SAMPLE MIDTERM FALL 2022

1. (15 points) Let $f(x) = x^3 - x$.

(a) List the intervals where the graph of $f(x)$ is increasing and decreasing.

$f'(x) = 3x^2 - 1 \Rightarrow 3x^2 - 1 = 0 \iff x = \sqrt{1/3}$ or $x = -\sqrt{1/3}$

$= 3(x^2 - 1/3)$

$\Rightarrow f'(x) = 3\underbrace{(x + \sqrt{1/3})}_{①}\underbrace{(x - \sqrt{1/3})}_{②}$

factors	$x < -\sqrt{1/3}$	$-\sqrt{1/3} < x < \sqrt{1/3}$	$\sqrt{1/3} < x$
①	$-$	$+$	$+$
②	$-$	$-$	$+$
$f'(x)$	$+$	$-$	$+$

- $f \nearrow$ on $(-\infty, -\sqrt{1/3}) \cup (\sqrt{1/3}, \infty)$
- $f \searrow$ on $(-\sqrt{1/3}, \sqrt{1/3})$.

(b) Find the local maximum and minimum values of $f(x)$.

- $f \nearrow$ on left & $f \searrow$ on the right

$\Rightarrow x = -\sqrt{1/3}$ is a local max.

local max. value is : $f(-\sqrt{1/3}) = -\left(\frac{1}{\sqrt{3}}\right)^3 + \frac{1}{\sqrt{3}}$

- $f \searrow$ on left & $f \nearrow$ right

$\Rightarrow x = \sqrt{1/3}$ is a local min.

$\Rightarrow$ loc. min. value is $f(\sqrt{1/3})$.

(c) List the intervals where the graph is concave up and concave down.

$f''(x) = 6x \Rightarrow 6x = 0 \Rightarrow x = 0$. (I.P.)

- $\underline{x < 0} \Rightarrow 6x < 0 \Rightarrow f''(x) < 0 \Rightarrow f \frown$ on $(-\infty, 0)$
- $\underline{x > 0} \Rightarrow 6x > 0 \Rightarrow f''(x) > 0 \Rightarrow f \smile$ on $(-\infty, 0)$

(d) Sketch the graph of the function.

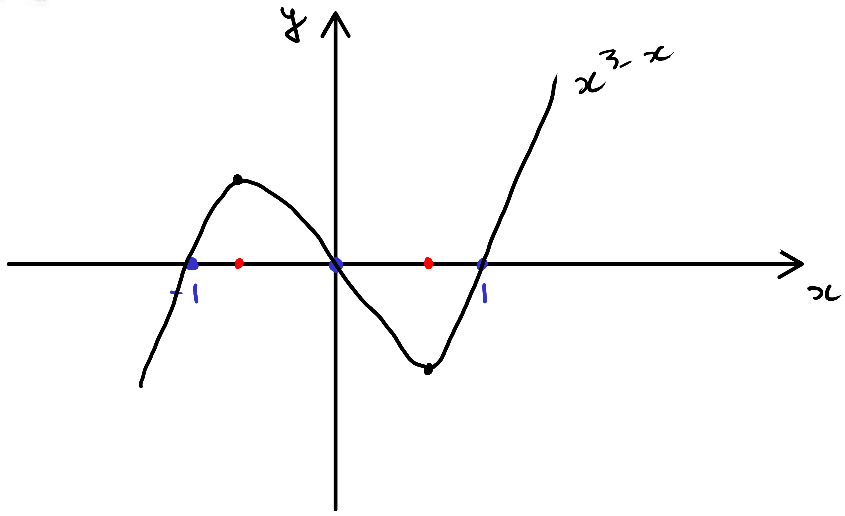
$x^3 - x = 0$

$\iff x(x^2 - 1) = 0$

$\iff x = 0$

$x = 1$

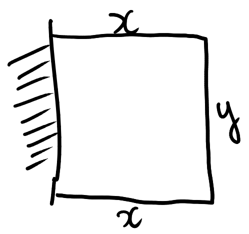
$x = -1$



2. (10 points) Emma wants to enclose a rectangular field with total area $200m^2$. Along one side of the field, she will use a pre-existing straight wall, but on the other three sides, she needs to buy fence.

