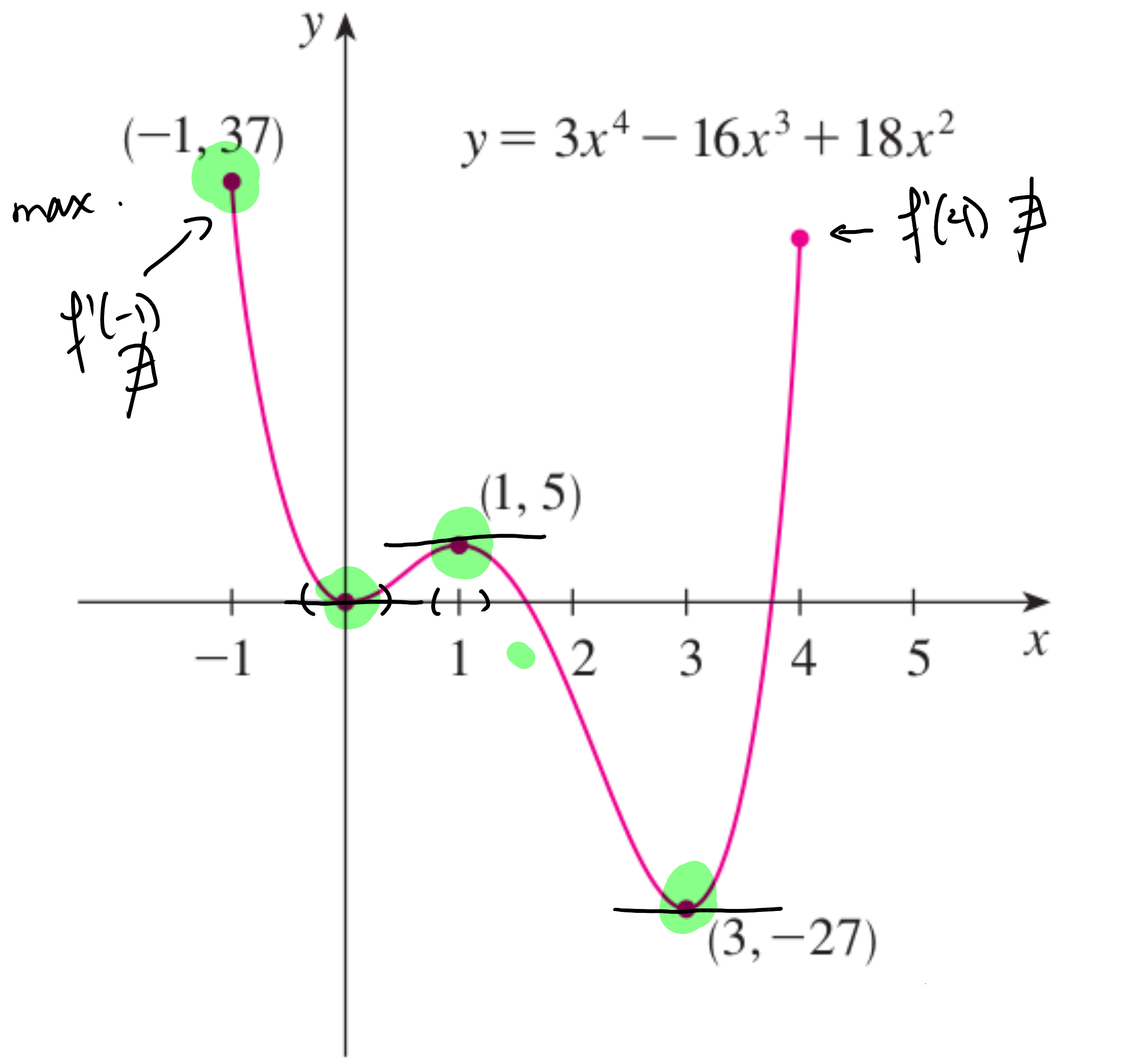
Chapter 3
Applications of Derivatives

3.1 Maximum and Minimum Values

Maximums and minimums.

What would be a maximum value or a minimum value of a function?



Suggestions/observations:

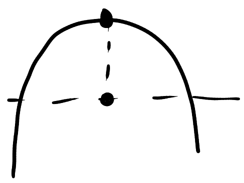
1) Max at −1 with $f(-1) = 37$ → abs. max.
Min at 3 with $f(3) = -27$ → abs. min.

