UNIT 5 COMPLEX NUMBERS

2.1 ALGEBRA OF COMPLEX NUMBERS:

Definition – Real and Imaginary parts, Conjugates, Modulus and amplitude form, Polar form of a complex number, multiplication and division of complex numbers (geometrical proof not needed) – Simple Problems. Argan Diagram – Collinear points, four points forming square, rectangle, rhombus and parallelogram only. Simple problems.

2.2 DE MOIVRE'S THEOREM

Demoivre's Theorem (Statement only) – related simple problems.

2.3 ROOTS OF COMPLEX NUMBERS

Finding the nth roots of unity - solving equation of the form $x^n \pm 1 = 0$ where $n \leq 7$. Simple problems.

2.1 ALGEBRA OF COMPLEX NUMBERS

Introduction:

Let us consider the quadratic equation $ax^2 + bx + c = 0$. The solution of this equation is given by the formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ which is meaningful only when $b^2 - 4ac > 0$. Because the square of a real number is always positive and it cannot be negative. If it is negative, then the solution for the equation extends the real number system to a new kind of number system that allows the square root of negative numbers. The square root of – 1 is denoted by the symbol i, called the imaginary unit, which was first introduced in mathematics by the famous Swiss mathematician, Leonhard Euler in 1748. Thus for any two real numbers a and b, we can form a new number a + ib is called a **complex number**. The set of all complex numbers denoted by C and the nomenclature of a complex number was introduced by a German mathematician C.F. Gauss.

Definition: Complex Number

A number which is of the form a + ib where a, b $\in$ R and $i^2 = -1$ is called a complex number and it is denoted by z. If z = a + ib then *a* is called the real part of z and *b* is called the imaginary part of z and are denoted by Re(z) and Im(z).

For example, if z = 3 + 4i then Re (z) = 3 and Im (z) = 4.

Note:

In the complex number z = a + ib we have,

(i) If a = 0 then z is purely imaginary

(ii) If b = 0 then z is purely real.

(iii) z = a + ib = (a, b) any complex number can be expressed as an ordered pair.

Conjugate of a complex number:

If z = a + ib then the conjugate of z is defined by a – ib and it is denoted by $\bar{z}$. Thus, if z = a + ib then $\bar{z} = a - ib$.

Results:

(i) $\overline{\overline{z}} = z$

(ii) $a = \text{Re}(z) = \dfrac{z + \overline{z}}{2}$ & $b = I_m(z) = \dfrac{z - \overline{z}}{2}$

(iii) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

(iv) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$

(v) $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$

(vi) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}$ where $z_2 \neq 0$

(vii) $\overline{z^n} = (\overline{z})^n$

Algebra of complex numbers:

(i) Addition of two complex numbers:

Let $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers then their sum is defined as

$z_1 + z_2 = a + ib + c + id = (a + c) + i\,(b + d) \in C$

$z + \overline{z} = 2a$ Real number.

(ii) Difference of two complex numbers:

Let $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers then their difference is defined as

$z_1 - z_2 = (a + ib) - (c + id) = (a - c) + i\,(b - d) \in C$

$z - \overline{z} = 2ib$ Imaginary number.

(iii) Multiplication of two compelx numbers:

Let $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers then their product is defined as,

$$
\begin{aligned}
z_1\, z_2 &= (a + ib)\,(c + id) \\
&= a\,c + i\,ad + i\,bc + i^2\,bd \\
&= (ac - bd) + i\,(ad + bc) \in C
\end{aligned}
$$

$z\,\overline{z} = (a + ib)\,(a - ib) = a^2 + b^2$

(iv) Division of two complex numbers:

Let $z_1 = a + ib$ and $z_2 = c + id \neq 0$ be any two complex numbers then their quotient is defined as

$$\frac{z_1}{z_2} = \frac{a + ib}{c + id} \times \frac{c - id}{c - id} = \left[\frac{ac + bd}{c^2 + d^2}\right] + i\left[\frac{bc - ad}{c^2 + d^2}\right]$$

Modulus of a complex number:

If $z = a + ib$ is a complex number then the modulus (or) absolute value of z is defined as $\sqrt{a^2 + b^2}$ and is denoted by $|z|$. Thus, if $z = a + ib$ then $|z| = \sqrt{a^2 + b^2}$.

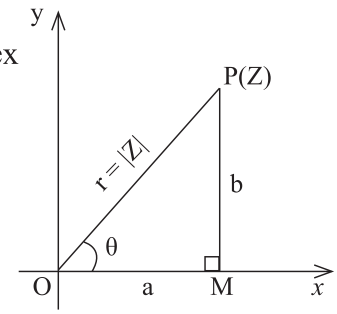
Note:

(i) $|\overline{z}| = |z| = \sqrt{a^2 + b^2}$

(ii) $|z| = \sqrt{z\overline{z}} = \sqrt{a^2 + b^2}$

(iii) $\text{Re}(z) \leq |z|$ and $\text{Im}(z) \leq |z|$

Polar form of a Complex Number:



Let (r, θ) be the Polar co-ordinates of the point P representing the complex number $z = a + ib$. Then from the fig. we get,

$$\cos\theta = \frac{OM}{OP} = \frac{a}{r} \text{ and } \sin\theta = \frac{PM}{OP} = \frac{b}{r}$$

$\Rightarrow a = r\cos\theta$ and $b = r\sin\theta$

where $r = \sqrt{a^2 + b^2} = |a + ib|$ is called the **modulus** of $z = a + ib$.

Also, $\tan\theta = \frac{b}{a} \Rightarrow \theta = \tan^{-1}\left(\frac{b}{a}\right)$ is called the **amplitude** or **argument** of $z = a + ib$ and denoted by amp(z) or arg(z) and is measured as the angle in positive sense. Thus, $\arg(z) = \theta = \tan^{-1}\left(\frac{b}{a}\right)$.

Hence $z = a + ib = r(\cos\theta + i\sin\theta)$ is called the Polar form or the modulus amplitude form of the complex number.

Theorems of Complex numbers:

1) The product of two complex numbers is a complex number whose modulus is the product of their modulii and whose amplitude is the sum of their amplitudes

   i.e, $|z_1 z_2| = |z_1||z_2|$

   and $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

2) The quotient of two complex numbers is a complex number whose modulus is the quotient of their modulii and whose amplitude is the difference of their amplitudes.

   i.e $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ wehre $Z_2 \neq 0$ and $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$.

Euler's formula:

The symbol $e^{i\theta}$ is defined by $e^{i\theta} = \cos\theta + i\sin\theta$ is known as Euler's formula.

If $z \neq 0$ then $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$. This is called the exponential form of the complex number z.

Note: If $z = re^{i\theta}$ then $\bar{z} = re^{-i\theta}$.

Multiplication and Division of complex numbers (Geometrical proof not needed)

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

be any two complex numbers in Polar form then their product is given by

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

Also the division of the above two complex numbers is given by

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] \text{ where } z_2 \neq 0.$$

WORKED EXAMPLES

PART – A

1. If $z_1 = 2 + 3i$ and $z_2 = 4 - 5i$ find $z_1 + z_2$.

Solution:

$$\begin{aligned}
\text{Given:} \quad z_1 &= 2 + 3i \quad \& \quad z_2 = 4 - 5i \\
z_1 + z_2 &= (2 + 3i) + (4 - 5i) \\
&= 2 + 3i + 4 - 5i \\
&= (2 + 4) + (3i - 5i)
\end{aligned}$$

$$\Rightarrow \boxed{z_1 + z_2 = 6 - 2i}$$

2. If $z_1 = 3 - 4i$ and $z_2 = -2 + 3i$ find the value of $2z_1 - 3z_2$.

Solution:

$$\begin{aligned}
\text{Given: } z_1 &= 3 - 4i \quad \& \quad z_2 = -2 + 3i \\
2z_1 - 3z_2 &= 2\,(3 - 4i) - 3\,(-2 + 3i) \\
&= 6 - 8i + 6 - 9i
\end{aligned}$$

$$\Rightarrow \boxed{2z_1 - 3z_2 = 12 - 17i}$$

3. Express: $(3 + 2i)\,(4 + 2i)$ in $a + ib$ form.

Solution:

$$\begin{aligned}
(3 + 2i)\,(4 + 2i) &= 12 + 6i + 8i + 4i^2 \\
&= 12 + 14i - 4 \\
&= 8 + 14i = a + ib \text{ form.}
\end{aligned}$$

4. Find the real and imaginary parts of $\dfrac{1}{3+2i}$.

Solution:

$$\begin{aligned}
\text{Let } z = \frac{1}{3+2i} &= \frac{1}{3+2i} \times \frac{3-2i}{3-2i} \\
&= \frac{3-2i}{(3)^2 - (2i)^2} \\
&= \frac{3-2i}{9+4} \\
&= \frac{3-2i}{13}
\end{aligned}$$

$$\Rightarrow \boxed{z = \frac{3}{13} - \frac{2i}{13}}$$

$$\therefore \text{Re}(z) = \frac{3}{13} \ \& \ \text{Im}(z) = \frac{-2}{13}$$

5. Find the conjugate of $\frac{1}{1+i}$.

Solution:

$$\text{Let } z = \frac{1}{1+i} = \frac{1}{1+i} \times \frac{1-i}{1-i}$$

$$= \frac{1-i}{(1)^2-(i)^2}$$

$$= \frac{1-i}{1+1}$$

$$= \frac{1-i}{2}$$

$$\Rightarrow \boxed{z = \frac{1}{2} - \frac{i}{2}}$$

$$\therefore \text{Conjugate}: \bar{z} = \frac{1}{2} + \frac{i}{2}$$

6. Find the modulus and amplitude of 1 + i.

Solution:

Let z = 1 + i

Here a = 1 & b = 1

$$\text{Modulus}: |z| = \sqrt{a^2+b^2} = \sqrt{(1)^2+(1)^2} = \sqrt{1+1} = \sqrt{2}$$

$$\text{and amp}(z) = \theta = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{1}{1}\right) = \tan^{-1}(1)$$

$$\Rightarrow \theta = 45^{o}$$

PART –B

1. Find the real and imaginary parts of $\frac{4+5i}{3-2i}$.

Solution:

$$\text{Let } z = \frac{4+5i}{3-2i} = \frac{4+5i}{3-2i} \times \frac{3+2i}{3+2i}$$

$$= \frac{12+8i+15i+10i^2}{(3)^2-(2i)^2}$$

$$= \frac{12+23i-10}{9+4}$$

$$= \frac{2+23i}{13}$$

$$z = \frac{2}{13} + \frac{23i}{13}$$

$$\therefore \text{Re}(z) = \frac{2}{13} \quad \& \quad \text{Im}(z) = \frac{23}{13}$$

2. Express the complex number $\frac{1}{3-2i}+\frac{1}{2-3i}$ in a + ib form.

Solution:

$$\text{Let } z=\frac{1}{3-2i}+\frac{1}{2-3i}$$

$$=\frac{1}{3-2i}\times\frac{3+2i}{3+2i}+\frac{1}{2-3i}+\frac{2+3i}{2+3i}$$

$$=\frac{3+2i}{3^2+2^2}+\frac{2+3i}{2^2+3^2}$$

$$=\frac{3+2i+2+3i}{13}$$

$$=\frac{5+5i}{13}$$

$$z=\frac{5}{13}+\frac{5}{13}i=a+ib \text{ form}$$

3. Find the modulus and argument of the complex number $\frac{1-i}{1+i}$.

Solution:

$$\text{Let } z=\frac{1-i}{1+i}=\frac{1-i}{1+i}\times\frac{1-i}{1-i}$$

$$=\frac{1-i-i+i^2}{(1)^2-(i)^2}$$

$$=\frac{1-2i-1}{1+1}$$

$$=\frac{-2i}{2}$$

$$z=-i \text{ where } a=0 \ \& \ b=-1$$

$$\text{Modulus}: |z|=\sqrt{a^2+b^2}=\sqrt{(0)^2+(-1)^2}=\sqrt{1}=1$$

$$\text{Argument}: \tan\theta=\frac{b}{a}=\frac{-1}{0}=\infty$$

$$\theta=\tan^{-1}(\infty)=90^o$$

The complex number – i = (0, – 1) lies IIIrd Quadrant.

Hence amplitude = $180^o + 90^o = 270^o$.

PART – C

1. Find the real and imaginary parts of the complex number $\frac{(1+i)\,(2-i)}{1+3i}$.

Solution:

$$\text{Let } z=\frac{(1+i)\,(2-i)}{1+3i}$$

$$=\frac{2-i+2i-i^2}{1+3i}$$

$$=\frac{2+i+1}{1+3i}$$

$$= \frac{3+i}{1+3i} \times \frac{1-3i}{1-3i}$$

$$= \frac{3-9i+i-3i^2}{(1)^2-(3i)^2}$$

$$= \frac{3-8i+3}{1+9}$$

$$= \frac{6-8i}{10}$$

$$= \frac{3-4i}{5}$$

$$z = \frac{3}{5} - \frac{4i}{5} = a+ib \text{ form.}$$

$$\therefore \text{Re}(z) = \frac{3}{5} \text{ \& } \text{Im}(z) = -\frac{4}{5}$$

2. Express the complex number $\frac{i-4}{3-2i} + \frac{4i+1}{2-3i}$ in a + ib form.

Solution:

$$\text{Let } z = \frac{i-4}{3-2i} + \frac{4i+1}{2-3i}$$

$$= \frac{i-4}{3-2i} \times \frac{3+2i}{3+2i} + \frac{4i+1}{2-3i} \times \frac{2+3i}{2+3i}$$

$$= \frac{3i-12+2i^2-8i}{3^2+2^2} + \frac{8i+2+12i^2+3i}{2^2+3^2}$$

$$= \frac{-5i-14}{13} + \frac{11i-10}{13}$$

$$= \frac{6i-24}{13}$$

$$= \frac{-24+6i}{13} = \frac{-24}{13} + \frac{6i}{13} = a+ib \text{ form}$$

3. Find the modulus and amplitude of $\frac{1+3\sqrt{3}i}{\sqrt{3}+2i}$.

Solution:

$$\text{Let } z = \frac{1+3\sqrt{3}i}{\sqrt{3}+2i}$$

$$= \frac{1+3\sqrt{3}i}{\sqrt{3}+2i} \times \frac{\sqrt{3}-2i}{\sqrt{3}-2i}$$

$$= \frac{\sqrt{3}-2i+9i-6\sqrt{3}i^2}{(\sqrt{3})^2-(2i)^2}$$

$$= \frac{\sqrt{3}+7i+6\sqrt{3}}{3+4}$$

$$= \frac{7\sqrt{3}+7i}{7}$$

$$= \frac{7(\sqrt{3}+i)}{7}$$

$$z = \sqrt{3}+i = a+ib \text{ form}$$

Here $a = \sqrt{3}$ & $b = 1$

$\therefore$ Modulus : $|z| = \sqrt{a^2 + b^2} = \sqrt{(3)^2 + (1)^2} = \sqrt{3+1} = \quad 4 = 2$

Amplitude : $\theta = \tan^{-1}\left(\dfrac{b}{a}\right) = \tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right) = 30^o$

4. Find the modulus and argument of the complex number $\dfrac{5-i}{2-3i}$.

Solution:

$$\text{Let } z = \frac{5-i}{2-3i}$$

$$= \frac{5-i}{2-3i} \times \frac{2+3i}{2+3i}$$

$$= \frac{10+15i-2i-3i^2}{(2)^2-(3i)^2}$$

$$= \frac{10+13i+3}{4+9}$$

$$= \frac{13+13i}{13}$$

$$= \frac{13(1+i)}{13}$$

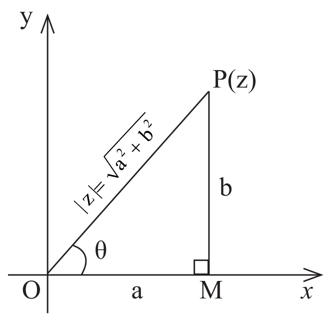
$z = 1 + i = a + ib$ form

Here $a = 1$ & $b = 1$

Modulus : $|z| = \sqrt{a^2 + b^2} = \sqrt{(1)^2 + (1)^2} = \sqrt{1+1} = \sqrt{2}$

Amplitude : $\theta = \tan^{-1}\left(\dfrac{b}{a}\right) = \tan^{-1}\left(\dfrac{1}{1}\right) = \tan^{-1}(1)$

$\Rightarrow \theta = 45^o$



Argand Diagram

Every complex number a + ib can be considered as an ordered pair (a, b) of real numbers, we can represent such number by a point in xy-plane called the complex plane and such a representation is also known as the argand diagram. The complex number z = a + ib represented by P(z) then the distance between z and the origin is the modulus. i.e $|z| = \sqrt{a^2 + b^2}$

Here the set of real numbers (x, 0) corresponds to the x-axis called real axis and the set of Imaginary numbers (0, y) correponds to the y-axis called the imaginary axis.

Result:

The distance between the two complex numbers z_1 and z_2 is $|z_1 - z_2|$. Thus, if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then $|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

Collinear Points:

If A, B and C are any three points representing the complex numbers $x_1 + iy_1$, $x_2 + iy_2$ and $x_3 + iy_3$ respectively, are collinear then the required condition is, the area of Δ ABC is zero.

i.e. $\dfrac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)] = 0$

2.2 DE-MOIVRE'S THEOREM

DeMoivre's Thoerem: (Statement only)

(i) If 'n' is an integer positive or negative then $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

(ii) If 'n' is a fraction, then $\cos n\theta + i\sin n\theta$ is one of the values of $(\cos\theta + i\sin\theta)^n$.

Results:

1) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

2) $(\cos\theta + i\sin\theta)^{-n} = (\cos n\theta - i\sin n\theta)$

3) $\dfrac{1}{\cos\theta + i\sin\theta} = (\cos\theta + i\sin\theta)^{-1} = \cos\theta - i\sin\theta$

4) $\dfrac{1}{\cos\theta - i\sin\theta} = (\cos\theta - i\sin\theta)^{-1} = \cos\theta + i\sin\theta$

5) $\sin\theta + i\cos\theta = \cos\left(\dfrac{\pi}{2} - \theta\right) + i\sin\left(\dfrac{\pi}{2} - \theta\right)$

Note:

1) $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)$

2) $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3) = \cos(\theta_1 + \theta_2 + \theta_3) + i\sin(\theta_1 + \theta_2 + \theta_3)$

3) $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \ldots (\cos\theta_n + i\sin\theta_n)$

$$= \cos(\theta_1 + \theta_2 + \ldots\ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \ldots\ldots + \theta_n)$$

WORKED EXAMPLES

PART – A

1. If $z = \cos 30^o + i\sin 30^o$ what is the value of z^3.

Solution:

$$\begin{aligned} z^3 &= [\cos 30^o + i\sin 30^o]^3 \\ &= \cos 90^o + i\sin 90^o \\ &= 0 + i(1) = i \end{aligned}$$

2. If $z = \cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}$ what is the value of z^8.

Solution:

$$\begin{aligned} z^8 &= \left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right]^8 \\ &= \cos 8\left(\frac{\pi}{2}\right) + i\sin 8\left(\frac{\pi}{2}\right) \\ &= \cos 4\pi + i\sin 4\pi \\ &= 1 + i(0) = 1 \end{aligned}$$

3. If $z = \cos 45^o - i\sin 45^o$ what is the value of $\dfrac{1}{z}$.

Solution:

$$\frac{1}{z} = z^{-1}$$

$= [\cos 45 - i \sin 45^o]^{-1}$

$= \cos 45^o + i \sin 45^o$

$= \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$

4. If $\frac{1}{z}$ = cos 60° + i sin 60° what is the value of z.

Solution:

$$z = \frac{1}{\frac{1}{z}}$$

$$= \frac{1}{\cos 60^o + i \sin 60^o} = (\cos 60^o + i \sin 60^o)^{-1}$$

$$= \cos 60^o - i \sin 60^o$$

$$= \frac{1}{2} - i\frac{\sqrt{3}}{2}$$

5. Find the value of $\frac{\cos 3\theta + i \sin 3\theta}{\cos\theta - i \sin\theta}$.

Solution:

$$\frac{\cos 3\theta + i \sin 3\theta}{\cos\theta - i \sin\theta} = (\cos 3\theta + i \sin 3\theta)(\cos\theta - i \sin\theta)^{-1}$$

$$= (\cos 3\theta + i \sin 3\theta)(\cos\theta + i \sin\theta)$$

$$= \cos(3\theta + \theta) + i \sin(3\theta + \theta)$$

$$= \cos 4\theta + i \sin 4\theta$$

6. Simplify: (cos 20° + i sin 20°) (cos 30° + i sin 30°) (cos 40° + i sin 40°)

Solution:

$(\cos 20^o + i \sin 20^o)(\cos 30^o + i \sin 30^o)(\cos 40^o + i \sin 40^o)$

$= \cos(20^o + 30^o + 40^o) + i \sin(20^o + 30^o + 40^o)$

$= \cos 90^o + i \sin 90^o$

$= 0 + i(1) = i$

7. If x = cos θ + i sin θ find $x + \frac{1}{x}$.

Solution:

Given: x = cos θ + i sin θ

$$\Rightarrow \quad \frac{1}{x} = \cos\theta - i \sin\theta$$

$$\therefore x + \frac{1}{x} = \cos\theta + i \sin\theta + \cos\theta - i \sin\theta$$

$$= 2\cos\theta$$

8. If x = cos α + i sin α and y = cos β + i sin β find xy.

Solution:

$$xy = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$$
$$= \cos(\alpha+\beta) + i\sin(\alpha+\beta)$$

9. If $a = \cos\alpha + i\sin\alpha$ and $b = \cos\beta + i\sin\beta$ find $\frac{a}{b}$.

Solution:

$$\frac{a}{b} = a(b)^{-1}$$
$$= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)^{-1}$$
$$= (\cos\alpha + i\sin\alpha)(\cos\beta - i\sin\beta)$$
$$= \cos(\alpha-\beta) + i\sin(\alpha-\beta)$$

10. Find the product of $3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ and $4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$.

Solution:

$$3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) \times 4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$
$$= 12\left[\cos\left(\frac{\pi}{3} + \frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{3} + \frac{\pi}{6}\right)\right]$$
$$= 12\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right]$$
$$= 12\,[1 + i(0)] = 12$$

PART – B

1. If $x = \cos\theta + i\sin\theta$ find the value of $x^m + \frac{1}{x^m}$.

Solution:

Given: $x = \cos\theta + i\sin\theta$

$\Rightarrow x^m = (\cos\theta + i\sin\theta)^m = \cos m\theta + i\sin m\theta$

also, $\frac{1}{x^m} = (x^m)^{-1}$

$$= [\cos m\theta + i\sin m\theta]^{-1}$$

$$\Rightarrow \quad \frac{1}{x^m} = \cos m\theta - i\sin m\theta$$

$$\therefore x^m + \frac{1}{x^m} = \cos m\theta + i\sin m\theta + \cos m\theta - i\sin m\theta = 2\cos m\theta$$

2. If $a = \cos\alpha + i\sin\alpha$ and $b = \cos\beta + i\sin\beta$ find $ab + \frac{1}{ab}$.

Solution:

Given: $a = \cos\alpha + i\sin\alpha$

& $b = \cos\beta + i\sin\beta$

$ab = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$

$\Rightarrow ab = \cos(\alpha+\beta) + i\sin(\alpha+\beta)$

and $\frac{1}{ab} = \cos(\alpha+\beta) - i\sin(\alpha+\beta)$

$$\therefore ab + \frac{1}{ab} = \cos(\alpha+\beta) + i\sin(\alpha+\beta) + \cos(\alpha+\beta) - i\sin(\alpha+\beta)$$
$$= 2\cos(\alpha+\beta)$$

3. Prove that $(\sin\theta + i\cos\theta)^n = \cos n\left(\frac{\pi}{2}-\theta\right) + i\sin\left(\frac{\pi}{2}-\theta\right)$.

Solution:

We have $\sin\theta = \cos\left(\frac{\pi}{2}-\theta\right)$

& $\cos\theta = \sin\left(\frac{\pi}{2}-\theta\right)$

LHS: $(\sin\theta + i\cos\theta)^n = \left[\cos\left(\frac{\pi}{2}-\theta\right) + i\sin\left(\frac{\pi}{2}-\theta\right)\right]^n$

$= \cos n\left(\frac{\pi}{2}-\theta\right) + i\sin n\left(\frac{\pi}{2}-\theta\right)$

4. If x = cos 3α + i sin 3α, y = cos 3β + i sin 3β find the value of $\sqrt[3]{xy}$.

Solution:

Given: x = cos 3α + i sin 3α

& y = cos 3β + i sin 3β

xy = (cos 3α + i sin 3α) (cos 3β + i sin 3β)

= cos (3α + 3β) + i sin (3α + 3β)

⇒ xy = cos 3 (α + β) + i sin 3 (α + β)

$\sqrt[3]{xy} = (xy)^{1/3}$

$= [\cos 3(\alpha+\beta) + i\sin 3(\alpha+\beta)]^{1/3}$

$= \cos\frac{1}{3}.3(\alpha+\beta) + i\sin\frac{1}{3}.3(\alpha+\beta)$

$= \cos(\alpha+\beta) + i\sin(\alpha+\beta)$

5. If a = cos α + i sin α, b = cos β + i sin β and c = cos γ + i sin γ find the value of $\frac{ab}{c}$.

Solution:

Given: a = cos α + i sin α

b = cos β + i sin β

& c = cos γ + i sin γ

$\therefore \frac{ab}{c} = ab(c)^{-1}$

= (cos α + i sin α) (cos β + i sin β) $(\cos\gamma + i\sin\gamma)^{-1}$

= (cos α + i sin α) (cos β + i sin β) (cos γ – i sin γ)

$\Rightarrow \frac{ab}{c} = \cos(\alpha+\beta-\gamma) + i\sin(\alpha+\beta-\gamma)$

PART – C

1. Simplify: $\frac{(\cos 2\theta + i\sin 2\theta)^3(\cos 3\theta - i\sin 3\theta)^4}{(\cos 3\theta + i\sin 3\theta)^2(\cos 4\theta + i\sin 4\theta)^{-3}}$

Solution:

$$\frac{(\cos 2\theta + i\sin 2\theta)^3 (\cos 3\theta - i\sin 3\theta)^4}{(\cos 3\theta + i\sin 3\theta)^2 (\cos 4\theta + i\sin 4\theta)^{-3}}$$

$$= \frac{(\cos\theta + i\sin\theta)^{3\times 2} (\cos\theta + i\sin\theta)^{4\times -3}}{(\cos\theta + i\sin\theta)^{2\times 3} (\cos\theta + i\sin\theta)^{-3\times 4}}$$

$$= \frac{(\cos\theta + i\sin\theta)^{6} (\cos\theta + i\sin\theta)^{-12}}{(\cos\theta + i\sin\theta)^{6} (\cos\theta + i\sin\theta)^{-12}}$$

$$= (\cos\theta + i\sin\theta)^{6-12-6+12}$$

$$= (\cos\theta + i\sin\theta)^{0}$$

$$= \cos 0 + i\sin 0$$

$$= 1 + i(0) = 1$$

2. Simplify : $\dfrac{(\cos 2\theta + i\sin 2\theta)^3 (\cos 4\theta - i\sin 4\theta)^3}{\cos 3\theta + i\sin 3\theta}$ when $\theta = \dfrac{\pi}{9}$.

Solution:

$$\frac{(\cos 2\theta + i\sin 2\theta)^3 (\cos 4\theta - i\sin 4\theta)^3}{\cos 3\theta + i\sin 3\theta}$$

$$= \frac{(\cos\theta + i\sin\theta)^{3\times 2} (\cos\theta + i\sin\theta)^{3\times -4}}{(\cos\theta + i\sin\theta)^{3}}$$

$$= \frac{(\cos\theta + i\sin\theta)^{6} (\cos\theta + i\sin\theta)^{-12}}{(\cos\theta + i\sin\theta)^{3}}$$

$$= (\cos\theta + i\sin\theta)^{6-12-3}$$

$$= (\cos\theta + i\sin\theta)^{-9}$$

$$= \cos 9\theta - i\sin 9\theta \qquad \text{when } \theta = \frac{\pi}{9}$$

$$= \cos 9\left(\frac{\pi}{9}\right) - i\sin 9\left(\frac{\pi}{9}\right)$$

$$= \cos\pi - i\sin\pi$$

$$= -1 - i(0) = -1$$

3. Prove that $\left(\dfrac{\cos\theta + i\sin\theta}{\sin\theta + i\cos\theta}\right)^4 = \cos 8\theta + i\sin 8\theta \cdot$

Solution:

$$\text{LHS}: \left(\frac{\cos\theta + i\sin\theta}{\sin\theta + i\cos\theta}\right)^4 = \left[\frac{\cos\theta + i\sin\theta}{\sin\theta + i\cos\theta} \times \frac{i}{i}\right]^4$$

$$= (i)^4 \left[\frac{\cos\theta + i\sin\theta}{i\sin\theta + i^2\cos\theta}\right]^4$$

$$= 1\left[\frac{\cos\theta + i\sin\theta}{-\cos\theta + i\sin\theta}\right]^4$$

$$= \left[\frac{\cos\theta + i\sin\theta}{-(\cos\theta - i\sin\theta)}\right]^4$$

$$= \left[\frac{\cos\theta + i\sin\theta}{(\cos\theta + i\sin\theta)^{-1}}\right]^4$$

$$= \left[\left(\cos\theta + i\sin\theta\right)^{1+1}\right]^4$$

$$= \left[(\cos\theta + i\sin\theta)^2\right]^4$$

$$= (\cos\theta + i\sin\theta)^8$$

$$= \cos 8\theta + i\sin 8\theta = \text{RHS}$$

4. Prove that $\left[\dfrac{1+\cos\theta + i\sin\theta}{1+\cos\theta - i\sin\theta}\right]^n = \cos n\theta + i\sin n\theta$.

Solution:

Let $z = \cos\theta + i\sin\theta$

$$\Rightarrow \frac{1}{z} = \cos\theta - i\sin\theta$$

$$\text{LHS}: \left[\frac{1+\cos\theta + i\sin\theta}{1+\cos\theta - i\sin\theta}\right]^n$$

$$= \left[\frac{1+z}{1+\frac{1}{z}}\right]^n$$

$$= \left[\frac{1+z}{\frac{z+1}{z}}\right]^n$$

$$= \left[\frac{z(1+z)}{(1+z)}\right]^n$$

$$= z^n$$

$$= (\cos\theta + i\sin\theta)^n$$

$$= \cos n\theta + i\sin n\theta = \text{RHS}$$

5. Show that $\left[\dfrac{1+\sin A + i\cos A}{1+\sin A - i\cos A}\right]^n = \cos n\left(\dfrac{\pi}{2} - A\right) + i\sin n\left(\dfrac{\pi}{2} - A\right)$.

Solution:

Let $z = \sin A + i\cos A$

$$\Rightarrow z = \cos\left(\frac{\pi}{2} - A\right) + i\sin\left(\frac{\pi}{2} - A\right)$$

$$\therefore \frac{1}{z} = \cos\left(\frac{\pi}{2} - A\right) - i\sin\left(\frac{\pi}{2} - A\right) = \sin A - i\cos A$$

$$\text{LHS}\left[\frac{1+\sin A+i\cos A}{1+\sin A-i\cos A}\right]^n$$

$$=\left[\frac{1+z}{1+\frac{1}{z}}\right]^n$$

$$=\left[\frac{1+z}{\frac{z+1}{z}}\right]^n$$

$$=\left[\frac{z(1+z)}{(1+z)}\right]^n$$

$$=(z)^n$$

$$=\left[\cos\left(\frac{\pi}{2}-A\right)+i\sin\left(\frac{\pi}{2}-A\right)\right]^n$$

$$=\cos n\left(\frac{\pi}{2}-A\right)+i\sin n\left(\frac{\pi}{2}-A\right)$$

6. If $a = \cos\theta + i\sin\theta$, $b = \cos\phi + i\sin\phi$ prove that

(i) $\cos(\theta+\phi)=\frac{1}{2}\left[ab+\frac{1}{ab}\right]$

(ii) $\sin(\theta-\phi)=\frac{1}{2i}\left[\frac{a}{b}-\frac{b}{a}\right]$

Solution:

Given $a = \cos\theta + i\sin\theta$

& $b = \cos\phi + i\sin\phi$

Now, $ab = (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)$

$\Rightarrow ab = \cos(\theta+\phi) + i\sin(\theta+\phi)$ .....(1)

also, $\frac{1}{ab}=(ab)^{-1}=[\cos(\theta+\phi)+i\sin(\theta+\phi)]$

$\Rightarrow \frac{1}{ab}=\cos(\theta+\phi)-i\sin(\theta+\phi)$.........(2)

