Problem 3.19

Assume gears and shafts have no inertia.

To find equivalent inertia, use kinetic energy:

$$KE = \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_2 \omega_2^2 + \frac{1}{2} m_2 v_2^2 + \frac{1}{2} m_3 v_3^2$$

But we want everything in terms of ω, so that the equivalent inertia is that "felt" at the input shaft:

$$\omega_2 = \frac{\omega_1}{2}$$

$$v_2 = R\omega_2 = \frac{R\omega_1}{2}$$

$$v_1 = R\omega_2 = \frac{R\omega_1}{2}$$

$$KE = \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_2 \left(\frac{\omega_1}{2}\right)^2 + \frac{1}{2} m_2 \left(\frac{R\omega_1}{2}\right)^2 + \frac{1}{2} m_3 \left(\frac{R\omega_1}{2}\right)^2$$

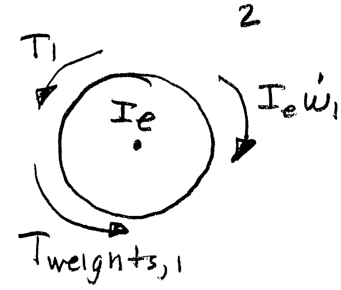
$$KE = \frac{1}{2} \underbrace{\left[I_1 + \frac{I_2}{4} + \frac{(m_2+m_3)R^2}{4}\right]}_{I_e} \omega_1^2$$

$$\boxed{I_e = I_1 + \frac{I_2}{4} + \frac{(m_2+m_3)R^2}{4}}$$ This is the inertia "felt" by the motor

The torque due to the two weights about pulley 2 is $T_{weights} = (m_3 - m_2) g R$

The torque due to the two weights about the motor shaft is $T_{weights,1} = \frac{1}{2} T_{weights} = \frac{(m_3-m_2)gR}{2}$

$$T_{weights,1} + T_1 - I_e \dot{\omega}_1 = 0$$



$$\boxed{I_e \dot{\omega}_1 = T_1 + \frac{(m_3-m_2)gR}{2}}$$

Problem 3.29

The expression for the kinetic energy is

$$KE = 2\left(\frac{1}{2}I_r\omega_r^2\right) + \frac{1}{2}I_f\omega_f^2 + \frac{1}{2}(m_b + 2m_r + m_f)v^2.$$

Also, the mass moments of inertia for the front and rear wheels are

$$I_f = \frac{1}{2}m_f R_f^2$$

$$I_r = \frac{1}{2}m_r R_r^2$$

and the angular velocities of the front and rear wheels can be written

$$\omega_f = \frac{v}{R_f} \qquad \omega_r = \frac{v}{R_r}.$$

This gives

$$KE = \frac{1}{2}\left(\frac{m_r R_r^2}{R_r^2} + \frac{1}{2}m_f\frac{R_f^2}{R_f^2} + m_b + 2m_r + m_f\right)v^2 = \frac{1}{2}(3m_r + m_b + 1.5m_f)v^2.$$

Hence,

$$m_e = 3m_r + m_b + 1.5m_f$$

This equivalent mass can now be used in Newton's second law:

$$m_e\dot{v} = \sum F.$$

The only force acting on the tractor is gravity; in the direction of the slope this force will have magnitude $m_T g\sin\theta$, where m_T is the total mass of the tractor, given by $m_T = m_b + m_f + 2m_r$. Therefore,

