D.I Distribution Function

PROBLEM 1. Let F_1 and F_2 be two distribution functions. Show that the function $F(x) = \alpha F_1(x) + (1-\alpha)F_2(x)$ is a distribution function, for any α satisfying $0 \leq \alpha \leq 1$.

PROBLEM 2. Given a random variable X, express the distribution function of $Y = \max\{0, X\}$ is terms of the distribution function of X.

PROBLEM 3. For which value of c is the function

$$F(x) = c\int_{-\infty}^{x} e^{-|t|}\,dt \quad (x \in \mathbb{R})$$

a distribution function?

D.II Continuous Random Variable

PROBLEM 4. If X has distribution function

$$F(x) = \begin{cases} \dfrac{1}{2(1+x^2)} & -\infty < x \leq 0 \\ \dfrac{1+2x^2}{2(1+x^2)} & 0 < x < \infty \end{cases}$$

Assuming that X is a continuous random variable, find the density function of X.

PROBLEM 5. Let the density function of a random variable X be given by

$$f_X(x) = \begin{cases} \dfrac{2}{\pi(1+x^2)} & -1 \leq x \leq 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Find the distribution function of f_X.

PROBLEM 6. A random variable X has density function

$$f_X(x) = cx(x-1) \quad (0 \leq x \leq 1)$$

and 0 elsewhere. Determine c so that F_X is a distribution function.

D.III Functions of Random Variables

PROBLEM 7. Let X be a random variable with the exponential distribution with parameter λ. Find the density function of

a) $Y = 2X + 5$.

b) $Y = (1+X)^{-1}$.

PROBLEM 8. Let X be a random variable whose distribution function F is a continuous function. Show that the random variable Y, defined by $Y = F(X)$, is uniformly distributed on the interval $(0, 1)$.

PROBLEM 9. The random variable X is uniformly distributed on the interval $[0, 1]$. Find the distribution and probability density function Y, where

$$Y = \frac{3X}{1-X}.$$

D.IV Expectation of Continuous Random Variables

PROBLEM 10. Find the expectation of the random variable X given in Problem 5.

PROBLEM 11. Find the expectation and variance of X given in Problem 6.

D.V Other Examples of Continuous Random Variables

PROBLEM 12. If $Z \sim N(0, 1)$ is a random variable that has the standard normal distribution, what is

a) $P(Z^2 < 1)$?

b) $P(Z^2 < 3.84146)$?

PROBLEM 13. A soft-drink machine can be regulated so that it discharges an average of μ ounces per cup. If the ounces of fill are normally distributed with stardard deviation 0.3 ounce, give the setting for μ so that 8-ounce cups will overflow only 1% of the time.

PROBLEM 14. Prove that

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}.$$

PROBLEM 15. If X has the normal distribution with mean 0 and variance 1, find the mean value of $Y = e^{2X}$.

PROBLEM 16. Show that if X has the normal distribution with parameters 0 and 1, then $Y = X^2$ has the χ^2 distribution with one degree of freedom.

PROBLEM 17. Suppose that X has an exponential distribution with parameter λ. Show that, if $a > 0$ and $b > 0$, then

$$P(X > a + b | X > a) = P(X > b).$$