$3\pi - \sqrt{5} + \frac{3}{7}$

1.) a.) chopping i.) $(3)(3.1415) - (2.2360) + (0.4285)$

$= 9.4245 - 2.2360 + 0.4285$

$= 7.1885 + 0.4285$

$= 7.6170$

$\begin{array}{r} 9.4245 \\ -2.2360 \\ \hline 7.1885 \end{array}$

rounding ii.) $(3)(3.1416) - (2.2361) + (0.4286)$

$= 9.4248 - 2.2361 + 0.4286$

$= 7.1887 + 0.4286$

$= 7.6173$

iii.) ~~(Interpreting "Exact Error" as "Absolute Error")~~

~~$|7.6170 - 7.6173| = 0.0003$~~

$|7.61728... - 7.61700| = \boxed{0.00028}$ chopping

$|7.61728... - 7.61730| = \boxed{0.00002}$ rounding $\rightarrow \boxed{-0.00002}$

b.) $e^{1.2} - \cos\left(\frac{\pi}{6}\right) = e^{1.2} - \frac{\sqrt{3}}{2}$

chopping i.) $(3.3201) - \frac{(1.7320)}{2}$

$= (3.3201) - (0.866)$

$= 2.4541$

rounding ii.) $(3.3201) - \frac{(1.7321)}{2}$

$= (3.3201) - (0.8661)$

$= 2.4540$

$\begin{array}{r} 3.3201 \\ 0.8661 \\ \hline \boxed{2.4540} \end{array}$

iii.) $|2.45409 - 2.45410| = \boxed{0.00001}$ chopping $\rightarrow \boxed{-0.00001}$

$|2.45409 - 2.45400| = \boxed{0.00009}$ rounding

2.)

$f(x) = e^{-2x}$

$x_0 = 0.1$

$n = 2$

$\sum_{k=0}^{n} \frac{f^{k}(x_0)}{k!}(x - x_0)^k = P_n(x)$

$\frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)^{(n+1)} = R_n(x)$

$f'(x) = -2e^{-2x}$

$f''(x) = 4e^{-2x}$

$f'''(x) = -8e^{-2x}$

$(x - 0.1)^3$

$= (x^2 - 0.2x + 0.01)(x - 0.1)$

$= \left(\begin{array}{l} x^3 - 0.2x^2 + 0.01x \\ -0.1x^2 + 0.02x - 0.001 \end{array}\right)$

$= \cancel{x^3 + 0.03x}$

$= (x^3 - 0.3x^2 + 0.03x - 0.001)$

$(x - 0.1)(x - 0.1) = x^2 - 0.1x - 0.1x + 0.01$

$= x^2 - 0.2x + 0.01$

$\frac{f^0(x_0)}{0!} + \frac{f'(x_0)}{1!}(x - x_0)^1 + \frac{f^2(x_0)}{2!}(x - x_0)^2$

$= f(0.1) + f'(0.1)(x - 0.1) + \frac{f^2(0.1)}{2}(x - 0.1)^2$

$= e^{-2(0.1)} + -2e^{-2(0.1)}(x - 0.1) + \frac{4e^{-2(0.1)}}{2}(x - 0.1)^2$

$= e^{-0.2} + -2e^{-0.2}(x - 0.1) + 2e^{-0.2}(x - 0.1)^2$

$=$

$2e^{-0.2}(x^2 - 0.2x + 0.01)$ $\quad -2e^{-0.2}(x - 0.1)$

$= e^{-0.2} + 2e^{-0.2}x^2 - 0.4e^{-0.2}x + 0.02e^{-0.2} + (-2e^{-0.2}x) + 0.2e^{-0.2}$

