1 Double Integrals

Ex.1 Find $\int_0^1 \int_0^y xy\,e^{-x^2}\,dx\,dy$ **[M.U. 2002]**

Solution: $$I = \int_0^1 y\left[\frac{e^{-x^2}}{-2}\right]_0^y dy = -\frac{1}{2}\int_0^1 \left(ye^{-y^2} - y\right)dy$$

$$= -\frac{1}{2}\left[\frac{e^{-y^2}}{-2} - \frac{y^2}{2}\right]_0^1 = \frac{1}{4}\left[\left(e^{-1}+1\right)-(1)\right] = \frac{1}{4e}$$

Ex.2 Evaluate $\int_0^1 \int_0^{x^2} e^{y/x}\,dy\,dx$ **[M.U. 1997, 2004]**

Solution: $$I = \int_0^1 \left[\int_0^{x^2} e^{y/x} dy\right]_0^y dx = \int_0^1 \left[\frac{e^{y/x}}{1/x}\right]_0^{x^2} dx$$

$$= \int_0^1 \frac{\left(e^x - 1\right)}{1/x} dx = \int_0^1 x\,e^x dx - \int_0^1 x\,dx$$

$$\therefore \quad I = \left[x\,e^x - e^x\right]_0^1 - \left[\frac{x^2}{2}\right]_0^1 = e^1 - e^1 + 1 - \frac{1}{2} = \frac{1}{2}$$

Ex.3 Evaluate $\int_0^1 \int_{x^2}^{x} xy\left(x+y\right)dy\,dx$ **[M.U. 1996]**

Solution: $$I = \int_0^1 \int_{x^2}^{x} \left(x^2y + xy^2\right)dy\,dx = \int_0^1 \left[\frac{x^2y^2}{2} + \frac{xy^3}{3}\right]_{x^2}^{x} dx$$

$$= \int_0^1 \left(\frac{x^4}{2} + \frac{x^4}{3} - \frac{x^6}{2} - \frac{x^7}{3}\right)dx = \left[\frac{5}{6}.\frac{x^5}{5} - \frac{x^7}{14} - \frac{x^8}{24}\right]_0^1$$

$$= \frac{1}{6} - \frac{1}{14} - \frac{1}{24} = \frac{3}{56}$$

Ex.4 Find $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2}\,dy\,dx$ **[M.U. 1999, 2002, 09]**

Solution: $$I = \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{y^2 + \left(1+x^2\right)}\,dy\,dx$$

$$= \int_0^1 \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1} \frac{y}{\sqrt{1+x^2}}\right]_0^{\sqrt{1+x^2}} dx$$

$$= \frac{\pi}{4}\int_0^1 \frac{dx}{\sqrt{1+x^2}}$$

$$\therefore \qquad I = \frac{\pi}{4}\left[\log\left(x+\sqrt{1+x^2}\right)\right]_0^1 = \frac{\pi}{4}\log\left(1+\sqrt{2}\right)$$

Ex.5 Evaluate $\int_0^1 \int_0^{\sqrt{\left(1-y^2\right)/2}} \frac{dx\,dy}{\sqrt{1-x^2-y^2}}$ **[M.U. 1999]**

Solution: Integrating w.r.t. x, treating y constant i.e. $1-y^2 = a^2$ say

$$I = \int_0^1 \int_0^{\sqrt{\left(1-y^2\right)/2}} \frac{dx\,dy}{\sqrt{\left(1-y^2\right)-x^2}}$$

$$= \int_0^1 \left[\sin^{-1}\left(\frac{x}{\sqrt{1-y^2}}\right)\right]_0^{\sqrt{\left(1-y^2\right)/2}} dy$$

$$= \int_0^1 \left[\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) - \sin^{-1} 0\right] dy = \int_0^1 \frac{\pi}{4}\,dy$$

$$= \frac{\pi}{4}[y]_0^1 = \frac{\pi}{4}$$

Ex.6 Evaluate $\int_0^{a\sqrt{3}} \int_0^{\sqrt{x^2+a^2}} \frac{x\,dy\,dx}{y^2+x^2+a^2}$ **[M.U. 1999, 2012]**

Solution: Treating x constant

$$I = \int_0^{a\sqrt{3}} \int_0^{\sqrt{x^2+a^2}} \frac{a}{y^2+\left(x^2+a^2\right)}\,dy\,dx$$

$$= \int_0^{a\sqrt{3}} \left[\frac{1}{\sqrt{x^2+a^2}}\tan^{-1}\left(\frac{y}{\sqrt{x^2+a^2}}\right)\right]_0^{\sqrt{x^2+a^2}} x\,dx$$

$$= \int_0^{a\sqrt{3}} \frac{1}{\sqrt{x^2+a^2}}.\left[\frac{\pi}{4}-0\right].x\,dx$$

$$= \frac{\pi}{4}\int_0^{a\sqrt{3}} \frac{x}{\sqrt{x^2+a^2}}\,dx = \frac{\pi}{4}\left[\sqrt{x^2+a^2}\right]_0^{a\sqrt{3}}$$

$$= \frac{\pi}{4}[2a-a] = \frac{\pi}{4}a$$

Ex.7 Evaluate $\int_0^\infty \int_0^\infty e^{-x^2\left(1+y^2\right)} x\,dx\,dy$ **[M.U. 2003]**

Solution: We put $x^2\left(1+y^2\right) = t \qquad \therefore \qquad 2x\left(1+y^2\right)dx = dt$

When $x=0, t=0$; when $x=\infty, t=\infty$

$$\therefore \quad I = \int_0^{\infty}\int_0^{\infty} e^{-t}.\frac{dt}{2\left(1+y^2\right)}dy$$

$$= \int_0^{\infty} \frac{1}{2\left(1+y^2\right)}\left[-e^{-t}\right]_0^{\infty} dy$$

$$= \int_0^{\infty} \frac{1}{2\left(1+y^2\right)}[0-1]dy$$

$$= \frac{1}{2}\int_0^{\infty}\frac{dy}{1+y^2} = \frac{1}{2}\left[\tan^{-1} y\right]_0^{\infty}$$

$$= \frac{1}{2}.\frac{\pi}{2} = \frac{\pi}{4}$$

Ex.8 Evaluate $\int_0^{\infty} dx \int_0^1 e^{-x^a y}\, dy$ **[M.U. 1997]**

Solution: Since the limits are constants and integration with respect to y first and then with respect to x leads to a complicated integral, we reverse the order of integration.

$$\therefore \quad I = \int_0^1 dy \int_0^{\infty} e^{-x^a y} dx$$

Now, put $x^a y = t$ $\quad\therefore\quad x = \left(\frac{t}{y}\right)^{1/a}$ $\quad\therefore\quad dx = \frac{1}{a}.\frac{t^{(1/a)-1}}{y^{1/a}}dt$

When x = 0, t = 0; when $x=\infty, t=\infty$

$$\therefore \quad I = \int_0^1 dt \int_0^{\infty} e^{-t}.\frac{1}{a.y^{1/a}}.t^{(1/a)-1}dt$$

$$= \frac{1}{a}\int_0^1 y^{-1/a}dy.\int_0^{\infty} e^{-t}.t^{(1/a)-1}dt$$

$$= \frac{1}{a}\left[\frac{y^{-(1/a)+1}}{(-1/a)+1}\right]_0^1 . \overline{\left|\frac{1}{a}\right.} = \frac{\overline{|1/a}}{a-1}$$ [By definition of $\overline{|n}$]

EXERCISE

Evaluate the following integrals :

- $\int_0^1\int_0^x \left(x^2+y^2\right)x\, dy\, dx$ **[M.U. 2002]**

Ans. $4/15$

- $\int_0^5\int_{2-x}^{2+x} dy\, dx$ **[M.U. 2002]**

Ans. 25

- $\int_1^2\int_{-(2-y)}^{(2-y)} 2x^2y^2 dx\, dy$ **[M.U. 2002]**

Ans. $22/45$

- $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2}$ **[M.U. 2002]**

Ans. $(\pi/4)\log\left(1+\sqrt{2}\right)$

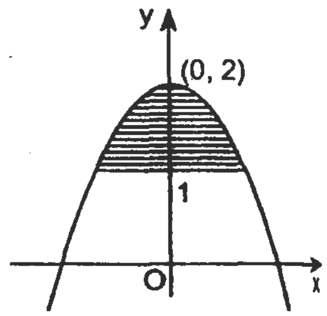
Ex.9 Sketch the area of integration and evaluate $\int_1^2 \int_{-\sqrt{2-y}}^{\sqrt{2-y}} 2x^2y^2 dx\,dy$

[M.U. 1987, 91, 2003]

Solution: We have $x = \pm\sqrt{2-y}$

$\therefore \quad x^2 = 2-y \quad$ i.e. $\quad y-2 = -x^2$

The curve is a parabola with vertex at (0, 2) as shown in the figure. And y varies from 1 to 2.



$$\therefore \quad I = \int_1^2 \int_{-\sqrt{2-y}}^{\sqrt{2-y}} 2x^2y^2 dx\,dy$$

$$= 2\int_1^2 \int_0^{\sqrt{2-y}} 2x^2y^2 dx\,dy$$

$$= 2\int_1^2 2y^2 \left[\frac{x^3}{3}\right]_0^{\sqrt{2-y}} dy$$

$$= \frac{4}{3}\int_1^2 y^2 (2-y)^{3/2} dy$$

Putting $2-y = t, dy = -dt$

When $y=1, t=1$; when $y=2, t=0$

$$\therefore \quad I = \frac{4}{3}\int_1^0 -(2-t)^2 .t^{3/2} dt$$

$$\therefore \quad I = \frac{4}{3}\int_0^1 \left(4-4t+t^2\right) t^{3/2} dt$$

$$= \frac{4}{3}\int_0^1 \left(4t^{3/2} - 4t^{5/2} + t^{7/2}\right) dt$$

$$= \frac{4}{3}\left[4.\frac{2}{5}t^{5/2} - 4.\frac{2}{7}t^{7/2} + \frac{2}{9}t^{9/2}\right]_0^1$$

$$= \frac{4}{3}\left[\frac{8}{5} - \frac{8}{7} + \frac{2}{9}\right] = \frac{856}{945}$$

Ex.10 Evaluate $\int_0^{\pi/4} \int_0^{\sqrt{\cos 2\theta}} \frac{r}{\left(1+r^2\right)^2} dr\,d\theta$ **[M.U. 2003]**

Solution: $I = \int_0^{\pi/4} \frac{1}{2}\left[-\frac{1}{1+r^2}\right]_0^{\sqrt{\cos 2\theta}} d\theta$

$$= \frac{1}{2}\int_0^{\pi/4}\left[1 - \frac{1}{1+\cos 2\theta}\right] d\theta = \frac{1}{2}\int_0^{\pi/4}\left(1 - \frac{1}{2}\sec^2\theta\right) d\theta$$

$$= \frac{1}{2}\left[\theta - \frac{1}{2}\tan\theta\right]_0^{\pi/4} = \frac{1}{2}\left(\frac{\pi}{4} - \frac{1}{2}\right) = \frac{1}{8}(\pi - 2)$$

EXERCISE

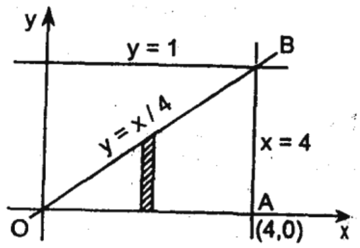
Evaluate the following polar integrals :

- $\displaystyle\int_0^{\pi/4}\int_0^{\sqrt{\cos 2\theta}} \frac{r}{\left(1+r^2\right)^2}\, dr\, d\theta$ **[M.U. 2008]**

Ans. $\dfrac{(\pi - 2)}{8}$

Ex.11 Change the order of integration and evaluate $\displaystyle\int_0^1\int_{4y}^4 e^{x^2}\, dx\, dy$ **[M.U. 1999]**

Solution: The region of integration is $x = 4y$, a line through the origin; $x = 4$, a line parallel to the y-axis; y = 0, the x-axis; and y = 1, the line parallel to the x-axis. Thus, the region is the triangle OAB.



To change the order of integration consider a strip parallel to the y-axis. On this strip y varies from y = 0 to y = x/4. Then, x varies from x = 0 to x = 4.

$$\therefore \quad I = \int_0^4\int_0^{x/4} e^{x^2}\, dy\, dx$$

$$= \int_0^4 e^{x^2}\left[y\right]_0^{x/4} dx = \int_0^4 e^{x^2} . \frac{x}{4}\, dx \qquad [\text{Put } x^2 = t]$$

$$= \frac{1}{4}\left[\frac{e^{x^2}}{2}\right]_0^2 = \frac{1}{8}\left[e^{16} - 1\right]$$

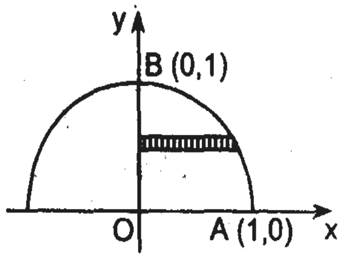
Ex.12 Change the order of integration and evaluate $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}} \frac{e^y}{\left(e^y+1\right)\sqrt{1-x^2-y^2}}\, dy\, dx$

[M.U. 1989, 93, 96, 2002]

Solution: The limits of y are 0 and $\sqrt{1-x^2}$ and those of x are 0 and 1. We therefore, draw the curve $y = \sqrt{1-x^2}$ i.e. the upper-half of the circle $x^2 + y^2 = 1$. The region of integration is OAB.

To change the order of integration we consider a strip parallel to x-axis. Now, x varies from 0 to $\sqrt{1-y^2}$ and y varies from 0 to 1.

$$\therefore \quad I = \int_0^1\int_0^{\sqrt{1-y^2}} \frac{e^y}{\left(e^y+1\right)\sqrt{\left(1-y^2\right)-x^2}}\, dx\, dy$$



$$= \int_0^1 \frac{e^y}{\left(e^y+1\right)}\left[\sin^{-1}\frac{x}{\sqrt{1-y^2}}\right]_0^{\sqrt{1-y^2}} dy$$

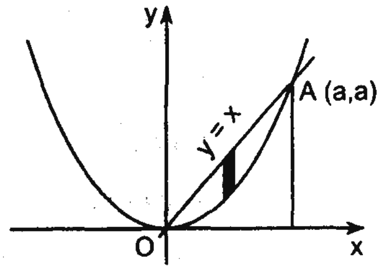
$$= \int_0^1 \frac{e^y}{e^y+1}.\frac{\pi}{2}.dy = \frac{\pi}{2}\left[\log\left(e^y+1\right)\right]_0^1$$

$$= \frac{\pi}{2}\log\left(\frac{e+1}{2}\right)$$

Ex.13 Change the order of integration and evaluate $\int_0^a\int_y^{\sqrt{ay}}\frac{x}{x^2+y^2}\,dx\,dy$ **[M.U. 1998]**

Solution: The region of integration is given by x = y, a line through the origin; $x=\sqrt{ay}$ i.e. $x^2=ay$, a parabola through the origin and opening upwards, y = 0, the x-axis and y = a, a line parallel to the y-axis.

