

# MATH 644

## PROBLEM SETS

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**PROBLEM 1.** Prove the parallelogram equality:

$$|z + w|^2 + |z - w|^2 = 2(|z|^2 + |w|^2).$$

**PROBLEM 2.** Let  $w$  be a non-zero complex number and let  $n \geq 1$  be a positive integer. Using the polar coordinates, find  $n$  solutions to  $z^n = w$ .

**PROBLEM 3.** Let  $z$  be a non-zero complex number. Show that  $0$ ,  $z$ ,  $iz$ , and  $iz + z$  are the vertices of a square.

**PROBLEM 4.** Prove that there is no complex number  $z$  so that

$$|z| - z = i.$$

**PROBLEM 5.** Find all complex numbers  $z$  satisfying the equation

$$4z - 3\bar{z} = \frac{1 - 18i}{2 - i}.$$

**PROBLEM 6.** Suppose that  $f$  is a continuous complex-valued function on a real interval  $[a, b]$ . Let

$$A = \frac{1}{b - a} \int_a^b f(x) dx.$$

- a) Show that if  $|f(x)| \leq |A|$  for all  $x \in [a, b]$ , then  $f \equiv A$ .
- b) Show that if  $|A| = \frac{1}{b-a} \int_a^b |f(x)| dx$ , then  $\arg f$  is constant modulo  $2\pi$  on  $\{z : f(z) \neq 0\}$ .

**PROBLEM 7.** Describe geometrically the following subsets:

- a)  $\operatorname{Re} z = \operatorname{Im} z$ .
- b)  $\operatorname{Re} z > 0$ .
- c)  $\operatorname{Im} z > 0$ .
- d)  $\frac{\pi}{6} < \arg z < \frac{\pi}{4}$ .

**PROBLEM 8.** Let  $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ . Prove that  $\mathbb{T}$  equipped with the complex multiplication is a commutative group.

**PROBLEM 9.** Suppose that  $\lim_{n \rightarrow \infty} w_n = w$ . Is it true then that also

$$\lim_{n \rightarrow \infty} \arg w_n = \arg w?$$

**PROBLEM 10.** Let  $\{z_n\}$  be a sequence of complex numbers such that  $\sum_{n=0}^{\infty} z_n$  converges and there is a  $\phi$  such that  $|\arg z_n| \leq \phi < \frac{\pi}{2}$  for any  $n \geq 0$ . Show that the series  $\sum_{n=0}^{\infty} z_n$  is absolutely convergent.

**PROBLEM 11.** Let  $\mathbb{C}^*$  be the extended plane, let  $\mathbb{S}^2$  be the sphere  $\{(X, Y, Z) : X^2 + Y^2 + Z^2 = 1\}$  and let  $\pi : \mathbb{C}^* \rightarrow \mathbb{S}^2$  be the stereographic projection with  $\pi(\infty) = (0, 0, 1)$ .

- a) Show that straight lines in  $\mathbb{C}$  correspond exactly to circles on  $\mathbb{S}^2$  passing through  $(0, 0, 1)$ .
- b) Show that if  $z \neq \infty$ , then

$$\chi(z, \infty) = \frac{2}{\sqrt{1 + |z|^2}}.$$

- c) Using the explicit formula of  $\chi$  in terms of  $z$  and  $w$ , show that, for any  $z, w \in \mathbb{C}^*$ ,

$$0 \leq \chi(z, w) \leq 2.$$

**PROBLEM 12.** For what values of  $z$  is

$$\sum_{n=0}^{\infty} \left( \frac{z}{1+z} \right)^n$$

convergent? Draw a picture of the region.

**PROBLEM 13.** Suppose that  $\sum_{n \geq 0} a_n (z - z_0)^n$  is a formal power series. Suppose that

$$R := \frac{1}{\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|}$$

exists and is finite.

- a) Show that the power series converges in  $\{z : |z - z_0| < R\}$ .
- b) Show that the power series diverges in  $\{z : |z - z_0| > R\}$ .

**PROBLEM 14.** Define  $e^z = \exp(z) := \sum_{n \geq 0} \frac{z^n}{n!}$ .

- a) Show that  $e^z e^w = e^{z+w}$  (using the power series definition).
- b) Show that  $|e^z| = e^{\operatorname{Re} z}$  and  $\arg e^z = \operatorname{Im} z$ .
- c) Show that  $\frac{d}{dz} e^z = e^z$ .
- d) Show that, for any non-zero integer  $n$ ,

$$\int_0^{2\pi} e^{int} dt = 0.$$

[Hint: Use Fundamental Theorem of Calculus.]

