

Portfolio Optimization

Graph-Based Portfolios

Daniel P. Palomar (2025). *Portfolio Optimization: Theory and Application*.
Cambridge University Press.

portfoliooptimizationbook.com

Outline

- 1 Introduction
- 2 Hierarchical clustering and dendrograms
- 3 Hierarchical clustering-based portfolios
 - Hierarchical $1/N$ portfolio
 - Hierarchical risk parity (HRP) portfolio
 - Hierarchical equal risk contribution (HERC) portfolio
- 4 Numerical experiments
- 5 Summary

Abstract

Graphs offer a compact representation of big data, enabling analysis of large networks and pattern extraction. For financial data, asset graphs provide crucial information for modern portfolio design, potentially enhancing the mean-variance portfolio formulation. However, the optimal incorporation of graph information in portfolio optimization remains an open question. These slides explore some attempts in the literature (Palomar 2025, chap. 12).

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- **Markowitz's mean-variance portfolio:**

- Balances expected return and risk.
- Optimization problem:

$$\begin{aligned} & \underset{\mathbf{w}}{\text{maximize}} && \mathbf{w}^T \boldsymbol{\mu} - \frac{\lambda}{2} \mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w} \\ & \text{subject to} && \mathbf{w} \in \mathcal{W}, \end{aligned}$$

- λ : risk-aversion hyper-parameter.
- \mathcal{W} : constraint set (e.g., $\mathcal{W} = \{\mathbf{w} \mid \mathbf{1}^T \mathbf{w} = 1, \mathbf{w} \geq \mathbf{0}\}$).

- **Challenges with estimation errors:**

- Mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$ are prone to errors.
- Estimation errors significantly impact portfolio performance.

- **Improvement using graph of assets:**

- Potential for enhancement by incorporating asset graph connectivity.
- Graph connectivity patterns may reveal key investment insights.

Graphs and distance matrices

- **Graph-based portfolio construction:**

- Utilizes graph information encoded as a distance matrix D .
- Distance reflects the relationship between asset pairs.

- **Correlation-based distance matrix:**

$$D_{ij} = \sqrt{\frac{1}{2}(1 - \rho_{ij})}$$

where ρ_{ij} is the correlation between assets i and j .

- **Connection with Euclidean distance:**

- Standardized data columns: $\tilde{\mathbf{x}}_i = (\mathbf{x}_i - \mu_i)/\sigma_i$
- Empirical correlation:

$$\rho_{ij} = \frac{1}{T} \tilde{\mathbf{x}}_i^\top \tilde{\mathbf{x}}_j$$

- Normalized squared Euclidean distance:

$$\frac{1}{T} \|\tilde{\mathbf{x}}_i - \tilde{\mathbf{x}}_j\|_2^2 = 2(1 - \rho_{ij})$$

- **Other distance functions:**

- Minkowski metric based on p -norm:

$$D_{ij} = \|\tilde{\mathbf{x}}_i - \tilde{\mathbf{x}}_j\|_p$$

where $\|\mathbf{a}\|_p = \left(\sum_{t=1}^T |a_t|^p\right)^{1/p}$

- Manhattan distance ($p = 1$)
- Euclidean distance ($p = 2$).

- **Holistic distance matrix approach:** Euclidean distance between distance vectors:

$$\tilde{D}_{ij} = \|\mathbf{d}_i - \mathbf{d}_j\|_2$$

- \mathbf{d}_i : i th column of \mathbf{D} .
- Reflects similarity of assets with the entire asset universe.
- Overcomes the limitation of pairwise-only information.

Toy example

- **Correlation matrix C :**

$$C = \begin{bmatrix} 1 & 0.7 & 0.2 \\ 0.7 & 1 & -0.2 \\ 0.2 & -0.2 & 1 \end{bmatrix}$$

- **Correlation-based distance matrix D :**

$$D = \begin{bmatrix} 0 & 0.3873 & 0.6325 \\ 0.3873 & 0 & 0.7746 \\ 0.6325 & 0.7746 & 0 \end{bmatrix}$$

- **Euclidean distance matrix of correlation distances \tilde{D} :**

$$\tilde{D} = \begin{bmatrix} 0 & 0.5659 & 0.9747 \\ 0.5659 & 0 & 1.1225 \\ 0.9747 & 1.1225 & 0 \end{bmatrix}.$$

Other advanced graph estimation methods (Palomar 2025, chap. 5):

- **Heavy-tailed Markov random field (MRF) with degree control:**

$$\begin{aligned} & \underset{\mathbf{w} \geq \mathbf{0}}{\text{maximize}} && \log \text{gdet}(\mathcal{L}(\mathbf{w})) - \frac{N + \nu}{T} \sum_{t=1}^T \log \left(\nu + (\mathbf{x}^{(t)})^\top \mathcal{L}(\mathbf{w}) \mathbf{x}^{(t)} \right) \\ & \text{subject to} && \mathfrak{d}(\mathbf{w}) = \mathbf{1}, \end{aligned}$$

where

- $\text{gdet}(\cdot)$: generalized determinant
- \mathbf{w} : graph weight vector
- $\mathcal{L}(\mathbf{w})$: Laplacian operator
- $\mathfrak{d}(\mathbf{w})$: degree operator
- ν : controls heavy-tailness.

- **k -component heavy-tailed MRF with degree control:** Aims for a k -component graph (graph with k clusters)

$$\begin{aligned} & \underset{\mathbf{w} \geq \mathbf{0}, \mathbf{F} \in \mathbb{R}^{N \times k}}{\text{maximize}} && \log \text{gdet}(\mathcal{L}(\mathbf{w})) - \frac{N + \nu}{T} \sum_{t=1}^T \log \left(\nu + (\mathbf{x}^{(t)})^\top \mathcal{L}(\mathbf{w}) \mathbf{x}^{(t)} \right) \\ & && + \gamma \text{Tr} \left(\mathbf{F}^\top \mathcal{L}(\mathbf{w}) \mathbf{F} \right) \\ & \text{subject to} && \mathbf{d}(\mathbf{w}) = \mathbf{1}, \quad \mathbf{F}^\top \mathbf{F} = \mathbf{I}, \end{aligned}$$

where

- γ : regularization hyper-parameter
- \mathbf{F} : enforces low-rank property.

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Hierarchical clustering and dendrograms

- **Clustering overview:**

- Multivariate statistical analysis technique.
- Used in machine learning, data mining, pattern recognition, bioinformatics, finance, etc.
- Groups elements into clusters based on similar characteristics.
- Unsupervised classification method.

- **Hierarchical clustering:**

- Forms a recursive nested clustering.
- Builds a binary tree of data points representing nested groups.
- Allows data exploration at different levels of granularity.
- Contrasts with partitional clustering, which finds all clusters simultaneously without a hierarchical structure.

