6.1220 LECTURE 12

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Contents

1 Linear Programming II 1
1.1 Review 1
1.2 Taking Duals 2

1. Linear Programming II

1.1. Review

Let us review some concepts about linear programming. Consider the following example linear program:

Example 1.1: Properties of LP

$$\mathcal{P}_5 : \max x_1 + x_2$$
$$\text{subject to}: \; -3x_1 + 2x_2 \leq 10$$
$$x_1 + x_2 \leq 15$$
$$x_1, x_2 \geq 0$$

We list some properties of this linear program:

LP or not?	Yes
# of variables?	2
# of constraints?	2(+2 non-neg)
Standard form?	Yes
Feasible or not?	Yes
Bounded feasible region or not?	Yes
Bounded or not?	Yes
Optimal solution?	$(x_1, x_2) = (4, 11)$
Optimal value?	26
Optimum unique or degenerate?	Unique

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We also review the intuition behind taking the dual.

Example 1.2: Dual of a LP

Consider multiplying equation 1 and 2 by the nonnegative coefficients y_1, y_2 respectively:

$$\begin{aligned} -y_1 3x_1 + 2y_1 x_2 &\leq 10y_1 \\ y_2 x_2 + y_2 x_2 &\leq 15y_1 \end{aligned}$$

If we add them up and regroup the terms, we have

$$x_1(-3y_1 + y_2) + x_2(2y_1 + y_2) \leq 10y_1 + 15y_2$$

If we ensure that $-3y_1 + y_2 \geq 1$ and $2y_1 + y_2 \geq 2$, we achieve an upper bound for the objective function of $\mathcal{P}_5$. We try to tighten up this bound by minimizing $10y_1 + 15y_2$, which gives us the dual form:

$$\begin{aligned} \mathcal{P}_6 :& \min 10y_1 + 15y_2 \\ \text{subject to :}& -3y_1 + y_2 \geq 1 \\ & 2y_1 + y_2 \geq 2 \\ & y_1, y_2 \geq 0 \end{aligned}$$

As a sidenote, the dual here is in minimization standard form.

1.2. Taking Duals

We go over taking dual with matrices again.

Definition 1.3: Duals with Matrices and Vectors

Suppose we have the linear program:

$$\begin{aligned} \mathcal{P} :& \max \vec{c}^{\,T} \vec{x} \\ \text{subject to: }& A\vec{x} \leq \vec{b} \\ & \vec{x} \geq \vec{0} \end{aligned}$$

The dual of this linear program is:

$$\begin{aligned} \mathcal{D} :& \min \vec{b}^{\,T} \vec{y} \\ \text{subject to: }& A^T \vec{y} \geq \vec{c} \\ & \vec{y} \geq \vec{0} \end{aligned}$$

Now, what if the linear program isn't in standard form? We can derive from first principles how to take the the dual.

Example 1.4: Taking the dual of a non-standard linear program

Consider the linear program:

$$\begin{aligned} \mathcal{P}_7 : &\max x_1 - x_2 + 2x_3 \\ \text{subject to: } &(-x_1 + 3x_2 \le -3)y_1 \\ &(x_1 + 3x_3 \ge 2)y_2 \\ &(x_2 + x_3 = 7)y_3 \\ &x_1, x_2 \ge 0 \end{aligned}$$

To be able to sum the inequalities up, we need to make sure the inequalities are in the same direction. Specifically, we need to ensure $y_2 \le 0$ to flip the direction of the inequality. Also, y_3 can be any number since it is an inequality. We get the constraints $y_1 \ge 0, y_2 \le 0, y_3$ unbounded. Summing the inequalities up, we get

$$x_1(-y_1 + y_2) + x_2(3y_1 + y_3) + x_3(3y_2 + y_3) \le -3y_1 + 2y_2 + 7y_3$$

Using the same reasoning as before, we get the dual to be

$$\begin{aligned} \mathcal{D}_7 : &\min -3y_1 + 2y_2 + 7y_3 \\ \text{subject to: } &-y_1 + y_2 \ge 1 \\ &3y_1 + y_3 \ge -1 \\ &3y_2 + y_3 = 2 \\ &y_1 \ge 0, y_2 \le 0 \end{aligned}$$

Notably, the third inequality is an equality because x_3 can be any number, hence equality is the only way to ensure the dual achieves a certain upperbound on the primal problem.

At this point, we might question why we are even thinking about duals, considering its a random upper bound we obtain. However, there are many important theorems in algorithms such as optimality of Kruskal's or min-flow max-cut that can be proven with duality, showing that it indeed is a useful tool. Now let us look at duality theorems again.

