Color-Reversal by Local Complementation

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Problem Statement

- **Input:** A connected bicolored graph G of order $n \geq 2$
- **Output:** Same underlying graph G, with color of its vertices reversed.

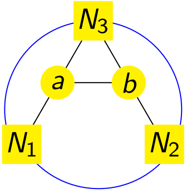


Figure: Input

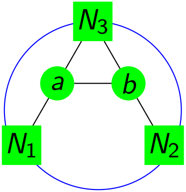
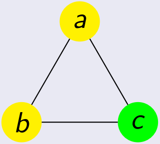


Figure: Output

Mathematical Preliminaries

Definitions

- Graph $G = (V, E)$.
- Bicoloration of a graph $\beta : V \mapsto \{-1, 1\}$



- Bicolored graph $B = (G, \beta)$.

Notation

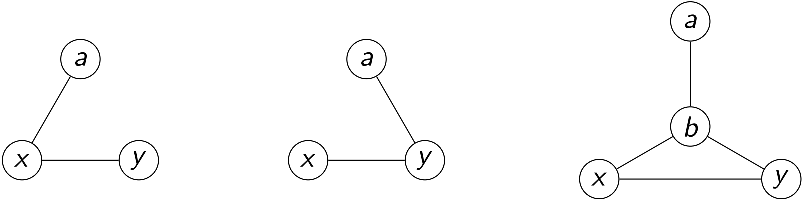
1. Given a graph $G = (V, E)$, we use $V(a; G)$ to denote the adjacent vertices of a in G.
2. We use $a \in V$ as pivot vertex or a distinguised vertex of G.
3. Induced subgraph.
    - Induced Subgraph: An induced subgraph of a graph G is a graph formed by subset $U \subset V$ and all edges among the vertices of U from the original graph. We use $G[U]$ to denote the subgraph induced by $U \subset A$.

Two relations

Relation 1

Let $G = (V, E)$ be a graph. Let x and y be any two different arbitrary vertices of G. We define a binary relation R_1 over the set V_G as follows: $(x, y) \in R_1$ if and only if

$$((x, y) \in E) \wedge ((x \notin V(a; G)) \vee (y \notin V(a; G))).$$

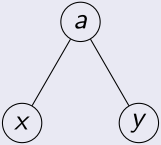


Two relations

Relation 2

Let $G = (V, E)$ be a graph. Let x and y be any two different arbitrary vertices of G. We define a binary relation R_2 over the set V_G as follows: $(x, y) \in R_2$ if and only if

$$((x, y) \notin E) \wedge ((x \in N(a; G)) \wedge (y \in N(a; G))).$$

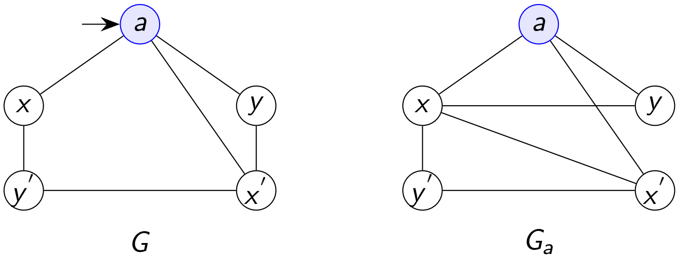


Local complement of a graph with respect to vertex a

Definition (Local complement of a graph with respect to a vertex)

