Coupled piano strings

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The admittance of the piano bridge has a crucial effect on piano tone by coupling together the strings belonging to one note into a single dynamical system. In this paper, we first develop theoretical expressions that show how the rate of energy transmission to the bridge as a function of time (including the phenomena of beats and "aftersound") depends on bridge admittance, hammer irregularities, and the exact state in which the piano is tuned. We then present experimental data showing the effects of mutual string coupling on beats and aftersound, as well as the great importance of the two polarizations of the string motion. The function of the *una corda* pedal in controlling the aftersound is explained, and the stylistic possibilities of a split damper are pointed out. The way in which an excellent tuner can use fine tuning of the unisons to make the aftersound more uniform is discussed.

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INTRODUCTION

With very few exceptions, and starting with the earliest instruments known, pianos have been built with more than one string per note; yet relatively little attention has been paid to this feature by acousticians. It was noted first by Martin and Ward,[1] and later in more detail by Kirk,[2] that the strings comprising a single "triplet"—that is, the group struck by a single hammer—were not tuned in perfect unison by excellent tuners; and that, indeed, listeners often preferred the sound produced when there was a small discrepancy among the individual frequencies. Setting aside the subjective "preferences," whose cause and nature—and, hence, significance—is difficult to assess, the practice of artist tuners must be taken with extreme seriousness. We note that in this context "discrepancies from unison" refers to frequency differences observed when the strings are sounded individually, with the other members of the same triplet damped. These differences are not equal to the beat frequencies heard when the strings sound together, due to the coupled nature of the string vibrations.

A number of years earlier, Martin[3] had reported on the characteristic "double decay" of piano tones, whereby the sound amplitude dies away with two distinct rates, breaking from an original "fast" decay to a later "slow" decay, or "aftersound." This feature is important in producing the characteristic piano tone. Martin did not at that time suggest any mechanism for the double decay, but in later work found that the phenomenon is connected with the presence of more than one string per note, and that the amount of aftersound is affected by the exact manner in which the unisons are tuned.[4]

The suggestion that the phase relations among the strings play an important role appears to have been made first by Hundley, Martin, and Benioff,[5] and in more explicit form by Benade.[6] Benade points out that when three strings vibrate in phase, the motion of the bridge is three times what it would be if one string were vibrating alone; hence, the rate of energy loss of each string is tripled. He suggests that, after some time, the strings lose their phase relationship, so that the decay rate becomes equal to that of a single string—hence the break in the decay curve. In fact, these relationships can have effects even more drastic than that: Not only can the decay rate be increased (up to threefold), but it can be decreased (up to infinityfold) if the strings vibrate in exactly opposing phases.

The central importance of phase relationships is only one manifestation of the fact that the strings are dynamically coupled through the bridge motion. Accordingly, we must analyze the system in terms of its normal modes, relating them in turn both to the initial conditions (hammer excitation) and to available radiation channels. Specifically, the concept of tuning the unisons to "zero beat" requires considerable elucidation, since the frequencies of coupled normal modes do not, in general, cross as adiabatic changes are made in the system parameters.

In our theoretical treatment, we generally restrict ourselves to sets of two, rather than three, strings. We also assume that the system is linear, so that a particular normal mode can be followed without regard to any other. Specifically, our analysis applies equally well to the fundamental vibrations of each string or to any of the higher-order modes of vibration. It should be added that no violations of this assumption have appeared in the data, as far as we have been able to recognize.

Our experimental results were all obtained on a single piano (Steinway model B, serial No. 216139), and were restricted to a few notes of the middle octave. We expect that these results are, nonetheless, typical as far as general properties of string triplets are concerned, at least in the middle range of the keyboard. On the other hand, a great deal of information about more specific features, such as the existence and properties of soundboard resonances, could be obtained by a more detailed survey.

I. NONCONSERVATIVELY COUPLED OSCILLATORS

A. Equations of motion

Although coupled harmonic oscillators occur often in physics, the case where the coupling mechanism is itself dissipative is less familiar. Accordingly, it is worthwhile to reivew some general properties which such a system has. [Similar dynamics apply to the K mesons, which are coupled by their decay mechanism[7]; certain atomic situations such as the Stark effect of degenerate (or almost degenerate) levels in hydrogen[8]; and many engineering problems involving the "pulling" of one oscillator by another.]

We approximate our system in zero order by a set of uncoupled and undamped harmonic oscillators all of the same angular frequency ω_0, whose equations of motion are

$$\ddot{q}_k + \omega_0^2 q_k = 0 \ , \quad k = 1, 2, \ldots, \tag{1}$$

where the q's are the displacements of the oscillators. Linearity being assumed, the perturbation consists of replacing the zeroes in Eq. (1) by linear homogeneous functions of the q's and their various time derivatives. We denote these functions by f_k.

Any such set of equations has solutions which can be written as a sum of (possibly complex) exponentials:

$$q_k = \sum_n A_{kn} \exp(i\beta_n t) \ . \tag{2}$$

Because of the form of the unperturbed equations, we know that all the β_n's will have values near $+\omega_0$ or near $-\omega_0$. Since the equations, as well as the variables, are by nature real, the coefficients of the terms whose frequencies are near $\pm\omega_0$ are connected by simple conditions; as a result we may write without loss of generality

$$q_k = \mathrm{Re}\{\psi_k\} \ , \tag{3}$$

where ψ_k are complex quantities which satisfy the same differential equations as the q's, and whose expansion in exponentials involves β_n's which are all close to $+\omega_0$ (and not to $-\omega_0$).

Equations (1) for the ψ's, with coupling terms included, can be factored into

$$\left(\frac{d}{dt} - i\omega_0\right)\left(\frac{d}{dt} + i\omega_0\right)\psi_k = f_k \ . \tag{4}$$

(The physical significance of the coupling will be discussed in the next section.) Since all frequencies are close to $+\omega_0$, we can replace the second factor by $2i\omega_0$ and divide both sides of Eq. (4) by it; the error made thereby is of order f in $2i\omega_0$, and will thus be second order in f when transferred to the right side. (We cannot do this with the factor containing the minus sign, since the differences between the frequencies and ω_0 are themselves proportional to the perturbation.) Similarly, we can replace all time derivatives which appear in f by factors of $i\omega_0$. This produces the set of equations

$$\left(\frac{d}{dt} - i\omega_0\right)\psi_k = i\sum_{k'} \Omega_{kk'}\psi_{k'} \ . \tag{5}$$

We note that if a factor of $\exp(i\omega_0 t)$ is removed from ψ_k, the $-i\omega_0$ in the parentheses disappears; this transformation is analogous to the "interaction picture" of quantum mechanics. The removal of this factor is of no consequence, since we will be interested in taking the absolute value squared of various linear combinations of the ψ_k's. Further, it is convenient to define a column vector Ψ whose components are ψ_k; the equation of motion of this vector is then

$$d\Psi/dt = i\Omega\Psi \ , \tag{6}$$

where the *dynamical matrix* Ω is made up of the elements $\Omega_{kk'}$.

