Math 644

Chapter 3

Section 3.1: The Maximum Principle

Contents

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THEOREM 1. Suppose f is analytic in a region Ω. If there exists a $z_0 \in \Omega$ such that

$$|f(z_0)| = \sup_{z\in\Omega} |f(z)|,$$

then f is constant in Ω.

LEMMA 2. If $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ which converges in $\{z : |z-z_0| < r_0\}$ for some $r_0 > 0$, then for $r < r_0$

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})\, dt. \longrightarrow \text{Mean Value Theorem. (MVP)}$$

Proof. Power series converges uniformly on $\{z : |z-z_0| = r\}$. So,

$$\frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{it})\,dt = \sum_{n=0}^{\infty} \frac{1}{2\pi}\int_0^{2\pi} a_n r^n e^{int}\,dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} a_0\,dt$$

$$= a_0 = f(z_0). \qquad \square$$

Proof of the Maximum Modulus Principle.

Suppose that $\exists z_0 \in \Omega$ s.t. $\sup_{\Omega} |f(z)| = |f(z_0)|$.

1) $|f(z_0)| = 0$, then $f \equiv 0$.

2) $|f(z_0)| \neq 0$ & set $\lambda := \dfrac{|f(z_0)|}{f(z_0)}$.

So, $\lambda f(z_0) = |f(z_0)|$

and from the MVP

$$\Rightarrow \frac{1}{2\pi}\int_0^{2\pi} \lambda f(z_0 + re^{it})\, dt = |f(z_0)|.$$

for any $0<r<r_1$ with $\{z : |z-z_0|<r_1\} \subseteq \Omega$.

$$\Rightarrow \frac{1}{2\pi}\int_0^{2\pi} |f(z_0)| - \mathrm{Re}\left(\lambda f(z_0+re^{it})\right) dt = 0$$

$$\Rightarrow |f(z_0)| = \mathrm{Re}\left(\lambda f(z_0 + re^{it})\right) \text{ in}$$

$$\{z : |z-z_0|<r_1\}.$$

Suppose $b = \mathrm{Im}\left[\lambda f(z_0 + re^{it})\right] \neq 0$ for some r & t. then

$$|\lambda f(z_0)| = \sqrt{|f(z_0)|^2 + |b|^2} > |f(z_0)| \quad \#.$$

So, $|f(z_0)| = \lambda f(z_0 + re^{it}) \quad \forall r<r_1$ &

$\forall t \in [0, 2\pi]$

$$\Rightarrow f \equiv f(z_0) \text{ in } \{z : |z-z_0|<r_1\}.$$

Identity Principle $\Rightarrow$ $f \equiv f(z_0)$ in Ω. $\square$

Note:

- If f is analytic in Ω and there is a $z_0 \in \Omega$ such that $|f(z_0)| = \inf_{z\in\Omega} |f(z)|$ and $|f(z_0)| \neq 0$, then f is constant in Ω.

Corollary 3. If f is a non-constant analytic function in a bounded region Ω, and if f is continuous on $\overline{\Omega} = \operatorname{clos}(\Omega)$, then

$$\max_{z \in \overline{\Omega}} |f(z)|$$

occurs on $\partial\Omega$, but not in Ω.

Proof. Since Ω is bounded, $\overline{\Omega}$ is bounded.

this means $\overline{\Omega}$ is compact, so $\exists z_0 \in \overline{\Omega}$ s.t.

$$|f(z_0)| = \max_{z \in \overline{\Omega}} |f(z)|.$$

If $z_0 \in \Omega$, then from the MVP (Version 1)

$f \equiv$ constant in Ω. #

So, $z_0 \in \partial\Omega$. $\square$

Note:

- The requirement that Ω is bounded is necessary: the function $f(z) = e^{-iz}$ is
  - analytic in the upper half-plane $\mathbb{H} := \{z : \operatorname{Im} z > 0\}$;
  - continuous on $\{z : \operatorname{Im} z \geq 0\}$ and;
  - has absolute value 1 on the real line $\mathbb{R}$.

  However, f is not bounded by 1 in $\mathbb{H}$.

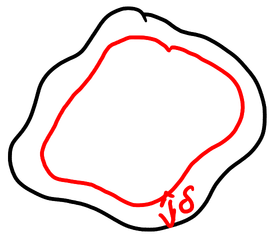
Let Ω be a region in $\mathbb{C}$.

- A sequence $(z_n)_{n\geq 1}$ tends to $\partial\Omega$ if for any compact subset $K \subset \Omega$, there exists an $N \in \mathbb{N}$ such that $z_n \notin K$, when $n \geq N$.
- The region Ω can be unbounded. In this case, we consider the region as lying in $\mathbb{C}^*$ and ∞ might be on $\partial\Omega$.
- If $f : \Omega \to \mathbb{C}$ is a continuous function, then

$$\limsup_{z\to\partial\Omega} |f(z)| := \sup\left\{ \limsup_{n\to\infty} |f(z_n)| \; : \; z_n \to \partial\Omega \right\}.$$

We can show that, if Ω is bounded, then

$$\limsup_{z\to\partial\Omega} |f(z)| = \lim_{\delta\to 0} \sup\{|f(z)| \; : \; z \in \Omega, \mathrm{dist}(z, \partial\Omega) = \delta\}$$



EXAMPLE 4.

a) Show that $z_n \to \partial\mathbb{D}$ if and only if $|z_n| \to 1$, as $n \to \infty$.

b) Let $\Omega := \{z \; : \; |z| > 2\}$. Compute $\displaystyle\limsup_{z\to\partial\Omega} \left|\frac{1+z}{1-z}\right|$.

