Chapter 3
Applications of Derivatives

3.4 Limits at Infinity; Horizontal Asymptotes

Example. What is the limit of $f(x) = \dfrac{x^2-1}{x^2+1}$ when x becomes large?

x	$f(x)$
10	≈ 0.99
100	≈ 0.9998
1000	≈ 0.999998
10 000	≈ 0.99999998

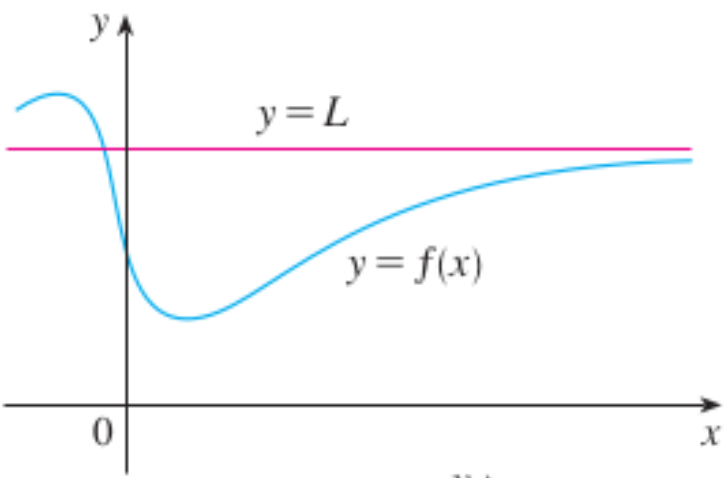
x	$f(x)$
100 000	≈ 0.9999999998
$\vdots$	$\vdots$

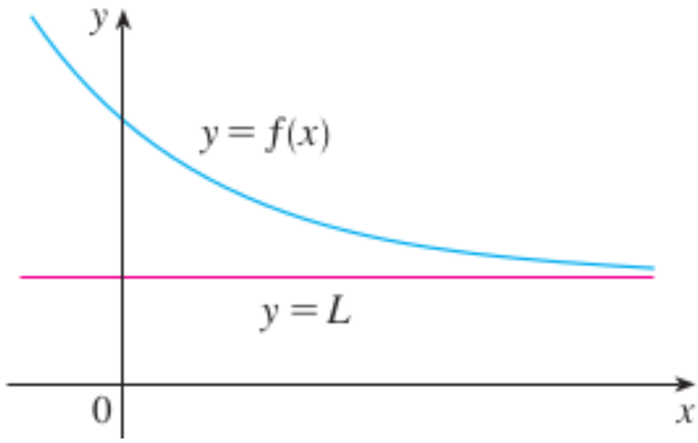
$$\frac{x^2-1}{x^2+1} \longrightarrow 1 \Rightarrow \lim_{x\to+\infty} \frac{x^2-1}{x^2+1} = 1 .$$

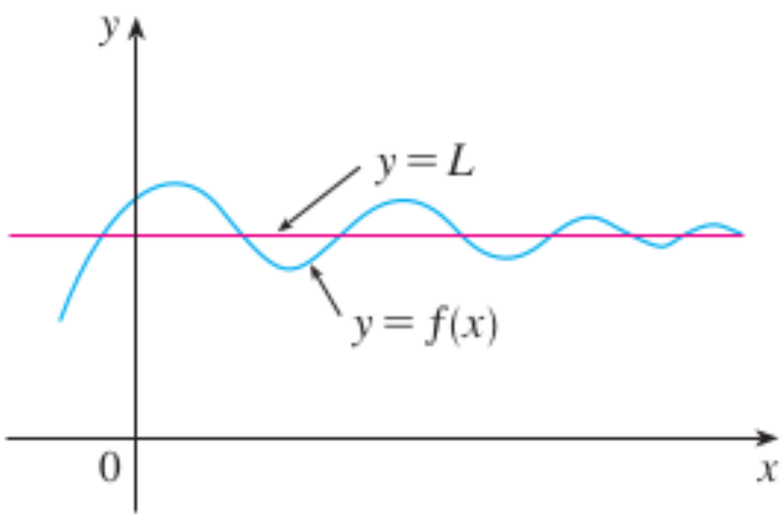
1 Intuitive Definition of a Limit at Infinity Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x\to\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large.







