MATH-241 Calculus 1 — Pierre-Olivier Parisé
Homework 03 — Fall 2021

Assigned date: 09/27/2021 9am
Due date: 10/04/2021 5pm

Last name: ______________________
First name: ______________________
Section: ______________________

Question:	1	2	3	4	5	6	Total
Points:	20	20	20	10	15	15	100
Score:							

Instructions: You must answer all the questions below and upload your solutions (in a PDF format) to Gradescope (go to `www.gradescope.com` with the Entry code GEK6Y4). Be sure that after you scan your copy, it is clear and readable. You must name your file like this:`LASTNAME_FIRSTNAME.pdf`. A homework may not be corrected if it's not readable and if it's not given the good name. No other type of files will be accepted (no PNG, no JPG, only PDF) and no late homework will be accepted. Good luck!

Question 1 **(20 points)**

Find the derivative of the following functions.

(a) (10 points) $f(x) = \frac{\cos x}{1+\cos x}$.

Solution: We have

$$\begin{aligned}\frac{d}{dx}\left(\frac{\cos x}{1+\cos x}\right) &= \frac{\frac{d}{dx}(\cos x)(1+\cos x) - \cos x \frac{d}{dx}(1+\cos x)}{(1+\cos x)^2} \\ &= \frac{-\sin x - \sin x \cos x + \cos x \sin x}{(1+\sin x)^2} \\ &= \frac{-\sin x}{(1+\sin x)^2}\end{aligned}$$

(b) (10 points) $f(x) = x \tan x$.

Solution: We have

$$\frac{d}{dx}(x \tan x) = \frac{d}{dx}(x)\tan x + x\frac{d}{dx}(\tan x) = \tan x + x\sec^2 x.$$

Question 2 (20 points)

Use the chain rule to find the derivative of the following functions.

(a) (10 points) $f(x) = (\frac{x}{x+1})^2$.

Solution: The inner function is $x/(x+1)$ and the outer function is x^2. So, using the chain rule, we get

$$\begin{aligned} f'(x) &= 2\left(\frac{x}{x+1}\right)^{2-1} \frac{d}{dx}\left(\frac{x}{x+1}\right) \\ &= \frac{1}{2}\left(\frac{x}{x+1}\right)\frac{d}{dx}\left(\frac{x}{x+1}\right). \end{aligned}$$

Now, we have $\frac{d}{dx}(\frac{x}{x+1}) = \frac{(x+1)-x}{(x+1)^2} = \frac{1}{(x+1)^2}$. Thus, we get

$$f'(x) = 2\left(\frac{x}{x+1}\right)\left(\frac{1}{(x+1)^2}\right) = \frac{2x}{(x+1)^3}.$$

(b) (10 points) $f(x) = \cos(\sin(2x))$.

Solution: We have three functions: $\cos t$ (the biggest box), $\sin t$ (the box in the middle), and $2t$ (the smallest box). Call each function $g(t) = \cos t$, $h(t) = \sin t$,and $k(t) = 2t$. So, by the chain rule (applied with three functoins), we get

$$\begin{aligned} f'(x) = g'(h(k(x)))h'(k(x))k'(t) &= -\sin(\sin(2x))\cos(2x)2 \\ &= -2\sin(\cos(2x))\cos(2x). \end{aligned}$$

Question 3 **(20 points)**

Use implicit differentiation to find the tangent line at the point $P = (3,3)$ to the curve[1] $y = y(x)$ given by the implicit equation

$$x^4 + y^3 = 36y.$$

Solution: By implicit differentiation, we can find an expression for y'. Differentiate each side of the implicit equation gives

$$4x^3 + 3y^2y' = 36y'$$

and so

$$4x^3 = (36 - 3y^2)y' \quad \Rightarrow \quad y' = \frac{4x^3}{36 - 3y^2}.$$

Plugging in the coordinates of the point $P = (3,3)$ in the last formula, we get

$$y' = \frac{108}{36 - 27} = \frac{108}{9} = 12.$$

The equation of the tangent line is $T(x) = mx + b$ with $m = 12$. So, $T(x) = 12x + b$. Knowing that $T(3) = 3$, we obtain the following equation that we can solve for b

$$3 = 36 + b \quad \Rightarrow \quad b = -33.$$

Thus, $T(x) = 12x - 33$.

[1] Use Desmos to get a sketch of the curve... The top portion of the curve looks like the shape of a Fuji apple (Miam)! What do you think, do you agree?

Question 4 (10 points)

The quantity of charge Q in coulombs (C) that has passed through a point in a wire up to time t (measured in seconds and $t \geq 0$) is given by $Q(t) = t^3 - \frac{3}{2}t^2 + 6t + 2$. The current I that passes through the wire is the derivative with respect to time t of the charge function.

(a) (5 points) Find the current at time $t = 0.5$ s.

Solution: We have that $I(t) = \frac{d}{dt}Q(t)$. Since $Q(t) = t^3 - \frac{3}{2}t^2 + 6t + 2$, we get

$$I(t) = 3t^2 - 3t + 6.$$

So, $I(0.5) = \frac{3}{4} - \frac{3}{2} + 6 = \frac{27}{4}$.

(b) (5 points) Find the possible time when the current is zero.

Solution: We have to solve the equation $I(t) = 0$, that is $3t^2 - 3t + 6 = 0$. We have $3t^2 - 3t + 6 = (3t + 3)(t - 2)$. So,

$$I(t) = 0 \quad \Longleftrightarrow \quad (3t + 3)(t - 2) = 0 \quad \Longleftrightarrow \quad t = -1 \text{ or } t = 2.$$

So, since time must be positive, we get $t = 2$.

Question 5 **(15 points)**

The height of a triangle is increasing at a rate of 2 cm/min while the area of the triangle is inscreasing at a rate of 2 cm^2 /min. At what rate is the base of the triangle changing when the altitude is 10 cm and the area is 10.

Solution: Let h be the height of the triangle, A be the area of the triangle, and b be the base of the triangle. Then, we have

$$A = \frac{bh}{2}.$$

Differentiate this last expression with respect to t and we get

$$2\frac{dA}{dt} = \frac{db}{dt}h + b\frac{d}{dt}h.$$

From the assumptions, we know that $\frac{dA}{dt} = 2$, $\frac{dh}{dt} = 2$, and $h = 10$. So, $b = \frac{2}{10} = 1/5$ and we get

$$4 = \frac{db}{dt}10 + (1/5)2$$

which is

$$\frac{9}{25} = \frac{db}{dt}.$$

So the rate of the base of the triangle is $\frac{9}{25}$ cm/min.

Question 6 (15 points)

(a) (10 points) Use a linear approximation (or differentials) to estimate the value of $\sqrt{4.1}$.

Solution: Take $\tilde{x} = x + dx = 4 + 0.1$. So $dx = 0.1$ and $x = 4$. Put $f(x) = \sqrt{x}$. Then we know $df = f'(x)dx$. Since $f'(x) = \frac{1}{2\sqrt{x}}$ and $x = 4$, we get

$$dx = \frac{1}{2\sqrt{4}}0.1 = 0.025.$$

So, using a linear approximation with the differential, we get

$$\sqrt{4.1} \approx f(2) + f'(2)dx = 2 + 0.025 = 2.025.$$

(b) (5 points) Compare your approximation with the value given by your computer.

Solution: Our approximation is quite good. The value given on a calculator is 2.0248456731. It's about ≈ 0.0015 off the value from your calculator.