Example 2

Using the sumbol $\sum$, write the sum $1 + 2 + 3 + 4$.

$$\sum_{n=k}^{m} a_n \quad , \qquad a_n = n$$

$$1+2+3+4 = a_1 + a_2 + a_3 + a_4$$

$$= \sum_{n=1}^{4} a_n$$

$$= \sum_{n=1}^{4} n$$

Example 3

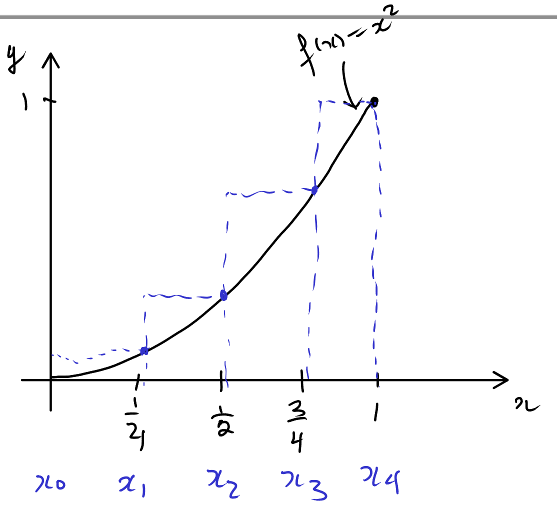
Using the symbol $\sum$, write the sum $1 + 1/2 + 1/3 + 1/4$.

$$S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \qquad \sum_{n=k}^{m} a_n \quad , \quad a_n = \frac{1}{n}$$

$$S = \underset{\uparrow \atop k}{a_1} + a_2 + a_3 + \underset{\uparrow \atop m}{a_4} = \sum_{n=1}^{4} a_n = \sum_{n=1}^{4} \frac{1}{n} .$$

Example 10

Approximate the area under the curve $y = x^2$ for $x \in [0, 1]$ with 4 rectangles.



$$\Delta x = \frac{1-0}{4} = \frac{1}{4}$$

$$n = 4 \qquad x_k = 0 + k \cdot \Delta x$$

$$x_0 = 0$$

$$x_1 = 0 + 1 \cdot \frac{1}{4} = \frac{1}{4}$$

$$x_2 = 0 + 2 \cdot \frac{1}{4} = \frac{1}{2}$$

$$x_3 = 0 + 3 \cdot \frac{1}{4} = \frac{3}{4}$$

$$x_4 = 0 + 4 \cdot \frac{1}{4} = 1$$

$$R_4 = \sum_{k=1}^{4} f(\underbrace{0 + k\Delta x}_{x_k})\, \Delta x = \sum_{k=1}^{4} f(k/4)\, \frac{1}{4}$$

$$= \sum_{k=1}^{4} \frac{1}{4} \left(\frac{k}{4}\right)^2$$

$$= \frac{1}{4} \sum_{k=1}^{4} \frac{k^2}{16}$$

$$= \frac{1}{64} \sum_{k=1}^{4} k^2$$

$$= \frac{1}{64} \frac{(4+1)(2 \cdot 4 + 1)(4)}{6}$$

$$= \frac{1}{64} \frac{5 \cdot 9 \cdot 4}{6}$$

$$= \frac{180}{64 \cdot 6}.$$

Example 12

Prove that $\lim_{n\to\infty} R_n = \frac{1}{3}$.

n rectangles

$f(x) = x^2$

$[a, b] = [0, 1]$

$\Delta x = \frac{1}{n}$

$x_k = 0 + k\Delta x = \frac{k}{n}$

$$R_n = \sum_{k=1}^{n} f(k/n)\,\Delta x$$
$$= \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 \cdot \frac{1}{n}$$
$$= \sum_{k=1}^{n} \frac{k^2}{n^3}$$
$$= \frac{1}{n^3} \sum_{k=1}^{n} k^2$$
$$= \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$$

$$\lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3} = \lim_{n\to\infty} \frac{n(2n^2+3n+1)}{6n^3}$$
$$= \lim_{n\to\infty} \frac{2n^3+3n^2+n}{6n^3} = \frac{2}{6} = \frac{1}{3}$$