Let $G = (V, E)$ be a graph and $a \in V$ be any arbitrary vertex of G. Then the *local complement* of G with respect to a on A, denoted as G_a, defined as reversing the relation of the adjacency between the neighbors. In other words, $E(G)$ is defined as follows:

$$\forall (x, y) \in V \times V, [((x, y) \in E(G_a)) \iff (R_1 \vee R_2)].$$



Some notations

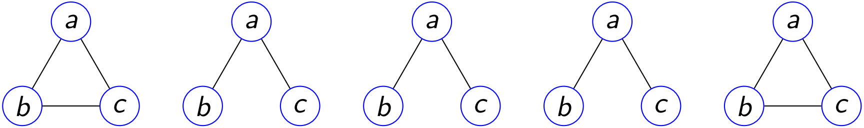
- Let $A = V(G)$.
- String over A: It is a finite sequence of elements from A.
- Kleene star of A: The Kleene star of A, denoted as A^*, is the set of all finite sequence of elements of A. In other words,

$$A^* = \bigcup_{i=0}^{\infty} A^i.$$

- We use ε to denote the empty string.
- Reverse of a string: reverse of $w = w_1 w_2 w_3 \cdots w_{n-2} w_{n-1} w_n$ is $w^R = w_n w_{n-1} w_{n-2} \cdots w_3 w_1 w_1$

Local complement of a graph with respect to a string

- Let $G = (A, E(G))$ be a graph and $a \in A$.
- Let $w \in A^*$ be any string.
- We define the local complement of G with respect to a string w, denoted as G_w, inductively as follows:
    - Base case: $G_\varepsilon = G$.
    - Inductive case: $G_{w'a} = (G_{w'})_a$.



G $\quad G_a \quad (G_a)_b \quad (G_{ab})_c \quad (G_{abc})_a$

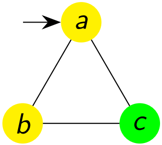
Local complement of a graph with respect to a string

- $u \sim w \iff G_u = G_w$ where $u, w \in A^*$
- $a^2 \sim \varepsilon$
- $ww^R \sim \varepsilon$ where $w \in A^*$ and w^R is reverse of w
- $a \sim \varepsilon \iff a$ is a pendant vertex

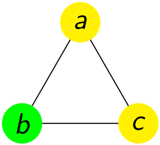
a new bicoloration β_a

1. Invert the color of a vertex x if and only if x is an adjacent vertex of a.
2. In other words, $\beta_a(x)$ is a function from the set A of vertices of G into a set of binary colors and is defined as follows:

$$\begin{aligned} \beta_a(x) &= -\beta(x), \text{ if } x \in V(a; G) \\ \beta_a(x) &= \beta(x), \text{ otherwise} \end{aligned}$$



β

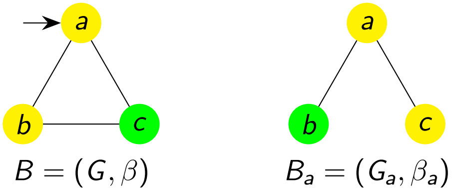


β_a

local inversion of a bicolored graph B

- Let $B = (G, \beta)$ be a bicolored graph
- Let a be a pivot vertex.
- Let G_a be the local complement of G with respect to a
- Let β_a be a new bicoloration.

Then the local inversion of B at a, denoted as B_a, is the 2-tuple $B_a = (G_a, \beta_a)$.



Complemented neighbors of a set of vertices with respect to a subset

- Let $G = (A, E(G))$ be a graph and $X \subset U \subset A$.
- The complemented neighbours of U with respect to X, denoted as $A_U(X, G)$, defined as the set of those vertices none of which belong to U but X is the largest common subset of both, the set of their adjacency vertices as well as U.
- Mathematically,

$$A_U(X, G) = \{x \in A \backslash U : V(x; G) \cap U = X\}.$$

Computing local inversion

- $U = \{a, b\}$
- $X \in \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$
- Let $N_1 = A_{ab}(a, G)$, $N_2 = A_{ab}(b, G)$, and $N_3 = A_{ab}(ab, G)$

