Malliavin-Letac II-III

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Exercise solutions and attempts

1 III - Fourier Analysis

Problems 1 to 5 serve to display the applications of radial measures in $\mathbb{R}^n$.

III-1 Characterizations of radial measures

Let G be the group O_d of $d \times d$ orthogonal matrices acting on $\mathbb{R}^d$. Let μ be a bounded complex measure on $\mathbb{R}^d$ with Fourier transform $\hat{\mu}$. Prove the equivalence of the following:
(i) μ is invariant under every element of G
(ii) $\exists \varphi : [0, \infty) \to \mathbb{C}$ s.t. $\hat{\mu}(t) = \varphi(||t||)$ for every t.
(iii) The image ν_a in $\mathbb{R}$ of μ under the mapping $x \to \langle a, x \rangle$ does not depend on a when a ranges over the unit sphere S_{d-1} of $\mathbb{R}^d$.

Proof:
(i) $\implies$ (ii).
Consider $t \in \mathbb{R}^d$ and $M \in O_d$. Note that, by invariance,

$$\begin{aligned}\hat{\mu}(Mt) &= \int_{\mathbb{R}^d} e^{i\langle Mt,x\rangle} = \int_{\mathbb{R}^d} e^{i\langle t,Mx\rangle} d\mu(x) \\ &= \int_{\mathbb{R}^d} e^{i\langle t,Mx\rangle} d\mu(Mx) = \int_{M\mathbb{R}^d=\mathbb{R}^d} e^{i\langle t,y\rangle} d\mu(y) \\ &= \hat{\mu}(t).\end{aligned}$$

There exists $M_t \in O_d$ s.t. $M_t t = ||t|| v_1$, where v_1 is chosen to be a fixed vector in the standard basis of $\mathbb{R}^d$. Let l_{v_1} denote the projection of $\mathbb{R}^d$ onto v_1. Then

$$\hat{\mu}(t) = \hat{\mu}(M_t t) = \int_{\mathbb{R}^d} e^{i\langle M_t t,x\rangle} d\mu(x) = \int_{\mathbb{R}^d} e^{i||t||l_{v_1}x} d\mu(x) = \int_{\mathbb{R}^{\geq 0}} e^{i||t||y} d\mu(l_{v_1}^{-1}(y)).$$

for arbitrary t. Define $\varphi(||t||)$ accordingly with the right-hand side.
(ii) $\implies$ (i).
Let $M \in O_d$ and let $t \in \mathbb{R}^d$ be arbitrary. Since $||Mt|| = ||t||$, we have that $\hat{\mu}(Mt) = \varphi(||Mt||) = \varphi(||t||) = \hat{\mu}(t)$. By the uniqueness of the Fourier transform of any measure, we have that μ is invariant under any $M \in O_d$.
(i) $\implies$ (iii).
For any $c \in S_{d-1}$, let $l_c : x \mapsto \langle c, x \rangle$. Let $a, b \in S_{d-1}$ and let B be in the Borel algebra

of $\mathbb{R}^{>0}$. We claim that $\mu(l_a^{-1}(B)) = \mu(l_b^{-1}(B))$. To see this, notice that if $M \in O_d$ is the transform sending a to b, then $l_b^{-1}(k) = \{x \in \mathbb{R}^d : ||x||cos(\theta(b,x)) = k\} = \{Mx \in \mathbb{R}^d : ||x||cos(\theta(a,x)) = k\} = Ml_a^{-1}(k)$. Taking the union over $k \in B$ yields $l_b^{-1}(B) = Ml_a^{-1}(B)$, whence we the apply invariance of μ.

(iii) $\implies$ (ii).

