Chapter 2

Functions and Limits

2.2 The Derivatives as a Function

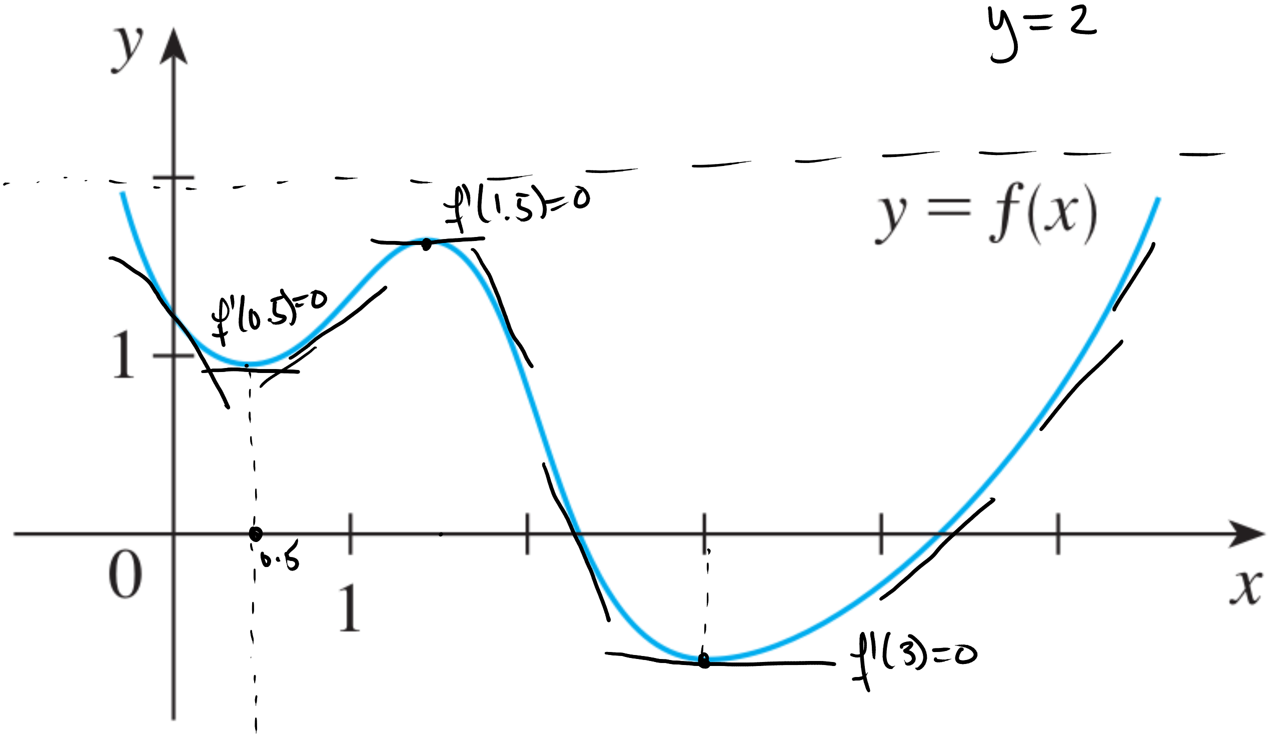
The derivative as a function.

