Applications of DFS

November 12, 2015

Outline

Applications of DFS

Basic Algorithm

```
vector<int> visited(n,0);
vector< vector<int> > adj; // adjacency list

void dfs(int v) {
  visited[v] = 1;
  for (int i = 0; i < adj[v].size(); i++) {
    if (!visited[adj[v][i]]) {
      dfs(adj[v][i]);
    }
  }
}
```

I. Topological Sorting

Problem

Given a directed acyclic graph (DAG), order the vertices as $v_1, \ldots, v_n$ such that for every edge $\overrightarrow{v_i v_j}$, we have $i \leq j$.

I. Topological Sorting

Equivalent formulation:

Problem

Given a partially ordered set $(V, \preceq)$, order the elements as $v_1, \ldots, v_n$ such that for every pair $v_i \preceq v_j$, we have $i \leq j$.

I. Topological Sorting

Equivalent formulation:

Problem

Given a partially ordered set $(V, \preceq)$, order the elements as $v_1, \ldots, v_n$ such that for every pair $v_i \preceq v_j$, we have $i \leq j$.

- The resulting order on V is called a linear extension of $\preceq$.

I. Topological Sorting

Equivalent formulation:

Problem

Given a partially ordered set $(V, \preceq)$, order the elements as $v_1, \ldots, v_n$ such that for every pair $v_i \preceq v_j$, we have $i \leq j$.

- The resulting order on V is called a linear extension of $\preceq$.
- Application: dependency resolution.

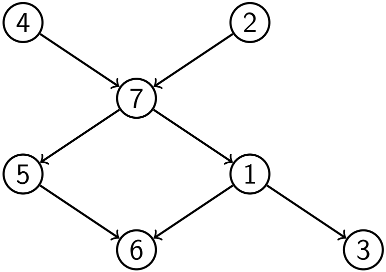
I. Topological Sorting

Algorithm (idea):

```
vector<int> result;

while (there is a  minimal element v) {
    result.push_back(v);
    remove v from the graph;
}
```

I. Topological Sorting



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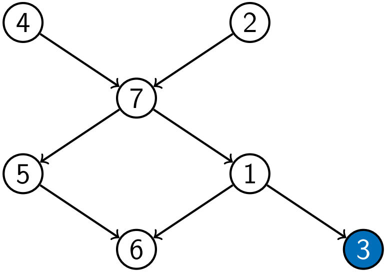

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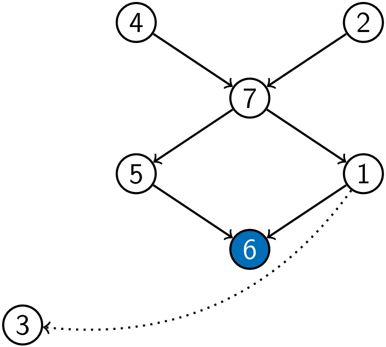

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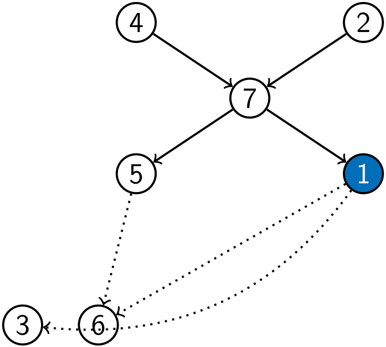

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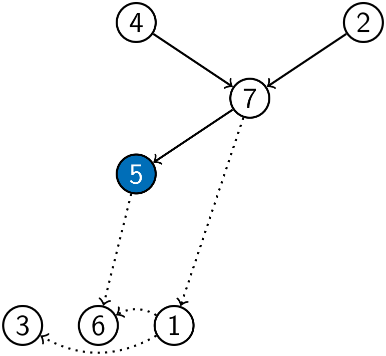

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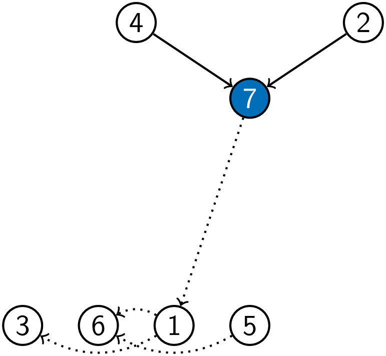

