MATH-241 Calculus 1 — Created by P.-O. Parisé
Homework 04 — Fall 2021

Assigned date: 10/11/2021 9am
Due date: 10/18/2021 5pm

Last name: ____________________
First name: ____________________
Section: ____________________

Question:	1	2	3	4	5	6	Total
Points:	20	20	20	20	10	10	100
Score:							

Instructions: You must answer all the questions below and upload your solutions (in a PDF format) to Gradescope (go to `www.gradescope.com` with the Entry code GEK6Y4). Be sure that after you scan your copy, it is clear and readable. You must name your file like this: `LASTNAME_FIRSTNAME.pdf`. A homework may not be corrected if it's not readable and if it's not given the good name. No other type of files will be accepted (no PNG, no JPG, only PDF) and no late homework will be accepted.

Make sure to show all your work!

Good luck!

Question 1 (20 points)

Find the absolute maximum and absolute minimum of the functions on the given interval.

(a) (10 points) $f(x) = (t^2 - 4)^3$ on $[-2, 3]$.

> **Solution:** The first step is to find the critical points of f. We have $f'(t) = 3(t^2-4)^2 2t$ by the chain rule. So,
>
> $$f'(t) = 0 \iff t^2 - 4 = 0 \text{ or } t = 0 \iff t = \pm 2 \text{ or } t = 0.$$
>
> The values at those critical points are $f(0) = -\sqrt[3]{4}$, $f(\pm 2) = 0$.
> The values of the function at the end points of the interval are $f(-2) = 0$ and $f(3) = \sqrt[3]{5}$.
> Thus the absolute minimum is $f(0) = -\sqrt[3]{4}$ and the absolute maximum is $f(3) = \sqrt[3]{5}$.

(b) (10 points) $f(u) = \sqrt{3}\cos x + \sin x$ on $[-\pi, \pi]$.

> **Solution:** The first step is to find the critical points of the function. We have $f'(x) = -\sqrt{3}\sin x + \cos x$. So,
>
> $$f'(x) = 0 \iff \sqrt{3}\sin x = \cos x \iff \tan x = \frac{1}{\sqrt{3}}.$$
>
> This occurs when $x = \pi/6$ or $x = 7\pi/6$. Put the tangent function is π peridic, so this happens when $x = \pi/6 + k\pi$ where $k \in \mathbb{Z}$. Since x must be between $-\pi$ and π, we have $x = \pi/6$ and $x = -5\pi/6$. The critical values are then $f(\pi/6) = 2$ and $f(-5\pi/6) = -2$.
> At the end points, we have $f(-\pi) = -\sqrt{3}$ and $f(\pi) = \sqrt{3}$.
> So the absolute maximum is $f(\pi/6) = 2$ and the absolute minimum is $f(-5\pi/2) = -2$.

Question 2 **(20 points)**

(a) (10 points) Show that the equation $2x + \cos x = 0$ has exactly one real root.

> **Solution:** Let $f(x) = 2x + \cos x$. Then $f(0) = 0 + 1 = 1$ and $f(-\pi) = -2\pi - 1$. So, we have $f(0) > 0$ and $f(-\pi) < 0$. So, by the intermediate value Theorem, there must be a $c \in (-\pi, 0)$ such that $f(c) = 0$.
>
> Now, suppose there is another root d. Then $f(c) = f(d) = 0$. Then, by Rolle's Theorem, there is a $x \in (c, d)$ such that $f'(x) = 0$. But, $f'(x) = 2 - \sin x$ and since $\sin x$ is always between -1 and 1, the function $f'(x) = 2 - \sin x$ can't be zero. This is a contradiction.
>
> In conclusion, the equation $2x + \cos x$ has only one real root.

(b) (10 points) Let $f(x) = \sqrt[3]{x}$. Find all numbers c inside the interval $[-8, 8]$ such that the slope of the secant line passing through the points $(-8, f(-8))$ and $(8, f(8))$ is equal to the slope of the tangent line at $(c, f(c))$.