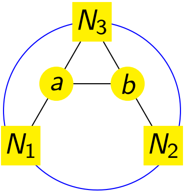


Figure: B

Computing B_{bab}

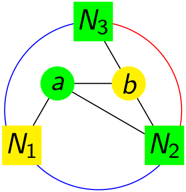


Figure: B_b

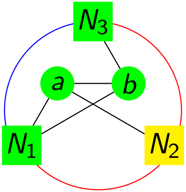


Figure: B_{ba}

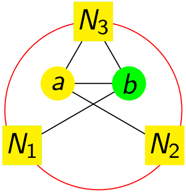


Figure: B_{bab}

Computing B_{aba}

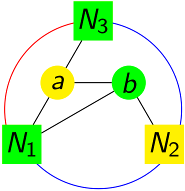


Figure: B_a

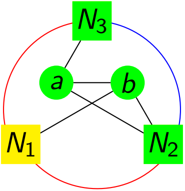


Figure: B_{ab}

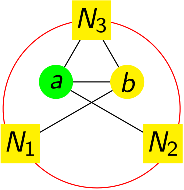


Figure: B_{aba}

Lemma 3

- Let $G = (A, E(G))$ be a graph.
- Let $(a, b) \in E(G)$.
- Then the two local complementations of G with respect to two strings *aba* and *bab* have the same underlying structure.
- The two graphs, namely G_{aba} and G_{bab} have the same underlying structure.

Lemma 3 continued

We define a function $h_u : V_G \mapsto \{1, -1\}$ as follows.

$$\begin{aligned} h_u(x) &= -1, \text{ if } x = u \\ &= 1, \text{ otherwise} \end{aligned}$$

We have the following expressions:

- $\beta_{aba} = \beta h_a$, and
- $\beta_{bab} = \beta h_b$.

Lemma 3 continued

- Lemma: Let $G = (A, E(G))$ be a graph. Let $(a, b) \in E(G)$. Then the two local complementations of G with respect to two strings *aba* and *bab* have the same underlying structure. In other words, the two graphs, namely G_{aba} and G_{bab} have the same underlying structure.
- $G_{aba} = G_{bab}$

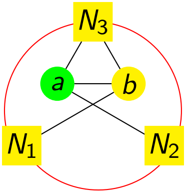


Figure: B_{aba}

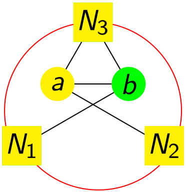


Figure: B_{bab}

Lemma 1

Lemma: Let $G = (A, E(G))$ be a graph. Let $(a, b) \in E(G)$. Then performing local inversion successively at a, b, a, b, a, b reproduces the graph G with colors of the vertex a and b are reversed and all other colors unchanged.

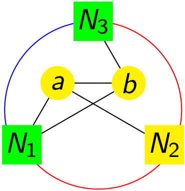


Figure: $B_{(aba)b}$

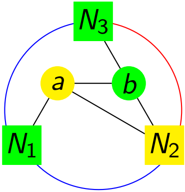


Figure: $B_{(aba)ba}$

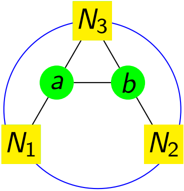


Figure: $B_{(aba)bab}$

Lemma 2

Lemma

Let $\Delta = abc$ be a triangle in G. Then performing local inversion successively at a, b, a, c, b, a, c reproduces the same graph G with the color of the vertex a reversed and all other colors unchanged.

Let a_0, a_1, a_2 be the three vertices of a triangle in G. Then $G_{(a_0a_1a_0)(a_1a_2a_1)(a_2a_0a_2)} = G_\varepsilon$.

Proof: Let $u_0 = a_0a_1a_0$, $u_1 = a_1a_2a_1$, and $u_2 = a_2a_0a_2$. Then we have the following results from Lemma-3:

1. $G_0 = G$,
2. $G_1 = (G_0)_{u_0}$,
3. $G_2 = (G_1)_{u_1}$,
4. $G_3 = G_0 = (G_2)_{u_2}$

Lemma-2 Proof continued...