For a arbitrary in S_{d-1}, let $\nu = \nu_a$ be the common symbol for all pushforwards (they are all equivalent). Let $t \in \mathbb{R}^d$. Observe that

$$\hat{\mu}(t) = \int_{\mathbb{R}^d} e^{i\langle t,x\rangle} d\mu(x) = \int_{\mathbb{R}^d} e^{i\langle t,x\rangle} d\mu(x) = \int_{\mathbb{R}^d} e^{i\langle \frac{t}{||t||},x\rangle ||t||} d\mu(x)$$

Let $c = t/||t||$ in S_{d-1}. We resume with the change of variables

$$\hat{\mu}(t) = \int_{\mathbb{R}^{>0}} e^{iy||t||} d\mu(l_c^{-1}(y)) = \int_{\mathbb{R}^{>0}} e^{iy||t||} d\nu(y).$$

The function φ is again defined accordingly with the RHS. $\square$

III-2 Radial measures and independence

Let T be a compact space, G a compact topological group, and let $(g,t) \mapsto gt$ be a continuous map from $G \times T$ to T s.t. $g \mapsto \{(g,t) \mapsto gt\}$ is a homomorphism from G to the group of bijections of T. Finally, suppose that G, T is a homogeneous space. Let dg denote the unique measure of total mass 1 on G which is invariant under left and right multiplication. (We accept without proof the existence and uniqueness of dg.)

(i) If f is continuous on T, show that $t \mapsto \int_G f(g^{-1}t)dg$ is constant (denote it $\sigma[f]$). Conclude that $\sigma[f]$ defines a probability measure on T which is invariant under the action of G.

Proof:

Take $t_1, t_2 \in T$ arbitrarily. Let $g_{12} \in G$ send t_1 to t_2. Then

$$\begin{aligned}\int_G f(g^{-1}t_2)dg &= \int_G f(g^{-1}g_{12}t_1)dg = \int_G f((gg_{12}^{-1})^{-1}t_1)dg \\ &= \int_{Gg_{12}^{-1}} f((g')^{-1}t_1)d(g'g_{12}) = \int_G f((g')^{-1}t_1 dg'.\end{aligned}$$

Hence indeed the mapping $t \mapsto \int_G f(g^{-1}t)dg$ has only one value $\sigma[f]$ when t ranges over T. Should f be positive, it follows naturally that $\sigma[f]$ is also positive. Therefore $\sigma : f \to \sigma[f]$ is a positive linear functional on $C_K(T) = C(T)$ *– however, the above argument also works for any integrable function over* T, *i.e.* $\sigma[\phi]$ *is well defined even if* ϕ *is not necessarily continuous.*

The Riesz theorem yields a positive Radon measure ν defined on a $\sigma-$algebra $\mathcal{B}$ of T containing its Borel algebra $\mathcal{B}_T$. Also,

$$\nu(T) = \int_T d\nu = \sigma[1_T] = \int_G dg = 1.$$

Take $g \in G$ and let $f \in C(T)$. If f_g is defined by $f_g(t) = f(gt)$, then it is routine to show that $\sigma[f_g] = \sigma[f]$, which in turn yields

$$\int_T f(t)d\nu(t) = \int_T f(gt)d\nu(t) = \int_T f(t')d\nu(g^{-1}t').$$

That is, $\nu(\cdot)$ and $\nu(g^{-1}\cdot)$ define the same pos. linear functional on $C(T)$ and are thus equivalent measures. Since g is arbitrary, it follows that ν is invariant under G. $\square$

(ii) If μ is a probability measure on T which is invariant under the action of G, show that $g \mapsto \int_T f[g^{-1}t]d\mu(t)$ is a constant. Integrate with respect to dg to conclude that $\mu = \sigma$

Proof:
Take $g_0 \in G$ arbitrarily. Then

$$\int_T f(g_0^{-1}t)d\mu(t) = \int_{g_0^{-1}T} f(t')d\mu(g_0t') = \int_T f(t')d\mu(t') = \int_T f d\mu.$$

Thus the mapping $g : g \mapsto \int_T f(g^{-1}t)d\mu(t)$ is constant. Keeping in mind that f is continuous on compact T, integrating w.r.t. dg_0 yields

$$\int_G \int_T f(g^{-1}t)d\mu(t)dg = \int_T f d\mu \int_G dg = \int_T f d\mu < +\infty.$$

Hence, we may apply Fubini's theorem to obtain

$$\begin{aligned}\int_T f d\mu = \int_G \int_T f(g^{-1}t)d\mu(t)dg &= \int_T \int_G f(g^{-1}t)dgd\mu(t)\\ &= \sigma[f]\int_T d\mu(t) = \sigma[f].\end{aligned}$$

By the uniqueness of σ's measure representation (Riesz theorem), this forces $\mu = \nu$. $\square$

(iii) If $(X, \mathcal{A})$ is an arbitrary measurable space and T is equipped with its Borel algebra, let $T \times X$ be given the product σ-algebra. Suppose that G acts on $T \times X$ by $g(t, x) = (gt, x)$. Show that every positive measure μ on $T \times X$ which is invariant under the action of G has the form $\sigma \otimes \nu$ where (by abuse of notation) σ is the measure invariant measure on T developped in (i), and ν is a measure ≥ 0 on $(X, \mathcal{A})$. Converse?

