Chapter 1

Functions and Limits

1.5 The Limit of a Function

Intuitive definition of a limit.

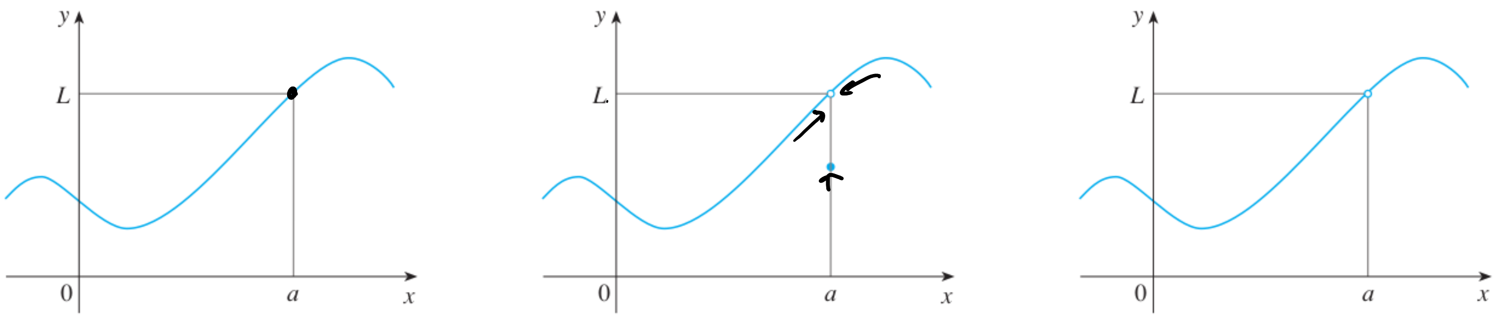
1 Intuitive Definition of a Limit Suppose $f(x)$ is defined when x is near the number a. (This means that f is defined on some open interval that contains a, except possibly at a itself.) Then we write

$$\lim_{x \to a} f(x) = L$$

and say "the limit of $f(x)$, as x approaches a, equals L"

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by restricting x to be sufficiently close to a (on either side of a) but not equal to a.

Three cases:



(a) $\lim_{x \to a} f(x) = f(a)$ ↙ L

(b) $\lim_{x \to a} f(x) = L \neq f(a)$

(c) $f(a) \nexists$ but $\lim_{x \to a} f(x) = L$

EXAMPLE 1 Guess the value of $\lim_{x \to 1} \dfrac{x-1}{x^2-1}$.

$f(x) = \dfrac{x-1}{x^2-1}$ → f is not defined at $x=1$
but f is defined around $x=1$

x	$f(x)$
0.8	0.555556
0.9	0.52632
0.99	0.50251
0.999	0.50025

↓ 0.5

x	$f(x)$
1.2	0.45454
1.1	0.47619
1.01	0.49751
1.001	0.49975

↓ 0.5

$$\Rightarrow \lim_{x \to 1} \frac{x-1}{x^2-1} = 0.5$$

$$\frac{x-1}{x^2-1} = \frac{\cancel{x-1}}{(x+1)\cancel{(x-1)}} = \frac{1}{x+1} \quad (x \neq 1).$$

→ $(x \neq -1)$

EXAMPLE 3 Guess the value of $\lim_{x \to 0} \frac{\sin x}{x}$.

$f(x) = \frac{\sin x}{x}$ & $a = 0$

<u>From the left</u>

x	$f(x)$
-0.5	0.95885
-0.1	0.99833
-0.01	0.99998
-0.001	$0.99999...$
$\downarrow$	$\downarrow$
0	1

<u>From the right.</u>

x	$f(x)$
0.5	0.95885
0.1	0.99833
0.01	0.99998
0.001	0.99999
$\downarrow$	$\downarrow$
0	1

Our guess: $\lim_{x \to 0} \frac{\sin x}{x} = 1$

EXAMPLE 4 Investigate $\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right)$.

