Electrical Systems

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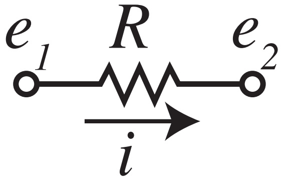
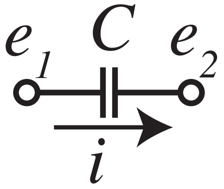
Basic elements

- Resistor



$$e_1 - e_2 = Ri$$

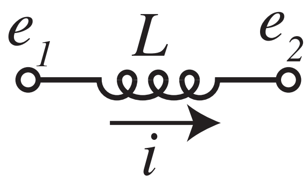
- Capacitor



$$e_1 - e_2 = \frac{1}{C}\int i\mathrm{d}t$$

$$i = C\frac{\mathrm{d}e_c}{\mathrm{d}t} = C\frac{\mathrm{d}}{\mathrm{d}t}(e_1 - e_2) = Cs(e_1 - e_2)$$

- Inductor

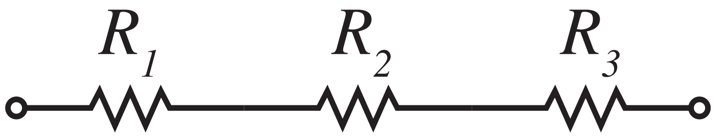


$$e_1 - e_2 = L\frac{\mathrm{d}i}{\mathrm{d}t}$$

$$si = \frac{\mathrm{d}i}{\mathrm{d}t} = \frac{1}{L}(e_1 - e_2)$$

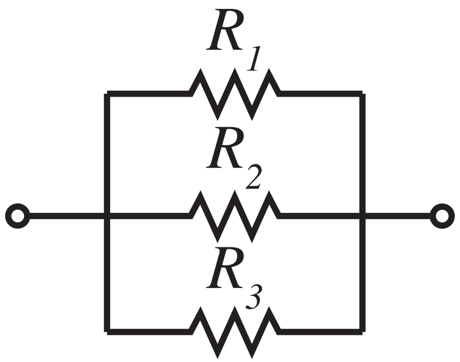
Resistors in series or parallel

- Resistors in series add



$$R_T = R_1 + R_2 + R_3$$

- Resistors in parallel add by recipricols



$$\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

Capacitors and inductors in series or parallel

- Capacitors in series add like resistors in parallel (recipricols)

$$\frac{1}{C_T} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$$

- Capacitors in parallel add

$$C_T = C_1 + C_2 + C_3$$

- Inductors in series add

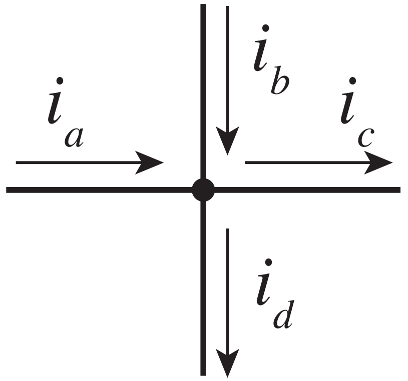
$$L_T = L_1 + L_2 + L_3$$

- Inductors in parallel add like resistors in parallel (recipricols)

$$\frac{1}{L_T} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}$$

Kirchoff’s current law (KCL)

- Node: A point of connection between two or more circuit components
- KCL: The sum of all currents at a node is equal to zero



$$\text{KCL} \rightarrow \quad i_a + i_b - i_c - i_d = 0$$

Deriving equations of “motion”

1. Draw a schematic of the circuit. Identify each component with a unique symbolic value. It is usually helpful to give each component a unique number.
2. Identify the nodes of the circuit and assign each one a letter to identify it.
3. For each component, assign the direction of positive current flow and indicate it on the circuit diagram. The current in the n^{th} component will be known as i_n.
4. Write an equation for the current in each component using the established physical relation for each type of component.
5. Define state variables to be *capacitor voltages* and *inductor currents*. Write component equations for capacitors and inductors in state-variable form.
6. Write node equations using KCL for each node that is not directly connected to a voltage source. <u>Where possible, express node voltages in terms of capacitor and input voltages.</u>
7. Use the remaining component equations and node equations to reduce the differential equations so that they contain only the state variables and inputs.

