Mechanical Rotation

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Deriving equations of motion for rotational systems

Five steps:

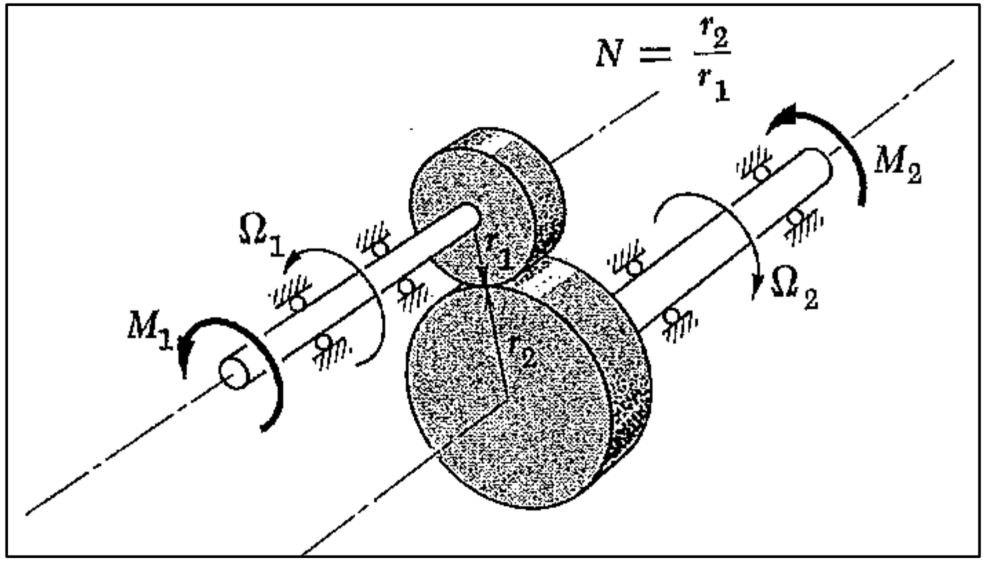
1. Define system geometry
2. Apply moment balance relations
3. Define kinematic relationships
4. Define constitutive relations for system elements
5. Combine relations to get EOM

Moment balance equations

For a rigid body in pure rotation about a fixed axis:

- Newton's 2nd law: $$\sum M_0 = J_0\ddot{\theta}$$
- D'Alembert's principle: $$\sum M_0^* = 0$$ About any axis!
- J is mass moment of inertia: $$J = \int r^2 dm$$

Kinematic relationships: gear trains



Cannon, *Dynamics of Physical Systems,* 1967

Ideal: massless, frictionless, rigid

$$P = M_1 \Omega_1 = M_2 \Omega_2$$

Geometric compatibility:

$$\Omega_2 r_2 = \Omega_1 r_1$$

$$\frac{\Omega_2}{\Omega_1} = \frac{r_1}{r_2} = \frac{1}{N}$$

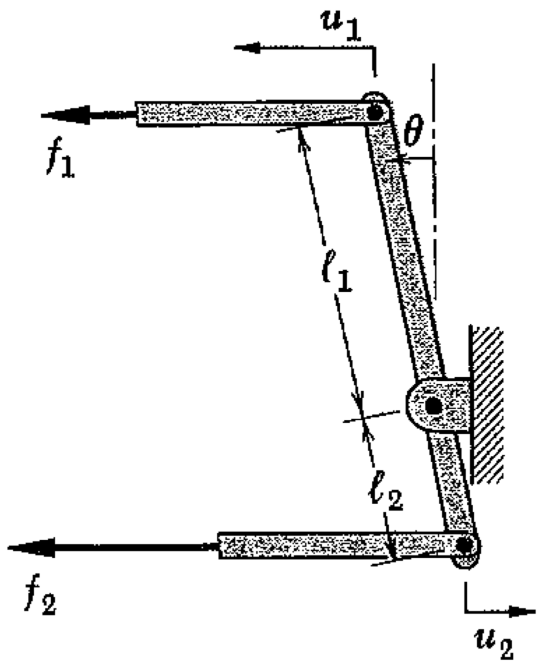
Force equilibrium:

$$M_1 = f r_1$$

$$M_2 = f r_2$$

$$\frac{M_2}{M_1} = \frac{r_2}{r_1} = N$$

Kinematic relationships: levers



Cannon, *Dynamics of Physical Systems,* 1967

Ideal: massless, frictionless, rigid

$$P = f_1 u_1 = f_2 u_2$$

Geometric relations:

$$\dot{\theta} = \frac{u_1}{\ell_1} = \frac{u_2}{\ell_2} \quad \text{(for small } \theta\text{)}$$

$$\frac{u_2}{u_1} = \frac{\ell_2}{\ell_1} = N$$

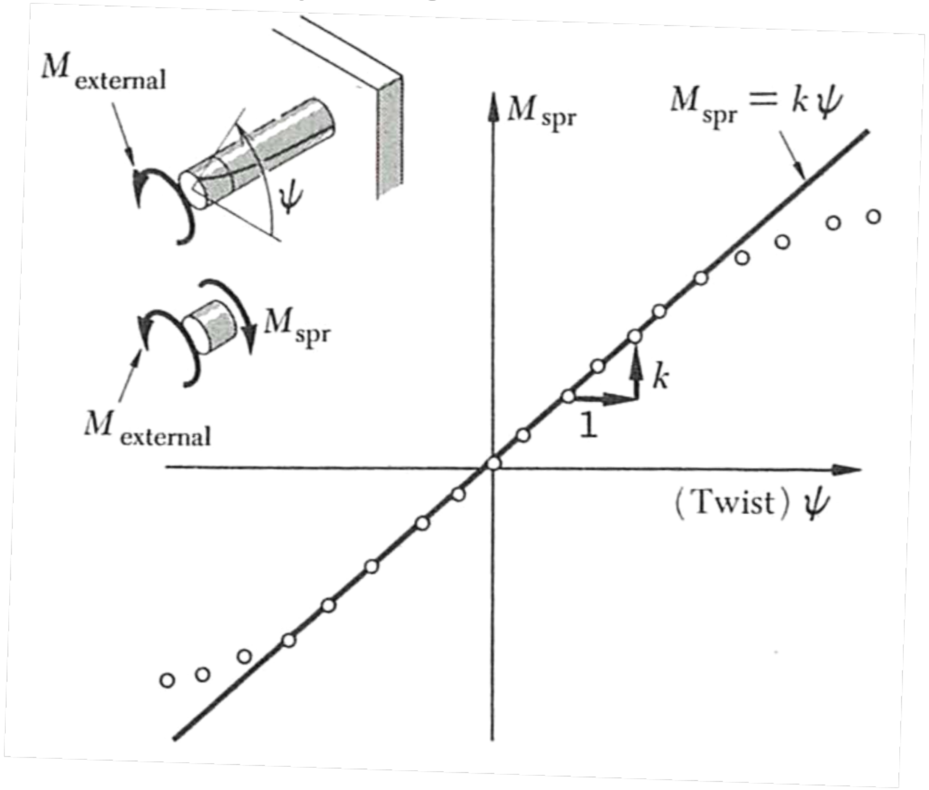
Force equilibrium:

$$f_1 \ell_1 = f_2 \ell_2$$

$$\frac{f_2}{f_1} = \frac{\ell_1}{\ell_2} = \frac{1}{N}$$

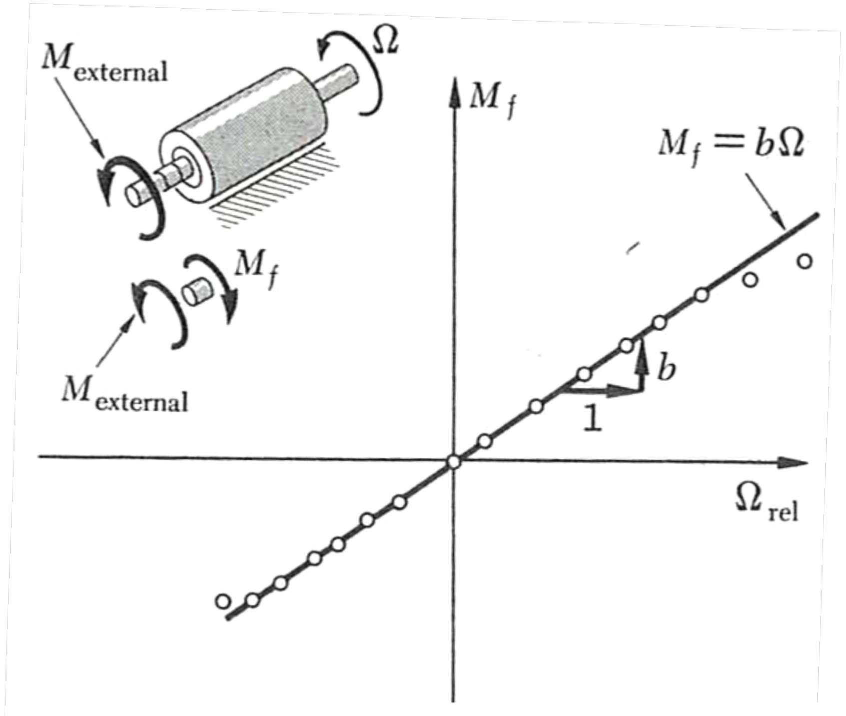
Constitutive relations

Torsional spring



$$T = k(\theta_2 - \theta_1)$$

Torsional friction/damper

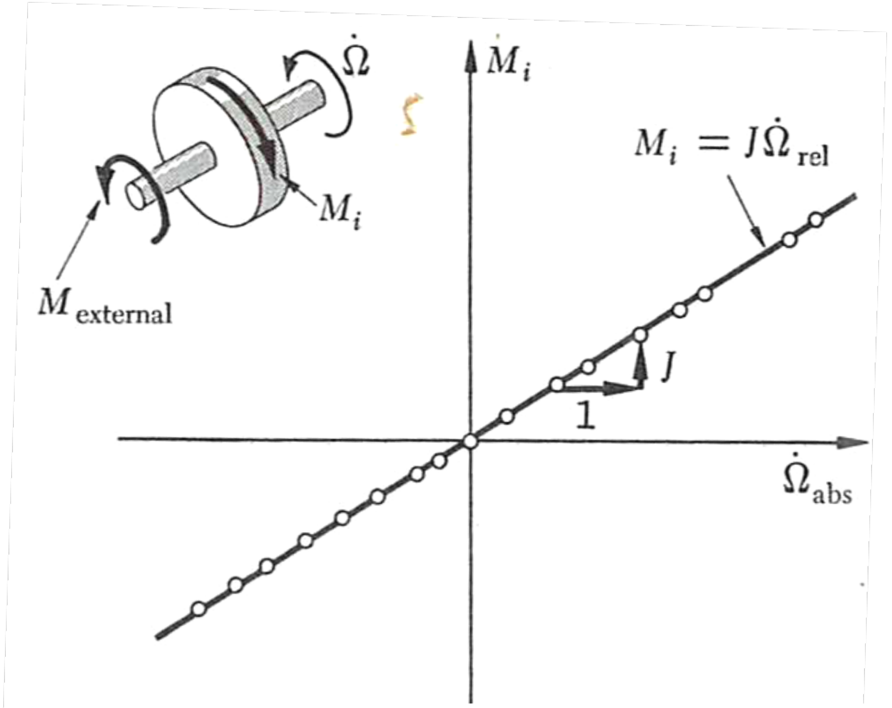