2) Derivation DNE at $x = -1$ & $x = 4$

3) loc. Max at $x = 1$ & loc. min at $x = 0$ & $x = 3$.

4) $f'(1) = 0$, $f'(0) = 0$ & $f'(3) = 0$.

$2 - x^2$



Important observations:

a) max or min when $f'(x) = 0$

b) max or min when $f'(x)$ ∄

1 Definition Let c be a number in the domain D of a function f. Then $f(c)$ is the

- **absolute maximum** value of f on D if $f(c) \geqslant f(x)$ for all x in D.
- **absolute minimum** value of f on D if $f(c) \leqslant f(x)$ for all x in D.

2 Definition The number $f(c)$ is a

- **local maximum** value of f if $f(c) \geqslant f(x)$ when x is near c.
- **local minimum** value of f if $f(c) \leqslant f(x)$ when x is near c.

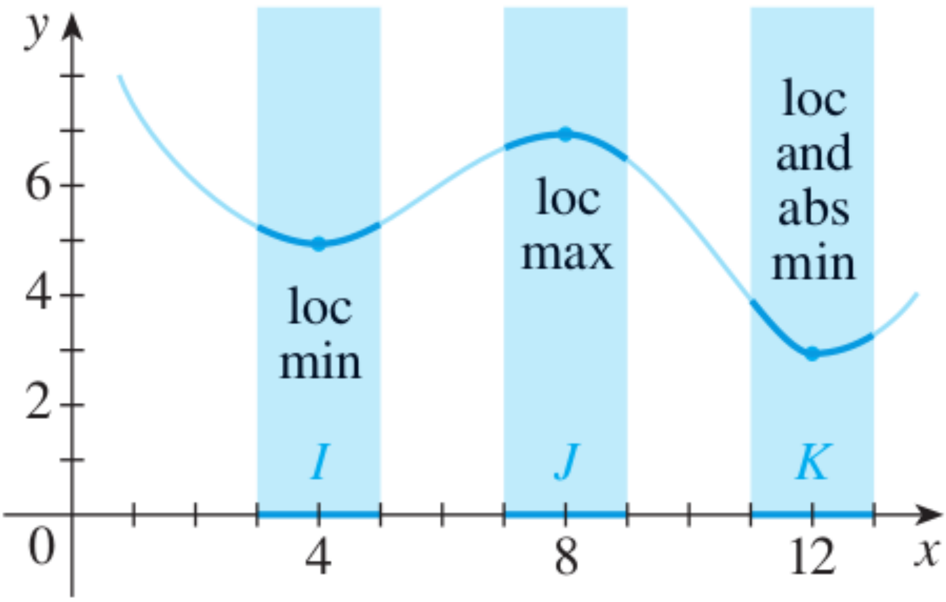


Illustration of the local and absolute max and min.

Remark: loc. max $\not\Rightarrow$ abs. max.
abs. max $\Rightarrow$ loc. max.

Terminology.

1) Global maximum or global minimum → for abs. max. or abs. min.

2) Extreme values for abs. max. and abs. min.

Extreme Values Theorem.

Which conditions garantee that extreme values exist?

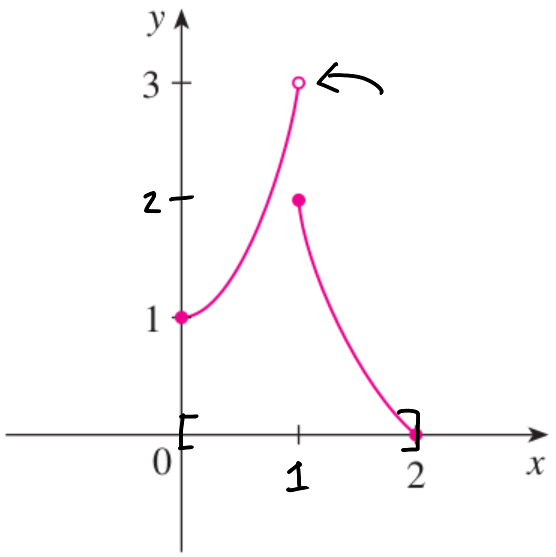


FIGURE 9
This function has minimum value $f(2) = 0$, but no maximum value.

