Chapter 3
Applications of Derivatives

3.5 Summary of Curve Sketching

A. Find the domain of the function.

B. Find the y-intercept and x-intercept, that is $f(0)$ and when $y = 0$.

C. Search for symmetries in the function (facultative)

- If $f(x) = f(-x)$, then the function is even.
- If $-f(x) = f(-x)$, then the function is odd.
- If $f(x+p) = f(x)$, then the function repeats itself after a period p (it is periodic).

D. Find the asymptotes of the function:

- The Horizontal asymptotes.
- The Vertical asymptotes.

E. Find the intervals of increase and decrease.

F. Find the local maximum and minimum values.

G. Find the concavity and the points of inflections.

H. Sketch.

EXAMPLE 1 Use the guidelines to sketch the curve $y = \dfrac{2x^2}{x^2 - 1}$.

A. All numbers except 1 & -1 $(x^2-1=0 \iff x=\pm 1)$

B. $x=0 \to y=0$ (y & x -intercepts).

C. • Do we have $f(-x)=f(x)$? Yes, $\dfrac{2(-x)^2}{(-x)^2-1} = \dfrac{2x^2}{x^2-1}$.

• Do we have $f(-x) = -f(x)$? No!

• Is it periodic? No!

D. 1) HA.

$$\lim_{x\to\infty} \frac{2x^2}{x^2-1} = \lim_{x\to\infty} \frac{2}{1-1/x^2} = 2$$

$$\lim_{x\to-\infty} \frac{2x^2}{x^2-1} = \lim_{x\to-\infty} \frac{2}{1-1/x^2} = 2.$$

So, $y=2$ is a HA at $\pm\infty$.

VA. $x=1$ VA!

$$\lim_{x\to1^+} \frac{2x^2}{(x-1)(x+1)} = \frac{2}{(1^+-1)(1+1)} = \frac{2}{(0^+)\cdot 2} = +\infty$$

$$\lim_{x\to1^-} \frac{2x^2}{(x-1)(x+1)} = \frac{2}{(1^--1)(1+1)} = \frac{2}{0^-\cdot 2} = -\infty.$$

$x=-1$ VA!

$$\lim_{x\to-1^+} \frac{2x^2}{(x-1)(x+1)} = \frac{2}{(-2)\,0^+} = -\infty, \quad \lim_{x\to-1^-} \frac{2x^2}{x^2-1} = +\infty.$$

E. $f'(x) = \left(\dfrac{2x^2}{x^2-1}\right)' = \dfrac{-4x}{(x^2-1)^2}$ → C.N.: $\boxed{0}$, -1, 1

remove -1

$f'(x) > 0 \iff -4x > 0 \iff x < 0$ → $f \nearrow$ on $(-\infty, 0)$

$f'(x) < 0 \iff -4x < 0 \iff x > 0$ → $f \searrow$ on $(0, \infty)$.

remove 1

F. By the 1st derivative test, f has a local max at $x=0$ with $f(0) = 0$.

G. $f''(x) = \dfrac{12x^2+4}{(x^2-1)^3}$ → $f''(x) = 0 \iff$ no such x because $\underbrace{12x^2}_{\text{positive}} + 4 \geq 4$.

concave up.

$f''(x) > 0$ iff. $x^2 - 1 > 0$

iff $x^2 > 1$ iff. $|x| > 1$ ($x < -1$ or $x > 1$)

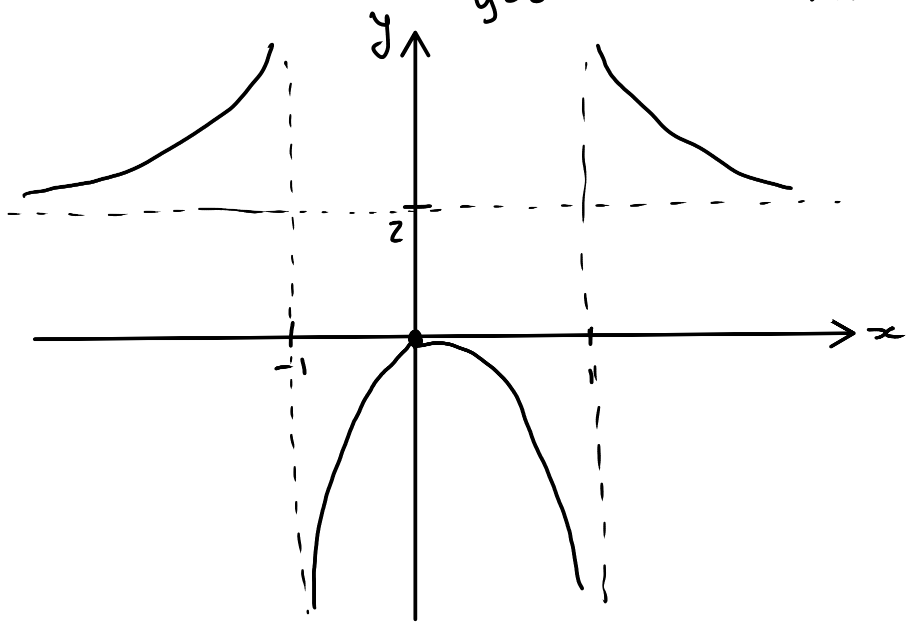
$f''(x) < 0$ iff $x^2 - 1 < 0$ iff. $|x| < 1$ $(-1 < x < 1)$.

So, f concave up on $(-\infty, -1)$ and $(1, \infty)$.

f concave down on $(-1, 1)$.

H.

x		-1		0		1	
f'	$+$	$\nexists$	$+$	0	$-$	$\nexists$	$-$
f''	$+$	$\nexists$	$-$	$*$	$-$	$\nexists$	$+$
f	$\nearrow$	$\nexists$ VA.	$\nearrow$	loc. Max. $y=0$	$\searrow$	$\nexists$ VA.	$\searrow$



EXAMPLE 2 Sketch the graph of $f(x) = \dfrac{x^2}{\sqrt{x + 1}}$.

EXAMPLE 3 Sketch the graph of $f(x) = \dfrac{\cos x}{2 + \sin x}$.