Geometry lecture 2

Lines in A^2

- we denote E^2 as A^2

- a line in A^2 is a set of pts. S s.t. the set of vectors which can be represented by pts. in S form a 1D vector subspace of V^2

- the map $\phi^2_Q : A^2 \to V^2$ identifies points with vectors when a point $Q \in A^2$ is fixed

- for $S \subseteq A$, S is a line $\Leftrightarrow$ for a point $Q \in S$,

$$\phi^2_Q(S) = \{\overrightarrow{QP} : B \in S\} \text{ is a 1D vector subspace of } V^2$$

- holds $\forall Q \in S$
- if S is a line, we call $\phi^2_Q(S)$ the <u>direction space</u> of the line S and denote it $D(S)$

Parametric equations

- S a line
- $\forall P, Q \in S$, $P \neq Q$, $\overrightarrow{QP}$ is the <u>direction vector</u> of S
- $D(S)$ is 1D $\Rightarrow$ all direction vectors are l.d.
- v a d.v. for $S \Leftrightarrow v$ l.d. on $\overrightarrow{QP} \Rightarrow \forall v$ a d.v. for S,

$$\exists!\, t \in \mathbb{R} \text{ s.t. } \overrightarrow{QP} = t \cdot v$$

- if P, Q are fixed, t will vary
- $\phi_Q^2 : A^2 \to V^2$ a bijection $\Bigg\} \Longrightarrow$

$\Longrightarrow$ S can be described as

$$S = \{P \in A^2 \mid \overrightarrow{QP} = t \cdot v,\ t \in \mathbb{R}\}$$

(Q arbitrary and fixed)

$$\forall O \in A^2, \quad \underbrace{\overrightarrow{OP} = \overrightarrow{OQ} + t \cdot v}_{}$$

vector equation of the line S (relative to O, having base point Q and d.v. v)

- fix a coord. system $K = (O, B)$

$$S: \underbrace{\begin{cases} x = x_Q + t \cdot v_x \\ y = y_Q + t \cdot v_y \end{cases}}_{\text{parametric equations of the line } S}$$

where $Q = (x_Q, y_Q)$ and $v = v(v_x, v_y)$

$$S: \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_Q \\ y_Q \end{bmatrix} + t \cdot \begin{bmatrix} v_x \\ v_y \end{bmatrix}$$

...rtesian equations

- by expressing t in both equations and we get

$$\frac{x-x_Q}{v_x} = \frac{y-y_Q}{v_y}$$

e.g.: $S: \begin{cases} x = 3+2\cdot t \\ y = 5+0\cdot t \end{cases}$

$\frac{x-3}{2} = \frac{y-5}{0} \Rightarrow S$ parallel to Qx, with eq. $y=5$

- any line S in A^2 can be described with

$$a\cdot x + b\cdot y + c = 0$$

- this eq. describes a line if $a \neq 0$ or $b \neq 0$

- $D(S)$ satisfies the eq. $D(S): a\cdot x + b\cdot y = 0$

Relative pos. of two lines in A^2

$\begin{cases} l_1: a_1\cdot x + b_1\cdot y + c_1 = 0 \\ l_2: a_2\cdot x + b_2\cdot y + c_2 = 0 \end{cases}$

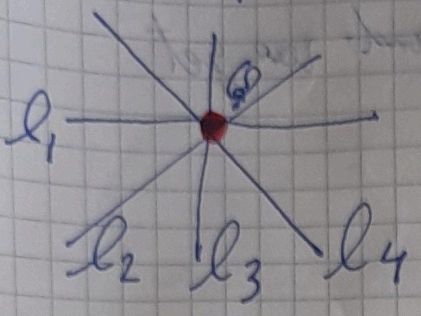
- we discuss the system:

a) $l_1 \cap l_2$ in a unique point O; x_0, y_0 are the sol. to the system

b) $l_1 \parallel l_2 \Leftrightarrow$ the system has no sol.

c) $l_1 = l_2 \Leftrightarrow$ we have infinite sol.