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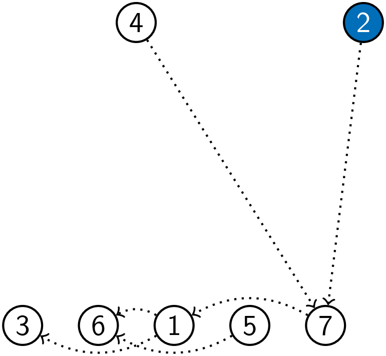

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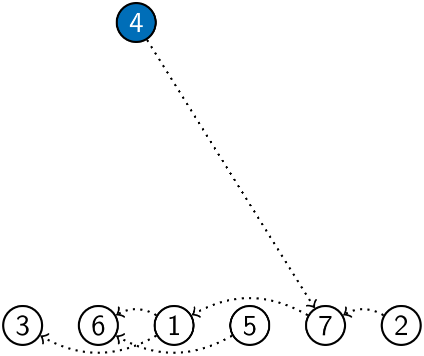

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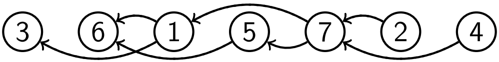

This idea can be implemented with multiple DFS's:

```
vector<int> result;
vector<int> visited(n,0);

for (int i = 0; i < n; i++) {
  if (!visited[i]) dfs(i);
}

void dfs(int v) {
    if (!visited[v]) {
        visited[v] = 1;
        for (int i = 0; i < adj[v].size(); i++) {
            if (!visited[adj[v][i]]) dfs(m);
        }
        result.push_back(v);
    }
}
```

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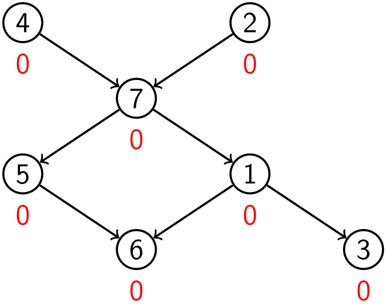

Result:

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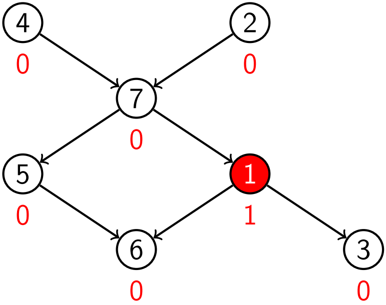

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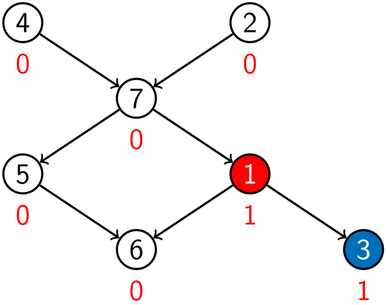

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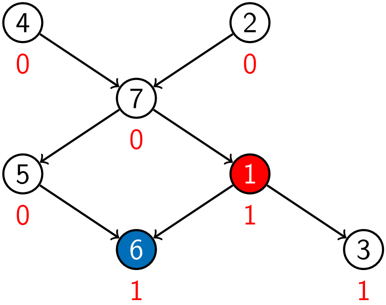

Result: 3

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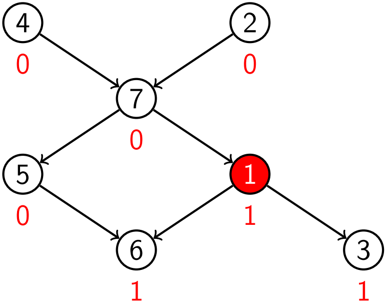

