Solving Equations of Motion

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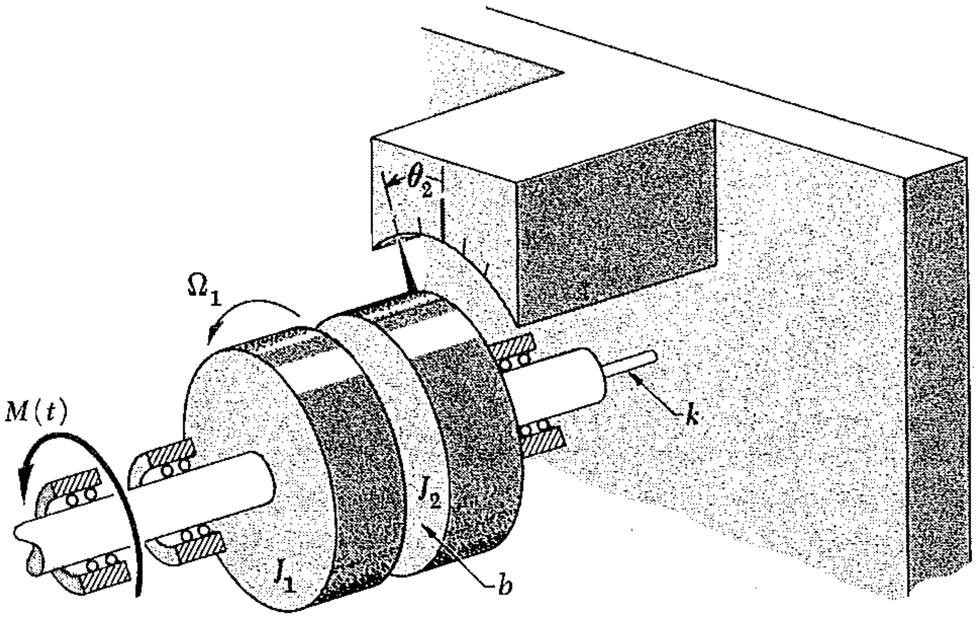
Introduction

- We have some experience deriving equations of motion for mechanical systems
- Now we want to do something useful with them
- Natural first step: solve equations for specified input to system
- Example - dynamometer