$f(x) = \sin\left(\frac{\pi}{x}\right)$, $a = 0$

x	$f(x)$
0.5	0
0.1	0
0.01	0
0.001	0
$\vdots$ $\downarrow$	$\vdots$ $\downarrow$
0	0

$\frac{\pi}{x} = \frac{\pi}{2} + 2k\pi \iff \frac{1}{x} = \frac{1+4k}{2}$

$\iff x = \frac{2}{1+4k}$, k any integ.

	x	$f(x)$
$k=1$	$2/5$	1
$k=2$	$2/9$	1
$k=3$	$2/13$	1
$k=4$	$2/17$	1
	$\downarrow$	$\downarrow$
	0	1

Fundamental thing about limits: the limit is unique.

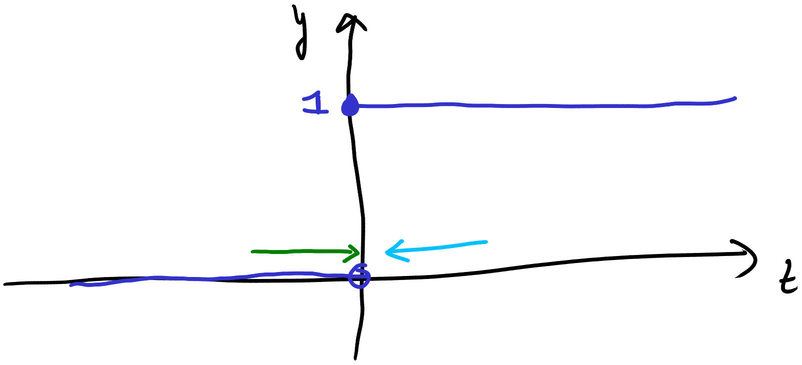
$\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right) \nexists$

One-sided Limits.

EXAMPLE 6 The Heaviside function H is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geqslant 0 \end{cases}$$

What is the limit when t approached 0 from the right and when t approaches 0 from the left.



Approach from the left $a=0$

If $t<0 \Rightarrow H(t)=0$

t	$H(t)$
-0.1	0
-0.001	0
-0.00001	0
	$\downarrow$ 0

$$\lim_{t \to 0^-} H(t) = 0$$

Approach from the right

$t>0, \quad H(t)=1 \Rightarrow \lim_{t \to 0^+} H(t) = 1$

So, $\lim_{t \to 0^-} H(t) \neq \lim_{t \to 0^+} H(t)$

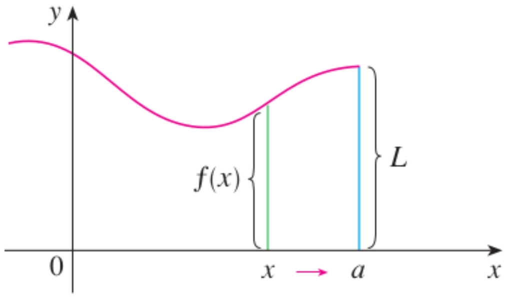
$$\Rightarrow \lim_{t \to 0} H(t) \nexists .$$

Left-hand limits.

2 Definition of One-Sided Limits We write

$$\lim_{x \to a^-} f(x) = L$$

and say the **left-hand limit of $f(x)$ as x approaches a** [or the **limit of $f(x)$ as x approaches a from the left**] is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a with x *less than* a.



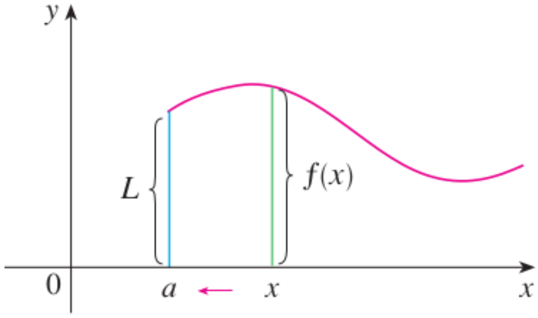
(a) $\lim_{x \to a^-} f(x) = L$

Right-hand limits.

