Chapter 4: Integrals
Week 12

Pierre-Olivier Parisé
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Upcoming this week

1. 4.4 Indefinite integral
2. 4.5 Substitution rule

In the previous sections, we were interested in the definite integral

$$\int_a^b f(x)\,dx$$

where the result is a number. With the fundamental Theorem of Calculus, we see that the antiderivatives of a function is an integral.

Indefinite integral

If F is an antiderivative of a function f, we note F by

$$\int f(x)\,dx.$$

Warning: You should distinguish carefully between definite and indefinite integrals. The indefinite integral is a *function* and the definite integral is a *number*.
The connection between definite and indefinite integral is given by Part 2 of the FTC:

$$\int_a^b f(x)\,dx = F(b) - F(a) = \int f(x)\,dx\Big|_a^b.$$

Here are some useful indefinite integrals:

- $\displaystyle\int cf(x)\,dx = c\int f(x)\,dx.$
- $\displaystyle\int f(x)+g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx.$
- $\displaystyle\int k\,dx = kx + C.$
- $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \ (n \neq -1)$
- $\displaystyle\int \sin x\,dx = -\cos x + C$
- $\displaystyle\int \cos x\,dx = \sin x + C$
- $\displaystyle\int \sec^2 x\,dx = \tan x + C$
- $\displaystyle\int \csc^2 x\,dx = -\cot x + C$
- $\displaystyle\int \sec x \tan x\,dx = \sec x + C$
- $\displaystyle\int \csc x \cot x\,dx = -\csc x + C$

Warning!! We have

$$\int \frac{1}{x}\,dx = \ln x + C.$$

Example 1

Find the general indefinite integral of

$$\int (10x^4 - 2\sec^2 x)\, dx.$$

Example 2

Evaluate the indefinite integral $\displaystyle\int \frac{\cos\theta}{\sin^2\theta}\, d\theta$.

Example 3

Evaluate $\displaystyle\int \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\, dt$.

The derivative of a function is the rate of change of the function. So, if F is an antiderivative of a function f, we say that $F(b) - F(a)$ is the **net change** of f on the interval $[a, b]$.

Net change

The integral of the rate of change is the net change:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

where F is an antiderivative of a function f.

Example 4

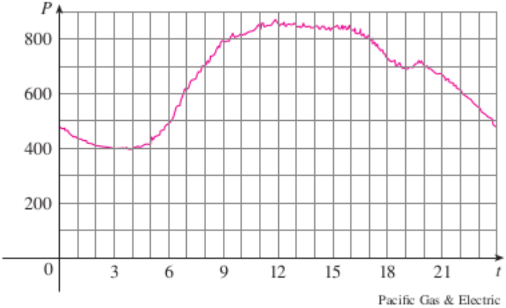
A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ (measured in meters per second).

a) Find the displacement of the particle during the time period $1 \leq t \leq 4$.

b) Find the distance traveled during this time period.

Example 5

The figure shows the power consumption in the city of San Francisco for a day in September (P is measured in megawatts; t is measured in hours starting at midnight). Estimate the energy used on that day.



Exercises: 1-4, 5-16, 19-42, 45, 48, 53, 56, 59, 61,

Example 6

Find the indefinite integral $\int 2x\sqrt{1+x^2}\,dx$.

Substitution Rule

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Example 7

Find the indefinite integrals:

a) $\int x^3\cos(x^4+2)\,dx$.

b) $\int \sqrt{2x+1}\,dx$.

c) $\int \sqrt{1+x^2}x^5\,dx$.

When we have found the indefinite integral of a function, we can use it to evaluate definite integral. This is essentially the Fundamental Theorem of Calculus.

Theorem 8 (FTC)

If F is the indefinite integral of $f(x)$, then

$$\int_a^b f(x) = F(b) - F(a).$$

Example 9

Compute the value of $\int_0^4 \sqrt{2x+1}\,dx$.

When we now that we used the substitution rule, there is a preferable method to evaluate the definite integral.

Substitution Rule for Definite Integrals

If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Example 10

Compute the value of the definite integrals.

a) $\displaystyle\int_0^4 \sqrt{2x+1}\,dx.$

b) $\displaystyle\int_1^2 \frac{dx}{(3-5x)^2}.$

Recall from Week-1 that

- a function f is even if $f(-x) = f(x)$.
- a function f is odd if $f(-x) = -f(x)$.

These functions are even:

- the function $f(x) = x^2$.
- the function $f(x) = \cos(x)$.

These functions are odd:

- the function $f(x) = x^3$.
- the function $f(x) = \sin(x)$.

When a function has symmetries (odd or even), then we can use this information to simplify the calculations of the definite integral.

Example 11

Compute the value of

- $\displaystyle\int_{-1}^{1} x^2\,dx.$
- $\displaystyle\int_{-1}^{1} x^3\,dx.$

Integrals of symmetric functions

Suppose f is continuous on $[-a, a]$.

- If f is even, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$
- If f is odd, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 0.$

Exercises: 1-6, 7-30, 31-34, 35-51, 64 and 85 (challenges).