Example – dynamometer

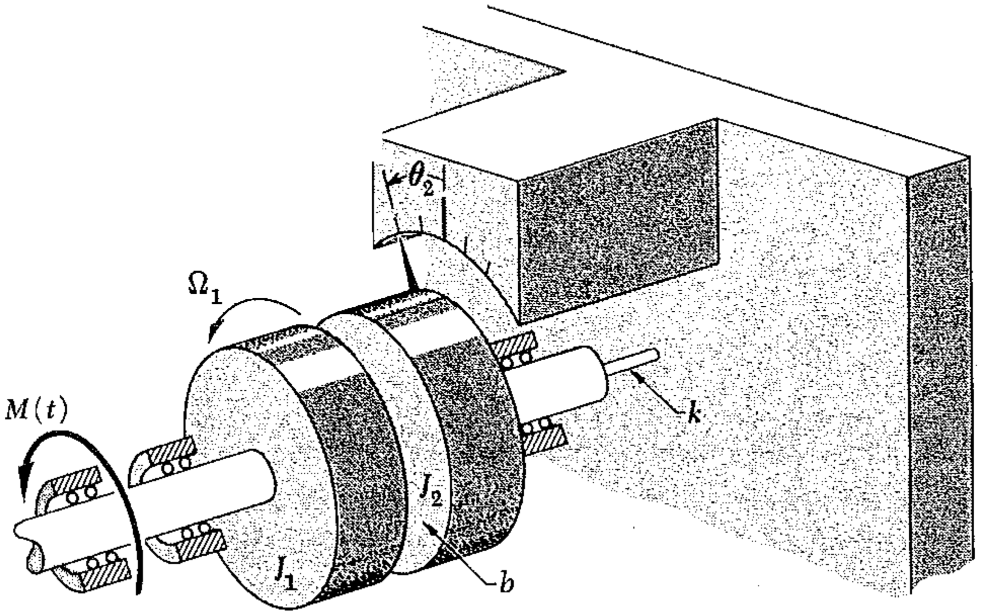


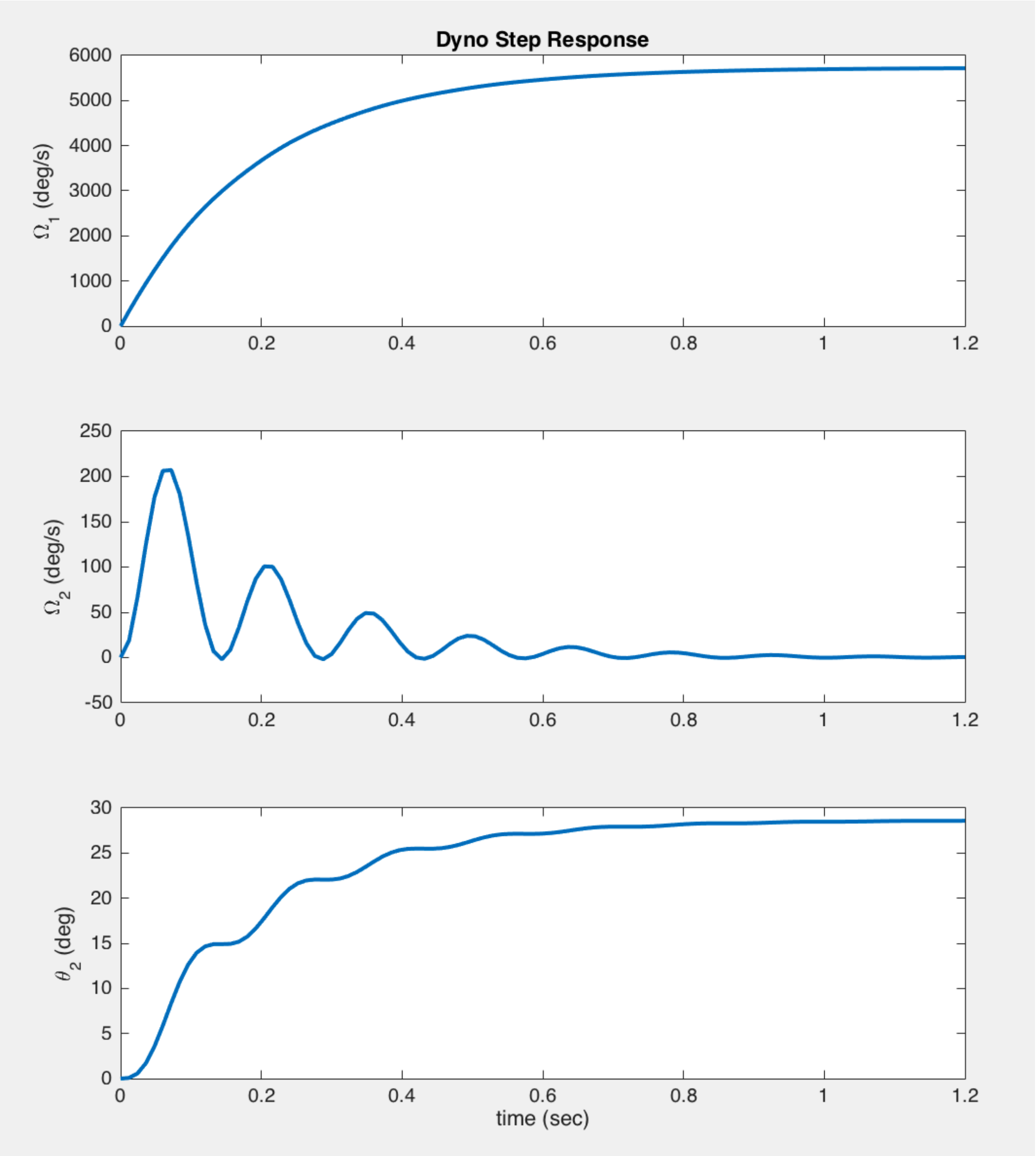
$$J_1\dot{\Omega}_1 + b(\Omega_1 - \Omega_2) = M(t)$$
$$J_2\dot{\Omega}_2 - b(\Omega_1 - \Omega_2) + k\theta_2 = 0$$

$$M_0 = 100 \text{ N-m}$$
$$J_1 = 0.2 \text{ kg-m}^2$$
$$J_2 = 0.1 \text{ kg-m}^2$$
$$b = 1 \text{ N-m-s/rad}$$
$$k = 200 \text{ N-m/rad}$$

What is it going to do?

Example – dynamometer





We want to solve EOM.
How do we do it?