Let us consider the following three partitions:

- $P_0 = A_{\{a_0,a_1,a_2\}}(X, G_0)$
- $P_1 = A_{\{a_0,a_1,a_2\}}(X, G_1)$
- $P_2 = A_{\{a_0,a_1,a_2\}}(X, G_2)$

- $X \subseteq \{a_0, a_1, a_2\} \implies X \subseteq 2^{|\{a_0,a_1,a_2\}|}$ Therefore, $X \in \{\emptyset, \{a_0\}, \{a_1\}, \{a_2\}, \{a_0, a_1\}, \{a_1, a_2\}, \{a_0, a_2\}, \{a_0, a_1, a_2\}\}$.
- $A_{\{a_0,a_1,a_2\}}(X, G_0)$ defines blocks of P_0.

Let us denote the blocks of P_0 as follows:

$$
\begin{aligned}
Y_0 &= A_{\{a_0,a_1,a_2\}}(\emptyset, G) = \{\emptyset\} \\
Y_1 &= A_{\{a_0,a_1,a_2\}}(\{a_0\}, G) = \{x\} \\
Y_2 &= A_{\{a_0,a_1,a_2\}}(\{a_1\}, G) = \{y\} \\
Y_3 &= A_{\{a_0,a_1,a_2\}}(\{a_2\}, G) = \{z\} \\
Y_4 &= A_{\{a_0,a_1,a_2\}}(\{a_0, a_1\}, G) = \{p\} \\
Y_5 &= A_{\{a_0,a_1,a_2\}}(\{a_0, a_2\}, G) = \{r\} \\
Y_6 &= A_{\{a_0,a_1,a_2\}}(\{a_1, a_2\}, G) = \{q\} \\
Y_7 &= A_{\{a_0,a_1,a_2\}}(\{a_0, a_1, a_2\}, G) = \{s\}
\end{aligned}
$$

Lemma-2 Proof continued...

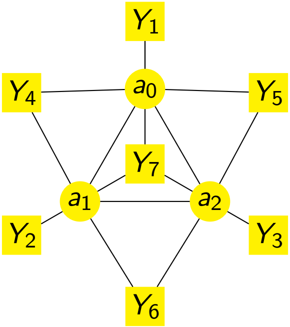


Figure: Blocks of P_0

Now we will apply lemma-3 on G_i and e_i, where $e_i = [a_i, a_{i+1}]$ where $i = 0, 1, 2$.

$G_0 \xrightarrow{a_0 a_1 a_0} G_1$

- Apply Lemma-3 on G_0 and $e_0 = [a_0, a_1]$. That is apply local inversion in order a_0, a_1, a_0.
- Note: $u_0 = (a_0, a_1, a_0)$
- $G_0 \xrightarrow{a_0, a_1, a_0} G_1$

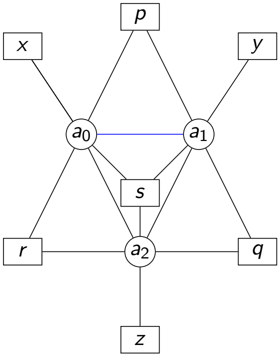


Figure: $G_0 = G$

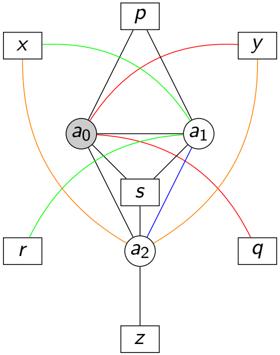


Figure: $G_1 = (G_0)_{a_0 a_1 a_0}$

Outcomes of $G_0 \xrightarrow{a_0 a_1 a_0} G_1$

- N_{01} : blocks adjacent to a_0 only $= \{x\} \cup \{r\}$
- N_{02} : blocks adjacent to a_1 only $= \{y\} \cup \{q\}$
- N_{03} : blocks adjacent to a_0 and a_1 simultaneously $= \{p\} \cup \{s\} \cup \{a_2\}$

Applying Local Inversion in sequence $a_0 a_1 a_0$ changes the adjacency as follows:

- By lemma-3, N_{01} becomes adjacent to a_1 and non-adjacent to a_0.
- N_{02} becomes adjacent to a_0 and non-adjacent to a_1.
- N_{03} remains adjacent to a_0, a_1.
- Adjacency between N_{01}, N_{02}, N_{03} are reversed. Therefore, x, y become adjacent to a_2.
- Color of a_0 is reversed.
- Colors of all other vertices are unchanged.

