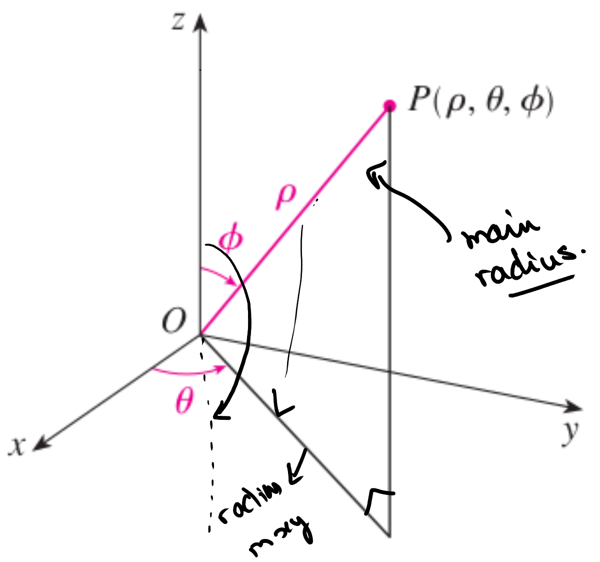
15.8 Integrals in spherical coordinates.



Basic settings.

ρ : distance from P to O.

θ : angle between the x-axis and the "radius in xy". ($0 \le \theta \le 2\pi$).

ϕ : angle between the z-axis and the main radius. ($0 \le \phi \le \pi$)

Relationships with cartesian coordinates.

Project ρ on xy-plane: $r = \rho \sin\phi$

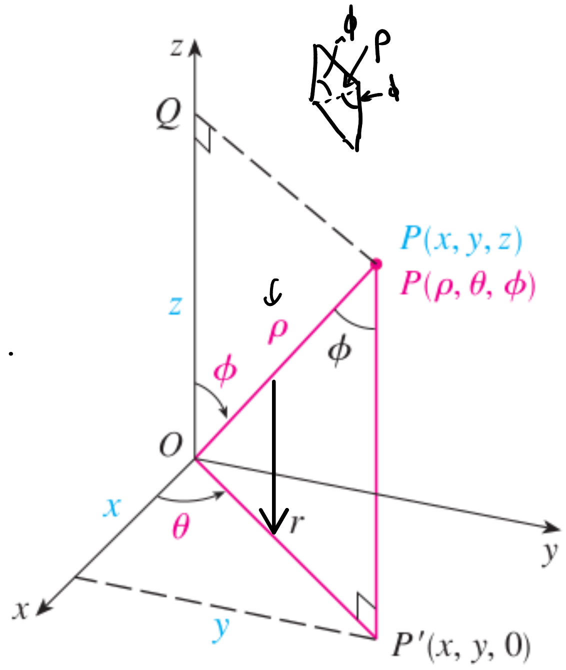
$x = r\cos\theta = \rho \sin\phi \cos\theta$

$y = r \sin\theta = \rho \sin\phi \sin\theta$

Now,

$z = \rho \cos\phi$

Spherical coord. to cart. coord.



EXAMPLE 1 The point $(2, \pi/4, \pi/3)$ is given in spherical coordinates. Plot the point and find its rectangular coordinates.

$\rho = 2$
$\theta = \pi/4$
$\phi = \pi/3$

$x = \rho \sin\phi \cos\theta = 2\cos(\pi/4)\sin(\pi/3)$
$= 2\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \sqrt{\frac{3}{2}}$

$y = \rho \sin\phi \sin\theta = 2\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)$
$= 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \sqrt{\frac{3}{2}}$

$z = \rho\cos\phi = 2\cos(\pi/3) = 2(1/2) = 1$

$(2, \pi/4, \pi/3) \longrightarrow (\sqrt{3/2}, \sqrt{3/2}, 1)$.

EXAMPLE 2 The point $(0, 2\sqrt{3}, -2)$ is given in rectangular coordinates. Find spherical coordinates for this point.