$\therefore$ (1) + (2) $\Rightarrow$

$$ab+\frac{1}{ab}=\cos(\theta+\phi)+i\sin(\theta+\phi)+\cos(\theta+\phi)-i\sin(\theta+\phi)$$

$$\Rightarrow ab+\frac{1}{ab}=2\cos(\theta+\phi)$$

$$\Rightarrow \cos(\theta+\phi)=\frac{1}{2}\left[ab+\frac{1}{ab}\right]$$

(ii) $\frac{a}{b}=a(b)^{-1}$

$=(\cos\theta+i\sin\theta)(\cos\phi+i\sin\phi)^{-1}$

$=(\cos\theta+i\sin\theta)(\cos\phi-i\sin\phi)$

$$\Rightarrow \frac{a}{b} = \cos(\theta - \phi) + i\sin(\theta - \phi) ....(3)$$

$$\text{also, } \frac{b}{a} = \left(\frac{a}{b}\right)^{-1} = \left[\cos(\theta - \phi) + i\sin(\theta - \phi)\right]^{-1}$$

$$\Rightarrow \frac{b}{a} = \cos(\theta - \phi) - i\sin(\theta - \phi) \quad ....(4)$$

$$(3) - (4) \Rightarrow$$

$$\frac{a}{b} - \frac{b}{a} = \cos(\theta - \phi) + i\sin(\theta - \phi) - \cos(\theta - \phi) + i\sin(\theta - \phi)$$

$$\Rightarrow \frac{a}{b} - \frac{b}{a} = 2i\sin(\theta - \phi)$$

$$\Rightarrow \sin(\theta - \phi) = \frac{1}{2i}\left[\frac{a}{b} - \frac{b}{a}\right]$$

7. If a = cos x + i sin x, b = cos y + i sin y prove that

(i) $\sqrt{ab} + \frac{1}{\sqrt{ab}} = 2\cos\left(\frac{x+y}{2}\right)$

(ii) $\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} = 2\cos\left(\frac{x-y}{2}\right)$

Solution:

Given: a = cos x + i sin x

& b = cos y + i sin y

(i) Now, ab = (cos x + i sin x) (cos y + i sin y)

$\Rightarrow$ ab = cos (x + y) + i sin (x + y)

$$\therefore \sqrt{ab} = (ab)^{1/2} = \left[\cos(x+y) + i\sin(x+y)\right]^{1/2}$$

$$\Rightarrow \sqrt{ab} = \cos\left(\frac{x+y}{2}\right) + i\sin\left(\frac{x+y}{2}\right) \quad ....(1)$$

$$\text{also, } \frac{1}{\sqrt{ab}} = \left(\sqrt{ab}\right)^{-1} = \left[\cos\left(\frac{x+y}{2}\right) + i\sin\left(\frac{x+y}{2}\right)\right]^{-1}$$

$$\Rightarrow \frac{1}{\sqrt{ab}} = \cos\left(\frac{x+y}{2}\right) - i\sin\left(\frac{x+y}{2}\right) \quad ......(2)$$

$$\therefore (1) + (2) \Rightarrow$$

$$\sqrt{ab} + \frac{1}{\sqrt{ab}} = \cos\left(\frac{x+y}{2}\right) + i\sin\left(\frac{x+y}{2}\right) + \cos\left(\frac{x+y}{2}\right) - i\sin\left(\frac{x+y}{2}\right)$$

$$\Rightarrow \sqrt{ab} + \frac{1}{\sqrt{ab}} = 2\cos\left(\frac{x+y}{2}\right)$$

$$\text{(ii) } \frac{a}{b} = a(b)^{-1} = (\cos x + i\sin x)(\cos y + i\sin y)^{-1}$$

$$= (\cos x + i\sin x)(\cos y - i\sin y)$$

$$\Rightarrow \quad \frac{a}{b} = \cos(x - y) + i\sin(x - y)$$

$$\therefore \sqrt{\frac{a}{b}} = \left(\frac{a}{b}\right)^{1/2} = \left[\cos(x-y) + i\sin(x-y)\right]^{1/2}$$

$$\Rightarrow \quad \sqrt{\frac{a}{b}} = \cos\left(\frac{x-y}{2}\right) + i\sin\left(\frac{x-y}{2}\right) \quad \text{........(3)}$$

also, $$\sqrt{\frac{b}{a}} = \left(\sqrt{\frac{a}{b}}\right)^{-1} = \left[\cos\left(\frac{x-y}{2}\right) + i\sin\left(\frac{x-y}{2}\right)\right]^{-1}$$

$$\Rightarrow \quad \sqrt{\frac{b}{a}} = \cos\left(\frac{x-y}{2}\right) - i\sin\left(\frac{x-y}{2}\right) \quad \text{......(4)}$$

$\therefore (3) + (4) \Rightarrow$

$$\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} = \cos\left(\frac{x-y}{2}\right) + i\sin\left(\frac{x-y}{2}\right) + \cos\left(\frac{x-y}{2}\right) - i\sin\left(\frac{x-y}{2}\right)$$

$$\Rightarrow \sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} = 2\cos\left(\frac{x-y}{2}\right)$$

8. If $a = \cos\alpha + i\sin\alpha$, $b = \cos\beta + i\sin\beta$ and $c = \cos\gamma + i\sin\gamma$ find the value of $\frac{ab}{c} - \frac{c}{ab}$.

Solution:

Given:

$a = \cos\alpha + i\sin\alpha$

$b = \cos\beta + i\sin\beta$

& $c = \cos\gamma + i\sin\gamma$

Now,

$$\frac{ab}{c} = ab(c)^{-1}$$

$$= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)^{-1}$$

$$= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma - i\sin\gamma)$$

$$\Rightarrow \quad \frac{ab}{c} = \cos(\alpha+\beta-\gamma) + i\sin(\alpha+\beta-\gamma) \quad \text{......(1)}$$

also, $$\frac{c}{ab} = \left[\frac{ab}{c}\right]^{-1}$$

$$= \left[\cos(\alpha+\beta-\gamma) + i\sin(\alpha+\beta-\gamma)\right]^{-1}$$

$$\Rightarrow \quad \frac{c}{ab} = \cos(\alpha+\beta-\gamma) - i\sin(\alpha+\beta-\gamma) \quad \text{........(2)}$$

$\therefore (1) - (2) \Rightarrow$

$$\frac{ab}{c} - \frac{c}{ab} = \cos(\alpha+\beta-\gamma) + i\sin(\alpha+\beta-\gamma) - \cos(\alpha+\beta-\gamma) + i\sin(\alpha+\beta-\gamma)$$

$$\Rightarrow \frac{ab}{c} - \frac{c}{ab} = 2i\sin(\alpha+\beta-\gamma)$$

9. If $x + \frac{1}{x} = 2\cos\theta$ prove that (i) $x^n + \frac{1}{x^n} = 2\cos n\theta$ (ii) $x^n - \frac{1}{x^n} = 2i\sin n\theta$.

Solution:

Given: $x+\frac{1}{x}=2\cos\theta$

$\frac{x^2+1}{x}=2\cos\theta$

$x^2+1=2x\cos\theta$

$x^2-2x\cos\theta+1=0$

Here a = 1, b = –2 cos θ & c = 1

$$\therefore x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{2\cos\theta\pm\sqrt{(-2\cos\theta)^2-4\times1\times1}}{2\times1}$$

$$=\frac{2\cos\theta\pm\sqrt{4\cos^2\theta-4}}{2}$$

$$=\frac{2\cos\theta\pm\sqrt{4(\cos^2\theta-1)}}{2}$$

$$=\frac{2\cos\theta\pm\sqrt{4(-\sin^2\theta)}}{2}$$

$$=\frac{2\cos\theta\pm i2\sin\theta}{2}$$

$$=\frac{2[\cos\theta\pm i\sin\theta]}{2}$$

$\Rightarrow x=\cos\theta\pm i\sin\theta$

Consider $x=\cos\theta+i\sin\theta$

$\therefore x^n=(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta$

& $\frac{1}{x^n}=\cos n\theta-i\sin n\theta$

(i) $x^n+\frac{1}{x^n}=\cos n\theta+i\sin n\theta+\cos n\theta-i\sin n\theta$

$\Rightarrow x^n+\frac{1}{x^n}=2\cos n\theta$

(ii) $x^n-\frac{1}{x^n}=\cos n\theta+i\sin n\theta-\cos n\theta+i\sin n\theta$

$\Rightarrow x^n-\frac{1}{x^n}=2i\sin n\theta$

10. Show that $(1+i)^n+(1-i)^n=2^{\frac{n+2}{2}}\cos\frac{n\pi}{4}$.

Solution:

Let $1+i=r(\cos\theta+i\sin\theta)=r\cos\theta+i\sin\theta$

Equating real & imaginary parts on both sides

$r\cos\theta = 1 \quad \& \quad r\sin\theta = 1$

Now, $(r\cos\theta)^2 + (r\sin\theta)^2 = (1)^2 + (1)^2$

$\Rightarrow r^2\cos^2\theta + r^2\sin^2\theta = 1 + 1$

$\Rightarrow r^2(\cos^2\theta + \sin^2\theta) = 2$

$\Rightarrow r^2(1) = 2$

$\Rightarrow r^2 = 2 \quad \Rightarrow \boxed{r = \sqrt{2}}$

Also, $\dfrac{r\sin\theta}{r\cos\theta} = \dfrac{1}{1}$

$\Rightarrow \tan\theta = 1$

$\Rightarrow \theta = \tan^{-1}(1)$

$\Rightarrow \boxed{\theta = \dfrac{\pi}{4}}$

$$\therefore 1 + i = \sqrt{2}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right] \quad ....(1)$$

Similarly we can prove that

$$1 - i = \sqrt{2}\left[\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right].......(2)$$

LHS: $(1 + i)^n + (1 - i)^n$

$$= \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^n + \left[\sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)\right]^n$$

$$= \left(\sqrt{2}\right)^n\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]^n + \left(\sqrt{2}\right)^n\left[\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right]^n$$

$$= \left(\sqrt{2}\right)^n\left[\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4} + \cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right]$$

$$= 2^{\frac{n}{2}}2\cos\frac{n\pi}{4}$$

$$= 2^{\frac{n}{2}+1}\cos\frac{n\pi}{4}$$

$$= 2^{\frac{n+2}{2}}\cos\frac{n\pi}{4} = \text{RHS}$$

2.3 ROOTS OF COMPLEX NUMBERS

Definition:

A number ω is called the n^{th} root of a complex number z, if $\omega^n = z$ and we write $\omega = z^{\frac{1}{n}}$.

Working rule to find the n^{th} roots of a complex numbers:

Step (I) : Write the given complex number in Polar form.

Step (II) : Add "$2k\pi$" to the argument.

Step (III) : Apply Demoivre's theorem

Step (IV) : Put $k = 0, 1, \ldots\ldots.$ upto $(n - 1)$.

