Portfolio Optimization

Portfolio Basics

Daniel P. Palomar (2025). *Portfolio Optimization: Theory and Application.* Cambridge University Press.

portfoliooptimizationbook.com

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Outline

Executive Summary

These slides, following (Palomar 2025, chap. 6),

- introduce core concepts of portfolio construction, including portfolio weights and rebalancing;
- review standard portfolio constraints used in practice;
- explain key performance metrics for evaluating portfolios;
- survey widely used portfolio types, focusing on both heuristic and risk-based constructions;
- present examples including:
  - equally-weighted ($1/N$) portfolio
  - quintile portfolio
  - global minimum variance portfolio.

Outline

Data Modeling

Asset Prices and Returns

- Denote prices of N assets at time t as $\boldsymbol{p}_t \in \mathbb{R}^N$.
- Linear returns defined as
$$\boldsymbol{r}_t^{\text{lin}} = \frac{\boldsymbol{p}_t}{\boldsymbol{p}_{t-1}} - \mathbf{1}.$$
- Log-returns defined as
$$\boldsymbol{r}_t^{\text{log}} = \log \boldsymbol{p}_t - \log \boldsymbol{p}_{t-1}.$$

Modeling Objectives

- Aim to forecast returns at time t using historical data up to $t-1$ ($\mathcal{F}_{t-1}$).
- Focus on conditional first- and second-order moments (mean vector and covariance matrix).

Mean Vector and Covariance Matrix

- Conditional mean vector for log-returns: $\boldsymbol{\mu}_t^{\text{log}} = \mathbb{E}\left[\boldsymbol{r}_t^{\text{log}} \mid \mathcal{F}_{t-1}\right]$.
- Conditional covariance matrix for log-returns: $\boldsymbol{\Sigma}_t^{\text{log}} = \text{Cov}\left[\boldsymbol{r}_t^{\text{log}} \mid \mathcal{F}_{t-1}\right]$.

Data Modeling

Preference for Log-Returns

- Financial models often prefer log-returns for modeling, despite needing linear returns for portfolio performance.
- Linear and log-returns are very similar for small return values.

Approximations Between Linear and Log-Returns

- Linear mean and covariance approximated by log counterparts: $\boldsymbol{\mu}_t^{\text{lin}} \approx \boldsymbol{\mu}_t^{\text{log}}$, $\boldsymbol{\Sigma}_t^{\text{lin}} \approx \boldsymbol{\Sigma}_t^{\text{log}}$.
- Exact mathematical relationships allow deriving linear moments from log moments.

Simplification for Econometric Models

- Time dependency of moments often dropped for simplicity.
- Linear moments ($\boldsymbol{\mu}$, $\boldsymbol{\Sigma}$) referred to by default, especially under i.i.d. model assumption with no time dependency.

Portfolio

Portfolio Definition

- Allocation of budget among N risky assets, with uninvested amount as cash.
- Represented by capital allocation $\boldsymbol{w}_t^{\text{cap}} \in \mathbb{R}^N$, where $w_{i,t}^{\text{cap}}$ is the dollar amount for asset i.
- Cash represented separately as $c_t^{\text{cap}} \in \mathbb{R}$ or included in $\boldsymbol{w}_t^{\text{cap}}$ as a riskless asset.
- Portfolio can also be represented by units held $\boldsymbol{w}_t^{\text{units}} \in \mathbb{R}^N$ (e.g., shares or coins).

Dollar Amount Variation

- Keeping unit amount constant, dollar amount changes with asset prices:

$$\boldsymbol{w}_t^{\text{cap}} = \boldsymbol{w}^{\text{units}} \odot \boldsymbol{p}_t.$$

- Portfolio time evolution:

$$\boldsymbol{w}_t^{\text{cap}} = \boldsymbol{w}_{t-1}^{\text{cap}} \odot \left(\mathbf{1} + \boldsymbol{r}_t^{\text{lin}}\right).$$

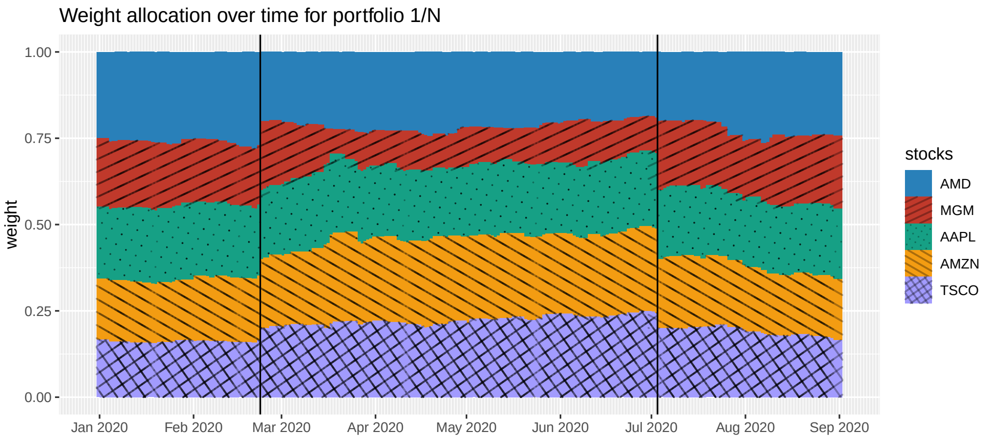
- Cash remains constant unless adjusted for contributions or withdrawals.

Rebalancing for Fixed Dollar Amount

- To maintain a fixed dollar allocation, portfolios require regular rebalancing.
- Rebalancing adjusts $\boldsymbol{w}_t^{\text{cap}}$ to original positions, incurring transaction costs.

Portfolio Rebalancing

Evolution of the $1/N$ portfolio over time with effect of rebalancing (vertical lines):



NAV and Return

Portfolio NAV Definition

- The portfolio *net asset value* (NAV) at time t includes current market valuation and cash:

$$\mathsf{NAV}_t \triangleq \mathbf{1}^\mathsf{T} \boldsymbol{w}_t^{\mathsf{cap}} + c_t^{\mathsf{cap}}.$$

NAV Evolution

- Evolution derived from portfolio time evolution:

$$\mathsf{NAV}_t = \mathsf{NAV}_{t-1} + \left(\boldsymbol{w}_{t-1}^{\mathsf{cap}}\right)^\mathsf{T} \boldsymbol{r}_t^{\mathsf{lin}}.$$

- Indicates NAV change depends on assets' returns.

Portfolio Return

- Defined as the change in NAV relative to previous NAV:

$$R_t^{\mathsf{portf}} \triangleq \frac{\mathsf{NAV}_t - \mathsf{NAV}_{t-1}}{\mathsf{NAV}_{t-1}} = \boldsymbol{w}_{t-1}^\mathsf{T} \boldsymbol{r}_t^{\mathsf{lin}}.$$

NAV and Return

Normalized Portfolio

- Portfolio normalized with respect to current NAV:

$$\boldsymbol{w}_t = \boldsymbol{w}_t^{\mathrm{cap}} / \mathrm{NAV}_t.$$

- Used in portfolio design and optimization.

Asset Additivity Property

Linear returns exhibit asset additivity, crucial for portfolio return calculation.