> **Solution:** The slope of the secant line is given by the expression
>
> $$\frac{f(8) - f(-8)}{8 - (-8)} = \frac{4}{16} = \frac{1}{4}.$$
>
> The derivative of f is $f'(x) = \frac{1}{3x^{2/3}}$. So, we want to find the numbers $c \in [-8, 8]$ such that
>
> $$\frac{1}{3c^{2/3}} = \frac{1}{4} \iff c^{2/3} = \frac{4}{3} \iff c^2 = \frac{64}{27}.$$
>
> Then, taking the square-root on each side, we find that
>
> $$c = \pm\frac{8}{3\sqrt{3}}.$$

Question 3 (20 points)

For the given function, find the where f is increasing or decreasing, find the local maximum and local minimum values of f, and find the intervals of cancavity and the inflection points. If possible, mention justify if you have a global maximum or minimum. Justify all your answers using the the derivatives.

(a) (10 points) $f(x) = x^4 - 2x^2 + 3$.

Solution: Derivative. We have $f'(x) = 4x^3 - 4x$. So, we found that

$$f'(x) = 0 \iff (x^2-1)x = 0 \iff x = \pm 1 \text{ or } x = 0.$$

We can construct a table for the derivative:

x		-1		0		1	
x	-	-	-	0	+	+	+
$x-1$	-	-	-	-	-	0	+
$x+1$	-	0	+	+	+	+	+
$f'(x)$	-	0	+	0	-	0	+

From the table, we see that

- f is decreasing on $(-\infty, -1)$ and on $(0, 1)$.
- f is increasing on $(-1, 0)$ and $(1, \infty)$.

From the first derivative test, we see that

- f has a local minimum at $x = -1$ and a local maximum at $x = 0$ and another local minimum at 1. The values are respectively $f(-1) = 2$, $f(0) = 3$ and $f(1) = 2$.

There is no global maximum and global minimum.

Second Derivative. We also have $f''(x) = 12x^2 - 4$ and $f''(x) = 0$ if and only if $x = \pm\frac{1}{\sqrt{3}}$. We can construct a table for the second derivative:

x		$-1/\sqrt{3}$		$1/\sqrt{3}$	
4	+	+	+	+	+
$\sqrt{3}x - 1$	-	-	-	0	+
$\sqrt{3}x + 1$	-	0	+	+	+
$f''(x)$	+	0	-	0	+

From the table, we see that

- f is concave downward on $(-\infty, -1/\sqrt{3})$ and $(1/\sqrt{3}, \infty)$.
- f is concave upward on $(-1/\sqrt{3}, 1/\sqrt{3})$.

(b) (10 points) $f(x) = \sin x + \cos x$ for $0 \leq x \leq 2\pi$. [You can use Desmos to know where the derivative and the second derivative are positive and negative. Make sure to insert the picture you obtained in your anwer.]

Solution: First Derivative. We have $f'(x) = \cos x - \sin x$. So, we have

$$f'(x) = 0 \iff \cos x = \sin x \iff \tan x = 1.$$

This happens when $x = \pi/4$ or $x = \pi/4 + k\pi$ for some $k \in \mathbb{Z}$. To stay within the interval $[0, 2\pi$, we must restrict the values of such x's to $x = \pi/4$ and $x = 5\pi/4$. We can then construct a table for the derivative:

x	0		$\pi/4$		$5\pi/4$		2π
$\cos x - \sin x$		+	0	-	0	+	

You can justify this table by plotting the function $\cos x - \sin x$ or just by trying some values in between. From the table, we see that

- f is increasing on $(0, \pi/4)$ and $(5\pi/4, 2\pi)$.
- f is decreasing on $(\pi/4, 5\pi/4)$.

From the first derivative Test, we see that

- f has a local minimum at $x = \pi/4$.
- f has a local maximum at $x = 5\pi/4$.

