Math 307

Chapter 5

Section 5.5: Similar Matrices, Diagonalization, and Jordan Canonical Form

Contents

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Motivation

Example 1. Let A be the 3×3 matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

Then, (a) compute A^5 (b) find the eigenvalues of A (c) find a basis for each eigenspace.

(a) $A \cdot A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 2^2 & 0 & 0 \\ 0 & 3^2 & 0 \\ 0 & 0 & 4^2 \end{bmatrix}$

$A^5 = A \cdot A \cdot A \cdot A \cdot A = \begin{bmatrix} 2^5 & 0 & 0 \\ 0 & 3^5 & 0 \\ 0 & 0 & 4^5 \end{bmatrix}$

(b) $\det(\lambda I - A) = 0 \iff \begin{vmatrix} \lambda - 2 & 0 & 0 \\ 0 & \lambda - 3 & 0 \\ 0 & 0 & \lambda - 4 \end{vmatrix} = 0 \iff (\lambda-2)(\lambda-3)(\lambda-4) = 0$

$\iff \lambda = 2, \lambda = 3, \lambda = 4.$

(c) $\underline{E_2 (\lambda = 2)}$ $\quad v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \to (\lambda I - A)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$

$$\begin{array}{cccc} & y & z & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right] \to \begin{matrix} x \text{ free} \\ y = 0 \\ z = 0 \end{matrix} \to \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

basis for E_2

$\underline{E_3 (\lambda = 3)}$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \to \begin{matrix} x = 0 \\ z = 0 \\ y \text{ free} \end{matrix}$$

$$\to \begin{bmatrix} x \\ y \\ z \end{bmatrix} = y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \leftarrow \text{basis for } E_3$$

$\underline{E_4 (\lambda = 4)}$

$$\left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \begin{matrix} x = 0 \\ y = 0 \\ z \text{ free} \end{matrix}$$

$$\to \begin{bmatrix} x \\ y \\ z \end{bmatrix} = z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \leftarrow \text{basis for } E_4.$$

<u>Remarks</u>

- It is pretty easy to deal with diagonal matrices.
- Our goal is to try to transform a general matrix into a diagonal matrix.

EXAMPLE 2. Let A be the following 3×3 matrix

$$A = \begin{bmatrix} 6 & -4 & -2 \\ 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Find (a) the eigenvalues of A (b) a basis for each eigenspace (c) compute A^5.

(a) $\det(\lambda I - A) = \lambda^3 - 9\lambda^2 + 26\lambda - 24 = (\lambda-2)(\lambda-3)(\lambda-4) = 0$

$\to \lambda = 2, \lambda = 3$, or $\lambda = 4$.

(b) $\underline{E_2 (\lambda=2)}$ $\quad 2I - A = \begin{bmatrix} -4 & 4 & 2 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix}$, solve $(2I-A)v = 0$, $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

$\to v = z \begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix}$ (z free var.) $\to w_1$

$\underline{E_3 (\lambda=3)}$ $\quad 3I - A = \begin{bmatrix} -3 & 4 & 2 \\ -1 & 1 & 1 \\ -1 & 0 & 2 \end{bmatrix}$, solve $(3I-A)v = 0$, $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

$\to v = z \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ (z free var.). $\to w_2$

$\underline{E_4 (\lambda=4)}$ $\quad 4I - A = \begin{bmatrix} -2 & 4 & 2 \\ -1 & 2 & 1 \\ -1 & 0 & 3 \end{bmatrix}$, solve $(4I-A)v = 0$, $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

$\to v = z \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$ $\to w_3$

Because eigenvectors associated to different eigenvalues, we have that w_1, w_2, w_3 are lin. independent

$\to w_1, w_2, w_3$ form a basis for $\mathbb{R}^3$!

(c) Change of basis from $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ to $\underbrace{w_1, w_2, w_3}_{\beta}$:

$$P = \begin{bmatrix} 1 & 2 & 3 \\ 1/2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \qquad P^{-1} = \begin{bmatrix} 0 & -2 & 2 \\ -1 & 4 & -1 \\ 1 & -2 & 0 \end{bmatrix}$$

Now,

$$[A]_\beta^\beta = P^{-1}AP = \begin{bmatrix} 0 & -2 & 2 \\ -1 & 4 & -1 \\ 1 & -2 & 0 \end{bmatrix}\begin{bmatrix} 6 & -4 & 2 \\ -1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 1/2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \qquad \underline{\text{diag. matrix!}}$$

