Chapter 4
Integrals

4.1 Areas and Distances

The summation symbol.

$$\boxed{1}+\underline{2}+\underline{3}+\underline{4}+\underline{5}+\boxed{6} \quad i \qquad i=1 \quad i=2 \quad i=6$$

Sigma notation $\longrightarrow$ $\sum_{i=1}^{6} i \longleftrightarrow \boxed{a_i = i}$

$f(x) = x$ $\qquad f(x) = x$

$$1 + 1/2 + 1/3 + 1/4 + 1/5 + 1/6$$

$1 \quad 1+1 \quad 2+1 \quad 3+1 \quad 4+1 \quad 5+1$

Sigma notation $\longrightarrow$ $\boxed{\sum_{i=1}^{6} \frac{1}{i}}$

$f(x) = \frac{1}{x}$

$$\boxed{1}/2 + \boxed{1}/(2\times 3) + \boxed{1}/(3\times 4) + \boxed{1}/(4\times 5)$$

1×2, $1 \to 1+1$ $\quad 2\times 3$, $2 \to 2+1$ $\quad 3\times 4$, $3 \to 3+1$ $\quad 4\times 5$, $4 \to 4+1$

Sigma notation $\longrightarrow$ $\sum_{i=1}^{4} \underbrace{\frac{1}{i(i+1)}}_{a_i}$

$\frac{1}{x(x+1)}$

DEFINITION (Sigma Notation)

If $a_1, a_2, a_3, \ldots, a_n$ are n numbers, then we can write their ordered sum with the Sigma notation:

1) $a_i = i$
2) $a_i = \frac{1}{i}$
3) $a_i = 1/i(i+1)$.

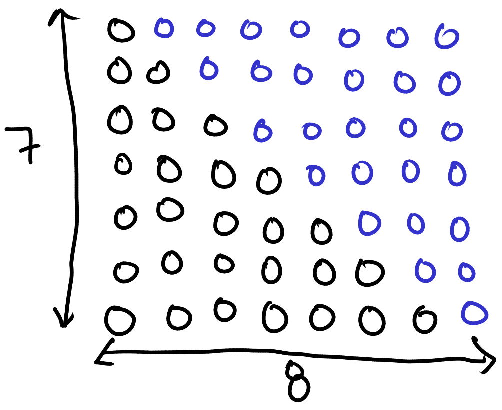
$$a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i.$$

Problem: What is the sum of 1 + 2 + 3 + 4 + 5 + 6 + 7? Can you compute it i~~n an efficient way~~?

$$S = 1 + 2 + 3 + 4 + 5 + 6 + 7$$
$$S = 7 + 6 + 5 + 4 + 3 + 2 + 1$$
$$\overline{\quad 8 + 8 + 8 + 8 + 8 + 8 + 8 \quad}$$

$\to 2S = 7 \times 8$

$\to S = \frac{7\times 8}{2}$



$\to$ Total of balls $= 7\times 8 = 56$

So, real Total of balls $= \frac{56}{2}$

$= \frac{7\times 8}{2}$

Carl Frederich Gauss (1777-1855). Also called the Prince of Mathematics. He discovered the tricks to compute the sum 1 + 2 + 3 + 4 + 5 + ... + 100 when he was around... 7 years old!

General formulas:

1) $$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

$\sum_{i=1}^{n} i^2$

2) $$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

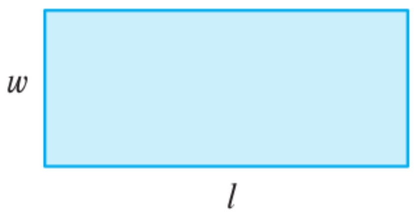
3) $$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

$\sum_{i=1}^{n} i^3$

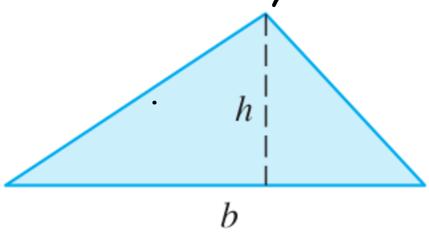
The Area Problem

What is the area of the following shapes?

