Chapter 4
Integrals

4.4 Indefinite Integrals and the Net Change Theorem

Indefinite Integral.

Previously on Calc I:

Fondamental Theorem of Calculus

$$\int_a^b f(x)dx = F(b) - F(a).$$

We introduce a notation for the antiderivatives:

$$\int f(x)\,dx = F(x) + C \quad \text{means} \quad F'(x) = f(x)$$

Example.

a) $\displaystyle\int x^2\,dx = \frac{x^3}{3} + C$.

b) $\displaystyle\int \cos x\,dx = \sin x + C$.

c) $\displaystyle\int \sec^2 x\,dx = \tan x + C$.

Table of Indefinite integrals

$\int cf(x)\,dx = c\int f(x)\,dx$	$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$
$\int k\,dx = kx + C$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$
$\int \sin x\,dx = -\cos x + C$	$\int \cos x\,dx = \sin x + C$
$\int \sec^2 x\,dx = \tan x + C$	$\int \csc^2 x\,dx = -\cot x + C$
$\int \sec x \tan x\,dx = \sec x + C$	$\int \csc x \cot x\,dx = -\csc x + C$

Remark: We adopt the convention that the general indefinite integral is valid on a given interval. Thus we write

$$\int \frac{1}{x^2}\,dx = -\frac{1}{x} + C$$

with the understanding that it is valid on the interval $(0,\infty)$ or on the interval $(-\infty,0)$.

EXAMPLE 1 Find the general indefinite integral

$$I = \int (10x^4 - 2\sec^2 x)\,dx$$

$$I = \int 10x^4\,dx - \int 2\sec^2 x\,dx$$

$$= 10\int x^4\,dx - 2\int \sec^2 x\,dx$$

$$= 10\left(\frac{x^5}{5} + C_1\right) - 2\left(\tan x + C_2\right)$$

$$= 2x^5 + 10C_1 - 2\tan x - 2C_2$$

Set $C = 10C_1 - 2C_2$

$$= 2x^5 - 2\tan x + C$$

EXAMPLE 2 Evaluate $\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$.

$$\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \int \frac{\cos\theta}{\sin\theta} \cdot \frac{1}{\sin\theta}\,d\theta$$

$$= -\int -\cot\theta\ \operatorname{cosec}\theta\,d\theta$$

$$= -\operatorname{cosec}\theta + C$$

$(\sec\theta)' = \sec\theta\tan\theta$

$(\operatorname{cosec}\theta)' = -\operatorname{cosec}\theta\cot\theta$

EXAMPLE 4 Find $\int_0^{12} (x - 12\sin x)\,dx$.

$$\int_0^{12} x - 12\sin x\,dx = \int_0^{12} x\,dx - 12\int_0^{12} \sin x\,dx$$

$$\int x\,dx = \frac{x^2}{2} + c$$

$$\int \sin x\,dx = -\cos x + c$$

so, $$\int_0^{12} x\,dx = \left.\frac{x^2}{2} + c\right|_0^{12} = \frac{12^2}{2} + \cancel{c} - (0 + \cancel{c})$$
$$= 72$$

$$\int_0^{12} \sin x\,dx = -\cos x\Big|_0^{12} = -\cos(12) + 1$$

Answer: $\boxed{72 + 12\cos(12) - 12}$

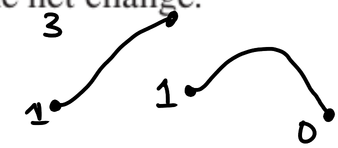
EXAMPLE 5 Evaluate $\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\,dt$. $\to I$

$$\frac{2t^2 + t^2\sqrt{t} - 1}{t^2} = 2 + \sqrt{t} - \frac{1}{t^2} = 2 + \sqrt{t} - t^{-2}$$

$$I = \int_1^9 2 + \sqrt{t} - t^{-2}\,dt$$
$$= \int_1^9 2\,dt + \int_1^9 \sqrt{t}\,dt - \int_1^9 t^{-2}\,dt$$
$$= 2\int_1^9 1\,dt + \int_1^9 t^{1/2}\,dt - \int_1^9 t^{-2}\,dt$$
$$= 2t\Big|_1^9 + \left.\frac{2}{3}t^{3/2}\right|_1^9 - \left.\frac{t^{-1}}{-1}\right|_1^9$$
$$= 2(8) + \frac{2}{3}(27 - 1) + \left(\frac{1}{9} - 1\right) = \boxed{32\tfrac{4}{9}}$$

Net Change Theorem The integral of a rate of change is the net change:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$



a) Displacement: $v(t) = s'(t)$ ($s(t)$: displacement)

$$\text{displ} = \int_a^b v(t)\,dt = s(b) - s(a).$$

b) Total distance traveled:

$$\text{total dist} = \int_a^b |v(t)|\,dt$$

c) Acceleration: net change velocity

$$\text{Net Accelerate} = \int_a^b a(t)\,dt = \int_a^b v'(t)\,dt = v(b) - v(a)$$

EXAMPLE 6 A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ (measured in meters per second).

(a) Find the displacement of the particle during the time period $\underset{a}{1} \leqslant t \leqslant \underset{b}{4}$.

(b) Find the distance traveled during this time period.

(a) $$\text{Displ.} = \int_1^4 v(t)\,dt = \int_1^4 t^2 - t - 6\,dt$$

$$= \frac{t^3}{3} - \frac{t^2}{2} - 6t \Big|_1^4$$

$$= \frac{4^3}{3} - \frac{4^2}{2} - 6(4) - \left(\frac{1}{3} - \frac{1}{2} - 6\right)$$

$$= -\frac{9}{2}\,m = -4.5\,m$$

(b) $$\text{tot. Dist.} = \int_1^4 |v(t)|\,dt = \int_1^4 |t^2 - t - 6|\,dt$$

$t^2 - t - 6 = (t-3)(t+2)$

$1 \le t \le 4$

- $t - 3 \geq 0$ when $t \geq 3$
- $t - 3 \leq 0$ when $t \leq 3$

- $t+2 \geq 0$ when $t \geq -2$ ←
- ~~$t+2 \leq 0$ when $t \leq -2$~~

$t^2-t-6 \geq 0$ when $t \geq 3$

& $t^2-t-6 \leq 0$ when $t \leq 3$

$$|t^2-t-6| = \begin{cases} t^2-t-6, & 3 < t \leq 4 \\ -t^2+t+6, & 1 \leq t \leq 3 \end{cases}$$

$$\int_1^4 |t^2-t-6|\,dt = \int_1^3 -t^2+t+6\,dt + \int_3^4 t^2-t-6\,dt$$

$$= \frac{61}{6} \approx \boxed{10.17\text{m}}$$