MUSICAL ACOUSTICS

FIRST HOMEWORK

Report
Helmholtz resonator and system impedance

Students

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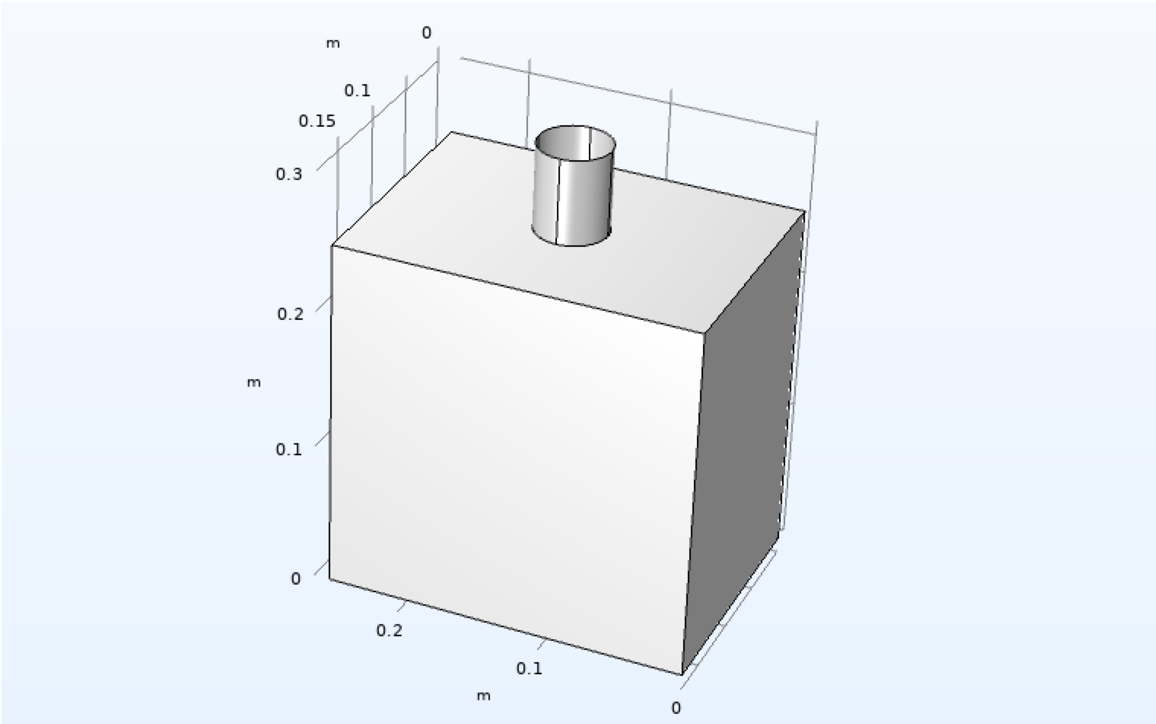

Figure 1: Helmoltz resonator

Introduction

In this report the acoustic properties of a system composed by an Helmholtz resonator are studied. The resonator is composed by a parallelepipedon shaped resonance cavity of dimensions: height $h = 25$ cm, width $w = 25$ cm and length $l = 18$ cm. The neck of the resonator is cylindrical with a length of $L = 6$ cm and radius $a = 2.5$ cm. The resonator is shown in figure 1. All the analysis has been done considering standard air conditions at 20°C like the air density $\rho = 1.2$ $[kg/m^3]$ and sound velocity $c = 343$ $[m/s]$.
Other important quantities to mention are:

- Neck Surface: $S = \pi \cdot a^2$
- Cavity Volume: $V = h \cdot w \cdot l$

a) Resonance Frequency of the resonator neglecting the virtual elongation of the neck

Knowing that an Helmholtz resonator can be seen as an equivalent mass-spring resonating system, as depicted in figure 2, the resonance frequency can be computed as:

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{K}{m}} \tag{1}$$

Where we need to compute the equivalent stiffness (corresponding to the internal volume of air) and equivalent mass (corresponding to the air contained in the neck) of the system.
The equivalent stiffness of the cavity is given by:

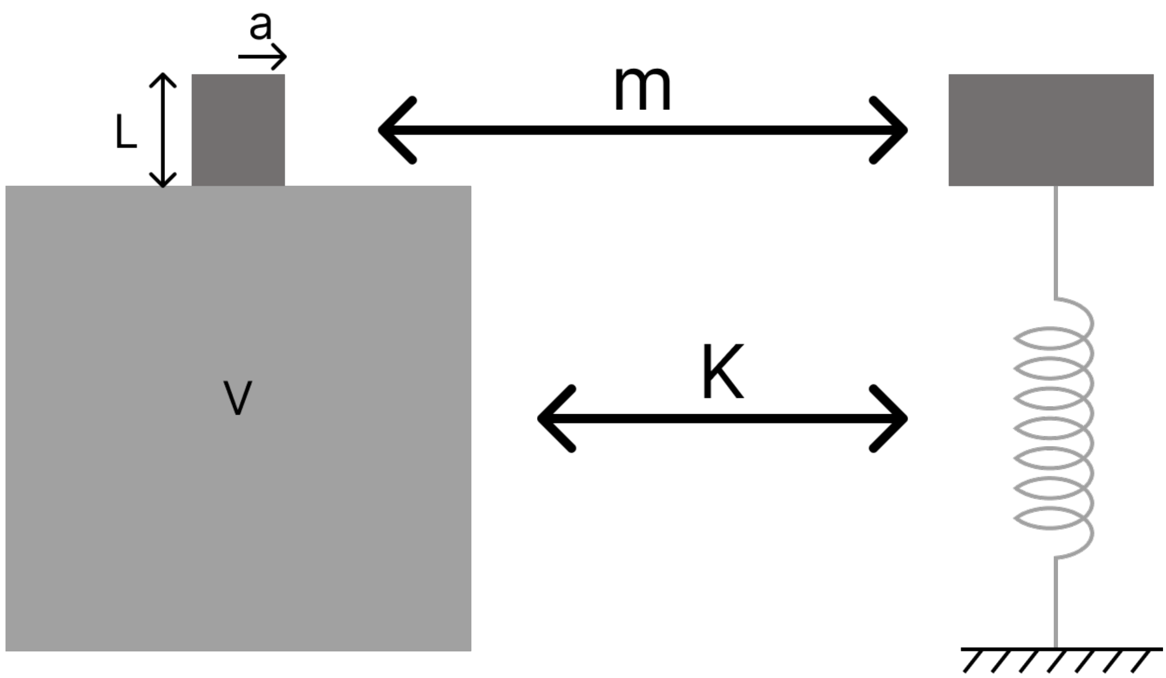

Figure 2: Mass-Spring equivalent system

$$K = \frac{\rho S^2 c^2}{V} \tag{2}$$

While the equivalent mass is given by:

$$m = \rho S L \tag{3}$$

Therefore we're able to compute the resonance frequency of the resonator:

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{K}{m}} = \frac{1}{2\pi}\sqrt{\frac{\rho S^2 c^2}{\rho S L V}} = \frac{c}{2\pi}\sqrt{\frac{S}{VL}} \tag{4}$$

$$\boxed{f_0 = \frac{c}{2\pi}\sqrt{\frac{S}{VL}}}$$

For the given geometry of the system the resonance ferquency is:

$$f_0 = 93.11 \ Hz$$

b) Resonance Frequency including virtual elongation of the neck

The air in proximity of the aperture of the neck actually interacts with the air that composes the vibrating mass, so it has to be taken into account when we're computing the resonance frequency. This is possibile thanks to the **end correction factor** δ, that corrects the length of the cylindrical neck, elongating it virtually, in order to consider the effects to the outer air of the neck. The **end correction factor** is different for the the flanged and unflanged pipes [1]. In the next lines will be discussed the value differences between factors. Therefore the neck becomes the one showed in figure 3.

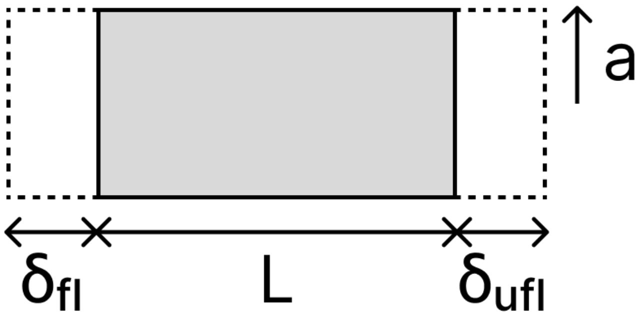

Figure 3: Neck corrected

The correction factor is given, for cylindrical apertures with a flange, by the formula that approximates it as:

$$\delta_{fl} = \frac{8}{3\pi} a \tag{5}$$

While for a cylindrical aperture without a flange, the value of correction factor is:

$$\delta_{ufl} \approx 0.61a \tag{6}$$

So the total length of the neck has to be considered as:

$$L_{corr} = L + \delta_{ufl} + \delta_{fl} \tag{7}$$

this fact allows us to compute the new resonance frequency simply substituting in the original formula the length of the neck:

$$\boxed{f_{0_C} = \frac{c}{2\pi}\sqrt{\frac{S}{VL_{corr}}}}$$

Increasing the length of the neck we're expecting a decreasing of the resonance frequency due to the inverse proportional relation with the length of the neck. This is confirmed by the numerical results:

$$f_{0_C} = 73.43\ Hz$$

c) Resistance for critically damped system

Now it is requested to consider the system critically damped for some resistance R and to compute it keeping all others parameters fixed. We need to remember that a vibrating system is critically damped when the condition $\alpha = \omega_0$ is verified. We also recall the definition of the damping factor α which is:

$$\alpha = \frac{R}{2m} \tag{8}$$

So, since we assumed that the system is critically damped, with the corrected resonance frequency, we can say that:

$$\alpha = \omega_{0_C} = \frac{R}{2m} \Rightarrow R = 2m\omega_{0_C} = 4\pi m f_{0_C} \tag{9}$$

$$\boxed{R = 4\pi m f_{0_C}}$$

Which, once computed has the value of:

$$R = 0.2097\ kg/s$$

d) Impedance of the system

Given a resistance of value $R = 5 \cdot 10^{-4}\ kg/s$ it is requested to derive the impedance of the system. The derivation of the impedance is given from the solution of the differential equation:

$$(-\omega^2 m + j\omega R + k)\tilde{X} = \tilde{F} \tag{10}$$

Since the impedance is defined as $Z = \frac{\tilde{F}}{\tilde{v}}$ we can find the time derivative of the displacement in order to obtain the velocity.

$$\tilde{X} = \frac{\tilde{F}}{-\omega^2 m + j\omega R + k} \tag{11}$$

$$\tilde{v} = \dot{\tilde{X}} = j\omega \frac{\tilde{F}}{-\omega^2 m + j\omega R + k} \tag{12}$$

$$Z = \frac{\tilde{F}}{\tilde{v}} = \frac{-\omega^2 m + j\omega R + k}{j\omega} = R + j(\omega m - \frac{k}{\omega}) \tag{13}$$

$$\boxed{Z = R + j(\omega m - \frac{k}{\omega})}$$

So, after feeding MATLAB these data we're able to plot the absolute value of the impedance frequency response shown in figure 4.

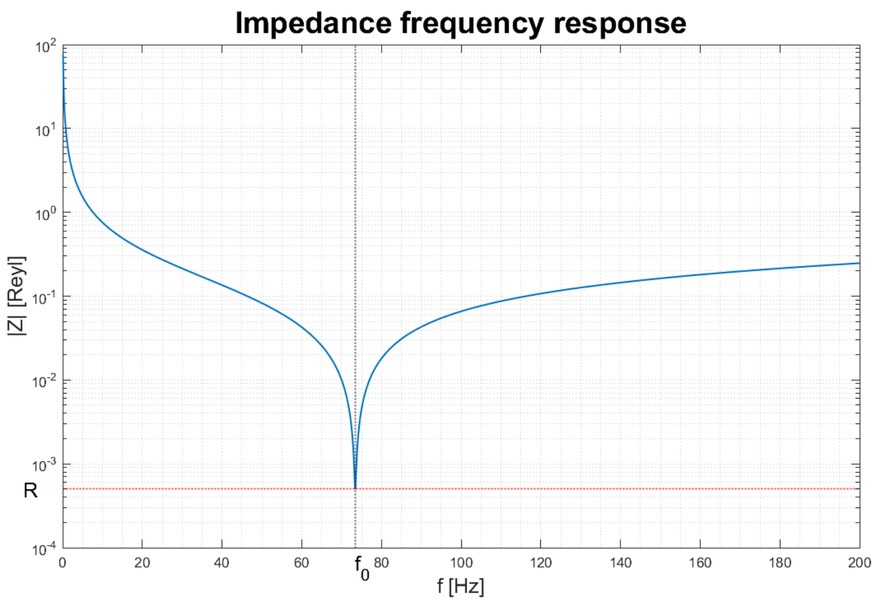

Figure 4: Impedance plot

It's easily visible that at the resonance frequency the system has the lower module of impedance equal to the resistive term R. In fact, at the resonance the imaginary part of the impedance, called *reactance* is nullified. Under the resonance condition the *spring like behaviour* of the impedance overgoes the *mass like behaviour*, and viceversa for frequencies higher than the resonance one.

