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This alternate proof attempts to clarify our problem's relation to the spectral graph theory. In what follows, we let $\cdot \restriction S$ denote restriction to the subset $S \subset \mathbb{N}$: for a matrix M, $M \restriction S$ is the $|S| \times |S|$ minor of M obtained by removing all rows and columns not indexed by S, for a vector v, $v \restriction S$ is the vector of length $|S|$ obtained by selecting all entries indexed by S. Also, for any infinite graph G, we identify its set of nodes with $\mathbb{N}$.

Theorem 0.1 *If $D = (d_{ij})_{i,j\in\mathbb{N}}$ is an infinite doubly stochastic (d.s.) matrix, then $||D||_{\ell^2\to\ell^2} = 1$ if and only if $\Phi(D) = \inf_{S\subset\mathbb{N}} \Phi_S(D) = 0$, where $\Phi_S(D) := (1/|S|)\sum_{i\in S, j\notin S} d_{ij}$*

Remark. D corresponds to the adjacency matrix of an infinite graph with degree 1 for all nodes. For symmetric D describing the adjacency matrix of an undirected graph, the value $\Phi_S(D)$ for a subset of nodes $S \subset \mathbb{N}$ corresponds to the edge expansion of S. and the value $\Phi(D)$ corresponds to the Cheeger constant of the infinite graph. To remain unambiguous, we use a different symbol for the Cheeger constant of finite graphs: for a finite graph G' with adjacency matrix A', let $\phi(A')$ denote the Cheeger constant of G' and let $\phi_{S'}(A')$ denote the edge expansion of a subset of nodes S' within G'.

Before proceeding with the proof, we record a lemma which will allow us to reason about the spectrum of infinite, undirected graphs from that of their finite subgraphs.

Lemma 0.2 *Let G be an infinite undirected graph having non-negative edge weights, with normalized Laplacian $\mathcal{L}$. For any node vector x with non-negative entries supported on $S \subset \mathbb{N}$, we have*

$$R(\mathcal{L}, x) \le R(\mathcal{L} \restriction S, x \restriction S),$$

where $R(M, x) := \langle x, Mx\rangle/\langle x, x\rangle$ denotes the Rayleigh quotient.

Proof. L0.2. Assume w.l.o.g. that all nodes of G have degree 1. Then $\mathcal{L} = I - A$ where $A = (a_{ij})_{i,j\in\mathbb{N}}$ is the adjacency matrix of G. We have

$$\begin{aligned}
\langle x, Ax\rangle &= \sum_{i\in\mathbb{N}, j\in\mathbb{N}} a_{ij}x_i x_j \\
&\ge \sum_{u\in S, v\in S} a_{uv}x_u x_v \\
&= \langle x \restriction S, (A \restriction S)x \restriction S\rangle
\end{aligned}$$

Divide now by $\langle x, x\rangle = \langle x \upharpoonright S, x \upharpoonright S\rangle$ on both sides, and subtract the result from 1 to obtain

$$\frac{\langle x, (I-A)x\rangle}{\langle x, x\rangle} \leq \frac{\langle x \upharpoonright S, ((I-A) \upharpoonright S)x \upharpoonright S\rangle}{\langle x \upharpoonright S, x \upharpoonright S\rangle},$$

which is what we sought to show. ////

We now proceed with the proof of Theorem 0.1.

Proof. T0.1.
In general, for infinite symmetric d.s. matrices M, we have $||M||_{\ell^2\to\ell^2} = 1$ if and only if $\lambda_{\min}(I-M) = 0$. That is, the minimum eigenvalue of the graph Laplacian associated with M is 0. For the forward direction, Bouthat had already shown that $||D||_{\ell^2\to\ell^2} = 1 \iff ||D^*D||_{\ell^2\to\ell^2} = 1 \implies \Phi(D^*D) = 0 \implies \Phi(D) = 0$. This second implication follows from the lower bound

$$R(I - D^*D, x) \geq \frac{1}{2}\Phi(D^*D).$$

Taking the infimum yields $0 = \lambda_{\min}(I - D^*D) \geq \Phi(D^*D)$. This last implication is a corollary to the bound $\Phi(D^*D) \geq 2\Phi(D)$, which was shown in a later proposition.

