SECTION 2.2: Limits and Continuity

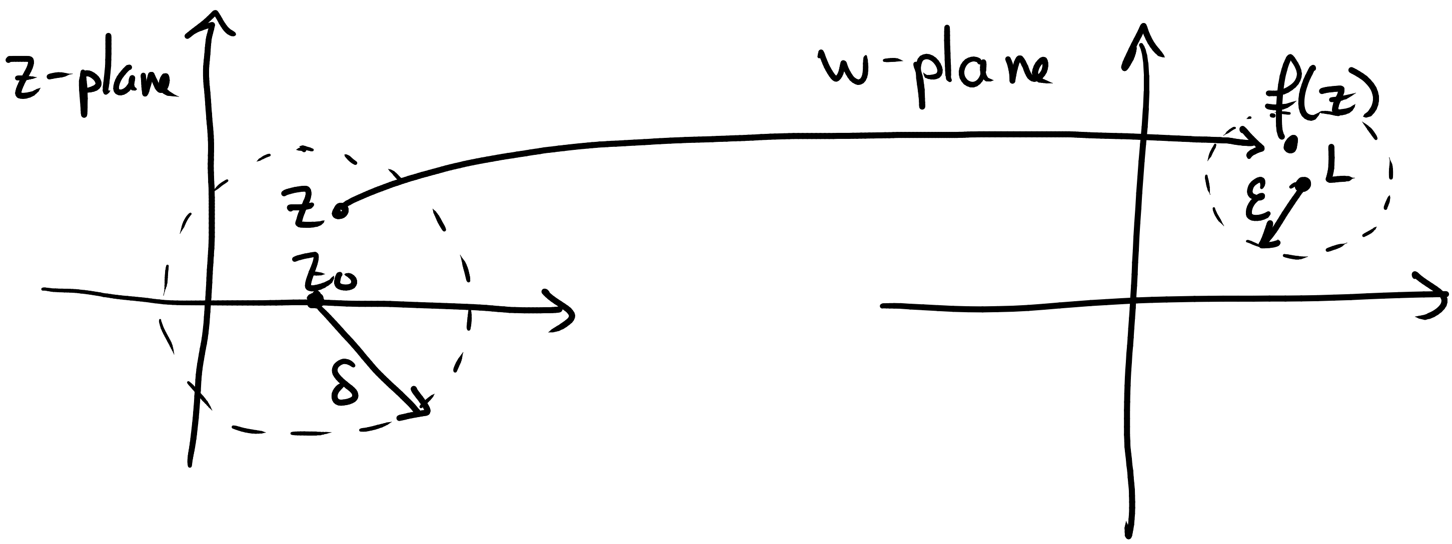
Limits

Def. Let $f: U \to \mathbb{C}$ where U is an open set.

We say L is the limit of f at $z_0 \in U$ if as z approaches z_0, $f(z)$ approaches L, that is

$\forall \varepsilon > 0, \exists \delta > 0$ s.t.

$$0 < |z - z_0| < \delta \Rightarrow |f(z) - L| < \varepsilon.$$



Prop. 2.2.2 (Uniqueness of limits)

If $f : U \to \mathbb{C}$ is a function defined on an open set U, and if f has limit L at $z_0 \in U$, then L is unique.

Proof. Assume that there are two limits L_1, L_2 at z_0 with $L_1 \neq L_2$. Then

$$|L_1 - L_2| \neq 0$$

Let $\varepsilon = \dfrac{|L_1 - L_2|}{4} > 0$. By def of limits, $\exists \delta_1 > 0$ and $\exists \delta_2 > 0$ such that

$$0 < |z - z_0| < \delta_1 \Rightarrow |f(z) - L_1| < \frac{|L_1 - L_2|}{4}$$

and

$$0 < |z - z_0| < \delta_2 \Rightarrow |f(z) - L_2| < \frac{|L_1 - L_2|}{4}$$

So, if $|z - z_0| < \min\{\delta_1, \delta_2\}$, then

$$\begin{aligned} |L_1 - L_2| &= |L_1 - f(z) + f(z) - L_2| \\ &\leq |L_1 - f(z)| + |f(z) - L_2| \\ &< \frac{|L_1 - L_2|}{4} + \frac{|L_1 - L_2|}{4} \\ &= \frac{|L_1 - L_2|}{2} \end{aligned}$$

$$\Rightarrow |L_1 - L_2| < \frac{|L_1 - L_2|}{2}$$

contradiction!