$$m_e\dot{v} = (m_b + m_f + 2m_r)g\sin\theta$$

or

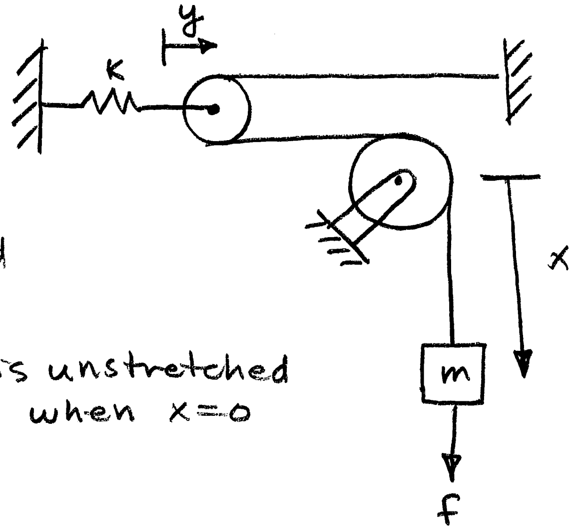
$$\dot{v} = \frac{(m_b + m_f + 2m_r)g\sin\theta}{3m_r + m_b + 1.5m + f} = \frac{[9000 + 2(500) + 800](32.2)\sin 10^\circ}{3(500) + 9000 + 1.5(800)} = 5.16\,\mathrm{ft/s^2}$$

so

$$v = 5.16t$$

where t is in seconds, and v is in ft/s.

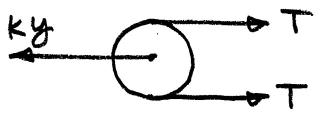
Problem 4.19



Note that x is not measured from the static equilibrium position, so we need to include gravity.

Assume the spring is unstretched when $x=0$. $\Rightarrow$ $y=0$ when $x=0$

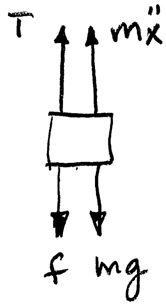
Upper pulley:



$\Rightarrow$ $ky = 2T$ because pulley is massless

$T = \frac{k}{2}y$

Mass:



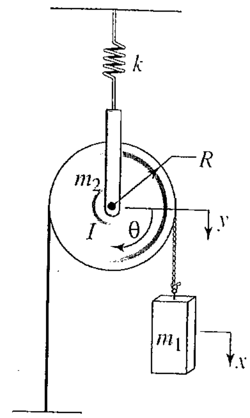
$\Rightarrow$ $m\ddot{x} = f + mg - T$

$m\ddot{x} = f + mg - \frac{k}{2}y$

But $x = 2y$ $\Rightarrow$ $\boxed{m\ddot{x} + \frac{k}{4}x = f + mg}$

$y = \frac{x}{2}$

Problem 4.36

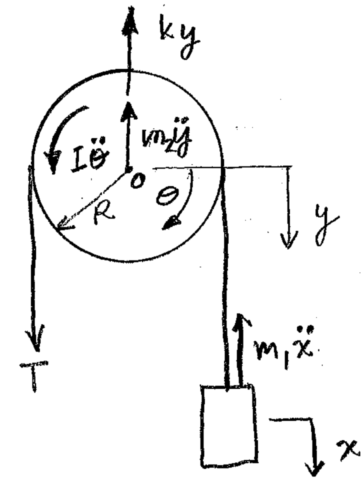


- Assume that the rope does not stretch and that it does not slip on the pulley.

Kinematic relations:

$$x = 2y$$
$$x - y = R\theta \implies y = R\theta$$

Free-body diagram:



- Assume displacements are measured relative to static equilibrium

$\Sigma F_y^* = 0$

$$-T + m_2\ddot{y} + ky + m_1\ddot{x} = 0 \qquad (\ddot{x} = 2\ddot{y})$$
$$-T + (2m_1 + m_2)\ddot{y} + ky = 0 \qquad (1)$$

$\Sigma M_O^* = 0$

For a cylinder, $I = \frac{1}{2}m_2R^2$

$$I\ddot{\theta} + TR + m_1R\ddot{x} = 0$$
$$\left(\tfrac{1}{2}m_2R^2\right)\frac{\ddot{y}}{R}\cdot\frac{1}{R} + T + m_1(2\ddot{y}) = 0$$

