

# MATH 311

## CHAPTER 6

### SECTION 6.2: LINEAR COMBINATION AND SUBSPACES

CONTENTS
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<b>Subspaces</b>	<b>2</b>
Important Examples . . . . .	4
More Examples . . . . .	7
Non-Examples . . . . .	8
<b>Spanning Sets</b>	<b>9</b>

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**EXAMPLE 1.** The solution to the homogeneous system

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 1 \\ 2 & 3 & 4 & 1 & 2 & -2 \\ 1 & 2 & 4 & 5 & 3 & -1 \\ 3 & 1 & 2 & 4 & 5 & 1 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

is

$$\mathbf{x} = t \begin{bmatrix} 1 \\ 9 \\ -7 \\ 3 \end{bmatrix} + s \begin{bmatrix} 0 \\ 3 \\ -3 \\ 1 \end{bmatrix} = t\mathbf{x}_1 + s\mathbf{x}_2, \quad s, t \in \mathbb{R}.$$

Notice that

S1.  $t = s = 0 \Rightarrow \vec{x} = \vec{0}$  is a solution.

S2.  $\vec{x} = 5\vec{x}_1 + 3\vec{x}_2$  and  $\vec{y} = -3\vec{x}_1 + \vec{x}_2$

$$\Rightarrow \vec{x} + \vec{y} = 2\vec{x}_1 + 4\vec{x}_2$$

$\Rightarrow \vec{x} + \vec{y}$  is still a solution.

S3.  $\vec{x} = 5\vec{x}_1 + 3\vec{x}_2$

$$\Rightarrow 2\vec{x} = 10\vec{x}_1 + 6\vec{x}_2 \text{ still a solution.}$$

If  $U = \{t\mathbf{x}_1 + s\mathbf{x}_2 : s, t \in \mathbb{R}\}$  is the set of all solutions, then  $U$  is called a **subspace**.

**DEFINITION 1.** A subset  $U$  of a vector space  $V$  is called a **subspace** of  $V$  if it satisfies the following properties:

[S1.] The zero vector  $\mathbf{0} \in U$ .

[S2.] If  $\mathbf{u}_1 \in U$  and  $\mathbf{u}_2 \in U$ , then  $\mathbf{u}_1 + \mathbf{u}_2 \in U$ .

[S3.] If  $\mathbf{u} \in U$  and  $a$  is a scalar, then  $a\mathbf{u} \in U$ .

Remarks:

① S2:  $U$  is said to be **closed under addition**.

② S3:  $U$  is said to be **closed under scalar multiplication**.

③ A subspace is a vector space itself.

**EXAMPLE 2.** Let  $V$  be a vector space. Show that  $U = \{\mathbf{0}\}$  is a subspace of  $V$ . This space is called the **zero subspace**.

**SOLUTION.**

S1.  $\vec{0}$  is in  $\{\vec{0}\}$ .

S2.  $\vec{u}_1 = \vec{0}$  and  $\vec{u}_2 = \vec{0} \rightarrow \vec{0} + \vec{0} = \vec{0} \in U$ .

S3.  $\vec{u} = \vec{0}$  and  $a \in \mathbb{R} \Rightarrow a\vec{0} = \vec{0} \in U$ .

Here S1-S3 are satisfied  $\Rightarrow U$  is a subspace.

Note: Any subspace  $U$  of  $V$  such that  $U \neq \{\mathbf{0}\}$  and  $U \neq V$  is called a **proper subspace**.

# Important Examples

**EXAMPLE 3.** Given an  $m \times n$  matrix  $A$ , define

$$U = \text{null} A := \{ \mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \vec{0}_{m \times 1} \}.$$

Show that  $\text{null} A$  is a subspace of  $\mathbb{R}^n$ .

**SOLUTION.**  $\text{null} A$  is a subset of  $\mathbb{R}^n$ .

S1.  $A \vec{0}_{n \times 1} = \vec{0}_{m \times 1} \Rightarrow \vec{0}_{n \times 1} \in \text{null} A$ .

