Math 311

Chapter 3

Section 3.1: The Cofactor Expansion

Contents

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Goal

Recall that if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det A = ad - bc$ and A is invertible if and only if $\det A \neq 0$.

Goal: To Generalize the determinant to $n \times n$ matrix.

A Basic Example

If A is a 3×3 square matrix and if A is invertible, then we know A can be carried to the identity matrix I.

Following the process of $A \to I$:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \to \begin{bmatrix} a & b & c \\ ad & ae & af \\ ag & ah & ai \end{bmatrix} \to \begin{bmatrix} a & b & c \\ 0 & ae - bd & af - cd \\ 0 & ah - bg & ai - cg \end{bmatrix}$$

Set $u = ae - bd$ and $v = ah - bg$. Then

$$\begin{bmatrix} a & b & c \\ 0 & u & af - cd \\ 0 & v & ai - cg \end{bmatrix} \to \begin{bmatrix} a & b & c \\ 0 & u & af - cd \\ 0 & vu & u(ai - cg) \end{bmatrix} \to \begin{bmatrix} a & b & c \\ 0 & vu & v(af - cd) \\ 0 & 0 & w \end{bmatrix}$$

$w = u(ai - cg) - v(af - cd)$. Hence, if we want to carry on the algorithm, we need that

$$w \neq 0$$

Definition 1. If A is a 3×3 matrix, then

$$\det A := w = aei + bfg + cdh - ceg - afh - bdi.$$

Remark:

- Notice that A is invertible if and only if $\det A \neq 0$.
- Notice that

$$\begin{aligned}\det A &= a(ei - fh) - b(di - fg) + c(dh - eg)\\ &= a \det \begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \det \begin{bmatrix} d & f \\ g & i \end{bmatrix} + c \det \begin{bmatrix} d & e \\ g & h \end{bmatrix}.\end{aligned}$$

- The terms $+a \det \begin{bmatrix} e & f \\ h & i \end{bmatrix}$, $-b \det \begin{bmatrix} d & f \\ g & i \end{bmatrix}$ and $+c \det \begin{bmatrix} d & e \\ g & h \end{bmatrix}$ are called **cofactors** of A and are denoted by $c_{11}(A)$, $c_{12}(A)$ and $c_{13}(A)$ respectively.

Example 1. Compute the determinant of $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$.

Solution.

$$\det A = 2 \det \begin{bmatrix} 2 & 3 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} - 3 \det \begin{bmatrix} 2 & 3 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} + 1 \det \begin{bmatrix} 2 & 3 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

$$= 2 \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - 3 \det \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} + \det \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = 5$$

Notice that

$$c_{12}(A) = (-1)^{1+2} \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = (-1)^{1+2} \det \begin{bmatrix} d & f \\ g & i \end{bmatrix}.$$

We denote by A_{ij} the $(n-1) \times (n-1)$ matrix obtained from A by deleting row i and column j.

Definition 2. Let A be an $n \times n$ matrix. The $\mathbf{(i, j)}$**-cofactor** $c_{ij}(A)$ is the scalar defined by

$$c_{ij}(A) = (-1)^{i+j} \det(A_{ij}).$$

Here, $(-1)^{i+j}$ is called the **sign** of the (i, j)-position.

Example 2. Find the cofactors of positions $(3, 2)$ of

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}.$$

$$c_{32}(A) = (-1)^{2+3} \det \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= (-1)^5 \big((2)(1) - (1)(1)\big) = \boxed{-1}$$

DEFINITION 3. Let $A = [a_{ij}]$ be an $n \times n$ matrix. The **determinant** of A is defined by

$$\det A = a_{11}c_{11}(A) + a_{12}c_{12}(A) + \cdots + a_{1n}c_{1n}(A).$$

Remark: This is called the **cofactor expansion** of $\det A$ along row 1.