- e) Compute the integral

$$\int e^{nt} \cos(mt) dt.$$

[Hint: Rewrite  $\cos(mt)$  as a complex exponential.]

**PROBLEM 15.** Prove the following assertions.

- a) If  $f$  and  $g$  are analytic at  $z_0$ , then  $(f + g)'(z_0) = f'(z_0) + g'(z_0)$  (Sum rule of differentiation for analytic functions).
- b) If  $f$  and  $g$  are analytic at  $z_0$ , then  $(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$  (Product Rule of differentiation for analytic functions).
- c) If  $f$  and  $g$  are analytic at  $z_0$  with  $g(z_0) \neq 0$ , then  $(f/g)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{(g(z_0))^2}$  (Quotient rule of differentiation for analytic functions).

- d) If  $f$  is analytic at  $z_0$  and  $g$  is analytic at  $f(z_0)$ , then  $(g \circ f)'(z_0) = g'(f(z_0))f'(z_0)$  (Chain Rule of differentiation for analytic functions).

Find the derivative of  $(z - a)^{-n}$ , where  $n$  is a positive integer and  $a \in \mathbb{C}$ .

**PROBLEM 16.** Let  $\Omega \subset \mathbb{C}$ . Show that  $\Omega$  is connected if and only if  $\Omega$  and  $\emptyset$  are the only open and closed subsets of  $\Omega$ .

**PROBLEM 17.** Suppose that  $f$  and  $g$  are two analytic functions on a region (open and connected)  $\Omega$ . Suppose there is a sequence  $(z_n)_{n=1}^{\infty}$  with  $z_n \in \Omega$  ( $\forall n \geq 1$ ) such that  $f(z_n) = g(z_n)$  ( $\forall n \geq 1$ ). If  $(z_n)$  has an accumulation point  $z_0 \in \Omega$ , then show that  $f \equiv g$  on  $\Omega$ .

**PROBLEM 18.** Show that  $\cos^2(z) + \sin^2(z) = 1$  for every  $z \in \mathbb{C}$ .

**PROBLEM 19.** Suppose  $f$  is analytic in a connected open set  $\Omega$  such that, for each  $z \in \Omega$ , there exists an  $n$  (depending on  $z$ ) such that  $f^{(n)}(z) = 0$ . Prove that  $f$  is a polynomial. [Hint: Use Baire's Theorem.]

**PROBLEM 20.** Let  $f$  be analytic in a region  $\Omega$  containing the point  $z = 0$ . Suppose  $|f(1/n)| < e^{-n}$  for  $n \geq n_0$ , for some integer  $n_0 \geq 0$ . Prove  $f \equiv 0$  in  $\Omega$ .

**PROBLEM 21.** Let  $f$  and  $g$  be analytic functions in a region  $\Omega$ .

- Show that if  $f'(z) = 0$  for all  $z$  in a neighborhood of some  $z_0 \in \Omega$ , then  $f$  is constant in  $\Omega$ , meaning there is a constant  $c \in \mathbb{C}$  such that  $f(z) = c$  for any  $z \in \Omega$ .
- Show that if  $f$  and  $g$  are analytic in a region  $\Omega$  with  $f'(z) = g'(z)$  for every  $z \in \Omega$ , then  $f - g$  is constant.

**PROBLEM 22.** Suppose that  $f(z) = az^3 + bz^2 + cz + d$ . In addition, suppose that for each  $z, w \in \mathbb{C}$  there exists a point  $\zeta$  on the line segment between  $z$  and  $w$  with

$$\frac{f(z) - f(w)}{z - w} = f'(\zeta).$$

Show that  $a = 0$ .

**PROBLEM 23.** Suppose that  $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$  converges in  $B = \{z : |z - z_0| < r\}$ . Show that the power series

$$F(z) := \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z - z_0)^{n+1}$$

converges in  $B$  and satisfies  $F'(z) = f(z)$  for all  $z \in B$ . Moreover, show that the radius of convergence of  $F$  is the same as the radius of convergence of  $f$ .

**PROBLEM 24.** Suppose  $\sum_{j=0}^{\infty} |a_j|^2 < \infty$ .

- Show that  $f(z) = \sum_{j=0}^{\infty} a_j z^j$  is analytic in  $\{z : |z| < 1\}$ .

b) Compute (with a proof) the following quantity:

$$\lim_{r \rightarrow 1^-} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi}.$$

**PROBLEM 25.** Suppose  $f$  has a power series expansion at 0 which converges in all of  $\mathbb{C}$ . Suppose also that  $\int_{\mathbb{C}} |f(x + iy)| dx dy < \infty$ . Prove that  $f \equiv 0$ .