B. Dynamical matrix for the simplified two-string model

The typical situation in a piano is that of three strings tuned to an almost complete unison; each string has two possible directions of polarization, to which we shall refer as "vertical" (perpendicular to soundboard) and "horizontal" (parallel to soundboard). We take the ψ_k to be the complex amplitudes of the first (i.e., fundamental) modes of the strings; thus Ψ should be viewed as a six-component vector, and Ω as a six-by-six matrix. In the following discussion we shall, however, restrict ourselves for the most part to a model in which there are only two strings, and horizontal polarizations are ignored, so that the dimensionality of the matrices is reduced to two. This must be viewed as a great oversimplification, as the discussion of Sec. II A, as well as the data of Secs. III B and III E, will indicate. Although many aspects of the system can be adequately discussed within this model, it should be understood that an analysis of the complete six-dimensional system will be required for a more detailed understanding.

Accordingly, we take Ψ to be a vector whose two components are the amplitudes of the fundamental vibrations of the two strings, with the polarization assumed vertical. In the unperturbed case, the matrix Ω is zero. In the actual case, it contains three contributions:

(a) the finite admittance of the bridge;

(b) the mistuning of the strings from unison; and

(c) the damping of the individual strings through mechanisms other than bridge motion.

We shall ignore the last factor; as we shall see, experiment shows it to be rather small, and if we assume (as seems reasonable) that it is the same for both strings, its effect is merely to superimpose a uniform slow decay upon the motion of the whole system.

To characterize the degree to which the two strings are not tuned identically, we imagine that (in the absence of any bridge motion) string two would have an angular frequency ω_0, and string one an angular frequency $\omega_0 + 2\epsilon$. We shall refer to the quantity ϵ thus defined as the angular frequency *mistuning*; intuitively, it is proportional to the angle through which the tuning pin has been rotated relative to its perfect-unison position. (Notice that mistuning as the difference in the ordinary frequencies is ϵ/π.) The corresponding equations of motion can be put into the form (5) by setting

$$\Omega = \begin{pmatrix} 2\epsilon & 0 \\ 0 & 0 \end{pmatrix} . \tag{7}$$

The contribution of the bridge admittance requires a little more thought. First, we note that the matrix ele-

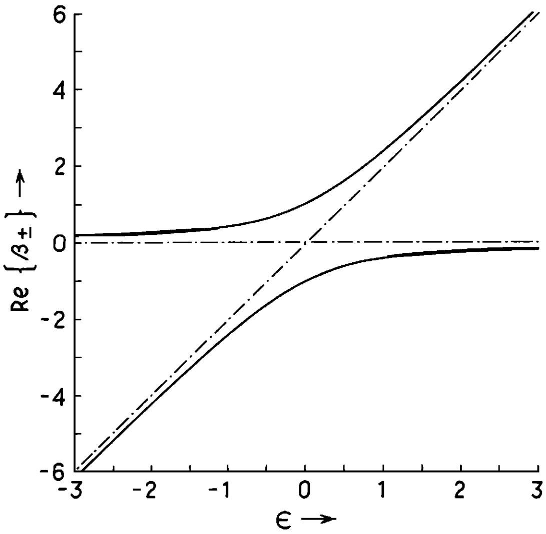

FIG. 1. Frequencies of the two normal modes of a two-string system, as a function of the angular frequency mistuning ϵ, when the coupling is purely reactive ($\xi = 1$, $\eta = 0$). The straight lines give the behavior in the absence of coupling.

ment $\Omega_{kk'}$ specifies (apart from constant factors) the force exerted on string k per unit motion of string k'. Now if we assume that both strings rest on the same place on the bridge, then a given motion of the bridge will exert equal forces on the two strings; conversely, the amount of motion of the bridge caused by a given motion of string one will be the same as is caused by an identical motion of string two. Translating this into matrix language, we conclude that the contribution of bridge admittance to all four elements of Ω is the same.

As to the size of this contribution, it is easy to see that the diagonal elements of this part of the matrix are simply the shift of frequency of either string alone due to bridge motion (assuming that the other string does not move). The latter problem is well known,[9] and has the solution

$$\delta\omega/\omega_0 = iZ_0 Y/\pi \;, \tag{8}$$

where Z_0 is the characteristic impedance of the string and Y is the admittance of the bridge. Letting

$$\zeta \equiv i\omega_0 Z_0 Y/\pi \;, \tag{9}$$

we thus obtain our complete dynamical matrix in the form

$$\Omega = \begin{pmatrix} 2\epsilon + \zeta & \zeta \\ \zeta & \zeta \end{pmatrix} . \tag{10}$$

Note that ζ is, in general, complex; we write it as $\zeta = \xi + i\eta$. The real part ξ may be positive or negative, depending on where ω_0 falls relative to soundboard resonances; a positive value corresponds to a masslike support, a negative value to a springlike support. The imaginary part η is a measure of dissipation at the support and must always be positive.

Experimentally, we shall often deal with the situation in which one of the strings is damped by a tuner's felt or rubber wedge, and the other string is allowed to sound freely. The primary effect of the wedge is to introduce a great deal of dissipation, so that (in the absence of bridge motion) the oscillation would be very highly damped. This corresponds to making the mistuning ϵ large and imaginary. Analytically, the behavior of the coupled system can then be analyzed by perturbation theory; in the limit of large ϵ—whether real or imaginary—the coupling ceases to have an effect, and the remaining string moves as though the other one were clamped, its amplitude decaying as $\exp(-\eta t)$, and its stored energy as $\exp(-2\eta t)$. For this reason, we shall refer to η as the *single-string decay rate.*

C. Eigenvalues of the dynamical matrix

At this point, it is convenient to replace ω_0 by $\omega_0 + \xi$, thus subtracting ξ from the dynamical matrix. For large values of ϵ, the real parts of the eigenvalues of Ω, Eq. (10), then approach 0 and 2ϵ. (This corresponds intuitively to the situation in which string two of a doublet remains under constant tension while the tuning pin of string one is rotated by an amount proportional to ϵ. The frequency reference is the frequency of string two when string one is tuned far away, or damped.) For arbitrary ϵ, the two eigenvalues of Ω are then

$$\beta_\pm = \zeta + \epsilon \pm (\epsilon^2 + \zeta^2)^{1/2} - \xi = \epsilon + i\eta \pm (\epsilon^2 + \xi^2 - \eta^2 + 2i\xi\eta)^{1/2} \; . \tag{11}$$

