1.6 EXERCISES

1. Given that

$$\lim_{x \to 2} f(x) = 4 \qquad \lim_{x \to 2} g(x) = -2 \qquad \lim_{x \to 2} h(x) = 0$$

find the limits that exist. If the limit does not exist, explain why.

(a) $\lim_{x \to 2} [f(x) + 5g(x)]$ (b) $\lim_{x \to 2} [g(x)]^3$

(c) $\lim_{x \to 2} \sqrt{f(x)}$ (d) $\lim_{x \to 2} \dfrac{3f(x)}{g(x)}$

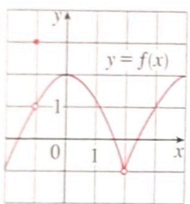
(e) $\lim_{x \to 2} \dfrac{g(x)}{h(x)}$ (f) $\lim_{x \to 2} \dfrac{g(x)h(x)}{f(x)}$

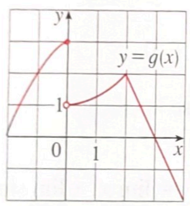
2. The graphs of f and g are given. Use them to evaluate each limit, if it exists. If the limit does not exist, explain why.

(a) $\lim_{x \to 2} [f(x) + g(x)]$ (b) $\lim_{x \to 0} [f(x) - g(x)]$

(c) $\lim_{x \to -1} [f(x)g(x)]$ (d) $\lim_{x \to 3} \dfrac{f(x)}{g(x)}$

(e) $\lim_{x \to 2} [x^2 f(x)]$ (f) $f(-1) + \lim_{x \to -1} g(x)$





3–9 Evaluate the limit and justify each step by indicating the appropriate Limit Law(s).

3. $\lim_{x \to 3} (5x^3 - 3x^2 + x - 6)$

4. $\lim_{x \to -1} (x^4 - 3x)(x^2 + 5x + 3)$

5. $\lim_{t \to -2} \dfrac{t^4 - 2}{2t^2 - 3t + 2}$

6. $\lim_{u \to -2} \sqrt{u^4 + 3u + 6}$

7. $\lim_{x \to 8} (1 + \sqrt[3]{x})(2 - 6x^2 + x^3)$

8. $\lim_{t \to 2} \left(\dfrac{t^2 - 2}{t^3 - 3t + 5} \right)^2$

9. $\lim_{x \to 2} \sqrt{\dfrac{2x^2 + 1}{3x - 2}}$

10. (a) What is wrong with the following equation?

$$\frac{x^2 + x - 6}{x - 2} = x + 3$$

(b) In view of part (a), explain why the equation

$$\lim_{x \to 2} \frac{x^2 + x - 6}{x - 2} = \lim_{x \to 2} (x + 3)$$

is correct.

11–32 Evaluate the limit, if it exists.

