Contemporary

Engineering Economics

Fourth Edition

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PREFACE

What is "Contemporary" About Engineering Economics?

Decisions made during the engineering design phase of product development determine the majority of the costs associated with the manufacturing of that product (some say that this value may be as high as 85%). As design and manufacturing processes become more complex, engineers are making decisions that involve money more than ever before. With more than 80% of the total GDP (Gross Domestic Product) in the United States provided by the service sector, engineers work on various economic decision problems in the service sector as well. Thus, the competent and successful engineer in the twenty-first century must have an improved understanding of the principles of science, engineering, and economics, coupled with relevant design experience. Increasingly, in the new world economy, successful businesses will rely on engineers with such expertise.

Economic and design issues are inextricably linked in the product/service life cycle. Therefore, one of my strongest motivations for writing this text was to bring the realities of economics and engineering design into the classroom and to help students integrate these issues when contemplating many engineering decisions. Of course my underlying motivation for writing this book was not simply to address contemporary needs, but to address as well the ageless goal of all educators: to help students to learn. Thus, thoroughness, clarity, and accuracy of presentation of essential engineering economics were my aim at every stage in the development of the text.

Changes in the Fourth Edition

Much of the content has been streamlined to provide materials in depth and to reflect the challenges in contemporary engineering economics. Some of the highlighted changes are as follows:

- Chapter 13 "Real Options Analysis" is new and provides a new perspective on how engineers should manage risk in their strategic economic decision problems. Traditionally, risk is avoided in project analysis, which is a passive way of handling the matter. The goal of the real options approach is to provide a contemporary tool that will assist engineers so that they can actively manage the risk involved in long-term projects.
- Chapter 12 has been significantly revised to provide more probabilistic materials for the analytical treatment of risk and uncertainty. Risk simulation has been introduced by way of using @RISK.
- Three chapters have been merged with various materials from other chapters. Chapter 3 on cost concepts and behaviors has been moved to Part III and now appears as Chapter 8 "Cost Concepts Relevant to Decision Making"; it is now part of project cash flow analysis. Chapter 6 on principles of investing is now part of Chapter 4 "Understanding Money and Its Management." Materials from various chapters have been merged into a single chapter and now appear as Chapter 9 "Depreciation and Corporate Income Taxes".
- The chapter on the economic analysis in the public sector has been expanded and now appears as Chapter 16 "Economic Analysis in the Service Sector"; this revised chapter now provides economic analysis unique to service sectors beyond the government

sector. Increasingly, engineers seek their career in the service sector, such as healthcare, financial institutions, transportation, and logistics. In this chapter, we present some unique features that must be considered when evaluating investment projects in the service sector.

- All the end-of-chapter problems are revised to reflect the materials changes in the main text.
- All the chapter opening vignettes—a trademark of *Contemporary Engineering Economics*—have been completely replaced with more current and thought-provoking case studies.
- Self-study problems and FE practice questions are available as interactive quizzes with instant feedback as part of the book's new OneKey CourseCompass site. OneKey is an online resource for instructors and students; more detailed information about OneKey can be found in the *OneKey* section of this Preface. OneKey can be accessed via www.prenhall.com/onekey.
- Various Excel spreadsheet modeling techniques are introduced throughout the chapters and the original Excel files are provided online at the OneKey site .

Overview of the Text

Although it contains little advanced math and few truly difficult concepts, the introductory engineering economics course is often a curiously challenging one for the sophomores, juniors, and seniors who take it. There are several likely explanations for this difficulty.

1. The course is the student's first analytical consideration of money (a resource with which he or she may have had little direct contact beyond paying for tuition, housing, food, and textbooks).
2. The emphasis on theory may obscure for the student the fact that the course aims, among other things, to develop a very practical set of analytical tools for measuring project worth. This is unfortunate since, at one time or another, virtually every engineer—not to mention every individual—is responsible for the wise allocation of limited financial resources.
3. The mixture of industrial, civil, mechanical, electrical, and manufacturing engineering, and other undergraduates who take the course often fail to "see themselves" in the skills the course and text are intended to foster. This is perhaps less true for industrial engineering students, whom many texts take as their primary audience, but other disciplines are often motivationally shortchanged by a text's lack of applications that appeal directly to them.

