**Politecnico di Milano**
M. Sc. Music and Acoustic Engineering
Musical Acoustic

# Assignment 03
Horn Design


Ceriola Federico 10786380
Zoboli Enrico 10800970

Professor:
Fabio Antonacci


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## Contents

# 1 Introduction

In the first part of this work an exponential horn in analyzed by computing its input impedance and comparing it with an approximated model. Different approaches of approximation are presented and the comparison is based on two error functions. The approximated model has to be composed by a sequence of conical section that must be optimised in order to resemble the most the exponential horn. It was our job to find a good method for the impedance computing.
In the second part a compund horn, composed by the exponential horn and a cylindrical part, is analyzed.

# 2 Exponential Horn Analytical Input Impedance

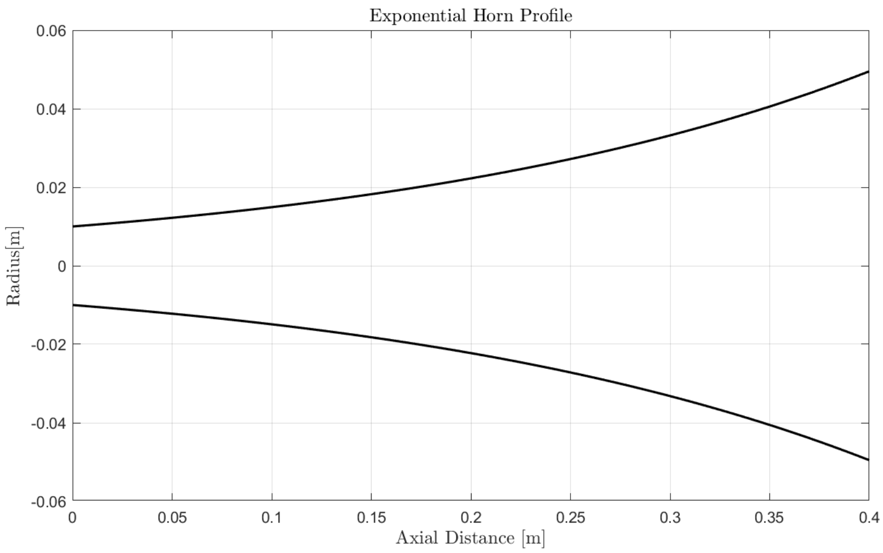


Figure 1: shape of the exponential horn given by the problem

The exponential horn is characterized by a length $L = 0.4\,\mathrm{m}$, its shape is given by the equivalent radius defined as $a = a_0 e^{mx}$, where $a_0 = 0.01\,\mathrm{m}$ ideally is the radius of the cylindrical pipe smoothly joined to the horn and $m = 4\,\mathrm{m}^{-1}$ is the flare coefficient of the horn. The range of frequency that are taken into account for the computation is between 0 Hz and 2000 Hz.

The analytical input impedance of the exponential horn is defined by the relation:

$$Z_{in} = \frac{\rho c}{S_1}\Big(\frac{Z_L \cos(bL+\theta) + j(\rho c/S_2 \sin(bL))}{jZ_L \sin(bL) + (\rho c/S_2 \cos(bL-\theta))}\Big) \tag{1}$$

where:

- $\rho$ : density of the air.
- $c$ : sound speed in the air.
- $S_1$ and $S_2$ : respectively the surfaces at the throat and at the mouth of the horn.
- L : the length of the horn.
- $b = \sqrt{k^2 - m^2}$
- $\theta = \tan^{-1}(m/b)$

- $k = \omega/c$
- $Z_L$: impedance load.

For this particular shape of the horn (Salmon horn), we can consider the wavefront passing through the horn planar, so the surfaces involved are represented by the area of the circle at the throat and at the mouth, $S_1 = \pi(a_0 e^{m\times 0})^2$ and $S_2 = \pi(a_0 e^{m\times L})^2$. Moreover, taken into account the temperature of 21 degrees, the sound velocity is $c = 343.21\,\mathrm{m/s}$ and air denisty $\rho = 1.2041\,\mathrm{kg/m^3}$ .

At the moment the radiation impedance at the mouth is neglected ($Z_L = 0$), and the following equation is taken in account :

$$Z_{in} = \frac{\rho c}{S_1}\Big(\frac{j(\rho c/S_2 \sin(bL))}{(\rho c/S_2 \cos(bL-\theta))}\Big) \tag{2}$$

In Fig.1 it is plotted the input impedance depending on the frequency.

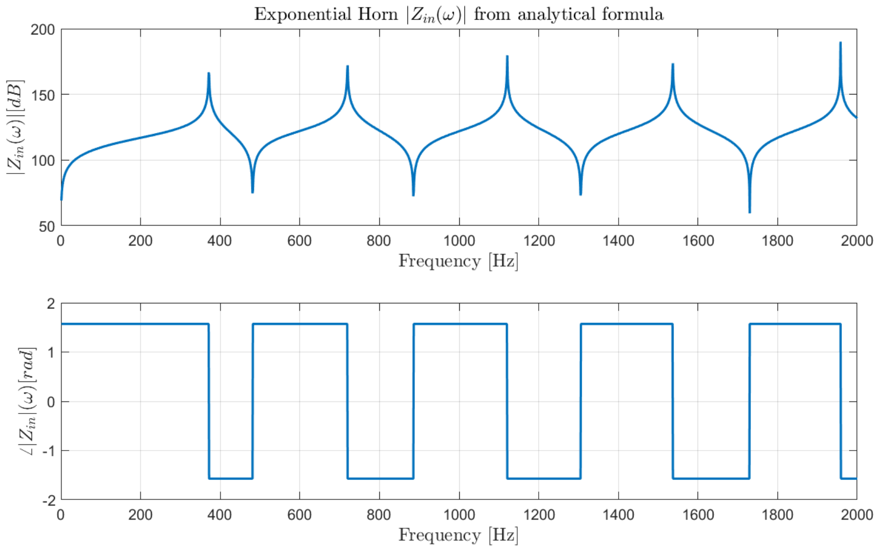


Figure 2: Magnitude and phase of input impedance of the exponential horn neglecting the radiation impedance

