Part 2: Sound propagation, radiation and bores - **Pipes and horns**

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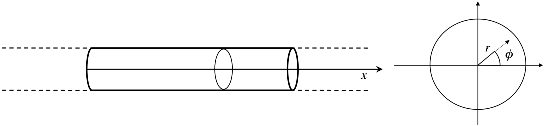
Outline

Learning objectives

- Modeling of the propagation and radiation from cavities such as horns and pipes, which is of utmost importance in wind instruments (covered in the second module).
- The propagation in cavities is also important for modeling the air resonance in guitar, violin and other instruments. No theoretical model exists in the case of complex boxes, though.

Infinite cylindrical pipes

Infinite cylindrical pipes



- Let us consider an infinite cylindrical pipe deployed along the x axis, with cross section S. Cylindrical coordinates (r, ϕ, x) are used.
- Ideal assumptions: walls are rigid (no displacement), smooth and thermally insulating.
- We assume that the pressure is uniform on the cross section, i.e.

$$p(r, \phi, x, t) = p(x, y) = \overline{p} e^{j(-kx+\omega t)},$$

- Acoustic volume flow: $U(x,t) \triangleq u(x,t)S = \left(\frac{S\overline{p}}{\rho c}\right) e^{j(-kx+\omega t)}$, where
  - ω is the frequency;
  - k is the related wavenumber;
  - S is the cross-section of the pipe;
  - ρ and c are the density and speed of sound in air.
- Acoustic impedance: $Z_0(x) = \frac{p(x,t)}{U(x,t)} = \frac{\rho c}{S}$ (notice the difference with the wave impedance).

Infinite cylindrical pipes

Before we assumed that the pressure and particle velocity distribution on the section is uniform. This is not true in general.
We have instead to assume that p depends on (r, ϕ, x, t), where
The wave equation in cylindrical coordinates is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 p}{\partial \phi^2} + \frac{\partial^2 p}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}.$$

Let a be the radius of the pipe. Under the assumption of insulating and rigid walls, we have to impose that the radial component of the velocity is zero at the walls:

$$u(r, \phi, t, x)|_{r=a} = 0$$

Recall that

$$\rho\frac{\partial u}{\partial t} = -\nabla p,$$

therefore

$$u(r, \phi, t, x)|_{r=a} = 0 \leftrightarrow \frac{\partial p(r, \phi, t, x)}{\partial r}|_{r=a} = 0.$$

Infinite cylindrical pipes

After the derivation, the mnth mode propagating in the tube is given by

$$p_{mn}(r,\phi,x,t) = \underbrace{p_{\cos}^{\sin}(m\phi)}_{\text{dependence on }\phi} \overbrace{J_m\left(\frac{\pi q_{mn} r}{a}\right)}^{\text{dependence on }r} \underbrace{e^{-jk_{mn}x}}_{\text{dependence on }x} e^{j\omega t}, \tag{1}$$

where

- J_m is the Bessel function of order m;
- In order to have $\frac{\partial p}{\partial r}|_{r=a} = 0$ the condition $J'_m(\pi q_{mn}) = 0$ must hold.
- This implies that $\pi q_{mn} = Z_n(J'_m)$, where $Z_n(J'_m) = n\text{th root of } J'_m$

Infinite cylindrical pipes

Zeros of the derivative of first orders of the Bessel function

n	$J'_0(x)$	$J'_1(x)$	$J'_2(x)$	$J'_3(x)$	$J'_4(x)$	$J'_5(x)$
0	0	1.8412	3.0542	4.2012	5.3175	6.4156
1	3.8317	5.3314	6.7061	8.0152	9.2824	10.5199
2	7.0156	8.5363	9.9695	11.3459	12.6819	13.9872
3	10.1735	11.7060	13.1704	14.5858	15.9641	17.3128
4	13.3237	14.8636	16.3475	17.7887	19.1960	20.5755

Notice that for $m = n = 0$ there exists a zero for $\pi q_{mn} = 0 \rightarrow q_{mn} = 0$. The mode (0,0) corresponds to the planar wave propagating in the tube.

Infinite cylindrical pipes

- The (m, n) mode has $m = 0, 1, 2, ..$ nodal diameters and $n = 0, 1, 2, ...$ nodal circles.
- By substituting the solution of the wave equation in the equation of motion, we obtain a condition on the wavenumber of the (mn)th mode:

$$k_{mn}^2 = \left(\frac{\omega}{c}\right)^2 - \left(\frac{\pi q_{mn}}{a}\right)^2$$

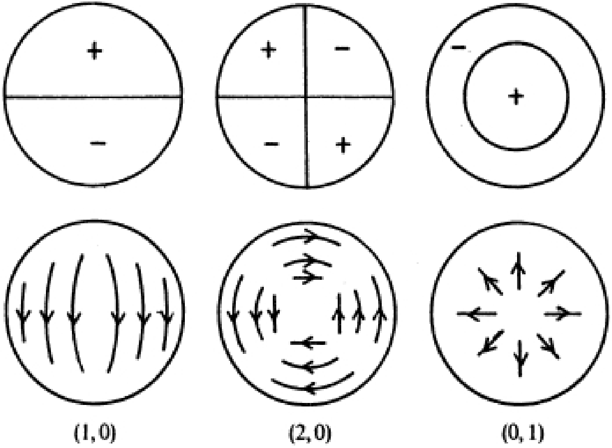
- For a mode to propagate k_{mn} must be real:
  - modes will propagate without radial attenuation above the cutoff frequency

$$\omega_c = \frac{\pi q_{mn} c}{a}.$$

  - modes below the cutoff frequency will experience an attenuation of $10\ \mathrm{dB}$ within a distance smaller than the pipe radius.
  - the only exception is the mode (0,0), which propagates at all the frequencies, since $q_{mn} = 0$.
  - Numerical exercise: compute the cutoff frequency for the mode (1,0) in a pipe with radius $a = 0.02\mathrm{m}$...

