MATH 644

CHAPTER 4

SECTION 4.3: APPROXIMATION BY RATIONAL FUNCTIONS

CONTENTS

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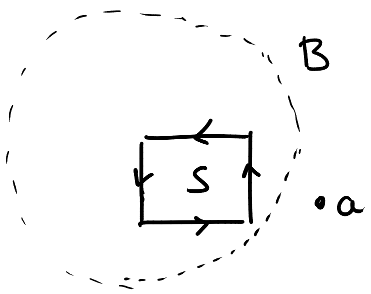
SPRING 2023

Theorem 1. If S is an open square with boundary ∂S parameterized in the counter-clockwise direction then

$$\frac{1}{2\pi i}\int_{\partial S}\frac{1}{z-a}\,dz = \begin{cases} 1 & \text{if } a \in S \\ 0 & \text{if } a \in \mathbb{C}\backslash\overline{S}. \end{cases}$$

Proof.

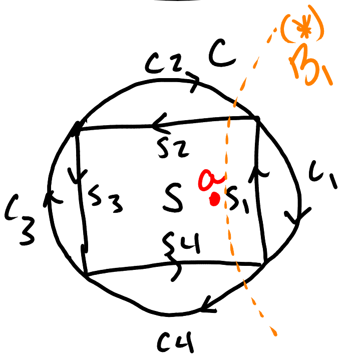
1) $a \in \mathbb{C}/\overline{S}$. There is a disk B s.t. $\overline{S} \subseteq B$ & $a \notin \overline{B}$.



Since $\frac{1}{z-a}$ is analytic in B, from Cor. 11 in sect. 4.2,

$$\int_{\partial S}\frac{1}{z-a}\,dz = 0.$$

2) $a \in S$ Let C be the circle circumscribed to S. Write



$$S = s_1 + s_2 + s_3 + s_4 \ \& \ C = c_1 + c_2 + c_3 + c_4$$

For $j = 1, 2, 3, 4$, $\sigma_j := s_j + c_j$ is a closed curve. For each j, we can find a disk B_j s.t. $\sigma_j \subseteq B_j$ & $a \notin B_j$. Therefore, $\int_{\sigma_j}\frac{1}{z-a}\,dz = 0$ by Cor. 4.11 in 4.2.

$$\Rightarrow \int_{S+(-C)}\frac{1}{z-a}\,dz = 0 \Rightarrow \int_S \frac{1}{z-a}\,dz = \int_C \frac{1}{z-a}\,dz = 2\pi i. \quad \square$$

(*) https://www.desmos.com/calculator/65bsuzfc9z

Theorem 2. If f is analytic in a neighborhood of the closure of $\overline{S}$ of an open square S, then, for $z \in S$,

$$f(z) = \frac{1}{2\pi i} \int_{\partial S} \frac{f(\zeta)}{\zeta - z}\, d\zeta,$$

where ∂S is parameterized in the counter-clockwise direction.

Proof. Fix $z \in S$, $\zeta \in \partial S$, $\quad \dfrac{f(\zeta) - f(z)}{\zeta - z} = \int_0^1 f'(z + t(\zeta - z))\, dt.$

So,

$$\frac{1}{2\pi i} \int_{\partial S} \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta = \lim_{\varepsilon \to 0} \int_\varepsilon^1 \int_{\partial S} \frac{df}{d\zeta}(z + t(\zeta - z))\, d\zeta\, \frac{dt}{t}$$

$$= 0.$$

From thm 1, $f(z) = (2\pi i)^{-1} \int_{\partial S} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$ $\square$

Corollary 3. If f is analytic in a neighborhood of the closure of $\overline{S}$ of an open square S, then

$$\int_{\partial S} f(\zeta)\, d\zeta = 0.$$

Proof.

Define $g(\zeta) = f(\zeta)(\zeta - z)$, $\zeta \in \overline{S}$.

So, g is analytic in $\overline{S}$.

