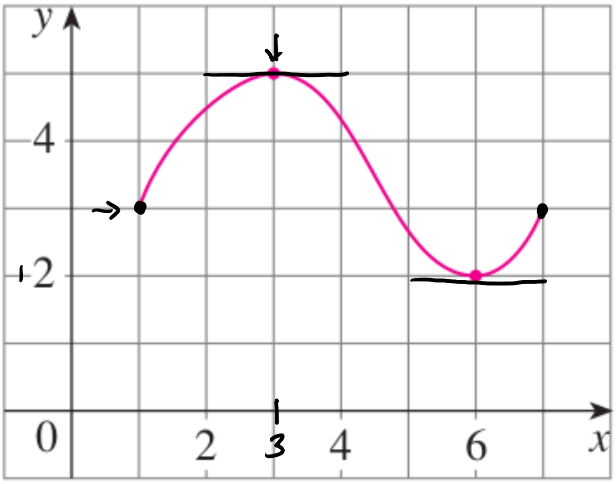
Chapter 3
Applications of Derivatives

3.1 Maximum and Minimum Values

Maximums and minimums.

What would be a maximum value or a minimum value of a function?



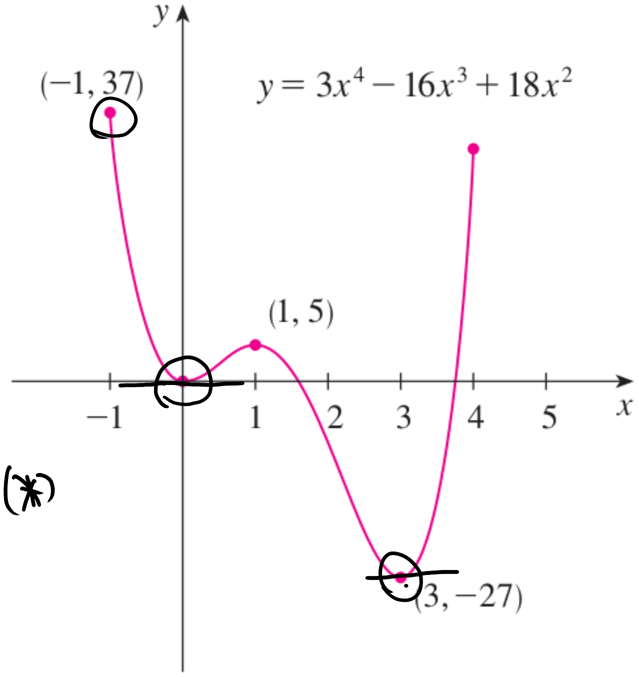
Suggestions/observations:

1) 5 → max.
   2 → min.

2) Slope of the tangent where a min occurs is 0.

3) Slope of the tangent where a max occurs is 0.

4) Derivative ∄ at the end points.



Suggestions/observations:

1) Even if the slope of the tangent is 0, it doesn't that we hit a max/min.

2) A max or min can occur when the derivative ∄.

3) —

4) —

Important observations:

a) max & min occurs when slope of tangent is 0

b) max & min occur when the derivative ∄.

1 Definition Let c be a number in the domain D of a function f. Then $f(c)$ is the

- **absolute maximum** value of f on D if $f(c) \geqslant f(x)$ for all x in D.
- **absolute minimum** value of f on D if $f(c) \leqslant f(x)$ for all x in D.

2 Definition The number $f(c)$ is a

- **local maximum** value of f if $f(c) \geqslant f(x)$ when x is near c.
- **local minimum** value of f if $f(c) \leqslant f(x)$ when x is near c.