Example. What is the limit of $f(x) = \dfrac{x^2-1}{x^2+1}$ when x becomes large?

x	$f(x)$
-10	0.98
$\vdots$	
-10000	0.99999998
$\vdots$	$\downarrow$
$\downarrow$	
$-\infty$	1

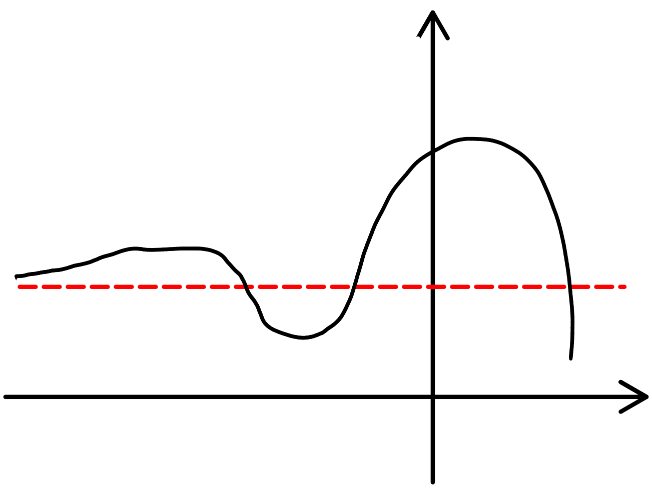
x	$f(x)$

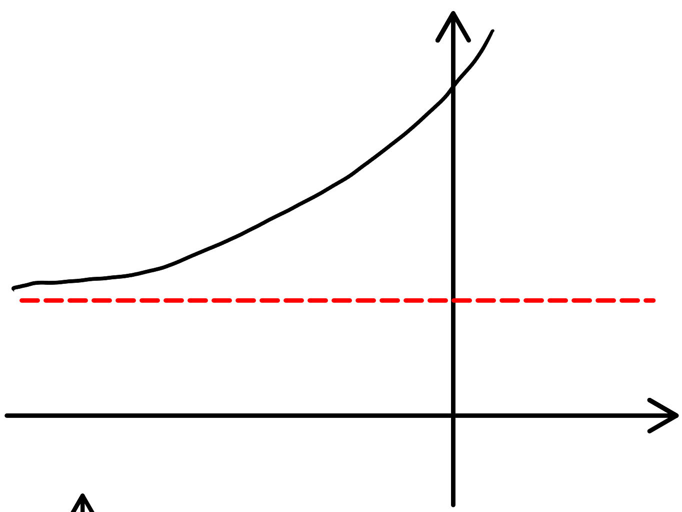
$$\Rightarrow \quad \lim_{x\to-\infty} \frac{x^2-1}{x^2+1} = 1$$

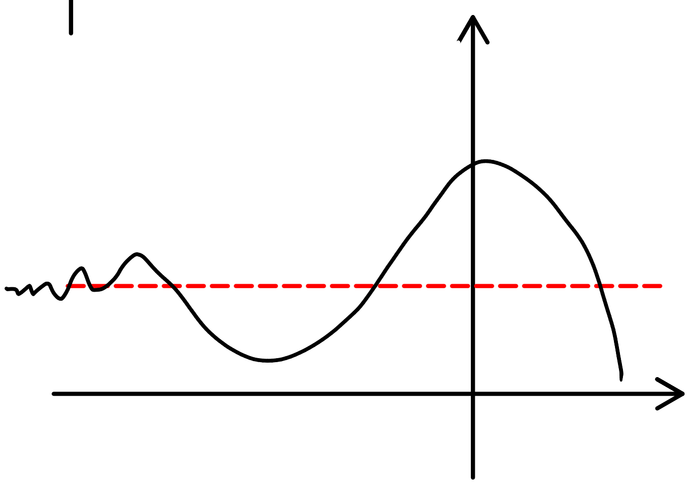
2 Definition Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x\to-\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large negative.







3 Definition The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = L$$

EXAMPLE 1 Find the infinite limits, limits at infinity, and asymptotes for the function f whose graph is shown in Figure 5.