11. $\lim_{x \to 5} \dfrac{x^2 - 6x + 5}{x - 5}$

12. $\lim_{x \to -3} \dfrac{x^2 + 3x}{x^2 - x - 12}$

13. $\lim_{x \to 5} \dfrac{x^2 - 5x + 6}{x - 5}$

14. $\lim_{x \to 4} \dfrac{x^2 + 3x}{x^2 - x - 12}$

15. $\lim_{t \to -3} \dfrac{t^2 - 9}{2t^2 + 7t + 3}$

16. $\lim_{x \to -1} \dfrac{2x^2 + 3x + 1}{x^2 - 2x - 3}$

17. $\lim_{h \to 0} \dfrac{(-5 + h)^2 - 25}{h}$

18. $\lim_{h \to 0} \dfrac{(2 + h)^3 - 8}{h}$

19. $\lim_{x \to -2} \dfrac{x + 2}{x^3 + 8}$

20. $\lim_{t \to 1} \dfrac{t^4 - 1}{t^3 - 1}$

21. $\lim_{h \to 0} \dfrac{\sqrt{9 + h} - 3}{h}$

22. $\lim_{u \to 2} \dfrac{\sqrt{4u + 1} - 3}{u - 2}$

23. $\lim_{x \to 3} \dfrac{\dfrac{1}{x} - \dfrac{1}{3}}{x - 3}$

24. $\lim_{h \to 0} \dfrac{(3 + h)^{-1} - 3^{-1}}{h}$

25. $\lim_{t \to 0} \dfrac{\sqrt{1 + t} - \sqrt{1 - t}}{t}$

26. $\lim_{t \to 0} \left(\dfrac{1}{t} - \dfrac{1}{t^2 + t} \right)$

27. $\lim_{x \to 16} \dfrac{4 - \sqrt{x}}{16x - x^2}$

28. $\lim_{x \to 2} \dfrac{x^2 - 4x + 4}{x^4 - 3x^2 - 4}$

29. $\lim_{t \to 0} \left(\dfrac{1}{t\sqrt{1 + t}} - \dfrac{1}{t} \right)$

30. $\lim_{x \to -4} \dfrac{\sqrt{x^2 + 9} - 5}{x + 4}$

31. $\lim_{h \to 0} \dfrac{(x + h)^3 - x^3}{h}$

32. $\lim_{h \to 0} \dfrac{\dfrac{1}{(x + h)^2} - \dfrac{1}{x^2}}{h}$

33. (a) Estimate the value of

$$\lim_{x \to 0} \frac{x}{\sqrt{1 + 3x} - 1}$$

by graphing the function $f(x) = x/(\sqrt{1 + 3x} - 1)$.

(b) Make a table of values of $f(x)$ for x close to 0 and guess the value of the limit.

(c) Use the Limit Laws to prove that your guess is correct.

34. (a) Use a graph of

$$f(x) = \frac{\sqrt{3 + x} - \sqrt{3}}{x}$$

to estimate the value of $\lim_{x \to 0} f(x)$ to two decimal places.

(b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.

(c) Use the Limit Laws to find the exact value of the limit.

35. Use the Squeeze Theorem to show that $\lim_{x\to 0}(x^2\cos 20\pi x) = 0$. Illustrate by graphing the functions $f(x) = -x^2$, $g(x) = x^2\cos 20\pi x$, and $h(x) = x^2$ on the same screen.

36. Use the Squeeze Theorem to show that

$$\lim_{x\to 0}\sqrt{x^3 + x^2}\,\sin\frac{\pi}{x} = 0$$

Illustrate by graphing the functions f, g, and h (in the notation of the Squeeze Theorem) on the same screen.

37. If $4x - 9 \leqslant f(x) \leqslant x^2 - 4x + 7$ for $x \geqslant 0$, find $\lim_{x\to 4} f(x)$.

38. If $2x \leqslant g(x) \leqslant x^4 - x^2 + 2$ for all x, evaluate $\lim_{x\to 1} g(x)$.

39. Prove that $\lim_{x\to 0} x^4\cos\dfrac{2}{x} = 0$.

40. Prove that $\lim_{x\to 0^+}\sqrt{x}\,[1 + \sin^2(2\pi/x)] = 0$.

41–46 Find the limit, if it exists. If the limit does not exist, explain why.

41. $\lim_{x\to 3}(2x + |x - 3|)$

42. $\lim_{x\to -6}\dfrac{2x + 12}{|x + 6|}$

43. $\lim_{x\to 0.5^-}\dfrac{2x - 1}{|2x^3 - x^2|}$

44. $\lim_{x\to -2}\dfrac{2 - |x|}{2 + x}$

45. $\lim_{x\to 0^-}\left(\dfrac{1}{x} - \dfrac{1}{|x|}\right)$

46. $\lim_{x\to 0^+}\left(\dfrac{1}{x} - \dfrac{1}{|x|}\right)$

47. The *signum* (or sign) *function*, denoted by sgn, is defined by

$$\operatorname{sgn} x = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

(a) Sketch the graph of this function.
(b) Find each of the following limits or explain why it does not exist.
(i) $\lim_{x\to 0^+}\operatorname{sgn} x$ (ii) $\lim_{x\to 0^-}\operatorname{sgn} x$
(iii) $\lim_{x\to 0}\operatorname{sgn} x$ (iv) $\lim_{x\to 0}|\operatorname{sgn} x|$