Theorem 1.5: Weak Duality

If $\vec{x}$ and $\vec{y}$ are primed and dual feasible solutions,

$$\vec{c}^T\vec{x} \le \vec{b}^T\vec{y}$$

Proof

$$\vec{c}^T\vec{\tilde{x}} = \vec{\tilde{x}}^T\vec{c} \leq \vec{\tilde{x}}^T A^T\vec{\tilde{y}} = (A\vec{\tilde{x}})^T\vec{\tilde{y}} \leq \vec{b}^T\vec{\tilde{y}}$$

Although this proof is somewhat terse, it is expected as we specifically constructed the dual to maintain an upper bound. Now let us look at strong duality.

If primal is unbounded, the dual is infeasible because we can't find any upperbound on the primal linear bound. If we the dual is unbounded, we can make the objective function as minimal as we want, meaning there is feasible solution for the primal linear program.

Theorem 1.6: Strong Duality

if either $\mathcal{P}$ or $\mathcal{D}$ are bounded, then both are bounded. Furthermore, their optimum values are equal.

$$\vec{c}^T\vec{x^*} = \vec{b}^T\vec{y^*}$$

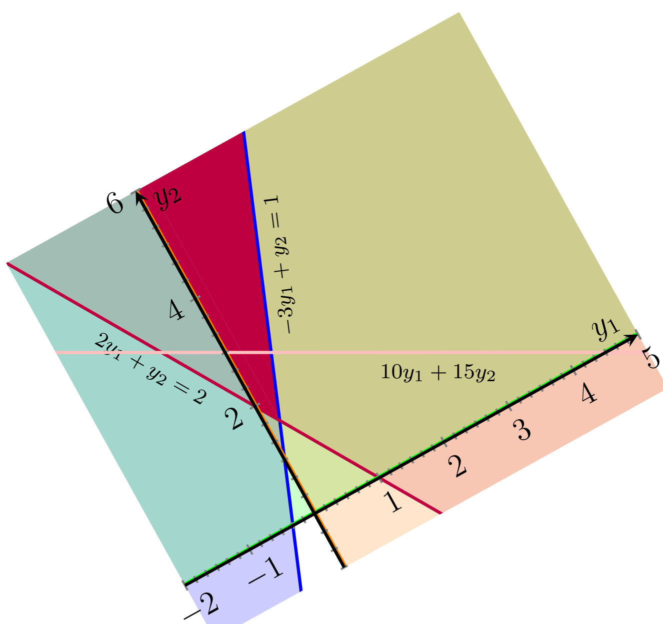

FIGURE 1. The rotated constraints, where the ball is dropped into

Proof of Strong Duality

We will prove using a generalizable example. Let us consider $\mathcal{P}_6$ again, which is the dual of $\mathcal{P}_5$.

$$\begin{aligned} \mathcal{P}_6 :\ & \min 10y_1 + 15y_2 \\ \text{subject to :}\ & -3y_1 + y_2 \geq 1 \\ & 2y_1 + y_2 \geq 2 \\ & y_1, y_2 \geq 0 \end{aligned}$$

Using the sweep line intuition, the optimal solution is where the line $10y_1+15y_2 = c$ lowest/most to the left. We can rephrase this intuition with physics. Consider each constraint to be a "wall". By rotating our diagram of $\mathcal{P}_6$ so that the line $10y_1 + 15y_2 = c$ is horizontal, if we drop a ball into the feasible region, it will eventually settle by gravity into some optimal position, held on by the walls. At that optimal position, the forces on the ball are gravity and the normal forces exerted by the walls. These forces are all normal to the constraints/function that they represent, so we can describe them as

$$\vec{F}_g = \begin{pmatrix} -10 \\ -15 \end{pmatrix}, \vec{F}_{n_1} = \tilde{x}_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix}, \vec{F}_{n_2} = \tilde{x}_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

after normalizing the gravitational force. Since the ball is at rest, the sum of the forces are 0, and we get this system of equations:

$$\begin{aligned} -3\tilde{x}_1 + 2\tilde{x}_2 &= 10 \\ \tilde{x}_1 + \tilde{x}_2 &= 15 \end{aligned}$$

However, the solution $\vec{x} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix}$ satisfies the linear program $\mathcal{P}_5$, meaning it is a primal feasible solution. From here, we note that since the ball is close to the walls that constrains it, the position y^* of the ball satisfies

$$\begin{aligned} -3y_1^* + y_2^* &= 1 \\ 2y_1^* + y_2^* &= 2 \end{aligned}$$

Multiplying by $\tilde{x}_1$ and $\tilde{x}_2$ and summing the equations, we get that $x_1^* + 2x_2^* = 10y_1^* + 15y_2^*$. This means that the objective values of the ball for both the primal and the dual linear program are the same. But since by weak duality, any primal feasible solution is upper bounded by the dual feasible solution we got, this must be the optimal feasible solution for the primal linear program.