Power Series Solution Near An Ordinary Point. / power series method.
(x=0 is an ordinary Point).

GTU. DEC 2009, DEC. 2011, Jan. 2013, Summer 2014, June 2014,

Ex:- Solve the eqn $\frac{d^2y}{dx^2}+y=0$. by Power series method.

$\Rightarrow$ $y''+y=0$ ------ (1)

Here $x=0$ is an ordinary Point.

$$y=\sum_{k=0}^{\infty} a_k \cdot x^k \quad \text{------ (2)}$$

$$= a_0+a_1x+a_2x^2+a_3x^3+\text{---}$$

$$y'=\sum_{k=1}^{\infty} a_k k x^{k-1} \quad \text{------ (3)}$$

$$= a_1+2a_2x+3a_3x^2+4a_4x^3+\text{---}$$

$$y''=\sum_{k=2}^{\infty} a_k k(k-1)x^{k-2} \quad \text{------ (4)}$$

$$= 2\cdot1a_2+3\cdot2a_3x+4\cdot3a_4x^2+\text{---}$$

Put, the values of eqn (2), (3), (4) in eqn (1);

$y''+y=0$

$$\therefore \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} + \sum_{k=0}^{\infty} a_k x^k = 0.$$

$$\boxed{k \to k+2}$$

$$\therefore \sum_{k=0}^{\infty} (k+1)(k+2) a_{k+2} \cdot x^k + \sum_{k=0}^{\infty} a_k x^k = 0.$$

$\therefore$ Equating the coefficient of $x^k$ with '0'.

$$(k+1)(k+2) a_{k+2} + a_k = 0$$

$$\checkmark \boxed{a_{k+2} = \frac{-a_k}{(k+2)(k+1)}}$$ Recurrence Relation

Put $k=0$.

$$a_2 = \frac{-a_0}{2 \cdot 1} = -\frac{a_0}{2!}$$

$$k=1 \Rightarrow a_3 = \frac{-a_1}{3 \cdot 2} = \frac{-a_1}{3!}$$

$$k=2 \Rightarrow a_4 = \frac{-a_2}{4 \cdot 3} = \frac{a_0}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{a_0}{4!}$$

$$k=3 \Rightarrow a_5 = \frac{-a_3}{5 \cdot 4} = \frac{a_1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{a_1}{5!}$$

$$k=4 \Rightarrow a_6 = \frac{-a_4}{6 \cdot 5} = -\frac{a_0}{6!}$$

$k=5 \Rightarrow a_7 = \frac{-a_5}{7.6} = \frac{-a_1}{7.6.5.4.3.2.1} = \frac{-a_1}{7!}$

& so on..

Put all values in eqn (1),

$\therefore y = \sum_{k=0}^{\infty} a_k x^k$

$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$

$= a_0 + a_1 x + \left[\frac{-a_0}{2!}\right]x^2 + \left[\frac{-a_1}{3!}\right]x^3 + \left[\frac{a_0}{4!}\right]x^4$

$+ \frac{a_1}{5!}x^5 + \left(\frac{-a_0}{6!}\right)x^6 + \left(\frac{-a_1}{7!}\right)x^7 + \cdots$

$y = a_0\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right] +$

$a_1\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right].$

GTU. winter 2013.

Ex:- $\frac{d^2y}{dx^2} + x^2 y = 0$.

Soln:- $x=0$ is an ordinary Point;

So series soln is given by,

$y = y'' + x^2 y = 0$ ——— (1)

$$y = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \quad \text{---- (2)}$$

$$y' = \sum_{k=1}^{\infty} k a_k x^{k-1} \quad \text{---- (3)}$$

$$= 1 \cdot a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$$

$$y'' = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} \quad \text{---- (4)}$$

$$= 2 \cdot 1 a_2 + 3 \cdot 2 a_3 x + 4 \cdot 3 a_4 x^2 + \cdots$$

Put all values in eqn (1).

$$y'' + x^2 y = 0$$

$$\therefore \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} + x^2 \sum_{k=0}^{\infty} a_k x^k = 0.$$

$$\therefore \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} + \sum_{k=0}^{\infty} a_k x^{k+2} = 0.$$

$$\boxed{k \rightarrow k+4}$$

$$\therefore \sum_{k=-2}^{\infty} (k+4)(k+3) a_{k+4} x^{k+2} + \sum_{k=0}^{\infty} a_k x^{k+2} = 0.$$

$$\therefore (-2+4)(-2+3) a_2 x^0 + (-1+4)(-1+3) a_3 x + \sum_{k=0}^{\infty} (k+4)(k+3) a_{k+4} x^{k+2} + \sum_{k=0}^{\infty} a_k x^{k+2} = 0.$$

$$2\cdot 1 a_2 x + 3\cdot 2 a_3 x + \sum_{k=0}^{\infty} (k+4)(k+3) a_{k+4} x^{k+2}$$

$$+ \sum_{k=0}^{\infty} a_k x^{k+2} = 0.$$

$$2a_2 = 0 \Rightarrow \boxed{a_2 = 0}$$

$$6a_3 = 0 \Rightarrow \boxed{a_3 = 0}$$

Equating the coefficient of $x^{k+2}$ with "0".

$$(k+4)(k+3)a_{k+4} + a_k = 0$$

$$\Rightarrow \boxed{a_{k+4} = \frac{-a_k}{(k+4)(k+3)}}$$

$$k=0 \Rightarrow a_4 = \frac{-a_0}{4\cdot 3}$$

$$k=1 \Rightarrow a_5 = \frac{-a_1}{4\cdot 5}$$

$$k=2 \Rightarrow a_6 = \frac{-a_2}{5\cdot 6} = 0$$

$$k=3 \Rightarrow a_7 = \frac{-a_3}{6\cdot 7} = 0$$

$$k=4 \Rightarrow a_8 = \frac{-a_4}{7\cdot 8} = \frac{a_0}{3\cdot 4\cdot 7\cdot 8}$$

$$k=5 \Rightarrow a_9 = \frac{-a_5}{8\cdot 9} = \frac{a_1}{4\cdot 5\cdot 8\cdot 9} \ \&$$

so on...

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$$

$$= a_0 + a_1 x - \frac{a_0}{3\cdot 4} x^4 - \frac{a_1}{4\cdot 5} x^5 + \frac{a_0}{3\cdot 4\cdot 7\cdot 8} x^8 + \frac{a_1}{4\cdot 5\cdot 8\cdot 9} x^9 + \cdots$$

$$y = a_0\left(1 - \frac{x^4}{3\cdot 4} + \frac{x^8}{3\cdot 4\cdot 7\cdot 8} \cdots\right) + a_1\left(x - \frac{x^5}{4\cdot 5} + \frac{x^9}{4\cdot 5\cdot 8\cdot 9} \cdots\right)$$

GTU March 2010, 2013.

Ex: $(1-x^2)y'' - 2xy' + 2y = 0$.

Soln:- $x=0$ is an ordinary point.

& except $x=-1$, $x=1$ all other points are regular points.

$\because (1-x^2)y'' - 2xy' + 2y = 0$ ------ (1)

$\therefore y = \sum_{k=0}^{\infty} a_k x^k$ ---- (2)

$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

$\therefore$ Differentiating $y' = dy/dx$

$$y' = \sum_{k=1}^{\infty} a_k k x^{k-1} \quad \text{------} \quad (3)$$

$$= a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \text{---}$$

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1) x^{k-2} \quad \text{------} \quad (4)$$

$$= 2a_2 + 3\cdot 2 a_3 x + 4\cdot 3 a_4 x^2 + \text{---}$$

Put all values in eqn (1)

$$(1-x^2) \sum_{k=2}^{\infty} a_k k(k-1) x^{k-2} - 2x \sum_{k=1}^{\infty} a_k k x^{k-1} + 2\sum_{k=0}^{\infty} a_k x^k = 0$$

$$\sum_{k=2}^{\infty} a_k k(k-1) x^{k-2} - \sum_{k=0}^{\infty} a_k k(k-1) x^k - 2\sum_{k=1}^{\infty} a_k k x^k + 2\sum_{k=0}^{\infty} a_k x^k = 0$$

$$\sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} - \sum_{k=0}^{\infty} \left[k(k-1) + 2k - 2\right] a_k x^k = 0$$

$$\sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} - \sum_{k=0}^{\infty} (k^2 - k + 2k - 2) a_k x^k = 0$$

$$\sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} - \sum_{k=0}^{\infty} (k+2)(k-1) a_k x^k = 0.$$



$\boxed{k \to k+2}$

$$\therefore \sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2}x^k - \sum_{k=0} (k+2)(k+1)a_k x^k = 0$$

equating the coefficients of $x^k$ by "0".

$$\therefore (k+2)(k+1)a_{k+2} - (k+2)(k-1)a_k = 0$$

$$\therefore a_{k+2} = \frac{(k+2)(k-1)a_k}{(k+2)(k+1)}$$

$$\therefore a_{k+2} = \frac{k-1}{k+1} a_k$$

For $k=0$, $a_2 = -a_0$

$k=1$, $a_3 = 0 \cdot a_1 = 0$.

$k=2$, $a_4 = \frac{1}{3} a_2 = -\frac{1}{3} a_0$

$k=3$, $a_5 = \frac{2}{4} a_3 = 0$.

$k=4$, $a_6 = \frac{3}{5} a_4 = \frac{3}{5}\left(-\frac{1}{3}\right) a_0$

$= -\frac{1}{5} a_0$

& so on ---

$\therefore$ Thus Required Power Series soln is

$$\therefore y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

$y = a_0 + a_1 x + (-a_0 x^2) + 0 + (-\frac{1}{3} a_0 x^4)$
$+ 0 + (-\frac{1}{5} a_0 x^6) + \cdots$

$\therefore y = a_1 x + a_0 \left(1 - x^2 - \frac{1}{3} x^4 - \frac{1}{5} x^6 - \frac{1}{7} x^8 + \cdots\right)$

Ex:- GTU. May 2011, Jan 2015

Sol:- $y' = 2xy$. $(-y(0) = 1)$

$\therefore x_0 = 0$ is Ordinary Point

$\therefore y' = 2xy$ ---- (1)

$\therefore y = \sum_{k=0}^{\infty} a_k x^k$ ---- (2)

$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

$\therefore y' = \sum_{k=1}^{\infty} k a_k x^{k-1}$ ---- (3)

$= a_1 + 2a_2 x + 3a_3 x^2 + \cdots$

$\therefore$ Put in eqn (1),

$y' - 2xy = 0$

$\sum_{k=1}^{\infty} k a_k x^{k-1} - 2x \sum_{k=0}^{\infty} a_k x^k$

$\therefore \sum_{k=1}^{\infty} k a_k x^{k-1} - 2 \sum_{k=0}^{\infty} a_k x^{k+1}$ ---- (4)



$k \to k+2$

$$\therefore \sum_{k=-1}^{\infty}(k+2)a_{k+2}x^{k+1} - 2\sum_{k=0}^{\infty}a_k x^{k+1} = 0$$

$$\Rightarrow a_1 x^0 + \sum_{k=0}^{\infty}(k+2)a_{k+2}x^{k+1} - 2\sum_{k=0}^{\infty}a_k x^{k+1} = 0$$

$$\therefore \boxed{a_1 = 0} \quad \& \quad (k+2)a_{k+2} - 2a_k = 0$$

$$a_{k+2} = \frac{2a_k}{k+2}$$

$k=0 \Rightarrow a_2 = \frac{\cancel{2}a_0}{\cancel{2}} = \boxed{a_0 = a_2}$

$k=1 \Rightarrow a_3 = \frac{2a_1}{3} = \boxed{0 = a_3}$

$k=2 \Rightarrow a_4 = \frac{\cancel{2}a_2}{\cancel{4}_2} = \boxed{\frac{1}{2!}a_0 = a_4}$

$k=3 \Rightarrow a_5 = \frac{2a_3}{3+2} = \boxed{0 = a_5}$

$k=4 \Rightarrow a_6 = \frac{\cancel{2}a_4}{\cancel{6}_3} = \frac{1}{3\times 2}a_0 = \boxed{\frac{1}{3!}a_0 = a_6}$

& so on....

$$\therefore y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$$

$$= a_0 + 0 + a_0 x^2 + 0 + \frac{1}{2!}a_0 x^4 + 0 + \frac{1}{3!}a_0 x^6 + \cdots$$

$$= a_0\left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots\right) = a_0 e^{x^2}$$



$y(0) = a_0(1 + 0 + 0 + 0 + \cdots)$

$\boxed{1 = a_0}$ $\quad y = \left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^n}{5!} + \cdots\right).$

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Similarly : $y' + 2xy = 0$ GTU. Dec. 2011.

Ex :- Summer 2014. Find Power series

Sol$^n$ : $\frac{d^2y}{dx^2} + y = 0$ about $x_0 = 0$.

Sol$^n$ :- $y'' + y = 0$ ——— (1)

$x_0 = 0$ is an ordinary Point

$y = \sum_{k=0}^{\infty} a_k x^k$ ——— (2)

$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

$y' = \sum_{k=1}^{\infty} k a_k x^{k-1}$ ——— (3)

$= a_1 + 2a_2 x + 3a_3 x^2 + \cdots$

$y'' = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}$

$2a_2 + 3\cdot 2 a_3 x + 4\cdot 3 a_4 x^2 \cdots$

Put in eq$^n$ (1)

$\therefore \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} + \sum_{k=0}^{\infty} a_k x^k = 0$

$\boxed{k \to k+2}$

$\therefore \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k + \sum_{k=0}^{\infty} a_k x^k = 0$.

Equating coefficients of $x^k$, with '0'.

$\therefore (k+2)(k+1)a_{k+2} + a_k = 0$

$\therefore a_{k+2} = \dfrac{-a_k}{(k+2)(k+1)}$ ; $k \geq 0$

$\therefore k=0 \Rightarrow a_2 = \dfrac{-a_0}{2\cdot 1} = -\dfrac{1}{2!}a_0$

$k=1 \Rightarrow a_3 = \dfrac{-a_1}{3\cdot 2} = -\dfrac{1}{3!}a_1$

$k=2 \Rightarrow a_4 = \dfrac{-a_2}{4\cdot 3} = \dfrac{a_0}{4!}$

$k=3 \Rightarrow a_5 = \dfrac{-a_3}{5\cdot 4} = \dfrac{a_1}{5!}$

& so on...

$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots$

$= a_0 + a_1x - \dfrac{1}{2!}a_0x^2 - \dfrac{1}{3!}a_1x^3 + \dfrac{a_0}{4!}x^4 + \dfrac{a_1}{5!}x^5 + \cdots$

$= a_0\left(1 - \dfrac{1}{2!}x^2 + \dfrac{1}{4!}x^4 - \cdots\right) + a_1\left(x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \cdots\right)$

$= a_0 \cos x + a_1 \sin x$.

Similarly, '$y'' = 2y'$. at $x_0 = 0$. winter 2012,

ex: $y'' + xy = 0$, $x_0 = 0$.

Jan 2013, Jan 2014



$\hookrightarrow a_{k+3} = \dfrac{-a_k}{(k+2)(k+3)}$, $k \geq 0$ $y = a_0\left(1 - \dfrac{x^3}{2\cdot 3} + \dfrac{x^6}{2\cdot 3\cdot 5\cdot 6} - \cdots\right) + a_1\left(x - \dfrac{x^4}{3\cdot 4} + \dfrac{x^7}{3\cdot 4\cdot 6\cdot 7}\right)$

Ex: $(x^2+1)y'' + xy' - xy = 0$ . ---(1) GTU May 2011, Jan. 2013, June 2013.

$\therefore x=0$ is an ordinary point

$\therefore y = \sum_{k=0}^{\infty} a_k x^k$ ----- (2)

$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

$\therefore y' = \sum_{k=1}^{\infty} k a_k x^{k-1}$ ----- (3)

$= a_1 + 2a_2 x + 3a_3 x^2 + \cdots$

$\therefore y'' = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}$ ----- (4)

$\therefore (x^2+1)\sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} + x\sum_{k=1}^{\infty} k a_k x^{k-1} - x\sum_{k=0}^{\infty} a_k x^k = 0.$

$\therefore \underbrace{\sum_{k=2}^{\infty} k(k-1)a_k x^k}_{\boxed{k \to k+1}} + \underbrace{\sum_{k=2}^{\infty} k(k-1)a_k x^{k-2}}_{\boxed{k \to k+3}}$

$+ \underbrace{\sum_{k=1}^{\infty} k a_k x^k}_{\boxed{k \to k+1}} + \sum_{k=0}^{\infty} a_k x^{k+1} = 0.$

$\therefore \sum_{k=2}^{\infty} k(k+1)a_{k+1} x^{k+1} + \sum_{k=-1}^{\infty} (k+3)(k+2) a_{k+3} x^{k+1}$

$+ \sum_{k=0}^{\infty} (k+1) a_{k+1} x^{k+1} + \sum_{k=0}^{\infty} a_k x^{k+1} = 0.$

$2(1)\,a_2 x^0 = 0 \qquad 2a_2 = 0 \qquad a_2 = 0$

$\therefore k(k+1)a_{k+1} + (k+3)(k+2)a_{k+3} + (k+1)a_{k+1} - a_k = 0$

$\therefore a_{k+1}\{(k+1)(k+1)\} + a_{k+3}(k+3)(k+2) = a_k$

$$\therefore a_{k+1} = \frac{a_k - a_{k+3}(k+3)(k+2)}{(k+1)^2}$$

$k=0 \qquad a_1 = \dfrac{a_0 - a_3(3)2}{1} = a_0 - 6a_3$

$k=1 \qquad a_2 = 0$

$k=2 \qquad a_3 = \dfrac{a_2 - a_5(5)(4)}{(3)^2} = -\dfrac{20}{9}a_5$

$$\therefore a_{k+3} = \frac{a_k - a_{k+1}(k+1)^2}{(k+3)(k+2)}$$

$k=0 \qquad a_3 = \dfrac{a_0 - a_1}{3\cdot 2} = \dfrac{1}{6}\{a_0 - a_1\}$

$k=1 \qquad a_4 = \dfrac{a_1 - a_2(4)}{4\cdot 3} = \dfrac{1}{12}a_1$

$k=2 \qquad a_5 = \dfrac{a_2 - a_3(3)^2}{5\cdot 4} = -\dfrac{\cancel{9}^{\,3}}{20}\left(\dfrac{a_0 - a_1}{\cancel{6}\,2}\right)$

$= \dfrac{3a_1 - 3a_0}{40}$

& so on...

$\therefore y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$

$= a_0 + a_1 x + 0 + \frac{1}{6}\{a_0 - a_1\} x^3 + \frac{1}{12} x^4 a_1 + \left(\frac{3}{40} a_1 - \frac{3}{40} a_0\right) x^5 + \cdots$

$y = a_0 \left\{1 + \frac{1}{6} x^3 - \frac{3}{40} x^5 + \cdots\right\} + a_1 \left\{x - \frac{1}{6} x^3 + \frac{1}{12} x^4 + \frac{3}{40} x^5 + \cdots\right\}.$

ex/ $(1+x^2) y'' + x y' - 9y = 0$ (S/W 2012)

$\therefore (1+x^2) \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} + x \sum_{k=1}^{\infty} k a_k x^{k-1} - 9 \sum_{k=0}^{\infty} a_k x^k = 0$

$\therefore \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} + \sum_{k=2}^{\infty} k(k-1) a_k x^k + \sum_{k=1}^{\infty} k a_k x^k - 9 \sum_{k=0}^{\infty} a_k x^k = 0.$

$\boxed{k \to k+2}$

$\therefore \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^{k} + \sum_{k=0}^{\infty} k(k-1) a_k x^k + \sum_{k=0}^{\infty} k a_k x^k - 9 \sum_{k=0}^{\infty} a_k x^k = 0$

$\therefore (k+2)(k+1) a_{k+2} + k(k-1) a_k + k a_k - 9 a_k = 0$

$(k+2)(k+1) a_{k+2} + a_k (k^2 - k + k - 9) = 0$

$a_{k+2} = \dfrac{-a_k (k^2 - 9)}{(k+2)(k+1)} \qquad k \geq 0.$

$k=0 \quad a_2 = +\dfrac{a_0 9}{2}$

$k=1 \quad a_3 = \dfrac{a_1 \cdot 4}{3}$

$k=2 \quad a_4 = \dfrac{+5 a_2}{3 \cdot 4} = \dfrac{15}{8} a_0$

$k=3 \quad a_5 = 0$

$k=4 \quad a_6 = -\frac{7}{16} a_0$

& so on.

$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$

$= a_0 + a_1 x + \frac{9}{2} a_0 x^2 + \frac{4}{3} a_1 x^3 + \frac{15}{8} a_0 x^4 + 0 - \frac{7}{16} a_0 x^6$

$= a_0 \left(1 + \frac{9x^2}{2} + \frac{15x^4}{8} - \frac{7x^6}{16} + \cdots\right) + a_1 \left(x + \frac{4x^3}{3} + \cdots\right)$

ex:- DEC2013. $(1-x^2) y'' - 2xy' + 2y = 0$ .. abt $x_0 = 0$.

$a_{k+2} = \frac{(k^2+k-2)a_k}{(k+1)(k+2)} = \frac{k-1}{k+1} a_k, \quad k \geq 0$

$y = a_1 x + a_0 \left(1 - x^2 - \frac{1}{3} x^4 - \frac{1}{5} x^6 + \cdots\right)$

ex:- winter2015 $(x-2) y'' - x^2 y' + 9y = 0$.

$y = a_0 \left(1 + \frac{9}{4} x^2 + \frac{3}{8} x^3 + \frac{15}{16} x^4 + \cdots\right) +$

$a_1 \left(x + \frac{3}{4} x^3 + \frac{7}{48} x^4 + \cdots\right)$.