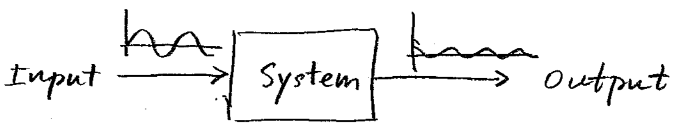
Frequency Response

- What is it?

  A measure of a system's response to sinusoidal inputs of varying frequencies

  Mag. ratio
  Phase difference



- Why do frequency response?

  + sinusoidal inputs are common in engineering systems
    - machine imbalance
    - A/C circuits
    - acoustic wave forms
    - wave forces on marine structures

  + Using Fourier analysis, any input signal can be viewed as a sum of sinusoids. Freq. response provides understanding of how a system will respond to general inputs.

  + In control systems, we can use freq. resp. to design the control system for arbitrary inputs

  + For sinusoidal inputs, freq. resp. characterizes system behavior concisely.

  + Can be determined experimentally or analytically

Some Review

If the input to a constant coefficient (time invariant) linear system is sinusoidal with amplitude U and frequency ω,

$$u(t) = U \sin \omega t$$

$$u \longrightarrow \boxed{G(s)} \longrightarrow y$$

then the steady-state output of the system will be a sinusoid of the same frequency, but possibly different amplitude and phase:

$$y(t) = Y \sin(\omega t + \phi)$$

For frequency response, we are interested in knowing about changes in the output magnitude and phase as we change the frequency of the input. Specifically, we want to know

$$\frac{Y}{U}(\omega) \quad , \quad \phi(\omega) \quad \text{as } \omega \text{ changes}$$

By definition, we know that the transfer function $G(s)$ is the ratio of the Laplace transform of $y(t)$ to the Laplace transform of the input $u(t)$:

$$G(s) = \frac{Y(s)}{U(s)}$$

If we restrict our attention to sinusoidal inputs, then $s = j\omega$ and we have

$$G(j\omega) = \frac{Y(j\omega)}{U(j\omega)}$$

Since $Y(j\omega)$, $U(j\omega)$, and $G(j\omega)$ all represent complex numbers,

$$\frac{Y}{U}(\omega) = \left| \frac{Y(j\omega)}{U(j\omega)} \right| = |G(j\omega)|$$

$$\phi(\omega) = \angle Y(j\omega) - \angle U(j\omega) = \angle \frac{Y(j\omega)}{U(j\omega)} = \angle G(j\omega)$$

mag ratio & phase vs. frequency

If we are interested in finding frequency response for an arbitrary dynamic system, one approach is to solve analytically ODE's for system with general sinusoidal input: $A \sin \omega t$

- Not too hard for 1st and 2nd-order systems
  - Laplace x-form → manipulate → inverse Laplace
  - Find ss-response (sinusoidal)
  - Compare input/output amplitude and phase vs. frequency

Calculate

- Very hard to impossible for higher order systems

The approaches used for finding $M(\omega)$ and $\phi(\omega)$ ~~used~~ in the text require solving the ODEOM for first & second-order systems with sinusoidal inputs.

This is not too hard for first and second-order systems. What about higher-order systems?

HARD, if not impossible to do.

Using the transfer-function approach, we can find the magnitude and phase response for higher order linear systems without great difficulty. Involves some algebra with complex numbers.

In general,

$$G(s) = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n}$$

Letting $s = j\omega$, $G(j\omega)$ can always be written as the ratio of two complex numbers

$$G(j\omega) = \frac{N(j\omega)}{D(j\omega)} = \frac{a + jb}{c + jd}$$

From this, we can evaluate

$$M(\omega) = |G(j\omega)|$$

and $$\phi(\omega) = \measuredangle G(j\omega)$$

as we have done for the first and second-order system examples

Transfer function approach to finding frequency response is more general and more powerful.

Transfer Functions and Frequency Response

If we know a system's transfer function, we can calculate its frequency response.

