Second-order Systems

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Free response

Consider the general form of the damped second-order equation of motion:

$$\ddot{x} + 2\sigma\dot{x} + \omega_n^2 x = 0 \qquad x(0) = x_0 \quad \dot{x}(0) = v_0$$

Taking the Laplace transform and solving for $X(s)$ gives

$$X(s) = \frac{s + 2\sigma}{s^2 + 2\sigma s + \omega_n^2} x_0 + \frac{1}{s^2 + 2\sigma s + \omega_n^2} v_0.$$

For solution to this equation, we will consider three cases of interest:

1. underdamped, $\zeta < 1$
2. critically damped, $\zeta = 1$
3. overdamped, $\zeta > 1$

Free response

Characteristic Equation:

$$s^2 + 2\sigma s + \omega_n^2 = 0$$

or

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

Roots:

$$s_{1,2} = -\sigma \pm \sigma\sqrt{1 - \frac{1}{\zeta^2}} \qquad \text{overdamped, } \zeta > 1$$

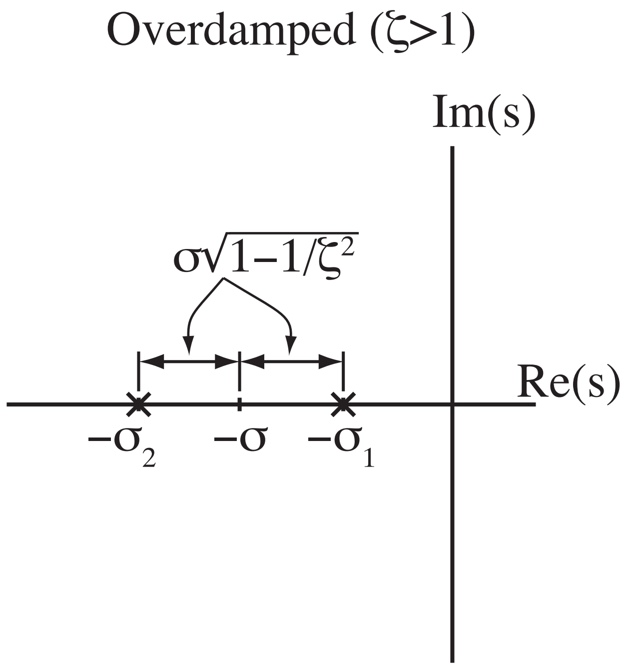
$$s_{1,2} = -\sigma,\ -\sigma \qquad \text{critically damped, } \zeta = 1$$

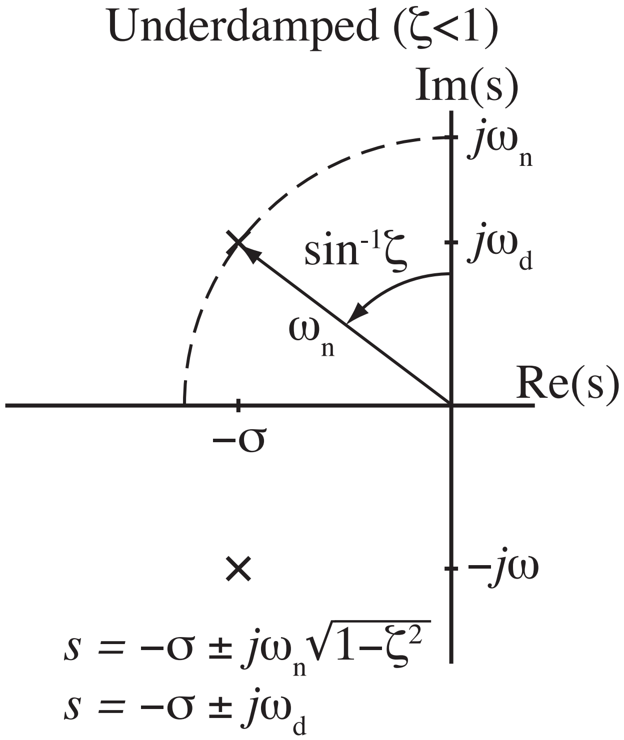
$$s_{1,2} = -\sigma \pm j\omega_d \qquad \text{underdamped, } \zeta < 1$$

$$\omega_d = \omega_n\sqrt{1 - \zeta^2} \qquad \text{damped natural frequency}$$

$$\zeta = \frac{\sigma}{\omega_n} \qquad \text{damping ratio}$$

Graphical Interpretation





$$s_{1,2} = -\sigma \pm \sigma\sqrt{1 - \frac{1}{\zeta^2}} \qquad \text{over damped, } \zeta > 1$$

