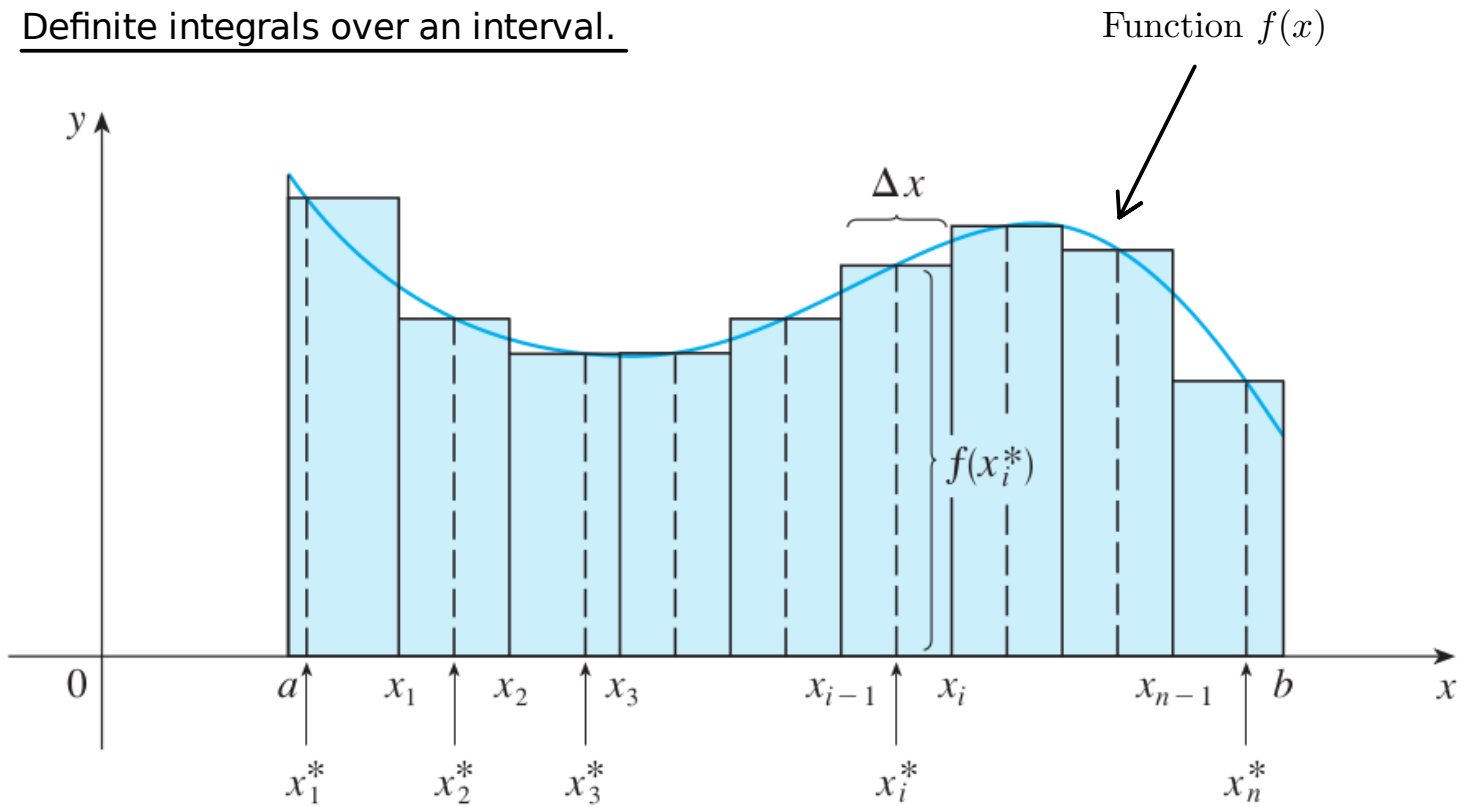


Chapter 15

Multiple Integrals

15.1 Double Integrals over a rectangle

Definite integrals over an interval.

