

MATH 311

CHAPTER 3

SECTION 3.3: DIAGONALIZATION AND EIGENVALUES

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WHY DIAGONALIZATION?

EXAMPLE 1.

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$. Compute A^{100} .

SOLUTION.

Way too long to compute directly.

Instead, we find

$$P = \begin{bmatrix} 1 & 2/3 \\ -1 & 1 \end{bmatrix}$$

$$P^{-1} = \frac{3}{5} \begin{bmatrix} 1 & -2/3 \\ 1 & 1 \end{bmatrix}$$

Then

$$P^{-1} A P = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} \xrightarrow{\text{Diagonal matrix}} A = P \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} P^{-1}$$

So,

$$\begin{aligned} A^2 &= A A = \left(P \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} P^{-1} \right) \left(P \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} P^{-1} \right) \\ &= P \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}^2 P^{-1} = P \begin{bmatrix} (-1)^2 & 0 \\ 0 & 4^2 \end{bmatrix} P^{-1} \end{aligned}$$

$$\Rightarrow A^{100} = P \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}^{100} P^{-1} = P \begin{bmatrix} (-1)^{100} & 0 \\ 0 & (4)^{100} \end{bmatrix} P^{-1}$$

Fact: If $A = P D P^{-1}$, then $A^k = P D^k P^{-1}$.

GOAL: Find the matrix P such that $P^{-1} A P$ is a diagonal matrix.

EIGENVALUES AND EIGENVECTORS

Exploration: Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}.$$

Set $\mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix}$ a 2×1 vector. Then

$$A\mathbf{x} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a + 2b \\ 3a + 2b \end{bmatrix}$$

Use Desmos¹ to explore and answer the following questions:

- Can you find an exceptional behavior of $A\mathbf{x}$ and \mathbf{x} for certain choices of \mathbf{x} ?
- Can you find a relation between $A\mathbf{x}$ and \mathbf{x} ?

Record your observations in the following blank space:

- ① Output and input lay on the same line .
- ② Output is a scalar multiple of the input .

¹<https://www.desmos.com/calculator/5xlrp9fd7g>

DEFINITION 1. Let A be an $n \times n$ matrix.

- a) A number λ is called an **eigenvalue** of A if there is a non-zero $n \times 1$ vector \mathbf{x} such that $A\mathbf{x} = \lambda\mathbf{x}$.
- b) The vector \mathbf{x} is called an **eigenvector** associated to λ .

EXAMPLE 2. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ and let $\mathbf{x} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$. Then

$$A\mathbf{x} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 + 4 \\ -6 + 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} = (-1) \begin{bmatrix} -2 \\ 2 \end{bmatrix} = -\vec{x}.$$

-1 : eigen value for A & \vec{x} eigenvector.

Finding eigenvalues

Notice that

$$\begin{aligned} \lambda \text{ is an eigenvalue of } A &\iff A\mathbf{x} = \lambda\mathbf{x} \text{ for some } \mathbf{x} \neq 0 \\ &\iff (\lambda I - A)\mathbf{x} = 0 \text{ for some } \mathbf{x} \neq 0. \end{aligned}$$

$$(\lambda I - A)^{-1}(\lambda I - A)\vec{x} = \vec{x} = \vec{0}$$

So

$$\begin{aligned} \lambda \text{ is an eigenvalue of } A &\iff (\lambda I - A) \text{ is not invertible} \\ &\iff \det(\lambda I - A) = 0 \end{aligned}$$

DEFINITION 2. The **characteristic polynomial** of an $n \times n$ matrix A is defined by

$$c_A(x) = \det(xI - A).$$

Conclusion:

$$\lambda \text{ is an eigenvalue of } A \iff \lambda \text{ is a root of } c_A(x).$$

EXAMPLE 3. Find all eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}.$$

SOLUTION.

