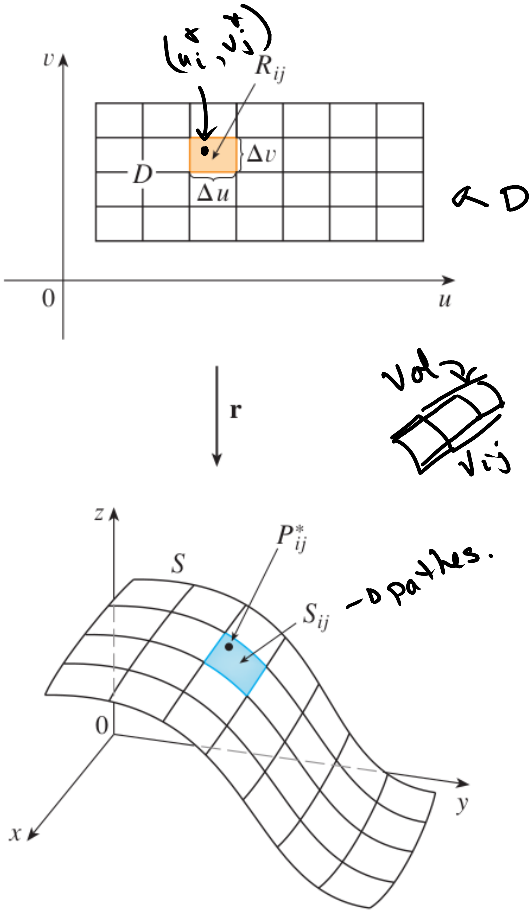
16.7 Surface Integrals.

Parametric surfaces.



f: function in 3 variables
S: surface with $\vec{r}(u,v)$ & domain D.

$$\mathrm{Vol}(V_{ij}) \cong A(S_{ij}) \cdot f(P_{ij}^*)$$
$$= \Delta S_{ij} \cdot f(P_{ij}^*)$$

So now, take lim on the number of divisions (number of patches)

$$\boxed{\iint_S f(x, y, z)\, dS = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}^*)\, \Delta S_{ij}}$$

But,

$$\Delta S_{ij} \cong |\vec{r}_u \times \vec{r}_v|\, \Delta u\, \Delta v.$$

So,

$$\Rightarrow \sum_i \sum_j f(P_{ij}^*)\, \Delta S_{ij} \cong \sum_i \sum_j \underbrace{f(\vec{r}(u_i^*, v_j^*))\, |\vec{r}_u \times \vec{r}_v|}_{g(u,v)}\, \Delta u\, \Delta v.$$

limit on number of patches

$$\to \iint_S f\, dS = \iint_D g(u,v)\, du\, dv.$$

$$\Longrightarrow \boxed{\iint_S f(x, y, z)\, dS = \iint_D \underbrace{f(\mathbf{r}(u, v))\, |\mathbf{r}_u \times \mathbf{r}_v|}_{g(u,v)}\, dA} \to$$

doesn't care about the orientation of the $\vec{r}_u \times \vec{r}_v$.

Mass and center of mass. An aluminum foil S with density $\rho(x,y,z)$.

$$m = \iint_S \rho(x,y,z)\, dS$$

$$\bar{x} = \frac{1}{m} \iint_S x\, \rho(x,y,z)\, dS$$

$$\bar{y} = \frac{1}{m} \iint_S y\, \rho(x,y,z)\, dS$$

$$\bar{z} = \frac{1}{m} \iint_S z\, \rho(x,y,z)\, dS.$$

center of mass:

$(\bar{x}, \bar{y}, \bar{z})$.

EXAMPLE 1 Compute the surface integral $\iint_S x^2\,dS$, where S is the unit sphere $x^2 + y^2 + z^2 = 1$.