Example 1

1. Draw a schematic of the circuit. Identify each component with a unique symbolic value. It is usually helpful to give each component a unique number.
2. Identify the nodes of the circuit and assign each one a letter to identify it.
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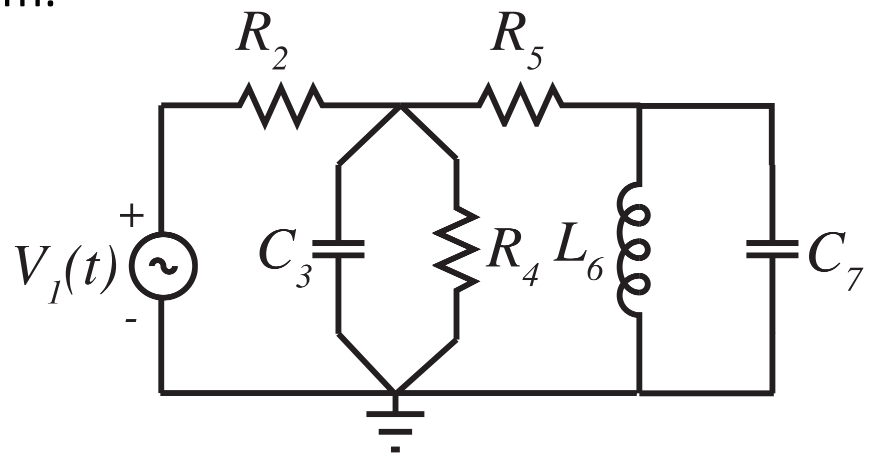
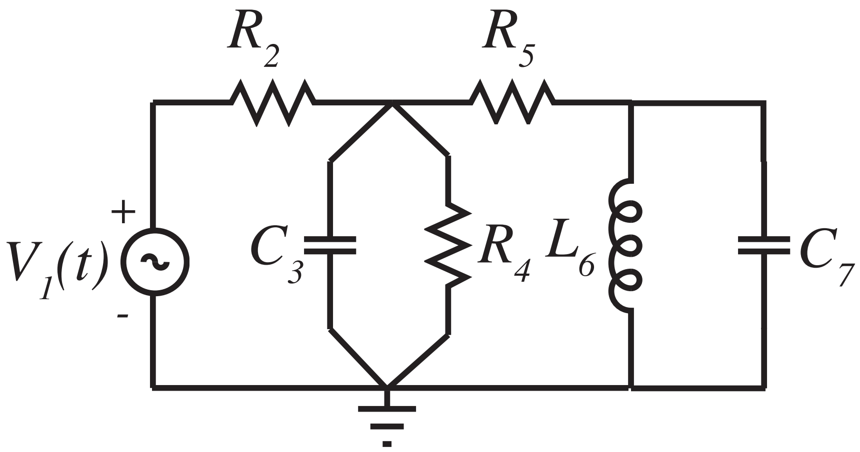


