$d(A, BC) = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2+b^2}}$

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Affine transformations

Parallel projection on a hyperplane

$H: a_1 x_1 + \dots a_n x_n + a_{n+1} = 0$, $\vec{a}(a_1 \dots a_n)$

$\vec{v}(v_1 \dots v_n)$

$$Pr_{H,\vec{v}}(P) = \left(I_n - \frac{\vec{v} \otimes \vec{a}}{\langle \vec{v}, \vec{a} \rangle}\right) P - \frac{a_{n+1}}{|\vec{a}|^2} \vec{a}$$

orthogonal : $$Pr^{\perp}_{H}(P) = \left(I_n - \frac{\vec{a} \otimes \vec{a}}{|\vec{a}|^2}\right) P - \frac{a_{n+1}}{|\vec{a}|^2} \cdot \vec{a}$$

Parallel projection on a line

Line l, H_p hyperplane containing P assoc with W

$H_p: a_1(x_1 - p_1) + \dots + a_n(x_n - p_n) = 0$, $P(p_1 \dots p_n)$

$\vec{a}(a_1 \dots a_n)$

$$Pr_{l,W}(P) = \frac{\vec{v} \otimes \vec{a}}{\langle \vec{v}, \vec{a} \rangle} P + \left(I_n - \frac{\vec{v} \otimes \vec{a}}{\langle \vec{v}, \vec{a} \rangle}\right) Q, \quad Q \in l$$

orthogonal: $$Pr^{\perp}_{l}(P) = \frac{\vec{a} \otimes \vec{a}}{|\vec{a}|^2} P + \left(I_n - \frac{\vec{a} \otimes \vec{a}}{|\vec{a}|^2}\right) Q$$

Reflections : $Refl_{H,\vec{v}}(P) = 2 Pr_{H,\vec{v}}(P) - P$

$$Refl_{l,W}(P) = 2 Pr_{l,W}(P) - P$$

$\boxed{\vec{v} \otimes \vec{u} = \vec{v}^T \cdot \vec{u}}$

1) Matrices of orthogonal projections on the coordinate hyperplanes and on the coord axes (Reflections)

• $\underline{H_z = xOy}$; $z = 0$; $\vec{v} = \vec{k}(0, 0, 1)$

$\vec{a} = \vec{k}$

$$Pr_{H_z,\vec{v}}(P) = \left(I_3 - \frac{\vec{k} \otimes \vec{k}}{|\vec{k}|^2}\right) P = (I_3 - \vec{k} \otimes \vec{k}) P$$

$$\vec{k} \otimes \vec{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$Pr_{H_z,k}(P) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} p_x \\ p_y \\ 0 \end{bmatrix}$$

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$\phi(x) = Ax + b$

$$\begin{bmatrix} \phi(x) \\ 1 \end{bmatrix} = \begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\underline{O_z}$: $\frac{x}{0} = \frac{y}{0} = \frac{z}{1}$ $\quad \vec{a}(0,0,1) = \vec{k}$

$$Pr^{\perp}_{O_z}(P) = \frac{\vec{a} \otimes \vec{a}}{|\vec{a}|^2} P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ P_z \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- $\underline{H_x}$: yOz : $x = 0 \Rightarrow \vec{v} = \vec{i}(1,0,0)$

$$H_{yOz} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} v \right.$$

$\vec{a} = (1,0,0)$

$$\vec{a} \otimes \vec{a} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} (1\ 0\ 0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$Pr_{H_x}(P) = \left(I_3 - \frac{\vec{i} \otimes \vec{i}}{|\vec{i}|^2} \right) P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \begin{bmatrix} 0 \\ P_y \\ P_z \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\underline{O_x}$: $\frac{x}{1} = \frac{y}{0} = \frac{z}{0} \Rightarrow \vec{a}(1,0,0)$

$$Pr^{\perp}_{O_x}(P) = \frac{\vec{a} \otimes \vec{a}}{|\vec{a}|^2} P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \begin{bmatrix} P_x \\ 0 \\ 0 \end{bmatrix}$$

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- $H_y : xOz : y = 0 \Rightarrow \vec{v} = \vec{j} = (0,1,0)$

$\vec{a}(0,1,0)$

$$\vec{a} \otimes \vec{a} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} (0\ 1\ 0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow Pr_{H_y}(P) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} p_x \\ 0 \\ p_z \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\underline{Oy} : \frac{x}{0} = \frac{y}{1} = \frac{z}{0} \Rightarrow \vec{a}(0,1,0)$

$$\Rightarrow Pr^{\perp}_{Oy}(P) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} 0 \\ p_y \\ 0 \end{bmatrix}$$

- $$Pr^{\perp}_{xOz}(P) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \Rightarrow Ref^{\perp}_{xOz}(P) = 2 Pr^{\perp}_{xOz}(P) - P$$

$$= 2 \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}}_{A} - \underbrace{\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}}_{b}$$

$$= \begin{bmatrix} p_x \\ -p_y \\ p_z \end{bmatrix}$$

$$Pr^{\perp}_{Oy}(P) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \Rightarrow Ref^{\perp}_{Oy}(P) = 2 \overbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}}^{A} - \overbrace{\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}}^{b}$$

$$= \begin{bmatrix} -p_x \\ p_y \\ -p_z \end{bmatrix}$$

2) Let $\vec{v}(2,1,1) \in V^3$

Give matrix form of the $\parallel \begin{cases} \text{proj on the plane } \pi : z = 0 \\ \text{refl} \quad \parallel \text{ to } \vec{v} \end{cases}$

$\vec{a}(0,0,1)$

$Pr_{\pi,\vec{v}} = (I_3$

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3) Orthonormal reflection of $P(6,-5,5)$ in the plane $2x-3y+z-4=0$ by determining the matrix form of the reflection.

$$Pr^{\perp}_{\pi}(P) = \left(I_3 - \frac{\vec{a}\otimes\vec{a}}{|\vec{a}|^2}\right)P - \frac{a_4}{|\vec{a}|^2}\vec{a}$$

