Math 644

Chapter 5

Section 5.5: The Argument Principle

Contents

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The Argument Principle

Theorem 1. Suppose f is meromorphic which not constant in a region Ω with zeros set $\{z_j\}$ and poles set $\{p_k\}$. Suppose γ is a cycle with $\gamma \sim 0$ in Ω, and suppose $\{z_j\} \cap \gamma = \varnothing$ and $\{p_k\} \cap \gamma = \varnothing$. Then

$$n(f(\gamma), 0) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz = \sum_j n(\gamma, z_j) - \sum_k n(\gamma, p_k).$$

Notes:

① The convention is if z is a zero of order k of f, then z appears k times in the list $\{z_j\}$.

② For the poles, we also have the same convention: if z is a pole of order k of f, then z appears k times in the list $\{p_k\}$.

Proof.

Theorem 2. Suppose γ is a closed curve in a region Ω with $\gamma \sim 0$ in Ω and $n(\gamma, z) = 0$ or $= 1$ for all $z \in \Omega \backslash \gamma$. If f and g are analytic in Ω and satisfy

$$|f(z) + g(z)| < |f(z)| + |g(z)|,$$

for all $z \in \gamma$, then f and g have the same number of zeros enclosed by γ.

Notes:

① Again, the number of zeros of f and g are counted according to their multiplicity.

Proof.

Example 3. Let $f(z) = z^9 - 2z^6 + z^2 - 8z - 2$.

(a) How many zeros does f have in $\{z : |z| < 1\}$?

(b) How many zeros does f have in $\{z : |z| < 2\}$?