Musical Acoustics

Fifth Homework

Report
Design of a recorder

Students

Stefano Donà - 10868267
Paolo Ostan - 10868276

POLITECNICO
MILANO 1863

Introduction

The report proposed discusses the design of a recorder flute in its main components. Hence the analysis comprises the dimensioning of the resonator and the process followed to determine the positions of two holes in it in the first section. In the second section the design of the flue channel is discussed.

a) First Component: Resonator

The design of the resonator is made following some technical specifications:

- The shape of the resonator has to be a truncated cone with a conical semiangle $\alpha = 0.75°$.
- The length is $L = 0.45\ m$, being the instrument aimed to be a treble recorder.

1) Cone diameters

The flute is requested to produce an *E4* ($f_0 = 329.63\ Hz$) when all the finger holes are closed. This specifications allow to determine the dimension of the truncated cone that constitutes the recorder. Specifically the head and foot diameters are computed. The recorder acoustic behaviour is associated to the one of an Helmoltz resonator. Therefore we need to consider the bore as an acoustic volume and find the value of r_1 and r_2 that guarantee a minimum on the input impedance. Knowing the length of the resonator L and conical semiangle α, we can consider the radius at the foot r_1 as independent variable and the radius at the mouth r_2 as dependent variable calculated as

$$r_2 = r_1 + L \cdot \tan \alpha \tag{1}$$

as shown in figure 1. In order to find the correct value of r_1, assuming the radiation load impedance $Z_{Load} = 0$ we compute the input impedance as

$$Z_{in} = \frac{j\rho c}{S_2} \times \frac{\sin(k_0 L')\sin(k_0\theta_1)}{\sin(k_0(L' + \theta_1))} + j\omega_0 M \tag{2}$$

where

- $\rho = 1.225\ kg/m^3$ air density.
- $c = 343\ m/s$ air propagation velocity.
- $L = 0.45\ m$ is the length of the bore.
- $L' = L + 0.85 \cdot r_1 = 0.48\ m$ length of the bore considering the end correction, we used the baffled end correction since in a recorder the resonator foot is baffled.
- $S_2 = \pi r_2^2$ is the input surface.
- $\omega_0 = 2\pi f_0 = 2071\ rad/s$
- $k_0 = \frac{\omega_0}{c} = 6.0383\ m^{-1}$ is the wavenumber.
- $j\omega_0 M$ is the series impedance which models the air volume at the mouth of the resonator.
- $M = \frac{\Delta L \cdot \rho}{S_2}$ is the inertance of the mouth.
- $\Delta L = 0.04\ m$ is the typical value assumed for an alto recorder [1]

The angle θ_1 is related to the distance x_1 from the open end to the truncated apex of the cone computed as

$$\theta_1 = \frac{\tan^{-1}(k_0 x_1)}{k_0} \tag{3}$$

where $x_1 = \frac{r_1}{\tan(\alpha)}$ [1].
Plotting the input impedance as function of r_1, as shown in figure 2, we can notice a minimum in the input impedance for $r_1 = 3.7\ cm$ and consequently $r_2 = 4.3\ cm$. The two diameters will be therefore:

$$\boxed{d_1 = 7.2\ [cm]}$$

$$\boxed{d_2 = 8.6\ [cm]}$$

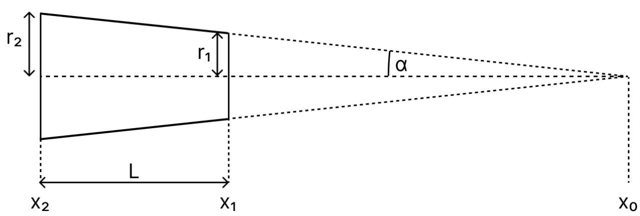

Figure 1: Truncated cone geometry

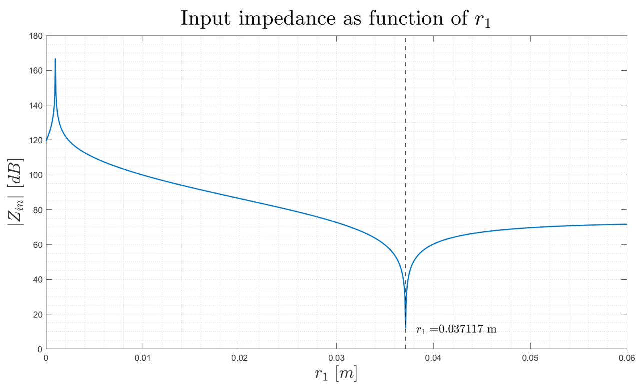

Figure 2: Input impedance as function of r_1

2) First Hole Position

Once the resonator geometry has been established, we aim to insert a fingerhole in order to produce a *F#4 (369.99 Hz)* when opened. Since a new acoustic component is inserted in the system, the acoustic behaviour of the resonator changes. To do so, some approximations are assumed:

- $S_h = S_1$: the hole surface is the same as the foot surface of the recorder;

- $l = \Delta$: the acoustic length of the hole is equal to the virtual elongation of the foot resonator.

The goal is to study the behaviour of the last part of the resonator and since the variaton of the radius is negligible we can approximate the cone with a straight pipe. Taking as reference the figure 3, first of all we study the impedance Z_1 at the connection of the hole D_1 from the foot of the resonator when the hole is closed:

$$Z_1^{(cl)} = j\frac{\rho c}{S_1} tan(k_1(D_1 + \Delta)) \tag{4}$$

Where $k_1 = \omega_1/c$, $\omega_1 = 2\pi f_1$, $f_1 = 369.99\ Hz$. When the hole is open, the impedance Z_1 changes and can be computed evaluating the impedance of the hole Z_{h1} and the final portion of pipe Z_{p1}. Since the two acoustic devices share the same pressure and the volume flow is splitted, the impedance Z_1 is computed considering the parallel configuration of the two impedances mentioned above:

$$Z_1^{(op)} = Z_{h1} // Z_{p1} \tag{5}$$

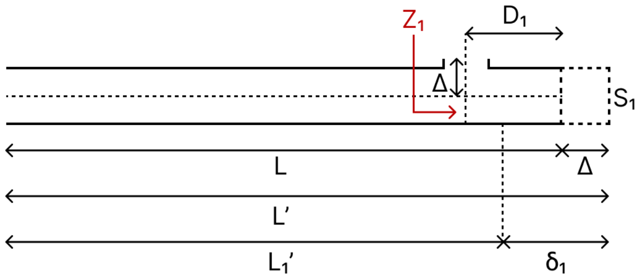

Figure 3: Resonator with first hole geometry

At this point it is convenient to switch to the admittance notation in order to simplify the computations. The admittances are computed as:

$$Y_{h1} = -j\frac{S_1}{\rho c} cot(k_1 \Delta) \tag{6}$$

$$Y_{p1} = -j\frac{S_1}{\rho c} cot(k_1(D1 + \Delta)) \tag{7}$$

The admittance $Y_1^{(op)} = 1/Z_1^{(op)}$ can be computed as the sum of the two reported above:

$$Y_1^{(op)} = -j\frac{S_1}{\rho c} \left[cot(k_1\Delta) + cot(k_1(D_1 + \Delta))\right] \tag{8}$$

And since $k(D_1 + \Delta) << \pi/2$ we can approximate the cotangent function as the inverse of its argument, obtaining:

$$Y_1^{(op)} \approx -j\frac{S_1}{\rho c k_1} \left[\frac{D_1 + 2\Delta}{D_1 + \Delta}\right] \tag{9}$$

From this admittance we can simply retrieve the impedance taking the inverse:

$$Z_1^{(op)} \approx j\frac{\rho c k_1}{S_1} \left[\frac{D_1 + \Delta}{D_1 + 2\Delta}\right] = j\frac{\rho c}{S_1}\Delta' \tag{10}$$

Since we know that in a pipe the impedance can be approximatively computed as in equation (10), where Δ' is the acoustic length, we can study how much the acoustic length of the resonator reduces after the opening of the fingerhole. We are going to study the *acoustic reduction* δ_1, recalling equations (4) and (10), defined as:

$$\delta_1 = L' - \Delta' = D_1 + \frac{\Delta^2}{D_1 + 2\Delta} \tag{11}$$

The acoustic length of the resonator is reduced of a quantity δ_1, so we introduce the notation, referencing to figure 3, to represent the new acoustic length when the hole is open:

$$L_1' = L' - \delta_1 \tag{12}$$

From this new acoustic length we can now compute the impendance of the cone of length L_1' with mouth surface S_2 as:

$$Z_{in,1}^{(op)} = j\frac{\rho c}{S_2}\frac{sin(k_1 L_1') sin(k_1 \theta_1')}{sin(k_1(L_1' + \theta_1'))} + j\omega_1 M \tag{13}$$

Where $\theta_1' = tan^{-1}(k_1 x')/k_1$ for which x' is the distance of the end of the modified acoustic length from the apex of the cone, it can be computed as $x' = x_1 + \delta_1 + \Delta$. Since all these formulas seen above can be written as function of D_1 we iterated che computation of the impedance $Z_{in}^{(op)}$ in order to find the value of D_1 for which the resonator presents a minimum, and therefore a resonance at the desired frequency. The impedance behaviour as function of D_1 can be seen in figure 4.

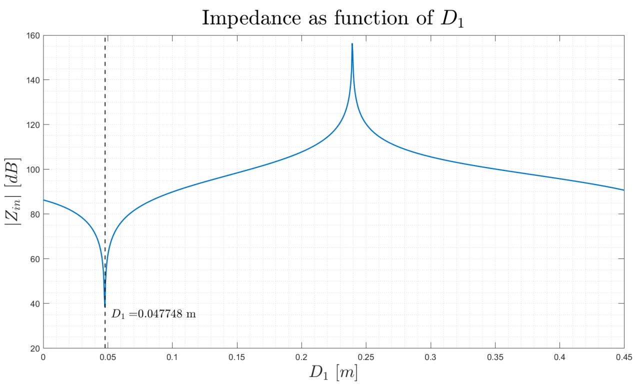

Figure 4: Impedance of the resonator as function of the distance D_1 of the hole from the foot

At the end of the computation we found the optimum value of D_1 of:

$$\boxed{D_1 = 0.0477\ [m]}$$

from which we can compute the coordinate of the first hole in relation to the length of the resonator:

$$\boxed{x_{h1} = L - D_1 = 0.4023\ [m]}$$

3) Second Hole Position

In this section we're are going to re-iterate the process seen in **section 2)** in order to find the position where is better to put a second hole, which allows, together with the first to produce a *G#4 (415.3 Hz)* when both are opened.

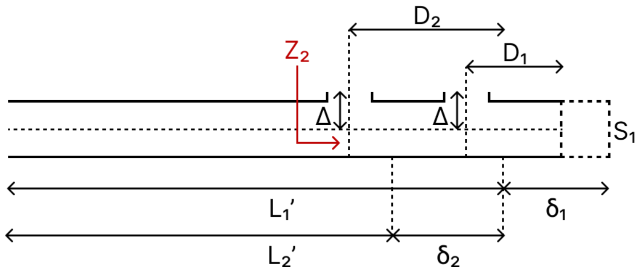

Figure 5: Resonator with two holes geometry

Taking as reference the geometry visible in figure 5 we can now consider, when the first hole H_1 is open, how changes the acoustic length of the resonator when also the second hole H_2 opens. As we seen in the previous section, when H_1 is open the acoustic length of the resonator is reduced to L_1' so now we can consider to have an acoustic pipe of that length. We write down the impedance formula of Z_2, when H_2 is closed:

$$Z_2^{(cl)} = j\frac{\rho c}{S_1} tan(k_2 D_2) \tag{14}$$

where $k_2 = \omega_2/c$, $\omega_2 = 2\pi f_2$, $f_2 = 415.3\ Hz$ and D_2 is the distance of H_2 from the foot of the acoustic length L_1'. Now, as done before we study the impedance, starting from the admittance, of the same pipe section when H_2 gets opened:

$$Y_{h2} = -j\frac{S_1}{\rho c} cot(k_2 \Delta) \tag{15}$$

$$Y_{p2} = -j\frac{S_1}{\rho c} cot(k_2 D_2) \tag{16}$$

and therefore the total admittance is:

$$Y_2^{(op)} \approx -j\frac{S_1}{\rho c k_2}\left[\frac{\Delta + D_2}{\Delta D_2}\right] \tag{17}$$

from which we derive the impedance:

$$Z_2^{(op)} \approx j\frac{\rho c k_2}{S_1}\left[\frac{\Delta D_2}{\Delta + D_2}\right] = j\frac{\rho c k_2}{S_1}\Delta'' \tag{18}$$

Now we can study the acoustic length reduction occurring in the resonator when H_2 is open, looking at the acoustic lengths in equations (14) and (18):

$$\delta_2 = D_2 - \Delta'' = \frac{D_2^2}{\Delta + D_2} \tag{19}$$

Once we obtain the formula of the acoustic length reduction we can retrieve the new acoustic length of the equivalent cone which compose the resonator as:

$$L_2' = L_1' - \delta_2 \tag{20}$$

And then we can put it in the equation of the input impedance of the resonator:

$$Z_{in,2}^{(op)} = j\frac{\rho c}{S_2}\frac{sin(k_2 L_2')sin(k_2 \theta_2')}{sin(k_2(L_2' + \theta_2'))} + j\omega_2 M \tag{21}$$

where $\theta_2' = tan^{-1}(k_2 x'')/k_2$, and $x'' = x' + \delta_2$ is the distance of the foot of L_2' from the apex of the cone. As done before, since all these equations can be written as function of D_2 we iterated the computation of the impedance over the variation of D_2 in order to find the value which allows the existance of a minimum in the impedance, visible in figure 6.

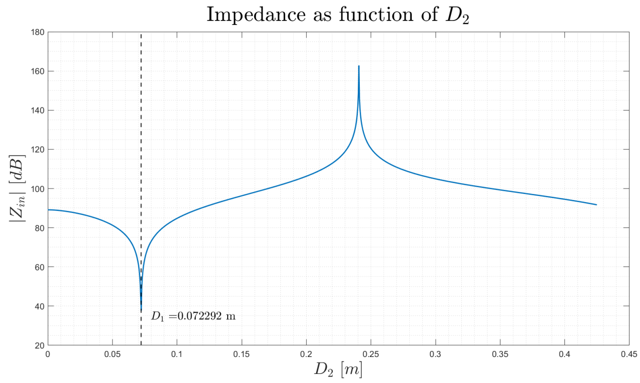

Figure 6: Impedance of the resonator as function of the distance D_2 of the hole from the foot of the acoustic length L_1'

The numerical solution given by the data extraction is:

$$\boxed{D_2 = 0.0723\ [m]}$$

from which we can retrieve the coordinate of the second hole with respect to the resonator length:

$$\boxed{x_{h2} = L_1' - D_2 = 0.3525\ [m]} \tag{22}$$

In figure 7 the final shape of the resonator obtained from the specifications given by the problem is visible.

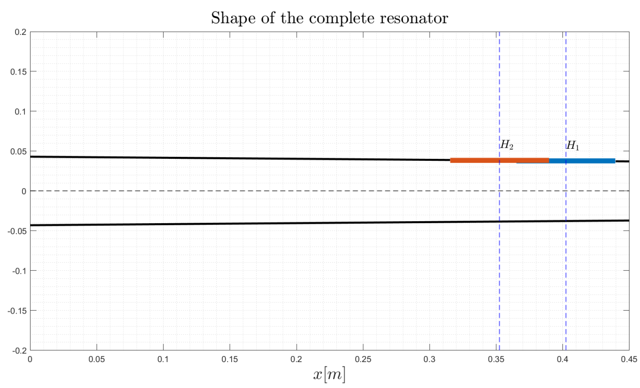

Figure 7: Shape of the resonator

We clearly notice how the two holes are superimposed, and it is obviously not a feasible result. This derives from the approximation of adopting holes that have the same surface of the resonator foot. Since the recorder is based on the Helmoltz resonator the possible solutions to bypass this problem are multiple, the simplest-one is to consider the fingerholes with diameters which are independent from the geometry of the resonator. Another simple solution could be to increase the volume of the resonator elongating it, looking for a narrower shape that allows to accept the hole surface approximation. In figure 8 can be visible the shape of the resonator considering the length of the resonator equal to $L = 47\ cm$.

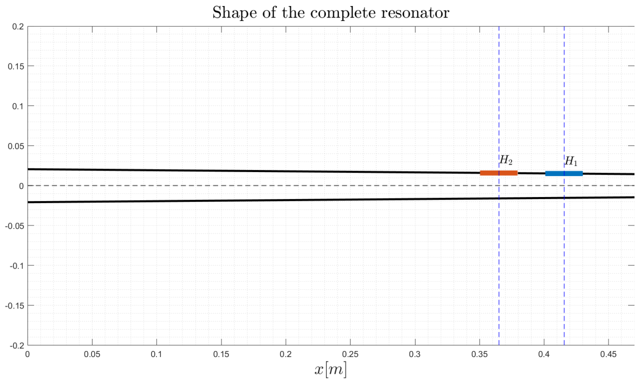

Figure 8: Shape of the resonator with L=47 cm

b) Second Component: Flue channel

In this section the design phase for the flue channel of the recorder is presented. The recorder is requested to present a centroid frequency $f_c = 1.7\ kHz$ when the pressure difference between the player mouth and the flue channel $\Delta p = 55\ Pa$.

4) Flue channel thickness

In order to find the flue channel thickness that complies with the above specifications, we need to compute first the central velocity at the channel exit U_j, computed as:

$$U_j = \sqrt{\frac{2\delta p}{\rho_0}} = 9.47\ [m/s] \tag{23}$$

The equation derives from the evaluation of the central velocity a jet produced by a reservoir in a state of overpressure with respect to the atmospheric pressure, representation of the player's mouth. The thickness of the flue channel becomes therefore:

$$h = \frac{0.3U_j}{f_c} = 1.6722\ [mm] \tag{24}$$

Then the *Reynolds number* Re is computed. It characterizes the structure of the jet in terms of the ratio between inertial and viscous forces. For values of the Reynolds number smaller than 2000, the jet remains laminar for a short distance, while for values above 3000, the jet becomes turbulent immediately downstream the flue exit. Estimations under playing conditions for different recorders indicate that the Reynolds number varies between 700 and 2000. The **Reynolds number** can be calculated as:

$$Re = \frac{U_j h}{\nu} = 1056.4226 \tag{25}$$

where $\nu = 1.5\ 10^{-5}\ m^2/s$ is the kinematic viscosity of air. This value of Re leads us to understand that for a short distance the jet produced by the player inside the instrument remains laminar.

5) Boundary layer thickness

In order to model correctly the behaviour of the flute we need to consider the presence of a boundary layer produced by the viscous interaction of the air in the channel with the walls. The thickness of this layer varies with the position x along the channel with the equation:

$$\delta(x) \approx \sqrt{\frac{\nu x}{U_j}} \tag{26}$$

Considering therefore that the lenght of the flue channel is $L = 25\ mm$ we find that the boundary layer at the exit is:

$$\delta(L) \approx \sqrt{\frac{\nu L}{U_j}} = 0.199\ [mm] \tag{27}$$

References

[1] N. H. Fletcher and T. D. Rossing, *The physics of musical instruments*, Springer Study Edition, Springer, New York, NY, 2 ed., Dec. 1998, pp. 533–535.