(a) Suppose $z_n \to \partial\mathbb{D}$. For any compact set $K \subseteq \mathbb{D}$, $\exists N \in \mathbb{N}$ s.t. $z_n \notin K$ $\forall n \geq N$.

Set $K = \{z : |z| \leq r\}$, with $0 < r < 1$.

$\exists N$ s.t. $|z_n| > r$ $\forall n \geq N$. In other words, $\displaystyle\lim_{n\to\infty} |z_n| = 1$.

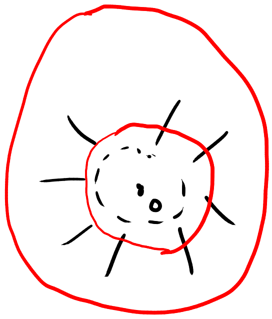
Suppose $|z_n| \to 1$ as $n \to \infty$. Let $K \subseteq \mathbb{D}$ be compact. There is a $R > 0$ s.t. $K \subseteq B_R \subsetneq \mathbb{D}$

Choose $\varepsilon = 1 - R$, then $\exists N \in \mathbb{N}$ s.t.

$1 - |z_n| < 1 - R$ $\forall n \geq N \iff |z_n| > R$ $\forall n \geq N$. $\square$

(b) Ω is unbounded, so $\Omega \subseteq \mathbb{C}^*$.

then, $\partial\Omega = \{z : |z| = 2\} \cup \{\infty\}$.



In this case, $z_n \to \partial\Omega$ if (*)

$$|z_n| \to 2 \quad \text{or} \quad |z_n| \to \infty$$ (two important cases to consider)

① $$\limsup_{|z|\to 2} |H(z)| = \max_{\{z : |z|=2\}} \left|\frac{1+z}{1-z}\right| .$$

Write $z = 2e^{i\theta}$, then

$$\left|\frac{1+z}{1-z}\right| = \sqrt{\frac{5+4\cos\theta}{5-4\cos\theta}}$$ (max occurs at $\theta = 0$)

$$\Rightarrow \limsup_{|z|\to 2} \left|\frac{1+z}{1-z}\right| = A = 3$$

② $$\limsup_{|z|\to\infty} \left|\frac{1+z}{1-z}\right| = 1 .$$

Now, $$\limsup_{z\to\partial\Omega} \left|\frac{1+z}{1-z}\right| = \max\{A, 1\} = A = 3 .$$

(*) $z_n \to \partial\Omega \iff \forall n \geq 0, \exists N$ s.t. $z_k \notin \{z \in \mathbb{C} : 2 + \frac{1}{n} \leq |z| \leq n\}$ $\forall k \geq N$.

Theorem 5. If f is analytic on a bounded region Ω, then

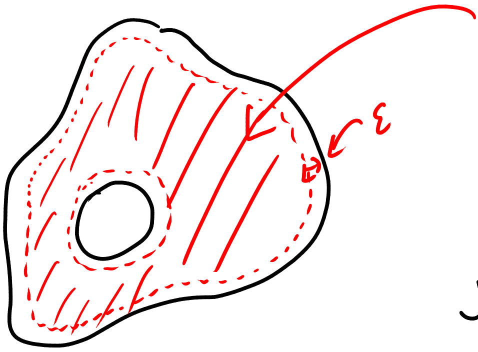
$$\limsup_{z \to \partial\Omega} |f(z)| = \sup_{\Omega} |f(z)|.$$

Proof. Suppose $f \not\equiv$ constant.

If $\sup_{\Omega} |f(z)| = \infty$, then there is a sequence $(z_n) \subseteq \Omega$ s.t. $|f(z_n)| \to \infty$ $(n \to \infty)$. Here, $z_n \to \partial\Omega$, and $\limsup_{z \to \partial\Omega} |f(z)| = \infty$.

Suppose $\sup_{\Omega} |f(z)| < \infty$. Define

$\Omega_\varepsilon := \{ z \in \Omega : \operatorname{dist}(z, \partial\Omega) \geq \varepsilon \}$



Since Ω is bounded, then Ω_ε is bounded and so compact.

Since f is analytic & continuous on Ω_ε, by the maximum modulus principle (version 2),

$$\sup_{z \in \Omega_\varepsilon} |f(z)| = \sup_{z \in \partial\Omega_\varepsilon} |f(z)|.$$

Let $\varepsilon \to 0$, so

$$\sup_{\Omega} |f(z)| = \lim_{\varepsilon \to 0} \sup_{z \in \partial \Omega_\varepsilon} |f(z)|$$

$$= \limsup_{z \to \partial \Omega} |f(z)| . \qquad \square$$

Note:

- If f is continuous on $\overline{\Omega}$, then we recover the second version of the Maximum Principle.