$u = x^2 \rightarrow \boxed{du = 2x\,dx}$

$$\int_0^1 x\cos(x^2)\,dx = \int_0^1 \cos(u)\,\frac{du}{2}$$

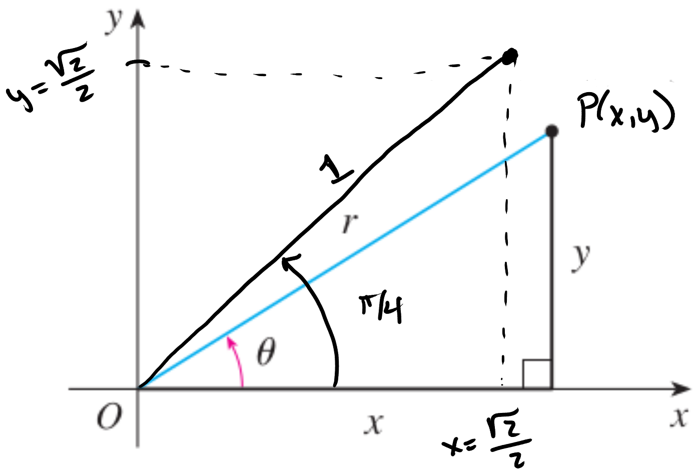
Chapter 15

Multiple Integrals

15.3 Double Integrals in polar coordinates

Polar coordinates

$$r = 1,\ \theta = \frac{\pi}{4} \longrightarrow x = 1\cos(\pi/4) = \sqrt{2}/2,\quad y = 1\sin(\pi/4) = \sqrt{2}/2$$



1) Polar to Cartesian:

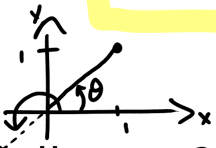
$$x = r\cos(\theta),\quad y = r\sin\theta$$

2) Cartesian to Polar:

$$r = \sqrt{x^2+y^2}$$

$$\tan\theta = \frac{y}{x} \Rightarrow \theta = \arctan\left(\frac{y}{x}\right)$$

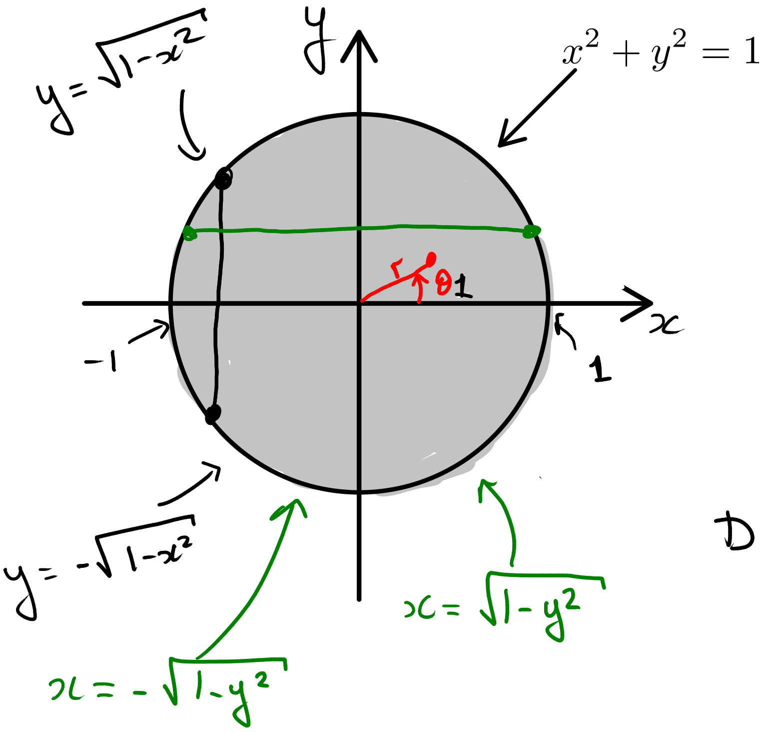
$$\left(\theta = \tan^{-1}\left(\frac{y}{x}\right)\right)$$