So, $A = P[A]_\beta^\beta P^{-1}$

$$\rightarrow A^5 = \left(P[A]_\beta^\beta \underbrace{P^{-1}\right)\left(P}_{=I}[A]_\beta^\beta \underbrace{P^{-1}\right)\left(P}_{=I}[A]_\beta^\beta \underbrace{P^{-1}\right)\left(P}_{=I}[A]_\beta^\beta \underbrace{P^{-1}\right)\left(P}_{I}[A]_\beta^\beta P^{-1}\right)$$

$$= P[A]_\beta^\beta \, I \, [A]_\beta^\beta \, I \, [A]_\beta^\beta \, I \, [A]_\beta^\beta \, I \, [A]_\beta^\beta P^{-1}$$

$$= P \left([A]_\beta^\beta\right)^5 P^{-1}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 1/2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2^5 & 0 & 0 \\ 0 & 3^5 & 0 \\ 0 & 0 & 4^5 \end{bmatrix}\begin{bmatrix} 0 & -2 & 2 \\ -1 & 4 & -1 \\ 1 & -2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2586 & -4264 & -422 \\ 781 & -1108 & -211 \\ 781 & -1140 & -179 \end{bmatrix}.$$

Definition

Diagonalizable Matrices:

diagonal matrix.

An $n \times n$ matrix A is *diagonalizable* if there is a matrix D and an invertible matrix P such that

$$A = \cancel{P^{-1}DP}$$

$$P D P^{-1}$$

Facts:

- Let A be an $n \times n$ matrix.
- Let λ_1, λ_2, …, λ_k be the eigenvalues of A.
- Let E_{λ_1}, E_{λ_2}, …, E_{λ_k} be the eigenspaces associated to each eigenvalue.

If $\dim(E_{\lambda_1}) + \dim(E_{\lambda_2}) + \cdots + \dim(E_{\lambda_k}) = n$, then A is diagonalizable.

Example 3. Is the matrix from Example 2 diagonalizable?

A: was a 3×3 matrix $\rightarrow$ $n = 3$

$\lambda_1 = 2$, $\lambda_2 = 3$, $\lambda_3 = 4$.

Also,

$\begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix}$ basis for E_2 $\rightarrow$ $\dim(E_2) = 1$

$+$

$\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ basis for E_3 $\rightarrow$ $\dim(E_3) = 1$

$+$

$\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$ basis for E_4 $\rightarrow$ $\dim(E_4) = 1$

$\overline{3}$

So, $\dim(E_1) + \dim(E_2) + \dim(E_3) = 3 = n$

$\rightarrow$ A is diagonalizable.

Example 4. Is the matrix

$$A = \begin{bmatrix} 1 & -2 & -6 \\ -2 & 2 & -5 \\ 2 & 1 & 8 \end{bmatrix}$$

diagonalizable? If so, determine the invertible matrix P such that $\cancel{P^{-1}AP}$ PAP^{-1} is a diagonal matrix.

1) Eigenvalues.

$\det(\lambda I - A) = \lambda^3 - 11\lambda^2 + 39\lambda - 45 = (\lambda-5)(\lambda-3)^2$

$\rightarrow \lambda = 5 \quad , \lambda = 3$ (mult. is 2)

2) EigenSpaces.

E_5 $(\lambda=5)$ Solve $(5I - A)v = 0 \qquad v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

$\rightarrow v = z \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ basis for E_5.

$\rightarrow \dim(E_5) = 1$

E_5 $(\lambda=3)$ Solve $(3I - A)v = 0$, $\quad v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

$\rightarrow v = z \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \longrightarrow \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$ basis for E_3

$\rightarrow \dim(E_3) = 1$

So,

$\dim(E_5) + \dim(E_3) = 2 \neq 3 (= n)$

$\rightarrow$ A is not diagonalizable

Example 5. Is the matrix

$1+0i \quad -1+0i$

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$1+0i \quad 1+0i$

diagonalizable? If so, determine the invertible matrix P such that ~~$P^{-1}AP$~~ PAP^{-1} is a diagonal matrix.

1) Eigenvalues.

$$\det(\lambda I - A) = \lambda^2 - 2\lambda + 2 \rightarrow \begin{array}{l} \lambda_1 = 1+i. \\ \lambda_2 = 1-i. \end{array} \quad \left(\begin{array}{l}\text{Ex. 7} \\ \text{in Sec. 5.4}\end{array}\right)$$

2) Eigenspaces. (Ex. 7).

E_{1+i} $\quad v = x\begin{bmatrix} 1 \\ -i \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ -i \end{bmatrix}$ basis for E_{1+i}.

↑ is complex

E_{1-i} $\quad v = x\begin{bmatrix} 1 \\ i \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ i \end{bmatrix}$ basis for E_{1-i}

↑ is complex.

$$\dim(E_{1+i}) + \dim(E_{1-i}) = 1 + 1 = 2 = n$$

$\rightarrow$ A is diagonalizable.

