NORM OF INFINITE DOUBLY STOCHASTIC MATRICES

In the following, the letter D will denote any semi-infinite doubly stochastic, that is a non-negative semi-infinite matrix

$$D = \begin{bmatrix} d_{11} & d_{12} & d_{13} & \cdots \\ d_{21} & d_{22} & d_{23} & \cdots \\ d_{31} & d_{32} & d_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

such that $\sum_{i=1}^{\infty} d_{ij} = 1$ for any $j \geq 1$ and $\sum_{j=1}^{\infty} d_{ij} = 1$ for any $i \geq 1$. We know from A.B. Eshkaftaki (2021) that any such semi-infinite matrix defines an operator from $\ell^p \to \ell^p$ for any $1 \leq p \leq \infty$. As such, we'll avoid the term semi-infinite matrix and simply use the terminology *doubly stochastic operator* (on ℓ^p) to refer to D.

In addition to proving that any doubly stochastic operator is an operator from $\ell^p \to \ell^p$ for any $1 \leq p \leq \infty$, Eshkaftaki also established that its norm is bounded above by 1, that is $\|D\|_{\ell^p \to \ell^p} \leq 1$. In the finite case, when D is simply a *doubly stochastic matrix*, that is a nonnegative square matrix with the property that each row and each column sums to 1, it turns out that we have an even stronger result. As an easy corollary to Birkhoff's theorem on doubly stochastic matrices, Bouthat (2022) proved in his master's thesis that the equality $\|D\|_{\ell^p \to \ell^p} = 1$ for any doubly stochastic matrix D and any $p \geq 1$.

In Eshkaftaki's paper, it is shown that for any $1 < p < \infty$, there exists some infinite doubly stochastic matrix for which the norm is strictly smaller than 1. This contrasts with the finite case, where all doubly stochastic matrices have a norm of exactly 1. Naturally, one may wonder when does the norm of an infinite doubly stochastic matrix have a norm smaller than 1.

The following theorem provides a complete characterization of the doubly stochastic matrices for which the operator p-norm is equal to 1. If we define

$$\Theta(D) := \sup_{S \subseteq \mathbb{N}} \frac{1}{|S|} \sum_{i,j \in S} d_{ij}, \tag{0.1}$$

then $\|D\|_{\ell^p \to \ell^p} = 1$ if and only if $\Theta(D^*D) = 1$, which intuitively means that D^*D contains a sequence of square submatrices which behaves more and more like a finite doubly stochastic matrix.

• Peut-on obtenir de meilleurs résultats en considérant $\sup_{\substack{S,V \subseteq \mathbb{N} \\ |S|=|V|}} \frac{1}{|S|} \sum_{\substack{i \in S \\ j \in V}} d_{ij}$?

Theorem 1. *Let $1 < p < \infty$, and let D be an infinite doubly stochastic matrix. Then $\|D\|_{\ell^p \to \ell^p} = 1$ if and only if $\Theta(D^*D) = 1$. Moreover, if D is symmetric then $\|D\|_{\ell^p \to \ell^p} = 1$ if and only if $\Theta(D) = 1$.*

Proof. First note that since we know that D is always a bounded operator on ℓ^2, we have $\|D\|^2_{\ell^2 \to \ell^2} = \|D^*D\|_{\ell^2 \to \ell^2}$. Hence, we may consider without loss of generality the self-adjoint doubly stochastic matrix $A := D^*D$ when $p = 2$. If D is already self-adjoint (symmetric), then we may simply consider $A = D$.

(Proof of $\Theta(A) = 1 \implies \|D\|_{\ell^p \to \ell^p} = 1$) Let us first consider the case $p = 2$. Let $S \subseteq \mathbb{N}$ be a finite, nonempty set, and let e_S be the sequence with 0 in every

• We should be extend $\mathbb{N}$ to any, possibly finite or uncountable, set I.

entries *not* in S, and $1/\sqrt{|S|}$ in the entries in S. Then observe that $\|e_S\|_2 = 1$ and that the nth entry of Ae_S is

$$(Ae_S)_n = \frac{1}{\sqrt{|S|}} \sum_{j \in S} a_{nj}.$$

Hence,

$$\begin{aligned}(Ae_S)_n - (e_S)_n &= \begin{cases} \frac{1}{\sqrt{|S|}} \sum_{j\in S} a_{nj} - 1 & \text{if } n \in S, \\ \frac{1}{\sqrt{|S|}} \sum_{j\in S} a_{nj} - 0 & \text{if } n \notin S \end{cases} \\ &= \begin{cases} -\frac{1}{\sqrt{|S|}} \sum_{j\notin S} a_{nj} & \text{if } n \in S, \\ \frac{1}{\sqrt{|S|}} \sum_{j\in S} a_{nj} & \text{if } n \notin S, \end{cases}\end{aligned}$$

since $\sum_{j\in S} a_{nj} + \sum_{j\notin S} a_{nj} = 1$ for all $n \in \mathbb{N}$. Therefore, we have

$$\|Ae_S - e_S\|_2^2 = \frac{1}{|S|} \sum_{i\in S} \left(\sum_{j\notin S} a_{ij} \right)^2 + \frac{1}{|S|} \sum_{i\notin S} \left(\sum_{j\in S} a_{ij} \right)^2.$$