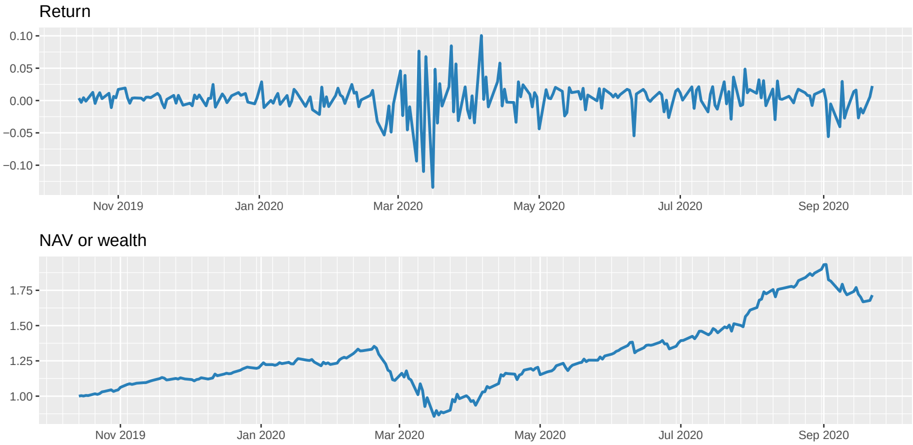
Adjustment for Cash Contributions

If cash contributions occur, adjust portfolio return calculation:

$$R_t^{\mathrm{portf}} \triangleq \frac{\mathrm{NAV}_t - \mathrm{NAV}_{t-1} - \mathrm{contribution}_{t-1}}{\mathrm{NAV}_{t-1}} = \boldsymbol{w}_{t-1}^{\mathsf{T}} \boldsymbol{r}_t^{\mathrm{lin}}.$$

NAV and Return

Return and NAV of the $1/N$ portfolio over time:



Cumulative Return and NAV

Cumulative Return and NAV

- Cumulative return: $\mathrm{NAV}_t/\mathrm{NAV}_0 - 1$.
- Does not include contributions or redemptions.
- Cumulative return plot starts at zero; NAV plot starts at 1 (if normalized).

Importance of Position Size

- Cumulative return and NAV depend on portfolio design and invested budget.
- Budget may include cash reserves.
- Capital invested is referred to as *position size*.

Cumulative Return and Sizing

Full Reinvesting

- Normalized portfolio $\boldsymbol{w}_t$ with full reinvestment of current NAV: $\boldsymbol{w}_t^{\text{cap}} = \text{NAV}_t \times \boldsymbol{w}_t$.
- NAV evolution:

$$\text{NAV}_t = \text{NAV}_{t-1} \times \left(1 + R_t^{\text{portf}}\right).$$

- Geometric growth:

$$\text{NAV}_t = \text{NAV}_0 \times \left(1 + R_1^{\text{portf}}\right) \times \cdots \times \left(1 + R_t^{\text{portf}}\right).$$

Constant Reinvesting

- Normalized portfolio $\boldsymbol{w}_t$ with constant reinvestment of initial NAV: $\boldsymbol{w}_t^{\text{cap}} = \text{NAV}_0 \times \boldsymbol{w}_t$.
- NAV evolution:

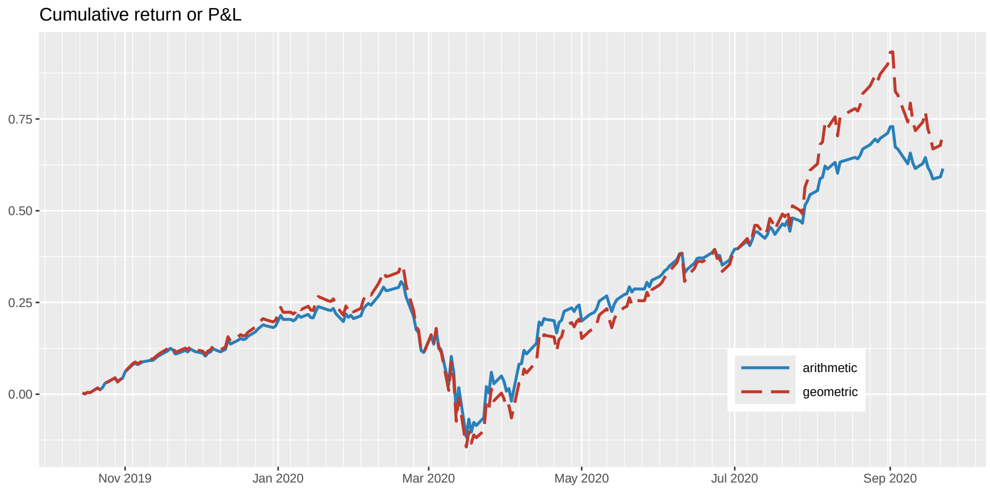
$$\text{NAV}_t = \text{NAV}_{t-1} + \text{NAV}_0 \times R_t^{\text{portf}}.$$

- Arithmetic growth:

$$\text{NAV}_t = \text{NAV}_0 \times \left(1 + R_1^{\text{portf}} + R_2^{\text{portf}} + \cdots + R_t^{\text{portf}}\right).$$

Cumulative return

Comparison of arithmetic and geometric cumulative returns:

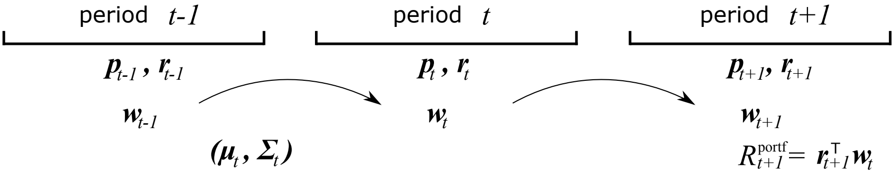


Portfolio Notation Over Time*

Portfolio Time Notation

- Portfolio at time t is denoted as $\boldsymbol{w}_t$.
- $\boldsymbol{w}_t$ uses information up to time $t-1$ (moments $\boldsymbol{\mu}_t$ and $\boldsymbol{\Sigma}_t$).
- Executed at time t, assuming instantaneous execution.

Portfolio Notation Over Time Periods



Portfolio Notation Over Time*

Incorrect Return Calculation

- Multiplying $\boldsymbol{w}_t$ with return at the same period $\boldsymbol{r}_t^{\text{lin}}$ is incorrect.
- This would incur look-ahead bias (refer to the topic of backtesting and its dangers).

Correct Return Calculation

- $\boldsymbol{w}_t$ is the portfolio executed and held at time t, with prices $\boldsymbol{p}_t$.
- Its return can be computed upon observing prices $\boldsymbol{p}_{t+1}$.
- Correct return calculation: $\boldsymbol{w}_t^\mathsf{T}\boldsymbol{r}_{t+1}^{\text{lin}}$.

Alternative Time Index Definition

- Some authors define returns with a shift in the time index.
- This allows $\boldsymbol{p}_t$ to be multiplied with $\boldsymbol{r}_t^{\text{lin}}$.
- Care must be taken to maintain consistency in notation and avoid look-ahead bias.

Portfolio Notation Over Time*

First- and Second-Order Moments in Portfolio Optimization

- Expected value and variance are crucial for making informed investment decisions.
- Optimization techniques aim to find portfolios with desirable expected returns and risk levels.

Expected Value of Portfolio Return

- Given by:

$$\mathbb{E}\left[R_t^{\text{portf}}\right] = \boldsymbol{w}_{t-1}^\mathsf{T}\boldsymbol{\mu}.$$

- Represents the mean return of the portfolio.

Variance of Portfolio Return

- Given by:

$$\text{Var}\left[R_t^{\text{portf}}\right] = \boldsymbol{w}_{t-1}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w}_{t-1}.$$

- Represents the risk or volatility of the portfolio return.

Transaction Costs

Transaction Costs Overview

- Consist of commission fee and slippage.
- Commission fee: payment to broker, varies by country and broker.
- Slippage: difference between expected and actual execution price (liquidity).