So, $L_1 = L_2$. $\square$

Notation:

$$L = \lim_{z \to z_0} f(z) \quad \text{or} \quad f(z) \to L \ (z \to z_0).$$

Thm. 2.2.9 Let $U \subset \mathbb{C}$ be an open set. Let $f: U \to \mathbb{C}$ be a function with

$$f(z) = u(z) + i v(z), \quad z \in U.$$

Then

$$\lim_{z \to z_0} f(z) = a + ib \iff \begin{cases} \lim_{z \to z_0} u(z) = a \\ \lim_{z \to z_0} v(z) = b \end{cases}$$

Proof.

($\Rightarrow$) Assume $\lim_{z \to z_0} f(z) = L = a + ib$.

WTS: $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$0 < |z - z_0| < \delta \Rightarrow |u(z) - a| < \varepsilon.$$

and

$\forall \varepsilon > 0$, $\exists \delta > 0$, such that

$$0 < |z - z_0| < \delta \Rightarrow |v(z) - b| < \varepsilon.$$

Let $\varepsilon > 0$. By def. of $\lim_{z \to z_0} f(z) = L$,

$\exists \delta > 0$ s.t.

$0 < |z - z_0| < \delta \Rightarrow |f(z) - L| < \varepsilon$. $(*)$

<u>Recall</u>:

$$|\operatorname{Re} w| \leq |w|.$$

Let $z \in U$ such that $0 < |z - z_0| < \delta$.

then

$$|u(z) - a| = |\operatorname{Re}(f(z) - \overbrace{(a+ib)}^{=L})|$$
$$\leq |f(z) - L|$$
$$< \varepsilon.$$

<u>Summary</u>: we found a $\delta > 0$ s.t.

$0 < |z - z_0| < \delta \Rightarrow |u(z) - a| < \varepsilon$.

Repeat same argument for $v(z)$.

($\Leftarrow$) Assume $\lim_{z \to z_0} u(z) = a$ and $\lim_{z \to z_0} v(z) = b$.

<u>WST</u> $\lim_{z \to z_0} f(z) = a + ib$.

i.e. $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$0 < |z - z_0| < \delta \implies |f(z) - (a + ib)| < \varepsilon$.

Let $\varepsilon > 0$. From the definition of limits, $\exists \delta_1 > 0$, $\exists \delta_2 > 0$ such that

$0 < |z - z_0| < \delta_1 \implies |u(z) - a| < \varepsilon/2$ (Δ)

& $0 < |z - z_0| < \delta_2 \implies |v(z) - b| < \varepsilon/2$. $(\circ)$

<u>Recall</u>:

$$|w| \leq |\operatorname{Re} w| + |\operatorname{Im} w|$$

Let $\delta := \min\{\delta_1, \delta_2\}$.

If $|z-z_0|<\delta$, then

$$|f(z)-(a+ib)| \le |u(z)-a|+|v(z)-b|$$
$$< \varepsilon/2 + \varepsilon/2$$
$$= \varepsilon$$

<u>Summary</u>: $\forall \varepsilon>0$, $\exists \delta>0$ s.t.

$$0<|z-z_0|<\delta \Rightarrow |f(z)-(a+ib)|<\varepsilon.$$

So, $\displaystyle\lim_{z\to z_0} f(z) = a+ib.$ $\square$

<u>Example</u> Compute

(a) $\displaystyle\lim_{z\to z_0} z$ (b) $\displaystyle\lim_{z\to z_0} z^2$.

<u>Solution</u>.

(a) $z\to z_0 \Leftrightarrow x\to x_0$ and $y\to y_0$.

(b) $z^2 = (x+iy)(x+iy)$

$$= \underbrace{x^2-y^2}_{u(z)} + \underbrace{(2xy)}_{v(z)} i$$

$$\lim_{z\to z_0} (x^2-y^2) = \lim_{(x,y)\to(x_0,y_0)} x^2-y^2$$

$$= x_0^2 - y_0^2 \quad \text{((Calculus)}.$$

$$\lim_{z\to z_0} 2xy = \lim_{(x,y)\to(x_0,y_0)} 2xy$$

$$= 2x_0y_0$$

Hence:

$$\lim_{z\to z_0} z^2 = \overbrace{(x_0^2-y_0^2)}^{a} + \overbrace{(2x_0y_0)}^{b} i$$

$$= z_0^2$$

In fact:

$$\lim_{z\to z_0} z^n = z_0^n \,.$$

Properties of limits

(1) $$\lim_{z \to z_0} \left(f(z) + g(z)\right) = \lim_{z \to z_0} f(z) + \lim_{z \to z_0} g(z)$$

(2) $$\lim_{z \to z_0} f(z)g(z) = \left(\lim_{z \to z_0} f(z)\right)\left(\lim_{z \to z_0} g(z)\right)$$

(3) $$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{\lim_{z \to z_0} f(z)}{\lim_{z \to z_0} g(z)}$$

if $\lim_{z \to z_0} g(z) \neq 0$.