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$$\omega_d = \omega_n\sqrt{1-\zeta^2} \qquad \text{damped natural frequency}$$

$$\zeta = \frac{\sigma}{\omega_n} \qquad \text{damping ratio}$$

Case 1: Underdamped, $\zeta < 1$

For $\zeta < 1$, $\sigma < \omega_n$, implying complex characteristic roots

$$s_{1,2} = -\sigma \pm j\omega_d$$

where ω_d is the damped natural frequency given by $\omega_n\sqrt{1-\zeta^2}$. Based on these characteristic roots, we can rewrite the general Laplace transform equation as

$$X(s) = \frac{s+2\sigma}{(s+\sigma)^2+\omega_d^2}x_0 + \frac{1}{(s+\sigma)^2+\omega_d^2}v_0$$

$$X(s) = \frac{(s+\sigma)x_0}{(s+\sigma)^2+\omega_d^2} + \frac{v_0+\sigma x_0}{(s+\sigma)^2+\omega_d^2}$$

Case 1: Underdamped, $\zeta < 1$

$$X(s) = \frac{(s+\sigma)x_0}{(s+\sigma)^2+\omega_d^2} + \frac{v_0+\sigma x_0}{(s+\sigma)^2+\omega_d^2}$$

Taking the inverse Laplace transform yields

$$x(t) = x_0 e^{-\sigma t}\cos\omega_d t + (v_0+\sigma x_0)\frac{e^{-\sigma t}}{\omega_d}\sin\omega_d t$$

$$x(t) = e^{-\sigma t}\left[x_0\cos\omega_d t + \frac{v_0+\sigma x_0}{\omega_d}\sin\omega_d t\right]$$

$$\boxed{x(t) = x_m e^{-\sigma t}\cos(\omega_d t - \psi)}$$

where

$$x_m = \sqrt{\left(\frac{v_0+\sigma x_0}{\omega_d}\right)^2 + x_0^2}$$

$$\psi = \tan^{-1}\left(\frac{v_0}{x_0\omega_d} + \frac{\sigma}{\omega_d}\right)$$

Case 2: Critically damped, $\zeta = 1$

For $\zeta = 1$, $\sigma = \omega_n$, which give two real roots

$$s_{1,2} = -\sigma$$

So we can rewrite the Laplace transform equation as

$$X(s) = \frac{s + 2\sigma}{(s+\sigma)^2} x_0 + \frac{1}{(s+\sigma)^2} v_0$$

Taking the inverse Laplace transform, and noting that

$$\mathcal{L}^{-1}\left\{\frac{s + A}{(s+\sigma)^2}\right\} = e^{-\sigma t} + (A - \sigma) t e^{-\sigma t}$$

gives

$$x(t) = [e^{-\sigma t} + \sigma t e^{-\sigma t}] x_0 + v_0 t e^{-\sigma t}$$

$$\boxed{x(t) = e^{-\sigma t}[x_0 + (\sigma x_0 + v_0) t]}$$

Case 3: Overdamped, $\zeta > 1$

For $\zeta > 1$, $\sigma > \omega_n$ which gives two real roots

$$s_{1,2} = -\sigma \pm \delta$$

where

$$\delta = \sigma\sqrt{1 - 1/\zeta^2}$$

We can then write the Laplace transform equation as

$$X(s) = \frac{s}{(s+\sigma+\delta)(s+\sigma-\delta)}x_0 + \frac{2\sigma x_0 + v_0}{(s+\sigma+\delta)(s+\sigma-\delta)}$$

Taking the inverse Laplace transform gives

$$x(t) = \frac{1}{(\sigma+\delta)-(\sigma-\delta)}[(\sigma+\delta)e^{(-\sigma-\delta)t} - (\sigma-\delta)e^{(-\sigma+\delta)t}]x_+ \frac{2\sigma x_0 + v_0}{(\sigma-\delta)-(\sigma+\delta)}[e^{(-\sigma-\delta)t} - e^{(-\sigma+\delta)t}]$$

