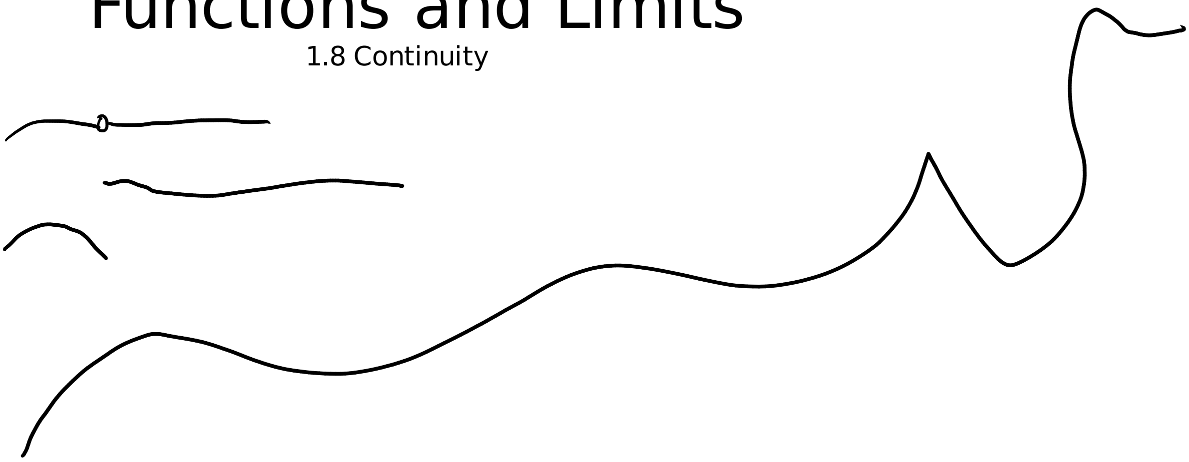
Chapter 1

Functions and Limits

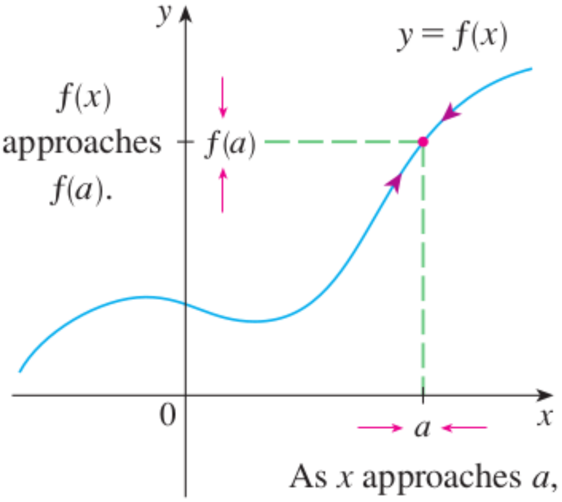
1.8 Continuity



Definition of continuity.

1 Definition A function f is **continuous at a number a** if

$$\lim_{x \to a} f(x) = f(a)$$



Three things to verify to show a function is continuous:

a) $f(a)$ must be defined at $x = a$.

b) $\lim_{x \to a} f(x)$ must exist.

c) $\lim_{x \to a} f(x)$ must agree with $f(a)$.

Discontinuity: not continuous at $x = a$.

This means a) or b) or c) is false.

EXAMPLE 1 Figure 2 shows the graph of a function f. At which numbers is f discontinuous? Why?

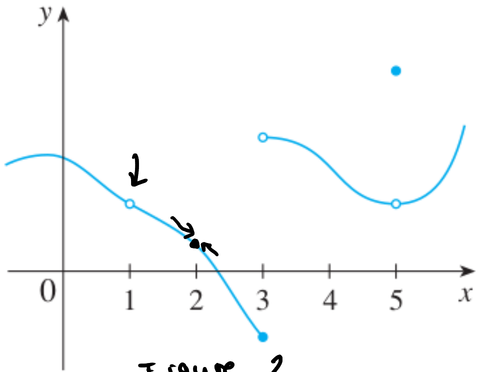


Figure 2.

