Math 308: Final Examination

by William Stein

Wednesday, June 12, 2013, 2:30–4:20 p.m.

Instructions:

- Your Name: ______________________________
  ID Number: ______________________
- Exactly as with the midterms, you may use one sheet of paper with notes on it, and you may use a calculator.
- Turn in your exam at 4:20pm, so you have 1:50 to complete this exam. Answer every question. Triple check your work! ☐ ☐ ☐

Questions (circle T or F):

1. **T** The following 2×5 matrix is in reduced row echelon form:

$$\begin{pmatrix} 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

2. **T** The following 1×6 matrix is in reduced row echelon form:

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$

3. **T** The span of the rows of the matrix $A = \begin{pmatrix} 1 & 0 & 3 \\ 4 & 0 & 6 \\ 7 & 0 & 9 \end{pmatrix}$ has dimension 2.

4. **F** The span of the rows of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ has dimension 3.

5. **F** The set of $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$ such that

$$\begin{aligned} x + 2\,y + 3\,z &= 1 \\ 4\,x + 5\,y + 6\,z &= 2 \\ 7\,x + 8\,y + 9\,z &= 3 \end{aligned}$$

is a vector space of dimension 1.

6. **F** The set of $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$ such that

$$\begin{aligned} x + 2\,y + 3\,z &= 0 \\ 4\,x + 5\,y + 6\,z &= 0 \\ 7\,x + 8\,y + 9\,z &= 0 \end{aligned}$$

is a vector space of dimension 2.

7. **T** The lower half plane, i.e., the set of points $(x, y) \in \mathbb{R}^2$ with $y \leq 0$, is not a subspace of $\mathbb{R}^2$.

8. **T** If A and B are any two square matrices (of the same size), then $(AB)^T = B^T A^T$ and $(A + B)^T = B^T + A^T$.

9. **F** If A and B are any two invertible square matrices (of the same size) such that $A + B$ is also invertible, then $(AB)^{-1} = B^{-1}A^{-1}$ and $(A + B)^{-1} = B^{-1} + A^{-1}$.

10. **T** Let $\mathbb{F}_2$ be the finite field with 2 elements and let V be a 2-dimensional vector space over $\mathbb{F}_2$. (In the book $\mathbb{F}_2$ is denoted $\mathbb{Z}_2$.) Then there are exactly 16 distinct linear transformations $V \to V$.

11. **T** If A is any $n \times n$ matrix and B is the reduced row echelon form of A, then $\det(A) = 0$ if and only if $\det(B) = 0$.

12. **F** If A is any $n \times n$ matrix and B is the reduced row echelon form of A, then $\det(A) = 1$ if and only if $\det(B) = 1$.

13. **T** Let $\mathcal{P}_3$ be the vector space over $\mathbb{R}$ of polynomials of degree at most 3 with real coefficients, and let $T : \mathcal{P}_3 \to \mathcal{P}_3$ be the linear transformation $T(ax^2+bx+c) = cx+b-a$. Then there is a basis $\mathcal{B}$ for $\mathcal{P}_3$ such that $[T]_{\mathcal{B},\mathcal{B}}$ is diagonal.

14. **T** Let $\mathcal{P}_3$ be the vector space over $\mathbb{R}$ of polynomials of degree at most 3 with real coefficients, and let $T : \mathcal{P}_3 \to \mathcal{P}_3$ be the linear transformation $T(ax^2+bx+c) = cx+b-a$. Then the kernel of T (i.e., the set of v with $T(v) = 0$) has dimension 1.

15. **F** Let $\mathcal{P}_2$ be the vector space over $\mathbb{R}$ of polynomials of degree at most 2 with real coefficients, and let $T : \mathcal{P}_2 \to \mathcal{P}_2$ be the linear transformation $T(bx+c) = cx - b$. Then the matrix $[T]_B$ of T with respect to the basis $1, x$ is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

16. **F** The matrix $A = \begin{pmatrix} 1 & 2013 \\ 0 & 1 \end{pmatrix}$ is diagonalizable over $\mathbb{C}$.

