Fall 2020 Midterm

Solutions.

#2. <u>Goal</u>: find the tangent line at $(0,1)$ (which is (x,y)).

Implicit differentiation:

$$\frac{d}{dx}\sin(\pi y^2) = \frac{d}{dx}(xy)$$

$$\Rightarrow \left(\cos(\pi y^2)\right)\cdot \pi 2y\cdot y' = y + xy'$$

$$\Rightarrow \left(\cos(\pi y^2)\right)\cdot \pi 2y\cdot y' - xy' = y$$

$$\Rightarrow \left(\left(\cos(\pi y^2)\right)\cdot 2\pi y - x\right)y' = y$$

$$\Rightarrow \quad y' = \frac{y}{\left(\cos(\pi y^2)\right)\cdot 2\pi y - x}.$$

So, the tangent line is given by

$$T(x) - 1 = m(x-0)$$

where $m = y'(0)$. We have

$$y' = \frac{1}{\cos(\pi)\cdot 2\pi - 0} = -\frac{1}{2\pi}.$$

So,

$$\boxed{T(x) = \frac{-1}{2\pi}x + 1 = -\frac{x}{2\pi} + 1}.$$

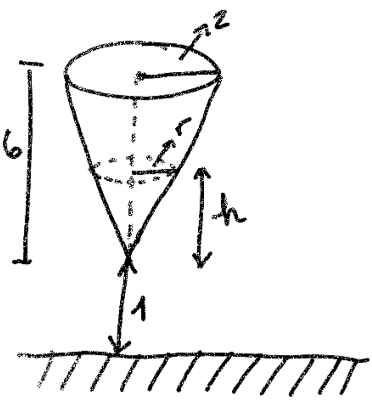
#3

r: radius (fct. of time)

h: height (fct. of time).

V: volume of water in the tank

Picture



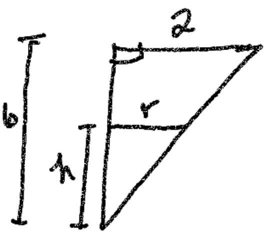
We have $V = \frac{\pi}{3} r^2 h$

change in h!

$\frac{dh}{dt}$: rate of change of the height.

$\frac{dV}{dt}$: rate of change of the volume

↳ $\frac{dV}{dt} = -0.5 \ m^3/\text{min}$.



$$\frac{r}{2} = \frac{h}{6} \Rightarrow r = \frac{h}{3}$$

So, $V(h) = \frac{\pi}{3}\left(\frac{h}{3}\right)^2 \cdot h = \frac{\pi}{27} \cdot h^3$.

Taking d/dt:

$$\frac{dV}{dt} = \frac{\pi}{27} \frac{d}{dt}(h^3) = \frac{\pi}{27} \cdot 3 \cdot h^2 \cdot \frac{dh}{dt}$$

$$\Rightarrow \frac{dV}{dt} = \frac{\pi}{9} \cdot h^2 \cdot \frac{dh}{dt}.$$

We want $\frac{dh}{dt}$ at $h = 6/2 = 3$. So,

$$-0.5 = \frac{\pi}{9} \cdot 3^2 \cdot \frac{dh}{dt} = \pi \cdot \frac{dh}{dt}$$

$$\Rightarrow \boxed{\frac{dh}{dt} = \frac{-0.5}{\pi} \text{ m/min}}$$

#4 (a) ① $f'(x) = \frac{1}{3}x^{-2/3}(x-4) + x^{1/3}$

$$= \frac{x-4}{3x^{2/3}} + x^{1/3}.$$

$$\Rightarrow f'(x) = \frac{x-4+3x}{3x^{2/3}} = \frac{4(x-1)}{3x^{2/3}}.$$

② Critical pts. $f'(x) = 0 \iff x = 1$

$f'(x) \nexists$ iff $x = 0$.

Also, $x = -1$ & $x = 4$ are the end-points.

③ Max & min.

$f(-1) = 5$, $f(0) = 0$, $f(1) = -3$ & $f(4) = 0$.

