Math 311

Chapter 6

Section 6.2: Linear Combination and Subspaces

Contents

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EXAMPLE 1. The solution to the homogeneous system

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 1 \\ 2 & 3 & 4 & 1 & 2 & -2 \\ 1 & 2 & 4 & 5 & 3 & -1 \\ 3 & 1 & 2 & 4 & 5 & 1 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

is

$$\mathbf{x} = t \begin{bmatrix} 1 \\ 9 \\ -7 \\ 3 \end{bmatrix} + s \begin{bmatrix} 0 \\ 3 \\ -3 \\ 1 \end{bmatrix} = t\mathbf{x_1} + s\mathbf{x_2}, \quad s, t \in \mathbb{R}.$$

Notice that

S1. $t = s = 0 \Rightarrow \vec{x} = \vec{0}$ is a solution.

S2. $\vec{x} = 5\vec{x}_1 + 3\vec{x}_2$ and $\vec{y} = -3\vec{x}_1 + \vec{x}_2$

$\Rightarrow \vec{x} + \vec{y} = 2\vec{x}_1 + 4\vec{x}_2$

$\Rightarrow \vec{x} + \vec{y}$ is still a solution.

S3. $\vec{x} = 5\vec{x}_1 + 3\vec{x}_2$

$\Rightarrow 2\vec{x} = 10\vec{x}_1 + 6\vec{x}_2$ still a solution.

If $U = \{t\mathbf{x_1} + s\mathbf{x_2} \ : \ s, t \in \mathbb{R}\}$ is the set of all solutions, then U is called a **subspace**.

DEFINITION 1. A subset U of a vector space V is called a **subspace** of V if it satisfies the following properties:

$\boxed{\text{S1.}}$ The zero vector $\mathbf{0} \in U$.

$\boxed{\text{S2.}}$ If $\mathbf{u_1} \in U$ and $\mathbf{u_2} \in U$, then $\mathbf{u_1} + \mathbf{u_2} \in U$.

$\boxed{\text{S3.}}$ If $\mathbf{u} \in U$ and a is a scalar, then $a\mathbf{u} \in U$.

Remarks:

① S2: U is said to be **closed under addition**.

② S3: U is said to be **closed under scalar multiplication**.

③ A subspace is a vector [redacted]space itself.

EXAMPLE 2. Let V be a vector space. Show that $U = \{\mathbf{0}\}$ is a subspace of V. This space is called the **zero subspace**.

SOLUTION.

S1. $\vec{0}$ is in $\{\vec{0}\}$.

S2. $\vec{u}_1 = \vec{0}$ and $\vec{u}_2 = \vec{0} \Rightarrow \vec{0} + \vec{0} = \vec{0} \in U$.

S3. $\vec{u} = \vec{0}$ and $a \in \mathbb{R} \Rightarrow a\vec{0} = \vec{0} \in U$.

Here S1–S3 are satisfied $\Rightarrow$ U is a subspace.

Note: Any subspace U of V such that $U \neq \{\mathbf{0}\}$ and $U \neq V$ is called a **proper subspace**.

Important Examples

Example 3. Given an $m \times n$ matrix A, define

$$U = \text{null}A := \{\mathbf{x} \in \mathbb{R}^n \ : \ A\mathbf{x} = \mathbf{0}\}. \quad \vec{0}_{m\times 1}$$

Show that null$A$ is a subspace of $\mathbb{R}^n$.

Solution. null A is a subset of $\mathbb{R}^n$.

S1. $A\vec{0}_{n\times 1} = \vec{0}_{m\times 1} \Rightarrow \vec{0}_{n\times 1} \in$ null A.

S2. Let $\vec{x}, \vec{y} \in$ null A. then

$$A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y} \quad \text{(Matrix mult. prop.)}$$
$$= \vec{0}_{m\times 1} + \vec{0}_{m\times 1}$$
$$= \vec{0}_{m\times 1} \quad \Rightarrow \vec{x}+\vec{y} \in \text{null} A.$$

S3. Let $\vec{x} \in$ null A and $a \in \mathbb{R}$. then

$$A(a\vec{x}) = a(A\vec{x}) = a\,\vec{0}_{m\times 1} = \vec{0}_{m\times 1}.$$

So, $a\vec{x} \in$ null A. <u>Conclusion</u>: null A a subspace.

<u>Note:</u> The subspace null$A$ is called the **null space** of a matrix A. It is the set of all solutions to the homogeneous system $A\mathbf{x} = \mathbf{0}$.

