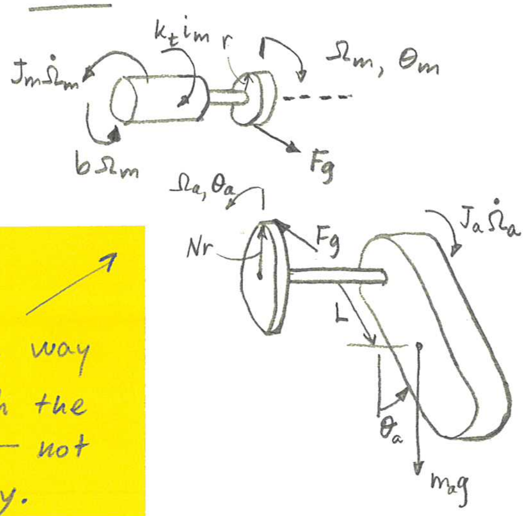
Problem 4

FBD



This is one way to deal with the gear ratio — not the only way.

Arm:

$$J_m\dot{\Omega}_m + b\Omega_m + F_g r = k_t i_m$$

$$\Rightarrow F_g r = -J_m\dot{\Omega}_m - b\Omega_m + k_t i_m \quad (*)$$

$$J_a\dot{\Omega}_a + m_a g L \sin\theta_a - N F_g r = 0$$

Subst. from (*)

$$J_a\dot{\Omega}_a + m_a g L \sin\theta_a + N\left(J_m\dot{\Omega}_m + b\Omega_m - k_t i_m\right) = 0$$

Note: $\Omega_m = N\Omega_a$, $\dot{\Omega}_m = N\dot{\Omega}_a$

$$J_a\dot{\Omega}_a + m_a g L \sin\theta_a + N\left(J_m N\dot{\Omega}_a + bN\Omega_a - k_t i_m\right) = 0$$

$$(J_a + N^2 J_m)\dot{\Omega}_a + N^2 b\,\Omega_a + m_a g L \sin\theta_a = N k_t i_m$$

True for parts (a) and (b)

$$\boxed{\dot{\Omega}_a = \frac{1}{J_a + N^2 J_m}\left[-N^2 b\,\Omega_a - m_a g L \sin\theta_a + N k_t i_m\right]} \quad (1)$$

$$\boxed{\dot{\theta}_a = \Omega_a} \quad (2)$$

Motor:

From notes:

$$\frac{di_m}{dt} = \frac{1}{L_m}\left[-R_m i_m - k_t\Omega_m + v_m(t)\right]$$

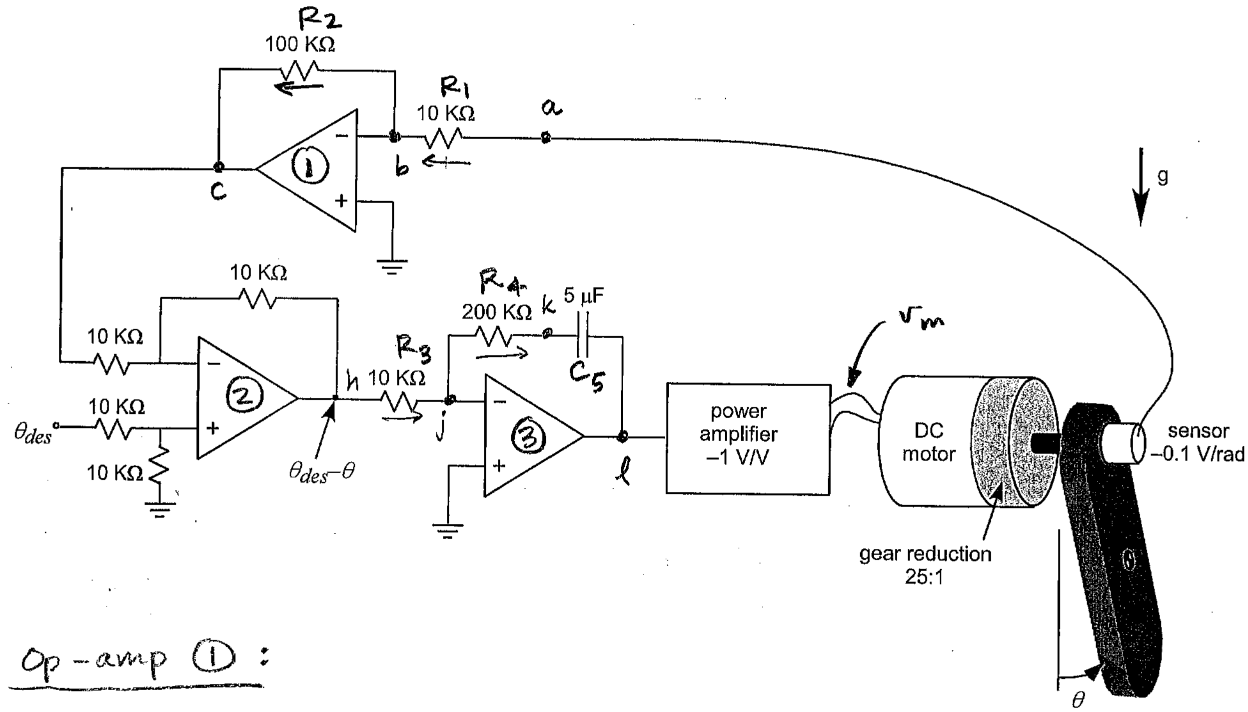
$$\frac{di_m}{dt} = \frac{1}{L_m}\left[-R_m i_m - K_t N\Omega_a + v_m\right]$$

must define in terms of states and input.