$x = 1$ $f(1)$ is not defined.

$x = 3$. $\lim_{x \to 3} f(x) \nexists$

$x = 5$. $\lim_{x \to 5} f(x) \neq f(5)$

EXAMPLE 2 Where are each of the following functions discontinuous?

(a) $f(x) = \frac{\overset{(x-2)(x+1)}{x^2 - x - 2}}{x-2} = x+1$ (b) $f(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } \boxed{x = 0} \end{cases}$ (c) $f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$

(a) $\underline{x=2}$ $f(2)$ is not defined. At all $x \neq 2$, f is continuous.

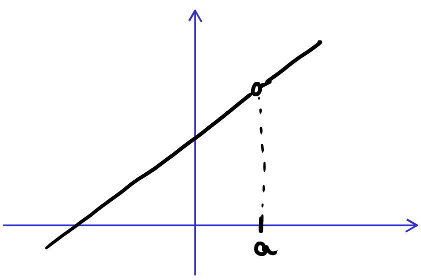
(b) $\underline{x=0}$ $\lim_{x \to 0} f(x) = +\infty$ ($\nexists$). At all $x \neq 0$, f is continuous.

(c) $\underline{x=0}$ $\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x) \to \lim_{x \to 0} f(x) \nexists$.

$\Rightarrow$ discontinuous at $x=0$.

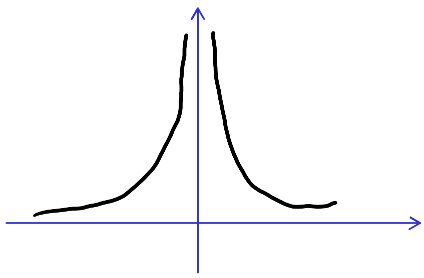
Continuous for all $x \neq 0$ because f is constant ($=1, x>0$; $=0, x<0$)

3 kinds of discontinuity.

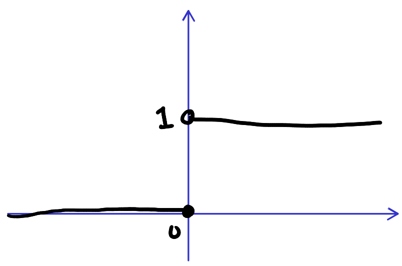


(a) Removable.

$f(a) := \lim_{x \to a} f(x)$



(b) Infinite discontinuity.



(c) Jump continuity.

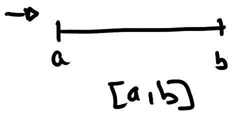
Right-continuous

$\lim_{x \to a^+} f(x) = f(a)$

Left-continuous

$\lim_{x \to a^-} f(x) = f(a)$

3 Definition A function f is **continuous on an interval** if it is continuous at every number in the interval. (If f is defined only on one side of an endpoint of the interval, we understand *continuous* at the endpoint to mean *continuous from the right* or *continuous from the left*.)



EXAMPLE 4 Show that the function $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on the interval $[-1, 1]$.

$1 - x^2 \geq 0 \iff -1 \leq x \leq 1$

three properties/conditions to verify.

Take a $-1 \leq x \leq 1$.

(a) $f(x)$ is well-defined.

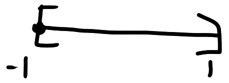
(b) + (c) $x \neq -1, 1$ $(-1 < x < 1)$

$$\lim_{t \to x} f(t) = \lim_{t \to x} \left(1 - \sqrt{1 - t^2}\right)$$

$$= \lim_{t \to x} 1 - \lim_{t \to x} \sqrt{1 - t^2}$$

$$= 1 - \sqrt{\lim_{t \to x} (1 - t^2)}$$

$$= 1 - \sqrt{1 - x^2} = f(x). \checkmark$$



$x = -1$ $\quad \lim_{t \to -1^+} f(t) = \lim_{t \to -1^+} \left(1 - \sqrt{1 - t^2}\right) = f(-1) \checkmark$

$x = 1$ $\quad \lim_{t \to +1^-} f(t) = \lim_{t \to 1^-} \left(1 - \sqrt{1 - t^2}\right) = f(1) \checkmark$

So, f is continuous on $[-1, 1]$.

