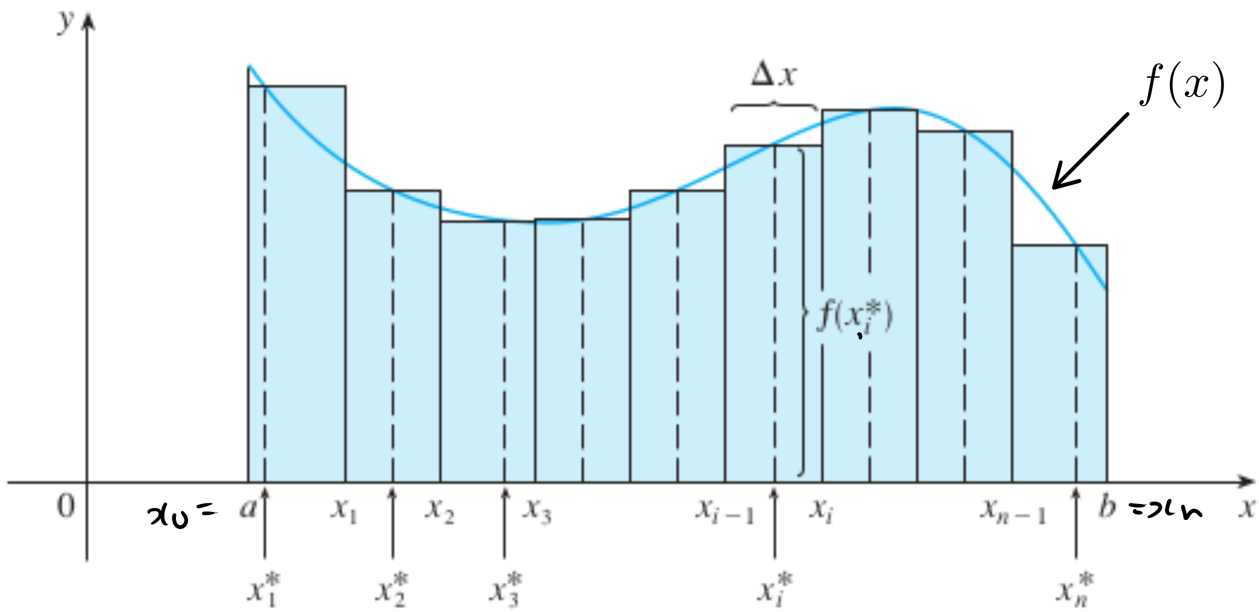
Chapter 4
Integrals

4.2 The Definite Integral

Sums with sample points.



1) Equidistributed numbers

Chose numbers $x_0 = a, x_1, x_2, \ldots, x_{n-1}$
$x_n = b$. $\Delta x =$ distance between x_i.

2) Sample points within $[x_{i-1}, x_i]$.

Choose points $x_1^*, x_2^*, \ldots, x_n^*$
within each subinterval.

Area using a random point in $[x_{i-1}, x_i]$.

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \, \Delta x$$

2 Definition of a Definite Integral If f is a function defined for $a \leqslant x \leqslant b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0 \,(= a), x_1, x_2, \ldots, x_n \,(= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any **sample points** in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the **definite integral of f from a to b** is

integral $\longrightarrow$
$$\boxed{\int_a^b f(x)\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\, \Delta x}$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is **integrable** on $[a, b]$.

Remarks:

1) Terminology.

$\int$: integral
a & b : lower & upper limits
$f(x)$: integrand
dx : independant variable.

2) Integral is a number!

$\int_a^b f(x)\,dx$ doesn't
any x (value)
$\Rightarrow$ number

3) Riemann Sums.

$$\sum_{i=1}^{n} f(x_i^*) \, \Delta x.$$

4) Net Area.

- $f(x) \geq 0 \to \int_a^b f(x)\,dx$ is the area.

