Fluid Systems

Thus far, we have studied mechanical and electrical systems. We have been exposed to three types of dynamic elements:

	Mechanical	Electrical
• Resistance	friction, damping	resistor
• Capacitance	springs, compliance	capacitor
• Inertia	mass, mass moment of inertia	inductance

Resistance: effort = R × flow

Capacitance: effort = $\frac{1}{C}$ × displacement

Inertia: flow = $\frac{1}{I}$ × momentum

These same dynamic elements exist for models of fluid systems. We'll spend some time talking about different examples.

Fluid Resistance

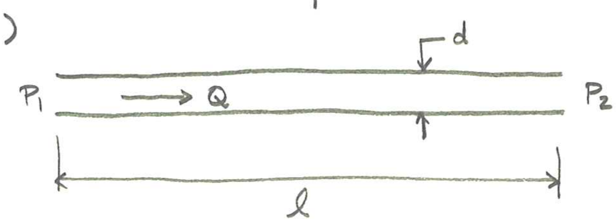
Fluid resistance can take a variety of forms and come from a variety of sources. As in electrical and mechanical systems, fluid resistance is modeled by a relationship between effort and flow variables — in this case pressure and flow rate.

Examples of fluid resistors include long tubes or pipes, orifices, and valves. In each of these cases, the pressure drop across the device is related to the flow through the device.

* Long tubes and pipes

Laminar flow ⟶ linear relationship
(for Re < 2000)

$$\underbrace{P_1 - P_2}_{\text{effort}} = \underbrace{\frac{128\mu l}{\pi d^4}}_{R} \underbrace{Q}_{\text{flow}}$$



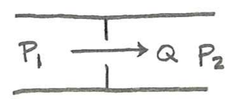
Turbulent flow $\longrightarrow$ nonlinear relationship (dependent on Re)

Example: $P_1 - P_2 = \underline{\underline{c}}\, Q\,|Q|^{3/4}$

experimentally identified

valid approximation for steady flow, $Re > 5000$

* Orifices



$$P_1 - P_2 = \frac{\rho}{2 C_d^2 A_o^2}\, Q\,|Q|$$

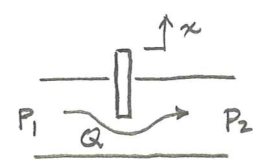
C_d = discharge coefficient = 0.62 (for round, sharp-edged orifices)

A_o = orifice area

of the form effort = $\underbrace{f\,(\text{flow})}_{\text{nonlinear function}}$

* why $Q\,|Q|$ instead of Q^2? (to account for direction of flow)

* Valves (modulated resistances)



$$P_1 - P_2 = \frac{\rho}{2 C_d^2(x)\, A^2(x)}\, Q\,|Q|$$

C_d and A are functions of valve position

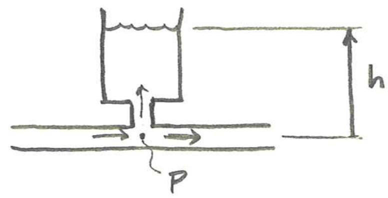
- Fluid resistance is a dissipative phenomena. It does not result in energy storage.

Fluid Capacitance

- Fluid capacitance is modeled by a relationship between the effort and displacement variables — in this case, pressure and volume. $\Delta P = \frac{1}{C}\Delta V$ — linear case

  Examples of fluid capacitance are tanks (e.g. a water tower), fluid compressibility, fluid line compliance, and accumulators.

* Tank



$P = \rho g h \qquad h = \frac{V}{A}$

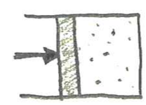
$P = \frac{\rho g}{A} V \qquad$ or $\qquad \underline{\underline{\Delta P = \frac{\rho g}{A}\Delta V}}$

$\Rightarrow C = \frac{A}{\rho g}$ ← gravity tank capacitance

* Fluid Compressibility

Hydraulic Systems (liquids)

Bulk modulus β, defined as



$$dP = -\beta \frac{dV}{V_0}$$

pressure decreases when fluid volume increases

isothermal

or

$$\underline{\underline{\Delta P = \frac{-\beta}{V_0}\Delta V}}$$

$C = -\frac{V_0}{\beta}$ ← liquid volume capacitance

Pneumatic Systems (gases)

$$\Delta P = -\frac{\rho_0 c_0^2}{V_0}\Delta V$$

subscript "o" indicates nominal value

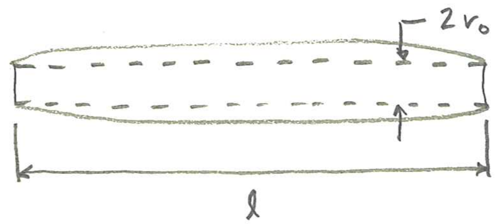
- ρ, c, V can change dramatically

$C = \frac{V_0}{\rho_0 c_0^2}$ ← gas volume capacitance

not valid for large changes in ρ_0, c_0, V_0.

