Section 3.4 — Problem 21 — 5 points

We multiply by the conjugate:

$$(\sqrt{9x^2+x}-3x)\Big(\frac{\sqrt{9x^2+x}+3x}{\sqrt{9x^2+x}+3x}\Big) = \frac{x}{\sqrt{9x^2+x}+3x}$$

and then factor x:

$$\frac{x}{\sqrt{9x^2+x}+3x} = \frac{x}{\sqrt{x^2}\sqrt{9+1/x}+3x}.$$

Since we are taking the limit as $x \to \infty$, we have $\sqrt{x^2} = x$. Therefore,

$$\lim_{x\to\infty}(\sqrt{9x^2+x}-3x) = \lim_{x\to\infty}\frac{x}{x\sqrt{9+1/x}+3x} = \lim_{x\to\infty}\frac{1}{\sqrt{9+1/x}+3}.$$

Since $\lim_{x\to\infty}\frac{1}{x} = 0$, then by the rule for limits, we get

$$\lim_{x\to\infty}(\sqrt{9x^2+x}-3x) = \frac{1}{\sqrt{9}+3} = \frac{1}{6}.$$

Section 3.4 — Problem 30 — 5 points

We factor x^2:

$$x^2 - x^4 = x^2(1 - x^2).$$

Therefore, since $\lim_{x\to\infty} x^2 = \infty$ and $\lim_{x\to\infty}(1 - x^2) = -\infty$, we obtain

$$\lim_{x\to\infty}(x^2 - x^4) = \lim_{x\to\infty} x^2 \lim_{x\to\infty}(1 - x^2) = \infty(-\infty) = -\infty.$$

Section 3.4 — Problem 46 — 5 points

We need the function to have a non-zero numerator and a zero denominator at $x = 1$ and $x = 3$. A good function for that would be

$$\frac{1}{(x-1)(x-3)}.$$

This last expression, however, does not have a horizontal asymptote $y = 1$ because

$$\lim_{x\to\infty} \frac{1}{(x-1)(x-3)} = 0$$

and

$$\lim_{x\to-\infty} \frac{1}{(x-1)(x-3)} = 0.$$

To change this, we can add 1 to the previous expression. The desire function is therefore

$$f(x) = 1 + \frac{1}{(x-1)(x-3)}.$$

Section 3.4 — Problem 63 — 5 points

We will use the Squeeze Theorem. First, we have

$$\lim_{x\to\infty} \frac{4x-1}{x} = 4.$$

Secondly, we have

$$\lim_{x\to\infty} \frac{4x^2+3x}{x^2} = 4.$$

Therefore, the function f is squeeze between two functions that have 4 as their limit at ∞. Using the Squeeze Theorem, we conclude that

$$\lim_{x\to\infty} f(x) = 4.$$

$\boxed{\textbf{Section 3.7 — Problem 2 — 10 points}}$

Let x and y be those two numbers. We therefore have

$$x - y = 100.$$

We want to minimize the product of x and y. From the equation above, we have

$$y = x - 100.$$

The function to optimize is

$$P(x) = xy = x(x - 100) = x^2 - 100x.$$

The derivative is

$$P'(x) = 2x - 100$$

and so $P'(x) = 0$ if $x = 50$. The second derivative is

$$P''(x) = 2.$$

Since $P''(x) > 0$ for any x, we conclude that $x = 50$ is an absolute minimum.

The numbers whose product is minimum are therefore $x = 50$ and $y = -50$.

Section 3.7 — Problem 14 — 15 points

Let

- V: volume of the box.
- b: width of the base.
- h: height of the box.
- A: total area of the box.

The equation of the volume of the box is

$$V = b^2 h$$

and since the volume is $32,000\text{cm}^3$, we therefore have

$$h = \frac{32000}{b^2}.$$

The function A is the total area of the box. The box has a square base of side length b with no top and four rectangular sides of sides length b and h. Therefore, the total area is

$$A = b^2 + 4bh$$

and using the expression of h, we obtain

$$A(b) = b^2 + \frac{128000}{b^2} = b^2 + (128000)b^{-2}.$$

We want to mimimize A. We have

$$A'(b) = 2b - \frac{256000}{b^3} = \frac{2b^4 - 256000}{b^3}.$$

We therefore have

$$A'(b) = 0 \iff 2b^4 - 256000 = 0 \iff b^4 = 128000.$$

Taking the fourth-root, we obtain

$$b = \pm\sqrt[4]{2^7 \times 10^3} = \pm\frac{40}{\sqrt[4]{20}}.$$

The length of the base is always positive, so we keep $b = 40/\sqrt[4]{20}$ Taking the second derivative, we have

$$A''(b) = 2 + \frac{768000}{b^4}.$$

We see that $A''(b) > 0$ for any b. Therefore, the number $b = 40/\sqrt[4]{20}$ minimize the function A (so minimize the amount of material used).

Using the link between b and h, we find that

$$b = \frac{40}{\sqrt[4]{20}}\,\text{cm} \quad \text{and} \quad h = \frac{32000 \cdot \sqrt{20}}{40^2} = 40\sqrt{5}\,\text{cm}.$$

Section 3.8 — Problem 6 — 5 points

We set $f(x) = 2x^3 - 3x^2 + 2$. The derivative of the function is

$$f'(x) = 6x^2 - 6x.$$

Newton's algorithm gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

We start we $x_1 = -1$. We have

$$f(x_1) = -3 \quad \text{and} \quad f'(x_1) = 12.$$

Therefore, we obtain

$$x_2 = -1 + \frac{3}{12} = -1 + \frac{1}{4} = -\frac{3}{4}.$$

We now use x_2 to obtain x_3. We have

$$f(x_2) = -0.53125 \quad \text{and} \quad f'(x_2) = 63/8.$$

Therefore, we obtain

$$x_3 = -\frac{3}{4} + \frac{0.53125}{63/8} \approx -0.6825$$

TOTAL (POINTS): 50.