Section 3.3 — Problem 12 — 20 points

(a) The derivative of f is

$$f'(x) = \frac{x^2 + 1 - 2x^2}{(x^2 + 1)^2} = \frac{1 - x^2}{(1 + x^2)^2} = \frac{(1 - x)(1 + x)}{(1 + x^2)^2}.$$

The critical points are at $x = 1$ and $x = -1$ where $f'(x)$ is zero. The derivative exists for any x.

When $x < -1$, then $(1 - x) > 0$ and $1 + x < 0$. The denominator is always positive and therefore $f'(x) < 0$. The function is decreasing when $x < -1$.

When $-1 < x < 1$, then $1 - x > 0$ and $1 + x > 0$. The denominator is always positive and therefore $f'(x) > 0$. The function is increasing for $-1 < x < 1$.

When $x > 1$, then $1 - x < 0$ and $1 + x > 0$. The denominator is always positive and therefore $f'(x) < 0$. The function is decreasing for $x > 1$.

(b) When $x < -1$, the function f is decreasing and when $-1 < x < 1$, the function f is increasing. By the first derivative test, $x = -1$ is a local minimum. The local minimum value of f there is therefore

$$f(-1) = -\frac{1}{2}.$$

When $-1 < x < 1$, the function f is increasing and when $x > 1$, the function f is decreasing. By the first derivative test, $x = 1$ is a local maximum. The local maximum value of f there is

$$f(1) = \frac{1}{2}.$$

(c) The second derivative is

$$\begin{aligned}
f''(x) &= \frac{-2x(1 + x^2)^2 - 4x(1 - x^2)(1 + x^2)}{(1 + x^2)^4} \\
&= \frac{-2x(1 + 2x^2 + x^4) - 4x(1 - x^4)}{(1 + x^2)^4} \\
&= \frac{-2x - 4x^3 - 4x^5 - 4x + 4x^5}{(1 + x^2)^4} \\
&= \frac{-6x - 4x^3}{(1 + x^2)^4} \\
&= -4\frac{x(3/2 - x^2)}{(1 + x^2)^4}
\end{aligned}$$

and therefore

$$f''(x) = -4\frac{x(\sqrt{3/2} - x)(\sqrt{3/2} + x)}{(1 + x^4)^4}.$$

The possible inflection points are when $f''(x) = 0$ or $f''(x)$ does not exist. There is no problem with the expression of $f''(x)$. The zeros are $x = -\sqrt{3/2}$, $x = 0$ and $x = \sqrt{3/2}$.

Here is a table summarizing all the information we need to answer the question. The detailed explanations are presented after the table.

Factors	$x <$	$-\sqrt{3/2}$	$< x <$	0	$< x <$	$\sqrt{3/2}$	$< x$
-4	$-$	$\cdot$	$-$	$\cdot$	$-$	$\cdot$	$-$
x	$-$	$\cdot$	$-$	$\cdot$	$+$	$\cdot$	$+$
$\sqrt{3/2} - x$	$+$	$\cdot$	$+$	$\cdot$	$+$	$\cdot$	$-$
$\sqrt{3/2} + x$	$-$	$\cdot$	$+$	$\cdot$	$+$	$\cdot$	$+$
$(1 + x^4)^4$	$+$	$\cdot$	$+$	$\cdot$	$+$	$\cdot$	$+$
$f''(x)$	$-$	0	$+$	0	$-$	0	$+$
$f(x)$	$\frown$	IP	$\smile$	IP	$\frown$	IP	$\smile$

When $x < -\sqrt{3/2}$, then $x < 0$, $\sqrt{3/2} - x > 0$, and $\sqrt{3/2} + x < 0$. Since $-4 < 0$ and the denominator is always positive, we conclude that $f''(x) < 0$. Therefore, the function is concave down for $x < -\sqrt{3/2}$.

When $-\sqrt{3/2} < x < 0$, then $x < 0$, $\sqrt{3/2} - x > 0$ and $\sqrt{3/2} + x > 0$. Since $-4 < 0$ and the denominator is always positive, we conclude that $f''(x) > 0$ there. Therefore, the function is concave up for $-\sqrt{3/2} < x < 0$.

Since f changes from concave down to concave up at $x = -\sqrt{3/2}$, the number $x = -\sqrt{3/2}$ is an inflection point.

When $0 < x < \sqrt{3/2}$, then $x > 0$, $\sqrt{3/2} - x > 0$, and $\sqrt{3/2} + x > 0$. Since $-4 < 0$ and the denominator is always positive, we conclude that $f''(x) < 0$ there. Therefore, the function is concave down for $0 < x < \sqrt{3/2}$.

Since f changes from concave up to concave down at $x = 0$, the number $x = 0$ is an inflection point.

Finally, when $x > \sqrt{3/2}$, then $x > 0$, $\sqrt{3/2} - x < 0$, $\sqrt{3/2} + x > 0$. Since $-4 < 0$ and the denominator is always positive, we conclude that $f''(x) > 0$ there. Therefore, the function is concave up for $x > \sqrt{3/2}$.