$a(2,-3,1)$

$|\vec{a}| = 4+9+1 = 14$

$$\vec{a}\otimes\vec{a} = \begin{pmatrix}2\\-3\\1\end{pmatrix}\begin{pmatrix}2 & -3 & 1\end{pmatrix} = \begin{pmatrix}4 & -6 & 2\\-6 & 9 & -3\\2 & -3 & 1\end{pmatrix}$$

$$\Rightarrow Pr^{\perp}_{\pi}(P) = \left[\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - \frac{1}{14}\begin{pmatrix}4&-6&2\\-6&9&-3\\2&-3&1\end{pmatrix}\right]\begin{bmatrix}x_P\\y_P\\z_P\end{bmatrix} - \frac{-4}{14}\begin{bmatrix}2\\-3\\1\end{bmatrix}$$

$$= \frac{1}{14}\left[\begin{pmatrix}10&+6&-2\\-6&5&3\\-2&3&13\end{pmatrix}\begin{pmatrix}x_P\\y_P\\z_P\end{pmatrix} + \begin{pmatrix}8\\-12\\4\end{pmatrix}\right]$$

$$= \frac{1}{14}\begin{pmatrix}10&6&-2\\-6&5&3\\-2&3&13\end{pmatrix}\begin{pmatrix}x_P\\y_P\\z_P\end{pmatrix} + \frac{4}{14}\begin{pmatrix}2\\-3\\1\end{pmatrix}$$

$$Ref^{\perp}_{\pi}(P) = 2Pr^{\perp}_{\pi}(P) - P = \frac{1}{7}\begin{pmatrix}10&6&-2\\-6&5&3\\-2&3&13\end{pmatrix}P + \frac{4}{14}\begin{pmatrix}2\\-3\\1\end{pmatrix} - P$$

$$\Rightarrow \frac{1}{7}\begin{bmatrix}10&6&-2&8-7p_x\\-6&5&3&-12-7p_y\\-2&3&13&4-7p_z\\0&0&0&7\end{bmatrix}\begin{bmatrix}6\\-5\\5\\1\end{bmatrix} = \frac{1}{7}\begin{bmatrix}-14\\48\\22\\7\end{bmatrix}$$

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$$\Rightarrow P'\left(\frac{-14}{7}, \frac{48}{7}, \frac{22}{7}\right)$$

4) $$Pr^{\perp}_{\vec{a}}(\vec{b}) = \frac{\langle\vec{a},\vec{b}\rangle}{\langle\vec{a},\vec{a}\rangle}\cdot\vec{a}$$, $\vec{a}(a_1,a_2,a_3)$, $\vec{b}(b_1,b_2,b_3)$

$$Pr^{\perp}_{\vec{a}}(\vec{b}) = \frac{\vec{a}\otimes\vec{a}}{|\vec{a}|^2}\vec{b}$$

$$(\vec{a}\otimes\vec{a})_{ij} = a_i \cdot a_j$$

$$\left[(\vec{a}\otimes\vec{a})\vec{b}\right]_i = \sum_{j=1}^{3} a_i a_j \cdot b_j = a_i\sum_{j=1}^{3} a_j b_j = a_i\langle\vec{a},\vec{b}\rangle$$

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$\Rightarrow \vec{a}\langle \vec{a}, \vec{b} \rangle = (\vec{a} \otimes \vec{a})\vec{b}$

$\langle \vec{a}, \vec{a} \rangle = |\vec{a}|^2$

$$\Rightarrow Pr^{\perp}_{\vec{a}}(b) = \frac{(\vec{a} \otimes \vec{b})\vec{a}}{\langle \vec{a}, \vec{a} \rangle}$$

5) $\pi : ax + by + c = 0$

$$l : \frac{x - x_0}{v_1} = \frac{y - y_0}{v_2} \quad s.t. \ \pi \,/\!/\, l$$

Deduce the matrix form of $Pr_{\pi, l}$, $Ref_{\pi, l}$

$\vec{v}(v_1, v_2)$, $\vec{a}(a, b)$

$$Pr_{\pi, v}(P) = \left(I_2 - \frac{\vec{v} \otimes \vec{a}}{\langle \vec{v}, \vec{a} \rangle} \right) P - \frac{c}{\langle \vec{v}, \vec{a} \rangle} \cdot \vec{v}$$

$$= \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{v_1 a + v_2 b} \begin{pmatrix} v_1 a & v_1 b \\ v_2 a & v_2 b \end{pmatrix} \right] P - \frac{c}{v_1 a + v_2 b} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\vec{v} \otimes \vec{a} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} (a \ b) = \begin{pmatrix} v_1 a & v_1 b \\ v_2 a & v_2 b \end{pmatrix}$$

$\langle \vec{v}, \vec{a} \rangle = v_1 a + v_2 b$

$$\Rightarrow Pr_{\pi, v}(P) = \frac{1}{v_1 a + v_2 b} \left[\begin{pmatrix} \cancel{v_1 a} + v_2 b - \cancel{v_1 a} & -v_1 b \\ -v_2 a & v_1 a + \cancel{v_2 b} - \cancel{v_2 b} \end{pmatrix} P - c \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right]$$

$$= \frac{1}{v_1 a + v_2 b} \left[\begin{pmatrix} v_2 b & -v_1 b \\ -v_2 a & v_1 a \end{pmatrix} P - \begin{pmatrix} c v_1 \\ c v_2 \end{pmatrix} \right]$$

$$\Rightarrow \frac{1}{v_1 a + v_2 b} \begin{bmatrix} v_2 b & -v_1 b & -c v_1 \\ -v_2 a & v_1 a & c v_2 \\ 0 & 0 & \frac{1}{v_1 a + v_2 b} \end{bmatrix}$$

$$Ref_{\pi, \vec{v}} = 2 Pr_{\pi, \vec{v}}(P) - P = \underbrace{\frac{2}{v_1 a + v_2 b} \begin{pmatrix} v_2 b & -v_1 b \\ -v_2 a & v_1 a \end{pmatrix}}_{A} P - \frac{2}{v_1 a + v_2 b} \begin{pmatrix} c v_1 \\ c v_2 \end{pmatrix} - P$$

$$\Rightarrow AP - \frac{2}{v_1 a + v_2 b} \begin{pmatrix} c v_1 \\ c v_2 \end{pmatrix} - P$$

$$= AP - \frac{2}{v_1 a + v_2 b} \begin{bmatrix} c v_1 + \frac{v_1 a + v_2 b}{2} P_x \\ c v_2 + \frac{v_1 a + v_2 b}{2} P_y \end{bmatrix}$$

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Seminar 9

13. Let $\varphi(\vec{x}) = A\vec{x} + \vec{b}$ be an affine transformation. Find the homogenous matrice of φ^{-1}

$\vec{y} = \varphi(\vec{x}) = A\vec{x} + \vec{b}$ $\qquad \varphi(x) = y \,|\, \varphi^{-1} \Rightarrow x = \varphi^{-1}(y)$

$\qquad y = \varphi(x)$

$A^{-1} \,|\, \vec{y} - \vec{b} = A\vec{x}$ $\qquad \varphi^{-1}(y) = x$

$A^{-1}(\vec{y} - \vec{b}) = \vec{x}$

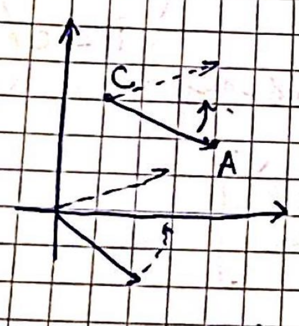
$A^{-1}\vec{y} - A^{-1}\vec{b} = \vec{x} \Rightarrow \varphi^{-1}(\vec{y}) = A^{-1}\vec{y} - A^{-1}\vec{b}$

Homogenous matrice of φ^{-1}: $\begin{bmatrix} A^{-1} & -A^{-1}b \\ 0 & 1 \end{bmatrix}$

14. $A(1,1)$, $B(4,1)$, $C(2,3)$ what is the image of ΔABC under a rotation ~~of~~ by $90°$ around C followed by an orthonormal reflexion relative to AB