Example 1

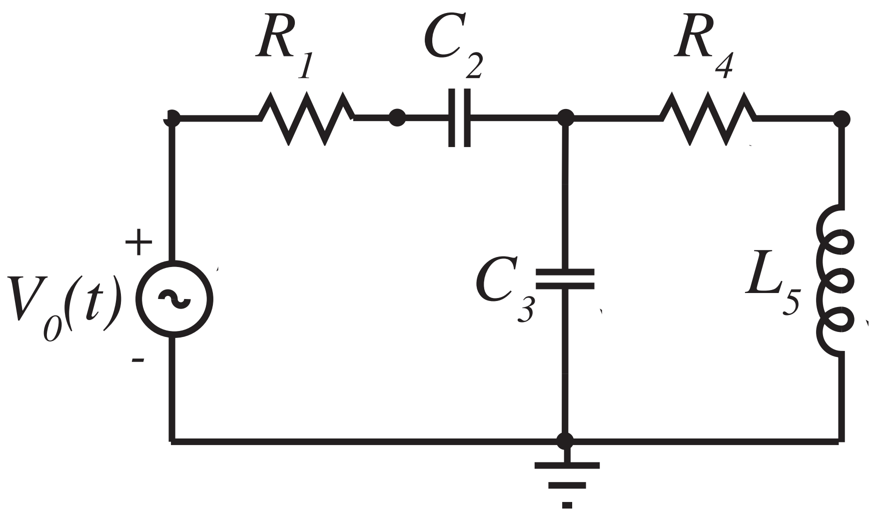


$$\boxed{se_3 = \frac{1}{C_3}\left[\frac{V_1(t)}{R_2} - \left(\frac{1}{R_2} + \frac{1}{R_4} + \frac{1}{R_5}\right)e_3 + \frac{1}{R_5}e_7\right]}$$

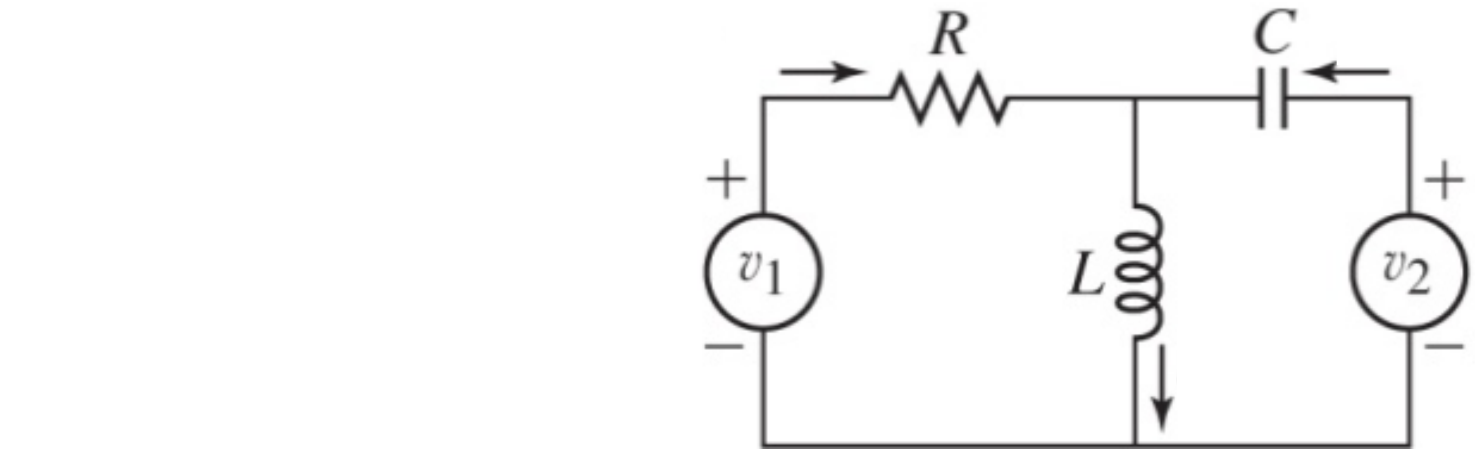
$$\boxed{se_7 = \frac{1}{C_7}\left[\frac{e_3}{R_5} - \frac{e_7}{R_5} - i_6\right]}$$

