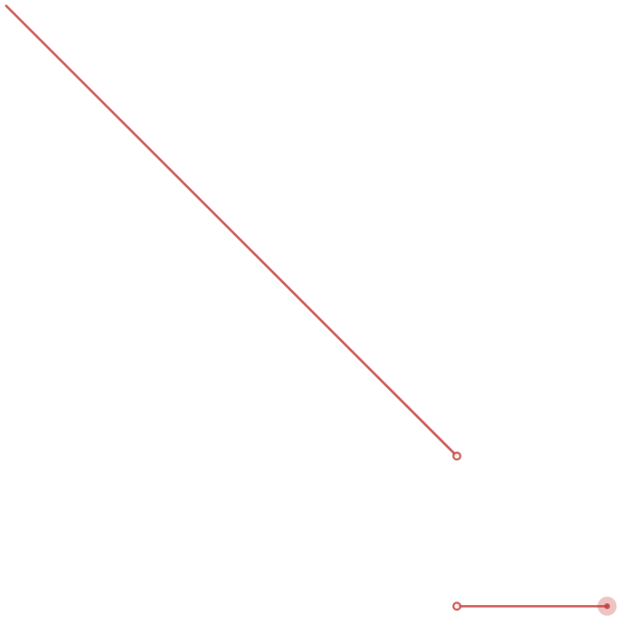
Chapter 1

Functions and Limits

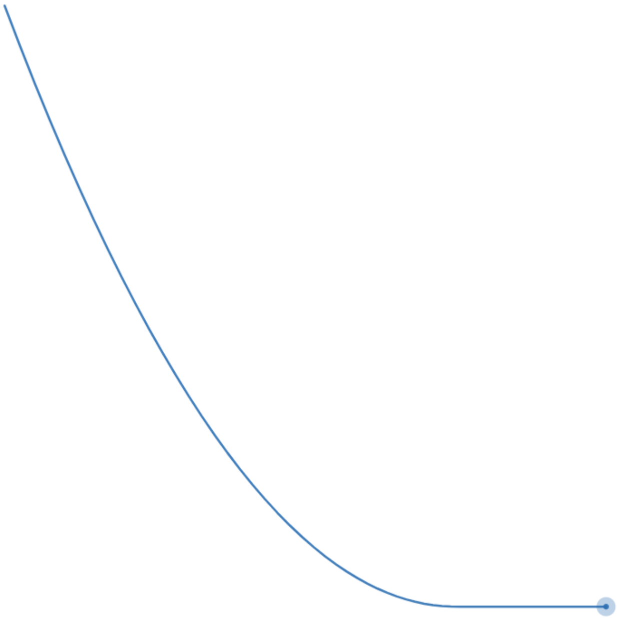
1.8 Continuity

Continuity

Example. What are the main difference(s) between the two following curves?
Illustration: https://www.desmos.com/calculator/mhaqtf2ord



(a) Graph of f(x).



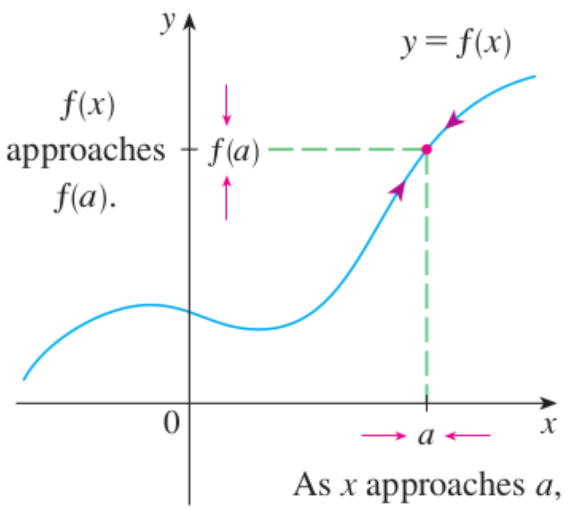
(b) Graph of g(x)

Example. Now, what are the differences between the two following functions?

(a) $f(x) = \begin{cases} 2 - x & \text{if } -2 \leq x < 1 \\ 0 & \text{if } 1 \leq x \leq 2 \end{cases}$ (b) $g(x) = \begin{cases} \frac{4}{9}(1-x)^2 & \text{if } -2 \leq x < 1 \\ 0 & \text{if } 1 \leq x \leq 2 \end{cases}$

1 Definition A function f is **continuous at a number a** if

$$\lim_{x \to a} f(x) = f(a)$$



Three things to verify to show a function is continuous:

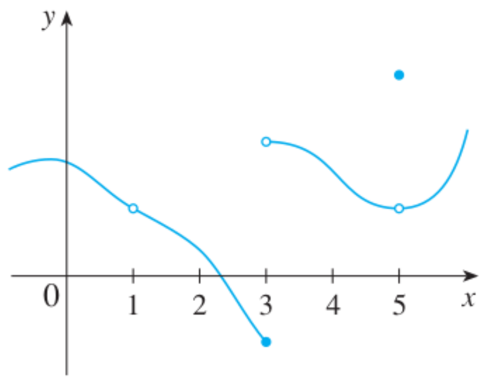
a) The function is defined at x = a.

b) The limit of the function exists at x = a.

c) The limit of the function at x = a equals the value of the function at x = a.