In order to approximate the exponential horn with a series of conical sections two methods were considered:

- Transmission line element
- Compound horn sequence

# 3 Exponential Horn as Transmission Line Approximation

A method that we took into account was the Transmission Line Method as presented in [1] and [2]. Thanks to this procedure it is possible to evaluate the input impedance of series of conical sections by the multiplication of singles transmission matrices of each conical element. The algorithm is presented as:

1. Divide the length axis in $N$ equi-spaced sections.
2. For each section compute the transmission matrix H (for a single frequency) with the elements $H_{11}$ $H_{12}$ $H_{21}$ $H_{22}$. The realtion for the computation of the element is proposed in [1] and in [2].
3. Compute the multiplication of each matrix H as: $\Pi_1^{N-1} H_i$, where $i$ denotes the $i$-th section.

4. Repeat the algorithm for each frequency of interest.
5. Compute the Impedance as: $Z_{in} = \frac{H_{12}+H_{11}Z_L}{H_{22}+H_{21}Z_L}$

The computed impedance has been plotted compared with the exponential horn using $N = 4$ sections.

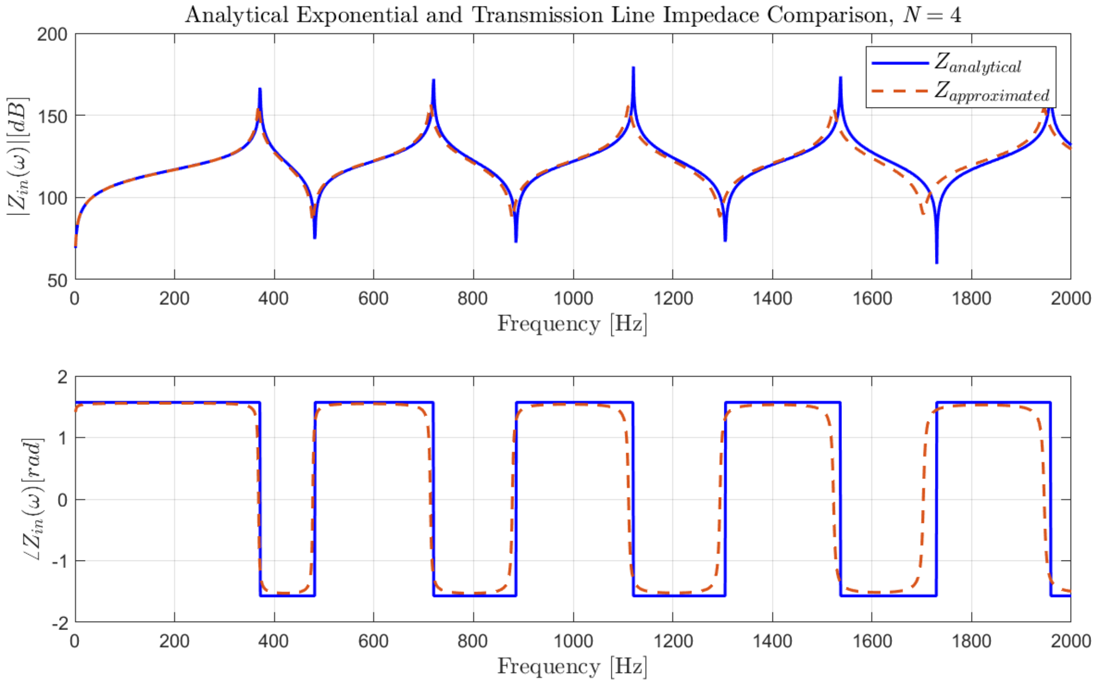


Figure 3: Comparison Analytical Exponential and Transmission Line Method

The location of frequency peaks shows a good a approximation, while for the amplitude the transmission line approximation is quite below the analytical formula. For this reason a further technique has been proposed.

# 4 Exponential Horn as Compound Horn Approximation

As a second method it is proposed considering the whole horn as a compound of single conical horns. In particular, considered a certain number of section, and considered the absence of a mouth radiation impedance, the total $Z_{in}$ is computed taking into account the section starting from the mouth (the farthest point from the throat) and going back to the beginning. With this method the$Z_{in}$ of the successive section will be take as the radiation load ($Z_L$ in the formula) the input impedance of the previous section. In this way, the model is treated as a compound horn with only conical section.

The input impedance for the conical horn is given by:

$$Z_{in} = \frac{\rho c}{S_1}\left\{\frac{jZ_L[\sin(kL-\theta_2)/\sin\theta_2] + (\rho/S_2)\sin kL}{Z_L[\sin(kl+\theta_1-\theta_2)/\sin\theta_1\sin\theta_2] - (j\rho c/S_2)[\sin(kL+\theta_1)/\sin\theta_1]}\right\} \tag{3}$$

where: $\theta_{1,2} = \tan^{-1}(kx_{1,2})$ and $x_{1,2}$ are the distances from the apex of the cone.
Iterating over the number of equi-spaced sections it is possible to obtain the input impedance at the throat. In Figure 4 it is possible to see the impedance computed with this method compared with the one of the exponential horn.

