Lecture 5

- affine morphism

$$\mathrm{lin}(\varphi)(\overrightarrow{OP}) = \overrightarrow{\varphi(O)\varphi(P)} = \overrightarrow{O'\varphi(P)} - \overrightarrow{O'\varphi(O)}$$

$$[\varphi(P)]_{K'} = \underbrace{M_{B'B}(\mathrm{lin}(\varphi))}_{A}[P]_K + \underbrace{[\varphi(O)]_{K'}}_{b}$$

$$\Leftrightarrow [\varphi(P)]_{K'} = A[P]_K + b$$

- homogenous matrix $\varphi: \mathbb{A}^n \to \mathbb{A}^m$, $\varphi(x) = Ax + b$

$$\Rightarrow \hat{M}_{K,K'}(\varphi) = \begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix}$$

- tensor product: $\vec{v}(v_1, \dots v_m)$, $\vec{w}(w_1, \dots w_m)$ $\Big| \Rightarrow \vec{v} \otimes \vec{w} = \vec{v}^T \cdot \vec{w} = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} [w_1 \dots w_m]$

$\rightsquigarrow (\vec{v} \otimes \vec{w})^T = \vec{w} \otimes \vec{v}$

$\rightsquigarrow$ B orthonormal $\Rightarrow (\vec{u} \otimes \vec{v})\vec{w} = \langle \vec{v}, \vec{w} \rangle \vec{u}$

- parallel projection on a hyperplane

$H: a_1x_1 + \dots + a_mx_m + a_{m+1} = 0$, $\vec{a}(a_1 \dots a_m)$, $\vec{v}(v_1 \dots v_m)$

$$Pr_{H,\vec{v}}(P) = \left(I_m - \frac{\vec{v} \otimes \vec{a}}{\langle \vec{v}, \vec{a} \rangle}\right)P - \frac{a_{m+1}}{\langle \vec{v}, \vec{a} \rangle}\vec{v}$$

orthogonal: $$Pr^{\perp}_{H}(P) = \left(I_m - \frac{\vec{a} \otimes \vec{a}}{|\vec{a}|^2}\right)P - \frac{a_{m+1}}{|\vec{a}|^2}\vec{a}$$

- parallel reflection on a hyperplane

$$Refl_{H,\vec{v}}(P) = 2Pr_{H,\vec{v}}(P) - P$$

- parallel projection on a line

$H_P: a_1(x_1 - p_1) + \dots + a_m(x_m - p_m) = 0$, $P(p_1 \dots p_m)$, $\vec{a}(a_1 \dots a_m)$

$$Pr_{\ell,W}(P) = \frac{\vec{v} \otimes \vec{a}}{\langle \vec{v}, \vec{a} \rangle}P + \left(I_m - \frac{\vec{v} \otimes \vec{a}}{\langle \vec{v}, \vec{a} \rangle}\right)Q, \quad Q \in \ell$$

orthogonal: $$Pr^{\perp}_{\ell}(P) = \frac{\vec{a} \otimes \vec{a}}{|\vec{a}|^2}P + \left(I_m - \frac{\vec{a} \otimes \vec{a}}{|\vec{a}|^2}\right)Q$$

- parallel reflection on a line

$$Refl_{\ell,W}(P) = 2Pr_{\ell,W}(P) - P$$

- isometries

$$A \in Mat_{m\times m}(\mathbb{R}) \text{ s.t } A^T A = I_m \iff A^T = A^{-1} \Rightarrow \text{orthogonal matrix}, \ O(n)$$

- special orthogonal $\iff \begin{cases} O(n) \\ \det = 1 \end{cases} \Rightarrow SO(n)$

$$\varphi(x) = Ax+b: \text{ if } \begin{cases} \det(A) = 1 & \Rightarrow \varphi \text{ direct isometry} \\ A \in O(n) & \\ \det(A) = -1 & \Rightarrow \varphi \text{ indirect isometry} \end{cases}$$

- rotations in dimension 2: $A \in SO(2) \iff \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$\Rightarrow \cos\theta = \frac{tr(lin(\varphi))}{2}$$

- rotations in dimension 3:

$$\cos\theta = \frac{tr(lin(\varphi))-1}{2}$$

Euler-Rodrigues: v unit vector $\Rightarrow$ rotation of θ and axis $\mathbb{R}v$

$\theta \in \mathbb{R}$

$$Rot_{v,\theta}(P) = \cos\theta \cdot P + \sin\theta(v \times P) + (1-\cos\theta)\langle v, P\rangle v$$

