Undamped 2nd-Order Systems:

Free Vibrations

Examples:

- pendulum
- LC circuit
- mass & spring
- cantilever
- free piston on an air column
- mercury in a U-tube

Equation of free motion (unforced) for an undamped 2nd-order system:

$$\ddot{\theta} + \omega^2\theta = 0$$

ω is the frequency of the free motion

Undamped 2nd order system —
- 2 different forms of energy storage
- No friction or damping to dissipate energy

- Motion characterized by sinusoidal oscillation:

$$\theta = \theta_m \cos(\omega t - \psi)$$

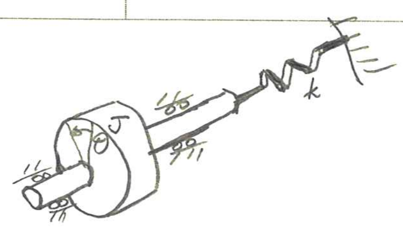
- Sinusoidal motion can also be represented by an equation of the form:

$$\theta = Ce^{st}$$

where s is a purely imaginary number

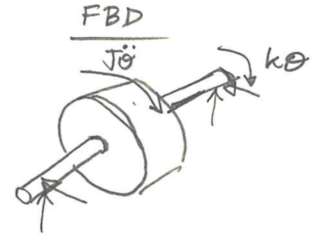
* Remember 1st-order response was given by $\theta = Ce^{st}$ where s was real.

Example:



$$\Sigma M^* = 0$$

$$J\ddot{\theta} + k\theta = 0$$



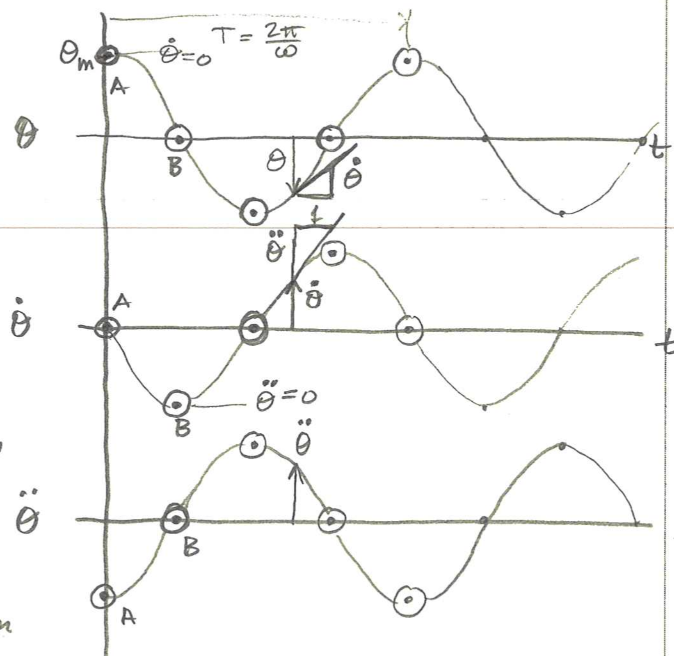
Physical Reasoning

- Give system initial displacement : θ_m
  - $\theta(0) = \theta_m$
  - $\dot{\theta}(0) = 0$
  - $\ddot{\theta}(0+) = -\frac{k}{J}\theta_m$

When $\theta(t) = 0$ for 1st time

$\dot{\theta}(t) = \min$

$\ddot{\theta}(t) = 0$



Energy : When spring is displaced, but system not moving,
- potential energy is stored in spring,
- no kinetic energy

When system moves through position of zero spring displacement,
- no potential energy
- kinetic energy is max

- Because system is conservative (no dissipative forces), KE at B must equal PE at A.
- PE converted to KE, KE to PE, and so on as system oscillates back & forth
- Max deflection of spring does not diminish with time.

Solving the equation of motion for the unforced response:

$$J\ddot{\theta} + k\theta = 0 \qquad \text{Assume} \quad \theta(0) = \theta_0, \quad \dot{\theta}(0) = \Omega_0$$

$$J\left[s^2\Theta(s) - s\theta(0) - \dot{\theta}(0)\right] + k\Theta(s) = 0$$

$$(Js^2 + k)\Theta(s) = Js\,\theta(0) + J\dot{\theta}(0) \qquad (*)$$

$$\Theta(s) = \frac{Js\,\theta(0)}{Js^2 + k} + \frac{J\dot{\theta}(0)}{Js^2 + k}$$

$$\Theta(s) = \frac{\theta_0 s}{s^2 + \frac{k}{J}} + \frac{\Omega_0}{s^2 + \frac{k}{J}}$$

