Problem 12

Let $x = r\cos\theta$ and $y = r\sin\theta$ (we change from cartesian to polar coordinates). The equation of the circle of radius 2 centered at the origin is simply $r = 2$. Also, in polar coordinates, we have $dA = rdrd\theta$ and so

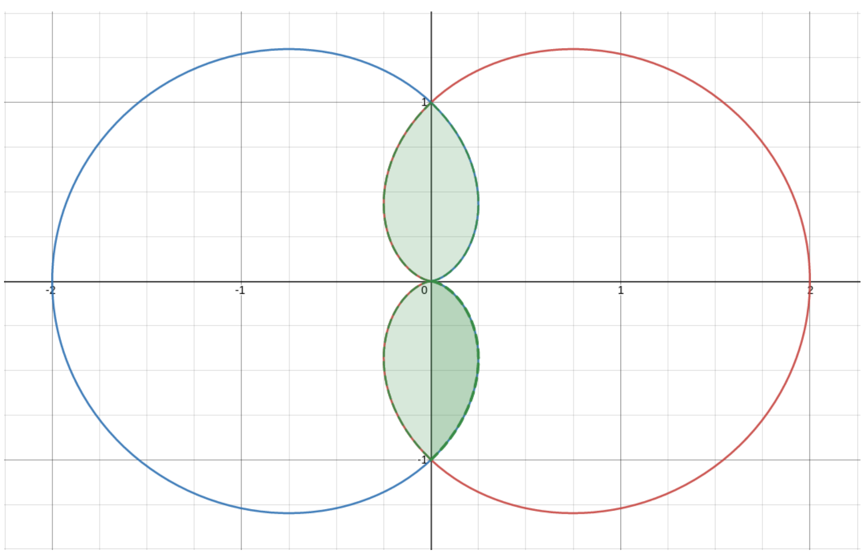
$$\iint_D \cos\sqrt{x^2+y^2}\,dA = \int_0^{2\pi}\int_0^2 (\cos r)r\,drd\theta = \Big(\int_0^{2\pi} d\theta\Big)\Big(\int_0^2 r\cos r\,dr\Big).$$

After an integration by parts, the value of the integral is

$$\iint_D \cos\sqrt{x^2+y^2}\,dA = 2\pi(-1+2\sin(2)+\cos(2)).$$

Problem 16

We draw the region between the two cardioids. Here are the two regions (in green) enclosed within the two cardiods: The two cardiods meet when $1+\cos\theta = 1-\cos\theta$. After rearranging, we



have to solve the equation $2\cos\theta = 0$. This occurs only when θ is $\pi/2 + k\pi$. We choose the values $\theta = \pi/2$ and $\theta = -\pi/2$. So, the polar coordinates of the two points of intersection are

$$(1, \pi/2) \quad \text{and} \quad (1, -\pi/2)$$

which corresponds to the following points in the cartesian plane:

$$(0, 1) \quad \text{and} \quad (0, -1).$$

We have now to setup the integral. Let D denote the region enclosed by the two cardioids. The area is given by

$$A(D) = \iint_D dA.$$

In polar coordinates, we have $dA = r dr d\theta$. Due to the symmetry of the domain, we can only compute the area of the petal with a positive (or zero) y coordinate (above the x-axis) and then multiply our result by 2. Call this region D_1.

The argument θ will vary from 0 to π. However, we have to split the interval $[0, \pi]$ into the intervals $[0, \pi/2]$ and $[\pi/2, \pi]$ because the cardioids intersect at $\theta = \pi/2$. We can apply again the symmetry argument because the region D_1 is symmetric with respect to the y axis and multiply by 2 to get the area of D_1. Denote half of the petal by D_2. So we have

$$A(D_2) = \int_0^{\pi/2} \int_0^{1-\cos(\theta)} r\, dr d\theta = \int_0^{\pi/2} \frac{(1-\cos\theta)^2}{2}\, d\theta = \frac{3\pi}{8} - 1.$$

Thus,

$$A(D) = 2A(D_1) = 4A(D_2) = \frac{3\pi}{2} - 4.$$

Problem 22

The surfaces that bounds the z-coordinate are $z = \sqrt{16 - x^2 - y^2}$ and $z = -\sqrt{16 - x^2 - y^2}$. We want the solid outside the cylinder $x^2 + y^2 = 2^2$. Let D be the domain of integration, then the volume of the solid is given by

$$V(S) = \iint_D \sqrt{16 - x^2 - y^2} - (-\sqrt{16 - x^2 - y^2})\, dA = \iint_D 2\sqrt{16 - x^2 - y^2}\, dA.$$

To find D, we will project on the XY-plane. The projections of the cylinder on the XY-plane is a circle of radius 2 and the projection of the sphere on the XY-plane is a circle of radius 4. Thus we want the region between these two circles. We will change in polar coordinates. The equation of the two circles are $r = 2$ and $r = 4$ and the angle ranges from 0 to 2π. So, in polar coordinates, we have