Impact on Portfolio NAV

- Rebalancing to $\boldsymbol{w}_t^{\text{cap,reb}}$ decreases NAV by transaction cost $\phi(\boldsymbol{w}_t^{\text{cap}} \to \boldsymbol{w}_t^{\text{cap,reb}})$.
- NAV after rebalancing and price change:

$$\text{NAV}_t = \text{NAV}_{t-1} + \left(\boldsymbol{w}_{t-1}^{\text{cap,reb}}\right)^{\mathsf{T}} \boldsymbol{r}_t^{\text{lin}} - \phi\left(\boldsymbol{w}_{t-1}^{\text{cap}} \to \boldsymbol{w}_{t-1}^{\text{cap,reb}}\right).$$

- Portfolio return with transaction costs:

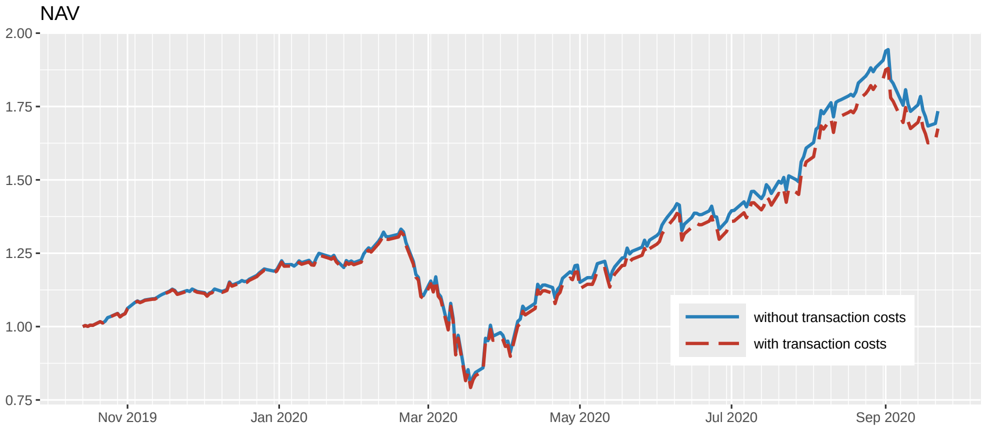
$$R_t^{\text{portf}} = \left(\boldsymbol{w}_{t-1}^{\text{reb}}\right)^{\mathsf{T}} \boldsymbol{r}_t^{\text{lin}} - \phi\left(\boldsymbol{w}_{t-1} \to \boldsymbol{w}_{t-1}^{\text{reb}}\right).$$

Practical Considerations

- Account for transaction costs in backtests for realistic performance assessment.
- Monitor strategy turnover and return on turnover due to transaction cost implications.

Transaction Costs

Effect of transaction costs on the NAV (with daily rebalancing and 90 bps of fees):



Transaction Costs*

Expected Value and Variance with Transaction Costs

- Expected value:
$$\mathbb{E}\left[R_t^{\text{portf}}\right] = \left(\boldsymbol{w}_{t-1}^{\text{reb}}\right)^{\mathsf{T}} \boldsymbol{\mu} - \phi\left(\boldsymbol{w}_{t-1} \rightarrow \boldsymbol{w}_{t-1}^{\text{reb}}\right).$$
- Variance:
$$\operatorname{Var}\left[R_t^{\text{portf}}\right] = \left(\boldsymbol{w}_{t-1}^{\text{reb}}\right)^{\mathsf{T}} \boldsymbol{\Sigma} \boldsymbol{w}_{t-1}^{\text{reb}}.$$

Transaction Cost Function: Decomposed into fees and slippage:
$$\phi\left(\boldsymbol{w}_t \rightarrow \boldsymbol{w}_t^{\text{reb}}\right) = \phi^{\text{fees}}\left(\boldsymbol{w}_t \rightarrow \boldsymbol{w}_t^{\text{reb}}\right) + \phi^{\text{slippage}}\left(\boldsymbol{w}_t \rightarrow \boldsymbol{w}_t^{\text{reb}}\right).$$

- Fees proportional to turnover:
$$\phi^{\text{fees}}\left(\boldsymbol{w}_t \rightarrow \boldsymbol{w}_t^{\text{reb}}\right) \approx \tau^{\text{fee}} \left\|\boldsymbol{w}_t^{\text{reb}} - \boldsymbol{w}_t\right\|_1.$$
- Slippage varies by asset:
$$\phi^{\text{slippage}}\left(\boldsymbol{w}_t \rightarrow \boldsymbol{w}_t^{\text{reb}}\right) \approx \sum_{i=1}^{N} \tau_i^{\text{slippage}} \left|w_{it}^{\text{reb}} - w_{it}\right|.$$

Portfolio Rebalancing

Portfolio Rebalancing Trade-off

- Frequent rebalancing aligns the market portfolio with the desired one.
- Less frequent rebalancing reduces transaction costs.

Rebalancing Schemes

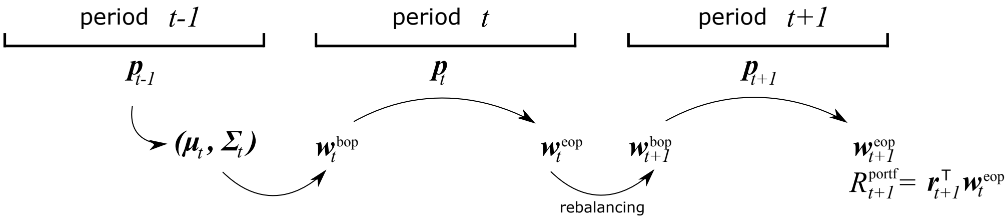
- Calendar-based: daily, weekly, or monthly rebalancing.
- Adaptive: rebalance only if portfolio distance exceeds a threshold.
- Measures of difference: turnover, Mahalanobis distance, tracking error squared distance.

Turnover Considerations

Turnover is often upper-bounded or penalized to avoid excessive rebalancing.

Portfolio Rebalancing*

Portfolio notation over time periods with rebalancing:



Time-index notation for rebalancing:

- Period t divided into beginning of period (bop) and end of period (eop).
- Portfolio $\boldsymbol{w}_t$ separated into $\boldsymbol{w}_t^{\text{bop}}$ and $\boldsymbol{w}_t^{\text{eop}}$.

Portfolio Rebalancing*

Price Change and Rebalancing

- Price change:

$$\boldsymbol{w}_t^{\text{eop}} = \boldsymbol{w}_t^{\text{bop}} \odot \left(\mathbf{1} + \boldsymbol{r}_t^{\text{lin}}\right).$$

- Rebalancing incurs transaction costs; if none, then $\boldsymbol{w}_{t+1}^{\text{bop}} = \boldsymbol{w}_t^{\text{eop}}$.

Fixed Portfolio Rebalancing

- After rebalancing (without transaction costs):

$$\boldsymbol{w}_t^{\text{bop}} = \boldsymbol{w}.$$

- After price change:

$$\boldsymbol{w}_t^{\text{eop}} = \boldsymbol{w} \odot \left(\mathbf{1} + \boldsymbol{r}_t^{\text{lin}}\right).$$

- Portfolio return:

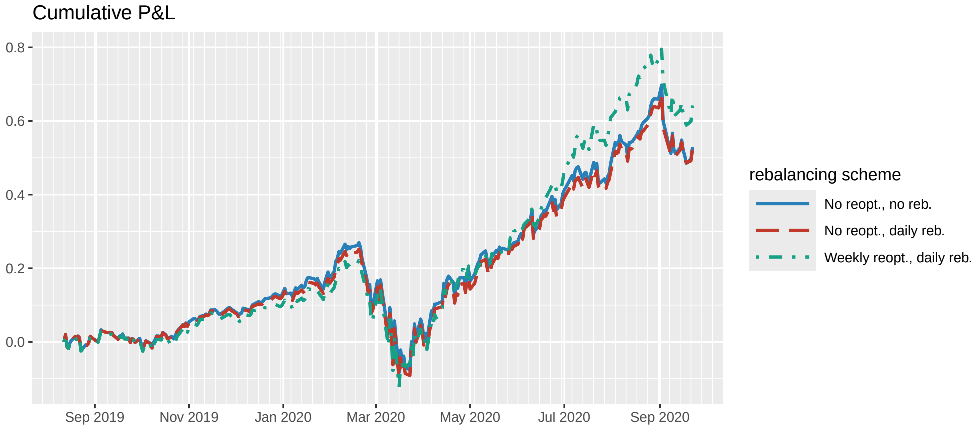
$$R_t^{\text{portf}} = \boldsymbol{w}^\mathsf{T}\boldsymbol{r}_t^{\text{lin}}.$$

Effect of Reoptimization and Rebalancing

- Different schemes affect performance, especially with transaction costs.
- Rebalancing vs. reoptimization: execution in the market vs. updating portfolio design.