$x \quad y \quad z$ cartesian

① $\rho = \sqrt{x^2 + y^2 + z^2}$

② $z = \rho\cos\phi \to \cos\phi = \frac{z}{\rho}$

③ $x = \rho \sin\phi \cos\theta \to \cos\theta = \frac{x}{\rho\sin\phi}$

$y = \rho \sin\phi \sin\theta \to \sin\theta = \frac{y}{\rho\sin\phi}$.

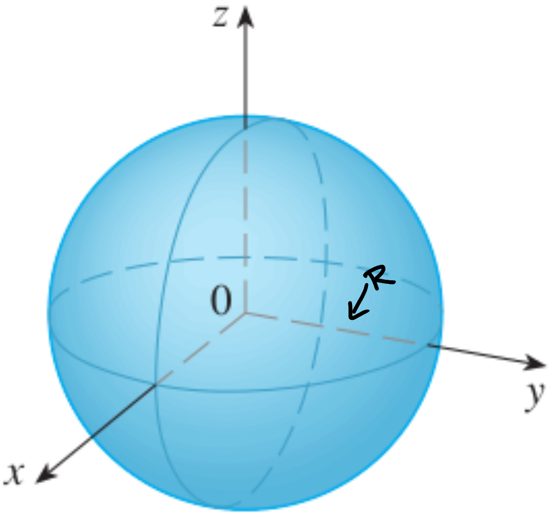
① $\rho = \sqrt{0^2 + (2\sqrt{3})^2 + (-2)^2}$
$= \sqrt{4 \cdot 3 + 4} = \sqrt{16} = 4$.

② $\cos\phi = \frac{-2}{4} = -\frac{1}{2} \to \phi = \frac{2\pi}{3}$.

③ $\cos\theta = \frac{0}{\rho\sin\phi} = 0 \to \boxed{\theta = \frac{\pi}{2}}$

because $y > 0$.

Important solids' equations.



Sphere.

$$x^2+y^2+z^2 = R^2$$

$x = \rho \sin\phi \cos\theta$ $\qquad$ $z = \rho \cos\phi$

$y = \rho \sin\phi \sin\theta$

$$\rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta + \rho^2 \cos^2\phi = R^2$$

$$\rightarrow \rho^2 \sin^2\phi\,(1) + \rho^2 \cos^2\phi = R^2$$

$$\rightarrow \rho^2 (\underbrace{\sin^2\phi + \cos^2\phi}_{=1}) = R^2$$

$$\rightarrow \boxed{\rho = R}$$

$$y = (\tan c)\, x$$

Half planes.

$$ax + by = 0 \qquad \tan c = -\frac{a}{b}$$

$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

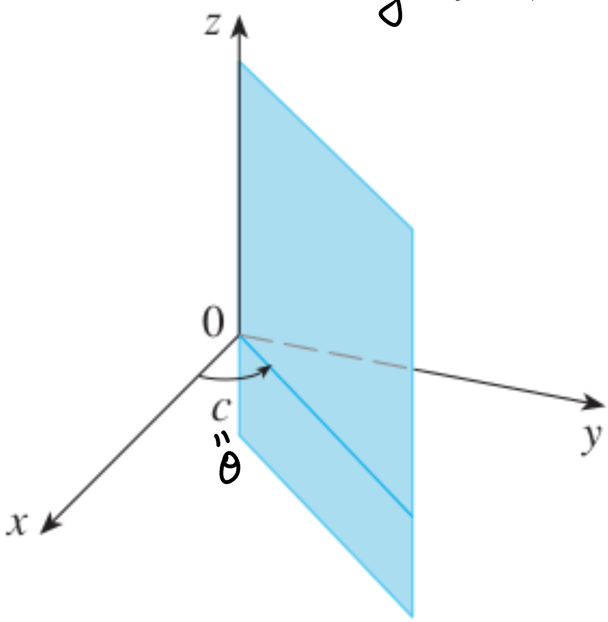
$$\rightarrow a\rho \sin\phi \cos\theta + b\rho \sin\phi \sin\theta = 0$$

