Applications of Markov Chains: Queueing Models

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under the supervision of

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Motivation

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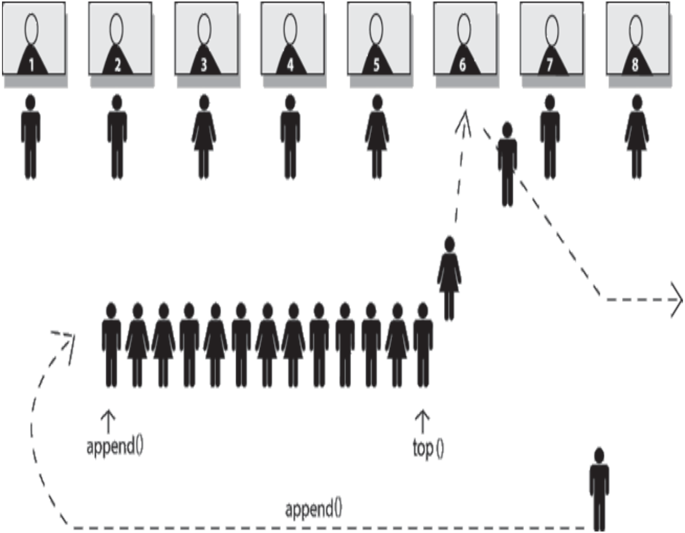


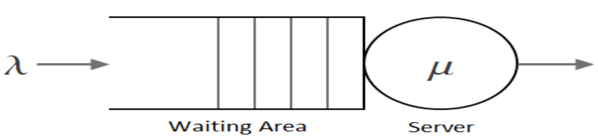
Basic concepts

- Generating function
  - The generating function $X(z)$ of a nonnegative discrete random variable, X, with $\mathbb{P}(X = n) = \pi_n$, $n = 0, 1, 2, \ldots$, is defined as
    - $X(z) = \mathbb{E}[z^X] = \sum_{n=0}^{\infty} \pi_n z^n$ for all $|z| \leq 1$
  - $X(1) = 1, X^{(k)}(1) = \mathbb{E}[X(X-1)\ldots(X-k+1)]$
- Laplace-Stieltjes transform (LST)
  - The Laplace-Stieltjes transform $L(\omega)$ of a nonnegative random variable, X, with density function $f(.)$, is defined as
    - $L(\omega) = \mathbb{E}[e^{-\omega X}] = \int_{x=0}^{\infty} e^{-\omega x} f(x)dx$ for all $\omega \geq 0$
  - $L(0) = 1, L^{(k)}(0) = (-1)^k \mathbb{E}[X^k]$

Basic concepts...

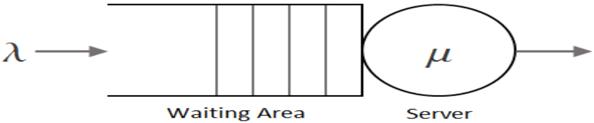
- Exponential distribution
  - The density of an exponential distribution with parameter λ is given by
    - $f(t) = \lambda e^{-\lambda t}$ for $t > 0$
  - Memoryless property: $\mathbb{P}(X > s + t | X > s) = \mathbb{P}(X > t)$
- Poisson process
  - N(t), the number of arrivals in $[0, t]$ for a Poisson process with rate λ, has a Poisson distribution with parameter λt,
    - $\mathbb{P}(N(t) = n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$ for $n = 0, 1, 2, \ldots$
  - Note that interarrival times are exponentially distributed with parameter λ

M/M/1 (Example 5.2.1)



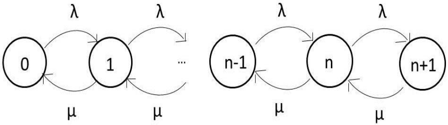
- Single server queue
- Customers arrive according to Poisson process with rate λ
  - exponential inter-arrival times with mean $1/\lambda$
- Exponential service times with mean $1/\mu$
- First-in-queue-first-out-of-queue (FIFO)

M/M/1 (Example 5.2.1)