Portfolio rebalancing*

Backtest cumulative P&L of a portfolio with different reoptimization and rebalancing schemes:



Outline

Portfolio Constraints

Common Portfolio Constraints

- Imposed by regulators or brokers: shorting constraints, leverage constraints, margin requirements.
- Optional (investor's views): market neutrality, portfolio sparsity, diversity enforcement.
- Intro reference: (Palomar 2025, chap. 6).

Examples

- Long-only or no-shorting constraint
- Capital budget constraint
- Holding constraints
- Cardinality constraint
- Turnover constraint
- Market-neutral constraint
- Dollar-neutral constraint
- Diversification constraint
- Leverage constraint
- Margin requirements

Portfolio Constraints

Long-Only or No-Shorting Constraint

- Investors typically take long positions.
- Short selling allowed in some markets, implying negative elements in $\boldsymbol{w}$.
- Constraint for no shorting:

$$\boldsymbol{w} \geq \mathbf{0},$$

  which is linear and convex.

Capital Budget Constraint

- Without shorting or leverage:

$$\mathbf{1}^{\mathsf{T}}\boldsymbol{w} + c = 1,$$

  linear and convex.
- Without cash variable:

$$\mathbf{1}^{\mathsf{T}}\boldsymbol{w} \leq 1.$$

- If equality is imposed, $\mathbf{1}^{\mathsf{T}}\boldsymbol{w} = 1$, the implication is that the cash is zero and the portfolio is fully invested in the risky assets.

Portfolio Constraints

Holding Constraints

- Limits on maximum positions for diversification.
- May include minimum positions for desired asset holdings.
- Holding constraints are imposed via lower and upper bounds:

$$\boldsymbol{l} \leq \boldsymbol{w} \leq \boldsymbol{u},$$

which is linear and convex.

Cardinality Constraint

- Limits the number of active positions to simplify operations:

$$\|\boldsymbol{w}\|_0 \leq K,$$

nonconvex and challenging to handle.

Turnover Constraint

- Controls portfolio turnover to manage transaction costs:

$$\|\boldsymbol{w} - \boldsymbol{w}_0\|_1 \leq u,$$

convex as norms are convex functions.

Portfolio Constraints

Market-Neutral Constraint

- Avoids market exposure, ensuring portfolio is orthogonal to market "beta":

$$\boldsymbol{\beta}^\mathsf{T}\boldsymbol{w} = \mathbf{0},$$

  linear and convex.

Dollar-Neutral Constraint

- Balances long and short positions:

$$\mathbf{1}^\mathsf{T}\boldsymbol{w} = \mathbf{0}.$$

- Should be used with a leverage constraint due to practical limitations on short selling.

Diversification Constraint

- Promotes allocation across multiple assets:

$$\|\boldsymbol{w}\|_2^2 \leq D,$$

  where $1/N \leq D < 1$.
- Uses the Herfindahl index as a diversification measure, convex constraint.

Portfolio Constraints

Leverage Constraint

- Limits borrowing for long and short positions.
- The ℓ_1-norm measures the total amount of long and short positions:

$$\|\boldsymbol{w}\|_1 \leq L,$$

where L indicates the leverage.

Margin Requirements

- Brokers impose margin requirements for borrowing:

$$m^{\text{long}} \times \mathbf{1}^{\mathsf{T}} \boldsymbol{w}^{+} + m^{\text{short}} \times \mathbf{1}^{\mathsf{T}} \boldsymbol{w}^{-} \leq 1, .$$

- Separates portfolio into longs ($\boldsymbol{w}^{+}$) and shorts ($\boldsymbol{w}^{-}$) with specific margin requirements for each.

Outline

Performance Measures

Portfolio Performance Assessment

- Recall the expression of the portfolio return:
$$R_t^{\text{portf}} = \boldsymbol{w}^\mathsf{T}\boldsymbol{r}_t.$$
- All the performance measures are derived from the portfolio return.
- References: (Bacon 2008), (Palomar 2025, chap. 6).

Examples

- Expected return
- Volatility
- Sharpe ratio and information ratio
- Downside risk and semivariance
- Gain-loss ratio
- Sortino ratio
- Value-at-risk (VaR) and Conditional value-at-risk (CVaR)
- Drawdown
- Calmar ratio and Sterling ratio

Performance Measures: Expected Return

Expected Return

- One-period expected return:

$$\mathbb{E}\left[R_t^{\text{portf}}\right] = \boldsymbol{w}^\mathsf{T}\boldsymbol{\mu}.$$

- Reflects average investment benefit, excluding risk.

Annualized Return

- Scaled by number of periods per year, γ:

$$\text{Annualized Return} = \gamma \times \boldsymbol{w}^\mathsf{T}\boldsymbol{\mu}.$$

- For daily prices: $\gamma = 252$ (stock market) or $\gamma = 365$ (cryptocurrency market).
- For hourly prices: $\gamma = 8,760$ (cryptocurrency market).

Performance Measures: Expected Return

Sample Mean Annualized Return

Arithmetic chaining (assuming T returns):

$$\frac{\gamma}{T}\sum_{t=1}^{T} R_t^{\text{portf}}.$$

Compound Annual Growth Rate (CAGR)

- Geometric chaining for compounding:

$$\left(\prod_{t=1}^{T}\left(1+R_t^{\text{portf}}\right)\right)^{\gamma/T} - 1.$$

- Geometric mean is lower than arithmetic mean before annualization.

Performance Measures: Volatility

Volatility

- Standard deviation of portfolio return:

$$\mathrm{Std}\left[R_t^{\mathrm{portf}}\right] = \sqrt{\boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w}}.$$

- Simplest risk measure, variance used by Markowitz.

Annualized Volatility

Scaled by $\sqrt{\gamma}$:

$$\text{Annualized Volatility} = \sqrt{\gamma} \times \sqrt{\boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w}}.$$

Performance Measures: Volatility-Adjusted Returns

Volatility-Adjusted Returns

- Adjusts returns for volatility comparison:

$$\bar{R}_t^{\text{portf}} = \boldsymbol{w}^\mathsf{T} \boldsymbol{r}_t \times \frac{\sigma^{\text{target}}}{\sigma_t},$$

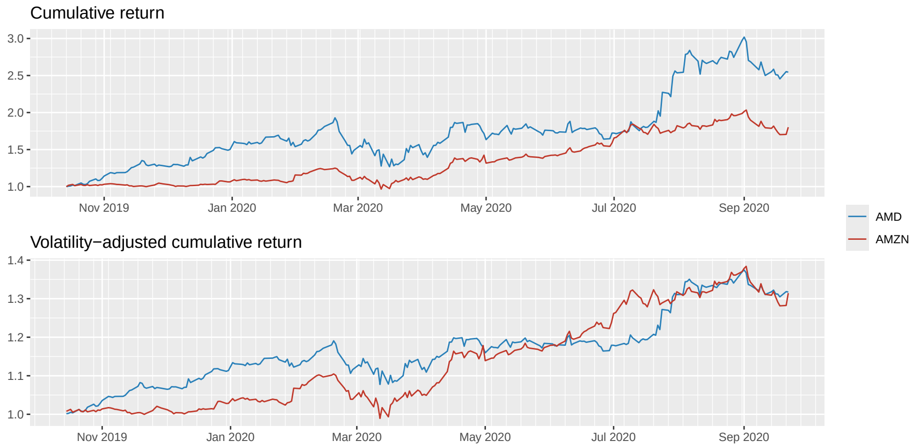
- σ_t: volatility of $\boldsymbol{w}^\mathsf{T} \boldsymbol{r}_t$.
- σ^{target}: desired target volatility.

Cumulative Returns Comparison

- Non-adjusted may mislead due to different volatilities.
- Volatility-adjusted provides a better return-volatility trade-off.
- Sharpe ratio captures this trade-off.

