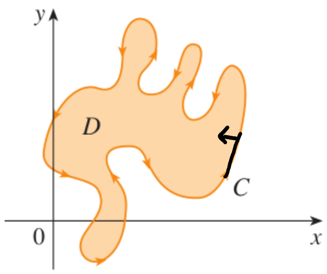
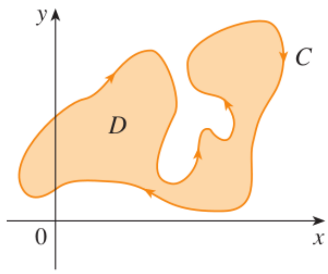
# 16.4 Green's Theorem.

Some terminology.




(a) Positive orientation




(b) Negative orientation

(a) Positive orientation.

The region D is always on your left when moving along the curve C.

(b) Negative orientation.

The region D is always on your right when moving along the curve C.

**Green's Theorem** Let $C$ be a positively oriented, piecewise-smooth, simple closed curve in the plane and let $D$ be the region bounded by $C$. If $P$ and $Q$ have continuous partial derivatives on an open region that contains $D$, then

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C P\,dx + Q\,dy = \iint_D \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{\text{circulation of } \vec{F}.} dA$$

$\vec{F} = \langle P, Q \rangle$

(x,y)

Remarks on notations.

① $\oint_C P\,dx + Q\,dy$ is used for integration over C with a pos. orient.

② Proof :


Do it for simpler regions of Type I & II.

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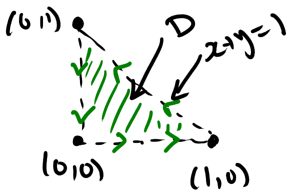
**EXAMPLE 1** Evaluate $\oint_C x^4\,dx + xy\,dy$, where $C$ is the triangular curve consisting of the line segments from $(0, 0)$ to $(1, 0)$, from $(1, 0)$ to $(0, 1)$, and from $(0, 1)$ to $(0, 0)$.

I

① Picture.




② Green's theorem.

$P = x^4$

$Q = xy$

$D = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 1-x\}$

$$I \overset{GT}{=} \iint_D y - 0\, dA = \iint_D y\, dA$$

$$= \int_0^1 \int_0^{1-x} y\, dy\, dx$$

$$= \boxed{1/6}$$

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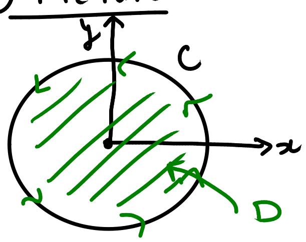
**EXAMPLE 2** Evaluate $\oint_C (3y - e^{\sin x})\,dx + \left(7x + \sqrt{y^4 + 1}\right) dy$, where $C$ is the circle $x^2 + y^2 = 9$.

I

① Picture.




$D = \{(x,y) : x^2 + y^2 \le 9\}$

② Green's Theorem.

$P = 3y - e^{\sin x}$

$Q = 7x + \sqrt{y^4 + 1}$

$$I \overset{GT}{=} \iint_D 7 - 3\, dA$$

$$= \iint_D 4\, dA = 4 \underbrace{\iint_D dA}_{\text{Area of } D}$$

So, $I = 4\pi 9 = \boxed{36\pi}$

**EXAMPLE 3** Find the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

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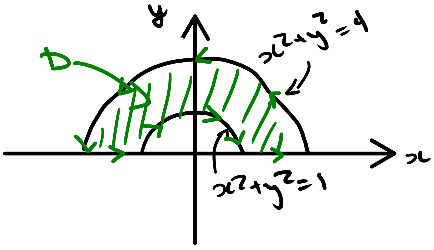
**EXAMPLE 4** Evaluate $\oint_C y^2\,dx + 3xy\,dy$, where $C$ is the boundary of the semiannular region $D$ in the upper half-plane between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

1 Picture.




2 Partial derivatives

$$\frac{\partial P}{\partial y} = 2y \qquad \frac{\partial Q}{\partial x} = 3y$$

3 Green's Theorem.

$x = r\cos\theta \quad y = r\sin\theta$

$$I = \iint_D 3y - 2y \, dA = \iint_D y \, dA.$$

$$D = \{(r,\theta) : 1 \le r \le 2,\ 0 \le \theta \le \pi\}$$

$$= \int_0^{\pi}\int_1^2 r\sin\theta \, r \, dr\, d\theta$$

$$= \left(\int_1^2 r^2 dr\right)\left(\int_0^{\pi} \sin\theta \, d\theta\right)$$

$$= \boxed{\frac{14}{3}}$$

Recall $A(D) = \iint_D 1 \, dA$.

