

MeEn 335: System Dynamics

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Chapter 1

Dynamic Systems and System Similarity

1.1 Static vs. Dynamic Systems

Most engineering classes up to this point have considered the steady-state values of a system's parameters. This is important for understanding systems that change slowly, or for predicting the response of systems that rarely change. However, many system phenomena depend on the *dynamic* properties of a system. For example, when a ball hits a bat, how much stress does the bat sustain? What loads will an artificial knee carry during walking? How will a car's tire respond to periodic bumps in a road? How fast can a sensor respond to changes in the parameters it's sensing? All of these questions, and many others, require a dynamic analysis of the system.

To start, we need a definition of a static and a dynamic system. A static system is **a system whose response depends only on the current values of any system variables**. For example, the vertical loads endured by a bridge are a function of the weights and locations of any vehicles, people, or other objects that are on the bridge at any instant. The vertical load at any instant in time does not depend strongly on the vehicles that were on the bridge before that instant. Therefore, in calculating the vertical loads on a bridge, it is often sufficient to consider it as a static system.

A dynamic system is **a system whose response depends on the current values of system variables as well as the time history of those variables**. For example, the famous failure of the Tacoma Narrows bridge occurred because lateral winds caused the bridge to begin vibrating at a rate known as the *resonant frequency* of the bridge. The lateral loads on the bridge depended not only on the current wind speed; they depended on the past values of the wind speed, the bridge's lateral deflection, and the bridge's lateral velocity (among other things).

Another example might be a normal household toilet. A static analysis would be sufficient to tell us the pressure at the valve that flushes the toilet; a dynamic analysis would be required to

determine whether the toilet bowl will fill fast enough after flushing to ensure that the suction will empty the contents of the toilet (more about this in class).

1.2 Conservation Laws for Engineering Systems

General Statement: For an engineering system,

$$\boxed{\text{Inflow} - \text{Outflow} = \text{Rate of Creation (or Storage)}}$$

In this class, we will apply the conservation laws (in different forms) to a variety of different systems (mechanical, electrical, fluid, etc.). The result will be *equations of motion* which describe the dynamic behavior of these systems. Equations of motion (EOM) will be ordinary differential equations (ODEs).

Standard forms for equations of motion:

- Single n^{th} -order ODE
- System of n first-order ODEs (coupled)—also called the “state variable form”
- Transfer function

1.2.1 Conservation Laws in Different Energy Domains

Mechanical – Linear Motion

The principle of linear-impulse/momentum states that momentum is equal to the applied impulse

$$\underbrace{\sum \int_0^t F dt}_{\text{impulses}} = \underbrace{mv}_{\text{momentum}} .$$

By differentiating with respect to time, we get the familiar statement of Newton’s Second Law

$$\sum F = \frac{d}{dt}(mv),$$

which for a constant mass gives

$$\sum F = ma.$$

Mechanical – Fixed-axis Rotation

The differential form of angular-impulse/momentum can be expressed as

$$\sum T = \frac{d}{dt}(J\Omega),$$

where T is torque, J is mass moment of inertia, and Ω is angular velocity. This is true for motion about a fixed axis or motion about an axis through the mass center of the rotating body. Again, if J is constant, we get the familiar form

$$\sum T = J \frac{d\Omega}{dt}.$$

Electrical

The principle of conservation of charge states that the charge in an electrical system is constant. This is commonly expressed as Kirchoff's current law (KCL): *The sum of currents at a node of a circuit is equal to the rate at which charge is stored at the node.*

$$\underbrace{\sum i}_{\text{inflow-outflow}} = \overbrace{\frac{dq}{dt}}^{\text{rate of storage}} = C \frac{dV}{dt}$$

If the node has no capacitance, then we get the familiar form of KCL: $\sum i = 0$.

Fluid

The principle of conservation of mass states that the mass of a fluid system is constant. In differential form, conservation of mass says that the net mass flow rate in and out of a control volume is equal to the time rate of change of the mass within the control volume. This can be expressed in equation form as

$$\underbrace{\sum \dot{m}}_{\text{inflow-outflow}} = \overbrace{\frac{d}{dt}}^{\text{rate of change}} (\rho V) = \rho \dot{V} + V \dot{\rho}.$$

Conservation of mass leads to the *continuity equation* which relates volumetric flow rates.

Thermal

Heat energy will also be conserved in a thermal system. Hence, the rate of heat flow into the system will equal the time rate of change of heat energy stored in the system. In equation form, this means

that, for a system with constant mass,

$$\underbrace{\sum q_h}_{\text{inflow} - \text{outflow}} = \overbrace{\frac{d}{dt}}^{\text{rate of change}} (mC_v T) = mC_v \frac{dT}{dt}.$$

Conservation of Energy – Multiple Domains

The principle of conservation of energy states that the energy of a system is constant. Expressed in differential form, it is also known as the *first law of thermodynamics* which says that the sum of power in and out of a system is equal to the rate at which energy is stored within the system. This can be expressed in equation form as

$$\underbrace{\sum \dot{Q}_h}_{\text{heat transfer}} - \overbrace{\dot{W}}^{\text{work}} + \underbrace{\dot{m} \left(h + \frac{v^2}{2g} + zg \right)}_{\text{mass flow across boundaries}} = \frac{d}{dt} \underbrace{\left(mu + \frac{mv^2}{2} + mzg \right)}_{\text{energy stored}}_{cv}.$$

When there is no heat transfer or work being done and when there is no energy being stored, this expression reduces to the familiar Bernoulli equation

$$\frac{P}{\rho} + \frac{v^2}{2} + zg = \text{constant along a streamline.}$$

1.3 Similarity Among Different Energy Domains

The systems that we have discussed up to this point are very different in that they represent different energy domains:

- mechanical translation
- mechanical rotation
- electrical
- fluid
- thermal

However all of them can be analyzed by applying conservation laws in a similar fashion. Therefore, it should come as no surprise that these systems from different energy domains have similarities among them. Let's investigate

Power Variables

Table 1.1: Power variables.

domain	effort variable	flow variable
mechanical translation	force - F	velocity - v
mechanical rotation	torque - T	angular velocity - Ω
electrical	voltage - V	current - i
fluid	pressure - P	volumetric flow rate - Q
thermal	temperature - T	heat flow rate - q

Effort and flow are called power variables because their product is power for each of the energy domains:

$$\text{power} = \text{effort} \times \text{flow}$$

EXCEPTION: This is not true for thermal systems! $P \neq T \times q$, rather $P = q$.

Energy Variables

Table 1.2: Energy variables.

domain	momentum variable	displacement variable
mechanical translation	linear momentum - p	linear displacement - x
mechanical rotation	angular momentum - H	angular displacement - θ
electrical	flux linkage - λ	charge - q
fluid	pressure momentum - p_P	volume - V
thermal	no correspondence	heat energy - Q

Momentum and displacement variables are referred to as energy variables because energy can be conveniently expressed in terms of momentum and displacement. For example, for a linear spring

$$PE = \int \text{power} \, dt = \int Fv \, dt = \int kx \, dx = \frac{1}{2}kx^2.$$

Similarly for a translating body,

$$KE = \int \text{power} \, dt = \int vF \, dt = \int v \, dp = \int mv \, dv = \frac{1}{2}mv^2.$$

These relationships between energy and momentum and displacement generalize to each of the different energy domains (except thermal).

In each of the energy domains with the exception of thermal systems, the power and energy variables are related:

$$\text{flow} = \frac{d}{dt}(\text{displacement})$$

$$\text{effort} = \frac{d}{dt}(\text{momentum}).$$

1.3.1 Dissipative (Resistance) Elements

As their name implies, dissipative elements dissipate energy. They do not store energy. Common examples include resistors in electrical systems or friction in a mechanical system. The linear relationship for a dissipative element is given by

$$\text{effort} = \text{resistance} \times \text{flow}$$

Table 1.3: Dissipative (Resistance) Elements.

domain	constitutive relation	physical description
mechanical translation	$F = bv$	friction, damping
mechanical rotation	$\tau = b\Omega$	friction, damping
electrical	$V = Ri$	electrical resistance
fluid	$P = RQ$	fluid drag, resistance
thermal	$T = Rq$	conduction, convection, radiation

Often the resistance constitutive relation is not linear. In its most general form, the resistance relation relating effort and flow is $\text{effort} = f(\text{flow})$. A common example of a nonlinear relation is coulomb or dry friction which can be described mathematically as

$$F_f = c \operatorname{sign}(v).$$

1.3.2 Energy Storage Elements

Unlike dissipative elements, energy storage elements do not dissipate energy. As their name implies, they store energy. We will consider two types of energy storage elements:

- Effort-based (capacitive)
- Flow-based (inertial)

Effort-based Energy Storage (Capacitance)

The linear constitutive relationship used to describe the behavior of a capacitive element is

$$\text{effort} = \frac{1}{\text{capacitance}} \times \text{displacement}$$

or

$$\text{capacitance} \times \frac{d}{dt} \text{effort} = \text{flow}$$

Table 1.4: Effort-based Energy Storage (Capacitance) Elements.

domain	constitutive relation	alt. const. relation	physical description
mechanical translation	$F = kx$	$\frac{1}{k} \frac{dF}{dt} = v$	linear spring
mechanical rotation	$\tau = k\theta$	$\frac{1}{k} \frac{d\tau}{dt} = \Omega$	torsional spring
electrical	$V = \frac{1}{C}q$	$C \frac{dV}{dt} = i$	capacitor
fluid	$\Delta P = \frac{1}{C}\Delta V$	$C \frac{dP}{dt} = Q$	compliance
thermal	$\Delta T = \frac{1}{mc_p}\Delta Q$	$mc_p \frac{dT}{dt} = q_h$	thermal capacitance

In its most general form, the constitutive relationship for a capacitance element is $\text{effort} = f(\text{displacement})$, or $\frac{d\text{effort}}{dt} = f(\text{flow})$.

Flow-based Energy Storage (Inertia)

The linear constitutive relationship used to describe the behavior of an inertial element is

$$\text{flow} = \frac{1}{\text{inertia}} \times \text{momentum}$$

or

$$\text{inertia} \times \frac{d}{dt} \text{flow} = \text{effort}$$

In its most general form, the constitutive relationship for an inertia element in any energy domain is $\text{flow} = f(\text{momentum})$, or $\frac{d\text{flow}}{dt} = f(\text{effort})$. Fortunately, the linear relations described above usually hold for inertial elements.

Table 1.5: Flow-based Energy Storage (Inertia) Elements.

domain	constitutive relation	alt. const. relation	physical description
mechanical translation	$v = \frac{1}{m}p$	$m \frac{dv}{dt} = F$	mass
mechanical rotation	$\Omega = \frac{1}{J}H$	$J \frac{d\Omega}{dt} = \tau$	mass moment of inertia
electrical	$i = \frac{1}{L}\lambda$	$L \frac{di}{dt} = V$	inductance
fluid	$Q = \frac{1}{I}p_P V$	$I \frac{dQ}{dt} = P$	fluid inertia
thermal	no correspondence		

1.3.3 Constitutive Relation Summary

$$\text{effort} = \text{resistance} \times \text{flow}$$

$$\text{effort} = \frac{1}{\text{capacitance}} \times \text{displacement} = \frac{1}{\text{capacitance}} \int \text{flow} dt$$

$$\text{flow} = \frac{1}{\text{inertia}} \times \text{momentum} = \frac{1}{\text{inertia}} \int \text{effort} dt$$

We will also sometimes use the D operator. The D operator performs the derivative operation. For example,

$$Dx \triangleq \frac{dx}{dt} \triangleq \dot{x}$$

$$\frac{1}{D} \triangleq \int x(t) dt.$$

The D operator is analogous in some respects to the Laplace variable s that you have seen before. Then

$$\text{capacitance} \times D \text{ effort} = \text{flow}$$

$$\text{inertia} \times D \text{ flow} = \text{effort}$$

1.4 Similar Systems

Because each type of system (mechanical, electrical, fluid, and thermal) has the same type of system elements (or system components), it should not be surprising that it is possible to construct systems in diverse energy domains but with similar behavior (meaning that the systems are modeled by equations that are qualitatively the same). This will occur when the same types of system components are used (resistive, effort-based storage, or flow-based storage), and when they are

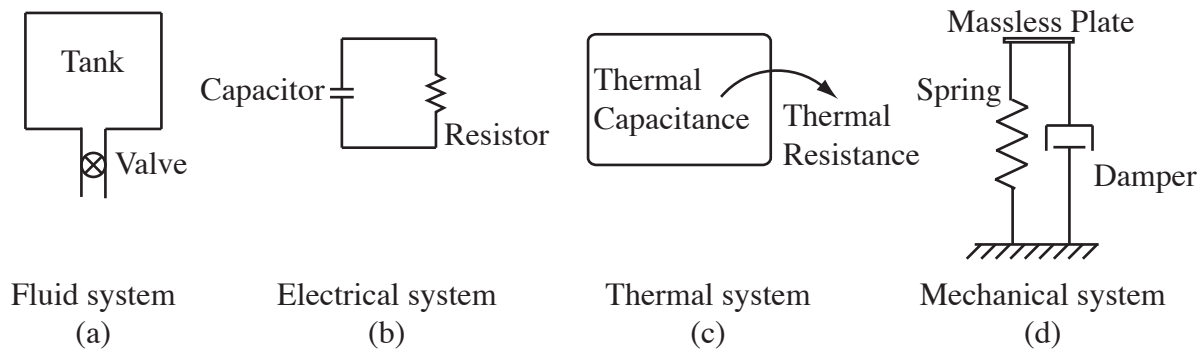


Figure 1.1: Similar systems in different energy domains. They might represent, for example, a water storage tank in a toilet (a), a camera flash (b), an oven (c), and a self-closing mechanism for a door (d).

connected in such a way that the effort and flow variables in each component are related in the same ways. While systems of arbitrary complexity could be found in diverse energy domains, it is most instructive to look at systems consisting of only two or three components each.

For example, consider the fluid system in Fig. 1.1(a), consisting of a fluid tank (an effort-based storage element) attached to a fluid resistance (such as a valve). This system might represent, for example, the water storage tank in a toilet, which stores fluid energy until it is released through the valve at its bottom. Note that, for this system, the effort variable (the pressure) is the same for both components, meaning that the pressure in the tank is the same as the pressure drop across the valve. Further, the flow variable in both components is the same — the volume flow rate leaving the tank is equal to the volume flow rate through the valve. Hence, to construct a similar mechanical, electrical, or thermal system, we need to use one effort storage component and one dissipative component, and we need to connect them in such a way that the effort variable and the flow variable in each is the same.

Therefore, a similar electrical system, shown in Fig. 1.1(b), consists of a capacitor and a resistor. They are wired so that they have the same voltage (effort) and current (flow). This system might represent a camera flash, in which electrical energy is stored in a capacitor prior to being released through the flash bulb (represented by the resistor).

A similar thermal system would consist of a thermal capacitance with a thermal resistance between it and the ambient temperature. The temperature difference (effort) between the capacitance and the ambient would then also be the temperature difference across the thermal resistor, and the heat flow rate (flow variable) leaving the capacitance would be equal to the heat flow through the thermal resistance. This system might represent an oven, which can store heat energy, and is thermally insulated from the environment.

A similar mechanical system would consist of a spring and a damper, connected so that the force (effort) exerted by the spring is opposed by the damper, and the velocity (flow) of the spring's and damper's ends is equivalent. Such a system might represent a door-closing mechanism in which the inertia of the door is negligible.

Similar systems could also be created using flow-based storage. For example, an electromagnet might consist of an inductor and a resistor. A similar mechanical system would consist of a mass and a damper. Such a system might model a box sliding on a thin lubricating film. A similar fluid system would consist of a very long pipe, in which the fluid inertia was important, and a valve. This might model, for example, an oil pipeline. Of course, no similar thermal system exists, since there is no flow-based storage for thermal systems.

Chapter 2

Mechanical Systems

2.1 Translational Motion

When talking about the conservation laws for different systems, we reintroduced the notion of conservation of momentum for mechanical systems. For a system with a constant mass undergoing translational motion, this simplifies to the familiar Newton's second law:

$$\begin{aligned}\sum F &= m \frac{dv}{dt} \\ &= ma \\ &= m\ddot{x}\end{aligned}$$

Newton's second law was originally derived to model the behavior of a point mass. We will apply it to mechanical translational systems of significant complexity.

The approach we will follow for deriving equations of motion for mechanical systems consists of five steps:

1. Define system geometry
2. Apply force balance relations
3. Develop kinematic relationships
4. Define constitutive relations for system elements
5. Combine relations to get equations of motion

2.1.1 Defining System Geometry

By defining the geometry of the system, we select the variables that will be used to describe mathematically the dynamics of the system. Two important questions to consider are:

1. Which variables interest us in this problem?
2. Which variables will be easiest to use in the analysis?

Usually the first step (and sometimes the only step) in this process is to define variables to *describe the configuration* of the system.

How do we do this?

- Draw a picture of the system in an arbitrary configuration
- Define position coordinates to describe the configuration of the system
- Definition of configuration variables defines positive directions for the system

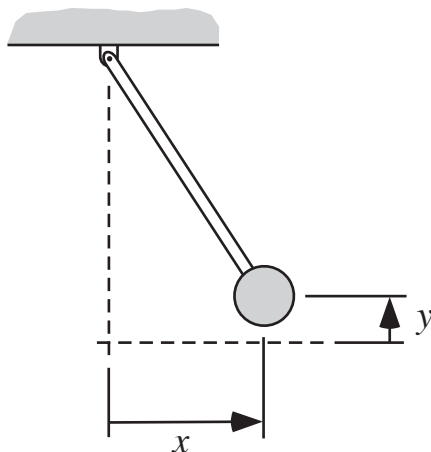
Figure 2.1 shows an example of defining the geometry of a system by choosing variables to define the system.

Degrees of Freedom (DOF)

The number of degrees of freedom of a mechanical system is defined as the minimum number of geometrically independent coordinates required to describe its configuration completely.

Q: In Fig. 2.1, how many degrees of freedom does the system have?

Q: In the pendulum example below, how many degrees of freedom are there?



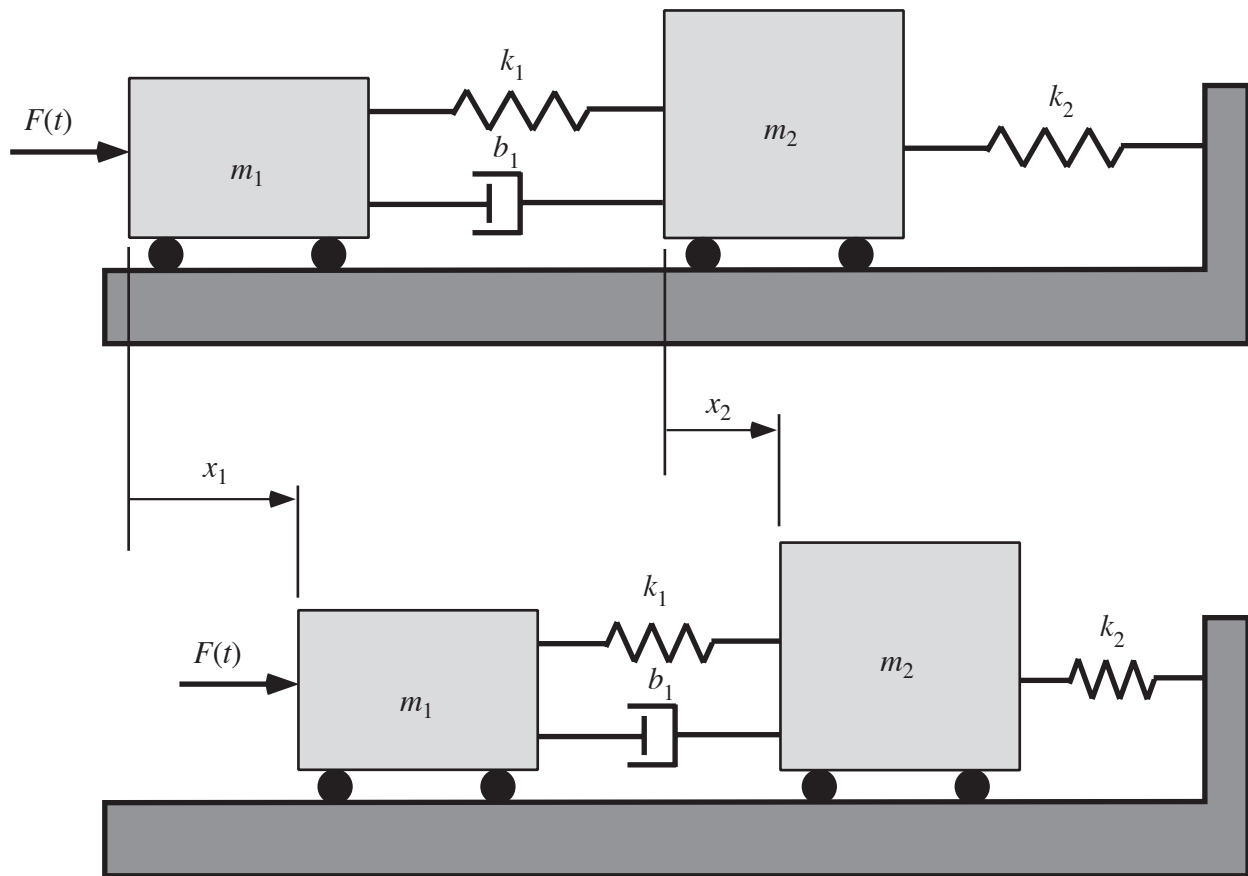


Figure 2.1: Step1: Define the configuration of the system.

Referring to Fig. 2.1, the configuration of the system is completely described by the displacement variables (x_1, x_2) and hence there are two degrees of freedom. The definition of displacements x_1 and x_2 also define positive direction for position, velocity, and accelerations of masses 1 and 2.

With the system geometry defined, we can define a set of state variables for our system. For a mechanical system,

$$\text{state variables} = \text{position coordinates} + \text{their derivatives}$$

Determining the set of state variables in this manner does not necessarily result in a “minimal” set of states. It is possible that the system could be described with fewer states. We’ll see an example of this later after we’ve learned to derive equations of motion.

Q: What do state variables describe?

- They describe how energy is stored within the system

- A history of the state variables with respect to time describes the behavior of the system completely.

Q: In the example problem, how is energy stored? (Kinetic energy is stored in the masses, and potential energy is stored in the springs.) What are the state variables? (x_1, x_2, v_1, v_2) Do they describe how energy is stored? (Energy = $\frac{1}{2} [m_1 v_1^2 + m_2 v_2^2 + k_1 (x_1 - x_2)^2 + k_2 x_2^2]$)

2.1.2 Apply Force Equilibrium Relations

In this step, we'll draw free-body diagrams and apply Newton's second law. To do this:

- Isolate the “free bodies” (masses)
- Replace connections to the environment or other bodies with forces acting from the connection
- Include inertial forces on the free body diagram (see below)

Back to the example problem:

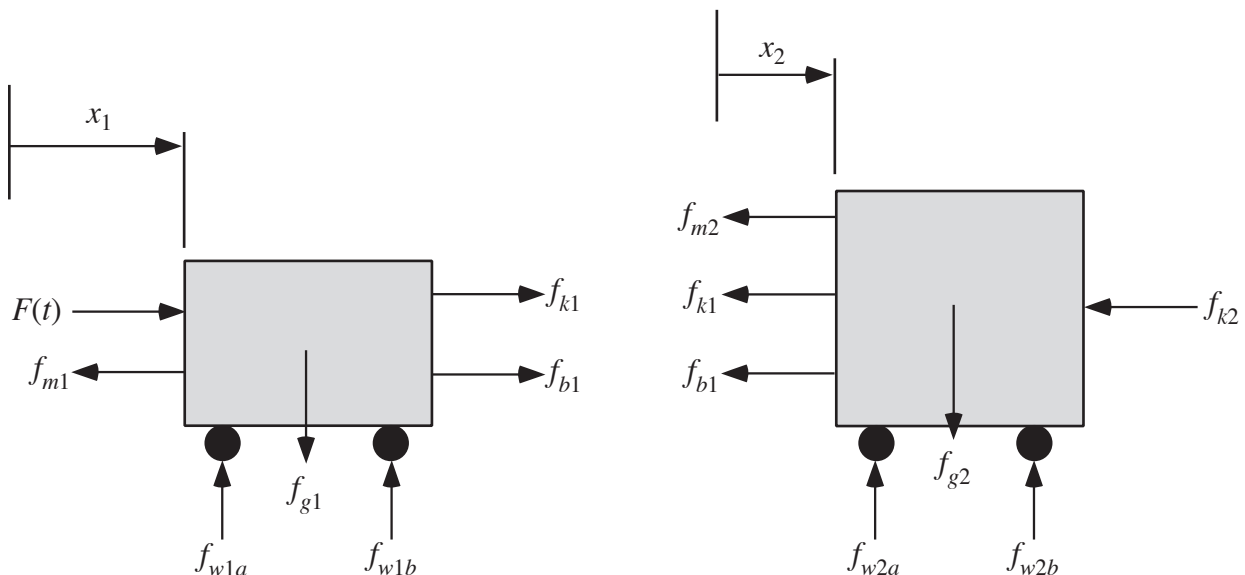


Figure 2.2: Free-body diagrams, including inertial forces.

D'Alembert's Principle

Newton's second law can be simply stated as $\sum F = m\ddot{x}$. Rearranging slightly gives $\sum F - m\ddot{x} = 0$. We can rename the left-hand side of this equation to arrive at D'Alembert's formulation of Newton's second law:

$$\sum F^* = 0.$$

Here, the “*” indicates that inertial forces have been included in the summation. In our example, D'Alembert's formulation is easy to apply.

There are several advantages to D'Alembert's formulation that make it an attractive alternative to direct implementation of Newton's second law in its traditional form, including:

- The force balance relation is identical to the static situation ($\sum F = 0$).
- It is systematic —“acceleration forces” are treated in the same way as those due to displacement and velocity.
- Physical intuition is strengthened by thinking that inertial forces oppose acceleration.
- It provides important analytical simplification in advanced problems (more on this later).

For the example (shown in Fig. 2.1 and Fig. 2.2), by examining forces in the horizontal direction, we get

$$F(t) - f_{m1} + f_{b1} + f_{k1} = 0$$

for mass 1 and

$$-f_{m2} - f_{b1} - f_{k1} - f_{k2} = 0.$$

for mass 2.

2.1.3 Define Constitutive Relations

Constitutive relations are simply mathematical descriptions of the physical laws that govern the behavior of system components. In problems involving mechanical translation, three different types of constitutive (physical) relations are found:

1. force-displacement (springs, compliance)
2. force-velocity (dampers, friction)
3. force-acceleration (mass)

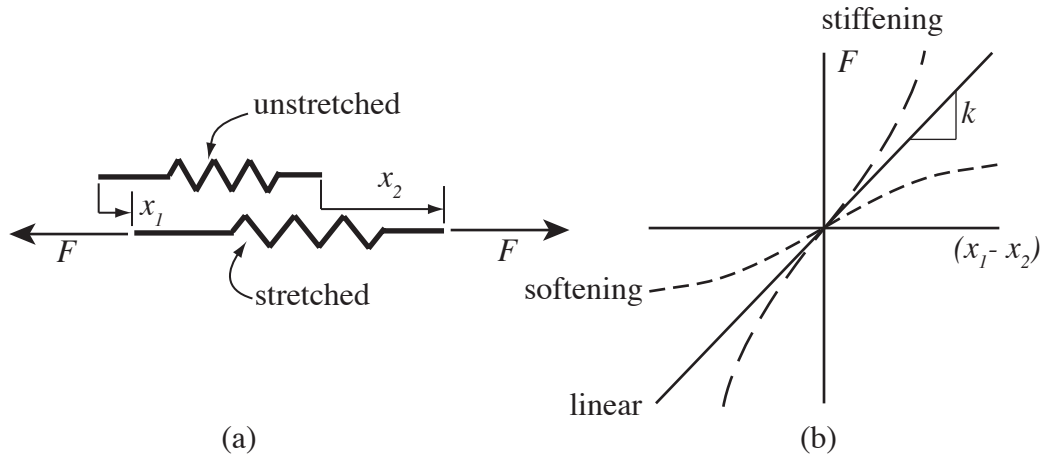


Figure 2.3: (a) A stretched spring. (b) A force-displacement graph showing a linear, stiffening, and softening spring

Force-displacement: Spring

Springs stretch or compress in response to an applied force. For the spring shown in Fig. 2.3a, three common types of responses are possible. The most common type is a linear relationship between force and displacement:

$$F = k(x_2 - x_1) \quad \text{linear}$$

Force is proportional to deflection. k is known as the spring constant or spring rate.

Nonlinear springs do not maintain a linear relationship between force and displacement:

$$F = f(x_1, x_2) \quad \text{nonlinear}$$

Two common types are stiffening and softening springs. Stiffening springs, also called hardening springs, have a larger slope for higher deflection. An example might be the deflection of a fixed-fixed beam with a force at its center. Stress stiffening causes the effective stiffness of the beam to increase with displacement. Softening springs have a smaller slope for higher deflection. A constant-force spring is an extreme example of a softening spring — for low deflections, the force increases rapidly, until it reaches a near-constant value over a wide range of displacements.

Force-velocity: Dampers, dashpots, and friction

These are the dissipative components for mechanical systems. For a linear or viscous damper, the force acting on the damper is a linear function of the relative velocity of its two ends, as shown in

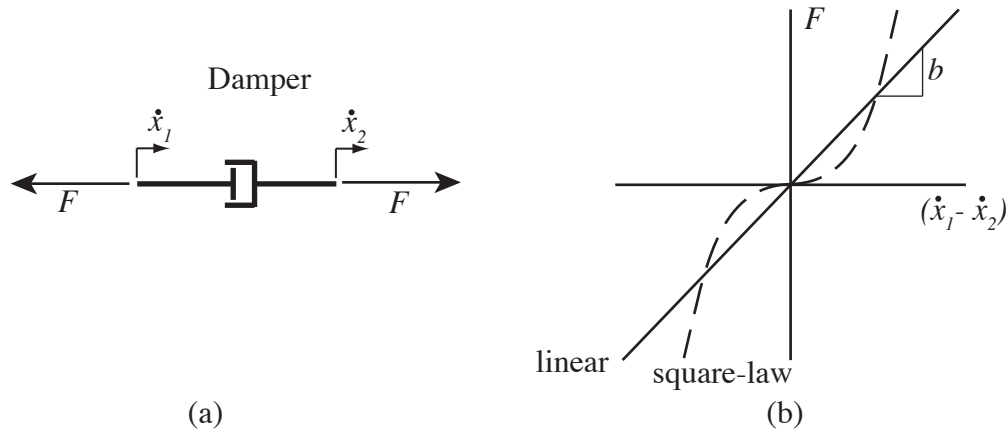


Figure 2.4: (a) A damper component. (b) Typical force-velocity profiles for some damper elements.

Fig. 2.4. The damper force-velocity relationship is thus

$$F = b(\dot{x}_2 - \dot{x}_1) \quad \text{linear or viscous}$$

Other common sources of damping are square-law damping and coulomb or dry friction. Square-law damping, which is frequently used to describe damping due to fluid resistance (such as the wind resistance on a bicyclist), is described by

$$F = b(\dot{x}_2 - \dot{x}_1)|\dot{x}_2 - \dot{x}_1| \quad \text{square law}$$

The damping is written this way (instead of with a simple square) in order to allow the sign of the force to change with the relative sign of the two velocities. Coulomb friction, on the other hand, describes the constant friction encountered when two surfaces rub together:

$$F = F_c \text{sign}(\dot{x}) \quad \text{coulomb or dry friction}$$

The sign of the velocity serves to describe the direction of the force, with the force always acting in the direction opposite the velocity. F_c is a constant force value giving the magnitude of the friction force. Coulomb force is illustrated graphically in Fig. 2.5.

Force-acceleration: Mass

$$F = m\ddot{x} \quad \text{Always linear! (as long as velocities are small relative to speed of light)}$$

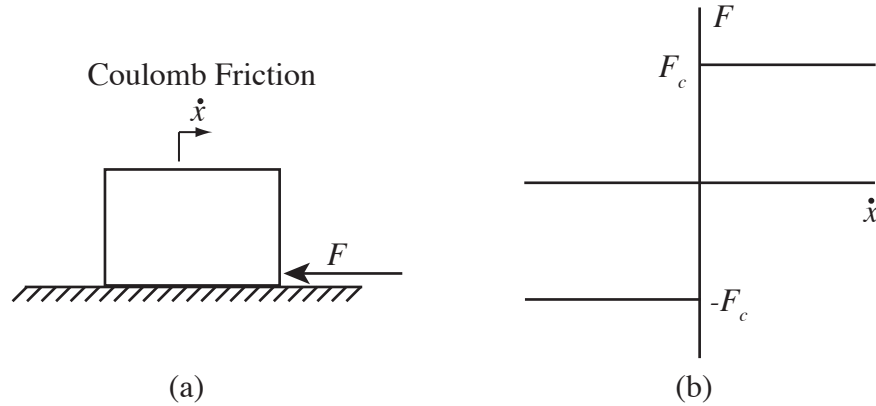


Figure 2.5: (a) A mass experiencing coulomb friction. (b) The force-velocity profile for coulomb friction.

Example cont. Assume linear springs and dampers.

If we consider again the example problem of Figures 2.1 and 2.2, we get for component relations

$$f_{m1} = m_1 \ddot{x}_1 \quad f_{m2} = m_2 \ddot{x}_2 \quad (2.1)$$

$$f_{k1} = k_1(x_2 - x_1) \quad f_{k2} = k_2 x_2 \quad (2.2)$$

$$f_{b1} = b_1(\dot{x}_2 - \dot{x}_1) \quad (2.3)$$

Note that we must carefully consider the signs of the displacements in the spring relationships, and the velocities in the damper relationships.

2.1.4 Combine Force-balance and Constitutive Relations

- Combine relations from steps 2, 3, and 4.
- Result: Equations of Motion (ODE's that describe the dynamic behavior of the system.)

The equations of motion must contain only the state variables, their derivatives, system inputs, and known parameters. If there is a term in the equations that contains an unknown parameter (especially an unknown changing parameter) that is not a state variable or one of its derivatives, you do not have the correct equations of motion!

Example cont.

Continuing the example from Figures 2.1 and 2.2, we obtain the equations of motion

$$\begin{aligned} F(t) - m_1 \ddot{x}_1 + b_1(\dot{x}_2 - \dot{x}_1) + k_1(x_2 - x_1) &= 0 \\ -m_2 \ddot{x}_2 - b_1(\dot{x}_2 - \dot{x}_1) - k_1(x_2 - x_1) - k_2 x_2 &= 0 \end{aligned}$$

We can rewrite this as

$$\begin{aligned} m_1 \ddot{x}_1 + b_1(\dot{x}_1 - \dot{x}_2) + k_1(x_1 - x_2) &= F(t) \\ m_2 \ddot{x}_2 + b_1(\dot{x}_2 - \dot{x}_1) + k_1(x_2 - x_1) + k_2 x_2 &= 0 \end{aligned}$$

2.1.5 Alternative Approach

- Write constitutive relations directly on the free-body diagrams by the corresponding force vector.
- Write equations of motion by inspection of the free-body diagrams.
- If kinematic relationships exist, write these next to the free-body diagram.

Revisiting the example:

The example problem is repeated in Fig. 2.6. Equations of motion (by inspection of the FBD) are then:

$$\begin{aligned} m_1 \ddot{x}_1 - b(\dot{x}_2 - \dot{x}_1) - k_2(x_2 - x_1) &= F(t) \\ m_2 \ddot{x}_2 + b(\dot{x}_2 - \dot{x}_1) + k_1(x_2 - x_1) + k_2 x_2 &= 0 \end{aligned}$$

2.1.6 Gravity Terms and Static Equilibrium

In some cases, gravity terms can be removed from the equations. For example, for the mass and spring shown in Fig. 2.7, if we write the equations of motion for the mass, we get:

$$mg + F(t) - m\ddot{x} - k(x + \Delta) = 0$$

However, if we choose x to be zero when $k\Delta = mg$, this simplifies to

$$m\ddot{x} + kx = F(t).$$

Hence, by careful choice of the system variables, we can remove gravity terms from the equations of motion!

