Portfolio Optimization

High-Order Portfolios

Daniel P. Palomar (2025). *Portfolio Optimization: Theory and Application.*
Cambridge University Press.

portfoliooptimizationbook.com

Latest update: 2025-10-13

Outline

Executive Summary

- Markowitz mean-variance theory struggles with real financial data due to non-Gaussian distributions (asymmetry and heavy tails).
- Higher-order moments (skewness and kurtosis) can address these limitations but introduce significant computational challenges.
- Complexity grows exponentially at rate of N^4 with number of assets, making computation, storage, and manipulation difficult.
- Resulting portfolio formulations are nonconvex, complicating optimization.
- These slides explore recent advances in computational power and techniques have made high-order portfolios feasible for large-scale applications (hundreds to thousands of assets) (Palomar 2025, chap. 9)

Outline

Introduction

Markowitz's Mean-Variance Portfolio Optimization

- Balances expected return and risk (variance).
- Optimization problem:

$$\begin{array}{ll} \underset{\boldsymbol{w}}{\text{maximize}} & \boldsymbol{w}^\mathsf{T}\boldsymbol{\mu} - \frac{\lambda}{2}\boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w} \\ \text{subject to} & \boldsymbol{w} \in \mathcal{W} \end{array}$$

  where
  - λ: risk-aversion hyper-parameter.
  - $\mathcal{W}$: constraint set, e.g., $\mathcal{W} = \{\boldsymbol{w} \mid \mathbf{1}^\mathsf{T}\boldsymbol{w} = 1, \boldsymbol{w} \geq \mathbf{0}\}$.

Beyond the Gaussian Assumption

- Empirical studies show financial data are not Gaussian.
- Portfolio optimization should include higher-order moments:
  - skewness (third moment)
  - kurtosis (fourth moment)
- Aim for higher skewness and lower kurtosis in portfolios.

Introduction

Skewed t Distribution

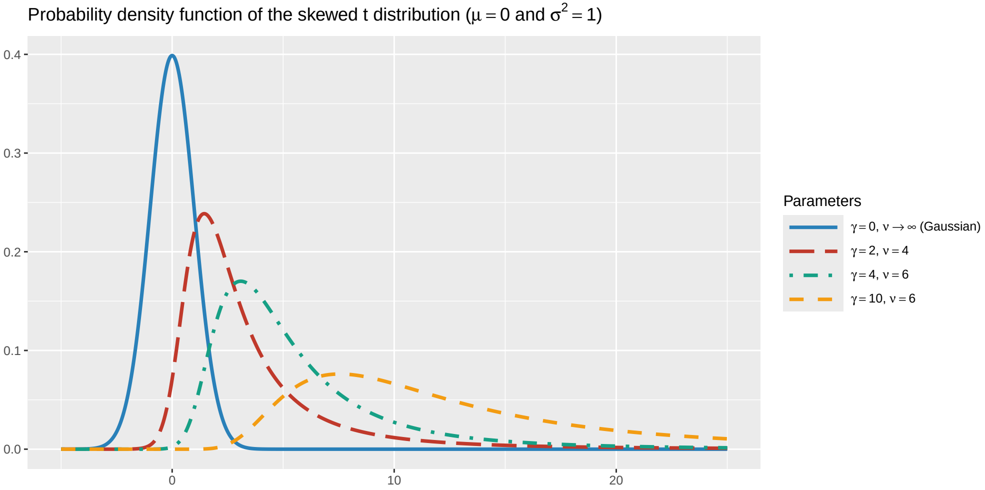
- Next figure shows skewed t distribution with varying skewness (γ) and kurtosis (ν).
- $\gamma = 0$: symmetric case.
- $\nu \to \infty$: non-heavy-tailed case.

Incorporating Higher Moments in Portfolio Optimization

- Investors may prefer to trade-off lower expected return/higher volatility for higher skewness/lower kurtosis.
- Measures can be skew-adjusted; example of Sharpe ratio:

$$\text{skew-adjusted-SR} = \text{SR} \times \sqrt{1 + \frac{\text{skewness}}{3}\text{SR}}.$$

Illustration of skewness and kurtosis with the skewed t distribution:



High-Order Portfolios Overview

- Third and fourth moments of a portfolio:
  - third moment (skewness): $\boldsymbol{w}^{\mathsf{T}}\boldsymbol{\Phi}(\boldsymbol{w}\otimes\boldsymbol{w})$.
  - fourth moment (kurtosis): $\boldsymbol{w}^{\mathsf{T}}\boldsymbol{\Psi}(\boldsymbol{w}\otimes\boldsymbol{w}\otimes\boldsymbol{w})$.
- Co-skewness and co-kurtosis matrices: $\boldsymbol{\Phi}\in\mathbb{R}^{N\times N^2}, \boldsymbol{\Psi}\in\mathbb{R}^{N\times N^3}$.
- Challenges:
  - computation, storage, and manipulation of high-order moments are difficult.
  - nonconvex nature of the third moment complicates portfolio formulations.

Historical Perspective (Palomar 2025, chap. 9)

- Early attempts in the 1960s to incorporate high-order moments (Young and Trent 1969; Jean 1971).
- Dimensionality and computational issues hindered progress.
- Skepticism due to the impracticality of modeling high-order cross-moments (Brandt, Santa-Clara, and Valkanov 2009).
- Recent advancements in estimation methods and computational techniques (Boudt, Lu, and Peeters 2014; Zhou and Palomar 2021; Wang et al. 2023).

Introduction

Estimation and Computational Challenges

- High-order portfolio design can negatively impact out-of-sample performance without improved estimators.
- Improved estimation methods introduce structure and shrinkage.
- Parametric multivariate distributions reduce parameter estimation complexity.

Algorithmic Developments

- Nonconvex problems can be addressed with meta-heuristic optimization, but have high computational cost.
- Local optimization methods offer practical solutions with acceptable costs.
- Difference-of-convex (DC) programming and DC-SOS decomposition techniques.
- Successive convex approximation (SCA) framework (Scutari et al. 2014) accelerates convergence for high-dimensional problems.

Advancements in High-Order Portfolio Optimization

- Significant reduction in computational cost through parametric models.
- Development of faster numerical methods for large-scale portfolio optimization.
- High-order portfolios now feasible for practical application.

Outline

High-Order Moments

Understanding High-Order Moments

- High-order moments are crucial for non-Gaussian distributions.
- First four moments of a random variable X:
  - mean (first moment): $\bar{X} \triangleq \mathbb{E}[X]$
  - variance (second moment): $\mathbb{E}[(X - \bar{X})^2]$
  - skewness (third moment): $\mathbb{E}[(X - \bar{X})^3]$
  - kurtosis (fourth moment): $\mathbb{E}[(X - \bar{X})^4]$

Interpretation

- Mean: indicates location.
- Variance: measures spread.
- Skewness: assesses asymmetry.
- Kurtosis: characterizes the tail thickness.

High-Order Moments: Non-Parametric Case

Portfolio Moments

- Portfolio return with N assets: $\boldsymbol{w}^\mathsf{T}\boldsymbol{r}$.

- First four moments of portfolio return:
  - mean: $\phi_1(\boldsymbol{w}) = \boldsymbol{w}^\mathsf{T}\boldsymbol{\mu}$
  - variance: $\phi_2(\boldsymbol{w}) = \boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w}$
  - skewness: $\phi_3(\boldsymbol{w}) = \boldsymbol{w}^\mathsf{T}\boldsymbol{\Phi}(\boldsymbol{w}\otimes\boldsymbol{w})$
  - kurtosis: $\phi_4(\boldsymbol{w}) = \boldsymbol{w}^\mathsf{T}\boldsymbol{\Psi}(\boldsymbol{w}\otimes\boldsymbol{w}\otimes\boldsymbol{w})$

- Parameters:
  - $\boldsymbol{\mu}$: mean vector
  - $\boldsymbol{\Sigma}$: covariance matrix
  - $\boldsymbol{\Phi}$: co-skewness matrix
  - $\boldsymbol{\Psi}$: co-kurtosis matrix

High-Order Moments: Non-Parametric Case

Computational Aspects

- We will optimize some combination of these four moments of the portfolio.
- This means that we will need to compute the gradient and Hessian of these moments.