If you compare the value at the endpoints and the values at the critical points, you see that there are in fact global minimum and global maximum.

Second Derivative. We have $f''(x) = -\sin x - \cos x = -f(x)$. So, in order to have $f''(x)$, it is necessary and sufficient for x so satisfy the equation $\tan x = -1$. This occurs for $x = -\pi/4$, or $x = -\pi/4 + k\pi$ for any $k \in \mathbb{Z}$. Since we have to stay within the interval $[0, 2\pi]$, we must confine ourself to $x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4}$. We can construct a table for the second derivative:

x	0		$3\pi/4$		$7\pi/4$		2π
$\sin x + \cos x$		+	0	-	0	+	
$f''(x)$		-	0	+	0	-	

From the table, we arrive at the following conclusions:

- f is concave upward on $(3\pi/4, 7\pi/4)$.
- f is concave downward on $(-\infty, 3\pi/4)$ and on $(7\pi/4, \infty)$.

Question 4 (20 points)

Find the horizontal asymptotes and vertical asymptotes of each curve.

(a) (10 points) $y = \frac{5+4x}{x+3}$.

Solution: We have $x+3=0$ if $x=-3$. We see that $5+4(-3)=-7\neq 0$. So

$$\lim_{x\to -3} \frac{5+4x}{x+3} = \pm\infty.$$

So, $x=-3$ is a vertical asymptote.
Also, we see that

$$\lim_{x\to\infty} \frac{5+4x}{x+3} = \lim_{x\to\infty} \frac{5/x+4}{1+3/x} = 4$$

and

$$\lim_{x\to-\infty} \frac{5+4x}{x+3} = \lim_{x\to-\infty} \frac{5/x+4}{1+3/x} = 4.$$

So, $y=4$ is a horizontal asymptote.

(b) (10 points) $y = \frac{1+x^4}{x^2-x^4}$.

Solution: First, we have that $x^2-x^4 = x^2(1-x^2)$ and so the denominator is zero when $x=0$ and $x=\pm 1$. Since $1+x^4\neq 0$ at those values of x, we see that

$$\lim_{x\to 0} \frac{1+x^4}{x^2-x^4} = \infty$$

and

$$\lim_{x\to \pm 1} \frac{1+x^4}{x^2-x^4} = \infty.$$

So, $x=0$, $x=1$, and $x=-1$ are vertical asymptotes.
Secondly, we have

$$\lim_{x\to\infty} \frac{1+x^4}{x^2-x^4} = \lim_{x\to\infty} \frac{1/x^4+1}{1/x^2-1} = -1.$$

and the same result is true for the limit at $-\infty$. So, the line $y=-1$ is a horizontal asymptote.

Question 5 (10 points)

By following the steps shown in the lecture notes, sketch the graph of the function

$$f(x) = \frac{x^2+1}{x^2-1}.$$

Solution:

1. The domain of the function is $\operatorname{dom} f := \mathbb{R}\backslash\{-1, 1\}$.
2. The y-intercept is $f(0) = -1$ and there is no x-intercept because x^2+1 is never zero.
3. We can see that
$$f(-x) = \frac{x^2+1}{x^2-1} = f(x).$$
So the function is even.
4. - There is one horizontal asymptote at ∞:
$$\lim_{x\to\infty} \frac{x^2+1}{x^2-1} = 1$$
and there is another one at $-\infty$
$$\lim_{x\to\infty} \frac{x^2+1}{x^2-1} = 1.$$
So $y = 1$ is an horizontal asymptote at ∞ and $-\infty$.
   - There are two vertical asymptotes. One at $x = 1$ because
$$\lim_{x\to 1^-} \frac{x^2+1}{x^2-1} = \frac{2}{0^-} = -\infty$$
and
$$\lim_{x\to 1^+} \frac{x^2+1}{x^2-1} = \frac{2}{0^+} = \infty.$$
One at $x = -1$ because
$$\lim_{x\to -1^+} \frac{x^2+1}{x^2-1} = \frac{2}{0^-} = -\infty$$
and
$$\lim_{x\to -1^-} \frac{x^2+1}{x^2-1} \frac{2}{0^+} = \infty.$$

