Musical Acoustics

Third Lab Homework

Report
Modeling Techniques

Students

Stefano Donà - 10868267
Paolo Ostan - 10868276

POLITECNICO
MILANO 1863

Introduction

Ex 1) Piano String Modeling

The main aim of the section is to model through *finite-difference method* a string of a piano when struck by the hammer. This method consists in the creation of a Space-time grid that allows to simpliy the behaviour of the continuous string in order to simulate its behaviour over time and compute the results derived directly from the equations that describe the physical behaviour of the system.

Hammer-String interaction

In order to model the interaction between the hammer and string a simplified model of the piano string mechanism is considered, represented in figure 1. The model does not take into account the action of the damper to the string. This allows to simplify the computation cost and the equation formulation used to develop the model, considering only the interaction between the hammer and the string.

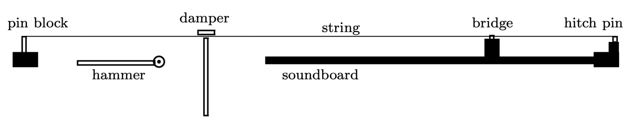


Figure 1: Piano string simplified mechanism

Hammer characterization

In order to model the interaction between hammer and string a time dependent force, exerted on a specific point x_0, is exploited. The force behaviour is specified by the equation below:

$$M_H \frac{d^2\eta}{dt^2} = -F_H(t) - b_H \frac{d\nu}{dt}$$

where M_H is the hammer mass, $\eta(t)$ is the hammer position with respect to the string equilibrium position, b_H is the air damping coefficient and $F_H(t)$ is the force imposed by the hammer. As shown in figure 2 the hammer is composed of an hard-wood cores of graduated sizes covered by two layers of felt with different thicknesses on the extension of the note range. The structure of the component allows adjusting the hammer hardness during the process of voicing. As a matter of fact the hammer response is influenced by the velocity of the strike performed. Due to the properties of the felt an higher velocity causes the hammer to harden and consequently solicites higher harmonics.
This kind of process is highly non-linear and has to be taken into account in order to correctly model the hammer behaviour. We can therefore simulate the hammer as a *lumped mass* attached to a *nonlinear*

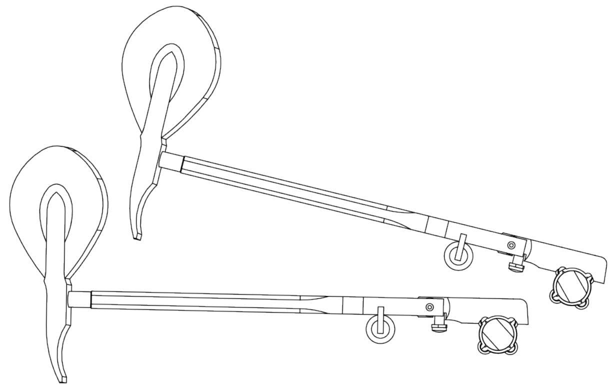


Figure 2: Piano hammer

spring to represent the behaviour of the felt. The force applied by the hammer and caused by the felt is:

$$F_H = K\xi^p$$

Where K is the stiffness, $\xi(t)$ is a time dependent function which describes the felt compression, and p is the stiffness exponent. The function $\xi(t)$ can be written as:

$$\xi(t) = |\eta(t) - y(x_0, t)|$$

where $\eta(t)$ is the hammer position with respect to the string equilibrium position, and $y(x_0, t)$ is the vertical position of the string in $x = x_0$ at the time instant t.
When considering the excitation the whole hammer on the string a force density needs to be considered. It is assumed that the force density term does not propagate along the string, so that the time and space dependences can be separated as shown in equation

$$f(x, x_0, t) = F_H(t) \cdot g(x, x_0)$$

where the time function is the one discussed previously while the space function is $g(x, x_0)$ is a dimensionless function which describes the force density over space. In the simulation computed below the space of the function is an *Hanning window* with width w and centered in *a: relative hammer striking position*, as shown in figure 3

