# Portfolio Optimization

## Robust Portfolios

Daniel P. Palomar (2025). *Portfolio Optimization: Theory and Application.*
Cambridge University Press.

portfoliooptimizationbook.com

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# Outline


1. Introduction
2. Robust Portfolio Optimization
   - Robust Optimization
   - Robust Worst-Case Portfolios
   - Numerical Experiments
3. Portfolio Resampling
   - Resampling Methods
   - Portfolio Resampling
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# Executive Summary

- Mean-variance portfolio optimization requires an estimation of the mean vector and covariance matrix.
- Estimation errors lead to unstable portfolio solutions that are highly sensitive to input parameters.
- This issues has been referred to as “Markowitz optimization enigma” and makes the approach less popular among practitioners.
- Two main solutions are overviewed in these slides (Palomar 2025, chap. 14):
  - **robust optimization**: takes into account parameter uncertainty
  - **resampling/bootstrapping**: aggregates solutions from resampled data

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## Introduction

**Markowitz's Mean-Variance Portfolio:** Trade-off between expected return $\boldsymbol{w}^\mathsf{T}\boldsymbol{\mu}$ and risk measured by variance $\boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w}$:

$$\begin{array}{ll} \underset{\boldsymbol{w}}{\text{maximize}} & \boldsymbol{w}^\mathsf{T}\boldsymbol{\mu} - \frac{\lambda}{2}\boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w} \\ \text{subject to} & \boldsymbol{w} \in \mathcal{W} \end{array}$$

where

- parameters $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ represent expected returns and covariance matrix
- $\lambda$ is a hyper-parameter controlling investor's risk aversion
- $\mathcal{W}$ denotes constraint set, e.g., $\mathcal{W} = \{\boldsymbol{w} \mid \mathbf{1}^\mathsf{T}\boldsymbol{w} = 1, \boldsymbol{w} \geq \mathbf{0}\}$.

**Parameter Estimation:**

- Parameters $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ are unknown and estimated using historical data $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_T$.
- Estimators range from simple sample estimators to sophisticated shrinkage heavy-tailed maximum likelihood estimators.
- Estimation error depends on the number of observations.
- Limited historical data and lack of stationarity lead to noisy estimates $\hat{\boldsymbol{\mu}}$ and $\hat{\boldsymbol{\Sigma}}$.
- Noisy estimates particularly affect $\hat{\boldsymbol{\mu}}$ (Michaud 1989; Chopra and Ziemba 1993).

# Introduction

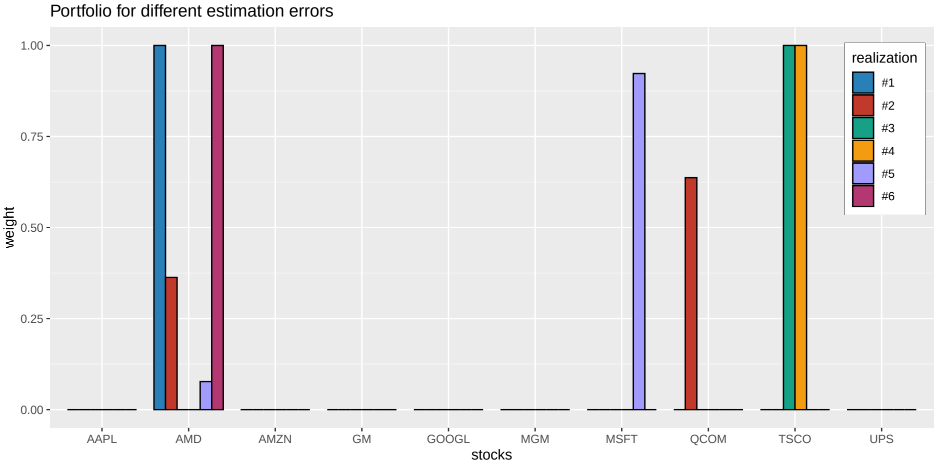
**Achilles' Heel of Portfolio Optimization:**

- Estimation noise in $\hat{\boldsymbol{\mu}}$ and $\hat{\boldsymbol{\Sigma}}$ leads to erratic portfolio designs.
- Known as "Markowitz optimization enigma" (Michaud 1989):
  - Portfolio optimization problems are "estimation-error maximizers."
  - "Optimal" portfolios are financially meaningless (absence of significant investment value).

**Sensitivity Illustration:**

- Next figure shows sensitivity of mean-variance portfolio for six different realizations of estimation error.
- Behavior is erratic due to sensitivity to parameter errors.
- Each realization differs significantly, which is impractical for portfolio allocation.

Sensitivity of the naive mean–variance portfolio:



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# Robust Optimization

**General Mathematical Optimization Problem:**
(with optimization variable $\boldsymbol{x}$ and parameter $\boldsymbol{\theta}$)

$$\begin{array}{lll} \underset{\boldsymbol{x}}{\text{minimize}} & f_0(\boldsymbol{x};\boldsymbol{\theta}) & \\ \text{subject to} & f_i(\boldsymbol{x};\boldsymbol{\theta}) \leq 0, & i = 1,\ldots,m \\ & h_i(\boldsymbol{x};\boldsymbol{\theta}) = 0, & i = 1,\ldots,p, \end{array}$$

where

- $f_0$: objective function
- $f_i,\ i = 1,\ldots,m$: inequality constraint functions
- $h_i,\ i = 1,\ldots,p$: equality constraint functions
- $\boldsymbol{\theta}$: external parameter (e.g., $\boldsymbol{\theta} = (\boldsymbol{\mu}, \boldsymbol{\Sigma})$ in mean-variance portfolio)
- solution denoted by $\boldsymbol{x}^\star(\boldsymbol{\theta})$

# Robust Optimization

**Parameter Estimation:**

- $\boldsymbol{\theta}$ is unknown and estimated as $\hat{\boldsymbol{\theta}}$.
- Solution using estimated parameter: $\boldsymbol{x}^\star(\hat{\boldsymbol{\theta}})$.
- Different from desired solution: $\boldsymbol{x}^\star(\hat{\boldsymbol{\theta}}) \neq \boldsymbol{x}^\star(\boldsymbol{\theta})$.
- Question of approximate equality: $\boldsymbol{x}^\star(\hat{\boldsymbol{\theta}}) \approx \boldsymbol{x}^\star(\boldsymbol{\theta})$.
- For mean-variance portfolio, solutions can be quite different as previously seen.

**Making the Problem Robust to Parameter Errors:**

- **Stochastic optimization:**
  - Relies on probabilistic modeling of the parameter (Prekopa 1995; Ruszczynski and Shapiro 2003; Birge and Louveaux 2011).
  - Includes: expectations (average behavior) and chance constraints (probabilistic constraints).
- **Worst-case robust optimization:**
  - Relies on defining an uncertainty set for the parameter (Ben-Tal, El Ghaoui, and Nemirovski 2009; Ben-Tal and Nemirovski 2008; D. Bertsimas and Caramanis 2011; Lobo 2000).

**Stochastic Robust Optimization:**

- Estimated parameter $\hat{\boldsymbol{\theta}}$ modeled as a random variable fluctuating around true value $\boldsymbol{\theta}$.
- True value modeled as:

$$\boldsymbol{\theta} = \hat{\boldsymbol{\theta}} + \boldsymbol{\delta},$$

  where $\boldsymbol{\delta}$ is the estimation error, a zero-mean random variable (e.g., Gaussian distribution).
- Importance of choosing the correct covariance matrix (or shape) for the error term.

**Average Constraint:**

- Instead of using the "naive constraint":

$$f(\boldsymbol{x}; \hat{\boldsymbol{\theta}}) \leq \alpha$$

  use the "average constraint":

$$\mathbb{E}_{\boldsymbol{\theta}}\left[f(\boldsymbol{x}; \boldsymbol{\theta})\right] \leq \alpha.$$

- Interpretation: constraint satisfied on average, a relaxation of the true constraint.
- Preserves convexity: if $f(\cdot; \boldsymbol{\theta})$ is convex for each $\boldsymbol{\theta}$, so is its expected value over $\boldsymbol{\theta}$.

# Stochastic Optimization: Implementation

- **Brute-Force Sampling:** Sample $S$ times the random variable $\boldsymbol{\theta}$ and use the constraint:

$$\frac{1}{S}\sum_{i=1}^{S} f(\boldsymbol{x};\boldsymbol{\theta}_i) \leq \alpha.$$

- **Adaptive Sampling:** Sample the random variable $\boldsymbol{\theta}$ efficiently at each iteration while solving the problem.

- **Closed-Form Expression:** Compute the expected value in closed form when possible.

- **Stochastic Programming:** Various numerical methods developed for stochastic optimization (Prekopa 1995; Ruszczynski and Shapiro 2003; Birge and Louveaux 2011).

