Sample problems (Rudin, Real and Complex Analysis)

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October 2023

1 Abstract integration

Exercise 1.9: Limits of log integrands

Suppose μ is a positive measure on X, $f\colon X \to [0,\infty]$ is measurable, $\int_X f d\mu = c$, where $0 < c < \infty$, and α is a constant. Prove that

$$\lim_{n\to\infty} \int_X n \log\left[1 + \left(\frac{f}{n}\right)^{\alpha}\right] d\mu = \begin{cases} \infty & \text{if } 0 < \alpha < 1 \\ c & \text{if } \alpha = 1 \\ 0 & \text{if } 1 < \alpha < \infty \end{cases}$$

Hint: If $\alpha \geq 1$, the integrands are dominated by αf. If $\alpha < 1$, Fatou's lemma can be applied.

Proof:
First, let $\alpha = 1$. Then for arbitrary n and $x \in X$,

$$n \log\left[1 + \left(\frac{f(x)}{n}\right)\right] = \log\left[\left(1 + \frac{f(x)}{n}\right)^n\right], \tag{1}$$

We know that the argument inside the log tends to $e^{f(x)}$ as $n \to \infty$. Moreover, writing out the binomial expansion and comparing it to the series expansion of $e^{f(x)}$ yields

$$\begin{aligned} \left(1 + \frac{f(x)}{n}\right)^n &= \sum_{k=0}^{n} \binom{n}{k} \left(\frac{f(x)}{n}\right)^k \\ &= \sum_{k=0}^{n} \frac{n!}{n^k (n-k)!} \left(\frac{f(x)^k}{k!}\right) \\ &< \sum_{k=0}^{\infty} \frac{f(x)^k}{k!} \\ &= e^{f(x)}. \end{aligned}$$

Hence, since log is strictly increasing and continuous everywhere, we have that,

(i) $\log\left[\left(1 + \frac{f(x)}{n}\right)^n\right] < \log(e^{f(x)}) = f(x)$
(ii) $\lim_{n\to\infty} \log\left[\left(1 + \frac{f(x)}{n}\right)^n\right] = \log\left[\lim_{n\to\infty} \left(1 + \frac{f(x)}{n}\right)^n\right] = \log(e^{f(x)}) = f(x)$.

In other words, the integrands in (1) are dominated by f and converge (at least pointwise) to f. By Lebesgue's dominated convergence theorem, the limit integral in this case is equal to $\int_X f d\mu = c$.

If $\alpha > 1$, a quick use of the generalized binomial theorem reveals that $1 + z^\alpha \leq (1+z)^\alpha$ for $z \geq 0$, so that

$$\begin{aligned} n \log\left[1 + \left(\frac{f(x)}{n}\right)^\alpha\right] &< n \log\left[\left(1 + \frac{f(x)}{n}\right)^\alpha\right] \\ &= \alpha \log\left[\left(1 + \frac{f(x)}{n}\right)^n\right] \\ &< \alpha f(x). \end{aligned}$$

Thus, αf dominates the integrands. We claim further that they converge pointwise to 0 at least a.e (we shall assume here that $f(x) < \infty$). To see this, first note that there exists N sufficiently large so that $n \geq N$ implies $(f(x)/n)^\alpha < 1$. As such, it is appropriate to use the Taylor expansion of $\log(1+z)$ for $|z| < 1$ to our advantage to obtain

$$\begin{aligned} \lim_{n\to\infty} n \log\left[1 + \left(\frac{f(x)}{n}\right)^\alpha\right] &= \lim_{n\to\infty} n \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left(\frac{f(x)^\alpha}{n^\alpha}\right)^k \\ &= \lim_{n\to\infty} \left(\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} f(x)^{k\alpha} n^{1-k\alpha}\right) \\ &= 0, \end{aligned}$$

given that $1 - k\alpha < 0$ for $k \geq 1$ (if instead $f(x) = \infty$ then the integrand converges to ∞ at x, but this is a non-issue since the set of all such x is negligible). With this, another application of the dominated convergence theorem yields the third case.

Finally, let $\alpha < 1$. We can use Fatou's lemma to show that

$$\infty = \int_X \liminf_{n\to\infty} n \log\left[1 + \left(\frac{f}{n}\right)^\alpha\right] d\mu \leq \liminf_{n\to\infty} \int_X n \log\left[1 + \left(\frac{f}{n}\right)^\alpha\right] d\mu.$$

To this end, we need only demonstrate that

$$\liminf_{n\to\infty} n \log\left[1 + \left(\frac{f}{n}\right)^\alpha\right] = \infty.$$

Similarly to the previous case, we can use the Taylor expansion of $\log(1+z)$. However, we also use the fact that there is an even M sufficiently large s.t. $n \geq M$ implies $1 - n\alpha < 0$. Then

$$\begin{aligned} \liminf_{n\to\infty} n \log\left[1 + \left(\frac{f}{n}\right)^\alpha\right] &= \liminf_{n\to\infty} \left(\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} f(x)^{k\alpha} n^{1-k\alpha}\right) \\ &\geq \liminf_{n\to\infty} \left(\sum_{k=1}^{M} \frac{(-1)^{k+1}}{k} f(x)^{k\alpha} n^{1-k\alpha}\right) + \liminf_{n\to\infty} \cancelto{0}{\left(\sum_{k=M+1}^{\infty} \frac{(-1)^{k+1}}{k} f(x)^{k\alpha} n^{1-k\alpha}\right)} \end{aligned}$$

$$\geq \liminf_{n\to\infty}\left(f(x)^{\alpha}n^{1-\alpha}-\frac{1}{2}f(x)^{2\alpha}n^{1-2\alpha}\right)+\cdots+\liminf_{n\to\infty}\left(\frac{1}{M-1}f(x)^{\alpha}n^{1-(M-1)\alpha}-\frac{1}{M}f(x)^{M\alpha}n^{1-M\alpha}\right).$$