Consider the first-order system:

$$\tau \dot{y} + y = F(t) \quad \longrightarrow \quad \tau s Y(s) + Y(s) = F(s)$$

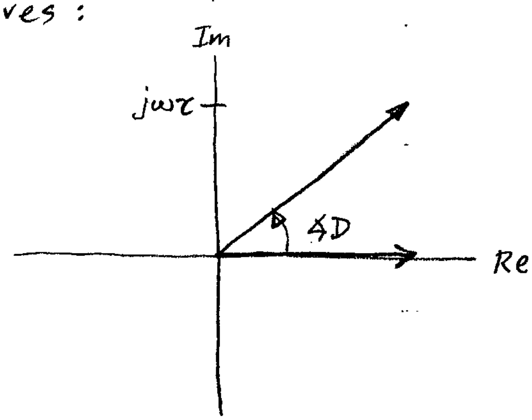
$$\frac{Y(s)}{F(s)} = G(s) = \frac{1}{\tau s + 1}$$

We can find the magnitude and phase response of the system by evaluating $G(s)$ at $s = j\omega$.

Recall that the Laplace variable is a complex number: $s = \sigma + j\omega$. Sinusoidal inputs correspond to $s = j\omega$ $(\sigma = 0)$.

$$G(j\omega) = \frac{1}{j\omega\tau + 1}$$

Plotting numerator and denominator in the complex plane gives:



$$|G(j\omega)| = \frac{|N(j\omega)|}{|D(j\omega)|} = \frac{1}{\sqrt{1 + (\omega\tau)^2}}$$

$$\measuredangle G(j\omega) = \measuredangle N(j\omega) - \measuredangle D(j\omega)$$

$$= 0 - \tan^{-1}(\omega\tau)$$

$$\measuredangle G(j\omega) = -\tan^{-1}(\omega\tau)$$

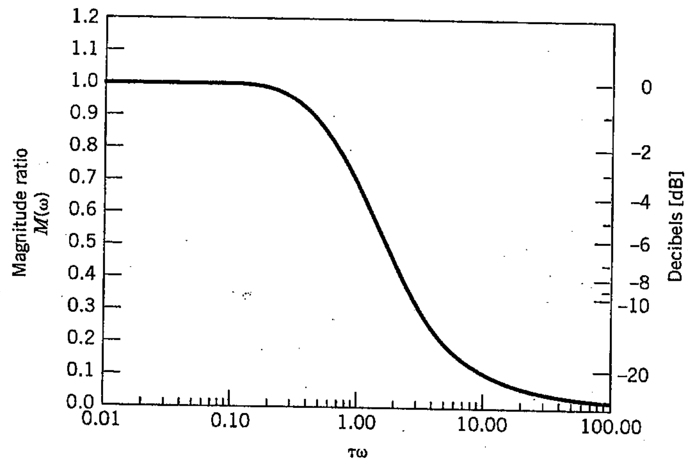


Figure 3.12 First-order system frequency response: magnitude ratio.

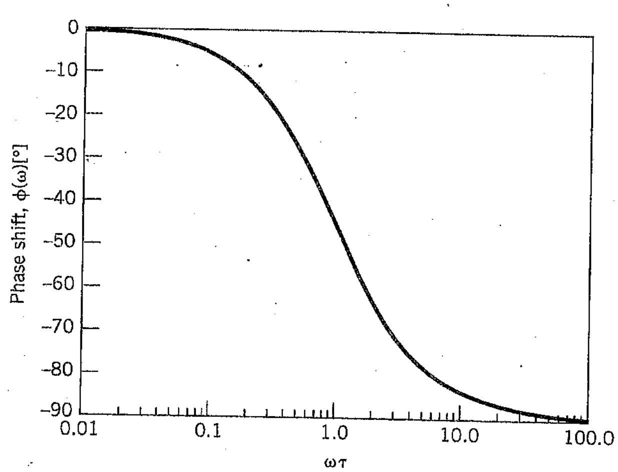


Figure 3.13 First-order system frequency response: phase shift.