$R \in SO(2) \Leftrightarrow R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \theta \in \mathbb{R}$

$(R^{-1} = R^{T}, \det R = 1)$



$R_{\theta,C}(P) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}(P - C) + C$

$\theta = 90° \Rightarrow R_{\theta,C}(P) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{pmatrix} x_p - 2 \\ y_p - 3 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$P(x_p, y_p)$

$C(2,3)$ $\qquad = \begin{pmatrix} -y_p + 3 \\ x_p - 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -y_p + 5 \\ x_p + 1 \end{pmatrix}$

$\Rightarrow R_{\theta,C}(A) = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \; R_{\theta,C}(B) = \begin{pmatrix} 4 \\ 5 \end{pmatrix}, \; R_{\theta,C}(C) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

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$\Rightarrow \Delta A'B'C'$

$A'(4,2),\ B'(4,5),\ C'(2,3)$

$$Pr_l^{\perp}(P) = \frac{\vec{a}\otimes\vec{a}}{|\vec{a}|^2}P + \left(I_2 - \frac{\vec{a}\otimes\vec{a}}{|\vec{a}|^2}\right)Q,\ Q\in l$$

$$Ref_l^{\perp}(P) = 2Pr_l^{\perp}(P) - P$$

$\overrightarrow{A'B'}\ (0,3)$

$$\overrightarrow{A'B'}\otimes\overrightarrow{A'B'} = \begin{pmatrix}0\\3\end{pmatrix}(0\ 3) = \begin{pmatrix}0&0\\0&9\end{pmatrix}$$

$|\overrightarrow{A'B'}|^2 = 9$

$$\Rightarrow Pr^{\perp}_{A'B'}(P) = \frac{1}{9}\begin{pmatrix}0&0\\0&9\end{pmatrix}P + \left(\begin{bmatrix}1&0\\0&1\end{bmatrix} - \frac{1}{9}\begin{bmatrix}0&0\\0&9\end{bmatrix}\right)A'$$

$$= \begin{pmatrix}0&0\\0&1\end{pmatrix}P + \begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}4\\2\end{pmatrix}$$

$$= \begin{pmatrix}0&0\\0&1\end{pmatrix}P + \begin{pmatrix}4\\0\end{pmatrix}$$

$$Pr^{\perp}_{A'B'}(C') = \begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}2\\3\end{pmatrix} + \begin{pmatrix}4\\0\end{pmatrix} = \begin{pmatrix}4\\3\end{pmatrix}$$

$$Ref^{\perp}_{A'B'}(P) = 2\cdot\begin{pmatrix}4\\3\end{pmatrix} - \begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}6\\3\end{pmatrix} = C''$$

$\Rightarrow \Delta ABC$ is mapped $\Delta A'B'C''$

Ex. Determine the matrix form of a rotation with angle $45°$ having the same center of rotation as

$$f(\vec{x}) = \frac{1}{\sqrt{13}}\begin{bmatrix}2&-3\\3&2\end{bmatrix}\vec{x} + \begin{bmatrix}1\\-2\end{bmatrix}$$

$$= R_\theta(\vec{x} - P) + P$$

$$= R_\theta\vec{x} - R_\theta P + P$$

$$= R_\theta\vec{x} + (I_2 - R_\theta)P$$

$$\Rightarrow R_\theta = \frac{1}{\sqrt{13}}\begin{bmatrix}2&-3\\3&2\end{bmatrix}$$

$$(I_2 - R_\theta)P = \begin{bmatrix}1\\-2\end{bmatrix} \ldots$$

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16. Let T be the isometry obtained by applying a rotation of angle $-\frac{\pi}{3}$ around the origin after a translation with vector $(-2,5)$. Determine T^{-1}

$$y = T(x) = R\left(x + \begin{bmatrix}-2\\5\end{bmatrix}\right)$$

$$\Rightarrow x + \begin{bmatrix}-2\\5\end{bmatrix} = R^{-1}y$$

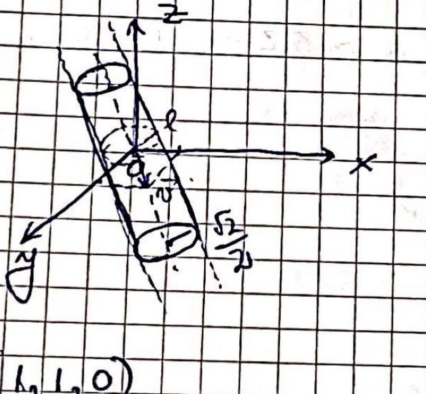
$$x = R^{-1}y - \begin{bmatrix}-2\\5\end{bmatrix} \quad \Rightarrow T^{-1}(y) = \begin{bmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix} y - \begin{bmatrix}-2\\5\end{bmatrix}$$

$R_\theta^{-1} = R_{-\theta},\ \theta \in \mathbb{R}$

21. Using the Euler-Rodrigues formula find the matrix forms of rotations around the coordinate axes

(*) $Rot_{\vec{v},\theta}(P) = \cos\theta P + \sin\theta(\vec{v} \times P) + (1-\cos\theta)\langle \vec{v}, P\rangle \vec{v}$ $(P = \vec{OP})$

22. Using formula (*), write the matrix form of a rotation around the axis $\mathbb{R}\vec{v}$, $\vec{v}(1,1,0)$. Give a parametrization of a cylinder with axis $\mathbb{R}\vec{v}$ and diameter $\sqrt{2}$



$$\mathbb{R}\vec{v} = \left\{(x,y,z) \in \mathbb{R}^3 : \begin{bmatrix}x\\y\\z\end{bmatrix} = t\begin{bmatrix}1\\1\\0\end{bmatrix}, t \in \mathbb{R}\right\}$$

$\vec{v}(1,1,0)$

$x_p + y_p - x_p\cos\theta - y_p\cos\theta$

$$Rot_{\vec{v},\theta} = \cos\theta \begin{bmatrix}x_p\\y_p\\z_p\end{bmatrix} + \sin\theta \begin{bmatrix}z_p\\-z_p\\y_p - x_p\end{bmatrix} + (1-\cos\theta)(x_p + y_p)\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$\vec{v} \times P = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 1 & 0\\ x_p & y_p & z_p\end{vmatrix} = z_p\vec{i} + y_p\vec{k} - x_p\vec{k} - z_p\vec{j}$$

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$$Rot_{\vec{v},\theta} = \begin{bmatrix} \cancel{x_p\cos\theta} + z_p\sin\theta + x_p + y_p - \cancel{x_p\cos\theta} - y_p\cos\theta \\ \cancel{y_p\cos\theta} - z_p\sin\theta + x_p + y_p - x_p\cos\theta - \cancel{y_p\cos\theta} \\ z_p\cos\theta + y_p\sin\theta - x_p\sin\theta \end{bmatrix}$$

$$Rot_{\vec{v},\theta} = \begin{bmatrix} 1 & 1-\cos\theta & \sin\theta \\ 1-\cos\theta & 1 & -\sin\theta \\ -\sin\theta & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix}$$

