M444 – Complex Analysis

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Chapter 4

Section 4.1: Weierstrass M-Test

Definition 1

Let f and f_n be complex-valued functions defined on a subset $E \subset \mathbb{C}$. We say that $(f_n)_{n\geq 1}$ **converges pointwise** to f on E if, for any $z \in E$, we have

$$\lim_{n\to\infty} f_n(z) = f(z).$$

Notes:

1. We use the notation $f_n \to f$ on E.
2. So $f_n \to f$ on E if and only if $\forall z \in E$, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$n \geq N \quad \Rightarrow \quad |f_n(z) - f(z)| < \varepsilon.$$

3. Here, the interger N depends on z and ε.

Example. Consider the sequence of functions

$$f_n(z) = z^n \qquad (|z| < 1).$$

Here $E := B_1(0)$.

Fix z such that $|z| < 1$. Then

$$\lim_{n \to \infty} f_n(z) = \lim_{n \to \infty} z^n = 0$$

because $|z| < 1$.

Hence, $f_n \to g$ on $B_1(0)$, where $g(z) = 0$ for any $z \in B_1(0)$.

In the previous example, notice that

$$|f_n(z)| = |z|^n \quad \Rightarrow \quad \sup_{|z|<1} |f_n(z)| = 1.$$

Hence $\lim_{n\to\infty} \sup_{|z|<1} |f_n(z)| \not\to 0$, as $n \to \infty$.

Definition 2

Let f and f_n be complex-valued functions defined on $E \subset \mathbb{C}$. We say that f_n **converges uniformly** to f on E if

$$\lim_{n\to\infty} \sup_{z\in E} |f_n(z) - f(z)| = 0.$$

Notes:

1. We use the notation $f_n \rightrightarrows f$ on E.
2. So $f_n \rightrightarrows f$ on E if and only if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that
$$n \geq N \quad \Rightarrow \quad |f_n(z) - f(z)| < \varepsilon, \forall z \in E.$$
3. Here, the integer N depends only on ε.

Example. Consider

$$f_n(z) = \frac{z^n}{n} \quad |z| \leq 1.$$

Here, $E = \overline{B_1(0)}$.

For any $|z| \leq 1$, we have

$$|f_n(z)| = \frac{|z|^n}{n} \leq \frac{1}{n}.$$

Hence,

$$\lim_{n\to\infty} \max_{|z|\leq 1} |f_n(z)| \leq \lim_{n\to\infty} \frac{1}{n} = 0.$$

Therefore $f_n \rightrightarrows 0$ on $\overline{B_1(0)}$.

Definition 3

A series of function $\sum_{n=1}^{\infty} u_n$ **converges uniformly** to u on $E \subset \mathbb{C}$ if

$$\sum_{k=1}^{n} u_k \rightrightarrows u \quad \text{on } E.$$

Notes:

1. We will abuse notation and write $\sum_{n=1}^{\infty} \rightrightarrows u$ on E.
2. With $s_n(z) = \sum_{k=1}^{n} u_n(z)$, we have

$$\sum_{n=1}^{\infty} u_n \rightrightarrows u \text{ on } E \iff \lim_{n \to \infty} \sup_{z \in E} |s_n(z) - u(z)| = 0.$$

3. More precisely, $\sum_{n=1}^{\infty} u_n \rightrightarrows u$ on E if and only if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$n \geq N \quad \Rightarrow \quad |s_n(z) - u(z)| \leq \varepsilon \quad \forall z \in E.$$

Example. Consider $\sum_{n=0}^{\infty} z^n$, for $|z| \leq \frac{1}{2}$. Here, we have $u_n(z) = z^n$.

We already know that, for a fixed z, $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$.

If n is fixed, we have, for $|z| \leq 1/2$,

$$s_n(z) = \sum_{k=0}^{n} z^n = \frac{1 - z^{n+1}}{1 - z} \quad \Rightarrow \quad \left| s_n(z) - \frac{1}{1 - z} \right| = \frac{|z|^{n+1}}{|1 - z|} \leq (1/2)^n.$$

Therefore

$$\lim_{n\to\infty} \max_{z \in B_{1/2}(0)} |s_n(z) - (1 - z)^{-1}| \leq \lim_{n\to\infty} (1/2)^n = 0.$$

Hence,

$$\sum_{k=0}^{\infty} z^k \rightrightarrows \frac{1}{1 - z} \quad \text{on } E.$$

Theorem 4.1.3

1. If $f_n \rightrightarrows f$ on E and each f_n is continuous on E, then f is continuous on E.
2. If $\sum_{n=1}^{\infty} u_n \rightrightarrows u$ on E and each u_n is continuous on E, then u is continuous on E.

Example. Consider

$$f_n(z) = \begin{cases} n|z| & \text{if } |z| < 1/n \\ 1 & \text{if } 1/n \leq |z| \leq 1 \end{cases}.$$

Then, we can show that

$$\lim_{n\to\infty} f_n(z) = g(z) = \begin{cases} 1 & \text{if } 0 < |z| \leq 1 \\ 0 & \text{if } z = 0 \end{cases}.$$

If $f_n \rightrightarrows g$ on $\overline{B_1(0)}$, then g should be continuous. However, g is not continuous. Therefore, $f_n \not\rightrightarrows g$ on $\overline{B_1(0)}$.