Goal of the Text

This text aims not only to provide sound and comprehensive coverage of the concepts of engineering economics but also to address the difficulties of students outlined above, all of which have their basis in inattentiveness to the practical concerns of engineering economics. More specifically, this text has the following chief goals:

1. To build a thorough understanding of the theoretical and conceptual basis upon which the practice of financial project analysis is built.

2. To satisfy the very practical needs of the engineer toward making informed financial decisions when acting as a team member or project manager for an engineering project.
3. To incorporate all critical decision-making tools—including the most contemporary, computer-oriented ones that engineers bring to the task of making informed financial decisions.
4. To appeal to the full range of engineering disciplines for which this course is often required: industrial, civil, mechanical, electrical, computer, aerospace, chemical, and manufacturing engineering, as well as engineering technology.

Prerequisites

The text is intended for undergraduate engineering students at the sophomore level or above. The only mathematical background required is elementary calculus. For Chapters 12 and 13, a first course in probability or statistics is helpful but not necessary, since the treatment of basic topics there is essentially self-contained.

Taking Advantage of the Internet

The integration of computer use is another important feature of *Contemporary Engineering Economics*. Students have greater access to and familiarity with the various spreadsheet tools, and instructors have greater inclination either to treat these topics explicitly in the course or to encourage students to experiment independently.

A remaining concern is that the use of computers will undermine true understanding of course concepts. This text does not promote the use of trivial spreadsheet applications as a replacement for genuine understanding of and skill in applying traditional solution methods. Rather, it focuses on the computer's productivity-enhancing benefits for complex project cash flow development and analysis. Specifically, *Contemporary Engineering Economics* includes a robust introduction to computer automation in the form of Computer Notes, which are included in the optional OneKey course (www.prenhall.com/onekey).

Additionally, spreadsheets are introduced via Microsoft Excel examples. For spreadsheet coverage, the emphasis is on demonstrating a chapter concept that embodies some complexity that can be much more efficiently resolved on a computer than by traditional longhand solutions.

OneKey

Available as a special package, OneKey is Prentice Hall's exclusive new resource for instructors and students. Instructors have access online to all available course supplements and can create and assign tests, quizzes, or graded homework assignments. Students have access to interactive exercises, quizzes, and more. The following resources are available when an instructor adopts the text plus OneKey package:

- Interactive self-study quizzes organized by chapter with instant feedback, plus interactive FE Exam practice questions
- Computer notes with Excel files of selected example problems from the text.
- Case Studies: A collection of actual cases, two personal-finance and six industry-based, is now available. The investment projects detailed in the cases relate to a