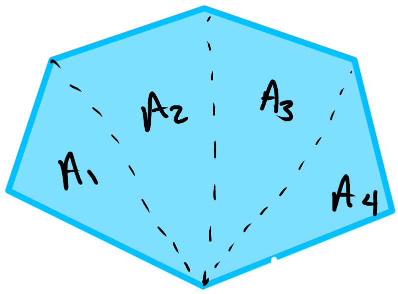
A = $w\,l$



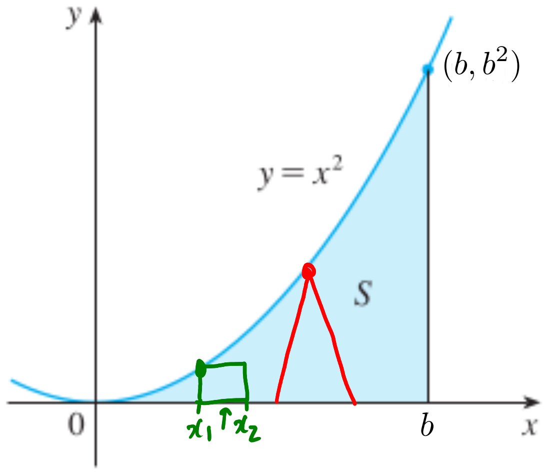
A = $h \cdot b / 2$



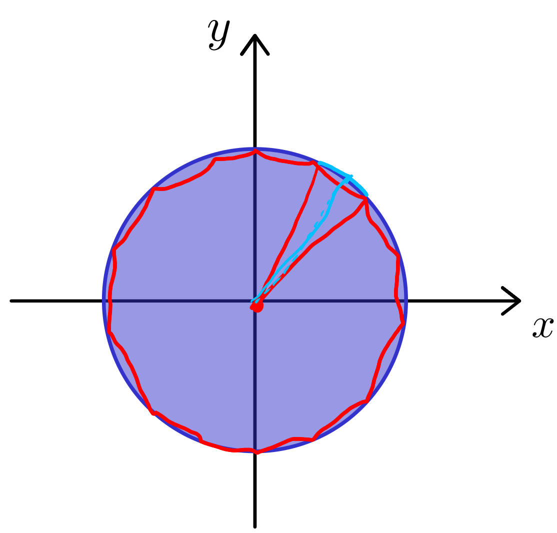
A = $A_1 + A_2 + A_3 + A_3$



What about the area of the following shapes



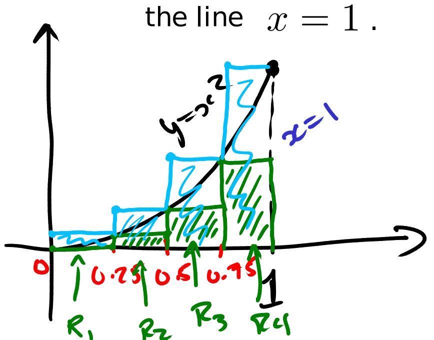
A =



A =

Remember the Youtube video that I showed you on the first day?

Example 1. Try to estimate the area of the region bounded by the curves $y = x^2$, the line $y = 0$ and the line $x = 1$.



1st) Divide in 4 equal parts the interval:

$\frac{1-0}{4} = 0.25$

0, 0.25, 0.5, 0.75, 1

2st) Choose a sample point inside each subinterval:

- Left end point: 0, 0.25, 0.5, 0.75.
- Right end point: 0.25, 0.5, 0.75, 1.

height of each rectangle is given by

height $y(0) = 0^2$, $y(0.25) = 1/16$, $y(0.5) = \frac{1}{4}$, $y(0.75) = 9/16$.

or height $y(0.25) = 1/16$, $y(0.5) = 1/4$, $y(0.75) = 9/16$ & $y(1) = 1$.

3st) Compute the area of each rectangle & sum the results.