48. Let $g(x) = \operatorname{sgn}(\sin x)$.
(a) Find each of the following limits or explain why it does not exist.
(i) $\lim_{x\to 0^+} g(x)$ (ii) $\lim_{x\to 0^-} g(x)$ (iii) $\lim_{x\to 0} g(x)$
(iv) $\lim_{x\to \pi^+} g(x)$ (v) $\lim_{x\to \pi^-} g(x)$ (vi) $\lim_{x\to \pi} g(x)$
(b) For which values of a does $\lim_{x\to a} g(x)$ not exist?
(c) Sketch a graph of g.

49. Let $g(x) = \dfrac{x^2 + x - 6}{|x - 2|}$.
(a) Find
(i) $\lim_{x\to 2^+} g(x)$ (ii) $\lim_{x\to 2^-} g(x)$
(b) Does $\lim_{x\to 2} g(x)$ exist?
(c) Sketch the graph of g.

50. Let

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1 \\ (x - 2)^2 & \text{if } x \geqslant 1 \end{cases}$$

(a) Find $\lim_{x\to 1^-} f(x)$ and $\lim_{x\to 1^+} f(x)$.
(b) Does $\lim_{x\to 1} f(x)$ exist?
(c) Sketch the graph of f.

51. Let

$$B(t) = \begin{cases} 4 - \frac{1}{2}t & \text{if } t < 2 \\ \sqrt{t + c} & \text{if } t \geqslant 2 \end{cases}$$

Find the value of c so that $\lim_{t\to 2} B(t)$ exists.

52. Let

$$g(x) = \begin{cases} x & \text{if } x < 1 \\ 3 & \text{if } x = 1 \\ 2 - x^2 & \text{if } 1 < x \leqslant 2 \\ x - 3 & \text{if } x > 2 \end{cases}$$

(a) Evaluate each of the following, if it exists.
(i) $\lim_{x\to 1^-} g(x)$ (ii) $\lim_{x\to 1} g(x)$ (iii) $g(1)$
(iv) $\lim_{x\to 2^-} g(x)$ (v) $\lim_{x\to 2^+} g(x)$ (vi) $\lim_{x\to 2} g(x)$
(b) Sketch the graph of g.

53. (a) If the symbol $[\![\ ]\!]$ denotes the greatest integer function defined in Example 10, evaluate
(i) $\lim_{x\to -2^+}[\![x]\!]$ (ii) $\lim_{x\to -2}[\![x]\!]$ (iii) $\lim_{x\to -2.4}[\![x]\!]$
(b) If n is an integer, evaluate
(i) $\lim_{x\to n^-}[\![x]\!]$ (ii) $\lim_{x\to n^+}[\![x]\!]$
(c) For what values of a does $\lim_{x\to a}[\![x]\!]$ exist?

54. Let $f(x) = [\![\cos x]\!]$, $-\pi \leqslant x \leqslant \pi$.
(a) Sketch the graph of f.
(b) Evaluate each limit, if it exists.
(i) $\lim_{x\to 0} f(x)$ (ii) $\lim_{x\to(\pi/2)^-} f(x)$
(iii) $\lim_{x\to(\pi/2)^+} f(x)$ (iv) $\lim_{x\to \pi/2} f(x)$
(c) For what values of a does $\lim_{x\to a} f(x)$ exist?

55. If $f(x) = [\![x]\!] + [\![-x]\!]$, show that $\lim_{x\to 2} f(x)$ exists but is not equal to $f(2)$.

56. In the theory of relativity, the Lorentz contraction formula

$$L = L_0\sqrt{1 - v^2/c^2}$$

expresses the length L of an object as a function of its velocity v with respect to an observer, where L_0 is the length of the object at rest and c is the speed of light. Find $\lim_{v\to c^-} L$ and interpret the result. Why is a left-hand limit necessary?

