Blending Bezier curve with continuous curvature

Portnov Ilya V.

January 6, 2023

1 Problem Statement

Let's consider two curves:

$$\varphi = \varphi(t), \quad t \in [0; 1]$$

and

$$\psi = \psi(t), \quad t \in [0; 1]$$

in 3D space. We want to derive such a curve

$$\gamma = \gamma(t), \quad t \in [0; 1]$$

that it has G2-continuous joints with the end of first curve (φ) and with the beginning of the second curve (ψ); more precisely, we want the curve γ to have the following proerties:

1. The end of φ coincides with the beginning of γ, i.e.,

$$\gamma(0) = \varphi(1) \tag{1}$$

2. Tangent vector of φ at it's end is equal to tangent vector of γ at it's beginning, i.e.,

$$\gamma'(0) = \varphi'(1) \tag{2}$$

3. Normal vector of φ at it's end is pointing in the same direction as the normal vector of γ at it's beginning. I.e.,

$$\mathbf{n}(\gamma)(0) \parallel \mathbf{n}(\varphi)(1) \tag{3}$$

(we use $\mathbf{n}(\gamma)$ to denote the normal vector of the curve).

4. γ curve has the same curvature at it's beginning, as φ has at it's end, i.e.,
$$\kappa(\gamma)(0) = \kappa(\varphi)(1) \tag{4}$$
(we use $\kappa(\gamma)$ to denote the curvature of the curve).

5. The same goes for joining of γ with ψ.

Obviously, there are a lot of curves that satisfy these conditions. But in addition, we want our curve γ to be a Bezier curve. We will show that to satisfy these conditions, it is enough to use Bezier curve of 5th degree (with 6 control points). We will search our curve in the following form:

$$\gamma(t) = A_1(1-t)^5 + B_1(1-t)^4 t + C_1(1-t)^3 t^2 + C_2(1-t)^2 t^3 + B_2(1-t)t^4 + A_2 t^5 \tag{5}$$

Here A_1, A_2, B_1, B_2, C_1 and C_2 are control points of our Bezier curve. They are to be found.

2 A note on problem statement

One may note, that it is always possible to replace our equations (3) and (4) with
$$\gamma''(0) = \varphi''(1) \tag{6}$$
(this is what called C2 continuity). Then from equations (1), (2) and (6) it would be quite simple to derive equations for our control points. However, the equation (6) is more strong than (3) and (4), as we will see in details later. If we experiment a bit with C2 continuity, we will see, that geometrically makes our blending curve follow the "inertia" of φ and ψ very strongly, and thus, the blending curve with C2 continuity can be bent too much in order to look "naturally". Figure 4 illustrates what we mean.

3 Determining first two pairs of control points

First, from (1) and (5) we can obviously state that

$$A_1 = \varphi(1), \quad A_2 = \psi(0)$$

To simplify the notation, we will just note that $\gamma(0)$, $\gamma(1)$, $\gamma'(0)$ and $\gamma'(1)$ are known to us, and later we will use these notations instead of φ and ψ. For example, the previous equation we may write as:

$$A_1 = \gamma(0), \quad A_2 = \gamma(1) \tag{7}$$

Now, if we write down the formula for the first derivative of our Bezier curve at $t = 0$, it will be pretty simple:

$$\gamma'(0) = 5(B_1 - A_1) \tag{8}$$

The formula for the first derivative at $t = 1$ will look similar:

$$\gamma'(1) = 5(A_2 - B_2) \tag{9}$$

So, now we know two more control points:

$$B_1 = A_1 + \frac{1}{5}\gamma'(0) \quad B_2 = A_2 - \frac{1}{5}\gamma'(1) \tag{10}$$

4 Determining C_1 and C_2

We have used two more or less trivial of our equations and found four control points. Now we have to use equations (3) and (4) to find C_1 and C_2. We will concentrate on finding C_1, as formulas for C_2 can be obviously found from symmetry. To simplify notation, we will omit the lower indexes, and write A, B and C for A_1, B_1 and C_1. We will write indexes only when it is necessary to avoid confusion.

Let's start with writing down formulas for curvature and normal:

$$\kappa = \frac{|\gamma' \times \gamma''|}{|\gamma'|^3} \tag{11}$$

$$\mathbf{n} = (\gamma' \times \gamma'') \times \gamma' \tag{12}$$

(we use $\times$ for vector cross product). Let's also write down the formula for the second derivative of our curve at $t = 0$:

$$\gamma''(0) = 20(A - 2B + C) \tag{13}$$

We see that both equations for κ and $\mathbf{n}$ contain $\gamma' \times \gamma''$. Let's calculate this product at $t = 0$:

$$\begin{aligned}
\gamma'(0) \times \gamma''(0) &= 5(B - A) \times 20(A - 2B + C) = \\
&= 100(B \times A - 2B \times B + B \times C - A \times A + 2A \times B - A \times C) = \\
&= 100(B \times A + B \times C + 2A \times B - A \times C) = \\
&= 100(A \times B + B \times C - A \times C)
\end{aligned}$$