So, when $f(x) \geq 0$ sometimes and $f(x) \leq 0$ some other times, then

$$\int_a^b f(x)\,dx = A_1 - A_2 + A_3$$

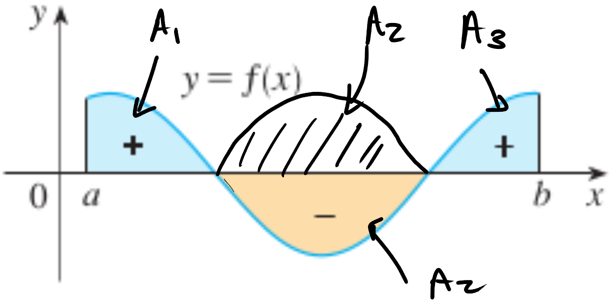


FIGURE 4

$\int_a^b f(x)\,dx$ is the net area.

5) Integrable functions.

> **3 Theorem** If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$; that is, the definite integral $\int_a^b f(x)\,dx$ exists.

Right endpoints formula.

> **4 Theorem** If f is integrable on $[a, b]$, then
>
> $$\int_a^b f(x)\,dx = \boxed{\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x}$$
>
> where $\boxed{\Delta x = \frac{b-a}{n}}$ and $x_i = a + i\,\Delta x$ $\;\; x_i^*$

EXAMPLE 1 Express

$$\lim_{n\to\infty} \boxed{\sum_{i=1}^{n} (x_i^3 + x_i \sin x_i)\,\Delta x}$$

$f(x)$? Δx?

as an integral on the interval $[0, \pi]$.

$a = 0$
$b = \pi$

$x_i^3 + x_i \sin(x_i) = f(x_i)$

where $f(x) = x^3 + x\sin x$

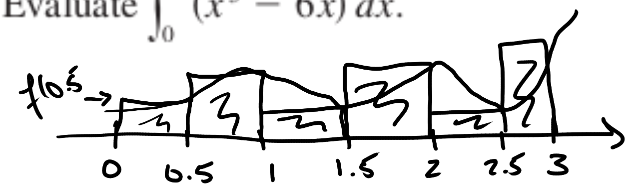
$$\lim_{n\to\infty} \sum_{i=1}^{n} (x_i^3 + x_i \sin(x_i))\,\Delta x = \int_0^{\pi} x^3 + x\sin x\,dx$$

EXAMPLE 2

(a) Evaluate the Riemann sum for $f(x) = x^3 - 6x$, taking the sample points to be right endpoints and $a = 0$, $b = 3$, and $n = 6$.

(b) Evaluate $\int_0^3 (x^3 - 6x)\,dx$.

(a)



$$\Delta x = \frac{3-0}{6} = \frac{1}{2} = 0.5$$

$x_1 = 0.5$ $\quad x_3 = 1.5$ $\quad x_5 = 2.5$

$x_2 = 1$ $\quad x_4 = 2$ $\quad x_6 = 3$

$$R_6 = f(0.5)\cdot 0.5 + f(1)\cdot 0.5 + f(1.5)\cdot 0.5 + f(2)\cdot 0.5 + f(2.5)\cdot 0.5 + f(3)\cdot 0.5$$

$$= -3.9375.$$

(b) R-E formula. $\int_0^3 x^3 - 6x\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x\, f(x_i)$.

$\Delta x = \frac{b-a}{n} = \frac{3}{n}$ & $x_i = a + i\,\Delta x = \frac{3i}{n}$.

So,

$$\sum_{i=1}^{n} \Delta x\, f(x_i) = \sum_{i=1}^{n} \left(\frac{3}{n}\right)\left[\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)\right]$$

$$= \sum_{i=1}^{n} \frac{3}{n}\left[\frac{27i^3}{n^3} - \frac{18i}{n}\right]$$

$$= \sum_{i=1}^{n} \left(\frac{81i^3}{n^4} - \frac{54i}{n^2}\right)$$

$\sum(a_i \pm b_i) = \sum a_i \pm \sum b_i$

$$= \sum_{i=1}^{n} \frac{81i^3}{n^4} - \sum_{i=1}^{n} \frac{54i}{n^2}$$

$\sum 2a_i = 2\sum a_i$

$$= \frac{81}{n^4} \underbrace{\sum_{i=1}^{n} i^3} - \frac{54}{n^2} \underbrace{\sum_{i=1}^{n} i}$$

$$= \frac{81}{n^4}\left(\frac{n(n+1)}{2}\right)^2 - \frac{54}{n^2}\,\frac{n(n+1)}{2}$$

$$= \frac{81}{4}\left(1 + \frac{1}{n}\right) - \frac{54}{2}\left(1 + \frac{1}{n}\right)$$

