6.1220 LECTURE 11

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Contents

1. Linear Programming I

1.1. Linear Programs

Definition 1.1: Linear Program

A linear program $\mathcal{P}$ takes the form

$$\{\max \text{ or } \min\} c_1 x_1 + \cdots + c_n x_n$$

This is whats called the objective function. Our linear program is also subject to the m linear restraints, with an additional non-negativity constraints:

$$\begin{gathered} A_{1,1}x_1 + \cdots + A_{1,n}x_n \ \{\leq or \geq or =\}\ b_1 \\ \vdots \\ A_{m,1}x_1 + \cdots + A_{m,n}x_n \ \{\leq or \geq or =\}\ b_m \\ x_1, \ldots, x_n \geq 0 \end{gathered}$$

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The variables/coefficeints in our definition are named as following, all of them in $\mathbb{R}$:

- $x_1, \ldots, x_n$: Decision variables.
- $c_1, \ldots, c_n$: Coefficients of objective function.
- $A_{1,1}, \ldots, A_{m,n}$: Coefficients of linear constraints.
- $b_1, \ldots, b_m$: Bounds of linear constraints.

Now we consider a toy problem.

Example 1.2

Consider two claw cranes (claw machine?) in an arcade. Label the cranes A and B. Suppose if we put x_1 dollars into crane A, we get toys having value of $\frac{x_1}{3}$. If we put x_2 dollars into crane B, we get toys having value of $\frac{x_2}{2}$. Since crane B is much more profitable for the customer, the crane owner puts the restraint that you can only put money into crane B as much as you put money into crane A. You only have 60 dollars given by your parents and anything you don't spend is confiscated at the end of the day. How do you maximize your profits?

Solution 1.3

Intuitively, we want to use crane B as much as possible to get the optimal solution. However, we can also model this problem as a linear programming problem:

$$\begin{aligned} \max & \frac{x_1}{3} + \frac{x_2}{2} \\ & x_1 - x_2 \geq 0 \\ & x_1 + x_2 \leq 60 \\ & x_1, x_2 \geq 0 \end{aligned}$$

1.2. Standard form

We want to make studying linear programs easier, so we reduce them to a "standard form".

Definition 1.4: Standard Form

(1) Max and Min objective functions are equivalent, since

$$\max f \iff \min(-f)$$

(2) The $\leq$ and $\geq$ constraints are equivalent.
(3) We can turn an equality $=$ into two inequalities of the form $\leq, \geq$
(4) We can convert inequalities of the form $\leq, \geq$ into equalities $=$, by introducing dummy variables. For example, we can turn $x_1 \leq 3 \to x_1 + x_1' = 3, x_1' \geq 0$.

Thus, we can assert in our linear program that we study either max or min and either all equality or inequalities of the same kind. For example, we can consider the Standard Maximization LP to be

$$\max \sum_{i=1}^{n} c_i x_i$$
$$\text{subject to } \sum_{i=1}^{n} A_{j,i} x_i \leq b_j, j \in \{0, \ldots, m\}$$
$$x_i \geq 0, i \in \{0, \ldots, n\}$$

To make our notation even more compact, we can use vector and matrix notation. We also introduce a couple of definitions.

Definition 1.5: Matrix-Vector Form of Linear Programming

We will define our LP to maximize $\vec{c}^T \vec{x}$, subject to $A\vec{x} \leq \vec{b}, \vec{x} \geq \vec{0}$.

Definition 1.6: Feasibility

Any set of solutions $x_1, \ldots x_n$ is feasible if they satisfy the constraints, otherwise they are unfeasible. The feasible region is the set of all feasible solutions. A linear program $\mathcal{P}$ is feasible if FEASIBLE($\mathcal{P}$) $\neq \emptyset$.

Definition 1.7: Boundedness

Let $\mathcal{P}$ be a feasible LP, then there exist a largest optimal value. If the optimal value is finite, then $\mathcal{P}$ is bounded. If it is infinite (make object function as large as you want), then $\mathcal{P}$ is unbounded.

We can also extend the definition of boundedness to feasible regions. The essential idea is if the feasible region is inside some sphere of radius r, then it is bounded. Otherwise, it is unbounded. Note here that while FEASIBLE($\mathcal{P}$) being bounded implies LP being bounded, the inverse is not true. FEASIBLE($\mathcal{P}$) being unbounded does not imply the LP is unbounded.

1.3. Geometric Visualization

LP is easy to think of for $n = 1$, and $n = 2$ tells us everything we need to know. Consider this $n = 2$ example:

Example 1.8

Consider this example LP $\mathcal{P}_5$:

$$\begin{aligned} \max x_1 + 2x_2 \\ -3x_1 + 2x_2 \leq 10 \\ x_1 + x_2 \leq 15 \\ x_1, x_2 \geq 0 \end{aligned}$$

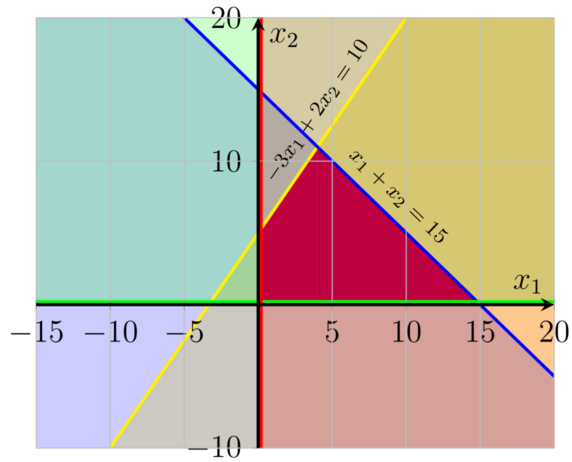

FIGURE 1. Graph of the constraints

Note that the feasible region must be convex. Now that we have the feasible region, we want to find the optimal point in the region. To do this, we can track the movement of the objective function with its corresponding line (the object function is linear). We can sweep the line over the feasible region to find the point where we finish sweeping/start sweeping, corresponding to our max/min optimal solution. The optimal solution is usually denoted as $\vec{x}^*$, with the star.

If there is more than optimal solution, then we say there is a degeneracy. Also, if constraints are useless/redundant/can be deduced from other constraints, we say that it is degenerate or redundant.

One of the algorithms that solves LP is the simplex algorithm, which checks point to point. Although this algorithm is exponential in nature, it is one of the best algorithms currently to solve LP.

1.4. Duality

The heart of duality is expressing the optimization function as a linear combination of the contraints. For example, continuing the previous example,

$$x_1 + 2x_2 = \frac{1}{5}(-3x_1 + 2x_2) + \frac{8}{5}(x_1 + x_2) \leq \frac{10}{5} + \frac{15 \cdot 8}{5} = 26$$

Writing this with notation abuse, we have

$$(-3x_1 + 2x_2 \leq 10)\frac{1}{5} + (x_1 + x_2 \leq 15)\frac{8}{5} = (x_1 + 2x_2 \leq 26)$$