$\boxed{\#1}$ $P = 0$ & $Q = x$ $\to$ $Q_x - P_y = 1 - 0 = 1$

$\boxed{\#2}$ $Q = 0$ & $P = -y$ $\to$ $Q_x - P_y = 0 - (-1) = 1$

$\boxed{\#3}$ $P = \frac{-y}{2}$ & $Q = \frac{x}{2}$ $\to$ $Q_x - P_y = \frac{1}{2} - \left(-\frac{1}{2}\right) = 1$

$$\boxed{A = \underset{\#1}{\oint_C \overset{Q}{x}\, dy} = \underset{\#2}{-\oint_C \overset{P}{y}\, dx} = \frac{1}{2}\oint_C \underbrace{\underset{Q}{x}\, dy - \underset{P}{y}\, dx}_{\#3}}$$

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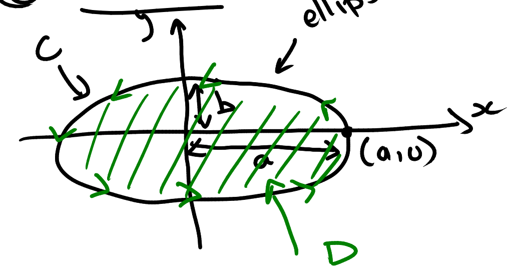
**EXAMPLE 3** Find the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

circle: $a = b$.

① Picture.




② Green's Theorem.

$$A = \oint_C x \, dy$$

③ Integrate

$\vec{r}(t) = \langle \overset{x}{a\cos t}, \overset{y}{b\sin t} \rangle$

$0 \le t \le 2\pi$

$y'(t) = b\cos t$

$$A = \int_0^{2\pi} a\cos t \; y'(t)\, dt$$

$$= \int_0^{2\pi} a b \cos^2 t \, dt$$

$$= ab \int_0^{2\pi} \cos^2 t \, dt \qquad \cos^2 t = \frac{1+\cos(2t)}{2}$$

$$= \boxed{ab\pi}$$

$$\frac{1}{2}\oint_C x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi} a\cos t \; b\cos t - (b\sin t\,(-a\sin t))\,dt$$

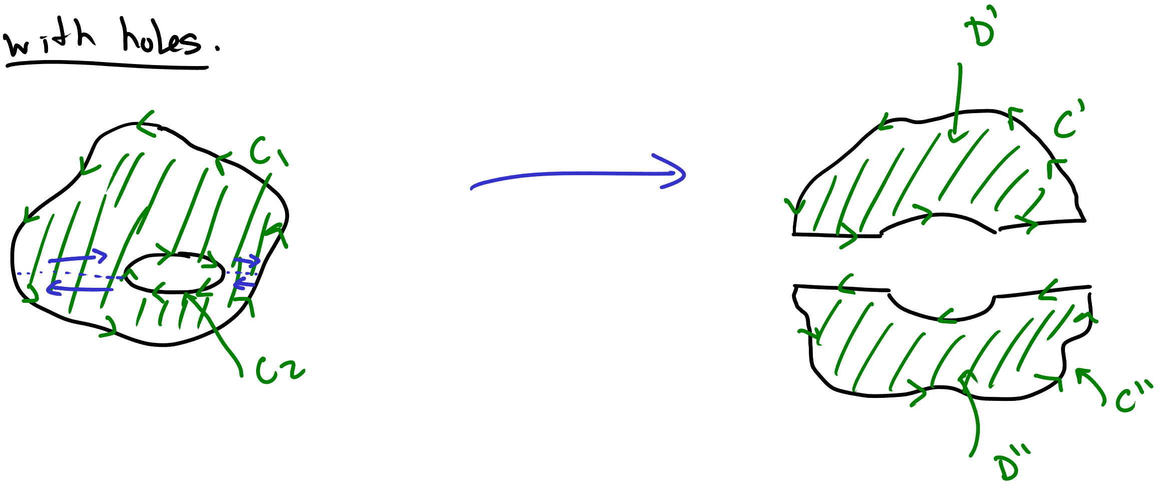
$$= \frac{1}{2}\int_0^{2\pi} ab\cos^2 t + ab\sin^2 t \, dt$$

$$= \frac{ab}{2}\int_0^{2\pi} \underbrace{\cos^2 t + \sin^2 t}_{1}\, dt$$

$$= \pi ab.$$

Extension to more general regions.

with holes.




$$\boxed{\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy = \int_C P\,dx + Q\,dy}$$

Now, $\displaystyle\iint_D Q_x - P_y\, dA = \iint_{D'} Q_x - P_y\, dA + \iint_{D''} Q_x - P_y\, dA$

$$= \int_{C'} P\,dx + Q\,dy + \int_{C''} P\,dx + Q\,dy$$

x2 $\rightleftarrows$ $$= \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy.$$