# Example: Stochastic Average Constraint in Closed Form

- Quadratic constraint $f(\boldsymbol{x};\boldsymbol{\theta}) = (\boldsymbol{c}^\mathsf{T}\boldsymbol{x})^2$, with parameter $\boldsymbol{\theta} = \boldsymbol{c}$.
- Modeled as $\boldsymbol{c} = \hat{\boldsymbol{c}} + \boldsymbol{\delta}$, where $\boldsymbol{\delta}$ is zero-mean with covariance matrix $\boldsymbol{Q}$.
- Expected value:

$$\begin{aligned}
\mathbb{E}_{\boldsymbol{\theta}}\left[f(\boldsymbol{x};\boldsymbol{\theta})\right] &= \mathbb{E}_{\delta}\left[\left((\hat{\boldsymbol{c}} + \delta)^\mathsf{T}\boldsymbol{x}\right)^2\right] \\
&= \mathbb{E}_{\delta}\left[(\hat{\boldsymbol{c}}^\mathsf{T}\boldsymbol{x})^2 + \boldsymbol{x}^\mathsf{T}\delta\boldsymbol{\delta}^\mathsf{T}\boldsymbol{x}\right] \\
&= (\hat{\boldsymbol{c}}^\mathsf{T}\boldsymbol{x})^2 + \boldsymbol{x}^\mathsf{T}\boldsymbol{Q}\boldsymbol{x}
\end{aligned}$$

- Interpretation: the robust expected value has the form of the naive version plus a quadratic regularization term.

# Stochastic Optimization

**Limitations of Average Constraints:**

- No control over specific realizations of the estimation error.
- Constraint may be violated in many instances.
- Worst-case approach addresses this but can be too conservative.

**Chance Constraints:**

- Compromise between being too relaxed or overly conservative.
- Focus on satisfying the constraint with high probability (e.g., 95%).
- Replace naive constraint with:

$$\Pr\left[f(\boldsymbol{x};\boldsymbol{\theta}) \leq \alpha\right] \geq \epsilon,$$

  where $\epsilon$ is the confidence level (e.g., $\epsilon = 0.95$ for 95%).
- Generally hard to deal with, often requiring approximations (Ben-Tal and Nemirovski 2008; Ben-Tal, El Ghaoui, and Nemirovski 2009).

**Worst-Case Robust Optimization:**

- Parameter $\boldsymbol{\theta}$ assumed to lie within an uncertainty region near the estimated value:

$$\boldsymbol{\theta} \in \mathcal{U}_{\boldsymbol{\theta}},$$

  where $\mathcal{U}_{\boldsymbol{\theta}}$ is the uncertainty set centered at $\hat{\boldsymbol{\theta}}$.
- Critical to choose the shape and size of the uncertainty set.

**Typical Choices for Uncertainty Set Shape (Size $\epsilon$):**

- **Spherical set:**

$$\mathcal{U}_{\boldsymbol{\theta}} = \left\{ \boldsymbol{\theta} \mid \|\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}\|_2 \leq \epsilon \right\}$$

- **Box set:**

$$\mathcal{U}_{\boldsymbol{\theta}} = \left\{ \boldsymbol{\theta} \mid \|\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}\|_\infty \leq \epsilon \right\}$$

- **Ellipsoidal set:**

$$\mathcal{U}_{\boldsymbol{\theta}} = \left\{ \boldsymbol{\theta} \mid \left(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}\right)^\mathsf{T} \boldsymbol{S}^{-1} \left(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}\right) \leq \epsilon^2 \right\},$$

  where $\boldsymbol{S} \succ \mathbf{0}$ defines the shape of the ellipsoid.

# Worst-Case Robust Optimization

**Worst-Case Constraint:**

- Instead of using:

$$f(\boldsymbol{x}; \hat{\boldsymbol{\theta}}) \leq \alpha,$$

  use the worst-case constraint:

$$\sup_{\boldsymbol{\theta} \in \mathcal{U}_{\hat{\boldsymbol{\theta}}}} f(\boldsymbol{x}; \boldsymbol{\theta}) \leq \alpha.$$

- Interpretation: constraint satisfied for any point inside the uncertainty set, a conservative approach.
- Preserves convexity: if $f(\cdot; \boldsymbol{\theta})$ is convex for each $\boldsymbol{\theta}$, so is its worst-case over $\boldsymbol{\theta}$.

**Variation: Distributionally Robust Optimization:**

- Applies worst-case uncertainty philosophy to probability distributions.
- Referred to as distributional uncertainty models or distributionally robust optimization (D. Bertsimas and Caramanis 2011).

# Worst-Case Robust Optimization: Implementation

- **Brute-Force Sampling:**
  - Sample $S$ times the uncertainty set $\mathcal{U}_{\boldsymbol{\theta}}$ and use the constraint:

$$\max_{i=1,\dots,S} f(\boldsymbol{x};\boldsymbol{\theta}_i) \leq \alpha$$

  - Or equivalently, include $S$ constraints:

$$f(\boldsymbol{x};\boldsymbol{\theta}_i) \leq \alpha, \qquad i = 1,\dots,S;$$

- **Adaptive Sampling Algorithms:** Sample the uncertainty set $\mathcal{U}_{\hat{\boldsymbol{\theta}}}$ efficiently at each iteration while solving the problem.
- **Closed-Form Expression:** Compute the supremum in closed form when possible.
- **Via Lagrange Duality:** Rewrite the worst-case supremum as an infimum that can be combined with the outer portfolio optimization layer.
- **Saddle-Point Optimization:** Use numerical methods designed for minimax problems or related saddle-point problems (Bertsekas 1999; Tütüncü and Koenig 2004).

# Example: Worst-Case Constraint in Closed Form

- Quadratic constraint $f(\boldsymbol{x};\boldsymbol{\theta}) = (\boldsymbol{c}^\mathsf{T}\boldsymbol{x})^2$, with parameter $\boldsymbol{\theta} = \boldsymbol{c}$.
- Belongs to a spherical uncertainty set:

$$\mathcal{U} = \{\boldsymbol{c} \mid \|\boldsymbol{c} - \hat{\boldsymbol{c}}\|_2 \leq \epsilon\}.$$

- Worst-case value:

$$\begin{aligned}\sup_{\boldsymbol{c}\in\mathcal{U}} |\boldsymbol{c}^\mathsf{T}\boldsymbol{x}| &= \sup_{\|\boldsymbol{\delta}\|\leq\epsilon} |(\hat{\boldsymbol{c}} + \boldsymbol{\delta})^\mathsf{T}\boldsymbol{x}| \\ &= |\hat{\boldsymbol{c}}^\mathsf{T}\boldsymbol{x}| + \sup_{\|\boldsymbol{\delta}\|\leq\epsilon} |\boldsymbol{\delta}^\mathsf{T}\boldsymbol{x}| \\ &= |\hat{\boldsymbol{c}}^\mathsf{T}\boldsymbol{x}| + \epsilon\|\boldsymbol{x}\|_2,\end{aligned}$$

  where we have used the triangle inequality and Cauchy-Schwarz's inequality (upper bound achieved by $\boldsymbol{\delta} = \epsilon\, \boldsymbol{x}/\|\boldsymbol{x}\|_2$).
- Interpretation: the worst-case expression has the form of the naive version plus a regularization term.

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# Worst-Case Mean Vector $\boldsymbol{\mu}$

**Global Maximum Return Portfolio (GMRP):** For an estimated mean vector $\hat{\boldsymbol{\mu}}$:

$$\begin{array}{ll} \underset{\boldsymbol{w}}{\text{maximize}} & \boldsymbol{w}^\mathsf{T}\hat{\boldsymbol{\mu}} \\ \text{subject to} & \mathbf{1}^\mathsf{T}\boldsymbol{w} = 1, \quad \boldsymbol{w} \geq \mathbf{0} \end{array}$$

- Known to be highly sensitive to estimation errors.

**Worst-Case Formulation:** Instead of the naive GMRP, use the worst-case formulation:

$$\begin{array}{ll} \underset{\boldsymbol{w}}{\text{maximize}} & \inf_{\boldsymbol{\mu}\in\mathcal{U}_{\boldsymbol{\mu}}} \boldsymbol{w}^\mathsf{T}\boldsymbol{\mu} \\ \text{subject to} & \mathbf{1}^\mathsf{T}\boldsymbol{w} = 1, \quad \boldsymbol{w} \geq \mathbf{0} \end{array}$$

- Typical choices for the uncertainty region $\mathcal{U}_{\boldsymbol{\mu}}$:
  - **Ellipsoidal set:**
  $$\mathcal{U}_{\boldsymbol{\mu}} = \left\{\boldsymbol{\mu} \mid (\boldsymbol{\mu} - \hat{\boldsymbol{\mu}})^\mathsf{T}\boldsymbol{S}^{-1}(\boldsymbol{\mu} - \hat{\boldsymbol{\mu}}) \leq \epsilon^2\right\}.$$
  - **Box set:**
  $$\mathcal{U}_{\boldsymbol{\mu}} = \{\boldsymbol{\mu} \mid \|\boldsymbol{\mu} - \hat{\boldsymbol{\mu}}\|_\infty \leq \epsilon\}.$$

# Worst-Case Mean Vector $\boldsymbol{\mu}$

## Worst-Case Mean Vector under Ellipsoidal Uncertainty Set

- **Ellipsoidal uncertainty set for $\mu$:**

$$\mathcal{U}_{\boldsymbol{\mu}} = \left\{ \boldsymbol{\mu} = \hat{\boldsymbol{\mu}} + \kappa \boldsymbol{S}^{1/2}\boldsymbol{u} \mid \|\boldsymbol{u}\|_2 \leq 1 \right\},$$

  where:
  - $\boldsymbol{S}^{1/2}$ is the symmetric square-root matrix of the shape $\boldsymbol{S}$, e.g., $\boldsymbol{S} = (1/T)\boldsymbol{\Sigma}$;
  - $\kappa$ determines the size of the ellipsoid.
- **Worst-case value of $\boldsymbol{w}^\mathsf{T}\mu$:**

$$\begin{aligned}
\inf_{\boldsymbol{\mu} \in \mathcal{U}_{\boldsymbol{\mu}}} \boldsymbol{w}^\mathsf{T}\boldsymbol{\mu} &= \inf_{\|\boldsymbol{u}\| \leq 1} \boldsymbol{w}^\mathsf{T}(\hat{\boldsymbol{\mu}} + \kappa \boldsymbol{S}^{1/2}\boldsymbol{u}) \\
&= \boldsymbol{w}^\mathsf{T}\hat{\boldsymbol{\mu}} + \kappa \inf_{\|\boldsymbol{u}\| \leq 1} \boldsymbol{w}^\mathsf{T}\boldsymbol{S}^{1/2}\boldsymbol{u} \\
&= \boldsymbol{w}^\mathsf{T}\hat{\boldsymbol{\mu}} - \kappa \left\|\boldsymbol{S}^{1/2}\boldsymbol{w}\right\|_2.
\end{aligned}$$

# Worst-Case Mean Vector $\boldsymbol{\mu}$

## Worst-Case Mean Vector under Box Uncertainty Set

- **Box uncertainty set for $\boldsymbol{\mu}$:**

$$\mathcal{U}_{\boldsymbol{\mu}} = \left\{\boldsymbol{\mu} \mid -\boldsymbol{\delta} \leq \boldsymbol{\mu} - \hat{\boldsymbol{\mu}} \leq \boldsymbol{\delta}\right\},$$

  where $\boldsymbol{\delta}$ is the half-width of the box in all dimensions.

- **Worst-case value of $\boldsymbol{w}^\mathsf{T}\boldsymbol{\mu}$:**

$$\inf_{\boldsymbol{\mu}\in\mathcal{U}_{\boldsymbol{\mu}}} \boldsymbol{w}^\mathsf{T}\boldsymbol{\mu} = \boldsymbol{w}^\mathsf{T}\hat{\boldsymbol{\mu}} - |\boldsymbol{w}|^\mathsf{T}\boldsymbol{\delta}.$$

# Worst-Case Mean Vector $\boldsymbol{\mu}$

**Robust GMRP with Ellipsoidal Uncertainty Set:**

- Robust version of the GMRP under ellipsoidal uncertainty set:

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{maximize}} & \mathbf{w}^T \hat{\boldsymbol{\mu}} - \kappa \left\| \mathbf{S}^{1/2} \mathbf{w} \right\|_2 \\ \text{subject to} & \mathbf{1}^T \mathbf{w} = 1, \quad \mathbf{w} \geq \mathbf{0} \end{array}$$

- Still a convex problem.
- Complexity increased to a second-order cone program (from a simple linear program in the naive formulation).

**Robust GMRP with Box Uncertainty Set:**

- Robust version of the GMRP under box uncertainty set:

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{maximize}} & \mathbf{w}^T \hat{\boldsymbol{\mu}} - |\mathbf{w}|^T \boldsymbol{\delta} \\ \text{subject to} & \mathbf{1}^T \mathbf{w} = 1, \quad \mathbf{w} \geq \mathbf{0} \end{array}$$

- Still a convex problem: can be rewritten as a linear program.
- Under constraints $\mathbf{1}^T \mathbf{w} = 1$ and $\mathbf{w} \geq \mathbf{0}$, the problem reduces to a naive GMRP where $\hat{\boldsymbol{\mu}}$ is replaced by $\hat{\boldsymbol{\mu}} - \boldsymbol{\delta}$.

# Quintile Portfolio as a Robust Portfolio

**Other Uncertainty Sets:**

- Various uncertainty sets can be considered, such as the $\ell_1$-norm ball.

**Example: Quintile Portfolio as a Robust Portfolio**

- **Quintile portfolio:**
  - A heuristic portfolio widely used by practitioners.
  - Selects the top fifth of the assets (or a different fraction) and equally allocates capital among them.
  - Common-sense heuristic portfolio.
- **Formal derivation as a robust portfolio:**
  - The quintile portfolio can be shown to be the optimal solution to the worst-case GMRP with an $\ell_1$-norm ball uncertainty set around the estimated mean vector (Zhou and Palomar 2020):

$$\mathcal{U}_{\boldsymbol{\mu}} = \left\{\hat{\boldsymbol{\mu}} + \boldsymbol{e} \mid \|\boldsymbol{e}\|_1 \leq \epsilon\right\}.$$

# Worst-Case Covariance Matrix $\boldsymbol{\Sigma}$

**Global Minimum Variance Portfolio (GMVP):** For an estimated covariance matrix $\hat{\boldsymbol{\Sigma}}$:

$$\begin{array}{ll} \underset{\boldsymbol{w}}{\text{minimize}} & \boldsymbol{w}^{\mathsf{T}}\hat{\boldsymbol{\Sigma}}\boldsymbol{w} \\ \text{subject to} & \mathbf{1}^{\mathsf{T}}\boldsymbol{w} = 1, \quad \boldsymbol{w} \geq \mathbf{0} \end{array}$$

- Sensitive to estimation errors (though less so than errors in $\boldsymbol{\mu}$).

**Worst-Case Formulation:** Instead of the naive GMVP, use the worst-case formulation:

$$\begin{array}{ll} \underset{\boldsymbol{w}}{\text{minimize}} & \sup_{\boldsymbol{\Sigma}\in\mathcal{U}_{\boldsymbol{\Sigma}}} \boldsymbol{w}^{\mathsf{T}}\boldsymbol{\Sigma}\boldsymbol{w} \\ \text{subject to} & \mathbf{1}^{\mathsf{T}}\boldsymbol{w} = 1, \quad \boldsymbol{w} \geq \mathbf{0} \end{array}$$

- Typical choices for the uncertainty region $\mathcal{U}_{\boldsymbol{\Sigma}}$:
  - **Spherical set:**
$$\mathcal{U}_{\boldsymbol{\Sigma}} = \left\{\boldsymbol{\Sigma} \mid \|\boldsymbol{\Sigma} - \hat{\boldsymbol{\Sigma}}\|_F \leq \epsilon\right\}$$
  - **Ellipsoidal set:**
$$\mathcal{U}_{\boldsymbol{\Sigma}} = \left\{\boldsymbol{\Sigma} \mid \text{vec}(\boldsymbol{\Sigma} - \hat{\boldsymbol{\Sigma}})^{\mathsf{T}}\boldsymbol{S}^{-1}\text{vec}(\boldsymbol{\Sigma} - \hat{\boldsymbol{\Sigma}}) \leq \epsilon^2\right\}$$
  - **Box set:**
$$\mathcal{U}_{\boldsymbol{\Sigma}} = \left\{\boldsymbol{\Sigma} \mid \|\boldsymbol{\Sigma} - \hat{\boldsymbol{\Sigma}}\|_\infty \leq \epsilon\right\}$$

# Worst-Case Covariance Matrix $\Sigma$

## Worst-Case Covariance Matrix under a Data Spherical Uncertainty Set

- **Spherical uncertainty set for data matrix $\boldsymbol{X}$:**

$$\mathcal{U}_{\boldsymbol{X}} = \left\{ \boldsymbol{X} \mid \|\boldsymbol{X} - \hat{\boldsymbol{X}}\|_{\mathsf{F}} \leq \epsilon \right\},$$

  where $\epsilon$ determines the size of the sphere.

- **Worst-case value of $\sqrt{\boldsymbol{w}^{\mathsf{T}}\boldsymbol{\Sigma}\boldsymbol{w}}$:** (assuming $\hat{\boldsymbol{\Sigma}} = \frac{1}{T}\hat{\boldsymbol{X}}^{\mathsf{T}}\hat{\boldsymbol{X}}$)

$$\begin{aligned}
\sup_{\boldsymbol{X}\in\mathcal{U}_{\boldsymbol{X}}} \sqrt{\boldsymbol{w}^{\mathsf{T}}\left(\frac{1}{T}\boldsymbol{X}^{\mathsf{T}}\boldsymbol{X}\right)\boldsymbol{w}} &= \sup_{\|\boldsymbol{\Delta}\|_{\mathsf{F}}\leq\epsilon} \frac{1}{\sqrt{T}} \left\|(\hat{\boldsymbol{X}} + \boldsymbol{\Delta})\boldsymbol{w}\right\|_2 \\
&= \frac{1}{\sqrt{T}} \left\|\hat{\boldsymbol{X}}\boldsymbol{w}\right\|_2 + \sup_{\|\boldsymbol{\Delta}\|_{\mathsf{F}}\leq\epsilon} \frac{1}{\sqrt{T}} \|\boldsymbol{\Delta}\boldsymbol{w}\|_2 \\
&= \frac{1}{\sqrt{T}} \left\|\hat{\boldsymbol{X}}\boldsymbol{w}\right\|_2 + \frac{1}{\sqrt{T}}\epsilon\|\boldsymbol{w}\|_2.
\end{aligned}$$

# Worst-Case Covariance Matrix $\boldsymbol{\Sigma}$

## Worst-Case Covariance Matrix under an Ellipsoidal Uncertainty Set

- **Ellipsoidal uncertainty set for $\boldsymbol{\Sigma}$:**

$$\mathcal{U}_{\boldsymbol{\Sigma}} = \left\{\boldsymbol{\Sigma} \succeq \mathbf{0} \mid \left(\mathrm{vec}(\boldsymbol{\Sigma}) - \mathrm{vec}(\hat{\boldsymbol{\Sigma}})\right)^{\mathsf{T}} \boldsymbol{S}^{-1}\left(\mathrm{vec}(\boldsymbol{\Sigma}) - \mathrm{vec}(\hat{\boldsymbol{\Sigma}})\right) \leq \epsilon^2\right\},$$

  where:
  - $\mathrm{vec}(\cdot)$ denotes the vec operator that stacks the matrix argument into a vector.
  - matrix $\boldsymbol{S}$ gives the shape of the ellipsoid.
  - $\epsilon$ determines the size.
- **Worst-case value of $\boldsymbol{w}^{\mathsf{T}}\boldsymbol{\Sigma}\boldsymbol{w}$:** Given by the (convex) SDP (Palomar 2025, chap. 14):

$$\begin{array}{ll} \underset{\boldsymbol{Z}}{\text{minimize}} & \mathrm{Tr}\left(\hat{\boldsymbol{\Sigma}}\left(\boldsymbol{w}\boldsymbol{w}^{\mathsf{T}} + \boldsymbol{Z}\right)\right) + \epsilon \left\|\boldsymbol{S}^{1/2}\left(\mathrm{vec}(\boldsymbol{w}\boldsymbol{w}^{\mathsf{T}}) + \mathrm{vec}(\boldsymbol{Z})\right)\right\|_2 \\ \text{subject to} & \boldsymbol{Z} \succeq \mathbf{0}. \end{array}$$

# Worst-Case Covariance Matrix $\boldsymbol{\Sigma}$

## Worst-Case Covariance Matrix under a Box Uncertainty Set

- **Box uncertainty set for $\boldsymbol{\Sigma}$:**

$$\mathcal{U}_{\boldsymbol{\Sigma}} = \left\{ \boldsymbol{\Sigma} \succeq \mathbf{0} \mid \underline{\boldsymbol{\Sigma}} \leq \boldsymbol{\Sigma} \leq \overline{\boldsymbol{\Sigma}} \right\},$$

  where $\underline{\boldsymbol{\Sigma}}$ and $\overline{\boldsymbol{\Sigma}}$ denote the elementwise lower and upper bounds, respectively.
- **Worst-case value of $\boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w}$:** Given by the (convex) SDP (Lobo 2000):

$$\begin{array}{ll}
\underset{\overline{\boldsymbol{\Lambda}}, \underline{\boldsymbol{\Lambda}}}{\text{minimize}} & \mathsf{Tr}(\overline{\boldsymbol{\Lambda}}\,\overline{\boldsymbol{\Sigma}}) - \mathsf{Tr}(\underline{\boldsymbol{\Lambda}}\,\underline{\boldsymbol{\Sigma}})\\
\text{subject to} & \begin{bmatrix} \overline{\boldsymbol{\Lambda}} - \underline{\boldsymbol{\Lambda}} & \boldsymbol{w}\\ \boldsymbol{w}^\mathsf{T} & 1 \end{bmatrix} \succeq \mathbf{0}\\
& \overline{\boldsymbol{\Lambda}} \geq \mathbf{0}, \quad \underline{\boldsymbol{\Lambda}} \geq \mathbf{0}.
\end{array}$$

# Worst-Case Covariance Matrix $\boldsymbol{\Sigma}$

**Robust GMVP with Spherical Uncertainty Set:**

- Robust version of the GMVP under a spherical uncertainty set for the data matrix:

$$\begin{array}{ll} \underset{\boldsymbol{w}}{\text{minimize}} & \left(\left\|\hat{\boldsymbol{X}}\boldsymbol{w}\right\|_2 + \epsilon\|\boldsymbol{w}\|_2\right)^2 \\ \text{subject to} & \mathbf{1}^\mathsf{T}\boldsymbol{w} = 1, \quad \boldsymbol{w} \geq \mathbf{0}. \end{array}$$

- Still a convex problem.
- Complexity increased to a second-order cone program (from a simple QP).

**Tikhonov Regularization:**

- Heuristic similar to the robust GMVP:

$$\begin{array}{ll} \underset{\boldsymbol{w}}{\text{minimize}} & \left\|\hat{\boldsymbol{X}}\boldsymbol{w}\right\|_2^2 + \epsilon\|\boldsymbol{w}\|_2^2 \\ \text{subject to} & \mathbf{1}^\mathsf{T}\boldsymbol{w} = 1, \quad \boldsymbol{w} \geq \mathbf{0}. \end{array}$$

- Objective function can be rewritten as:

$$\boldsymbol{w}^\mathsf{T}\frac{1}{T}\left(\hat{\boldsymbol{X}}^\mathsf{T}\hat{\boldsymbol{X}} + \epsilon\boldsymbol{I}\right)\boldsymbol{w}.$$

- Leads to a regularized sample covariance matrix.

# Worst-Case Covariance Matrix $\Sigma$

**Robust GMVP with Ellipsoidal Uncertainty Set:**

- Robust version of the GMVP under an ellipsoidal uncertainty set for the covariance matrix:

$$
\begin{array}{ll}
\underset{\boldsymbol{w},\boldsymbol{W},\boldsymbol{Z}}{\text{minimize}} & \mathrm{Tr}\left(\hat{\Sigma}\left(\boldsymbol{W}+\boldsymbol{Z}\right)\right)+\epsilon\left\|\boldsymbol{S}^{1/2}\left(\mathrm{vec}(\boldsymbol{W})+\mathrm{vec}(\boldsymbol{Z})\right)\right\|_2 \\
\text{subject to} & \begin{bmatrix}\boldsymbol{W} & \boldsymbol{w}\\ \boldsymbol{w}^\mathsf{T} & 1\end{bmatrix}\succeq\mathbf{0} \\
& \mathbf{1}^\mathsf{T}\boldsymbol{w}=1,\quad \boldsymbol{w}\geq\mathbf{0} \\
& \boldsymbol{Z}\succeq\mathbf{0}.
\end{array}
$$

- Still a convex problem.
- Complexity increased to a semidefinite program (from a simple QP in the naive formulation).
- Note: the first matrix inequality is equivalent to $\boldsymbol{W}\succeq\boldsymbol{w}\boldsymbol{w}^\mathsf{T}$ and, at an optimal point, it can be shown to be satisfied with equality $\boldsymbol{W}=\boldsymbol{w}\boldsymbol{w}^\mathsf{T}$.

# Worst-Case Mean Vector $\boldsymbol{\mu}$ and Covariance Matrix $\boldsymbol{\Sigma}$

**Combined Worst-Case Mean-Variance Portfolio:**

- Combines uncertainty in the mean vector $\boldsymbol{\mu}$ and the covariance matrix $\boldsymbol{\Sigma}$ under the mean-variance portfolio formulation.
- For illustration, consider the mean-variance worst-case portfolio formulation under box uncertainty sets for $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$.

**Formulation:**

$$\begin{array}{ll} \underset{\boldsymbol{w},\overline{\boldsymbol{\Lambda}},\underline{\boldsymbol{\Lambda}}}{\text{maximize}} & \boldsymbol{w}^\mathsf{T}\hat{\boldsymbol{\mu}} - |\boldsymbol{w}|^\mathsf{T}\boldsymbol{\delta} - \frac{\lambda}{2}\left(\mathsf{Tr}(\overline{\boldsymbol{\Lambda}}\,\overline{\boldsymbol{\Sigma}}) - \mathsf{Tr}(\underline{\boldsymbol{\Lambda}}\,\underline{\boldsymbol{\Sigma}})\right) \\ \text{subject to} & \mathbf{1}^\mathsf{T}\boldsymbol{w} = 1, \quad \boldsymbol{w} \geq \mathbf{0}, \\ & \begin{bmatrix} \overline{\boldsymbol{\Lambda}} - \underline{\boldsymbol{\Lambda}} & \boldsymbol{w} \\ \boldsymbol{w}^\mathsf{T} & 1 \end{bmatrix} \succeq \mathbf{0} \\ & \overline{\boldsymbol{\Lambda}} \geq \mathbf{0}, \quad \underline{\boldsymbol{\Lambda}} \geq \mathbf{0}. \end{array}$$

- This is a convex semidefinite problem (SDP).

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# Numerical Experiments

**Effectiveness of Robust Portfolio Formulations:**

- Depends on the shape and size of the uncertainty region.
- Parameters must be properly chosen and tuned to the type of data and nature of the financial market.
- Risk of overfitting the model to the training data; extra care is needed.

**Goal of Robust Design:**

- Aim to make the solution more stable and less sensitive to error realization.
- Not necessarily to improve performance.
- Focus on gaining robustness rather than achieving the best performance in a given backtest compared to a naive design.

**Evaluation of Robust Portfolios Under Errors in Mean Vector $\boldsymbol{\mu}$:**

- We next consider robustness against errors in mean vector $\boldsymbol{\mu}$, which is paramount.
- Robustness for the covariance matrix $\boldsymbol{\Sigma}$ is less critical, but should be also assessed.

# Numerical Experiments

**Sensitivity Analysis:**

- Sensitivity of a robust portfolio under an ellipsoidal uncertainty set for the mean vector $\boldsymbol{\mu}$ over six different realizations of the estimation error.
- Compared to naive portfolio design, it is more stable and less sensitive.
- Similar results can be observed with other variations in the robust formulation.

**Performance Assessment:**

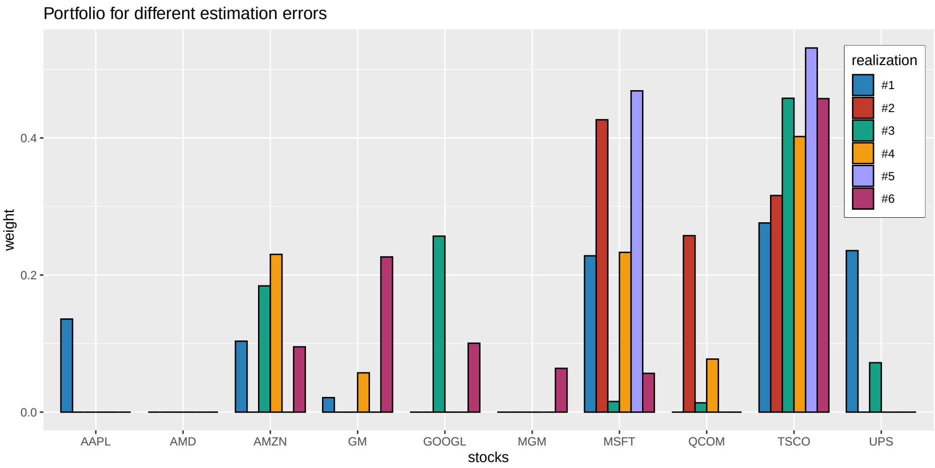
- Assess performance of mean-variance robust designs (with box and ellipsoidal uncertainty sets for the mean vector $\boldsymbol{\mu}$) compared to the naive design.
- Backtests conducted for 50 randomly chosen stocks from the S&P 500 during 2017-2020.

**Empirical Distribution Analysis:**

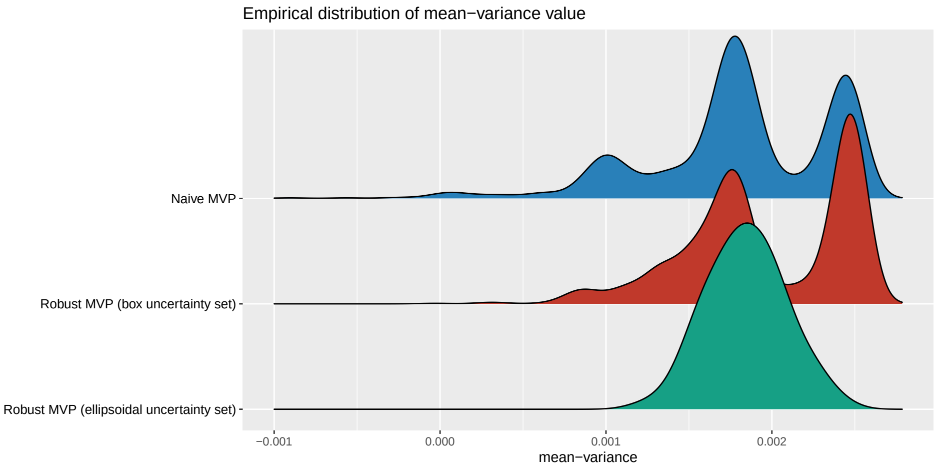
- Empirical distribution of the achieved mean-variance objective and the Sharpe ratio, calculated over 1,000 Monte Carlo noisy observations.
- Robust designs avoid the worst-case realizations (left tail in the distributions) at the expense of not achieving the best-case realizations (right tail).
- Robust designs are more stable and robust as expected.

# Numerical Experiments

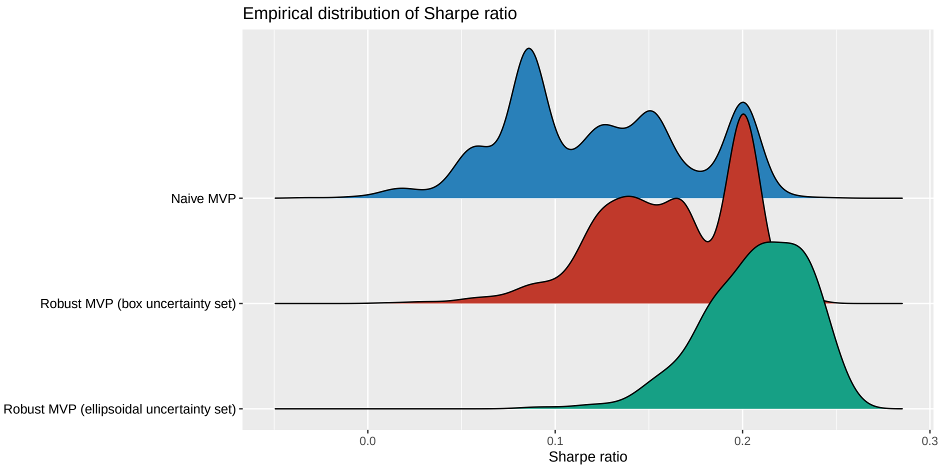
Sensitivity of the robust mean–variance portfolio under an ellipsoidal uncertainty set for $\boldsymbol{\mu}$:



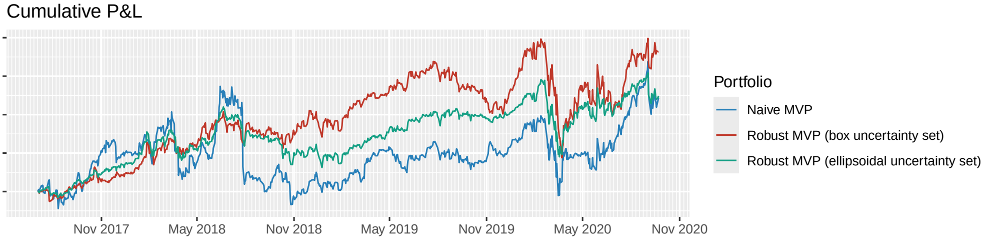
Empirical performance distribution of naive versus robust mean–variance portfolios:

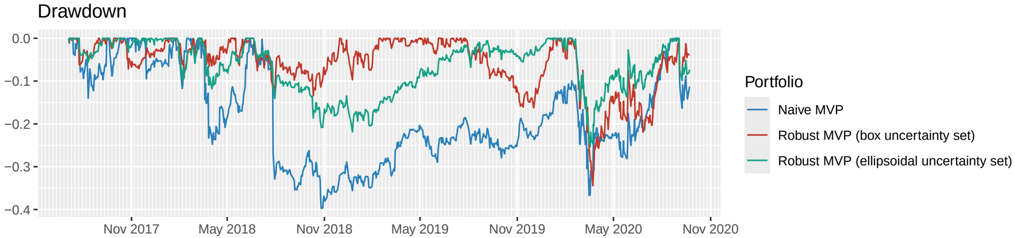


Empirical performance distribution of naive versus robust mean–variance portfolios:



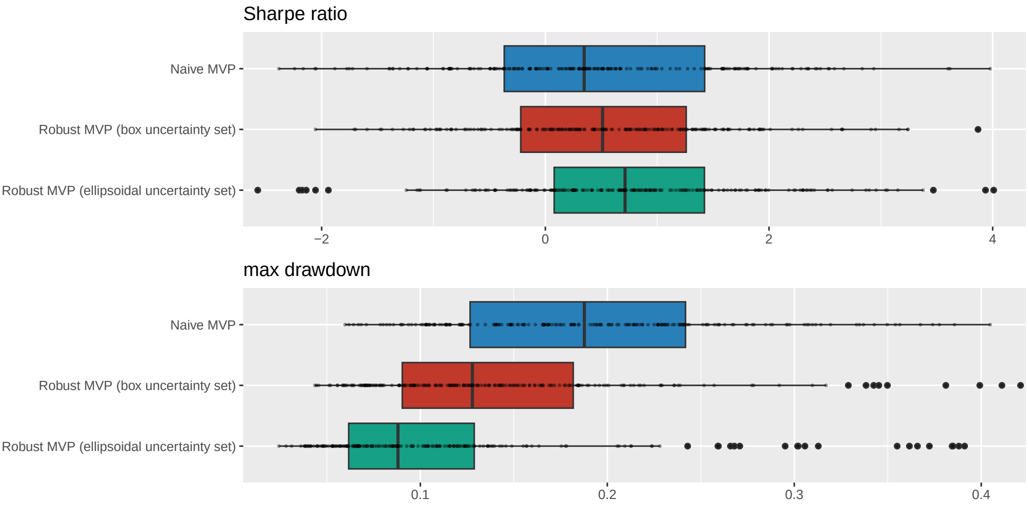
Backtest of naive versus robust mean–variance portfolios:



# Numerical Experiments

Multiple backtests of naive versus resampled mean–variance portfolios:



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# What Are Resampling Methods?

**Importance of Estimating Parameter Accuracy:**

- Estimating a parameter $\boldsymbol{\theta}$ is of little use without knowing the accuracy of the estimate.
- Confidence intervals are key in statistical inference, allowing localization of the true parameter with a certain confidence level (e.g., 95%).
- Traditionally, confidence intervals were derived using theoretical mathematics.
- Resampling methods use computer-based numerical techniques to assess statistical accuracy without formulas (Efron and Tibshirani 1993).

**Resampling Methods:**

- Creation of new samples based on a single observed sample block.
- Suppose we have $n$ observations, $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$, of a random variable $\boldsymbol{x}$ from which we estimate some parameters $\boldsymbol{\theta}$ as:

$$\hat{\boldsymbol{\theta}} = f(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$$

- The estimation $\hat{\boldsymbol{\theta}}$ is a random variable because it is based on $n$ random variables.
- Resampling methods help characterize the distribution of the estimation without needing more realizations of the random variable $\boldsymbol{x}$.

**Cross-Validation:**

- Widely used in portfolio backtesting and machine learning.
- Divides $n$ observations into two groups: a training set for fitting the estimator $f(\cdot)$ and a validation set for assessing its performance.
- Repeated multiple times to provide multiple realizations of the performance value.
- ***k*-fold cross-validation:** divides the set into $k$ subsets, each held out in turn as the validation set while using the others for training.
- **Leave-one-out cross-validation:** extreme case where the original dataset of $n$ observations is divided into $k = n$ subsets, with each subset holding out a single observation for validation.

**The Bootstrap:**

- Proposed in 1979 by Efron, based on sound statistical theory (Efron 1979).
- Mimics the original sampling process by sampling $n$ times with replacement from the original $n$ observations.
- Repeated $B$ times to obtain bootstrap samples:

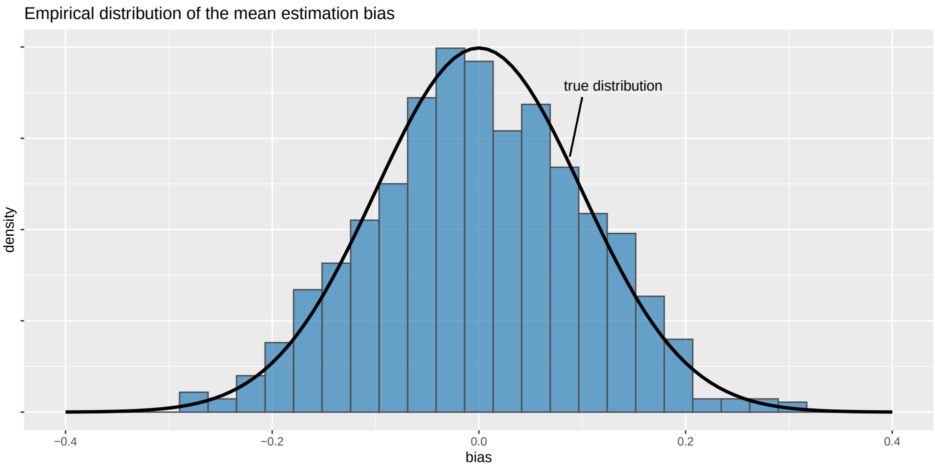
$$(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) \rightarrow \left(\boldsymbol{x}_1^{*(b)}, \ldots, \boldsymbol{x}_n^{*(b)}\right), \quad b = 1, \ldots, B.$$

- Leads to different realizations of the estimation (bootstrap replicates):

$$\hat{\boldsymbol{\theta}}^{*(b)} = f\left(\boldsymbol{x}_1^{*(b)}, \ldots, \boldsymbol{x}_n^{*(b)}\right), \quad b = 1, \ldots, B.$$

- Measures of accuracy (bias, variance, confidence intervals, etc.) can be empirically obtained.
- Theoretical result: statistical behavior of the random resampled estimates $\hat{\boldsymbol{\theta}}^{*(b)}$ compared to $\hat{\boldsymbol{\theta}}$ faithfully represents the statistics of the random estimates $\hat{\boldsymbol{\theta}}$ compared to the true parameter $\boldsymbol{\theta}$ (Efron and Tibshirani 1993).
- Estimations of accuracy are asymptotically consistent as $B \rightarrow \infty$.

Empirical distribution of the sample mean bias via the bootstrap:



# Resampling Methods

Several variations and extensions of the basic bootstrap have been developed over the years:

**Parametric Bootstrap:**

- Assumes a specific distribution for the data (e.g., Gaussian) (Efron and Tibshirani 1993).
- Steps:
  1. Assume a parametric distribution for the data.
  2. Estimate the distribution parameters from the observed data.
  3. Generate data using the estimated parametric distribution.

**Block Bootstrap:**

- Addresses structural dependency in data (Lahiri 1999).
- Involves resampling blocks of data rather than individual observations.

**Bag of Little Bootstraps:**

- Designed for large datasets with many observations (Kleiner et al. 2014).
- Steps:
  1. Divide the dataset into smaller subsets.
  2. Apply the bootstrap method to each subset.
  3. Aggregate results to assess estimator quality.

**The Jackknife:**

- Proposed in the mid-1950s by M. Quenouille.
- Derived for estimating biases and standard errors of sample estimators.
- Given $n$ observations $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$, the $i$th jackknife sample is obtained by removing the $i$th data point:

$$\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{i-1}, \boldsymbol{x}_{i+1}, \ldots, \boldsymbol{x}_n.$$

- Produces $B = n$ bootstrap samples each with $n - 1$ observations.
- An approximation to the bootstrap; makes a linear approximation to the bootstrap.
- Accuracy depends on how "smooth" the estimator is; for highly nonlinear functions, the jackknife can be inefficient.

