Math 241

Chapter 4

Section 4.1: Areas and Distances

Contents

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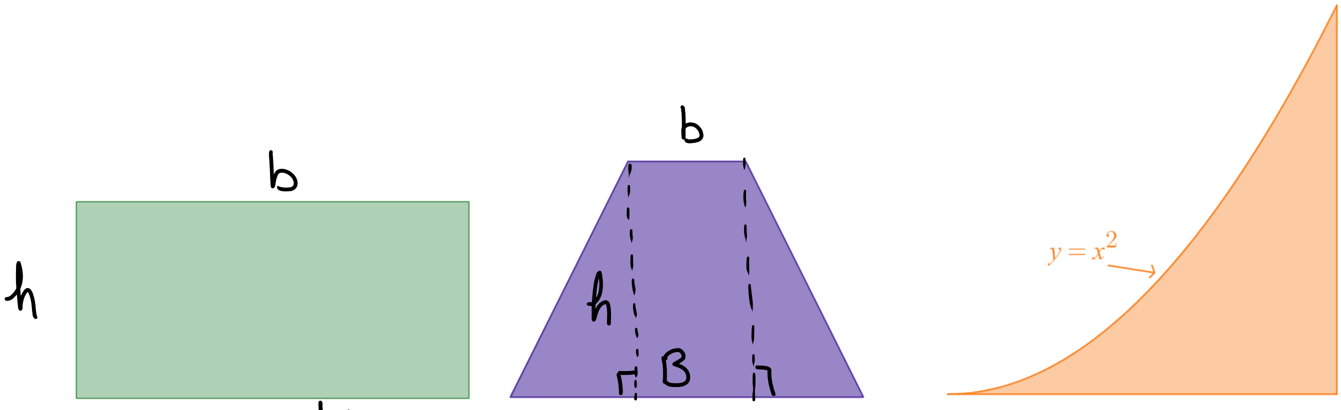
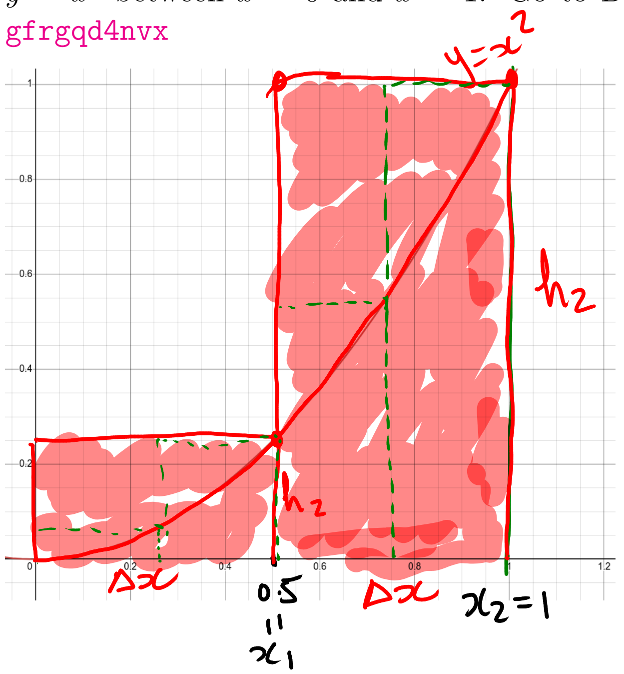
What is the area of the following shapes?



(a) Area = hb (b) Area = $\left(\frac{B+b}{2}\right)h$ (c) Area = $\frac{1}{3}$

Trick: Use simpler shapes, such as rectangles, to approximate the area.

Example 1. Using rectangles, approximate the area of the region S under the graph of $y = x^2$ between $x = 0$ and $x = 1$. Go to Desmos: https://www.desmos.com/calculator/gfrgqd4nvx



Step 1) $n=2$ squares

↳ Divide $[0,1]$ into 2 subintervals.

$$\Delta x = \frac{1-0}{2} = 0.5$$

$[0, 0.5]$ & $[0.5, 1]$

Step 2) Choose Rightend point

$x_1 = 0.5$, $x_2 = 0.5 + 0.5 = 1$

Step 3) Draw rectangles.