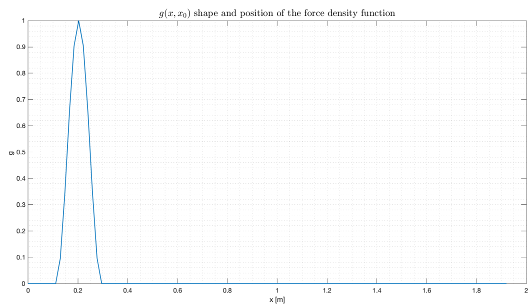


Figure 3: $g(x, x_0)$ Force density function shape

In order to be able to fully characterize the interaction between hammer and string we need to define the contact duration between them. The influence of the hammer on the movement of the string is imposed until the following condition is valid:

$$\eta(t) < y(x_o, t)$$

The hammer parameters considered for the simulations are listed below:

- $M_H = 4.9 \cdot 10^{-3}$ Mass of the hammer
- $w = 0.2$ $[m]$ width of $g(x, x_0)$
- $p = 2.3$ Stiffness exponent: it describes how the stiffness changes with the force
- $V_{H0} = 2.5$ $[m/s]$ Initial Hammer Velocity
- $b_H = 1 \cdot 10^{-4}$ air damping coefficient
- $K = 4 \cdot 10^8$ Hammer felt stiffness
- $a = 0.12$ Relative striking position

String modeling

Considering the string behaviour we want to analyze a string whose behaviour resembles the one of a string in pseudo-real conditions, therefore when introducing the equation of motion which describes it we consider the losses and stiffness of the string:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} - \kappa^2 \frac{\partial^4 y}{\partial x^4} - 2b_1 \frac{\partial y}{\partial t} + 2b_2 \frac{\partial^3 y}{\partial x^2 \partial t} + \rho^{-1} f(x, x_0, t)$$

where the fourth order term derives from the analysis of the stiff string behaviour and κ is the defined as string stiffness coefficient.
While the term $-2b_1 \frac{\partial y}{\partial t} + 2b_2 \frac{\partial^3 y}{\partial x^2 \partial t}$ introduces frequency dependent losses in the equation, being b_1 the air damping coefficient for the string and b_2 the string internal friction coefficient. The term $\rho^{-1} f(x, x_0, t)$ indicates the acceleration imposed to the string by the hammer where ρ is the linear mass density of the string and $f(x, x_0, t)$ is the force density function discussed in the previous section.

Boundary Conditions

Considering now the boundary conditions at the bridge given as:

$$\frac{\partial^3 y}{\partial x^3} = -\frac{c^2}{\kappa^2} \frac{\partial y}{\partial y} + \frac{\zeta_b c}{\kappa^2} \frac{\partial y}{\partial t} \quad \zeta_b = \frac{R_b}{\rho c} \quad \left. \frac{\partial^2 y}{\partial x^2} \right|_{x_M, t_n} = 0$$

where ζ_b is the normalized bridge impedance, with R_b being the impedance of the bridge, x_M is the coordinate at the bridge end, where the string actually cannot move. But produces a force F_b on the bridge with F_b being specified by the equation

$$R_b \frac{\partial y}{\partial t} = F_b \quad F_b = -T_e \frac{\partial y}{\partial x} + \kappa^2 \rho \frac{\partial^3 y}{\partial x^3}$$

$$R_b \frac{\partial y}{\partial t} = -T_e \frac{\partial y}{\partial x} + \kappa^2 \rho \frac{\partial^3 y}{\partial x^3}$$

While the boundary conditions for the left hinged end are given by the equation

$$\frac{\partial^3 y}{\partial x^3} = -\frac{c^2}{\kappa^2} \frac{\partial y}{\partial y} + \frac{\zeta_l c}{\kappa^2} \frac{\partial y}{\partial t} \quad \zeta_l = \frac{R_b}{\rho c} \quad \left. \frac{\partial^2 y}{\partial x^2} \right|_{x_0, t_n} = 0$$

being ζ_l the normalized impedance at the left end and x_0 is the coordinate at the left end, where, as in the previous case, the string actually cannot move.

