Math 644

Chapter 4

Section 4.3: Approximation by Rational Functions

Contents

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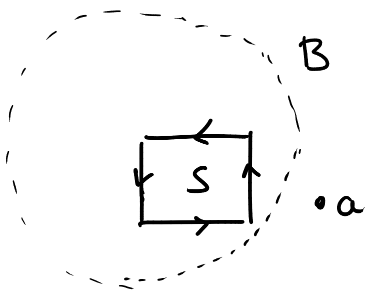
Cauchy Integral Formula For a Square

Theorem 1. If S is an open square with boundary ∂S parameterized in the counter-clockwise direction then

$$\frac{1}{2\pi i}\int_{\partial S}\frac{1}{z-a}\,dz = \begin{cases} 1 & \text{if } a\in S \\ 0 & \text{if } a\in \mathbb{C}\backslash\overline{S}. \end{cases}$$

Proof.

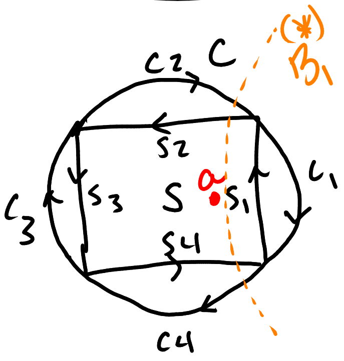
1) $a \in \mathbb{C}\backslash\overline{S}$. There is a disk B s.t. $\overline{S} \subseteq B$ & $a \notin \overline{B}$.



Since $\frac{1}{z-a}$ is analytic in B, from Cor. 11 in sect. 4.2,

$$\int_{\partial S}\frac{1}{z-a}\,dz = 0.$$

2) $a \in S$ Let C be the circle circumscribed to S. Write



$S = s_1+s_2+s_3+s_4$ & $C = c_1+c_2+c_3+c_4$

For $j=1,2,3,4$, $\sigma_j := s_j + c_j$ is a closed curve. For each j, we can find a disk B_j s.t. $\sigma_j \subseteq B_j$ & $a \notin B_j$. Therefore, $\int_{\sigma_j}\frac{1}{z-a}\,dz = 0$ by Cor. 4.11 in 4.2.

$$\Rightarrow \int_{S+(-C)}\frac{1}{z-a}\,dz = 0 \quad \Rightarrow \int_S \frac{1}{z-a}\,dz = \int_C \frac{1}{z-a}\,dz = 2\pi i. \quad \square$$

(*) https://www.desmos.com/calculator/65bsuzfc9z

Theorem 2. If f is analytic in a neighborhood of the closure of $\overline{S}$ of an open square S, then, for $z \in S$,

$$f(z) = \frac{1}{2\pi i} \int_{\partial S} \frac{f(\zeta)}{\zeta - z} \, d\zeta,$$

where ∂S is parameterized in the counter-clockwise direction.

Proof. Fix $z \in S$, $\zeta \in \partial S$, $\displaystyle\frac{f(\zeta) - f(z)}{\zeta - z} = \int_0^1 f'(z + t(\zeta - z))\, dt.$

So,

$$\frac{1}{2\pi i} \int_{\partial S} \frac{f(\zeta) - f(z)}{\zeta - z} \, d\zeta = \lim_{\varepsilon \to 0} \int_\varepsilon^1 \int_{\partial S} \frac{df}{d\zeta}(z + t(\zeta - z))\, d\zeta \, \frac{dt}{t}$$

$$= 0.$$

From thm 1, $\displaystyle f(z) = (2\pi i)^{-1} \int_{\partial S} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$ □

Corollary 3. If f is analytic in a neighborhood of the closure of $\overline{S}$ of an open square S, then

$$\int_{\partial S} f(\zeta) \, d\zeta = 0.$$

Proof.

Define $g(\zeta) = f(\zeta)(\zeta - z)$, $\zeta \in \overline{S}$.

So, g is analytic in $\overline{S}$.

Apply Corollary 2:

$$0 = g(z) = \frac{1}{2\pi} \int_{\partial S} \frac{f(\zeta)(\zeta - z)}{\zeta - z}\, d\zeta = \frac{1}{2\pi} \int_{\partial S} f(\zeta)\, d\zeta.$$ □

First Version of Runge's Theorem

Theorem 4. If f is analytic on a compact set K, and if $\varepsilon > 0$, then there is a rational function r so that

$$\sup_{z \in K} |f(z) - r(z)| < \varepsilon.$$

Proof.

Second Version With Control On the Poles

Definition 5. Let $r(z) = p(z)/q(z)$ be a rational function where p and q are two polynomials with no common zeros. The zeros of q are called the **poles** of the rational function r.

Note:

- If b is a pole of a rational function r, then $|r(z)| \to \infty$ as $|z| \to \infty$.

Lemma 6. Suppose that U is a region and suppose $b \in U$. Then a rational function with poles in U can be uniformly approximated on $\mathbb{C} \backslash U$ by a rational function with poles only at b.

Proof.

Corollary 7. Suppose U is a region and suppose $\{z : |z| > R\} \subset U$ for some $R < \infty$. Then a rational function with poles only in U can be uniformly approximated on $\mathbb{C} \backslash U$ by a polynomial.

Proof.

Components

DEFINITION 8. Let U be an open set.

a) A polygonal curve in U is a curve consisting of a finite union of line segments.

b) For $a, b \in U$, define $a \sim_U b$ if and only if there is a polynomial curve contained in U with edges parallel to the axis and joining a to b.

THEOREM 9. Let $U \subset \mathbb{C}$ be an open set.

a) Show that the equivalence classes of $\sim_U$ are closed and open (relative to U) and connected.

b) Show that there are countably many equivalent classes.

Note:

- The equivalence classes are called the **components** of U. They are the maximal connected subsets of U.

Closed Components

DEFINITION 10. Suppose $K \subset \mathbb{C}$ is a compact set.

a) For $a, b \in K$, define $a \sim_K b$ if and only if there is a connected subset of K containing a and b.

THEOREM 11. Let $K \subset \mathbb{C}$ be a compact set.

a) Show that the equivalence classes of $\sim_K$ are connected and closed.

b) Show that there might be infinitely many equivalence classes.

Note:

- The equivalence classes are called the **(closed) components** of K.

THEOREM 12. [Runge] Suppose K is a compact set. Choose one point a_n in each bounded component of U_n in $\mathbb{C}\backslash K$. If f is analytic on K and $\varepsilon > 0$, then we can find a rational function r with poles only in the set $\{a_n\}$ such that

$$\sup_{z \in K} |f(z) - r(z)| < \varepsilon.$$

If $\mathbb{C}\backslash K$ has no bounded components, then we may take r to be a polynomial.

Proof.

Third Version of Runge's Theorem: Poles On The Boundary

Theorem 13. If f is analytic on an open set $\Omega \neq \mathbb{C}$, then there is a sequence of rational functions r_n with poles in $\partial\Omega$ so that r_n converges to f uniformly on compact subsets of Ω.

Proof.

Note:

- The improvement of Theorem 9 over Theorem 4 is that the poles of r_n are outside of Ω, not just outside the compact set K on which r_n is close to f.