CHAPTER ONE

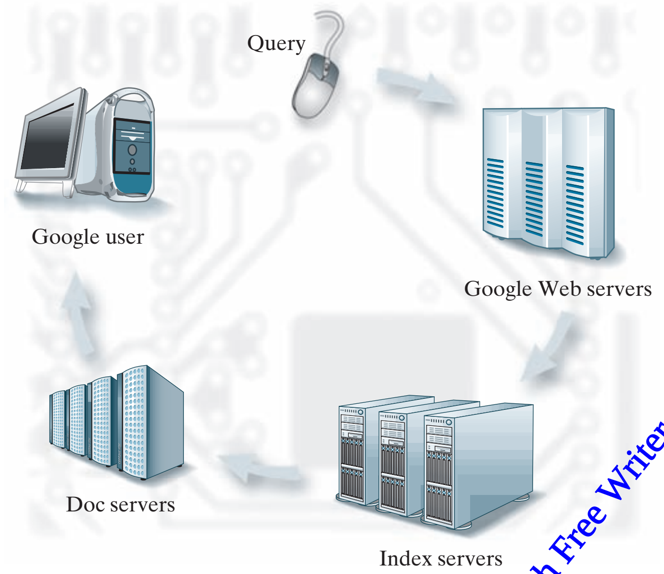
Engineering Economic Decisions

Google Cofounder Sergey Brin Comes to Class at Berkeley[1] Sergey Brin, cofounder of Google, showed up for class as a guest speaker at Berkeley on October 3, 2005. Casual and relaxed, Brin talked about how Google came to be, answered students' questions, and showed that someone worth $11 billion (give or take a billion) still can be comfortable in an old pair of blue jeans. Indistinguishable in dress, age, and demeanor from many of the students in the class, Brin covered a lot of ground in his remarks, but ultimately it was his unspoken message that was most powerful: To those with focus and passion, all things are possible. In his remarks to the class, Brin stressed simplicity. Simple ideas sometimes can change the world, he said. Likewise, Google started out with the simplest of ideas, with a global audience in mind. In the mid-1990s, Brin and Larry Page were Stanford students pursuing doctorates in computer science. Brin recalled that at that time there were some five major Internet search engines, the importance of searching was being de-emphasized, and the owners of these major search sites were focusing on creating portals with increased content offerings. "We believed we could build a better search. We had a simple idea—that not all pages are created equal. Some are more important," related Brin. Eventually, they developed a unique approach to solving one of computing's biggest challenges: retrieving relevant information from a massive set of data.

According to Google lore,[1] by January of 1996 Larry and Sergey had begun collaboration on a search engine called BackRub, named for its unique ability to analyze the "back links" pointing to a given website.

[1] *UC Berkeley News*, Oct. 4, 2005, UC Regents and Google Corporate Information: http://www.google.com/corporate/history.html.

How Google Works



1. The Web server sends the query to the index servers—it tells which pages contain the words that match the query.
2. The query travels to the Doc servers (which retrieve the stored documents) and snippets are generated to describe each search result.
3. The search results are returned to the user in a fraction of a second.

Larry, who had always enjoyed tinkering with machinery and had gained some "notoriety" for building a working printer out of Lego® bricks, took on the task of creating a new kind of server environment that used low-end PCs instead of big expensive machines. Afflicted by the perennial shortage of cash common to graduate students everywhere, the pair took to haunting the department's loading docks in hopes of tracking down newly arrived computers that they could borrow for their network. A year later, their unique approach to link analysis was earning BackRub a growing reputation among those who had seen it. Buzz about the new search technology began to build as word spread around campus. Eventually, in 1998 they decided to create a company named "Google" by raising $25 million from venture capital firms Kleiner Perkins Caufield & Byers and Sequoia Capital. Since taking their Internet search engine public in August 2004, the dynamic duo behind Google has seen their combined fortune soar to $22 billion. At a recent $400, Google trades at 90 times trailing earnings, after starting out at $85. The success has vaulted both Larry and Sergey into *Forbes* magazine's list of the 400 wealthiest Americans. The net worth of the pair is estimated at $11 billion each.

A Little Google History

- 1995
  - Developed in dorm room of Larry Page and Sergey Brin, graduate students at Stanford University
  - Nicknamed BackRub
- 1998
  - Raised $25 million to set up Google, Inc.
  - Ran 100,000 queries a day out of a garage in Menlo Park
- 2005
  - Over 4,000 employees worldwide
  - Over 8 billion pages indexed

The story of how the Google founders got motivated to invent a search engine and eventually transformed their invention to a multibillion-dollar business is a typical one. Companies such as Dell, Microsoft, and Yahoo all produce computer-related products and have market values of several billion dollars. These companies were all started by highly motivated young college students just like Brin. One thing that is also common to all these successful businesses is that they have capable and imaginative engineers who constantly generate good ideas for capital investment, execute them well, and obtain good results. You might wonder about what kind of role these engineers play in making such business decisions. In other words, what specific tasks are assigned to these engineers, and what tools and techniques are available to them for making such capital investment decisions? We answer these questions and explore related issues throughout this book.

CHAPTER LEARNING OBJECTIVES

After completing this chapter, you should understand the following concepts:

- The role of engineers in business.
- Types of business organization.
- The nature and types of engineering economic decisions.
- What makes the engineering economic decisions difficult.
- How a typical engineering project idea evolves in business.
- Fundamental principles of engineering economics.

1.1 Role of Engineers in Business

Yahoo, Apple Computer, Microsoft Corporation, and Sun Microsystems produce computer products and have a market value of several billion dollars each. These companies were all started by young college students with technical backgrounds. When they went into the computer business, these students initially organized their companies as proprietorships. As the businesses grew, they became partnerships and were eventually converted to corporations. This chapter begins by introducing the three primary forms of business organization and briefly discusses the role of engineers in business.

