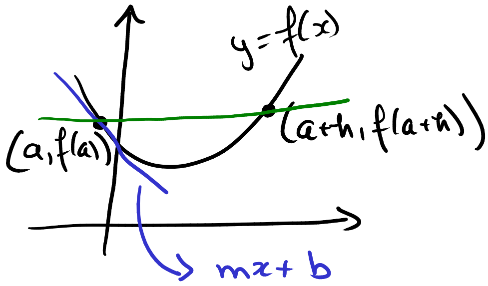
Calculus I: Summary

Chapter 1

- Tangent problem:

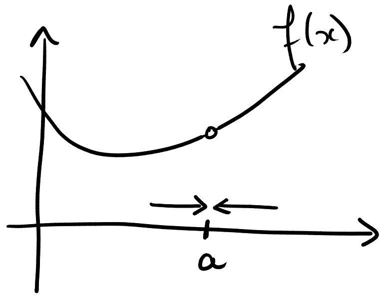


$$m = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

$$m \approx \frac{f(a+h)-f(a)}{h}$$

for h small or close to 0.

- limits:



$$\lim_{x\to a^-} f(x) = L$$

$$\lim_{x\to a^+} f(x) = R$$

$$\lim_{x\to a} f(x) = M \iff \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x).$$

1) $\displaystyle \lim_{x\to a} cf(x) = c\lim_{x\to a} f(x)$

2) $\displaystyle \lim_{x\to a} \big(f(x)+g(x)\big) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

3) $\displaystyle \lim_{x\to a} \big(f(x)\,g(x)\big) = \Big(\lim_{x\to a} f(x)\Big)\Big(\lim_{x\to a} g(x)\Big)$

4) $\displaystyle \lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \quad \Big(\lim_{x\to a} g(x) \neq 0\Big).$

5) $\boxed{\frac{0}{0}}$ $\displaystyle \lim_{x\to 1} \frac{x-1}{x^2-1} = \lim_{x\to 1} \frac{\cancel{x-1}}{(\cancel{x-1})(x+1)}$

$$= \lim_{x\to 1} \frac{1}{x+1} = \frac{1}{2}$$

6) $\displaystyle \lim_{x\to a} x^n = \Big(\lim_{x\to a} x\Big)^n.$ 7) $\displaystyle \lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{\lim_{x\to a} x}.$

8) $\displaystyle \lim_{x\to a} [f(x)]^n = \Big(\lim_{x\to a} f(x)\Big)^n.$ 8) $\displaystyle \lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}.$

- Continuous functions.

f is continuous at a if

① f is defined at a.

② $\lim_{x \to a} f(x)$ exists.

③ $\lim_{x \to a} f(x) = f(a)$.

Note f is continuous on (a,b) if it is continuous at every point of (a,b).

1) $f+g$ continuous if f,g cont.

2) cf continuous if f cont.

3) fg cont. if f,g cont.

4) $\frac{f}{g}$ cont. if f,g cont. ($g(a) \neq 0$).

5) f^n cont. if f cont.

6) $\sqrt[n]{f}$ cont. if f cont. ($f(x) \geq 0$, n even).

7) $\cos$, $\sin$, x^n, $\tan$, $\cot$, $\sec$, cosec, $\sqrt[n]{x}$, $x^2+1+\cos x$, $\sqrt[3]{x^2+1}$.

I.V.T.

f cont. on $[a,b]$ & $f(a) \neq f(b)$, then for any y between $f(a)$ & $f(b)$, there is a $a<c<b$ s.t. $f(c) = y$.

1) $\cos x = x$ has solution $[0, \pi/2]$

Squeeze Thm.

$L = \lim_{x \to a} f(x) = \lim_{x \to a} g(x)$ & $f(x) \leq h(x) \leq g(x) \Rightarrow \lim_{x \to a} h(x) = L$

(Note: works also for $a = \infty$).

Chapter 2. Derivatives.