Note: Gases are more compressible than liquids by a factor of 1000 to 10,000.

* Fluid Line Compliance



$$\Delta V = \frac{2\pi \ell r_o^3}{E\, t_w}\,\Delta P$$

$$V_o = \pi r_o^2 \ell$$

t_w = wall thickness

E = elastic modulus of line material

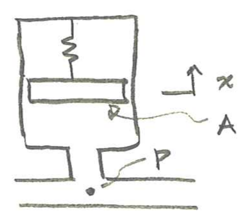
$$\Rightarrow \Delta V = \frac{2r_o}{E t_w} V_o\, \Delta P$$

or

$$\underline{\underline{\Delta P = \frac{E t_w}{2 r_o V_o}\,\Delta V}} \quad \Rightarrow \quad \underline{C = \frac{2 r_o V_o}{E t_w}}$$

* Accumulators

Spring-type:

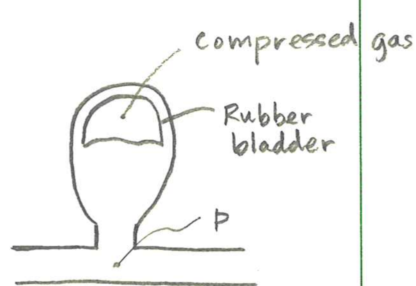


$$\Delta F = k\,\Delta x$$

$$\Delta P = \frac{\Delta F}{A} \qquad \Delta V = \Delta x \cdot A$$

$$\Delta P = \frac{\Delta F}{A} = \frac{k\,\Delta x}{A} = \frac{k\,\Delta x\, A}{A^2}$$

$$\underline{\underline{\Delta P = \frac{k}{A^2}\,\Delta V}}$$



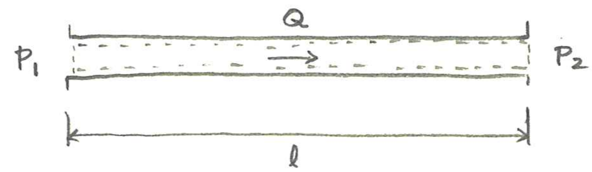
Gas bladder-type:

PV^γ = constant   adiabatic process

γ = ratio of specific heats $\frac{c_p}{c_v}$

$$\underline{\underline{\Delta P = \frac{P_o \gamma}{V_o}\,\Delta V}}$$

Fluid Inertia



$$\Sigma F_{ext} = m\dot{v}$$

$$P_1 A - P_2 A = m\dot{v} \qquad \text{but} \quad v = \frac{Q}{A}$$

$$\dot{v} = \frac{\dot{Q}}{A}$$

$$= \rho A L \cdot \frac{\dot{Q}}{A}$$

$$m = \rho A L$$

$$\Rightarrow \Delta P = \frac{\rho l}{A} \dot{Q}$$

For inertia elements,

$$\text{effort} = I \cdot \frac{d}{dt}(\text{flow})$$

$$\Delta P = \frac{\rho l}{A} \frac{d}{dt} Q \quad \Rightarrow \quad \underline{\underline{I = \frac{\rho l}{A}}}$$

Most significant for long slender tubes!

- A slender tube has greater fluid inertia than a fat tube! Why?

Consider two masses with equivalent force applied:

$$F = m\ddot{x}$$

mass : a measure of resistance to acceleration

$$m = \frac{F}{\ddot{x}} \longrightarrow \text{small acceleration} \rightarrow \text{large mass}$$

Consider two pipes with equivalent pressure drops:

$$\Delta P = I\dot{Q}$$

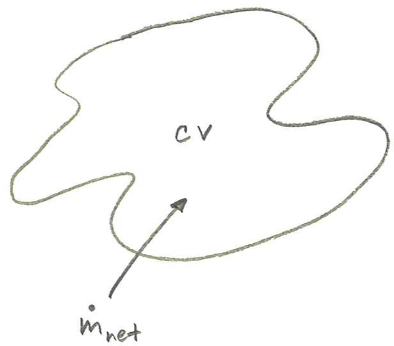
$$I = \frac{\Delta P}{\dot{Q}}$$

For a given ΔP, which pipe will have the smallest $\dot{Q}$?

