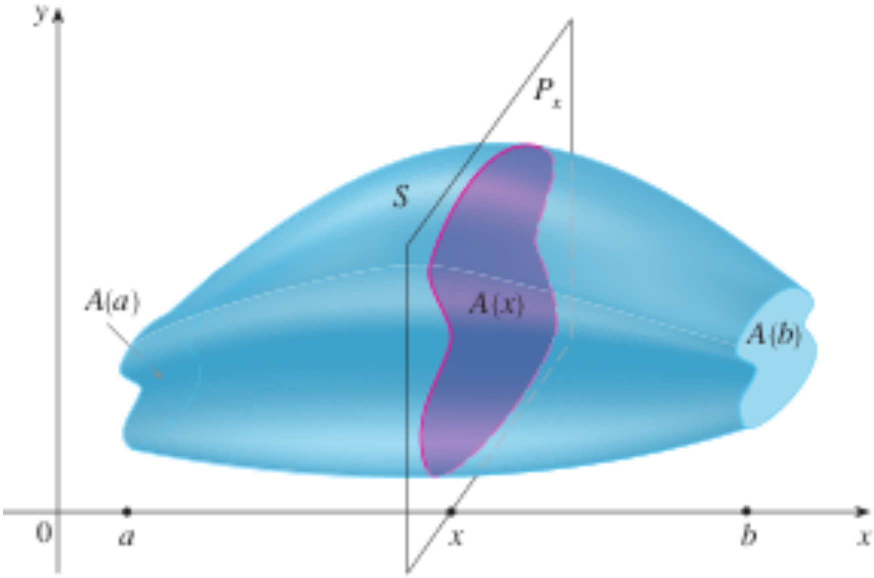
# Chapter 5
# Applications in integration

5.2 Volumes




S: Object (or solid)

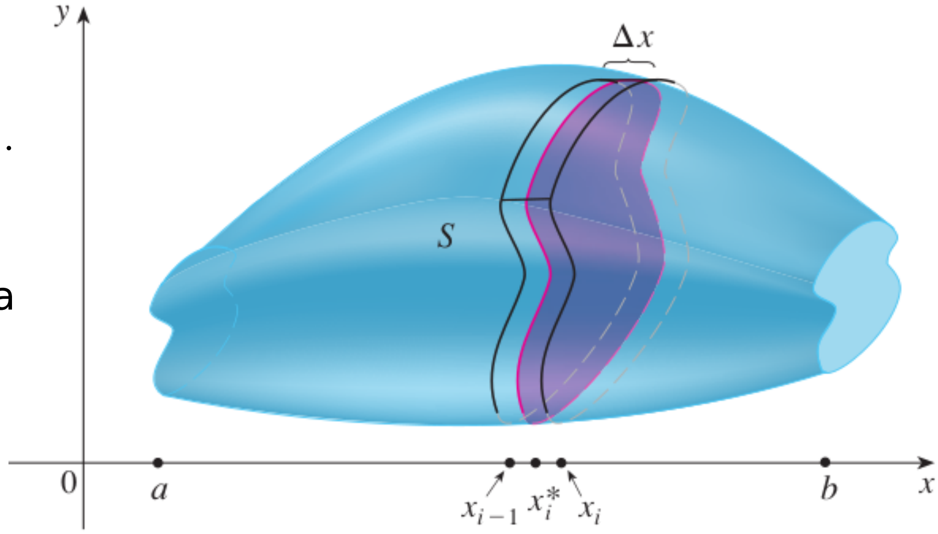
$P_x$: Cross-section at x.

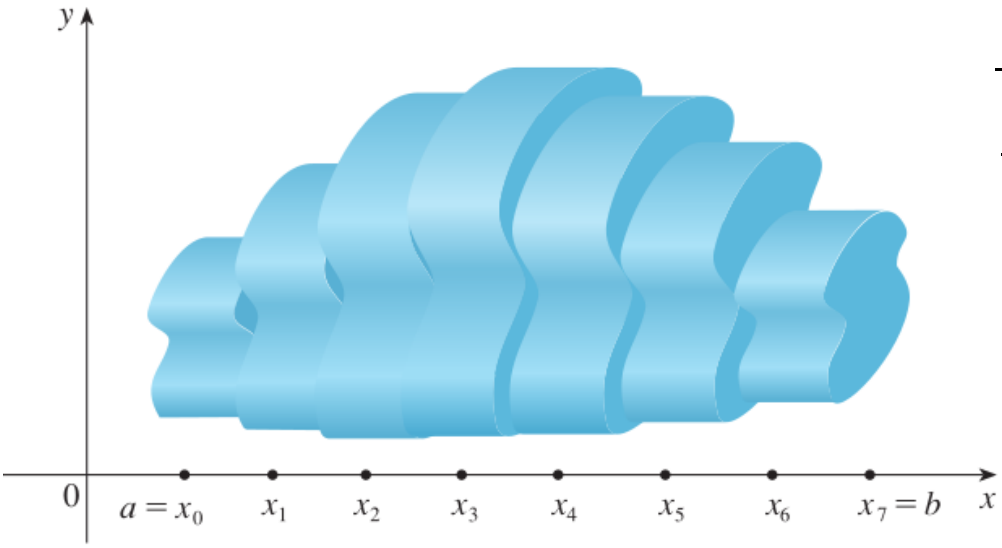
A(x): Area of the cross-section.

- Cut the solid in n slices

$$P_{x_1}, P_{x_2}, \ldots, P_{x_n}.$$

- Select sample points in each subinterval.
- Approximate the slice by a bunch of cylinders:




- $\mathrm{Vol(Slide)} = A(x_i^*)\Delta x.$
- Sum the volume of all slices:

$$V \approx \sum_{i=1}^{n} A(x_i^*)\Delta x$$

- Taking $n \to \infty$

Formula for the volume of a generic solid:

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)\,dx.$$

**EXAMPLE 1** Show that the volume of a sphere of radius $r$ is $V = \frac{4}{3}\pi r^3$.

Solid of revolution.

Rotation about the x-axis.

**EXAMPLE 2** Find the volume of the solid obtained by rotating about the $x$-axis the region under the curve $y = \sqrt{x}$ from 0 to 1. Illustrate the definition of volume by sketching a typical approximating cylinder.

## Rotation around the y-axis.

**EXAMPLE 3** Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$, and $x = 0$ about the $y$-axis.

Cross-section as a washer.

**EXAMPLE 4** The region $\mathscr{R}$ enclosed by the curves $y = x$ and $y = x^2$ is rotated about the $x$-axis. Find the volume of the resulting solid.

## Rotation about another line.

**EXAMPLE 5** Find the volume of the solid obtained by rotating the region in Example 4 about the line $y = 2$.

# In summary.

- If the cross-section is a disk (as in Examples 1–3), we find the radius of the disk (in terms of $x$ or $y$) and use

$$A = \pi(\text{radius})^2$$

- If the cross-section is a washer (as in Examples 4 and 5), we find the inner radius $r_{\text{in}}$ and outer radius $r_{\text{out}}$ from a sketch (as in Figures 8, 9, and 10) and compute the area of the washer by subtracting the area of the inner disk from the area of the outer disk:

$$A = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2$$