Math 471 | Examples Sheet Chapter E | Fall 2023

E.I Bivariate Distributions

Example 1. Let $S = \{$⚀, ⚁, ⚂, ⚃, ⚄, ⚅$\}$ with $\mathcal{A} = \mathcal{P}(S)$ be the probability space associated to throwing a fair dice. Assume that we throw two such die and consider the following random variable:

1. X: "The number of dots on the first dice".
2. Y: "The number of dots on the second dice".

Assuming the two dice are independent, what is the probability that $X \leq 2$ and $Y \leq 2$?

Example 2. Assume that X and Y are random variables with joint distribution function

$$F_{X,Y} = \begin{cases} 1 - e^{-x} - e^{-y} + e^{-x-y} & \text{if } x, y \geq 0. \\ 0 & \text{otherwise.} \end{cases}$$

Find the marginals of this joint distribution.

E.II Continuous Random Vectors

Example 3. Suppose that a radioactive particle is randomly located in a square with sides of unit length. That is, if two regions within the unit square and of equal area are considered, the particle is equally likely to be in either region. Let X and Y denote the coordinates of the particle's location. A reasonable model for the joint probability density function of X and Y is

$$f_{X,Y}(x, y) = \begin{cases} 1 & \text{if } (x, y) \in [0, 1] \times [0, 1] \\ 0 & \text{elsewhere.} \end{cases}$$

a) Sketch the probability density function.

b) Find $F(0.2, 0.4)$.

c) Find $P(0.1 \leq X \leq 0.3, 0 \leq Y \leq 0.5)$.

d) Find $P(X + Y > 0.5)$.

E.III Marginals and Independence

Example 4. Let (X, Y) be a random vector with probability density function

$$f_{X,Y}(x, y) = \begin{cases} 2x & \text{if } (x, y) \in [0, 1] \times [0, 1] \\ 0 & \text{elsewhere.} \end{cases}$$

a) Sketch $f_{X,Y}$.

b) Find the marginal density of X and Y.

Example 5. Assume that X and Y are random variables with joint distribution function

$$F_{X,Y}(x,y) = \begin{cases} 1 - e^{-x} - e^{-y} + e^{-x-y} & \text{if } x, y \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

a) Find the joint probability density function of X and Y.

b) Are X and Y independent?

Example 6. Let X and Y be two random variables having joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} 2e^{-x-y} & \text{if } 0 < x < y \\ 0 & \text{elsewhere.} \end{cases}$$

Example 7. Find $f_{X|Y}$ if X and Y have joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} 2e^{-x-y} & \text{if } 0 < x < y < \infty \\ 0 & \text{otherwise.} \end{cases}$$

E.IV Important Measurements

Example 8. Compute the expected value of the random variable $Z = X + Y$, if X and Y have joint probability density function $f_{X,Y}$.

Example 9. Let X, Y be two uniformly distributed on the unit disc, so that

$$f_{X,Y}(x,y) = \begin{cases} \pi^{-1} & \text{if } x^2 + y^2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $\mathrm{Exp}(\sqrt{X^2 + Y^2})$.

Example 10. If X and Y have joint density function

$$f(x,y) = \begin{cases} e^{-y} & \text{if } 0 < x < y < \infty \\ 0 & \text{otherwise.} \end{cases}$$

Find $\mathrm{Exp}(Y|X = x)$.

Example 11. Assume that X and Y are uniformly distributed on the rectangle with vertices $(-1,0)$, $(0,1)$, and $(1,0)$.

a) Find $\mathrm{Cov}(X,Y)$.

b) Are X and Y independent?

c) Find the coefficient of correlation for X and Y.

d) Does your answer to part (b) lead you to doubt your answer to part (a)? Why or why not.