The one with the smaller $\dot{Q}$ has the greater fluid inertia.

This supports $I = \rho\frac{l}{A}$

Continuity Equation for a Control Volume



$$\dot{m}_{net} = \frac{d}{dt} m_{cv} = \frac{d}{dt}(\rho_{cv} V)$$

$$\rho Q_{net} = \rho_{cv} \dot{V} + \dot{\rho}_{cv} V \qquad \text{where } Q_{net} = Q_{in} - Q_{out}$$

If $\rho = \rho_{cv}$, then

$$Q_{net} = \dot{V} + \frac{V}{\rho}\dot{\rho}$$

But $\rho = \rho(P,T) \Rightarrow \dot{\rho} = \frac{\partial \rho}{\partial P}\dot{P} + \cancel{\frac{\partial \rho}{\partial T}\dot{T}}$ $\nearrow 0$ assume isothermal

Also $\beta = \rho_0 \left.\frac{\partial P}{\partial \rho}\right|_{P_0, T_0} \Rightarrow \frac{\partial \rho}{\partial P} = \frac{\rho_0}{\beta}$

Combining gives $\dot{\rho} = \frac{\rho_0}{\beta}\dot{P}$

$$Q_{net} = \dot{V} + \frac{V}{\rho} \cdot \frac{\rho_0}{\beta}\dot{P}$$

$$Q_{net} = \dot{V} + \frac{V_0}{\beta}\dot{P}$$

or

$$\boxed{Q_{in} - Q_{out} = \dot{V} + \frac{V_0}{\beta}\dot{P}}$$

Volume change due to motion, line compliance, accumulator, gravity tank

fluid compliance

Analyzing Fluid Systems

1) Define distinct pressure nodes

2) Establish control volumes around pressure nodes

3) Write continuity equation for each pressure node

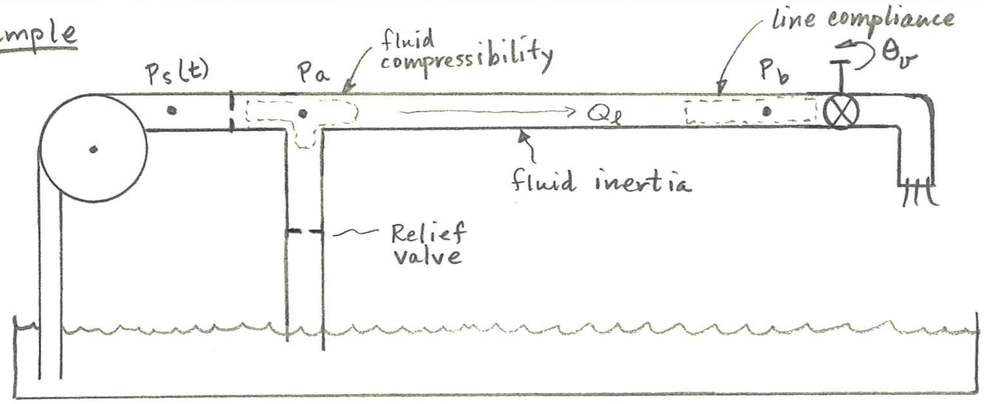
4) Define physical relations for $\dot{V}$ terms in continuity equation

5) Model pressure drops due to fluid resistances and inertias between pressure nodes. Write physical relations to model these pressure drops

6) Model mechanical portions of the system by drawing free-body diagrams and applying Newton's 2nd Law. Establish relations that describe power transfer between fluid and mechanical domains.

7) Combine relations to give the equations of motion in state-variable form. Compliances will yield a pressure state, while inertias will give a flow state.

