Graphs Theory

Walk: a set of edges s.t. $\forall e_i \in E$, the terminal vertex of e_i is the initial vertex of e_{i+1}

Trail: walk with no duplicate edges

Path: walk with no duplicate vertices

Cycle path where the initial and terminal vertex of the path are the same

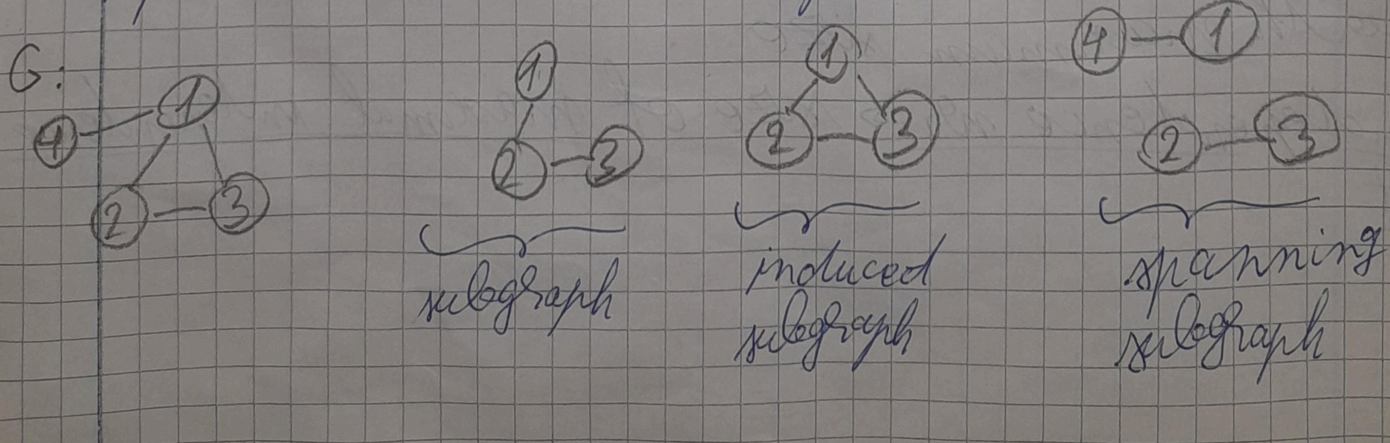
Circuit: ~~walk~~ trail where –11–

Subgraph

$G = (V, E)$

$G' = (V', E')$ is a subgraph of G if $V' \subseteq V$ and $E' \subseteq E$

- if E' contains all edges from E which have vertices of V' as endpoints, then G' is an induced subgraph
- if $V' = V \Longrightarrow G'$ is a spanning subgraph



Eccentricity

- the ecc. of a vertex is the max. dist. it can have between any other node
- if ∄ a path that connects two nodes, the dist. between them is ∞

Radius

- the radius of a graph is the smallest eccentricity

Diameter

- the greatest eccentricity

Center

- the set of vertices with minimum eccentricity

Independent set

- for $G=(V,E)$, $S \subseteq V$ is an independent set if $\forall i,j \in S$, $(i,j) \notin E$.
- maximal independent set → independent set of G with maximum size
- independence nr. → size of maximal independence set

Dominating set

- $S \subseteq V$ is a dominating set if $\forall i \in V$, $i \in S$ or $\exists v \in S$ s.t. $(i, v) \in E$
- dominating nr $\Rightarrow$ size of the smallest dominating set

Tree

- connected
- no cycles
- $m = n - 1$
- each node has one parent
- removing an edge $\Rightarrow$ undirected graph