4 Theorem If f and g are continuous at a and if c is a constant, then the following functions are also continuous at a:

1. $f+g$ 2. $f-g$ 3. cf
4. fg 5. $\frac{f}{g}$ if $g(a) \neq 0$

$f(x) = x$
$g(x) = x^2$
$\lim_{x \to a} x = a$
$\lim_{x \to a} x^2 = a^2$

$f(x)+g(x) = x + x^2$

Application: Any polynomial is continuous on $(-\infty, \infty)$ and any rational function is continuous on its domain.

Proof. Take $-\infty < a < \infty$. We want to show that (a), (b) & (c) are satisfied. Call $P(x)$ the polynomial

(a) the domain of P is all $(-\infty, \infty) \to P(a)$ is well-defined.

(b)+(c)

$$\lim_{x \to a} P(x) = \lim_{x \to a} (a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0)$$

$$= \lim_{x \to a} a_n x^n + \lim_{x \to a} a_{n-1}x^{n-1} + \dots + \lim_{x \to a} a_1 x + \lim_{x \to a} a_0$$

$$= a_n \lim_{x \to a} x^n + a_{n-1} \lim_{x \to a} x^{n-1} + \dots + a_1 \lim_{x \to a} x + a_0$$

$$= a_n (a)^n + a_{n-1}(a)^{n-1} + \dots + a_1 (a) + a_0$$

$$= P(a) \checkmark$$

EXAMPLE 5 Find $\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}$. "$f(x)$"

$5 - 3x = 0 \to x = \frac{5}{3} \to f$ is continuous except at $x = \frac{5}{3}$.

f is continuous at -2

$$\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} = f(-2) = \frac{-8 + 8 - 1}{5 + 6} = \boxed{\frac{-1}{11}}$$

7 Theorem The following types of functions are continuous at every number in their domains:

- polynomials
- rational functions
- root functions
- trigonometric functions

EXAMPLE 6 On what intervals is each function continuous?

(a) $f(x) = x^{100} - 2x^{37} + 75$ (b) $g(x) = \dfrac{x^2 + 2x + 17}{x^2 - 1}$

(c) $h(x) = \sqrt{x} + \dfrac{x+1}{x-1} - \dfrac{x+1}{x^2+1}$

(a) $(-\infty, \infty)$ (b) $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$ | $(-\infty, \infty) \setminus \{-1, 1\}$

(c) $[0, \infty) \setminus \{1\}$ | $[0, 1) \cup (1, \infty)$.

EXAMPLE 7 Evaluate $\lim_{x \to \pi} \dfrac{\sin x}{2 + \cos x}$. $= f(x)$

$2 + \cos(\pi) = 2 - 1 = 1 \neq 0$

f is continuous at $x = \pi$, so

$$\lim_{x \to \pi} \frac{\sin x}{2 + \cos x} = \frac{\sin \pi}{2 + \cos \pi} = \frac{0}{1} = \boxed{0}.$$

Composition of Continuous Functions.

8 Theorem If f is continuous at b and $\lim_{x \to a} g(x) = b$, then $\lim_{x \to a} f(g(x)) = f(b)$. In other words,

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$$

9 Theorem If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a.

EXAMPLE 8 Where are the following functions continuous?