States: $\Omega_a, \theta_a, i_m, e_c$

Input: θ_{des}

v_m is different in parts (a) and (b)



Op-amp ① :

Input: $v_a = -0.1\,\theta$

Output: v_c States: none

Component Relations:
$$i_1 = \frac{1}{R_1}(v_a - \cancel{v_b}^{\approx 0}) = \frac{1}{R_1} v_a$$
$$i_2 = \frac{1}{R_2}(\cancel{v_b} - v_c) = -\frac{1}{R_2} v_c$$

KCL: ⓑ $\rightarrow$ $i_1 = i_2$

$$\frac{1}{R_1} v_a = -\frac{1}{R_2} v_c \implies v_c = -\frac{R_2}{R_1} v_a$$
$$= -\frac{100\,k\Omega}{10\,k\Omega}(-0.1\,\theta)$$
$$\underline{v_c = \theta_a}$$

Op-amp ② :

From problem 1 of this homework,

$$v_h = \frac{10\,k\Omega}{10\,k\Omega}(\theta_{des} - v_c)$$
$$\underline{v_h = \theta_{des} - \theta_a}$$

Op-amp (3):

(a) Capacitor C_5 short circuited

Input: $v_h = \theta_{des} - \theta$

Output: v_ℓ States: none

Component Relations: $i_3 = \frac{1}{R_3}(v_h - \cancel{v_j}^{\,0}) = \frac{1}{R_3} v_h$

$$i_4 = \frac{1}{R_4}(\cancel{v_j}^{\,0} - v_\ell) = -\frac{1}{R_4} v_\ell$$

KCL: $i_3 = i_4$

$$\frac{1}{R_3} v_h = -\frac{1}{R_4} v_\ell \implies v_\ell = -\frac{R_4}{R_3} v_h$$

$$v_m = -v_\ell = \frac{R_4}{R_3} v_h = \frac{R_4}{R_3}(\theta_{des} - \theta)$$

$$\boxed{v_m(t) = 20(\theta_{des} - \theta_a)}$$

(b) Component Relations: $i_3 = \frac{1}{R_3} v_h$, $i_4 = -\frac{1}{R_4} v_k$

$$\dot{v}_5 = \frac{1}{C_5} i_5 \qquad v_5 = v_k - v_\ell$$

KCL: $i_3 = i_4$, $i_4 = i_5$ $\qquad v_\ell = v_k - v_5$

State eqn: $\dot{v}_5 = \frac{1}{C_5} i_3 = \frac{1}{R_3 C_5} v_h$

$$\boxed{\dot{v}_5 = \frac{1}{R_3 C_5}(\theta_{des} - \theta_a)}$$

$$v_m = -v_\ell = v_5 - v_k = v_5 + R_4 i_4$$

$$= v_5 + \frac{R_4}{R_3} v_h$$

$$\boxed{v_m(t) = v_5 + \frac{R_4}{R_3}(\theta_{des} - \theta_a)}$$

Summary

(a) Arm dynamics given by (1) and (2)

Motor/feedback dynamics:

