Math 644

Chapter 1

Section 1.2: Estimates

Contents

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Spring 2023

Theorem 1. [Triangle Inequality] For $z, w \in \mathbb{C}$, we have

- $\star$ $|z + w| \leq |z| + |w|$;
- $\star$ $||z| - |w|| \leq |z - w|$;
- $\star$ $||z| - |w|| \leq |z + w|$.

Proof. Prove the above inequalities.

We also have the following inequalities:

- $-|z| \leq \operatorname{Re} z \leq |z|$;
- $-|z| \leq \operatorname{Im} z \leq |z|$;
- $|z| \leq |\operatorname{Re} z| + |\operatorname{Im} z|$.

Proof. Prove the above inequalities.

Sequences

A sequence is a function $\mathbb{N} \to \mathbb{C}$. We usually denote a sequence of complex numbers by

$$(z_n)_{n=1}^{\infty} \quad \text{or} \quad \{z_n\}_{n=1}^{\infty}.$$

A shortcut notation is simply (z_n) or $\{z_n\}$.

Definition 2. A sequence (z_n) converges to some complex number a if

$$\forall \varepsilon > 0,\ \exists N = N(\varepsilon) \in \mathbb{N} \quad \text{s.t. } |z_n - a| < \varepsilon,\ \forall n \geq N.$$

When a sequence (z_n) converges to a, we denote this by

- $z_n \to a$ (as $n \to \infty$);
- $\lim_{n\to\infty} z_n = a$.

Theorem 3. Let (z_n) be a sequence.

- $z_n \to a \iff \operatorname{Re} z_n \to \operatorname{Re} a$ and $\operatorname{Im} z_n \to \operatorname{Im} a$;
- If $z_n \to a$, then $|z_n| \to |a|$.

Proof. Prove the two above statements.

Cauchy Sequences

A Cauchy sequence is a sequence (z_n) satisfying the following properties:

$$\forall \varepsilon > 0,\ \exists N = N(\varepsilon) \in \mathbb{N},\ \forall n, m \geq N, \quad |z_n - z_m| < \varepsilon.$$

Theorem 4. Let (z_n) be a sequence.

- If (z_n) is a Cauchy sequence, then (z_n) is bounded, meaning that there is a finite positive number M such that $|z_n| \leq M$ for any $n \geq 1$.
- (z_n) converges if and only if (z_n) is a Cauchy sequence.

Proof. Prove these assertions.

To each sequence (a_n) of complex numbers, we associate an infinite series

$$\sum_{n=1}^{\infty} a_n.$$

The value of $\sum_{n=1}^{\infty} a_n$ might not exists and this is why we introduce the following definition of the value of a series.

Given a sequence (a_n) of complex numbers, we define its m-th partial sums as

$$S_m := \sum_{n=1}^{m} a_n.$$

Definition 5. We say that $\sum_{n=1}^{\infty} a_n$ exists, or converge, if the sequence of partial sums (S_m) converges.

We have other important related notions of convergent series:

- A series $\sum_{n=1}^{\infty} a_n$ converges absolutely if $\sum_{n=1}^{\infty} |a_n|$ converges.
- A series $\sum_{n=1}^{\infty} a_n$ diverges if it does not converge.

Example 6.

- The series $\sum_{n=1}^{\infty} \frac{i^n}{n^2}$ converges and converges absolutely.
- The series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Theorem 7.

a) A series $\sum_{n=1}^{\infty} a_n$ converges if and only if
- $\sum_{n=1}^{\infty} \operatorname{Re} a_n$ converges and;
- $\sum_{n=1}^{\infty} \operatorname{Im} a_n$ converges.

b) A series $\sum_{n=1}^{\infty} a_n$ converges absolutely if and only if
- $\sum_{n=1}^{\infty} \operatorname{Re} a_n$ converges absolutely and;
- $\sum_{n=1}^{\infty} \operatorname{Im} a_n$ converges absolutely.

c) If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n$ converges.

d) If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$.

Proof. Prove the above assertions.

(c) $\sum_{n=1}^{\infty} |a_n|$ conv. . if $A_m := \sum_{n=1}^{m} |a_n|$

$\Rightarrow (A_m)_{m=1}^{\infty}$ is conv. $\Rightarrow (A_m)$ is Cauchy.

$\underline{m>n}$ $\quad |S_m - S_n| = \left|\sum_{k=n+1}^{m} a_k\right| \leq \sum_{k=n+1}^{m} |a_k| = |A_m - A_n|$

So, (S_m) is Cauchy $\Rightarrow$ (S_m) is conv.