$$\rightarrow a \cos\theta + b \sin\theta = 0$$

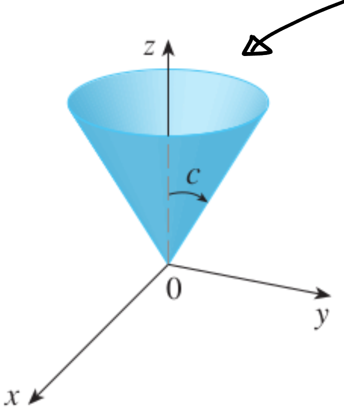
$$\rightarrow -\frac{a}{b} = \frac{\sin\theta}{\cos\theta}$$

$$\rightarrow \tan c = \tan\theta$$

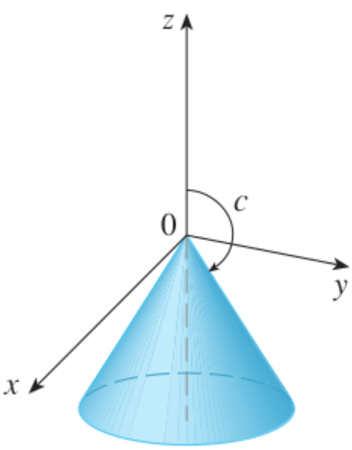
$$\rightarrow \boxed{c = \theta}$$



Cones. $z = \left(\sqrt{x^2+y^2}\right) R \rightarrow$ radius cylinder



$0 < c < \pi/2$



$\pi/2 < c < \pi$

$z = \rho \cos\phi$

$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

$$\rho \cos\phi = \sqrt{\rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta}$$

$$\rightarrow \rho \cos\phi = \rho \sin\phi \sqrt{\cos^2\theta + \sin^2\theta}$$

$$\rightarrow \cancel{\rho} \cos\phi = \cancel{\rho} \sin\phi$$

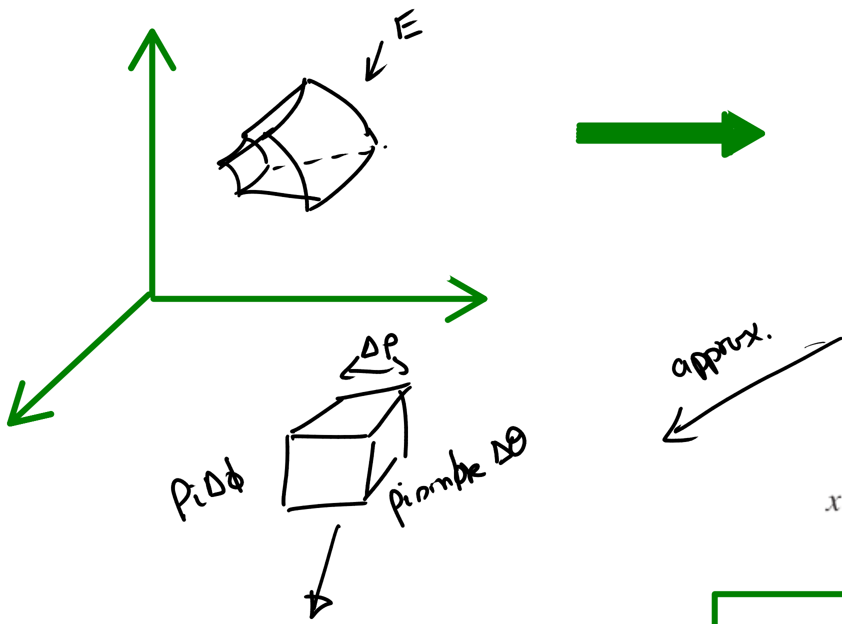
$$\rightarrow R = \frac{\sin\phi}{\cos\phi} \rightarrow R = \tan\phi$$