$= 2e^{-0.2}x^2 - 0.4e^{-0.2}x - 2e^{-0.2}x + e^{-0.2} + 0.02e^{-0.2}$

$= 2e^{-0.2}x^2 - 2.4e^{-0.2}x + 1.02e^{-0.2}$

$\overset{alt}{=} e^{-0.2} + (-2e^{-0.2}x) + (2e^{-0.2}x^2)$

$P_2(x) = e^{-0.2}(2x^2 - 2.4x + 1.02)$

$P_{2,alt}(x) = e^{-0.2}(2x^2 - 2x + 1)$

$\rightarrow P_{2,alt}(x) = e^{-0.2}(2(0.1)^2 - 2(0.1) + 1)$

$= e^{-0.2}(0.02 - 0.20 + 1)$

$\boxed{P_{2,alt}(0.1) = e^{-0.2}(0.82)}$

$R_2(x) = \frac{-8e^{-2(\xi(x))}}{3!}(x - 0.1)^3$

$R_2(x) = \frac{-4}{3}e^{-2(\xi(x))}(x^3 - 0.3x^2 + 0.03x - 0.001)$

$\boxed{R_{2,alt}(x) = -\frac{4}{3}e^{-2(\xi(x))}(x^3)}$

3.)

$-1 = \lim_{h\to 0} \frac{1 - e^h}{h} = \lim_{h\to 0}\left(\frac{1}{h} - \frac{e^h}{h}\right) = \lim_{h\to 0}\left(\frac{1}{h} - \frac{1}{h}\left[\sum_{k=0}^{\infty}\frac{h^k}{k!}\right]\right) = \lim_{h\to 0}\left(\frac{1}{h} - \frac{1}{h}\left(1 + h + \frac{h^2}{2} + \frac{h^3}{6} + \ldots\right)\right)$

$L = \lim_{h\to 0} F(h)$

$= \lim_{h\to 0}\left(\frac{1}{h} - \sum_{k=0}^{\infty}\frac{h^{k-1}}{k!}\right) = \lim_{h\to 0}\left(\frac{1}{h} - \frac{1}{h} - \frac{h}{h} - \frac{h^2}{h(2)} - \frac{h^3}{6h} + \ldots\right)$

$= \lim_{h\to 0}\left(-1 - \frac{h}{2} - \frac{h^2}{6} + \ldots\right)$

$= -1 + \lim_{h\to 0}\left(-\frac{h}{2} - \frac{h^2}{6} + \ldots\right) = -1$

$= \lim_{h\to 0}\left(-\frac{h}{2} - \frac{h^2}{6} + \ldots\right) = 0$

$= \lim_{h\to 0} G(h) = 0$

The rate of convergence is $O(h)$.

4.) $\boxed{P_{n,\cos(x)}(x) = 1 + \left(-\frac{1}{2}\right)x^2 + \left(\frac{1}{24}\right)x^4 + \left(\frac{-1}{720}\right)x^6 + \ldots}$

$$P_n(x) = \sum_{k=0}^{n} \frac{f^k(x_0)}{k!}(x-x_0)^k$$

$n = n$, $x_0 = 0$, $f_1(x) = \cos(x)$, $f_2(x) = \sin(x)$

$$P_n(x) = \frac{f^0(0)}{0!}x^0 + \frac{f^1(0)}{1!}x^1 + \frac{f^2(0)}{2!}x^2 + \frac{f^3(0)}{3!}x^3 + \frac{f^4(0)}{4!}x^4 + \frac{f^5(0)}{5!}x^5 + \frac{f^6(0)}{6!}x^6$$

$$= f^0(0) + f^1(0)x + \frac{f^2(0)}{2}x^2 + \frac{f^3(0)x^3}{6} + \frac{f^4(0)}{24}x^4 + \frac{f^5(0)}{120}x^5 + \frac{f^6(0)}{720}x^6 + \ldots$$