Proof:
For any $A \in \mathcal{A}$, define $\nu(A) = \mu(T \times A)$. It is routine to show that ν is a positive measure on $(X, \mathcal{A})$. Let $\sigma \otimes \nu$ be the product measure on $T \times X$. We claim that $\mu = \sigma \otimes \nu$. It suffices to show that $\mu(B \times A) = \sigma(B)\nu(A)$ for $\sigma(B), \nu(A) < +\infty$, given that this same property is unique to $\sigma \otimes \nu$.
To this end, note that for arbitrary $g \in G$, we have that

$$\int_{T\times X} 1_{B\times A}(g^{-1}t, x)d\mu(t, x) = \int_{T\times X} 1_{B\times A}(t, x)d\mu(t, x) = \mu(B \times A)$$

using the invariance of μ. Integrating w.r.t. dg yields

$$\mu(A \times B) = \int_G \int_{T\times X} 1_{B\times A}(g^{-1}t, x)d\mu(t, x)dg.$$

Knowing that $\mu(B \times A) < +\infty$, we apply Fubini's theorem to obtain

$$\mu(B \times A) = \int_{T\times X} \int_G 1_{B\times A}(g^{-1}t, x)dgd\mu(t, x)$$

$$
\begin{aligned}
&= \int_{T\times X}\int_G 1_B(g^{-1}t)1_A(x)dgd\mu(t,x) \\
&= \int_G 1_B(g^{-1}t)dg \int_{T\times X} 1_A(x)d\mu(t,x) \\
&= \sigma(A)\int_{T\times X} 1_{T\times A}(t,x)d\mu(t,x) \\
&= \sigma(A)\nu(B).
\end{aligned}
$$

The converse follows naturally. □

(iv) Remark (Letac): applying the preceding results for $T = S_d$ (the unit sphere of $\mathbb{R}^{d+1}$, $G = O_{d+1}$ (the group of $(d+1)\times(d+1)$ orthogonal matrices), and $X = [0,\infty)$ tells us that a probability measure P on $\mathbb{R}^{d+1}$ is invariant under G if and only if $P = \sigma\otimes\nu$, where σ is (necessarily) the image of P under $x\mapsto x/||x||$ and ν is the image of P under $x\mapsto ||x||$. That is, $x\mapsto x/||x||$ and $x\mapsto ||x||$ are independent w.r.t. P. Moreover, $x\mapsto x/||x||$ has uniform distribution.

III-3

In the Euclidean space $\mathbb{R}^d$ equipped with the norm $||x||$, let m be the Lebesgue measure.
(i) If ν_0 and ν_1 are the images of m in $[0,+\infty)$ under the mappings $\phi : x\mapsto ||x||$ and $\varphi : x\mapsto \frac{||x||^2}{2}$ respectively, show that

$$\nu_1(d\gamma) = \frac{(\sqrt{2\pi})^d}{\Gamma(d/2)}\gamma^{\frac{d}{2}-1}d\gamma.$$

where Γ is the usual Euler function. Use this to find $\nu_0(d\rho)$.

Proof: More explicitly, the goal is to show that there exists a measure on $[0,+\infty)$ whose value on a Lebesgue-measurable A corresponds to the volume of the $\mathbb{R}^d$ spherical "shell", whose cross-section w.r.t the positive real line yields A. Also, we want to find its density function w.r.t. the Lebesgue measure on $\mathbb{R}^{\geq 0}$. We provide a path which deviates from the above instructions.

