NUMERICAL MODELING AND SIMULATION FOR ACOUSTICS

Homework Number 1: *Webster's Equation in a longitudinal domain.*

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1 Weak Formulation of the Webster's Equation

1.1 Problem definition

It is intended to solve the following Cauchy problem:

$$\begin{cases} S(x)\psi_{tt}(x,t) - \gamma^2\left[S(x)\psi_x(x,t)\right]_x = f(x,t), & (x,t) \in (0,1)\times(0,T] \\ \psi(x,0) = u_0(x), & x \in (0,1) \\ \psi_t(x,0) = v_0(x), & x \in (0,1) \\ \psi(0,t) = g(t), & t \in (0,T] \\ \psi_x(1,t) = 0, & t \in (0,T] \end{cases} \tag{1.1}$$

where f, u_0, v_0, g are given regular functions and $\gamma = c/L$ is a positive constant, with c representing the wave velocity inside the medium and L the length of the spatial domain.

The one-dimensional time-harmonic equation presented in Equation (1.1) is known as Webster's equation or Webster's horn equation, and is used to model the vibrations of bars or the propagation of acoustic waves in tubes with profile $S(x)$ that are filled with perfect fluids. Moreover, the Webster's equation may be a relevant approximation of two- or three-dimensional physical models, e.g. to describe the acoustic field in axisymmetric waveguides such as the ones presented in Figure 1. As a matter of fact, in these contexts, the solution ψ is also known as acoustic potential, from which it is possible to derive the pressure and velocity field that characterise the acoustic features of the system.

In this paper, it is is our goal to solve Webster's equation with finite element method (FEM) by considering different boundary conditions.

1.2 Weak formulation of the problem

The classic weak formulation of a problem is obtained by multiplying the differential equation subject of interest by a test function v, integrating over the space domain Ω of the solution and integrating by parts. In this way, the aim of the problem is to look for a solution u belonging to a *trial space* V (or space of kinematically admissible displacements) that satisfies the resulting equality for any test function v belonging to a *test space* V_0. In our case - or more generally when a non-homogeneous displacement boundary condition is required - the *trial space* differs from the *test space* and so it is convenient to come back to the comfortable situation where $V = V_0$ using a lifting function.

In our case, then, we consider a lifting function $R_g(x,t)$ such that the new trial function $w = \psi - R_g$ solves the following Cauchy problem:

$$\begin{cases} S(x)w_{tt}(x,t) - \gamma^2\left[S(x)w_x(x,t)\right]_x = f(x,t) - S(x)R_{g,tt}(x,t) + \gamma^2\left[S(x)R_{g,x}(x,t)\right]_x, & (x,t) \in (0,1)\times(0,T] \\ w(x,0) = w_0(x), & x \in (0,1) \\ w_t(x,0) = \dot{w}_0(x), & x \in (0,1) \\ w(0,t) = 0, & t \in (0,T] \\ w_x(1,t) = 0, & t \in (0,T] \end{cases} \tag{1.2}$$

where the conditions $w(0,t) \equiv \psi(0,t) - R_g(0,t) = 0$ and $w_x(1,t) \equiv \psi_x(1,t) - R_{g,x}(1,t) = 0$ impose that the lifting function must be such that

$$\begin{cases} R_g(0,t) = g(t), & t \in (0,T] \\ R_{g,x}(1,t) = 0, & t \in (0,T] \end{cases} \tag{1.3}$$

The Sobolev space in which the solution w is searched is defined as follows:

$$\boxed{V = H^1_\star(0,1) = \{v \in (0,1) \subseteq \mathbb{R} \to \mathbb{R},\ v, v_x \in L^2(0,1),\ v(0) = 0\}} \tag{1.4}$$

where $L^2(0,1)$ is the Lebesgue function space with norm $\|v\|^2_{L^2(0,1)} = \int_0^1 v^2(x)dx,\ \forall v \in L^2(0,1)$.

In this way, the norm of $H^1_\star(0,1)$ can be formulated as:

$$\|v\|^2_{H^1_\star(0,1)} = \|v\|^2_{L^2(0,1)} + \|v_x\|^2_{L^2(0,1)},\ \forall v \in H^1_\star(0,1) \tag{1.5}$$

With these considerations, the weak formulation (WF) can be stated:

(WF.1)

Find $w \in H^1_\star(0,1)$ such that $\forall t \in (0,T]$

$$\int_0^1 S w_{tt} v dx - \int_0^1 \gamma^2 (S w_x)_x v dx = \int_0^1 f v dx - \int_0^1 S R_{g,tt} v dx + \int_0^1 \gamma^2 (S R_{g,x})_x v dx \quad \forall v \in H^1_\star(0,1)$$

and such that $\begin{cases} w(x,0) = w_0(x), & x \in (0,1) \\ w_t(x,0) = \dot{w}_0(x), & x \in (0,1) \end{cases}$

Integrating by parts the elements with a space-derivative dependency we notice that:

$$\begin{aligned} -\int_0^1 \gamma^2 (S w_x)_x v dx &= -\Big[\gamma^2 S w_x v\Big]_0^1 + \int_0^1 \gamma^2 S w_x v_x dx \\ &= -\Big[\gamma^2 S(1) \underbrace{w_x(1)}_{=0,\ \text{B.C. of Eq. (1.2)}} v(1) - \gamma^2 S(0) w_x(0) \underbrace{v(0)}_{=0,\ v \in H^1_\star(0,1)}\Big] + \int_0^1 \gamma^2 S w_x v_x dx \equiv \int_0^1 \gamma^2 S w_x v_x dx \end{aligned} \tag{1.6}$$

and that

$$\begin{aligned} \int_0^1 \gamma^2 (S R_{g,x})_x v dx &= \Big[\gamma^2 S R_{g,x} v\Big]_0^1 - \int_0^1 \gamma^2 S R_{g,x} v_x dx \\ &= \Big[\gamma^2 S(1) \underbrace{R_{g,x}(1)}_{=0,\ \text{B.C. of Eq. (1.3)}} v(1) - \gamma^2 S(0) R_{g,x}(0) \underbrace{v(0)}_{=0,\ v \in H^1_\star(0,1)}\Big] - \int_0^1 \gamma^2 S R_{g,x} v_x dx \equiv -\int_0^1 \gamma^2 S R_{g,x} v_x dx \end{aligned} \tag{1.7}$$

The two presented results allow to reformulate the weak formulation:

(WF.2)

Find $w \in H^1_\star(0,1)$ such that $\forall t \in (0,T]$

$$\int_0^1 S w_{tt} v dx + \int_0^1 \gamma^2 S w_x v_x dx = \int_0^1 f v dx - \int_0^1 S R_{g,tt} v dx - \int_0^1 \gamma^2 S R_{g,x} v_x dx \quad \forall v \in H^1_\star(0,1)$$

and such that $\begin{cases} w(x,0) = w_0(x), & x \in (0,1) \\ w_t(x,0) = \dot{w}_0(x), & x \in (0,1) \end{cases}$

Remark 1 - On the choice of the lifting function. The choice of lifting function $R_g(x,t)$ does not affect the generality of the presented result as long as it is chosen as a continuous function in $(0,1)$ $\forall t \in (0,T]$ that satisfies the boundary conditions shown in Equation (1.3). Hence, different functions can be conceived that meet such conditions. For example:

- Constant lifting function, $R_g(x,t) = g(t)$ for $(x,t) \in (0,1) \times (0,T]$;
- Cubic lifting function, $R_g(x,t) = g(t)(2x^3 - 3x^2 + 1)$ for $(x,t) \in (0,1) \times (0,T]$.

From now on, although the choice of one of these functions may simplify the maths - especially if the substitution $R_g(x,t) = g(t)$ were applied - to preserve the generality of the weak formulation we will not choose a specific lifting function. The choice will be postponed once the Finite Element formulation will be formulated.

Remark 2 - Linear and bilinear forms for the weak formulation. The weak formulation can also be stated and shrunk by means of real and continuous linear and bilinear operators defined on the *trial space* $V = H^1_\star(0,1)$.

Indeed, by defining

- Bilinear form $a : H^1_\star(0,1) \times H^1_\star(0,1) \to \mathbb{R}$ such that

$$a(u,v) = \int_0^1 \gamma^2 S u_x v_x dx \quad \forall u, v \in H^1_\star(0,1) \tag{1.8}$$

- Bilinear form $m : H^1_\star(0,1) \times H^1_\star(0,1) \to \mathbb{R}$ such that

$$m(u,v) = \int_0^1 S u v dx \quad \forall u, v \in H^1_\star(0,1) \tag{1.9}$$

- Linear form $\mathcal{F} : H^1_\star(0,1) \to \mathbb{R}$ such that

$$\mathcal{F}(u) = \int_0^1 f u dx \quad \forall u \in H^1_\star(0,1) \tag{1.10}$$

then, provided that $S \in \left\{S \in C^0(0,1) : S(x) \neq 0 \ \forall x \in [0,1]\right\}$, the weak formulation can be stated as:

(WF.3)

Find $w \in H^1_\star(0,1)$ such that $\forall t \in (0,T]$

$$m(w_{tt}, v) + a(w,v) = \mathcal{F}(v) - m(R_{g,tt}, v) - a(R_g, v) \quad \forall v \in H^1_\star(0,1)$$

and such that $\begin{cases} w(x,0) = w_0(x), & x \in (0,1) \\ w_t(x,0) = \dot{w}_0(x), & x \in (0,1) \end{cases}$

Remark 3 - On the well posedness of the weak formulation. The weak formulation (WF.3) is well posed if the bilinear form $a(\cdot,\cdot)$ defined in Equation (1.9) is coercive and continuous and the linear form $\mathcal{F}(\cdot)$ defined in Equation (1.10) is continuous (Lax-Milgram Lemma).

2 Galerkin Formulation of the Webster's Equation

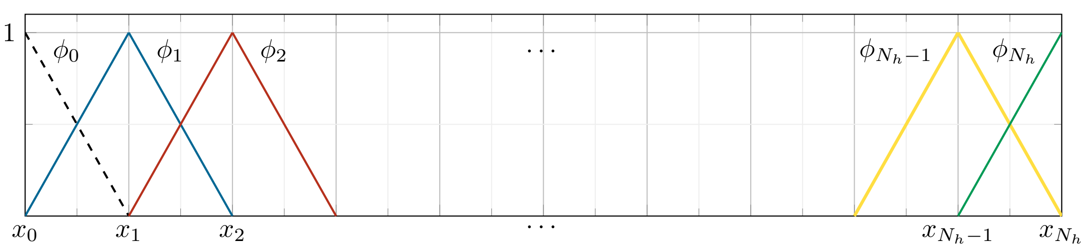
As mentioned in the introduction, the focus of the study is to solve the Webster problem (1.2) by means of the finite element method (FEM), which relies on an approximation of the *trial space* $V = H^1_\star(0,1)$ by a finite-dimension subspace $V_h \subset V$ of dimension N_h.

2.1 On the choice of the *trial* subspace

The *trial* subspace V_h we choose to consider to derive the Galerkin formulation is the set of continuous functions defined in $(0,1) \subset \mathbb{R}$ such that they are piece-wise and linear ($\mathbb{P}^1$) in each interval k_j of the partition (mesh) $\mathcal{T}_h$ of $(0,1) \subset \mathbb{R}$ with dimension N_h, i.e.

$$V_h = \mathcal{X}^1_{h,\star} = \left\{v \in C^0(0,1) : v\big|_{k_j} \in \mathbb{P}^1 \quad \forall k_j \in \mathcal{T}_h, \ v(0) = 0\right\} \tag{2.1}$$

A convenient basis of the *trial* subspace $\mathcal{X}^1_{h,\star}$ is the set $\{\phi_j(x)\}_{j=0}^{N_h}$ of "Hat" functions, which can be appreciated in the following graph.



