Stochastic calculus, Ramon van Handel

R.D.P. Vo

July 2024

1 Black-Scholes

Answer to homework 3, question 2 of Ramon van Handel's ACM 217 course on stochastic calculus and stochastic control.

1.1 Total balance expression

We write

$$X_{t_n+1} = X_0 + \sum_{k=0}^{n}(X_{k+1} - X_k) \tag{1}$$

However, note that for arbitrary $j = 1, ..., n$ we have $X_{t_{j+1}} = \alpha_{t_{j+1}} S_{t_{j+1}} + \beta_{t_{j+1}} R_{t_j} = \alpha_{t_j} S_{t_{j+1}} + \beta_{t_j} R_{t_j}$, and $X_{t_j} = \alpha_{t_j} S_{t_j} + \beta_{t_j} R_{t_j}$. Hence

$$X_{j+1} - X_{t_j} = \alpha_{t_j}(S_{t_{j+1}} - S_{t_j}) + \beta_{t_j}(R_{t_{j+1}} - R_{t_j}).$$

Moreover, we naturally have $X_0 = \alpha_0 + \beta_0$, whence we may substitute into (1) to obtain

$$X_{t_{n+1}} = \alpha_0 + \beta_0 + \sum_{k=0}^{n} \left(\alpha_{t_k}(S_{t_{k+1}} - S_{t_k}) + \beta_{t_k}(R_{t_{k+1}} - R_{t_k})\right),$$

which is the desired equality.

The integral form is motivated by a generalization to continuous processes. More precisely, we would like to integrate the $\mathcal{F}_t-$adapted stochastic processes α_t and β_t with, respectively, dS_t and dR_t. The latter differential is given by the chain rule: $dR_t = rR_t dt$, whereas the former differential requires an application of the Itô rule.
Let $Z_t \coloneqq (\mu - \sigma^2/2)t + \sigma W_t$ – it is an Itô process (with constant drift and volatility), and so the transformation $S_t = e^{Z_t}$ admits the following decomposition (in symbolic differential form):

$$\begin{aligned}
dS_t &= \frac{d}{dZ_t} e^{Z_t} dZ_t + \frac{1}{2}\left(\frac{d^2}{dZ_t^2} e^{Z_t}(dZ_t)^2\right) \\
&= e^{Z_t} dZ_t + \frac{1}{2} e^{Z_t}(dZ_t)^2, \;\; dZ_t = (\mu - \sigma^2/2)dt + \sigma dW_t, (dZ_t)^2 = \sigma^2 dt \text{ [1]} \\
&= e^{Z_t}\left((\mu - \sigma^2/2)dt + \sigma dW_t\right) + \frac{1}{2} e^{Z_t} \sigma^2 dt \\
&= \mu S_t dt + \sigma S_t dW_t
\end{aligned}$$

In formal notation, this translates to

$$S_{t_{i+1}} - S_{t_i} = \int_{t_i}^{t_{i+1}} \mu S_s ds + \int_{t_i}^{t_{i+1}} \sigma S_t dW_t,$$

from which we obtain

$$\begin{aligned}
X_{t_{n+1}} &= X_0 + \sum_{i=0}^{n} \alpha_{t_i}(S_{t_{i+1}} - S_{t_i}) + \sum_{i=0} \beta_{t_i}(R_{t_{i+1}} - R_{t_i}) \\
&= X_0 + \sum_{i=0}^{n} \alpha_{t_i} \left(\int_{t_i}^{t_{i+1}} \mu S_s ds + \int_{t_i}^{t_{i+1}} \sigma S_t dW_s \right) + \sum_{i=0}^{n} \beta_{t_i} \int_{t_i}^{t_{i+1}} r R_s ds \\
&= X_0 + \int_0^{t_{n+1}} \alpha_s \mu S_s ds + \int_0^{t_{n+1}} \alpha_s \sigma S_s dW_s + \int_0^{t_{n+1}} \beta_s r R_s ds,
\end{aligned}$$

for simple, $\{\mathcal{F}_{t_i}\}_{i=1}^{n+1}$-adapted processes α_t and β_t. We can write, for more general $\mathcal{F}_t$-adapted processes α_t, β_t,

$$\begin{aligned}
X_t &= X_0 + \int_0^t \alpha_s dS_s + \int_0^t \beta_s dR_s && (2) \\
&= X_0 + \int_0^t \alpha_s \mu S_s ds + \int_0^t \alpha_s \sigma S_s dW_s + \int_0^t \beta_s r R_s ds. && (3) \\
&= X_0 + \int_0^t (\alpha_s \mu S_s + \beta_s r R_s) ds + \int_0^t \alpha_s \sigma S_s dW_s && (4)
\end{aligned}$$

