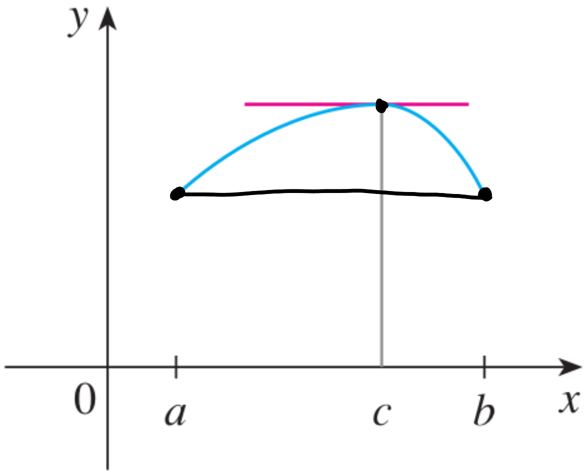
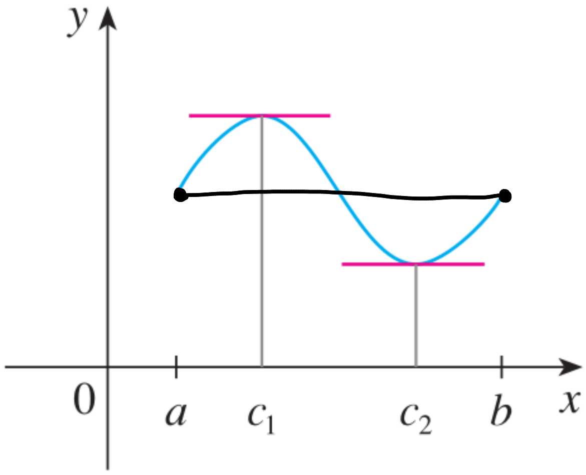
Chapter 3
Applications of Derivatives

3.2 The Mean Value Theorem

The following graphs have a commun geometric property.



(b)



(c)

Is there a condition that garantees the graph of a function has horizontal tangents?

Rolle's Theorem Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
3. $f(a) = f(b)$

Then there is a number c in (a, b) such that $f'(c) = 0$.

EXAMPLE 2 Prove that the equation $x^3 + x - 1 = 0$ has exactly one real root.

IVT:

$f(x) = x^3 + x - 1$ → continuous.

$f(0) = -1 < 0$ & $f(1) = 1 > 0$

So, there must be a c s.t.

$c^3 + c - 1 = 0$ (by the IVT)

Rolle's Thm:

Suppose there is another $c_2 < c$ s.t.

$c_2^3 + c_2 - 1 = 0$

$\Rightarrow$ $f(c) = 0$ & $f(c_2) = 0$

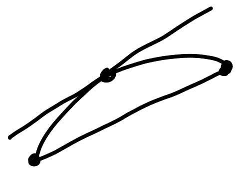
$\Rightarrow$ by Rolle's Thm, there is a $c_2 < d < c$ s.t. $f'(d) = 0$

$f'(x) = \underbrace{3x^2}_{\geq 0} + 1 \geq 1 \Rightarrow$ can't be zero

contradiction.

The Mean Value Theorem Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).



Then there is a number c in (a, b) such that

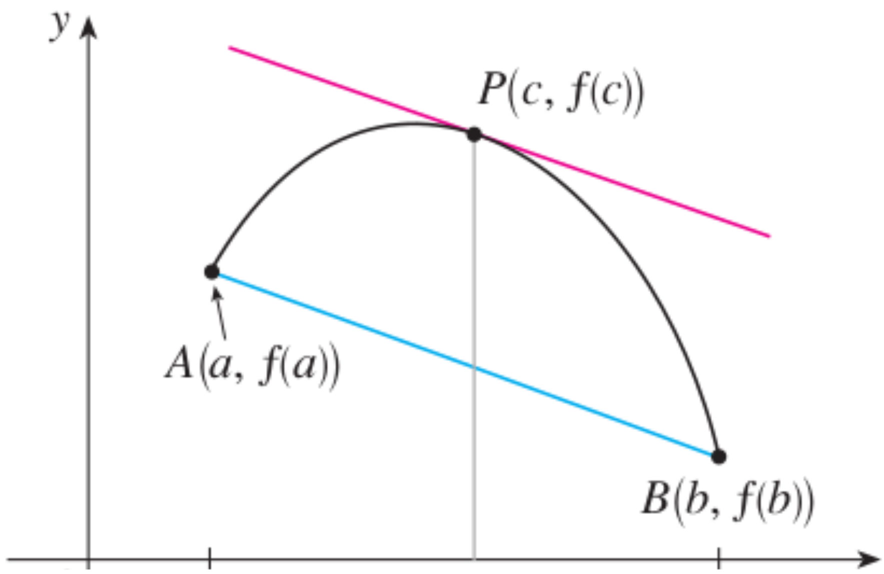
1 slope of tangent line. ← $$f'(c) = \frac{f(b) - f(a)}{b - a}$$ → slope of the secant line