- a rooted tree is a tree with a root

Forest

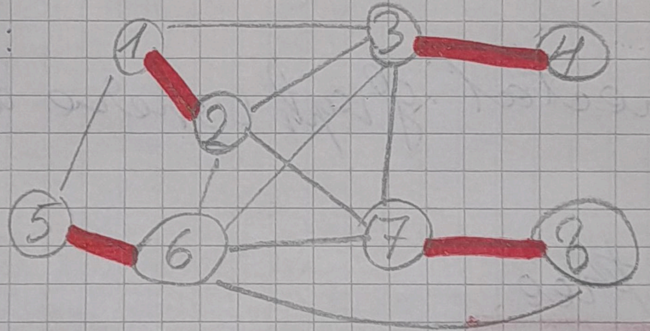
- disconnected graph where all components are trees

Spanning tree:

- spanning subgraph that is a tree
- minimum spanning ~~tree~~ subtree: spanning tree for which the cost of the path between any two vertices is minimal

Matchings

- $G(V,E)$ undirected graph
- $M \subset E$ is a matching if ~~$\forall$ 2 edges~~ $\forall e_1, e_2 \in M$, e_1 and e_2 do not share incident vertices
- if ~~and~~ a vertex is incident to an edge from M, it is saturated by M
- a perfect matching saturates all vertices of a graph
- an unweighted maximum matching is the biggest matching possible
- perfect matching:



Alternating path

- a path whose edges alternate between edges from M and edges from E/M

Augmenting path

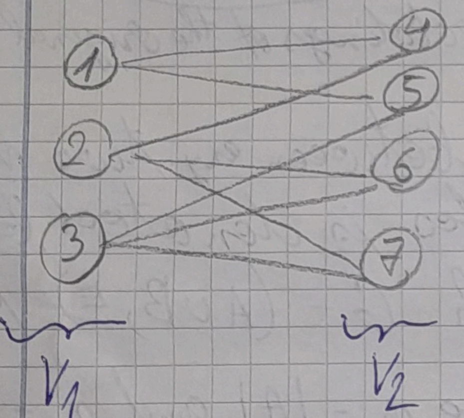
- ~~an altern~~ an alternating path where the first and last edge are not from M, and the start and end vertices are not saturated

Berge's theorem

- M is a maximum matching $\iff \nexists$ an augmenting path w.r.t. M

Bipartite graphs

- $G(V,E)$ is bipartite $\iff \exists\, V_1, V_2$ s.t. $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \emptyset$, and $\forall (u,v) \in E$, $u \in V_1$ and $v \in V_2$ or viceversa

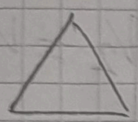
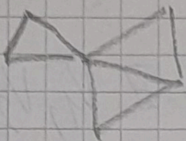


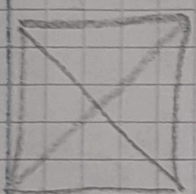
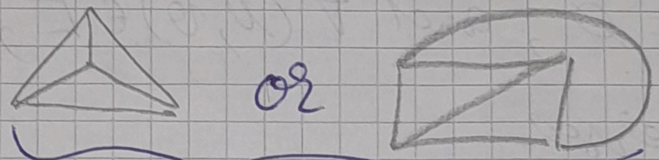
Eulerian graphs

- $G(V,E)$ connected graph
- $(l_1, l_2 .. l_k)$ trail, if $l_1, l_2 .. l_k$ are all the edges in $E_X \Rightarrow (l_1, l_2 .. l_k)$ is an Eulerian trail
- if the trail starts and ends in the same vertex $\Rightarrow$
$\Rightarrow$ Eulerian circuit
- a graph with an Eulerian circuit is an Eulerian graph
- a graph is Eulerian $\iff$ the degree of any vertex is even

Planar graphs

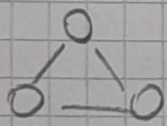
- only for undirected graphs
- a graph is **planar** $\iff$ ~~is~~ it can be drawn in a plane s.t. edges only intersect at endpoints (it can be **embedded in a plane**)