EXAMPLE 3. compute the determinant of $A = \begin{bmatrix} 3 & 4 & 5 & 0 \\ 1 & 7 & 2 & 0 \\ 9 & 8 & -6 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

SOLUTION.

$$\det(A) = 3c_{11}(A) + 4c_{12}(A) + 5c_{13}(A) + 0 \cdot c_{14}(A)$$

$$= 3(-1)^{1+1}\det\begin{bmatrix} 7 & 2 & 0 \\ 8 & -6 & 3 \\ 1 & 1 & 1 \end{bmatrix} + 4(-1)^{1+2}\det\begin{bmatrix} 1 & 2 & 0 \\ 9 & -6 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

$$+ 5(-1)^{1+3}\det\begin{bmatrix} 1 & 7 & 0 \\ 9 & 8 & 3 \\ 1 & 1 & 1 \end{bmatrix} + 0 \cdot (-1)^{1+4}\det\begin{bmatrix} 1 & 7 & 2 \\ 9 & 8 & -6 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= 3(-73) - 4(-21) + 5(-37) - 0$$

$$= \boxed{-320}$$

Theorem 1. [Proved by Pierre-Simon de Laplace (1749-1827)] The determinant of an $n \times n$ matrix A can be computed by using the cofactor expansion along any row or column of A.

Example 4. Compute $\det A$ if $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 2 & 3 & 1 \end{bmatrix}$.

Solution. Use the 1st column.

$$\det A = (1)C_{11}(A) + \cancel{(0)C_{21}(A)} + \cancel{(0)C_{31}(A)} + \cancel{(0)C_{41}(A)}$$

$$= (1)(-1)^{1+1} \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}$$

$$= (1)\Big[\cancel{(0)(-1)^{2+1} \det \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}} \to 0$$

$$+ (1)(-1)^{2+2} \det \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

$$+ (1)(-1)^{2+3} \det \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \Big]$$

$$= (1)\big(0 + (1)(1)(-1) + (1)(-1)(1) \big) = \boxed{-2}$$

Interchanging two rows

EXAMPLE 5. Show that

$$\det \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = -\det \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

SOLUTION.

THEOREM 2. If B is an $n \times n$ matrix obtained from interchanging two rows of an $n \times n$ matrix A, then

$$\det(B) = -\det(A).$$

Remark: This fact is still true if we interchange two *columns* (instead of rows).

Scaling a row

EXAMPLE 6. Show that

$$\det \begin{bmatrix} 2 & 6 & 8 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = 2 \det \begin{bmatrix} 1 & 3 & 4 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

SOLUTION.

THEOREM 3. If B is an $n \times n$ matrix for which column j is obtained by multiplying k times the column j of an $n \times n$ matrix A, with $k \neq 0$, then

$$\det(B) = k \det(A).$$

Remark: This fact is still true if a column j of a matrix B is obtained by multiplying the column j of a given matrix A by a nonzero scalar

Subtracting a Multiple of a Row

EXAMPLE 7. Show that

$$\det \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} = \det \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 3 \\ 1 & 2 & 3 \end{bmatrix}.$$

SOLUTION.

THEOREM 4. If a the row j of a matrix B is obtained by subtracting a multiple of a row of a matrix A to the row j of A, then

$$\det(B) = \det(A).$$

Remark: This remains true if we replace the row operation by the corresponding column operation.

Theorem 5. Let A be an $n \times n$ matrix.

1. If A has a row (or column) of zero, then $\det(A) = 0$.
2. If A has two identical rows (or columns), then $\det(A) = 0$.

Proof.

1. Developing $\det(A)$ along the row of zero, then $\det(A) = 0$.
2. Assume that the two identical rows have index p and q. Let B be the matrix obtained by interchanging rows p and q of A. Then, $A = B$. But, $\det(B) = -\det(A)$, which implies that $2\det(A) = 0$, hence $\det(A) = 0$. $\square$

Example 8. Find the values of x for which $\det(A) = 0$, where
$$A = \begin{bmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{bmatrix}.$$

Solution.

Example 9. Compute $\det(A)$ if $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 3 & 4 & 5 & 0 \\ 4 & 3 & 2 & 10 \end{bmatrix}$.

Solution.

Definition 4. A matrix A is

1. **lower triangle** if all the entries above the main diagonal are zero.
2. **upper triangle** if all the entries below the main diagonal are zero.
3. **triangular** if it is lower triangle or upper triangle.

Theorem 6. If $A = [a_{ij}]$ is an $n \times n$ triangular matrix, then $\det(A) = a_{11}a_{22}\cdots a_{nn}$.