$A(R_1) = 0 \cdot 0.25$

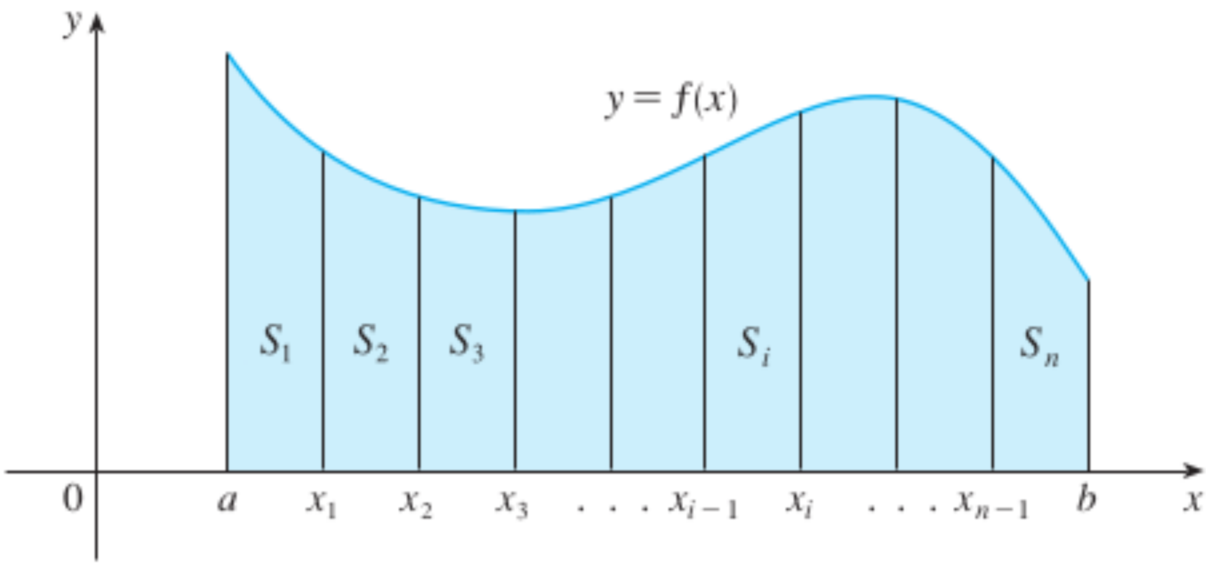
$A(R_2) = \frac{1}{16} \cdot 0.25 = \frac{1}{64}$

$A(R_3) = \frac{1}{4} \cdot 0.25 = \frac{1}{16}$

$A(R_4) = 9/16 \cdot 0.25 = 9/64$

total $= \boxed{0.21875}$

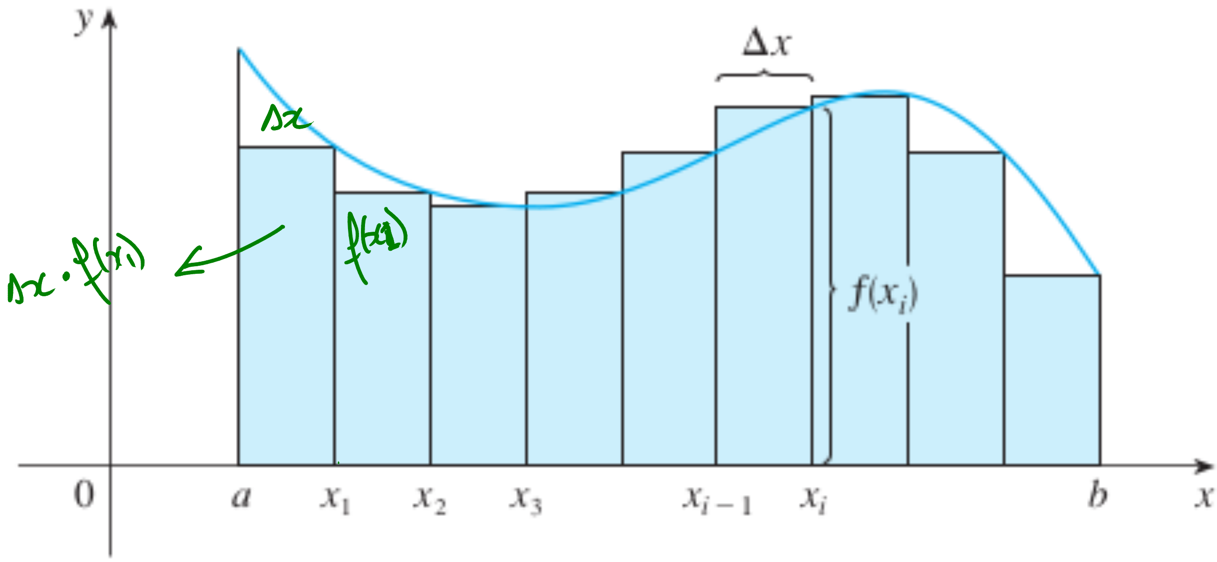
The key idea: divide and conquer!



1) Divide the region S into n strips.

Choose $x_1, x_2, \dots, x_{n-1}$ to create n equal parts.