If it costs \$2 for each meter of fence, what is the least amount she can spend to enclose her field? (Simplify your answer.)

① Sketch.



② Notations.

x: side length. (width). (meters)

y: side length (heigth). (meters).

P: perimeter (fencing) (meters).

C: total cost.

Goal: minimize C.

③ Formula for cost.

$$C(x,y) = 2(2x) + 2y = 4x + 2y.$$

④ Eliminate a variable.

$$xy = 200 \Rightarrow y = \frac{200}{x}$$

so, $C(x) = 4x + \frac{400}{x}$. $(x > 0)$

⑤ Optimize!

$$C'(x) = 4 - \frac{400}{x^2} = 0 \iff x^2 = 100$$

$$\iff x = \pm 10$$

$$\Rightarrow x = 10$$

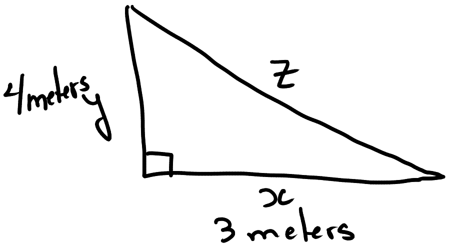
$$C''(x) = 4 + \frac{800}{x^3} > 0 \Rightarrow$$ $x = 10$ is an absolute minimum.

⑥ Answer

$$C(10) = 40 + 40 = \boxed{80\,\$}$$

3. (7 points) A right triangle is changing shape. If the base is 3 meters and expanding at 0.2 meters per minute, and the height is 4 meters and <u>shrinking</u> at 0.1 meters per minute, at what rate is the length of the hypotenuse changing?

① Sketch.



z: hypothenuse. $\to z(t)$
x: base $\to x(t)$
y: heigth. $\to y(t)$

Find $\left.\frac{dz}{dt}\right|_{\substack{x=3\\y=4}}$

② Link.

Pyth. thm: $x^2 + y^2 = z^2$

③ Differentiate

$$\frac{d}{dt}(x^2) + \frac{d}{dt}(y^2) = \frac{d}{dt}(z^2)$$

$$2x \cdot \frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

Know: $x=3$, $y=4$ $\quad \frac{dx}{dt} = 0.2$ & $\frac{dy}{dt} = -0.1$

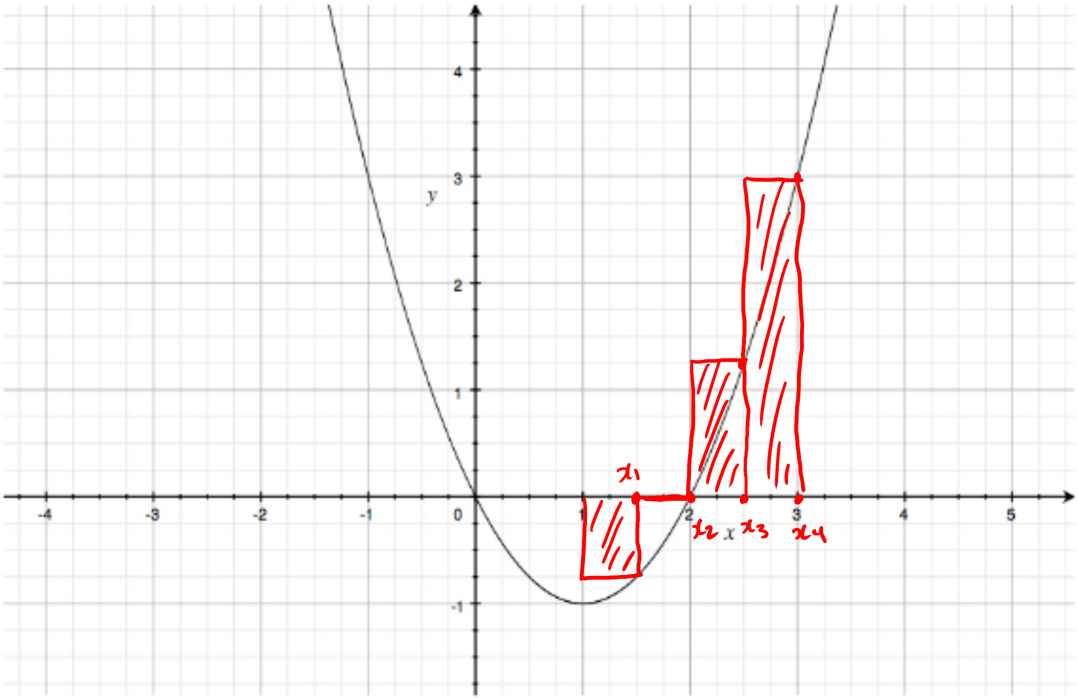
$$z^2 = 3^2 + 4^2 = 9+16 = 25 \Rightarrow z = \sqrt{25} = 5$$

