Math 302

Chapter 7

Section 7.2: Series Solutions Near an Ordinary Point

Contents

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Main goal:

- Solve a second order ODE
$$A(x)y'' + B(x)y' + C(x)y = 0$$
where $A(x)$, $B(x)$, and $C(x)$ are polynomials.
- Use power series to obtain the solution $y(x)$. Such a solution is called a **power series solution** to the ODE.

Recall from the previous section that

- $y(x) = \sum_{n=0}^{\infty} a_n x^n$.
- $y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$.
- $y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$.

Remark:

- We denote the left-hand side by
$$L(y) := A(x)y'' + B(x)y' + C(x)y.$$
- The application $y \mapsto L(y)$ is called a **differential operator** in the litterature.

Example 1. Find a power series solution to $y'' + y = 0$.

① Left-hand side as a Power Series.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad , \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\Rightarrow y'' + y = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} \left((n+2)(n+1) a_{n+2} + a_n \right) x^n$$

② Find Recurrence Relation.

$$y'' + y = 0 \implies \sum_{n=0}^{\infty} \left((n+2)(n+1)a_{n+2} + a_n \right) x^n = \sum_{n=0}^{\infty} 0 x^n$$

$$\implies (n+2)(n+1)a_{n+2} + a_n = 0 \quad . \quad (n \geq 0)$$

$$\implies a_{n+2} = -\frac{a_n}{(n+2)(n+1)} \quad (n \geq 0)$$

③ Find an expression for a_n.

$\underline{n=0}$ $\quad a_0$ is arbitrary

$$2\cdot 1 \leftarrow \implies \underline{n=2} \quad a_2 = a_{0+2} = -\frac{a_0}{(0+2)(0+1)} = \overset{(-1)^1}{-}\frac{a_0}{\underbrace{2\cdot 1}_{2!}}$$

$$2\cdot 2 \leftarrow \underline{n=4} \quad a_4 = a_{2+2} = -\frac{a_2}{(2+2)(2+1)} = -\frac{a_2}{4\cdot 3} = \overset{(-1)^2}{} \frac{a_0}{\underbrace{4\cdot 3\cdot 2\cdot 1}_{4!}}$$

$\vdots$

$$\underline{n=2k} \to a_{2k} = (-1)^k \frac{a_0}{(2k)!}$$

$\underline{n=1}$ $\quad a_1$ arbitrary.

$$\implies \underline{n=3} \quad a_3 = a_{1+2} = -\frac{a_1}{(1+2)(1+1)} = \overset{(-1)^1}{-}\frac{a_1}{\underbrace{3\cdot 2\cdot 1}_{3!}}$$

$2\cdot \underset{3}{1}+1$

$2\cdot \underset{5}{2}+1$

$$\underline{n=5} \quad a_5 = a_{3+2} = -\frac{a_3}{(3+2)(3+1)} = -\frac{a_3}{5\cdot 4}$$

$$(-1)^2 \leftarrow \; = \frac{a_1}{\underbrace{5\cdot 4\cdot 3\cdot 2\cdot 1}_{5!}}$$

$$\underline{n=2k+1} \quad a_{2k+1} = (-1)^k \frac{a_1}{(2k+1)!}$$

③ General Solution.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

$$= a_0 + a_1 x - \frac{a_0}{2!} x^2 - \frac{a_1}{3!} x^3 + \frac{a_0}{4!} x^4$$

$$+ \frac{a_1}{5!} x^5 + \cdots$$

$$= a_0 - \frac{a_0}{2!} x^2 + \frac{a_0}{4!} x^4 + \cdots$$

$$+ a_1 x - \frac{a_1}{3!} x^3 + \frac{a_1}{5!} x^5 + \cdots$$

$$= a_0 \left(1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 + \cdots \right)$$

$$+ a_1 \left(x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 + \cdots \right)$$

$$= a_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$= a_0 \cos(x) + a_1 \sin(x).$$

Recurrence Relation:

Solving ODE with power series involves a lot of recurrence relations. In the above problems, we encountered:

$$a_{n+2} = -\frac{a_n}{(n+2)(n+1)}$$

Example 2. Find a power series solution to $x^2y'' + y = 0$.