In the region of integration, consider a strip parallel to the y-axis. On this strip y varies from $y=x^2/a$ to y = x and then x varies from x = 0 to x = a.



$$\therefore \quad I = \int_0^a\int_{\frac{x^2}{a}}^{x}\frac{x}{x^2+y^2}\,dy\,dx$$

$$= \int_0^a\left[\frac{x}{x}\tan^{-1}\frac{y}{x}\right]_{x^2/a}^{x} dx$$

$$= \int_0^a\left[\tan^{-1}1-\tan^{-1}\frac{x}{a}\right]dx$$

$$= \int_0^a\left(\frac{\pi}{4}-\tan^{-1}\frac{x}{a}\right)dx$$

$$= \frac{\pi}{4}.[x]_0^a-\left[x.\tan^{-1}\frac{x}{a}-\int x.\frac{1}{1+\left(x^2/a^2\right)}.\frac{1}{a}dx\right]_0^a$$

$$= \frac{\pi a}{4}-\left[x\tan^{-1}\frac{x}{a}-\int\frac{ax}{x^2+a^2}dx\right]_0^a$$

$$= \frac{\pi a}{4}-\left[x\tan^{-1}\frac{x}{a}-\frac{a}{2}\log\left(x^2+a^2\right)\right]_0^a$$

$$= \frac{\pi a}{4}-\left[a\frac{\pi}{4}-\frac{a}{2}\log\left(2a^2\right)-0+\frac{a}{2}\log a^2\right]$$

$$\therefore \quad I = \frac{a}{2}\log 2$$

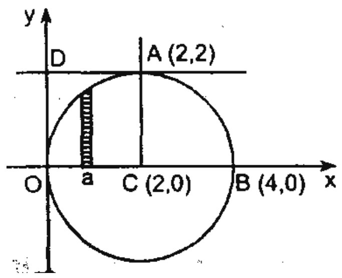
Ex.14 Change the order of integration and evaluate $\int_0^2\int_{2-\sqrt{4-y^2}}^{2+\sqrt{4-y^2}}dx\,dy$ **[M.U. 2002]**

Solution: The limits for x are $2-\sqrt{4-y^2}$ and $2+\sqrt{4-y^2}$ and those for y are 0 and 2.

We therefore, draw the curves $x=2-\sqrt{4-y^2}$ and $x=2+\sqrt{4-y^2}$

i.e. the circle $(x-2)^2+$ which has the centre at (2, 0) radius 2.

Now, $x=2-\sqrt{4-y^2}$ is the arc OA and $x=2+\sqrt{4-y^2}$ is the arc AB.



Since y = 0 is the x-axis and y = 2 is the line AD the region of integration is the semi-circle OAB.

To change the order of integration we consider a strip parallel to the y-axis. On this strip y varies from 0 to $\sqrt{4-(x-2)^2}$. This strip sweeps the region when x varies from 0 to 4.

$$\therefore \quad I=\int_0^4\int_0^{\sqrt{4-(x-2)^2}} dy\,dx$$

$$=\int_0^4 [y]_0^{\sqrt{4-(x-2)^2}}\,dx$$

$$=\int_0^4 \sqrt{4-(x-2)^2}.dx$$

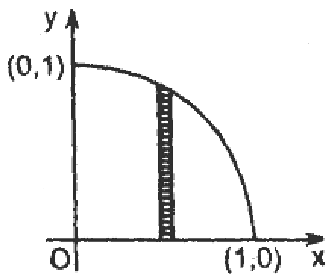
$$=\left[\frac{(x-2)}{2}\sqrt{4-(x-2)^2}+\frac{4}{2}\sin^{-1}\frac{(x-2)}{2}\right]_0^4$$

$$=\left(2.\frac{\pi}{2}\right)-\left(-2.\frac{\pi}{2}\right)=2\pi$$

Ex.15 Change the order of integration and evaluate $\int_0^1\int_0^{\sqrt{1-y^2}}\frac{\cos^{-1}x}{\sqrt{1-x^2}\sqrt{1-x^2-y^2}}\,dx\,dy$

[M.U. 1992, 99, 2007]

Solution: The limits of y are 0 and 1 and for x are 0 and $x=\sqrt{1-y^2}$ i.e. $x^2+y^2=1$. Hence, the region of integration is the first quadrant of the circle $x^2+y^2=1$



Now, if we change the order of integration y varies from 0 to $\sqrt{1-x^2}$ and x varies from 0 to 1. Hence,

$$I=\int_0^1\int_0^{\sqrt{1-x^2}}\frac{\cos^{-1}x}{\sqrt{1-x^2}\sqrt{1-x^2-y^2}}\,dy\,dx$$

$$= \int_0^1 \frac{\cos^{-1} x}{\sqrt{1-x^2}} \left[\sin^{-1} \frac{y}{\sqrt{1-x^2}} \right]_0^{\sqrt{1-x^2}} dx$$

$$I = \frac{\pi}{2} \int_0^1 \frac{\cos^{-1} x}{\sqrt{1-x^2}}$$

Put $\cos^{-1} x = t . \dfrac{dx}{\sqrt{1-x^2}} = -dt$

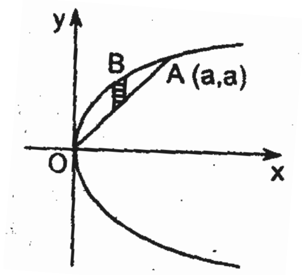
$$\therefore \quad I = -\frac{\pi}{2} \int_{\pi/2}^{0} t\, dt = \frac{\pi}{2} \int_0^{\pi/2} t\, dt$$

$$= \frac{\pi}{2} \left[\frac{t^2}{2} \right]_0^{\pi/2} = \frac{\pi^3}{16}$$

Ex.16 Change the order of integration and evaluate $\displaystyle\int_0^a \int_{y^2/a}^{y} \frac{y\, dx\, dy}{(a-x)\sqrt{ax-y^2}}$

[M.U. 1991, 98]

Solution: In the region x varies from y^2/a (i.e. $y^2 = ax$) to x = y and y varies from 0 to a. Thus, the region is OAB bounded by the line x = y and the arc of the parabola $y^2 = ax$. When the order is reversed y varies from x to $\sqrt{ax}$ and x varies from 0 to a.



$$\therefore \quad I = \int_0^a \int_x^{\sqrt{ax}} \frac{y\, dy\, dx}{(a-x)\sqrt{ax-y^2}}$$

$$= -\int_0^a \frac{1}{(a-x)} \left[\sqrt{ax} - y^2 \right]_x^{\sqrt{ax}} dx$$

$$= \int_0^a \frac{1}{(a-x)} . \sqrt{x} . \sqrt{a-x} dx$$

$$= \int_0^a \sqrt{\left(\frac{x}{a-x} \right)} dx$$

Now, put $x = a\cos^2 \theta$

$$\therefore \quad dx = -2a \cos\theta \sin\theta\, d\theta$$

$$\therefore \quad I = \int_0^{\pi/2} \frac{\cos\theta}{\sin\theta} . 2a \sin\theta \cos\theta\, d\theta$$

$$= a \int_0^{\pi/2} 2\cos^2 \theta\, d\theta$$

$$= a \int_0^{\pi/2} (1 + \cos 2\theta)\, d\theta$$

$$= a \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\pi/2} = \frac{\pi}{2} . a$$

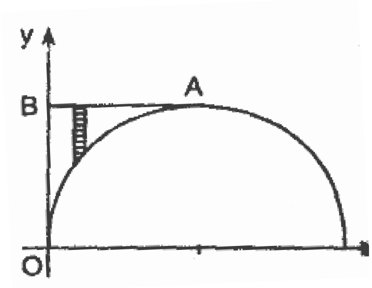
Ex.17 Evaluate $\int_0^a dy \int_0^{a-\sqrt{a^2-y^2}} \frac{xy\log(x+a)}{(x-a)^2}dx$ **[M.U. 1994, 2000, 09]**

Solution: In the given form, the integral is to be evaluated w.r.t. x first, which obviously is complicated. We, therefore, change the order of integration.

The limits for x are 0 and $a-\sqrt{a^2-y^2}$ and for y are 0 and a. We therefore, draw the curves x = 0 i.e. the y-axis and $x = a-\sqrt{a^2-y^2}$ i.e. the left half of the circle $(x-a)^2+y^2=a^2$.

It is clear that A is (a, a) and B is (0, a). The region of integration is OAB.

Now, to change the order of integration if we consider a strip parallel to the y-axis, y varies from $\sqrt{a^2-(x-a)^2}$ i.e. $\sqrt{2ax-x^2}$ to a and x varies from 0 to a.



$$\therefore \quad I = \int_0^a dx \int_{\sqrt{2ax-x^2}}^a \frac{xy\log(x+a)}{(x-a)^2}dy$$

$$= \int_0^a \frac{x\log(x+a)}{(x-a)^2}\left[\frac{y^2}{2}\right]_{\sqrt{2ax-x^2}}^a dx$$

$$= \int_0^a \frac{x\log(x+a)}{2(x-a)^2}\left[a^2-2ax+x^2\right]dx$$

$$= \frac{1}{2}\int_0^a x\log(x+a)\,dx$$

$$= \frac{1}{2}\left[\log(x+a).\frac{x^2}{2} - \int \frac{x^2}{2(x-a)}dx\right]_0^a$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\log(x+a) - \frac{1}{2}\int \frac{\left(x^2-a^2\right)+a^2}{(x+a)}dx\right]_0^a$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\log(x+a) - \frac{1}{2}\int(x-a) - \frac{a^2}{2}\int\frac{dx}{x+a}\right]_0^a$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\log(x+a) - \frac{1}{2}\left(\frac{x^2}{2}-ax\right) - \frac{a^2}{2}\log(x+a)\right]_0^a$$

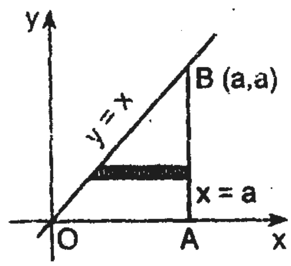
$$= \frac{1}{2}\left[\frac{a^2}{2}\log(a+a) - \frac{1}{2}\left(\frac{a^2}{2}-a^2\right) - \frac{a^2}{2}\log(a+a) + \frac{a^2}{2}\log a\right]$$

$$= \frac{a^2}{8}\left[1+2\log a\right]$$

Ex.18 Evaluate $\int_0^a \int_0^x \frac{e^y}{\sqrt{(a-x)(x-y)}}\, dy\, dx$ **[M.U. 1995, 2005, 10]**

Solution: Since integration with respect to y is complicated we change the order of integration.

The limits for y are y = 0 to y = x and for x are x = 0 to x = a. The region of integration is the triangle OAB.



Now, consider a strip parallel to the x-axis. On this strip x varies from x = y to x = a and for the strip y varies from y = 0 to y = a.

$$\therefore \quad I = \int_0^a \int_y^a \frac{e^y}{\sqrt{(a-x)(x-y)}}\, dx\, dy$$

$$= \int_0^a \int_y^a \frac{e^y}{\sqrt{-ay-\left[x^2(a+y)x\right]}}\, dx\, dy$$

$$= \int_0^a \int_y^a \frac{e^y}{\sqrt{\left(\frac{a-y}{2}\right)^2 - \left(x-\frac{a+y}{2}\right)^2}}\, dx\, dy$$

$$= \int_0^a e^y \left[\sin^{-1}\left\{\frac{x-(a+y)/2}{(a-y)/2}\right\}\right]_y^a dy$$

$$= \int_0^a e^y \left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] dy = \pi \int_0^a e^y\, dy$$

$$= \pi\left[e^y\right]_0^a = \pi\left(e^a - 1\right)$$

Ex.19 Change the order of integration and evaluate $\int_0^a \int_0^x \frac{dy\, dx}{(y+a)\sqrt{(a-x)(x-y)}}$

[M.U. 1995, 99, 2003]

Solution: The given region of integration is the same as above Ex. 18.

$$\therefore \quad I = \int_0^a \int_0^{\sqrt{a-y}} \frac{dy}{(y+a)} \cdot \frac{2t\, dt}{\sqrt{\left(a-y-t^2\right).t}}$$

$$= 2\int_0^a \int_0^{\sqrt{a-y}} \frac{dy}{(y+a)} \cdot \frac{dt}{\sqrt{(a-y)-t^2}}$$

$$= 2\int_0^a \frac{dy}{(y+a)} \left[\sin^{-1} \frac{t}{\sqrt{a-y}}\right]_0^{\sqrt{a-y}}$$

$$= 2\int_0^a \frac{dy}{y+a} \cdot \frac{\pi}{2} = \pi \int_0^a \frac{dy}{y+a}$$

$$= \pi \left[\log(y+a) \right]_0^a$$

$$= \pi \left[\log 2a - \log a \right] = \pi \log 2$$

Ex.20 Change the order of integration and evaluate $\int_0^\pi \int_0^x \frac{\sin y \, dy \, dx}{\sqrt{(\pi - x)(x-y)}}$ **[M.U. 1997]**

Solution: The region of integration is same as above Ex. 18. Hence,

$$\therefore \quad I = \int_0^\pi \int_y^\pi \frac{\sin y \, dx \, dy}{\sqrt{(\pi - x)(x-y)}}$$

Put $x - y = t^2 \quad \therefore \quad dx = 2t\,dt$

When $x = y, t = 0$; when $x = \pi, t = \sqrt{\pi - y}$

$$\therefore \quad I = \int_0^\pi \int_0^{\sqrt{\pi - y}} \sin y . \frac{2t\,dt}{\sqrt{(\pi - y) - t^2}.t}$$

$$= 2\int_0^\pi \int_0^{\sqrt{\pi - y}} \sin y . \frac{dt}{\sqrt{(\pi - y) - t^2}} dy$$

$$= 2\int_0^\pi \sin y \left[\sin^{-1}\left(\frac{t}{\sqrt{\pi - y}} \right) \right]_0^{\sqrt{\pi - y}} dy$$

$$= 2\int_0^\pi \frac{\pi}{2} \sin y \, dy = \pi \left[-\cos y \right]_0^\pi = 2\pi$$

Ex.21 Change the order of integration and evaluate $\int_0^a \int_0^y \frac{x \, dx \, dy}{\sqrt{(a^2 - x^2)(a-y)(y-x)}}$

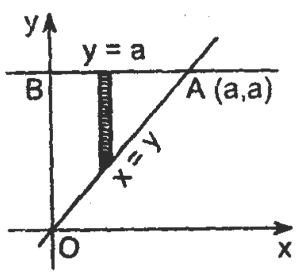
[M.U. 1997, 2002, 04]

Solution: The region of integration is bounded by x = 0, the y-axis, x = y, the line, y = 0, the x-axis and x = a, a line parallel to the x-axis.

To change the order of integration consider a strip to the y-axis.

