D.I Distribution Function

PROBLEM 1.

Let α be a number between 0 and 1. If $x_1 \leq x_2$, then

$$F(x_1) = \alpha F_1(x_1) + (1-\alpha)F_2(x_2) \leq \alpha F_1(x_2) + (1-\alpha)F_2(x) = F(x_2).$$

Therefore, F is inscreasing. We also have

$$\lim_{x\to\infty} F(x) = \lim_{x\to\infty} \alpha F_1(x) + (1-\alpha)F_2(x) = \alpha \lim_{x\to\infty} F_1(x) + (1-\alpha)\lim_{x\to\infty} F_2(x) = \alpha(1) + (1-\alpha)(1) = 1.$$

Therefore, F is distribution function. □

PROBLEM 2. The values of Y are always positive, so if $y < 0$, then $F_Y(y) = 0$. If $y \geq 0$, then $Y = X$ and therefore $F_Y(y) = F_X(y)$. Therefore, $F_Y = \max\{0, F_X\}$.

PROBLEM 3. For which value of c is the function

$$F(x) = c\int_{-\infty}^{x} e^{-|t|}\,dt \quad (x \in \mathbb{R})$$

a distribution function?

The total integral should be 1. We have

$$\int_{-\infty}^{\infty} e^{-|t|}\,dt = 2$$

so that

$$c(2) = 1 \quad \Rightarrow \quad c = 1/2.$$ □

D.II Continuous Random Variable

PROBLEM 4. For $x < 0$, we can differentiate and get

$$f_X(x) = \frac{d}{dx}F_X(x) = \frac{d}{dx}\Big(\frac{1}{2(1+x^2)}\Big) = \frac{-x}{(1+x^2)^2}\Big).$$

For $x > 0$, we can differentiate and get

$$f_X(x) = \frac{d}{dx}F_X(x) = \frac{d}{dx}\left(\frac{1+2x^2}{2(1+x^2)}\right) = \frac{x}{(1+x^2)^2}.$$

At $x = 0$, $f_X(0)$ is not defined. □

Problem 5. When $x < -1$, then $F_X(x) = 0$ because $f_X(x) = 0$. When $-1 \leq x < 1$, then

$$F_X(x) = \int_{-1}^{x} \frac{2}{\pi(1+t^2)}\, dt = \frac{2}{\pi} \arctan(x) + 1/2$$

When $x \geq 1$, then

$$F_X(x) = \frac{2}{\pi} \arctan(1) + \frac{1}{2} = 1.$$

Hence

$$F_X(x) = \begin{cases} 0 & x < -1 \\ \frac{2}{\pi} \arctan(x) + \frac{1}{2} & -1 \leq x \leq 1 \\ 1 & x > 1. \end{cases}$$

□

Problem 6. We must have $\lim_{x\to\infty} F_X(x) = 1$, and so

$$c \int_0^1 x(x-1)\, dx = 1 \iff -\frac{c}{6} = 1.$$

Hence we must have $c = -6$. □

D.III Functions of Random Variables

Problem 7. When $Y = g(X)$ and g is increasing, then we use the formula

$$f_Y(y) = f_X(g^{-1}(y)) \frac{d}{dy}(g^{-1}(y)).$$

Since X has the exponential distribution,

$$f_Y(y) = \lambda e^{-\lambda g^{-1}(y)} \frac{d}{dy}(g^{-1}(y)).$$

When g is decreasing, then

$$f_Y(y) = -\lambda e^{-\lambda g^{-1}(y)} \frac{d}{dy}(g^{-1}(y)).$$

a) We have $g^{-1}(y) = (y-5)/2$. Since g is increasing, with $\frac{d}{dy}(g^{-1}(y)) = 1/2$, we obtain

$$f_Y(y) = \frac{\lambda}{2} e^{-\frac{\lambda}{2}(y-5)}.$$

b) The inverse is found in the following way. Set $y = (1+x)^{-1}$ and then

$$y(1+x) = 1 \iff y + yx = 1 \iff x = \frac{1-y}{y}.$$

Therefore $g^{-1}(y) = (1-y)/y$. Since g is decreasing, with $\frac{d}{dy}(g^{-1}(y)) = -1/(1-y)^2$, we obtain

$$f_Y(y) = \frac{\lambda e^{-\frac{\lambda(1-y)}{y}}(1-y)}{y}.$$

□

Problem 8. Since the range of F is between 0 and 1, then $\operatorname{Im} Y \in [0,1]$ with each number between 0 and 1 being attained because of the continuity of F.