1.1.1 Types of Business Organization

As an engineer, you should understand the nature of the business organization with which you are associated. This section will present some basic information about the type of organization you should choose should you decide to go into business for yourself.

The three legal forms of business, each having certain advantages and disadvantages, are proprietorships, partnerships, and corporations.

Proprietorships

A **proprietorship** is a business owned by one individual. This person is responsible for the firm's policies, owns all its assets, and is personally liable for its debts. A proprietorship has two major advantages. First, it can be formed easily and inexpensively. No legal and organizational requirements are associated with setting up a proprietorship, and organizational costs are therefore virtually nil. Second, the earnings of a proprietorship are taxed at the owner's personal tax rate, which may be lower than the rate at which corporate income is taxed. Apart from personal liability considerations, the major disadvantage of a proprietorship is that it cannot issue stocks and bonds, making it difficult to raise capital for any business expansion.

Partnerships

A **partnership** is similar to a proprietorship, except that it has more than one owner. Most partnerships are established by a written contract between the partners. The contract normally specifies salaries, contributions to capital, and the distribution of profits and losses. A partnership has many advantages, among which are its low cost and ease of formation. Because more than one person makes contributions, a partnership typically has a larger amount of capital available for business use. Since the personal assets of all the partners stand behind the business, a partnership can borrow money more easily from a bank. Each partner pays only personal income tax on his or her share of a partnership's taxable income.

On the negative side, under partnership law each partner is liable for a business's debts. This means that the partners must risk all their personal assets—even those not invested in the business. And while each partner is responsible for his or her portion of the debts in the event of bankruptcy, if any partners cannot meet their pro rata claims, the remaining partners must take over the unresolved claims. Finally, a partnership has a limited life, insofar as it must be dissolved and reorganized if one of the partners quits.

Corporations

A **corporation** is a legal entity created under provincial or federal law. It is separate from its owners and managers. This separation gives the corporation four major advantages: (1) It can raise capital from a large number of investors by issuing stocks and bonds; (2) it permits easy transfer of ownership interest by trading shares of stock; (3) it allows limited liability—personal liability is limited to the amount of the individual's investment in the business; and (4) it is taxed differently than proprietorships and partnerships, and under certain conditions, the tax laws favor corporations. On the negative side, it is expensive to establish a corporation. Furthermore, a corporation is subject to numerous governmental requirements and regulations.

As a firm grows, it may need to change its legal form because the form of a business affects the extent to which it has control of its own operations and its ability to acquire funds. The legal form of an organization also affects the risk borne by its owners in case of bankruptcy and the manner in which the firm is taxed. Apple Computer, for example, started out as a two-man garage operation. As the business grew, the owners felt constricted by this form of organization: It was difficult to raise capital for business expansion; they felt that the risk of bankruptcy was too high to bear; and as their business income grew, their tax burden grew as well. Eventually, they found it necessary to convert the partnership into a corporation.

In the United States, the overwhelming majority of business firms are proprietorships, followed by corporations and partnerships. However, in terms of total business volume (dollars of sales), the quantity of business transacted by proprietorships and partnerships is several times less than that of corporations. Since most business is conducted by corporations, this text will generally address economic decisions encountered in that form of ownership.

1.1.2 Engineering Economic Decisions

What role do engineers play within a firm? What specific tasks are assigned to the engineering staff, and what tools and techniques are available to it to improve a firm's profits? Engineers are called upon to participate in a variety of decisions, ranging from manufacturing, through marketing, to financing decisions. We will restrict our focus, however, to various economic decisions related to engineering projects. We refer to these decisions as **engineering economic decisions**.

In manufacturing, engineering is involved in every detail of a product's production, from conceptual design to shipping. In fact, engineering decisions account for the majority (some say 85%) of product costs. Engineers must consider the effective use of capital assets such as buildings and machinery. One of the engineer's primary tasks is to plan for the acquisition of equipment (**capital expenditure**) that will enable the firm to design and produce products economically.