- $X(t)$ denote the number of customers in the system at time t, where 'system' means waiting plus service area, such that
  - $\pi_n(t) = \mathbb{P}(X(t) = n)$ for $n = 0, 1, 2, \ldots$
- $X(t)$ jumps up by amount 1 at an arrival time
- $X(t)$ jumps down by amount 1 at a departure time

M/M/1 (Example 5.2.1)



- $X(t)$ is a continuous time Markov chain with state space $\{0, 1, 2, \ldots\}$ and transition rates $q_{ij} = \begin{cases} \lambda, & j = i + 1 \\ \mu, & j = i - 1 \\ 0, & \text{otherwise.} \end{cases}$
- Limiting (invariant) distribution $\pi_n = \lim_{t \to \infty} \pi_n(t)$
- Equilibrium (balance) equations:

$$\lambda \pi_0 = \mu \pi_1$$
$$(\lambda + \mu)\pi_n = \lambda \pi_{n-1} + \mu \pi_{n+1}, \quad n = 1, 2, \ldots$$

- Normalization equation: $\sum_{n=0}^{\infty} \pi_n = 1$

M/M/1 (Example 5.2.1)

- Therefore, the limiting probability is given by

$$\pi_n = (1-\rho)\rho^n \tag{1}$$

where $\rho = \frac{\lambda}{\mu} < 1$ (condition for the positive recurrent)

- Probability generation function of the number of customers in the system is obtained as

$$\begin{aligned} G(z) &= \sum_{n=0}^{\infty} \pi_n z^n = \sum_{n=0}^{\infty} (1-\rho)\rho^n z^n \\ &= \frac{(1-\rho)}{1-\rho z} \quad \text{for } |z| \le 1 \end{aligned} \tag{2}$$

M/M/1 (Example 5.2.1)

- W denote the steady-state waiting time of a customer
- T denote the service time of a customer
- $S = W + T$ is the steady-state sojourn time of a customer
- Therefore, $S = \sum_{k=1}^{X^a+1} T_k$, where
  - X^a is the number of customers in the system at the arrival time of a customer
  - T_k is the service time of the k-th customer ($T_k \sim T$)
- Remark: PASTA property implies that

$$\mathbb{P}(X^a = n) = \pi_n = (1-\rho)\rho^n \tag{3}$$

M/M/1 (Example 5.2.1)

- LST of the sojourn time of a customer is obtained by conditioning on X^a, which is given by

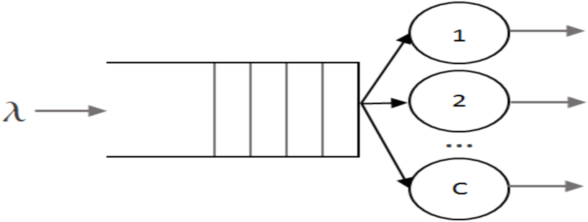
$$\begin{aligned} S(\omega) &= \mathbb{E}[e^{-\omega S}] \\ &= \frac{\mu(1-\rho)}{\mu(1-\rho)+\omega} \end{aligned} \tag{4}$$

which is the LST of the exponentially distributed random variable

- Hence LST of the waiting time of a customer is given by

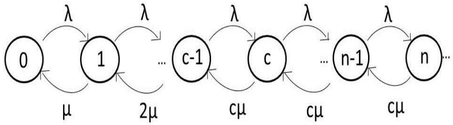
$$W(\omega) = \frac{\mu+\omega}{\mu} S(\omega)$$

M/M/c (Example 5.2.2)



- c parallel identical servers
- Customers arrive according to Poisson process with rate λ
- Exponential service times with mean $1/\mu$
- Customers are served in order of arrival (FIFO)

M/M/c (Example 5.2.2)



- $X(t)$ is a continuous time Markov chain with state space $\{0, 1, 2, \ldots\}$ and transition rates $q_{ij} = \begin{cases} \lambda, & j = i+1 \\ \min(c,i)\mu, & j = i-1 \\ 0, & \text{otherwise.} \end{cases}$
- Balance equations:

$$\begin{aligned} \lambda\pi_0 &= \mu\pi_1 \\ (\lambda + n\mu)\pi_n &= \lambda\pi_{n-1} + (n+1)\mu\pi_{n+1}, \quad 1 \le n \le c-1 \\ (\lambda + c\mu)\pi_n &= \lambda\pi_{n-1} + c\mu\pi_{n+1}, \quad n \ge c \end{aligned}$$

