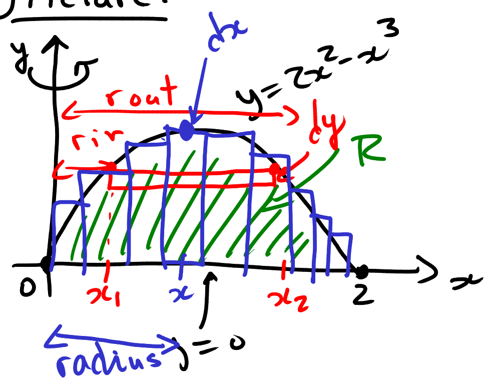
Chapter 5
Applications in integration

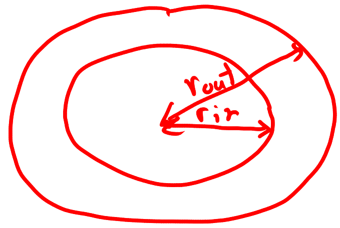
5.3 Volumes by Cylindrical Shells

DIY.

EXAMPLE 1 Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$.

1 Picture.

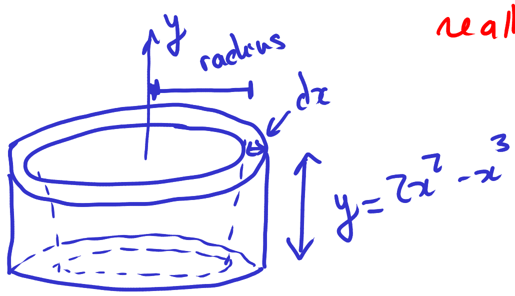




$r_{in} = x_1 \longrightarrow$ isolate x_1 in $\underbrace{y = 2x_1^2 - x_1^3}_{\text{really complicated}}$.

$r_{out} = x_2$

Rotate  $\longrightarrow$



radius $= x$

height $= y = 2x^2 - x^3$

thickness $= dx$

2 Volume.

$$V = \int_0^2 2\pi x(2x^2 - x^3)\,dx$$

$$= \int_0^2 2\pi(2x^3 - x^4)\,dx$$

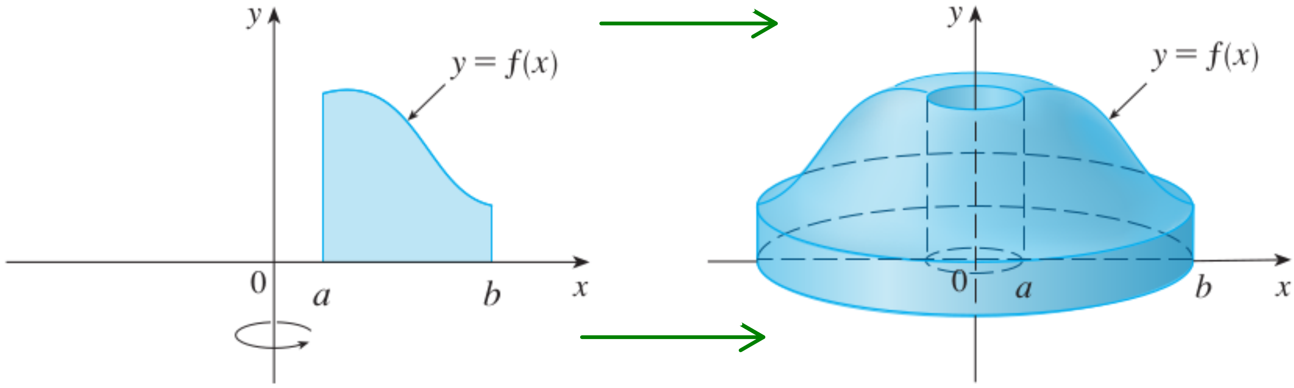
$$= 2\pi \int_0^2 2x^3\,dx - 2\pi \int_0^2 x^4\,dx$$