Example



- Pressure Nodes: $P_s(t)$, P_a, P_b ← relative to P_{atm}

- continuity Eqns:

ⓐ $$Q_{in,a} - Q_{out,a} = \cancel{\dot{V}_a}^{\,0} + \frac{V_{0,a}}{\beta}\dot{P}_a$$

ⓑ $$Q_{in,b} - Q_{out,b} = \dot{V}_b + \cancel{\frac{V_{0,b}}{\beta}\dot{P}_b}^{\,0}$$

- Define $\dot{V}$'s: $\dot{V}_a = 0$, $\dot{V}_b = C_\ell \dot{P}_b$ $\quad C_\ell = \frac{2 r_0 V_0}{E t_w}$

- Flow Relations:

$$Q_{in,a} = k_1 \sqrt{|P_s(t) - P_a|}\ \mathrm{sign}\,(P_s(t) - P_a)$$

$$Q_{out,a} = Q_\ell + k_3 \sqrt{P_a}$$

Recall for orifice:

$$P_1 - P_2 = \frac{\rho}{2 C_d^2 A_0^2} Q|Q|$$

$$\Rightarrow Q = k\sqrt{P_1 - P_2}\ \mathrm{sign}\,(P_1 - P_2)$$

$$Q_{in,b} = Q_\ell$$

Q_ℓ is flow state variable associated with fluid inertia

$$Q_{out,b} = k_4 \theta_v \sqrt{P_b}$$

Also: $$\boxed{\dot{Q}_\ell = \frac{A}{\rho \ell}(P_a - P_b)}$$ where $Q_\ell = Q_{in,b}$

- Combine

From ⓐ: $$\dot{P}_a = \frac{\beta}{V_{0,a}}\left[Q_{in,a} - Q_{out,a}\right]$$

$$\boxed{\dot{P}_a = \frac{\beta}{V_{0,a}}\left[-Q_\ell - k_3\sqrt{P_a} + k_1\sqrt{|P_s(t) - P_a|}\ \mathrm{sign}\,(P_s(t) - P_a)\right.}$$

(cont.)

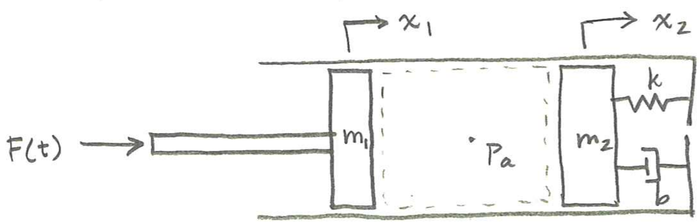
From (b)

$$\dot{V}_b = Q_{in,b} - Q_{out,b}$$

$$\dot{P}_b = \frac{1}{C_\ell}\left[Q_\ell - k_4 \theta_v \sqrt{P_b}\right]$$

$$\boxed{\dot{P}_b = \frac{E\, t_w}{2 r_o V_{o,b}}\left[Q_\ell - k_4 \theta_v \sqrt{P_b}\right]}$$

Example: Fluid Spring

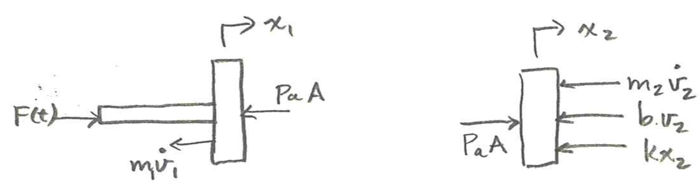


- Pressure node: P_a
- Continuity Eqn: $\cancel{Q_{in}}^{0} - \cancel{Q_{out}}^{0} = \dot{V} + \frac{V_o}{\beta}\dot{P}_a$ (a)
- Define $\dot{V}$:

$$\dot{V} = A\dot{x}_1 - A\dot{x}_2$$

$$\dot{V} = A(v_1 - v_2)$$

- Flow relations: none
- Mechanical sys:



$$m_1\dot{v}_1 + P_a A = F(t) \qquad m_2\dot{v}_2 + b v_2 + k x_2 = P_a A$$

$$\boxed{\begin{aligned}\dot{v}_1 &= -\frac{A}{m_1}P_a + \frac{1}{m_1}F(t)\\ \dot{x}_1 &= v_1\end{aligned}} \qquad \boxed{\begin{aligned}\dot{v}_2 &= -\frac{b}{m_2}v_2 - \frac{k}{m_2}x_2 - \frac{A}{m_2}P_a\\ \dot{x}_2 &= v_2\end{aligned}}$$

- Combine: From (a) $\dot{P}_a = \frac{\beta}{V_o}\dot{V}$

$$\boxed{\dot{P}_a = \frac{\beta A}{V_o}(v_1 - v_2)}$$