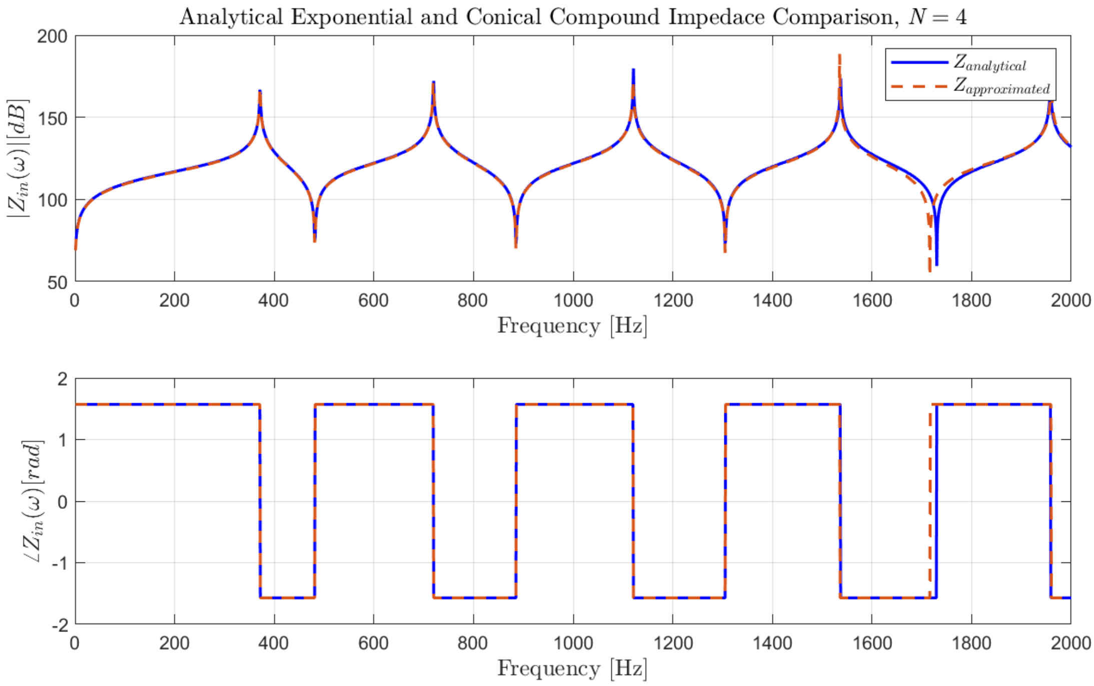


Figure 4: Magnitude and phase comparison Analytical Exponential and Compound Method

The procedure shows a good approximation in terms of frequency peaks location and amplitude. This method worked better than the transmission line method presented before using the same number of linear spaced sections. The compound horn has been chosen as method to approximate the exponential horn.

In Figure 5 it is possible to see the comparison between the profile of the exponential horn and the series of conical sections. It can be seen that near the throat of the horn the conical sections tends to resemble the shape of the exponential while near the mouth, visually, the error introduced by the approximation increases.

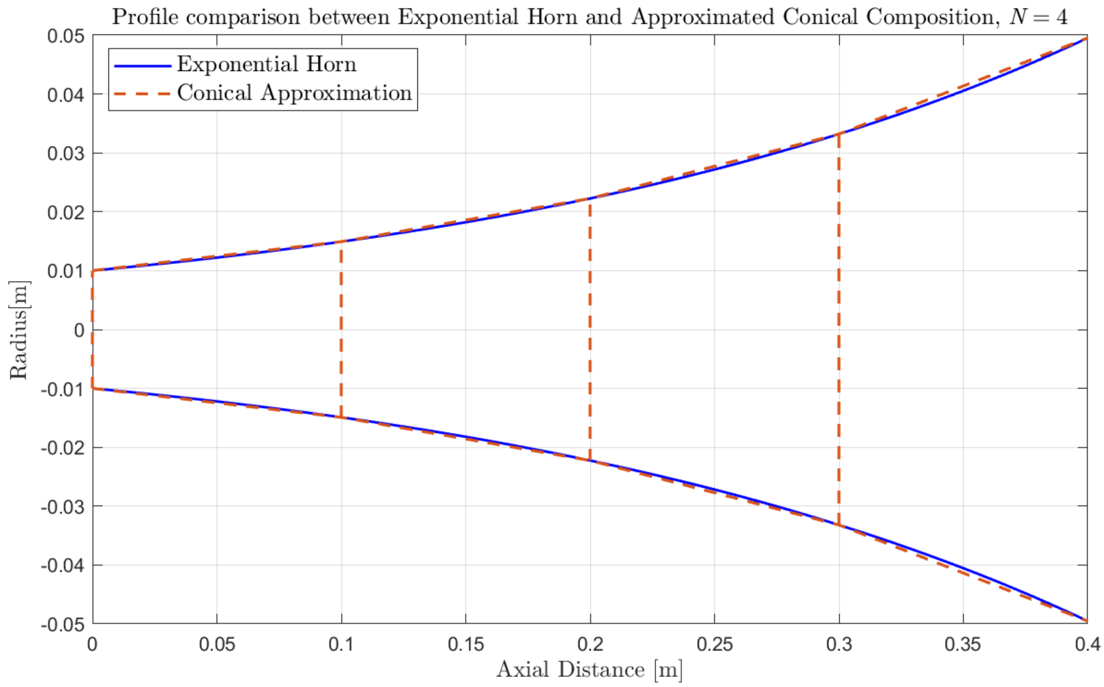


Figure 5: Profile comparison between exponential horn and conical series

# 5 Errors $e_1$ and $e_2$

The main objective of this homework is to compute an evaluation on the method provided through the value of two parameter called $e_1$ and $e_2$. By looking at this two parameters it is possible to tune the model in order to minimize the error and select an optimal number of section in which the model slice the exponential horn.