① Parametrization & dS.

$$\vec{r}(\theta,\phi) = \langle \underbrace{\cos\theta\sin\phi}_{x},\ \sin\theta\sin\phi,\ \cos\phi \rangle \qquad \begin{array}{l} 0 \le \theta \le 2\pi \\ 0 \le \phi \le \pi . \end{array}$$

$$\vec{r}_\theta = \langle -\sin\theta\sin\phi,\ \cos\theta\sin\phi,\ 0 \rangle$$

$$\vec{r}_\phi = \langle \cos\theta\cos\phi,\ \sin\theta\cos\phi,\ -\sin\phi \rangle$$

$$\Rightarrow \vec{r}_\theta \times \vec{r}_\phi = \langle -\cos\theta\sin^2\phi,\ -\sin\theta\sin^2\phi,\ -\sin\phi\cos\phi \rangle$$

$$\Rightarrow |\vec{r}_\theta \times \vec{r}_\phi| = \sin\phi \quad \rightarrow \quad dS = \sin\phi\ d\phi\, d\theta$$

② Integrate.

$$\iint_S x^2\,dS = \iint_D (\cos\theta\sin\phi)^2 \sin\phi\ d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \cos^2\theta\ \sin^3\phi\ d\phi\, d\theta$$

$$= \Big(\int_0^{2\pi} \underbrace{\cos^2\theta}_{\frac{1+\cos 2\theta}{2}}\, d\theta\Big)\underbrace{\Big(\int_0^{\pi} \sin^3\phi\, d\phi\Big)}_{\text{MT02} \Rightarrow 4/3}$$

$$= \pi \cdot \frac{4}{3}$$

$$= \boxed{\frac{4\pi}{3}}$$

Graphs of functions. $z=g(x,y)$ with $(x,y)\in D$.

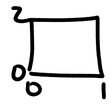
$\vec{r}(x,y)=\langle x,y,g(x,y)\rangle$

$\vec{r}_x=\langle 1,0,g_x\rangle$ $\to$ $|\vec{r}_x\times\vec{r}_y|=\sqrt{1+g_x^2+g_y^2}$

$\vec{r}_y=\langle 0,1,g_y\rangle$

$$\boxed{\iint_S f(x,y,z)\,dS=\iint_D f(x,y,g(x,y))\underbrace{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}\,dA}_{dS}}$$

EXAMPLE 2 Evaluate $\iint_S y\,dS$, where S is the surface $z=x+y^2$, $0\leqslant x\leqslant 1$, $0\leqslant y\leqslant 2$. (See Figure 2.)

D= 

$$\iint_S y\,dS=\iint_D y\sqrt{z_x^2+z_y^2+1}\,dA$$

$$=\int_0^2\int_0^1 y\sqrt{1+4y^2+1}\,dx\,dy$$

$$=\int_0^2\int_0^1 y\sqrt{2+4y^2}\,dx\,dy$$

$$=\int_0^2 y\sqrt{2+4y^2}\,dy$$

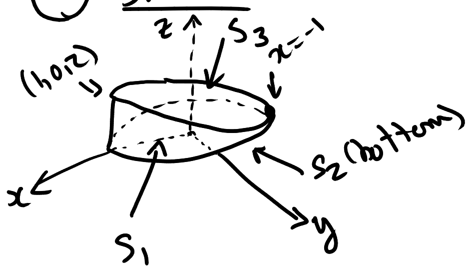
$u=2+4y^2$

$du=8y\,dy$

$$=\boxed{\frac{13\sqrt{2}}{3}}$$

EXAMPLE 3 Evaluate $\iint_S z\, dS$, where S is the surface whose sides S_1 are given by the cylinder $x^2 + y^2 = 1$, whose bottom S_2 is the disk $x^2 + y^2 \leqslant 1$ in the plane $z = 0$, and whose top S_3 is the part of the plane $z = 1 + x$ that lies above S_2.