On the strip y varies from y = x to y = a and then x varies from x = 0 to x = a.

$$\therefore \quad I = \int_0^a \int_x^a \frac{x \, dy \, dx}{\sqrt{a^2 - x^2}\sqrt{(a-y)(y-x)}}$$



Put $y - x = t^2 \quad \therefore \quad dy = 2t\,dt$

When $y = x, t = 0$; when $y = a, t = \sqrt{a - x}$

$$\therefore \quad I = \int_0^a \int_0^{\sqrt{a-x}} \frac{x\,dx}{\sqrt{a^2 - x^2}} \frac{2t\,dt}{\sqrt{(a-x) - t^2}.t}$$

$$= 2\int_0^a \frac{x\,dx}{\sqrt{a^2 - x^2}} \left[\sin^{-1}\left(\frac{t}{\sqrt{a-x}} \right) \right]_0^{\sqrt{a-x}}$$

$$= 2\int_0^a \frac{x}{\sqrt{a^2-x^2}}\frac{\pi}{2}dx = \pi\left[-\sqrt{a^2-x^2}\right]_0^a$$

$$= \pi[-0+a] = \pi a$$

Ex.22 Change the order of integration and evaluate $\int_0^a\int_0^x \frac{\sin y\,dy\,dx}{\sqrt{[(a-x)(x-y)]}(4-5\cos y)^2}$

[M.U. 1996, 2002]

Solution: The region of integration is the same as above Ex. 18.

$$\therefore \quad I = \int_0^a\int_y^a \frac{\sin y}{(4-5\cos y)}\cdot\frac{dx}{\sqrt{[(a-x)(x-y)]}}dy$$

Put $x-y=t^2 \quad \therefore \quad dx = 2t\,dt$

When $x=y, t=0;$ when $x=a, t=\sqrt{a-y}$

$$\therefore \quad I = \int_0^a\int_0^{\sqrt{a-y}} \frac{\sin y\,dy}{(4-5\cos y)}\cdot\frac{2t\,dt}{\sqrt{(a-y)-t^2}.t}$$

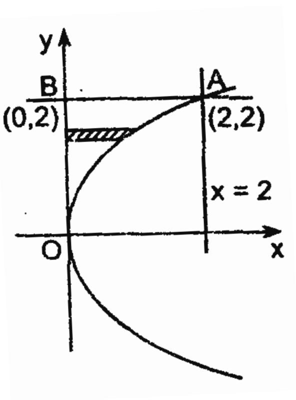
$$= 2\int_0^a \frac{\sin y}{4-5\cos y}\cdot\left[\sin^{-1}\left(\frac{t}{\sqrt{a-y}}\right)\right]_0^{\sqrt{a-y}} dy$$

$$= \pi\int_0^a \frac{\sin y}{4-5\cos y}dy = \pi\left[-\frac{1}{5}\log(4-5\cos y)\right]_0^a$$

$$= \frac{\pi}{5}[-\log(4-5\cos a)+0] = -\frac{\pi}{5}\log(4-5\cos a)$$

Ex.23 Change the order of integration and evaluate $\int_0^2\int_{\sqrt{2x}}^2 \frac{y^2\,dy\,dx}{\sqrt{y^4-4x^2}}$ **[M.U. 1995]**

Solution: Here, the region of integration is bounded by $y=\sqrt{2x}$ i.e. $y^2=2x$ a parabola; y = 2, a line parallel to the x-axis; x = 0, the y-axis and x = 2, the line parallel to the y-axis.



If we consider a strip parallel to the x-axis, on this strip x varies x = 0 to $x=y^2/2$ and then y varies from y = 0 to y = 2.

$$\therefore \quad I = \int_0^2\int_0^{y^2/2} \frac{y^2\,dx\,dy}{\sqrt{y^4-4x^2}}$$

$$= \frac{1}{2}\int_0^2\int_0^{y^2/2} \frac{y^2}{\sqrt{(y^2/2)^2-x^2}}dx\,dy$$

$$= \frac{1}{2}\int_0^2 y^2\left[\sin^{-1}\left(\frac{x}{y^2/2}\right)\right]_0^{y^2/2} dy$$

$$= \frac{1}{2}\int_0^2 y^2\left[\sin^{-1}1 - \sin^{-1}0\right]dy$$

$$\therefore \qquad I = \frac{1}{2}\int_0^2 \frac{\pi}{2}.y^2 dy = \frac{\pi}{4}\left[\frac{y^3}{3}\right]_0^2 = \frac{2\pi}{3}$$

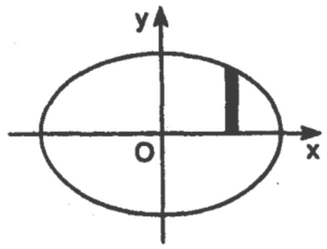
Ex.24 Change the order of integration and evaluate $\int_0^{1/2}\int_0^{\sqrt{1-4y^2}} \frac{1+x^2}{\sqrt{1-x^2}\sqrt{1-x^2-y^2}}\,dx\,dy$

[M.U. 1998, 2001]

Solution: Here, the region of integration is bounded by x = 0 i.e. the y-axis.

$$x = \sqrt{1-4y^2} \qquad \therefore \qquad x^2 = 1 - 4y^2$$

$$\therefore \qquad x^2 + \frac{y^2}{(1/4)} = 1$$



It is an ellipse with semi-major axis 1 and semi-minor axis 1/2, y = 0 and y = 1/2. Thus, the region of integration is the first quadrant of the above ellipse.

To change the order of integration, consider a strip parallel to the y-axis. On this strip y varies from y = 0 to $y = \frac{\sqrt{1-x^2}}{2}$. Then x varies from 0 to 1.

$$\therefore \qquad I = \int_0^1\int_0^{\frac{\sqrt{1-x^2}}{2}} \frac{1+x^2}{\sqrt{1-x^2}}.\frac{dy}{\sqrt{\left(1-x^2\right)-y^2}}dx$$

$$= \int_0^1 \frac{1+x^2}{\sqrt{1-x^2}}.\left[\sin^{-1}\left(\frac{y}{\sqrt{1-x^2}}\right)\right]_0^{\sqrt{1-x^2}/2} dx$$

$$= \int_0^1 \frac{1+x^2}{\sqrt{1-x^2}}.\left[\sin^{-1}\frac{1}{2} - \sin^{-1}0\right]dx$$

$$= \frac{\pi}{6}\int_0^1 \frac{1+x^2}{\sqrt{1-x^2}}dx$$

To find the integral, put $x = \sin\theta, dx = \cos\theta\, d\theta$

$$\therefore \qquad I = \frac{\pi}{6}\int_0^{\pi/2} \frac{1+\sin^2\theta}{\cos\theta}.\cos\theta\, d\theta$$

$$= \frac{\pi}{6}\int_0^{\pi/2}\left(1+\sin^2\theta\right)d\theta$$

$$= \frac{\pi}{6}\left[\{\theta\}_0^{\pi/2} + \frac{1}{2}.\frac{\pi}{2}\right] = \frac{\pi}{6}\left[\frac{\pi}{2}+\frac{\pi}{4}\right] = \frac{\pi^2}{8}$$

Ex.25 Change the order of integration and evaluate $\int_{x=0}^{1} dx \int_{y=1}^{\infty} e^{-y} y^x \log y \, dy$

[M.U. 1997, 2005]

Solution: Since all the limits of integration are constants, the order can be changed directly without taking the help of a diagram.

$$\therefore \quad I = \int_{y=1}^{\infty} e^{-y} \log y \, dy . \int_{x=0}^{1} y^x dx$$

$$= \int_{y=1}^{\infty} e^{-y} \log y \left[\frac{y^x}{\log y} \right]_0^1 dy = \int_{y=1}^{\infty} e^{-y} (y-1) \, dy$$

$$= \int_1^{\infty} \left(ye^{-y} - e^{-y} \right) dy = \left[-ye^{-y} - e^{-y} + e^{-y} \right]_1^{\infty}$$

$$= \left[-ye^{-y} \right]_1^{\infty} = e^{-1} = \frac{1}{e}$$

Ex.26 Change the order of integration and evaluate $\int_0^5 \int_{2-x}^{2+x} dy \, dx$ **[M.U. 1997, 2008]**

Solution: The region of integration is ABC. When the order of integration is changed the region splits into two.

In the region ABD, x varies from 2 – y to 5 and y varies from –3 to 2. In the region ADC, x varies from y – 2 to 5 and y varies from 2 to 7. Hence,

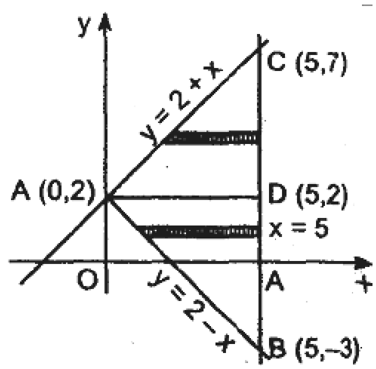
$$\therefore \quad I = \int_{-3}^{2} \int_{2-y}^{5} dx \, dy + \int_2^7 \int_{y-2}^{5} dx \, dy$$

$$= \int_{-3}^{2} [x]_{2-y}^{5} dy + \int_2^7 [x]_{y-2}^{5} dy$$

$$= \int_{-3}^{2} (3+y) \, dy + \int_2^7 (7-y) \, dy$$

$$= \left[3y + \frac{y^2}{2} \right]_{-3}^{2} + \left[7y - \frac{y^2}{2} \right]_2^7$$

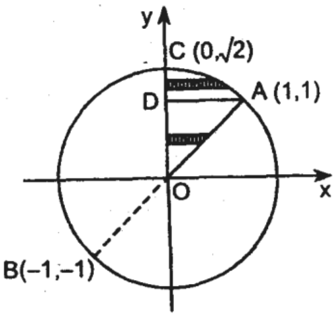
$$= \left(17 - \frac{9}{2} \right) + \left(37 - \frac{49}{2} \right) = 25$$



Ex.27 Change the order of integration and evaluate $\int_0^1 dx \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}} dy$

[M.U. 2003, 07, 08]

Solution: The limits for y are x and $\sqrt{2-x^2}$ and those for x are 0 and 1. We therefore, draw the curves y = x which is a straight line and $y = \sqrt{2-x^2}$ which is the upper half of the circle $x^2 + y^2 = 2$. The region of integration is OACD. Solving the equations y = x and $x^2 + y^2 = 2$, we get the points of intersection A(1, 1) and B(–1, –1).



If we consider a strip parallel to the x-axis the region has to be divided into two parts OAD and ADC.

In the region ODA, x varies from 0 to y and y varies from 0 to 1.

In the region ADC, x varies from 0 to $\sqrt{2-y^2}$ and y varies from 1 to $\sqrt{2}$.

$$\therefore \quad I=\int_0^1 dy\int_0^y \frac{x}{\sqrt{x^2+y^2}}\,dx+\int_1^{\sqrt{2}} dy\int_0^{\sqrt{2-y^2}} \frac{x}{\sqrt{x^2+y^2}}\,dx$$

Now, $I_1=\int_0^1 dy\left[\sqrt{x^2+y^2}\right]_0^y=\int_0^1\left(\sqrt{2}.y-y\right)dy$

$$=\left(\sqrt{2}-1\right)\left[\frac{y^2}{2}\right]_0^1=\frac{1}{2}\left(\sqrt{2}-1\right)$$

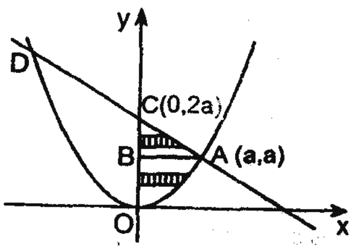
And $I_2=\int_1^{\sqrt{2}} dy\left[\sqrt{x^2+y^2}\right]_0^{\sqrt{2-y^2}}=\int_1^{\sqrt{2}}\left(\sqrt{2}-y\right)dy$

$$\therefore \quad I_2=\left[\sqrt{2}y-\frac{y^2}{2}\right]_1^{\sqrt{2}}=\frac{3}{2}-\sqrt{2}$$

$$\therefore \quad I=I_1+I_2=1-\frac{1}{\sqrt{2}}$$

Ex.28 Change the order of integration and evaluate $\int_0^a\int_{x^2/a}^{2a-x} xy\,dy\,dx$ **[M.U. 1992]**

Solution: The limits for y are x^2/a and $2a-x$ and those for x are 0 and a. We, therefore, draw the curves $y=x^2/a$. i.e. the parabola $x^2=ay$ and $y=2a-x$ i.e. the straight line $x+y=2a$.



The region of integration is OACB. Solving the equations $x^2=ay$ and $x+y=2a$ we get the points of intersection as A (a, a) and D(–2a, 4a).

Now, to change the order if we consider a strip parallel to the x-axis, the region has to be divided into two parts OAB and BCA.

In the region OBA, x varies from 0 to $\sqrt{ay}$ and y varies from 0 to a.

In the region BAC, x varies from 0 to 2a – y and y varies from a to 2a.

$$\therefore \quad I=\int_0^a\int_0^{\sqrt{ay}} xy\,dx\,dy+\int_a^{2a}\int_0^{2a-y} xy\,dx\,dy$$

Now, $I_1=\int_0^a y\left[\frac{x^2}{2}\right]_0^{\sqrt{ay}} dy=\int_0^a \frac{a}{2}y^2dy=\frac{a}{2}\left[\frac{y^3}{3}\right]_0^a=\frac{a^4}{6}$

$$I_2 = \int_a^{2a} y\left[\frac{x^2}{2}\right]_0^{2a-y} dy = \int_a^{2a} \frac{1}{2}\left(4a^2y - 4ay^2 + y^3\right)dy$$

$$= \frac{1}{2}\left[2a^2y^2 - \frac{4a}{3}y^3 + \frac{y^4}{4}\right]_a^{2a}$$

$$= \frac{a^4}{2}\left[\left(8 - \frac{32}{3} + 4\right) - \left(2 - \frac{4}{3} + \frac{1}{4}\right)\right]$$

$$= \frac{a^4}{2}\left(\frac{4}{3} - \frac{11}{12}\right) = \frac{a^4}{2}\cdot\frac{15}{36} = \frac{5a^4}{24}$$

$$\therefore \quad I = \frac{a^4}{6} - \frac{5a^4}{24} = \frac{3}{8}a^4$$

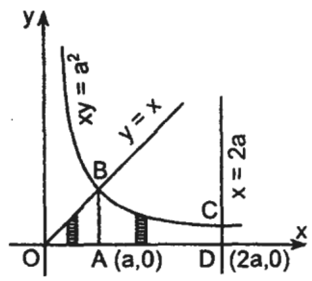
Ex.29 Change the order of integration and evaluate $\iint\limits_R x^2 dx\, dy$ where T is the region in the first quadrant bounded by $xy = a^2, x = 2a, y = 0$ and $y = x$ **[M.U. 1994]**

Solution: The region of integration is bounded by $xy = a^2$, a rectangular hyperbola; x = 2a, a line parallel to the y-axis; y = 0, the x-axis and y = x, a line passing through the origin. The region is OBCDO.

If we change the order of integration, the region is split into two parts. OAB and ABCD.

Now, consider a strip in the region OAB, parallel to the y-axis. On this strip y-varies from y = 0 to y = x. Then x varies from x = 0 to x = a.