It is easy to see that the first term in this last expression is unbounded, while the rest are either unbounded as well or converging to 0. ■

Exercise 1.10: Uniform convergence and integral convergence

Suppose $\mu(X)<\infty$, $\{f_n\}$ is a sequence of bounded complex measurable functions on X, and $f_n \to f$ uniformly on X. Prove that

$$\lim_{n\to\infty}\int_X f_n d\mu = \int_X f d\mu.$$

Proof:
Let $\epsilon>0$. Since $\{f_n\}\to f$ uniformly, there exists some $N\in\mathbb{N}$ s.t. $\sup_{x\in X}|f_n(x)-f|<\frac{\epsilon}{\mu(X)}$. for any $n\geq N$. This implies that

$$\int_X |f_n-f|d\mu \leq \sup_{x\in X}|f_n(x)-f(x)|\cdot\mu(X) < \frac{\epsilon}{\mu(X)}\mu(X)=\epsilon.$$

We conclude the proof, for we have shown that

$$\lim_{n\to\infty}\int_X (f_n-f)\,d\mu = 0.$$

■

Exercise 1.11: Borel-Cantelli lemma alternate proof

For a sequence of measurable sets $\{E_k\}$ in X satisfying

$$\sum_{n=1}^{\infty}\mu(E_k)<\infty,$$

let $A=\{x\in X : x \text{ is in infinitely many sets } E_k\}$. Prove that

$$A=\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}E_k,$$

and with this, show that $\mu(A)=0$.

Proof:
First, we verify the set equality. Let $x\in A$, and let $n\geq 1$ be arbitrary. it is guaranteed that $x\in E_j$ for some $j\geq n$, hence $x\in\bigcup_{k=n}^{\infty}E_k$, so $A\subset\bigcap\bigcup E_k$ follows. If instead $x\in\bigcap\bigcup E_k$, then similarly, by definition, for any $n\geq 1$ we can find a set $E_j, j\geq n$ with $x\in E_j$. Hence, of all the sets containing x, there is none of greatest index. The equivalence follows.

We now prove that $\mu(A)=0$. First, notice that the sequence of partial sums $\{\sum_{k=1}^{n}\mu(E_k)\}$ converges to $\sum_{n=1}^{\infty}\mu(E_k)$, that is,

$$\lim_{n\to\infty}\sum_{k=n}^{\infty}\mu(E_k)=0.$$

Now, observe that

$$A = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k \subset \bigcup_{k=N}^{\infty} E_k$$

for any $N \geq 1$, thus

$$\mu(A) \leq \lim_{N\to\infty} \mu\left(\bigcup_{k=N}^{\infty} E_k\right) \leq \lim_{N\to\infty} \sum_{k=N}^{\infty} \mu(E_k) = 0.$$

■

Exercise 1.12: Integral can be made arbitrarily small

Suppose $f \in L^1(\mu)$. Prove that to each $\epsilon > 0$ there exists a $\delta > 0$ such that $\int_E |f| d\mu < \epsilon$ whenever $\mu(E) < \delta$.

Proof:

If there are some $x \in X$ s.t. $|f(x)| = \infty$, then we must have at least $\mu(\{x \in X : |f(x)| = \infty\} = 0$ for $f \in L^1(\mu)$ – label this set A.

Observe that for $\beta > 0$ we have that $\{x \in X : |f(x)| > \beta\}$ is measurable, and so consider the sequence of sets $A_n = \{x \in X : |f(x)| > n\}$. Notice that $A_1 \supset A_2 \supset \cdots$ and that $\bigcap_{n=1}^{\infty} A_n = A$. Hence for any positive measure ϕ defined on the sigma algebra of X, we have that $\phi(A_n) \to \phi(A)$ as $n \to \infty$.

By an earlier result, defining $\phi(E) = \int_E |f| d\mu$ for any E in the sigma algebra does yield a positive measure. Hence

$$\lim_{n\to\infty} \int_{A_n} |f| d\mu = \int_A |f| d\mu = 0 \cdot \infty = 0.$$

We can then choose N large enough so that $\int_{A_N} |f| d\mu < \epsilon/2$. Set $\delta = \epsilon/2N$ and choose a measurable S s.t. $\mu(S) < \delta$. Then

$$\begin{aligned}
\int_S |f| d\mu &= \int_{S\cap A_N^c} |f| d\mu + \int_{S\cap A_N} |f| d\mu \\
&< \frac{\epsilon}{2} + N \cdot \frac{\epsilon}{2N} \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

■

2 Positive Borel Measures

Exercise 2.9: Constructing an exotic sequence f_n

Construct a sequence of continuous functions f_n on $[0, 1]$ s.t. $0 \le f_n \le 1$ s.t.

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0,$$

but such that the sequence $\{f_n(x)\}$ converges for no $x \in [0, 1]$.

Proof:
Let $n = 1$. Then, there exists balls $B_1, B_2, ..., B_{k(n)}$ all of radius $1/n$ with centers $\{x_1, x_2, ..., x_{k(n)}\}$ covering $[0, 1]$. Reiterate for $n = 2, 3, ...$ and concatenate the corresponding sets of centers to get a sequence $\{x_1, x_2, ...\}$ with the particular property that

$$\{B_1(x_1), ..., B_1(x_{k(1)})\},\ \{B_{1/2}(x_{k(1)+1}), ..., B_{1/2}(x_{k(2)})\},$$

and so on all cover $[0, 1]$.

Now consider an arbitrary x_n with its corresponding ball $B(x_n)$: let $B'(x_n)$ be the ball about x_n with 2 times the radius of $B(x_n)$. By Urysohn's lemma, $\exists f_n \in C_c(\mathbb{R})$ which is identically 1 on $B(x_n)$, whose compact support lies in $B'(x_n)$, and whose image lies in $[0, 1]$ (more commonly abbreviated $B'(x_n) \prec f_n \prec B(x_n)$). Now restrict f_n to $[0, 1]$.