① Sketch



Property: $S = S_1 \cup S_2 \cup S_3$

$$\iint_S f\,dS = \iint_{S_1} f\,dS + \iint_{S_2} f\,dS + \iint_{S_3} f\,dS$$

② S_1

$$\vec{r}(\theta, z) = \langle \cos\theta,\ \sin\theta,\ z \rangle$$

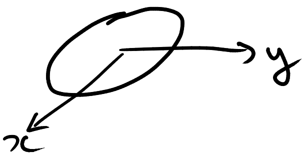
$$0 \le \theta \le 2\pi \quad , \quad 0 \le z \le 1+x = 1+\cos\theta$$

$$\vec{r}_\theta = \langle -\sin\theta,\ \cos\theta,\ 0 \rangle$$
$$\vec{r}_z = \langle 0,\ 0,\ 1 \rangle$$

$$\vec{r}_\theta \times \vec{r}_z = \langle \cos\theta,\ \sin\theta,\ 0 \rangle \Rightarrow |\vec{r}_\theta \times \vec{r}_z| = 1$$

$$\Rightarrow \iint_{S_1} z\,dS = \int_0^{2\pi} \int_0^{1+\cos\theta} z\,dz\,d\theta = \frac{3\pi}{2}$$

③ S_2



$$\vec{r}(r,\theta) = \langle r\cos\theta,\ r\sin\theta,\ 0 \rangle$$

$$\iint_{S_2} z\,dS = \iint_{S_2} 0\,dS = 0 .$$

④ S_3 $\quad z = 1+x$

$$\vec{r}(\rho,\theta) = \langle \overset{x}{\rho\cos\theta},\ \overset{y}{\rho\sin\theta},\ \overset{z}{1+\rho\cos\theta} \rangle$$

$$\iint_{S_3} z\,dS = \iint_{x^2+y^2\le 1} (1+x)\sqrt{1+1^2+0^2}\,dA$$

$$= \sqrt{2} \iint_{x^2+y^2\le 1} (1+x)\,dA$$

$x = r\cos\theta$
$y = r\sin\theta$

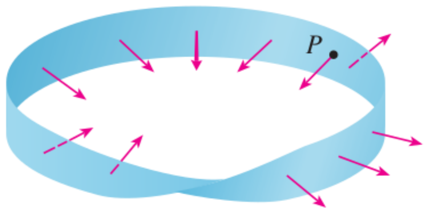
$$= \sqrt{2} \int_0^{2\pi} \int_0^1 (1+r\cos\theta)\,r\,dr\,d\theta$$

$$= \sqrt{2} \int_0^{2\pi} \int_0^1 r + r^2\cos\theta\,dr\,d\theta = \sqrt{2}\pi$$

$$\iint_S z\,dS = \frac{3\pi}{2} + 0 + \sqrt{2}\pi = \boxed{\frac{3+2\sqrt{2}}{2}\pi}$$

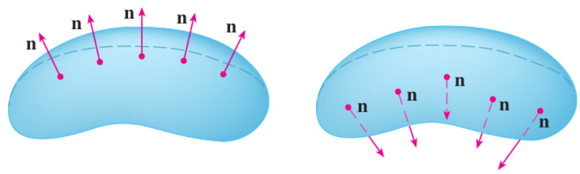
Surface integral of Vector Fields.

Non-orientable surfaces.



Surface has only one side.
This means there is no way of defining a normal properly

Orientable surface.



Surface has two sides.
We can define two normals
- one pointing "outward"
- one pointing "inward".

Special orientations:

1. Graph of a function.

$z = g(x,y)$ then

$$\vec{n} = \frac{\langle -g_x, -g_y, 1\rangle}{\sqrt{1+g_x^2+g_y^2}} \to \vec{r}_x \times \vec{r}_y$$

$\vec{k}$ - component of $\vec{n} > 0 \to$ upward

$\vec{k}$ - component of $\vec{n} < 0 \to$ downward

2. Parametric surface.

S is given by $\vec{r}(u,v)$, then

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

Example with a sphere.

closed surface: a sphere of radius a.

