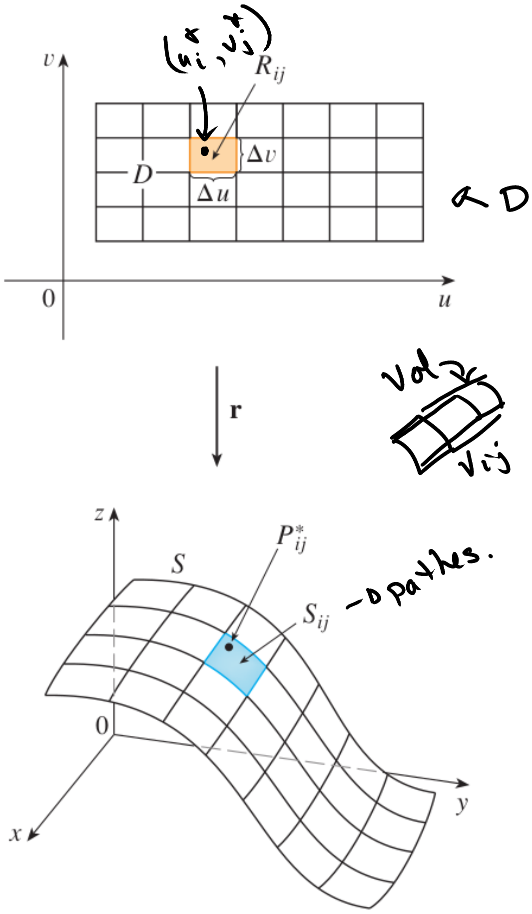
16.7 Surface Integrals.

Parametric surfaces.



f: function in 3 variables

S: surface with $\vec{r}(u,v)$ & domain D.

$$\text{Vol}(V_{ij}) \cong A(S_{ij}) \cdot f(P_{ij}^*)$$
$$= \Delta S_{ij} \cdot f(P_{ij}^*)$$

So now, take lim on the number of divisions (number of patches)

$$\boxed{\iint_S f(x, y, z)\, dS = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}^*)\, \Delta S_{ij}}$$

But,

$$\Delta S_{ij} \cong |\vec{r}_u \times \vec{r}_v|\, \Delta u\, \Delta v.$$

So,

$$\Rightarrow \sum_i \sum_j f(P_{ij}^*)\, \Delta S_{ij} \cong \sum_i \sum_j \underbrace{f(\vec{r}(u_i^*, v_j^*))\, |\vec{r}_u \times \vec{r}_v|}_{g(u,v)}\, \Delta u\, \Delta v.$$

limit on number of patches $\to \iint_S f\, dS = \iint_D g(u,v)\, du\, dv$.

$$\Longrightarrow \boxed{\iint_S f(x, y, z)\, dS = \iint_D \underbrace{f(\mathbf{r}(u, v))\, |\mathbf{r}_u \times \mathbf{r}_v|}_{g(u,v)}\, dA} \to \text{doesn't care about the orientation of } \vec{r}_u \times \vec{r}_v.$$

Mass and center of mass. An aluminum foil S with density $\rho(x,y,z)$.

$$m = \iint_S \rho(x,y,z)\, dS$$

$$\bar{x} = \frac{1}{m} \iint_S x\, \rho(x,y,z)\, dS$$

$$\bar{y} = \frac{1}{m} \iint_S y\, \rho(x,y,z)\, dS$$

$$\bar{z} = \frac{1}{m} \iint_S z\, \rho(x,y,z)\, dS.$$

center of mass: $(\bar{x}, \bar{y}, \bar{z})$.

EXAMPLE 1 Compute the surface integral $\iint_S x^2\, dS$, where S is the unit sphere $x^2 + y^2 + z^2 = 1$.

