Lecture 4: Sampling and Graphical Models 2

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This class was more of an interaction session rather than a lecture. So, we have presented it as a Q&A.

1 Assumptions made:

We can directly sample from uniform and gaussian distributions using libraries of common programming languages such as Python.

2 Questions:

2.1 Sample uniformly on a line segment (0 to a)

- **Solution**

  Firstly we can sample w (say) uniformly in the range [0, 1] using standard libraries. After this we multiply w with a to get x . This puts it in the range [0, a]. Since there is a one to one mapping between w and x, the latter will be uniformly sampled in [0, a].

Algorithm 1 Pseudo Code

$u \sim U(0, 1)$
$x \leftarrow a * u$
$return\ x$

2.2 Sample uniformly on the circumference of a unit circle

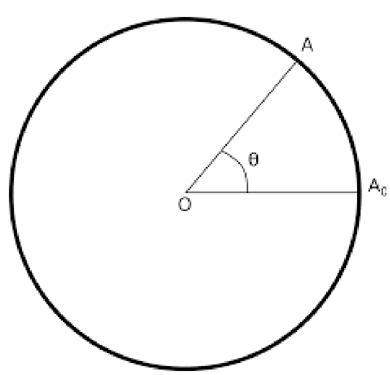


Figure 1: Central angle θ

- **Solution**

  First we sample θ (as shown in the figure above) uniformly in the range [0, 2π]. Clearly each point on the circumference is uniquely mapped to a corresponding value of θ. So, using the relations (keeping in mind r is 1),

$x = \cos\,\theta$
$y = \sin\,\theta$
we get a uniform sample pair (x, y)

Algorithm 2 Pseudo Code

$u \sim U(0, 1)$
$\theta \leftarrow 2\pi * u$
// In cartesian coordinates
$x \leftarrow cos\theta$
$y \leftarrow sin\theta$
// In polar coordinates
$x \leftarrow 1$
$y \leftarrow \theta$
$return\ (x, y)$

2.3 Sample uniformly on the surface of a square of unit side

- **Rudimentary Approach (Incorrect)**
  Perform ancestral sampling. First sample x uniformly in [0, 1]. Then sample y from a certain conditional probability distribution $p(y|x)$.

- **Solution**
  We note that to choose any point (x, y) on the surface of a unit square (with bottom-left corner as (0, 0)) we can independently select x and y from the range [0, 1]. So, as we did in the rudimentary approach, we sample x uniformly in [0, 1]. In addition we also sample y uniformly in [0, 1] and finally obtain (x, y).

Algorithm 3 Pseudo Code

$x \sim U(0, 1)$
$y \sim U(0, 1)$
return (x, y)

2.4 Sample uniformly on the surface of a rectangle of sides l and b

- **Solution**
  This approach is almost exactly similar to the one above where we sampled from a square. The only difference is the domain of the uniform distributions. As usual we sample x and y uniformly in [0, 1]. But as seen in question 0.1 (sampling uniformly on a line segment) we multiply x with l to uniformly sample in [0, l], and multiply y with b to uniformly sample in [0, b]. The obtained samples here are the required ones.

Algorithm 4 Pseudo Code

$u1 \sim U(0, 1)$
$x \leftarrow l * u1$
$u2 \sim U(0, 1)$
$y \leftarrow b * u2$
return (x, y)

2.5 Sample uniformly on the surface of a circle

- **Solution**
  We assume a unit circle. Let us visualize a square of side length 2 circumscribing this circle. Now, as seen in question 0.3, we uniformly sample on the surface of the square. Thus, we have

achieved our result of uniform samples on the surface of the circle too - although with the problem of some extra samples lying outside the circle. Its solution? Just remove them, i.e, we make the condition - For a sample point (x, y) if $x^2 + y^2 > 1$, reject this point and sample another one. We do this until we get a point within the circle.

Algorithm 5 Pseudo Code

$x \leftarrow 10$
$y \leftarrow 10$
while $x^2 + y^2 > 1$ **do**
  $u1 \sim U(0,1)$
  $x \leftarrow 2 * u1 - 1$
  $u2 \sim U(0,1)$
  $y \leftarrow 2 * u2 - 1$
end while
$return\ (x, y)$

- **Note**
  Rejections leads to inefficiency.

