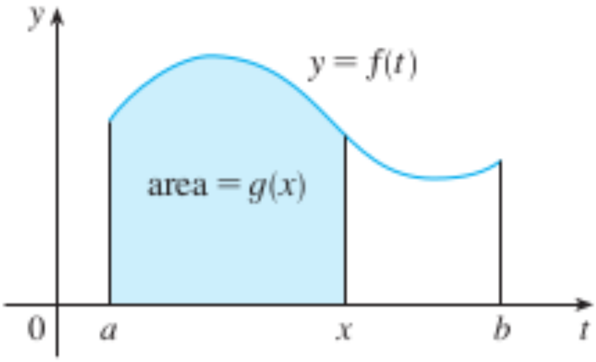
Chapter 4
Integrals

4.3 The Fundamental Theorem of Calculus

Area Function.



EXAMPLE 1 If f is the function whose graph is shown in Figure 2 and $g(x) = \int_0^x f(t)\,dt$, find the values of $g(0)$, $g(1)$, $g(2)$, $g(3)$, $g(4)$, and $g(5)$. Then sketch a rough graph of g.

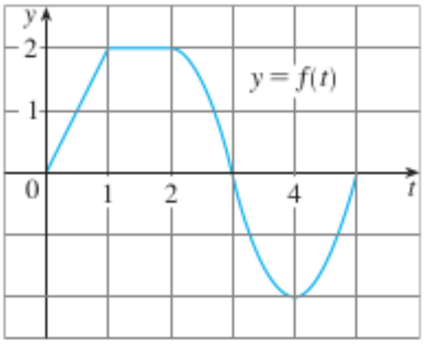
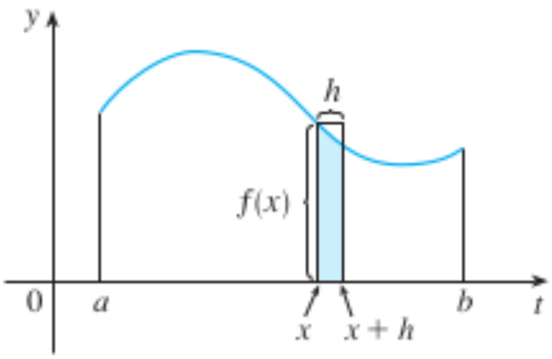


FIGURE 2



The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\,dt \qquad a \leqslant x \leqslant b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

EXAMPLE 2 Find the derivative of the function $g(x) = \int_0^x \sqrt{1 + t^2}\,dt$.

EXAMPLE 4 Find $\dfrac{d}{dx}\displaystyle\int_1^{x^4} \sec t\, dt.$

Example. Find the derivative of the function $f(x) = \displaystyle\int_{\sin x}^{1} \sqrt{1+t^2}\, dt$.

Second part of the Fundamental Theorem of Calculus.

Example. Compute the integral $\displaystyle\int_a^b x\,dx$ where a and b are two numbers such that a < b.

Definition A function F is called an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

General Antiderivative:

Example. Find the general antiderivative of each of the following functions.

(a) $f(x) = x$ (b) $f(x) = \sqrt{x}$ (c) $f(x) = \sin x$ (d) $f(x) = 2x\sin(x^2)$

Table of Antiderivaties of some functions.

Function	Particular antiderivative	Function	Particular antiderivative
$cf(x)$	$cF(x)$	$\cos x$	$\sin x$
$f(x) + g(x)$	$F(x) + G(x)$	$\sin x$	$-\cos x$
$x^n \ (n \neq -1)$	$\dfrac{x^{n+1}}{n+1}$	$\sec^2 x$	$\tan x$
		$\sec x \tan x$	$\sec x$

EXAMPLE Find f if $f'(x) = x\sqrt{x}$ and $f(1) = 2$.

The Fundamental Theorem of Calculus, Part 2 If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function F such that $F' = f$.

Consequence on the distance problem:

EXAMPLE 5 Evaluate the integral $\int_{-2}^{1} x^3\,dx$.

EXAMPLE 7 Find the area under the cosine curve from 0 to b, where $0 \leqslant b \leqslant \pi/2$.

EXAMPLE 8 What is wrong with the following calculation?

$$\int_{-1}^{3} \frac{1}{x^2}\,dx = \frac{x^{-1}}{-1}\Bigg]_{-1}^{3} = -\frac{1}{3} - 1 = -\frac{4}{3}$$

Differentiation and Integration as Inverse Processes.

The Fundamental Theorem of Calculus Suppose f is continuous on $[a, b]$.

1. If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$.

2. $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f, that is, $F' = f$.