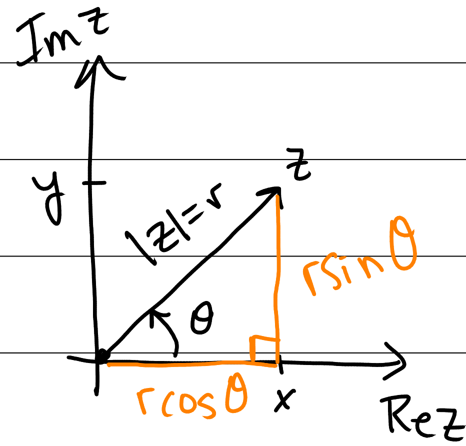
SECTION 1.3: POLAR FORM



DEF. Let $z = x+iy$, $z \neq 0$.

Let

$$r := \sqrt{x^2+y^2} = |z|$$

and θ be an angle such that

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

$$\left(\frac{x}{r} = \cos\theta\right) \qquad \left(\frac{y}{r} = \sin\theta\right).$$

The polar form (or representation) of z is

$$z = r(\cos\theta + i\sin\theta)$$
$$= r\cos\theta + i\,r\sin\theta.$$

Here r: modulus.

θ: argument.

Remark • The polar representation of 0 is undefined because $\arg z$ is undefined.

• We can find the argument θ by using the $\tan^{-1}$:

$$\theta = \tan^{-1}(y/x), \quad x \neq 0.$$

• $|z| = 1 \Rightarrow z = \cos\theta + i\sin\theta$.

These numbers are called unimodular.

DEF 1.3.2 The principal value of the argument of $z = x+iy$ $(z \neq 0)$ is the unique number $Arg(z)$ such that

- $-\pi < Arg(z) \leq \pi$
- $\cos(Arg\, z) = x/r$
- $\sin(Arg\, z) = y/r$

$\Big\}$ $Arg(z)$ is an argument for z.

The set of all arguments is

$$arg(z) = \{ Arg(z) + 2k\pi : k \in \mathbb{Z} \}.$$

Remark Here $arg(z)$ is multi-valued.

Example 1.3.3 Find the modulus, the argument and polar form of

(d) $z_4 = 1+i$ (e) $z_5 = 1-i$ (f) $z_6 = -1-i$

Sol. (d) $|z_4| = \sqrt{2}$.

$$arg(z) = \{ Arg(z) + 2k\pi : k \in \mathbb{Z} \}.$$

Here,

$$\theta = \tan^{-1}(1/1) = \pi/4 = Arg(z)$$

$$\Rightarrow z_4 = \sqrt{2}\left(\cos(\pi/4) + i \sin(\pi/4) \right).$$

(e) $|z_5| = \sqrt{2}$

Here $\theta = \tan^{-1}(-1/1) = -\pi/4 = Arg(z)$

$\Rightarrow \quad z_5 = \sqrt{2}\left(\cos\left(-\pi/4\right) + i\sin\left(-\pi/4\right)\right)$

(f) $|z_6| = \sqrt{2}$

Here

$$\theta = \tan^{-1}(-1/-1) + \pi = \frac{\pi}{4} + \pi = \frac{5\pi}{4}$$

Here,

$$Arg(z) = \frac{5\pi}{4} - 2\pi = -\frac{3\pi}{4}$$

So,

$$z_6 = \sqrt{2}\left(\cos\left(5\pi/4\right) + i\sin\left(5\pi/4\right)\right).$$