The special case where the bridge admittance is purely reactive ($\eta = 0$) leads to the well-known behavior shown in Fig. 1. When ϵ is large, the eigenfrequencies approach the values they would have in the absence of interaction; as the mistuning is decreased, the two "repel" each other, that is, the frequencies remain further apart than they would be in the absence of interaction. It should be especially emphasized that the eigenfrequencies *never cross.*

In the other extreme situation, where the bridge admittance is completely resistive ($\xi = 0$), the two eigenvalues are

$$\beta_\pm = i\eta \pm (\epsilon^2 - \eta^2)^{1/2} + \epsilon \; . \tag{12}$$

The behavior of the system then falls into two régimes, depending on whether ϵ is greater than or less than η. In the first case, both normal modes have the same decay rate, equal to that of a single string; the frequency difference is $2(\epsilon^2 - \eta^2)^{1/2}$. In the second case, the frequency difference is pure imaginary. This means that there is no beat; the modes differ only in their rates of decay. As the mistuning ϵ goes to zero, the "fast" mode decay rate becomes twice the single-string rate, while the "slow" mode loses its damping altogether. This is a natural result of the coherent way in which the two strings together force the bridge to move. In the model we are using, where bridge motion is the only source of dissipation, the decay rate must always be zero when the two strings are moving with equal amplitudes and opposite phases. It is only at $\epsilon = 0$, however, that such a totally antisymmetric motion is an exact normal mode.

The dependence of eigenfrequencies on mistuning is shown in Fig. 2. Note that in this case the frequencies *pull on each other* instead of repelling; that is, they are closer together than would be the case in the absence of

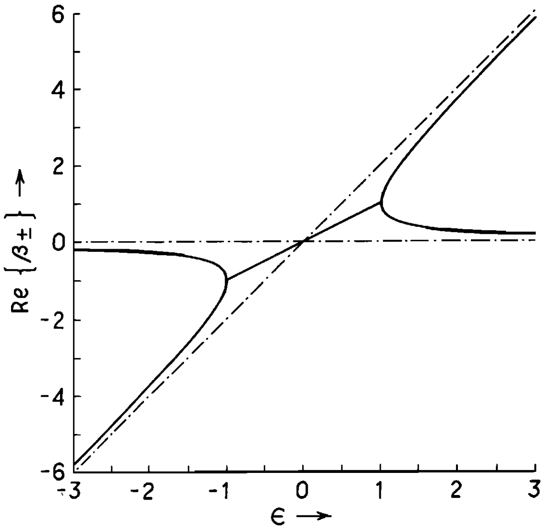

FIG. 2. Frequencies of the two normal modes of a two-string system, as a function of the angular frequency mistuning ϵ, when the coupling is purely resistive ($\xi=0$, $\eta=1$). The straight lines give the behavior in the absence of coupling.

interaction. When the magnitude of mistuning ϵ is equal to the single-string damping rate η, the two eigenfrequencies have coalesced. For even smaller mistuning, it is only the damping rates of the two modes that change In this special case of purely resistive coupling, the two eigenfrequencies do cross.

If ζ has both a real and an imaginary part, the behavior of the eigenvalues is (as one might expect) intermediate between the other two cases. In general, the two modes differ both in their frequencies and in their decay rates. This has the striking consequence that the decay rate of the beat is not the same as the decay rate

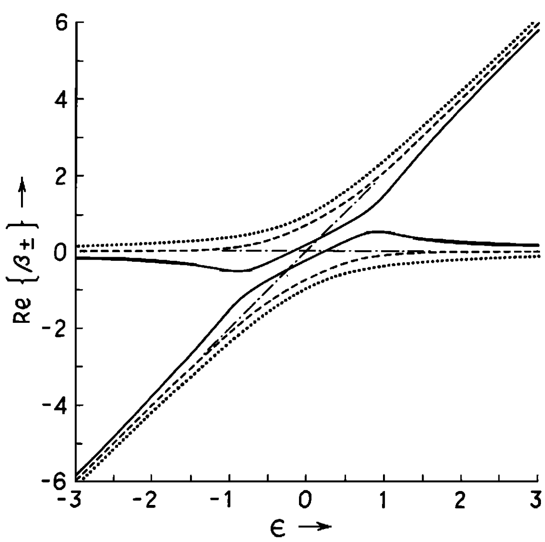

FIG. 3. Frequencies of the two normal modes of a two-string system, as a function of the angular frequency mistuning ϵ, for various phases of the coupling admittance. Dotted: $\xi=0.98$, $\eta=0.2$; dashed: $\xi=0.71$, $\eta=0.71$; solid: $\xi=0.2$, $\eta=0.98$. The straight lines give the behavior in the absence of coupling.

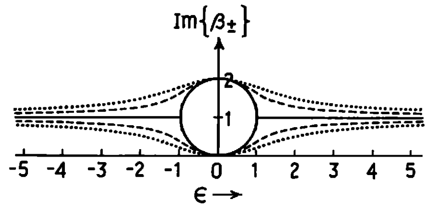

FIG. 4. Damping rates of the two normal modes of a two-string system, as a function of the angular frequency mistuning ϵ, for various values of the coupling admittance. Solid: $\xi=0$, $\eta=1$; dashed: $\xi=0.5$, $\eta=1$; dotted: $\xi=1$, $\eta=1$.

of the sound; in fact, the amplitude of the beat generally *grows* before it decays (Sec. II C).

Figures 3 and 4 show some curves of the dependence of frequencies and decay rates upon angular frequency mistuning, for various phases of the coupling admittance.

II. THEORY OF BEATS AND AFTERSOUND

A. Integration of equations of motion

We continue our calculation for the simplified two-string model introduced in Sec. IB, in which $\Psi(t)$ is interpreted as a two-component vector whose components $\psi_1(t)$ and $\psi_2(t)$ are complex numbers representing the amplitudes and phases of the vertical displacements of the two strings. In this model, the rate of energy transmission from strings to bridge is proportional to $|\psi_1(t)+\psi_2(t)|^2$, since an antisymmetric string motion would exert no net force on the bridge. (Some shortcomings of this formulation will be discussed in Sec. III B.) We shall for the moment also assume that the initial conditions of the two strings are identical, as they would be immediately after being struck by an ideally aligned hammer.

It is convenient to introduce a dimensionless quantity $R(t)$ defined by

$$R(t) \equiv |\psi_1(t)+\psi_2(t)|^2/4|b|^2 , \tag{13}$$

where b is the initial value of ψ_1 and ψ_2. This quantity is equal to unity at $t=0$, and gives the subsequent relative time variation of the rate of energy transmission from strings to bridge.