From the Laplace x-form tables:

$$\boxed{\theta(t) = \theta_0 \cos\sqrt{\tfrac{k}{J}}\,t + \Omega_0\sqrt{\tfrac{J}{k}}\,\sin\sqrt{\tfrac{k}{J}}\,t}$$

or

$$\boxed{\theta(t) = C\cos\left(\sqrt{\tfrac{k}{J}}\,t - \psi\right)}$$

where $C = \sqrt{\theta_0^2 + \left(\Omega_0\sqrt{\frac{J}{k}}\right)^2}$

$$\psi = \tan^{-1}\left(\frac{\Omega_0\sqrt{J/k}}{\theta_0}\right)$$

see pp 127-128 of R&k for details

$$\psi = \tan^{-1}\left(\frac{\Omega_0}{\theta_0}\sqrt{J/k}\right)$$

Note: If the initial conditions on $\theta, \dot{\theta}$ are zero, Equation (*) reduces to:

$$(Js^2 + k)\Theta(s) = 0$$

$\Theta(s) = 0$ is a trivial solution

$$\Rightarrow \boxed{Js^2 + k = 0}$$

Characteristic Equation

Characteristic Roots (eigenvalues): $\boxed{s = \pm j\sqrt{\frac{k}{J}}}$ $\quad j = \sqrt{-1}$

natural frequency

Damped 2nd-Order Systems

Def: systems having 2 separate energy-storage elements plus a mechanism for energy dissipation.

Equation of motion has the form:

$$\ddot{\theta} + 2\sigma\dot{\theta} + \omega_n^2\theta = \alpha(t)$$

or

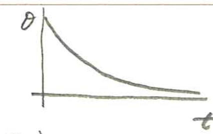
$$\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = \alpha(t)$$

ω_n = undamped natural frequency of oscillation

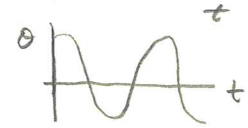
ζ = damping ratio

Natural Motion

- 1st-order system: exponential

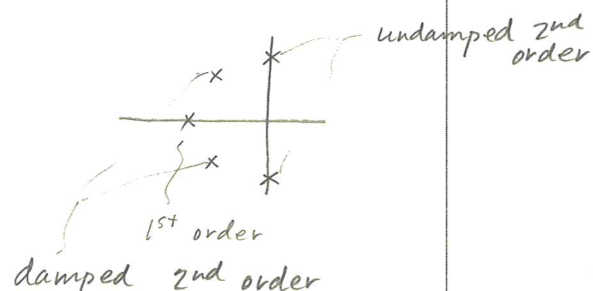


- undamped 2nd-order: sinusoidal

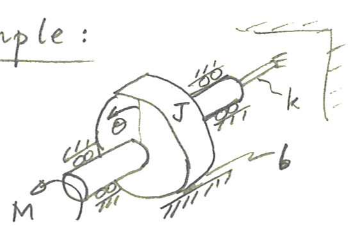


- damped 2nd-order: combination of exponential & sinusoidal

All can be represented by an equation of the form: $\theta = Ce^{st}$



Example:



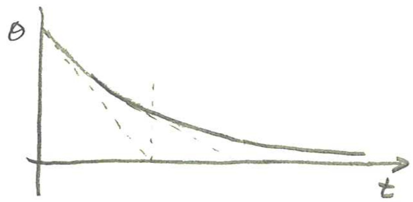
$$J\ddot{\theta} + b\dot{\theta} + k\theta = M(t)$$

for natural motion $M(t) = 0$

$$J\ddot{\theta} + b\dot{\theta} + k\theta = 0$$

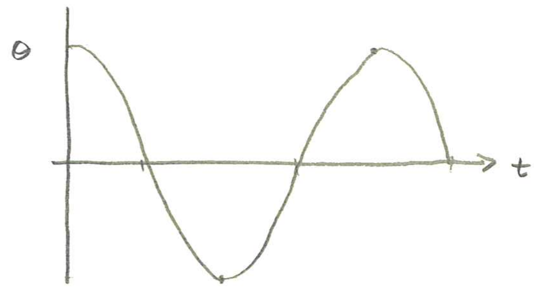
Types of Motion:

Small J



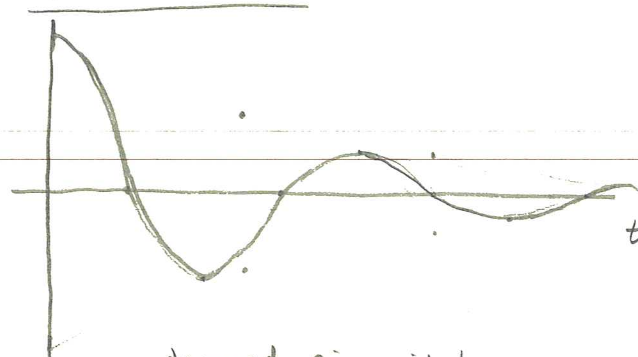
pure exponential

Small b



pure sinusoidal

$J\ddot{\theta} \approx b\dot{\theta} \approx k\theta$



damped sinusoidal

$\theta = Ce^{st}$ can represent each of these types of motion

Characteristic equation

$$\theta = \Theta e^{st} \qquad \dot{\theta} = s\Theta e^{st} \qquad \ddot{\theta} = s^2 \Theta e^{st}$$

$$J\ddot{\theta} + b\dot{\theta} + k\theta = 0$$

$$Js^2 \Theta e^{st} + bs\Theta e^{st} + k\Theta e^{st} = 0$$

$$\boxed{Js^2 + bs + k = 0}$$ characteristic equation

quadratic eqn in s.

