Chapter 2
Derivatives

2.5 Chain Rule

How do you differentiate the function $F(x) = \sqrt{x^2+1}$?

$$(x^2+1)^{1/2} \rightarrow \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x = \frac{2x}{2\sqrt{x^2+1}}$$

$$F'(x) = \lim_{h\to 0} \frac{\sqrt{(x+h)^2+1} - \sqrt{x^2+1}}{h} = \lim_{h\to 0} \frac{2xh+h^2}{h\left(\sqrt{(x+h)^2+1}+\sqrt{x^2+1}\right)}$$

$$= \lim_{h\to 0} \frac{2x+h}{\sqrt{(x+h)^2+1}+\sqrt{x^2+1}}$$

$$\frac{d}{dx}(\sqrt{x}) \leftarrow = \frac{2x}{2\sqrt{x^2+1}} \rightarrow \frac{d}{dx}(x^2+1)$$

The Chain Rule If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Example. Redo the first example with the Chain Rule.

$f(x) = \sqrt{x}$

$g(x) = x^2+1$

$F'(x) = f'(g(x)) \cdot g'(x)$

1 $f'(x) = \frac{1}{2}x^{1/2-1} = \frac{1}{2}x^{-1/2}$

2 $g'(x) = 2x$

3 $F'(x) = f'(x^2+1) \cdot g'(x)$

$$= \frac{1}{\cancel{2}}(x^2+1)^{-1/2} \cdot \cancel{2}x = \boxed{\frac{x}{(x^2+1)^{1/2}}}$$

$$x^{-1/2} = \frac{1}{x^{1/2}} \qquad \frac{x^3}{x^1} = x^3 \cdot x^{-1} = x^{3-1} = x^2$$

Main idea:

$$\frac{d}{dx} \underbrace{f}_{\text{outer function}} \underbrace{(g(x))}_{\text{evaluated at inner function}} = \underbrace{f'}_{\text{derivative of outer function}} \underbrace{(g(x))}_{\text{evaluated at inner function}} \cdot \underbrace{g'(x)}_{\text{derivative of inner function}}$$

EXAMPLE 2 Differentiate (a) $y = \sin(x^2)$ and (b) $y = \sin^2 x$. $= (\sin x)^2$

(a) $\frac{dy}{dx} = \cos(x^2) \cdot 2x$

(b) $\frac{dy}{dx} = 2(\sin x) \cdot \cos x$

EXAMPLE 4 Find $f'(x)$ if $f(x) = \frac{1}{\sqrt[3]{x^2 + x + 1}}$. $= (x^2+x+1)^{-1/3}$

$f'(x) = -\frac{1}{3}(x^2+x+1)^{-1/3-1} \cdot (2x+1)$

$= \boxed{-\frac{1}{3}(x^2+x+1)^{-4/3}(2x+1)}$

EXAMPLE 6 Differentiate $y = \underbrace{(2x+1)^5}\underbrace{(x^3 - x + 1)^4}$.

$$\frac{dy}{dx} = \frac{d}{dx}\left((2x+1)^5\right)(x^3-x+1)^4 + (2x+1)^5 \frac{d}{dx}\left((x^3-x+1)^4\right)$$

$$= \left[5(2x+1)^4 \cdot 2\right](x^3-x+1)^4 + (2x+1)^5\, 4(x^3-x+1)^3 \cdot (3x^2-1)$$

$$= 10(2x+1)^4(x^3-x+1)^4 + 4(2x+1)^5(x^3-x+1)^3(3x^2-1).$$