To compute $R(t)$, we note that Eq. (6) can be integrated to

$$\Psi(t)=\exp(i\Omega t)\,\Psi(0)=\exp(i\Omega t)\begin{pmatrix} b \\ b \end{pmatrix} . \tag{14}$$

Now the dynamical matrix Ω can, from Eq. (10), be written as

$$\Omega=\zeta+\epsilon+\mu\Sigma , \tag{15}$$

where

$$\mu \equiv (\epsilon^2+\zeta^2)^{1/2} \tag{16}$$

and

$$\Sigma \equiv \begin{pmatrix} \epsilon/\mu & \zeta/\mu \\ \zeta/\mu & -\epsilon/\mu \end{pmatrix} . \tag{17}$$

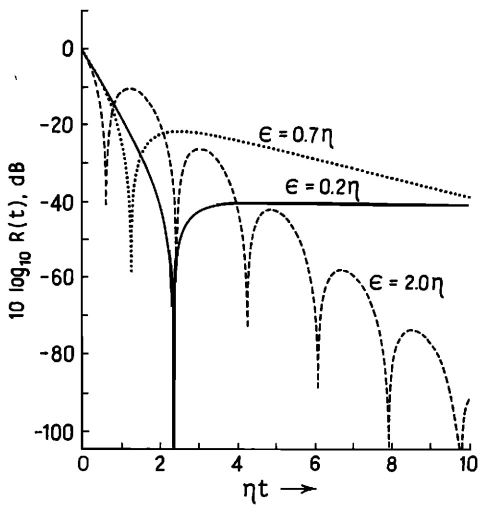

FIG. 5. Theoretical relative level of energy transmission to the bridge of a two-string system as a function of time, for various values of angular frequency mistuning ϵ, when the coupling is resistive. $R(t)$ is defined in Eq. (13).

We have defined Σ so that its square is unity. Using this fact in the power series expansion of the exponential, we easily establish that $\exp(i\theta\Sigma)=\cos\theta+i\Sigma\sin\theta$ for any θ; hence, from Eq. (15),

$$\exp(i\Omega t)=\exp[i(\zeta+\epsilon)t]\times[\cos\mu t+i\Sigma\sin\mu t] \ . \tag{18}$$

By substituting this expression into Eq. (14), and using the definition of $R(t)$ given in Eq. (13), we finally obtain the solution

$$R(t)=\left|\frac{(\zeta+\mu)\exp[i(\zeta+\mu)t]-(\zeta-\mu)\exp[i(\zeta-\mu)t]}{2\mu}\right|^2 \ . \tag{19}$$

Both ζ and μ are, in general, complex.

B. Purely resistive coupling

As we saw in Sec. I C, the behavior of the system for $\zeta=i\eta$ falls into two régimes: $\epsilon>\eta$ and $\epsilon<\eta$. For the first case, Eq. (19) reduces to

$$R(t)=\exp(-2\eta t)\times(\mu\cos\mu t-\eta\sin\mu t)^2/\mu^2 \ , \tag{20}$$

where now

$$\mu=(\epsilon^2-\eta^2)^{1/2} \ . \tag{21}$$

Equation (20) represents a beat at frequency 2μ with an overall decay rate of 2η. Thus the beat decays at the single-string rate, maintaining a constant amplitude relative to the signal, just as though the two strings were independent—except, of course, that the beat frequency is somewhat smaller than would be expected on the basis of measuring the individual strings. It is also interesting that the slope of Eq. (20) at $t=0$ is identically 4η, twice the single-string rate. This is the effect discussed by Benade,[6] caused by the two strings initially forcing the soundboard in a coherent fashion.

When ϵ is less than η, Eq. (19) reduces to

$$R(t)=\left(\frac{(\eta+\nu)\exp[-(\eta+\nu)t]-(\eta-\nu)\exp[-(\eta-\nu)t]}{2\nu}\right)^2 \ , \tag{22}$$

where

$$\nu\equiv(\eta^2-\epsilon^2)^{1/2} \ . \tag{23}$$

There is now no beat, except for a single null; but there is a decided aftersound, in that there are two components decaying at different exponential rates. The relative amount of aftersound is given by $(\eta-\nu)^2/(\eta+\nu)^2$, and is thus a strong function of the mistuning ϵ.

The computer-generated curves of Fig. 5 show a few examples of the behavior of $R(t)$ for the case of purely resistive coupling.

C. More general coupling

If the bridge admittance contains both a resistive and a reactive part, Eq. (19) must be evaluated as a complex function. Although the algebra is then more involved, certain general statements can be made. The two normal modes differ both in the real and the imaginary parts of the frequency; hence, if the sound is conceived as extending both ways in time, one mode will dominate at sufficiently early times, and the other at sufficiently late times. The beat between them will be strongest when their amplitudes are approximately equal, that is, near the crossing point. Thus the beat amplitude will grow, and then decay, at a rate equal to the difference of the damping rates of the normal modes. In actuality, of course, the sound begins at $t=0$, so that in cases where the crossing point falls at a negative time the initial growth of the beat would not be observed. Incidentally, the sum of the two damping rates is always twice the single-string rate, as is seen from the trace of Eq. (10).

At large mistunings, the imaginary parts of the two frequencies approach each other. This gives the familiar prolonged and relatively rapid beat pattern. At the other extreme, near $\epsilon=0$, the sound pattern is strongly dependent on the initial conditions. This is so because only a small fraction of a beat period may fit into the time that the two amplitudes are comparable.

Figure 6 shows a few computer-generated curves of this type. In all cases, the ideal initial conditions are assumed.

D. The *una corda* pedal

In a grand piano, the *una corda* pedal (the one on the left) mechanically shifts the action sideways so that only two out of three, or one out of two, strings are struck by the hammer. This changes the sound pattern drastically.

