

Section 16.6, Problem 40

We have

$$\begin{aligned}\vec{r}_u &= \langle 1, -3, 1 \rangle \\ \vec{r}_v &= \langle 1, 0, -1 \rangle.\end{aligned}$$

So, $\vec{r}_u \times \vec{r}_v = \langle 3, 2, -3 \rangle$. Thus,

$$\iint_S dS = \iint_D |\vec{r}_u \times \vec{r}_v| dA = \int_{-1}^1 \int_0^2 \sqrt{21} du dv = 4\sqrt{21}.$$

Section 16.6, Problem 44

The domain D of the parameters x, y are inside the triangle. More precisely, we have

$$D = \{(x, y) : 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1 - x\}.$$

By the formula viewed in the lecture notes for surfaces obtained from a graph, we can use the following formula

$$A(S) = \iint_D \sqrt{1 + z_x^2 + z_y^2} dA.$$

We have $z_x = -4x$ and $z_y = 1$. Thus, we obtain

$$A(S) = \int_0^1 \int_0^{1-x} \sqrt{2 + 16x^2} dy dx = \sqrt{2} \int_0^1 (1-x) \sqrt{1 + 8x^2} dx \approx 0.72828$$

Section 16.7, Problem 10

Intersection the plane with the xy -plane, we get our domain:

$$D = \{(x, y) : 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1 - x\}.$$

Now, we have $z = 4 - 2x - 2y$, so that

$$dS = \sqrt{1 + z_x^2 + z_y^2} dA = \sqrt{1 + 4 + 4} dA = 3 dA.$$

Thus we get

$$\iint_S xz dS = \int_0^1 \int_0^{1-x} 3x(4 - 2x - 2y) dy dx = 3 \int_0^1 x(1 - x) - 2x^2(1 - x) - x(1 - x)^2 dx = -1/12.$$

Section 16.7, Problem 16

In spherical coordinates, a cone is $\phi = \pi/4$. Thus, we have

$$S = \{(\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi) : 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi/4\}.$$

We have $dS = |\vec{r}_\theta \times \vec{r}_\phi| dA$. So

$$\begin{aligned}\vec{r}_\theta &= \langle -\sin \theta \sin \phi, \cos \theta \sin \phi, 0 \rangle \\ \vec{r}_\phi &= \langle \cos \theta \cos \phi, \sin \theta \cos \phi, -\sin \phi \rangle.\end{aligned}$$

So, with these tangent vectors, we see that

$$|\vec{r}_\theta \times \vec{r}_\phi| = \sin \phi.$$

Thus, $dS = \sin \phi dA$ and

$$\begin{aligned}\iint_S y^2 dS &= \int_0^{\pi/4} \int_0^{2\pi} \sin^2 \theta \sin^2 \phi \sin \phi d\phi d\theta \\ &= \int_0^{\pi/4} \sin^2 \theta d\theta (4/3) \\ &= (1/8)(\pi - 2)(4/3) \\ &= \frac{\pi - 2}{6}.\end{aligned}$$

Section 16.7, Problem 44

A parametrization of the hemisphere is

$$\vec{r}(\phi, \theta) = \langle 3 \cos \theta \sin \phi, 3 \sin \theta \sin \phi, 3 \cos \phi \rangle$$

where $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi/2$. The cross product of the tangent vectors is

$$\vec{r}_\phi \times \vec{r}_\theta = \langle 9 \sin^2 \phi \cos \theta, 9 \sin^2 \phi \sin \theta, 9 \sin \phi \cos \phi \rangle = 9 \sin \phi \langle \cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi \rangle.$$

The vector $\vec{r}_\phi \times \vec{r}_\theta$ is pointing outward from the sphere (because it's pointing in the same direction as the position vector). Thus we get

$$\begin{aligned} \rho \vec{v} \cdot (\vec{r}_\phi \times \vec{r}_\theta) &= 9\rho \sin \phi \langle 3 \sin \theta \sin \phi, 3 \cos \theta \sin \phi, 0 \rangle \cdot \langle \cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi \rangle \\ &= 9\rho \sin \phi (3 \sin \theta \cos \theta \sin^2 \phi + 3 \sin \theta \cos \theta \sin^2 \phi) \\ &= 54\rho \sin \theta \cos \theta \sin^3 \phi. \end{aligned}$$

Thus, we have

$$\begin{aligned} \iint_S \rho \vec{v} \cdot d\vec{S} &= \iint_D \rho \vec{v} \cdot (\vec{r}_\phi \times \vec{r}_\theta) dA \\ &= 54\rho \int_0^{2\pi} \int_0^{\pi/2} \sin \theta \cos \theta \sin^3 \phi d\phi d\theta \\ &= 54\rho \left(\int_0^{2\pi} \sin \theta \cos \theta d\theta \right) \left(\int_0^{\pi/2} \sin^3 \phi d\phi \right) \\ &= 0. \end{aligned}$$

Thus, the net flow is 0 kg/s.