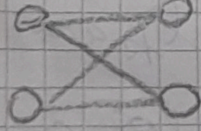



embeddings of the original graph

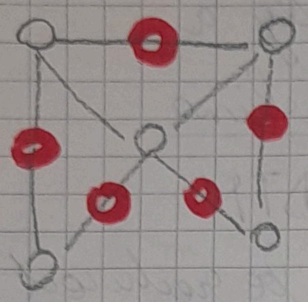
- a complete graph has an edge between any two nodes
- a complete graph with n nodes is denoted by K_n
- a complete bipartite graph $G = (A \cup B, E)$ is denoted by K_{n_1, n_2}, where $n_1 = |A|$ and $n_2 = |B|$

$K_3 \rightarrow$ 

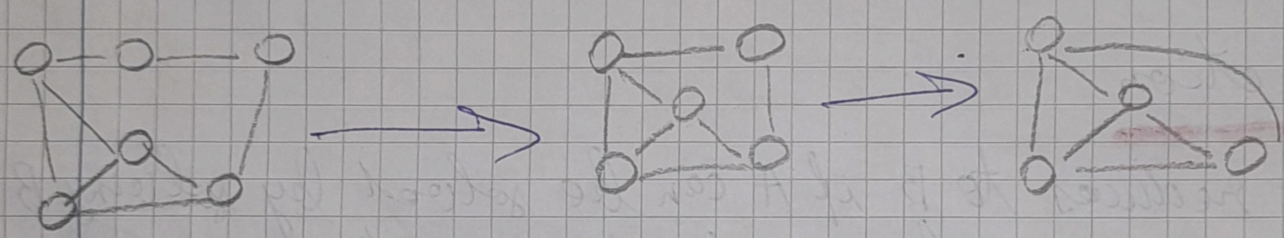
$K_{2,2} \rightarrow$ 

- ~~Obs~~ Theorem: $\left.\begin{matrix} K_5 \\ K_{3,3} \end{matrix}\right\}$ not planar graphs

- two graphs G_1 and G_2 are <u>homeomorphic</u> if we can obtain one from the other by adding or removing a vertex with degree 2



- <u>contracting</u>: transforming a graph into another <u>homeomorphic</u> graph by removing nodes



- a graph is planar $\Longleftrightarrow$ it contains no subgraphs which are homeomorphic to K_5 or $K_{3,3}$

<u>Planar graphs (cont.)</u>

- an embedding of a ~~the~~ planar graph partitions the plane into parts ~~(fa ban~~ bounded by the cycles of the graph called <u>faces</u>
- the <u>unbounded face</u> is called the <u>exterior face</u>

<u>Euler's formula</u>

- G a planar graph with n nodes and m edges
- E_G an embedding of G with f faces
- then $\boxed{n - m + f = 2}$

Complexity classes

- P → can be solved in polinomial time by a deterministic Turing machine (DTM)
- NP → solveable in polinomial time by a ~~non dete~~ non-deterministic Turing machine
  → verifiable in pol. time by a DTM
- NP-Hard → all problems in NP can be reduced to this
- NP-Complete → ~~p~~ NP-Hard problems which are in NP

Reduction

- A reduces to B if A can be solved by solving B:
  1) input of A can be turned into input of B in pol. time
  2) B can be solved and return an output with the given input
  3) output of B can be turned into output of A in pol. time
  4) the result is correct

Propositional satisfiability problem (SAT)

$$\underbrace{(a \vee b) \wedge (c \vee \neg d \wedge f) \wedge \dots}_{\text{NP-Complete}}$$

- any NP-problem can be reduced to SAT
- SAT is about logical expressions in CNF

3-SAT: SAT problem with $k = 3$

e.g. exp $= (x_1 \vee \bar{x}_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee x_3)$

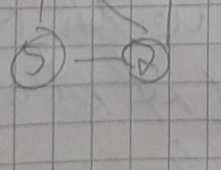
- any SAT problem can be reduced to a 3-SAT problem