We have

$$\begin{aligned} C_A(x) &= \det(xI - A) \\ &= \det\left(\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} x-1 & -2 \\ -3 & x-2 \end{bmatrix}\right) \\ &= (x-1)(x-2) - 6 = x^2 - 3x - 4 \\ &= (x+1)(x-4) \end{aligned}$$

Hence

$$\begin{aligned} C_A(x) = 0 &\iff (x+1)(x-4) = 0 \\ &\iff x = -1 \text{ or } x = 4 \end{aligned}$$

Eigen values: $\boxed{\lambda_1 = -1, \lambda_2 = 4}$

$\lambda = 4$ Write

$$(\lambda I - A)\vec{x} = 0 \Leftrightarrow \left(\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} 3 & -2 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} 3x - 2y = 0 \\ -3x + 2y = 0 \end{cases}$$

$$\rightarrow 3x - 2y = 0 \Rightarrow x = \frac{2}{3}y. \text{ So}$$

$$\vec{x}_2 = \begin{bmatrix} \frac{2}{3}y \\ y \end{bmatrix} = y \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$$

For $\lambda = -1$, we can set $x = 1$

$$\Rightarrow \vec{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

For $\lambda = 4$, we can set $y = 1$

$$\Rightarrow \vec{x}_2 = \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}.$$

Finding Eigenvectors

For a given eigenvalue λ , the eigenvectors associated to λ are the solutions \mathbf{x} to the system

$$(\lambda I - A)\mathbf{x} = \vec{0}.$$

EXAMPLE 4. Find the eigenvectors associated to the each eigenvalue of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}.$$

$\lambda = -1$ Let $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, so that

$$((-1)I - A)\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow \left(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} -2 & -2 \\ -3 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} -2x - 2y = 0 \\ -3x - 3y = 0 \end{cases}$$

$$\Rightarrow x + y = 0 \Rightarrow y = -x \Rightarrow \vec{x}_1 = \begin{bmatrix} x \\ -x \end{bmatrix} = x \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

EXAMPLE 5.

Find all eigenvalues and associated eigenvectors of the matrix

$$A = \begin{bmatrix} 7 & 0 & -4 \\ 0 & 5 & 0 \\ 5 & 0 & -2 \end{bmatrix}.$$

SOLUTION.**① Eigenvalues**

$$\begin{aligned} C_A(x) &= \det(xI - A) \\ &= \begin{vmatrix} x-7 & 0 & 4 \\ 0 & x-5 & 0 \\ -5 & 0 & x+2 \end{vmatrix} \end{aligned}$$

$$\Rightarrow C_A(x) = (x-2)(x-3)(x-5) = 0$$

$$\text{So, } \lambda_1 = 2, \lambda_2 = 3, \lambda_3 = 5.$$

② Eigen vectors

$$\underline{\lambda_1 = 2} \quad (2I - A)\vec{x} = \vec{0} \Rightarrow \begin{bmatrix} -5 & 0 & 4 \\ 0 & -3 & 0 \\ -5 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -5x + 4z = 0 \text{ and } -3y = 0$$

$$\Rightarrow x = \frac{4}{5}z \text{ and } y = 0. \Rightarrow \vec{x}_1 = z \begin{bmatrix} 4/5 \\ 0 \\ 1 \end{bmatrix}.$$

EXAMPLE 6. Find a matrix P such that

$$P^{-1}AP$$

is a diagonal matrix, where A is from Example 1.

SOLUTION.

THEOREM 1. Let A be an $n \times n$ matrix. Then if all eigenvalues of A are distinct, then A is diagonalizable.

Notice that if A is diagonalizable and if we let $P = [\mathbf{x}_1 \ \mathbf{x}_2 \ \cdots \ \mathbf{x}_n]$:

$$\begin{aligned} P^{-1}AP &= D \iff AP = PD \\ &\iff A[\mathbf{x}_1 \ \mathbf{x}_2 \ \cdots \ \mathbf{x}_n] = [\mathbf{x}_1 \ \mathbf{x}_2 \ \cdots \ \mathbf{x}_n] D \\ &\iff A\mathbf{x}_1 = \lambda_1\mathbf{x}_1, A\mathbf{x}_2 = \lambda_2\mathbf{x}_2, \dots, A\mathbf{x}_n = \lambda_n\mathbf{x}_n. \end{aligned}$$

ALGORITHM 1. Let A be an $n \times n$ matrix with distinct eigenvalues.

- ① Find all distinct eigenvalues of A .
- ② For each eigenvalue of A , find the corresponding set of eigenvectors.
- ③ If $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ are a set of n distinct eigenvectors, then set

$$P = [\mathbf{x}_1 \ \mathbf{x}_2 \ \cdots \ \mathbf{x}_n].$$

WARNING!

Not every matrix is diagonalizable. For instance, the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

is not diagonalizable.

For a more general algorithm, see *Jordan Canonical Form*, Chapter 11 from the textbook. Complex numbers are required.