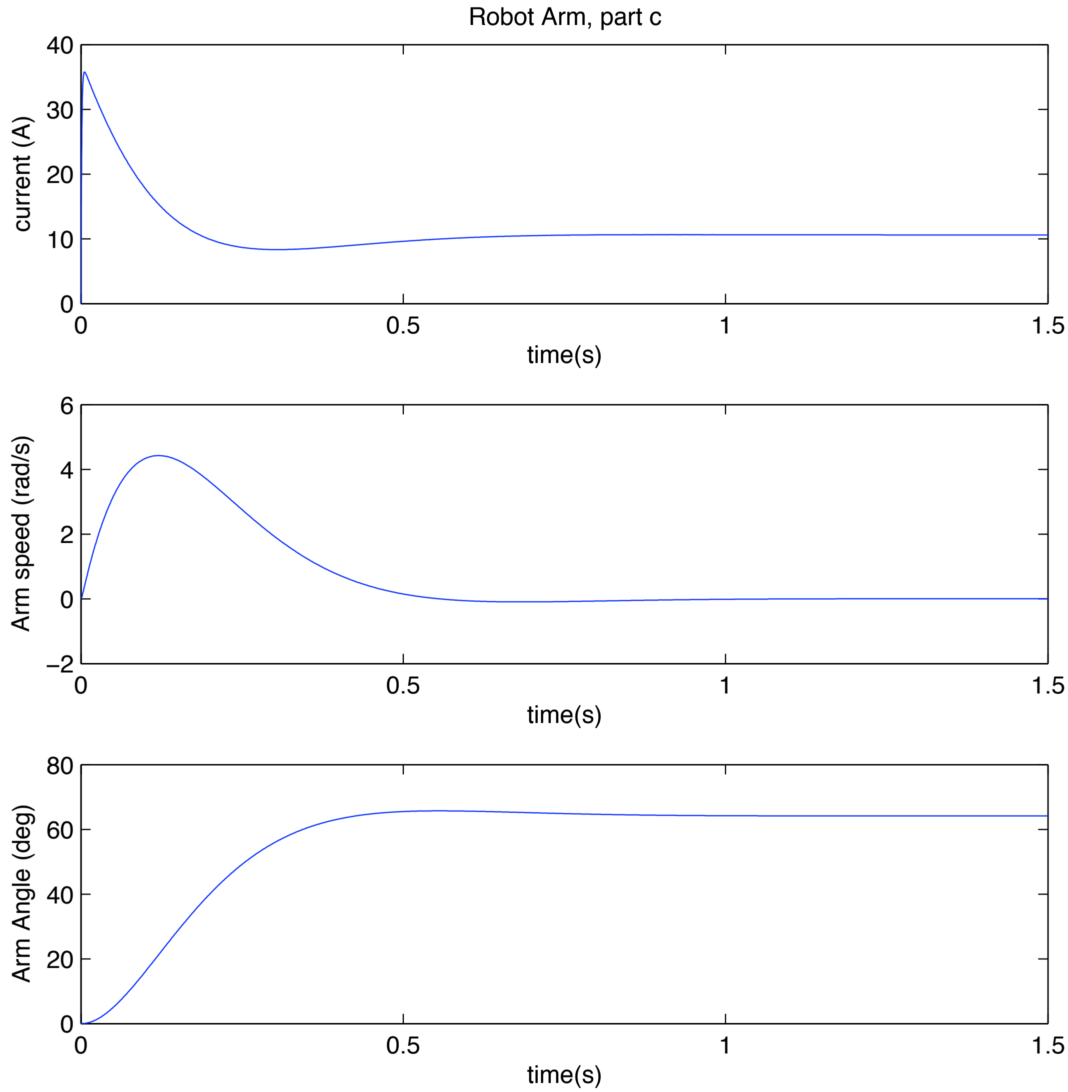
$$\boxed{\frac{di_m}{dt} = \frac{1}{L_m}\left[-R_m i_m - K_t N \Omega_a + \frac{R_4}{R_3}(\theta_{des} - \theta_a)\right]}$$

(b) Arm dynamics given by (1) and (2)

Motor/feedback dynamics:

$$\boxed{\begin{aligned} \frac{di_m}{dt} &= \frac{1}{L_m}\left[-R_m i_m - K_t N \Omega_a + v_5 + \frac{R_4}{R_3}(\theta_{des} - \theta_a)\right] \\ \dot{v}_5 &= \frac{1}{R_3 C_5}(\theta_{des} - \theta_a) \end{aligned}}$$

Plots of the state variables:



Note that the robot arm only comes to about 64 degrees! This is far short of the desired 90 degrees. This is because the steady-state solution to the equations of motion results in just two equations:

$$20(\theta_{des} - \theta \ ) = R_m i_m$$

$$n k_t i_m = m \ \ gl \ \ \sin\theta$$

The simultaneous solution to these equations cannot result in the arm angle equaling the desired angle, because that would require the motor current to be zero (by the first equation), resulting in an arm angle of zero (by the second equation). Hence, the system cannot result in zero steady-state error without considering the capacitor.

The MATLAB code:

```
%robotc.m - robot arm problem part c

t0 = 0;%s - initial time
tf = 1.5;%s - final time
X0 = [0 0 0];% define initial conditions

%options = odeset('AbsTol',0.0001);
[t,X] = ode45(@robotarm,[t0 tf],X0);%,options);

%extract state variables for convenience
im = X(:,1);
omegaa = X(:,2);
thetaa = X(:,3);

%plot the results
figure(1)
clf
subplot 311
plot(t,im)
ylabel('current (A)')
xlabel('time(s)')
title('Robot Arm, part c')

subplot 312
plot(t,omegaa)
ylabel('Arm speed (rad/s)')
xlabel('time(s)')

subplot 313
plot(t,thetaa*180/pi)
ylabel('Arm Angle (deg)')
xlabel('time(s)')
```

And the function:

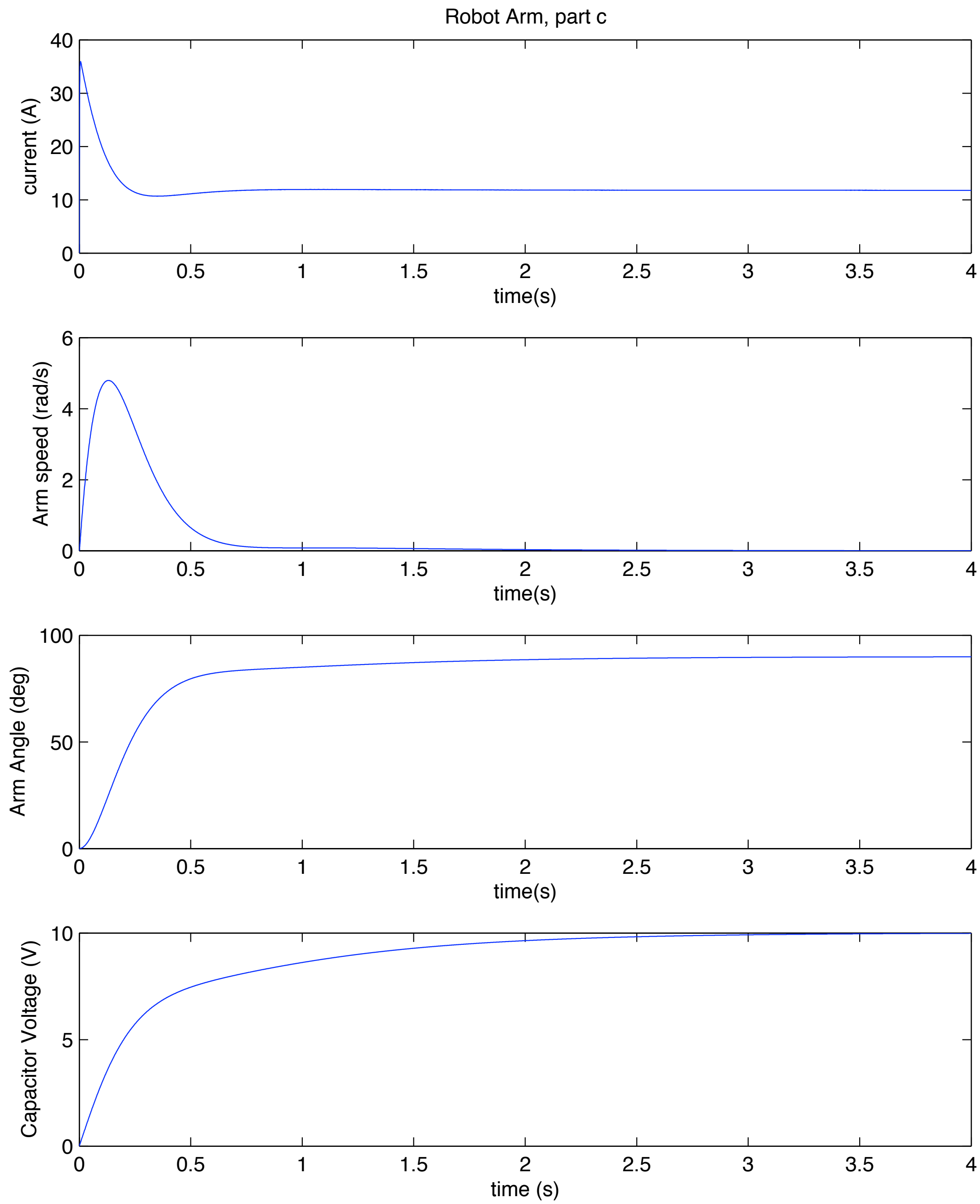
```
function xdot = robatarm(t,x)

%this function gives the equations of motion for the closed-loop
position
%control problem for homework 7.
%state variabes are:    x(1) = im, the motor current
%                       x(2) = omegaa, the arm angular velocity
%                       x(3) = thetaa, the angle of the robot
arm
% This simulation ignores the capacitor.

%initialize the output
xdot = zeros(length(x),1);
%define constants
Lm = 0.001;% H
Rm = 0.85;% Ohm
n = 25;% gear ratio
Ke = 0.1;%N*m/A
Kt = Ke;
bm = 0.01;%N*m*s
ma = 10;%kg
g = 9.81;%m/s^2
la = 0.3;%m
Jm = 4e-5;%N*m*s^2
Ja = 1;%N*m*s^2

%define input
thetades = pi/2;%rad
%now for the state equations
xdot(1) = (20*(thetades - x(3)) - Rm*x(1) - n*Ke*x(2))/Lm;
xdot(2) = (n*Kt*x(1) - bm*n^2*x(2) - ma*g*la*sin(x(3)))/
(Jm*n^2+Ja);
xdot(3) = x(2);
```

State Variable Plots:



The arm does come to the horizontal now. Notice that the state equations reduce to the following three equations in steady-state:

$$20(\theta_{des} - \theta\ \ ) + e_3 = R_m i_m$$

$$n k_t i_m = m\ \ gl\ \ \sin\theta$$

$$\theta\ \ = \theta_{des}$$

The extra equation specifies that the arm angle will be equal to the desired angle in steady-state! Moreover, the first equation is now modified to include the voltage on the capacitor, allowing the motor current to be non-zero even when the arm angle is equal to the desired arm angle.

