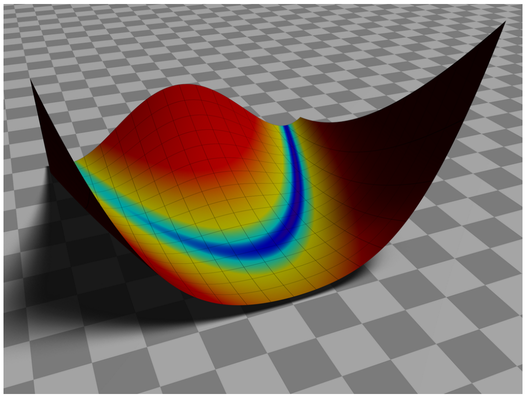
Sequential Quadratic Programming 1

Lecture 18



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Outline

Equality Constrained Quadratic Programming

Equality Constrained Sequential Quadratic Programming

Quadratic Program (QP): an optimization problem with a quadratic objective function and linear constraints.

Equality Constrained Quadratic Programming

Pull out a sheet of paper, and try to solve the following analytically (don't look ahead):

$$\begin{array}{rl} \text{minimize} & x^2 - 8x + y^2 - 12y + 48 \\ \text{subject to} & x + y = 8 \end{array}$$

Consider a general equality-constrained QP:

$$\begin{array}{rl} \text{minimize} & \frac{1}{2} x^T Q x + x^T c \\ \text{subject to} & Ax = b \end{array}$$

Form the Lagrangian:

$$\mathcal{L}(x, \lambda) = \frac{1}{2}x^T Q x + x^T c + \lambda^T (Ax - b)$$

Take derivatives and set equal to zero:

$$\nabla_x \mathcal{L} = Qx + c + A^T \lambda = 0$$
$$\nabla_\lambda \mathcal{L} = Ax - b = 0$$

Rewrite:

$$\begin{bmatrix} Q & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} -c \\ b \end{bmatrix}$$

Note: we can solve for x^*, λ^* just by solving one linear system of equations.

As long as Q is positive definite, this always has a solution, and it is the global minimum of the QP. In other words, this is a convex problem.

Important: We can directly find the global minimum of any equality-constrained QP just by solving a linear system.

* As long as Q is positive definite.

Example: Controls

Linear Dynamical System:

$$x_{t+1} = Ax_t + Bu_t$$

x_t : state at time t

u_t : control input at time t

Define a cost function:

$$J = \sum_{0}^{N} x_t^T Q x_t + u_t^T R u_t$$

(cost often represents deviations in the states from target values plus energy expended by control action)

Objective is quadratic: $J = \sum_0^N x_t^T Q x_t + u_t^T R u_t$

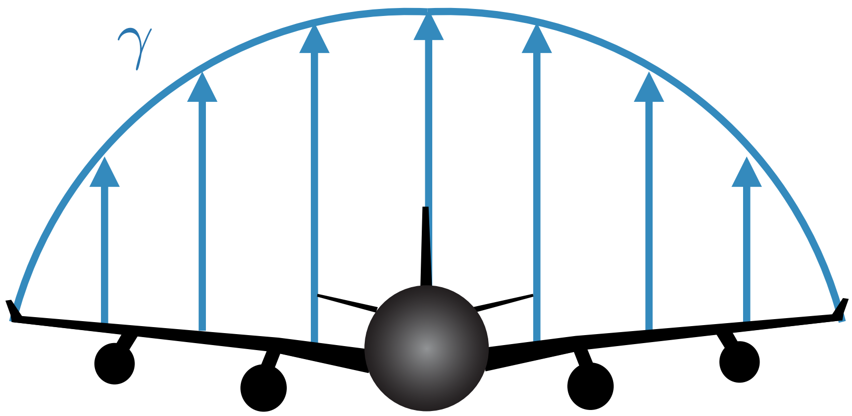
Constraints are linear: $x_{t+1} = Ax_t + Bu_t$

This is a QP, and thus we can efficiently minimize J to find the optimal sequence control inputs u_t.

This feedback controller is called a linear-quadratic regulator or LQR.

Example: Aerodynamics

Consider a variation on the wing example I showed last time. Let's optimize the lift distribution γ.



Minimize Drag:

$$D = \gamma^T DIC\gamma + \gamma^T D_2\gamma + D_1^T\gamma + D_0$$

Constrain Lift:

$$L = LIC^T\gamma$$

Constrain bending weight:

$$W = WIC^T\gamma$$

Quadratic objective with linear constraints.

The actual problem I demoed in class had inequality constraints (which is still a QP but not as simple), and we optimized planform shape rather than just the lift distribution and so it was a general nonlinear problem and not a QP.

Equality Constrained Sequential Quadratic Programming

Now assume we have a general nonlinear equality-constrained optimization problem.

$$
\begin{array}{rl}
\text{minimize} & f(x) \\
\text{subject to} & \hat{c}_j(x) = 0, \quad j = 1, \ldots, \hat{m}
\end{array}
$$

Want to minimize Lagrangian:

$$\mathcal{L}(x, \hat{\lambda}) = f(x) + \hat{\lambda}^T \hat{c}(x)$$

Constraints:

$$\hat{c}_j(x) = 0$$

Define, for convenience, the Jacobian of the constraints as

$$A(x) = \nabla \hat{c}(x) = \begin{bmatrix} \nabla \hat{c}_1(x)^T \\ \nabla \hat{c}_2(x)^T \\ \vdots \\ \nabla \hat{c}_{\hat{m}}(x)^T \end{bmatrix}$$

Let's take a quadratic approximation of our objective and a linear approximation of our constraints (thus forming a QP) for some step p near our current point.

Thus, we locally approximation our problem as:

$$
\begin{aligned}
\text{minimize} \quad & \frac{1}{2} p^T \nabla_{xx} \mathcal{L} p + \nabla_x \mathcal{L}^T p + \mathcal{L}(x) \\
\text{with respect to} \quad & p \\
\text{subject to} \quad & A_k p + c_k = 0
\end{aligned}
$$

After simplification:

$$\begin{bmatrix} \nabla_{xx}\mathcal{L}_k & A_k^T \\ A_k & 0 \end{bmatrix} \begin{bmatrix} p_k \\ \hat{\lambda}_{k+1} \end{bmatrix} = \begin{bmatrix} -g_k \\ -\hat{c}_k \end{bmatrix}$$

SQP Procedure

Choose initial pair (x_0, λ_0)
Repeat until convergence

1. Evaluate f_k, g_k, c_k, A_k, $\nabla_{xx}\mathcal{L}_k$
2. Solve linear system
3. Update step: $x_{k+1} = x_k + \alpha p_k$, and update Lagrange multiplier λ_{k+1} (note α implying a line search)

SQP can be thought of as *iteratively* minimizing a local quadratic approximation of the Lagrangian with a local linear approximation of the constraints.

An alternative derivation of SQP (now shown here) shows that it is simply an application of Newton's method (root finding) to the KKT conditions.