6.1220 LECTURE 9

BRENDON JIANG

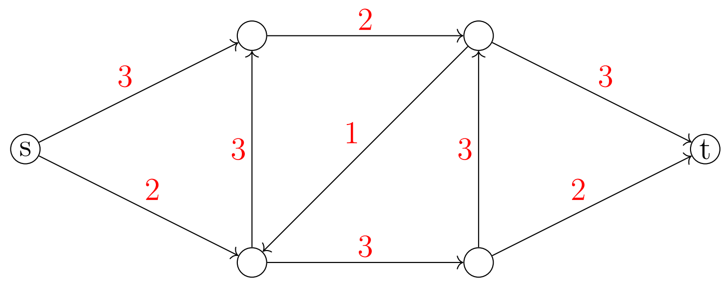
Contents

1. Max Flow I

1.1. Flow Networks

Definition 1.1: Flow Network

A **flow network** is a directed graph $G = (V, E)$ with two distinguished vertices: a source vertex s and a sink t.
Each directed edge $(u, v) \in E$ has a positive capacity $c(u, v)$. If $(u, v) \notin E$, then we define $c(u, v) = 0$.



Sample flow network, with capacities in red

We want to view flow as a rate, not a quantity. The for all source/sink nodes, each node should have the same incoming rate and outcoming rate. This is called flow conservation. Moreover, the flow can not exceed the capacity specified by the edge.

Date: September 2024

1.2. Maximum-flow problem

Definition 1.2: Maximum-flow problem

Given a flow network G, find the a flow of maximum value on G.

Basically, we want use the capacities so we have maximum outgoing flow from the source node and maximum incoming flow to the sink node.

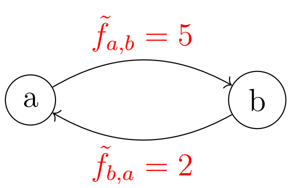
Definition 1.3: Gross and Net flow

A (gross) flow on G is a function $\tilde{f} : V \times V \to \mathbb{R}_{\geq 0}$ satisfying:

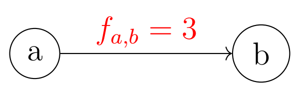
- Capacity constraint: For all $u, v \in V : \tilde{f}(u, v) \leq c(u, v)$.
- Flow conservation: For all $u \notin \{s, t\} : \displaystyle\sum_{v \in V} \tilde{f}(v, u) = \sum_{v \in V} \tilde{f}(u, v)$.

Given a gross flow $\tilde{f}$ we define its net flow $f : V \times V \to \mathbb{R}$ by taking for every edge $(u, v) \in E$:

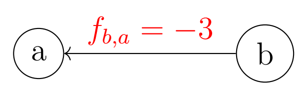
- $f(u, v) = \tilde{f}(u, v) - \tilde{f}(v, u)$
- $f(v, u) = -f(v, u)$



Gross flow



Net flow



Net flow

In other words, the net flow between two points is what we get when we cancel out their gross flows. If vertex a has a gross flow of 5 to b, and b has a gross flow of 2 to a, then in total vertex a has a net flow of 3 to b, and b has a net flow of -3 to a. We now show some properties of net flow.

Lemma 1.4

A net flow is a function $f : V \times V \to \mathbb{R}$ which satisfies:

- Capacity constraint: For all $u, v \in V : f(u, v) \leq c(u, v)$.
- Flow conservation: For all $u \notin \{s, t\} : \displaystyle\sum_{v \in V} f(u, v) = 0$.
- Skew symmetry: For all $u, v \in V : f(u, v) = -f(v, u)$.

We make these remarks:

(1) The capacity and skew symmetry constraints imply

$$f(u, v) > 0 \Rightarrow (u, v) \in E$$

(2) given f satisfying above constraints, we can define a gross flow $\tilde{f}$ that satisfies capacity and flow conservation setting:

$$\tilde{f}(u,v) = max(f(u,v),0)$$

Essentially, this take the "edge" that net flow corresponds to and turns it into a gross flow.

For each net flow, there are many ways we can implement gross flows for it.

Definition 1.5: Value of Net Flow

The value of a net flow, denoted by $|f|$, is given by $|f| = \sum_{v \in V} f(s,v)$.

From now on, we use implicit summation notation. A set used in an arithmetic formula represents the sum over the elements of the set. For example,

$$\text{value of flow} = |f| = \sum_{v \in V} f(s,v) = f(s,V)$$

Another example: the net flow conservation property can be written as

$$f(u,V) = 0 \text{ for all } u \notin \{s,t\}$$

We present some more properties of net flow.

