SECTION 1.5: Sequences & Series of $\mathbb{C}$-numbers.

A sequence of complex numbers is an ordered list $a_1, a_2, a_3, \dots, a_n, \dots$ where $a_n \in \mathbb{C}$ $(a : \mathbb{N} \to \mathbb{C})$.

Notations: $\{a_n\}_{n=1}^{\infty}$ and $(a_n)_{n=1}^{\infty}$.

Examples

- $a_n = \dfrac{1}{n}$, $n \in \mathbb{N}$. So

$$\{a_n\}_{n=1}^{\infty} = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\}.$$

$$(a_n)_{n=1}^{\infty} = (1, 1/2, 1/3, \dots).$$

- $a_n = i^n$, $n \in \mathbb{N}$. So

$$\{a_n\}_{n=1}^{\infty} = \{i, -1, -i, 1, i, -1, -i, 1, \dots\}$$

Convergence of sequences

Example: For $\{(1+i)/n\}_{n=1}^{\infty}$, as n gets bigger and bigger, $\dfrac{1+i}{n}$ gets closer and closer to 0. How big n should be to get $|a_n| < 0.001$?

$$\Rightarrow \quad \frac{\sqrt{2}}{n} < \frac{1}{1000} \quad \Leftrightarrow \quad 1000\sqrt{2} < n .$$

We would require $n \geq \lfloor 1000\sqrt{2} \rfloor + 1 = 1414 + 1$

$$\Leftrightarrow \quad n \geq 1415 .$$

<u>Def.</u> A sequence $\{a_n\}_{n=1}^{\infty}$ **converges** to a $a \in \mathbb{C}$ if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$\text{if } n \geq N, \text{ then } |a_n - a| < \varepsilon .$$

If $\{a_n\}_{n=1}^{\infty}$ does not converge, we say it **diverges**.

<u>Remarks</u>:

1. Notation: $a_n \longrightarrow a$ or $\lim\limits_{n \to \infty} a_n = a$.

2. Divergent: $a_n \not\to a$.

   Negation: $\exists \varepsilon > 0$, $\forall N \in \mathbb{N}$, $\exists n \geq N$ s.t. $|a_n - a| \geq \varepsilon$.

<u>THM</u> (thm 1.5.8) Let $a_n = x_n + i y_n$.

$$a_n \longrightarrow x + iy \quad \Leftrightarrow \quad x_n \to x \ \& \ y_n \to y .$$

Proof.

($\Longrightarrow$) Assume that $a_n \longrightarrow x+iy$. Let $\varepsilon > 0$.

Notice that

$$|x_n - x| \leq |a_n - (x+iy)| \quad \forall n \in \mathbb{N}.$$

From the def. of $a_n \longrightarrow x+iy$, there's an $N \in \mathbb{N}$ s.t. $|a_n - (x+iy)| < \varepsilon$, $\forall n \geq N$.

So, if $n \geq N$, then

$$|x_n - x| \leq |a_n - (x+iy)| < \varepsilon.$$

So, $x_n \longrightarrow x$ by def. Similarly, you get $y_n \longrightarrow y$.