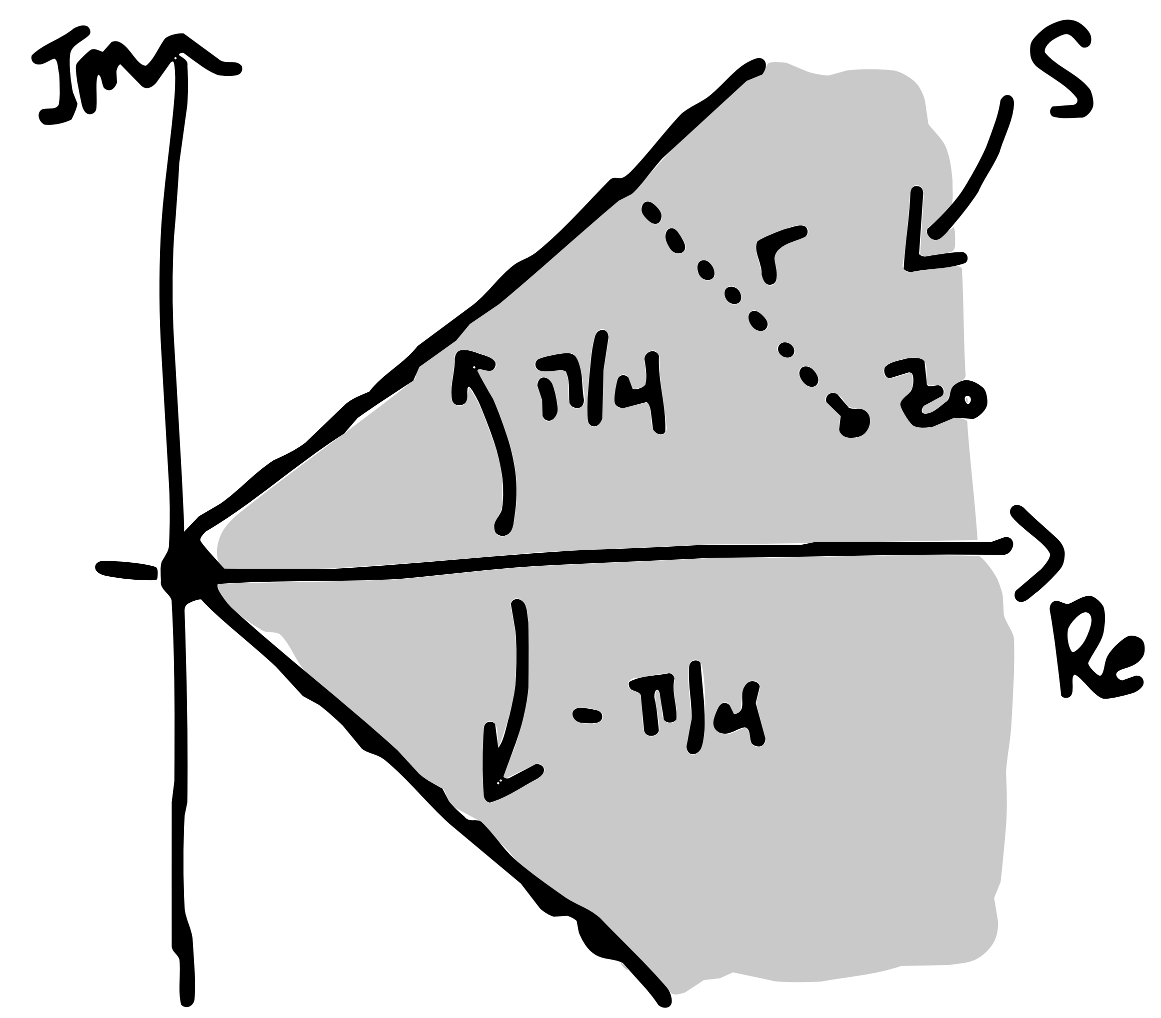
Section 2.1 M444

Problems Solution P.-O. Parise

Problem 8 $S := \{ z : z \neq 0 \text{ and } |\mathrm{Arg}\, z| < \pi/4 \}$.