Hamiltonian ~~&~~ & Cycle

- graph $G(V,E)$
- a path $L (a_1, a_2 \dots a_k)$ is a hamiltonian path if it contains all vertices from V
- if $a_1 = a_k \Rightarrow$ hamiltonian ~~cycl~~ cycle $\Longrightarrow$ G is ~~a graph with~~ a hamiltonian graph

Cliques

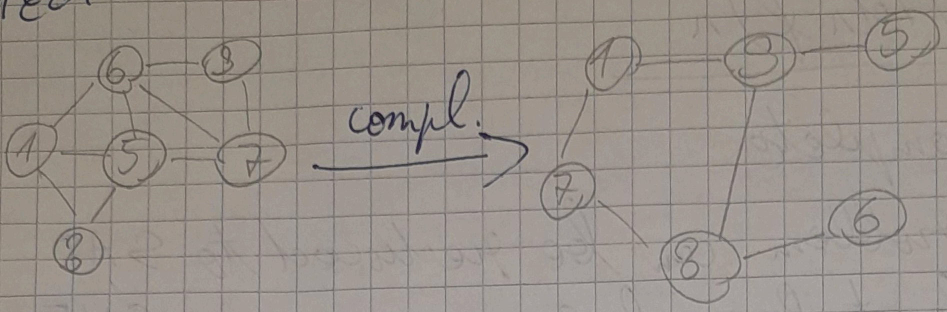
- C is a clique of graph G if C is a subgraph of G and C is complete



$\longrightarrow$ 6,5,7 is a clique

- a complement of G is the graph with the edges inverted

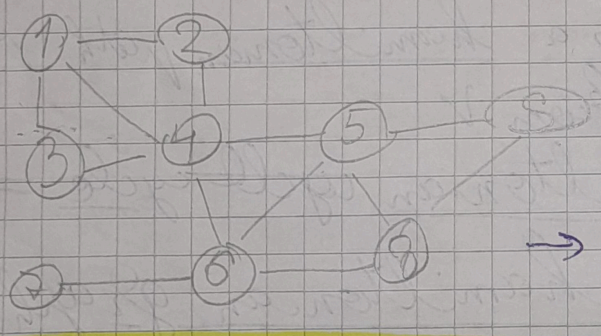
e.g.



- any clique in a graph is an independent set in its complement

Vertex cover

- a subset of vertices s.t. all edges are incident to one of them

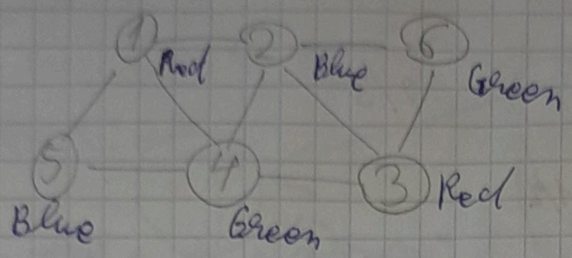


→ 4, 8, 7, 1, 5 is a vertex cover

Vertex coloring

- associate a color to each vertex s.t. no two ~~and~~ adjacent vertices ~~or~~ have the same color
- colors are usually natural nrs.
- a K-coloring is a coloring with K-colors
- a graph is K- ~~co~~ colored if it admits a K-coloring
- all vertices of the same color will form an independent set

- $\chi(G)$ is called the chromatic nr. of graph G and is the size of the smallest coloring that G admits
- $\chi(G) \geq$ the ~~s~~ size of the biggest clique



Network flows

- a dir. graph
- each edge has a capacity
- we have a source and a sink
- a flow network (transport network) $N = (G, s, t, C)$ has:
  - a dir graph G
  - $s \in V$ the source
  - $t \in V$ the sink
  - $C : V \times V \to \mathbb{R}^+$ the capacity:
    - if $(v_1, v_2) \in E \Rightarrow C(v_1, v_2) > 0$