- **Dendrogram:**

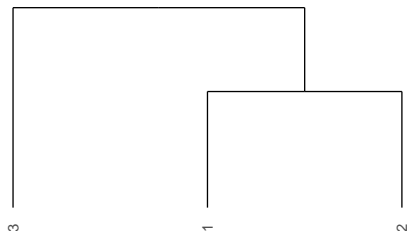
- Visual representation of the hierarchical clustering tree.
- Encodes the successive or hierarchical clustering process.
- Provides a complete, interpretable description of the clustering in graphical format.
- Popular due to its high interpretability.

Hierarchical clustering and dendrograms

Consider a toy example with distance matrix:

$$\tilde{\mathbf{D}} = \begin{bmatrix} 0 & 0.5659 & 0.9747 \\ 0.5659 & 0 & 1.1225 \\ 0.9747 & 1.1225 & 0 \end{bmatrix}$$

The dendrogram groups first the first and second elements since they have the smallest distance:

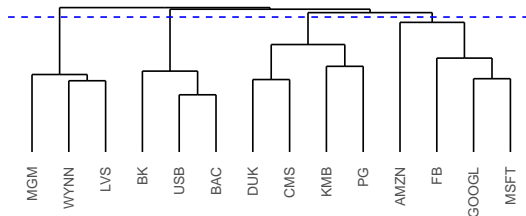


- **Hierarchical clustering process:**
 - Requires a distance matrix D .
 - Sequentially clusters items based on distance.
- **Methods for hierarchical clustering:**
 - **Agglomerative (bottom-up):**
 - Starts with each item as a singleton cluster.
 - Merges the closest clusters sequentially.
 - Continues until one cluster remains.
 - **Divisive (top-down):**
 - Starts with all items in one cluster.
 - Recursively divides each cluster into smaller ones.
- **Levels of hierarchy:**
 - Each level represents a grouping into disjoint clusters.
 - The entire hierarchy is an ordered sequence of groupings.

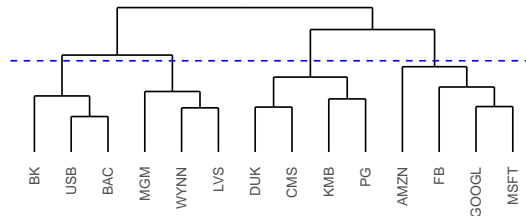
- **Linkage clustering methods:** measure of dissimilarity between clusters:
 - **Single linkage:**
 - Distance is the minimum distance between any two points in the clusters.
 - Related to the minimum spanning tree (MST).
 - **Complete linkage:**
 - Distance is the maximum distance between any two points in the clusters.
 - **Average linkage:**
 - Distance is the average distance between any two points in the clusters.
 - **Ward's method:**
 - Distance is the increase in squared error when merging clusters.
 - Related to distances between cluster centroids.
- **Effects of Linkage Method:**
 - Significantly impacts the resulting hierarchical clustering.
 - Single linkage may cause a “chaining” effect and imbalanced groups.
 - Complete linkage tends to produce more balanced groups.
 - Average linkage is an intermediate case.
 - Ward's method often yields results similar to average linkage.

Dendrograms of S&P 500 stocks

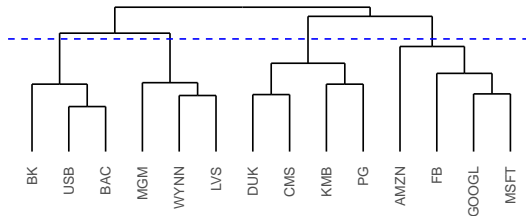
Single linkage



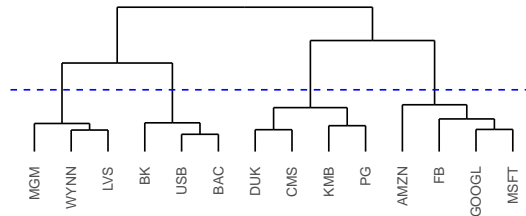
Complete linkage



Average linkage



Ward's method



- **Determining the number of clusters:**

- Traversing the dendrogram from top to bottom transitions from one giant cluster to N singleton clusters.
- In practice, dealing with N singleton clusters may lead to overfitting.

- **Simplification vs. detail:**

- Fewer clusters simplify the data but lose fine details, too many clusters might identify spurious patterns.
- The challenge lies in choosing the optimal number of clusters.

- **Automatic detection of optimal clusters:**

- Essential to avoid overfitting.
- Aids in identifying the most appropriate number of clusters.

- **Gap statistic:**

- Determines the optimal number of clusters.
- Compares empirical within-cluster dissimilarity to uniformly distributed data.
- Identifies the balance between simplification and preserving significant patterns.

Quasi-diagonalization of correlation matrix

- **Quasi-diagonalization of correlation matrix:**

- Hierarchical clustering reorders items in the correlation matrix.
- Groups similar assets closer and dissimilar assets farther apart.
- Known as *matrix seriation* or *matrix quasi-diagonalization*.
- An old statistical technique for revealing inherent clusters.

- **Benefits of quasi-diagonalization:**

- Rearranges the correlation matrix into a quasi-diagonal form.
- Reveals similar assets as blocks along the main diagonal.
- Enhances visual pattern recognition compared to a randomly ordered matrix.

- **Visualization through heatmaps:**

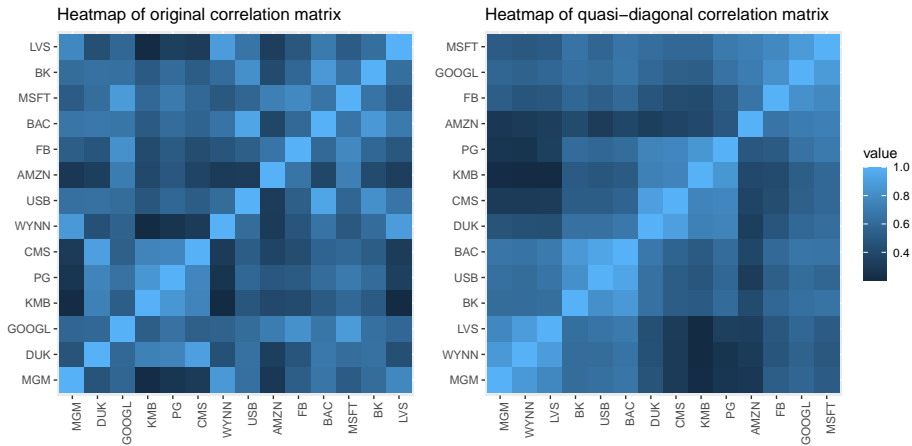
- Heatmaps can display the original and quasi-diagonal correlation matrices.
- Original matrix with randomly ordered stocks shows no clear pattern.
- Quasi-diagonal matrix, after reordering, clearly shows correlated stocks in diagonal blocks.

