Musical Acoustics

Second Lab Homework

Report

Helmotz Resonators Tree

Students

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Introduction

Ex 1) Model of the response of a single Helmotz Resonator

In this section we aim to model a given Helmholtz resonator response through the electrical analogies of an acoustic system. The resonator has the following dimensions:

- air volume $V_0 = 0.1\ m^3$
- neck length $l = 10\ cm$
- neck surface $S = 100\ m^2$

The standard air conditions are considered. The physical quantities needed are the sound speed $c = 343\ m/s$ and the air density $\rho = 1.2\ kg/m^3$.

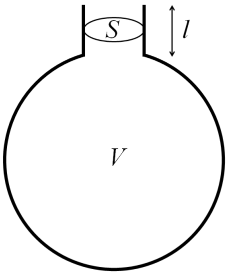


Figure 1: Helmholtz resonator

In this model are taken into account also the effects of the virtual elongation of the neck, which are different for each end of the pipe since one opening is baffled and the other one is unbaffled. So the total length of the neck is from now on considered as:

$$l_{corr} = l + \left(\frac{8}{3\pi} + 0.6\right)\sqrt{\frac{S}{\pi}} \tag{1}$$

For semplicity we will use the notation l instead of l_{corr}.

a) Simscape simulation and frequency response

The electrical modeling of the system is based on the relations between the physical quantities. The two main variables we're looking for are the pressure, represented by voltage, and the air volume flow, represented by the current. Elements which share the same pressure are posed in parallel, while, if the volume flow is shared they're posed in series. Acoustical elements are represented by electrical components that respect their impedance behaviour: air cavities can be modeled with *capacitors* and open pipes with *inductors*. The resistance in this system is given by the wall losses of the pipe that composes the neck, and it is assumed frequency-independent. In order to compute the values needed for the simulations we need to understand the behaviour of the elements' impedances:

$$Z_A^{pipe} = j\omega\frac{\rho l}{S} = j\omega L \qquad L = \frac{\rho l}{S} \tag{2}$$

$$Z_A^{cav} = \frac{1}{j\omega}\frac{\rho c^2}{V} = \frac{1}{j\omega C} \qquad C = \frac{V}{\rho c^2} \tag{3}$$

$$R_A^{pipe} = \frac{\rho c}{S} \tag{4}$$

Since in the system there's only one mass element we'll obtain a single loop in the circuit composed by an RLC series, supplied by a controlled voltage generator. The scheme is visible in figure 2.

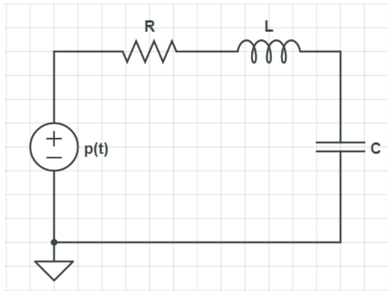


Figure 2: Electrical model of the resonator

Once the topography of model has been obtained, we proceeded with the simulation setup in *Simulink*, through the tools provided by the *Electrical Modules of Simscape*. The aim of this phase is to obtain the mobility of the system, also known as the admittance, defined as:

$$H(\omega) = \frac{u(\omega)}{p(\omega)} \tag{5}$$

where $u(\omega)$ is the volume flow velocity and $p(\omega)$ is the pressure applied as input. As said before, in the model, the pressure is represented by the voltage and volume flow by current, so we're looking for the ratio:

$$H(\omega) = \frac{I(\omega)}{V(\omega)}$$

The electrical scheme in *simulink* has been set up as shown in figure 3.
Since we want to obtain the *transfer function* we simply compute the impulsive response of the system and then we apply the fourier transform.

$$H(\omega) = \mathcal{F}\left[h(t)\right]$$

The impulse response is easily achievable supplying the system with an impulsive voltage input and reading the output with an amperometer. The impulse is provided in simulink simply by the subraction of two step functions delayed by one single sample. Since the circuit is composed by a single loop, Kirchhoff's laws allow us to assume that in each component the same current value can be measured.
For this reason the point of measurement can be placed in any position of the circuit.