Also consider a strip in the region ABCD, parallel to the y-axis. On this strip, y varies from y = 0 to $y = a^2/x$. Then x varies from x = a to x = 2a.



$$\therefore \quad I = \int_0^a \int_0^x x^2 dy\, dx + \int_a^{2a} \int_0^{a^2/x} x^2 dy\, dx$$

$$= \int_0^a x^2 [y]_0^x dx + \int_a^{2a} x^2 [y]_0^{a^2/x} dx$$

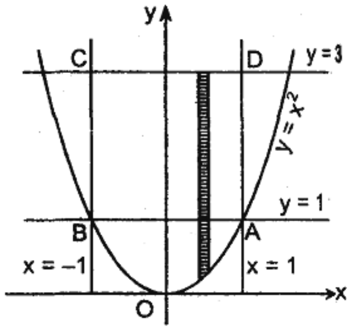
$$\therefore \quad I = \int_0^a x^3 dx + \int_a^{2a} a^2 x dx = \left[\frac{x^4}{4}\right]_0^a + \left[\frac{a^2x^2}{2}\right]_a^{2a}$$

$$= \frac{a^4}{4} + 2a^4 - \frac{a^4}{2} = \frac{7a^4}{4}$$

Ex.30 Express as a single integral and evaluate $I = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx\, dy + \int_1^3 dy \int_{-1}^1 dx$ **[M.U. 2000]**

Solution: Let $I = I_1 + I_2$

Now, for I_1, the limits are $x=-\sqrt{y}$ and $x=\sqrt{y}$ i.e. $x^2=y$, a parabola with vertex at the origin and opening upwards. The limits for y are y = 0 to y = 1. **The region is OAB.** For I_2, x varies from x = –1 to x = 1 and y varies from y = 1 to y = 3. **The region is ABCD.** We have to sweep both the regions i.e. the region OABDCBO. Now, consider a strip parallel to the x-axis extending from the parabola to the line CD. On this strip y varies from $y=x^2$ to y = 3. To sweep the whole area the strip has to move from x = –1 to x = 1.



$$\therefore \qquad I=\int_{-1}^{1}\int_{x^2}^{3} dy\,dx=\int_{-1}^{1}[y]_{x^2}^{3}\,dx$$

$$-\int_{-1}^{1}\left[3-x^2\right]dx=2\int_{0}^{1}\left(3-x^2\right)dx \qquad [\because \text{ Even function}]$$

$$=2\left[3x-\frac{x^3}{3}\right]_0^1=2\left[3-\frac{1}{3}\right]=\frac{16}{3}$$

EXERCISE

Change the order of integration and evaluate :

- $\int_0^{\infty}\int_0^{x} xe^{-x^2/y}\,dy\,dx$ **[M.U. 2009, 11]**

Ans. $\frac{1}{2}$

- $\int_0^{1}\int_x^{2-x} \frac{x}{y}\,dy\,dx$ **[M.U. 1988]**

Ans. $\log\left(\frac{4}{e}\right)$

- $\int_0^{2}\int_{\sqrt{2y}}^{2} \frac{x^2 dx\,dy}{\sqrt{x^4-4y^2}}$ **[M.U. 1995]**

Ans. $\frac{2\pi}{3}$

- $\int_0^{1}\int_x^{1/x} \frac{y}{(1+xy)^2\left(1+y^2\right)}\,dy\,dx$ **[M.U. 2000]**

Ans. $\frac{\pi-1}{4}$

- $\int_0^{1}\int_{x^2}^{2-x} \frac{x}{y}\,dy\,dx$ **[M.U. 1988]**

Ans. $2\log 2-\frac{3}{4}$

- $\int_0^a \int_0^y \frac{dx\,dy}{\sqrt{\left(a^2+x^2\right)(a-y)(y-x)}}$ **[M.U. 1997, 2003, 04]**

Ans. $\pi.\log\left(1+\sqrt{2}\right)$

- $\int_0^{\pi/2} \int_0^y \cos 2y.\sqrt{1-a^2\sin^2 x}.dx\,dy$ **[M.U. 2003]**

Ans. $\frac{1}{3a^2}\left[\left(1-a^2\right)^{3/2}-1\right]$

- Express as a single integral and them evaluate $\int_0^1 \int_0^y \left(x^2+y^2\right)dx\,dy+\int_1^2 \int_0^{2-y}\left(x^2+y^2\right)dx\,dy$ **[M.U. 2004]**

Ans. $\frac{4}{3}$

Ex.31 Change to polar co-ordinates and evaluate $\int_0^a \int_y^a \frac{x^2}{\sqrt{x^2+y^2}}dx\,dy$ **[M.U. 1997, 2006]**

Solution: The region of integration is the same as Ex. 38 but with $x = a$ intersecting $x = y$ at (a,a) instead of $x = 1$. Hence, putting $x = r\cos\theta, y = r\sin\theta, dx\,dy = r\,dr\,d\theta$, we get

$$I=\int_0^{\pi/4}\int_0^{a/\cos\theta}\frac{r^2\cos^2\theta}{r}.r\,d\,r\,d\theta$$

$$=\int_0^{\pi/4}\int_0^{a/\cos\theta} r^2\cos^2\theta\,dr\,d\theta$$

$$\therefore \quad I=\int_0^{\pi/4}\left[\frac{r^3}{3}\right]_0^{a/\cos\theta}.\cos^2\theta\,d\theta$$

$$=\int_0^{\pi/4}\frac{a^3}{3}.\frac{1}{\cos^3\theta}.\cos^2\theta\,d\theta$$

$$=\frac{a^3}{3}\int_0^{\pi/4}\sec\theta\,d\theta$$

$$=\frac{a^3}{3}\left[\log(\sec\theta+\tan\theta)\right]_0^{\pi/4}$$

$$=\frac{a^3}{3}\left[\log\left(\sqrt{2}+1\right)-\log 1\right]$$

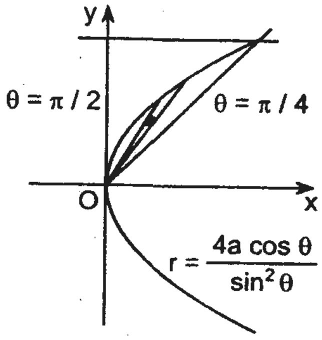
$$=\frac{a^3}{3}\log\left(1+\sqrt{2}\right)$$

Ex.32 Express in polar coordinates and evaluate $\int_0^{4a}\int_{y^2/4a}^{y}\left(\frac{x^2-y^2}{x^2+y^2}\right)dxdy$

[M.U. 1994, 96, 99, 2006]

Solution: The limits of x are $y^2 = 4ax$ and x = y. Putting $x = r\cos\theta, y = r\sin\theta, y^2 = 4ax$ gives $r^2\sin^2\theta = 4ar\cos\theta$ i.e. $r = 4a\cos\theta/\sin^2\theta$ and $y = x$ gives $r\sin\theta = r\cos\theta$ i.e. $\theta = \pi/4$. Further, $\left(x^2-y^2\right)/\left(x^2+y^2\right) = \cos^2\theta - \sin^2\theta$ and dx dy changes to $r\,d\theta\,dr$.

In the given region bounded by the arc of the parabola and the line, r varies from 0 to $4a\cos\theta/\sin^2\theta$ and θ varies from $\pi/4$ to $\pi/2$.



$$\therefore \quad I = \int_{\pi/4}^{\pi/2}\int_0^{4a\cos\theta/\sin^2\theta}\left(\cos^2\theta - \sin^2\theta\right) r\,dr\,d\theta$$

$$= \int_{\pi/4}^{\pi/2}\left(\cos^2\theta - \sin^2\theta\right)\left[\frac{r^2}{2}\right]_0^{4a\cos\theta/\sin^2\theta} d\theta$$

$$= 8a^2\int_{\pi/4}^{\pi/2}\left(\cos^2\theta - \sin^2\theta\right)\frac{\cos^2\theta}{\sin^4\theta}\,d\theta$$

$$= 8a^2\int_{\pi/4}^{\pi/2}\left(\cos^4\theta - \cot^2\theta\right)d\theta$$

$$= 8a^2\int_{\pi/4}^{\pi/2}\left(\cot^2\theta\,\mathrm{cosec}^2\theta - 2\,\mathrm{cosec}^2\theta + 2\right)d\theta$$

$$= 8a^2\left[-\frac{\cot^3\theta}{3} + 2\cot\theta + 2\theta\right]_{\pi/4}^{\pi/2}$$

$$= 8a^2\left[\frac{\pi}{2} - \frac{5}{3}\right]$$

Ex.33 Express the following integral in polar coordinates and evaluate

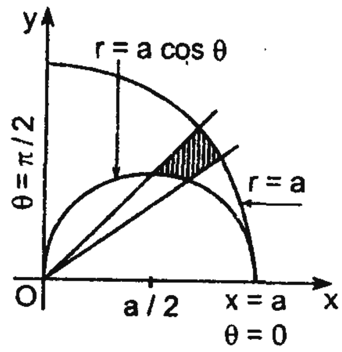
$$\int_0^a\int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}}\frac{dy\,dx}{\sqrt{a^2-x^2-y^2}}$$ **[M.U. 1989, 92, 2000, 04]**

Solution: The limits of y are $\sqrt{ax-x^2}$ and $\sqrt{a^2-x^2}$ i.e. the upper halves of the circles

i) $x^2 + y^2 - ax = 0;$

$\left[x - (a/2)\right]^2 + y^2 = a^2/4$ and

ii) $x^2 + y^2 = a^2$



To change the given integral to polar coordinates we put $x = r\cos\theta, y = r\sin\theta$ and $dx\,dy = r\,d\theta\,dr$. The equations of the circles now become

i) $r^2 - ar\cos\theta = 0$ i.e. $r = a\cos\theta$

ii) $r^2 = a^2$ i.e. $r = a$

Hence, r changes from $a\cos\theta$ to a and θ changes from 0 to $\pi/2$

$$\therefore \qquad I=\int_0^{\pi/2}\int_{a\cos\theta}^{a}\frac{r\,dr\,d\theta}{\sqrt{a^2-r^2}}$$

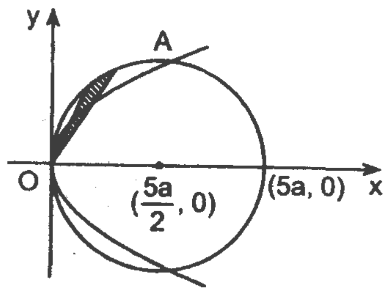
$$=\int_0^{\pi/2}\left[-\sqrt{a^2-r^2}\right]_{a\cos\theta}^{a}d\theta$$

$$=\int_0^{\pi/2}a\sin\theta\,d\theta=[-a\cos\theta]_0^{\pi/2}=a$$

Ex.34 Evaluate $\int_0^a\int_{2\sqrt{ax}}^{\sqrt{5ax-x^2}}\frac{\sqrt{x^2+y^2}}{y^2}dy\,dx$ **[M.U. 2004]**

Solution: The region of integration is bounded by the parabola $y^2=4ax$ and the circle $\left(x-\frac{5a}{2}\right)^2+y^2=\left(\frac{5a}{2}\right)^2$. The point of intersection of the circle and the parabola is (a, 2a).

Putting $x=r\cos\theta, y=\sin\theta$, the parabola $y^2=4ax$ becomes $r^2\sin^2\theta=4a\cos\theta$ i.e. $r=4a\cos\theta/\sin^2\theta$ and the circle $y^2=5ax-x^2$ i.e. $x^2+y^2=5ax$ becomes $r^2=5ar\cos\theta$ i.e. $r=5a\cos\theta$. Hence, r varies from $4a\cos\theta/\sin^2\theta$ to $5a\cos\theta$.



Now, at $A(a,2a), x=r\cos\theta=a,$ and $y=r\sin\theta=2a$

$\therefore \quad \tan\theta=2 \quad \therefore \quad \theta=\tan^{-1}2$ and at 0, $\theta=\pi/2$

$$\therefore \qquad I=\int_{\tan^{-1}2}^{\pi/2}\int_{\frac{4a\cos\theta}{\sin^2\theta}}^{5a\cos\theta}\frac{r}{r^2\sin^2\theta}r\,dr\,d\theta$$

$$=\int_{\tan^{-1}2}^{\pi/2}[r]_{4a\cos\theta/\sin^2\theta}^{5a\cos\theta}\frac{d\theta}{\sin^2\theta}$$

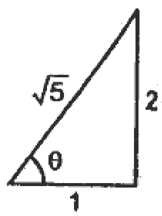
$$=\int_{\tan^{-1}2}^{\pi/2}\left[5a\cos\theta-\frac{4a\cos\theta}{\sin^2\theta}\right]\frac{d\theta}{\sin^2\theta}$$

$$=\int_{\tan^{-1}2}^{\pi/2}\left[5a\frac{\cos\theta}{\sin^2\theta}-\frac{4a\cos\theta}{\sin^4\theta}\right]d\theta$$

Now, put $\sin\theta=t$.

$$\therefore \qquad I=5a\int_{2/\sqrt{5}}^{1}\frac{dt}{t^2}-4a\int_{2/\sqrt{5}}^{1}\frac{dt}{t^4}$$

$$=5a\left[-\frac{1}{t}\right]_{2/\sqrt{5}}^{1}+\frac{4a}{3}\left[\frac{1}{t^3}\right]_{2/\sqrt{5}}^{1}$$

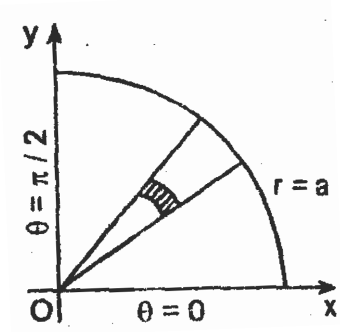


$$= 5a\left[\frac{\sqrt{5}}{2} - 1\right] + \frac{4a}{3}\left[1 - \frac{5\sqrt{5}}{8}\right] = \frac{a}{3}\left(5\sqrt{5} - 11\right)$$

Ex.35 Change into polar coordinates and evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} e^{-\left(x^2+y^2\right)} dy\, dx$

[M.U. 1990, 2002]

Solution: Putting $x = r\cos\theta, y = r\sin\theta$ the given limit $y^2 = a^2 - x^2$ i.e. the circle $x^2 + y^2 = a^2$, changes to $r = a$ and $y = 0$ i.e. the x-axis changes to the initial line $\theta = 0$; $x = 0$, the y-axis becomes $\theta = \pi/2$. Hence, in the given region r changes from 0 to a and θ changes from 0 to $\pi/2$.



$$\therefore \quad I = \int_0^{\pi/2} \int_0^a e^{-r^2} r\, dr\, d\theta$$

$$= \int_0^{\pi/2} \left[-\frac{1}{2} e^{-r^2}\right]_0^a d\theta$$

$$= -\frac{1}{2} \int_0^{\pi/2} \left(e^{-a^2} - 1\right) d\theta$$

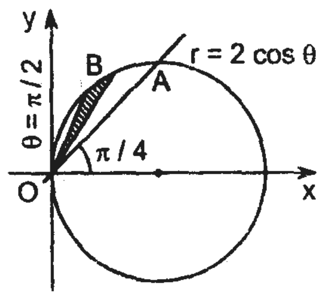
$$= -\frac{1}{2}\left(e^{-a^2} - 1\right)[\theta]_0^{\pi/2} = \frac{\pi}{4}\left(1 - e^{-a^2}\right)$$

Ex.36 Evaluate by changing to polar coordinates $\int_0^1 \int_x^{\sqrt{2x-x^2}} \left(x^2 + y^2\right) dy\, dx$

[M.U. 1998, 2005, 12]

Solution: The region of integration is bounded by y = x, a line; $y = \sqrt{2x - x^2}$ i.e. $x^2 = -2x + y^2 = 0$ i.e. $(x-1)^2 + y^2 = 1$ i.e. a circle with centre at (1, 0) and radius 1. x = 0, the y-axis and x = 1, a line parallel to the y-axis.

If we put $x = r\cos\theta, y = r\sin\theta, y = \sqrt{2x - x^2}$ i.e. $x^2 + y^2 = 2x$ becomes $r^2\cos^2\theta + r^2\sin^2\theta = 2r\cos\theta$ i.e. $r = 2\cos\theta$. The line y = x becomes $r\sin\theta = r\cos\theta$ i.e. $\tan\theta = 1$ i.e. $\theta = \pi/4, x^2 + y^2$ becomes r^2.



