Brots without holes

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Introduction

The beauty of Maths

At the end of my master thesis, I generated an image that was processed by iterated a rickiki polynomial.

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I made a frame to install it in my living-room :D

Figure: My living-room

Introduction
Original image

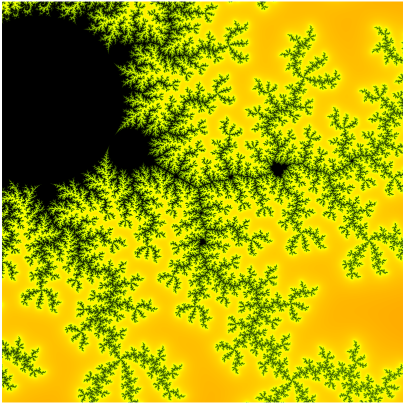

Figure: Original image

Beautiful preliminaries from Complex Analysis

Connected components: Definition

Give a set $X \subset \mathbb{C}$

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Connected components: Definition

Give a set $X \subset \mathbb{C}$ and an equivalence relation $\sim$:

$$x \sim y \iff \exists\ C \subset X \text{ connected which contains } x \text{ and } y.$$

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If X is open, then each connected components of X is open.

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Question: How many components may have a set X?

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Holomorphic Functions: Definition

By the way:

- U is a open subset of $\mathbb{C}$;
- $f : U \to \mathbb{C}$ a complex-valued function.
- f is differentiable at $z_0 \in U$ if $\frac{f(z)-f(z_0)}{z-z_0}$ converges to a certain number $f'(z_0)$ as $z \to z_0$.

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Definition 3

f is **holomorphic** in U if f is differentiable at every point of U.

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Holomorphic Functions: Examples

Some examples :

- Sums, multiplications, divisions and compositions of holomorphic functions are holomorphic;
- Polynomials are holomorphic on $\mathbb{C}$;
- sin and cos are holomorphic on $\mathbb{C}$;
- Exponential function is holomorphic on $\mathbb{C}$;
- Principal branch of log is holomorphic on $\mathbb{C}\backslash[-\infty, 0]$.

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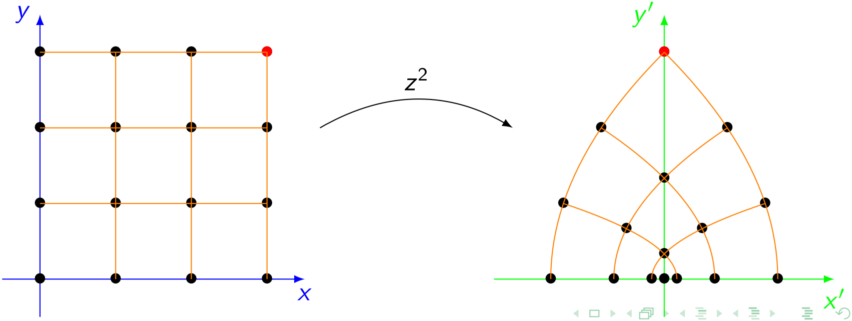
Holomorphic Functions: Picture

$$z = x + y\mathbf{i_1}$$

$$w = z^2 = x^2 - y^2 + 2xy\mathbf{i_1}$$



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Maximum Modulus Principles: Statement

Let K be the closure of a bounded connected subset U of $\mathbb{C}$.

Theorem 4

If f is holomorphic on U, non constant and continuous on K, then $|f|$ attains its maximum at some point of ∂K.

Multibrot sets
Definition

First,

- $p \geq 2$ is an integer;
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Definition 5

The Multibrot of order p is the set

$$\mathcal{M}^p := \left\{ c \in \mathbb{C} \,:\, \{f_c^m(0)\}_{m \in \mathbb{N}} \text{ is bounded} \right\}$$

Multibrot sets

Definition: Properties

Some properties :

- $\mathcal{M}^p \subset \overline{D}(0, 2^{1/(p-1)})$;
- $c \in \mathcal{M}^p \iff |f_c^m(0)| \leq 2^{1/(p-1)} \ \forall m \in \mathbb{N}$;
- $\mathcal{M}^p$ is a compact set;
- $\mathcal{M}^p$ is a connected set (that is not trivial).

