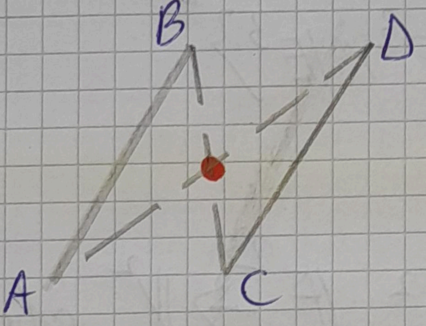
Geometry lecture 1

two pts A, B can be assembled in an ordered pair:
(A;B) which contains:

- distance } from A to B
- direction }
- location (the segment [A;B])

(A,B) and (C,D) are equipollent ((A,B) ~ (C,D)) if
[A,D] and [B C] have the same midpoint



these statements are equiv:

- (A,B) ~ (C,D)
- ~~ABCD~~ ABDC parallelogram
- (A,B) and (C,D) define the same distance and direction

$\forall (A,B)$ and a point O, $\exists! X$ s.t. $(A,B) \sim (O, X)$

$\overrightarrow{AB} = \{$ ordered pairs (X,Y) s.t. $(X,Y) \sim (A,B)\}$

the vector containing the ordered pair (A,B)

- the set of vectors defined with pts in $\mathbb{E}^2$ and $\mathbb{E}^3$ are the equivalence classes:

$\mathbb{V}^2 = \{\overrightarrow{AB} : (A,B) \in \mathbb{E}^2 \times \mathbb{E}^2\}$ or

$\mathbb{V}^3 = \{\overrightarrow{AB} : (A,B) \in \mathbb{E}^3 \times \mathbb{E}^3\}$

$\forall O$, the map $\phi_O^2 : \mathbb{E}^2 \to \mathbb{V}^2$ defined by

$\phi_O^2(A) = \overrightarrow{OA}$

is a bijection between pts and vectors

-//- for $\mathbb{E}^3$

for $a, b \in \mathbb{V}^3$ and $x, y \in \mathbb{R}$ we have:

- $(x+y) \cdot a = x \cdot a + y \cdot a$
- $x \cdot (a+b) = x \cdot a + x \cdot b$
- $x \cdot (y \cdot a) = (x \cdot y) \cdot a$
- $1 \cdot a = a$

$$\left.\begin{array}{l} (\mathbb{V}^2, +, \cdot) \\ (\mathbb{V}^3, +, \cdot) \end{array}\right\} \text{vector spaces}$$

$+ : \mathbb{V}^3 \times \mathbb{V}^3 \to \mathbb{V}^3$

$\cdot : \mathbb{R} \times \mathbb{V}^3 \to \mathbb{V}^3$

- we can define an addition of vectors with points, for $a \in \mathbb{V}^3$ and a point O $\exists!$ point X s.t. $a = \overrightarrow{OX} \Rightarrow$

$\Rightarrow$ we have a map:

$$+ \mathbb{V}^3 \times \mathbb{E}^3 \to \mathbb{E}^3 \cancel{a +}$$

given by $a + O = X$

- the vectors in $\mathbb{V}^3$ act on the set of points $\mathbb{E}^3$ by translations

Def:

- let ~~V~~ $\mathbb{V}^n$ be an n-dimensional real vector space
- an n-dimensional real affine space over $\mathbb{V}^n$ is

$A^n \neq \emptyset$, whose elem. are called points of A, tgtr. with a map

$$t : \mathbb{V}^n \times A^n \to A^n$$

which satisfies two axioms:

~~AS1:~~ ~~$\forall$~~ AS1: $\forall A, B \in A^n$, $\exists !\, a \in \mathbb{V}^n$ s.t. $B = t(a, A)$

AS2: $\forall A \in A^n$
$\forall a, b \in \mathbb{V}^n$ we have $t(a, t(b, a)) = t(a+b, A)$

Obs: if we fix a point O, then by AS1,
$\forall P \in A^n$, $\exists !\, v \in \mathbb{V}^n$ s.t. $P = t(v, O)$; we denote by $\overrightarrow{OP}$ and get a bijection $\phi_O^n : A^n \to \mathbb{V}^n$

Theorem

$S \subseteq E^3$, ~~$O \in S$~~ O a point in S

Then:

- S is a line $\Leftrightarrow$ $\phi_O^3(S)$ is a 1 dimensional vector subspace

$\vec{OA}, \vec{OB}$ are l.d $\Leftrightarrow$ O, A, B colin.