## 5.1 Mean Square Error $e_1$

The first parameter is defined as:

$$e_1 = \frac{1}{\omega_{max} - \omega_{min}} \int_{\omega_{min}}^{\omega_{max}} |Z_1(\omega) - Z_2(\omega)|^2 d\omega \tag{4}$$

This represents the mean square error between the two impedance. The value of $Z_1(\omega)$ and $Z_2(\omega)$ are respectively the impedance obtained by the approximation method and the analytical one. To evaluate the error as function of the segment length, resulting from the sampling of "length" axis of the exponential horn, a `for` loop has been used. The loop iterates over the number of sections used to slice the horn, starting from 1 up to 100 sections. For each iteration the new approximated impedance is evaluated and the error is computed. The MSE as function of the segment length $\delta$ has been plotted in a logarithm fashion in Figure 6. It can be observed a decreasing behaviour by reducing the dimensions of each conical horn starting from $\delta = 0.05\,\mathrm{m}$ where the horn is divided in 8 sections. For some values of $\delta$ between 0.1 and 0.05 the error increased more than considering the exponential horn as a single conical horn ($\delta = 0.4\,\mathrm{m}$).

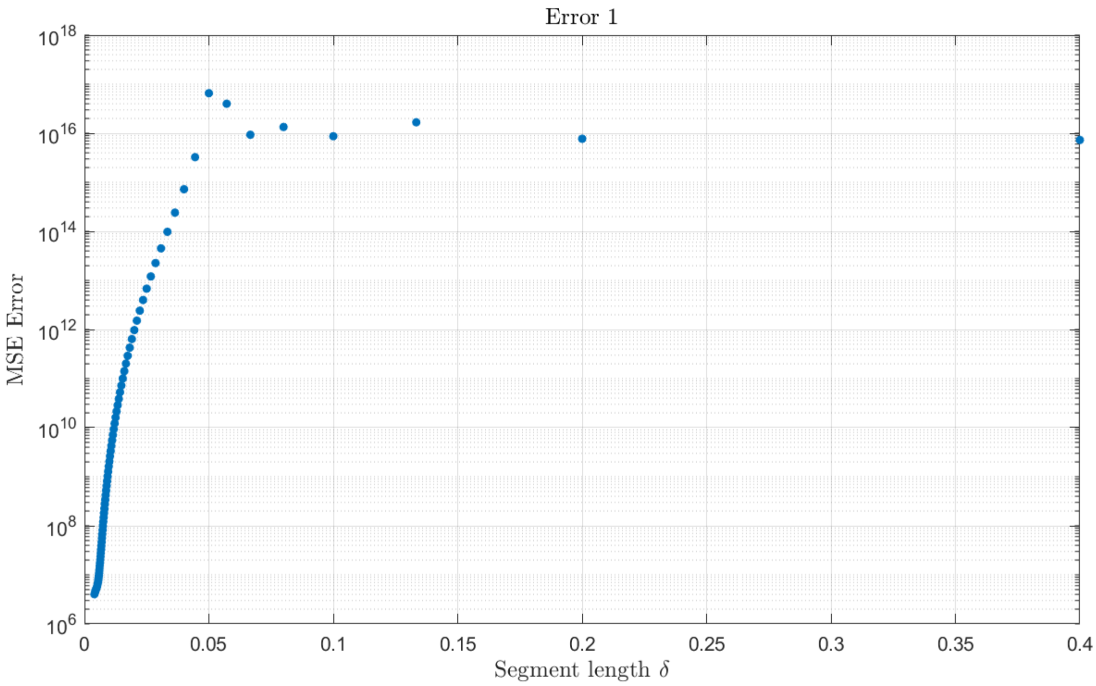


Figure 6: Mean Square error as function of the segment length

### 5.2 Frequency Peaks Location Error $e_2$

As second measurement parameter the error $e_2$ has been defined as:

$$e_2 = \sum_{1}^{N_{peaks}} \min |argmax_\omega Re(Z_1(\omega)) - argmax_\omega Re(Z_2(\omega))| \tag{5}$$

This type of function measure the difference in frequency between the peaks of the impedance of the the exponential horn and its conical approximation. Practically, it has been implemented by applying the function `findpeaks` of MATLAB over the magnitude of the two impedance. The error as function of the segment length has been plotted in Figure 7. It can be observed that decreasing the segment length (increasing the number of sections) the error sets to zero near $\delta = 0.06\,\mathrm{m}$ which means dividing the horn in 6 sections.

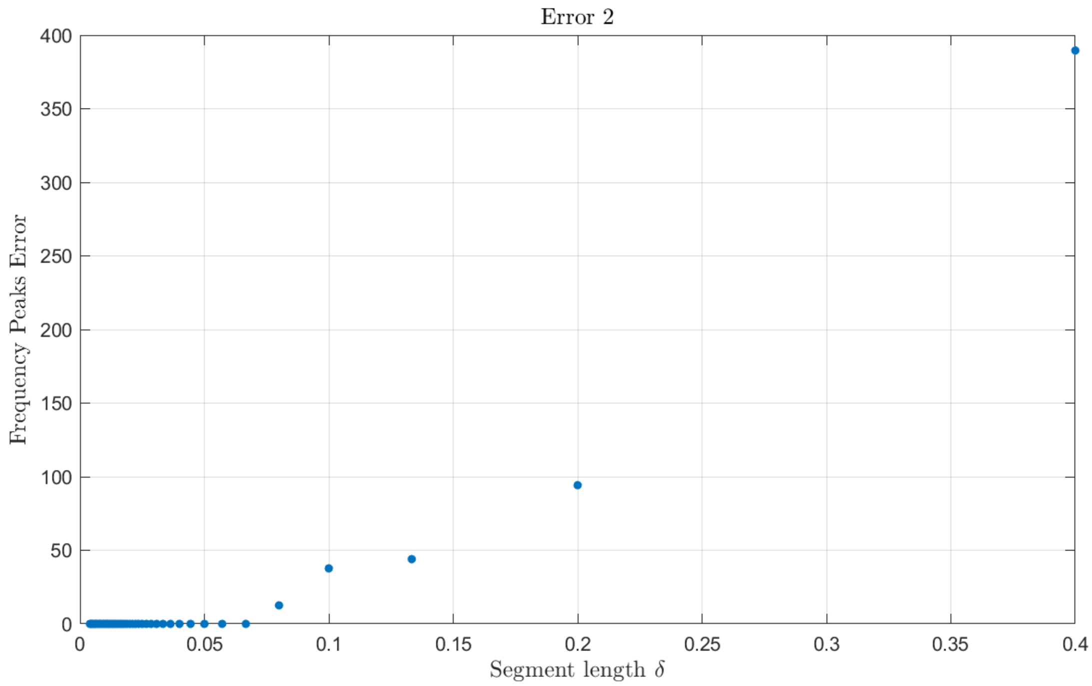


Figure 7: Frequency peaks error as function of the segment length

From the errors $e_1$ and $e_2$ it can seen that starting dividing the horn from 6 sections the all frequency peaks are almost perfectly positioned by the approximation. The issue of the conical slicing still the magnitude, represented by the MSE, that still decreasing in the range of sections considered.

# 6 Non-Linear Sampled Sections

As described before, linearly sampling the exponential horn introduce, visually, an error approaching the mouth of the horn. The idea of a non-uniform sampling is to decrease the distance between the slicing points approaching the mouth. Two approaches has been considered:

1. Hyperbolic tangent sampling.
2. Fminsearch sampling.

## 6.1 Hyperbolic tangent sampling

The idea behind the *Tanh* sampling is to map a linear spaced vector into a non-linearly spaced with a thickening of the slicing approaching the mouth. As shown in the Figure 8, a linearly spaced input vector has been used as input to the *Tanh* function. The result is then normalized and scaled to be constrained in the limits of the length of the horn. The resulted slicing can be seen in Figure 9 where 6 sections have been used. Approaching the mouth the distance between the sections decreasing trying to approximate the exponential profile. This spacing ratio can be modulated by a parameter called “gain”, that scales the input vector to the *Tanh* function. Changing the gain parameter the cuts near the mouth will be closer and closer to each others.

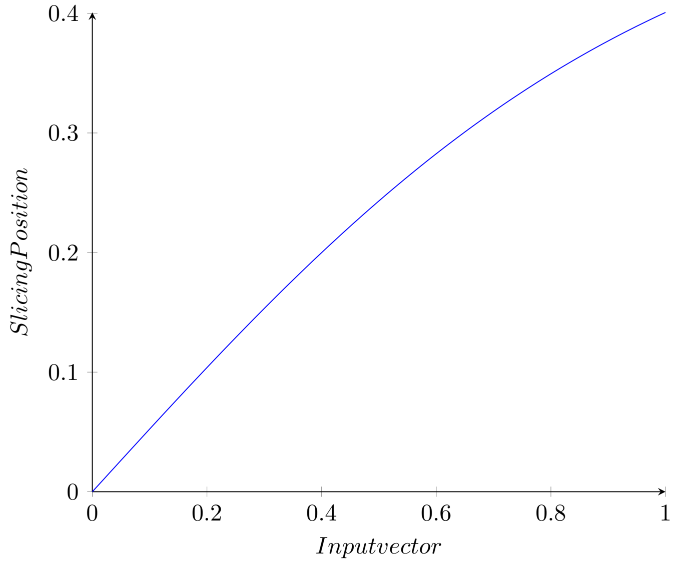


Figure 8: Hyperbolic tangent mapping

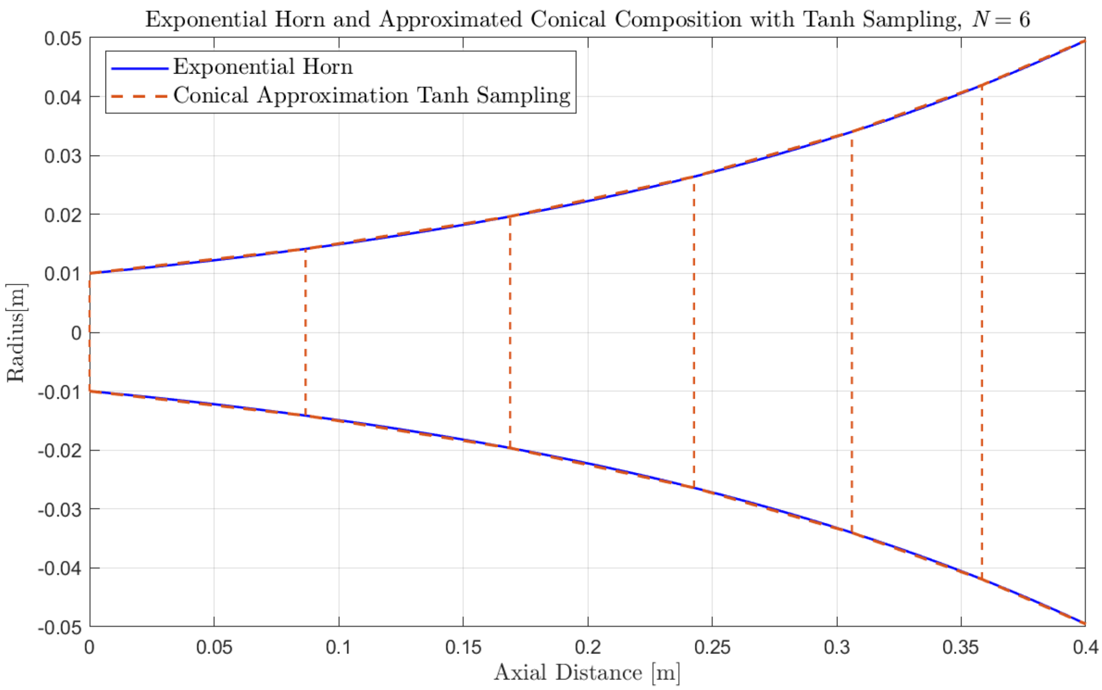


Figure 9: Slicing position of Tanh Sampling, 6 sections, gain = 1

### 6.2 Fminsearch Sampling

The second approach has been done using a minimization algorithm as `fminsearch` from MATLAB. In particular, given $N$ sections, the function minimize the error $e_1$ by changing $N-1$ slicing positions. The process is expensive in terms of computation time and become unfeasible by a large number of sections. The slicing position found for 6 sections can be seen in Figure 10. The cutting position are no more linearly spaced and they are closer and closer both to the throat and the mouth of the horn.