Result: 3 6

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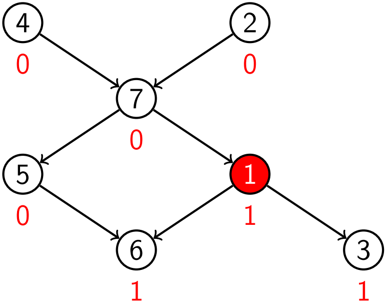
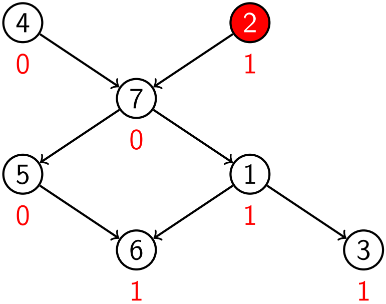


Result: 3 6 1



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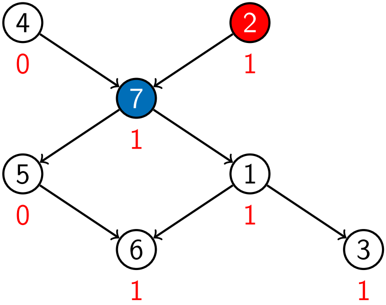

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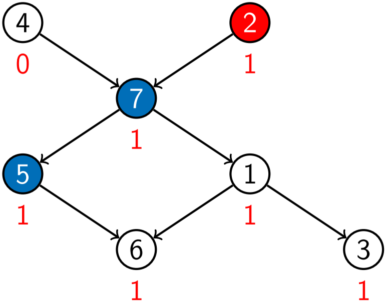

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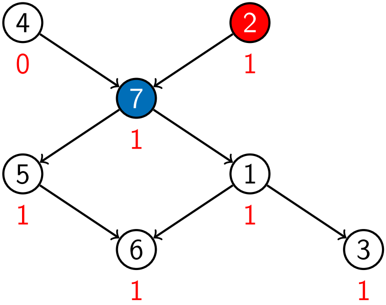

Result: 3 6 1 5

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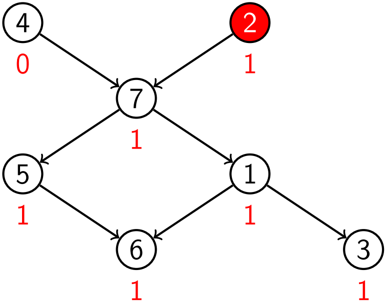

Result: 3 6 1 5 7

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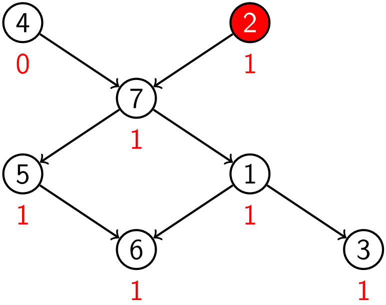

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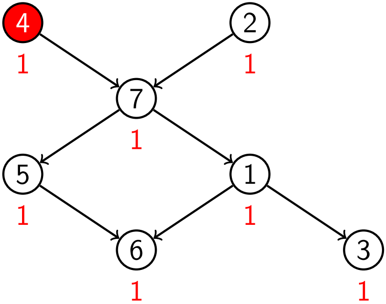

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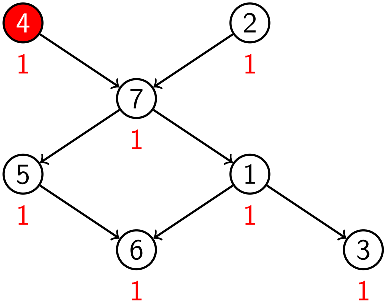

Result: 3 6 1 5 7 2 4

I. Topological Sorting

- Each vertex and edge is considered only once.
- So the running time of this algorithm is linear (more precisely: $O(|V| + |E|)$).

II. BCC/Articulation Points/Bridges

Definition

Let G be a simple graph. An articulation point is a vertex of G whose removal separates G into at least two components.

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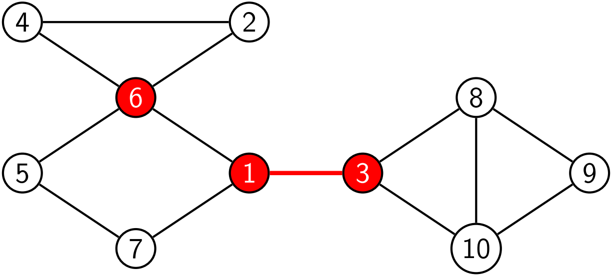
Definition

Let G be a simple graph. A bridge is an edge of G whose removal separates G into two components.