- **Identification of clusters:**

- Quasi-diagonal matrix allows for easy identification of asset clusters.
- Corresponding dendrograms can confirm the number and composition of these clusters.

Quasi-diagonalization of correlation matrix

Effect of seriation in the correlation matrix of S&P 500 stocks:



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Hierarchical clustering-based portfolios

- **Portfolio design based on graph of assets:**

- Aims to create robust, diversified portfolios with better risk-adjusted performance.
- Less reliance on noisy estimates of mean vector μ and covariance matrix Σ .

- **Hierarchical clustering for diversification:**

- Distributes capital weights across hierarchically nested clusters.
- Identifies isolated stocks contributing to diversification.
- Visualized using the hierarchical tree layout.

- **Capital allocation in hierarchical clustering-based portfolios:**

- Total capital starts at the top of the dendrogram.
- Capital is allocated top-down through the hierarchy.
- Each division of a cluster into sub-clusters splits the capital accordingly.
- Portfolios for sub-clusters are designed at each split.

Hierarchical clustering-based portfolios

- **Characteristics of hierarchical clustering-based portfolios:**

- ➊ **Distance matrix:** Defines the graph (e.g., correlation-based, distance matrix of columns, sophisticated graph learning).
- ➋ **Linkage method:** Employed in the clustering process (e.g., single, complete, average, Ward).
- ➌ **Clustering stopping criterion:** Determines when to stop clustering (e.g., single-item clusters, gap statistic).
- ➍ **Splitting criterion:** Recursively splits the assets (e.g., bisection, dendrogram-based).
- ➎ **Intra-weight allocation:** Allocation of weights within clusters.
- ➏ **Inter-weight allocation:** Allocation of weights across clusters.

- **We will explore:**

- Hierarchical $1/N$ Portfolio.
- Hierarchical Risk Parity Portfolio.
- Hierarchical Equal Risk Contribution Portfolio.

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- **Cluster-based waterfall portfolio overview:**

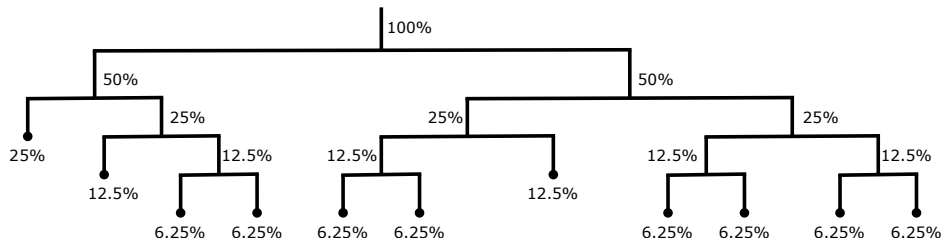
- Introduced by (Papenbrock 2011) in his PhD thesis.
- Utilizes hierarchical tree from correlation-based distance matrix.
- Allocation process splits weights equally at each dendrogram split.

- **Allocation process:**

- Proceeds in a top-down manner through the dendrogram.
- Splits weights equally at each splitting point.
- Illustration provided in the next figure.

Hierarchical 1/ N portfolio

Illustration of the hierarchical 1/ N portfolio construction in a top-down manner:

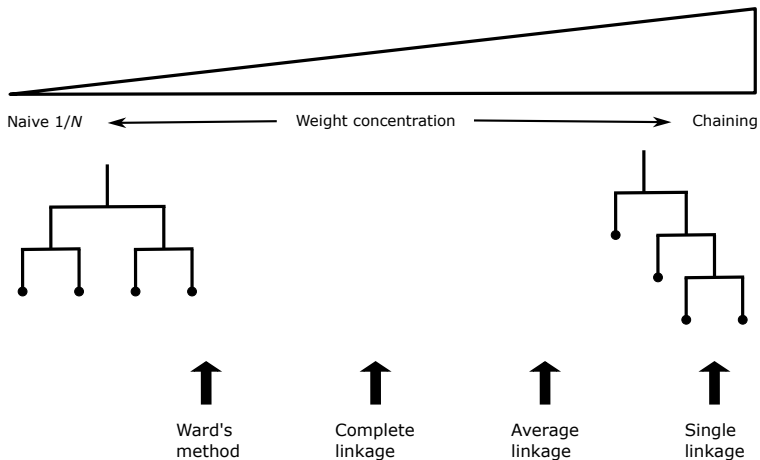


Impact of linkage method on weight allocation:

- **Single linkage:**
 - Suffers from “chaining” effect, leading to high weights on some stocks.
- **Complete linkage:**
 - Produces more even groups and weights.
- **Average linkage:**
 - Intermediate between single and complete linkage.
- **Ward’s method:**
 - Similar to complete linkage but with even more balanced groups and weights.
- **Regular $1/N$ portfolio:**
 - Represents equalized weights without using graph information, termed *naive* $1/N$ portfolio.

Hierarchical $1/N$ portfolio

Chaining effect of different linkage methods on the hierarchical $1/N$ allocation:



Summary:

① Distance matrix:

- Correlation-based: $D_{ij} = \sqrt{\frac{1}{2}(1 - \rho_{ij})}$.

② Linkage method:

- Single linkage for high-risk investors.
- Ward's method for risk-averse investors.

③ Clustering stopping criterion:

- Continues to single-item clusters.

④ Splitting criterion:

- Follows the dendrogram.

⑤ Intra-weight allocation:

- $1/N$ portfolio strategy.

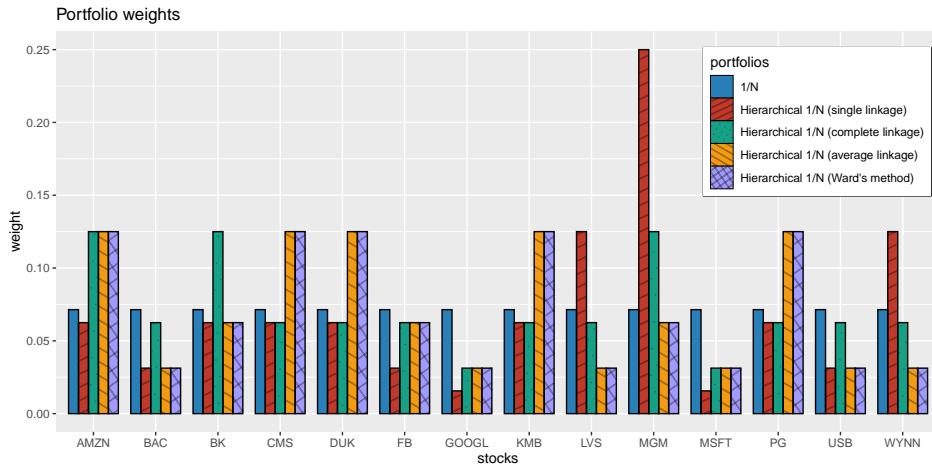
⑥ Inter-weight allocation:

- $1/N$ portfolio with $N = 2$, i.e., 50% - 50% split at each branching.