~~**2 Definition of One-Sided Limits** We write~~

$$\cancel{\lim_{x \to a^-} f(x) = L}$$

~~and say the **left-hand limit of $f(x)$ as x approaches a** [or the **limit of $f(x)$ as x approaches a from the left**] is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a with x *less than* a.~~



(b) $\lim_{x \to a^+} f(x) = L$

$\lim_{x \to a^+} f(x) = L$ if $f(x)$ approaches L when x approaches a from the right.

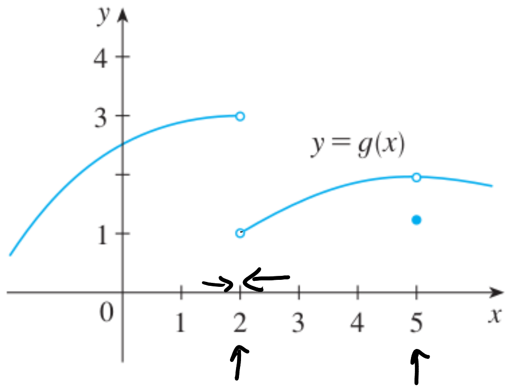
Fundamental Property:

3 $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$

EXAMPLE 7 The graph of a function g is shown in Figure 10. Use it to state the values (if they exist) of the following:

(a) $\lim_{x \to 2^-} g(x)$ (b) $\lim_{x \to 2^+} g(x)$ (c) $\lim_{x \to 2} g(x)$

(d) $\lim_{x \to 5^-} g(x)$ (e) $\lim_{x \to 5^+} g(x)$ (f) $\lim_{x \to 5} g(x)$



(a) $\lim_{x \to 2^-} g(x) = 3$

(b) $\lim_{x \to 2^+} g(x) = 1$

(c) $\lim_{x \to 2} g(x)$ $\nexists$ because of (a) & (b)

(d) $\lim_{x \to 5^-} g(x) = 2$

(e) $\lim_{x \to 5^+} g(x) = 2$

(f) $\lim_{x \to 5} g(x) = 2$

Infinite limits.

EXAMPLE 8 Find $\lim_{x \to 0} \frac{1}{x^2}$ if it exists.

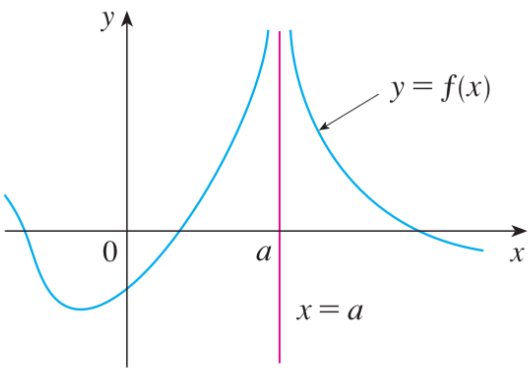
If x approaches 0 $\Rightarrow$ $\frac{1}{x^2}$ approaches $+\infty$

Positive infinity.

4 Intuitive Definition of an Infinite Limit Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a, but not equal to a.

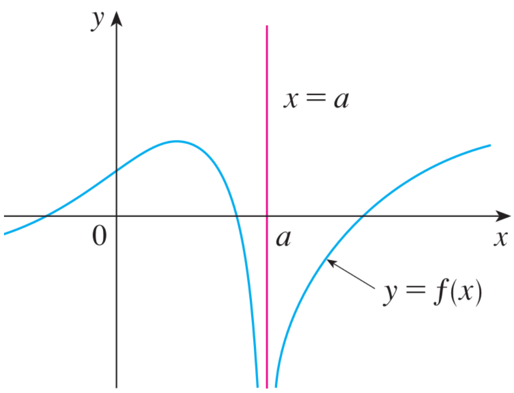


Negative Infinity

5 Definition Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = -\infty$$

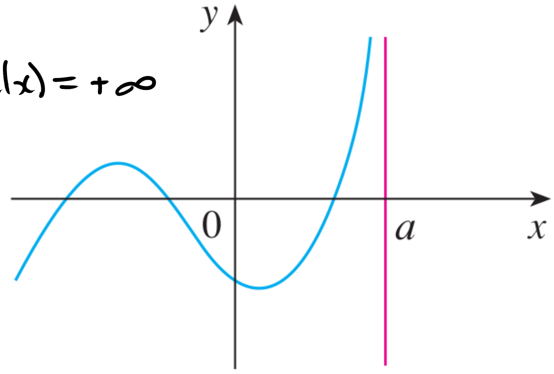
means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a, but not equal to a.