$G_1 \xrightarrow{a_2,a_1,a_2} G_2$

- Apply Lemma-3 on G_1 and $e_1 = [a_1, a_2]$. That is apply local inversion in order a_1, a_2, a_1.
- Note: $u_1 = (a_1, a_2, a_1)$
- Note: $a_1, a_2, a_1 \sim a_2, a_1, a_2$. **Therefore we will apply lemma-3 in order** a_2, a_1, a_2**.**
- $G_1 \xrightarrow{a_2,a_1,a_2} G_2$

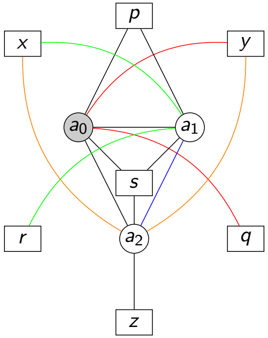


Figure: G_1

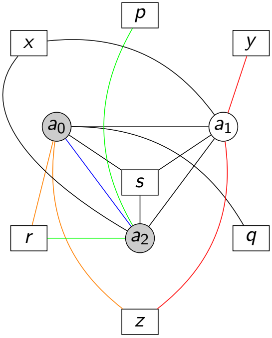


Figure: $G_2 = (G_1)_{a_2 a_1 a_2}$

Outcomes of $G_1 \xrightarrow{a_2 a_1 a_2} G_2$

- N_{11} : blocks adjacent to a_2 only $= \{z\} \cup \{y\}$
- N_{12} : blocks adjacent to a_1 only $= \{r\} \cup \{p\}$
- N_{13} : blocks adjacent to a_2 and a_1 simultaneously $= \{s\} \cup \{x\} \cup \{a_0\}$

Applying Local Inversion in sequence $a_2 a_1 a_2$ changes the adjacency as follows:

- By lemma-3, N_{11} becomes adjacent to a_1 and non-adjacent to a_2.
- N_{12} becomes adjacent to a_2 and non-adjacent to a_1.
- N_{13} remains adjacent to a_2, a_1.
- Adjacency between N_{11}, N_{12}, N_{13} are reversed. Therefore, z, r become adjacent to a_0 and y, p become non-adjacent to a_0.
- Color of a_2 is reversed.
- Colors of all other vertices are unchanged.

Outcomes of $G_2 \xrightarrow{a_2 a_0 a_2} G_3$

- Apply Lemma-3 on G_2 and $e_2 = [a_2, a_0]$. That is apply local inversion in order a_2, a_0, a_2.
- We will apply lemma-3 in order a_2, a_0, a_2.
- $G_2 \xrightarrow{a_2, a_0, a_2} G_3$