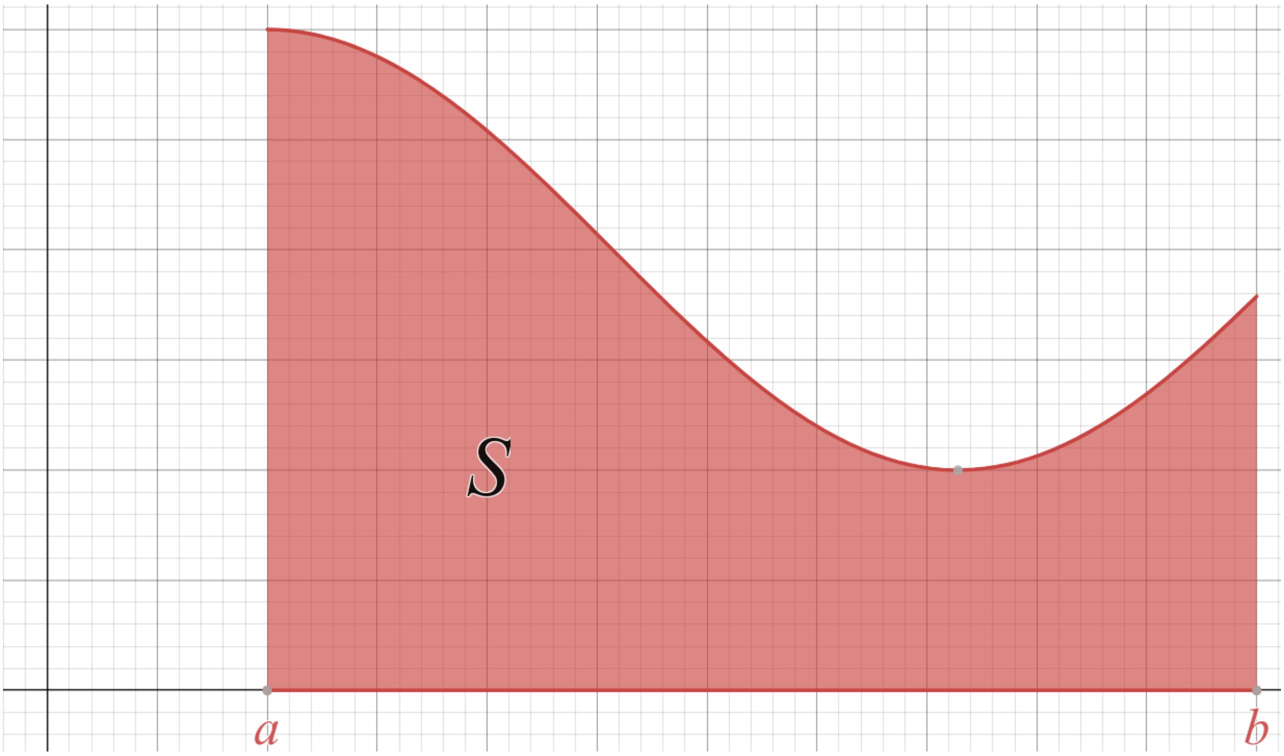
$h_1 = f(x_1) = 0.5^2 = 0.25$, $h_2 = f(x_2) = 1^2 = 1$

Step 4) Add area rectangles:

$$A \approx h_1 \cdot \Delta x + h_2 \cdot \Delta x = 0.25 \cdot 0.5 + 1 \cdot 0.5$$
$$= 0.625$$

Divide and Conquer With the Right Endpoint Rule!

Suppose we want to compute the area of a region S bounded by the graph of some function $y = f(x)$.



STEP I Subdivide the region S into n strips of equal width $\Delta x = (b - a)/n$.