Now, since we have both $\sum_{j\in S} a_{ij} \le \sum_{j\in\mathbb{N}} a_{ij} = 1$ and $\sum_{j\notin S} a_{ij} \le \sum_{j\in\mathbb{N}} a_{ij} = 1$, it follows that

$$\begin{aligned}\|Ae_S - e_S\|_2^2 &= \frac{1}{|S|} \sum_{i\in S} \left(\sum_{j\notin S} a_{ij} \right)^2 + \frac{1}{|S|} \sum_{i\notin S} \left(\sum_{j\in S} a_{ij} \right)^2 \\ &\le \frac{1}{|S|} \sum_{i\in S} \left(\sum_{j\notin S} a_{ij} \right) + \frac{1}{|S|} \sum_{i\notin S} \left(\sum_{j\in S} a_{ij} \right) \\ &= \frac{2}{|S|} \sum_{i\in S} \sum_{j\notin S} a_{ij},\end{aligned}$$

where the last identity follows from the fact that A is doubly stochastic since

$$\begin{aligned}\sum_{i\notin S} \sum_{j\in S} a_{ij} &= \sum_{i\in\mathbb{N}} \sum_{j\in S} a_{ij} - \sum_{i\in S} \sum_{j\in S} a_{ij} = 1 - \sum_{i\in S} \sum_{j\in S} a_{ij} \\ &= 1 - \sum_{i\in S} \sum_{j\in\mathbb{N}} a_{ij} + \sum_{i\in S} \sum_{j\notin S} a_{ij} = 1 - 1 + \sum_{i\in S} \sum_{j\notin S} a_{ij} \\ &= \sum_{i\in S} \sum_{j\notin S} a_{ij}.\end{aligned}$$

Moreover, observe that

$$\frac{1}{|S|} \sum_{i\in S} \sum_{j\notin S} a_{ij} = \frac{1}{|S|} \sum_{i\in S} \sum_{j\in\mathbb{N}} a_{ij} - \frac{1}{|S|} \sum_{i\in S} \sum_{j\in S} a_{ij} =: 1 - \Theta_S(A).$$

Additionally, observe that

$$\|Ae_S - e_S\|_2^2 \ge \big(\|e_S\|_2 - \|Ae_S\|_2\big)^2 = \big(1 - \|Ae_S\|_2\big)^2 \ge \big(1 - \|A\|_{\ell^2\to\ell^2}\big)^2,$$

since $\|A\|_{\ell^2\to\ell^2} \le \|Ae_S\|_2 \le 1$ and $\|e_S\|_2 = 1$. Therefore, if $\Theta(A) = \sup_{S\subseteq\mathbb{N}} \Theta_S(A) = 1$, then we have

$$\big(1 - \|A\|_{\ell^2\to\ell^2}\big)^2 \le 2\big(1 - \Theta(A)\big) = 0,$$

which implies that $\|D\|_{\ell^2\to\ell^2} = \|A\|_{\ell^2\to\ell^2} = 1$.

Now, by Riesz–Thorin interpolation theorem, we have when $1 \le p \le 2$ that

$$1 = \|D\|_{\ell^2\to\ell^2} \le \|D\|_{\ell^p\to\ell^p}^{p/2}\|D\|_{\ell^\infty\to\ell^\infty}^{1-p/2} = \|D\|_{\ell^p\to\ell^p}^{p/2} \le 1,$$

which then implies that $\|D\|_{\ell^p\to\ell^p} = 1$. Similarly, if $2 \le p \le \infty$, then

$$1 = \|D\|_{\ell^2\to\ell^2} \le \|D\|_{\ell^1\to\ell^1}^{1-\frac{p}{2(p-1)}}\|D\|_{\ell^p\to\ell^p}^{\frac{p}{2(p-1)}} = \|D\|_{\ell^p\to\ell^p}^{\frac{p}{2(p-1)}} \le 1,$$

which once again implies that $\|D\|_{\ell^p\to\ell^p} = 1$.