Cannon, *Dynamics of Physical Systems,* 1967

$$T = b(\dot{\theta}_2 - \dot{\theta}_1)$$

Constitutive relations

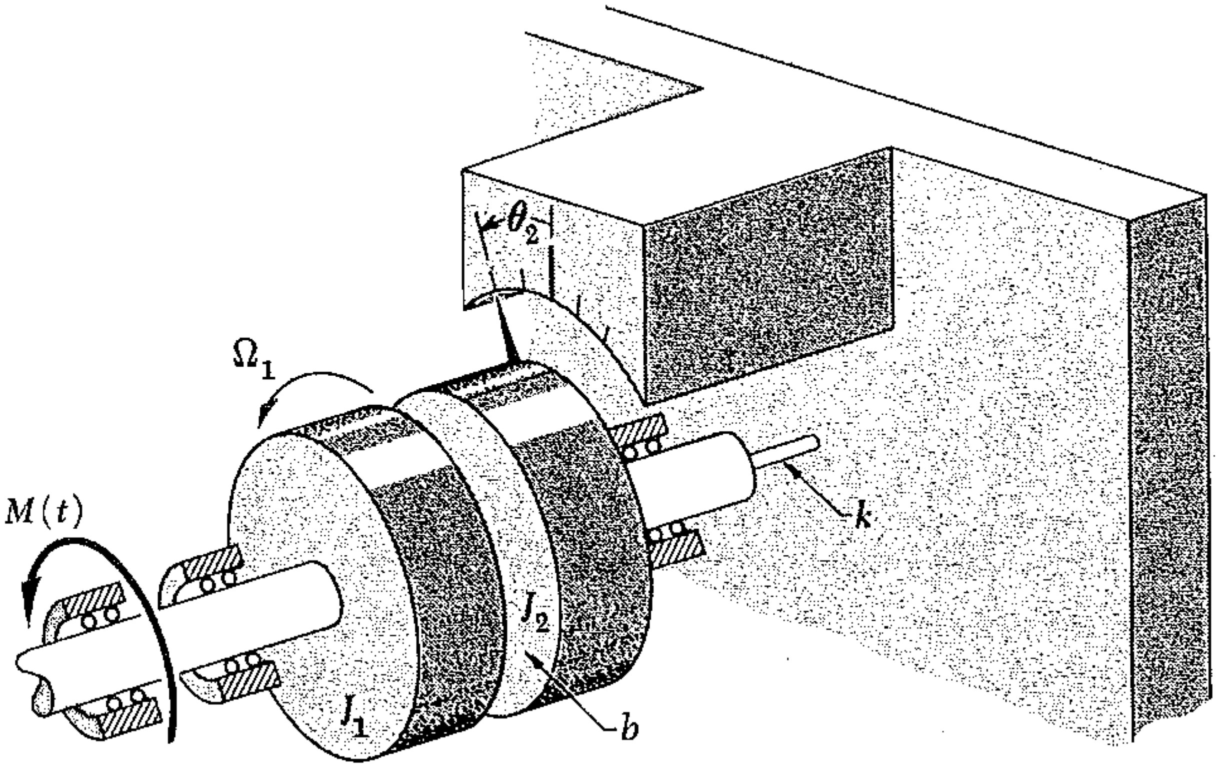
Inertia under acceleration



Cannon, *Dynamics of Physical Systems*, 1967