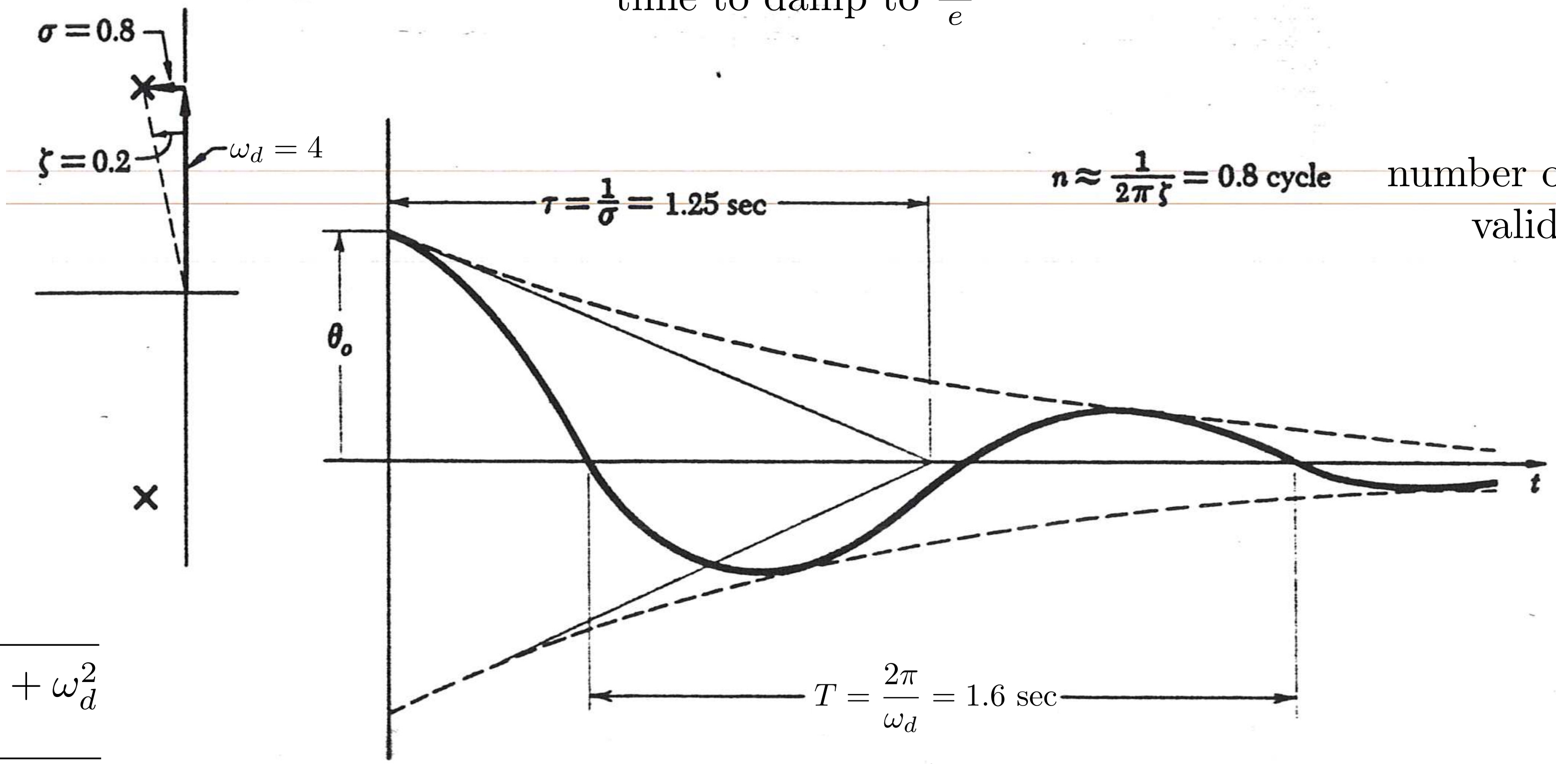
And some algebraic manipulation gives

$$\boxed{x(t) = \frac{-(\sigma-\delta)x_0 - v_0}{2\delta}e^{(-\sigma-\delta)t} + \frac{(\sigma+\delta)x_0 + v_0}{2\delta}e^{(-\sigma+\delta)t}}$$