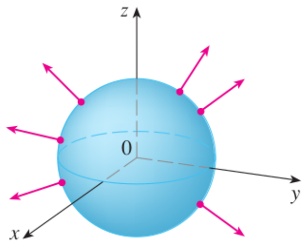
$\vec{r}(\theta,\phi) = \langle a\cos\theta \sin\phi, \; a\sin\theta \sin\phi, \; a\sin\phi\rangle$

$\vec{r}_\theta \times \vec{r}_\phi = \langle -a^2\cos\theta \sin^2\phi, \; -a^2\sin\theta\sin^2\phi, \; -a^2\sin\phi\cos\phi\rangle$

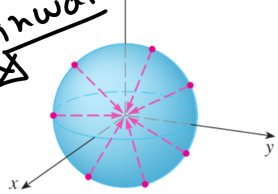
$|\vec{r}_\theta \times \vec{r}_\phi| = a^2 \sin\phi$

$\Rightarrow \quad \vec{n} = \langle -\cos\theta\sin\phi, \; -\sin\theta\sin\phi, \; -\cos\phi\rangle$

$\to \quad \vec{n} = \frac{-1}{a}\langle a\cos\theta\sin\phi, \; a\sin\theta\sin\phi, \; a\cos\phi\rangle = \frac{-1}{a}\vec{r}$



inward

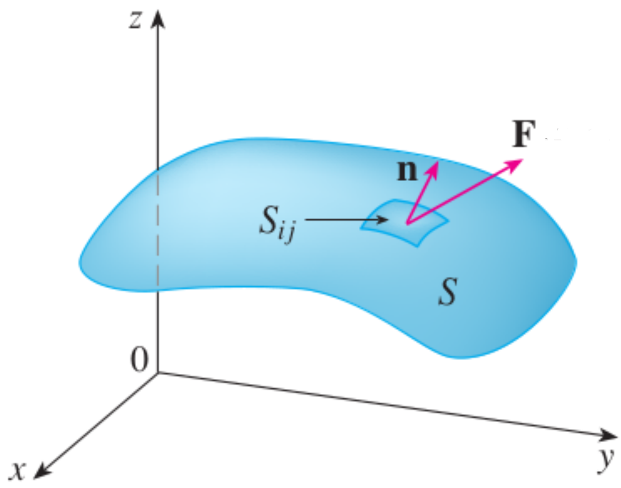


Positive orientation. When $\vec{n}$ points outward.

Negative orientation: $\vec{n}$ points inward.

The convention is to take the positive orientation.

Flux integral (or Surface integral).



Suppose S is a surface &
$\vec{F} = \rho \vec{v}$ ($\vec{v}$ speed & ρ: density).
S_{ij} : patches of the surface S.
Approximate the mass of fluid passing through S_{ij} by
$(\vec{n} \cdot \vec{F})\, A(S_{ij}) \to$ (kg/s).
Add all the contributions & take the limit on the number of patches then

8 Definition If **F** is a continuous vector field defined on an oriented surface S with unit normal vector **n**, then the **surface integral of F over S** is

$$\boxed{\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS}$$

This integral is also called the **flux** of **F** across S.

- Parametric surface: Integral formula.

If S is given by $\vec{r}(u,v)$, then

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

&

$$dS = |\vec{r}_u \times \vec{r}_v|\, dA$$

Then,

$$d\vec{S} = (\vec{r}_u \times \vec{r}_v)\, dA$$

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA$$

x, y, z by the coordinates of $\vec{r}(u,v)$.

- Graph of a function: Integral formula.

$z = g(x,y)$, then $\vec{r}_x \times \vec{r}_y = \langle -g_x, -g_y, 1 \rangle$

Also, if $\vec{F} = \langle P, Q, R \rangle$, then

$$\vec{F} \cdot (\vec{r}_x \times \vec{r}_y) = -Pg_x - Qg_y + R$$

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) dA$$

EXAMPLE 4 Find the flux of the vector field $\mathbf{F}(x, y, z) = z\,\mathbf{i} + y\,\mathbf{j} + x\,\mathbf{k}$ across the unit sphere $x^2 + y^2 + z^2 = 1$.