Apply Corollary 2:

$$0 = g(z) = \frac{1}{2\pi} \int_{\partial S} \frac{f(\zeta)(\zeta - z)}{\zeta - z}\, d\zeta = \frac{1}{2\pi} \int_{\partial S} f(\zeta)\, d\zeta. \quad \square$$

THEOREM 4. If f is analytic on a compact set K, and if $\varepsilon > 0$, then there is a rational function r so that

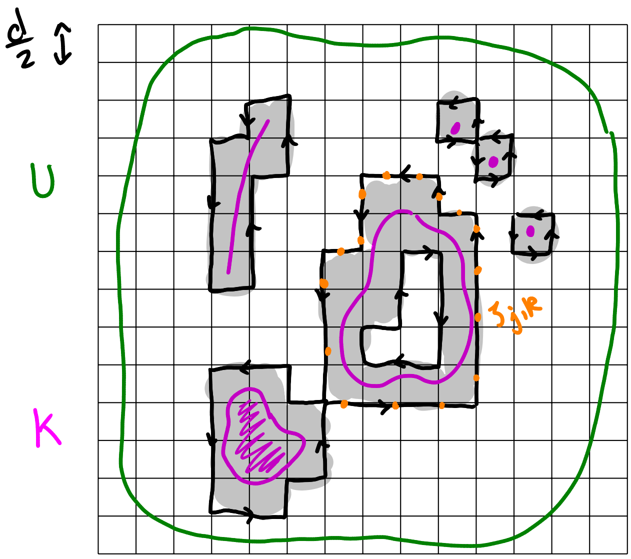
$$\sup_{z\in K} |f(z) - r(z)| < \varepsilon.$$

Proof. Let $U \supseteq K$ be bounded open s.t. f is analytic on U.

Let

$$d := \text{dist}(\partial U, K) := \min\{|z-w| : z \in \partial U, w \in K\}.$$

Construct a grid of closed squares of side length $d/2$



1) Shade the squares intersecting K.

2) Each closed shaded squares $\subseteq U$ (because diam $= \frac{d}{\sqrt{2}}$).

3) $\{S_k\}$ be the coll. of shaded closed squares

4) $\Gamma := \partial(\cup_j S_j)$, ∂S_j param. counter-clock.

Then Γ is a cycle and $\Gamma \cap K = \emptyset$.

Fix $z \in \text{int}(S_{j_0})$, some $j_0 \Rightarrow \zeta \longmapsto \frac{f(\zeta)}{\zeta - z}$ is analytic in S_j, $\forall j \neq j_0$. From Thm 2 and Cor. 3

$$f(z) = \frac{1}{2\pi i}\int_{\partial S_{j_0}} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{2\pi i}\int_{\Gamma} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

This equality holds $\forall z \in \bigcup S_j \setminus \Gamma$ (continuity).

Fix $z_0 \in K$ and write $\Gamma = \sum_{j=1}^{N} \Gamma_j$, where Γ_j are cont. diff. closed curves. Then

$$f(z_0) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{\zeta - z_0}\, d\zeta = \sum_{j=1}^{N} \frac{1}{2\pi i} \int_{\Gamma_j} \frac{f(\zeta)}{\zeta - z_0}\, d\zeta.$$

Let $0 = t_{j,0} < t_{j,1} < \dots < t_{j,m_j} = 1$ & $\zeta_{j,k} := \zeta(t_{j,k}) \in \Gamma_j$ for $j = 1, \dots, N$, $m_j \in \mathbb{N}$ and $k = 1, \dots, m_j$.

Now, use a Rieman Sum with m_j big enough so that, given $\varepsilon > 0$,

$$(*) \quad \Bigg| f(z_0) - \underbrace{\sum_{j=1}^{N} \sum_{k=0}^{m_j} \frac{f(\zeta_{j,k})(\zeta_{j,k+1} - \zeta_{j,k})}{(\zeta_{j,k} - z_0)\, 2\pi i}}_{r_P(z_0).\ (\text{some partition } P)} \Bigg| < \varepsilon$$

From Rie. Int. Theory, $(*)$ holds for any refinements P_1 of the partition $P_j = \{\zeta_{j,k}\}_{k=1}^{m_j}$, $j = 1, \dots, N$.

<u>Claim</u>: Moreover $(*)$ holds for any z in a small disk centered at z_0 and any refinements of the partition if ε is replaced by 4ε.

1) Fix j. Consider a segment $I = [a,b]$ in Γ_j.

