Math 308: Last Homework Assignment – SOLUTIONS

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True or False:

1. **F** A system of linear equations over $\mathbf{R}$ has either no solutions or infinitely many solutions. [Hint: The possibilities for a system of linear equations with coefficients real numbers are: no solutions, a unique solution, infinitely many solutions.]

2. **F** The span of the rows of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ has dimension 3. [Hint: False, because when you put this matrix in echelon form, the last row consists of 0's.]

3. **T** There are 4 linearly independent vectors in $\mathbf{R}^5$. [Hint: Yes, for example, the first four standard basis vectors e_1, e_2, e_3, e_4.]

4. **F** The upper half plane, i.e., the set of points $(x, y) \in \mathbf{R}^2$ with $y \geq 0$, is a subspace of $\mathbf{R}^2$. [Hint: No the upper half plane is not a subspace, because $(1, 0)$ is in the upper half plane, but $-1 \cdot (1, 0) = (-1, 0)$ is not, and subspaces are closed under scalar multiplication.]

5. **T** If A and B are any two square matrices (of the same size), then $\det(AB) = \det(BA)$ and $\det(A + B) = \det(B + A)$. [Hint: This is true; the first fact follows from $\det(AB) = \det(A)\det(B)$, which is a somewhat difficult theorem. The second fact just follows from commutativity of matrix addition.]

6. **T** Let $\mathbf{F}_3$ be the finite field with 3 elements and let V be a 2-dimensional vector space over $\mathbf{F}_3$. Then there are 81 linear transformations $V \to V$. [Hint: Yes, the linear transformations correspond to 2×2 matrices with entries in $\mathbf{F}_3 = \{0, 1, 2\}$. The number of such matrices is $3 \cdot 3 \cdot 3 \cdot 3 = 81$, since there are three choices for each entry.]

7. **F** If A is any $n \times n$ matrix and B is the reduced row echelon form of A, then $\det(A) = \det(B^t)$, where B^t is the transpose of B. [Hint: This is false, with the transpose being a red herring. Replacing a matrix by its reduced row echelon form dramatically changes the determinant; in fact, the determinant of a matrix in reduced row echelon form is either 0 or 1!]

8. **T** Suppose A is a 3×3 matrix with eigenvalues $1, 2, 3$. Then A must be diagonalizable. [Hint: Yes, because each eigenspace has dimension at least 1, and there are three of them, so their dimensions must add up to 3. That is, there is a basis of eigenvectors.]

9. **F** Suppose A is a 3×3 matrix with eigenvalues 1 and -1. Then $A^2 = I_3$. [Hint: This is false. A counterexample is the matrix $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.]

10. **T** Let $\mathcal{P}$ be the vector space over $\mathbf{R}$ of polynomials of degree at most 3 with real coefficients, and let $T : \mathcal{P} \to \mathcal{P}$ be the linear transformation $T(ax^2 + bx + c) = a + b + c$. Then there is a basis

$\mathcal{B}$ for $\mathcal{P}$ such that $[T]_{\mathcal{P},\mathcal{P}}$ is diagonal. [Hint: This is true, as we can see by first writing down the matrix $[T]_{\mathcal{C}}$ for *some* choice of basis, then computing the characteristic polynomial and eigenvectors. We make the arbitrary choice of basis $\mathcal{C}$ to be $1, x, x^2$, and find that $A = [T]_{\mathcal{C}} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. The characteristic polynomial is $(x-1)x^2$, the nullspace of $A-1$ has dimension 1, and the nullspace of $A-0$ has dimension 2, so A is diagonalizable, hence T is as well.]

11. **F** Let $\mathcal{P}$ be the vector space over $\mathbf{R}$ of polynomials of degree at most 3 with real coefficients, and let $T : \mathcal{P} \to \mathbf{R}^1$ be the linear transformation $T(ax^2+bx+c) = a+b+c$. Then the kernel of T (i.e., the set of v with $T(v) = 0$) has dimension 1. [Hint: Choose a basis for both vector spaces, then compute the matrix of T with respect to this choice of bases. For $\mathcal{P}$ choose $1, x, x^2$, and for $\mathbf{R}^1$ choose (1). Then the corresponding matrix is $A = [1,1,1]$. The nullspace of A has dimension 2, since the rank is 1 and rank + nullity = number of columns = 3; or, just compute the nullspace and get dimension 2. Since $2 \neq 1$, this is false.]

12. **T** Suppose V and W are vector spaces over $\mathbf{R}$ of dimensions 2 and 3, respectively. Then there is a linear transformation $T : V \to W$ such that $\ker(T) = 0$. [Hint: Yes, since $2 \leq 3$, you can find such a linear transformation.]