(The α_t, β_t need to satisfy $\alpha_t S_t, \beta_t R_t \in L^2(\mu_T \times \mathbb{P})$ so that the above Lebesgue and Itô integrals are well defined.)

The previous equality shows that this generalization at least correctly reduces to the discrete case.

1.2 Discounting

To obtain the integral form of the discounted wealth, observe that

$$\bar{S}_t = e^{\bar{Z}_t} = e^{[(\mu - r) - \sigma^2/2]t + \sigma W_t} \tag{5}$$

Once more, we apply the Itô rule in a very similar manner to the above, which gives

$$\begin{aligned}
d\bar{S}_t &= \bar{S}_t \left((\mu - r)dt - \frac{\sigma^2}{2} dt + \sigma dW_t \right) + \frac{1}{2} \bar{S}_t \sigma^2 dt \\
&= (\mu - r)\bar{S}_t dt + \sigma \bar{S}_t dW_t,
\end{aligned}$$

and consequently

$$\bar{X}_t = X_0 + \int_0^t \alpha_s (\mu - r)\bar{S}_s ds + \int_0^t \alpha_s \bar{S}_s \sigma dW_s. \tag{6}$$

1.3 The E.M.M.

The advantage of discounting the wealth by R_t (apparently called a "numéraire") becomes apparent when we rewrite $\bar{X}_t$ as

$$\bar{X}_t = X_0 + \int_0^t \alpha_s d\bar{S}_s.$$

Indeed, the term $\int_0^t \beta_s dR_s$ (apparently, the "bond wealth") is gone, effectively making $\bar{X}_t$ an integral transform of $\bar{S}_t$. Thus, it suffices to show that $\bar{S}_t$ is a martingale under $\mathbb{Q}$. Then it will follow that (6) is a (continuous) martingale transform, which is itself a martingale. [2] Consider now the following form of $\bar{S}_t$:

$$\bar{S}_t = \int_0^t (\mu - r)\bar{S}_s ds + \int_0^t \sigma \bar{S}_s dW_s. \tag{7}$$

When taking a conditional expectation $\mathbb{E}_{\mathbb{P}}[\cdot|\mathcal{F}_u]$ on (7), the only obstacle is the first term
$\int_0^t (\mu - r)\bar{S}_s ds$, since the second term is already an Itô integral which is always a local martingale (for appropriate α_t, it is a true martingale). A little detective work shows that this first term has its roots from the drift of the dynamics of the underlying process

$$\bar{Z}_t = \left[(\mu - r) - \frac{\sigma^2}{2}\right] t + \sigma W_t.$$

Add $\frac{\sigma^2}{2}t$ to both sides to account for the Itô correction term.[3] After normalization, we have

$$\frac{\bar{Z}_t}{\sigma} + \frac{\sigma}{2}t = \left[\frac{\mu - r}{\sigma}\right] t + W_t$$

However, by Girsanov's theorem, there exists a change of measure $\mathbb{P} \to \mathbb{Q}$ driven by the Radon-Nikodym derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left\{-\int_0^T \frac{\mu - r}{\sigma} dW_s - \frac{1}{2}\int_0^T \left(\frac{\mu - r}{\sigma}\right)^2 ds\right\} = e^{-(\frac{\mu - r}{\sigma})W_T - \frac{1}{2}(\frac{\mu - r}{\sigma})^2 T}$$

such that, under $\mathbb{Q}$, $V_t := \bar{Z}_t/\sigma + \frac{\sigma}{2}t$ is now the Wiener process. Applying the Itô rule to $\bar{S}_t = e^{\bar{Z}_t} = e^{\sigma V_t - \frac{\sigma^2}{2}t}$ (adapted to V_t) now yields

$$\begin{aligned}
d\bar{S}_t &= \bar{S}_t d\left(\sigma V_t - \frac{\sigma^2}{2}t\right) + \frac{1}{2}\bar{S}_t\left(d\left(\sigma V_t - \frac{\sigma^2}{2}t\right)\right)^2 \\
d\bar{S}_t &= \bar{S}_t \sigma dV_t - \frac{\sigma^2}{2}\bar{S}_t dt + \frac{1}{2}\bar{S}_t \sigma^2 dt \\
d\bar{S}_t &= \bar{S}_t \sigma dV_t
\end{aligned}$$

