

Chapter 15

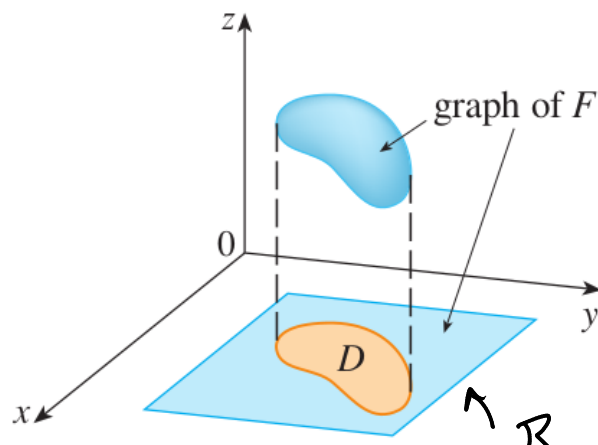
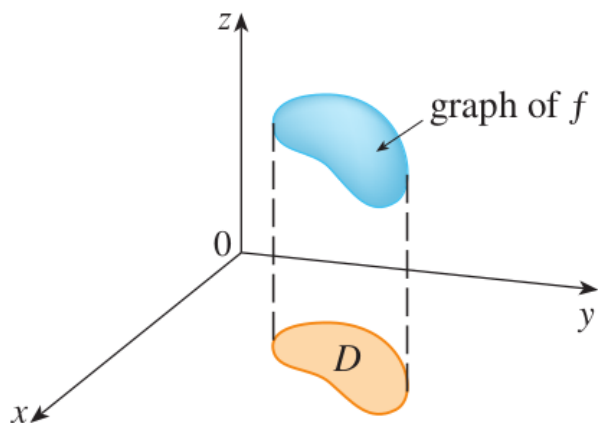
Multiple Integrals

15.2 Double Integrals over general regions

Definition.

Given: A function f defined on D

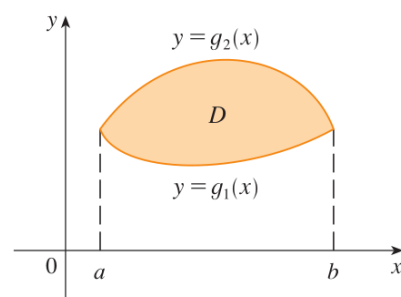
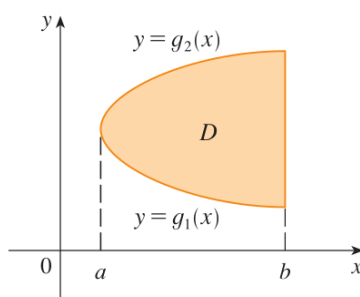
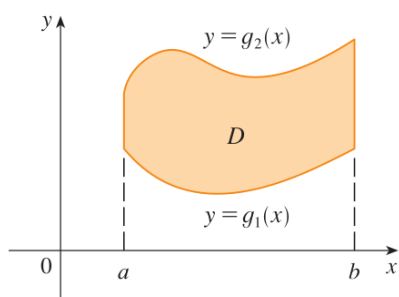
Extend f to a rectangle containing D



$$F(x, y) = \begin{cases} f(x, y), & (x, y) \in D \\ 0, & (x, y) \in R \text{ but } (x, y) \notin D \end{cases}$$

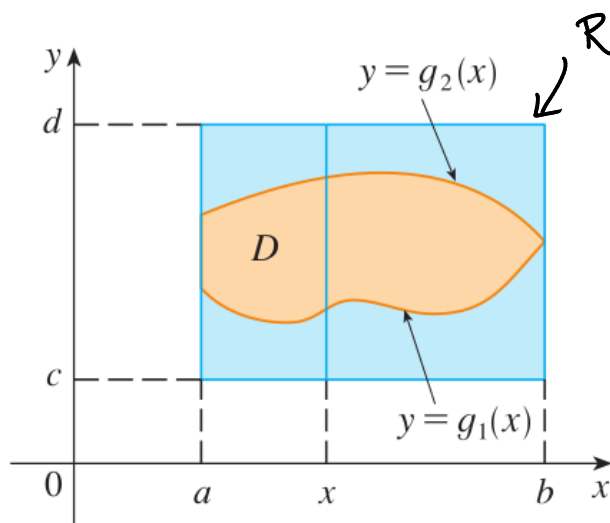
$$\iint_D f(x, y) dA = \iint_R F(x, y) dA$$

