

S.I Terminology

PROBLEM 1. Let $x \in A$. Then either $x = 1$ or $x = 2$. When $x = 1$, we see that $1 \in B$, so $x \in B$. When $x = 2$, we see that $2 \in B$, so that $x \in B$. Therefore, $\forall x \in A, x \in B$ and $A \subset B$.

PROBLEM 2. Let $U = \{1, 2, 3, 4\}$. Show that U has 2^4 subsets. Bonus: In general, show that if U has a finite number of elements, then U has $2^{(\#U)}$ subsets.

The power set of U is

$$2^U = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \\ \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, U\}.$$

Counting the number of subsets in the power set will give the total number of subsets of U . There are 16 items in the power set 2^U and $16 = 2^4$.

In general, define a subset A of U as a function $f_A : U \rightarrow \{0, 1\}$ in the following way:

$$f_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A. \end{cases}$$

Then the number of subsets of U will be determined by the number of functions from U into $\{0, 1\}$. There are $2^{\#U}$ such functions because for each element $x \in U$, you have two possible choice for $f_A(x)$.

PROBLEM 3. By definition $\emptyset \subset A$ if $x \in \emptyset$, then $x \in A$. However, the assumption is always false and therefore the implication $x \in \emptyset \Rightarrow x \in A$ is always true. Thus $\emptyset \subset A$.

PROBLEM 4. Let $x \in A$. Since $A \subset B$, then $x \in B$. Now, since $B \subset C$, then $x \in C$. The element x was arbitrary chosen, so $A \subset C$.

S.II Operations With Sets

PROBLEM 5. Let A and B be two subsets of a universal set U .

- Let $x \in A \cap B$. By definition of intersection, we then have $x \in A$ and $x \in B$. In particular, we have $x \in A$. Since x was arbitrary chosen, then $A \cap B \subset A$.
- Let $x \in A$. Then the disjunction statement " $x \in A$ or $x \in B$ " is true because $x \in A$ is assume to be true. Therefore, $x \in A \cup B$ by definition of the union of two sets. Since x was arbitrary, then $A \subset A \cup B$.
- Assume that $A \subset B$. We start by showing that $A \cup B \subset B$. If $x \in A \cup B$, then $x \in A$ or $x \in B$ by definition of union. Two cases:
 - If $x \in A$, then $x \in B$ because $A \subset B$.

- If $x \in B$, then $x \in B$ (right?).

In both cases, we obtain that $x \in B$. Since x was arbitrary, we have $A \cup B \subset B$. We now show that $B \subset A \cup B$. In fact, this comes from a) applied to B in place of A so that $B \subset A \cup B$. Since $A \cup B \subset B$ and $B \subset A \cup B$, we have $A \cup B = B$.

- d) Assume that $A \subset B$. Let $x \in A \cap B$. Then $x \in A$ and $x \in B$. In particular $x \in A$. So $A \cap B \subset A$. Let $x \in A$. Then $x \in B$ also because $A \subset B$. Therefore $x \in A$ and $x \in B$ and so $x \in A \cap B$. So $A \subset A \cap B$. Since $A \cap B \subset A$ and $A \subset A \cap B$, we get $A \cap B = A$.

PROBLEM 6. Let A and B be two subsets of a universal set U .

- a) We have to show that $A \cup \bar{A} \subset U$ and $U \subset A \cup \bar{A}$.

Let $x \in A \cup \bar{A}$. There are two cases because of the definition of union:

- If $x \in A$, then $x \in U$ because $A \subset U$.
- If $x \in \bar{A}$, then $x \in U$ because $\bar{A} \subset U$.

Each case implies that $x \in U$. Therefore, $A \cup \bar{A} \subset U$.

Let $x \in U$. Then $x \in A$ or $x \notin A$ which implies that $x \in A \cup \bar{A}$. Therefore $U \subset A \cup \bar{A}$.

- b) Assume that $A \subset B$. Let $x \in \bar{B}$. We will argue by contradiction. Suppose that $x \notin A$, so that $x \in A$. By assumption, this implies that $x \in B$. But $x \in \bar{B}$. In other words, $x \in B$ and $x \notin B$. This is not possible and therefore $x \in \bar{A}$. [Note: You can also take the contrapositive for a direct proof! The contrapositive of $A \subset B$ is $x \notin B \Rightarrow x \notin A$ which is equivalent to $x \in \bar{B} \Rightarrow x \in \bar{A}$.]

PROBLEM 7. Since $\emptyset \subset A \cap B$, we only need to prove that $A \cap B \subset \emptyset$. We will argue by contradiction. For $A \cap B$ to not be a subset of \emptyset , it must contain at least one element. So, assume that $x \in A \cap B$. This implies that $x \in A$ and $x \in B$. By definition of A , x is an odd integer. By definition of B , x is also an even integer. But an integer can't be odd and even at the same time and this is a contradiction. Therefore, $A \cap B \subset \emptyset$.

Since $\emptyset \subset A \cap B$ and $A \cap B \subset \emptyset$, we conclude that $A \cap B = \emptyset$.

S.III Important Laws For Set Algebra

PROBLEM 8. Let A , B , and C be subsets of a universal set U .

- a) We have

$$\begin{aligned} x \in A \cap B &\iff x \in A \text{ and } x \in B && \text{[Definition of } A \cap B\text{]} \\ &\iff x \in B \text{ and } x \in A && \text{[Order of words don't matter]} \\ &\iff x \in B \cap A && \text{[Definition of } B \cap A\text{]} \end{aligned}$$

- b) We start by showing that $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$. Let $x \in A \cup (B \cap C)$. Then by definition of the union, we have $x \in A$ or $x \in B \cap C$. The fact $x \in B \cap C$ implies that $x \in B$ and $x \in C$. In particular, $x \in B$. Combined with $x \in A$, we obtain $x \in A$ or $x \in B$. Similarly, we obtain $x \in A$ or $x \in C$. We can then create the conjunction of " $x \in A$ or $x \in B$ " with " $x \in A$ or $x \in C$ ", that is $(x \in A \vee x \in B) \wedge (x \in A \vee x \in C)$. Using the

definition of union, the last statement is rewritten as $(x \in A \cup B) \wedge (x \in A \cup C)$. From the definition of intersection, we can rewrite the last statement as $x \in (A \cup B) \cap (A \cup C)$.

To show that $(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$ is similarly. We just start from the end of the previous paragraph and make our way back to the first sentence of the paragraph (so read the last paragraph from the last sentence to the first sentence to obtain the proof).

Therefore, we just proved equality between the two sets.

c) We have

$$\begin{aligned}x \in \overline{A \cup B} &\iff x \notin A \cup B \\&\iff x \notin A \wedge x \notin B \quad [\text{Negation of } A \cup B] \\&\iff x \in \overline{A} \wedge x \in \overline{B} \quad [\text{Definition of complement}] \\&\iff x \in \overline{A} \cap \overline{B} \quad [\text{Definition of intersection}].\end{aligned}$$

Therefore, $\overline{A \cup B} = \overline{A} \cap \overline{B}$.