

## Chapter 26 : Integration of single variable functions

**Example 1 :**  $\int x^6 dx + c$

**Ans :**  $\frac{x^7}{7} + C$

**Example 2 :**  $\int (2x)^5 dx$

**Ans :**  $\frac{(2x)^6}{12} + C$

**Example 3 :**  $\int \frac{5}{x} dx$

**Ans :**  $5 \log |x|$

**Example 4 :**  $\int \frac{1}{x+2} dx$

**Ans :**  $\log |x+2|$

**Example 5 :**  $\int \frac{6}{(9x-2)} dx$

**Ans :**  $\frac{2}{3} \log |9x-2| + C$

**Example 6 :**  $\int \cos 5x dx$

**Ans :**  $\frac{1}{5} \sin 5x + C$

**Example 7 :**  $\int \sin(2x-1) dx$

**Ans :**  $-\frac{1}{2} \cos(2x-1) + C$

**Example 8 :**  $\int e^{(5x-3)} dx$

**Ans :**  $\frac{1}{5} e^{5x-3} + C$

**Example 9 :**  $\int 2^x dx$

**Ans :**  $\frac{2^x}{\log_e 2} + C$

**Example 10 :**  $\int 3^{5x+2} dx$

**Ans :**  $\int \frac{3^{5x+2}}{\log 3} \times \frac{1}{5} + C$

**Example 11 :**  $\int \left( \frac{1}{3x^2+5} \right) dx$

**Ans :**  $\frac{1}{\sqrt{15}} \tan^{-1} \frac{\sqrt{3}x}{\sqrt{5}} + C$

**Example 12 :**  $\int \frac{1}{x^2+4x+9} dx$

**Ans :**  $\frac{1}{\sqrt{5}} \tan^{-1} \frac{x+2}{\sqrt{5}} + C$

**Example 13 :**  $\int \frac{1}{x^2-4x+3} dx$

**Ans :**  $\frac{1}{2} \log \left| \frac{x-3}{x-1} \right| + C$

**Example 14 :**  $\int \left( \frac{1}{7-2x^2} \right) dx$

**Ans :**  $\frac{1}{2 \times \sqrt{14}} \log \left| \frac{\sqrt{7} + \sqrt{2x}}{\sqrt{7} - \sqrt{2x}} \right| + C$

**Example 15 :**  $\int \left( \frac{1}{\sqrt{3x^2+5}} \right) dx$

**Ans :**  $\frac{1}{\sqrt{3}} \log \left| x + \sqrt{x^2 + \frac{5}{3}} \right| + C$

**Example 16 :**  $\int \frac{1}{\sqrt{9x^2-4}} dx$

**Ans :**  $\frac{1}{3} \log \left| x + \sqrt{\frac{9x^2-4}{9}} \right| + C$

**Example 17 :**  $\int \frac{1}{\sqrt{4x^2-4x-5}} dx$

**Ans :**  $\frac{1}{2} \log |(2x-1) + \sqrt{4x^2-4x-5}| + C$

**Example 18 :**  $\int \frac{1}{\sqrt{8-2x-x^2}} dx$

**Ans :**  $\sin^{-1} \frac{x+1}{3} + C$

**Example 19 :**  $\int \sqrt{x^2+9} dx$

**Ans :**  $\frac{x}{2} \sqrt{x^2+3^2} + \frac{9}{2} \log |x + \sqrt{x^2+9}| + C$

**Example 20 :**  $\int \sqrt{x^2-4} dx$

**Ans :**  $\frac{x}{2} \sqrt{x^2-(2)^2} - \frac{4}{2} \log |x + \sqrt{x^2-(2)^2}| + C$

**Example 21 :**  $\int \sqrt{25-x^2} dx$

**Ans :**  $\frac{x}{2} \sqrt{(5)^2-x^2} + \frac{25}{2} \sin^{-1} \left( \frac{x}{5} \right) + C$

**Example 22 :**  $\int \sqrt{4x^2+4x-15} dx$

**Ans :**  $\frac{1}{4} (2x+1) \sqrt{4x^2+4x-15} - 4 \log |(2x+1) + \sqrt{4x^2+4x-15}| + C$

**Example 23 :**  $\int \sqrt{2x-x^2} dx$

**Ans :**  $\frac{(x-1)}{2} \sqrt{2x-x^2} + \frac{1}{2} \sin^{-1} (x-1) + C$

**Example 24 :**  $\int \sqrt{(3x-1)(x+2)} dx$

**Ans :**  $\frac{6x+5}{12} \times \sqrt{(3x-1)(x+2)} - \frac{49}{24\sqrt{3}} \log \left| \frac{6x+5}{2\sqrt{3}} + \sqrt{(3x-1)(x+2)} \right| + C$

**Example 25 :**  $\int \frac{2x+3}{x^2+4x+9} dx$

**Ans :**  $\log |x^2+4x+9| - \frac{1}{\sqrt{5}} \tan^{-1} \frac{x+2}{\sqrt{5}} + C$

**Example 26 :**  $\int \frac{x^2-3}{x^2+3} dx$

**Ans :**  $x - 2\sqrt{3} \tan^{-1} \frac{x}{\sqrt{3}} + C$

**Example 27 :**  $\int \frac{2x^2-3x+1}{x^2+x+1} dx$

**Ans :**  $2x - \frac{5}{2} \log |x^2+x+1| + \sqrt{3} \tan^{-1} \frac{2x+1}{\sqrt{3}} + C$

**Example 28 :**  $\int \frac{2x^3+18x-1}{x^2+9} dx$

**Ans :**  $x^2 - \frac{1}{3} \tan^{-1} \frac{x}{3} + C$

**Example 29 :**  $\int \frac{x^3}{x^2-a^2} dx$

**Ans :**  $\frac{x^2}{2} + \frac{a^2}{2} \log |x^2-a^2| + C$

**Remember :**

$$(1) \int x^n dx = \frac{x^{n+1}}{n+1} + C$$

$$(2) \int \frac{1}{x} dx = \log_e |x| + C = \ln |x| + C$$

$$(3) \int \cos x dx = \sin x + C$$

$$(4) \int \sin x dx = -\cos x + C$$

$$(5) \int e^x dx = e^x + C$$

$$(6) \int a^x dx = \frac{a^x}{\log_e a} + C = \frac{a^x}{\ln a} + C$$

$$[1A] \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$$

$$[1B] \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C$$

$$[1C] \int \frac{1}{a^2 - x^2} dx = - \int \frac{1}{x^2 - a^2} dx = - \frac{1}{2a} \log_e \left| \frac{x-a}{x+a} \right| + C = \frac{1}{2a} \ln \left| \frac{x+a}{x-a} \right| + C$$

$$[2A] \int \frac{1}{\sqrt{x^2 + a^2}} dx = \log_e |x + \sqrt{x^2 + a^2}| + C$$

$$[2B] \int \frac{1}{\sqrt{x^2 - a^2}} dx = \log_e |x + \sqrt{x^2 - a^2}| + C$$

$$[2C] \int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a} + C = -\cos^{-1} \frac{x}{a} + C$$

$$[3] \int \frac{1}{|x|\sqrt{x^2 - a^2}} dx = \frac{1}{a} \sec^{-1} \frac{x}{a} + C = -\frac{1}{a} \operatorname{cosec}^{-1} \frac{x}{a} + C$$

$$[4A] \int \sqrt{x^2 + a^2} dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \log_e |x + \sqrt{x^2 + a^2}| + C$$

$$[4B] \int \sqrt{x^2 - a^2} dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log_e |x + \sqrt{x^2 - a^2}| + C$$

