Chapter 2: Derivatives
Week 5

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Calculus I (MATH-241 01/02)

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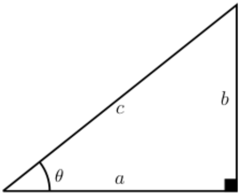
Upcoming this week

1. 2.4 Derivative of Trigonometric Functions
2. 2.5 Chain Rule
3. 2.6 Implicit differentiation

Recall the definitions of the trigonometric functions:



$$\begin{aligned}
\sin\theta &= \frac{\text{opposite side of } \theta}{\text{hypothenuse}} = \frac{b}{c} & \csc\theta &= \frac{\text{hypothenuse}}{\text{opposite side of } \theta} = \frac{c}{b} \\
\cos\theta &= \frac{\text{adjacent side of } \theta}{\text{hypothenuse}} = \frac{a}{c} & \sec\theta &= \frac{\text{hypothenuse}}{\text{adjacent side of } \theta} = \frac{b}{c} \\
\tan\theta &= \frac{\text{opposite side of } \theta}{\text{adjacent side of } \theta} = \frac{b}{a} & \cot\theta &= \frac{\text{adjacent side of } \theta}{\text{opposite side of } \theta} = \frac{a}{b}
\end{aligned}$$

Pythagore's formula : $a^2 + b^2 = c^2$

Remarks : $\tan x = \dfrac{\sin x}{\cos x} \quad \cot x = \dfrac{\cos x}{\sin x} \quad \sec x = \dfrac{1}{\cos x} \quad \csc x = \dfrac{1}{\sin x}.$

Question 1

What would be the derivative of the function $f(x) = \sin x$?

Some hints:

- try to draw the function $\sin x$ with Desmos.
- try to find the slope of the tangent line to the curve $y = \sin(x)$ at some remarkable points with the help of the graph.
- try to draw a table of values of the quotient $\frac{\sin(x+h)-\sin(x)}{h}$ for values of h near 0.
- try to compute exactly the limit using trigonometric identities (use the Trigonometric sheet on the course website).

Question 2

Now, what would be the derivative of $\cos x$? of $\tan x$?

Theorem 3

We have

- $\frac{d}{dx} \sin x = \cos x.$
- $\frac{d}{dx} \cos x = -\sin x.$
- $\frac{d}{dx} \tan x = \sec^2 x.$

Example 4

Compute the derivative of $f(x) = x^2 \sin(x)$.

Example 5

Compute the derivative of

- $f(x) = \frac{1}{\sin x}$.
- $f(x) = \frac{1}{\cos x}$.
- $f(x) = \frac{1}{\tan x}$.

Theorem 6

We have

- $\frac{d}{dx}(\csc x) = -\csc x \cot x.$
- $\frac{d}{dx}(\sec x) = \sec x \tan x.$
- $\frac{d}{dx}(\cot x) = -\csc^2 x.$

Example 7

Compute the derivative of $f(x) = \frac{\sec x}{1+\tan x}$.

Exercises: 1-25, 32, 34, 35, 39-50, 55, 56.

Example 8

Suppose that a the volume of a balloon is given by $V(r) := \frac{4\pi}{3} r^3$ where r is the radius of the balloon. You inflate air in such a way that $r(t) = (t^2 + 1)$ where t is the time (in seconds) after you started to inflate the balloon.

- What is the speed at which the volume increases?

Theorem 9

If f and g are two differentiable functions where g is defined on the range of f, then

$$(g \circ f)'(x) = g'(f(x))f'(x).$$

Notations: If $y = g(f(x))$ and $u = f(x)$ (the intermediate function), then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

Example 10

Find $F'(x)$ if $F(x) = \sqrt{x^2+1}$.

Here is exactly what's happening in the chain rule:

$$\frac{d}{dx}\underbrace{f}_{\text{Outer function}}\big(\underbrace{g(x)}_{\text{Inner function}}\big) = \underbrace{f'}_{\text{derivative of outer function}}\big(\underbrace{g(x)}_{\text{evaluate at inner function}}\big)\cdot\underbrace{g'(x)}_{\text{derivative of inner function}}.$$

When Stark (the function) uses the HulkBuster (the Power rule), we get something remarkable!

Theorem 11 (The Powain rule)

If b is any real number and $u = g(x)$ is a differentiable function, then

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}.$$

Alternatively, $\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1}g'(x)$.

Example 12

Find y' if $y = (x^3 - 1)^{100}$.

Exercises: 1-50, 52, 55, 63, 67, 76, 77, 88.

Most of the functions are expressed in terms of $y = f(x)$. For example,

$$y = \sqrt{x^2 + 1} \quad \text{or} \quad y = x \sin x.$$

Not all relations can be defined explicitly like that... Some are defined implicitly by a relation between x and y:

$$x^2 + y^2 = 25 \quad \text{or} \quad x^3 + y^3 = 6xy.$$

- In some cases, we can solve the equation and express y in terms of x: $y = \pm\sqrt{25 - x^2}$.
- In other cases, we are not able to easily solve the equation and express y in terms of x: folium of Descartes. Implicit Functions

Question 13

Can you find the slope of the tangent line to the

- circle given by the formula $x^2 + y^2 = 25$ at $P = (3, 4)$.
- to the folium of Descartes given by the equation $x^3 + y^3 = 6xy$ at $P = (3, 3)$.

Definition 14

By differentiate implicitly, we mean to differentiate both sides of an equation with respect to x.

Example 15

Let $x^3 + y^3 = 6xy$ be the folium of Descartes.

a) Find y'.

b) Find the equation of the tangent line passing through the point $P = (3, 3)$.

Example 16

Reprove that if $y = \tan x$, then $y' = \sec^2 x$.

Exercises: 1-21, 23, 25-32, 34, 42, 49, 50, 55, 62.