We may use an important result from calculus. The volume of the d-ball with radius r can be shown to be

$$V_d(R) = \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)}R^d \tag{1}$$

Let $\iota : \mathbb{R}^d \to S_{d-1}\times\mathbb{R}^{\geq 0}$ denote the natural isometry between the "cartesian coordinates" and the "polar coordinates". Let $I = [a,b]$ be an interval in $\mathbb{R}^{\geq 0}$. Note that

$$\nu_0(I) = m(\phi^{-1}(I)) = m(\iota^{-1}(S_{d-1}\times I)),$$

i.e., the volume of the d-dimensional spherical shell with outer radius b and inner radius a. We resume by applying (1):

$$
\begin{aligned}
\nu_0(I) &= V_d(b) - V_d(a) \\
&= \int_I V_d'(\gamma)\mathrm{d}\gamma \qquad \text{(F.T.C.)}
\end{aligned}
$$

$$= \int_I \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)} d\gamma^{d-1}\, \mathrm{d}\gamma$$
$$= \int_I \frac{2\pi^{\frac{d}{2}}}{\left(\frac{d}{2}\right)\Gamma(\frac{d}{2})} \left(\frac{d}{2}\right) \gamma^{d-1} \mathrm{d}\gamma \qquad (\Gamma \text{ recurrence relation})$$
$$= \int_I \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \gamma^{d-1} \mathrm{d}\gamma$$

So far, we have obtained the radial measure on closed intervals. We aim to extend this result to the much broader family of Lebesgue measurable sets. First, the equality clearly holds for any connected set (intervals which are open, half-open half-closed, closed, as well as $\mathbb{R}^{\geq 0}$ and $\emptyset$). The same is true for finite unions and finite intersections of connected sets and we omit the routine demonstration. This set which we have just described also corresponds to the Boolean algebra of open sets on $\mathbb{R}^{\geq 0}$, denote it $\mathcal{B}_b$.

In order to pass from finite to uncountable unions/intersections, we use a limiting process. If $A_1 \subset A_2 \subset ...$ is a sequence of increasing sets, then

$$\nu_0(A_n) = \int_{A_n} V_d'(\gamma)\mathrm{d}\gamma$$
$$\implies \lim_{n\to\infty} \nu_0(A_n) = \lim_{n\to\infty} \int_{\mathbb{R}^{\geq 0}} 1_{A_n} V_d'(\gamma)\mathrm{d}\gamma.$$

The limit of measures on the LHS corresponds to the measure of $\cup A_n$. For the RHS, we use the Fatou Beppo Levi theorem to pass the limit under the integral, which gives

$$\nu_0(\cup A_n) = \int_{\cup A_n} V_d'(\gamma)\mathrm{d}\gamma.$$

If $B_1, B_2, ...$ is a sequence of decreasing sets, we use the dominated convergence theorem, yielding

$$\nu_0(\cap B_n) = \int_{\cap B_n} V_d'(\gamma)\mathrm{d}\gamma.$$

(This is done with the assumption that there is one B_j for which $\nu_0(B_j) < +\infty$.)
As such, the result holds on the monotone class $\mathcal{M}(\mathcal{B}_b)$, which also corresponds to the σ-algebra generated by $\mathcal{B}_b$, which in turn is actually the Borel algebra $\mathcal{B}$ of $\mathbb{R}^{\geq 0}$.

Finally, suppose that C is a Lebesgue-measurable set on the non-negative real line. Let λ be the the Lebesgue measure on $\mathbb{R}^{\geq 0}$. One can provide $A \in \mathcal{F}_\sigma \subset \mathcal{B}$ and $B \in \mathcal{G}_\delta \subset \mathcal{B}$ (respectively, the set of countable unions of closed sets and the set of countable intersection of open sets) such that $\lambda(B - A) = 0$. Then $\lambda(B) = \lambda(B - C) + \lambda(C)$, but $B - C \subset B - A$, so that $\lambda(B - C) \leq \lambda(B - A) = 0$, whence $\lambda(B) = \lambda(C)$.
Of course, this argument parallels on $\mathbb{R}^d$: $m(\iota^{-1}(S_{d-1} \times B)) = m(\iota^{-1}(S_{d-1} \times C)$ knowing that $S_{d-1} \times A \subset S_{d-1} \times C \subset S_{d-1} \times B$ and that $S_{d-1} \times (B - A)$, akin to $B - A$, is a negligible set.