**Bagging (Bootstrap Aggregating):**

- Method for generating multiple versions of an estimator or predictor via the bootstrap and then using these to get an aggregated version (Breiman 1996; Hastie, Tibshirani, and Friedman 2009).
- Improves accuracy of the basic estimator or predictor, which typically suffers from sensitivity to the realization of the random data.
- Mathematically, bagging is a simple average of the bootstrap replicates:

$$\hat{\boldsymbol{\theta}}^{\text{bag}} = \frac{1}{B} \sum_{b=1}^{B} \hat{\boldsymbol{\theta}}^{*(b)}.$$

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# Portfolio Resampling

**Optimization Problem in Portfolio Design:**

- Formulated based on $T$ observations of assets' returns $\boldsymbol{x}_1, \dots, \boldsymbol{x}_T$.
- Solution is an optimal portfolio $\boldsymbol{w}$.
- Sensitive to noise in observed data and estimated parameters ($\hat{\boldsymbol{\mu}}$ and $\hat{\boldsymbol{\Sigma}}$).

**Resampling Techniques:**

- Utilize statistical methods from the past half-century.
- Techniques include bootstrap and bagging.
- Improve portfolio design by reducing sensitivity to data noise.

**Historical Context:**

- Resampling proposed in the 1990s to assess portfolio accuracy.
- Naive approach: use available data to design mean-variance portfolios and obtain the efficient frontier.
- Issue: the computed efficient frontier is unreliable due to high sensitivity.

**Resampled Efficient Frontier:** (Jorion 1992; Michaud and Michaud 1998)

- Resampling allows computation of a more reliable efficient frontier.
- Identifies statistically equivalent portfolios.

# Portfolio Bagging

**Aggregating Portfolios via Bagging:** Technique considered in 1998 for portfolio aggregation using a bagging procedure (Michaud and Michaud 1998, 2008; Scherer 2002).

**Steps in the Bagging Procedure:**

1. **Resample the original data:** resample the original data $(\boldsymbol{x}_1, \dots, \boldsymbol{x}_T)$ $B$ times using the bootstrap method and estimate $B$ different versions of the mean vector and covariance matrix: $\hat{\boldsymbol{\mu}}^{*(b)}$ and $\hat{\Sigma}^{*(b)}$.
2. **Solve the optimal portfolio:** solve the optimal portfolio $\boldsymbol{w}^{*(b)}$ for each bootstrap sample.
3. **Average the portfolios via bagging:**

$$\boldsymbol{w}^{\text{bag}} = \frac{1}{B} \sum_{b=1}^{B} \boldsymbol{w}^{*(b)}.$$

**Observations:**

- The bagging procedure for portfolio aggregation is straightforward.
- The main bottleneck is the increase in computational cost by a factor of the number of bootstraps $B$ compared to the naive approach.

# Portfolio Subset Resampling

**Idea:** Sample the asset dimension rather than the observation (temporal) dimension.

**Procedure:**

1. **Select subset size:**
   - Choose the subset size using the rule-of-thumb $\lceil N^{0.7} \rceil$ or $\lceil N^{0.8} \rceil$.
   - Example: For $N = 50$, subset sizes would be 16 or 23 assets, respectively.
2. **Generate random subset:**
   - Randomly select a subset of assets according to the chosen size.
   - This reduces the portfolio dimensionality and computational cost.
3. **Design portfolio:**
   - Solve the portfolio optimization problem using only the selected subset of assets.
   - The resulting portfolio has weights only for the selected assets (other weights are zero).
4. **Repeat:**
   - Repeat steps 2-3 multiple times with different random subsets.
5. **Aggregate portfolios:**
   - Average all computed portfolios to obtain the final portfolio.
   - Since each portfolio is sparse (non-zero weights only for selected assets), the averaging naturally handles the different dimensions.

# Portfolio Subset Resampling

**Benefits:**

- **Reduced computational cost:** significant reduction in computational cost due to smaller subset size.
- **Improved parameter estimation:** better estimation of parameters because the ratio of observations-to-dimensionality has increased.
  - Example: for $T = 252$ daily observations and $N = 50$ assets, the nominal ratio would be $T/N \approx 5$, but the subset resampling ratio is $T/N^{0.7} \approx 16$ or $T/N^{0.8} \approx 11$.
  - Higher ratio leads to more reliable parameter estimates.

**Numerical Experiments:**

- Subsequent experiments will show that subset resampling is an effective technique in practice.

# Portfolio Subset Bagging

**Combination of Techniques:**

- Combine random subset resampling along the asset domain with the bootstrap along the temporal domain (Shen et al. 2019).
- Each bootstrap sample contains only a subset of the $N$ assets.
- This combination leverages the benefits of both resampling techniques to create more robust and computationally efficient portfolios.

**Summary:**

- **Computational savings and improved robustness:** portfolio subset resampling and portfolio subset bagging achieve significant computational savings and improve the robustness and reliability of portfolios.
- **Mitigating sensitivity to noise:** help mitigate sensitivity to noise and variability in the data, leading to more stable investment strategies.

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# Numerical Experiments

**Sensitivity Analysis:**

- **Naive mean-variance portfolio (MVP):** extremely sensitive to estimation errors as previously seen.
- **Robust portfolio optimization:** less sensitive to estimation errors as shown next as previously seen.
- **Bagged portfolios:** more stable and less sensitive with $B = 200$ bootstrap samples.

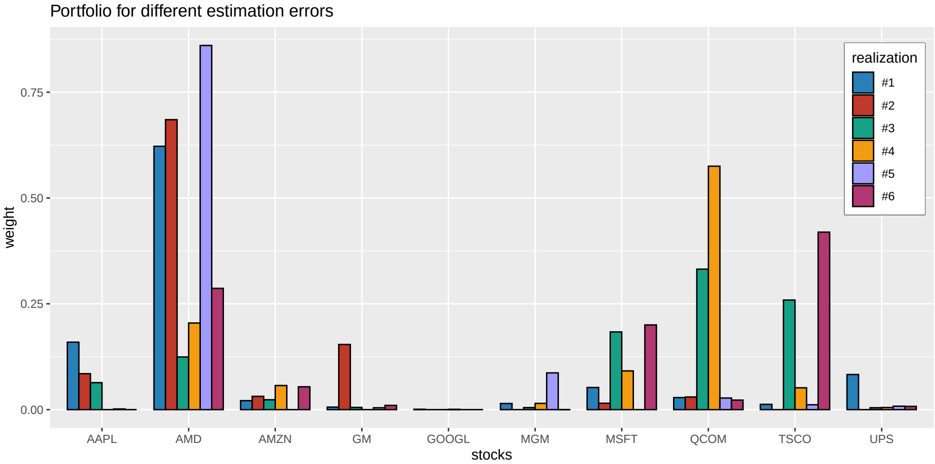
**Performance Assessment:**

- Assess performance of resampled portfolios (bagging, subset resampling, subset bagging) compared with naive design.
- Backtests conducted for 50 random stocks from the S&P 500 during 2017-2020.

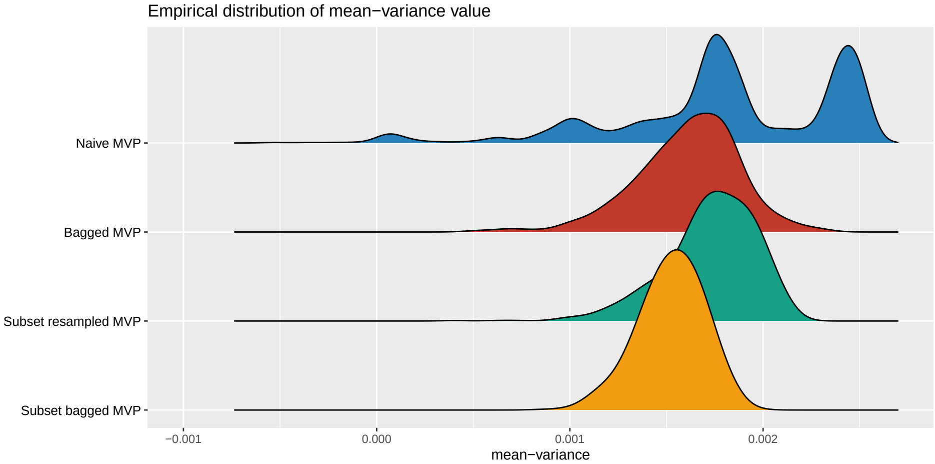
**Empirical Distribution Analysis:**

- Empirical distribution of the achieved mean-variance objective and the Sharpe ratio, calculated over 1,000 Monte Carlo noisy observations.
- Resampled portfolios are more stable, avoid extreme bad realizations (although the naive design can be superior in some cases).
- Resampled portfolios seem to be superior in Sharpe ratio and drawdown.

Sensitivity of the bagged mean–variance portfolio:



Empirical performance distribution of naive versus resampled mean–variance portfolios:



# Numerical Experiments

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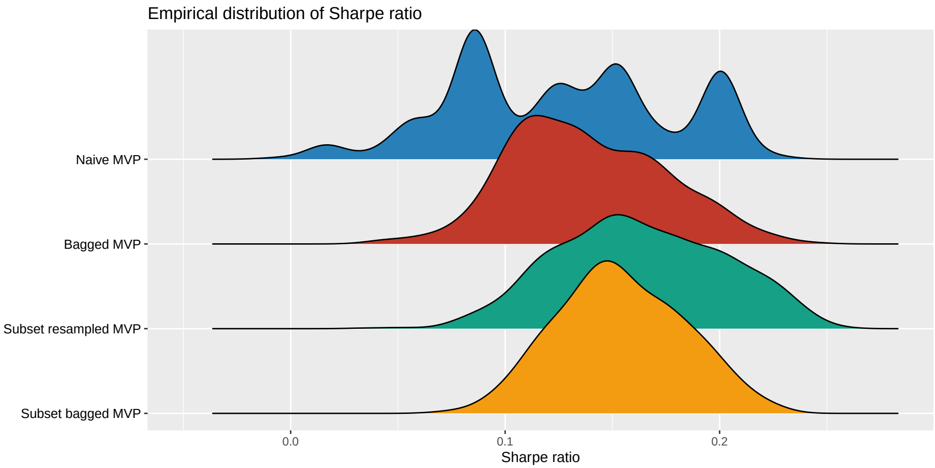
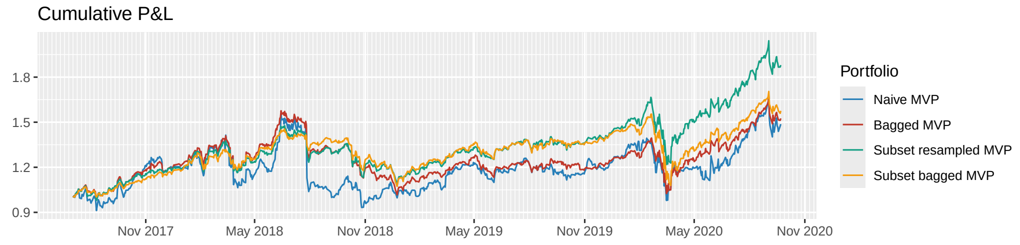
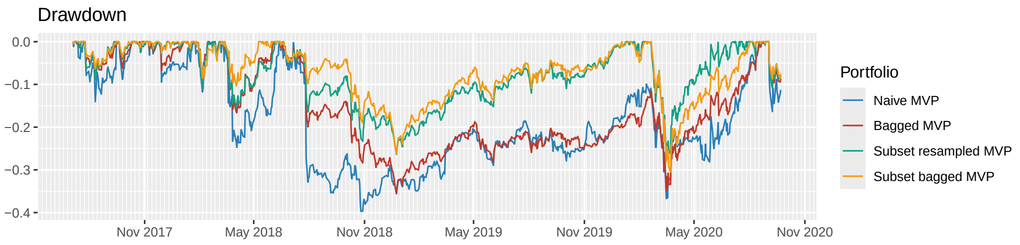


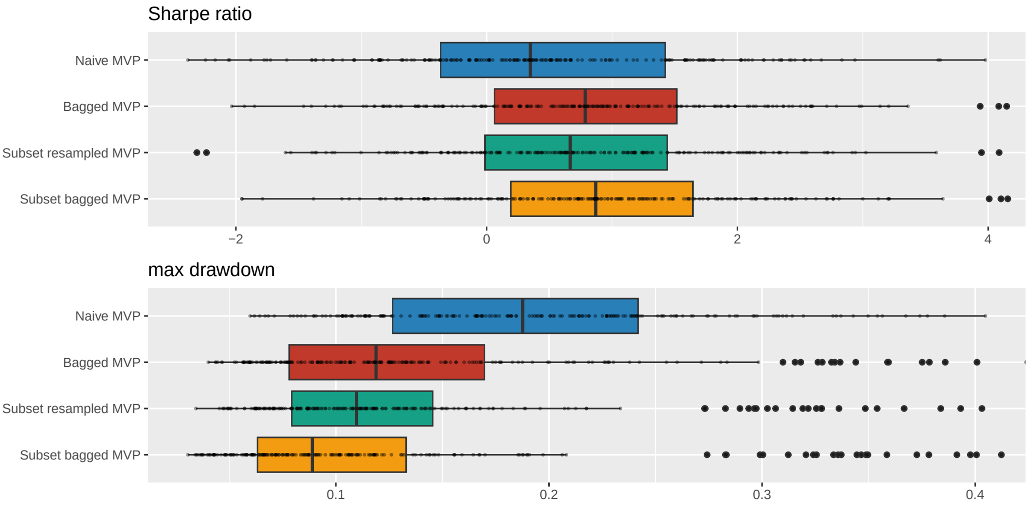
Backtest of naive versus resampled mean–variance portfolios:



# Numerical Experiments

Multiple backtests of naive versus resampled mean–variance portfolios:



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- Ideally, a portfolio optimization solution should meet the desired objective within constraints.
- In practice, it often fails due to reliance on estimated parameters like the mean vector and covariance matrix, which contain errors from noisy and limited data.
- Ignoring these estimation errors can lead to disastrous results, earning the term "estimation-error maximizers" for such portfolio problems.
- Effective approaches to mitigate naive solutions include:
  - **Robust portfolios**: Incorporate parameter errors using robust optimization, a well-developed method in portfolio optimization.
  - **Resampled portfolios**: Use bootstrapping and resampling to aggregate multiple naive solutions into a more stable and reliable portfolio.

# References I


Ben-Tal, A., L. El Ghaoui, and A. Nemirovski. 2009. *Robust Optimization*. Princeton University Press.

Ben-Tal, A., and A. Nemirovski. 2008. "Selected Topics in Robust Convex Optimization." *Mathematical Programming* 112 (1).

Bertsekas, D. P. 1999. *Nonlinear Programming*. Athena Scientific.

Birge, J. R., and F. V. Louveaux. 2011. *Introduction to Stochastic Programming*. Springer.

Breiman, L. 1996. "Bagging Predictors." *Machine Learning* 24: 123–40.

Chopra, V., and W. Ziemba. 1993. "The Effect of Errors in Means, Variances and Covariances on Optimal Portfolio Choice." *Journal of Portfolio Management* 19 (2): 6–11.

# References II


D. Bertsimas, D. B. Brown, and C. Caramanis. 2011. "Theory and Applications of Robust Optimization." *SIAM Review* 53 (3).

Efron, B. 1979. "Bootstrap Methods: Another Look at the Jackknife." *The Annals of Statistics* 7 (1): 1–26.

Efron, B., and R. J. Tibshirani. 1993. *An Introduction to the Bootstrap*. Springer.

Hastie, T., R. Tibshirani, and J. Friedman. 2009. *The Elements of Statistical Learning*. 2nd ed. Springer.

Jorion, P. 1992. "Portfolio Optimization in Practice." *Financial Analysts Journal* 48 (1): 68–74.

Kleiner, A., A. Talwalkar, P. Sarkar, and M. I. Jordan. 2014. "A Scalable Bootstrap for Massive Data." *Journal of the Royal Statistical Society. Series B: Statistical Methodology* 76 (4): 795–816.

# References III


Lahiri, S. N. 1999. "Theoretical Comparisons of Block Bootstrap Methods." *The Annals of Statistics* 27 (1): 386–404.

Lobo, M. S. 2000. "Robust and Convex Optimization with Applications in Finance." PhD thesis, Stanford University.

Michaud, R. O. 1989. "The Markowitz Optimization Enigma: Is 'Optimized' Optimal?" *Financial Analysts Journal* 45 (1): 31–42.

Michaud, R. O., and R. O. Michaud. 1998. *Efficient Asset Management: A Practical Guide to Stock Portfolio Optimization and Asset Allocation*. Harvard Business School Press.

———. 2008. *Efficient Asset Management: A Practical Guide to Stock Portfolio Optimization and Asset Allocation*. 2nd ed. Oxford University Press.

Palomar, D. P. 2025. *Portfolio Optimization: Theory and Application*. Cambridge University Press.

# References IV


Prekopa, A. 1995. *Stochastic Programming*. Kluwer Academic Publishers.

Ruszczynski, A. P., and A. Shapiro. 2003. *Stochastic Programming*. Elsevier.

Scherer, B. 2002. "Portfolio Resampling: Review and Critique." *Financial Analysts Journal* 58 (6): 98–109.

Shen, W., B. Wang, J. Pu, and J. Wang. 2019. "The Kelly Growth Optimal Portfolio with Ensemble Learning." In *Proceedings of the AAAI Conference on Artificial Intelligence*, 33:1134–41. 1.

Tütüncü, R. H., and M. Koenig. 2004. "Robust Asset Allocation." *Annals of Operations Research* 132 (1): 157–87.

Zhou, R., and D. P. Palomar. 2020. "Understanding the Quintile Portfolio." *IEEE Transactions on Signal Processing* 68 (July): 4030–40.