## Question 1 (10 pts)

If a spherical snowball melts so that its surface area decreases at a rate of $1\,\text{cm}^2/\text{min}$, find, using **Calculus**, the rate at which the diameter decreases when the diameter is 10 cm.

*Note: The surface area of a sphere is $A = 4\pi r^2$.*

4 pts for "assembling" info

$$A = 4\pi r^2$$

$$\frac{dA}{dt} = -1 \text{ cm}^2/\text{min}$$

$r$ = radius of snowball

$D$ = diameter = $2r$

$$A = \pi D^2$$ 1 pt

$$\frac{dA}{dt} = 2\pi D \frac{dD}{dt}$$ 3 pts

Substitute

$$-1 = 2\pi(10)\frac{dD}{dt}$$

$$\frac{-1}{20\pi} = \frac{dD}{dt}$$ 2 pts

## QUESTION 2 **(10 pts)**

Let $f(x) = \sqrt{1+x}$.

(a) (5 points) Find the linearization of the function $f$ at the point $a = 0$.

$$L(x) = f(a) + f'(a)(x-a)$$

$a = 0$

$$L(x) = f(0) + f'(0)(x-0)$$

2 pts

$f(0) = 1$

$$f'(0) = \frac{1}{2}(1+0)^{-1/2} = \frac{1}{2}$$

2 pts

$$L(x) = 1 + \frac{1}{2}x$$

(b) (5 points) Using the linearization, estimate the value of $\sqrt{1.1}$. Explain clearly how you obtained your answer and leave it in decimal form.

Realize that we are being asked for $f(0.1)$.

2pts

$$f(0.1) \approx L(0.1) = 1 + \frac{1}{2}(0.1)$$

$$= 1.05$$

3 pts

## QUESTION 3 **(22 pts)**

Let $f(x) = \dfrac{3x^2 - 3}{x^2 + 3}$.

(a) (4 points) Using **Calculus**, find the vertical asymptotes (if any) and horizontal asymptotes (if any) of the function $f(x)$.

$x^2 + 3 > 0$ for all $x$, so no vertical asymptotes +1pt

$$\lim_{x \to \infty} \frac{3x^2 - 3}{x^2 + 3} = \lim_{x \to \infty} \frac{(3x^2 - 3) \cdot \frac{1}{x^2}}{(x^2 + 3) \cdot \frac{1}{x^2}} \overset{+1pt}{=} \lim_{x \to \infty} \frac{3 - \frac{3}{x^2}}{1 + \frac{3}{x^2}} = \frac{3}{1} = 3 \quad +1pt$$

$\frac{3}{x^2} \to 0$

Same when $x \to -\infty$, so only horizontal asymptote is

$\boxed{y = 3}$ +1pt

(b) (4 points) The first derivative of $f$ is $f'(x) = \frac{24x}{(x^2+3)^2}$. Find the critical numbers (if any) and the open interval(s) of increase and decrease.

$(x^2 + 3)^2 > 0$ for all $x$, so ignore. +1pt

$24x = 0$ when $x = 0$, so $x = 0$ is the only critical number +1pt

$-$ $+$

0

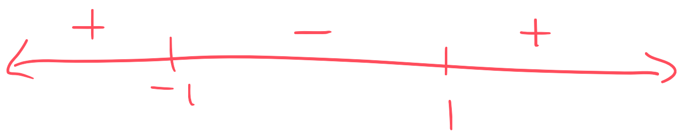
int. of decrease: $(-\infty, 0)$ +1pt

int. of increase: $(0, \infty)$ +1pt

(c) (6 points) The second derivative of $f$ is $f''(x) = \frac{-72(x^2-1)}{(x^2+3)^3}$. Find the $x$-coordinate of the inflection points (if any) and the open interval(s) of concavity.

$(x^2+3)^3 > 0$ for all $x$, so ignore +1pt

$-72(x^2-1) = 0$ when $x = \pm 1$. +2pts




if $x < -1$, $-72(x^2-1) > 0$

if $-1 < x < 1$, $-72(x^2-1) < 0$

if $x > 1$, $-72(x^2-1) > 0$

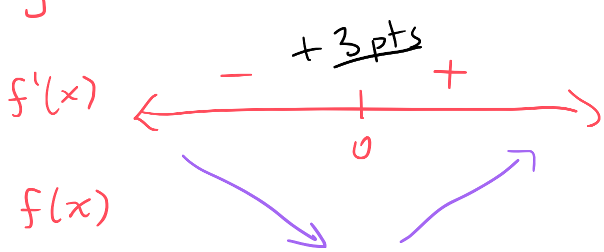
Thus, $x = 1$, $x = -1$ are $x$-values of inflection points.

int. of concave up: $(-\infty, -1), (1, \infty)$ +2pts

int. of concave down: $(-1, 1)$ +1pt

(d) (4 points) Using one of the derivative tests, find the local maximum(s) and/or local minimum(s) of the function.