Roots (eigen values):

$$s = -\frac{b}{2J} \pm \sqrt{\left(\frac{b}{2J}\right)^2 - \frac{k}{J}} \qquad \frac{k}{J} < \left(\frac{b}{2J}\right)^2$$ real roots

or

$$s = -\frac{b}{2J} \pm j\sqrt{\frac{k}{J} - \left(\frac{b}{2J}\right)^2} \qquad \frac{k}{J} > \left(\frac{b}{2J}\right)^2$$ complex roots (conjugate pair)

General form:

$$\ddot{\theta} + 2\sigma\dot{\theta} + \omega_n^2\theta = 0$$

Roots:

$$s = -\sigma\left(1 - \sqrt{1 - \frac{1}{\zeta^2}}\right), -\sigma\left(1 + \sqrt{1 - \frac{1}{\zeta^2}}\right) \qquad \underline{\zeta > 1} \quad \text{overdamped}$$

$$s = -\sigma, -\sigma \qquad \underline{\zeta = 1} \quad \text{critically damped}$$

(fastest return to static equilibrium)

$$s = -\sigma + j\omega_d, -\sigma - j\omega_d \qquad \underline{\zeta < 1} \quad \text{underdamped}$$

Comparing w/ Example:

$$\sigma = \frac{b}{2J} \qquad \omega_n = \sqrt{\frac{k}{J}}$$

undamped natural frequency (observed freq. if $\zeta = 0$)

$$\omega_d = \sqrt{\omega_n^2 - \sigma^2}$$

damped natural frequency

$$\omega_d = \omega_n\sqrt{1 - \zeta^2}$$

$$\zeta = \frac{\sigma}{\omega_n}$$

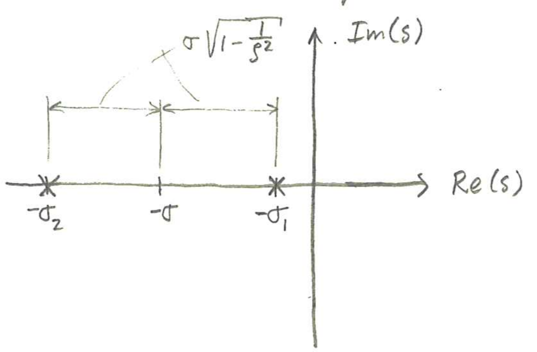
damping ratio

Graphical Interpretation:

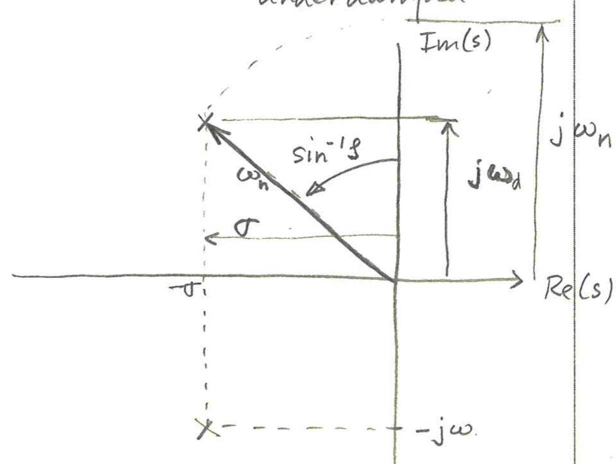
the s-plane

the complex plane

overdamped $\zeta > 1$



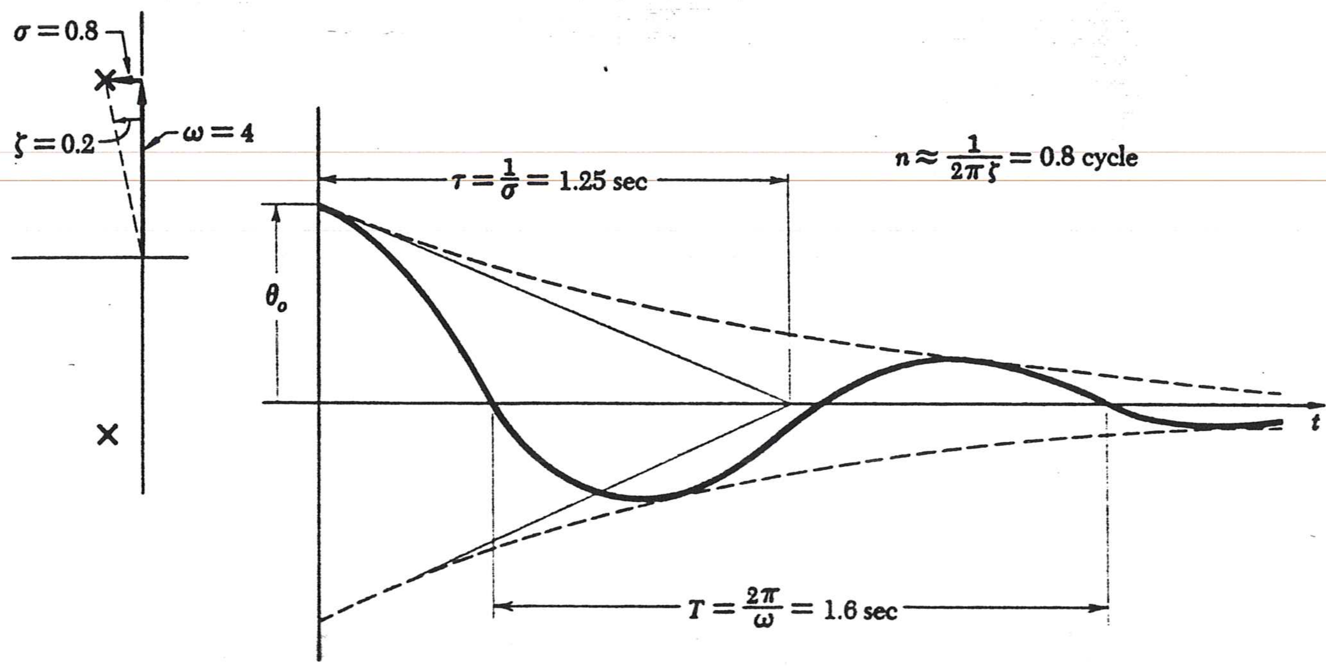
underdamped



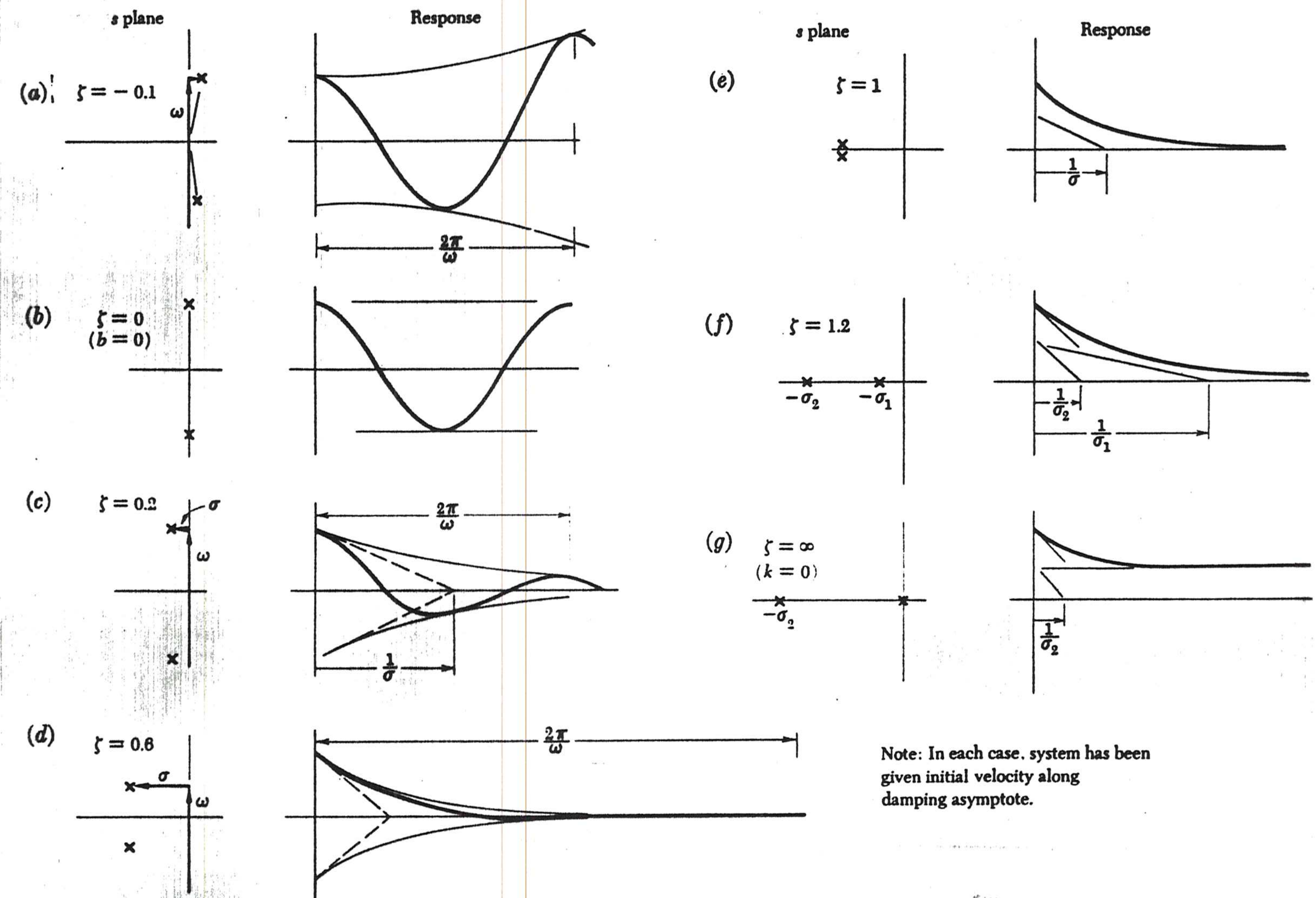
$$s = -\sigma \pm j\omega_n\sqrt{1 - \zeta^2}$$

$$s = -\sigma \pm j\omega_d$$

From "Dynamics of Physical Systems", Cannon, 1967.