3) Change of basis.

$$P = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \quad \& \quad P^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ i/2 & -i/2 \end{bmatrix}$$

$$D = [A]_\beta^\beta = P^{-1}AP = \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix}.$$

In general:

An $n \times n$ matrix A is *similar* to an $n \times n$ matrix B if there is an invertible $n \times n$ matrix P such that

$$B = P^{-1}AP.$$

Notation: $A \sim B$ means that A is similar to B.

Facts:

- If A is similar to B and B is similar to C, then A is similar to C.
- If P is the change of bases matrix from α to β and T is a linear transformation, then $[T]_\beta^\beta = P^{-1}[T]_\alpha^\alpha P$. So $[T]_\beta^\beta \sim [T]_\alpha^\alpha$.

Question:

For non-diagonalizable matrices, can we reduce them to a simple form?

In other words, can we find a matrix B, as simple as possible, such that $B \sim A$?

Answer: Yes! We will replace the diagonal form by the Jordan canonical form.

Jordan blocks

A Jordan block is a square matrix A taking the following shape:

$$A = \begin{bmatrix} \mu & 1 & 0 & \cdots & 0 & 0 \\ 0 & \mu & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \mu & 1 \\ 0 & 0 & 0 & \cdots & 0 & \mu \end{bmatrix}.$$

Why are these type of matrices important?

EXAMPLE 6. Let A be the matrix

$$A = \begin{bmatrix} \mu & 1 & 0 \\ 0 & \mu & 1 \\ 0 & 0 & \mu \end{bmatrix}.$$

(a) Compute $\det(\lambda I - A)$. (b) Find the dimension of the eigenspaces.

(a) $\det(\lambda I - A) = \begin{vmatrix} \lambda-\mu & -1 & 0 \\ 0 & \lambda-\mu & -1 \\ 0 & 0 & \lambda-\mu \end{vmatrix} = (\lambda-\mu)^3 = 0$

$\to \quad \lambda = \mu \quad$ (alg. mult. = 3)

(b) $\underline{E_\mu}$. $\quad (\mu I - A) v = 0, \quad v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$\mu I - A = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \left[\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \begin{array}{l} y=0 \\ z=0 \\ x \text{ free} \end{array}$

$\to \quad v = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \quad \to \quad \dim(E_\mu) = 1.$

<u>Remark</u>:

- a $n \times n$ Jordan block associated to a number μ has only one eigenvalue.
- The algebraic multiplicity of this eigenvalue is necessarily <u>equal to n</u>.
- geo. mult. We always have $\dim(E_\mu) = 1$ for an $n \times n$ Jordan block.
- Jordan blocks are the building blocks for the set of matrices that can't be diagonalizable.

Reduction to Jordan Blocks

EXAMPLE 7. We know that the matrix

$$A = \begin{bmatrix} 1 & -2 & -6 \\ -2 & 2 & -5 \\ 2 & 1 & 8 \end{bmatrix}$$

is not diagonalizable. Find a matrix B, not necessarily a diagonal matrix, such that A is similar to B.

<u>General Procedure</u>: Suppose A is an $n \times n$ matrix.

- Express $\det(\lambda I - A)$ as

$$\det(\lambda I - A) = (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_k)^{m_k}$$

  where m_1 is the multiplicity of λ_1, m_2 is the multiplicity of λ_2, …, m_k is the multiplicity of λ_k.

- For each λ_j, write

$$A_j = \begin{bmatrix} J_{m_{j-1}+1} & 0 & \cdots & 0 \\ 0 & J_{m_{j-1}+2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_{m_j} \end{bmatrix}.$$

  where each J_p, for $p = m_{j-1} + 1, \ldots m_j$, is a Jordan block

$$J_p = \begin{bmatrix} \lambda_j & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_j & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_j & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_j \end{bmatrix}.$$

- Then the Jordan Canonical Form (JCF) is

$$B = \begin{bmatrix} A_1 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ 0 & 0 & A_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_k \end{bmatrix}$$

- The invertible matrix P such that $B = P^{-1}AP$ is more complicated to find. In theory, the method to find P uses the notion of a **generalized eigenvector**. In our situation, we will use Python to find this matrix P.

If you want to know more on the generalized eigenvectors and the Jordan Canonical Form, I suggest to take a look at the following references:

- A more math article: *Down With Determinants!* by Sheldon Axler, `https://www.maa.org/sites/default/files/pdf/awards/Axler-Ford-1996.pdf`.
- A Youtube video: `https://www.youtube.com/watch?v=GVixvieNnyc`.

Example 8. Let A be an 7×7 matrix with the following eigenvalues:

$$\{1, 1, 1, 1, 2, 2, 3\}.$$

Give the possible Jordan canonical form B of the matrix A.