**PROBLEM 26.** [Hard] Let  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  have radius of convergence 1 and suppose that  $a_n \geq 0$  for all  $n$ . Prove that  $z = 1$  is a singular point of  $f$ . That is, there is no function  $g$  analytic in a ball  $B$  containing  $z = 1$  such that  $f = g$  on  $B \cap D$ .

**PROBLEM 27.** If  $f$  is analytic in a region  $\Omega$  and if there is a  $z_0 \in \Omega$  such that

$$|f(z_0)| = \inf_{z \in \Omega} |f(z)|,$$

and if  $f(z_0) \neq 0$ , then  $f$  is constant in  $\Omega$ .

**PROBLEM 28.** Let  $\Omega$  be a region in  $\mathbb{C}$ . Show that if  $f : \Omega \rightarrow \mathbb{C}$  is an open map, then  $f$  satisfies the maximum modulus principle.

**PROBLEM 29.**

- a) Show geometrically why the maximum principle holds using a “walking the dog” argument. Make it rigorous by following the steps of the proof of the Fundamental Theorem of Algebra.
- b) Use the maximum modulus principle to prove the Fundamental Theorem of Algebra.

**PROBLEM 30.** Let  $f$  be an analytic function defined on some bounded region  $\Omega \subset \mathbb{C}$ . Show that

$$\limsup_{z \rightarrow \partial\Omega} |f(z)| = \limsup_{\delta \rightarrow 0} \{ |f(z)| : z \in \Omega, \text{dist}(z, \partial\Omega) = \delta \}.$$

**PROBLEM 31.** Let  $\Omega \subset \mathbb{C}$  be a region. Show that  $z_n \rightarrow \partial\Omega$  as  $n \rightarrow \infty$  if and only if  $(z_n)_{n \geq 1}$  has no subsequence converging to a point  $z_0 \in \Omega$ .

**PROBLEM 32.** Suppose that  $f$  is analytic in a connected open (region) set  $\Omega$ .

- a) Prove that if  $|f(z)|$  is constant on  $\Omega$ , then  $f$  is constant on  $\Omega$ .
- b) Prove that if  $\text{Re } f$  is constant on  $\Omega$ , then  $f$  is constant on  $\Omega$ .

**PROBLEM 33.**

- a) Prove that if  $f$  is analytic in  $\mathbb{C}$ , then  $f(z) = \sum_{n \geq 0} a_n z^n$  for any  $z \in \mathbb{C}$ . In other words, the radius of convergence of the power series  $\sum_{n \geq 0} a_n z^n$  representing  $f$  at  $z = 0$  is  $R = \infty$ .
- b) Suppose that  $f$  is analytic in  $\mathbb{C}$  and  $|f(z)| \leq C|z|^n$ , for some  $|z| > M$  and  $n \geq 0$ . Show that  $f$  must be a polynomial.
- c) Suppose that  $f$  and  $g$  are analytic in  $\mathbb{C}$  with  $|f(z)| \leq |g(z)|$  for all  $z \in \mathbb{C}$ . Prove that there exists a constant  $c \in \mathbb{C}$  such that  $f(z) = cg(z)$  for all  $z \in \mathbb{C}$ .

**PROBLEM 34.** Prove that if  $f$  is non-constant and analytic on all of  $\mathbb{C}$ , then  $f(\mathbb{C})$  is dense in  $\mathbb{C}$ .

**PROBLEM 35.** Let  $f$  be analytic in  $\mathbb{D}$  and suppose  $|f(z)| < 1$  on  $\mathbb{D}$ . Let  $a = f(0)$ . Show that  $f$  does not vanish in  $\{z : |z| < |a|\}$ .

**PROBLEM 36.**

- a) Prove that  $\varphi$  is a one-to-one analytic map of  $\mathbb{D}$  onto  $\mathbb{D}$  if and only if

$$\varphi(z) = c \left( \frac{z - a}{1 - \bar{a}z} \right) \quad (z \in \mathbb{D}),$$

for some constants  $c$  and  $a$ , with  $|c| = 1$  and  $|a| < 1$ .

- b) Let  $f$  be analytic in  $\mathbb{D}$  and satisfy  $|f(z)| \rightarrow 1$  as  $|z| \rightarrow 1$ . Prove that  $f$  is rational.