Why would we use polar coordinates?

$$x = 1,\ y = 1 \longrightarrow r = \sqrt{2},\quad \theta = \boxed{\pi/4},\ \cancel{5\pi/4},\ \ldots$$

Example. Describe the following region:



TYPE I:

$$D = \{(x,y) : -1 \le x \le 1 \text{ and } -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}\}$$

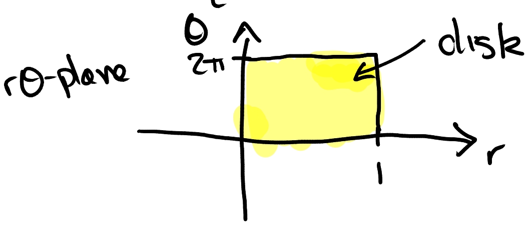
TYPE II:

$$D = \{(x,y) : -1 \le y \le 1 \text{ and } -\sqrt{1-y^2} \le x \le \sqrt{1-y^2}\}$$

Polar coordinates $\quad 0 \le \underbrace{\text{Distance from origin}}_{= r} \le 1$

$$0 \le \text{angle} = \theta \le 2\pi$$

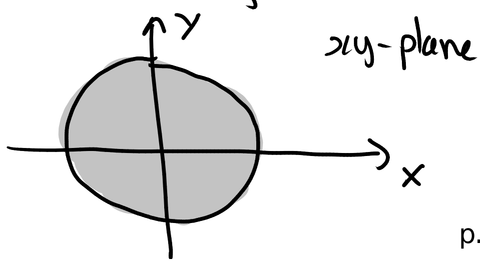
$$D = \{(r,\theta) : 0 \le r \le 1,\ 0 \le \theta \le 2\pi\} \longrightarrow \text{rectangle.}$$