We let m denote the Lebesgue measure on $\mathbb{R}$. If $\epsilon > 0$, then $\exists N \in \mathbb{N}$ large enough so that $n \ge N$ implies

$$m\left(B'(x_n)\right) = \frac{2}{n} < \epsilon.$$

Since $m\left(B'(x_n)\right) \ge \int_0^1 f_n(x)dx$, we have proven that $\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0$. Moreover, if $x \in [0, 1]$ then there are infinitely many balls $B(x_n)$ which contain x, and infinitely many $B'(x_n)$ which do not contain x, hence $\{f_n(x)\}$, while bounded by $[0, 1]$, never converges.

Exercise 2.11: Carrier of compact Hausdorff X ***with measure 1***

Let μ be a regular Borel measure on a compact Hausdorff space X; assume $\mu(X) = 1$. Prove that there is a compact set $K \subset X$ such that $\mu(K) = 1$ but $\mu(H) < 1$ for every proper compact subset H of K. *Hint*: Regularity of μ is needed.

Proof:
Let $\{K_\alpha\}$ denote the set of all compact subsets of X with measure 1, and take $K = \bigcap_\alpha K_\alpha$. If $H \subsetneq K$ and H is compact, then necessarily $\mu(H) < 1$; if not, then $\mu(H) = 1$ so that $K \subset H$, a contradiction. We now show that $\mu(K) = 1$. (this is where the hint is used)

It would suffice, by the regularity of μ, to prove that for an arbitrary open set $V \supset K$, there is $K_\alpha \subset V$; for then $\mu(K) = \inf\{\mu(V) : V \supset K, V \text{ open }\} \ge 1$. To this end, let $F_\alpha = (K_\alpha - V)$. Every F_α is compact (they are closed and a subset of the compact X), and that

$$\bigcap_\alpha F_\alpha = \bigcap_\alpha (K_\alpha - V) = \bigcap_\alpha K_\alpha - V = \emptyset.$$

By an earlier lemma, some finite subcollection of $\{F_\alpha\}$ also has empty intersection: let $\{F_{\alpha_1}, ..., F_{\alpha_n}\}$ denote the finite set. We get

$$\bigcap_{i=1}^{n} F_{\alpha_i} = \emptyset \implies \bigcap_{i=1}^{n} K_{\alpha_i} - V = \emptyset \implies \bigcap_{i=1}^{n} K_{\alpha_i} \subset V.$$

This finite intersection is the compact subset of V which we seek: we need only prove that $\mu\left(\bigcap_{i=1}^{n} K_{\alpha_i}\right) = 1$.

First, let $K_1, K_2 \in \{K_\alpha\}$. Note $K_1 \subset (K_1 - K_2) \cup K_2$, so that

$$1 = \mu(K_1) \leq \mu((K_1 - K_2) \cup K_2) = \mu(K_1 - K_2) + \mu(K_2) \implies \mu(K_1 - K_2) + \mu(K_2) = 1.$$

Since $\mu(K_2) = 1$, we now see that $\mu(K_1 - K_2) = 0$. In other words, the difference between any two sets in $\{K_\alpha\}$ is negligible. Now, consider $\mu(K_1 \cap K_2)$. We have

$$\begin{aligned} &K_1 \cap K_2 = K_1 \cup K_2 - (K_1 - K_2) \cup (K_2 - K_1) \\ &\implies \mu(K_1 \cap K_2) = \mu(K_1 \cup K_2) - \mu(K_1 - K_2) - \mu(K_2 - K_1) \\ &\implies \mu(K_1 \cap K_2) = 1. \end{aligned}$$

Applying induction, we have that the measure of any finite intersection of sets from $\{K_\alpha\}$ is 1, whence $\mu(\cap_{i=1}^{n} K_{\alpha_i}) = 1$. ■

Exercise 2.12: Carriers of a Borel measure on $\mathbb{R}$

Prove that every compact $K \subset \mathbb{R}$ is the support of a Borel measure.
(Other literature suggests that the author wants us to construct a measure μ which is nonzero only on K, i.e. that $\mu(E) = 0$ for any $E \subset K^c$, but $\mu(N(x)) > 0$ if $N(x)$ is the neighborhood of some $x \in K$.)

Proof:
Consider the Lebesgue measure m on $\mathbb{R}$, defined on the usually accompanying σ-algebra $\mathfrak{M} = \{E \subset \mathbb{R} : F_\sigma \subset E \subset G_\delta, m(G_\delta - F_\delta) = 0\}$ which contains all Borel sets. Define a new function μ on $\mathfrak{M}$ by $\mu(E) = \mu(K \cap E)$. A routine verification shows that μ is not identically zero and satisfies countable additivity on $\mathfrak{M}$, and is hence a positive Borel measure on $\mathbb{R}$. If $x \in X$, then any neighborhood $N(x)$ forms an intersection $K \cap N(x)$ containing a closed, connected segment I in K for which $\mu(I) = m(I) > 0$. On the other hand, if $E \subset K^c$ then $\mu(K \cap E) = \mu(\emptyset) = 0$. ■

Exercise 2.17: An exotic metric space

Define the distance between two points (x_1, y_1) and (x_2, y_2) in the plane to be

$$|y_1 - y_2| \text{ if } x_1 = x_2, \quad 1 + |y_1 - y_2| \text{ if } x_1 \neq x_2.$$

Show that this is indeed a metric, and that the resulting metric space X is locally compact. If $f \in C_c(X)$, let $x_1, ..., x_n$ be those values of x for which $f(x, y) \neq 0$ for at least one y (there are only finitely many such x, and define

$$\Lambda f = \sum_{j=1}^{n} \int_{-\infty}^{\infty} f(x_j, y)dy.$$

Let μ be the measure associated with Λ in the sense of the Riesz representation theorem (Theorem 2.14). If E is the $x-$axis, show that $\mu(E) = \infty$, although $\mu(K) = 0$ for every compact $K \subset E$.