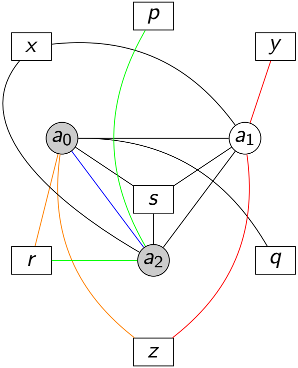


Figure: G_2

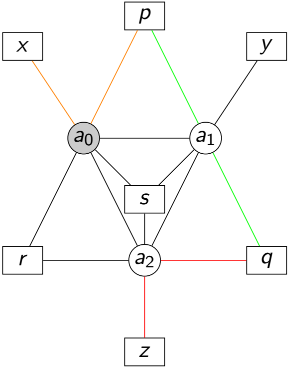


Figure: $G_3 = (G_2)_{a_2 a_0 a_2} = G_0 = G$

Outcomes of $G_2 \xrightarrow{a_2 a_0 a_2} G_3$

- N_{21} : blocks adjacent to a_2 only $= \{p\} \cup \{x\}$
- N_{22} : blocks adjacent to a_0 only $= \{q\} \cup \{z\}$
- N_{23} : blocks adjacent to a_2 and a_0 simultaneously $= \{s\} \cup \{r\} \cup \{a_1\}$

Applying Local Inversion in sequence $a_2 a_0 a_2$ changes the adjacency as follows:

- By lemma-3, N_{21} becomes adjacent to a_0 and non-adjacent to a_2.
- N_{22} becomes adjacent to a_2 and non-adjacent to a_0.
- N_{23} remains adjacent to a_2, a_0.
- Adjacency between N_{21}, N_{22}, N_{23} are reversed. Therefore, p, q become adjacent to a_1 and x, z become non-adjacent to a_1.
- Color of a_2 is reversed.
- Colors of all other vertices are unchanged.

Lemma-2 Proof continued...

- $u_0 u_1 u_2 = (a_0 a_1 a_0)(a_1 a_2 a_1)(a_2 a_0 a_2) \sim (a_0 a_1 a_0)(a_2 a_1 a_2)(a_2 a_0 a_2) \sim a_0 a_1 a_0 a_2 a_1 a_0 a_2 = w = \varepsilon$
- $a_0 \rightarrow a, a_1 \rightarrow b, a_2 \rightarrow c$
- $a_0 a_1 a_0 a_2 a_1 a_0 a_2 = a, b, a, c, b, a, c \sim \varepsilon$
- From Lemma-3, $\beta_w = \beta h_{a_0} h_{a_2}^2 = \beta h_{a_0}$
- That means, after applying local inversion in sequence w, color of vertex a is reversed, with colors of other vertices unchanged.

Evolution of adjacency between vertices in different blocks of P

Consider the sets:
$N_{i1} = A_i(a_i, G_i), N_{i2} = A_i(a_i, G_i), N_{i3} = A_i(a_i, G_i)$
where A_i is short for $A_{a_i a_{i+1}}$. We have the following equations:

- $N_{i1} = A(a_i, G_i) \cup A(a_i a_{i+2}, G_i)$
- $N_{i2} = A(a_{i+1}, G_i) \cup A(a_{i+1} a_{i+2}, G_i)$
- $N_{i1} = A(a_i a_{i+1}, G_i) \cup A(\Delta, G_i) \cup a_{i+2}$

G_0	G_1	G_2
$N_{01} = Y_1 \cup Y_5 = \{x\} \cup \{r\}$	$N_{11} = Y_5 \cup Y_4 = \{z\} \cup \{y\}$	$N_{21} = Y_4 \cup Y_1 = \{p\} \cup \{x\}$
$N_{02} = Y_2 \cup Y_6 = \{y\} \cup \{q\}$	$N_{12} = Y_3 \cup Y_2 = \{r\} \cup \{p\}$	$N_{21} = Y_6 \cup Y_3 = \{z\} \cup \{q\}$
$N_{03} = Y_4 \cup Y_7 \cup a_2 = \{p\} \cup \{s\} \cup \{a_2\}$	$N_{13} = Y_1 \cup Y_7 \cup a_0 = \{x\} \cup \{s\} \cup \{a_0\}$	$N_{23} = Y_5 \cup Y_7 \cup a_1 = \{r\} \cup \{s\} \cup \{a_1\}$

- Let $x \in Y_r, y \in Y_s, r \neq s$. Then we have that $[x, y] \notin E(G_{i+1}) \iff [x, y] \in E(G_i)$ and $Y_r \subset N_{ij}, Y_s \subset N_{ik}, j \neq k$.
- The number of times the adjacency between x and y is changed to its complement is either zero or two.
- Therefore, adjacency between different blocks remains same as original graph.