By the above, $1_B(\gamma)\gamma^{d-1}$ and $1_C(\gamma)\gamma^{d-1}$ are equal λ−a.e., and so

$$\nu_0(C) = \nu_0(B) = \int_{\mathbb{R}^{\geq 0}} \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}\gamma^{d-1} 1_B(\gamma)\mathrm{d}\gamma = \int_{\mathbb{R}^{\geq 0}} \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}\gamma^{d-1} 1_C(\gamma)\mathrm{d}\gamma = \int_C \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}\gamma^{d-1}\mathrm{d}\gamma. \square$$

*Throughout this proof (attempt), I've resorted to a lot of abuse of notation. By "the

Lebesgue measure on $\mathbb{R}^{\geq 0}$, I truly ment "the restriction of the Lebesgue measure on $\mathbb{R}^{\geq 0}$ from $\mathbb{R}$", and idem for "the Boolean/Borel algebra on $\mathbb{R}^{\geq 0}$, and so on.
Also, Letac originally intended for the reader to use the normal distribution on $\mathbb{R}^d$ to simplify the Laplace transform of ν_1, which, by a uniqueness argument, would have provided the density function of ν_1 – from there, one can retrieve ν_0 using u-substitution. I decide to keep the exercise question as it is for integrity.

We leave section III for a little while to pay a visit to some pertinent (and interesting) section II problems.

Chapter II-18: Poisson kernel on the half-space (Attempt)

Let $\mathbb{R}_+^{n+1}$ denote the set of pairs (a,p) with $p>0$ and $a \in \mathbb{R}^n$. Let

$$K(a,p) = K_n p[||a||^2 + p^2]^{-(n+1)/2}$$

where K_n is the constant such that $\int_{\mathbb{R}^n} K(x,1)dx = 1$. The goal of this problem is to calculate

$$I_t(a,p) = \int_{\mathbb{R}^n} \exp i\langle x,t\rangle K(x-a,p)dx.$$

where $t \in \mathbb{R}^n$.
First, we keep in mind that $K(x,1) \in L^p(\mathbb{R}^n)$ as a function of x. If for arbitrary $(a,p) \in \mathbb{R}_+^{n+1}, a = (a_1, ..., a_n)$, $f : \mathbb{R}_+^{n+1} \to \mathbb{C}$ satifies

$$\nabla^2 f(a,p) = \frac{\partial^2}{\partial^2 p} f(a,p) + \sum_{j=1}^{n} \frac{\partial^2}{\partial^2 a_j} f(a,p) = 0,$$

f is called harmonic.
(1) Show that K is harmonic in $\mathbb{R}_+^{n+1}$. Show that if, $p_0 > 0$ and $V = (\frac{p_0}{2}, \frac{3p_0}{2})$, there exists a constant C such that all first and double partials are less than $C(1+||a||^2)^{-(n+1)/2}$ for all $(a,p) \in \mathbb{R}^n \times V$.

Proof:
First, we calculate all first partials to be

$$\frac{\partial}{\partial p} K(a,p) = -K_n p(||a||^2 + p^2)^{-(n+1)/2-1}(np^2 - ||a||^2) = -\frac{(np^2 - ||a||^2)}{(||a||^2 + p^2)} K(a,p),$$

$$\frac{\partial}{\partial a_j} K(a,p) = -K_n (n+1) p a_j \left(||a||^2 + p^2\right)^{-(n+1)/2-1} = -\frac{(n+1)a_j}{||a||^2 + p^2} K(a,p),$$

and all second partials to be

$$\frac{\partial^2}{\partial p^2} K(a,p) = K_n(n+1)p(||a||^2 + p^2)^{-(n+1)/2-2}(np^2 - 3||a|^2) = \frac{(n+1)(np^2 - 3||a||^2)}{(||a||^2 + p^2)^2} K(a,p),$$

$$\frac{\partial^2}{\partial a_j^2} K(a,p) = K_n(n+1)p(||a||^2 + p^2)^{-(n+1)/2-2}((n+3)a_j^2 - ||a||^2 - p^2)$$

