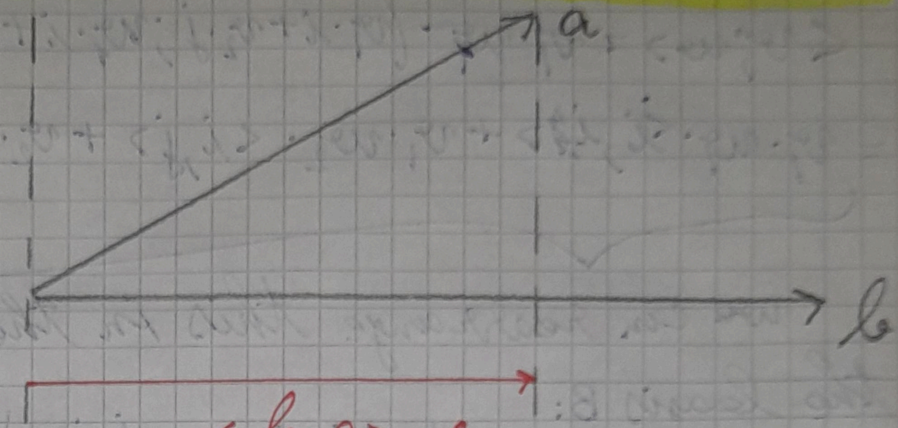
Geometry chapter 4

Scalar product in E^2

$$\langle a ; b \rangle = |a| \cdot |b| \cdot \cos \sphericalangle (a ; b)$$

$$d(A,B) = |\vec{AB}| = \sqrt{\langle \vec{AB}, \vec{AB} \rangle}$$

Projections in E^2



$$\frac{\langle b, a \rangle}{\langle b, b \rangle} \cdot b$$

Properties:

a) bilinear:

- $\forall a, b \in \mathbb{R}, \forall v, w, u \in V^2$ we have:

$$\langle a \cdot v + b \cdot w, u \rangle = a \cdot \langle v, u \rangle + b \langle w, u \rangle \quad \text{and}$$

$$\langle w, a \cdot v + b \cdot u \rangle = a \langle w, v \rangle + b \cdot \langle w, u \rangle$$

b) symmetric:

- $\forall v, w \in V^2$: $\quad \langle v, w \rangle = \langle w, v \rangle$

c) positive definite:

if $v \neq 0 \Rightarrow \langle v, v \rangle > 0$

- let $B(i,j)$ a basis of V^2
- $\cancel{\forall}\ v = v_1 \cdot i + v_2 \cdot j$
- $w = w_1 \cdot i + w_2 \cdot j$
- then by bilinearity we have:

$$\langle v, w \rangle = \cancel{v_1 \cdot w_1} (v_1 \cdot i + v_2 \cdot j ; w_1 \cdot i + w_2 \cdot j) =$$

$$= \underbrace{v_1 \cdot w_1 \cdot \langle i, i \rangle + v_1 \cdot w_2 \cdot \langle i, j \rangle + v_2 \cdot w_1 \cdot \langle j, i \rangle + v_2 \cdot w_2 \cdot \langle j, j \rangle}$$

we can rearrange this in the <u>Gram matrix</u> of the basis B:

$$G_B = \begin{vmatrix} \langle i, i \rangle & \langle i, j \rangle \\ \langle j, i \rangle & \langle j, j \rangle \end{vmatrix}$$

- we will then have:

$$\langle v, w \rangle = [v]_B^{T} \cdot G_B \cdot [w]_B = \cancel{[v_1 \ v_2] \begin{matrix} \langle i, i \rangle \\ \langle j, i \rangle \end{matrix}}$$

$$= [v_1 \ v_2] \cdot \begin{bmatrix} \langle i, i \rangle & \langle i, j \rangle \\ \langle j, i \rangle & \langle j, j \rangle \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