- Iterating implies that $\lambda\pi_{n-1} = \min(c,n)\mu\pi_n$

M/M/c (Example 5.2.2)

- After simplification, we obtain

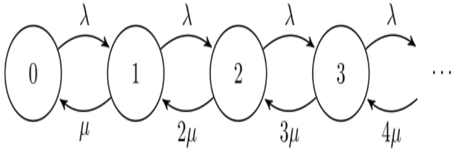
$$\pi_n = \frac{(c\rho)^n}{n!}\pi_0, \quad n = 1, 2, \ldots, c \tag{5}$$

$$\pi_{c+m} = \rho^m \frac{(c\rho)^c}{c!}\pi_0, \quad m = 0, 1, 2, \ldots, \tag{6}$$

where $\pi_0 = \left(\sum_{n=0}^{c-1} \frac{(c\rho)^n}{n!} + \frac{(c\rho)^c}{c!}\frac{1}{1-\rho}\right)^{-1}$ provided $\rho = \frac{\lambda}{c\mu} < 1$

- Stability condition : $\rho = \frac{\lambda}{c\mu} < 1$

M/M/∞



- $X(t)$ is a continuous time Markov chain with state space $\{0, 1, 2, \dots\}$ and transition rates $q_{ij} = \begin{cases} \lambda, & j = i+1 \\ i\mu, & j = i-1 \\ 0, & \text{otherwise.} \end{cases}$
- Balance equations:

$$\lambda \pi_0 = \mu \pi_1$$
$$(\lambda + n\mu)\pi_n = \lambda \pi_{n-1} + (n+1)\mu \pi_{n+1}, \quad n = 1, 2, \dots$$

- Iterating implies that $\lambda \pi_{n-1} = n\mu\pi_n$

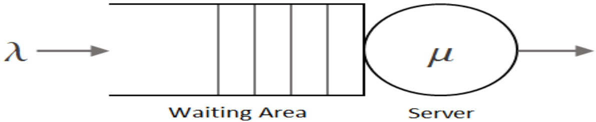
M/M/∞

- After simplification, we obtain

$$\pi_n = \frac{\rho^n}{n!} e^{-\rho}, \quad n = 1, 2, \dots \tag{7}$$

  where $\rho = \frac{\lambda}{\mu}$
- Hence, the number of customers in the system has a Poisson distribution with mean ρ.

M/G/1 (Example 5.2.7)



- Single server queue
- Customers arrive according to Poisson process with rate λ
- The service times are independent and identically distributed with general probability density function $f(.)$
- Here, the number of customers in the system at time t, $X(t)$, is not a continuous time Markov chain

M/G/1 (Example 5.2.7)

- Recurrence relations:

$$X_k^d = \begin{cases} X_{k-1}^d - 1 + Y_k & \text{if } X_{k-1}^d \geq 1 \\ Y_k & \text{if } X_{k-1}^d = 0 \end{cases}, \quad k = 1, 2, 3, \dots, \quad (8)$$

  - X_k^d is the number of customers at the departure time of the kth customer
  - Y_k is the number of arrivals during the service time of the kth customer
- The sequence X_k^d forms a Markov chain
- As we look at the departure times (embedded points), the Markov chain is called the embedded Markov chain

M/G/1 (Example 5.2.7)

- Transition probabilities:

$$
\begin{aligned}
p_{ij} &= \mathbb{P}(X_k^d = j | X_{k-1}^d = i) \\
&= \begin{cases} \alpha_j, & \text{if } i = 0 \\ \alpha_{j-i+1}, & \text{if } j \geq i-1, i > 0 \\ 0, & \text{otherwise,} \end{cases}
\end{aligned}
$$

  where α_n is the probability that during the service time of a customer n customers arrive

- As the number of customers, that arrive during the service time, is Poisson distributed with parameter λt, α_n is given by

$$\alpha_n = \int_{t=0}^{\infty} \frac{(\lambda t)^n}{n!} e^{-\lambda t} f(t) \mathrm{d}t \tag{9}$$

M/G/1 (Example 5.2.7)

