Example 1 $x = y^2/2$ ← $y = \pm\sqrt{2x}$

Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$.

distance between two points

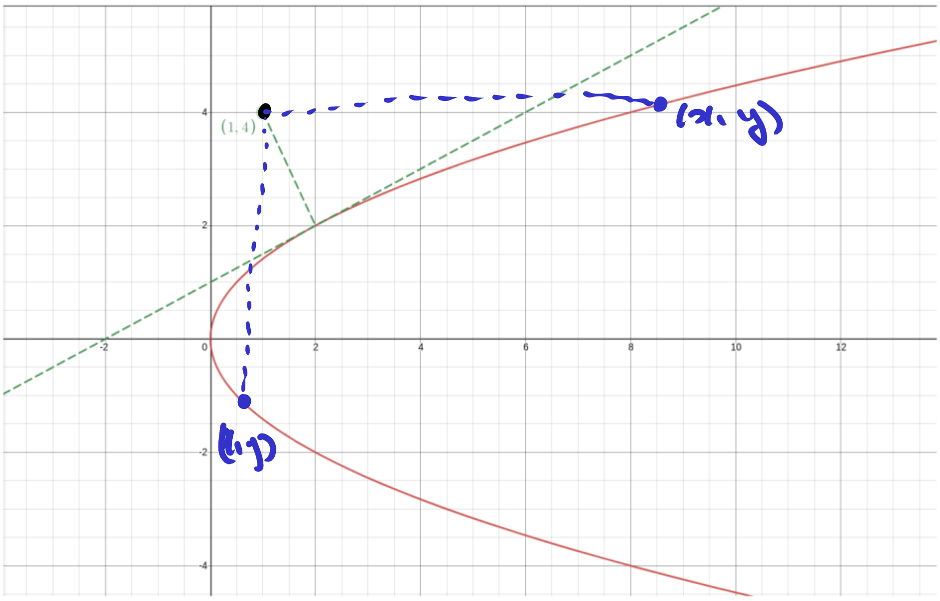
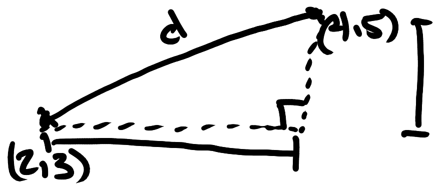
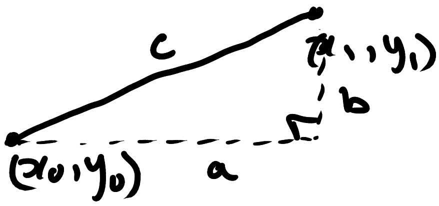


Figure: Drawing of the situation



$$d = \sqrt{(4-2)^2 + (5-3)^2}$$
$$= \sqrt{2^2 + 2^2}$$
$$= \sqrt{8}.$$

$$d = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$$



$$c = \sqrt{a^2 + b^2}$$

- (x, y) pt. on the curve.
- coordinates for the pt. not on the curve: $(1, 4)$
- Formula of the curve: $y^2 = 2x$.
- $d = \sqrt{(1-x)^2 + (4-y)^2}$

Goal: find the min of d.

Trick: $D = d^2 = (1-x)^2 + (4-y)^2$

New goal: find the minimum of D.

$$x = y^2/2 \Rightarrow D(y) = \left(1 - \frac{y^2}{2}\right)^2 + (4-y)^2$$

$$D'(y) = 2\left(1 - \frac{y^2}{2}\right) \cdot (-y) + 2(4-y) \cdot (-1)$$

$$= -\cancel{2y} + y^3 - 8 + \cancel{2y}$$

$$= y^3 - 8$$

So, $D'(y) = y^3 - 8 = 0 \iff y^3 = 8$

$\iff y = \sqrt[3]{8} = 2$

y		2	
$y^3 - 8$	$-$	0	$+$
D	$\searrow$	C.P.	$\nearrow$

So, $y = 2$ is an abs. minimum.

We know that $y = 2 \Rightarrow 2x = 2^2$

$\Rightarrow x = 2.$

$\boxed{\text{So, the point on the curve } 2x = y^2 \text{ closest to the point } (1,4) \text{ is } (2,2).}$

Example 2

Find the area of the largest rectangle that can be inscribed in a semicircle of radius 2.