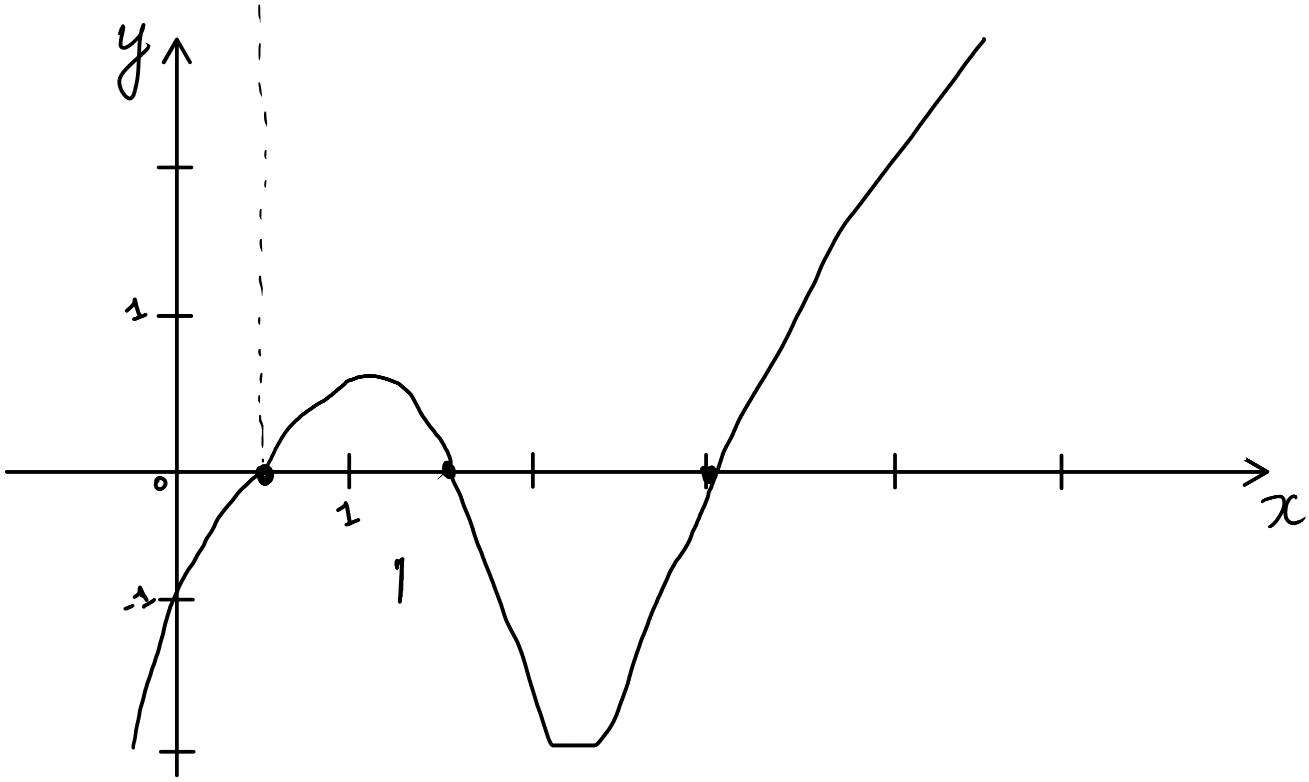
$$\overset{\downarrow}{f'(x)} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Derivative function

Dom of f': x such that $\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$ exists

EXAMPLE 1 The graph of a function f is given in Figure 1. Use it to sketch the graph of the derivative f'.





EXAMPLE 3 If $f(x) = \sqrt{x}$, find the derivative of f. State the domain of f'.
~~Illustrate this formula by comparing the graphs of f and f'.~~

$$f'(x) = \lim_{h \to 0} \frac{\overset{a}{\sqrt{x+h}} - \overset{b}{\sqrt{x}}}{h} \cdot \frac{\overset{a}{\sqrt{x+h}} + \overset{b}{\sqrt{x}}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{\cancel{x} + h - \cancel{x}}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x} + \sqrt{x}} = \boxed{\frac{1}{2\sqrt{x}}}$$

EXAMPLE 4 Find f' if $f(x) = \dfrac{1-x}{2+x}$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{1-(x+h)}{2+x+h} - \dfrac{1-x}{2+x}}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{(1-(x+h))(2+x) - (1-x)(2+x+h)}{(2+x+h)(2+x)}}{h}$$

$$= \lim_{h \to 0} \frac{(1-x-h)(2+x) - (1-x)(2+x) - (1-x)h}{h(2+x+h)(2+x)}$$

$$= \lim_{h \to 0} \frac{(1-x)\cancel{(2+x)} - h(2+x) - (1-x)\cancel{(2+x)} - (1-x)h}{h(2+x+h)(2+x)}$$

$$= \lim_{h \to 0} \frac{-2h - \cancel{xh} - h + \cancel{xh}}{h(2+x+h)(2+x)}$$

$$= \lim_{h \to 0} \frac{-3\cancel{h}}{\cancel{h}(2+x+h)(2+x)}$$

$$= \lim_{h \to 0} \frac{-3}{(2+x+h)(2+x)}$$

$$= \boxed{\frac{-3}{(2+x)^2}}$$

Other notations for the derivative.