$$\Rightarrow 2\cdot 3 \cdot \frac{2}{10} - 2\cdot 4 \cdot \frac{1}{10} = 2\cdot 5\,\frac{dz}{dt}$$

$$\Rightarrow \frac{12-8}{10} = 10 \cdot \frac{dz}{dt}$$

$$\Rightarrow \frac{2}{50} = \frac{dz}{dt} \Rightarrow \boxed{\frac{dz}{dt} = \frac{1}{25}\ \text{meters/min}}$$

4. Consider the function $f(x) = x^2 - 2x$ as pictured below.



(a) (6 points) Compute a Riemann sum for this function that approximates the integral $\int_1^3 f(x)dx$. Use four equal-width intervals for your Riemann sum, and use the right end-point of each interval to determine the height of the corresponding rectangle. You do not have to simplify your answer.

$a = 1$

$b = 3$

$f(x) = x^2 - 2x$

$\Delta x = \frac{b-a}{n} = \frac{3-1}{4} = \frac{2}{4} = \frac{1}{2}.$

$x_1 = a + \Delta x = 1 + \frac{1}{2} = \frac{3}{2} = 1.5$

$x_2 = a + 2\Delta x = 1 + 1 = 2$

$x_3 = a + 3\Delta x = 1 + \frac{3}{2} = \frac{5}{2} = 2.5$

$x_4 = a + 4\Delta x = 1 + \frac{4}{2} = 1 + 2 = 3$

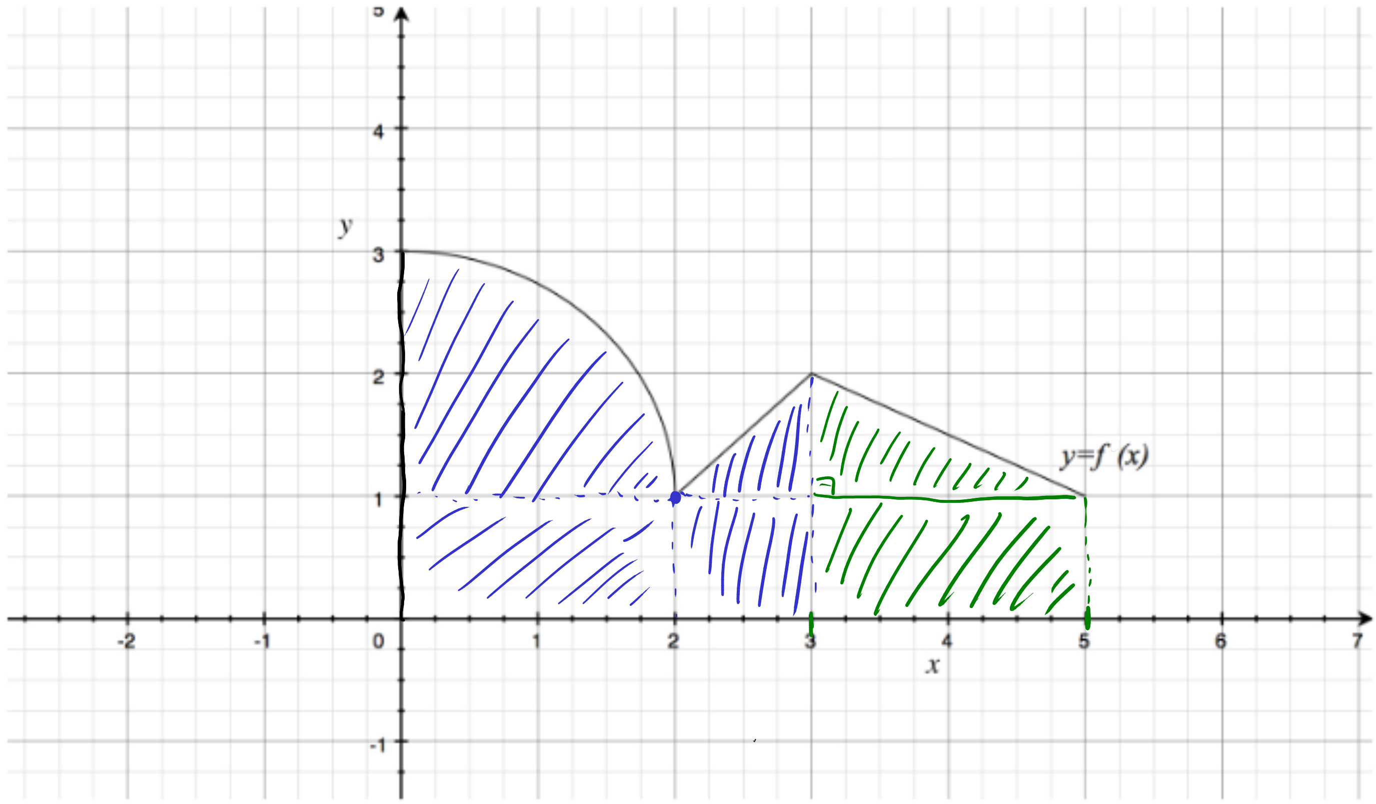
$$\int_1^3 f(x)dx \approx R_4 = \Delta x f(x_1) + \Delta x f(x_2) + \Delta x f(x_3) + \Delta x f(x_4)$$

$$= \frac{1}{2}(1.5^2 - 3) + \frac{1}{2}\cancel{(2^2 - 4)}^{0} + \frac{1}{2}(2.5^2 - 5)$$

