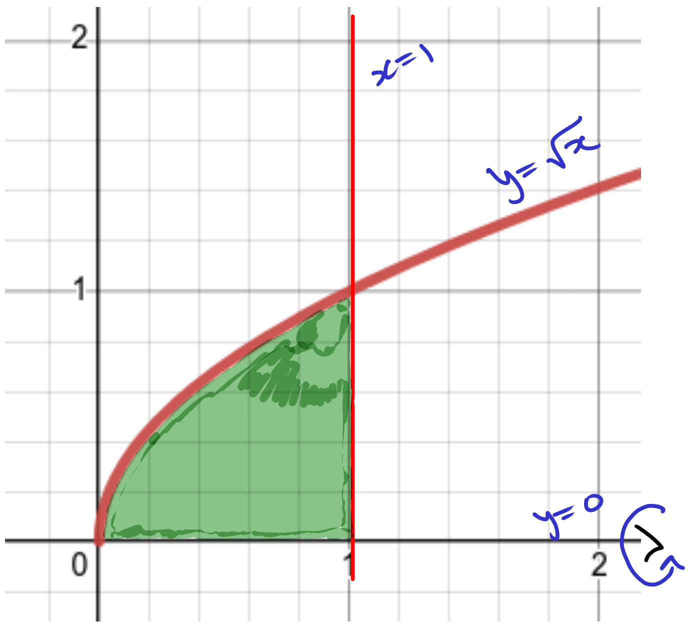
Chapter 5
Applications in integration

5.2 Volumes



$f(x) = \sqrt{x}$

- Consider the region enclosed by

$$x = 0 \ , \qquad x = 1 \ ,$$

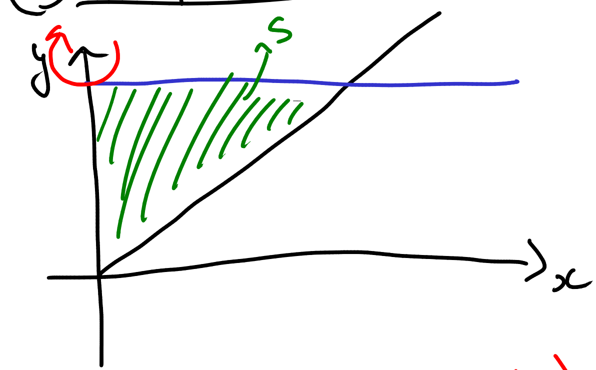
$$y = 0 \quad \text{and} \quad y = \sqrt{x}$$

- Rotate the region about one of the axis.
  - About x-axis
  - About y-axis

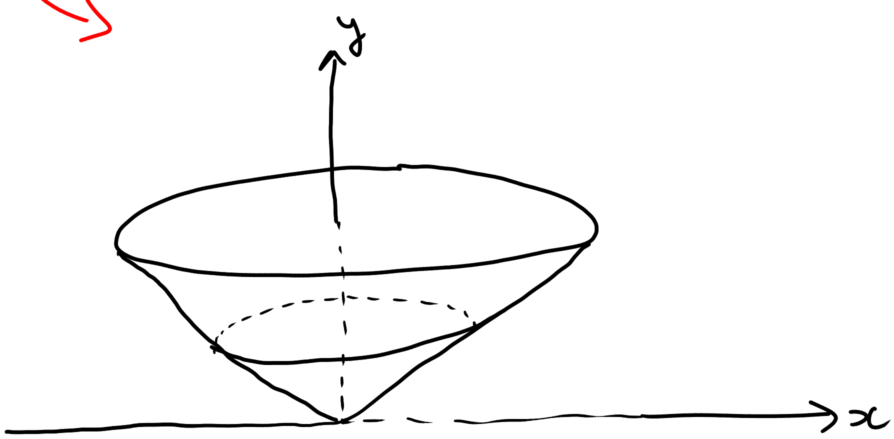
Example.

Rotate the region enclosed by $y = x, y = 1, x = 0$ about the y-axis.

① 2D picture

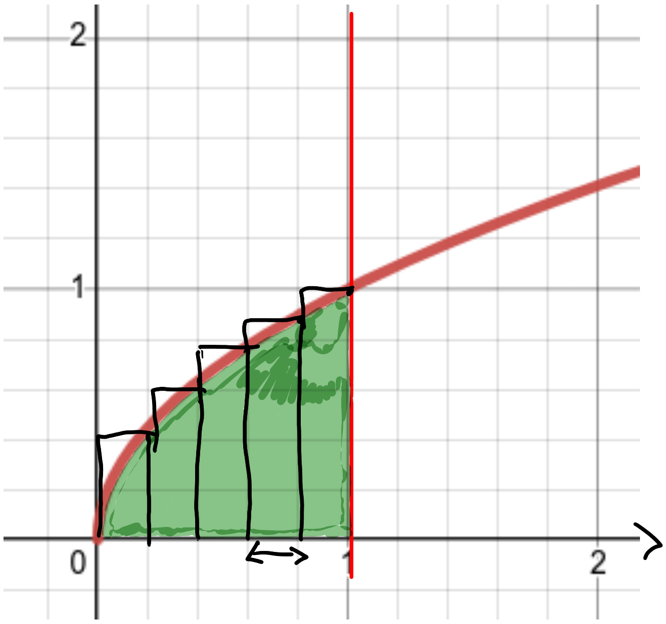


rotate.



