Math 644

Chapter 5

Section 5.1: Cauchy's Theorem

Contents

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PROPOSITION 1. If r is a rational function with poles p_1, p_2, ..., p_N and if γ is a closed curve for which $p_k \notin \gamma$, for any $k = 1, 2, \ldots, N$, then

$$\int_\gamma r(\zeta)\, d\zeta = \sum_{k=1}^{N} c_{k,1} \int_\gamma \frac{1}{\zeta - p_k}\, d\zeta,$$

where $c_{k,1} \in \mathbb{C}$, for $k = 1, 2, \ldots, N$.

Proof.

Theorem 2. Suppose γ is a cycle contained in a region Ω, and suppose

$$\int_\gamma \frac{d\zeta}{\zeta - a} = 0 \quad (\forall a \notin \Omega).$$

If f is analytic on Ω, then

$$\int_\gamma f(\zeta)\, d\zeta = 0.$$

Proof.

Theorem 3. Suppose γ is a cycle contained in a region Ω, and suppose

$$\int_\gamma \frac{d\zeta}{\zeta - a} = 0 \quad (\forall a \notin \Omega).$$

If f is analytic on Ω and $z \in \mathbb{C} \backslash \gamma$, then

$$\frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta = f(z) \cdot \frac{1}{2\pi i} \int_\gamma \frac{1}{\zeta - z}\, d\zeta.$$

Proof.

Example 4. Let $\gamma = \gamma_1 + \gamma_2$ be the cycle formed by $\gamma_1(t) = z_0 + re^{it}$ (clockwise direction) and $\gamma_2(t) = z_0 + Re^{it}$ (counter-clockwise direction), where $0 \le t \le 2\pi$ and $r < R$. Let Ω be the region bounded by γ. If f is analytic on the closure of Ω, then

$$\frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta = \begin{cases} 0 & \text{if } |z - z_0| < r, \\ f(z) & \text{if } r < |z - z_0| < R, \\ 0 & \text{if } |z - z_0| > R. \end{cases}$$