The MATLAB code:

```
%robotd.m - robot arm problem part d

t0 = 0;%s - initial time
tf = 4;%s - final time
X0 = [0 0 0 0];% define initial conditions

%options = odeset('AbsTol',0.0001);
[t,X] = ode45(@robotarmd,[t0 tf],X0);%,options);

%extract state variables for convenience
im = X(:,1);
omegaa = X(:,2);
thetaa = X(:,3);
e3 = X(:,4);

%plot the results
figure(1)
clf
subplot 411
plot(t,im)
ylabel('current (A)')
xlabel('time(s)')
title('Robot Arm, part c')

subplot 412
plot(t,omegaa)
```

```
ylabel('Arm speed (rad/s)')
xlabel('time(s)')

subplot 413
plot(t,thetaa*180/pi)
ylabel('Arm Angle (deg)')
xlabel('time(s)')

subplot 414
plot(t,e3)
ylabel('Capacitor Voltage (V)')
xlabel('time (s)')
```

And the function:

```
function xdot = robatarmd(t,x)

%this function gives the equations of motion for the closed-loop
position
%control problem for homework 7.
%state variabes are:    x(1) = im, the motor current
%                       x(2) = omegaa, the arm angular velocity
%                       x(3) = thetaa, the angle of the robot
arm
%                       x(4) = e3, the voltage on the capacitor

%initialize the output
xdot = zeros(length(x),1);
%define constants
Lm = 0.001;% H
Rm = 0.85;% Ohm
n = 25;% gear ratio
Ke = 0.1;%N*m/A
Kt = Ke;
bm = 0.01;%N*m*s
ma = 10;%kg
g = 9.81;%m/s^2
la = 0.3;%m
Jm = 4e-5;%N*m*s^2
Ja = 1;%N*m*s^2
```

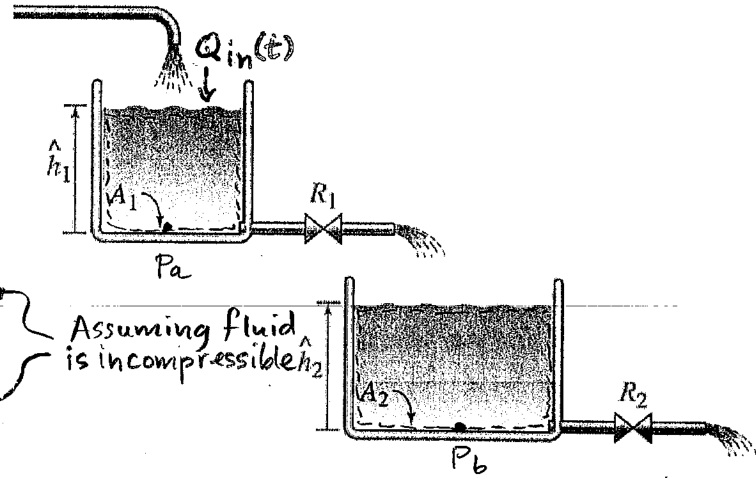
```
C = 5e-6;%F
R1 = 10000;%ohm

%define input
thetades = pi/2;%rad
%now for the state equations
xdot(1) = (20*(thetades - x(3)) - Rm*x(1) - n*Ke*x(2) + x(4))/
Lm;
xdot(2) = (n*Kt*x(1) - bm*n^2*x(2) - ma*g*la*sin(x(3)))/
(Jm*n^2+Ja);
xdot(3) = x(2);
xdot(4) = (thetades - x(3))/(R1*C);
```

Problem 2

Distinct Pressures : P_a, P_b

Continuity Eqns :



ⓐ $Q_{in,a} - Q_{out,a} = \dot{V}_a + \cancel{\dfrac{V_{0,a}}{\beta}\dot{P}_a}^{\,0}$

ⓑ $Q_{in,b} - Q_{out,b} = \dot{V}_b + \cancel{\dfrac{V_{0,b}}{\beta}\dot{P}_b}^{\,0}$

Assuming fluid is incompressible

Define $\dot{V}$ terms :

$$\dot{V}_a = \frac{A_1}{\rho g}\dot{P}_a \qquad \dot{V}_b = \frac{A_2}{\rho g}\dot{P}_b$$

for a gravity tank

Define flows :

$$Q_{in,a} = Q_{in}(t) \qquad Q_{out,a} = k_1\sqrt{P_a}$$

$$Q_{in,b} = k_1\sqrt{P_a} \qquad Q_{out,b} = k_2\sqrt{P_b}$$

Combine relations :

$$\boxed{\begin{aligned} \dot{P}_a &= \frac{\rho g}{A_1}\left[Q_{in}(t) - k_1\sqrt{P_a}\right] \\ \dot{P}_b &= \frac{\rho g}{A_2}\left[k_1\sqrt{P_a} - k_2\sqrt{P_b}\right] \end{aligned}}$$

Part (a)

Part (b)

