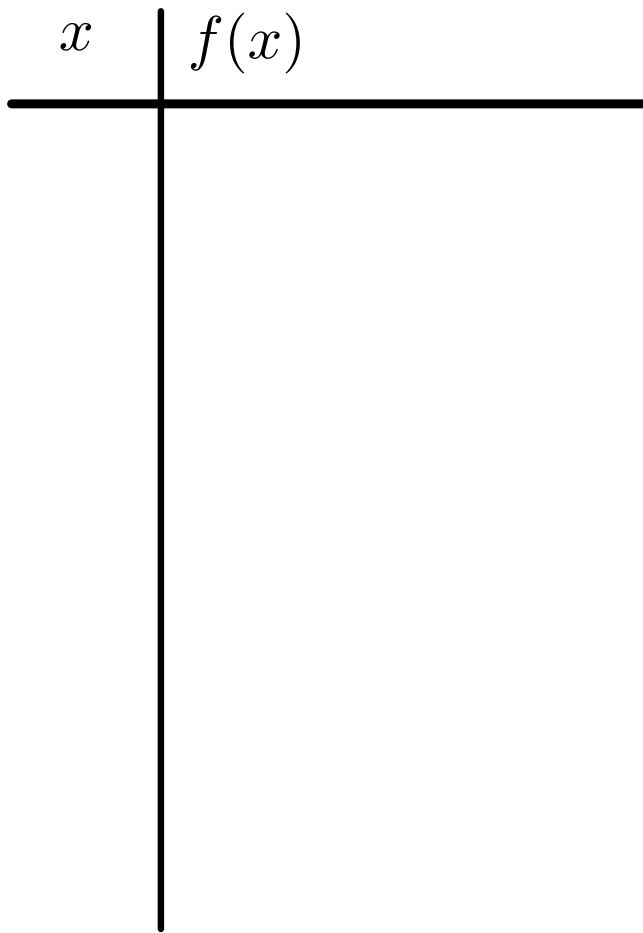


# Chapter 3

## Applications of Derivatives

3.4 Limits at Infinity; Horizontal Asymptotes

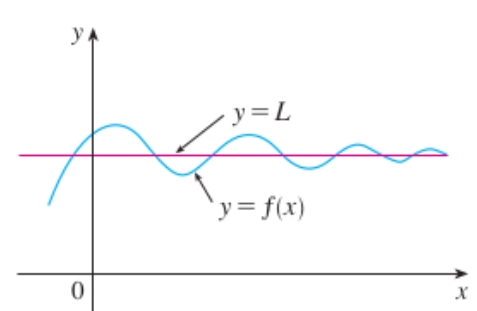
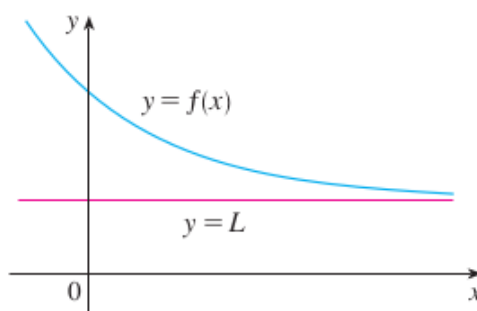
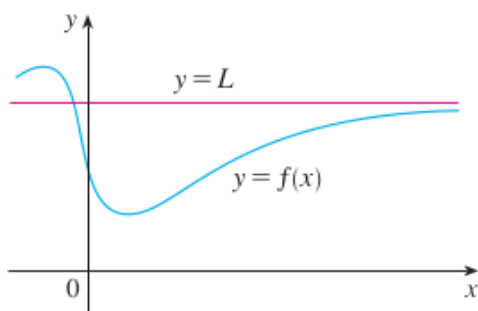
**Example.** What is the limit of  $f(x) = \frac{x^2 - 1}{x^2 + 1}$  when  $x$  becomes large?