- if we choose $i \perp j \Rightarrow G_B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and

$$\langle v, w \rangle = v_1 \cdot w_1 + v_2 \cdot w_2$$

E^n

$<v, w> = v_1 \cdot w_1 + v_2 \cdot w_2 + \ldots + v_n \cdot w_n$ → for orthonormal coord. system

$B = (e_1 \ldots e_n)$ a basis of V^n

$$<v, w> = [v]_B^T \cdot G_B \cdot [w]_B = [v_1 \ldots v_n] \cdot G_B \cdot \begin{bmatrix} w_1 \\ \vdots \\ v_n \end{bmatrix}$$

where $G_B = (g_{ij})$ is defined by $g_{ij} = <e_i, e_j>$

Orthonormal coord. system

- v is a unit vector if $|v| = 1$
- to normalize w: replace with $\frac{w}{|w|}$

$$\measuredangle(a; b) = \arccos\left(\frac{<a, b>}{|a| \cdot |b|}\right) \in [0; \pi)$$

$$|a| = \sqrt{a_1^2 + a_2^2 + \ldots + a_n^2}$$

$$\cos \measuredangle(a, b) = \frac{a_1 \cdot b_1 + a_2 \cdot b_2 + \ldots + a_n \cdot b_n}{\sqrt{a_1^2 + a_2^2 + \ldots + a_n^2} \cdot \sqrt{b_1^2 + b_2^2 + \ldots + b_n^2}}$$

$$a \perp b \Leftrightarrow \cancel{a_1 \cdot b_n +} \; a_1 \cdot b_1 + a_2 \cdot b_2 + \ldots + a_n \cdot b_n$$

$$d(P, Q) = |\overrightarrow{PQ}| = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2 + \ldots + (p_n - q_n)^2}$$

- a basis B of V^n is orthonormal $\Leftrightarrow G_B = I_n$

- $A \in M_n^2$ is orthonormal $\Leftrightarrow A^T \cdot A = I_n$

Gram-Schmidt process

- let $B(e_1 \ldots e_n)$ be a basis of V^3
- we want to build B' an orthonormal basis starting from B
- recall that the projection of e_i on $e_j = \frac{\langle e_j ; e_i \rangle}{\langle e_j ; e_j \rangle} \cdot e_j$
- we have two steps:

I. construct an orthogonal basis $e_1', e_2' \ldots e_n'$:

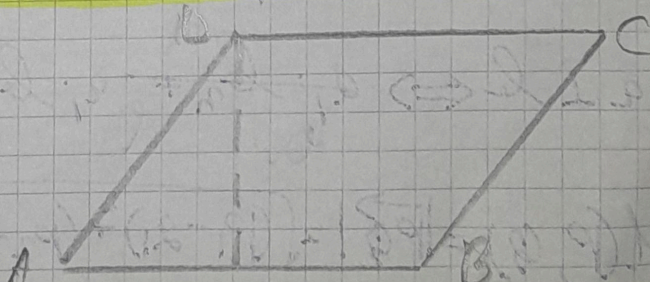
- $e_1' = e_1$
- $e_2' = e_2 - \frac{\langle e_1', e_2 \rangle}{\langle e_1', e_1' \rangle} \cdot e_1'$
- $e_3' = e_3 - \frac{\langle e_1', e_3 \rangle}{\langle e_1', e_1' \rangle} \cdot e_1' - \frac{\langle e_2', e_3 \rangle}{\langle e_2', e_2' \rangle} \cdot e_2'$

$\vdots$

II. normalize:

$$B' = \left\{ \frac{e_1'}{|e_1'|}, \frac{e_2'}{|e_2'|}, \ldots \frac{e_n'}{|e_n'|} \right\}$$