Now, consider a radial strip in the region OAB. On this strip r varies from r = 0 to $r = 2\cos\theta$. Then θ varies from $\pi/4$ to $\pi/2$.

$$\therefore \quad I = \int_{\pi/4}^{\pi/2} \int_0^{2\cos\theta} r^2 r\, dr\, d\theta = \int_{\pi/4}^{\pi/2} \int_0^{2\cos\theta} r^3 dr\, d\theta$$

$$= \int_{\pi/4}^{\pi/2} \left[\frac{r^4}{4}\right]_0^{2\cos\theta} d\theta = \frac{1}{4}\int_{\pi/4}^{\pi/2} 2^4 \cos^4\theta\, d\theta$$

$$= 4\int_{\pi/4}^{\pi/2} \cos^4\theta\, d\theta = 4\int_{\pi/4}^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

$$= \int_{\pi/4}^{\pi/2}\left(1+2\cos 2\theta+\cos^2 2\theta\right)d\theta$$

$$= \int_{\pi/4}^{\pi/2}\left(1+2\cos 2\theta+\frac{1+\cos 4\theta}{2}\right)d\theta$$

$$= \int_{\pi/4}^{\pi/2}\left(\frac{3}{2}+2\cos 2\theta+\frac{\cos 4\theta}{2}\right)d\theta$$

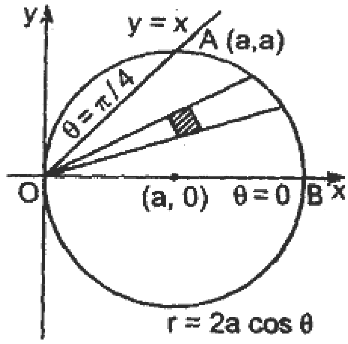
$$= \left[\frac{3}{2}\theta+\sin 2\theta+\frac{\sin 4\theta}{8}\right]_{\pi/4}^{\pi/2}$$

$$= \frac{3}{2}.\frac{\pi}{2}-\frac{3}{2}.\frac{\pi}{4}-1 = \frac{3\pi}{8}-1$$

Ex.37 Evaluate $\int_0^a \int_y^{a+\sqrt{a^2-y^2}} \frac{dx\,dy}{\left(4a^2+x^2+y^2\right)^2}$ **[M.U. 1992, 2002]**

Solution: The curve $x = a+\sqrt{a^2-y^2}$ is the circle $(x-a)^2+y^2=a^2$. We evaluate this integral by changing the co-ordinates to polar. In polar form the above circle i.e. $x^2+y^2=2ax$ becomes $r=2a\cos\theta$. The line x = y in polar form becomes $\theta=\pi/4$.

The region of integration is OAB. Hence,



$$I = \int_0^{\pi/4}\int_0^{2a\cos\theta} \frac{r\,d\theta\,dr}{\left(4a^2+r^2\right)^2}$$

$$= -\frac{1}{2}\int_0^{\pi/4}\left[\frac{1}{4a^2+r^2}\right]_0^{2a\cos\theta} d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/4}\left[\frac{1}{4a^2+4a^2\cos^2\theta}-\frac{1}{4a^2}\right]d\theta$$

$$= \frac{1}{8a^2}\int_0^{\pi/4}\left[1-\frac{1}{1+\cos^2\theta}\right]d\theta$$

$$= \frac{1}{8a^2}\int_0^{\pi/4}\left[1-\frac{\sec^2\theta}{\sec^2\theta+1}\right]d\theta$$

$$= \frac{1}{8a^2}\left[\int_0^{\pi/4}d\theta-\int_0^1\frac{dt}{2+t^2}\right] \qquad \text{(Put } t=\tan\theta\text{)}$$

$$= \frac{1}{8a^2}\left[\{\theta\}_0^{\pi/4}-\left\{\frac{1}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}}\right\}_0^1\right]$$

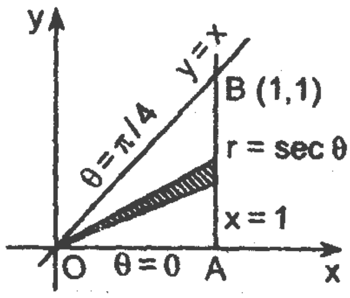
$$\therefore \qquad I = \frac{1}{8a^2}\left[\frac{\pi}{4}-\frac{1}{\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}}\right]$$

Ex.38 Change to polar coordinate and evaluate $\int_0^1 \int_0^x (x+y)\,dy\,dx$

[M.U. 1997, 99, 2003, 05, 10]

Solution: The region of integration is bounded by the line y = 0 i.e. the x-axis, the line y = x, the line x = 0 i.e. the y-axis and the line x = 1.

To change the coordinate system, we put $x = r\cos\theta, y = r\sin\theta$. Then the line y = x becomes $r\cos\theta = r\sin\theta$. i.e. $\theta = \pi/4$ and the line x = 1 becomes $r\cos\theta = 1$ i.e. $r = \sec\theta$.



Now, consider a radial strip as shown in the figure. On this strip r varies from r = 0 to $r = \sec\theta$ and then θ varies from $\theta = 0$ to $\pi/4$.

$$I = \int_0^{\pi/4} \int_0^{\sec\theta} (r\cos\theta + r\sin\theta) r\,dr\,d\theta$$

$$= \int_0^{\pi/4} \int_0^{\sec\theta} (\cos\theta + \sin\theta) r^2 dr\,d\theta$$

$$= \int_0^{\pi/4} (\cos\theta + \sin\theta) \left[\frac{r^3}{3}\right]_0^{\sec\theta} d\theta$$

$$= \frac{1}{3} \int_0^{\pi/4} (\cos\theta + \sin\theta) \sec^3\theta\,d\theta$$

$$= \frac{1}{3} \left[\int_0^{\pi/4} \sec^2\theta\,d\theta + \int_0^{\pi/4} \frac{1}{\cos^3\theta}.\sin\theta\,d\theta\right]$$

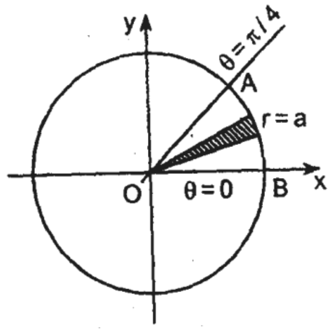
$$= \frac{1}{3} \left[\{\tan\theta\}_0^{\pi/4} + \left\{\frac{1}{2\cos^2\theta}\right\}_0^{\pi/4}\right] \qquad \text{Put} \quad \cos\theta = t$$

$$= \frac{1}{3}\left[1 + \frac{1}{2}(2-1)\right] = \frac{1}{3}\left(1 + \frac{1}{2}\right) = \frac{1}{2}$$

Ex.39 Change to polar coordinates and evaluate $\int_0^{a/\sqrt{2}} \int_y^{\sqrt{a^2-y^2}} \log\left(x^2+y^2\right) dx\,dy$

[M.U. 2002, 11, 12]

Solution: The region of integration is bounded by x = y, a line through the origin; $x = \sqrt{a^2 - y^2}$ i.e. $x^2 + y^2 = a^2$, a circle with centre at the origin and radius a; y = 0, the x-axis and $y = a/\sqrt{2}$, a line parallel to the x-axis. When x = y, $x^2 + y^2 = a^2$ gives $2x^2 = a^2$ i.e. $x = a/\sqrt{2}$ and then $y = a/\sqrt{2}$. Thus, the line and the circle intersect in $A\left(a/\sqrt{2}, a/\sqrt{2}\right)$



To change to polar, we put $x = r\cos\theta, y = r\sin\theta$.

Then x^2+y^2 changes to r^2 and x = y changes to $r\cos\theta = r\sin\theta$ i.e. $\cos\theta = \sin\theta$ i.e. $\theta = \pi/4$.

Now, consider a radial strip in the region of integration OAB. On the strip r varies from r = 0 to r = a. Then θ varies from $\theta = 0$ to $\theta = \pi/4$

$$\therefore \quad I = \int_0^{\pi/4}\int_0^a \log r^2 . r\,dr\,d\theta = \int_0^{\pi/4}\int_0^a 2\log r . r\,dr\,d\theta$$

$$= 2\int_0^{\pi/4}\left[\log r . \frac{r^2}{2} - \int \frac{r^2}{2} . \frac{1}{r}\,dr\right]_0^a d\theta$$

$$\therefore \quad I = 2\int_0^{\pi/4}\left[\log r . \frac{r^2}{2} - \frac{r^2}{4}\right]_0^a d\theta$$

$$= 2\int_0^{\pi/4}\left(\frac{a^2}{2}\log a - \frac{a^2}{4}\right) d\theta = \left(a^2\log a - \frac{a^2}{2}\right)\int_0^{\pi/4} d\theta$$

$$= \left(a^2\log a - \frac{a^2}{2}\right)[\theta]_0^{\pi/4} = a^2\left(\log a - \frac{1}{2}\right) . \frac{\pi}{4}$$

EXERCISE

Change to polar coordinates and evaluate :

- $\displaystyle\int_0^{2a}\int_0^{\sqrt{(2ax-x^2)}} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}}$ **[M.U. 2009, 11]**

Ans. $\dfrac{4a^2}{3}$

- $\displaystyle\int_0^a\int_y^a \frac{x^2}{\sqrt{x^2+y^2}}\,dx\,dy$ **[M.U. 1994]**

Ans. $\dfrac{a^3}{3}\log\left(1+\sqrt{2}\right)$

- $\displaystyle\int_0^1\int_0^{\sqrt{x-x^2}} \frac{4xy}{x^2+y^2}\,e^{-\left(x^2+y^2\right)}\,dy\,dx$ **[M.U. 2010]**

Ans. $\dfrac{1}{e}$

- $\displaystyle\int_0^4\int_{\sqrt{4x-x^2}}^{\sqrt{16-x^2}} \frac{dy\,dx}{\sqrt{16-x^2-y^2}}$ **[M.U. 1989]**

Ans. 4

- $\int_0^2 \int_{\sqrt{2x-x^2}}^{\sqrt{4-x^2}} \frac{dy\,dx}{\sqrt{4-x^2-y^2}}$ **[M.U. 1992]**

Ans. 2

- $\int_0^a \int_0^{\sqrt{a^2-x^2}} \left(x^2+y^2\right) dy\,dx$ **[M.U. 2003]**

Ans. $\frac{\pi a^4}{8}$

- $\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} \frac{xy\,e^{-\left(x^2+y^2\right)}}{x^2+y^2} dy\,dx$ **[M.U. 2001]**

Ans. $\frac{1}{4a^2}\left\{\left(1+a^2\right)e^{-a^2}-1\right\}$

- $\int_0^a \int_0^{\sqrt{a^2-x^2}} y\sqrt{\left(x^2+y^2\right)}dy\,dx$ **[M.U. 2005]**

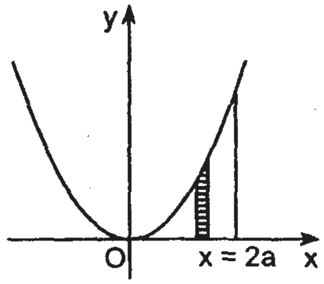
Ans. $\frac{a^4}{4}$

- $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dx\,dy}{\left(a^2+x^2+y^2\right)^{3/2}}$ **[M.U. 2002]**

Ans. $\frac{2\pi}{a}$

Ex.40 Evaluate $\iint xy\,dx\,dy$ over the region bounded by the x-axis, ordinate at x = 2a and the parabola $x^2 = 4ay$ **[M.U. 2003]**

Solution: In the region y varies from 0 to $x^2/4a$ and x varies from 0 to 2a.



$$\therefore \quad I = \int_0^{2a} \int_0^{x^2/4a} xy\,dy\,dx$$

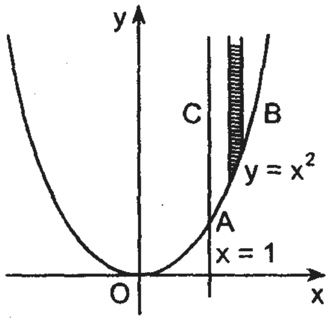
$$= \int_0^{2a} \left[x.\frac{y^2}{2}\right]_0^{x^2/4a} dx$$

$$= \int_0^{2a} \frac{x}{2}.\frac{x^4}{16a^2} dx = \frac{1}{32a^2}\int_0^{2a} x^5 dx$$

$$= \frac{1}{32a^2}\left[\frac{x^6}{6}\right]_0^{2a} = \frac{1}{32a^2}.\frac{64a^6}{6} = \frac{a^4}{3}$$

Ex.41 Evaluate $\iint_R \frac{1}{x^4+y^2} dx\,dy$ **[M.U. 1996, 2002, 07]**

Solution: The boundaries of the region are $y = x^2 a$ parabola with vertex at the origin and opening upwards. The line x = 1 is a line parallel to the y-axis. The region of integration is the region between the line x = 1 and the branch of the parabola in the first quadrant.



In this region consider a strip parallel to the y-axis. On this strip y varies from $y = x^2$ to $y = \infty$. Then x varies from x = 1 to $x = \infty$.

$$\therefore \quad I = \int_1^{\infty}\int_{x^2}^{\infty} \frac{dy}{y^2 + \left(x^2\right)^2}\,dx$$

$$= \int_1^{\infty}\left[\frac{1}{x^2}\tan^{-1}\frac{y}{x^2}\right]_{x^2}^{\infty} dx = \int_1^{\infty}\frac{1}{x^2}\left[\frac{\pi}{2} - \frac{\pi}{4}\right]dx$$

$$= \frac{\pi}{4}\int_1^{\infty}\frac{dx}{x^2} = \frac{\pi}{4}\left[-\frac{1}{x}\right]_1^{\infty} = \frac{\pi}{4}[-0+1] = \frac{\pi}{4}$$

Ex.42 Evaluate $\iint xy\,dx\,dy$ over the area bounded by the parabolas $y = x^2$ and $x = -y^2$

[M.U. 1997, 2005]

Solution: The two parabolas are as shown in the figure. They intersect in O(0, 0) and A(–1, 1).

In the region OAB, consider a strip parallel to the x-axis. On this strip x varies from $x = -\sqrt{y}$ to $x = -y^2$ and then y varies from y = 0 to 1.

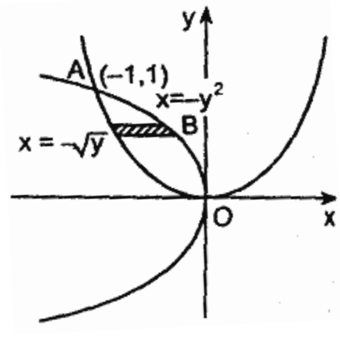
$$\therefore \quad I = \int_0^1\int_{-\sqrt{y}}^{-y^2} y.x\,dx\,dy$$

$$= \int_0^1 y\left[\frac{x^2}{2}\right]_{-\sqrt{y}}^{-y^2} dy$$

$$= \frac{1}{2}\int_0^1 y\left[y^4 - y\right]dy$$

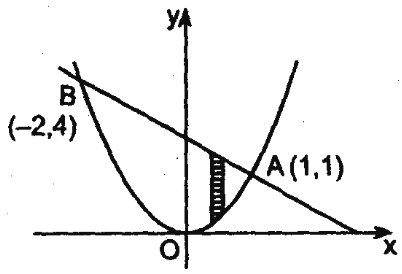
$$= \frac{1}{2}\int_0^1\left(y^5 - y^2\right)dy = \frac{1}{2}\left[\frac{y^6}{6} - \frac{y^3}{3}\right]_0^1$$

$$= \frac{1}{2}\left[\frac{1}{6} - \frac{1}{3}\right] = -\frac{1}{12}$$



Ex.43 Evaluate $\iint y\,dx\,dy$ over the area bounded by $x = 0, y = x^2, x + y = 2$ in the first quadrant. **[M.U. 2003, 05]**

Solution: We have to find the integral over the region OAB.



$$\therefore \quad I = \int_0^1 \int_{x^2}^{2-x} y\,dy\,dx = \int_0^1 \left[\frac{y^2}{2}\right]_{x^2}^{2-x} dx$$

$$= \frac{1}{2}\int_0^1 \left[(2-x)^2 - x^4\right] dx$$

$$= \frac{1}{2}\int_0^1 \left(4 - 4x + x^2 - x^4\right) dx$$

$$= \frac{1}{2}\left[4x - 2x^2 + \frac{x^3}{3} - \frac{x^5}{5}\right]_0^1$$

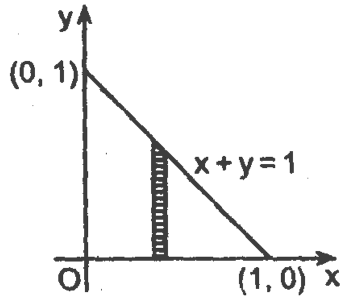
$$= \frac{1}{2}\left[4 - 2 + \frac{1}{3} - \frac{1}{5}\right] = \frac{16}{15}$$

Ex.44 Evaluate $\iint \sqrt{xy(1-x-y)}\,dx\,dy$ over the area bounded by x = 0, y = 0 and x + y = 1.

