Proof of Primorial Recurrence Equivalence

Supplementary Material

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1 Statement of Result

This document provides a rigorous proof that the iterative recurrence formulation is equivalent to the direct alternating factorial sum for computing primorials.

Theorem 1 (Recurrence Equivalence). *Define the recurrence relation starting from $S_0 = \{0, 0, 1\}$:*

$$S_{n+1} = \left\{ n+1,\, b_n,\, b_n + (a_n - b_n)\left(n + \frac{1}{3+2n}\right) \right\} \tag{1}$$

where $S_n = \{n, a_n, b_n\}$ denotes the state after n iterations.

Then for all $n \geq 0$:

$$b_n = 1 + 2\sum_{k=1}^{n} \frac{(-1)^k \cdot k!}{2k+1} \tag{2}$$

2 Proof

2.1 Base Case

For $n = 0$:

- From initialization: $b_0 = 1$
- From equation (2): $1 + 2 \cdot (\text{empty sum}) = 1 + 0 = 1$

Thus the base case holds.

2.2 Recurrence Structure

Lemma 2. *For all $n \geq 1$, we have $a_n = b_{n-1}$.*

Proof. This follows directly from the recurrence update: $S_{n+1} = \{n+1, b_n, \ldots\}$ sets a_{n+1} to the previous value of b_n. □

2.3 Inductive Step

Assume equation (2) holds for n, i.e., $b_n = 1 + 2S_n$ where

$$S_n = \sum_{k=1}^{n} \frac{(-1)^k \cdot k!}{2k+1}$$

By Lemma 2, we also have:

$$a_n = b_{n-1} = 1 + 2S_{n-1}$$

From the recurrence (1):

$$\begin{aligned} b_{n+1} &= b_n + (a_n - b_n)\left(n + \frac{1}{3+2n}\right) \\ &= (1 + 2S_n) + (1 + 2S_{n-1} - 1 - 2S_n)\left(n + \frac{1}{3+2n}\right) \\ &= 1 + 2S_n + 2(S_{n-1} - S_n)\left(n + \frac{1}{3+2n}\right) \end{aligned} \tag{3}$$

Lemma 3. *The difference of consecutive partial sums is:*

$$S_n - S_{n-1} = \frac{(-1)^n \cdot n!}{2n+1}$$

Proof. By definition, S_n includes all terms from $k = 1$ to $k = n$, while S_{n-1} includes only $k = 1$ to $k = n - 1$. The difference is precisely the n-th term. □

Substituting Lemma 3 into equation (3):

$$b_{n+1} = 1 + 2S_n - 2 \cdot \frac{(-1)^n \cdot n!}{2n+1} \cdot \left(n + \frac{1}{3+2n}\right) \tag{4}$$

The key is to show that the update term equals the next term in the doubled sum:

$$2(S_{n+1} - S_n) = 2 \cdot \frac{(-1)^{n+1} \cdot (n+1)!}{2(n+1)+1} = 2 \cdot \frac{(-1)^{n+1} \cdot (n+1)!}{2n+3}$$

2.4 Core Algebraic Identity

Lemma 4. *For all $n \geq 0$:*

$$-2 \cdot \frac{(-1)^n \cdot n!}{2n+1} \cdot \left(n + \frac{1}{3+2n}\right) = 2 \cdot \frac{(-1)^{n+1} \cdot (n+1)!}{2n+3}$$

Proof. First, note that $(-1)^{n+1} = -(-1)^n$, so we can factor out $-2(-1)^n$ from both sides:

$$\frac{n!}{2n+1} \cdot \left(n + \frac{1}{3+2n}\right) = \frac{(n+1)!}{2n+3} \tag{5}$$

Simplify the left-hand side:

$$n + \frac{1}{3+2n} = \frac{n(3+2n)+1}{3+2n} = \frac{3n+2n^2+1}{3+2n} = \frac{1+3n+2n^2}{3+2n}$$

Therefore:

$$\begin{aligned} \text{LHS} &= \frac{n!}{2n+1} \cdot \frac{1+3n+2n^2}{3+2n} \\ &= \frac{n! \cdot (1+3n+2n^2)}{(2n+1)(3+2n)} \end{aligned}$$

Note that $(2n+1)(3+2n) = 6n+4n^2+3+2n = 3+8n+4n^2$ and $1+3n+2n^2 = (n+1)(2n+1)$:

$$\begin{aligned} \text{LHS} &= \frac{n! \cdot (n+1)(2n+1)}{(2n+1)(2n+3)} \\ &= \frac{n! \cdot (n+1)}{2n+3} \\ &= \frac{(n+1)!}{2n+3} = \text{RHS} \end{aligned}$$

This completes the proof. □

2.5 Conclusion of Inductive Step

By Lemma 4, equation (4) becomes:

$$\begin{aligned} b_{n+1} &= 1 + 2S_n + 2 \cdot \frac{(-1)^{n+1} \cdot (n+1)!}{2n+3} \\ &= 1 + 2S_n + 2(S_{n+1} - S_n) \\ &= 1 + 2S_{n+1} \end{aligned}$$

This completes the induction.

3 Extraction Formula

Theorem 5 (Primorial Extraction)**.** *After $h = \lfloor (m-1)/2 \rfloor$ iterations, the primorial of m is given by:*

$$\mathit{Primorial}(m) = 2 \cdot \mathit{den}(b_h - 1)$$

where $\mathit{den}(q)$ denotes the denominator of rational q in lowest terms.

Proof. From Theorem 1, $b_h = 1 + 2S_h$, so:

$$b_h - 1 = 2S_h = 2 \cdot \frac{1}{2} \sum_{k=1}^{h} \frac{(-1)^k \cdot k!}{2k+1} = \sum_{k=1}^{h} \frac{(-1)^k \cdot k!}{2k+1}$$

Since primorials always contain the factor 2, and $\text{den}(S_h) = \text{Primorial}(m)$ (the main result to be proved), we have:

$$\text{den}(2S_h) = \frac{\text{Primorial}(m)}{2}$$

Therefore:

$$2 \cdot \text{den}(b_h - 1) = 2 \cdot \frac{\text{Primorial}(m)}{2} = \text{Primorial}(m)$$

□