6.1220 LECTURE 20

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Contents

1. Markov Chains/Random Walks II

Our main task today is given a probability distribution π, efficiently sample from π. In our main setting of interest, the state space of π is exponentially large. We can not hope to enumerate all possible states and pick a random one. Instead, we design a Markov chain. Our meta algorithm looks like this:

Algorithm 1: Meta-Algorithm

```
Sample X_0 ~ π_0
for t ∈ {1, ..., T} do
    Sample X_t ~ W(X_{t-1}, ·)
return X_T
```

We assume that

- $\pi(x) = \dfrac{F(x)}{\sum_y F(y)}$ and we can compute $F(x)$ given x.
- We can also sample from simple and well-known distributions.

1.1. Gerrymandering

Lawmakers can carve up the state into equal population congressional districts, each of which elects a single representative to the US House by popular vote. Gerrymandering is the political manipulation of electoral district boundaries to advantage a party, group, or socioeconomic class within the constituency. Techniques include crushing or packing, dividing up the districts to suit each parties goal. How do we detect gerrymandering?

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Example 1.1

We can first define some nonpartisan distribution π over a geometrically natural looking redistricting plan. We can sample a bunch of random plans from π, and compare statistics. The challenge with policy is defining π and getting every to agree. The algorithmic challenge is sampling, as there is an astronomical number of district plans.

1.2. Simulating Statistical Mechanics

Consider a discretized plane of hexagons. Each hexagon can possible be colored in, represeting a gas particle. We want to ensure that the hexagons form an independent set in the graph where edges connect the hexagons.

Suppose π is uniform over all independent sets. Given G, the number of possible independent set is exponentially huge in $V!$. Note that we are able to compute $F(s) = \mathbb{I}[\text{S is independent}]$ effeciently.

Algorithm 2: Gibbs Sampler Markov Chain

Sample $v \in V$ uniformly
if $v \in I$ **then**
  Remove v with probability $\frac{1}{2}$
else if $N[v] \cap I =$ **then**
  Add v with probability $\frac{1}{2}$
else
  Do nothing

Theorem 1.2

Let W denote the transition probability matrix of the Gibbs sampler with respect to the uniform distribution π over all independent sets.

- π is the unique stationary distribution of W, i.e. $\pi = \pi W$, and
- $\forall$ initial distribution $\pi_0, \pi_0 W^T \to \pi$ as $T \to \infty$

Proof

We can use the fundamental theorem of Markov Chains to prove the first and second part if we prove the graph is strongly connected. This strongly connected property is proven by the fact that every independent set communicates with the empty set. Now, all that is left is to verify that π is a stationary distribution of W. We prove this using a technique called reversibility (see below). We need to verify W is a symmetric matrix to complete the proof. This can be intuited by the fact that for whatever path we take from x to y, we can take the same path back from y to x with the same probability, so $W(x, y) = W(y, x)$.

Definition 1.3

We say W is reversible with respect to π if it satisfies the detailed balance equations:

$$\forall \text{ states } x, y, \pi(x) \cdot W(x,y) = \pi(y) \cdot W(y,x)$$

Following claims are left as exercises:

Claim 1.4

If W is reversible with respect to π, then $\pi = \pi W$.

Claim 1.5

$\forall t, \forall \text{states} a, b$, (assuming reversibility),

$$\Pr[X_t = a, X_{t+1} = b] = \Pr[X_{t+1} = a, X_t = b]$$

Claim 1.6

Random walk on weighted undirected graphs are reversible.

1.3. Metropolis-hastings

Definition 1.7: Metropolis-Hastings

We are given the distribution $\pi(x) = \dfrac{F(x)}{\sum_y F(y)}$. We start with any markov chain g. We define $W(x,y) = (x,y) \cdot P_{accept}(x,y)$ where $P_{accept}(x,y) = \min\left\{1, \dfrac{\pi(y) \cdot g(y,x)}{\pi(x) \cdot g(x,y)}\right\}$

Theorem 1.8

$\forall g, W(x,y) = g(x,y) P_{accept}(x,y)$ is reversible.

Proof

$$\pi(x) \cdot W(x,y) = \pi(x) g(x,y) \cdot \min\left\{1, \frac{\pi(y) \cdot g(y,x)}{\pi(x) \cdot g(x,y)}\right\} = \min\left\{\pi(x) g(x,y), \pi(y) g(y,x)\right\}$$

Theorem 1.9

Let W denote the transition probability matrix of the Metropolis-Hastings chain with respect the uniform distribution π over all q-colorings. If $q > \deg(v) + 1$ for all $v \in V$, then

- π is the unique stationary distribution of W, i.e. $\pi = \pi W$, and
- $\forall$ initial distribution $\pi_0, \pi_0 W^T \to \pi$ as $T \to \infty$.

Algorithm 3: Metropolis-Hastings for Graph Colorings

```
Sample v ∈ V uniformly
Sample c ∈ {1, ..., q} uniformly
if c is available to v then
    χ(v) ← c
else
    Do nothing
```