Bundle of lines in A^2



$L_Q = \{l_1, l_2, l_3, l_4\} \to$ bundle of lines

$Q \to$ center of the bundle L_Q

$l_1: a_1 \cdot x + b_1 \cdot y + c_1 = 0$

$l_2: a_2 \cdot x + b_2 \cdot y + c_2 = 0$

$l_1, l_2 \in L_Q,\ l_1 \neq l_2$

$\cancel{\text{- then } \forall l\ L_Q = \{l_{\lambda,\beta} = \lambda \cdot (a_1 \cdot x + b_1 \cdot y + c_1) + \beta \cdot (a_2 \cdot x + b_2 \cdot y + c_2 \mid \lambda, \beta \in \mathbb{R}\} \ \lambda, \beta \text{ not both } 0}$

- then $L_Q = \{l_{\lambda,\beta} = \lambda \cdot (a_1 \cdot x + b_1 \cdot y + c_1) + \beta \cdot (a_2 \cdot x + b_2 \cdot y + c_2) = 0 \mid \lambda, \beta \in \mathbb{R}, \lambda \neq 0 \text{ or } \beta \neq 0\}$

- a reduced bundle is a bundle from which we remove one line (e.g. l_2):

$L_Q = \{l_{1,t}: (a_1 \cdot x + b_1 \cdot y + c_1) + t \cdot (a_2 \cdot x + b_2 \cdot y + c_2) = 0 \mid t \in \mathbb{R}\}$

$(t = \frac{\lambda}{\beta} \in \mathbb{R})$

- improper bundle of lines L_v = the set of all lines in A^2 with d. v. $v \to$ direction vector of L_v

…anes in A^3

- consider the bijection $\phi_Q^3 : A^3 \to V^3$, $Q \in A^3$
- S is a plane ~~1~~ $\iff \forall Q \in S$, $(S \subseteq A^3)$

  $\phi_Q^3(S) = \{\overrightarrow{QP} \mid P \in S\}$ is a 2D vector subspace of V^3

  $\hookrightarrow$ the direction space of the plane S: $D(S)$

- (v, w) basis of $D(S)$ $\quad \left.\begin{matrix} -w \\ -\cancel{w\times} v \end{matrix}\right\}$ direction vectors for S
- fix Q, let P vary in S

  $S = \{P \in A^3 \mid \overrightarrow{QP} = s \cdot v + t \cdot w \,;\, s, t \in \mathbb{R}\}$

  $Q \to$ arbitrary but fixed; base point

$$\underbrace{\overrightarrow{OP} = \overrightarrow{OQ} + s \cdot v + t \cdot w}_{\text{vector eq. of } S}$$

$$S: \begin{cases} x = x_A + s \cdot v_x + t \cdot w_x \\ y = y_A + s \cdot v_y + t \cdot w_y \\ z = z_A + s \cdot v_z + t \cdot w_z \end{cases}$$

~~v_x~~ - we can eliminate s and t and obtain:

$$\left(\frac{v_x}{w_x} - \frac{v_z}{w_z}\right) \cdot \left(\frac{x - x_Q}{w_x} - \frac{y - y_Q}{w_y}\right) = \left(\frac{v_x}{w_x} - \frac{v_y}{w_y}\right) \cdot \left(\frac{x - x_Q}{w_x} - \frac{z - z_Q}{w_z}\right)$$

- * an easier way of describing S:

$P(x, y, z) \in S \Longleftrightarrow$

$$(*) \quad \begin{vmatrix} x - x_Q & y - y_Q & z - z_Q \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = 0$$

- every plane in A^3 can be described with:

$$a \cdot x + b \cdot y + c \cdot z + d = 0 \Rightarrow \text{cartesian eq. of the plane}$$