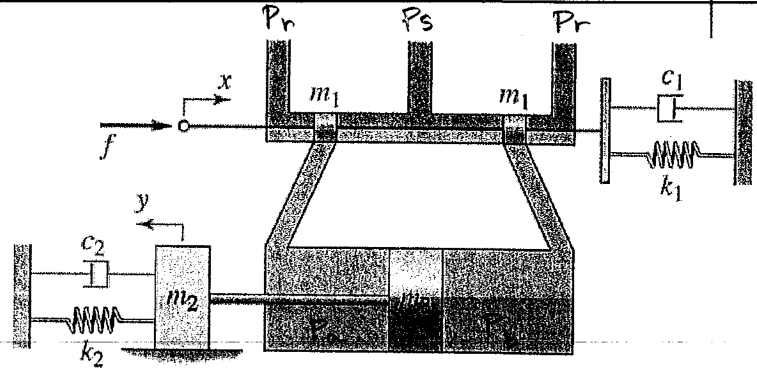
$$P_a = \rho g h_1 \qquad P_b = \rho g h_2$$

$$\dot{P}_a = \rho g \dot{h}_1 \qquad \dot{P}_b = \rho g \dot{h}_2$$

Substituting :

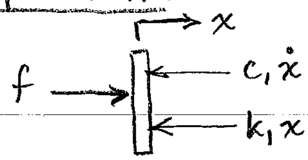
$$\boxed{\begin{aligned} \dot{h}_1 &= \frac{1}{A_1}\left[Q_{in}(t) - k_1\sqrt{\rho g h_1}\right] \\ \dot{h}_2 &= \frac{1}{A_2}\left[k_1\sqrt{\rho g h_1} - k_2\sqrt{\rho g h_2}\right] \end{aligned}}$$

Problem 3



Mechanical models :

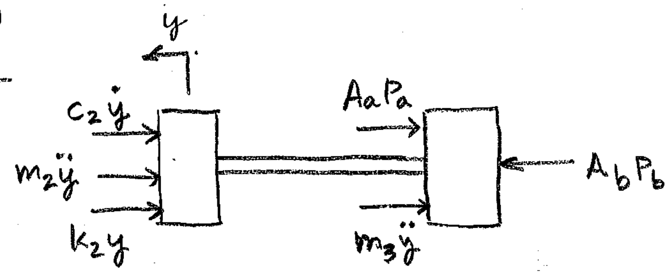
Spool valve



$$c_1 \dot{x} + k_1 x = f(t)$$

$$\dot{x} = -\frac{k_1}{c_1} x + \frac{1}{c_1} f(t)$$

Piston and load



$$(m_2+m_3)\ddot{y} + c_2\dot{y} + k_2 y + A_a P_a - A_b P_b = 0$$

$$\ddot{y} = \frac{1}{m_2+m_3}\left[-c_2\dot{y} - k_2 y - A_a P_a + A_b P_b\right]$$

Distinct Pressures : P_a, P_b, P_s, P_r — inputs

Continuity Eqns : (assume $x > 0$)

$$\cancel{Q_{in,a}}^{\,0} - Q_{out,a} = \dot{V}_a + \frac{V_{0,a}}{\beta}\dot{P}_a$$

$$Q_{in,b} - \cancel{Q_{out,b}}^{\,0} = \dot{V}_b + \frac{V_{0,b}}{\beta}\dot{P}_b$$

$\dot{V}$ terms : $\dot{V}_a = -A_a \dot{y}$ $\qquad$ $\dot{V}_b = A_b \dot{y}$

Flows : $Q_{out,a} = k(x)\sqrt{|P_a - P_r|}\,\text{sign}(P_a - P_r)$

$\qquad Q_{in,b} = k(x)\sqrt{|P_s - P_b|}\,\text{sign}(P_s - P_b)$

combine :

a: $$-k(x)\sqrt{|P_a-P_r|}\,\text{sign}(P_a-P_r) = -A_a\dot{y} + \frac{V_{0,a}}{\beta}\dot{P}_a$$

b: $$k(x)\sqrt{|P_s-P_b|}\,\text{sign}(P_s-P_b) = A_b\dot{y} + \frac{V_{0,b}}{\beta}\dot{P}_b$$

(cont)

(Prob 3 cont)

States :

$$P_a, P_b, y, v_y, x$$

State equations:

$$\dot{P}_a = \frac{\beta}{V_{o,a}}\left[A_a v_y - k(x)\sqrt{|P_a - P_r|}\,\text{sign}(P_a - P_r)\right]$$

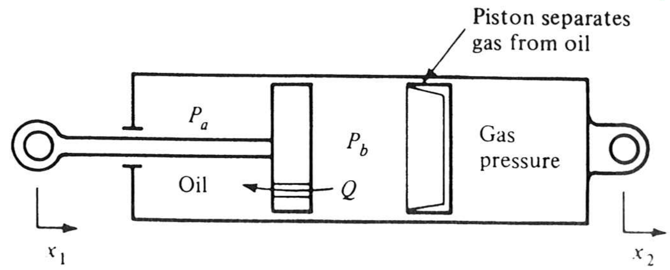
$$\dot{P}_b = \frac{\beta}{V_{o,b}}\left[k(x)\sqrt{|P_s - P_b|}\,\text{sign}(P_s - P_b) - A_b v_y\right]$$