$$= \frac{(n+1)((n+3)a_j^2 - ||a||^2 - p^2)}{(||a||^2+p^2)^2} K(a,p),$$

We follow with showing that K is harmonic. Observe that

$$\begin{aligned}
\sum_{j=1}^{n} \frac{\partial^2}{\partial a_j^2} K(a,p) &= K_n(n+1)p(||a||^2+p^2)^{-(n+1)/2-2} \sum_{j=1}^{n}((n+3)a_j^2 - ||a||^2 - p^2) \\
&= K_n(n+1)p(||a||^2+p^2)^{-(n+1)/2-2}(3||a||^2 - np^2) \\
&= -\frac{\partial^2}{\partial p^2} K(a,p),
\end{aligned}$$

which shows that $\nabla^2 K(a,p) = 0$. Next, we set up the bounding work in order to pass the nabla operator under the integral. First, let $P(a) = (||a||^2+1)^{-(n+1)/2}$. If $p_0 > 0$, then let $p_1 = \frac{p_0}{2}$, and $p_2 = \frac{3p_0}{2}$. Note that for arbitrary $p \in V = (p_1, p_2)$,

$$K(a,p) = \frac{K_n p}{(||a||^2+p^2)^{(n+1)/2}} \leq \frac{K_n p_2}{(||a||^2+p_1^2)^{(n+1)/2}} \leq K_n p_2 C_0 P(a), \tag{2}$$

where $C_0 > 0$ is such that

$$\frac{(||a||^2+1)^{(n+1)/2}}{C_0} \leq (||a||^2+p_1^2)^{(n+1)/2} \ \ \forall a \in \mathbb{R}^n.$$

Now, we may use (2) to obtain

$$\begin{aligned}
|\frac{\partial}{\partial p} K(a,p)| &= \frac{|(np^2 - ||a||^2)|}{(||a||^2+p^2)} K(a,p) \\
&\leq nK(a,p) \\
&\leq nK_n p_2 C_0 P(a).
\end{aligned}$$

Now, for some arbitrary $j = 1, ..., n$ we have

$$\begin{aligned}
|\frac{\partial}{\partial a_j} K(a,p)| &= \frac{(n+1)|a_j|}{||a||^2+p^2} K(a,p) \\
&\leq \frac{(n+1)|a_j|}{||a||^2+p_1^2} K(a,p)
\end{aligned}$$

Note that if $||a|| > n+1$, then $(n+1)|a_j| \leq (n+1)||a|| < ||a||^2 < ||a||^2 + p_1^2$, which would imply that

$$\frac{(n+1)|a_j|}{||a||^2+p_1^2} < 1.$$

On the other hand, if $||a|| \leq n+1$, observe that

$$\frac{(n+1)|a_j|}{||a||^2+p_1^2} \leq \frac{(n+1)|a_j|}{p_1^2} \leq \frac{(n+1)^2}{p_1^2},$$

hence, setting $C_1 = \max\left\{\frac{(n+1)^2}{p_1^2}, 1\right\}$ yields

$$|\frac{\partial}{\partial a_j} K(a,p)| \leq K_n p_2 C_0 C_1 P(a).$$

We move on to providing an upper bound for the second partials (which is surprisingly easier than doing so for the first partials):

$$\begin{aligned}
|\frac{\partial^2}{\partial p^2}K(a,p)| &= \frac{(n+1)|(np^2-3||a||^2)|}{(||a||^2+p^2)^2}K(a,p) \\
&\leq \frac{3n(n+1)|(||a||^2+p^2)|}{(||a||^2+p^2)^2}K(a,p) \\
&= \frac{3n(n+1)}{||a||^2+p^2}K(a,p) \\
&\leq \frac{3n(n+1)}{p_1^2}K_n p_2 C_0 P(a).
\end{aligned}$$

with

$$\begin{aligned}
|\frac{\partial^2}{\partial a_j^2}K(a,p)| &= \frac{(n+1)|(||a||^2+p^2-(n+3)a_j^2)|}{(||a||^2+p^2)^2}K(a,p) \\
&\leq \frac{(n+1)}{||a||^2+p^2}K(a,p) \\
&\leq \frac{n+1}{p_1^2}K_n p_2 C_0 P(a).
\end{aligned}$$

We now define C to be the maximal value of all preceding factors by which we've scaled $P(a)$. $\square$

(ii) Let μ be a Radon measure on $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} (||x||^2+1)^{-(n+1)/2} d|\mu|(x) < +\infty \tag{3}$$

and let $F_\mu(a,p) = \int_{\mathbb{R}^n} K(x-a,p)d\mu(x)$. Show that F_μ is harmonic and that $\lim_{p\to+\infty} F_\mu(a,p) = 0$.