or, equivalently,

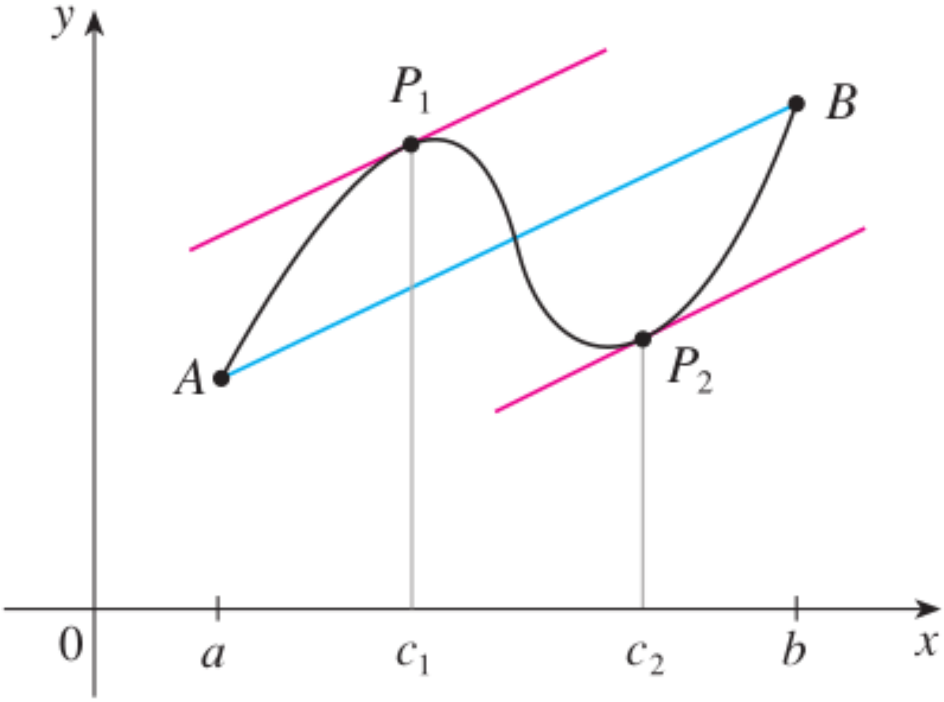
2 $$f(b) - f(a) = f'(c)(b - a)$$

The Meaning: Find an equivalent slope of the secant line with the slope of one of the tangent line.

Only one c.



Multiple c.



Example

Let $f(x) = \sqrt{x}$. Find the number c that satisfies the conclusion of the Mean Value Theorem on the interval [0, 4].

$a = 0$
$b = 4$

Goal: Find c s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Derivative: $f'(x) = \frac{1}{2\sqrt{x}}$.

So,

$$\frac{1}{2\sqrt{c}} = \frac{\sqrt{4} - \sqrt{0}}{4 - 0}$$

$$\Rightarrow \quad \frac{1}{2\sqrt{c}} = \frac{2 - 0}{4 - 0}$$

$$\Rightarrow \quad \frac{1}{2\sqrt{c}} = \frac{1}{2}$$

$$\Rightarrow \quad 1 = \sqrt{c} \quad \Rightarrow \quad \boxed{c = 1}$$

Consequences of the Mean Value Theorem.

5 Theorem If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).

7 Corollary If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b); that is, $f(x) = g(x) + c$ where c is a constant.

EXAMPLE 5 Suppose that $f(0) = -3$ and $f'(x) \leqslant 5$ for all values of x. How large can $f(2)$ possibly be?

Assume that f is cont & diff.

Take $a=0$ & $b=2$ in the MVT.

$\Rightarrow \quad f(2) - f(0) = f'(c)\,(b-a)$

for some c between $a=0$ & $b=2$.

So,

$$f(2) - f(0) = \underbrace{f'(c)}_{\leq 5}\,(2-0)$$

$$\leq 5 \cdot 2 = 10$$

$$\Rightarrow \quad f(2) - \underbrace{f(0)}_{-3} \leq 10$$

$$\Rightarrow \quad f(2) + 3 \leq 10$$

$$\Rightarrow \quad \boxed{f(2) \leq 7}$$