Also, $\ddot{\theta} = \frac{\ddot{y}}{R}$

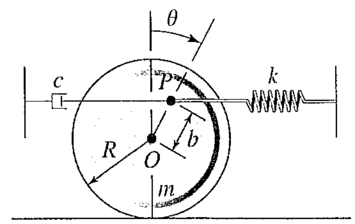
$$(2m_1 + \tfrac{1}{2}m_2)\ddot{y} = -T \qquad (2)$$

Substitute (2) → (1) to give

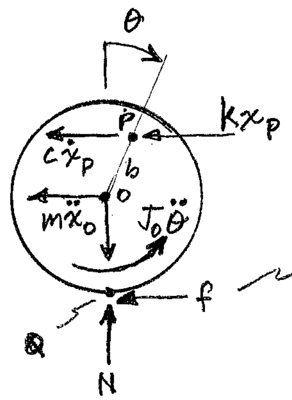
$$\boxed{(4m_1 + \tfrac{3}{2}m_2)\ddot{y} + ky = 0}$$

Problem 4.58

Assume small angles $\Rightarrow x_p = (R+b)\theta$



Assume displacement is measured relative to static equilibrium.



$x_o = R\theta$

No slipping implies there is a friction force acting.

$J_o = \frac{1}{2} m R^2$

$\Sigma M^*_Q = 0$

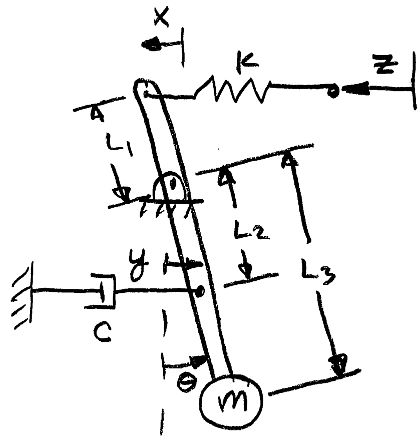
$$J_o \ddot{\theta} + (m\ddot{x}_o)R + (c\dot{x}_p)(R+b) + kx_p(R+b) = 0$$

$$\frac{1}{2} mR^2 \ddot{\theta} + mR(R\ddot{\theta}) + c(R+b)^2\dot{\theta} + k(R+b)^2\theta = 0$$

$$\boxed{\frac{3}{2} mR^2\ddot{\theta} + c(R+b)^2\dot{\theta} + k(R+b)^2\theta = 0}$$

Problem 4.62

θ small



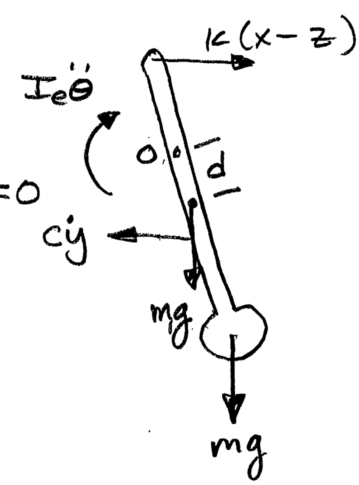
Let x = displacement of the upper end of the rod

y = displacement of the point of attachment of the damper

The total inertia of the system about the pivot is $I_e = I + mL_3^2$

Assume $x > z$

$[\Sigma M_o = 0]$



$$-I_e\ddot{\theta} - c\dot{y}L_2\cos\theta - m_r g d\sin\theta - mgL_3\sin\theta - k(x-z)L_1\cos\theta = 0$$

Small angle $\theta \Rightarrow \sin\theta \approx \theta$, $\cos\theta \approx 1$

$$-I_e\ddot{\theta} - c\dot{y}L_2 - m_r g d\theta - mgL_3\theta - k(x-z)L_1 = 0$$