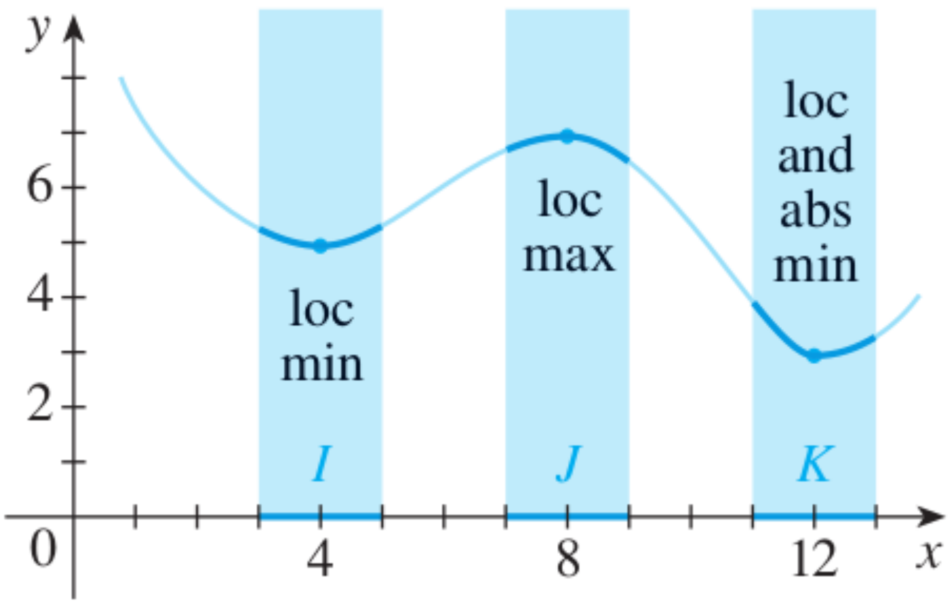


Illustration of the local and absolute max and min.

Terminology.

loc. min. = local minimum.
loc. max. = local maximum.
abs. min. = absolute minimum
abs. max. = absolute maximum.

1) Global maximum or global minimum

2) Extreme values for abs. max. and abs. min.
loc. loc.

Example 4. Identify the extremums of the function $f(x) = 3x^4 - 16x^3 + 18x^2$ using the graph of the function. $-1 \le x \le 4$

See (*) abs. max: at $x=-1$ with $f(-1) = 37$
abs. min.: at $x=3$ with $f(3) = -27$
loc. max.: at $x=-1$ with $f(-1)=37$ $x=4$ with $f(4)=32$.
at $x=1$ with $f(1)=5$
loc. min.: at $x=0$ with $f(0)=0$
at $x=3$ with $f(3)=-27$.

Extreme Values Theorem.

Which conditions garantee that extreme values exist?

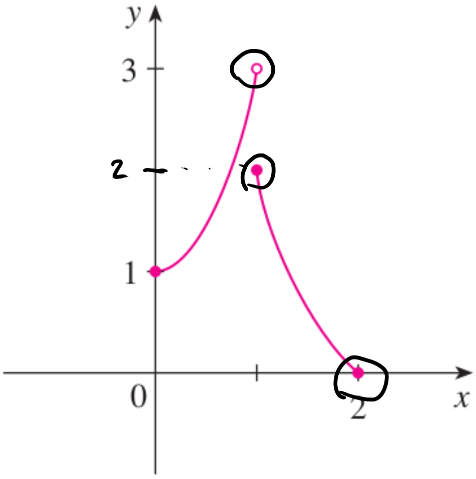


FIGURE 9
This function has minimum value $f(2) = 0$, but no maximum value.

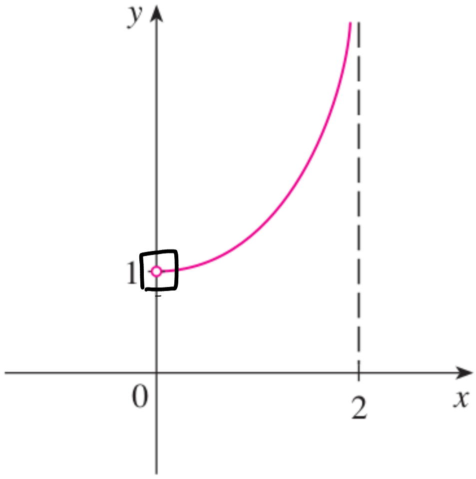
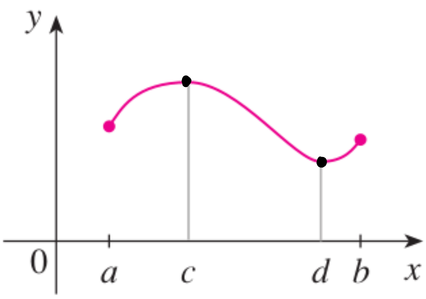
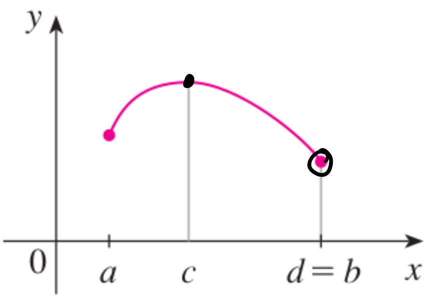