$$\dot{x} = -\frac{k_1}{c_1} + \frac{1}{c_1} f(t)$$

$$\dot{v}_y = \frac{1}{m_1 + m_2}\left[-c_2 v_y - k_2 y - A_a P_a + A_b P_b\right]$$

$$\dot{y} = v_y$$

Problem 4



- distinct pressures : P_a, P_b, P_c

- continuity eqn's :

$$Q_{in,a} - \cancel{Q_{out,a}}^{\,0} = \dot{V}_a + \frac{V_{o,a}}{\beta}\dot{P}_a$$

$$\cancel{Q_{in,b}}^{\,0} - Q_{out,b} = \dot{V}_b + \frac{V_{o,b}}{\beta}\dot{P}_b$$

$$\cancel{Q_{in,c}}^{\,0} - \cancel{Q_{out,c}}^{\,0} = \dot{V}_c + \frac{V_{o,c}}{\rho_o c_o^2}\dot{P}_c$$

Inputs : x_1, x_2 (v_1, v_2)

- $\dot{V}$ terms :

$$\dot{V}_a = A_a(v_1 - v_2) \qquad \dot{V}_b = A_b(v_3 - v_1) \qquad \dot{V}_c = A_b(v_2 - v_3)$$

- flow terms :

$$Q_{in,a} = k_\ell \sqrt{|P_b - P_a|}\, \text{sign}(P_b - P_a)$$

$$Q_{out,b} = k_\ell \sqrt{|P_b - P_a|}\, \text{sign}(P_b - P_a)$$

- mechanical :

$P_b A_b \rightarrow$ [mass] $\leftarrow P_c A_b$, $\leftarrow m_3 \dot{v}_3$

$$m_3 \dot{v}_3 + A_b P_c - A_b P_b = 0$$

$$\boxed{\begin{aligned} \dot{v}_3 &= \frac{A_b}{m_3}(P_b - P_c) \\ \dot{x}_3 &= v_3 \end{aligned}}$$

- combine relations :

$$\dot{P}_a = \frac{\beta}{V_{o,a}}\left(Q_{in,a} - \dot{V}_a\right)$$

$$\boxed{\dot{P}_a = \frac{\beta}{V_{o,a}}\left[k_\ell \sqrt{|P_b - P_a|}\, \text{sign}(P_b - P_a) - A_a\left(v_1(t) - v_2(t)\right)\right]}$$

$$\dot{P}_b = \frac{\beta}{V_{o,b}}\left(-Q_{out,b} - \dot{V}_b\right)$$

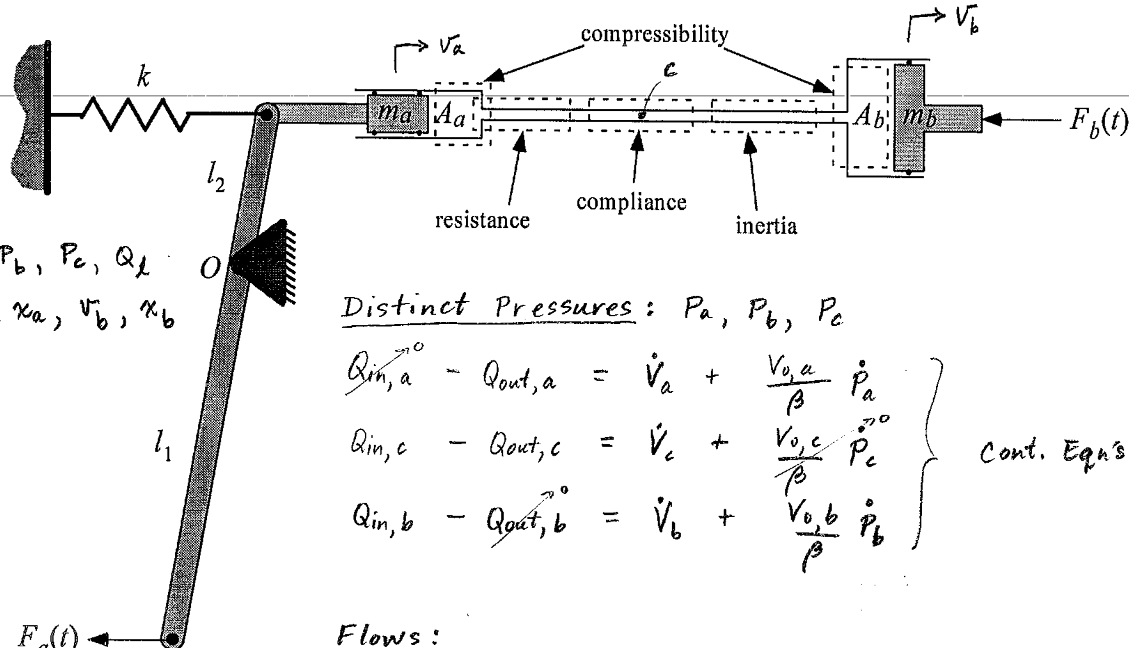
$$\boxed{\dot{P}_b = \frac{\beta}{V_{o,b}}\left[-k_\ell \sqrt{|P_b - P_a|}\, \text{sign}(P_b - P_a) - A_b\left(v_3 - v_1(t)\right)\right]}$$

$$\dot{P}_c = -\frac{\rho_o c_o^2}{V_{o,c}}\dot{V}_c$$

$$\boxed{\dot{P}_c = \frac{\rho_o c_o^2 A_b}{V_{o,c}}\left(v_3 - v_2(t)\right)}$$

Problem 5

Pictured below is a schematic of a hydraulic braking system. Model the system using the techniques demonstrated in class. Use your engineering judgement (and the material in the notes and text) to come up with physical relations for the elements of your model. Derive equations of motion to describe the dynamics of the braking system.