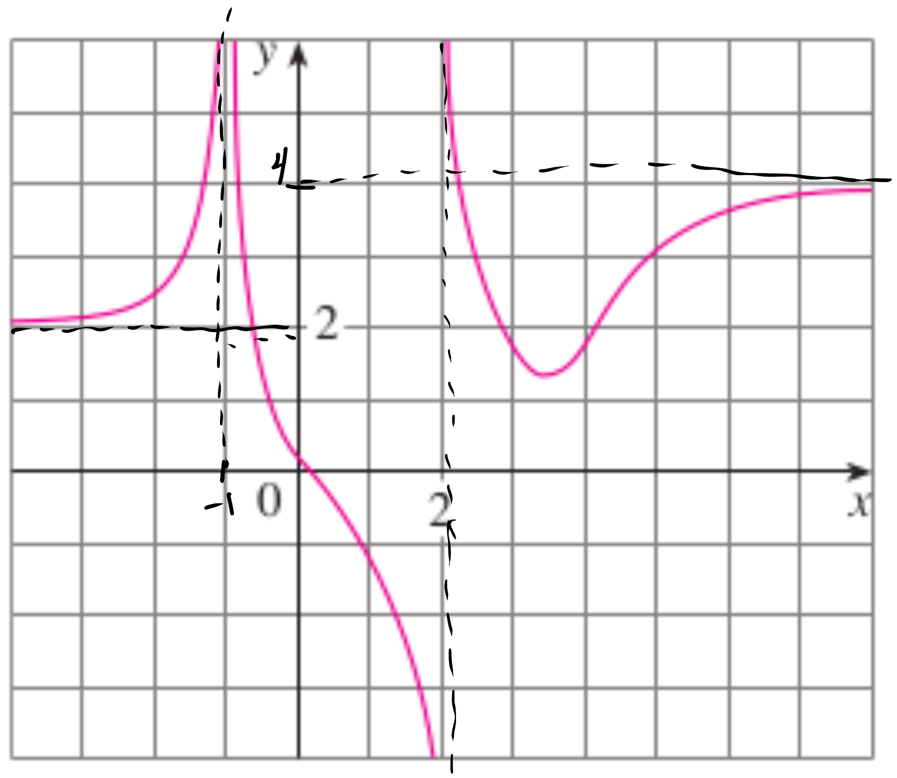


FIGURE 5

A) Infinite limits

$\lim_{x \to -1} f(x) = +\infty$

$\lim_{x \to 2^-} f(x) = -\infty$

$\lim_{x \to 2^+} f(x) = +\infty$.

$x = -1$ is a VA.

$x = 2$ is a VA.

limits At infinity

$\lim_{x \to +\infty} f(x) = 4 \quad \Rightarrow \quad y = 4$ is a HA.

$\lim_{x \to -\infty} f(x) = 2 \quad \Rightarrow \quad y = 2$ is a HA.

Rules for Limits at infinity.

4 Theorem If $r > 0$ is a rational number, then

$$\lim_{x \to \infty} \frac{1}{x^r} = 0 \qquad \frac{1}{\sqrt{x}}$$

If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x \to -\infty} \frac{1}{x^r} = 0$$

EXAMPLE 3 Evaluate

$$\lim_{x \to \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$$

$x = \frac{x^2}{x}$

$-2 = \frac{-2x^2}{x^2}$

$$\frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \frac{\cancel{x^2}\left(3 - \frac{1}{x} - \frac{2}{x^2}\right)}{\cancel{x^2}\left(5 + \frac{4}{x} + \frac{1}{x^2}\right)}$$

$$= \frac{3 - \frac{1}{x} - \frac{2}{x^2}}{5 + \frac{4}{x} + \frac{1}{x^2}}$$

$$\lim_{x \to \infty}\left(3 - \frac{1}{x} - \frac{2}{x^2}\right) = \lim_{x \to \infty} 3 - \lim_{x \to \infty} \frac{1}{x} - 2 \lim_{x \to \infty} \frac{1}{x^2}$$

$$= 3 - 0 - 2 \cdot 0$$

$$= 3$$

$$\lim_{x \to \infty}\left(5 + \frac{4}{x} + \frac{1}{x^2}\right) = \lim_{x \to \infty} 5 + 4 \lim_{x \to \infty} \frac{1}{x} + \lim_{x \to \infty} \frac{1}{x^2}$$

$$= 5 + 4 \cdot 0 + 0 = 5$$

By quotient rule:

$$\lim_{x \to \infty} \frac{3 - \frac{1}{x} - \frac{2}{x^2}}{5 + \frac{4}{x} + \frac{1}{x^2}} = \boxed{\frac{3}{5}}$$

EXAMPLE 4 Find the horizontal and vertical asymptotes of the graph of the function

$$f(x) = \frac{\sqrt{2x^2+1}}{3x-5}$$

$$f(x) = \frac{\sqrt{x^2(2+1/x^2)}}{3x-5}$$

$$= \frac{\sqrt{x^2}\,(2+1/x^2)^{1/2}}{3x-5}$$

$$x^2 = 4$$

$$\sqrt{x^2} = \sqrt{4}$$

$$\pm x = 2$$

$$\sqrt{x^2} = \pm x$$

$+ \to x > 0.$

$- \to x < 0.$

HA:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\sqrt{x^2}\,(2+1/x^2)^{1/2}}{3x-5}$$

$$= \lim_{x\to\infty} \frac{\cancel{x}\,(2+1/x^2)^{1/2}}{\cancel{x}(3-5/x)}$$

$$= \lim_{x\to\infty} \frac{(2+1/x^2)^{1/2}}{3-5/x} = \frac{(2+0)^{1/2}}{3-0}$$

$$= \frac{\sqrt{2}}{3}.$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{\sqrt{x^2}\,(2+1/x^2)^{1/2}}{3x-5}$$