"Hat" functions defined from the mesh $\{x_0, \ldots, x_{N_h}\}$

We want to point out that, to describe the elements of the *trial* subspace $\mathcal{X}^1_{h,\star}$, the basis function $\phi_0(x)$ may not be needed and could then be discarded. Despite this, all the N_h+1 elements of the basis are anyway considered. Hence, any $u_h \in \mathcal{X}^1_{h,\star}$ can be expressed by means of the following superposition:

$$\begin{cases} u_h(x,t) = \sum_{j=0}^{N_h} u_j(t)\phi_j(x), \quad \forall t \in (0,T] \\ u_{h,t}(x,t) = \dfrac{\partial u_h(x,t)}{\partial t} = \sum_{j=0}^{N_h} \dot{u}_j(t)\phi_j(x), \quad \forall t \in (0,T] \\ u_{h,tt}(x,t) = \dfrac{\partial^2 u_h(x,t)}{\partial t^2} = \sum_{j=0}^{N_h} \ddot{u}_j(t)\phi_j(x), \quad \forall t \in (0,T] \end{cases} \tag{2.2}$$

2.2 Derivation of the Galerkin formulation

The Galerkin formulation of the Webster's equation can be derived by reducing the formulation of (WF.3) to the *trial* subspace $\mathcal{X}^1_{h,\star}$ by considering the *trial* solution w_h, a piece-wise expression of the lifting function R_g and by considering any *test* function v_h. Hence, the Galerkin formulation (GF) can be stated:

(GF.1)

Find $w_h \in \mathcal{X}^1_{h,\star}$ such that $\forall t \in (0,T]$

$$m(w_{h,tt}, v_h) + a(w_h, v_h) = \mathcal{F}(v_h) - m(R_{g,h,tt}, v_h) - a(R_{g,h}, v_h) \quad \forall v_h \in \mathcal{X}^1_{h,\star}$$

and such that $\begin{cases} w_h(x,0) = w_0(x), & x \in (0,1) \\ w_{h,t}(x,0) = \dot{w}_0(x), & x \in (0,1) \end{cases}$

Since the Galerkin formulation (GF.1) is valid $\forall v_h \in \mathcal{X}^1_{h,\star}$, then we can express (GF.1) by choosing $v_h \equiv \phi_i$ and by considering the superposition presented in Equation (2.2) to describe w_h, $R_{g,h}$ and their time derivatives.

In this way

$$m(w_{h,tt}, v_h) = m\left(\sum_{j=0}^{N_h} \ddot{w}_j(t)\phi_j(x), \phi_i(x)\right) \underset{\substack{\uparrow \\ m(\cdot,\cdot) \text{ is bilinear}}}{\equiv} \sum_{j=0}^{N_h} \ddot{w}_j(t) \cdot \underbrace{m\big(\phi_j(x), \phi_i(x)\big)}_{m(\phi_j(x),\phi_i(x)) =: m_{i,j}} = \sum_{j=0}^{N_h} m_{i,j} \cdot \ddot{w}_j(t) \tag{2.3}$$

$$a(w_h, v_h) = a\left(\sum_{j=0}^{N_h} w_j(t)\phi_j(x), \phi_i(x)\right) \underset{\substack{\uparrow \\ a(\cdot,\cdot) \text{ is bilinear}}}{\equiv} \sum_{j=0}^{N_h} w_j(t) \cdot \underbrace{a\big(\phi_j(x), \phi_i(x)\big)}_{a(\phi_j(x),\phi_i(x)) =: a_{i,j}} = \sum_{j=0}^{N_h} a_{i,j} \cdot w_j(t) \tag{2.4}$$

$$\mathcal{F}(v_h) = \mathcal{F}(\phi_i) =: F_i(t) \tag{2.5}$$

$$m(R_{g,h,tt}, v_h) = m\left(\sum_{j=0}^{N_h} \ddot{R}_{g,j}(t)\phi_j(x), \phi_i(x)\right) \underset{\substack{\uparrow \\ m(\cdot,\cdot) \text{ is bilinear}}}{\equiv} \sum_{j=0}^{N_h} \ddot{R}_{g,j}(t) \cdot m\big(\phi_j(x), \phi_i(x)\big) \equiv \sum_{j=0}^{N_h} m_{i,j} \cdot \ddot{R}_{g,j}(t) \tag{2.6}$$

$$a(R_{g,h}, v_h) = a\left(\sum_{j=0}^{N_h} R_{g,j}(t)\phi_j(x), \phi_i(x)\right) \underset{\substack{\uparrow \\ a(\cdot,\cdot) \text{ is bilinear}}}{\equiv} \sum_{j=0}^{N_h} R_{g,j}(t) \cdot a\big(\phi_j(x), \phi_i(x)\big) \equiv \sum_{j=0}^{N_h} a_{i,j} \cdot R_{g,j}(t) \tag{2.7}$$

and so, if we introduce the $\mathbb{R}^{N_h+1}$ vectors

$$\underline{w}(t) = \{w_0(t), \dots, w_{N_h}(t)\}^T \;\Rightarrow\; \underline{\ddot{w}}(t) = \{\ddot{w}_0(t), \dots, \ddot{w}_{N_h}(t)\}^T$$

$$\underline{R}_g(t) = \{R_{g,0}(t), \dots, R_{g,N_h}(t)\}^T \;\Rightarrow\; \underline{\ddot{R}}_g(t) = \{\ddot{R}_{g,0}(t), \dots, \ddot{R}_{g,N_h}(t)\}^T$$

$$\underline{F}(t) = \{F_0(t), \dots, F_{N_h}(t)\}^T$$

and the $(N_h+1)\times(N_h+1)$ matrices

$$M = [m_{i,j}]_{i,j=0}^{N_h} \underset{\substack{\uparrow \\ \text{Eq. (1.8)}}}{=} \left[\int_0^1 S\phi_j\phi_i dx\right]_{i,j=0}^{N_h} \qquad A = [a_{i,j}]_{i,j=0}^{N_h} \underset{\substack{\uparrow \\ \text{Eq. (1.9)}}}{=} \left[\int_0^1 \gamma^2 S\frac{\partial\phi_j}{\partial x}\frac{\partial\phi_i}{\partial x} dx\right]_{i,j=0}^{N_h} \tag{2.8}$$

we can express (GF.1) as an algebraic vectorial, or finite element (FE), formulation in the following manner:

(FE.1)

Find $\underline{w}(t) \in \mathbb{R}^{N_h+1}$ such that $\forall t \in (0,T]$

$M\underline{\ddot{w}}(t) + A\underline{w}(t) = \underline{F}(t) - M\underline{\ddot{R}}_g(t) - A\underline{R}_g(t)$

and such that $\begin{cases} \underline{w}(0) = \underline{w}_0 = \{w_0(0),\dots,w_{N_h}(0)\}^T \\ \underline{\dot{w}}(0) = \underline{\dot{w}}_0 = \{\dot{w}_1(0),\dots,\dot{w}_{N_h}(0)\}^T \end{cases}$

where $\underline{w}_0$, $\underline{\dot{w}}_0$ are vectors containing the projections of the initial conditions $w_0(x)$, $\dot{w}_0(t)$ into the *trial* subspace $\mathcal{X}^1_{h,\star}$.

Remark 1 - On the tridiagonality of the mass and stiffness matrices. The mass and stiffness matrices M,A are symmetric, defined positive and tridiagonal, which means that they are characterised by nonzero elements only on the diagonal and on the slots horizontally or vertically adjacent the diagonal, i.e. along the subdiagonal and superdiagonal. This is a result of the properties of the $\mathbb{P}^1$ Hat functions and can be shown - even for non-constant profiles $S(x)$ - by means of the Reference Element Technique (RET).

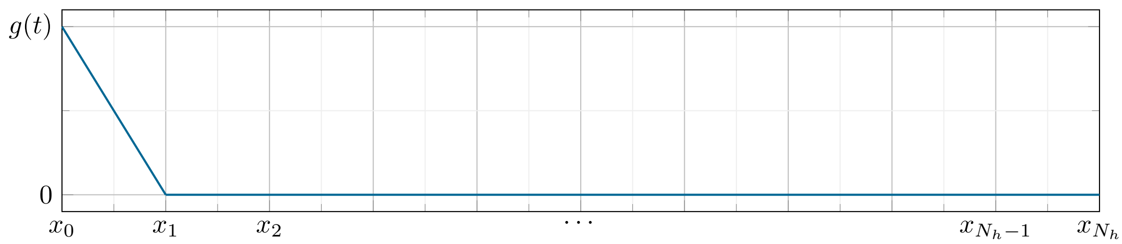
Remark 2 - On the integration and redefinition of the external force. Each element of the vector $\underline{F}(t)$ must be determined by computing the following integral:

$$F_i(t) = \mathcal{F}(\phi_i) \underset{\substack{\uparrow \\ \text{Eq. (1.10)}}}{=} \int_0^1 f(x,t)\phi_i(x)dx \equiv \int_{x_{i-1}}^{x_i} f(x,t)\phi_i(x)dx + \int_{x_i}^{x_{i+1}} f(x,t)\phi_i(x)dx \tag{2.9}$$

Since, in general, the forcing term $f(x,t)$ is unknown, the two previous integrals are usually computed with quadrature formulas, i.e. by means of numerical algorithms.

Remark 3 - On the choice of the lifting function vector. The Galerkin formulation allows us to choose a "simple" lifting function $R_g(x,t)$. Hence, we select a continuous piece-wise function such that in $\mathcal{T}_h$:

$$\underline{R}_g(t) = \{g(t), 0, \dots, 0\}^T \ \forall t \in (0,T] \tag{2.10}$$



Lifting function vector $\forall t \in (0,T]$.

Remark 4 - Finite element final formulation. Once the vector $\underline{F}(t)$ is computed for a given time instant and the vector $\underline{R}_g(t)$ associated to the lifting function is chosen, then the right-hand of (FE.1) can be treated as a single $\mathbb{R}^{N_h+1}$ vector:

$$\underline{F}_g(t) := \underline{F}(t) - M\underline{\ddot{R}}_g(t) - A\underline{R}_g(t), \quad \forall t \in (0,T] \tag{2.11}$$

Thanks to this considerations, (FE.1) can be newly stated:

(FE.2)

Find $\underline{w}(t) \in \mathbb{R}^{N_h+1}$ such that $\forall t \in (0, T]$

$M\underline{\ddot{w}}(t) + A\underline{w}(t) = \underline{F}_g(t)$

and such that $\begin{cases} \underline{w}(0) = \underline{w}_0 \\ \underline{\dot{w}}(0) = \underline{\dot{w}}_0 \end{cases}$

and compute the actual solution $\underline{\psi}(t) = \underline{w}(t) + \underline{R}_g(t)$, where

$\underline{\psi}(t) = \{g(t), w_1(t), \dots, w_{N_h}(t)\}^T \ \forall t \in (0, T]$

3 Convergence test for different section profiles

The (FE.2) represents a system of ordinary linear differential equations with time dependency, whose solution could be numerically computed by considering an arbitrary integration scheme.

3.1 Time partition and time integration scheme

In order to discretize the general integral associated to (FE.2) we consider a time partition of the time integration interval $(0, T]$ such that, for an integer $M > 0$,

$$t_k = k\Delta t,\ \forall k = 0, \dots, M \ \Rightarrow \ t_M = M\Delta t \equiv T \tag{3.1}$$

In this way, we can define the *trial* solution $\underline{w}(t)$ at time $t_k \in (0, T]$ as $\underline{w}_k = \underline{w}(t_k)$ and approximate $\underline{\ddot{w}}(t)$ in terms of $\underline{w}_k$ and $\underline{w}_{k\pm1}$ $\forall k = 1, \dots, M-1$.

For the problem under consideration, we intend to compute the solution of (FE.2) by implementing the following integration scheme, which is different for the value $k = 0$ so that the initial conditions can be properly implemented.

$$\begin{cases} \left(M + \dfrac{\Delta t^2}{2} A\right)\underline{w}_1 = \dfrac{\Delta t^2}{2}\underline{F}_{g,1} + M\underline{w}_0 + \Delta t M\underline{\dot{w}}_0, \quad k = 0 \\ \left(M + \Delta t^2 A\right)\underline{w}_{k+1} = \Delta t^2 \underline{F}_{g,k+1} + 2M\underline{w}_k - M\underline{w}_{k-1}, \quad k \geq 1 \end{cases} \tag{3.2}$$

In this way, the solution of (FE.2) can be determined $\forall t_k \in (0, T]$.