Performance Measures: Volatility-Adjusted Returns

Cumulative returns and volatility-adjusted cumulative returns:



Performance Measures: Sharpe Ratio

Sharpe Ratio (SR)

- Risk-adjusted expected excess return:

$$\mathrm{SR} = \frac{\boldsymbol{w}^\mathsf{T}\boldsymbol{\mu} - r_\mathrm{f}}{\sqrt{\boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w}}},$$

- r_f: risk-free rate.
- Measures return per unit of risk.

Annualized Sharpe Ratio

Adjusted for number of periods per year:

$$\text{Annualized SR} = \sqrt{\gamma} \times \frac{\boldsymbol{w}^\mathsf{T}\boldsymbol{\mu} - r_\mathrm{f}}{\sqrt{\boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w}}}.$$

Performance Measures: Sharpe Ratio

Geometric Sharpe Ratio

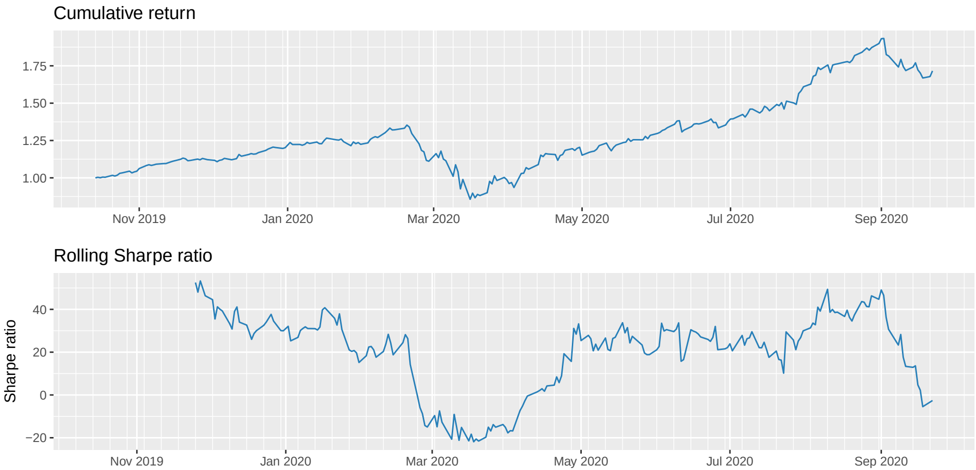
- Uses compounded annualized return in the numerator.
- Based on geometric mean of returns.

Rolling Sharpe Ratio

- SR computed on a rolling-window basis.
- Useful for assessing performance over time.

Performance measures: Rolling Sharpe ratio

Cumulative returns and rolling annualized Sharpe ratio (overall SR of 1.65):



Performance Measures: Information Ratio

Information Ratio (IR)

- Compares portfolio return to a benchmark:

$$\mathrm{IR} = \frac{\mathbb{E}\left[\boldsymbol{w}^\mathsf{T}\boldsymbol{r}_t - r_t^\mathrm{b}\right]}{\mathrm{Std}\left[\boldsymbol{w}^\mathsf{T}\boldsymbol{r}_t - r_t^\mathrm{b}\right]},$$

- r_t^b: return of the benchmark.
- Measures excess return per unit of risk relative to the benchmark.

Performance Measures: Downside Risk and Semivariance

Downside Risk and Semivariance

- Variance measures deviation from mean return:

$$\mathrm{Var}\left[\mathbf{w}^{\mathsf{T}}\mathbf{r}_t\right] = \mathbb{E}\left[\left(\mathbf{w}^{\mathsf{T}}\mathbf{r}_t - \mathbf{w}^{\mathsf{T}}\boldsymbol{\mu}\right)^2\right].$$

- Downside risk focuses on underperformance relative to a reference level.

Semivariance

- Measures risk of return falling below the mean:

$$\mathrm{SemiVar}\left[\mathbf{w}^{\mathsf{T}}\mathbf{r}_t\right] = \mathbb{E}\left[\left(\left(\mathbf{w}^{\mathsf{T}}\boldsymbol{\mu} - \mathbf{w}^{\mathsf{T}}\mathbf{r}_t\right)^+\right)^2\right],$$

- $(\cdot)^+$ avoids penalizing above-mean performance.

Performance Measures: Downside Risk and Semivariance*

Semi-Deviation/Downside Deviation

Square root of semivariance, analogous to volatility.

Lower Partial Moments (LPM)

- General downside risk measure:

$$\mathrm{LPM}_{\alpha}(\tau) = \mathbb{E}\left[\left(\left(\tau - \boldsymbol{w}^{\mathsf{T}}\boldsymbol{r}_t\right)^{+}\right)^{\alpha}\right],$$

- τ: disaster level.
- α: reflects investor's risk attitude (risk-averse, neutral, risk-seeking).
- Can form various downside measures by choosing α and τ (e.g., semivariance with $\alpha = 2$ and $\tau = \boldsymbol{w}^{\mathsf{T}}\boldsymbol{\mu}$).

Gain-Loss Ratio (GLR)

- Downside risk measure comparing positive and negative returns.
- Formula:

$$\text{GLR} = \frac{\mathbb{E}\left[R_t^{\text{portf}} \mid R_t^{\text{portf}} > 0\right]}{-\mathbb{E}\left[R_t^{\text{portf}} \mid R_t^{\text{portf}} < 0\right]}.$$

- Reflects the average gain to average loss ratio.
- Note: the denominator is made positive for a meaningful ratio by taking its absolute value.

Performance Measures: Sortino Ratio*

Sortino Ratio

- Adjusts for downside risk instead of total volatility.
- Formula:

$$\text{SoR} = \frac{\boldsymbol{w}^\mathsf{T}\boldsymbol{\mu} - r_\mathsf{f}}{\sqrt{\text{SemiVar}\left[\boldsymbol{w}^\mathsf{T}\boldsymbol{r}_t\right]}},$$

- Focuses on undesirable volatility (downside risk).
- More appropriate for investors concerned primarily with downside risk.

Performance Measures: Value-at-Risk (VaR)

Value-at-Risk (VaR)

- VaR focuses on the tail of the distribution, indicating potential large losses.
- Measures maximum loss at a specified confidence level (e.g., 95%).
- Defined as the quantile of the loss distribution:

$$\mathrm{VaR}_\alpha = \inf \left\{ \xi_0 \mid \Pr\left(\xi \leq \xi_0\right) \geq \alpha \right\},$$

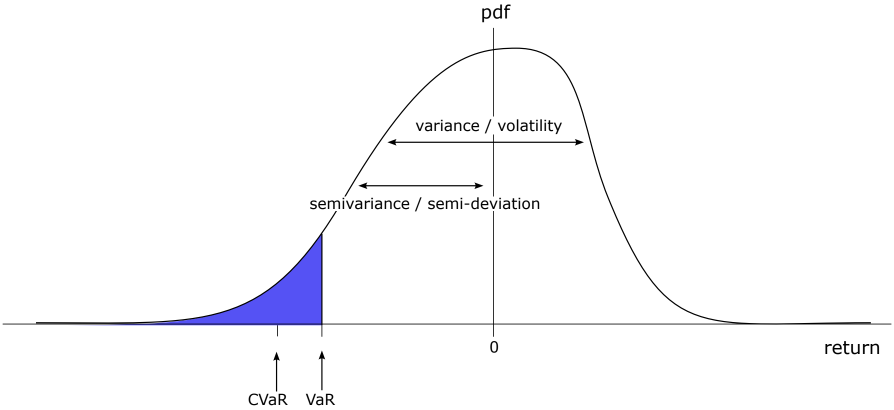
  where $\xi_t = -\boldsymbol{w}^\mathsf{T}\boldsymbol{r}_t$ represents the loss.
- α is the confidence level (e.g., $\alpha = 0.95$ for 95% confidence).

Drawbacks of VaR

- Does not quantify the shape of the tail beyond the VaR threshold.
- Cannot indicate the magnitude of losses exceeding the VaR.
- Not subadditive, challenging the principle of diversification (merging two portfolios can increase perceived risk).