Proof:
Let p_0 remain arbitrary, and choose whatever $a_0 \in \mathbb{R}^n$ is to your liking. Observe that $\forall p \in V = (\frac{p_0}{2}, \frac{3p_0}{2})$, the function of x, $K(x-a_0,p)$, is dominated by $K_n\frac{3p_0}{2}C_0P(x-a_0)$, and as such we have that $F_\mu(a_0,p) < +\infty$, whence the function is well defined w.r.t. $p \in V$.

Also, the derivative $\frac{\partial}{\partial p}K(x-a_0,p)$ exists and is continuous (as a function of p) at p_0 for all $x \in \mathbb{R}^n$. Finally, this same derivative, now as a function of x, is dominated by $CP(x-a_0)$ for all $p \in (\frac{p_0}{2}, \frac{3p_0}{2})$. We apply Proposition 7.8.4 (differentiability of an integral depending on a parameter) to obtain

$$\frac{\partial}{\partial p}F_\mu\bigg|_{(a_0,p_0)} = \int_{\mathbb{R}^n} \frac{\partial}{\partial p}K(x-a_0,p_0)d\mu(x).$$

We apply a very similar argument to pass the second derivative under the integral:

$$\frac{\partial^2}{\partial p^2}F_\mu\bigg|_{(a_0,p_0)} = \int_{\mathbb{R}^n} \frac{\partial^2}{\partial p^2}K(x-a_0,p_0)d\mu(x).$$

We repeat the process, this time w.r.t. a_j. In contrast to the above, we now treat p_0 as a constant variable and only focus on the pair of $\mathbb{R}^n$-variables x and a. Notice that $\forall a$, $F_\mu(a,p_0) < +\infty$ (dominance of $K_n\frac{3p_0}{2}C_0P(x-a)$ over $K(x-a,p_0)$). Also, $\frac{\partial}{\partial a_j}K(x-a,p_0)$ exists and is continuous in a for all x. Again finally, this same derivative (function in x) is dominated by $CP(x-a)$ for all a. Apply Proposition 7.8.4, then do it again for the second partial to obtain

$$\frac{\partial^2}{\partial a_j^2}F_\mu\bigg|_{(a_0,p_0)} = \int_{\mathbb{R}^n}\frac{\partial^2}{\partial a_j^2}K(x-a_0,p_0)d\mu(x).$$

Therefore,

$$\nabla^2 F_\mu(a_0,p_0) = \int_{\mathbb{R}^n}\nabla^2 K(x-a_0,p_0)d\mu(x),$$

but a two-fold application of the chain rule yields that $\nabla^2 K(x-a_0,p_0)=0$, similarly to the case with no "inverse translation" by x. Hence the gradient of F_μ does evaluate to 0 for any pair $(a_0,p_0)\in\mathbb{R}^n$.

For arbitrary a, the integrand $K(x-a,p)$ converges to 0 pointwise. As mentioned above, it is dominated by $K_np_2C_0P(a)$ and hence $\lim_{p\to+\infty}F_\mu(a,p)=0$.

(iii) Show that there exists a function $g:\mathbb{R}^n\to\mathbb{C}$ such that $I_t(a,p)=g(pt)\exp(i\langle a,t\rangle)$. Use (2) to calculate g.