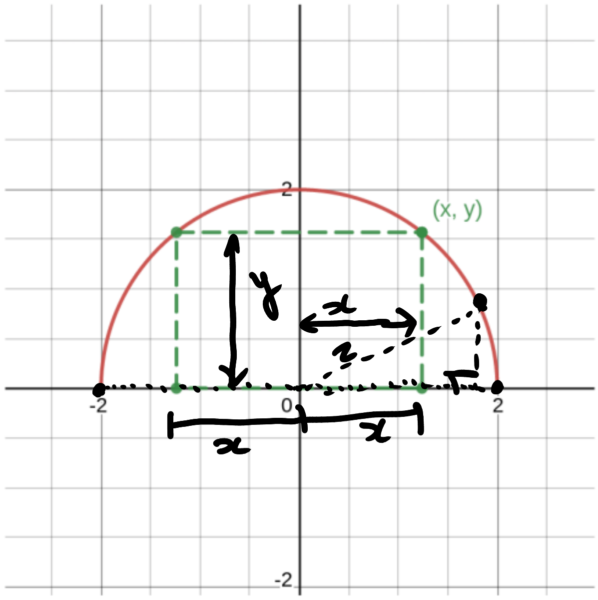


Figure: Drawing of the situation

x : ~~variable~~ ~~for width~~ → distance from the y-axis

y: variable for heigth

A : Area of the rectangle.

Info. • radius of semi-circle : 2

↳ $x^2 + y^2 = (\text{radius})^2 = 4$.

• rectangle Area: $A = (\text{width})(\text{heigth}) = 2x \cdot y$

Goal. Find the max of A.

Trick. $-2 \le x \le 2 \to 0 \le x \le 2$.

$y = \sqrt{4-x^2}$ $(y \ge 0)$

$$A = 2x \cdot y = \underbrace{2x}_{f} \underbrace{\sqrt{4-x^2}}_{g}$$

So, we have

$$A'(x) = 2\sqrt{4-x^2} + 2x \frac{-2x}{2\sqrt{4-x^2}}$$

$$= 2\sqrt{4-x^2} - 2x^2/\sqrt{4-x^2}$$

$$= \frac{2(4-x^2) - 2x^2}{\sqrt{4-x^2}}$$

$$\Rightarrow \quad A'(x) = \frac{8 - 4x^2}{\sqrt{4-x^2}} = \frac{4(2-x^2)}{\sqrt{4-x^2}}$$

So, $A'(x) = 0 \Leftrightarrow 4(2-x^2) = 0 \Leftrightarrow x = \pm\sqrt{2}$

$\Leftrightarrow x = \sqrt{2}$.

x	0		$\sqrt{2}$		2
$2-x^2$		$+$	0	$-$	
$A'(x)$		$+$	0	$-$	
$A(x)$		$\nearrow$	C.P.	$\searrow$	

So, $A(x)$ has a local. max at $x = \sqrt{2}$.

- $A(0) = 0$
- $A(\sqrt{2}) = 4 \longrightarrow$ $\boxed{\text{4 is the biggest area of the rectangle inside the semi-circle}}$
- $A(2) = 0$

Example 3

Find an approximation to the root of

$$x^5 - 2x^4 - 5x^3 + 0.1 = 0.$$

① Take an initial guess

$a = x_0 = 0.5 \quad \to \quad f(0.5) \cong -0.61875$

② Compute x_1

2.1 Tangent line T_1

$$T_1(x) = \underbrace{m}_{f'(a)}(x - a) + f(a)$$

$$f'(x) = 5x^4 - 8x^3 - 15x^2 \to f'(0.5) \cong -4.4375$$

$$T_1(x) = -4.4375(x - 0.5) - 0.61875$$

So, $T_1(x) = 0 \iff \dfrac{0.61875}{-4.4375} + 0.5 = x$

$\iff x_1 \cong 0.3605.$

③ Find x_2

Tangent line: $T_2(x) = -2.2396(x - 0.3605) - 0.1619$

So $x_2 \cong 0.2882.$

④ Find x_3 $\quad x_3 \cong 0.2657$

Example 4

Do three steps of the Newton's method to find an approximation of $\sqrt[6]{2}$. How many more interations are needed to get an approximation of $\sqrt[6]{2}$ right for 8 decimal places.

$$f(x) = x^6 - 2 = 0 \iff x^6 = 2$$
$$f'(x) = 6x^5 \qquad \iff x = \boxed{\pm\sqrt[6]{2}}$$

① Guess. $x_0 = 1$.

② Compute x_1

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$
$$= 1 - \frac{f(1)}{f'(1)} = 1 - \frac{(1^6 - 2)}{6 \cdot 1^5}$$
$$= 1 + \frac{1}{6}$$

$$\Rightarrow x_1 \approx 1.16666667.$$

③ Find x_2

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \cong 1.1264437$$

④ Find x_3

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \cong 1.1224970.$$

Midterm.

Material: Week 5 – Week 9.

Derivatives → rules (chain, product, sum, difference, quotient)

Tangent & approximations → Linear approx.

Rates of change → problems. (differentials)

Shape of a fct → increasing, decreasing, concave up, down, critical pts, max, min, sketching of graph.

Optimization → derivative to optimize.

Newton's method.