2. [8 pts] Consider the 3×3 matrix

$$B = \begin{bmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{bmatrix}, \tag{1}$$

where x, y, z are scalars.
(a) Use row operations to verify that $\det B = (y-x)(z-x)(z-y)$.
(b) What conditions must x, y, z satisfy in order for B to be invertible?

(a) $\det B = \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} = \begin{vmatrix} x & x^2 & 1 \\ y-x & y^2-x^2 & 0 \\ z-x & z^2-x^2 & 0 \end{vmatrix}$

$= (-1)^{1+3} \, 1 \cdot \begin{vmatrix} y-x & y^2-x^2 \\ z-x & z^2-x^2 \end{vmatrix} = (y-x)(z^2-x^2) - (z-x)(y^2-x^2)$

$= (y-x)(z-x)(z+x) - (z-x)(y-x)(y+x) = (y-x)(z-x)\left[(z+x)-(y+x)\right]$

$= (y-x)(z-x)(z-y)$.

(b) For B to be invertible, $\det B \neq 0$, i.e., x, y, z must be distinct.

3. [3 pts] Let A be an $m \times m$ matrix and $A^5 = O_m$, where O_m is the $m \times m$ zero matrix. Find $\det A$.

Since $\det A^5 = \det O_m = 0$ and $\det A^5 = (\det A)^5$, $(\det A)^5 = 0$. Therefore, $\boxed{\det A = 0}$

Quiz 6 Key P2

3. [6 pts] The matrix A and an echelon form of A are shown:

$$A = \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 4 & 7 & -4 & -3 & 9 \\ 6 & 9 & -5 & 2 & 4 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a). Find a basis for Col A.
(b). Find a basis for Nul A.

(a) As the echelon form shows that columns 1, 2, & 4 are pivot columns, we obtain a basis for col A:

$$\boxed{\left\{ \begin{bmatrix} 2 \\ 4 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 7 \\ 9 \\ -9 \end{bmatrix}, \begin{bmatrix} -4 \\ -3 \\ 2 \\ 5 \end{bmatrix} \right\}}$$

(b) Continue to use Row Reduction on Augmented matrix:

$$\begin{bmatrix} 2 & 5 & -3 & -4 & 8 & 0 \\ 0 & -3 & 2 & 5 & -7 & 0 \\ 0 & 0 & 0 & 4 & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 5 & -3 & -4 & 8 & 0 \\ 0 & -3 & 2 & 5 & -7 & 0 \\ 0 & 0 & 0 & 1 & -3/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 5 & -3 & 0 & 2 & 0 \\ 0 & -3 & 2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 & -3/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 5 & -3 & 0 & 2 & 0 \\ 0 & 1 & -2/3 & 0 & -1/6 & 0 \\ 0 & 0 & 0 & 1 & -3/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 0 & 1/3 & 0 & 17/6 & 0 \\ 0 & 1 & -2/3 & 0 & -1/6 & 0 \\ 0 & 0 & 0 & 1 & -3/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/6 & 0 & 17/12 & 0 \\ 0 & 1 & -2/3 & 0 & -1/6 & 0 \\ 0 & 0 & 0 & 1 & -3/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ Reduced echelon.

General solution to $A\vec{x} = \vec{0}$ is

$$\begin{cases} x_1 = -\frac{1}{6}x_3 - \frac{17}{12}x_5 \\ x_2 = \frac{2}{3}x_3 + \frac{1}{6}x_5 \\ x_4 = \quad \frac{3}{2}x_5 \\ x_3, x_5 \text{ are free} \end{cases}$$

i.e.

$$\vec{x} = x_3 \begin{bmatrix} -1/6 \\ 2/3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -\frac{17}{12} \\ 1/6 \\ 0 \\ 3/2 \\ 1 \end{bmatrix}.$$

A basis for Nul A is: $$\boxed{\left\{ \begin{bmatrix} -1/6 \\ 2/3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{17}{12} \\ 1/6 \\ 0 \\ 3/2 \\ 1 \end{bmatrix} \right\}}$$

General solution to $A\vec{x}=\vec{0}$ is

$$\begin{cases} x_1 = -3x_3 \\ x_2 = 3x_3 + 7x_5 \\ x_4 = 2x_5 \\ x_3, x_5 \text{ are free} \end{cases} \quad \text{i.e.} \quad \vec{x} = x_3\begin{bmatrix}-3\\3\\1\\0\\0\end{bmatrix} + x_5\begin{bmatrix}0\\7\\0\\2\\1\end{bmatrix}$$

A basis for Nul A is $\boxed{\left\{\begin{bmatrix}-3\\3\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\7\\0\\2\\1\end{bmatrix}\right\}}$

(30) dim Col A = $\boxed{3}$; dim Nul A = $\boxed{2}$.

#3. (10) Col A $\neq \mathbb{R}^3$. Reason: Since A is 4×7, every vector in Col A must have 4 entries. However, each vector in $\mathbb{R}^3$ has 3 entries. Thus, Col A $\neq \mathbb{R}^3$.

(20) dim Nul A = $\boxed{4}$ Reason: Since A has 3 pivot columns, dim Col A must be 3. Thus, dim Nul A = 7 − dim Col A
= 7 − 3 = 4.