Fix $\varepsilon > 0$. Since $z_0 \notin \Gamma$, there is a $\delta > 0$ s.t.

$$|\zeta - \eta| < \delta \;,\; \eta, \zeta \in I \;\Rightarrow\; \left| \frac{f(\zeta)}{\zeta - z_0} - \frac{f(\eta)}{\eta - z_0} \right| < \varepsilon .$$

Take a partition $\{I_\ell\}$ of I with $I_\ell = [\mu_{\ell-1}, \mu_\ell]$

s.t. $I = \bigcup_{\ell=1}^{m} I_\ell$ and $|I_\ell| < \delta$. Then, for any

$\zeta \in I_\ell$, $$\left| \frac{f(\zeta)}{\zeta - z_0} - \frac{f(\mu_\ell)}{\mu_\ell - \zeta} \right| < \varepsilon .$$

If $[\eta_{p-1}, \eta_p] \subseteq [\mu_{\ell-1}, \mu_\ell]$ is a refinement,

then

$$\zeta \in [\eta_{p-1}, \eta_p] \;\Rightarrow\; \left| \frac{f(\zeta)}{\zeta - z_0} - \frac{f(\eta_p)}{\eta_p - z_0} \right| < 2\varepsilon .$$

3) Since $\mathrm{dist}(z_0, \Gamma) > 0$, then for $|z - z_0| \leq \tilde{\delta} < \frac{\mathrm{dist}(z_0, \Gamma)}{2}$

$$\left| \frac{f(\zeta)}{\zeta - z} - \frac{f(\zeta)}{\zeta - z_0} \right| < \varepsilon \;,\; \forall \zeta \in I .$$

If $\zeta \in [\eta_{p-1}, \eta_p] \subseteq [\mu_{\ell-1}, \mu_\ell]$ (a refinement):

$$\left| \frac{f(\zeta)}{\zeta - z} - \frac{f(\eta_p)}{\eta_p - z} \right| \leq \left| \frac{f(\zeta)}{\zeta - z} - \frac{f(\zeta)}{\zeta - z_0} \right| + \left| \frac{f(\zeta)}{\zeta - z_0} - \frac{f(\eta_p)}{\eta_p - z} \right|$$

$$< \varepsilon + \left| \frac{f(\zeta)}{\zeta - z_0} - \frac{f(\eta_p)}{\eta_p - z_0} \right| + \left| \frac{f(\eta_p)}{\eta_p - z_0} - \frac{f(\eta_p)}{\eta_p - z} \right|$$

$$< \varepsilon + 2\varepsilon + \varepsilon = 4\varepsilon . \; (\ddot\smile)$$

4) Now,

$$\left| \int_I \frac{f(\zeta)}{\zeta - z}\, d\zeta - \sum_{p=1}^{M} \frac{f(\eta_p)}{\eta_p - z} (\eta_p - \eta_{p-1}) \right|$$

$$\leq \sum_{p=1}^{M} \int_{[\eta_{p-1}, \eta_p]} \left| \frac{f(\zeta)}{\zeta - z} - \frac{f(\eta_p)}{\eta_p - z} \right| d\zeta \leq 4\varepsilon \quad \left(\varepsilon \leftarrow \frac{\varepsilon}{|I|} \right)$$

5) Since Γ_j is a sum of segment I:

$$\left| \int_{\Gamma_j} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \sum_{\substack{I \subseteq \Gamma_j \\ I \text{ segment}}} \sum_{p=1}^{M} \frac{f(\eta_p)}{\eta_{p_1} - z} (\eta_p - \eta_{p-1}) \right| < 4\varepsilon$$

if $\varepsilon \leftarrow \dfrac{\varepsilon}{|\Gamma_j|}$ and $|z - z_0| < \tilde{\delta}$.

6) Therefore, from 1) - 5), (*) remains true for any z s.t. $|z - z_0| < \tilde{\delta}$ and any refinement of the partition $\{t_{j,k}\}_{k=1}^{m_j}$, $j = 1, \dots, N$.

This means $\quad |f(z) - r_p(z)| < 4\varepsilon$

for any refinement $\mathcal{P}$ and $|z-z_0| < \tilde{\delta}$. $\triangle$

Cover $K \subseteq \bigcup_{\ell=1}^{J} B_\ell$, where $B_\ell = \{ z : |z-z_\ell| < \tilde{\delta}_\ell \}$.