Proof:
Here, let $\hat{z}$ denote an element of $\mathbb{R}^2$, and let $\hat{z}_x$ and $\hat{z}_y$ denote its x,y-coordinates (as in the sense of the cartesian plane) respectively. For brevity, we don't include the demonstration that d is a metric. We do show that $(\mathbb{R}^2, d)$ is locally compact: first, consider an arbitrary point $\hat{x} \in (\mathbb{R}^2, d)$. We will provide a neighborhood around it whose closure is compact. To this end, take the ball

$$B_{1/2}^{(\mathbb{R}^2,d)}(\hat{x}) = \{\hat{z} \in \mathbb{R}^2, d) : d(\hat{x}, \hat{z}) < \frac{1}{2}\} = \{\hat{z} \in \mathbb{R}^2 : \hat{x}_x = \hat{z}_x, |\hat{x}_y - \hat{z}_y| < \frac{1}{2}\}, \tag{2}$$

and observe that its closure w.r.t. to the metric d is the set $\{\hat{z} \in \mathbb{R}^2 : \hat{x}_x = \hat{z}_x, |\hat{x}_y - \hat{z}_y| \leq \frac{1}{2}\}$. This closure is closed and bounded in $\mathbb{R}^2$, and is hence compact, given that the regular Euclidean distance $||\cdot||_2$ and metric d induce the same topology on $\mathbb{R}^2$ (using arbitrary unions, the set of every open ball w.r.t. d can generate the set of every open ball w.r.t. $||\cdot||_2$, and vice versa.)

We now verify the assertion that the compact support of a function lies on a collection of finitely many vertical lines. We tackle the general case: let $K \subset (\mathbb{R}^2, d)$ be a compact set. Then, the collection of 1/2-balls with centers ranging over the elements of K (denoted $\{B_{1/2}^{(\mathbb{R}^2,d)}(\hat{x})\}_{\hat{x}\in K}$) has a finite subset $\{B_{1/2}^{(\mathbb{R}^2,d)}(\hat{x}_i)\}_{i=1}^n$ which covers K. That is,

$$K \subset \bigcup_{i=1}^n B_{1/2}^{(\mathbb{R}^2,d)}(\hat{x}_i)$$

which, by (2), shows that the set of points in K is indeed determined using finitely many distinct x-coordinates. Now let μ be the Radon measure representing Λ as in the hypothesis – any compact set K is automatically measurable. Let $E = \{\hat{x} \in \mathbb{R}^2 : \hat{x}_y = 0\}$ be the real line: it is also measurable, for if

$$U_n := \bigcup_{\hat{x}\in E} B_{1/n}^{(\mathbb{R}^2,d)}(\hat{x}),$$

then notice that $E = \bigcap_{n=1}^\infty U_n$, which lies in the Borel algebra of $\mathbb{R}^2$.

Moving on, let $K \subset E$: we claim that $\mu(K) = 0$. By the above, K consists of finitely many points $\{\hat{z}_1, ..., \hat{z}_k\}$ on the real line, also $\mu(K) = \inf\{\Lambda f | K \prec f\}$. Choose $\epsilon > 0$: let $E_i = \{\hat{x}_i \in \mathbb{R}^2 : \hat{z}_{i_x} = \hat{x}_x, |\hat{z}_{i_y} - \hat{x}_y| < \frac{\epsilon}{k}\}$. We can choose a function f satisfying $K \prec f$ as well as $\text{supp}(f) \subset \bigcup_{i=1}^m E_i$ with $0 \leq f \leq 1$. Then, if m denotes the Lebesgue measure on $\mathbb{R}^2$, we have

$$\begin{aligned}\Lambda f &= \sum_{j=1}^m \int_{-\infty}^\infty f(\hat{z}_{j_x}, y)dy \\ &< \sum_{j=1}^k m(E_j)\end{aligned}$$

$$= \epsilon.$$

We conclude that $\mu(K) = 0$.

We now show that $\mu(E) = \infty$. Note that $\mu(E) = \inf\{\mu(V) | V \supset E, V \text{ open}\}$ by the (pre-defined) outer regularity of μ. We need to prove, for an arbitrary $V \supset E$, that $\mu(V)$ is unbounded. First, V contains an arbitrary union of open balls w.r.t. d centered on the real line: more precisely, we can write

$$V \supset \bigcup_{\hat{x} \in E} B_{r_{\hat{x}}}^{(\mathbb{R}^2, d)}(\hat{x})$$

Take $N \in \mathbb{N}$ large enough such that $2 \cdot r_{\hat{x}} \geq 1/N$ for uncountably many $\hat{x} \in E$, and then consider an infinite sequence $\{\hat{x}_1, \hat{x}_2, ...\}$ of such points. Now, if $M > 0$ is an arbitrarily large integer, then we can choose some function $g \in C_c((\mathbb{R}^2, d))$ s.t.

$$\bigcup_{i=1}^{2MN} B_{r_{\hat{x}_i}/2}^{(\mathbb{R}^2, d)}(\hat{x}_i) \prec g \prec V.$$

Then

$$\begin{aligned}
\Lambda g &= \sum_{i=1}^{2MN} \int_{-\infty}^{\infty} f(\hat{x}_{i_x}, y) dy \\
&\geq \sum_{i=1}^{2MN} m\left(B_{r_{\hat{x}_i}/2}^{(\mathbb{R}^2, d)}(\hat{x}_i)\right) \\
&= \sum_{i=1}^{2MN} 2 \cdot (r_{\hat{x}_i}/2) \\
&= \sum_{i=1}^{2MN} \frac{1}{2N} \\
&= M.
\end{aligned}$$

Hence, we have that $\mu(V) = \infty$, which forces $\mu(E) = \infty$. ■
(This exercise gives us an example of a locally compact metric (hence, Hausdorff) space equipped with a Radon measure μ containing a Borel set E that is not inner regular w.r.t. μ.)