Gradients and Hessians

- Gradients:
  - $\nabla \phi_1(\boldsymbol{w}) = \boldsymbol{\mu}$
  - $\nabla \phi_2(\boldsymbol{w}) = 2\boldsymbol{\Sigma}\boldsymbol{w}$
  - $\nabla \phi_3(\boldsymbol{w}) = 3\boldsymbol{\Phi}(\boldsymbol{w} \otimes \boldsymbol{w})$
  - $\nabla \phi_4(\boldsymbol{w}) = 4\boldsymbol{\Psi}(\boldsymbol{w} \otimes \boldsymbol{w} \otimes \boldsymbol{w})$
- Hessians:
  - $\nabla^2 \phi_1(\boldsymbol{w}) = \boldsymbol{0}$
  - $\nabla^2 \phi_2(\boldsymbol{w}) = 2\boldsymbol{\Sigma}$
  - $\nabla^2 \phi_3(\boldsymbol{w}) = 6\boldsymbol{\Phi}(\boldsymbol{I} \otimes \boldsymbol{w})$
  - $\nabla^2 \phi_4(\boldsymbol{w}) = 12\boldsymbol{\Psi}(\boldsymbol{I} \otimes \boldsymbol{w} \otimes \boldsymbol{w})$

High-Order Moments: Non-Parametric Case

Complexity Analysis

- Complexity of parameters and their computation grows with N:
  - $\boldsymbol{\mu}$: $O(N)$
  - $\boldsymbol{\Sigma}$: $O(N^2)$
  - $\boldsymbol{\Phi}$: $O(N^3)$
  - $\boldsymbol{\Psi}$: $O(N^4)$
- Gradients and Hessians complexity:
  - $\nabla\phi_1(\boldsymbol{w})$ and $\nabla^2\phi_1(\boldsymbol{w})$: $O(1)$ and $O(1)$
  - $\nabla\phi_2(\boldsymbol{w})$ and $\nabla^2\phi_2(\boldsymbol{w})$: $O(N^2)$ and $O(1)$
  - $\nabla\phi_3(\boldsymbol{w})$ and $\nabla^2\phi_3(\boldsymbol{w})$: $O(N^3)$ and $O(N^3)$
  - $\nabla\phi_4(\boldsymbol{w})$ and $\nabla^2\phi_4(\boldsymbol{w})$: $O(N^4)$ and $O(N^4)$
- Memory requirement example: storing $\boldsymbol{\Psi}$ for $N = 200$ needs $\sim$12 GB.

Implications for Practical Application

- Markowitz's portfolio complexity: $O(N^2)$.
- Incorporating third and fourth moments increases complexity to $O(N^4)$.
- High complexity limits practical application to portfolios with a small number of assets.

High-Order Moments: Structured Moments

Structured Moments in Portfolio Analysis

- Introducing structure to high-order moment matrices can reduce parameter estimation.
- Factor modeling introduces structure at the expense of higher complexity in the estimation procedure due to intricate matrix structures.

Single Market-Factor Model

- Returns modeled as:

$$\boldsymbol{r}_t = \boldsymbol{\alpha} + \boldsymbol{\beta} r_t^{\mathsf{mkt}} + \boldsymbol{\epsilon}_t$$

- Moments expressed as:
  - mean vector: $\boldsymbol{\mu} = \boldsymbol{\alpha} + \boldsymbol{\beta}\phi_1^{\mathsf{mkt}}$
  - covariance matrix: $\boldsymbol{\Sigma} = \boldsymbol{\beta}\boldsymbol{\beta}^{\mathsf{T}}\phi_2^{\mathsf{mkt}} + \boldsymbol{\Sigma}_\epsilon$
  - co-skewness matrix: $\boldsymbol{\Phi} = \boldsymbol{\beta}\left(\boldsymbol{\beta}^{\mathsf{T}} \otimes \boldsymbol{\beta}^{\mathsf{T}}\right)\phi_3^{\mathsf{mkt}} + \boldsymbol{\Phi}_\epsilon$
  - co-kurtosis matrix: $\boldsymbol{\Psi} = \boldsymbol{\beta}\left(\boldsymbol{\beta}^{\mathsf{T}} \otimes \boldsymbol{\beta}^{\mathsf{T}} \otimes \boldsymbol{\beta}^{\mathsf{T}}\right)\phi_4^{\mathsf{mkt}} + \boldsymbol{\Psi}_\epsilon$
  - ϕ_i^{mkt}: ith moment of the market factor
  - $\boldsymbol{\Sigma}_\epsilon$, $\boldsymbol{\Phi}_\epsilon$, $\boldsymbol{\Psi}_\epsilon$: covariance, co-skewness, and co-kurtosis matrices of residuals

High-Order Moments: Parametric Case

Overview of Multivariate Distributions

- Multivariate normal distribution characterized by mean $\boldsymbol{\mu}$ and covariance $\boldsymbol{\Sigma}$.
- Financial data often exhibit non-Gaussian features like skewness and kurtosis.

Multivariate Normal Mixture Distributions

- Introduce randomness into covariance and mean for more general distributions.
- Variance mixtures affect covariance but not mean.
- Mean-variance mixtures affect both mean and covariance.

Normal Variance Mixture

- Example: multivariate t distribution with inverse gamma distribution for w.
- Models heavy tails but not asymmetry.
- Hierarchical structure:

$$\boldsymbol{x} \mid \tau \sim \mathcal{N}\left(\boldsymbol{\mu}, \frac{1}{\tau}\boldsymbol{\Sigma}\right),$$

$$\tau \sim \text{Gamma}\left(\frac{\nu}{2}, \frac{\nu}{2}\right).$$

High-Order Moments: Parametric Case

Normal Mean-Variance Mixture

- Example: multivariate generalized hyperbolic (GH) distribution.
- Models both heavy tails and asymmetry.
- Hierarchical structure for skewed t distribution:

$$\boldsymbol{x} \mid \tau \sim \mathcal{N}\left(\boldsymbol{\mu} + \frac{1}{\tau}\boldsymbol{\gamma}, \frac{1}{\tau}\boldsymbol{\Sigma}\right),$$
$$\tau \sim \text{Gamma}\left(\frac{\nu}{2}, \frac{\nu}{2}\right).$$

Complexity in Fitting Distributions

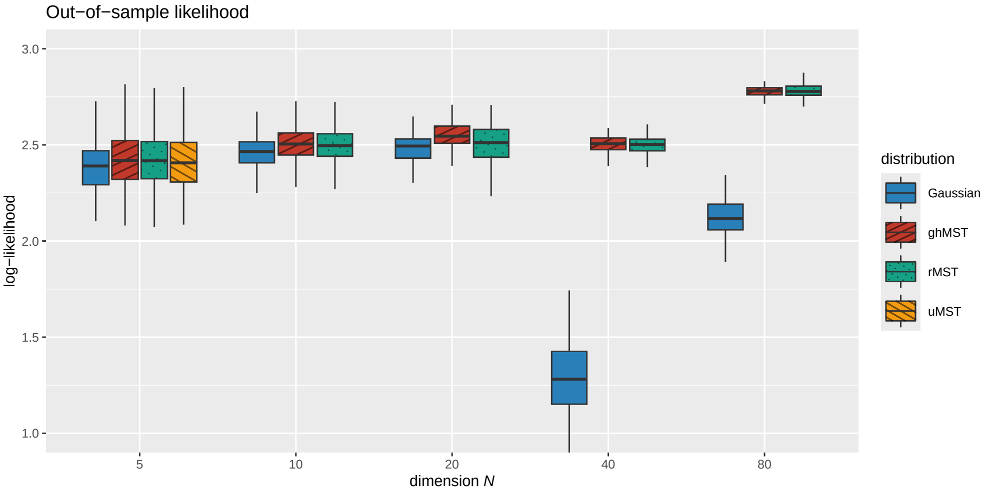
- More complex distributions like unrestricted multivariate skewed t (uMST) are hard to fit due to computational complexity.
- Skewed t distribution offers a good balance between fitting financial data asymmetries and computational simplicity.

