MATH-244 Calculus IV
Practice Midterm 02

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Spring 2022

Last name: ______________________________

First name: ______________________________

Question:	1	2	3	4	5	Total
Points:	20	20	20	20	20	100
Score:						

Instructions: Make sure to write your complete name on your copy. You must answer all the questions below and write your answers directly on the questionnaire. At the end of the 75 minutes, hand out your copy.

No devices such as a smart phone, cell phone, laptop, or tablet can be used during the exam. You are not allowed to use the lecture notes, the textbook. You may bring one 2-sided cheat sheet of handwriting notes. You may use a digital calculator (no graphical calculator or symbolic calculator will be allowed).

You must show ALL your work to have full credit. An answer without justification worth no point.

Good luck!

Pierre-Olivier Parisé

Your Signature: ______________________________

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QUESTION 1 **(20 pts)**

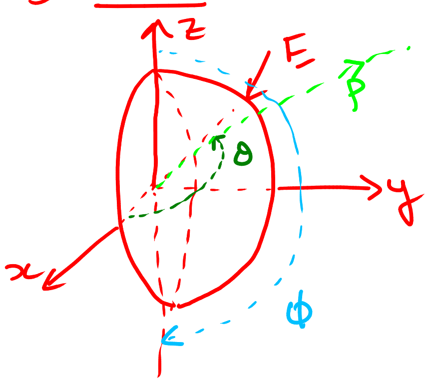
Evaluate the integral

$$\iiint_E z^2\,dV,$$

where E is the solid hemisphere

$$x^2+y^2+z^2 \le 9 \quad \text{and} \quad y \ge 0.$$

① Picture.



$E = \{(x,y,z) : x^2+y^2+z^2 \le 9,\ y \ge 0\}.$

spherical coordinates:

$E = \{(\rho,\theta,\phi) : 0 \le \rho \le 3,\ 0 \le \theta \le \pi,\ 0 \le \phi \le \pi\}$

② Integrate.

$x = \rho\cos\theta\sin\phi$
$y = \rho\sin\theta\sin\phi$
$z = \rho\cos\phi$

$dV = \rho^2\sin\phi\,d\rho\,d\theta\,d\phi$

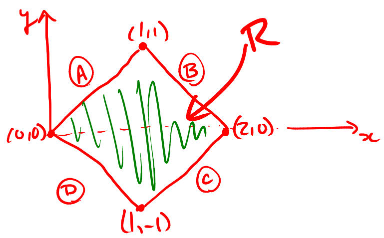
So,

$$\iiint_E z^2\,dV = \int_0^\pi\int_0^\pi\int_0^3 \rho^2\cos^2\phi\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$
$$= \left(\int_0^\pi \cos^2\phi\sin\phi\,d\phi\right)\left(\int_0^\pi d\theta\right)\left(\int_0^3 \rho^4\,d\rho\right)$$
$$= \left(\int_{-1}^1 u^2\,du\right)\pi\left(\frac{3^5}{5}\right) = \frac{2\cdot 81\pi}{5} = \boxed{\frac{162\pi}{5}}$$

Consider the square R with vertices $(0,0)$, $(1,1)$, $(2,0)$, and $(1,-1)$.

(a) (10 points) Find the image of R under the transformation

$$u = y - x \quad \text{and} \quad v = y + x.$$



Ⓐ $0 \le x \le 1$, $y = x$
Ⓑ $1 \le x \le 2$, $y = 2 - x$
Ⓒ $1 \le x \le 2$, $y = x - 2$
Ⓓ $0 \le x \le 1$, $y = -x$

image of $R = D$

Transformation.

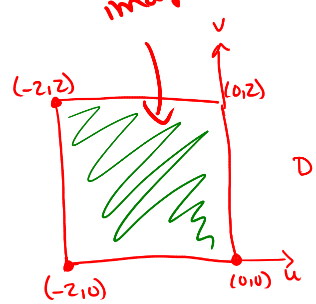
Ⓐ ($0 \le x \le 1$, $y = x$) $\to$ $u = y - x = 0 \to u = 0$; $v = x + y = 2x$, $0 \le v \le 2$

Ⓑ ($1 \le x \le 2$, $y = 2 - x$) $\to$ $u = 2 - x - x = 2 - 2x \to -2 \le u \le 0$; $v = x + 2 - x = 2$, $v = 2$

Ⓒ ($1 \le x \le 2$, $y = x - 2$) $\to$ $u = x - 2 - x = -2 \to u = -2$; $v = x + x - 2 = 2x - 2$, $-2 \le v \le 0$

Ⓓ ($0 \le x \le 1$, $y = -x$) $\to$ $u = -x - x = -2x \to -2 \le u \le 0$; $v = x - x = 0$, $v = 0$

$\to$



$D = [-2,0] \times [0,2]$

(b) (10 points) Use this transformation to evaluate the integral

$$\iint_R x \, dA.$$

Need $(u,v) \xrightarrow{T} (x,y)$. Linear algebra gives

$$x = \frac{v-u}{2} \qquad y = \frac{u+v}{2}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{2}$$

So, $$\iint_R x \, dA = \iint_D \frac{v-u}{2} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

$$= \frac{1}{2} \int_0^2 \int_{-2}^0 \frac{v-u}{2} \, du\,dv$$

$$= \frac{1}{2} \int_0^2 \frac{vu}{2} - \frac{u^2}{4} \Big|_{-2}^0 \, dv$$

$$= \frac{1}{2} \int_0^2 1 + v \, dv = \boxed{2}$$

QUESTION 3 **(20 pts)**

Consider the vector field

$$\vec{F}(x,y) := \left\langle 4x^3y^2 - 2xy^3, 2x^4y - 3x^2y^2 + 4y^3 \right\rangle.$$

(a) Is $\vec{F}$ conservative? Explain.