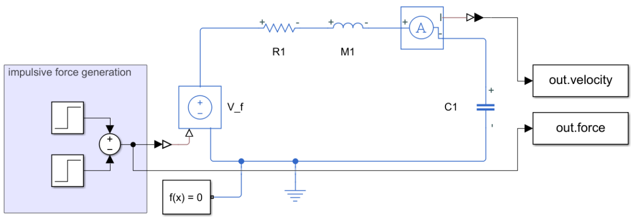


Figure 3: Simulink scheme

Both the input and the output are sent from simulink to MATLAB workspace in order to manipulate the data. The frequency response of the resonator has been computed as:

$$H(\omega) = \mathcal{F}\left[\frac{u(t)}{p(t)}\right] \tag{6}$$

and the result is visible in figure 4.

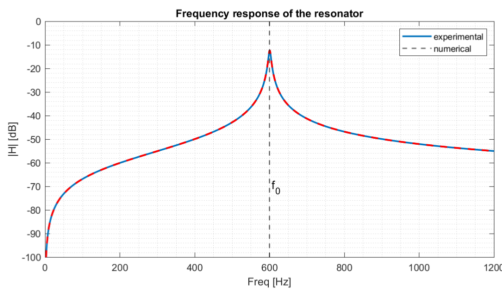


Figure 4: Frequency response of the resonator

The curve obtained via the simulation (blue) perfectly overlaps the numerical curve (red) computed as follow:

$$H^{(num)}(\omega) = Z(\omega)^{-1} = \left[R + j\left(\omega m - \frac{k}{\omega}\right)\right]^{-1} \tag{7}$$

where $k = 1/C$ is the stiffness of the air volume and $m = L$ is the mass of the air contained in the neck.
The peak of the experimentala curve, called resonance, is found at $f_0 = 600\ Hz$.

b) Natural frequency verification

The numerical natural frequency can be easily retrieved through the relation:

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{\rho c^2 S}{V\rho l}} = c\sqrt{\frac{S}{lV}} \tag{8}$$

so applying the angular frequency-to-frequency relation:

$$f_0^{(num)} = \frac{c}{2\pi}\sqrt{\frac{S}{lV}} = 600.14\ Hz \tag{9}$$

Ex 2) Combined resonators

a) Helmotz resonators tree

Using the basic RLC circuit for the Helmotz resonator built in the previous exercise, a tree is now built with multiple Helmotz resonators. The parameters for each type of component are assumed to be the same along the whole tree. In order to compute the response of the system we consider the ratio between the *volume acoustic flow u*, which takes the role of the *electric current*, and the *input pressure which* represents *electric voltage*. This ratio can be called *acoustic mobility or admittance*. In the specific, the measurement points for the voltage and the current are on the input of the root of the tree. In order to calculate the *mobility* of the system we can recall that it is the inverse of the *impedance* of the analog circuit:

$$H_0 = H_{in}(\omega) = \frac{u_{in}(\omega)}{p_{in}(\omega)} = \frac{I_{in}(\omega)}{V_{in}(\omega)} = Z_{in}^{-1}(\omega) \tag{10}$$

The first tree implemented is a full binary tree with height 2, using the notation suggested by the reference we call it a *2×2 tree*, its topology is represented in figure 5. The analog components circuit is represented in figure 6, through *simulink* components representation. The subsystem which represent the single Helmotz resonator circuit is represented in figure 7.
The impedance between the pressure applied on the input neck of the root and the volume acoustic flow measured on the same point can be analitically computed as:

$$Z_0(\omega) = j\omega L + R + \frac{1}{j\omega C + \sum_{n=1}^{N}\frac{1}{Z_{1,n}}} \tag{11}$$