$$[4C] \int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C$$

$$[5A] \int \sec^2 x dx = \tan x + C$$

$$[5B] \int \operatorname{cosec}^2 x dx = -\cot x + C$$

$$[5C] \int \sec \tan x dx = \sec x + C$$

$$[5D] \int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x + C$$

$$[5E] \int \tan x dx = \log |\sec x| + C$$

$$[5F] \int \cot x dx = \log |\sin x| + C$$

$$[5G] \int \operatorname{cosec} x dx = \log \left| \tan \frac{x}{2} \right| + C$$

$$= \log |\operatorname{cosec} x - \cot x| + C$$

$$[5H] \int \sec x dx = \log \left| \tan \left( \frac{\pi}{4} + \frac{x}{2} \right) \right| + C$$

$$= \log |\sec x + \tan x| + C$$

Examples based on formula :  $\int x^n dx = \frac{x^{n+1}}{(n+1)} + C$

**Example 1 :**  $\int x^6 dx = \frac{x^{6+1}}{6+1} + C = \frac{x^7}{7} + C$

**Example 2 :**  $\int (2x)^5 dx = \frac{(2x)^{5+1}}{5+1} \times \frac{1}{2} + C$   
 $= \frac{(2x)^6}{6} \times \frac{1}{2} = \frac{(2x)^6}{12} + C$

Examples based on formula :  $\int \frac{1}{x} dx = \log_e |x| + C$

**Example 3 :**  $\int \frac{5}{x} dx = 5 \int \frac{1}{x} dx = 5 \log_e |x|$

**Example 4 :**  $\int \frac{1}{x+2} dx = \log_e |x+2|$

**Example 5 :**  $\int \frac{6}{(9x-2)} dx = 6 \int \frac{1}{(9x-2)} dx$   
 $= 6 \log_e |9x-2| \times \frac{1}{9} + C$   
 $= \frac{2}{3} \log_e |9x-2| + C$

Examples based on formula :  $\int \cos x dx = \sin x + C$

**Example 6 :**  $\int \cos 5x dx = \sin 5x \times \frac{1}{5} + C$   
 $= \frac{1}{5} \sin 5x + C$

Examples based on formula :  $\int \sin x dx = -\cos x + C$

**Example 7 :**  $\int \sin (2x-1) dx = -\cos (2x-1) \times \frac{1}{2} + C$   
 $= -\frac{1}{2} \cos (2x-1) + C$

**CAUTION :**  $\int \cos^2 x dx \neq \frac{\cos^3 x}{3} \times \frac{1}{\sin x}$

Examples based on formula :  $\int e^x dx = e^x + C$

**Example 8 :**  $\int e^{(5x-3)} dx = e^{5x-3} \times \frac{1}{5} + C$   
 $= \frac{1}{5} e^{5x-3} + C$

**CAUTION :**  $\int e^{(x^2)} dx \neq e^{(x^2)} \times \frac{1}{2x}$

Examples based on formula :  $\int a^x dx = \frac{a^x}{\log_e a} + C$

**Example 9 :**  $\int 2^x dx = \frac{2^x}{\log_e 2} + C$

**Example 10 :**  $\int 3^{5x+2} dx = \int \frac{3^{5x+2}}{\log_e 3} \times \frac{1}{5} + C$

$$[1 \text{ A}] \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$$

$$[1 \text{ B}] \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log_e \left| \frac{x-a}{x+a} \right| + C$$

$$[1 \text{ C}] \int \frac{1}{a^2 - x^2} dx = - \int \frac{1}{x^2 - a^2} dx = - \frac{1}{2a} \log_e \left| \frac{x-a}{x+a} \right| + C = \frac{1}{2a} \log_e \left| \frac{x+a}{x-a} \right| + C$$