Remark - Properties of the integration scheme. The integration scheme under consideration is implicit and conditionally stable. The former term refers to the fact that, in Equation (3.2), the solution at time t_{k+1} is computed by means of contributions evaluated at the "implicit" time t_{k+1}. The latter term, instead, is a consequence of the Courant-Friedrichs-Lewy (CFL) conditions, which states that an integration scheme is conditionally stable ($\exists L > 0 : \ |\underline{w}_k| < L \ \forall k$) if and only if

$$\Delta t \leq C\frac{h}{c} \tag{3.3}$$

where $C > 0$ is a constant, c is the speed of the sound wave in the medium under consideration and h is the mesh dimension.

3.2 Verification test on a given toy problem

The solution presented in Equation (FE.2) shall be tested with a proper toy problem, so that the reliability of the integration scheme and the solution obtained with FEM can be verified by evaluating, for different choices of the discretization parameters h and Δt, the behaviour of the norm $\|\psi - \psi_{EX}\|_{L^2(0,1)}$.

The toy problem that we intend to consider is the following:

$$\begin{cases} (0, L) = (0, 1) \text{ (space domain)} \\ (0, T] = (0, 1] \text{ (time domain)} \\ c = 1 \Rightarrow \gamma = 1 \\ \psi_{EX}(x, t) = \cos(2\pi x)\sin(2\pi t),\ (x, t) \in (0, L) \times (0, T] \end{cases} \tag{3.4}$$

Furthermore, we want to implement the proposed toy problem for three different profiles $S(x)$.

$$\begin{cases} \text{Case } (a)\text{: } S(x) = 1 \\ \text{Case } (b)\text{: } S(x) = (1+2x)^2 \\ \text{Case } (c)\text{: } S(x) = 1 - \dfrac{3}{4}\sin(5\pi x) \end{cases}$$

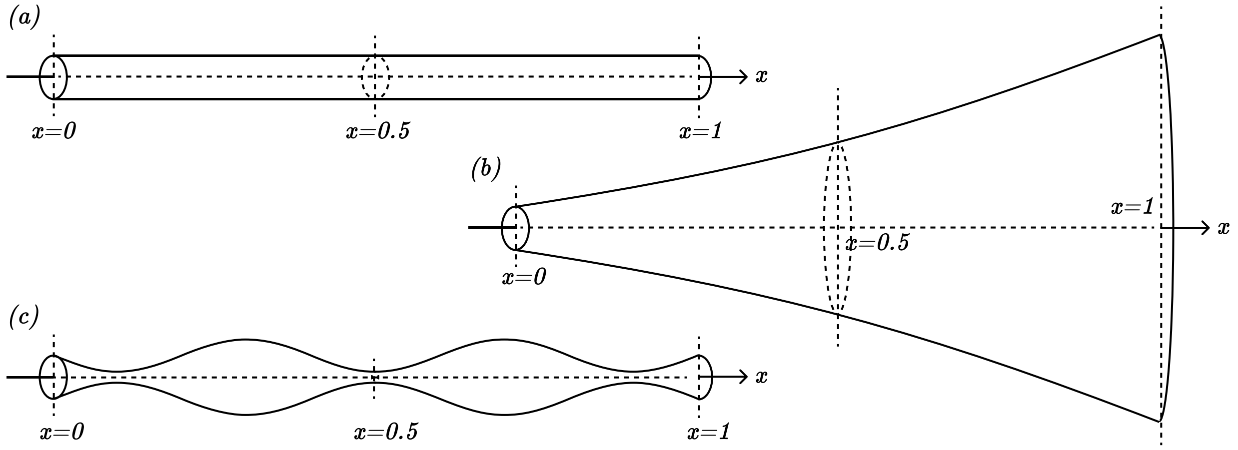


Figure 1: Section profiles that are object of study.

Remark - On the chosen toy model. From the expression of the exact solution $\psi_{EX}(x,t)$, it is straightforward to derive that

$$\psi_{t,EX}(x,t) = \frac{\partial \psi_{EX}(x,t)}{\partial t} = -2\pi \sin(2\pi x)\sin(2\pi t) \tag{3.5}$$

and so that, independently of the choice of $S(x)$, Equation (1.1) leads to

$$\begin{cases} u_0(x) = \psi_{EX}(x,0) = 0 \\ v_0(x) = \psi_{t,EX}(x,0) = 2\pi\cos(2\pi x) \\ g(t) = \psi_{EX}(0,t) = \sin(2\pi t) \end{cases} \tag{3.6}$$

Hence, the section profile $S(x)$ influences only the forcing term $f(x,t)$.

3.2.1 Verification test on Case *(a)*

In this first case, the Webster's equation leads to

$$f(x,t) = 0 \tag{3.7}$$

The behaviour of the norms $\|\psi - \psi_{EX}\|_{L^2(0,1)}$, $\|\psi - \psi_{EX}\|_{H^1_\star(0,1)}$ is studied as a function of the mesh dimension h and for two different time integration steps Δt. In particular, the values $N_h = 2^7,\ 2^6,\ 2^5,\ 2^4$ are chosen.

In Figure 2 it is reported the behaviour of the two norms for $\Delta t = 5 \times 10^{-4}$. In Figure 3, instead, it is reported the behaviour of the two norms for $\Delta t = 10^{-4}$. The errors behave as expected, the L^2 error is not always decreasing as h decreases, this is probably due to the integration scheme that was used. This trend appears to be dependent on the time-step, while the H^1 error is not affected by it.

3.2.2 Verification test on Case *(b)*

In this second case, the Webster's equation leads to

$$f(x,t) = 8\pi(1+2x)\sin(2\pi x)\sin(2\pi t) \tag{3.8}$$

The behaviour of the norms $\|\psi - \psi_{EX}\|_{L^2(0,1)}$, $\|\psi - \psi_{EX}\|_{H^1_\star(0,1)}$ is studied as a function of the mesh dimension h and for two different time integration steps Δt. Again, the values of N_h are chosen as $N_h = 2^7,\ 2^6,\ 2^5,\ 2^4$.

In Figure 4 it is reported the behaviour of the two norms for $\Delta t = 5 \times 10^{-4}$.

In Figure 5, instead, it is reported the behaviour of the two norms for $\Delta t = 10^{-4}$. Similar considerations on the behaviour of the errors can be made.

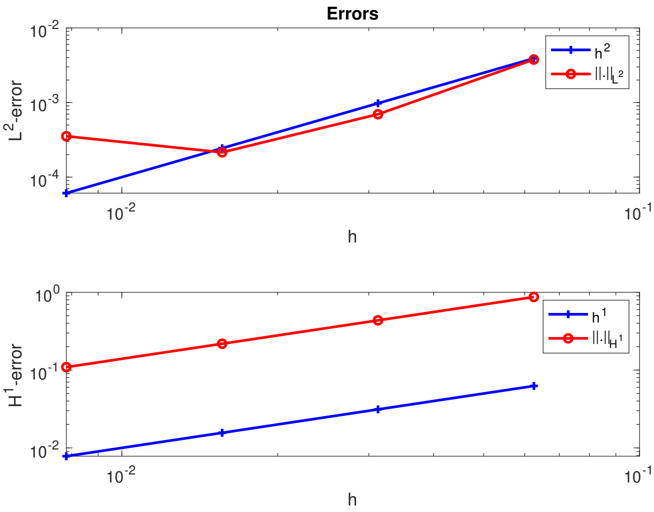


Figure 2: Behaviour of the norms $\|\psi - \psi_{EX}\|_{L^2(0,1)}$, $\|\psi - \psi_{EX}\|_{H^1_\star(0,1)}$ as a function of the mesh dimension h and for $S(x) = 1$ and $\Delta t = 5 \times 10^{-4}$.

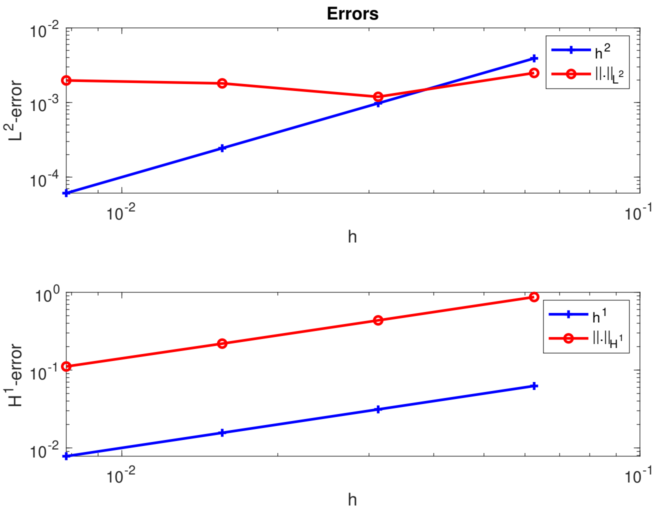


Figure 3: Behaviour of the norms $\|\psi - \psi_{EX}\|_{L^2(0,1)}$, $\|\psi - \psi_{EX}\|_{H^1_\star(0,1)}$ as a function of the mesh dimension h and for $S(x) = 1$ and $\Delta t = 10^{-4}$.

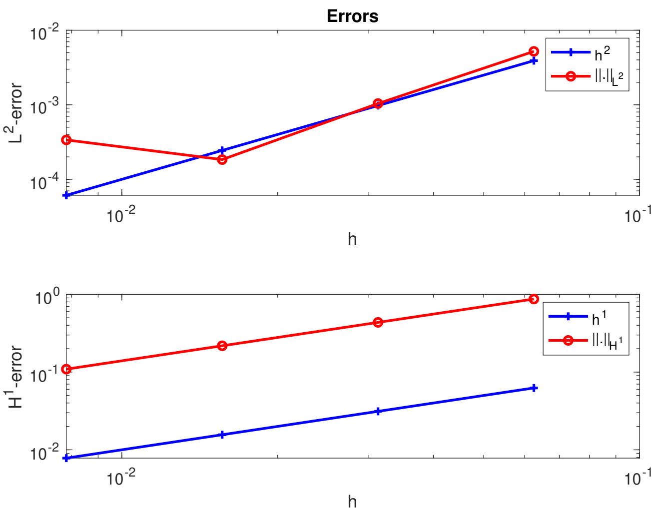


Figure 4: Behaviour of the norms $\|\psi - \psi_{EX}\|_{L^2(0,1)}$, $\|\psi - \psi_{EX}\|_{H^1_\star(0,1)}$ as a function of the mesh dimension h and for $S(x) = (1+2x)^2$ and $\Delta t = 5 \times 10^{-4}$.

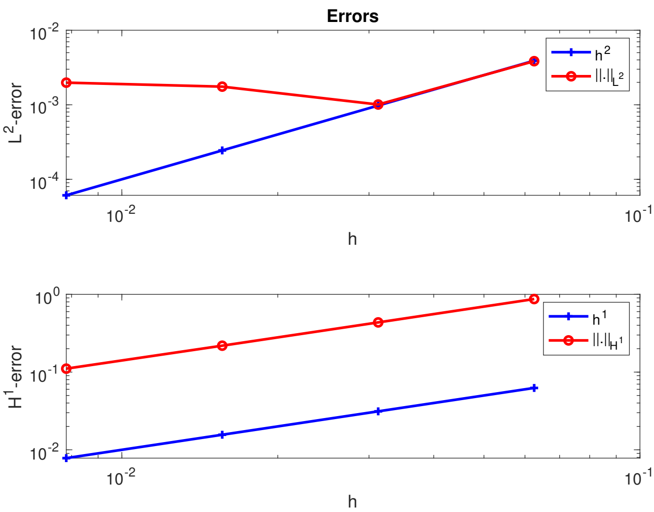


Figure 5: Behaviour of the norms $\|\psi - \psi_{EX}\|_{L^2(0,1)}$, $\|\psi - \psi_{EX}\|_{H^1_\star(0,1)}$ as a function of the mesh dimension h and for $S(x) = (1+2x)^2$ and $\Delta t = 10^{-4}$.

