Example 4

Does the function

✓

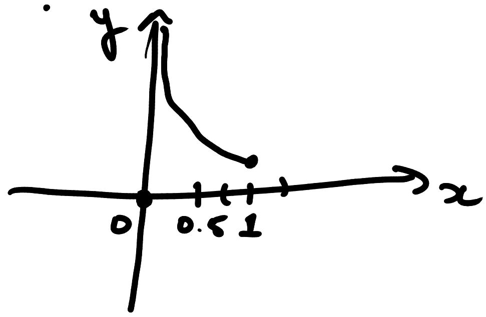
$$f(x) = \begin{cases} 1/x & 0 < x \le 1 \\ 0 & x = 0 \end{cases}$$

2

have a maximum?

x	$f(x)$
0.5	2
0.25	4
0.10	10
0.01	100

x	$f(x)$
1	1
0.5	2
0.25	4
0.1	10
0.01	100



- One at $x = 1$ → $f(x) = 1 \le \frac{1}{x}$ $\forall x \le 1$. $(x \ne 0)$
- Another one at $x = 0$ → $f(0) = 0 \le 1 \le \frac{1}{x}$ $\forall x$

Example 7

Let $f(x) = x^3$. What is $f'(0)$? Is $f(0)$ a local maximum or local minimum?

$f'(x) = 3x^2$

The solution to $f'(x) = 0$

$\iff 3x^2 = 0$

$\iff x = 0$

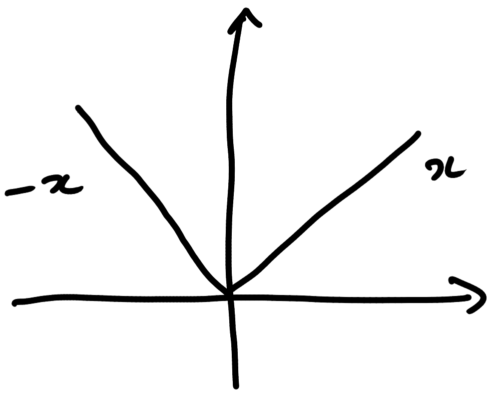
It seems that $x = 0$ is a local max or local min.

x	$f(x)$
2	8
1	1
0	0
-1	-1
-2	-8

x is not a maximum and not a minimum!

Example 8

Let $f(x) = |x|$. Is $f(0)$ a local maximum, global maximum, local minimum, or global minimum?



Here $f'(0) \nexists$

But $f(0) = 0$ is an absolute minimum.

x	$f(x)$
-2	2
-1	1
0	0
1	1
2	2

Example 10

Find the critical numbers of $f(x) = x^{3/5}(4 - x)$.

1) Find the derivative

$$f'(x) = (x^{3/5})'(4-x) + x^{3/5}\cdot(-1)$$

$$= \frac{3}{5}x^{-2/5}(4-x) - x^{3/5}$$

$$= \frac{3\cdot 4}{5}x^{-2/5} - \frac{3}{5}x^{3/5} - x^{3/5}$$

$$= \frac{12}{5}x^{-2/5} - x^{3/5}\left(\frac{8}{5}\right)$$

$$\frac{12}{5x^{2/5}} - \frac{8x^{3/5}}{5}$$

$$= \frac{12-8x}{5x^{2/5}}$$

$$= \frac{4(3-2x)}{5x^{2/5}}$$

2) Solve $f'(x) = 0$

$$f'(x) = 0 \iff \frac{4(3-2x)}{5x^{2/5}} = 0$$

$$\iff x = \frac{3}{2}$$

3) Find $f'(x)$ where $\nexists$

$f'(x)$ $\nexists$ when $x = 0$

Answer

critical numbers are $3/2$ & 0

Example 11

Find the absolute maximum and minimum values of the function $f(x) = x^3 - 3x^2 + 1$ on the interval $[-1/2, 4]$.

1) Find the critical numbers.

$f'(x) = 3x^2 - 6x$

Solve $f'(x) = 0$ (for x).

$\Leftrightarrow \quad 3x^2 - 6x = 0$

$\Leftrightarrow \quad (3x - 6)x = 0$

$\Leftrightarrow \quad 3x - 6 = 0 \quad$ or $\quad x = 0$

$\Leftrightarrow \quad x = 2 \quad$ or $\quad x = 0$

We have $f(0) = 1$ & $f(2) = -3$.

2) Compute the values of f at endpoints.

$a = \frac{-1}{2}$ & $b = 4$

$f(-1/2) = 1/8$

$f(4) = 17$.

3) Take max & min

$\max\{1, -3, 1/8, 17\} = 17$

$\min\{1, -3, 1/8, 17\} = -3$

Answers: abs maximum is 17 abs minimum is -3

Example 12

Show that the equation $x^3 + x - 1 = 0$ has at least one root.

Here $f(x) = x^3 + x - 1$.

$f(0) = 0 + 0 - 1 = -1$ $\quad a = 0$

$f(1) = 1$ $\quad b = 1$

So, $f(0) < 0$ & $f(1) > 0$.

So, $\exists c \in (0,1)$ s.t. $f(c) = 0$.

In other words, $c^3 + c - 1 = 0$.

Example 13

$f(x) = x^3 + x - 1.$

Show that the equation $x^3 + x - 1 = 0$ has exactly one root.

Suppose that there are two different roots a & b $(a \neq b)$.

So, $f(a) = 0 = f(b)$.

- f is cont. on $[a,b]$
- f is diff. on (a,b).

So, by Rolle's theorem, $f'(c) = 0$ for some $c \in (a,b)$.

contradiction.

$f'(x) = 3x^2 + 1 \geq 1$, so $f'(x) \neq 0$.

So, we must conclude that there is only one root.

Example 14

Consider $f(x) = x^2$.

$x^2 = 0^2$ $\quad$ $x^2 = 2^2$

- Find the slope of the secant line passing through $Q = (0, 0)$ and $P = (2, 4)$.
- Can you find a tangent line to $y = x^2$ with the same slope?

- the slope of PQ

$$m_{PQ} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 0}{2 - 0} = 2$$

- Here $f(x) = x^2$. We have $f'(x) = 2x$

So,

$$f'(x) = 2 \iff 2x = 2$$
$$\iff \boxed{x = 1}$$

Example 15

Find the numbers $c \in [0, 2]$ such that the average of the function $f(x) = x^3 - x$ on the interval $[0, 2]$ is attained by $f'(c)$.

Here, $a = 0$, $b = 2$.

$f(x) = x^3 - x \to$
- continuous on $[0,2]$
- differentiable on $(0,2)$.

So, $\exists c \in (0,2)$ s.t.

$$f'(c) = \frac{f(2) - f(0)}{2 - 0}$$

$f'(x) = 3x^2 - 1$

$$\frac{f(2) - f(0)}{2 - 0} = \frac{6 - 0}{2} = 3$$

So,

$$3c^2 - 1 = 3$$

$$\Rightarrow \quad 3c^2 = 4$$

$$\Rightarrow \quad \sqrt{c^2} = \sqrt{4/3}$$

$$\Rightarrow \quad |c| = \sqrt{4/3}$$

$$\Rightarrow \quad c = +\sqrt{4/3} \text{ or } -\sqrt{4/3}$$

The slope of the tangent line is the same as the slope of the secant if

$\boxed{x = 2/\sqrt{3}}$ or $x = \cancel{-2/\sqrt{3}}$ → does not belong to $[0,2]$.