1. Eulerian Graphs

Reminder:

- A **trail** is a walk with no repeated edges.
- A *trail* that starts and ends at the same vertex is called a **circuit**.
- A **path** is a walk with no repeated vertices (except the initial and terminal vertices).
- A *path* that starts and ends at the same vertex is called a **cycle**.
- The number of edges a vertex is incident to is called the **degree** of the vertex.

Def 1: Let $G = (V, E)$ be a connected graph and a trail T in this graph, with edge sequence (e_1, e_2, …, e_k), where $e_i = e_j$ if and only if $i = j$. If T uses all edges from E, then it is called an **Eulerian trail**. If the starting vertex of e_1 and terminal vertex of e_k are the same (starting and terminal vertices of the trail are the same), then it is called an **Eulerian circuit**.

In other words, an **Eulerian circuit** is a circuit which uses all edges in a graph.

Def 2: an **Eulerian graph** is a graph that has an Eulerian circuit.

For example, consider the graph in Figure 1. The trail (e1 e7 e5 e6 e2 e4 e3) is an Eulerian circuit. The graph in Figure 1 is an Eulerian graph.

In an Eulerian graph, all vertices have an even degree. Intuitively this must be true because:

1) for any vertex not the start of the trail, the trail must reach the vertex on one edge and leave on a different edge (since it is a trail edges cannot repeat), and this must happen every time the trail reaches that vertex => the degree of the vertex must be a multiple of 2, therefore an even.
2) For the first vertex, there must be an edge for the trail to start and a different edge for the trail to finish. Any other time the trail reaches the first vertex, the same thing happens as the other vertices: it must reach in one edge and leave by another. Therefor for the first vertex also, the degree is even.

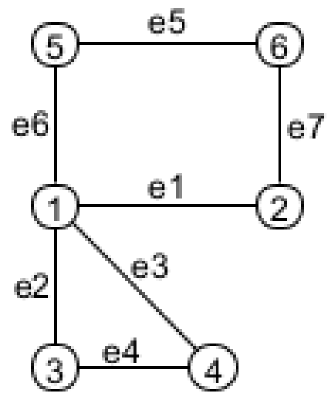

Figure 1

The converse is also true, if G is a connected graph with all vertices having even degrees, then it is an Eulerian graph.

=> it is easy to check if a graph is eulerian (Theta(n), depending on implementation). Just check all if all the vertices have an even degree.

To find an Eulerian trail we can use the Hierholzer algorithm. The idea of the algorithm is that it continuously searches for trails in the graph and merges them until all edges are used.

Hierholzer algorithm (short description):

a. Start from a random vertex.
b. Follow unused edges from that vertex onward until you reach the vertex again.
c. Find a vertex on the circuit that still has unused edges.
d. Follow unused edges from that vertex onward until you reach the vertex again.
e. Add the new circuit to the old one.
f. Repeat steps c to e until there are no other vertices with unused edges.

For example, on graph from figure 1, the algorithm might start at vertex 5. It might find the trail (e6, e1, e7, e5). Then, it sees that vertex 1 still has an unused edge and it will find the trail (e2, e4, e3). Then it merges them to the final trail: (e6, e2, e4, e3, e1, e7, e5).

A great visualization of how this algorithm works can be found at:

https://algorithms.discrete.ma.tum.de/graph-algorithms/hierholzer/index_en.html

The complexity of the algorithm depends on how we find a new vertex with unused edges and how we merge the trails. The algorithm can run in Theta(n+m) if we use the following:

a. For each vertex a set of incident edges that are not used – basically a set of sets. When an edge (v1, v2) is added to the circuit remove it from the set of incident edges of both v1 and v2. Initialization of this set of sets (adding all edges to the correct vertex) can be done in Theta(m+n) with any traversal.
b. A set of vertices in the circuit that still have unvisited edges. When adding an edge (v1,v2) to the circuit, after removing the edge from unused edges, if v2 is not in this set, add v to set. If v1 has no more unused edges (theta 1 if check size with structure from a), remove v1 from the set.
c. With the Hierholzer algorithm we find multiple circuits. In order to connect them efficiently, we can use another set that says what circuit is connected to what circuit at what point. Only at the end we create the whole circuit by traversing once these circuits and connecting where necessary.

2. Planar graphs

Def 3: In Graph Theory, a **planar graph** is a graph which can be embedded in the plane, i.e. it can be drawn on the plane in such a manner that its edges intersect only at their endpoints.