With the purchase of any fixed asset—equipment, for instance—we need to estimate the profits (more precisely, cash flows) that the asset will generate during its period of service. In other words, we have to make capital expenditure decisions based on predictions about the future. Suppose, for example, you are considering the purchase of a deburring machine to meet the anticipated demand for hubs and sleeves used in the production of gear couplings. You expect the machine to last 10 years. This decision thus involves an implicit 10-year sales forecast for the gear couplings, which means that a long waiting period will be required before you will know whether the purchase was justified.

An inaccurate estimate of the need for assets can have serious consequences. If you invest too much in assets, you incur unnecessarily heavy expenses. Spending too little on fixed assets, however, is also harmful, for then the firm's equipment may be too obsolete to produce products competitively, and without an adequate capacity, you may lose a portion of your market share to rival firms. Regaining lost customers involves heavy marketing expenses and may even require price reductions or product improvements, both of which are costly.

1.1.3 Personal Economic Decisions

In the same way that an engineer can play a role in the effective utilization of corporate financial assets, each of us is responsible for managing our personal financial affairs. After we have paid for nondiscretionary or essential needs, such as housing, food, clothing, and transportation, any remaining money is available for discretionary expenditures on items such as entertainment, travel, and investment. For money we choose to invest, we want to maximize the economic benefit at some acceptable risk. The investment choices are unlimited and include savings accounts, guaranteed investment certificates, stocks, bonds, mutual funds, registered retirement savings plans, rental properties, land, business ownership, and more.

How do you choose? The analysis of one's personal investment opportunities utilizes the same techniques that are used for engineering economic decisions. Again, the challenge is predicting the performance of an investment into the future. Choosing wisely can be very rewarding, while choosing poorly can be disastrous. Some investors in the energy stock Enron who sold prior to the fraud investigation became millionaires. Others, who did not sell, lost everything.

A wise investment strategy is a strategy that manages risk by diversifying investments. With such an approach, you have a number of different investments ranging from very low to very high risk and are in a number of business sectors. Since you do not have all your money in one place, the risk of losing everything is significantly reduced. (We discuss some of these important issues in Chapters 12 and 13.)

1.2 What Makes the Engineering Economic Decision Difficult?

The economic decisions that engineers make in business differ very little from the financial decisions made by individuals, except for the scale of the concern. Suppose, for example, that a firm is using a lathe that was purchased 12 years ago to produce pump shafts. As the production engineer in charge of this product, you expect demand to continue into the foreseeable future. However, the lathe has begun to show its age: It has broken frequently during the last 2 years and has finally stopped operating altogether. Now you have to decide whether to replace or repair it. If you expect a more efficient lathe to be available in the next 1 or 2 years, you might repair the old lathe instead of replacing it. The major issue is whether you should make the considerable investment in a new lathe now or later. As an added complication, if demand for your product begins to decline, you may have to conduct an economic analysis to determine whether declining profits from the project offset the cost of a new lathe.

Let us consider a real-world engineering decision problem on a much larger scale, as taken from an article from *The Washington Post*.[2] Ever since Hurricane Katrina hit the city of New Orleans in August 2005, the U.S. federal government has been under pressure to show strong support for rebuilding levees in order to encourage homeowners and businesses to return to neighborhoods that were flooded when the city's levees crumbled under Katrina's storm surge. Many evacuees have expressed reluctance to rebuild without assurances that New Orleans will be made safe from future hurricanes, including Category 5 storms, the most severe. Some U.S. Army Corps of Engineers officers estimated that it would cost more than $1.6 billion to restore the levee system merely to its design strength—that is, to withstand a Category 3 storm. New design features would include floodgates on several key canals, as well as stone-and-concrete fortification of some of

[2] Joby Warrick and Peter Baker, "Bush Pledges $1.5 Billion for New Orleans—Proposal Would Double Aid From U.S. for Flood Protection," *The Washington Post*, Dec. 16, 2005, p. A03.