1. $$y(x) = \sum_{n=0}^{\infty} a_n x^n \longrightarrow y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

$$x^2y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^n$$

$$\Rightarrow x^2y'' + y = \sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=2}^{\infty} a_n x^n + a_0 + a_1 x$$

$$= \sum_{n=2}^{\infty} \left(n(n-1)a_n + a_n\right)x^n + a_0 + a_1 x$$

2. $$x^2y'' + y = 0$$

$$\Rightarrow a_0 + a_1 x + \sum_{n=2}^{\infty} [n(n-1)a_n + a_n] x^n = 0$$

$$\Rightarrow \begin{cases} a_0 = 0 \\ a_1 = 0 \\ n(n-1)a_n + a_n = 0 & (n \geq 2) \end{cases}$$

$$\Rightarrow \begin{cases} a_0 = 0 \\ a_1 = 0 \\ (n^2 - n + 1)a_n = 0 & (n \geq 2) \end{cases}$$

$$\Rightarrow a_0 = a_1 = 0, \quad a_n = 0 \quad (n \geq 2).$$

(3) General Solu.

$$y(x) = \sum_{n=0}^{\infty} a_n x = 0$$

No solution.

$$y(x) = c_1 \sqrt{x} \cos\left(\frac{\sqrt{3}}{2} \ln|x| \right)$$

$$+ c_2 \sqrt{x} \sin\left(\frac{\sqrt{3}}{2} \ln|x| \right).$$

$A(x)y'' + \cdots$

Ordinary and Singular Points

- A number x_0 is called an **ordinary point** if $A(x_0) \neq 0$.
- A number x_0 is called a **singular point** if $A(x_0) = 0$.

We will mainly focus ~~s~~ on power series solutions centered at ordinary points.

Example 3. For each of the following ODEs, find the singular points.

(a) $(1 - x^2)y'' + y = 0$.

(b) $(1 + 2x + x^2)y'' + y' + (2 + x)y = 0$.

(c) $(2x + 3x^2 + x^3)y'' + (x + 1)y' + (x^2 + 1)y = 0$.

(a) $1 - x^2 = 0 \iff x = \pm 1$

sing. Pts. are $x = -1$ & $x = 1$.

(b) $1 + 2x + x^2 = 0 \iff (x+1)^2 = 0$

$\iff x = -1$. ← sing Pts.

(c) $2x + 3x^2 + x^3 = 0 \iff (2 + 3x + x^2)x = 0$

$\iff (x+2)(x+1)x = 0$

$\iff x = -2, \ x = -1, \ x = 0$

Sing. Pts: $-2, -1, 0$.

Remark:

- A power series solution must be centered at an ordinary point, that is, if x_0 is an ordinary point, then the form of the solution is

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

- In Example 2, we see why we can't solve: The power series used was centered at $x_0 = 0$, a singular point.
- In the case of a singular points, we need the Frobenius method. This is covered in a second class in ODE.

Example 4. $\to \Sigma a_n x^n$

(a) Find a power series solution of

$$(x^2-4)y'' + 3xy' + y = 0.$$

(b) Find the solution to the IVP

$$(x^2-4)y'' + 3xy' + y = 0, \quad y(0) = 4,\ y'(0) = 1.$$

(a) ① Singular Points $\quad x^2-4=0 \iff x=-2$ or $x=2$.

② $y(x) = \sum_{n=0}^{\infty} a_n x^n \to y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$

$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$.

• $(x^2-4)y'' = x^2 y'' - 4y''$

$= \sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=2}^{\infty} 4n(n-1) a_n x^{n-2}$

• $3xy' = \sum_{n=1}^{\infty} 3n a_n x^n$

$$\text{LHS} = \sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=2}^{\infty} 4n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} 3n a_n x^n + \sum_{n=0}^{\infty} a_n x^n$$

$$= (a_0 - 8a_2) + (4a_1 - 24a_3)x + \sum_{n=2}^{\infty} \Big[n(n-1)a_n - 4(n+2)(n+1)a_{n+2} + 3na_n + a_n \Big] x^n$$

