4. Orientation in E^n

- consider two bases $\mathcal{E}$ and F of V^n; then $\det(M_{\mathcal{E},F}) \neq 0$
  - if $\det(M_{\mathcal{E},F}) > 0 \Rightarrow F$ has the same orientation as $\mathcal{E}$
  - if $\det(M_{\mathcal{E},F}) < 0 \Rightarrow F$ and $\mathcal{E}$ have opposite orientations

Def: E^n is called <u>oriented</u> if $\exists$ a choice of a coord. system $K = (O, B)$ which is called <u>right oriented</u>; all other bases of V^n with the same orientation as B are also <u>right oriented</u>, while all other bases with opposite orientation are called <u>left oriented</u>

Box product in E^n

- let $v_1 \dots v_n$ be vectors in V^n with components $v_i(v_{i,1}, \dots v_{i,n})$ relative to a right oriented orthonormal basis B of V^n
- the (n-fold) box product of these vectors is

$$[v_1 \dots v_n] = \begin{vmatrix} v_{1,1} & v_{2,1} & \dots & v_{n,1} \\ v_{1,2} & v_{2,2} & \dots & v_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ v_{1,n} & v_{2,n} & \dots & v_{n,n} \end{vmatrix}$$

Def: B basis of V^3; $Vol_{or}(B)$ (oriented volume of B) is the box product of the vectors of B; the volume of B is $|Vol_{or}(B)|$ and we denote it by $Vol(B)$; in V^2 we refer to it as (oriented) area of B ($Area_{or}(B)$)

Properties

- E^3: $[a, b, c] = [b, c, a] = [c, a, b] = -[b, a, c] = -[a, c, b] = -[c, b, a]$

- ~~Obs: v_1, v_2, v_3 coplanar if~~

  Obs: $v_1 \dots v_n$ l.i. if $\begin{vmatrix} v_{1,1} & v_{2,1} & \dots & v_{n,1} \\ v_{1,2} & v_{2,2} & \dots & v_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ v_{1,n} & v_{2,n} & \dots & v_{n,n} \end{vmatrix} \neq 0$

- the orientation of the basis (a, b, c) is determined as follows:

$$[a\ b\ c] \begin{cases} > 0 & \text{then } (a, b, c) \text{ is right oriented;} \\ = 0 & \text{then } (a, b, c) \text{ is not a basis} \\ < 0 & \text{then } (a, b, c) \text{ is left oriented} \end{cases}$$

J - operator in E^2

- for $v \in V^2$ we define $J(v)$ to be the (unique) vector in V^2 ~~saty~~ satisfying the following properties:

  - $J(v) \perp v$
  - $|J(v)| = |v|$
  - $(v, J(v))$ is a right oriented basis of V^2

- the J-operator is given by $J(v) = -\Lambda_B(v)$, where B is ~~an~~ any right oriented ~~orthor~~ orthonormal basis of V^2; $J : V^2 \to V^2$ is a linear map

Cross product (vector product) in E^3

- let $a, b \in V^3$, the cross product of a and b is denoted by $a \times b$ and given the following properties

1. if $a \parallel b$ then $a \times b = 0$
2. if $a \nparallel b$ then:
   - $|a \times b| = |a| \cdot |b| \cdot \sin \sphericalangle(a, b)$
   - $a \times b \perp a$ and $a \times b \perp b$
   - $(a, b, a \times b)$ is a right oriented basis of V^3

$a \times b = a \wedge_B b$, where B is any right oriented orthonormal basis of V^3

Properties

- bilinear: $\forall a, b \in \mathbb{R}$ and $\forall v, w, u \in V^2$ we have:

$$(a \cdot v + b \cdot w) \times u = a (v \times u) + b \cdot (w \times u)$$

$$v \times (a \cdot w + b \cdot u) = a \cdot (v \times w) + b \cdot (v \times u)$$

- skew-symmetric: $\forall v, w \in V^2$ we have:

$$v \times w = - w \times v$$

$$\underbrace{\langle a \times b, c \rangle}_{\text{the mixed product of } a, b, c} = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = [a, b, c]$$

Non assoc.
$\times$ - let (i, j, k) be a right oriented orthonormal basis; then:

$$(i \times j) \times j = k \times j = -i$$

$$i \times (j \times j) = 0$$

Jacobi identity

$$(a \times b) \times c + (b \times c) \times a + (c \times a) \times b = 0, \quad \forall a, b, c \in V^3$$