      else $C(v_1, v_2) = 0$

Flow

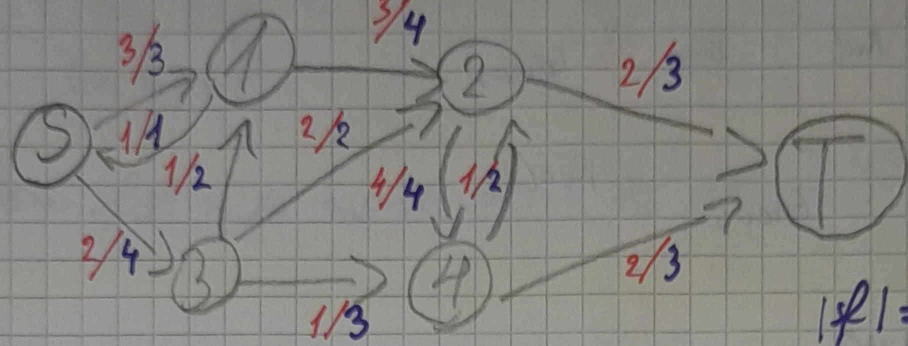
$f : V \times V \to \mathbb{R}^+$

- capacity rule: $f(v_1, v_2) \leq C(v_1, v_2)$
- flow conservation rule: only the source can create flow and only the sink can destroy flow

Total flow

not. $|f| = \sum_{v \in V} f(S, v) - \sum_{v \in V} f(v, S) = \sum_{v \in V} f(v, T) - \sum_{v \in V} f(T, v)$

e.g.



$|f| = 4$

Residual network

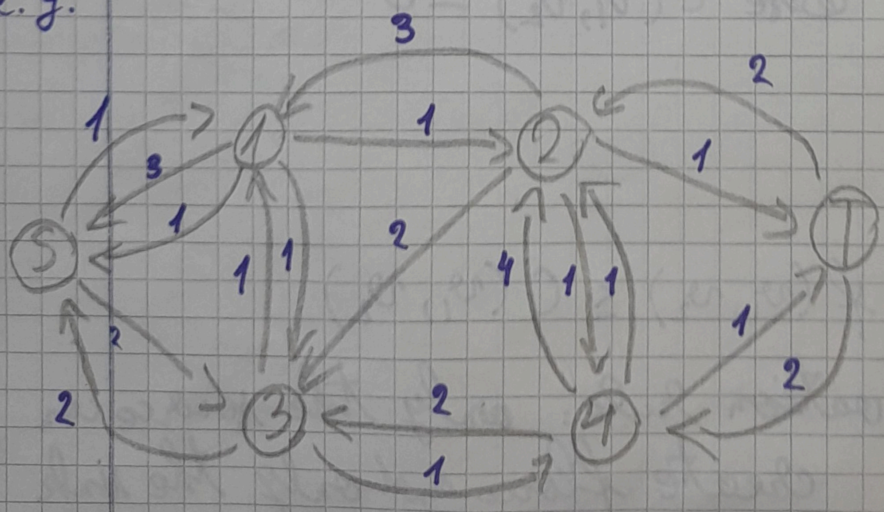
- $N = (G = (V, E), S, T, c)$ a flow network
- $N_f = (G_f = (V, E_f), S, T, c_f)$ is the residual network

w. r. t. flow f of N if:

~~f~~(v_1, v_2) is a forward edge: 1) if $f(v_1, v_2) < c(v_1, v_2)$ then $(v_1, v_2) \in E_f$ and $c_f(v_1, v_2) = c(v_1, v_2) - f(v_1, v_2)$

(v_2, v_1) is a backwards edge: 2) if $f(v_1, v_2) > 0$ then $(v_2, v_1) \in E_f$ and ~~$c_f(v_2, v_1) = f(v_1, v_2)$~~ $c_f(v_2, v_1) = f(v_1, v_2)$

e.g.