Performance Measures: Value-at-Risk (VaR)

Illustration of distribution of returns and measures of risk:



Performance Measures: Conditional Value-at-Risk (CVaR)

Conditional Value-at-Risk (CVaR)

- Expected value of the loss tail:

$$\mathrm{CVaR}_\alpha = \mathbb{E}\left[\xi \mid \xi \geq \mathrm{VaR}_\alpha\right].$$

- Also known as Expected Shortfall (ES).
- Takes into account the average value of losses exceeding the VaR.

Advantages of CVaR

- Provides insight into the magnitude of losses beyond the VaR threshold.
- Subadditive, aligning with the principle of diversification.

Performance Measures: Drawdown

Drawdown (DD)

- Measures decline from a historical peak in cumulative profit or NAV:

$$D_t = \mathrm{HWM}_t - \mathrm{NAV}_t, .$$

- High Water Mark (HWM):

$$\mathrm{HWM}_t = \max_{\tau \in [0,t]} \mathrm{NAV}_\tau .$$

Normalized Drawdown

- Relative decline from the peak:

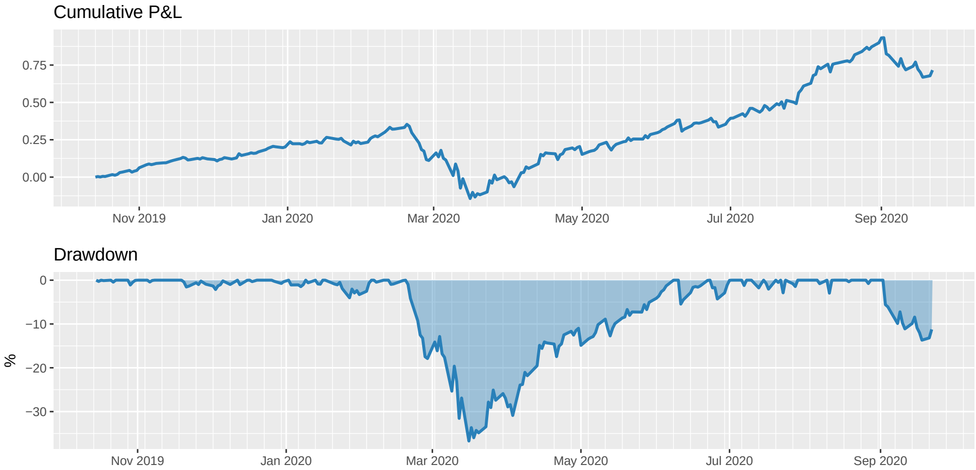
$$\bar{D}_t = \frac{\mathrm{HWM}_t - \mathrm{NAV}_t}{\mathrm{HWM}_t} .$$

Drawdown as a Path-Dependent Measure

Depends on the sequence of returns, unlike other performance measures.

Performance Measures: Drawdown

Cumulative P&L and corresponding drawdown of a portfolio:



Performance Measures: Drawdown*

Maximum Drawdown (Max-DD)

- Largest observed drawdown:

$$\text{Max-DD} = \max_{1 \leq t \leq T} \bar{D}_t;$$

Average Drawdown (Ave-DD)

- Mean of drawdowns over time:

$$\text{Ave-DD} = \frac{1}{T} \sum_{1 \leq t \leq T} \bar{D}_t;$$

Conditional Drawdown-at-Risk (CDaR)

- Expected drawdown in the worst cases:

$$\text{CDaR}_\alpha = \mathbb{E}\left[\bar{D}_t \mid \bar{D}_t \geq \text{VaR}_\alpha\right],$$

- VaR_α: value at risk of drawdown at confidence level α.

Performance Measures: Calmar and Sterling Ratios*

Calmar Ratio

- Adjusts Sharpe ratio by using maximum drawdown:

$$\text{Calmar ratio} = \frac{\boldsymbol{w}^\mathsf{T}\boldsymbol{\mu} - r_\mathrm{f}}{\text{Max-DD}}.$$

- Focuses on return per unit of maximum drawdown risk.

Sterling Ratio

- Similar to Calmar ratio, with an added risk measure:

$$\text{Sterling ratio} = \frac{\boldsymbol{w}^\mathsf{T}\boldsymbol{\mu} - r_\mathrm{f}}{\text{Max-DD} + 0.1}.$$

- Incorporates an excess risk measure to account for additional risk considerations.

Outline

Heuristic Portfolios

Heuristic Portfolios Overview

- Not mathematically derived but widely used in practice.
- Serve as benchmarks for portfolio performance.
- Include simple, rule-based strategies for asset allocation.

Examples

- Buy&hold (B&H)
- Global maximum return portfolio (GMRP)
- $1/N$ portfolio
- Quintile portfolios

Basic Reference: (Palomar 2025, chap. 6).

Heuristic Portfolios: B&H

Buy&Hold (B&H) Portfolio

- Simple investment strategy focusing on a single asset or a set of assets.
- Allocation:
$$\boldsymbol{w} = \boldsymbol{e}_i,$$
where $\boldsymbol{e}_i$ is an all-zero vector except for a one at the ith element.
- Lacks diversification in its single-asset form.
- Aimed at long-term investment without adjustments over short horizons.
- Based on the belief that the chosen asset(s) will appreciate over time.
- Asset selection can be guided by fundamental or technical analysis.

Heuristic Portfolios: GMRP

Global Maximum Return Portfolio (GMRP)

- Formulation:

$$\begin{array}{ll} \underset{\boldsymbol{w}}{\text{maximize}} & \boldsymbol{w}^{\mathsf{T}}\boldsymbol{\mu} \\ \text{subject to} & \mathbf{1}^{\mathsf{T}}\boldsymbol{w} = 1, \quad \boldsymbol{w} \geq \mathbf{0} \end{array}$$

- The problem can be solved with an LP solver.
- However, in this case the solution is trivial: it allocates all the budget to the asset with the highest forecast return.

Cautions with GMRP

- Lacks diversification, potentially leading to poor performance.
- Past performance is not indicative of future results.
- Estimation of $\boldsymbol{\mu}$ is often noisy and unreliable.

Heuristic Portfolios: $1/N$

Equally Weighted Portfolio (EWP) or $1/N$ Portfolio

- Allocates capital equally across all N assets:

$$\boldsymbol{w} = \frac{1}{N} \times \mathbf{1},$$

  where $\mathbf{1} \in \mathbb{R}^N$ is the all-one vector.
- Emphasizes diversification: "do not put all your eggs in the same basket."

Historical Context and Performance

- Also known as "Talmudic rule" based on ancient recommendation.
- Has shown superior historical performance and inspired equally weighted ETFs.
- S&P 500 equally weighted indices developed by Standard & Poor.
- Claims of outperforming Markowitz's mean-variance portfolio, though debated.

Heuristic Portfolios: Quintile Portfolios

Quintile Portfolio

- Allocates equally to top 20% of assets based on performance ranking.
- Can be adjusted for different fractions (e.g., quartile, decile).
- Long-short version involves short-selling bottom-ranked assets.

Portfolio Composition

- Assets ranked from top to bottom performers.
- Allocation:

$$\boldsymbol{w} = \frac{1}{N/5} \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} 1 \\ \vdots \\ 1 \end{matrix}}\right\} 20\% \\ \left.\vphantom{\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}}\right\} 80\% \end{matrix}$$

Heuristic Portfolios: Quintile Portfolios

Ranking and Factor Investing

- Ranking based on various factors (Value, Low Size, Low Volatility, Momentum, Quality).
- Factors derived from characteristics like book-to-price ratio, market capitalization, standard deviation, past returns, return on equity.
- Empirical evidence of excess returns for these factors.

Public Data Ranking Methods

- Expected performance:
$$\text{score} = \boldsymbol{\mu}.$$
- Sharpe ratios:
$$\text{score} = \frac{\boldsymbol{\mu}}{\sqrt{\operatorname{diag}(\boldsymbol{\Sigma})}}.$$
- Mean-variance ratios:
$$\text{score} = \frac{\boldsymbol{\mu}}{\operatorname{diag}(\boldsymbol{\Sigma})}.$$

Heuristic Portfolios as Robust Portfolios

The $1/N$ and Quintile Portfolios as Robust Portfolios

- In principle, they are just heuristic portfolios.
- Nevertheless, they can be formally derived as portfolios robust to estimation errors in $\boldsymbol{\mu}$.