3 L^p spaces and convexity

Exercise 3.3: Weaker convexity condition for continuous φ

Assume that φ is a continuous real function on (a, b) s.t.

$$\varphi\left(\frac{x+y}{2}\right) \leq \frac{1}{2}\varphi(x) + \frac{1}{2}\varphi(y) \tag{3}$$

for all $x, y \in (a, b)$. Prove that φ is convex.

Proof:
Let $u, v \in \mathbb{R}$, $a < u < v < b$. Denote the line which intersects $(u, \varphi(u))$ and $(v, \varphi(v))$ on the graph by the function s. We wish to prove that

$$\varphi(x) \leq s(x) \tag{4}$$

for any $u \leq x \leq v$. This is true for the point $x = (u+v)/2$, and if we reapply (3) to the two endpoints u and $(u+v)/2$, we obtain (4) for $x = (u+v)/4$, and with the same approach we can get (4) for $x = 3(u+v)/4$, and so on. In essence, the inequality (4) holds for all x in a dense subset A of $[u, v]$: formally, we write

$$A = \bigcup_{n=1}^{\infty} A_n, \;\; A_n = \{u, u + 2^{-n}(v-u), ..., v\},$$

that is, A_n is a partition of $[u, v]$ into 2^n equal parts. Ultimately, if x is arbitrary in $[u, v]$, we can find a sequence $x_k \to x$ for which

$$\varphi(x_k) \leq s(x_k) \;\; \forall k = 1, 2, 3, ...$$

holds. Then, taking this inequality to the limit and using the continuity of both φ and s on (a, b), we have that

$$\lim_{k\to\infty} \varphi(x_k) \leq \lim_{k\to\infty} s(x_k) \implies \varphi(x) \leq s(x).$$

■

Exercise 3.4: log***-convexity in the*** p***-norm***

Let f be a complex measurable function on X. Define the function $\varphi : \mathbb{R} \to \mathbb{R}$ by

$$\varphi(p) = ||f||_p^p.$$

w.r.t. μ. Let $E = \{p \in \mathbb{R} : \varphi(p) < \infty\}$, and lastly, suppose that $||f||_\infty > 0$. Prove that:
(a) $r < p < s$ with $r, s \in E$ implies that $p \in E$.
(b) $\log \varphi$ is convex in the interior of E, and so φ is continuous on E.
(c) By (a), E is connected. E can be open, closed, or consist of a single point. Lastly, any connected interval in $\mathbb{R}^{>0}$ can be E.
(d) $r < p < s$ implies $||f||_p \leq \max(||f||_r, ||f||_s)$, and hence shows the inclusion $L^r(\mu) \cap L^s(\mu) \subset L^p(\mu)$.
(e) $||f||_r$ for some $r < \infty$ implies that

$$||f||_p \to ||f||_\infty$$

as $p \to \infty$.

Proof:

For (a) and (b), we assume w.l.o.g. that $|f|$ is not infinity anywhere (the fact that a power of $|f|$ is Lebesgue integrable implies that it evaluates to infinity only on a negligible set).

(a) Given $f \in L^r(\mu)$ and $L^s(\mu)$, we must prove that $f \in L^p(\mu)$. Let $A = \{x \in X : |f(x)| < 1\}$. Then $X = A \cup A^c$, $r < p \implies |f|^p < |f|^r$ on A, and $p < s \implies |f|^p \leq |f|^s$ on A^c. Hence

$$\begin{aligned}\int_X |f|^p d\mu &\leq \int_A |f|^p d\mu + \int_{A^c} |f|^p d\mu \\ &\leq \int_A |f|^r d\mu + \int_{A^c} |f|^s d\mu \\ &\leq \int_X |f|^r d\mu + \int_X |f|^s d\mu \\ &< \infty,\end{aligned}$$

so that $f \in L^p(\mu)$.

(b) Let m, n be in the interior of A and assume that $m < n$. We now prove convexity on $[m, n]$. If $\lambda \in (0,1)$, then $(1-\lambda) + \lambda = 1$. We also have that

$$\log \varphi\left((1-\lambda)m + \lambda n\right) = \log\left(\int_X |f|^{(1-\lambda)m+\lambda n} d\mu\right) = \log\left(\int_X |f|^{(1-\lambda)m} |f|^{\lambda n} d\mu\right).$$

An application of Hölder's inequality to the product of functions $|f|^{(1-\lambda)m}|f|^{\lambda n}$ with the conjugate exponents $(1-\lambda)^{-1}$ and λ^{-1} yields

$$\begin{aligned}\int_X |f|^{(1-\lambda)m}|f|^{\lambda n} d\mu &\leq \left[\int_X |f|^{(1-\lambda)m/(1-\lambda)} d\mu\right]^{1-\lambda} \left[\int_X |f|^{\lambda n/\lambda} d\mu\right]^{\lambda} \\ &= \left[\int_X |f|^m d\mu\right]^{1-\lambda} \left[\int_X |f|^n d\mu\right]^{\lambda}\end{aligned}$$

$$\implies \log\left(\int_X |f|^{(1-\lambda)m+\lambda n} d\mu\right) \leq \log\left(\left[\int_X |f|^m d\mu\right]^{1-\lambda} \left[\int_X |f|^n d\mu\right]^{\lambda}\right)$$

$$\implies \log \varphi\left((1-\lambda)m + \lambda n\right) \leq (1-\lambda)\log\varphi(m) + \lambda \log\varphi(n).$$

