Math 644

Chapter 2

Section 2.5: Elementary Operations

Contents

Created by: Pierre-Olivier Parisé
Spring 2023

Theorem 1. Let f and g be analytic at z_0. Then,

a) $f+g$ is analytic at z_0;

b) $f-g$ is analytic at z_0;

c) cf is analytic at z_0, for any $c \in \mathbb{C}$.

Proof.

(a) Analytic at $z_0 \Rightarrow f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$, $|z-z_0|<r_1$

& $g(z) = \sum_{n=0}^{\infty} b_n (z-z_0)$, $|z-z_0|<r_2$

So, for $r := \min\{r_1, r_2\}$, we have

$$f(z)+g(z) = \sum_{n=0}^{\infty} (a_n+b_n)(z-z_0)^n, \quad |z-z_0|<r.$$

So, $f+g$ is analytic at z_0.

Repeat for (b) & (c). $\square$

- If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are absolutely convergent series with sums A and B respectively, then their Cauchy Product

$$\left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} b_n\right) = \sum_{n=1}^{\infty}\left(\sum_{m=1}^{n-1} a_m b_{n-m}\right)$$

converges absolutely to AB. [See Problem]

Theorem 2. Let f and g be two analytic functions at z_0. Then, the function $h = fg$ is analytic at z_0.

Proof.

Write, for $0 < r < \infty$ small enough,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \& \quad g(z) = \sum_{n=0}^{\infty} b_n (z-z_0)^n$$

on $\{z : |z-z_0| < r\}$.

Since the power series of f & g converge absolutely on $\{z : |z-z_0| < r\}$, we have

$$f(z)\,g(z) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} a_m b_{n-m} (z-z_0)^n$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} a_m b_{n-m}\right) (z-z_0)^n$$

$$= \sum_{n=0}^{\infty} c_n (z-z_0)^n$$

converges in $\{z : |z-z_0| < r\}$. So $h = fg$ is analytic at z_0. $\square$

THEOREM 3. If f is analytic at z_0 and g is analytic at $a_0 = f(z_0)$, then the function $h = g \circ f$ is analytic at z_0.

Proof.

Suppose $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$, $|z-z_0| < r$

& $g(z) = \sum_{n=0}^{\infty} b_n (z-a_0)^n$, $|z-a_0| < \rho$.

The series

$$(*) \qquad \sum_{m=1}^{\infty} |a_m| \, |z-z_0|^{m-1}$$

converges in $\{z : 0 < |z-z_0| < r\}$. By the root test, the power series in $(*)$ conv. uniformly on $\{z : |z-z_0| \leq r_1\}$, $r_1 < r$.

Therefore, it is bounded:

$$\exists M > 0, \quad \sum_{m=1}^{\infty} |a_m| |z-z_0|^m \leq M |z-z_0|$$

for $|z-z_0| \leq r_1$. Therefore,

$$\sum_{n=0}^{\infty} |b_n| \left(\sum_{m=1}^{\infty} |a_m| \, |z-z_0|^m \right)^n \leq \sum_{n=0}^{\infty} |b_n| \, M^n |z-z_0|^n$$

and the RHS converges if

$$|z-z_0| < \min\left\{ \frac{r_1}{M}, \frac{\rho}{M} \right\}.$$

So,

$$\sum_{n=0}^{\infty} b_n \left(\sum_{m=0}^{\infty} a_m (z-z_0)^m - a_0 \right)$$

conv. abs. in $|z-z_0| < \min\{\frac{r_1}{M}, \frac{\rho}{M}\}$.

We can therefore rearrange the doubly indexed series so that

$$h(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n, \quad |z-z_0| < \min\{\frac{r_1}{M}, \frac{\rho}{M}\}.$$

$\square$

Consequences:

- If f is analytic at z_0 with $f(z_0) \neq 0$, then $1/f$ is analytic at z_0.
- If $r = p/q$ is a rational function, then r is analytic on $\{z : q(z) \neq 0\}$.

Definition 4. If f is defined in a disk (neighborhood) of z, then

$$f'(z) := \lim_{w \to z} \frac{f(w) - f(z)}{w - z}$$

is called the **(complex) derivative** of f, provided the limit exists.

Note:

- The function $f(z) = \overline{z}$ does not have a complex derivative.
- If n is a non-negative integer, then

$$(z^n)' = nz^{n-1}.$$

Theorem 5. If $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges in $B = \{z : |z - z_0| < r\}$, then

a) $f'(z)$ exists for all $z \in B$ and

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} (z - z_0)^n \quad (\forall z \in B).$$

b) Moreover, the series for f' based at z_0 has the same radius of convergence as the series for f.