States: P_a, P_b, P_c, Q_ℓ

v_a, x_a, v_b, x_b

Distinct Pressures: P_a, P_b, P_c

$$\left.\begin{aligned}
\cancel{Q_{in,a}}^{\to 0} - Q_{out,a} &= \dot{V}_a + \frac{V_{0,a}}{\beta}\dot{P}_a \\
Q_{in,c} - Q_{out,c} &= \dot{V}_c + \cancel{\frac{V_{0,c}}{\beta}\dot{P}_c}^{\to 0} \\
Q_{in,b} - \cancel{Q_{out,b}}^{\to 0} &= \dot{V}_b + \frac{V_{0,b}}{\beta}\dot{P}_b
\end{aligned}\right\}\ \text{Cont. Eqn's}$$

Flows:

$$\boxed{\dot{Q}_\ell = \frac{A}{\rho \ell}\left[P_c - P_b\right]}$$

$$Q_{out,c} = Q_{in,b} = Q_\ell$$

$$Q_{out,a} = k_1\sqrt{|P_a - P_c|}\ \text{sign}(P_a - P_c)$$

$$Q_{in,c} = k_1\sqrt{|P_a - P_c|}\ \text{sign}(P_a - P_c)$$

$\dot{V}$ terms: $\dot{V}_a = -A_a v_a$, $\dot{V}_b = A_b v_b$, $\dot{V}_c = \frac{2 r_0 V_0}{E t w}\dot{P}_c$

Combining:

$$\dot{P}_a = \frac{\beta}{V_{0,a}}\left[-\dot{V}_a - Q_{out,a}\right]$$

$$\boxed{\dot{P}_a = \frac{\beta}{V_{0,a}}\left[A_a v_a - k_1\sqrt{|P_a - P_c|}\ \text{sign}(P_a - P_c)\right]}$$

$$\dot{P}_b = \frac{\beta}{V_{0,b}}\left[Q_{in,b} - \dot{V}_b\right]$$

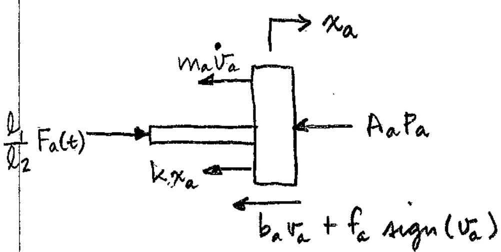
$$\boxed{\dot{P}_b = \frac{\beta}{V_{0,b}}\left[Q_\ell - A_b v_b\right]}$$

(cont.)

$$\dot{P}_c = \frac{E\, t_w}{2 r_0 V_0}\left[Q_{in,c} - Q_{out,c} \right]$$

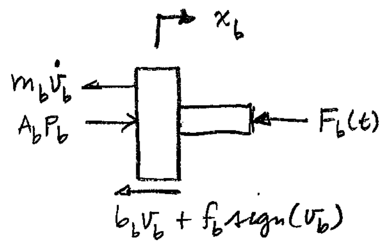
$$\boxed{\dot{P}_c = \frac{E\, t_w}{2 r_0 V_0}\left[k_1 \sqrt{|P_a - P_c|}\; \mathrm{sign}(P_a - P_c) - Q_\ell \right]}$$

<u>Mechanical:</u>



$$m_a \dot{v}_a + b_a v_a + f_a\, \mathrm{sign}(v_a) + k x_a + A_a P_a - \frac{l_1}{l_2} F_a(t) = 0$$

$$\boxed{\begin{aligned} \dot{v}_a &= -\frac{b_a}{m_a} v_a - \frac{f_a}{m_a}\, \mathrm{sign}(v_a) - \frac{k}{m_a} x_a - \frac{A_a}{m_a} P_a + \frac{l_1}{m_a l_2} F_a(t) \\ \dot{x}_a &= v_a \end{aligned}}$$



$$m_b \dot{v}_b + b_b v_b + f_b\, \mathrm{sign}(v_b) - A_b P_b + F_b(t) = 0$$

$$\boxed{\begin{aligned} \dot{v}_b &= -\frac{b_b}{m_b} v_b - \frac{f_b}{m_b}\, \mathrm{sign}(v_b) + \frac{A_b}{m_b} P_b - \frac{1}{m_b} F_b(t) \\ \dot{x}_b &= v_b \end{aligned}}$$