Math 307

Chapter 1

Section 1.1: Systems of Linear Equations

Contents

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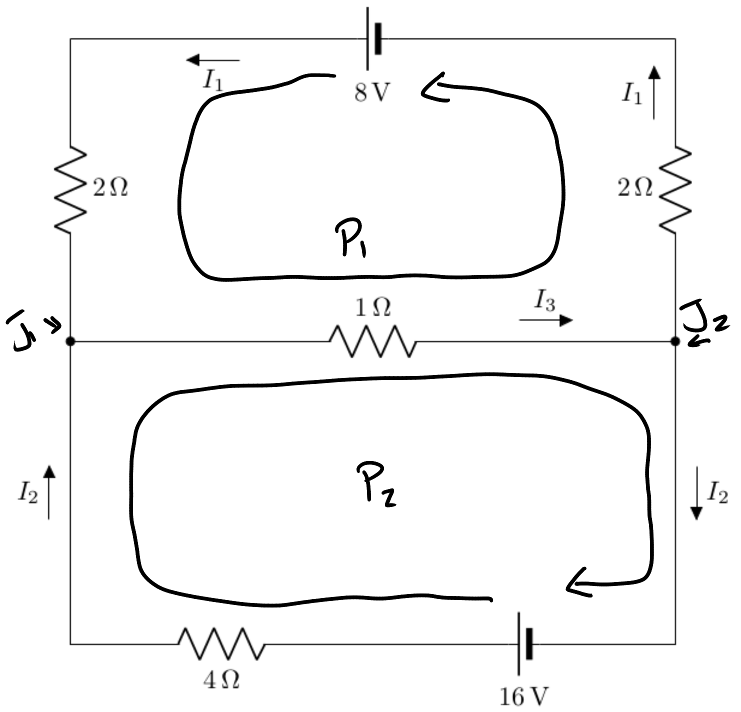
Why do we care about Systems of Linear Equations?



Ohm's Law

- Voltage drop at a resistor is given by $V = IR$.

Kirchhorff's Laws

- Junction: Current flowing into a junction must flow out of it.
- Path: Sum of IR terms in any direction around a closed path is equal to the total voltage in the path in that direction.

Goal: Find the values of I_1, I_2, I_3

J1) $I_1 + I_2 = I_3$

J2) $I_3 = I_1 + I_2$ $\longrightarrow$ $I_1 + I_2 - I_3 = 0$

PATH. P1) $2I_1 + I_3 + 2I_1 = 8$ $\rightarrow$ $4I_1 + I_3 = 8$

P2) $4I_2 + I_3 = 16$

To find I_1, I_2, I_3, we must solve the system of lin. eqs.:

$$\begin{cases} I_1 + I_2 - I_3 = 0 \\ 4I_1 \qquad + I_3 = 8 \\ \qquad 4I_2 + I_3 = 16 \end{cases}$$

Linear Equation

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b$$

where

- a_1, a_2, …, a_n are constants.
- n is the number of variables.
- x_1, x_2, …, x_n are the variables (unknowns).
- b is the right-hand side constant term.

Systems of Linear Equations

last example
n=3
m=3

$$\begin{array}{rl}
\text{no 1)} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
\text{no 2)} & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
 & \vdots \\
\text{no m)} & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m
\end{array}$$

where

- m is the number of linear equations.
- n is the number of variables.
- a_{11}, …, a_{mn} are constants.
- b_1, b_2, …, b_m are the right-hand side constant terms.
- x_1, …, x_n are the variables (unknowns).

Solution of a System of Linear Equations

A list $(x_1^*, x_2^*, \ldots, x_n^*)$ is a solution to a system of linear equations if it satisfies each equation of the system.

Going back to our previous example

$(1, 3, 4)$ is a solution to our system in the last example.

$\uparrow$ $\uparrow$ $\uparrow$
x_1 x_2 x_3

$1 + 3 - 4 = 0$ ✓

$4 \cdot 1 + 4 = 8$ ✓

$4 \cdot 3 + 4 = 16$ ✓

Systems of two linear equations with two variables

$$\begin{aligned} x_1 + x_2 &= 0 \\ 2x_1 + x_2 &= 1. \end{aligned}$$

Method 1 (Isolate)

1) $x_2 = 1 - 2x_1$

2) $x_1 + 1 - 2x_1 = 0$

$\to 1 - x_1 = 0$

$\to 1 = x_1$

3) $x_2 = 1 - 2 \cdot 1 = -1$

solution: $(\underset{x_1}{\underset{\uparrow}{1}}, \underset{x_2}{\underset{\uparrow}{-1}})$