$$= 4\pi \int_0^2 x^3\,dx - 2\pi \left.\frac{x^5}{5}\right|_0^2$$

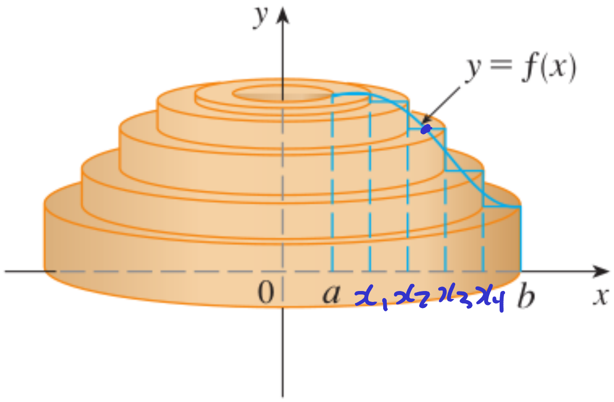
$$= \cancel{4}\pi \left.\frac{x^4}{\cancel{4}}\right|_0^2 - 2\pi\left(\frac{32}{5}\right)$$

$$= \boxed{\pi\, 16 - \frac{2\pi}{5}\,32}$$

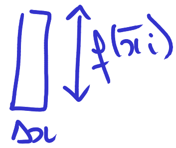
Method with Cylindrical Shells. (Rotation about the y-axis)



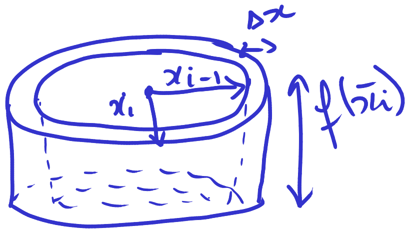
Approximation by spherical Shells.



Divide $[a,b]$ in n subintervals of length Δx.
Let $\bar{x}_i$ be the midpoint of each subinterval $[x_{i-1}, x_i]$
Create a rectangle:



Rotate rectangle:



$\longrightarrow$

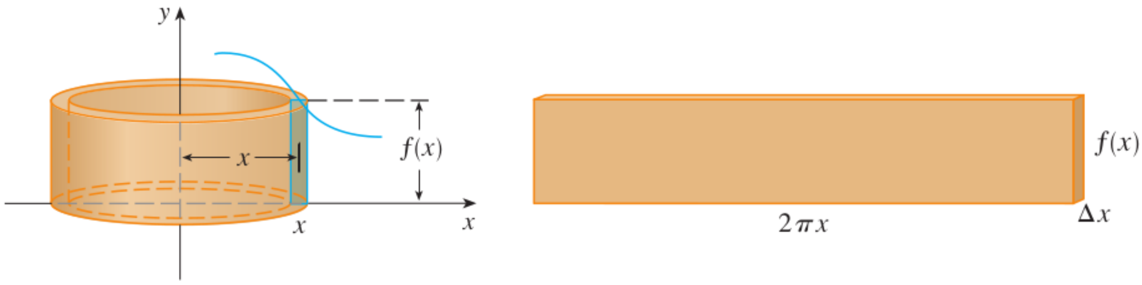
$$
\begin{aligned}
Vol_i &= f(\bar{x}_i)\cdot \pi x_i^2 - f(\bar{x}_i)\, \pi x_{i-1}^2 \\
&= f(\bar{x}_i)\, \pi \left(x_i^2 - x_{i-1}^2\right) \\
&= f(\bar{x}_i)\, \pi\, (x_i + x_{i-1})(x_i - x_{i-1}) \\
&= f(\bar{x}_i)\ 2\pi \bar{x}_i \qquad \Delta x
\end{aligned}
$$

$$\Rightarrow \text{Tot Vol} \cong \sum_{i=1}^{n} Vol_i = \sum_{i=1}^{n} \underbrace{2\pi \bar{x}_i f(\bar{x}_i)}_{2\pi x f(x)} \Delta x$$