This last inequality is trivial if $\lambda = 0, 1$. We conclude that $\log\varphi$ is convex in the interior of E. Hence φ is also convex, and so continuous on the interior of E. Should p be a limit point of E which is also in E, let $p_n \to p$ be a sequence in E approaching p. Then

$$\begin{aligned}\lim_{n\to\infty} \varphi(p_n) = \lim_{n\to\infty} \int_X |f|^{p_n} d\mu &= \lim_{n\to\infty} \int_A |f|^{p_n} d\mu + \lim_{n\to\infty} \int_{A^c} |f|^{p_n} d\mu \\ &= \int_A \lim_{n\to\infty} |f|^{p_n} d\mu + \int_{A^c} \lim_{n\to\infty} |f|^{p_n} d\mu \\ &= \int_A |f|^p d\mu + \int_{A^c} |f|^p d\mu \\ &= \varphi(p).\end{aligned}$$

The passage of the limit under the integral sign on A and A^c uses the dominated convergence theorem with $|f|^{\inf_n\{p_n\}}$ and $|f|^{\sup_n\{p_n\}}$, respectively, as the Lebesgue integrable dominating functions. We conclude that φ is continuous on not only the interior of E, but also on all of E.

(c) **Attempt**
We first show that any open interval can be E. In general, if f is a function, we let E_f denote the set $\{p > 0 : ||f||_p < \infty\}$. Now, consider $X = \mathbb{R}$ equipped with m, the Lebesgue measure, and $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$. Define $g = f(x) \cdot \chi_{[1,\infty)}$ and $h = f(x) \cdot \chi_{(0,1]}$. First, applying the integral test to g^q for some $q > 0$, we have that

$$||g^q||_p^p = \int_{\mathbb{R}} (g^q)^p dm = \int_1^\infty f(x)^{qp} dx = \int_1^\infty \frac{1}{x^{qp}} < \infty \iff \sum_{n=1}^\infty \frac{1}{n^{qp}} \text{ converges } \iff qp > 1.$$

In other words, the p-norm of g^q is finite if and only if $p > \frac{1}{q}$. Hence $E_{g^q} = (1/q, \infty)$.
What is E_{h^q}? We can examine it in a similar fashion:

$$||h^q||_p^p = \int_{\mathbb{R}} (h^q)^p dm = \int_0^1 f(x)^{qp} dx = \int_0^1 \frac{1}{x^{qp}} dx = \frac{(1)^{1-qp}}{1-qp} - \lim_{x\to 0} \frac{x^{1-qp}}{1-qp},$$

which converges only when $qp < 1 \implies p < 1/q$, i.e. $E_{h^q} = (0, 1/q)$. Now take $m, n > 0$ with $m < n$. Define the function $j\colon \mathbb{R} \to \mathbb{R}$ by

$$j(x) = \begin{cases} 0 & \text{if } x \le 0 \\ \frac{1}{x^{1/m}} & \text{if } 0 < x < 1 \\ \frac{1}{x^{1/n}} & \text{if } 1 \le x < \infty. \end{cases}$$

It follows that

$$||j||_p^p = ||h^{\frac{1}{m}}||_p^p + ||g^{\frac{1}{n}}||_p^p,$$

whence $E_j = (m, n)$. We have just proven that any open interval in positive $\mathbb{R}$ can be E.

(d) Let $\lambda \in (0, 1)$ so that $p = (1-\lambda)r + \lambda s$. Applying the convexity of $\log\varphi$ yields

$$\begin{aligned} &\log\varphi(p) \le (1-\lambda)\log\varphi(r) + \lambda\log\varphi(s) \\ \implies &\varphi(p) \le \varphi(r)^{(1-\lambda)}\varphi(s)^\lambda \\ \implies &||f||_p \le ||f||_r^{(1-\lambda)r/p}||f||_s^{\lambda s/p} \end{aligned}$$

Either $||f||_r \le ||f||_s$ or $||f||_s > ||f||_s$. If we assume the former, we have that

$$||f||_p \le ||f||_r^{(1-\lambda)r/p}||f||_s^{\lambda s/p} \le ||f||_s^{(1-\lambda)s/p+\lambda s/p} = ||f||_s.$$

If we assume the latter, we get $||f||_p \le ||f||_r$. In other words, $||f||_p \le ||f||_s$ or $||f||_r$, hence $||f||_p \le \max(||f||_r, ||f||_s)$.

(e) Assume first that $||f||_\infty < \infty$. Let $p > r$; since $|f| \le ||f||_\infty$ $\mu-$a.e., we have that $|f|^p = |f|^r|f|^{p-r} \le |f|^r||f||_\infty^{p-r}$ $\mu-$a.e. Hence

$$\int_X |f|^p d\mu \le ||f||_\infty^{p-r} \int_X |f|^r d\mu < \infty, \tag{5}$$

so that $f \in L^p(\mu)$ for all $r \le p \le \infty$, or rather $E_f \supset [r, \infty)$. Moreover, let $\epsilon > 0$: taking the $p - th$ root on both sides of (5) yields

$$||f||_p \le ||f||_\infty^{1-r/p}||f||_r^{r/p}$$

$$\implies ||f||_p \leq ||f||_\infty \left(\frac{||f||_r^r}{||f||_\infty^r}\right)^{1/p}.$$

We can take Q large enough s.t. $p \geq Q$ implies $(||f||_r^r/||f||_\infty^r)^{1/p} \leq (1+\epsilon)$, so that

$$||f||_p \leq ||f||_\infty + \epsilon \ (p \geq Q)$$

We now digress and prove the result for the particular case of simple measurable functions. Let s be such a function which is Lebesgue integrable on X. If $\{\alpha_1, ..., \alpha_n\}$ is the image of s over X, let $A_i = \{x \in X | s(x) = \alpha_i\}$ (the A_i are measurable). Then

$$s = \sum_{i=1}^{n} \alpha_i \chi_{A_i} \text{ and } \int_X s d\mu = \sum_{i=1}^{n} \alpha_i \mu(A_i).$$