**Remember :**  $a^2 + 2ab + b^2 = (a+b)^2$  . The last term  $= b^2 = \frac{(2ab)^2}{4a^2} = \frac{(\text{Middle term})^2}{4 \times \text{First term}}$

$$\begin{aligned} 4x^2 + 9x + 3 &= (2x)^2 + 9x + 3 \\ &= (2x)^2 + 9x + [ \quad ] - [ \quad ] + 3 \\ &= (2x)^2 + 9x + \left[ \frac{(9x)^2}{4 \times 4x^2} \right] - \left[ \frac{(9x)^2}{4 \times 4x^2} \right] + 3 \quad \left[ \because \text{last term} = \frac{(\text{Middle term})^2}{4 \times \text{First term}} \right] \\ &= (2x)^2 + 9x + \left[ \frac{81}{16} \right] - \left[ \frac{81}{16} \right] + 3 \\ &= \left( 2x + \frac{9}{4} \right)^2 + \left( 3 - \frac{81}{16} \right) \\ &= \left( 2x + \frac{9}{4} \right)^2 + \left( \frac{-65}{16} \right) \\ &= \left( 2x + \frac{9}{4} \right)^2 - \left( \frac{\sqrt{65}}{4} \right)^2 \end{aligned}$$

**Example 11 :**  $\int \left( \frac{1}{3x^2 + 5} \right) dx$

$$\begin{aligned} \text{Ans : } &\int \left( \frac{1}{3x^2 + 5} \right) dx \\ &= \int \left[ \frac{1}{3 \left( x^2 + \frac{5}{3} \right)} \right] dx \\ &= \frac{1}{3} \int \left[ \frac{1}{x^2 + \frac{5}{3}} \right] dx \\ &= \frac{1}{3} \int \frac{1}{x^2 + \left[ \frac{\sqrt{5}}{\sqrt{3}} \right]^2} dx \\ &= \frac{1}{3} \times \frac{1}{\frac{\sqrt{5}}{\sqrt{3}}} \tan^{-1} \frac{x}{\frac{\sqrt{5}}{\sqrt{3}}} + C \quad \left[ \text{Remember : } \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} \right] \\ &= \frac{1}{\sqrt{15}} \tan^{-1} \frac{\sqrt{3}x}{\sqrt{5}} + C \end{aligned}$$

**Example 12 :**  $\int \frac{1}{x^2 + 4x + 9} dx$

$$\begin{aligned} \text{Ans : } &\int \frac{1}{x^2 + 4x + 9} dx \\ &= \int \frac{1}{x^2 + 4x + 4 + 5} dx \\ &= \int \frac{1}{(x+2)^2 + (\sqrt{5})^2} dx \\ &= \frac{1}{\sqrt{5}} \tan^{-1} \frac{x+2}{\sqrt{5}} + C \quad \left[ \text{Remember : } \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} \right] \end{aligned}$$

**Example 13 :**  $\int \frac{1}{x^2 - 4x + 3} dx$

**Ans :**  $\int \frac{1}{x^2 - 4x + 3} dx$

$$= \int \frac{1}{x^2 - 4x + 4 - 1} dx$$
$$= \int \frac{1}{(x - 2)^2 - (1)^2} dx$$
$$= \frac{1}{2(1)} \log \left| \frac{(x-2)-1}{(x-2)+1} \right| + C \quad \left[ \text{Remember : } \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| \right]$$
$$= \frac{1}{2} \log \left| \frac{x-3}{x-1} \right| + C$$

**Example 14 :**  $\int \left( \frac{1}{7 - 2x^2} \right) dx$

**Ans :**  $\int \left( \frac{1}{7 - 2x^2} \right) dx$

$$= \int \frac{1}{2 \left[ \frac{7}{2} - x^2 \right]} dx$$
$$= \frac{1}{2} \int \frac{1}{\frac{7}{2} - x^2} dx$$
$$= \frac{1}{2} \int \frac{1}{\left( \frac{\sqrt{7}}{\sqrt{2}} \right)^2 - x^2} dx$$
$$= \frac{1}{2} \times \frac{1}{2 \times \frac{\sqrt{7}}{\sqrt{2}}} \log \left| \frac{x + \frac{\sqrt{7}}{\sqrt{2}}}{x - \frac{\sqrt{7}}{\sqrt{2}}} \right| + C \quad \left[ \text{Remember : } \int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log_e \left| \frac{x+a}{x-a} \right| \right]$$
$$= \frac{1}{2 \times \sqrt{14}} \log \left| \frac{\sqrt{2}x + \sqrt{7}}{\sqrt{2}x - \sqrt{7}} \right| + C$$

$$[2 \text{ A}] \int \frac{1}{\sqrt{x^2 + a^2}} dx = \log_e |x + \sqrt{x^2 + a^2}| + C$$

$$[2 \text{ B}] \int \frac{1}{\sqrt{x^2 - a^2}} dx = \log_e |x + \sqrt{x^2 - a^2}| + C$$

$$[2 \text{ C}] \int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a} + C$$

**Example 15 :**  $\int \left( \frac{1}{\sqrt{3x^2 + 5}} \right) dx$

**Ans :**  $\int \left( \frac{1}{\sqrt{3x^2 + 5}} \right) dx$

$$= \int \left[ \frac{1}{\sqrt{3 \left( x^2 + \frac{5}{3} \right)}} \right] dx$$

$$= \int \left[ \frac{1}{\sqrt{3} \sqrt{x^2 + \frac{5}{3}}} \right] dx$$

$$= \frac{1}{\sqrt{3}} \int \left[ \frac{1}{\sqrt{x^2 + \frac{5}{3}}} \right] dx$$

$$= \frac{1}{\sqrt{3}} \int \frac{1}{\sqrt{x^2 + \left( \frac{\sqrt{5}}{\sqrt{3}} \right)^2}} dx$$

$$= \frac{1}{\sqrt{3}} \log \left| x + \sqrt{x^2 + \left( \frac{\sqrt{5}}{\sqrt{3}} \right)^2} \right| + C \quad \left[ \text{Remember : } \int \frac{1}{\sqrt{x^2 + a^2}} dx = \log |x + \sqrt{x^2 + a^2}| \right]$$

$$= \frac{1}{\sqrt{3}} \log \left| x + \sqrt{x^2 + \frac{5}{3}} \right| + C$$

**Example 16 :**  $\int \frac{1}{\sqrt{9x^2 - 4}} dx$

**Ans :**  $\int \frac{1}{\sqrt{9x^2 - 4}} dx$

$$= \int \frac{1}{\sqrt{9 \left( x^2 - \frac{4}{9} \right)}} dx$$

$$= \int \frac{1}{3 \sqrt{x^2 - \frac{4}{9}}} dx$$

$$= \frac{1}{3} \int \frac{1}{\sqrt{x^2 - \frac{4}{9}}} dx$$

$$= \frac{1}{3} \int \frac{1}{\sqrt{x^2 - \left( \frac{2}{3} \right)^2}} dx$$

$$= \frac{1}{3} \times \log \left| x + \sqrt{x^2 - \left( \frac{2}{3} \right)^2} \right| + C \quad \left[ \text{Remember : } \int \frac{1}{\sqrt{x^2 - a^2}} dx = \log |x + \sqrt{x^2 - a^2}| \right]$$

$$= \frac{1}{3} \log \left| x + \sqrt{\frac{9x^2 - 4}{9}} \right| + C$$

**Example 17 :**  $\int \frac{1}{\sqrt{4x^2 - 4x - 5}} dx$

**Ans :**  $\int \frac{1}{\sqrt{4x^2 - 4x - 5}} dx$   
 $= \int \frac{1}{\sqrt{4x^2 - 4x + 1 - 6}} dx$   
 $= \int \frac{1}{\sqrt{(4x^2 - 4x + 1) - (6)}} dx$   
 $= \frac{1}{\sqrt{(2x - 1)^2 - (\sqrt{6})^2}} dx$   
 $= \log \left| (2x - 1) + \sqrt{(2x - 1)^2 - (\sqrt{6})^2} \right| \times \frac{1}{2} + C$   
 $= \frac{1}{2} \log \left| (2x - 1) + \sqrt{4x^2 - 4x - 5} \right| + C$