3.2.3 Verification test on Case *(c)*

In this last case, the Webster's equation leads to

$$f(x,t) = -\frac{15}{2}\pi^2 \cos(5\pi x)\sin(2\pi x)\sin(2\pi t) \tag{3.9}$$

The behaviour of the norms $\|\psi - \psi_{EX}\|_{L^2(0,1)}$, $\|\psi - \psi_{EX}\|_{H^1_\star(0,1)}$ is studied as a function of the mesh dimension h and for two different time integration steps Δt. Also in this case, the values $N_h = 2^7,\ 2^6,\ 2^5,\ 2^4$ are chosen.

In Figure 6 it is reported the behaviour of the two norms for $\Delta t = 5 \times 10^{-4}$. In Figure 7, instead, it is reported the behaviour of the two norms for $\Delta t = 10^{-4}$. Once again, the said considerations are valid.

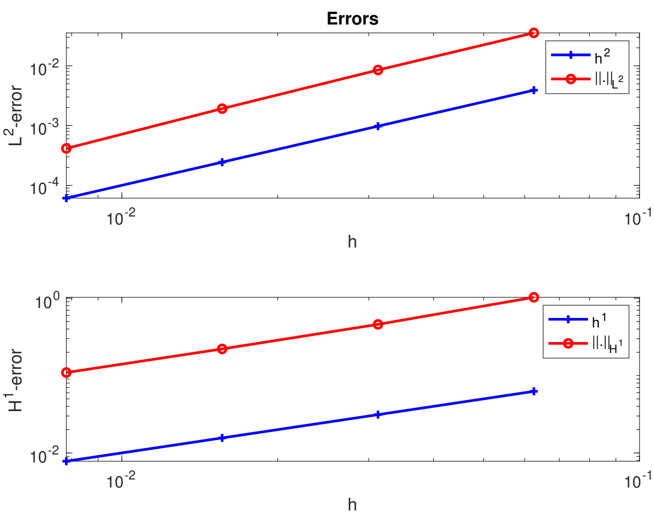


Figure 6: Behaviour of the norms $\|\psi - \psi_{EX}\|_{L^2(0,1)}$, $\|\psi - \psi_{EX}\|_{H^1_\star(0,1)}$ as a function of the mesh dimension h and for $S(x) = 1 - 3/4\sin(5\pi x)$ and $\Delta t = 5 \times 10^{-4}$.

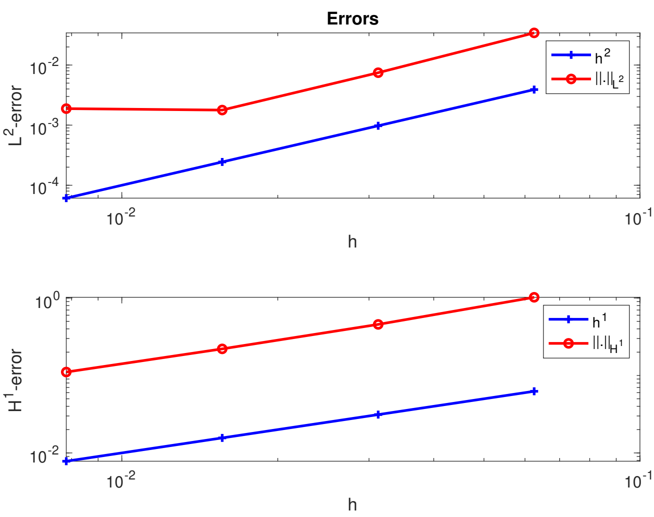


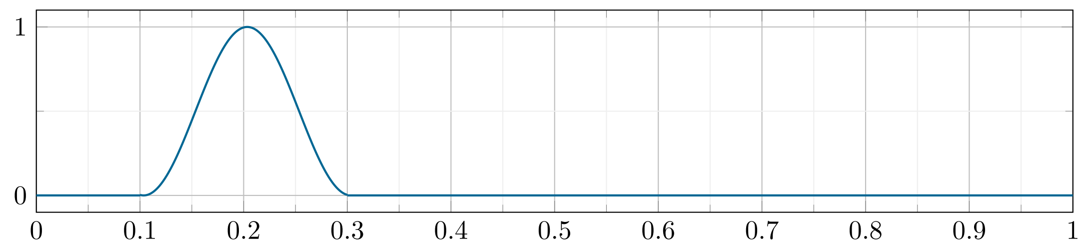
Figure 7: Behaviour of the norms $\|\psi - \psi_{EX}\|_{L^2(0,1)}$, $\|\psi - \psi_{EX}\|_{H^1_\star(0,1)}$ as a function of the mesh dimension h and for $S(x) = 1 - 3/4\sin(5\pi x)$ and $\Delta t = 10^{-4}$.

4 Webster's Equation with impulsive boundary condition

The verification test that was studied and discussed in the previous Section made it possible to demonstrate that (FE.2) formulation converges to the exact solution and can therefore be implemented to determine the solution of new problems, which is actually what we intend to do in this Section. In more detail, we intend to determine the solution associated to the following problem:

$$\begin{cases} (0,L) \times (0,T] = (0,1) \times (0,1) \\ c = 2 \Rightarrow \gamma = 2 \\ u_0(x) = 0, \;\; \forall x \in (0,1) \\ v_0(x) = 0, \;\; \forall x \in (0,1) \\ f(x,t) = 0, \;\; \forall (x,t) \in (0,L) \times (0,T) \\ g(t) = \dfrac{1}{2}\left[1 + \cos\left(\dfrac{t-0.2}{0.1}\pi\right)\right] \text{ if } |t - 0.2| \leq 0.1 \text{ (otherwise } g(t) = 0) \end{cases} \tag{4.1}$$

Hence, we are concerned to evaluate the response of the system in the case in which the leftmost extreme of the space domain is excited by a causal half-cosine pulse of unit amplitude and width 0.2.



Half-cosine pulse $g(t)$ for $t \in (0,1]$.

As in the previous Section, the solution $\psi(x,t)$ is determined for the same three different profiles $S(x)$ which were examined also previously (ref. Figure 1).

Remark - On the properties of the acoustic potential ψ. As mentioned in the introduction, the ψ function - in an acoustic context - is also known as acoustic potential as, from it, it is possible to derive, with space or time derivation, the pressure or velocity field that characterise the system's acoustics. Indeed, the pressure $p(x,t)$ and velocity field $u(x,t)$ are such that

$$\begin{cases} p(x,t) = \dfrac{1}{\gamma}\psi_t(x,t) \\ u(x,t) = -S(x)\psi_x(x,t) \end{cases} \tag{4.2}$$

4.1 Acoustic potential and pressure wave in time-space domain for section profile *(a)*

The response of the system when excited by the pulse $g(t)$ is examined, in terms of the acoustic potential and in terms of the corresponding pressure wave, in time-space domain by properly choosing mesh and integration parameters.

In particular, taking into consideration the results reported in Figures 2 and 3, it is decided to study the response of the system defined by the profile $S(x) = 1$ by selecting:

$$N_h = 2^6, \ \Delta t = 1.0 \times 10^{-4}$$

In this way, the dissipation error is limited and the solution is determined with reasonable accuracy in an affordable amount of time.

In Figure 8 and in Figure 11 is respectively reported the time-space behaviour of the acoustic potential and of the pressure wave for the profile section under consideration.

In Figure 8 it can be noticed that the half-cosine pulse propagates with constant speed. Moreover, in correspondence of the right-most edge of the domain, a reflection occurs without a phase inversion due to the Neumann boundary conditions. A similar trend is observed also in Figure 11 for the pressure.

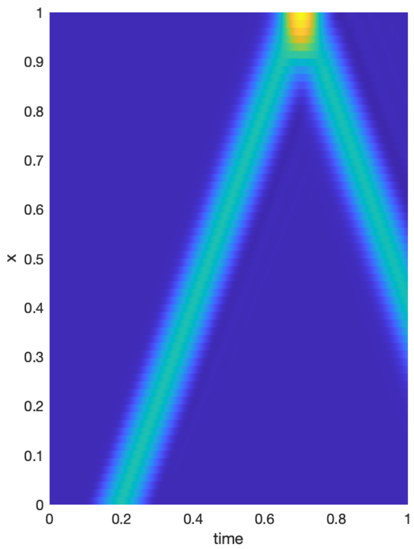


Figure 8: Time-space evolution of the acoustic potential inside the horn with profile: $S(x) = 1$.

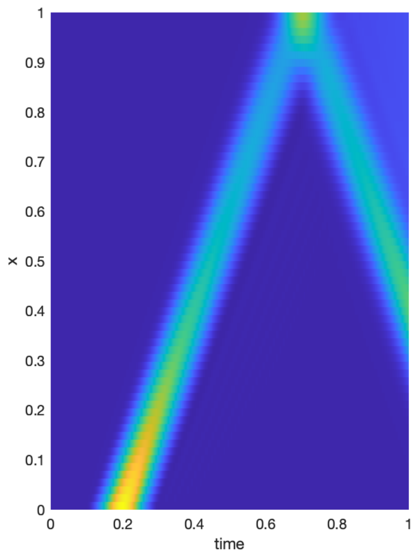


Figure 9: Time-space evolution of the acoustic potential inside the horn with profile: $S(x) = (1 + 2x)^2$.

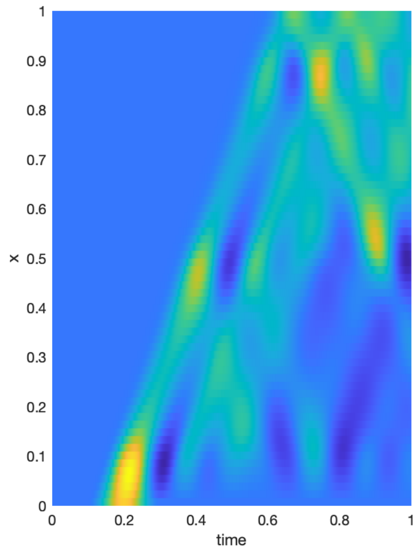


Figure 10: Time-space evolution of the acoustic potential inside the horn with profile: $S(x) = 1 - \frac{3}{4}\sin(5\pi x)$.

4.2 Acoustic potential and pressure wave in time-space domain for section profile *(b)*

The response of the system when excited by the pulse $g(t)$ is examined, in terms of the acoustic potential and in terms of the corresponding pressure wave, in time-space domain by properly choosing mesh and integration parameters.

In particular, taking into consideration the results reported in Figures 4 and 5, it is decided to study the response of the system defined by the profile $S(x) = (1 + 2x)^2$ by selecting:

$$N_h = 2^6 \text{ m}, \ \Delta t = 1.0 \times 10^{-4} \text{ s}$$

In this way, the dissipation error is limited and the solution is determined with reasonable accuracy in an affordable amount of time.

In Figure 9 and in Figure 12 is respectively reported the time-space behaviour of the acoustic potential and of the pressure wave for the profile section under consideration.

In Figure 9 it can be noticed that the half-cosine pulse propagates with constant speed despite being reduced in amplitude due to the profile shape. Moreover, in correspondence of the right-most edge of the domain, a reflection occurs without a phase inversion due to the Neumann boundary conditions. A similar trend is observed also in Figure 12 for the pressure.