In general, if $p > 0$ then

$$s^p = \sum_{i=1}^{n} \alpha_i^p \chi_{A_i} \text{ and } \int_X s^p d\mu = \sum_{i=1}^{n} \alpha_i^p \mu(A_i)$$

given that the sets $A_1, A_2, ..., A_n$ are mutually disjoint. Now let $\alpha_j = \max\{\alpha_1, ..., \alpha_n\}$ for some $1 \leq j \leq n$. Should 0 be in the image of s, let α_k be the indexed element taking its value. Then

$$\begin{aligned}
||s||_p &= \left(\sum_{i=1}^{n} \alpha_i^p \mu(A_i)\right)^{\frac{1}{p}} \\
&= \alpha_j \left(\sum_{i=1}^{n} \left(\frac{\alpha_i}{\alpha_j}\right)^p \mu(A_i)\right)^{\frac{1}{p}} \\
\implies \lim_{p\to\infty} ||s||_p &= \alpha_j \lim_{p\to\infty} \left(\sum_{i=1}^{n} \left(\frac{\alpha_i}{\alpha_j}\right)^p \mu(A_i)\right)^{\frac{1}{p}}.
\end{aligned}$$

However, observe that $0 < (\alpha_i/\alpha_j)^p < 1$ for all $i = 1, ..., n,\ i \neq j, i \neq k$ – also $(\alpha_j/\alpha_j)^p = 1, (\alpha_k/\alpha_j)^p = 0$. Thus

$$\mu(A_j)^{1/p} \leq \left(\sum_{i=1}^{n} \left(\frac{\alpha_i}{\alpha_j}\right)^p \mu(A_i)\right)^{\frac{1}{p}} \leq \left(\sum_{i=1, i\neq k}^{n} \mu(A_i)\right)^{\frac{1}{p}}.$$

An application of the sandwich theorem yields

$$\lim_{p\to\infty} \left(\sum_{i=1}^{n} \left(\frac{\alpha_i}{\alpha_j}\right)^p \mu(A_i)\right)^{\frac{1}{p}} = 1,$$

so that

$$\lim_{p\to\infty} ||s||_p = \alpha_j = \operatorname*{ess\,sup}_{x\in X} s(x).$$

We resume with the proof for f. First, let $A = \{x \in X : |f(x)| \leq ||f||_\infty\}$, and let s_n be a sequence of simple measurable functions such that $s_1 \leq s_2 \leq ...$ and $s_n \to |f|$ on A as in Theorem 1.17 of Chapter 1. Define $s_n(x) = 0$ for all $x \in A^c$, $n \in \mathbb{N}$ (this removes the issue that some s_n must evaluate to a value greater than $||f||_\infty$ on a non-negligible set). Now take $x \in A$ s.t. $||f||_\infty - \epsilon/2 < |f(x)| \leq ||f||_\infty$, and N large enough such that $n \geq N$ implies $|f(x)| - \epsilon/2 < s_n(x) \leq f(x)$. Then

$$||f||_\infty - \epsilon < s_n(x) \leq ||f||_\infty \ (n \geq N).$$

In particular, it follows that

$$||f||_\infty - \epsilon < s_N(x) \leq ||s_N||_\infty$$

$$\implies \text{ for large enough } P > 0, \ ||f||_\infty - \epsilon \leq ||s_N||_p \text{ for all } p \geq P.$$

Knowing that $s_n \leq |f|$ on X, we have that $||s_n||_q \leq ||f||_q$ for all n and all $q > 0$. Hence

$$||f||_\infty - \epsilon \leq ||f||_p \ (p \geq P).$$

Taking $J = \max\{P, Q\}$, we have that

$$\text{abs}(||f||_\infty - ||f||_p) < \epsilon \ \text{ for all } p \geq J.$$

If $||f||_\infty = \infty$, then the result is immediate if E_f is bounded. Hence, suppose $E_f \supset [r, \infty)$. Let $M \in \mathbb{R}^{>0}$, and let $E_M = \{x \in X : |f(x)| \geq M\}$. By our assumptions, for no M is E_M of negligible measure. Hence, if $\{s_n\} \to |f|$ as in Th. 1.17. of Chapter 1, then $||s_n||_\infty = n$, so that

$$\lim_{n\to\infty} ||s_n||_\infty = \infty \implies \lim_{n\to\infty}\left(\lim_{p\to\infty} ||s_n||_p\right) = \infty.$$

Fixing M in positive $\mathbb{R}$, we realize that we can take P and N large enough so that $||f||_p \geq ||s_N||_p \geq M$ for all $p \geq P$, showing that the limit of the p-norm of f diverges to $+\infty$. ■

Exercise 3.5: log***-convexity in the*** p***-norm on a set of measure 1***

Assume, in addition to the hypotheses of Exercise 3.4., that

$$\mu(X) = 1.$$

(a) Prove that $||f||_r \leq ||f||_s$ if $0 < r < s \leq \infty$.
(b) Under what conditions does it happen that $0 < r < s \leq \infty$ and $||f||_r = ||f||_s < \infty$?
(c) Prove that $L^r(\mu) \supset L^s(\mu)$ if $0 < r < s$. Under what conditions do these two spaces contain the same functions?
(d) Assume that $||f||_r < \infty$ for some $r > 0$, and prove that

$$\lim_{p\to 0} ||f||_p = \exp\left[\int_X \log|f| d\mu\right]$$

if $\exp(-\infty)$ is defined to be 0.