Since F is an increasing function, we can apply the formula to find the density function of Y. For $y \in [0,1]$, we have

$$f_Y(y) = f_X(F^{-1}(y))\frac{d}{dy}(F^{-1}(y)).$$

Since X is a continuous random variable, we have

$$F(x) = \int_{-\infty}^{x} f_X(t)\,dt.$$

A formula for the derivative of the inverse in terms of the initial function F is

$$\frac{d}{dy}(F^{-1}(y)) = \frac{1}{F'(F^{-1}(y))}.$$

Since $F'(x) = f_X(x)$, we therefore obtain

$$f_Y(y) = \frac{f_X(F^{-1}(y))}{f_X(F^{-1}(y))} = 1.$$

Therefore, the density function of Y is 1 identically on $[0,1]$. Thus, Y has a uniform distribution on $[0,1]$. $\square$

Problem 9. Set $g(x) = \frac{3x}{1-x}$. Then, $g'(x) = \frac{1}{(1-x)^2}$. The derivative is always positive, so g is increasing. The density function of Y is then given by the following formula:

$$f_Y(y) = f_X(g^{-1}(y))\frac{d}{dy}(g^{-1}(y)).$$

After some calculations, we obtain

$$g^{-1}(y) = \frac{y}{y+3} \quad \text{and} \quad \frac{d}{dy}(g^{-1}(y)) = \frac{3}{(3+y)^2}.$$

Hence,

$$f_Y(y) = f_X\left(\frac{y}{y+3}\right)\frac{3}{(3+y)^2}.$$

When $-3 < y < 0$, then $\frac{y}{y+3} < 0$ because $y+3 > 0$. Therefore $f_X(y/(y+3)) = 0$.

If $y \geq 0$, then $0 \leq \frac{y}{y+3} \leq 1$, because $y \leq y+3$ and $y+3$ is positive. Therefore, $f_X(y/(y+3)) = 1$ and then

$$f_Y(y) = \frac{3}{(3+y)^2},$$

when $y \geq 0$.

If $y < -3$, then $y < y+3$ implies that $1 > \frac{y+3}{y}$ and therefore $\frac{y}{y+3} > 1$. Therefore, $f_Y(y) = 0$.

Hence, we get

$$f_Y(y) = \begin{cases} 0 & y < 0 \\ \frac{3}{(3+y)^2} & y \geq 0. \end{cases}$$

We can now find the distribution of Y. When $y < 0$, $F_Y(y) = 0$ because $f_Y(y) = 0$. When $y \geq 0$, then

$$F_Y(y) = \int_0^y \frac{3}{(3+t)^2}\,dt = \left(\frac{-3}{3+t}\right)\Big|_0^y = 1 - \frac{3}{3+y}.$$

Notice that $F_Y(y) = \frac{y}{y+3} = g^{-1}(y)$, when $y \geq 0$. ☐

D.IV Expectation of Continuous Random Variables

Problem 10. The expectation is

$$\begin{aligned}
\mathrm{Exp}(X) &= \int_{-\infty}^{\infty} x f_X(x)\,dx \\
&= \int_{-1}^{1} \frac{2x}{\pi}\arctan(x) + \frac{x}{2}\,dx \\
&= \int_{-1}^{1} \frac{2x}{\pi}\arctan(x)\,dx \\
&= \frac{4}{\pi}\int_0^1 x\arctan(x)\,dx \\
&= \left(\frac{4}{\pi}\right)\left(\frac{1}{4}(\pi-2)\right) \\
&= \frac{\pi-2}{\pi}.
\end{aligned}$$

The expected value of X is $(\pi - 2)/\pi$.