- fix a coord. system $K(O, B)$ for the plane π
- $D(\pi)$ is the 2D subspace of V^3 which, relative to B, satisfies the eq:

$$D(\pi): a \cdot x + b y + c \cdot z = 0$$

2-fold wedge product

- if we expand $(*)$, we see that the coeff. of x, y, z are

$$a = \begin{vmatrix} v_y & v_z \\ w_y & w_z \end{vmatrix}$$

$$b = - \begin{vmatrix} v_x & v_z \\ w_x & w_z \end{vmatrix}$$

$$c = \begin{vmatrix} v_x & v_y \\ w_x & w_y \end{vmatrix}$$

~~of~~ Def: let $v(v_x, v_y, v_z)$, $w(w_x, w_y, w_z)$ $\in V^3$; basis ~~A~~ $B = (e_1, e_2, e_3)$

- the <u>wedge product</u> of v and w is:

$$v \wedge_B w = \begin{vmatrix} e_1 & e_2 & e_3 \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

?

~~ob~~ <u>Obs</u>: v, w are l.d. $\Leftrightarrow v \wedge_B w = 0$

<u>Relative pos. of 2 planes</u>

$$\begin{cases} \pi_1 : a_1 \cdot x + b_1 \cdot y + c_1 \cdot z + d_1 = 0 \\ \pi_2 : a_2 \cdot x + b_2 \cdot y + c_2 \cdot z + d_2 = 0 \end{cases}$$

- let M be the matrix of the system and $\bar{M}$ the extended matrix

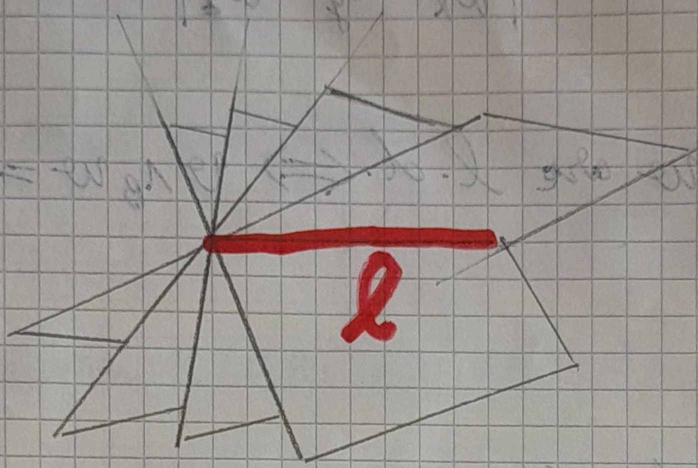
a) $R(M) = R(\bar{M}) = 2 \Rightarrow \pi_1 \cap \pi_2 = \{l\}$, l a line, the points on l are the sol.

b) $\pi_1 \cap \pi_2 = \cancel{\{\emptyset\}}\ \emptyset \Rightarrow \pi_1 \parallel \pi_2$; only if $R(M) < R(\bar{M})$

c) the sol. depend on two params $\Rightarrow \pi_1 = \pi_2$; only if $R(M) = R(\bar{M}) = 1$

Bundle of planes in A^3

- let $l \subseteq A^3$ be a line
- Π_l of all planes in A^3 containing l is called a <u>bundle of planes</u>; l is called the <u>axis</u> (<u>carrier line</u>) of Π_l



- let $\left.\begin{array}{l}\pi_1 = a_1 \cdot x + b_1 \cdot y + c_1 \cdot z + d_1 = 0 \\ \pi_2 = a_2 \cdot x + b_2 \cdot y + c_2 \cdot z + d_2 = 0\end{array}\right\} \in \Pi_l$

- then $\Pi_l = \{\pi_{\lambda,\beta} \mid \pi_{\lambda,\beta} = \lambda \cdot (a_1 \cdot x + b_1 \cdot y + c_1 \cdot z + d_1) +$
$+ \beta \cdot (a_2 \cdot x + b_2 \cdot y + c_2 \cdot z + d_2) = 0;\ \lambda, \beta \in \mathbb{R}$, not both $0\}$