For this problem we only consider undirected graphs since having directed edges does not really change anything: if 2 vertices have 2 edges between them, in opposite directions, we can just draw them infinitely close together, basically simulating one undirected edge => directed edges cannot change any planarity properties of the graph.

Since undirected graphs are easier to study, we will only talk about undirected graphs (although everything applies the same to directed graphs, just ignore orientation and one edge in case of parallel edges).

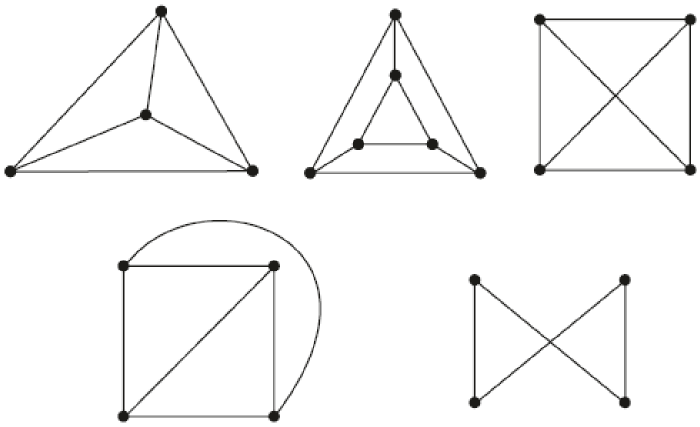

Figure 2

Consider the graphs in Figure 2. All of these graphs are planar graphs. Yes, even the third graph and the last graph. See a planar graph is a graph which **can** be embedded in the plane. If it **can** be drawn without edges intersecting it is a planar graph, no matter if the current drawing has crossing edges.

Actually, the third graph is the same as the fourth graph, just with a different drawing (and you can see that it *can* be drawn without edges crossing). Drawing the las graph without the edges crossing will be left as an exercise for the reader.

Def 4: A simple graph that has edges between every vertex pair is called **complete**. A complete graph that has n vertices is denoted by K_n, for example, K_4 is a complete graph with four vertices. A complete undirected graph has n(n-1)/2 edges (while a directed one has n(n-1)).

In Figure 3 you can se an example complete graph with 5 vertices: K_5.

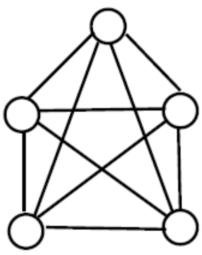

Figure 3 – graph K_5

Def 5: A graph $G = (V, E)$ is called a **bipartite graph** if the vertex set V can be partitioned into two disjoint subsets V_1 and V_2 such that any edge of G connects a vertex in V_1 to a vertex in V_2.

Formally, Graph $G = (V, E)$ is **bipartite** if $\exists\, V_1, V_2 \subset V, V_1 \cup V_2 = V, V_1 \cap V_2 \neq \emptyset$ such that $\forall (v_1, v_2) \in E$, either $v_1 \in V_1$ and $v_2 \in V_2$, or $v_1 \in V_2$ and $v_2 \in V_1$.

Notation: usually a bipartite graph is noted as $G(V_1 \cup V_2, E)$.

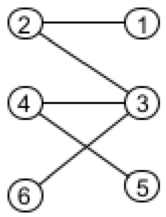

Figure 4

In figure 4 you can see an example bipartite graph. The partitions are $V_1=\{2,4,6\}$ and $V_2=\{1,3,5\}$. As you can see, there are no edges between the vertices of any partition.

Def 6: Let $G(V_1 \cup V_2, E)$ be a bipartite graph. If every vertex in V_1 is adjacent to every vertex in V_2, G is a **complete bipartite graph**.

If |V1| = x and |V2| = y, then G is denoted by $K_{x,y}$. A complete bipartite graph has x*y edges.

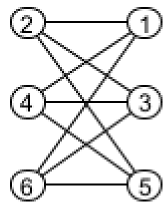

Figure 5 – graph $K_{3,3}$

In Figure 5 an example complete bipartite graph is given: $K_{3,3}$.

K_4 is a planar graph (actually the third and fourth graphs in Figure 2 are K_4).