**PROBLEM 37.**

- a) Suppose  $p$  is a non-constant polynomial with all its zeros in the upper half-plane  $\mathbb{H} := \{z : \operatorname{Im} z > 0\}$ . Prove that all the zeros of  $p'$  are contained in  $\mathbb{H}$ . [*Hint: Look at the partial fraction expansion of  $p'/p$ .*]
- b) Use a) to prove that if  $p$  is a polynomial, then the zeros of  $p'$  are contained in the (closed) convex hull of the zeros of  $p$ . (The closed convex hull is the intersection of all half-planes containing the zeros.)

**PROBLEM 38.** Suppose  $f$  is analytic in  $\mathbb{D}$  and  $|f(z)| \leq 1$  in  $\mathbb{D}$  and  $f(0) = 1/2$ . Prove that  $|f(1/3)| \geq 1/5$ .

**PROBLEM 39.** Suppose  $f$  is analytic and non-constant in  $\mathbb{D}$  and  $|f(z)| \leq M$  on  $\mathbb{D}$ . Prove that the number of zeros of  $f$  in a disk of radius  $1/4$ , centered at 0, does not exceed

$$\frac{1}{\ln 4} \ln \left| \frac{M}{f(0)} \right|.$$



**PROBLEM 40.** Find a parametrization for each set.

- (a) A square of height  $h$  and length  $l$ .
- (b) The arc of the unit circle starting at 1 and ending at  $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ .
- (c) The parabola starting at  $(0, 0)$  and ending at  $(1, 1)$ .
- (d) An ellipse centered at  $z_0 \in \mathbb{C}$  with minor axis  $m$  and major axis  $M$ .

**PROBLEM 41.** Which of the curves in the previous problem are (i) an arc (ii) closed (ii) simple?

**PROBLEM 42.** Find the value of

$$\int_{\gamma} x - y + ix^2 dz$$

if  $\gamma$  is

- (a) the straight line joining 0 to  $1 + i$ ;
- (b) the imaginary axis from 0 to  $i$ ;
- (c) the line parallel to the real axis from  $i$  to  $1 + i$ .

**PROBLEM 43.** Compute  $\int_{\gamma} \frac{1}{z^3} dz$ , where  $\gamma$  is the circle of radius  $1/2$  centered at the origin. Compare with Example 9 from Section 4.1.

**PROBLEM 44.** Let  $\gamma$  denote a circular path with center 1 and radius 1, described once counter-clockwise and starting at the point 2.

- (a) Find a parametrization of  $\gamma$ .
- (b) Find the value of  $\int_{\gamma} |z|^2 dz$ .

**PROBLEM 45.** Suppose that  $f$  and  $g$  are two holomorphic function on a region  $\Omega$  containing a piecewise continuously differentiable curve  $\gamma : [a, b] \rightarrow \mathbb{C}$ . Prove the complex analogue of the integration by parts formula:

$$\int_{\gamma} f(z)g'(z) dz = f(z_1)g(z_1) - f(z_0)g(z_0) - \int_{\gamma} f'(z)g(z) dz$$

where  $z_1 = \gamma(b)$  and  $z_0 = \gamma(a)$ .

**PROBLEM 46.** Let  $\gamma$  be a piecewise continuously differentiable curve. Prove that

$$\int_{-\gamma} f(z) dz = - \int_{\gamma} f(z) dz.$$

**PROBLEM 47.** Let  $\alpha, \beta : [0, 1] \rightarrow \mathbb{C}$  be two curves. Show that  $\alpha + \beta = \beta + \alpha$  (in the sense of Definition 12 in Section 4.1).

**PROBLEM 48.** Prove Corollary 16 in Section 4.1.

**PROBLEM 49.** Prove all of the properties on page 9 in section 4.1.

**PROBLEM 50.** Let  $B := \{z : |z - 1| < 1\}$ .

- (a) Let  $n$  be a positive integer. Define a function  $g(z) = \sqrt[n]{z}$  on  $B$ .
- (b) Show that  $g(z) = \sqrt[n]{z}$  is analytic on  $B$ .

**PROBLEM 51.** Let  $f$  be analytic in a neighborhood of the closure of a bounded convex set  $S$  with piecewise continuously differentiable boundary  $\partial S$ . Show that for any  $z \in S$ ,

$$f(z) = \frac{1}{2\pi} \int_{\partial S} \frac{f(\zeta)}{\zeta - z} d\zeta,$$

where  $\partial S$  is parameterized in the counter-clockwise direction.

**PROBLEM 52.** Let  $f$  be analytic in a neighborhood of the closure of a bounded convex set  $S$  with piecewise continuously differentiable boundary  $\partial S$ . Show that

$$\frac{1}{2\pi i} \int_{\partial S} f(\zeta) d\zeta = 0.$$

**PROBLEM 53.** Show that there is no function  $f$  analytic in a neighborhood  $\Omega$  of  $\partial\mathbb{D}$  such that  $f'(z) = 1/z$  for any  $z \in \Omega$ .