Now we are going to simplify things even further. We know that Bezier curves have the property of translational invariance; i.e., if we move all control points by some vector, we will have the same curve, just moved by the same vector. So, let's substract the vector A from all our control points (we will just have to remember to add it back in the end). We thus will replace A with 0, B with $\overrightarrow{AB}$, and C with $\overrightarrow{AC}$. This way we will have a simple formula:

$$\gamma'(0) \times \gamma''(0) = 100(\overrightarrow{AB} \times \overrightarrow{AC}) \tag{14}$$

Now, if we remember the definition of vector cross product in 3D space, we can rewrite (11) as

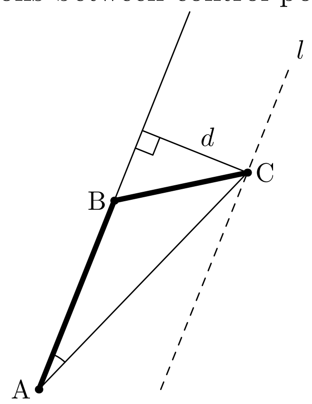
$$100|\overrightarrow{AB}||\overrightarrow{AC}| \sin \angle CAB = \kappa(\gamma)(0)|\gamma'(0)|^3$$

or, if we remember that $|\overrightarrow{AB}| = |\gamma'(0)|/5$,

$$|\overrightarrow{AC}| \sin \angle CAB = \frac{1}{20}\kappa(\gamma)(0)|\gamma'(0)|^2 \tag{15}$$

Here we know everything on the right side, and we do not know $|\overrightarrow{AC}|$ and $\angle CAB$ on the left side. This equation has interesting geometric meaning: it means that C lies on some line l, which is parallel to (AB), and distance between l and (AB) is $d = \frac{1}{20}\kappa(\gamma)(0)|\gamma'(0)|^2$ (this should be clear from fig.1).

Figure 1: Relations between control points A, B and C



Also we know that C lies in the plane, which goes through point A and B, and contains the $\mathbf{n}(\gamma)(0)$ vector (this is clear from (13)). So we already know how to draw the line l: we should draw a line which is parallel to (AB), lies in the plane $(A, B, A + \mathbf{n}(\gamma)(0))$, and has distance from (AB) equal to d. Now the remaining question is where exactly on that line should we place our point C. Any point on line l will satisfy conditions (1) – (4). To select a specific point, we should add some additional condition.

Obviously we can invent different conditions for location of C. For example,

1. we can simply let $\overrightarrow{BC}$ be parallel to $\mathbf{n}(\gamma)(0)$, i.e. let $C = B + d\mathbf{n}(\gamma)(0)$. There is no clear geometrical reason for this, but this is computationally the simplest idea.

2. Or we can say that we want to minimize $|\gamma''(0)|$.

3. Or we may want C to lie somewhere near the line (B_1B_2) – as near as possible. Some experiments show that this tends to give nice enough curves.

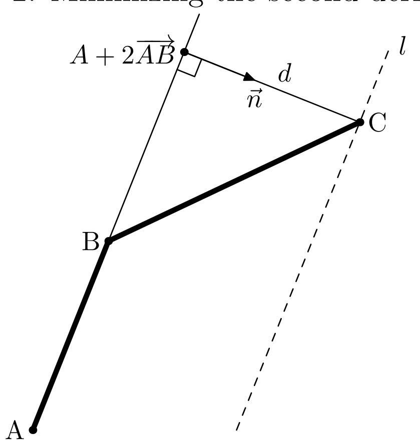
5 Minimizing $|\gamma''(0)|$

Let's consider the second condition. From (13), we can write the second derivative of γ at $t = 0$ as

$$\gamma''(0) = 20(\overrightarrow{AC} - 2\overrightarrow{AB}).$$

From this equation it is clear that if we want to minimize $|\gamma''(0)|$, we need to make C as close to $A + 2\overrightarrow{AB}$ as possible, while remaining on the line l. This is achieved when $A + 2\overrightarrow{AB}$ is the projection of C to (AB) line (refer to fig. 2).

Figure 2: Minimizing the second derivative



From the same picture it is now clear that in such a case we can write C as

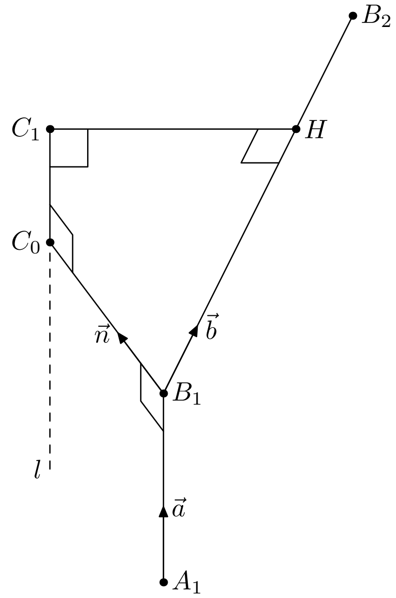
$$C = B + \overrightarrow{AB} + d\mathbf{n}(\gamma)(0) \tag{16}$$

Figure 5 is the example of what these formulas give for the same initial curves that were used for fig.4.