Example 4. Given an $m \times n$ matrix A, define

$$\mathrm{im}A := \{\underbrace{A\mathbf{x}}_{m\times 1} : \mathbf{x} \in \mathbb{R}^n\}.$$

Show that $\mathrm{im}A$ is a subspace of $\mathbb{R}^m$.

Solution. $\mathrm{im}\,A$ is a subset of $\mathbb{R}^m$.

<u>S1</u>. If $\vec{x} = \vec{0} \in \mathbb{R}^n$, then

$$A\vec{x} = A\vec{0} = \vec{0} \in \mathbb{R}^m$$

$$\Rightarrow \quad \vec{0}_{m\times 1} \in \mathrm{im}\,A.$$

<u>S2</u>. Assume $\vec{y}_1, \vec{y}_2 \in \mathrm{im}\,A$.

Goal: $\vec{y}_1 + \vec{y}_2 \in \mathrm{im}\,A$. $(\vec{y}_1 + \vec{y}_2 = A\vec{x})$.

Here, $\vec{y}_1 = A\vec{x}_1$ and $\vec{y}_2 = A\vec{x}_2$

$$\Rightarrow \quad \vec{y}_1 + \vec{y}_2 = A\vec{x}_1 + A\vec{x}_2 = A(\vec{x}_1 + \vec{x}_2)$$

So, if $\vec{x} = \vec{x}_1 + \vec{x}_2 \Rightarrow \vec{y}_1 + \vec{y}_2 = A\vec{x}$. ✓

<u>S3</u>. Assume $\vec{y} \in \mathrm{im}\,A$ and $a \in \mathbb{R}$.

$$a\vec{y} = aA\vec{x} = A(\underbrace{a\vec{x}}_{=\vec{u}}) = A\vec{u} \Rightarrow a\vec{y} \in \mathrm{im}\,A. ✓$$

<u>Note:</u> The subspace $\mathrm{im}A$ is called the **image space** (or **range space**) of the matrix A. It is the set of all vectors $\mathbf{b}$ such that $A\mathbf{x} = \mathbf{b}$ has a solution.

Example 5. For an $n \times n$ matrix A and a number λ, define

$$E_\lambda(A) := \{\mathbf{x} \in \mathbb{R}^n \,:\, A\mathbf{x} = \lambda\mathbf{x}\}.$$

Show that $E_\lambda(A)$ is a subspace of $\mathbb{R}^n$.

Solution.

<u>Note:</u> When λ is an eigenvalue of A, the subspace $\mathrm{E}_\lambda(A)$ is called the **eigenspace** associated to λ.

More Examples

Example 6. Let $\mathbf{M}_{nn}$ be the vector space of $n \times n$ matrices. Show that $U = \{A : A^\top = A\}$ is a subspace of $\mathbf{M}_{nn}$.

Solution.

Note: The set U is the subspace of all symmetric matrices.

Non-Examples

EXAMPLE 7. Show that the set

$$U = \{p : p \in \mathbf{P}_3 \text{ and } p(2) = 1\}$$

is not a subspace of $\mathbf{P}_3$.

SOLUTION.

EXAMPLE 8. The solutions set to the system $A\mathbf{x} = \mathbf{0}$ given in Example 1 is given by the linear combination

$$t\mathbf{x_1} + s\mathbf{x_2}, \quad t, s \in \mathbb{R}.$$

The set $\{t\mathbf{x_1} + s\mathbf{x_2} \,:\, t, s \in \mathbb{R}\}$ is called the **span** of $\mathbf{x_1}$ and $\mathbf{x_2}$.

DEFINITION 2. Let $\{\mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v_n}\}$ be a collection of vectors in a vector space V.

① a **linear combination** of the vectors $\mathbf{v_1}$, $\mathbf{v_2}$, ..., $\mathbf{v_n}$ is an expression of the form

$$a_1\mathbf{v_1} + a_2\mathbf{v_2} + \cdots + a_n\mathbf{v_n}$$

where a_1, a_2, ..., a_n are scalars called the **coefficients** of each vector.

② The set of all linear combinations of these vectors is called their **span**.

③ If it happens that $V = \operatorname{span}\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$, then the vectors are called a **spanning set** for V.

Remarks:

- $\operatorname{span}\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ is a subspace of V.

Example 9. Consider $p_1 = 1 + x + 4x^2$ and $p_2 = 1 + 5x + x^2$, two polynomials in $\mathbf{P}_2$.

a) Is p_1 in the span$\{1 + 2x - x^2, 3 + 5x + 2x^2\}$.

b) Is p_2 in the span$\{1 + 2x - x^2, 3 + 5x + 2x^2\}$.

Solution.