Other types of infinite limits.

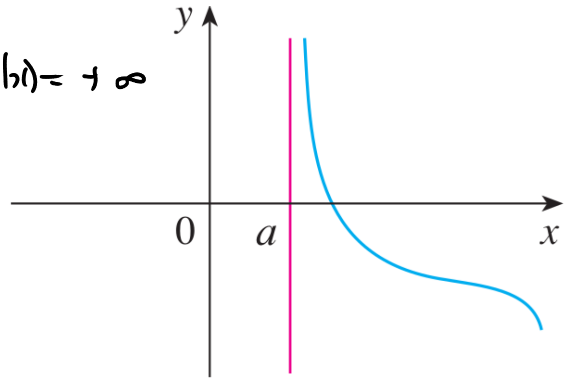
a)

$\lim_{x\to a^-} f(x) = +\infty$

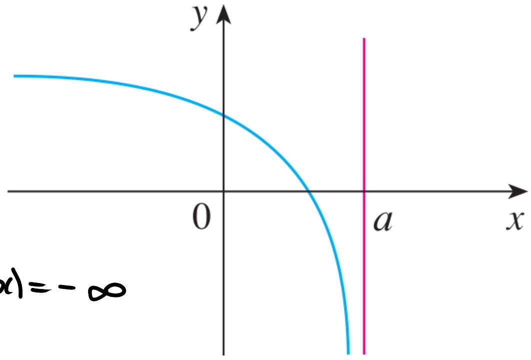


b)

$\lim_{x\to a^+} f(x) = +\infty$

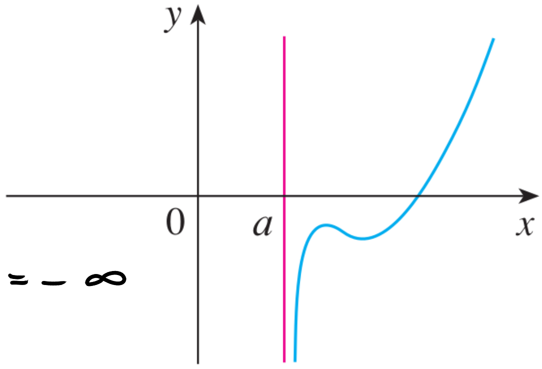


c)



$\lim_{x\to a^-} f(x) = -\infty$

d)



$\lim_{x\to a^+} f(x) = -\infty$

EXAMPLE 9 Find $\lim_{x\to 3^+} \frac{2x}{x-3}$ and $\lim_{x\to 3^-} \frac{2x}{x-3}$.

A more straightforward way:

$$\lim_{x\to 3^+} \frac{\overset{\to 6}{2x}}{\underset{\to 0^+}{x-3}} = \frac{6}{0^+} = +\infty$$

$$\lim_{x\to 3^-} \frac{\overset{\to 6}{2x}}{\underset{\to 0^-}{x-3}} = \frac{6}{0^-} = -\infty$$

Warning!

It doesn't for $\frac{0^\pm}{0^\pm}$

→ undefined.

6 Definition The vertical line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x \to a} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

EXAMPLE 10 Find the vertical asymptotes of $f(x) = \tan x$.

$$f(x) = \tan x = \frac{\sin x}{\cos x}.$$

$\tan x$ will explode $\iff \cos x = 0$

$\iff x = \frac{\pi}{2} + k\pi$

Verify for $x = \frac{\pi}{2}$

$$\lim_{x \to (\frac{\pi}{2})^-} \tan x = \lim_{x \to (\frac{\pi}{2})^-} \frac{\sin x \to 1^-}{\cos x \to 0^+}$$

$$= \frac{1^-}{0^+}$$

$$= +\infty$$

So $x = \frac{\pi}{2}$ is a $\boxed{\text{VA}}$