<u>No closed</u>

$\partial S = \{ z : |\mathrm{Arg}\, z| = \frac{\pi}{4} \text{ and } \mathrm{Re}\, z \geq 0 \}$

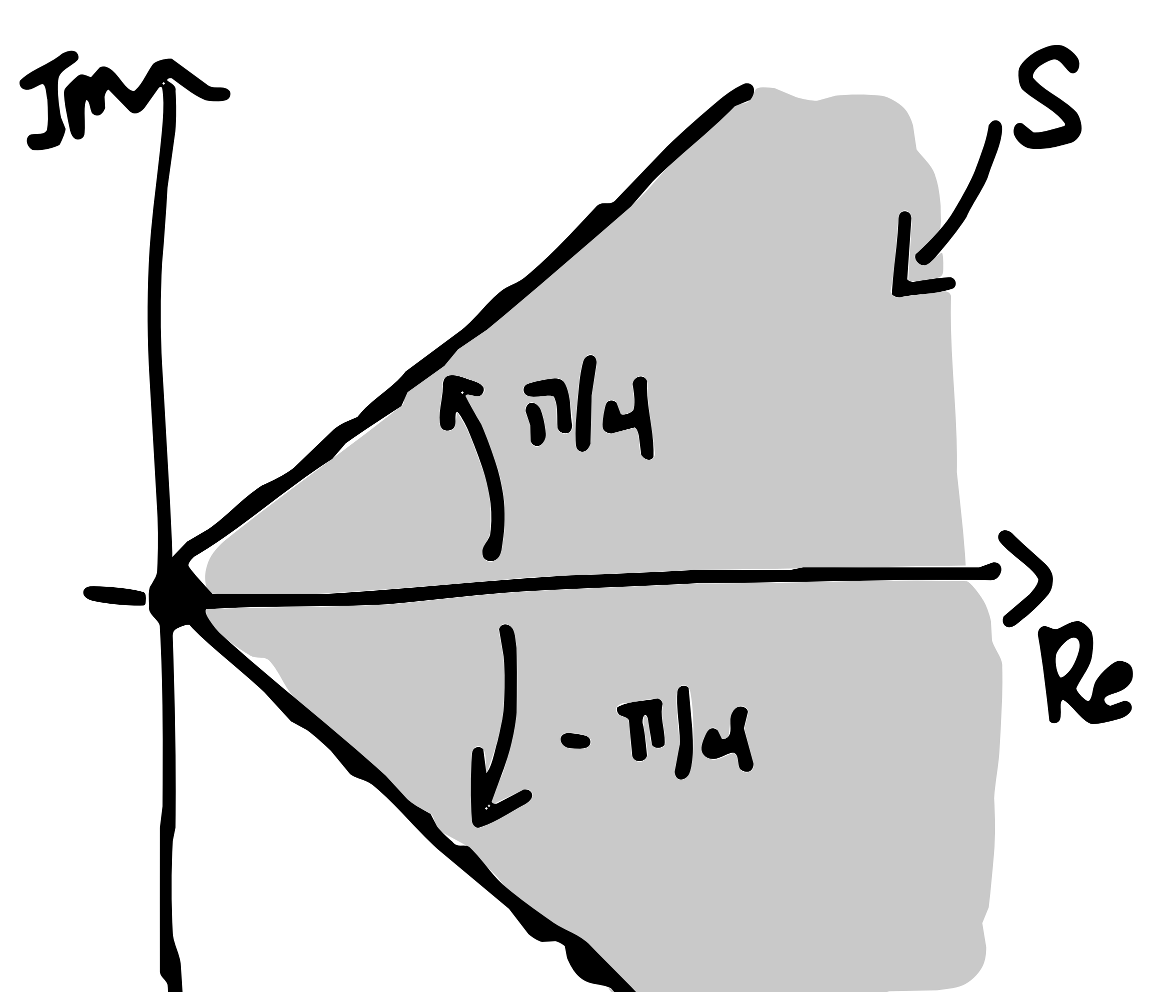
In this case $0 \in \partial S$ but $0 \notin S$.

<u>Open</u>

Let $z_0 \in S$. Let r be the distance from z_0 to the cone (see picture). Then the disk $B(z_0, r/2) \subset S$.

Since z_0 was arbitrary, S is open.