Parameters for string modeling

Before introducing the finite differences technique to approximate the string behaviour some parameters need to be defined and computed:

- **Tension**: For the specific string the tension is calculated tuning the string to a fundamental frequency f_1. The formula to retrieve the tension from the frequency tuning is: $T = 4 \cdot L^2 \rho f_1^2$. Where L is the length of the string, and ρ is the linear mass density.
- **String Propagation velocity**: The string propagation velocity can be calculated as $c = \sqrt{\frac{T}{\rho}}$

The specific parameters values considered for the string in the assignment are listed below:

- $f_1 = 65.4$ $[m/s]$ Fundamental frequency of the string
- $L = 1.92$ $[m]$ Length of the string
- $M_s = 35 \cdot 10^{-3}$ $[m]$ Length of the string
- $\rho = \frac{M_s}{L} = 18.2 \cdot 10^{-3}$ $[m]$ Length of the string
- $b_1 = 0.5$ $[s^{-1}]$ Air damping coefficient for the string
- $b_2 = 6.25 \cdot 10^{-9}$ $[s]$ Air damping coefficient for the string
- $\kappa = \epsilon$ String stiffness coefficient
- $\epsilon = 7.5 \cdot 10^{-6}$ String stiffness parameter
- $\zeta_l = 1 \cdot 10^{20}$ Left end normalized impedance
- $\zeta_b = 1000$ Right end normalized impedance
- $T = 4 \cdot L^2 \rho f_1^2 = 1149.7$ $[N]$ Tension applied to the string
- $c = \sqrt{\frac{T}{\rho}} = 251.136$ $[m/s]$ Wave propagation velocity

Finite difference stability

In order guarantee that the max number of spatial samples guarantees the spatial stability the condition given by the equation found in the article [1]:

$$M_{max} = \sqrt{\frac{-1 + \sqrt{1 + 16\epsilon\gamma^2}}{8\epsilon}}$$

where $\gamma = \frac{f_s}{2f_1}$ and ϵ is the stiffness parameter defined before.
In order to comply CFL conditions the following condition must be true:

$$\lambda = \frac{cT}{X} \leq 1$$

where T is the time-sample length, calculated as $\frac{1}{Fs}$. While X is the spatial-sample length, calculated as $\frac{L}{M_{max}}$ Considering that value obtained for the specific case is:

$$M_max = 105.83$$

Considering M as:

$$M = floor(M_{max}) = 105$$

Substituting the value of M for calculating courant number the value obtained is $\lambda = 0.85$ and respects the stability conditions. In order to demonstrate that the CFL condition guarantees the stability of the simulation the maximum number of samples that guarantees stability has been calculated. It is calculated as:

$$M_{maxCFL} = \frac{L\lambda}{cT}$$

And in the specific case it is equivalent to $M_{maxCFL} = 122$. We noticed during simulations that exciding this number the simulations don't lead to a convergent solution.

Hammer force function discretization

The hammer force functions $F_H(t)$ and $g(x, x_0)$ are now discretized in time and space, obtaining $F_H(n)$ and $g(m, m_0)$. The contact duration time is discretized too:

$$\eta(n+1) < y(m_o, n+1)$$

Finite difference

After all the equations and the parameters for the problem have been defined the finite difference technique is now implemented.
In order to compute the string behaviour the finite difference implemented is:

$$\begin{aligned} y_m^{n+1} &= a_1(y_{m+2}^n + y_{m-2}^n) + a_2(y_{m+1}^n + y_{m-1}^n) \\ &+ a_3 y_m^n + a_4 y_m^{n-1} + a_5(y_{m+1}^{n-1} + y_{m-1}^{n-1}) + a_F F_m^n \end{aligned} \tag{1}$$

where:

- $a_1 = \frac{-\lambda^2\mu}{1+b_t1}$
- $a_2 = \frac{\lambda^2+4\lambda^2\mu+\nu}{1+b_t1}$
- $a_3 = \frac{2-2\lambda^2+6\lambda^2\mu+2\nu}{1+b_t1}$
- $a_4 = \frac{-1+b_1T+2\nu}{1+b_t1}$
- $a_5 = \frac{-\nu}{1+b_t1}$

m is the evaluated spatial index and n is the evaluated time index.