$$\boxed{si_6 = \frac{e_7}{L_6}}$$

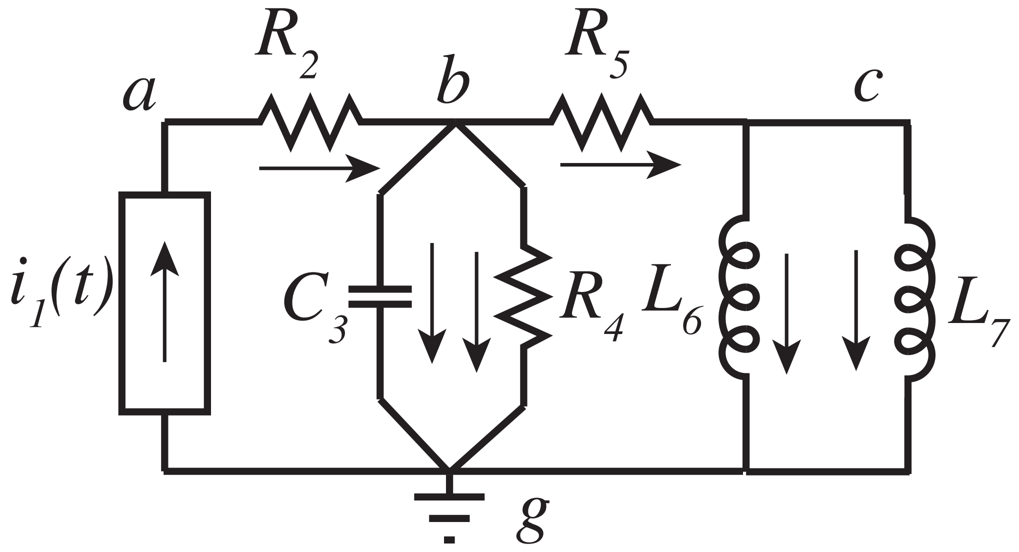
Example 2



Example 6.2.13

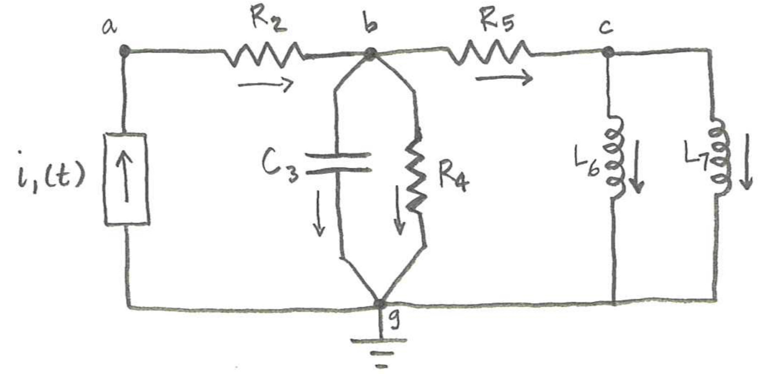


Example 3



Example 3 – Energy storage elements that are not independent

Example 3



Component Relations:

$$i_2 = \frac{e_a - e_b}{R_2} \qquad i_4 = \frac{e_b}{R_4} \qquad i_5 = \frac{e_b - e_c}{R_5}$$

$$i_3 = C_3 s e_3 \qquad i_6 = \frac{1}{sL_6} e_c \qquad i_7 = \frac{1}{sL_7} e_c$$

$$e_3 = e_b$$

Node Equations:

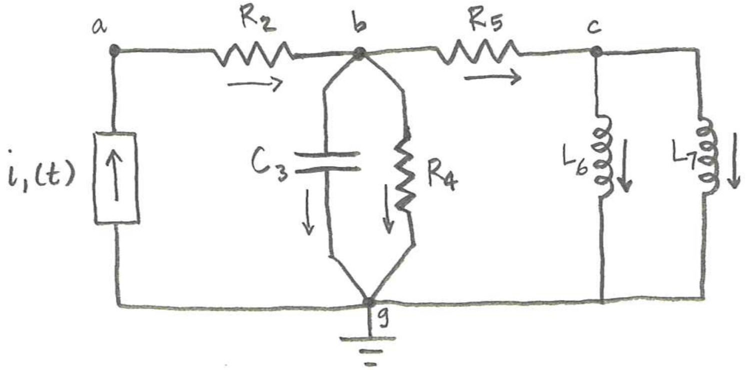
ⓐ $i_1 = i_2$

ⓑ $i_2 = i_3 + i_4 + i_5$

ⓒ $i_5 = i_6 + i_7$

State Variables: e_3, i_6, i_7 Inputs: $i_1(t)$

Example 3



State Variable Equations:

$$se_3 = \frac{1}{C_3} i_3 = \frac{1}{C_3}(i_2 - i_4 - i_5) = \frac{1}{C_3}\left(i_1(t) - \frac{e_b}{R_4} - i_6 - i_7\right)$$

$$se_3 = \frac{1}{C_3}\left[i_1(t) - \frac{e_3}{R_4} - (i_6 + i_7)\right]$$

$$si_6 = \frac{1}{L_6} e_c$$

$$si_6 = \frac{1}{L_6}\left[e_3 - R_5(i_6 + i_7)\right]$$

$$e_b - e_c = R_5 i_5$$

$$e_c = e_b - R_5 i_5$$

$$= e_3 - R_5(i_6 + i_7)$$

$$si_7 = \frac{1}{L_7} e_c$$

$$si_7 = \frac{1}{L_7}\left[e_3 - R_5(i_6 + i_7)\right]$$

Note: L_6 and L_7 are not independent!

