Portfolio Optimization

Modern Portfolio Theory

Daniel P. Palomar (2025). *Portfolio Optimization: Theory and Application.*
Cambridge University Press.

portfoliooptimizationbook.com

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Outline

Abstract

Modern portfolio theory (MPT) started with Harry Markowitz's 1952 seminal paper Portfolio Selection, for which he would later receive the Nobel prize in 1990. He put forth the idea that risk-adverse investors should optimize their portfolio based on a combination of two objectives: expected return and risk. Until today, that idea has remained central to portfolio optimization. In practice, however, the vanilla Markowitz portfolio formulation has some issues and drawbacks; as a consequence most practitioners tend to combine it with several heuristics or avoid it altogether. In these slides, we explore the mean–variance Markowitz portfolio in its many facets (Palomar 2025, chap. 7).

Outline

Mean–variance portfolio (MVP)

- **Mean–variance portfolio (MVP):**
  - Concept introduced by Markowitz in his 1952 paper (Markowitz 1952).
  - Further discussed in monographs by Rubinstein(Rubinstein 2002) and Kolm, Tutuncu, & Fabozzi (Kolm, Tütüncü, and Fabozzi 2014)] with a retrospective view.
  - Markowitz received the Nobel Prize for this work.

- **Importance of risk in portfolio management:**
  - Expected return $\boldsymbol{w}^\mathsf{T}\boldsymbol{\mu}$ measures the average benefit.
  - Risk is crucial to avoid bankruptcy.
  - Risk measure quantifies investment strategy risk.

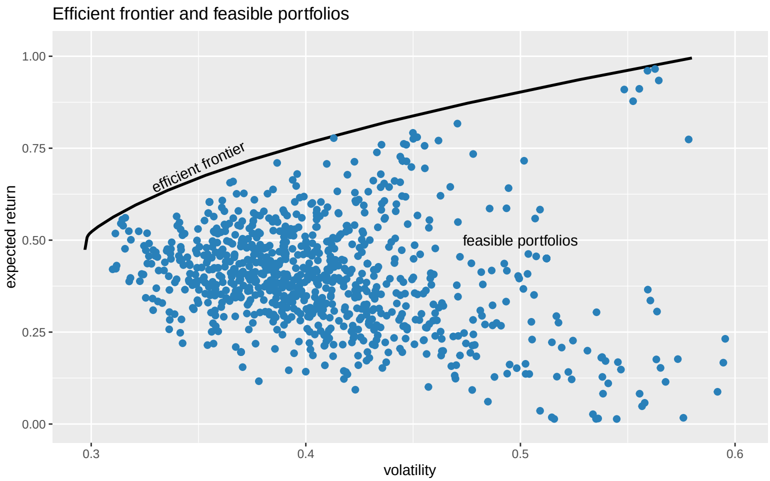
- **Basic risk measures:**
  - Volatility: $\sqrt{\boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w}}$
  - Variance: $\boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w}$
  - Higher variance indicates potential for larger losses.

Return–risk trade-off

- **Sophisticated risk measures:**
  - Downside risk measures (e.g., semivariance)
  - Value-at-Risk (VaR)
  - Conditional Value-at-Risk (CVaR)
- **Markowitz's risk-return optimization:**
  - Investors should consider both expected return and risk.
  - Trade-off: higher expected return comes with higher risk, and vice versa.
  - Multi-objective optimization problem.
- **Efficient frontier:**
  - Optimal trade-off curve of expected return and volatility.
  - Represents best possible expected return-volatility pairs for feasible portfolios.
  - Investor's choice on the curve depends on their risk appetite.
  - Figure shows the trade-off between expected return and volatility.

Return–volatility trade-off

Efficient frontier and 1,000 random feasible portfolios:



MVP formulation

- **Bi-objective optimization:**
  - Objectives: expected return $\boldsymbol{w}^\mathsf{T}\boldsymbol{\mu}$ and risk (volatility $\sqrt{\boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w}}$ or variance $\boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w}$).
  - Variance is computationally more efficient than volatility.
  - Variance involves quadratic programming; volatility involves second-order cone programming.

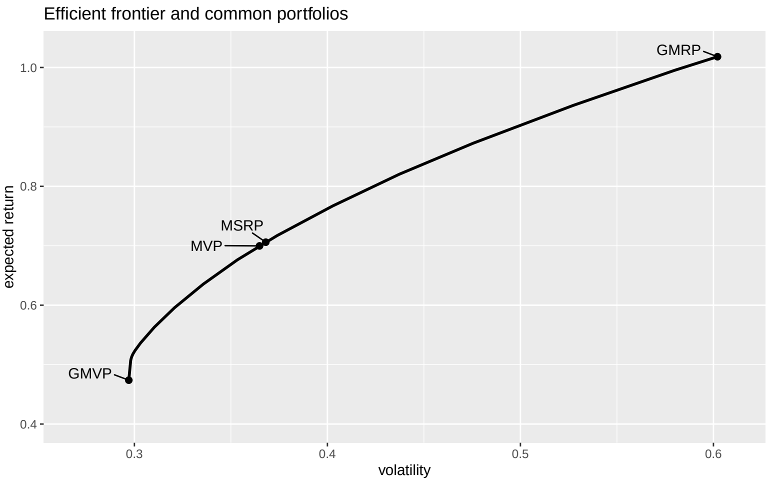
- **Scalarization approach:**
  - Combines objectives into a single weighted sum.
  - Optimization problem:

$$\begin{array}{ll} \underset{\boldsymbol{w}}{\text{maximize}} & \boldsymbol{w}^\mathsf{T}\boldsymbol{\mu} - \frac{\lambda}{2}\boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w} \\ \text{subject to} & \mathbf{1}^\mathsf{T}\boldsymbol{w} = 1, \quad \boldsymbol{w} \geq \mathbf{0}, \end{array}$$

  - λ is a risk-aversion hyper-parameter.
  - Constraints ensure full investment and non-negativity of asset weights.

MVP formulation

Efficient frontier and common portfolios:



MVP formulation: Impact of λ

- Varying λ produces portfolios along the efficient frontier.
  - $\lambda = 0$: Global Maximum Return Portfolio (GMRP), focusing solely on expected return.
  - $\lambda \to \infty$: Global Minimum Variance Portfolio (GMVP), focusing solely on minimizing variance.
- Portfolio allocation varies significantly with λ in the original MVP formulation.
- Backtest results show that smaller λ values may lead to worse Sharpe ratios and more severe drawdowns compared to larger values.

Quadratic problem (QP) solution for MVP

- **MVP as a QP:**
  - The MVP formulation is solvable using a QP solver.
  - The vanilla formulation ignoring no-shorting constraint $\boldsymbol{w} \geq \mathbf{0}$ allows for a closed-form solution.

- **Closed-form solution:**
  - Solution without shorting constraint:

$$\boldsymbol{w} = \frac{1}{\lambda}\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\mu} + \nu\mathbf{1}\right),$$

  - Optimal dual variable ν:

$$\nu = \frac{\lambda - \mathbf{1}^\mathsf{T}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}{\mathbf{1}^\mathsf{T}\boldsymbol{\Sigma}^{-1}\mathbf{1}},$$

  - ν ensures the normalization constraint $\mathbf{1}^\mathsf{T}\boldsymbol{w} = 1$ is satisfied.

Example: Optimum investment sizing

- **Investment sizing problem:**
  - Determine budget allocation between a risky asset and cash.
  - Optimal sizing derived from MVP for a single asset.

- **MVP formulation for single asset:**
  - Optimization problem for $N = 1$:

$$\begin{array}{ll} \underset{w}{\text{maximize}} & w\mu - \frac{\lambda}{2} w^2 \sigma^2 \\ \text{subject to} & 0 \leq w \leq 1, \end{array}$$

  - Solution for investment sizing:

$$w = \left[\frac{1}{\lambda} \frac{\mu}{\sigma^2} \right]_0^1,$$

    where $[\cdot]_0^1$ is a projection to ensures w is within the interval $[0, 1]$.