Discontinuity:

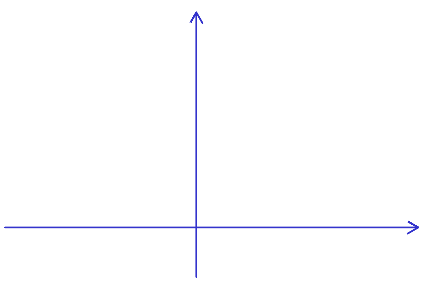
EXAMPLE 1 Figure 2 shows the graph of a function f. At which numbers is f discontinuous? Why?



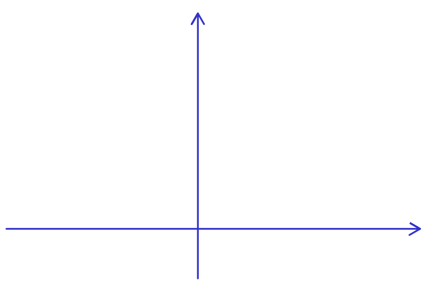
EXAMPLE 2 Where are each of the following functions discontinuous?

(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$ (b) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ (c) $f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$

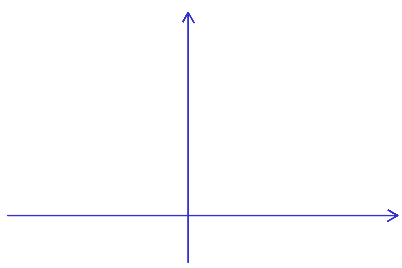
3 kinds of discontinuity.



(a) Removable.



(b) Infinite discontinuity.



(c) Jump discontinuity.

EXAMPLE 4 Show that the function $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on the interval $[-1, 1]$.

Continuity on an interval:

3 Definition A function f is **continuous on an interval** if it is continuous at every number in the interval. (If f is defined only on one side of an endpoint of the interval, we understand *continuous* at the endpoint to mean *continuous from the right* or *continuous from the left.*)

Properties of Continuous Functions.

4 Theorem If f and g are continuous at a and if c is a constant, then the following functions are also continuous at a:

1. $f + g$
2. $f - g$
3. cf
4. fg
5. $\dfrac{f}{g}$ if $g(a) \neq 0$

Consequences:

7 Theorem The following types of functions are continuous at every number in their domains:

- polynomials
- rational functions
- root functions
- trigonometric functions

Substitution Rule Revisited.

EXAMPLE 5 Find $\displaystyle\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}$.

EXAMPLE 7 Evaluate $\displaystyle\lim_{x \to \pi} \frac{\sin x}{2 + \cos x}$.

Composition of Continuous Functions.

8 Theorem If f is continuous at b and $\lim_{x \to a} g(x) = b$, then $\lim_{x \to a} f(g(x)) = f(b)$. In other words,

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$$

9 Theorem If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a.

EXAMPLE 8 Where are the following functions continuous?

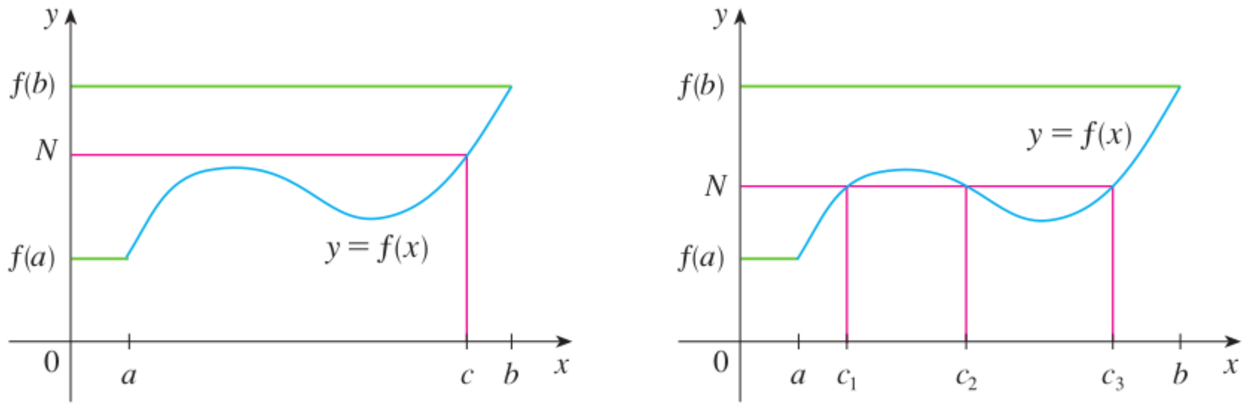
(a) $h(x) = \sin(x^2)$ (b) $F(x) = \dfrac{1}{\sqrt{x^2 + 7} - 4}$

Example. Suppose we have a function

$$f(x) = x^2 - 1.$$

Does the graph of the function f cross the horizontal line y = 3?

10 The Intermediate Value Theorem Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.



(a) (b)

EXAMPLE 9 Show that there is a root of the equation

$$4x^3 - 6x^2 + 3x - 2 = 0$$

between 1 and 2.