Numerical Solution of EOM

First step: write EOM in state-variable form

$$\dot{x} = f(x, u)$$

standard form for numerical integration

Equations of motion can always be expressed as a set of coupled first-order ordinary differential equations

$$\dot{x}_1 = f_1(x_1, x_2, \cdots, x_n, u_1, u_2, \cdots, u_m)$$

$$\dot{x}_2 = f_2(x_1, x_2, \cdots, x_n, u_1, u_2, \cdots, u_m)$$

$$\vdots$$

$$\dot{x}_n = f_n(x_1, x_2, \cdots, x_n, u_1, u_2, \cdots, u_m)$$

$$\longrightarrow \quad \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$$

State-variable form

1. Define the state variables. These will normally correspond to the energy storage elements in the system. In mechanical systems, they are often the velocities and positions associated with the configuration variables.
2. Solve the equation(s) for the highest-order derivative. These will define state derivatives.
3. Identify state variables and inputs on right-hand side of equations.
4. Write equations for state derivatives in terms of states and inputs.

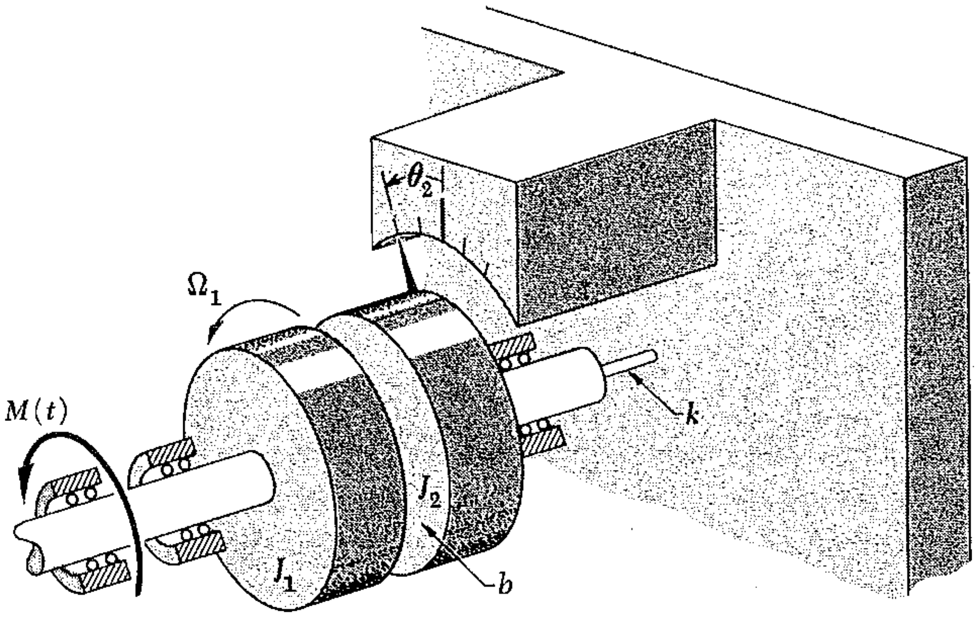
$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$$

Simple pendulum

Equation of motion:

$$mL^2\ddot{\theta} + b\dot{\theta} + mgL\sin\theta = 0$$

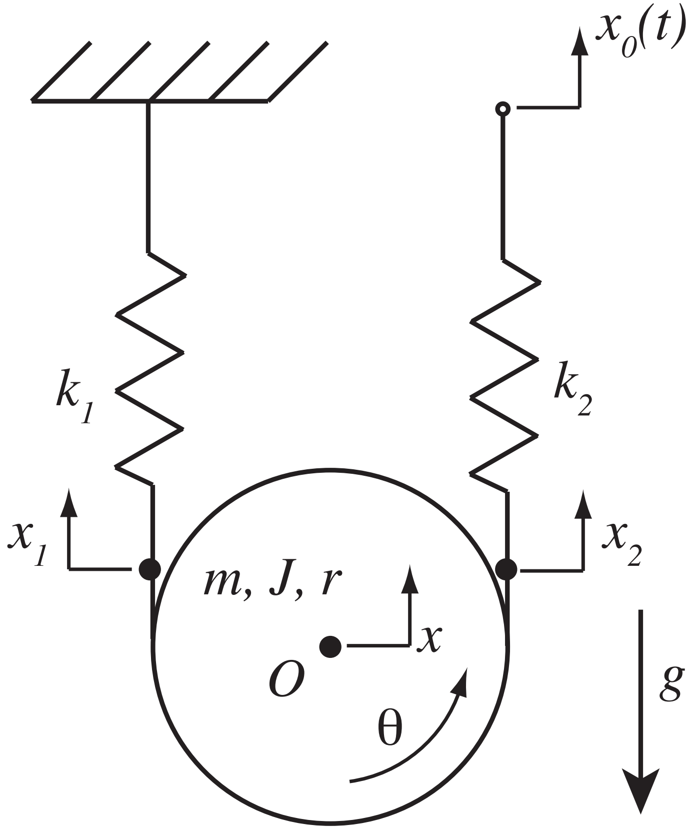
Example – dynamometer



$$J_1\dot{\Omega}_1 + b(\Omega_1 - \Omega_2) = M(t)$$

$$J_2\dot{\Omega}_2 - b(\Omega_1 - \Omega_2) + k\theta_2 = 0$$

Example – Thingamajig



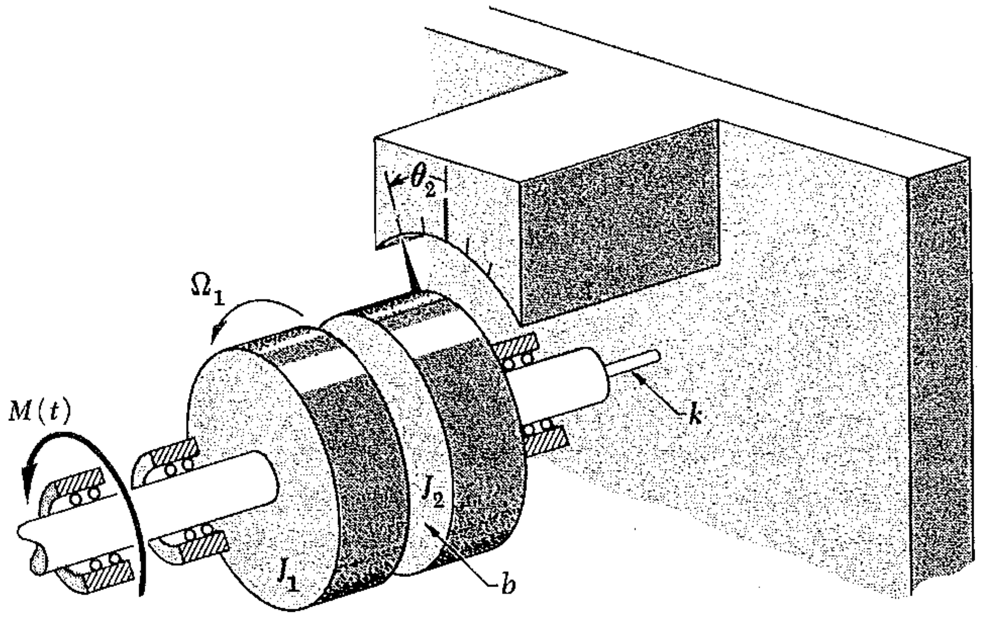
$$m\ddot{x} + k_1(x - r\theta) + k_2[x + r\theta - x_0(t)] = 0$$

$$J\ddot{\theta} + k_1 r(r\theta - x) + k_2 r[x + r\theta - x_0(t)] = 0$$

Solving equations numerically in Matlab

- General approach:
  - Use nonlinear solvers – Runge Kutta (e.g., ode45 in Matlab)
  - Works for either linear or nonlinear systems
  - Equations must be in state-variable form
  - We’ll take this approach for now
- Linear systems
  - Can utilize linear solution method – matrix exponential (e.g., step, lsim in Matlab)
  - Numerically much faster
  - Can utilize state-space or transfer function models

Example – dynamometer



$$\dot{\Omega}_1 = -\frac{b}{J_1}\Omega_1 + \frac{b}{J_1}\Omega_2 + \frac{1}{J_1}M(t)$$

$$\dot{\Omega}_2 = -\frac{b}{J_2}\Omega_2 + \frac{b}{J_2}\Omega_1 - \frac{k}{J_2}\theta_2$$

$$\dot{\theta}_1 = \Omega_1$$

$$\dot{\theta}_2 = \Omega_2$$

$$\mathbf{x} = [\Omega_1 \ \Omega_2 \ \theta_1 \ \theta_2]^T$$

$$\mathbf{u} = M(t)$$

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$$