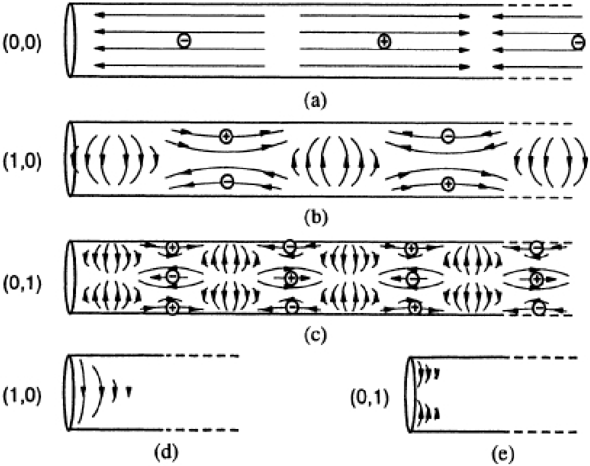
Infinite cylindrical pipes

Pressure and transverse flow patterns for the lowest propagating modes of a cylindrical pipe. The (0,0) mode is not shown.



Infinite cylindrical pipes

Cross section view of the particle velocity patterns for the lowest propagating modes of a cylindrical pipe. (d) and (e) refer to the propagation below the cutoff frequency for the modes (1,0) and (0,1).



Wall losses

Wall Losses

Let's assume that only the (0,0) mode propagates. u_x and p should be uniform on the cross-section. However, it is observed that both have a maximum along the axis of the pipe and a minimum close to the walls. These are the wall losses. Two reasons:

1. viscous drag operated by the walls on the air;
2. thermal exchange between the walls and the air.

Viscous drag

- Viscous drag is responsible for a reduction of the velocity profile, with respect to the ideal case, going from the centre of the pipe to the walls.
- Viscous boundary layer: region in which the viscous drag impacts on the particle velocity.

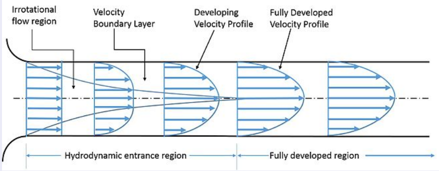


Figure: Development of the boundary layer of the velocity profile (for mode (0,0))

Wall losses
Viscous drag

- It is possible to study the impact of the viscous drag by analyzing the extension of the boundary layer compared to the radius of the pipe.
- Ratio between the pipe radius and the boundary layer thickness:

$$r_{\mathrm{v}} = 632.8\, af^{1/2}(1 - 0.0029\Delta T),$$

where $\Delta T = T - 300^{\circ}\ \mathrm{K}$.

We can notice that the viscous drag is more relevant for small pipes.

Wall losses

Thermal losses

- Thermal boundary layer: section of the pipe where thermal exchanges between the wall and the air are possible. This modifies the compressibility of the air in contact with the walls.
- Ratio between the pipe radius and the boundary layer thickness:

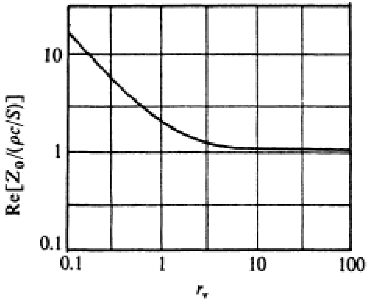
$$r_{\mathrm{t}} = 532.8\, af^{1/2}(1 - 0.0031\Delta T).$$

- Effect of the losses (both thermal and related to the viscous drag): the characteristic impedance Z_0 is different from $\frac{\rho c}{S}$ and becomes complex.
- The wavenumber becomes complex as well $\rightarrow$ attenuation as the wave propagates along the pipe.

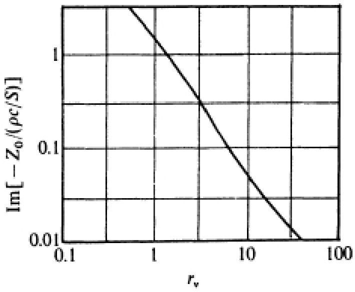
Wall losses
Impact on the impedance

Real and imaginary parts of the characteristic impedance in units of $\rho c/S$ as a function of r_{v}.