Discretized boundary conditions

The boundary conditions are now discretized through finite difference analysis, the general equation introduced in equation 1 is valid only for the spatial samples from $m >= 2$ and $m <= M-2$. Therefore the boundary conditions are defined for $m = 0, 1, M-1, M$ Considering the left side we need to take into account two spatial sample as boundaries to allow the computation to correctly happen, on the left side being sample with m = 0 and m = 1.
For m = 0:

$$\begin{aligned} y_m^{n+1} &= b_{L1} y_m^n + b_{L2} y_{m+1}^n + b_{L3} y_{m+2}^n \\ &+ b_{L4} y_m^{n-1} + b_{LF} F_m^n \end{aligned}$$

where:

- $b_{L1} = \frac{2-2\lambda^2\mu-2\lambda^2}{1+b_1T+\zeta_l}$
- $b_{L2} = \frac{4\lambda^2\mu+2\lambda^2}{1+b_1T+\zeta_l\lambda}$
- $b_{L3} = \frac{-2\lambda^2\mu}{1+b_1T+\zeta_l\lambda}$
- $b_{L4} = \frac{-1-b_1T+\zeta_l\lambda}{1+b_1T+\zeta_l\lambda}$
- $b_{LF} = \frac{T^2/\rho}{1+b_1T+\zeta_l\lambda}$

For m = 1:

$$\begin{aligned} y_m^{n+1} &= a_1(y_{m+2}^n - y_m^n + y_{m-1}^n) + a_2(y_{m+1}^n + y_{m-1}^n) \\ &+ a_3 y_m^n + a_4 y_m^{n-1} + a_5(y_{m+1}^{n-1} + y_{m-1}^{n-1}) + a_F F_m^n \end{aligned}$$

While, considering the right side of the string, the equation on the bridge become:
For m = M:

$$\begin{aligned} y_m^{n+1} &= b_{R1} y_m^n + b_{R2} y_{m-1}^n + b_{R3} y_{m-2}^n \\ &+ b_{R4} y_m^{n-1} + b_{RF} F_m^n \end{aligned}$$

while for m = M-1:

$$\begin{aligned} y_m^{n+1} &= a_1(2y_{m+1}^n - y_m^n + y_{m-2}^n) + a_2(y_{m+1}^n + y_{m-1}^n) \\ &+ a_3 y_m^n + a_4 y_m^{n-1} + a_5(y_{m+1}^{n-1} + y_{m-1}^{n-1}) + a_F F_m^n \end{aligned}$$

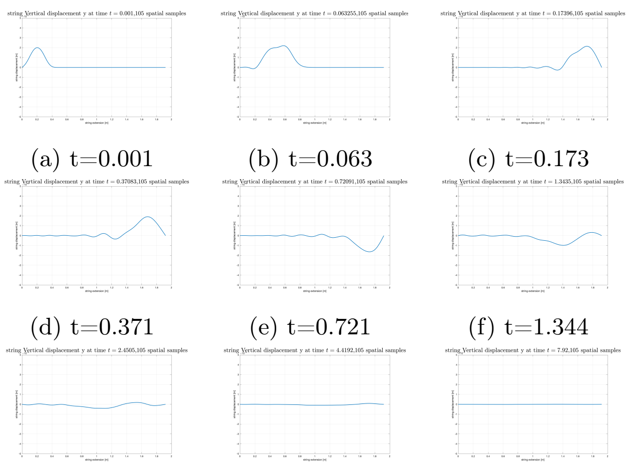
The coefficients b_{R1}, b_{R2}, b_{R3}, b_{R4}, b_{RF} can be computed as those computed before, just substituting the value of ζ_l with ζ_b.

Initial Conditions

- At istant t = 0, therefore for n = 0 the string is not moving and therefore the speed is 0.
- For n = 0 the string is in its equilbrium position.
- For n = 0 the hammer is moving and its initial velocity is $V_{H0} = 2.5\ [m/s]$.
- For n = 0 the hammer is set to be in contact with the string. Therefore its displacement is $\eta(0) = 0$.
- For n = 1 the hammer position depends on the initial velocity V_{H0} and on the time resolution between n and $n+1$ (ΔT) which can be calculated as $\frac{1}{Fs}$. The value $\eta(1) = \frac{V_{H0}}{Fs}$. The formula for the hammer displacement is valid only for $n \geq 2$.
- For n = 0 the hammer force is 0, being $\eta(0) = 0$ and $y(m, 0) = 0$

Numerical simulation results

The finite difference method is now implemented and the values of the displacement of the string are simulated for eight seconds. Below are listed the results obtained through the numerical simulation setup discussed previously. In figure 4 the nine frames plotted are an example of the displacement at different times used to notice the behaviour of the string during the time simulation.