Finally [4], we have

$$\bar{S}_t = \int_0^t \sigma \bar{S}_t dV_t,$$

which shows that $\bar{S}_t$ is an Itô integral w.r.t. under $\mathbb{Q}$. As such, we have shown that $\bar{S}_t$ is a true martingale, assuming that α_t is such that $\alpha_t \bar{S}_t \in L^2(\mu_T \times \mathbb{P}) \implies \bar{S}_t \in L^2(\mu_T \times \mathbb{P})$.

1.4 No arbitrage

Suppose there is arbitrage in the market. Then, there exists $\bar{X}_t$ s.t. $X_0 = a, \bar{X}_t \geq a\mathbb{P}$ almost surely and $\mathbb{P}(\bar{X}_t > a) > 0$. Notice that, by absolute contiuity, we have $\mathbb{Q}(\bar{X}_t > a) > 0$ and $\bar{X}_t \geq a$ $\mathbb{Q}-$almost surely. Now take $\epsilon > 0$ small enough s.t. $\mathbb{P}(\bar{X}_t \geq a + \epsilon) > 0$. Then

$$\mathbb{E}_{\mathbb{Q}}[\bar{X}_t] \geq (a+\epsilon)\mathbb{Q}(\bar{X}_t \geq a+\epsilon) + a\mathbb{Q}(a+\epsilon > \bar{X}_t \geq a) > a = X_0.$$

This is a contradiction (under $\mathbb{Q}$, we must have $\mathbb{E}_{\mathbb{Q}}[\bar{X}_t] = X_0$).

1.5 Existence of hedging strategy

The discounted call option payoff $(\bar{S}_T - Ke^{-rT})^+$ is $\mathcal{F}_T^W-$measurable and $\mathbb{P}-$square integrable because composition with $(\cdot - K)^+$ preserves exactly those properties of $\bar{S}_T = \exp((\mu - r - \sigma^2/2)T + \sigma W_T)$. Hence, apply the Itô representation theorem to $(S_T - K)^+$ to obtain an $\mathcal{F}_t^W-$adapted *unique* process H_t which satisfies[1]

$$\bar{P}_T \coloneqq (\bar{S}_T - Ke^{-rT})^+ = \mathbb{E}[(\bar{S}_T - Ke^{-rT})^+] + \int_0^T H_s dW_s$$

Indeed, we can now define and examine the process $\bar{P}_t = \mathbb{E}[(\bar{S}_T - Ke^{-rT})^+] + \int_0^t H_s dW_s$; upon closer look, it is a martingale with mean $\mathbb{E}[(\bar{S}_T - Ke^{-rT})^+]$. We would like to replicate its dynamics with a discounted strategy $\bar{X}_t$.

However, this is not evident in the $\mathbb{P}-$world. Following our noses, we can smell an opportunity to exploit the fact that all strategies in the $\mathbb{Q}-$world are automatically martingales of the form $X_0 + \int_0^t \sigma\alpha_s S_s dV_s$. Thus, we redo all of the above work, this time under $\mathbb{Q}$:

$(\bar{S}_T - Ke^{-rT})^+$ is $\mathcal{F}_T^V$-measurable and $\mathbb{Q}$-square integrable ($\bar{S}_t = \exp(\sigma V_t - \frac{\sigma^2}{2}t)$). Recycling the above notation, we write

$$\bar{P}_T \coloneqq \mathbb{E}_{\mathbb{Q}}[(\bar{S}_T - Ke^{-rT}] + \int_0^T H_s dV_s$$

where H_s now symbolizes a different, $\mathcal{F}_t^V-$adapted *unique* process. We recover the strategy by setting $\alpha_s\sigma_s\bar{S}_s = H_s \implies \alpha_s = H_s/(\sigma_s\bar{S}_s)$, and then use the self-financing condition to get $\bar{P}_s = \alpha_s\bar{S}_s + \beta_s \implies \beta_s = \bar{P}_s - \alpha_s\bar{S}_s$. Finally, we obtain

$$\bar{X}_s = \alpha_s\bar{S}_s + \beta_s.$$

1.6 Argument for fair price

Let us return to the $\mathbb{P}-$world. In this setting, we claim that the fair price for the call option must be $\pi = \mathbb{E}[(\bar{S}_T - Ke^{-rT})^+]$. From the perspective of the seller, we need to appeal to the investors by ensuring that our call option can at least bring in as much money as the bank account's interest rate. Suppose it were priced differently at $P > \pi$, and suppose a buy-side agent had, at time $t = 0$, exactly $X_0 = P$\$. Instead of spending P to buy the option, the agent could

alternatively invest it in the bond (the account with risk-free interest rate), and at the maturity obtain $Pe^{rT} > \pi e^{rT} = \mathbb{E}[(S_T - K)^+]$, which beats the option payoff.
On the other hand, if the option is priced at $P < \pi$ and a buy-side agent starts with $X_0 = P$\$, then selling the option is not appealing anymore. After the transaction and come the time of maturity, it follows that $Pe^{rT} < \mathbb{E}[(S_T - K)^+]$, which shows that the buyer is expected to have made a net gain and beat the market yet again.

1.7 Price calculation

An expression for the price of the call option at time t under $\mathbb{Q}$, which corresponds to $X_t = e^{rt}\bar{X}_t = e^{rt}\mathbb{E}_{\mathbb{Q}}[(\bar{S}_T - Ke^{-rT})^+|\mathcal{F}_t]$. The conditional expectation naturally reduces to an integral expression of the form

$$\int_{-\infty}^{\infty} (y - Ke^{-rT})^+ \varphi_{\bar{S}_T|\bar{S}_t=x}(y)dy,$$

where $\varphi_{\bar{S}_T|\bar{S}_t=x}(y)dy$ denotes the distribution of $\bar{S}_T$ given that $\bar{S}_t = x$. Finding this conditioned distribution is not evident; however, we can use the clever trick of "embedding" the extra information into the expression of which we are taking the expectation. To wit,

$$\begin{aligned} \bar{S}_T &= \bar{S}_t \frac{\bar{S}_T}{\bar{S}_t} \\ &= \bar{S}_t e^{\sigma(V_T - V_t) - \frac{\sigma^2}{2}(T-t)} \end{aligned}$$

If $\bar{S}_t = x$, then we can write $\bar{S}_T = xe^{\sigma(V_T - V_t) - \frac{\sigma^2}{2}(T-t)}$, and as such we only need to worry about the distribution of the exponential on the RHS. Finally, instead of writing the log-normal distribution in the integrand, we set the normalized Wiener process difference $\frac{V_T - V_t}{\sqrt{T-t}} \approx \mathcal{N}(0,1)$ as our integration variable y. This gives

$$X_t = e^{rt} \int_{-\infty}^{\infty} (xe^{\sigma y\sqrt{T-t} - \frac{\sigma^2}{2}(T-t)} - Ke^{-rT})^+ \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$$

Therefore, the price at time $t = 0$ corresponds to the following integral:

$$X_0 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (e^{\sigma y\sqrt{T} - \frac{\sigma^2}{2}T} - Ke^{-rT})^+ e^{-\frac{y^2}{2}} dy$$

The condition for $\bar{S}_T - Ke^{-rT} > 0$ is equivalent to

$$\begin{aligned} &\sigma y\sqrt{T} - \frac{\sigma^2}{2}T > \log Ke^{-rT} = \log K - rT \\ \iff &y > \frac{\log K - (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} =: \gamma_0. \end{aligned}$$