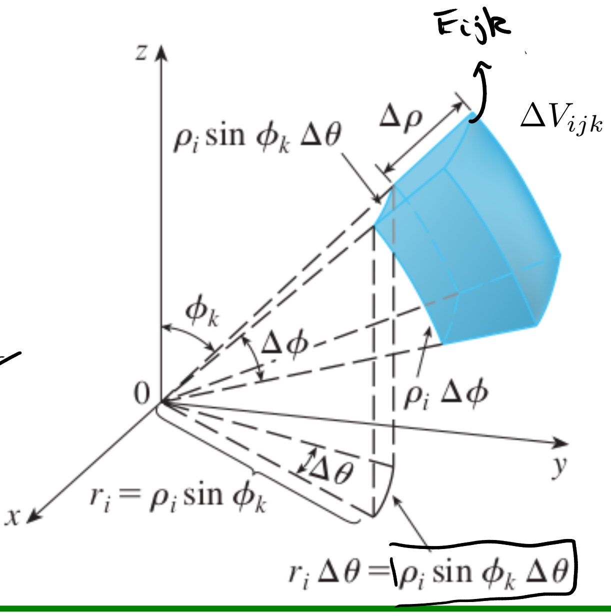
$$\rightarrow \phi = \arctan(R)$$

$$\boxed{\phi = c}$$

Evaluating integrals.

$E = \{(\rho, \theta, \phi) : a \le \rho \le b, \quad \alpha \le \theta \le \beta, \quad c \le \phi \le d\}$





$$\boxed{\Delta V_{ijk} \approx \rho_i^2 \sin(\phi_k)\Delta\rho\Delta\theta\Delta\phi} \longrightarrow \boxed{I \approx \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}, y_{ijk}, z_{ijk})\underline{\Delta V_{ijk}}}$$

$x_{ijk} = \rho_i \sin\phi_k \cos\theta_j$
$y_{ijk} = \rho_i \sin\phi_k \sin\theta_j$
$z_{ijk} = \rho_i \cos\phi_k$

$$I = \iiint_E f(x,y,z)\, dV$$

$$\approx \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} \underbrace{f(\rho_i \sin\phi_k \cos\theta_j, \rho_i \sin\phi_k \sin\theta_j, \rho_i \cos\phi_k) \cdot \rho_i^2 \sin\phi_k}_{g(\rho_i, \theta_j, \phi_k)} \Delta\rho\Delta\theta\Delta\phi.$$

Riemann sum for g $\xrightarrow[m,n,l\to\infty]{}$ $\int_a^b \int_\alpha^\beta \int_c^d g(\rho, \theta, \phi)\, \underline{d\phi\, d\theta\, d\rho}$

Formula for the change of variable (in polar coordinates). polar: $dA = r\,dr\,d\theta$

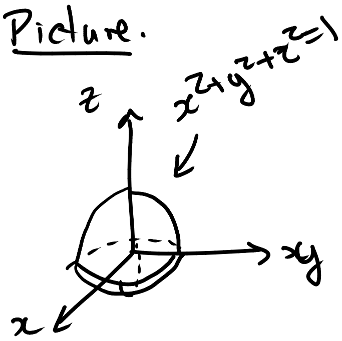
$$\boxed{\iiint_E f(x,y,z)\,dV = \int_c^d \int_\alpha^\beta \int_a^b f(\underbrace{\rho\sin(\phi)\cos(\theta)}_{x}, \underbrace{\rho\sin(\phi)\sin(\theta)}_{y}, \underbrace{\rho\cos(\phi)}_{z})\, \underbrace{\rho^2\sin(\phi)\,d\rho\,d\theta\,d\phi}_{dV}}$$

$E = \{(\rho,\theta,\phi)\,|\,a \le \rho \le b,\ \alpha \le \theta \le \beta,\ c \le \phi \le d\}$