- theorem Chasles: direct isometry of E^2: $\begin{cases} \text{identity} \\ \text{translation} \\ \text{rotation} \end{cases}$

- theorem: indirect isometry of E^2 fixes a line and is either $\begin{cases} \text{reflection in } l \\ \underbrace{\text{reflection in } l + \text{translation} \parallel l}_{= \text{glide-reflection}} \end{cases}$

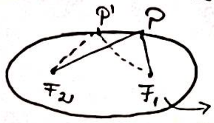
- rotation in E^3 + translation $\parallel$ rotation axis $\Rightarrow$ glide-rotation (helical displacement)

- theorem Chasles: direct isometry of E^3: $\begin{cases} \text{identity} \\ \text{translation} \\ \text{rotation} \\ \text{glide-rotation} \end{cases}$

- theorem: indirect isometry of E^3 fixes a plane π: $\begin{cases} \text{reflection in } \pi \\ \underbrace{\text{reflection in } \pi + \text{translation} \parallel \pi}_{\text{glide reflection}} \\ \underbrace{\text{reflection in } \pi + \text{rotation of axis} \perp \pi}_{\cancel{\text{glide}} \text{ rotation reflection}} \end{cases}$

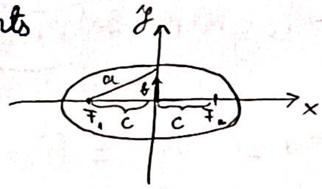
LECTURE 6

- quadratic curve : $ax^2 + bxy + cy^2 + dx + ey + f = 0$
- ellipse



focal points

$$\mathcal{E}_{a,b} = \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



$$|F_1F_2| = 2c \Rightarrow b^2 = a^2 - c^2$$

$$\mathcal{E}_{a,b} \cap Ox = \{(\pm a, 0)\}$$

$$\mathcal{E}_{a,b} \cap Oy = \{(0, \pm b)\}$$

$\varepsilon = \frac{c}{a} = \sqrt{1 - \frac{b^2}{a^2}}$ = eccentricity of $\mathcal{E}_{a,b}$ (roundness)

$$M(x_M, y_M) \in \mathcal{E}_{a,b} \Leftrightarrow (\pm x_M, \pm y_M) \in \mathcal{E}_{a,b}$$

- relative position of a line $\begin{cases} \mathcal{E}_{a,b}: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \\ y = kx + m \end{cases}$

$\ldots \Rightarrow \Delta = 4a^2b^2(a^2k^2 + b^2 - m^2) \Rightarrow \Delta > 0 \Leftrightarrow a^2k^2 + b^2 > m^2$

$\Leftrightarrow -\sqrt{a^2k^2+b^2} < m < \sqrt{a^2k^2+b^2}$

$\Leftrightarrow$ 2 intersection points

$\Rightarrow \Delta = 0 \Leftrightarrow a^2k^2 + b^2 = m^2$

$\Leftrightarrow m = \pm\sqrt{a^2k^2+b^2}$

$\Leftrightarrow$ 1 intersection point

$\Rightarrow \Delta < 0 \Leftrightarrow a^2k^2 + b^2 < m^2 \Leftrightarrow \begin{cases} m > \sqrt{a^2k^2+b^2} \\ m < -\sqrt{a^2k^2+b^2} \end{cases}$

$\Leftrightarrow$ no intersection points

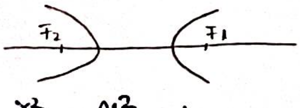
- for a given k (slope) $\Rightarrow y = kx \pm \sqrt{a^2k^2+b^2}$ tangent lines
- tangent line - via gradient: $\gamma: \mathbb{E}^2 \to \mathbb{R}$, $\gamma(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

$$\mathcal{E}_{a,b} = \gamma^{-1}(1)$$

$$\nabla_{(x_0,y_0)}(\gamma) = \left(2\frac{x}{a^2}, 2\frac{y}{b^2}\right)_{(x_0,y_0)} = 2\left(\frac{x_0}{a^2}, \frac{y_0}{b^2}\right)$$

$$\nabla_{(x_0,y_0)}(\gamma) \perp (x_0,y_0) \in \mathcal{E}_{a,b} \Rightarrow T_{(x_0,y_0)}\mathcal{E}_{a,b}: \frac{x_0x}{a^2} + \frac{y_0y}{b^2} = 1$$