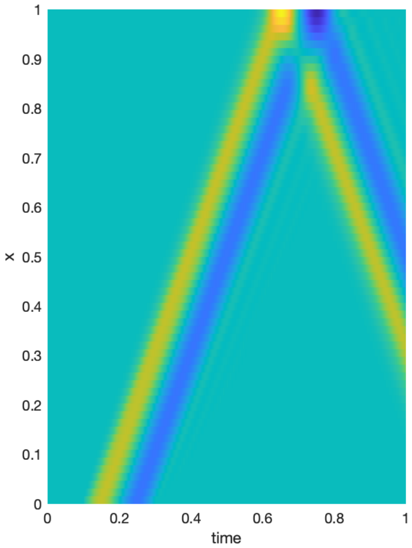


Figure 11: Time-space evolution of the pressure wave inside the horn with profile: $S(x) = 1$.

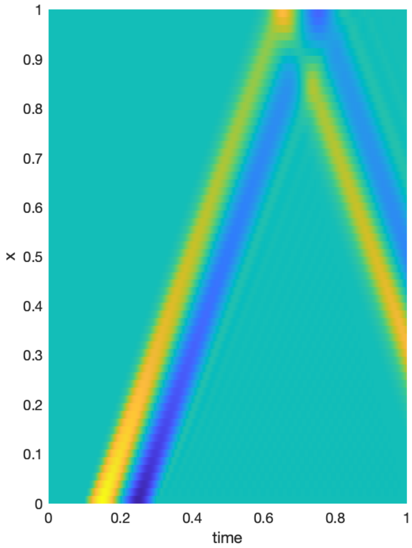


Figure 12: Time-space evolution of the pressure wave inside the horn with profile: $S(x) = (1 + 2x)^2$.

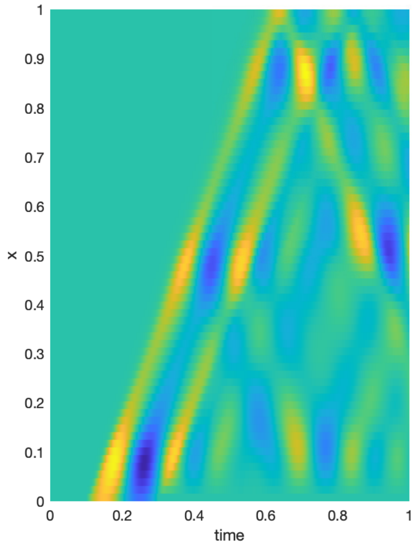


Figure 13: Time-space evolution of the pressure wave inside the horn with profile: $S(x) = 1 - \frac{3}{4}\sin(5\pi x)$.

4.3 Acoustic potential and pressure wave in time-space domain for section profile *(c)*

The response of the system when excited by the pulse $g(t)$ is examined, in terms of the acoustic potential and in terms of the corresponding pressure wave, in time-space domain by properly choosing mesh and integration parameters.

In particular, taking into consideration the results reported in Figures 6 and 7, it is decided to study the response of the system defined by the profile $S(x) = 1 - \frac{3}{4}\sin(5\pi x)$ by selecting:

$$N_h = 2^6 \text{ m}, \;\; \Delta t = 1.0 \times 10^{-4} \text{ s}$$

In this way, the dissipation error is limited and the solution is determined with reasonable accuracy in an affordable amount of time.

In Figure 10 and in Figure 13 is respectively reported the time-space behaviour of the acoustic potential and of the pressure wave for the profile section under consideration.

In Figure 10 it can be noticed that the half-cosine pulse propagates with constant speed despite being reduced in amplitude due to the profile shape and by the inner reflections in correspondence of the three throats. Moreover, in correspondence of the right-most edge of the domain, reflections occur without a phase inversion due to the Neumann boundary conditions. A similar trend is observed also in Figure 13 for the pressure.

5 Webster's Equation with Robin and absorbing boundary condition

5.1 Problem definition

In this Section we intend to solve again the Webster's equation but with new boundary conditions. Hereafter, we report the Cauchy Problem that is object of interest:

$$\begin{cases} S(x)\psi_{tt}(x,t) - \gamma^2\left[S(x)\psi_x(x,t)\right]_x = f(x,t), \;\; (x,t) \in (0,1) \times (0,T] \\ \psi(x,0) = u_0(x), \;\; x \in (0,1) \\ \psi_t(x,0) = v_0(x), \;\; x \in (0,1) \\ \psi(0,t) = g(t), \;\; t \in (0,T] \\ \psi_x(1,t) = -\alpha_1\psi_t(1,t) - \alpha_2\psi(1,t), \;\; t \in (0,T] \end{cases} \tag{5.1}$$

where f, u_0, v_0, g, S and $\gamma = c/L$ are the same functions and parameters there were defined in the previous Section, while α_1 and α_2 are two constants such that

$$\begin{cases} \alpha_1 = \dfrac{1}{2(0.8216)^2\gamma} \\ \alpha_2 = \dfrac{L}{0.8216\sqrt{S(0)S(1)/\pi}} \end{cases} \tag{5.2}$$

With the presented boundary condition, the horn end located at $x = 1$ is subject to both Robin and Absorbing boundary conditions. Indeed, the former are defined as a linear combination of the value of ψ at the boundary with the value of its space derivative ψ_x, while the latter are conceived to be a linear superposition of the flux of ψ (i.e., the time derivative ψ_t) at the boundary with the value of the space derivative ψ_x.

Hence, with the combination of this two boundary conditions, the acoustic potential ψ is related the motion of a pressure wave that, in correspondence of the end located at $x = 1$, is reflected (Robin conditions) but not completely (Absorbing conditions), since a part of it is transmitted to the outer air.

Once again, its is our goal to solve Webster's equation with finite element method (FEM) by considering the new boundary conditions.

5.2 Weak formulation of the problem

The weak formulation of the problem is obtained by multiplying the differential equation subject of interest by a test function v, integrating over the space domain Ω of the solution and integrating by parts. Hence, as presented also in Section 1.2, the aim of the problem is to look for a solution u which belongs to a *trial space* V that satisfies the resulting equality for any test function v belonging to a *test space* V_0. In our case, again, the *trial space* differs from the *test space* and so it is convenient to consider a lifting function to come back to the comfortable situation where $V = V_0$.

Hence, we consider a lifting function $R_g(x,t)$ such that the new trial function $w = \psi - R_g$ solves the following Cauchy problem:

$$\begin{cases} S(x)w_{tt}(x,t) - \gamma^2\big[S(x)w_x(x,t)\big]_x = f(x,t) - S(x)R_{g,tt}(x,t) + \gamma^2\big[S(x)R_{g,x}(x,t)\big]_x, \;\; (x,t) \in (0,1)\times(0,T] \\ w(x,0) = w_0(x), \;\; x \in (0,1) \\ w_t(x,0) = \dot{w}_0(x), \;\; x \in (0,1) \\ w(0,t) = 0, \;\; t \in (0,T] \\ w_x(1,t) = -\alpha_1 w_t(1,t) - \alpha_2 w(1,t), \;\; t \in (0,T] \end{cases} \tag{5.3}$$

where the conditions $w(0,t) \equiv \psi(0,t) - R_g(0,t) = 0$ and $w_x(1,t) \equiv \psi_x(1,t) - R_{g,x}(1,t)$ impose that the lifting function must be such that

$$\begin{cases} R_g(0,t) = g(t), \;\; t \in (0,T] \\ R_{g,x}(1,t) = -\alpha_1 R_{g,t}(1,t) - \alpha_2 R_g(1,t), \;\; t \in (0,T] \end{cases} \tag{5.4}$$

Also in this case, the Sobolev space in which the solution w is searched is $V = H^1_\star(0,1)$, defined with the norm reported in Equation (1.5).

With these considerations, the weak formulation (WF′) can be stated:

(WF′.1)

Find $w \in H^1_\star(0,1)$ such that $\forall t \in (0,T]$

$$\int_0^1 S w_{tt} v dx - \int_0^1 \gamma^2 (S w_x)_x v dx = \int_0^1 f v dx - \int_0^1 S R_{g,tt} v dx + \int_0^1 \gamma^2 (S R_{g,x})_x v dx \quad \forall v \in H^1_\star(0,1)$$

and such that $\begin{cases} w(x,0) = w_0(x), \;\; x \in (0,1) \\ w_t(x,0) = \dot{w}_0(x), \;\; x \in (0,1) \end{cases}$

Integrating by parts the elements with a space-derivative dependency we notice that:

$$
\begin{aligned}
-\int_0^1 \gamma^2 (Sw_x)_x v dx &= -\Big[\gamma^2 S w_x v\Big]_0^1 + \int_0^1 \gamma^2 S w_x v_x dx \\
&= -\Big[\gamma^2 S(1)\underbrace{w_x(1)}_{= -\alpha_1 w_t(1) - \alpha_2 w(1),\ \text{Eq. (5.3)}} v(1) - \gamma^2 S(0) w_x(0) \underbrace{v(0)}_{= 0,\ v \in H^1_\star(0,1)}\Big] + \int_0^1 \gamma^2 S w_x v_x dx = \\
&= \gamma^2 S(1)\alpha_1 w_t(1)v(1) + \gamma^2 S(1)\alpha_2 w(1) v(1) + \int_0^1 \gamma^2 S w_x v_x dx
\end{aligned}
\tag{5.5}
$$

and that:

$$
\begin{aligned}
\int_0^1 \gamma^2 (SR_{g,x})_x v dx &= \Big[\gamma^2 S R_{g,x} v\Big]_0^1 - \int_0^1 \gamma^2 S R_{g,x} v_x dx \\
&= \Big[\gamma^2 S(1)\underbrace{R_{g,x}(1)}_{= -\alpha_1 R_{g,t}(1) - \alpha_2 R_g(1),\ \text{Eq. (5.4)}} v(1) - \gamma^2 S(0) R_{g,x}(0) \underbrace{v(0)}_{= 0,\ v \in H^1_\star(0,1)}\Big] - \int_0^1 \gamma^2 S R_{g,x} v_x dx = \\
&= -\gamma^2 S(1)\alpha_1 R_{g,t}(1)v(1) - \gamma^2 S(1)\alpha_2 R_g(1) v(1) - \int_0^1 \gamma^2 S R_{g,x} v_x dx
\end{aligned}
\tag{5.6}
$$

The two presented results allow to reformulate the weak formulation:

(WF′.2)

Find $w \in H^1_\star(0,1)$ such that $\forall t \in (0,T]$

$$
\begin{aligned}
&\int_0^1 S w_{tt} v dx + \int_0^1 \gamma^2 S w_x v_x dx + \gamma^2 S(1)\alpha_1 w_t(1) v(1) + \gamma^2 S(1)\alpha_2 w(1) v(1) = \\
&\quad \int_0^1 f v dx - \int_0^1 S R_{g,tt} v dx - \int_0^1 \gamma^2 S R_{g,x} v_x dx - \gamma^2 S(1)\alpha_1 R_{g,t}(1) v(1) - \gamma^2 S(1)\alpha_2 R_g(1) v(1) \quad \forall v \in H^1_\star(0,1)
\end{aligned}
$$

and such that $\begin{cases} w(x,0) = w_0(x), & x \in (0,1) \\ w_t(x,0) = \dot{w}_0(x), & x \in (0,1) \end{cases}$

Remark - Linear and bilinear forms for the weak formulation. The weak formulation can be rewritten by means of the real and continuous linear and bilinear operators $a(\cdot,\cdot)$, $m(\cdot,\cdot)$ and $\mathcal{F}(\cdot)$ that were defined previously in Equations (1.8), (1.9), (1.10).

Hence, provided that $S \in \big\{S \in C^0(0,1) : S(x) \neq 0 \ \forall x \in [0,1]\big\}$, the weak formulation can be stated as:

(WF′.3)

Find $w \in H^1_\star(0,1)$ such that $\forall t \in (0,T]$

$$
\begin{aligned}
&m(w_{tt}, v) + a(w,v) + \gamma^2 S(1)\alpha_1 w_t(1) v(1) + \gamma^2 S(1)\alpha_2 w(1) v(1) = \\
&\quad = \mathcal{F}(v) - m(R_{g,tt}, v) - a(R_g, v) - \gamma^2 S(1)\alpha_1 R_{g,t}(1) v(1) - \gamma^2 S(1)\alpha_2 R_g(1) v(1) \quad \forall v \in H^1_\star(0,1)
\end{aligned}
$$

and such that $\begin{cases} w(x,0) = w_0(x), & x \in (0,1) \\ w_t(x,0) = \dot{w}_0(x), & x \in (0,1) \end{cases}$

Remark 2 - On the well posedness of the weak formulation. As stated in the previous Section, the weak formulation (WF′.3) is well posed if the bilinear form $a(\cdot,\cdot)$ is coercive and continuous and the linear form $\mathcal{F}(\cdot)$ is continuous (Lax-Milgram Lemma).