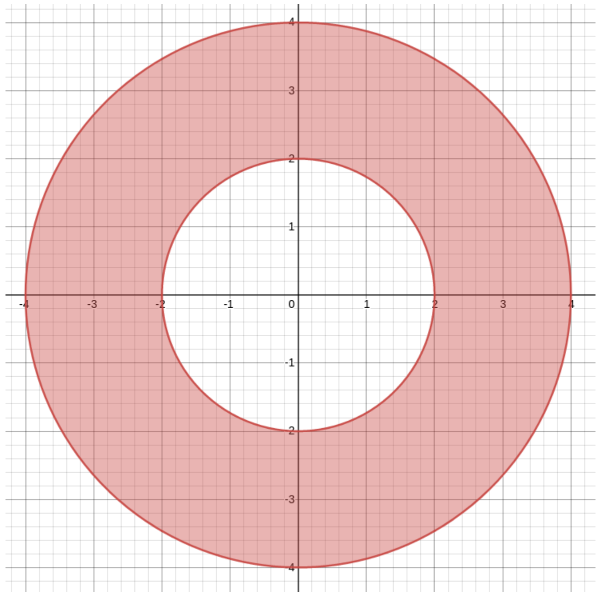
$$D = \{(r, \theta) : 2 \le r \le 4 \text{ and } 0 \le \theta \le 2\pi\}.$$

Thus, using the change of variable formula, we get

$$V(S) = 2\int_0^{2\pi} \int_2^4 (\sqrt{16 - r^2}) r\, dr d\theta = 4\pi \int_2^4 r\sqrt{16 - r^2}\, dr.$$

Setting $u = 16 - r^2$ and completing the calculations for the integral, we get the value

$$V(S) = 32\pi\sqrt{3}.$$



Problem 32

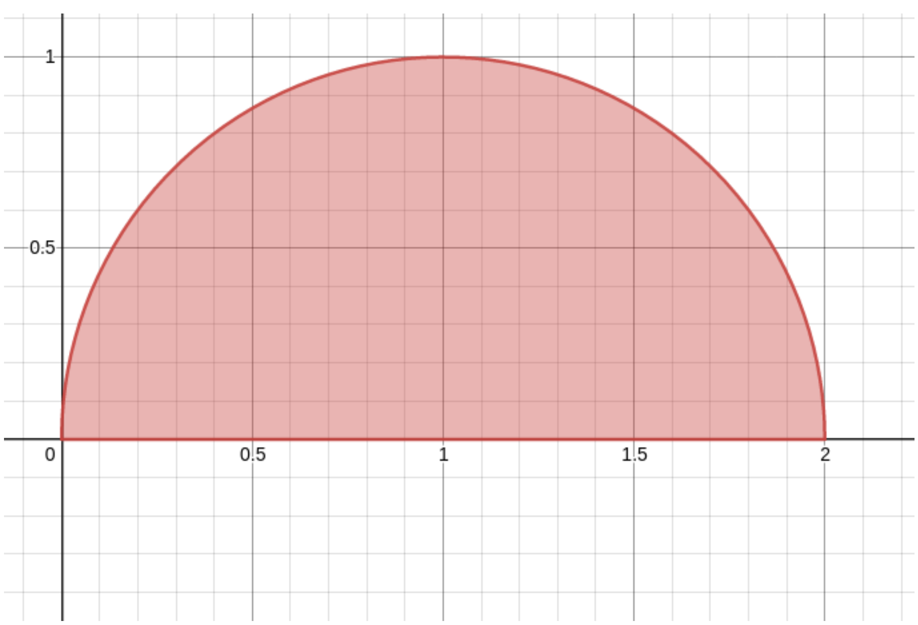
The bounds in the integrals give us

$$D = \{(x, y) \,:\, 0 \le x \le 2 \text{ and } 0 \le y \le \sqrt{2x - x^2}\}$$

So, the upper bound of y is half of a circle of radius 1 and center $(1, 0)$ because

$$y = \sqrt{2x - x^2} \iff (x - 1)^2 + y^2 = 1.$$

So the region looks like this



Let's describe the domain D in polar coordinates. Let $x = r\cos\theta$ and $y = r\sin\theta$. The equation of the circle in polar coordinate is $r = 2\cos\theta$ (replace x and y by $r\cos\theta$ and $r\sin\theta$ in the equation $x^2 + y^2 = 2x$). Also, the angle θ will vary from 0 to $\pi/2$ so we cover the upper half of the circle (and its interior). So

$$D = \{(r, \theta) \,:\, 0 \leq r \leq 2\cos\theta \text{ and } 0 \leq \theta \leq \pi/2\}.$$

We can now compute the integral, call it I, in polar coordinates:

$$I = \int_0^{\pi/2} \int_0^{2\cos\theta} r^2 \, dr d\theta = \frac{8}{3} \int_0^{\pi/2} \cos^3\theta \, d\theta = \frac{2}{3} \int_0^{\pi/2} 3\cos(\theta) + \cos(3\theta) \, d\theta.$$

After finding the value of the integral, we get $I = 16/9$.