$\Rightarrow$ $\sum_{n=1}^{\infty} a_n$ is conv.

(d) $\sum a_n$ conv. $\Rightarrow$ (S_m) conv.

$\Rightarrow$ (S_m) Cauchy.

Let $\varepsilon > 0$. $\exists N \in \mathbb{N}$ s.t.

$n, m \geq N$, $\quad |S_m - S_n| < \varepsilon$.

Take $m = n+1$ $\Rightarrow$ $\quad |S_m - S_n| = |a_{n+1}|$

So, $\forall n \geq N$, $\quad |a_{n+1}| < \varepsilon$.

So, $\quad a_n \to 0$. $\quad \square$

For Finite sequences

Given $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \in \mathbb{C}$, we have

$$\Big|\sum_{j=1}^{n} a_j \bar{b}_j\Big| \leq \Big(\sum_{j=1}^{n} |a_j|^2\Big)^{1/2} \Big(\sum_{j=1}^{n} |b_j|^2\Big)^{1/2}.$$

Equality occurs if and only if

- $a_j = cb_j$ for some $c \in \mathbb{C}$ or;
- $b_j = 0$ for any $j \geq 1$.

Note: Can you extend the Cauchy-Schwarz inequality to series?

if $\sum_{n=1}^{\infty} |a_n|^2 < \infty$ & $\sum_{n=1}^{\infty} |b_n|^2 < \infty$, then.

1) $\sum_{j=1}^{\infty} a_j \bar{b}_j$ conv.

2) $\Big| \sum_{j=1}^{\infty} a_j \bar{b}_j \Big| \leq \Big(\sum_{j=1}^{\infty} |a_j|^2\Big)^{1/2} \Big(\sum_{j=1}^{\infty} |b_j|^2\Big)^{1/2}.$

For Functions

- A function $f : [a, b] \to \mathbb{C}$ is said to be continuous if

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x, y \in [a, b], \quad |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

- An important fact: A function $f : [a, b] \to \mathbb{C}$ is continuous if and only if
  - $\operatorname{Re} f : [a, b] \to \mathbb{R}$ is continuous and;
  - $\operatorname{Im} f : [a, b] \to \mathbb{R}$ is continuous.

DEFINITION 8. For a continuous function $f : [a, b] \to \mathbb{C}$, we define its integral on $[a, b]$ by

$$\int_a^b f(t)\, dt := \int_a^b \operatorname{Re} f(t)\, dt + i \int_a^b \operatorname{Im} f(t)\, dt.$$

Note: The integrals of $\operatorname{Re} f(t)$ and $\operatorname{Im} f(t)$ are the Riemann integral. We will only use the Riemann integral.

THEOREM 9. If $f, g : [a, b] \to \mathbb{C}$ are two continuous functions, then

$$\left| \int_a^b f(t)\overline{g(t)}\, dt \right| \leq \Big(\int_a^b |f(t)|^2\, dt \Big)^{1/2} \Big(\int_a^b |g(t)|^2\, dt \Big)^{1/2}$$

Proof. Prove the Cauchy-Schwarz inequality for functions.

$$\int_a^b |f(t)|^2\, dt = \lim_{n\to\infty} \frac{b-a}{n} \sum_{j=1}^n |f(t_j)|^2$$

&

$$\int_a^b |g(t)|^2\, dt = \lim_{n\to\infty} \frac{b-a}{n} \sum_{j=1}^n |g(t_j)|^2$$

where $t_j := a + \frac{b-a}{n} j$, $j = 1, 2, 3, \dots, n$.

Now, similarly,

$$\int_a^b f(t)\overline{g(t)}\, dt = \lim_{n\to\infty} \frac{b-a}{n} \sum_{j=1}^n f(t_j)\overline{g(t_j)}$$

$$\frac{b-a}{n} \left| \sum_{j=1}^n f(t_j)\overline{g(t_j)} \right| \overset{\text{C.S.}}{\leq} \frac{b-a}{n} \Big(\sum_{j=1}^n |f(t_j)|^2 \Big)^{1/2} \cdot \Big(\sum_{j=1}^n |g(t_j)|^2 \Big)^{1/2}$$

$$= \Big(\frac{b-a}{n} \sum_{j=1}^n |f(t_j)|^2 \Big)^{1/2} \cdot \Big(\frac{b-a}{n} \sum_{j=1}^n |g(t_j)|^2 \Big)^{1/2}$$

Take $n \to \infty$ to obtain the inequality. $\square$