(4) $$\lim_{z \to z_0} |z| = |z_0|$$

(5) $$\lim_{z \to z_0} \overline{z} = \overline{\lim_{z \to z_0} z}.$$

THM (Squeeze Theorem; Thm 2.2.5)

Let f, g be defined on an open set $U \subset \mathbb{C}$ and let $z_0 \in U$.

a) $\lim_{z \to z_0} f(z) = 0$ and $|g(z)| \leq |f(z)|$ in a deleted neighborhood of z_0, then $\lim_{z \to z_0} g(z) = 0$

b) $\lim_{z \to z_0} f(z) = 0$ and g is bounded in a deleted neighborhood of z_0, then $\lim_{z \to z_0} f(z) g(z) = 0$.

Remark: g is bounded in a deleted neighborhood of z_0 means $\exists r > 0$ and $\exists M > 0$ s.t.

$$|g(z)| \leq M \quad , \forall z \in B_r'(z_0)$$

Proof.

(a) Assume $\lim_{z \to z_0} f(z) = 0$ and

$|g(z)| \le |f(z)|$, for all $z \in B'_r(z_0)$.

Let $\varepsilon > 0$. From the assumption,

$\exists \delta_1 > 0$ s.t.

$$0 < |z - z_0| < \delta_1 \Rightarrow |f(z)| < \varepsilon .$$

Let $\delta = \min\{\delta_1, r\}$. If $0 < |z - z_0| < \delta$,

then

$$|g(z)| \le |f(z)| < \varepsilon .$$

So, $\lim_{z \to z_0} g(z) = 0$.

(b) From assumption, $\exists r > 0$, $\exists M > 0$

s.t. $\quad |g(z)| \le M \quad , \forall z \in B'_r(z_0)$

Now, for $z \in B'_r(z_0)$

$$|f(z)g(z)| \le |f(z)||g(z)|$$

$$\le M|f(z)|$$

for any $z \in B_r'(z_0)$.

Also,

$$\lim_{z\to z_0} Mf(z) = M \lim_{z\to z_0} f(z)$$

$$= M \cdot 0 = 0.$$

From part (a) applied to $f(z)g(z)$:

$$\lim_{z\to z_0} f(z)g(z) = 0. \qquad \square$$

Limits at infinity

We assume f is at least defined in a neighborhood of $z_0 \in \mathbb{C}$

or on $\{ z \in \mathbb{C} : |z| > R \}$.

① $\lim_{z \to z_0} f(z) = \infty$ if $\forall M > 0, \exists \delta > 0$

$0 < |z - z_0| < \delta \Rightarrow |f(z)| > M$.

② $\lim_{z \to \infty} f(z) = L$ if $\forall \varepsilon > 0, \exists R > 0$

$|z| > R \Rightarrow |f(z) - L| < \varepsilon$.

③ $\lim_{z \to \infty} f(z) = \infty$ if $\forall M > 0, \exists R > 0$

$|z| > R \Rightarrow |f(z)| > M$.

<u>Remark:</u>

$$\lim_{z \to \infty} f(z) = L \Leftrightarrow \lim_{z \to 0} f(1/z) = L.$$

Using this trick,

$$\lim_{z\to\infty} \frac{1}{z^n} = \lim_{z\to 0} \frac{1}{(1/z)^n} = \lim_{z\to 0} z^n = 0 .$$

Continuous Functions

<u>DEF</u> (modif. of Def. 2.2.12)

A function f is continuous at z_0 if

(a) f is defined in a neighborhood of z_0.

(b) $\lim_{z\to z_0} f(z) = f(z_0)$.

A function f defined on an open set U is continuous on U if it is continuous $\forall z_0 \in U$.

Consequences:

① f, g continuous at z_0, then $af + bg$ is continuous at z_0 fn any $a, b \in \mathbb{C}$ and fg is continuous at z_0.

② f, g are continuous at z_0, then $\frac{f}{g}$ is continuous at z_0 provided that $g(z_0) \neq 0$.

③ Any polynomial $p(z) = a_n z^n + \dots + a_0$ is continuous on $\mathbb{C}$.

④ Any rational function

$$f(z) = \frac{p(z)}{q(z)} = \frac{a_n z^n + \dots + a_0}{b_m z^m + \dots + b_0}$$

is continuous on $\mathbb{C} \setminus \{z : q(z) = 0\}$.

(5) The function $f(z) = \bar{z}$ is continuous on $\mathbb{C}$.

<u>Thm</u> Let f be continuous at z_0 and h be continuous at $f(z_0)$.

(a) $h \circ f$ is continuous at z_0.

(b) $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are continuous at z_0

(c) The function $g(z) = |f(z)|$ is continuous at z_0 .