5.3 Galerkin Formulation of the Webster's Equation

As done for the Cauchy problem presented in Section 2.2, the focus of the study is to solve the Webster problem (5.3) by means of the finite element method (FEM), which relies on an approximation of the *trial space* $V = H^1_\star(0,1)$ by a finite-dimension subspace $V_h \subset V$ of dimension N_h.

To this end, we first determine the Galerkin formulation of the Webster's equation, which is obtained by reducing the formulation of (WF′.3) to the *trial* subspace $V_h = \mathcal{X}^1_{h,\star}$, i.e. the subspace of continuous functions defined in Equation (2.1) whose elements can be defined with a linear superposition of the basis"Hat" functions $\{\phi_j(x)\}_{j=0}^{N_h}$.

Hence, by taking into account the *trial* solution w_h, a piece-wise expression of the lifting function R_g and by considering any *test* function v_h, the Galerkin formulation (GF′) can be stated:

(GF′.1)

Find $w_h \in \mathcal{X}^1_{h,\star}$ such that $\forall t \in (0,T]$

$$m(w_{h,tt}, v_h) + a(w_h, v_h) + \gamma^2 S(1)\alpha_1 w_{h,t}(1)v_h(1) + \gamma^2 S(1)\alpha_2 w_h(1)v_h(1) =$$
$$\mathcal{F}(v_h) - m(R_{g,h,tt}, v_h) - a(R_{g,h}, v_h) - \gamma^2 S(1)\alpha_1 R_{g,h,t}(1)v_h(1) - \gamma^2 S(1)\alpha_2 R_{g,h}(1)v_h(1) \quad \forall v_h \in \mathcal{X}^1_{h,\star}$$

and such that $\begin{cases} w_h(x,0) = w_0(x), & x \in (0,1) \\ w_{h,t}(x,0) = \dot{w}_0(x), & x \in (0,1) \end{cases}$

Since the Galerkin formulation (GF′.1) is valid $\forall v_h \in \mathcal{X}^1_{h,\star}$, then we can express (GF′.1) by choosing $v_h \equiv \phi_i$ and by considering the superposition presented in Equation (2.2) to describe w_h, $R_{g,h}$ and their time derivatives as shown in Equations (2.3), (2.4), (2.5), (2.6), (2.7).

Hence, by introducing the $\mathbb{R}^{N_h+1}$ vectors:

$$\underline{w}(t) = \{w_0(t), \ldots, w_{N_h}(t)\}^T \;\Rightarrow\; \underline{\dot{w}}(t) = \{\dot{w}_0(t), \ldots, \dot{w}_{N_h}(t)\}^T \;\Rightarrow\; \underline{\ddot{w}}(t) = \{\ddot{w}_0(t), \ldots, \ddot{w}_{N_h}(t)\}^T$$
$$\underline{R}_g(t) = \{R_{g,0}(t), \ldots, R_{g,N_h}(t)\}^T \;\Rightarrow\; \underline{\dot{R}}_g(t) = \{\dot{R}_{g,0}(t), \ldots, \dot{R}_{g,N_h}(t)\}^T \;\Rightarrow\; \underline{\ddot{R}}_g(t) = \{\ddot{R}_{g,0}(t), \ldots, \ddot{R}_{g,N_h}(t)\}^T$$
$$\underline{F}(t) = \{F_0(t), \ldots, F_{N_h}(t)\}^T$$

and the $(N_h+1) \times (N_h+1)$ matrices:

$$M = \left[m_{i,j}\right]_{i,j=0}^{N_h} \underset{\substack{\uparrow \\ \text{Eq. (1.8)}}}{=} \left[\int_0^1 S\phi_j\phi_i dx\right]_{i,j=0}^{N_h} \qquad A = \left[a_{i,j}\right]_{i,j=0}^{N_h} \underset{\substack{\uparrow \\ \text{Eq. (1.9)}}}{=} \left[\int_0^1 \gamma^2 S \frac{\partial\phi_j}{\partial x}\frac{\partial\phi_i}{\partial x} dx\right]_{i,j=0}^{N_h} \tag{5.7}$$

$$I = \begin{bmatrix} 0 & \ldots & 0 & 0 \\ 0 & \ldots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \ldots & 0 & \gamma^2 S(1)\alpha_1 \end{bmatrix} \qquad R = \begin{bmatrix} 0 & \ldots & 0 & 0 \\ 0 & \ldots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \ldots & 0 & \gamma^2 S(1)\alpha_2 \end{bmatrix} \tag{5.8}$$

we can express (GF′.1) as an algebraic vectorial, or finite element (FE′), formulation in the following manner:

(FE′.1)

Find $\underline{w}(t) \in \mathbb{R}^{N_h+1}$ such that $\forall t \in (0,T]$

$$M\underline{\ddot{w}}(t) + I\underline{\dot{w}}(t) + (R+A)\underline{w}(t) = \underline{F}(t) - M\underline{\ddot{R}}_g(t) - I\underline{\dot{R}}_g(t) - (R+A)\underline{R}_g(t)$$

and such that $\begin{cases} \underline{w}(0) = \underline{w}_0 = \{w_0(0), \ldots, w_{N_h}(0)\}^T \\ \underline{\dot{w}}(0) = \underline{\dot{w}}_0 = \{\dot{w}_0(0), \ldots, \dot{w}_{N_h}(0)\}^T \end{cases}$

where $\underline{w}_0$, $\underline{\dot{w}}_0$ are vectors containing the projections of the initial conditions $w_0(x)$, $\dot{w}_0(t)$ into the *trial* subspace $\mathcal{X}^1_{h,\star}$.

Remark 1 - On the choice of the lifting function vector. The Galerkin formulation allows us to choose a "simple" lifting function $R_g(x,t)$. Hence, as done in the previous case, we select the continuous piece-wise function such that in $\mathcal{T}_h$:

$$\underline{R}_g(t) = \{g(t), 0, \ldots, 0\}^T \ \forall t \in (0,T] \tag{5.9}$$

Remark 2 - Finite element final formulation. Once the vector $\underline{F}(t)$ is computed for a given time instant and the vector $\underline{R}_g(t)$ associated to the lifting function is chosen, then the right-hand of (FE′.1) can be treated as a single $\mathbb{R}^{N_h+1}$ vector:

$$\underline{F}_g(t) := \underline{F}(t) - M\ddot{\underline{R}}_g(t) - I\dot{\underline{R}}_g(t) - (R+A)\underline{R}_g(t), \quad \forall t \in (0,T] \tag{5.10}$$

Thanks to this considerations, (FE′.1) can be newly stated:

(FE′.2)

Find $\underline{w}(t) \in \mathbb{R}^{N_h+1}$ such that $\forall t \in (0,T]$

$$M\ddot{\underline{w}}(t) + I\dot{\underline{w}}(t) + (R+A)\underline{w}(t) = \underline{F}_g(t)$$

and such that $\begin{cases} \underline{w}(0) = \underline{w}_0 \\ \dot{\underline{w}}(0) = \dot{\underline{w}}_0 \end{cases}$

and compute the actual solution $\underline{\psi}(t) = \underline{w}(t) + \underline{R}_g(t)$, where

$$\underline{\psi}(t) = \{g(t), w_1(t), \ldots, w_{N_h}(t)\}^T \ \forall t \in (0,T]$$

5.4 Webster's Equation with impulsive, Robin and absorbing boundary conditions

Hereafter, as done in Section 4, we intend to employ the (FE′.2) formulation to determine the solution associated to the following problem:

$$\begin{cases} (0,L) \times (0,T] = (0,1) \times (0,1) \\ c = 2 \Rightarrow \gamma = 2 \\ u_0(x) = 0, \ \forall x \in (0,1) \\ v_0(x) = 0, \ \forall x \in (0,1) \\ f(x,t) = 0, \ \forall (x,t) \in (0,L) \times (0,T) \\ g(t) = \dfrac{1}{2}\left[1 + \cos\left(\dfrac{t-0.2}{0.1}\pi\right)\right] \text{ if } |t-0.2| \leq 0.1 \text{ (otherwise } g(t) = 0) \end{cases} \tag{5.11}$$

In particular, we are concerned to evaluate, for the same three different profiles $S(x)$ shown in Figure 1, the potential $\psi(x,t)$ and the pressure field $p(x,t) = \frac{1}{\gamma}\psi_t(x,t)$ of the system in the case in which the leftmost extreme of the space domain is excited by the given causal half-cosine pulse $g(t)$.

5.4.1 Acoustic potential and pressure wave in time-space domain for section profile *(a)*

The response of the system when excited by the pulse $g(t)$ is examined, in terms of the acoustic potential and in terms of the corresponding pressure wave, in time-space domain by properly choosing mesh and integration parameters.

In particular, as done in Section 4.1, it is decided to study the response of the system defined by the profile $S(x) = 1$ by selecting:

$$N_h = 2^6, \ \Delta t = 1.0 \times 10^{-4} \text{ s}$$

In this way, the dissipation error is limited and the solution is determined with reasonable accuracy in an affordable amount of time.

In Figure 14 and in Figure 17 is respectively reported the time-space behaviour of the acoustic potential and of the pressure wave for the profile section under consideration.

In Figure 14 it can be noticed that the half-cosine pulse propagates with constant speed. Moreover, in correspondence of the right-most edge of the domain, a reflection occurs with a phase inversion due to the Robin boundary conditions and the overall amplitude is reduced due to Absorbing boundary conditions. A similar trend is observed also in Figure 17 for the pressure. The only difference concerns the phase conservation at $x = 1$.

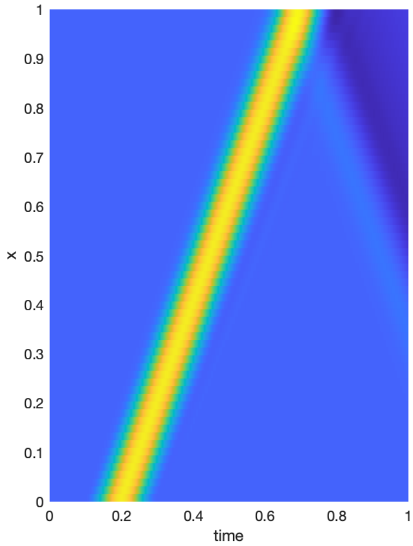


Figure 14: Time-space evolution of the acoustic potential inside the horn with profile: $S(x) = 1$.

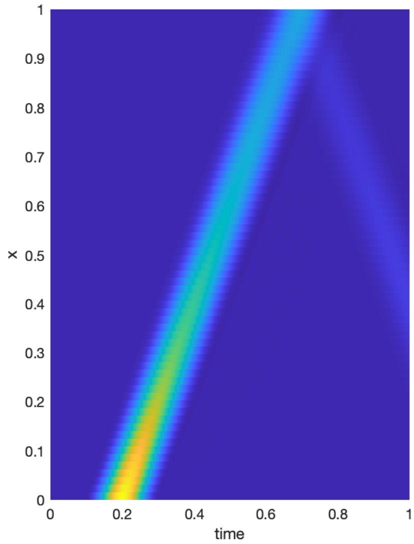


Figure 15: Time-space evolution of the acoustic potential inside the horn with profile: $S(x) = (1 + 2x)^2$.

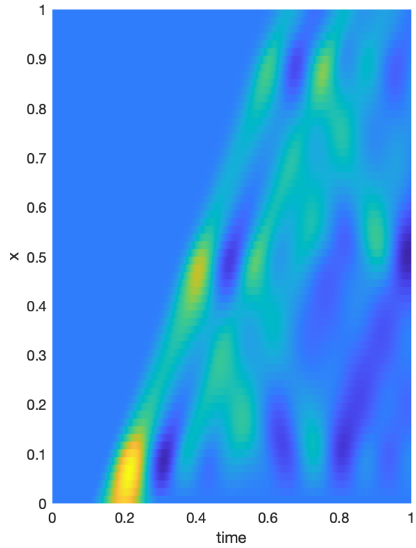


Figure 16: Time-space evolution of the acoustic potential inside the horn with profile: $S(x) = 1 - \frac{3}{4}\sin(5\pi x)$.

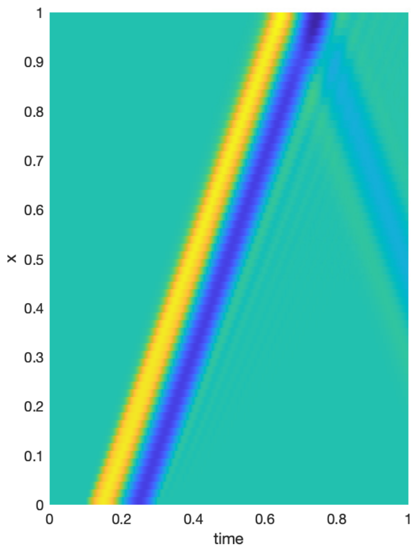


Figure 17: Time-space evolution of the pressure wave inside the horn with profile: $S(x) = 1$.

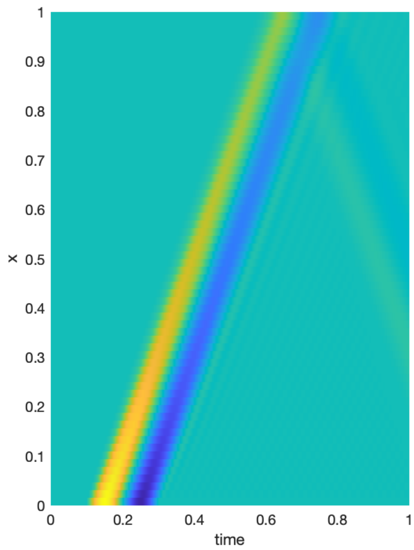


Figure 18: Time-space evolution of the pressure wave inside the horn with profile: $S(x) = (1 + 2x)^2$.

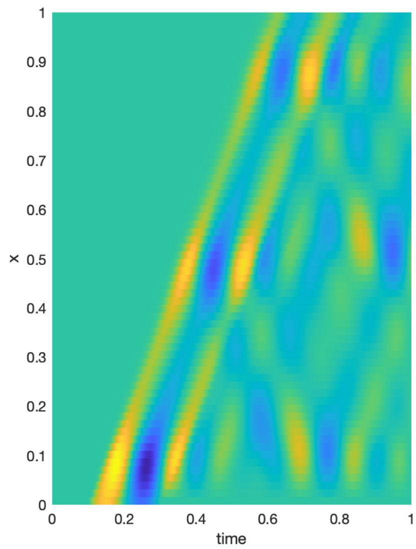


Figure 19: Time-space evolution of the pressure wave inside the horn with profile: $S(x) = 1 - \frac{3}{4}\sin(5\pi x)$.

5.4.2 Acoustic potential and pressure wave in time-space domain for section profile *(b)*

The response of the system when excited by the pulse $g(t)$ is examined, in terms of the acoustic potential and in terms of the corresponding pressure wave, in time-space domain by properly choosing mesh and integration parameters.

In particular, as done in Section 4.2, it is decided to study the response of the system defined by the profile $S(x) = (1 + 2x)^2$ by selecting:

$$N_h = 2^6 \text{ m}, \ \Delta t = 1.0 \times 10^{-4}$$

In this way, the dissipation error is limited and the solution is determined with reasonable accuracy in an affordable amount of time.

In Figure 15 and in Figure 18 is respectively reported the time-space behaviour of the acoustic potential and of the pressure wave for the profile section under consideration.

In Figure 15 it can be noticed that the half-cosine pulse propagates with constant speed despite being reduced in amplitude by the section profile. Moreover, in correspondence of the right-most edge of the domain, a reflection occurs with a phase inversion due to the Robin boundary conditions and the overall amplitude is reduced due to Absorbing boundary conditions. A similar trend is observed also in Figure 18 for the pressure. The only difference concerns the phase conservation at $x = 1$.

5.4.3 Acoustic potential and pressure wave in time-space domain for section profile *(c)*

The response of the system when excited by the pulse $g(t)$ is examined, in terms of the acoustic potential and in terms of the corresponding pressure wave, in time-space domain by properly choosing mesh and integration parameters.

In particular, as done in Section 4.3, it is decided to study the response of the system defined by the profile $S(x) = 1 - \frac{3}{4}\sin(5\pi x)$ by selecting:

$$N_h = 2^6 \text{ m}, \ \Delta t = 1.0 \times 10^{-4}$$

In this way, the dissipation error is limited and the solution is determined with reasonable accuracy in an affordable amount of time.

In Figure 16 and in Figure 19 is respectively reported the time-space behaviour of the acoustic potential and of the pressure wave for the profile section under consideration.

In Figure 16 it can be noticed that the half-cosine pulse propagates with constant speed despite being reduced in amplitude by the section profile and by the inner reflections in correspondence of the three throats. Moreover, in correspondence of the right-most edge of the domain, a reflection occurs with a phase inversion due to the Robin boundary conditions and the overall amplitude is reduced due to Absorbing boundary conditions. A similar trend is observed also in Figure 19 for the pressure. The only difference concerns the phase conservation at $x = 1$.

6 Webster's Equation in space-frequency domain

The Webster's Equation

$$S(x)\psi_{tt}(x,t) - \gamma^2\big[S(x)\psi_x(x,t)\big]_x = f(x,t) \tag{6.1}$$

can be studied and analysed in space-frequency domain by assuming that the acoustic potential $\psi(x,t)$ and the external force $f(x,t)$ are complex propagating waves with angular frequency ω, i.e. complex functions harmonically modulated in time which can be written as

$$\begin{cases} \psi(x,t) = \hat{\psi}(x)e^{-i\omega t} \\ f(x,t) = \hat{f}(x)e^{-i\omega t} \end{cases} \tag{6.2}$$

In this way:

$$\psi_{tt}(x,t) \equiv -\omega^2\hat{\psi}(x)e^{-i\omega t}, \quad \big[S(x)\psi_x(x,t)\big]_x \equiv \big[S(x)\hat{\psi}_x(x)\big]_x e^{-i\omega t} \tag{6.3}$$

and so, in space-frequency domain, Equation (6.1) can be re-written as

$$-\omega^2 S(x)\hat{\psi}(x)e^{-i\omega t} - \gamma^2\big[S(x)\hat{\psi}_x(x)\big]_x e^{-i\omega t} = \hat{f}(x)e^{-i\omega t} \tag{6.4}$$

or equivalently as

$$-\omega^2 S(x)\hat{\psi}(x) - \gamma^2\big[S(x)\hat{\psi}_x(x)\big]_x = \hat{f}(x) \tag{6.5}$$

since the time-dependent terms can be neglected ($e^{-i\omega t} \neq 0 \ \forall t$).

In the relevant case in which $\hat{f}(x) = 0$, Equation (6.5) reduces to the following:

$$k^2 S(x)\hat{\psi}(x) + \big[S(x)\hat{\psi}_x(x)\big]_x = 0 \tag{6.6}$$

which is also known as Helmholtz Webster's equation, where the generalised wave number $k = \omega/\gamma$ has been introduced.

In the following sections, it will be our goal to study how the solution of the Helmholtz Webster's equation changes as a function of ω by employing finite element method (FEM) and by considering Dirichlet (sound soft) and Neumann (sound hard) boundary conditions:

$$\begin{cases} k^2 S(x)\hat{\psi}(x) + \big[S(x)\hat{\psi}_x(x)\big]_x = 0, \;\; x \in (0,1) \\ \hat{\psi}(0) = 0 \\ \hat{\psi}_x(1) = 0 \end{cases} \tag{6.7}$$

Remark - On the time-frequency duality of the solutions. The acoustic potential $\psi(x,t)$ is a solution of Equation (6.1) if and only if the acoustic space-potential $\hat{\psi}(x)$ is a solution of Equation (6.6). Such a result, which could be demonstrated by considering that $\hat{\psi}(x) = \hat{\psi}(x,\omega)$ is the Fourier transform of $\psi(x,t)$, i.e.

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{\psi}(x,\omega) e^{i\omega t} d\omega, \;\; \forall x \tag{6.8}$$

states that the two solutions show identical behavior patterns except for an interchange of the roles played by time and frequency.

6.1 Weak formulation of the Helmholtz Webster's equation

The weak formulation of the Helmholtz Webster's equation with the provided boundary conditions is derived in the same manner it was computed in the previous Sections, i.e. by multiplying the Helmholtz equation subject of interest by a test function q, integrating over the space domain Ω of the solution and integrating by parts. Hence, the aim of the problem is to look for the solution $\hat{\psi}$ which belongs to a *trial space* V that satisfies the resulting equality for any test function q belonging to a *test space* V_0. In our case, the boundary conditions allow to choose the comfortable situation in which $V = V_0$.

The Sobolev space in which the solution $\hat{\psi}$ is searched is again $V = H^1_\star(0,1)$, as defined in Equation 1.4, with the norm reported in Equation (1.5).

With these considerations, the weak formulation (WF_H) can be stated:

Find $\hat{\psi} \in H^1_\star(0,1)$ such that

$$\int_0^1 k^2 S\hat{\psi} q dx + \int_0^1 (S\hat{\psi}_x)_x q dx = 0 \;\; \forall q \in H^1_\star(0,1) \tag{WF_H.1}$$

Integrating by parts the elements with a space-derivative dependency we notice that:

$$\begin{aligned} \int_0^1 (S\hat{\psi}_x)_x q dx &= \Big[S\hat{\psi}_x q\Big]_0^1 - \int_0^1 S\hat{\psi}_x q_x dx = \\ &= \Big[\underbrace{S(1)\hat{\psi}_x(1)q(1)}_{=0,\ \text{B.C. of Eq. (6.7)}} - \underbrace{S(0)\hat{\psi}_x(0)q(0)}_{=0,\ q \in H^1_\star(0,1)}\Big] - \int_0^1 S\hat{\psi}_x q_x dx \equiv -\int_0^1 S\hat{\psi}_x q_x dx \end{aligned} \tag{6.9}$$

Thanks to the presented result, the weak formulation can be newly stated:

Find $\hat{\psi} \in H^1_\star(0,1)$ such that

$$\int_0^1 k^2 S\hat{\psi} q dx - \int_0^1 S\hat{\psi}_x q_x dx = 0 \;\; \forall q \in H^1_\star(0,1) \tag{WF_H.2}$$

Remark - Bilinear forms for the weak formulation. The weak formulation can be rewritten by means of real and continuous bilinear operators $a(\cdot,\cdot)$, $m(\cdot,\cdot)$ similar to those that were defined in Equations (1.8), (1.9).

Indeed, by defining

- Bilinear form $a : H^1_\star(0,1) \times H^1_\star(0,1) \to \mathbb{R}$ such that

$$a(p,q) = \int_0^1 Sp_x q_x dx \quad \forall p,q \in H^1_\star(0,1) \tag{6.10}$$

- Bilinear form $m : H^1_\star(0,1) \times H^1_\star(0,1) \to \mathbb{R}$ such that

$$m(p,q) = \int_0^1 Spqdx \quad \forall p,q \in H^1_\star(0,1) \tag{6.11}$$

and provided that $S \in \{S \in C^0(0,1) : S(x) \neq 0 \ \forall x \in [0,1]\}$, the weak formulation can be stated as:

(WF$_H$.3)

Find $\hat{\psi} \in H^1_\star(0,1)$ such that

$k^2 m(\hat{\psi},q) - a(\hat{\psi},q) = 0 \quad \forall q \in H^1_\star(0,1)$

Remark 2 - On the well posedness of the weak formulation. As stated in the previous Section, the weak formulation (WF$_H$.3) is well posed if the bilinear form $a(\cdot,\cdot)$ is coercive and continuous (Lax-Milgram Lemma).