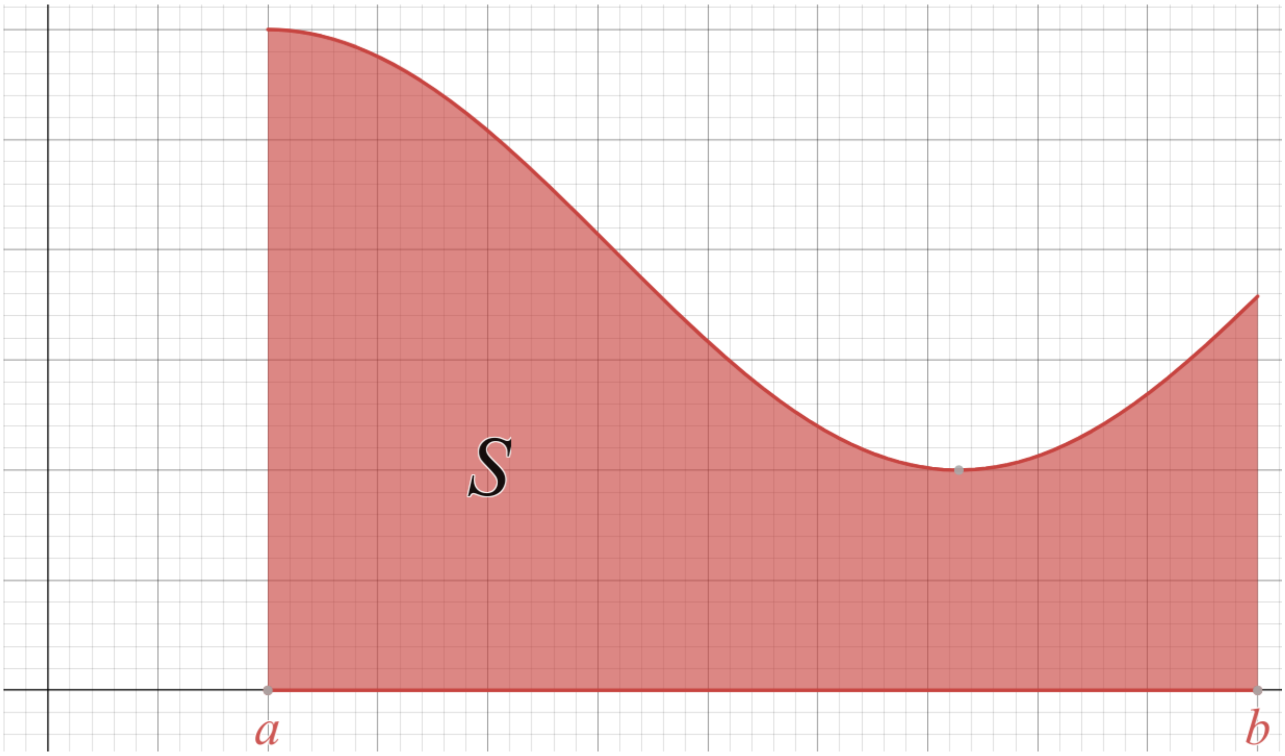
STEP II Choose the right-end point for all subintervals:
$x_1 = a + \Delta x$, $x_2 = a + 2\Delta x$, $\ldots$, $x_{n-1} = a + (n-1)\Delta x$, $x_n = b$.

STEP III Approximate by adding the area of each rectangle:

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x.$$

Divide and Conquer With the Left Endpoint Rule!

Suppose we want to compute the area of a region S bounded by the graph of some function $y = f(x)$ from $x = a$ to $x = b$.



<u>STEP I</u> Subdivide the region S into n strips of equal width $\Delta x = (b-a)/n$.

<u>STEP II</u> Choose the left-end point for all subintervals:
$x_0 = a$, $x_1 = a + \Delta x$, …, $x_{n-2} = a + (n-2)\Delta x$, $x_{n-1} = a + (n-1)\Delta x$.

<u>STEP III</u> Approximate by adding the area of each rectangle:

$$L_n = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x.$$

Sigma Notation

We use the symbol $\sum$ to write a summation of numbers compactly:

$$\sum_{i=k}^{n} a_i$$

EXAMPLE 2.

a) Expand $\sum_{i=1}^{7} i$.

b) Write $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}$ with the Sigma notation.

c) Write $1 + 3 + 5 + 7 + 9 + 11 + 13$ with the Sigma notation.

Useful Sum Formulas:

- $\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$;
- $\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$;
- $\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \cdots + n^3 = \left(\dfrac{n(n+1)}{2}\right)^2$.

Taking the Limit!