(a) $h(x) = \sin(x^2)$ (b) $F(x) = \dfrac{1}{\sqrt{x^2 + 7} - 4}$

(a) $f(x) = \sin x \to$ cont.
$g(x) = x^2 \to$ cont on $(-\infty, \infty)$
$\to h(x) = \underbrace{f \circ g}_{f(g(x))}(x) \to \sin(x^2)$ is cont. on $(-\infty, \infty)$

(b) $f(x) = \dfrac{1}{x} \to (-\infty, \infty) \setminus \{0\}$, $g(x) = \sqrt{x} - 4 \to [0, \infty)$, $h(x) = x^2 + 7 \to (-\infty, \infty)$

$$F(x) = f(g(h(x))) = \frac{1}{\sqrt{x^2 + 7} - 4}$$

$f \to$ cont on $(-\infty, \infty) \setminus \{0\}$
$g \to$ cont. on $[0, \infty)$
$h \to$ cont. on $(-\infty, \infty)$

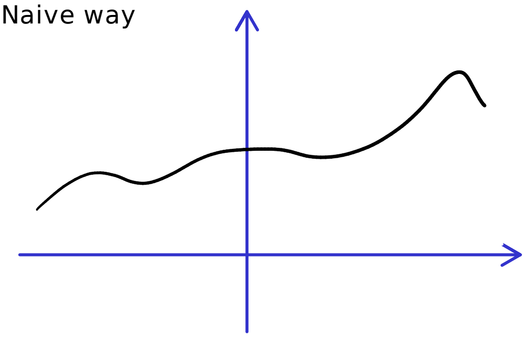
$\to \sqrt{x^2 + 7} - 4 = 0 \iff x = \pm 3$

$\to g \circ h$ is cont. on $(-\infty, \infty)$

$\to (-\infty, \infty) \setminus \{-3, 3\}$

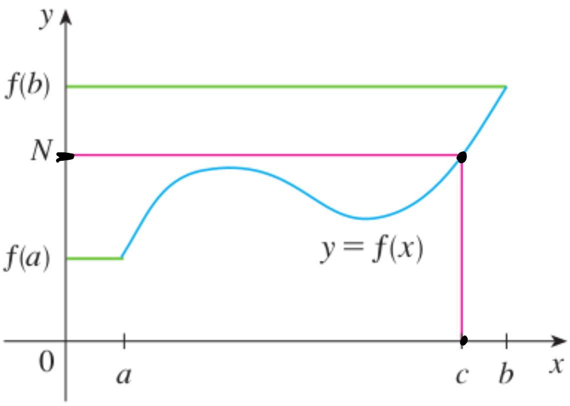
The Intemediate Theorem.

Naive way

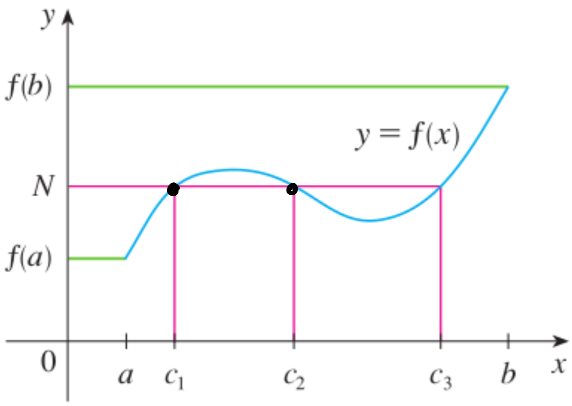


Rigorously put

> **10 The Intermediate Value Theorem** Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.



(a) attained once



(b) Attained multiple times.

EXAMPLE 9 Show that there is a root of the equation

$$\underbrace{4x^3 - 6x^2 + 3x - 2 = 0}$$

between 1 and 2.

root?? $\exists\, x$ s.t. $y = f(x) = 0$

$f(x) = 4x^3 - 6x^2 + 3x - 2$ $\qquad N = 0 \qquad a = 1 \qquad b = 2$

$f(1) = 4 - 6 + 3 - 2 = -1$

$f(2) = 32 - 24 + 6 - 2 = 12$

$\rightarrow \quad \underset{-1}{\underset{\shortparallel}{f(1)}} < 0 < \underset{12}{\underset{\shortparallel}{f(2)}}$

By the IVT, $\exists c$ between 1 and 2 s.t.

$f(c) = 0$.