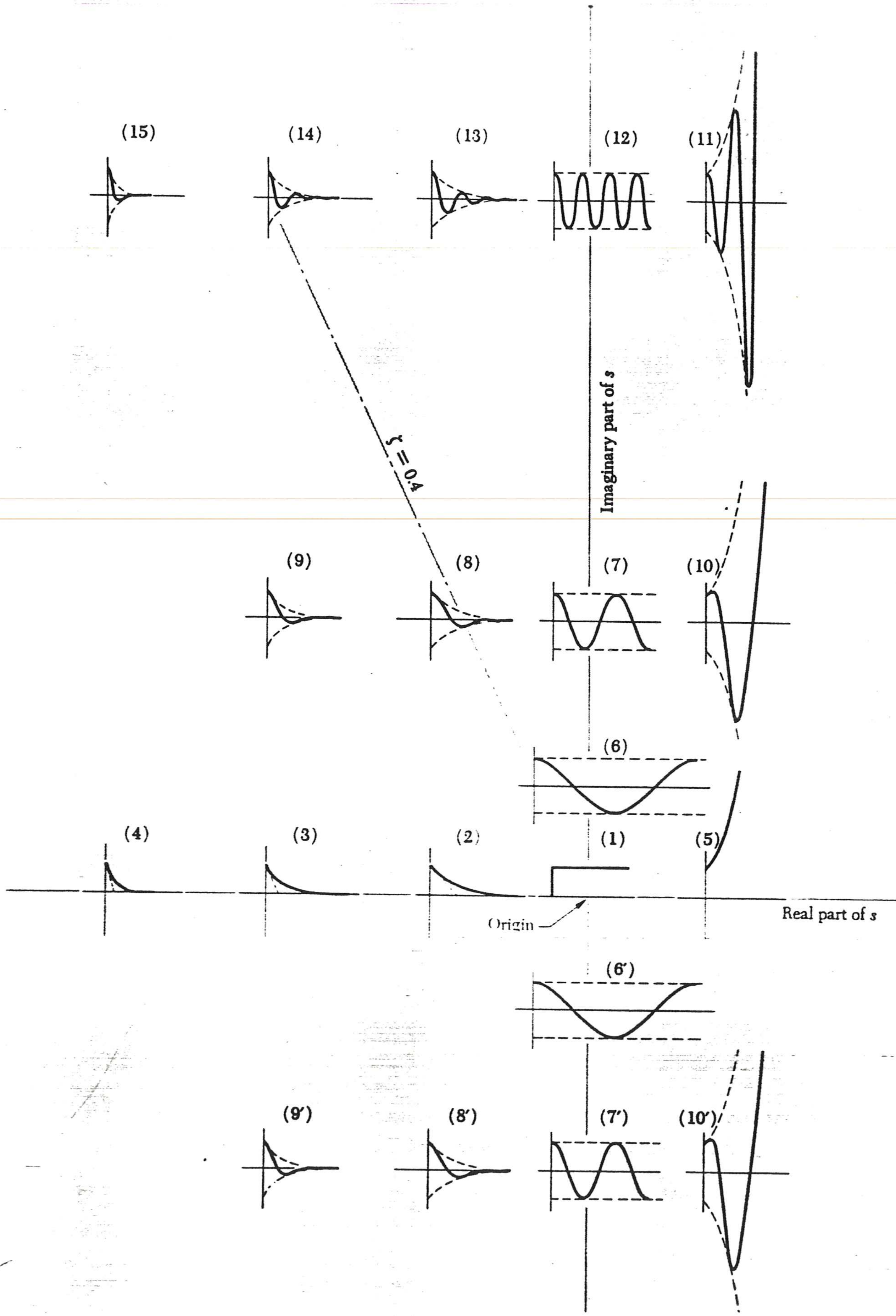


From "Dynamics of Physical Systems", Cannon, 1967.



Note: In each case, system has been given initial velocity along damping asymptote.

From "Dynamics of Physical Systems", Cannon, 1967



Natural Response of Damped 2nd-order Systems

EOM : $\ddot{\theta} + 2\sigma\dot{\theta} + \omega_n^2\theta = 0$ $\qquad \theta(0) = \Theta_0$

$\dot{\theta}(0) = \Omega_0$

Taking Laplace Transform :

$$s^2\Theta - s\Theta_0 - \Omega_0 + 2\sigma\left[s\Theta - \Theta_0\right] + \omega_n^2\Theta = 0$$

$$\left(s^2 + 2\sigma s + \omega_n^2\right)\Theta = (s + 2\sigma)\Theta_0 + \Omega_0$$

$$\Theta = \frac{s + 2\sigma}{s^2 + 2\sigma s + \omega_n^2}\Theta_0 + \frac{1}{s^2 + 2\sigma s + \omega_n^2}\Omega_0 \qquad (*)$$

Three Cases of Interest : ① Underdamped, $\zeta < 1$

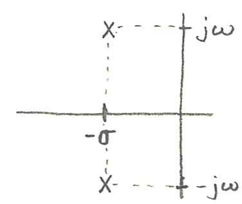
② Critically damped, $\zeta = 1$

③ Overdamped, $\zeta > 1$

* To solve for time response each case must be treated separately.

