2 BETA, GAMMA FUNCTIONS

Theory

- Prove that $\int_0^{\pi/2} \sin^p\theta\cos^q\theta\, d\theta = \frac{1}{2}B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$. **[M.U. 1992]**
- State and prove duplication formula. **[M.U. 1994, 96, 97, 02, 11]**
- Prove that $\Gamma(1/4).\Gamma(3/4) = \pi\sqrt{2}$. **[M.U. 1998, 2005]**
- Prove that $\Gamma(1/2) = \sqrt{\pi}$ **[M.U. 2002]**
- Show that $\Gamma(n+1) = n\Gamma(n)$ **[M.U. 2003]**
- Using duplication formula, prove that

  $B(n,n).B\left(n+\frac{1}{2}, n+\frac{1}{2}\right) = \frac{\pi}{n}.2^{1-4n}$. **[M.U. 2005]**

GAMMA FUNCTION

Ex.1 Evaluate $\int_0^\infty \sqrt{x}e^{-\sqrt[3]{x}}.dx$ **[M.U. 1991]**

Solution: Put $x^{1/3} = t$.

$\therefore \quad x = t^3 \quad \therefore \quad dx = 3t^2dt$

$$\therefore \int_0^\infty \sqrt{x}e^{-\sqrt[3]{x}}dx = \int_0^\infty t^{3/2}.c^{-t}3.t^2dt$$

$$= 3\int_0^\infty e^{-t}t^{7/2}dt = 3\Gamma(9/2)$$

$$= 3.\frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{315}{16}\sqrt{\pi}$$

Ex.2 Evaluate $\int_0^\infty x^{1/4}e^{-\sqrt{x}}dx$ **[M.U. 1992]**

Solution: Put $\sqrt{x} = t$.

$\therefore \quad x = t^2 \quad \therefore dx = 2tdt$

$$\int_0^\infty x^{1/4}e^{-\sqrt{x}}dx = \int_0^\infty t^{1/2}.e^{-t}.2t.dt = \int_0^\infty 2e^{-t}.t^{3/2}dt$$

$$= 2\Gamma\left(\frac{5}{2}\right) = 2.\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = 2.\frac{3}{2}.\frac{1}{2}\sqrt{\pi} = \frac{3}{2}\sqrt{\pi}$$

Ex.3 Show that $\int_0^\infty xe^{-x^8}dx.\int_0^\infty x^2e^{-x^4}dx = \frac{\pi}{16\sqrt{2}}$ **[M.U. 1991, 98, 2003, 07]**

Solution: Let $I_1 = \int_0^\infty xe^{-x^8}dx$ and $I_2 = \int_0^\infty x^2e^{-x^4}dx$

Putting $x^8 = t$ i.e. $x = t^{1/8}; dx = \frac{1}{8}t^{-7/8}dt$

$$\therefore \quad I_1 = \int_0^\infty t^{1/8} e^{-t} . \frac{1}{8} t^{-7/8} dt = \frac{1}{8} \int_0^\infty e^{-t} t^{-6/8} dt$$

$$= \frac{1}{8} \int_0^\infty e^{-t} t^{-3/4} dt = \frac{1}{8} \Gamma\left(\frac{1}{4}\right)$$

Putting $\quad x^4 = t$ i.e. $x = t^{1/4}; dx = \frac{1}{4} t^{-3/4} dt$

$$\therefore \quad I_2 = \int_0^\infty t^{1/2} e^{-t} \frac{1}{4} t^{-3/4} dt$$

$$\therefore \quad I_2 = \frac{1}{4} \int_0^\infty e^{-t} t^{-1/4} dt = \frac{1}{4} \Gamma\left(\frac{3}{4}\right)$$

$$\therefore \quad I_1 . I_2 = \frac{1}{8} \Gamma\left(\frac{1}{4}\right) . \frac{1}{4} \Gamma\left(\frac{3}{4}\right)$$

$$= \frac{1}{32} \sqrt{2} . \pi = \frac{\pi}{16\sqrt{2}}$$

Ex.4 Prove that $\int_0^\infty \frac{e^{-x^3}}{\sqrt{x}} dx . \int_0^\infty y^4 e^{-y^6} dy = \frac{\pi}{9}$ **[M.U. 1998, 2000, 04, 07]**

Solution: In I_1, put $x^3 = t, \quad \therefore x = t^{1/3} \therefore dx = \frac{1}{3} t^{-2/3} dt$

$$\therefore \quad I_1 = \int_0^\infty e^{-t} . t^{-1/6} . \frac{1}{3} . t^{-2/3} dt$$

$$= \frac{1}{3} \int_0^\infty e^{-t} . t^{-5/6} dt = \frac{1}{3} \Gamma\left(\frac{1}{6}\right)$$

In I_2, put $y^6 = t, \quad \therefore y = t^{1/6} \quad \therefore dy = \frac{1}{6} t^{-5/6} dt$

$$\therefore \quad I_2 = \int_0^\infty t^{4/6} . e^{-t} . \frac{1}{6} t^{-5/6} dt$$

$$\frac{1}{6} \int_0^\infty e^{-t} . t^{-1/6} dt = \frac{1}{6} . \Gamma\left(\frac{5}{6}\right)$$

$$\therefore \quad I = I_1 \times I_2 = \frac{1}{3} \Gamma\left(\frac{1}{6}\right) . \frac{1}{6} \Gamma\left(\frac{5}{6}\right)$$

$$\frac{1}{18} \Gamma\left(\frac{1}{6}\right) . \Gamma\left(\frac{5}{6}\right) = \frac{1}{18} . 2\pi = \frac{\pi}{9}$$

Ex.5 Prove that $\int_0^\infty \sqrt{y} . e^{-y^2} dy . \int_0^\infty \frac{e^{-y^2}}{\sqrt{y}} dy = \frac{\pi}{2\sqrt{2}}$ **[M.U. 1990, 96]**

Solution: Put $\quad t = y^2, y = t^{1/2}, dy = \frac{1}{2} t^{-1/2} dt.$

$$\therefore \qquad I_1 = \int_0^\infty t^{1/4} e^{-t} . \frac{1}{2} t^{-1/2} dt = \frac{1}{2} \int_0^\infty t^{-1/4} e^{-t} dt = \frac{1}{2} \Gamma\left(\frac{3}{4}\right)$$

$$\therefore \qquad I_2 = \int_0^\infty t^{-1/4} e^{-t} . \frac{1}{2} t^{-1/2} dt = \frac{1}{2} \int_0^\infty t^{-3/4} e^{-t} dt = \frac{1}{2} \Gamma\left(\frac{1}{4}\right)$$

$$\therefore \qquad I_1 + I_2 = \frac{1}{4} \Gamma\left(\frac{3}{4}\right) . \Gamma\left(\frac{1}{4}\right) = \frac{1}{4} \sqrt{2} . \pi = \frac{\pi}{2\sqrt{2}}$$

EXERCISE

Evaluate the following integrals

- $\int_0^\infty \sqrt{x} . e^{-x^2} dx$ **[M.U. 2003]**

Ans. $\frac{1}{2} \Gamma\left(\frac{3}{4}\right)$

- $\int_0^\infty \left(x^2 + 4\right) e^{-2x^2} dx$ **[M.U.1990]**

Ans. $\frac{9\sqrt{\pi}}{4\sqrt{2}}$

- $\int_0^\infty x^2 e^{-x^4} dx . \int_0^\infty e^{-x^4} dx$ **[M.U.1997]**

Ans. $\frac{\pi}{8\sqrt{2}}$

- $\int_0^\infty x e^{-x^2} dx . \int_0^\infty \frac{e^{-x^2}}{\sqrt{x}} dx$ **[M.U.2008]**

Ans. $\frac{\pi}{4\sqrt{2}}$

- $\int_0^\infty x e^{-x^8} dx . \int_0^\infty x^2 e^{-x^8} dx$ (Hint: Put $x^8 = t$) **[M.U.1998, 2007]**

Ans. $\frac{\pi}{32\sqrt{2}}$

- $\int_0^\infty x^n . e^{-\sqrt{ax}} dx$ **[M.U.2002, 2010]**

Ans. $\frac{2\Gamma(2n+2)}{a^n + 1}$

Ex.6 Evaluate $\int_0^1 x^m \left(\log \frac{1}{x}\right)^n dx.$ **[M.U. 1992, 99]**

Solution: $\int_0^1 x^m \left(\log\frac{1}{x}\right)^n dx = \int_0^1 x^m (\log 1 - \log x)^n dx$

$= (-1)^n \int_0^1 x^m (\log x)^n dx = (-1)^n (-1)^n \frac{\left\lceil n+1 \right.}{(m+1)^{n+1}}$ **(put** ***log x*** **= -*t*** **)**

$= \frac{\left\lceil n+1 \right.}{(m+1)^{n+1}}$

Ex.7 Evaluate $\int_0^1 x^{q-1} \left(\log\frac{1}{x}\right)^{p-1} dx.$ **[M.U. 1999]**

Ex.8 Evaluate $\int_0^1 (\log x)^4 dx$ **[M.U.2001]**

Solution: try by putting $\log x = -t$

Ex.9 Evaluate $\int_0^1 (x \log x)^3 dx$ **[M.U.2003]**

Solution: **refer class notes**

EXERCISE

Evaluate the following integrals

- $\int_0^1 \frac{dx}{\sqrt{-\log x}}$ **[M.U. 2003]**

Ans. $\sqrt{x}$

- $\int_0^1 (x \log x)^4 dx$ **[M.U.2009, 11]**

Ans. $\frac{4!}{5^5}$

- $\int_0^1 \frac{dx}{\sqrt{x.\log(1/x)}}$ **[M.U.2000, 05]**

Ans. $\sqrt{2\pi}$

- $\int_0^1 \sqrt{\log(1/x)}.dx$ **[M.U. 1995]**

Ans. $\frac{\sqrt{\pi}}{2}$

Ex.10 Evaluate $\int_0^\infty \frac{x^7}{7^x} dx$ **[M.U.1997]**

Solution: Put $7^x = e^t$

$\therefore$ $t = x \log 7$ $\therefore$ $dt = \log 7.dx$

When $x=0, t=0$; when $x=\infty, t=\infty$.

$$\therefore \quad \int_0^\infty \frac{x^7}{7^x}\,dx = \int_0^\infty \left(\frac{t}{\log 7}\right)^7 e^{-t}.\frac{1}{(\log 7)}\,dt$$

$$= \frac{1}{(\log 7)^8}\int_0^\infty t^7 e^{-t}\,dt = \frac{\Gamma 8}{(\log 7)^8} = \frac{7!}{(\log 7)^8}.$$

EXERCISE

Evaluate the following integrals

- $\displaystyle\int_0^\infty \frac{x^4}{4^x}\,dx$ **[M.U. 1997]**

Ans. $\dfrac{24}{(\log 4)^5}$

Ex.11 Prove that $\Gamma\left(n+\frac{1}{2}\right) = \dfrac{1.3.5....(2n-1)}{2^n}\sqrt{\pi}$ **[M.U.1997]**

Hence or otherwise prove that

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n!)\sqrt{\pi}}{n!4^n}$$

Solution: Clearly n must be positive integer

$$\therefore \quad \Gamma\left(n+\frac{1}{2}\right) = \left(n-\frac{1}{2}\right)\Gamma\left(n-\frac{1}{2}\right)$$

$$= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\Gamma\left(n-\frac{3}{2}\right) \text{ and so on}$$

$$= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)....\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right).......\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\sqrt{\pi}$$

$$= \frac{(2n-1)(2n-3).....5.3.1\sqrt{\pi}}{2^n}$$

Further multiply the numerator and denominator by

$2n(2n-2)(2n-4)....6\,.\,4\,.\,2$

$$\therefore \quad \Gamma\left(n+\frac{1}{2}\right) = \frac{2n(2n-2)(2n-2)....5\,.\,4\,.\,3\,.\,2\,.\,1\sqrt{\pi}}{2^n.2n(2n-2)(2n-4)......6\,.\,4\,.\,2}$$

$$= \frac{2n(2n-1)(2n-2).....3.2.1\sqrt{\pi}}{2^n.2^n.n(n-1)(n-2)....3.2.1} = \frac{(2n)!}{4^n.n!}\sqrt{\pi}$$

Ex.12 If $I_n = \dfrac{\frac{\sqrt{\pi}}{2}\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}$, show that $I_{n+2} = \dfrac{n+1}{n+2} I_n$ and hence, find I_5 **[M.U.1990, 2000]**

Solution: Replacing n by n + 2 in I_n.

$$I_{n+2} = \frac{\frac{\sqrt{\pi}}{2}\Gamma\left(\frac{n+3}{2}\right)}{\Gamma\left(\frac{n+2}{2}+1\right)} = \frac{\frac{\sqrt{\pi}}{2}\Gamma\left(\frac{n+1}{2}+1\right)}{\Gamma\left(\frac{n+2}{2}+1\right)}$$

$$= \frac{\frac{\sqrt{\pi}}{2}\cdot\frac{n+1}{2}\Gamma\left(\frac{n+1}{2}\right)}{\frac{n+2}{2}\Gamma\left(\frac{n+2}{2}\right)}$$

$$= \left(\frac{n+1}{n+2}\right)\cdot\frac{\frac{\sqrt{\pi}}{2}\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)} = \frac{n+1}{n+2}.I_n$$

Putting n = 3, we get,

$$I_5 = \frac{4}{5}I_3 = \frac{4}{5}.\frac{2}{3}I_1 = \frac{8}{15}\frac{\left(\sqrt{\pi}/2\right).1}{\Gamma(3/2)}$$

$$= \frac{8}{15}.\frac{\left(\sqrt{\pi}/2\right)}{(1/2)\Gamma(1/2)} = \frac{8}{15}.\frac{\sqrt{\pi}}{\sqrt{\pi}} = \frac{8}{15}.$$

Ex.13 Show that $\int_0^\infty x^{m-1}\cos ax\,dx = \dfrac{\Gamma(m)}{a^m}\cos\left(\dfrac{m\pi}{2}\right)$. **[M.U.2000, 06, 08, 09]**

Solution: Since $e^{-iax} = \cos ax - i\sin ax$, we consider the real part of

$I = \int_0^\infty x^{m-1}e^{-iax}dx.$ Put $iax = t, dx = \dfrac{dt}{ia}$

$$\therefore \quad I = \int_0^\infty \frac{t^{m-1}}{(ia)^{m-1}}.e^{-t}.\frac{dt}{ia} = \frac{1}{i^m a^m}\int_0^\infty e^{-t}t^{m+1}dt$$

$$= \frac{\Gamma(m)}{a^m}.\frac{1}{i^m}. \qquad \text{But} \qquad i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$$

$$= \frac{\Gamma(m)}{a^m}\left(\cos m\frac{\pi}{2} - i\sin m\frac{\pi}{2}\right) \qquad \text{[By Eulers Formula]}$$

$$\therefore \quad I = \int_0^\infty \frac{t^{m-1}}{(ia)^{m-1}}.e^{-t}.\frac{dt}{ia} = \frac{1}{i^m a^m}\int_0^\infty e^{-t}t^{m+1}dt$$

$$= \frac{\Gamma(m)}{a^m}.\frac{1}{i^m}. \qquad \text{But} \qquad i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$$

$$= \frac{\Gamma m}{a^m}\left(\cos m\frac{\pi}{2} - i\sin m\frac{\pi}{2}\right)$$

$$\therefore \int_0^\infty x^{m-1}\cos ax\,dx = \text{ Real part of I}$$

$$= \frac{\Gamma m}{a^m}\cos\frac{\pi}{2}$$

(Equating Imaginary part, we get $\int_0^\infty x^{m-1}\sin ax\,dx = -\frac{\Gamma m}{a^m}\sin\frac{m\pi}{2}$)

Ex.14 Prove that $\int_0^\infty xe^{-ax}\sin bx\,dx = \frac{2ab}{\left(a^2+b^2\right)^2}$. **[M.U.2004]**

Solution: Consider $\int_0^\infty xe^{-(a-ib)x}.dx$

Put $(a-ib)x = t \qquad \therefore (a-ib)dx = dt$

$$\therefore \qquad I = \int_0^\infty e^{-t}.\frac{t}{(a-ib)^2}dt = \frac{1}{(a-ib)^2}\Gamma 2 = \frac{1}{(a-ib)^2}$$

Now, $$\frac{1}{(a-ib)^2} = \frac{1}{\left(a^2-b^2\right)-2aib}.\frac{\left(a^2-b^2\right)+2aib}{\left(a^2-b^2\right)+2aib}$$

$$= \frac{\left(a^2-b^2\right)+2aib}{\left(a^2+b^2\right)^2}$$

Equating real and imaginary parts, we get

$$\int_0^\infty xe^{-ax}\cos bx\,dx = \frac{a^2-b^2}{\left(a^2+b^2\right)^2}$$

$$\int_0^\infty xe^{-ax}\sin bx\,dx = \frac{2ab}{\left(a^2+b^2\right)^2}.$$

BETA FUNCTIONS

Ex.15 Evaluate $\int_0^3 \frac{x^{3/2}}{\sqrt{3-x}}dx\int_0^1 \frac{dx}{\sqrt{1-x^{1/4}}}$. **[M.U.1997, 2002]**

Solution: In I_1, Put $x = 3t, dx = 3dt$

$$\therefore \qquad I_1 = \int_0^1 (3t)^{3/2}\frac{3dt}{\sqrt{3}\sqrt{1-t}} = 9\int_0^1 t^{3/2}(1-t)^{-1/2}dt$$

$$= 9B\left(\frac{5}{2},\frac{1}{2}\right)$$

In I_2, put $x^{1/4} = t$ i.e. $x = t^4$ $\quad \therefore\ dx = 4t^3 dt$

$$\therefore \quad I_2 = \int_0^1 4t^3 \frac{dt}{\sqrt{1-t}} = 4\int_0^1 t^3 (1-t)^{-1/2}\, dt = 4B\left(4, \frac{1}{2}\right)$$

$$\therefore \quad I = I_1 . I_2 = 9B\left(\frac{5}{2}, \frac{1}{2}\right) . 4B\left(4, \frac{1}{2}\right)$$

$$= 36 . \frac{\Gamma(5/2)\Gamma(1/2)}{\Gamma(3)} . \frac{\Gamma(4)\Gamma(1/2)}{\Gamma(9/2)}$$

$$= 36 . \frac{\Gamma(5/2)\Gamma(1/2)}{2!} . \frac{3!\,\Gamma(1/2)}{(7/2)(5/2)\Gamma(5/2)}$$

$$\therefore \quad I = 36 . \frac{3.2.1}{2.1} . \left(\Gamma\left(\frac{1}{2}\right)\right)^2 . \frac{2}{7} . \frac{2}{5}\pi = \frac{432}{35}\pi.$$

Ex.16 Prove that $\int_0^1 \sqrt{1-\sqrt{x}}\,dx . \int_0^{1/2} \sqrt{2y - 4y^2}\,dy = \frac{\pi}{30}$. **[M.U.1998, 2001]**

Solution: In I_1, Put $\sqrt{x} = t$ i.e. $x = t^2$ $\therefore dx = 2t\,dt$

$$\therefore \quad I_1 = \int_0^1 \sqrt{1-t} . 2t dt = 2\int_0^1 t(1-t)^{1/2}\, dt = 2B\left(2, \frac{3}{2}\right)$$

In I_2, put 2y = t $\quad \therefore$ 2y = dt.

$$\therefore \quad I_2 = \int_0^{1/2} \sqrt{2y}\sqrt{1-2y}\,dy = \int_0^1 t^{1/2}(1-t)^{1/2} . \frac{1}{2} dt$$

$$= \frac{1}{2}\int_0^1 t^{1/2}(1-t)^{1/2}\, dt = \frac{1}{2} . B\left(\frac{3}{2}, \frac{3}{2}\right)$$

$$I = I_1 . I_2 = 2B\left(2, \frac{3}{2}\right) . \frac{1}{2} B\left(\frac{3}{2}, \frac{3}{2}\right)$$

$$= B\left(2, \frac{3}{2}\right) . B\left(\frac{3}{2}, \frac{3}{2}\right)$$

$$= \frac{\Gamma(2)\Gamma(3/2)}{\Gamma(7/2)} . \frac{\Gamma(3/2)\Gamma(3/2)}{\Gamma(3)}$$

$$= \frac{1!\,\Gamma(3/2)}{(5/2)(3/2)\Gamma(3/2)} \frac{\left[(1/2)\Gamma(1/2)\right]^2}{2!}$$

$$= \frac{4}{15} . \frac{1}{4} . \frac{1}{2} . \pi = \frac{\pi}{30}.$$

EXERCISE

Evaluate the following

- $\int_0^1 x^6 (1-x)^{1/2}\, dx$ **[M.U. 2003]**

Ans. $B(7,3/2)$

Ex.17 Evaluate $\int_0^{2a} x^2\sqrt{2ax-x^2}\,dx$ **[M.U.2002, 04]**

Solution: Put x = 2at, dx = 2a dt. (same as taking $2ax$ common and then substituting)

$$I=\int_0^{2a} x^2.x^{1/2}\sqrt{2a-x}.dx$$

$$=\int_0^1 (2a)^{2+(1/2)}.t^{2+(1/2)}.\sqrt{2a}.(1-t)^{1/2}.2adt$$

$$=(2a)^4\int_0^1 t^{5/2}(1-t)^{1/2}\,dt=(2a)^4 B\left(\frac{7}{2},\frac{3}{2}\right)$$

$$=16a^4\frac{\Gamma(7/2)\Gamma(3/2)}{\Gamma 5}=\frac{16a^4}{4!}\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right).\Gamma\frac{1}{2}.\left(\frac{1}{2}\right)\Gamma\frac{1}{2}$$

$$=\frac{16a^4}{4!}.\frac{15}{16}.\left(\Gamma(1/2)\right)^2=\frac{5}{8}a^4.\pi$$

EXERCISE

Evaluate the following

- $\int_0^1 \frac{x^2dx}{\sqrt{1-x^4}}\int_0^1\frac{dx}{\sqrt{1-x^4}}$ **[M.U. 1994, 95]**

Ans. $\frac{\pi}{4}$

- $\int_0^1 \sqrt{\left[\sqrt{x}-x\right]}dx$ **[M.U. 1998, 02]**

Ans. $\frac{\pi}{8}$

Ex.18 Show that $\int_0^1 \frac{dx}{\sqrt{1-x^n}}=\frac{2^{(2-n)/n}\left(\Gamma(1/n)\right)^2}{n\Gamma(2/n)}$ **[M.U.1997]**

Solution: put $x^n=t$ to show that $\int_0^1\frac{dx}{\sqrt{1-x^n}}=\frac{1}{n}B\left(\frac{1}{n},\frac{1}{2}\right)=\frac{1}{n}\frac{\Gamma(1/n)\Gamma(1/2)}{\Gamma\left((1/n)+(1/2)\right)}$

To get the required form, we need to use duplication formula.
Putting m = 1/n in (10), we get,

$$2^{(2/n)-1}.\Gamma\frac{1}{n}.\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)=\sqrt{\pi}\,\Gamma\frac{2}{n}$$

$$\therefore\quad \Gamma\left(\frac{1}{n}+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{(2-n)/n}.\Gamma(1/n)}.\Gamma\frac{2}{n}$$

$$\therefore \quad I=\frac{1}{n}.\frac{\Gamma(1/n)\Gamma(1/2)}{\sqrt{\pi}\,\Gamma(2/n)}.2^{(2-n)/n}\Gamma\left(\frac{1}{n}\right)=\frac{2^{(2-n)/n}.\left(\Gamma(1/n)\right)^2}{n\,\Gamma(2/n)}$$

Ex.19 Show that $\int_0^1 \sqrt{1-x^4}\,dx=\frac{\sqrt{\pi}}{6}.\frac{\Gamma(1/4)}{\Gamma(3/4)}$ **[M.U. 2008]**

Solution: Put $x^4=t$

$\therefore x=t^{1/4}$

$$\therefore \quad dx=\frac{1}{4}t^{-3/4}dt$$

$$\therefore \quad I=\int_0^1 \frac{1}{4}.t^{-3/4}(1-t)^{-1/2}\,dt=\frac{1}{4}B\left(\frac{1}{4},\frac{3}{2}\right)$$

$$=\frac{1}{4}.\frac{\Gamma(1/4)\Gamma(3/2)}{\Gamma(7/4)}=\frac{1}{4}.\Gamma\left(\frac{1}{4}\right).\frac{1}{2}.\Gamma\left(\frac{1}{2}\right).\frac{1}{(3/4)\Gamma(3/4)}$$

$$\therefore \quad I=\frac{1}{6}.\frac{\Gamma(1/4)}{\Gamma(3/4)}.\sqrt{\pi}$$

[But by the result $\Gamma(3/4)=\sqrt{2}.\pi/\Gamma(1/4)$

$$\therefore \quad I=\frac{1}{6}.\frac{\Gamma(1/4)}{\sqrt{2}.\pi}.\Gamma(1/4)\sqrt{\pi}=\frac{1}{6\sqrt{2.\pi}}\left(\Gamma(1/4)\right)^2 \text{ also]}$$

EXERCISE

Evaluate the following integrals.

- $\int_0^{\pi}\frac{\sin^4\theta}{(1+\cos\theta)^2}.d\theta$ **[M.U. 2005]**

Ans. $\frac{3}{2}\pi$

- $\int_0^1 x^4\cos^{-1}x\,dx$ **[M.U. 2006]**

Ans. $\frac{8}{75}$

- Prove that $\int_0^{\pi/2}\sin^p x\,dx\int_0^{\pi/2}\sin^{p+1}x\,dx=\frac{1}{(p+1)}.\frac{\pi}{2}$ **[M.U. 1999]**

Ex.20 Evaluate $\int_0^{\pi/6}\cos^3 3\theta\sin^2 6\theta\,d\theta$ **[M.U. 2005, 11]**

Solution : Put $3\theta=t$ $\therefore$ $d\theta=\frac{dt}{3}$

When $\theta=0, t=0$; when $\theta=\pi/6, t=\pi/2$

$$\therefore \quad \text{I} = \int_0^{\pi/2}\cos^3 t.\sin^2 2t.\frac{dt}{3}$$

$$= \frac{1}{3}\int_0^{\pi/2} \cos^3 t\,(2\sin t\cos t)^2\,dt$$

$$= \frac{4}{3}\int_0^{\pi/2} \cos^3 t .4\sin^2 t\cos^2 t\,dt$$

$$= \frac{4}{3}\int_0^{\pi/2} \cos^5 t\sin^2 t\,dt$$

$$\therefore \quad I = \frac{4}{3}.\frac{4.2.1}{7.5.3.2} = \frac{32}{315}$$

Ex.21 Evaluate $\int_{-\pi}^{\pi} \sin^2 x\cos^4 x\,dx$ **[M.U. 2008]**

Solution : We have $I = \int_{-\pi}^{\pi} \sin^2 x\cos^4 x\,dx$

$$= 2\int_0^{\pi} \sin^2 x\cos^4 x\,dx$$

$$\left[\begin{array}{ll} \because \int_{-a}^{a} f(x)\,dx = 2\int_0^{a} f(x)\,dx & \text{if } f(x) \text{ is even} \\ \quad = 0 & \text{if } f(x) \text{ is odd} \end{array}\right]$$

$$\therefore \quad I = 2\left[\int_0^{\pi/2} \sin^2 x\cos^4 x\,dx + \int_0^{\pi/2} \sin^2(\pi-x)\cos^4(\pi-x)\,dx\right]$$

$$\left[\because \int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_0^{a} f(2a-x)\,dx\right]$$

$$\therefore \quad I = 2\left[\int_0^{\pi/2} \sin^2 x\cos^4 x + \int_0^{\pi/2} \sin^2 x\cos^4 x\,dx\right]$$

$$= 4\int_0^{\pi/2} \sin^2 x\cos^4 x\,dx$$

$$= 4.\frac{1.3.1}{6.4.2}.\frac{\pi}{2} = \frac{\pi}{8}$$

Ex.22 Prove that $\int_0^{\pi} x\sin^5 x\cos^4 x\,dx = \frac{8\pi}{315}$ **[M.U. 2008]**

Solution : $I = \int_0^{\pi} (\pi-x)\sin^5(\pi-x)\cos^4(\pi-x)\,dx$

$$\left[\because \int_0^{a} f(x)\,dx = \int_0^{a} f(a-x)\,dx\right]$$

$$= \int_0^{\pi} (\pi-x)\sin^5 x\cos^4 x\,dx$$

$$= \pi \int_0^{\pi} x \cos^4 x \, dx - \int_0^{\pi} x \sin^5 x \cos^4 x \, dx$$

$$\therefore \quad 2I = \pi \int_0^{\pi} \sin^5 x \cos^4 x dx$$

$$= \pi \int_0^{\pi/2} \sin^5 x \cos^4 x \, dx + \pi \int_0^{\pi/2} \sin^5 (\pi - x) \cos^4 (\pi - x) \, dx$$

$$\left[\because \int_0^{2a} f(x) \, dx = \int_0^{a} f(x) \, dx + \int_0^{a} f(2a - x) \, dx \right]$$

$$= 2\pi \int_0^{\pi/2} \sin^5 x \cos^4 x \, dx$$

$$\therefore \quad I = \pi . \frac{4.2.3.1}{9.7.5.3.1} \qquad \text{[By formula (27)]}$$

$$= \frac{8\pi}{315}$$

EXERCISE

Evaluate the following integrals.

- $\int_0^{2\pi} \sin^2 \theta . (1 + \cos\theta)^4 \, d\theta$ **[M.U. 2004]**

Ans. $\frac{21\pi}{8}$

Ex.23 Express $\int_0^{\pi/2} \sqrt{\tan\theta} d\theta$ as Gamma Function **[M.U. 1998]**

Solution : $\int_0^{\pi/2} \sqrt{\tan\theta} d\theta = \int_0^{\pi/2} \sin^{1/2} \theta \cos^{-1/2} \theta d\theta$

$$= \frac{1}{2} B\left(\frac{3}{4}, \frac{1}{4}\right) = \frac{1}{2} . \frac{\overline{|3/4}\,\overline{|1/4}}{\overline{|1}} = \frac{1}{2} \overline{|\tfrac{3}{4}}\,\overline{|\tfrac{1}{4}}$$

But by standard result , we have

$$\overline{|3/4} . \overline{|1/4} = \sqrt{2} . \pi$$

$$\therefore \quad \int_0^{\pi/2} \sqrt{\tan\theta} d\theta = \frac{1}{2} . \sqrt{2} . \pi = \frac{\pi}{\sqrt{s2}} .]$$

Ex.24 Prove that $\int_0^{\pi/2} \sqrt{\tan\theta} d\theta \int_0^{\pi/2} \sqrt{\cot\theta} d\theta = \frac{\pi^2}{2}$ **[M.U. 2002]**

Solution: Self Study

Ex.25 Evaluate $\int_0^{\infty} \left(\frac{t}{1 + t^2} \right)^4 dt.$ **[M.U. 2000]**

Solution : Note: this can be solved using $t^2 = y$ and using result of Ex. 30.

Put $t = \tan\theta, \sec^2\theta d\theta = dt$

$$\therefore \quad I = \int_0^{\pi/2} \left(\frac{\tan\theta}{\sec^2\theta}\right)^4 .\sec^2\theta d\theta$$

$$= \int_0^{\pi/2} \sin^4\theta.\cos^4\theta.\sec^2\theta d\theta$$

$$= \int_0^{\pi/2} \sin^4\theta.\cos^2\theta d\theta$$

$$= \frac{3.1.1}{6.4.2}.\frac{\pi}{2} = \frac{\pi}{32}$$

Ex.26 Evaluate $\int_0^{\infty} \frac{dx}{1+x^4}$. **[M.U. 1993, 2000, 02]**

Solution : Note: this can be solved using $x^4 = y$ and using result of Ex. 30.

Put $x^2 = \tan\theta \qquad \therefore 2xdx = \sec^2\theta d\theta$

$$\therefore \quad dx = \frac{1}{2x}.\sec^2\theta d\theta = \frac{1}{2\sqrt{\tan\theta}}.\sec^2\theta d\theta$$

$$\therefore \quad I = \int_0^{\pi/2} \frac{1}{1+\tan^2\theta}.\frac{1}{2\sqrt{\tan\theta}}.\sec^2\theta d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \sqrt{\left(\frac{\cos\theta}{\sin\theta}\right)} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^{-1/2}\theta.\cos^{1/2}\theta d\theta = \frac{1}{4}B\left(\frac{1}{4},\frac{3}{4}\right)$$

$$\therefore \quad I = \frac{1}{4}.\frac{\Gamma(1/4)\Gamma(3/4)}{\Gamma(1)} = \frac{1}{4}\Gamma(1/4).\Gamma(3/4)$$

But by the particular case of the duplication formula,

$$\Gamma(1/4).\Gamma(3/4) = \sqrt{2}.\pi$$

$$\therefore \quad I = \frac{1}{4}.\sqrt{2}.\pi = \frac{\pi}{2\sqrt{2}}$$

Ex.27 Evaluate $\int_0^{\pi/2} \frac{d\Phi}{\sqrt{1-(1/2)\sin^2\Phi}}$. **[M.U. 2002, 03, 10]**

Solution : Put $\frac{1}{2}\sin^2\Phi = \sin^2\theta \qquad \therefore \sin\Phi = \sqrt{2}.\sin\theta$

$$\therefore \quad \cos\Phi d\Phi = \sqrt{2}.\cos\theta d\theta$$

$$\therefore \quad d\Phi = \sqrt{2}.\frac{\cos\theta}{\sqrt{1-\sin^2\Phi}} d\theta$$

$$= \frac{\sqrt{2}.\cos\theta}{\sqrt{1-2\sin^2\theta}} d\theta = \frac{\sqrt{2}.\cos\theta}{\sqrt{\cos 2\theta}} d\theta$$

When $\Phi = 0, \theta = 0$

When $\Phi = \pi/2, \theta = \dfrac{\pi}{4}$

$$\therefore \qquad I = \int_0^{\pi/4} \frac{1}{\sqrt{1-\sin^2\theta}} \cdot \frac{\sqrt{2}.\cos\theta}{\sqrt{\cos 2\theta}}\, d\theta$$

$$= \sqrt{2}\int_0^{\pi/2} (\cos 2\theta)^{-1/2}\, d\theta$$

Put $2\theta = t$,

$$\therefore \qquad I = \sqrt{2}\int_0^{\pi/2} (\cos t)^{-1/2} \cdot \frac{dt}{2}$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{1}{2} B\left(\frac{1}{2}, \frac{1}{4}\right)$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{1}{2} \frac{\Gamma(1/2)\,\Gamma(1/4)}{\Gamma(3/4)}$$

[But by the particular case of the duplication formula,

$$\Gamma(3/4) = \sqrt{2}.\pi / \Gamma(1/4)$$

$$\therefore \qquad I = \frac{1}{2\sqrt{2}} \frac{\sqrt{\pi}.\left(\Gamma(1/4)\right)^2}{\sqrt{2}.\pi} = \frac{1}{4} \cdot \frac{\left(\Gamma(1/4)\right)^2}{\sqrt{\pi}}.]$$

Ex.28 Express $\displaystyle\int_{-\pi/4}^{\pi/4} (\sin\theta + \cos\theta)^{1/3}\, d\theta$ as a Gamma function. **[M.U. 1997, 08]**

Solution : $$I = \int_{-\pi/4}^{\pi/4} 2^{1/6}.\left(\sin\theta.\frac{1}{\sqrt{2}} + \cos\theta.\frac{1}{\sqrt{2}}\right)^{1/3} d\theta$$

$$= 2^{1/6} \int_{-\pi/4}^{\pi/4} \left[\sin\left(\frac{\pi}{4} + \theta\right)\right]^{1/3} d\theta$$

Now, put $\dfrac{\pi}{4} + \theta = t \quad \therefore d\theta = dt$

When $\theta = -\dfrac{\pi}{4}, t = 0$; when $\theta = \dfrac{\pi}{4}, t = \dfrac{\pi}{2}$.

$$\therefore \qquad I = 2^{1/6}\int_0^{\pi/2} \sin^{1/3} t\,dt = 2^{1/6}\int_0^{\pi/2} \sin^{1/3} t \cos^{0} t\, dt$$

$$= 2^{1/6}.\frac{1}{2} B\left(\frac{2}{3}, \frac{1}{2}\right) = 2^{(1/6)-1} \frac{\Gamma(2/3)\,\Gamma(1/2)}{\Gamma(7/6)}$$

$$= \frac{1}{2^{5/6}} \cdot \frac{\Gamma(2/3)}{\Gamma(7/6)}.\sqrt{\pi}$$

EXERCISE

Evaluate the following:

- $\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} . \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta$ **[M.U. 2006]**

Ans. π

- $\int_0^{\infty} \left(\frac{x}{1+x^2}\right)^6 dx$ **[M.U. 1994]**

Ans. $\frac{3\pi}{512}$.

Ex.29 Prove that $\int_a^b (x-a)^m (b-x)^n \, dx = (b-a)^{m+n+1} B(m+1, n+1)$ **[M.U. 1988, 95, 05]**

Proof: Put $(x-a)=(b-a)t$ $\quad\therefore dx=(b-a)dt$

$$\therefore \quad b-x=b-a-(b-a)t=(b-a)(1-t)$$

$$\therefore \quad I=\int_0^1 (b-a)^m t^m (b-a)^n (1-t)^n (b-a)\,dt$$

$$=(b-a)^{m+n+1}\int_0^1 t^m (1-t)^n \, dt$$

$$=(b-a)^{m+n+1} B(m+1, n+1)$$

EXERCISE

Prove that

- $\int_{-1}^{1} (1+x)^m (1-x)^n \, dx = 2^{m+n+1} B(m+1, n+1)$ Hence, evaluate $\int_{-1}^{1} \sqrt{\left(\frac{1+x}{1-x}\right)} dx$.

[M.U. 1990, 97, 98, 2002, 05]

Ans. $2B\left(\frac{3}{2}, \frac{1}{2}\right)$

- $\int_3^7 \sqrt[4]{(x-3)(7-x)}.dx = \frac{2\left(\Gamma(1/4)\right)^2}{3\sqrt{\pi}}$ **[M.U. 1997, 2000]**

Ex.30 Prove that $\int_0^{\infty} \frac{x^{m-1}}{(a+bx)^{m+n}} dx = \frac{1}{a^n b^m} B(m, n)$ **[M.U. 1992, 95, 06, 10]**

Proof: Put $\quad bx=\frac{at}{1-t}$ $\quad\therefore$ when $x=0, t=0,$ when $x=\infty, t=1$

$$\therefore \quad b\,dx = a\left[\frac{(1-t)+t}{(1-t)^2}\right]dt = \frac{a}{(1-t)^2}dt$$

And $\quad a+bx=a+\frac{at}{1-t}=\frac{a}{1-t}$

$$I=\int_0^1\left(\frac{a}{b}\right)^{m-1}\cdot\frac{t^{m-1}}{(1-t)^{m-1}}\cdot\frac{1}{a^{m+n}}\cdot(1-t)^{m+n}\cdot\frac{1}{b}\cdot\frac{adt}{(1-t)^2}$$

$$=\frac{1}{a^n b^m}\cdot\int_0^1 t^{m-1}(1-t)^{n-1}\,dt=\frac{1}{a^n b^m}B(m,n)$$

Note : the method discussed in class can also be used.

Cor. Putting a = 1, b = 1, we get

$$\boxed{\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx=B(m,n)}$$

Poof: To prove the corollary above independently, put $x=\frac{t}{1-t}$. Try it like Ex. 30.

Ex.31 Evaluate $\int_0^\infty \frac{\sqrt{x}}{a^2+2ax+x^2}\cdot dx$ **[M.U. 2004]**

Solution: $I=\int_0^\infty \frac{x^{1/2}}{(a+x)^2}.dx$ …(1)

Put $x=\frac{at}{1-t}$. When x = 0, t = 0; when $x=\infty, t=1$

Now, $a+x=a+\frac{at}{1-t}=\frac{a}{1-t}, dx=\frac{adt}{(1-t)^2}$

$$\therefore \quad I=\int_0^1 \frac{a^{1/2}t^{1/2}}{(1-t)^{1/2}}\cdot\frac{(1-t)^2}{a^2}\cdot\frac{adt}{(1-t)^2}$$

$$=\frac{1}{\sqrt{a}}\int_0^1 t^{1/2}(1-t)^{-1/2}=\sqrt{a}B\left(\frac{3}{2},\frac{1}{2}\right)$$

$$=\frac{1}{\sqrt{a}}\cdot\frac{(1/2)\overline{|1/2}\,\overline{|1/2}}{1}=\frac{\pi}{2\sqrt{a}}$$

Or comparing (1) with the Ex. 30, we see that,

$$m-1=\frac{1}{2}, m+n=2, b=1$$

$$\therefore \quad I=\frac{1}{\sqrt{a}}B\left(\frac{1}{2}+1,2-\frac{1}{2}-1\right)=\frac{1}{\sqrt{a}}B\left(\frac{3}{2},\frac{1}{2}\right)$$

$$=\frac{1}{\sqrt{a}}\cdot\frac{\left(\frac{1}{2}\right)\overline{\left|\frac{1}{2}\right.}\,\overline{\left|\frac{1}{2}\right.}}{1}=\frac{\pi}{2\sqrt{a}}$$

Ex.32 Evaluate $\int_0^\infty \frac{x^{10}-x^{18}}{(1+x)^{30}}dx.$ **[M.U. 2005, 09]**

Solution: Let $I=\int_0^\infty \frac{x^{10}}{(1+x)^{30}}dx-\int_0^\infty \frac{x^{18}}{(1+x)^{30}}dx=I_1-I_2$ …(1)

Now, Put $x=\dfrac{t}{1-t}$. When x = 0, t = 0; when $x=\infty, t=1$

$$1+x=1+\frac{t}{1-t}=\frac{1}{1-t} \qquad \therefore dx=\frac{1}{(1-t)^2}dt$$

$$\therefore \quad I_1=\int_0^\infty \frac{t^{10}}{(1-t)^{10}}.(1-t)^{30}.\frac{1}{(1-t)^2}dt$$

$$=\int_0^\infty t^{10}(1-t)^{18}\,dt=B(11,19)$$

And $$I_2=\int_0^\infty \frac{t^{18}}{(1-t)^{18}}.(1-t)^{30}.\frac{1}{(1-t)^2}dt$$

$$=\int_0^\infty t^{18}(1-t)^{10}\,dt=B(19,11)=B(11,19)$$

$$\therefore \quad I=B(11,19)-B(11,19)=0$$

We see that in $I_1,=m=10, n=30.$

$$\therefore \quad I_1=B(m+1,n-m-1) = B(10+1,30-10-1)$$
$$= \text{B}(11, 19)$$

In $I_2, m=18, n=30.$

$$\therefore \quad I_2=B(m+1,n-m-1) = B(18+1,30-18-1)$$
$$= \text{B}(19, 11) = \text{B}(11, 19)$$

$$\therefore \quad I=B(11,19)-B(11,19)=0$$

EXERCISE

Prove that

- $\displaystyle\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}=\frac{1}{a^n b^m}B(m,n)$ and hence, evaluate

  i) $\displaystyle\int_0^\infty \frac{\sqrt{x}}{\left(4+4x+x^2\right)}dx$ **[M.U. 1995]**

  Ans. $\dfrac{\pi}{2\sqrt{2}}$

  ii) $\displaystyle\int_0^\infty \frac{\sqrt{x}}{1+2x+x^2}dx$ **[M.U. 2004]**

  Ans. $\dfrac{\pi}{2}$

- $\displaystyle\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx=B(m,n)$ **[M.U. 1990, 95, 2005]**

(Hint: $x = \dfrac{t}{(1-t)}$)

- $\displaystyle\int_0^\infty \frac{x^8\left(1-x^6\right)}{(1+x)^{24}}dx = 0$ **[M.U. 1993, 2002]**

- $\displaystyle\int_0^\infty \frac{x^4\left(1+x^5\right)}{(1+x)^{15}}dx = \frac{1}{5005}$ **[M.U. 2008]**

$\text{Ans} = 2.\dfrac{\overline{|5}\,\overline{|10}}{\overline{|15}} = 2.\dfrac{9!4!}{14!} = \dfrac{1}{5005}$)

- $\displaystyle\int_0^\infty \frac{x^8 - x^5}{\left(1+x^3\right)^5}x^2 dx = 0$ **[M.U.1996, 02]**

- $\displaystyle\int_0^\infty \frac{x^5}{(2+3x)^{16}} = \frac{1}{2^{10}3^6}.\frac{\overline{|6}\,\overline{|10}}{\overline{|16}}$ **[M.U.1987, 03]**

Ex.33 Prove that $\displaystyle\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}}dx = \frac{B(m,n)}{(a+b)^m .a^n}$ **[M.U. 1991, 02]**

Solution: Put $x = \dfrac{at}{a+b-bt}$

$\therefore \quad dx = \dfrac{(a+b-bt)a - at(-b)}{(a+b-bt)^2} = \dfrac{a(a+b)}{(a+b-bt)^2}dt$

$\therefore \quad 1 - x = 1 - \dfrac{at}{a+b-bt} = \dfrac{(a+b)(1-t)}{(a+b-bt)}$,

And $\quad a + bx = a + \dfrac{b.at}{(a+b-bt)} = \dfrac{a(a+b)}{(a+b-bt)}$

$$\therefore \quad I = \int_0^1 \frac{a^{m-1}t^{m-1}}{(a+b-bt)^{m-1}}.\frac{(a+b)^{n-1}.(1-t)^{n-1}}{(a+b-bt)^{n-1}}.\frac{(a+b-bt)^{m+n}}{a^{m+n}(a+b)^{m+n}}.\frac{a(a+b)}{(a+b-bt)^2}dt$$

$$\therefore \quad I = \frac{1}{(a+b)^m .a^n}\int_0^1 t^{m-1}(1-t)^{n-1}dt = \frac{B(m,n)}{(a+b)^m .a^n}$$

Ex.34 Prove that $\displaystyle\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(1+x)^{m+n}} = \frac{B(m,n)}{2^m}$ and hence, evaluate

i) $\displaystyle\int_0^1 \frac{x^3 - 2x^4 + x^5}{(1+x)^7}dx$ **[M.U. 1996, 08]**

Solution: Putting a = 1, b = 1 in Ex. 33 above or try putting $x = \dfrac{t}{2-t}$ you can prove the first result.

For the deduction observe that

i) $x^3 - 2x^4 + x^5 = x^3(1-x)^2$

$$\therefore \quad I = \int_0^1 \frac{x^3(1-x)^2}{(1+x)^7} = \frac{B(4,3)}{2^4}$$

$$= \frac{\boxed{4}\,\boxed{3}}{2^4 \boxed{7}} = \frac{3!2!}{2^4 6!} = \frac{1}{960}.$$

Ex.35 Prove that $\displaystyle\int_0^1 \frac{x^{-1/3}(1-x)^{-2/3}}{(1+2x)}dx = \frac{1}{\sqrt[3]{9}} B\left(\frac{2}{3}, \frac{1}{3}\right)$ **[M.U. 1999]**

Solution: Comparing with the above Ex. 33, we see that a = 1, b = 2.

Hence, we put $x = \dfrac{t}{3-2t}$

$$\therefore \quad dx = \frac{(3-2t) - t(-2)}{(3-2t)^2} dt = \frac{3}{(3-2t)^2} dt$$

When x = 0, t = 0; when x = 1, t = 1.

Further, $1 - x = 1 - \dfrac{t}{3-2t} = \dfrac{3(1-t)}{3-2t}$

$$1 + 2x = 1 + \frac{2t}{3-2t} = \frac{3}{3-2t}$$

$$\therefore \quad I = \int_0^1 \frac{t^{-1/3}}{(3-2t)^{-1/3}} \cdot \frac{3^{-2/3}(1-t)^{-2/3}}{(3-2t)^{-2/3}} \cdot \frac{(3-2t)}{3} \cdot \frac{3dt}{(3-2t)^2}$$

$$= \int_0^1 \frac{1}{3^{2/3}} . t^{-1/3}(1-t)^{-2/3} dt = \frac{1}{\sqrt[3]{3}} . B\left(\frac{2}{3}, \frac{1}{3}\right)$$

Ex.36 Prove that $\displaystyle\int_0^1 \frac{\left(1-x^4\right)^{3/4}}{\left(1+x^4\right)^2} dx = \frac{1}{4} \cdot \frac{1}{2^{1/4}} B\left(\frac{7}{4}, \frac{1}{4}\right).$ **[M.U. 1999, 07]**

Solution: Put $x^4 = t \quad \therefore x = t^{1/4}, dx = \dfrac{1}{4} t^{-3/4} dt$

$$\therefore \quad I = \int_0^1 \frac{(1-t)^{3/4}}{(1+t)^2} . \frac{1}{4} t^{-3/4} dt$$

Now, Put $t = \dfrac{y}{2-y}$. When t = 0, y = 0; when t = 1,

$$1 = \frac{y}{2-y} \qquad \therefore 2 - y = y \qquad \therefore 2 = 2y \qquad \therefore y = 1$$

$$dt = \frac{(2-y)1 - y(-1)}{(2-y)^2} dy = \frac{2}{(2-y)^2} dy$$

$$\therefore \quad 1 - t = 1 - \frac{y}{2-y} = \frac{2(1-y)}{2-y} \text{ and } 1 + t = 1 + \frac{y}{2-y} = \frac{2}{2-y}$$

$$\therefore \qquad I=\int_0^1 \frac{2^{3/4}(1-y)^{3/4}}{(2-y)^{3/4}}\cdot\frac{(2-y)^2}{2^2}\cdot\frac{1}{4}\left(\frac{y}{2-y}\right)^{-3/4}\cdot\frac{2dy}{(2-y)^2}$$

$$=\frac{2^{3/4}}{8}\int_0^1 y^{-3/4}(1-y)^{3/4}\,dy$$

$$=\frac{1}{2^{3-(3/4)}}\int_0^1 y^{-3/4}(1-y)^{3/4}\,dy$$

$$=\frac{1}{2^{9/4}}B\left(\frac{1}{4},\frac{7}{4}\right)=\frac{1}{2^{2+(1/4)}}B\left(\frac{1}{4},\frac{7}{4}\right)$$

$$=\frac{1}{4}\cdot\frac{1}{2^{1/4}}B\left(\frac{7}{4},\frac{1}{4}\right)$$

Ex.37 Prove that $B(m,m).B\left(m+\frac{1}{2}.m+\frac{1}{2}\right)=\frac{\pi}{m}.2^{1-4m}$ **[M.U. 1994, 06, 08]**

Solution: $B(m,m).B\left(m+\frac{1}{2},m+\frac{1}{2}\right)=\frac{\Gamma(m)\Gamma(m)}{\Gamma(2m)}\cdot\frac{\Gamma(m+(1/2))\Gamma(m+(1/2))}{\Gamma(2m+1)}$

$$=\left[\frac{\Gamma(m)\Gamma(m+(1/2))}{\Gamma(2m)}\right]^2\cdot\frac{1}{2m} \qquad \left(\because \Gamma(2m+1)=2m\Gamma(2m)\right)$$

$$=\frac{\pi}{2^{4m-2}}\cdot\frac{1}{2m}=\frac{\pi}{2^{4m-1}}\cdot\frac{1}{m}=\frac{\pi}{m}.2^{1-4m}$$

Ex.38 Prove that $B(x,x)=\frac{1}{2^{2x-1}}B\left(x,\frac{1}{2}\right)$ **[M.U. 1996, 97, 02]**

Solution: $B(x,x)=\frac{\Gamma(x)\Gamma(x)}{\Gamma(2x)}$

But duplication formula gives $\Gamma(m)\Gamma(m+(1/2))=\frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)$

$$\therefore \qquad \frac{\Gamma(m)}{\Gamma(2m)}=\frac{\sqrt{\pi}}{2^{2m-1}\Gamma(m+(1/2))}$$

$$\therefore \qquad B(x,x)=\frac{1}{2^{2x-1}}\cdot\frac{\sqrt{\pi}}{\Gamma(x+(1/2))}\Gamma(x)=\frac{1}{2^{2x-1}}\cdot\frac{\Gamma(x)\Gamma(1/2)}{\Gamma(x+(1/2))}$$

$$=\frac{1}{2^{2x-1}}B\left(x,\frac{1}{2}\right)$$

Ex.39 If $B(n,3)=\frac{1}{105}$ and n is a positive integer, find n. **[M.U. 2002]**

Solution: $B(n,3)=\frac{\Gamma(n)\Gamma(3)}{(n+2)(n+1)n\Gamma(n)}$ $\qquad\left[\because \Gamma(n+1)=n\Gamma(n)\right]$

$$=\frac{2!}{(n+2)(n+1)n} \qquad \left[\because \Gamma(n)=(n-1)!\right]$$

By data this is equal to $\frac{1}{105}$.

$\therefore \quad \frac{2}{(n+2)(n+1)n} = \frac{1}{105}$

$\therefore \quad (n+2)(n+1)n = 210 = 7.6.5 \qquad \therefore n = 5$

Or $\quad n^3 + 3n^2 + 2n - 210 = 0$

$\therefore \quad (n-5)\left(n^2 + 8n - 42\right) = 0$

$\therefore$ n = 5 since n is an integer.

Ex.40 Given $\int_0^\infty \frac{x^{p-1}}{1+x}dx = \frac{\pi}{\sin p\pi}$,

Prove that, $\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin p\pi}\left(0 < p < 1\right)$.

Hence, evaluate $\int_0^\infty \frac{dy}{1+y^4}$ **[M.U. 1999, 02]**

Solution: Putting $x = \tan^2\theta$, we get, (also $y^4 = t$ also works here)

$$I = \int_0^{\pi/2} \frac{\tan^{2p-2}\theta.2\tan\theta\sec^2\theta d\theta}{1+\tan^2\theta}$$

$$= 2\int_0^{\pi/2} \tan^{2p-1}\theta d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2p-1}\theta.\cos^{1-2p}\theta d\theta$$

$$= 2.\frac{1}{2}.\frac{\Gamma\left(\frac{2p-1+1}{2}\right)\Gamma\left(\frac{1-2p+1}{2}\right)}{\Gamma\left(\frac{2p-1+1-2p+2}{2}\right)} = \Gamma(p)\Gamma(1-p)$$

But $\quad I = \frac{\pi}{\sin p\pi} \qquad \therefore \Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin p\pi}$

For deduction put $\quad y^4 = x$,

$\therefore \quad y = x^{1/4} \qquad \therefore dy = \frac{1}{4}x^{-3/4}dx$

$$\therefore \quad \int_0^\infty \frac{dy}{1+y^4} = \int_0^\infty \frac{1}{4}.\frac{x^{-3/4}}{1+x}dx = \frac{1}{4}.\int_0^\infty \frac{x^{(1/4)-1}}{1+x}dx$$

$$\therefore \quad \int_0^\infty \frac{dy}{1+y^4} = \frac{1}{4}.\frac{\pi}{\sin(\pi/4)} = \frac{\sqrt{2}\pi}{4} = \frac{\pi}{2\sqrt{2}}.$$

Ex.41 State true or false with proper justification.

$\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{2}{6}\right)\Gamma\left(\frac{3}{6}\right)\Gamma\left(\frac{4}{6}\right)\Gamma\left(\frac{5}{6}\right) = 4\pi^2\sqrt{\frac{\pi}{3}}$ **[M.U. 1998]**

Solution: $\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{2}{6}\right)\Gamma\left(\frac{4}{6}\right)\Gamma\left(\frac{3}{6}\right)=\left(\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{5}{6}\right)\right)\left(\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)\right)\Gamma\left(\frac{1}{2}\right)$

$$=2\pi.\frac{2\pi}{\sqrt{3}}.\sqrt{\pi}=4\pi^2.\sqrt{\frac{\pi}{3}}$$

$\therefore$ The statement is true.

Ex.42 Prove that $\int_0^1 \sqrt{1-x^4}\,dx=\frac{\left(\Gamma(1/4)\right)^2}{6\sqrt{2}\pi}$. **[M.U. 2001]**

Solution: Put $x^4=t, x=t^{1/4}$ $\quad\therefore dx=\frac{1}{4}t^{-3/4}dt$

$$\therefore\quad I=\int_0^1(1-t)^{1/2}.\frac{1}{4}.t^{-3/4}dt=\frac{1}{4}\int_0^1 t^{-3/4}(1-t)^{1/2}\,dt$$

$$=\frac{1}{4}B\left(\frac{1}{4},\frac{3}{2}\right)=\frac{1}{4}.\frac{\Gamma(1/4)\Gamma(3/2)}{\Gamma(7/4)}$$

$$=\frac{1}{4}.\frac{\Gamma(1/4)(1/2)\Gamma(1/2)}{(3/4)\Gamma(3/4)}$$

But $\Gamma(1/4)\Gamma(3/4)=\sqrt{2}.\pi$ $\quad\therefore\Gamma(3/4)=\frac{\sqrt{2}\pi}{\Gamma(1/4)}$

$$\therefore\quad I=\frac{1}{4}.\frac{1}{2}.\frac{4}{3}.\frac{\Gamma(1/4)}{\sqrt{2}\pi}.\Gamma(1/4).\sqrt{\pi}$$

$$=\frac{1}{6}.\frac{1}{\sqrt{2\pi}}.\left(\Gamma(1/4)\right)^2$$

Ex.43 Prove that $\int_0^\infty \frac{x}{\left(1+x^4\right)^{5/4}}dx.\int_0^\infty\frac{dx}{\sqrt{1+x^4}}=\frac{\pi}{2\sqrt{2}}$. **[M.U. 1995, 2006]**

Solution: Put $x^4=t\ \therefore x=t^{1/4}$ $\quad\therefore dx=\frac{1}{4}t^{-3/4}dt$

When $x=0,t=0;\ when\ x=\infty,t=\infty.$

$$\therefore\quad I_1=\int_0^\infty\frac{1}{(1+t)^{5/4}}.t^{1/4}.\frac{1}{4}.t^{-3/4}dt$$

$$=\frac{1}{4}\int_0^\infty\frac{t^{-1/2}}{(1+t)^{5/4}}dt=\frac{1}{4}\int_0^\infty\frac{t^{(1/2)-1}}{(1+t)^{(1/2)+(3/4)}}dt$$

$$=\frac{1}{4}.B\left(\frac{1}{2},\frac{3}{4}\right)$$

$$I_2=\int_0^\infty\frac{1}{(1+t)^{1/2}}.\frac{1}{4}.t^{-3/4}dt=\frac{1}{4}\int_0^\infty\frac{t^{(1/4)-1}}{(1+t)^{(1/4)+(1/4)}}dt$$

$$=\frac{1}{4}.B\left(\frac{1}{4},\frac{1}{4}\right)$$

$$\therefore \quad I = I_1 \times I_2$$

$$= \frac{1}{4}.B\left(\frac{1}{2},\frac{3}{4}\right) \times \frac{1}{4}.B\left(\frac{1}{4},\frac{1}{4}\right)$$

$$\therefore \quad I = \frac{1}{16},\frac{\Gamma(1/2)\Gamma(3/4)}{\Gamma(5/4)}.\frac{\Gamma(1/4)\Gamma(1/4)}{\Gamma(1/2)}$$

$$= \frac{1}{16}.\frac{\Gamma(3/4)\left(\Gamma(1/4)\right)^2}{(1/4)\Gamma(1/4)} = \frac{1}{4}.\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)$$

$$= \frac{1}{4}.\sqrt{2}.\pi = \frac{\pi}{2\sqrt{2}}$$

Alternatively: We may put $x^2 = \tan\theta, 2xdx = \sec^2\theta d\theta$

And $\quad dx = \frac{\sec^2\theta}{2x}d\theta = \frac{1}{2}.\frac{\sec^2\theta}{\sqrt{\tan\theta}}d\theta$

When $x = 0, \theta = 0$; when $x = \infty, \theta = \pi/2$

$$\therefore \quad I_1 = \int_0^{\pi/2} \frac{1}{2}.\frac{\sec^2\theta}{\left(\sec^2\theta\right)^{5/4}} = \frac{1}{2}\int_0^{\pi/2} \sec^{-1/2}\theta d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^0\theta \cos^{1/2}\theta d\theta = \frac{1}{2}.\frac{1}{2}.B\left(\frac{1}{2}.\frac{3}{4}\right)$$

$$= \frac{1}{4}.B\left(\frac{1}{2}.\frac{3}{4}\right)$$

$$I_2 = \int_0^{\pi/2} \frac{1}{2}.\frac{1}{\sec\theta}.\frac{\sec^2\theta}{\sqrt{\tan\theta}}d\theta = \frac{1}{2}\int_0^{\pi/2} \frac{\sec\theta}{\sqrt{\tan\theta}}d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^{-1/2}\theta \cos^{-1/2} d\theta = \frac{1}{2}.\frac{1}{2}.B\left(\frac{1}{4},\frac{1}{4}\right)$$

$= \frac{1}{4}.B\left(\frac{1}{4},\frac{1}{4}\right)$.Now, proceed as above.

Ex.44 Prove that $B(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx.$

Hence, evaluate $\quad \int_0^1 \frac{x^2+x^3}{(1+x)^7}dx$ **[M.U. 1991, 2000, 02, 03, 06]**

Solution: Let $\quad I_1 = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}dx, I_2 = \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}dx$

Note: An alternate method is suggested below to prove the above result. The method described in the class is better and should be used.

In I_1 put $x=\dfrac{t}{1-t}$ $\quad\therefore\ 1+x=\dfrac{1}{1-t},\ dx=\dfrac{1}{(1-t)^2}dt$

When $x=0, t=0;\ when\ x=1, t=\dfrac{1}{2}.$

$$\therefore \qquad I_1=\int_0^{1/2}\left(\frac{t}{1-t}\right)^{m-1}.(1-t)^{m+n}.\frac{dt}{(1-t)^2}$$

$$=\int_0^{1/2} t^{m-1}.(1-t)^{n-1}\,dt$$

Similarly, $\quad I_2=\displaystyle\int_0^{1/2} t^{n-1}.(1-t)^{m-1}\,dt$

Now, Put $\quad t=1-z\ in\ I_2$

$$\therefore \qquad I_2=\int_1^{1/2}(1-z)^{n-1}z^{m-1}(-dz)=\int_{1/2}^{1} t^{m-1}(1-t)^{n-1}\,dt$$

$$I=I_1+I_2$$

$$=\int_0^{1/2} t^{m-1}.(1-t)^{n-1}\,dt+\int_{1/2}^{1} t^{m-1}.(1-t)^{n-1}\,dt$$

$$=\int_0^{1} t^{m-1}.(1-t)^{n-1}\,dt=B(m,n)$$

Putting the particular values of m, n

$$\int_0^1\frac{x^2+x^3}{(1+x)^7}dx=B(3,4)$$

Ex.45 Show that $\displaystyle\int_0^{\pi/2}\frac{\cos^{2m-1}\theta.\sin^{2n-1}\theta}{\left(a^2\cos^2\theta+b^2\sin^2\theta\right)^{m+n}}dx=\frac{B(m,n)}{2.a^{2n}.b^{2m}}.$ **[M.U. 1997, 99, 02, 04]**

Solution: Dividing the numerator and denominator by $\cos^{2m+2n}\theta,$ we get

$$I=\int_0^{\pi/2}\frac{\cos^{2m-1}\theta}{\cos^{2m}\theta}.\frac{\sin^{2n-1}\theta}{\cos^{2n}\theta}.\frac{1}{\left(a^2+b^2\tan^2\theta\right)^{m+n}}d\theta$$

$$=\int_0^{\pi/2}\frac{1}{\cos\theta}.\frac{\sin^{2n-1}\theta}{\cos^{2n-1}\theta}.\frac{1}{\cos\theta}.\frac{1}{\left(a^2+b^2\tan^2\theta\right)^{m+n}}d\theta$$

$$=\int_0^{\pi/2}\frac{\tan^{2n-1}\theta}{\left(a^2+b^2\tan^2\theta\right)^{m+n}}.\sec^2\theta d\theta$$

$$\therefore \qquad I=\int_0^{\pi/2}\frac{(\tan\theta)^{2m-2}.\tan\theta\sec^2\theta}{\left(a^2+b^2\tan^2\theta\right)^{m+n}}.d\theta$$

Now put $b^2 \tan^2 \theta = a^2 y,$

$\therefore \quad b^2 .2\tan\theta \sec^2 \theta d\theta = a^2 dy$

When $\theta = 0, y = 0;$ *when* $\theta = \frac{\pi}{2}, y = \infty$

$$\therefore \quad I = \int_0^\infty \left(\frac{a^2 y}{b^2}\right)^{m-1} \cdot \frac{1}{\left(a^2 + a^2 y\right)^{m+n}} \cdot \frac{a^2}{2b^2} dy$$

$$= \frac{1}{2.a^{2n}.b^{2m}} \cdot \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}} dy$$

$$= \frac{1}{2.a^{2n}.b^{2m}} .B(m,n)$$

Ex.46 Prove that $\int_0^\infty \frac{dx}{\left(e^x + e^{-x}\right)^n} = \frac{1}{4} B\left(\frac{n}{2}, \frac{n}{2}\right)$ and hence

Evaluate $\int_0^\infty \operatorname{sec} h^8 x dx.$ **[M.U. 1998, 02, 03, 06, 07, 11]**

Solution: **An alnernate method is used below. The method described in class is much shorter and must be used.**

We have $I = \int_0^\infty \frac{dx}{\left(e^x + e^{-x}\right)^n} = \frac{1}{2} \int_{-\infty}^\infty \frac{dx}{\left(e^x + e^{-x}\right)^n}$

Put $e^x = \tan\theta \qquad \therefore e^x dx = \sec^2 \theta d\theta \qquad \therefore dx = \frac{\sec^2 \theta d\theta}{\tan\theta}$

When $x = \infty, e^x = \infty, \tan\theta = \infty \qquad \therefore \theta = \frac{\pi}{2}$

When $x = -\infty, e^x = 0, \tan\theta = 0 \qquad \therefore \theta = 0$

$$\therefore \quad I = \frac{1}{2} \int_0^{\pi/2} \frac{1}{(\tan\theta + \cot\theta)^n} \cdot \frac{\sec^2 \theta}{\tan\theta} .d\theta$$

$$\therefore \quad I = \frac{1}{2} \int_0^{\pi/2} \frac{1}{\left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)^n} \cdot \frac{1}{\cos^2 \theta} \cdot \frac{\cos\theta}{\sin\theta} .d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} \frac{\sin^n \theta \cos^n \theta}{\sin\theta \cos\theta} .d\theta = \frac{1}{2} \int_0^{\pi/2} \sin^{n-1} \theta \cos^{n-1} \theta .d\theta$$

$$= \frac{1}{2} \cdot \frac{1}{2} .B\left(\frac{n-1+1}{2} \cdot \frac{n-1+1}{2}\right) = \frac{1}{4} B\left(\frac{n}{2}, \frac{n}{2}\right)$$

Since, $\frac{e^x + e^{-x}}{2} = \cosh x, e^x + e^{-x} = 2\cosh x$

Putting n = 8 in the integral,

$$\therefore \quad \int_0^\infty \frac{dx}{\left(e^x+e^{-x}\right)^8} = \int_0^\infty \frac{dx}{2^8 \cosh^8 x} = \frac{1}{4} B(4,4)$$

$$\therefore \quad \int_0^\infty \operatorname{sec}h^8 x dex = \frac{2^8}{4} \cdot \frac{\Gamma 4 \Gamma 4}{\Gamma 8} = 2^6 \cdot \frac{3!.3!}{7!} = \frac{16}{35}.$$

Ex.47 Prove that $\int_0^\infty \frac{dx}{x^{p+1}(x-1)^q} = B(p+q, 1-q), -p<q<1$ **[M.U. 1997, 99]**

Solution: Let x – 1 = t $\therefore$ dx = dt

When x = 1, t = 0; when $x=\infty, t=\infty$

$$\therefore \quad I = \int_0^\infty \frac{dt}{(1+t)^{p+1}.t^q} = \int_0^\infty \frac{t^{-q}}{(1+t)^{p+1}} dt$$

Comparing this with

$$\int_0^\infty \frac{x^m}{(1+x)^n} dx = B(m+1, n-m-1)$$

We get, $$I = \int_0^\infty \frac{t^{-q}}{(1+t)^{p+1}} dt = B(-q+t, p+1+q-1)$$

$$= B(p+q, 1-q).$$

Alternatively: Putting $x=\frac{1}{t}, dx = -\frac{1}{t^2} dt,$ we get,

$$I = \int_0^0 \frac{1}{\frac{1}{t^{p+1}}\left(\frac{1}{t}-1\right)^q} \cdot \left(-\frac{1}{t^2}\right) dt = \int_0^1 \frac{t^{p+q-1}}{(1-t)^q} dt$$

$$= \int_0^1 t^{p+q-1}.(1-t)^{-q}\, dt = B(p+q, 1-q).$$

Ex.48 Prove that $B\left(n+\frac{1}{2}, n+\frac{1}{2}\right) = \frac{1}{2^{2n}} \cdot \frac{\Gamma\left(n+(1/2)\right)}{\Gamma(n+1)} . \sqrt{\pi}$

Hence, deduce that $2^n \Gamma\left(n+(1/2)\right) = 1.3.5.....(2n-1)\sqrt{\pi}$ **[M.U. 2002, 07]**

Solution: We have, by definition,

$$B\left(n+\frac{1}{2}, n+\frac{1}{2}\right) = \int_0^1 x^{n-(1/2)}.(1-x)^{n-(1/2)}\, dx$$

Putting $x = \sin^2\theta, dx = 2\sin\theta\cos\theta\, d\theta$

$$B\left(n+\frac{1}{2}, n+\frac{1}{2}\right) = \int_0^{\pi/2} \sin^{2n-1}\theta \cos^{2n-1}\theta . 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2n}\theta \cos^{2n}\theta\, d\theta$$

$$B\left(n+\frac{1}{2}.n+\frac{1}{2}\right)=2\int_0^{\pi/2}\left(\frac{\sin 2\theta}{2}\right)^{2n} d\theta$$

$$=\frac{1}{2^{2n-1}}\int_0^{\pi/2}(\sin 2\theta)^{2n}\, d\theta$$

$$=\frac{1}{2^{2n}}\int_0^{\pi}\sin^{2n}\phi d\phi \; where\; \phi=2\theta$$

$$=\frac{2}{2^{2n}}\int_0^{\pi/2}\sin^{2n}\phi\, d\phi$$

$$\left[\because \int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx \; if \; f(2a-x)=f(x)\right]$$

$$=\frac{1}{2^{2n}}.2.\frac{1}{2}\frac{\Gamma\left(\frac{2n+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{2n+2}{2}\right)}=\frac{1}{2^{2n}}.\frac{\Gamma\left(n+\frac{1}{2}\right).\sqrt{\pi}}{\Gamma(n+1)}$$

But $\quad B\left(n+\frac{1}{2},n+\frac{1}{2}\right)=\frac{\left[\Gamma(n+(1/2))\right]^2}{\Gamma(2n+1)}$

Equating the two results, we get,

$$\frac{\left[\Gamma(n+(1/2))\right]^2}{\Gamma(2n+1)}=\frac{1}{2^{2n}}.\frac{\Gamma(n+(1/2)).\sqrt{\pi}}{\Gamma(n+1)}$$

$$\therefore \quad 2^{2n}\Gamma(n+(1/2))=\frac{\Gamma(2n+1)}{\Gamma(n+1)}.\sqrt{\pi}=\frac{2n(2n-1)....3.2.1\sqrt{\pi}}{n(n-1)(n-2).....3.2.1}$$

$$=\frac{2n(2n-1).2(n-1)(2n-3).....3.2.1\sqrt{\pi}}{n(n-1)(n-2).....3.2.1}$$

$$=2^n.1.3.5.....(2n-1)\sqrt{\pi}$$

$$\therefore \quad 2^n\Gamma(n+(1/2))=1.3.5.....(2n-1)\sqrt{\pi}\,.$$

Ex.49 Show that $\int_0^a \frac{dx}{\left(a^n-x^n\right)^{1/n}}=\frac{\pi}{n}\operatorname{cosec}\left(\frac{\pi}{n}\right).$ **[M.U. 2003, 08, 09]**

Solution: Putting $x^n=a^n\sin^2\theta$ (take $x^n/a^n = t$ also works here)

i.e. $\quad x=a\sin^{2/n}\theta, dx=\frac{2a}{n}\sin^{(2/n)-1}\theta.\cos\theta.d\theta$

[or put $x^n=a^n t$ i.e. $x=at^{1/n}$

$\therefore I=\int_0^1\frac{1}{a(1-t)^{1/n}}.\frac{a}{n}.t^{(1/n)-1}dt.$ Now put $t=\sin^2\theta$]

$$I=\int_0^{\pi/2}\frac{2a}{n}.\frac{1}{a\cos^{2/n}\theta}.\sin^{(2/n)-1}\theta.\cos\theta.d\theta$$

$$= \frac{2}{n}\int_0^{\pi/2} \sin^{(2/n)-1}\theta.\cos^{1-(2/n)}\theta.d\theta$$

$$= \frac{2}{n}.\frac{1}{2}.\frac{\Gamma\left(\frac{(2/n)-1+1}{2}\right)\Gamma\left(\frac{1-(2/n)+1}{2}\right)}{\Gamma\left(\frac{(2/n)-(2/n)+2}{2}\right)}$$

$$= \frac{1}{n}.\frac{\Gamma(1/n)\Gamma(1-(1/n))}{\Gamma(1)} = \frac{1}{n}.\frac{\pi}{\sin(\pi/n)}$$

$$= \frac{\pi}{n}\operatorname{cosec}\left(\frac{\pi}{n}\right).$$

Ex.50 Show that $\int_0^{\pi/2} \tan^n x dx = \frac{\pi}{2}\sec\left(\frac{\pi n}{2}\right)$ **[M.U. 1996, 97]**

Deduce that $\int_0^{\pi/2} \cot^n x dx = \frac{\pi}{2}\sec\left(\frac{\pi n}{2}\right)$.

Solution: $I = \int_0^{\pi/2} \sin^n x \cos^{-n} x dx$

$$= \frac{1}{2}B\left(\frac{n+1}{2}.\frac{-n+1}{2}\right)$$

$$= \frac{1}{2}.\frac{\Gamma((n+1)/2)\Gamma((-n+1)/2)}{\Gamma(1)}$$

$$= \frac{1}{2}\Gamma(p)\Gamma(1-p) = \frac{1}{2}.\frac{\pi}{\sin p\pi} \quad where\ p = \frac{n+1}{2}$$

$$= \frac{1}{2}.\frac{\pi}{\sin\left(\frac{n+1}{2}.\pi\right)} = \frac{\pi}{2}\sec\left(\frac{n\pi}{2}\right)$$

Cor. $\int_0^{\pi/2} \cot^n x dx = \int_0^{\pi/2} \cot^n\left(\frac{\pi}{2} - x\right)dx$

$$= \int_0^{\pi/2} \tan^n x dx = \frac{\pi}{2}\sec\left(\frac{\pi n}{2}\right)$$

Ex.51 Prove that $\int_0^{\pi} \frac{\sin^{n-1} x}{(a + b\cos x)^n} dx = \frac{2^{n-1}}{\left(a^2 - b^2\right)^{n/2}} B\left(\frac{n}{2},\frac{n}{2}\right)$. **[M.U. 2007]**

Solution: Put $t = \tan\frac{x}{2}, \sin x = \frac{2t}{\left(1+t^2\right)}, \cos x = \frac{\left(1-t^2\right)}{\left(1+t^2\right)}, dx = \frac{2dt}{\left(1+t^2\right)}$.

$$\therefore \quad I=\int_0^\infty \frac{\left[2t/\left(1+t^2\right)\right]^{n-1}}{\left[a+b.\dfrac{\left(1-t^2\right)}{\left(1+t^2\right)}\right]^n}\cdot\frac{2dt}{\left(1+t^2\right)}$$

$$=2^n\int_0^\infty \frac{t^{n-1}}{\left[(a+b)+(a-b)t^2\right]^n}dt$$

Put $\quad t=\sqrt{\dfrac{a+b}{a-b}}.\tan\theta \qquad \therefore dt=\sqrt{\dfrac{a+b}{a-b}}.\sec^2\theta\, d\theta$

$$\therefore \quad I=2^n\int_0^{\pi/2}\frac{\left[\sqrt{(a+b)/(a-b)}\right]^{n-1}\tan^{n-1}\theta}{(a+b)^n.\left(\sec^2\theta\right)^n}.\sqrt{\frac{a+b}{a-b}}\sec^2\theta\, d\theta$$

$$=\frac{2^n}{\left(a^2-b^2\right)^{n/2}}\int_0^{\pi/2}\sin^{n-1}\theta.\cos^{n-1}\theta\, d\theta$$

$$=\frac{2^n}{\left(a^2-b^2\right)^{n/2}}\cdot\frac{1}{2}.B\left(\frac{n}{2},\frac{n}{2}\right)=\frac{2^{n-1}}{\left(a^2-b^2\right)^{n/2}}B\left(\frac{n}{2},\frac{n}{2}\right)$$

Ex.52 Prove that $\displaystyle\int_0^\pi \frac{\sqrt{\sin x}}{(5+3\cos x)^{3/2}}dx=\frac{\left(\left|\overline{3/4}\right.\right)^2}{2\sqrt{2\pi}}$ **[M.U. 1989, 2001, 09]**

Solution: In the above Ex. 51, Put n = 3/2, a = 5, b = 3

or independently as above put $t=\tan\dfrac{x}{2},\sin x=\dfrac{2t}{\left(1+t^2\right)},\cos x=\dfrac{\left(1-t^2\right)}{\left(1+t^2\right)},dx=\dfrac{2dt}{\left(1+t^2\right)}$

$$I=\int_0^\infty \frac{\sqrt{2t/\left(1+t^2\right)}}{\left[5+3.\left(\dfrac{1-t^2}{1+t^2}\right)\right]^{3/2}}\cdot\frac{2dt}{\left(1+t^2\right)}=\int_0^\infty\frac{2\sqrt{2}.\sqrt{t}dt}{\left(8+2t^2\right)^{3/2}}$$

$$=\int_0^\infty\frac{\sqrt{t}}{\left(4+t^2\right)^{3/2}}dt$$

Putting $\quad t^2=4y,t=2\sqrt{y} \qquad \therefore dt=\dfrac{dy}{\sqrt{y}}$

$$\therefore \quad I=\frac{1}{8}\int_0^\infty\frac{\sqrt{2}.y^{1/4}}{(1+y)^{3/2}}\cdot\frac{dy}{\sqrt{y}}=\frac{1}{4\sqrt{2}}\int_0^\infty\frac{y^{-1/4}}{(1+y)^{3/2}}dy$$

$$=\frac{1}{4\sqrt{2}}B\left(\frac{3}{4},\frac{3}{4}\right) \qquad \left[\because\int_0^\infty\frac{x^{m-1}}{(1+x)^{m+n}}dx=B(m,n)\right]$$

$$=\frac{1}{4\sqrt{2}}\cdot\frac{\overline{|3/4}\,\overline{|3/4}}{\overline{|3/2}}=\frac{1}{4\sqrt{2}}\cdot\frac{\left(\overline{|3/4}\right)^2}{(1/2)\overline{|1/2}}$$

$$=\frac{\left(\overline{|3/4}\right)^2}{2\sqrt{2\pi}}$$

Ex.53 Given $\int_0^\infty \frac{x^{p-1}}{1+x}dx=\frac{\pi}{\sin p\pi}$, prove that

$$\int_0^1 \frac{x^{n-1}}{(1+cx)(1-x)^n}dx=\frac{1}{(1+c)^n}\cdot\frac{\pi}{\sin(n\pi)},0<p<1.$$ **[M.U. 1996]**

Solution: We put $\frac{x}{1+cx}=\frac{y}{1+c}\quad \therefore (1+c)x=y(1+cx)$

$\therefore \quad x(1+c-cy)=y \qquad \therefore x=\frac{y}{1+c-cy}$

$\therefore \quad x=\frac{(1+c-cy)1-y(-c)}{(1+c-cy)^2}dy=\frac{1+c}{(1+c-cy)^2}dy$

Further $\quad 1-x=1-\frac{y}{1+c-cy}=\frac{(1+c)(1-y)}{1+c-cy}$

And $\quad 1+cx=1+\frac{cy}{1+c-cy}=\frac{1+c}{1+c-cy}$

$$\therefore \quad I=\int_0^1 \frac{y^{n-1}}{(1+c-cy)^{n-1}}\cdot\frac{(1+c-cy)}{(1+c)}\cdot\frac{(1+c-cy)^n}{(1+c)^n(1-y)^n}\cdot\frac{1+c}{(1+c-cy)^2}dy$$

$$=\int_0^1 \frac{y^{n-1}}{(1+c)^n(1-y)^n}dy$$

To get the limits 0 to ∞, we put $\frac{y}{1-y}=t \;\therefore y=t-ty$

$\therefore \quad y(1+t)=t \qquad \therefore y=\frac{t}{1+t}$

$\therefore \quad dy=\frac{(1+t).1-t.1}{(1+t)^2}dt=\frac{1}{(1+t)^2}dt$

And $\quad 1-y=1-\frac{t}{1+t}=\frac{1}{1+t}$

$$\therefore \quad I=\int_0^\infty \frac{t^{n-1}}{(1+t)^{n-1}}\cdot\frac{1}{(1+c)^n}\cdot\frac{(1+t)^n}{1}\cdot\frac{dt}{(1+t)^2}$$

$$=\int_0^\infty \frac{t^{n-1}}{(1+c)^n.(1+t)}dt$$

$$= \frac{1}{(1+c)^n}\int_0^\infty \frac{t^{n-1}}{(1+t)}\,dt = \frac{1}{(1+c)^n}\cdot\frac{\pi}{\sin(n\pi)} \quad \text{[By data]}$$

EXERCISE

- Show that $\Gamma\left(\frac{3}{2}-n\right)\Gamma\left(\frac{3}{2}+n\right) = \left(\frac{1}{4}-n^2\right)\pi\sec n\pi, (-1<2n<1)$. **[M.U. 2007]**

  (Hint: l.h.s $= \left[(1/2)-n\right]\Gamma\left((1/2)-n\right).\left[(1/2)+n\right]\Gamma\left((1/2)+n\right)$

  $= \left[(1/4)-n^2\right]\Gamma\left(n+(1/2)\right)\Gamma\left(1-\left[n+(1/2)\right]\right)$

  $= \left(\frac{1}{4}-n^2\right)\frac{\pi}{\sin\left[n+(1/2)\pi\right]} = \left(\frac{1}{4}-n^2\right).\frac{\pi}{\cos n\pi}$)

- Prove that $\int_0^1 \frac{x^2}{\left(1-x^4\right)^{1/2}}dx.\int_0^\infty \frac{1}{\left(1+x^4\right)^{1/2}}dx = \frac{\pi}{4\sqrt{2}}$ **[M.U. 2005]**

- Prove that $\int_0^\infty \frac{e^{2mx}+e^{-2mx}}{\left(e^x+e^{-x}\right)^{2n}}dx = \frac{1}{2}B(m+n, n-m)$. **[M.U. 1990, 02]**

  (Hint: Multiply the numerator and denominator by e^{2nx} and put $e^{2x}=t$. Then put $t=1/y$.)

- Prove that $\int_0^\infty \frac{x^2dx}{\left(1+x^4\right)^{3/2}}.\int_0^\infty \frac{dx}{\sqrt{1+x^4}} = \frac{\pi}{4}$. **[M.U. 1989]**

  (Hint: Put $x^4=t$.)

- Express $\int_{-1}^{1}(1+x)^m(1-x)^n\,dx$ as a Beta Function **[M.U. 1997]**

  (Hint: Put 1 + x = 2t.)

Ans. $2^{m+n+1}B(m+1, n+1)$

- Prove that $\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{2^{(2-n)/n}.\left(\Gamma(1/n)\right)^2}{n.\Gamma(2/n)}$ **[M.U. 1997]**

  (Hint: Put $x^n=t$ and then use duplication formula)

- Prove that $\int_0^\infty \sec h^6 x dx = \frac{8}{15}$. **[M.U. 1998]**

- Prove that $\int_0^2 x.\sqrt[3]{8-x^3}\,dx = \frac{16\pi}{9\sqrt{3}}$. **[M.U. 2005]**

- Prove that $B(n,n) = 2\int_0^{1/2}\left(t-t^2\right)^{n-1}dt$. **[M.U. 2002]**

(Hint: $B(n,n) = \int_0^{1/2} t^n (1-t)^{n-1}\, dt + \int_{1/2}^{1} t^n (1-t)^{n-1}\, dt.$ In I_2 put 1 – t = x.)

- Prove that $\int_0^1 \frac{\left(1-x^4\right)^{3/4}}{\left(1+x^4\right)^2}\, dx = \frac{3\pi}{2^{15/4}}$. **[M.U. 1999]**

- Prove that $\int_a^b (x-a)^m (b-x)^n\, dx = (b-a)^{m+n+1} B(m+1, n+1)$

  Hence deduce that

  $\int_0^n x^n (1-x)^p\, dx = n^{p+n+1} B(n+1, p+1)$ **[M.U. 2005]**