Seminar 10

Quadratic curves

1) Find the equation of the circle:

a) of diameter $[AB]$ with $A(1,2)$, $B(-3,-1)$

b) with the center $I(2,-3)$ and radius $R=7$

c) with the center $I(-1,2)$ and passing through $A(2,6)$

d) centered at the origin and tangent to $l: 3x-4y+20=0$

e) passing through $A(3,1)$, $B(-1,3)$ and having the center on the line $l: 3x-y-2=0$

f) passing through $A(1,1)$, $B(1,-1)$, $C(2,0)$

g) tangent to both $l_1: 2x+y-5=0$, $l_2: 2x+y+15=0$ if one tangency point is $M(3,-1)$

c) $R=\sqrt{(-1-2)^2+(2-6)^2} = \sqrt{9+16} = 5$

$(x+1)^2+(y-2)^2 = 25$

d) $I(0,0)$

$$\begin{cases} l: 3x-4y+20=0 \Rightarrow x = \dfrac{4(y-5)}{3} \\ x^2+y^2=R^2 \Rightarrow \dfrac{16(y-5)^2}{9} + y^2 = R^2 \end{cases}$$

$16(y^2-10y+25)+9y^2 = 9R^2$

$25y^2-160y+400 = 9R^2$

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$25y^2 - 160y + 400 - 9R^2 = 0$

one solution $\Rightarrow \Delta = 0 \Rightarrow \Delta = 160^2 - 4(400 - 9R^2)\,25 = 0 \mid :100$

$\Leftrightarrow 16^2 - 400 + 9R^2 = 0$

$\Leftrightarrow 9R^2 - 144 = 0$

$\Leftrightarrow R = \frac{12}{3} = 4$

e) $(x - x_J)^2 + (y - y_J)^2 = R^2$

$\left.\begin{array}{l}(3 - x_J)^2 + (1 - y_J)^2 = R^2 \\ (-1 - x_J)^2 + (3 - y_J)^2 = R^2\end{array}\right| \Rightarrow (3 - x_J)^2 + (1 - y_J)^2 = (-1 - x_J)^2 + (3 - y_J)^2$

$y_J = 3x_J - 2 \Rightarrow (3 - x_J)^2 + (3 - 3x_J)^2 = (-1 - x_J)^2 + (5 - 3x_J)^2$

$9 - 6x_J + \cancel{x_J^2} + 9 - 18x_J + 9x_J^2 = 1 + 2x_J + \cancel{x_J^2} + 25 - 30x_J + \cancel{9x_J^2}$

$\cancel{10x_J^2} - 24x_J + 18 = \cancel{10x_J^2} - 28x_J + 26$

$4x_J = 8 \Rightarrow x_J = 2 \Rightarrow y_J = 4$

$\Rightarrow R = 1 + 9 = 10$

$(x - 2)^2 + (y - 4)^2 = 10$

f) $\begin{cases}(1 - x_J)^2 + (1 - y_J)^2 = R^2 \\ (1 - x_J)^2 + (-1 - y_J)^2 = R^2 \\ (2 - x_J)^2 + (0 - y_J)^2 = R^2\end{cases}$

$\begin{cases}x_J^2 - 2x_J + 1 + y_J^2 - 2y_J + 1 = R^2 \\ x_J^2 - 2x_J + 1 + y_J^2 + 2y_J + 1 = R^2 \\ x_J^2 + 4x_J + 4 + y_J^2 = R^2\end{cases} \stackrel{(-)}{\Rightarrow} -4y_J = 0 \Rightarrow y_J = 0$

$\Rightarrow \begin{cases}x_J^2 - 2x_J + 2 = R^2 \\ x_J^2 + 4x_J + 4 = R^2\end{cases} \stackrel{(-)}{\Rightarrow} -6x_J + 2 = 0$

$x_J = -\frac{1}{3}$

$\Rightarrow R^2 = \left(1 + \frac{1}{3}\right)^2 + 1 = \frac{16}{9} + 1 = \frac{25}{9} \Rightarrow R = \frac{5}{3}$

$\Rightarrow \left(x + \frac{1}{3}\right)^2 + y^2 = \frac{25}{9}$

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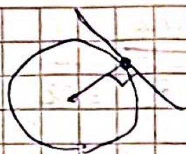
g) $(3-x_J)^2+(-1-y_J)^2=R^2$ $(*)$



$y=-2x-15$

$(x-x_J)^2+(-2x-15-y_J)^2=R^2$

- $JM \perp l_1$, $\vec{JM}(3-x_J,-1-y_J)$

Let $A(0,5), B(5,-5) \in l_1 \Rightarrow \vec{AB}(5,-10) \Rightarrow \vec{v}(1,-2)$

$3-x_J+2+2y_J=0 \Rightarrow x_J=5+2y_J$

$(*) \Rightarrow (2+2y_J)^2+(y_J+1)^2=R^2$

$\Leftrightarrow 5(y_J+1)^2=R^2 \Rightarrow (y_J+1)^2=\frac{R^2}{5} \Rightarrow y_J+1=\pm\frac{R}{\sqrt{5}}$

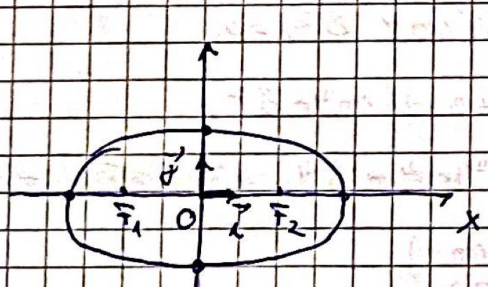
- $JN \perp l_2$,

Ellipse: Canonical equation. Relative position of a line

Def: (ellipse): Geometric locus of points $P \in E^2$ for which the $|PF_1|+|PF_2|=$ const, where $F_1, F_2 \in E^2$ are called ~~fol~~ focal points (foci)

Canonical equation: Let $a, b>0$, $F_1, F_2 \in E^2$

Consider the frame $(O,\vec{i},\vec{j})$ s.t. $\begin{cases} F_1, F_2 \in Ox \\ \vec{F_1F_2} \parallel \vec{i} \\ O \text{ is the midpoint of } [F_1F_2] \end{cases}$



Then, the ellipse with foci F_1, F_2 for which the sum of the distances from F_1, F_2 is $2a$ has the equation:

$$\boxed{\mathcal{E}_{a,b}: \frac{x^2}{a^2}+\frac{y^2}{b^2}=1}$$

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Remarks:

- If $|F_1F_2| = 2c$, then $b^2 = a^2 - c^2$ $(b \leq a)$
- $\mathcal{E}_{a,b} \cap Ox = \{(\pm a, 0)\}$

  $\mathcal{E}_{a,b} \cap Oy = \{(0, \pm b)\}$
- $e = \frac{c}{a} = \sqrt{1 - \frac{b^2}{a^2}}$ is called the eccentricity of $\mathcal{E}_{a,b}$
- $M(x_M, y_M) \in \mathcal{E}_{a,b} \Leftrightarrow (\pm x_M, \pm y_M) \in \mathcal{E}_{a,b}$