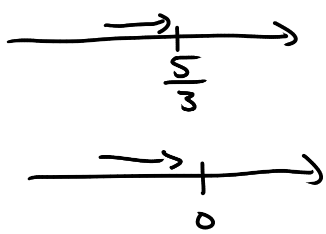
$$= \lim_{x\to-\infty} \frac{-\cancel{x}(2+1/x^2)^{1/2}}{\cancel{x}(3-5/x)} = \lim_{x\to-\infty} \frac{-(2+1/x^2)^{1/2}}{3-5/x}$$

$$= \frac{-(2+0)^{1/2}}{3-0} = -\frac{\sqrt{2}}{3}$$

the HA are $y = \frac{\sqrt{2}}{3}$ & $y = -\frac{\sqrt{2}}{3}$.

② VA Problem when $3x-5=0 \iff x = \frac{5}{3}$

$$\lim_{x\to\frac{5}{3}^-} \frac{\sqrt{2x^2+1}}{3x-5} = \frac{\sqrt{\frac{50}{9}+1}}{0^-} = -\infty$$



V.A. at $x = \frac{5}{3}$.

EXAMPLE 5 Compute $\lim_{x\to\infty}(\sqrt{x^2+1}-x)$.

$\cancel{\infty - \infty = 0}$ $\cancel{\frac{0}{0} = 0}$

$$\lim_{x\to\infty}(\sqrt{x^2+1}-x) \;\cancel{=}\; \infty - \infty$$

WRONG MATH

Simplify.

$$(\sqrt{x^2+1}-x)\cdot\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x} = \frac{x^2+1-x^2}{\sqrt{x^2+1}+x}$$

$$= \frac{1}{\sqrt{x^2+1}+x}$$

$$= \frac{1}{\sqrt{x^2}\left(\sqrt{1+1/x^2}\right)+x}$$

So,

$$\lim_{x\to\infty}(\sqrt{x^2+1}-x) = \lim_{x\to\infty}\frac{1}{x\left(\sqrt{1+1/x^2}\right)+x}$$

$x+xy$

$=x(1+y)$

$$= \lim_{x\to\infty}\frac{1}{x\left(\sqrt{1+1/x^2}+1\right)}$$

$$= \lim_{x\to\infty}\frac{1}{x}\cdot\frac{1}{\sqrt{1+1/x^2}+1}$$

$$= \lim_{x\to\infty}\frac{1}{x}\cdot\lim_{x\to\infty}\frac{1}{\sqrt{1+1/x^2}+1}$$

$$= 0\cdot\frac{1}{\sqrt{1+0}+1}$$

$$= 0$$

Infinite Limits at Infinity.

The notation

$$\lim_{x\to\infty} f(x) = \infty$$

means that the values of $f(x)$ become larger and larger as the values of x becomes larger and larger. Similar meanings are attached to the following symbols:

$$\lim_{x\to-\infty} f(x) = \infty, \qquad \lim_{x\to\infty} f(x) = -\infty \quad \text{and} \quad \lim_{x\to-\infty} f(x) = -\infty.$$

WARNING!!

$\infty - \infty$ is undefined !!

EXAMPLE 8 Find $\lim\limits_{x\to\infty} x^3$ and $\lim\limits_{x\to-\infty} x^3$.

$$\lim_{x\to\infty} x^3 = +\infty$$

$x = 10 \to x^3 = 1000$

$x = 100 \to x^3 = 1\,000\,000$

$$\lim_{x\to-\infty} x^3 = -\infty$$

$x = -10 \to x^3 = -1000$

$x = -100 \to x^3 = -1000\,000$

$$\lim_{x\to-\infty} x^2 = +\infty$$

EXAMPLE 9 Find $\lim_{x\to\infty} (x^2 - x)$. $\longrightarrow$

$$\lim_{x\to\infty} (x^2 - x) = \lim_{x\to\infty} x(x-1)$$

$$= \lim_{x\to\infty} x \lim_{x\to\infty} (x-1)$$

$$= \infty \cdot \infty = \infty$$

Rules with infinite limits at infinity.

$$\lim_{x\to\infty} f(x) = \infty \quad \& \quad \lim_{x\to\infty} g(x) = \infty$$

- $\lim_{x\to\infty} f(x) \cdot g(x) = \infty$
- $\lim_{x\to\infty} (f(x) + c) = \infty$
- $\lim_{x\to\infty} (f(x) - c) = \infty$