# SECTION 1.5: Sequences & Series of $\mathbb{C}$-numbers.

A sequence of complex numbers is an ordered list $a_1, a_2, a_3, \dots, a_n, \dots$ where $a_n \in \mathbb{C}$ $(a : \mathbb{N} \to \mathbb{C})$.

Notations: $\{a_n\}_{n=1}^{\infty}$ and $(a_n)_{n=1}^{\infty}$.

Examples

- $a_n = \frac{1}{n}$, $n \in \mathbb{N}$. So

$$\{a_n\}_{n=1}^{\infty} = \{1, \tfrac{1}{2}, \tfrac{1}{3}, \dots\}.$$

$$(a_n)_{n=1}^{\infty} = (1, 1/2, 1/3, \dots).$$

- $a_n = i^n$, $n \in \mathbb{N}$. So

$$\{a_n\}_{n=1}^{\infty} = \{i, -1, -i, 1, i, -1, -i, 1, \dots\}$$

## Convergence of sequences

Example: For $\{(1+i)/n\}_{n=1}^{\infty}$, as $n$ gets bigger and bigger, $\frac{1+i}{n}$ gets closer and closer to $0$. How big $n$ should be to get $|a_n| < 0.001$?

$$\Rightarrow \quad \frac{\sqrt{2}}{n} < \frac{1}{1000} \quad \Leftrightarrow \quad 1000\sqrt{2} < n .$$

We would require $n \geq \lfloor 1000\sqrt{2} \rfloor + 1 = 1414 + 1$

$$\Leftrightarrow \quad n \geq 1415 .$$

<u>Def.</u> A sequence $\{a_n\}_{n=1}^{\infty}$ **converges** to a $a \in \mathbb{C}$ if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$\text{if } n \geq N, \text{ then } |a_n - a| < \varepsilon .$$

If $\{a_n\}_{n=1}^{\infty}$ does not converge, we say it **diverges**.

<u>Remarks</u>:

① Notation: $a_n \to a$ or $\lim_{n\to\infty} a_n = a$.

② Divergent: $a_n \not\to a$.

Negation: $\exists \varepsilon > 0$, $\forall N \in \mathbb{N}$, $\exists n \geq N$ s.t. $|a_n - a| \geq \varepsilon$.

<u>THM</u> (thm 1.5.8) Let $a_n = x_n + iy_n$.

$$a_n \to x + iy \quad \Leftrightarrow \quad x_n \to x \ \& \ y_n \to y .$$

Proof.

($\Rightarrow$) Assume that $a_n \longrightarrow x+iy$. Let $\varepsilon > 0$.
Notice that
$$|x_n - x| \leq |a_n - (x+iy)| \quad \forall n \in \mathbb{N}.$$

From the def. of $a_n \longrightarrow x+iy$, there's an $N \in \mathbb{N}$ s.t. $|a_n - (x+iy)| < \varepsilon$, $\forall n \geq N$.
So, if $n \geq N$, then
$$|x_n - x| \leq |a_n - (x+iy)| < \varepsilon.$$
So, $x_n \longrightarrow x$ by def. Similarly, you get $y_n \longrightarrow y$.

($\Leftarrow$) Assume $x_n \longrightarrow x$ and $y_n \longrightarrow y$. Recall:
$$|z| \leq |\mathrm{Re}\, z| + |\mathrm{Im}\, z|, \quad \forall z \in \mathbb{C}.$$
Let $\varepsilon > 0$. then
$$|a_n - (x+iy)| \leq |x_n - x| + |y_n - y|$$
Let $N_1 \in \mathbb{N}$ s.t. if $n \geq N_1$, then
$$|x_n - x| < \varepsilon/2$$
Let $N_2 \in \mathbb{N}$ s.t. if $n \geq N_2$, then
$$|y_n - y| < \varepsilon/2.$$

Let $N = \max\{N_1, N_2\}$. If $n \geq N$, then

$$|a_n - (x+iy)| \leq |x_n - x| + |y_n - y| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

So, $a_n \longrightarrow x+iy$. $\square$

## Properties:

① Prop. 1.5.2: $a_n \to a \Rightarrow a$ is unique.