Hence, we have

$$
\begin{aligned}
X_0 &= \frac{1}{\sqrt{2\pi}} \int_{\gamma_0}^{\infty} (e^{\sigma y\sqrt{T}-\frac{\sigma^2}{2}T} - Ke^{-rT})e^{-\frac{y^2}{2}}\, dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{\gamma_0}^{\infty} e^{\sigma y\sqrt{T}-\frac{\sigma^2}{2}T} e^{-\frac{y^2}{2}}\, dy - Ke^{-rT} \int_{\gamma_0}^{\infty} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy \\
&= \int_{Ke^{-rT}}^{\infty} z\varphi_{\bar{S}_T}(z)dz - Ke^{-rT}\left(1-\Phi(\gamma_0)\right),
\end{aligned}
$$

where Φ denotes the C.D.F. of the standard normal distribution and φ denotes the distribution of $\bar{S}_T$. Note the change in integration variables $y \mapsto z$, going from the normalized Wiener process difference to the discounted stock. We resume with

$$
X_0 = (1 - \text{C.D.F.}_{\bar{S}_T}(\gamma_0)) - Ke^{-rT}(1-\Phi(\gamma_0)).
$$

The C.D.F. can be made explicit. Indeed, $\bar{S}_T \approx \text{Lognormal}\left(-\frac{\sigma^2}{2}T, \sigma^2 T\right)$ so that we now have

$$
\begin{aligned}
X_0 &= \left(1 - \Phi\left(\frac{\log(Ke^{-rT}) + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right)\right) - Ke^{-rT}(1-\Phi(\gamma_0)). \\
X_0 &= (1-\Phi(\gamma_1)) - Ke^{-rT}(1-\Phi(\gamma_0)),
\end{aligned}
$$

where $\gamma_1 := \frac{\log(Ke^{-rT})+\frac{\sigma^2}{2}T}{\sigma\sqrt{T}}$.

1.8 Notes for P1

[1]. Refer to the Itô table
[2]. For the unconvinced reader, one could show that the martingale property holds for simple $\bar{X}_t = X_0 + \sum_{i=0}^{n} \alpha_{t_i}(\bar{S}_{t_{i+1}} - \bar{S}t_i)$. Then, approximate the conditional expectation of more general processes in $L^2(\mu_T \times \mathbb{Q})$.
[3] I hadn't caught this at first. Initially, I set the Wiener process as $V_t = \bar{Z}_t/\sigma$ and was confused as to why the drift hadn't completely disappeared. In this case, I had

$$
d\bar{S}_t = \sigma\bar{S}_t dV_t + \frac{1}{2}\sigma^2\bar{S}_t dt.
$$

[4]. I was stuck on this 3rd question for a whole day trying to figure out how to get the right Radon-Nikodym derivative. On my first attempt, I used the fact that a conditional expectation under $\mathbb{Q}$ could be expressed in terms of $\mathbb{P}$ as

$$
\mathbb{E}_{\mathbb{Q}}[\bar{S}_t|\mathcal{F}_u] = \frac{\mathbb{E}_{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\bar{S}_t|\mathcal{F}_u\right]}{\mathbb{E}_{\mathbb{P}}[\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_u]}.
$$

By equating this expression to $\bar{S}_u$, we obtain

$$
\frac{\mathbb{E}_{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}e^{[(\mu-r)-\sigma^2/2]t+\sigma W_t}|\mathcal{F}_u\right]}{\mathbb{E}_{\mathbb{P}}[\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_u]} = e^{[(\mu-r)-\sigma^2/2]u+\sigma W_u},
$$

which suggested that the derivative was of the form

$$\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_t = e^{-[(\mu-r)t-\sigma^2/2]t},$$

with, ideally,

$$\mathbb{E}_{\mathbb{P}}\left[\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_t |\mathcal{F}_u\right] = \left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_u$$

for all $u \leq t$. However, this is not possible because as it stands, $(\frac{d\mathbb{Q}}{d\mathbb{P}})_t$ is a deterministic function of t. That's when I tried to introduce some "appropriate randomness" by introducing a stopping time. The idea was to guarantee that, knowledge of upto $\mathcal{F}_u$ would trigger the stopping time to freeze at u, i.e. something like

$$\mathbb{E}_{\mathbb{P}}\left[\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_{t\wedge\tau} |\mathcal{F}_u\right] = \left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_u .$$

To my knowledge, such a mechanism couldn't exist.