Case 1 : Underdamped, $\zeta < 1$

For $\zeta < 1$, $\sigma < \omega_n$ which implies complex characteristic roots



$$s_{1,2} = -\sigma + j\omega, \quad -\sigma - j\omega$$

where ω = damped natural frequency

$$= \omega_n\sqrt{1 - \zeta^2}$$

Based on characteristic root locations, re-write $(*)$:

$$\Theta = \frac{s + 2\sigma}{(s + \sigma)^2 + \omega^2}\Theta_0 + \frac{1}{(s + \sigma)^2 + \omega^2}\Omega_0$$

$$\Theta = \frac{(s + \sigma)\Theta_0}{(s + \sigma)^2 + \omega^2} + \frac{\Omega_0 + \sigma\Theta_0}{(s + \sigma)^2 + \omega^2}$$

(cont).

Taking $\mathcal{L}^{-1}\{\Theta\}$:

$$\theta(t) = \theta_0 e^{-\sigma t} \cos \omega t + (\Omega_0 + \sigma\theta_0)\frac{e^{-\sigma t}}{\omega} \sin \omega t$$

$$= e^{-\sigma t}\left[\theta_0 \cos \omega t + \frac{\Omega_0 + \sigma\theta_0}{\omega} \sin \omega t\right]$$

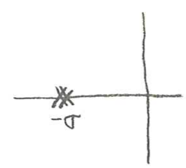
$$\boxed{\theta(t) = \theta_m e^{-\sigma t} \cos(\omega t - \psi)}$$

where $\theta_m = \sqrt{\left(\frac{\Omega_0 + \sigma\theta_0}{\omega}\right)^2 + \theta_0^2}$

$$\psi = \tan^{-1}\left(\frac{\Omega_0}{\theta_0 \omega} + \frac{\sigma}{\omega}\right)$$

Case 2 : Critically damped, $\zeta = 1$

For $\zeta = 1$, $\sigma = \omega_n$ which gives two real roots at $s = -\sigma$



$$s_{1,2} = -\sigma, -\sigma$$

Rewriting (*) :

$$\Theta = \frac{s + 2\sigma}{(s+\sigma)^2}\theta_0 + \frac{1}{(s+\sigma)^2}\Omega_0$$

Taking $\mathcal{L}^{-1}\{\Theta\}$:

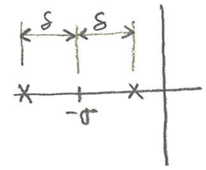
$$\theta(t) = \left[e^{-\sigma t} + \sigma t e^{-\sigma t}\right]\theta_0 + \left[t e^{-\sigma t}\right]\Omega_0$$

$$\boxed{\theta(t) = e^{-\sigma t}\left[\theta_0 + (\sigma\theta_0 + \Omega_0)t\right]}$$

Note : $\mathcal{L}^{-1}\left\{\frac{s + A}{(s+\sigma)^2}\right\} = e^{-\sigma t} + (A - \sigma)te^{-\sigma t}$

Case 3 : Overdamped , $\zeta > 1$

For $\zeta > 1$, $\sigma > \omega_n$ which gives two real roots



$$s_{1,2} = -\sigma + \delta \quad , \quad s = -\sigma - \delta$$

$$\text{where } \delta = \sigma\sqrt{1 - \frac{1}{\zeta^2}}$$

Rewriting $(*)$:

$$\Theta = \frac{s + 2\sigma}{(s+\sigma+\delta)(s+\sigma-\delta)}\,\theta_0 + \frac{1}{(s+\sigma+\delta)(s+\sigma-\delta)}\,\Omega_0$$

$$\Theta = \frac{s}{(s+\sigma+\delta)(s+\sigma-\delta)}\,\theta_0 + \frac{2\sigma\theta_0 + \Omega_0}{(s+\sigma+\delta)(s+\sigma-\delta)}$$

Taking $\mathcal{L}^{-1}\{\Theta\}$:

$$\theta(t) = \frac{1}{(-\sigma-\delta)-(-\sigma+\delta)}\left[(-\sigma-\delta)e^{(-\sigma-\delta)t} - (-\sigma+\delta)e^{(-\sigma+\delta)t}\right]\theta_0$$

$$+ \frac{1}{(-\sigma-\delta)-(-\sigma+\delta)}\left[e^{(-\sigma-\delta)t} - e^{(-\sigma+\delta)t}\right](2\sigma\theta_0 + \Omega_0)$$

After some algebra...

$$\boxed{\theta(t) = \frac{-(\sigma-\delta)\theta_0 - \Omega_0}{2\delta}\,e^{-(\sigma+\delta)t} + \frac{(\sigma+\delta)\theta_0 + \Omega_0}{2\delta}\,e^{-(\sigma-\delta)t}}$$

