Chapter 2

Functions and Limits

2.1 Derivatives and Rates of Change

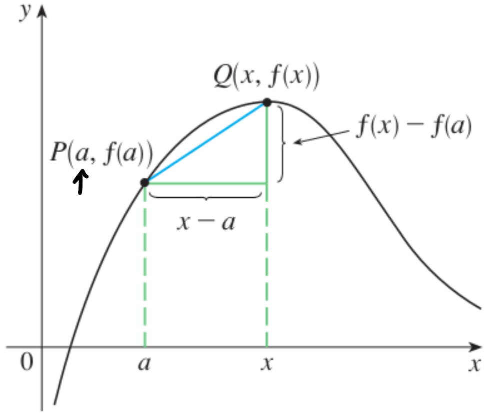
Tangents.

> How do we find the tangent at a point P on a curve given by the graph of a function?

Answer:

1) Find the slope of the secant line passing to two points P and Q on the curve:

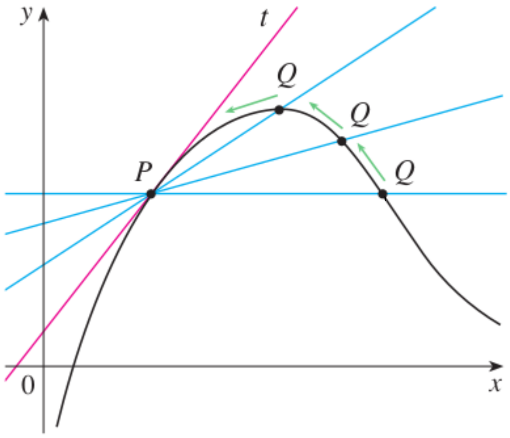
$$m_{PQ} = \frac{f(x)-f(a)}{x-a}$$



2) Taking the limit as Q approached P.

$$m = \lim_{x\to a} \overbrace{\frac{f(x)-f(a)}{x-a}}^{m_{PQ}}$$

make sure that the limit exists



1 Definition The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

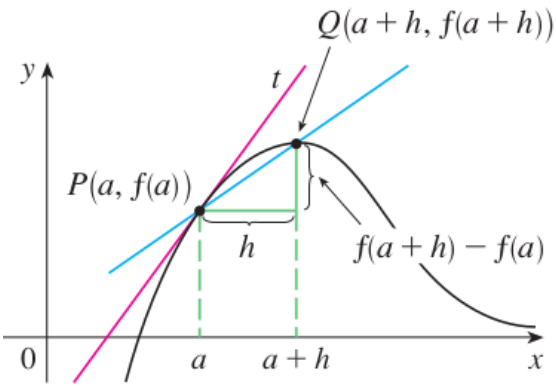
$$m = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

provided that this limit exists.

Another expression for calculating the slope of the tangent line.

$x - a = h$

$x = a + h$



$$m = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

EXAMPLE 2 Find an equation of the tangent line to the hyperbola $y = 3/x$ at the point (3, 1).

LDP $a = 3$
$f(a) = 1$

Eq. line $\quad y - \underset{f(a)}{y_0} = m(x - \underset{a}{x_0})$

① Compute m.

$f(x) = 3/x$

$$m = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h\to 0} \frac{\frac{3}{3+h} - 1}{h}$$

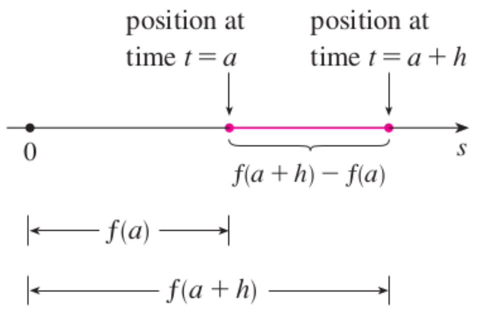
$$= \lim_{h\to 0} \frac{\frac{3 - (3+h)}{3+h}}{h}$$

$$= \lim_{h\to 0} \frac{-\cancel{h}}{(3+h)\cancel{h}}$$

$$= \lim_{h\to 0} \frac{-1}{3+h}$$

$$= \frac{-1}{3}$$

$$y - 1 = \frac{-1}{3}(x - 3) \Rightarrow \boxed{y = -\frac{x}{3} + 2}$$



Position function: $s = f(t)$ f: position function.

Average Velocity:

$$\frac{\text{distance traveled}}{\Delta \text{ time}} = \frac{f(a+h) - f(a)}{h}$$

Instantaneous Velocity.

average velocity.

$$\boxed{v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$

EXAMPLE 3 Suppose that a ball is dropped from the upper observation deck of the CN Tower, 450 m above the ground.
(a) What is the velocity of the ball after 5 seconds?
(b) How fast is the ball traveling when it hits the ground?

Recall Galileo:

$f(t) = s(t) = 4.9t^2$

$2 \cdot 4.9 \cdot t^{2-1}$

$= 9.8t$

(a) $a = 5$

$f(t) = 4.9t^2$

$$v(5) = \lim_{h \to 0} \frac{f(5+h) - f(5)}{h} \quad f(5) \to 4.9 \cdot (5)^2$$

$$= \lim_{h \to 0} \frac{4.9(5+h)^2 - 4.9 \cdot 25}{h}$$

$$= \lim_{h \to 0} \left[\frac{4.9}{h}\left((5+h)^2 - 25\right)\right]$$

$$= \lim_{h \to 0} \frac{4.9}{h}\left(\cancel{25} + 10h + h^2 - \cancel{25}\right)$$

$$= \lim_{h \to 0} \frac{4.9(10+h)\cancel{h}}{\cancel{h}}$$

$$= \lim_{h \to 0} 4.9(10+h)$$

$$= 4.9 \cdot \underset{2 \cdot 5}{10} = \boxed{49 \text{ m/s}}$$

(b) Which t s.t. $s(t) = 450$

$\Leftrightarrow 4.9t^2 = 450$

$\Leftrightarrow t^2 = \frac{450}{4.9}$

$\Leftrightarrow t = \sqrt{450/4.9} \approx 9.6$ s.

