Chapter 5: Applications of Integration
Week 14

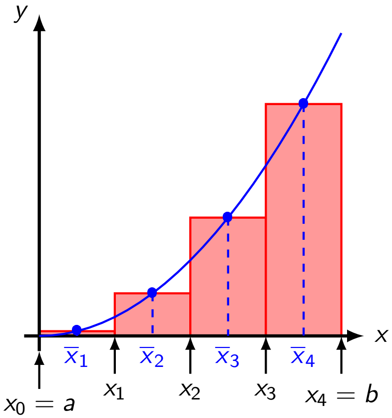
Pierre-Olivier Parisé
Calculus I (MATH-241 01/02)

University of Hawai'i
Fall 2021

Upcoming this week

1. The return of the integral
2. 5.3 Volumes by cylindrical shells

Let's go back to the integral. There is a way to compute the integral by taking the midpoints of the partition of an interval $[a, b]$.



- We partition the interval $[a, b]$ with equidistributed points x_i for $i = 0, 1, 2, \ldots, n$ where $x_0 = a$ and $x_n = b$.
- The midpoint of the interval $[x_{i-1}, x_i]$ is $\overline{x}_i = \frac{x_i + x_{i-1}}{2}$

Mid-point rule

The definite integral of a function $f(x)$ from $x = a$ to $x = b$ can be obtained by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(\overline{x}_i)\Delta x_i$$

where $\Delta x_i = x_i - x_{i-1}$ and $\overline{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ given by $\overline{x}_i = \frac{x_i + x_{i-1}}{2}$.

If we try to use the "washer" method from the previous section to compute the volume of the following solids of revolution, then we would have to solve for x the cubic equation $y = 2x^2 - x^3$.

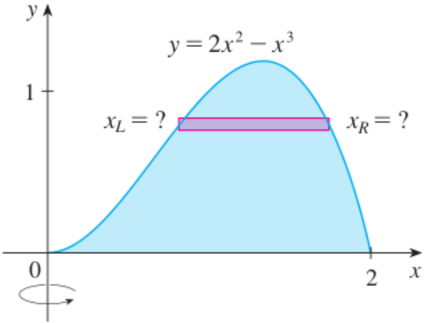


Figure: The region between $y = 2x^2 - x^3$ and $y = 0$ to rotate around $x = 0$.

Fortunately there is a better way: the method of cylindrical shells.

Instead of considering washers, we consider cylinders like in the following picture.

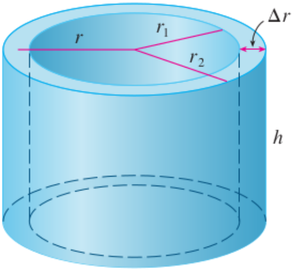


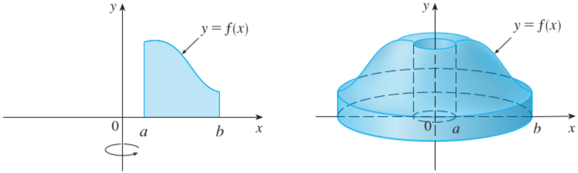
Figure: A cylindrical shell of inner radius r_1, outer radius r_2 and height h.

The volume of the cylindrical shell is then $V = V_2 - V_1$ which equals

$$V = 2\pi r h \Delta r.$$

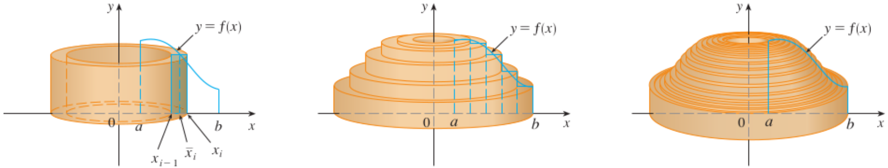
where $r = \frac{r_1+r_2}{2}$ and $\Delta r = r_2 - r_1$.

Consider an arbitrary shape that we rotate by the line $x = 0$.



We then approximate the volume of the solid by several cylindrical shells of volume

$$V_i = (2\pi \overline{x}_i) f(\overline{x}_i) \Delta x.$$



In this situation, the volume of the solid is approximated by

$$V \approx \sum_{i=1}^{n} (2\pi \overline{x}_i) f(\overline{x}_i) \Delta x.$$

By letting $n \to \infty$, we obtain an integral formula for the volume of the solid of revolution.

Method with cylindrical shells

The volume of a solid of revolution, obtained by rotating aroung the y-axis the region under the curve $y = f(x)$ from a to b, is

$$V = \int_a^b 2\pi x f(x)\, dx.$$

Example 1

Find the volume of the solid obtained by rotating around the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$.

Example 2

Find the volume of the solid obtained by rotating around the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.

Exercises: 3-7, 9-14, 15-20, 29-32.