- **Comparing hierarchical $1/N$ portfolios with different linkage methods:**
 - Hierarchical $1/N$ portfolios are compared using single, complete, average, and Ward's linkage methods.
 - Naive $1/N$ portfolio serves as a benchmark.
- **Backtest results:**
 - Ward's method seems to be a good choice for hierarchical $1/N$ portfolio construction.
 - The original publication (Papenbrock 2011) supports the use of Ward's method for subsequent analysis.

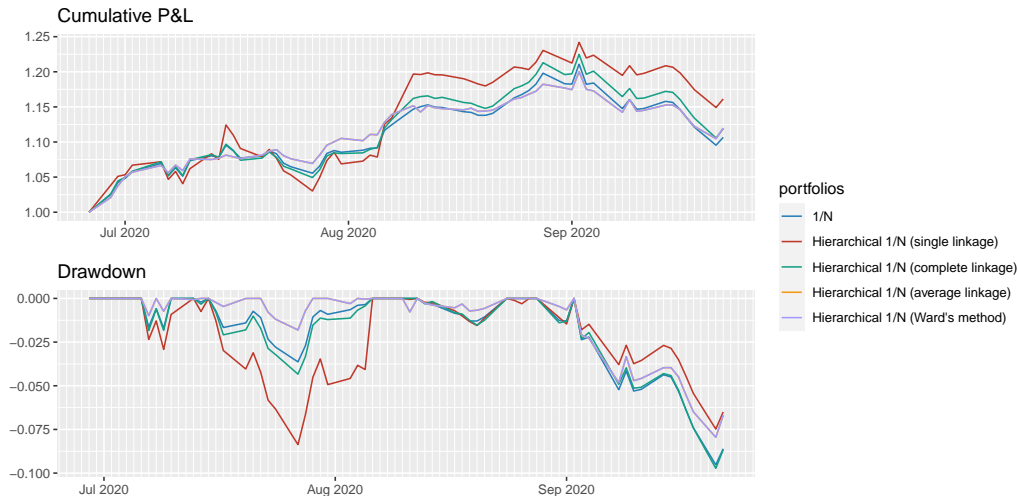
Numerical results

Portfolio allocation of hierarchical $1/N$ portfolios with different linkage methods:



Numerical results

Backtest performance of hierarchical 1/ N portfolios with different linkage methods:



- **Comparison of hierarchical $1/N$ portfolio using different distance matrices:**

- ① **Correlation-based distance matrix:**

- As per (Papenbrock 2011):

$$D_{ij} = \sqrt{\frac{1}{2}(1 - \rho_{ij})}$$

- ② **Correlation-based distance-of-distance matrix:**

- As used in (López de Prado 2016):

$$\tilde{D}_{ij} = \|\mathbf{d}_i - \mathbf{d}_j\|_2$$

- ③ **Graphs estimated via heavy-tailed MRF:**

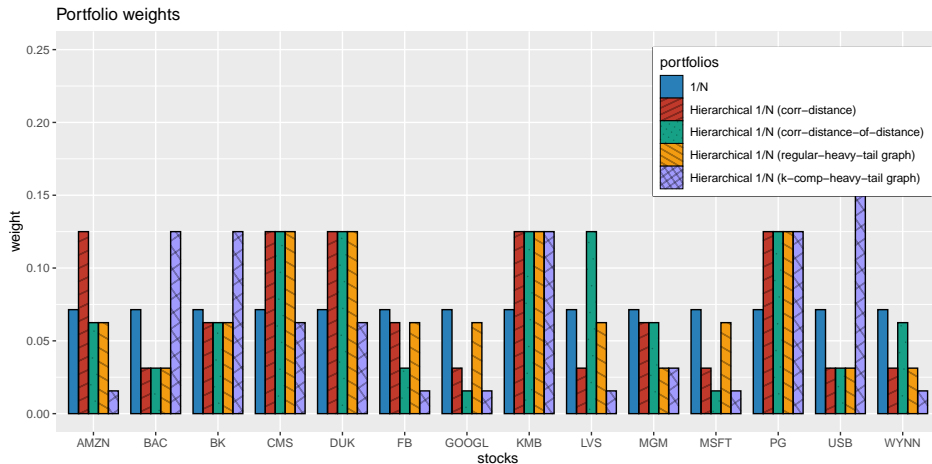
- Regular heavy-tailed MRF.
 - k -component heavy-tailed MRF.

- **Backtest observations:**

- The simple correlation-based distance-of-distance matrix appears to have a better drawdown profile.
 - More exhaustive backtests are recommended to draw definitive conclusions.

Numerical results

Portfolio allocation of hierarchical $1/N$ portfolios with different distance matrices:



Numerical results

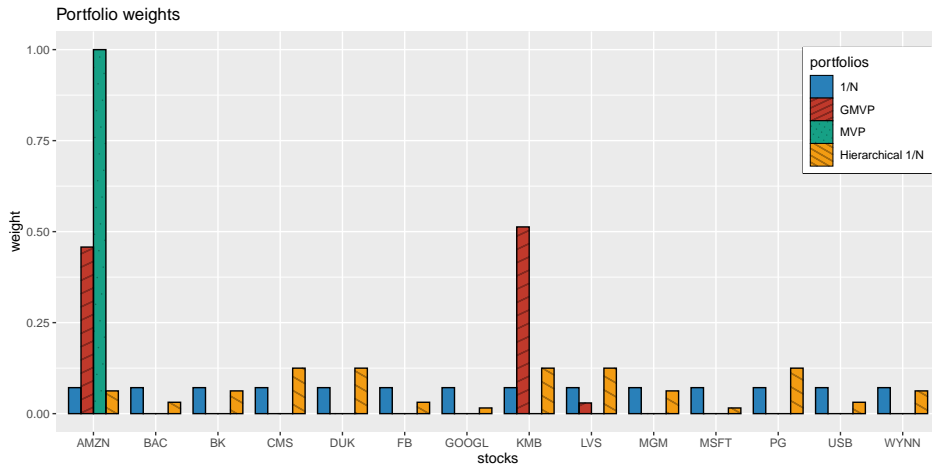
Backtest performance of hierarchical $1/N$ portfolios with different distance matrices:



- **Final comparison of hierarchical $1/N$ portfolio:**
 - Selected version uses Ward's method for linkage and correlation-based distance-of-distance matrix.
 - Benchmarks: naive $1/N$ portfolio, global minimum variance portfolio (GMVP), and Markowitz mean-variance portfolio (MVP).
- **Observations:**
 - Hierarchical $1/N$ portfolio shows a distinct allocation pattern, emphasizing diversification.
 - MVP exhibits the worst drawdown due to sensitivity in estimating μ .
 - GMVP and naive $1/N$ portfolio show better performance than MVP.
 - Hierarchical $1/N$ portfolio demonstrates the mildest drawdown, indicating superior risk management.
- **Considerations and further evaluation:**
 - The presented backtest is anecdotal and not sufficient for definitive conclusions.
 - A proper empirical evaluation requires multiple randomized backtests.
 - Further analysis is necessary to robustly assess the performance of the hierarchical $1/N$ portfolio against benchmarks.