VOLUME Of SOLIDS OF REVOLUTION.

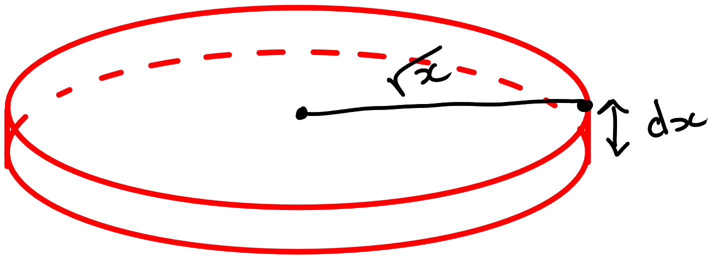
Rotation about the x-axis.



$f(x) = \sqrt{x}$

- Divide into subintervals.
- Create rectangles.
- Rotate those rectangles.
- Generate small cylinders

Typical cylinder:



- Radius: $\sqrt{x}$ $\quad (f(x))$
- Heigth: $\Delta x \to dx$

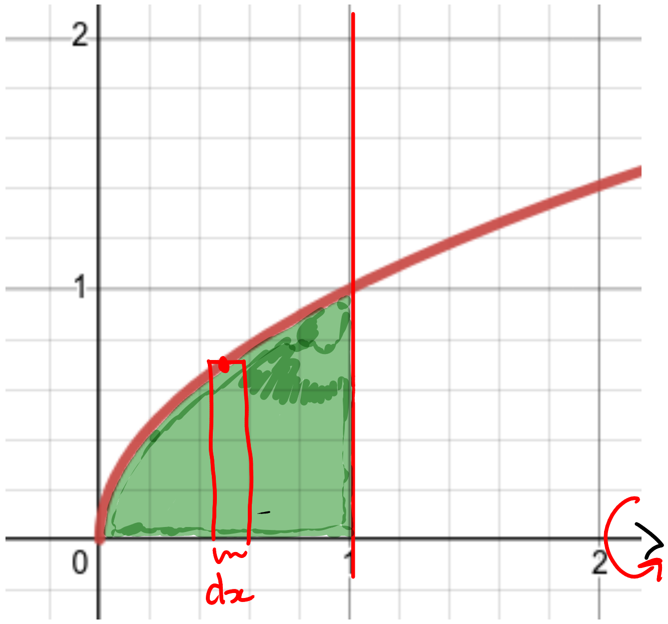
Volume of typical cylinder:

$$V = \pi \left(\sqrt{x}\right)^2 dx$$
$$= \pi (\text{radius})^2 dx \quad \to \quad V(\text{Solid}) = \int_0^1 \pi \left(\sqrt{x}\right)^2 dx$$

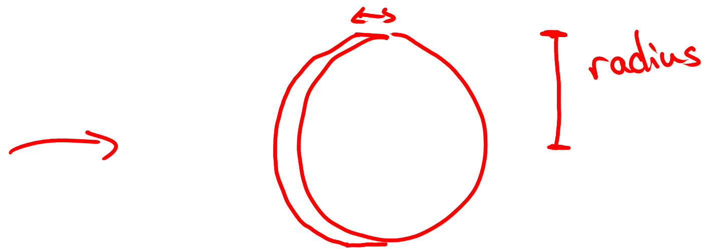
$$\boxed{\text{Vol(Solid)} = \int_a^b \pi (\text{radius})^2 \, dx}$$

EXAMPLE 2 Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1. Illustrate the definition of volume by sketching a typical approximating cylinder.