13. **F** Suppose V and W are vector spaces over $\mathbf{R}$ of dimensions 3 and 2, respectively. Then there is a linear transformation $T : V \to W$ such that $\ker(T) = 0$. [Hint: No, this isn't possible, since any 2×3 matrix will have a nonzero nullspace.]

14. **F** Every 2×2 matrix A is diagonalizable (over the complex numbers $\mathbf{C}$). [Hint: No, for example $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable.]

15. **T** The matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is diagonalizable over $\mathbf{C}$. [Hint: Yes, as you can see by computing the characteristic polynomial and verifying that it has two distinct roots. The roots happen to be in $\mathbf{R}$, but that's fine, since $\mathbf{R}$ is a subset of $\mathbf{C}$.]

16. **T** Let V be the vector space of all differentiable functions $\mathbf{R} \to \mathbf{R}$. The function that send f to its derivative is a linear transformation $V \to V$. [Hint: Yes, this is a basic fact you know from Calculus, namely that $(af+bg)' = af' + bg'$.]

17. **T** The dimension of the kernel of the derivative transformation (defined in the previous problem) is 1. [Hint: True, since $f' = 0$ if and only if f is a constant, and the constant functions are a space of dimension 1, with basis the function $f(x) = 1$.]

18. **T** Let V be the set of all integrable functions $\mathbf{R} \to \mathbf{R}$, i.e., functions that have some antiderivative. Then V is a vector space. [Hint: Yes, since the sum and scalar multiples of a function with an antiderivative also has an antiderivative.]

19. **T** Let V be the set of all integrable functions $\mathbf{R} \to \mathbf{R}$. The function that send f to "the antiderivative of f that sends 0 to 0" is a linear transformation of V. [Hint: Yes, it satisfies the properties. The key is that we make the choice of constant so that 0 goes to 0.]

20. **F** Let V be the set of all integrable functions $\mathbf{R} \to \mathbf{R}$. The function that f to "the antiderivative of f that sends 0 to 1" is a linear transformation of V. [Hint: This is definitely not a linear transformation, since it sends the 0 function to the constant function 1, which is not 0, and linear transformations send 0 to 0.]

21. **T** The determinant of $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}$ is 0. [Hint: Yes, since the last two rows are linear combinations of the first two.]

22. **T** If B and C are basis for a (finite dimensional) vector space V, then there must be some linear transformation $T : V \to V$ such that $[T]_{B,C}$ is the identity matrix. [Hint: Yes, just take T to be the transformation that sends b_i to c_i.]

23. **T** Let V be the set of functions $\mathbf{R} \to \mathbf{R}$. Then the function $V \to \mathbf{R}^2$ that sends f to $(f(1), f(\pi))$ is a linear transformation. [Hint: Yes, it is, as you can check from the definitions.]

24. **F** Let V be the vector space over $\mathbf{R}$ that is the span of $\sin(x)$, $\cos(x)$, and $\cos(3x)$. Let $T : V \to \mathbf{R}^1$ be the linear transformation that sends $f \in V$ to $f(0)$. Then the kernel of T has dimension 1. [Hint: Simply compute the matrix of T with respect to the basis $\sin(x), \cos(x), \cos(3x)$ for the domain and (1) for the codomain. That matrix is $A = [0, 1, 1]$, which has nullspace of dimension 2, sincek there are two nonpivot columns, namely the first and third. Thus the kernel has dimension 2.]

25. **T** Let V be the vector space over $\mathbf{R}$ that is the span of $\sin(x)$, $\cos(x)$, and $\cos(3x)$. Let $T : V \to \mathbf{R}^1$ be the linear transformation that sends $f \in V$ to $f(0)$. Then T is surjective, i.e., for every $a \in \mathbf{R}^1$ there is $v \in V$ such that $T(v) = a$. [Hint: This is true, since $aT(\cos(x)) = (a)$.]

26. **T** Suppose A is a matrix with characteristic polynomial $x(x-1)(x-2)(x-3)$. Then there is a basis $\mathcal{B}$ of $\mathbf{R}^4$ such that $[A]_{\mathcal{B},\mathcal{B}} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. [Hint: Yes, since the roots of the characteristic polynomial all have algebraic multiplicity 1.]

27. **T** Suppose A has characteristic polynomial $x^7 + x^3 - 3$. Then $\det(A) \neq 0$. [Hint: Yes. The idea is to notice that $\det(xI - A) = x^7 + x^3 - 3$, then substitute $x = 0$ into both sides, so get $\det(-A) = -3$. Since $\det(-A) = \pm\det(A)$, we see that $\det(A) \neq 0$.]