① Parametrization.

$$\vec{r}(\theta, \phi) = \langle \cos\theta \sin\phi,\ \sin\theta \sin\phi,\ \cos\phi \rangle \qquad \begin{aligned} 0 &\le \theta \le 2\pi \\ 0 &\le \phi \le \pi \end{aligned}$$

so,

$$\vec{r}_\theta \times \vec{r}_\phi = \langle -\cos\theta \sin^2\phi,\ -\sin\theta \sin^2\phi,\ -\cos\phi \sin\phi \rangle.$$

② Integrate.

$$\begin{aligned} F(\vec{r}(\theta,\phi)) &= \vec{F}(\cos\theta \sin\phi,\ \sin\theta \sin\phi,\ \cos\phi) \\ &= \langle \cos\phi,\ \sin\theta \sin\phi,\ \cos\theta \sin\phi \rangle \end{aligned}$$

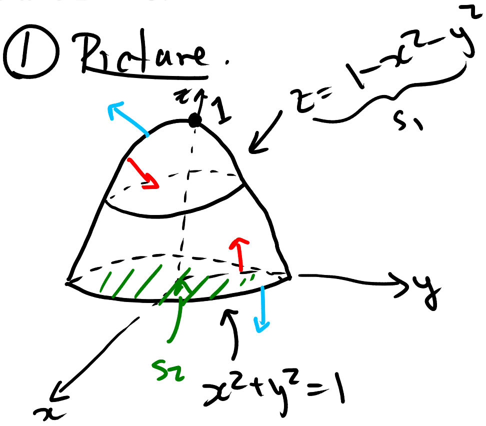
$$\begin{aligned} \vec{F}(\vec{r}(\theta,\phi)) \cdot (\vec{r}_\theta \times \vec{r}_\phi) &= -\cos\theta \cos\phi \sin^2\phi - \sin^2\theta \sin^3\phi \\ &\quad - \cos\theta \cos\phi \sin^2\phi \\ &= -2\cos\theta \cos\phi \sin^2\phi - \sin^2\theta \sin^3\phi. \end{aligned}$$

so,

$$\begin{aligned} \iint_S \vec{F} \cdot d\vec{S} &= \iint_D \vec{F} \cdot (\vec{r}_\theta \times \vec{r}_\phi)\, dA \\ &= \int_0^{\pi}\int_0^{2\pi} -2\cos\theta \cos\phi \sin^2\phi - \sin^2\theta \sin^3\phi \; d\theta\, d\phi \\ &= -2\left(\int_0^{2\pi} \cos\theta\, d\theta\right)\left(\int_0^{\pi} \cos\phi \sin^2\phi\, d\phi\right) \\ &\quad - \left(\int_0^{2\pi} \sin^2\theta\, d\theta\right)\left(\int_0^{\pi} \sin^3\phi\, d\phi\right) \\ &= 0 - \frac{4\pi}{3} \\ &= \boxed{-\frac{4\pi}{3}} \end{aligned}$$

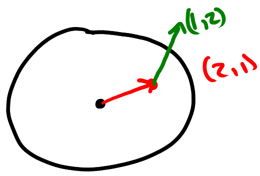
EXAMPLE 5 Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = y\,\mathbf{i} + x\,\mathbf{j} + z\,\mathbf{k}$ and S is the boundary of the solid region E enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.

① Picture.



S_1 : $z = 1 - x^2 - y^2$

S_2 : $x^2 + y^2 \le 1$



② Parametrize.

S_1 : $\vec{r}(x,y) = \langle x, y, 1-x^2-y^2 \rangle \qquad x^2+y^2 \le 1$

$\vec{r}_x \times \vec{r}_y = \langle -z_x, -z_y, 1 \rangle = \langle 2x, 2y, 1 \rangle$ (Pointing outward)

S_2 : $\vec{r}(x,y) = \langle x, y, 0 \rangle \qquad x^2+y^2 \le 1$

$\vec{r}_x \times \vec{r}_y = \langle 0,0,1 \rangle \longrightarrow \underbrace{-\vec{r}_x \times \vec{r}_y}_{} = \langle 0,0,-1 \rangle.$ (Pointing outward)

$\vec{r}(x,y) = \langle -x, y, 0 \rangle \longrightarrow$ new $\vec{r}_x \times \vec{r}_y$

