Math 644

Chapter 2

Section 2.2: Fundamental Theorem of Algebra and Partial Fractions

Contents

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The local behavior of a polynomial (Walking a Dog picture) is really helpful to give a proof of the FTA.

Theorem 1. Every non-constant polynomial has a zero.

Some precision:

- A function $f : \Omega \subseteq \mathbb{C} \to \mathbb{C}$ has a zero at $a \in \Omega$ if $f(a) = 0$.

Lemma 2. If $n := \deg p \geq 1$, then $|p(z)| \to \infty$, as $|z| \to \infty$.

Proof.

Lemma 3. If $p(z)$ is a polynomial with no zero, then

$$M := \inf\{|p(z)| \,:\, z \in \mathbb{C}\} \in (0, \infty).$$

Proof.

Proof of the FTA.

COROLLARY 4. If p is a polynomial of degree $n \geq 1$, then there are complex numbers z_1, z_2, ..., z_n and a compact constant c such that

$$p(z) = c \prod_{k=1}^{n} (z - z_k).$$

Proof.

Example 5. Find the zeros of $p(z) = z^n - 1$, $n \geq 1$.

Rational Functions

A **rational function** is a quotient of two polynomials. From the FTA, we can write

$$r(z) = \frac{p(z)}{\prod_{j=1}^{N}(z - z_j)^{n_j}}$$

for some $N, n_j \in \mathbb{C}$ and $z_1, z_2, \ldots, z_N \in \mathbb{C}$.

Corollary 6. Let p be a polynomial. Then there is a polynomial $q(z)$ and complex constants $c_{k,j}$ such that

$$\frac{p(z)}{\prod_{j=1}^{N}(z - z_j)^{n_j}} = q(z) + \sum_{j=1}^{N} \sum_{k=1}^{n_j} \frac{c_{k,j}}{(z - z_j)^k}.$$

A simple case: