Forier Integrals

$$A(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos(\lambda x)\, dx$$

$$B(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin(\lambda x)\, dx$$

$$f(x) = \int_{0}^{\infty} \left[A(\lambda) \cos(\lambda x) + B(\lambda) \sin(\lambda x) \right] d\lambda$$

- cosine Integral :-

$$B(\lambda) = 0$$

$$A(\lambda) = \frac{2}{\pi} \int_{0}^{\infty} f(x) \cos(\lambda x)\, dx$$

cosine Integral $f(x) = \int_{0}^{\infty} A(\lambda) \cos(\lambda x)\, d\lambda$

- sine Integral :-

$$A(\lambda) = 0$$

$$B(\lambda) = \frac{2}{\pi} \int_{0}^{\infty} f(x) \sin(\lambda x)\, dx$$

$$f(x) = \int_{0}^{\infty} B(\lambda) \sin(\lambda x)\, d\lambda$$

$\rightarrow \int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

$\rightarrow \int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$

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Ex: Show that $\int_0^{\infty} \frac{\cos(\lambda x) + \lambda \sin(\lambda x)}{1+\lambda^2} = \begin{cases} 0 & x<0 \\ \pi/2 & x=0 \\ \pi e^{-x} & x>0 \end{cases}$

$\rightarrow \quad A(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos(\lambda x) \, dx$

$= \frac{1}{\pi}\left[\int_{-\infty}^{0} 0 \, dx + \int_0^{\infty} \pi e^{-x} \cos(\lambda x) \, dx\right.$

$= \frac{1}{\pi} \cdot \pi \int_0^{\infty} e^{-x} \cos(\lambda x) \, dx$

$= \int_0^{\infty} e^{-x} \cos(\lambda x) \, dx$

$a = -1 \quad b = \cancel{x}\lambda$

$= \left[\frac{e^{-\lambda}}{1+\lambda^2}\left[-(\cos(\lambda x) + \lambda \sin(\lambda x)\right]\right]_0^{\infty}$

$= \left[0 - \frac{1}{1+\lambda^2}(-1+0)\right]$

$\boxed{A(\lambda) = \frac{1}{1+\lambda^2}}$

$\rightarrow \quad B(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin(\lambda x) \, dx$

$= \frac{1}{\pi}\left[\int_{-\infty}^{0} 0 \, dx + \int_0^{\infty} \pi e^{-x} \sin(\lambda x) \, dx\right.$

$= \frac{1}{\pi} \cdot \pi \int_0^{\infty} e^{-x} \sin(\lambda x) \, dx$

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$$= \int_0^\infty e^{-x} \sin(\lambda x)\, dx$$

$= a = -1 \quad b = \lambda$

$$= \left[\frac{e^{-x}}{a^2+\lambda^2} \left(a \sin(\lambda x) - \lambda \cos(\lambda x)\right) \right]_0^\infty$$

$$= \left[0 - \frac{1}{1+\lambda^2} (0 - \lambda) \right]$$

$$\boxed{B(\lambda) = \frac{\lambda}{1+\lambda^2}}$$

$$f(x) = \int_0^\infty \left[A(\lambda) \cos(\lambda x) + B(\lambda) \sin(\lambda x) \right] d\lambda$$

$$= \int_0^\infty \left[\frac{\cancel{x} 1}{1+\lambda^2} \cos(\lambda x) + \frac{\lambda}{1+\lambda^2} \sin(\lambda x) \right] d\lambda$$

$$= \int_0^\infty \frac{\cos(\lambda x) + \lambda \sin(\lambda x)}{1+\lambda^2}\, d\lambda$$

Ex: find cosin and sine Integral of the function

$f(x) = e^{-kx}$ where $k > 0,\ x > 0$

cosine Integral ~~f(x)~~ $B(\lambda) = 0$

$$A(\lambda) = \frac{2}{\pi} \int_0^\infty f(x) \cos(\lambda x)\, dx$$

$$= \frac{2}{\pi} \int_0^\infty e^{-kx} \cos(\lambda x)\, dx$$

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$$= \frac{2}{\pi} \int_0^\infty e^{-kx} \cos(\lambda x)\, dx$$

$a = -k \qquad b = \lambda$

$$= \frac{2}{\pi} \left[\frac{e^{-kx}}{k^2+\lambda^2} -k\cos(\lambda x) - k\sin(\lambda x) \right]_0^\infty$$

$$= \frac{2}{\pi} \left[\cancel{0} + 0 - \frac{1}{k^2+\lambda^2} (-k - 0) \right]$$

$$\boxed{A(\lambda) = \frac{2}{\pi} \left[\frac{k}{k^2+\lambda^2} \right]}$$

$$f(x) = \int_0^\infty A(\lambda) \cos(\lambda x)\, d\lambda$$

$$\boxed{f(x) = \frac{2}{\pi} \int_0^\infty \frac{k}{k^2+\lambda^2} \cos(\lambda x)\, d\lambda}$$

→ sine Integral

$A(\lambda) = 0$

$$B(\lambda) = \frac{2}{\pi} \int_0^\infty e^{-kx} \sin(\lambda x)\, dx$$

$$= \frac{2}{\pi} \left[\frac{e^{-kx}}{k^2+\lambda^2} \{-k\sin(\lambda x) - \lambda\cos(\lambda x) \right]_0^\infty$$

$$\boxed{B(\lambda) = \frac{2}{\pi} \left[\frac{\lambda}{k^2+\lambda^2} \right]}$$

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Ex: $f(x) = \frac{\pi}{2} e^{-x}$

Cosine Integral

$B(\lambda) = 0$

$\rightarrow$ $A(\lambda) = \frac{2}{\pi} \int_0^\infty \frac{\pi}{2} e^{-x} \cos(\lambda x)\, dx$

$= \int_0^\infty e^{-x} \cos(\lambda x)$

$a = -1 \quad b = \lambda$

$= \left[\frac{e^{-x}}{1+\lambda^2} \left(-\cos(\lambda x) + \lambda \sin(\lambda x) \right) \right]_0^\infty$

$= \left[0 - \frac{1}{1+\lambda^2} (-1) \right]$

$\boxed{A(\lambda) = \frac{1}{1+\lambda^2}}$

$f(x) = \int_0^\infty \frac{1}{1+\lambda^2} \cos(\lambda x)\, d\lambda$

$\rightarrow$ sine Integral

$A(\lambda) = 0$

$B(\lambda) = \frac{2}{\pi} \int_0^\infty \frac{\pi}{2} e^{-x} \sin(\lambda x)\, dx$

$= \int_0^\infty e^{-x} \sin(\lambda x)\, dx$

$= \left[\frac{e^{-x}}{1+\lambda^2} \left(-\sin(\lambda x) - \lambda \cos(\lambda x) \right) \right]_0^\infty$

$= \left[0 - \frac{1}{1+\lambda^2} (0 - \lambda) \right.$

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$$B(\lambda) = \frac{\lambda}{1+\lambda^2}$$

$$f(x) = \int_0^\infty \frac{\lambda}{1+\lambda^2} \sin(\lambda x)\, d\lambda$$

ex: show that $\displaystyle\int_0^\infty \left(\frac{1-\cos(\pi\lambda)}{\lambda}\right) \sin(\lambda x)\, d\lambda = \begin{cases} \pi/2 & ; 0<x<\pi \\ 0 & ; x>\pi \end{cases}$

→ given function is sine Integral

$$A(\lambda) = 0$$

$$B(\lambda) = \frac{2}{\pi} \int_0^\pi \left(\cancel{\frac{\pi}{2}/\cos(\pi\lambda)}\right) \frac{\pi}{2} \sin(\lambda x)\, dx + \int_\pi^\infty 0\, dx$$

$$= \frac{2}{\pi}\cdot\frac{\pi}{2}\left[\int_0^\pi \sin(\lambda x)\, dx\right]$$

$$= \left[\frac{-\cos(\lambda x)}{\lambda}\right]_0^\pi$$

$$= \frac{1}{\lambda}\left[-\cos(\pi\lambda) + 1\right]$$

$$\boxed{B(\lambda) = \frac{1}{\lambda}\left[1 - \cos(\pi\lambda)\right]}$$

$$f(x) = \int_0^\infty \left(\frac{1-\cos(\pi\lambda)}{\lambda}\right) \sin(\lambda x)\, d\lambda$$

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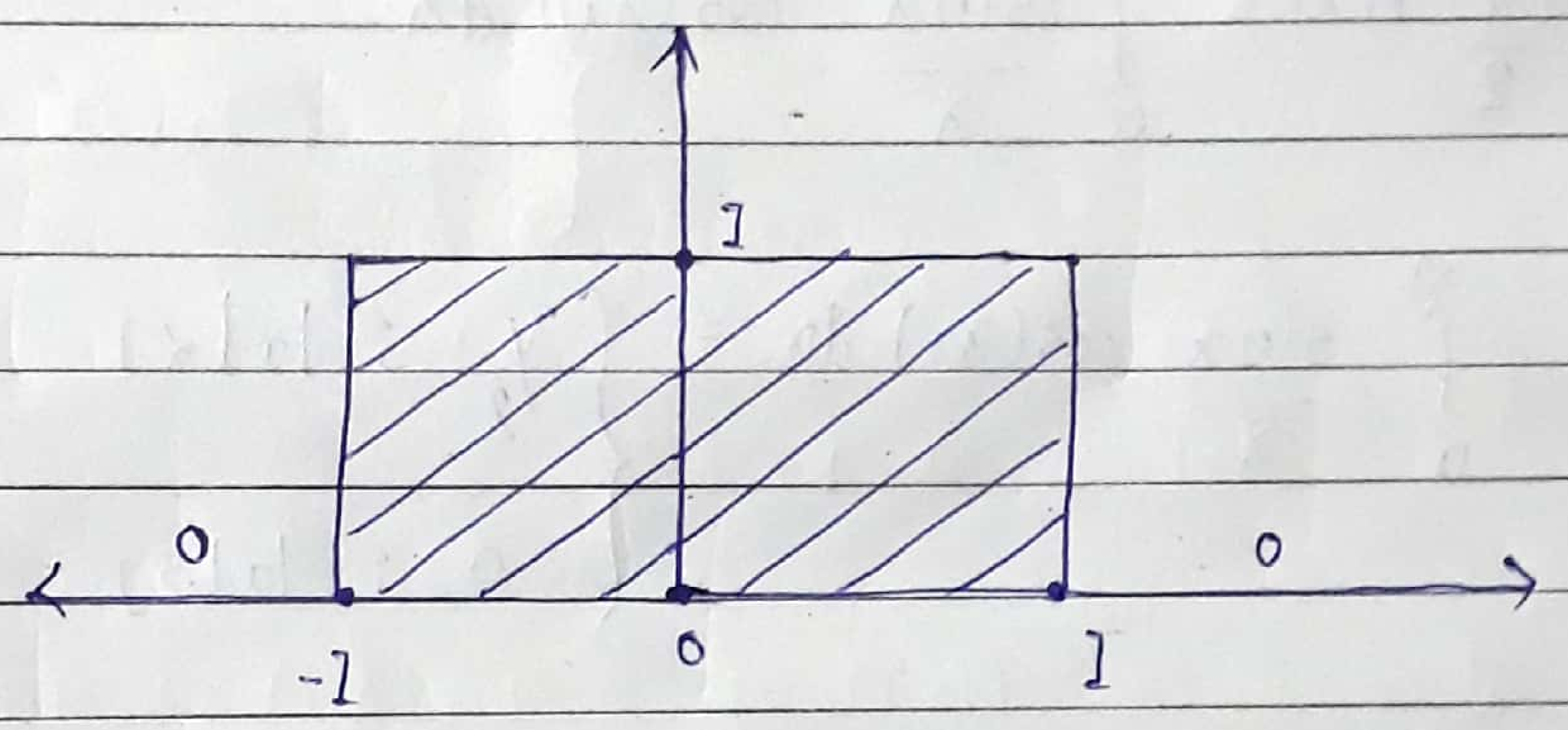
Ex: Find the Fourier Integral representation of the fun^n

$$f(x) = 1 \quad ; \quad |x| < 1$$
$$= 0 \quad : \quad |x| > 1$$

Hence evaluate

(i) $\displaystyle\int_0^\infty \frac{\sin\lambda \cos\lambda x}{\lambda} \, d\lambda$

(ii) $\displaystyle\int_0^\infty \frac{\sin\lambda}{\lambda} \, d\lambda$



given function is even cosine Integral

$$B(\lambda) = 0$$

$$A(\lambda) = \frac{2}{\pi} \int_0^\infty f(x) \cos(\lambda x) \, dx$$

$$= \frac{2}{\pi} \left[\int_0^1 1 \cos(\lambda x) \, dx + \int_1^\infty 0 \, dx \right]$$

$$= \frac{2}{\pi} \left[\int_0^1 \cos(\lambda x) \, dx \right]$$

$$= \frac{2}{\pi} \left[\frac{\sin \lambda x}{\lambda} \right]_0^1$$

$$= \frac{2}{\pi} \left[\frac{\sin\lambda}{\lambda} - 0 \right]$$

$$\boxed{A(\lambda) = \frac{2}{\pi} \left(\frac{\sin\lambda}{\lambda} \right)}$$

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$$f(x) = \int_0^\infty A(\lambda) \cos(\lambda x)\, d\lambda$$

$$f(x) = \int_0^\infty \frac{2}{\pi} \left(\frac{\sin\lambda}{\lambda}\right) \cos(\lambda x)\, d\lambda$$

$$f(x) = \frac{2}{\pi} \cdot \frac{\cancel{\pi}}{\cancel{\pi}} \int_0^\infty \frac{\sin\lambda}{\lambda} \cos(\lambda x)\, d\lambda$$

$$\frac{\pi}{2} f(x) = \int_0^\infty \frac{\sin\lambda}{\lambda} \cos(\lambda x)\, d\lambda$$

$$\int_0^\infty \frac{\sin\lambda}{\lambda} \cos(\lambda x)\, d\lambda = \begin{cases} \pi/2 & ; |x| < 1 \\ 0 & ; |x| > 1 \end{cases} \quad \text{--- (1)}$$

$\longrightarrow$ $|x| = 1 \Rightarrow x = \pm 1$

given fun^n is discontinuous

at $x = 1$ $\quad f(x) = \frac{1}{2}\left\{ \lim_{x\to 1^+} f(x) + \lim_{x \to 1^-} f(x) \right\}$

$= \frac{1}{2}\{1 + 0\}$

$= 1/2$

at $x = -1$ $\quad f(x) = \frac{1}{2}\left\{ \lim_{x\to 1^-} f(x) + \lim_{x \to 1^+} f(x) \right\}$

$= \frac{1}{2}\{0 + 1\}$

$= 1/2$

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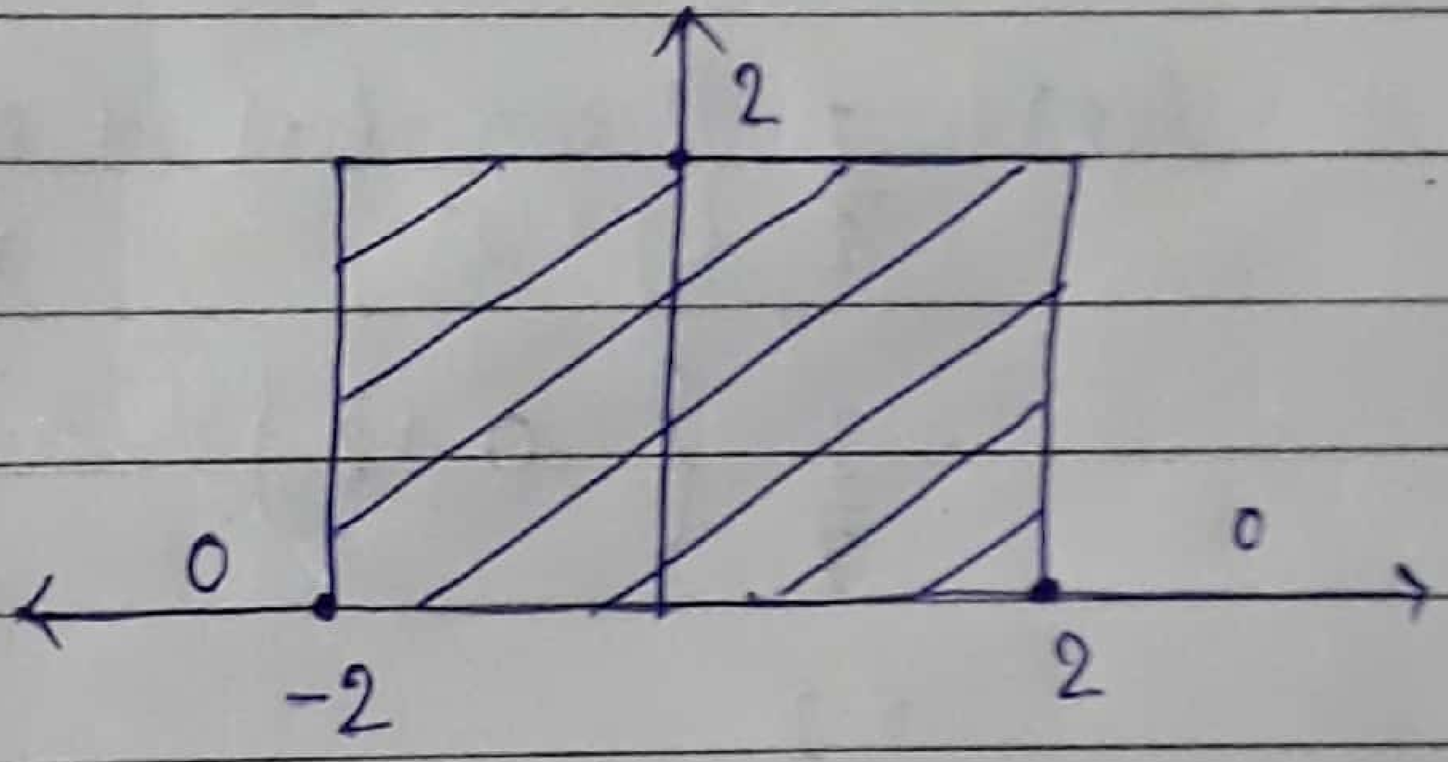
$$\rightarrow \int_0^\infty \frac{\sin\lambda}{\lambda}\cos(\lambda x)\,d\lambda = \begin{cases} \pi/2 & ; |x|<1 \\ \pi/4 & ; |x|=1 \\ 0 & ; |x|>1 \end{cases}$$

then putting $x=0$ in eq (1)

$$\rightarrow \int_0^\infty \frac{\sin\lambda}{\lambda}\,d\lambda = \frac{\pi}{2} \quad ; \ (f(0)=1)$$

Ex: $f(x) = 2 \ ; \ |x|<2$
$\quad = 0 \ ; \ |x|>2$



given function is even

cosine Integral

$$A(\lambda) = \frac{2}{\pi}\int_0^\infty f(x)\cos(\lambda x)\,dx$$

$$= \frac{2}{\pi}\left[\int_0^2 2\cos(\lambda x)\,dx + \int_2^\infty 0\,dx\right]$$

$$= \frac{2\times 2}{\pi}\left[\int_0^2 \cos(\lambda x)\,dx\right]$$

$$= \frac{4}{\pi}\left[\frac{\sin\lambda x}{\lambda}\right]_0^2$$

$$= 4/\pi\left[\frac{\sin 2\lambda}{\lambda} - 0\right]$$

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$$A(\lambda) = \frac{\sin 2\lambda}{\lambda} \cdot \frac{4}{\pi}$$

$$f(x) = \int_0^\infty \frac{4}{\pi} \frac{\sin 2\lambda}{\lambda} \cos(\lambda x)\, d\lambda$$

$$\frac{\pi}{4} f(x) = \int_0^\infty \frac{\sin 2\lambda}{\lambda} \cos(\lambda x)\, d\lambda$$

$$\int_0^\infty \frac{\sin 2\lambda}{\lambda} \cos(\lambda x)\, d\lambda = \begin{cases} \frac{\pi}{4} ; & |x| < 2 \\ 0 ; & |x| > 2 \end{cases}$$

→ given fun is discontinuous
$|x| = 2 \Rightarrow x = \pm 2$

at $x = 2$ $\quad f(x) = \frac{1}{2} \left\{ \lim_{x \to 2^+} f(x) + \lim_{x \to 2^-} f(x) \right\}$

$$= \frac{1}{2} \{ 0 + 2 \}$$

$$= 1$$

$x = -2$ $\quad f(x) = \frac{1}{2} \left\{ \lim_{x \to 2^-} f(x) + \lim_{x \to 2^+} f(x) \right\}$

$$= \frac{1}{2} \{ 2 + 0 \}$$

$$= 1$$

$$\int_0^\infty \frac{\sin 2\lambda}{\lambda} \cos(\lambda x)\, d\lambda = \begin{cases} \pi/4 \cdot \frac{?}{2} ; & |x| < 1 \\ \pi/4 ; & |x| = 1 \\ 0 ; & |x| > 1 \end{cases}$$

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ex: Winter 2015

show that $\int_0^\infty \frac{\lambda^3}{\lambda^4+4} \sin(\lambda x)\, dx = \pi/2\, e^{-x} \cos x \quad x>0$

given function is sine integral

$A(\lambda) = 0$

$B(\lambda) = \frac{2}{\pi} \int_0^\infty f(x) \cdot \cancel{\cos} \sin(\lambda x)\, dx$

$= \frac{2}{\pi} \int \frac{\pi}{2} e^{-x} \cos x \cdot \sin(\lambda x)\, dx$

$= \frac{2}{\pi} \cdot \frac{\pi}{2} \cdot \frac{1}{2} \int e^{-x}\, 2 \sin(\lambda x) \cdot \cos x\, dx$

$= \frac{1}{2} \int_0^\infty e^{-x} \left(\sin((\lambda+1)x) + \sin((\lambda-1)x) \right] dx$

$= \frac{1}{2} \int_0^\infty e^{-x} \sin(\lambda+1)x\, dx + \frac{1}{2} \int_0^\infty (\sin(\lambda-1)x) e^{-x}\, dx$

$= \frac{1}{2} \left[\frac{e^{-x}}{1+(\lambda+1)^2} \left[-\sin(\lambda+1)x + (\lambda-1) \cos(\lambda+1)x \right] \right]_0^\infty$

$+ \frac{1}{2} \left[\frac{e^{-x}}{1+(\lambda-1)^2} \left[-\sin(\lambda-1)x + (\lambda-1) \cos(\lambda+1)x \right] \right]_0^\infty$

$= \frac{1}{2} \left[0 - \frac{1}{(1+(\lambda+1)^2} - (\lambda+1) \right] + \left[0 - \frac{1}{1+(\lambda-1)^2} - (\lambda-1) \right]$

$= \frac{1}{2} \left[\frac{\lambda+1}{1+\lambda^2+2\lambda+1} + \frac{\lambda-1}{1+\lambda^2-2\lambda+1} \right]$

$= \frac{1}{2} \left[\frac{\lambda+1}{2+\lambda^2+2\lambda} + \frac{\lambda-1}{\lambda^2-2\lambda+2} \right]$

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$$= \frac{1}{2}\left[\frac{\cancel{\lambda^3}-\cancel{2\lambda^2}+\cancel{2\lambda}+\cancel{\lambda^2}-\cancel{2\lambda}+\cancel{2}+\lambda^3+\cancel{2\lambda^2}+\cancel{\lambda}-\cancel{\lambda^2}-\cancel{2\lambda}-\cancel{2}}{[(\lambda^2+2)+2\lambda]-[(\lambda^2+2)-2\lambda]}\right]$$

$$= \frac{1}{2}\left[\frac{2\lambda^3}{(\lambda^2+2)^2-4\lambda^2}\right]$$

$$= \frac{\lambda^3}{\lambda^4+4\lambda^2+4-4\lambda^2}$$

$$\boxed{B(\lambda) = \frac{\lambda^3}{\lambda^4+4}}$$

$$\boxed{f(x) = \int_0^{\infty} \frac{\lambda^3}{\lambda^4+4} \sin(\lambda x)\, d\lambda}$$

ex: express the function $f(x) = \begin{cases} \sin x & ; 0 \le x \le \pi \\ 0 & ; x > \pi \end{cases}$ sine Integral

$A(\lambda) = 0$

$$B(\lambda) = \int_0^{\infty} f(x) \sin(\lambda x)\, dx$$

$$\frac{2}{\pi}\left[\int_0^{\pi} \frac{\sin x\, dx}{\sin(\lambda x)} + \int_{\pi} 0\, dx\right]$$

$$= \frac{2}{\pi}\left[\left[-\cos x\right]_0^{\pi} + \cancel{0}\right]$$

$$= \frac{2 \times 2}{\pi}\left[\frac{1}{-2}\int_0^{\pi} -2\sin x \sin(\lambda x)\, dx\right]$$

$$= \frac{2}{\pi}[-\cos\pi + \cos 0]$$

$$= \frac{2}{\pi}\left[\frac{-1}{2}\int_0^{\infty} \cos(\lambda+1)x - \cos(\lambda-1)x\right]$$

$$= \frac{2}{\pi}[1+1]$$

$$= \frac{-1}{\pi}$$

$$= \frac{4}{\pi}$$

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$$= \frac{-1}{\pi}\left[\frac{\sin(\lambda+1)x}{\lambda+1} - \frac{\sin(\lambda-1)x}{\lambda-1}\right]_0^{\pi}$$

$$= \frac{-1}{\pi}\left[(0+0) - (0+0)\right]$$

$$= 0\ \frac{-1}{\pi}\left[\left(\frac{\sin(\lambda+1)\pi}{(\lambda+1)} - \frac{\sin(\lambda-1)\pi}{(\lambda-1)}\right) - (0-0)\right]$$

$$B(\lambda) = \left(\frac{-\sin(\lambda+1)}{(\lambda+1)} + \frac{\sin(\lambda-1)}{(\lambda-1)}\right)\frac{1}{\pi}$$

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\left(\frac{\sin(\lambda-1)\pi}{\lambda-1} - \frac{\sin(\lambda+1)\pi}{(\lambda+1)}\right)\sin(\lambda x)\,d\lambda$$

$$= \frac{1}{\pi}\int\left(-\frac{\sin(\pi\lambda-\pi)}{\lambda-1} + \frac{\sin(-(\pi-\lambda\pi))}{(\lambda+1)}\right)\sin(\lambda x)\,d\lambda$$

$$= \frac{1}{\pi}\int\left(\frac{\sin\lambda\pi}{\lambda-1} - \frac{\sin\lambda\pi}{\lambda+1}\right)\sin(\lambda x)\,d\lambda$$

$$= \frac{1}{\pi}\int\left(\frac{(\lambda+1)\sin\lambda\pi - (\lambda-1)\sin\pi\lambda}{\lambda^2-1}\right)\sin(\lambda x)\,d\lambda$$

$$= \frac{1}{\pi}\int\left(\frac{\cancel{\lambda\sin\lambda\pi} + \sin\lambda\pi - \cancel{\lambda\sin\pi\lambda} + \sin\pi\lambda}{\lambda^2-1}\right)\sin(\lambda x)\,d\lambda$$

$$f(x) = \frac{1}{\pi}\int\left(\frac{2\sin(\pi\lambda)}{\lambda^2-1}\right)\sin(\lambda x)\,d\lambda$$

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