Relative position of a line

$\mathcal{E}_{a,b} : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\ell : y = kx + m$

The intersection $\mathcal{E}_{a,b} \cap \ell$ is the set of solutions of

$$\begin{cases} \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \\ y = kx + m \end{cases} \Leftrightarrow \begin{cases} \frac{x^2}{a^2} + \frac{(kx+m)^2}{b^2} = 1 \quad (*) \\ y = kx + m \end{cases}$$

$(*) \Leftrightarrow b^2x^2 + a^2(kx+m)^2 = a^2b^2$

$b^2x^2 + a^2(k^2x^2 + 2kmx + m^2) - a^2b^2 = 0$

$(b^2 + a^2k^2)x^2 + 2kma^2x + a^2(m^2 - b^2) = 0$

$\Rightarrow \Delta = 4k^2m^2a^4 - 4a^2(b^2 + a^2k^2)(m^2 - b^2)$

$= \cancel{4k^2m^2a^4} - 4a^2(b^2m^2 - b^4 + \cancel{a^2m^2k^2} - a^2b^2k^2)$

$= \underbrace{4a^2b^2}_{>0}(a^2k^2 + b^2 - m^2)$

- If $\Delta > 0 \Leftrightarrow a^2k^2 + b^2 > m^2 \Leftrightarrow -\sqrt{a^2k^2+b^2} < m < \sqrt{a^2k^2+b^2}$

  $\rightarrow$ 2 intersection points
- If $\Delta = 0 \Leftrightarrow a^2k^2 + b^2 = m^2 \Leftrightarrow m = \pm\sqrt{a^2k^2+b^2}$

  $\Rightarrow$ 1 intersection point

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- If $\Delta < 0 \Leftrightarrow a^2k^2+b^2 < m^2 \Leftrightarrow \begin{cases} m > \sqrt{a^2k^2+b^2} \\ m < -\sqrt{a^2k^2+b^2} \end{cases}$

$\Rightarrow$ no intersection points

Remark : For a given k (slope), there are 2 tangent lines to $\mathcal{E}_{a,b}$

$\mathcal{E}_{a,b} : y = kx \pm \sqrt{a^2k^2+b^2}$

3) Determine the focal points of the ellipse

$9x^2+25y^2-225=0$

$9x^2+25y^2=225 \,|: 225$

$\frac{x^2}{25}+\frac{y^2}{9}=1 \Rightarrow \begin{matrix} a=5 \\ b=3 \end{matrix}$

$b^2=a^2-c^2 \Rightarrow c^2=a^2-b^2=25-9=16 \Rightarrow c=4$

$|F_1F_2|=2c=8$

$F_1(c,0),\ F_2(-c,0)$

$F_1(4,0),\ F_2(-4,0)$

4) $\ell \cap \mathcal{E} = ?$

$\ell : 2x+y-10=0 \Rightarrow y=-2x+10 \Rightarrow k=-2,\ m=10$

$\mathcal{E} : \frac{x^2}{9}+\frac{y^2}{4}-1=0 \Rightarrow \frac{x^2}{9}+\frac{y^2}{4}=1 \Rightarrow \begin{cases} a=3 \\ b=2 \end{cases}$

$m^2=100$

$a^2k^2+b^2=9\cdot 4+4=40$

$\Rightarrow a^2k^2+b^2 < m^2$

$\Rightarrow \ell \cap \mathcal{E} = \emptyset$

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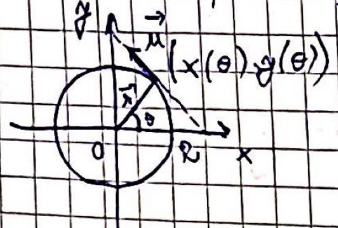
Seminar 11

2). For a circle $\mathcal{C}$ of radius R, use the parametrization

$\theta \longmapsto (R\cos\theta, R\sin\theta)$ to deduce a param of tangent lines to $\mathcal{C}$



$\vec{r}\,(x(\theta), y(\theta))$

$\vec{r} \cdot \vec{u} = 0 \Leftrightarrow x(\theta)u_1 + y(\theta)u_2 = 0$

$u_1 = -\dfrac{y(\theta)}{x(\theta)}u_2$, let $u_2 = x(\theta)$

$u_1 = -\dfrac{y(\theta)}{x(\theta)} \cdot x(\theta) = -y(\theta)$

$l_t: \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} R\cos\theta \\ R\sin\theta \end{bmatrix} + t\begin{bmatrix} -R\sin\theta \\ R\cos\theta \end{bmatrix}$

Hw: $x \longmapsto \left(x, \pm\sqrt{R^2 - x^2}\right)$

8) $\mathcal{E}_a : \dfrac{x^2}{a^2} + \dfrac{y^2}{16} = 1$ For what values of $a \in \mathbb{R}$ is $\mathcal{E}_a$ tangent to the line $l: x - y + 5 = 0$?

9) $l_c : \sqrt{5}x - y + c = 0$. For what values of $c \in \mathbb{R}$ is l_c tangent to $\mathcal{E}: x^2 + \dfrac{y^2}{4} = 1$?

$l_c : y = \sqrt{5}x + c \Rightarrow m = c,\ k = \sqrt{5}$

$\Rightarrow c = \pm\sqrt{\sqrt{5}^2 \cdot 1^2 + 4}$

$c = \pm\sqrt{5+4}$

$c = \pm\sqrt{9} = \pm 3$

10) Determine the common tangents to:

$\dfrac{x^2}{45} + \dfrac{y^2}{9} = 1, \quad \dfrac{x^2}{9} + \dfrac{y^2}{18} = 1$

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$a_1 = \sqrt{45} = 3\sqrt{5}$, $a_2 = 3$

$b_1 = 3$ $\quad b_2 = 3\sqrt{2}$

$y = kx \pm m = kx \pm \sqrt{k^2a^2 + b^2} \Rightarrow y = kx \pm \sqrt{k^2 45 + 9}$

$y = kx \pm \sqrt{k^2 \cdot 9 + 18}$

$\Downarrow$

$45k^2 + 9 = 9k^2 + 18$

$36k^2 = 9 \Rightarrow k^2 = \frac{9}{36} \Rightarrow k = \pm\frac{3}{6} = \pm\frac{1}{2}$

$k = \frac{1}{2} \Rightarrow m = \sqrt{\frac{45}{4} + 9} \Rightarrow m = \sqrt{\frac{81}{4}} = \frac{9}{2} \Rightarrow l: y = \frac{1}{2}x \pm \frac{9}{2}$

$k = \frac{1}{2} \Rightarrow m = \sqrt{\frac{9}{4} + 18} \Rightarrow m = \sqrt{\frac{81}{4}} \Rightarrow m = \frac{9}{2} \Rightarrow l: y = -\frac{1}{2}x \pm \frac{9}{2}$