EXAMPLE 3 Evaluate $\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\, dV$, where B is the unit ball:

$$B = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$$

① Picture.



$$B = \{ (x,y,z) : -1 \leq x \leq 1 ,\ -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2},\ -\sqrt{1-x^2-y^2} \leq z \leq \sqrt{1-x^2-y^2} \}$$

Eq. sphere : $\rho = 1 \rightarrow 0 \leq \rho \leq 1$ inside

Rotate around z-axis : $0 \leq \theta \leq 2\pi$ (full turn)

Along the latitude : $0 \leq \phi \leq \pi$

$$B = \{ (\rho, \theta, \phi) : 0 \leq \rho \leq 1 ,\ 0 \leq \theta \leq 2\pi ,\ 0 \leq \phi \leq \pi \}$$

② Integrate.

$\rho = \sqrt{x^2+y^2+z^2}$, so

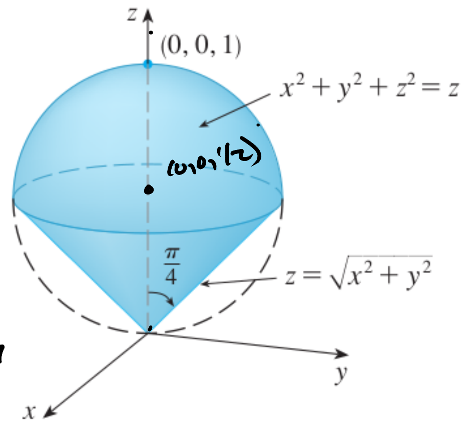
$$\iiint_E e^{(x^2+y^2+z^2)^{3/2}}\, dV = \int_0^{\pi}\int_0^{2\pi}\int_0^1 e^{\rho^3}\rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$$

$\rightarrow u = \rho^3$

$$= \left(\int_0^{\pi} \sin\phi \, d\phi\right)\left(\int_0^{2\pi} d\theta\right)\left(\int_0^1 \rho^2 e^{\rho^3} d\rho\right)$$

$$= \boxed{\frac{4}{3}\pi(e-1)}$$

EXAMPLE 4 Use spherical coordinates to find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$.



① Picture

$$x^2+y^2+z^2 = z \quad \to \quad x^2+y^2+\left(z-\frac{1}{2}\right)^2 = \frac{1}{4}$$

$$-\sqrt{\frac{1}{4}-x^2-y^2} \le z-\frac{1}{2} \le \sqrt{\frac{1}{4}-x^2-y^2}$$

$$\Rightarrow \quad \sqrt{x^2+y^2} \le z \le \frac{1}{2}+\sqrt{\frac{1}{4}-x^2-y^2}$$

$$E = \left\{ (x,y,z) : (x,y) \text{ circle radius } \frac{1}{2} \ \& \ \sqrt{x^2+y^2} \le z \le \frac{1}{2}+\sqrt{\frac{1}{4}-x^2-y^2} \right\}$$

$x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$ & $z = \rho\cos\phi$

$$\rho^2\sin^2\phi + \rho^2\cos^2\phi = \rho\cos\phi \quad \to \quad \rho^2 = \rho\cos\phi$$

$$\to \quad \rho = \cos\phi .$$

$$E = \left\{ (\rho, \theta, \phi) : 0 \le \rho \le \cos\phi, \ 0 \le \theta \le 2\pi, \ 0 \le \phi \ \frac{\pi}{4} \right\}$$

② Volume.

$$V(E) = \iiint_E dV = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi/4}\int_0^{\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi\right)$$

$$= 2\pi \left(\int_0^{\pi/4} \frac{\rho^3}{3}\Big|_0^{\cos\phi} \sin\phi \, d\phi \right)$$

$$= 2\pi \left(\int_0^{\pi/4} \frac{\cos^3\phi \sin\phi}{3} \, d\phi \right)$$

$$= \boxed{\frac{\pi}{8}}$$