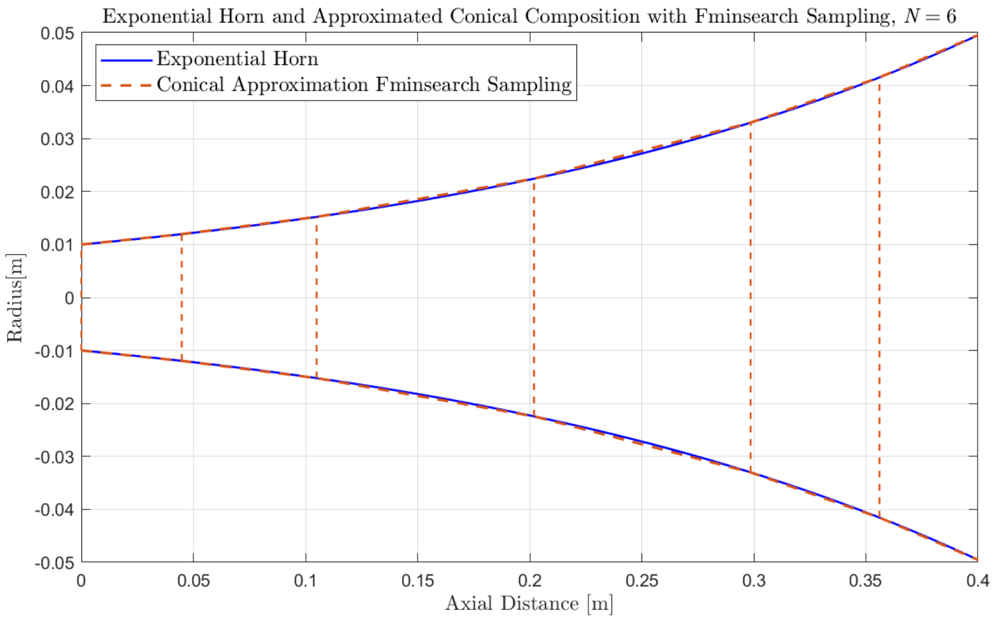


Figure 10: Slicing position of Fminsearch Sampling

To measure the error introduced by the different types of sampling the $e_1$ and $e_2$ has been evaluated as function of the number of sections. The results of $e_1$ can be seen in Figure 11 while the $e_2$ in Figure 12.

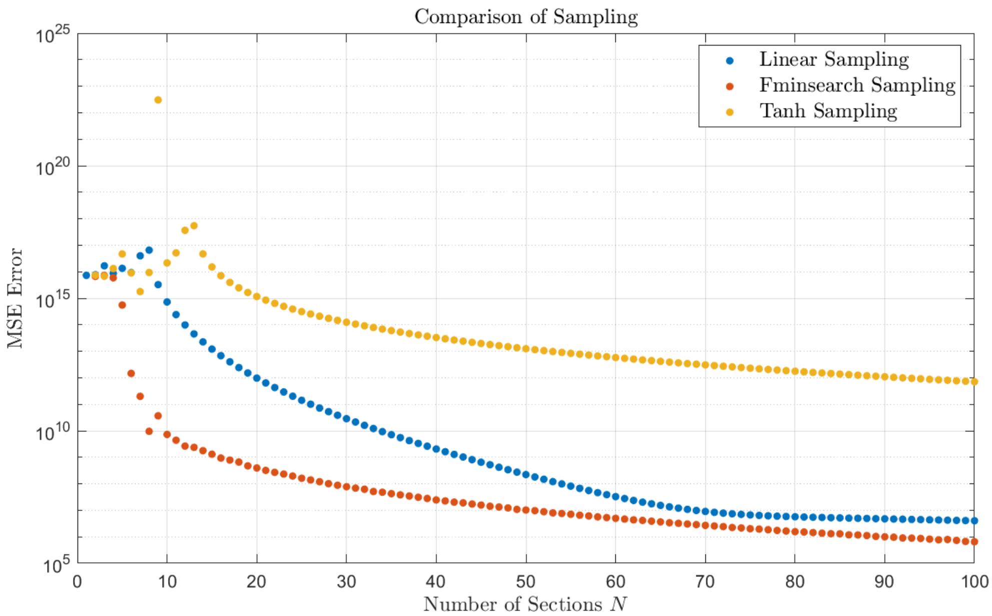


Figure 11: MSE of the three different type of sampling as function of the number of sections

