Section 16.4, Problem 6

We have $P(x,y) = x^2 + y^2$ and $Q(x,y) = x^2 - y^2$. By Green's Theorem, we have

$$\oint (x^2 + y^2)dx + (x^2 - y^2)dy = \iint_D Q_x - P_y \, dA.$$

The domain D is the triangle with vertices $(0,0)$, $(2,1)$, and $(0,1)$. The positive orientation is a parametrization that passes to all the points in the following order: $(0,0)$ to $(2,1)$ to $(0,1)$ and then coming back to $(0,0)$. We can write

$$D = \{(x,y) \,:\, 0 \leq x \leq 2, x/2 \leq y \leq 1\}.$$

Thus,

$$\iint_D 2x - 2y \, dA = \int_0^1 \int_{x/2}^1 2x - 2y \, dy dx = 0.$$

So the line integral is zero.

Section 16.4, Problem 12

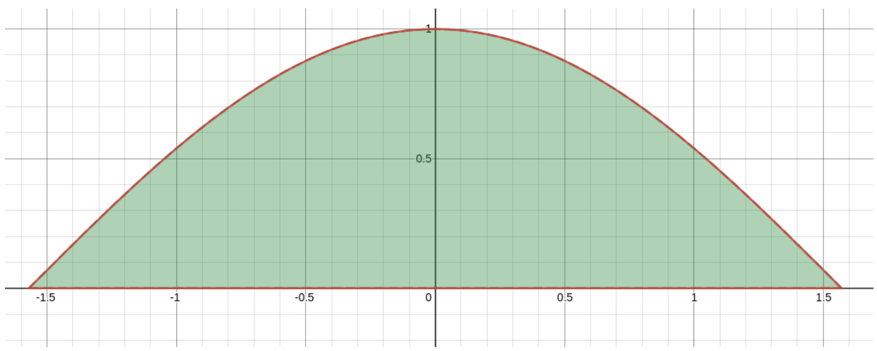
We see that $P(x, y) = e^{-x} + y^2$ and $Q(x, y) = e^{-y} + x^2$ and so

$$Q_x - P_y = 2x - 2y.$$

We want to compute

$$\int_C \vec{F} \cdot d\vec{r} = \int_C e^{-x} + y^2\,dx + e^{-y} + x^2\,dy.$$

The curve (in red) and the domain (in green) bounded by the curve is represented below. Taking



the counterclockwise orientation on the curve, we can write

$$\int_C e^{-x} + y^2\,dx + e^{-y} + x^2\,dy = \oint_C e^{-x} + y^2\,dx + e^{-y} + x^2\,dy$$

and by Green's Theorem, we obtain

$$\oint_C e^{-x} + y^2\,dx + e^{-y} + x^2\,dy = \iint_D Q_x - P_y\,dA = \iint_D 2x - 2y\,dA.$$

The description of D is

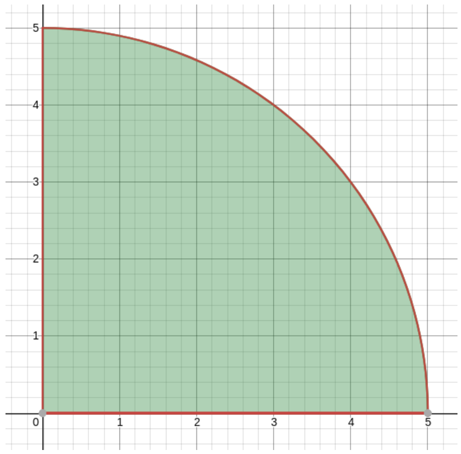
$$D = \{(x, y) \,:\, -\pi/2 \le x \le \pi/2,\ 0 \le y \le \cos x\}.$$

So, we obtain

$$\iint_D 2x - 2y\,dA = \int_{-\pi/2}^{\pi/2} \int_0^{\cos x} 2x - 2y\,dydx = \pi/2.$$

$\boxed{\textbf{Section 16.4, Problem 18}}$

Here is the path traced by the particle and the region D enclosed by the curve C. The work



is given by

$$W = \int_C \vec{F} \cdot d\vec{r}$$

where $\vec{F}(x, y) = \left\langle \sin x, \sin y + xy^2 + \frac{1}{3}x^3 \right\rangle$. Taking the counterclockwise orientation, we see from Green's Theorem that

$$W = \iint_D Q_x - P_y \, dA.$$

We have

$$D = \{(x, y) \,:\, 0 \leq x \leq 5,\ 0 \leq y \leq \sqrt{25 - x^2}\},$$

$Q_x = y^2 + x^2$ and $P_y = 0$, so

$$W = \int_0^5 \int_0^{\sqrt{25-x^2}} x^2 + y^2 \, dy dx.$$

We change to polar coordinates. We have

$$W = \int_0^5 \int_0^{\pi/2} r^2 \, d\theta dr = \frac{125\pi}{6} \approx 65.4498.$$

Section 16.5, Problem 2

a) By definition, the curl is

$$\text{curl}\vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix}$$

We have $P = 0$, $Q = x^3yz^2$, and $R = y^4z^3$. So, we see that

$$\begin{aligned} \text{curl}\vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & x^3yz^2 & y^4z^3 \end{vmatrix} \\ &= (4y^3z^3 - 2x^3yz)\vec{i} - (0)\vec{j} + (3x^2yz^2)\vec{k} \\ &= (4y^3z^3 - 2x^3yz)\vec{i} + (3x^2yz^2)\vec{k}. \end{aligned}$$

b) The divergence is

$$\text{div}\vec{F} = \vec{\nabla} \cdot \vec{F} = P_x + Q_y + R_z.$$

So, we get

$$\text{div}\vec{F} = 0 + x^3z^2 + 3y^4z^2 = (x^3 + 3y^4)z^2.$$

Section 16.5, Problem 14

For $\vec{F}$ to be conservative, we have to verify that $\text{curl}\vec{F} = 0$. The curl of $\vec{F}$ is

$$\begin{aligned}\text{curl}\vec{F} &= (4x^2z^3 - 4x^2z^3)\vec{i} - (8xyz^3 - 4xyz^3)\vec{j} + (2xz^4 - xz^4)\vec{k} \\ &= -8xyz^3\vec{j} + xz^4\vec{k} \neq \vec{0}.\end{aligned}$$

So the vector field is not conservative.