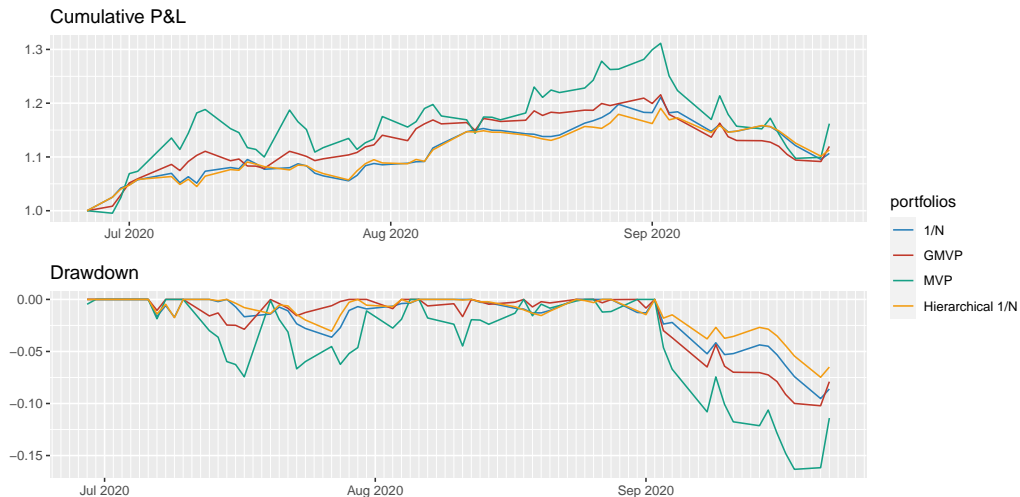
Numerical results

Portfolio allocation of hierarchical $1/N$ portfolio along benchmarks:



Numerical results

Backtest performance of hierarchical $1/N$ portfolio along benchmarks:



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Hierarchical risk parity (HRP) portfolio

- **Hierarchical risk parity (HRP) portfolio overview:**

- Introduced by (López de Prado 2016).
- Based on hierarchical tree from correlation-based distance-of-distance matrix.
- Utilizes single linkage method for clustering.
- Allocation process uses inverse-variance portfolio (IVarP) for weight splitting.

- **Global minimum variance portfolio (GMVP) recap:**

- Minimizes portfolio variance subject to budget constraint.
- Solution simplifies to IVarP if covariance matrix Σ is diagonal:

$$\mathbf{w} = \frac{\boldsymbol{\sigma}^{-2}}{\mathbf{1}^T \boldsymbol{\sigma}^{-2}}.$$

- **Inverse-variance portfolio (IVarP) for $N = 2$ assets:**

- Weight allocation based on inverse of variances:

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} \sigma_2^2 / (\sigma_1^2 + \sigma_2^2) \\ \sigma_1^2 / (\sigma_1^2 + \sigma_2^2) \end{bmatrix}.$$

Hierarchical risk parity (HRP) portfolio

- **HRP portfolio design process:**

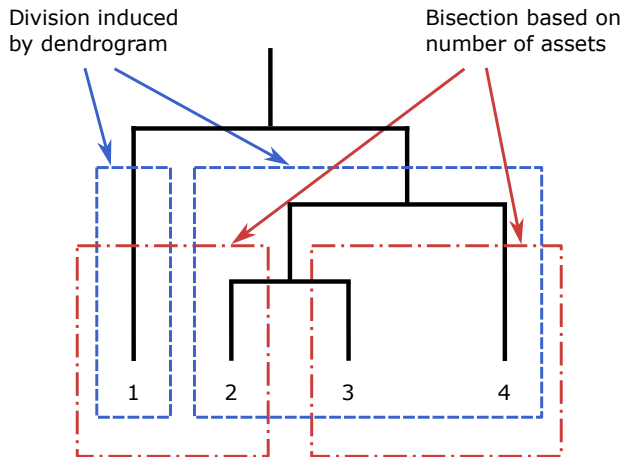
- Similar to hierarchical $1/N$ portfolio, allocation proceeds top-down through the dendrogram.
- **Differences:**
 - Uses bisection for splitting, not the dendrogram's natural structure.
 - Weight splitting based on IVarP for $N = 2$ assets.
- Empirical evaluation required to compare performance with hierarchical $1/N$ portfolio.

- **Interpretation and connection to GMVP:**

- HRP can be seen as a refined version of the IVarP.
- At each step, weights are scaled based on inverse variances, ignoring correlations between subsets.
- Correlations considered only in variance computation of subsets.
- When covariance matrix is diagonal, IVarP, GMVP, and HRP coincide.
- Connection between HRP and GMVP explored further in subsequent sections.

Hierarchical risk parity (HRP) portfolio

Comparison of bisection splitting and dendrogram-based splitting:



Summary:

- ① **Distance matrix:**
 - Correlation-based distance-of-distance matrix.
- ② **Linkage method:**
 - Single linkage.
- ③ **Clustering stopping criterion:**
 - Continues to single-item clusters.
- ④ **Splitting criterion:**
 - Bisection, ignoring dendrogram grouping sizes.
- ⑤ **Intra-weight allocation:**
 - IVarP.
- ⑥ **Inter-weight allocation:**
 - IVarP for $N = 2$.

- **Comparison of HRP portfolios with benchmarks:**

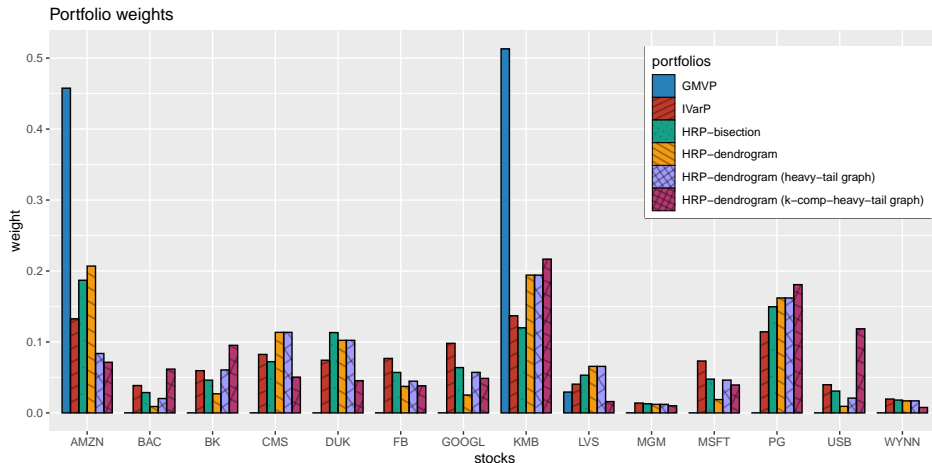
- HRP portfolios compared with global minimum variance portfolio (GMVP) and inverse-variance portfolio (IVarP).
- Two versions of HRP: one with bisection split and another with dendrogram split.