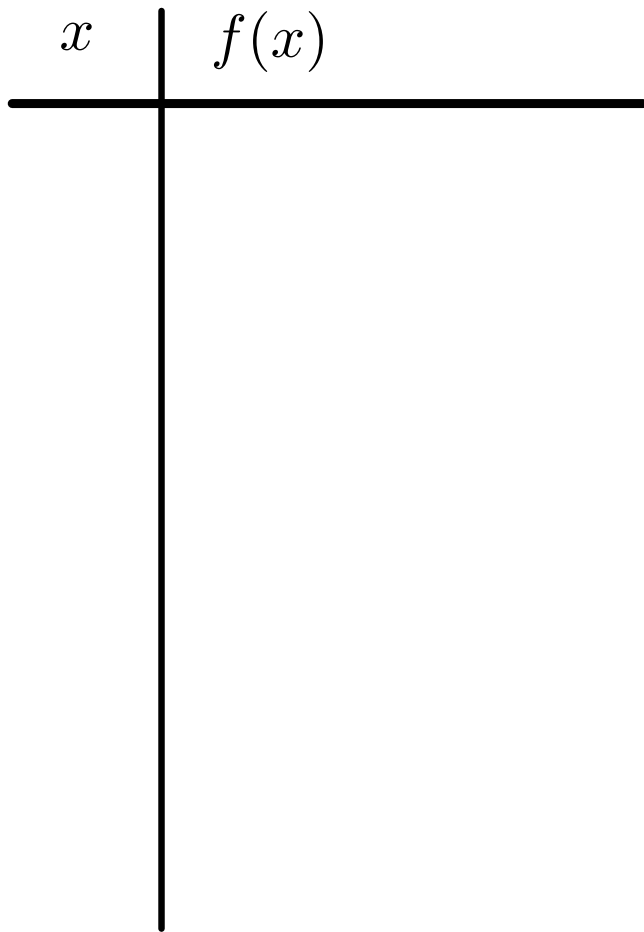
**1 Intuitive Definition of a Limit at Infinity** Let  $f$  be a function defined on some interval  $(a, \infty)$ . Then

$$\lim_{x \rightarrow \infty} f(x) = L$$

means that the values of  $f(x)$  can be made arbitrarily close to  $L$  by requiring  $x$  to be sufficiently large.



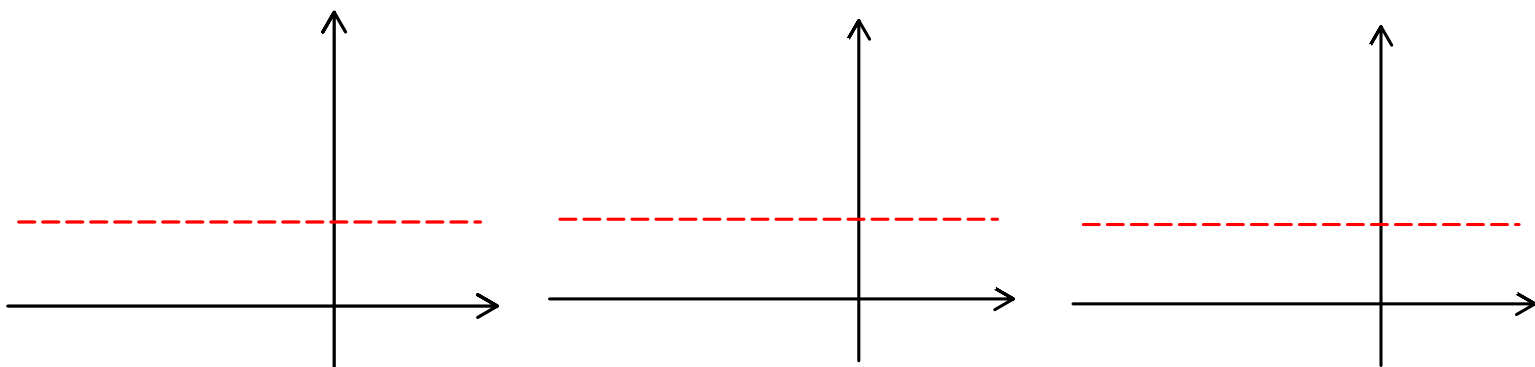
**Example.** What is the limit of  $f(x) = \frac{x^2 - 1}{x^2 + 1}$  when  $x$  becomes large?



