Math 311

Chapter 6

Section 6.3: Linear Independence and Dimension

Contents

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EXAMPLE 1. Let $\mathbf{v} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$. Let $\mathbf{u_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u_2} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u_3} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{v_3} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$

a) Consider the vectors. Can you write $\mathbf{v}$ as a unique linear combination of the vectors $\mathbf{u_1}$, $\mathbf{u_2}$, and $\mathbf{u_3}$?

b) Consider the vectors. Can you write the vector $\mathbf{v}$ as a unique linear combination of the vectors $\mathbf{v_1}$, $\mathbf{v_2}$, $\mathbf{v_3}$?

SOLUTION.

(a) Let $\vec{v} = a_1\vec{u}_1 + a_2\vec{u}_2 + a_3\vec{u}_3$. The solution is $a_1 = 2-t$, $a_2 = -1-t$ and $a_3 = t$.

$$\left.\begin{array}{l} t=0 \to \vec{v} = 2\vec{u}_1 + (-1)\vec{u}_2 + 0\vec{u}_3 \\ t=1 \to \vec{v} = 1\vec{u}_1 + (-2)\vec{u}_2 + 1\vec{u}_3 \end{array}\right\} \text{ not unique.}$$

(b) Let $\vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3$. The solution $a_1 = 3$, $a_2 = 2$, $a_3 = -3$.

$\to \vec{v} = 3\vec{v}_1 + 2\vec{v}_2 + (-3)\vec{v}_3$ $\boxed{\text{Unique!}}$

DEFINITION 1. A set of vectors $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ in a vector space V is called **linearly independent** (or simply **independent**) if

$$s_1\mathbf{v_1} + s_2\mathbf{v_2} + \cdots + s_n\mathbf{v_n} = \mathbf{0} \quad \Rightarrow \quad s_1 = s_2 = \cdots = s_n = 0.$$

A set of vectors that is not independent is said to be **linearly dependent** (or simply **dependent**).

Note:

- The **trivial linear combination** of the vectors $\mathbf{v_1}$, $\mathbf{v_2}$, ..., $\mathbf{v_n}$ is the one with every coefficient zero:

$$0\mathbf{v_1} + 0\mathbf{v_2} + \cdots + 0\mathbf{v_n}.$$

- So the vectors $\mathbf{v_1}$, $\mathbf{v_2}$, ..., $\mathbf{v_n}$ are linearly independent if and only if the only way to write $\mathbf{0}$ is using the trivial linear combination.

EXAMPLE 2. Show that the set

$$\left\{\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 1 & -1\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}\right\}$$

is independent.

SOLUTION.

Example 3. Let $\{\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}\}$ be an independent set in a vector space V. Which of the following set is indepedent?

a) $\{\mathbf{x} - \mathbf{y}, \mathbf{y} - \mathbf{z}, \mathbf{z} - \mathbf{x}\}$.

b) $\{\mathbf{x} - \mathbf{y}, \mathbf{y} - \mathbf{z}, \mathbf{z} - \mathbf{w}, \mathbf{w} - \mathbf{x}\}$.

Solution.

Definition 2. A set $\{\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}\}$ of vectors in a vector space V is called a basis of V if it satisfies the following two conditions:

① $\{\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}\}$ is linearly independent.

② $V = \operatorname{span}\{\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}\}$.

Example 4. Let $V = \mathbb{R}^3$. Verify the following.

a) If $\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{e_3}$ are the columns of I_3, then $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ is a basis for $\mathbb{R}^3$.

b) $\left\{\begin{bmatrix}1 & -1 & 0\end{bmatrix}^\top, \begin{bmatrix}3 & 2 & -1\end{bmatrix}^\top, \begin{bmatrix}3 & 5 & -2\end{bmatrix}^\top\right\}$ is a basis for $\mathbb{R}^3$.

Observations:

- Invariance Theorem (p.347): If $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is a basis for a vector space V and if $\{\mathbf{f_1}, \mathbf{f_2}, \dots, \mathbf{f_m}\}$ is a basis for a vector space V, then $m = n$.

Definition 3. If $\{\mathbf{e_1}, \mathbf{e_2}, \dots, \mathbf{e_n}\}$ is a basis of a nonzero vector space V, the number n of vectors in the basis is called the **dimension**, and we write

$$\dim V = n.$$

In the case of the zero vector space, we define $\dim\{\mathbf{0}\} = 0$.

Note:

① We have $\dim \mathbb{R}^m = m$ because the n columns of the identity matrix I_m is a basis.

② We have $\dim \mathbf{M_{mn}} = mn$. Let E_{ij} be the matrix with a 1 in the (i, j)-entry and 0 elsewhere. A basis for $\mathbf{M_{mn}}$ is

$$B = \{M_{ij} \, : \, 1 \leq i \leq m, \, 1 \leq j \leq n\}.$$

This is called the **canonical basis** or **standard basis** of $\mathbf{M_{mn}}$. For instance, if $m = n = 2$, then a basis for $\mathbf{M_{22}}$ is

$$\begin{aligned} B &= \{M_{11}, M_{12}, M_{21}, M_{22}\} \\ &= \left\{ \qquad\qquad\qquad\qquad \right\} \end{aligned}$$

③ We have $\dim \mathbf{P_n} = \qquad$.

④ Any subspace U of a vector space V is a vector space. Therefore, we can find the dimension of U.

Subspaces of $\mathbb{R}^m$

For subspaces of $\mathbb{R}^m$, there is a really nice way to determine a basis and the dimension of a spanning set. Let

$$U = \text{span}\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}.$$

Let

- $A = \begin{bmatrix} \mathbf{v_1} & \mathbf{v_2} & \cdots & \mathbf{v_n} \end{bmatrix}$.
- R be the RREF of A.

Then

- $\dim U$ = number of pivots in R.
- A basis for U is given by the vector in the same column as the pivots.

Example 5. Find a basis and calculate the dimension for the following subspace of $\mathbb{R}^4$:

$$U = \text{span}\{(1, -1, 2, 0), (2, 3, 0, 3), (1, 9, -6, 6)\}.$$

Solution.

Note: This trick also works for subspaces of the space of polynomials $\mathbf{P_n}$.

Subspaces of Matrices

Example 6. Define the subspace of $\mathbf{M_{22}}$ as

$$U = \left\{ X \in \mathbf{M_{22}} \, : \, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} X = X \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \right\}.$$

Find a basis of U and its dimension.

Solution.