x^2$$

$$\Rightarrow \quad x^2 = x^2 \qquad \checkmark$$

The solution y is valid on $(0, \infty)$ or $(-\infty, 0)$.

Solution and Integral Curves

$x^2+y^2=1$
$y = \pm\sqrt{1-x^2}$

- The graph of a solution of an ODE is a **solution curve**.
- More generally, a curve C in the plane is said to be an **integral curve** of an ODE if every function $y = y(x)$ whose graph is a segment of C is a solution of the ODE.

EXAMPLE 3. Plot the solutions obtained in Example 2. Are they solution curves of the ODE?

Yes! Graph of a fct.

EXAMPLE 4. If a is any positive constant, check that the circle

$$x^2 + y^2 = a^2$$

$y' = f(x,y)$

is an integral curve of $y' = -x/y$.

Find an expression for $y = y(x)$

$$x^2+y^2 = a^2 \Rightarrow y^2 = a^2 - x^2$$
$$\Rightarrow y = \pm\sqrt{a^2-x^2} \qquad -a \leq x \leq a$$

$$y_+(x) = \sqrt{a^2-x^2} \quad \& \quad y_-(x) = -\sqrt{a^2-x^2}$$

Use y_+.

$$y_+ = \frac{1}{\cancel{2}\sqrt{a^2-x^2}} \cdot -\cancel{2}x = -\frac{x}{\sqrt{a^2-x^2}} = -\frac{x}{y_+}$$

$$\Rightarrow y_+' = -\frac{x}{y_+}$$ satisfies the ODE on $(-a,a)$.

Can do the same calculations for y_-:

$$\checkmark \; y_-' = -\frac{x}{y_-}.$$

EXAMPLE 5. Find a solution of

$$y' = x^3$$

satisfying the additional condition $y(1) = 2$.

Find solution to ODE. integrate $\Rightarrow$ $y(x) = \frac{x^4}{4} + c$

Find c. $y(1) = 2 = \frac{1}{4} + c \Rightarrow c = \frac{7}{4}$

$$\Rightarrow y(x) = \frac{x^4}{4} + \frac{7}{4}.$$

EXAMPLE 6. All the solutions to

$$y'' - 2y' + 3y = 0$$

are the functions

$$y(x) = c_1 e^x + c_2 e^{-3x}$$

where c_1, c_2 are arbitrary constants. Find the solution that satisfies $y(0) = 1$ and $y'(0) = 0$.

$y(0) = c_1 e^0 + c_2 e^0 = c_1 + c_2$ & $y(0) = 1$

$y'(x) = c_1 e^x - 3c_2 e^{-3x} \rightarrow y'(0) = c_1 - 3c_2$

& $y'(0) = 0$

$$\Rightarrow \begin{cases} c_1 + c_2 = 1 \\ c_1 - 3c_2 = 0 \end{cases} \Rightarrow \begin{matrix} c_1 = 3/4 \\ c_2 = 1/4 \end{matrix}$$

$$\boxed{y(x) = \frac{3}{4} e^x + \frac{1}{4} e^{-3x}}$$

An **Initial Value Problem** (abbreviated by IVP) is an ODE with additional **Initial conditions**. The general form of an IVP is

$$y^{(n)}(x) = f(x, y(x), y'(x), \ldots, y^{(n-1)}(x)), \quad y(x_0) = k_0,\ y'(x_0) = k_1, \ldots,\ y^{(n-1)}(x_0) = k_{n-1}.$$

- The largest open interval that contains x_0 on which $y(x)$ is defined and satisfies the ODE is called the **interval of validity** of y.

EXAMPLE 7. Find the interval of validity of the solution to

$$y' = x^3,\ y(\overset{x_0}{1}) = 2.$$

We know that $y(x) = \frac{x^4}{4} + \frac{7}{4}$.

$\frac{x^4}{4} + \frac{7}{4}$ is well-defined for each number x

$\Rightarrow$ interval of validity is $(-\infty, \infty)$.

Example 8. Find the interval of validity of the solution to the following IVPs:

1. $xy' + y = x^2$, $y(\overset{\downarrow x_0}{1}) = 4/3$.
2. $xy' + y = x^2$, $y(-1) = -2/3$.

1) From ex. 2, $y(x) = \frac{x^2}{3} + \frac{1}{x}$ is a solution to the ODE.

$y(1) = \frac{1}{3} + 1 = 4/3 \rightarrow$ Yes y is a solution to the IVP.

Domain: $(-\infty, 0) \cup (0, \infty)$

$x_0 = 1 \rightarrow$ interval of validity: $(0, \infty)$

2) From ex. 2, $y(x) = \frac{x^2}{3} + \frac{1}{x}$ is a solution to the ODE.

$y(-1) = \frac{1}{3} - 1 = -\frac{2}{3}$ ✓.

$x_0 = -1 \rightarrow$ interval of validity: $(-\infty, 0)$.