Problem 9 $S = \{ z : z \neq 0 \text{ and } |\mathrm{Arg}\, z| < \frac{\pi}{4} \} \cup \{0\}$.

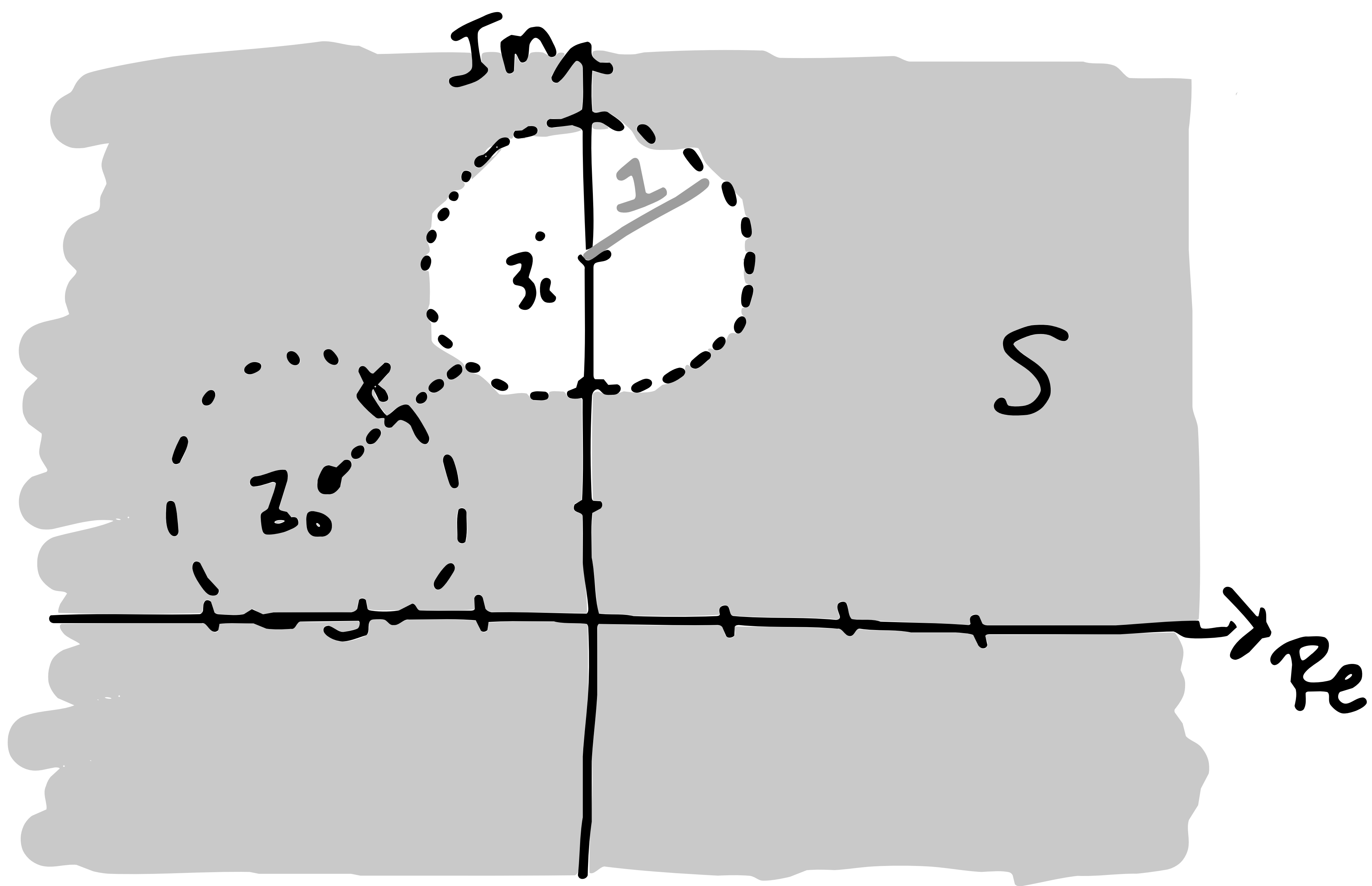


<u>Closed</u>

$\partial S = \{ z : |\mathrm{Arg}\, z| = \frac{\pi}{4} \text{ and } \mathrm{Re}\, z \geq 0 \}$

We see that $\partial S \subset S$ and so S is closed.