- **Observations:**

- GMVP shows concentration in two assets, while others are more diversified.
- HRP portfolios show similar diversification to IVarP.
- HRP portfolios exhibit slight improvement over IVarP.
- Graphs estimated via heavy-tailed MRF methods may offer better performance than correlation-based methods.
- HRP portfolios aim to balance diversification and risk management.
- The choice of splitting method in HRP (bisection vs. dendrogram) may not significantly alter the diversification profile compared to IVarP.
- The performance of HRP portfolios in terms of drawdown and P&L suggests potential advantages over traditional IVarP, especially when using advanced graph estimation methods.

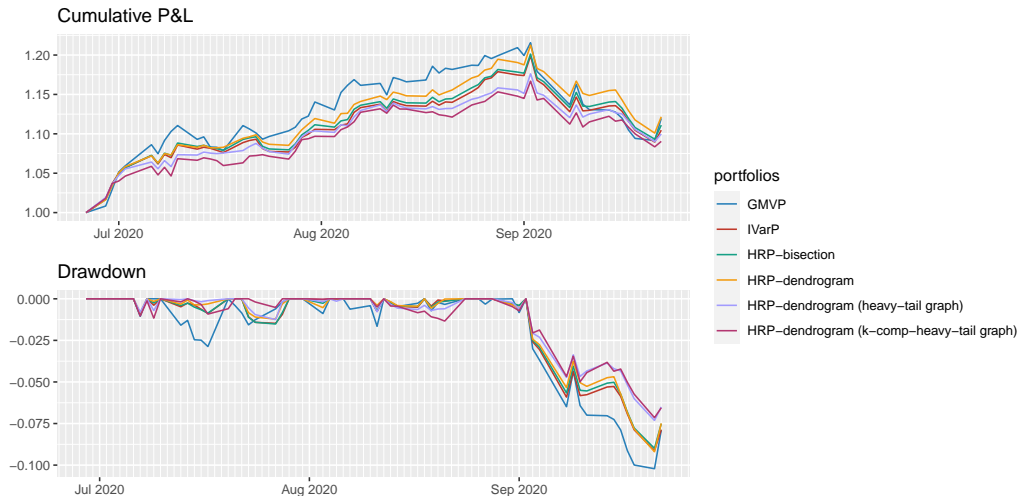
Numerical results

Portfolio allocation of HRP portfolios and benchmarks:



Numerical results

Backtest performance of HRP portfolios and benchmarks:



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Hierarchical equal risk contribution (HERC) portfolio

- **Hierarchical equal risk contribution (HERC) portfolio overview:**
 - Introduced by (Raffinot 2018).
 - Refines and extends the hierarchical 1/ N and HRP portfolios.
 - Incorporates early stopping based on the gap statistic for cluster selection.
 - Utilizes equal risk contribution (ERC) for weight allocation among clusters.
- **Key differences from previous approaches:**
 - **Early stopping with Gap statistic:**
 - Automatically selects the appropriate number of clusters.
 - Avoids clustering down to single assets.
 - **General equal risk contribution:**
 - Splits weights based on alternative risk measures (e.g., standard deviation, conditional value-at-risk).
 - Formula for two clusters:

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} RC_1 / (RC_1 + RC_2) \\ RC_2 / (RC_1 + RC_2) \end{bmatrix},$$

where RC_i is the risk contribution of the i th cluster.

Hierarchical equal risk contribution (HERC) portfolio

- **Main findings:**

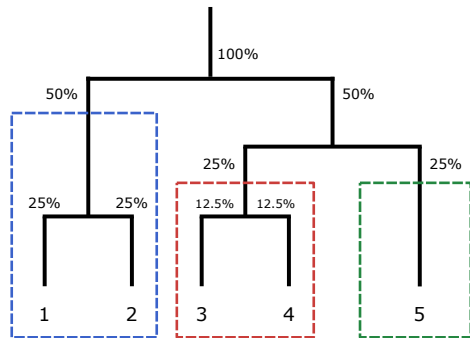
- Hierarchical $1/N$ portfolio is a strong baseline.
- HERC portfolios based on downside risk measures (especially conditional drawdown-at-risk) show statistically better risk-adjusted performances.

- **Illustration of early stopping:**

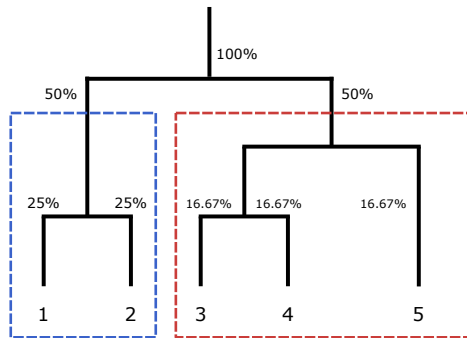
- The next figure demonstrates early stopping in hierarchical clustering with a toy dendrogram.
- Groups assets into clusters based on the gap statistic, avoiding overly granular clustering.

Hierarchical equal risk contribution (HERC) portfolio

Effect of early stopping in the hierarchical clustering process:



Three clusters



Two clusters

Hierarchical equal risk contribution (HERC) portfolio

Summary:

- ① **Distance matrix:**
 - Correlation-based distance-of-distance matrix.
- ② **Linkage method:**
 - Ward's method.
- ③ **Clustering stopping criterion:**
 - Gap statistic for optimal cluster selection.
- ④ **Splitting criterion:**
 - Follows the dendrogram structure.
- ⑤ **Intra-weight allocation:**
 - $1/N$ portfolio strategy.
- ⑥ **Inter-weight allocation:**
 - Equal risk contribution based on various risk measures.

Conclusion: HERC portfolio represents a sophisticated approach to portfolio construction, balancing risk across clusters for improved risk-adjusted returns.