- reduced bundle: same as for lines

- let $W \subseteq V^3$ a vector subspace of 2D; the set Π_W of planes in A^3 which admit W as their d.s. is called an <u>improper bundle of planes</u>
- W is called the vector subspace associated with Π_W

...nes in A^3

- S is a line if $\forall Q \in A^3 S$ we have:

$$\underbrace{\phi^3_Q(S)}_{D(S)} = \{\overrightarrow{QP} : P \in S\} \text{ a 1D subspace of } V^3$$

Parametric equations

$\overrightarrow{QP} = t \cdot v$, ~~for~~ $t \in \mathbb{R}$, $\forall v$ a d. v. of S

- S can be described as:

$$S = \{P \in A^3 \mid \overrightarrow{QP} = t \cdot v, \text{ for some } t \in \mathbb{R}\}$$

$$\underbrace{\overrightarrow{OP} = \overrightarrow{OQ} + t \cdot v}_{\text{vector eq. of } S}$$

- fix a coord. system $K = (O, B)$

$$\begin{cases} x = x_Q + t \cdot v_x \\ y = y_Q + t \cdot v_y \\ z = z_Q + t \cdot v_z \end{cases} \rightarrow \text{parametric eq. for } \cancel{\text{the}}\ S$$

Cartesian eq.

$$\underbrace{\frac{x - x_Q}{v_x} = \frac{y - y_Q}{v_y} = \frac{z - z_Q}{v_z}}_{\text{symmetric equations}}$$

- every line in A^3 can be described with:

$$\begin{cases} a_1 \cdot x + b_1 \cdot y + c_1 \cdot z + d_1 = 0 \\ a_2 \cdot x + b_2 \cdot y + c_2 \cdot z + d_2 = 0 \end{cases} \rightarrow$$ describe the line ℓ relative to $K = (O, B)$

- the direction space $D(\ell)$ of the line ℓ is the 1D subspace of V^3 which, relative to B, satisfies

$$D(\ell): \begin{cases} a_1 \cdot x + b_1 \cdot y + c_1 \cdot z = 0 \\ a_2 \cdot x + b_2 \cdot y + c_2 \cdot z = 0 \end{cases} \rightarrow$$ cartesian eq. of ℓ

- these eq. describe a line as the intersection of two planes

Relative pos. of two lines in A^3

$$\ell_1: \begin{cases} a_1 \cdot x + b_1 \cdot y + c_1 \cdot z + d_1 = 0 \\ a_2 \cdot x + b_2 \cdot y + c_2 \cdot z + d_2 = 0 \end{cases}$$

$$\ell_2: \begin{cases} a_3 \cdot x + b_3 \cdot y + c_3 \cdot z + d_3 = 0 \\ a_4 \cdot x + b_4 \cdot y + \cancel{b_4 z}\ c_4 \cdot z + d_4 = 0 \end{cases}$$

- we discuss the system comprised of the 4 eq.
- it is easier to use parametric eq.:

$$\ell_1 \neq: \begin{cases} x = x_1 + t \cdot v_x \\ y = y_1 + t \cdot v_y \\ z = z_1 + t \cdot v_z \end{cases}$$

$$\ell_2 \neq: \begin{cases} x = x_2 + t \cdot u_x \\ y = y_2 + t \cdot u_y \\ z = z_2 + t \cdot u_z \end{cases}$$

a) if v and u are d. p. then $l_1 \parallel l_2$

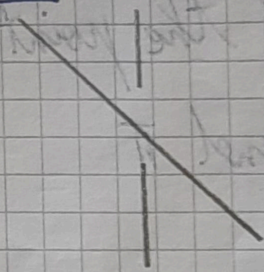
b) if $l_1 \parallel l_2$ and they have a point in common $\Rightarrow l_1 = l_2$

c) if $l_1 \nparallel l_2$ they are coplanar if

$$\begin{vmatrix} x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ v_x & v_y & v_z \\ u_x & u_y & u_z \end{vmatrix} = 0$$

in which case they intersect in one point

d) $\left.\begin{array}{l} l_1 \nparallel l_2 \\ l_1, l_2 \text{ not coplanar} \end{array}\right\} \Rightarrow l_1, l_2$ are <u>skew</u> relative to each other:



Relative pos. of a line and plane in A^3

- the plane π: $a \cdot x + b \cdot y + c \cdot z + d = 0$
- the line l: $\begin{cases} x = x_0 + t \cdot v_x \\ y = y_0 + t \cdot v_y \\ z = z_0 + t \cdot v_z \end{cases}$
- we check to see which points in l satisfy π:

$$a \cdot (x_0 + t \cdot v_x) + b \cdot (y_0 + t \cdot v_y) + c \cdot (z_0 + t \cdot v_z) + d = 0 \Longleftrightarrow$$

$$\Longleftrightarrow (a \cdot v_x + b \cdot v_y + c \cdot v_z) \cdot t + a \cdot x_0 + b \cdot y_0 + c \cdot z_0 + d = 0$$

a) $a \cdot v_x + b \cdot v_y + c \cdot v_z = 0$

$a \cdot x_0 + b \cdot y_0 + c \cdot z_0 + d \neq 0$

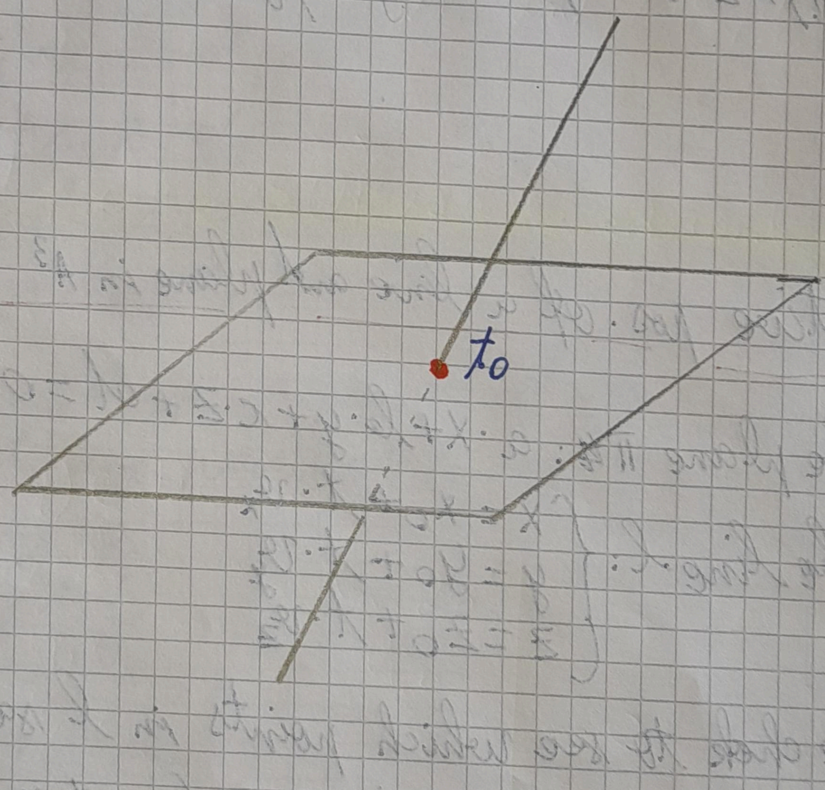
- the eq. has no sol. $\Rightarrow \Pi \cap l = \emptyset \Rightarrow$ they are parallels

b) $a \cdot v_x + b \cdot v_y + c \cdot v_z = 0$

$a \cdot x_0 + b \cdot y_0 + c \cdot z_0 + d = 0$

- $t \in \mathbb{R}$ is a sol. $\Rightarrow l \subseteq \Pi$

c) $a \cdot v_x + b \cdot v_y + c \cdot v_z \neq 0 \Rightarrow$ the eq. has a unique sol. t_0, and the point corresponding to t_0 is the intersection of l and Π