[M.U. 1996, 97, 2000, 02, 03]

Solution: We shall first integrate w.r.t. y. Now y varies 0 to 1 – x

We shall firs find



$$\therefore \quad I_1 = \int_0^{1-x} \sqrt{y(1-x-y)}\,dy$$

Suppose $1 - x = a$

$$= \int_0^a \sqrt{y(a-y)}\,dy \qquad \text{Put} \quad y = at$$

$$= \int_0^1 a^{1/2} t^{1/2} a^{1/2} (1-t)^{1/2}\, a\,dt$$

$$= a^2 \int_0^1 t^{1/2} (1-t)^{1/2}\, dt$$

Now put $t = \sin^2\theta$ $\quad \therefore \quad dt = 2\sin\theta\cos\theta\,d\theta$

$$\therefore \quad I_1 = a^2 \int_0^{\pi/2} \sin\theta.\cos\theta.2\sin\theta\cos\theta\,d\theta$$

$$= 2a^2 \int_0^{\pi/2} \sin^2\theta\cos^2\theta\,d\theta$$

$$= 2a^2.\frac{1.1}{4.2}.\frac{\pi}{2} = \frac{\pi a^2}{8}$$

$$\therefore \quad I = \int_0^1 x^{1/2}.a^2 \frac{\pi}{8}\,dx = \frac{\pi}{8}\int_0^1 x^{1/2}(1-x)^2\,dx$$

Again put $x = \sin^2\theta$, $\quad dx = 2\sin\theta\cos\theta\,d\theta$

$$\therefore \quad I = \frac{\pi}{8}\int_0^{\pi/2} \sin\theta.\cos^4\theta.2\sin\theta\cos\theta\,d\theta$$

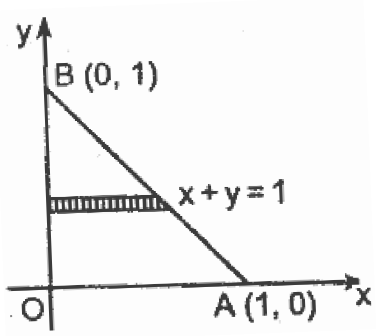
$$= \frac{\pi}{4}\int_0^{\pi/2} \sin^2\theta\cos^5\theta\,d\theta$$

$$= \frac{\pi}{4}.\frac{1.4.2.1}{7.5.3.1} = \frac{2\pi}{105}$$

Ex.45 Evaluate $\iint_R xy\sqrt{1-x-y}\,dx\,dy$ over the area of the triangle formed by $x=0, y=0, x+y=1$. **[M.U. 1996, 2001, 10]**

Solution: The region of integration is shown in the figure. Now, consider a strip parallel to the x-axis. On this strip x varies from x = 0 to $x=1-y$. Then y varies from y = 0 to y = 1.

$$\therefore \quad I=\int_0^1\int_0^{1-y} y.x\sqrt{(1-y)-x}.dx\,dy$$



Now, put $x=(1-y)t \therefore \quad dx=(1-y)dt$

When $x=0, t=0$; when $x=1-y, t=1$

$$\therefore \quad I=\int_0^1\int_0^1 y(1-y).t\sqrt{(1-y)-(1-y)t}.(1-y)\,dt\,dy$$

$$=\int_0^1\int_0^1 y(1-y)^2.\sqrt{1-y}.t\sqrt{1-t}.dt\,dy$$

$$=\int_0^1\int_0^1 y(1-y)^{5/2}.\left[t(1-t)^{1/2}\right].dt\,dy$$

$$=\int_0^1 y(1-y)^{5/2}\,dy\int_0^1 t(1-t)^{1/2}\,dt$$

Put $t=\sin^2\theta$ and $y=\sin^2\theta$

$$\therefore \quad dt=2\sin\theta\cos\theta\,d\theta$$

$$dy=2\sin\theta\cos\theta\,d\theta$$

$$\therefore \quad I=\int_0^{\pi/2}\sin^2\theta\cos^5\theta.2\sin\theta\cos\theta\,d\theta\times\int_0^{\pi/2}\sin^2\theta\cos\theta.2\sin\theta\cos\theta\,d\theta$$

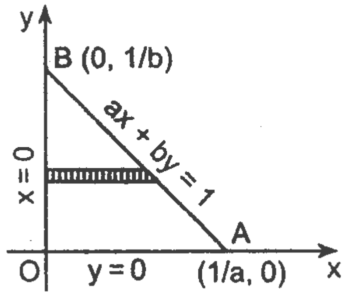
$$=2\int_0^{\pi/2}\sin^3\theta.\cos^6\theta\,d\theta\times 2\int_0^{\pi/2}\sin^3\theta\cos^2\theta\,d\theta$$

$$=4.\frac{2.5.3.1}{9.7.5.3}.\frac{2}{5.3.1}=\frac{16}{945}$$

Ex.46 Prove that $\iint_R e^{ax+by}\,dx\,dy=2R$ where R is area of the triangle whose boundaries are x = 0, y = 0 and $ax+by=1$. **[M.U. 1996, 98]**

Solution: Consider a strip parallel to the x-axis. On this strip x varies from x = 0 to $x=(1-by)/a$. Then y varies from y = 0 to y = 1/b.

$$\therefore \quad I=\int_0^{1/b}\int_0^{(1-by)/a} e^{ax+by}\,dx\,dy$$



$$=\int_0^{1/b} e^{by}\left[\frac{e^{ax}}{a}\right]_0^{(1-by)/a} dy$$

$$=\int_0^{1/b}\frac{e^{by}}{a}\left[e^{1-by}-1\right]dy=\frac{1}{a}\int_0^{1/b}\left(e-e^{by}\right)dy$$

$$= \frac{1}{a}\left[ey - \frac{e^{by}}{b}\right]_0^{1/b} = \frac{1}{a}\left[e\left(\frac{1}{b}\right) - \frac{e}{b} + \frac{1}{b}\right]$$

$$= \frac{1}{ab} = 2\left[\frac{1}{2}\left(\frac{1}{a}\right)\left(\frac{1}{b}\right)\right]$$

$= 2R$ where R is the area of the triangle OAB.

Ex.47 Evaluate $\iint_R \frac{y\,dx\,dy}{(a-x)\sqrt{ax-y^2}}$ where R is the ragion bounded by and y = x.

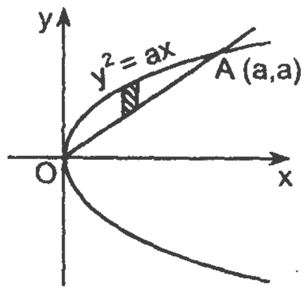
[M.U. 1998, 2012]

Solution: We shall first integrate with respect to y and then with respect to x.

The curve $y^2 = ax$ is a parabola with vertex at the origin and opening on the right; y = x is a line through the origin.

In the region of integration consider a strip parallel to the y-axis. On the strip y varies from y = x to $y = \sqrt{ax}$. Then x varies from x = 0 to x = a.

$$\therefore \quad I = \int_0^1 \int_x^{\sqrt{ax}} \frac{1}{(a-x)} \cdot \frac{y\,dy}{\sqrt{ax-y^2}}\,dx$$



$$= \int_0^a \frac{1}{(a-x)}\left[-\sqrt{ax} - y^2\right]_x^{\sqrt{ax}} dx$$

$$= \int_0^a \frac{1}{(a-x)}\left[0 + \sqrt{ax-x^2}\right] dx$$

$$= \int_0^a \frac{\sqrt{x}}{\sqrt{a-x}}\,dx$$

Now, put $x = a\sin^2\theta \qquad \therefore \qquad dx = 2a\sin\theta\cos\theta\,d\theta$

$$\therefore \quad I = \int_0^{\pi/2} \frac{\sqrt{a}\sin\theta}{\sqrt{a}\cos\theta} .2a\sin\theta\cos\theta\,d\theta$$

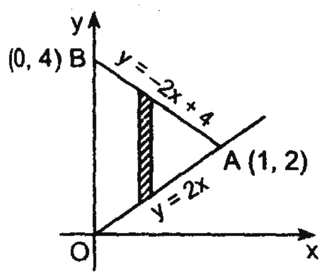
$$= 2a\int_0^{\pi/2} \sin^2\theta\,d\theta = 2a.\frac{1}{2}.\frac{\pi}{2} = \frac{\pi a}{2}$$

Ex.48 Evaluate $\iint_R x(x-y)\,dx\,dy$ where R is the triangle with vertices (0, 0), (1, 2), (0, 4).

[M.U. 1997]

Solution: Let O(0, 0), A(1, 2), B(0, 4) be the vertices of the triangle OAB. The equation of the line OA is $\frac{y-0}{0-2} = \frac{x-0}{0-0}$ i.e. $y = 2x$. The equation of the line AB is $\frac{y-2}{2-4} = \frac{x-1}{1-0}$ i.e. $y = -2x + 4$

Now, consider a strip parallel to the y-axis. On this strip y varies from y = 2x to $y = -2x + 4$ and then x varies from x = 0 to x = 1.



$$\therefore \quad I=\int_0^1\int_{2x}^{-2x+4}\left(x^2-xy\right)dy\,dx=\int_0^1\left[x^2y-\frac{xy^2}{2}\right]_{2x}^{-2x+4}dx$$

$$=\int_0^1\left[x^2(-2x+4)-\frac{x}{2}(-2x+4)^2-x^2.2x+\frac{x}{2}(2x)^2\right]dx$$

$$=\int_0^1\left[-2x^3+4x^2-2x^3+8x^2-8x-2x^3+2x^3\right]dx$$

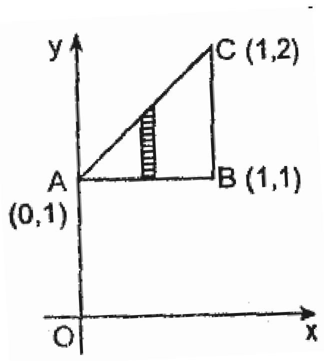
$$=\int_0^1\left(-4x^3+12x^2-8x\right)dx$$

$$=\left[-x^4+4x^2-4x^2\right]_0^1=-1$$

Ex.49 Evaluate $\iint\left(x^2+y^2\right)dx\,dy$ over the area of the triangle whose vertices are (0, 1), (1, 1), (1, 2). **[M.U. 1997]**

Solution: The equation of the line AC is

$$\frac{y-2}{1}=\frac{x-1}{1} \text{ i.e. } y=x+1$$



$$\therefore \quad I=\int_0^1\int_1^{x+1}\left(x^2+y^2\right)dy\,dx$$

$$=\int_0^1\left[x^2y+\frac{y^3}{3}\right]_1^{x+1}dx$$

$$=\int_0^1\left[\left\{x^2(x+1)+\frac{(x+1)^3}{3}\right\}-\left\{x^2+\frac{1}{3}\right\}\right]dx$$

$$=\frac{1}{3}\int_0^1\left(4x^3+3x^2+3x\right)dx=\frac{1}{3}\left[x^4+x^3+\frac{3x^2}{2}\right]_0^1$$

$$=\frac{1}{3}\left[1+1+\frac{3}{2}\right]=\frac{7}{6}$$

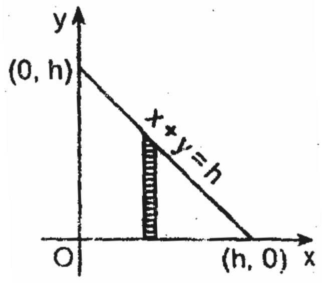
Ex.50 Evaluate $\iint x^{m-1}y^{n-1}\,dx\,dy$ over the region bounded by $x+y=h, x=0, y=0$

[M.U. 2002]

Solution: The region is bounded by the x-axis, y-axis and the line $x+y=h$.

On the strip y varies from 0 to h – x and then strip moves from x = 0 to x = h.

$$\therefore \quad I=\int_0^h\int_0^{h-x}x^{m-1}y^{n-1}dy\,dx$$



Let $\quad I_1=\int_0^{h-x}y^{n-1}dy=\left[\frac{y^n}{n}\right]_0^{h-x}$

$$=\frac{1}{n}(h-x)^n$$

Now, $I=\int_0^h x^{m-1}.\frac{1}{n}(h-x)^n\,dx \quad$ Put $\quad x=ht$

$$= \int_0^1 h^{m-1}.t^{m-1}.\frac{1}{n} h^n (1-t)^n .h\,dt$$

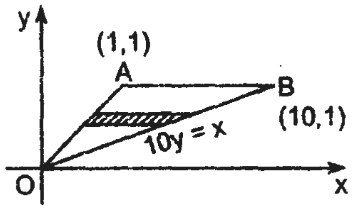
$$= \frac{h^{m+n}}{n} \int_0^1 t^{m-1} (1-t)^n \, dt$$

$$= \frac{h^{m+n}}{n} . \frac{\overline{|m}\overline{|n+1}}{\overline{|m+n+1}} = \frac{h^{m+n}\overline{|m}\overline{|n}}{(m+n)\overline{|m+n}}$$

Ex.51 Evaluate $\iint_R \sqrt{xy - y^2}\,dx\,dy$ where R is a triangle whose vertices are (0, 0), (10, 1) and (1, 1) **[M.U. 1997, 98, 2002, 04]**

Solution: The triangle OAB is as shown in the figure. The equation of the line OA is

$$\frac{y-1}{1-0} = \frac{x-1}{1-0} \text{ i.e. } y = x$$



The equation of the line OB is

$$\frac{y-1}{1-0} = \frac{x-10}{10-0} \text{ i.e. } x = 10y$$

Now, consider a strip parallel to the y-axis. On this strip x varies from x = y to x = 10y. Then y varies from y = 0 to y = 1.

$$\therefore \quad I = \int_0^1 \int_y^{10y} \sqrt{xy - y^2}\,dx\,dy$$

$$= \int_0^1 \left[\frac{\left(xy - y^2\right)^{3/2}}{(3/2)y} \right]_y^{10y} dy = \int_0^1 \frac{\left(9y^2 - 0\right)^{3/2}}{(3/2)y} dy$$

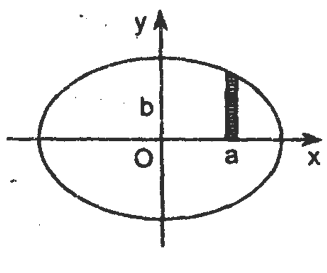
$$= \frac{2}{3} \int_0^1 27.y^2 dy = 18 \left[\frac{y^3}{3} \right]_0^1 = 6$$

Ex.52 Show that $\iint_R x^{m-1} y^{n-1} dy\,dx = \frac{1}{2n} a^m b^n B\left(\frac{m}{2}, \frac{n}{2} + 1\right)$ where R is the positive quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ **[M.U. 1997]**

Solution: To evaluate the integral consider a strip parallel to the y-axis.

