Section 6.1, Problem 2

We have $Y_1(x) = c_1e^{2x} + c_2e^{3x} - x/6 - 11/36$ and $Y_2(x) = 2c_1e^{2x} + c_2e^{3x} - 2x/3 - 1/18$. Therefore,

$$Y' = \begin{bmatrix} Y_1' \\ Y_2' \end{bmatrix} = \begin{bmatrix} 2c_1e^{2x} + 3c_2e^{3x} - 1/6 \\ 4c_1e^{2x} + 3c_2e^{3x} - 2/3 \end{bmatrix}$$

and

$$\begin{aligned} AY + \begin{bmatrix} 1 \\ x \end{bmatrix} &= \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} + \begin{bmatrix} 1 \\ x \end{bmatrix} \\ &= \begin{bmatrix} 2c_1e^{2x} + 3c_2e^{3x} - 1/6 \\ 4c_1e^{2x} + 3c_2e^{3x} - 2/3 \end{bmatrix}. \end{aligned}$$

So Y is a solution to the system of ODEs.

Section 6.1, Problem 4

We compute the Wronskien. We have

$$\begin{aligned} W(Y_1(x), Y_2(x), Y_3(x)) &= \begin{vmatrix} e^{-2x} & 0 & 0 \\ 0 & 3\cos 5x & \sin 5x \\ 0 & -3\sin 5x & \cos 5x \end{vmatrix} \\ &= e^{-2x} \begin{vmatrix} 3\cos 5x & \sin 5x \\ -3\sin 5x & \cos 5x \end{vmatrix} \\ &= e^{-2x}(3\cos^2 5x + 3\sin^2 5x) \\ &= 3e^{-2x} \end{aligned}$$

where in the last equality, we used the identity $\sin^2(5x) + \cos^2(5x) = 1$. Therefore, since the exponential function is never zero, this means that $3e^{-2x} \neq 0$ for at least one x. Therefore, the Wronskian is not zero for at least one x and the vector of functions are linearly independent.

Section 6.1, Problem 5

The solution is

$$Y(x) = \begin{bmatrix} c_1 e^{x} \\ c_2 e^{-2x} \end{bmatrix} = c_1 \begin{bmatrix} e^{x} \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ e^{-2x} \end{bmatrix}.$$

A foundamental set of solutions for the system of ODEs is therefore

$$\begin{bmatrix} e^{x} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ e^{-2x} \end{bmatrix}.$$

Section 6.1, Problem 11

The general solution is

$$Y(x) = \begin{bmatrix} c_1 e^{-x} \\ c_2 \\ c_3 e^{4x} \end{bmatrix}.$$

From the initial condition, we must have

$$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = Y(0) = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

and therefore $c_1 = 2$, $c_2 = 1$ and $c_3 = 0$. The solution to the initial value problem is

$$Y(x) = \begin{bmatrix} 2e^{-x} \\ 1 \\ 0 \end{bmatrix}.$$

Section 6.2, Problem 2

The system to solve is

$$Y' = \begin{bmatrix} 6 & -8 \\ 4 & -6 \end{bmatrix} Y$$

The eigenvalues of A are -2 and 2. The diagonal matrix similar to A is

$$D = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$$

and the change of basis P such that $D = P^{-1}AP$ is

$$P = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

Solve the diagonal system.
The system $Y' = AY$ becomes $Y' = PDP^{-1}Y$ and multiplying by P^{-1}, we obtain the system

$$P^{-1}Y' = DP^{-1}Y.$$

By letting $Z = P^{-1}Y$, the diagonal system is then $Z' = DZ$. The solution is therefore

$$Z = \begin{bmatrix} c_1 e^{-2x} \\ c_2 e^{2x} \end{bmatrix}.$$

Solve the general system.
We know that $Z = P^{-1}Y$ and therefore $Y = PZ$. By multiplying Z by P, we obtain

$$Y = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{-2x} \\ c_2 e^{2x} \end{bmatrix} = \begin{bmatrix} c_1 e^{-2x} + 2c_2 e^{2x} \\ c_1 e^{-2x} + c_2 e^{2x} \end{bmatrix}.$$

Section 6.2, Problem 14

From the problem 2, the general solution is

$$Y = \begin{bmatrix} c_1 e^{-2x} + 2c_2 e^{2x} \\ c_1 e^{-2x} + c_2 e^{2x} \end{bmatrix}.$$

Therefore, we must have

$$\begin{bmatrix} 0 \\ -1 \end{bmatrix} = Y(0) = \begin{bmatrix} c_1 + 2c_2 \\ c_1 + c_2 \end{bmatrix}.$$

This is a system of linear equations in the unknown c_1 and c_2. After solving it, we obtain $c_1 = -2$ and $c_2 = 1$. The solution to the initial value problem is

$$Y(x) = \begin{bmatrix} -2e^{-2x} + 2e^{2x} \\ -2e^{-2x} + e^{2x} \end{bmatrix}.$$

Section 6.3, Problem 2

The system of ODEs is

$$Y' = \begin{bmatrix} 4 & -4 \\ 1 & 0 \end{bmatrix} Y.$$

The eigenvalue of A is 2 with algebraic multiplicity two, but the matrix A is not diagonalizable. Using Python, we obtain the Jordan Canonical Form and the change of basis:

$$B = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

Solve the upper-triagular System.
Using the matrix P, we have to solve the system of ODEs

$$Z' = BZ = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} \quad \longleftrightarrow \quad \begin{cases} Z_1' = 2Z_1 + Z_2 \\ Z_2' = 2Z_2 \end{cases}$$

The solution to the second equation is $Z_2(z) = c_2e^{2x}$. We now have to solve $Z_1' = 2Z_1 + c_2e^{2x}$. The solution to the homogeneous part is $Z_{1,H}(x) = c_1e^{2x}$. The particular solution is

$$Z_{1,P}(x) = e^{2x} \int e^{-2x} c_2 e^{2x}\, dx = c_2xe^{2x}.$$

Therefore, we obtain

$$Z_1 = Z_{1,H} + Z_{1,P} = c_1e^{2x} + c_2xe^{2x}.$$

So, we have

$$Z = \begin{bmatrix} c_1e^{2x} + c_2xe^{2x} \\ c_2e^{2x} \end{bmatrix}.$$

Solve the general System.
We know that $Z = P^{-1}Y$ and therefore

$$Y = PZ = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_1e^{2x} + c_2xe^{2x} \\ c_2e^{2x} \end{bmatrix} = \begin{bmatrix} 2c_1e^{2x} + c_2(1+2x)e^{2x} \\ c_1e^{2x} + c_2xe^{2x} \end{bmatrix}.$$

Section 6.3, Problem 10

From the Problem 2, the general solution is

$$Y = \begin{bmatrix} 2c_1e^{2x} + c_2(1+2x)e^{2x} \\ c_1e^{2x} + c_2xe^{2x} \end{bmatrix}.$$

The initial condition is $Y(0) = \begin{bmatrix} 0 & -1 \end{bmatrix}^\top$ and this gives the following system for c_1 and c_2:

$$\begin{bmatrix} 2c_1 + c_2 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

We obtain $c_1 = -1$ and $c_2 = 2$ adn therefore

$$Y = \begin{bmatrix} -2e^{2x} + 2(1+2x)e^{2x} \\ -e^{2x} + 2xe^{2x} \end{bmatrix}.$$