G.I Mean-Square Law of Large Numbers

PROBLEM 1. We have

$$\mathrm{Exp}((aX_n + b - (aX + b))^2) = \mathrm{Exp}(a^2(X_n - X)^2) = a^2\mathrm{Exp}((X_n - X)^2).$$

By assumption, $\lim_{n\to\infty} \mathrm{Exp}((X_n - X)^2) = 0$, therefore

$$\lim_{n\to\infty} \mathrm{Exp}((aX_n + b - (aX + b))^2) = a^2 \lim_{n\to\infty} \mathrm{Exp}((X_n - X)^2) = 0.$$

Hence, $aX_n + b \to aX + b$ in mean-square. △

PROBLEM 2. Let N_m be the number of occurences of 5 or 6 in m throws of a fair die. Show that

$$\frac{1}{m}N_m \to \frac{1}{3} \quad \text{in mean square}$$

as $m \to \infty$.

Let X_i be the random variable with output 1 if the die lands on 5 or 6 and output 0 if the die lands on $1, 2, 3$, or 4. Then, we have

$$\mathrm{Exp}(X_i) = 0 \times \frac{2}{3} + 1 \times \frac{1}{3} = \frac{1}{3}$$

and

$$\mathrm{Var}(X_i) = \mathrm{Exp}(X_i^2) - (\mathrm{Exp}(X_i))^2 = \frac{1}{3} - \frac{1}{9} = \frac{2}{9}.$$

Therefore, we get $\mathrm{Exp}(N_m) = \frac{m}{3}$ and $\mathrm{Var}(N_m) = \frac{2m}{9}$. Hence, we compute

$$\mathrm{Exp}\Big(\Big(\frac{N_m}{m} - \frac{1}{3}\Big)^2\Big) = \mathrm{Exp}\Big(\frac{(N_m - \frac{m}{3})^2}{m^2}\Big) = \frac{1}{m^2}\mathrm{Var}(N_m) = \frac{2}{9m}.$$

As $m \to \infty$, $\frac{2}{9m} \to 0$ and therefore $N_m/m \to 1/3$ in mean-square, as $m \to \infty$. △

G.II Central Limit Theorem

PROBLEM 3.

a) Let X_i be a random variable with output the facture strenght of the i-th piece. Then, we have $\mu = \mathrm{Exp}(X_i) = 14$, for any i and $\sigma^2 = \mathrm{Var}(X_i) = 4$ for any i. We have $n = 100$, the size of the sample and let $S_n/n = (X_1 + X_2 + \ldots + X_n)/n$ represents the average of the fracture strength in the sample.

We will use the Central Limit Theorem to estimate the probability

$$P\Big(\frac{S_{100}}{100} > 14.5\Big).$$

The standardized version of S_{100} is

$$Z_{100} = \frac{S_{100} - 100\mu}{\sqrt{100}\sigma} = \frac{S_{100} - 1400}{20}.$$

We see that

$$\frac{S_{100}}{100} > 14.5 \iff S_{100} > 1450 \iff S_{100} - 1400 > 50 \iff \frac{S_{100} - 1400}{20} > 2.5.$$

Therefore,

$$P\Big(\frac{S_{100}}{100} > 14.5\Big) = P(Z_{100} > 2.5).$$

From the Central Limit Theorem,

$$P\Big(\frac{S_{100}}{100} > 14.5\Big) = P(Z_{100} > 2.5) \approx P(Z > 2.5)$$

where $Z \sim N(0,1)$. Using the table of the normal distribution, we find that

$$P(Z > 2.5) = 1 - P(Z \leq 2.5) = 1 - 0.99379 = 0.00731.$$

b) Since the standardized version is centered at the average, we will try to find a such that $S_n/n \in [\mu - a, \mu + a]$ in 95% of the chances. We want to find $a > 0$ such that

$$P\Big(\Big|\frac{S_{100}}{100} - \mu\Big| < a\Big) = 0.95.$$

Rearranging the left-hand side:

$$\frac{S_{100}}{100} - 14 = \frac{S_{100} - 1400}{100} = \frac{2}{10}\Big(\frac{S_{100} - 1400}{20}\Big) = \frac{2}{10} Z_n$$

and therefore

$$P\Big(\Big|\frac{S_{100}}{100} - 1400\Big| < a\Big) = P(|Z_n| < 5a).$$

Using the Central Limit Theorem, $P(|Z_n| < 5a) \approx P(|Z| < 5a)$, for $Z \sim N(0,1)$ and we have to find a such that $P(|Z| < 5a) = 0.95$. Now, since the normal density of $N(0,1)$ is symmetric with respect to the y-axis, we have $P(Z > 5a) = 0.025 = P(Z < -5a)$. Therefore,

$$P(|Z| < 5a) = 0.95 \iff P(Z < 5a) = 0.975.$$

Using the table, we find a $z = 1.96$ and therefore $a = 1.96/5 = 0.392$. Hence, the interval containing $S_{100}/100$ in 95% of the chances is $[13.608, 14.392]$.