

Seminars 12 and 13 - 2024

1. Consider the sequence of independent identically distributed random variables $(X_n)_{n \geq 1}$ such that $X_n \sim \text{Unif}[1, 3]$ for each $n \geq 1$. Compute the a.s. limit of the sequence which is
 - i) the arithmetic mean of X_1, \dots, X_n , as $n \rightarrow \infty$;
 - ii) the geometric mean of X_1, \dots, X_n , as $n \rightarrow \infty$;
 - iii) the harmonic mean of X_1, \dots, X_n , as $n \rightarrow \infty$.
2. Let $(X_n)_{n \geq 1}$ be a sequence of random variables such that $P(X_n = n^2) = \frac{1}{n}$ and $P(X_n = 0) = 1 - \frac{1}{n}$, for all $n \geq 1$. Prove that:
 - a) $X_n \xrightarrow{P} 0$.
 - b) $(X_n)_{n \geq 1}$ does not converge in mean square.
3. Consider a binary communication channel transmitting codes of n bits each. Assume that the probability of successful transmission of a single bit is $p \in (0, 1)$ and that the probability of an error is $1 - p$. Assume also that the channel is capable of correcting up to m errors, where $0 < m < n$. If we assume that the transmission of successive bits is independent, compute the probability of successful code transmission.
4. The measurement error (in millimeters) of a certain object produced in a factory is a continuous random variable X with the cumulative distribution function $F : \mathbb{R} \rightarrow [0, 1]$,

$$F(x) = \begin{cases} 0, & x < -1 \\ \frac{1}{4}(2 + 3x - x^3), & x \in [-1, 1] \\ 1, & x > 1. \end{cases}$$

Find: $P(-\frac{1}{2} < X < \frac{1}{2})$, $P(X < \frac{1}{2} | X > -\frac{1}{2})$, $E(X)$.

5. A random number generator returns the value Y according to the distribution given by $\begin{pmatrix} -2 & -1 & 0 & 1 \\ \frac{3}{10} & \frac{2}{10} & \frac{1}{10} & \frac{4}{10} \end{pmatrix}$. Find the mean value and the standard deviation of the generator.

6. A random value X is generated according to the density function $f_X : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = \frac{1}{2}e^{-|x|}$, for all $x \in \mathbb{R}$.

Compute:

- a) the cumulative distribution function of X ;
- b) the cumulative distribution function of the random value X^2 ;
- c) $P(X^2 \geq 1)$;
- d) the mean value and the variance of X .

7. Let X and Y be two jointly continuous random variables with joint probability density function $f_{(X,Y)} : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by

$$f_{(X,Y)}(x, y) = \begin{cases} 6xy, & \text{if } 0 \leq x \leq 1, 0 \leq y \leq \sqrt{x} \\ 0, & \text{otherwise.} \end{cases}$$

- (a) Find f_X and f_Y .
- (b) Are X and Y independent?
- (c) Compute $E(2X + 3Y)$.