Proof.

Let $0 < |h| < r$. Then

$$\frac{f(z_0+h) - f(z_0)}{h} - a_1 = \frac{\sum_{n=0}^{\infty} a_n h^n - a_0}{h} - a_1$$

$$= \sum_{n=1}^{\infty} a_{n+1} h^n .$$

By root test, $\sum_{n=1}^{\infty} a_{n+1} h^n$ conv. unif. on some disk around 0.

By continuity, $\lim_{h \to 0} \sum_{n=1}^{\infty} a_{n+1} h^n = 0$.

Therefore, $f'(z_0)$ exists & $f'(z_0) = a_1$.

Now, from thm 3 in Section 2.4, for fixed z,

$$(*) \quad f(w) = \sum_{k=0}^{\infty} \left[\sum_{n=k}^{\infty} a_n \binom{n}{k} (z-z_0)^{n-k} \right] (w-z)^k$$

& in some $\{w : |w-z| < \rho\} \subseteq \{w : |w-z_0| < r\}$.

From the first step,

$$\begin{aligned} f'(z) &= 2^{nd} \text{ coef of } (w-z) \text{ in } (*) \\ &= \sum_{n=1}^{\infty} a_n \binom{n}{1} (z-z_0)^{n-1} \\ &= \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}. \end{aligned}$$

For part (b), by the root test & the fact that $\liminf n^{-1/n} = 1$, the radius of convergence is the same. $\square$

Note: The rules of differentiation hold:

- $(f+g)'(z) = f'(z) + g'(z)$;
- $(cf)'(z) = cf'(z)$;
- $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$;
- $(\frac{f}{g})'(z) = (f'(z)g(z) - f(z)g'(z))/(g(z))^2$;
- $(g \circ f)(z) = g'(f(z))f'(z)$.

COROLLARY 6. An analytic function f has derivatives of all orders. Moreover, if f is equal to a convergent power series on $B = \{z : |z - z_0| < r\}$, then the power series is given by

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n \quad (\forall z \in B).$$

Proof.

If $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$, then from thm. 5

$$f'(z_0) = a_1 .$$

Again, by thm. 5,

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}$$

& from the proof of thm. 5,

$$f''(z_0) = 2a_2 \Rightarrow a_2 = \frac{f''(z_0)}{2} .$$

By induction,

$$\frac{f^{(n)}(z_0)}{n!} = a_n . \qquad \square$$

Consequences:

- If f is analytic in a region Ω with $f'(z) = 0$ for all z in a neighborhood of $z_0 \in \Omega$, then f is constant in Ω.
- If f and g are analytic in a region Ω with $f' = g'$, then $f - g$ is constant.
- If $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges in $B = \{z : |z - z_0| < r\}$, then the power series

$$F(z) := \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z - z_0)^{n+1}$$

converges in B and satisfies $F'(z) = f(z)$ for all $z \in B$.

Corollary 7. If $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges in $B = \{z : |z-z_0| < r\}$, then

$$f'(z_0) = \lim_{z,w \to z_0} \frac{f(z)-f(w)}{z-w}.$$

__Proof.__ Set $z = z_0 + h$ & $w = z_0 + k$, $|h|, |k| < r$.

Then, for $\max\{|h|, |k|\} \leq \varepsilon < r$,

$$\frac{f(z_0+h) - f(z_0+k)}{h-k} - a_1 = \sum_{n=2}^{\infty} a_n \frac{h^n - k^n}{h-k}$$

$$= \sum_{n=2}^{\infty} a_n \left(\sum_{j=0}^{n-1} h^j k^{n-j-1} \right).$$

We have

$$\sum_{n=N}^{M} |a_n| \left(\sum_{j=0}^{n-1} |h|^j |k|^{n-j-1} \right) \leq \sum_{n=N}^{M} n |a_n| \varepsilon^{n-1}$$

By the root test, $\sum_{n=1}^{\infty} n |a_n| \varepsilon^{n-1}$ is abs. convergent. Thus,

$$\sum_{n=2}^{\infty} a_n \left(\sum_{j=0}^{n-1} h^j k^{n-j-1} \right)$$ conv. unif. & abs. in h & k for $|h|, |k| \leq \varepsilon$.

By uniform convergent, (*) is continuous at $(0,0)$ & vanishes at $(0,0)$

$$\Rightarrow \lim_{h,k \to 0} \left[\frac{f(z_0+h) - f(z_0+k)}{h-k} - a_1 \right] = \lim_{h,k \to 0} \sum_{n=2}^{\infty} a_n \left(\sum_{j=0}^{n-1} h^j k^{n-j-1} \right)$$

$$= 0.$$

$\square$