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$

Leibniz notation

Evaluating in the Leibniz notation: $\left.\frac{dy}{dx}\right|_{x=a} \approx f'(a)$

Example. What is the value of $\left.\frac{dy}{dx}\right|_{x=2}$ if $y = f(x) = x^2$.

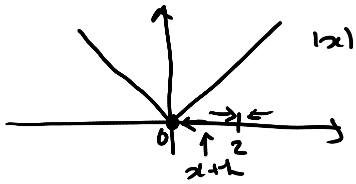
$$\frac{dy}{dx} = f'(x) = 2x^{2-1} = 2x \qquad \left(\frac{dy}{dx} = \frac{d}{dx}(x^2) \right)$$

$$\left.\frac{dy}{dx}\right|_{x=2} = 2^2 = \boxed{4}.$$

Differentiable Functions.

3 Definition A function f is **differentiable at a** if $f'(a)$ exists. It is **differentiable on an open interval** (a, b) [or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

EXAMPLE 5 Where is the function $f(x) = |x|$ differentiable? $\rightarrow (-\infty, \infty) \setminus \{0\}$.



$$f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$$\left.\frac{dy}{dx}\right|_{x=2} = \lim_{h \to 0} \frac{|x+h| - |x|}{h} = \lim_{h \to 0} \frac{2+h-2}{h} = \lim_{h \to 0} \frac{h}{h} = 1$$

$\frac{dy}{dx} = 1$ for any $x > 0$

$\frac{dy}{dx} = -1$ for any $x < 0$

$$\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{h}{h} = 1$$

$$\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = -1 \quad \neq$$

$\Rightarrow f'(0) \nexists$.

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4 Theorem If f is differentiable at a, then f is continuous at a.

Proof.

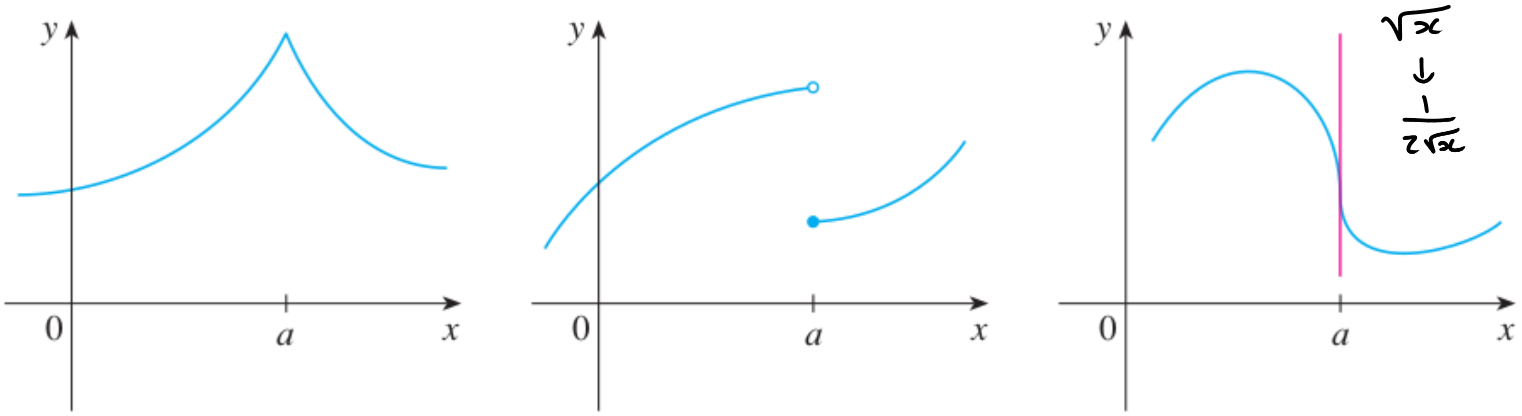
$$f(x) - f(a) = \frac{f(x) - f(a)}{x - a} \cdot (x - a)$$

Goal

$$\lim_{x \to a} f(x) = f(a)$$

Remark: "If f continuous at a then f differentiable at a" is false
$f(x) = |x|$ is continuous at 0, but not diff. at $x = 0$.