Real part:



Imaginary part:



Wall losses

Impact on the wavenumber

- If we write $k = \omega/v - j\alpha$, it is important to characterize the terms v/c and α/f as a function of r_v.
- For large tubes: approximate equations (valid for $r_\mathrm{v} > 10$ and usable down to $r_\mathrm{v} = 3$):

$$v \approx c\left[1 - \frac{1.65 \times 10^{-3}}{af^{1/2}}\right],$$

$$\alpha \approx \frac{3 \times 10^{-5} f^{1/2}}{a}.$$

- For small tubes such as fingerholes, see next figures

Wall losses

Impact on the wavenumber

Phase velocity:

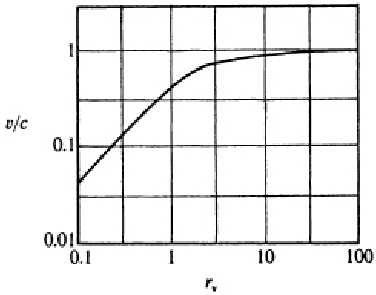


Figure: Phase velocity relative to the free-air sound velocity c as a function of the parameter r_v.

Attenuation coefficient:

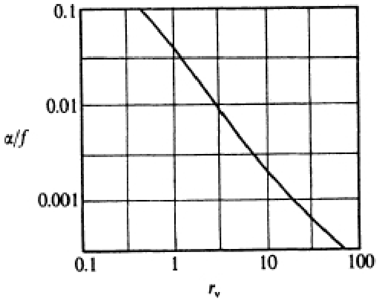


Figure: Attenuation coefficient in meters^{-1} at frequency f as a function of the parameter r_v.

Finite cylindrical pipes

Finite cylindrical pipes

Input impedance

From now on, we assume the only presence of the mode $m = 0, n = 0$.

- Pressure wave: $p(x,t) = [Ae^{-jkx} + Be^{jkx}]e^{j\omega t}$.
- Through $\rho \frac{\partial u}{\partial t} = -\nabla p$ we get: $U(x,t) = Su(x,t) = \frac{S}{\rho c}[Ae^{-jkx} - Be^{jkx}]e^{j\omega t}$.
- At $x = L$ there is a terminating impedance Z_L so that

$$\frac{p(L,t)}{U(L,t)} = Z_L.$$

- Imposing the above condition yields

$$\frac{B}{A} = e^{-2jkL}\left[\frac{Z_L - Z_0}{Z_L + Z_0}\right],$$

  that is a condition on the ratio of the magnitudes of ingoing and outgoing waves:

- The input impedance is found as $\frac{p(0,t)}{U(0,t)}$ and yields

$$Z_{\text{IN}} = Z_0\left[\frac{Z_L\cos(kL) + jZ_0\sin(kL)}{jZ_L\sin(kL) + Z_0\cos(kL)}\right]. \quad (2)$$

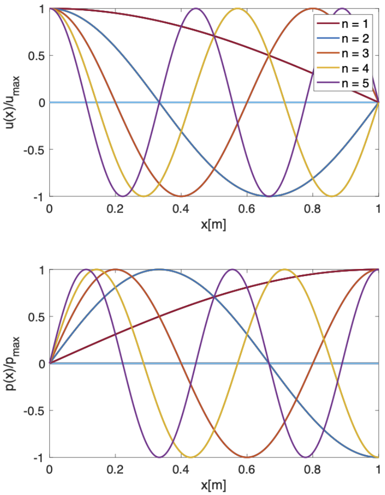
Finite cylindrical pipes
Input impedance

Stopped end

- With a stopped end $Z_L = \infty$ and $Z_{\mathrm{IN}}^{\mathrm{stopped}} = -jZ_0 \cot(kL)$.
- Impedance minima found by imposing that $Z_{\mathrm{IN}}^{\mathrm{stopped}}|_{x=0}$, to obtain

$$\omega^{\mathrm{stopped}} = \frac{(2n-1)\pi c}{2L}.$$

- It corresponds to an odd number of quarter wavelengths in the pipe



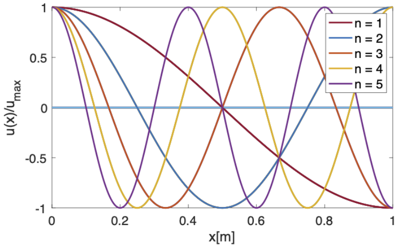
Finite cylindrical pipes
Input impedance

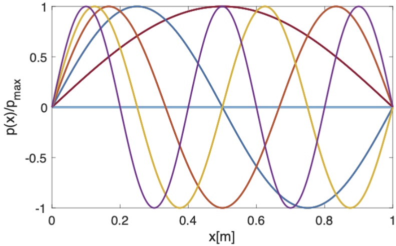
Open end

- With an open end $Z_L = 0$ and $Z_{\mathrm{IN}}^{\mathrm{open}} = jZ_0 \tan(kL)$.
- Notice that for $kL \ll 1$, $Z_{\mathrm{IN}}^{\mathrm{open}} \approx jZ_0 kL$
- Impedance minima found by imposing that at $x = 0$ the pipe is open, to obtain

$$\omega^{\mathrm{open}} = \frac{n\pi c}{L}.$$

- It corresponds to an even number of quarter wavelengths.
- However open ends are unfeasible: some of the energy in the pipe will be radiated outside, and therefore $p(L,t) \neq 0$.