Using First Derivative Test:




$f$ must have loc. min at $x = 0$ +1pt

$\therefore$ loc. min is $f(0) = -1$.

— OR —

Using Second Derivative Test:

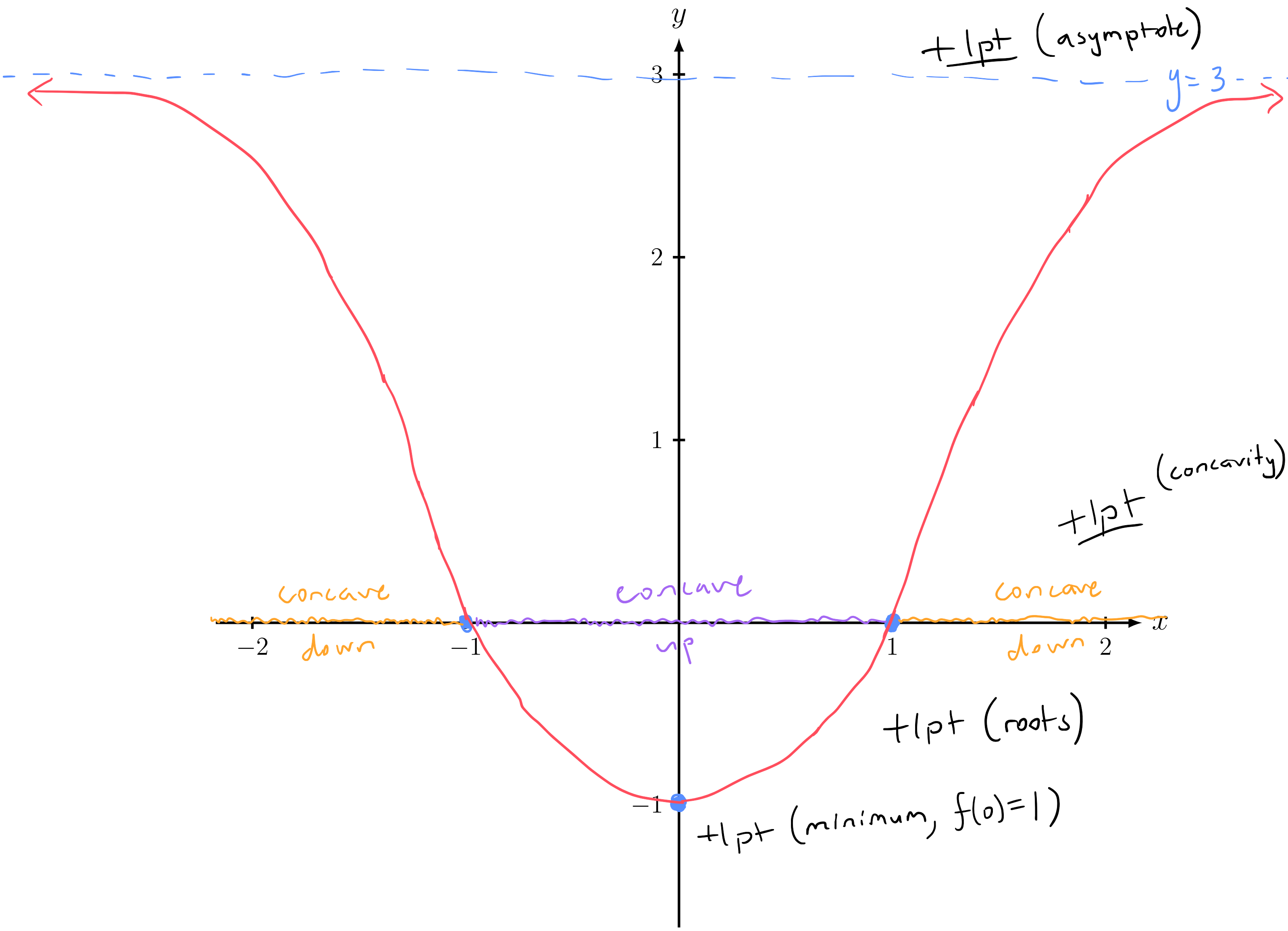
critical number: $x = 0$.