$l_1: \frac{1}{2}x - \frac{9}{2}y = 0$

$l_2: \frac{1}{2}x + \frac{9}{2}y = 0$

18) Determine the tangents to the hyperbola $\mathcal{H}: x^2 - y^2 = 16$ which contain $M(-1, 7)$

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$x^2 - y^2 = 16 \mid : 16 \Rightarrow \frac{x^2}{16} - \frac{y^2}{16} = 1 \Rightarrow a = b = 4$

$y = kx \pm \sqrt{a^2k^2 - b^2}$

$\Leftrightarrow y = kx \pm \sqrt{16k^2 - 16}$

$\Leftrightarrow y = kx \pm 4\sqrt{k^2 - 1} \Rightarrow 7 = -k \pm 4\sqrt{k^2-1}$

$7 + k = \pm 4\sqrt{k^2 - 1} \mid ()^2$

$49 + 14k + k^2 = 16k^2 - 16$

$15k^2 - 14k - 65 = 0$

$\Delta = 196 + 3900 = 4096 \Rightarrow k_{1,2} = \frac{14 \pm 64}{30}$

$k_1 = \frac{78}{30} = \frac{13}{5}$, $k_2 = -\frac{50}{30} = -\frac{5}{3}$

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$l_1: -5y = -\frac{5}{3}x + 4\sqrt{\left(-\frac{5}{3}\right)^2 - 1} =$

$l_2: \quad y = \frac{13}{x} + 4\sqrt{\frac{13}{25} - 1}$

20) Find the area of the triangle determined by the asymptotes of the hyperbola $\mathcal{H}: \frac{x^2}{4} - \frac{y^2}{9} - 1 = 0$ and the line $l: 9x + 2y - 24 = 0$

Asymptotes: $y = \pm \frac{b}{a}x$

$\mathcal{H}: \frac{x^2}{4} - \frac{y^2}{9} = 1 \Rightarrow a = 2$
$b = 3$

$y = \pm \frac{3}{2}x$

$l: \begin{cases} 9x + 2y - 24 = 0 \\ 3x - 2y = 0 \end{cases}$

$12x = 24 \Rightarrow x = 2$

$6 - 2y = 0 \Rightarrow y = 3 \quad \Big| \Rightarrow A(2, 3)$

$\begin{cases} 9x + 2y - 24 = 0 \\ 3x + 2y = 0 \end{cases} \quad (-)$

$6x \quad = 24 \Rightarrow x = 4$

$12 + 2y = 0 \Rightarrow y = -6 \quad \Big| \Rightarrow B(4, -6)$

$\mathcal{A}_{\triangle AOB} = \frac{1}{2}\left|\begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_O & y_O & 1 \end{vmatrix}\right| = \frac{1}{2}\left|\begin{vmatrix} 2 & 3 & 1 \\ 4 & -6 & 1 \\ 0 & 0 & 1 \end{vmatrix}\right| = \frac{1}{2}\left|-12 - 12\right| = 12$

26. For which the value of k is the line $y = kx + 2$ tangent to the parabola $P: y^2 = 4x$?

$P_p: y^2 = 2px \Rightarrow p = 2$

$y = kx + \frac{p}{2k}$ is tangent to P_p

$\Rightarrow \frac{2}{2k} = 2 \Rightarrow k = \frac{1}{2}$

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27. $P: y^2 = 16x$. Find the tangents to P which are:

a) parallel to $l: 3x - 2y + 30 = 0$

b) perpendicular to $l: 4x + 2y + 7 = 0$

Seminar 12

<u>Classification of quadratic curves</u>

2) $A = \begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix}$. Write a quadratic equation associated with A and find $M \in SO(2)$ which diagonalizes A.

I $[x \ \ y] A \begin{bmatrix} x \\ y \end{bmatrix} = 0$

II M: change of basis matrix $M_{e'e}$ ← initial basis; e' → orthonormal basis of eigenvectors

$$[x \ \ y] \begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\Leftrightarrow \cancel{\begin{bmatrix} 6x + 2y \\ 2x + 9y \end{bmatrix}} \quad [6x + 2y \quad 2x + 9y] \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$x(6x + 2y) + y(2x + 9y) = 0$$

$$6x^2 + 2xy + 2xy + 9y^2 = 0$$

$$6x^2 + 4xy + 9y^2 = 0$$

$$\begin{vmatrix} 6-\lambda & 2 \\ 2 & 9-\lambda \end{vmatrix} = 0 \Leftrightarrow (6-\lambda)(9-\lambda) - 4 = 0$$

$$54 - 15\lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 - 15\lambda + 50 = 0$$

$$\Delta = 225 - 200 = 25 \Rightarrow \lambda_{1,2} = \frac{15 \pm 5}{2} \begin{cases} 5 \\ 10 \end{cases}$$

• $\lambda_1 = 5$; $(A - 5I)\vec{v} = 0$

$$\left[\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} - \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}\right] \vec{v} = 0 \Leftrightarrow \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \vec{v} = \vec{0}$$

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$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x_v \\ y_v \end{pmatrix} = 0 \iff \begin{cases} x_v + 2y_v = 0 \\ 2x_v + 4y_v = 0 \end{cases} \Rightarrow x_v = -2y_v, \; y_v \in \mathbb{R}$$

$$V(\lambda_1) = \langle (-2, 1) \rangle$$

$$\lambda_2 : \; (A - 10I)\vec{v} = 0$$

$$\iff \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x_v \\ y_v \end{pmatrix} = 0 \iff \begin{cases} -4x_v + 2y_v = 0 \\ 2x_v - y_v = 0 \end{cases} \Rightarrow y_v = 2x_v, \; x_v \in \mathbb{R}$$

$$V(\lambda_2) = \langle (1, 2) \rangle$$

$$\vec{v}_1 = (-2, 1) \Rightarrow |\vec{v}_1| = \sqrt{5} \qquad \Rightarrow B\left\{\left(\frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right), \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)\right\}$$

$$\vec{v}_2 = (1, 2) \Rightarrow |\vec{v}_2| = \sqrt{5}$$

$$\left(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right) = -\frac{2}{\sqrt{5}}\vec{e_1} + \frac{1}{\sqrt{5}}\vec{e_2} = \frac{1}{\sqrt{5}}\left(-2\vec{e_1} + \vec{e_2}\right) \Rightarrow M = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\Rightarrow M^T = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$MM^T = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} \cancel{5} & 0 \\ 0 & \cancel{5} \end{bmatrix} = \cancel{5} I_2$$

$$\det M = \frac{1}{5}(-4 - 1) = -1$$

$$M^T \cdot A \cdot M = \frac{1}{5}\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} -10 & 5 \\ 10 & 20 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{1}{5}\cancel{5}\begin{bmatrix} -2 & 1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \cancel{5}\begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}$$

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4) $-x^2+xy-y^2=0$

Write the assoc matrix

Bring to the cannonical form

$[x \ y] A \begin{bmatrix} x \\ y \end{bmatrix} = 0$

$1\cdot x^2 - xy + 1\cdot y^2 = 0 \Rightarrow A = \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}$