① Parametrization & dS.

$$\vec{r}(\theta,\phi) = \langle \underbrace{\cos\theta \sin\phi}_{x},\ \sin\theta \sin\phi,\ \cos\phi \rangle \qquad \begin{array}{l} 0 \le \theta \le 2\pi \\ 0 \le \phi \le \pi . \end{array}$$

$$\vec{r}_\theta = \langle -\sin\theta \sin\phi,\ \cos\theta \sin\phi,\ 0 \rangle$$

$$\vec{r}_\phi = \langle \cos\theta \cos\phi,\ \sin\theta \cos\phi,\ -\sin\phi \rangle$$

$$\Rightarrow \vec{r}_\theta \times \vec{r}_\phi = \langle -\cos\theta \sin^2\phi,\ -\sin\theta \sin^2\phi,\ -\sin\phi \cos\phi \rangle$$

$$\Rightarrow |\vec{r}_\theta \times \vec{r}_\phi| = \sin\phi \quad \rightarrow \quad dS = \sin\phi\, d\phi\, d\theta$$

② Integrate.

$$\iint_S x^2\, dS = \iint_D (\cos\theta \sin\phi)^2 \sin\phi\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \cos^2\theta \sin^3\phi\, d\phi\, d\theta$$

$$= \Big(\int_0^{2\pi} \underbrace{\cos^2\theta}_{\frac{1+\cos 2\theta}{2}}\, d\theta\Big)\underbrace{\Big(\int_0^{\pi} \sin^3\phi\, d\phi\Big)}_{\text{MT02} \Rightarrow 4/3}$$

$$= \pi \cdot \frac{4}{3}$$

$$= \boxed{\frac{4\pi}{3}}$$

Graphs of functions. $z = g(x,y)$ with $(x,y) \in D$.

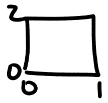
$$\vec{r}(x,y) = \langle x, y, g(x,y) \rangle$$

$$\vec{r}_x = \langle 1, 0, g_x \rangle \quad \to \quad |\vec{r}_x \times \vec{r}_y| = \sqrt{1 + g_x^2 + g_y^2}$$

$$\vec{r}_y = \langle 0, 1, g_y \rangle$$

$$\boxed{\iint_S f(x,y,z)\, dS = \iint_D f(x,y,g(x,y)) \underbrace{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA}_{dS}}$$

EXAMPLE 2 Evaluate $\iint_S y\, dS$, where S is the surface $z = x + y^2$, $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 2$. (See Figure 2.)

$D =$ 

$$\iint_S y\, dS = \iint_D y \sqrt{z_x^2 + z_y^2 + 1}\, dA$$

$$= \int_0^2 \int_0^1 y \sqrt{1 + 4y^2 + 1}\, dx\, dy$$

$$= \int_0^2 \int_0^1 y \sqrt{2 + 4y^2}\, dx\, dy$$

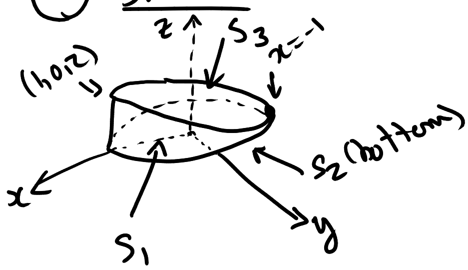
$$= \int_0^2 y \sqrt{2 + 4y^2}\, dy$$

$u = 2 + 4y^2$
$du = 8y\, dy$

$$= \boxed{\frac{13\sqrt{2}}{3}}$$

EXAMPLE 3 Evaluate $\iint_S z\, dS$, where S is the surface whose sides S_1 are given by the cylinder $x^2 + y^2 = 1$, whose bottom S_2 is the disk $x^2 + y^2 \leqslant 1$ in the plane $z = 0$, and whose top S_3 is the part of the plane $z = 1 + x$ that lies above S_2.

(1) Sketch



Property: $S = S_1 \cup S_2 \cup S_3$

$$\iint_S f\, dS = \iint_{S_1} f\, dS + \iint_{S_2} f\, dS + \iint_{S_3} f\, dS$$

(2) S_1

$$\vec{r}(\theta, z) = \langle \cos\theta,\ \sin\theta,\ z \rangle$$

$$0 \le \theta \le 2\pi, \qquad 0 \le z \le 1 + x = 1 + \cos\theta$$

$$\vec{r}_\theta = \langle -\sin\theta,\ \cos\theta,\ 0 \rangle$$

$$\vec{r}_z = \langle 0,\ 0,\ 1 \rangle$$

$$\vec{r}_\theta \times \vec{r}_z = \langle \cos\theta,\ \sin\theta,\ 0 \rangle \Rightarrow |\vec{r}_\theta \times \vec{r}_z| = 1$$

$$\Rightarrow \iint_{S_1} z\, dS = \int_0^{2\pi} \int_0^{1+\cos\theta} z\, dz\, d\theta = \frac{3\pi}{2}$$