Underdamped second-order system

$$x(t) = x_m e^{-\sigma t}\cos(\omega_d t - \psi)$$

τ: time constant
time to damp to $\frac{\theta_0}{e}$



number of cycles to damp to $\frac{\theta_0}{e}$
valid for $\zeta << 1$

Period of oscillation

$$\omega_n = \sqrt{\sigma^2 + \omega_d^2}$$

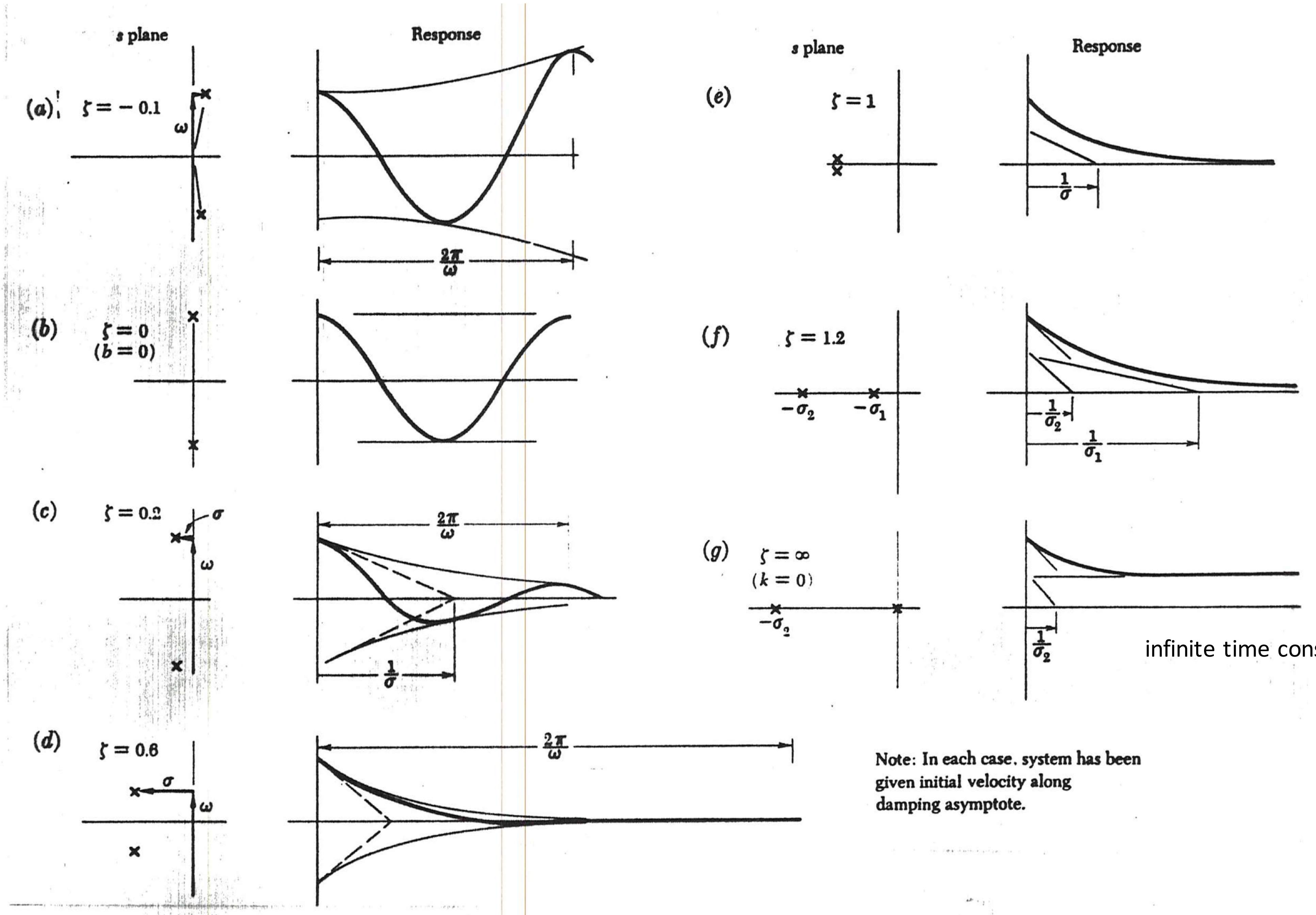
$$\omega_d = \sqrt{\omega_n^2 - \sigma^2}$$