(a) t=0.001 (b) t=0.063 (c) t=0.173

(d) t=0.371 (e) t=0.721 (f) t=1.344

(g) t=2.451 (h) t=4.420 (i) t=7.920

Figure 4: Vertical displacement of the string at different simulations time

The sound of the struck string is now approximated by averaging the string displacement over 12 spatial samples around the position m_0, center of the hammer strike. Given that the number of samples considered for the mean has to be even the mean window is position from the sample m_0-5 to the sample m_0+6. In figure 5 the waveform of the approximated signal is plotted. In figure 6 the *Fourier Transform* of the waveform is plotted. This plot allows us to notice how the fundamental

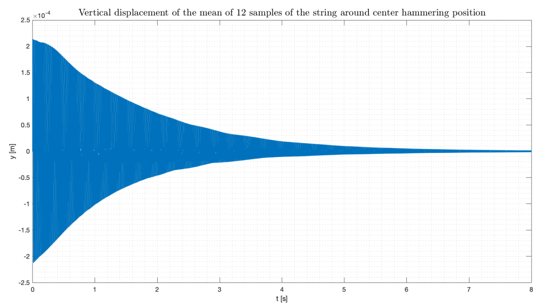


Figure 5: 12 samples mean displacement waveform of the string around m_0

frequency of the struck string is the one expected of 65.4 Hz. Moreover the spectrogram of the signal is plotted in figure 7 to visualize how the spectrum decays over time. The harmonic content of the model, visible both in figures 6 and 7 validates the theorical assumptions: the spectrum presents minima at harmonic numbers multiples of approximately 9 where the string is struck at a fraction $0.12 \approx 1/9$ of its length.

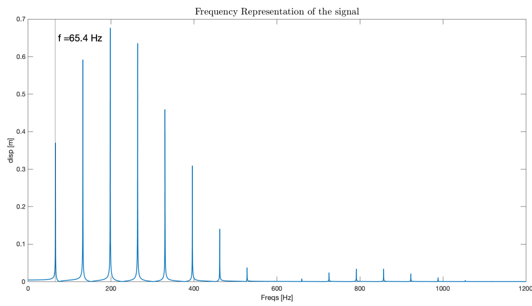


Figure 6: Frequency representation of the exported signal

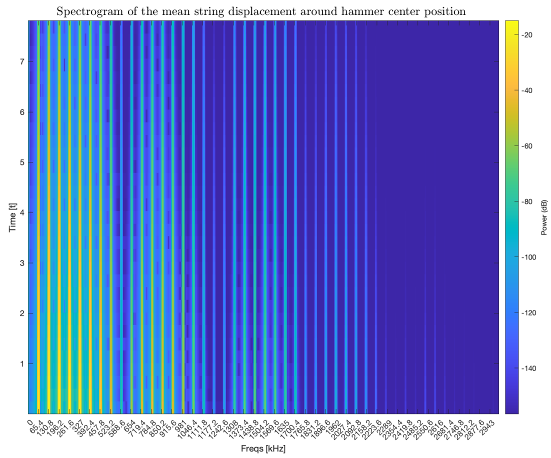


Figure 7: Spectrogram of the signal computed

Ex 2) Guitar Modeling

Guitar model with square wave excitement

In this section we aim to obtain the time signal of the plucked E1 string of a classical guitar. In order to do that we exploit the electrical equivalence of the classical guitar acoustic model. We used the rigid back plate model, that allows us to consider as acoustic acting components only the top plate, the hole and the air volume of the resonating chamber. The equivalent acoustic model is visible in figure 8.