Since f changes from concave down to concave up at $x = \sqrt{3/2}$, the number $x = \sqrt{3/2}$ is an inflection point.

Section 3.3 — Problem 30 — 10 points

(a) The derivative and the second derivative are positive at B. The reasons are that, at B, the slope of the tangent line is positive and the graph of the function is concave up.

(b) The derivative and the second derivative are negative at E. The reasons for that are, at E, the slope of the tangent line is negative and the graph of the function is concave down.

(c) The derivative is negative and the second derivative is positive at A. The reasons for that are, at A, the slope of the tangent line is negative and the graph of the function is concave up.

Section 3.4 — Problem 12 — 5 points

Factoring x^3, we have

$$\frac{4x^3+6x^2-2}{2x^3-4x+5} = \frac{4+6/x-2/x^3}{2-4/x^2+5/x^3}$$

Using the fact that $\lim_{x\to-\infty} \frac{1}{x^r} = 0$, we obtain

$$\lim_{x\to-\infty}\left(4+\frac{6}{x}-\frac{2}{x^3}\right) = 4 \quad \text{and} \quad \lim_{x\to-\infty}\left(2-\frac{4}{x^2}+\frac{5}{x^3}\right) = 2.$$

By the quotient rule, we see that

$$\lim_{x\to-\infty}\frac{4x^3+6x^2-2}{2x^3-4x+5} = \frac{4+6/x-2/x^3}{2-4/x^2+5/x^3} = \frac{4}{2} = 2.$$

Section 3.4 — Problem 14 — 5 points

Factoring $t^{3/2}$, we have

$$\frac{t - t^{3/2}}{2t^{3/2} + 3t - 5} = \frac{1/t^{1/2} - 1}{2 + 3/t^{1/2} - 5/t^{3/2}}.$$

Using the fact that $\lim_{t\to\infty} \frac{1}{t^r} = 0$, we obtain

$$\lim_{t\to\infty} \left(\frac{1}{t^{1/2}} - 1\right) = -1 \quad \text{and} \quad \lim_{t\to\infty} \left(2 + \frac{3}{t^{1/2}} - \frac{5}{t^{3/2}}\right) = 2.$$

By the quotient rule, we see that

$$\lim_{t\to\infty} \frac{t - t^{3/2}}{2t^{3/2} + 3t - 5} = \frac{-1}{2}.$$

Section 3.4 — Problem 18 — 5 points

Factoring x^4, we see that

$$\sqrt{x^4+1} = \sqrt{x^4}\sqrt{1+1/x^4}$$

Since $x \to \infty$, we must have that $x > 0$ eventually and therefore $\sqrt{x^4} = x^2$. Then, we can write

$$\frac{x^2}{\sqrt{x^4+1}} = \frac{1}{\sqrt{1+1/x^4}}.$$

Using the fact that $\lim_{x\to\infty} \frac{1}{x^r} = 0$, we see that

$$\lim_{x\to\infty} \left(1 + \frac{1}{x^4}\right) = 1$$

and by the root law for limits, we conclude that

$$\lim_{x\to\infty} \sqrt{1+\frac{1}{x^4}} = 1.$$

By the quotient law, we obtain

$$\lim_{x\to\infty} \frac{x^2}{\sqrt{x^4+1}} = \frac{1}{1} = 1.$$

Section 3.4 — Problem 22 — 5 points

Since $x \to -\infty$, we have that $x < 0$ eventually. Let's multiply by the conjugate:

$$\left(\sqrt{4x^2+3x}+2x\right)\left(\frac{\sqrt{4x^2+3x}-2x}{\sqrt{4x^2-3x}-2x}\right) = \frac{4x^2+3x-4x^2}{\sqrt{4x^2+3x}-2x} = \frac{3x}{\sqrt{4x^2+3x}-2x}.$$

Factoring x^2 in the root, we find

$$\sqrt{4x^2+3x} = \sqrt{x^2}\sqrt{4+3/x}$$

and since $x < 0$, we have $\sqrt{x^2} = -x$. This means we can rewrite the above expression as followed:

$$\sqrt{4x^2+3x} = -x\sqrt{4+3/x}.$$

Replacing this last expression in the quotient above, we find out that

$$\sqrt{4x^2+3x}+2x = \frac{3}{\sqrt{4+3/x}-2}.$$

We therefore see that

$$\lim_{x\to-\infty}\left(\sqrt{4+\frac{3}{x}}-2\right) = 0^-.$$

We get a zero minus because $x < 0$ and therefore $4+3/x < 4$. Therefore, we obtain

$$\lim_{x\to-\infty}\frac{3}{\sqrt{4+3/x}-2} = \frac{3}{0^-} = -\infty.$$

The limit does not exist.

TOTAL (POINTS): 50.