2 The volume of the solid in Figure 3, obtained by rotating about the y-axis the region under the curve $y = f(x)$ from a to b, is

$$V = \int_a^b 2\pi x f(x)\, dx \qquad \text{where } 0 \leqslant a < b$$

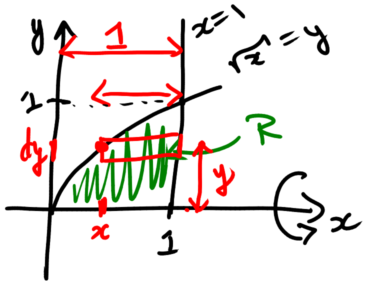
$$\int_a^b \underbrace{(2\pi \overset{\text{radius}}{x})}_{\text{circumference}}\ \underbrace{[f(x)]}_{\text{height}}\ \underbrace{dx}_{\text{thickness}}$$

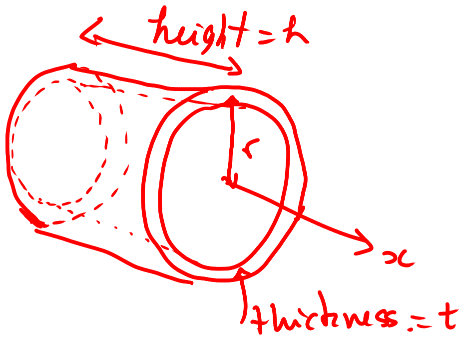


Rotation about the x-axis. (Find the radius of the cylindrical shell)

EXAMPLE 3 Use cylindrical shells to find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.

① Picture.





$r = y$

$h = 1 - x = 1 - y^2$

$t = dy$

② Volume.

$$V = \int_0^1 2\pi(\text{radius})\,\text{height}\,dy$$

$$= \int_0^1 2\pi\, y\,(1-y^2)\,dy$$

$$= 2\pi \int_0^1 y - y^3\,dy$$

$$= 2\pi \int_0^1 y\,dy - 2\pi \int_0^1 y^3\,dy$$

$$= \cancel{2}\pi\, \frac{y^2}{\cancel{2}}\Big|_0^1 - \cancel{2}\pi\, \frac{y^4}{2\cancel{4}}\Big|_0^1$$

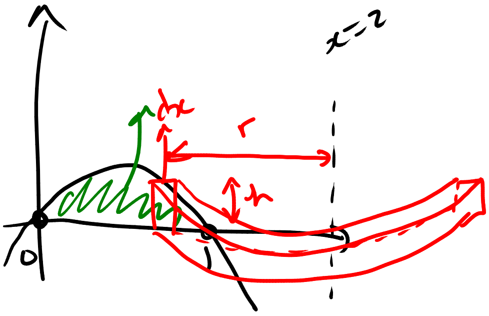
$$= \pi - \frac{\pi}{2}(1)$$

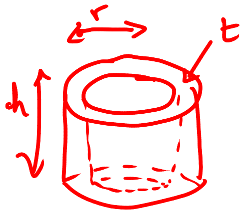
$$= \boxed{\frac{\pi}{2}}$$

Rotation about another axis.

EXAMPLE 4 Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 2$.

① Picture.





$r = 2 - x$

$h = y = x - x^2$

$t = dx$

② Volume.

$$V = \int_0^1 2\pi r h t = \int_0^1 2\pi (2-x)(x-x^2)\, dx$$

$$= 2\pi \int_0^1 2x - 2x^2 - x^2 + x^3\, dx$$

$$= 2\pi \int_0^1 2x - 3x^2 + x^3\, dx$$

$$= 2\pi \int_0^1 2x\, dx - 2\pi \int_0^1 3x^2\, dx + 2\pi \int_0^1 x^3\, dx$$

$$= 4\pi \int_0^1 x\, dx - 6\pi \int_0^1 x^2\, dx + 2\pi \left. \frac{x^4}{4} \right|_0^1$$

$$= \overset{2}{\cancel{4}}\pi \left. \frac{x^2}{\cancel{2}} \right|_0^1 - \overset{2}{\cancel{6}}\pi \left. \frac{x^3}{\cancel{3}} \right|_0^1 + \frac{\pi}{2}$$

$$= 2\pi - 2\pi + \frac{\pi}{2}$$

$$= \boxed{\frac{\pi}{2}}$$