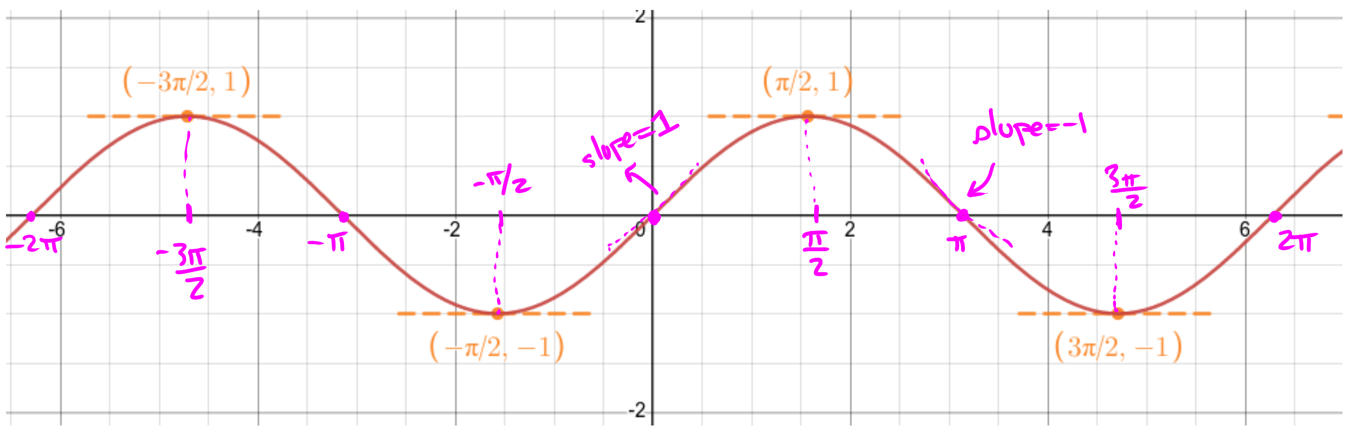
Chapter 2

Derivatives

2.4 Derivatives of Trigonometric Functions

Derivative of the Sine function.



Desmos: https://www.desmos.com/calculator/mhbl7c2hzy

$$\boxed{\frac{d}{dx}(\sin x) = \cos x}$$

Proof.

By def.:

$$\frac{d}{dx}(\sin x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h} \to \frac{0}{0}$$

Trig identity:

$$\sin(x+h) = \sin(x)\cos(h) + \sin(h)\cos(x)$$

$$\Rightarrow \frac{\sin(x+h) - \sin(x)}{h} = \frac{\sin(x)\cos(h) + \sin(h)\cos(x) - \sin(x)}{h}$$

$$= \frac{\sin(x)(\cos(h)-1) + \cos(x)\sin(h)}{h}$$

$$= \sin(x)\frac{\cos(h)-1}{h} + \cos(x)\frac{\sin(h)}{h}$$

So,

$$\lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} = \lim_{h\to 0} \sin(x)\left(\frac{\cos h - 1}{h}\right) \; 2$$

$$+ \lim_{h \to 0} \cos(x)\frac{\sin h}{h} \; 1$$

① We can prove that

$$\cos h \le \frac{\sin h}{h} \le 1$$

We have $\lim_{h\to 0} \cos h = \cos(0) = 1$

& $\lim_{h\to 0} 1 = 1$

Squeeze Theorem $\Downarrow$

$$\lim_{h\to 0} \frac{\sin h}{h} = 1$$

② $\cos h - 1 = -2\left(\frac{1-\cos h}{2}\right) = -2\sin^2(h/2)$

$$\Rightarrow \lim_{h\to 0} \frac{\cos h - 1}{h} = \lim_{h\to 0} \frac{-2\sin^2(h/2)}{h}$$

$$= \lim_{h\to 0} -2\,\frac{\sin(h/2)}{h}\,\sin(h/2)$$

$$= \lim_{h\to 0} -\frac{\sin(h/2)}{(h/2)}\cdot\sin(h/2)$$

$$= -\lim_{h\to 0}\frac{\sin(h/2)}{(h/2)}\cdot\lim_{h\to 0}\sin(h/2)$$

$$= -\lim_{h\to 0}\frac{\sin(h)}{h}\cdot\sin(0)$$

$$= -1\cdot 0 = 0$$

So

$$\lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} = \sin(x)\cdot 0 + \cos(x)\cdot 1$$

$$= \boxed{\cos(x)}$$

Trigonometric Functions (reminder).