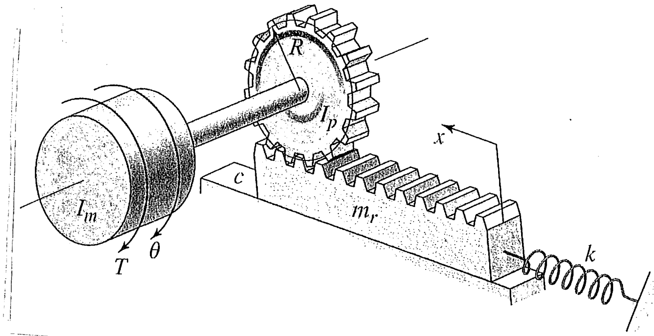
But $\dot{y} = L_2\dot{\theta}$ and $x = L_1\theta$

$$-I_e\ddot{\theta} - L_2^2 c\dot{\theta} - m_r g d\theta - mgL_3\theta - L_1^2 k\theta + L_1 k z = 0$$

Collect and rearrange:

$$\boxed{I_e\ddot{\theta} + L_2^2 c\dot{\theta} + \left[L_1^2 k + (mL_3 + m_r d)g\right]\theta = L_1 k z}$$

Problem 4.65



Since this is a 1-DOF problem, I can find an equivalent inertia as seen by the torque T.

First, what is x in terms of θ?

$x = R\theta \Rightarrow \dot{x} = R\dot{\theta}$

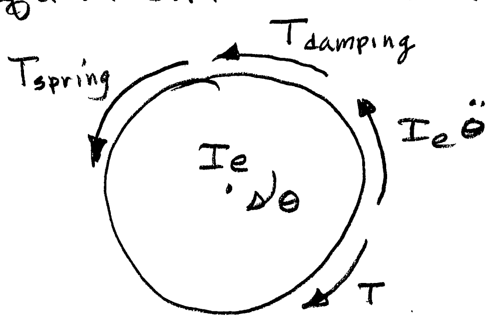
Kinetic energy:

$$KE = \frac{1}{2}(I_m + I_p)\dot{\theta}^2 + \frac{1}{2}m_r\dot{x}^2 = \frac{1}{2}(I_m + I_p)\dot{\theta}^2 + \frac{1}{2}m_r R^2\dot{\theta}^2$$

$$KE = \frac{1}{2}\left(I_m + I_p + m_r R^2\right)\dot{\theta}^2$$

$$I_e = I_m + I_p + m_r R^2$$

The equivalent rotational model is:



$$T - I_e\ddot{\theta} - T_{damping} - T_{spring} = 0$$

But what are the spring and damping torques?

The <u>forces</u> are $F_{spring} = kx$ and $F_{damping} = c\dot{x}$

The torques are $T_{spring} = Rkx$ and $T_{damping} = Rc\dot{x}$

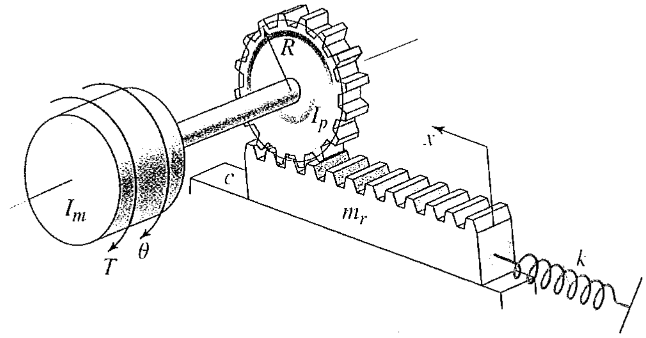
But $x = R\theta$ and $\dot{x} = R\dot{\theta}$, so

$$T_{spring} = R^2 k\theta, \quad T_{damping} = R^2 c\dot{\theta}$$

$$\boxed{I_e\ddot{\theta} + R^2 c\dot{\theta} + R^2 k\theta = T}$$

where $I_e = I_m + I_p + m_r R^2$

Problem 4.65

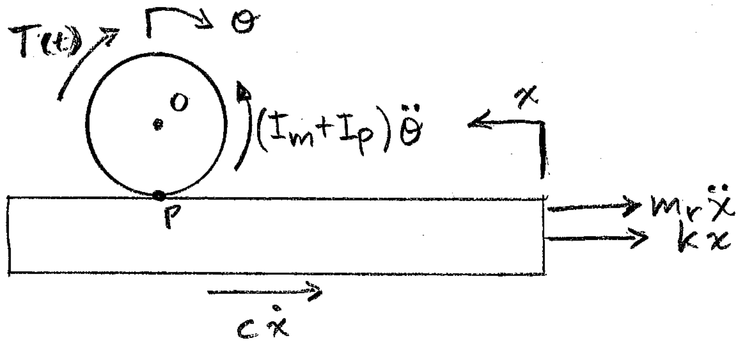


Solution using D'Alembert's principle.