Intro: $\quad m = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h} = f'(a)$

1) $f(x) = x \longrightarrow f'(x) = 1$
2) $f(x) = x^n \longrightarrow f'(x) = n x^{n-1}$
3) $f(x) = \cos x \longrightarrow f'(x) = -\sin x$
4) $f(x) = \sin x \longrightarrow f'(x) = \cos x$

6) $\dfrac{d}{dx}\left(f(x) + g(x)\right) = \dfrac{d}{dx}\left(f(x)\right) + \dfrac{d}{dx}\left(g(x)\right)$

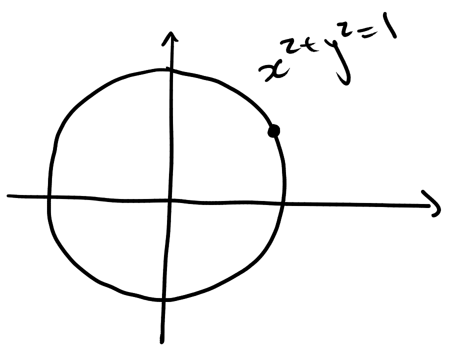
7) $\dfrac{d}{dx}\left(c f(x)\right) = c \dfrac{d}{dx}\left(f(x)\right)$

8) $\dfrac{d}{dx}\left(f(x) g(x)\right) = \dfrac{d}{dx}\left(f(x)\right) g(x) + f(x) \dfrac{d}{dx}\left(g(x)\right)$

9) $\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{\dfrac{d}{dx}\left(f(x)\right) g(x) - f(x) \dfrac{d}{dx}\left(g(x)\right)}{\left(g(x)\right)^2}$

10) $\dfrac{d}{dx}\left(g(f(x))\right) = \dfrac{d}{dx}\left(g(f(x))\right) \cdot \dfrac{d}{dx}\left(f(x)\right).$

Implicit differentiation



$y = f(x)$

$x^2 + y^2 = 1$

$\Rightarrow \dfrac{d}{dx}\left(x^2 + y^2\right) = \dfrac{d}{dx}(1)$

$\Rightarrow \dfrac{d}{dx}(x^2) + \dfrac{d}{dx}(y^2) = 0$

$\Rightarrow 2x + 2y \cdot \dfrac{dy}{dx} = 0$

Note: $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$.

Chapter 3: "Applications"

Max, min

1) Critical numbers: $f'(c) = 0$ or $f'(c) \nexists$.
2) Closed interval method:
   - → Find C.N. in $[a,b]$
   - → Find values of f at C.N.
   - → Abs Max = max between $f(a)$, $f(b)$ & values of f at C.N.

     Abs Min. = min. between $f(a)$, $f(b)$ & values of f at C.N.

Incr./Decr.

1) Find C.N.
2) $f'(x) > 0 \longrightarrow f \nearrow$

   $f'(x) < 0 \longrightarrow f \searrow$
3) 1^{st} Derivative test:

C.N.	$x <$	a	$< x$
$f'(x)$	$\searrow$	0 or $\nexists$	$\nearrow$
$f(x)$		loc. min	

C.N.	$x <$	a	$< x$
$f'(x)$	$\nearrow$	0 or $\nexists$	$\searrow$
$f(x)$		loc. max.	

Concavity

1) Inflection Points: $f''(x) = 0$.
2) $f''(x) > 0 \longrightarrow f \cup$

   $f''(x) < 0 \longrightarrow f \cap$
3) 2^{nd} test:

   $f''(a) > 0 \to$ loc. min

   $f''(a) < 0 \to$ loc. max.

Limits at infinity

1) Horizontal Asymptote: $\lim_{x \to +\infty} f(x) = L$ (exists)

or $\lim_{x \to -\infty} f(x) = M$ (exists)

2) Squeeze Theorem for limits at ∞:

$f(x) \le h(x) \le g(x)$ for $a < x < \infty$

& $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x) = L$

then $\lim_{x \to \infty} h(x) = L$.

(Same for $\lim_{x \to -\infty}$).

Ex: Compute $\lim_{x \to \infty} \frac{\sin x}{x^2}$.

We know that $-1 \le \sin x \le 1$

$$\Rightarrow \quad \frac{-1}{x^2} \le \frac{\sin x}{x^2} \le \frac{1}{x^2}$$

We have $\lim_{x \to \infty} \frac{-1}{x^2} = 0$

& $\lim_{x \to \infty} \frac{1}{x^2} = 0$.

By Squeeze Theorem for limits at ∞:

$$\lim_{x \to \infty} \frac{\sin x}{x^2} = 0.$$