$$i_6 = \frac{e_c}{s} L_6 \qquad i_7 = \frac{e_c}{s} L_7 \quad \Rightarrow \quad \frac{e_c}{s} = \frac{i_7}{L_7}$$

$$i_6 = \frac{L_6}{L_7} i_7$$

State variables related by an algebraic expression rather than an ODE.

Only 2 states required to represent this system!

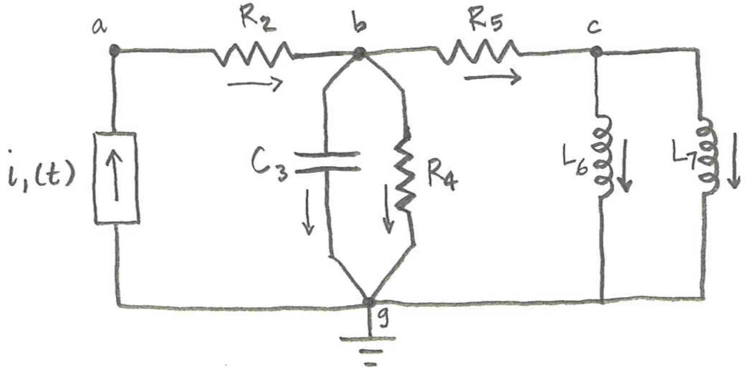
Example 3

* $i_6 + i_7 = \frac{L_6}{L_7} i_7 + i_7 = \left(\frac{L_6}{L_7} + 1\right) i_7$

State equations can be simplified to:

$$s e_3 = \frac{1}{C_3}\left[i_1(t) - \frac{e_3}{R_4} - \left(\frac{L_6}{L_7} + 1\right) i_7 \right]$$

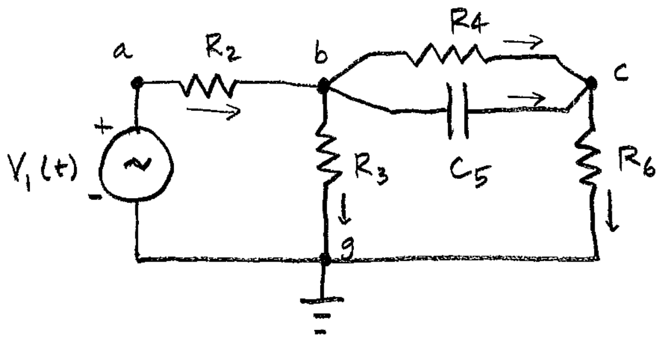
$$s i_7 = \frac{1}{L_7}\left[e_3 - R_5 \left(\frac{L_6}{L_7} + 1\right) i_7 \right]$$



* Simplified equations could have also been obtained by recognizing that L_6 and L_7 are in parallel and by replacing them with a single inductance:

$$\boxed{L_{eq} = \frac{L_6 L_7}{L_6 + L_7}}$$

Example 4 – Coupled resistances



Component Relations

$$i_2 = \frac{V_1(t) - e_b}{R_2} \qquad i_3 = \frac{e_b}{R_3} \qquad i_4 = \frac{e_b - e_c}{R_4} \qquad i_6 = \frac{e_c}{R_6}$$

$$i_5 = C_5 s e_5 \quad , \quad e_5 = e_b - e_c \quad \Rightarrow \quad e_c = e_b - e_5$$