Continuing, for the moment, to limit our discussion to two strings, we have seen that when an ideal hammer strikes an ideally tuned pair of strings, a completely symmetric motion is excited; that is, the two strings continue to vibrate with equal amplitudes and in the same phase. In such a case we would expect no aftersound at all. In fact, a real hammer does not strike the two strings in a completely identical manner. The resulting difference in initial amplitudes is equivalent to a small admixture of the antisymmetric mode; in other words, the *sum* of the two amplitudes decays rapidly (at twice

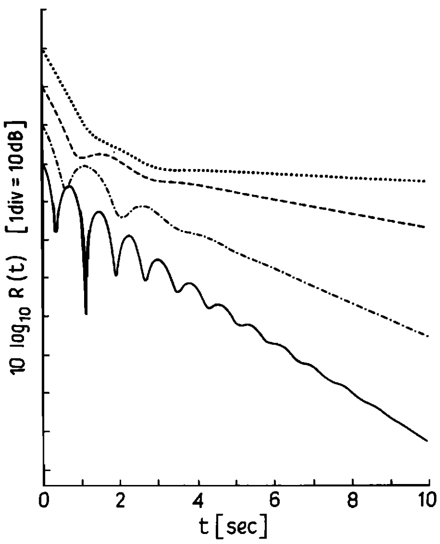

FIG. 6. Theoretical relative level of energy transmission to the bridge of a two-string system as a function of time, for various values of the angular frequency mistuning ϵ, when the coupling admittance has equal resistive and reactive parts ($\xi=\eta=1$). All curves begin at 0 dB, but are offset for visual clarity. Dotted: $\epsilon=0.5$; dashed: $\epsilon=1$; dot-dash: $\epsilon=2$; solid: $\epsilon=4$. $R(t)$ is defined in Eq. (13).

the single-string rate), while the *difference* persists, providing a small amount of aftersound. Another small amount is contributed by any residual mistuning, which causes the normal modes to deviate from perfect symmetry and antisymmetry.

With the *una corda* pedal depressed, only one string is initially excited. It is still correct to say that the sum of the two string amplitudes decays quickly while their difference decays slowly; but now the difference is as big as the sum, resulting in a large increase of aftersound relative to attack. As is well known, this pedal is used in playing soft passages, and it has generally been thought that its main function is to decrease the loudness. Since, however, this decrease is small compared to the dynamic range of the instrument, the pedal's perceptual effect is more appropriately described as giving a soft sound a more "lyrical" quality, by allowing a subdued attack without concomitant loss of aftersound.

It should be noted that this explanation remains valid even if all three strings are considered. There is then one completely symmetric mode, in which the three strings vibrate in the same phase; and *two* antisymmetric modes, for each of which the sum of the three displacements is zero. If the hammer is ideal and the unison tuning perfect, only the completely symmetric mode will be excited, so that again the amount of aftersound is proportional to the "imperfections" and hence small; but with the *una corda* pedal depressed, the excitation of the antisymmetric modes becomes comparable to that of the symmetric one.

III. EXPERIMENTAL STUDIES

A. Method of recording time-variation of amplitudes

All experimentally observed signals, whether coming from a microphone or a string vibration pickup, were recorded on a Tandberg model 9100X tape deck; when two simultaneous signals were needed, both tape channels were used. The signals were then processed using a Hewlett–Packard model 3580A spectrum analyzer. This instrument is generally designed to sweep through a frequency range, but can be set so that the range is zero, in which case the amplitude of some fixed-frequency signal is obtained as a function of time. In this way we picked out the fundamental component of each of the signals. The bandwidth of the instrument was normally set at 30 Hz, which is wide enough to transmit all observed transients, but narrow enough for considerable noise reduction.

The figures that follow were drawn by an XY recorder connected to the spectrum analyzer output, which was set to a logarithmic scale. In all cases, the reference level is arbitrary. Specifically, it should be noted that we made no great effort to standardize the strength of hammer impact from one data graph to the next, except to convince ourselves that results obtained with different impacts differed only in a uniform vertical shift on the level scale. This is the behavior to be expected of a linear system.

B. Vertical and horizontal polarizations

Our first observations concern the sound produced by the vibration of a single string, the remaining two strings of the triplet being damped by felt wedges. Figure 7 shows the relative sound pressure level, at the fundamental frequency, as recorded by a microphone placed near the piano. A definite "double decay" is observed, which we interpret as a superposition of two vibrations with two different decay rates, as indicated by the dashed lines.

To establish that the two vibrations correspond to the two polarizations of string vibration, the pickup shown in Fig. 8 was built. It consists of two electrodes which view the string at 45° angles with the vertical. In using this device, we put the piano at a potential of − 25 V with respect to ground, and connected the probes to a two-channel preamplifier, so as to obtain two independent projections of the string's motion. The preamplifier output may, at any one frequency, be regarded as proportional to the displacement of the string. Undoubtedly some phase shifts are present, but they do not affect our interpretation of the results so long as only equal-frequency signals are compared to each other.

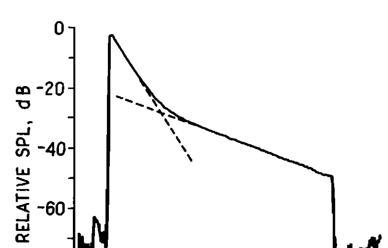

FIG. 7. Relative sound pressure level from a single string at its fundamental frequency (D$\#_4$, 311 Hz).

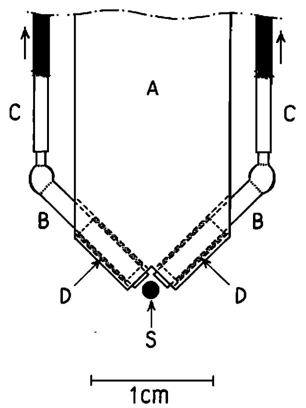

FIG. 8. Vibration probe for piano string. A, aluminum body; BB, electrodes; CC, shielded amplifier leads; DD, insulating sleeves; S, string.

It is apparent from the symmetry of the system that the sum of the two electrical signals will be proportional to the vertical displacement of the string, and the difference to the horizontal. (The constants of proportionality need not be the same, however; see Sec. III D.) This separation cannot, however, be relied on in an absolute way unless the placement of the probe, and the general geometry, can be controlled with great precision. What we did instead was to feed the two signals into a mixing amplifier which combined them in adjustable proportions, and set these proportions so as to get outputs whose decays show a single exponential dependence upon time. The appropriate linear combinations did turn out to be in the vicinity of the sum and difference; but if the normal modes were tilted from the vertical and horizontal by a few degrees, there would be no way for our apparatus to detect this. Henceforth, we shall refer to "horizontal" and "vertical" motions with the understanding that these are nominal descriptions (within, perhaps, 15°).

Figures 9 and 10 show, respectively, the displacement levels of the two polarizations as a function of time. It is clear that the vertical motion has the fast decay. (A similar behavior of harpsichord strings was reported by Pyle and Schultz.[10])

It may seem obvious that, if the dissipation is due to bridge motion, it would be much greater for the vertical

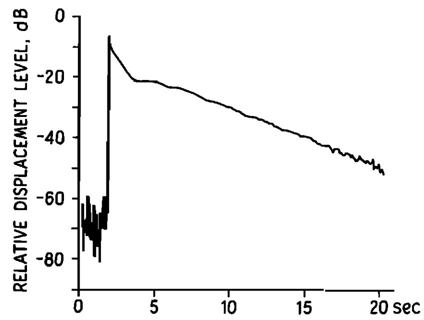

FIG. 10. Relative displacement level of single string as a function of time, horizontal polarization (D#4, 311 Hz).

mode, since the "give" of the bridge is much greater in that direction. As we shall see in the next section, however, the situation is more complex: It appears that while the resistive (i.e., dissipative) part of the bridge admittance is much greater in the vertical direction, the reactive part, which is of comparable magnitude, is approximately the same for both polarizations.

Some interesting conclusions can be drawn from Fig. 11, obtained under circumstances identical to those of Fig. 7 *except for microphone placement in the room.* As the dashed lines indicate, we again see a transition from a stronger and more rapidly decaying component (presumably due to vertical string motion) to a weaker and longer-lived one (due to horizontal motion). We interpret the behavior in the region where the two components cross as indicating that their sound pressures are approximately in phase in Fig. 7 but approximately in opposite phase in Fig. 11. This shows that the two modes of string motion do not radiate sound through the same single-channel "antenna"; if they did, the *relative* phase of the two signals could not be changed merely by moving the microphone.

In Sec. II A, it was assumed that only vertical motion of the bridge is involved in transmitting energy from the strings, consistent with the model in which the strings themselves move only vertically. We now see that, when two-dimensional string motion is discussed, the two polarizations must be thought of as driving two independent "antennas." There can then be no single scalar quantity [such as $R(t)$] which governs the time variation of sound amplitude at all locations in the room.

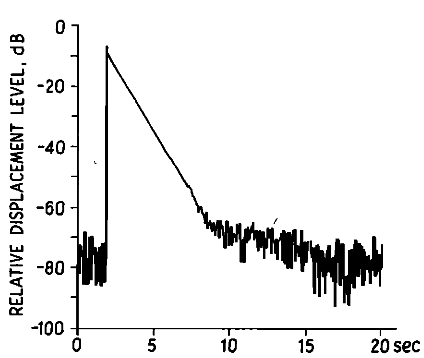

FIG. 9. Relative displacement level of single string as a function of time, vertical polarization (D#4, 311 Hz).

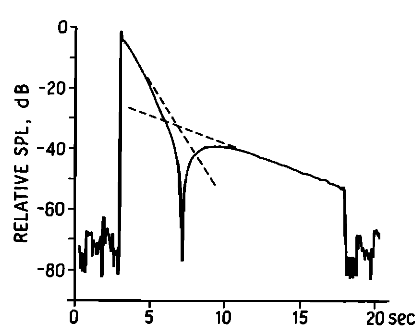

FIG. 11. Same as Fig. 7, but with a different placement of the microphone.

C. Polarization dependence of reactive component

The resistive part of the bridge admittance causes a decay in the string vibration; the reactive part, by contrast, gives rise to a frequency shift. While it is easy to measure absolute decay rates, it would be difficult to measure absolute frequency shifts, since no comparison with the "unperturbed situation" is experimentally available. It is, however, possible to measure the *difference* in reactive admittance for the different polarizations, by looking for a variation in the (real part of the) frequency. That this variation must be quite small is already apparent from the absence of beats in the sound of Fig. 7, or if the signal from *one* of the 45° probes is examined. This sets an upper limit on the beat frequency of, perhaps, $\frac{1}{4}$ Hz, or (for the $D\#_4$ of Fig. 7) an interval between string frequencies of about $1\frac{1}{2}$ cents.

To extend the accuracy of this limit, electronic measurements of the two frequencies were made using a Hewlett-Packard model 5302A frequency counter. In the "period average" mode, this instrument counts cycles of an internal 10-MHz clock, starting at a positive-going zero crossing of the signal, and ending at the tenth following positive-going zero crossing; thus the elapsed time of ten periods is measured, which would imply a precision of 10 ns in the period—that is, about 3 parts per million. In fact, such accuracy is not realizable for a number of reasons. Apart from the obvious factor of noise, the vibration of a piano string is not periodic—both because of the decay and because the normal frequencies are inharmonic. The exponential decay alone would not affect the time between zero crossings, but the inharmonicity clearly does: The instant of zero crossing is shifted by the changing phase of the upper partials. In addition, the first few cycles of oscillation after the hammer blow exhibit a great deal of irregularity which, although repeatable, makes it difficult to assign a meaningful value to the period.

Accordingly, we supplemented the frequency-counting circuit in two ways. First, we added a time-delay circuit which was triggered by the onset of the signal, and which reset the frequency counter a predetermined time later. In this way any selectable group of ten periods, rather than the first ten, could be measured. The delay time was set to half a second or so for the rapidly decaying mode; for the long-lived one it could be set anywhere up to 6 or 8 sec without undue loss of signal-to-noise ratio. Secondly, the signal was fed through a narrow-band amplifier to suppress higher partials. It must be noted that the bandwidth of this amplifier cannot be made too small, lest its own transient introduce more trouble than it eliminates. Specifically, this transient must have become negligible by the time the counter is reset by the delay circuit. We found a Q between 5 and 10 to be generally satisfactory. With these two modifications, it was possible to obtain period measurements with a consistency and accuracy of approximately 0.1 cent.

To this accuracy, we were unable to detect any difference in the period of the vertical and horizontal modes, that is, in the real part of the angular frequency. By contrast, the difference in the imaginary part of the angular frequency—that is, in the decay rates—of the two modes is, of course, clearly observable, and is in the vicinity of 1 $\sec^{-1}$, which corresponds to approximately 1 cent. We conclude that the angular variation of the reactive part of the bridge admittance is at least a factor of ten smaller than the variation in the resistive part.

It is tempting to interpret this result as indicating that the bridge admittance to horizontal motion is much smaller altogether than the admittance to vertical motion; and that the latter is almost entirely resistive. However, further experiments on the vibration of pairs of strings seem to contradict this, as we shall see in Sec. III E.

D. Radiation efficiencies

According to our interpretation, the presence of a double decay in Fig. 7 shows that both polarizations are capable of radiating sound. A measurement of the relative efficiencies can be obtained by comparing the proportion of the two modes in the sound with that in the string vibration.

The electrostatic pickup of Fig. 8, when ideally aligned, has a vertical plane of symmetry. Therefore, the relation of the electrical signals e_1, e_2 to the horizontal and vertical string displacements x, y must be of the form

$$\begin{aligned} e_1 &= Ax+By \ , \\ e_2 &= -Ax+By \ . \end{aligned} \tag{24}$$

There is no reason, however, to assume that the coefficients A and B are equal, that is, that a motion at 45° with the vertical will induce a signal in only one probe. To calibrate these coefficients relative to each other, a monochord was constructed consisting of a single piano string under realistic tension, terminated by two metal bridges, with no other structure to interfere with the mounting of the pickup at an arbitrary angle. The monochord string, like the piano string, had differing decay rates for the horizontal and vertical modes, but here it was the horizontal one that was short lived; the damping being due to back-and-forth swaying of the whole apparatus on its support. (This effect is negligible in the piano because of the massive frame.) We mounted the pickup on the monochord in such a way that its axis made a 45° angle with the vertical; in other words, one of its probes was exactly vertical, the other exactly horizontal. The monochord string was then struck manually and allowed to decay for a second or two, after which the amplitude of the rapidly decaying mode was negligible, the remaining motion being exactly vertical. At this moment, the signals in the two probes could be compared.