- hyperbola



$$\mathcal{H}_{a,b}: \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$|F_1F_2| = 2c \Rightarrow b^2 = c^2 - a^2$$

$$\mathcal{H}_{a,b} \cap Ox = \mathcal{H}_{a,b} \cap Oy = \{(\pm a, 0)\}$$

$\varepsilon = \frac{c}{a} = \sqrt{1 + \frac{b^2}{a^2}}$ (how open or how close)

$M(x_M, y_M) \in \mathcal{H}_{a,b} \Leftrightarrow (\pm x_M, \pm y_M) \in \mathcal{H}_{a,b}$

• relative position of a line : $\begin{cases} \mathcal{H}_{a,b}: \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \\ y = kx + m \end{cases}$

quadratic ec.

$\dots \Rightarrow \Delta = 4a^2b^2(m^2 + b^2 - a^2k^2) \Rightarrow \Delta > 0 \Rightarrow \begin{cases} m < -\sqrt{a^2k^2-b^2} \\ m > \sqrt{a^2k^2-b^2} \end{cases} \Rightarrow$ 2 distinct points

$\Delta = 0 \Rightarrow m = \pm\sqrt{a^2k^2-b^2} \Rightarrow$ 1 point

$\Delta < 0 \Rightarrow -\sqrt{a^2k^2-b^2} < m < \sqrt{a^2k^2-b^2} \Rightarrow$ no point

• for a given k (slope) $\Rightarrow y = kx \pm \sqrt{a^2k^2-b^2}$ tangent line $\Rightarrow T_{(x_0,y_0)}\mathcal{H}_{a,b}: \frac{x_0 x}{a^2} - \frac{y_0 y}{b^2} = 1$

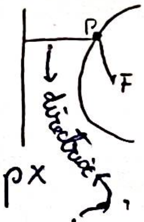
linear ec.

$-2kma^2x - a^2(m^2+b^2) = 0$

$m \neq 0 \Rightarrow x = -\frac{m^2+b^2}{2a^2} \Rightarrow$ unique intersection point

$m = 0 \Rightarrow y = \pm\frac{b}{a}x \Rightarrow$ no intersection point

• parabola



$\mathcal{P}_p: y^2 = 2px$

$F\left(\frac{p}{2}, 0\right) \Rightarrow d: x = -\frac{p}{2}$

$\mathcal{P}_p \cap Ox = \mathcal{P}_p \cap Ox = \{(0,0)\}$

$M(x_M, y_M) \in \mathcal{P}_p \Leftrightarrow (x_M, \pm y_M) \in \mathcal{P}_p$

• relative position of a line : $\begin{cases} \mathcal{P}_p: y^2 = 2px \\ y = kx + m \end{cases}$

$\dots \Rightarrow \Delta = 4p(p - 2km) \Rightarrow \Delta > 0 \Rightarrow km < \frac{p}{2} \Rightarrow$ 2 distinct points

$\Delta = 0 \Rightarrow km = \frac{p}{2} \Rightarrow$ unique point

$\Delta < 0 \Rightarrow km > \frac{p}{2} \Rightarrow$ no point

• for a given k (slope) $\Rightarrow y = kx + \frac{p}{2k}$ tangent line $\Rightarrow T_{(x_0,y_0)}\mathcal{P}_p: yy_0 = p(x + x_0)$

LECTURE 7

- hyperquadrics : $Q: \sum_{i,j=1}^{m} q_{ij} x_i x_j + \sum_{i}^{m} b_i x_i + c = 0$

$$q_{ii} = a_{ii}, \quad q_{ij} = q_{ji} = \frac{a_{ij} + a_{ji}}{2} \Rightarrow \text{symmetric matrix}$$

$$L = Q: X^T \cdot Q \cdot X + b^T \cdot X + c = 0$$

- reducing to canonical form : ~~I rotation~~

(rotation) $\mathcal{C} : q_{11} x^2 + 2 q_{12} xy + q_{22} y^2 + b_1 x + b_2 y + c = 0$

$$\Leftrightarrow [x \ y] \begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + [b_1 \ b_2] \begin{bmatrix} x \\ y \end{bmatrix} + c = 0$$

(translation)

$$\det \begin{bmatrix} q_{11} - \lambda & q_{12} \\ q_{12} & q_{22} - \lambda \end{bmatrix} = 0 \Rightarrow \lambda_1, \lambda_2$$