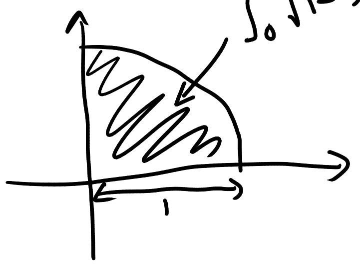
So, $\lim \frac{81}{4}\left(1 + \cancel{\frac{1}{n}}^{\,0}\right) - \frac{54}{2}\left(1 + \cancel{\frac{1}{n}}^{\,0}\right) = \frac{81}{4} - \frac{54}{2} = \boxed{-\frac{27}{4}}$

EXAMPLE 4 Evaluate the following integrals by interpreting each in terms of areas.

(a) $\int_{0=a}^{1=b} \underbrace{\sqrt{1-x^2}}_{f(x)}\,dx$ (b) $\int_0^3 (x-1)\,dx$ $f(x) = x-1$, $a=0$, $b=3$

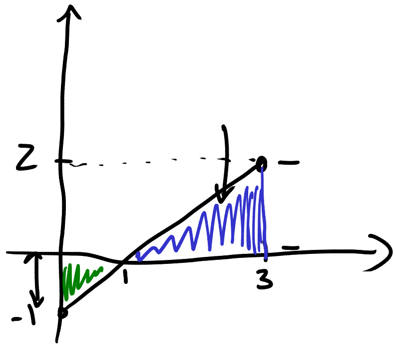
(a) $f(x) = \sqrt{1-x^2} \geq 0$

$\int_0^1 \sqrt{1-x^2}\,dx$



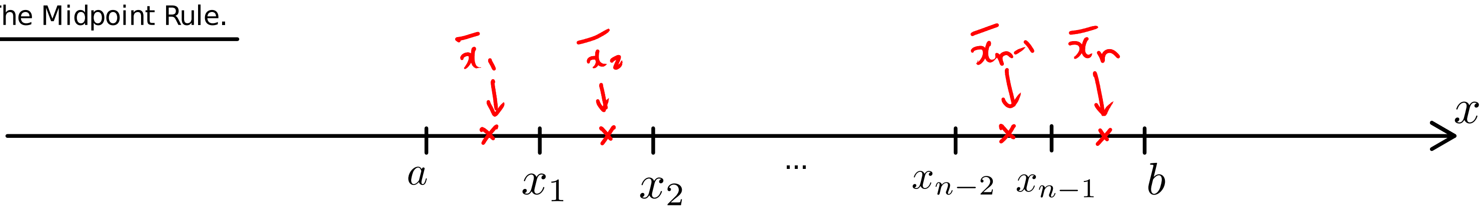
$\rightarrow \int_0^1 \sqrt{1-x^2}\,dx = \boxed{\frac{\pi}{4}}$

(b)



$$\int_0^3 x-1\,dx = A(\triangle) - A(\triangledown)$$
$$= \frac{2\times 2}{2} - \frac{1.1}{2}$$
$$= 2 - \frac{1}{2}$$
$$= \frac{3}{2} = \boxed{1.5}$$

The Midpoint Rule.



Your sample points are $x_i^* = \dfrac{x_{i-1} + x_i}{2} = \bar{x}_i$

Midpoint Rule

$$\int_a^b f(x)\,dx \approx \boxed{\sum_{i=1}^{n} f(\bar{x}_i)\,\Delta x} = \Delta x\,[f(\bar{x}_1) + \cdots + f(\bar{x}_n)]$$

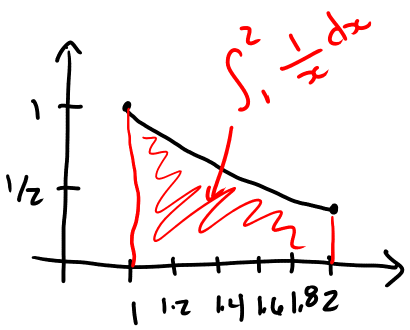
where $\Delta x = \dfrac{b-a}{n}$

and $\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i) = \text{midpoint of } [x_{i-1}, x_i]$

EXAMPLE 5 Use the Midpoint Rule with $n = 5$ to approximate $\int_1^2 \frac{1}{x}\,dx$. $\to \ln(2)$

$\hookrightarrow f(x) = \frac{1}{x}$

① Sketch.