③ Integrate on S_1

$$\iint_{S_1} \vec{F} \cdot d\vec{S} = \iint_{x^2+y^2\le 1} \langle y, x, 1-x^2-y^2 \rangle \cdot \langle 2x, 2y, 1 \rangle \, dA$$

$x = r\cos\theta$

$y = r\sin\theta$

$$= \iint_{x^2+y^2\le 1} 2xy + 2xy + 1 - x^2 - y^2 \, dA$$

$$= \int_0^1 \int_0^{2\pi} \left(4r^2 \cos\theta \sin\theta + 1 - r^2\right) r \, d\theta \, dr = \pi/2$$

④ Integrate on S_2

$$\iint_{S_2} \vec{F} \cdot d\vec{S} = \iint_{x^2+y^2\le 1} \langle y, x, 0 \rangle \cdot \langle 0,0,-1 \rangle \, dA$$

$$= \iint_{x^2+y^2\le 1} 0 \, dA = 0$$

Answer.

$$\iint_S \vec{F} \cdot d\vec{S} = \frac{\pi}{2} + 0 = \boxed{\frac{\pi}{2}}$$

Applications to Physics.

Electric Flux.

"Amount of electricity passing though S"

$\vec{E}$: electric field $\rightarrow$ $\boxed{\iint_S \mathbf{E}\cdot d\mathbf{S}}$

Gauss' Law.

" net charge enclosed by a closed surface S"

$$\boxed{Q = \varepsilon_0 \iint_S \mathbf{E}\cdot d\mathbf{S}}$$

Heat flow.

u: temperature of a body at (x,y,z)

$\vec{F}: -K\vec{\nabla}u$ (flowing warm $\rightarrow$ cold).

The rate of heat flow passing though S is:

$$\boxed{-K\iint_S \nabla u \cdot d\mathbf{S}}$$

EXAMPLE 6 The temperature u in a metal ball is proportional to the square of the distance from the center of the ball. Find the rate of heat flow across a sphere S of radius a with center at the center of the ball.

$u = C(x^2+y^2+z^2)$ $\quad C$ constant

S: sphere of radius a & center $(0,0,0)$. $\quad 0\le\phi\le\pi$, $0\le\theta\le 2\pi$

$\vec{r}(\phi,\theta) = \langle a\cos\theta\sin\phi,\ a\sin\theta\sin\phi,\ a\cos\phi\rangle$

$\vec{r}_\phi\times\vec{r}_\theta = \langle a^2\cos\theta\sin^2\phi,\ a^2\sin\theta\sin^2\phi,\ a^2\sin\phi\cos\phi\rangle$

$= a\sin\phi\ \vec{r}(\phi,\theta)$

$\vec{\nabla}u = C\langle 2x, 2y, 2z\rangle$. So

$$-K\iint_S \vec{\nabla}u\cdot d\vec{S} = -K\iint_D C\langle 2x,2y,2z\rangle\cdot a\sin\phi\,\vec{r}(\phi,\theta)\,dA$$

$$= -2KCa\sin\phi\int_0^{2\pi}\int_0^{\pi}\vec{r}(\phi,\theta)\cdot\vec{r}(\phi,\theta)\,d\phi\,d\theta$$

$$= -2KCa\int_0^{2\pi}\int_0^{\pi}|\vec{r}(\phi,\theta)|^2\sin\phi\,d\phi\,d\theta$$

$$= -2kCa^3 \int_0^{2\pi}\int_0^{\pi} \sin\phi \, d\phi \, d\theta$$

$$= -2kCa \int_0^{2\pi}\int_0^{\pi} \underbrace{a^2 \sin\phi}_{|\vec{r}_\phi \times \vec{r}_\theta|} \, d\phi \, d\theta$$

$$= -2kCa \iint_S dS$$

$$= -2kCa \; A(S)$$

$$= -2kCa \; 4\pi a^2 = \boxed{-8kCa^3\pi}$$

<u>16.7</u>

$\rho = C$ (constant)

$\vec{v}$: velocity field

$$\iint_S \underset{\vec{F}}{\underset{\uparrow}{\rho \vec{v}}} \cdot \vec{dS}$$