Theorem 4.1.5 and Corollary 4.1.6

Let Ω be a region and γ be a path in Ω.

① If each f_n is continuous on Ω and $f_n \rightrightarrows f$ on γ, then

$$\lim_{n\to\infty} \int_\gamma f_n(z)\, dz = \int_\gamma f(z)\, dz.$$

② If each u_n is continous on Ω and $\sum_{n=1}^\infty u_n \rightrightarrows u$ on γ, then

$$\int_\gamma \Big(\sum_{n=1}^\infty u_n(z) \Big)\, dz = \sum_{n=1}^\infty \int_\gamma u_n(z)\, dz.$$

Proof. Let $M_n := \max_{z\in\gamma} |f_n(z) - f(z)|$. Then, by assumption, $M_n \to 0$.

Now, we have

$$\left| \int_\gamma f_n(z)\, dz - \int_\gamma f(z)\, dz \right| \leq \left| \int_\gamma (f_n(z) - f(z))\, dz \right| \leq \ell(\gamma) M_n \to 0.$$

This shows ①. To get part ②, apply ① to $s_n(z)$. $\square$

Theorem

Let u_n be functions defined on $E \subset \mathbb{C}$ and M_n be numbers such that

1. $|u_n(z)| \leq M_n$ for all $z \in E$
2. $\sum_{n=1}^{\infty} M_n < \infty$.

Then $\sum_{n=1}^{\infty} u_n$ converges uniformly and absolutely on E.

Notes:

- Converges absolutely means that $\sum_{n=1}^{\infty} |u_n(z)| < \infty$ for any $z \in E$. In particular, $u(z) := \sum_{n=1}^{\infty} u_n(z)$ exists for every $z \in E$.
- Uniform converges: $\sum_{n=1}^{\infty} u_n \rightrightarrows u$ on E.

Example. Consider, for $|z| < 1$, $\sum_{n=0}^{\infty} z^n$. Here $u_n(z) = z^n$ with $|z| < 1$.

1. Assume that $|z| \leq r$, for some $0 < r < 1$. Then, in this case

$$|u_n(z)| = |z|^n \leq r^n.$$

Since $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} < \infty$, by the Weierstrass M-test

$$\sum_{n=0}^{\infty} z^n \rightrightarrows \frac{1}{1-z} \text{ on } \overline{B_r(0)}.$$

In other words, $\sum_{n=0}^{\infty} z^n$ convergs uniformly on every disks $\overline{B_r(0)}$.

2. However, if $|z| < 1$, then

$$\lim_{z\to 1} \sum_{n=0}^{\infty} z^n = \lim_{z\to 1} \frac{1}{1-z} = \infty$$

and hence $\sum_{n=0}^{\infty} z^n$ does not converge uniformly on $B_1(0)$.

Theorem 4.1.10

Let f_n be analytic on a region Ω for every n. Assume that $f_n \rightrightarrows f$ on every closed disk containned in Ω. Then

1. f is analytic on Ω.
2. $f_n^{(k)} \rightrightarrows f^{(k)}$ on every closed disks contained in Ω.

Consequences:

- Since z is included in a closed disk, we deduce that $f_n^{(k)} \to f^{(k)}$ on Ω.
- Applying this result on the partial sums of $\sum_{n=1}^{\infty} u_n$ with u_n analytic on Ω, we get
$$\frac{d^k}{dz^k} \sum_{n=1}^{\infty} u_n(z) = \sum_{n=1}^{\infty} \frac{d^k u_n}{dz^k}(z)$$
for every $z \in \Omega$.

Proof. We will only prove ②.

Let $\overline{B_r(z_0)}$ be a closed disk in Ω and $C_r(z_0) := \partial B_r(z_0)$. Let $d > 0$ be the minimum distance from any point of $C_r(z_0)$ to $\partial\Omega$. Let $R = r + d/2$.

By Cauchy's integral formula, for any $w \in \overline{B_r(z)}$, we have

$$f_n^{(k)}(w) - f^{(k)}(w) = \frac{k!}{2\pi i} \int_{C_R(z_0)} \frac{f_n(z) - f(z)}{z - w}\, dz.$$

Therefore, for any $w \in \overline{B_r(z)}$

$$|f_n^{(k)}(w) - f^{(k)}(w)| \leq \frac{\ell(C_R(z_0)) M_n}{d/2} = \frac{4\pi R}{d} M_n$$

where $M_n := \max_{|z-z_0|=R} |f_n(z) - f(z)| \to 0$.

Hence,

$$\lim_{n\to\infty} \max_{|w-z_0|\leq r} |f_n^{(k)}(w) - f^{(k)}(w)| \leq \frac{4\pi R}{d} \lim_{n\to\infty} M_n = 0.$$

meaning $f_n^{(k)} \rightrightarrows f^{(k)}$ on $\overline{B_r(z_0)}$. □