K_5 **is not** a planar graph. In Figure 6 is given a visual proof of this. We need to link all vertices, 10 edges in total. First, we make a cycle graph by uniting all vertices one after the other, drawing 5 edges. Then we unite as many vertices in the inside as possible – only 2 are possible. We do the same for the outside – still only 2 are possible. That makes 9 edges, but there are 10 edges that need to be drawn. Indeed, in Figure 6 on the right it can be seen that there is no way to draw edge (b, d) without intersecting any other edge. Therefore K_5 is not planar.

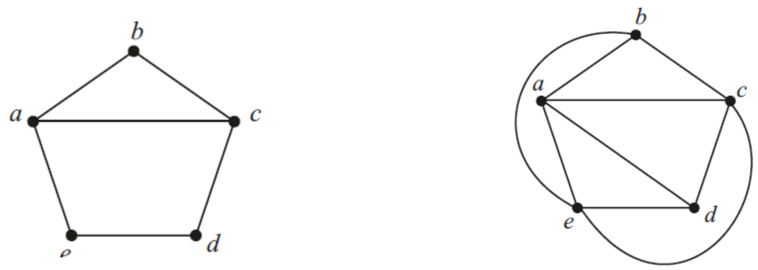

Figure 6

$K_{3,3}$ is also not a planar graph (nonplanar). Visual proof will be left as an exercise for the reader.

Def 7: two graphs G_1 and G_2 are said to be **homeomorphic** if and only if G_2 can be obtained from G_1 (and the other way around) by the insertion or deletion of a number of vertices of **degree two**.

Example: Graphs in Figure 7 are homeomorphic.

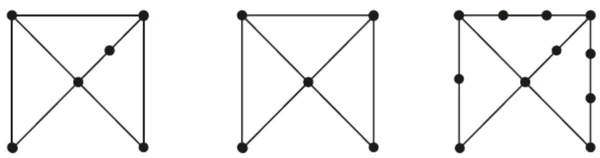

Figure 7

One can think of homeomorphic graphs as being the same shape. Adding or deleting a vertex of degree two does not change the shape of the edges but simply replaces a single edge by a pair of edges taking the same shape (or vice versa).

Def 8: contraction (or smoothing) is the process by which vertices of degree 2 are continuously removed from a graph *G* to find a homeomorphic graph with fewer vertices.

In Figure 8 is given an example of contraction, up to a graph that cannot be contracted anymore (it does not have any vertices of degree 2).

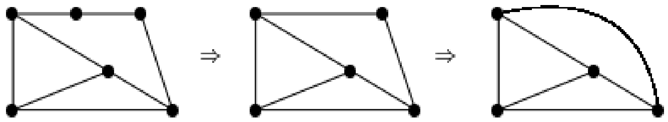

Figure 8

Theorem: A graph is planar if and only if it has no subgraph homeomorphic to K_5 or $K_{3,3}$. I.e if it does not contain any subgraphs which can *contract* to K_5 or $K_{3,3}$.

Def 9: A planar graph which has already been drawn in the plane *without edge intersection*s* is called a **plane graph** or planar **embedding of the graph.**

**we can draw it with intersections but then it is not a planar embedding (or a plane graph)*

For example, in figure 1, the 4^{th} picture is an embedding of K_4 while the third picture is **not** an embedding of K_4.

An embedding of a graph can be defined as a mapping from every vertex of the graph to a point in 2D space, and from every edge to a curve, such that the extreme points of each curve are the points mapped from its end nodes, and all the curves are disjoint except on their extreme points.

Def 10: An embedding of a graph partitions the plane in multiple regions, bounded by cycles of the graph. Such a region of the plane is called a **face** of the graph. Each plane graph has exactly one unbounded face, called the **exterior face**.

A connected, undirected graph with no cycles has only one face, the exterior face. All trees are planar graphs.

Eulers formula: Let $G(V,E)$ be a connected, undirected, planar graph and E_G an embedding of G with f faces then $\mathbf{n - m + f = 2}$.

Corollary: All planar embeddings of a given connected planar graph have the same number of faces.

Proof: let E_G be an embedding of G with f faces. Then $n - m + f = 2 \Rightarrow f = 2 + m - n$. Therefore, all undirected, connected, planar graphs with n vertices and m edges have $f = 2 + m - n$ faces on any of their plane embeddings.

Corollary 2: Let G be an undirected, connected planar graph. If $n >= 3$, then $m <= 3n-6$. This can be used to check quickly if a graph is **not** a planar graph (it cannot be used to check if it is a planar graph).