Problem 12 $S = \{ z : |z - 3i| > 1 \}$



<u>Not closed</u>

$\partial S = \{ z : |z - 3i| = 1 \}$.

Therefore, it is not closed because $\partial S \not\subset S$.

<u>Open.</u> Let $z_0 \in S$ and r be the distance from z_0 to the circle $\{ z : |z - z_0| = 1 \}$. Consider $B(z_0, \frac{r}{2})$. Then

$B(z_0, r/2) \subset S$, as shown on the picture.

Since z_0 was arbitrary, S is open.

Problem 15

Consider the disk

$$B_1(0) = \{ z = x+iy : \ x^2+y^2 < 1 \}$$

and the set

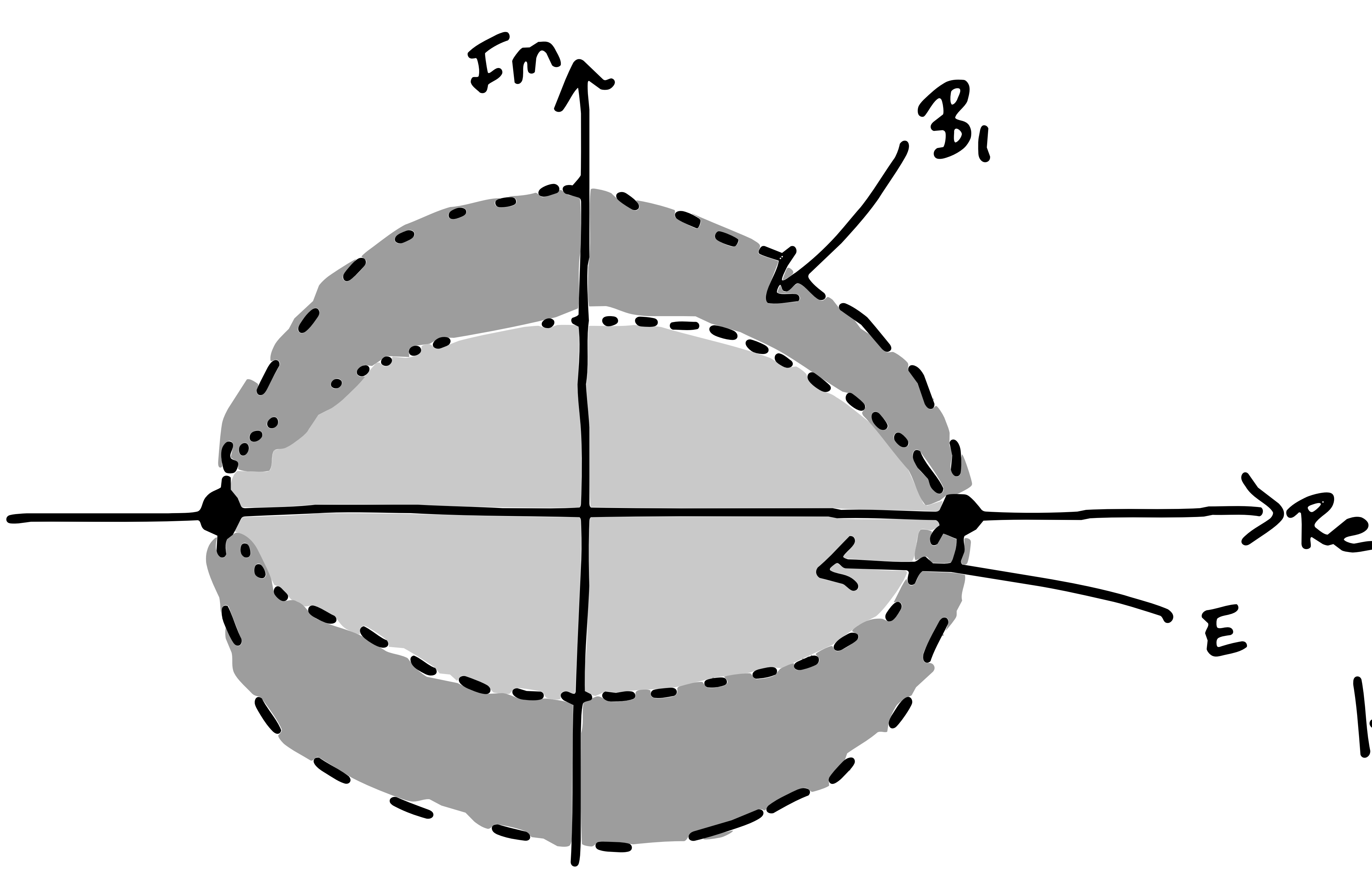
$$E = \{ z = x+iy : \ x^2 + 4y^2 < 1 \}$$