- $(Q - \lambda_1 I)\vec{v} = 0 \Rightarrow V(\lambda_1)$
- $(Q - \lambda_2 I)\vec{v} = 0 \Rightarrow V(\lambda_2)$

we normalize them $\Rightarrow M = [V(\hat{\lambda_1}) \ V(\hat{\lambda_2})]$

$\Downarrow$

$M^T = M^{-1}$

$$\Rightarrow [x' \ y'] \cdot \underbrace{M^T \cdot Q \cdot M}_{\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}} \begin{bmatrix} x' \\ y' \end{bmatrix} + [b_1 \ b_2] \cdot M \begin{bmatrix} x' \\ y' \end{bmatrix} + c = 0$$

- isometric invariants

$\mathcal{C} : q_{11} x^2 + 2 q_{12} xy + q_{22} y^2 + 2b_1 x + 2b_2 y + c = 0$

$$[x \ y \ 1] \underbrace{\begin{bmatrix} q_{11} & q_{12} & b_1 \\ q_{21} & q_{22} & b_2 \\ b_1 & b_2 & c \end{bmatrix}}_{\hat{Q}} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

$$Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}$$

$\hat{\Delta} = \det(\hat{Q})$

$\Delta = \det(Q)$

$T = tr(Q)$

$\hat{\Delta}$	Δ	T	curve C
$\hat{\Delta} = 0$	$\Delta > 0$	-	a point
	$\Delta = 0$	-	2 lines or the empty set
	$\Delta < 0$	-	2 lines
$\hat{\Delta} \neq 0$	$\Delta > 0$	$\Delta T < 0$	an ellipse
	$\Delta > 0$	$\Delta T > 0$	the empty set
	$\Delta = 0$	-	a parabola
	$\Delta < 0$	-	a hyperbola

- affine classification in E^2

$r = \text{rank}\, Q$	$(p, r-p)$ → positive eigenvalues	equation	name
2	(0,2) or (2,0)	$x^2+y^2+1=0$	imaginary ellipse
2	(1,1)	$x^2-y^2-1=0$	hyperbola
2	(0,2) or (2,0)	$x^2+y^2-1=0$	ellipse
2	(0,2) or (2,0)	$x^2+y^2=0$	2 complex lines
2	(1,1)	$x^2-y^2=0$	2 real lines
1	(0,1) or (1,0)	$x^2+1=0$	2 complex lines
1	(1,1)	$x^2-1=0$	2 real lines
1	(1,1)	$x^2=0$	a real double-line
1	(0,1) or (1,0)	$x^2-y=0$	parabola

- Lagrange method

$\mathcal{C}: g_{11}x^2+2g_{12}xy+g_{22}y^2+b_1x+b_2y+c=0$

- eliminate mixed terms by completing squares
- eliminate linear terms by completing squares
- equation $ax^2+by^2+c=0$ or $ax^2+by+c=0 \Rightarrow$ table of affine classification

- classification of quadrics

$\mathcal{C}: g_{11}x^2+g_{22}y^2+g_{33}z^2+2g_{12}xy+2g_{12}yz+2g_{12}xz+b_1x+b_2y+b_3z+c=0$

- bring to canonical form

$r = \text{rank}\, Q$	$(p, r-p)$	equation	name
3	(3,0) or (0,3)	$x^2+y^2+z^2-1=0$	ellipsoid
3	(2,1) or (1,2)	$x^2+y^2-z^2-1=0$	hyperboloid of one sheet
3	(2,1) or (1,2)	$x^2-y^2-z^2-1=0$	hyperboloid of two sheets
3	(3,0) or (0,3)	$x^2+y^2+z^2+1=0$	imaginary ellipsoid
3	(3,0) or (0,3)	$x^2+y^2+z^2=0$	imaginary cone
3	(2,1) or (1,2)	$x^2+y^2-z^2=0$	(real, elliptic) cone
2	(2,0) or (0,2)	$x^2+y^2+1=0$	cylinder on imaginary ellipse
2	(1,1)	$x^2-y^2-1=0$	cylinder on hyperbola
2	(2,0) or (0,2)	$x^2+y^2-1=0$	cylinder on ellipse
2	(2,0) or (0,2)	$x^2+y^2=0$	cylinder on two complex lines
2	(1,1)	$x^2-y^2=0$	cylinder on two real lines
1	(1,0) or (0,1)	$x^2+1=0$	two complex planes
1	(1,0) or (0,1)	$x^2-1=0$	two real planes
1	(1,0) or (0,1)	$x^2=0$	a double plane
1	(1,0) or (0,1)	$x^2+1=0$	two complex planes
1	(1,0) or (0,1)	$x^2-1=0$	two real planes
1	(1,0) or (0,1)	$x^2=0$	a double plane
2	(2,0) or (0,2)	$x^2+y^2-z=0$	elliptic paraboloid
2	(1,1)	$x^2-y^2-z=0$	hyperbolic paraboloid
1	(1,0) or (0,1)	$x^2+y=0$	cylinder on parabola