Theorem

Let G be a connected bicolored graph of order $n \geq 2$. Then G can be color-reversed by local inversion in $6n + 3$ moves. More generally, given two arbitrary bicolorations β and β' of G, then β' can be obtained from β in $\leq 9n$ moves.

Proof:

- From lemma-1, that simultaneous color reversal of two adjacent vertices takes 6 steps.
- Let G be a graph of even number of vertices. By using Perfect Forest Theorem, we can always get spanning forests of graph G such that every nodes of this forest have odd degree.
- Therefore, if the spanning forests of graph G have m edges, then reversing the color of vertices of G using lemma-1 will take $6m$ moves.

Theorem

Proof continued:

- If graph G has odd number of vertices say n, then we can split this graph into two induces subgraphs say G_1 having $n-1$ vertices (even number of vertices) and G_2 having remaining one vertex.
- We can obtain perfect forests out of induced subgraph G_1 and reverse the colors of these $n-1$ vertices using lemma-1 in $\leq 6(n-1)$ moves.
- After reversing the color of G_1, consider the original graph G, then perform lemma-2 operations on vertex of G_2 in G which will take at most 9 *(7 moves if this vertex is a part of a triangle already, else make it a part of triangle by using local complement, which will take atmost 2 more moves to do and undo at the end)* moves.
- Therefore, if G has odd number of vertices then its vertices can be color reversed in $\leq 6(n-1)+9$ i.e. $\leq 6n+3$ steps.

Sagartanu's Thesis

Local complementation is an equivalence relation.

- $G \xrightarrow{\varepsilon} G$: Reflexive
- If $G \xrightarrow{w} G' then G' \xrightarrow{w^R} G$: Symmetric
- If $G \xrightarrow{w_1} G' and G' \xrightarrow{w_2} G'' then G \xrightarrow{w_1 w_2} G''$: Transitive

Local complement equivalent graphs

- G and G' are equivalent if one can be generated from the other.
- Each equivalence class is known as an orbit.
- For a graph class $\mathcal{G}$, we refer to an orbit as a $\mathcal{G}$ orbit if every graph G in the orbit belongs to $\mathcal{G}$.

Sagartanu's Thesis

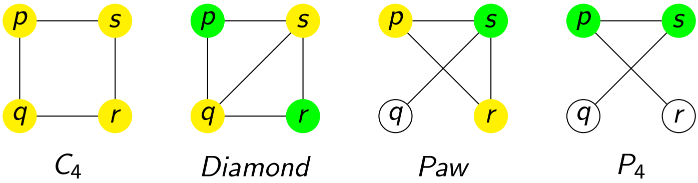


Figure: Equivalent Graphs

Observations

1. *A graph G in an $H - free$ orbit is $\mathcal{H} - free$, where $\mathcal{H}$ is the orbit of H.*
2. K_t and $K_{1,t-1}$ form an orbit.
3. A graph G belongs to a $K_2 - \textit{free}$ orbit if and only if G is an empty graph.
4. A graph G in a $\textit{triangle} - \textit{free}$ orbit is a disjoint union of K_2s and K_1s.
5. A graph G in a complete bipartite orbit is either a K_2 or an empty graph.

Lemma

A connected component G of a graph X in $K_t - \textit{free}$ orbit is $P_{2t-3} - \textit{free}$ for $t \geq 4$.

Applications

Local complementation has significant applications in quantum computing:

- **Graph states:** Quantum states represented by graphs where vertices represent qubits and edges represent entanglement
- **Local complementation:** Corresponds to local Clifford operations on graph states
- **Color reversal:** Relates to phase flips across a set of qubits

References

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Thank you