② Prop. 1.5.4: $a_n \to a \Rightarrow (a_n)$ is bounded.

(bounded: $\exists M > 0$ s.t. $|a_n| \leq M, \forall n$).

③ Prop. 1.5.6: Let $(a_n), (b_n)$ be two seq.

(i) $a_n \to 0$ and $|b_n| \leq |a_n| \Rightarrow b_n \to 0$.

(ii) $a_n \to 0$ and $(b_n)$ bounded $\Rightarrow a_n b_n \to 0$.

④ Prop. 1.5.7: Assume $a_n \to a$ and $b_n \to b$.

(i) $\alpha a_n + \beta b_n \longrightarrow \alpha a + \beta b$ $\quad \alpha, \beta \in \mathbb{C}$.

(ii) $a_n b_n \longrightarrow ab$.

(iii) If $b \neq 0$, $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{a}{b}$.

(iv) $\bar{a}_n \longrightarrow \bar{a}$ $\left( \lim\limits_{n\to\infty} \bar{a}_n = \overline{\lim\limits_{n\to\infty} a_n} \right)$.

(v) $|a_n| \longrightarrow |a|$ $\left( \lim\limits_{n\to\infty} |a_n| = \left| \lim\limits_{n\to\infty} a_n \right| \right)$.

**Example** Compute $\lim_{n\to\infty}\left(\frac{(3+2i)^2}{n+1}+\frac{n+n^2i}{n^3i}\right)$.

From the properties:

$$= (3+2i)^2 \lim_{n\to\infty}\frac{1}{n+1} + \lim_{n\to\infty}\frac{n+n^2i}{n^3i}$$

$$= 0 + \lim_{n\to\infty}\frac{n+n^2i}{n^3i}$$

$$= \lim_{n\to\infty}\left(\frac{1}{n^2i}+\frac{1}{n}\right)$$

$$= \frac{1}{i}\lim_{n\to\infty}\frac{1}{n^2} + \lim_{n\to\infty}\frac{1}{n} = 0.$$

**Example 1.5.9**

(a) If $|z|<1$, then compute $\lim_{n\to\infty} z^n$.

(b) If $z\neq 1$ and $|z|\geq 1$, then show that $\lim_{n\to\infty} z^n$ does not exist.

**SoL.**

(a) If we want to show that $z^n\to 0$, then we have to consider:

$$|z^n-0| = |z|^n .$$

From Calculus, $\lim_{n\to\infty} r^n = 0$, $0 \le r < 1$.

Put $r = |z| \Rightarrow \lim_{n\to\infty} |z|^n = 0$.

(b) Assume $z \neq 1$ and $|z| \ge 1$.

For a proof by contradiction, assume $\lim_{n\to\infty} z^n = L$, for some $L \in \mathbb{C}$.

We have

$$L = \lim_{n\to\infty} z^n = z \lim_{n\to\infty} z^{n-1} = zL.$$

$$\Rightarrow \quad L = zL$$

Since $|z| \ge 1 \Rightarrow |z|^n \ge 1 \xRightarrow{n\to\infty} |L| \ge 1$

So, $L \neq 0 \Rightarrow \frac{L}{L} = \frac{zL}{L}$

$$\Rightarrow \quad 1 = z \quad \#.$$

So, $\lim_{n\to\infty} z^n \nexists$.

1.5.10
<u>DEF.</u> A sequence $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that if $n, m \ge N$, then $|a_n - a_m| < \varepsilon$.

**THM 1.5.11** Let $\{a_n\}_{n=1}^{\infty}$ be a sequence, $a_n \in \mathbb{C}$.

(i) If $\{a_n\}$ is convergent, then it is Cauchy.

(ii) If $\{a_n\}$ is Cauchy, then $\{a_n\}$ converges.

## Series of complex numbers

An infinite series is an expression of the form

nth term.

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots$$

Partial sums: $s_n = \sum_{j=0}^{n} a_j = a_0 + a_1 + \cdots + a_n$

**DEF. 1.5.12** $\sum_{n=0}^{\infty} a_n$ converges to some $A \in \mathbb{C}$ if $\lim_{n\to\infty} s_n$ exists and

$\lim_{n\to\infty} s_n = A$.