In this way, we found the ratio B/A to be approximately 1.6. Naturally, this number varies strongly with the position of the string relative to the probe, so its application to piano measurements depends on duplicating this position, which is difficult. The presence of the other strings may also introduce a complicating factor through the mechanism of electrostatic

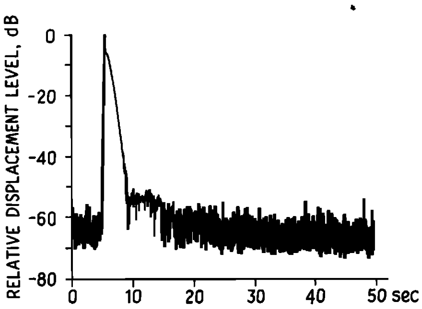

FIG. 12. Relative displacement level of string as a function of time, vertical polarization, other members of triplet damped (C_4, 262 Hz).

images. If we nonetheless use this number, we find that the relative level of horizontal displacement in the vibration of the string is larger by about 10 dB than the relative level of long-lived component in the sound pressure; in other words, the radiation efficiency of the vertical mode appears to be greater by about 10 dB than that of the horizontal mode. More reliable measurements of this quantity could be obtained by constructing a probe with a horizontal as well as a vertical plane of symmetry, circumventing the uncertainty in calibration.

E. Beats and aftersound from pairs of strings

The appearance of a slowly decaying vertical motion as a result of coupling between two strings is experimentally demonstrated by Figs. 12 and 13. The first shows the vertical motion of one string, with the other two strings of the triplet damped. In Fig. 13 the probe has not been moved, but one other string of the triplet is allowed to sound at the same time. (The hammer strikes both strings in either case.)

Figure 14 shows the same situation as Fig. 13, but with varying degrees of mistuning between the two strings which are sounded. (The amount of mistuning was not independently measured.) In cases (a) and (b), the interaction of two normal modes is clearly recognizable. Such data, together with a knowledge of the single-string decay rate (which is easily measured), can in principle be used to obtain both the resistive and the reactive parts of the bridge admittance. Specifically, it is easily shown that if η denotes the single-string decay rate, η' the slower of the coupled decay rates (that is, the one after the beat), and τ the beat period, then the coupling ζ is given by

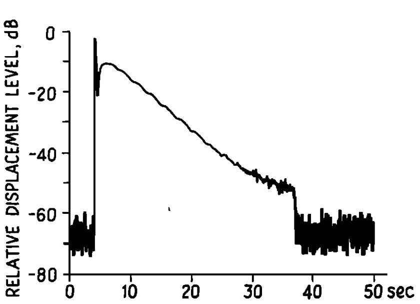

FIG. 13. Relative displacement level of string as a function of time, vertical polarization, one other member of triplet allowed to sound (C_4, 262 Hz).

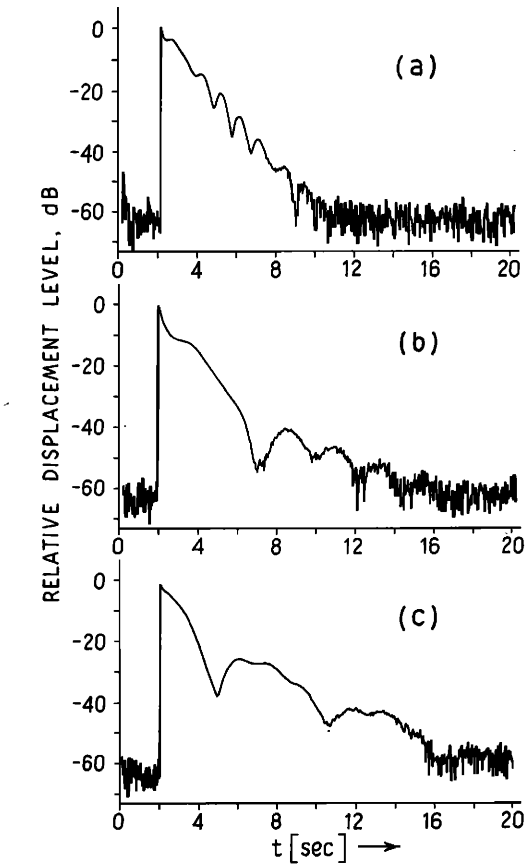

FIG. 14. Relative displacement level of string as a function of time, vertical polarization, one other member of triplet allowed to sound, variable mistuning. (a) and (c): A_4, 440 Hz; (b): C_4, 262 Hz.

$$\zeta=\pi(\eta-\eta')/\eta\tau+i\eta \ . \tag{25}$$

(This formula is valuable because it does not require a knowledge of the mistuning ϵ, which is more difficult to measure.) However, we did not try to carry this program out numerically, since some of the curves [such as Fig. 14(c)] clearly show the appearance of more than two normal modes. Presumably, this is due to some leakage of the horizontal motion, whose frequencies in the coupled case are not the same as those of the vertical motion.

Nonetheless, the fact that a pronounced beat appears together with a pronounced aftersound shows that the resistive and reactive parts of ζ are of comparable size. Since the resistive part is easily measured from the decay rate of a single string, we conclude that the real and imaginary parts of ζ are, in the middle range of the keyboard, each around 1 $\sec^{-1}$.

The characteristic impedance of a piano string in the middle octave is in the vicinity of 2 kg/sec. Substituting these values into Eq. (9), we obtain the order of magnitude of the bridge admittance as $Y \simeq 10^{-3}$ sec/kg.