Another example: Vehicle Suspension System (Quarter-Car Model)

We choose y_1 and y_2 to be zero when the springs are in static equilibrium. By inspection from the free body diagrams, we get:

$$\begin{aligned} m_1 \ddot{y}_1 + b(\dot{y}_1 - \dot{y}_2) + k_2(y_1 - y_2) + k_1(y_1 - y_0(t)) &= 0 \\ m_2 \ddot{y}_2 + b(\dot{y}_2 - \dot{y}_1) + k_2(y_2 - y_1) &= 0 \end{aligned}$$

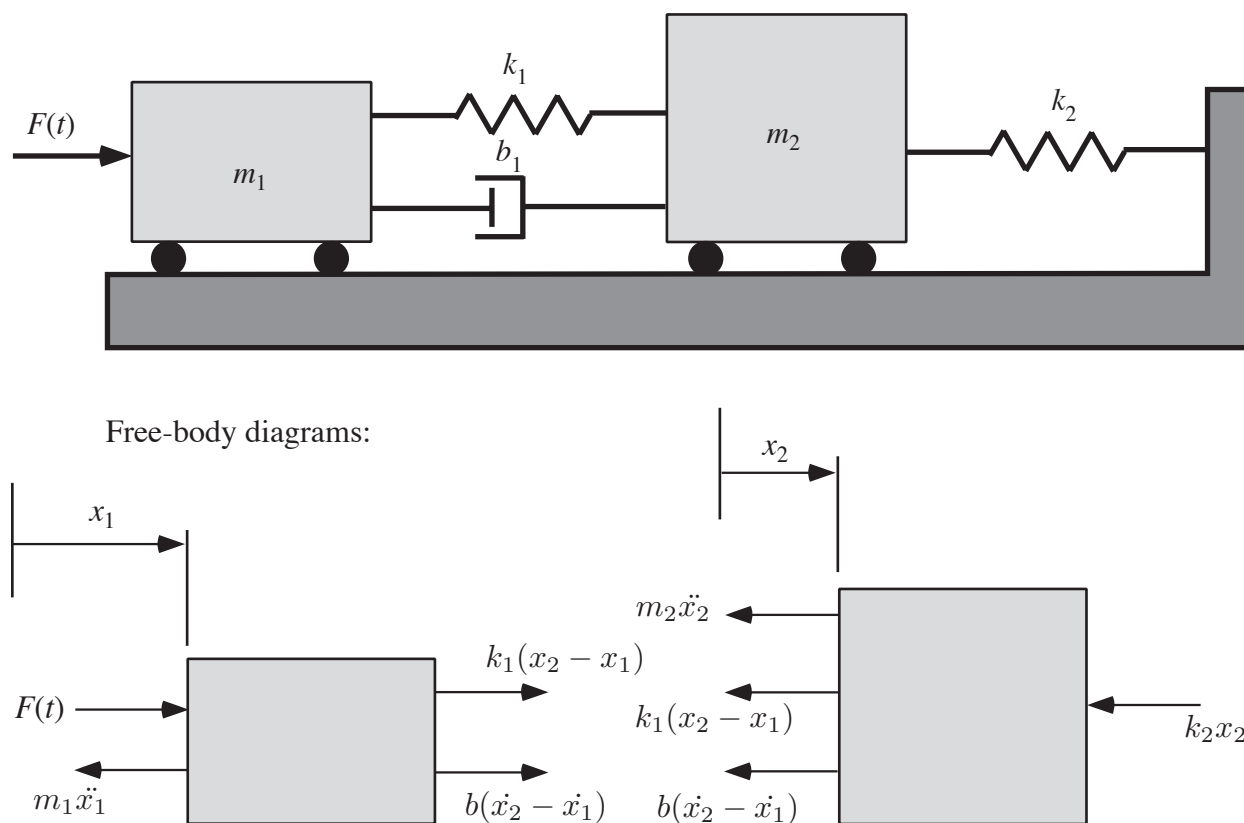


Figure 2.6: Revised free-body diagrams showing the constitutive equations drawn directly on the diagrams

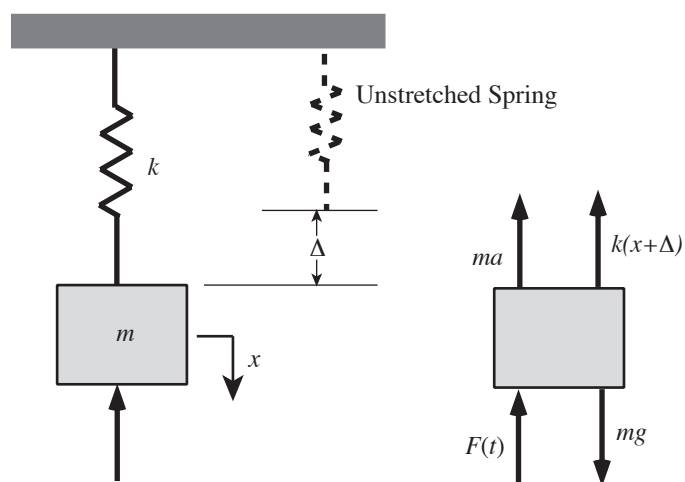


Figure 2.7: Example showing removal of gravity terms

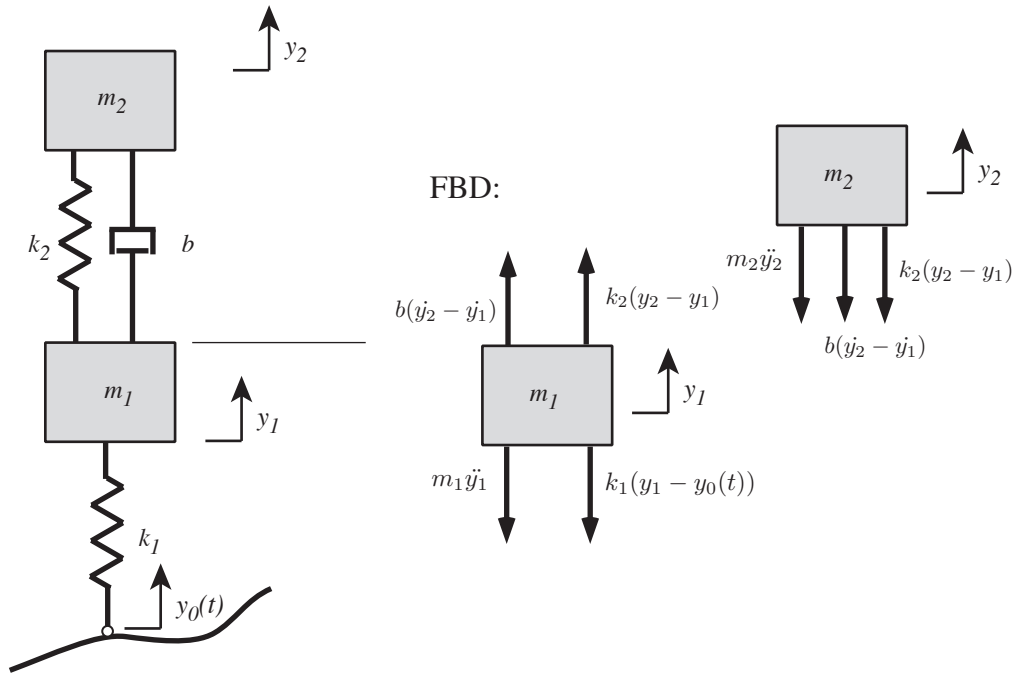


Figure 2.8: Model of a vehicle suspension system

2.1.7 Develop Kinematic Relationships

A kinematic relationship is a static equation relating displacement variables to each other. It applies in cases where motion is constrained, such as by a pin joint, gears, or a system of pulleys and cables. For systems with only translation, pulleys are a common example of components that may use kinematic relationships.

Consider the pulley system shown in Fig. 2.9. For this system, as long as the cable is inextensible and does not slip,

$$x + y = \text{constant}.$$

This equation may then be differentiated with respect to time to obtain relationships between the velocities and accelerations of the two masses:

$$\dot{x} + \dot{y} = 0 \quad \rightarrow \quad \dot{x} = -\dot{y}$$

$$\ddot{x} + \ddot{y} = 0 \quad \rightarrow \quad \ddot{x} = -\ddot{y}$$

The same method also applies to more complicated arrangements of pulleys. For example, in the system shown in Fig. 2.10a, we need to develop a kinematic relationship between x and y . In this system, if the mass m_1 goes up, the mass m_2 will go down, so that if x increases, y will also

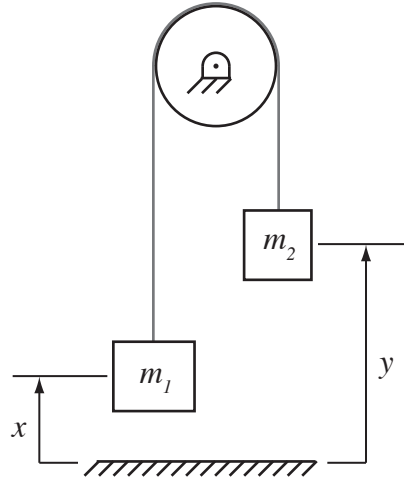


Figure 2.9: A pulley attached by an inextensible rope to two masses

increase. Also, because the cable is inextensible, any increase in x must correspond to half as much increase in y , since two parts of the cable contribute to the change in y . Hence,

$$x - 2y = \text{constant}$$

so that

$$\dot{x} = 2\dot{y}$$

$$\ddot{x} = 2\ddot{y}$$

Obtaining Force Relationships from Kinematic Relationships

In many cases, relationships between forces applied to different bodies in a system can also be obtained from kinematic relationships. This can be done by considering conservation of energy:

$$P_{in} - P_{out} = P_{loss} + \frac{dE}{dt}$$

where P_{in} is the power being put into a system, P_{out} is the power being used to do external work, P_{loss} is the power being lost by the system, and $\frac{dE}{dt}$ represents the power contributing to energy storage in the system. For the pulley system, if we assume the pulleys turn with negligible friction, then P_{loss} is zero. If we ignore the mass of the pulleys, then they cannot store energy. (Recall that we have already assumed that the cable is inextensible and does not slip.) Therefore, for the system consisting only of the cable and the pulleys,

$$P_{in} = P_{out}.$$

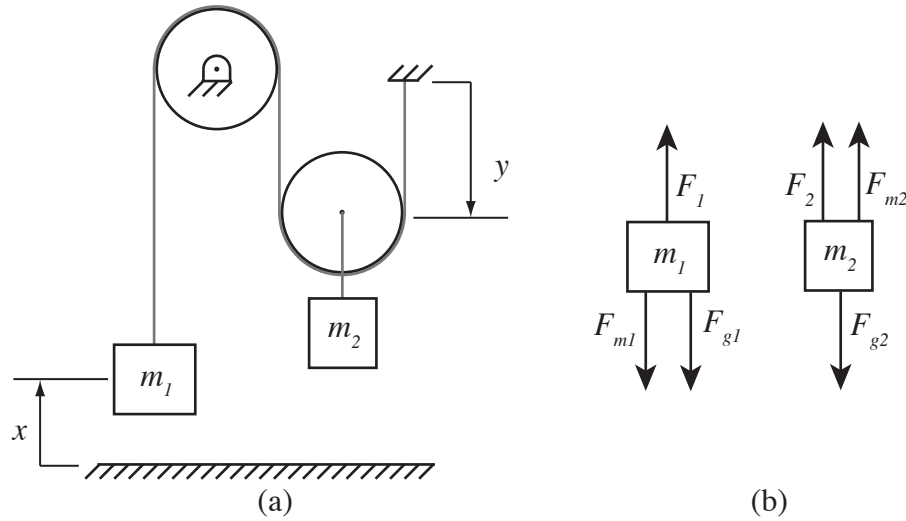


Figure 2.10: (a) A more complicated arrangement of pulleys and masses. (b) Free body diagrams for each mass.

For a translational system, power is equal to an applied force multiplied by the velocity of the body to which the force is applied.

The full free-body diagrams for masses 1 and 2 are shown in Fig. 2.10b. Notice that the inertial force is shown acting in the negative direction for each mass; because positive motion is defined in the opposite sense for masses 1 and 2, their inertial forces are also shown acting in opposite directions. From the law of conservation of energy, we can write

$$F_1 \dot{x} = F_2 \dot{y} \quad \rightarrow \quad F_1 = \frac{F_2}{2}.$$

An example requiring kinematic relations

For the system shown in Fig. 2.11, find an equation of motion to describe the system.

Answer The free-body diagrams and kinematic relations are shown in Fig. 2.11b. Writing equations from the free body diagrams gives

$$m_1 \ddot{x} + F(t) = F_1 \quad (2.4)$$

$$m_2 \ddot{y} + b \dot{y} + ky = F_2 \quad (2.5)$$

Since we are free to choose zero values for both x and y , we choose them to be zero when the spring is in static equilibrium, so that we don't need to consider any gravity forces. (Note that when both x and y are zero, the spring will be stretched by an amount equal to the force required

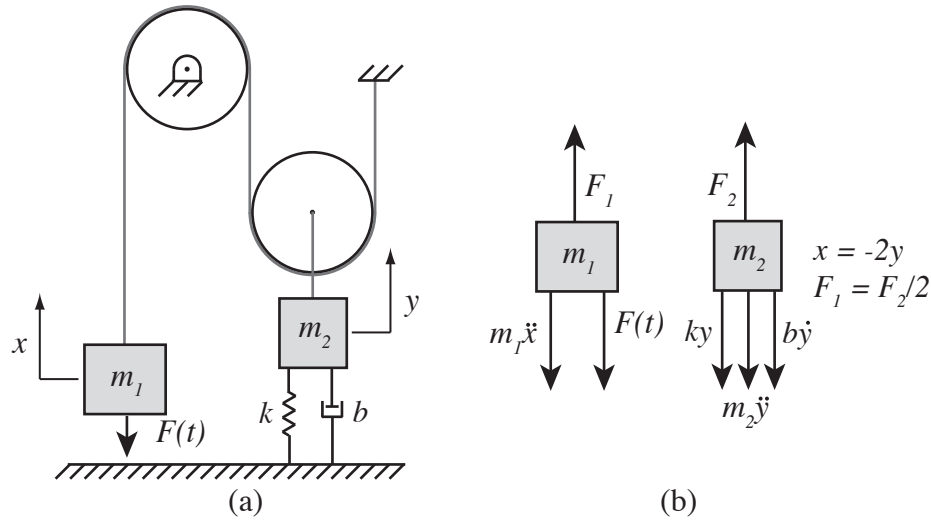


Figure 2.11: (a) The pulley system under consideration. (b) The free body diagrams and kinematic relations

to counteract gravity for both masses.) Then, with both x and y equal to zero at the same location, we find from the kinematic relation

$$x + 2y = \text{constant} = 0 \quad \rightarrow \quad x = -2y$$

and it follows that

$$\dot{x} = -2\dot{y} \quad \ddot{x} = -2\ddot{y}$$

If we assume that the mass of the pulleys is negligible, we can further write that

$$F_1 = \frac{F_2}{2} \tag{2.6}$$

We can then substitute both (2.4) and (2.5) into (2.6), giving

$$2m_1\ddot{x} + 2F(t) = m_2\ddot{y} + b\dot{y} + ky$$

or

$$(4m_1 + m_2)\ddot{y} + b\dot{y} + ky = 2F(t)$$

Note that we could also write this equation in terms of x :

$$\left(m_1 + \frac{m_2}{4}\right)\ddot{x} + \frac{b\dot{x}}{4} + \frac{kx}{4} = -F(t)$$

2.2 Rotational Motion

Equations of motion for rotational systems can be derived using the same five-step procedure as for translational systems:

1. Define geometry
2. Apply moment-balance equations
3. Develop kinematic relationships
4. Define physical relations for elements
5. Combine relations

There are slight differences in steps 2, 3, and 4.

2.2.1 Step 2 (Moment-Balance Equations)

Newton's second law for a body rotating about a fixed axis:

$$\sum M_0 = J_0 \ddot{\theta}$$

where J_0 is taken about the axis of rotation. Or, using D'Alembert's formulation:

$$\sum M_0^* = 0$$

about any axis! (Note that M_0^* includes inertial moments and moments due to inertial forces.)

In these equations, J is the *mass moment of inertia*

$$J = \int r^2 dm$$

where r is the radius from some reference axis to a differential mass dm . J for rotating systems is analogous to m for translation, with one key difference: J is different depending on the reference axis chosen. It will be a minimum for an axis taken through the center of gravity. For any other axis, the parallel axis theorem applies:

$$J = J_g + md^2$$

where J_g is the mass moment of inertia taken about an axis through the center of gravity, m is the mass of the rotating body, and d is the distance of the reference axis from the axis through the center of gravity.

2.2.2 Step 3 (Develop Kinematic Relationships)

A common kinematic element in rotating systems is gears. For a gear train, a gear ratio N exists that describes the relationship between the input and output shaft speeds:

$$\omega_{in} = N\omega_{out}$$

For many systems, N is larger than 1, meaning that the output speed is slower than the input speed. As before with pulley systems, this equation can be differentiated to give

$$\alpha_{in} = N\alpha_{out}$$

Similarly, if the gears have negligible mass and friction, then the input power equals the output power, so that

$$\tau_{in}\omega_{in} = \tau_{out}\omega_{out}$$

and

$$\tau_{out} = N\tau_{in}$$

describes the relationship between the output torque τ_{out} and the input torque τ_{in} .

2.2.3 Step 4 (Physical Relations)

Physical relations:

Springs: $T = k(\theta_2 - \theta_1)$

Dampers: $T = b(\dot{\theta}_2 - \dot{\theta}_1)$

Mass moment of inertia: $T = J\ddot{\theta}$

Example: Fluid Dynamometer

Fig. 2.12a shows a dynamometer used to measure the speed ω_1 of the input shaft. Fluid coupling between the two flywheels acts like a linear viscous damper. Develop the equations of motion.

Fig. 2.12b gives the free-body diagrams for each flywheel. From these, the equations of motion can be written

$$J_1 \dot{\omega}_1 + b(\omega_1 - \dot{\theta}_2) = T(t)$$

$$J_2 \ddot{\theta}_2 - b(\omega_1 - \dot{\theta}_2) + k\theta_2 = 0$$

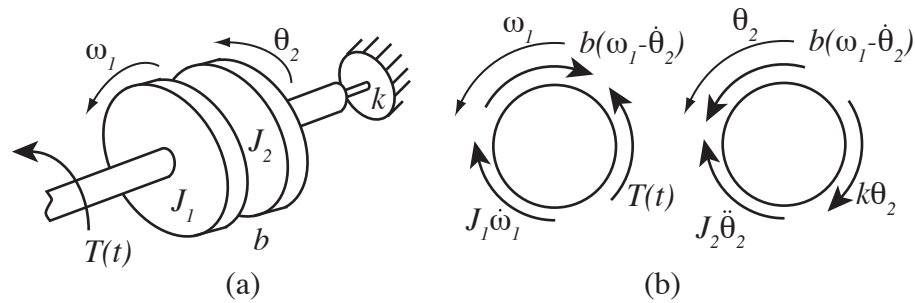


Figure 2.12: (a) A dynamometer using fluid coupling. (b) Free body diagrams showing torques on the two flywheels.

2.3 Combined Translational and Rotational Motion

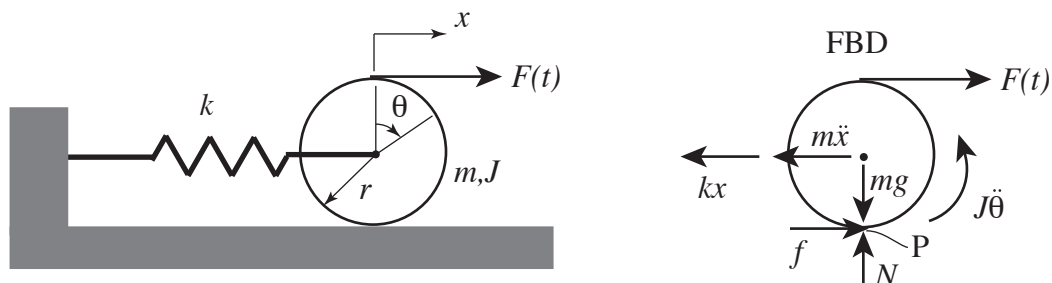
The five-step approach is still valid:

1. Define system geometry
2. Apply force and moment balance relations
3. Develop kinematic relationships
4. Define physical relations for system elements
5. Combine relations to get equations of motion

The application of this approach is, in general, more complicated since problems involving mixed motion are more complex.

The method is best demonstrated using a simple example.

Example: Rolling Wheel



From the kinematics of the problem, we find $x = r\theta$.

$$\begin{aligned} \sum M_P^* = 0 \quad \Rightarrow \quad & J\ddot{\theta} + mr\ddot{x} + kxr = 2rF(t) \\ & J\frac{\ddot{x}}{r} + mr\ddot{x} + kxr = 2rF(t) \\ & \left(\frac{J}{r^2} + m\right)\ddot{x} + kx = 2F(t) \end{aligned}$$

For a circular disk, $J = \frac{1}{2}mr^2$

$$\Rightarrow \quad \boxed{\frac{3}{2}m\ddot{x} + kx = 2F(t)}$$

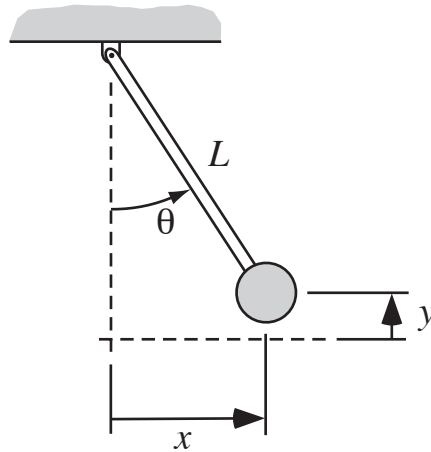
Notice that we do not need to resort to the parallel axis theorem since we have used D'Alembert's principle!

2.3.1 Kinematic Relationships for Pendulum Systems

The accelerations of a pendulum system create inertial forces that deserve further study. For the pendulum system shown below,

$$x = L \sin \theta \quad (2.7)$$

$$y = L(1 - \cos \theta) \quad (2.8)$$



Once you have developed the kinematic relationships, you can take their derivatives with re-

spect to time to arrive at relationships between velocities and accelerations. For the example above

$$\dot{x} = L\dot{\theta} \cos \theta \quad (2.9)$$

$$\dot{y} = L\dot{\theta} \sin \theta \quad (2.10)$$

$$\ddot{x} = L(\ddot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta) \quad (2.11)$$

$$\ddot{y} = L(\ddot{\theta} \sin \theta + \dot{\theta}^2 \cos \theta) \quad (2.12)$$

The acceleration equations illustrate the tangential and normal components of acceleration expected in a rotating system. These are often written

$$A_t = L\ddot{\theta} \quad (2.13)$$

$$A_n = L\dot{\theta}^2 \quad (2.14)$$

and their directions are illustrated on the drawing in Fig. 2.13.

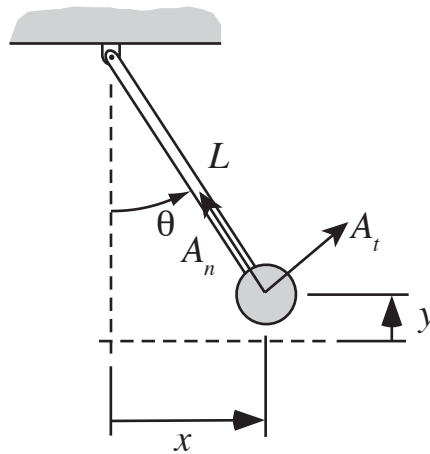


Figure 2.13: Illustration of the tangential and normal acceleration vectors of a pendulum

The use of these kinematic relationships in a pendulum system is best shown by an example.

Gantry Crane Example

Figure 2.14 shows a mass that moves horizontally on frictionless, massless rollers. It has a pendulum hanging from it with a point mass at its end. Find the equations of motion for this system.

This problem is most easily solved by using the kinematics of the problem along with D'Alembert's method. Since the motion of the pendulum is constrained by the pivot, its accelerations relative to the mass are given by (2.13) and (2.14). The free-body diagram for the pendulum is shown in

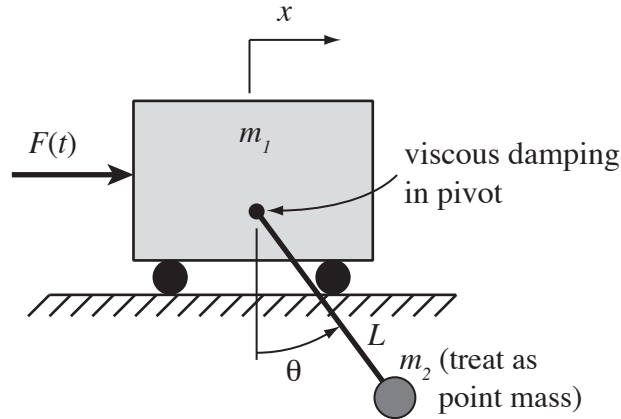


Figure 2.14: Gantry crane illustration

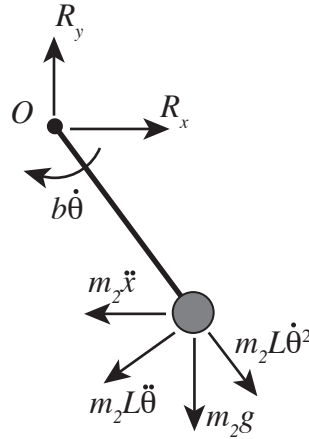


Figure 2.15: Free body diagram for the pendulum

Fig. 2.15. The forces R_x and R_y are reaction forces acting between the mass and the pendulum. Since these are unknowns, it is convenient to take moments about point O:

$$m_2\ddot{x}L\cos\theta + m_2L^2\ddot{\theta} + m_2gL\sin\theta + b\dot{\theta} = 0$$

This is one equation of motion. To get the other, we will sum the forces acting on the whole system. This is most easily done by drawing a free-body diagram showing all of the x -direction forces acting on the whole crane. This method ignores the internal reaction forces (the R 's). Fig. 2.16 shows the resulting free-body diagram. From this diagram, the second equation of motion is derived:

$$m_1\ddot{x} + m_2\ddot{x} + m_2L\ddot{\theta}\cos\theta - m_2L\dot{\theta}^2\sin\theta = F(t)$$

Solving this problem using direct application of Newton's second law would be significantly

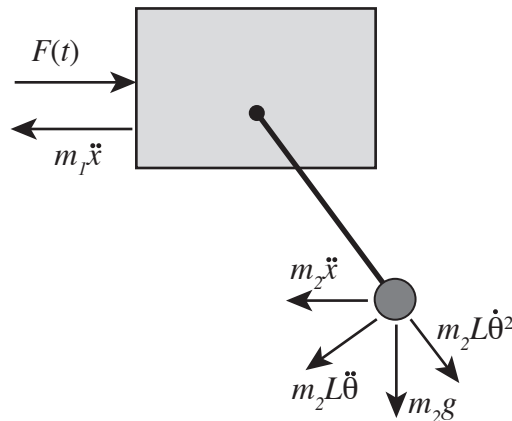


Figure 2.16: Free body diagram for the whole crane

more difficult. We would have had to solve for reaction forces in the pivot, apply $F = ma$ in multiple directions, combine relations, and so on. Using D'Alembert's method provides a significant reduction in the time required to generate the equations!

Another example

Figure 2.17 shows a cylinder supported by two springs. The input to the system is the motion of one of the springs. Develop the equations of motion. Assume that x_1 , x_2 , x , and θ are all zero when the system is in equilibrium and $x_0(t)$ is zero.

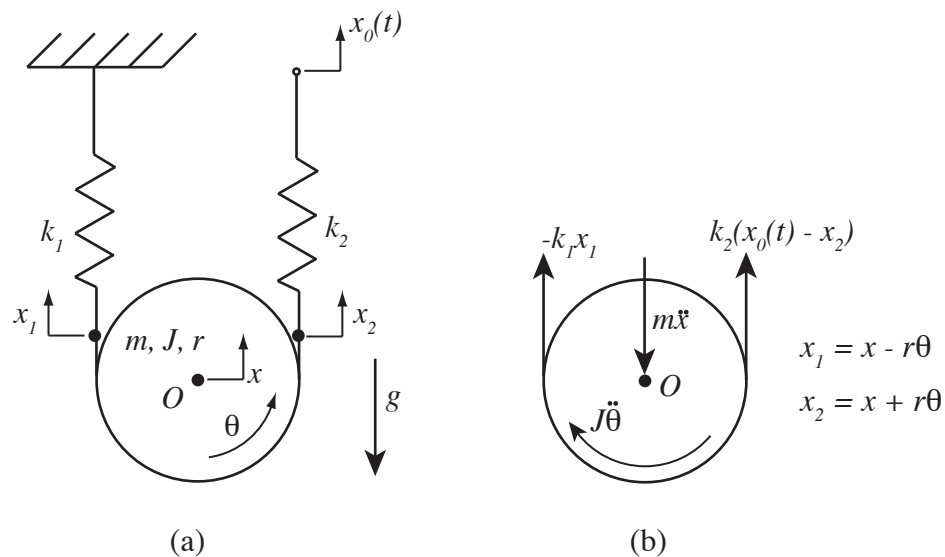


Figure 2.17: A cylinder supported by two springs

Based on the free-body diagram shown in Fig. 2.17b, we can write two equations of motion by summing forces in the y -direction and by taking moments about the center of the cylinder. First, summing forces:

$$m\ddot{x} + k_1x_1 + k_2(x_2 - x_0(t)) = 0$$

or

$$m\ddot{x} + k_1(x - r\theta) + k_2(x + r\theta - x_0(t)) = 0$$

Summing moments about point O gives

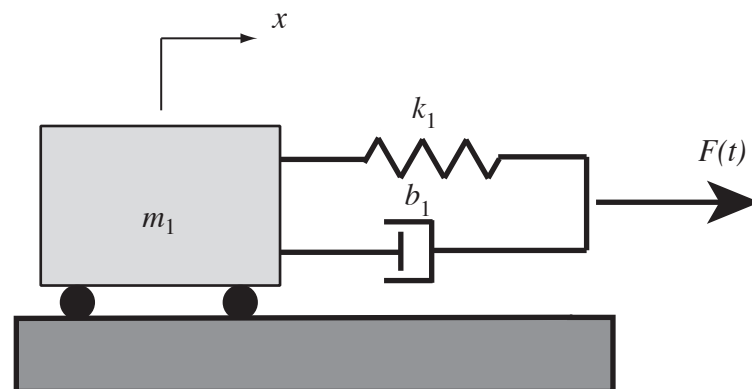
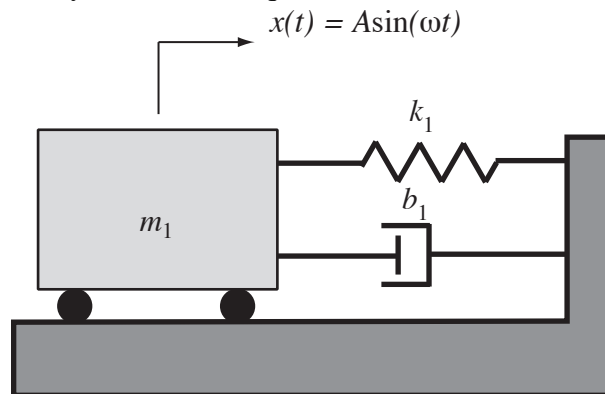
$$J\ddot{\theta} - k_1rx_1 - k_2r(x_0(t) - x_2) = 0$$

or

$$J\ddot{\theta} - k_1r(x - r\theta) - k_2r(x_0(t) - x - r\theta) = 0$$

Some Tricky Problems

Can you write the equations of motion for the two systems below?




Chapter 3

Electrical Systems

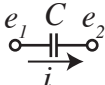
3.1 Review of Basic Elements

Recall that there are three basic elements of passive electrical systems: the resistor, the capacitor, and the inductor.

- Dissipative element: the resistor:  For a resistor with resistance R and voltages e_1 and e_2 on either side of the resistor,

$$e_1 - e_2 = Ri$$

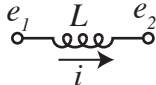
where i is the current flowing from the side at e_1 toward the side at e_2 .

- Effort storage element: the capacitor:  For a capacitor with capacitance C with current i flowing from a voltage on one side of the capacitor of e_1 to the voltage on the other side of e_2 ,

$$e_1 - e_2 = \frac{1}{C} \int i dt$$

or


$$i = C \frac{de_c}{dt} = C \frac{d}{dt}(e_1 - e_2) = Cs(e_1 - e_2)$$



- Flow storage elements: the inductor:  For an inductor of inductance L , with voltage and current defined as with the other two elements,

$$e_1 - e_2 = L \frac{di}{dt}$$

or

$$si = \frac{di}{dt} = \frac{1}{L}(e_1 - e_2)$$

Here, we have made use of the Laplace variable s , which corresponds to taking the derivative of a quantity. In addition, sources define inputs to the system. A voltage source:  supplies a


voltage $V(t)$, and a current source:  or  supplies a current $I(t)$. Either may be constant or a different specified function of time.

3.1.1 Impedance

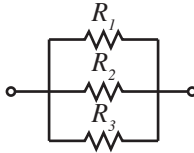
The concept of impedance allows us to represent any passive component (resistor, capacitor, or inductor) using a single variable. The impedance Z of a component is the ratio of its voltage difference to the current flowing through the component. Hence, the impedance for each type of component is

- Resistor: $Z = R$.
- Inductor: $Z = Ls$.
- Capacitor: $Z = 1/(Cs)$.

3.1.2 Components Acting in Series or Parallel

Resistors in series add their resistances directly: 

$$R_T = R_1 + R_2 + R_3$$

Resistors in parallel add by reciprocals. For three parallel resistors: 

$$\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

Capacitors in series add like resistors in parallel. For three capacitors in series:

$$\frac{1}{C_T} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$$

Capacitors in parallel add directly:

$$C_T = C_1 + C_2 + C_3$$

Inductors in Series add like resistors:

$$L_T = L_1 + L_2 + L_3$$

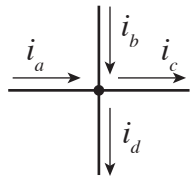
Inductors in Parallel add like resistors in parallel:

$$\frac{1}{L_T} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}$$

In general, impedances in series add by adding each impedance; impedances in parallel add using reciprocals, just like resistances.

3.2 Passive Circuit Analysis

In passive circuit analysis, Kirchoff's Current Law (KCL) is applied to nodes in the system. A *node* is the point of connection of two or more components. The component ends which are connected to a node are assumed to all have the same voltage. Kirchoff's Current Law states that the sum of all currents at a node is equal to zero:



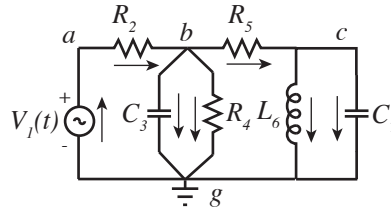
$$\text{KCL} \rightarrow i_a + i_b - i_c - i_d = 0$$

Using KCL, we can define a systematic method for deriving the equations of motion of an electrical system. The steps are:

1. Draw a schematic of the circuit. Identify each component with a unique symbolic value (e.g. R_1, R_2, C_3, L_4 , etc.). It is usually helpful to give each component a unique number.
2. Identify the nodes of the circuit and assign each one a letter to identify it.
3. For each component, assign the direction of positive current flow and indicate it on the circuit diagram. The current in the n th component will be known as i_n (e.g. the current in R_1 is i_1 , the current in C_3 is i_3 , and so forth).
4. Write an equation for the current in each component using the established physical relation for each type of component (e.g. $i_1 = (e_a - e_b)/R_1$, $i_3 = C_3 se_3$, $si_4 = \frac{1}{L_4} e_4$).

5. Write node equations using KCL for each node that is not directly connected to a voltage source.
6. Define state variables to be capacitor voltages and inductor currents. Write component equations for capacitors and inductors in state-variable form. Use the remaining component equations and node equations to reduce the differential equations so that they contain only the state variables and inputs.

Example 1



Component relations:

$$i_2 = \frac{e_a - e_b}{R_2}$$

$$i_4 = \frac{e_b - 0}{R_4} = \frac{e_b}{R_4}$$

$$i_5 = \frac{e_b - e_c}{R_5}$$

$$i_3 = C_3 s e_3$$

$$i_7 = C_7 s e_7$$

$$s i_6 = \frac{1}{L_6} e_c$$

$$e_3 = e_b$$

$$e_7 = e_c$$

Node equations:

$$\textcircled{b} \quad i_2 = i_3 + i_4 + i_5$$

$$\textcircled{c} \quad i_5 = i_6 + i_7$$

State variables: e_3, e_7, i_6 Inputs: $V_1(t)$

We proceed by developing the state variable equations:

$$\begin{aligned} s e_3 &= \frac{1}{C_3} i_3 \\ &= \frac{1}{C_3} [i_2 - i_4 - i_5] \\ &= \frac{1}{C_3} \left[\frac{e_a - e_b}{R_2} - \frac{e_b}{R_4} - \frac{e_b - e_c}{R_5} \right] \end{aligned}$$

But

$$e_b = e_3$$

$$e_c = e_7$$

$$e_a = V_1(t)$$

So

$$se_3 = \frac{1}{C_3} \left[\frac{V_1(t)}{R_2} - \left(\frac{1}{R_2} + \frac{1}{R_4} + \frac{1}{R_5} \right) e_3 + \frac{1}{R_5} e_7 \right]$$

Similarly,

$$\begin{aligned} se_7 &= \frac{1}{C_7} i_7 \\ &= \frac{1}{C_7} (i_5 - i_6) \\ &= \frac{1}{C_7} \left[\frac{e_b - e_c}{R_5} - i_6 \right] \end{aligned}$$

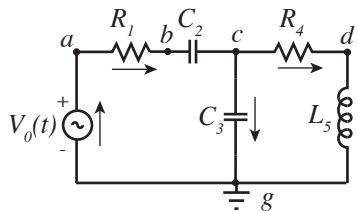
$$se_7 = \frac{1}{C_7} \left[\frac{e_3}{R_5} - \frac{e_7}{R_5} - i_6 \right]$$

And

$$si_6 = \frac{e_c}{L_6}$$

$$si_6 = \frac{e_7}{L_6}$$

Example 2



Component relations:

$$i_1 = \frac{e_a - e_b}{R_1}$$

$$i_4 = \frac{e_c - e_d}{R_4}$$

$$i_2 = C_2 se_2$$

$$i_3 = C_3 se_3$$

$$i_5 = \frac{1}{L_5 s} e_d$$

$$e_2 = e_b - e_c$$

$$e_c = e_3$$

$$e_b = e_2 + e_3$$

Node equations:

⑥ $i_1 = i_2$

© $i_2 = i_3 + i_4$

④ $i_4 = i_5$

State variables: e_2, e_3, i_5 Inputs: $V_0(t)$

We proceed by developing the state variable equations:

$$\begin{aligned} se_2 &= \frac{1}{C_2} i_2 \\ &= \frac{1}{C_2} i_1 \\ &= \frac{1}{C_2} \left(\frac{e_a - e_b}{R_1} \right) \end{aligned}$$

$$se_2 = \frac{1}{R_1 C_2} [V_0(t) - e_2 - e_3]$$

$$\begin{aligned} se_3 &= \frac{1}{C_3} i_3 \\ &= \frac{1}{C_3} (i_2 - i_4) \\ &= \frac{1}{C_3} (i_1 - i_5) \end{aligned}$$

$$se_3 = \frac{1}{C_3} \left[\frac{1}{R_1} (V_0(t) - e_2 - e_3) - i_5 \right]$$

$$si_5 = \frac{1}{L_5} e_d$$

Also,

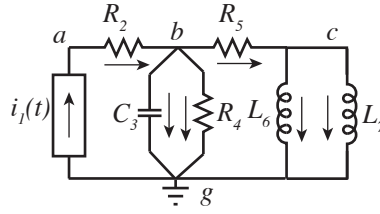
$$e_c - e_d = R_4 i_4$$

$$e_d = e_c - R_4 i_4$$

$$e_d = e_3 - R_4 i_5$$

$$si_5 = \frac{1}{L_5} (e_3 - R_4 i_5)$$

Example 3



This one is tricky. Note that the two inductors are in parallel with each other - in effect, they act like a single inductor with equivalent inductance $L_{eq} = \frac{L_6 L_7}{L_6 + L_7}$. As a result, i_6 and i_7 are not independent, so that the two inductors together give one state variable. This also means that applying KCL at node c gives

$$i_5 = i_{eq}$$

where i_{eq} is the equivalent current passing through the equivalent inductor.

Hence, for component relations we get

$$\begin{aligned} i_2 &= \frac{e_a - e_b}{R_2} & i_4 &= \frac{e_b}{R_4} & i_5 &= \frac{e_b - e_c}{R_5} \\ i_3 &= C_3 s e_3 & i_{eq} &= \frac{e_c}{s L_{eq}} & e_3 &= e_b \end{aligned}$$

and the node equations are

- Ⓐ $i_1(t) = i_2$
- Ⓑ $i_2 = i_3 + i_4 + i_5$
- Ⓒ $i_5 = i_{eq}$

The state variables are e_3 and i_{eq} , and the input is $i_1(t)$.

To get the state variable equations, we write

$$s e_3 = \frac{i_e}{C_3} = \frac{1}{C_3} (i_2 - i_4 - i_5) = \frac{1}{C_3} \left(i_1(t) - \frac{e_b}{R_4} - i_{eq} \right)$$

$$s e_3 = \frac{1}{C_3} \left(i_1(t) - \frac{e_3}{R_4} - i_{eq} \right)$$

and

$$s i_{eq} = \frac{1}{L_{eq}} e_c = \frac{1}{L_{eq}} (e_b - i_5 R_5)$$

$$s i_{eq} = \frac{1}{L_{eq}} (e_3 - i_{eq} R_5)$$

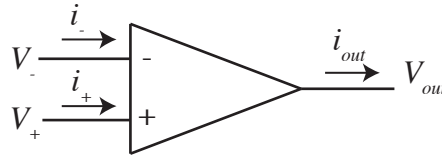


Figure 3.1: Schematic of an op-amp

3.3 Active Circuit Analysis

Passive circuits contain only passive components — resistors, capacitors, inductors, and sources. Active circuits employ these components as well as amplifying devices. We will confine our attention to operational amplifiers and how to analyze circuits that use them.

3.3.1 Operational Amplifiers — the Ideal Model

Figure 3.1 shows a schematic diagram of an operational amplifier, or op-amp. For analyzing op-amps, we make three assumptions:

1. Both inputs have infinite input impedance. Practically speaking, this means that

$$i_- = i_+ = 0$$

and the op-amp does not draw current from any input.

2. The op-amp has infinite gain. This means that, for

$$G(V_+ - V_-) = V_{out}$$

the gain G is infinite. If the op-amp is wired using feedback (meaning that the output voltage is connected somehow to either of the inputs), then this implies that

$$V_+ - V_- = 0 \quad \rightarrow \quad V_+ = V_-$$

3. The op-amp has zero output impedance. Hence, the output voltage does not depend at all on the output current. Any devices that draw current from the output will have no effect on the value of V_{out} . (In other words, for a device using feedback, rules 1 and 2 can be used to find V_{out} . The value of i_{out} will then depend on whatever devices are attached to the output.)

Each of these assumptions is an approximation for true op-amp behavior, but as long as the op-amp is operated within reasonable limits, the approximations are very good (for example, most op-amps cannot provide more than about 10 mA of output current without reducing V_{out}).

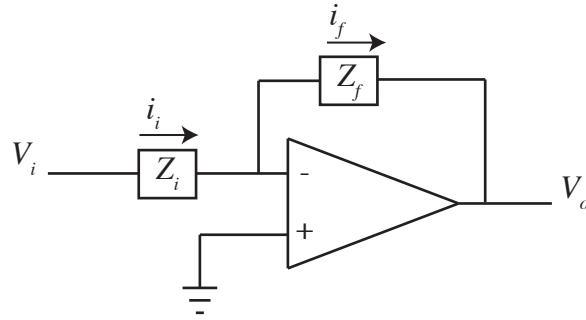


Figure 3.2: A very common op-amp circuit

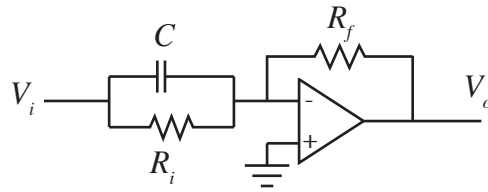


Figure 3.3: A lead filter

One of the most common types of op-amp circuits is illustrated in Fig. 3.2. In this type of circuit, the non-inverting input is grounded, and there are two impedances in the circuit: an input impedance Z_i and a feedback impedance Z_f .

From assumption 1, we know that $i_+ = i_- = 0$, so that $i_i = i_f$. From assumption 2, we know that $V_+ = V_- = 0$ because the non-inverting input is grounded. The component relations are then

$$i_i = \frac{V_i}{Z_i} \qquad i_f = \frac{-V_o}{Z_f}$$

and combining gives

$$\frac{V_i}{Z_i} = -\frac{V_o}{Z_f}$$

$$\boxed{V_o = -\frac{Z_f}{Z_i} V_i}$$

Example — Lead Filter

For the lead filter circuit shown in Fig. 3.3, find the transfer function V_o/V_i .

The input impedance is found by adding impedances in parallel:

$$Z_i = \frac{Z_C Z_R}{Z_C + Z_R} = \frac{\frac{1}{C_s} R_i}{\frac{1}{C_s} + R_i}$$

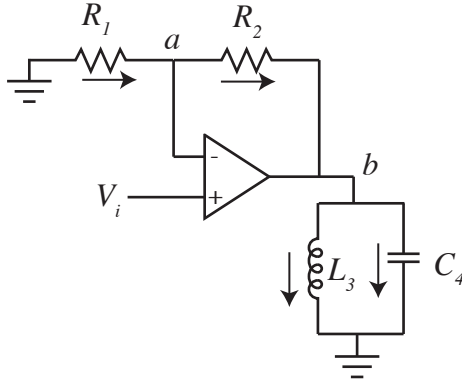


Figure 3.4: A circuit including a noninverting amplifier

$$Z_i = \frac{R_i}{R_i Cs + 1}$$

The feedback impedance is simply $Z_f = R_f$. Then

$$V_o = -\frac{Z_f}{Z_i} V_i = -\frac{R_f}{R_i} (R_i Cs + 1) V_i$$

$$\boxed{\frac{V_o}{V_i} = -\frac{R_f}{R_i} (R_i Cs + 1)}$$

However, not all op-amp circuits are so simple. The non-inverting input is not always tied to ground. In this case, application of the three rules will lead to a solution.

Example — Non-Inverting Amplifier

For the circuit shown in Fig. 3.4, find the transfer function i_3/V_i . Also, find a state variable representation for the system.

To solve this problem, we will apply the three op-amp rules. By the first rule, $i_+ = i_- = 0$. Hence, KCL applied at node a gives

$$i_1 = i_2$$

By the second rule,

$$V_a = V_i$$

We also write the component relations for each component:

$$\begin{aligned} i_1 &= \frac{-V_i}{R_1} & i_2 &= \frac{V_i - V_b}{R_2} \\ si_3 &= \frac{V_b}{L_3} & i_4 &= C_4 s V_b \end{aligned}$$

Note that we do not apply KCL at node b because we do not know the output current from the op-amp.

Since $i_1 = i_2$, we set these two component relations equal to each other:

$$-\frac{V_i}{R_1} = \frac{V_i - V_b}{R_2}$$

so

$$V_b = \left(1 + \frac{R_2}{R_1}\right) V_i$$

Therefore, by the third component relation,

$$si_3 = \frac{1}{L_3} \left(1 + \frac{R_2}{R_1}\right) V_i$$

$$\boxed{\frac{i_3}{V_i} = \frac{1}{L_3 s} \left(1 + \frac{R_2}{R_1}\right) = \frac{R_1 + R_2}{L_3 R_1 s}}$$

For state variable form, we first define the state variables. Since there is an inductor and a capacitor in the system, it appears that we require two state variables. However, this problem is tricky. Since the voltage on the capacitor is V_b , and V_b is a static function of V_i , we cannot treat the capacitor voltage as a state variable. Essentially, the op-amp will supply whatever current is necessary to satisfy the relation $V_b = (1 + R_2/R_1)V_i$, making the capacitor voltage a static function of the input voltage. Therefore, the only state variable is the current i_3 through the inductor. We can rewrite the transfer function above to give

$$\boxed{si_3 = \frac{R_1 + R_2}{L_3 R_1} V_i}$$

Note that we can use the component relationship for the capacitor to find the capacitor current:

$$i_4 = C_4 \left(1 + \frac{R_2}{R_1}\right) sV_i$$

and applying KCL at node b then gives the current supplied by the op-amp

$$i_2 + i_o = i_3 + i_4$$

$$i_o = i_3 + i_4 - i_2$$

$$i_o = \left[\frac{R_1 + R_2}{L_3 R_1 s} + \frac{C_4 s (R_1 + R_2)}{R_1} - \frac{1}{R_2} + \frac{R_1 + R_2}{R_1 R_2} \right] V_i$$

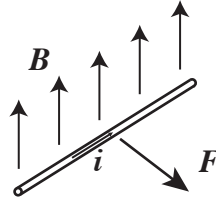


Figure 3.5: Force created when current flows through a wire in a magnetic field

3.4 Electromechanical Systems

Electromechanical systems are composed of electrical and mechanical components. They can be modeled by applying what we already know about electrical circuits and mechanical dynamics. There are many types of electromechanical systems, including many different types of motors, servos, generators, and linear motors (speakers). Here, we'll focus on DC motors and speakers, which have many similarities.

A critical point in modeling multi-disciplinary systems is understanding how power is transferred between energy domains. Motors typically involve wires moving through magnetic fields. Recall that the force \mathbf{F} on a wire of length l in a magnetic field of strength \mathbf{B} is

$$\mathbf{F} = i\mathbf{l} \times \mathbf{B}$$

where i is the current passing through the wire. Note that bold face is used to represent a vector; this is a vector equation. Figure 3.5 shows the wire, magnetic field, current flow, and resulting force.

Similarly, if a wire travels with a velocity v normal to a magnetic field, a voltage difference will be induced between the two ends of the wire, given by

$$V_i = Blv$$

Most electromechanical systems, including motors and generators, function under these simple principles.

3.4.1 DC Motors

The dynamic behavior of a DC motor can be modeled by considering the electrical and mechanical dynamics of the system and how power is transferred between the electrical and mechanical domains. Figure 3.6 depicts a schematic diagram model of a voltage-driven DC motor. The coils of the motor are modeled as having resistance and inductance. The back-emf voltage drop in the

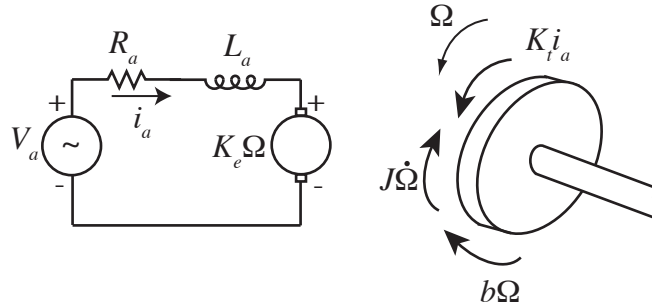


Figure 3.6: Schematic diagram of a DC motor

motor coils is also modeled. The back-emf voltage is the voltage induced as the coils move through the magnetic field and is equal to $K_e \Omega$. K_e is called the back-emf constant and has units of V·s or V/krpm.

The mechanical portion of the motor model includes the inertia of the motor rotor (the coils, shaft, and so on), as well as friction and the torque produced by the motor. The motor torque is equal to $K_t i_a$, where i_a is the armature current flowing through the motor. K_t is called the motor torque constant and has units of N·m/A.

Torque and Back-emf Constants

The torque and back-emf constants model the transduction of power between the electrical and mechanical domains. If SI units are used, the numerical values of K_t and K_e will be the same:

$$1 \text{ V} \cdot \text{s} = 1 \frac{\text{J}}{\text{C}} \cdot \text{s} = 1 \text{ N} \cdot \text{m/A}.$$

Notice that the torque constant converts the flow variable (i_a) in the electrical domain to the effort variable (τ) in the mechanical domain. Similarly, the back-emf constant converts the flow variable (Ω) in the mechanical domain to the effort variable (V_e) in the electrical domain.

Notice that power is conserved in the transduction process:

$$\text{power} = \text{effort} \times \text{flow}$$

Electrical

$$P_e = V_e i$$

Mechanical

$$P_m = \tau \Omega$$

$$\tau = K_t i$$

$$V_e = K_e \Omega \rightarrow \Omega = \frac{V_e}{K_e}$$

$$P_m = K_t i \frac{V_e}{K_e}$$

$$P_m = \frac{K_t}{K_e} P_e$$

With consistent units, $K_t = K_e$; therefore

$$P_m = P_e$$

and power is conserved.

DC Motor Model

Summing voltages about the loop in the circuit schematic shown in Fig. 3.6 gives

$$V_a(t) - R_a i_a - L_a \dot{i}_a - K_e \Omega = 0$$

$$\frac{di_a}{dt} = \frac{1}{L_a} [V_a(t) - R_a i_a - K_e \Omega]$$

and summing torques on the motor armature gives

$$J\dot{\Omega} + b\Omega - K_t i_a = 0$$

$$\dot{\Omega} = \frac{K_t}{J} i_a - \frac{b}{J} \Omega$$

Hence, a DC motor powered by a voltage source is a second-order system. Depending on the model parameters, it may be underdamped, critically damped, or overdamped.

DC Motor Powered by a Current Source

If a DC motor is driven by a current source rather than a voltage source, then $i_a(t)$ is the input, rather than a state variable. In this case, only one state variable equation is needed to model the motor:

$$\dot{\Omega} = \frac{K_t}{J} i_a - \frac{b}{J} \Omega$$

Effectively, the electrical dynamics are eliminated by using the current source.

3.4.2 Linear Motors

Linear motors behave much like DC motors, except that they move in a line instead of rotating. An extremely common example of a linear motor is a sound speaker. The model for a linear motor is shown in Fig. 3.7. The electrical part of the system consists of a coil, with resistance and inductance. When current flows through it, it creates a magnetic field that produces a force on the mass equal to $K_f i_a$, where K_f is the force constant. The mass is suspended by a spring and damper;

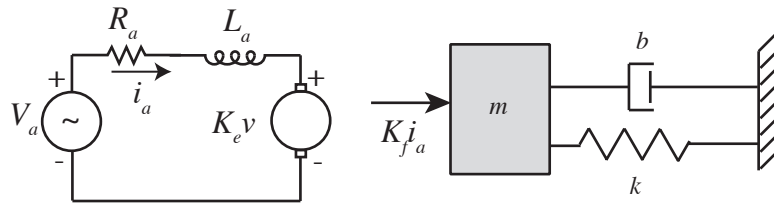


Figure 3.7: Schematic diagram of a linear motor

as it moves, it creates a back-emf equal to $K_e v$, where v is the velocity of the mass. Hence, the equations of motion for the linear motor are

$$\frac{di_a}{dt} = \frac{1}{L_a} [V_a(t) - R_a i_a - K_e v]$$

$$\dot{v} = \frac{K_f}{m} i_a - \frac{b}{m} v - \frac{k}{m} x$$

$$\dot{x} = v$$

As before, K_f and K_e will be equal to each other if they use consistent units. Because of the addition of the spring, this system is third order.

Chapter 4

Fluid and Thermal Systems

4.1 Fluid Systems

In mechanical and electrical systems, we saw that there were three basic types of components for dynamic systems:

- Resistance — friction, damping, or electrical resistors. Effort = $R \times \text{Flow}$
- Capacitance — springs, mechanical stiffness, or electrical capacitors. Effort = $1/C \times \int \text{Flow } dt$
- Inertia — mass, mass moment of inertia, or electrical inductance. Effort = $I \times \frac{d}{dt} \text{Flow}$

These same dynamic elements exist for models of fluid systems. As discussed in Chapter 1, the effort variable for fluid systems is pressure, and the flow variable is volume flow rate. These variables work very well for hydraulic systems. However, for pneumatic systems, mass flow rate is a more meaningful flow variable. Since your book uses mass flow rate for both hydraulic and pneumatic systems, and since mass and volume flow rates are directly related to each other for hydraulic systems ($\rho q_v = q_m$), we will substitute mass flow rate for volume flow rate here.

4.1.1 Fluid Resistance

Fluid resistance can take a variety of forms and come from a variety of sources. As in electrical and mechanical systems, fluid resistance is modeled by a static relationship between effort and flow variables. Fluid resistance dissipates energy, normally as heat. No energy is stored due to fluid resistance.

Examples of fluid resistors include long tubes or pipes, orifices, and valves. In each of these cases, the pressure drop across the device is related to the flow through the device.

Long Tubes and Pipes

Laminar flow occurs when the Reynolds number (R_e) is less than approximately 2000. The Reynolds number is given by

$$R_e = \frac{\rho v d}{\mu}$$

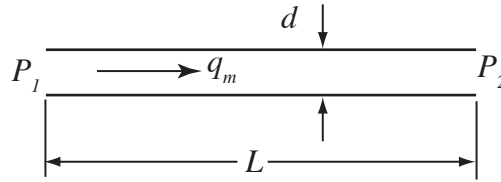
where μ is the viscosity of the fluid, ρ is the fluid density, v is the average velocity, and d is a characteristic length (diameter for circular pipes). For flow in pipes with circular cross section of area A , the average velocity is given by $v = q_v/A = 4q_v/(\pi d^2)$, so the Reynolds number reduces to

$$R_e = \frac{4\rho q_v}{\pi d \mu} = \frac{4q_m}{\pi d \mu}$$

For laminar flow, there is a linear relationship between pressure drop and flow rate. For a circular pipe, this relationship is

$$P_1 - P_2 = \frac{128\mu L}{\pi \rho d^4} q_m$$

where L is the length of the pipe, d is the internal diameter of the pipe, and q_m is the mass flow rate.



Turbulent flow occurs when R_e is greater than about 2000. Turbulent flow is characterized by a nonlinear relationship between pressure and flow rate. This nonlinear relationship is typically dependent on R_e .

$$P_1 - P_2 = f \frac{8L}{\pi^2 d^5 \rho} q_m |q_m|$$

where f is the friction factor, which is a function of R_e . Note that R_e is a function of flow rate, so that it can be quite challenging to solve problems in turbulent flow. Note that the absolute value allows the flow rate to be squared, while still considering the direction of flow.

Orifice Flow

An orifice restricts flow by reducing the area through which fluid may flow. For an orifice with cross-sectional area A_0 ,

$$P_1 - P_2 = \frac{1}{2\rho C_d^2 A_0^2} q_m |q_m|$$

where C_d is the discharge coefficient, equal to 0.62 for round, sharp-edged orifices. Practically speaking, C_d is normally experimentally determined.

$$P_1 \xrightarrow{\quad q_m \quad} P_2$$

Note that this equation can also be solved for q_m :

$$q_m = \sqrt{2\rho C_d^2 A_0^2} \sqrt{|P_1 - P_2|} \text{sign}(P_1 - P_2)$$

Valves

$$P_1 \xrightarrow{\quad q_m \quad} P_2$$

↑
 x

A valve acts much like an orifice, except that the size and shape of the orifice are controllable. If the valve input variable is x ,

$$P_1 - P_2 = \frac{1}{2\rho C_d^2(x) A^2(x)} q_m |q_m|$$

4.1.2 Fluid Capacitance

Fluid capacitance is modeled by a relationship between the derivative of the effort and the flow. Examples of fluid capacitance are tanks (like a water tower), fluid compressibility, fluid line compliance, and accumulators.

Tanks

The pressure at the bottom of a tank is given by $P = \rho gh$, where g is the acceleration due to gravity, and h is the height of the fluid in the tank. Taking the derivative of this expression gives

$$\frac{dP}{dt} = \rho g \frac{dh}{dt}$$

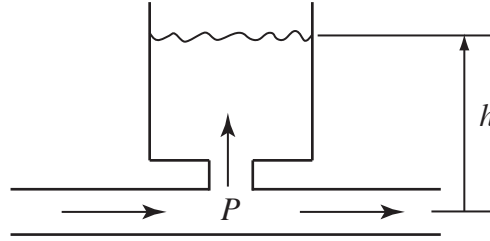
but the derivative of the height is simply the volume flow rate divided by the cross-sectional area of the tank (which may be a function of fluid height). However, for a cylindrical tank, the area A is constant, giving

$$\frac{dh}{dt} = \frac{q_m}{\rho A}$$

and so

$$\frac{dP}{dt} = \frac{g}{A} q_m = \frac{1}{C} q_m$$

and the capacitance C is simply A/g .



Fluid Compressibility

For hydraulic systems (using liquids), a bulk modulus β is defined as

$$\frac{dP}{dt} = -\frac{\beta}{V_0} \frac{dV}{dt}$$

where V is the volume of the fluid and V_0 is a reference volume. Note that this equation applies to cases with a constant amount of fluid being compressed or expanded. It is negative because increasing volume leads to decreasing pressure. For the case where fluid is compressed or expanded by flowing into or out of a constant volume,

$$\frac{dP}{dt} = \frac{\beta}{\rho V_0} q_m \quad (4.1)$$

and the capacitance C is $\rho V_0 / \beta$. In this case, there is a positive relationship because fluid flowing into the volume causes the pressure to rise. (Note that the bulk modulus is typically quite large for liquids. Water, for example, has a bulk modulus of about 2.15×10^9 N/m².)

For pneumatic systems, the relationship between pressure and volume can be quite nonlinear. For this class, we will assume that (4.1) applies for very small changes in pressure. (This is known as a linearizing assumption — any relationship can be made to appear linear if the domain is small enough.)

Fluid Line Compliance

Some types of fluid lines, such as rubber tubing or plastic pipe, can have significant wall compliance (think about a water weenie). In such a case,

$$\Delta V = \frac{2\pi l r_0^3}{E t_w} \Delta P$$

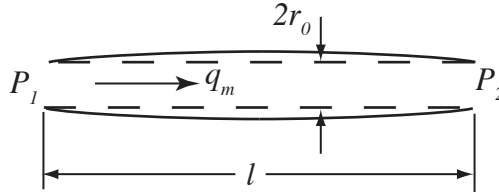
where l is the length of the line, r_0 is its nominal radius (before being stretched), E is the Young's modulus of the pipe material, and t_w is its wall thickness. Note that this equation assumes that t_w is quite small compared to r_0 . Taking the derivative gives

$$\frac{dV}{dt} = \frac{2\pi l r_0^3}{E t_w} \frac{dP}{dt}$$

and rearranging gives

$$\frac{dP}{dt} = \frac{Et_w}{2\pi l r_0^3} \frac{dV}{dt} = \frac{Et_w}{2\pi \rho l r_0^3} q_m$$

and so $C = 2\pi \rho l r_0^3 / (Et_w)$.



Accumulators

A spring-type accumulator works by having fluid push against a spring-loaded piston to store energy. If the mass of the piston can be neglected, then we can write

$$P = \frac{F}{A} = \frac{kx}{A}$$

where F is the force applied to the piston, A is the cross-sectional area of the piston, k is the spring constant, and x is the spring deflection. The total volume change may be written

$$V_2 - V_0 = Ax$$

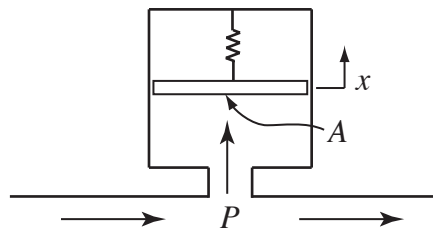
so

$$P = \frac{k(V_2 - V_0)}{A^2}$$

and taking the derivative gives

$$\frac{dP}{dt} = \frac{k}{A^2} \frac{dV}{dt} = \frac{k}{\rho A^2} q_m$$

so $C = \rho A^2 / k$.



4.1.3 Fluid Inertia

For fluid flowing through a pipe, Newton's second law holds:

$$\sum F_{ext} = m\dot{v}$$

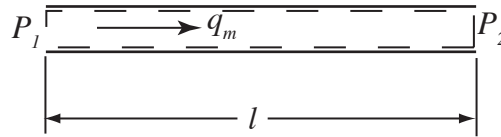
$$P_1A - P_2A = m\dot{v}$$

but $v = q_m/(\rho A)$ so

$$P_1 - P_2 = \frac{\rho A l}{A} \frac{\dot{q}_m}{\rho A}$$

$$\Delta P = \frac{l}{A} \dot{q}_m.$$

Therefore, the fluid inertia is $I = l/A$. Hence, fluid inertia is most significant in long, slender tubes.



4.1.4 Continuity Equation for a Control Volume

For a control volume, fluid mass must be conserved. This means

$$q_{m,net} = \frac{d}{dt}m_{cv} = \frac{d}{dt}(\rho V).$$

Essentially, this equation says that the total fluid mass flowing into or out of the control volume must equal the change in mass within the control volume. This equation can be expanded as

$$q_{m,in} - q_{m,out} = \rho\dot{V} + \dot{\rho}V$$

The first term on the right side of the equation describes mass change due to a change in volume of the control volume, perhaps due to motion of a piston, fluid height change in a gravity tank, line compliance, or other motion. The second term describes mass change due to compression of the fluid. Recall that fluid compressibility for constant volume may be described using the equation

$$\dot{P} = \frac{\beta}{\rho V_0} \frac{dm}{dt}$$

where V_0 is a reference volume. But, for constant volume,

$$\frac{dm}{dt} = \dot{\rho}V$$

so

$$\dot{\rho}V = \frac{\rho V_0}{\beta} \dot{P}$$

and

$$q_{m,in} - q_{m,out} = \rho \dot{V} + \frac{\rho V_0}{\beta} \dot{P}$$

This is the form of the continuity equation we will use to develop equations of motion for fluid systems.

4.1.5 Analyzing Fluid Systems

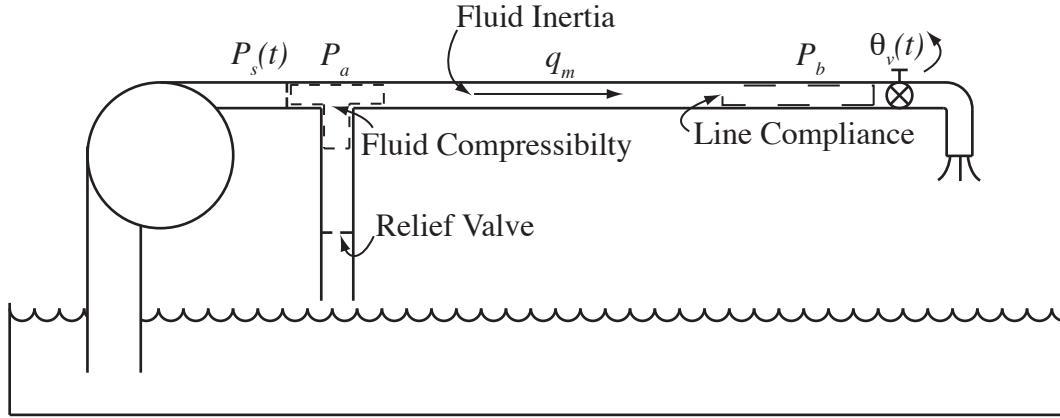
We will apply the following process to analyze fluid systems.

1. Define distinct pressure nodes.
2. Establish control volumes around pressure nodes.
3. Write the continuity equation for each pressure node.
4. Define physical relations for the \dot{V} terms in the continuity equations.
5. Model pressure drops due to fluid resistances or inertias between pressure nodes. Write physical relations to model these pressure drops.
6. Model mechanical portions of the system by drawing free-body diagrams and applying Newton's second law. Establish relations that describe power transfer between fluid and mechanical domains.
7. Combine relations to give the equations of motion in state-variable form. Compliances (capacitances) will yield a pressure state, while inertias will give a flow state.

Several examples show how to apply the process.

Example

The figure below shows a water distribution system for delivering water to a faucet. Develop the equations of motion.



1. Pressure nodes are assigned as shown in the figure. Three nodes are assigned: P_s , P_a , and P_b .
2. The control volumes are also shown.

3. Continuity equations:

$$\textcircled{a}: q_{in,a} - q_{out,a} = \rho \dot{V}_a + \frac{\rho V_0}{\beta} \dot{P}_a$$

$$\textcircled{b}: q_{in,b} - q_{out,b} = \rho \dot{V}_b + \frac{\rho V_0}{\beta} \dot{P}_b$$

4. $\dot{V}_a = 0$, $\dot{V}_b = \frac{2\pi l r_o^3}{E t_w} \dot{P}_b$. Also, since β is large for a liquid, we can ignore the second term on the right-hand side in Equation \textcircled{b} .

5. Pressure drops:

$$q_{in,a} = k_1 \sqrt{|P_s(t) - P_a|} \text{sign}(P_s(t) - P_a)$$

where

$$k_1 = \sqrt{2\rho C_{d,1}^2 A_1^2}$$

Also,

$$q_{out,a} = q_m + k_3 \sqrt{|P_a|} \text{sign}(P_a)$$

$$q_{in,b} = q_m$$

$$q_{out,b} = k_4(\theta_v) \sqrt{|P_b|} \text{sign}(P_b)$$

Here, k_3 and k_4 are defined similarly to k_1 . Also,

$$q_m = \frac{A}{l} (P_a - P_b)$$

6. There are no mechanical elements to model in this system.

7. Combine relations:

$$\dot{P}_a = \frac{\beta}{\rho V_0} (q_{in,a} - q_{out,a})$$

$$\dot{P}_a = \frac{\beta}{\rho V_0} [k_1 \sqrt{|P_s(t) - P_a|} \text{sign}(P_s(t) - P_a) - q_m - k_3 \sqrt{P_a}]$$

where we have assumed that $P_a > 0$. Also,

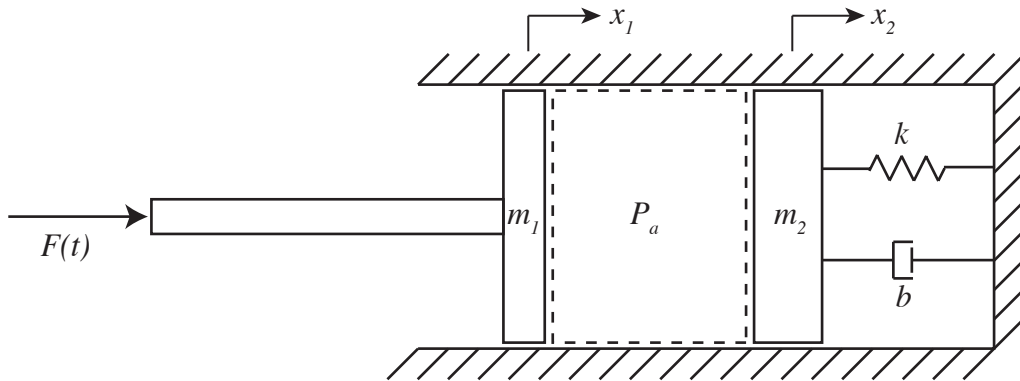
$$\dot{V}_b = \frac{1}{\rho} (q_{in,b} - q_{out,b})$$

$$\dot{P}_b = \frac{Et_w}{2\pi\rho l r_o^3} (q_m - k_4(\theta_v) \sqrt{P_b})$$

where we have assumed $P_b > 0$.

Example: Fluid Spring

The figure below shows a fluid spring which uses fluid compressibility. Develop the equations of motion for this system. There should be 4 states: 3 mechanical states and 1 fluid state.



1. There is one pressure node, P_a , as shown in the figure.
2. The control volume encompasses all of the fluid in the figure.
3. Continuity equation:

$$q_{in} - q_{out} = \rho \dot{V} + \frac{\rho V_0}{\beta} \dot{P}_a$$

$$0 = \rho \dot{V} + \frac{\rho V_0}{\beta} \dot{P}_a$$

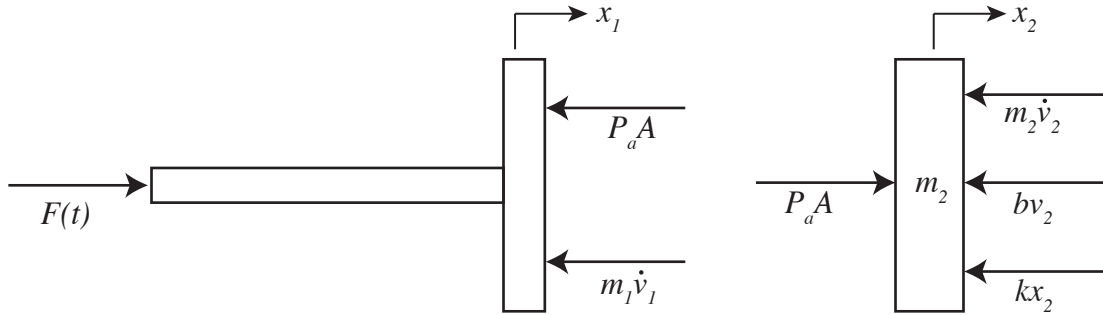
4. Define \dot{V}

$$\dot{V} = A(\dot{x}_2 - \dot{x}_1)$$

$$\dot{V} = A(v_2 - v_1)$$

5. Pressure drops: none.

6. Mechanical system:



$$m_1 \dot{v}_1 + P_a A = F(t)$$

$$\dot{v}_1 = -\frac{A}{m_1} P_a + \frac{1}{m_1} F(t)$$

$$m_2 \dot{v}_2 + b v_2 + k x_2 = P_a A$$

$$\dot{v}_2 = -\frac{b}{m_2} v_2 - \frac{k}{m_2} x_2 + \frac{A}{m_2} P_a$$

$$\dot{x}_2 = v_2$$

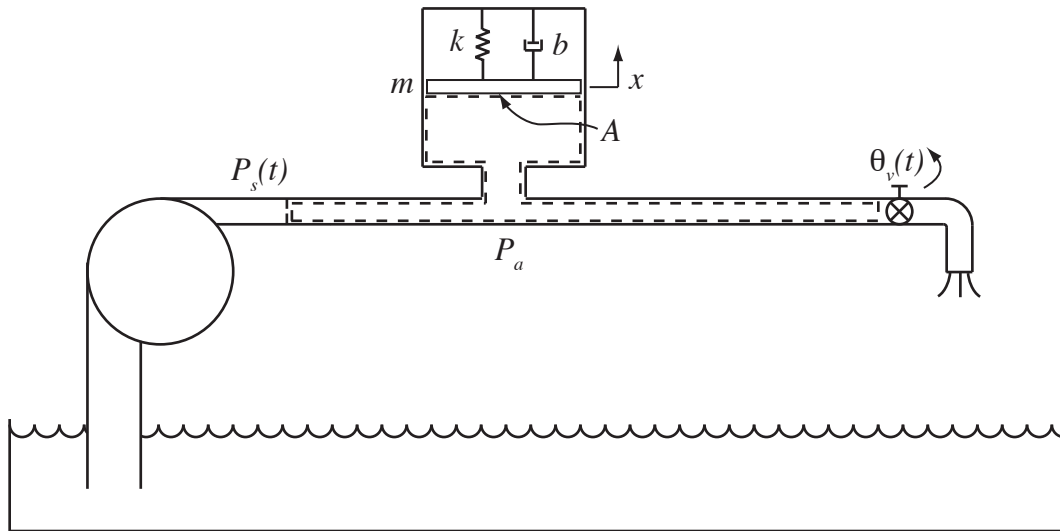
7. Combine relations:

$$\dot{P}_a = -\frac{\beta}{V_0} \dot{V}$$

$$\dot{P}_a = \frac{\beta A}{V_0} (v_1 - v_2)$$

Example: Pumping System

The figure below shows a pumping system. Assume that the pipe is long enough that fluid compressibility becomes significant. Develop the equations of motion for this system. There should be 3 states: 2 mechanical states and 1 fluid state.



1. Pressure nodes $P_s(t)$ and P_a are shown in the figure.

2. The control volume is drawn as shown.

3. Continuity equation:

$$q_{in} - q_{out} = \rho \dot{V} + \frac{\rho V_0}{\beta} \dot{P}_a$$

4. Define \dot{V}

$$\dot{V} = Av$$

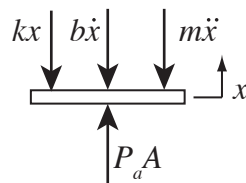
5. Pressure drops:

$$q_{in} = k_1 \sqrt{P_s(t) - P_a}$$

$$q_{out} = k_2(\theta_v) \sqrt{P_a}$$

where we have assumed $P_s(t) > P_a > 0$.

6. Mechanical system:



$$m\dot{v} + bv + kx = P_a A$$

$$\dot{v} = -\frac{b}{m}v - \frac{k}{m}x + \frac{A}{m}P_a$$

$$\dot{x} = v$$

7. Combine relations:

$$k_1 \sqrt{P_s(t) - P_a} - k_2(\theta_v) \sqrt{P_a} = \rho A v + \frac{\rho V_0}{\beta} \dot{P}_a$$

$$\dot{P}_a = \frac{\beta}{\rho V_0} \left[k_1 \sqrt{P_s(t) - P_a} - k_2(\theta_v) \sqrt{P_a} - \rho A v \right]$$

4.2 Thermal Systems

We will model thermal systems using continuity principles, just as we have each of the other systems we have studied. The main difference is that there is no analog to inertia or flow storage components in thermal systems. Instead, we consider just thermal resistance and thermal capacitance.

4.2.1 Thermal Resistance

Thermal resistances are characterized by a relationship between temperature and heat flux, the effort and flow variables for thermal systems. Note that, unlike each of the other types of systems, thermal resistances do not dissipate energy; they merely limit the flow of heat from one part of a system to another. There are three types of thermal resistances: conduction, convection, and radiation.

Conduction

Thermal conduction is characterized by heat flow through a material. The temperature difference is given by

$$\Delta T = R q_h$$

where R is the thermal resistance and q_h is the heat flux. For a rectangular solid of length L , cross-sectional area A , and conductivity κ , the resistance is

$$R = \frac{L}{\kappa A}$$

Convection

Convection occurs when heat is carried by a moving fluid. The heat flux from a solid to a fluid is often written

$$q_h = hA\Delta T$$

where h is the convection coefficient and A is the area over which convective heat transfer occurs. h is large for forced convection and for liquids, and small for free convection and for gases.

Radiation

Radiative heat transfer occurs when heated bodies radiate energy in the form of electromagnetic radiation. Unlike conduction and convection, it is a nonlinear phenomenon.

$$q_h = \beta(T_1^4 - T_2^4)$$

where β is the radiative heat transfer coefficient, which depends on the materials involved, geometry, and sometimes even on the temperatures T_1 and T_2 . The temperatures must be given on an absolute scale (Rankine or Kelvin), and they represent the temperatures of the two bodies exchanging heat due to radiation.

4.2.2 Thermal Capacitance

All materials store heat, and their temperature is a measure of how much heat they store. The thermal capacitance of a body results in the relation

$$q_h = C \frac{dT}{dt} = \rho V c_p \frac{dT}{dt} = m c_p \frac{dT}{dt}$$

where c_p is the heat capacity of the material.

4.2.3 Thermal Sources

Thermal sources are normally of two types: a prescribed temperature or a prescribed heat flux. A temperature source represents the case of a thermal reservoir. A heat flux source represents a heater, such as an electrical heater, in which case

$$q_{h,s} = i^2 R$$

where i and R are the electrical current and resistance of the heater.

4.2.4 Biot Number

The Biot number is the ratio of the convective heat transfer to the conductive heat transfer.

$$N_B = \frac{hA\Delta T}{\frac{\kappa A\Delta T}{L}} = \frac{hL}{\kappa}$$

L is a representative length of the system — often the volume to surface ratio.

If N_B is small (less than 0.1 or so), the body can be treated as having a single temperature (a lumped model). Otherwise, it will need to be split into multiple “lumps,” or a distributed model must be used. In addition, a Biot number analysis can be used to determine the relative importance of convective and conductive heat transfer.

4.2.5 Conservation of Energy

The rate of change of the energy stored in a system is governed by the principle of conservation of energy, which is also known as the first law of thermodynamics. In words, this law states that

$$\left[\begin{array}{l} \text{Rate at which} \\ \text{the energy of the} \\ \text{system changes} \end{array} \right] = \left[\begin{array}{l} \text{Net rate at which} \\ \text{heat is transferred} \\ \text{into the system} \end{array} \right] - \left[\begin{array}{l} \text{Nate rate at which} \\ \text{work is done by} \\ \text{the system} \end{array} \right]$$

Assuming there are no work interactions and the system is stationary, the first law simplifies to

$$mc_p \frac{dT}{dt} = q_{h,in} - q_{h,out}$$

4.2.6 Analyzing Thermal Systems

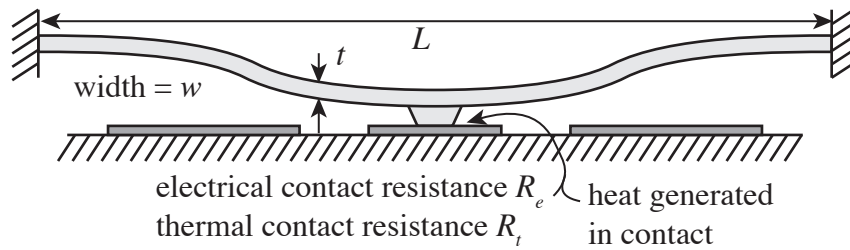
To analyze thermal systems,

1. Define temperature nodes in the system. If necessary, use a Biot number analysis to determine important temperature nodes.
2. Write the conservation of energy equation for each temperature node.
3. Use thermal resistance relations to define $q_{h,in}$ and $q_{h,out}$ for each node. Again, a Biot number analysis may be required to determine the important heat transfer modes.
4. Combine relations to give a set of state equations. State variables are simply the temperatures at each node.

Example

The figure below shows a micromachined switch that carries current. The current flowing through the electrical contact resistance at the contact spot causes heating of the switch. This heat flows through the switch beam, as well as the thermal contact resistance, to the environment at a temperature of T_0 . The dimensions of the gold beam and other parameters are:

$$\begin{aligned} L &= 300 \mu\text{m} & w &= 100 \mu\text{m} & t &= 3 \mu\text{m} \\ h &= 1 \text{ W/m}^2\cdot\text{K} & \kappa &= 317 \text{ W/m}\cdot\text{K} & R_t &= 1000 \text{ K/W} \\ \rho &= 19,000 \text{ kg/m}^3 & c_p &= 129 \text{ J/kg}\cdot\text{K} \end{aligned}$$



Determine the equations of motion, and use them to determine the thermal time constant of the beam.

Solution: First, we will calculate the Biot number for the beam. The effective length for calculating the Biot number is

$$L_e = \frac{V}{A} = \frac{Ltw}{2Lw + 2Lt}$$

where V is the volume of the beam and A is its surface area. Since $w \gg t$,

$$L_e = \frac{Ltw}{2Lw} = \frac{t}{2}$$

and the Biot number is

$$N_B = \frac{hL_e}{\kappa} = \frac{ht}{2\kappa} = 2.37 \times 10^{-9}$$

Since the Biot number is so small, we can treat the beam as having a single temperature node, and we can ignore convective heat transfer. Hence, we will define a single temperature node, at the center of the beam. The heat flux into this node is the electrical heating $q_{h,e}(t)$. There are three paths for heat to leave this node: conduction through both sides of the beam, and conduction through the thermal contact resistance.

The conservation of energy equation is

$$q_{h,in} - q_{h,out} = mc_p \frac{dT}{dt}$$

The heat flux into the node is

$$q_{h,in} = q_{h,e}(t)$$

where $q_{h,e}(t)$ is the heat generated in the node. The heat conduction through one half of the beam is

$$q_{h,L/2} = \frac{\kappa w t}{L/2} (T - T_0)$$

and the heat conduction through the thermal contact resistance is

$$q_{h,R} = \frac{1}{R_t} (T - T_0).$$

Since the total heat leaving the node is the sum of the heat traveling through each half of the beam and the contact,

$$q_{h,out} = 2q_{h,L/2} + q_{h,R}$$

so that

$$q_{h,e}(t) - 2\frac{2\kappa w t}{L} (T - T_0) - \frac{1}{R_t} (T - T_0) = mc_p \frac{dT}{dt}$$

$$\boxed{\frac{dT}{dt} = \frac{q_{h,e}(t)}{mc_p} - \left[\frac{4\kappa w t}{Lmc_p} + \frac{1}{R_t mc_p} \right] (T - T_0)}$$

To find the thermal time constant, we write $\Delta T = T - T_0$, and then

$$mc_p s \Delta T + \left[\frac{4\kappa w t}{L} + \frac{1}{R_t} \right] \Delta T = q_{h,e}(t)$$

$$\frac{\Delta T}{q_{h,e}} = \frac{1}{mc_p s + \left[\frac{4\kappa w t}{L} + \frac{1}{R_t} \right]}$$

So the characteristic equation is

$$mc_p s + \left[\frac{4\kappa w t}{L} + \frac{1}{R_t} \right] = 0$$

and the root is

$$s = - \frac{\left[\frac{4\kappa w t}{L} + \frac{1}{R_t} \right]}{mc_p}$$

Therefore, the time constant is

$$\tau = -\frac{1}{s} = \frac{mc_p}{\left[\frac{4\kappa w t}{L} + \frac{1}{R_t} \right]} = 97 \mu s$$

Chapter 5

Laplace Transform Analysis

The Laplace transform can be used as a tool to study linear, constant-coefficient systems. To do this, we will:

1. Find the equations of motion that describe the system dynamic behavior.
2. Take the Laplace transform of the equations of motion.
3. Manipulate the transformed equations to obtain the transfer function, the characteristic equation and eigenvalues, and the final (steady-state) values.
4. Apply the inverse Laplace transform if the time response is desired.

This chapter will discuss steps 2, 3, and 4 to show how they are done.

5.1 Review of the Laplace Transform

Laplace transforms are presented here with an object toward using them to analyze dynamics systems. Many references exist to further explore the mathematical theory behind Laplace transforms. The one-sided Laplace transform of a function $f(t)$ is given by:

$$\mathcal{L}_-\{f(t)\} = F(s) \equiv \int_{0^-}^{\infty} f(t)e^{-st} dt$$

This is called the “one-sided” Laplace transform since it has as its lower limit 0^- instead of $-\infty$ for the two-sided transform. 0^- means the instant before $t = 0$. Using 0^- as the lower bound allows the impulse function to be considered. Usually there is no distinction between 0 and 0^- . Here, we will simply write the one-sided Laplace transform as $\mathcal{L}\{\}$. Also, note that $\mathcal{L}\{f(t)\}$ is a

function of the complex variable s , which essentially represents frequency. For this reason, taking the Laplace transform is sometimes referred to as taking a system into the frequency domain.

The time function corresponding to a given Laplace transform can be found using the inverse Laplace transform. However, practically speaking, both the Laplace transform and the inverse Laplace transform are found by breaking the functions down into simpler parts and using tables to find the transforms of those parts. A fairly extensive table is included in your text, pages 115–117.

5.2 Properties of Laplace Transforms

Four properties of Laplace transforms are highlighted here because of their common use in dynamic systems problems. The first is superposition. Because the Laplace transform is a linear operation, the following principle holds:

$$\begin{aligned}\mathcal{L}\{\alpha f_1(t) + \beta f_2(t)\} &= \int_0^\infty [\alpha f_1(t) + \beta f_2(t)]e^{-st} dt \\ &= \alpha \int_0^\infty f_1(t)e^{-st} dt + \beta \int_0^\infty f_2(t)e^{-st} dt \\ &= \alpha F_1(s) + \beta F_2(s)\end{aligned}$$

Secondly, to take the Laplace transform of a derivative, we write

$$\mathcal{L}\left\{\frac{df}{dt}\right\} = \int_{0^-}^\infty \left(\frac{df}{dt}\right) e^{-st} dt.$$

We can integrate by parts:

$$\begin{aligned}u &= e^{-st} & dv &= \frac{df}{dt} dt \\ \mathcal{L}\left\{\frac{df}{dt}\right\} &= e^{-st} f(t)|_{0^-}^\infty + s \int_{0^-}^\infty f(t) e^{-st} dt.\end{aligned}$$

$e^{-s\infty}$ will go to zero, so this gives

$$\mathcal{L}\left\{\frac{df}{dt}\right\} = -f(0^-) + sF(s).$$

Notice that this means that derivatives in the original equations of motion will become algebraic relations in the transformed equations. Further, we can write

$$\begin{aligned}\mathcal{L}\{\ddot{f}\} &= s^2 F(s) - sf(0^-) - \dot{f}(0^-) \\ \mathcal{L}\{f^{(m)}\} &= s^m F(s) - s^{m-1} f(0^-) - s^{m-2} \dot{f}(0^-) - \dots - f^{(m-1)}(0^-).\end{aligned}$$

The proof of these last statements is left as an exercise for the reader.

The third property is the property of integration, which is given without proof.

$$\mathcal{L} \left\{ \int f(t) dt \right\} = \frac{F(s)}{s}$$

where $F(s)$ is the Laplace transform of $f(t)$.

Finally, a shift in time may be described using

$$\mathcal{L}\{F(t-T)u_s(t-T)\} = e^{-Ts}F(s)$$

where $u_s(t)$ is the unit step function, and T is the time shift.

5.2.1 Example

For the first-order linear system given by

$$m\dot{v} + bv = u(t)$$

where $u(t)$ is an arbitrary input force, the Laplace transform yields

$$\mathcal{L}\{m\dot{v} + bv\} = \mathcal{L}\{u(t)\}$$

$$msV(s) - mv(0^-) + bV(s) = U(s)$$

here $V(s)$ and $U(s)$ are the Laplace transforms of $v(t)$ and $u(t)$. Further manipulation yields

$$V(s) = \frac{U(s) + mv_0}{ms + b},$$

where v_0 is the initial value of $v(t)$. (Note that if the initial condition is taken to be zero, this equation represents the *transfer function* for the system. This will be discussed in more detail later.) If the form of $U(s)$ is known, the inverse Laplace transform can now be taken to yield a solution to the differential equation. For example, if $u(t) = 0$, then $U(s) = 0$, and

$$V(s) = \frac{v_0}{s - (-\frac{b}{m})}.$$

Using the tables, the inverse Laplace transform can now be identified as

$$v(t) = v_0 e^{-\frac{b}{m}t}.$$

Note that this *solution* to the original differential equation represents the response due to the initial conditions, since the input was taken to be zero. Hence, it is called the *free* or *natural* response of the system. In addition, because it goes to zero as t becomes large, it is also called the *transient* system response.

5.3 Final Value Theorem

The preceding section showed how to perform steps 2 and 4 from the process described at the beginning of the chapter - taking the Laplace transform, manipulating the result, and taking the inverse transform to find a solution to the differential equations of motion. This section will describe manipulating the transformed equations to yield the final values of the system.

The final or steady-state values of a system can be found using the final value theorem. This theorem is given here without proof. It states that, for a function $x(t)$ and its Laplace transform $X(s)$, the final value of x may be found from

$$x(t \rightarrow \infty) = \lim_{s \rightarrow 0} sX(s).$$

This equation will only hold, however, if all roots of the denominator of $X(s)$ have negative real parts except for a single root at the origin.

5.3.1 Example

For the previous example problem, we have

$$V(s) = \frac{U(s) + mv_0}{ms + b}.$$

If we take $u(t) = 1$ (u is a step function; $U(s) = 1/s$), then

$$V(s) = \frac{1 + smv_0}{s(ms + b)}.$$

The final value of v is then

$$v(t \rightarrow \infty) = \lim_{s \rightarrow 0} \frac{s + s^2mv_0}{s(ms + b)} = \frac{1}{b}.$$

Note that the solution to the differential equation

$$m\dot{v} + bv = 1$$

with initial condition $v(0) = v_0$ is

$$v(t) = \frac{1}{b} \left(1 - e^{-\frac{b}{m}t} \right) + v_0 e^{-\frac{b}{m}t}$$

so that it may be confirmed that the final (steady-state) value of v is $1/b$. Note that in this solution, we have the *free* response $v_0 e^{-\frac{b}{m}t}$, as well as the *forced* response (the part of the solution due to the system inputs) $\frac{1}{b} \left(1 - e^{-\frac{b}{m}t} \right)$. Also, we have a *steady-state* response of $1/b$ (the part that doesn't disappear with time), and a *transient* response of $(v_0 - \frac{1}{b})e^{-\frac{b}{m}t}$.

5.4 Initial Value Theorem

The initial value theorem can be used to determine the value of a state variable at time 0^+ , a time infinitesimally after 0. This theorem states that

$$x(0^+) = \lim_{s \rightarrow \infty} sX(s).$$

For this theorem to hold, the limit must exist, and the Laplace transforms of $x(t)$ and $\frac{dx}{dt}$ must also exist. In addition, if $X(s)$ is a rational function, the degree of its numerator must be less than the degree of its denominator.

For example, for the Laplace transform

$$V(s) = \frac{mv_0}{ms + b}$$

the initial value theorem gives

$$v(0^+) = \lim_{s \rightarrow \infty} \frac{smv_0}{ms + b} = v_0$$

5.5 Transfer Functions

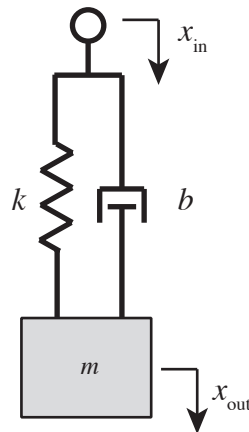
When we speak of transfer functions and frequency response, we are dealing with the *forced response* of a system, or the output in response to a driving input. This is separate from the natural response, which is the transient response of the system to initial conditions. The transfer function is an analytical expression, obtained from the equations of motion, that describes the ratio between the input and the output of the system. It applies only to linear systems with constant coefficients—nonlinear systems cannot be represented by a transfer function.

The *transfer function* of a system gives a relationship between the Laplace transform of the output and an input. For example, for the problem given in the previous example, the transfer function is

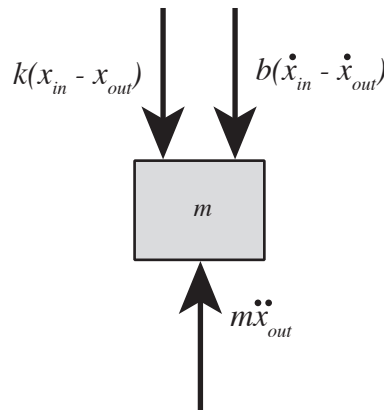
$$\frac{V(s)}{U(s)} = \frac{1}{ms + b}.$$

To find the transfer function, set all of the initial conditions equal to zero, and solve for the ratio of the output to the input. If there is more than one input, set all of the inputs equal to zero except for the one for which you are finding the transfer function.

For example, consider a mass-spring-damper system being driven by an input displacement as illustrated below:



The system has the free-body diagram



The equation of motion for this system is

$$m\ddot{x}_{out} + b(\dot{x}_{out} - \dot{x}_{in}) + k(x_{out} - x_{in}) = 0$$

$$m\ddot{x}_{out} + b\dot{x}_{out} + kx_{out} = b\dot{x}_{in} + kx_{in}$$

We can find the transfer function of the system by taking the Laplace transform:

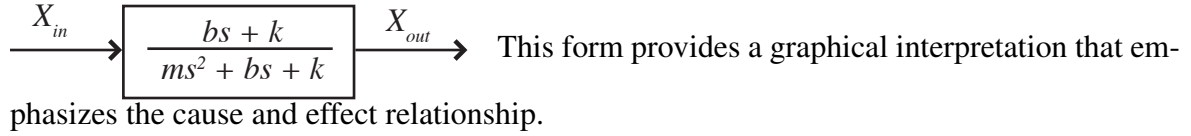
$$ms^2X_{out}(s) + bsX_{out}(s) + kX_{out}(s) = bsX_{in}(s) + kX_{in}(s)$$

$$\frac{X_{out}(s)}{X_{in}(s)} = \frac{bs + k}{ms^2 + bs + k}$$

This is the transfer function for the mass-spring-damper system driven by a displacement input. A few comments on transfer functions:

- They describe how a specific input signal is altered by the dynamics of the system. It is the ratio of the effect to the cause.

- They are often used in a block diagram system representation:



- Even though the transfer function represents how the system responds to a forcing input, the natural response characteristics are described by the characteristic function which is in the denominator:

$$D(s) = ms^2 + bs + k$$

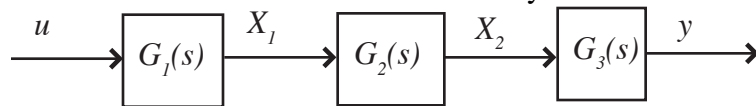
and setting equal to zero gives the characteristic equation:

$$ms^2 + bs + k = 0$$

- The system's natural response is described by the roots of the characteristic equation. These are the eigenvalues or poles of the system.
- The roots of the numerator polynomial are called the zeros of the transfer function. ($s = -k/b$ is the zero for this example.) Physically, these correspond to values of s for which a non-zero input of the form $u = Ue^{st}$ gives a zero output.

5.5.1 Transfer functions for groups of subsystems

The transfer function response for a group of subsystems acting one after another is just the product of the transfer function for each subsystem:



$$\frac{X_1}{u} = G_1(s) \quad \frac{X_2}{X_1} = G_2(s) \quad \frac{y}{X_2} = G_3(s)$$

$$\frac{y}{u} = \frac{X_1}{u} \frac{X_2}{X_1} \frac{y}{X_2}$$

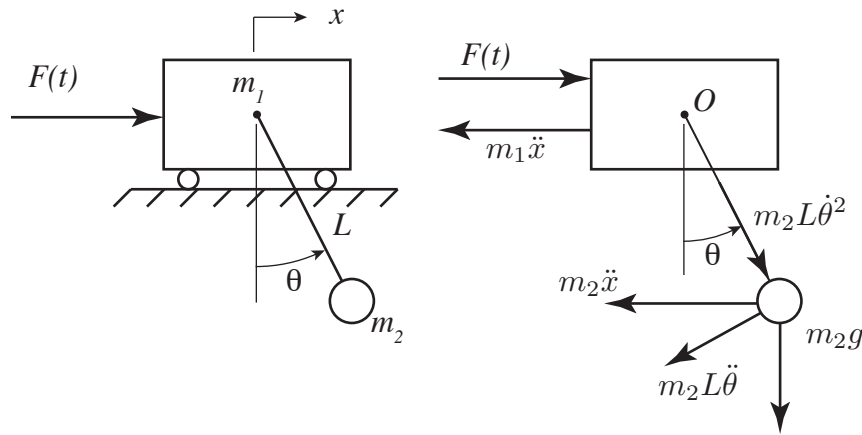
$$\frac{y}{u} = G_1(s)G_2(s)G_3(s)$$

5.5.2 Transfer Functions for Dynamically Coupled Systems

Many systems consist of several coupled states. In such systems, Laplace transform methods, coupled with matrix algebra, provide an easy way to isolate the transfer function of just one state. An example illustrates the method.

Example

A crane moves horizontally in response to a force input $F(t)$. As it does so, the cable and hook at its end can be modeled as a pendulum with a point mass (negligible J) swinging at its end. A schematic and free-body diagram are shown below.



From the free-body diagram, we derive the following equations:

$$\begin{aligned}\sum M_0^* &= 0: \quad m_2 \ddot{x} L \cos \theta + m_2 L^2 \ddot{\theta} + m_2 g L \sin \theta = 0 \\ \sum F_x^* &= 0: \quad (m_1 + m_2) \ddot{x} + m_2 L \ddot{\theta} \cos \theta - m_2 L \dot{\theta}^2 \sin \theta = F(t)\end{aligned}$$

To linearize these equations, we assume small values of θ . This allows us to make the approximations $\cos \theta = 1$, $\sin \theta = \theta$, and higher order terms are zero.

$$\begin{aligned}\ddot{x} + L \ddot{\theta} + g \theta &= 0 \\ (m_1 + m_2) \ddot{x} + m_2 L \ddot{\theta} &= F(t)\end{aligned}$$

Taking Laplace transforms and ignoring initial conditions gives

$$\begin{aligned}(Ls^2 + g)\Theta + s^2 X &= 0 \\ (m_1 + m_2)s^2 X + m_2 L s^2 \Theta &= F\end{aligned}$$

or

$$\begin{bmatrix} (m_1 + m_2)s^2 & m_2 L s^2 \\ s^2 & Ls^2 + g \end{bmatrix} \begin{bmatrix} X \\ \Theta \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix}$$

We can use matrix algebra to solve for the transfer function of either X or Θ in terms of F . Specifically, we can use Cramer's rule. Cramer's rule proceeds like this: if we define the coefficient matrix as \mathbf{A} , we can write the determinant of \mathbf{A} as $\det \mathbf{A}$. Similarly, we can create a matrix \mathbf{A}' by replacing

the column of \mathbf{A} corresponding to the variable of interest with the vector on the right-hand side of the equation. For example, if X is the variable of interest, then

$$\mathbf{A}' = \begin{bmatrix} F & m_2 L s^2 \\ 0 & L s^2 + g \end{bmatrix}.$$

The value of X is then found by dividing the determinant of \mathbf{A}' by $\det \mathbf{A}$:

$$X = \frac{\det \mathbf{A}'}{\det \mathbf{A}}$$

For example, in this case,

$$\det \mathbf{A} = (m_1 + m_2)(L s^4 + g s^2) - m_2 L s^4 = s^2 [m_1 L s^2 + (m_1 + m_2)g]$$

so that

$$X = \frac{F(L s^2 + g)}{s^2 [m_1 L s^2 + (m_1 + m_2)g]}$$

and

$$\frac{X}{F} = \frac{L s^2 + g}{s^2 [m_1 L s^2 + (m_1 + m_2)g]}.$$

Similarly, for Θ ,

$$\Theta = \frac{-s^2 F}{s^2 [m_1 L s^2 + (m_1 + m_2)g]}$$

and

$$\frac{\Theta}{F} = \frac{-1}{m_1 L s^2 + (m_1 + m_2)g}.$$

Hence, if we consider X the system output, the system has zeros at $s = \pm j\sqrt{g/L}$, corresponding to the natural frequency of the pendulum, and poles at $s = 0$, $s = 0$, and $s = \pm j\sqrt{(m_1 + m_2)g/(m_1 L)}$. Looking at Θ , the poles are simply $s = \pm j\sqrt{(m_1 + m_2)g/(m_1 L)}$.

5.5.3 Characteristic Equation and Roots

As has already been mentioned, the denominator of the transfer function is called the *characteristic polynomial* of the system. If it is set equal to zero, the *characteristic equation* results. Hence, for system given by

$$m\dot{v} + bv = u(t)$$

the characteristic equation is

$$ms + b = 0.$$

The *roots* or *eigenvalues* (also called *poles*) of the system are simply the solutions to the characteristic equation. For the example problem, there is a single root at $s = -\frac{b}{m}$. As we will soon see, the poles of a system have an enormous effect on its behavior; hence, the ability to find the system poles is very important.

5.6 Common System Inputs

System inputs can be any function of time. However, there are certain types of inputs that are frequently used to study system behavior. This section briefly discusses each of these.

5.6.1 Impulse Input

The impulse input represents an impact or zero-length pulse to the system. Mathematically, it takes the form of the Dirac delta function δ , for which the Laplace transform is simply 1. The most interesting feature of impulse inputs is that they can *instantaneously* change the values of the system variables. Obviously, this is not a real effect, but it closely models real behavior for many systems.

5.6.2 Step Input

A step input represents a sudden change from one value to another. Mathematically, it is described by the Heaviside function $u_s(t)$, which is zero for all times before $t = 0$ and 1 for all times after 0. A step input is usually assumed when the system input is constant. The Laplace transform of the Heaviside function is $1/s$.

5.6.3 Ramp Input

A ramp input starts at 0 for all times less than zero, and increases linearly (with slope 1) after 0. The Laplace transform of a ramp input is $1/s^2$.

5.6.4 Sinusoidal Input

A sinusoidal input is just a sine wave at a given frequency. For linear systems (meaning any system of which you can take a Laplace transform) the steady-state response to a sinusoidal input will also be a sinusoid at the same frequency. This is the basis for the principle of frequency response, which will be covered in Chapter 8.

Chapter 6

Time-Domain Analysis

6.1 First Order Systems

Taking Laplace transforms of equations of motion allows us to find transfer functions, characteristic equations, and eigenvalues for the system. To explore these topics, let us first examine the behavior of first-order systems. A first-order system can only store energy in one form and location. The mathematical equation describing its motion (its equation of motion) can be written in terms of a single variable and its first derivative only. Several examples are:

- a single mass moving against friction (such as a motor rotor)
- a single spring moving a body of negligible mass against friction
- a single electrical capacitance with resistors
- a single electrical inductance with resistors
- a fluid tank with flow out of the tank impeded by a fluidic resistance
- a thermal capacitance with thermal resistance (like a thermocouple)

Remember that masses, springs, electrical capacitors, inductors, fluid tanks, fluid inertias, and thermal capacitances store energy. Friction, electrical resistors, fluid resistances, and thermal resistances dissipate energy or impede its flow. For all first-order systems with constant input, the system will return naturally to a state of static equilibrium. This means that the state variable will go to a steady-state value and stop changing.

In general, a first-order system may be described by the equation

$$\tau \dot{x} + x = u(t)$$

where x is the state variable and $u(t)$ is the system input. As we will see, τ is the “time constant” of the system.

6.1.1 Example of a first-order system

A typical mechanism for closing a door consists of a spring that works to close the door and a resistance that acts to slow down the closing of the door. If we ignore the mass moment of inertia of the door, and if we consider only the natural (unforced) motion, we can write

$$b\dot{\theta} + k\theta = 0$$

where θ is the angle of the door.

Motion of the System by Physical Reasoning

At any instant, two torques are acting on the door: the spring torque $k\theta$, which opposes displacement, and the damping torque $b\dot{\theta}$, which opposes velocity. At any instant, these torques are equal and opposite to each other:

$$b\dot{\theta} = -k\theta.$$

The spring torque is attempting to return the door to the neutral (closed) position, where the spring force is zero. The damping torque opposes this motion, so that the velocity at any instant is proportional to the twist of the spring. Hence, as the door approaches the closed position, the velocity goes to zero, so that the door closes without overshooting the closed position. Mathematically,

$$\dot{\theta} = -\frac{k}{b}\theta$$

so that the rate of return toward zero is always proportional to the displacement from zero.

We can get more insight into the system by taking the Laplace transform.

$$b\dot{\theta} + k\theta = 0$$

$$b[s\Theta(s) - \theta(0)] + k\Theta(s) = 0$$

where

$$\mathcal{L}\{\theta(t)\} = \Theta(s).$$

Hence,

$$\Theta(s) = \frac{\theta(0)}{s + \frac{k}{b}}.$$

Note that, if $\theta(0) = 0$ (zero initial conditions), this reduces to

$$(bs + k)\Theta(s) = 0.$$

The first part of this expression gives us the characteristic equation of the system:

$$bs + k = 0.$$

The roots of the characteristic equation are called the eigenvalues of the system, and they determine completely the dynamic behavior of natural system behavior. In other words, the eigenvalues tell us a lot about how the system will act. The sole eigenvalue for this system is

$$s = -\frac{k}{b}.$$

The positive reciprocal of the system pole is known as the time constant of the system: $\tau = b/k$. We will study transfer functions, the characteristic equation, and eigenvalues in more depth later.

Using Laplace transform tables, we can take the inverse Laplace transform of $\Theta(s)$:

$$\theta(t) = \mathcal{L}^{-1}\{\Theta(s)\} = \mathcal{L}^{-1}\left\{\frac{\theta(0)}{s + \frac{k}{b}}\right\}$$

And defining the initial angle $\theta_0 = \theta(0)$ gives

$$\theta(t) = \theta_0 e^{-\frac{k}{b}t}.$$

Note that, if this system had been put in the form of the general equation, it would have been written

$$\frac{b}{k}\dot{\theta} + \theta = 0$$

so that τ would be b/k . The solution could then be written

$$\theta(t) = \theta_0 e^{-\frac{t}{\tau}}$$

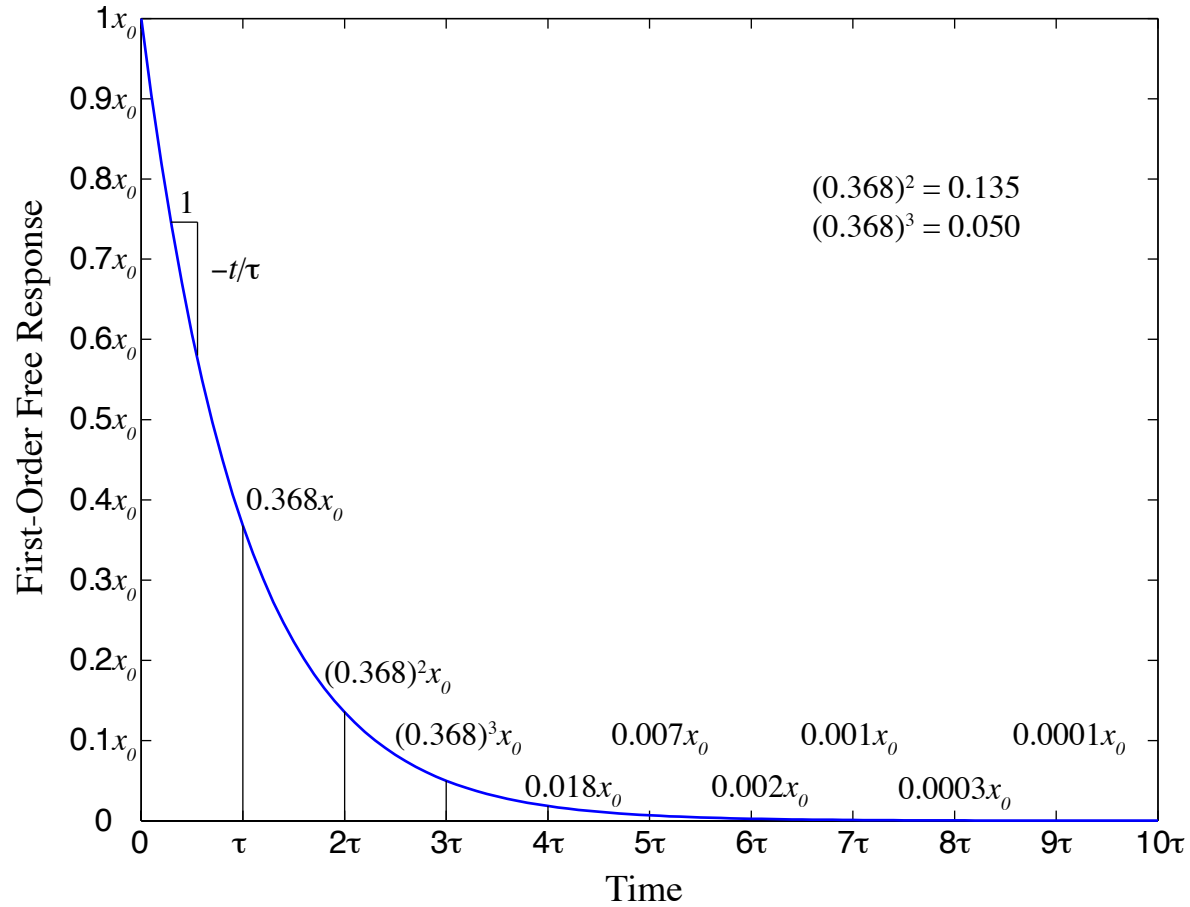
6.1.2 Plotting Natural Response

At the end of one time constant, the “decay” in the state variable is known. For instance, in the previous example, at the end of one time constant, $t = \tau = b/k$,

$$\theta(b/k) = \theta_0 e^{-\frac{k}{b} \cdot \frac{b}{k}} = \theta_0 e^{-1} = \frac{\theta_0}{e}$$

$$\boxed{\theta(\tau) \approx 0.368\theta_0}$$

Hence, the plotted behavior will look like the following graph:



Note that τ , the time constant for the system, depends only on the system parameters and not on the initial values of the states or on any inputs to the system. In other words, it is a characteristic of the system, and it characterizes the free motion of the system—its natural response.

6.2 Stability

The response of a first-order system is stable if its characteristic root or eigenvalue is less than 0. This will ensure that the motion will decay, as illustrated in the plot above. If the solution to the characteristic equation (the eigenvalue) is positive, this will mean that the time-domain equation will look like

$$\theta(t) = \theta_0 e^{st}$$

where s is the positive real eigenvalue. This implies that the response will grow without bound, and we say that the system is *unstable*.

6.3 Forced Motion

Forced motion can be considered generally by applying the Laplace transform to the function describing the input, or by straightforward solution of the particular solution to the differential equation. Here, we illustrate a Laplace transform solution using a step input to the door system presented earlier.

6.3.1 Example

For a step input:

$$b\dot{\theta} + k\theta = M_0 u(t)$$

where M_0 is the magnitude of the moment step, and $u(t)$ is the step function (0 for $t < 0$ and 1 for $t \geq 0$). From what we know about the door, it will reach steady-state at a new position where the spring torque is equal and opposite to M_0 . Hence, the mathematical solution must contain a constant (to represent the final displacement) and an exponential term (to represent the transient motion). If we take the Laplace transform, we get

$$b[s\Theta(s) - \theta(0)] + k\Theta(s) = \frac{M_0}{s}.$$

And, taking $\theta_0 = 0$, we get

$$\begin{aligned} (bs + k)\Theta(s) &= \frac{M_0}{s} \\ \Theta(s) &= \frac{M_0/b}{s(s + k/b)} = \frac{M_0}{k} \frac{k/b}{s(s + k/b)} \end{aligned}$$

where the result has been explicitly factored to facilitate the inverse Laplace transform. From the transform table, we get

$$\theta(t) = \frac{M_0}{k} \left[1 - e^{-(k/b)t} \right].$$

Hence, the final value will be M_0/k .

6.4 Undamped Second-Order Systems: Free Vibrations

Undamped second order systems represent well the behavior of the following example systems:

- pendulum
- LC circuit

- a mass attached to a spring
- a vibrating cantilever
- a piston vibrating on an air column
- mercury in a u-shaped tube

A second-order, undamped system is characterized by having two different forms of energy storage, with no friction or damping to dissipate the energy. It approximates well the behavior of systems with very small energy dissipation compared to the energy stored in the two energy storage components.

The equation of free or unforced motion for an *undamped* second-order system is

$$\ddot{x} + \omega_n^2 x = 0$$

where ω_n is the frequency of free motion. The solution to this equation reveals that the motion is characterized by sinusoidal oscillation:

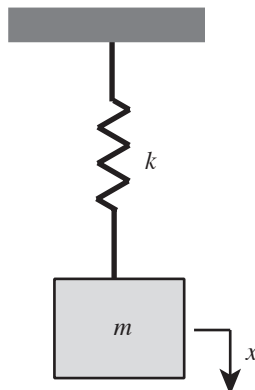
$$x = x_m \cos(\omega_n t - \psi)$$

where x_m represents the magnitude of the oscillation and ψ is the phase of the oscillation, and both are determined by the initial conditions and system parameters. Purely sinusoidal motion can also be represented by an equation of the form

$$x = C e^{st}$$

where s is a purely imaginary number. (Recall that a first-order system was described by the same equation, but s was real.)

6.4.1 Example



$$m\ddot{x} + kx = 0$$

Analysis of the system by physical reasoning

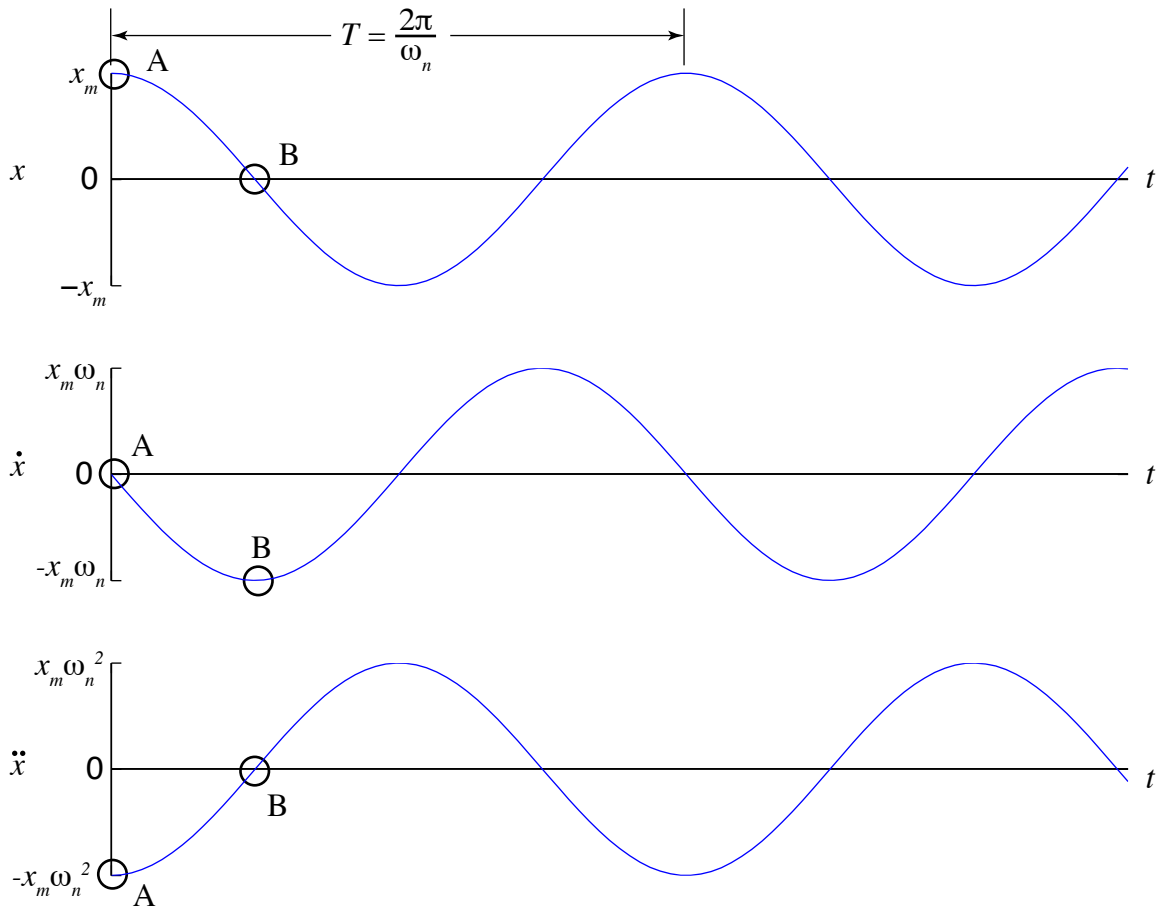
If the system is given an initial displacement x_m :

$$x(0) = x_m$$

$$\dot{x}(0) = 0$$

$$\ddot{x}(0) = -\frac{k}{m}x_m$$

Also, when $x(t) = 0$ the first time, $\dot{x}(t)$ will be a minimum, and $\ddot{x}(t)$ will also be zero, according to the following figure:



An energy analysis gives more insight into the system. When the spring is displaced, but the mass is not moving (point A in the figure), the spring stores potential energy, while the mass stores no kinetic energy. When the system moves through the zero position (point B in the figure), the

spring stores no potential energy, but the kinetic energy is a maximum. Because the system is conservative (no energy dissipation), the potential energy at A must equal the kinetic energy at B. Therefore, the system converts potential energy to kinetic energy, then kinetic energy back to potential energy, and so on as the system oscillates. Because there is no damping, the maximum spring deflection and the maximum velocity both remain the same in each cycle.

To solve the equation of motion, we take the most general condition, with $x(0) = x_m$ and $\dot{x}(0) = v_m$. Then

$$m[s^2X(s) - sx_m - v_m] + kX(s) = 0$$

$$(ms^2 + k)X(s) = msx_m + mv_m$$

$$X(s) = \frac{msx_m}{ms^2 + k} + \frac{mv_m}{ms^2 + k}$$

$$X(s) = \frac{sx_m}{s^2 + k/m} + \frac{v_m}{s^2 + k/m}$$

From the Laplace transform tables, we get

$$x(t) = x_m \cos\left(t\sqrt{k/m}\right) + v_m \sqrt{\frac{m}{k}} \sin\left(t\sqrt{k/m}\right)$$

or

$$x(t) = x_m \cos(t\omega_n) + \frac{v_m}{\omega_n} \sin(t\omega_n).$$

As an alternative, we could also write

$$x(t) = C \cos\left(t\sqrt{\frac{k}{m}} - \psi\right)$$

where

$$C = \sqrt{x_m^2 + v_m^2 \frac{m}{k}}$$

$$\psi = \tan^{-1}\left(\frac{v_m \sqrt{m/k}}{x_m}\right)$$

$$\psi = \tan^{-1}\left(\frac{v_m}{x_m} \sqrt{\frac{m}{k}}\right)$$

Finally, we can extract the characteristic equation by setting the denominator of the Laplace transform of x equal to zero:

$$s^2 + \frac{k}{m} = 0.$$

And the solutions to this equation give the characteristic roots or eigenvalues of the system:

$$s = \pm j\sqrt{\frac{k}{m}}$$

where $j = \sqrt{-1}$. Note that the eigenvalues for an undamped second-order system are purely imaginary, in contrast to the eigenvalue of a first-order system, which is always real.

6.5 Damped Second-Order Systems

A system having two separate energy storage elements as well as a mechanism for energy dissipation is a damped second-order system. The equation of motion for a damped second-order system has the form

$$\ddot{x} + 2\sigma\dot{x} + \omega_n^2x = f(t)$$

or

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = f(t)$$

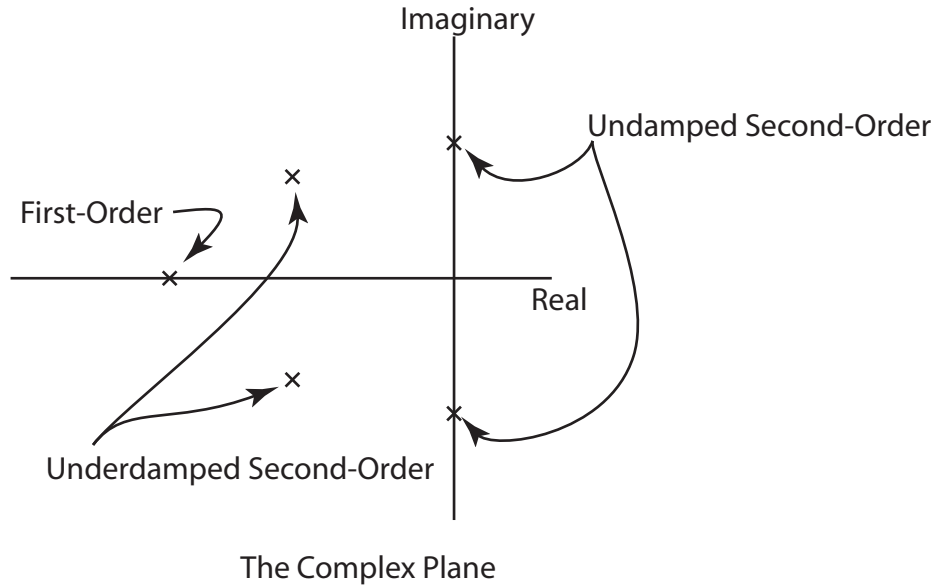
where ω_n is the undamped natural frequency, and ζ is the damping ratio. Note that $\sigma = \zeta\omega_n$.

6.5.1 Natural Motion

Recall that for a first-order system, the natural response was an exponential. For an undamped second-order system, the natural response was sinusoidal. For a damped second-order system, the response will be a combination of exponential and sinusoidal behavior. However, all three responses can be represented by an equation of the form

$$x = Ce^{st}.$$

Note that for first-order systems (exponential response), s is real, while for undamped second-order systems (sinusoidal response) s is purely imaginary, as shown on the figure below. Damped second-order systems are characterized by values of s that are either complex (for underdamped systems) or real (for critically damped or overdamped systems).



6.5.2 Free Response of Damped Second-Order Systems

Consider the natural motion of a spring-mass-damper system. The differential equation is given by

$$m\ddot{x} + b\dot{x} + kx = 0.$$

If either m or k is small compared to the other terms, then the system behavior will be similar to a first-order system (exponential response). If b is small, then the system will be nearly undamped, and the response will be sinusoidal. However, if all of the terms have similar magnitudes, the response will be a damped sinusoid. To explore further, we find the characteristic equation of the system.

Since we know that the solution will be of the form $x = Ce^{st}$, we can substitute this into the equation of motion above. Since $\dot{x} = Cse^{st}$ and $\ddot{x} = Cs^2e^{st}$,

$$ms^2Ce^{st} + bsCe^{st} + kCe^{st} = 0$$

$$ms^2 + bs + k = 0.$$

Note that this is the same characteristic equation we get by setting the denominator of the transfer function equal to zero; that is, $ms^2 + bs + k$ is the denominator of the transfer function. The roots or eigenvalues of the system can be found using the quadratic equation:

$$s = \frac{-b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}}, \quad \frac{k}{m} < \left(\frac{b}{2m}\right)^2 \quad \text{real roots}$$

or

$$s = \frac{-b}{2m} \pm j\sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}, \quad \frac{k}{m} > \left(\frac{b}{2m}\right)^2 \quad \begin{array}{l} \text{complex roots} \\ \text{(conjugate pairs)} \end{array}$$

We can also solve for the eigenvalues using the general form

$$\ddot{x} + 2\sigma\dot{x} + \omega_n^2 x = 0$$

giving

$$\begin{aligned} s_{1,2} &= -\sigma \left(1 \pm \sqrt{1 - 1/\zeta^2} \right), & \zeta > 1 & \text{ overdamped} \\ s_1 &= -\sigma, s_2 = -\sigma \text{ (two identical roots),} & \zeta &= 1 \text{ critically damped} \\ s_{1,2} &= -\sigma \pm j\omega_n \sqrt{1 - \zeta^2} = -\sigma \pm j\omega_d, & \zeta < 1 & \text{ underdamped} \end{aligned}$$

If we carry out the full Laplace transform solution (see below), we find that the free response solutions corresponding to each case are

$$\begin{aligned} x(t) &= Ae^{s_1 t} + Be^{s_2 t}, & \zeta > 1 & \text{ overdamped} \\ x(t) &= Ae^{-\sigma t} + Bte^{-\sigma t}, & \zeta &= 1 \text{ critically damped} \\ x(t) &= Ae^{-\sigma t} \cos(\omega_d t - \psi), & \zeta < 1 & \text{ underdamped} \end{aligned}$$

where A , B , and ψ are constants that will depend on the values of the initial conditions.

By comparison with the example, we find that, for this particular system (*not* in general),

$$\sigma = \frac{b}{2m} \qquad \omega_n = \sqrt{\frac{k}{m}}$$

Also, for an underdamped system, the frequency of vibration is given by

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} = \sqrt{\omega_n^2 - \sigma^2}$$

More Detail on the Calculation of the Free Response

Consider the general form of the damped second-order equation of motion:

$$\ddot{x} + 2\sigma\dot{x} + \omega_n^2 x = 0 \quad x(0) = x_0 \quad \dot{x}(0) = v_0$$

Taking the Laplace transform and solving for $X(s)$ gives

$$X(s) = \frac{s + 2\sigma}{s^2 + 2\sigma s + \omega_n^2} x_0 + \frac{1}{s^2 + 2\sigma s + \omega_n^2} v_0.$$

For solution to this equation, we will consider three cases of interest:

1. underdamped, $\zeta < 1$
2. critically damped, $\zeta = 1$
3. overdamped, $\zeta > 1$

Case 1: Underdamped, $\zeta < 1$

For $\zeta < 1$, $\sigma < \omega_n$, implying complex characteristic roots

$$s_{1,2} = -\sigma \pm j\omega_d$$

where ω_d is the damped natural frequency given by $\omega_n \sqrt{1 - \zeta^2}$. Based on these characteristic roots, we can rewrite the general Laplace transform equation as

$$X(s) = \frac{s + 2\sigma}{(s + \sigma)^2 + \omega_d^2} x_0 + \frac{1}{(s + \sigma)^2 + \omega_d^2} v_0$$

$$X(s) = \frac{(s + \sigma)x_0}{(s + \sigma)^2 + \omega_d^2} + \frac{v_0 + \sigma x_0}{(s + \sigma)^2 + \omega_d^2}$$

and taking the inverse Laplace transform yields

$$x(t) = x_0 e^{-\sigma t} \cos \omega_d t + (v_0 + \sigma x_0) \frac{e^{-\sigma t}}{\omega_d} \sin \omega_d t$$

$$x(t) = e^{-\sigma t} \left[x_0 \cos \omega_d t + \frac{v_0 + \sigma x_0}{\omega_d} \sin \omega_d t \right]$$

$$x(t) = x_m e^{-\sigma t} \cos(\omega_d t - \psi)$$

where

$$x_m = \sqrt{\left(\frac{v_0 + \sigma x_0}{\omega_d} \right)^2 + x_0^2}$$

$$\psi = \tan^{-1} \left(\frac{v_0}{x_0 \omega_d} + \frac{\sigma}{\omega_d} \right)$$

Case 2: Critically damped, $\zeta = 1$

For $\zeta = 1$, $\sigma = \omega_n$, which give two real roots

$$s_{1,2} = -\sigma$$

So we can rewrite the Laplace transform equation as

$$X(s) = \frac{s + 2\sigma}{(s + \sigma)^2} x_0 + \frac{1}{(s + \sigma)^2} v_0$$

Taking the inverse Laplace transform, and noting that

$$\mathcal{L}^{-1} \left\{ \frac{s + A}{(s + \sigma)^2} \right\} = e^{-\sigma t} + (A - \sigma)te^{-\sigma t}$$

gives

$$x(t) = [e^{-\sigma t} + \sigma te^{-\sigma t}]x_0 + v_0 te^{-\sigma t}$$

$$x(t) = e^{-\sigma t} [x_0 + (\sigma x_0 + v_0)t]$$

Overdamped, $\zeta > 1$

For $\zeta > 1$, $\sigma > \omega_n$ which gives two real roots

$$s_{1,2} = -\sigma \pm \delta$$

where

$$\delta = \sigma \sqrt{1 - 1/\zeta^2}$$

We can then write the Laplace transform equation as

$$X(s) = \frac{s}{(s + \sigma + \delta)(s + \sigma - \delta)} x_0 + \frac{2\sigma x_0 + v_0}{(s + \sigma + \delta)(s + \sigma - \delta)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned} x(t) = \frac{1}{(\sigma + \delta) - (\sigma - \delta)} [(\sigma + \delta)e^{(-\sigma - \delta)t} - (\sigma - \delta)e^{(-\sigma + \delta)t}] x_0 \\ + \frac{2\sigma x_0 + v_0}{(\sigma - \delta) - (\sigma + \delta)} [e^{(-\sigma - \delta)t} - e^{(-\sigma + \delta)t}] \end{aligned}$$

And some algebraic manipulation gives

$$x(t) = \frac{-(\sigma - \delta)x_0 - v_0}{2\delta} e^{(-\sigma - \delta)t} + \frac{(\sigma + \delta)x_0 + v_0}{2\delta} e^{(-\sigma + \delta)t}$$

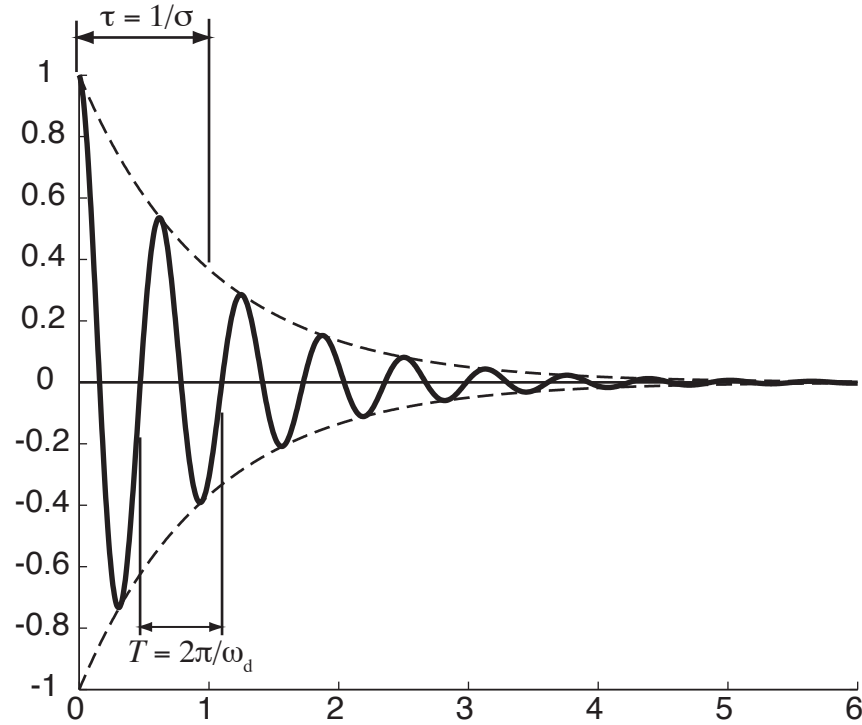


Figure 6.1: Sample sketch of an underdamped second-order natural response

Plotting Natural Response

Once σ and ω_d are known, the natural response may easily be sketched. Figure 6.1 shows how this is done for an underdamped system. First, plot the exponential functions $e^{-\sigma t}$ and $-e^{-\sigma t}$. Note that this is done using the methods for exponentials described in the section on first-order systems, with $\tau = 1/\sigma$. Next, find the period of vibration $T = 2\pi/\omega_d$. Plot a sinusoid with this period between the two exponentials, as shown in the figure. The sinusoid should start at the initial position, with slope equal to the initial velocity. This damped sinusoid is the natural response.

For an overdamped system, the response is just a sum of two exponentials. Typically, one of these will have a smaller time constant than the other. The one with the larger time constant will dominate, and the plot will look much like an exponential with the larger time constant.

6.6 Graphical Interpretation

It is helpful to look at the effect the location of poles has on system response. First, consider the plot of the real vs. imaginary parts of poles for overdamped or underdamped systems shown in Fig. 6.2. For overdamped systems, the two poles will be on the real axis, equidistant from $-\sigma$.

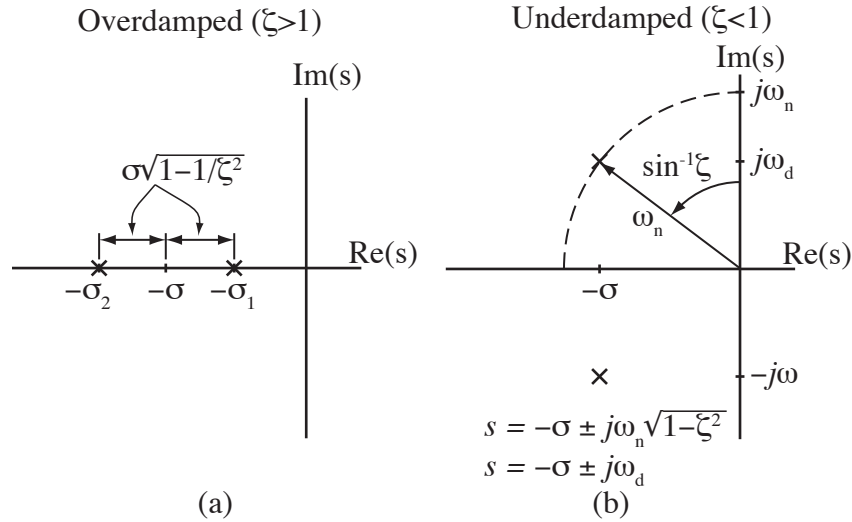


Figure 6.2: (a) Plot of poles for an overdamped system. (b) Plot of poles for an underdamped system

Their distance from $-\sigma$ will depend on ζ .

For underdamped systems, the two poles will mirror each other about the real axis. The distance from the origin to each pole is equal to ω_n . The imaginary part of each pole is simply $j\omega_d$, and the real part is simply $-\sigma$. Recall that the time constant τ and period of vibration T are both easily found from σ and ω_d . In addition, if a line (of length ω_n) is drawn from the origin to the pole, it will make an angle with respect to the imaginary axis of

$$\sin^{-1}\left(\frac{\sigma}{\omega_n}\right) = \sin^{-1}\zeta$$

If poles are plotted on this “s-plane,” their locations have direct relationship to the nature of the free response, as shown in Fig. 6.3. As poles move up or down from the real axis, the frequency of vibration increases (and the period of vibration decreases). As poles move to the left from the imaginary axis, the time constant for exponential decay becomes shorter. As poles move to the right from the imaginary axis, the system becomes unstable, and the time constant for exponential increase becomes shorter. You should spend some time studying Fig. 6.3. It embodies one of the most powerful concepts taught in this class.

6.7 Impulse Response

So far, we have considered the free response of first-order systems, undamped second-order systems, and damped second-order systems. The impulse response is the forced response of the sys-

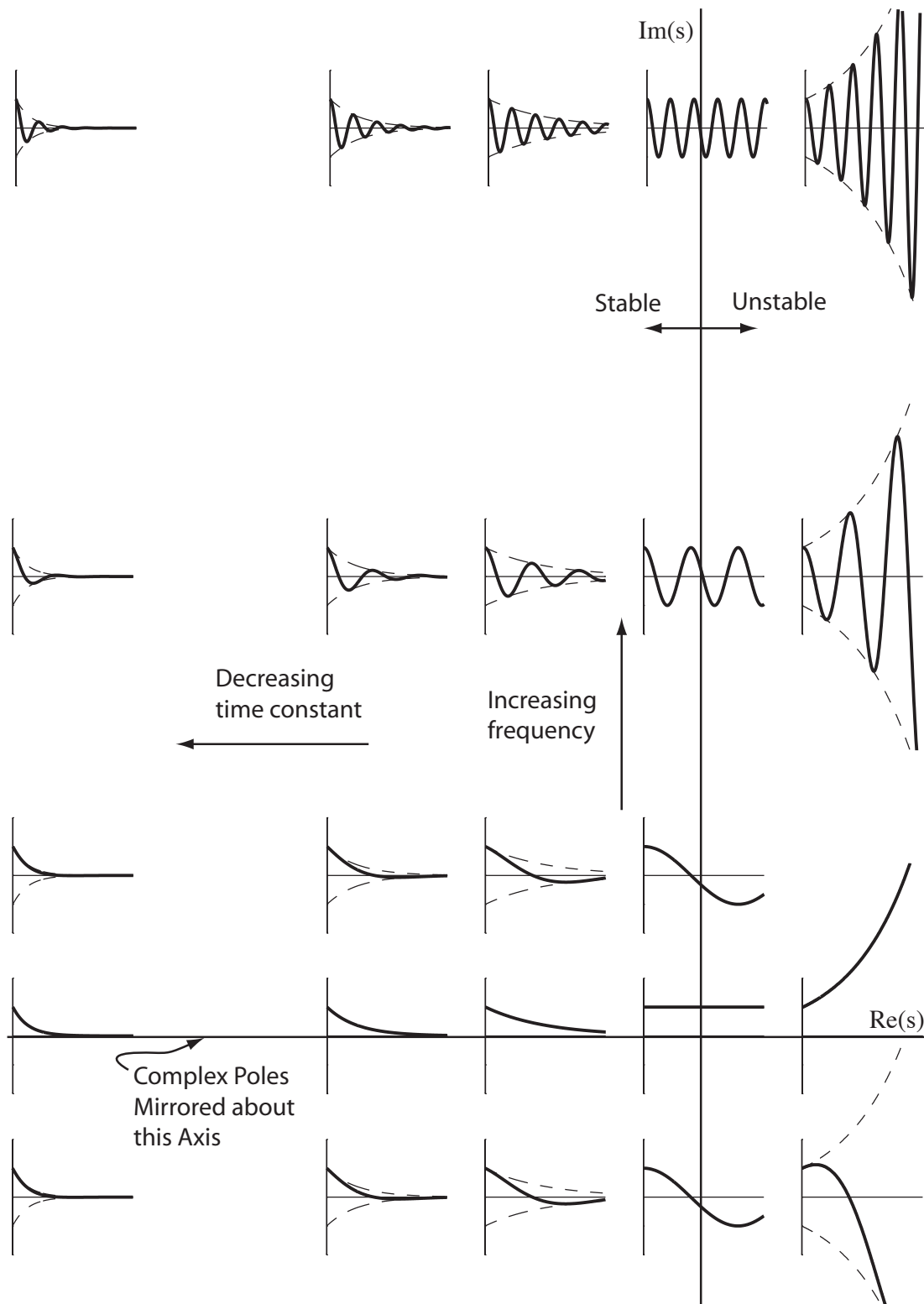


Figure 6.3: Effect of pole locations on system response

tem to an impulse input. Mathematically, an impulse input is the Dirac delta function. Physically, it corresponds to a pulse input that lasts negligibly long, such as an impact load (like a hammer blow to a mass, for example).

Consider the second-order system

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = \frac{1}{m}F(t) = A\delta(t)$$

where $\delta(t)$ is the Dirac delta function and A is a constant. Since the Laplace transform of the Dirac delta is 1, the transform of the entire equation, ignoring initial conditions, is

$$s^2X(s) + \frac{b}{m}sX(s) + \frac{k}{m}X(s) = A$$

or

$$X(s) = \frac{A}{s^2 + bs/m + k/m}.$$

But this is identical to the Laplace transform we would get for the free response due to an initial velocity of A and an initial displacement of 0! Hence, the impulse response of any second-order system will have the same form as its free response, with initial displacement of 0 and initial velocity equal to the constant multiplier of the Dirac delta function. For a first-order system, the impulse response is the same as the free response with initial condition A .

6.8 Step Response

The step response of a system is its response to a unit step function $u_s(t)$. By definition, the step function is 0 for all times less than zero, and 1 for all times greater than zero.

6.8.1 First-Order System

Consider the system

$$\tau\dot{x} + x = F(t) = u_s(t).$$

The Laplace transform, ignoring initial conditions, is

$$\tau sX(s) + X(s) = \frac{1}{s}$$

and

$$X(s) = \frac{1}{s(\tau s + 1)}.$$

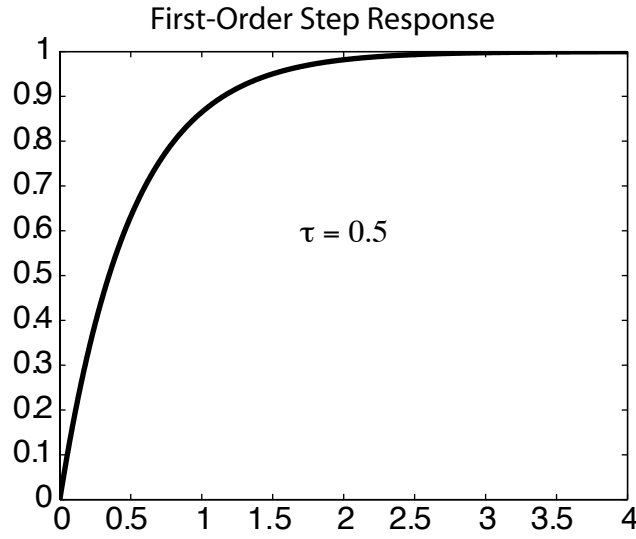


Figure 6.4: First-order step response with $\tau = 0.5$ s.

Taking the inverse Laplace transform gives

$$x(t) = 1 - e^{-t/\tau}$$

Therefore, the step response for a first-order system has a steady-state value of 1, which it approaches exponentially depending on the time constant of the system. The step response of a first-order system with time constant 0.5 s is shown in Fig. 6.4.

6.8.2 Second-Order System

For a second-order system in standard form

$$\ddot{x} + 2\sigma\dot{x} + \omega_n^2 x = F(t) = u_s(t)$$

the Laplace transform gives

$$X(s) = \frac{1}{s(s^2 + 2\sigma s + \omega_n^2)}.$$

Overdamped

If the system is overdamped, the Laplace transform of x can be written

$$X(s) = \frac{1}{s(s + \sigma_1)(s + \sigma_2)} = \frac{A}{s} + \frac{B}{s + \sigma_1} + \frac{C}{s + \sigma_2}$$

where A , B , and C are found using the method of partial fractions. The solution is

$$x(t) = A + Be^{-\sigma_1 t} + Ce^{-\sigma_2 t}.$$

As with the first-order system, the steady-state behavior is constant, in this case with $x_{ss} = A$, with exponential rise to this point. Hence, the response will look much like first-order response with time constant equal to the longer of the two time constants in the system.

Critically Damped

For a critically damped system, the Laplace transform of x is

$$X(s) = \frac{1}{s(s + \sigma)^2} = \frac{A}{s} + \frac{B}{s + \sigma} + \frac{C}{(s + \sigma)^2}$$

and the time response is

$$x(t) = A + Be^{-\sigma t} + Cte^{-\sigma t}.$$

As before, the response rises exponentially to a constant value, with time constant equal to $1/\sigma$.

Underdamped

For an underdamped system, the Laplace transform is

$$X(s) = \frac{1}{s(s^2 + 2\sigma s + \omega_n^2)} = \frac{A}{s} + \frac{Bs + C}{(s + \sigma)^2 + \omega_d^2}$$

and the time response is

$$x(t) = A + e^{-\sigma t}(D \sin \omega_d t + E \cos \omega_d t)$$

where D and E are found as part of the solution process. In this case, the steady-state solution reaches a constant value, but it oscillates about this final value. Hence, the response will *overshoot* — that is, it will exceed the steady-state value one or more times during the oscillation, before finally settling to the steady-state value. Note, however, that the time constant for exponential decay to the steady-state value is still $1/\sigma$, and the frequency of vibration about the steady-state value is still ω_d . A typical step response for an underdamped system is shown in Fig. 6.5.

Undamped

The Laplace transform of $x(t)$ for this case is

$$X(s) = \frac{1}{s(s^2 + \omega_n^2)} = \frac{1}{\omega_n^2} \left(\frac{1}{s} - \frac{s}{s^2 + \omega_n^2} \right)$$

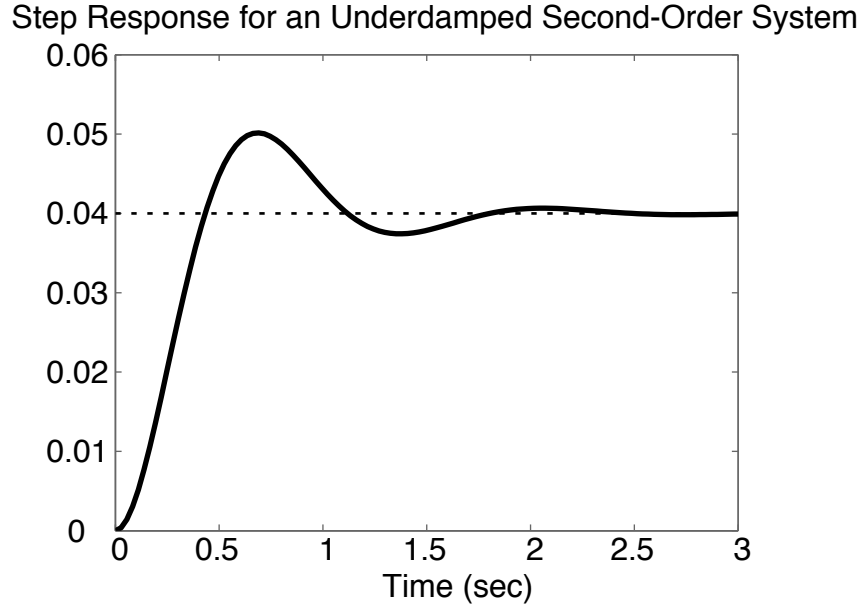


Figure 6.5: Underdamped second-order step response with $\sigma = 2$ s and $\omega_n = 5$ (so $\zeta = 0.4$)

and the time response is

$$x(t) = \frac{1}{\omega_n^2}(1 - \cos \omega_n t).$$

Hence, in this case, no constant value is reached. The response oscillates continuously between 0 and $2/\omega_n^2$.

6.9 Higher-Order Systems

So far, this chapter has discussed only first and second order systems explicitly. If a system is third, fourth, or higher order, it will have additional poles — one pole per order of the system. These poles may be real or complex. Because complex poles always occur in pairs (as conjugates of each other), the distribution of real and complex poles in a higher-order system is constrained by whether the system order is even or odd. Hence, a third order system will always have at least one real pole. The other two poles may be either real or complex conjugates of each other. A fourth order system will either have four real poles, two real poles and two complex poles, or two pairs of complex poles. A fifth order system will have either five real poles, three real poles and two complex poles, or one real pole and two pairs of complex poles. Even higher-order systems will continue these trends.

In addition, recall that the free response of a system is given by the sum of the solutions due to

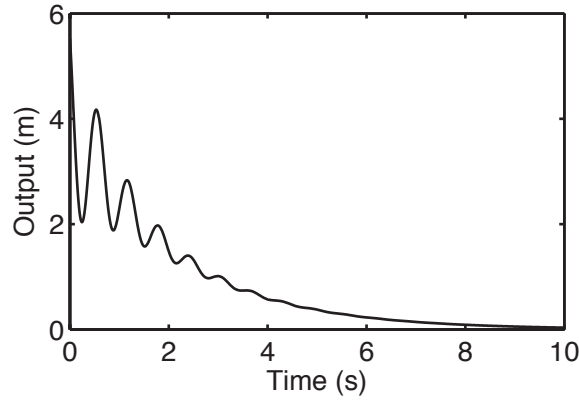


Figure 6.6: An example of a possible third order free response, consisting of an exponential added to a damped sinusoid.

each system pole. For example, an overdamped second-order system's free response is the sum of two exponentials, with each exponential corresponding to one of the two system poles. This will be true for higher-order systems as well. For example, the free response of a third order system with one real pole and two complex poles will be

$$x(t) = Ae^{s_1 t} + Be^{s_{2,r} t} \cos(s_{2,i} t - \psi)$$

where A , B , and ψ are constants depending on the initial conditions, $s_{2,r}$ is the real part of the second (or third!) pole, and $s_{2,i}$ is the imaginary part of the second pole (or the absolute value of the imaginary part of the third pole). Such a free response might look like the one shown in Fig. 6.6.

6.10 System Identification

It is sometimes necessary to use experimental data of system response to characterize the system parameters. One way to do this uses the free response of an underdamped second-order system. This method, called the “logarithmic decrement” method, allows determination of σ , ζ , ω_n , and ω_d . Thus, for a mass-spring-damper system, it allows determination of any two of the mass, spring constant, or damping constant, as long as the third value is known.

Consider the sample free response of an underdamped second-order system shown in Fig. 6.7. The values of the response at successive peaks of the response are called x_1 , x_2 , and so on to x_n . The times at which each peak occurs are called t_1 , t_2 , and so on to t_n . From the magnitudes x_1 and

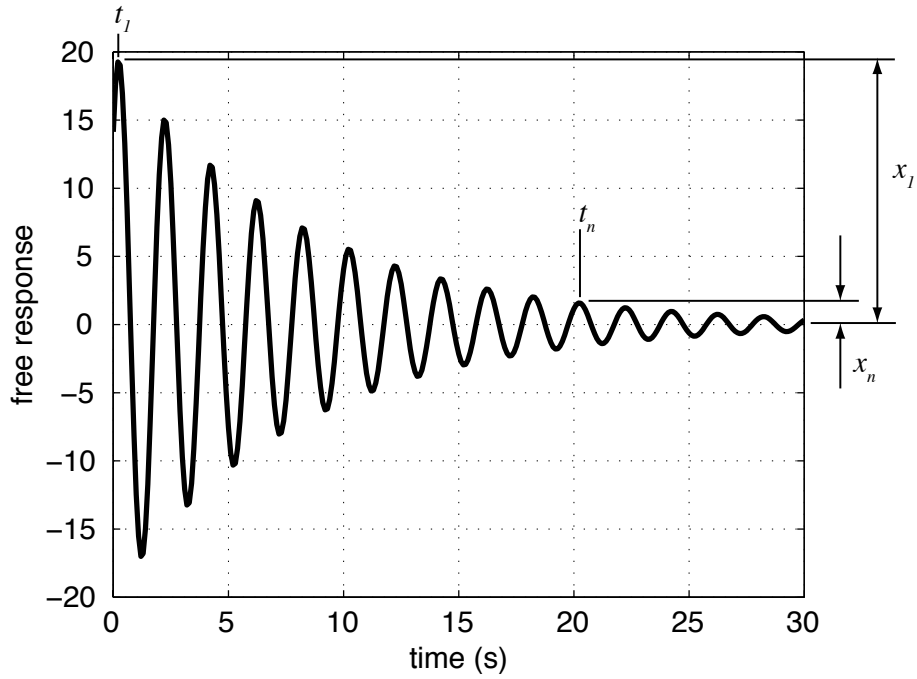


Figure 6.7: Free response of an underdamped second-order system showing how to use the logarithmic decrement method to determine system parameters.

x_n , the damping ratio ζ can be calculated using the equation

$$\zeta = \frac{\frac{1}{n-1} \ln \left(\frac{x_1}{x_n} \right)}{\sqrt{4\pi^2 + \left[\frac{1}{n-1} \ln \left(\frac{x_1}{x_n} \right) \right]^2}}$$

To find the damped natural frequency ω_d , we measure the period T . This is most accurately done over several cycles using

$$T = \frac{t_n - t_1}{n - 1}.$$

Then

$$\omega_d = \frac{2\pi}{T}.$$

With both ω_d and ζ known, we can calculate the natural frequency ω_n as

$$\omega_n = \frac{\omega_d}{\sqrt{1 - \zeta^2}}$$

and the value of σ is found from

$$\sigma = \omega_n \zeta.$$

Knowing σ and ω_n allows calculation of unknown physical parameters in the system, such as spring stiffness, mass, damping constant, and so on.

6.11 Some Examples

For the mechanical spring-mass-damper system with equation of motion

$$m\ddot{x} + b\dot{x} + kx = f(t)$$

$$x(0) = 1, \quad \dot{x}(0) = 1$$

taking the Laplace transform gives

$$ms^2X(s) - ms - m + bsX(s) - b + kX(s) = F(s)$$

$$X(s) = \frac{F(s)}{ms^2 + bs + k} + \frac{ms}{ms^2 + bs + k} + \frac{m + b}{ms^2 + bs + k}$$

If we let $\sigma = b/(2m)$ and $\omega_n = \sqrt{k/m}$, then this may be written

$$X(s) = \frac{F(s)/m}{s^2 + 2\sigma s + \omega_n^2} + \frac{s}{s^2 + 2\sigma s + \omega_n^2} + \frac{1 + 2\sigma}{s^2 + 2\sigma s + \omega_n^2}$$

We will consider two inputs: $f(t) = u_s(t)$, the step input function, and $f(t) = 0$, the natural response. Further we will consider model parameters that yield an underdamped system, a critically damped system, and an overdamped system.

Underdamped

Let $m = 1$, $b = 1$, and $k = 25$. Then $\omega_n = 5$, $\sigma = 0.5$, and $\zeta = 0.1$. Since this is an underdamped system, the eigenvalues are $-0.5 \pm j\sqrt{5^2 - 0.5^2} = -0.5 \pm j4.97$. The Laplace transform equation is then

$$X(s) = \frac{F(s)}{(s + 0.5)^2 + 24.75} + \frac{s}{(s + 0.5)^2 + 24.75} + \frac{1 + 2(0.5)}{(s + 0.5)^2 + 24.75}$$

For a step input, $F(s) = 1/s$, so

$$X(s) = \frac{1}{s[(s + 0.5)^2 + 24.75]} + \frac{s}{(s + 0.5)^2 + 24.75} + \frac{2}{(s + 0.5)^2 + 24.75}$$

and the inverse Laplace transform is

$$x(t) = \frac{1 - e^{-0.5t}[\cos(4.97t) + 0.5/4.97 \sin(4.97t)]}{.5^2 + 24.75} + e^{-0.5t}[\cos(4.97t) - 0.5/4.97 \sin(4.97t)] + 2e^{-0.5t} \frac{\sin(4.97t)}{4.97}$$

and collecting terms gives

$$x(t) = \frac{1}{25} + e^{-0.5t} \left[\left(-\frac{1}{25} + 1 \right) \cos(4.97t) + \left(-\frac{0.5}{(4.97)(25)} - \frac{0.5}{4.97} + \frac{2}{4.97} \right) \sin(4.97t) \right]$$

$$\boxed{x(t) = \frac{1}{25} + e^{-0.5t} \left[\frac{24}{25} \cos(4.97t) + \frac{74}{(4.97)(50)} \sin(4.97t) \right]} \quad \text{step response}$$

Similarly, if there is no input ($f(t) = 0$), then the first term of the Laplace transform equations would be zero, giving

$$X(s) = \frac{s}{(s+0.5)^2 + 24.75} + \frac{2}{(s+0.5)^2 + 24.75}$$

and the inverse Laplace transform

$$x(t) = e^{-0.5t} [\cos(4.97t) - 0.5/4.97 \sin(4.97t)] + 2e^{-0.5t} \frac{\sin(4.97t)}{4.97}$$

and collecting terms gives

$$\boxed{x(t) = e^{-0.5t} \left[\cos(4.97t) + \frac{1.5}{4.97} \sin(4.97t) \right]} \quad \text{natural response}$$

Notice that the only difference between the solution for a step input and the natural response is the presence of the constant in the step response, and the values of the coefficients of the sin and cos terms.

Critically Damped

If $m = 1$, $b = 10$, and $k = 25$, then $\omega_n = 5$, $\sigma = 5$, and $\zeta = 1$. The eigenvalues are both real, at $s_{1,2} = -\sigma = -5$. The Laplace transform equation is

$$X(s) = \frac{F(s)}{(s+5)^2} + \frac{s}{(s+5)^2} + \frac{1+2(5)}{(s+5)^2}$$

For a step input, this is

$$X(s) = \frac{1}{s(s+5)^2} + \frac{s}{(s+5)^2} + \frac{11}{(s+5)^2}$$

To take the inverse Laplace transform, we can apply the method of partial fractions to the first term. A general result gives

$$\mathcal{L}^{-1} \left\{ \frac{1}{s(s+r_1)^2} \right\} = \frac{1}{r_1^2} [1 - e^{-r_1 t} - r_1 t e^{-r_1 t}]$$

So the solution is

$$x(t) = \frac{1}{25}[1 - e^{-5t} - 5te^{-5t}] + e^{-5t} + (11 - 5)te^{-5t}$$

$$\boxed{x(t) = \frac{1}{25} + \frac{24}{25}e^{-5t} + \frac{29}{5}te^{-5t}} \quad \text{step response}$$

As with the underdamped system, the natural response just leaves out the first term:

$$X(s) = \frac{s}{(s+5)^2} + \frac{11}{(s+5)^2}$$

$$x(t) = e^{-5t} + (11 - 5)te^{-5t}$$

$$\boxed{x(t) = e^{-5t} + (6)te^{-5t}} \quad \text{natural response}$$

Overdamped

If $m = 1$, $b = 10$, and $k = 16$, then $\omega_n = 4$, $\sigma = 5$, and $\zeta = 1.25$. The eigenvalues are both real, at $s_{1,2} = -\sigma \pm \sqrt{\sigma^2 - \omega_n^2} = -5 \pm \sqrt{9} = -8, -2$. The Laplace transform equation is

$$X(s) = \frac{F(s)}{(s+8)(s+2)} + \frac{s}{(s+8)(s+2)} + \frac{1+2(5)}{(s+8)(s+2)}$$

and, for a step input

$$X(s) = \frac{1}{s(s+8)(s+2)} + \frac{s}{(s+8)(s+2)} + \frac{11}{(s+8)(s+2)}$$

Taking the inverse Laplace transform gives

$$x(t) = \frac{1}{16} \left[1 - \left(\frac{2e^{-8t} - 8e^{-2t}}{2-8} \right) \right] + \frac{8e^{-8t} - 2e^{-2t}}{8-2} + 11 \frac{e^{-8t} - e^{-2t}}{2-8}$$

and combining terms gives

$$\boxed{x(t) = \frac{1}{16} - \frac{23}{48}e^{-8t} + \frac{17}{12}e^{-2t}} \quad \text{step response}$$

As with the previous two cases, the natural response results when the first term in the Laplace transform equation is ignored:

$$X(s) = \frac{s}{(s+8)(s+2)} + \frac{11}{(s+8)(s+2)}$$

$$x(t) = \frac{8e^{-8t} - 2e^{-2t}}{8-2} + 11 \frac{e^{-8t} - e^{-2t}}{2-8}$$

$$\boxed{x(t) = -\frac{1}{2}e^{-8t} + \frac{3}{2}e^{-2t}} \quad \text{natural response}$$

Chapter 7

Solving Equations of Motion to Obtain Transient Response

We have used traditional methods to derive equations of motion for simple mechanical systems:

- Translational Systems
- Rotational Systems
- Combined Translation and Rotation

With the equations of motion generated, we need to convert them to a form suitable for numerical solution, called the “state-variable form.” This form consists of coupled first-order differential equations, one for each state variable in the system:

$$\dot{x} = f(x, u)$$

Equations of this form can easily be solved using Runge-Kutta integration routines.

The *state* of a dynamic system is the smallest set of variables (called state variables) such that the knowledge of these variables at $t = t_0$, together with the knowledge of the input for $t \geq t_0$, completely determines the system behavior for any time $t \geq t_0$. The state of a system is related to the energy stored in the elements of a system at any given instant. Hence, for a minimal representation, we will have one state variable for each energy storage element of the system. For mechanical systems, for example, energy storage elements are springs and masses. Hence, for a minimal representation, we will have one state variable for each spring and mass in the model. In the state-variable form, we will express the system using one first-order differential equation for each state variable.

7.1 Putting Equations of Motion in State-Variable Form

Equations of motion can always be expressed as a set of coupled first-order differential equations:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, \dots, x_n, u) \\ \dot{x}_2 &= f_2(x_1, x_2, \dots, x_n, u) \\ &\vdots \\ \dot{x}_n &= f_n(x_1, x_2, \dots, x_n, u) \end{aligned}$$

Or, using vector notation:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}).$$

This form is completely general, allowing any number of state variables and inputs, as well as non-linear relationships. \mathbf{x} is a column vector having n elements for an n th order system. It is called the “state vector.” $\dot{\mathbf{x}}$ is then called the state derivative, and is also a vector of length n . \mathbf{u} is the input vector, and can be any length. \mathbf{f} is a vector function, mapping the state vector and the input vector to the state derivative.

For higher-order differential equations, it is normally straightforward to get the equations of motion into state variable form.

Example

For a simple pendulum with damping at the pivot:

$$mL^2\ddot{\theta} + b\dot{\theta} + mgL\sin\theta = 0$$

Step 1: Define the state variables. These will normally correspond to the energy storage elements in the system.

For this case, there are 2 state variables, corresponding to the kinetic and potential energy stored in the pendulum. A convenient set of state variables is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \theta \\ \Omega \end{bmatrix}$$

Step 2: Solve the equation(s) for the highest-order derivative. This will then become a state derivative:

$$\ddot{\theta} = -\frac{b}{mL^2}\dot{\theta} - \frac{g}{L}\sin\theta$$

Or, since $\Omega = \dot{\theta}$

$$\dot{\Omega} = -\frac{b}{mL^2}\Omega - \frac{g}{L}\sin\theta$$

Notice that the right-hand side of this equation contains only state variables (not their derivatives) and system parameters, which are constant.

Step 3: Write equations for the state derivatives in terms of states and inputs.

$$\dot{\theta} = \Omega \quad (7.1)$$

$$\dot{\Omega} = -\frac{b}{mL^2}\Omega - \frac{g}{L}\sin\theta \quad (7.2)$$

With the system in state-variable form, it is ready to be solved using numerical solution.

Example

For the equations of motion given below, find a state variable representation.

$$a(m_1\ddot{x}_1 + b_1\dot{x}_1) + b \left[k \left(\frac{b}{a}x_1 - x_2 \right) \right] = aF(t)$$

$$m_2\ddot{x}_2 + b\dot{x}_2 + k \left(x_2 - \frac{b}{a}x_1 \right) = 0$$

Step 1: Choose state variables. Inspection of both equations shows that they contain x_1 , x_2 , and their first and second derivatives. Hence, a convenient set of state variables is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \end{bmatrix}$$

Step 2: Solve the equations for the highest-order derivative. This results in

$$\dot{v}_1 = \frac{aF(t) - ab_1v_1 - b \left[k \left(\frac{b}{a}x_1 - x_2 \right) \right]}{am_1}$$

$$\dot{v}_2 = \frac{-bv_2 - k \left(x_2 - \frac{b}{a}x_1 \right)}{m_2}$$

Step 3: The final step is to write two additional state equations by considering the relationships between the state variables. The result is

$$\dot{x}_1 = v_1$$

$$\dot{x}_2 = v_2$$

7.2 Numerical Integration

Detailed explanation of the methods of numerical integration is beyond the scope of this class. Instead, we will use numerical integration to study system behaviors. The basic idea behind numerical solution of differential equations with initial values is to use derivative information to predict the value of the state variables at successive time steps after the initial time. State variable form is normally used because it explicitly gives derivative information as a function of current values of the state variables, inputs, and system parameters.

The simplest method of numerical prediction is called Euler's method. Though it is not very accurate, it is useful in describing the general approach of numerical solution. In Euler's method, the value of a state variable x at a time $t + \Delta t$ is approximated as $x(t) + \Delta t \dot{x}(t)$. In other words, the state variable is assumed to vary linearly, with slope equal to the derivative with respect to time. Values of the state variable at successive time steps are found by iteration.

More complicated methods, such as the Runge-Kutta methods, are based on the same idea, though they use a more accurate way to predict future values of the state variables. However, in every case, these predictions are based on values of the derivatives of state variables.

7.3 Matlab Introduction

In this class, we will use Matlab for numerical analysis of dynamic systems. This section introduces Matlab for those less familiar with it.

- To start Matlab, type "matlab" at the Unix prompt for Unix. In Windows, under CAEDM, you will find the icon to start Matlab under the Math Programs directory in the start menu.
- The main Matlab window is called the command window. The Matlab prompt is `>>`. You type commands at the prompt, and Matlab returns responses in the command window.
- To get help, type `helpdesk` or `helpwin`. You can also access help from the menus at the top of the command window. To get help on a particular command, type `help <command name>`.
- Type `quit` or `exit` at the prompt to exit, or use the menu commands.
- The basic philosophy of Matlab is that every variable is a matrix. A scalar is simply a 1×1 matrix, a vector is a $1 \times n$ or $n \times 1$ matrix, and so on.

- To enter a matrix and assign it to a variable *A* type:

```
A = [1 2 3; 4 5 6; 7 8 9]; or
```

```
b = [4; 6; 8];
```

(The semicolon at the end of each command simply represses any responses so that Matlab won't print anything to the command screen as a result of the command.)

- Some basic operations:

```
c = A*b
```

```
help inv
```

```
D = inv(A)
```

```
I = A*D
```

- To create a vector of uniformly spaced values, type

```
t = 0:0.01:1;
```

This creates a vector of 101 points from 0 to 1 with increments of 0.01. *t*(11) references the 11th element of the vector *t*. Typing *t*(11:15) references the 11th through the 15th values.

- Other functions:

```
y = sin(t) creates a vector y by taking the sin of all elements of t
```

```
z = cos(t)
```

```
w = y+z
```

```
w = y*z ← This doesn't work! Why not?
```

We need to use *w* = *y*'**z* or *w* = *y***z*'. If we want *w* to be a vector where each element is the product of the corresponding elements of *y* and *z*, then we type

```
w = y.*z
```

- To plot:

```
plot(t,y)
```

```
plot(t,y,t,z)
```

```
plot(t,y,'-',t,z,'--',t,w,'-.')
```

```
help plot
```

To write programs, we use m-files (files saved with a .m extension). The first type we'll look at are called scripts. These are:

- a list of Matlab commands, to be executed in order, saved in a file
- usually written by the user

- executed repeatedly simply by typing the name of the .m file at the command prompt
- run in the global workspace.

- A sample script:

```
% This is a test
t = 0:0.01:1;
y = 2*sin(2*t);
z = 5*cos(5*t);
w = y.*z;
figure(1)
clf
plot(t,y,t,z,t,w);
figure(2)
clf
subplot(311); plot(t,y);
subplot(312); plot(t,z);
subplot(313); plot(t,w);
```

(Note that the % sign makes the rest of the characters in that line into a comment.)

The other type of program is called a function. For a function:

- can be called like a C function
- parameters are passed in and out
- runs with only local variables unless global variables are explicitly defined
- many, many built-in Matlab functions
- users also write their own
- first line (other than comments) is a function declaration
- sample function:

```
% Add bias and gain to a signal
function out = calibrate(in,gain,bias)
out = gain*in + bias;
```


- A function can be called many times with different arguments:

```
out1 = calibrate(in1,g1,b1);  
out2 = calibrate(in2,g2,b2);  
out3 = calibrate(in3,g3,b3);
```

Example

For the pendulum problem above, the state equations were given by (7.1) and (7.2). They are repeated here for convenience:

$$\begin{aligned}\dot{\theta} &= \Omega \\ \dot{\Omega} &= -\frac{b}{mL^2}\Omega - \frac{g}{L}\sin\theta\end{aligned}$$

To solve this problem, we will take $L = 0.8$ ft, $g = 32.2$ ft/s², $b = 0.5$ lb·s·ft, and $m = 1.0$ slugs. In class, we will show some Matlab code, consisting of a script and a function, to numerically simulate this system. The code will also be available on the course web site.

Chapter 8

Frequency Response

8.1 Introduction

The frequency response of a system is a measure of that system's response to sinusoidal inputs of varying frequencies. This is an important topic because many engineering systems have sinusoidal inputs. For example, rotating machines, AC circuits, acoustic waves, and wave forces acting on marine structures all typically have sinusoidal inputs. In addition, using Fourier analysis, any arbitrary input signal can be represented as a sum of sinusoids. Hence, understanding the frequency response of the system provides an understanding of how the system will respond to general inputs.

In addition, for sinusoidal inputs, understanding the frequency response of the system allows concise, rapid characterization of the system's behavior. The frequency response of the system can also be determined either analytically or experimentally, making this a valuable tool in understanding real systems.

Recall that if the input to a constant-coefficient (time-invariant) linear system is sinusoidal with amplitude A and frequency ω

$$u(t) = A \sin \omega t$$

then, after any transients have died away, the forced response of the system will also be a sinusoid of the *same frequency*, but with possibly different amplitude and phase. Hence, the response can be written

$$y(t) = B \sin(\omega t + \phi).$$

From this reasoning, we see that we will completely understand the long-term system response if we know just two things: the amplitude B of the output and the phase difference ϕ . If we write the

system in transfer-function form, we will get

$$\frac{Y(s)}{U(s)} = G(s)$$

Sinusoidal behavior is represented by $s = j\omega$, giving

$$\frac{Y(j\omega)}{U(j\omega)} = G(j\omega)$$

So we can find the magnitude ratio as

$$\left| \frac{B}{A} \right| = \left| \frac{Y(j\omega)}{U(j\omega)} \right| = |G(j\omega)|$$

and the phase difference as

$$\phi = \angle[Y(j\omega) - U(j\omega)] = \angle G(j\omega).$$

These two relations show that if we can find the magnitude $|G(j\omega)|$ and angle $\angle G(j\omega)$ as functions of frequency, then we'll know how $y(t)$ behaves for any sinusoidal input.

8.2 Graphical Method

Hence, the question reduces to how to find $|G|$ and $\angle G$. If we know the poles and zeros of $G(s)$, then we can find $|G|$ and $\angle G$ graphically. For example,

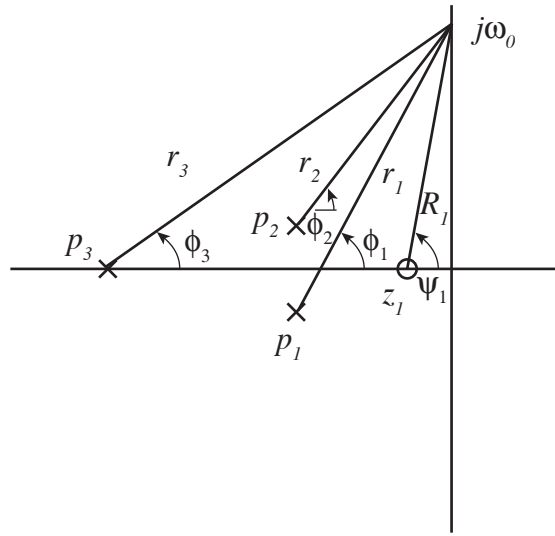
$$G(s) = \frac{s + z_1}{(s + p_1)(s + p_2)(s + p_3)}$$

$$G(j\omega_0) = \frac{j\omega_0 + z_1}{(j\omega_0 + p_1)(j\omega_0 + p_2)(j\omega_0 + p_3)}$$

$$G(j\omega_0) = \frac{R_1 e^{j\psi_1}}{r_1 e^{j\phi_1} r_2 e^{j\phi_2} r_3 e^{j\phi_3}}$$

$$G(j\omega_0) = \frac{R_1}{r_1 r_2 r_3} e^{j(\psi_1 - \phi_1 - \phi_2 - \phi_3)}$$

where $R_1, r_1, r_2, r_3, \psi_1, \phi_1, \phi_2$, and ϕ_3 are illustrated in the following graph:



Then

$$|G(j\omega_0)| = \frac{R_1}{r_1 r_2 r_3}$$

$$\angle G(j\omega_0) = \psi_1 - \phi_1 - \phi_2 - \phi_3$$

Plots of the magnitude and phase ratio for $G(s)$ could be determined by making these calculations for many values of ω_0 as it varied from $0 \rightarrow \infty$. However, this process is too hard. Instead, we look analytically at the transfer function.

8.3 Analytical Method

If we know a system's transfer function, we can calculate its frequency response. Consider the first-order system:

$$\tau \dot{y} + y = F(t)$$

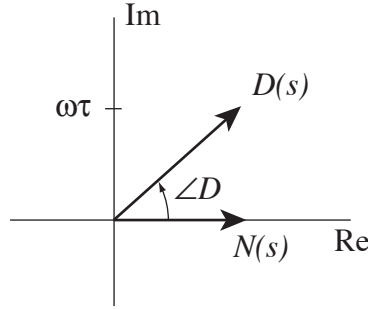
with transfer function

$$\frac{Y(s)}{F(s)} = G(s) = \frac{1}{\tau s + 1}.$$

We can find the magnitude and phase response of the system by evaluating $G(s)$ at $s = j\omega$. Recall that the Laplace variable s is a complex number: $s = \sigma + j\omega$. Sinusoidal inputs correspond to $s = j\omega$, with $\sigma = 0$. Then

$$G(j\omega) = \frac{1}{j\omega\tau + 1}.$$

Plotting the numerator and denominator in the complex plane gives



Then

$$|G(j\omega)| = \frac{|N(j\omega)|}{|D(j\omega)|} = \frac{1}{\sqrt{1 + (\omega\tau)^2}}$$

$$\angle G(j\omega) = \angle N(j\omega) - \angle D(j\omega) = 0 - \tan^{-1}(\omega\tau) = -\tan^{-1}(\omega\tau)$$

We can apply the same approach to a second-order system.

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2y = \omega_n^2F(t)$$

Taking the Laplace transform and assuming zero initial conditions gives

$$s^2Y(s) + 2\zeta\omega_nsY(s) + \omega_n^2Y(s) = \omega_n^2F(s)$$

$$G(s) = \frac{Y(s)}{F(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_ns + \omega_n^2}$$

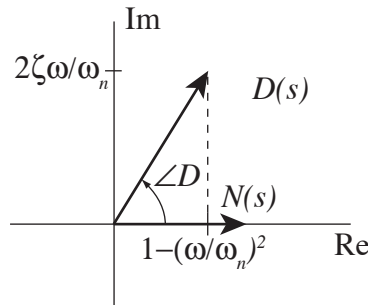
And taking $s = j\omega$ gives

$$G(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2}$$

$$G(j\omega) = \frac{\omega_n^2}{\omega_n^2 - \omega^2 + j(2\zeta\omega\omega_n)}$$

$$G(j\omega) = \frac{1}{1 - (\omega/\omega_n)^2 + j[2\zeta(\omega/\omega_n)]} = \frac{N(j\omega)}{D(j\omega)}$$

Plotting $N(j\omega)$ and $D(j\omega)$ in the complex plane gives



$$|G(j\omega)| = \frac{|N(j\omega)|}{|D(j\omega)|} = \frac{1}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + (2\zeta\omega/\omega_n)^2}}$$

$$\angle G(j\omega) = \angle N(j\omega) - \angle D(j\omega) = 0 - \tan^{-1} \left(\frac{2\zeta\omega/\omega_n}{1 - (\omega/\omega_n)^2} \right)$$

$$\angle G(j\omega) = -\tan^{-1} \left(\frac{2\zeta\omega/\omega_n}{1 - (\omega/\omega_n)^2} \right)$$

The transfer-function approach illustrated by these examples allows us to find the magnitude and phase response for higher-order linear systems without great difficulty. In general,

$$G(s) = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n}$$

Letting $s = j\omega$ allows us to write $G(j\omega)$ as the ratio of two complex numbers:

$$G(j\omega) = \frac{N(j\omega)}{D(j\omega)} = \frac{a + jb}{c + jd}$$

From this result, we can evaluate

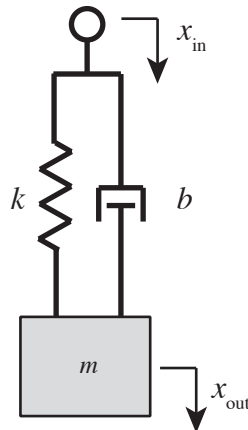
$$M(\omega) = |G(j\omega)| = \frac{\sqrt{a^2 + b^2}}{\sqrt{c^2 + d^2}}$$

$$\phi(\omega) = \angle G(j\omega) = \tan^{-1} \left(\frac{b}{a} \right) - \tan^{-1} \left(\frac{d}{c} \right)$$

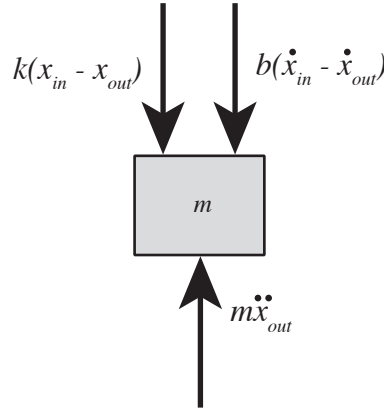
This method is very general and powerful.

8.3.1 Example

For example, consider a mass-spring-damper system being driven by an input displacement as illustrated below:



The system has the free-body diagram



The equation of motion for this system is

$$m\ddot{x}_{out} + b(\dot{x}_{out} - \dot{x}_{in}) + k(x_{out} - x_{in}) = 0$$

$$m\ddot{x}_{out} + b\dot{x}_{out} + kx_{out} = b\dot{x}_{in} + kx_{in}$$

We can find the transfer function of the system by taking the Laplace transform:

$$ms^2X_{out}(s) + bsX_{out}(s) + kX_{out}(s) = bsX_{in}(s) + kX_{in}(s)$$

$$\frac{X_{out}(s)}{X_{in}(s)} = \frac{bs + k}{ms^2 + bs + k}$$

This is the transfer function for the mass-spring-damper system driven by a displacement input.

The frequency response of the system can be studied by replacing s in the transfer function by $j\omega$. Then

$$G(j\omega) = \frac{bj\omega + k}{-m\omega^2 + bj\omega + k} = \frac{k + jb\omega}{k - m\omega^2 + jb\omega}$$

Then the magnitude is

$$|G(j\omega)| = \frac{\sqrt{k^2 + b^2\omega^2}}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}}$$

and the phase is

$$\angle G(j\omega) = \tan^{-1}\left(\frac{b\omega}{k}\right) - \tan^{-1}\left(\frac{b\omega}{k - m\omega^2}\right)$$

These equations can be plotted using a computer to create *Bode plots*. A Bode plot is simply a set of two graphs - one showing the magnitude vs. angular frequency, and the other showing phase vs. angular frequency. The magnitude plot is normally shown on a log-log scale, while the phase plot is normally drawn with a linear scale for phase and a log scale for angular frequency. In class, we will see a demonstration of using the equations developed in this example for the creation of Bode plots. The next section gives considerably more detail about generating Bode plots.

8.4 Bode Plotting Techniques

Plots of the frequency response magnitude and phase are most easily generated using a computer. However, using the techniques of Bode, we can generate a quick sketch of the frequency response that can be very valuable in understanding the behavior of a system. The first step is to factor the transfer function so that it is composed of the following 7 types of terms:

1. k (constant)
2. $(j\omega)^n$
3. $(j\omega)^{-n}$
4. $j\omega\tau + 1$
5. $\frac{1}{j\omega\tau + 1}$
6. $\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\left(\frac{j\omega}{\omega_n}\right) + 1$
7. $\frac{1}{\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\left(\frac{j\omega}{\omega_n}\right) + 1}$

When factored in this way, the transfer function is said to be in Bode form. We will describe how to plot the magnitude and phase for each type of term and then demonstrate how to generate composite plots for general transfer functions.

1. k

- magnitude plot is a horizontal line having magnitude k for all ω
- phase plot is $\phi = 0$ for all ω

The magnitude and phase plots for this case are shown in Fig. 8.1.

2. $(j\omega)^n$

- magnitude plot is a straight line with slope n passing through 1 at $\omega = 1$ rad/sec
- phase plot is $\phi = n \times 90^\circ$ for all ω

The magnitude and phase plots for this case are shown in Fig. 8.2.

3. $(j\omega)^{-n}$

- magnitude plot is a straight line with slope $-n$ passing through 1 at $\omega = 1$ rad/sec
- phase plot is $\phi = -n \times 90^\circ$ for all ω

The magnitude and phase plots for this case are shown in Fig. 8.3.

4. $j\omega\tau + 1$

- magnitude plot:

$$\text{For } \omega \ll \frac{1}{\tau}, \quad |j\omega\tau + 1| \approx 1$$

$$\text{For } \omega \gg \frac{1}{\tau}, \quad |j\omega\tau + 1| \approx \tau|j\omega| \leftarrow +1 \text{ slope}$$

These two expressions give asymptotes for small and large frequencies. Moreover, when $\omega = 1/\tau$, $\text{mag} = \sqrt{2} = 1.414$. This is the break point, or transition point, between the two asymptotes.

- phase plot:

$$\text{For } \omega \ll \frac{1}{\tau}, \quad \angle(j\omega\tau + 1) \approx \angle 1 = 0^\circ$$

$$\text{For } \omega \gg \frac{1}{\tau}, \quad \angle(j\omega\tau + 1) \approx \angle(j\omega\tau) = 90^\circ$$

There is also an asymptote between 0° at $\omega = 0.2/\tau$ and 90° at $\omega = 5/\tau$, passing through 45° at $1/\tau$.

The magnitude and phase plots for this case are shown in Fig. 8.4.

5. $\frac{1}{j\omega\tau + 1}$

- magnitude plot:

$$\text{For } \omega \ll \frac{1}{\tau}, \quad \left| \frac{1}{j\omega\tau + 1} \right| \approx 1$$

$$\text{For } \omega \gg \frac{1}{\tau}, \quad \left| \frac{1}{j\omega\tau + 1} \right| \approx \frac{1}{\tau|j\omega|} \leftarrow -1 \text{ slope}$$

These two expressions give asymptotes for small and large frequencies. When $\omega = 1/\tau$, $\text{mag} = 1/\sqrt{2} = 0.707$. This is the break point, or transition point, between the two asymptotes.

- phase plot:

$$\text{For } \omega \ll \frac{1}{\tau}, \quad \angle \left(\frac{1}{j\omega\tau + 1} \right) \approx \angle 1 = 0^\circ$$

$$\text{For } \omega \gg \frac{1}{\tau}, \quad \angle \left(\frac{1}{j\omega\tau + 1} \right) \approx \angle \frac{1}{j\omega\tau} = -90^\circ$$

There is also an asymptote between 0° at $\omega = 0.2/\tau$ and -90° at $\omega = 5/\tau$, passing through -45° at $1/\tau$.

The magnitude and phase plots for this case are shown in Fig. 8.5.

$$6. \left(\frac{j\omega}{\omega_n} \right)^2 + 2\zeta \left(\frac{j\omega}{\omega_n} \right) + 1$$

- magnitude plot:

$$\text{For } \omega \ll \omega_n, \quad \left| \left(\frac{j\omega}{\omega_n} \right)^2 + 2\zeta \left(\frac{j\omega}{\omega_n} \right) + 1 \right| \approx 1$$

$$\text{For } \omega \gg \omega_n, \quad \left| \left(\frac{j\omega}{\omega_n} \right)^2 + 2\zeta \left(\frac{j\omega}{\omega_n} \right) + 1 \right| \approx \left| \left(\frac{j\omega}{\omega_n} \right)^2 \right| \leftarrow +2 \text{ slope}$$

Again, these expressions give asymptotes for small and high frequencies. The break point between them occurs at $\omega = \omega_n$, at which

$$\left| \left(\frac{j\omega}{\omega_n} \right)^2 + 2\zeta \left(\frac{j\omega}{\omega_n} \right) + 1 \right| = 2\zeta$$

Note that the size of the magnitude ratio at $\omega = \omega_n$ depends entirely on the damping ratio ζ .

- phase plot:

$$\text{For } \omega \ll \omega_n, \quad \angle \left[\left(\frac{j\omega}{\omega_n} \right)^2 + 2\zeta \left(\frac{j\omega}{\omega_n} \right) + 1 \right] \approx \angle 1 = 0^\circ$$

$$\text{For } \omega \gg \omega_n, \quad \angle \left[\left(\frac{j\omega}{\omega_n} \right)^2 + 2\zeta \left(\frac{j\omega}{\omega_n} \right) + 1 \right] \approx \angle \left(\frac{j\omega}{\omega_n} \right)^2 = 180^\circ$$

The transition between 0° and 180° depends on ζ . Smaller ζ will have a more rapid transition, while larger ζ will cause a slower transition. However, at $\omega = \omega_n$, the phase angle will always be 90° , halfway between the angles at low and high frequencies.

The magnitude and phase plots for this case are shown in Fig. 8.6.

$$7. \frac{1}{\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\left(\frac{j\omega}{\omega_n}\right) + 1}$$

- magnitude plot:

$$\begin{aligned} \text{For } \omega \ll \omega_n, \quad & \left| \frac{1}{\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\left(\frac{j\omega}{\omega_n}\right) + 1} \right| \approx 1 \\ \text{For } \omega \gg \omega_n, \quad & \left| \frac{1}{\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\left(\frac{j\omega}{\omega_n}\right) + 1} \right| \approx \left| \left(\frac{j\omega}{\omega_n}\right)^{-2} \right| \leftarrow -2 \text{ slope} \end{aligned}$$

Again, these expressions give asymptotes for small and high frequencies. The break point between them occurs at $\omega = \omega_n$, at which

$$\left| \frac{1}{\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\left(\frac{j\omega}{\omega_n}\right) + 1} \right| = \frac{1}{2\zeta}$$

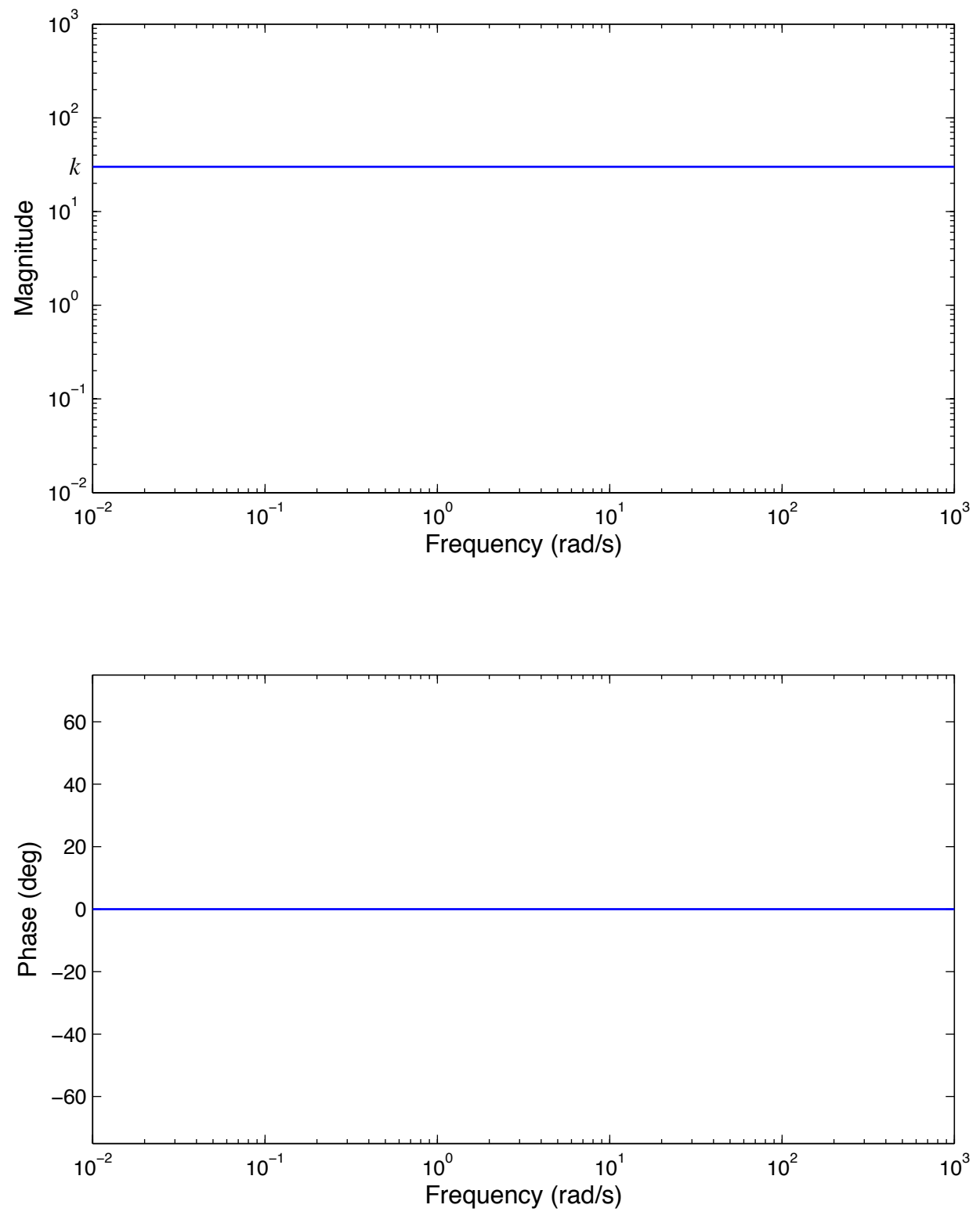
Again, the magnitude ratio at $\omega = \omega_n$ depends entirely on the damping ratio ζ .

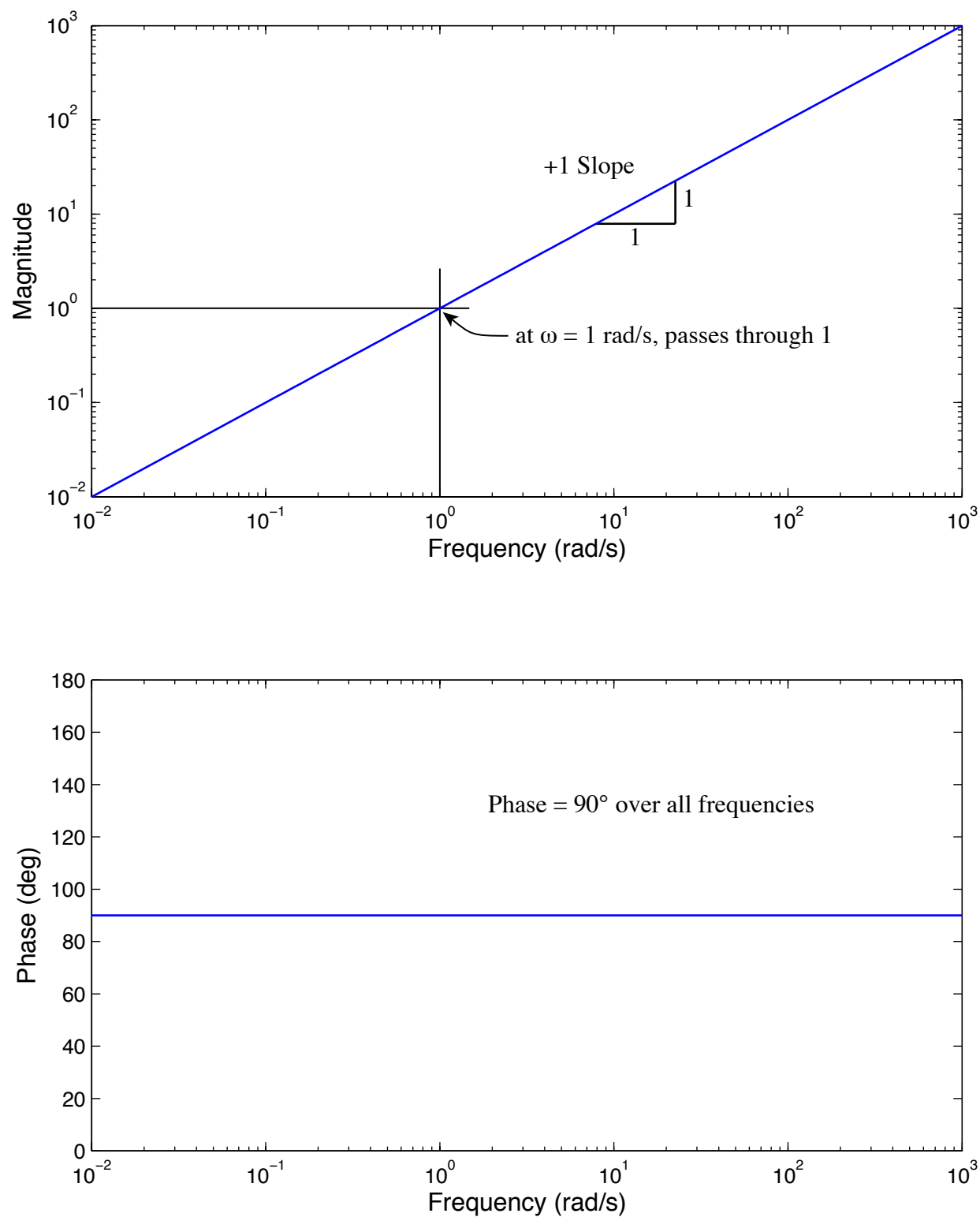
- phase plot:

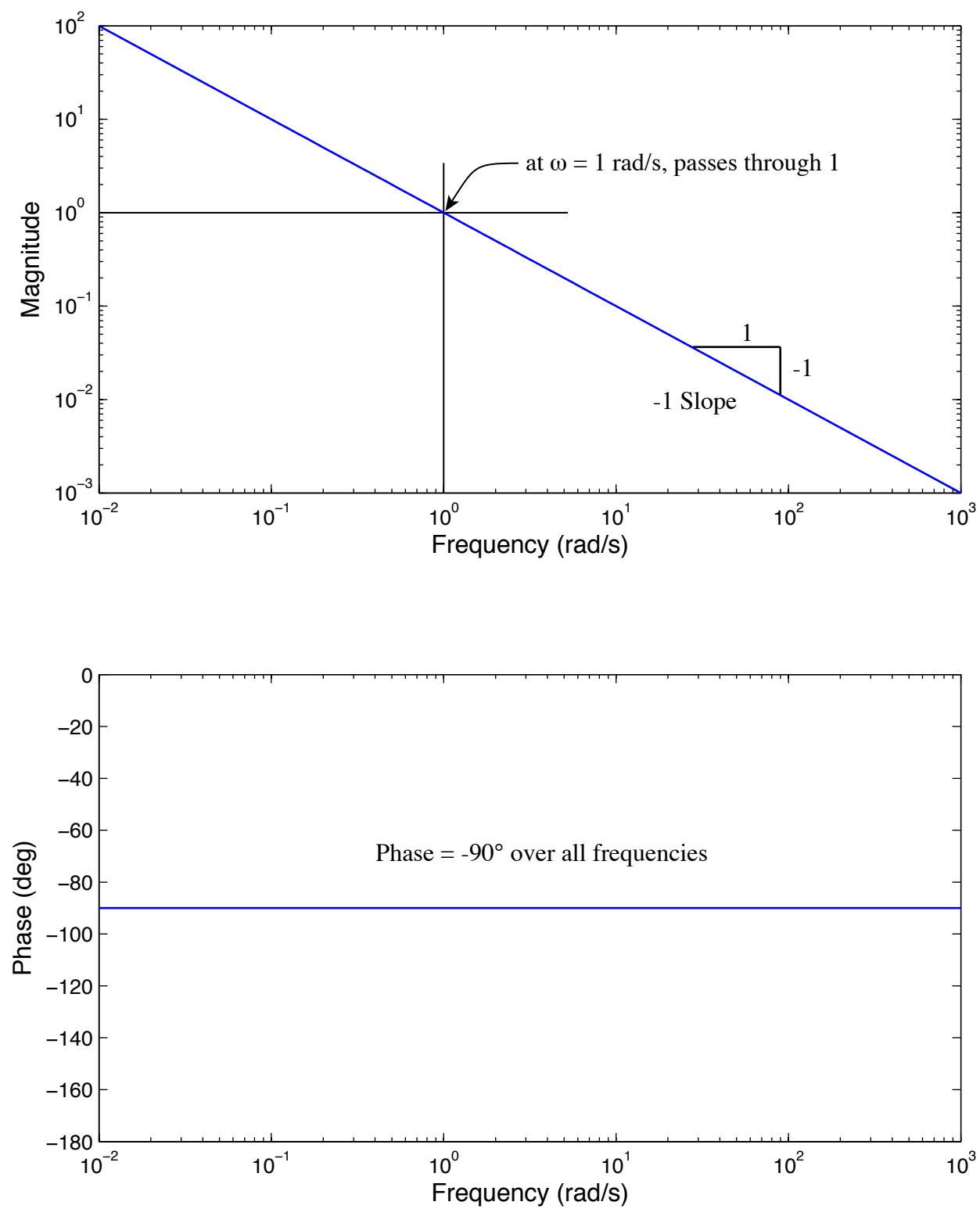
$$\begin{aligned} \text{For } \omega \ll \omega_n, \quad & \angle \left[\frac{1}{\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\left(\frac{j\omega}{\omega_n}\right) + 1} \right] \approx \angle 1 = 0^\circ \\ \text{For } \omega \gg \omega_n, \quad & \angle \left[\frac{1}{\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\left(\frac{j\omega}{\omega_n}\right) + 1} \right] \approx \angle \left(\frac{j\omega}{\omega_n}\right)^{-2} = -180^\circ \end{aligned}$$

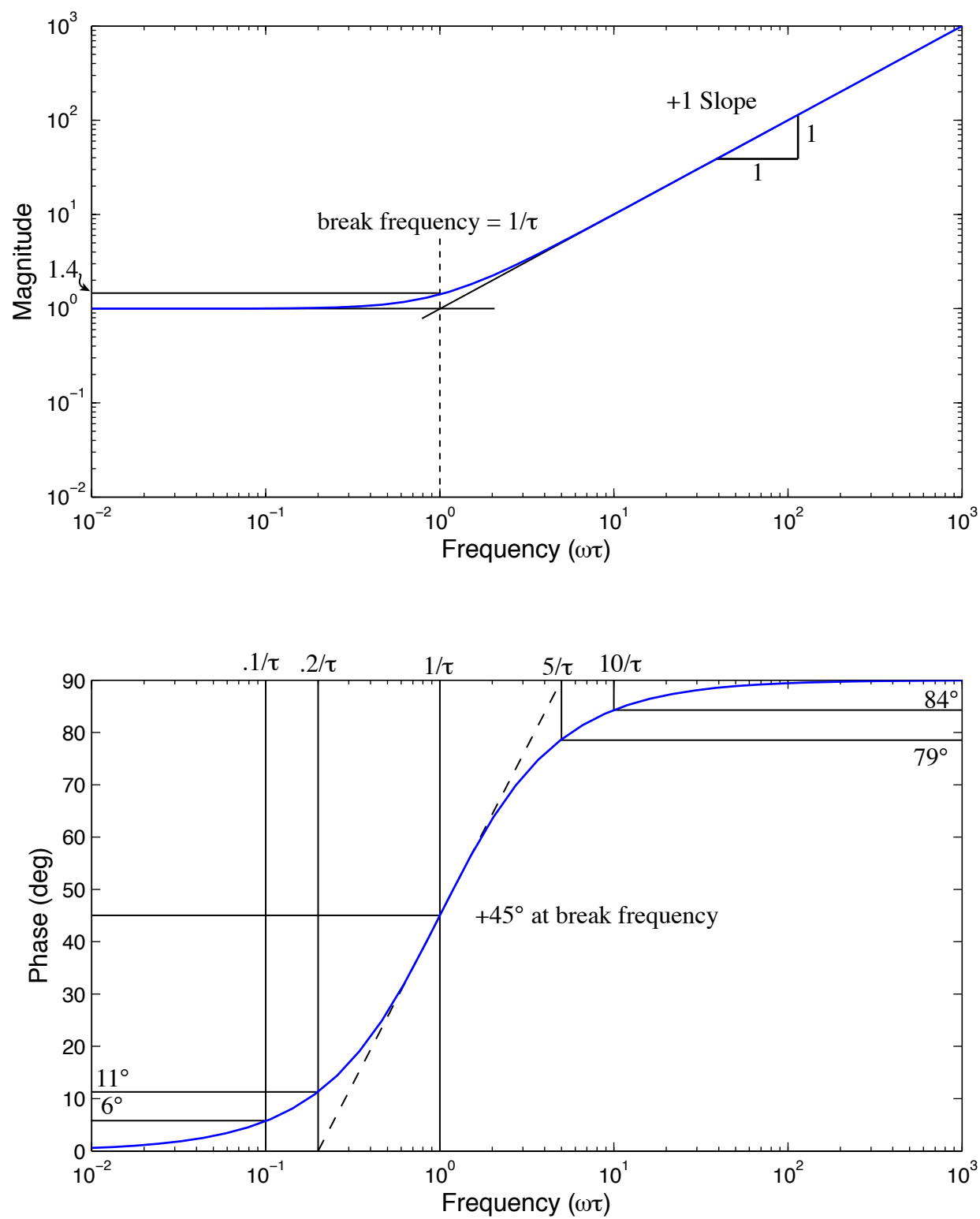
As before, the transition between 0° and -180° depends on ζ . Smaller ζ will have a more rapid transition, while larger ζ will cause a slower transition. However, at $\omega = \omega_n$, the phase angle will always be -90° , halfway between the angles at low and high frequencies.

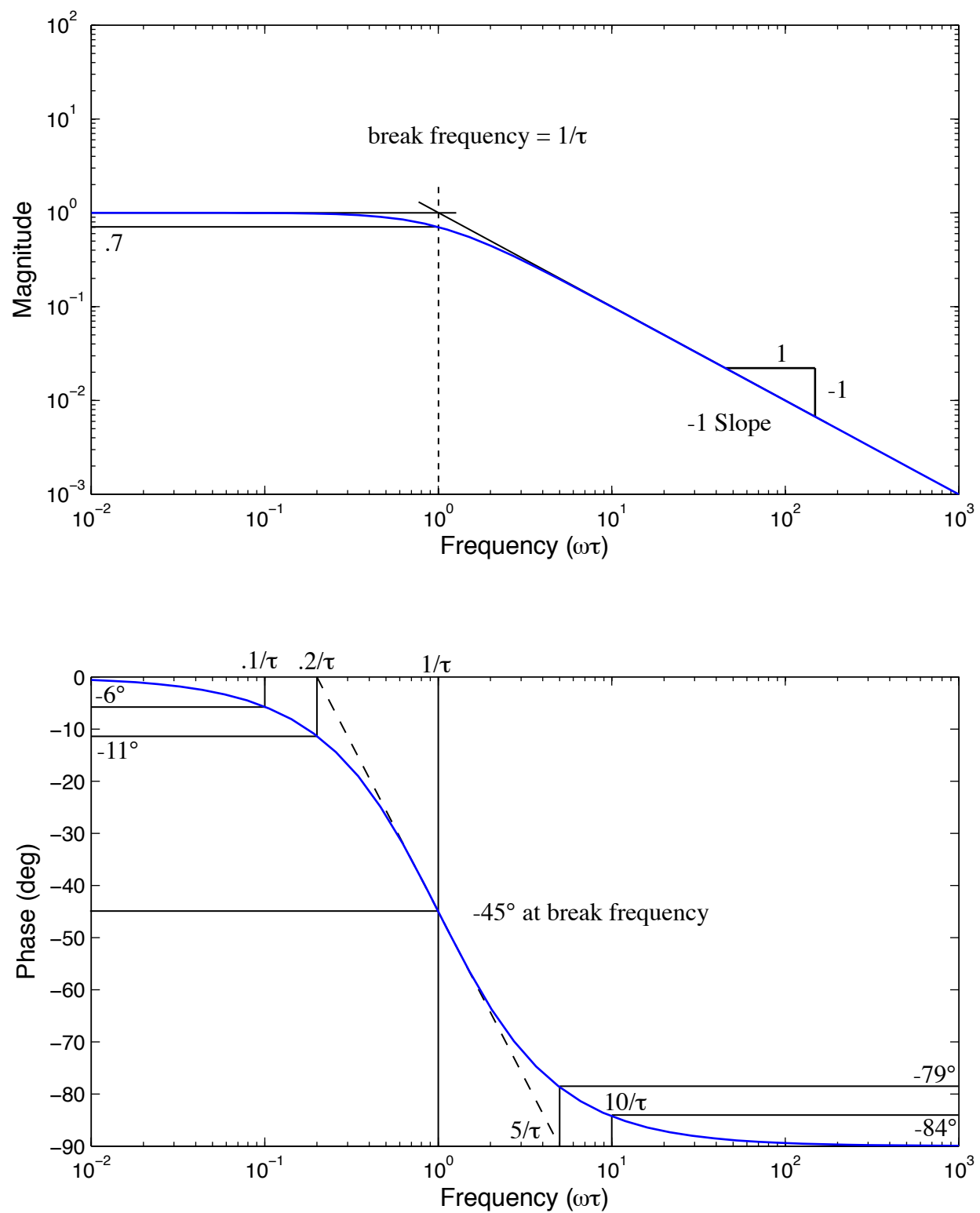
The magnitude and phase plots for this case are shown in Fig. 8.7. In addition, Fig. 8.8 shows how the magnitude and phase plots change for varying values of ζ .

Figure 8.1: Magnitude and phase plots for a constant k

Figure 8.2: Magnitude and phase plots for $(j\omega)^1$

Figure 8.3: Magnitude and phase plots for $(j\omega)^{-1}$

Figure 8.4: Magnitude and phase plots for $(j\omega\tau + 1)$

Figure 8.5: Magnitude and phase plots for $1/(j\omega\tau + 1)$

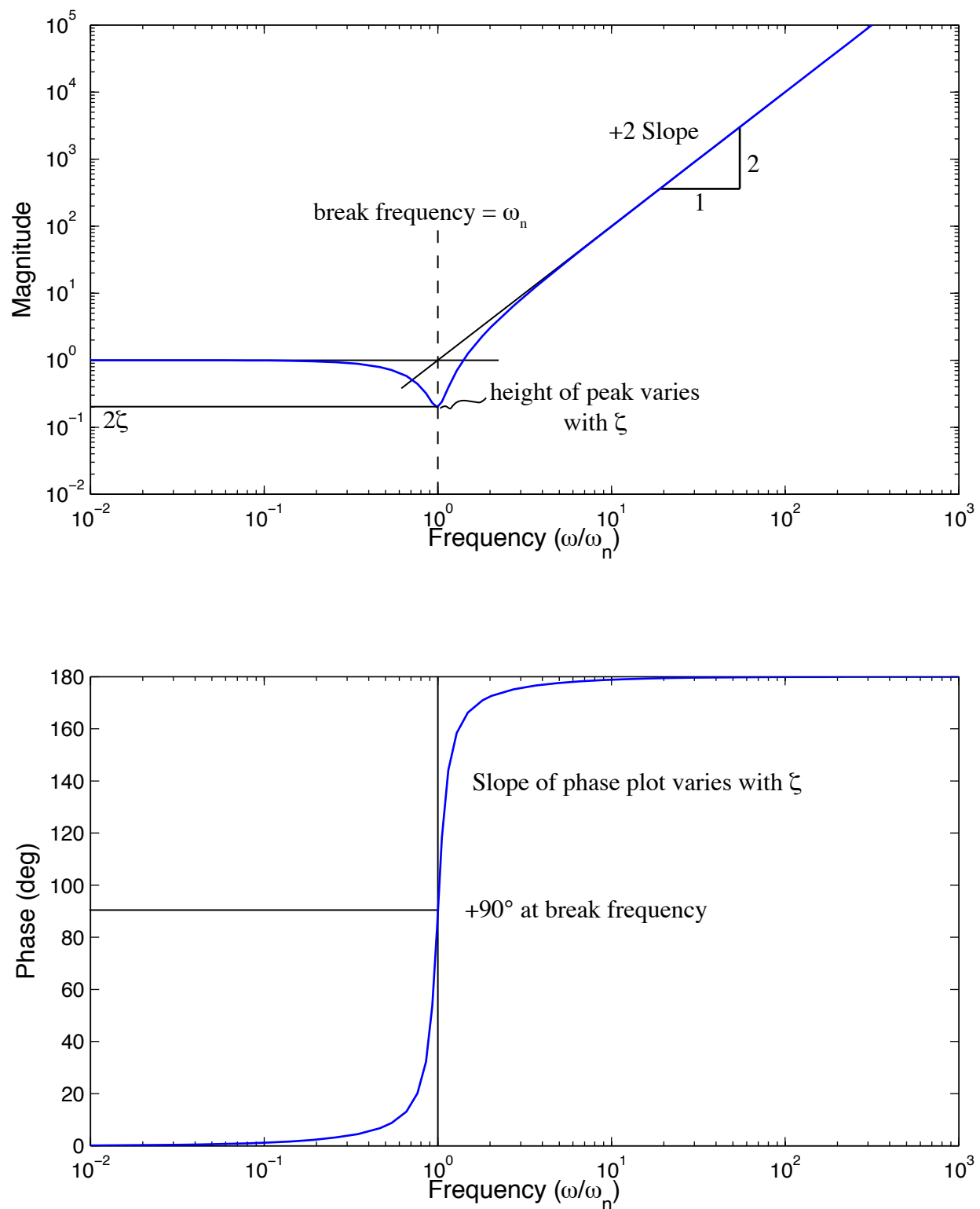


Figure 8.6: Magnitude and phase plots for $(j\omega/\omega_n)^2 + 2\zeta(j\omega/\omega_n) + 1$; $\zeta = 0.1$ for this case.

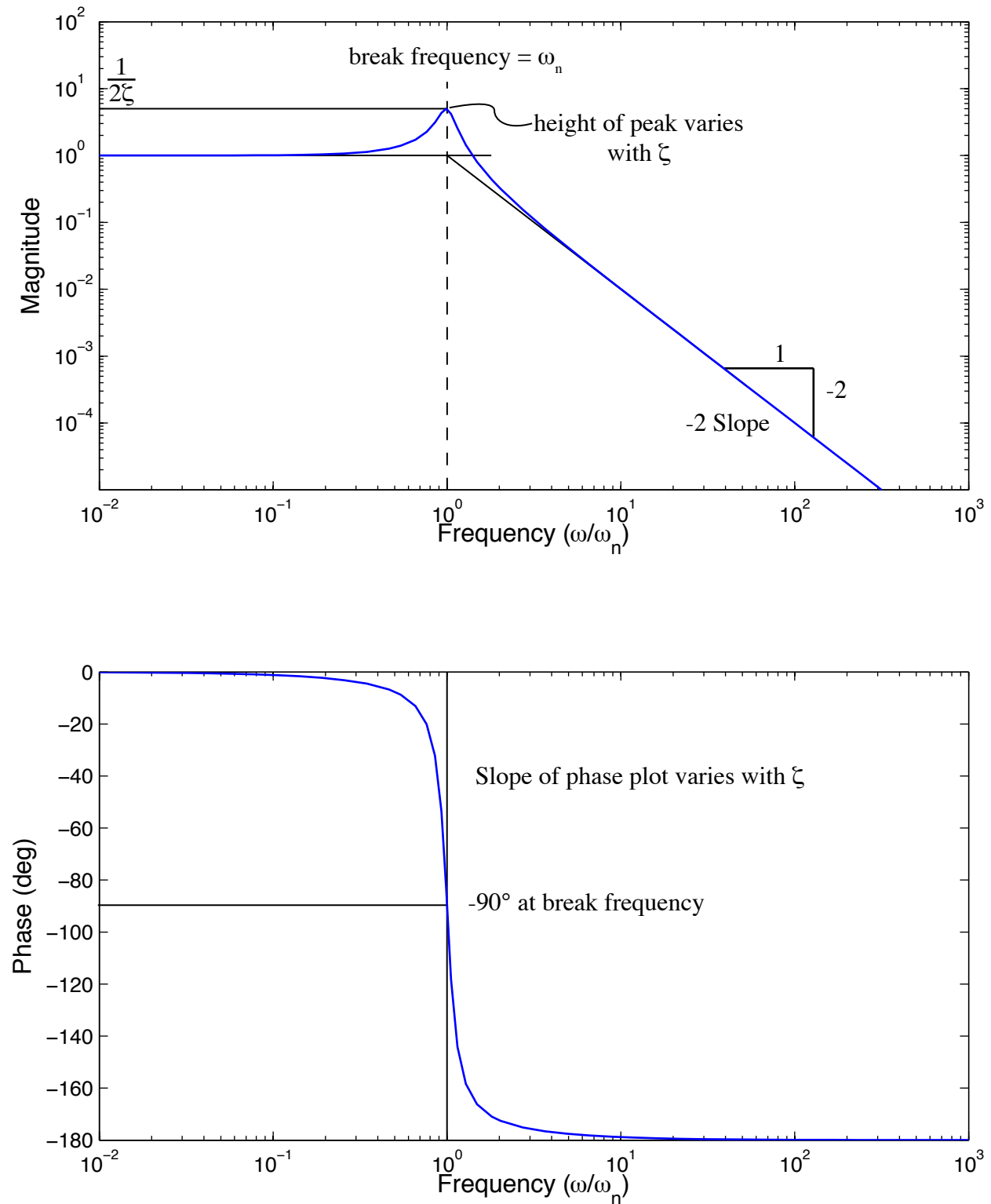


Figure 8.7: Magnitude and phase plots for $1/[(j\omega/\omega_n)^2 + 2\zeta(j\omega/\omega_n) + 1]$; $\zeta = 0.1$ for this case.

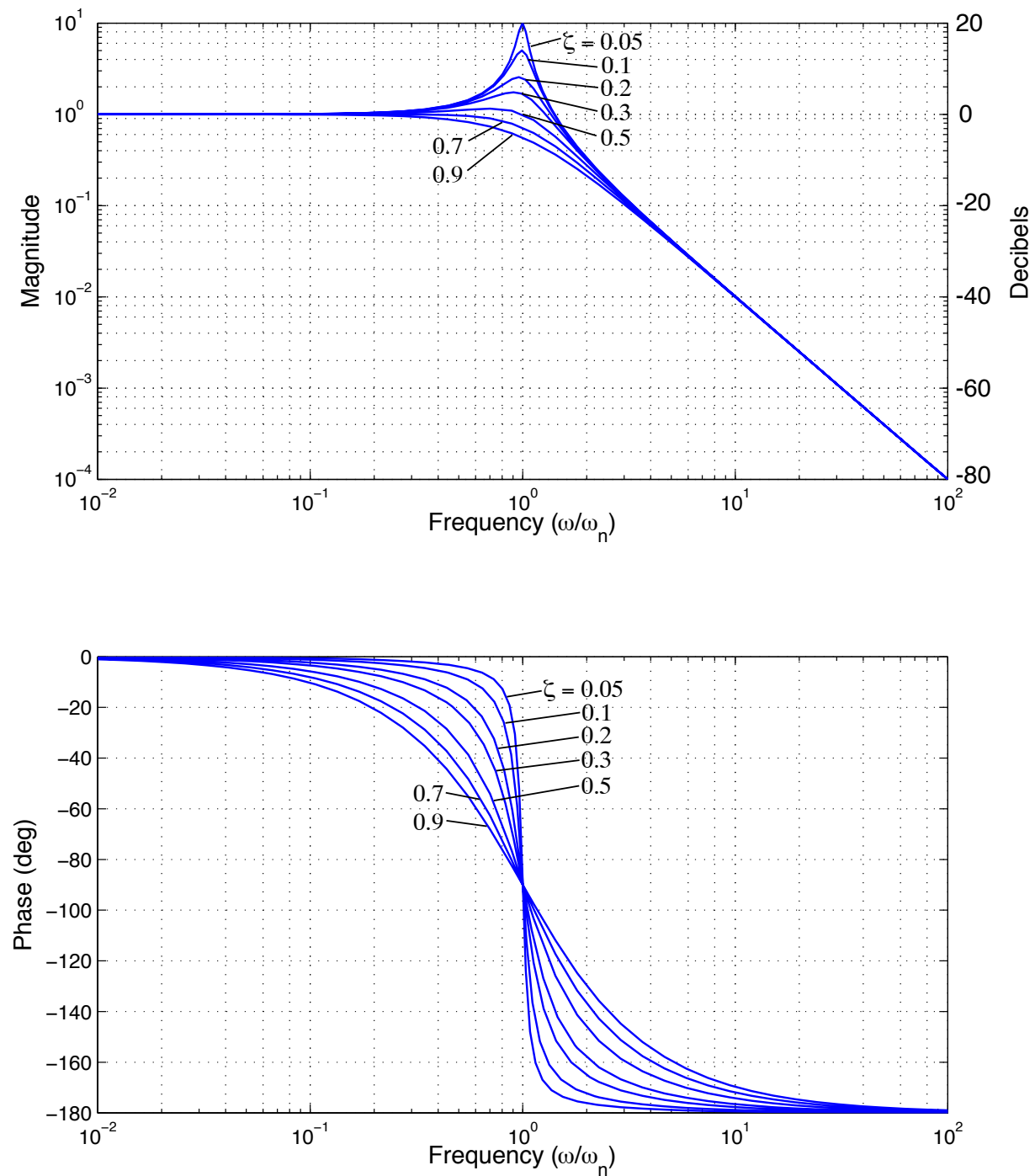


Figure 8.8: Illustration of variation of magnitude and phase with ζ for a second-order system

Given any transfer function $G(s)$, we can always factor it into a form composed only of these seven terms. In other words, any transfer function can be factored to put it in Bode form. For example,

$$G(s) = k_0 \frac{b_2 s^2 + b_1 s}{a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$

$$G(j\omega) = k_0 \frac{b_2 (j\omega)^2 + b_1 (j\omega)}{a_3 (j\omega)^3 + a_2 (j\omega)^2 + a_1 (j\omega) + a_0}$$

$$G(j\omega) = k_0 \frac{b_1 (j\omega) \left[\frac{b_2}{b_1} (j\omega) + 1 \right]}{a_0 [c_1 (j\omega) + 1] [d_2 (j\omega)^2 + d_1 (j\omega) + 1]}$$

which is of the form

$$G(j\omega) = K \frac{(j\omega)(j\omega\tau_1 + 1)}{(j\omega\tau_2 + 1) \left[\left(\frac{j\omega}{\omega_n} \right)^2 + 2\zeta \left(\frac{j\omega}{\omega_n} \right) + 1 \right]}.$$

Sketching the Magnitude and Phase Plots

Once the transfer function has been factored, the magnitude and phase plots for each term can be plotted. The magnitude and phase plots for the whole transfer function are then found by simply adding the plots from each term. We can see this by considering an example transfer function,

$$G(s) = \frac{N_1(s)N_2(s)}{D_1(s)D_2(s)D_3(s)}$$

Then

$$\log_{10} |G| = \log_{10} \left| \frac{N_1 N_2}{D_1 D_2 D_3} \right|$$

$$\log_{10} |G| = \log_{10} |N_1| + \log_{10} |N_2| + \log_{10} \left| \frac{1}{D_1} \right| + \log_{10} \left| \frac{1}{D_2} \right| + \log_{10} \left| \frac{1}{D_3} \right|$$

Similarly, the phase of the total system comes from adding the phases of each individual component. Therefore, to generate the composite sketch, we:

1. Put the transfer function in Bode form, identifying terms according to the 7 types presented above.
2. Plot the magnitude for each individual term on a log-log scale.
3. Add the individual magnitude plots to generate the composite magnitude curve. Add the distances on the plot, not the values. Work from low to high frequencies.

4. Plot the phase associated with $(j\omega)^n$ or $(j\omega)^{-n}$ terms. This will be the low-frequency asymptote for the phase plot.
5. Sketch the approximate phase curve by stepping $\pm 90^\circ$ or $\pm 180^\circ$ at the break points. Again, work from low to high frequencies.
6. Draw transition asymptotes.
7. Sketch the composite phase curve.

Example 1

$$G(s) = \frac{2000(s+0.5)}{s(s+10)(s+50)}$$

$$G(s) = \frac{2\left(\frac{s}{0.5} + 1\right)}{s\left(\frac{s}{10} + 1\right)\left(\frac{s}{50} + 1\right)}$$

or

$$G(s) = \frac{k(\tau_1 s + 1)}{s(\tau_2 s + 1)(\tau_3 s + 1)}$$

where $k = 2$, $1/\tau_1 = 0.5 \text{ s}^{-1}$, $1/\tau_2 = 10 \text{ s}^{-1}$, $1/\tau_3 = 50 \text{ s}^{-1}$. Then

$$G(s) = k \cdot (\tau_1 s + 1) \cdot \frac{1}{s} \cdot \frac{1}{\tau_2 s + 1} \cdot \frac{1}{\tau_3 s + 1}$$

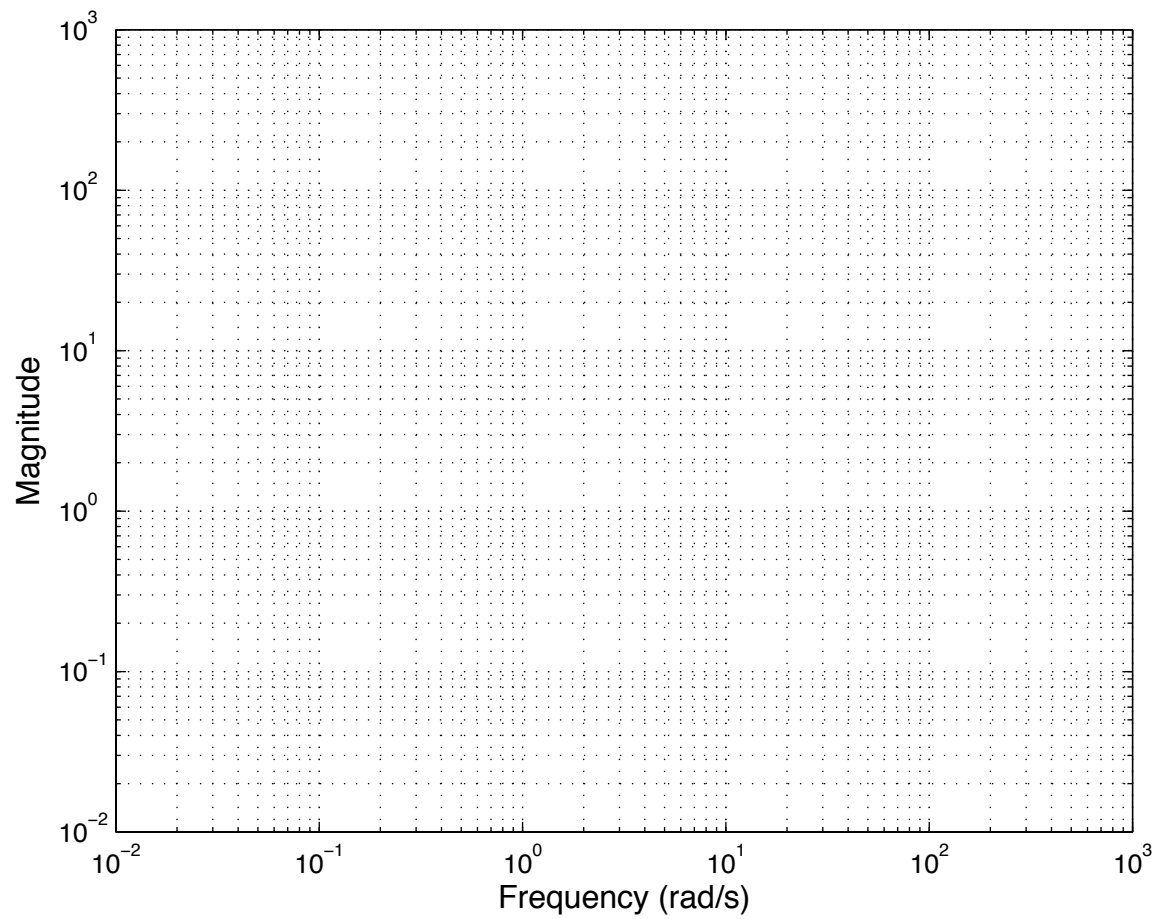
so

$$|G(s)| = |k| \cdot |\tau_1 s + 1| \cdot \left| \frac{1}{s} \right| \cdot \left| \frac{1}{\tau_2 s + 1} \right| \cdot \left| \frac{1}{\tau_3 s + 1} \right|$$

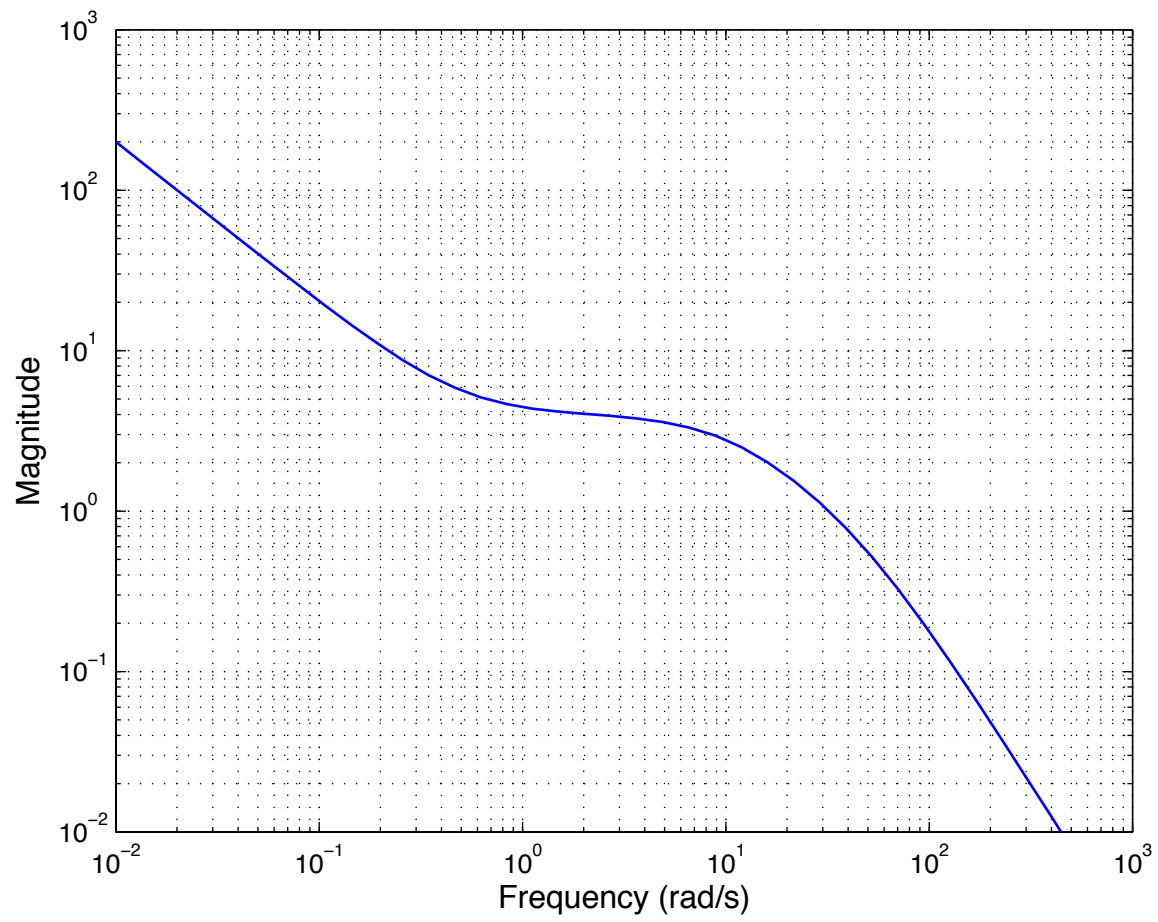
On the log scale, each of these terms adds. Also,

$$\angle G(s) = \angle k + \angle(\tau_1 s + 1) + \angle\left(\frac{1}{s}\right) + \angle\left(\frac{1}{\tau_2 s + 1}\right) + \angle\left(\frac{1}{\tau_3 s + 1}\right)$$

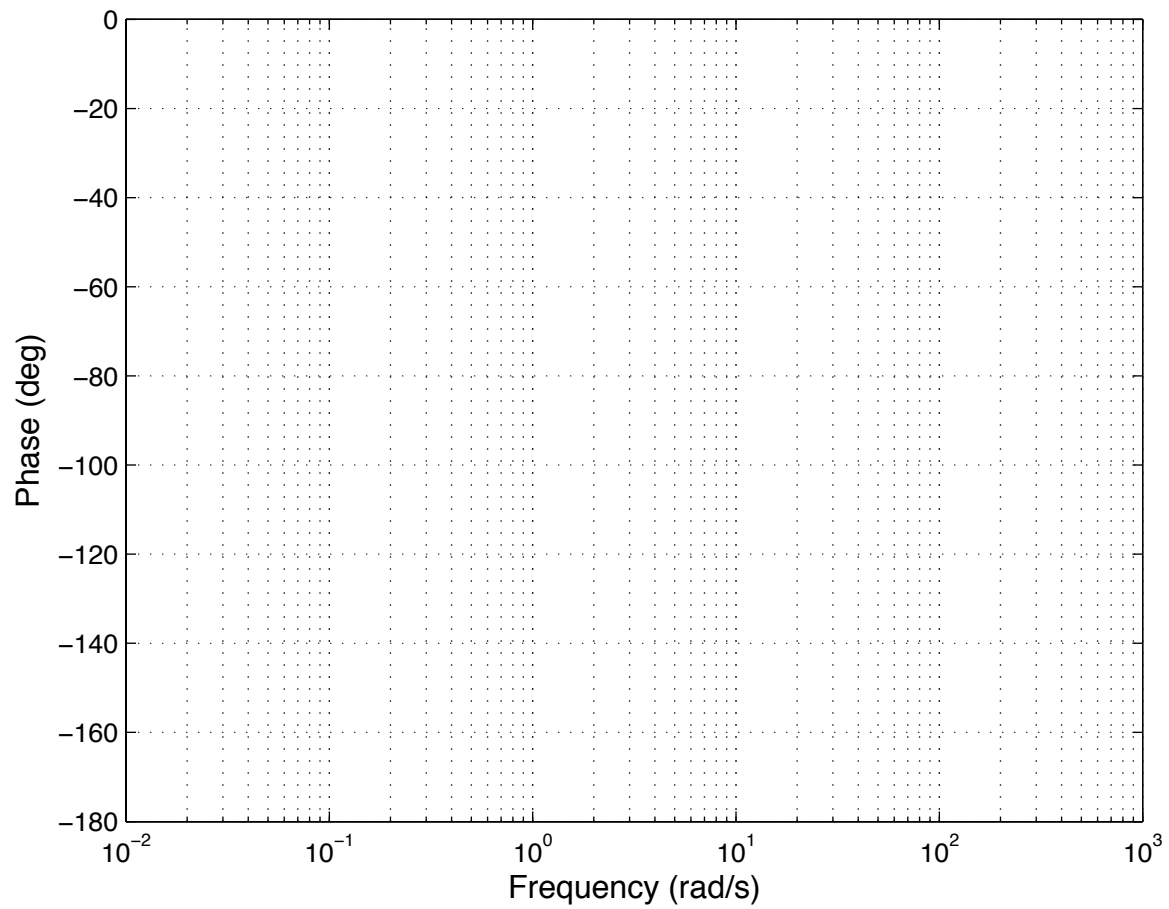
Blank magnitude and phase plots are found on the next pages for you to practice sketching; the correct answers are also shown.



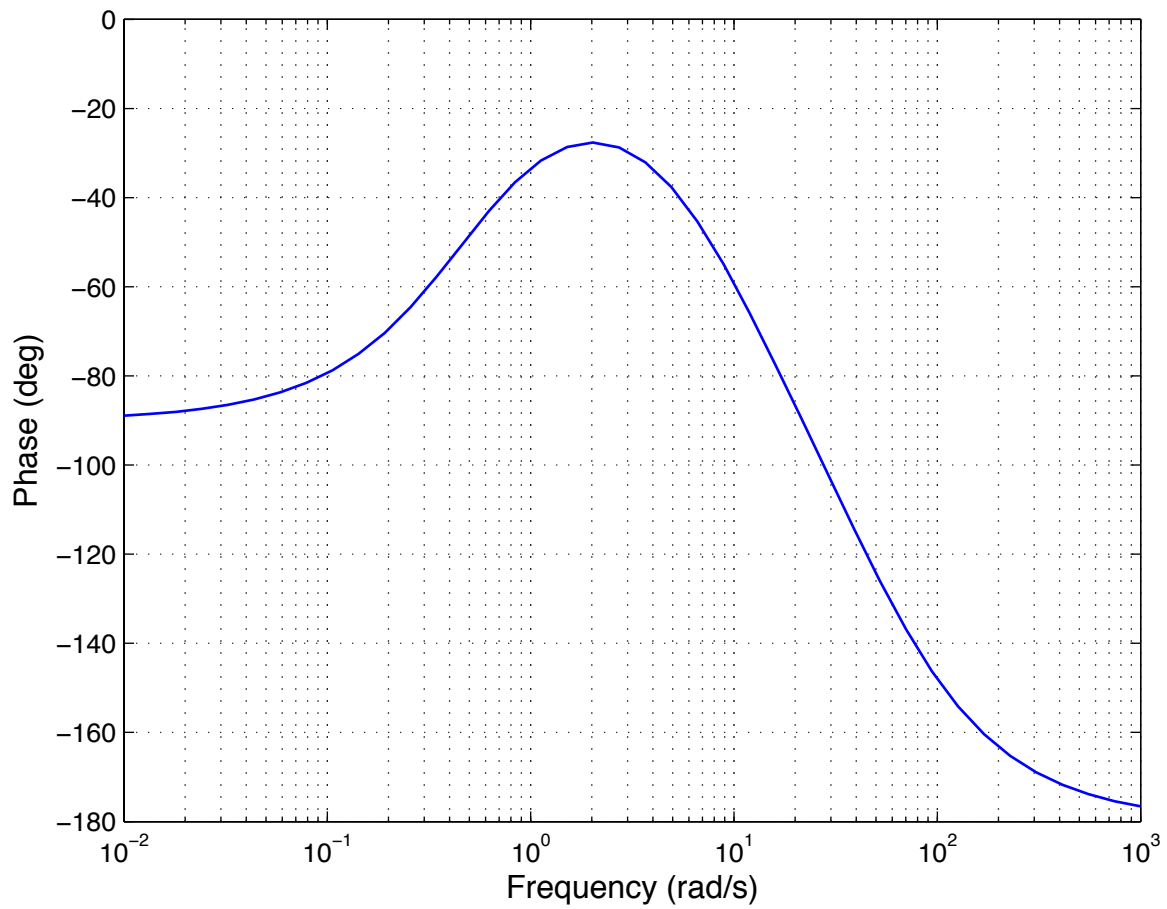
$$G(s) = \frac{2000(s + 0.5)}{s(s + 10)(s + 50)}$$



$$G(s) = \frac{2000(s + 0.5)}{s(s + 10)(s + 50)}$$



$$G(s) = \frac{2000(s+0.5)}{s(s+10)(s+50)}$$



$$G(s) = \frac{2000(s + 0.5)}{s(s + 10)(s + 50)}$$

Example2

$$G(s) = \frac{20,000s + 20,000}{s^3 + 40s^2 + 10,000s}$$

We place the transfer function in Bode form:

$$\begin{aligned} G(j\omega) &= \frac{20,000(j\omega + 1)}{j\omega[(j\omega)^2 + 40j\omega + 10,000]} \\ &= \frac{20,000(j\omega + 1)}{10,000j\omega \left[\left(\frac{j\omega}{100} \right)^2 + 0.4 \frac{j\omega}{100} + 1 \right]} \end{aligned}$$

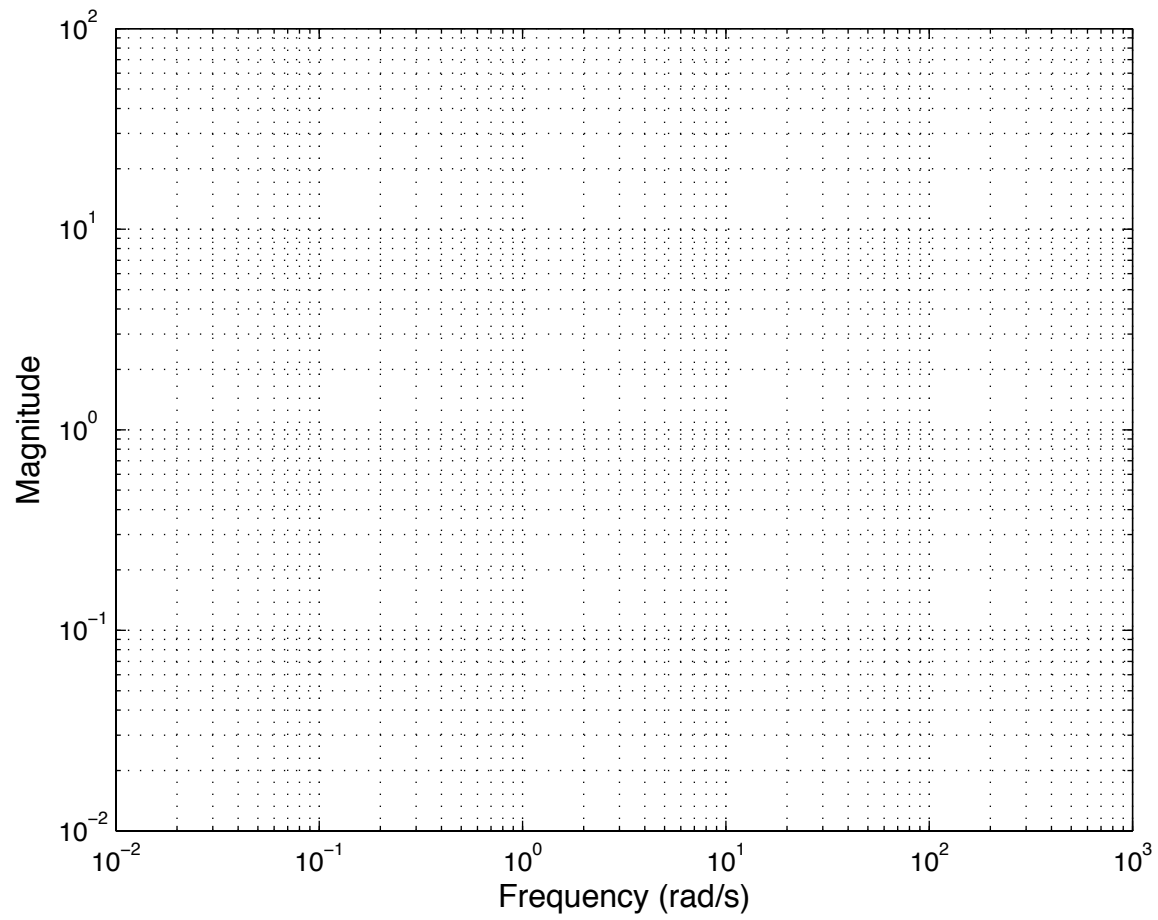
$$G(j\omega) = 2(j\omega)^{-1} \frac{j\omega + 1}{\left[\left(\frac{j\omega}{100} \right)^2 + 2(0.2) \left(\frac{j\omega}{100} \right) + 1 \right]}$$

This is in Bode form if

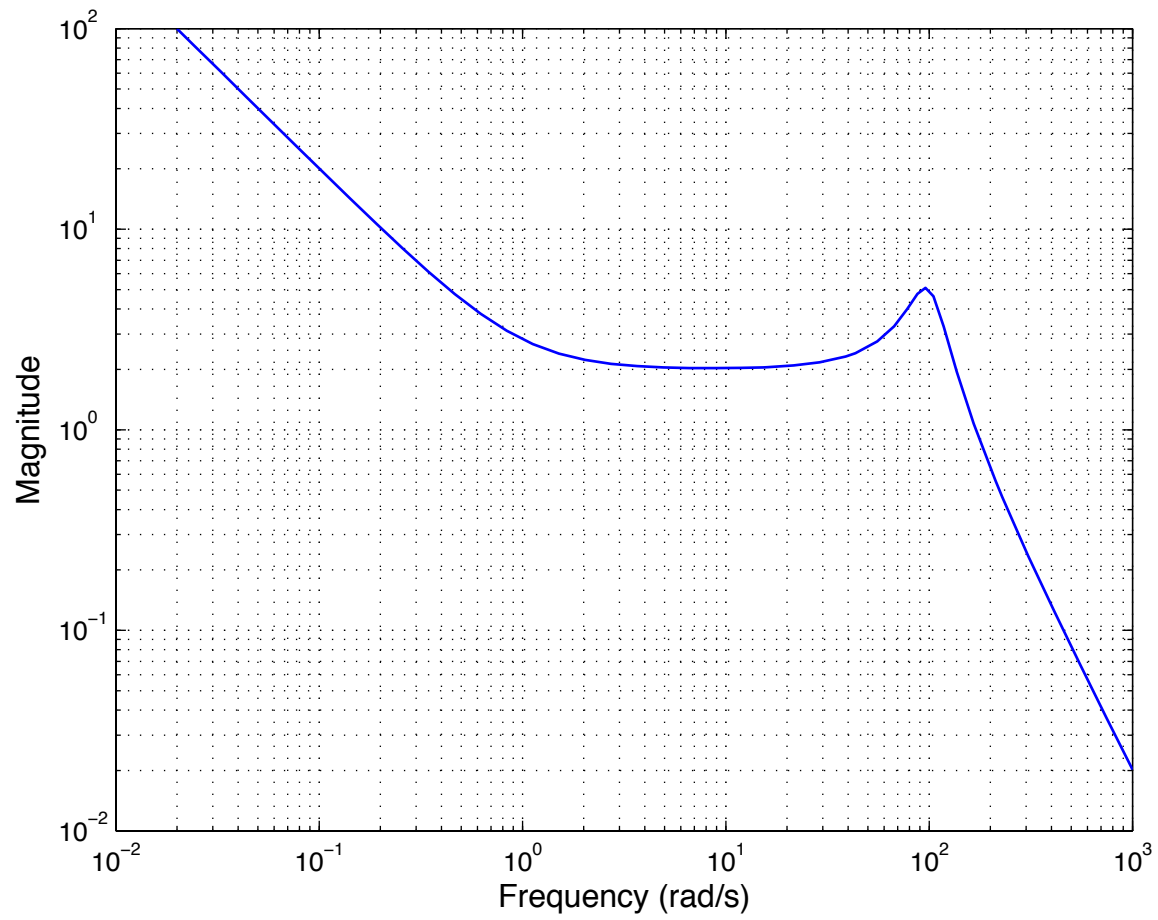
$$k = 2 \quad n = -1 \quad \tau = 1 \text{ rad/s} \quad \omega_n = 100 \text{ rad/s} \quad \zeta = 0.2$$

Blank magnitude and phase plots are located on the following pages for you to sketch these curves.

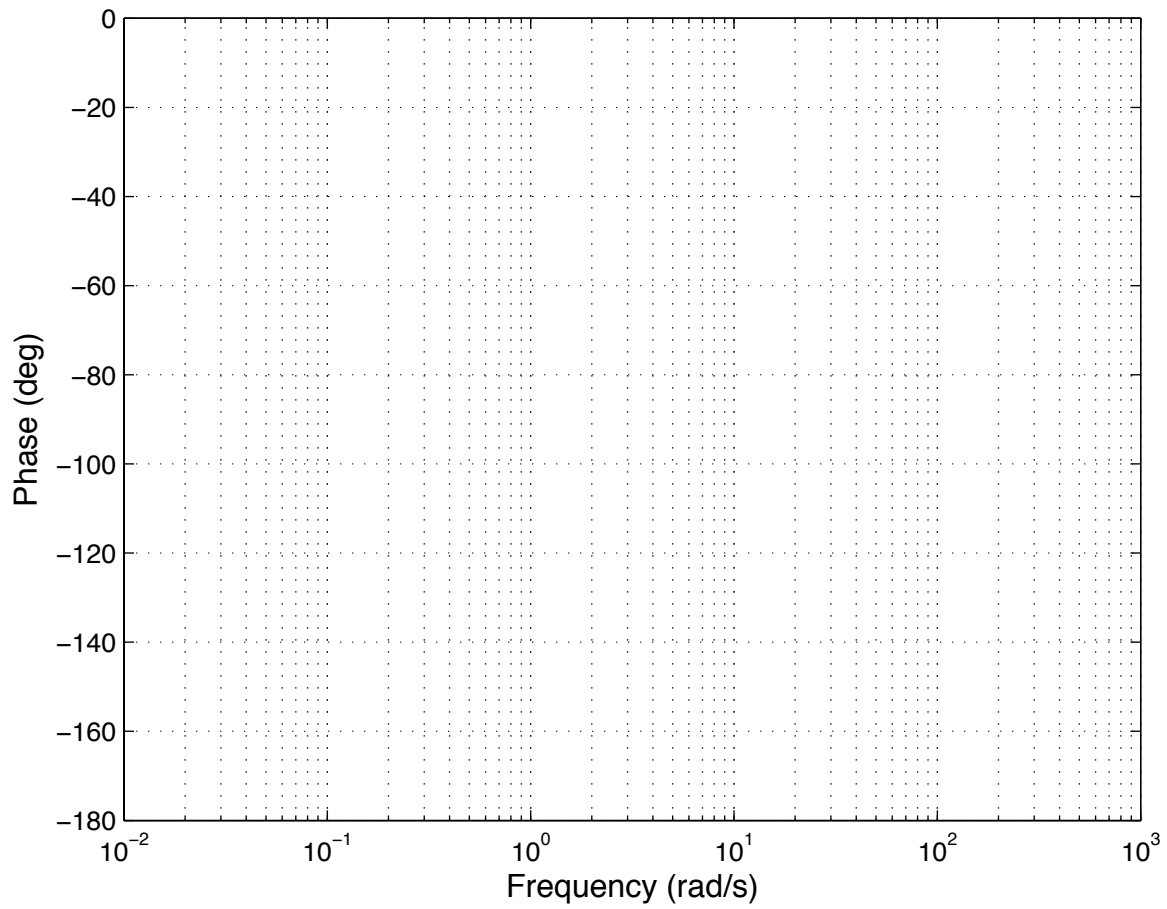
The correct answers are also given.



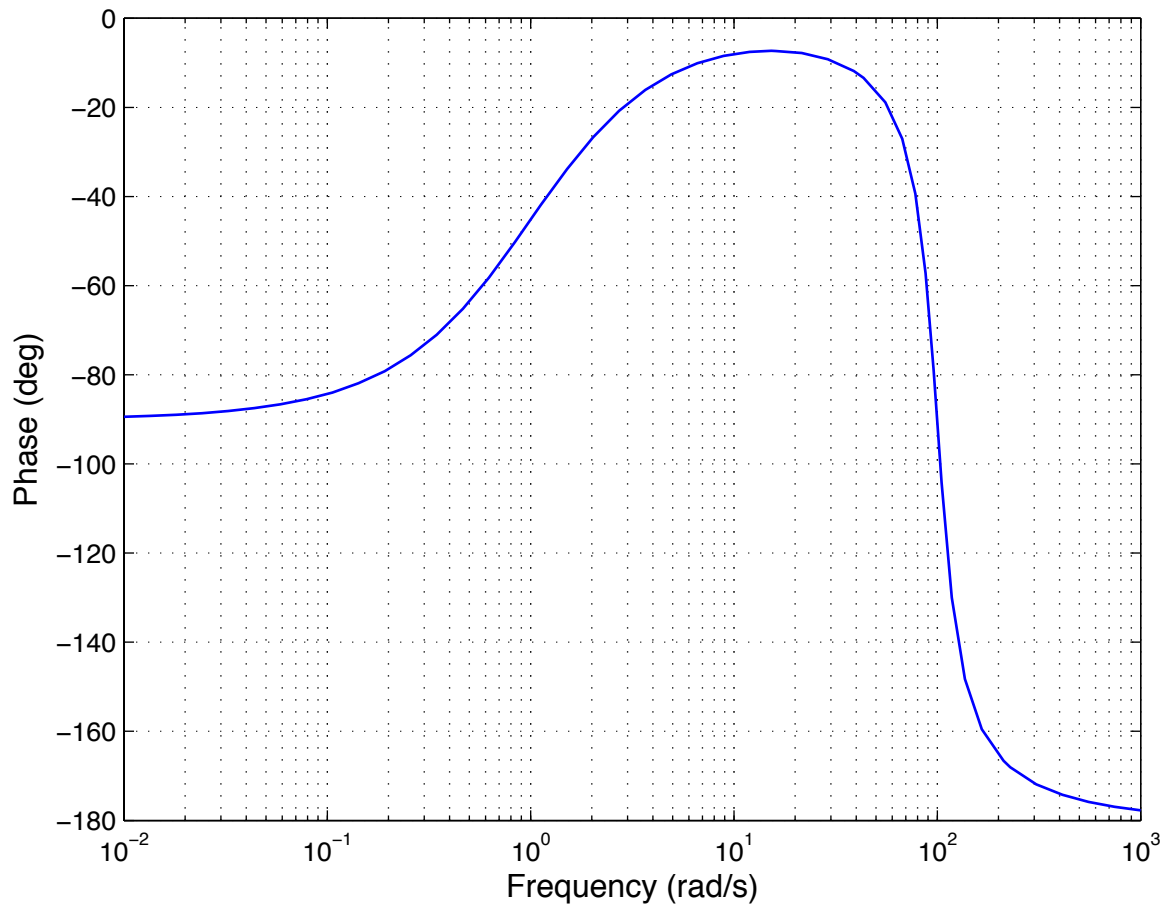
$$G(s) = \frac{20,000s + 20,000}{s^3 + 40s^2 + 10,000s}$$



$$G(s) = \frac{20,000s + 20,000}{s^3 + 40s^2 + 10,000s}$$



$$G(s) = \frac{20,000s + 20,000}{s^3 + 40s^2 + 10,000s}$$



$$G(s) = \frac{20,000s + 20,000}{s^3 + 40s^2 + 10,000s}$$

8.5 Frequency Response Sketching Rules of Thumb

The low-frequency gain (magnitude ratio) can be obtained by setting $s = j\omega = 0$. For example, if

$$G(s) = \frac{2000(s + 0.5)}{(s + 10)(s + 50)}$$

then the low-frequency magnitude ratio is

$$G(s)|_{s=0} = \frac{2000 \cdot 0.5}{10 \cdot 50} = 2$$

When a transfer function is factored into factors of s , $\tau s + 1$, and $(\frac{s^2}{\omega_n^2} + 2\zeta \frac{s}{\omega_n} + 1)$, the order n of the s term tells about low-frequency magnitude and phase response. Specifically, at low frequencies,

- slope of magnitude plot = n
- phase = $n \times 90^\circ$

For example, if

$$G(s) = \frac{2000(s + 0.5)}{s(s^2 + 25s + 625)}$$

$$G(s) = \frac{2000 \cdot 0.5}{625} \frac{(\frac{s}{0.5} + 1)}{s \left[(\frac{s}{25})^2 + 2(0.5)(\frac{s}{25}) + 1 \right]}$$

Since the order of the s term is -1, the low-frequency slope and phase are -1 and -90° .

If the order of the numerator is m and the order of the denominator is p , then the slope of the magnitude plot at high frequencies (beyond all break points) is $m - p$ and the phase is $(m - p) \times 90^\circ$.

For example, for

$$G(s) = \frac{2000(s + 0.5)}{s(s^2 + 25s + 625)}$$

the order of the numerator is 1 ($m = 1$) and the order of the denominator is 3 ($p = 3$). The slope of the magnitude plot at high frequencies is then $m - p = -2$, and the phase is $(m - p) \times 90^\circ = -180^\circ$.

These simple rules can be used as a quick check of a sketch or MATLAB output.

8.5.1 Evaluating Magnitude Ratio and Phase Difference at a Single Frequency

If we have the transfer function for a system, we can evaluate the magnitude ratio and phase difference between the input and output at a specific frequency. For example, we will find the

magnitude ratio and phase difference at $\omega = 100$ rad/s for

$$G(s) = \frac{2000(s + 0.5)}{s(s + 10)(s + 50)}$$

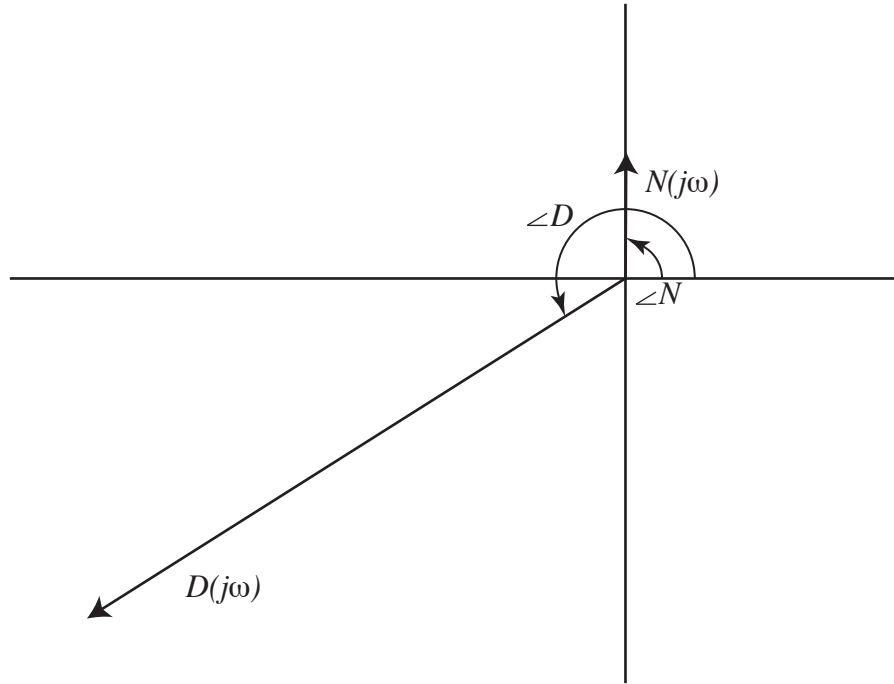
Then

$$G(j\omega) = \frac{2000j\omega + 1000}{(j\omega)^3 + 60(j\omega)^2 + 500(j\omega)} = \frac{1000 + j2000\omega}{-60\omega^2 + j(500\omega - \omega^3)}$$

For $\omega = 100$ rad/s, we get

$$G(j\omega)|_{\omega=100\text{rad/s}} = \frac{1,000 + j200,000}{-600,000 - j950,000}$$

The plot below shows a graphical representation of the numerator and denominator for this case.



$$|G(j100)| = \frac{|N|}{|D|} = \frac{\sqrt{1,000^2 + 200,000^2}}{\sqrt{600,000^2 + 950,000^2}}$$

$$|G(j100)| = .178$$

For phase,

$$\angle G(j100) = \angle N - \angle D = \tan^{-1} \left(\frac{200,000}{1,000} \right) - \tan^{-1} \left(\frac{-950,000}{-600,000} \right)$$

$$\angle G(j100) = 89.71^\circ - 237.72^\circ = -148.01^\circ$$

Bibliography