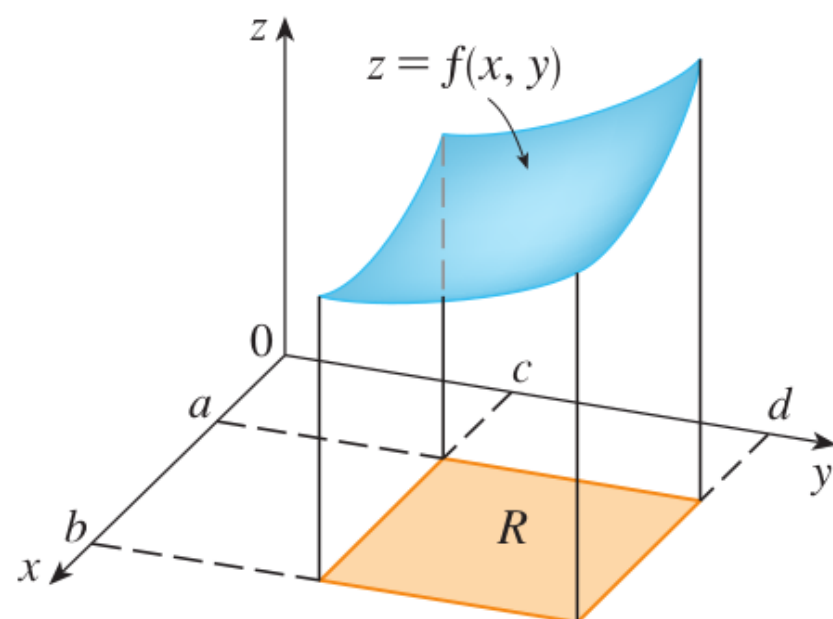


- 1) Divide the interval in n parts of equal length Δx
- 2) Name each subinterval $[a, x_1], [x_1, x_2], \dots, [x_{n-1}, b]$
- 3) Choose some point x_1^* in $[a, x_1], x_2^*$ in $[x_1, x_2], \dots, x_n^*$ in $[x_{n-1}, b]$
 \Rightarrow Total Area of rectangles $= f(x_1^*)\Delta x + f(x_2^*)\Delta x + \dots + f(x_n^*)\Delta x$
- 4) Take the limit as $n \rightarrow \infty$

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x$$

Useful Fact:

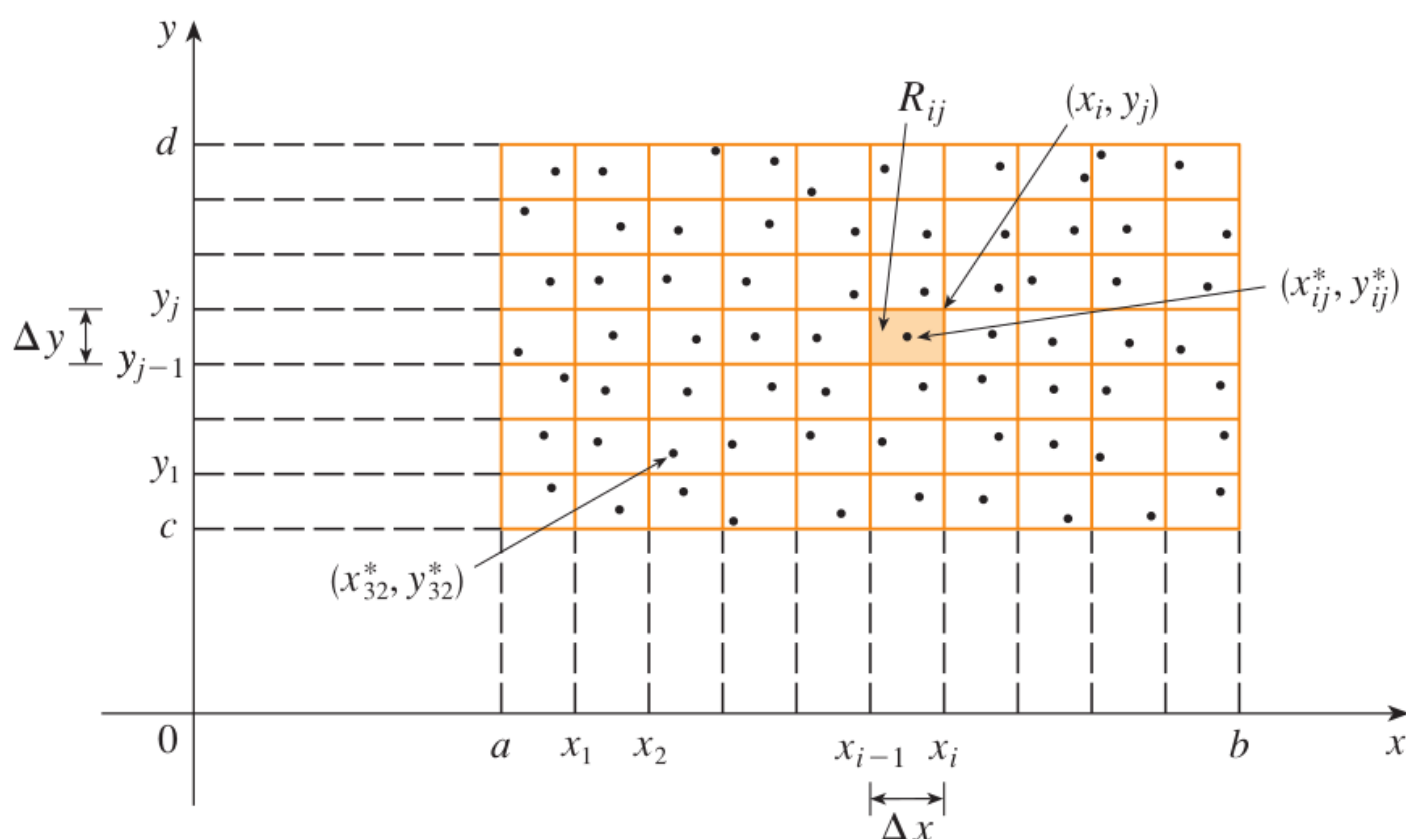
Volumes and Double Integrals.