$\det(A - \lambda I_2) = 0$

$\begin{vmatrix} 1-\lambda & -\frac{1}{2} \\ -\frac{1}{2} & 1-\lambda \end{vmatrix} = 0$

$1 - 2\lambda + \lambda^2 - \frac{1}{4} = 0$

$\lambda^2 - 2\lambda + \frac{3}{4} = 0 \Rightarrow \lambda_1 = \frac{1}{2}, \lambda_2 = \frac{3}{2}$

• $\lambda = \frac{1}{2} \Rightarrow$ $(A - \lambda I_2)\vec{v} = 0$

$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$

$\begin{cases} \frac{x}{2} - \frac{y}{2} = 0 \\ -\frac{x}{2} + \frac{y}{2} = 0 \end{cases} \Rightarrow x = y \Rightarrow V(\lambda_1) = \langle (1,1) \rangle$

• $\lambda = \frac{3}{2} \Rightarrow \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$

$\begin{cases} -\frac{x}{2} - \frac{y}{2} = 0 \\ -\frac{x}{2} - \frac{y}{2} = 0 \end{cases} \Rightarrow x = -y \Rightarrow V(\lambda_2) = \langle (1,-1) \rangle$

Let $v_1 = (1,1) \Rightarrow \vec{v}_1' = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$v_2 = (1,-1) \Rightarrow \vec{v}_2' = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

$\Rightarrow M = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

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$\cancel{[x' \ y']}$

$$M^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$M^T \cdot A \cdot M = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{3}{2} \end{bmatrix}$$

$$[x' \ y'] \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = 0$$

$$\Leftrightarrow \begin{bmatrix} \frac{x'}{2} & \frac{3y'}{2} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = 0$$

$$\Leftrightarrow \frac{x'^2}{2} + \frac{3y'^2}{2} = 0$$

5) Decide the type of quadratic curve based on a6112:

a) $x^2 - 4xy + y^2 = 0$

$$C : [x \ y \ 1] \underbrace{\begin{bmatrix} q_{11} & q_{12} & b_1 \\ q_{21} & q_{22} & b_2 \\ b_1 & b_2 & c \end{bmatrix}}_{\hat{Q}} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

$$C : q_{11}x^2 + 2q_{12}xy + q_{22}y^2 + 2b_1x + 2b_2y$$

$$Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} + c = 0$$

$\hat{\Delta} = \det(\hat{Q})$, $\Delta = \det(Q)$, $T = tr(Q)$

isometric invariants

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6) $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

a) Give the algebraic form of $R_{90°}$, $T_{\vec{v}}$, $T_{\vec{v}} \circ R_{90°}$

b) Determine the eq of $\mathcal{H}: \frac{x^2}{4} - \frac{y^2}{9} - 1 = 0$ after transforming it with $T_{\vec{v}} \circ R_{90°}$

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$T_{\vec{v}}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} + \vec{v} = \begin{bmatrix} x+1 \\ y \end{bmatrix}$$

$$T_{\vec{v}} \circ R_{90°}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1-y \\ x \end{bmatrix}$$

$$\begin{bmatrix} x & y \end{bmatrix} A \begin{bmatrix} x \\ y \end{bmatrix} = 1$$

$$\frac{x^2}{4} - \frac{y^2}{9} = 1$$

$$\Rightarrow A = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & -\frac{1}{9} \end{bmatrix}$$

$$(T_v \circ T_{90})^{-1}\left(\begin{bmatrix} x' \\ y' \end{bmatrix}\right) = \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{-1}}_{\text{transpose cause } \in SO_2}\begin{bmatrix} x'-1 \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x'-1 \\ y' \end{bmatrix}$$

$$= \begin{bmatrix} y' \\ 1-x' \end{bmatrix}$$

$$\begin{bmatrix} y' & 1-x' \end{bmatrix}\begin{bmatrix} \frac{1}{4} & 0 \\ 0 & -\frac{1}{9} \end{bmatrix}\begin{bmatrix} y' \\ 1-x' \end{bmatrix} = 0$$

$$\cancel{\begin{bmatrix} x'-1 & y' \end{bmatrix}\begin{bmatrix} y' \\ 1-x' \end{bmatrix} = 0}$$

$$\cancel{y'(x'-1) + y'(1-x') = 0}$$

$$y'x' - y' + y' - x'y' = 0 \Leftrightarrow 0 = 0$$

$$\begin{bmatrix} \frac{1}{4}y' & \frac{x'-1}{9} \end{bmatrix}\begin{bmatrix} y' \\ 1-x' \end{bmatrix}$$

$$= \frac{y'^2}{4} - \frac{(1-x')^2}{9} = 1$$

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Seminar 13

Quadratic curves & surfaces

7.7) Find the canonical equation: a) $5x^2+4xy+8y^2-32x-56y+80=0$

a) $5x^2+4xy+8y^2-32x-56y+80=0$

~~$(x+2y)^2+4x^2+4y^2-32x-56y+80=0$~~

~~$2(2y+x)^2$~~

$8\left(y+\frac{x}{4}\right)^2+5x^2-2x^2-32x-56y+80=0$

Let $y'=y+\frac{x}{4} \Rightarrow y=y'-\frac{x}{4}$

$\Rightarrow 8y'^2+3x^2-32x-56\left(y'-\frac{x}{4}\right)+80=0$

~~$8y'^2+3x^2-4x-56y'+80=0$~~

~~$3(x-2)^2+8($~~

~~$48y'^2+\frac{9}{2}x^2-18x-56y'+80=0$~~

$8y'^2-\frac{9}{2}x^2-18x-56y'+80=0$

$\frac{9}{2}(x-2)^2-18+8\left(y'-\frac{7}{2}\right)^2-\frac{49\cdot 8}{4}+80=0$

Let $x'=x-2$

$y''=y'-\frac{7}{2}$

~~$\frac{9}{2}x'^2-18(x'$~~

$\frac{9}{2}x'^2+8y''^2-36=0$

$\frac{x'^2}{\frac{2}{9}}+\frac{y''^2}{\frac{1}{8}}=36 \Rightarrow$ ellipse

8.8) $x'=x-2 \Rightarrow x=x'+2$

$y''=y+\frac{x}{4}-\frac{7}{2}$

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$$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix} + b$$

$$M = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix}$$

$$b = \begin{bmatrix} -2 \\ \frac{7}{2} \end{bmatrix}$$

7.9 Discuss the types of curves in terms of $\lambda \in \mathbb{R}$

$$x^2 + \lambda xy - 6x - 16 = 0$$

7.10

a) $x^2 + 2y^2 + z^2 + xy + yz + zx = 1$

b) $xy + yz + zx = 1$

$$\mathcal{C}: a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{23}yz + 2a_{13}xz + b_1x + b_2y + b_3z + c = 0$$

$$M = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 2 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}$$

$$\det(M - \lambda I_3) = 0$$

$$\Leftrightarrow \begin{vmatrix} 1-\lambda & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 2-\lambda & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1-\lambda \end{vmatrix} = \cancel{\frac{1}{8}(1-\lambda)^2(2-\lambda)}\ \frac{1}{8}\begin{vmatrix} 2-2\lambda & 1 & 1 \\ 1 & 4-2\lambda & 1 \\ 1 & 1 & 2-2\lambda \end{vmatrix} = 0$$

$$\Rightarrow 8(1-\lambda)^2(2-\lambda) + 1 + 1 - (4-2\lambda) - (2-2\lambda) - (2-2\lambda) = 0$$

$$8(1-\lambda)^2(2-\lambda) + \cancel{2} - 4 + 2\lambda - \cancel{2} + 2\lambda - 2 + 2\lambda = 0$$

$$8(1-\lambda)^2(2-\lambda) + 8\lambda - 6 = 0$$

$$8(1-2\lambda+\lambda^2)(2-\lambda) + 8\lambda - 6 = 0$$

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$8(2-\lambda-4\lambda+2\lambda^2+2\lambda^2-\lambda^3)+8\lambda-6=0$