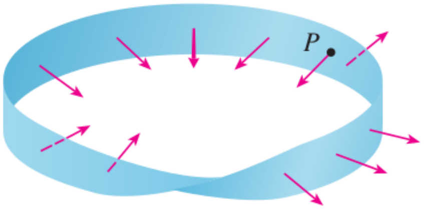
(3) S_2

$$\vec{r}(r, \theta) = \langle r\cos\theta,\ r\sin\theta,\ 0 \rangle$$

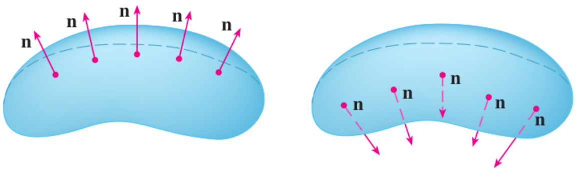
$$\iint_{S_2} z\, dS = \iint_{S_2} 0\, dS = 0.$$

Surface integral of Vector Fields.

Non-orientable surfaces.



Orientable surface.

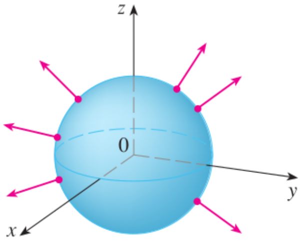


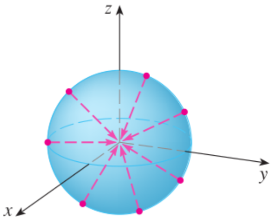
Special orientations:

1. Graph of a function.

2. Parametric surface.

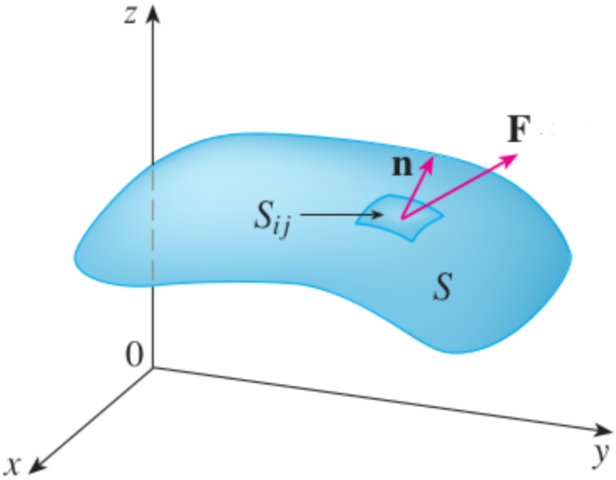
Example with a sphere.





Positive orientation.

Flux integral (or Surface integral).



8 Definition If $\mathbf{F}$ is a continuous vector field defined on an oriented surface S with unit normal vector $\mathbf{n}$, then the **surface integral of F over S** is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

This integral is also called the **flux** of $\mathbf{F}$ across S.

- Parametric surface: Integral formula.

$$\boxed{\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA}$$

- Graph of a function: Integral formula.

$$\boxed{\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(-P \frac{\partial g}{\partial x} - Q \frac{\partial g}{\partial y} + R \right) dA}$$

EXAMPLE 4 Find the flux of the vector field $\mathbf{F}(x, y, z) = z\,\mathbf{i} + y\,\mathbf{j} + x\,\mathbf{k}$ across the unit sphere $x^2 + y^2 + z^2 = 1$.

EXAMPLE 5 Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = y\,\mathbf{i} + x\,\mathbf{j} + z\,\mathbf{k}$ and S is the boundary of the solid region E enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.

Applications to Physics.

Electric Flux.

$$\boxed{\iint_S \mathbf{E} \cdot d\mathbf{S}}$$

Gauss' Law.

$$\boxed{Q = \varepsilon_0 \iint_S \mathbf{E} \cdot d\mathbf{S}}$$

Heat flow.

$$\boxed{-K \iint_S \nabla u \cdot d\mathbf{S}}$$

EXAMPLE 6 The temperature u in a metal ball is proportional to the square of the distance from the center of the ball. Find the rate of heat flow across a sphere S of radius a with center at the center of the ball.