Analysis of Damped Second-order Systems

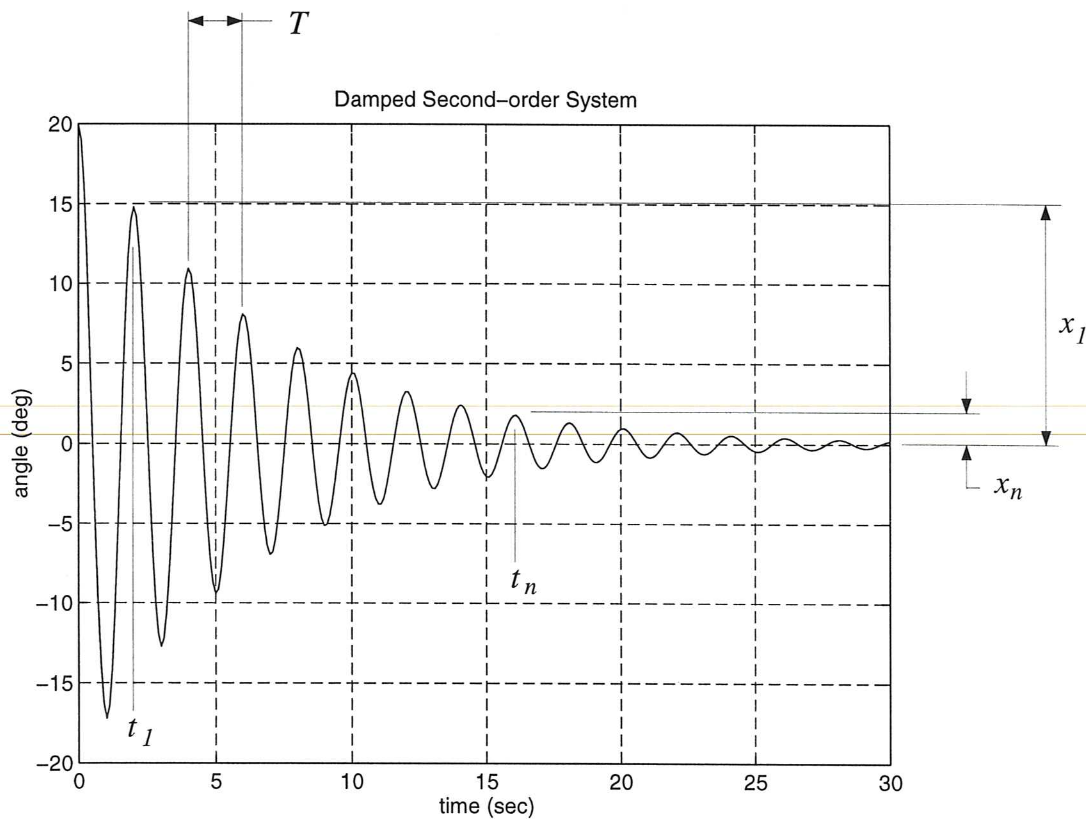


Figure 1:

From the amplitudes x_1 and x_n, the damping ratio, ζ can be easily calculated:

$$\zeta = \frac{\frac{1}{n-1}\ln\left(\frac{x_1}{x_n}\right)}{\sqrt{4\pi^2 + \left[\frac{1}{n-1}\ln\left(\frac{x_1}{x_n}\right)\right]^2}}. \tag{1}$$

Furthermore, the damped natural frequency can be determine by measuring the period. An accurate measurement of the period T is obtained by considering several periods:

$$T = \frac{t_n - t_1}{n-1}. \tag{2}$$

$$\omega_d = \frac{2\pi}{T}. \tag{3}$$

Once we know both ζ and ω_d, we can then calculate the natural frequency of oscillation ω_n.

$$\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} \tag{4}$$

and finally the value of σ:

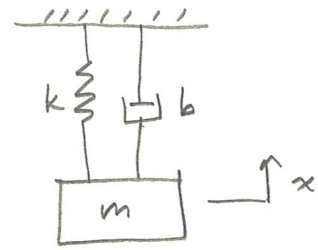
$$\sigma = \omega_n \zeta. \tag{5}$$

These values of σ and ω_n can be used to determine unknown physical parameters in the system. For example, for the system of section 8.2 of Cannon, if any one of the parameters J, b, or k are known, then the remaining two can be calculated from a knowledge of σ and ω_n, where σ and ω_n are determined from the experimental time response of the system.

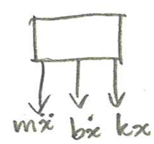
Parameter ID for 2nd-Order system:

Example:

$m = 1$ kg
$b = 5 \frac{N\text{-}sec}{m}$
$k = 100$ N/m



FBD



EOM: $m\ddot{x} + b\dot{x} + kx = 0$

Assume measurement of mass is available: $m = 1$ kg

① Do log decrement to find ζ:

$n = 3$, $x_1 = 0.7$, $x_3 = 0.03$

$$\rightarrow \zeta = \frac{\frac{1}{2}\ln\left(\frac{x_1}{x_3}\right)}{\sqrt{4\pi^2 + \left(\frac{1}{2}\ln\left(\frac{x_1}{x_3}\right)\right)^2}}$$

$$= \frac{\frac{1}{2}\ln\left(\frac{0.7}{0.03}\right)}{\sqrt{4\pi^2 + \left(\frac{1}{2}\ln\left(\frac{0.7}{0.03}\right)\right)^2}}$$

$$\boxed{\zeta = 0.24}$$

② Measure ω from plot:

$$T = \frac{1.42\text{ sec} - 0.125\text{ sec}}{2} = 0.648\text{ sec}$$

$$\omega = \frac{2\pi}{T} = \boxed{9.70 \text{ rad/sec}}$$

③ Calculate ω_n: $\omega_n = \frac{\omega}{\sqrt{1-\zeta^2}}$

$$\omega_n = \frac{9.70 \text{ rad/sec}}{\sqrt{1 - 0.24^2}}$$

$$\boxed{\omega_n = 9.996 \text{ rad/sec}}$$

④ Calculate k:

$$\omega_n^2 = \frac{k}{m} \rightarrow k = m\omega_n^2$$

$$= (1\ \text{kg})(9.996\ \text{rad/sec})^2$$

$$\boxed{k = 99.92\ \frac{\text{N}}{\text{m}}}$$

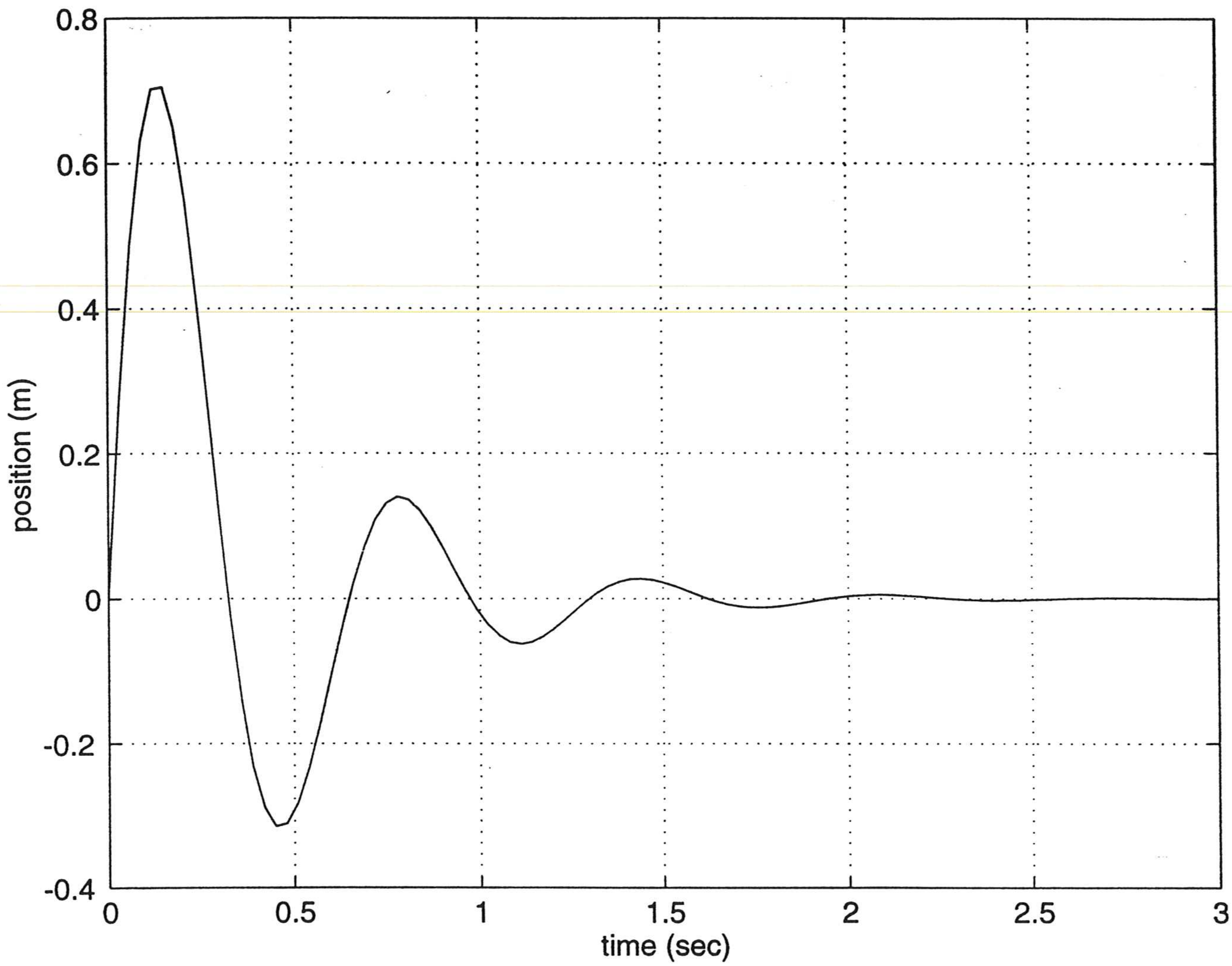
⑤ Calculate b:

$$\sigma = \zeta\omega_n = \frac{b}{2m}$$

$$\rightarrow b = 2m\zeta\omega_n$$

$$= 2(1\text{kg})(0.24)(9.996\ \text{rad/sec})$$

$$\boxed{b = 4.80\ \frac{\text{N-s}}{\text{m}}}$$


0.8
0.6
0.4
0.2
0
-0.2
-0.4
position (m)
0
0.5
1
1.5
2
2.5
3
time (sec)