Affine subspaces of A^n

- a d-dimensional affine subspace of the affine space is a subset $S \subseteq A^n$ s.t. the set $D(S)$ (vectors repr. by points in S) form a d-D vector subspace of V^n ($D(S)$ is the direction space of S)

- given two affine subspaces S_1 and S_2 in A^n, we say: $S_1 \parallel S_2 \iff D(S_1) \subseteq D(S_2)$ or vice versa
- $\dim(S) = \dim(D(S))$

- fix a point $O \in A^n$, $Q \in S$ and a basis $(v_1, v_2 \ldots v_d)$ of $D(S)$
- then S is a d-D affine subspace $\iff$

$$\iff S = \{P \in A^n \mid \vec{OP} = \vec{OQ} + t_1 v_1 + \ldots + t_d v_d;\ t_1, \ldots t_d \in \mathbb{R}\}$$

$$S: \cancel{\begin{bmatrix} x_1 \\ \vdots \\ x_d \end{bmatrix}} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} + t_1 \begin{bmatrix} v_{1,1} \\ v_{1,2} \\ \vdots \\ v_{1,n} \end{bmatrix} + \ldots + t_d \begin{bmatrix} v_{d,1} \\ \vdots \\ v_{d,n} \end{bmatrix}$$

Hyperplanes

Def: affine subspaces in A^n ~~with~~ whose dimension is $n-1$ are called hyperplanes

- let H be a hyperplane and $(v_1 \dots v_{n-1})$ a basis of $D(H)$ with respect to $K(O,(e_1,\dots e_n))$ of A^n; the parametric eq. of H are of the form

$$H: \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} + t_1 \begin{bmatrix} v_{1,1} \\ v_{1,2} \\ \vdots \\ v_{1,n} \end{bmatrix} + \dots + t_{n-1} \cdot \begin{bmatrix} v_{n-1,1} \\ v_{n-1,2} \\ \vdots \\ v_{n-1,n} \end{bmatrix}$$

- where $v_i\,(v_{i,1},\dots v_{i,n})$ and $Q(q_1 \dots q_n)$ is a point in H, $t_i \in \mathbb{R}\ \forall\, i = \overline{1,n-1}$

~~a point~~ - a point $P \in H \iff \overrightarrow{QP}$ is a linear comb. of $v_1 \dots v_{n-1}$

- reformulate:

$P(x_1, \dots x_{n-1}) \in H \iff$

$$\iff \begin{vmatrix} x_1 - q_1 & x_2 - q_2 & \dots & x_n - q_n \\ v_{1,1} & v_{1,2} & \dots & v_{1,n} \\ v_{2,1} & v_{2,2} & \dots & v_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{n-1,1} & v_{n-1,2} & \dots & v_{n-1,n} \end{vmatrix} = 0$$

$\longrightarrow$ cartesian eq. of H

(n-1)-fold wedge product

$H: a_1 \cdot x_1 + a_2 \cdot x_2 + \dots + a_n \cdot x_n + b = 0 \Rightarrow$ the eq. of H

where a_i corresponds to an $n-1$ minor in the determinant

Def: let $v_1 \dots v_{n-1}$ be $n-1$ vectors in V^n

fix a basis $B = (e_1, e_2 \dots e_n)$ of V^n

let $v_i(v_{i,1}, v_{i,2} \dots v_{i,n})$ with respect to B

- the wedge product of $v_1 \dots v_{n-1}$ is the vector

$$v_1 \wedge_B \dots \wedge_B v_{n-1} = \begin{vmatrix} e_1 & e_2 & \dots & e_n \\ v_{1,1} & v_{1,2} & \dots & v_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{n-1,1} & v_{n-1,2} & \dots & v_{n-1,n} \end{vmatrix}$$