Lemma 1.6

Let $X, Y, Z \subseteq V$ be sets of vertices. Then

- $f(X,X) = 0$. A simple proof:

$$f(X,X) = \sum_{x \in X} \sum_{x' \in X} f(x,x') = \sum_{x \in X} \sum_{x' \in X} \frac{f(x,x') - f(x',x)}{2} = 0$$

  by skew symmetry.
- $f(X,Y) = -f(Y,X)$
- $f(X \cup Y, Z) = f(X,Z) + f(Y,Z)$ if $X \cap Y = \emptyset$.

With these lemma, we can prove that the outgoing flow to the source node is the incoming flow to the sink node.

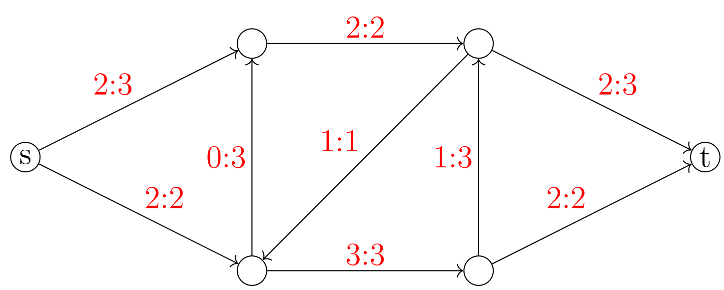
Theorem 1.7: Flow of source and sink node

A net flow f satisfies $|f| = f(V,t)$.

Proof

$$
\begin{aligned}
|f| &= f(s, V) \\
&= f(V, V) - f(V - \{s\}, V) \\
&= f(V, V - \{s\}) \\
&= f(V, t) + f(V, V - \{s\} - \{t\}) \\
&= f(V, t) + \sum_{u \notin \{s,t\}} f(V, u) \\
&= f(V, t)
\end{aligned}
$$

In the example below, the value of flow is $f(s, V) = 4$, and the flow into the sink is $f(V, t) = 4$. The edges are in the format $\{\text{Gross flow} : \text{Capacity}\} =$



Sample flow, flow out of source is equal to flow into sink

1.3. Cuts

Definition 1.8: Cut on flow network

A cut (S, T) of a flow network $G = (V, E)$ is a partition of V such that $s \in S$ and $t \in T$. If f is a net flow on G, then we call $f(S, T)$ the net flow across the cut.

Intuitively, the the set S must give the same flow as the source node to the set T. We can phrase this into a lemma:

Lemma 1.9

For any net flow f and any cut (S, T), we have $|f| = f(S, T)$.

Proof

$$
\begin{aligned}
f(S,T) &= f(S,V) - f(S,S) \\
&= f(S,V) \\
&= f(s,V) + f(S - \{s\}, V) \\
&= f(s,V)
\end{aligned}
$$

We can extend the definition of capacity to cuts:

Definition 1.10: Capacity of a cut

The capacity of a cut (S,T) is $c(S,T) = \sum_{x \in S} \sum_{y \in T} c(x,y)$.

Similarly, the capacity bounds the flow for cuts.

Theorem 1.11

The value of any net flow is bounded above by the capacity of any cut. In other words, for all cuts (S,T):

$$|f| \leq c(S,T)$$

Proof

$$
\begin{aligned}
|f| &= f(S,T) \\
&= \sum_{x \in S} \sum_{y \in T} f(x,y) \\
&\leq \sum_{x \in S} \sum_{y \in T} c(x,y) \\
&= c(S,T)
\end{aligned}
$$

Theorem 1.12: Max-flow Min-cut

Suppose f is net flow of maximum possible value. Then:

- there is some cut (S,T) that has $|f| = c(S,T)$.

Proof is delayed to the next lecture. Essentially, this theorem restates the fact that the flow is bounded above by capacity of cuts, and there is a cut that meets the maximum flow.

1.4. Residual Networks

Definition 1.13

Let f be a net flow on $G = (V, E, c)$. The residual network $G_f(V, E_f, c_f)$ is a weighted directed graph defined as flows:

- set $c_f(u, v) = c(u, v) - f(u, v)$
- E_f contains all pairs (u, v) such that $c_f(u, v) > 0$.

Essentially, we decrease the capacity of each edge by how much the flow already uses it, so we get opportunity to send c_f more flow to that edge.