Method 2 (Operations)

1) $\begin{aligned} x_1 + x_2 &= 0 \quad E_1 \\ 2x_1 + x_2 &= 1 \quad E_2 \end{aligned}$

2) $E_1 - E_2 \to E_1$

$$\begin{array}{r} x_1 + x_2 = 0 \\ - \quad 2x_1 + x_2 = 1 \\ \hline -x_1 + \quad = -1 \end{array}$$

$\to -x_1 = -1$

$\begin{cases} -x_1 = -1 & E_1 \\ 2x_1 + x_2 = 1 & E_2 \end{cases}$

3) $-E_1 \to E_1$

$-(-x_1) = -1 \to x_1 = 1$

$\begin{cases} x_1 = 1 & E_1 \\ 2x_1 + x_2 = 1 & E_2 \end{cases}$

4) $2E_1 - E_2 \to E_2$

$$\begin{array}{r} 2x_1 \quad = 2 \\ - \quad 2x_1 + x_2 = 1 \\ \hline -x_2 = 1 \end{array}$$

$\begin{cases} x_1 = 1 & E_1 \\ -x_2 = 1 & E_2 \end{cases}$

$\downarrow$

$\boxed{\begin{cases} x_1 = 1 \\ x_2 = -1 \end{cases}}$

Gauss-Jordan Elimination

Based on three *elementary operations* on the equations:

- Interchange two equations in the system.
- Replace an equation by a multiple of itself.
- Replace an equation by itself plus a multiple of another equation.

Main GOAL: transform our system into

$$\begin{aligned} x + 0y + 0z &= \tilde{b}_1 \\ 0x + y + 0z &= \tilde{b}_2 \\ 0x + 0y + z &= \tilde{b}_3. \end{aligned}$$

Example 1. Find the solution(s) to the following system of linear equations:

$$
\begin{aligned}
1x - y + z &= 0 && E_1\\
2x - 3y + 4z &= -2 && E_2\\
-2x - y + z &= 7. && E_3
\end{aligned}
$$

1) $-2E_1 + E_2 \to E_2$

$$
\begin{array}{rl}
 & -2x + 2y - 2z = 0\\
+ & 2x - 3y + 4z = -2\\
\hline
 & -y + 2z = -2
\end{array}
$$

$2E_1 + E_3 \to E_3$

$$
\begin{array}{rl}
 & 2x - 2y + 2z = 0\\
+ & -2x - y + z = 7\\
\hline
 & -3y + 3z = 7
\end{array}
$$

$$
\left\{
\begin{aligned}
x - y + z &= 0 && E_1\\
-y + 2z &= -2 && E_2\\
-3y + 3z &= 7 && E_3
\end{aligned}
\right.
$$

2) $E_1 - E_2 \to E_1$

$$
\begin{array}{rl}
 & x - y + z = 0\\
- & -y + 2z = -2\\
\hline
 & x - z = 2
\end{array}
$$

$-3E_2 + E_3 \to E_3$

$$
\begin{array}{rl}
 & 3y - 6z = 6\\
+ & -3y + 3z = 7\\
\hline
 & -3z = 13
\end{array}
$$

$$
\left\{
\begin{aligned}
x - z &= 2 && E_1\\
-y + 2z &= -2 && E_2\\
-3z &= 13 && E_3
\end{aligned}
\right.
$$

3). $2E_3 + 3E_2 \to E_2$

$$
\begin{array}{rl}
 & -6z = 26\\
+ & -3y + 6z = -6\\
\hline
 & -3y = 20
\end{array}
$$

$3E_1 - E_3 \to E_1$

$$
\begin{array}{rl}
 & 3x - 3z = 6\\
- & -3z = 13\\
\hline
 & 3x = -7
\end{array}
$$

$$
\left\{
\begin{aligned}
3x &= -7\\
-3y &= 20\\
-3z &= 13
\end{aligned}
\right.
$$

4)

$$
\frac{3x}{3} = \frac{-7}{3} \to \qquad \frac{-3y}{-3} = \frac{20}{-3} \to \qquad \frac{-3z}{-3} = \frac{13}{-3} \to
$$

$$
\boxed{\begin{aligned} x &= -\frac{7}{3}\\ y &= -\frac{20}{3}\\ z &= -\frac{13}{3}\end{aligned}}
$$

Augmented Matrix

More efficient way: transform the system in an **augmented matrix**.