Then $\partial B_1(0) = \{ z = x+iy \ : x^2+y^2 = 1 \}$

which is a circle and

$$\partial E = \{ z = x+iy : \ x^2+4y^2 = 1 \}$$

which is an ellipse. See picture.



The boundary ∂E contains ± 1 which is on $\partial B_1(0)$ but not in $B_1(0)$.

Hence, $\partial E \not\subset B_1(0)$.

Problem 16

The empty set $\emptyset$. We have $\partial \emptyset = \emptyset$.
Assume, for a contradiction, that $\partial\emptyset \neq \emptyset$.
Then, there is some $z \in \partial \emptyset$. By definition, $\forall r > 0$,

$$\emptyset \cap B_r(z) \neq \emptyset \quad \text{and} \quad \emptyset \cap (B_r(z))^c \neq \emptyset.$$

However, $\emptyset$ has no element

$$\Rightarrow \forall r > 0, \quad \emptyset \cap B_r(z) = \emptyset \text{ and}$$
$$\emptyset \cap (B_r(z))^c = \emptyset.$$

This is a contradiction, thus, $\partial\emptyset = \emptyset$.

Problem 17

($\Rightarrow$) Assume that S is open. We want to show that $\partial(\mathbb{C}\setminus S) \subset \mathbb{C}\setminus S$. Let $z_0 \in \partial(\mathbb{C}\setminus S)$
We want to show that $z_0 \in \mathbb{C}\setminus S$.
Assume the contrary, that is $z \notin \mathbb{C}\setminus S$.

Then, $z_0 \notin S^c \Rightarrow z_0 \in S$.

By assumption, S is open. Therefore, $\exists r>0$ such that $B_r(z_0) \subset S$. Hence

$$B_r(z_0) \cap S = \emptyset .$$

But $z_0 \in \partial S$, so $B_r(z_0) \cap S \neq \emptyset$.
This is a contradiction. We must conclude that $z_0 \in \mathbb{C} \setminus S = S^c$.

($\Leftarrow$) Assume $\mathbb{C} \setminus S = S^c$ is closed. We want to show that S is open. Let $z_0 \in S$. We want to show that $\exists r>0$ s.t.

$$B_r(z_0) \subset S .$$

Assume the opposite, that is $\forall r>0$

$$B_r(z_0) \not\subset S .$$

This means $\forall r>0$, $B_r(z_0) \cap S^c \neq \emptyset$
But, since $z_0 \in S$, we also have

$\forall r>0, \quad B(z_0, r) \cap S = \{z_0\} \neq \emptyset.$

Hence, $\forall r > 0$

$$B_r(z_0) \cap S^c \neq 0 \quad \text{and} \quad B_r(z_0) \cap S \neq \emptyset.$$

$\Leftrightarrow \quad \forall r>0$

$$B_r(z_0) \cap S^c \neq 0 \quad \text{and} \quad B_r(z_0) \cap (S^c)^c \neq \emptyset.$$

Thus, $z_0 \in \partial S^c$. Since $\partial S^c \subset S^c$ because S^c is closed, we conclude that $z_0 \in S^c$. Hence