II. BCC/Articulation Points/Bridges

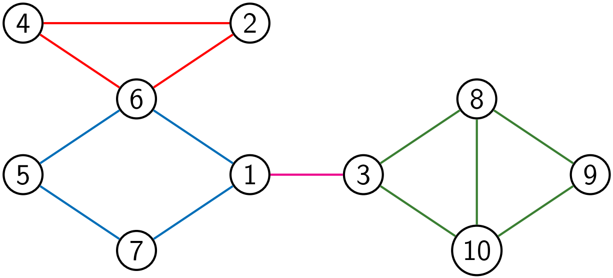


II. BCC/Articulation Points/Bridges

Definition

A graph G is biconnected if it has no articulation points.
A biconnected component of a graph is a maximally biconnected subgraph.

II. BCC/Articulation Points/Bridges



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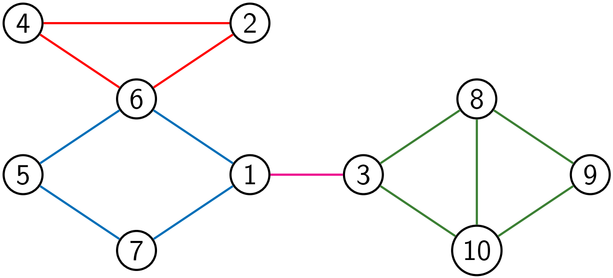
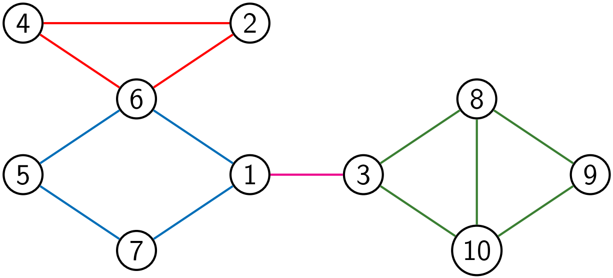


- Biconnected components intersect in articulation points.

II. BCC/Articulation Points/Bridges



- Biconnected components intersect in articulation points.
- Bridges are biconnected components of size two.

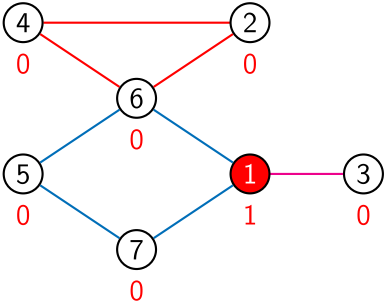
II. BCC/Articulation Points/Bridges

Using DFS, we can find biconnected components, articulation points, and bridges in linear time.

II. BCC/Articulation Points/Bridges

First, observe that running a DFS from a node creates a spanning tree (the *DFS tree*), as well as some back-edges.

Example:



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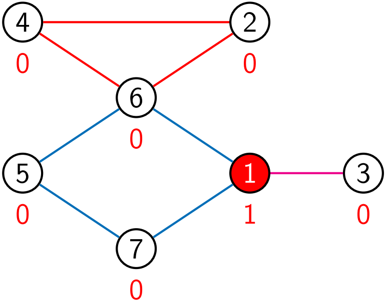

II. BCC/Articulation Points/Bridges

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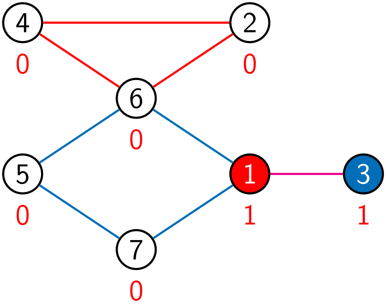

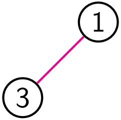

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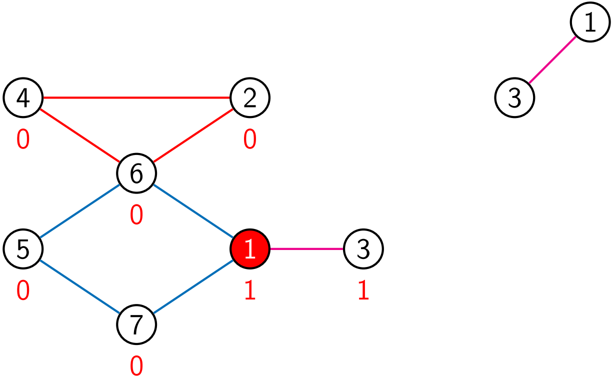

II. BCC/Articulation Points/Bridges

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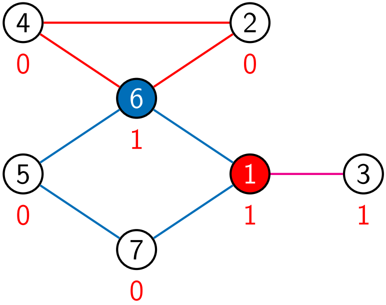

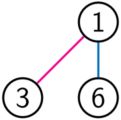

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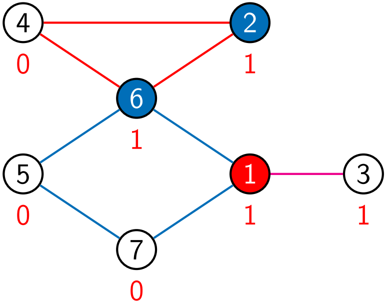

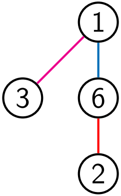

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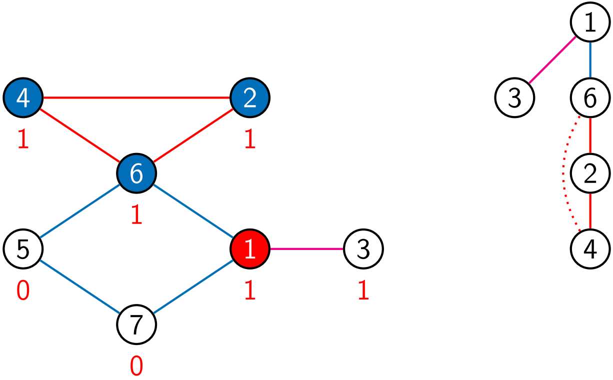

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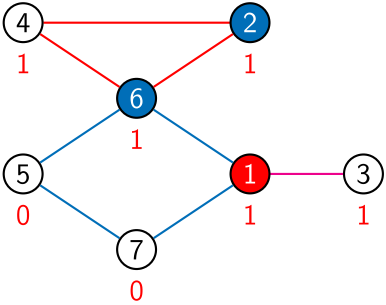

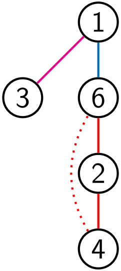

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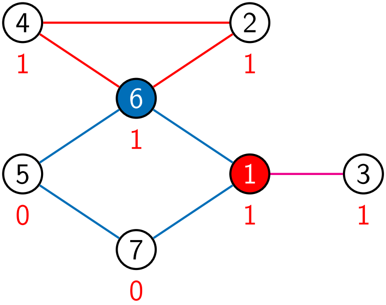

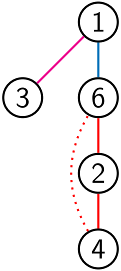

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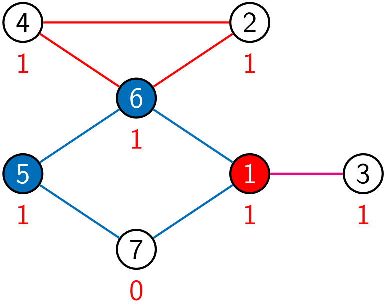