For a second-order system

$$\ddot{y} + 2\zeta\omega_n \dot{y} + \omega_n^2 y = \omega_n^2 F(t)$$

Taking Laplace x-form: (zero IC's)

$$s^2 Y(s) + 2\zeta\omega_n s Y(s) + \omega_n^2 Y(s) = \omega_n^2 F(s)$$

$$G(s) = \frac{Y(s)}{F(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

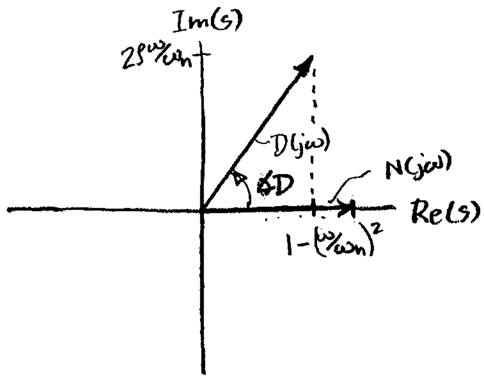
Consider sinusoidal signals $\Rightarrow s = j\omega$

$$G(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n (j\omega) + \omega_n^2}$$

$$= \frac{\omega_n^2}{(\omega_n^2 - \omega^2) + j(2\zeta\omega\omega_n)}$$

$$G(j\omega) = \frac{1}{\left[1 - (\omega/\omega_n)^2\right] + j\left[2\zeta(\omega/\omega_n)\right]} = \frac{N(j\omega)}{D(j\omega)}$$

Plotting $N(j\omega)$, $D(j\omega)$ in the complex plane:



$$|G(j\omega)| = \frac{|N(j\omega)|}{|D(j\omega)|} = \underline{\underline{\frac{1}{\sqrt{\left[1 - (\omega/\omega_n)^2\right]^2 + \left(2\zeta\omega/\omega_n\right)^2}}}}$$

$$\tan \measuredangle D = \frac{2\zeta\omega/\omega_n}{1 - (\omega/\omega_n)^2}$$

$$\measuredangle G(j\omega) = \measuredangle N(j\omega) - \measuredangle D(j\omega)$$

$$= 0 - \tan^{-1}\left(\frac{2\zeta\omega/\omega_n}{1 - (\omega/\omega_n)^2}\right)$$

$$\measuredangle G(j\omega) = \underline{\underline{-\tan^{-1}\left(\frac{2\zeta\,\omega/\omega_n}{1 - (\omega/\omega_n)^2}\right)}}$$

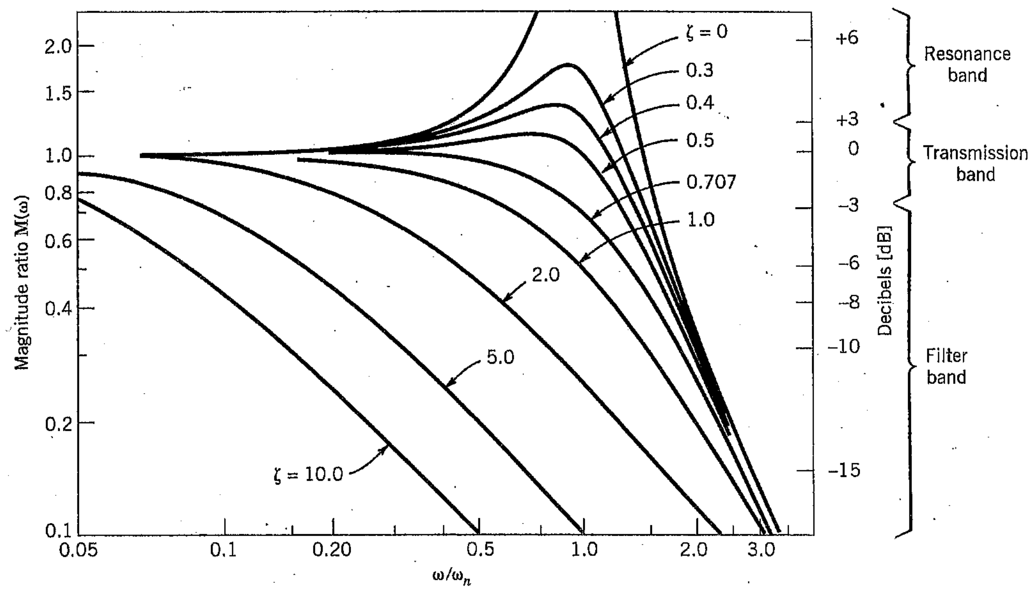


Figure 3.16 Second-order system frequency response: magnitude ratio.