SKETCH



$f(x) = \sqrt{x}$



radius $= \sqrt{x}$

height $= dx$

$$\text{Vol(Solid)} = \int_0^1 \pi \,(\text{radius})^2\, dx$$

$$= \int_0^1 \pi \, x \, dx$$

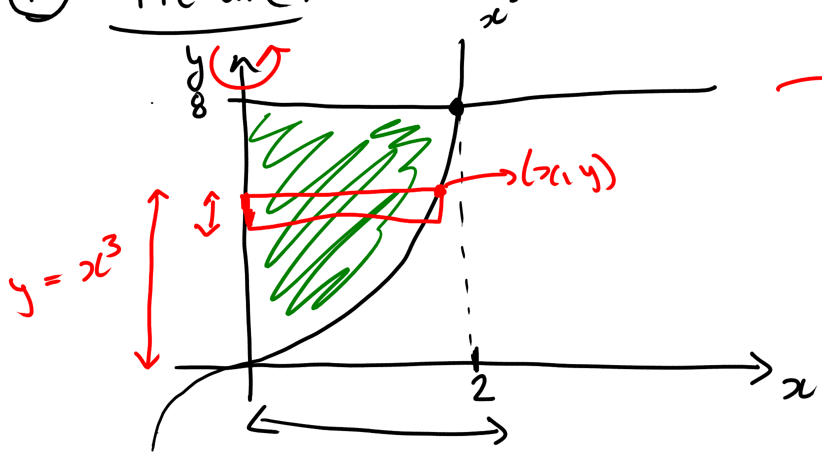
$$= \boxed{\frac{\pi}{2}}$$

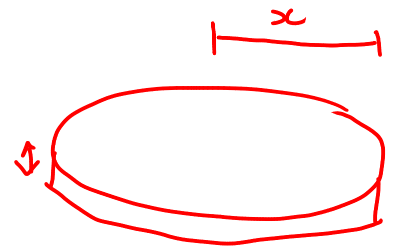
Rotation around the y-axis.

$$\text{Vol(Solid)} = \int_a^b \pi(\text{radius})^2 \, dy$$

EXAMPLE 3 Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$, and $x = 0$ about the y-axis.

① Picture.





radius $= x = y^{1/3}$

height $= dy$

② Volume

$$\text{Vol(Solid)} = \int_0^8 \pi (\text{radius})^2 \text{ height}$$

$$= \int_0^8 \pi (\text{radius})^2 \, dy$$

$$= \int_0^8 \pi x^2 \, dy$$

$y = x^3$

$\Rightarrow y^{1/3} = x$

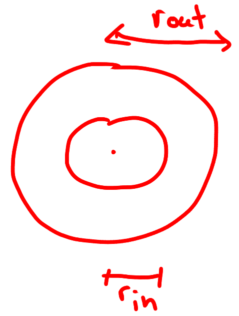
$$= \int_0^8 \pi y^{2/3} \, dy$$

$$= \boxed{\frac{96\pi}{5}}$$

Cross-section as a washer.

Rotation about x-axis

$$\mathrm{Vol(Solid)} = \int_a^b \pi(r_{\text{out}}^2 - r_{\text{in}}^2)\, dx$$

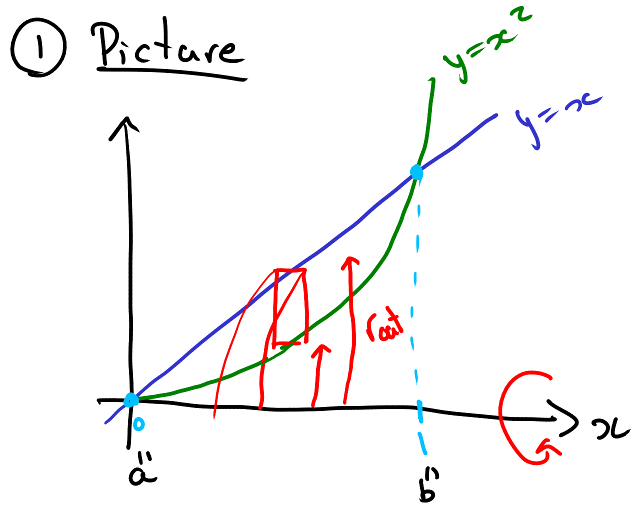


Rotation about y-axis

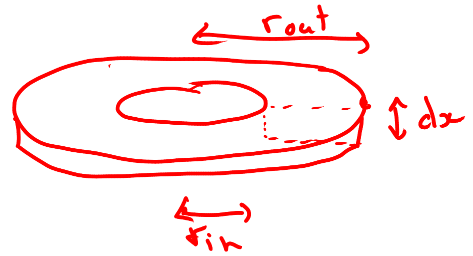
$$\mathrm{Vol(Solid)} = \int_a^b \pi(r_{\text{out}}^2 - r_{\text{in}}^2)\, dy$$

EXAMPLE 4 The region $\mathscr{R}$ enclosed by the curves $y = x$ and $y = x^2$ is rotated about the x-axis. Find the volume of the resulting solid.

① Picture



$x = x^2 \Rightarrow x = 0$ & $x = 1$



$r_{out} = y_1 = x$

$r_{in} = y_2 = x^2$

height

② Volume

$$\mathrm{Vol(Solid)} = \int_0^1 \pi\left(r_{out}^2 - r_{in}^2\right) dx$$

$$= \int_0^1 \pi\left(x^2 - (x^2)^2\right) dx$$

$$= \int_0^1 \pi x^2 - \pi x^4 \, dx = \boxed{\frac{2\pi}{15}}$$