Defining $Z_{1,n}$ as the impedance of the n-th child of the node of height *0*. In the specified case all the nodes of height 1 are leaves and their impedance can be calculated as:

$$Z_1(\omega) = \frac{(j\omega)^2 LC + j\omega RC + 1}{j\omega C} \tag{12}$$

Recalling equation 10 we can find mobility H_0 from impedance. The mobility on the input H_0 is also computed through simulink simulation, moreover the FRF H_1 is computed between the *volume acoustic flow* on the input of one of the children and the pressure imposed on the input of the root. The results are plotted in figure 8. It is noticeable that the analytical response $H_{0numerical}$ and the simulated response H_0 assume the same values along the frequency range considered.
In order to compute the analytical value of the impedance on the input of a generic tree we can exploit the generalized iterative formula for a Helmotz tree of height K:

$$Z_K(\omega) = \frac{-\omega^2 LC + j\omega RC + 1}{j\omega C + \sum_{n=1}^{N}\frac{1}{Z_{K-1,n}}} \tag{13}$$

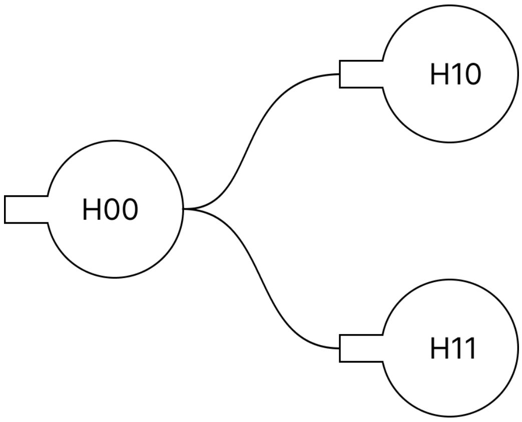


Figure 5: Helmotz Tree 2x2 Full

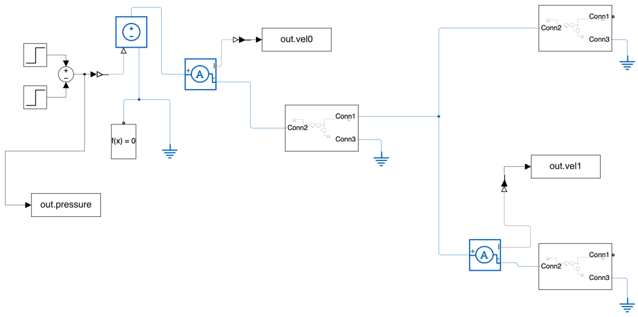


Figure 6: Helmotz Tree 2x2 Full circuit analog

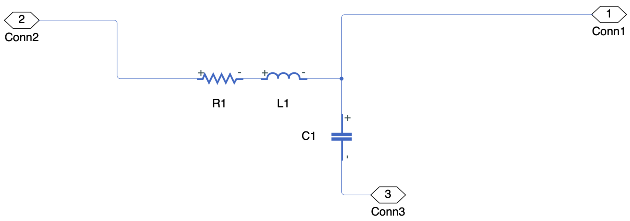


Figure 7: Helmotz Resonator circuit analog used in tree

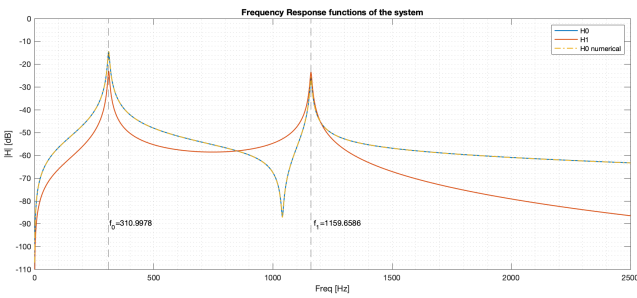


Figure 8: Helmotz Tree Frequency response functions

Where $Z_{K-1,n}$ is the n-th child of the node on which the impedance is calculated. In this notation the root has impedance Z_K and the farthest leaf from the root has impedance Z_1.

b) Different K N combinations

In this section different combinations of *K: height of the whole tree* and *N: number of children-per-parent* are presented and evaluated. From now on, the trees will be defined with the *KxN* notation, adding the term *full* if all the leaves are positioned at the same height, which results to be the height of the tree. Moreover in general it is assumed that all the nodes, other than the leaves, have N children.
For every tree are shown three figures which characterize it:

- **Tree scheme**: containing specified topology.
- **Tree circuit**: where each analog circuit is represented. In all the trees the single Helmotz resonator is represented through a subsystem, which is the same for all the systems considered. Its specific topology is represented in figure 7. In the circuit the generator and the measuring points represented through ammeters are also shown.
- **Mobility FRF**: for each tree the plot of multiple FRFs is plotted. Each function represents the ratio between the current measured on some relevant nodes for the tree and the voltage measured on the input of the root. The number and position of the measuring points chosen allows to have the best representation of the response of the system avoiding redundances caused by symmetry effects.

Tree 2x3

The first example of expansion of the simple tree presented in the previous section is a tree of height 2 where the root has 3 children. The topology of the tree is represented in figure 9. The topology chosen yields to the circuit represented in figure 10. The FRFs measured on the input of the Helmotz resonators on each layer is represented in figure 11, the FRFs represented show how the addition of a leaf doesn't change the number of resonances but only contributes by shifting the lower resonance down and the upper resonance up.

Full Tree 3x2

A full 3x2 tree is now considered, the topology is represented in figure 12, the resulting circuit is represented in figure 13 and the Frequency response functions measured on each layer of the tree are presented in figure 14. The addition of a layer to the tree simulated previously yields to 4 resonances for the whole system. We can notice how, considering the FRF of the 2x2 Tree in figure 8 with respect to the one in figure 14, we can't assume the resonances as a series of second order filters, as a matter of fact there are no resonance frequencies repeated between the two graphs, although the basic components are the same. This is because the Helmotz resonators influence one another and therefore the addition of new nodes in the tree needs a new study of the whole system considered.