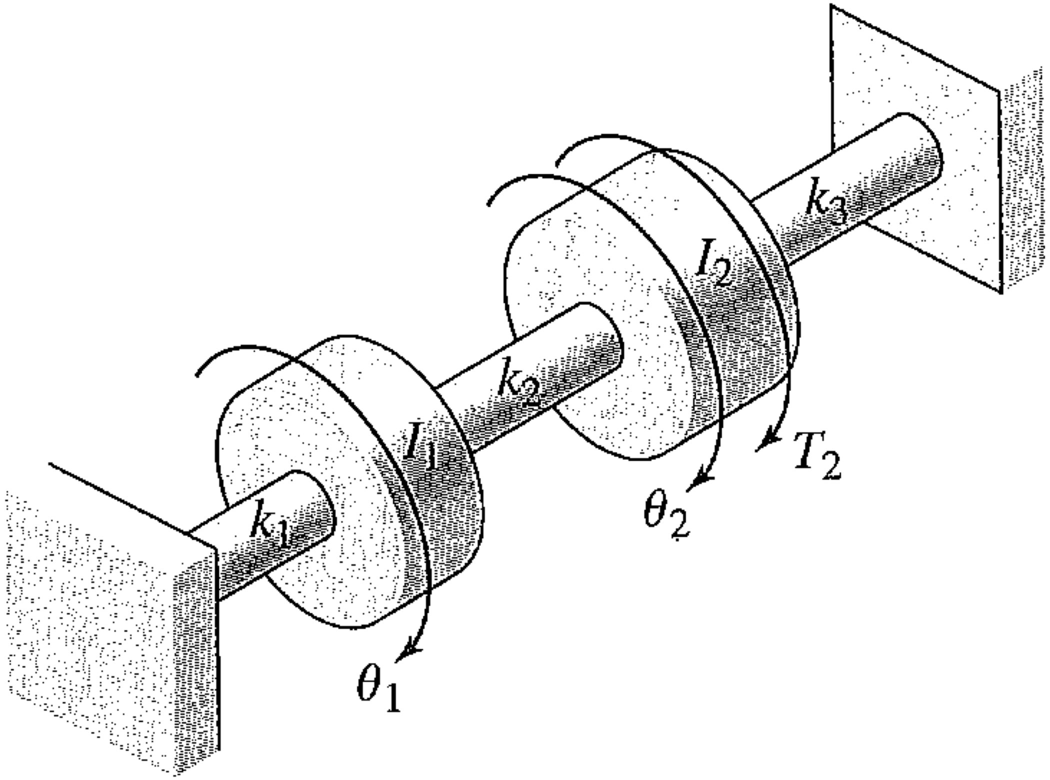
$$T = J\ddot{\theta}$$

Example – dynamometer



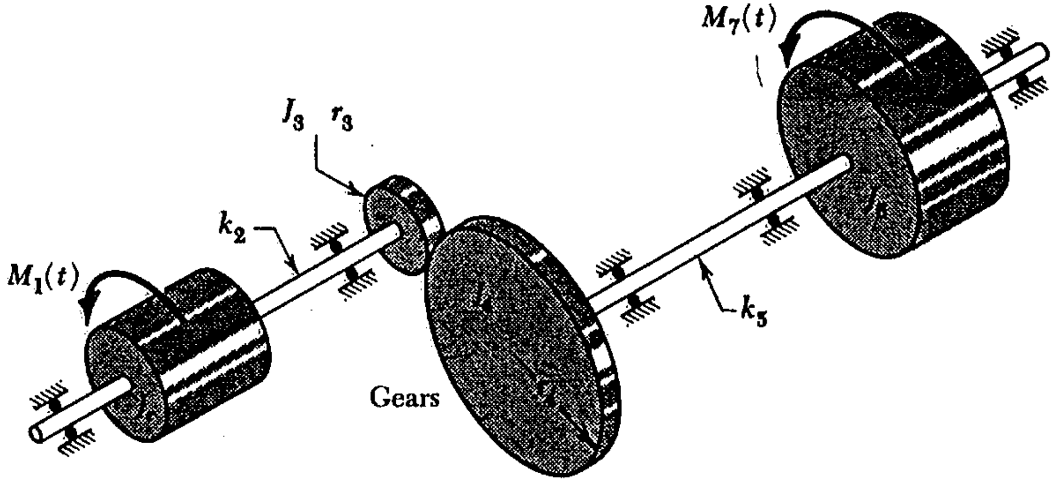
$$J_1\dot{\Omega}_1 + b(\Omega_1 - \dot{\theta}_2) = M(t)$$
$$J_2\ddot{\theta}_2 - b(\Omega_1 - \dot{\theta}_2) + k\theta_2 = 0$$