Illustration:

Let $z = r(\cos\theta + i\sin\theta)$

$\Rightarrow z = r[\cos(2k\pi + \theta) + i\sin(2k\pi + \theta)]$ where $k \in I$

$$\therefore z^{\frac{1}{n}} = \{r[\cos(2k\pi + \theta) + i\sin(2k\pi + \theta)]\}^{\frac{1}{n}}$$

$$= r^{\frac{1}{n}}[\cos(2k\pi + \theta) + i\sin(2k\pi + \theta)]^{\frac{1}{n}}$$

$$= r^{\frac{1}{n}}\left[\cos\left(\frac{2k\pi + \theta}{n}\right) + i\sin\left(\frac{2k\pi + \theta}{n}\right)\right] \text{ where } K = 0, 1, 2, \ldots\ldots, n - 1.$$

Only these values of k will give 'n' different values of $z^{\frac{1}{n}}$ provided $z \neq 0$.

To find the n^{th} roots of unity

$1 = \cos 0 + i\sin 0 = \cos 2k\pi + i\sin 2k\pi$

$$\therefore n^{th} \text{ roots of unity} = 1^{1/n} = (\cos 2k\pi + i\sin 2k\pi)^{1/n}$$

$$= \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right) \text{ where } k = 0, 1, 2, \ldots\ldots\ldots\ldots\ldots, n - 1$$

The roots are,

for $k = 0$; $R_1 = \cos 0 + i\sin 0 = 1 + i0 = 1 = e^{i0}$

$$k = 1;\ R_2 = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n} = e^{i\frac{2\pi}{n}} = \omega \text{ (say)}$$

$$k = 2;\ R_3 = \cos\frac{4\pi}{n} + i\sin\frac{4\pi}{n} = e^{i\frac{4\pi}{n}} = \left[e^{i\frac{2\pi}{n}}\right]^2 = \omega^2$$

$$k = n - 1;\ R_n = \cos\frac{2(n-1)\pi}{n} + i\sin\frac{2(n-1)\pi}{n} = e^{i\frac{2(n-1)\pi}{n}} = \omega^{n-1}$$

$\therefore$ The n^{th} roots of unity are

$$e^{i0}, e^{i\frac{2\pi}{n}}, e^{i\frac{4\pi}{n}}, \ldots\ldots\ldots\ldots, e^{i\frac{2(n-1)\pi}{n}}$$

i.e., $1, \omega, \omega^2, \ldots\ldots\ldots\ldots, \omega^{n-1}$.

Result:

If ω is n^{th} roots of unity then

(i) $\omega^n = 1$

(ii) Sum of the roots is zero.

i.e $1 + \omega + \omega^2 + \ldots\ldots\ldots + \omega^{n-1} = 0$

(iii) The roots are in G.P with common ratio ω.

(iv) The arguments are in A.P with common difference $\frac{2\pi}{n}$.

(v) The product of the roots is $(-1)^{n+1}$.

To find cube roots of unity

Let $x = (1)^{1/3}$

$$= (\cos 0 + i\sin 0)^{1/3}$$

$$= (\cos 2k\pi + i\sin 2k\pi)^{1/3}$$

$$= \cos\left(\frac{2k\pi}{3}\right) + i\sin\left(\frac{2k\pi}{3}\right) \text{ where } k = 0, 1, 2$$

$\therefore$ The roots are

for $k = 0$; $R_1 = \cos 0 + i \sin 0 = 1 + i0 = 1$

$$k = 1;\quad R_2 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$k = 2;\quad R_3 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

The cube roots of unity are $1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}$.

Result:

If we denote the second root $R_2 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$ by ω then the other root ,

$R_3 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$ becomes ω^2

$$\text{Thus, } R_2 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2} = \omega$$

$$R_3 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2} = \omega^2$$

$\therefore$ The cube roots of unit are $1, \omega, \omega^2$

Note:

If ω is cube roots of unity then (i) $\omega^3 = 1$, (ii) $1 + \omega + \omega^2 = 0$

Fourth roots of unity

Let $x = (1)^{1/4}$

$$= (\cos 0 + i\sin 0)^{1/4}$$

$$= (\cos 2k\pi + i\sin 2k\pi)^{1/4}$$

$$= \cos\left(\frac{2k\pi}{4}\right) + i\sin\left(\frac{2k\pi}{4}\right) \text{ where } k = 0, 1, 2, 3$$

$\therefore$ The roots are,

for $k = 0$; $R_1 = \cos 0 + i \sin 0 = 1 + i0 = 1$

$$k = 1; \quad R_2 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = 0 + i(1) = i = \omega \text{ (say)}$$

$$k = 2; \quad R_3 = \cos\pi + i\sin\pi = -1 + i(0) = -1 = \omega^2$$

$$k = 3; \quad R_4 = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = 0 + i(-1) = -i = \omega$$

The fourth roots of unity are 1, i, – 1, – i (i.e.) 1, ω, ω^2, ω^3.

Note:

(i) The sum of the fourth roots of unity is zero. i.e $1 + \omega + \omega^2 + \omega^3 = 0$ and $\omega^4 = 1$.

(ii) The value of ω used in cube roots of untiy and in fourth roots of unity are different.

Sixth roots of unity

$$\text{Let } x = (1)^{1/6}$$

$$= (\cos 0 + i\sin 0)^{1/6}$$

$$= (\cos 2k\pi + i\sin 2k\pi)^{1/6}$$

$$= \cos\left(\frac{2k\pi}{6}\right) + i\sin\left(\frac{2k\pi}{6}\right) \text{ where } k = 0, 1, 2, 3, 4, 5$$

$\therefore$ The six roots are

for k = 0; $R_1 = \cos 0 + i \sin 0 = e^{i0} = 1$

$$k = 1; \quad R_2 = \cos\frac{2\pi}{6} + i\sin\frac{2\pi}{6} = e^{i\frac{2\pi}{6}} = \omega$$

$$k = 2; \quad R_3 = \cos\frac{4\pi}{6} + i\sin\frac{4\pi}{6} = e^{i\frac{4\pi}{6}} = \omega^2$$

$$k = 3; \quad R_4 = \cos\frac{6\pi}{6} + i\sin\frac{6\pi}{6} = e^{i\frac{6\pi}{6}} = \omega^3$$

$$k = 4; \quad R_5 = \cos\frac{8\pi}{6} + i\sin\frac{8\pi}{6} = e^{i\frac{8\pi}{6}} = \omega^4$$

$$k = 5; \quad R_6 = \cos\frac{10\pi}{6} + i\sin\frac{10\pi}{6} = e^{i\frac{10\pi}{6}} = \omega^5$$

$\therefore$ The sixth roots of unity are $e^{i0}, e^{i\frac{2\pi}{6}}, e^{i\frac{4\pi}{6}}, e^{i\frac{6\pi}{6}}, e^{i\frac{8\pi}{6}}, e^{i\frac{10\pi}{6}}$

i.e, 1, ω, ω^2, ω^3, ω^4, ω^5.

Note:

The sum of the sixth roots of unity is zero. i.e $1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 = 0$ and $\omega^6 = 1$

Note: $1 = \cos 0 + i \sin 0$

$-1 = \cos \pi + i \sin \pi$

$$i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$$

$$-i = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}$$

WORKED EXAMPLES

PART – A

1. If ω is a cube roots of unity, find the value of $\omega^4 + \omega^5 + \omega^6$.

Solution:

If ω is cube roots of unity then $\omega^3 = 1$.

$$\therefore \omega^4 + \omega^5 + \omega^6 = \omega^3 . \omega + \omega^3 . \omega^2 + \omega^3 . \omega^3$$
$$= (1)\, \omega + (1)\, \omega^2 + (1)\,(1)$$
$$= \omega + \omega^2 + 1$$
$$= 1 + \omega + \omega^2 = 0$$

2. Simplify: $(1 + \omega)(1 + \omega^2)$ where ω is cube roots of unity.

Solution:

$$(1 + \omega)(1 + \omega^2) = 1 + \omega^2 + \omega + \omega^3$$
$$= [1 + \omega + \omega^2] + \omega^3$$
$$= 0 + 1$$
$$= 1$$

3. Solve: $x^2 - 1 = 0$

Solution:

Given: $x^2 - 1 = 0$

$$x^2 = 1$$
$$x = (1)^{1/2} = [\cos 0 + i \sin 0]^{1/2}$$
$$= [\cos 2k\pi + i \sin 2k\pi]^{1/2}$$
$$= \cos\left(\frac{2k\pi}{2}\right) + i \sin\left(\frac{2k\pi}{2}\right) \text{ where } k = 0,1$$

4. Find the value of $(i)^{1/3}$.

Solution:

Let $x = (i)^{1/3}$

$$= \left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right]^{1/3}$$
$$= \left[\cos\left(2k\pi + \frac{\pi}{2}\right) + i\sin\left(2k\pi + \frac{\pi}{2}\right)\right]^{1/3}$$
$$= \cos\frac{1}{3}\left(2k\pi + \frac{\pi}{2}\right) + i\sin\frac{1}{3}\left(2k\pi + \frac{\pi}{2}\right) \quad \text{where } k = 0, 1, 2.$$

5. Find the value of $(-1)^{1/3}$.

Solution:

Let $x = (-1)^{1/3}$

$$= (\cos\pi + i\sin\pi)^{1/3}$$
$$= [\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)]^{1/3}$$
$$= \cos\frac{1}{3}(2k\pi + \pi) + i\sin\frac{1}{3}(2k\pi + \pi) \text{ where } k = 0, 1, 2$$

6. Find the value of $\left(\frac{-1+i\sqrt{3}}{2}\right)^3$.

Solution:

We have $\frac{-1+i\sqrt{3}}{2} = \frac{-1}{2} + i\frac{\sqrt{3}}{2}$ = cos 120° + i sin 120°

$$\left(\frac{-1+i\sqrt{3}}{2}\right)^3 = (\cos 120^o + i\sin 120^o)^3$$

$$= \cos 360^o + i\sin 360^o$$

$$= 1 + i(0)$$

$$= 1$$

PART – B

1. Find the cube roots of unity.

Solution:

Let 'x' be the cube roots of unity.

i.e. $x^3 = 1$

$$x = (1)^{1/3}$$

$$= (\cos 0 + i\sin 0)^{1/3}$$

$$= (\cos 2k\pi + i\sin 2k\pi)^{1/3}$$

$$= \cos\frac{2k\pi}{3} + i\sin\frac{2k\pi}{3} \quad \text{where } k = 0,1,2$$

For k = 0; x = cos 0 + i sin 0 = 1 + i0 = 1

$$k = 1;\ \ x = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$$

$$k = 2;\ \ x = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$$

2. Find all the values of $(i)^{2/3}$.

Solution:

Let $x = (i)^{2/3}$

$$= \left[\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\right]^{2/3}$$

$$= \left[\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^2\right]^{1/3}$$

$$= \left[\cos 2\left(\frac{\pi}{2}\right) + i\sin 2\left(\frac{\pi}{2}\right)\right]^{1/3}$$

$$= [\cos\pi + i\sin\pi]^{1/3}$$

$$= [\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)]^{1/3}$$

$$= \cos\left(\frac{2k\pi + \pi}{3}\right) + i\sin\left(\frac{2k\pi + \pi}{3}\right) \quad \text{where } k = 0, 1, 2$$

when $k = 0$; $x = \cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}$

$k = 1$; $x = \cos\dfrac{3\pi}{3} + i\sin\dfrac{3\pi}{3}$

$k = 2$; $x = \cos\dfrac{5\pi}{3} + i\sin\dfrac{5\pi}{3}$

3. Solve: $x^2 + 16 = 0$

Solution:

Given: $x^2 + 16 = 0$

$$x^2 = -16 = 16 \times -1$$

$$x = (16)^{1/2}(-1)^{1/2}$$

$$= 4[\cos\pi + i\sin\pi]^{1/2}$$

$$= 4[\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)]^{1/2}$$

$$= 4\left[\cos\left(\frac{2k\pi + \pi}{2}\right) + i\sin\left(\frac{2k\pi + \pi}{2}\right)\right] \quad \text{where } k = 0, 1$$

when $k = 0$; $x = 4\left[\cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}\right]$

$k = 1$; $x = 4\left[\cos\dfrac{3\pi}{2} + i\sin\dfrac{3\pi}{2}\right]$

4. If ω is the cube roots of unity then prove that $(1 - \omega + \omega^2)^5 + (1 + \omega - \omega^2)^5 = 32$.

Solution:

If ω is the cube roots of unity then $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$.