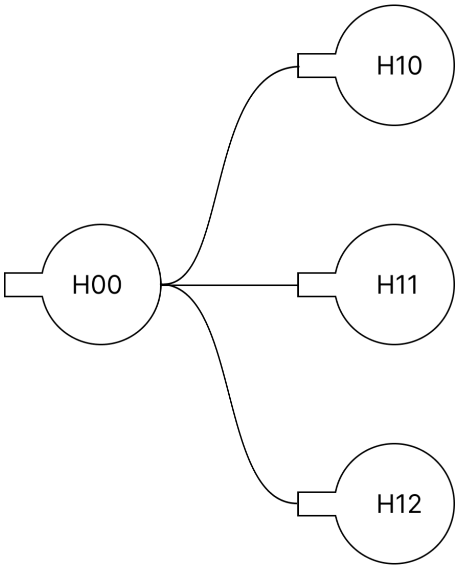


Figure 9: Full Tree 2x3

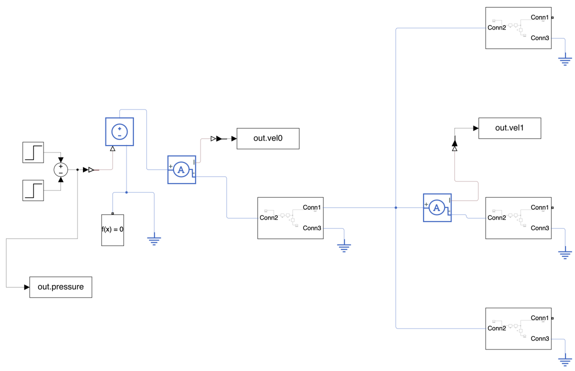


Figure 10: Full Tree 2x3 Circuit

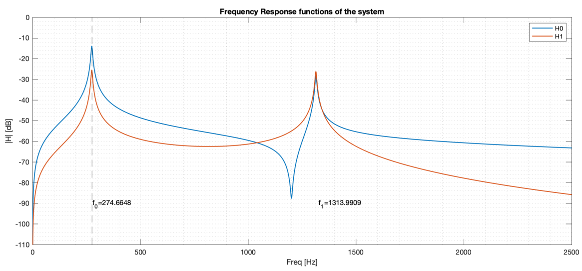


Figure 11: Full Tree 2x3 FRF

Tree 3x3

Considering now a 3x3 tree with topology specified in figure 15. The corresponding circuit is represented in figure 16. The tree is 3 levels high and only one node on level 1 has three children, which are three leaves. This configuration produces 4 resonance frequencies, as noticeable in figure 17. Therefore two more resonances are produced with respect to the basis configuration considered in figure 9. This behaviour does not allow to simply deduce the number of resonance frequencies from the numbers height of the tree K or the number of children per node N. On the contrary, as a cause of

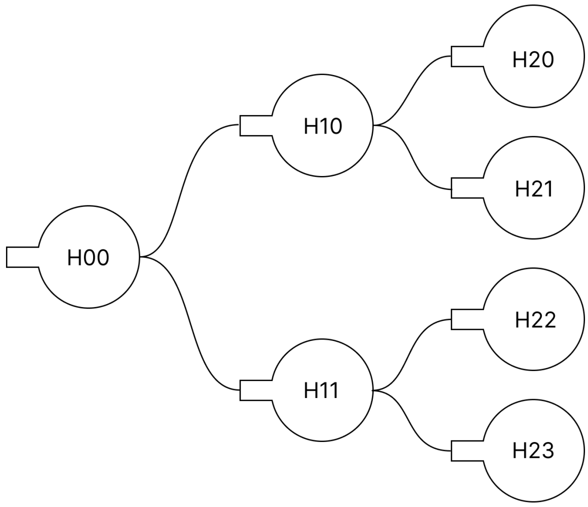


Figure 12: Full Tree 3x2

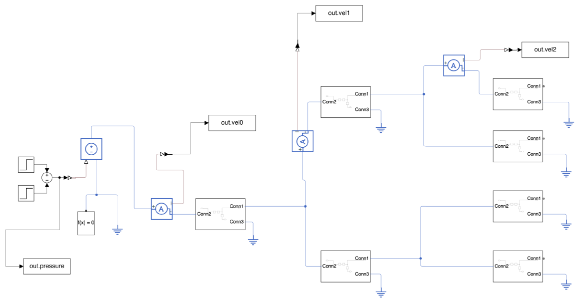


Figure 13: Full Tree 3x2 Circuit

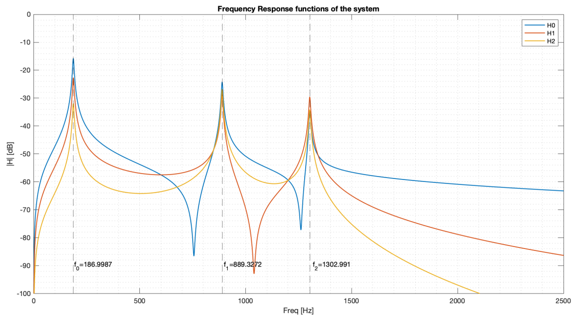


Figure 14: Full Tree 3x2 FRF

the particular topology of the circuit generated, it is necessary to evaluate the particular structure of each tree to give at least an evaluation by inspection of the number of resonant frequencies for each tree.

Inspection evaluation of the number of resonant frequencies

In order to evaluate the number of resonant frequencies by inspection, a practical rule has been found for analyzing them from the tree topology: starting from the root, for each level of height, two nodes are grouped together if they have subtrees with equal topology. On the level under evaluation, for each group composed by nodes with equal subtrees or single nodes which can't