Example – flexible shaft



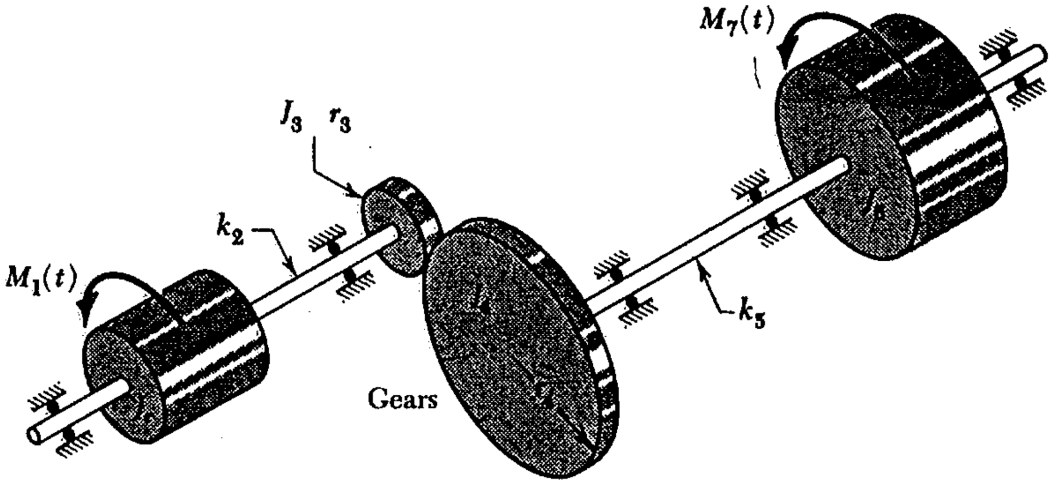
Example – motor/gear system

2.47 A torque $M_1(t)$ is applied electrically to the rotor of the motor (figure), whose moment of inertia is J_r. The radii and moments of inertia of the gears are as labeled. The load has moment of inertia J_6 and is subjected to an independent load torque $M_7(t)$. The shafts are elastic, as indicated. Write the equations of motion, neglecting gear friction. (How many degrees of freedom has the system?)



Prob. 2.47

Example – motor/gear system

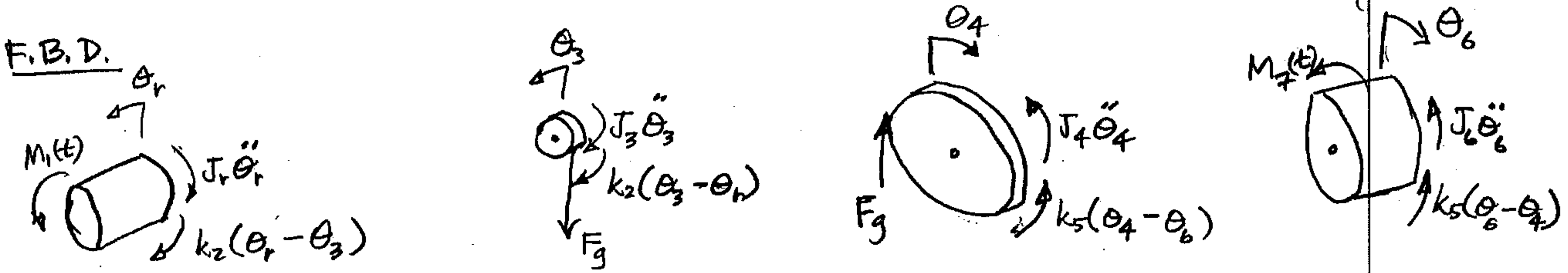


Geometric constraint: $r_3\theta_3 = r_4\theta_4 \rightarrow \theta_4 = \frac{r_3}{r_4}\theta_3$

$$\dot{\theta}_4 = \frac{r_3}{r_4}\dot{\theta}_3$$

$$\ddot{\theta}_4 = \frac{r_3}{r_4}\ddot{\theta}_3$$

F.B.D.