LHS : $(1 - \omega + \omega^2)^5 + (1 + \omega - \omega^2)^5$

$= (1 + \omega^2 - \omega)^5 + (1 + \omega - \omega^2)^5$

$= (-\omega - \omega)^5 + (-\omega^2 - \omega^2)^5$

$= (-2\omega)^5 + (-2\omega^2)^5$

$= (-2)^5\,\omega^5 + (-2)^5\,(\omega^2)^5$

$= -32\omega^2 - 32\omega$

$= -32\,(\omega^2 + \omega)$

$= -32\,(-1) = 32 = \text{RHS}$

PART – C

1. Solve: $x^7 + 1 = 0$

Solution:

Given: $x^7 + 1 = 0$

$$x^7 = -1$$

$$x = (-1)^{1/7}$$

$$= (\cos\pi + i\sin\pi)^{1/7}$$

$$= [\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)]^{1/7}$$

$$= \cos\left(\frac{2k\pi + \pi}{7}\right) + i\sin\left(\frac{2k\pi + \pi}{7}\right) \quad \text{where } k = 0, 1, 2, 3, 4, 5, 6,$$

$\therefore$ The values are

$$\text{when } k = 0;\ x = \cos\frac{\pi}{7} + i\sin\frac{\pi}{7}$$

$$k = 1;\ x = \cos\frac{3\pi}{7} + i\sin\frac{3\pi}{7}$$

$$k = 2;\ x = \cos\frac{5\pi}{7} + i\sin\frac{5\pi}{7}$$

$$k = 3;\ x = \cos\frac{7\pi}{7} + i\sin\frac{7\pi}{7}$$

$$k = 4;\ x = \cos\frac{9\pi}{7} + i\sin\frac{9\pi}{7}$$

$$k = 5;\ x = \cos\frac{11\pi}{7} + i\sin\frac{11\pi}{7}$$

$$k = 6;\ x = \cos\frac{13\pi}{7} + i\sin\frac{13\pi}{7}$$

2. Solve: $x^6 - 1 = 0$

Solution:

Given: $x^6 - 1 = 0$

$$x^6 = 1$$

$$x = (1)^{1/6}$$

$$= (\cos 0 + i\sin 0)^{1/6}$$

$$= (\cos 2k\pi + i\sin 2k\pi)^{1/6}$$

$$= \cos\frac{2k\pi}{6} + i\sin\frac{2k\pi}{6} \quad \text{where } k = 0, 1, 2, 3, 4, 5$$

$\therefore$ The values are,

when $k = 0;\ x = \cos 0 + i\sin 0$

$$k = 1;\ x = \cos\frac{2\pi}{6} + i\sin\frac{2\pi}{6}$$

$$k = 2;\ x = \cos\frac{4\pi}{6} + i\sin\frac{4\pi}{6}$$

$$k = 3;\ x = \cos\frac{6\pi}{6} + i\sin\frac{6\pi}{6}$$

$$k = 4;\ x = \cos\frac{8\pi}{6} + i\sin\frac{8\pi}{6}$$

$$k = 5;\ x = \cos\frac{10\pi}{6} + i\sin\frac{10\pi}{6}$$

3. Solve: $x^8 + x^5 + x^3 + 1 = 0$

Solution:

Given: $x^8 + x^5 + x^3 + 1 = 0$

$$x^5(x^3 + 1) + 1(x^3 + 1) = 0$$

$$(x^5 + 1)(x^3 + 1) = 0$$

$$x^5 + 1 = 0 \quad ; x^3 + 1 = 0$$

<u>Case (i)</u>

$$x^5 + 1 = 0$$

$$x = (-1)^{1/5}$$

$$= (\cos\pi + i\sin\pi)^{1/5}$$

$$= [\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)]^{1/5}$$

$$= \cos\left(\frac{2k\pi + \pi}{5}\right) + i\sin\left(\frac{2k\pi + \pi}{5}\right) \quad \text{where } k = 0, 1, 2, 3, 4$$

The roots are,

when $k = 0;\ x = \cos\frac{\pi}{5} + i\sin\frac{\pi}{5}$

$$k = 1;\ x = \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}$$

$$k = 2;\ x = \cos\frac{5\pi}{5} + i\sin\frac{5\pi}{5}$$

$$k = 3;\ x = \cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5}$$

$$k = 4;\ x = \cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5}$$

Case (ii) :

$x^3 + 1 = 0$

$x = (-1)^{1/3}$

$= (\cos\pi + i\sin\pi)^{1/3}$

$= [\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)]^{1/3}$

$= \cos\left(\frac{2k\pi+\pi}{3}\right) + i\sin\left(\frac{2k\pi+\pi}{3}\right)$ where k = 0, 1, 2

when $k = 0$; $x = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

$k = 1$; $x = \cos\frac{3\pi}{3} + i\sin\frac{3\pi}{3}$

$k = 2$; $x = \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}$

4. Find all the values of $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{3/4}$ and also prove that the product of the four values is 1.

Solution:

Let $a + ib = \frac{1}{2} + i\frac{\sqrt{3}}{2} = r(\cos\theta + i\sin\theta)$ ....(1)

Here $a = \frac{1}{2}$ & $b = \frac{\sqrt{3}}{2}$

Modulus:

$$r = \sqrt{a^2 + b^2} = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{1} = 1$$

Arugument:

$$\theta = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left[\frac{\sqrt{3}/2}{1/2}\right] = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$$

$\therefore$ (1) becomes,

$$\frac{1}{2} + i\frac{\sqrt{3}}{2} = 1\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$\Rightarrow \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{3/4} = \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{3/4}$$

$$= \left[\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^3\right]^{\frac{1}{4}}$$

$$= \left[\cos 3\left(\frac{\pi}{3}\right) + i\sin 3\left(\frac{\pi}{3}\right)\right]^{1/4}$$

$$= [\cos\pi + i\sin\pi]^{1/4}$$

$$= [\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)]^{1/4}$$

$$= \cos\left(\frac{2k\pi + \pi}{4}\right) + i\sin\left(\frac{2k\pi + \pi}{4}\right) \quad \text{where } k = 0, 1, 2, 3$$

$\therefore$ The values are,

$$\text{when } k = 0;\ R_1 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$$

$$k = 1;\ R_2 = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}$$

$$k = 2;\ R_3 = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}$$

$$k = 3;\ R_4 = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}$$

Product of the four values

$R_1 \times R_2 \times R_3 \times R$

$$= \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$$

$$= \cos\left(\frac{\pi}{4} + \frac{3\pi}{4} + \frac{5\pi}{4} + \frac{7\pi}{4}\right) + i\sin\left(\frac{\pi}{4} + \frac{3\pi}{4} + \frac{5\pi}{4} + \frac{7\pi}{4}\right)$$

$$= \cos\frac{16\pi}{4} + i\sin\frac{16\pi}{4}$$

$$= \cos 4\pi + i\sin 4\pi$$

$$= 1 + i(0) = 1$$

EXERCISE

PART – A

1. If $z_1 = -1 + 2i$ and $z_2 = -3 + 4i$ find $3z, - 4z_2$.
2. If $z_1 = (2, 3)$ and $z = (5, 7)$ find $4z_1 + 3z_2$.
3. If $z_1 = (-3, 5)$ and $z_2 = (1, -2)$ find $z_1\ z_2$.
4. If $z_1 = 1 + i$ and $z_2 = 1 - i$ find z_1 / z_2.
5. If $z_1 = 2 + i$ and $z_2 = 1 + i$ find z_2 / z_1.
6. Express the following complex numbers in a + ib form.

   (i) $\frac{1}{4+3i}$ (ii) $\frac{2}{3-i}$ (iii) $(4+5i)(5+7i)$
7. Find the real and imaginary parts of the following complex numbers

   (i) $\frac{1}{2-i}$ (ii) $\frac{1}{2+3i}$ (iii) $\frac{1}{i-3}$ (iv) $\frac{1+i}{1-i}$
8. Find the complex conjugate of the following:

   (i) $(2-3i)(7+11i)$ (ii) $\frac{4}{1-i}$ (iii) $\frac{1-i}{1+i}$ (iv) $\frac{2}{i-5}$
9. Find the modulus and argument (or) amplitude of the following:

   (i) $\sqrt{3}+i$ (ii) $-1+i$ (iii) $\sqrt{3}-i$ (iv) $1-\sqrt{-3}$ (v) $\frac{1}{2}+i\frac{\sqrt{3}}{2}$

   (vi) $-\frac{1}{2}-i\frac{\sqrt{3}}{2}$ (vii) $1-i\sqrt{3}$
10. Find the distance between the following two complex numbers

    (i) $2 + 3i$ and $3 - 2i$ (ii) $4 + 3i$ and $5 - 6i$ (iii) $2 - 3i$ and $5 + 7i$ (iv) $1 + i$ and $3 - 2i$
11. State DeMoivre's theorem.
12. Simplify the following:

    (i) $(\cos\theta + i\sin\theta)^3 (\cos\theta + i\sin\theta)^{-4}$

    (ii) $(\cos\phi + i\sin\phi)^5 (\cos\phi + i\sin\phi)^{-6}$

    (iii) $(\cos\theta - i\sin\theta)^4 (\cos\theta + i\sin\theta)^7$
13. Find the value of the following:

    (i) $\frac{\cos 5\theta + i\sin 5\theta}{\cos 3\theta - i\sin 3\theta}$ (ii) $\frac{\cos 3\theta + i\sin 3\theta}{\cos 2\theta - i\sin 2\theta}$

    (iii) $\frac{\cos 10\theta + i\sin 10\theta}{\cos 7\theta + i\sin 7\theta}$ (iv) $\frac{\cos 4\theta + i\sin 4\theta}{\cos\theta + i\sin\theta}$

    (v) $\frac{(\cos\theta + i\sin\theta)^8}{(\cos\theta + i\sin\theta)^4}$ (vi) $\frac{\cos 6\theta + i\sin 6\theta}{(\cos\theta - i\sin\theta)^4}$
14. Simplify: $\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)\left(\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\right)\left(\cos\frac{5\pi}{8} + i\sin\frac{5\pi}{8}\right)$

15. Simplify: $\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

16. Find the product of $5\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$ and $2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$.