- $\sec x = \dfrac{1}{\cos x}$
- $\tan x = \dfrac{\sin x}{\cos x}$
- $\csc x = \dfrac{1}{\sin x}$
- $\cot x = \dfrac{1}{\tan x} = \dfrac{\cos x}{\sin x}$

Derivatives of Other Trigonometric Functions.

Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

Proof for the formula for f(x) = tan(x).

Here

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

$$= \frac{\frac{d}{dx}(\sin x)\cdot\cos x - \sin x\,\frac{d}{dx}(\cos x)}{\cos^2 x}$$

$$= \frac{\cos x\cdot\cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \left(\frac{1}{\cos x}\right)^2$$

$$= \boxed{\sec^2 x}$$

EXAMPLE 2 Differentiate $f(x) = \dfrac{\sec x}{1 + \tan x}$. For what values of x does the graph of f have a horizontal tangent?

① Compute $f'(x)$

$$f'(x) = \frac{(\sec x)'(1+\tan x) - (\sec x)(1+\tan x)'}{(1+\tan x)^2}$$

$$= \frac{\sec x \tan x (1+\tan x) - \sec x (1' + (\tan x)')}{(1+\tan x)^2}$$

$$= \frac{\sec x \tan x + \overset{A}{\boxed{\sec x}} \tan^2 x - \overset{A}{\boxed{\sec x}} \sec^2 x}{(1+\tan x)^2}$$

$2x + x^2$
$= x(2+x)$

$$= \frac{\sec x \tan x + \sec x (\tan^2 x - \sec^2 x)}{(1+\tan x)^2}$$

$\to \dfrac{\sin^2 x - 1}{\cos^2 x}$

$$= \frac{\sec x \tan x + \sec x \left(\frac{\sin^2 x}{\cos^2 x} - \frac{1}{\cos^2 x} \right)}{(1+\tan x)^2}$$

$\cos^2 x + \sin^2 x = 1$
$\sin^2 - 1 = -\cos^2 x$

$$= \frac{\sec x \tan x + \sec x \left(\frac{-\cos^2 x}{\cos^2 x} \right)}{(1+\tan x)^2}$$

$f'(x)$

$$= \frac{\sec x \tan x - \sec x}{(1+\tan x)^2} = \frac{\sec x (\tan x - 1)}{(1+\tan x)^2}$$

② Zeros of $f'(x)$

$f'(x) = 0 \iff \sec x (\tan x - 1) = 0 \iff \dfrac{\overset{\to\, = 0}{\tan x - 1}}{\cos x} = 0$

$\Rightarrow \tan x = 1 \Rightarrow \boxed{x = \frac{\pi}{4} + n\pi}$

EXAMPLE 6 Calculate $\lim_{x\to 0} x\cot x$.

(1) $x\cot x = x\cdot\frac{\cos x}{\sin x} = \frac{x}{\sin x}\cos x = \frac{\cos x}{\sin x}\cdot\frac{1}{\left(\frac{1}{x}\right)}$

$x = \frac{1}{x^{-1}} = \frac{1}{(1/x)}$ $\quad = \frac{\cos x}{\left(\frac{\sin x}{x}\right)}$

$$\lim_{x\to 0} x\cot x = \lim_{x\to 0}\frac{\cos x}{\left(\frac{\sin x}{x}\right)} = \frac{\lim_{x\to 0}\cos x}{\lim_{x\to 0}\frac{\sin x}{x}}$$

$$= \frac{\cos(0)}{1} = \boxed{1}$$

(2) See it as the derivative of some function.

$$x\cot(x) = \frac{x}{\tan(x)} \Rightarrow \frac{1}{x\cot(x)} = \frac{\tan(x)}{x}$$

$$= \frac{\tan(x) - 0}{x - 0}$$

derivative of $\tan x$ at $x=0$.

$$= \frac{\tan(x) - \tan(0)}{x - 0}$$

So, $\lim_{x\to 0}\frac{\tan(x)-\tan(0)}{x-0} = \frac{d}{dx}(\tan x)\Big|_{x=0}$

$$= \sec^2(0) = 1$$

So, $\lim_{x\to 0} x\cot x = \lim_{x\to 0}\frac{1}{\left(\frac{1}{x\cot x}\right)}$

$$= \frac{\lim_{x\to 0} 1}{\lim_{x\to 0}\frac{1}{x\cot x}} = \frac{1}{1} = \boxed{1}$$