S2. Let  $\vec{x}, \vec{y} \in \text{null} A$ . then

$$\begin{aligned} A(\vec{x} + \vec{y}) &= A\vec{x} + A\vec{y} \quad (\text{Matrix mult. prop.}) \\ &= \vec{0}_{m \times 1} + \vec{0}_{m \times 1} \\ &= \vec{0}_{m \times 1} \Rightarrow \vec{x} + \vec{y} \in \text{null} A. \end{aligned}$$

S3. Let  $\vec{x} \in \text{null} A$  and  $a \in \mathbb{R}$ . then

$$A(a\vec{x}) = a(A\vec{x}) = a\vec{0}_{m \times 1} = \vec{0}_{m \times 1}.$$

So,  $a\vec{x} \in \text{null} A$ . Conclusion:  $\text{null} A$  a subspace.

Note: The subspace  $\text{null} A$  is called the **null space** of a matrix  $A$ . It is the set of all solutions to the homogeneous system  $A\mathbf{x} = \mathbf{0}$ .

**EXAMPLE 4.** Given an  $m \times n$  matrix  $A$ , define

$$\text{im}A := \overbrace{\{A\mathbf{x} : \mathbf{x} \in \mathbb{R}^n\}}^{m \times 1}.$$

Show that  $\text{im}A$  is a subspace of  $\mathbb{R}^m$ .

**SOLUTION.**  $\text{im}A$  is a subset of  $\mathbb{R}^m$ .

S1. If  $\vec{x} = \vec{0} \in \mathbb{R}^n$ , then

$$A\vec{x} = A\vec{0} = \vec{0} \in \mathbb{R}^m$$

$$\Rightarrow \vec{0}_{m \times 1} \in \text{im}A.$$

S2. Assume  $\vec{y}_1, \vec{y}_2 \in \text{im}A$ .

Goal:  $\vec{y}_1 + \vec{y}_2 \in \text{im}A$ . ( $\vec{y}_1 + \vec{y}_2 = A\vec{x}$ ).

Here,  $\vec{y}_1 = A\vec{x}_1$  and  $\vec{y}_2 = A\vec{x}_2$

$$\Rightarrow \vec{y}_1 + \vec{y}_2 = A\vec{x}_1 + A\vec{x}_2 = A(\vec{x}_1 + \vec{x}_2)$$

So, if  $\vec{x} = \vec{x}_1 + \vec{x}_2 \Rightarrow \vec{y}_1 + \vec{y}_2 = A\vec{x}$ . ✓

S3. Assume  $\vec{y} \in \text{im}A$  and  $a \in \mathbb{R}$ .

$$a\vec{y} = aA\vec{x} = A(\underbrace{a\vec{x}}_{=\vec{u}}) = A\vec{u} \Rightarrow a\vec{y} \in \text{im}A. \checkmark$$

Note: The subspace  $\text{im}A$  is called the **image space** (or **range space**) of the matrix  $A$ . It is the set of all vectors  $\mathbf{b}$  such that  $A\mathbf{x} = \mathbf{b}$  has a solution.

**EXAMPLE 5.** For an  $n \times n$  matrix  $A$  and a number  $\lambda$ , define

$$E_\lambda(A) := \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \lambda\mathbf{x}\}.$$

Show that  $E_\lambda(A)$  is a subspace of  $\mathbb{R}^n$ .

**SOLUTION.** We see that  $E_\lambda(A) \subseteq \mathbb{R}^n$

$$\begin{aligned} \underline{S1.} \quad A\vec{0} &= \vec{0} \Rightarrow A\vec{0} = \lambda\vec{0} \\ \lambda\vec{0} &= \vec{0} \Rightarrow \vec{0} \in E_\lambda(A) \checkmark \end{aligned}$$

S2.  $\vec{x}, \vec{y} \in E_\lambda(A)$ . Goal:  $\vec{x} + \vec{y} \in E_\lambda(A)$ .

$$\text{So, } A\vec{x} = \lambda\vec{x} \text{ and } A\vec{y} = \lambda\vec{y}.$$

$$\begin{aligned} A(\vec{x} + \vec{y}) &= A\vec{x} + A\vec{y} = \lambda\vec{x} + \lambda\vec{y} \\ &= \lambda(\vec{x} + \vec{y}) \checkmark \end{aligned}$$

Then,  $\vec{x} + \vec{y} \in E_\lambda(A)$ .