Robust Formulation of GMRP

- Maximizes the worst-case return considering uncertainty in $\boldsymbol{\mu}$:

$$\begin{array}{ll} \underset{\boldsymbol{w}}{\text{maximize}} & \underset{\boldsymbol{\mu}\in\mathcal{S}}{\min}\; \boldsymbol{w}^\mathsf{T}\boldsymbol{\mu} \\ \text{subject to} & \mathbf{1}^\mathsf{T}\boldsymbol{w} = 1, \quad \boldsymbol{w} \geq \mathbf{0}, \end{array}$$

- Uncertainty set $\mathcal{S}$:

$$\mathcal{S} = \left\{\hat{\boldsymbol{\mu}} + \boldsymbol{e} \mid \|\boldsymbol{e}\|_1 \leq \epsilon\right\},$$

  where ϵ is the size of the uncertainty region.

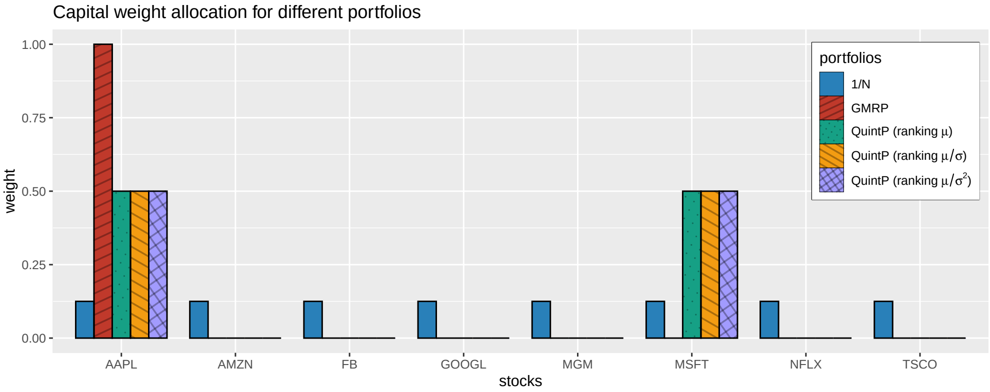
Connection to Quintile and 1/N Portfolios

- The quintile portfolio emerges as a solution for a specific ϵ.
- The $1/N$ portfolio arises as a solution for a sufficiently large ϵ.
- Explains the practical performance of these portfolios due to their robustness.
- Further details in (Palomar 2025, chap. 14).

Heuristic Portfolios: Numerical Experiments

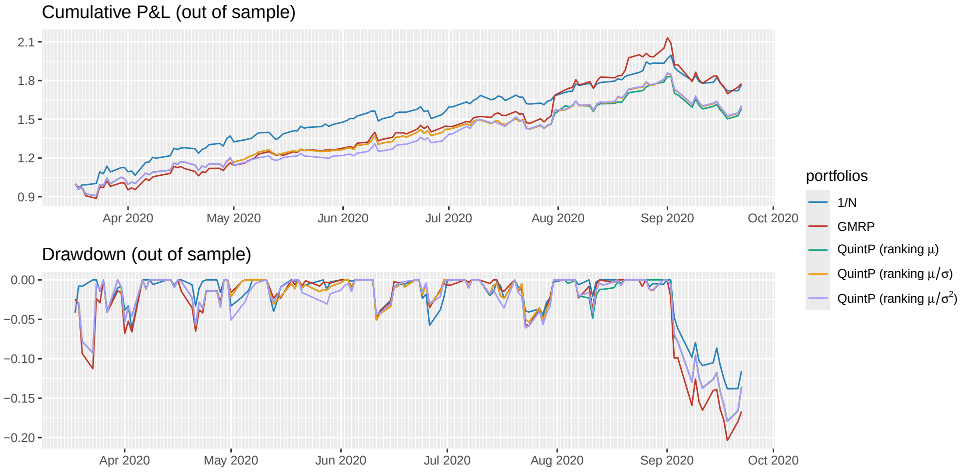
Comparison of heuristic portfolios:

- Portfolios: GMRP, $1/N$, and quintile portfolios (ranked by μ, μ/σ, μ/σ^2).
- Data: Eight S&P 500 stocks, 2019-2020.
- Backtest: 70% training data, 30% test data.



Heuristic Portfolios: Numerical Experiments

Backtest cumulative P&L and drawdown:



Heuristic Portfolios: Numerical Experiments

Backtest Performance Analysis

- GMRP shows high returns but increased volatility and drawdown.
- $1/N$ portfolio exhibits the best drawdown performance, indicating good diversification.

Considerations for Realistic Backtesting

- Multiple rolling-window backtests recommended for robust assessment.
- Chapter on backtesting (Palomar 2025, chap. 8) provides detailed methodologies.

Outline

Risk-Based Portfolios

Risk-Based Portfolios Overview

- Focus on risk considerations, utilizing the covariance matrix $\boldsymbol{\Sigma}$.
- Ignore return forecasts $\boldsymbol{\mu}$ due to their unreliable estimation.
- Aim to achieve diversification and risk management without relying on expected returns.

Characteristics

- Based on the premise that risk parameters ($\boldsymbol{\Sigma}$) are more reliably estimated than returns ($\boldsymbol{\mu}$).
- Seek to optimize portfolio construction through risk minimization or distribution.

Significance

- Address the challenge of high uncertainty in return forecasts.
- Provide a framework for portfolio construction when return predictions are deemed too speculative or volatile.

Risk-Based Portfolios

Examples

- Global minimum variance portfolio (GMVP)
- Inverse volatility portfolio (IVolP)
- Inverse variance portfolio (IVarP)
- Risk parity portfolio (RPP)
- Most diversified portfolio (MDP)
- Maximum decorrelation portfolio (MDCP)

References

- (Ardia et al. 2017) for detailed exploration of risk-based portfolios.
- (Chopra and Ziemba 1993) highlighting the issues with estimating $\boldsymbol{\mu}$ in practice.
- (Palomar 2025, chap. 6) for a detailed coverage of these slides.

Global Minimum Variance Portfolio (GMVP)

- Objective: Minimize portfolio variance.
- Formulation:

$$\begin{array}{ll} \underset{\boldsymbol{w}}{\text{minimize}} & \boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w} \\ \text{subject to} & \mathbf{1}^\mathsf{T}\boldsymbol{w} = 1, \quad \boldsymbol{w} \geq \mathbf{0}. \end{array}$$

- Convex quadratic program (QP) solvable with QP solvers.

Solution Without Shorting Constraints

The portfolio has the closed-form solution:

$$\boldsymbol{w} = \frac{1}{\mathbf{1}^\mathsf{T}\boldsymbol{\Sigma}^{-1}\mathbf{1}}\boldsymbol{\Sigma}^{-1}\mathbf{1}.$$

Significance

- Utilized for its simplicity and as a benchmark in academic research.
- Focuses on risk minimization through variance reduction.
- Often used to evaluate covariance matrix estimators, excluding $\boldsymbol{\mu}$ estimation effects.

Inverse Volatility Portfolio (IVolP)

- Aims to control risk by inversely weighting assets based on their volatility.
- Defined as:

$$\boldsymbol{w} = \frac{\boldsymbol{\sigma}^{-1}}{\mathbf{1}^\mathsf{T}\boldsymbol{\sigma}^{-1}},$$

where $\boldsymbol{\sigma}^2 = \mathrm{diag}\left(\boldsymbol{\Sigma}\right)$ and $\boldsymbol{\sigma}$ are the assets' volatilities.

Characteristics

- Lower weight to high volatility assets, higher weight to low volatility assets.
- Also known as *equal volatility portfolio*.
- Ensures weighted constituent assets have equal volatility contribution to the portfolio.