(a) Sps. first that $s < \infty$. Then $(s-r)+r = s \implies (s-r)/s + r/s = 1$ which yields the conjugate exponent pair $s/(s-r)$ and s/r. Applying Hölder's inequality with this pair to the product $|f|^r = |f|^r \cdot 1$ yields

$$\int_X |f|^r d\mu \leq \left[\int_X (|f|^r)^{\frac{s}{r}} d\mu\right]^{r/s} \left[\int_X 1 d\mu\right]^{(s-r)/s}$$

$$\implies \left[\int_X |f|^r d\mu\right]^{\frac{1}{r}} \leq \left[\int_X (|f|^r)^{\frac{s}{r}}\, d\mu\right]^{\frac{1}{s}} (\mu(X))^{(s-r)/sr}$$
$$\implies ||f||_r \leq ||f||_s \mu(X)^{\frac{1}{r}-\frac{1}{s}}$$

Which yields $||f||_r \leq ||f||_s$ in our case.

For the case $s = \infty$, we have that $|f| \leq ||f||_\infty$ a.e. If $||f||_\infty = \infty$, then the equality $||f||_r \leq ||f||_\infty$ is immediate. If $||f||_\infty < \infty$, then

$$\left[\int_X |f|^r d\mu\right]^{\frac{1}{r}} \leq \left[\int_X ||f||_\infty d\mu\right]^{\frac{1}{r}} \tag{6}$$
$$\implies ||f||_r \leq ||f||_\infty \mu(X)^{\frac{1}{r}} \tag{7}$$
$$\implies ||f||_r \leq ||f||_\infty. \tag{8}$$

(b) We again split this into cases. If $s < \infty$, then as shown above, the result $||f||_r \leq ||f||_s$ is equivalent to the application of Hölder's inequality to $|f|^r \cdot 1$ with the conjugate exponents $p = s/r$ and $q = s/(s-r)$. Letting $g = |f|^r, h = 1$, we have that equality holds when

$$\left(\frac{g}{||g||_p}\right)^p = \left(\frac{h}{||h||_q}\right)^q \quad \mu\text{-a.e.} \tag{9}$$
$$\implies \left(\frac{|f|^{rp}}{||\,|f|^r\,||_p^p}\right) = \left(\frac{1}{||1||_q^q}\right) \quad \mu\text{-a.e..} \tag{10}$$

Notice that

$$||\,|f|^r\,||_p^p = \int_X |f|^{r(s/r)} d\mu = \int_X |f|^s d\mu = ||f||_s^s$$

and that

$$||1||_q^q = \int_X |1|^q d\mu = \mu(X).$$

We resume with deriving the necessary conditions for equality from (10):

$$\implies \left(\frac{|f|^s}{||f||_s^s}\right) = \left(\frac{1}{\mu(X)}\right) \quad \mu\text{-a.e.}$$
$$\implies |f|^s = ||f||_s^s \quad \mu\text{-a.e.}$$
$$\implies |f| = ||f||_s \quad \mu\text{-a.e.},$$

which in our case, happens if and only if $|f|$ is a constant a.e. Indeed, assume $|f| = c \in \mathbb{R}^{\geq 0}$. Then

$$||f||_r = \left[\int_X c^r d\mu\right]^{1/r} = c\mu(X)^{1/r} = c = c\mu(X)^{1/s} = \left[\int_X ||f||_s^s d\mu\right]^{1/s} = ||f||_s.$$

If $s = \infty$, then (6), with the inequality instead being an equality, implies that $|f| = ||f||_\infty^{1/r}$ μ-a.e., which is only true if $|f| = 1$ or $|f| = \infty$ a.e.

(c) **Attempt**
By (a), if $0 < r < s \leq \infty$ then $L^s(\mu) \subset L^r(\mu)$. We want to derive a broad enough condition

which grants us the reverse inclusion: $L^r(\mu) \subset L^s(\mu)$. We have control over the domain space, its measure, and the values of r and s.

Exercise 3.4 (c) shows that (X, μ) can not be $(\mathbb{R}, m)$: no matter what values we take for r, s, the condition that $r < s$ makes it possible for us to define an open interval containing r but not s which corresponds to the set E_f of a certain function $f\colon \mathbb{R} \to \mathbb{R}$. That is, we can get $f \in L^r(\mu)$ without $f \in L^s(\mu)$ for any $0 < r < s$.

(d) **Attempt**
Observe that $f \in L^p(\mu)$ for all $r < 0$. An approach using simple functions:
On X equipped with the measure μ, define the function $s = \sum_{i=1}^n \alpha_i \chi_{A_i}$, with $\{\alpha_i, ...\alpha_n\}$ being the image of s and $A_i = \{x \in X : s(x) = \alpha_i\}$. Notice that s also has a multiplicative form:

$$s = \prod_{i=1}^n \alpha_i^{\chi_{A_i}},$$

so that

$$\left(\exp\left[\int_X \log s^p d\mu\right]\right)^{1/p} = \left(\exp\left[\int_X \log\left(\prod_{i=1}^n \alpha_i^{p\chi_{A_i}}\right) d\mu\right]\right)^{1/p}$$
$$= \left(\exp\left[p\sum_{i=1}^n \mu(A_i)\log\alpha_i\right]\right)^{1/p} = \alpha_1^{\mu(A_1)} \cdots \alpha_n^{\mu(A_n)}.$$

This reduces the statement to a claim that

$$\lim_{p\to 0}\left[\left(\sum_{i=1}^n \alpha_i \mu(A_i)\right)^{1/p}\right] = \alpha_1^{\mu(A_1)} \cdots \alpha_n^{\mu(A_n)}.$$

Moreover, applying the general AM-GM inequality (i.e. $\exp\int \log f \leq \int f$) along with the above identities shows that

$$\alpha_1^{\mu(A_1)} \cdots \alpha_n^{\mu(A_n)} \leq \left[\int_X s^p d\mu\right]^{1/p} = ||f||_p$$

for all $p > 0$.