- consider B' another basis in V^n
- if $M_{BB'} = M^T_{B'B}$ then

$$v_1 \wedge_B \dots \wedge_B v_{n-1} = v_1 \wedge_{B'} \dots \wedge_{B'} v_{n-1}$$

- we can write the eq. of H as

$$H: [v_1 \wedge_B \dots \wedge_B v_{n-1}] \cdot \left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \right) = 0$$

where $v_1 \dots v_{n-1}$ form a basis of $D(H)$ and $Q_K(q_1 \dots q_n)$ is a point in H

J Lines

- a line in A^n is a 1-D ~~vs~~ affine subspace

$$l = \{P \in A^n \mid \vec{OP} = \vec{OQ} + t \cdot v,\ t \in \mathbb{R}\},\ \forall O \in A^n,\ Q \in l$$

$$l: \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} + t \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

- eliminate t and we obtain:

$$l: \frac{x_1 - q_1}{v_1} = \frac{x_2 - q_2}{v_2} = \dots = \frac{x_n - q_n}{v_n}$$

- a line in A^n can be described as the intersection of $n-1$ hyperlanes

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Relative pos.

- let S, T be 2 affine subspaces of A^n
- if $S \parallel T \Rightarrow$ they are disjoint or one is included in the other
- if $\dim(S) = \dim(T)$, then $S \parallel T \Leftrightarrow D(S) = D(T)$
- two hyperlanes are parallel if the coeff. of the unknowns in their eq. are proportional

- let S, T be parallel affine subspaces of A^n, $\dim(S) \le \dim(T)$

  a) if S and T have a point in common $\Rightarrow S \subseteq T$

  b) if a) and $\dim(S) = \dim(T) \Rightarrow S = T$

Corollary

- S an affine subspace of A^n, ~~$P \in A^n$~~
- $P \in A^n$
- $\exists!$ affine subspace T of A^n s.t. $P \in T$, $T \parallel S$ and $\dim(T) = \dim(S)$

Def: if two affine subspaces S and T of A^n are not parallel $\Rightarrow$ they are skew if they do not meet or incident if they have a point in common

$\cancel{S.\ \Sigma}$

Suppose S and T are given by:

$$S: \sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = \overline{1, n-s}$$

$$T: \sum_{j=1}^{n} c_{kj} x_j = d_k, \quad k = \overline{1, n-t}$$

$$S \cap T: \begin{cases} \sum_{j=1}^{n} a_{ij} \cdot x_j = b_i & \text{for } i = \overline{1, n-s} \\ \sum_{j=1}^{n} c_{kj} \cdot x_j = d_k & k = \overline{1, n-t} \end{cases}$$

- if $S \cap T \neq \emptyset$, then it is an affine subspace of A^n

Prop. 2.23

- if $S \cap T$ is an affine subspace of A^n, then we have:

$$\dim(S) + \dim(T) - \dim(A^n) \leq \dim(S \cap T) \leq \min\{\dim(S), \dim(T)\}$$

Prop. 2.24

- S, T two affine subspaces of A^n
- $V^n = D(S) + D(T) \iff S \cap T \neq \emptyset$ and

$$\dim(S \cap T) = \dim(S) + \dim(T) - \dim(A^n)$$

Changing the reference frame

- S an affine subspace of A^n with respect to $K=(O,B)$ via the eq. from 2.23

- if $K'=(O',B')$ is another coord. system, we have:

$$S: \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix} = M_{K'K} \cdot \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + [O]_{K'} + t_1 \cdot M_{K'K} \cdot \begin{bmatrix} v_{1,1} \\ v_{1,2} \\ \vdots \\ v_{1,n} \end{bmatrix} + \dots +$$

$$+ t_d \cdot M_{K'K} \cdot \begin{bmatrix} v_{d,1} \\ v_{d,2} \\ \vdots \\ v_{d,n} \end{bmatrix}$$

~~- we can translate the system as:~~

$\cancel{S: A \cdot M_{K,K'}}$