- S is a plane $\Leftrightarrow \phi_0^3(S)$ is a 2-dimensional vector subspace
- $\vec{OA}, \vec{OB}, \vec{OC}$ are l.d. $\Leftrightarrow$ O, A, B, C are coplanar
- if S is a line or plane, the vector subspace $\phi_0^3(S)$ is independent of the choice of O in S

Cartesian coordinate system

Def: a coord. system for E^2 (reference frame) is a tuple $K = (O, B)$, where O is the origin, and B is a basis of V^2, $B = (i, j)$

$$\forall P \in E^2, \exists! \, x_P, y_P \text{ s.t. }$$

$$\vec{OP} = x_P \cdot i + y_P \cdot j$$

$$[P]_K = P(x_P, y_P) = \begin{bmatrix} x_P \\ y_P \end{bmatrix}$$

Changing the basis in a vector space

- the matrix $M_{\varepsilon,F}$ is the change of basis matrix from the basis F to the basis ε; its columns are the coords. of the vectors in F with respect to ε

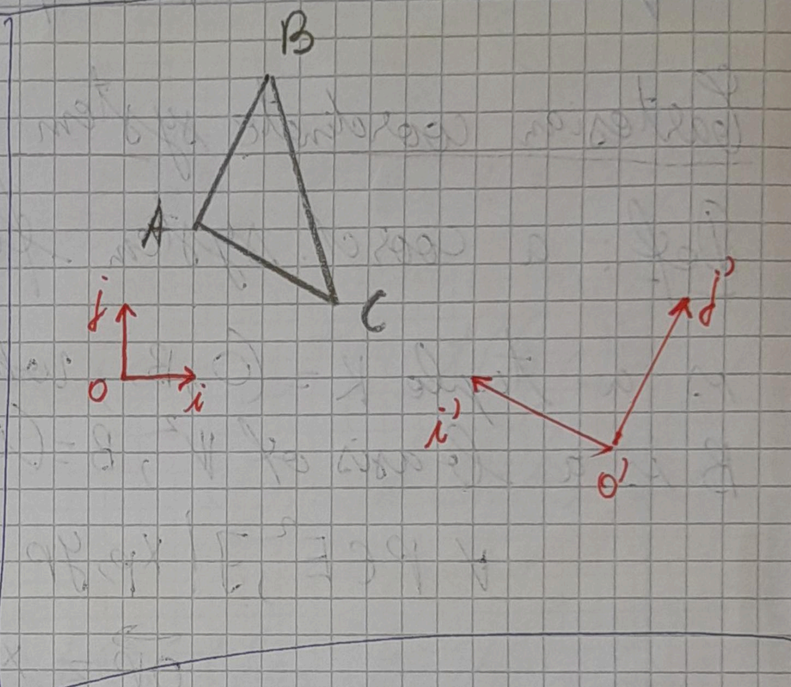
Changing ~~cos~~ coord. systems

let $K=(O,(i,j))$

$K'=(O',(i',j'))$ two coord. systems of E^2

suppose we know:

$[O']_K=\begin{bmatrix}7\\-1\end{bmatrix}_K$, $[i']_K=\begin{bmatrix}-2\\1\end{bmatrix}_K$, $[j']_K=\begin{bmatrix}1\\2\end{bmatrix}_K$



let $A\in E^2$

Step 1:

- change the origin

$\{\overrightarrow{O'A}\}=\overrightarrow{OA}-\overrightarrow{OO'}$

Step 2

change the coordinate axes

$[\overrightarrow{OA}]_{K'}=M_{K'K}\cdot[\overrightarrow{OA}]_K$

thus we obtain

$[\overrightarrow{O'A}]_{K'}=M_{K'K}\cdot\left([\overrightarrow{OA}]_K-[\overrightarrow{OO'}]_K\right)=M_{K'K}[\overrightarrow{OA}]_K-[\overrightarrow{OO'}]_{K'}$

therefore

$$\boxed{[A]_{K'}=\cancel{M_{K'K}\cdot[A]_K+[O]_{K'}}\ M_{K'K}\cdot\left([A]_K-[O']_K\right)= =M_{K'K}\cdot[A]_K+[O]_{K'}}$$