(Proof of $\|D\|_{\ell^p\to\ell^p} = 1 \implies \Theta(A) = 1$) By Riesz–Thorin interpolation theorem, for any $2 \le p \le \infty$, we have

$$\|D\|_{\ell^p\to\ell^p} \le \|D\|_{\ell^2\to\ell^2}^{2/p}\|D\|_{\ell^\infty\to\ell^\infty}^{1-2/p} = \|D\|_{\ell^2\to\ell^2}^{2/p} \le 1,$$

since $\|D\|_{\ell^\infty\to\ell^\infty} = 1$ for all infinite doubly stochastic matrix A. Similarly, for $1 \le p \le 2$ we have

$$\|D\|_{\ell^p\to\ell^p} \le \|D\|_{\ell^2\to\ell^2}^{2(1-1/p)} \le 1.$$

In both cases, if $\|D\|_{\ell^p\to\ell^p} = 1$, then we also have $\|D\|_{\ell^2\to\ell^2} = 1$. Hence, consider in the following the case $p = 2$. Once again, since $\|A\|_{\ell^2\to\ell^2} = \|D\|_{\ell^2\to\ell^2} = 1$, where $A = D^*D$ (or $A = D$ is D is symmetric), we may simply consider the matrix A instead of D.

Now, since A is self-adjoint, we can apply the lower bound of [**?**, Theorem 3.1] with $L = I - A$, π the counting measure, $\mu(i,j) = a_{ij}$ and $K = 0$. More specifically, apply [**?**, equation (3.18)] which states that

$$\langle x, (I-A)x\rangle_2 \ge \frac{1}{2}\cdot\Phi(A)\cdot\|x\|_2^2,$$

where $\Phi(A) := 1 - \Theta(A)$. But observe that we have

$$\langle x, (I-A)x\rangle_2 = \langle x, x\rangle_2 - \langle x, Ax\rangle_2 = \|x\|_2^2 - \|A\|_2^2.$$

Hence, dividing by $\|x\|_2^2$ yields

$$1 - \frac{\|A\|_2^2}{\|x\|_2^2} \ge \frac{1}{2}\cdot\Phi(A),$$

and taking the infimum on the $x \in \ell^2$ then gives

$$1 - \|A\|_{\ell^2\to\ell^2}^2 \ge \frac{1}{2}\cdot\Phi(A).$$

Since we have shown that $\|D\|_{\ell^p\to\ell^p} = 1$ implies that $\|A\|_{\ell^2\to\ell^2} = 1$, it follows that $1 - \Theta(A) = \Phi(A) = 0$, i.e., that $\Theta(A) = 1$, which concludes the proof. □

Remark 1. The quantity $\Phi(A)$ is called Cheeger's isoperimetric constant, and is one of the main motivation that started the field of spectral graph theory. This constant essentially tells whether or not the graph associated to the matrix A has a "bottleneck". Hence, the main theorem says that $\|D\|_{\ell^2\to\ell^2} = 1$ if and only if there is a positive isoperimetric Cheeger constant, i.e., the graph is *non-amenable*.

If D is symmetric, we have seen that our main theorem simplifies to $\|D\|_{\ell^2\to\ell^2} = 1$ if and only if $\Theta(D) = 1$. This naturally raises the question of relating $\Theta(D)$ and $\Theta(D^*D)$. More specifically, do we have $\Theta(D) = 1$ if and only if $\Theta(D^*D) = 1$? Once again, we do not have an answer to this question, although we do have the following inequality.

Proposition 1. *Let D be a doubly stochastic matrix and set $A := D^*D$. Then*

$$1 - \Theta(A) \geq 2(1 - \Theta(D)).$$

In particular, $\Theta(A) = 1 \implies \Theta(D) = 1$.

Proof. For a finite $S \subset V$ define $t_k := \sum_{i \in S} d_{ki} \in [0, 1]$. A direct computation gives $a_{ij} = \sum_{k=1}^{\infty} d_{ki} d_{kj}$, from which it follows that

$$\begin{aligned}
|S|\Phi_S(A) &= \sum_{i \in S,\, j \notin S} a_{ij} = \sum_{i \in S,\, j \notin S} \sum_{k=1}^{\infty} d_{ki} d_{kj} = \sum_{k \in \mathbb{N}} t_k (1 - t_k) \\
&= \sum_{k \notin S} t_k (1 - t_k) + \sum_{k \in S} t_k (1 - t_k) \leq \sum_{k \notin S} t_k + \sum_{k \in S} (1 - t_k) \\
&= \sum_{k \in \mathbb{N}} t_k - \sum_{k \in S} t_k + \sum_{k \in S} (1 - t_k) = 2 \sum_{k \in S} (1 - t_k) \\
&= 2 \sum_{k \in S} \sum_{i \notin S} a_{ki} = 2|S|\Phi_S(D).
\end{aligned}$$

Dividing by $|S|$ and taking the infimum over S shows $\Phi(A) \leq 2\,\Phi(D)$,which concludes the proof. $\square$