$\omega = \omega_n$. This behavior is characteristic of system resonance. Real systems possess some amount of damping, which modifies the abruptness and magnitude of resonance, but underdamped systems may still achieve resonance. This region on Figures 3.16 and 3.17 is called the *resonance band* of the system, referring to the range

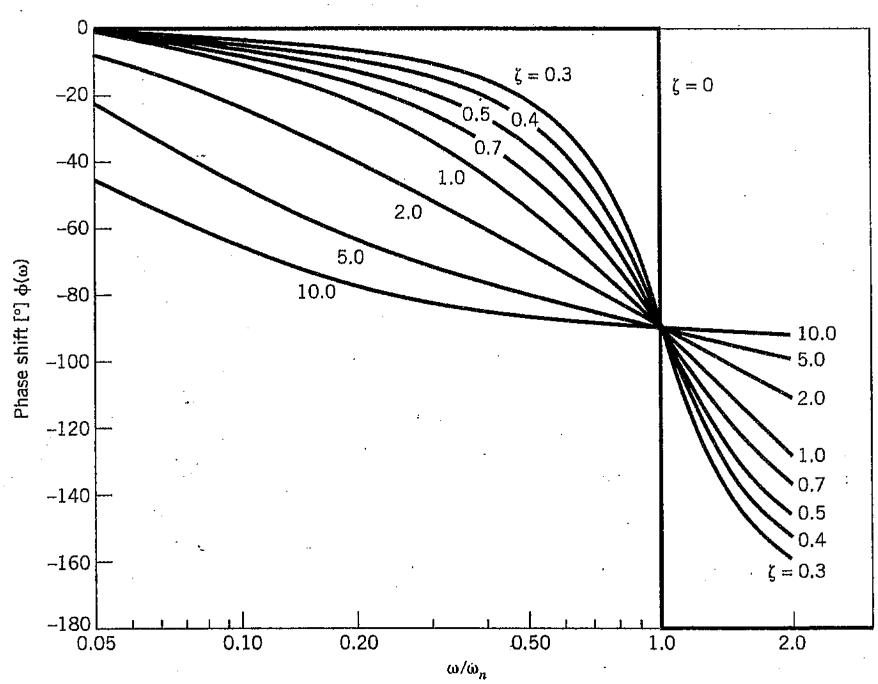


Figure 3.17 Second-order system frequency response: phase shift.

Frequency Response Example

$$G(s) = \frac{2(s+2)}{(s+5)(s^2+6s+13)}$$

$$G(j\omega) = \frac{2(j\omega+2)}{(j\omega+5)((j\omega)^2+6(j\omega)+13)}$$

$$= \frac{4+j(2\omega)}{(5+j\omega)((-\omega^2+13)+j(6\omega))} \longleftarrow \begin{array}{l} -5\omega^2+65+j(30\omega) \\ +j(13\omega)-6\omega^2-j\omega^3 \end{array}$$

$$G(j\omega) = \frac{4+j(2\omega)}{(-11\omega^2+65)+j(43\omega-\omega^3)}$$

$$|G(j\omega)| = \frac{\sqrt{4^2+(2\omega)^2}}{\sqrt{(-11\omega^2+65)^2+(43\omega-\omega^3)^2}}$$

$$\measuredangle G(j\omega) = \tan^{-1}\left(\frac{2\omega}{4}\right) - \tan^{-1}\left[\frac{43\omega-\omega^3}{(-11\omega^2+65)}\right]$$

For $\omega = 10$ rad/s,

$$|G(j10)| = 0.0173$$

$$\measuredangle G(j10) = -130.2 \text{ deg}$$

Show FRexample.m

Show FRmassspring.m

Frequency Response — How to find it.

Analytical - from the transfer function model

- Bode's methods for hand-sketched approximations of magnitude and phase plots
  - Builds intuition and understanding
    Quick approximation
- Numerical evaluation of magnitude and phase characteristics.
  Example: bode command in Matlab
  - Accurate, easy pretty plots

Experimental - from the physical system

- Apply sinusoidal inputs to system one frequency at a time. Measure magnitude ratio, phase difference at numerous frequencies
  - Very accurate at specific frequencies of interest. Can be a bit tedious
- Input signal with broad spectral content. Use Fast-Fourier transform methods to calculate magnitude ratio and phase difference over broad range of frequencies.
  - Fast, but lots of data required

system
↓
TF model
↓
freq. resp.

system
↓
freq. resp.
⇣
TF model