Probability Assignment -III

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Question: Let X represent the difference between the number of heads and the number of tails obtained when a coin is tossed 6 times. What are possible values of X?

Solution:
Let H be a random variable which represents the number of Heads obtained in 6 coin tosses.
And T be a random variable which represents the number of Tails obtained in 6 coin tosses.
Then,
$H \in \{0, 1, 2, 3, 4, 5, 6\}$
Similarly,
$T \in \{0, 1, 2, 3, 4, 5, 6\}$

$$H + T = 6 \tag{1}$$
$$X = |H - T| \tag{2}$$
$$X = |H - (6 - H)| \tag{3}$$
$$X = |2H - 6| \tag{4}$$
$$X = \begin{cases} 6 & H \in \{0, 6\} \\ 4 & H \in \{1, 5\} \\ 2 & H \in \{2, 4\} \\ 0 & H \in \{3\} \end{cases} \tag{5}$$

Hence, The possible values of X are {0,2,4,6}

$$\Pr(H = k) = {}^6C_k \left(\frac{1}{2}\right)^6, 0 \le k \le 6 \tag{6}$$
$$\tag{7}$$
$$\Pr(X = i) = \begin{cases} \Pr(H = 3) & i = 0 \\ \Pr(H = 2) + \Pr(H = 4) & i = 2 \\ \Pr(H = 1) + \Pr(H = 5) & i = 4 \\ \Pr(H = 0) + \Pr(H = 6) & i = 6 \end{cases} \tag{8}$$

The distribution of X is

$$\Pr(X = i) = \begin{cases} {}^6C_3 \left(\frac{1}{2}\right)^6 & i = 0 \\ {}^6C_2 \left(\frac{1}{2}\right)^6 + {}^6C_4 \left(\frac{1}{2}\right)^6 & i = 2 \\ {}^6C_1 \left(\frac{1}{2}\right)^6 + {}^6C_5 \left(\frac{1}{2}\right)^6 & i = 4 \\ {}^6C_0 \left(\frac{1}{2}\right)^6 + {}^6C_6 \left(\frac{1}{2}\right)^6 & i = 6 \end{cases} \tag{9}$$

The CDF of Random Variable X can be written as

$$F_X(k) = \Pr(X \le k) = \sum_{i=0}^{k} \Pr(X = i) \tag{10}$$

Therefore,

$$F_X(0) = \Pr(X = 0) = {}^6C_3 \left(\frac{1}{2}\right)^6 = \frac{5}{16} \tag{11}$$
$$F_X(2) = \sum_{i=0}^{2} \Pr(X = i) = {}^6C_3 \left(\frac{1}{2}\right)^6 + {}^6C_2 \left(\frac{1}{2}\right)^6 + {}^6C_4 \left(\frac{1}{2}\right)^6 = \frac{25}{32} \tag{12}$$
$$\begin{aligned} F_X(4) &= \sum_{i=0}^{4} \Pr(X = i) \\ &= {}^6C_3 \left(\frac{1}{2}\right)^6 + {}^6C_2 \left(\frac{1}{2}\right)^6 + {}^6C_4 \left(\frac{1}{2}\right)^6 + {}^6C_1 \left(\frac{1}{2}\right)^6 + {}^6C_5 \left(\frac{1}{2}\right)^6 \\ &= \frac{31}{32} \end{aligned} \tag{13}$$
$$\begin{aligned} F_X(6) &= \sum_{i=0}^{6} \Pr(X = i) \\ &= {}^6C_3 \left(\frac{1}{2}\right)^6 + {}^6C_2 \left(\frac{1}{2}\right)^6 + {}^6C_4 \left(\frac{1}{2}\right)^6 + {}^6C_1 \left(\frac{1}{2}\right)^6 + {}^6C_5 \left(\frac{1}{2}\right)^6 \\ &+ {}^6C_0 \left(\frac{1}{2}\right)^6 + {}^6C_6 \left(\frac{1}{2}\right)^6 \end{aligned} \tag{14}$$
$$= 1 \tag{15}$$

X	0	2	4	6
$F_X(x)$	$\frac{5}{16}$	$\frac{25}{32}$	$\frac{3}{32}$	1