6 Putting C near (B_1B_2) line

We will now concentrate on the latest condition.

Figure 3: Location of point C near (B_1B_2) line



In fact, what we want is C to be the nearest point on line l to the line (B_1B_2). For this to take place, there should be the following conditions met:

1. $\angle B_1C_0C$, where C_0 is defined as the nearest point on l to (AB), should be a right angle;

2. $\angle B_1HC$, where H is defined as the nearest point on (B_1B_2) to l, should be a right angle.

We can rewrite these conditions in terms of vector scalar products:

$$(C_0C, HC) = 0 \tag{17}$$

and

$$(HC, B_1H) = 0 \tag{18}$$

These equations can be solved algebraically. Let's define vector $\vec{a}$ as a unit vector in the same direction as $\overrightarrow{AB}$, and vector $\vec{b}$ as a unit vector in the same direction as $\overrightarrow{B_1B_2}$. Then we can state that there exist such numbers t_1 and t_2, that

$$C = C_0 + t_1\vec{a} \tag{19}$$

and

$$H = B_1 + t_2\vec{b}$$

Also we should remember that

$$C_0 = B_1 + d\mathbf{n}(\gamma)(0)$$

Now we can rewrite (17) as

$$(t_1\vec{a}, d\mathbf{n} + t_1\vec{a} - t_2\vec{b}) = 0$$

Expanding the parentheses in the left-hand side, we will have

$$(t_1\vec{a}, d\mathbf{n}) + (t_1\vec{a}, t_1\vec{a}) - (t_1\vec{a}, t_2\vec{b}) = 0$$

or, remembering that $\vec{a}$ is orthogonal to $\mathbf{n}$,

$$(t_1\vec{a}, t_1\vec{a}) = (t_1\vec{a}, t_2\vec{b}).$$

Now if we expand the definition of scalar product and remember that $\vec{a}$ and $\vec{b}$ are unit vectors, we will have

$$t_1^2 = t_1 t_2 \cos(\widehat{\vec{a}, \vec{b}}) = t_1 t_2 (\vec{a}, \vec{b})$$

or, dividing by t_1, simply

$$t_1 = t_2(\vec{a}, \vec{b}). \tag{20}$$

Now let's return to equation (18). Similarly, we will have

$$(d\mathbf{n} + t_1\vec{a} - t_2\vec{b}, t_2\vec{b}) = 0,$$

or

$$(d\mathbf{n}, t_2\vec{b}) + (t_1\vec{a}, t_2\vec{b}) - (t_2\vec{b}, t_2\vec{b}) = 0,$$

or (remembering that $\vec{b}$ is a unit vector)

$$dt_2(\mathbf{n}, \vec{b}) + t_1 t_2 (\vec{a}, \vec{b}) = t_2^2,$$

or (dividing by t_2)

$$d(\mathbf{n}, \vec{b}) + t_1(\vec{a}, \vec{b}) = t_2. \tag{21}$$

Subsitituting t_1 from (20) into (21), we will have

$$d(\mathbf{n}, \vec{b}) + t_2(\vec{a}, \vec{b})^2 = t_2.$$

Now we can express t_2 as

$$t_2 = \frac{d(\mathbf{n}, \vec{b})}{1 - (\vec{a}, \vec{b})^2}. \tag{22}$$

(Side note: the denominator in the last equation is exactly $\sin^2(\widehat{\vec{a}, \vec{b}})$).

So, finally, subsituting (22) and (20) into (19), we have

$$C = B_1 + d\mathbf{n} + d\frac{(\vec{a}, \vec{b})(\mathbf{n}, \vec{b})}{1 - (\vec{a}, \vec{b})^2}\vec{a} \tag{23}$$

Figure 6 is the example of what these formulas give for the same initial curves that were used for fig.4.

7 Resulting blending curves

The following figures show what kind of curves are received by using different approaches we discussed above.

Figure 4: C2-continuous blending of two curves

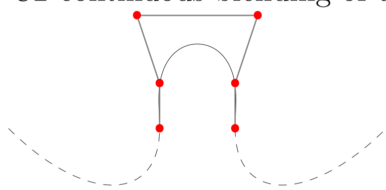


Figure 5: G2-continous blending of two curves, second derivative minimized

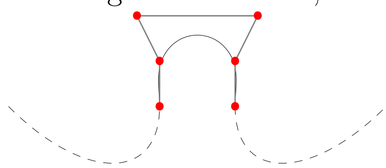


Figure 6: G2-continous blending of two curves, C points aligned by $(B_1 B_2)$