Area of a parallelogram



$v = \overrightarrow{AD}$

$w = \overrightarrow{HD}$

$u = \overrightarrow{AB}$

$$A(ABCD) = \sqrt{|u|^2 \cdot |v|^2 - \langle u, v \rangle^2}$$

Normal vectors of hyperlanes

- recall that, with respect to $K=(O,B)$, a hyperlane H is given by:

$$H: a_1 \cdot x_1 + a_2 \cdot x_2 + \ldots + a_n \cdot x_n = b$$

- assume K is orthonormal; fix a point $Q(q_1, q_2 \ldots q_n) \in H$
- ~~since~~ $Q \in H \Rightarrow a_1 \cdot q_1 + a_2 \cdot q_2 + \ldots + a_n \cdot q_n = b$
- hence $P(p_1 \ldots p_n) \in H \Leftrightarrow a_1(q_1 - p_1) + \ldots + a_n(q_n - p_n) = 0$
- if we denote by n the vector $(a_1 \ldots a_n)$ then

$P \in H \Leftrightarrow n \perp \overrightarrow{PQ}$

- we say n is orthogonal to H

- a vector v is called a <u>normal vector</u> of H if it is orthogonal to H

- let H be a hyperlane
- let $v_1 \ldots v_{n-1} \in D(H)$ be l. i. vectors
- any normal vector of H is a scalar multiple of $v_1 \wedge_B v_2 \wedge_B \ldots \wedge_B v_{n-1}$ for any orthonormal basis B of V^n

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- let S, H be parallel affine subspaces of E^n

$$d(S,H) = d(A,H), \ \forall A \in S$$

Distance from point to hyperplane

$H: a_1 \cdot x_1 + \ldots + a_n \cdot x_n = b$, $Q \in H$

$P \to$ a point $\notin H$

choose $h \in H$ s. t. $Ph \perp H$

~~$d(P, H) = |PH|$~~ $d(P,H) = d(P,h)$

- consider the normal vector $n(a_1, a_2 \ldots a_n)$
- then $|Ph|$ is the length of the projection of $\overrightarrow{QP}$ on n

~~- let N be s. t. $n = \overrightarrow{QN}$~~

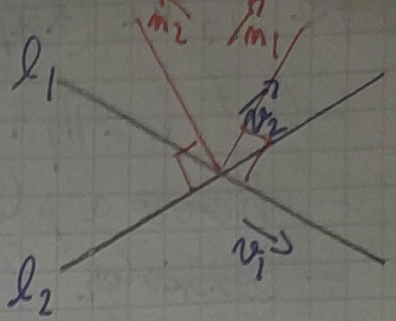
$$d(P,H) = \left| \frac{\langle n, \overrightarrow{QP} \rangle}{\langle n, n \rangle} \cdot n \right| = \frac{|\langle n, \overrightarrow{QP} \rangle|}{|n|} = \frac{|\langle n, \overrightarrow{OP} - \overrightarrow{OQ} \rangle|}{|n|} =$$

$$= \frac{|\langle n, \overrightarrow{OP} \rangle - b|}{|n|}$$

- in coords. we obtain

$$d(P,H) = \frac{|a_1 \cdot p_1 + \ldots + a_n \cdot p_n + b|}{\sqrt{a_1^2 + a_2^2 + \ldots + a_n^2}}$$

Angles between lines and hyperplanes



$$\cos \measuredangle (v_1, v_2) = \frac{\langle v_1, v_2 \rangle}{|v_1| \cdot |v_2|}$$

$$\cos \measuredangle (n_1, n_2) = \frac{\langle n_1, n_2 \rangle}{|n_1| \cdot |n_2|}$$

- if we know v_1 a dir. vector for l_1 and n_2 a normal vector for l_2 then

$$\measuredangle (l_1, l_2) = \frac{\pi}{2} - \arccos\left(\left|\frac{\langle v_1, n_2 \rangle}{|v_1| \cdot |n_2|}\right|\right) \in \left[0; \frac{\pi}{2}\right)$$

Generalization

- consider in E^n two lines l_1 and l_2 with dir. vectors v_1 and v_2, and two hyperplanes H_1 and H_2 with normals vectors n_1 and n_2

- $\cos \measuredangle (v_1, v_2) = \dfrac{\langle v_1, v_2 \rangle}{|v_1| \cdot |v_2|}$