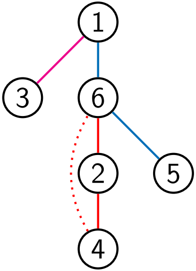

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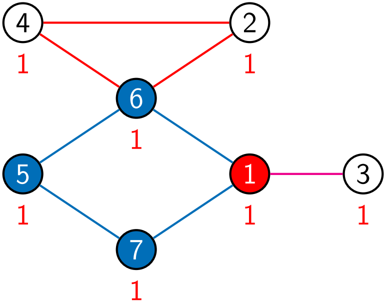
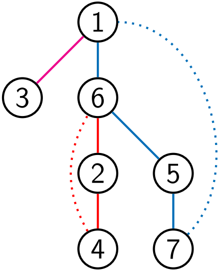


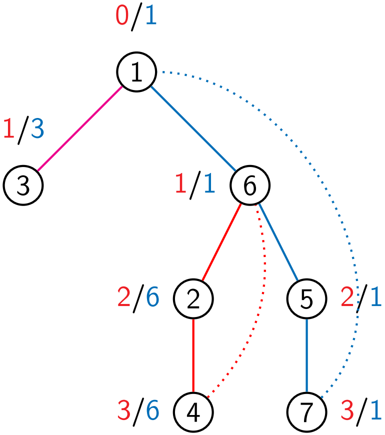


II. BCC/Articulation Points/Bridges

We will compute

- `depth[v]`: the depth of vertex v in the DFS tree
- `low[v]`: the lowest-depth neighbor of v or a descendant of v via back-edges in the DFS tree (or v itself if there is no lower-depth neighbor)

In our example, `depth` and `low` are as follows:



We can find articulation points:

- The root vertex is an articulation point iff it has more than one child.
- A non-root vertex `u` is an articulation point iff it has a child `v` with `depth[low[v]] >= depth[u]`.

II. BCC/Articulation Points/Bridges

We can find bridges:

- All bridges must be edges of the DFS tree.
- A vertex `u` forms a bridge with a child `v` iff `depth[low[v]] > depth[u]`.

II. BCC/Articulation Points/Bridges

We can find biconnected components:

- follow the DFS tree from the leaves upwards
- put vertices into the component of the leaf along the way
- stop at articulation points.

II. BCC/Articulation Points/Bridges

It is easy to compute depth and low during the DFS.

```
void dfs(int v, int d) {
  visited[v] = 1;
  depth[v] = d; // set depth
  low[v] = v;
  for (each child u of v) {
    if (!visited[u]) {
      dfs(u, d+1);
      if (depth[low[u]] <= depth[l]) low[v] = low[u];
    } else if (depth[u] < depth[v]-1) low[v] = u;
  }
}
dfs(0,0); // call like this
```

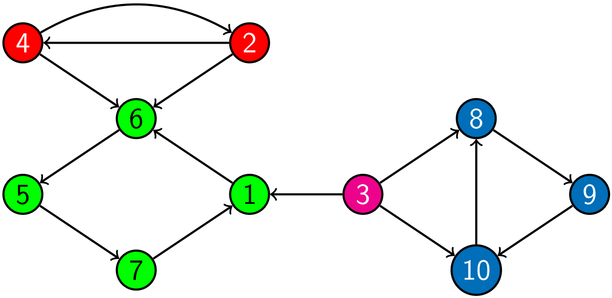
III. Strongly Connected Components

Definition

Let D be a directed graph. Write $u \sim v$ if vertex u can be reached from v and vice-versa. Then

- $\sim$ is an equivalence relation on $V(D)$, and
- the equivalence classes are called strongly connected components of D.

III. Strongly Connected Components (SCC)



- After contracting each SCC, we are left with a DAG!

III. Strongly Connected Components

How to compute strongly connected components?

- Do a DFS and compute finishig times.
- Compute transposed graph, by reverting all edges.
- Do DFS on transposed graph, prioritizing large finishing times.

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The complexity is $\mathcal{O}(|V| \log |V| + |E|)$.

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The complexity is $\mathcal{O}(|V| \log |V| + |E|)$.
To get $\mathcal{O}(|V| + |E|)$ you can use Tarjans algorithm which uses `low[v]`.

That's all.