$16-40\lambda+32\lambda^2-16\lambda^3+8\lambda-6=0$

$-16\lambda^3+32\lambda^2-32\lambda+10=0 \mid :2$

$-8\lambda^3+16\lambda^2-16\lambda+5=0$

$\cancel{\lambda_1\ \lambda=\frac{1}{2},\lambda_3=\frac{6}{2}}$

	λ^3	λ^2	λ^1	λ^0
	-8	16	-16	5
1		-8	8	

...

$\Rightarrow \lambda_1=1,\ \lambda_2=\frac{5}{2},\lambda_3=\frac{1}{2}$

$(M-\lambda I_3)\vec{v}=0$

I $\lambda=1 \Rightarrow \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}\vec{v}=0 \Rightarrow \begin{bmatrix} \frac{1}{2}y+\frac{1}{2}z \\ \frac{1}{2}x+y+\frac{1}{2}z \\ \frac{1}{2}x+\frac{1}{2}y \end{bmatrix}=0,$

$\Rightarrow \begin{cases} y+z=0 \Rightarrow y=-z \\ x+2y+z=0 \\ x+y=0 \Rightarrow x=-y=z \end{cases} \Rightarrow V(1)=\langle(1,-1,1)\rangle$
$=\langle(-1,1,-1)\rangle$

II $\lambda=\frac{1}{2} \Rightarrow \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}\vec{v}=0 \Rightarrow \begin{cases} x+y+z=0 \\ x+3y+z=0 \\ x+y+z=0 \Rightarrow x=-z \end{cases} \Rightarrow y=0$

$\Rightarrow V\left(\frac{1}{2}\right)=\langle(-1,0,1)\rangle$

III $\lambda=\frac{5}{2} \Rightarrow \begin{bmatrix} -\frac{3}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{3}{2} \end{bmatrix}\vec{v}=0 \Rightarrow \begin{cases} -3x+y+z=0 \Rightarrow y=2x \\ x-y+z=0 \\ x+y-3z=0 \end{cases}$

$2x-2z=0 \Rightarrow x=z$

$\Rightarrow V\left(\frac{5}{2}\right)=\langle(1,2,1)\rangle$

rank $M=3$, $\lambda_1,\lambda_2,\lambda_3>0 \Rightarrow$ signature is $(3,0) \Rightarrow$ ellipsoid

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8) $\mathcal{E}_{2,\sqrt{3},3}: \frac{x^2}{4}+\frac{y^2}{3}+\frac{z^2}{9}=1$

$\ell: x=y=z$

8.4. $\mathcal{E}_{2,3,2\sqrt{2}}: \frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{8}=1$

Find the tangent planes to $\mathcal{E}_{2,3,2\sqrt{3}}$ which are $\parallel$ to

$\pi: 3x-2y+5z+1=0$

$$\frac{x_0x}{a^2}+\frac{y_0y}{b^2}+\frac{z_0z}{c^2}=1$$

$$\frac{x_0x}{4}+\frac{y_0y}{9}+\frac{z_0z}{8}=1$$

$$\begin{cases} \frac{\frac{x_0}{4}}{3}=\frac{\frac{y_0}{9}}{-2}=\frac{\frac{z_0}{8}}{5}=t \Rightarrow \begin{array}{l} x_0=12t \\ y_0=-18t \\ z_0=40t \end{array} \\ \frac{x_0^2}{4}+\frac{y_0^2}{9}+\frac{z_0^2}{8}=1 \end{cases}$$

$$\frac{144t^2}{4}+\frac{324t^2}{9}+\frac{1600t^2}{8}=1$$

$$36t^2+36t^2+200t^2=1$$

$$272t^2=1 \Rightarrow t^2=\frac{1}{272} \Rightarrow t=\pm\frac{1}{\sqrt{272}}$$

$$\Rightarrow (x_0,y_0,z_0)\in\left\{\pm\left(\frac{12}{\sqrt{272}}, -\frac{18}{\sqrt{272}}, \frac{40}{\sqrt{272}}\right)\right\}$$

8.5. $\mathcal{E}: \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. Find $P_{(x_0,y_0,z_0)}\in\mathbb{E}^3$ s.t. $T_P\mathcal{E}$ intersect the coordonate axes in congruent segments

$\cancel{(\alpha,0,0)}$
$\cancel{(0,\alpha,0)}$
$\cancel{(0,0,\alpha)}$

$T\mathcal{E}\cap O_x \Rightarrow \frac{x_0x}{a^2}=1 \to x=\frac{a^2}{x_0}$

$T\mathcal{E}\cap O_y \to \frac{y_0y}{b^2}=1 \to y=\frac{b^2}{y_0}$

$T\mathcal{E}\cap O_z \Rightarrow \frac{z_0z}{c^2}=1 \Rightarrow z=\frac{c^2}{z_0}$

$\cancel{x_0\neq y_0\neq z_0}$ $\left|\frac{a^2}{x_0}\right|=\left|\frac{b^2}{y_0}\right|=\left|\frac{c^2}{z_0}\right|$

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$$|x_0| = \frac{|z_0| \cdot a^2}{c^2} \Rightarrow x_0 = \pm \frac{z_0 a^2}{c^2}$$

$$|y_0| = \frac{|z_0| b^2}{c^2} \Rightarrow y_0 = \pm \frac{z_0 b^2}{c^2}$$

$$\frac{z_0^2 a^{\cancel{4}}}{c^4 \cancel{a^2}} + \frac{z_0^2 b^{\cancel{4}}}{c^4 \cancel{b^2}} + \frac{z_0^2}{c^2} = 1$$

$$\frac{z_0^2}{c^2}\left(\frac{a^2}{c^2} + \frac{b^2}{c^2} + 1\right) = 1 \quad \ldots \quad z_0 = \pm \frac{c^2}{\sqrt{a^2+b^2+c^2}}$$

$$P = \frac{1}{\sqrt{a^2+b^2+c^2}}\left(\pm a^2, \pm b^2, \pm c^2\right)$$