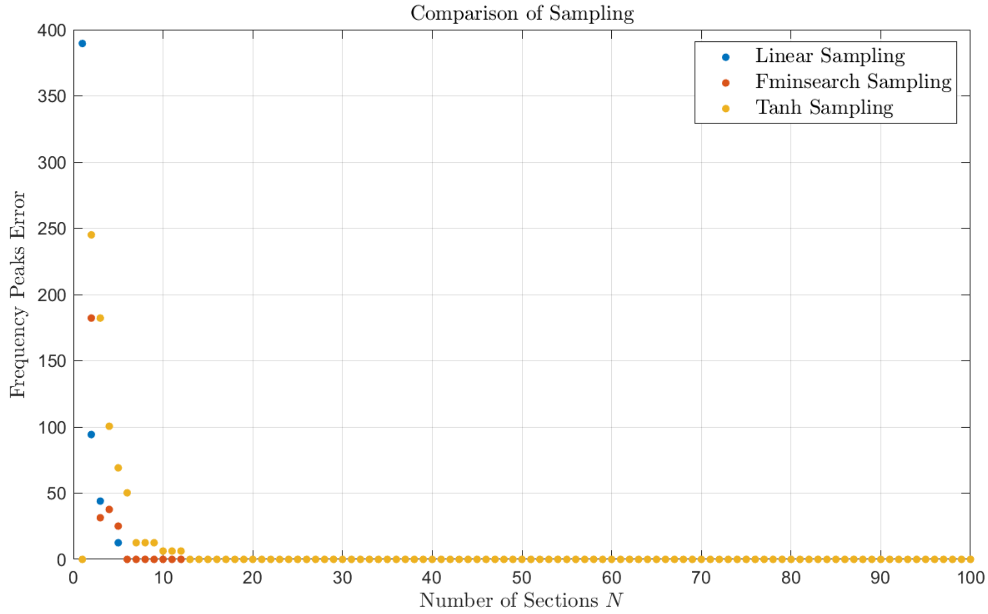


Figure 12: Frequency peak error of the three different type of sampling as function of the number of sections

From Figure 11, representing the MSE, the *Tanh* sampling can be compared at a low number of sections to the *Linear* sampling with a improvement of the order $10^3$ at 7 sections. After 10 sections the *Tanh* sampling introduce more error with respect to the *Linear* sampling, decreasing slower by increasing *N*.
The *Fminsearch* sampling shows better results with respect both the linear and *Tanh* sampling. It decreases faster below 10 sections and then tends to reduce the slope due to the tolerances and number of iterations used for the convergence of the algorithm.

From the Figure 12, representing the frequency peaks error, the *Fminsearch* and *Linear* sampling reach the zero at the same number of sections (6 sections), while the *Tanh* sampling takes around 12 sections to reach the same point. The *Fminsearch* perform a slope steeper compared to the *Linear* sampling, meaning that below 6 section the *Fminseach* perfom better than the *Linear* sampling. Further techniques can be applied using the gradient of the impedance as described in [2].

Since both the *Fminsearch* and the Linear sampling reached the zero using the same number of section, the linearly spaced model with 6 sections will be considered from this moment.

# 7 Input Impedance with Load

For what has been said before, the model used in this the one with a linear sampling of the conical section with $n = 6$ sections. To compute the radiation impedance, the formula of an unflanged cylindrical pipe with radius $a = a_0^{m \times L}$:

$$Z_L(\omega) = Z_{L0}(\omega)\frac{S_p}{S_s} \tag{6}$$

where:

- $Z_{L0}(\omega) = 0.25\frac{\omega^2\rho}{\pi c} + 0.61j\frac{\rho\omega}{\pi a}$
- $S_p = \pi a^2$
- $S_s = \frac{2S_p}{1+cos\theta}$
- $\theta$ is the flaring angle of the last conical section.

At this point we compute again the impedance with the same method used before but starting by imposing the new value of $Z_L$, so the last conical section will be loaded with $Z_L$ retrieved from the formula above and its input impedance will be considered as the load for the previous conical section. In the Figure 13 it is possible to notice a shifting of the peaks towards the lower frequencies and a smoothing in the magnitude and phase with respect to the input impedance of the unloaded horn. At high frequency the peaks appear to be smoother and phase changes less steeper.

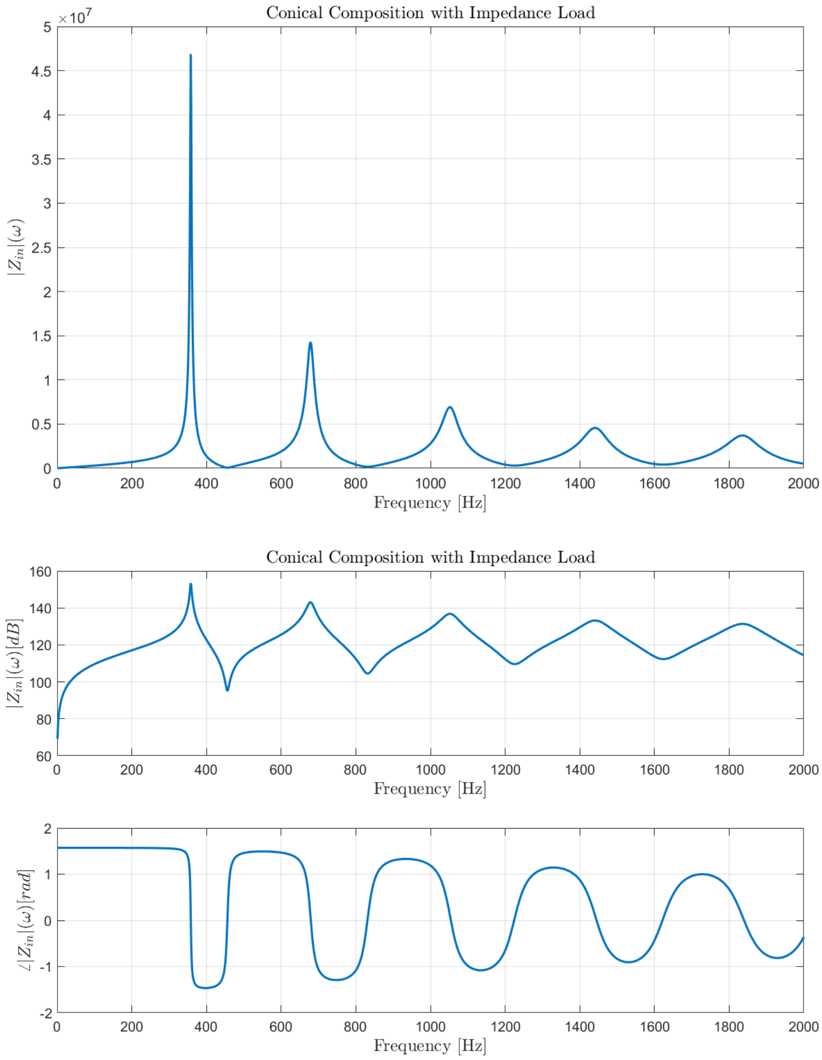


Figure 13: Input Impedance of the approximated model considering the Radiation Load, respectively: magnitude, dB magnitude, phase