On each B_ℓ, there is a refinement $\mathcal{P}_\ell$ such that

$$\left| f(z) - r_{\mathcal{P}_\ell}(z) \right| < 4\varepsilon, \quad \forall z \in B_\ell.$$

Now, consider $\mathcal{P}$, the union of all refinements $\mathcal{P}_\ell$. $\mathcal{P}$ is a refinement of each $\mathcal{P}_\ell$ and

$$\left| f(z) - r_{\mathcal{P}}(z) \right| < 4\varepsilon \qquad \forall z \in \bigcup_{\ell=1}^{J} B_\ell. \quad \square$$

DEFINITION 5. Let $r(z) = p(z)/q(z)$ be a rational function where p and q are two polynomials with no common zeros. The zeros of q are called the **poles** of the rational function r.

Note:

- If b is a pole of a rational function r, then $|r(z)| \to \infty$ as $z \to b$.

LEMMA 6. Suppose that U is a region (connected & open) and suppose $b \in U$. Then a rational function with poles in U can be uniformly approximated on $\mathbb{C}\backslash U$ by a rational function with poles only at b.

Proof. Let $a_1, a_2, \ldots, a_n \in U$ be the poles of r.

We know that $r(z) = p(z) + \sum_{j=1}^{N} \sum_{k=1}^{n_j} \frac{c_{k,j}}{(z-a_j)^k}$

We just need to show $(z-a_j)^k$ can be approx. uniformly on $\mathbb{C}\backslash U$ by rat. fct. with pole at b only.

1) Consider $c, a \in U$ s.t. $|c-a| < \text{dist}(a, \partial U) = \delta$.

Then,

$$(*) \quad \frac{1}{z-c} = \sum_{n=0}^{\infty} \frac{(c-a)^n}{(z-a)^{n+1}} = \lim_{N\to\infty} \sum_{n=0}^{N} \frac{(c-a)^n}{(z-a)^{n+1}} \quad (\text{unif. on } \mathbb{C}\backslash U)$$

since $|z-c| \geq \text{dist}(\partial U, a) > |c-a|$ for $z \in \mathbb{C}\backslash U$.

Taking product, $(z-c)^{-m}$, $m \geq 2$ can be unif. approx. by a rational function with poles at b only.

Taking linear comb (finite), rat. funct. unif approx. by rat. funct. with poles at b only.
(poles at c.)

So, there is a disk $B : \{c : |c-a| < \delta\} \subseteq \mathbb{C} \setminus U$ such that any rat. fct. with poles at c is unif. approx. by rat. fct. poles at b.

2) Say $c \in R_d$ if every rat. funct. with poles at c can be unif. approx. by rat. funct. with with poles at d. (unif. on $\mathbb{C} \setminus U$)

Transitive: $c \in R_d$, $d \in R_e \Rightarrow c \in R_e$.

Define $E := \{a \in U : a \in R_b\}$

i) From step 1, E is open.

ii) $a_n \to a_\infty \in U$ and $a_n \in E$, $n \in \mathbb{N}$.

Choose $N \in \mathbb{N}$ s.t. $|a_N - a_\infty| < \dfrac{\text{dist}(\partial U, a_\infty)}{2}$.

$$\Rightarrow |a_N - a_\infty| < \text{dist}(\partial U, a_\infty) - |a_N - a_\infty|$$

If $z \notin U$,
$$\begin{aligned} |z - a_N| &\geq |z - a_\infty| - |a_\infty - a_N| \\ &\geq \text{dist}(\partial U, a_\infty) - |a_\infty - a_N| \\ &> |a_\infty - a_N| \end{aligned}$$

So, using step (1), $a_\infty \in R_{a_N}$ and $a_N \in R_b$

by transitivity, $a_\infty \in R_b \Rightarrow a_\infty \in E$. E is closed.

By connectedness, $E = \emptyset$ or $E = U$. Since $b \in E$,

$E = U$. $\square$

Corollary 7. Suppose U is a region and suppose $\{z : |z| > R\} \subset U$ for some $R < \infty$. Then a rational function with poles only in U can be uniformly approximated on $\mathbb{C}\backslash U$ by a polynomial.

Proof.

From Lemma 6, a rat. funct. with poles in U can be uniformly approx. by rat. functions with poles at some $b \in U$.

Choose b s.t. $|b| > R$. Then for $z \in \mathbb{C}\backslash U$

$$\frac{1}{z-b} = \frac{1}{-b\left(1-\frac{z}{b}\right)} = -\sum_{n=0}^{\infty} \frac{z^n}{b^{n+1}} \quad (\text{unif. on } \mathbb{C}\backslash U).$$

So, $(z-b)^{-1}$ is unif. approximated by a polynomial on $\mathbb{C}\backslash U$.