So $\boxed{\begin{aligned} \text{max} &= 5 \\ \text{min} &= -3 \end{aligned}}$

(b) ① $g''(x)=0 \iff x=0$ or $x=4$.

x		0		4	
$x^{1/3}$	$-$	0	$+$	$+$	$+$
$x-4$	$-$	$-$	$-$	0	$+$
$g''(x)$	$+$	0	$-$	0	$+$

So, $\boxed{\begin{array}{l} g \text{ concave-up on } (-\infty,0) \,\&\, (4,\infty) \\ g \text{ concave-down on } (0,4) \end{array}}$

#5 (a) Approximate by the tangent line:

$g(6) \approx L(6)$ where

$$L(x) = m(x-5) + g(5).$$

$m = g'(5) = 5$. So,

$$L(x) = 5(x-5) + 24$$

$$\Rightarrow L(x) = 5x - 1.$$

So, $\boxed{g(6) \approx L(6) = 29}$

(b) HA at ∞

$$\lim_{x\to\infty} \frac{4x^2-3}{2-x-x^2} = \lim_{x\to\infty} \frac{4-3/x^2}{2/x^2-1/x-1}$$

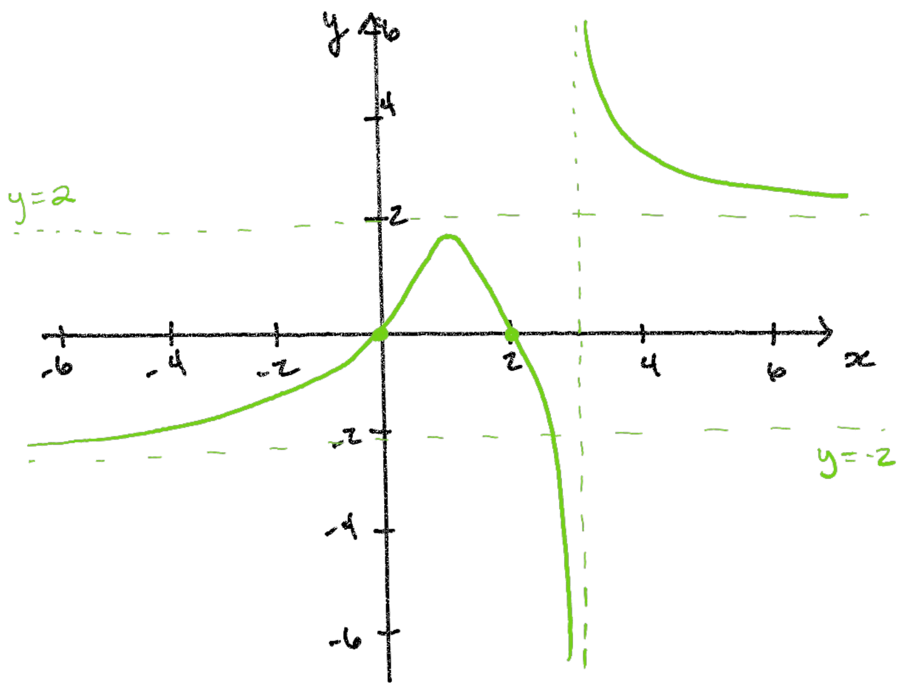
$$= -4.$$

HA at $-\infty$

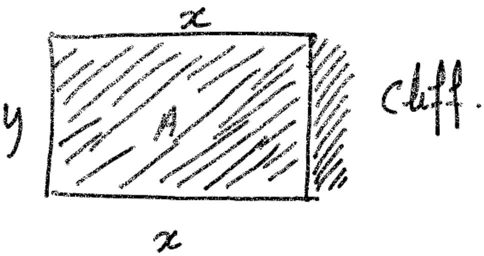
$$\lim_{x\to-\infty} \frac{4x^2-3}{2-x-x^2} = -4.$$

So, $y=-4$ is a HA at ∞ & $-\infty$.

#6



#7.



A: area of the field

x: width of the field.

y: legth of the field.

$\underline{A = xy}$.

We know that $60 = 2x + y$

$$\Rightarrow \quad y = 60 - 2x.$$

then,

$$A(x) = x(60 - 2x) = 60x - 2x^2.$$

thus,

$$A'(x) = 60 - 4x = 0$$

$$\Leftrightarrow \quad x = 15.$$

x		15	
$A'(x)$	+	0	−

So, $x = 15$ is a global max.

Thus,

$$\boxed{A = 15 \cdot 30 = 450\ m^2}$$

#8 (a) x_2 is less than the zero because the tangent line lies above the graph of the fct.

(b) x_2 is greater than the zero because the tangent line lies below the graph.

(c) With $x_1 = 0$ because the slope of the tangent line at that point is steeper that the slope of the tangent line at the point $x_1 = 1$.