$$
\begin{aligned}
a_{11}\cancel{x_1} + a_{12}\cancel{x_2} + \cdots + a_{1n}\cancel{x_n} &= b_1 \\
a_{21}\cancel{x_1} + a_{22}\cancel{x_2} + \cdots + a_{2n}\cancel{x_n} &= b_2 \\
&\vdots \\
a_{m1}\cancel{x_1} + a_{m2}\cancel{x_2} + \cdots + a_{mn}\cancel{x_n} &= b_m
\end{aligned}
\implies
\begin{array}{c}
\begin{array}{ccccc} x_1 & x_2 & & x_n & b \end{array} \\
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn} & b_m
\end{bmatrix}
\end{array}
$$

Example 2. Find the augmented matrix of the system of Example 1.

$$
\begin{aligned}
1x - y + z &= 0 \\
2x - 3y + 4z &= -2 \\
-2x - y + z &= 7
\end{aligned}
\implies
\begin{array}{c}
\begin{array}{cccc} x & y & z & b \end{array} \\
\begin{bmatrix}
1 & -1 & 1 & 0 \\
2 & -3 & 4 & -2 \\
-2 & -1 & 1 & 7
\end{bmatrix}
\end{array}
$$

Elementary operations revisited

Elementary operations on linear equations become elementary operations on the rows of the augmented matrix:

- Interchange two rows.
- Replace a row by a multiple of itself.
- Replace a row by itself plus a multiple of another row.

Example 3. Solve the system:

$$\begin{aligned} 2x + 3y - z &= 3 \\ -x - y + 3z &= 0 \\ x + 2y + 2z &= 3 \\ y + 5z &= 3. \end{aligned}$$

$$\begin{bmatrix} 2 & 3 & -1 & 3 \\ -1 & -1 & 3 & 0 \\ 1 & 2 & 2 & 3 \\ 0 & 1 & 5 & 3 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & -1 & 3 \\ \boxed{0} & 1 & 5 & 3 \\ 0 & -1 & -5 & -3 \\ 0 & 1 & 5 & 3 \end{bmatrix} \begin{array}{l} \\ R_1 + 2R_2 \to R_2 \\ R_1 - 2R_3 \to R_3 \\ \\ \end{array}$$

$$\sim \begin{bmatrix} 2 & 0 & -16 & -6 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} R_1 - 3R_2 \to R_1 \\ \\ R_2 + R_3 \to R_3 \\ R_2 - R_4 \to R_4 \end{array}$$

$$\sim \begin{array}{cccc} x & y & z & b \\ \hline \end{array}$$

$$\begin{bmatrix} 1 & 0 & -8 & -3 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \tfrac{1}{2}R_1 \to R_1$$

$$x - 8z = -3 \quad \& \quad y + 5z = 3$$

$$\Rightarrow \quad x = -3 + 8z \quad \& \quad y = 3 - 5z .$$

• Remarks.

- z is a free variable or free parameter.
- x & y are called dependent variables.
- Geometry: solutions is a line in 3D

$z = t$ $\quad x = -3 + 8t$ $\quad$ direction is
$y = 3 - 5t$ $\quad \langle 8, -5, 1 \rangle$.
$z = t$

- There are infinitely many solutions!.

$\langle 5, -2, 1 \rangle_{z=1}$ & $\langle 13, -7, 2 \rangle_{z=2}$ are solutions

EXAMPLE 4. Solve the system:

$$\begin{aligned} 4x_1 - 8x_2 - x_3 + x_4 + 3x_5 &= 0 \\ 5x_1 - 10x_2 - x_3 + 2x_4 + 3x_5 &= 0 \\ 3x_1 - 6x_2 - x_3 + x_4 + 2x_5 &= 0. \end{aligned}$$

$n = 5$

$m = 3$

$$\begin{array}{c} \\ R_1 \\ R_2 \\ R_3 \end{array}\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & b \end{array} \\ \left[\begin{array}{cccccc} 4 & -8 & -1 & 1 & 3 & 0 \\ 5 & -10 & -1 & 2 & 3 & 0 \\ 3 & -6 & -1 & 1 & 2 & 0 \end{array}\right]\end{array} \sim \left[\begin{array}{cccccc} 4 & -8 & -1 & 1 & 3 & 0 \\ 0 & 0 & -1 & -3 & 3 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right] \begin{array}{l} \\ 5R_1 - 4R_2 \to R_2 \\ 3R_1 - 4R_3 \to R_3 \end{array}$$

