Homework 4 Solution

Problem 1

The following statements are correct or not. No justification needed.

1. Weighted Interval Scheduling $\leq_P$ Interval Scheduling.
2. If NP $\neq$ P, then every problem in NP requires exponential time.
3. If P = NP, then all the computational problems can be solved in polynomial time.
4. If 3-SAT $\leq_P$ Independent Set, then P = NP.

Solution:

1. True. Any problem in P is polynomial time reducible to any other problem.
2. False. P is a subset of NP.
3. False. NP only contains decision problems that can be verified in polynomial time.
4. False. Both problems are NP-complete problems. Their reduction does not mean anything.

Problem 2

We define the decision version of the Shortest Path problem as follows: given an undirected graph $G = (V, E)$, two distinct vertices $s, t \in V$, and a positive integer k, return yes if and only if there exists a path from s to t of length at most k. Otherwise, return no.

For each of the two questions below, decide whether the answer is (i) "Yes", (ii) "No", or (iii) "Unknown, because it would resolve the question of whether P=NP." Give an explanation of your answers.

1. Question: Is it the case that Shortest Path $\leq_P$ Independent Set?
2. Question: Is it the case that Independent Set $\leq_P$ Shortest Path?

Solution:

Question 1: Yes. Shortest Path has a polynomial time algorithm, so Shortest Path is polynomial time reducible to any other problem.

Question 2: Unknown, because it would resolve the question of whether P=NP. Since Shortest Path has a polynomial time algorithm, any problem that is polynomial time reducible to Shortest Path implies that the problem can be solved in polynomial time.

Problem 3

3-Color problem is defined as follows: Given a graph $G = (V, E)$, does it have a 3-coloring?
4-Color problem is defined as follows: Given a graph $G = (V, E)$, does it have a 4-coloring?
Prove that 3-Color $\leq_P$ 4-Color.

Solution:

Given a graph $G = (V, E)$, we construct $G' = (V', E')$ based on G by adding a new vertex x to V and adding edges between x and every vertex of V. More formally, we set $V' = V \cup \{x\}$ where $x \notin V$, and $E' = E \cup \{(x, v)$ for every $v \in V\}$. We prove that G has a 3-coloring if and only if G' has a 4-coloring.

We show that if G has a 3-coloring, then G' has a 4-coloring. Let $f : V \to \{1, 2, 3\}$ be a 3-coloring of G. We define $f'(v) = f(v)$ if $v \in V$ and $f'(x) = 4$. G' is a 4-coloring, because all edges in E have distinct colors of the two end points by the 3-coloring f and all the other edges have distinct colors by $f(x) = 4$ and $f(v) \neq 4$ for every $v \in V$.

We show that if G' has a 4-coloring, then G has a 3-coloring. Let $g : V' \to \{1, 2, 3, 4\}$ be a 4-coloring of G'. We define $g'(v) = g(v)$ if $v \in V$. and $f'(x) = 4$. Because x is adjacent to every vertex in V, so the vertices of V have only three colors. Hence, g' is a 3-coloring.

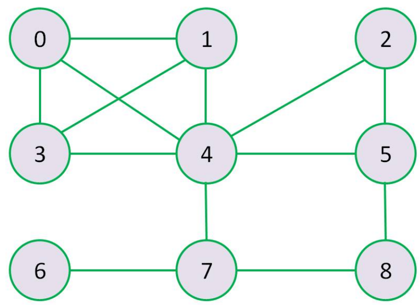
So, we can solve 3-coloring in polynomial time using a subroutine 4-coloring for 4-Color problem as follows:

1. Construct $G' = (V', E')$ such that $V' = V \cup \{x\}$ where $x \notin V$, and $E' = E \cup \{(x, v)$ for every $v \in V\}$.
2. Return the solution of 4-coloring using G' as input.

Problem 4

Complete Subgraph problem is defined as follows: Given a graph $G = (V, E)$ and an integer k, output yes if and only if there is a subset of vertices $S \subseteq V$ such that $|S| = k$, and every pair of vertices in S are adjacent (there is an edge between any pair of vertices).