**2 Definition** Let  $f$  be a function defined on some interval  $(-\infty, a)$ . Then

$$\lim_{x \rightarrow -\infty} f(x) = L$$

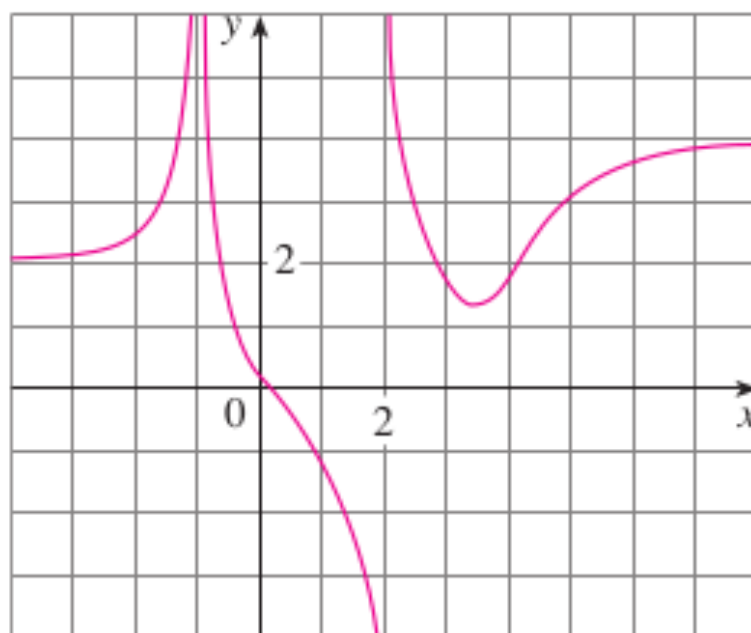
means that the values of  $f(x)$  can be made arbitrarily close to  $L$  by requiring  $x$  to be sufficiently large negative.



**3 Definition** The line  $y = L$  is called a **horizontal asymptote** of the curve  $y = f(x)$  if either

$$\lim_{x \rightarrow \infty} f(x) = L \quad \text{or} \quad \lim_{x \rightarrow -\infty} f(x) = L$$

**EXAMPLE 1** Find the infinite limits, limits at infinity, and asymptotes for the function  $f$  whose graph is shown in Figure 5.



**FIGURE 5**

**4 Theorem** If  $r > 0$  is a rational number, then

$$\lim_{x \rightarrow \infty} \frac{1}{x^r} = 0$$

If  $r > 0$  is a rational number such that  $x^r$  is defined for all  $x$ , then

$$\lim_{x \rightarrow -\infty} \frac{1}{x^r} = 0$$

---

**EXAMPLE 3** Evaluate

$$\lim_{x \rightarrow \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$$

**EXAMPLE 4** Find the horizontal and vertical asymptotes of the graph of the function

$$f(x) = \frac{\sqrt{2x^2 + 1}}{3x - 5}$$

**EXAMPLE 5** Compute  $\lim_{x \rightarrow \infty} (\sqrt{x^2 + 1} - x)$ .

The notation

$$\lim_{x \rightarrow \infty} f(x) = \infty$$

means that the values of  $f(x)$  become larger and larger as the values of  $x$  becomes larger and larger. Similar meanings are attached to the following symbols:

$$\lim_{x \rightarrow -\infty} f(x) = \infty, \quad \lim_{x \rightarrow \infty} f(x) = -\infty \quad \text{and} \quad \lim_{x \rightarrow -\infty} f(x) = -\infty.$$

**WARNING!!**

---

**EXAMPLE 8** Find  $\lim_{x \rightarrow \infty} x^3$  and  $\lim_{x \rightarrow -\infty} x^3$ .



**EXAMPLE 9** Find  $\lim_{x \rightarrow \infty} (x^2 - x)$ .