57. If p is a polynomial, show that $\lim_{x\to a} p(x) = p(a)$.

58. If r is a rational function, use Exercise 57 to show that $\lim_{x\to a} r(x) = r(a)$ for every number a in the domain of r.

59. If $\lim_{x \to 1} \dfrac{f(x) - 8}{x - 1} = 10$, find $\lim_{x \to 1} f(x)$.

60. If $\lim_{x \to 0} \dfrac{f(x)}{x^2} = 5$, find the following limits.

(a) $\lim_{x \to 0} f(x)$ (b) $\lim_{x \to 0} \dfrac{f(x)}{x}$

61. If

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

prove that $\lim_{x \to 0} f(x) = 0$.

62. Show by means of an example that $\lim_{x \to a} [f(x) + g(x)]$ may exist even though neither $\lim_{x \to a} f(x)$ nor $\lim_{x \to a} g(x)$ exists.

63. Show by means of an example that $\lim_{x \to a} [f(x)\, g(x)]$ may exist even though neither $\lim_{x \to a} f(x)$ nor $\lim_{x \to a} g(x)$ exists.

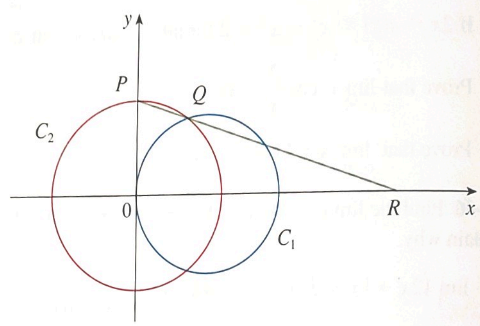
64. Evaluate $\lim_{x \to 2} \dfrac{\sqrt{6 - x} - 2}{\sqrt{3 - x} - 1}$.

65. Is there a number a such that

$$\lim_{x \to -2} \frac{3x^2 + ax + a + 3}{x^2 + x - 2}$$

exists? If so, find the value of a and the value of the limit.

66. The figure shows a fixed circle C_1 with equation $(x - 1)^2 + y^2 = 1$ and a shrinking circle C_2 with radius r and center the origin. P is the point $(0, r)$, Q is the upper point of intersection of the two circles, and R is the point of intersection of the line PQ and the x-axis. What happens to R as C_2 shrinks, that is, as $r \to 0^+$?



1.7 The Precise Definition of a Limit

The intuitive definition of a limit given in Section 1.5 is inadequate for some purposes because such phrases as "x is close to 2" and "$f(x)$ gets closer and closer to L" are vague. In order to be able to prove conclusively that

$$\lim_{x \to 0} \left(x^3 + \frac{\cos 5x}{10{,}000} \right) = 0.0001 \qquad \text{or} \qquad \lim_{x \to 0} \frac{\sin x}{x} = 1$$

we must make the definition of a limit precise.

To motivate the precise definition of a limit, let's consider the function

$$f(x) = \begin{cases} 2x - 1 & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$$

Intuitively, it is clear that when x is close to 3 but $x \neq 3$, then $f(x)$ is close to 5, and so $\lim_{x \to 3} f(x) = 5$.

To obtain more detailed information about how $f(x)$ varies when x is close to 3, we ask the following question:

How close to 3 does x have to be so that $f(x)$ differs from 5 by less than 0.1?

The distance from x to 3 is $|x - 3|$ and the distance from $f(x)$ to 5 is $|f(x) - 5|$, so our problem is to find a number δ such that

It is traditional to use the Greek letter δ (delta) in this situation.

$$|f(x) - 5| < 0.1 \qquad \text{if} \qquad |x - 3| < \delta \quad \text{but } x \neq 3$$

If $|x - 3| > 0$, then $x \neq 3$, so an equivalent formulation of our problem is to find a number δ such that

$$|f(x) - 5| < 0.1 \qquad \text{if} \qquad 0 < |x - 3| < \delta$$