Chapter 2

Derivatives

2.3 Differentiation Formulas

Constant Function.

$$f(x) = c \longrightarrow f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{c - c}{h} = 0$$

Derivative of a Constant Function

$$\frac{d}{dx}(c) = 0$$

Power Functions.

n = 1. $y = x$ $\boxed{\frac{dy}{dx} = 1}$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{x + h - x}{h}$$

$$= \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1$$

$$= 1$$

n = 2. $y = x^2$ $\boxed{\frac{dy}{dx} = 2x}$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{2x\cancel{h} + h^{\cancel{2}1}}{\cancel{h}}$$

$$= \lim_{h \to 0} 2x + h = 2x$$

n = 3. $y = x^3$ $\boxed{dy/dx = 3x^2}$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h \to 0} \frac{3x^2\cancel{h} + 3xh^{\cancel{2}1} + h^{\cancel{3}2}}{\cancel{h}}$$

$$= \lim_{h \to 0} 3x^2 + 3xh + h^2$$

$$= 3x^2$$

The Power Rule If n is a positive integer, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

EXAMPLE 4 Find the points on the curve $y = x^4 - 6x^2 + 4$ where the tangent line is horizontal.

$$y' = (x^4)' - 6(x^2)' + (4)'$$
$$= 4x^3 - 12x + 0$$

Goal: Find where y' is zero.

$$y' = \lim_{h\to 0} \frac{(x+h)^4 - 6(x+h)^2 + 4 - x^4 + 6x^2 - 4}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^4 - x^4 - 6(x+h)^2 + 6x^2 + 4 - 4}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^4 - x^4}{h} + \lim_{h\to 0} -6\left(\frac{(x+h)^2 - x^2}{h}\right)$$
$$+ \lim_{h\to 0} \frac{4-4}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^4 - x^4}{h} - 6\lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} + \lim_{h\to 0} \frac{4-4}{h}$$

$$= \frac{d}{dx}(x^4) - 6\frac{d}{dx}(x^2) + \frac{d}{dx}(4)$$

$$= 4x^3 - 6\cdot 2x + 0$$

$$\Rightarrow y' = 4x^3 - 12x$$

Solution:

$$y' = 0 \iff 4x^3 - 12x = 0$$
$$\iff (4x^2 - 12)x = 0$$
$$\iff \boxed{x = 0 \text{ or } x = \pm\sqrt{3}}$$

Multiplication by a constant.

The Constant Multiple Rule If c is a constant and f is a differentiable function, then

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$$

Sum.

The Sum Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

Difference.

The Difference Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

Product.

The Product Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

Caution!!!

$$\frac{d}{dx}(fg) \neq \frac{d}{dx}(f)\frac{d}{dx}(g).$$

Example.

$f(x) = x$ & $g(x) = x$

$f(x)\,g(x) = x^2 \to 2x$

$\cancel{f'(x) = 1}$

$\cancel{g'(x) = 1}$ $\to \cancel{1 \cdot 1 = 1}$

Example. Find the derivative of the function $f(x) = (5x^2 - 2)(x^3 + 3x)$.

$$f'(x) = (5x^2-2)\frac{d}{dx}(x^3+3x) + (x^3+3x)\frac{d}{dx}(5x^2-2)$$

$$= (5x^2-2)(3x^2+3) + (x^3+3x)(10x - 0)$$

$$= (5x^2-2)(3x^2+3) + 10x(x^3+3x).$$

$$\to = 15x^4 + 39x^2 - 6$$

Quotient.

The Quotient Rule If f and g are differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

Caution !!

$$\frac{d}{dx}\left(\frac{f}{g}\right) \neq \frac{\frac{d}{dx}(f)}{\frac{d}{dx}(g)}$$

Example.

$$x^2 = \frac{x^3}{x} \quad x^3 \to f(x), \quad x \to g(x)$$

$$\Rightarrow \cancel{(x^2)' \stackrel{?}{=} \frac{3x^2}{1}} = \cancel{3x^2} \neq 2x$$

✓

EXAMPLE 8 Let $y = \dfrac{x^2 + x - 2}{x^3 + 6}$. Compute the derivative.

$f(x) = x^2 + x - 2 \longrightarrow f'(x) = 2x + 1$

$g(x) = x^3 + 6 \longrightarrow g'(x) = 3x^2$

$$y' = \frac{g(x)\, f'(x) - f(x)\, g'(x)}{[g(x)]^2}$$

$$= \frac{(x^3+6)(2x+1) - (x^2+x-2)(3x^2)}{(x^3+6)^2}$$ ✓

$$= \frac{2x^4 + x^3 + 12x + 6 - 3x^4 - 3x^3 + 6x^2}{(x^3+6)^2}$$

$$= \boxed{\frac{-x^4 - 2x^3 + 6x^2 + 12x + 6}{(x^3+6)^2}}$$

General Power rule.

The Power Rule (General Version) If n is any real number, then $(n \neq 0)$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Case n = 0: $\frac{d}{dx}(x^0) = \frac{d}{dx}(1) = 0$

Example. Find the derivative of the function $f(x) = x^{2/3}$.

$$f'(x) = \frac{2}{3} x^{\frac{2}{3}-1} = \boxed{\frac{2}{3} x^{-1/3}}$$

$$= \frac{2}{3x^{1/3}}$$

$$\frac{2}{3} - 1 = \frac{2}{3} - \frac{3}{3} = -\frac{1}{3}$$

$$g(x) = x^{-1/2} \longrightarrow g'(x) = \left(-\frac{1}{2}\right) x^{-\frac{1}{2}-1} = -\frac{1}{2} x^{-3/2}$$

EXAMPLE 13 At what points on the hyperbola $\underbrace{xy = 12}_{y = \frac{12}{x}}$ is the tangent line parallel to the line $3x + y = 0$?

① tangent line

② Parallel to another line. $\rightarrow$ $y = m_1 x + b_1$: L_1

$y = m_2 x + b_2$: L_2

$L_1 \parallel L_2$ if $m_1 = m_2$.

Goal: Find x s.t. $f'(x)$ is equal to the slope of $3x+y=0$.

Here, $3x+y=0 \rightarrow y = -3x \rightarrow$ slope $= -3$.

Here, $y = \frac{12}{x} \rightarrow y' = -12x^{-1-1} = -12x^{-2}$

$= 12x^{-1}$ $\Rightarrow y' = \frac{-12}{x^2}$

Now, $\frac{-12}{x^2} = y' = -3$

$\Rightarrow (\div -3)\ \frac{-12}{x^2} = -3 \Rightarrow 4 = x^2 \Rightarrow x = \pm 2$

Answer: the tangent is parallel to $3x+y=0$ at $(-2,-6)$ & $(2,6)$.

Summary of Differentiation Formulas.

Table of Differentiation Formulas

$\frac{d}{dx}(c) = 0$	$\frac{d}{dx}(x^n) = nx^{n-1}$	
$(cf)' = cf'$	$(f+g)' = f' + g'$	$(f-g)' = f' - g'$
$(fg)' = fg' + gf'$	$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$	