SECTION 1.6 : Exponential Functions

Calculus:

$$x \in \mathbb{R} \longrightarrow e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Complex: replace x by $z \in \mathbb{C}$ and

$$\lim_{n\to\infty} \frac{\left|\frac{z^{n+1}}{(n+1)!}\right|}{\left|\frac{z^n}{n!}\right|} = \lim_{n\to\infty} \frac{|z|}{n+1} = 0$$

By the ratio test:

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad (*)$$

converges absolutely $\forall z \in \mathbb{C}$.

__DEF 1.6.1__ The complex exponential function $\exp(z)$ or e^z is defined as the series $(*)$, that is

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad \forall z \in \mathbb{C}.$$

THM 1.6.2 Let $z, w \in \mathbb{C}$. Then

① $e^{z+w} = e^z e^w$.

② $e^z \neq 0$ and $e^{-z} = \dfrac{1}{e^z}$.

③ $e^{z-w} = \dfrac{e^z}{e^w}$.

Proof. Assume $z, w \in \mathbb{C}$.

① LHS is well-defined.

RHS is also well-defined by the product of two abs. conv. series.

RHS:

$$e^z e^w = \left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right)\left(\sum_{m=0}^{\infty} \frac{w^m}{m!}\right) = \sum_{n=0}^{\infty} c_n$$

where

$$c_n = \sum_{j=0}^{n} \frac{z^j}{j!} \frac{w^{n-j}}{(n-j)!}$$

$$= \frac{1}{n!} \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} z^j w^{n-j}$$

$$= \frac{(z+w)^n}{n!} \quad \text{[Binomial formula]}$$

So,

$$e^z e^w = \sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} = e^{z+w}.$$

② Notice $e^0 = 1$

$$\Rightarrow e^{z-z} = 1$$

$$\Rightarrow e^z e^{-z} = 1.$$

Hence, $e^z \neq 0$ and

$$e^{-z} = 1/e^z.$$

③ We have

$$e^{z-w} = e^z \cdot e^{-w} = e^z/e^w. \quad \square$$

<u>Prop. 1.6.3</u> Let $z = i\theta$, $\theta \in \mathbb{R}$. Then

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Proof. Here,

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}$$

$$= 1 + i\theta + \frac{(-1)\theta^2}{2!} + \frac{(-1)i\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} + \cdots$$

$$= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots + i\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right]$$

$$= \cos\theta + i\sin\theta. \qquad \square$$

Corollary 1.6.4 Let $z = x+iy \in \mathbb{C}$. Then

$$e^z = e^{x+iy} = e^x\cos y + ie^x\sin y.$$

Proof. Write $e^z = e^x e^{iy}$

$$\Rightarrow e^z = e^x\cos y + ie^x\sin y. \qquad \square$$