Grassman cross product

$$(a \times b) \times c = \langle a, c \rangle \cdot b - \langle b, c \rangle \cdot a, \quad \forall a, b, c \in V^3$$

Lagrange's identity

$\langle a \times b, c \times d \rangle = \langle a, c \rangle \cdot \langle b, d \rangle - \langle b, c \rangle \cdot \langle a, d \rangle =$

$= \begin{vmatrix} \langle a, c \rangle & \langle a, d \rangle \\ \langle b, c \rangle & \langle b, d \rangle \end{vmatrix}, \forall a, b, c, d \in V^3$

Triple cross product

$(a \times b) \times (c \times d) = b \cdot [a, c, d] - a \cdot [b, c, d] =$

$= c \cdot [a, b, d] - d \cdot [a, b, c], \forall a, b, c, d \in V^3$

* (n-1)-fold wedge product in E^3

- for each of the involved vectors v_i, the orthogonal complement $v_i^{\perp}$ is an $(n-1)$ dimensional vector ~~space~~ subspace of V^n; ~~ie~~ if $v_1 \dots v_{n-1}$ are l. i. then

$W = \bigcap_{i=1}^{n-1} v_i^{\perp}$ is a 1 dimensional vector subspace of V^n

- the $(n-1)$-fold cross product is the $(n-1)$-fold wedge product with respect to a right oriented orthonormal basis
- if B is a right oriented orthonormal basis, then

$$\langle v_1 \wedge_B v_2 \wedge_B \dots \wedge_B v_{n-1}, v_n \rangle = [v_n, v_1 \dots v_{n-1}]$$

Area of parallelogram

- $B = (a, b)$ a basis of V^2
- $P \to$ parallelogram spanned by a and b

$$Area_{or}(P) = Area_{or}(B) = \det(M_{\mathcal{E},B}) = [a; b]$$

- consider $a(a_1, a_2, a_3)$ and $b(b_1, b_2, b_3)$ in V^3
- $P \to$ parallelogram in E^3 spanned by a and b

$$Area(P) = |a \times b| = \sqrt{\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}^2}$$

Area of triangle

ΔABC s.t. $a = \vec{CA}$ and $b = \vec{CB}$, $A_{ABC} = \frac{1}{2}|a \times b|$

- A, B, C ~~line~~ lie in the O_{xy} plane, (i.e. $A(x_A, y_A, 0)$, $B(x_B, y_B, 0)$, $C(x_C, y_C, 0)$) then

$$a \times b = \begin{vmatrix} i & j & k \\ x_B - x_A & y_B - y_A & 0 \\ x_C - x_A & y_C - y_A & 0 \end{vmatrix} = k \cdot \begin{vmatrix} x_B - x_A & y_B - y_A \\ x_C - x_A & y_C - y_A \end{vmatrix}$$

oriented angle

- let a, b be two vectors in E^2
- the oriented angle of a and b is

$$\measuredangle_{or}(a, b) = \begin{cases} \measuredangle(a, b) & \text{if } [a; b] \geq 0 \\ -\measuredangle(a; b) & \text{if } [a; b] < 0 \end{cases}$$

Obs: $[a; b] > 0 \iff (a; b)$ is a right oriented basis of V^2

$\Longrightarrow$ • $-\pi < \measuredangle_{or}(a; b) \leq \pi$

• $0 < \measuredangle_{or}(a; b) < \pi \iff (a; b)$ is a right oriented basis

• $\measuredangle_{or}(a, J(a)) = \frac{\pi}{2}$

• $\measuredangle_{or}(b, a) = -\measuredangle_{or}(a, b)$

- if a and b are two non-zero vectors in E^2, then

$$\cos(\measuredangle_{or}(a, b)) = \frac{\langle a, b \rangle}{|a| \cdot |b|}$$

$$\sin(\measuredangle_{or}(a, b)) = \frac{[a; b]}{|a| \cdot |b|}$$

- if a is a non-zero vector in E^2, and if $c^2 + s^2 = 1 \Rightarrow \exists$ a vector b s.t.

$$\cos(\measuredangle_{or}(a, b)) = c \quad \text{and} \quad \sin(\measuredangle_{or}(a, b)) = s$$