$$\xrightarrow{\qquad}$$

$x = r\cos\theta$

$y = r\sin\theta$



How does it affect the double integral

$u = x^2$
$du = \boxed{2x}\,dx$

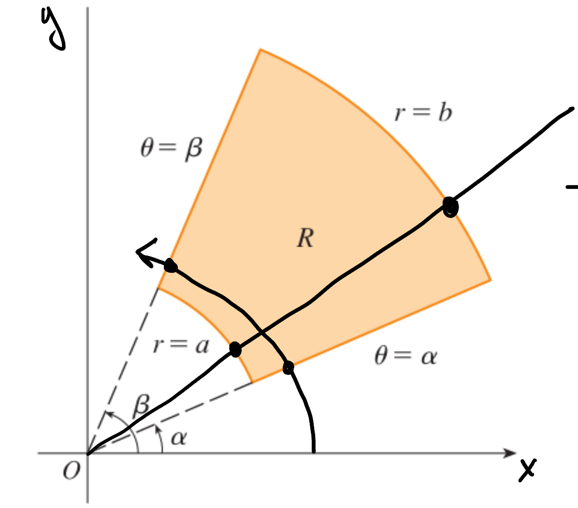
$dA = ??$
$\boxed{}\,dr\,d\theta$

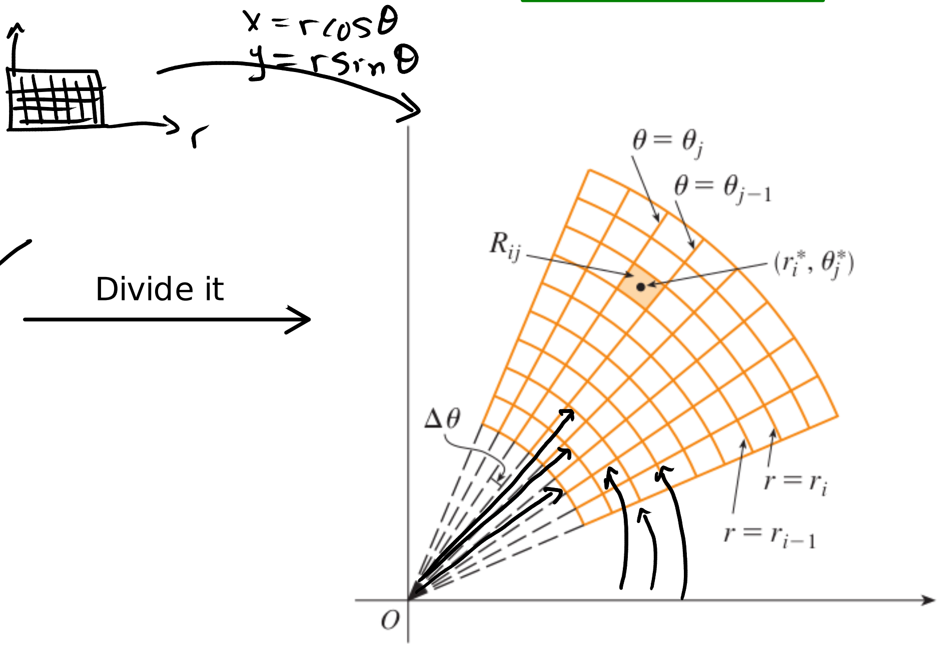
Recall:

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dx\,dy \longrightarrow \boxed{dA = dx\,dy}$$

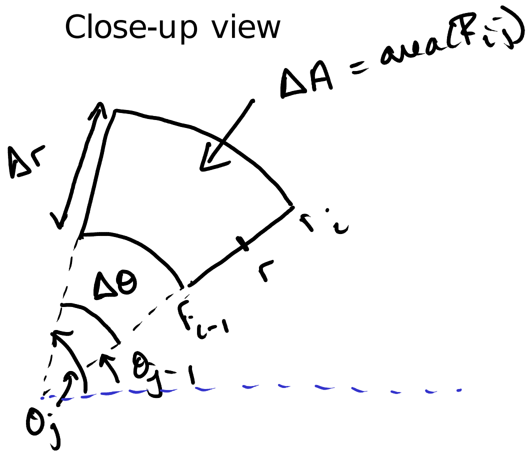
$$= \int_c^d \int_a^b f(x,y)\,dy\,dx \longrightarrow \boxed{dA = dy\,dx}$$

Polar rectangle:





Close-up view



$$\Delta A = \frac{\Delta\theta \cdot r_i^2}{2} - \frac{\Delta\theta \cdot r_{i-1}^2}{2}$$

$$= \frac{\Delta\theta}{2}\left(r_i^2 - r_{i-1}^2\right)$$

$$= \frac{\Delta\theta}{2}(r_i - r_{i-1})(r_i + r_{i-1})$$

$$= \Delta\theta\,\Delta r\left(\frac{r_i + r_{i-1}}{2}\right)$$

$$= \Delta\theta\,\Delta r \cdot r$$

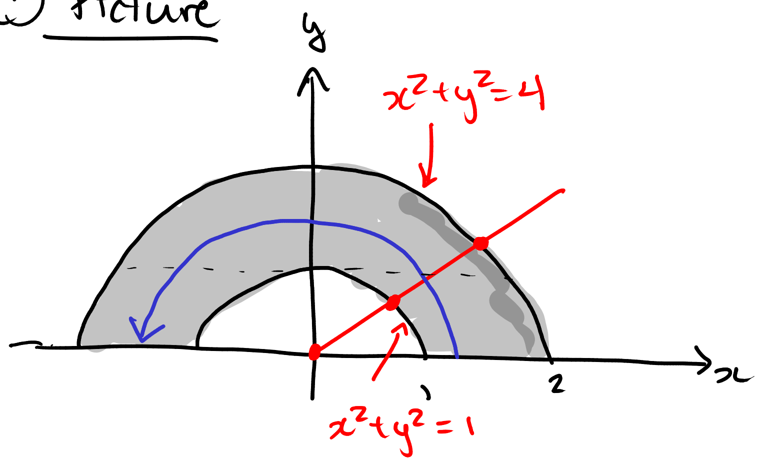
$$\Rightarrow \boxed{\Delta A = r\,\Delta\theta\,\Delta r} \longrightarrow \boxed{\begin{array}{l} dA = r\,d\theta\,dr \\ dA = r\,dr\,d\theta \end{array}}$$

$$\boxed{\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta}$$

R is a polar rectangle given by $a \le r \le b$ and $\alpha \le \theta \le \beta$, with $\beta - \alpha \le 2\pi$.