unstable

undamped

underdamped

underdamped

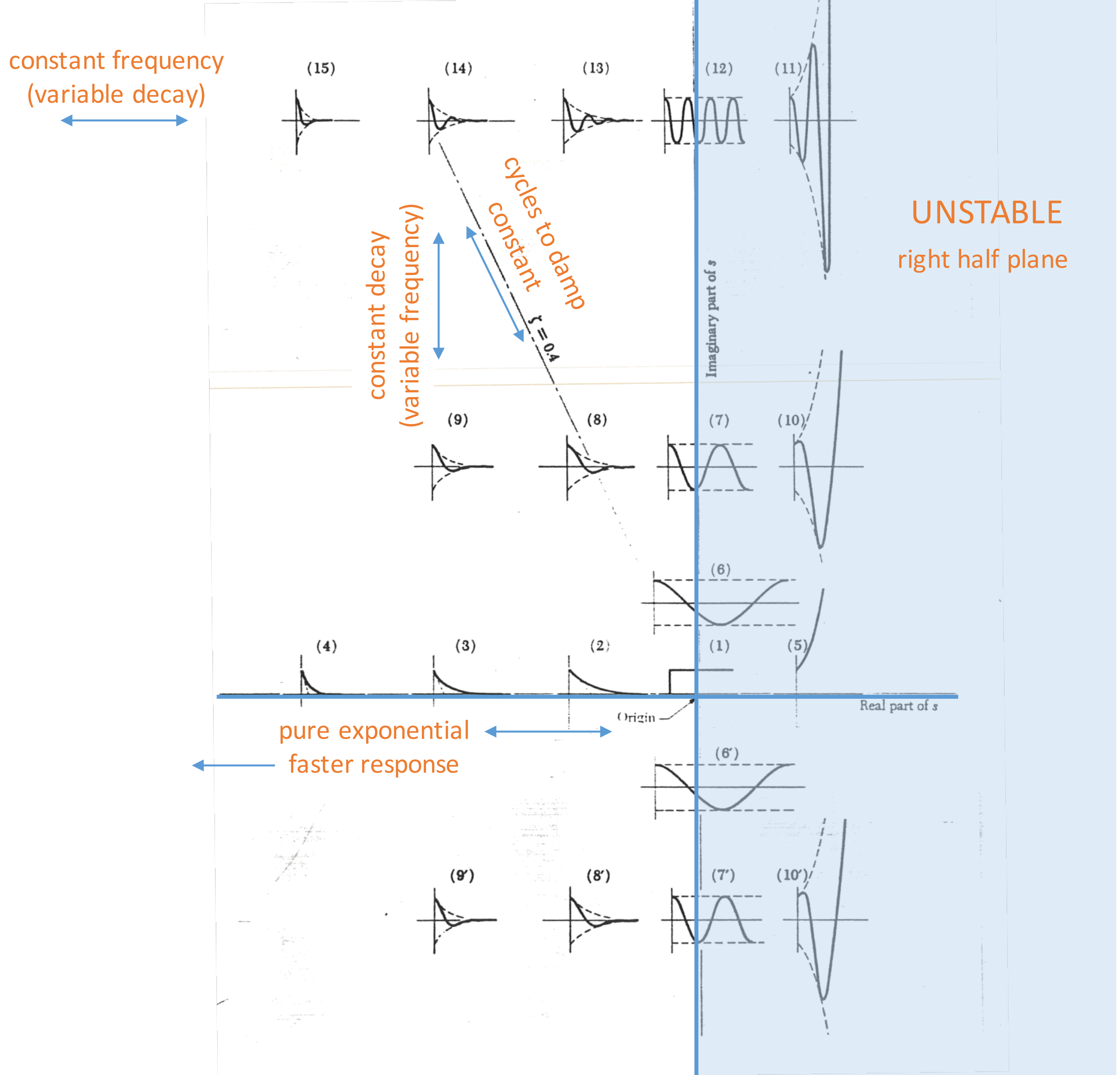


critically damped

overdamped

spring removed

infinite time constant



constant frequency
(variable decay)
(15)
(14)
(13)
(12)
(11)
cycles to damp
constant
$\zeta = 0.4$
constant decay
(variable frequency)
UNSTABLE
right half plane
Imaginary part of s
(9)
(8)
(7)
(10)
(6)
(4)
(3)
(2)
(1)
(5)
Origin
Real part of s
pure exponential
faster response
(6′)
(9′)
(8′)
(7′)
(10′)

General form