e) Q Factor and Time Decay Factor

The *Q factor* (or merit factor) gives us a qualitative/quantitative measure to describe how thin and how high is the resonance peak with respect to the resonance frequency. It also compares the spring and the dampening forces. In fact it can be obtained as follows:

$$Q = \frac{F_k}{F_R} = \frac{Kx_0}{R\omega_0 x_0} = \frac{K}{R\omega_0} = \frac{K}{R\omega_0} \cdot \frac{m}{m} = \frac{\omega_0}{2\alpha} \tag{14}$$

$$\boxed{Q = \frac{\omega_0}{2\alpha}}$$

and for the under study system it has value of:

$$Q = 209.74$$

While the *time decay factor* is the time quantity that tells us how quickly the amplitude is decreased about a factor of $1/e$. This is easy to derive because we know that the amplitude decrease with $e^{-\alpha t}$. So if we put (neglecting the oscillations):

$$X(\tau) = X_0 e^{-1} = X_0 e^{-\tau\alpha} \tag{15}$$

So we need:

$$\tau\alpha = 1 \;\Rightarrow\; \tau = \frac{1}{\alpha} \tag{16}$$

$$\boxed{\tau = \frac{1}{\alpha}}$$

In our case the computed value is:

$$\tau = 0.9092 \ s$$

f) Resonance Frequency and Q factor as a function of the resistance R

In order to compute the Q factor in relation to the resistivity from expression 14 we exploit the formula:

$$Q = \frac{K}{R\omega_0} \tag{17}$$

This allows to plot the merit factor as a function of the Resistivity, as shown in the first plot of figure 5. We can notice how the Q-factor is inversely proportional to resistivity, therefore the increasing of resistivity leads to a decreasing of the value of the merit factor.

For the computation of the damped resonance frequency we exploit the formula:

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} \tag{18}$$

We can compute alpha as function of the merit factor Q calculated as:

$$\alpha = \frac{\omega_0}{2Q} \tag{19}$$

Recalling expression 17 we can rewrite α as:

$$\alpha = \frac{R\omega_0^2}{2K} \tag{20}$$

From this we derived the value of the damped frequency as a function of the Resistivity as:

$$\omega_d = \sqrt{\omega_0^2 - \left(\frac{R\omega_0^2}{2K}\right)^2} \tag{21}$$

The second plot below, in figure 5, shows the Real and Imaginary part of the damped resonance frequency of the system. We can notice how, below the value of resistance that corresponds to the critically damped system the damped resonance frequency is real. Therefore we can assume the system is in underdamped condition and is vibrating. While, for values of resistance above the critically damped condition, the damped resonance frequency is purely imaginary and when inserted in the complex exponential form leads to a real negative exponential. For those values of resistance we can assume that the system is not vibrating.

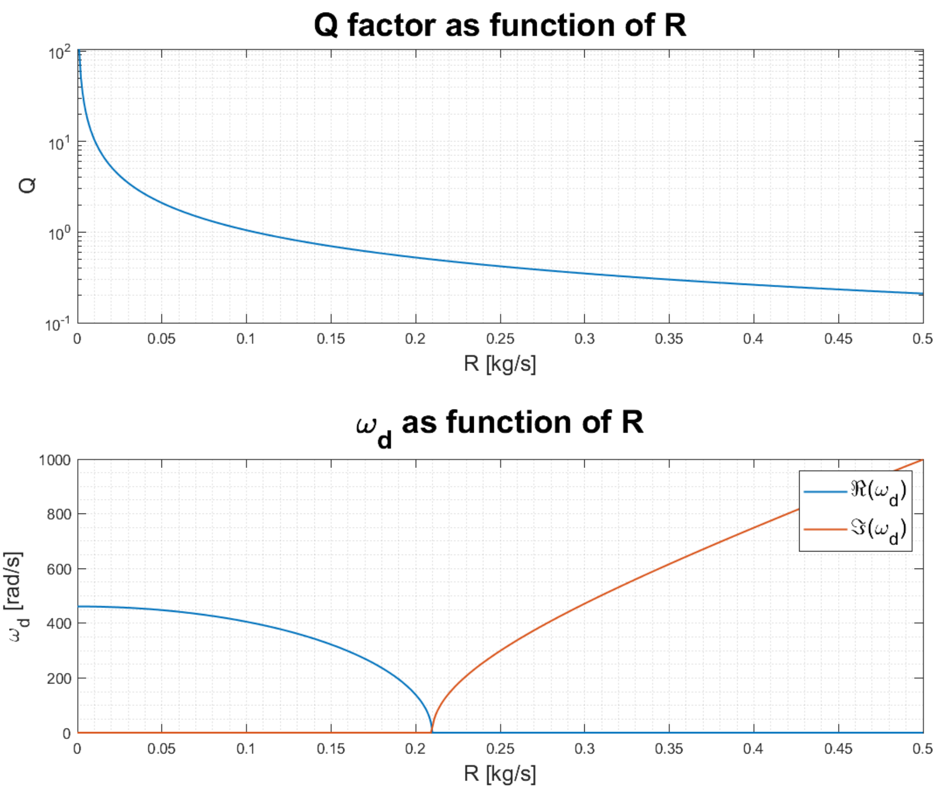

Figure 5: Q-Factor and damped resonance Frequency as a function of resistivity

Bibliography

[1] N. Fletcher and T. Rossing, *The Physics of Musical Instruments*, Springer New York, 2008.