Solution

Worksheet 01

Question 1.

(a) The function is continuous everywhere on $(-\infty, 0) \cup (0,1) \cup (1, \infty)$.

We have to verify if it is at $x = 0, 1$.

$\underline{x=0}$ $\quad \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x+2 = 2$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 2x^2 = 0.$

The limit doesn't exist, so the function is discontinuous at the point $x=0$.

$\underline{x=1}$ $\quad \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 2x^2 = 2$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 2-x = 1.$

The limit doesn't exist at $x=1$.

So, the function is continuous on $(-\infty, 0) \cup (0,1) \cup (1, \infty)$.

(b) We have the composition of two functions:

- $x+2x^3$ is continuous on $\mathbb{R}$
- x^4 is continuous on $\mathbb{R}$.

So, $(x+2x^3)^4$ is continuous on $\mathbb{R}$.

(c) The $\sqrt{x}$ is continuous on $[0,\infty)$.

The $\sin x$ is continuous on $\mathbb{R}$.

So, $\sin(\sqrt{x})$ is continuous on $[0,\infty)$.

(d) $1+\cos x = 0 \iff \cos x = -1$

$\iff x = (2n+1)\pi \quad n \in \mathbb{Z}$.

So, f is continuous on $\mathbb{R} \setminus \{(2n+1)\pi : n \in \mathbb{Z}\}$.

Question 2.

(a) The function $\dfrac{\sin x + \cos x}{(x-1)(x^2+1)}$ is continuous at $x=0$ (Quotient of two continuous fcts. and $(x-1)(x^2+1)\Big|_{x=0} \neq 0$).

So,

$$\lim_{x \to 0} \frac{\sin x + \cos x}{(x-1)(x^2+1)} = \frac{\sin(0) + \cos(0)}{(0-1)(0^2+1)}$$

$$= \frac{1}{-1} = -1.$$

(b) The function is continuous because

- $\sin x$ is continuous
- $x + \sin x$ is continuous.

So,

$$\lim_{x \to \pi} \sin(x + \sin x) = \sin(\pi + \sin \pi)$$

$$= \sin(\pi) = 0.$$

(c) The function in the limit is continuous because it's a rational function and $x=20$ is in the domain of the rational function. We have

$$\lim_{x\to 20} \frac{x-19}{\sqrt{x+5}} = \frac{20-19}{\sqrt{20+5}} = \frac{1}{\sqrt{25}} = \frac{1}{5}$$

(d) As you can see, the limit is $\frac{0}{0}$. We rewrite the denominator as

$$\sin\left(\frac{\pi}{2}x\right) = \sin\left[\frac{\pi}{2}(x+2-2)\right]$$
$$= \sin\left(\frac{\pi}{2}(x+2) - \pi\right)$$
$$= -\sin\left(\frac{\pi}{2}(x+2)\right).$$

So, make the change $u = x+2$:

$$\lim_{x\to -2} \frac{x+2}{\sin\left(\frac{\pi}{2}x\right)} = -\lim_{u\to 0} \frac{u}{\sin \frac{\pi}{2}u} = -\frac{2}{\pi} \lim_{u\to 0} \frac{\pi/2\, u}{\sin \frac{\pi}{2}u}.$$

Make the change $v = \pi/2\, u$. So

$$\lim_{x\to -2} \frac{x+2}{\sin\left(\frac{\pi}{2}x\right)} = -\frac{2}{\pi} \lim_{v\to 0} \frac{v}{\sin v}.$$

We know that $\lim\limits_{v \to 0} \frac{\sin v}{v} = 1$. So,

$$\lim_{v \to 0} \frac{v}{\sin v} = \lim_{v \to 0} 1/(\sin v/v)$$

$$= \frac{1}{\lim\limits_{v \to 0} \frac{\sin v}{v}} \quad \text{[quotient rule]}.$$

$$= 1$$

So,

$$\lim_{x \to -2} \frac{x+2}{\sin \frac{\pi}{2} x} = -\frac{2}{\pi} \cdot 1 = \frac{-2}{\pi}.$$