Kinematic relations:

$$x = R\theta$$
$$\dot{x} = R\dot{\theta}$$
$$\ddot{x} = R\ddot{\theta}$$

FBD



Key point:

Recognize that forces acting on the rack are transferred to the pinion at point P.

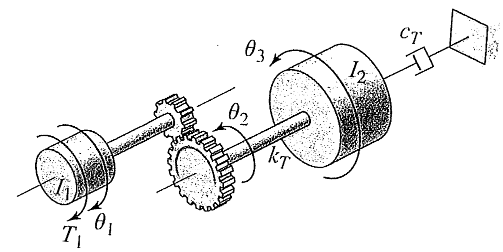
$\underline{\Sigma M_0^* = 0}$

$$T(t) - (I_m + I_p)\ddot{\theta} - R(m_r\ddot{x} + c\dot{x} + kx) = 0$$

$$(I_m + I_p)\ddot{\theta} + R^2(m_r\ddot{\theta} + c\dot{\theta} + k\theta) = T(t)$$

$$\boxed{(I_m + I_p + m_r R^2)\ddot{\theta} + cR^2\dot{\theta} + kR^2\theta = T(t)}$$

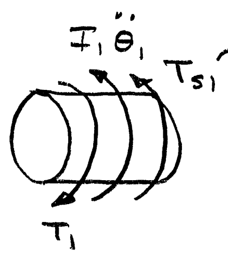
Problem 4.66



This is a 2-DOF problem represented by θ_1 and θ_3

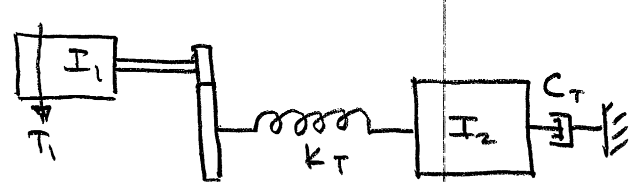
$\theta_1 = N\theta_2$ Assume $\theta_2 > \theta_3$

Body 1:



Torque from spring on body 1

Model:



$$I_1\ddot{\theta}_1 + T_{s1} = T_1$$

But $T_{s1} = \frac{1}{N}\underbrace{K_T(\theta_2-\theta_3)}_{\text{Spring torque on gear 2}}$

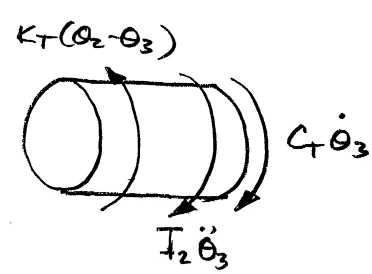
To convert to torque on gear 1

$$I_1\ddot{\theta}_1 + \frac{K_T}{N}(\theta_2-\theta_3) = T_1$$

But $\theta_2 = \frac{1}{N}\theta_1$

$$\boxed{I_1\ddot{\theta}_1 + \frac{K_T}{N^2}\theta_1 - \frac{K_T}{N}\theta_3 = T_1}$$

Body 2:



$$I_2\ddot{\theta}_3 + C_T\dot{\theta}_3 - K_T(\theta_2-\theta_3) = 0$$

But $\theta_2 = \frac{1}{N}\theta_1$

$$\boxed{I_2\ddot{\theta}_3 + C_T\dot{\theta}_3 + K_T\theta_3 - \frac{K_T}{N}\theta_1 = 0}$$