## 8 Cylindrical Compound Horn

In this part it is considered a compound horn composed by a cylindrical pipe and the exponential horn. The length of the cylindrical pipe is equal to $L_2 = 0.6\,\mathrm{m}$ and its radius is equal to the radius of the smaller section of the horn. This is because the horn is considered as smoothly joined with the cylindrical pipe. So the radius is $r = a_0$. Moreover, the radiation load is kept in account for the calculation of the input impedance of the exponential horn. The formula to compute the impedance of a finite cylinder is given by:

$$Z_{IN} = Z_0 \left[ \frac{Z_L cos(kL_2) + jZ_0 sin(kL_2)}{jZ_L sin(kL_2) + Z_0 cos(kL_2)} \right] \tag{7}$$

Where:

1. $L_2$: is the length of the cylindrical part.
2. $Z_0$: is the characteristic impedance computed as $Z_0 = \rho c/S$, where $S$ is the cross section of the pipe.
3. $k$: the wavenumber considering the wall losses in the cylindrical pipe as $k = \omega/v - j\alpha$
4. $v = c(1 - \frac{1.65\times10^{-3}}{a_0 f^{1/2}})$
5. $\alpha = \frac{3\times10^{-5} f^{1/2}}{a_0}$
6. $Z_L$: is the impedance of the exponential part, computed with the formula 1.

The impedance of the compound horn has been plotted in Figure 14. The impedance, in this case, presents more peaks compared with the impedance of the single exponential horn, due to the elongation of the overall system. Also the fundamental frequency has been shifted toward the low frequencies.

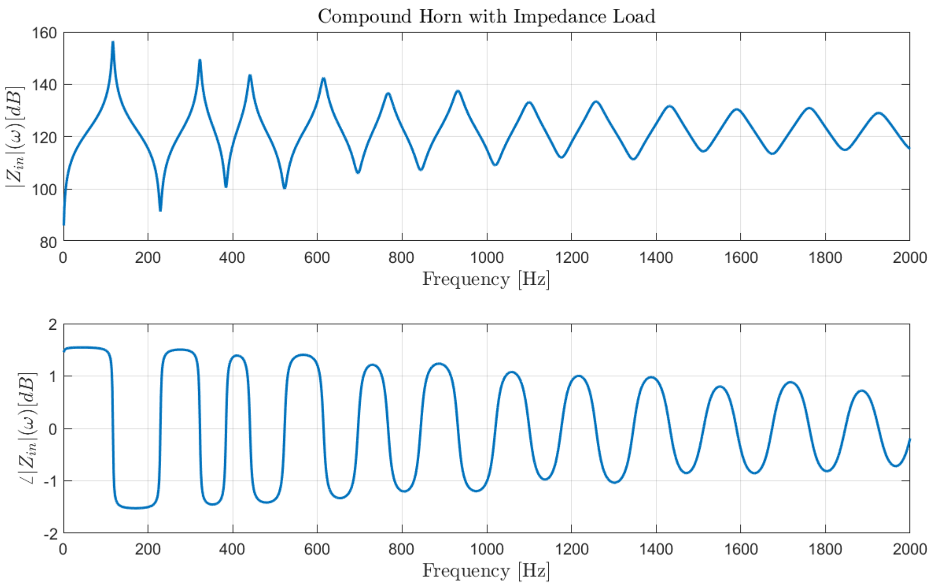


Figure 14: Impedance of the compound horn with a cylindrical and an exponential part

## 8.1 First ten maxima of the impedance

| First ten maxima of the impedance | |
|---|---|
| 1 | 116 Hz |
| 2 | 321 Hz |
| 3 | 440 Hz |
| 4 | 613 Hz |
| 5 | 766 Hz |
| 6 | 931 Hz |
| 7 | 1099 Hz |
| 8 | 1257 Hz |
| 9 | 1431 Hz |
| 10 | 1590 Hz |

In the table besides is possible to see the frequencies of the first ten peaks of the input impedance, found with `findpeaks` of MATLAB. They can also be retrieved, with some discards, by the zeros of the equation:

$$\tan kL_2 - \cot kL_1 - \frac{m}{k} = 0 \qquad (8)$$

where $b = \sqrt{k^2 - m^2}$ and $L_1, L_2$ are respectively the length of the horn and of the pipe.

# 9 Conclusions

It has been shown as a very good approximation of the exponential horns can be obtained with a sequence of conical horns by properly choosing the sampling strategy and the number of samples. The Transmission Line Method has not been so useful, it's results give errors way bigger than the compound horn method. As regard the *Fminsearch* sampling strategy, trying to find the optimal section time by time is a very energy demanding process but the results lowered the mean square error by five order of magnitude, which is relevant. The *Tanh* sampling performed worse than we expected, producing relevant results just for specific number of sections. Taking into account the radiation load the results coincide with what expected, that is to say a smoothing of the magnitude and phase and a shifting towards lower frequencies of the peaks. Also the behaviour of the compound horn with a cylindrical pipe confirms our predictions of the appearance of more peaks in the considered frequency range.

# References

[1] D. Mapes-Riordan. "Horn Modeling with Conical and Cylindrical Transmission Line Element". In: *Audio Engineering Society* (1991).

[2] J. O. Noreland. "A GRADIENT BASED OPTIMISATION ALGORITHM FOR THE DESIGN OF BRASS-WIND INSTRUMENTS". In: (Nov. 2021).