For the reverse direction, we use the symmetrization $A = (D + D^{\mathrm{tr}})/2$, which is also an infinite d.s. matrix. An important fact remains that $||A||_{\ell^2\to\ell^2} = 1 \implies ||D||_{\ell^2\to\ell^2} = 1$ from an application of the triangle inequality:

$$||D||_{\ell^2\to\ell^2} = \frac{1}{2}(||D||_{\ell^2\to\ell^2} + ||D^{\mathrm{tr}}||_{\ell^2\to\ell^2}) \geq ||(D + D^{\mathrm{tr}})/2||_{\ell^2\to\ell^2}.$$

That $||A||_{\ell^2\to\ell^2} = 1$, or rather, $\lambda_{\min}(I - A) = 0$, will follow naturally from a constraint on the Cheeger constant of A, namely $\Phi(A) = 0$. This last equality in turn follows from the fact that symmetrization preserves the edge expansion of any subset $S \subset \mathbb{N}$. To wit, observe that for any subset $S \subset \mathbb{N}$, we have

$$\begin{aligned}
\sum_{i\in S, j\notin S} d_{ij} &= \sum_{i\in S, j\in\mathbb{N}} d_{ij} - \sum_{i\in S, j\in S} d_{ij} \\
&= |D| - \sum_{i\in S, j\in S} d_{ji} \\
&= \sum_{i\in S, j\in\mathbb{N}} d_{ji} - \sum_{i\in S, j\in S} d_{ji} \\
&= \sum_{i\in S, j\notin S} d_{ji}.
\end{aligned}$$

From here it is not hard to see that $\Phi_S(A) = \Phi_S(D)$, and thus $\Phi(A) = \Phi(D) = 0$. Suppose that the infimum $0 = \inf_{S\subset\mathbb{N}} \Phi_S(A)$ is attained by a finite subset of nodes $V \subset \mathbb{N}$; then the undirected graph associated to A is a disconnected graph with isolated component V. What's more, by Lemma 0.2, we know that $R(I - A, x) \leq R((I - A) \upharpoonright V, x \upharpoonright V)$. Plugging in $x = \mathbf{1}_V$ defined by $\mathbf{1}_V(i) = 1, i \in V$ and 0 otherwise yields

$$\lambda_{\min}(I - A) \leq R(I - A, \mathbf{1}_V) \leq R((I - A) \upharpoonright V, \mathbf{1}_V \upharpoonright V) = 0.$$

The last equality follows from the fact that V is isolated; as a consequence, restricting to V yields the normalized Laplacian of an undirected connected graph, which always has a minimum eigenvalue of 0 attained by the uniform frequency component. If on the other hand the infimum is not attained by a finite subset of nodes, then the graph associated with A is connected, and we can construct a sequence of subgraphs with adjacency given by $(A \restriction V_n), V_n \subset \mathbb{N}$ for $n \in \mathbb{N}$ with decreasing edge expansion $\Phi_{V_n}(A)$. Then let D_n be the degree matrix associated to $A \restriction V_n$. Restricting to V_n yields

$$\begin{aligned}
\lambda_{\min}(I - A) &\leq R(I - A, \mathbf{1}_{V_n}) \\
&\leq R((I - A) \restriction V_n, \mathbf{1}_{V_n} \restriction V_n) \\
&\leq c_n R(D_n^{-1/2}((I - A) \restriction V_n) D_n^{-1/2}, \mathbf{1}_{V_n} \restriction V_n) \\
&= c_n R(I \restriction V_n - D_n^{-1/2}(A \restriction V_n) D_n^{-1/2}, \mathbf{1}_{V_n} \restriction V_n) + c_n R(D_n^{-1} - I \restriction V_n, \mathbf{1}_{V_n} \restriction V_n) \\
&= c_n R(D_n^{-1} - I \restriction V_n, \mathbf{1}_{V_n} \restriction V_n)
\end{aligned}$$

In the third inequality, we used a correction factor c_n to bound above by a Rayleigh quotient of the normalized Laplacian associated with $A \restriction V_n$, plus an error term. We compute, for $L_n = I - A \restriction V_n$

$$\begin{aligned}
R(L_n, x) &\leq R(D_n^{-1/2} L_n D_n^{-1/2}, D_n^{1/2} x) \\
&\leq \frac{1}{\min_i D_n(i)} R(D_n^{-1/2} L_n D_n^{-1/2}, x),
\end{aligned}$$

so that we may use $c_n := 1/\min_i D_n(i)$. Because the edge expansion $\Phi_{V_n}(A)$ tends to zero, the degrees of each node in V_n within the subgraph given by the adjacency $A \restriction V_n$ tends to 1 (keeping a fixed ordering of the nodes), and thus the error term $c_n R(D_n^{-1} - I \restriction V_n, \mathbf{1}_{V_n} \restriction V_n)$ tends to zero.

Remark. The necessary and sufficient condition that $\Phi(D^*D) = 0$ can be lifted to $\Phi(D) = 0$. The key is to use the symmetric matrix D^*D for the forward direction, and use the other symmetrization $(D + D^{\mathrm{tr}})/2$ for the reverse direction.

Remark. There is maybe a way to generalize this characterization to the case where our object of interest are doubly stochastic operators acting on $L^p, 1 \leq p \leq \infty$.