**Example 18 :**  $\int \frac{1}{\sqrt{8 - 2x - x^2}} dx$

**Ans :**  $\int \frac{1}{\sqrt{8 - 2x - x^2}} dx$   
 $= \int \frac{1}{\sqrt{9 - 1 - 2x - x^2}} dx$   
 $= \int \frac{1}{\sqrt{9 - x^2 - 2x - 1}} dx$   
 $= \int \frac{1}{\sqrt{9 - (x^2 + 2x + 1)}} dx$   
 $= \int \frac{1}{\sqrt{(3)^2 - (x + 1)^2}} dx$   
 $= \sin^{-1} \frac{x + 1}{3} + C \quad \left[ \text{Remember : } \int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a} \right]$

$$[4 \text{ A}] \int \sqrt{x^2 + a^2} \, dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \log_e |x + \sqrt{x^2 + a^2}| + C$$

$$[4 \text{ B}] \int \sqrt{x^2 - a^2} \, dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log_e |x + \sqrt{x^2 - a^2}| + C$$

$$[4 \text{ C}] \int \sqrt{a^2 - x^2} \, dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C$$

**Example 19 :**  $\int \sqrt{x^2 + 9} \, dx$

$$\begin{aligned} \text{Ans : } & \int \sqrt{x^2 + 9} \, dx \\ &= \int \sqrt{x^2 + (3)^2} \, dx \\ &= \frac{x}{2} \sqrt{x^2 + (3)^2} + \frac{(3)^2}{2} \log_e |x + \sqrt{x^2 + 9}| + C \\ &= \frac{x}{2} \sqrt{x^2 + 3^2} + \frac{9}{2} \log |x + \sqrt{x^2 + 9}| + C \end{aligned}$$

**Example 20 :**  $\int \sqrt{x^2 - 4} \, dx$

$$\begin{aligned} \text{Ans : } & \int \sqrt{x^2 - 4} \, dx \\ &= \int \sqrt{x^2 - (2)^2} \, dx \\ &= \frac{x}{2} \sqrt{x^2 - (2)^2} - \frac{4}{2} \log |x + \sqrt{x^2 - (2)^2}| + C \end{aligned}$$

**Example 21 :**  $\int \sqrt{25 - x^2} \, dx$

$$\begin{aligned} \text{Ans : } & \int \sqrt{25 - x^2} \, dx \\ &= \int \sqrt{(5)^2 - x^2} \, dx \\ &= \frac{x}{2} \sqrt{(5)^2 - x^2} + \frac{25}{2} \sin^{-1} \left( \frac{x}{5} \right) + C \end{aligned}$$

**Example 22 :**  $\int \sqrt{4x^2 + 4x - 15} \, dx$

$$\begin{aligned} \text{Ans : } & \int \sqrt{4x^2 + 4x - 15} \, dx \\ &= \int \sqrt{4x^2 + 4x + 1 - 16} \, dx \\ &= \int \sqrt{(2x + 1)^2 - (4)^2} \, dx \\ &= \left\{ \frac{(2x+1)}{2} \sqrt{(2x+1)^2 - (4)^2} - \frac{16}{2} \log |(2x+1) + \sqrt{(2x+1)^2 - 16}| \right\} \times \frac{1}{2} + C \\ &= \frac{1}{4} (2x + 1) \sqrt{4x^2 + 4x - 15} - 4 \log |(2x + 1) + \sqrt{4x^2 + 4x - 15}| + C \end{aligned}$$