6.1220 LECTURE 12

BRENDON JIANG

Contents

1. Game Theory

1.1. Presidential Elections

We set up an overarching example that we will revisit in the last part of lecture.

Example 1.1: Presidential Elections

Suppose two candidates are competing for the presidency. Each has two topics appealing to their base, and wants to decide where to focus their campaign.

- Candidate X: focus on economy, society, or some mixture of the two
- Candidate Y: focus on morality, tax-cuts, or some mixture of the two

For each case, we can describe the outcome with a table, where a pair of values denotes the gains of each candidate in millions.

	Morality	Tax-cuts
Economy	3, -3	-1, 1
Society	-2, 2	1, -1

1.2. Game Theory

Let us consider a simpler game.

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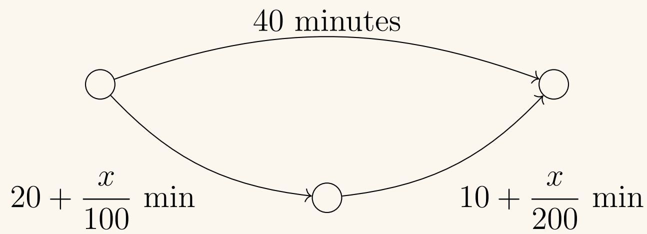
Example 1.2: Road Congestion

1000 drivers drive at 5pm each day from Boston to Lexington. There are 2 possible routes they can take. They either take MA-2A, or a combo of MA-2 and I-95. MA-2 takes a constant 40 minute to traverse, while MA-2 and I-95 take $20 + \frac{x}{100}$ and $10 + \frac{x}{200}$ minutes respectively, where x is the number of drivers on the road.



Prediction

Suppose x drivers use bottom route at steady state. The time it takes to travel the bottom route is then $30 + \frac{3x}{200}$ minute. If the cost is greater than 40 minute (aka $x \geq 667$), then the bottom route is suboptimal. Vice versa, if $30 + \frac{3x}{200} < 40$, then the top route is suboptimal. therefore $x = 666$ is the steady state which the population will strive to achieve.

Example 1.3: Prisoner's Dilemma

Suppose two members of a criminal gang are arrested and imprisoned. Each prisoner is in solitary confinement with no means of speaking to or exchanging messages with the other. The police admit they don't have evidence to convict the pair on the principal charge. They plan to sentence both to $\frac{1}{2}$ a year in prison on a lesser charge. Simultaneously, the police offer each prisoner a bargain, where they can betray their partner for no time in prison or 5 years depending on if the partner also stays silent.

	Silent	Betray
Silent	$-\frac{1}{2}, -\frac{1}{2}$	-10, 0
Betray	0, -10	-5, -5

What will happen?

From each row or column's perspective, choosing the betray strategy always yield a better result no matter what (betray dominates silent). Therefore, the rational strategy for both players is to betray. This is a dominant strategy equilibrium.

Games are thought experiments to help us learn how to predict rational behavior in situations of conflict.

Definition 1.4: Situation of conflict

Each player's utility/payoff is affected by their own as well as other players' actions.

We also assume rational behavior, where each player want to maximize their own utility. We assume there is no unmodeled altruism, envy, masochism, etc. If we want to introduce these, they have to be in the model.

Example 1.5: Rock-Paper-Scissors

	Rock	Paper	Scissors
Rock	0,0	-1, 1	1, -1
Paper	1,-1	0,0	-1, 1
Scissors	-1,1	1,-1	0,0

No pair of strategy is stable, because there is always a strategy that can best a strategy. However, it turns out there is a pair of stable randomized strategy. It turns out the uniform strategy (1/3, 1/3, 1/3) is optimal against itself. To show this, note that playing one of rock, scissor, paper has expected payoff of 0 always. Thus, any randomized combination of these cases is always 0. Therefore, any strategy is optimal against the uniforms strategy.

However, note that the uniform strategy is not optimal against every strategy. For example, when facing an all-rock strategy, it will be better to play paper everytime.

1.3. Nash's Theorem

Definition 1.6: Nash Equilibrium

There exists a collection of potentially randomized strategies for each player of the game such that, given the strategies of the other players, no player had an incentive to unilaterally change their strategy.

Does every game have an Nash Equilibrium? Nash in 1950 showed that every finite game (finite players and finite strategies), there exists at least one collection of randomized strategies such that no player has an incentive to unilaterally change their own strategy given the strategies of the others.

We'll prove Nash's theorem for the special case of 2-player 0-sum games.

Definition 1.7: 0-sum game

A game where the sum of players' payoffs is 0 for any outcome.

1.4. Min-Max Theorem via LP duality

Let us go back to the Presidential Election game. Not that no set of deterministic strategies is optimal against each other. We assume each candidate randomizes their strategy.

	Morality	Tax-cuts
Economy	3, -3	-1, 1
Society	-2, 2	1, -1

Suppose candidate X focus with probability x_1 on economy and with probability x_2 on society, where $x_1 + x_2 = 1$ and $x_1, x_2 \geq 0$. Define y_1, y_2 similarly, where y_1 is for morality and y_2 is for tax-cuts. Then, the expected gains of the two candidates are

$$u_X(x, y) = 3x_1y_1 - 1x_1y_2 - 2x_2y_1 + 1x_2y_2$$
$$u_Y(x, y) = -3x_1y_1 + 1x_2y_2 + 2x_2y_1 - 1x_2y_2 = -u_X(x, y)$$

We do a thought experiment. Suppose Candidate X is forced to commit to a strategy $x = (x_1, x_2)$ and announce it. Then, Y will think they would get $-3x_1 + 2x_2$ for morality, and they would get $x_1 - x_2$ for playing tax-cuts. Thus, candidate Y wants to take $\max(-3x_1 + 2x_2, x_1 - x_2)$. This means the payoff for X would be $-\max(-3x_1 + 2x_2, x_1 - x_2) = \min(-3x_1 - 2x_2, -x_1 + x_2)$. Thus if X was forced to commit, they would choose x to maximize their payoff:

$$\min(3x_1 - 2x_2, -x_1 + x_2)$$

To formulate this problem as a linear program, we consider a dummy variable z that represents the minimum. We then get the linear program

$$\begin{aligned} \max : z& \\ \text{s.t. } & 3x_1 - 2x_2 \geq z \\ & -x_1 + x_2 \geq z \\ & x_1 + x_2 = 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

We think about this thought experiment for Candidate Y. By following the exact same logic, X will think they could get $3y_1 - y_2$ for playing economy and $-2y_1 + y_2$ for playing society for a best score of $\max(3y_1 - y_2, -2y_1 + y_2)$. Y then wants to minimize $X's$ payoff of:

$$\max(3y_1 - y_2, -2y_1 + y_2)$$

We can also set up a LP for this, with dummy variable w:

$$\begin{aligned} \min : & \, w \\ \text{s.t. } & 3y_1 - y_2 \le w \\ & -2y_1 + y_2 \le w \\ & y_1 + y_2 = 1 \\ & y_1, y_2 \ge 0 \end{aligned}$$

Both of these solutions have an optimal objective value of $\frac{1}{7}$. In the first LP, we see that X wins at least $\frac{1}{7}$, while the second strategy gives that Y loses at most $\frac{1}{7}$. This set of solution is a Nash equilibrium, because the strategies are optimal against each other. This is also not a coincidence, because the linear programs are actually duals (we can convert to standard form by taking $z = z^+ - z^-$ and $w = w^+ - w^-$ to make variables non-negative), so by strong duality they must have the same objective value.