17. Find the product of $\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$ and $\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$.

17.(a) If $z_1 = 5\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$ and $z_2 = 3\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)$ find Z_1Z_2.

18. If $x = \cos\theta + i\sin\theta$ find $x-\frac{1}{x}$.

19. If $a = \cos\theta + i\sin\theta$, $b = \cos\phi + i\sin\phi$ find ab.

20. If $a = \cos\alpha + i\sin\alpha$, $b = \cos\beta + i\sin\beta$ find $\frac{b}{a}$.

21. If $a = \cos x + i\sin x$, $b = \cos y + i\sin y$ find $\sqrt{ab}$.

22. If ω is the cube root of unity find the value of $1 + \omega^2 + \omega^4$.

23. If ω is the fourth root of unity find the value $\omega^4 + \omega^5 + \omega^6 + \omega^7$.

24. If ω is the six root of unity find the value of $\omega^2 + \omega^4 + \omega^6$.

25. Solve: $x^2 + 1 = 0$

26. Find the value of (i) $(i)^{1/2}$ (ii) $(1)^{1/3}$ (iii) $(-1)^{1/2}$

27. Find the value of $\left(\frac{-1-i\sqrt{3}}{2}\right)^3$.

PART – B

1. Express the following complex numbers in a + ib form.

   (i) $\frac{1}{1+i}+\frac{1}{1-i}$ (ii) $\frac{2+3i}{4-i}$ (iii) $\frac{1+i}{(1-i)^2}$ (iv) $\frac{4+3i}{1-i}$

2. Find the real and imaginary parts of the following complex numbers

   (i) $\frac{1+2i}{1-i}$ (ii) $\frac{(2-i)^2}{1+i}$ (iii) $\frac{2+i}{1+4i}$

3. Find the conjugate the following complex numbers

   (i) $\frac{13}{11+12i}$ (ii) $\frac{1+i}{1-i}$ (iii) $\frac{1}{2-i}+\frac{1}{2+i}$

4. Find the modulus and amplitude of the following complex numbers

   (i) $\frac{1+i}{1-i}$ (ii) $\frac{\sqrt{3}}{2}+i\frac{\sqrt{3}}{2}$ (iii) $2+2\sqrt{3}i$ (iv) $-\sqrt{2}+\sqrt{2}i$

5. Show that the following complex numbers are collinear.

   (i) $1 + 3i,\ 5 + i,\ 3 + 2i$

   (ii) $4 + 2i,\ 7 + 5i,\ 9 + 7i$

   (iii) $1 + 3i,\ 2 + 7i,\ -2 - 9i$

6. Show that the following complex numbers form a parallelogram.

(i) 2 – 2i, 8 + 4i, 5 + 7i, – 1 + i

(ii) 3 + i, 2 + 2i, – 2 + i, – 1

(iii) 1 – 2i, –1 + 4i, 5 + 8i, 7 + 2i

7. If x = cos θ + i sin θ find the value of

(i) $x^n + \frac{1}{x^n}$ (ii) $x^m - \frac{1}{x^m}$ (iii) $x^3 + \frac{1}{x^3}$ (iv) $x^5 + \frac{1}{x^5}$ (v) $x^2 - \frac{1}{x^2}$ (vi) $x^8 - \frac{1}{x^8}$

8. If a = cos x + i sin x, b = cos y + i sin y find $ab - \frac{1}{ab}$.

9. If x = cos 2α + i sin 2α, y = cos 2β + i sin 2β find $\sqrt{xy}$.

10. If x = cos α + i sin α, y = cos β + i sin β find $x^m y^n$.

11. Find the cube roots of 8.

12. Find the all the values of $(-1)^{2/3}$.

PART – C

1. Find the real and imaginary parts of the following complex numbers

(i) $\frac{(1+i)(1+2i)}{1+3i}$ (ii) $\frac{(1+2i)^3}{(1+i)(2-i)}$ (iii) $\left(\frac{1-i}{1+i}\right)^3$ (iv) $\frac{3}{4+3i} + \frac{i}{3-4i}$

(vii) $\frac{3}{3+4i} + \frac{i}{5-2i}$ (viii) $\frac{4}{3+2i} + \frac{2}{5-4i}$

(ix) $\frac{1+3\sqrt{3}i}{\sqrt{3}+2i}$ (x) $\frac{1}{1+\cos\theta + i\sin\theta}$

2. Express the following complex numbers in a + ib form.

(i) $\frac{(1+i)(1-2i)}{(1+3i)}$ (ii) $\frac{(1+i)(1+2i)}{(1+4i)}$ (iii) $\frac{(1+i)(3+i)^2}{(2-i)^2}$

(iv) $\frac{3}{4+3i} + \frac{i}{3-4i}$ (v) $\frac{2+3i}{1-i}$ (vi) $\frac{7-5i}{(2+3i)^2}$

3. Find the conjugate of the following complex numbers

(i) $\frac{(1+i)(2-i)}{(2+i)^2}$ (ii) $\frac{(1+i)(2+i)}{(3+i)}$ (iii) $\frac{1-i}{3+2i}$ (iv) $\frac{3+i}{2+5i}$ (v) $\frac{5-i}{2-3i}$

4. Find the modulus and argument of the following complex numbers

(i) $\frac{1+\sqrt{3}i}{1+i}$ (ii) $\frac{2-i}{3+7i}$ (iii) $\frac{1+i\sqrt{3}}{1-i}$

(iv) $\frac{(1+i)(1+2i)}{1+3i}$ (v) $\frac{i-3}{i-1}$ (vi) $\frac{1-i}{1+i}$

5. Prove that the following complex numbers form a square.

(i) 9 + i, 4 + 13i, – 8 + 8i, – 3 – 4i (ii) 2 + i, 4 + 3i, 2 + 5i, 3i

(iii) – 1, 3i, 3 + 2i, 2 – i (iv) 4 + 5i, 1 + 2i, 4 – i, 7 + 2i

6. Show that the following complex numbers form a rectangle.

(i) 1 + 2i, – 2 + 5i, 7i, 3 + 4i (ii) 4 + 3i, 12 + 9i, 15 + 5i, 7 – i

(iii) 1 + i, 3 + 5i, 4 + 4i, 2i (iv) 8 + 4i, 5 + 7i, – 1 + i, 2 – 2i

7. Show that the following complex number form a rhombus.

(i) 8 + 5i, 16 + 11i, 10 + 3i, 2 – 3i (ii) 6 + 4i, 4 + 5i, 6 + 3i, 8 + i

(iii) 1 + i, 2 + i, 2 + 2i, 1 + 2i

8. Simplify the following using De Moivre's theorem

(i) $\dfrac{(\cos 2\theta - i\sin 2\theta)^4(\cos 4\theta + i\sin 4\theta)^{-5}}{(\cos 3\theta + i\sin 3\theta)^2(\cos 5\theta - i\sin 5\theta)^{-3}}$

(ii) $\dfrac{(\cos 2\theta - i\sin 2\theta)^3(\cos 3\theta + i\sin 3\theta)^4}{(\cos 3\theta + i\sin 3\theta)^2(\cos 5\theta - i\sin 5\theta)^{-3}}$

(iii) $\dfrac{(\cos 2\theta - i\sin 2\theta)^7(\cos 3\theta + i\sin 3\theta)^{-5}}{(\cos 4\theta + i\sin 4\theta)^2(\cos 5\theta - i\sin 5\theta)^{-6}}$

(iv) $\dfrac{(\cos 3\theta + i\sin 3\theta)^2(\cos 4\theta - i\sin 4\theta)^3}{(\cos\theta + i\sin\theta)^3}$ when $\theta = \dfrac{\pi}{9}$

(v) $\dfrac{(\cos x - i\sin x)^3(\cos 3x + i\sin 3x)^5}{(\cos 2x - i\sin 2x)^5(\cos 5x + i\sin 5x)^7}$ when $x = \dfrac{2\pi}{13}$

(vi) $\dfrac{(\cos 5\theta - i\sin 5\theta)\,(\cos 2\theta - i\sin 2\theta)^{-3}}{(\cos\theta + i\sin\theta)^5(\cos 3\theta + i\sin 3\theta)^{-5}}$ when $\theta = \dfrac{2\pi}{11}$

9. Show that $\left[\dfrac{\cos\theta + i\sin\theta}{\sin\theta - i\cos\theta}\right]^4 = 1$

10. Show that $\left[\dfrac{1+\cos\theta + i\sin\theta}{1+\cos\theta - i\sin\theta}\right]^3 = \cos 3\theta + i\sin 3\theta$

11. Prove that $\left[\dfrac{1+\sin\dfrac{\pi}{8} + i\cos\dfrac{\pi}{8}}{1+\sin\dfrac{\pi}{8} - i\cos\dfrac{\pi}{8}}\right]^8 = 1$.

12. If a = cos α + i sin α, b = cos β + i sin β and c = cos γ + i sin γ find the value of $\dfrac{ab}{c} + \dfrac{c}{ab}$.

13. If x = cos 3α + i sin 3α, y = cos 3β + i sin 3β prove that

(i) $\sqrt[3]{xy} + \dfrac{1}{\sqrt[3]{xy}} = 2\cos(\alpha + \beta)$ (ii) $\sqrt[3]{xy} - \dfrac{1}{\sqrt[3]{xy}} = 2i\sin(\alpha + \beta)$

14. If x = cos α + i sin α, y = cos β + i sin β, z = cos γ + i sin γ and if x + y + z = 0 show that

(i) cos 3α + cos 3β + cos 3γ = 3 cos (α + β + γ)

(ii) sin 3α + sin 3β + sin 3γ = 3 sin (α + β + γ)

15. If x = cos α + i sin α, y = cos β + i sin β and z = cos γ + i sin γ and if x + y + z = 0 prove that $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 0$

16. If $x + \dfrac{1}{x} = 2\cos\theta$ and $y + \dfrac{1}{y} = 2\cos\phi$ show that $\dfrac{x^m}{y^m} + \dfrac{y^n}{x^m} = 2\cos(m\theta - n\phi)$.

17. If x = cos α + i sin α, y = cos β + i sin β prove that $x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\alpha + n\beta)$.

18. If 'n' is a positive integer, prove that $(1+i\sqrt{3})^n + (1-i\sqrt{3})^n = 2^{n+1}\cos\frac{n\pi}{3}$.

19. If 'n' is a positive integer prove that $(\sqrt{3}+i)^n - (\sqrt{3}-i)^n = 2^{n+1}\cos\frac{n\pi}{6}$.

20. Prove that $(1 + \cos\theta + i\sin\theta)^n + (1 + \cos\theta - i\sin\theta)^n = 2^{n+1}\cos^n\left(\frac{\theta}{2}\right)\cos\left(\frac{n\theta}{2}\right)$.