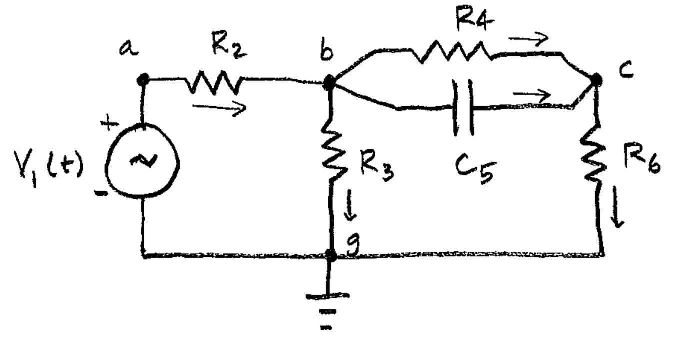
State Variables : e_5

Node Eqns :

(b) $i_2 = i_3 + i_4 + i_5$

(c) $i_4 + i_5 = i_6$

Example 4 – Coupled resistances



Component Relations

$$i_2 = \frac{V_1(t) - e_b}{R_2} \qquad i_3 = \frac{e_b}{R_3} \qquad i_4 = \frac{e_b - e_c}{R_4} \qquad i_6 = \frac{e_c}{R_6}$$

$$i_5 = C_5 s e_5 \quad , \quad e_5 = e_b - e_c \quad \Rightarrow \quad e_c = e_b - e_5$$

EOM:

$$\dot{e}_5 = \frac{1}{C_5} i_5$$

$$= \frac{1}{C_5}\left[i_2 - i_3 - i_4 \right]$$

$$= \frac{1}{C_5}\left[\frac{V_1(t) - e_b}{R_2} - \frac{e_b}{R_3} - \frac{e_5}{R_4} \right]$$

$e_b = ?$ want in terms of e_5, $V_1(t)$

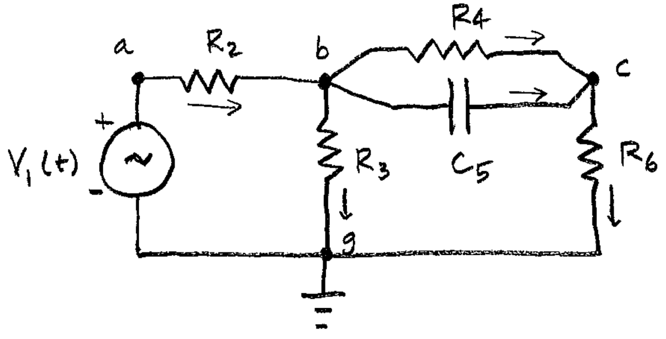
use 2^{nd} node equation

$$i_4 + i_5 = i_6$$

$$\frac{e_5}{R_4} + C_5 \dot{e}_5 = \frac{e_c}{R_6} = \frac{e_b - e_5}{R_6}$$

$$\Rightarrow e_b = R_6\left[\frac{e_5}{R_4} + \frac{e_5}{R_6} + C_5 \dot{e}_5 \right]$$

Example 4 – Coupled resistances



Substituting

$$C_5\dot{e}_5 = \frac{V_1(t)}{R_2} - \underbrace{\left(\frac{R_6}{R_2} + \frac{R_6}{R_3}\right)}_{\frac{R_6(R_2+R_3)}{R_2R_3}}\left[\frac{e_5}{R_4} + \frac{e_5}{R_6} + C_5\dot{e}_5\right] - \frac{e_5}{R_4}$$

Solve for $\dot{e}_5$:

$$\left[1 + \frac{R_6(R_2+R_3)}{R_2R_3}\right]C_5\dot{e}_5 = \frac{V_1(t)}{R_2} - \frac{R_6(R_2+R_3)}{R_2R_3}\left(\frac{e_5}{R_4} + \frac{e_5}{R_6}\right) - \frac{e_5}{R_4}$$

$$\boxed{\dot{e}_5 = \frac{R_2R_3}{R_2R_3 + R_6(R_2+R_3)}\left[\frac{V_1(t)}{R_2} - \frac{1}{R_4}\left(1 + \frac{(R_2+R_3)(R_4+R_6)}{R_2R_3R_4}\right)e_5\right]}$$

Some messy algebra, but nothing too hard ...