EXAMPLE 3. Show that the area of the region S in Example 1 is $1/3$. In other words, show that

$$\text{Area}(S) = \lim_{n\to\infty} R_n = 1/3.$$

General definition of Area: The area of the region S lying under the graph of a function $y = f(x)$ from $x = a$ to $x = b$ is given by

- $\text{Area}(S) = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \Big(f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x \Big)$
- $\text{Area}(S) = \lim_{n\to\infty} L_n = \lim_{n\to\infty} \Big(f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x \Big)$

The Distance Problem

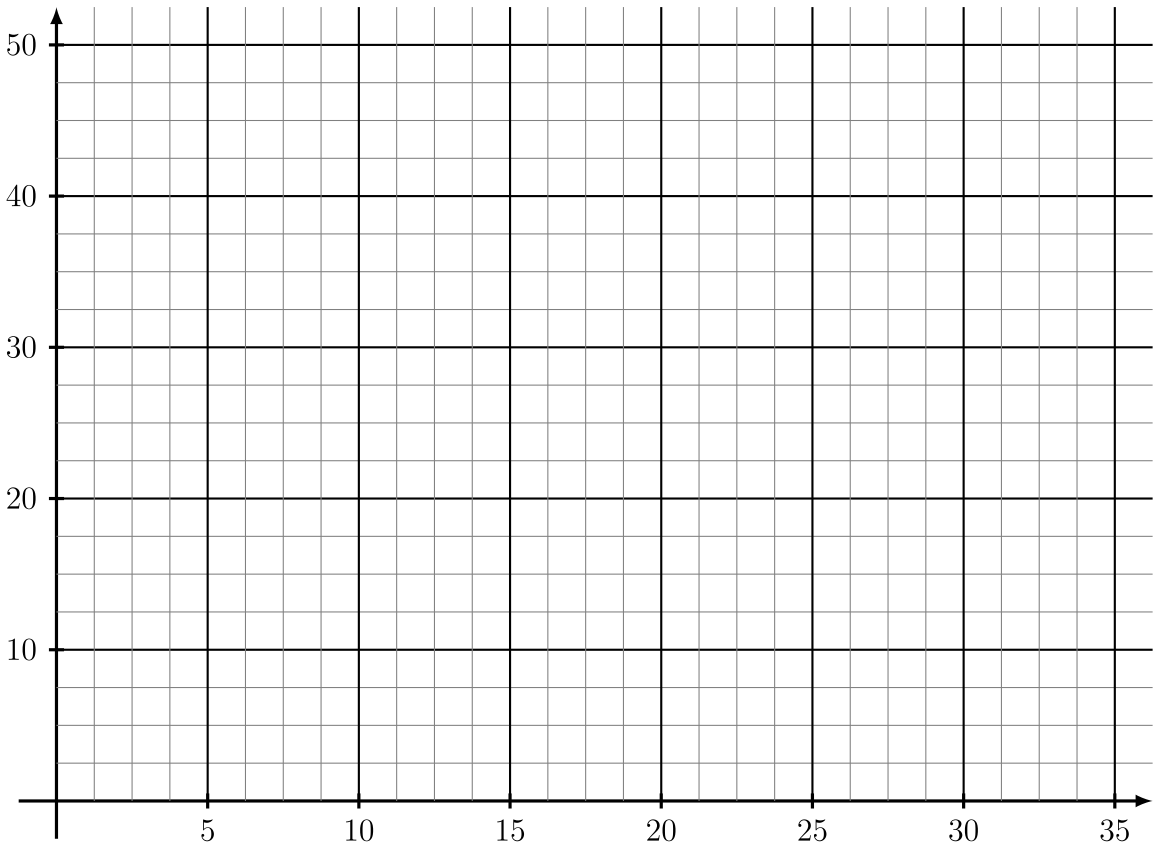
If an object move at constant velocity, then the distance between the start and finish line is easy to compute:

$$\text{Distance} = \text{Velocity} \times \Delta\text{Time} .$$

What do we do if the velocity is not constant?

Example 4. Suppose the odometer on our car is broken and we want to estimate the distance driven over a 30-second time interval. We take speedometer readings every five seconds and record them in the following table:

Time (s)	0	5	10	15	20	25	30
Velocity (ft/s)	25	31	35	43	47	45	41



Remark:

- The total distance is given by the area under the curve of the velocity function!