21. Solve: $x^4 + 1 = 0$

22. Solve: $x^5 + 1 = 0$

23. Solve: $x^6 + 1 = 0$

24. Solve: $x^4 - 1 = 0$

25. Solve: $x^5 - 1 = 0$

26. Solve: $x^7 - 1 = 0$

27. Solve: $x^5 + x^3 + x^2 + 1 = 0$

28. Solve: $x^8 - x^5 + x^3 - 1 = 0$

29. Solve: $x^7 + x^4 + x^3 + 1 = 0$

30. Solve: $x^7 - x^4 + x^3 - 1 = 0$

ANSWERS

PART – A

(1) $9 - 10i$ (2) (23, 33) (3) $7 - i$ (4) i (5) $\frac{3+i}{5}$ (6) (i) $\frac{4}{25} - \frac{3i}{25}$ (ii) $\frac{3}{5} + \frac{i}{5}$ (iii) $-15 + 53i$

(7) (i) $\mathrm{Re}(z) = \frac{1}{5}, \mathrm{Im}(z) = \frac{1}{5}$ (ii) $\mathrm{Re}(z) = \frac{2}{13}, \mathrm{Im}(z) = \frac{-3}{13}$

(iii) $\mathrm{Re}(z) = \frac{-3}{10}, \mathrm{Im}(z) = \frac{-1}{10}$ (iv) $\mathrm{Re}(Z) = 0, \mathrm{Im}(z) = 1$

(8) (i) $\overline{Z} = 47 - i$ (ii) $\overline{Z} = 2(1-i)$ (iii) $\overline{Z} = i$ (iv) $\overline{Z} = \frac{-5+i}{13}$

(9)

(i) $r = 2; \theta = \frac{\pi}{6}$ (ii) $r = \sqrt{2};\ \theta = -\frac{\pi}{4}$ (iii) $r = 2;\ \theta = -\frac{\pi}{6}$

(iv) $r = 2; \theta = -\frac{\pi}{3}$ (v) $r = 1;\ \theta = \frac{\pi}{3}$ (vi) $r = 1;\ \theta = \frac{\pi}{3}$ (vii) $r = 2;\ \theta = -\frac{\pi}{3}$

10) (i) $\sqrt{26}$ (ii) $\sqrt{82}$ (iii) $\sqrt{109}$ (iv) $\sqrt{13}$

12) (i) $\cos\theta - i\sin\theta$ (ii) $\cos 11\phi + i\sin 11\phi$ (iii) $\cos 3\theta + i\sin 3\theta$

13) (i) $\cos 8\theta + i\sin 8\theta$ (ii) $\cos 5\theta + i\sin 5\theta$ (iii) $\cos 3\theta + i\sin 3\theta$

(iv) $\cos 3\theta + i\sin 3\theta$ (v) $\cos 4\theta + i\sin 4\theta$ (vi) $\cos 10\theta + i\sin 10\theta$

14) – 1 15) –1 16) $5(-1 - i\sqrt{3})$ 17) i 17(a) – 15 18) $2i\sin\theta$

19) $\cos(\theta+\phi)+i\sin(\theta+\phi)$ 20) $\cos(\beta-\alpha)+i\sin(\beta-\alpha)$

21) $\cos\left(\frac{x+y}{2}\right)+i\sin\left(\frac{x+y}{2}\right)$ 22) 0 23) 0 24) $-\omega-\omega^3-\omega^5$

25) $x=\cos\left(\frac{2k\pi+\pi}{2}\right)+i\sin\left(\frac{2k\pi+\pi}{2}\right)$ where k = 0, 1

26) (i) $\cos\frac{\pi}{4}+i\sin\frac{\pi}{4},\ \cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}$

(ii) $\cos 0+i\sin 0,\ \cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3},\ \cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}$

(iii) $\cos\frac{\pi}{2}+i\sin\frac{\pi}{2},\ \cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}$

27) 1

PART – B

1) (i) 1 (ii) $\frac{11+10i}{17}$ (iii) $\frac{-1+i}{2}$ (iv) $\frac{1+7i}{2}$

2) (i) $\text{Re}(Z)=-\frac{1}{2},\quad I_m(Z)=\frac{3}{2}$

(ii) $\text{Re}(Z)=\frac{1}{2},\quad I_m(Z)=\frac{-9}{2}$

(iii) $\text{Re}(Z)=\frac{6}{17},\quad I_m(Z)=-\frac{7}{17}$

3) (i) $\overline{Z}=\frac{13}{23}(11+12i)$ (ii) $\overline{Z}=-i$ (iii) $\overline{Z}=\frac{4}{5}$

4) (i) $r=1,\ \theta=\infty$ (ii) $r=\frac{3}{2},\ \theta=45^\circ$

(iii) $r=4;\ \theta=\frac{\pi}{3}$ (iv) $r=2,\quad \theta=\frac{-\pi}{4}$

7) (i) $2\cos n\theta$ (ii) $2i\sin m\theta$ (iii) $2\cos 3\theta$ (iv) $2\cos 5\theta$ (v) $2i\sin 2\theta$ (vi) $2i\sin 8\theta$

8) $2i\sin(x-y)$ 9) $\cos(x+y)+i\sin(x-y)$ 10) $\cos(m\alpha+n\beta)+i\sin(m\alpha+n\beta)$

11) $2[\cos 0+i\sin 0],\ 2\left[\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right],\ 2\left[\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right]$

12) $\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3},\ \cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3},\ \cos\frac{6\pi}{3}+i\sin\frac{6\pi}{3}$

PART – C

1) (i) $Re(z)=\frac{4}{5}, Im(z)=\frac{3}{5}$ (ii) $Re(z)=\frac{-7}{2}, Im(z)=\frac{1}{2}$

(iii) $Re(z)=0,\ Im(z)=1$ (iv) $Re(z)=\frac{8}{25},\ Im(z)=-\frac{6}{25}$

(v) $Re(z)=1,\ Im(z)=\frac{1}{2}$ (vi) $Re(z)=\frac{26}{41},\ Im(z)=-\frac{29}{41}$

(vii) $Re(z)=\frac{366}{725}, Im(z)=\frac{-298}{725}$ (viii) $Re(z)=\frac{622}{533},\ Im(z)=\frac{-224}{533}$

(ix) $Re(z)=\sqrt{3},\ Im(z)=1$ (x) $Re(z)=\frac{1}{2},\ Im(z)=-\frac{1}{2}\tan\frac{\theta}{2}$

2) (i) $\frac{8-9i}{29}$ (ii) $\frac{193+149i}{41}$ (iii) $\frac{1}{2}-i$

(iii) $\frac{23-24i}{65}$ (v) $\frac{17+7i}{26}$ (vi) -1

3) (i) $\overline{Z}=\frac{13}{25}+\frac{9i}{25}$ (ii) $\overline{Z}=\frac{3}{5}-\frac{4i}{5}$ (iii) $\overline{Z}=\frac{1}{13}+\frac{5i}{13}$

(iv) $\overline{Z}=\frac{11}{29}-\frac{13}{29}i$ (v) $\overline{Z}=1-i$

4) (i) $|Z|=2,\ \theta=\tan^{-1}(1-\sqrt{3})$ (ii) $|Z|=\frac{\sqrt{290}}{58},\ \theta=\tan^{-1}(17)$

(iii) $|Z|=\sqrt{2},\ \theta=\tan^{-1}\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)$ (iv) $|Z|=\sqrt{\frac{5}{2}},\ \theta=\tan^{-1}=(9)$

(v) $|Z|=\sqrt{5}$, $\theta=\tan^{-1}\left(\frac{1}{2}\right)$ (vi) $|Z|=1, \theta=-\frac{\pi}{2}$

8) (i) $\cos 19\theta - i\sin 19\theta$ (ii) $\cos 15\theta - i\sin 15\theta$ (iii) $\cos 107\theta - i\sin 107\theta$

(iv) -1 (v) 1 (vi) 1

12) $2\cos(\alpha+\beta-\gamma)$

21) $\cos\frac{\pi}{4}+i\sin\frac{\pi}{4},\ \cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4},\ \cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4},\ \cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}$

22) $\cos\frac{\pi}{5}+i\sin\frac{\pi}{5},\ \cos\frac{3\pi}{5}+i\sin\frac{3\pi}{5},\ \cos\frac{5\pi}{5}+i\sin\frac{5\pi}{5},\ \cos\frac{7\pi}{5}+i\sin\frac{7\pi}{5}, \cos\frac{9\pi}{5}+i\sin\frac{9\pi}{5}$

23) $\cos\frac{\pi}{6}+i\sin\frac{\pi}{6},\ \cos\frac{3\pi}{6}+i\sin\frac{3\pi}{6},\ \cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6},\ \cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6},$

$\cos\frac{9\pi}{6}+i\sin\frac{9\pi}{6},\ \cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6}$

24) $\cos 0+i\sin 0, \cos\frac{2\pi}{4}+i\sin\frac{2\pi}{4}, \cos\frac{4\pi}{4}, i\sin\frac{4\pi}{4},\ \cos\frac{6\pi}{4}+i\sin\frac{6\pi}{4}$

25) $\cos 0+i\sin 0, \cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}, \cos\frac{4\pi}{5}, i\sin\frac{4\pi}{5},\ \cos\frac{6\pi}{5}+i\sin\frac{6\pi}{5}, \cos\frac{8\pi}{5}+i\sin\frac{8\pi}{5}$

26) $\cos 0+i\sin 0, \cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}, \cos\frac{4\pi}{7}, i\sin\frac{4\pi}{7},\ \cos\frac{6\pi}{7}+i\sin\frac{6\pi}{7}, \cos\frac{8\pi}{7}+i\sin\frac{8\pi}{7}, \cos\frac{9\pi}{7}+i\sin\frac{9\pi}{7}$

27) Case (i) $\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$, $\cos\frac{3\pi}{3}+i\sin\frac{3\pi}{3}$, $\cos\frac{9\pi}{7}+i\sin\frac{9\pi}{7}$

Case (ii) $\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}$, $\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}$

28) Case (i) $\cos 0+i\sin 0$, $\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$, $\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}$

Case (ii) $\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}$, $\cos\frac{3\pi}{5}+i\sin\frac{3\pi}{5}$, $\cos\frac{5\pi}{5}+i\sin\frac{5\pi}{5}$,

$\cos\frac{7\pi}{5}+i\sin\frac{7\pi}{5}$, $\cos\frac{9\pi}{5}+i\sin\frac{9\pi}{5}$

29) $\cos\left(\frac{2k\pi+\pi}{4}\right)+i\sin\left(\frac{2k\pi+\pi}{4}\right)$, $k=0,1,2,3$.

$\cos\left(\frac{2k\pi+\pi}{3}\right)+i\sin\left(\frac{2k\pi+\pi}{3}\right)$, $k=0,1,2$

30) $\cos\left(\frac{2k\pi+\pi}{4}\right)+i\sin\left(\frac{2k\pi+\pi}{4}\right)$, $k=0,1,2,3$

$\cos\left(\frac{2k\pi+\pi}{3}\right)+i\sin\left(\frac{2k\pi+\pi}{3}\right)$, $k=0,1,2$