4 Linear Differential Equations with constant Coefficients

Ex.1 Solve $\dfrac{d^3y}{dx^3}-5\dfrac{d^2y}{dx^2}+8\dfrac{dy}{dx}-4y=0$ **[M.U. 2002]**

Solution: The auxiliary equation is $D^3-5D^2+8D-4=0$

$\therefore \quad D^3-2D^2-3D^2+6D+2D-4=0$

$\therefore \quad (D-2)\left(D^2-3D+2\right)=0$

$\therefore \quad (D-2)(D-2)(D-1)=0 \quad \therefore \quad D=1,2,2$

$\therefore$ The solution is $y=(c_1+c_2x)e^{2x}+c_3e^x$

Ex.2 Solve $\dfrac{d^4y}{dx^4}+k^4y=0$ **[M.U. 2003]**

Solution: The auxiliary equation is $D^4+k^4=0$

$\therefore \quad \left(D^4+2D^2k^2+k^4\right)-\left(2D^2k^2\right)=0$

$\therefore \quad \left(D^2+k^2\right)^2-\left(\sqrt{2}.Dk\right)^2=0$

$\therefore \quad \left(D^2-\sqrt{2}.Dk+k^2\right)\left(D^2+\sqrt{2}.Dk+k^2\right)=0$

Now, $D^2-\sqrt{2}.Dk+k^2=0$ gives $D=\dfrac{k\pm ik}{\sqrt{2}}$

$D^2+\sqrt{2}.Dk+k^2=0$ gives $D=\dfrac{-k\pm ik}{\sqrt{2}}$

Since, we have two pairs of complex roots, the solution is

$$y=e^{(k/\sqrt{2})x}\left[c_1\cos\left(k/\sqrt{2}\right)x+c_2\sin\left(k/\sqrt{2}\right)x\right]$$
$$+e^{(-k/\sqrt{2})x}\left[c_3\cos\left(k/\sqrt{2}\right)x+c_4\sin\left(k/\sqrt{2}\right)x\right]$$

EXERCISE

Find the solutions using complimentary functions:

- $\left\{\left(D^2+1\right)^3\left(D^2+D+1\right)^2\right\}y=0$ **[M.U. 2002]**

Ans. $y=\left(c_1+c_2x+c_3x^2\right)\cos x+(c_4+c_5x+c_6x)\sin x$
$+e^{-x/2}\left[(c_7+c_8x)\cos\left(\sqrt{3}.x/2\right)+(c_9+c_{10}x)\sin\left(\sqrt{3}.x/2\right)\right]$

- $\left(D^4+8D^2+16\right)y=0$ **[M.U. 2003]**

Ans. $y=(c_1+c_2x)\cos 2x+(c_3+c_4x)\sin 2x$

Kalpana Classes

Ex.3 Solve $\left(D^3-2D^2-5D+6\right)y=e^{3x}+8$ **[M.U. 1991]**

Solution: The auxiliary equation is $D^3-2D^2-5D+6=0$

$\therefore \quad (D-1)(D-3)(D+2)=0$

$\therefore \quad D=1,-2,3$

$\therefore \quad$ C.F. is $y=c_1e^{x}+c_2e^{-2x}+c_3e^{3x}$

P.I. $$=\frac{1}{(D-1)(D+2)(D-3)}.e^{3x}+\frac{8}{(D-1)(D+2)(D-3)}.e^{0x}$$

$$=\frac{1}{(2)(5)}.\frac{1}{D-3}.e^{3x}+\frac{8}{(-1)(2)(-3)}.e^{0x}$$

$$=\frac{1}{10}.x.e^{3x}+\frac{4}{3}$$

$\therefore$ The complete solution is

$$y=c_1e^{x}+c_2e^{-2x}+c_3e^{3x}+\frac{x}{10}e^{3x}+\frac{4}{3}$$

Ex.4 Solve $\left(D^3-2D^2-5D+6\right)y=\left(e^{2x}+3\right)^2$ **[M.U. 1993]**

Solution: The auxiliary equation is $D^3-2D^2-5D+6=0$

As in the above example

$\therefore \quad (D-1)(D-3)(D+2)=0$

$\therefore \quad D=1,-2,3$

$\therefore \quad$ C.F. is $y=c_1e^{x}+c_2e^{-2x}+c_3e^{3x}$

P.I. $$=\frac{1}{D^3-2D^2-5D+6}\left(e^{4x}+6e^{2x}+9\right)$$

$$=\frac{1}{D^3-2D^2-5D+6}e^{4x}+6\frac{1}{D^3-2D^2-5D+6}e^{2x}+9\frac{1}{D^3-2D^2-5D+6}e^{0x}$$

$$=\frac{e^{4x}}{18}-\frac{3}{2}e^{2x}+\frac{3}{2}$$

$\therefore$ The complete solution is

$$y=c_1e^{x}+c_2e^{-2x}+c_3e^{3x}+\frac{e^{4x}}{18}-\frac{3}{2}e^{2x}+\frac{3}{2}$$

Ex.5 Solve $\dfrac{d^3y}{dx^3}-4\dfrac{dy}{dx}=2\cosh^2 2x$ **[M.U. 1993, 94, 2002, 09]**

Solution: The auxiliary equation is $D^3-4D=0$

$\therefore \quad D\left(D^2-4\right)=0 \qquad \therefore \quad D=0,2,-2$

$\therefore \quad$ C.F. is $y=c_1e^{x}+c_2e^{2x}+c_3e^{-2x}$

Kalpana Classes

P.I. $= \frac{1}{D^3 - 4D} 2\cosh^2(2x) = \frac{1}{D^3 - 4D} 2\left(\frac{e^{2x} + e^{-2x}}{2}\right)^2$

$= \frac{1}{2} \cdot \frac{1}{D^3 - 4D}\left(e^{4x} + 2 + e^{-4x}\right)$

$= \frac{1}{2}\left[\frac{1}{D^3 - 4D} e^{4x} + 2\frac{1}{D\left(D^2 - 4\right)} e^{0x} + \frac{1}{D^3 - 4D} e^{-4x}\right]$

$= \frac{1}{2}\left[\frac{1}{48} e^{4x} - \frac{x}{2} - \frac{1}{48} e^{-4x}\right]$

$\therefore$ P.I. $= -\frac{x}{4} + \frac{1}{48}\left(\frac{e^{4x} - e^{-4x}}{2}\right) = -\frac{x}{4} + \frac{1}{48}\sinh 4x$

$\therefore$ The complete solution is

$y = c_1 + c_2 e^{2x} + c_3 e^{-2x} - \frac{x}{4} + \frac{1}{48}\sin h\, 4x$

Ex.6 Solve $6\frac{d^2y}{dx^2} + 17\frac{dy}{dx} + 12y = e^{-3x/2} + 2^x$ **[M.U. 1999]**

Solution: The auxiliary equation is $6D^2 + 17D + 12 = 0$

$\therefore$ $(3D+4)(2D+3) = 0$

$\therefore$ $D = -4/3, D = -3/2$

$\therefore$ C.F. is $y = c_1 e^{-4x/3} + c_2 e^{-3x/2}$

P.I. $= \frac{1}{(3D+4)(2D+3)}\left(e^{-3x/2} + 2^x\right)$

$= \frac{1}{(3D+4)(2D+3)} e^{-3x/2} + \frac{1}{(3D+4)(2D+3)} e^{x\log 2}$

$\left[\because 2^x = e^{x\log 2}\right]$

$\therefore$ P.I. $= \frac{1}{\left[-(9/2)+4\right]} .x.e^{-3x/2} + \frac{e^{x\log 2}}{6(\log 2)^2 + 17\log 2 + 12}$

$= -2xe^{-3x/2} + \frac{2^x}{6(\log 2)^2 + 17\log 2 + 12}$

$\therefore$ The complete solution is

$y = c_1 + e^{-4x/3} + c_2 e^{-3x/2} - 2x\, e^{-3x/2} + \frac{2^x}{6(\log 2)^2 + 17\log 2 + 12}$

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EXERCISE

Solve the following differential equations :

- $\left(D^2+4D+4\right)y=\cosh 2x$ **[M.U. 1988, 93, 97]**

Ans. Hint : $\cos h\,2x=\left(e^{2x}+e^{-2x}\right)/2$

$$\therefore y=(c_1+c_2x)e^{-2x}+\frac{1}{32}e^{2x}+\frac{x^2}{4}.e^{-2x}$$

- $\dfrac{d^2y}{dx^2}+3\dfrac{dy}{dx}+2y=e^{-x}$ **[M.U. 1994]**

Ans. $y=c_1e^{-x}+c_2e^{-2x}+xe^{-x}$

- $\left(D^4+1\right)y=\cosh 4x\sinh 3x$ **[M.U. 2003]**

Ans. Hint: $D^4+1=\left(D^2+1\right)^2-\left(\sqrt{2}.D\right)^2$

$$y=e^{x/\sqrt{2}}\left\{c_1\cos\left(x/\sqrt{2}\right)+c_2\sin\left(x/\sqrt{2}\right)\right\}$$

$$+e^{-x/\sqrt{2}}\left\{c_3\cos\left(x/\sqrt{2}\right)+c_4\sin\left(x/\sqrt{2}\right)\right\}+\frac{1}{9608}\left(e^{7x}-e^{-7x}\right)-\frac{1}{8}\left(e^{x}-e^{-x}\right)$$

- $\left(D^2-2D+1\right)y=e^x+1$ **[M.U. 1989]**

Ans. $y=(c_1+c_2x)e^x+\dfrac{x^2}{2}e^x+1$

- $\left(D^3-4D\right)y=2\cos h\,2x$ **[M.U. 1989, 90]**

Ans. $y=c_1+c_2e^{2x}+c_3e^{-2x}+\dfrac{x}{8}\left(e^{2x}+e^{-2x}\right)$

- $\left(D^4-4D^3+8D^2-8D+4\right)y=e^x+1$ **[M.U. 2011]**

Ans. $y=e^{2x}\left[(C_1+C_2x)\cos x+(C_3+C_4x)\sin x\right]+e^x+\dfrac{1}{4}$

Ex.7 Solve $\dfrac{d^3y}{dx^3}-3\dfrac{d^2y}{dx^2}+9\dfrac{dy}{dx}-27y=\cos 3x$ **[M.U. 2005]**

Solution: The auxiliary equation is $D^3-3D^2+9D-27=0$

$\therefore \quad D^2(D-3)+9(D-3)=0$

$\therefore \quad (D-3)\left(D^2+9\right)=0 \quad \therefore \quad D=3,3i,-3i$

$\therefore \quad$ The C.F. is $y=c_1e^{3x}+\left(c_2\cos 3x+c_3\sin 3x\right)$

Now, P.I. $=\dfrac{1}{(D-3)\left(D^2+9\right)}\cos 3x$

Kalpana Classes

Since, D^2+9 is a factor of $\Phi\left(D^2\right)$, the general method falls.

$\therefore$ $$\text{P.I.} = \frac{1}{D^2+9}.\frac{D+3}{D^2-9}.\cos 3x$$

$$= \frac{1}{D^2+9}.\frac{1}{-9-9}.(D+3)\cos 3x$$

$$= \frac{1}{D^2+9}.\frac{(-3\sin 3x+3\cos 3x)}{-18}$$

$$= \frac{1}{6}.\frac{1}{D^2+9}.(\sin 3x-\cos 3x)$$

Now, by using the formulae of 10 (a)

$$\frac{1}{D^2+9}\sin 3x = x.\frac{1}{2.D}\sin 3x = \frac{x}{2}\int \sin 3x\,dx = -\frac{x}{6}\cos 3x$$

and $$\frac{1}{D^2+9}\cos 3x = x.\frac{1}{2.D}\cos 3x = \frac{x}{2}\int \cos 3x\,dx\frac{x}{6}\sin 3x$$

$\therefore$ The complete solution is

$$y = c_1e^{3x} + (c_2\cos 3x + c_3\sin 3x) - \frac{x}{36}\cos 3x - \frac{x}{36}\sin 3x$$

Ex.8 Solve $\dfrac{d^2y}{dx^2}+9y = e^x - \cos 2x$ **[M.U. 1992]**

Solution: The auxiliary equation is $D^2+9=0$ $\therefore$ $D = 3i, -3i$

$\therefore$ The C.F. is $y = c_1\cos 3x + c_2\sin 3x$

$$\text{P.I.} = \frac{1}{D^2+9}\left(e^x - \cos 2x\right)$$

$$= \frac{1}{D^2+9}e^x - \frac{1}{D^2+9}\cos 2x$$

$$= \frac{1}{10}e^x - \frac{1}{5}\cos 2x$$

$\therefore$ The complete solution is

$$y = c_1\cos 3x + c_2\sin 3x + \frac{1}{10}e^x - \frac{1}{5}\cos 2x$$

Ex.9 Solve $\left(D^4-1\right)y = e^x + \cos x\cos 3x$ **[M.U. 1993]**

Solution: The auxiliary equation is $D^4-1=0$

$\therefore$ $\left(D^2-1\right)\left(D^2+1\right)=0$ $\therefore$ $D = 1, -1, +i, -i$

$\therefore$ The C.F. is $y = c_1e^x + c_2e^{-x} + c_3\cos x + c_3\sin x$

$$\text{P.I.} = \frac{1}{D^4-1}\left(e^x + \cos x\cos 3x\right)$$

$$= \frac{1}{D^4-1}\left[e^x + \frac{1}{2}(\cos 4x + \cos 2x)\right]$$

Kalpana Classes

$$=\frac{1}{(D-1)(D+1)\left(D^2+1\right)}e^x+\frac{1}{2}\cdot\frac{1}{\left(D^4-1\right)}\cos 4x+\frac{1}{2\left(D^4-1\right)}\cos 2x$$

$$=\frac{x}{4}e^x+\frac{1}{510}\cos 4x+\frac{1}{30}\cos 2x$$

Ex.10 Solve $(D-1)^2\left(D^2+1\right)y=e^x+\sin^2(x/2)$ **[M.U. 2008, 12]**

Solution: The auxiliary equation is $(D-1)^2\left(D^2+1\right)=0$

$\therefore \quad D=1,1,+i,-i$

$\therefore$ The C.F. is $y=(c_1+c_2x)e^x+(c_3\cos x+c_4\sin x)$

$$\text{P.I.}=\frac{1}{(D-1)^2\left(D^2+1\right)}\left[e^x+\sin^2\frac{x}{2}\right]$$

Now, $\dfrac{1}{(D-1)^2\left(D^2+1\right)}e^x=\dfrac{x^2}{2!}\cdot\dfrac{1}{2}e^x$

and $\dfrac{1}{(D-1)^2\left(D^2+1\right)}\sin^2\dfrac{x}{2}=\dfrac{1}{(D-1)^2\left(D^2+1\right)}\left[\dfrac{1-\cos x}{2}\right]$

$$=\frac{1}{(D-1)^2\left(D^2+1\right)}\left(\frac{1}{2}e^{0x}\right)-\frac{1}{(D-1)^2\left(D^2+1\right)}\left(\frac{1}{2}\cos x\right)$$

$$=\frac{1}{(-1)^2(1)}\cdot\frac{1}{2}-\frac{1}{\left(D^2-2D+1\right)\left(D^2+1\right)}\left(\frac{1}{2}\cos x\right)$$

$$=\frac{1}{2}-\frac{1}{-2D}\cdot\frac{1}{\left(D^2+1\right)}\left(\frac{\cos x}{2}\right)$$

$$=\frac{1}{2}-\frac{1}{\left(D^2+1\right)}\cdot\frac{D}{\left(-2D^2\right)}\left(\frac{\cos x}{2}\right)$$

$$=\frac{1}{2}+\frac{1}{4}\cdot\frac{1}{\left(D^2+1\right)(-1)}(-\sin x)=\frac{1}{2}+\frac{1}{4}\cdot\frac{1}{\left(D^2+1\right)}(\sin x)$$

$$=\frac{1}{2}+\frac{1}{4}\cdot\frac{x}{2D}\sin x$$

$$=\frac{1}{2}+\frac{x}{8}\int\sin x\,dx=\frac{1}{2}-\frac{x}{8}\cos x$$

Hence, the complete solution is

$$y=(c_1+c_2x)e^x+c_3\cos x+c_4\sin x+\frac{1}{2}\cdot\frac{x^2}{2!}e^x+\frac{1}{2}-\frac{x}{8}\cos x$$

Ex.11 Solve $\left(D^4+8D^2+16\right)y=\sin^2 x$ **[M.U. 2002, 03]**

Solution: The auxiliary equation is $D^4+8D^2+16=0$

Kalpana Classes

$\therefore \quad \left(D^2+4\right)^2=0 \qquad \therefore \quad D=2i,-2i,2i,-2i$

$\therefore$ The C.F. is $y=(c_1+c_2x)(c_3\cos 2x+c_4\sin 2x)$

$$\text{P.I.} = \frac{1}{\left(D^2+4\right)^2}\sin^2 x=\frac{1}{\left(D^2+4\right)^2}\left(\frac{1-\cos 2x}{2}\right)$$

Now, $\dfrac{1}{\left(D^2+4\right)^2}.\dfrac{1}{2}=\dfrac{1}{2}.\dfrac{1}{(0+4)^2}=\dfrac{1}{32}$

and $\dfrac{1}{\left(D^2+4\right)^2}\left(-\dfrac{1}{2}\cos 2x\right)=-\dfrac{1}{2}.\dfrac{x}{2\left(D^2+4\right).2D}\cos 2x$

$$=-\frac{x}{8\left(D^2+4\right)}\int\cos 2x\,dx=-\frac{x}{8\left(D^2+4\right)}.\frac{\sin 2x}{2}$$

$$=-\frac{x}{16}.\frac{x}{2D}\sin 2x=-\frac{x^2}{32}\int\sin 2x\,dx$$

$$=-\frac{x^2}{32}.\left(-\frac{\cos 2x}{2}\right)=\frac{x^2}{64}\cos 2x$$

$\therefore$ The complete solution is

$$y=(c_1+c_2x)(c_3\cos 2x+c_4\sin 2x)+\frac{1}{32}+\frac{x^2}{64}\cos 2x$$

Ex.12 Solve $\left(D^2+D+1\right)y=(1+\sin x)^2$ **[M.U. 2006]**

Solution: The auxiliary equation is $D^2+D+1=0 \quad \therefore \quad D=\dfrac{-1\pm\sqrt{3}.i}{2}$

$\therefore$ C.F. is $y=e^{-x/2}\left(c_1\cos\dfrac{\sqrt{3}}{2}x+c_2\sin\dfrac{\sqrt{3}}{2}x\right)$

$\therefore$ $\text{P.I.} =\dfrac{1}{D^2+D+1}(1+\sin x)^2$

$$=\frac{1}{D^2+D+1}\left(1+2\sin x+\sin^2 x\right)$$

$$=\frac{1}{D^2+D+1}\left(1+2\sin x+\frac{1-\cos 2x}{2}\right)$$

$$=\frac{1}{D^2+D+1}\left(\frac{3}{2}+2\sin x-\frac{1}{2}\cos 2x\right)$$

Now, $\dfrac{1}{D^2+D+1}.\left(\dfrac{3}{2}\right)=\dfrac{3}{2}.\dfrac{1}{D^2+D+1}e^{0x}$

$$=\frac{3}{2}.\frac{1}{0+0+1}.e^{0x}=\frac{3}{2}$$

Kalpana Classes

$$\frac{1}{D^2+D+1}\sin x = \frac{1}{-1+D+1}\sin x$$
$$= \frac{1}{D}\sin x$$
$$= \int \sin x\, dx = -\cos x$$

$$\frac{1}{D^2+D+1}\cos 2x = \frac{1}{-4+D+1}\cos 2x$$
$$= \frac{D+3}{D^2-9}\cos 2x$$
$$= \frac{-2\sin 2x + 3\cos 2x}{-13}$$

$\therefore$ The complete solution is

$$y = e^{-x/2}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right)$$
$$+\frac{1}{3} - 2\cos x - \frac{1}{26}(2\sin 2x - 3\cos 2x)$$

EXERCISE

Solve the following differential equations :

- $\frac{d^4y}{dx^4} - a^4 y = \sin ax$ **[M.U. 1988, 2008]**

Ans. $y = c_1 e^{ax} + c_2 e^{-ax} + c_3 \cos ax + c_4 \sin ax + \frac{1}{4a^3}.x\cos ax$

- $(D-1)^2\left(D^2+1\right)^2 y = \sin^2\frac{x}{2} + e^x$ **[M.U. 2001, 10]**

Ans. $y = (c_1 + c_2 x)e^x + (c_3 + c_4 x)(c_5 \cos x + c_6 \sin x) + \frac{1}{2} - \frac{1}{32}x^2 \sin x + \frac{1}{8}x^2 e^x$

- $\left(D^4 + 10D^2 + 9\right)y = \cos(2x+3)$ **[M.U. 1988, 2004]**

Ans. $y = c_1 \cos x + c_2 \sin x + c_3 \cos 3x + c_4 \sin 3x - \frac{1}{15}\cos(2x+3)$

- $\left(D^2 - 4\right)y = \sin^2 x$ **[M.U. 1988]**

Ans. $y = c_1 e^x + c_2 e^{2x} - \frac{x}{8}\sin 2x - \frac{1}{8}$

- $\left(D^2+4\right)y = \cos 2x$ **[M.U. 2003]**

Ans. $y = c_1 \cos 2x + c_2 \sin 2x + \frac{x}{4}\sin 2x$

Ex.13 Solve $\frac{d^3y}{dx^3} - 2\frac{dy}{dx} + 4y = 3x^2 - 5x + 2$ **[M.U. 1996, 99]**

Solution: The auxiliary equation is $D^3 - 2D + 4 = 0$

$\therefore \quad D^3 + 2D^2 - 2D^2 - 4D + 2D + 4 = 0$

$\therefore \quad (D+2)\left(D^2 - 2D + 2\right) = 0 \quad \therefore \quad D = -2, 1 \pm i$

$\therefore$ The C.F. is $y = c_1 e^{-2x} + e^x \left(c_2 \cos x + c_3 \sin x\right)$

$$\text{P.I.} = \frac{1}{D^3 - 2D + 4}\left(3x^2 - 5x + 2\right)$$

$$= \frac{1}{4\left[1 - \dfrac{2D - D^3}{4}\right]}\left(3x^2 - 5x + 2\right)$$

$$= \frac{1}{4}\left[1 - \frac{2D - D^3}{4}\right]^{-1}\left(3x^2 - 5x + 2\right)$$

$$= \frac{1}{4}\left[1 + \frac{2D - D^3}{4} + \frac{4D^2}{16} + \ldots\right]\left(3x^2 - 5x + 2\right)$$

$$= \frac{1}{4}\left[3x^2 - 5x + 2 + \frac{1}{2}(6x - 5) + \frac{1}{4}(6)\right]$$

$$= \frac{1}{4}\left[3x^2 - 2x + 1\right]$$

$\therefore$ The complete solution is

$$y = c_1 e^{-2x} + e^x \left(c_2 \cos x + c_3 \sin x\right) + \frac{1}{4}\left[3x^2 - 2x + 1\right]$$

Ex.14 Solve $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = x^2 + e^x + \cos 2x$ **[M.U. 1995, 2005, 10, 11]**

Solution: The auxiliary equation is $D^2 - 4D + 4 = 0$

$\therefore \quad (D-2)^2 = 0 \quad \therefore \quad D = 2, 2$

$\therefore$ The C.F. is $y = \left(c_1 + c_2 x\right)e^{2x}$

$$\text{P.I.} = \frac{1}{(D-2)^2}\left(x^2 + e^x + \cos 2x\right)$$

Now, $$\frac{1}{D^2 - 4D + 4}x^2 = \frac{1}{4\left[1 - \dfrac{4D - D^2}{4}\right]}x^2$$

$$= \frac{1}{4}\left[1 - \left(\frac{4D - D^2}{4}\right)\right]^{-1} x^2 = \frac{1}{4}\left[1 + \left(\frac{4D - D^2}{4}\right) + D^2\right]x^2$$

$$= \frac{1}{4}\left[x^2 + \frac{1}{4}(8x - 2) + 2\right] = \frac{1}{4}\left[x^2 + 2x + \frac{3}{2}\right]$$

$$\frac{1}{D^2 - 4D + 4}e^x = \frac{1}{1 - 4 + 4} = e^x$$

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$$\frac{1}{D^2-4D+4}\cos 2x = -\frac{1}{4D}\cos 2x$$

$$= -\frac{1}{4}\int \cos 2x\, dx = -\frac{1}{8}\sin 2x$$

$\therefore$ The completion solution is

$$y = (c_1 + c_2 x)e^{2x} + \frac{1}{4}\left[x^2 + 2x + \frac{3}{2}\right] + e^x - \frac{1}{8}\sin 2x$$

Ex.15 Solve $\left(D^3 - 2D^2 + D\right)y = x^2 + x$ **[M.U. 1992]**

Solution: The auxiliary equation is $D\left(D^2 - 2D + 1\right) = 0$

$\therefore$ $D(D-1)^2 = 0$ $\therefore$ $D = 0, 1, 1$

$\therefore$ The C.F. is $y = c_1 + (c_2 + c_3 x)e^x$

$$\text{P.I.} = \frac{1}{D - 2D^2 - D^3}\left(x^2 + x\right) = \frac{1}{D\left(1 - 2D + D^2\right)}\left(x^2 + x\right)$$

$$\therefore \quad \text{P.I.} = \frac{1}{D}\left[1 + \left(2D - D^2\right) + 4D^2 \ldots\right]\left(x^2 + x\right)$$

$$= \frac{1}{D}\left[1 + 2D + 3D^2 \ldots\right]\left(x^2 + x\right)$$

$$= \frac{1}{D}\left[\left(x^2 + x\right) + 2(2x+1) + 3(2)\right]$$

$$= \frac{1}{D}\left[x^2 + 5x + 8\right]$$

$$= \int\left(x^2 + 5x + 8\right)dx = \frac{x^3}{3} + \frac{5x^2}{2} + 8x$$

$\therefore$ The completion solution is

$$y = c_1 + (c_2 + c_3 x)e^x + \frac{x^3}{3} + \frac{5x^2}{2} + 8x$$

Ex.16 Solve $\frac{d^3y}{dt^3} + \frac{dy}{dt} = \cos t + t^2 + 3$ **[M.U. 1992]**

Solution: The auxiliary equation is $D\left(D^2 + 1\right) = 0$ $\therefore$ $D = 0, i, -i$

$\therefore$ The C.F. is $y = c_1 + c_2\cos t + c_3\sin t$

$$\text{P.I.} = \frac{1}{D + D^3}\left(\cos t + t^2 + 3\right)$$

$$\frac{1}{D + D^3}\cos t = \frac{1}{D\left(1 + D^2\right)}\cos t = \frac{1}{D}\cdot\frac{t}{2}\sin t$$

$$= \frac{1}{2}\int t\sin t\,dt = \frac{1}{2}\left[-t\cos t + \sin t\right]$$

$$\frac{1}{D+D^3}t^2 = \frac{1}{D\left(1+D^2\right)}t^2 = \frac{1}{D}\left(1-D^2+....\right)t^2$$

$$= \frac{1}{D}\left[t^2-2\right] = \int\left(t^2-2\right)dt = \frac{t^3}{3}-2t$$

$$\frac{1}{D+D^3}.3 = 3.\frac{1}{D\left(1+D^2\right)}e^{0t} = 3.\frac{1}{D}.1 = 3\int dt = 3t$$

$\therefore$ The complete solution is

$$y = c_1 + c_2\cos t + c_3\sin t + \frac{1}{2}\left[-t\cos t+\sin t\right] + \frac{t^3}{3} + t$$

Ex.17 Solve $\left(D^3-D^2-6D\right)y = x^2+1$ **[M.U. 2009]**

Solution: The auxiliary equation is $D^3-D^2-6D=0$

$\therefore$ $D\left(D^2-D-6\right)=0$

$\therefore$ $D(D+2)(D-3)=0$ $\therefore$ $D=0,-2,3$

$\therefore$ The C.F. is $y = c_1 + c_2e^{-2x} + c_3e^{3x}$

$$\text{P.I.} = \frac{1}{D^3-D^2-6D}\left(x^2+1\right)$$

$$= -\frac{1}{6D}.\frac{1}{\left\{1+\left[\left(D-D^2\right)/6\right]\right\}}\left(x^2+1\right)$$

$$= -\frac{1}{6D}.\left[1+\frac{D-D^2}{6}\right]^{-1}\left(x^2+1\right)$$

$$= -\frac{1}{6D}\left[1-\frac{\left(D-D^2\right)}{6}+\left\{\frac{D-D^2}{6}\right\}^2-....\right]\left(x^2+1\right)$$

$$= -\frac{1}{6D}\left[1-\frac{D}{6}+\frac{D^2}{6}+\frac{D^2}{36}+....\right]\left(x^2+1\right)$$

$$= -\frac{1}{6D}\left[x^2+1-\frac{x}{3}+\frac{1}{3}+\frac{1}{18}\right]$$

$$= -\frac{1}{6D}\left[x^2-\frac{x}{3}+\frac{25}{18}\right] = -\frac{1}{6}\int\left(x^2-\frac{x}{3}+\frac{25}{18}\right)dx$$

$$= -\frac{1}{6}\left[\frac{x^3}{3}-\frac{x^2}{6}+\frac{25x}{18}\right]$$

$\therefore$ The complete solution is

$$y = c_1 + c_2e^{-2x} + c_3e^{3x} - \frac{1}{6}\left[\frac{x^3}{3}-\frac{x^2}{6}+\frac{25x}{18}\right]$$

Kalpana Classes

EXERCISE

Solve the following differential equations :

- $\left(D^4-2D^3+D^2\right)y=x^3$ **[M.U. 1996]**

Ans. $y=c_1+c_2x+\left(c_3+c_4x\right)e^x+\dfrac{x^5}{20}+\dfrac{x^4}{2}+3x^3-12x^2$

- $\left(D^2-4D+4\right)y=8\left(x^2+\sin 2x+e^{2x}\right)$ **[M.U. 1997]**

Ans. $y=\left(c_1+c_2x\right)e^{2x}+2x^2+4x+3+\cos 2x+4x^2e^{2x}$

- $\left(D^3-D\right)y=2e^x+2x+1-4\cos x$ **[M.U. 2006]**

Ans. $y=c_1+c_2e^x+c_3e^{-x}-x^2-x+2\sin x+xe^x$

- $\left(D^2+4\right)y=x^2+\sin 2x$ **[M.U. 1998]**

Ans. $y=c_1\cos 2x+c_2\sin 2x-\dfrac{x}{4}\cos 2x+\dfrac{1}{4}\left(x^2-\dfrac{1}{2}\right)$

- $\left(D^2+2D+2\right)y=x^2+1$ **[M.U. 2004]**

Ans. $y=\left(c_1\cos x+c_2\sin x\right)e^{-x}+\dfrac{1}{2}\left(x^2-2x+2\right)$

Ex.18 Solve $\left(D^2-3D+2\right)y=x^2e^{2x}$ **[M.U. 1994]**

Solution: The auxiliary equation is $D^2-3D+2=0$

$\therefore \quad (D-1)(D-2)=0 \quad \therefore \quad D=1,2$

$\therefore$ The C.F. is $y=c_1e^x+c_2e^{2x}$

$$\text{P.I.}=\frac{1}{D^2-3D+2}x^2e^{2x}=e^{2x}.\frac{1}{(D+2)^2-3(D+2)+2}.x^2$$

$$=e^{2x}\frac{1}{D^2+D}x^2=e^{2x}.\frac{1}{D.(1+D)}x^2=e^{2x}\frac{1}{D}(1+D)^{-1}x^2$$

$$=e^{2x}\frac{1}{D}\left[1-D+D^2-D^3+....\right]x^2$$

$$=e^{2x}\frac{1}{D}\left[x^2-2x+2\right]=e^{2x}\int\left(x^2-2x+2\right)dx$$

$$=e^{2x}\left(\frac{x^3}{3}-x^2+2x\right)$$

$\therefore$ The complete solution is

$$y=c_1e^x+c_2e^{2x}+e^{2x}\left(\frac{x^3}{3}-x^2+2x\right)$$

Kalpana Classes

Ex.19 Solve $\frac{d^2y}{dx^2}+2y=x^2e^{3x}+e^x-\cos 2x$ **[M.U. 1993, 2003, 06]**

Solution: The auxiliary equation is $\left(D^2+2\right)=0$

$\therefore \quad D=\sqrt{2}.i,-\sqrt{2}.i$

$\therefore$ The C.F. is $y=c_1\cos\sqrt{2}.x+c_2\sin\sqrt{2}.x$

$$\text{P.I.} =\frac{1}{D^2+2}\left(x^2e^{3x}+e^x-\cos 2x\right)$$

Now, $\frac{1}{D^2+2}e^{3x}x^2=e^{3x}.\frac{1}{(D+3)^2+2}.x^2$

$$=e^{3x}.\frac{1}{D^2+6D+11}x^2=\frac{e^{3x}}{11}\left[1+\frac{6D+D^2}{11}\right]^{-1}x^2$$

$$=\frac{e^{3x}}{11}\left[1-\frac{\left(6D+D^2\right)}{11}+\frac{36D^2}{121}+...\right]x^2$$

$$=\frac{e^{3x}}{11}\left[x^2-\frac{12x}{11}-\frac{2}{11}+\frac{72}{121}\right]$$

$$=\frac{e^{3x}}{11}\left[x^2-\frac{12x}{11}+\frac{50}{121}\right]$$

$$\frac{1}{D^2+2}e^x=\frac{1}{3}e^x$$

$$\frac{1}{D^2+2}\cos 2x=-\frac{1}{2}\cos 2x$$

$\therefore$ The complete solution is

$$y=c_1\cos\sqrt{2}.x+c_2\sin\sqrt{2}.x+\frac{e^{3x}}{11}\left[x^2-\frac{12x}{11}+\frac{50}{121}\right]+\frac{1}{3}e^x+\frac{1}{2}\cos 2x$$

Ex.20 Solve $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=2e^x\cos\frac{x}{2}$ **[M.U. 1995, 2005, 10]**

Solution: The auxiliary equation is $D^2-3D+2=0$

$\therefore \quad (D-1)(D-2)=0 \quad \therefore \quad D=1,2$

$\therefore$ The C.F. is $y=c_1e^x+c_2e^{2x}$

$$\text{P.I.} =2.\frac{1}{D^2-3D+2}.e^x\cos\left(\frac{x}{2}\right)$$

$$=2.e^x\frac{1}{(D+1)^2-3(D+1)+2}.\cos\left(\frac{x}{2}\right)$$

$$=2.e^x\frac{1}{D^2-D}\cos\left(\frac{x}{2}\right)$$

Kalpana Classes

$$= 2.e^{x} \frac{1}{-(1/4)-D} \cos\left(\frac{x}{2}\right)$$

$$= -8e^{x} . \frac{1}{4D+1} \cos\left(\frac{x}{2}\right)$$

$$= -8e^{x} . \frac{4D-1}{16D^{2}-1} . \cos\left(\frac{x}{2}\right)$$

$$= \frac{8}{5} e^{x} \left[-2\sin\left(\frac{x}{2}\right) - \cos\left(\frac{x}{2}\right) \right]$$

$\therefore$ The complete solution is

$$y = c_1 e^{x} + c_2 e^{2x} - \frac{8}{5} e^{x} \left[2\sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{2}\right) \right]$$

Ex.21 Solve $\left(D^2-1\right) = \cosh x \cos x$ **[M.U. 2002]**

Solution: The auxiliary equation is $D^2 - 1 = 0$

$\therefore$ $(D-1)(D+1) = 0$ $\therefore$ $D = 1, -1$

$\therefore$ The C. F. is $y = c_1 e^{x} + c_2 e^{-x}$

$$\text{P.I.} = \frac{1}{D^2-1} \cosh x \cos x = \frac{1}{D^2-1} \left(\frac{e^{x}+e^{-x}}{2} \right) \cos x$$

$$= \frac{1}{2} \left[\frac{1}{D^2-1} e^{x} \cos x + \frac{1}{D^2-1} e^{-x} \cos x \right]$$

$$= \frac{1}{2} \left[e^{x} . \frac{1}{(D+1)^2-1} \cos x + e^{-x} . \frac{1}{(D+1)^2-1} \cos x \right]$$

$$= \frac{1}{2} \left[e^{x} . \frac{1}{D^2+2D} \cos x + e^{-x} . \frac{1}{D^2-2D} \cos x \right]$$

$$= \frac{1}{2} \left[e^{x} . \frac{1}{2D-1} \cos x - e^{-x} . \frac{1}{2D+1} \cos x \right]$$

$$= \frac{1}{2} \left[e^{x} . \frac{2D+1}{4D^2-1} \cos x - e^{-x} . \frac{2D-1}{4D^2-1} \cos x \right]$$

$$= \frac{1}{2} \left[-\frac{e^{x}}{5} (-2\sin x + \cos x) + \frac{e^{x}}{5} (-2\sin x - \cos x) \right]$$

$$= \frac{1}{5} \left[2\sin x \left(\frac{e^{x}-e^{-x}}{2} \right) - \cos x \left(\frac{e^{x}+e^{-x}}{2} \right) \right]$$

$$\text{P.I.} = \frac{1}{5} [2\sin x \sinh x - \cos x \cosh x]$$

$\therefore$ The complete solution is

$$y = c_1 e^{x} + c_2 e^{-x} + \frac{1}{5} [2\sin x \sinh x - \cos x \cosh x]$$

Kalpana Classes

Ex.22 Solve $\left(D^2+2\right)y=e^x\cos x+x^2e^{3x}$ **[M.U. 2001, 08, 12]**

Solution: The auxiliary equation is $D^2+2=0$

$\therefore \quad D=+\sqrt{2}.i,-\sqrt{2}.i$

$\therefore$ The C.F. is $y=c_1\cos\sqrt{2}.x+c_2\sin\sqrt{2}.x$

$$\text{P.I.}=\frac{1}{D^2+2}e^x\cos x=e^x.\frac{1}{(D+1)^2+2}\cos x$$

$$=e^x.\frac{1}{D^2+2D+3}.\cos x=e^x.\frac{1}{2D+2}\cos x$$

$$=e^x.\frac{1}{2}.\frac{D-1}{D^2-1}.\cos x=e^x.\frac{1}{2}.\frac{1}{-2}.(-\sin x-\cos x)$$

$$=e^x.\frac{1}{4}(\sin x+\cos x)$$

$$=\frac{e^{3x}}{11}\left(x^2-\frac{12x}{11}+\frac{50}{121}\right)$$

$\therefore$ The complete solution is

$$y=c_1\cos\sqrt{2}.x+c_2\sin\sqrt{2}.x+e^x.\frac{1}{4}(\sin x+\cos x)+\frac{e^{3x}}{11}\left(x^2-\frac{12}{x}+\frac{50}{121}\right)$$

Ex.23 Solve $\left(D^3-7D-6\right)y=\cosh x\cos x$ **[M.U. 2002]**

Solution: The auxiliary equation is $D^3-7D-6=0$

$\therefore \quad D^3+D^2-D^2-D-6D-6=0$

$\therefore \quad (D+1)\left(D^2-D-6\right)=0$

$\therefore \quad (D+1)(D+2)(D-3)=0 \quad \therefore \quad D=-1,-2,3$

$\therefore$ The C.F. is $y=c_1e^{-x}+c_2e^{-2x}+c_3e^{3x}$

$$\text{P.I.}=\frac{1}{D^3-7D-6}\cosh x\cos x$$

$$\text{P.I.}=\frac{1}{D^3-7D-6}\left(\frac{e^x+e^{-x}}{2}\right).\cos x$$

Now, $\dfrac{1}{D^3-7D-6}.e^x\cos x=e^x.\dfrac{1}{(D+1)^3-7(D+1)-6}\cos x$

$$=e^x.\frac{1}{D^3+3D^2-4D-12}\cos x$$

$$=e^x.\frac{1}{-D-3-4D-12}\cos x \qquad (\text{Putting } D^2=-1)$$

$$=-\frac{1}{5}e^x.\frac{1}{D+3}\cos x=-\frac{1}{5}e^x.\frac{(D-3)}{\left(D^2-9\right)}\cos x$$

Kalpana Classes

$$= -\frac{1}{5}e^x.\frac{1}{(-1-9)}.(D-3)\cos x$$

$$= \frac{e^x}{50}(-\sin x - 3\cos x)$$

Similarly, we find that

$$\frac{1}{D^3 - 7D - 6}.e^{-x}\cos x = \frac{e^{-x}}{34}(3\cos x - 5\sin x)$$

$\therefore$ The complete solution is

$$y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x} - \frac{1}{100}.e^x(\sin x + 3\cos x)$$

$$+ \frac{1}{68}.e^{-x}(3\cos x - 5\sin x)$$

EXERCISE

Solve the following differential equations :

- $\left(D^3 - 7D - 6\right)y = e^{2x}(x+1)$ **[M.U. 1992, 96]**

Ans. $y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x} - e^{2x}.\frac{1}{12}\left(x + \frac{17}{12}\right)$

- $\left(D^3 - 7D - 6\right)y = \left(1 + x^2\right)e^{2x}$ **[M.U. 1999, 07]**

Ans. $y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x} - e^{2x}.\frac{1}{12}.\left(x^2 + \frac{5}{6}x + \frac{169}{72}\right)$

- $\left(D^2 + 4D + 4\right)y = \frac{e^{-2x}}{x^5}$ **[M.U. 2004]**

Ans. $y = (c_1 + c_2 x)e^{-2x} + \frac{e^{-2x}}{12x^3}$

- $\left(D^2 + D - 6\right)y = e^{2x}\sin 3x$ **[M.U. 1996]**

Ans. $y = c_1 e^{2x} + c_2 e^{-3x} - \frac{e^{2x}}{102}(5\cos 3x + 3\sin 3x)$

- $\left(D^2 - 4\right)y = x^2 e^{3x}$ **[M.U. 1997]**

Ans. $y = c_1 e^{2x} + c_2 e^{-2x} + \frac{e^{3x}}{5}\left(x^2 - \frac{12x}{5} + \frac{62}{25}\right)$

- $\left(D^2 - 1\right)y = x\sinh x$ **[M.U. 2003]**

Ans. $y = c_1 e^x + c_2 e^{-x} + \frac{x^2}{4}\cosh x - \frac{x}{4}\sinh x$

- $\left(D^2 - 2D + 4\right)y = e^x\cos^2 x$ **[M.U. 1999]**

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Ans. $y = e^{x}\left(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x\right) + \frac{1}{8}.e^{x} - \frac{1}{2}.e^{x}\cos 2x$

- $\left(D^2 - 3D + 2\right)y = 2e^{x}\sin\left(\frac{x}{2}\right)$ **[M.U. 2004, 07]**

Ans. $y = c_1 e^{x} + c_2 e^{2x} - \frac{8}{5}e^{x}\left(\sin\frac{x}{2} - 2\cos\frac{x}{2}\right)$

- $\left(D^4 - 1\right)y = \cos x \cosh x$ **[M.U. 2002]**

Ans. $y = c_1 e^{x} + c_2 e^{-x} + c_3 \cos x + c_4 \sin x - \frac{1}{5}\cos x \cosh x$

Ex.24 Solve $\left(D^2 + 4\right) = x\sin x$ **[M.U. 2005]**

Solution: The auxiliary equation is $D^2 + 4 = 0 \qquad \therefore \qquad D = 2i, -2i$

$\therefore$ The C.F. is $y = c_1 \cos 2x + c_2 \sin 2x$

$$\text{P.I.} = \frac{1}{D^2+4}.(x\sin x) = \left\{x - \frac{1}{D^2+4}.2D\right\}.\frac{1}{D^2+4}\sin x$$

$$= \left\{x - \frac{1}{D^2+4}.2D\right\}.\frac{1}{3}\sin x = \frac{x}{3}.\sin x - \frac{1}{D^2+4}.\frac{2}{3}\cos x$$

$$= \frac{x}{3}.\sin x - \frac{2}{3}.\frac{1}{3}\cos x$$

$\therefore$ The complete solution is

$$y = c_1 \cos 2x + c_2 \sin 2x + \frac{x}{3}\sin x - \frac{2}{9}.\cos x$$

Ex.25 Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^{x}\sin x$ **[M.U. 2011]**

Solution: The auxiliary equation is $D^2 - 2D + 1 = 0$

$\therefore \quad (D-1)^2 = 0 \quad \therefore \quad D = 1, 1$

$\therefore$ The C.F. is $y = (c_1 + c_2 x)e^{x}$

$$\therefore \quad \text{P.I.} = \frac{1}{(D-1)^2}.e^{x}.x\sin x$$

$$= e^{x}.\frac{1}{\left[(D+1)-1\right]^2}x\sin x$$

$$= e^{x}.\frac{1}{D^2}.x\sin x = e^{x}\left[x - \frac{1}{D^2}.2D\right].\frac{1}{D^2}\sin x$$

$$= e^{x}\left[x - \frac{1}{D^2}.2D\right]\left(\frac{1}{-1}\right)\sin x = -e^{x}\left[x - \frac{1}{D^2}.2D\right]\sin x$$

$$= -e^{x}\left[x\sin x - \frac{1}{D^2}.2\cos x\right] = -e^{x}\left[x\sin x - \frac{2}{(-1)}\cos x\right]$$

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$\therefore \quad$ P.I. $= -e^{x}\left[x\sin x + 2\cos x\right]$

$\therefore \quad$ The complete solution is

$$y = (c_1 + c_2 x)e^{x} - e^{-x}(x\sin x + 2\cos x)$$

Ex.26 Solve $\left(D^2 - 4\right)y = x\sinh x$ **[M.U. 1991]**

Solution: The auxiliary equation is $D^2 - 4 = 0 \quad \therefore \quad D = 2, -2$

$\therefore \quad$ The C.F. is $y = c_1 e^{2x} + c_2 e^{-2x}$

$$\text{P.I.} = \frac{1}{D^2-4}x\sinh x = \frac{1}{D^2-4}x.\left(\frac{e^{x}-e^{-x}}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{D^2-4}.xe^{x} - \frac{1}{D^2-4}x.e^{-x}\right]$$

$$= \frac{1}{2}\left[x - \frac{1}{D^2-4}.2D\right]\frac{1}{D^2-4}e^{x} - \frac{1}{2}\left[x - \frac{1}{D^2-4}.2D\right]\frac{1}{D^2-4}e^{-x}$$

$$= \frac{1}{2}\left[x - \frac{1}{D^2-4}.2D\right]\left(-\frac{1}{3}e^{x}\right) - \frac{1}{2}\left[x - \frac{1}{D^2-4}.2D\right]\left(-\frac{1}{3}e^{-x}\right)$$

$$= -\frac{1}{6}\left[x.e^{x} - \frac{1}{D^2-4}.2e^{x}\right] + \frac{1}{6}\left[x.e^{-x} - \frac{1}{D^2-4}.2\left(-e^{-x}\right)\right]$$

$$= -\frac{1}{6}\left[x.e^{x} + \frac{2}{3}e^{x}\right] + \frac{1}{6}\left[x.e^{-x} - \frac{2}{3}.e^{-x}\right]$$

$$= -\frac{x}{6}\left(e^{x} - e^{-x}\right) - \frac{1}{6}.\frac{2}{3}\left(e^{x} + e^{-x}\right)$$

$$= -\frac{x}{3}\left(\frac{e^{x}-e^{-x}}{2}\right) - \frac{2}{9}\left(\frac{e^{x}+e^{-x}}{2}\right) = -\frac{x}{3}\sinh x - \frac{2}{9}\cosh x$$

$\therefore \quad$ The complete solution is

$$y = c_1 e^{2x} + c_2 e^{-2x} - \frac{x}{3}\sinh x - \frac{2}{9}\cosh x$$

Ex.27 Solve $\left(D^2 - 1\right)y = x\sin x + \cos x$ **[M.U. 1987]**

Solution: The auxiliary equation is $D^2 - 1 = 0 \quad \therefore \quad D = +1, -1$

$\therefore \quad$ The C.F. is $y = c_1 e^{x} + c_2 e^{-x}$

$$\text{P.I.} = \frac{1}{D^2-1}\left[x\sin 3x + \cos x\right]$$

$$= \left[x - \frac{1}{D^2-1}.2D\right]\frac{1}{D^2-1}\sin 3x + \frac{1}{D^2-1}\cos x$$

$$= \left[x - \frac{1}{D^2-1}.2D\right]\left(-\frac{1}{10}\right)\sin 3x - \frac{1}{2}\cos x$$

$$= -\frac{1}{10}\left[x\sin 3x - \frac{1}{D^2-1}6\cos 3x\right] - \frac{1}{2}\cos x$$

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$$= -\frac{1}{10}\left[x\sin 3x + \frac{6}{10}\cos 3x\right] - \frac{1}{2}\cos x$$

$\therefore$ P.I. $= -\frac{1}{10}\left[x\sin 3x + \frac{3}{5}\cos 3x\right] - \frac{1}{2}\cos x$

$\therefore$ The complete solution is

$$y = c_1 e^x + c_2 e^{-x} - \frac{1}{10}\left[x\sin 3x + \frac{3}{5}\cos x\right] - \frac{1}{2}\cos x$$

Ex.28 Solve $\left(D^2 - 1\right)y = x^2 \sin 3x$ **[M.U. 2002]**

Solution: The auxiliary equation is $D^2 - 1 = 0$ $\therefore$ $D = 1, -1$

$\therefore$ The C.F. is $y = c_1 e^x + c_2 e^{-x}$

P.I.. = Imaginary Part of $\frac{1}{D^2 - 1} . x^2 e^{3ix}$

$$= \text{I.P. of } e^{3ix} . \left\{\frac{1}{(D+3i)^2 - 1}\right\} x^2$$

$$= \text{I.P. of } e^{3ix} . \left\{\frac{1}{D^2 + 6Di - 10}\right\} x^2$$

$$= \text{I.P. of } e^{3ix} . \frac{1}{(-10)}\left\{1 - \frac{6Di + D^2}{10}\right\}^{-1} x^2$$

$$= \text{I.P. of } e^{3ix} . \frac{1}{(-10)}\left\{1 + \left(\frac{6Di + D^2}{10}\right) + \frac{36D^2 i^2}{100} \ldots\right\} x^2$$

$$= \text{I.P. of } e^{3ix} . \frac{1}{(-10)}\left\{1 + \frac{6Di}{10} - \frac{26}{100}D^2\right\} x^2$$

$$= \text{I.P. of } e^{3ix} . \frac{1}{-10}\left[x^2 + \frac{6}{5}xi - \frac{13}{25}\right]$$

$$= \text{I.P. of } (\cos 3x + i\sin 3x) . \frac{1}{(-10)} . \left[x^2 + \frac{6}{5}xi - \frac{13}{25}\right]$$

$\therefore$ P.I.. $= \frac{1}{-10}\left\{x^2 \sin 3x + \frac{6}{5}x\cos 3x - \frac{13}{25}\sin 3x\right\}$

$\therefore$ The complete solution is

$$y = c_1 e^x + c_2 e^{-x} - \frac{1}{10}\left\{x^2 \sin 3x + \frac{6}{5}x\cos 3x - \frac{13}{25}\sin 3x\right\}$$

Ex.29 Solve $\left(D^4 + 2D^2 + 1\right)y = x^2 \cos x$ **[M.U. 2012]**

Solution: The auxiliary equation is $D^4 + 2D^2 + 1 = 0$ $\therefore$ $\left(D^2 + 1\right)^2 = 0$

$\therefore$ $D = i, i, -i, -i$

$\therefore$ The C.F. is $y = (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x$

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$$\therefore \quad \text{P.I.} = \frac{1}{D^4 + 2D^2 + 1} . x^2 \cos x$$

$$= \text{Real Part of } \frac{1}{\left(D^2+1\right)^2} . x^2 e^{ix}$$

$$= \text{R.P. of } e^{ix} \frac{1}{\left[(D+i)^2+1\right]^2} x^2$$

$$= \text{R.P. of } e^{ix} \frac{1}{\left(D^2+2Di\right)^2} x^2$$

$$= \text{R.P. of } e^{ix} \frac{1}{D^2 (D+2i)^2} x^2$$

$$= \text{R.P. of } e^{ix} \frac{1}{D^2} . \frac{1}{-4+4iD+D^2} x^2$$

$$\text{P.I.} = \text{R.P. of } e^{ix} . \frac{1}{D^2}\left(-\frac{1}{4}\right)\left[1 - \frac{4iD+D^2}{4}\right]^{-1} x^2$$

$$= \text{R.P. of } e^{ix} \frac{1}{D^2} . \left(-\frac{1}{4}\right)\left[1 + \frac{4iD+D^2}{4} - D^2 + ...\right] x^2$$

$$= \text{R.P. of } e^{ix} . \left(-\frac{1}{4D^2}\right)\left[1 + iD - \frac{3}{4}D^2 + ...\right] x^2$$

$$= \text{R.P. of } e^{ix} . \left(-\frac{1}{4D^2}\right)\left(x^2 + 2ix - \frac{3}{2}\right)$$

$$= \text{R.P. of } e^{ix} . \left(-\frac{1}{4D}\right)\int\left(x^2 + 2ix - \frac{3}{2}\right) dx$$

$$= \text{R.P. of } e^{ix} . \left(-\frac{1}{4D}\right)\left(\frac{x^3}{3} + ix^2 - \frac{3}{2}x\right)$$

$$= \text{R.P. of } e^{ix} . \left(-\frac{1}{4}\right)\int\left(\frac{x^3}{3} + ix^2 - \frac{3}{2}x\right) dx$$

$$= \text{R.P. of } \left(-\frac{e^{ix}}{4}\right)\left(\frac{x^4}{12} + \frac{ix^3}{3} - \frac{3}{4}x^2\right)$$

$$= \text{R.P. of } \left(-\frac{1}{4}\right)(\cos x + i \sin x)\left(\frac{x^4}{12} + \frac{ix^3}{3} - \frac{3}{4}x^2\right)$$

$$= -\frac{1}{4}\left(\frac{x^4}{12}\cos x - \frac{3}{4}x^2 \cos x - \frac{x^3}{3}\sin x\right)$$

$$\therefore \quad \text{P.I.} = -\frac{1}{48}\left(x^4 - 9x^2\right)\cos x + \frac{x^3}{12}\sin x$$

Kalpana Classes

$\therefore$ The complete solution is

$$y=(c_1+c_2x)\cos x+(c_3+c_4x)\sin x-\frac{1}{48}\left(x^4-9x^2\right)\cos x+\frac{x^3}{12}\sin x$$

Ex.30 Solve $\left(D^2+4\right)y=x\sin^2 x$ **[M.U. 2003, 08]**

Solution: The auxiliary equation is $D^2+4=0$ $\therefore$ $D=2i,-2i$

$\therefore$ The C.F. is $y=c_1\cos 2x+c_2\sin 2x$

$$\text{P.I.} =\frac{1}{D^2+4}\left(x\sin^2 x\right)=\frac{1}{D^2+4}x\left(\frac{1-\cos 2x}{2}\right)$$

$$=\frac{1}{2}\cdot\frac{1}{D^2+4}x-\frac{1}{2}\cdot\frac{1}{D^2+4}x\cos 2x$$

Now, $\frac{1}{2}\cdot\frac{1}{D^2+4}x=\frac{1}{2}\cdot\frac{1}{4}\left(1-\frac{D^2}{4}\ldots\right)x=\frac{1}{8}x$

and $\frac{1}{2}\cdot\frac{1}{D^2+4}x\cos 2x=\frac{1}{2}$ R.P. of $\frac{1}{D^2+4}x.e^{2ix}$

$$=\frac{1}{2}\text{ R.P. of } e^{2ix}\cdot\frac{1}{(D+2i)^2+4}x$$

$$=\frac{1}{2}\text{ R.P. of } e^{2ix}\cdot\frac{1}{D^2+4iD}x$$

$$=\frac{1}{2}\text{ R.P. of } e^{2ix}\cdot\frac{1}{4iD}\cdot\left[1-\frac{D}{4i}\ldots\right]x$$

$$=\frac{1}{2}\text{ R.P. of } e^{2ix}\cdot\frac{1}{4iD}\cdot\left[x-\frac{1}{4i}\right]$$

$$=\frac{1}{2}\text{ R.P. of } e^{2ix}\cdot\frac{1}{4i}\left[\frac{x^2}{2}-\frac{x}{4i}\right] \qquad \left[\because \frac{1}{D}=\int dx\right]$$

$$=\frac{1}{2}\text{ R.P. of } e^{2ix}\left(\frac{x^2}{8i}+\frac{x}{16}\right)$$

$$=\frac{1}{2}\text{ R.P. of } (\cos 2x+i\sin 2x)\left(\frac{x^2}{8i}+\frac{x}{16}\right)$$

$$=\frac{x}{32}\cos 2x+\frac{x^2}{16}\sin 2x$$

$\therefore$ The complete solution is

$$y=c_1\cos 2x+c_2\sin 2x+\frac{x}{8}-\frac{x}{32}\cos 2x-\frac{x^2}{16}\sin 2x$$

Example 31 to 34 and 39 to 41 is not expected in exam using general particular integrals. But the latter sums can be asked as a Variation of Parameters problem.

Kalpana Classes

Ex.31 Solve $\left(D^2+a^2\right)y=\sec ax$ **[M.U. 1991]**

Solution: The auxiliary equation is $D^2+a^2=0 \quad \therefore \quad D=+ai,-ai$

$\therefore$ The C.F. is $y=c_1\cos ax+c_2\sin ax$

$$\text{P.I.} =\frac{1}{D^2+a^2}\sec ax$$

$$=\frac{1}{(D+ai)(D-ai)}\sec ax$$

$$=\frac{1}{2ai}\left[\frac{1}{D-ai}-\frac{1}{D+ai}\right]\sec ax$$

$$=\frac{1}{2ai}\left[\frac{1}{D-ai}\sec ax-\frac{1}{D+ai}\sec ax\right]$$

$$=\frac{1}{2ai}\left[e^{aix}\int e^{-aix}\sec ax\,dx-e^{-aix}\int e^{aix}\sec ax\,dx\right]$$

$\therefore$
$$\text{P.I.} =\frac{1}{2ai}\left[e^{aix}\int(\cos ax-i\sin ax)\sec ax\,dx\right.$$
$$\left.-e^{-aix}\int(\cos ax+i\sin ax)\sec ax\,dx\right]$$

$$=\frac{1}{2ai}\left[e^{aix}\int(1-i\tan ax)\,dx-e^{-aix}\int(1+i\tan ax)\,dx\right]$$

$$=\frac{1}{2ai}\left[e^{aix}\left\{x-\frac{i}{a}\log\sec ax\right\}-e^{-aix}\left\{x+\frac{i}{a}\log\sec ax\right\}\right]$$

$$=\frac{1}{2ai}\left[(\cos ax+i\sin ax)\left\{x-\frac{i}{a}\log\sec ax\right\}\right.$$
$$\left.-(\cos ax-i\sin ax)\left\{x+\frac{i}{a}\log\sec ax\right\}\right]$$

$$=\frac{1}{2ai}\left\{2ix\sin ax-\frac{2i}{a}\cos ax\log\sec ax\right\}$$

$$=\frac{x}{a}\sin ax-\frac{1}{a^2}\cos ax\log\sec ax$$

$\therefore$
$$\text{P.I.} =\frac{x}{a}\sin ax+\frac{1}{a^2}\cos ax\log\cos ax$$

$\therefore$ The complete solution is

$$y=c_1\cos ax+c_2\sin ax+\frac{x}{a}\sin ax+\frac{1}{a^2}\cos ax\log\cos ax$$

Ex.32 Solve $\left(D^2+a^2\right)=2a\tan ax$ **[M.U. 2003]**

Solution: The auxiliary equation is $D^2+a^2=0 \quad \therefore \quad D=ai,-ai$

$\therefore$ The C.F. is $y=c_1\cos ax+c_2\sin ax$

Kalpana Classes

$$\text{P.I.} = \frac{2a}{D^2+a^2}\tan ax = \frac{1}{i}\left[\frac{1}{D-ai}-\frac{1}{D+ai}\right]\tan ax$$

Now, $\frac{1}{D-ai}\tan ax = e^{aix}\int e^{-aix}\tan ax\,dx$

$$= e^{aix}\int(\cos ax - i\sin ax)\tan ax\,dx$$

$$= e^{aix}\int\left(\sin ax - i\frac{\sin^2 ax}{\cos ax}\right)dx$$

$$= e^{aix}\int\left(\sin ax - i\frac{\left(1-\cos^2 ax\right)}{\cos ax}\right)dx$$

$$= e^{aix}\int(\sin ax - i\sec ax + i\cos ax)\,dx$$

$$\frac{1}{D-ai}\tan ax = e^{aix}\left[-\frac{\cos ax}{a}-\frac{i}{a}\log\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)+\frac{i}{a}\sin ax\right]$$

$$= -e^{aix}\left[\frac{1}{a}(\cos ax - i\sin ax)+\frac{i}{a}\log\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)\right]$$

$$= -\frac{e^{aix}}{a}\left[e^{-aix}+\frac{i}{a}\log\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)\right]$$

$$= -\frac{1}{a}\left[1+ie^{aix}\log\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)\right]$$

Changing i to –i

$$\frac{1}{D+ai}\tan ax = -\frac{1}{a}\left[1-ie^{-aix}\log\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)\right]$$

$$\text{P.I.} = \frac{1}{i}\left[-\frac{i}{a}\left(e^{aix}+e^{-aix}\right)\log\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)\right]$$

$$= -\frac{2}{a}\left(\frac{e^{aix}+e^{-aix}}{2}\right)\log\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)$$

$$= -\frac{2}{a}\cos ax\log\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)$$

$\therefore$ The complete solution is

$$y = c_1\cos ax + c_2\sin ax - \frac{2}{a}\cos ax\log\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)$$

Ex.33 Solve $\left(D^2+3D+2\right)y = \sin e^x$ **[M.U. 1997, 2000, 05]**

Solution: The auxiliary equation is $D^2+3D+2=0$ $\therefore$ $(D+2)(D+1)=0$

$\therefore$ The C.F. is $y = c_1e^{-x}+c_2e^{-2x}$

$$\text{P.I.} = \frac{1}{(D+2)(D+1)}\sin e^x = \left(\frac{1}{D+1}-\frac{1}{D+2}\right)\sin e^x$$

Kalpana Classes

$$= \frac{1}{(D+1)}\sin e^x - \frac{1}{D+2}\sin e^x$$

$$= e^{-x}\int e^x \sin e^x .dx - e^{-2x}\int e^{2x}\sin e^x .dx$$

To evaluate the integrals put $e^x = t, e^x dx = dt$

$\therefore$ P.I. $= e^{-x}\int \sin t\, dt - e^{-2x}\int t \sin t\, dt$

$$= e^{-x}(-\cos t) - e^{-2x}\left[(t)(-\cos t) - (1)(-\sin t)\right]$$

$\therefore$ P.I. $= -e^{-x}\cos e^x - e^{-2x}\left[-e^x \cos e^x + \sin e^x\right]$

$$= -e^{-2x}\sin e^x$$

$\therefore$ The complete solution is

$$y = c_1 e^{-x} + c_2 e^{-2x} - e^{-2x}\sin e^x$$

Ex.34 Solve $\left(D^2 + 5D + 6\right)y = e^{-2x}\sec^2 x(1 + 2\tan x)$ **[M.U. 1996, 97, 2002, 05]**

Solution: The auxiliary equation is $D^2 + 5D + 6 = 0$

$\therefore$ $(D+3)(D+2) = 0$ $\therefore$ $D = -2, -3$

$\therefore$ The C.F. is $y = c_1 e^{-2x} + c_2 e^{-3x}$

$\therefore$ P.I. $= \frac{1}{(D+3)}.\frac{1}{(D+2)}e^{-2x}\sec^2 x(1 + 2\tan x)dx$

$$= \frac{1}{(D+3)}.e^{-2x}\int e^{2x}.e^{-2x}\sec^2 x(1 + 2\tan x)dx$$

$$= \frac{1}{D+3}.e^{-2x}\int \sec^2 x(1 + 2\tan x)dx$$

$$= \frac{1}{D+3}.e^{-2x}\left(\tan x + \tan^2 x\right)$$

$$= e^{-3x}\int e^{3x}.e^{-2x}\left(\tan x + \tan^2 x\right)dx$$

$$= e^{-3x}\int e^x\left\{\left(\tan x + \sec^2 x\right) - 1\right\}dx$$

$$= e^{-3x}\left[\int e^x\left(\tan x + \sec^2 x\right)dx - \int e^x dx\right]$$

$$= e^{-3x}\left[e^x \tan x - e^x\right] = e^{-2x}[\tan x - 1]$$

$$\left[\because \int e^x\left[f(x) + f'(x)\right]dx = e^x f(x)\right]$$

$\therefore$ The complete solution is

$$y = c_1 e^{-2x} + c_2 e^{-3x} + e^{-2x}[\tan x - 1]$$

Ex.35 Solve $\left(D^2 - 6D + 9\right)y = e^{3x}(1 + x)$ **[M.U. 1990]**

Solution: The auxiliary equation is $D^2 - 6D + 9 = 0$

$\therefore$ $(D-3)^2 = 0$ $\therefore$ D = 3, 3

Kalpana Classes

$\therefore$ The C.F. is $y=(c_1+c_2x)e^{3x}$

$$\text{P.I.} =\frac{1}{D^2-6D+9}e^{3x}(1+x)$$

$$\therefore \quad \text{P.I.} =\frac{1}{(D-3)^2}e^{3x}+\frac{1}{(D-3)^2}e^{3x}.x$$

$$=\frac{x^2}{2!}e^{3x}+e^{3x}.\frac{1}{(D+3-3)^2}x$$

$$=\frac{x^2}{2}e^{3x}+e^{3x}.\frac{1}{D^2}x$$

But, $\frac{1}{D^2}x=\frac{1}{D}\int x\,dx=\frac{1}{D}\frac{x^2}{2}=\int\frac{x^2}{2}dx=\frac{x^3}{6}$

$\therefore$ The complete solution is

$$y=(c_1+c_2x)e^{3x}+\frac{x^2}{2}e^{3x}+\frac{x^3}{6}.e^{3x}$$

Ex.36 Solve $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=xe^x\sin x$ **[M.U. 1987, 95, 2008]**

Solution: The auxiliary equation is $D^2-2D+1=0$

$\therefore$ $(D-1)^2=0$ $\therefore$ D = 1, 1

$\therefore$ The C.F. is $y=(c_1+c_2x)e^x$

$$\text{P.I.} =\frac{1}{(D-1)^2}e^x(x\sin x)=e^x\frac{1}{(D+1-1)^2}x\sin x$$

$$=e^x\frac{1}{D^2}x\sin x=e^x\frac{1}{D}\int x\sin x\,dx$$

$$=e^x\frac{1}{D}\left[x(-\cos x)-\int(-\cos x).1.dx\right]$$

$$=e^x\frac{1}{D}\left[-x\cos x+\sin x\right]dx$$

$$=e^x\int\left[-x\cos x+\sin x\right]dx$$

$$=e^x\left[(-x)\sin x-\int\sin x(-1)\,dx-\cos x\right]$$

$$=e^x\left[-x\sin x-\cos x-\cos x\right]$$

$\therefore$ The complete solution is

$$y=(c_1+c_2x)e^x-e^x(x\sin x+2\cos x)$$

$$\therefore \quad \text{P.I.} =\frac{1}{(D-1)^2}e^x.(x\sin x)$$

$$=e^x\frac{1}{(D+1-1)^2}x\sin x=e^x.\frac{1}{D^2}x\sin x$$

Kalpana Classes

$$= e^{x}\left[x - \frac{1}{D^2}.2D\right]\frac{1}{D^2}\sin x = e^{x}\left[x - \frac{2}{D}\right](-\sin x)$$

$$= e^{x}\left[-x\sin x - 2\cos x\right]$$

Ex.37 Solve $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = xe^{-x}\cos x$ **[M.U. 1997, 2009]**

Solution: The auxiliary equation is $D^2 + 2D + 1 = 0$

$\therefore \quad (D+1)^2 = 0 \qquad \therefore \quad D = -1, -1$

$\therefore$ The C.F. is $y = (c_1 + c_2 x)e^{x}$

$$\text{P.I.} = \frac{1}{(D+1)^2}e^{-x}x\cos x$$

$$= e^{-x}.\frac{1}{(D-1+1)^2}x\cos x$$

$$= e^{-x}.\frac{1}{D^2}x\cos x = e^{-x}\frac{1}{D}\int x\cos x\,dx$$

$$= e^{-x}.\frac{1}{D}\left[x\sin x + \cos x.1\right]$$

(By generalized rule of integration by parts)

$$= e^{-x}\int\left[x\sin x + \cos x\right]dx$$

$$= e^{-x}\left[x(-\cos x) - (-\sin x).1 + \sin x\right]$$

$$= e^{-x}\left[-x\cos x + 2\sin x\right]$$

$\therefore$ The complete solution is

$$y = (c_1 + c_2 x)e^{-x} + e^{-x}(-x\cos x + 2\sin x)$$

$$= e^{-x}(c_1 + c_2 x - x\cos x + 2\sin x)$$

$$\text{P.I.} = \frac{1}{(D+1)^2}e^{-x}.x\cos x$$

$$= e^{-x}.\frac{1}{(D-1+1)^2}x\cos x = e^{-x}.\frac{1}{D^2}x\cos x$$

$\therefore$ $$\text{P.I.} = e^{-x}\left[x - \frac{1}{D^2}.2D\right]\frac{1}{D^2}\cos x$$

$$= e^{-x}\left[x - \frac{1}{D^2}.2D\right](-1)\cos x$$

$$= e^{-x}\left[-x\cos x - \frac{1}{D^2}2\sin x\right]$$

$$= e^{-x}\left[-x\cos x + 2\sin x\right]$$

Ex.38 Solve $\left(D^2 + 4D + 4\right)y = e^{-2x}x\cos x$ **[M.U. 1990, 93]**

Kalpana Classes

Solution: The auxiliary equation is $D^2+4D+4=0$

$\therefore \quad (D+2)^2=0 \qquad \therefore \quad D=-2,2$

$\therefore$ The C.F. is $y=(c_1+c_2x)e^{-2x}$

$$\text{P.I.} = \frac{1}{(D+2)^2}e^{-2x}x\cos x = e^{-2x}\frac{1}{(D-2+2)^2}x\cos x$$

$$= e^{-2x}\frac{1}{D^2}.x\cos x = e^{-2x}.\frac{1}{D}\int x\cos x\,dx$$

$$= e^{-2x}.\frac{1}{D}\left[x(\sin x)+\cos x\right]$$

$$= e^{-2x}\int(x\sin x+\cos x)\,dx$$

$$= e^{-2x}\left[x(-\cos x)-(-\sin x)+\sin x\right]$$

$\therefore$ P.I. $= e^{-2x}\left[-x\cos x+2\sin x\right]$

Alternatively: P.I. $= e^{-2x}.\dfrac{1}{D^2}x\cos x$

$$= e^{-2x}\left[x-\frac{1}{D^2}.2D\right]\frac{1}{D^2}\cos x$$

$$= e^{-2x}\left[-x\cos x+2\sin x\right]$$

$\therefore$ The complete solution is

$$y=(c_1+c_2x)e^{-2x}+e^{-2x}(-x\cos x+2\sin x)$$

Ex.39 Solve $\dfrac{d^2y}{dx^2}+3\dfrac{dy}{dx}+2y=e^{e^x}$ **[M.U. 1996, 99, 2002]**

Solution: The auxiliary equation is $D^2+3D+2=0$

$\therefore \quad (D+1)(D+2)=0 \qquad \therefore \quad D=-1,-2$

$\therefore$ The C.F. is $y=c_1e^{-x}+c_2e^{-2x}$

$$\text{P.I.} = \frac{1}{(D+2)(D+1)}e^{e^x}$$

$$= \frac{1}{D+2}.e^{-x}\int e^{e^x}e^x\,dx$$

To find the integral, put $e^x=t \qquad \therefore \qquad e^x dx=dt$

$$\therefore \quad \int e^{e^x}e^x\,dx=\int e^t\,dt=e^t=e^{e^x}$$

$$\therefore \quad \frac{1}{D+2}e^{-x}\int e^{e^x}e^x\,dx=\frac{1}{D+2}e^{-x}.e^{e^x}$$

$$= e^{-2x}\int e^{e^x}.e^{2x}.e^{-x}\,dx$$

$$= e^{-2x}\int e^{e^x}e^x\,dx$$

$$= e^{-2x}.e^{e^x}$$

Kalpana Classes

$\therefore$ The complete solution is

$$y = c_1 e^{-x} + c_2 e^{-2x} + e^{-2x}.e^{e^x}$$

Ex.40 Solve $\left(D^2 + D\right)y = \dfrac{1}{1+e^x}$ **[M.U. 2009]**

Solution: The auxiliary equation is $D(D+1) = 0 \quad \therefore \quad D = 0, -1$

$\therefore$ The C.F. is $y = c_1 + c_2 e^{-x}$

$$\text{P.I.} = \frac{1}{D+1}.\frac{1}{D}.\frac{1}{1+e^x} = \frac{1}{D+1}\int \frac{dx}{1+e^x}$$

$$= \frac{1}{D+1}.\int \frac{e^{-x}}{e^{-x}+1}dx \qquad [\text{Put } e^{-x}+1 = t]$$

$$= \frac{1}{D+1}\left[-\log\left(e^{-x}+1\right)\right]$$

$$= -e^{-x}\int e^x\left[\log\left(e^{-x}+1\right).e^x - \int e^x.\frac{\left(-e^{-x}\right)}{e^{-x}+1}dx\right]$$

(By integrating by parts)

$$= -e^{-x}\left[e^x \log\left(e^{-x}+1\right) + \int \frac{dx}{e^{-x}+1}\right]$$

$$= -e^{-x}\left[e^x \log\left(e^{-x}+1\right) + \int \frac{e^x}{e^x+1}dx\right]$$

$$= -e^{-x}\left[e^x \log\left(e^{-x}+1\right) + \log\left(1+e^x\right)\right]$$

$\therefore$ The complete solution is

$$y = c_1 + c_2 e^{-x} - e^{-x}\left[e^x \log\left(e^{-x}+1\right) + \log\left(1+e^x\right)\right]$$

Ex.41 Solve $\left(D^2 - D - 2\right)y = 2\log x + \dfrac{1}{x} + \dfrac{1}{x^2}$ **[M.U. 2000, 08, 10, 11]**

Solution: The auxiliary equation is $\left(D^2 - D - 2\right) = 0$

$\therefore \quad (D-2)(D+1) = 0 \quad \therefore \quad D = -1, 2$

$\therefore$ The C.F. is $y = c_1 e^{-x} + c_2 e^{2x}$

$$\text{P.I.} = \frac{1}{(D-2)(D+1)}.\left(2\log x + \frac{1}{x} + \frac{1}{x^2}\right)$$

$$= \frac{1}{D-2}.e^{-x}\int e^x\left(2\log x + \frac{1}{x} + \frac{1}{x^2}\right)dx$$

$$= \frac{1}{D-2}.e^{-x}\left[\int e^x\left(2\log x + \frac{2}{x}\right)dx + \int e^x\left(-\frac{1}{x} + \frac{1}{x^2}\right)dx\right]$$

Kalpana Classes

$$= \frac{1}{D-2}.e^{-x}.\left[e^x 2\log x - e^x.\frac{1}{x}\right]$$

$$\left[\because \int e^x\left[f(x)+f'(x)\right]dx = e^x f(x)\right]$$

$$= \frac{1}{D-2}.\left[2\log x - \frac{1}{x}\right] = e^{2x}\int e^{-2x}\left(2\log x - \frac{1}{x}\right)dx$$

$$= e^{2x}\left[2\log x\left(-\frac{e^{-2x}}{2}\right) - \int\left(-\frac{e^{-2x}}{2}.\frac{2}{x}\right)dx - \int e^{-2x}\frac{1}{x}dx\right]$$

[Or you may use $\int e^{ax}\left[af(x)+f'(x)\right]dx = e^{ax}f(x)$]

$$= -e^{2x}.e^{-2x}.\log x = -\log x$$

$\therefore$ The complete solution is

$$y = c_1 e^{-x} + c_2 e^{2x} - \log x$$

EXERCISE

Solve the following differential equations :

- $\left(D^2 + a^2\right)y = \operatorname{cosec} ax$ **[M.U. 1997]**

Ans. $y = c_1\cos ax + c_2\sin ax + \frac{1}{a^2}\log(\sin ax) - \frac{x}{a}\cos ax$

- $\left(D^2 + 2D + 1\right)y = 4e^{-x}\log x$ **[M.U. 1997, 99]**

Ans. $y = (c_1 + c_2 x)e^{-x} + e^{-x}x^2(2\log x - 3)$

- $\left(D^2 - 1\right)y = e^{-x}\sin\left(e^{-x}\right) + \cos\left(e^{-x}\right)$ **[M.U. 2002, 06]**

Ans. $y = c_1 e^{-x} + c_2 e^x - e^x \sin e^{-x}$

(**Hint:** P.I. $= \frac{1}{D-1}.\frac{1}{D+1}\left[\cos\left(e^{-x}\right) + e^{-x}\sin\left(e^{-x}\right)\right]$

P.I. $= \frac{1}{D-1}.e^{-x}\int e^x\left[\cos\left(e^{-x}\right) + e^{-x}\sin\left(e^{-x}\right)\right]dx$

$= \frac{1}{D-1}e^{-x}.e^x\cos\left(e^{-x}\right)$ $\left[\because \int e^x\left[f(x)+f'(x)\right]dx = e^x f(x)\right]$

$= \frac{1}{D-1}\cos\left(e^{-x}\right) = e^x\int e^{-x}\cos\left(e^{-x}\right)dx = -e^x\sin\left(e^{-x}\right)$

- $\left(D^2 - 1\right)y = \frac{2}{1+e^x}$ **[M.U. 2001]**

Ans. $y = c_1 e^{-x} + c_2 e^x - e^{-x}\log\left(1+e^x\right) - 1 + e^x\log\left(e^{-x}+1\right)$

(**Hint:** The part e^{-x} coming from P.I. can be absorbed in c_2 of C.F.)

Ex.42 Apply the method of variation of parameters to solve $\frac{d^2y}{dx^2} + a^2 y = \sec ax$

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[M.U. 1995, 99, 2003]

Solution: The auxiliary equation is $D^2 + a^2 = 0 \quad \therefore \quad D = ai, -ai$

$\therefore$ The C.F. is $y = c_1 \cos ax + c_2 \sin ax$

Here, $y_1 = \cos ax, y_2 = \sin ax, X = \sec ax$

Let P.I. be $y = uy_1 + vy_2$

Now, $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} = a$

$\therefore \quad u = -\int \frac{y_2 X}{W} dx = -\frac{1}{a}\int \sin ax . \sec ax \, dx$

$= -\frac{1}{a}\int \tan ax \, dx = \frac{1}{a^2} \log \cos ax$

And $\quad v = \int \frac{y_1 X}{W} dx = \frac{1}{a}\int \cos x . \sec ax \, dx$

$= \frac{1}{a}\int dx = \frac{x}{a}$

$\therefore$ P.I. $= \frac{1}{a^2} \log \cos ax . \cos ax + \frac{x}{a} . \sin ax$

$\therefore$ The complete solution is

$$y = c_1 \cos ax + c_2 \sin ax + \frac{1}{a^2} \log \cos ax . \cos ax + \frac{x}{a} . \sin ax$$

Ex.43 Apply the method of variation of parameters to Solve $\left(D^2 - 2D + 2\right) y = e^x \tan x$

[M.U. 2002, 09, 11, 12]

Solution: The auxiliary equation is $D^2 - 2D + 2 = 0 \quad \therefore \quad D = 1 \pm i$

$\therefore$ The C.F. is $y = e^x \left(c_1 \cos x + c_2 \sin x\right)$

Here, $y_1 = e^x \cos x, y_2 = e^x \sin x, X = e^x \tan x$

Let P.I. be $y = uy_1 + vy_2$

Now, $W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} e^x \cos x & e^x \sin x \\ e^x (\cos x - \sin x) & e^x (\sin x + \cos x) \end{vmatrix}$

$= e^x \cos x . e^x (\sin x + \cos x) - e^x \sin x . e^x (\cos x - \sin x)$

$\therefore \quad W = e^{2x}\left(\sin^2 x + \cos^2 x\right) = e^{2x}$

$\therefore \quad u = -\int \frac{y_2 X}{W} dx = -\int \frac{e^x \sin x . e^x \tan x}{e^{2x}} dx$

$= -\int \frac{\sin^2 x}{\cos x} dx - \int \frac{\left(1 - \cos^2 x\right)}{\cos x} dx$

$= -\int \sec x \, dx + \int \cos x \, dx = -\log(\sec x + \tan x) + \sin x$

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And $\quad v=\int \frac{y_1 X}{W}dx=\int \frac{e^x \cos x . e^x \tan x}{e^{2x}}dx$

$$=\int \sin x\, dx=-\cos x$$

$\therefore \quad$ P.I. $=-\log(\sec x+\tan x).e^x \cos x+e^x \sin x \cos x-e^x \cos x \sin x$

$\therefore \quad$ The complete solution is

$$y=e^x(c_1 \cos x+c_2 \sin x)-e^x \cos x.\log(\sec x+\tan x)$$

Ex.44 Use the method of variation of parameters to Solve $\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=e^{e^x}$

[M.U. 1995, 96, 99, 2002, 05, 09]

Solution: The auxiliary equation is $D^2+3D+2=0$

$\therefore \quad (D+1)(D+2)=0 \quad \therefore \quad D=-1,2$

$\therefore \quad$ The C.F. is $y=c_1e^{-x}+c_2e^{-2x}$

Here $\quad y_1=e^{-x}, y_2=e^{-2x}, X=e^{e^x}$

Let P.I. be $y=uy_1+vy_2$

Now, $\quad W=\begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}=\begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix}=-e^{-3x}$

$\therefore \quad u=-\int \frac{y_2X}{W}dx=-\int \frac{e^{-2x}.e^{e^x}}{-e^{-3x}}dx$

$$=\int e^{e^x}e^x dx=e^{e^x} \qquad [\text{Put } e^x=t]$$

And $\quad v=\int \frac{y_1X}{W}dx=\int \frac{e^{-x}.e^{e^x}}{e^{-3x}}dx=\int e^{2x}e^{e^x}dx$

Putting $\quad e^x=t, v=\int e^t.t\,dt=t\,e^t-e^t$

$\therefore \quad v=e^xe^{e^x}-e^{e^x}$

$\therefore \quad$ P.I. $=e^{e^x}.e^{-x}-\left(e^xe^{e^x}-e^{e^x}\right).e^{-2x}$

$$=e^{-2x}.e^{e^x}$$

$\therefore \quad$ The complete solution is

$$y=c_1e^x+c_2e^{-2x}+e^{-2x}.e^{e^x}$$

Ex.45 Solve the following by the method of variation of parameters

$$\frac{d^2y}{dx^2}-y=e^{-x}\sin\left(e^{-x}\right)+\cos\left(e^{-x}\right)$$ **[M.U. 2003]**

Solution: The auxiliary equation is $D^2-1=0$

$\therefore \quad D=-1,1$

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$\therefore$ The C.F. is $y = c_1 e^{-x} + c_2 e^{x}$

Here $y_1 = e^{-x}, y_2 = e^{x}, X = e^{-x}\sin\left(e^{-x}\right) + \cos\left(e^{-x}\right)$

Let P.I. be $y = uy_1 + vy_2$

Now, $W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{x} \\ -e^{-x} & e^{x} \end{vmatrix} = e^0 + e^0 = 2$

$$\therefore \quad u = -\int \frac{y_2 X}{W} dx = -\frac{1}{2}\int e^{x}\left[\cos\left(e^{-x}\right) + e^{-x}\sin\left(e^{-x}\right)\right] dx$$

$$= -\frac{1}{2} e^{x}\cos\left(e^{-x}\right) \quad \left[\because \int e^{x}\left[f(x) + f'(x)\right] dx = e^{x} f(x)\right]$$

and $v = \int \frac{y_1 X}{W} dx = \frac{1}{2}\int e^{-x}\left[e^{-x}\sin\left(e^{-x}\right) + \cos\left(e^{-x}\right)\right] dx$

For integration, put $e^{-x} = t \therefore -e^{-x} dx = dt$

$$v = -\frac{1}{2}\int (t\sin t + \cos t)\, dt$$

$$= -\frac{1}{2}\left[t(-\cos t) - (1)(-\sin t) + \sin t\right]$$

$$= \frac{1}{2} t\cos t - \sin t = \frac{1}{2} e^{-x}\cos\left(e^{-x}\right) - \sin\left(e^{-x}\right)$$

$$\therefore \text{P.I.} \quad = -\frac{1}{2} e^{x}\cos\left(e^{-x}\right).e^{-x} + \left[\frac{1}{2}.e^{-x}\cos\left(e^{-x}\right) - \sin\left(e^{-x}\right)\right] e^{x}$$

$$= -e^{x}.\sin\left(e^{-x}\right)$$

$\therefore$ The complete solution is

$$y = c_1 e^{x} + c_2 e^{-x} - e^{x}.\sin\left(e^{-x}\right)$$

Ex.46 Solve by the method of variation of parameters

$$\frac{d^2 y}{dx^2} + y = \frac{1}{1 + \sin x}$$ **[M.U. 2005, 07]**

Solution: The auxiliary equation is $D^2 + 1 = 0$ $\quad \therefore D = i, -i$

$\therefore$ The C.F. is $y = c_1\cos x + c_2\sin x$

Here $y_1 = \cos x, y_2 = \sin x, X = \dfrac{1}{1 + \sin x}$

Let P.I. be $y = uy_1 + vy_2$

Now, $W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$

$$\therefore \quad u = -\int \frac{y_2 X}{W} dx = -\int \frac{\sin x}{1}.\frac{1}{1 + \sin x} dx$$

$$= -\int \frac{\sin x}{1 + \sin x}.\frac{(1 - \sin x)}{(1 - \sin x)} dx = -\int \frac{\sin x(1 - \sin x)}{\cos^2 x} dx$$

$$= -\int\left(\sec x\tan x - \tan^2 x\right)dx$$

$$= -\int\left(\sec x\tan x - \sec^2 x + 1\right)dx$$

$$= -[\sec x - \tan x + x]$$

and $\quad V = \int \frac{y_1 X}{W}\,dx = \int \frac{\cos x}{1}.\frac{1}{(1+\sin x)}\,dx = \log(1+\sin x)$

$\therefore$ P.I. $= -[\sec x - \tan x + x]\cos x + \log(1+\sin x).\sin x$

$\therefore$ The complete solution is

$y = c_1\cos x + c_2\sin x - [1 - \sin x + x\cos x] + \sin x.\log(1+\sin x)$

Ex.47 Solve by the method of variation of parameters

$$\left(D^2 - 6D + 9\right)y = \frac{e^{3x}}{x^2}.$$ **[M.U. 2000, 08]**

Solution: The auxiliary equation is $(D-3)^2 = 0 \quad \therefore D = 3, 3$

$\therefore$ The C.F. is $y = (c_1 + c_2 x)e^{3x} = c_1 e^{3x} + c_2 x e^{3x}$

Here $\quad y_1 = e^{3x}, y_2 = xe^{3x}, X = e^{3x}/x^2$

Let P.I. be $y = uy_1 + vy_2$

Now, $W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3x} + 3xe^{3x} \end{vmatrix} = e^{6x}$

$\therefore \quad u = -\int \frac{y_2 X}{W}\,dx = -\int \frac{xe^{3x}.\left(e^{3x}/x^2\right)}{e^{6x}}\,dx$

$$= -\int \frac{dx}{x} = -\log x$$

and $\quad v = \int \frac{y_1 X}{W}\,dx = \int \frac{e^{3x}.\left(e^{3x}/x^2\right)}{e^{6x}} = \int \frac{dx}{x^2} = -\frac{1}{x}$

$\therefore$ P.I. $= -e^{3x}.\log x - xe^{3x}.\frac{1}{x} = -e^{3x}(\log x + 1)$

$\therefore$ The complete solution is

$y = c_1 e^{3x} + c_2 + xe^{3x} - e^{3x}(\log x + 1)$

Ex.48 Solve by the method of variation of parameters

$$\left(D^2 - 4D + 4\right)y = e^{2x}\sec^2 x$$ **[M.U. 2008, 10]**

Solution: The auxiliary equation is $(D-2)^2 = 0 \quad \therefore D = 2, 2$

$\therefore$ The C.F. is $y = (c_1 + c_2 x)e^{2x} = c_1 e^{2x} + c_2 x e^{2x}$

Here $\quad y_1 = e^{2x}, y_2 = xe^{2x}, X = e^{2x}\sec^2 x.$

Let P.I. be $y = uy_1 + vy_2$

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Now, $W=\begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}=\begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x}+2xe^{2x} \end{vmatrix}=e^{4x}$

$\therefore \quad u=-\int \dfrac{y_2 X}{W}dx$

$=-\int \dfrac{xe^{2x}.e^{2x}\sec^2 x}{e^{4x}}dx=-\int x\sec^2 x\,dx$

$=-\left[x\tan x-\int \tan x.1.dx\right]$

$=-x\tan x+\log\sec x$

and $\quad v=\int \dfrac{y_1 X}{W}dx=\int \dfrac{e^{2x}.e^{2x}\sec^2 x}{e^{4x}}dx$

$=\int \sec^2 x dx=\tan x$

$\therefore$ P.I. $\quad =-xe^{2x}\tan x+e^{2x}.\log\sec x-xe^{2x}\tan x$

$=e^{2x}.\log\sec x$

$\therefore \quad$ The complete solution is

$y=c_1e^{2x}+c_2xe^{2x}+e^{2x}.\log\sec x$

Ex.49 Solve $\left(D^2-1\right)y=\dfrac{2}{\sqrt{1-e^{-2x}}}$ **[M.U. 2007]**

Solution: The auxiliary equation is

$D^2-1=0 \qquad \therefore D=+1,-1$

$\therefore \quad$ The C.F. $y=c_1e^{x}+c_2e^{-x}$

$\therefore \quad y_1=e^{x}, y_2=e^{-x}, X=\dfrac{2}{\sqrt{1-e^{-2x}}}$

$\therefore \quad W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=\begin{vmatrix} e^{x} & e^{-x} \\ e^{x} & -e^{-x} \end{vmatrix}=-2$

$\therefore \quad u=-\int \dfrac{y_2 X}{W}dx=-\int e^{-x}.\dfrac{2}{\sqrt{1-e^{-2x}}}.\dfrac{1}{-2}dx$

$=\int \dfrac{e^{-x}}{\sqrt{1-e^{-2x}}}dx=\int \dfrac{-dt}{\sqrt{1-t^2}} \qquad$ Put $e^{-x}=t$

$=-\sin^{-1}(t)=-\sin^{-1}\left(e^{-x}\right)$

$\therefore \quad uy_1=-e^{x}\sin^{-1}\left(e^{-x}\right)$

$v=\int \dfrac{y_1 X}{W}dx=\int e^{x}.\dfrac{2}{\sqrt{1-e^{-2x}}}.\dfrac{1}{-2}dx$

$=\int \dfrac{e^{x}}{\sqrt{1-e^{-2x}}}dx=\int \dfrac{e^{x}.e^{x}}{\sqrt{e^{2x}+1}}dx$

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(Multiply by e^x in the numerator and denominator)

Put $e^x = t$ $\quad \therefore I = \int \frac{tdt}{\sqrt{t^2+1}} = \sqrt{t^2+1} = \sqrt{e^{2x}+1}$

$$v.y_2 = e^{-x}\sqrt{e^{2x}+1} = \sqrt{1+e^{-2x}}$$

$\therefore$ The complete solution is

$$y = c_1e^x + c_2e^{-x} - e^x \sin\left(e^{-x}\right) + \sqrt{1+e^{-2x}}$$

Ex.50 Use the method of variation of parameters to solve the equation $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = e^{-2x}.\sec^2 x(1+2\tan x)$ **[M.U. 2010]**

Solution: The auxiliary equation is $D^2 + 5D + 6 = 0$

$\therefore$ $(D+2)(D+3) = 0 \quad \therefore D = -2, -3$

$\therefore$ C.F. is $y = c_1e^{-2x} + c_2e^{-3x}$

$\therefore$ $y_1 = e^{-2x}, y_2 = e^{-3x}, X = e^{-2x}\sec^2 x(1+2\tan x)$

$\therefore$ $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-2x} & e^{-3x} \\ -2e^{-2x} & -3e^{-3x} \end{vmatrix} = -e^{-5x}$

$\therefore$ $u = -\int \frac{y_2X}{W}dx$

$$= -\int \frac{e^{-3x}.e^{-2x}}{-e^{-5x}}\sec^2 x.(1+2\tan x)dx$$

$$= \int (1+2\tan x)\sec^2 x dx$$

$$= \frac{1}{4}(1+2\tan x)^2$$

$$v = \int \frac{y_1X}{W}dx$$

$$= \int \frac{e^{-2x}.e^{-2x}.\sec^2 x(1+2\tan x)}{-e^{-5x}}$$

$$= -\int e^{-x}.[1+2\tan x].\sec^2 x dx$$

Let $f(x) = \left(\frac{1+2\tan x}{2}\right) \therefore f'(x) = \sec^2 x$

$\therefore$ $\int e^x\left[f(x) + f'(x)\right]dx = e^x f(x)$

$\therefore$ $v = -e^x.\frac{(1+2\tan x)}{2}$

$\therefore$ The complete solution is

$$y = c_1e^{-2x} + c_2e^{-3x} + \frac{1}{4}(1+2\tan x)^2 - \frac{e^x}{2}(1+2\tan x)$$

Kalpana Classes

Ex.51 Apply the method of variation of parameters to solve $\left(D^3+D\right)y=\cos ecx$

[M.U. 1997, 2005, 08]

Solution: The auxiliary equation is $D\left(D^2+1\right)=0$

$\therefore$ $D=0,i,-i$

$\therefore$ The C.F. is $y=c_1+c_2\cos x+c_3\sin x$

Here $y_1=1, y_2=\cos x, y_3=\sin x, X=\cos ecx$

Let P.I. be $y=uy_1+vy_2+wy_3$

Now, $W=\begin{vmatrix} y_1 & y_2 & y_3 \\ y'_1 & y'_2 & y'_3 \\ y''_1 & y''_2 & y''_3 \end{vmatrix}=\begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix}$

$=\sin^2 x+\cos^2 x=1$

$\therefore$ $u=\int\frac{(y_2y'_3-y_3y'_2)X}{W}dx$

$=\int\left(\cos^2 x+\sin^2 x\right)\cos ec\,x\,dx$

$=\int\cos ecxdx=\log(\cos ecx-\cot x)$

$v=\int\frac{(y_3y'_1-y_1y'_3)X}{W}dx$

$=\int(\sin x.0-1.\cos x).\cos ecx\,dx$

$=-\int\cot xdx=-\log\sin x$

and $w=\int\frac{(y_1y'_2-y_2y'_1)X}{W}dx$

$=\int\left[1.(-\sin x)-0.\cos x\right]\cos ecxdx$

$=\int -dx=-x$

$\therefore$ P.I. $=\log(\cos ecx-\cot x)1-\log\sin x.\cos x-x\sin x$

$\therefore$ The complete solution is

$y=c_1+c_2\cos x+c_3\sin x+\log(\cos ecx-\cot x)-\log\sin x.\cos x-x\sin x$

Kalpana Classes

EXERCISE

Solve the following differential equations by the method of variation of parameters

- $\frac{d^2y}{dx^2}+k^2y=\tan kx$ **[M.U. 1998, 04]**

Ans. $y=c_1\cos kx+c_2\sin kx-\frac{1}{k^2}\cos kx.\log(\sec kx+\tan kx)$

- $\left(D^2-1\right)y=\frac{2}{1+e^x}$ **[M.U. 1997, 02, 03]**

Ans. $y=c_1e^x+c_2e^{-x}-1+\log\left(1+e^{-x}\right)e^x-\left[\log\left(1+e^{-x}\right)\right]e^{-x}$

or $\quad y = c_1 e^x + c_2 e^{-x} - 1 - xe^x + \left(e^x - e^{-x}\right)\log\left(1+e^x\right)$

- $\left(D^2 + D\right)y = \dfrac{1}{1+e^x}$ **[M.U. 1997, 2003]**

Ans. $y = c_1 + c_2 e^{-x} - \log\left(1+e^{-x}\right) - e^{-x}\log\left(1+e^x\right)$

or $\quad y = c_1 + c_2 e^{-x} - \left(1+e^{-x}\right)\log\left(1+e^x\right) + x$

- $\left(D^2 + a^2\right)y = a^2 \sec^2 ax$ **[M.U. 1997, 2003]**

Ans. $y = c_1 \cos ax + c_2 \sin ax - 1 + \sin ax.\log(\sec ax + \tan ax)$

- $\left(D^2 + 3D + 2\right)y = \dfrac{1}{1+e^x}$ **[M.U. 2011]**

Ans. $y = c_1 e^{-x} + c_2 e^{-x} + \left(e^{-x} - e^{-2x}\right)\log\left(1+e^x\right) + e^{-2x}\left(1+e^x\right)$

Ex.52 Solve $\left(D^3 + 1\right) = e^{x/2} \sin\left(\dfrac{\sqrt{3}}{2}x\right)$ **[M.U. 2007]**

Solution: The auxiliary equation is $D^3 + 1 = 0$

$\therefore \quad (D+1)\left(D^2 - D + 1\right) = 0 \qquad \therefore D = -1, \dfrac{1 \pm \sqrt{3}.i}{2}$

$\therefore$ The C.F. is $y = c_1 e^{-x} + e^{x/2}\left(c_2 \cos\dfrac{\sqrt{3}}{2}x + c_3 \sin\dfrac{\sqrt{3}}{2}x\right)$.

$$\text{P.I.} = \frac{1}{D^3+1} e^{x/2}.\sin\frac{\sqrt{3}x}{2}$$

$$= e^{x/2} \frac{1}{\left[D+(1/2)\right]^3 + 1} \sin\frac{\sqrt{3}}{2}x$$

$$= e^{x/2} \frac{1}{D^3 + (3/2)D^2 + (3/4)D + (9/8)} \sin\frac{\sqrt{3}}{2}x$$

If we put $D^2 = -3/4$ the denominator vanishes. Since $\Phi'(D)^2 = 3D^2 + 3D + (3/4).$, we get,

$$P.I. = e^{x/2} \frac{x}{3(-3/4) + 3D + (3/4)} \sin\frac{\sqrt{3}}{2}x$$

$$= e^{x/2} \frac{x}{3D - (3/2)} \sin\frac{\sqrt{3}}{2}x$$

$$\therefore \quad \text{P.I.} = e^{x/2}.x \frac{3D + (3/2)}{9D^2 - (9/4)} \sin\frac{\sqrt{3}}{2}x$$

$$= \frac{e^{x/2}.x.\left[3.\left(\sqrt{3}/2\right)\cos\left(\sqrt{3}/2\right)x + (3/2)\sin\left(\sqrt{3}/2\right)x\right]}{-9}$$

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$$= -\frac{xe^{x/2}}{6}\left[\sqrt{3}\cos\left(\sqrt{3}/2\right)x + \sin\left(\sqrt{3}/2\right)x\right]$$

$\therefore$ The complete solution is

$$y = c_1 e^{-x} + e^{x/2}\left(c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right)$$

$$-\frac{xe^{x/2}}{6}\left[\sqrt{3}\cos\left(\sqrt{3}/2\right)x + \sin\left(\sqrt{3}/2\right)x\right]$$

Ex.53 Solve $\left(D^3 + D^2 + D + 1\right) = \sin^2 x$ **[M.U. 1990, 08]**

Solution: The auxiliary equation is $D^3 + D^2 + D + 1 = 0$

$\therefore$ $\left(D^2 + 1\right)(D+1) = 0 \quad \therefore D = \pm i, -1$

$\therefore$ The C.F. is $y = c_1 \cos x + c_2 \sin x + c_3 e^{-x}$

P.I
$$= \frac{1}{D^3 + D^2 + D + 1}\sin^2 x + \frac{1}{D^3 + D^2 + D + 1}.\frac{(1-\cos 2x)}{2}$$

$$= \frac{1}{D^3 + D^2 + D + 1}.\frac{1}{2}e^0 x + \frac{1}{D^3 + D^2 + D + 1}\left(-\frac{1}{2}\right)\cos 2x$$

$$= \frac{1}{2} - \frac{1}{2}.\frac{1}{-4D - 4 + D + 1}\cos 2x$$

$$= \frac{1}{2} + \frac{1}{2}.\frac{1}{3D + 3}\cos 2x$$

$$= \frac{1}{2} + \frac{1}{6}.\frac{1}{D+1}.\frac{D-1}{D-1}\cos 2x$$

$$= \frac{1}{2} + \frac{1}{6}.\frac{D-1}{D^2 - 1}\cos 2x$$

$$= \frac{1}{2} + \frac{1}{6}.\frac{-2\sin 2x - \cos 2x}{-4-1}$$

$$= \frac{1}{2} + \frac{1}{30}(2\sin 2x + \cos 2x)$$

$\therefore$ The complete solution is

$$y = c_1 \cos x + c_2 \sin x + c_3 e^{-x} + \frac{1}{2} + \frac{1}{30}(2\sin 2x + \cos 2x)$$

Ex.54 Solve $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = \left(x^2 e^x\right)^2$ **[M.U. 1992, 02]**

Solution: The auxiliary equation is $D^2 - 4D + 3 = 0$

$\therefore$ $(D-1)(D-3) = 0 \quad \therefore D = 1, 3$

$\therefore$ The C.F is $y = c_1 e^x + c_2 e^{3x}$

P.I. $= \frac{1}{D^2 - 4D + 3}e^{2x}.x^4$

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$$= e^{2x}.\frac{1}{(D+2)^2 - 4(D+2) + 3}.x^4$$

$$= e^{2x}.\frac{1}{D^2 - 1}x^4 = -e^{2x}\left(1 - D^2\right)^{-1}.x^4$$

$$= -e^{2x}.\left[1 + D^2 + D^4 + ...\right].x^4$$

$$= -e^{2x}.\left(x^4 + 12x^2 + 24\right)$$

$\therefore$ The complete solution is

$$y = c_1 e^x + c_2 e^{3x} - e^{2x}.\left(x^4 + 12x^2 + 24\right).$$

Ex.55 Solve $\frac{d^2y}{dx^2} + y = \sin x \sin 2x + 2^x$. **[M.U. 1992]**

Solution: The auxiliary equation is $D^2 + 1 = 0$ $\therefore D = i, -i$

$\therefore$ The C.F. is $y = c_1 \cos x + c_2 \sin x$

P.I. $= \frac{1}{D^2+1}(\sin x \sin 2x) = \frac{1}{D^2+1}\left[-\frac{1}{2}(\cos 3x - \cos x)\right]$

Now, $\frac{1}{D^2+1}(\sin x \sin 2x) = \frac{1}{D^2+1}\left[-\frac{1}{2}(\cos 3x - \cos x)\right]$

$$= -\frac{1}{2}.\frac{1}{D^2+1}\cos 3x + \frac{1}{2}.\frac{1}{D^2+1}\cos x$$

$$= -\frac{1}{2}.\frac{1}{(-8)}\cos 3x + \frac{1}{2}.\frac{x}{2}\sin x$$

And $\frac{1}{D^2+1}.2^x = \frac{1}{D^2+1}e^{x\log 2}$

$$= \frac{1}{(\log 2)^2 + 1}e^{x\log 2} = \frac{1}{(\log 2)^2 + 1}.2^x$$

$\therefore$ The complete solution is

$$y = c_1 \cos x + c_2 \sin x + \frac{1}{16}\cos 3x + \frac{x}{4}\sin x + \frac{1}{(\log 2)^2 + 1}.2^x.$$

EXERCISE

Solve the following differential equations

- $\left(D^2 + 1\right)y = \sin x \sin 2x$ **[M.U. 2009]**

Ans. $y = c_1 \cos x + c_2 \sin x + \frac{1}{4}x \sin x + \frac{1}{16}\cos 3x$

- $\left(D^2 - (a+b)D + ab\right)y = e^{ax} + e^{bx}$ **[M.U. 1998, 01, 09]**

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Ans. $y = c_1 e^{ax} + c_2 e^{-ax} + \frac{x}{a-b}\left[e^{ax} - e^{bx}\right]$

- $\left(D^4 - 2D^3 + D^2\right)y = x^3$ **[M.U. 1994]**

Ans. $y = (c_1 + c_2 x) + (c_3 + c_4 x)e^x + \frac{x^5}{20} + \frac{x^4}{2} + 3x^3 + 12x^2$

- $\left(D^2 - 5D + 6\right)y = x\left(x + e^x\right)$ **[M.U. 1991]**

Ans. $y = c_1 e^{2x} + c_2 e^{3x} + \frac{1}{6}\left(x^2 + \frac{5}{3}x + \frac{19}{18}\right) + \frac{xe^x}{2} + \frac{3}{4}e^x$

- $\left(D^2 - 2D + 1\right)y = e^x + \sin\left(\sqrt{3}\right)x$ **[M.U. 1992]**

Ans. $y = (c_1 + c_2 x)e^x + \frac{1}{8}\left(\sqrt{3}.\cos\sqrt{3}.x - \sin\sqrt{3}.x\right)$

- $\left(D^2 + D - 6\right)y = e^{2x}\sin 3x$ **[M.U. 1997]**

Ans. $y = c_1 e^{2x} + c_2 e^{-3x} - \frac{1}{306}e^{2x}\left(15\cos 3x + 9\sin 3x\right)$

- $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 5y = e^x\cos 3x$ **[M.U. 1996]**

Ans. $y = c_1 e^{-x} + e^x\left(c_2\cos 2x + c_3\sin 2x\right) - \frac{e^x}{65}\left(3\sin 3x + 2\cos 3x\right)$

- $\frac{d^3y}{dx^3} - y = \left(1 + e^x\right)^2$ **[M.U. 1995]**

Ans. $y = c_1 e^x + e^{-x/2}\left[c_2\cos\left(\sqrt{3}/2\right)x + c_3\sin\left(\sqrt{3}/2\right)x\right] - 1 + \frac{2}{3}xe^x + \frac{1}{7}e^{2x}$

- $\frac{d^2y}{dx^2} + 2y = x^2 e^{3x} + e^x\cos 3x$ **[M.U. 1993]**

Ans. $y = c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x + \frac{e^{3x}}{11}\left[x^2 - \frac{12x}{11} + \frac{50}{121}\right] + \frac{e^x}{6}\left(\sin 3x - \cos 3x\right)$

- $\left(D^2 - 8D + 16\right)y = \frac{e^{4x}}{x^2}$ **[M.U. 1994]**

Ans. $y = c_1\cos 4x + c_2\sin 4x - e^{4x}\log x$

- $\left(D^2 - 6D + 9\right)y = \frac{e^{3x}}{x^2}$ **[M.U. 2002]**

Ans. $y = (c_1 + c_2 x)e^{3x} - e^{3x}\log x$

- $\left(D^2 + 6D + 9\right)y = \frac{1}{x^3}e^{-3x} + 2^x$ **[M.U. 2006]**

Ans. $y = (c_1 + c_2 x)e^{-3x} + \frac{1}{2x}.e^{-3x} + \frac{1}{\left(3 + \log 2\right)^2}.2^x$

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- $\left(D^2-2D+2\right)y=e^x(x+\sin x)$ **[M.U. 1991]**

Ans. $y=e^x(c_1\cos x+c_2\sin x)+xe^x-\frac{x}{2}e^x\cos x$

- Find $\frac{1}{D^2-2D+2}e^x(x+\sin x)$ **[M.U. 1991]**

Ans. $y=xe^x\left(1-\frac{1}{2}\sin x\right)$

- Find $\frac{1}{D^2+a^2}(\sin ax+\cos ax)$ **[M.U. 1991]**

Ans. $y=\frac{x}{2a}(\sin ax-\cos ax)$

- $\left(D^2-4D+4\right)y=\frac{e^{2x}}{1+x^2}$ **[M.U. 2004]**

Ans. $y=(c_1+c_2x)e^{2x}\left[x\tan^{-1}x-\frac{1}{2}\log\left(1+x^2\right)\right]$

- $\left(D^2+6D+9\right)y=\sinh 3x$ **[M.U. 2004]**

Ans. $y=(c_1+c_2x)e^{-3x}+\frac{1}{2}\left[\frac{e^{3x}}{36}+\frac{x^2}{2}e^{-3x}\right]$

- $(D-2)^2y=8\left[e^{2x}+\sin 2x+x^2\right]$ **[M.U. 2002]**

Ans. $y=(c_1+c_2x)e^{2x}+4x^2e^{2x}+\cos 2x+2x^2+4x+3$

- $\left(D^3-7D-6\right)y=\left(1+x^2\right)e^{2x}$ **[M.U. 2007]**

Ans. $y=c_1e^{-x}+c_2e^{-2x}+c_3e^{3x}-\frac{1}{12}.e^{2x}\left(\frac{169}{72}+\frac{5x}{6}+x^2\right)$

- $\left(D^2-4D+4\right)y=e^{2x}+x^3+\cos 2x$ **[M.U. 2003]**

Ans. $y=(c_1+c_2x)e^{2x}+\frac{x^2}{2}e^{2x}+\frac{1}{4}\left[x^3+3x^2+\frac{9x}{2}-3\right]-\frac{\sin 2x}{8}$

Kalpana Classes