For example, for the following graph and $k = 4$, the answer is yes, because $S = \{0, 1, 3, 4\}$ satisfies the requirement. But if $k \geq 5$, the answer is no.



1. (15 points) Show that the Complete Subgraph problem is in NP.
2. (25 points) Show that Complete Subgraph problem is NP-Complete.

   (hint 1: the **Independent Set** problem is a NP-Complete problem.)

   (hint 2: You can also use other NP-Complete problems to prove NP-Complete of Complete Subgraph.)

Solution:
(1) Certificate: A set S of k vertices.
Certifier: Check if every pair of vertices in S are adjacent in the graph.
(2) We show INDEPENDENT-SET$\leq_P$COMPLETE-SUBGRAPH.
Algorithm: (graph $G = (V, E)$ and k as an INDEPENDENT-SET instance)

1. Let G' be the complement of G (two vertices are adjacent in G' iff they are not adjacent in G).
2. Run the algorithm for COMPLETE-SUBGRAPH using G' and k as input, and return the result by COMPLETE-SUBGRAPH.

One can show that if S is an independent set of G, then S is a complete subgraph of G'.

Problem 5:
The ***Number Partition*** problem asks, given a collection of non-negative integers $S = \{x_1, \ldots, x_n\}$ whether or not the integers can be partitioned into two sets A and $\overline{A} = S - A$ such that

$$\sum_{x \in A} x = \sum_{x \in \overline{A}} x.$$

1. Show that *Number Partition* is in NP. Define your certificate, and describe the certificate algorithm using the certificate you defined.

2. Show that *Number Partition* is NP-Complete. Describe your reduction. **Make sure the direction of reduction is correct!**

   (hint 1: It is known that the ***Subset Sum*** problem is a NP-Complete problem. In the *Subset Sum* problem, we are given a collection of non-negative integers $Y = \{y_1, y_2, \ldots, y_m\}$ and a target integer t, we want to see if it is possible find a subset $Z \subseteq Y$ such that the sum of integers in Z equals to t

   $$\sum_{x \in Z} x = t.$$

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   (hint 2: Add an integer for the reduction.)

Solution
(1) Certificate: A set $X \subset S$ such that $\sum_{x \in X} x = \sum_{x \in S-X} x$.
Certifier: Check if $\sum_{x \in X} x = \sum_{x \in S-X} x$.
(2) We need to show Subset-Sum$\leq_P$Number-Partition.
Given an instance of subset sum problem: a set of integers Y and a parameter t, we want to construct an instance of number partition problem such that the solution for the instance of number partition problem is same to the solution for the instance of the subset sum problem. Let $v = \sum_{x \in Y} x - 2t$. We let $S = Y \cup \{v\}$. We show that the instance of subset sum is a yes instance iff S for number partition problem is an yes instance.
If (Y, t) is a yes instance for subset sum, then we have $Z \subseteq Y$ such that $\sum_{x \in Z} x = t$. Let $A = Z \cup \{v\}$, and $\overline{A} = S - A$. We have $\overline{A} = Y - Z$, hence

$$\sum_{x \in A} x = \sum_{x \in Z \cup \{v\}} x = \sum_{x \in Y} x - t = \sum_{x \in Y - Z} x = \sum_{x \in \overline{A}} x.$$

So, S is a yes instance for number partition problem.
If S is a yes instance for number partition problem, then there exists $A \subset S$ such that

$$\sum_{x \in A} x = \sum_{x \in \overline{A}} x = (\sum_{x \in Y} x + v)/2 = \sum_{x \in Y} x - t.$$

Since v is an element of S, so v is in either A or $\overline{A}$. Without loss of generality, assume $v \in A$. We have $\sum_{x \in A - \{v\}} x = \sum_{x \in A} x - v = t$. Since $A - \{v\}$ is a subset of Y, so (Y, t) is a yes instance of subset sum problem.