Augmenting path

- an AP in a residual network is any path between S and T

Residual capacity

$$c_f(P) = \min_{(v_1, v_2) \in P} \{ c_f(v_1, v_2) \}$$

- a flow for n is max. $\Leftrightarrow \nexists$ an AP in N_f

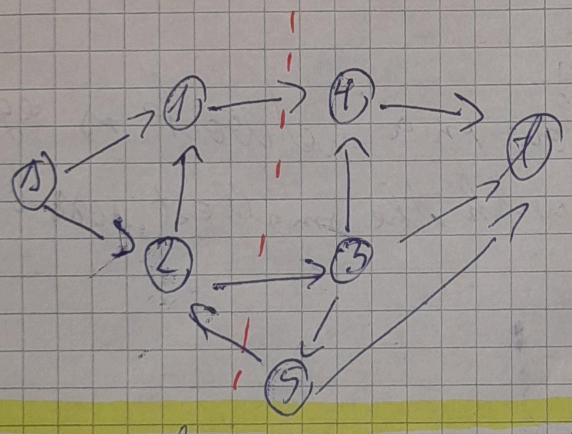
S-T cuts

$N = (G(V,E), s, t, c)$

- a cut $[S, T]$ is a partition of V s.t. $s \in S$ and $t \in T$
- we can also define it by edges: $(v_1, v_2) \in C$ if $v_1 \in S$, $v_2 \in T$ and $(v_1, v_2) \in E$

e.g.

N:



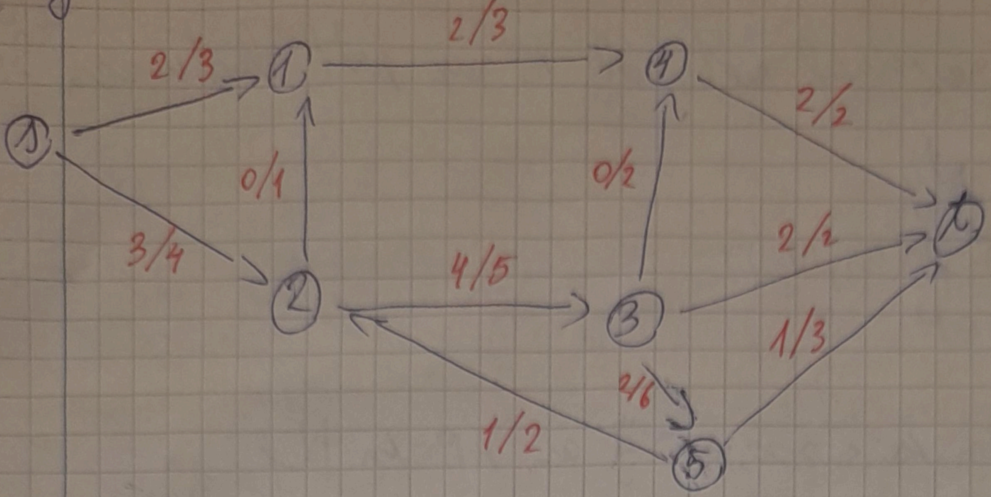
cut C: $s, 1, 2 / t, 3, 4, 5$

cut: C: $\{(1,4), (2,3), (5,2)\}$

- (v_1, v_2) is a: forward edge in cut $[S,T]$ if $v_1 \in S$ and $v_2 \in T$
  backwards edge in cut $[S,T]$ if $v_1 \in T$ and $v_2 \in S$
- the capacity of $[S,T]$ is the sum of the capacity of all forward edges $\xrightarrow{not.} C(S,T)$
- the flow in $[S,T]$ is the sum of the flow of all forward edges minus the flow of all backward edges $\xrightarrow{not.} f[S,T]$

$$\boxed{f(S,T) \leq C(S,T)}$$

e.g.



cut: s, 1, 2 / t, 3, 4, 5

$C(S,T) = 3 + 5 = 8$

$f(S,T) = 2 + 4 - 1 = 5$

$\boxed{f(S,T) = |f|}$

Min-cut Max-flow theorem

- the value of the max flow in a network is equal to the capacity of the cut with the smallest possible capacity