(b) If you answer to a) is yes, find a potention function f for $\vec{F}$.

(c) Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$, where C is the curve parametrized by

$$\vec{r}(t) := \langle t + \sin(\pi t), 2t + \cos(\pi t) \rangle, \quad 0 \le t \le 1.$$

a) $P_y = 8x^3y^2 - 6xy^2$ $\qquad Q_x = 8x^3y - 6xy^2$

$P_y = Q_x \Rightarrow \vec{F}$ is conservative.

b) $\vec{F} = \vec{\nabla} f \Rightarrow \begin{cases} 4x^3y^2 - 2xy^3 = f_x \\ 2x^4y - 3x^2y^2 + 4y^3 = f_y \end{cases}$

Integrate $\rightarrow$ $f(x,y)$ "=" $x^4y^2 - x^2y^3$

$f(x,y)$ "=" $x^4y^2 - x^2y^3 + y^4$.

$\rightarrow$ $f(x,y) = x^4y^2 - x^2y^3 + y^4 + C$

c) FTLI $\rightarrow$ $\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{\nabla} f \cdot d\vec{r} = f(\vec{r}(1)) - f(\vec{r}(0))$

$\vec{r}(1) = \langle 1, 1 \rangle$

$\vec{r}(0) = \langle 0, 1 \rangle$

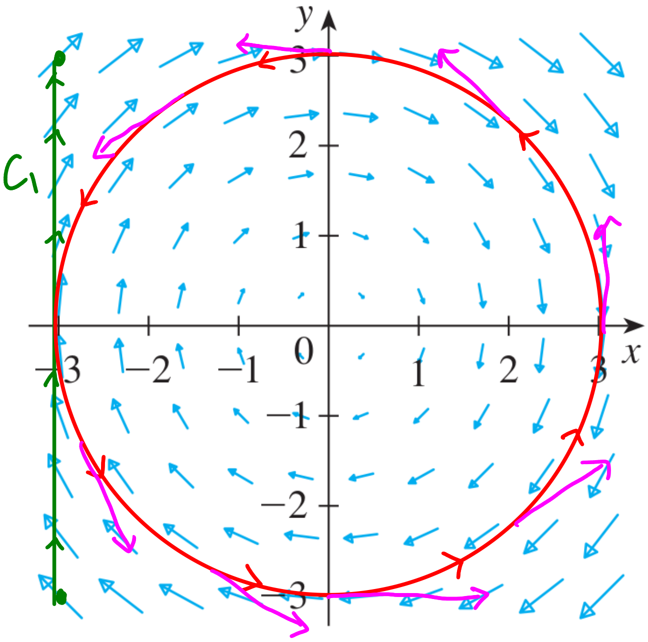
$\rightarrow \int_C \vec{F} \cdot d\vec{r} = f(1,1) - f(0,1)$

$= 1 + C - (1 + C)$

$= \boxed{0}$

Question 4 **(20 pts)**

Let $\vec{F}$ be the vector field shown in the figure below



(a) (10 points) If C_1 is the vertical line segment from $(-3,-3)$ to $(-3,3)$, determine whether $\int_{C_1} \vec{F}\cdot d\vec{r}$ is positive, negative, or zero.

By definition, $\int_{C_1} \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}\cdot\vec{r}\,'(t)\,dt$

The tangent vector $\vec{r}\,'(t)$ is always pointing upward $\uparrow$

The angle between $\vec{r}\,'(t)$ & the vectors of $\vec{F}$ is always between 0 & $\pi/2$, so $\cos\theta \geq 0$.

So, $\vec{F}\cdot\vec{r}\,'(t) = |\vec{F}|\,|\vec{r}\,'|\cos\theta$ is always positive

$\Rightarrow \int_{C_1} \vec{F}\cdot d\vec{r} > 0$.

(b) (10 points) If C_2 is the counterclockwise-oriented circle with radius 3 and center the origin, determine whether $\int_{C_2} \vec{F}\cdot d\vec{r}$ is positive, negative, or zero.

Again, by definition, $\int_{C_2} \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}\cdot\vec{r}\,'(t)\,dt$.

We see that the tangent vector to the circle ($\vec{r}\,'(t)$ in color ■) are always in the opposite direction of the vectors in $\vec{F}$. So, if a particle is moving along C_2 in $\vec{F}$, it is moving against $\vec{F}$. This means that overall, $\int_C \vec{F}\cdot d\vec{r} < 0$ (the work will be negative).

QUESTION 5 **(20 pts)**

Suppose you're asked to determine the curve that requires the least work for a force field $\vec{F}$ to move a particle from one point to another point. You decide to check first whether $\vec{F}$ is conservative, and indeed it turns out that it is. How would you reply to the request?

Since $\vec{F}$ is conversative, the line integral of $\vec{F}$ is indepent of the path, this means

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$$

for any two curves starting at A & B.

Since the line integral represent the work, we would say that any curve will do!