LECTURE 8

- ellipsoid $\mathcal{E}_{a,b,c}: \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

tangent planes $T_p\mathcal{E}_{a,b,c}: \frac{x_p x}{a^2}+\frac{y_p y}{b^2}+\frac{z_p z}{c^2}=1$

parametrization $\begin{cases} x(\theta_1,\theta_2)=a\cos\theta_1\cos\theta_2 \\ y(\theta_1,\theta_2)=b\sin\theta_1\cos\theta_2 \\ z(\theta_1,\theta_2)=c\sin\theta_2 \end{cases}$ $\theta_1\in[0,2\pi], \theta_2\in[-\frac{\pi}{2},\frac{\pi}{2}]$

- elliptic cone : $\mathcal{C}_{a,b,c}: \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$

tangent planes $T_p\mathcal{C}_{a,b,c}: \frac{x_p x}{a^2}+\frac{y_p y}{b^2}-\frac{z_p z}{c^2}=0$

parametrizations $\begin{cases} x(\theta,h)=ha\cos\theta \\ y(\theta,h)=hb\sin\theta \\ z(\theta,h)=hc \end{cases}$

- hyperboloid of one sheet: $\mathcal{H}_{a,b,c}: \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$

tangent planes $T_p\mathcal{H}_{a,b,c}: \frac{x_p x}{a^2}+\frac{y_p y}{b^2}-\frac{z_p z}{c^2}=1$

parametrizations $\sigma_1(\theta_1,\theta_2)=\begin{bmatrix} a\sqrt{1+\theta_2^2}\cos\theta_1 \\ b\sqrt{1+\theta_2^2}\sin\theta_1 \\ c\theta_2 \end{bmatrix}$ and $\sigma_2(\theta_1,\theta_2)=\begin{bmatrix} a\cosh(\theta_2)\cos\theta_1 \\ b\cosh\theta_2\sin\theta_1 \\ c\sinh\theta_2 \end{bmatrix}$

- hyperboloid of two sheets: $\mathcal{H}^2_{a,b,c}: \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$

tangent planes : $T_p\mathcal{H}^2_{a,b,c}: \frac{x_p x}{a^2}+\frac{y_p y}{b^2}-\frac{z_p z}{c^2}=-1$

parametrizations : $\sigma_1(\theta_1,\theta_2)=\begin{bmatrix} a\sqrt{\theta_2^2-1}\cos\theta_1 \\ b\sqrt{\theta_2^2-1}\sin\theta_1 \\ c\theta_2 \end{bmatrix}$ and $\sigma_2(\theta_1,\theta_2)=\begin{bmatrix} a\sinh\theta_2\cos\theta_1 \\ b\sinh\theta_2\sin\theta_1 \\ \varepsilon c\cosh(\theta_2) \end{bmatrix}$

- elliptic paraboloid : $P^e_{a,b}: \frac{x^2}{a}+\frac{y^2}{b}-2z=0$

tangent planes: $T_pP^e_{a,b}: \frac{x_p x}{a}+\frac{y_p y}{b}-z_p-z=0$

parametrization $\sigma(\theta_1,\theta_2)=\begin{bmatrix} \sqrt{a\theta_2}\cos\theta_1 \\ \sqrt{b\theta_2}\sin\theta_1 \\ \frac{\theta_2}{2} \end{bmatrix}$

- hyperbolic paraboloid $P^h_{a,b} : \frac{x^2}{a} - \frac{y^2}{b} - 2z = 0$

tangent planes: $T_p P^h_{a,b} : \frac{x_p x}{a} - \frac{y_p y}{b} - z_p - z = 0$

parametrizations: $r_2(\theta_1, \theta_2) = \begin{bmatrix} \sqrt{a}\,\theta_1 \\ \sqrt{b}\,\theta_2 \\ \frac{1}{2}(\theta_1^2 - \theta_2^2) \end{bmatrix}$