Math 302

Chapter 5

Section 5.2: Constant Coefficient Homogeneous Equations

Contents

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What is a Constant Coefficient Homogeneous ODE?

We restrict even further the second order ODE. A **second order constant coefficient ODE** is an ODE of the form

$$ay'' + by' + cy = f(x) \tag{1}$$

where a, b, c are fixed numbers and f is a continuous function.

<u>Goal:</u>

Find the solutions to

$$ay'' + by' + cy = 0.$$

We call this the **constant coefficient homogeneous ODE**.

<u>Trick</u>:

Distinct Real Roots: $\sqrt{b^2 - 4ac} > 0$

Example 1.

a) Find the general solution of

$$y'' + 6y' + 5y = 0.$$

b) Solve the following IVP:

$$y'' + 6y' + 5y = 0,$$

General Fact:

- If the roots of the characteristic polynomial are r_1 and r_2, then $y_1(x) = e^{r_1 x}$ and $y_2 = e^{r_2 x}$ are solutions to the ODE.
- The general solutions is given by

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

Example 2.

a) Find the general solution of

$$y'' + 6y' + 9y = 0.$$

b) Solve the following IVP:

$$y'' + 6y' + 9y = 0, \quad y(0) = 3,\ y'(0) = -1.$$

General Facts:

- If the root of the characteristic polynomial is r_1, then $y_1(x) = e^{r_1 x}$ and $y_2(x) = xe^{r_1 x}$ are solutions to the ODE.
- The general solution is given by

$$y(x) = e^{r_1 x}(c_1 + c_2 x).$$

Example 3.

a) Find the general solution of

$$y'' + 4y' + 13y = 0.$$

b) Solve the following IVP:

$$y'' + 4y + 13y = 0, \quad y(0) = 2,\, y'(0) = -3.$$

Complex Numbers

A complex number is an expression of the form

$$z = \alpha + i\beta$$

where α, β are real numbers and $i^2 = -1$ ($i = \sqrt{-1}$).

Consider $z = \alpha + i\beta$ and $w = \gamma + i\mu$.

- $z = w$ if and only if $\alpha = \gamma$ and $\beta = \mu$.
- $z + w = (\alpha + \gamma) + i(\beta + \mu)$.
- $zw = (\alpha\gamma - \beta\mu) + i(\alpha\mu + \beta\gamma)$.
- $z/w = \frac{\alpha\gamma+\beta\mu}{\gamma^2+\mu^2} + i\left(\frac{\alpha\gamma-\beta\mu}{\gamma^2+\mu^2}\right)$, if $w \neq 0$.

Example 4. If $z = 1 + i$ and $w = 1 - i$, find

a) $z + w$. b) zw. c) z/w.

Example 5. Complete the previous example.

General Facts:

- If $r_1 = \alpha + \beta i$ and $r_2 = \alpha - \beta i$ are the roots of the characteristic polynomial, then $y_1(x) = e^{\alpha x}\cos(\beta x)$ and $y_2(x) = e^{\alpha x}\sin(\beta x)$ are solutions to the ODE.
- The general solution has the form

$$y(x) = e^{\alpha x}(c_1\cos(\beta x) + c_2\sin(\beta x)).$$