Region of type I.



$$D = \{(x, y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

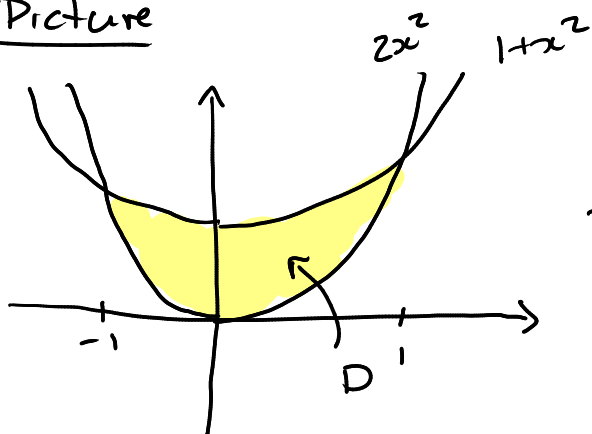
$$\begin{aligned} \iint_D f(x, y) dA &= \iint_R F(x, y) dA \\ &= \int_a^b \int_c^d F(x, y) dy dx \\ &= \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx \end{aligned}$$



$$\iint_D f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx.$$

EXAMPLE 1 Evaluate $\iint_D (x + 2y) dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

① Picture



$$1 + x^2 = 2x^2$$

$$1 = x^2 \Rightarrow x = \pm 1$$

$$D = \{ (x, y) : -1 \leq x \leq 1, 2x^2 \leq y \leq 1 + x^2 \}.$$

② Integrate

$$g_1(x) = 2x^2 \quad \& \quad g_2(x) = 1 + x^2$$

$$\iint_D x + 2y dA = \int_{-1}^1 \int_{2x^2}^{1+x^2} x + 2y dy dx$$

$$= \int_{-1}^1 xy \Big|_{2x^2}^{1+x^2} + y^2 \Big|_{2x^2}^{1+x^2} dx$$

$$= \int_{-1}^1 x \left(1 + x^2 - 2x^2 \right) + \left(1 + x^2 \right)^2 - \left(2x^2 \right)^2 dx$$

$$= \int_{-1}^1 x - x^3 + 1 + 2x^2 + x^4 - 4x^4 dx$$

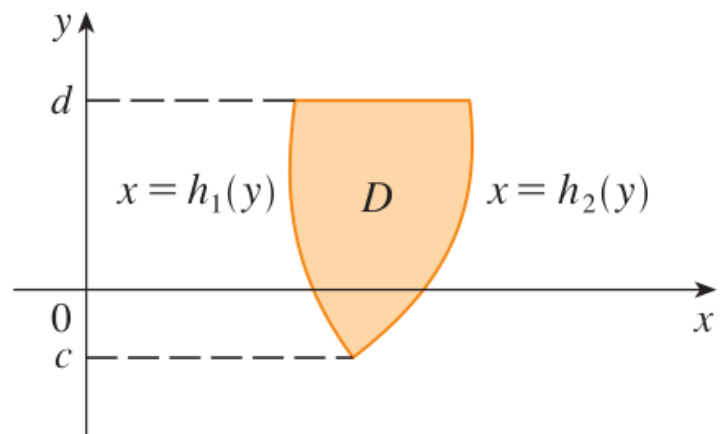
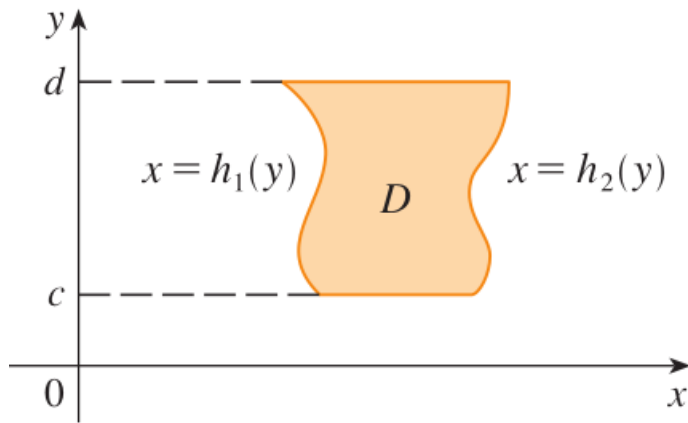
$$= \int_{-1}^1 1 + x + 2x^2 - x^3 - 3x^4 dx$$

$$= \boxed{\frac{32}{15}} \approx 1.3167.$$

$$\left(x + \frac{x^2}{2} + \frac{2x^3}{3} - \frac{x^4}{4} - \frac{3x^5}{5} \right) \Big|_{-1}^1$$

Region of Type II.

