Dyno Example :

Objectives : - Find $\frac{\Omega_1}{M}$, $\frac{\theta_2}{M}$ transfer functions

- Simulate response using step , lsim

EOM : $J_1 \dot{\Omega}_1 + b(\Omega_1 - \Omega_2) = M(t)$

$J_2 \dot{\Omega}_2 - b(\Omega_1 - \Omega_2) \; k\theta_2 = 0$

Remember that $\dot{\theta}_1 = \Omega_1$, $\dot{\theta}_2 = \Omega_2$

$\ddot{\theta}_1 = \dot{\Omega}_1$, $\ddot{\theta}_2 = \Omega_2$

Can write EOM in terms of θ_1 and θ_2 and their derivatives

$$J_1 \ddot{\theta}_1 + b(\dot{\theta}_1 - \dot{\theta}_2) = M(t)$$

$$J_2 \ddot{\theta}_2 + b(\dot{\theta}_2 - \dot{\theta}_1) + k\theta_2 = 0$$

Take Laplace x-form :

$$J_1 s^2 \Theta_1(s) + bs\,\Theta_1(s) - bs\,\Theta_2(s) = M(s)$$

$$J_2 s^2 \Theta_2(s) + bs\,\Theta_2(s) - bs\,\Theta_1(s) + k\,\Theta_2(s) = 0$$

$$s(J_1 s + b)\Theta_1(s) - bs\,\Theta_2(s) = M(s)$$

$$-bs\,\Theta_1(s) + (J_2 s^2 + bs + k)\Theta_2(s) = 0$$

$$\underbrace{\begin{bmatrix} s(J_1 s + b) & -bs \\ -bs & (J_2 s^2 + bs + k) \end{bmatrix}}_{A} \begin{bmatrix} \Theta_1(s) \\ \Theta_2(s) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} M(s)$$

By Cramer's rule,

$$\frac{\Theta_1}{M} = \frac{1}{\det A} \cdot \det \begin{bmatrix} 1 & -bs \\ 0 & J_2 s^2 + bs + k \end{bmatrix}$$

$$\frac{\Theta_2}{M} = \frac{1}{\det A} \cdot \det \begin{bmatrix} s(J_1 s + b) & 1 \\ -bs & 0 \end{bmatrix}$$

$$\begin{aligned} \det A &= (J_1 s^2 + bs)(J_2 s^2 + bs + k) - b^2 s^2 \\ &= J_1 J_2 s^4 + J_1 b s^3 + J_1 k s^2 + J_2 b s^3 + \cancel{b^2 s^2} + bks - \cancel{b^2 s^2} \\ \det A &= s\left[J_1 J_2 s^3 + (J_1 + J_2) b s^2 + J_1 k s + bk\right] \end{aligned}$$

$$\frac{\Theta_1}{M} = \frac{J_2 s^2 + bs + k}{s\left[J_1 J_2 s^3 + (J_1 + J_2) b s^2 + J_1 k s + bk\right]}$$

$$\frac{\Theta_2}{M} = \frac{b}{J_1 J_2 s^3 + (J_1 + J_2) b s^2 + J_1 k s + bk}$$

To Matlab : dyno_tf.m

Working with LTI objects in Matlab:

Tranfer functions and state-space matrices are two examples of LTI objects in Matlab.

LTI: linear, time-invariant

- Same as LCC: linear, constant coefficient

We have looked at a transfer function example —

Let's consider a state-space example — the dyno

From before, equations of motion:

$$\left.\begin{aligned} J_1\dot{\Omega}_1 + b(\Omega_1 - \Omega_2) &= M(t) \\ J_2\dot{\Omega}_2 + b(\Omega_2 - \Omega_1) + k\theta_2 &= 0 \\ \dot{\theta}_1 = \Omega_1 \qquad \dot{\theta}_2 &= \Omega_2 \end{aligned}\right\} \quad (1)$$

can be multiple

Define states, input, output of interest

order not critical

$$x = [\Omega_1 \ \ \Omega_2 \ \ \theta_1 \ \ \theta_2]^T \qquad u = M(t)$$

$$y = [\Omega_1 \ \ \theta_2]$$ — for example

Write equations (1) in state-space form

$$\dot{x} = Ax + Bu \qquad y = Cx + Du$$

$$\dot{\Omega}_1 = -\frac{b}{J_1}\Omega_1 + \frac{b}{J_1}\Omega_2 + \frac{1}{J_1}M(t)$$

$$\dot{\Omega}_2 = \frac{b}{J_2}\Omega_1 - \frac{b}{J_2}\Omega_2 - \frac{k}{J_2}\theta_2$$

$$\dot{\theta}_1 = \Omega_1 \qquad \dot{\theta}_2 = \Omega_2$$

(cont.)

$$\begin{bmatrix} \dot{\Omega}_1 \\ \dot{\Omega}_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{b}{J_1} & \frac{b}{J_1} & 0 & 0 \\ \frac{b}{J_2} & -\frac{b}{J_2} & 0 & -\frac{k}{J_2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \theta_1 \\ \theta_2 \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{J_1} \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{B} M(t)$$

$$y = \begin{bmatrix} \Omega_1 \\ \theta_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{C} \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \theta_1 \\ \theta_2 \end{bmatrix} + \underbrace{0}_{D} M(t)$$

The matrices (A, B, C, D) define the dynamics in state-space form

The state-space and transfer function representations for a system are related. Starting with the state-space equations:

$$\dot{x} = Ax + Bu \qquad y = Cx + Du$$

Taking the laplace x-form and manipulating:

$$sX(s) = AX(s) + B\mathcal{U}(s) \qquad Y(s) = CX(s) + D\mathcal{U}(s)$$

$$(sI - A)X(s) = B\mathcal{U}(s)$$

$$X(s) = (sI-A)^{-1}B\mathcal{U}(s)$$

$$Y(s) = C(sI-A)^{-1}B\mathcal{U}(s) + D\mathcal{U}(s)$$

$$\underline{\frac{Y(s)}{\mathcal{U}(s)} = C(sI-A)^{-1}B + D}$$

transfer function calculated from state-space matrices

Matlab will do this for us.

See dyno_ss.m