Variational Autoencoders - 2

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1 Recap

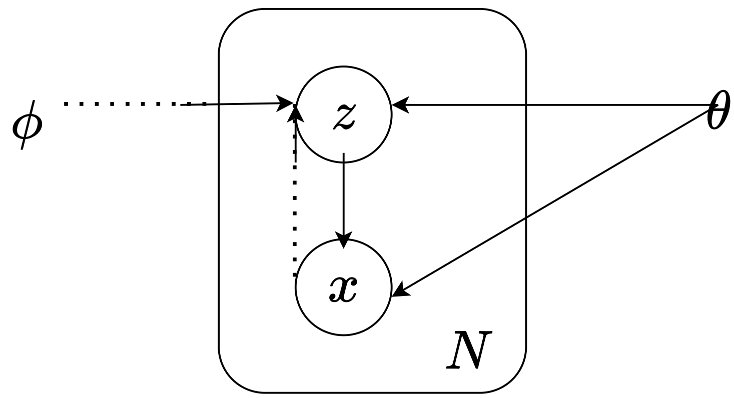

Figure 1: Variational inference

The goal is to generate N samples of the data X, assuming a latent variable z using the process. For example, we want to generate images similar to the images in the MNIST dataset. X is modelled assuming z. There are also other parameter θ, which is kept outside the plate. The latent variable z is also dependent on θ, therefore we also have a connection between z and θ in the plate notation. We use a latent variable z to model x, by calculating the probability $p(x|z)$ and integrate it for all z. If we take a *log* on both the sides, the problem becomes difficult because the log of integral is intractable. Here, we use the method of **variational inference**.

$$p_\theta(x) = \int p_\theta(x, z)\, dz$$

$$log(p_\theta(x)) = log\left(\int p_\theta(x, z)\, dz\right)$$

We define some $q_\phi(z|x)$, and multiply as well as divide $p_\theta(x, z)$ with it.

$$\begin{aligned} log(p_\theta(x)) &= log\left(\int p_\theta(x, z) dz\right) \\ &= log\left(\int \frac{p_\theta(x, z)}{q_\phi(z|x)} \times q_\phi(z|x) dz\right) \end{aligned}$$

Now we use the Jensen's inequality,

$$\begin{aligned} log(p_\theta(x)) &= log\left(\int \frac{p_\theta(x, z)}{q_\phi(z|x)} \times q_\phi(z|x) dz\right) \\ &\leq q_\phi(z|x) \times \left(\int \frac{p_\theta(x, z)}{q_\phi(z|x)} dz\right) \\ &= q_\phi(z|x) \times \left(\int \frac{p_\theta(x, z)}{q_\phi(z|x)} dz\right) \end{aligned}$$

This lower bound is called the variational lower bound or ELBO (Evidence Lower Bound). We try to maximize this quantity with respect to θ and ϕ. Integration over a pdf is difficult to compute. Hence, we use the Monte Carlo Approximation.

$$L(\theta, \phi; x) = E_{z\ q_\phi(z|x)}\left[log\left(\frac{p_\theta(x|z) \times p_\theta(z)}{q_\phi(z|x)}\right)\right]$$

2 Computing $L(\theta, \phi; x)$

2.1 $p_\theta(x|z)$

- To compute $p_\theta(x|z)$ we need z. So we will first generate z. We can get z by sampling from $q_\phi(z|x)$.
  - To get $q_\phi(z|x)$ feed x to NN_ϕ, to get μ_z and σ_z. Then $q_\phi(z|x) = Normal(z; \mu_z, \sigma_z)$
  - Sample z using $z = \epsilon \times \sigma_z + \mu_z$, with ϵ sampled from $Normal(0, I)$
- Feed z into NN_θ to get μ_x and σ_x

$$p_\theta(x|z) = Normal\,(x; \mu_x, \sigma_x)$$

- To get the value for the given $x^{(s)}$, evaluate $Normal(x^{(s)}; \mu_x, \sigma_x)$

2.2 $p_\theta(z)$

- We say that the sample $z^{(0)}$ is sampled from $q_\phi(z|x)$
- Evaluate $Normal(z^{(0)}; 0, 1)$

2.3 $q_\phi(z|x)$

- Sample $z^{(0)}$ from $q_\phi(z|x)$
- Evaluate $q_\phi(z^{(0)}|x^{(0)}) = Normal(z^{(0)}|\mu_z, \sigma_z)$

3 Training

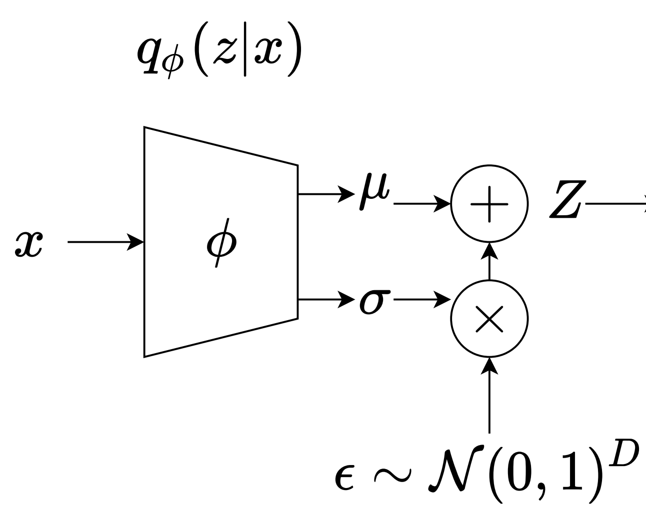

Figure 2: Variational Autoencoder training

To train this model we have to modify the differentiate the loss function with respect to the parameters θ and ϕ. Differentiating the term z sampled from $q_\phi(z|x)$ with respect to ϕ is difficult. Therefore we use the reparameterization trick.

$$z = \epsilon \cdot \sigma + \mu$$

Claim:

$$\nabla_\phi E_{z\ q_\phi(z|x)}\left[log\left(\frac{p_\theta(x|z) \times p_\theta(z)}{q_\phi(z|x)}\right)\right] = \nabla_\phi E_{z\ Normal(\epsilon;0,1)}\left[log\left(\frac{p_\theta(x|z) \times p_\theta(z)}{q_\phi(z|x)}\right)\right], z = \epsilon\sigma + \mu$$

Proof:

$$\begin{aligned} E_{z\ q_\phi(z)}\left[f(z)\right] &= \int q_\phi(z) \cdot f(z) dz, \epsilon \sim p(\epsilon) \\ &= \int q_\phi(\epsilon\sigma + \mu) \cdot f(\epsilon\sigma + \mu)\sigma d\epsilon \end{aligned}$$

We now have to get rid of $q_\phi(z)$

$$\begin{aligned} &= \int \frac{p(\epsilon)}{\sigma} \cdot f(\epsilon\sigma + \mu)\sigma d\epsilon \\ &= E_{\epsilon\sim p(\epsilon)}\left[f(\epsilon\sigma + \mu)\right] \end{aligned}$$

Reparameterization trick is useful because $q_\phi(z)\left|\frac{\partial z}{\partial \epsilon}\right|$ is not a function of ϕ.