S3. Let  $\vec{x} \in E_\lambda(A)$  and  $\mu \in \mathbb{R}$ .

Goal:  $\mu\vec{x} \in E_\lambda(A)$ .

$$\Rightarrow A(\mu\vec{x}) = \mu(A\vec{x}) = \mu(\lambda\vec{x}) = \lambda(\mu\vec{x}) \checkmark$$

$\Rightarrow \mu\vec{x} \in E_\lambda(A)$ . Conclusion:  $E_\lambda(A)$  subspace.

Note: When  $\lambda$  is an eigenvalue of  $A$ , the subspace  $E_\lambda(A)$  is called the **eigenspace** associated to  $\lambda$ .

## More Examples

**EXAMPLE 6.** Let  $\mathbf{M}_{nn}$  be the vector space of  $n \times n$  matrices. Show that  $U = \{A : A^T = A\}$  is a subspace of  $\mathbf{M}_{nn}$ .

**SOLUTION.** For sure  $U \subseteq \mathbf{M}_{nn}$ .

S1.  $0^T = 0 \Rightarrow 0 \in U$ .

S2.  $A, B \in U$ . Goal:  $(A+B)^T = A+B$ .

From properties in Chapter 2:

$$\begin{aligned}(A+B)^T &= A^T + B^T \\ &= A + B \Rightarrow A+B \in U.\end{aligned}$$

S3.  $A \in U$ ,  $\lambda \in \mathbb{R}$ . Goal:  $(\lambda A)^T = \lambda A$ .

From properties in Chapter 2:

$$(\lambda A)^T = \lambda A^T = \lambda A \Rightarrow \lambda A \in U.$$

Conclusion:  $U$  is a subspace.

Note: The set  $U$  is the subspace of all symmetric matrices.

# Non-Examples

**EXAMPLE 7.** Show that the set

$$U = \{p : p \in \mathbf{P}_3 \text{ and } p(2) = 1\}$$

is not a subspace of  $\mathbf{P}_3$ .

**SOLUTION.**

Sl. Recall that

$$0(x) = 0x^3 + 0x^2 + 0x + 0 = 0$$

$$\Rightarrow 0(2) = 0 \neq 1 \Rightarrow 0(x) \notin U.$$

Conclusion:  $U$  is not a subspace  
of  $\mathbf{P}_3$ .



**EXAMPLE 8.** The solutions set to the system  $A\mathbf{x} = \mathbf{0}$  given in Example 1 is given by the linear combination

$$t\mathbf{x}_1 + s\mathbf{x}_2, \quad t, s \in \mathbb{R}.$$

The set  $\{t\mathbf{x}_1 + s\mathbf{x}_2 : t, s \in \mathbb{R}\}$  is called the **span** of  $\mathbf{x}_1$  and  $\mathbf{x}_2$ .

**DEFINITION 2.** Let  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  be a collection of vectors in a vector space  $V$ .

- ① a **linear combination** of the vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  is an expression of the form

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$$

where  $a_1, a_2, \dots, a_n$  are scalars called the **coefficients** of each vector.

- ② The set of all linear combinations of these vectors is called their **span**.
- ③ If it happens that  $V = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ , then the vectors are called a **spanning set** for  $V$ .

Remarks:

- $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  is a subspace of  $V$ .

**EXAMPLE 9.** Consider  $p_1 = 1 + x + 4x^2$  and  $p_2 = 1 + 5x + x^2$ , two polynomials in  $\mathbf{P}_2$ .

a) Is  $p_1$  in the  $\text{span}\{1 + 2x - x^2, 3 + 5x + 2x^2\}$ .

b) Is  $p_2$  in the  $\text{span}\{1 + 2x - x^2, 3 + 5x + 2x^2\}$ .

**SOLUTION.**

