

MATH 644

CHAPTER 2

SECTION 2.5: ELEMENTARY OPERATIONS

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THEOREM 1. Let f and g be analytic at z_0 . Then,

- a) $f + g$ is analytic at z_0 ;
- b) $f - g$ is analytic at z_0 ;
- c) cf is analytic at z_0 , for any $c \in \mathbb{C}$.

Proof.

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- If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are absolutely convergent series with sums A and B respectively, then their Cauchy Product

$$\left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} b_n\right) = \sum_{n=1}^{\infty} \left(\sum_{m=1}^n a_m b_{n-m}\right)$$

converges absolutely to AB . [See Problem]

THEOREM 2. Let f and g be two analytic functions at z_0 . Then, the function $h = fg$ is analytic at z_0 .

Proof.

THEOREM 3. If f is analytic at z_0 and g is analytic at $a_0 = f(z_0)$, then the function $h = g \circ f$ is analytic at z_0 .

Proof.

Consequences:

- If f is analytic at z_0 with $f(z_0) \neq 0$, then $1/f$ is analytic at z_0 .
- If $r = p/q$ is a rational function, then r is analytic on $\{z : q(z) \neq 0\}$.

DEFINITION 4. If f is defined in a disk (neighborhood) of z , then

$$f'(z) := \lim_{w \rightarrow z} \frac{f(w) - f(z)}{w - z}$$

is called the **(complex) derivative** of f , provided the limit exists.

Note:

- The function $f(z) = \bar{z}$ does not have a complex derivative.
- If n is a non-negative integer, then

$$(z^n)' = nz^{n-1}.$$

THEOREM 5. If $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges in $B = \{z : |z - z_0| < r\}$, then

a) $f'(z)$ exists for all $z \in B$ and

$$f'(z) = \sum_{n=1}^{\infty} na_n(z - z_0)^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}(z - z_0)^n \quad (\forall z \in B).$$

b) Moreover, the series for f' based at z_0 has the same radius of convergence as the series for f .

Proof.

Note: The rules of differentiation hold:

- $(f + g)'(z) = f'(z) + g'(z);$
- $(cf)'(z) = cf'(z);$
- $(fg)'(z) = f'(z)g(z) + f(z)g'(z);$
- $(\frac{f}{g})'(z) = (f'(z)g(z) - f(z)g'(z))/(g(z))^2;$
- $(g \circ f)(z) = g'(f(z))f'(z).$

COROLLARY 6. An analytic function f has derivatives of all orders. Moreover, if f is equal to a convergent power series on $B = \{z : |z - z_0| < r\}$, then the power series is given by

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n \quad (\forall z \in B).$$

Proof.

Consequences:

- If f is analytic in a region Ω with $f'(z) = 0$ for all z in a neighborhood of $z_0 \in \Omega$, then f is constant in Ω .
- If f and g are analytic in a region Ω with $f' = g'$, then $f - g$ is constant.
- If $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges in $B = \{z : |z - z_0| < r\}$, then the power series

$$F(z) := \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z - z_0)^{n+1}$$

converges in B and satisfies $F'(z) = f(z)$ for all $z \in B$.

COROLLARY 7. If $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges in $B = \{z : |z - z_0| < r\}$, then

$$f'(z_0) = \lim_{z, w \rightarrow z_0} \frac{f(z) - f(w)}{z - w}.$$

Proof.