$f''(0) < 0$ implies $f(x)$ has loc. min at $x = 0$. +3pts

$\therefore$ loc. min is $f(0) = -1$. +1pt

*...Question 3 continued...*

(e) (4 points) Sketch the graph of the function $f$ in the axes below. Note that the $y$-intercept is $-1$ and the $x$-intercepts are $x = -1$ and $x = 1$.



QUESTION 4 **(10 pts)**

Compute the following limits. If the limit does not exist, write explicitly DNE. Make sure to write all the details of your calculations.

(a) (5 points) $\lim_{x\to\infty} \frac{3x-2}{2x+1}$.

$$\lim_{x\to\infty} \frac{3x-2}{2x+1} = \lim_{x\to\infty} \frac{x(3-2/x)}{x(2+1/x)}$$ (2 pts)

$$= \lim_{x\to\infty} \frac{3-2/x}{2+1/x}$$ (1 pt)

$$= \frac{3-0}{2+0}$$ (1 pt)

$$= 3/2$$ (1 pt)

No points if just $\frac{3x}{2x} = \frac{3}{2}$ is used. Up to you, though

(b) (5 points) $\lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5}$.

$$\lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to\infty} \frac{\sqrt{x^2}\sqrt{2+1/x^2}}{x(3-5/x)}$$ (1 pt)

$$= \lim_{x\to\infty} \frac{|x|\sqrt{2+1/x^2}}{x(3-5/x)}$$ (1 pt)

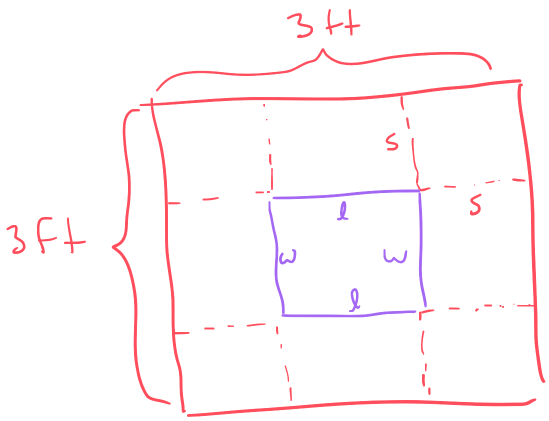
$$= \lim_{x\to\infty} \frac{-x\sqrt{2+1/x^2}}{x(3-5/x)}$$ (1 pt)

$$= \lim_{x\to-\infty} \frac{-\sqrt{2+1/x^2}}{3-5/x}$$ (1 pt)

$$= \frac{-\sqrt{2+0}}{3-0} = -\frac{\sqrt{2}}{3}$$ (1 pt)

## QUESTION 5 (15 pts)

A box with an open top is to be constructed from a square piece of cardboard of side length 3 ft by cutting out a square from each of the four corners and bending up the sides. **Using calculus**, find the largest volume that such a box can have. Make sure to justify clearly your answer.




+1pt (picture with labels)

$V = l \cdot w \cdot h$ +1pt (volume formula)

$l = 3-2s$

$w = 3-2s$ +2pts (variables in terms of s)

$h = s$

$V(s) = (3-2s)(3-2s)s$ +1pt (substitution in V formula)

$= 4s^3 - 12s^2 + 9s$

$V'(s) = 12s^2 - 24s + 9$ +2pts (derivative)

$0 = 12s^2 - 24s + 9$

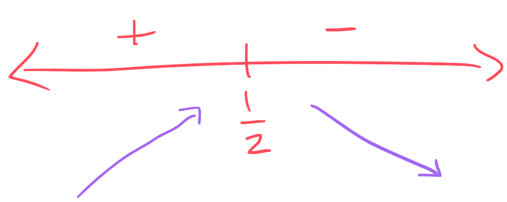
quadratic formula or factor to get +2pts (solving technique)

$s = \frac{1}{2}$ and $s = \frac{3}{2}$ +1pt (roots)

outside of domain of problem (only want $0 < s < \frac{3}{2}$) +1pt

$V(0) = 9 > 0$

$V(1) = 12 - 24 + 9 < 0$




...check it's a maximum

+2pts

$\therefore$ Largest volume is $V(\frac{1}{2}) = 2\ ft^3$ +1pt (units!)