- **Growth rate maximization:**
  - Maximum growth rate when $\lambda = 1$.
  - Optimal sizing is the projected mean-to-variance ratio: $w = [\mu/\sigma^2]_0^1$.

Alternative MVP formulations

- Markowitz's mean-variance portfolio (MVP) can be reformulated in two other widely used ways, each offering a different approach to balancing the trade-off between risk and return.
- **Variance as a constraint:**

$$\begin{array}{ll} \underset{\boldsymbol{w}}{\text{maximize}} & \boldsymbol{w}^\mathsf{T}\boldsymbol{\mu} \\ \text{subject to} & \boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w} \leq \alpha \\ & \mathbf{1}^\mathsf{T}\boldsymbol{w} = 1, \quad \boldsymbol{w} \geq \mathbf{0}, \end{array}$$

  where the hyper-parameter α controls the maximum level of variance accepted.
- **Characteristics:**
  - Intuitive interpretation: maximum accepted variance.
  - Can recover the entire efficient frontier by adjusting α.
  - May be infeasible if α is not properly chosen; e.g., safe choice is: $\alpha = \frac{1}{N^2}\mathbf{1}^\mathsf{T}\boldsymbol{\Sigma}\mathbf{1}$ (based on variance of $1/N$ portfolio).
  - Quadratically-constrained QP (QCQP), requiring more complex solvers.

Alternative MVP formulations

- **Expected return as a constraint:**

$$\begin{array}{ll} \underset{\boldsymbol{w}}{\text{minimize}} & \boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w} \\ \text{subject to} & \boldsymbol{w}^\mathsf{T}\boldsymbol{\mu} \geq \beta \\ & \mathbf{1}^\mathsf{T}\boldsymbol{w} = 1, \quad \boldsymbol{w} \geq \mathbf{0}, \end{array}$$

  where the hyper-parameter β controls the minimum level of expected return accepted.

- **Characteristics:**
  - Intuitive interpretation: minimum accepted expected return.
  - Can recover the entire efficient frontier by adjusting β.
  - May be infeasible if β is not properly chosen; e.g., safe choice is: $\beta = \frac{1}{N}\mathbf{1}^\mathsf{T}\boldsymbol{\mu}$ (based on expected return of $1/N$ portfolio).
  - Still a QP, efficiently solvable with a QP solver.

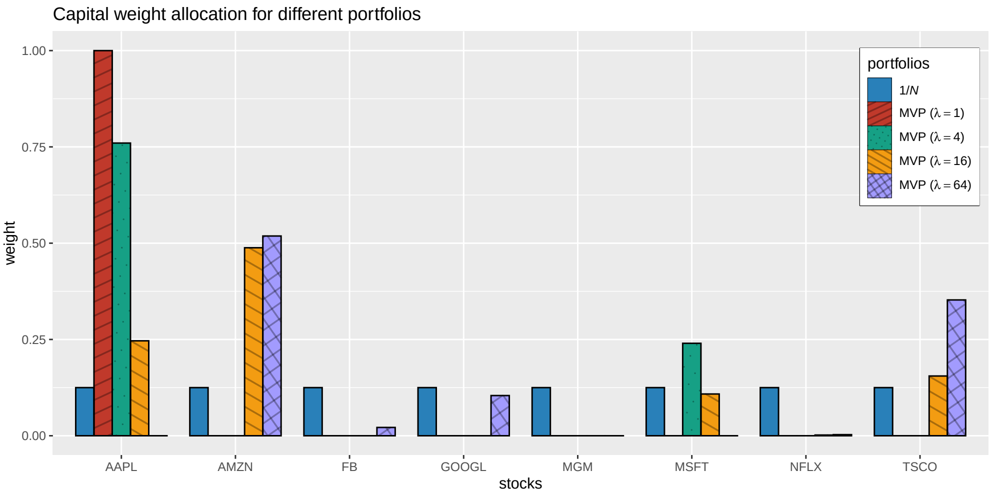
Portfolio optimization software

- **R package `fPortfolio`:** Offers a wide variety of portfolio optimization formulations and constraints.

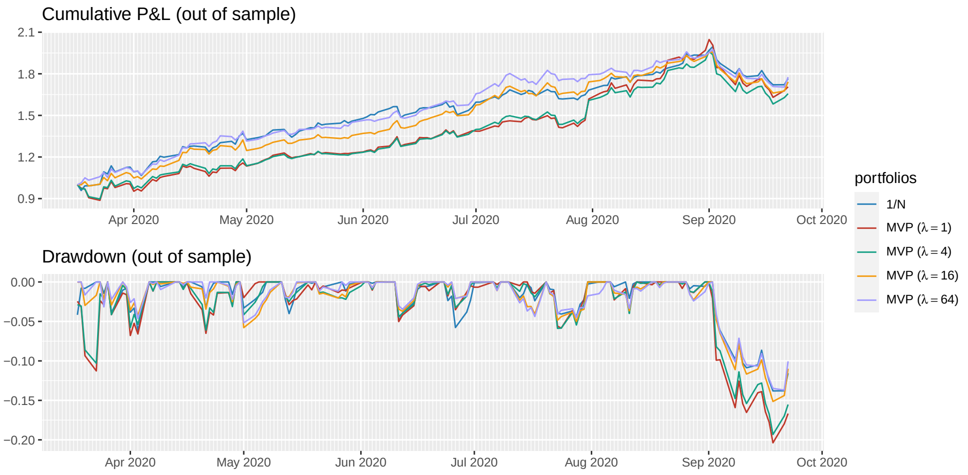
- **Python library `Riskfolio-Lib`:** Specializes in risk parity portfolio optimization with an extensive range of risk measures.

MVP: Numerical experiments

Portfolio allocation of MVP with different values of hyper-parameter λ:



Backtest performance of MVP with different values of hyper-parameter λ:



MVP: Numerical experiments

Backtest performance of MVP with different values of hyper-parameter λ:

Portfolio	Sharpe ratio	annual return	annual volatility	max drawdown
$1/N$	3.34	115%	35%	14%
MVP ($\lambda = 1$)	2.60	112%	43%	20%
MVP ($\lambda = 4$)	2.57	106%	41%	19%
MVP ($\lambda = 16$)	3.37	113%	33%	15%
MVP ($\lambda = 64$)	3.65	116%	32%	14%

MVP as a regression*

- The Mean-Variance Portfolio (MVP) formulation can intriguingly be viewed through the lens of regression analysis.
- This perspective hinges on interpreting the portfolio's variance as an ℓ_2-norm error term, offering a novel way to understand portfolio optimization.
- **Variance as ℓ_2-norm error:**

$$\begin{aligned} \boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w} &= \boldsymbol{w}^\mathsf{T}\mathbb{E}\left[(\boldsymbol{r}_t - \boldsymbol{\mu})(\boldsymbol{r}_t - \boldsymbol{\mu})^\mathsf{T}\right]\boldsymbol{w} \\ &= \mathbb{E}\left[\left(\boldsymbol{w}^\mathsf{T}(\boldsymbol{r}_t - \boldsymbol{\mu})\right)^2\right] \\ &= \mathbb{E}\left[\left(\boldsymbol{w}^\mathsf{T}\boldsymbol{r}_t - \rho\right)^2\right], \end{aligned}$$

where $\rho = \boldsymbol{w}^\mathsf{T}\boldsymbol{\mu}$.

MVP as a regression*

- **Sample approximation:**

$$\mathbb{E}\left[\left(\boldsymbol{w}^\mathsf{T}\boldsymbol{r}_t - \rho\right)^2\right] \approx \frac{1}{T}\sum_{t=1}^{T}\left(\boldsymbol{w}^\mathsf{T}\boldsymbol{r}_t - \rho\right)^2 = \frac{1}{T}\left\|\boldsymbol{R}\boldsymbol{w} - \rho\boldsymbol{1}\right\|_2^2$$

where $\boldsymbol{R} = [\boldsymbol{r}_1, \ldots, \boldsymbol{r}_T]^\mathsf{T}$.

- **Reformulated optimization:**

$$\begin{array}{ll} \underset{\boldsymbol{w}}{\text{minimize}} & \boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w} - \frac{2}{\lambda}\boldsymbol{w}^\mathsf{T}\boldsymbol{\mu} \\ \text{subject to} & \boldsymbol{1}^\mathsf{T}\boldsymbol{w} = 1, \quad \boldsymbol{w} \geq \boldsymbol{0} \end{array}$$

- **Substituting variance with ℓ_2-norm:**

$$\begin{array}{ll} \underset{\boldsymbol{w},\rho}{\text{minimize}} & \frac{1}{T}\left\|\boldsymbol{R}\boldsymbol{w} - \rho\boldsymbol{1}\right\|_2^2 - \frac{2}{\lambda}\rho \\ \text{subject to} & \rho = \boldsymbol{w}^\mathsf{T}\boldsymbol{\mu} \\ & \boldsymbol{1}^\mathsf{T}\boldsymbol{w} = 1, \quad \boldsymbol{w} \geq \boldsymbol{0}. \end{array}$$

MVP as a regression: Interpretation and insights*

- Recall the formulation:

$$\begin{array}{ll}\underset{\boldsymbol{w},\rho}{\text{minimize}} & \frac{1}{T}\|\boldsymbol{R}\boldsymbol{w} - \rho\mathbf{1}\|_2^2 - \frac{2}{\lambda}\rho\\ \text{subject to} & \rho = \boldsymbol{w}^\mathsf{T}\boldsymbol{\mu}\\ & \mathbf{1}^\mathsf{T}\boldsymbol{w} = 1, \quad \boldsymbol{w} \geq \mathbf{0}.\end{array}$$

- **Objective:** The portfolio $\boldsymbol{w}$ aims to achieve returns as constant as possible over time, ideally equal to ρ, thereby minimizing variance as quantified by the ℓ_2-norm.
- **Expected Return as a Variable:** The expected return, ρ, is an optimization variable, though it can also be fixed to a predetermined value.
- **Relation to Index Tracking:** This regression-based interpretation of MVP aligns with the concept of index tracking, where the goal is to closely follow a benchmark index's performance with minimal deviation.

MVP with practical constraints

- The Mean-Variance Portfolio (MVP) framework can be extended to incorporate a variety of practical constraints beyond the basic budget constraint $\mathbf{1}^\mathsf{T}\boldsymbol{w} = 1$ and no-shorting constraint $\boldsymbol{w} \geq \mathbf{0}$.

- These additional constraints reflect more realistic trading conditions and investor preferences, such as
  - managing risk exposure,
  - controlling transaction costs, and adhering to regulatory or self-imposed investment guidelines.

- This flexibility makes the MVP framework a powerful tool for constructing portfolios that are not only theoretically optimal but also practically feasible and aligned with investor preferences.

Extended MVP formulation with practical constraints

- **Formulation:**

$$
\begin{array}{lll}
\underset{\boldsymbol{w}}{\text{maximize}} & \boldsymbol{w}^{\mathsf{T}}\boldsymbol{\mu} - \frac{\lambda}{2}\boldsymbol{w}^{\mathsf{T}}\boldsymbol{\Sigma}\boldsymbol{w} & \\
\text{subject to} & \|\boldsymbol{w}\|_1 \leq \gamma & \text{leverage} \\
 & \|\boldsymbol{w} - \boldsymbol{w}_0\|_1 \leq \tau & \text{turnover} \\
 & |\boldsymbol{w}| \leq \boldsymbol{u} & \text{max positions} \\
 & \boldsymbol{\beta}^{\mathsf{T}}\boldsymbol{w} = 0 & \text{market neutral} \\
 & \|\boldsymbol{w}\|_0 \leq K & \text{sparsity}
\end{array}
$$

- **Constraints explained:**
  - **Leverage:** $\|\boldsymbol{w}\|_1 \leq \gamma$ limits the total absolute weight, controlling shorting and leverage.
  - **Turnover:** $\|\boldsymbol{w} - \boldsymbol{w}_0\|_1 \leq \tau$ limits the total change in weights, reducing transaction costs.
  - **Max Positions:** $|\boldsymbol{w}| \leq \boldsymbol{u}$ sets upper bounds on individual asset weights.
  - **Market Neutral:** $\boldsymbol{\beta}^{\mathsf{T}}\boldsymbol{w} = 0$ ensures the portfolio's beta relative to a benchmark (e.g., a market index) is zero.
  - **Sparsity:** $\|\boldsymbol{w}\|_0 \leq K$ limits the number of non-zero weights, selecting a subset of assets.

Convexity and solvability

- **General formulation:**

$$\begin{array}{ll} \underset{\boldsymbol{w}}{\text{maximize}} & \boldsymbol{w}^\mathsf{T}\boldsymbol{\mu} - \frac{\lambda}{2}\boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w} \\ \text{subject to} & \boldsymbol{w} \in \mathcal{W}. \end{array}$$

  where $\mathcal{W}$ represents a general set of convex constraints.

- As long as the constraints in $\mathcal{W}$ are convex, the optimization problem remains convex and solvable.
- The cardinality constraint $\|\boldsymbol{w}\|_0 \leq K$ is nonconvex, making it a notable exception.

Improving the MVP with heuristics

- The mean-variance portfolio (MVP) can be enhanced with various heuristics to address its tendency towards insufficient diversification.
- These heuristics include no-shorting constraints, upper bound constraints, and ℓ_2-norm constraints, which have shown practical effectiveness.

- **No-shorting constraint:** $\boldsymbol{w} \geq \mathbf{0}$
  - Reduces noise amplification from the estimated covariance matrix.
  - Performs comparably to sophisticated covariance estimators.
- **Upper bound constraints:** $\mathbf{0} \leq \boldsymbol{w} \leq \boldsymbol{u}$
  - Acts as a regularizer for the covariance matrix.
- **Diversification constraint:** $\|\boldsymbol{w}\|_2^2 \leq D$
  - Maximum diversity level D is bounded by $1/N$ and can be set based on a benchmark portfolio.

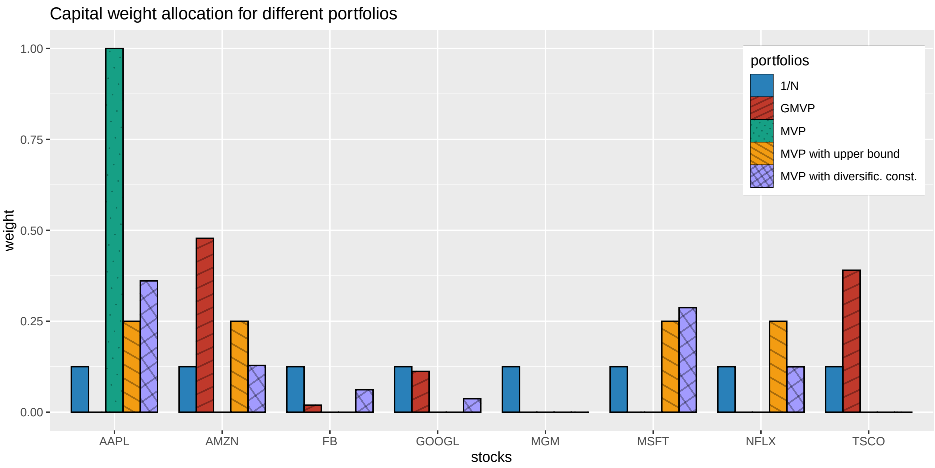
Improving the MVP with heuristics: Numerical experiments

- **Portfolio allocation:**
  - Diversification heuristics lead to improved portfolio diversification.
  - Example heuristics: upper bound $\|\boldsymbol{w}\|_\infty \leq 0.25$ and diversification constraint $\|\boldsymbol{w}\|_2^2 \leq 0.25$.
- **Backtest results:**
  - More diversified MVPs exhibit better Sharpe ratios and drawdowns.

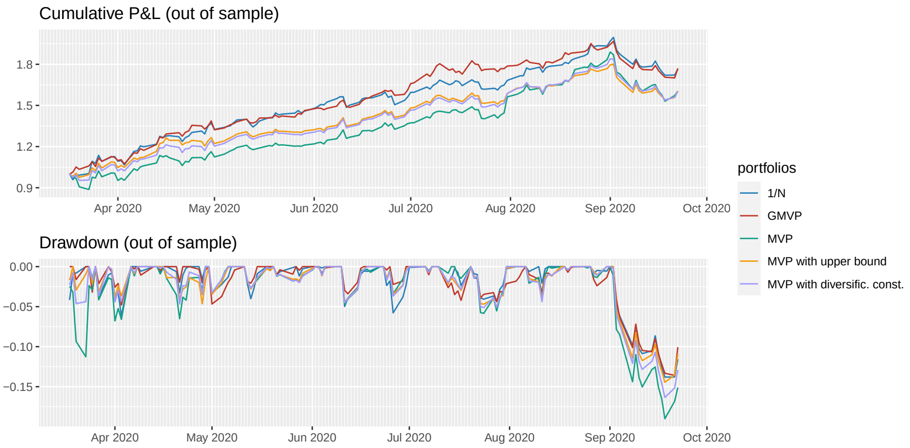
- Overall, these heuristics serve to mitigate one of the MVP's key limitations by promoting diversification, which is a fundamental principle in portfolio management.

Improving the MVP with heuristics: Numerical experiments

Portfolio allocation of MVP under diversification heuristics:



Backtest performance of MVP under diversification heuristics:



Improving the MVP with heuristics: Numerical experiments

Backtest performance of MVP under diversification heuristics:

Portfolio	Sharpe ratio	annual return	annual volatility	max drawdown
1/N	3.34	115%	35%	14%
GMVP	3.67	115%	31%	14%
MVP	2.44	99%	41%	19%
MVP with upper bound	2.98	96%	32%	14%
MVP with diversific. const.	2.79	97%	35%	16%

Outline

1. Mean–variance portfolio (MVP)
2. Maximum Sharpe ratio portfolio (MSRP)
3. Utility-based portfolios
   - Kelly criterion portfolio
   - Expected utility theory
4. Universal algorithm*
5. Drawbacks
6. Summary

Maximum Sharpe ratio portfolio (MSRP)

- The Sharpe ratio of a portfolio is the excess return per unit of risk.
- The Maximum Sharpe Ratio Portfolio (MSRP) is a key concept within Markowitz's mean-variance framework, aiming to identify the portfolio on the efficient frontier that offers the highest Sharpe ratio.
- **Formulation:**

$$\begin{array}{ll} \underset{\boldsymbol{w}}{\text{maximize}} & \dfrac{\boldsymbol{w}^\mathsf{T}\boldsymbol{\mu} - r_\mathrm{f}}{\sqrt{\boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w}}} \\ \text{subject to} & \mathbf{1}^\mathsf{T}\boldsymbol{w} = 1, \quad \boldsymbol{w} \geq \mathbf{0}, \end{array}$$

  where r_f is the return of the risk-free asset.
- **Problem type:** Non-convex, but a fractional program (FP) solvable by various methods (Palomar 2025, chap. 7).

Solving the MSRP: Bisection method

- **Approach:** Solve a sequence of convex feasibility problems.
- **Convex feasibility problem for MSRP:**

$$\begin{array}{ll}\underset{\boldsymbol{w}}{\text{find}} & \boldsymbol{w}\\ \text{subject to} & t\sqrt{\boldsymbol{w}^{\mathsf{T}}\boldsymbol{\Sigma}\boldsymbol{w}} \leq \boldsymbol{w}^{\mathsf{T}}\boldsymbol{\mu} - r_{\mathrm{f}}\\ & \mathbf{1}^{\mathsf{T}}\boldsymbol{w} = 1, \quad \boldsymbol{w} \geq \mathbf{0},\end{array}$$

  where t is a fixed parameter, not an optimization variable!
- **Type of problem:** Second-order cone program (SOCP).
- **Note:** Feasibility or infeasibility depends on the value of the parameter t.

Solving the MSRP: Bisection method

The bisection method provides a systematic approach to finding the MSRP by iteratively narrowing down the range within which the maximum Sharpe ratio lies.

Bisection method

Initialization:

- Choose an interval $[l, u]$ containing the optimal Sharpe ratio and a tolerance $\epsilon > 0$.

Repeat:

1. Set $t \leftarrow (l + u)/2$.
2. Solve the convex feasibility problem.
3. If feasible, set $l \leftarrow t$ and keep solution $\boldsymbol{w}$; else set $u \leftarrow t$.

Until: $u - l \leq \epsilon$.

Solving the MSRP: Dinkelbach method

- The Dinkelbach method offers an efficient approach to solving non-convex concave-convex fractional programs (FPs), such as the Maximum Sharpe Ratio Portfolio (MSRP).

- This method iteratively solves a series of manageable convex optimization problems, specifically second-order cone programs (SOCPs), to find the optimal solution.

- **Convex problem sequence:** At each iteration k:

$$\begin{array}{ll}\underset{\boldsymbol{w}}{\text{maximize}} & \boldsymbol{w}^\mathsf{T}\boldsymbol{\mu} - r_\mathrm{f} - y^k\sqrt{\boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w}}\\ \text{subject to} & \mathbf{1}^\mathsf{T}\boldsymbol{w} = 1, \quad \boldsymbol{w} \geq \mathbf{0},\end{array}$$

- **Parameter update:**

$$y^k = \frac{(\boldsymbol{w}^k)^\mathsf{T}\boldsymbol{\mu} - r_\mathrm{f}}{\sqrt{(\boldsymbol{w}^k)^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w}^k}}$$

Solving the MSRP: Dinkelbach method

Dinkelbach method

Initialization:

- Choose an initial portfolio $\boldsymbol{w}^0$.
- Set iteration counter $k \leftarrow 0$.

Repeat (kth iteration):

1. Update y^k.
2. Solve the convex SOCP to obtain the solution $\boldsymbol{w}^{k+1}$.
3. Increment the iteration counter: $k \leftarrow k + 1$.

Until: The solution converges to the optimal portfolio.

Solving the MSRP: Schaible transform method

- The Schaible transform method provides an efficient way to solve concave-convex fractional programs (FPs) without resorting to iterative schemes.
- This method is particularly useful for the Maximum Sharpe Ratio Portfolio (MSRP) problem, allowing for a direct approach to optimization.
- **Transformed MSRP problem:**

$$\begin{array}{ll}\underset{\mathbf{y}}{\text{maximize}} & \mathbf{y}^\mathsf{T}\left(\boldsymbol{\mu} - r_\mathrm{f}\mathbf{1}\right)\\ \text{subject to} & \mathbf{y}^\mathsf{T}\boldsymbol{\Sigma}\mathbf{y} \leq 1\\ & \mathbf{1}^\mathsf{T}\mathbf{y} > 0, \quad \mathbf{y} \geq \mathbf{0},\end{array}$$

- **Recovery of original variables:** $\mathbf{w} = \mathbf{y}/\left(\mathbf{1}^\mathsf{T}\mathbf{y}\right)$.
- **Note:** The constraint $\mathbf{1}^\mathsf{T}\mathbf{y} > 0$ can often be ignored with interior-point methods.
- **Problem type:** A convex quadratically-constrained quadratic program (QCQP).

Solving the MSRP: Schaible transform method*

- **Alternative Schaible Transform Minimization Form:**

$$\begin{array}{ll} \underset{\boldsymbol{y}}{\text{minimize}} & \boldsymbol{y}^{\mathsf{T}} \boldsymbol{\Sigma} \boldsymbol{y} \\ \text{subject to} & \boldsymbol{y}^{\mathsf{T}} \left(\boldsymbol{\mu} - r_{\mathrm{f}} \mathbf{1} \right) \geq 1 \\ & \mathbf{1}^{\mathsf{T}} \boldsymbol{y} > 0, \quad \boldsymbol{y} \geq \mathbf{0}, \end{array}$$

- **Feasibility:** Requires $\boldsymbol{y}^{\mathsf{T}} \left(\boldsymbol{\mu} - r_{\mathrm{f}} \mathbf{1} \right) > 0$.

- **Problem Type:** A simpler quadratic program (QP), preferred for its efficiency.

Example: MSRP with return and upper bound constraints*

- **Original formulation:**

$$\begin{array}{ll}\underset{\mathbf{w}}{\text{maximize}} & \dfrac{\mathbf{w}^{\mathsf{T}}\boldsymbol{\mu}}{\sqrt{\mathbf{w}^{\mathsf{T}}\boldsymbol{\Sigma}\mathbf{w}}}\\ \text{subject to} & \mathbf{w}^{\mathsf{T}}\boldsymbol{\mu}\geq\beta, \quad \mathbf{w}\leq\mathbf{u}\\ & \mathbf{1}^{\mathsf{T}}\mathbf{w}=1, \quad \mathbf{w}\geq\mathbf{0}\end{array}$$

- **Reformulated problem as a QP:**

$$\begin{array}{ll}\underset{\mathbf{y}}{\text{minimize}} & \mathbf{y}^{\mathsf{T}}\boldsymbol{\Sigma}\mathbf{y}\\ \text{subject to} & \mathbf{y}^{\mathsf{T}}\boldsymbol{\mu}\geq 1\\ & 0<\mathbf{1}^{\mathsf{T}}\mathbf{y}\leq\beta^{-1}, \quad \mathbf{0}\leq\mathbf{y}\leq\mathbf{u}\cdot\left(\mathbf{1}^{\mathsf{T}}\mathbf{y}\right)\end{array}$$

Example: MSRP with shorting and return constraint*

- **Original formulation:**

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{maximize}} & \dfrac{\mathbf{w}^T \boldsymbol{\mu}}{\sqrt{\mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w}}} \\ \text{subject to} & \mathbf{w}^T \boldsymbol{\mu} \geq \beta \\ & \|\mathbf{w}\|_1 = 1. \end{array}$$

- **Reformulated problem as a QP:**

$$\begin{array}{ll} \underset{\mathbf{y}}{\text{minimize}} & \mathbf{y}^T \boldsymbol{\Sigma} \mathbf{y} \\ \text{subject to} & \mathbf{y}^T \boldsymbol{\mu} \geq 1 \\ & 0 < \|\mathbf{y}\|_1 \leq \beta^{-1}, \end{array}$$

Outline

Utility-Based Portfolios

- Markowitz's mean–variance portfolio is based on the mean and the variance trade-off.
- Utility-based portfolios offer a more general approach to expressing investor preferences through utility functions, which can encompass a variety of risk and return considerations beyond mean and variance.
- We explore the specific Kelly criterion portfolio and the general expected utility theory (Palomar 2025, chap. 7).

Outline

Kelly criterion portfolio

- **Origin:** Introduced by John Larry Kelly, Jr. in 1956, combining game theory and information theory.
- **Concept:** Maximizes the expected logarithmic growth of wealth.
- **Application:** Applied to portfolio design by Markowitz and popularized in gambling and investment strategies.
- **Portfolio returns:** $R_t^{\text{portf}} = \boldsymbol{w}^\mathsf{T}\boldsymbol{r}_t$
- **Wealth accumulation formula:** $W_T = W_0 \prod_{t=1}^{T}\left(1 + \boldsymbol{w}^\mathsf{T}\boldsymbol{r}_t\right)$
- **Exponential growth:** $W_t \sim e^{t \times G}$
- **Growth rate:** $G = \lim_{T\to\infty} \log\left(\frac{W_T}{W_0}\right)^{1/T}$
- **Estimation:** $G = \lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} \log\left(1 + \boldsymbol{w}^\mathsf{T}\boldsymbol{r}_t\right) = \mathbb{E}\left[\log\left(1 + \boldsymbol{w}^\mathsf{T}\boldsymbol{r}\right)\right],$

Kelly criterion portfolio formulation

- **Objective:** Maximize the growth rate for long-term wealth.
- **Formulation:**

$$\begin{array}{ll} \underset{\boldsymbol{w}}{\text{maximize}} & \mathbb{E}\left[\log\left(1+\boldsymbol{w}^\mathsf{T}\boldsymbol{r}\right)\right] \\ \text{subject to} & \mathbf{1}^\mathsf{T}\boldsymbol{w}=1, \quad \boldsymbol{w}\geq\mathbf{0}. \end{array}$$

- **Convexity:** The problem is convex due to the concavity of the logarithm function.
- **Resolution methods:**
  - via sample average
  - via exponential cone programming
  - via mean–variance approximations

Kelly criterion portfolio via sample average

- **Sample average approximation:** Replace the expectation with a sample mean:

$$\mathbb{E}\left[\log\left(1+\boldsymbol{w}^\mathsf{T}\boldsymbol{r}\right)\right] \approx \frac{1}{T}\sum_{t=1}^{T}\log\left(1+\boldsymbol{w}^\mathsf{T}\boldsymbol{r}_t\right).$$

- **Practical challenges:** Solvers that can handle the logarithm function directly may be limited.

Kelly criterion portfolio via mean–variance approximations

- The Kelly criterion portfolio, which aims to maximize the expected logarithmic growth of wealth, can be approached through mean-variance approximations.
- **First-order Taylor approximation:**

$$\mathbb{E}\left[\log\left(1+\boldsymbol{w}^\mathsf{T}\boldsymbol{r}\right)\right] \approx \boldsymbol{w}^\mathsf{T}\boldsymbol{\mu} - \frac{1}{2}\boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w},$$

- **Justification for mean-variance formulation:** This approximation aligns with Markowitz's mean-variance portfolio formulation, effectively justifying its use with $\lambda = 1$.

Kelly criterion portfolio via mean–variance approximations*

- **Better Higher-Order Approximations:** Further refinements can be made by approximating around the point $\boldsymbol{r} = \boldsymbol{\mu}$:

$$\mathbb{E}\left[\log\left(1 + \boldsymbol{w}^\mathsf{T}\boldsymbol{r}\right)\right] \approx \log\left(1 + \boldsymbol{w}^\mathsf{T}\boldsymbol{\mu}\right) - \frac{1}{2}\frac{\boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w}}{(1 + \boldsymbol{w}^\mathsf{T}\boldsymbol{\mu})^2}$$

or

$$\mathbb{E}\left[\log\left(1 + \boldsymbol{w}^\mathsf{T}\boldsymbol{r}\right)\right] \approx \boldsymbol{w}^\mathsf{T}\boldsymbol{\mu} - \frac{1}{2}(\boldsymbol{w}^\mathsf{T}\boldsymbol{\mu})^2 - \frac{1}{2}\frac{\boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w}}{1 + 2\boldsymbol{w}^\mathsf{T}\boldsymbol{\mu}}$$

- **Levy-Markowitz Interval Approximation:**

$$\mathbb{E}\left[\log\left(1 + \boldsymbol{w}^\mathsf{T}\boldsymbol{r}\right)\right] \approx \frac{1}{2\kappa^2}\log\left(\left(1 + \boldsymbol{w}^\mathsf{T}\boldsymbol{\mu}\right)^2 - \kappa^2\boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w}\right) + \left(1 - \frac{1}{\kappa^2}\right)\log\left(1 + \boldsymbol{w}^\mathsf{T}\boldsymbol{\mu}\right)$$

where κ measures the width of the approximating interval in standard deviations.

Kelly criterion portfolio via mean–variance approximations*

- **Nonconvexity and estimation errors:** While these approximations offer theoretical refinements, their practical benefits may be limited due to nonconvexity in the formulations and the magnitude of estimation errors in $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$.
- **Historical perspective:** Markowitz provides a historical overview of mean-variance approximations and their implications for portfolio design.
- **Beyond mean-variance:** Higher order moments, such as skewness and kurtosis, can be used to achieve better approximations for portfolio design, addressing the limitations of mean-variance approximations.

Outline

Expected utility theory

- Expected utility theory forms the cornerstone of rational decision-making in economics and statistics, providing a framework for modeling choices under uncertainty.
- It was axiomatized by von Neumann and Morgenstern, and further developed by Savage, to formalize how individuals make choices that maximize their satisfaction or "utility."
- In portfolio design, the objective is to maximize some expected utility of portfolio returns.

Expected utility portfolios

- **Formulation:**

$$\begin{array}{ll} \underset{\boldsymbol{w}}{\text{maximize}} & \mathbb{E}\left[U(\boldsymbol{w}^\mathsf{T}\boldsymbol{r})\right] \\ \text{subject to} & \mathbf{1}^\mathsf{T}\boldsymbol{w} = 1, \quad \boldsymbol{w} \geq \mathbf{0}. \end{array}$$

- **Utility function $U(\cdot)$:** Represents the investor's preferences over different outcomes.
- **Examples of utility functions:**
  - Logarithmic: $U(x) = \log(1 + x)$
  - Square Root: $U(x) = \sqrt{1 + x}$
  - Inverse: $U(x) = -\frac{1}{x}$
  - Power: $U(x) = -p\frac{1}{x^p}$ for $p > 0$
  - Exponential: $U(x) = 1 - \exp(-\lambda x)$, where $\lambda > 0$ is the risk aversion parameter

Expected utility portfolios

- **Practical considerations:** While expected utility theory offers a theoretically robust framework for portfolio optimization, its practical application can be challenging.
- The choice of utility function significantly influences the resulting portfolio, and the direct application of this theory may not always provide actionable advice for investors.
- **Convexity:** The optimization problem is convex if the utility function $U(\cdot)$ is concave.
- **Kelly criterion as a special case:** If the utility function is chosen as the expected logarithmic growth of wealth, then the formulation becomes the Kelly portfolio.
- **Resolution methods:**
  - via sample average
  - via mean–variance approximations

Expected utility portfolios via sample average

- **Sample average approximation:** The expectation in the objective can be approximated by the sample mean:

$$\mathbb{E}\left[U(\boldsymbol{w}^\mathsf{T}\boldsymbol{r})\right] \approx \frac{1}{T}\sum_{t=1}^{T} U\left(\boldsymbol{w}^\mathsf{T}\boldsymbol{r}_t\right).$$

- **Solver challenges:** Finding solvers capable of directly handling various utility functions may be difficult, even when these functions are concave.

Expected utility portfolios via mean–variance approximations*

- The expected utility maximization problem can be approached through mean-variance approximations, making it more tractable for practical applications.

- **Second-order Taylor approximation around $\boldsymbol{r} = \mathbf{0}$:**

$$\mathbb{E}\left[U(\boldsymbol{w}^\mathsf{T}\boldsymbol{r})\right] \approx U(0) + U'(0)\ \boldsymbol{w}^\mathsf{T}\boldsymbol{\mu} + \frac{1}{2}U''(0)(\boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w} + (\boldsymbol{w}^\mathsf{T}\boldsymbol{\mu})^2),$$

- **Second-order Taylor approximation around $\boldsymbol{r} = \boldsymbol{\mu}$:**

$$\mathbb{E}\left[U(\boldsymbol{w}^\mathsf{T}\boldsymbol{r})\right] \approx U(\boldsymbol{w}^\mathsf{T}\boldsymbol{\mu}) + \frac{1}{2}U''(\boldsymbol{w}^\mathsf{T}\boldsymbol{\mu})\boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w}.$$

Expected utility portfolios via mean–variance approximations*

- **Three-point interval Levy-Markowitz approximation:**

$$\mathbb{E}\left[U(\boldsymbol{w}^\mathsf{T}\boldsymbol{r})\right] \approx U(\boldsymbol{w}^\mathsf{T}\boldsymbol{\mu}) + \frac{U\left(\boldsymbol{w}^\mathsf{T}\boldsymbol{\mu} + \kappa\sqrt{\boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w}}\right) + U\left(\boldsymbol{w}^\mathsf{T}\boldsymbol{\mu} - \kappa\sqrt{\boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w}}\right) - 2U\left(\boldsymbol{w}^\mathsf{T}\boldsymbol{\mu}\right)}{2\kappa^2}.$$

  where κ measures the width of the approximating interval centered at the mean.
- **Empirical performance:** These mean-variance approximations have been shown to perform well in practice, with negligible differences among them for real data.
- **Simplification for optimization:** By reducing the expected utility problem to a mean-variance framework, these approximations facilitate the use of conventional optimization solvers and techniques.
- **Beyond mean-variance:** Further exploration of higher order moments, such as skewness and kurtosis, is discussed for refining these approximations and enhancing portfolio design.

Outline

Universal algorithm for portfolio optimization

- All previous portfolio formulations, including those based on the Kelly criterion and utility-based portfolios, can be expressed in terms of mean and variance.

- However, each formulation results in a different type of optimization problems, requiring a specific numerical method or solver:
  - **Scalarized MVP:** Requires a quadratic programming (QP) solver.
  - **Mean-constrained MVP:** Also needs a QP solver.
  - **Variance-constrained MVP:** Requires a quadratically constrained QP (QCQP) solver.
  - **MSRP:** An FP that can be solved via bisection of SOCPs, Dinkelbach sequence of SOCPs, or one-shot Schaible transformed QP.
  - **Kelly portfolio:** After approximation, it can be solved with a QP solver.
  - **Utility-based portfolios:** Can be solved with a QP solver after mean-variance approximation.

Universal algorithm for portfolio optimization

- Rather than solving each of these portfolios in a different way, what if we could simply solve the basic mean–variance formulation with a properly chosen value of the hyper-parameter λ?:

$$\begin{array}{ll} \underset{\boldsymbol{w}}{\text{maximize}} & \boldsymbol{w}^\mathsf{T}\boldsymbol{\mu} - \dfrac{\lambda}{2}\boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w} \\ \text{subject to} & \boldsymbol{w} \in \mathcal{W} \end{array}$$

- The challenge naturally lies in the determination of the appropriate value of the hyper-parameter λ.

- It turns out that it is possible to iteratively adjust the hyper-parameter λ_k, where k denotes iterations (Xiu, Wang, and Palomar 2023), (Palomar 2025, chap. 7).

Universal algorithm for portfolio optimization

- **Universal framework:**

$$\begin{array}{ll}\underset{\boldsymbol{w}}{\text{minimize}} & f\left(\boldsymbol{w}^\mathsf{T}\boldsymbol{\mu}, \boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w}\right)\\ \text{subject to} & g\left(\boldsymbol{w}^\mathsf{T}\boldsymbol{\mu}, \boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w}\right) \leq 0\\ & \boldsymbol{w} \in \mathcal{W}\end{array}$$

- The functions $f(x, y)$ and $g(x, y)$ consider a trade-off between mean x and variance y.
- The next table summarizes how the functions $f(x, y)$ and $g(x, y)$ particularize to various mean-variance formulations.

Universal algorithm for portfolio optimization

Portfolio formulations with the corresponding functions $f(x, y)$ and $g(x, y)$ in the general mean–variance formulation:

Portfolio	$f(x, y)$	$g(x, y)$
MVP	$-x + \frac{\lambda}{2} y$	—
mean–volatility portfolio	$-x + \kappa\sqrt{y}$	—
mean–constrained MVP	y	$\beta - x$
variance-constrained MVP	$-x$	$y - \alpha$
MSRP	$-\dfrac{x - r_\mathrm{f}}{\sqrt{y}}$	—
Kelly portfolio	$-x + \frac{1}{2} y$	—
Kelly portfolio	$-\log(1 + x) + \frac{1}{2} \dfrac{y}{(1 + x)^2}$	—
expected utility portfolio	$-U(0) - U'(0)x - \frac{1}{2} U''(0)(y + x^2)$	—

Universal algorithm for portfolio optimization

- A universal algorithm can be developed for mean-variance portfolio formulations, offering computational efficiency and code reusability.

- This algorithm is based on the successive convex approximation (SCA) method (Scutari et al. 2014), which iteratively solves a sequence of simpler surrogate problems.

- **Advantages of the universal algorithm:**
  1. **Computational efficiency:** Solving a quadratic program (QP) is generally more efficient than solving more complex problems like quadratically constrained QP (QCQP) or second-order cone programs (SOCP).
  2. **Code reusability:** Implementing a solver for one mean-variance problem allows the same code to be reused for different formulations by adjusting the hyper-parameter λ.
  3. **Code specialization:** Advanced users can create tailored numerical algorithms that exploit specific features of the portfolio formulation, such as sparsity or other numerical optimizations.

Successive Convex Approximation (SCA) method

The SCA method, also known as SQP in this context, solves the universal mean-variance formulation by approximating it with a sequence of quadratic programs. The surrogate problems are defined as:

$$\begin{array}{ll} \underset{\boldsymbol{w}}{\text{minimize}} & \tilde{f}\left(\boldsymbol{w};\boldsymbol{w}^k\right) + \frac{\tau^k}{2}\|\boldsymbol{w}-\boldsymbol{w}^k\|_2^2 \\ \text{subject to} & \boldsymbol{w} \in \mathcal{W}, \end{array}$$

- $\tilde{f}\left(\boldsymbol{w};\boldsymbol{w}^k\right)$**:** Quadratic approximation of the original function f around the previous iterate $\boldsymbol{w}^k$.
- τ^k**:** Proximal term coefficient, taken as $\tau^k = 0$ for convergence.
- $\mathcal{W}$**:** Set of constraints for the portfolio.

Surrogate quadratic function

- The surrogate function $\tilde{f}$ is obtained by linearizing the original function f:

$$\tilde{f}\left(\mathbf{w};\mathbf{w}^k\right) = -\alpha^k\mathbf{w}^\mathsf{T}\boldsymbol{\mu} + \frac{\beta^k}{2}\mathbf{w}^\mathsf{T}\boldsymbol{\Sigma}\mathbf{w},$$

where α^k and β^k are coefficients derived from the partial derivatives of f with respect to the mean and variance at the current iterate:

$$\begin{aligned}
\alpha^k &= -\frac{\partial f}{\partial x}\left(x^k = \left(\mathbf{w}^k\right)^\mathsf{T}\boldsymbol{\mu}, y^k = \left(\mathbf{w}^k\right)^\mathsf{T}\boldsymbol{\Sigma}\mathbf{w}^k\right)\\
\beta^k &= 2\frac{\partial f}{\partial y}\left(x^k = \left(\mathbf{w}^k\right)^\mathsf{T}\boldsymbol{\mu}, y^k = \left(\mathbf{w}^k\right)^\mathsf{T}\boldsymbol{\Sigma}\mathbf{w}^k\right).
\end{aligned}$$

Surrogate mean-variance problems

- Summarizing, the surrogate problems become:

$$\begin{array}{ll} \underset{\boldsymbol{w}}{\text{maximize}} & \boldsymbol{w}^\mathsf{T}\boldsymbol{\mu}^k - \dfrac{\lambda^k}{2}\boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}^k\boldsymbol{w}\\ \text{subject to} & \boldsymbol{w} \in \mathcal{W} \end{array}$$

with

$$\begin{aligned} \boldsymbol{\mu}^k &= \boldsymbol{\mu} + \frac{\tau^k}{\alpha^k}\boldsymbol{w}^k\\ \boldsymbol{\Sigma}^k &= \boldsymbol{\Sigma} + \frac{\tau^k}{\beta^k}\boldsymbol{I}\\ \lambda^k &= \frac{\beta^k}{\alpha^k}. \end{aligned}$$

- The next table summarizes the expressions of α^k and β^k for various portfolio formulations.

Universal algorithm for portfolio optimization

Portfolio formulations with the corresponding expressions for α^k and β^k:

Portfolio	$f(x,y)$	$\frac{\partial f}{\partial x}$	$\frac{\partial f}{\partial y}$	α^k	β^k
MVP	$-x+\frac{\lambda}{2}y$	-1	$\lambda/2$	1	λ
mean–volatility portfolio	$-x+\kappa\sqrt{y}$	-1	$\frac{\kappa}{2\sqrt{y}}$	1	$\frac{\kappa}{\sqrt{(\boldsymbol{w}^k)^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w}^k}}$
MSRP	$-\frac{x-r_\mathrm{f}}{\sqrt{y}}$	$-\frac{1}{\sqrt{y}}$	$\frac{x-r_\mathrm{f}}{2y^{3/2}}$	$\frac{1}{\sqrt{(\boldsymbol{w}^k)^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w}^k}}$	$\frac{(\boldsymbol{w}^k)^\mathsf{T}\boldsymbol{\mu}-r_\mathrm{f}}{\left((\boldsymbol{w}^k)^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w}^k\right)^{3/2}}$
Kelly portfolio	$-x+\frac{1}{2}y$	-1	$1/2$	1	1

Universal algorithm

Universal SQP-MVP algorithm (Xiu, Wang, and Palomar 2023)

Initialization:

- Start with an initial portfolio $\boldsymbol{w}^0$ within the feasible set $\mathcal{W}$.
- Define sequences $\{\tau^k\}$ and $\{\gamma^k\}$ for the algorithm.

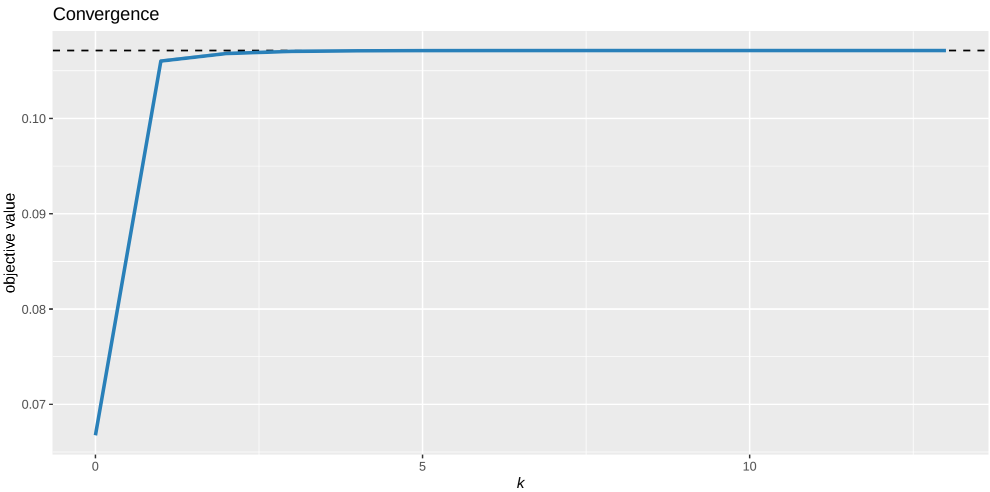
Repeat (kth iteration):

1. Calculate α^k and β^k.
2. Calculate $\boldsymbol{\mu}^k$, $\boldsymbol{\Sigma}^k$, and λ^k.
3. Solve the QP and denote the solution as $\boldsymbol{w}^{k+1/2}$.
4. Update the portfolio as $\boldsymbol{w}^{k+1} = \boldsymbol{w}^k + \gamma_t \left(\boldsymbol{w}^{k+1/2} - \boldsymbol{w}^k\right)$.
5. $k \leftarrow k + 1$

Until: The solution converges to the optimal portfolio.

Convergence of universal algorithm

Convergence of the SQP-MVP algorithm for the MSRP formulation:



Example: MSRP

- **MSRP formulation:**

$$\begin{array}{ll}\underset{\mathbf{w}}{\text{maximize}} & \dfrac{\mathbf{w}^{\mathsf{T}}\boldsymbol{\mu}-r_{\mathsf{f}}}{\sqrt{\mathbf{w}^{\mathsf{T}}\boldsymbol{\Sigma}\mathbf{w}}}\\ \text{subject to} & \mathbf{1}^{\mathsf{T}}\mathbf{w}=1, \quad \mathbf{w}\geq\mathbf{0}.\end{array}$$

- **Universal algorithm :** solve iteratively the mean-variance surrogate problems

$$\begin{array}{ll}\underset{\mathbf{w}}{\text{maximize}} & \mathbf{w}^{\mathsf{T}}\boldsymbol{\mu}-\dfrac{\lambda^{k}}{2}\mathbf{w}^{\mathsf{T}}\boldsymbol{\Sigma}\mathbf{w}\\ \text{subject to} & \mathbf{1}^{\mathsf{T}}\mathbf{w}=1, \quad \mathbf{w}\geq\mathbf{0},\end{array}$$

where $\lambda^{k}=\beta^{k}/\alpha^{k}=\dfrac{(\mathbf{w}^{k})^{\mathsf{T}}\boldsymbol{\mu}-r_{\mathsf{f}}}{(\mathbf{w}^{k})^{\mathsf{T}}\boldsymbol{\Sigma}\mathbf{w}^{k}}$.

Example: Mean–volatility portfolio

- **Mean–volatility formulation:**

$$\begin{array}{ll} \underset{\boldsymbol{w}}{\text{maximize}} & \boldsymbol{w}^T\boldsymbol{\mu} - \kappa\sqrt{\boldsymbol{w}^T\boldsymbol{\Sigma}\boldsymbol{w}} \\ \text{subject to} & \mathbf{1}^T\boldsymbol{w} = 1, \quad \boldsymbol{w} \geq \mathbf{0}, \end{array}$$

- **Universal algorithm :** solve iteratively the mean-variance surrogate problems

$$\begin{array}{ll} \underset{\boldsymbol{w}}{\text{maximize}} & \boldsymbol{w}^T\boldsymbol{\mu} - \frac{\lambda^k}{2}\boldsymbol{w}^T\boldsymbol{\Sigma}\boldsymbol{w} \\ \text{subject to} & \mathbf{1}^T\boldsymbol{w} = 1, \quad \boldsymbol{w} \geq \mathbf{0}, \end{array}$$

where $\lambda^k = \beta^k/\alpha^k = \kappa/\sqrt{(\boldsymbol{w}^k)^T\boldsymbol{\Sigma}\boldsymbol{w}^k}$.

Outline

Drawbacks

- **Markowitz's mean–variance portfolio criticisms:**
  - Despite its theoretical appeal, it's considered risky in practice.
  - Referred to as "Markowitz optimization enigma" and "error maximizer" due to its drawbacks.
- **Some reasons for limited acceptance:**
  - noisy estimation of the expected returns
  - variance or volatility as measure of risk
  - single-number measure of risk

Drawbacks: Sensitivity to estimation errors

- **Noisy estimation of expected returns:**
  - Sample means are imprecise estimators of the population mean, leading to poor out-of-sample performance.
  - Estimation errors in expected returns significantly impact portfolio optimization, more so than errors in covariance matrix estimation.
  - Ignoring expected returns when no additional information is available can sometimes yield better out-of-sample performance.

- **Pragmatic approaches to portfolio construction:**
  - **Risk-based portfolios:** Avoid reliance on $\boldsymbol{\mu}$, focusing instead on risk characteristics.
  - $1/N$ **portfolio:** Simple equal-weighting scheme that doesn't require estimation of expected returns.
  - **Heuristic constraints:** Implementing constraints to stabilize the MVP and improve practical performance.

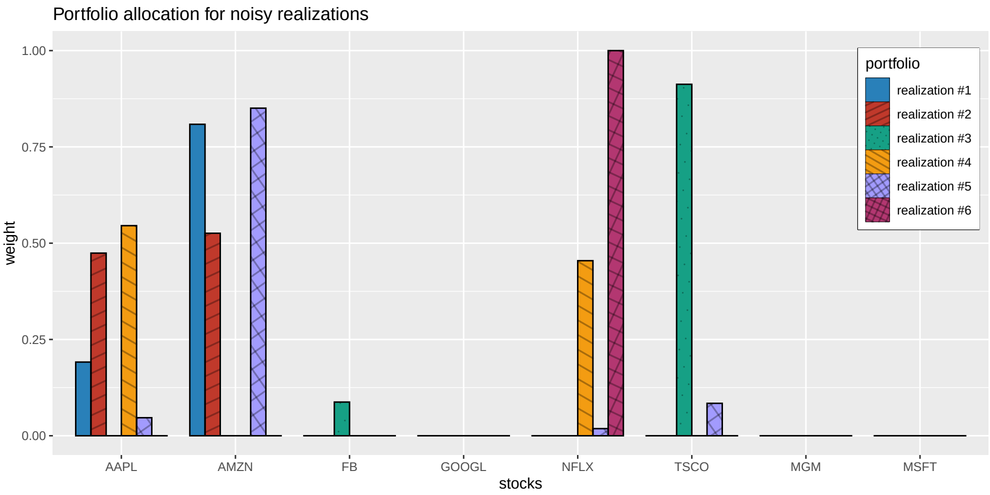
Drawbacks: Sensitivity to estimation errors

- **Strategies to address estimation noise:**
  - **Improved estimators:** Enhance estimation process using prior information, shrinkage, or better statistical models.
  - **Robust optimization:** Acknowledge parameter noise and use techniques like bootstrapping or robust optimization methods.

- **Illustration of MVP instability:** The instability of the MVP is highlighted by showing how portfolio allocations vary significantly under different samples of returns used for estimating expected returns. This demonstrates the sensitivity of the MVP to estimation errors, particularly in expected returns.

Drawbacks

Effect of parameter estimation noise in the MVP allocation:



Drawbacks: Use of variance as measure of risk

- **Variance as a measure of risk:**
  - Criticized for penalizing both gains and losses equally.
  - Does not adequately capture tail risk, which is crucial for understanding potential large losses.

- **Alternative Risk Measures:** Consider measures that focus on downside risk, such as
  - semivariance
  - VaR
  - CVaR
  - drawdown

Drawbacks: Single-number measure of risk

- **Single-number risk characterization:**
  - A single risk measure may not fully capture the risk contributions from individual assets.
  - Emphasizes the importance of risk diversification to avoid concentration in a few assets.

- **Risk diversification and risk parity:**
  - **Risk parity portfolio:**
    - Decomposes overall risk into contributions from each asset for balanced risk distribution.
    - Aims for equal risk contribution from all assets, enhancing diversification.

Outline

Summary

- Markowitz's 1952 paper introduced modern portfolio theory (MPT), focusing on portfolios designed around expected return and risk variance.
- The mean–variance model, a convex problem, creates an efficient frontier of portfolios with varying risk levels.
- Its practical performance is hindered by sensitivity to market parameter errors (expected returns and covariance) and a simplistic risk measure (variance/volatility).
- Practitioners have developed solutions, including heuristic constraints, improved market parameter estimators (e.g., shrinkage, robust estimators), alternative risk measures, and refined risk profiles.
- The Sharpe ratio-maximizing portfolio on the efficient frontier poses a nonconvex challenge, yet practical numerical methods can find optimal solutions.
- The Kelly criterion and expected utility portfolios extend the trade-off between return and risk; they can be effectively approximated by the mean–variance model with efficient algorithms.

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