$$+ \frac{1}{2}(3^2 - 6)$$

(b) (2 points) Sketch the rectangles that correspond to part (a) on the graph above.

5. A function f of a single variable x is defined on the interval $[0,5]$
The following picture shows the graph of $f(x)$.



In the picture, the portion of the graph on the interval $(0,2)$ identifies with a quarter-circle of radius 2 and center $(0,1)$; the portions of the graph on the intervals $[2,3]$ and $[3,5]$ are line segments.

(a) (2 points) What is the value of $\int_0^0 f(x)\,dx$?

$$\int_0^0 f(x)\,dx = 0$$

(b) (4 points) What is the value of $\int_0^3 f(x)\,dx$?

$$\int_0^3 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^3 f(x)\,dx$$
$$= \text{Area}(\ ) + \text{Area}(\ ) + \text{Area}(\ ) + \text{Area}(\ )$$
$$= \frac{\pi\, 2^2}{4} + 2\cdot 1 + \frac{1\cdot 1}{2} + 1^2$$
$$= \pi + 2 + \frac{1}{2} + 1 = \boxed{\pi + \frac{7}{2}}$$

(c) (4 points) What is the value of $\int_3^5 f(x)\,dx$?

$$\int_3^5 f(x)\,dx = \text{Area}(\ ) + \text{Area}(\ )$$
$$= 2\cdot 1 + \frac{1\cdot 2}{2}$$
$$= \boxed{3}$$

6. (6 points) Use linear approximation to estimate the number $(.95)^{10}$.