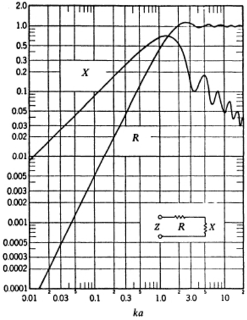
Finite cylindrical pipes

Input impedance

We model the radiation on the part of the pipe with a radiation load. In the case that the pipe terminates in a plane flange (baffle), the radiation load is: $Z^{\text{flanged}} = R + jX$, where

$$R = Z_0\left[\frac{(ka)^2}{2} - \frac{(ka)^4}{2^2\cdot 3} + \frac{(ka)^6}{2^2\cdot 3^2\cdot 4} - \dots\right]$$

$$X = \frac{Z_0}{\pi k^2 a^2}\left[\frac{(2ka)^3}{3} - \frac{(2ka)^5}{3^2\cdot 5} + \frac{(2ka)^7}{3^2\cdot 5^2\cdot 7} - \dots\right]$$



Finite cylindrical pipes

Input impedance

In the region $ka \ll 1$ the radiation load can be approximated by

$$Z^{\text{flanged}} \approx jZ_0 k(\frac{8a}{3\pi}).$$

As highlighted before, when the length Δ of a pipe is small, the impedance of the open end pipe can be approximated by

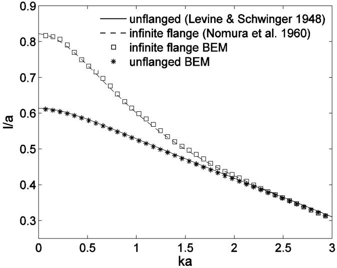
$$Z_{\text{in}} = jZ_0 \tan(k\Delta) \approx jZ_0 k\Delta.$$

We discover, therefore, that the radiation load impedance of the open pipe can be assimilated to the impedance of a short pipe with length $\Delta^{\text{flanged}} = \dfrac{8a}{3\pi} \approx 0.85a$.
The real open end pipe can be therefore assimilated to an ideal open pipe of length $L + \Delta^{\text{flanged}}$, so that the radiation loss is included in the picture.

Finite cylindrical pipes

Input impedance

- In the case of an unflanged pipe, the virtual pipe elongation is $\Delta^{\text{Unflanged}} \approx 0.6a$, valid in the region $ka \ll 1$.
- It is possible to extend the validity of the end corrections model up to $ka = 3$.
- The approximation of a short virtual pipe that is attached to the original pipe still holds. The length of the end correction, normalized by the radius of the pipe, is plotted below, for the two cases (flanged and unflanged).

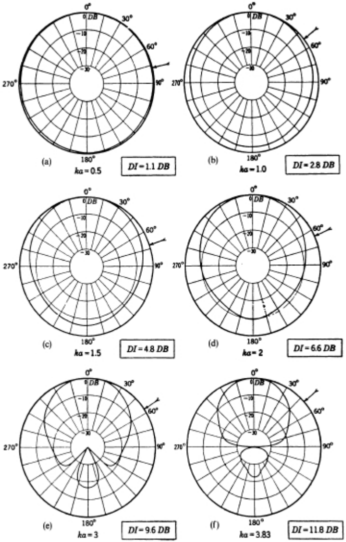


- For $ka > 3$ also the radiation losses must be kept into account, as the radiation load cannot be approximated as purely reactive, and therefore the virtual elongation model does not work.

Finite cylindrical pipes

- It is important to study also the directional properties of the radiation.
- Intensity radiated towards the angle θ is proportional to $\left[\frac{2J_1(ka\sin(\theta))}{ka\sin(\theta)}\right]^2$.
- For an unflanged pipe: only from 0 to 180°.

Finite cylindrical pipes



DI: sound intensity on the axis compared with that of an isotropic source.

Impedance curves

Impedance curves
With wall losses

Characteristic impedance of an open pipe with wall losses:

$$Z_{\mathrm{IN}} = Z_0 \left[\frac{\tanh \alpha L + j \tan(\omega L/v)}{1 + j \tanh \alpha L \tan(\omega L/v)} \right].$$

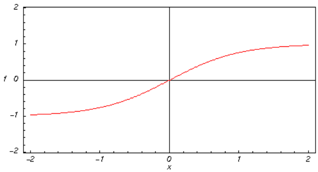


Figure: Hyperbolic tangent

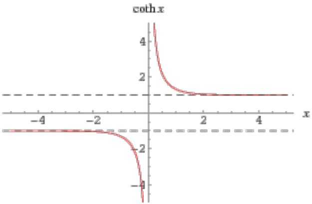


Figure: Hyperbolic co-tangent

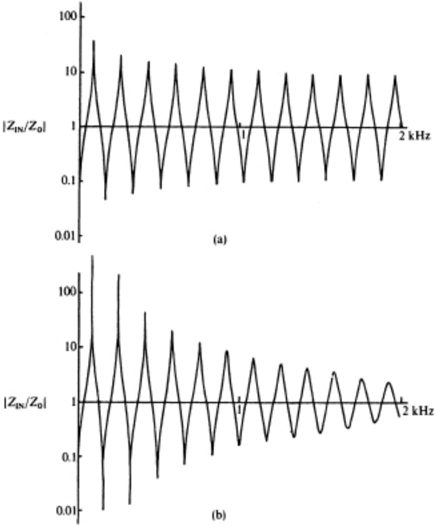
Impedance curves
With wall losses

- Location of maxima and minima unmodified with respect to the ideal case without losses
- Magnitude of the impedance at maxima: $Z_0 \coth \alpha L$
- Magnitude of the impedance at minima: $Z_0 \tanh \alpha L$.
- α increases with frequency $\rightarrow$ at high frequencies maxima of Z_{IN} converge to Z_0
- As a function of the pipe size:
  - in narrow pipes the wall losses dominate at low frequencies;
  - in wide pipes the radiation losses from the end become more important at high frequencies.