6.2 Galerkin Formulation of the Helmholtz Webster's Equation

As presented in the introduction, the focus of the study is to solve the Helmholtz Webster problem (6.7) by means of the finite element method (FEM), which relies on an approximation of the *trial space* $V = H^1_\star(0,1)$ by a finite-dimension subspace $V_h \subset V$ of dimension N_h. To this end, we first determine the Galerkin formulation of the Helmholtz Webster's equation, which is obtained by reducing the formulation of (WF$_H$.3) to the *trial* subspace $V_h = \mathcal{X}^1_{h,\star}$, i.e. the subspace of continuous functions defined in Equation (2.1) whose elements can be defined with a linear superposition of the basis"Hat" functions $\{\phi_j(x)\}_{j=0}^{N_h}$.

Hence, by taking into account the *trial* solution $\hat{\psi}_h$ and by considering any *test* function q_h, the Galerkin formulation (GF$_H$) can be stated:

(GF$_H$.1)

Find $\hat{\psi}_h \in \mathcal{X}^1_{h,\star}$ such that

$k^2 m(\hat{\psi}_h, q_h) - a(\hat{\psi}_h, q_h) = 0, \quad \forall q_h \in \mathcal{X}^1_{h,\star}$

Since the Galerkin formulation (GF$_H$.1) is valid $\forall q_h \in \mathcal{X}^1_{h,\star}$, then we can express (GF$_H$.1) by choosing $q_h \equiv \phi_i$ and by employing the linear superposition principle to describe $\hat{\psi}_h$ as

$$\hat{\psi}_h(x) = \sum_{j=0}^{N_h} \psi_j \phi_j(x) \tag{6.12}$$

In this way,

$$m(\hat{\psi}_h, q_h) = m\left(\sum_{j=0}^{N_h} \psi_j \phi_j(x), \phi_i(x)\right) \underset{\substack{\uparrow \\ m(\cdot,\cdot)\text{ bilinear}}}{\equiv} \sum_{j=0}^{N_h} \psi_j \cdot \underbrace{m\big(\phi_j(x), \phi_i(x)\big)}_{m(\phi_j(x),\phi_i(x)) =: m_{i,j}} = \sum_{j=0}^{N_h} \psi_j m_{i,j} \tag{6.13}$$

$$a(\hat{\psi}_h, q_h) = a\left(\sum_{j=0}^{N_h} \psi_j \phi_j(x), \phi_i(x)\right) \underset{\substack{\uparrow \\ a(\cdot,\cdot)\text{ bilinear}}}{\equiv} \sum_{j=0}^{N_h} \psi_j \cdot \underbrace{a\big(\phi_j(x), \phi_i(x)\big)}_{a(\phi_j(x),\phi_i(x)) =: a_{i,j}} = \sum_{j=0}^{N_h} \psi_j a_{i,j} \tag{6.14}$$

Hence, by introducing the $\mathbb{R}^{N_h+1}$ vector:

$$\underline{\psi} = \{\psi_0, \dots, \psi_{N_h}\}^T$$

and the $N_h \times N_h$ matrices:

$$M = [m_{i,j}]_{i,j=0}^{N_h} \underset{\substack{\uparrow \\ \text{Eq. (6.10)}}}{=} \left[\int_0^1 S\phi_j\phi_i dx \right]_{i,j=0}^{N_h} \qquad A = [a_{i,j}]_{i,j=0}^{N_h} \underset{\substack{\uparrow \\ \text{Eq. (6.11)}}}{=} \left[\int_0^1 S\frac{\partial \phi_j}{\partial x}\frac{\partial \phi_i}{\partial x} dx \right]_{i,j=0}^{N_h} \tag{6.15}$$

we can express (GF$_H$.1) as an algebraic vectorial, or finite element (FE$_H$), formulation in the following manner:

(FE$_H$)

Find $\underline{\psi} \in \mathbb{R}^{N_h+1}$ such that

$$k^2 M\underline{\psi} - A\underline{\psi} = 0$$

Remark 1 - On the tridiagonality of the mass and stiffness matrices. The mass and stiffness matrices M and A are symmetric, defined positive and tridiagonal. This last fundamental property is sufficient to assure that $\exists M^{-1}$. In this way, an eigenvector equation in the eigenvalues $\lambda = k^2$ can be written:

$$k^2 M\underline{\psi} - A\underline{\psi} = 0 \;\Leftrightarrow\; A\underline{\psi} = k^2 M\underline{\psi} \;\Leftrightarrow\; M^{-1}A\underline{\psi} = k^2\underline{\psi} \;\Leftrightarrow\; \det(M^{-1}A - \lambda) = 0, \;\; \underline{\psi} \neq \underline{0} \tag{6.16}$$

Remark 2 - On the orthogonality of the mode shapes. The $N_h + 1$ eigenvectors $\underline{X}_j$ that solve the equation

$$(M^{-1}A)\underline{X}_j = \lambda_j \underline{X}_j, \quad \forall j = 0, \dots, N_h \tag{6.17}$$

which are also known as mode shapes, are orthogonal and do realise a basis of $\mathbb{R}^{N_h+1}$.

6.3 Modal Analysis applied to the Helmholtz Webster Equation

The result obtained in (FE$_H$) encourages to make use of Modal Analysis to write, in space-frequency domain, the solution ψ by means of the mode shapes $\underline{X}_j$ collected in the modal matrix $\Phi = [\underline{X}_0 \quad \underline{X}_2 \quad \dots \underline{X}_{N_h}]$.

In fact, Modal Analysis ensures that the *Frequency Response Modal Matrix* $H_\psi(\omega)$ of the solution ψ can be obtained with the following superposition:

$$H_\psi(\omega) = \mathrm{Tr}\big\{[-\omega^2 M_m + A_m]^{-1}\big\} \equiv \sum_{j=0}^{N_h} \frac{1}{-\omega^2 m_{j,j} + a_{j,j}} \tag{6.18}$$

where the modal mass matrix $M_m = [m_{i,j}]_{i,j=0}^{N_h}$ and the modal stiffness matrix $A_m = [a_{i,j}]_{i,j=0}^{N_h}$, diagonal by definition, are given, in terms of the mass and stiffness matrices M and A, by the following relations:

$$M_m = \Phi^T M \Phi, \quad A_m = \Phi^T A \Phi \tag{6.19}$$

Remark 1 - On the location of the resonant frequencies. The Frequency Response Modal Matrix of Equation (6.18) is characterised by different poles, i.e. frequency values for which the denominator $-\omega^2 m_{j,j} + a_{j,j}$ is driven to zero. Such frequencies are known as resonant frequencies and their value is calculable from the eigenvalues of Equation (6.17). Indeed, each resonant frequency ω_j is real and such that:

$$\omega_j = \gamma k_j = \gamma\sqrt{\lambda_j}, \quad \forall j = 0, \dots, N_h \tag{6.20}$$

Remark 2 - On the significance of the Frequency Response Modal Matrix. The Helmholtz-Webster system under consideration is characterised by the real-valued modal frequency response $H_\psi(\omega)$, whose importance can be appreciated if Equation (6.8) is considered. Indeed, the frequency response $H_\psi(\omega)$ is by definition the Fourier Transform of the space-time function $\psi(x,t)$ and so:

$$\exists \hat{x} \in (0,1): \; H_\psi(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \psi(\hat{x},t) e^{-i\omega t} dt \;\Leftrightarrow\; \psi(\hat{x},t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} H_\psi(\omega) e^{i\omega t} d\omega \tag{6.21}$$

where $\hat{x}$ is a point in the domain where the modeshape has a maximum. From this result, it is straightforward to deduce that $H_\psi(\omega) \equiv \hat{\psi}(\hat{x},\omega)$.

6.3.1 Behaviour of the acoustic space-potential $\hat{\psi}$

In conclusion, the normalised magnitude of the acoustic space-potential $\hat{\psi} = H_\psi$ is computed, in the range $(0.1,\ 5000)$ and with the choice of $c = 340$, with Equation (6.18) for the three different section profile $S(x)$ that were presented in Section 3.2 and shown in Figure 1. The results obtained are reported in Figure 20.

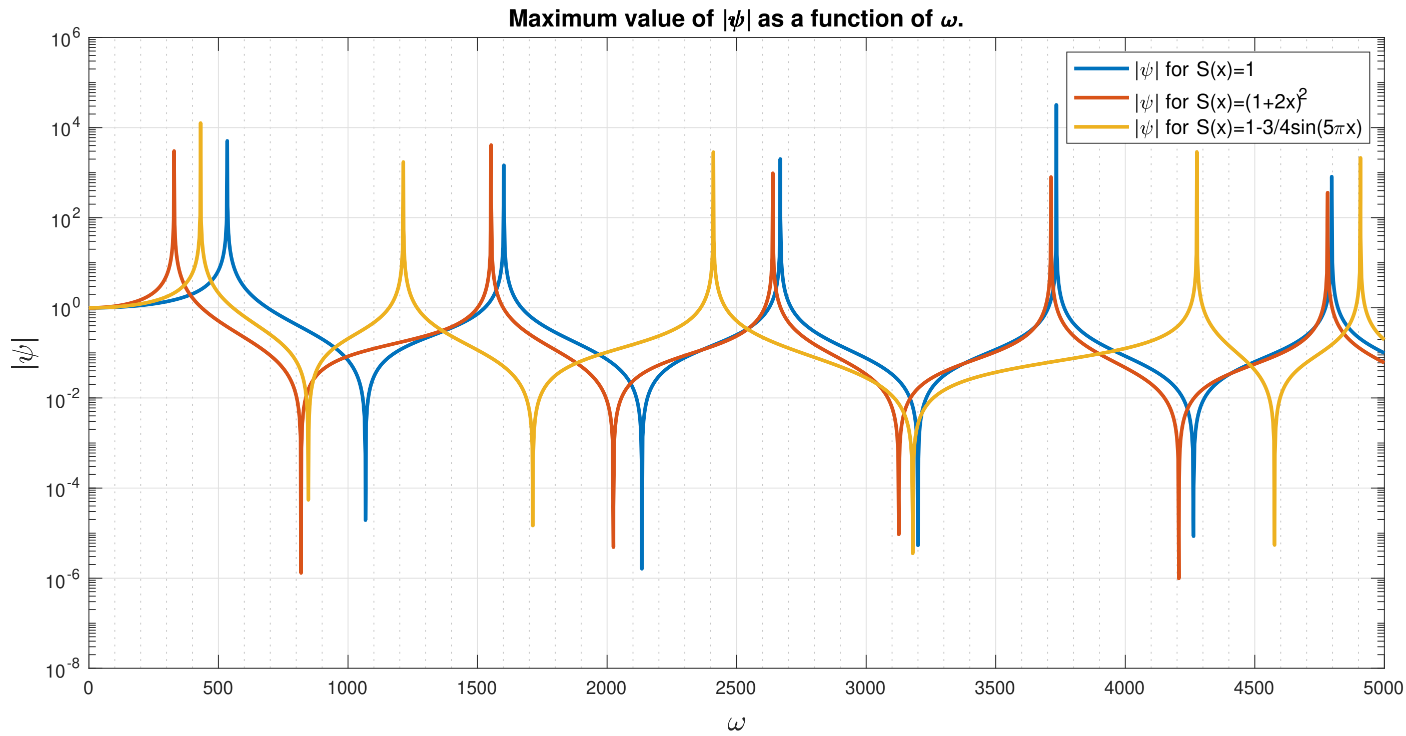


Figure 20: Normalised magnitude of the acoustic space-potential $\hat{\psi}$ as a function of the angular frequency ω for three different section profiles $S(x)$.