Comparison with GMVP

- The GMVP with $\boldsymbol{\Sigma}$ diagonal leads to the inverse variance portfolio (IVarP):

$$\boldsymbol{w} = \frac{\boldsymbol{\sigma}^{-2}}{\mathbf{1}^\mathsf{T}\boldsymbol{\sigma}^{-2}},$$

- Both strategies allocate inversely based on volatility, but the IVarP squares the volatilities (variance).

Risk-Based Portfolios: Risk Parity Portfolio (RPP)

Risk Parity Portfolio (RPP) or Equal Risk Portfolio (ERP)

- Aims to equalize risk contributions from all assets to the total portfolio risk.
- Considers both asset volatilities and correlations.
- Detailed exploration in (Palomar 2025, chap. 11).

Comparison with IVolP

- IVolP focuses on inversely weighting assets based on their volatilities, ignoring correlations.
- RPP advances this concept by incorporating asset correlations, offering a more comprehensive risk management approach.

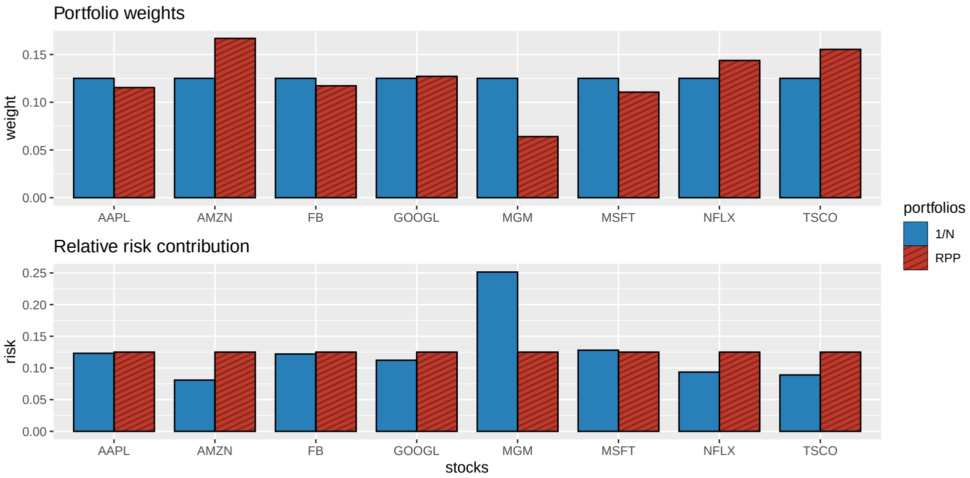
Objective

- Distribute risk evenly across assets, unlike $1/N$ portfolio which equalizes capital allocation.
- Ensures no single asset disproportionately affects the portfolio's overall risk.

Significance

- Recognizes and addresses the impact of asset correlations on portfolio risk.
- Provides a balanced approach to risk management, suitable for diversified investments.

From dollar diversification ($1/N$ portfolio) to risk diversification (RPP):



Risk-Based Portfolios: Most Diversified Portfolio (MDP)*

Most Diversified Portfolio (MDP)

- Utilizes the diversification ratio (DR) as a proxy for risk efficiency.
- Defined as:

$$\mathrm{DR} = \frac{\boldsymbol{w}^\mathsf{T}\boldsymbol{\sigma}}{\sqrt{\boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w}}}.$$

- Long-only portfolios have $\mathrm{DR} \geq 1$.

MDP Formulation

The formulation maximizes the DR:

$$\begin{array}{ll} \underset{\boldsymbol{w}}{\text{maximize}} & \dfrac{\boldsymbol{w}^\mathsf{T}\boldsymbol{\sigma}}{\sqrt{\boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w}}} \\ \text{subject to} & \mathbf{1}^\mathsf{T}\boldsymbol{w} = 1, \quad \boldsymbol{w} \geq \mathbf{0}. \end{array}$$

Significance

- The MDP aims to maximize diversification benefits by considering asset volatilities and correlations.
- It offers a portfolio construction approach that aligns with the concept of risk efficiency in markets.

Risk-Based Portfolios: Maximum Decorrelation Portfolio (MDCP)*

Maximum Decorrelation Portfolio (MDCP)

- Minimizes the portfolio correlation:

$$\begin{array}{ll} \underset{\boldsymbol{w}}{\text{minimize}} & \boldsymbol{w}^\mathsf{T}\boldsymbol{C}\boldsymbol{w} \\ \text{subject to} & \mathbf{1}^\mathsf{T}\boldsymbol{w} = 1, \quad \boldsymbol{w} \geq \mathbf{0}. \end{array}$$

- Correlation matrix $\boldsymbol{C}$ represents covariance of normalized assets with unit volatility.

Relation to GMVP and MDP

- MDCP is akin to GMVP with correlation matrix instead of covariance matrix.
- MDCP weights, when adjusted for asset volatilities, yield MDP weights.

Properties of MDCP

- Maximizes the Diversification Ratio (DR) when assets have equal volatility.
- Maximizes the Sharpe Ratio (SR) when assets have equal risks and returns.

Significance

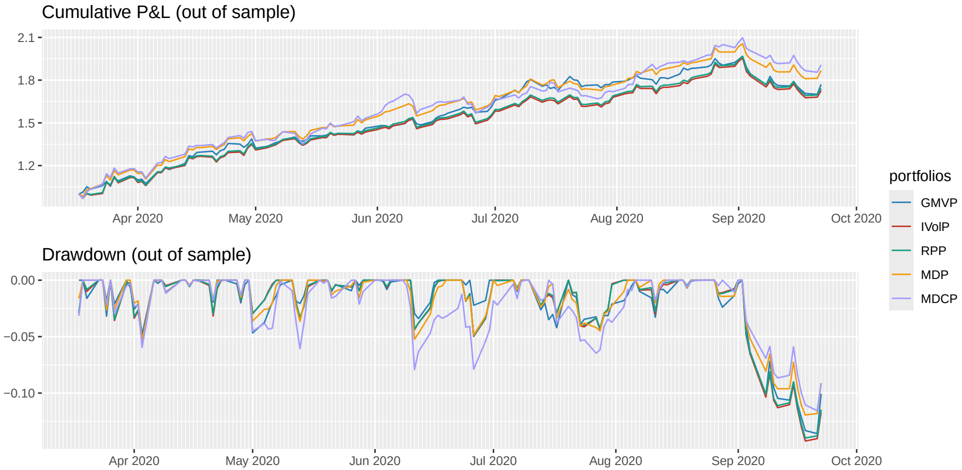
- MDCP focuses on reducing portfolio correlation, enhancing diversification.
- Offers an alternative risk management strategy by considering asset correlations.

Risk-Based Portfolios: : Numerical Experiments

Comparison of risk-based portfolios:

- Portfolios: GMVP, IVolP, RPP, MDP, and MDCP.
- Data: Eight S&P 500 stocks, 2019-2020.

Backtest cumulative P&L and drawdown:



Risk-Based Portfolios: Numerical Experiments

Backtest Performance Analysis

- Emphasizes the effectiveness of risk-based portfolios in minimizing drawdowns compared to heuristic portfolios.
- Showcases the impact of different risk minimization strategies on portfolio performance.

Observations

- Risk-based portfolios generally exhibit smaller drawdowns, highlighting their focus on risk minimization.
- The allocation and performance of these portfolios demonstrate the practical implications of different risk management approaches in portfolio construction.

Outline

Summary

- A portfolio of N assets is represented by vector $\boldsymbol{w}$, with return $\boldsymbol{w}^\mathsf{T}\boldsymbol{r}$ where $\boldsymbol{r}$ is the asset returns vector.
- Portfolios require periodic rebalancing, which incurs transaction costs that reduce returns.
- Portfolios must meet various constraints, including regulatory, broker-imposed, and investor-specific ones, most of which are convex for easy optimization, except sparsity control.
- Portfolio performance is assessed using metrics like Sharpe ratio and CVaR, and incorporating these into optimization creates complex formulations.
- Simple heuristic portfolios like the $1/N$ and quintile portfolios are easy to compute and historically effective.
- Risk-based portfolios aim to minimize return variability without market trend speculation, offering ease of computation and practical robustness.

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