Impedance curves

With wall losses

Input impedance for a pipe of length $1\ \mathrm{m}$ and wide (a)$2\ \mathrm{cm}$ and (b)$10\ \mathrm{cm}$

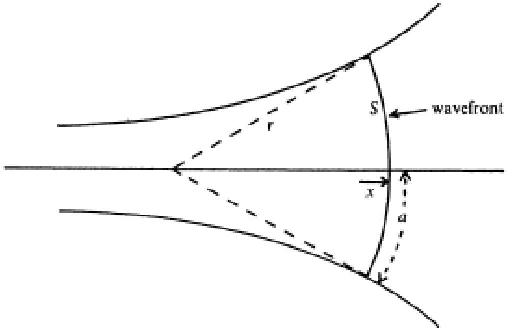


Horns

Horns

- In horns the cross section is not constant throughout the length.
- A good approximation can be obtained under the assumption that the horn does not flare too rapidly:
  - wavefronts can be considered spherical;
  - they must be orthogonal to the walls of the horn.

In a horn, the wavefront has approximately the form of a spherical cap of area S and the curvature radius is r. The radius of the equivalent cross section is a.



Horns

Let x be the coordinate along the axis of the horn.
Moreover, notice that p will decrease with x due to the spreading of the horn. In order to compensate this decrease, let us introduce $\psi = pS^{-1/2}$.

Wave equation for a circular horn:

$$\frac{\partial^2 \psi}{\partial x^2} + \left(k^2 - \frac{1}{a}\frac{\partial^2 a}{\partial x^2} \right) \psi = 0, \; k = \omega / c. \tag{3}$$

The solution of this equation is a propagating wave if $\left(k^2 - \frac{1}{a}\frac{\partial^2 a}{\partial x^2} \right) > 0$. $F = \frac{1}{a}\frac{\partial^2 a}{\partial x^2}$ is the barrier function for a horn and:

- frequencies for which $k^2 > F$ can propagate;
- frequencies for which $k^2 < F$ cannot propagate.

Horns

For horns that do not flare too rapidly, the function F can be approximated by $F \approx \dfrac{1}{R_L R_T}$. We assume here that the wavefront takes a planar form.

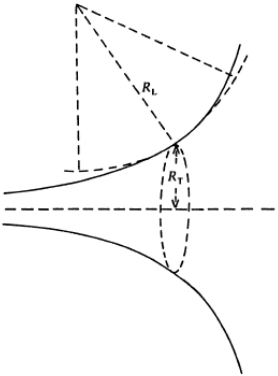


Figure: Longitudinal radius of curvature R_L and internal transverse radius of curvature R_T.

Horns - Salmon horns

In Salmon horns the function F is constant throughout the length of the horn, and therefore also the cutoff frequency benefits of the same property.

- Equivalent radius in Salmon horns:

$$a = a_0[\cosh mx + T\sinh(mx)].$$

- Pressure wave:

$$p = \left(\frac{p_0}{a}\right) e^{j\omega t} e^{-j\sqrt{k^2-m^2}x},$$

  Does not propagate if $k < m$.
- Different shapes can be obtained from different values of T.

Horns - Salmon horns

- If $T = 1$ then we have the exponential horn:

$$a = a_0 e^{mx}.$$

- If $T = 0$ we have a catenoidal horn, i.e.

$$a = a_0 \cosh(mx),$$

  it smoothly joins a cylindrical pipe extending along the negative axis to the origin.

- If $T = 1/mx_0$ and $m \to 0$ we have the conical form, for which

$$a = a_0 \left(1 + \frac{x}{x_0}\right),$$

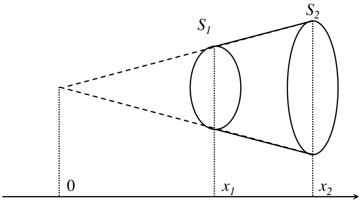
  with vertex at $-x_0$ and a semiangle of $\tan^{-1}(a_0/x_0)$. In this case $F = 0$, so the conical horn has no cutoff.

Finite horns

Finite conical and exponential horns

Let us consider a conical horn with

- a throat of area S_1 at position x_1,
- mouth of area S_2 at position x_2



The impedance is

$$Z_{\mathrm{IN}} = \frac{\rho c}{S_1}\left\{\frac{jZ_L[\sin(kL-\theta_2)/\sin\theta_2] + (\rho c/S_2)\sin kL}{Z_L[\sin(kL+\theta_1-\theta_2)/\sin\theta_1\sin\theta_2] - (j\rho c/S_2)[\sin(kL+\theta_1)/\sin\theta_1]}\right\}$$

$$\theta_{1/2} = \tan^{-1}(kx_{1/2})$$

where $x_{1/2}$ is the distance from the apex of the cone.
For an exponential horn of length L

$$Z_{\mathrm{IN}} = \frac{\rho c}{S_1}\left[\frac{Z_L\cos(bL+\theta) + j(\rho c/S_2)\sin bL}{jZ_L\sin bL + (\rho c/S_2\cos(bL-\theta))}\right],$$

where $b^2 = k^2 - m^2$; and $\theta = \tan^{-1}(m/b)$.

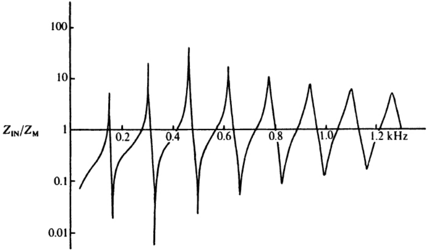
Conical horns

For an open conical horn ($Z_L = 0$), we obtain that

$$Z_{\text{IN}} = j\left(\frac{\rho c}{S_1}\right)\frac{\sin kL \sin\theta_1}{\sin(kL+\theta_1)}.$$

- Zeros occur for the frequencies where $\sin kL = 0$, i.e. the same frequencies for a pipe of the same length with an open end.
- Maxima occur where the denominator is zero, i.e. $kL = n\pi - \tan^{-1} kx_1$.

Magnitude of the input impedance of a conical horn of length $1\ \text{m}$, throat diameter $1\ \text{cm}$ and mouth diameter $10\ \text{cm}$ relative to the impedance $\rho c/S_1$.



Horns - Bessel horns

In Bessel horns $S = Bx^{-2\epsilon}$ and $a = bx^{-\epsilon}$, with x distance measured from a reference point $x = 0$.
When $\epsilon > 0$ we obtain the case of rapidly flaring horns.

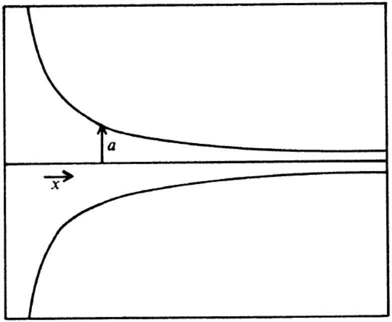


Figure: The form of a Bessel horn with $\epsilon > 0$.

Horns - Bessel horns

It is important to understand how the barrier function F works for exponential horns. In theory F for x close to zero is infinite $\rightarrow$ no propagation could occur at this end. However...

- the plane-wave approximation is not usable here: the horn rapidly flares and therefore F cannot be defined as before.
- Instead, close to the open end it is more reasonable to use the the spherical wave approximation, for which the F function is shown in the bottom figure.

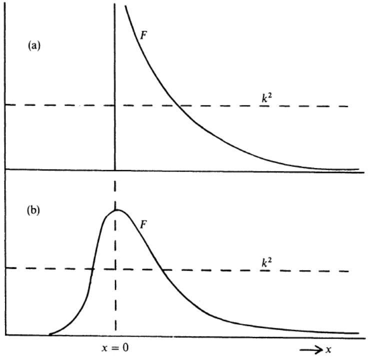


Figure: The horn function for a Bessel cone based on (a) plane-wave and (b) spherical wave approximations.

Compound horns

Compound horns

- Most brass instruments are based on the combination of different shapes: a part has a constant section, and another one has an expanding cross section. Examples:
  - Conical horn smoothly joined to a cylindrical section (suitable for old instruments).
  - Conical horn smoothly joined to an exponential section (suitable for modern instruments).

Compound horns

Let us consider a cylindrical section (throat) joined to a conical or exponential one (mouth). We find the overall input impedance using as load impedance of the cylindrical section the input impedance of the conical/exponential one.

- Input impedance of the conical section:

$$Z_c = \frac{j\rho c}{S_1}\left(\cot kL_1 + \frac{1}{kx_1}\right)^{-1}.$$

  This is the terminating impedance of the cylindrical section.

- Replacing into the impedance of the cylindrical section we obtain that maxima of the input impedance are where

$$\tan kL_2 - \cot kL_1 - \left(\frac{1}{kx_1}\right) = 0.$$

- For the case of a cylinder linked to an exponential horn, we obtain

$$\tan kL_2 - \frac{b}{k}\cot bL_1 - \frac{m}{k} = 0,$$

  where $b = (k^2 - m^2)^{1/2}$.

Compound horns

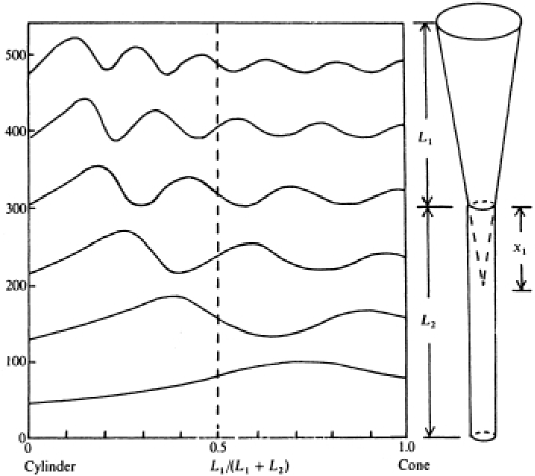


Figure: Maxima of the input impedance for a compound horn for the shape shown on the right.