Problem 11. The expectation is

$$\int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^1 -6x^2(x-1)\,dx = \frac{1}{2}.$$

The variance is

$$\mathrm{Exp}(X^2) - (\mathrm{Exp}(X))^2 = \int_{-0}^{1} -6x^3(x-1)\ \ dx - \frac{1}{4} = \frac{3}{10} - \frac{1}{4} = \frac{1}{20}.$$ ☐

D.V Other Examples of Continuous Random Variables

Problem 12. If $Z \sim N(0,1)$ is a random variable that has the standard normal distribution, what is

a) We have $Z^2 < 1$ if and only if $-1 < Z < 1$. Therefore,

$$P(Z^2 < 1) = P(-1 < Z < 1) = P(Z < 1) - P(Z \leq -1) = P(Z \leq 1) - P(Z \leq -1).$$

Using the table,

$$P(Z^2 < 1) = 0.84134 - 0.15866 = 0.68268.$$

b) Similar calculations:

$$P(Z^2 < 3.84146) = P(Z < 3.84146) - P(Z < -3.84146) = 0.99994 - 0.00006 = 0.99988.$$ ☐

Problem 13.

Define $Z = \frac{X-\mu}{0.3}$, so that $Z \sim N(0,1)$. We want to know for which μ, $P(X > 8) = 0.01$. Using Z, we want to find μ such that

$$P(Z > \frac{8-\mu}{0.3}) = 0.01 \iff P(Z \leq \frac{8-\mu}{0.3}) = 0.99$$

The z-score corresponding to a probability of 0.1 is $z = 2.325$. Therefore

$$\frac{8-\mu}{0.3} = 2.325 \iff \mu = 7.3025. \qquad \square$$

Problem 14. This is a little trick from Calculus IV. Let I denote the integral we want to compute. Consider

$$I^2 = \Big(\int_{-\infty}^{\infty} e^{-x^2} dx\Big)\Big(\int_{-\infty}^{\infty} e^{-y^2}\, dy\Big) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\, dydx.$$

Let $R = \{(x,y) : -\infty < x < \infty, -\infty < y < \infty\}$. Using polar coordinates, we see that $R = \{(r,\theta) : 0 \leq r < \infty, 0 \leq \theta \leq 2\pi\}$. Therefore,

$$I^2 = \iint_R e^{-x^2-y^2}\, dA = \int_0^{2\pi}\int_0^{\infty} re^{-r^2}\, drd\theta$$

Using $u = r^2$, we see that

$$\int_0^{\infty} re^{-r^2}\, dr = \frac{1}{2}\int_0^{\infty} e^{-u}\, du = \frac{1}{2}$$

and hence

$$I^2 = \pi \quad \Longrightarrow I = \sqrt{\pi}. \qquad \triangle$$

Problem 15. Here, we have $g(x) = e^{2x}$. Since it is an increasing function with $g^{-1}(y) = \frac{1}{2}\ln x$, using the formula, the density function of Y is

$$f_Y(y) = \frac{e^{-\frac{1}{4}(\ln(y))^2}}{2y} \quad (\text{for } y > 0).$$

Therefore, the expected value is

$$\mathrm{Exp}(Y) = \int_{-\infty}^{\infty} yf_Y(y)\, dy = \int_0^{\infty} e^{-(\frac{\ln(y)}{2})^2}\, dy.$$

Let $u = \ln(y)/2$, so that $du = \frac{1}{2y}\, dy$. This means

$$\mathrm{Exp}(Y) = \int_{-\infty}^{\infty} 2e^{-u^2}e^{2u}\, du = 2e\int_{-\infty}^{\infty} e^{-(u-1)^2}\, du = 2e\sqrt{\pi}. \qquad \triangle$$

Problem 16. Show that if X has the normal distribution with parameters 0 and 1, then $Y = X^2$ has the χ^2 distribution with one degree of freedom.

Problem 17. If X has the exponential distribution, then

$$F_X(x) = \int_0^x \lambda e^{-\lambda t}\, dt = 1 - e^{-\lambda x} \quad (x > 0).$$

Since $a, b > 0$, notice that

$$\{X > a + b\} \cap \{X > a\} = \{X > a + b\}.$$

Therefore, by definition of the conditional probability,

$$P(X > a + b | X > a) = \frac{P(X > a + b)}{P(X > a)} = \frac{1 - (1 - e^{-\lambda(a+b)})}{1 - (1 - e^{-\lambda a})} = e^{-\lambda b} = P(X > b). \qquad \triangle$$