# Math 302

## Chapter 1

### Section 1.2: Basic Concepts

Contents




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Fall 2022

## What's a DE?

- A **differential equation** (abbreviated by DE) is an equation that contains one or more derivatives of an unknown function.
  - Examples: $T' = -k(T - T_m)$, $y' = x^2$, $x^2 y'' + xy' + 2 = 0$.
- The **order** of a DE is the order of the highest derivatives that it contains.
  - Example: $y' = x^2$ is of order ______.
  - Example: $x^2 y'' + xy' + 2 = 0$ is of order ______.
- An **Ordinary Differential Equation** (abbreviated ODE) is a DE involving an unknown function of only one variable.
- An **Partial Differential Equation** (abbreviated PDE) is a DE involving an unknown function of more than one variable.

The simplest ODE is of the form

$$y' = f(x) \quad \text{or} \quad y^{(n)} = f(x)$$

where $f$ is a known function of $x$.

Example 1. Find functions $y = y(x)$ satisfying

1. $y' = x^2$.
2. $y'' = \cos(x)$.

Our goal is to study general ODEs of the form

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}).$$

# What Is a Solution to an ODE?

A **solution** to the ODE

$$y^{(n)}(x) = f(x, y(x), y'(x), \ldots, y^{(n-1)}(x))$$

is a function $y = y(x)$ that verifies the ODE for any $x$ in some open interval $(a, b)$.

Remark:

- Functions that satisfy an ODE at isolated points are not considered solutions.

**Example 2.** Verify that

$$y = \frac{x^2}{3} + \frac{1}{x}$$

is a solution of

$$xy' + y = x^2$$

on $(-\infty, 0)$ and $(0, \infty)$.

## Solution and Integral Curves

- The graph of a solution of an ODE is a **solution curve**.
- More generally, a curve $C$ in the plane is said to be an **integral curve** of an ODE if every function $y = y(x)$ whose graph is a segment of $C$ is a solution of the ODE.

EXAMPLE 3. Plot the solutions obtained in Example 2. Are they solution curves of the ODE?

EXAMPLE 4. If $a$ is any positive constant, check that the circle

$$x^2 + y^2 = a^2$$

is an integral curve of $y' = -x/y$.

**Example 5.** Find a solution of

$$y' = x^3$$

satisfying the additional condition $y(1) = 2$.

**Example 6.** All the solutions to

$$y'' - 2y' + 3y = 0$$

are the functions

$$y(x) = c_1 e^x + c_2 e^{-3x}$$

where $c_1$, $c_2$ are arbitrary constants. Find the solution that satisfies $y'(0) = 1$ and $y'(0) = 0$.

An **Initial Value Problem** (abbreviated by IVP) is an ODE with additional **Initial conditions**. The general form of an IVP is

$$y^{(n)}(x) = f(x, y(x), y'(x), \ldots, y^{(n-1)}(x)), \quad y(x_0) = k_0,\ y'(x_0) = k_1, \ldots,\ y^{(n-1)}(x_0) = k_{n-1}.$$

- The largest open interval that contains $x_0$ on which $y(x)$ is defined and satisfies the ODE is called the **interval of validity** of $y$.

EXAMPLE 7. Find the interval of validity of the solution to

$$y' = x^3,\ y(1) = 1.$$

EXAMPLE 8. Find the interval of validity of the solution to the following IVPs:

1. $xy' + y = x^2$, $y(1) = 4/3$.
2. $xy' + y = x^2$, $y(-1) = -2/3$.