- $\cos \measuredangle (n_1, n_2) = \dfrac{\langle n_1, n_2 \rangle}{|n_1| \cdot |n_2|}$

- ~~l_1 and H_1 define two suppl.~~

  if $\cos \measuredangle (v_1, n_1) \geq 0 \Rightarrow \measuredangle (v_1, n_1)$ is acute and

  $$\measuredangle (l_1, H_1) = \frac{\pi}{2} - \arccos\left(\frac{\langle v_1, n_1 \rangle}{|v_1| \cdot |n_1|}\right)$$

- if $\cos \measuredangle (v_1, n_1) < 0$ replace n_1 with $-n_1$

- for Π_1, Π_2 (hyper)planes we have dihedral angle:
  - let $l = \Pi_1 \cap \Pi_2$
  - choose a plane $\Pi \perp l$
  - consider $l_1 = \Pi \cap \Pi_1$, $l_2 = \Pi \cap \Pi_2$
  - calculate $\sphericalangle(l_1, l_2)$ $\rightarrow$ see page 56 of the course

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Spectral theorem

- B a basis of V^n
- $n \times n$ matrix M
- we then have the linear map

$$\phi : V^n \to V^n \text{ defined by } \phi(v) = M \cdot [v]_B$$

- and the bilinear map

$$\psi : V^n \times V^n \to \mathbb{R} \text{ defined by } \cancel{\psi(v,w)} =$$

$$\psi(v, w) = [v]_B^T \cdot M \cdot [w]_B$$

- given a l.m. $\phi : V^n \to V^n$, it has an assoc. matrix $M_{B,B}(\phi)$
- given a bl. m. $\psi : V^n \times V^n \to \mathbb{R}$ we assoc. to it the Gram matrix $G_B(\psi)$
- let B' be another basis of V^n
  - $M_{B,B'}$ is the change matrix from B' to B

$$M_{B',B'}(\phi) = M_{B,B'}^{-1} \cdot M_{B,B}(\phi) \cdot M_{B,B'}$$

$$G_{B'}(\phi) = M_{B,B'}^{T} \cdot G_B(\psi) \cdot M_{B,B'}$$

- basis B
- a bilinear map: $\psi : V^n \times V^n \to \mathbb{R}$ is called symmetric if the Gram matrix $G_B(\psi)$ is symmetric

- consider $A \in M_n^2$ s. t. $A^T = A$
- let ϕ be the linear map assoc. ~~to~~ to A in B and let ψ be the bilinear map assoc. to M in B

$$\psi(v, w) = [v]_B^T \cdot A \cdot [w]_B = \langle v, M \cdot [w]_B \rangle = \langle v, \phi(w) \rangle$$

- since A is symmetric, we also have

$$[v]_B^T \cdot A \cdot [w]_B = (A^T \cdot [v]_B)^T \cdot [w]_B = (A \cdot [v]_B)^T \cdot [w]_B$$

therefore

$$\langle \phi(v), w \rangle = \psi(v, w) = \langle v, \phi(w) \rangle$$

Orthogonal complement

- let $v \in V^n$
- the orthogonal compl. of v denoted by $v^\perp$, is the set of all vectors in V^n $\perp$ with v

$$v^\perp = \{ w \in V^n \mid \langle v, w \rangle = 0 \}$$

$$\mathrm{Dim}(v^\perp) = n - 1$$

- the characteristic polynomial of a symmetric matrix $A \in M_n^2(\mathbb{R})$ has only real roots

Spectral theorem

- let $\phi : V^n \to V^3$ be a symmetric linear map; then $\exists$ an orthonormal basis B' s.t. $M_{B',B'}(\phi)$ is a diagonal matrix

- let M be a symmetric matrix
- let $\varphi : V^n \times V^n \to \mathbb{R}$ be the bilinear map assoc. with M relative to the orthonormal basis B

- then $\exists$ an orthonormal basis B' s.t.

$$G_{B'}(\varphi) = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

where $\lambda_1 \ldots \lambda_n$ are the eigenvalues of M