FIGURE 10
This continuous function g has no maximum or minimum.

3 **The Extreme Value Theorem** If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.



attained inside



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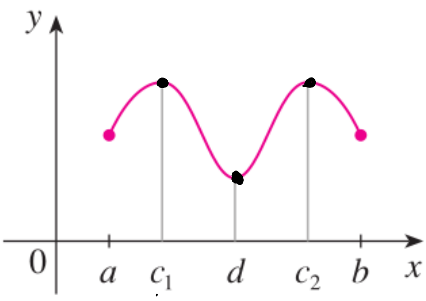
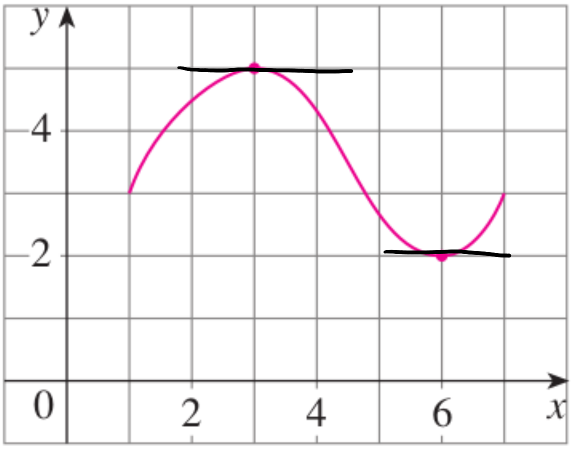


Attained multiple times

Fermat's Theorem.

An observation:

when we have a (loc) max or abs. max or (loc) min or abs. min, then the slope of the tangent is $\underline{0}$