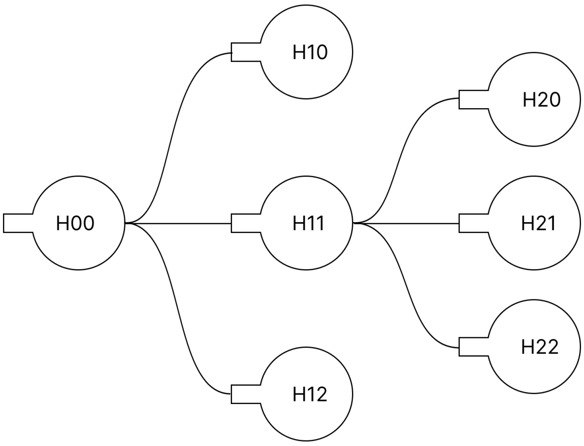


Figure 15: 3x3 Tree

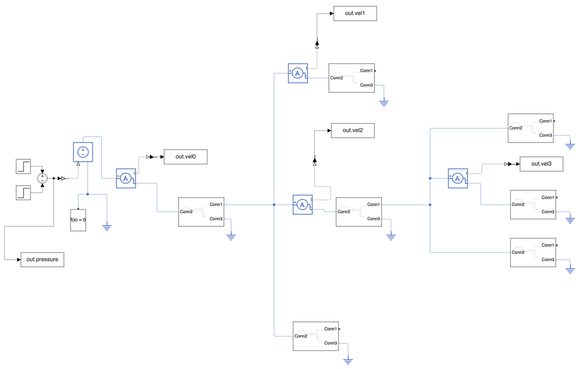


Figure 16: 3x3 Tree Circuit

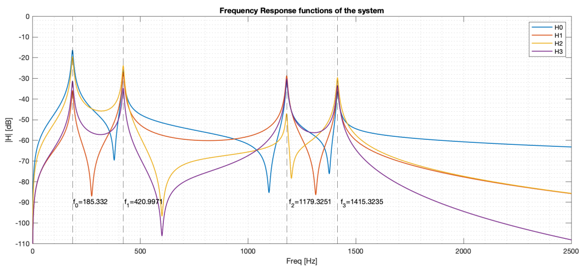


Figure 17: 3x3 Tree FRF

be grouped, a new resonant frequency is added. Then the evaluation is applied on the nodes of the next level and the process is reiterated separating the subtrees of the nodes which can't be grouped together. In the next section this rule is applied on an example in order to understand the procedure.

Tree 4x2

The tree considered has height 4 and each node which is not a leaf has 2 children and the tree topology is depicted in figure 18. The resulting circuit is represented in figure 19. As we can notice in figure 20 the FRF of the mobility presents seven resonances, the pattern for the resonances can be noticed in all the plots that compose the graph.
The method, discussed before, to count them by inspec-