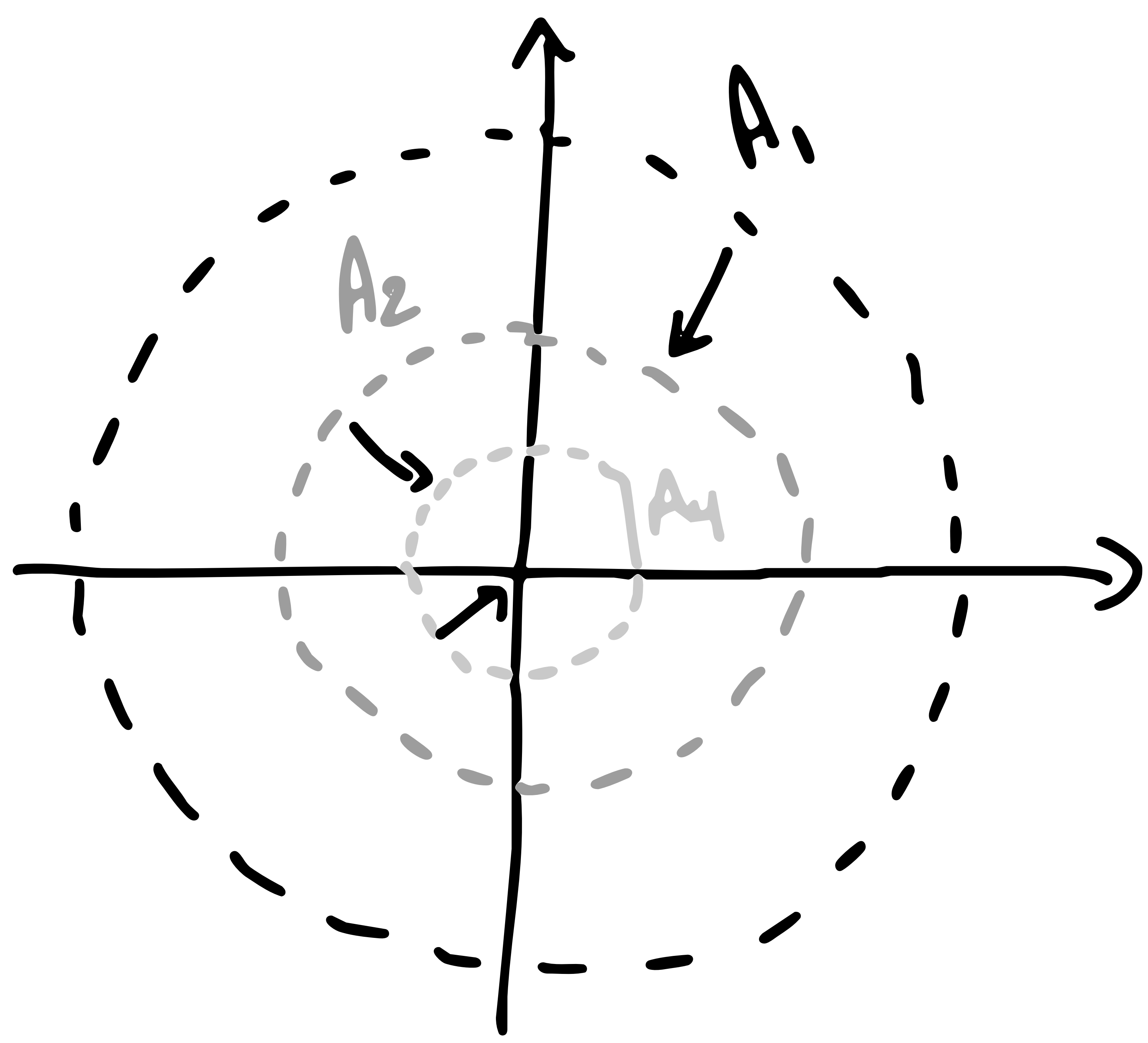
$$z_0 \in S^c \quad \text{and} \quad z_0 \in S.$$

A contradiction. We must have that S is open. $\square$

Problem 20B

Consider $A_n = \left\{ z : |z| < \frac{1}{n} \right\}$.

Since $\frac{1}{n} \to 0$, A_n shrink to $\{0\}$.



Since $A_{n+1} \subset A_n$

then

$$\bigcap_{k=1}^{n} A_k \subset \bigcap_{k=2}^{n} A_k \subset \bigcap_{k=3}^{n} A_k \subset \dots \subset A_n$$

Since A_n shrinks to $\{0\}$, we have

$$\bigcap_{k=1}^{\infty} A_k = \{0\}.$$

A_n are all open (disks) but $\{0\}$ is closed (which is not open in this case). $\square$