Math 644

Chapter 6

Section 6.1: Conformal Maps

Contents

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Definition 1. A function $f : \Omega \to \mathbb{C}$ is conformal on Ω if

(a) f is analytic in Ω and;

(b) f is one-to-one in Ω. $\to f'(z_0) \neq 0 \Rightarrow f(z) = a_0 + a_1 z + o(z^2)$

Example 2. Find an conformal map f from the unit disk $\mathbb{D}$ onto the unit disk $\mathbb{D}$, with $f(0) = \frac{1}{2}$.

Formula analytic automorphisms:

$$\varphi(z) = c\,\frac{z-a}{1-\bar{a}z}, \quad |c| = 1, \ a \in \mathbb{D}, \ z \in \mathbb{D}.$$

Set $a = \frac{1}{2}$, $c = -1$

$$\Rightarrow \varphi(z) = \frac{\frac{1}{2} - z}{1 - \frac{z}{2}} = \frac{1-2z}{2-z}.$$

We know φ is injective (and analytic).

Theorem 3. If there exists a conformal map of a region Ω onto $\mathbb{D}$, then, given any $z_0 \in \Omega$, there exists a unique conformal map f of Ω onto $\mathbb{D}$ such that

$$f(z_0) = 0 \text{ and } f'(z_0) > 0.$$

Proof.

Let $z_0 \in \mathbb{D}$ and $g : \Omega \to \mathbb{D}$ be a conformal map.

Set $a := g(z_0) \in \mathbb{D}$.

Set $\varphi(z) = c \dfrac{z-a}{1-\bar{a}z}$, then

$$f(z) := \varphi(g(z)) \quad \& \quad f(z_0) = 0.$$

Also,

$$f'(z_0) = \varphi'(g(z_0)) \cdot g'(z_0)$$
$$= \frac{c}{1-|g(z_0)|^2} \cdot g'(z_0)$$

Set $c = \dfrac{|g'(z_0)|}{g'(z_0)}$. $\Rightarrow f'(z_0) = \dfrac{|g'(z_0)|}{1-|g(z_0)|^2} > 0$. $\square$

THEOREM 4. If φ is a conformal map of a region Ω onto $\mathbb{D}$, then Ω must be simply-connected.

Proof.

Let $\gamma \subseteq \Omega$ be a closed curve and $a \notin \Omega$.

Goal: show $n(\gamma, a) = 0$.

Since $\varphi(\Omega) = \mathbb{D}$ & φ is injective on Ω, the inverse $\varphi^{-1} : \mathbb{D} \to \Omega$ exists & is a conformal map from $\mathbb{D} \to \Omega$.

Let $\gamma_{\mathbb{D}} \subseteq \mathbb{D}$ be a closed curve s.t.

$$\varphi^{-1}(\gamma_{\mathbb{D}}) = \gamma.$$

Then,

$$n(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{1}{\zeta - a}\, d\zeta$$

$$= \frac{1}{2\pi i} \int_{\varphi^{-1}(\gamma_{\mathbb{D}})} \frac{1}{\zeta - a}\, d\zeta$$

$\zeta = \varphi^{-1}(\gamma_{\mathbb{D}}(t))$

$\leadsto d\zeta = (\varphi^{-1})'(\gamma_{\mathbb{D}}(t))\, \gamma_{\mathbb{D}}'(t)$

$$= \frac{1}{2\pi i} \int_\alpha^\beta \frac{(\varphi^{-1})'(\gamma_{\mathbb{D}}(t))\, \gamma_{\mathbb{D}}'(t)}{\varphi^{-1}(\gamma_{\mathbb{D}}) - a}\, dt$$

$$= \frac{1}{2\pi i} \int_{\gamma_{\mathbb{D}}} \frac{(\varphi^{-1})'(\zeta)}{\varphi^{-1}(\zeta) - a}\, d\zeta$$

The map $z \longmapsto \dfrac{(\varphi^{-1})'(z)}{\varphi^{-1}(z) - a}$ is analytic on $\mathbb{D}$

Since $\gamma_{\mathbb{D}} \sim \{0\}$, from Cauchy's theorem:

$$\frac{1}{2\pi i}\int_{\gamma_{\mathbb{D}}} \frac{(\varphi^{-1})'(z)}{\varphi^{-1}(z)-a}\,dz = 0 .$$

$$\Rightarrow \qquad n(\gamma, a) = 0 .$$

$\square$