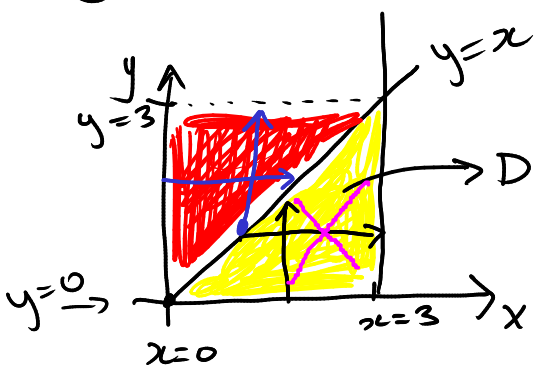
$$D = \{(x, y) \mid c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$$



$$\iint_D f(x, y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy$$

EXAMPLE. Evaluate $\iint_D e^{-y^2} dA$, where D is the region bounded by the lines $x = 0$, $y = 3$ and $x = y$.

① Picture.



$$D = \{(x, y) : 0 \leq x \leq 3, x \leq y \leq 3\}$$

$$= \{(x, y) : 0 \leq x \leq y, 0 \leq y \leq 3\}$$

TYPE I

TYPE II

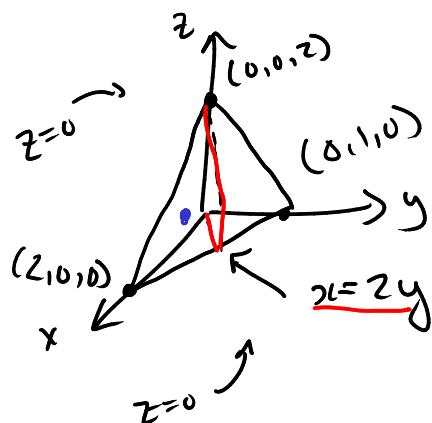
② Integrate

$$c=0, d=3, h_1(y)=0, h_2(y)=y.$$

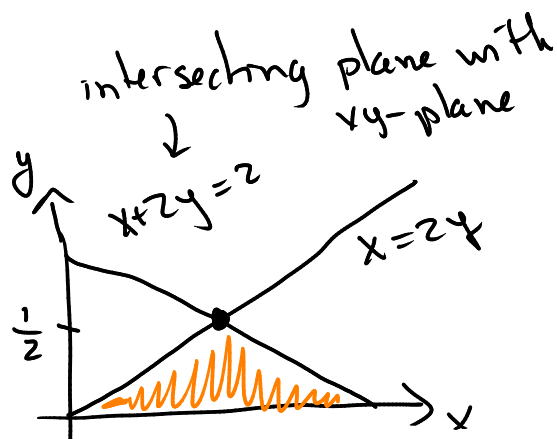
$$\begin{aligned} \iint_D e^{-y^2} dA &= \int_0^3 \int_0^y e^{-y^2} dx dy \\ &= \int_0^3 (ye^{-y^2}) dy = \frac{1}{2} \int_0^9 e^{-u} du \quad (u=y^2) \\ &= \left[\frac{1}{2} (1 - e^{-9}) \right] \end{aligned}$$

EXAMPLE. Find the volume of the tetrahedron bounded by the planes $x + 2y + z = 2$, $x = 2y$, $y = 0$, and $z = 0$.

① Picture



Squeeze →



TYPE II

$$D = \{(x, y) : 0 \leq y \leq \frac{1}{2}, 2y \leq x \leq 2-2y\}.$$

② Integrate.

$$\begin{aligned} \text{height} &= z \\ &= 2 - x - 2y \end{aligned}$$

$$\text{Vol}(S) = \iint_D \text{height } dA$$

$$= \int_0^{1/2} \int_{2y}^{2-2y} (2 - x - 2y) dx dy$$

$$= \int_0^{1/2} \left(2x - \frac{x^2}{2} - 2yx \right) \Big|_{2y}^{2-2y} dy$$

$$\begin{aligned} &= \int_0^{1/2} \left(2(2-2y) - \frac{(2-2y)^2}{2} - 2y(2-2y) \right) \\ &\quad - \left(4y - 2y^2 - 4y^2 \right) dy \end{aligned}$$

$$= \boxed{\frac{1}{3}} \approx 0.33 \dots$$

EXAMPLE 5 Evaluate the iterated integral $\int_0^1 \int_x^1 \sin(y^2) dy dx$.