$v(9.6) = 9.8 \cdot 9.6 \approx \boxed{94 \text{ m/s}}$

4 Definition The **derivative of a function f at a number a**, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

Another notation:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

EXAMPLE 4 Find the derivative of the function $f(x) = x^2 - 8x + 9$ at the number a.

$f'(a)$???

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{(a+h)^2 - 8(a+h) + 9 - (a^2 - 8a + 9)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{a^2} + 2ah + h^2 \cancel{- 8a} - 8h \cancel{+ 9} - \cancel{a^2} \cancel{+ 8a} \cancel{- 9}}{h}$$

$$= \lim_{h \to 0} \frac{2ah + h^2 - 8h}{h}$$

$$= \lim_{h \to 0} (2a + h - 8)$$

$$= 2a + 0 - 8$$

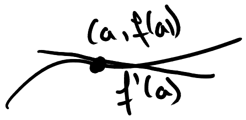
$x^2 - 8x' + \cancel{9}$

$2x$

$$= 2a - 8 \quad \to \quad \boxed{f'(a) = 2a - 8}$$

Tangent line to a curve.

The tangent line to $y = f(x)$ at $(a, f(a))$ is the line through $(a, f(a))$ whose slope is equal to $f'(a)$, the derivative of f at a.



eq. of the tangent line

$$\boxed{y - f(a) = f'(a)(x - a)}$$

$m = f'(a)$

EXAMPLE 5 Find an equation of the tangent line to the parabola $\underline{y = x^2 - 8x + 9}$ at the point $(3, -6)$.

$f'(a) = 2a - 8$

$a = 3$
$f(a) = -6$

$$y - (-6) = f'(3)(x - 3)$$

$$\Rightarrow \quad y + 6 = (2 \cdot 3 - 8)(x - 3)$$

$$\Rightarrow \quad y + 6 = -2(x - 3)$$

$$\Rightarrow \quad y = -2x + 6 - 6 \Rightarrow \boxed{y = -2x}$$

Rates of Change.

Increment in x.

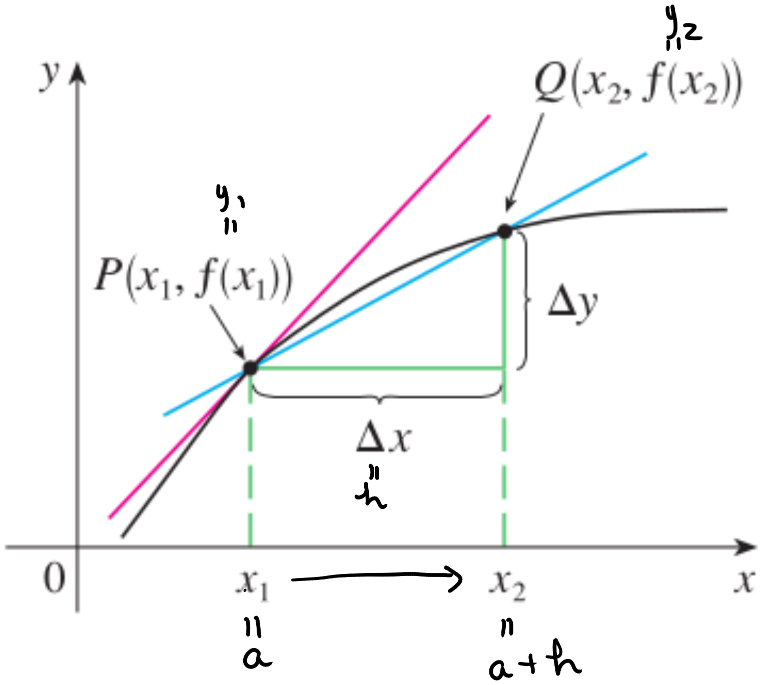
$$\Delta x = x_2 - x_1$$

Increment in y.

$$\Delta y = y_2 - y_1$$
$$= f(x_2) - f(x_1)$$

Average Change.

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$



$$\boxed{\textbf{6} \qquad \text{instantaneous rate of change} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(a)}$$

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14. If a rock is thrown upward on the planet Mars with a velocity of 10 m/s, its height (in meters) after t seconds is given by $H = 10t - 1.86t^2$.
(a) Find the velocity of the rock after one second.
(b) Find the velocity of the rock when $t = a$.
(c) When will the rock hit the surface?
(d) With what velocity will the rock hit the surface?

(a) $H(t) = 10t - 1.86t^2$, $v(1) = \lim\limits_{h \to 0} \dfrac{H(1+h) - H(1)}{h}$

$v(t) = H'(t) = 1 \cdot 10(t^{1-1}) - 2 \cdot (1.86) \cdot t^{2-1}$

$= 10 - 3.72t \longrightarrow v(1) = 10 - 3.72$

$= 6.28$ m/s.

(b) $v(a) = 10 - 3.72a$.

(c) $H(t) = 0$ iff $10t - 1.86t^2 = 0$

iff $(10 - 1.86t)\,t = 0$

beginning

iff $10 - 1.86t = 0$ or $\cancel{t = 0}$

iff $t = 10/1.86$.

(d) $v(10/1.86) = \dfrac{10}{1.86} - \dfrac{2 \cdot 1.86 \cdot 10}{1.86} = \dfrac{10}{1.86} - 20 \approx -14.623$ m/s