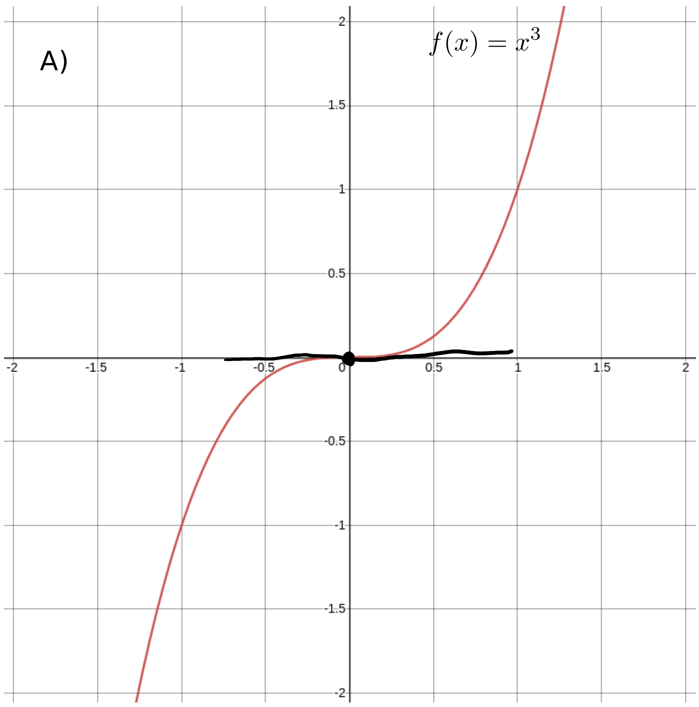


$x^n+y^n=z^n \quad (n>2)$

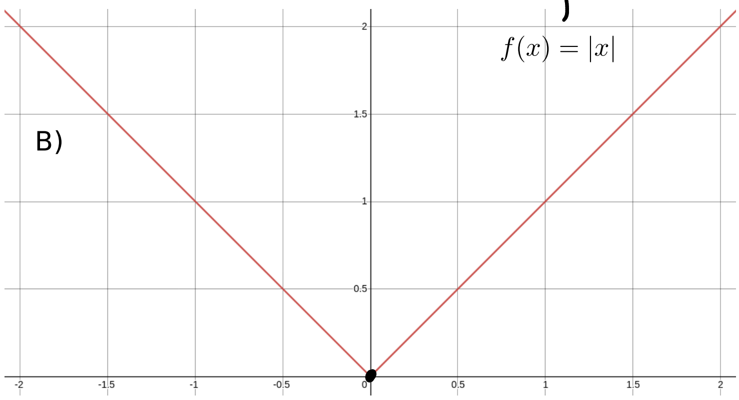
4 Fermat's Theorem If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c)=0$.

Interested in the proof: see page 207 in the textbook.

BE CAREFUL!!



$f(0)=|0|=0$



A) Even if $f'(0)=0$, $f(0)$ is not a max or a min in the fct $f(x)$.

B) When $f'(0)$ ∄, it doesn't that there are no max or no min. (Here, $f(0)=0$ is an abs. max.).

6 Definition A **critical number** of a function f is a number c in the domain of f such that either $f'(c)=0$ or $f'(c)$ does not exist.

EXAMPLE 7 Find the critical numbers of $f(x) = \underbrace{x^{3/5}(4-x)}_{\text{Product}}$. $= \underbrace{4x^{3/5} - x^{8/5}}_{\text{sum of 2 fcts.}}$

① Compute $f'(x)$

$$f'(x) = 4 \cdot \frac{3}{5} x^{-2/5} - \frac{8}{5} x^{3/5} = \frac{12}{5x^{2/5}} - \frac{8}{5} x^{3/5}$$

$$(x^{3/5})' = \frac{3}{5} x^{3/5 - 1} = \frac{3}{5} x^{3/5 - 5/5} = \frac{3}{5} x^{-2/5}$$

$$= \frac{12 - 40x}{5x^{2/5}}$$

$$= \frac{4(3-10x)}{5x^{2/5}}.$$

② Find the zeros of $f'(x)$

$$f'(x) = 0 \iff \frac{4}{5}\left(\frac{3-10x}{x^{2/5}}\right) = 0$$

$$\iff \frac{\boxed{3-10x}}{x^{2/5}} = 0$$

$$\iff 3 - 10x = 0$$

$$\iff x = \frac{3}{10}$$ zero of f.

③ Find where $f'(x)$ $\exists$

$f'(x)$ $\exists$ if $x \neq 0$ because

$f'(0) = \frac{12}{0}$ number $= \pm\infty \to f'(0) \nexists$.

$x = 0$ is a number where $f'(x) \nexists$.

Critical numbers: Dom $f = (-\infty, \infty)$

So, the C.N. are $x = \frac{3}{10}$ & $x = 0$.

The Closed Interval Method To find the *absolute* maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

EXAMPLE 8 Find the absolute maximum and minimum values of the function

$$f(x) = x^3 - 3x^2 + 1 \qquad -\tfrac{1}{2} \leqslant x \leqslant 4$$

① Critical numbers.

$f'(x) = 3x^2 - 6x$

$\downarrow$

C.N. $x=0$, $x=2$

$\downarrow$

$f(0) = 1$

$f(2) = -3$

(1.1) Zeros.

$f'(x)=0 \iff 3x^2-6x=0$

$\iff (3x-6)x=0$

$\iff 3x-6=0$ or $x=0$

$\iff x=2$ or $x=0$

(1.2) $f'(x) \not\exists$

$f'(x)$ is a poly. $\to$ always well defined.

② Endpoints

$f(-1/2) = 1/8$ & $f(4) = 17$.

③ Largest/smallest of ① & ②

Max of $f = \max\{1, -3, 1/8, 17\} = \boxed{17.}$

Min of $f = \min\{1, -3, 1/8, 17\} = \boxed{-3.}$