Laplace Transform Analysis

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Approach

- Up to this point, we have learned how to derive equations of motion for a variety of systems: mechanical, electrical, fluid
- We will now shift our attention to analyzing these equations to better understand the behavior of these systems
- We will use the Laplace transform
  - Solve EOM for low-order, linear, constant-coefficient systems
  - Create transfer-function-based models
  - Study in detail first-order and second-order systems

Application of the Laplace transform

- Find the equations of motion that describe the system dynamic behavior
- Take the Laplace transform of the equations of motion
- Manipulate transformed equations to obtain transfer function, characteristic equation, eigenvalues, and the final (steady-state) values
- Apply the inverse Laplace transform if the time response is desired

Laplace transform

- Definition $$\mathcal{L}_{-}\{f(t)\} = F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt$$
- One-sided Laplace transform since lower bound is 0^-
- 0^- means the instant before $t = 0$
  - Allows us to include impulses that occur at $t = 0$
- For simplicity, we will usually drop the minus notation
- Laplace transform of a time-domain function is a function of the complex variable s, which can be thought of as frequency. Taking the Laplace transform is sometimes referred to as taking a system into the frequency domain

Properties of Laplace transform

- Superposition – Laplace transform is a linear operation

$$
\begin{aligned}
\mathcal{L}\{\alpha f_1(t) + \beta f_2(t)\} &= \int_0^\infty [\alpha f_1(t) + \beta f_2(t)] e^{-st} dt \\
&= \alpha \int_0^\infty f_1(t) e^{-st} dt + \beta \int_0^\infty f_2(t) e^{-st} dt \\
&= \alpha F_1(s) + \beta F_2(s)
\end{aligned}
$$

Properties of Laplace transform

- Laplace transform of time derivative of function

$$\mathcal{L}\left\{\frac{df}{dt}\right\} = -f(0^-) + sF(s)$$

- Higher order derivatives

$$\mathcal{L}\left\{\ddot{f}\right\} = s^2F(s) - sf(0^-) - \dot{f}(0^-)$$

$$\mathcal{L}\left\{f^{(m)}\right\} = s^mF(s) - s^{m-1}f(0^-) - s^{m-2}\dot{f}(0^-) - \cdots - f^{(m-1)}(0^-)$$

Properties of Laplace transform

- Laplace transform of time integral of function

$$\mathcal{L}\left\{\int f(t)\,dt\right\} = \frac{F(s)}{s}$$

- Laplace transform of function shifted in time

$$\mathcal{L}\{f(t-T)u_s(t-T)\} = e^{-Ts}F(s)$$

Laplace transform tables

Table 2.2.1 Table of Laplace transform pairs.

$X(s)$	$x(t), t \geq 0$
1. 1	$\delta(t)$, unit impulse
2. $\frac{1}{s}$	$u_s(t)$, unit step
3. $\frac{c}{s}$	constant, c
4. $\frac{e^{-sD}}{s}$	$u_s(t-D)$, shifted unit step
5. $\frac{n!}{s^{n+1}}$	t^n
6. $\frac{1}{s+a}$	e^{-at}
7. $\frac{1}{(s+a)^n}$	$\frac{1}{(n-1)!}t^{n-1}e^{-at}$
8. $\frac{b}{s^2+b^2}$	$\sin bt$
9. $\frac{s}{s^2+b^2}$	$\cos bt$
10. $\frac{b}{(s+a)^2+b^2}$	$e^{-at}\sin bt$
11. $\frac{s+a}{(s+a)^2+b^2}$	$e^{-at}\cos bt$
12. $\frac{a}{s(s+a)}$	$1-e^{-at}$

Laplace transform tables

13.	$\dfrac{1}{(s+a)(s+b)}$	$\dfrac{1}{b-a}(e^{-at}-e^{-bt})$
14.	$\dfrac{s+p}{(s+a)(s+b)}$	$\dfrac{1}{b-a}[(p-a)e^{-at}-(p-b)e^{-bt}]$
15.	$\dfrac{1}{(s+a)(s+b)(s+c)}$	$\dfrac{e^{-at}}{(b-a)(c-a)}+\dfrac{e^{-bt}}{(c-b)(a-b)}+\dfrac{e^{-ct}}{(a-c)(b-c)}$
16.	$\dfrac{s+p}{(s+a)(s+b)(s+c)}$	$\dfrac{(p-a)e^{-at}}{(b-a)(c-a)}+\dfrac{(p-b)e^{-bt}}{(c-b)(a-b)}+\dfrac{(p-c)e^{-ct}}{(a-c)(b-c)}$
17.	$\dfrac{b}{s^2-b^2}$	$\sinh bt$
18.	$\dfrac{s}{s^2+b^2}$	$\cosh bt$
19.	$\dfrac{a^2}{s^2(s+a)}$	$at-1+e^{-at}$
20.	$\dfrac{a^2}{s(s+a)^2}$	$1-(at+1)e^{-at}$
21.	$\dfrac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$	$\dfrac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\omega_n\sqrt{1-\zeta^2}t$
22.	$\dfrac{s}{s^2+2\zeta\omega_n s+\omega_n^2}$	$-\dfrac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}t-\phi\right), \phi=\tan^{-1}\dfrac{\sqrt{1-\zeta^2}}{\zeta}$
23.	$\dfrac{\omega_n^2}{s\left(s^2+2\zeta\omega_n s+\omega_n^2\right)}$	$1-\dfrac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}t+\phi\right)$

Laplace transform tables

Table 2.2.1 *(Continued)*

	$X(s)$	$x(t),\ t \geq 0$
24.	$\dfrac{1}{s[(s+a)^2+b^2]}$	$\dfrac{1}{a^2+b^2}\left[1-\left(\dfrac{a}{b}\sin bt+\cos bt\right)e^{-at}\right],\ \phi=\tan^{-1}\dfrac{\sqrt{1-\zeta^2}}{\zeta}$
25.	$\dfrac{b^2}{s(s^2+b^2)}$	$1-\cos bt$
26.	$\dfrac{b^3}{s^2(s^2+b^2)}$	$bt-\sin bt$
27.	$\dfrac{2b^3}{(s^2+b^2)^2}$	$\sin bt - bt\cos bt$
28.	$\dfrac{2bs}{(s^2+b^2)^2}$	$t\sin bt$
29.	$\dfrac{s^2-b^2}{(s^2+b^2)^2}$	$t\cos bt$
30.	$\dfrac{s}{\left(s^2+b_1^2\right)\left(s^2+b_2^2\right)}$	$\dfrac{1}{b_2^2-b_1^2}(\cos b_1t-\cos b_2t),\quad \left(b_1^2\neq b_2^2\right)$
31.	$\dfrac{s^2}{(s^2+b^2)^2}$	$\dfrac{1}{2b}(\sin bt+bt\cos bt)$

Example

- Find the Laplace transform of this EOM

$$m\dot{v} + bv = f(t)$$

- Find the solution for $v(t)$ if $f(t) = 0$
- Find the solution for $v(t)$ if $f(t)$ is a step function of size F_0

Final Value Theorem

$$x(t \to \infty) = \lim_{s \to 0} sX(s)$$

- Apply to our previous example where

$$V(s) = \frac{F(s) + mv_0}{ms + b}$$