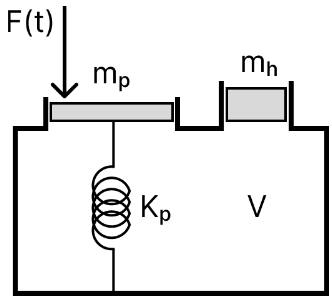


Figure 8: Equivalent Model of the Guitar

The model takes into account the mass of the top plate m_p and its stiffness K_p and also the resonator through the air mass in the hole m_h and the volume V. The force coming from the string is applied to the top plate through the bridge and it's modeled with $F(t)$. Since every mass can be modelled electrically with an inductor, stiffness and air volumes with capacitance and mechanical/acoustical losses with resistances we can build the electrical circuit visible in figure 9. The application of the force $F(t)$ is modelled via a controlled voltage generator. Moreover, since we're dealing with an acoustic problem each mechanical impedance has to be divided by the surface it occupies.

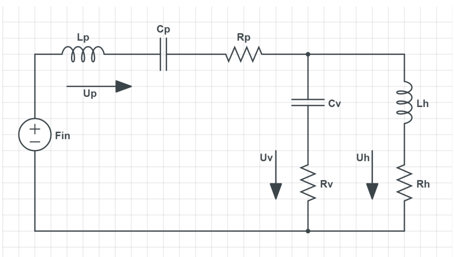


Figure 9: Equivalent Circuit of the Guitar

The three components L_p, C_p and R_p represent the totality of the plate and present only one resonance at $\omega_{0.p} = 1/\sqrt{L_p C_p}$. In the next model, however, the first 20 resonances of the plate are taken into account. This is possible simply adding in parallel a filter bank, where each branch is composed by a RLC circuit tuned to match the empirically measured resonance frequencies. The equivalent circuit is depicted in figure 10.

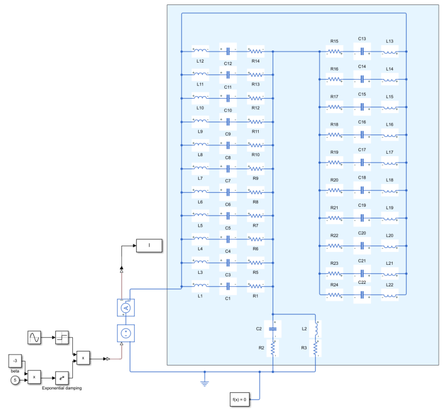


Figure 10: Equivalent Circuit of the Guitar considering first 20 resonances of the top plate

Through the impulse response of the system we're able to extract via the fourier transform the Frequency Response Function. The FRF of this model guitar body presents indeed 22 resonances. It is visible in figure 11.

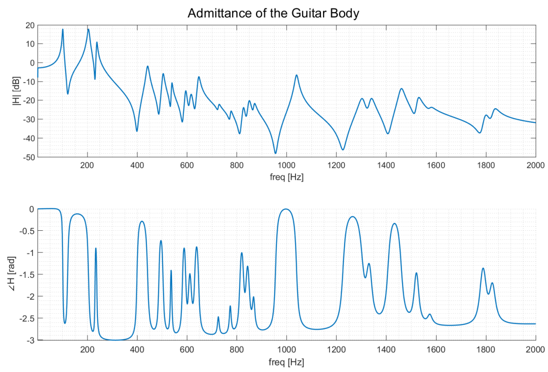


Figure 11: FRF of the guitar body

The resonanse extracted from the system are visible in table 1.

EigenModes [Hz]							
1	101.00	**7**	595.00	**13**	850.20	**19**	1530.81
2	204.20	**8**	616.60	**14**	871.40	**20**	1580.81
3	238.00	**9**	647.40	**15**	1039.60	**21**	1800.81
4	441.80	**10**	729.20	**16**	1302.41	**22**	1838.01
5	504.40	**11**	777.00	**17**	1340.81		
6	539.60	**12**	827.00	**18**	1461.01		

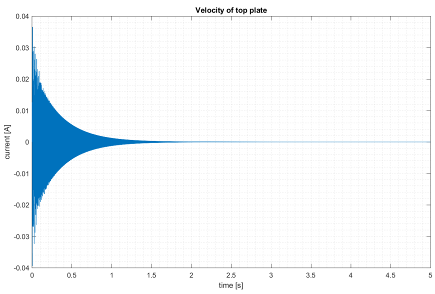
Table 1: Eigenmodes of the guitar body

In this simulation the force applied to the top plate has been obtained through the use of a signal controlled voltage generator which is fed with a time decaying square wave of frequency $f_F = 300\ Hz$:

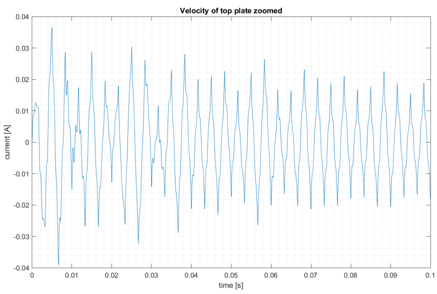
$$F(t) = sgn(cos(2\pi f_F t))e^{-3t}$$

. The output of the system is the velocity of the top plate, the sound radiator of the guitar, which is represented by the current flowing out the voltage generator.

The signal is captured by an amperometer, imported in MATLAB and the time plot is visible in figures 12a and 12b.



(a) Velocity of top plate



(b) Velocity of top plate zoomed

Figure 12: Time Signal of Top Plate Velocity

We also took the fourier transform of the velocity to obtain their frequency responses. It is visible in figure 13.

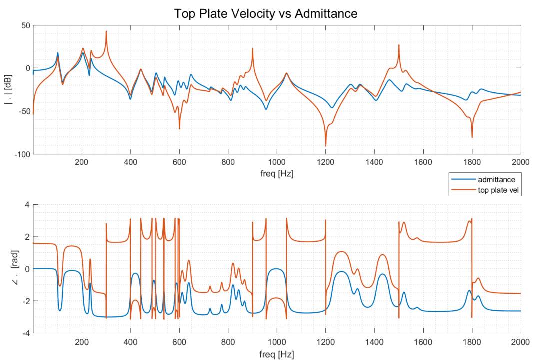


Figure 13: Top Plate Velocity vs Admittance frequency response

It is easily understandable from the figure 13 that the velocity of the top plate respects the expected behaviour given by the admittance response. The peaks due to the exciter, that must present maxima at odd harmonics of the fundamental, are also present at $f_0 = 300\ Hz$, $f_1 = 900\ Hz$, $f_2 = 1500\ Hz$ and so on, while even harmonics aren't present.

Guitar model with string excitement

Now, another component is added to the simulation: the model a real plucked string is added. The string taken into account is the E1 string, the thinnest string of a guitar, which has frequency $f_{E1} = 329.5\ Hz$. In order to simulate it we exploit two properties of the string: we need to model the standing wave that is created in fixed-fixed strings and the initial deformation due to the plucking action. Since the standing waves are created by the presence of reflections, the latters are taken into account through the use of a transmission line. As a matter of fact a travelling wave into a transmission line can be reflected and/or transmitted according to the relation between the characteristic impedance and the load impedance. Moreover, recalling the analogy of a mechanical system to a electrical circuit, since we want to simulate a rigid end that reflects back the incident wave with negative sign we need to end the transmission line with a short circuit.

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = -1 \quad \text{since } Z_L = 0$$

Where Γ is the reflection coefficient. Given that the string is fixed-fixed, after the initial excitement the transmission line must end on both bounds with short circuits.
The initial excitement of the string is given by the model of the plucking of the string at one fifth (1/5) of it's length. In order to do it, the transmission line delay has been set to 1.515 ms, that is half of the period of a wave with frequency f_{E1} and at the extremities are posed two symmetrical triangular waves of duration 1.515 ms. For the generator I_1 in the interval 0-0.303 ms the signal is rising while in the interval 0.303-1.515ms the signal is falling. The generator I_2 behaves symmetrically in time. Moreover, after 1.515 ms four switches commutes in order to detach the generators from the transmission line and leave the system resonate, as visible in figure 14. Doing so, after 1.515 ms the voltages created by I_1 and I_2 coincide along the transmission line, which corresponds to the plucking of the string at 1/5 of its length. The generated wave is depicted in figure 15.