Goodness of Fit

- Empirical comparison shows skewed t distribution a good choice for financial data.
- Maintains simplicity while fitting data well, unlike more complex distributions.

High-Order Moments: Parametric Case

Likelihood of different fitted multivariate distributions for S&P 500 daily stock returns:



High-Order Moments: Parametric Case

Simplified Moments under Multivariate Skewed t Distribution

- The parametric model simplifies moment computations significantly.
- First four moments (Wang et al. 2023):
    - $\phi_1(\boldsymbol{w}) = \boldsymbol{w}^\mathsf{T}\boldsymbol{\mu} + a_1\boldsymbol{w}^\mathsf{T}\gamma$
    - $\phi_2(\boldsymbol{w}) = a_{21}\boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w} + a_{22}(\boldsymbol{w}^\mathsf{T}\gamma)^2$
    - $\phi_3(\boldsymbol{w}) = a_{31}(\boldsymbol{w}^\mathsf{T}\gamma)^3 + a_{32}(\boldsymbol{w}^\mathsf{T}\gamma)\boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w}$
    - $\phi_4(\boldsymbol{w}) = a_{41}(\boldsymbol{w}^\mathsf{T}\gamma)^4 + a_{42}(\boldsymbol{w}^\mathsf{T}\gamma)^2\boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w} + a_{43}(\boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w})^2$
- Coefficients a_1, a_{21}, a_{22}, etc., are functions of the degrees of freedom ν.

Clarification on Parameters

- $\boldsymbol{\mu}$: location vector, not the mean.
- $\boldsymbol{\Sigma}$: scatter matrix, not the covariance matrix.

High-Order Moments: Parametric Case

Gradients and Hessians

- Gradients:
  - $\nabla \phi_1(\boldsymbol{w}) = \boldsymbol{\mu} + a_1 \gamma$
  - $\nabla \phi_2(\boldsymbol{w}) = 2a_{21}\boldsymbol{\Sigma}\boldsymbol{w} + 2a_{22}(\boldsymbol{w}^\mathsf{T}\gamma)\gamma$
  - $\nabla \phi_3(\boldsymbol{w}) = 3a_{31}(\boldsymbol{w}^\mathsf{T}\gamma)^2\gamma + a_{32}\left((\boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w})\gamma + 2(\boldsymbol{w}^\mathsf{T}\gamma)\boldsymbol{\Sigma}\boldsymbol{w}\right).$
  - $\nabla \phi_4(\boldsymbol{w}) = 4a_{41}(\boldsymbol{w}^\mathsf{T}\gamma)^3\gamma + 2a_{42}\left((\boldsymbol{w}^\mathsf{T}\gamma)^2\boldsymbol{\Sigma}\boldsymbol{w} + (\boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w})(\boldsymbol{w}^\mathsf{T}\gamma)\gamma\right) + 4a_{43}(\boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w})\boldsymbol{\Sigma}\boldsymbol{w}$
- Hessians:
  - $\nabla^2 \phi_1(\boldsymbol{w}) = \boldsymbol{0}$
  - $\nabla^2 \phi_2(\boldsymbol{w}) = 2a_{21}\boldsymbol{\Sigma} + 2a_{22}\gamma\gamma^\mathsf{T}$
  - $\nabla^2 \phi_3(\boldsymbol{w}) = 6a_{31}(\boldsymbol{w}^\mathsf{T}\gamma)\gamma\gamma^\mathsf{T} + 2a_{32}\left(\gamma\boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma} + \boldsymbol{\Sigma}\boldsymbol{w}\gamma^\mathsf{T} + (\boldsymbol{w}^\mathsf{T}\gamma)\boldsymbol{\Sigma}\right)$
  - $\nabla^2 \phi_4(\boldsymbol{w}) =$
    $12a_{41}(\boldsymbol{w}^\mathsf{T}\gamma)^2\gamma\gamma^\mathsf{T} + 2a_{42}\left(2(\boldsymbol{w}^\mathsf{T}\gamma)\boldsymbol{\Sigma}\boldsymbol{w}\gamma^\mathsf{T} + (\boldsymbol{w}^\mathsf{T}\gamma)^2\boldsymbol{\Sigma} + 2(\boldsymbol{w}^\mathsf{T}\gamma)\gamma\boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma} + (\boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w})\gamma\gamma^\mathsf{T}\right) +$
    $4a_{43}\left(2\boldsymbol{\Sigma}\boldsymbol{w}\boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma} + (\boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w})\boldsymbol{\Sigma}\right)$

Take-Home Message under Parametric Modeling

- No need to compute, store, and manipulate huge co-skewness and co-kurtosis matrices.
- Can cheaply compute gradients and Hessians based on the parameters: $\boldsymbol{\mu}$, $\boldsymbol{\Sigma}$, ν, and γ.

High-Order Moments: L-Moments

L-Moments Overview

- L-moments characterize the distribution of a random variable and describe its properties such as location, dispersion, asymmetry, and tail thickness.
- They are linear functions of order statistics, making them easier to estimate than traditional moments.

Definition of L-Moments

- Let X be a random variable and $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ be the *order statistics* of a random sample of size n drawn from the distribution of X.
- L-moments for a random variable X are defined as:

$$\lambda_r = \frac{1}{r} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \mathbb{E}[X_{r-k:r}], \quad r = 1, 2, \ldots$$

- First four L-moments:
  - $\lambda_1 = \mathbb{E}[X]$
  - $\lambda_2 = \frac{1}{2}\mathbb{E}[X_{2:2} - X_{1:2}]$
  - $\lambda_3 = \frac{1}{3}\mathbb{E}[X_{3:3} - 2X_{2:3} + X_{1:3}]$
  - $\lambda_4 = \frac{1}{4}\mathbb{E}[X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4}]$

High-Order Moments: L-Moments

Descriptive Information Provided by L-Moments

- L-location (λ_1) is identical to the mean.
- L-scale (λ_2) measures expected difference between any two realizations (like variance).
- L-skewness (λ_3) provides a measure of asymmetry less sensitive to extreme tails.
- L-kurtosis (λ_4) measures tail thickness, less sensitive to extreme tails.

Estimation of L-Moments

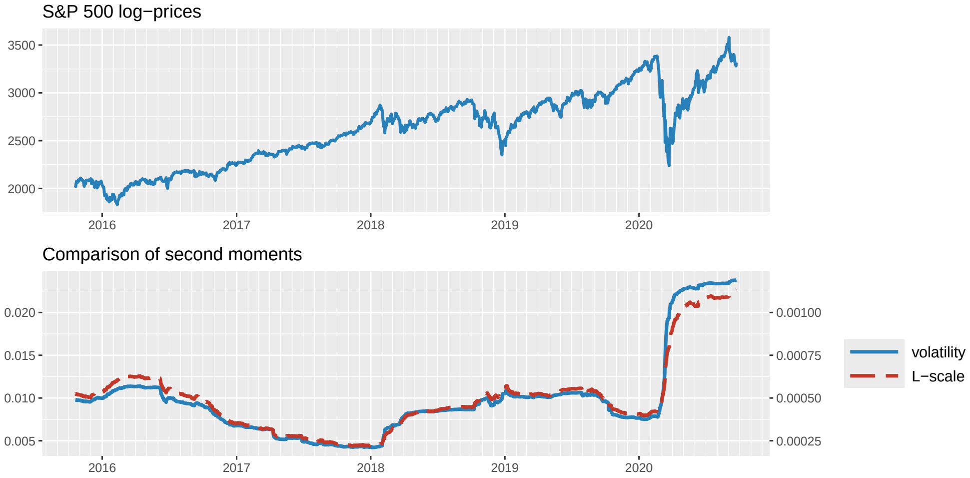
- Direct estimation from observations is computationally demanding.
- Simplified estimators in terms of sample values in ascending order $x_{(i)}$:
  - $\hat{\lambda}_1 = \frac{1}{n} \sum_{i=1}^{n} x_{(i)}$
  - $\hat{\lambda}_2 = \frac{1}{2} \frac{1}{C_2^n} \sum_{i=1}^{n} \left(C_1^{i-1} - C_1^{n-i} \right) x_{(i)}$
  - $\hat{\lambda}_3 = \frac{1}{3} \frac{1}{C_3^n} \sum_{i=1}^{n} \left(C_2^{i-1} - 2C_1^{i-1} C_1^{n-i} + C_2^{n-i} \right) x_{(i)}$
  - $\hat{\lambda}_4 = \frac{1}{4} \frac{1}{C_4^n} \sum_{i=1}^{n} \left(C_3^{i-1} - 3C_2^{i-1} C_1^{n-i} + 3C_1^{i-1} C_2^{n-i} - C_3^{n-i} \right) x_{(i)}$

Comparison with Traditional Moments

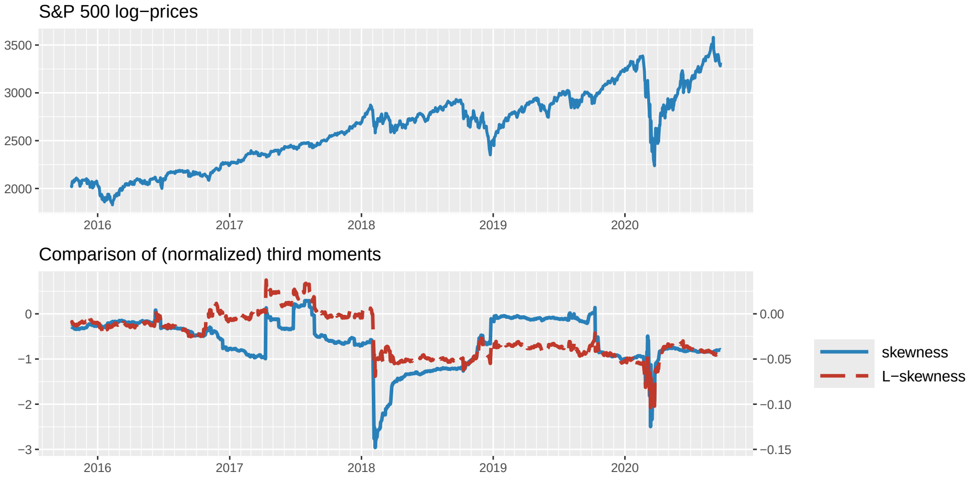
- L-moments convey similar information to traditional moments but are more stable.
- This stability makes L-moments particularly useful for analyzing financial data, where they exhibit fewer jumps and provide a clearer picture.

High-Order Moments: L-Moments

Moments and L-moments of the S&P 500 index in a rolling-window fashion:

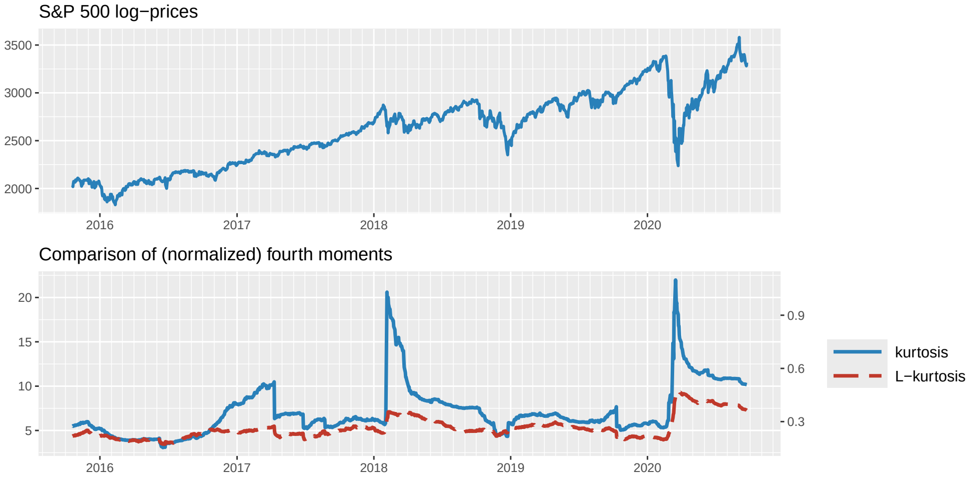


Moments and L-moments of the S&P 500 index in a rolling-window fashion:



High-Order Moments: L-Moments

Moments and L-moments of the S&P 500 index in a rolling-window fashion:



Outline

Portfolio Formulations

Formulations Involving High-Order Moments

- Higher-order moments introduce nonconvexity into portfolio formulations.
- Different expressions for moments $\phi_1(\boldsymbol{w})$, $\phi_2(\boldsymbol{w})$, $\phi_3(\boldsymbol{w})$, and $\phi_4(\boldsymbol{w})$ offer various formulation options.
- MVSK portfolio: mean-variance-skewness-kurtosis.

Moment Expression Options

- **Non-parametric:** direct calculations from returns; computationally demanding and may need large data.
- **Factor model structured:** utilizes market or multiple factors to structure moments, reducing parameter count.
- **Parametric:** uses multivariate skewed t distribution, easing moment computations and balancing distributional capture with feasibility.
- **L-moments:** linear in order statistics, offering easier estimation and robustness to extremes, while conveying distribution insights.

MVSK Portfolios

MVSK Portfolio Formulation

Optimize a weighted combination of the first four moments:

$$\begin{array}{ll} \underset{\boldsymbol{w}}{\text{minimize}} & -\lambda_1\phi_1(\boldsymbol{w}) + \lambda_2\phi_2(\boldsymbol{w}) - \lambda_3\phi_3(\boldsymbol{w}) + \lambda_4\phi_4(\boldsymbol{w}) \\ \text{subject to} & \boldsymbol{w} \in \mathcal{W} \end{array}$$

where λ_i are hyper-parameters reflecting risk aversion.

Investor Preferences

- Seek higher mean and skewness: $\phi_1(\boldsymbol{w})$, $\phi_3(\boldsymbol{w})$.
- Prefer lower variance and kurtosis: $\phi_2(\boldsymbol{w})$, $\phi_4(\boldsymbol{w})$.

MVSK Portfolios

Alternative Formulations

Constraints on moments:

$$\begin{array}{ll} \underset{\boldsymbol{w}}{\text{find}} & \boldsymbol{w} \\ \text{subject to} & \phi_1(\boldsymbol{w}) \geq \alpha_1 \\ & \phi_2(\boldsymbol{w}) \leq \alpha_2 \\ & \phi_3(\boldsymbol{w}) \geq \alpha_3 \\ & \phi_4(\boldsymbol{w}) \leq \alpha_4, \end{array}$$

where the α_i's are hyper-parameters denoting investor's targets.

Numerical Algorithms

- General-purpose solvers are always an option.
- Specialized algorithms for solving MVSK have been recently developed (Zhou and Palomar 2021; Wang et al. 2023).

MVSK Portfolios

Expected Utility in Portfolio Design

- Focuses on maximizing the expected value of a utility function $\mathbb{E}\left[U(\mathbf{w}^\mathsf{T}\mathbf{r})\right]$.
- Moves beyond the mean-variance objective.

Historical Context

- High-order portfolios considered since 1969.
- Geometric mean approximation of returns:

$$\mathbb{E}\left[\log\left(1+\mathbf{w}^\mathsf{T}\mathbf{r}\right)\right] \approx \log\left(1+\phi_1(\mathbf{w})\right) - \frac{\phi_2(\mathbf{w})}{2\phi_1^2(\mathbf{w})} + \frac{\phi_3(\mathbf{w})}{3\phi_1^3(\mathbf{w})} - \frac{\phi_4(\mathbf{w})}{4\phi_1^4(\mathbf{w})}$$

- Mean-variance portfolio arises from using the first two terms of the log-approximation.

Advancements in High-Order Approximations

- High-order expansions for arbitrary expected utilities can be now considered.
- Recent work includes high-order approximations with structured estimators for moments.

Making Portfolios Efficient

Shortage Function in Multi-Objective Optimization

- Measures the distance between a portfolio's moments and the efficient frontier.
- Utilized to optimize portfolios towards Pareto-optimal points.

Optimization Using the Shortage Function

- Given a reference portfolio $\boldsymbol{w}^0$ and direction vector $\boldsymbol{g}$.
- Objective: maximize δ, representing movement towards the efficient frontier.
- Constraints ensure improvement in desired moments along $\boldsymbol{g}$.
- Formulation:

$$\begin{array}{ll} \underset{\boldsymbol{w},\delta\geq 0}{\text{maximize}} & \delta \\ \text{subject to} & \phi_1(\boldsymbol{w}) \geq \phi_1(\boldsymbol{w}^0) + \delta g_1 \\ & \phi_2(\boldsymbol{w}) \leq \phi_2(\boldsymbol{w}^0) - \delta g_2 \\ & \phi_3(\boldsymbol{w}) \geq \phi_3(\boldsymbol{w}^0) + \delta g_3 \\ & \phi_4(\boldsymbol{w}) \leq \phi_4(\boldsymbol{w}^0) - \delta g_4 \end{array}$$

Feasibility and Efficiency

- Formulation is always feasible.
- If $\boldsymbol{w}^0$ is on the efficient frontier, the solution is $\boldsymbol{w} = \boldsymbol{w}^0$ and $\delta = 0$.

Portfolio Tilting

Extending Portfolio Improvement with Optimality Measure

- Obtained by minimizing a cost function $\xi(\boldsymbol{w})$.
- Examples of $\xi(\cdot)$ include Herfindahl index, risk contributions equalization, diversification ratio, and tracking error.

MVSK Portfolio Tilting Formulation

- Objective: maximize δ to make the reference portfolio $\boldsymbol{w}^0$ more optimal.
- Constraints include maintaining or improving moment values and allowing a controlled loss of optimality (κ) for closer efficient frontier alignment.
- Formulation:

$$\begin{array}{ll}\underset{\boldsymbol{w},\delta\geq 0}{\text{maximize}} & \delta\\ \text{subject to} & \xi(\boldsymbol{w}) \leq \xi(\boldsymbol{w}^0) + \kappa\\ & \phi_1(\boldsymbol{w}) \geq \phi_1(\boldsymbol{w}^0) + g_1(\delta)\\ & \phi_2(\boldsymbol{w}) \leq \phi_2(\boldsymbol{w}^0) - g_2(\delta)\\ & \phi_3(\boldsymbol{w}) \geq \phi_3(\boldsymbol{w}^0) + g_3(\delta)\\ & \phi_4(\boldsymbol{w}) \leq \phi_4(\boldsymbol{w}^0) - g_4(\delta)\end{array}$$

Portfolio Tilting

Cost Function Examples for Portfolio Tilting

- **Herfindahl index:** $\xi(\boldsymbol{w}) = \sum_{i=1}^{N} w_i^2$; encourages diversity.
- **Risk parity:** $\xi(\boldsymbol{w}) = \sum_{i=1}^{N} \left(\frac{w_i(\boldsymbol{\Sigma}\boldsymbol{w})_i}{\boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w}} - \frac{1}{N}\right)^2$; equalizes risk contributions.
- **Diversification ratio:** $\xi(\boldsymbol{w}) = -\dfrac{\boldsymbol{w}^\mathsf{T}\boldsymbol{\sigma}}{\sqrt{\boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w}}}$; maximizes return-to-risk ratio.
- **Tracking error:** $\xi(\boldsymbol{w}) = \sqrt{(\boldsymbol{w} - \boldsymbol{w}^{\text{benchmark}})^\mathsf{T}\boldsymbol{\Sigma}(\boldsymbol{w} - \boldsymbol{w}^{\text{benchmark}})}$; minimizes benchmark deviation.

Optimization Parameters

- $g_i(\delta)$: functions increasing with δ.
- κ: allows optimality loss for efficient frontier approach, set as $0.01 \times \xi(\boldsymbol{w}^0)$.

Numerical Algorithms

- General-purpose solvers are always an option.
- Efficient algorithms developed for solving the MVSK tilting portfolio formulation (Zhou and Palomar 2021).

Polynomial Goal Programming MVSK Portfolio

Polynomial Goal Programming for Moment Trade-Off

- Objective: minimize distance to reference moments with a polynomial.
- Formulation:

$$
\begin{array}{ll}
\underset{\boldsymbol{w},\boldsymbol{d}\geq \boldsymbol{0}}{\text{minimize}} & \left|\frac{d_1}{\phi_1^0}\right|^{\lambda_1} + \left|\frac{d_2}{\phi_2^0}\right|^{\lambda_2} + \left|\frac{d_3}{\phi_3^0}\right|^{\lambda_3} + \left|\frac{d_4}{\phi_4^0}\right|^{\lambda_4} \\
\text{subject to} & \phi_1(\boldsymbol{w}) + d_1 \geq \phi_1^0 \\
& \phi_2(\boldsymbol{w}) - d_2 \leq \phi_2^0 \\
& \phi_3(\boldsymbol{w}) + d_3 \geq \phi_3^0 \\
& \phi_4(\boldsymbol{w}) - d_4 \leq \phi_4^0
\end{array}
$$

- $\boldsymbol{d}$: deviation from aspired moment levels ϕ_i^0.
- Aspired levels represent extreme values and are not jointly achievable.

Minkovski Distance Case

- Special case of polynomial goal programming with exponents $\lambda_i = 1/p$.
- Objective becomes a Minkovski distance minimization.

Outline

Algorithms

MVSK Portfolio Formulation and Objective Function

- Focus on MVSK portfolio with nonconvex objective due to higher-order moments.
- Objective function:

$$f(\boldsymbol{w}) = -\lambda_1\phi_1(\boldsymbol{w}) + \lambda_2\phi_2(\boldsymbol{w}) - \lambda_3\phi_3(\boldsymbol{w}) + \lambda_4\phi_4(\boldsymbol{w})$$

Decomposing MVSK Objective into Convex and Nonconvex Terms

$$\begin{aligned} f_{\text{cvx}}(\boldsymbol{w}) &= -\lambda_1\phi_1(\boldsymbol{w}) + \lambda_2\phi_2(\boldsymbol{w}) \\ f_{\text{ncvx}}(\boldsymbol{w}) &= -\lambda_3\phi_3(\boldsymbol{w}) + \lambda_4\phi_4(\boldsymbol{w}). \end{aligned}$$

Efficient Numerical Methods beyond General Nonlinear Solvers

- Developing ad-hoc methods for greater efficiency based on the *successive convex approximation* (SCA) framework (Scutari et al. 2014) and the *majorization-minimization* (MM) framework (Sun, Babu, and Palomar 2017).
- R package `highOrderPortfolios` (Zhou and Palomar 2021; Wang et al. 2023).

SCA: Preliminaries

Successive Convex Approximation (SCA) Method (Scutari et al. 2014)

- SCA approximates a difficult optimization problem with a series of simpler convex problems.
- Iteratively produces a sequence $\boldsymbol{x}^0, \boldsymbol{x}^1, \boldsymbol{x}^2, \ldots$ converging towards a solution $\boldsymbol{x}^\star$.

SCA Iterative Process

- At each iteration k, uses a surrogate function $\tilde{f}\left(\boldsymbol{x}; \boldsymbol{x}^k\right)$ to approximate objective $f(\boldsymbol{x})$.
- The update rule includes a smoothing step to ensure convergence:

$$\left.\begin{aligned} \hat{\boldsymbol{x}}^{k+1} &= \underset{\boldsymbol{x}\in\mathcal{X}}{\operatorname{argmin}}\ \tilde{f}\left(\boldsymbol{x}; \boldsymbol{x}^k\right) \\ \boldsymbol{x}^{k+1} &= \boldsymbol{x}^k + \gamma^k\left(\hat{\boldsymbol{x}}^{k+1} - \boldsymbol{x}^k\right) \end{aligned}\right. \qquad k = 0, 1, 2, \ldots$$

  where $\{\gamma^k\}$ is a sequence with $\gamma^k \in (0, 1]$.

Convergence Conditions

- Surrogate function $\tilde{f}\left(\boldsymbol{x}; \boldsymbol{x}^k\right)$ must be strongly convex on $\mathcal{X}$.
- Surrogate function must be differentiable with gradient equal to that of $f(\boldsymbol{x})$ at $\boldsymbol{x}^k$.

SCA Framework for Nonconvex Optimization

- Leave convex term $f_{\text{cvx}}(\boldsymbol{w})$ unchanged.
- Approximate nonconvex term $f_{\text{ncvx}}(\boldsymbol{w})$ quadratically around $\boldsymbol{w} = \boldsymbol{w}^k$:

$$\begin{aligned}\tilde{f}_{\text{ncvx}}\left(\boldsymbol{w}; \boldsymbol{w}^k\right) =& f_{\text{ncvx}}\left(\boldsymbol{w}^k\right) + \nabla f_{\text{ncvx}}\left(\boldsymbol{w}^k\right)^{\mathsf{T}}\left(\boldsymbol{w} - \boldsymbol{w}^k\right) \\ &+ \frac{1}{2}\left(\boldsymbol{w} - \boldsymbol{w}^k\right)^{\mathsf{T}}\left[\nabla^2 f_{\text{ncvx}}\left(\boldsymbol{w}^k\right)\right]_{\text{PSD}}\left(\boldsymbol{w} - \boldsymbol{w}^k\right)\end{aligned}$$

- Projection onto positive semidefinite matrices: $[\boldsymbol{\Xi}]_{\text{PSD}} = \boldsymbol{U}\text{Diag}\left(\boldsymbol{\lambda}^{+}\right)\boldsymbol{U}^{\mathsf{T}}$.

Quadratic Convex Approximation of $f(\boldsymbol{w})$

$$\tilde{f}\left(\boldsymbol{w}; \boldsymbol{w}^k\right) = \frac{1}{2}\boldsymbol{w}^{\mathsf{T}}\boldsymbol{Q}^k\boldsymbol{w} + \boldsymbol{w}^{\mathsf{T}}\boldsymbol{q}^k + \text{constant},$$

where:

$$\begin{aligned}\boldsymbol{Q}^k &= \lambda_2\nabla^2\phi_2(\boldsymbol{w}) + \left[\nabla^2 f_{\text{ncvx}}\left(\boldsymbol{w}^k\right)\right]_{\text{PSD}} \\ \boldsymbol{q}^k &= -\lambda_1\nabla\phi_1(\boldsymbol{w}) + \nabla f_{\text{ncvx}}\left(\boldsymbol{w}^k\right) - \left[\nabla^2 f_{\text{ncvx}}\left(\boldsymbol{w}^k\right)\right]_{\text{PSD}}\boldsymbol{w}^k.\end{aligned}$$

SCA: Algorithm

SCA-Q-MVSK method for MVSK portfolio optimization (Zhou and Palomar 2021)

Initialization:

- Choose initial point $\mathbf{w}^0 \in \mathcal{W}$ and sequence $\{\gamma^k\}$.
- Set iteration counter $k \leftarrow 0$.

Repeat (kth iteration):

1. Calculate $\nabla f_{\text{ncvx}}\left(\mathbf{w}^k\right)$ and $\left[\nabla^2 f_{\text{ncvx}}\left(\mathbf{w}^k\right)\right]_{\text{PSD}}$.
2. Solve the QP approximation problem and keep solution as $\hat{\mathbf{w}}^{k+1}$.
3. $\mathbf{w}^{k+1} \leftarrow \mathbf{w}^k + \gamma^k(\hat{\mathbf{w}}^{k+1} - \mathbf{w}^k)$
4. $k \leftarrow k+1$

Until: convergence

MM: Preliminaries

Majorization-Minimization (MM) Method Overview (Hunter and Lange 2004; Sun, Babu, and Palomar 2017) (Palomar 2025, Appendix B)

- MM simplifies complex optimization problems through iterative surrogate minimization.
- Iteratively produces a sequence $\boldsymbol{x}^0, \boldsymbol{x}^1, \boldsymbol{x}^2, \ldots$ converging towards a solution $\boldsymbol{x}^\star$.

MM Iterative Process

- At each iteration k, MM uses a surrogate function $u(\boldsymbol{x}; \boldsymbol{x}^k)$ to approximate $f(\boldsymbol{x})$.
- The update rule is:

$$\boldsymbol{x}^{k+1} = \underset{\boldsymbol{x}\in\mathcal{X}}{\operatorname{argmin}}\; u\left(\boldsymbol{x}; \boldsymbol{x}^k\right) \qquad k = 0, 1, 2, \ldots$$

Convergence Conditions for MM: Surrogate function $u(\boldsymbol{x}; \boldsymbol{x}^k)$ must satisfy:

- *Upper bound property*: $u(\boldsymbol{x}; \boldsymbol{x}^k) \geq f(\boldsymbol{x})$.
- *Touching property*: $u(\boldsymbol{x}^k; \boldsymbol{x}^k) = f(\boldsymbol{x}^k)$.
- *Tangent property*: differentiable with $\nabla u(\boldsymbol{x}; \boldsymbol{x}^k) = \nabla f(\boldsymbol{x})$.

Role of the Surrogate Function

- Acts as a majorizer, providing an upper bound to the original function.
- The name MM stems from the process of constructing and minimizing the majorizer.

MM Framework for Nonconvex Optimization

- Convex term $f_{\text{cvx}}(\boldsymbol{w})$ remains as is.
- Construct a majorizer for nonconvex term $f_{\text{ncvx}}(\boldsymbol{w})$ around $\boldsymbol{w} = \boldsymbol{w}^k$:

$$\tilde{f}_{\text{ncvx}}\left(\boldsymbol{w}; \boldsymbol{w}^k\right) = f_{\text{ncvx}}\left(\boldsymbol{w}^k\right) + \nabla f_{\text{ncvx}}\left(\boldsymbol{w}^k\right)^{\mathsf{T}}\left(\boldsymbol{w} - \boldsymbol{w}^k\right) + \frac{\tau_{\text{MM}}}{2}\|\boldsymbol{w} - \boldsymbol{w}^k\|_2^2$$

where τ_{MM} is a positive constant ensuring $\tilde{f}_{\text{ncvx}}\left(\boldsymbol{w}; \boldsymbol{w}^k\right)$ is an upper-bound of $f_{\text{ncvx}}(\boldsymbol{w})$.

Quadratic Convex Approximation of $f(\boldsymbol{w})$

Formulated as:

$$\tilde{f}\left(\boldsymbol{w}; \boldsymbol{w}^k\right) = -\lambda_1\phi_1(\boldsymbol{w}) + \lambda_2\phi_2(\boldsymbol{w}) + \nabla f_{\text{ncvx}}\left(\boldsymbol{w}^k\right)^{\mathsf{T}}\boldsymbol{w} + \frac{\tau_{\text{MM}}}{2}\|\boldsymbol{w} - \boldsymbol{w}^k\|_2^2 + \text{constant},$$

gradient of f_{ncvx} obtained from gradients of $\phi_3(\boldsymbol{w})$ and $\phi_4(\boldsymbol{w})$.

MM: Algorithm

MM-Based Algorithm Implementation

Solve convex problems successively:

$$\underset{\boldsymbol{w}}{\text{minimize}} - \lambda_1 \phi_1(\boldsymbol{w}) + \lambda_2 \phi_2(\boldsymbol{w}) + \nabla f_{\text{ncvx}} \left(\boldsymbol{w}^k\right)^\top \boldsymbol{w} + \frac{\tau_{\text{MM}}}{2} \|\boldsymbol{w} - \boldsymbol{w}^k\|_2^2,$$

denote solution as $\text{MM}(\boldsymbol{w}^k)$, with $\boldsymbol{w}^{k+1} = \text{MM}(\boldsymbol{w}^k)$.

Acceleration Technique - SQUAREM

Instead of direct update, use two steps and combine:

$$\begin{aligned}
\text{difference first update:} \quad & \boldsymbol{r}^k = \text{MM}(\boldsymbol{w}^k) - \boldsymbol{w}^k, \\
\text{difference of differences:} \quad & \boldsymbol{v}^k = R(\text{MM}(\boldsymbol{w}^k)) - R(\boldsymbol{w}^k), \\
\text{stepsize:} \quad & \alpha^k = -\max\left(1, \|\boldsymbol{r}^k\|_2 / \|\boldsymbol{v}^k\|_2\right), \\
\text{actual step taken:} \quad & \boldsymbol{y}^k = \boldsymbol{w}^k - \alpha^k \boldsymbol{r}^k, \\
\text{final update on actual step:} \quad & \boldsymbol{w}^{k+1} = \text{MM}(\boldsymbol{y}^k).
\end{aligned}$$

Stepsize α^k can be refined for robustness and faster convergence.

MM: Algorithm

Acc-MM-L-MVSK method for MVSK portfolio optimization (Wang et al. 2023)

Initialization:

- Choose initial point $\boldsymbol{w}^0 \in \mathcal{W}$ and proper constant τ_{MM} for the majorized problem.
- Set iteration counter $k \leftarrow 0$.

Repeat (kth iteration):

1. Calculate $\nabla f_{\mathsf{ncvx}}\left(\boldsymbol{w}^k\right)$.
2. Compute the quantities $\boldsymbol{r}^k$, $\boldsymbol{v}^k$, α^k, $\boldsymbol{y}^k$, and current solution $\boldsymbol{w}^{k+1}$, which requires solving the majorized problem three times.
3. $k \leftarrow k+1$

Until: convergence

Numerical Experiments: Convergence and Computation Time

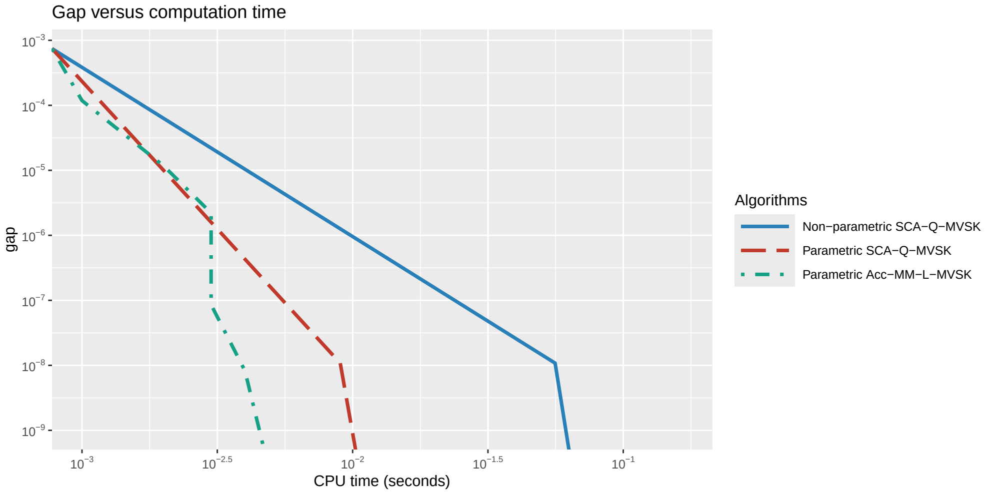
Empirical Study Summary

- Assess computational efficiency of SCA-Q-MVSK versus Acc-MM-L-MVSK methods.
- Both non-parametric and parametric calculations for gradients and Hessians were used.

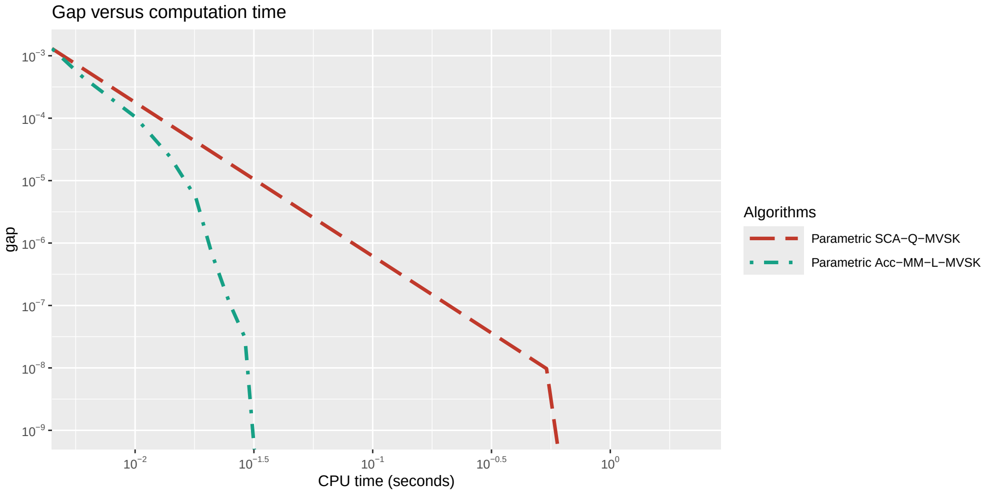
Findings

- Parametric calculations are faster and necessary for large N.
- At $N = 100$, parametric is quicker (0.5s) than non-parametric (5s).
- At $N = 400$, only parametric (1 min) is viable.
- Enhanced Acc-MM-L-MVSK method omits τ_{MM} calculation, offering the quickest convergence.

Convergence of different MVSK portfolio optimization algorithms for $N = 100$:

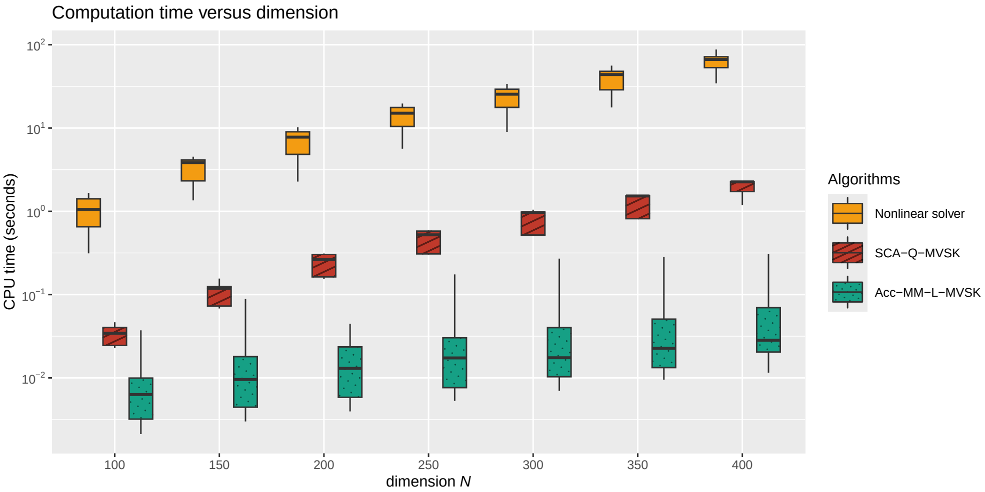


Convergence of different MVSK portfolio optimization algorithms for $N = 400$:



Numerical Experiments: Convergence and Computation Time

Computation time of different (parametric) MVSK portfolio optimization algorithms:



Numerical Experiments: Portfolio Backtest

Portfolios

- Global Maximum Return (GMRP).
- Global Minimum Variance (GMVP).
- Modified MVSK (like GMVP with $\lambda_1 = 0$).

Evaluation

- Period: 2016-2019.
- Universe: 20 S&P 500 stocks.

Metrics

- Cumulative P&L.
- Drawdown.

Results

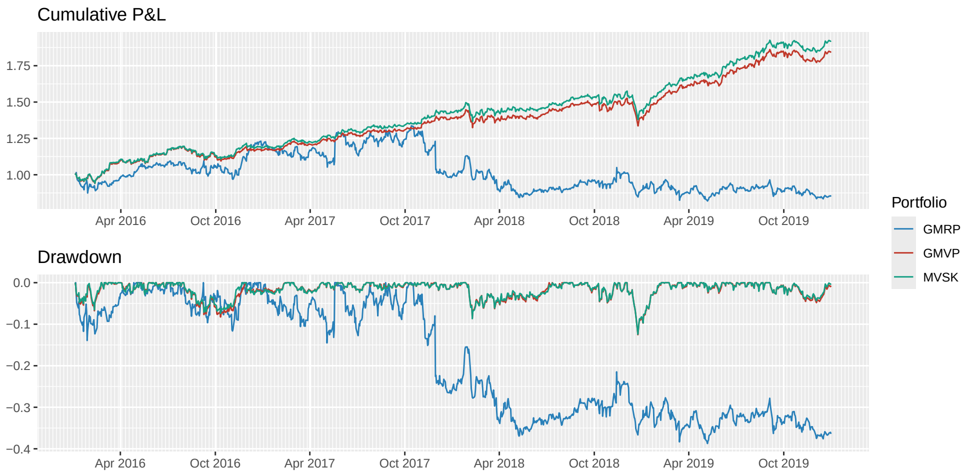
- Cumulative P&L and drawdown shown in figure and table.
- MVSK shows slight improvement over GMVP.

Insights

- Incorporating skewness and kurtosis enhances performance.
- Highlights the value of higher-order moments in optimization.

Numerical Experiments: Portfolio Backtest

Backtest of high-order portfolios: cumulative P&L and drawdown:



Backtest of high-order portfolios: performance measures:

Portfolio	Sharpe ratio	annual return	annual volatility	Sortino ratio	max drawdown	CVaR (0.95)
GMRP	-0.01	0%	27%	-0.02	39%	4%
GMVP	1.47	16%	11%	2.12	13%	2%
MVSK	1.56	17%	11%	2.26	12%	2%

Numerical Experiments: Multiple Portfolio Backtests

Multiple Randomized Backtests Overview

- Dataset: $N = 20$ stocks, 2015-2020.
- Method: 100 resamples with $N = 8$ stocks and random 2-year periods.
- Backtest: walk-forward with 1-year lookback, monthly reoptimization.

Results Presentation

- Performance measures for each portfolio across all backtests in next table.
- Boxplots of Sharpe ratio and maximum drawdown.

Key Observations

- Modest performance improvement of MVSK portfolio over GMVP.
- Highlights the potential benefits of incorporating higher-order moments in portfolio optimization strategies.

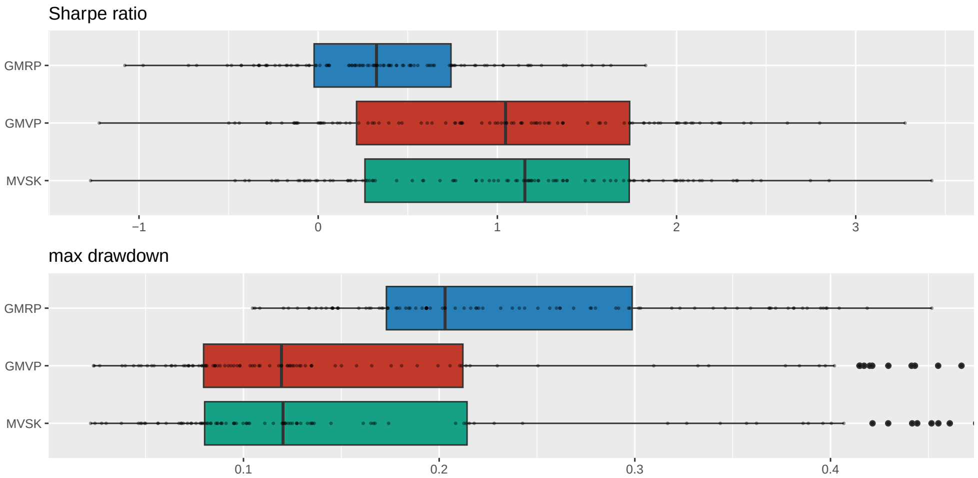
Numerical Experiments: Multiple Portfolio Backtests

Multiple randomized backtest of high-order portfolios: performance measures:

Portfolio	Sharpe ratio	annual return	annual volatility	Sortino ratio	max drawdown	CVaR (0.95)
GMRP	0.33	9%	26%	0.45	20%	4%
GMVP	1.05	13%	13%	1.48	12%	2%
MVSK	1.15	14%	13%	1.58	12%	2%

Numerical Experiments: Multiple Portfolio Backtests

Multiple randomized backtest of high-order portfolios: Sharpe ratio and maximum drawdown:



Outline

Summary

- Markowitz's portfolio, based on mean and variance, may not fully capture financial data's non-Gaussian traits, suggesting a need for higher moments.
- High-order portfolios include skewness and kurtosis to address the asymmetry and heavy tails in financial distributions.
- Initially conceptualized in the 1960s, high-order portfolios faced challenges due to the exponential increase in parameters (N^4) and nonconvex formulations, making early estimation and optimization difficult.
- Various high-order portfolio strategies exist, including MVSK portfolios and polynomial-goal formulations.
- Modern algorithms now facilitate efficient high-order portfolio management.
- Decades of research have made high-order portfolio design feasible for managing extensive asset collections, leaving the adoption of higher moments to traders' discretion.

References I

Boudt, K., W. Lu, and B. Peeters. 2014. “Higher Order Comoments of Multifactor Models and Asset Allocation.” *Finance Research Letters* 13: 225–33.

Brandt, M. W., P. Santa-Clara, and R. Valkanov. 2009. “Parametric Portfolio Policies: Exploiting Characteristics in the Cross Section of Equity Returns.” *Review of Financial Studies* 22 (9): 3411–47.

Hunter, D. R., and K. Lange. 2004. “A Tutorial on MM Algorithms.” *The American Statistician* 58: 30–37.

Jean, W. H. 1971. “The Extension of Portfolio Analysis to Three or More Parameters.” *Journal of Financial and Quantitative Analysis* 6 (1): 505–15.

Palomar, D. P. 2025. *Portfolio Optimization: Theory and Application*. Cambridge University Press.

References II

Scutari, G., F. Facchinei, P. Song, D. P. Palomar, and J.-S. Pang. 2014. "Decomposition by Partial Linearization: Parallel Optimization of Multi-Agent Systems." *IEEE Transactions on Signal Processing* 62 (3): 641–56.

Sun, Y., P. Babu, and D. P. Palomar. 2017. "Majorization-Minimization Algorithms in Signal Processing, Communications, and Machine Learning." *IEEE Transactions on Signal Processing* 65 (3): 794–816.

Wang, X., R. Zhou, J. Ying, and D. P. Palomar. 2023. "Efficient and Scalable Parametric High-Order Portfolios Design via the Skew-t Distribution." *IEEE Transactions on Signal Processing* 71 (October): 3726–40.

Young, W. E., and R. H. Trent. 1969. "Geometric Mean Approximations of Individual Security and Portfolio Performance." *Journal of Financial and Quantitative Analysis* 4 (2): 179–99.

References III

Zhou, R., and D. P. Palomar. 2021. "Solving High-Order Portfolios via Successive Convex Approximation Algorithms." *IEEE Transactions on Signal Processing* 69 (February): 892–904.