- if a, b, c are three non-zero vectors in E^2, then

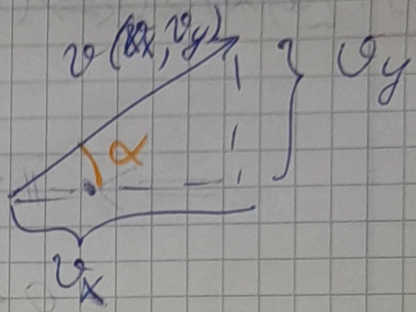
$$\measuredangle_{or}(a,b) = \measuredangle_{or}(a,c) + \measuredangle_{or}(c,b) \mod 2\pi$$

Angular coefficient of a line in E^2

$$\frac{x - x_A}{v_x} = \frac{y - y_A}{v_y}$$

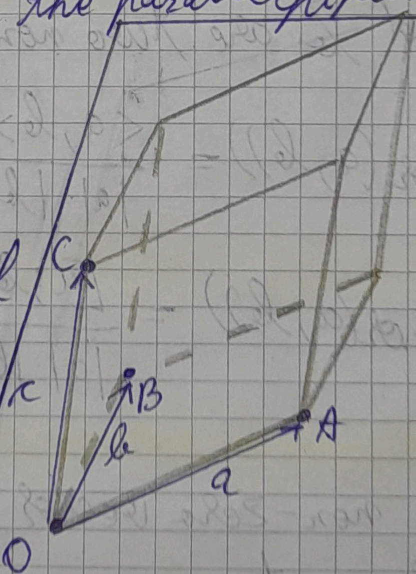
can be rearranged as

$$y - y_A = tg(\alpha) \cdot (x - x_A)$$



Volume of an n-simplex

- E^3: a 3-simplex is a tetrahedron
- let $a = \vec{OA}$, $b = \vec{OB}$, $c = \vec{OC}$ be non-collinear vectors in V^3 and let P be the parallelpiped spanned by a, b, c



- $[a, b, c]$ is the volume of P if (a, b, c) is right oriented, it is minus the volume of P if (a, b, c) is left oriented

...ence $[a, b, c] = \langle a \times b, c \rangle$ we have that

$$\frac{[a, b, c]}{|a \times b|^2} (a \times b) = \underbrace{\frac{\langle a \times b, c \rangle}{|a \times b|^2} \cdot (a \times b)}_{\text{the proj. of } c \text{ on } a \times b}$$

$$\Rightarrow \left| \frac{\langle a \times b, c \rangle}{|a \times b|^2} (a \times b) \right| = \left| \frac{[a, b, c]}{|a \times b|} \right|$$ is the height of P

$$\Longrightarrow Vol(P) = |[a, b, c]|$$

- we can generalize to n dimensions:

- let P be a ~~hyper~~ hyperparallelipiped spanned by the vectors $v_1, \dots v_n$
- with respect to an orthonormal basis ~~B~~ B, let $w = v_1 \wedge_B v_2 \wedge_B \dots \wedge_B v_{n-1}$
- consider the proj. of v_n on w

$$\frac{[v_1, v_2, \dots v_n]}{|w|^2} \cdot w = \underbrace{\frac{\langle w, v_n \rangle}{|w|^2} \cdot w}_{\text{height of } P}$$

$$\Rightarrow Vol(P) = \left| \frac{[v_1, v_2 \dots v_n]}{|w|} \right| \cdot Vol(F)$$

where F is the hyperparallelipiped spanned by $v_1, v_2 \dots v_{n-1}$

- to calculate $|w|$ we may choose an orthonormal basis $\mathcal{E}(e_1 \dots$
s.t. e_n orthogonal to $v_1 \dots v_{n-1}$; then

$$w = v_1 \wedge_B v_2 \wedge_B \dots \wedge_B v_{n-1} = \pm v_1 \wedge_{\mathcal{E}} v_2 \wedge_{\mathcal{E}} \dots \wedge_{\mathcal{E}} v_{n-1} =$$

$$= \begin{vmatrix} e_1 & e_2 & \dots & e_n \\ \leftarrow & v_1' & \rightarrow & 0 \\ \leftarrow & v_2' & \rightarrow & 0 \\ & \vdots & & \vdots \\ \leftarrow & v_n' & \rightarrow & 0 \end{vmatrix} = \pm \begin{vmatrix} \leftarrow v_1' \rightarrow \\ \vdots \\ \leftarrow v_n' \rightarrow \end{vmatrix} e_n$$

where v_i' is the row matrix obtained from v_i with the last component (zero) removed