Taking products & lin. comb., any rational function with poles in U are uniformly approx. by polynomial. $\square$

Components

DEFINITION 8. Let U be an open set.

a) A polygonal curve in U is a curve consisting of a finite union of line segments.

b) For $a, b \in U$, define $a \sim_U b$ if and only if there is a polynomial curve contained in U with edges parallel to the axis and joining a to b.

THEOREM 9. Let $U \subset \mathbb{C}$ be an open set.

a) Show that the equivalence classes of $\sim_U$ are closed and open (relative to U) and connected.

b) Show that there are countably many equivalent classes.

Note:

- The equivalence classes are called the **components** of U. They are the maximal connected subsets of U.

Closed Components

DEFINITION 10. Suppose $K \subset \mathbb{C}$ is a compact set.

a) For $a, b \in K$, define $a \sim_K b$ if and only if there is a connected subset of K containing a and b.

THEOREM 11. Let $K \subset \mathbb{C}$ be a compact set.

a) Show that the equivalence classes of $\sim_K$ are connected and closed.

b) Show that there might be ~~infinitely~~ uncountably many equivalence classes.

Note:

- The equivalence classes are called the **(closed) components** of K.

THEOREM 12. [Runge] Suppose K is a compact set. Choose one point a_n in each bounded component of U_n in $\mathbb{C}\backslash K$. If f is analytic on K and $\varepsilon > 0$, then we can find a rational function r with poles only in the set $\{a_n\}$ such that

$$\sup_{z \in K} |f(z) - r(z)| < \varepsilon.$$

If $\mathbb{C}\backslash K$ has no bounded components, then we may take r to be a polynomial.

Proof. From Thm. 4, $\sup_K |f(z) - r(z)| < \varepsilon$, for some rat. fct. r.

By Lemma 6, we may suppose that the poles of r are at $a_1, a_2, \ldots$

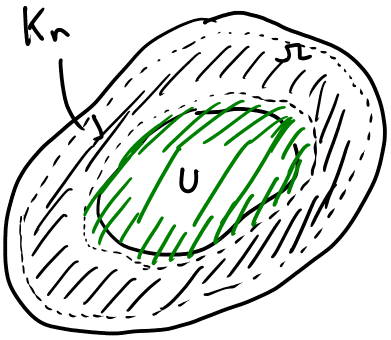
If $\mathbb{C} \backslash K$ has no bounded components, then use Cor. 7 to change r into a polynomial. $\square$

Theorem 13. If f is analytic on an open set $\Omega \neq \mathbb{C}$, then there is a sequence of rational functions r_n with poles in $\partial\Omega$ so that r_n converges to f uniformly on compact subsets of Ω.

Proof. × Fix $K \subseteq U$ a compact set.

Set $K_n := \{ z \in \Omega : \mathrm{dist}(z, \partial\Omega) \geq \frac{1}{n} \ \& \ |z| \leq n \}$.

Each bouded component U of $\mathbb{C} \setminus K_n$ contains a point of $\partial\Omega$ because



$\partial U \subseteq K_n \subseteq \Omega$ so $U \cap \Omega \neq \emptyset$.

Now, if $z \in U \cap \Omega$, then

$|z| < n$ & $|z - \zeta| < \dfrac{1}{n}$, some $\zeta \in \partial\Omega$.

Let L be a segment joining z to ζ.

If $\alpha \in L$ then

$$|\alpha - \zeta| < \frac{1}{n} \Rightarrow \alpha \notin K_n$$

Since L is connected & $L \subseteq \mathbb{C} \setminus K_n$, L must be in one of the component of $\mathbb{C} \setminus K_n$.

Because $z \in L \cap U$, then $L \subseteq U$. However

$$\zeta \in L \cap \partial\Omega \subseteq U \Rightarrow \zeta \in U.$$

From Thm. 8 applied to K_n, we can select (a_n) to be on $\partial\Omega$. Therefore, we can find r_n (rat. funct.) with poles on $\partial\Omega$ s.t. $|r_n - f| < \varepsilon$ unif. on K_n

Since each compact $K \subseteq \Omega$ is included in some K_n, $r_n \longrightarrow f$ unif. on each compact subsets of Ω. $\square$

Note:

- The improvement of Theorem 9 over Theorem 4 is that the poles of r_n are outside of Ω, not just outside the compact set K on which r_n is close to f.