② Data

$a = 1$, $b = 2$ $\qquad \Delta x = \dfrac{2-1}{5} = \dfrac{1}{5} = 0.2$

$x_1 = 1 + 0.2 = 1.2 \qquad x_3 = 1.6 \qquad x_5 = 2$

$x_2 = 1.4 \qquad x_4 = 1.8$

$\bar{x}_1 = \dfrac{1 + 1.2}{2} = 1.1 \qquad \bar{x}_3 = 1.5 \qquad \bar{x}_5 = 1.9$

$\bar{x}_2 = 1.3 \qquad \bar{x}_4 = 1.7$

③ Rieman Sum midpoint rule.

$$\int_1^2 \frac{1}{x}\,dx \approx \Delta x f(\bar{x}_1) + \Delta x f(\bar{x}_2) + \Delta x f(\bar{x}_3) + \Delta x f(\bar{x}_4) + \Delta x f(\bar{x}_5)$$

$\Delta x = 0.2$

$$= \boxed{0.6919.}$$

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

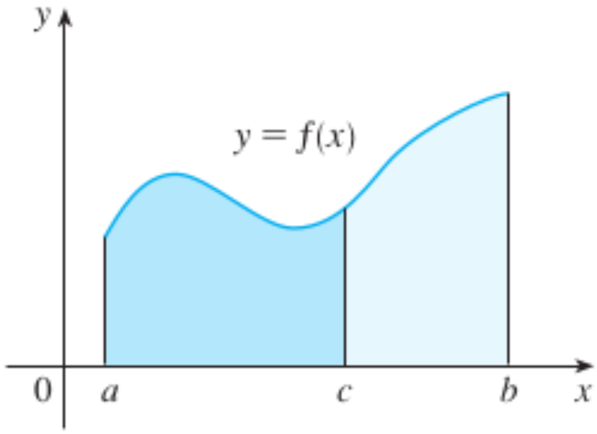
$$\int_a^a f(x)\,dx = 0$$

Properties of the Integral

1. $\int_a^b c\,dx = c(b-a)$, where c is any constant
2. $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, where c is any constant
4. $\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

EXAMPLE 6 Use the properties of integrals to evaluate $\int_0^1 (4 + 3x^2)\,dx$.

5. $$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$



EXAMPLE 7 If it is known that $\int_0^{10} f(x)\,dx = 17$ and $\int_0^8 f(x)\,dx = 12$, find $\int_8^{10} f(x)\,dx$.

Comparison Properties of the Integral

6. If $f(x) \geqslant 0$ for $a \leqslant x \leqslant b$, then $\int_a^b f(x)\,dx \geqslant 0$.

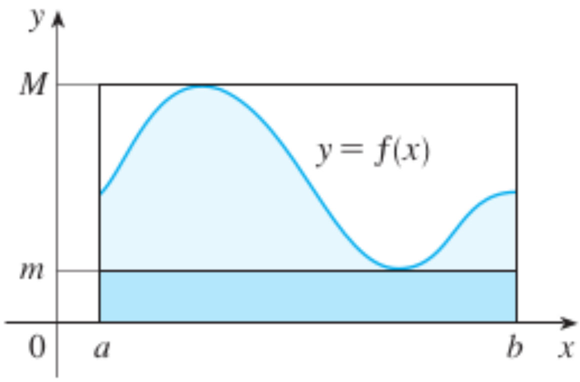
7. If $f(x) \geqslant g(x)$ for $a \leqslant x \leqslant b$, then $\int_a^b f(x)\,dx \geqslant \int_a^b g(x)\,dx$.

8. If $m \leqslant f(x) \leqslant M$ for $a \leqslant x \leqslant b$, then

$$m(b-a) \leqslant \int_a^b f(x)\,dx \leqslant M(b-a)$$

7.

8.



EXAMPLE 8 Use Property 8 to estimate $\int_1^4 \sqrt{x}\,dx$.