EXAMPLE 1 Evaluate $\iint_R (3x + 4y^2)\, dA$, where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

① Picture



Polar coord.

$D = \{(r,\theta) : 1 \le r \le 2,\ 0 \le \theta \le \pi\}$

$= [1,2] \times [0,\pi]$

$x^2+y^2=4 \to r^2=4 \to r=2$

$x^2+y^2=1 \to r^2=1 \to r=1$

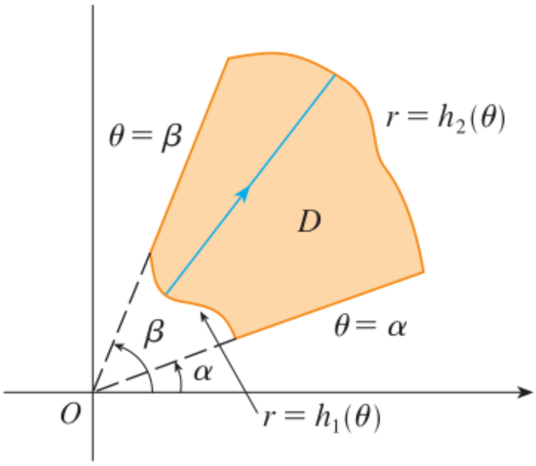
② Integrate

$$\iint_R 3x + 4y^2\, dA = \int_0^{\pi}\int_1^2 (3r\cos\theta + 4r^2\sin^2\theta)\, r\, dr\, d\theta$$

$x = r\cos\theta$ $\quad$ $y = r\sin\theta$

EXAMPLE 2 Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.

More complicated region:



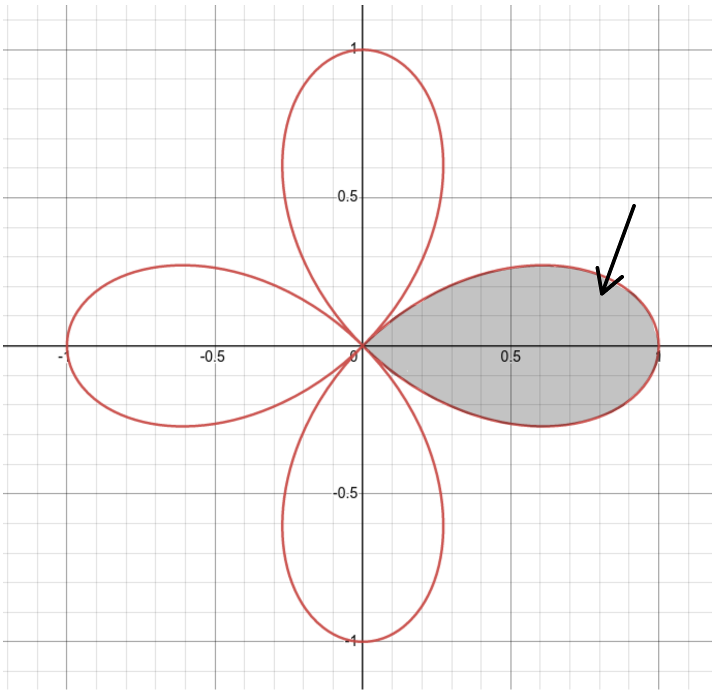
3 If f is continuous on a polar region of the form

$$D = \{(r, \theta) \mid \alpha \leqslant \theta \leqslant \beta,\ h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$$

then
$$\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

EXAMPLE 3 Use a double integral to find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

$\boxed{1}$ PICTURE



EXAMPLE 4 Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.