- Limiting distributions, $\pi_n = \lim_{k\to\infty} \mathbb{P}(X_k^d = n)$, are the solutions of the linear system

$$\pi P = \pi$$

  - $\pi = (\pi_0, \pi_1, \pi_2, \ldots, \pi_n, \ldots)$
  - $P = [P_{ij}]$ is the transition probability matrix
- Balance equations:

$$\begin{aligned}\pi_n =& \pi_0\alpha_n + \pi_1\alpha_n + \pi_2\alpha_{n-1} + \cdots + \pi_n\alpha_1 + \pi_{n+1}\alpha_0 \\ =& \pi_0\alpha_n + \sum_{k=0}^{n} \pi_{n+1-k}\alpha_k, \quad n = 0, 1, 2, \ldots\end{aligned}$$

- To solve the balance equations, we will use the generating function approach

M/G/1 (Example 5.2.7)

- Define the probability generating functions: $G(z) = \mathbb{E}[z^{X^d}], A(z) = \mathbb{E}[z^Y]$ for $|z| \leq 1$,

$$
\begin{aligned}
G(z) =& \sum_{n=0}^{\infty} \left(\pi_0 \alpha_n + \sum_{k=0}^{n} \pi_{n+1-k} \alpha_k \right) z^n \\
=& \pi_0 \sum_{n=0}^{\infty} \alpha_n z^n + z^{-1} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \pi_{n+1-k} z^{n+k-1} \alpha_k z^k \\
=& \pi_0 A(z) + z^{-1} \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \pi_{n+1-k} z^{n+k-1} \alpha_k z^k \\
=& \pi_0 A(z) + z^{-1} A(z)(G(z) - \pi_0) \\
\Longrightarrow \quad G(z) =& \frac{\pi_0 (z-1) A(z)}{z - A(z)} \qquad (10)
\end{aligned}
$$

M/G/1 (Example 5.2.7)

- Note that $G(1) = 1, A(1) = 1$ and $A'(1) = \mathbb{E}[A] = \rho$, this implies that

$$
\begin{aligned}
G(1) &= \lim_{z\to 1} \frac{\pi_0 (z-1) A(z)}{z - A(z)} \\
\implies 1 &= \lim_{z\to 1} \frac{\pi_0 \Big((z-1)A'(z) + A(z) \Big)}{1 - A'(z)} \\
\implies 1 &= \frac{\pi_0}{1 - \mathbb{E}[A]} \\
\implies \pi_0 &= 1 - \rho \qquad (11)
\end{aligned}
$$

M/G/1 (Example 5.2.7)

- Using (9), we can obtain $A(z)$ as

$$\begin{aligned}
A(z) &= \sum_{n=0}^{\infty} \alpha_n z^n \\
&= \sum_{n=0}^{\infty} \int_{t=0}^{\infty} \frac{(\lambda t)^n}{n!} e^{-\lambda t} f(t) \mathrm{d} t z^n \\
&= \int_{t=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\lambda t z)^n}{n!} e^{-\lambda t} f(t) \mathrm{d} t \\
&= \int_{t=0}^{\infty} e^{\lambda t z} e^{-\lambda t} f(t) \mathrm{d} t \\
&= \int_{t=0}^{\infty} e^{-\lambda(1-z) t} f(t) \mathrm{d} t \\
&= L(\lambda(1-z))
\end{aligned} \tag{12}$$

M/G/1 (Example 5.2.7)

- Hence, the probability generating function $G(z)$ is given by

$$G(z) = \frac{(1-\rho)(z-1)L(\lambda(1-z))}{z - L(\lambda(1-z))} \text{ provided } \rho < 1 \tag{13}$$

- Remark: using PASTA and the transition diagram, we can write

$$\lim_{k\to\infty} \mathbb{P}(X_k^a = n) = \lim_{t\to\infty} \mathbb{P}(X(t) = n) = \pi_n \tag{14}$$

  - X_k^a is the number of customers in the system at the arrival time of the kth customer
  - $X(t)$ is the number of customers in the system at time t

Summary

- Waiting time to $M/M/c$ model is a nice exercise
- Embedded Markov chain in $M/G/1$ is important
- Markov chains can also be used in Queueing networks

THANKS