#4. [There are many possible answers.] Here is an example:

$$\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\\0&0&0\end{bmatrix}.$$

(Another 'more interesting' example is $\begin{bmatrix}1&2&3\\1&2&3\\1&2&3\\1&2&3\end{bmatrix}$.)

3. [8 pts] Let

$$A = \begin{bmatrix} 1 & 3 & -3 & 4 \\ 2 & 8 & -6 & 8 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

(a) Describe all solutions of $A\mathbf{x} = \mathbf{0}$ in parametric vector form.
(b) Describe all solutions of $A\mathbf{x} = \mathbf{b}$ in parametric vector form.

(a) $\left[\begin{array}{cccc|c} 1 & 3 & -3 & 4 & 0 \\ 2 & 8 & -6 & 8 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 3 & -3 & 4 & 0 \\ 0 & 2 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 3 & -3 & 4 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & -3 & 4 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right]$

$A\vec{x} = \vec{0}$ has general solution: $\begin{cases} x_1 = 3x_3 - 4x_4 \\ x_2 = 0 \\ x_3, x_4 \text{ are free} \end{cases}$. Thus, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_3 - 4x_4 \\ 0 \\ x_3 \\ x_4 \end{bmatrix}$

$= \begin{bmatrix} 3x_3 \\ 0 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} -4x_4 \\ 0 \\ 0 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$

Solution in Parametric form: $\vec{x} = s\vec{u} + t\vec{v}$, for all real s, t, where $\vec{u} = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

(b) $\left[\begin{array}{cccc|c} 1 & 3 & -3 & 4 & 3 \\ 2 & 8 & -6 & 8 & 4 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 3 & -3 & 4 & 3 \\ 0 & 2 & 0 & 0 & -2 \end{array}\right] \sim \begin{bmatrix} 1 & 3 & -3 & 4 & 3 \\ 0 & 1 & 0 & 0 & -1 \end{bmatrix} \sim \left[\begin{array}{cccc|c} 1 & 0 & -3 & 4 & 4 \\ 0 & 1 & 0 & 0 & -1 \end{array}\right]$

general solution: $\begin{cases} x_1 = 4 + 3x_3 - 4x_4 \\ x_2 = -1 \\ x_3, x_4 \text{ are free} \end{cases}$ Thus, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 + 3x_3 - 4x_4 \\ -1 \\ x_3 \\ x_4 \end{bmatrix}$

$= \begin{bmatrix} 4 \\ -1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$ Solution in parametric form $\vec{x} = \vec{p} + s\vec{u} + t\vec{v}$ for all real s, t, where $\vec{p} = \begin{bmatrix} 4 \\ -1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

4. [3 pts] Do # 26 on Page 56 (in Section 1.5).

[First note that 'precisely when' means 'if and only if'.]

Since $A\vec{x} = \vec{b}$ has a solution, the solution is unique if and only if there are no free variables in $A\vec{x} = \vec{b}$, that is, if and only if every column of A is a pivot column. This happens if and only if the eqn $A\vec{x} = \vec{0}$ has only the trivial solution

3. [6 pts] Do # 6 on Page 80.

Consider $\left[\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 3 & -4 & 5 & 9 \\ 0 & 1 & 1 & 3 \\ -3 & 5 & -4 & -6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 2 & 2 & 6 \\ 0 & 1 & 1 & 3 \\ 0 & -1 & -1 & -3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 2 & 2 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ Thus, $\begin{cases} x_1 + 3x_3 = 7 \\ x_2 + x_3 = 3 \\ 0 = 0 \\ 0 = 0 \end{cases}$ i.e. $\begin{cases} x_1 = 7 - 3x_3 \\ x_2 = 3 - x_3 \\ x_3 \text{ is free} \end{cases}$

If we choose $x_3 = 1$, then $x_1 = 4$, $x_2 = 2$. Thus, $\vec{x} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$'s image under T is $\vec{b}$.

Apparently, $\vec{x}$ is not unique since x_3 is a free variable.

4. [7 pts] Consider the linear transformation $T : R^3 \to R^4$ defined by

$$T(x_1, x_2, x_3) = (x_1 - 2x_2 + 3x_3, 3x_1 - 2x_3, x_1 + x_2 + x_3, 2x_1 - 3x_2 + x_3)$$

(a) Find its standard matrix.
(b) Determine whether T is one-to-one. Justify your answer.

(a) $T(\vec{x}) = \begin{bmatrix} x_1 - 2x_2 + 3x_3 \\ 3x_1 - 2x_3 \\ x_1 + x_2 + x_3 \\ 2x_1 - 3x_2 + x_3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 3 \\ 3 & 0 & -2 \\ 1 & 1 & 1 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$

Thus, its standard matrix is $A = \begin{bmatrix} 1 & -2 & 3 \\ 3 & 0 & -2 \\ 1 & 1 & 1 \\ 2 & -3 & 1 \end{bmatrix}$.

(b) T is one-to-one if and only if $A\vec{x} = \vec{0}$ has only the trivial solution.

$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 3 & 0 & -2 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & -3 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & 6 & -11 & 0 \\ 0 & 3 & -2 & 0 \\ 0 & 1 & -5 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 6 & -11 & 0 \\ 0 & 3 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 19 & 0 \\ 0 & 0 & 13 & 0 \end{array}\right]$

Each column is a pivot column. Thus, $A\vec{x} = \vec{0}$ has only the trivial solution $\vec{0}$. Hence, T is one-to-one.