EOR.
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1. Proportional fairness $U(x) = \log x_r$ yields the following problem:

$$\max_x \sum_{r=0}^{L} U(x_r) = \max_x \sum_{r=0}^{L} \log x_r$$

with constraints

$$x_0 + x_l \leq 1 \quad \forall l = 0, 1 \ldots, L$$

$$x \geq 0$$

x is vector of x_0 through x_L. $\log x \to -\infty$ as $x \to 0$ $\therefore$ $x_r \geq 0$ $\forall r = 0, \ldots L$. Last constraint can be ignored.

p_l: Lagrangian multiplier associated with capacity constraint at link l and p = vector of lagrangian multiplier.

$$L(x, p) = \sum_{r=0}^{L} \log x_r - \sum_{l=1}^{L} p_l (x_0 + x_l - 1)$$

$$\frac{\partial L}{\partial x_r} = 0$$

$$\therefore \quad x_0 = \frac{1}{\sum_{l=1}^{L} p_l}, \quad x_l = \frac{1}{p_l} \quad \forall l \geq 1$$

KKT condition:

$$p_l (x_0 + x_l - 1) = 0 \quad \& \quad p_l \geq 0 \quad \forall l \geq 1$$

$$\therefore \quad p_l = \frac{L+1}{L} \quad \forall l \geq 1$$

$$\therefore \quad x_0 = \frac{1}{L+1}, \quad x_r = \frac{L}{L+1} \quad \forall r \geq 1$$

2. Minimum potential delay

Problem:

$$\max_{x} \sum_{r=0}^{L} -\frac{1}{x_r}$$

with constraints

$$x_0 + x_l \leq 1 \quad \forall\, l = 0, \ldots, L$$
$$x \geq 0$$

$$L(x,p) = \sum_{r=0}^{L} -\frac{1}{x_r} - \sum_{l=1}^{L} p_l (x_0 + x_l - 1)$$

$\frac{\partial L}{\partial x_r} = 0$ for each r gives

$$x_0 = \frac{1}{\sqrt{\sum_{l=1}^{L} p_l}}, \quad x_l = \frac{1}{\sqrt{p_l}} \quad \forall\, l \geq 1$$

KKT:

$$p_l (x_0 + x_l - 1) = 0 \quad \text{and} \quad p_l \geq 0 \quad \forall\, l \geq 1$$

Solving, $\cancel{p_l =}$ $\sqrt{\sum_{l=1}^{L} p_l} = 1 + \sqrt{L}$

$$\therefore\ x_0 = \frac{1}{1+\sqrt{L}}, \quad x_l = \frac{\sqrt{L}}{1+\sqrt{L}}, \quad p_l = \frac{(1+\sqrt{L})^2}{L} \quad \forall\, l \geq 1$$

3. Sum-rate

$$\max_x \sum_{r=0}^{L} U(x_r) = \max_x \sum_{r=0}^{L} x_r$$

with constraints

$$x_0 + x_l \leq 1 \qquad \forall l = 0, \dots L$$

$$x \geq 0$$

$$L(x, p) = \sum_{r=0}^{L} x_r - \sum_{l=1}^{L} p_l (x_0 + x_l - 1)$$

$$\frac{\partial L}{\partial x_r} = 0$$

Lagrangian doesn't give any result.

~~Both~~ KTT : $x_l = 1 - x_0 \quad \forall l \geq 1$

By observation, $x_r = 1$, $x_0 = 0$ is the optimal solutions as utility monotonically ~~decreases~~ decreases with x_0 increase.