MATH 131: Numerical Methods for Scientists and Engineers

Homework 2 Solutions

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1.) Suppose $f(x) = (x-1)^2 - 1$, $a = 1$, $b = 3$ and $K = 0$. Prove that it will take at least 5 iterations of Bisection method to get an absolute error less than 10^{-5}.

Proof. f is continuous on $[a, b]$. $K \in (f(1), f(3))$. By IVT, there exists a $p \in (1, 3)$ where $f(p) = K$. We will call this p the *root p of f*. $f(1) \cdot f(3) = -3 < 0$. Therefore, Bisection method generates a sequence $\{p_n\}_{n=1}^{\infty}$ approximating a root p of f with an absolute error $|p_n - p| \leq \frac{b-a}{2^n}$. Suppose that $|p_n - p| < 10^{-5}$ and $\frac{b-a}{2^n} \leq 10^{-5}$. We can now say the following:

$$
\begin{aligned}
& \tfrac{3-1}{2^n} \leq 10^{-5} \\
\equiv\ & \tfrac{1}{2^{n-1}} \leq 10^{-5} \\
\equiv\ & \log_2 2^{-(n-1)} \leq \log_2 10^{-5} \\
\equiv & -(n-1) \leq -5 \log_2 10 \\
\equiv & -n \leq -5 \log_2 10 - 1 \\
\equiv\ & n \geq 5 \log_2 10 + 1 \approx 4.322 \\
\equiv\ & n \geq 5
\end{aligned}
$$

Therefore, it will take at least 5 iterations of the Bisection method to get an absolute error less than 10^{-5}. $\square$

2.) (Process)

$f(a) = f(0) = 0^2 - 2(0) - 3 = -3$

(Iteration 1)

$c = (0 + 5)/2 = 2.5$
$f(c) = f(2.5) \ = \ 2.5^2 - 2(2.5) - 3 \ = \ 6.25 - 5 - 3 \ = \ 1.25 - 3 \ = \ -1.75$
$f(c) \cdot f(a) > 0$ is true so the root is between c and b right now.
$a \leftarrow c$ and $f(a) \leftarrow f(c)$
$\therefore\ a = 2.5,\ b = 5,\ c = 2.5,\ f(a) = -1.75$

(Iteration 2)

$c = (2.5 + 5)/2 = 3.75$
$f(c) = f(3.75) \ = \ 3.75^2 - 2(3.75) - 3 \ = \ 14.065 - 7.5 - 3 \ = \ 3.5625$
$f(c) \cdot f(a) > 0$ is false, so the root is between a and c.
$b \leftarrow c$
$\therefore\ a = 2.5,\ b = 3.75,\ c = 3.75,\ f(a) = -1.75$

(Iteration 3)

$c = (2.5 + 3.75)/2 = 3.125$
$f(c) = f(3.125) \ = \ 3.125^2 - 2(3.125) - 3 \ = \ 9.765625 - 6.25 - 2 \ = \ 1.515625$
$f(c) \cdot f(a) > 0$ is false, so the root is between a and c.
$b \leftarrow c$
$\therefore \ a = 2.5, \ b = 3.125, \ c = 3.125, \ f(a) = -1.75$

(Iterations Complete)

We are out of iterations, so the final c as an approximation to the 2nd root is $c = 3.125$.

3.) Recall Fixed-Point Theorem. Suppose we have a function $f(x) = x^2 - 2x - 8 = 0$ with roots $c_1 = -2$ and $c_2 = 4$. $[a, b] = [3, 5]$.

a.) $g_a(x) = 2 + \frac{8}{x}$
$g_a'(x) = -8x^{-2}$
$\max\limits_{x\in[3,5]} |-8x^{-2}| < 1$
$\equiv \frac{1}{9} < 1$
This is true. This function would be a great choice for the Fixed-Point method.

b.) $g_b(x) = \frac{(x^3-2x^2)}{8}$
$g_b'(x) = \frac{(3x^2-4x)}{8}$
$\max\limits_{x\in[3,5]} |\frac{(3x^2-4x)}{8}| < 1$
$\equiv \frac{55}{8} < 1$
This is false. We cannot guarantee convergence under Fixed-Point Theorem.

c.) $g_c(x) = \frac{(x^2-8)}{2}$
$g_c'(x) = x$
$\max\limits_{x\in[3,5]} |x| < 1$
$\equiv 5 < 1$
This is false. We cannot guarantee convergence under Fixed-Point Theorem.

d.) $g_d(x) = x^2 - x - 8$
$g_d'(x) = 2x - 1$
$\max\limits_{x\in[3,5]} |2x-1| < 1$
$\equiv 9 < 1$
This is false. We cannot guarantee convergence under Fixed-Point Theorem.

7.) Consider two pairs of points $(x_n, f(x_n))$ and $(x_{n+1}, 0)$ as well as $(x_n, f(x_n))$ and $(x_{n-1}, f(x_{n-1}))$ as we connect a line along both of these pairs of points. Remember point-slope form: $y - y_0 = m(x - x_0)$. The point-slope form for both of these newly made lines are the following:

$$m(x_{n+1} - x_n) = 0 - f(x_n)$$
$$\equiv m = \frac{f(x_n)}{(x_{n+1} - x_n)}$$

$$m(x_{n-1} - x_n) = f(x_{n-1}) - f(x_n)$$
$$\equiv m = \frac{f(x_{n-1}) - f(x_n)}{(x_{n-1} - x_n)}$$

Using these equations, we can derive the formula for the Secant method:

$$\frac{f(x_{n-1}) - f(x_n)}{(x_{n-1} - x_n)} = \frac{f(x_n)}{(x_{n+1} - x_n)}$$
$$\equiv x_{n+1} - x_n = \frac{f(x_n)(x_{n-1} - x_n)}{f(x_{n-1}) - f(x_n)}$$
$$\equiv x_{n+1} = x_n + \frac{f(x_n)(x_{n-1} - x_n)}{f(x_{n-1}) - f(x_n)}$$
$$\equiv x_{n+1} = x_n + \frac{f(x_n)}{\frac{f(x_{n-1}) - f(x_n)}{x_{n-1} - x_n}}$$

Thus, we have derived the formula for the Secant method.