**Section 16.8, Problem 2**

By Stokes' Theorem, we have

$$\iint_S \operatorname{curl} \vec{F} \, d\vec{S} = \int_C \vec{F} \cdot d\vec{r}.$$

The surface $S$ is the part of the paraboloïd oriented upward. The boundary of $S$ is the circle $x^2 + y^2 = 1$ (when we let $z = 0$ in the equation of the paraboloïd). We parametrize the circle with

$$\vec{r}(\theta) = \langle \cos\theta, \sin\theta, 0 \rangle \quad (0 \leq \theta \leq 2\pi).$$

The surface induces the counterclockwise orientation on $C$. Thus, we get

$$\begin{aligned} \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} &= \int_0^{2\pi} \left\langle \cos^2\theta \sin(0), \sin^2\theta, \cos\theta \sin\theta \right\rangle \cdot \langle -\sin\theta, \cos\theta, 0 \rangle \; dt \\ &= \int_0^{2\pi} \sin^2\theta \cos\theta \, dt = 0. \end{aligned}$$

## Section 16.8, Problem 8

By Stokes' Theorem, we have

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}$$

The curve $C$ is the boundary of the surface $S$ and a parametrization for the surface $S$ is

$$\vec{r}(x,y) = \langle x, y, 1 - 3x - 2y \rangle \quad (0 \le x \le 1/3,\ 0 \le y \le (1-3x)/2\ ).$$

Since $C$ is oriented counterclockwise, $S$ must be positively oriented. So, a normal vector to $S$ would be

$$\vec{r}_x \times \vec{r}_y = \langle 3, 3, 1 \rangle\,.$$

The curl $\vec{F}$ is

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 1 & (x+yz) & xy - \sqrt{z} \end{vmatrix} = \langle x - y, -y, 1 \rangle\,.$$

So, we obtain

$$\operatorname{curl} \vec{F} \cdot (\vec{r}_x \times \vec{r}_y) = 3x - 3y - 3y + 1 = 3x - 6y + 1.$$

Thus, we finally obtain

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{1/3} \int_0^{(1-3x)/2} 3x - 6y + 1\, dy dx = 1/36.$$

## Section 16.9, Problem 6

By the Divergence Theorem, we have

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F}\, dV.$$

The surface is a rectangular box $S$. The inside of the box is

$$E = \{(x, y, z) \,:\, 0 \le x \le a, 0 \le y \le b, 0 \le z \le c\}.$$

We have

$$\operatorname{div} \vec{F} = 2xyz + 2xyz + 2xyz = 6xyz.$$

So, we obtain

$$\iint_S \vec{F} \cdot d\vec{S} = \int_0^c \int_0^b \int_0^a 6xyz\, dxdydz = 3a^2b^2c^2/4.$$

**Section 16.9, Problem 18**

Let $S_1$ be the bottom of the paraboloid. This is the disk

$$S_1 = \{(x, y, z) : x^2 + y^2 \leq 1, z = 1\}.$$

Let $\tilde{S} := S \cup S_1$, with the outward orientation. By the Divergence Theorem, we have

$$\iint_{\tilde{S}} \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} \, dV.$$

The solid $E$ bounded by $\tilde{S}$ is

$$E = \{(x, y, z) : x^2 + y^2 \leq 1, 1 \leq z \leq 2 - x^2 - y^2\}.$$

The divergence of $\vec{F}$ is

$$\operatorname{div} \vec{F} = 0 + 0 + 1 = 1.$$

So, we obtain

$$\iint_{\tilde{S}} \vec{F} \cdot d\vec{S} = \iint_D \Big( \int_1^{2-x^2-y^2} 1 \, dz \Big) dA = \iint_D 1 - x^2 - y^2 \, dA,$$

where $D := \{(x, y) : x^2 + y^2 \leq 1\}$. We can compute this integral by passing to polar coordinates. We then obtain

$$\int_0^{2\pi} \int_0^1 (1 - r^2) r \, dr d\theta = \pi/2.$$

The flux of $\vec{F}$ through $\tilde{S}$ is then $\pi/2$.

This is not exactly the flux of $\vec{F}$ through $S$. To obtain the flux through $S$, we have to write

$$\pi/2 = \iint_{\tilde{S}} \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot d\vec{S} + \iint_{S_1} \vec{F} \cdot d\vec{S}$$

and so

$$\iint_S \vec{F} \cdot d\vec{S} = \pi/2 - \iint_{S_1} \vec{F} \cdot d\vec{S}.$$

Recall that $\tilde{S}$ had the outward orientation, so to be consistent with that choice, $S_1$ has the downward orientation. A normal vector to $S_1$ pointing downward is $\vec{n} = \langle 0, 0, -1 \rangle$ and so

$$\iint_{S_1} \vec{F} \cdot d\vec{S} = \iint_{S_1} \vec{F} \cdot \langle 0, 0, -1 \rangle \, dS = \iint_{S_1} -z \, dS = -\iint_{S_1} z \, dS$$

But, when on $S_1$, we have $z = 1$, and so the double integral represents the area of the disk. The disk has radius 1 and we then obtain

$$\iint_S \vec{F} \cdot d\vec{S} = \pi/2 + \pi = 3\pi/2.$$