17. **F** The subset $\mathrm{GL}_3(\mathbb{C})$ of all invertible 3×3 complex matrices is a vector subspace of the vector space $M_{3,3}(\mathbb{C})$ of all 3×3 complex matrices.

18. **T** The matrix $A = \begin{pmatrix} 1 & 2013 \\ 2013 & 1 \end{pmatrix}$ is diagonalizable over $\mathbb{C}$.

19. **F** Let V be the vector space of all infinitely differentiable functions $\mathbb{R} \to \mathbb{R}$. The function that send f to its third derivative is a linear transformation $V \to V$ whose kernel has dimension 2.

20. **T** Let V be the vector space of all infinitely differentiable functions $\mathbb{R} \to \mathbb{R}$. The kernel of the linear transformation that sends f to its third derivative is the space $\mathcal{P}_2$ of polynomials of degree at most 2.

21. **T** The determinant of A is 0, where

$$A = \begin{pmatrix} -\frac{12825}{11504} & \frac{90625}{88467} & \frac{49455}{35626} & -\frac{6151}{38520} \\ \frac{48955}{10471} & \frac{60835}{89249} & \frac{17564}{97411} & -\frac{7103}{3700} \\ -\frac{12825}{11504} & \frac{90625}{88467} & \frac{49455}{35626} & -\frac{6151}{38520} \\ -\frac{2275}{7108} & \frac{49132}{79339} & -\frac{49791}{670} & \frac{6115}{4868} \end{pmatrix}.$$

22. **T** Let V be the vector space of all functions $\mathbb{R} \to \mathbb{R}$. Then the function $V \to \mathbb{R}^2$ that sends f to $(0, f(0))$ is a linear transformation.

23. **T** Let V be the vector space of all functions $\mathbb{R} \to \mathbb{R}$. Then the function $V \to \mathbb{R}^5$ that sends f to $(f(-1), f(1), f(2), f(\pi), f(4))$ is a linear transformation.

24. **T** There are infinitely many distinct orthonormal bases for $\mathbb{R}^2$.

25. **F** There is exactly one orthonormal basis for $\mathbb{R}^1$.

26. **F** If A is any orthogonal matrix, then $\det(A) = 1$.

27. **F** If A is any 2×2 matrix with real entries such that $\det(A) = 1$, then A must be an orthogonal matrix.

28. **F** If A is any 2×2 matrix with real entries such that $A^2 = I_2$, then A must be an orthogonal matrix.

29. **T** Suppose A is a 2×2 matrix with characteristic polynomial $x^2 + 3x$. Then A is not invertible.

30. **T** The vectors $v_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ -1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 4 \end{pmatrix}$ are an orthogonal basis for a subspace of $\mathbb{R}^4$.

31. **T** Suppose A is a 3×5 matrix of rank 2 (with real entries). Then the dimension of the nullspace of A must be 3.

32. **F** Suppose A is a 3×5 matrix with real entries that looks like this (where the entries marked with * can be absolutely anything): $A = \begin{pmatrix} 1 & 2 & 1 & * & * \\ -1 & 1 & 1 & * & * \\ 2 & 0 & 1 & * & * \end{pmatrix}$. Then the dimension of the nullspace of A must be 3.

33. **F** Suppose A is any orthogonal 3×3 matrix. Then A must be in reduced row echelon form.

34. **F** Suppose A is any orthogonal 3×3 matrix. Then $A = A^T$.

35. **T** Suppose A is any diagonalizable 3×3 matrix over the complex numbers with eigenvalues a, b, c and corresponding eigenvectors v_1, v_2, v_3. Let B be the matrix with columns v_1, v_2, v_3 and D the diagonal matrix with entries a, b, c. Then

$$D^{-1} = B^{-1}A^{-1}B.$$

36. **T** Let V be a finite dimensional vector space, $b_1, \ldots, b_n$ a basis B for V and $c_1, \ldots, c_n$ a basis C for V. Let T be the unique linear transformation such that $T(b_i) = c_i$ for each i. Then $[T]_{B,C} = I_n$ is the identity matrix.