On this strip y varies from y = 0 to $y = \frac{b}{a}\sqrt{a^2 - x^2}$ and then x varies from x = 0 to x = a.

$$\therefore \quad I = \int_0^a \int_0^{b\sqrt{a^2-x^2}/a} x^{m-1} y^{n-1} dy\,dx$$



$$= \int_0^a x^{m-1} \left[\frac{y^n}{n} \right]_0^{b\sqrt{a^2-x^2}/a} dx$$

$$= \int_0^a x^{m-1} . \frac{1}{n} . \frac{b^n}{a^n} \left(a^2 - x^2\right)^{n/2} dx$$

Now, put $x = a\sin\theta$ $\quad\therefore\quad$ $dx = a\cos\theta\, d\theta$

When $x = 0, \theta = 0$; when $x = a, \theta = \pi/2$

$$\therefore \quad I = \int_0^{\pi/2} \frac{b^n}{n.a^n} a^{m-1} \sin^{m-1}\theta . a^n \cos^n \theta . a\cos\theta\, d\theta$$

$$= \frac{1}{n} . \frac{b^n}{a^n} . a^{m-1} . a^n . a . \int_0^{\pi/2} \sin^{m-1}\theta . \cos^{n+1}\theta\, d\theta$$

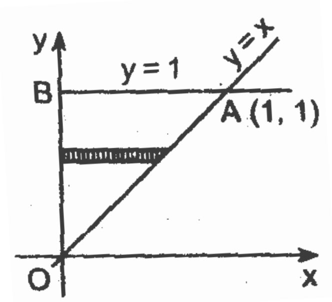
$$= \frac{a^m . b^n}{n} . \frac{1}{2} B\left(\frac{m}{2} . \frac{n}{2} + 1\right)$$

Ex.53 Evaluate $\iint_R \frac{2xy^5}{\sqrt{1 + x^2y^2 - y^4}} dx\, dy$ where R is the region of the triangle whose vertices are (0, 0), (1, 1), (0, 1). **[M.U. 1995, 2003, 05]**

Solution: Let O(0, 0), A(1, 1), B(0, 1) be the vertices of the triangle OAB. Now, the equation of the line AB is y = 1 and the equation of the line OA is

$$\frac{x - 0}{0 - 1} = \frac{y - 0}{0 - 1} \text{ i.e. } x = y$$

Now, consider a strip parallel to the x-axis. On this strip x varies from x = 0 to x = y. The strip moves from y = 0 to y = 1.



$$\therefore \quad I = \int_0^1 \int_0^y \frac{2y^5 . x}{\sqrt{\left(1 - y^4\right) + x^2y^2}} dx\, dy$$

$$= \int_0^1 \int_0^y \frac{1}{y} . \frac{2y^5 . x}{\sqrt{\frac{1 - y^4}{y^2} + x^2}} dx\, dy$$

$$= \int_0^1 2y^4 \left[\sqrt{\frac{1 - y^4}{y^2} + x^2}\right]_0^y dy$$

$$= \int_0^1 2y^4 \left[\sqrt{\frac{1 - y^4}{y^2} + y^2} - \sqrt{\frac{1 - y^4}{y^2}}\right] dy$$

$$= \int_0^1 2y^4 \left[\frac{1}{y} - \frac{\sqrt{1 - y^4}}{y}\right] dy$$

$$= 2\int_0^1 \left[y^3 - \sqrt{1 - y^4} . y^3\right] dy = 2\left[\frac{y^4}{4} + \frac{1}{4} . \frac{\left(1 - y^4\right)^{3/2}}{3/2}\right]_0^1$$

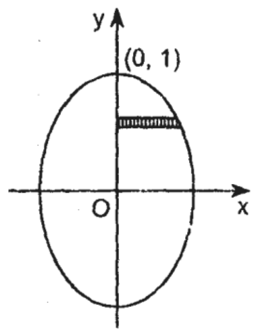
$$\therefore \qquad I = 2\left[\frac{1}{4}-\frac{1}{4}.\frac{2}{3}\right] = \frac{1}{2}-\frac{1}{3} = \frac{1}{6}$$

Ex.54 Prove that $\iint\limits_R \frac{dx\,dy}{\sqrt{1-x^2-y^2}} = \frac{\pi}{4}$ where R is the region of the first quadrant of the ellipse $2x^2+y^2=1$. **[M.U. 1995]**

Solution: The ellipse $2x^2+y^2=1$ i.e. $\frac{x^2}{1/2}+\frac{y^2}{1}=1$ has semi-major axis $a=\frac{1}{\sqrt{2}}$ and semi-minor axis b = 1.

If we consider a strip parallel to the x-axis, on this strip x varies from x = 0 to $x=\sqrt{1-y^2}/\sqrt{2}$. This strip moves from y = 0 to y = 1.

$$\therefore \qquad I = \int_0^1 \int_0^{\sqrt{\left(1-y^2\right)/2}} \frac{dx\,dy}{\sqrt{\left(1-y^2\right)-x^2}}$$



$$= \int_0^1 \sin^{-1}\left[\frac{x}{\sqrt{1-y^2}}\right]_0^{\sqrt{\left(1-y^2\right)/2}} dy$$

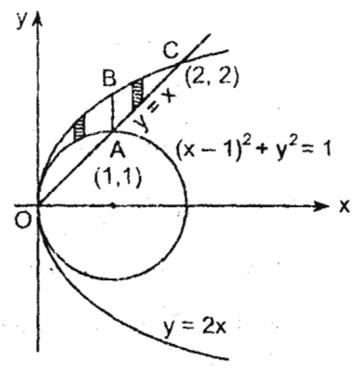
$$= \int_0^1 \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) dy = \int_0^1 \frac{\pi}{4}\,dy$$

$$= \frac{\pi}{4}\left[y\right]_0^1 = \frac{\pi}{4}$$

Ex.55 Evaluate $\iint\limits_R xy\,dx\,dy$ over the region R given by $x^2+y^2-2x=0, y^2=2x, y=x$

[M.U. 1994, 97, 2005, 07, 10]

Solution: The given region is bounded by the line y = x, the parabola $y^2=2x$, the circle $(x-1)^2+y^2=1$ with centre (1, 0) and radius = 1. For evaluating the integral we see that the region is divided into two parts, OAB and ABC.



In the region OAB. Consider a strip parallel to the y-axis. On this strip y varies from $y=\sqrt{2x-x^2}$ to $y=\sqrt{2x}$ and then x varies from x = 0 to x = 1.

In the region ABC, consider again a strip parallel to the y-axis. On this strip y varies from y = x to $y=\sqrt{2x}$, and then x varies from x = 1 to x = 2.

$$\therefore \qquad I = \int_0^1 \int_{\sqrt{2x-x^2}}^{\sqrt{2x}} xy\,dy\,dx + \int_1^2 \int_x^{\sqrt{2x}} xy\,dy\,dx$$

$$= \int_0^1 x.\left[\frac{y^2}{2}\right]_{\sqrt{2x-x^2}}^{\sqrt{2x}} dx + \int_1^2 x\left[\frac{y^2}{2}\right]_x^{\sqrt{2x}} dx$$

$$= \frac{1}{2}\int_0^1 x.\left[2x-\left(2x-x^2\right)\right]dx + \frac{1}{2}\int_1^2 x\left[2x-x^2\right]dx$$

$$= \frac{1}{2}\int_0^1 x^3 dx + \frac{1}{2}\int_1^2 \left(2x^2 - x^3\right)dx$$

$$= \frac{1}{2}\left[\frac{x^4}{4}\right]_0^1 + \frac{1}{2}\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_1^2$$

$$= \frac{1}{8} + \frac{1}{2}\left[\frac{16}{3} - \frac{16}{4} - \frac{2}{3} + \frac{1}{4}\right]$$

$$= \frac{1}{8} + \frac{1}{2}\left[\frac{64-48-8+3}{12}\right] = \frac{1}{8} + \frac{11}{24} = \frac{14}{24} = \frac{7}{12}$$

EXERCISE

Solve the following examples :

- Evaluate $\iint\left(x^2 - y^2\right)dx\,dy$ over the area of the triangle whose vertices are at the points (0, 1), (1, 1), (1, 2). **Ans.** $-2/3$ **[M.U. 2011]**
- Evaluate $\iint x^{m-1}y^{n-1}dx\,dy$ over the positive quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. **[M.U. 1997]**

Ans. $\frac{a^m b^n}{n} B\left(\frac{m}{2}, \frac{n}{2} + 1\right)$

- Evaluate $\iint\limits_R \frac{y\,e^{2y}}{\sqrt{(1-x)(x-y)}}dx\,dy$ where R is the region of the triangle whose vertices are (0, 0), (1, 0) and (1, 1). **[M.U. 1998]**

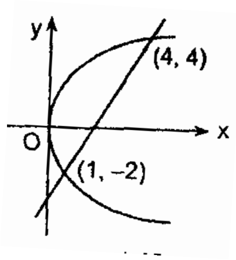
Ans. $\frac{\pi}{4}\left(e^2 + 1\right)$

- Evaluate $\iint\left(x^2 + y^2\right)dx\,dy$ over the area of the triangle whose vertices are (0, 0), (1, 0), (1, 2). **[M.U. 2002]**

Ans. $7/6$

- Evaluate $\iint xy\,dx\,dy$ over the area bounded by $y^2 = 4x$ and $y = 2x - 4$ **[M.U. 2002]**

Ans. $45/2$



- Evaluate $\iint_R \left(x^2+y^2\right)dx\,dy$ over the area enclosed by the curves $y=4x, x+y=3, y=0, y=2$ **[M.U. 1989]**

Ans. 463/48

- Evaluate $\iint \frac{dy\,dx}{\sqrt{1-2x^2-y^2}}$ over the first quadrant of the ellipse $2x^2+y^2=1$

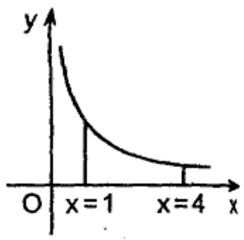
[M.U. 1995, 2003]

Ans. $\pi/2\sqrt{2}$

- Evaluate $\iint_R xy(x-1)\,dx\,dy$ where R is the region bounded by $xy=4, y=0, x=1, x=4$

[M.U. 1995, 2009]

Ans. $8(3-\log 4)$



- Evaluate $\iint_R xy(x+y)\,dx\,dy$ where R is the region bounded by $x^2=y$ and x = y.

[M.U. 1988, 2000, 02, 08]

Ans. 3/56

Ex.56 Calculate $\iint r^3\,dr\,d\theta$ over the area included between the circles $r=2\sin\theta$ and $r=4\sin\theta$. **[M.U. 1988, 90, 2002]**

Solution: The circle $r=2\sin\theta$ i.e. $r^2=2r\sin\theta$ becomes in Cartesian system $x^2+y^2=2y$ i.e. $x^2+(y-1)^2=1$. Similarly, the circle $r=4\sin\theta$ i.e. $r^2=4r\sin\theta$ becomes in Cartesian system $x^2+y^2=4y$ i.e. $x^2+(y-2)^2=4$.

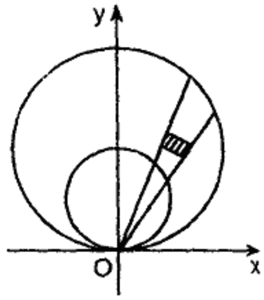
In the given region r varies from $2\sin\theta$ to $4\sin\theta$ and θ varies from 0 to π.

$$\therefore \quad I=\int_0^{\pi}\int_{2\sin\theta}^{4\sin\theta} r^3 dr\,d\theta=\int_0^{\pi}\left[\frac{r^4}{4}\right]_{2\sin\theta}^{4\sin\theta} d\theta$$

$$=\frac{1}{4}\int_0^{\pi}\left[4^4\sin^4\theta-2^4\sin^4\theta\right]d\theta$$

$$=60\int_0^{\pi}\sin^4\theta\,d\theta=120\int_0^{\pi/2}\sin^4\theta\,d\theta$$

$$=120.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}=\frac{45}{2}.\pi$$



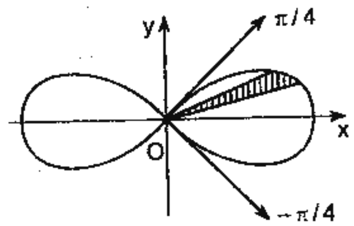
Ex.56 Evaluate $\iint \frac{r\, d\theta\, dr}{\sqrt{r^2+a^2}}$ over the loop of the lemniscates $r^2 = a^2\cos 2\theta$

[M.U. 1993, 2000]

Solution: In the given region r varies from 0 to $a\sqrt{\cos 2\theta}$ and θ varies from $-\frac{\pi}{4}$ to $\frac{\pi}{4}$.

$$\therefore \quad I = \int_{-\pi/4}^{\pi/4}\int_0^{a\sqrt{\cos 2\theta}} \frac{r\, d\theta\, dr}{\sqrt{a^2+r^2}}$$

$$= \int_{-\pi/4}^{\pi/4}\left[\left(r^2+a^2\right)^{1/2}\right]_0^{a\sqrt{\cos 2\theta}} d\theta$$



$$= \int_{-\pi/4}^{\pi/4}\left[\sqrt{a^2+a^2\cos 2\theta}-a\right]d\theta$$

$$= a\int_{-\pi/4}^{\pi/4}\left[\sqrt{2}.\cos\theta - 1\right]d\theta$$

$$= 2a\left[\sqrt{2}.\sin\theta - \theta\right]_0^{\pi/4}$$

$$= 2a\left[1-\frac{\pi}{4}\right] = \frac{a}{2}(4-\pi)$$

EXERCISE

Solve the following examples :

- Evaluate $\iint r\sqrt{a^2-r^2}.dr\, d\theta$ over upper half of the circle $r = a\cos\theta$ **[M.U. 1988]**

Ans. $\frac{a^3}{18}(3\pi-4)$

- Evaluate $\iint r^2 dr\, d\theta$ over the area between the circles $r = 2a\cos\theta$ **[M.U. 1988]**

Ans. $\frac{28}{9}a^3$

- Evaluate $\iint \frac{r\, dr\, d\theta}{\sqrt{r^2+4}}$ over the loop of $r^2 = 4\cos 2\theta$ **[M.U. 1989]**

Ans. $(4-\pi)$

- Evaluate $\iint r\sin\theta\, dA$ over the cardioide $r = a(1+\cos\theta)$ above the initial line.

[M.U. 2010]

Ans. $(4/3)a^3$

Ex.57 Evaluate $\iint y^2 dx\,dy$ over the area outside $x^2+y^2-ax=0$ and inside $x^2+y^2-2ax=0$ **[M.U. 1991, 93, 2006]**

Solution: First we note that $x^2+y^2-ax=0$ i.e. $\left[x-(a/2)^2+y^2=(a/2)^2\right]$ and $x^2+y^2-2ax=0$ i.e. $(x-a)^2+y^2=a^2$ are circles as shown in the figure.

We change the given integral into polar co-ordinates. Putting $x=r\cos\theta, y=r\sin\theta$ in $x^2+y^2-2ax=0$, we get $r^2-ar\cos\theta=0$ i.e. $r=a\cos\theta$ and in $x^2+y^2-2ax=0$, we get $r^2-2ar\cos\theta=0$ i.e. $r=2a\cos\theta$.