$\boxed{P_{n,\sin(x)}(x) = x + \left(-\frac{1}{6}\right)x^3 + \left(\frac{1}{120}\right)x^5 + \ldots}$

$$P_{n,\cos(x)}(x) = \cos(0) + (-\sin(0))\cancel{x} + \frac{(-\cos(0))}{2}x^2 + \frac{\sin(0)}{6}x^3 + \frac{(\cos(0))}{24}x^4 + \frac{(-\sin(0))}{120}x^5 + \frac{(-\cos(0))}{720}x^6$$

$$= 1 + 0 + \left(-\frac{1}{2}\right)x^2 + 0 + \left(\frac{1}{24}\right)x^4 + 0 + \left(-\frac{1}{720}\right)x^6 + \ldots$$

$$P_{n,\sin(x)}(x) = \sin(0) + (\cos(0))x + \left(\frac{-\sin(0)}{2}\right)x^2 + \left(\frac{-\cos(0)}{6}\right)x^3 + \left(\frac{\sin(0)}{24}\right)x^4 + \left(\frac{\cos(0)}{120}\right)x^5 + \left(\frac{-\sin(0)}{720}\right)x^6$$

$$= 0 + x + 0 + \left(-\frac{1}{6}\right)x^3 + 0 + \left(\frac{1}{120}\right)x^5 + 0 + \ldots$$

For cos(x), the even function, its taylor polynomial only has even powers.
For sin(x), the odd function, its taylor polynomial only has odd powers.

5.) Syntax: I wrote "function" without completing ~~by~~ my function name or anything else.
The temporary fix was commenting out the line, so I could run the code.
The permanent fix was

Runtime: "Unrecognized function or variable 'Sum'" & 2-4 more error lines.
The error mainly came from this line: $\boxed{\text{Sum = Sum+k;}}$
Here, I tried to use Sum to add onto itself without giving it a value first.
I fixed this by adding the line $\boxed{\text{Sum = 0;}}$ beforehand.

Logic: The output for questions 6, 7, & 8 didn't show up.
I called the fprintf function with the format specifier allowing the functions' output to show.

9.)

$$P_n(x) = \sum_{k=0}^{n} \left(\frac{d^k}{dx}e^x\Big|_{x_0}\right)\left(\frac{1}{k!}\right)x^k = \sum_{k=0}^{n}\left(\frac{d^k}{dx}e^x\right)\left(\frac{x^k}{k!}\right) =$$

$x_0 = 0$

$$= \sum_{k=0}^{n}\left((1)\left(\frac{x^k}{k!}\right)\right) = \boxed{\sum_{k=0}^{n}\frac{x^k}{k!} = P_{n,e^x}(x)}$$

Show that $\sum_{k=0}^{n}\frac{x^k}{k!}$ is the derivative of e^x. (Informal) ~~(Formal would require Γ(x))~~

$$e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \ldots$$

$$e^x = \frac{d}{dx}e^x = \frac{d}{dx}\left[\sum_{k=0}^{\infty}\frac{x^k}{k!}\right] = \frac{d}{dx}(1) + \frac{d}{dx}(x) + \frac{d}{dx}\left(\frac{x^2}{2}\right) + \frac{d}{dx}\left(\frac{x^3}{6}\right) + \frac{d}{dx}\left(\frac{x^4}{24}\right) + \frac{d}{dx}\left(\frac{x^5}{120}\right) + \ldots$$

$$\boxed{\frac{d}{dx}e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \ldots}$$

$$\sum_{k=0}^{\infty}\frac{x^k}{k!} \equiv \sum_{k=1}^{\infty}\frac{x^{(k-1)}}{(k-1)!} \qquad \sum_{k=0}^{\infty}\frac{x^k}{k!} = 1 + \sum_{k=1}^{\infty}\frac{x^k}{k!}$$

$$\frac{d}{dx}(e^x) = \frac{d}{dx}\left[\sum_{k=0}^{\infty}\frac{x^k}{k!}\right] = \frac{d}{dx}\left[1 + \sum_{k=1}^{\infty}\frac{x^k}{k!}\right] = \frac{d}{dx}(1) + \frac{d}{dx}\sum_{k=1}^{\infty}\frac{x^k}{k!} = 0 + \sum_{k=1}^{\infty}\frac{d}{dx}\left(\frac{x^k}{k!}\right) = \sum_{k=1}^{\infty}\frac{kx^{(k-1)}}{k!} = \sum_{k=1}^{\infty}\frac{x^{(k-1)}}{(k-1)!} = \sum_{k=0}^{\infty}\frac{x^k}{k!} = e^x$$