- **Simplification in weight splitting for HERC portfolios:**

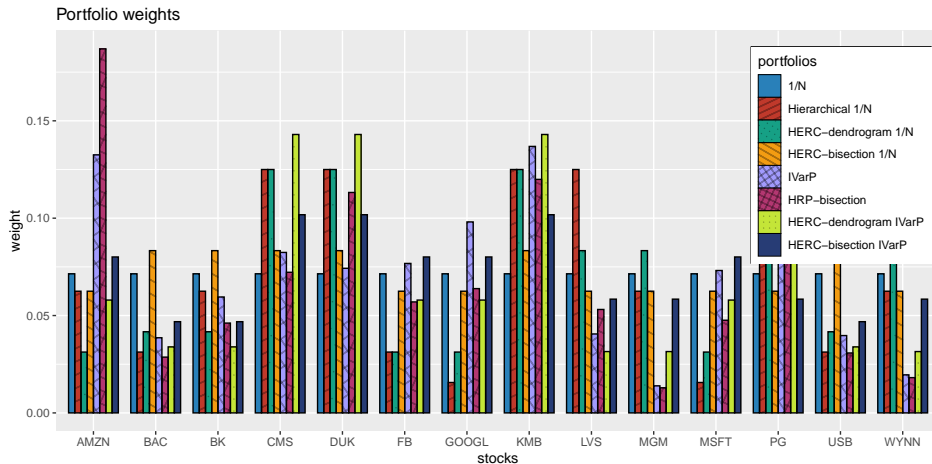
- For weight allocation, two risk contribution measures are considered:
 - $RC_i = 1$: Leads to a 50% - 50% split, similar to the hierarchical $1/N$ portfolio.
 - $RC_i = 1/\sigma_i^2$: Aligns with the inverse-variance portfolio (IVarP) formula, akin to the HRP portfolio.

- **Comparison of HERC portfolios with benchmarks:**

- Benchmarks include: $1/N$ portfolio, hierarchical $1/N$ portfolio, inverse-variance portfolio (IVarP), and HRP portfolio.
- Two versions of HERC portfolios are evaluated: one with bisection split and another with dendrogram split.

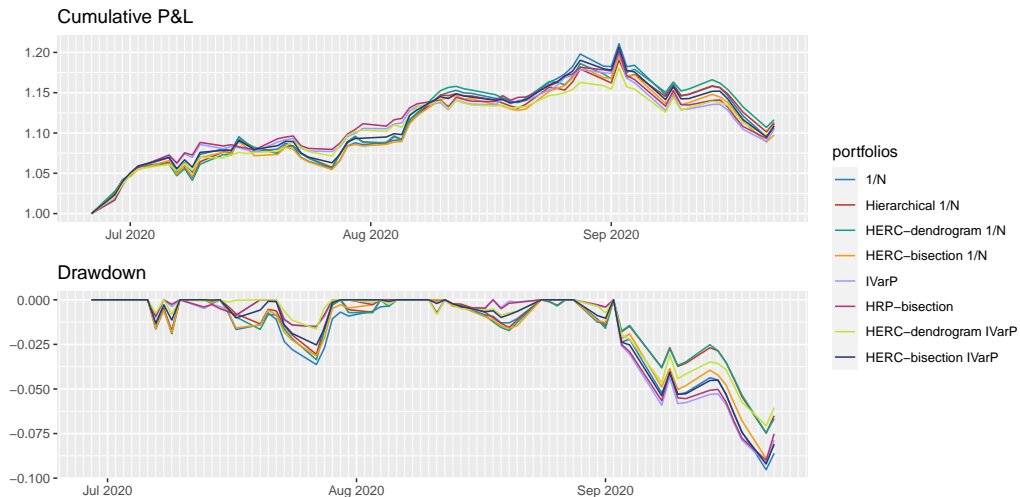
Numerical results

Portfolio allocation of HERC portfolios and benchmarks:



Numerical results

Backtest performance of HERC portfolios and benchmarks:



- **Observations and further evaluation:**

- Difficult to draw definitive conclusions from a single backtest.
- More exhaustive backtests are necessary to robustly assess the performance of HERC portfolios.
- Future analysis should aim to evaluate the risk-adjusted returns and drawdown characteristics of HERC portfolios in various market conditions.

- **Conclusion:**

- The HERC portfolio introduces a nuanced approach to portfolio construction by incorporating risk contributions and early stopping based on the gap statistic.
- Its performance relative to traditional and hierarchical portfolio strategies warrants further empirical investigation to fully understand its benefits and limitations.

From portfolio risk minimization to hierarchical portfolios

- The basic structure of hierarchical portfolios is heuristic and suboptimal, which is understandable since the motivation was not optimality but stability against estimation errors.
- On the other hand, portfolios designed based on the minimization of some properly chosen measure of risk are not heuristic by definition but optimal according to the design criterion.
- Can we make an explicit connection between the two paradigms?
- Indeed, it is possible to design a continuum between hierarchical portfolios and optimally designed portfolios (Palomar 2025, sec. 12.3.4).

Outline

- 1 Introduction
- 2 Hierarchical clustering and dendrograms
- 3 Hierarchical clustering-based portfolios
 - Hierarchical $1/N$ portfolio
 - Hierarchical risk parity (HRP) portfolio
 - Hierarchical equal risk contribution (HERC) portfolio
- 4 Numerical experiments**
- 5 Summary

- **Overview:**

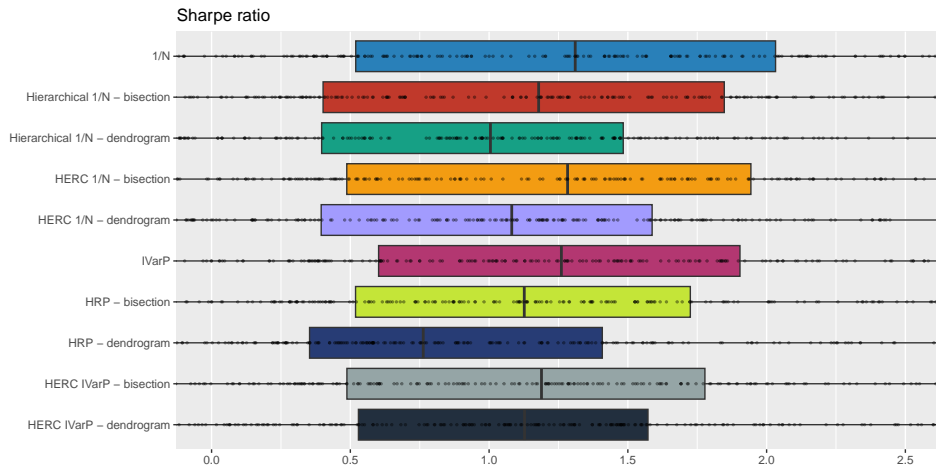
- Conducted multiple randomized backtests using S&P 500 stocks from 2015-2020.
- Generated 200 resamples with $N = 50$ stocks and a random two-year period.
- Walk-forward backtest with a 1-year lookback, reoptimizing monthly.
- Caution: Results are indicative and should be supplemented with more exhaustive backtests.