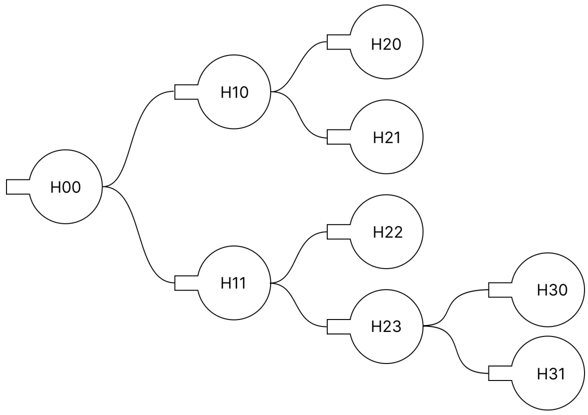


Figure 18: 4x2 Tree

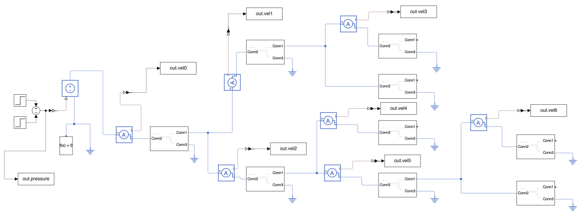


Figure 19: 4x2 Tree Circuit

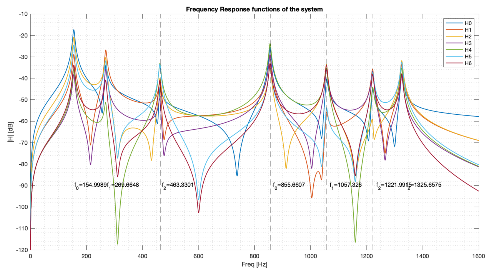


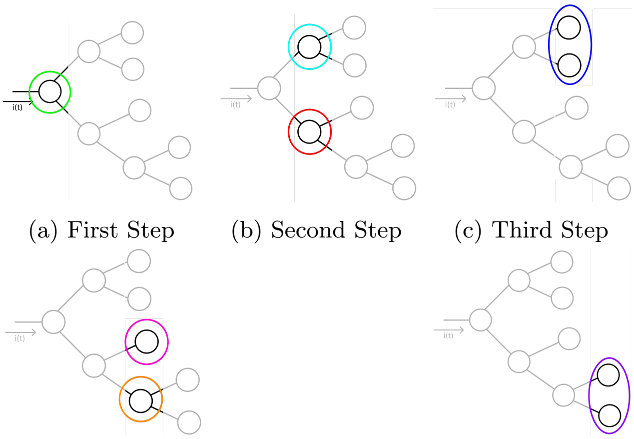
Figure 20: 4x2 Tree FRF

tion is now presented:

- Formally the root is first evaluated and a resonance is added. Figure 21a.
- The nodes on the second level of the height *H10, H11* are then evaluated, given that the relative subtrees are different they can't be grouped and represent a resonance each. The respective children are then evaluated separately. Figure 21b.
- Considering now the nodes H20 and H21. Children of the node H10, we can notice how they are roots of two null subtrees, therefore they can be grouped and represent a single resonance. Figure 21c.
- Considering now the nodes H22 and H23. Children of the node H11, they have different subtrees, therefore they can't be grouped and represent two resonances. Figure 21d.
- Considering now the nodes H30 and H31. Children of the node H23, they have the same null subtree

and can be grouped, we add a single resonance. Figure 21e.

Therefore we find a total of 7 resonances, as represented by the figure 20. This iter has been tested on all the trees considered in the report and on other trees with different topologies but which share the same rules considering K and N.



(a) First Step (b) Second Step (c) Third Step

(d) Fourth Step (e) Fifth Step

Figure 21: Steps to evaluate the number of resonances of the tree

c) Evaluation position

It can be seen that if we consider different points of measure we obtain different curves of the frequency response. This is due to the structure of the circuit. In fact the first property visible is that, despite the measure point position all the curves will have the same number of resonances, given by the number of poles of the admittance. As a matter of fact, this number is position-independent because it is a fixed property given by the topology of the circuit. What changes, instead is the number and the central frequency of the antiresonances. They're in fact given by the number of zeros of the circuit, which is a local property of the circuit. After some computations and simulations it has been observed that we can determine by inspection the number of antiresonances only in complete trees. This is because the circuits created by complete trees are symmetric and each node contributes equally in the complessive load effect. As a general rule of thumb the number of antiresonances in a complete tree of height K measured at the level h can be computed as:

$$n_h = K - h - 1 \tag{14}$$

As we can see, in a complete tree, n_h is *N-independent*, and a result of this is visible in the FRF of the 3×2 *complete tree* in figure 14. At the leaves level $h = 2$ we have $n_h = 3 - 2 - 1 = 0$ resonance, at level $h = 1$ we have $n_h = 1$ and at the root $h = 0$ we have $n_h = 2$. Another way to compute n_h can be found using a bottom-up approach and fixing $n_h = 0$ at the leaves level and adding $+1$ each time one moves up to the upper level. Nevertheless, these inspective approaches are not applicable to uncomplete trees because for each branch the load effects are different for each topology. So the easiest way to determine the number of the antiresonances is simulating the circuit with simulink. Another behaviour has been observed: the number n_h obeserved at the root is always equal to the number of resonances minus one, in order to maintain the system stable.