Math 644

Chapter 5

Section 5.3: Removable Singularities

Contents

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THEOREM 1. Suppose f is analytic in $\Omega = \{z : 0 < |z-a| < \delta\}$ and suppose

$$\lim_{z \to a} (z-a) f(z) = 0.$$

Then f extends to be analytic in $\{z : |z-a| < \delta\}$.

Proof.

Let $0 < \varepsilon < |z-a| < r < \delta$. Then, by Cauchy's integral formula:

$$f(z) = \frac{1}{2\pi i} \int_{C_r} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i} \int_{C_\varepsilon} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

where C_r & C_ε are circles of radii r & ε and centered at a.

We have,

$$\left| \int_{C_\varepsilon} \frac{f(\zeta)}{\zeta - z}\, d\zeta \right| \leq \sup_{\zeta \in C_\varepsilon} |f(\zeta)| \int_{C_\varepsilon} \frac{|d\zeta|}{|z-a| - \varepsilon}$$

$$= \sup_{\zeta \in C_\varepsilon} |f(\zeta)| \frac{2\pi\varepsilon}{|z-a| - \varepsilon}$$

For $\zeta \in C_\varepsilon$, $\lim\limits_{\varepsilon \to 0} |f(\zeta)|\, \varepsilon = \lim\limits_{\zeta \to a} |f(\zeta)(\zeta - a)| = 0$

and since z is fixed

$$\limsup_{\varepsilon \to 0} \left| \int_{C_\varepsilon} \frac{f(\zeta)}{\zeta - z}\, d\zeta \right| = 0$$

Note:

- Important case: If f is bounded and analytic in a punctured neighborhood of a, then f extends to be analytic in a neighborhood of a.

So, $$f(z) = \frac{1}{2\pi i} \int_{C_r} \frac{f(\zeta)}{\zeta - z} \, d\zeta \quad (*)$$

The right-hand side is analytic by Lemma 2 in section 4.4 (in $\{z : |z-a| \le r\}$.

Extend f at a to be

$$f(a) = \frac{1}{2\pi i} \int_{C_r} \frac{f(\zeta)}{\zeta - a} \, d\zeta .$$

This extension is analytic by $(*)$. $\square$

DEFINITION 2. A compact set $E \subset \mathbb{C}$ has **one-dimensional Hausdorff measure equal to** 0 if for every $\varepsilon > 0$ there are finitely many disks D_j with radius r_j so that

$$E \subset \cup_j D_j \quad \text{and} \quad \sum_j r_j < \varepsilon.$$

THEOREM 3. Suppose $E \subset \mathbb{C}$ is a compact set with one-dimensional Hausdorff measure 0. If f is bounded and analytic on $U \backslash E$, where U is open and $E \subset U$, then f extends to be analytic in U.

Proof. Fix $U \supseteq E$ open & f analytic on $U \backslash E$.

Repeat the construction in Runge's theorem to find a cycle $\gamma \subseteq U \backslash E$ s.t.

1) γ is a finite union of poly. curves γ_j & the boundary of closed squares $\{S_j\}$.

2) $n(\gamma, a) = 0$ or 1 $\forall a \notin \gamma$.

3) $n(\gamma, b) = 1$, $\forall b \in \cup S_j \backslash \gamma \supseteq E$.

4) $n(\gamma, b) = 0$, $\forall b \notin \cup S_j$

From 4), we have $n(\gamma, b) = 0$, $\forall b \in \mathbb{C} \backslash U$.

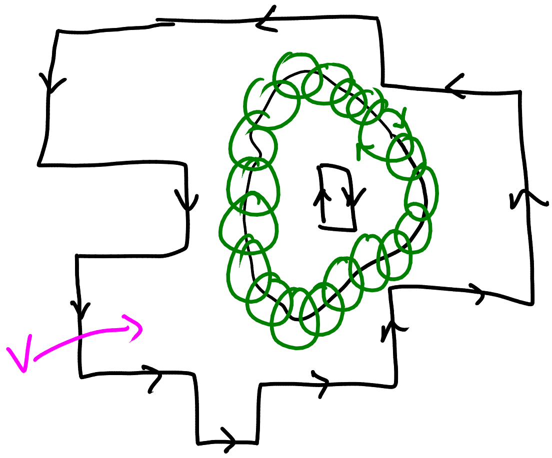
Take finitely many disks D_k such that

$E \subseteq \cup D_k$ & $\sum_k r_k < \varepsilon$ (given $\varepsilon > 0$).

Assume further that $D_k \cap E \neq \emptyset$.

Take ε small enough so that

$$D_k \subseteq \cup S_j \setminus \gamma .$$



Define $\quad V := \{z : n(\gamma, z) = 1\}$

$$\sigma := \partial(\cup D_k)$$

$$\Omega := V \setminus \cup \overline{D_k}$$

Then, $\partial\Omega = \sigma + \gamma$ parametrized so that $\partial\Omega$ has positive orientation. Therefore, since $U \setminus \cup_k \overline{D_k}$ is a region containing Ω & $\partial\Omega$ has pos. orientation

$$\Rightarrow n(\partial\Omega, a) = 0, \quad \forall a \notin \left(\overline{U \setminus \cup_k \overline{D_k}}\right)^c$$

Therefore, $\partial\Omega \sim 0$ in $U \setminus \cup_k \overline{D_k}$.

By Cauchy's integral formula:

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta - \frac{1}{2\pi i}\int_{-\sigma} \frac{f(\zeta)}{\zeta - z}\,d\zeta .$$

However, f is bounded on $\sigma \subseteq U \setminus E$

$$\Rightarrow \left|\int_\sigma \frac{f(\zeta)}{\zeta - z}\,d\zeta\right| \leq C(z) \sup_{\zeta\in\sigma}|f(\zeta)| \sum_j 2\pi r_j$$

$$\leq C(z) \sup_{\zeta\in\sigma}|f(\zeta)|\, 2\pi\, \varepsilon$$

therefore, as $\varepsilon \to 0$,

$$\int_{\sigma} \frac{f(\zeta)}{\zeta - z}\, d\zeta = 0$$

So,

$$f(z) = \underbrace{\frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z}\, d\zeta}_{\text{analytic on } \mathbb{C} \setminus \gamma \text{ (Lem. 4 sect. 4.4)}}$$

Extend f on E with the RHS. $\square$

Formula For the Inverse