Perturbations

Perturbations

- **Goals:**
  - study the modifications introduced by the irregularities of the shape (e.g. fingerholes);
  - exploit these modifications to alter the behavior of the instrument (e.g. modify the resonance frequencies).
- Altered section of the horn: $S(x) = S_0(x) + \delta S(x)$,
- New pressure distribution $p(x,t) = \beta p_0(x,t) + p_1(x,t)$, with $\beta \approx 1$ under the assumption that the perturbation is small.
- New resonance frequency $\omega = \omega_0 + \delta\omega$,

After replacing the above expressions into the wave equation and substitution of the ideal forms, and integration over the length of the horn we obtain

$$\delta\omega = -\frac{c^2}{2\omega_0 N} \int S_0 \frac{d}{dx}\left(\frac{\delta S}{S_0} p_0 \frac{dp_0}{dx}\right) dx,$$

where

$$N = \int S_0 p_0^2 dx.$$

Perturbations

Let us consider $\delta S(x) = \Delta\delta(x - x_0)$. After integration we obtain that

$$\delta\omega = \frac{c^2\Delta}{2\omega_0 N}\left[\frac{d}{dx}\left(p_0\frac{dp_0}{dx}\right)\right]_{x=x_0}.$$

If $p_0(x,t) = \sin(kx)\sin(\omega t)$ then $\delta\omega \propto \cos(2kx_0)$.

- If the enlargement is close a maximum of the pressure wave, we obtain that $\delta\omega \propto \cos(2kx_0) = -1$, i.e. the resonance frequency is lowered.
- If the enlargement is close a minimum of the pressure wave, we obtain that $\delta\omega \propto \cos(2kx_0) = +1$, i.e. the resonance frequency is increased.
- In case of a reduction of the cross-section opposite considerations apply.

Curved horns

Curved horns

Bending is used to keep the extension of the instrument into a reasonable range.

- Bending parameterized by $B = r/R$, where r is the radius of the circular cross-section of the horn, and R is the curvature radius.
- Impedance and sound speed modifications: $\dfrac{\delta v}{c} = -\dfrac{\delta Z}{Z_0} = \left(\dfrac{2I}{\pi B}\right)^{1/2} - 1$, where

$$I = \int_0^{\pi/2} \cos\theta \ln\left(\frac{1 + B\cos\theta}{1 - B\cos\theta}\right) d\theta.$$

Curved horns

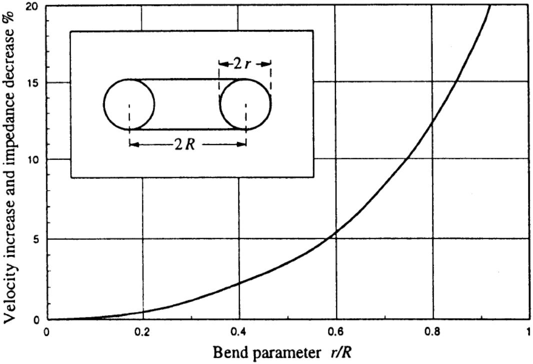


Figure: Percentage increase δv of the velocity and decrease $-\delta Z$ of duct impedance as a function of the curvature parameter $B = r/R$.

For more information on the laminar and turbulent flows in tubes, see `http://www.bg.ic.ac.uk/research/k.parker/homepage/Mechanics%20of%20the%20Circulation/Chap_05/_Chapter_05.htm`.

Analysis in the time domain

Analysis in the time domain

- The analysis in the time domain is important to identify the transient behavior of the sound of the instrument.
- The time behavior is characterized by the Inverse Fourier transform of the impedance, called as Green's function.
- The pressure in the time domain is obtained as

$$p(t) = \int_{-\infty}^{t} G(t-t')U(t')dt' = G(t) \star U(t),$$

  where $G(t)$ is the Green's function and $U(t)$ is the input signal.
- Problem: $G(t)$ has a large extension in time $\rightarrow$ implementation of the above equation not convenient.

Analysis in time domain

- The plane-wave reflection function $r(\omega)$ at the input of the instrument can be written as

$$r(\omega) = \frac{Z(\omega) - Z_0}{Z(\omega) + Z_0},$$

- Then

$$Z(\omega) = Z_0 + Z_0 r(\omega) + r(\omega) Z(\omega),$$

- By taking the inverse Fourier Transform we obtain that

$$G(t) = Z_0 \delta(t) + Z_0 \delta(t) \star r(t) + r(t) \star G(t),$$

- and then

$$p(t) = Z_0 U(t) + Z_0 r(t) \star U(t) + r(t) \star p(t) = \tag{4}$$

$$= Z_0 U(t) + \int_0^{\infty} r(t') [Z_0 U(t - t') + p(t - t')] dt'.$$

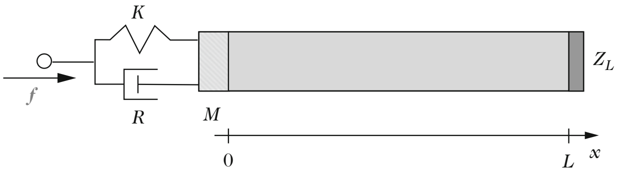
Analysis in the time domain

- Notice that $p(t)$ is zero for t less than the wave-return transit time τ along the cylindrical part of the bore.
- Moreover, $p(t)$ is smaller than $G(t)$ since returning waves are absorbed by the matching input termination.
- The term $p(t)$ can therefore be neglected in the integral, which makes its computation feasible.