Multibrot sets

Definition: Pictures

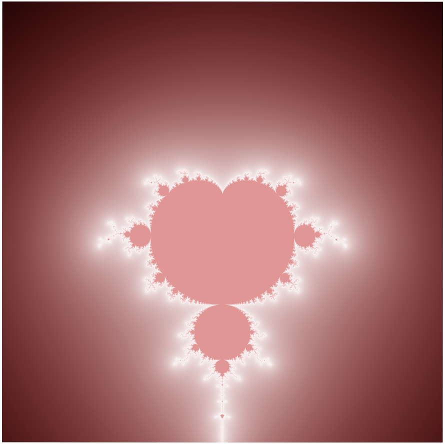

Figure: $\mathcal{M}^2$

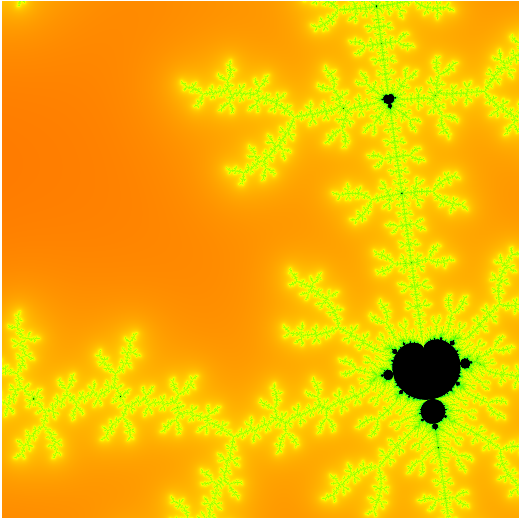

Figure: Region of $\mathcal{M}^2$

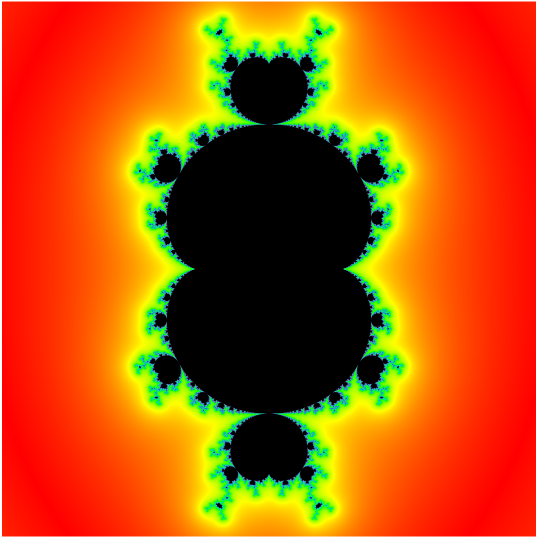

Figure: $\mathcal{M}^3$

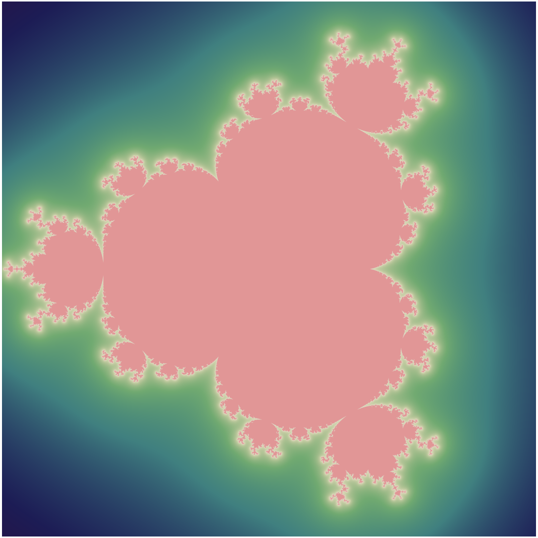

Figure: $\mathcal{M}^4$

Multibrot sets

Simply connected theorem: Statement

Now, we want to proof that

Theorem 6

The Multibrot of order p is a simply connected set

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A set X is **simply connected** if any simple closed curve in X can be reduced to a point without leaving the set.

Multibrot sets

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Theorem 6

The Multibrot of order p is a simply connected set

A set X is **simply connected** if any simple closed curve in X can be reduced to a point without leaving the set.
Loosely speaking, that means that there is no hole in the set X.

Multibrot sets

Simply connected theorem: proof without holes

We will prove the theorem by contradiction.

Multibrot sets

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- Then, from the maximum modulus principle (OMG!!) , $|f_c^m(0)|$ attains its maximum on ∂E, say at $c_0 \in \partial E$. Since $\partial E \subset \mathcal{M}^p$, $|f_{c_0}^m(0)| \leq 2^{1/(p-1)}$.

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- Thus, $\forall c \in E$, $|f_c^m(0)| \leq 2^{1/(p-1)}$ for each $m \in \mathbb{N}$. This implies that $c \in \mathcal{M}^p$. A contradiction. $\square$

Conclusion

- **Now we proved (definitely!!):**

  Complex Analysis has the most beautiful spirit.

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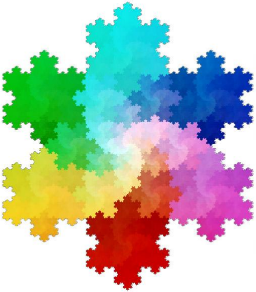
- **A citation:**

  Á cheval sur mes maths, je calope sur les plaines de la découverte

Thanks for your attention!



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- L. Carleson and T. W. Gamelin, *Complex Dynamics*, Springer-Verlag New York (1993).