$\Rightarrow$ ~~for~~ $|w| = |[v_1', v_2' \dots v_{n-1}']| = Vol(F)$ of the hyperparallelipiped F of dimension $n-1$

- the hyperparallelipiped P can be divided into $n!$ simplices which have the same volume:

$$Vol(S) = \frac{1}{n!} \cdot Vol(P)$$

- the n-simplex spanned by $v_1 \dots v_n$ at the origin is the set

$$S = \{ B \in E^n \,/\, \vec{OB} = t_1 \cdot v_1 + \dots + t_n \cdot v_n, \text{ with } t_1 \dots t_n \geq 0$$
$$\text{and } t_1 + \dots + t_n \leq 1 \}$$

~~istance between lines in E^3~~

~~I. i~~

- consider two lines l_1, l_2 in E^3

I. if $l_1 \cap l_2 \neq \emptyset \Rightarrow d(l_1, l_2) = 0$

II. if l_1 and l_2 are skew relative $\Rightarrow \exists!\, \Pi_1$ containing l_1 s.t. $\Pi_1 \parallel l_2$ ~~and~~ and $\exists!\, \Pi_2$ containing l_2 s.t. $\Pi_2 \parallel l_1$

$$\Rightarrow d(l_1, l_2) = d(\Pi_1, P_2) = d(P_1, \Pi_2),\ \forall P_1 \in l_1, \forall P_2 \in l_2$$

- more concretely:

$$l_1: \begin{cases} x = x_1 + t \cdot v_x \\ y = y_1 + t \cdot v_y \\ z = z_1 + t \cdot v_z \end{cases} \quad \text{and} \quad l_2: \begin{cases} x = x_2 + t \cdot u_x \\ y = y_2 + t \cdot u_y \\ z = z_2 + t \cdot u_z \end{cases}$$

then Π_1: $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ v_x & v_y & v_z \\ u_x & u_y & u_z \end{vmatrix} = 0$

- if we let $a(a_x, a_y, a_z) = v \times u$, we obtain:

$$d(l_1, l_2) = d(\Pi_1, P_2) = \frac{|a_x(x_2 - x_1) + a_y(y_2 - y_1) + a_z(z_2 - z_1)|}{\sqrt{a_x^2 + a_y^2 + a_z^2}}$$

Ex - suppose you have a line

$$l_1 : \begin{cases} x = x_A + t \cdot v_x \\ y = y_A + t \cdot v_y \\ z = z_a + t \cdot v_z \end{cases}$$ and a point $P = (x_P, y_P, z_P) \in E^3$

that $d(P, l) = \dfrac{|\vec{PA} \times v|}{|v|}$

Common perpendicular line of two skew lines in E^3

- two skew lines l_1 and l_2 in E^3

$$l_1 : \begin{cases} x = x_1 + t \cdot v_x \\ y = y_1 + t \cdot v_y \\ z = z_1 + t \cdot v_z \end{cases} \qquad l_2 : \begin{cases} x = x_2 + t \cdot u_x \\ y = y_2 + t \cdot u_y \\ z = z_2 + t \cdot u_z \end{cases}$$

- take $a = v \times u$, a non-zero (l_1 and l_2 are skew) and $\perp$ to v and u

$$\cancel{l : \Pi_1' : \begin{vmatrix} x - x_1 & y - y_1 \\ v_x & v_y \\ a_x & a_y \end{vmatrix} = 0}$$

$$\cancel{\Pi_2' : x_2 - x_2}$$

$$l : \begin{cases} \Pi_1' : \begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ v_x & v_y & v_z \\ a_x & a_y & a_z \end{vmatrix} = 0 \\ \Pi_2' : \begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ u_x & u_y & u_z \\ a_x & a_y & a_z \end{vmatrix} = 0 \end{cases}$$

common perpendicular of l_1 and l_2 : $l \perp l_1$ and $l \perp l_2$

$l \cap l_1 \neq \emptyset$ and $l \cap l_2 \neq \emptyset$