- **Exercise**
  Please check whether the following algorithm samples uniformly on the surface of a circle:

Algorithm 6 Exercise

$u1 \sim U(0,1)$
$\theta \leftarrow 2\pi * u1$
$u2 \sim U(0,1)$
$r \leftarrow \sqrt{u}2$
$return\ (rcos\theta, rsin\theta)$

2.6 How to sample points on the surface of a unit sphere?

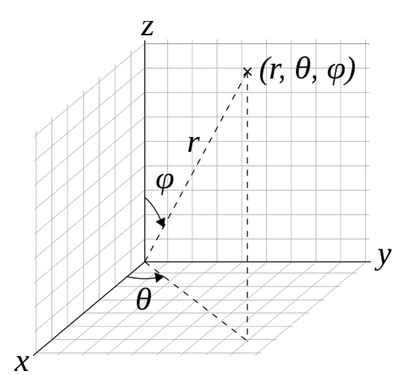


Figure 2: Polar coordinates

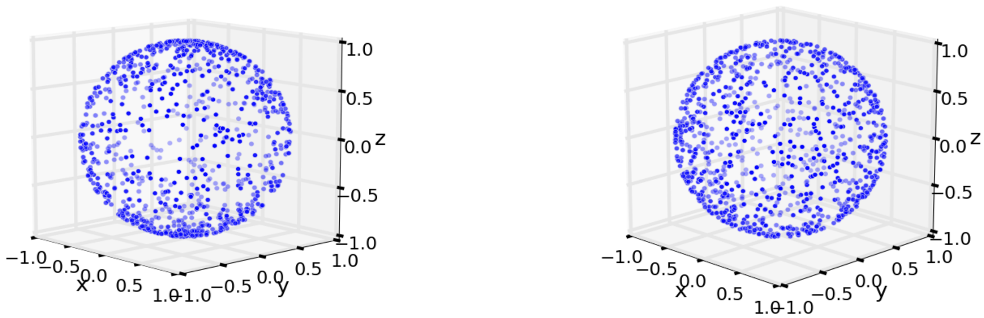
We look at possible approaches, taking inspiration from the previous questions posed.

2.6.1 Possible Approaches/Pitfalls

- **Independently sampling each polar coordinate.**

  Here, we independently sample ϕ and θ from appropriately scaled uniform distributions. Unfortunately, this suffers from the same problem as was seen in sampling points from a unit circle's area. ϕ corresponding to points closer to the poles has the same probability of being sampled when compared to the ϕ corresponding to points near the equator.

This results in the density of points to be greater at the poles when compared to the equator. An elegant way of observing this is to look at a globe with latitude and longitude lines. We can see that the lines are more congested towards the poles.



(a) Non-uniform Distribution

(b) Uniform Distribution

Figure 3: We sample the polar coordinates uniformly in (a) leading to a non-uniform distribution of points. We project points onto the surface after sampling from a Spherically Symmetric distribution in (b) leading to a uniform distribution on the surface

- **Sample uniformly on the circumference of the sphere and then sample z uniformly**

  In this approach, we uniformly sample θ(which is equivalent to uniformly sampling from the circumference of a circle), and then uniformly and independently sample z.

  This suffers from the same pitfall as the previous approach. Values of z closer to the poles would lead to greater concentration of points in these regions even when z is sampled uniformly.

- **Rejection Sampling by rejecting points not lying on a unit sphere**

  This is a method that should theoretically work. The pitfall is that the probability of a point lying on the surface of a unit sphere when sampling points from a cube is precisely 0. Hence, this method will be incredibly inefficient.

- **Sampling from a cube and projecting the points on the surface of a unit sphere**

  The points may be uniform in the cube, but the projection would not be uniform. This is because points in the cube close to the edges and corners of the cube create a canopy structure when being projected, which make the projections focused on certain parts in the unit sphere.

- **Rejection Sampling to sample from a unit sphere and then projecting the points on the surface**

  This is similar to the previous approach. The only difference is that we reject points that do not lie in the unit circle. This makes the distribution spherically symmetric. Hence, the projection of these points lead to an uniform distribution on the surface of the unit sphere.

Algorithm 7 Rejection Sampling

$x \leftarrow 1,\ y \leftarrow 1,\ z \leftarrow 1$
while $x^2 + y^2 + z^2 > 1$ **do**
  $x \sim U(0,1)$
  $y \sim U(0,1)$
  $z \sim U(0,1)$
end while
$r, \theta, \phi = euc2pol(x, y, z)$
$return\ (1, \theta, \phi)$

- **Sampling from a Gaussian distribution and then projecting the points on the surface**

  The previous approach worked because the underlying distribution before the projection was spherically symmetric. As it turns out, the standard Gaussian is Spherically Symmetric as well. Hence, we can just sample from a Gaussian and then project the points on the surface of a unit sphere. This will lead to a Uniform distribution on the surface.

Algorithm 8 Gaussian Sampling

$x \sim N(0,1)$
$y \sim N(0,1)$
$z \sim N(0,1)$
$r, \theta, \phi = euc2pol(x, y, z)$
$return\ (1, \theta, \phi)$