+1pt (finding $V(\frac{1}{2})$)

## QUESTION 6 (15 pts)

Find the most general antiderivative of the following functions.

(a) (5 points) $f(x) = 4\sqrt{x} - 6x^2 + 3$.

$$\therefore \int f(x)\,dx = \int (4\sqrt{x} - 6x^2 + 3)\,dx$$

$$= 4\int x^{1/2}dx - 6\int x^2 dx + 3\cdot\int 1\cdot dx \Big\}\ \text{1-pt}$$

$$= 4\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} - 6\frac{x^{2+1}}{2+1} + 3x + C \Big\}\ \text{each term 1-pt}$$

$$= \boxed{\frac{8}{3}x^{3/2} - 2x^3 + 3x + C} \leftarrow \text{1 pt (final answer + "C")}$$

(b) (5 points) $f(x) = \cos(x) + 2\sec^2(x)$.

$$\therefore \int f(x)\,dx = \int \cos(x) + 2\sec^2(x)\,dx$$

$$= \int \cos(x)\,dx + 2\int \sec^2(x)\,dx \Big\}\ \text{2 pts}$$

$$= \boxed{\sin(x) + 2\tan(x) + C} \leftarrow \text{2 pt correct expression, 1 pt for the "+C"}$$

1 pt 1 pt 1 pt

(c) (5 points) $f(x) = x\sqrt{x} + \dfrac{x^2+x}{x}$.

$$= x\sqrt{x} + x + 1 \Big\}\ \text{1 pt for algebraic Simplification}$$

$$\therefore \int f(x)\,dx = \int (x\sqrt{x} + x + 1)\,dx$$

$$= \int x\cdot x^{1/2}dx + \int x\,dx + \int 1\,dx \Big\}\ \text{1 point for separation}$$

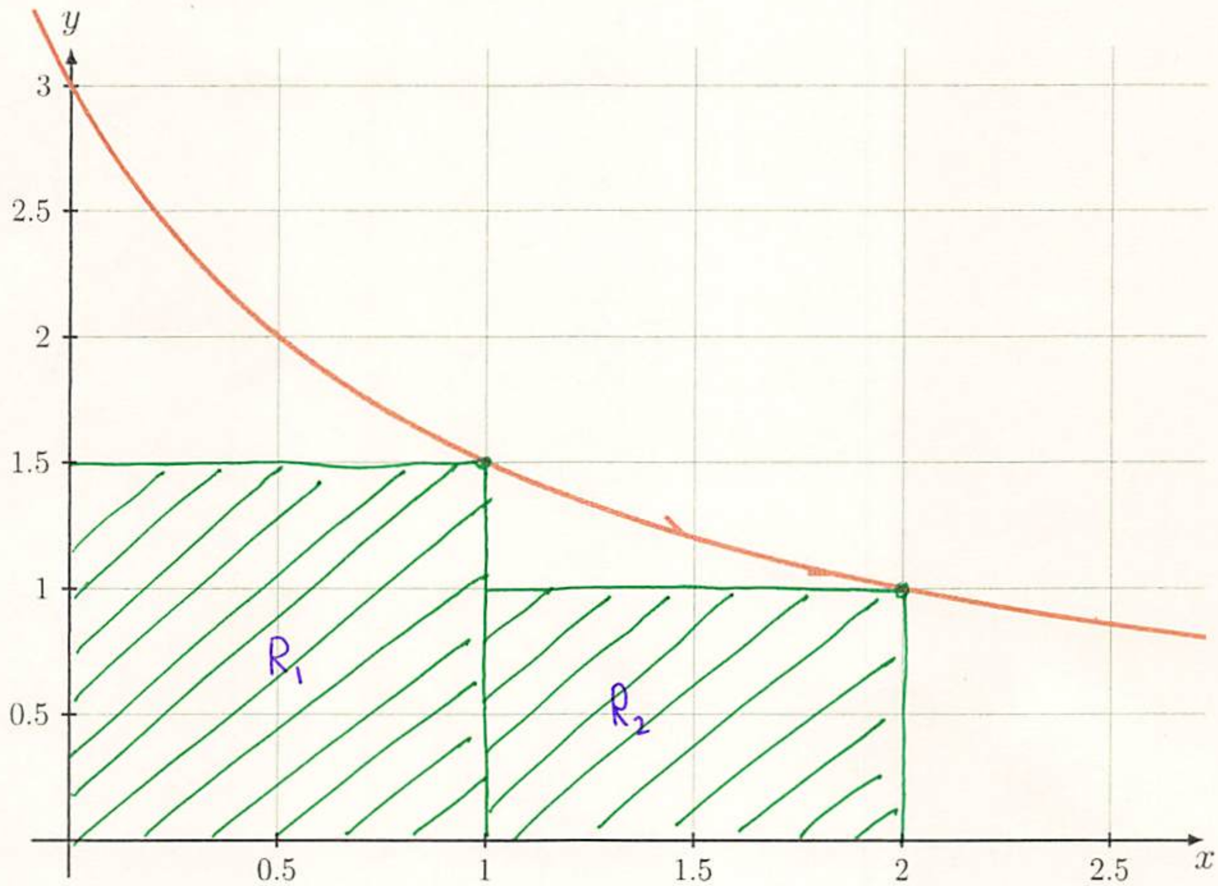
$$= \int x^{3/2}dx + \frac{1}{2}x^2 + x + C$$