$$n(n-1)a_n - 4(n+2)(n+1)a_{n+2} + 3na_n + a_n$$

$$= (n^2 - n + 3n + 1)a_n - 4(n+2)(n+1)a_{n+2}$$

$$= (n^2 + 2n + 1)a_n - 4(n+2)(n+1)a_{n+2}$$

$$= (n+1)^2 a_n - 4(n+2)(n+1)a_{n+2}.$$

③ <u>LHS = 0</u>

$$\text{LHS} = 0 \iff \begin{aligned} a_0 - 8a_2 &= 0 \\ 4a_1 - 24a_3 &= 0 \\ (n+1)^2 a_n - 4(n+2)(n+1)a_{n+2} &= 0 \end{aligned}$$

$$\Rightarrow \quad a_2 = \frac{a_0}{8}, \quad a_3 = \frac{a_1}{6}$$

$$\& \quad a_{n+2} = \frac{n+1}{4(n+2)} a_n$$

<u>Start recurrence.</u>

a_0 arbitrary $\longrightarrow$ $a_2 = \frac{a_0}{8}$

$$a_4 = a_{2+2} = \frac{2+1}{4(2+2)} a_2 = \frac{3}{16} \cdot \frac{a_0}{8}$$
$$= \frac{3a_0}{128}$$

$$a_6 = a_{4+2} = \frac{4+1}{4(4+2)} a_4 = \frac{5}{4(6)} \frac{3a_0}{128}$$
$$= \frac{15}{1024} a_0$$

$\underline{a_1 \text{ arbitrary.}}$ $\rightarrow$ $a_3 = \dfrac{a_1}{6}$

$$\rightarrow a_5 = a_{3+2} = \frac{3+1}{4(3+2)} a_3 = \frac{4}{4(5)} \cdot \frac{a_1}{6}$$

$$= \frac{a_1}{30}$$

$$\rightarrow a_7 = a_{5+2} = \frac{5+1}{4(5+2)} a_5 = \frac{6}{4 \cdot 7} \cdot \frac{a_1}{30}$$

$$= \frac{a_1}{140}$$

4 General Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$$

$$= a_0 + a_1 x + \frac{a_0}{8} x^2 + \frac{a_1}{6} x^3 + \frac{3a_0}{128} x^4$$

$$+ \frac{a_1}{30} x^5 + \cdots$$

Example 5. Find a power series solution to the following IVP:

$$(t^2 - 2t - 3)\frac{d^2y}{dt^2} + 3(t-1)\frac{dy}{dt} + y = 0, \quad y(1) = 4,\ y'(1) = -1.$$

Radius of Convergence

It is important to know where our solution is valid.

- The **radius of convergence** of a power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is the number R such that
  - $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges for any x such that $|x-x_0| < R$.
  - $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ diverges for all x such that $|x-x_0| > R$.
- If the limit

$$L := \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

  exists, then the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$ is $R = \frac{1}{L}$.

Example 6. Find the radius of convergence of

(a) $f(x) = \sum_{n=0}^{\infty} x^n$.

(b) $g(x) = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$.

Theorem 7. Suppose that x_0 is an ordinary point of the ODE

$$A(x)y'' + B(x)y' + C(x)y = 0.$$

Then the ODE has a general solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

The radius of convergence of any such series solution is at least as large as the distance from x_0 to the nearest (real or complex) singular point of the ODE.

Example 8. Determine the radius of convergence guaranteed by the last Theorem of a series solution of

$$(x^2 + 9)y'' + xy' + x^2 y = 0$$

(a) in powers of x.

(b) in powers of $x - 4$.

Taylor Polynomial

When we have a solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

of an ODE

$$A(x)y'' + B(x)y' + C(x)y = 0,$$

we can draw an approximation of the solution.

- The **Taylor polynomial** $T_N(x)$, where $N \geq 0$ is an integer, is given by the expression

$$T_N(x) = \sum_{n=0}^{N} a_n (x - x_0)^n = a_0 + a_1(x - x_0) + \cdots + a_N(x - x_0)^N.$$

- When the power series of $y(x)$ converges on a given interval I, we have

$$y(x) \approx T_N(x)$$

for a sufficiently large integer N.

Example 9.

(a) Plot the graph of $T_4(x)$, $T_{10}(x)$, and $T_{20}(X)$ of the power series representation of $f(x) = \cos(x)$.

(b) Plot the graph of $T_4(x)$, $T_{10}(x)$, $T_{20}(x)$ for the power series solution of Example 5.