Given:

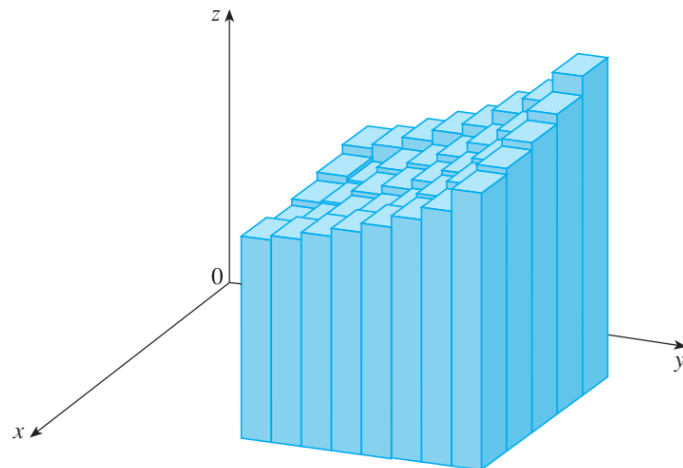
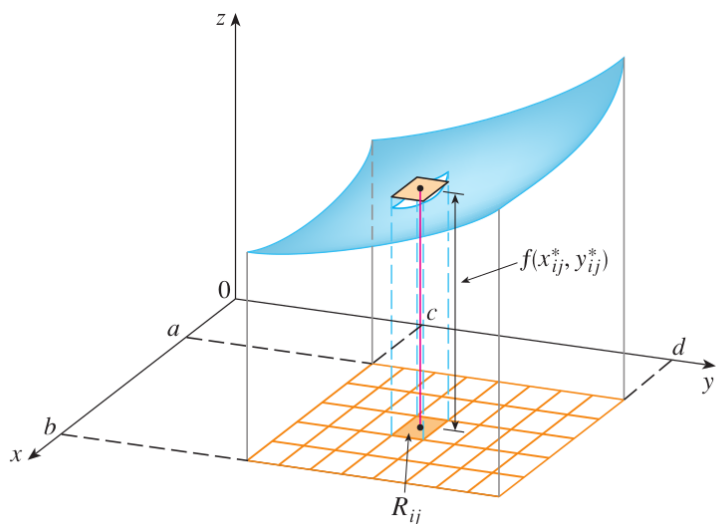
- A function $z = f(x, y)$
- The domain $R = [a, b] \times [c, d]$

1st Step: Divide the domain to create a grid.



- 1) Divide $[a, b]$ in m equal parts Δx
- 2) Divide $[c, d]$ in n equal parts Δy
- 3) Create the rectangle $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$
- 4) Select a point (x_{ij}^*, y_{ij}^*) in R_{ij}

2nd Step: Approximate the volume by "buildings"



1) Volume of a building: $\Delta A \cdot f(x_{ij}^*)$

2) Total volume:

$$V = \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \cdot \Delta A$$

3) Take the limit as $m, n \rightarrow \infty$:

$$\iint_R f(x, y) dA = \lim_{n, m \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A$$

Useful Fact:

EXAMPLE 1 Estimate the volume of the solid that lies above the square $R = [0, 2] \times [0, 2]$ and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$. Divide R into four equal squares and choose the sample point to be the upper right corner of each square R_{ij} . Sketch the solid and the approximating rectangular boxes.

Fubini's Theorem:

$$\iint_R f(x, y) dA = \int_c^d \int_a^b f(x, y) dx dy = \int_a^b \int_c^d f(x, y) dy dx$$

EXAMPLE 4 Evaluate the iterated integrals.

(a) $\int_0^3 \int_1^2 x^2 y \, dy \, dx$

(b) $\int_1^2 \int_0^3 x^2 y \, dx \, dy$

Example. Evaluate the following integral:

$$\int_0^1 \int_0^1 v(u^2 + v^2)^4 \, du \, dv$$

EXAMPLE 7 Find the volume of the solid S that is bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$, and the three coordinate planes.

EXAMPLE 8 If $R = [0, \pi/2] \times [0, \pi/2]$, then compute $\iint_R \sin x \cos y \, dA$.

Useful Fact:

$$\iint_R g(x)h(y) \, dA = \left(\int_a^b g(x) \, dx \right) \left(\int_c^d h(y) \, dy \right)$$

where $R = [a, b] \times [c, d]$