Δx : length of the subintervals (constant)



2) Approximate the area of S by the sum of the area of each rectangle created.

Right endpoints

If we choose $x_1, x_2, \dots, x_{n-1}, b$ then

$$A(S) \approx \sum_{i=1}^{n} f(x_i)\,\Delta x \to R_n$$

$$= f(x_1)\,\Delta x + f(x_2)\,\Delta x + \dots + f(x_n)\,\Delta x$$

here $x_n = b$.

Left endpoints

If we choose the left-end points $a, x_1, x_2, \dots, x_{n-1}$ then

$$A(S) \approx \sum_{i=1}^{n} f(x_{i-1})\,\Delta x \to L_n$$

$$= f(x_0)\,\Delta x + f(x_1)\,\Delta x + \dots + f(x_{n-1})\,\Delta x$$

here $x_0 = a$.

2 DEFINITION. The AREA A of the region S that lies under the graph of the continuous function f is the limit of of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x]$$

or $$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} [f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x]$$

Example 2. Show that the area of the region S in example 1 is 1/3. In other words, show that

$$\lim_{n\to\infty} R_n = 1/3.$$

Desmos: see the illustration.

$[a,b]=[0,1]$, $\Delta x = \frac{1-0}{n} = \frac{1}{n}$, and

$x_1 = 0 + \Delta x = 0 + \frac{1}{n} = \frac{1}{n}$ $\qquad x_i = 0 + i\,\Delta x = \frac{i}{n}$

$\vdots$

$x_2 = 0 + 2\Delta x = 0 + \frac{2}{n} = \frac{2}{n}$ $\qquad x_n = n \cdot \Delta x = \frac{n}{n} = 1 = b$

So,

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_i)\Delta x + \cdots + f(x_n)\Delta x$$

$f(x) = x^2$

$$= x_1^2 \Delta x + x_2^2 \Delta x + \cdots + x_i^2 \Delta x + \cdots + x_n^2 \Delta x$$

$$= \left(\frac{1}{n}\right)^2\left(\frac{1}{n}\right) + \left(\frac{2}{n}\right)^2 \frac{1}{n} + \cdots + \left(\frac{i}{n}\right)^2 \frac{1}{n} + \cdots + \left(\frac{n}{n}\right)\left(\frac{1}{n}\right)$$

$$= \frac{1}{n^3} + \frac{2^2}{n^3} + \cdots + \frac{i^2}{n^3} + \cdots + \frac{n^2}{n^3}$$

$$= \frac{1}{n^3}\left(1^2 + 2^2 + \cdots + i^2 + \cdots + n^2\right)$$

$$= \frac{1}{n^3} \sum_{i=1}^{n} i^2 = \frac{1}{n^3} \frac{n(n+1)(2n+1)}{6}$$

So, $$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{(n^2+n)(2n+1)}{6n^3} = \lim_{n\to\infty} \frac{2n^3+3n^2+n}{6n^3}$$

$$= \lim_{n\to\infty} \frac{2 + 3/n + 1/n^2}{6} = \boxed{\frac{1}{3}}$$

The distance problem.

If the velocity remains constant, then the distance between the start and the finish line is easy to compute:

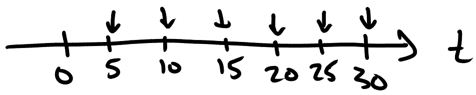
$$\text{DISTANCE} = \text{VELOCITY} \times \text{TIME}.$$

What do we do if the velocity varies??

EXAMPLE 4 Suppose the odometer on our car is broken and we want to estimate the distance driven over a 30-second time interval. We take speedometer readings every five seconds and record them in the following table:

Time (s)	0	5	10	15	20	25	30
Velocity (ft/s)	25	31	35	43	47	45	41

velocities were converted from mi/h to ft/s.



Right-end points

$\Delta t = 5$ $\quad t_1 = 5 \quad t_2 = 10 \quad t_3 = 15 \quad t_4 = 20 \quad t_5 = 25 \quad t_6 = 30$

velocity at 5s.

$$\text{distance} \approx \Delta t \cdot v(5) + \Delta t \cdot v(10) + \Delta t \cdot v(15) + \Delta t \cdot v(20) + \Delta t \cdot v(25) + \Delta t \cdot v(30)$$

$$= 5(31 + 35 + 43 + 47 + 45 + 41)$$

$$= 5 \cdot 242$$

$$= \boxed{1210 \text{ ft}}$$