Proof:

Our method yields a function $g(p,t)$ rather than one of the above form. First, let $K_p(x)=K(x,p)$. Then, notice that $I_t(a,p)=\widehat{\tau_a K_p}(t)$, i.e. the Fourier transform of the Poisson kernel translated by a evaluated at the character $\chi_t=e^{i\langle t,\cdot\rangle}$. As such, we can extract the translation like so (Theorem III-1.7.2 on the trivialization of the translation operator):

$$I_t(a,p)=e^{i\langle t,a\rangle}\hat{K}_p(x)=e^{i\langle t,a\rangle}\hat{K}_p(t)=e^{i\langle t,a\rangle}\int_{\mathbb{R}^n}e^{i\langle t,x\rangle}K(x,p)dx.$$

Whence, $g(p,t):=\int_{\mathbb{R}^n}e^{i\langle t,x\rangle}K(x,p)dx$. We now apply (2): notice that the measure $\chi_t(x)dx=e^{i\langle t,x\rangle dx}$ is Radon and satisfies (3). Therefore

$$\begin{aligned}
I_t(a,p) &= F_{\chi_t}(a,p)\\
\nabla^2\, e^{i\langle t,a\rangle}g(p,t) &= 0\\
e^{i\langle t,a\rangle}\frac{\partial^2}{\partial p^2}g(p,t)+g(p,t)\sum_{j=1}^n\frac{\partial^2}{\partial a_j^2}e^{i\langle t,a\rangle} &= 0\\
g(p,t)\sum_{j=1}^n\frac{\partial^2}{\partial a_j^2}e^{i\langle t,a\rangle} &= -e^{i\langle t,a\rangle}\int_{\mathbb{R}^n}e^{i\langle t,x\rangle}\frac{\partial^2}{\partial p^2}K(x,p)dx
\end{aligned}$$

The second partial w.r.t. p is passed under the integral using arguments similar to the above. We now turn our focus solely on said integral. Observe that

$$\int_{\mathbb{R}^n}e^{i\langle t,x\rangle}\frac{\partial^2}{\partial p^2}K(x,p)dx = K_n(n+1)\int_{\mathbb{R}^n}e^{i\langle t,x\rangle}\frac{p(np^2-3||x||^2)}{(||x||^2+p^2)^{(n+1)/2+2}}dx$$

******

Not sure how to tackle this integral. I think we need to use the radial measure to simplify

this to an integral on the positive real line, but I'd have to account for the exp evaluation. I tried the case for $n = 1$, and obtained $g(p,t) = e^{-pt}$, which yields $I_t(a,p) = e^{ita-pt}$. Also, I think that it would suffice to use the fact that the upper-half space Poisson kernel corresponds to the Fourier transform of the Abel transform to calculate $g(p,t)$ using the Fourier inversion theorem, which would have eliminated the need to show that I_t was harmonic – though this feels so much like cheating.
****

We can try using Problem III-1. Notice that for $p > 0$ fixed, $\mu_K = K_p(x)dx$ defines a measure on $\mathbb{R}^d$ which is bounded and invariant under the group O_d. There may be a way to eliminate the inner product inside the Fourier integral. Using the notation of III-1 ,

$$g(p,t) = \int_{\mathbb{R}^d} e^{i\langle t,x\rangle} d\mu_K(x) = \int_0^\infty e^{i||t||y} d\mu_K(l_{v_1}^{-1}(y))$$

(to have a good understanding of $d\mu_K$ I probably need to learn about Radon-Nikodym derivatives. Dead end!)

********** New idea: we can try a substitution in the hopes of eliminating as many p terms as possible – set $u = x/p$. Then

$$\begin{aligned}
\int_{\mathbb{R}^n} e^{i\langle t,x\rangle} \frac{p(np^2 - 3||x||^2)}{(||x||^2 + p^2)^{(n+1)/2+2}} dx &= \int_{\mathbb{R}^n} e^{i\langle t,pu\rangle} \frac{p(np^2 - 3||pu||^2)}{(||pu||^2 + p^2)^{(n+1)/2+2}} p du \\
&= \int_{\mathbb{R}^n} e^{ip\langle t,u\rangle} \frac{p^4(n - 3||u||^2)}{[p^2(||u||^2 + 1)]^{(n+5)/2}} du \\
&= \frac{1}{p^{n+1}} \int_{\mathbb{R}^n} e^{i\langle pt,u\rangle} \frac{n - 3||u||^2}{(||u||^2 + 1)^{(n+5)/2}} du
\end{aligned}$$

We can reduce this integral to a product of several integrals on $\mathbb{R}^n$, one term for each component. ********* TO DO.