$$\sim \left[\begin{array}{cccccc} 4 & -8 & 0 & 4 & 0 & 0 \\ 0 & 0 & -1 & -3 & 3 & 0 \\ 0 & 0 & 0 & -4 & 4 & 0 \end{array}\right] \begin{array}{l} R_1 - R_2 \to R_1 \\ \\ R_2 + R_3 \to R_3 \end{array}$$

$$\sim \left[\begin{array}{cccccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & -3 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{array}\right] \begin{array}{l} 1/4\, R_1 \to R_1 \\ -1 R_2 \to R_2 \\ -\frac{1}{4} R_3 \to R_3 \end{array}$$

$$\sim \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & b \end{array} \\ \left[\begin{array}{cccccc} \underline{1} & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & \underline{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \underline{1} & -1 & 0 \end{array}\right]\end{array} \begin{array}{l} R_1 - R_3 \to R_1 \\ R_2 - 3R_3 \to R_2 \\ \\ \end{array}$$

$$\begin{aligned} x_1 - 2x_2 + x_5 &= 0 \\ x_3 &= 0 \\ x_4 - x_5 &= 0 \end{aligned} \quad \to \quad \boxed{\begin{aligned} x_1 &= 2x_2 - x_5 \\ x_3 &= 0 \\ x_4 &= x_5 \end{aligned}}$$

Here x_2 & x_5 are free parameters.

x_1, x_3 & x_4 are dependent variables.

Reduced row-echelon form (RREF)

Transformed augmented matrix after row operations:

- Any rows of zero (called zero rows) appear at the bottom.
- The first nonzero entry of a nonzero row is 1 (called a leading 1).
- The leading 1 of a nonzero row appears to the right of the leading 1 of any preceding row.
- All the other entries of a column containing a leading 1 are zero.

Consistent Systems vs Inconsistent Systems

- Consistent: means the system of equations has at least one solution.
  - How to recognize that a system is consistent?

    (1) RREF has the form
    $$\begin{bmatrix} 1 & 0 & 0 & \cdots & * \\ 0 & 1 & 0 & \cdots & * \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & 1 \quad * \end{bmatrix}$$

    (2) RREF has the form
    $$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & * \\ 0 & 1 & 0 & \cdots & 0 & * \\ \vdots & \vdots & \vdots & & & \\ 0 & 0 & 0 & \cdots & 1 & * \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \to$$ contain rows of zeros.

- Inconsistent: means the system of equations has no solution.
  - How to recognize that a system is inconsistent?

    (1)
    $$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & * \\ 0 & 1 & 0 & \cdots & 0 & * \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \tilde{b} \end{bmatrix} \to$$ line of zeros except the last entry which is $\neq 0$ ($\tilde{b} \neq 0$)

    This means $0 = \tilde{b}$ with $\tilde{b} \neq 0$.

Homogeneous System

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

- Trivial solution: $x_1 = x_2 = \cdots = x_n = 0$.

Theorem 5. A homogeneous system of m linear equations in n variables has

- infinitely many nontrivial solutions if $m < n$; less eq. than var.
- exactly one (trivial solution) if $m = n$;
- no solution if $m > n$. more eq. than var.

Goal. Transform the augmented matrix into an new augmented matrix with the following properties:

- any zero rows appear at the bottom.
- The first nonzero entry of a nonzero row is 1.
- The leading 1 of a nonzero row appears to the right of the leading 1 of any preceding row.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$ Example of a valid reduced Gaussian Form.

EXAMPLE 6. Determine the values of a, b, and c so that the system

$$\begin{aligned} x - y + 2z &= a \\ 2x + y - z &= b \\ x + 2y - 3z &= c \end{aligned}$$

has solutions.

$$\begin{bmatrix} 1 & -1 & 2 & a \\ 2 & 1 & -1 & b \\ 1 & 2 & -3 & c \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & a \\ 0 & -3 & 5 & 2a-b \\ 0 & -3 & 5 & a-c \end{bmatrix} \begin{array}{l} 2R_1 - R_2 \to R_2 \\ R_1 - R_3 \to R_3 \end{array}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & a \\ 0 & -3 & 5 & 2a-b \\ 0 & 0 & 0 & a-b+c \end{bmatrix} R_2 - R_3$$

$a = 1$
$b = 1$ $\quad 1-1+0=0$ ✓ $\quad \to \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & -3 & 5 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ ✓
$c = 0$

Here, if $\boxed{a-b+c=0}$, the system is consistent!