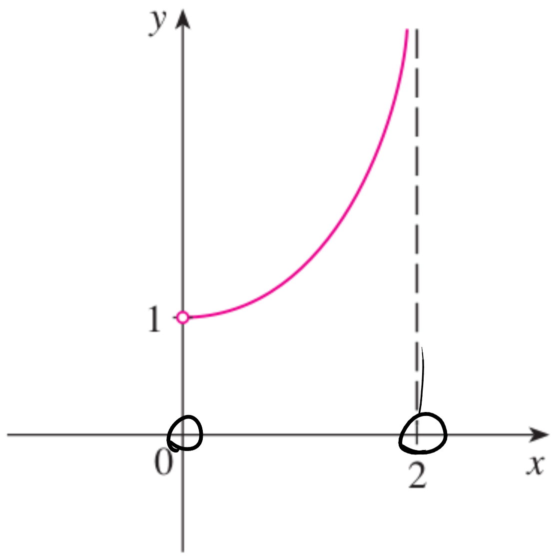
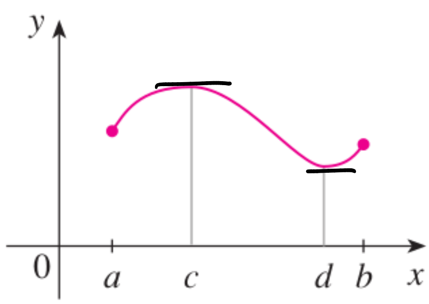


FIGURE 10
This continuous function g has no maximum or minimum.

3 The Extreme Value Theorem If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.



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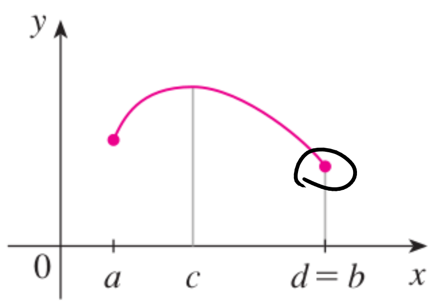

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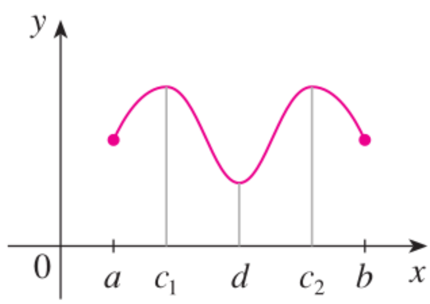


Attained multiple times

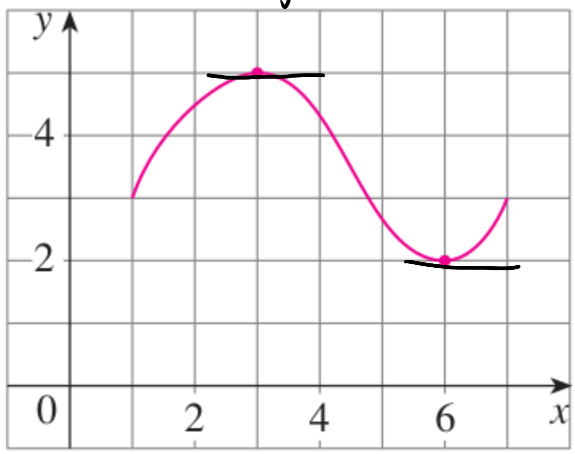
Fermat's Theorem.

$x^2+y^2=z^2$ $\quad 3^2+4^2=5^2$
$x^3+y^3=z^3$ $\quad x^4+y^4=z^4$ $\quad x^n+y^n=z^n$

An observation:

When we have a (loc) max or (loc) min at c:

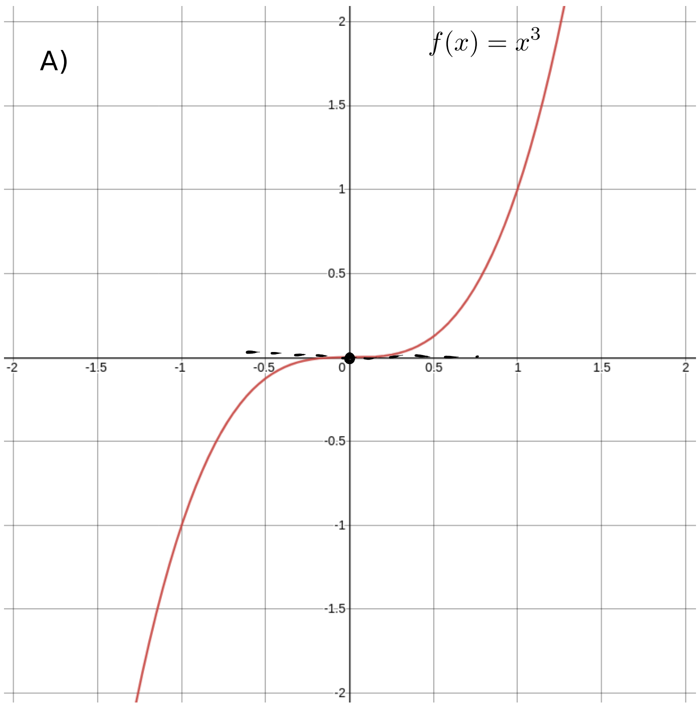
$$f'(c)=0$$

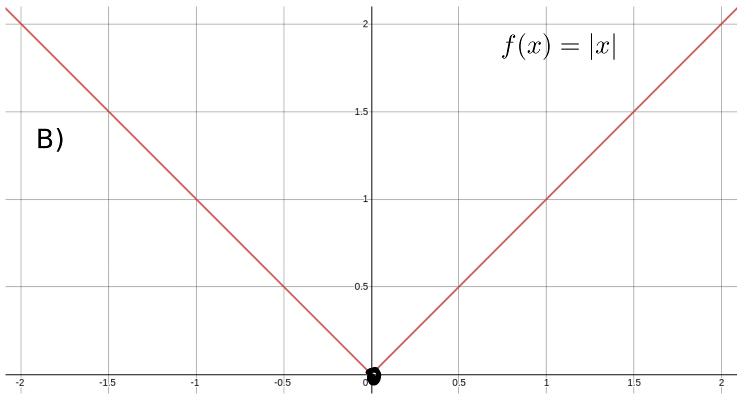


4 Fermat's Theorem If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Interested in the proof: see page 207 in the textbook.

BE CAREFUL!!





A) $f'(0) = 0$ but no max & no min.

B) $f'(0) \nexists$ but there is a min at $x = 0$.

6 Definition A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

EXAMPLE 7 Find the critical numbers of $f(x) = x^{3/5}(4 - x)$.

① Derivative.

$$f'(x) = \frac{4(3-10x)}{5x^{2/5}}.$$

② Find zeros of $f'(x)$

$$f'(x) = 0 \iff \frac{4(3-10x)}{5x^{2/5}} = 0$$

$$\iff 4(3-10x) = 0$$

$$\iff x = \frac{3}{10}$$

③ Find where $f'(x)$ DNE

Denominator $= 0 \Rightarrow 5x^{2/5} = 0$

$\Rightarrow x = 0$.

Answer: Critical numbers $\rightarrow$ C.N. are

$$\boxed{0, \quad \frac{3}{10}}$$

Finding Extremum Values on closed intervals.

The Closed Interval Method To find the *absolute* maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

EXAMPLE 8 Find the absolute maximum and minimum values of the function

$$f(x) = x^3 - 3x^2 + 1 \qquad \underbrace{-\tfrac{1}{2} \leqslant x \leqslant 4}_{[-\frac{1}{2},\, 4]}$$

① C.N. in $(-\frac{1}{2}, 4)$

$f'(x) = 3x^2 - 6x$

$\hookrightarrow f'(x) = 0$ $\qquad\qquad$ $\hookrightarrow f'(x) \nexists$

always exists ✓

$\iff 3x^2 - 6x = 0$

$\iff 3x(x-2) = 0$

$\iff x = 0$ or $x = 2$

We have $f(0) = 1$ & $f(2) = -3$

② Evaluate f at the endpoints

We have $f(-\frac{1}{2}) = \frac{1}{8}$ & $f(4) = 17$

③ Find abs. max & abs. min

abs. max $= \max\{1, -3, \frac{1}{8}, 17\} = \boxed{17}$

abs. min $= \min\{1, -3, \frac{1}{8}, 17\} = \boxed{-3}$