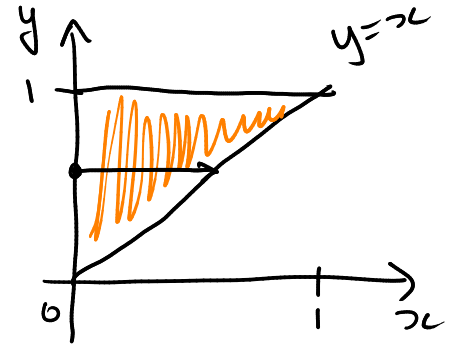
① Picture

→ TYPE I.

$$D = \{ (x, y) : 0 \leq x \leq 1, x \leq y \leq 1 \}.$$

$$= \{ (x, y) : 0 \leq x \leq y, 0 \leq y \leq 1 \}$$

→ TYPE II.



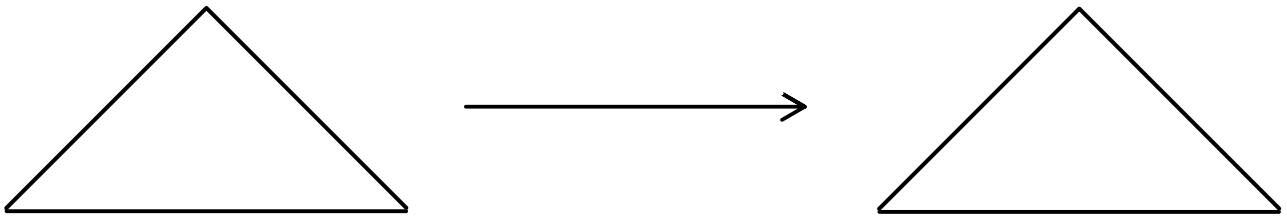
$$\boxed{6} \quad \iint_D (f(x, y) + g(x, y)) \, dA = \iint_D f(x, y) \, dA + \iint_D g(x, y) \, dA$$

$$\boxed{7} \quad \iint_D cf(x, y) \, dA = c \iint_D f(x, y) \, dA$$

$$\boxed{8} \quad \text{If } f(x, y) \geq g(x, y) \text{ on } D, \text{ then } \iint_D f(x, y) \, dA \geq \iint_D g(x, y) \, dA$$

$$\boxed{9} \quad \text{If } D = D_1 \cup D_2, \text{ with } D_1 \cap D_2 = \emptyset, \text{ then}$$

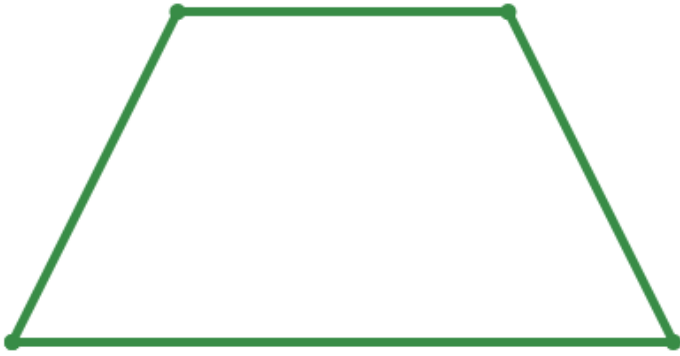
$$\iint_D f(x, y) \, dA = \iint_{D_1} f(x, y) \, dA + \iint_{D_2} f(x, y) \, dA$$



$$\boxed{10} \quad \text{Area}(D) = \iint_D 1 \, dA$$

$$\boxed{11} \quad \text{If } m \leq f(x, y) \leq M, \text{ then } m \cdot \text{Area}(D) \leq \iint_D f(x, y) \, dA \leq M \cdot \text{Area}(D)$$

Example. Find the area of the trapezoid below:



Challenge. Find the area of the hexagone below using properties 9 and 10:

