Transfer Functions

Tim McLain

Introduction to Transfer Functions

- Until now, we have modeled dynamic behavior of systems using ODEs in time domain
  - Modeling methods produce ODEs
  - Very general
  - Accommodates nonlinearities and time-varying parameters
- As we have seen, dynamic systems have two components to their response
  - Natural response (response to initial conditions)
  - Forced response (response to inputs)
- When we are dealing with linear, constant coefficient systems, *transfer function* represents alternative approach for modeling forced response of system

Example - rubber band-mass system

- Natural response
- Forced response

- Thinking back to ODE course, equations had two parts to their solutions:
  - Homogeneous solution (natural response)
  - Particular solution (forced response)

What is the transfer function?

- A transfer function is an analytical expression obtained from the time domain equations of motion
- It describes the ratio of input and output of the system
- Transfer function concept cannot be applied to nonlinear systems
- Transfer functions are used to represent linear time-invariant systems only

How do we find the transfer function?

- A common method for finding the transfer function of a system is to use the Laplace transform
- Take the Laplace transform of the equations of motion
  - Assume initial conditions are zero (we are only concerned with forced response)
- Algebraic equations in Laplace variable *s* are then manipulated to form the transfer function
- The transfer function is the ratio of the system output over the system input
- Most easily demonstrated by example: rubber band-mass system

Example - rubber band-mass system

- Equations of motion: $$m\ddot{x}_o + b(\dot{x}_o - \dot{x}_i) + k(x_o - x_i) = 0$$

- Transformed: $$(ms^2 + bs + k)X_o(s) = (bs + k)X_i(s)$$

- Transfer function: $$\frac{X_o(s)}{X_i(s)} = \frac{bs + k}{ms^2 + bs + k}$$

Transfer function interpretation

- Transfer function describes how specific input signal is altered by dynamics of system to produce output signal
- Transfer functions are often used in block diagram representation

$$X_i(s) \longrightarrow \boxed{\frac{bs+k}{ms^2+bs+k}} \longrightarrow X_o(s)$$

- Can think of them like a filter

Natural response

- Even though transfer function represents system’s response to forcing inputs, natural response characteristics can also be determined from the transfer function
- Natural response is described by *characteristic function* of the system, which is the denominator polynomial
- For the rubber band-mass system: $D(s) = ms^2 + bs + k$
- For a n^{th}-order system, denominator polynomial will be of degree n

Natural response

- Setting the characteristic function to zero gives the characteristic equation

$$ms^2 + bs + k = 0$$

- System's natural response is described by roots of characteristic equation, which are also known as the eigenvalues or poles of the transfer function

poles: $$s_{1,2} = -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}}$$

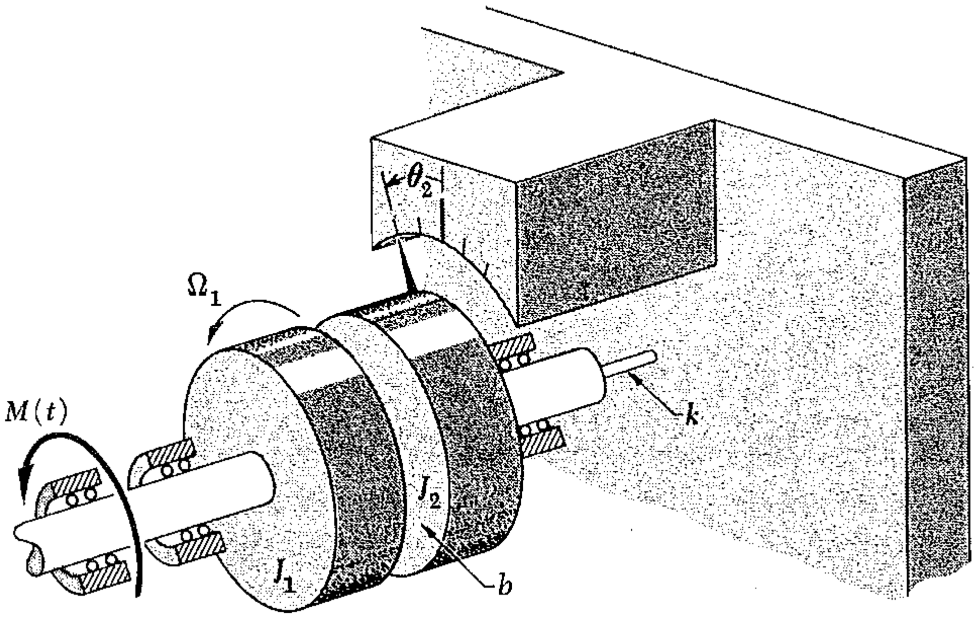
- The poles of the transfer function tell what kind of natural motions the system can have regardless of how the natural motions are excited

Natural response

- The roots of the numerator polynomial are called the zeros of the transfer function. The poles and zeros together describe how the system will respond to forcing inputs

zero: $$s = -\frac{k}{b}$$

Example – dynamometer



$$J_1\dot{\Omega}_1 + b(\Omega_1 - \Omega_2) = M(t)$$

$$J_2\dot{\Omega}_2 - b(\Omega_1 - \Omega_2) + k\theta_2 = 0$$

$$M_0 = 100 \text{ N-m}$$

$$J_1 = 0.2 \text{ kg-m}^2$$

$$J_2 = 0.1 \text{ kg-m}^2$$

$$b = 1 \text{ N-m-s/rad}$$

$$k = 200 \text{ N-m/rad}$$