How can a Function Fail to be diffentiable?



(a) A corner (b) A discontinuity (c) A vertical tangent

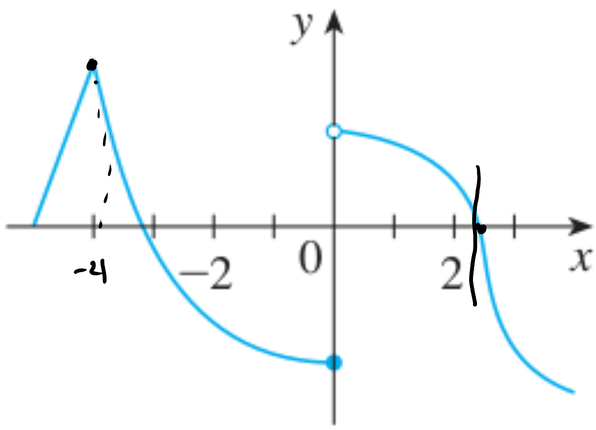
(a) It happens when $\lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h} \neq \lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h}$.

(b) When f has a discontinuity at $x=a$.

(c) It happens when $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \pm\infty$

39–42 The graph of f is given. State, with reasons, the numbers at which f is *not* differentiable.

39.



$x=-4$. A corner!

$x=0$ Discontinuous.

$x=2.5$ Vertical tangent.

Higher Derivatives.

Second derivative:

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$$

$$\underbrace{\frac{d}{dx}}_{\text{derivative of}} \overbrace{\underbrace{\left(\frac{dy}{dx}\right)}_{\text{first derivative}}}^{f'(x)} = \overbrace{\underbrace{\frac{d^2y}{dx^2}}_{\text{second derivative}}}^{f''(x)}$$

→ Leibniz Notation

Other notations:

$f''(x)$ or $f^{(2)}(x)$

EXAMPLE 6 If $f(x) = x^3 - x$, find and interpret $f''(x)$.

$$f'(x) = \frac{d}{dx} f(x) = \frac{d}{dx}(x^3 - x) = \frac{d}{dx}(x^3) - \frac{d}{dx}(x)$$

$$= 3x^2 - 1$$

$$f''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(3x^2 - 1)$$

$$= 3\frac{d}{dx}(x^2) - \frac{d}{dx}(1)$$

$$= 3 \cdot 2 \cdot x - 0 = \boxed{6x}$$

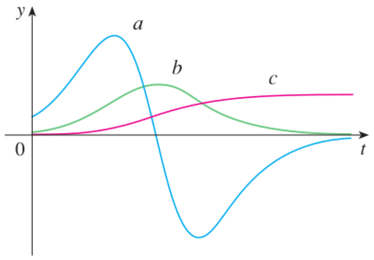
Acceleration:

$$a(t) = \frac{d}{dt} v(t) = \frac{d}{dt}\left(\frac{ds}{dt}\right)$$

where $s(t)$ is the position function.

Example

49. The figure shows the graphs of three functions. One is the position function of a car, one is the velocity of the car, and one is its acceleration. Identify each curve, and explain your choices.



a → suspect for the derivative of b.

b → suspect for the derivative of c

fct → c

1st derivative → b

2nd derivative → a

Third Derivative.

$$\boxed{y''' = f'''(x) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}}$$

Jerk: $j = \frac{da}{dt} = \frac{d^3s}{dt^3}$

n-th Derivative.

$$\boxed{y^{(n)} = f^{(n)}(x) = \frac{d^ny}{dx^n}}$$

EXAMPLE 7 If $f(x) = x^3 - x$, find $f'''(x)$ and $f^{(4)}(x)$.

$f''(x) = 6x$ $\quad f'''(x) = 6$ $\quad f^{(4)}(x) = 0$.