$$= \frac{x^{3/2+1}}{3/2+1} + \frac{1}{2}x^2 + x + C = \boxed{\frac{2}{5}x^{5/2} + \frac{1}{2}x^2 + x + C}$$

1 pt 1 pt 1 pt

QUESTION 7 **(8 pts)**

The graph of the function $f(x) = \dfrac{3}{1+x}$ is given below.




(a) (4 points) Estimate $\int_0^2 f(x)\,dx$ using two rectangles and right endpoints.

$$\int_0^2 f(x)\,dx \simeq R_1 + R_2$$

$$= \Delta x\, f(1) + \Delta x\, f(2)$$

$$= \Delta x\,[f(1) + f(2)] \quad \text{or} \quad 1 \text{ pt}$$

$$= \underbrace{1 \cdot \left[\tfrac{3}{2} + 1\right] = \boxed{\tfrac{5}{2}}}_{1 \text{ pt}}$$

$$\left.\begin{aligned} \Delta x &= \frac{2-0}{2} = 1 \\ f(1) &= \frac{3}{1+1} = \frac{3}{2} \\ f(2) &= \frac{3}{1+2} = 1 \end{aligned}\right\} 2 \text{ pts.}$$

(b) (2 points) Draw the two rectangles from part (a) on the above picture of the graph of $f(x)$. 1 pt each Rectangle – Make sure RH top point sits on graph.

(c) (2 points) Is your answer over or under approximating the actual value of the integral?

under estimate of true value of Integral as Rectangles below graph.
2 pts

## Question 8 (10 pts)

Answer the following questions.

(a) (5 points) Using a comparison property for the definite integral, find a lower bound and an upper bound for the value of the following definite integral:

$$\int_0^1 \sqrt{1+3x}\, dx.$$

$f(x)=\sqrt{1+3x}$

$f'(x)=\dfrac{3}{2\sqrt{1+3x}} > 0$ for $x$ in $[0,1]$

$m=f(0)=\sqrt{1+0}=1$ (1 pt)

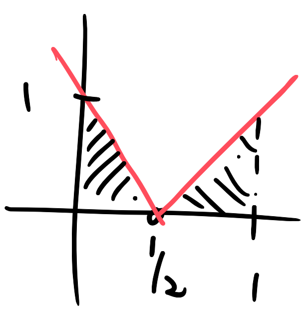
$M=f(1)=\sqrt{1+3}=2$ (1 pt)

(1 pt) $1 \leq \sqrt{1+3x} \leq 2 \Rightarrow 1\cdot(1-0) \leq \int_0^1 \sqrt{1+3x}\,dx \leq 2(1-0)$ (1 pt)

$1 \leq \int_0^1 \sqrt{1+3x}\,dx \leq 2$ (1 pt)

(b) (5 points) Using a geometric approach, find the value of the following definite integral:

$$\int_0^1 |2x-1|\, dx.$$




$\int_0^1 |2x-1|\,dx$

$= \underbrace{\frac{1}{2}\cdot\frac{1}{2}\cdot 1}_{2\text{pts}} + \underbrace{\frac{1}{2}\cdot\frac{1}{2}\cdot 1}_{2\text{pts}}$

$= \frac{1}{2}$ (1 pt)