**PROBLEM 54.**

- (a) Use Cauchy's estimate to prove Liouville's Theorem.
- (b) Use Cauchy's estimate to compute a lower bounded on the radius of convergence of the power series representation of a bounded holomorphic function.

**PROBLEM 55.** Suppose  $\Omega$  is a region which is symmetric with respect to  $\mathbb{R}$ , meaning that  $z \in \Omega$  if and only if  $\bar{z} \in \Omega$ . Set  $\Omega := \Omega \cap \mathbb{H}$  and  $\Omega^- := \Omega \cap (\mathbb{C} \setminus \overline{\mathbb{H}})$ , where  $\mathbb{H} := \{z : \operatorname{Im} z > 0\}$ . If  $f$  is continuous on  $\Omega^+$ , continuous on  $\Omega^+ \cup (\Omega \cup \mathbb{R})$ , and  $\operatorname{Im} f = 0$  on  $\Omega \cap \mathbb{R}$ , then show that the function defined by

$$F(z) := \begin{cases} f(z) & \text{if } z \in \Omega \setminus \Omega^- \\ \overline{f(\bar{z})} & \text{if } z \in \Omega^- \end{cases}$$

is analytic on  $\Omega$ .

**PROBLEM 56.** Let  $\gamma : [a, b] \rightarrow \Omega$  be a curve, where  $\Omega$  is a region in  $\mathbb{C}$ . Assume that  $f$  is analytic in  $\Omega$ . The goal of this problem is to generalize the integral of  $f$  along a curve (not necessarily piecewise continuously differentiable).

- (a) Given  $0 < \varepsilon < \text{dist}(\gamma, \partial\Omega)$ , show that there is a finite partition  $0 = t_0 < t_1 < \cdots < t_n = 1$  so that  $\gamma([t_{j-1}, t_j]) \subset B_j := \{z : |z - \gamma(t_j)| < \varepsilon\} \subset \Omega$  for any  $j = 1, 2, \dots, n$ .
- (b) For a given  $0 < \varepsilon < \text{dist}(\gamma, \partial\Omega)$ , take a partition from (a). Let  $\sigma := \sum_{j=1}^n \sigma_j$  be the polygonal curve, where  $\sigma_j$  is the line segment joining  $\gamma(t_{j-1})$  to  $\gamma(t_j)$ . Define

$$\int_{\gamma} f(\zeta) d\zeta := \int_{\sigma} f(\zeta) d\zeta.$$

Show that this definition does not depend on the choice of the polygonal curve  $\sigma$ .

- (c) Show that the previous definition agrees for a piecewise continuously differentiable curve  $\gamma$ .

**PROBLEM 57.**

- (a) For  $x > 0$ , define  $x^{-z} := e^{-z \ln(x)}$ . Prove that the Riemann Zeta Function

$$\zeta(z) := \sum_{n=1}^{\infty} n^{-z}$$

converges and is analytic in  $\{z : \text{Re } z > 1\}$ .

- (b) Show that, for  $\text{Re } z > 1$ ,

$$\zeta(z) - \int_1^{\infty} x^{-z} dx = \sum_{n=1}^{\infty} \int_n^{n+1} \int_n^x z t^{-z-1} dt dx.$$

- (c) Use (b) to prove that  $(z-1)\zeta(z)$  has a unique analytic extension to  $\{z : \text{Re } z > 0\}$ .
- (d) Use the fundamental theorem of calculus to give a series for  $\zeta(z) - 1/(z-1)$ , valid in  $\text{Re } z > 0$ , which does not involve an integral.

**PROBLEM 58.** Show that there is a constant  $C < \infty$  so that if  $f$  is analytic on  $\mathbb{D}$ , then

$$|f'(z)| \leq C \int_{\mathbb{D}} |f(x+iy)| dx dy$$

for all  $|z| \leq 1/2$ . [Hint: Use Cauchy's integral formula.]

**PROBLEM 59.** Prove that there exists a sequence of polynomials  $p_k$  such that

$$\lim_{k \rightarrow \infty} p_k(z) = \begin{cases} 1 & \text{if } \text{Re } z > 0 \\ 0 & \text{if } \text{Re } z = 0 \\ -1 & \text{if } \text{Re } z < 0. \end{cases}$$

**PROBLEM 60.** Suppose  $f$  has a complex derivative at each point of a region  $\Omega$ . Prove that  $f$  is analytic in  $\Omega$ .