8.2.28 Compass-and-Ruler Constructions

(i) Given a triangle, locate the centroid.

(j) Given a circle, with two given points P. Q in its interior, inscribe a right angle in this circle, such that one leg passes through P and one leg passes through Q. The construction may not be possible, depending on the placement of P, Q.

**\*Note**: In other words, the problem is to find a point C such that C is on the circumference of the circle and ∠PQC = 90 . I figure out the construction as follow: Let call the given circle . Draw a circle with diameter PQ called , if at only one point then the intersection is C, else it is impossible to find C that satisfy all the condition. Couldn’t prove the case for 2 intersect, got the intuition why it is wrong though.

(k) Given two circles, draw the lines tangent to them.

**\*Note:** Let call the two circles . WLOG, assume R > R’. There will be four ways to construct the tangent lines. The idea is to figure out the point lies on the circumference of the bigger circle. It is based on the observation that when the tangent line of two circle is construct, we can construct a rectangle with two points of tangency, center of the smaller circle and an interior/exterior point of the bigger circle. The discovery of the last point will lead to two points of tangency. Take a point on the circumference of called A and a point with distance to = R’ called B, then draw the parallel line to AB from the center of called O’ intersect with the circumference of at C. Since AB = O’C, ∠ABO’ = ∠CO’B, BC = BC gives . Then we can conclude that AO’ and BC are parallel (since ∠AO’B = ∠CBO’). Which give ABO’C a parallelogram, now the trick is: If one there is one right angle in the parallelogram then all the other angles are right (two opposite angles are supplement), so we can choose B such that ABO’ = 90. In conclusion, the requirements for such B are AB = R’ and ABO’ = 90.

8.2.29 The triangle formed by joining the midpoints of the sides of a given triangle is called the medial triangle.

(a) Prove that the medial triangle and the original triangle have the same centroid. (For a much harder variation on this, see Problem 8.3.41.)

(b) Prove that the orthocenter (intersection of altitudes) of the medial triangle is the circumcenter of the original triangle.

8.2.30 Let e, and e 2 be parallel lines, and let OJ and y be two circles lying between these lines so that PI is tangent to OJ, OJ is tangent to y, and y is tangent to P2. Prove that the three points of tangency are collinear, i.e., lie on the same line.

8.2.36 (Bay Area Mathematical Olympiad 2000) Let ABC be a triangle with D the midpoint of side AB, E the midpoint of side BC, and F the midpoint of side AC. Let kl be the circle passing through points A, D, and F; let k2 be the circle passing through points B, E, and D; and let k3 be the circle passing through C, F, and E. Prove that circles k I , k2' and k3 intersect in a point.

**\*Note**: This can be solved with transformation.