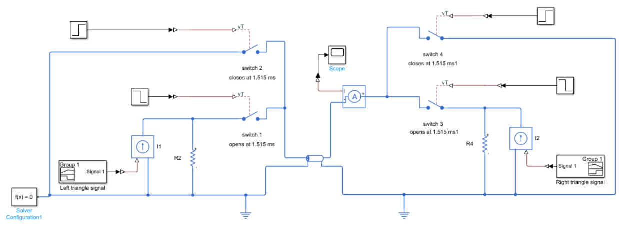


Figure 14: Plucked String Electrical Model

Once the string is modelled, the connection between it and the body is evaluated. This connection is im-

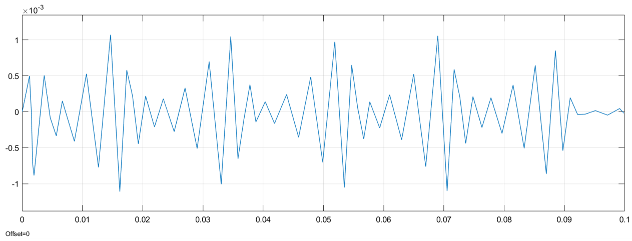


Figure 15: Plucked String Time Signal

plemented substituting one of the short circuit of the transmission line with the circuit of figure 10. The resulting circuit is visible in figure 16.

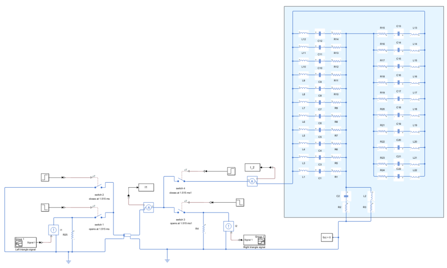


Figure 16: Complete Circuit of the Guitar Model

Once the system is set up the simulation is run and the data of the current flowing in the filter bank branch are collected. The current represents the velocity of the top plate and allows us to hear what the model of the plucked string of the guitar sounds like. In figure 17 the time signal is plotted while in figure 18 the frequency response can be evaluated.

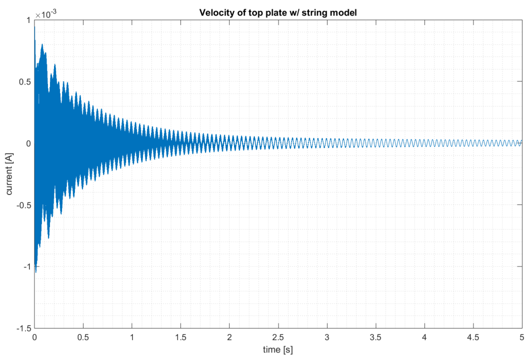


Figure 17: Time Signal of Top Plate Velocity

Results comments

From the frequency response it is clearly visible that the frequency content isn't equally distributed along the spectrum, but some harmonics are missing. This is perfectly coherent with the theory of the plucked string in guitars which confirms that if a string is plucked at a fraction $1/n$ of its length then all the harmonics number kn with $k = 1, 2, \ldots$ are missing due to the presence of a node in the plucking position that does not allows

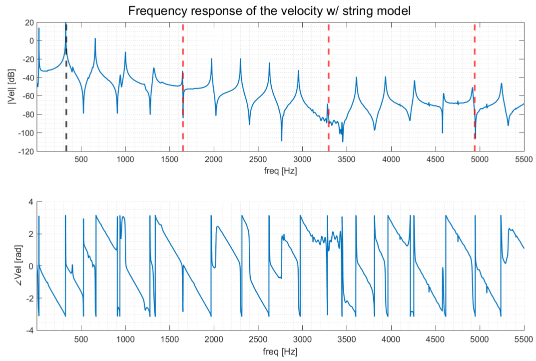


Figure 18: Frequency Response of Top Plate Velocity

them to resonate. In figure 18 they're highlighted with red vertical lines and are $f = 5f_0$, $f = 10f_0$, $f = 15f_0$, and so on, while the fundamental f_0 is highlighted with a black line. The sound synthesized from the simulation and present in the delivery is quite likely the one produced from a real guitar so the validity of the model is demonstrated.

References

[1] A. Chaigne and A. Askenfelt, *Numerical simulations of piano strings. i. a physical model for a struck string using finite difference methods*, The Journal of the Acoustical Society of America, 95 (1994), pp. 1112–1118.