5. We have $f'(x) = -\frac{4x}{(x^2-1)^2}$. It is zero when $x = 0$. We see that f is increasing when $x < 0$ because $f'(x) > 0$ there and f is decreasing when $x > 0$ because $f'(x) < 0$ there. Also, the derivative doesn't exist if $x = \pm 1$. We can draw a table for the derivative:

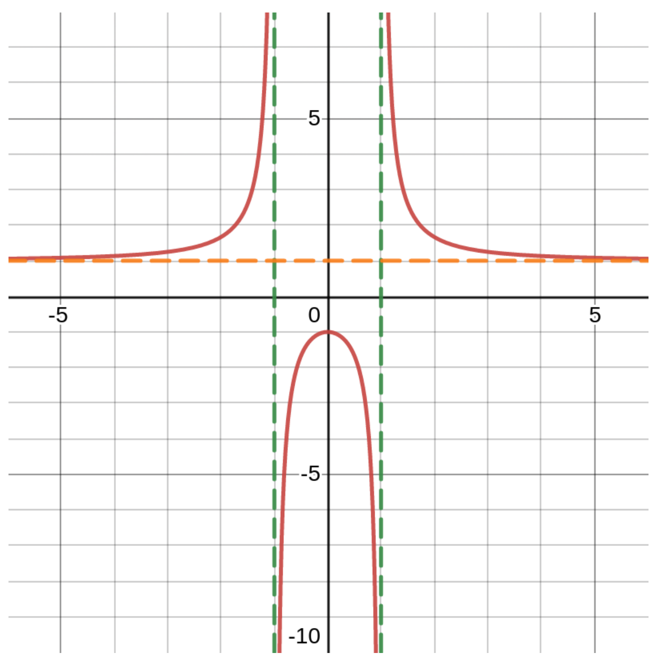
x		-1		0		1	
$-4x$	-	-	-	0	-	-	-
$\frac{1}{(x^2-1)^2}$	+	$\nexists$	+	+	+	$\nexists$	+
$A'(x)$	-	$\nexists$	-	0	+	$\nexists$	+

6. From the previous constatations, we see that $x = 0$ is a local maximum and $f(0) = -1$.

7. We find that $f''(x) = \frac{4(3x^2+1)}{(x-1)^3(x+1)^3}$. There is no zero. But, there is a change in the sign of $f''(x)$ at some points.

x		-1		0		1	
$4(3x^2+1)$	+	+	+	0	+	+	+
$\frac{1}{(x-1)^3}$	-	-	-	-	-	$\nexists$	+
$\frac{1}{(x+1)^3}$	-	$\nexists$	+	+	+	+	+
$A''(x)$	+	$\nexists$	-	0	-	$\nexists$	+

8. The graph of the function should look like this:



Question 6 (10 points)

The sum of two positive numbers is 16. What is the smallest possible value of the sum of their squares?

Solution: Let x and y be two positive numbers such that $x+y=16$. The sum of their squares, denoted by S, is x^2+y^2. Since $x+y=16$, we can write

$$S(x) = x^2 + (16-x)^2.$$

Take the derivative to find $S'(x) = 2x - 2(16-x)$. Equate this last expression to 0 to find out that $x = 8$.

Take the second derivative to find out that $S''(x) = 4$. So, $S''(8) > 0$ which implies that $x = 8$ gives a local minimum, but since $S''(x) > 0$ for any x, this gives a global minimum. So, the smallest possible value for the sum of the squares of x and y with $x = 8$ is

$$S(x) = 8^2 + 8^2 = 128.$$