Coupling between pipe and mechanical oscillator

Coupling between pipe and mechanical oscillator ⋆

- **Goal:** illustrate the influence of the coupling between an air cavity and an elastic structure.
- This situation is encountered in almost all stringed instruments and can also be found in a significant number of percussion instruments, as the membranophones (drums and timpani).
- Simplification of the problem: the structure is modeled here by a single-degree-of freedom oscillator, while the cavity is assumed to be one-dimensional. The coupling between a plate and a volume of air will be analyzed in the second module of the course (guitar).
- The system is composed by a pipe with cross-section S and length L excited at one end by a mechanical oscillator of mass M, angular frequency ω_0, and reduced damping ζ_0. The term $f(t)$ is the excitation force of the oscillator.



Coupling between pipe and mechanical oscillator ⋆

- It is further assumed that the pipe is closed at the other end by a mechanical impedance Z_L, defined as $P(L, j\omega) = Z_L(j\omega)V(L, j\omega)$, where $P(L, j\omega)$ and $V(L, j\omega)$ are the Fourier transforms of pressure and velocity, respectively.
- The equations of the systems are

$$\begin{cases} \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = \frac{\partial^2 p}{\partial x^2} \\ \rho\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} \\ M\left(\ddot{\xi} + 2\zeta_0\omega_0\dot{\xi} + \omega_0^2\xi\right) = -Sp(x=0,t) + f(t) \\ u(0,t) = \dot{\xi}(t) \end{cases} \tag{5}$$

- We will now examine the influence of the coupling on the eigenfrequencies of the system

Coupling between pipe and mechanical oscillator $\star$

- The continuity of the displacement at $x = 0$ implies that $u(0,t) = \dot{\xi}(t)$. The formula for the transfer impedance leads to the expression for the input impedance of the pipe acting on the oscillator

$$P(x = 0, j\omega) = \rho c u(0,t) z(j\omega) = j\omega\rho c \Xi(j\omega) z(j\omega), \tag{6}$$

where $z(j\omega) = \dfrac{\tanh \frac{j\omega L}{c} + z_L}{1 + z_L \tanh \frac{j\omega L}{c}}$ and $z_L = Z_L/\rho c S$ and Ξ is the Fourier transform of the displacement ξ.

- Using the third equation of the system we can write the equation of the motion of the oscillator

$$\left[-\omega^2 + 2j\omega\zeta_0\omega_0 + \omega_0^2\right]\Xi(j\omega) = \frac{F(j\omega)}{M} - j\omega\frac{\rho c S}{M}\Xi(j\omega) z(j\omega),$$

which leads to

$$\Xi(j\omega) = \frac{F(j\omega)}{M}\frac{1}{-\omega^2 + 2j\omega\omega_0\left[\zeta_0 + \zeta_a z(j\omega)\right] + \omega_0^2}, \tag{7}$$

with $2\zeta_a\omega_0 = \dfrac{\rho c S}{M}$.

Coupling between pipe and mechanical oscillator ⋆

Some considerations on particular cases:

1. If the loading of the pipe at position $x = L$ is $Z_L = \rho c S$, there is no reflection, and $z(j\omega) = 1$. This also corresponds to the case of an infinite pipe. From the expression of $\Xi(j\omega)$, notice that acoustic propagation in the pipe has the effect of increasing the damping of the oscillator. It is the acoustic radiation that causes this effect.

2. If the pipe is closed at $x = L$, then Z_L tends to infinity and $z(j\omega) = 1/\tanh j\omega L/c$. In addition, if the length L of the pipe is assumed to be small enough so that the approximation $\tanh j\omega L/c \approxeq j\omega L/c$ can be made, then the displacement of the oscillator becomes

$$\Xi(j\omega) = \frac{F(j\omega)}{M} \frac{1}{-\omega^2 + 2j\omega\omega_0\zeta_0 + \omega_0^2 + 2\omega_0\zeta_a c/L}. \tag{8}$$

   Here the pipe can be considered as a lumped element system and it acts as an added stiffness $K_a = \dfrac{2\omega_0 M\zeta_a c}{L} = \dfrac{\rho c^2 S^2}{\mathcal{V}^2}$, where $\mathcal{V}$ is the volume of air enclosed in the pipe. This added stiffness has effect only if it is comparable or greater than the stiffness $M\omega_0^2$ of the oscillator.

Coupling between pipe and mechanical oscillator $\star$

3. For a pipe open at $x = L$, and ignoring the open end radiation, we can approximate $Z_L = 0$, which yields $z(j\omega) = \tanh(j\omega L/c)$.
   If the pipe is short enough and we limit to low frequencies, we can write $\tanh(j\omega L/c) \approx j\omega L/c$, to obtain

$$\Xi(j\omega) = \frac{F(j\omega)}{M} \frac{1}{-\omega^2 + 2j\omega\omega_0\zeta_0 + \omega_0^2 - 2\omega^2\omega_0^2\zeta_a L/c}.$$

   In this case, the pipe will act as an added mass

$$M_a = \frac{2M\omega_0\zeta_a L}{c} = \rho\mathcal{V}.$$