Example – motor/gear system

Geometric Constraint: $r_3 \theta_3 = r_4 \theta_4 \rightarrow$

$$\theta_4 = \frac{r_3}{r_4}\theta_3$$

$$\dot{\theta}_4 = \frac{r_3}{r_4}\dot{\theta}_3$$

$$\ddot{\theta}_4 = \frac{r_3}{r_4}\ddot{\theta}_3$$

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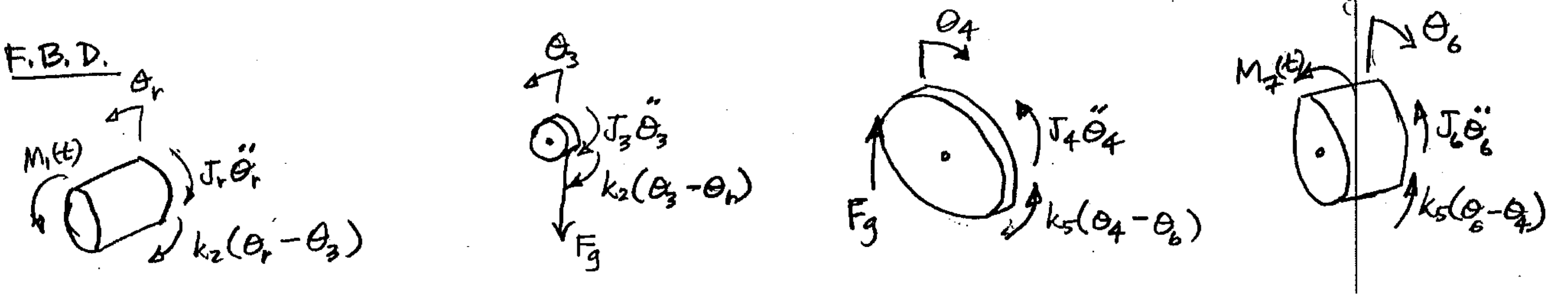

EoM

$$\boxed{\begin{aligned} J_r\ddot{\theta}_r + k_2(\theta_r - \theta_3) &= M_1(t) \\ J_6\ddot{\theta}_6 + k_5(\theta_6 - \theta_4) &= -M_7(t) \end{aligned}}$$

Example – motor/gear system

$$J_3 \ddot{\theta}_3 + k_2(\theta_3 - \theta_r) + F_g r_3 = 0$$

$$J_4 \ddot{\theta}_4 + k_5(\theta_4 - \theta_6) - F_g r_4 = 0 \longrightarrow F_g = \frac{1}{r_4}\left[J_4 \ddot{\theta}_4 + k_5(\theta_4 - \theta_6)\right]$$

$$J_3 \ddot{\theta}_3 + k_2(\theta_3 - \theta_r) + \frac{r_3}{r_4}\left[J_4 \ddot{\theta}_4 + k_5(\theta_4 - \theta_6)\right] = 0$$

$$\ddot{\theta}_4 = \frac{r_3}{r_4}\ddot{\theta}_3 \quad , \quad \theta_4 = \frac{r_3}{r_4}\theta_3$$

$$\rightarrow J_3 \ddot{\theta}_3 + k_2(\theta_3 - \theta_r) + \left(\frac{r_3}{r_4}\right)^2 J_4 \ddot{\theta}_3 + \left(\frac{r_3}{r_4}\right)^2 k_5 \theta_3 - \left(\frac{r_3}{r_4}\right) k_5 \theta_6 = 0$$

$$\boxed{\left[J_3 + \left(\frac{r_3}{r_4}\right)^2 J_4\right]\ddot{\theta}_3 + \left[k_2 + \left(\frac{r_3}{r_4}\right)^2 k_5\right]\theta_3 - k_2\theta_r - \frac{r_3}{r_4}k_5\theta_6 = 0}$$