F. Reconversion and "antisymmetric radiation"

An especially interesting phenomenon occurs when a string doublet is allowed to sound for a few seconds, and then one of the strings is suddenly damped. With

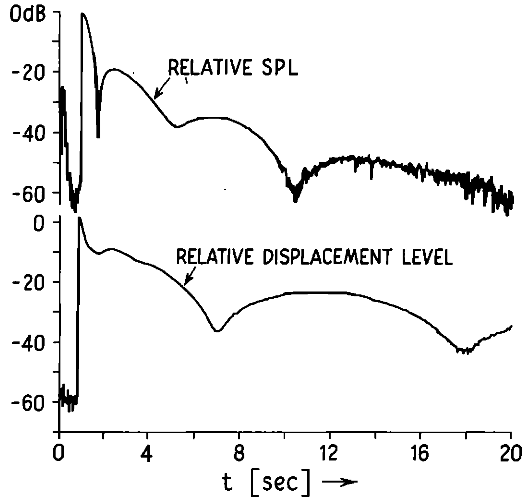

FIG. 15. Two strings sounding (C_4, 262 Hz). Upper curve: relative sound pressure level. Lower curve: relative vertical displacement level of one of the strings.

the tuning near unison, we have seen that the antisymmetric mode is the long-lived one; its decay rate is small because the two strings force the bridge in opposite directions. When one string is damped, the other shifts to the much faster single-string rate of decay. At the same time, the radiated sound pressure *rises*, because the rate of energy transmission to the bridge increases. We refer to this phenomenon as *reconversion*. (The exactly analogous K-meson process is called "regeneration,"[7] but we prefer to avoid this term because of its positive-feedback connotations.)

Figures 15 and 16 illustrate reconversion experimentally. In Fig. 15 we see two curves, of which the bottom one is the (vertical) displacement level of one string of a doublet, and the top one the sound pressure level picked up by a nearby microphone. It is worth commenting that, although the two curves were recorded simultaneously, the sound pressure level shows beats at a frequency which is just about absent in the displacement level. This is, presumably, due to the contribution of the horizontal string motion to the sound.

Figure 16 was taken under identical conditions up to a time of about 12 sec, when a felt wedge was inserted so as to damp the other of the two sounding strings—that is, the one whose vibration was not being monitored by the pickup. The top curve shows an immediate increase in pressure level of close to 20 dB, followed on both curves by the faster decay characteristic of single-string motion. (The slight increase in beat frequency between Figs. 15 and 16 is due to the drift of tuning between the two sets of data.)

It might be thought that the increase of sound pressure level at the moment of damping would provide a figure for the relative radiation efficiencies of the symmetric and antisymmetric motions, but it is hard to see how a perfectly antisymmetric motion could radiate at all—surely the sound amplitude which it produces must be down from that of symmetric motion by a great many orders of magnitude. Rather, it is probable that the slight remaining mistuning makes the long-lived mode deviate from perfect antisymmetry (in analogy to the way that a small electric field "quenches" the metastable state of the hydrogen atom).

Reconversion could have interesting applications in the construction of pianos, in that a split damper, which damps some but not all strings of a multiplet, could be operated by a separate pedal or other device so as to put an accent into the middle of an otherwise sustained note.

IV. CONCLUSIONS

We have shown that the finite admittance of the bridge has a crucial effect on piano tone, and that in the range of ordinary "good" tuning the individual strings cannot be viewed as independent dynamical systems. The statement that the unisons are tuned "to zero beat," or (for that matter) to a slow but nonzero rate of beating, cannot be accepted at face value. Perhaps one should add that these restrictions apply only to unisons. In setting the temperament, when a tuner adjusts very slow beat rates between upper partials of strings mounted in different places on the bridge, the effects of dynamical coupling are presumably negligible.

It appears that the fine tuning of the unisons is not so much a matter of regulating the beat rate as of regulating the amount of aftersound. We saw in Fig. 5 how the amount of aftersound can depend on the degree of mistuning, being affected strongly by a change of even 1 cent in the frequency of a string. Clearly, the aftersound will also be affected by irregularities of the hammer, which cause one string to be hit harder than another, and which may cause a greater or lesser excitation of the horizontal vibration. It is our conjecture that an excellent tuner adjusts the unisons so as to compensate for hammer irregularities, making the total aftersound uniform from note to note; this gives the piano the characteristic "singing" quality that less talented technicians are unable to produce. According to this

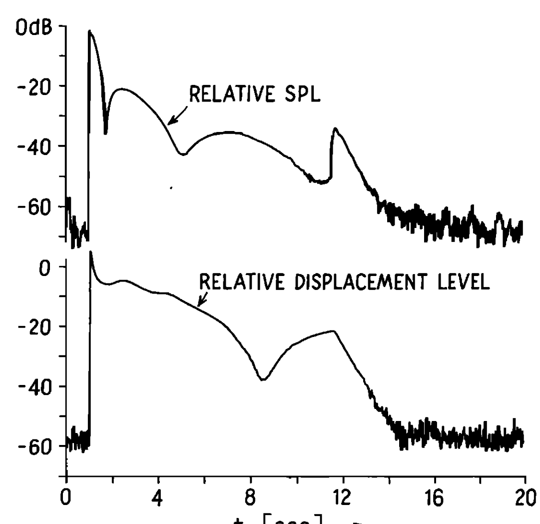

FIG. 16. Demonstration of reconversion. The situation is the same as in Fig. 15 up to about $t = 12$ sec, at which time the other string is damped with a felt wedge.

conjecture, the seemingly random irregularities of tuning, observed by Kirk,[2] are really correlated with variations of hammer shape (which *are* random).

One would expect the admittance of the bridge to depend strongly on soundboard resonances, with the reactive part changing sign each time that a resonance frequency is passed. Because our measurements covered only a small range, we did not look for such effects. It does seem puzzling that the reactive part of the bridge admittance was as isotropic as it seemed to be; one would hardly expect this if resonances are important, and it is possible that the isotropy, in the few cases measured, was accidental.

In any case, one could expect to build a better piano if the aftersound could be more deliberately controlled through an appropriate understanding of the soundboard. The approach we have presented has, we believe, considerable promise for the pursuit of this complex topic.

ACKNOWLEDGMENTS

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[1]D. W. Martin and W. D. Ward, J. Acoust. Soc. Am. **26**, 932(A) (1954); **33**, 582–585 (1961).
[2]R. E. Kirk, J. Acoust. Soc. Am. **31**, 1644–1648 (1959).
[3]D. W. Martin, J. Acoust. Soc. Am. **19**, 535–541 (1947).
[4]D. W. Martin, private communication (1977).
[5]T. C. Hundley, D. W. Martin, and H. Benioff, J. Acoust. Soc. Am. **28**, 769(A) (1956).
[6]A. H. Benade, *Fundamentals of Musical Acoustics* (Oxford University, New York, 1976), Chap. 17.
[7]See, for example, P. K. Kabir, *The CP Puzzle* (Academic, New York, 1968), Chap. 1.
[8]See, for example, R. K. Wangsness, Phys. Rev. **149**, 60–61 (1966).
[9]See, for example, P. M. Morse, *Vibration and Sound* (McGraw–Hill, New York, 1948), Sec. III. 13.
[10]R. W. Pyle, Jr., and T. J. Schultz, J. Acoust. Soc. Am. **39**, 1220(A) (1966).