

MATH 644

CHAPTER 4

SECTION 4.3: APPROXIMATION BY RATIONAL FUNCTIONS

CONTENTS

Cauchy Integral Formula For a Square	2
First Version of Runge's Theorem	4
Second Version With Control On the Poles	7
Components	10
Closed Components	10
Third Version of Runge's Theorem: Poles On The Boundary	11

THEOREM 1. If S is an open square with boundary ∂S parameterized in the counter-clockwise direction then

$$\frac{1}{2\pi i} \int_{\partial S} \frac{1}{z-a} dz = \begin{cases} 1 & \text{if } a \in S \\ 0 & \text{if } a \in \mathbb{C} \setminus \overline{S}. \end{cases}$$

Proof.

THEOREM 2. If f is analytic in a neighborhood of the closure of \overline{S} of an open square S , then, for $z \in S$,

$$f(z) = \frac{1}{2\pi i} \int_{\partial S} \frac{f(\zeta)}{\zeta - z} d\zeta,$$

where ∂S is parameterized in the counter-clockwise direction.

Proof.

COROLLARY 3. If f is analytic in a neighborhood of the closure of \overline{S} of an open square S , then

$$\int_{\partial S} f(\zeta) d\zeta = 0.$$

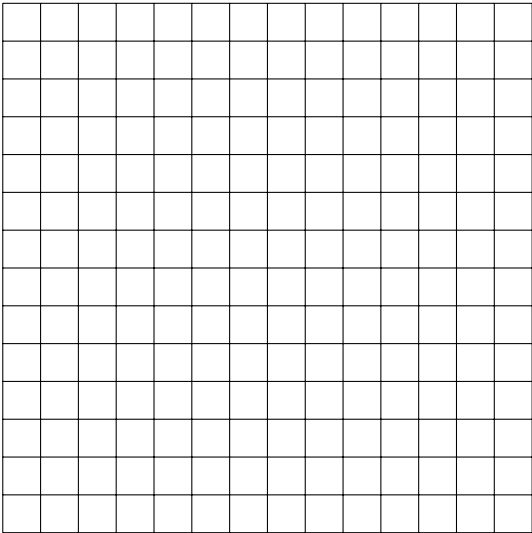
Proof.

FIRST VERSION OF RUNGE'S THEOREM

THEOREM 4. If f is analytic on a compact set K , and if $\varepsilon > 0$, then there is a rational function r so that

$$\sup_{z \in K} |f(z) - r(z)| < \varepsilon.$$

Proof.



DEFINITION 5. Let $r(z) = p(z)/q(z)$ be a rational function where p and q are two polynomials with no common zeros. The zeros of q are called the **poles** of the rational function r .

Note:

- If b is a pole of a rational function r , then $|r(z)| \rightarrow \infty$ as $|z| \rightarrow \infty$.

LEMMA 6. Suppose that U is a region and suppose $b \in U$. Then a rational function with poles in U can be uniformly approximated on $\mathbb{C} \setminus U$ by a rational function with poles only at b .

Proof.

COROLLARY 7. Suppose U is a region and suppose $\{z : |z| > R\} \subset U$ for some $R < \infty$. Then a rational function with poles only in U can be uniformly approximated on $\mathbb{C} \setminus U$ by a polynomial.

Proof.

Components

DEFINITION 8. Let U be an open set.

- a) A polygonal curve in U is a curve consisting of a finite union of line segments.
- b) For $a, b \in U$, define $a \sim_U b$ if and only if there is a polygonal curve contained in U with edges parallel to the axis and joining a to b .

THEOREM 9. Let $U \subset \mathbb{C}$ be an open set.

- a) Show that the equivalence classes of \sim_U are closed and open (relative to U) and connected.
- b) Show that there are countably many equivalent classes.

Note:

- The equivalence classes are called the **components** of U . They are the maximal connected subsets of U .

Closed Components

DEFINITION 10. Suppose $K \subset \mathbb{C}$ is a compact set.

- a) For $a, b \in K$, define $a \sim_K b$ if and only if there is a connected subset of K containing a and b .

THEOREM 11. Let $K \subset \mathbb{C}$ be a compact set.

- a) Show that the equivalence classes of \sim_K are connected and closed.
- b) Show that there might be infinitely many equivalence classes.

Note:

- The equivalence classes are called the **(closed) components** of K .

THEOREM 12. [Runge] Suppose K is a compact set. Choose one point a_n in each bounded component of U_n in $\mathbb{C} \setminus K$. If f is analytic on K and $\varepsilon > 0$, then we can find a rational function r with poles only in the set $\{a_n\}$ such that

$$\sup_{z \in K} |f(z) - r(z)| < \varepsilon.$$

If $\mathbb{C} \setminus K$ has no bounded components, then we may take r to be a polynomial.

Proof.

THEOREM 13. If f is analytic on an open set $\Omega \neq \mathbb{C}$, then there is a sequence of rational functions r_n with poles in $\partial\Omega$ so that r_n converges to f uniformly on compact subsets of Ω .

Proof.

Note:

- The improvement of Theorem 9 over Theorem 4 is that the poles of r_n are outside of Ω , not just outside the compact set K on which r_n is close to f .