- **Observations:**

- **Splitting:** Natural dendrogram splits might be expected to outperform, but it seems that bisection might provide more balanced clusters (while utilizing dendrogram ordering).
- **Graph learning:** Sophisticated methods do not clearly outperform simple graph-based approaches.

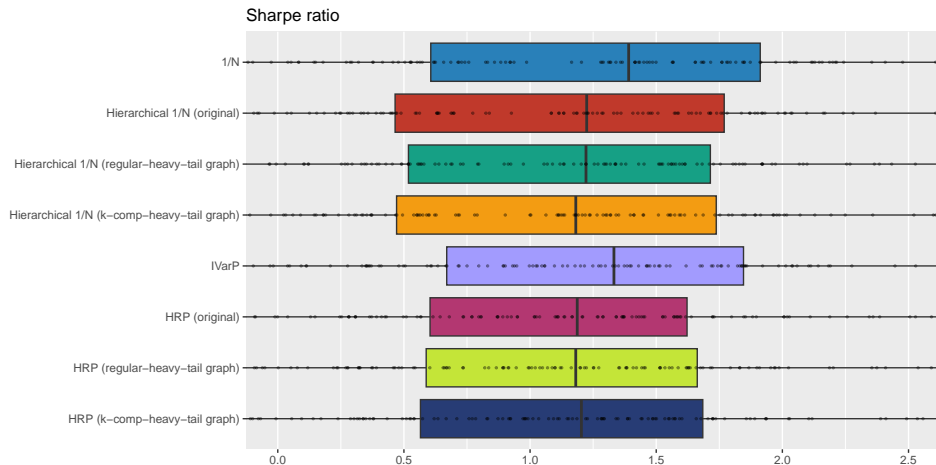
Numerical experiments - Splitting: bisection versus dendrogram

Comparison of graph-based portfolios: bisection versus dendrogram splitting:



Numerical experiments - Graph estimation: simple versus sophisticated

Comparison of graph-based portfolios: simple versus sophisticated graph learning methods:



Numerical experiments - Final comparison

- **Portfolios compared:**

- Hierarchical $1/N$ portfolio.
- HERC $1/N$ portfolio.
- HRP portfolio.
- HERC IVarP.

- **Benchmarks:**

- $1/N$ portfolio.
- IVarP.

- **Empirical results:**

- The following table and figures show no significant differences among methods.

- **Conclusion:**

- Further exhaustive comparison needed to draw clear conclusions.
- Current analysis does not favor one graph-based portfolio method over others.

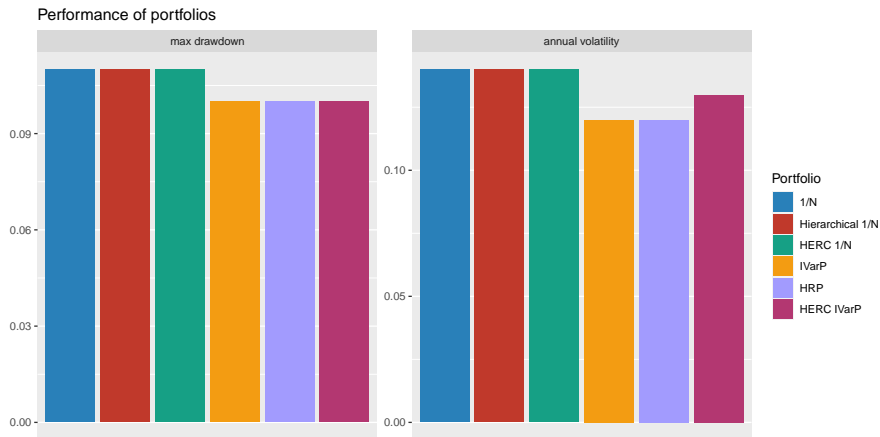
Numerical experiments - Final comparison

Comparison of selected graph-based portfolios: performance measures:

Portfolio	Sharpe ratio	annual return	annual volatility	max drawdown
1/N	1.01	14%	14%	11%
Hierarchical 1/N	0.81	12%	14%	11%
HERC 1/N	0.99	14%	14%	11%
IVarP	1.04	13%	12%	10%
HRP	0.91	11%	12%	10%
HERC IVarP	0.89	12%	13%	10%

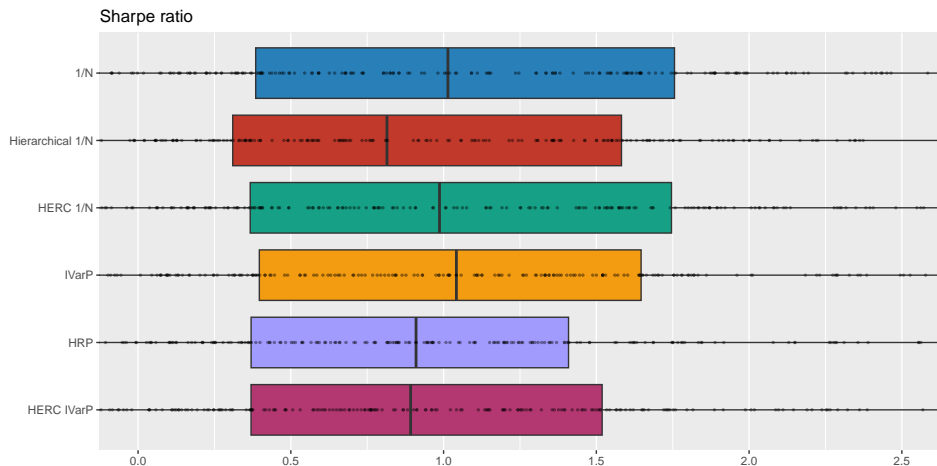
Numerical experiments - Final comparison

Comparison of selected graph-based portfolios: barplots of maximum drawdown and annualized volatility:



Numerical experiments - Final comparison

Comparison of selected graph-based portfolios: boxplots of Sharpe ratio:



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Graphs compactly represent big data, revealing underlying structure and patterns.

Key takeaways for portfolio design using graphs:

- Graphs represent asset relationships: nodes are assets, edges are pairwise relationships.
- Financial graphs can be learned from data, e.g., based on heavy-tailed Markov random fields or k -component versions for clustered graphs (Palomar 2025, chap. 5).
- Hierarchical clustering partitions assets into clusters at different levels of detail.
- Graph information should be incorporated into portfolio formulation, with notable examples being hierarchical $1/N$, risk parity, and equal risk contribution portfolios.

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- Papenbrock, J. 2011. “Asset Clusters and Asset Networks in Financial Risk Management and Portfolio Optimization.” PhD thesis, Karlsruher Institute für Technologie.
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<https://dx.doi.org/10.2139/ssrn.3237540>.