$f(x) = x^{10}$. $\quad L(x) = f'(a)(x-a) + f(a)$

1 is close to 0.95 & $1^{10} = 1 \Rightarrow a = 1$

$f'(x) = 10x^9 \Rightarrow f'(1) = 10$

$f(1) = 1$

$\Rightarrow L(x) = 10(x-1) + 1$

Here, $x = 0.95$

$$\Rightarrow (0.95)^{10} \approx L(0.95) = 10(0.95-1)+1$$

$$= 10(-0.05)+1$$

$$= \frac{-10 \cdot 5}{100} + 1$$

$$= -\frac{5}{10} + 1$$

$$= -\frac{1}{2} + 1 = \frac{1}{2}$$

So, $\boxed{(0.95)^{10} \approx 0.5}$

7. Compute the following limits.

(a) $\lim\limits_{x\to\infty} \dfrac{4-7x^2}{(x+5)^2}$.

$$(x+5)^2 = x^2+10x+25$$

$$\Rightarrow \lim_{x\to\infty} \frac{4-7x^2}{x^2+10x+5} = \frac{-7}{1} = \boxed{-7}$$

(b) $\lim\limits_{x\to\infty} \dfrac{7-\sqrt{x}}{7+\sqrt{x}}$.

$$\lim_{x\to\infty} \frac{7-\sqrt{x}}{7+\sqrt{x}} = \lim_{x\to\infty} \frac{\cancel{\sqrt{x}}\left(\frac{7}{\sqrt{x}}-1\right)}{\cancel{\sqrt{x}}\left(\frac{7}{\sqrt{x}}+1\right)}$$

$$= \lim_{x\to\infty} \frac{\frac{7}{\sqrt{x}}-1}{\frac{7}{\sqrt{x}}+1}$$

$$= \frac{\lim\limits_{x\to\infty} \cancel{\frac{7}{\sqrt{x}}}^{\to 0} - 1}{\lim\limits_{x\to\infty} \cancel{\frac{7}{\sqrt{x}}}^{\to 0} + 1} = \frac{-1}{1} = \boxed{-1}$$

8. Answer the following.

(a) Given that $\frac{1}{2} \leq \frac{x}{x+1} \leq \frac{2}{3}$ for any x such that $1 \leq x \leq 2$, give an estimate of the following integral:

$$\int_1^2 \frac{x}{x+1}\,dx.$$

From the comparison properties

$$\frac{1}{2}(2-1) \leq \int_1^2 \frac{x}{x+1}\,dx \leq \frac{2}{3}(2-1)$$

$$\Rightarrow \quad \frac{1}{2} \leq \int_1^2 \frac{x}{x+1}\,dx \leq \frac{2}{3}$$

take the average :

$$\int_1^2 \frac{x}{x+1}\,dx \approx \frac{1/2 + 2/3}{2} = \boxed{\frac{7}{12}}$$

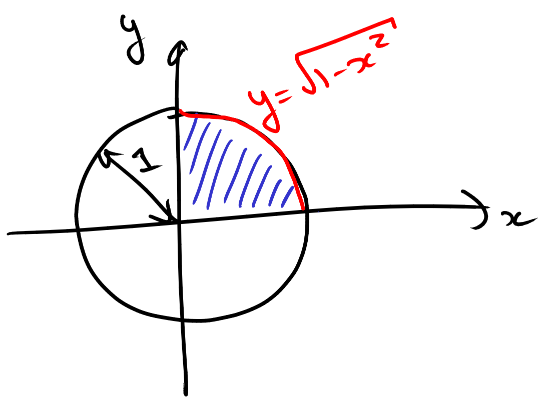
(b) Find the value of the following integral by intepreting it geometrically:

$$\int_0^1 \sqrt{1-x^2}\,dx.$$

$y = \sqrt{1-x^2} \geq 0 \quad \Rightarrow$ integral represents area under some curve.

square

$y^2 = 1 - x^2 \quad \Rightarrow \quad x^2 + y^2 = 1 \quad \rightarrow$ circle of radius 1.



$$\Rightarrow \quad \int_0^1 \sqrt{1-x^2}\,dx = \text{Area}(\;) = \boxed{\frac{\pi}{4}}$$