Replacing $dx\,dy$, by $r\,d\theta\,dr$, we get

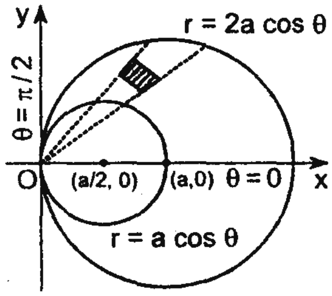
$$I=2\int_0^{\pi/2}\int_{a\cos\theta}^{2a\cos\theta} r^2\sin^2\theta.r\,dr\,d\theta$$

$$=2\int_0^{\pi/2}\sin^2\theta\left[\frac{r^4}{4}\right]_{a\cos\theta}^{2a\cos\theta}d\theta$$

$$=\frac{1}{2}\int_0^{\pi/2}\sin^2\theta\left[16a^4\cos^4\theta-a^4\cos^4\theta\right]d\theta$$

$$=\frac{15a^4}{2}\int_0^{\pi/2}\cos^4\theta\sin^2\theta\,d\theta$$

$$=\frac{15a^4}{2}.\frac{3.1.1}{6.4.2}.\frac{\pi}{2}=\frac{15\pi a^4}{64}$$



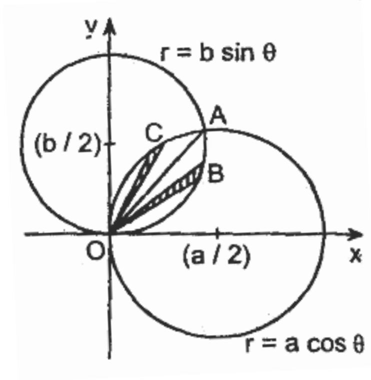
Ex.58 Evaluate $\iint \frac{\left(x^2+y^2\right)^2}{x^2y^2}dx\,dy$ over the area common to $x^2+y^2=ax$ and $x^2+y^2=by, a,b>0$ **[M.U. 2002, 03, 08]**

Solution: First we note that $x^2+y^2=ax$ i.e. $\left[x-(a/2)\right]^2+y^2=a^2/4$ and $x^2+y^2=by$ i.e. $x^2+\left[y-(b/2)\right]^2=b^2/4$ are the circles as shown in the figure.

If we change to polar coordinates by putting $x=r\cos\theta; y=r\sin\theta$, the equations of the circles become $r=a\cos\theta, r=b\sin\theta$. At the point of intersection $a\cos\theta=b\sin\theta$

$\therefore \quad \tan\theta=\frac{a}{b}$

i.e. $\quad \theta=\tan^{-1}\left(\frac{a}{b}\right)=\alpha$ say.



Also, $\frac{\left(x^2+y^2\right)^2}{x^2y^2}=\frac{r^4}{r^4\sin^2\theta\cos^2\theta}$

Now, integral over the region OBA.

$$I_1 = \int_0^{\alpha}\int_0^{b\sin\theta} \frac{r^4}{r^4 \sin^2\theta\cos^2\theta} r\,dr\,d\theta$$

$$= \int_0^{\alpha}\int_0^{b\sin\theta} \frac{1}{\sin^2\theta\cos^2\theta} r\,dr\,d\theta$$

$$= \int_0^{\alpha} \frac{1}{\sin^2\theta\cos^2\theta}\left[\frac{r^2}{2}\right]_0^{b\sin\theta} d\theta$$

$$= \frac{1}{2}b^2\int_0^{\alpha}\sec^2\theta\,d\theta = \frac{b^2}{2}[\tan\theta]_0^{\alpha}$$

$$= \frac{1}{2}b^2\tan\alpha = \frac{1}{2}b^2\frac{a}{b} = \frac{ab}{2}.$$

And integral over the region OCA.

$$I_2 = \int_{\alpha}^{\pi/2}\int_0^{a\cos\theta} \frac{1}{\sin^2\theta\cos^2\theta} r\,dr\,d\theta$$

$$= \int_{\alpha}^{\pi/2} \frac{1}{\sin^2\theta\cos^2\theta}\left[\frac{r^2}{2}\right]_0^{a\cos\theta} d\theta$$

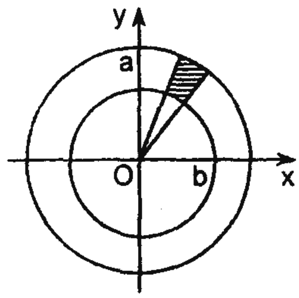
$$= \frac{a^2}{2}\int_{\alpha}^{\pi/2}\operatorname{cosec}^2\theta\,d\theta = \frac{a^2}{2}[-\cot\theta]_{\alpha}^{\pi/2}$$

$$= -\frac{a^2}{2}[0-\cot\alpha] = \frac{1}{2}a^2\frac{b}{a} = \frac{1}{2}ab$$

$\therefore$ Required integral = ab.

Ex.59 Evaluate $\iint \frac{x^2y^2}{x^2+y^2}$ over the annular region between circles $x^2+y^2=a^2$ and $x^2+y^2=b^2; a>b(>0)$ **[M.U. 2002]**

Solution: Changing to polar coordinates, we see that



$$I = 4\int_0^{\pi/2}\int_b^{a} \frac{r^4\sin^2\theta\cos^2\theta}{r^2} r\,dr\,d\theta$$

$$= 4\int_0^{\pi/2}\sin^2\theta\cos^2\theta\left[\frac{r^4}{4}\right]_b^{a} d\theta$$

$$= \left(b^4-a^4\right)\int_0^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta$$

$$= \left(b^4-a^4\right).\frac{1.1}{4.2}.\frac{\pi}{2} = \left(b^4-a^4\right).\frac{\pi}{16}$$

Ex.60 Evaluate $\iint \frac{dx\,dy}{\left(1+x^2+y^2\right)^2}$ over the loop of the lemniscates $\left(x^2+y^2\right)^2 = x^2-y^2$

[M.U. 1985, 2003, 07]

Solution: Changing to polar coordinates by putting $x = r\cos\theta, y = r\sin\theta, dx\,dy = r\,d\theta\,dr$ the equation of the lemniscates becomes $r^4 = r^2\left(\cos^2\theta - \sin^2\theta\right)$ i.e. $r^2 = \cos 2\theta$. The loop is shown in example 63.

$$I = \int_{-\pi/4}^{\pi/4}\int_0^{\sqrt{\cos 2\theta}} \frac{r\,dr\,d\theta}{\left(1+r^2\right)^2} = \int_{-\pi/4}^{\pi/4} -\frac{1}{2}\left[\frac{1}{\left(1+r^2\right)}\right]_0^{\sqrt{\cos 2\theta}} d\theta$$

$$= -\frac{1}{2}.2\int_0^{\pi/4}\left[\frac{1}{1+\cos 2\theta} - 1\right] d\theta$$

$$= -\int_0^{\pi/4}\left(\frac{1}{2}\sec^2\theta - 1\right) d\theta = -\left[\frac{1}{2}\tan\theta - \theta\right]_0^{\pi/4}$$

$$= -\left[\frac{1}{2} - \frac{\pi}{4}\right] = \frac{\pi}{4} - \frac{1}{2}$$

Ex.61 Evaluate $\iint \frac{dx\,dy}{\sqrt{4+x^2+y^2}}$ over one loop of lemniscates $\left(x^2+y^2\right) = 4\left(x^2-y^2\right)$

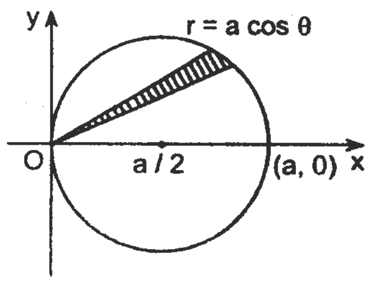
[M.U. 1992]

Solution: In Ex. 56, put a = 2

$\therefore \quad I = 4 - \pi$

Ex.62 Evaluate $\iint_R \sqrt{a^2 - x^2 - y^2}\,.dx\,dy$ where R is the area of the upper half of the circle $x^2 + y^2 = ax$. **[M.U. 1988, 95, 2008]**

Solution: The region is upper half of the circle with centre at $\left[(a/2), 0\right]$ and radius a. Now, in this region r varies from 0 to $a\cos\theta$ and θ varies from 0 to $\pi/2$.



$$I = \int_0^{\pi/2}\int_0^{a\cos\theta} \sqrt{a^2 - r^2}\,.r\,dr\,d\theta$$

Now, put $a^2 - r^2 = t$ $\quad\therefore\quad -2r\,dr = dt$

When $r = a\cos\theta, t = a^2\sin^2\theta$

When $r = 0, t = a^2$

$$\therefore \quad I = \int_{a^2}^{a^2\sin^2\theta} t^{1/2}\left(-\frac{1}{2}\right) dt\,d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2}\left[\frac{t^{3/2}}{3/2}\right]_{a^2}^{a^2\sin^2\theta} d\theta$$

$$= -\frac{1}{3}\int_0^{\pi/2}\left(a^3\sin^3\theta - a^3\right) d\theta = \frac{a^3}{3}\int_0^{\pi/2}\left(1 - \sin^3\theta\right) d\theta$$

$$= \frac{a^3}{3}\left[\int_0^{\pi/2} d\theta - \int_0^{\pi/2} \sin^3\theta\,d\theta\right]$$

$$=\frac{a^3}{3}\left[\frac{\pi}{2}-\frac{2}{3}.1\right]=\frac{a^3}{18}(3\pi-4)$$

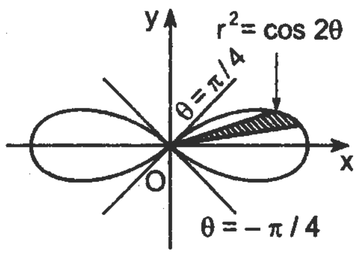
Ex.63 Change to polar coordinates and evaluate $\iint_R \frac{dx\,dy}{\left(1+x^2+y^2\right)^2}$ over one loop of the lemniscates $\left(x^2+y^2\right)^2=x^2-y^2$ **[M.U. 1995, 2007]**

Solution: If we put $x=r\cos\theta, y=r\sin\theta, \left(x^2+y^2\right)^2=x^2-y^2$ becomes

$$r^4=r^2\left(\cos^2\theta-\sin^2\theta\right) \quad \text{i.e.} \quad r^2=\cos 2\theta$$

$$\frac{1}{\left(1+x^2+y^2\right)^2}=\frac{1}{\left(1+r^2\right)^2}$$

Now, on the loop r varies from 0 to $\sqrt{\cos 2\theta}$ and θ varies from $-\pi/4$ to $\pi/4$.



$$\therefore \quad I=\int_{-\pi/4}^{\pi/4}\int_0^{\sqrt{\cos 2\theta}}\frac{r\,dr\,d\theta}{\left(1+r^2\right)^2}$$

$$=2\int_0^{\pi/4}\int_0^{\sqrt{\cos 2\theta}}\frac{r\,dr\,d\theta}{\left(1+r^2\right)^2}$$

$$=2\int_0^{\pi/4}\frac{1}{2}\left[-\frac{1}{1+r^2}\right]_0^{\sqrt{\cos 2\theta}}d\theta$$

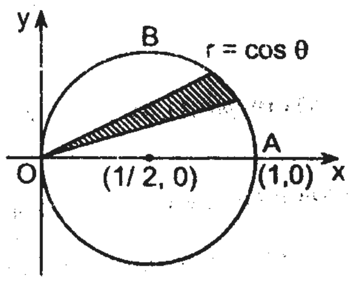
$$=-\int_0^{\pi/4}\left[\frac{1}{1+\cos 2\theta}-1\right]d\theta=\int_0^{\pi/4}\left[1-\frac{1}{1+\cos 2\theta}\right]d\theta$$

$$=\int_0^{\pi/4}\left[1-\frac{\sec^2\theta}{2}\right]d\theta=\left[\theta-\frac{\tan\theta}{2}\right]_0^{\pi/4}$$

$$=\frac{\pi}{4}-\frac{1}{2}=\frac{\pi-2}{4}$$

Ex.64 Change to polar coordinates and evaluate $\iint_R \frac{1}{\sqrt{xy}}dx\,dy$ where R is the region of integration bounded by $x^2+y^2-x=0$ and $y\ge 0$ **[M.U. 2001, 06, 07]**

Solution: The curve $x^2+y^2-x=0$ i.e. $\left[x-(1/2)^2+y^2=(1/2)^2\right]$ is a circle with centre $\left[(1/2,0)\right]$ and radius 1/2. The line y = 0 is the x-axis. The region of integration is the upper semi-circle OAB.



To change to polar put $x = r\cos\theta, y = r\sin\theta$. Then $x^2 + y^2 - x = 0$ changes to $r^2\cos^2\theta + r^2\sin^2\theta = r\cos\theta$ i.e. $r^2 = r\cos\theta$ i.e. $r = \cos\theta$.

Now, consider a radial strip. On this strip r varies from r = 0 to $r = \cos\theta$. Then θ varies from $\theta = 0$ to $\theta = \pi/2$.

$$\therefore \quad I = \int_0^{\pi/2}\int_0^{\cos\theta} \frac{1}{r\sqrt{\sin\theta\cos\theta}}\, r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^{\cos\theta} \frac{dr\,d\theta}{\sqrt{\sin\theta\cos\theta}}$$

$$= \int_0^{\pi/2} \frac{1}{\sqrt{\sin\theta\cos\theta}}[r]_0^{\cos\theta}\,d\theta$$

$$= \int_0^{\pi/2} \frac{1}{\sqrt{\sin\theta\cos\theta}}.\cos\theta\,d\theta$$

$$= \int_0^{\pi/2} \sin^{-1/2}\theta\cos^{1/2}\theta\,d\theta$$

$$= \frac{1}{2}B\left(\frac{1}{4},\frac{3}{4}\right) = \frac{1}{2}.\frac{\Gamma(1/4)\Gamma(3/4)}{\Gamma(1)}$$

$$= \frac{1}{2}.\sqrt{2}.\pi = \frac{\pi}{\sqrt{2}}$$

Ex.65 Evaluate $\iint\sqrt{\left[\frac{a^2b^2 - b^2x^2 - a^2y^2}{a^2b^2 + b^2x^2 + a^2y^2}\right]}dx\,dy$ where R is the region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. **[M.U. 1997, 2002, 09]**

Solution: We use elliptical polar coordinates i.e. we put $x = ar\cos\theta, y = br\sin\theta, dx\,dy = ab\,r\,dr\,d\theta$. The ellipse $\left(x^2/a^2\right) + \left(y^2/b^2\right) = 1$ is transformed to circle $r^2 = 1$ i.e. $r = 1$.

$$\therefore \quad I = \iint\sqrt{\left[\frac{a^2b^2\left(1-r^2\right)}{a^2b^2\left(1+r^2\right)}\right]}\,a\,b\,r\,dr\,d\theta$$

$$= 4\int_0^{\pi/2}\int_0^1 \sqrt{\frac{1-r^2}{1+r^2}}\,a\,b\,r\,dr\,d\theta$$

$$= 4ab\int_0^{\pi/2}\int_0^1 \frac{1-r^2}{\sqrt{1-r^4}}\,r\,dr\,d\theta. \qquad \text{Put} \quad r^2 = \sin t$$

$$= 4ab\int_0^{\pi/2}\int_0^{\pi/2} \frac{1-\sin t}{\cos t}.\frac{1}{2}\cos t\,dt\,d\theta$$

$$= 2ab\int_0^{\pi/2}\int_0^{\pi/2} (1-\sin t)\,dt\,d\theta$$

$$= 2ab\int_0^{\pi/2}[t - \cos t]_0^{\pi/2}\,d\theta$$

$$= 2ab\int_0^{\pi/2}\left[\frac{\pi}{2}-1\right]d\theta = 2ab\left(\frac{\pi}{2}-1\right)\int_0^{\pi/2} d\theta$$

$$= 2ab\left(\frac{\pi}{2}-1\right)[\theta]_0^{\pi/2} = \pi\, ab\left(\frac{\pi}{2}-1\right)$$

Ex.66 Evaluate $\iint\sqrt{\left\{\frac{1-x^2-y^2}{1+x^2+y^2}\right\}}$ over the area of the positive quadrant of the circle $x^2+y^2=1$ by changing to polar co-ordinates. **[M.U. 1997, 98]**

Solution: Putting a = 1, b = 1 in the above example, we get

$$I = \pi\left(\frac{\pi}{2}-1\right)$$

EXERCISE

Evaluate the following integrals over the region stated, by changing to polar coordinates:

- $\iint xy\left(x^2+y^2\right)^{3/2} dx\,dy$ over the first quadrant of the circle $x^2+y^2=a^2$

[M.U. 1995, 2003]

Ans. $a^2/4$