Math 308: Last Homework Assignment – SOLUTIONS

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True or False:

1. **F** A system of linear equations over $\mathbf{R}$ has either no solutions or infinitely many solutions.

2. **F** The span of the rows of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ has dimension 3.

3. **T** There are 4 linearly independent vectors in $\mathbf{R}^5$.

4. **F** The upper half plane, i.e., the set of points $(x, y) \in \mathbf{R}^2$ with $y \geq 0$, is a subspace of $\mathbf{R}^2$.

5. **T** If A and B are any two square matrices (of the same size), then $\det(AB) = \det(BA)$ and $\det(A + B) = \det(B + A)$.

6. **T** Let $\mathbf{F}_3$ be the finite field with 3 elements and let V be a 2-dimensional vector space over $\mathbf{F}_3$. Then there are 81 linear transformations $V \to V$.

7. **F** If A is any $n \times n$ matrix and B is the reduced row echelon form of A, then $\det(A) = \det(B^t)$, where B^t is the transpose of B.

8. **T** Suppose A is a 3×3 matrix with eigenvalues $1, 2, 3$. Then A must be diagonalizable.

9. **F** Suppose A is a 3×3 matrix with eigenvalues 1 and -1. Then $A^2 = I_3$.

10. **T** Let $\mathcal{P}$ be the vector space over $\mathbf{R}$ of polynomials of degree at most 3 with real coefficients, and let $T : \mathcal{P} \to \mathcal{P}$ be the linear transformation $T(ax^2 + bx + c) = a + b + c$. Then there is a basis $\mathcal{B}$ for $\mathcal{P}$ such that $[T]_{\mathcal{P},\mathcal{P}}$ is diagonal.

11. **F** Let $\mathcal{P}$ be the vector space over $\mathbf{R}$ of polynomials of degree at most 3 with real coefficients, and let $T : \mathcal{P} \to \mathbf{R}^1$ be the linear transformation $T(ax^2 + bx + c) = a + b + c$. Then the kernel of T (i.e., the set of v with $T(v) = 0$) has dimension 1.

12. **T** Suppose V and W are vector spaces over $\mathbf{R}$ of dimensions 2 and 3, respectively. Then there is a linear transformation $T : V \to W$ such that $\ker(T) = 0$.

13. **F** Suppose V and W are vector spaces over $\mathbf{R}$ of dimensions 3 and 2, respectively. Then there is a linear transformation $T : V \to W$ such that $\ker(T) = 0$.

14. **F** Every 2×2 matrix A is diagonalizable (over the complex numbers $\mathbf{C}$).

15. **T** The matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is diagonalizable over $\mathbf{C}$.

16. **T** Let V be the vector space of all differentiable functions $\mathbf{R} \to \mathbf{R}$. The function that send f to its derivative is a linear transformation $V \to V$.

17. **T** The dimension of the kernel of the derivative transformation (defined in the previous problem) is 1.

18. **T** Let V be the set of all integrable functions $\mathbf{R} \to \mathbf{R}$, i.e., functions that have some antiderivative. Then V is a vector space.

19. **T** Let V be the set of all integrable functions $\mathbf{R} \to \mathbf{R}$. The function that f to "the antiderivative of f that sends 0 to 0" is a linear transformation of V.

20. **F** Let V be the set of all integrable functions $\mathbf{R} \to \mathbf{R}$. The function that f to "the antiderivative of f that sends 0 to 1" is a linear transformation of V.

21. **T** The determinant of $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}$ is 0.

22. **T** If B and C are basis for a (finite dimensional) vector space V, then there must be some linear transformation $T : V \to V$ such that $[T]_{B,C}$ is the identity matrix.

23. **T** Let V be the set of functions $\mathbf{R} \to \mathbf{R}$. Then the function $V \to \mathbf{R}^2$ that sends f to $(f(1), f(\pi))$ is a linear transformation.

24. **F** Let V be the vector space over $\mathbf{R}$ that is the span of $\sin(x)$, $\cos(x)$, and $\cos(3x)$. Let $T : V \to \mathbf{R}^1$ be the linear transformation that sends $f \in V$ to $f(0)$. Then the kernel of T has dimension 1.

25. **T** Let V be the vector space over $\mathbf{R}$ that is the span of $\sin(x)$, $\cos(x)$, and $\cos(3x)$. Let $T : V \to \mathbf{R}^1$ be the linear transformation that sends $f \in V$ to $f(0)$. Then T is surjective, i.e., for every $a \in \mathbf{R}^1$ there is $v \in V$ such that $T(v) = a$.

26. **T** Suppose A is a matrix with characteristic polynomial $x(x-1)(x-2)(x-3)$. Then there is a basis $\mathcal{B}$ of $\mathbf{R}^4$ such that $[A]_{\mathcal{B},\mathcal{B}} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

27. **T** Suppose A has characteristic polynomial $x^7 + x^3 - 3$. Then $\det(A) \neq 0$.