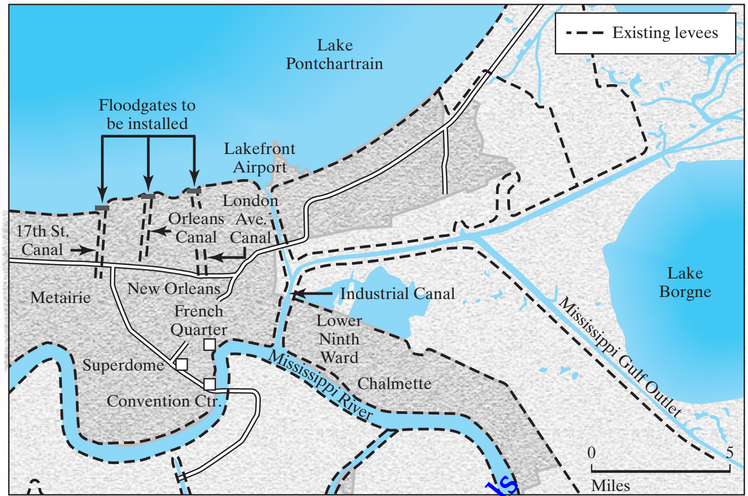


Figure 1.1 The White House pledged $1.5 billion to armor existing New Orleans levees with concrete and stone, build floodgates on three canals, and upgrade the city's pumping system.

the city's earthen levees, as illustrated in Figure 1.1. Donald E. Powell, the administration's coordinator of post-Katrina recovery, insisted that the improvements would make the levee system much stronger than it had been in the past. But he declined to say whether the administration would support further upgrades of the system to Category 5 protection, which would require substantial reengineering of existing levees at a cost that could, by many estimates, exceed $30 billion.

Obviously, this level of engineering decision is far more complex and more significant than a business decision about when to introduce a new product. Projects of this nature involve large sums of money over long periods of time, and it is difficult to justify the cost of the system purely on the basis of economic reasoning, since we do not know when another Katrina-strength storm will be on the horizon. Even if we decide to rebuild the levee systems, should we build just enough to withstand a Category 3 storm, or should we build to withstand a Category 5 storm? Any engineering economic decision pertaining to this type of extreme event will be extremely difficult to make.

In this book, we will consider many types of investments—personal investments as well as business investments. The focus, however, will be on evaluating engineering projects on the basis of their economic desirability and on dealing with investment situations that a typical firm faces.

1.3 Economic Decisions versus Design Decisions

Economic decisions differ in a fundamental way from the types of decisions typically encountered in engineering design. In a design situation, the engineer utilizes known physical properties, the principles of chemistry and physics, engineering design correlations, and engineering judgment to arrive at a workable and optimal design. If the judgment is sound, the calculations are done correctly, and we ignore technological advances, the design is time invariant. In other words, if the engineering design to meet a particular need is done today, next year, or in five years' time, the final design would not change significantly.

In considering economic decisions, the measurement of investment attractiveness, which is the subject of this book, is relatively straightforward. However, the information required in such evaluations always involves predicting or forecasting product sales, product selling prices, and various costs over some future time frame—5 years, 10 years, 25 years, etc.

All such forecasts have two things in common. First, they are never completely accurate compared with the actual values realized at future times. Second, a prediction or forecast made today is likely to be different from one made at some point in the future. It is this ever-changing view of the future that can make it necessary to revisit and even change previous economic decisions. Thus, unlike engineering design, the conclusions reached through economic evaluation are not necessarily time invariant. Economic decisions have to be based on the best information available at the time of the decision and a thorough understanding of the uncertainties in the forecasted data.

1.4 Large-Scale Engineering Projects

In the development of any product, a company's engineers are called upon to translate an idea into reality. A firm's growth and development depend largely upon a constant flow of ideas for new products, and for the firm to remain competitive, it has to make existing products better or produce them at a lower cost. In the next section, we present an example of how a design engineer's idea eventually turned into an innovative automotive product.

1.4.1 How a Typical Project Idea Evolves

The Toyota Motor Corporation introduced the world's first mass-produced car powered by a combination of gasoline and electricity. Known as the Prius, this vehicle is the first of a new generation of Toyota cars whose engines cut air pollution dramatically and boost fuel efficiency to significant levels. Toyota, in short, wants to launch and dominate a new "green" era for automobiles—and is spending \$1.5 billion to do it. Developed for the Japanese market initially, the Prius uses both a gasoline engine and an electric motor as its motive power source. The Prius emits less pollution than ordinary cars, and it gets more mileage, which means less output of carbon dioxide. Moreover, the Prius gives a highly responsive performance and smooth acceleration. The following information from *BusinessWeek*[3] illustrates how a typical strategic business decision is made by the engineering staff of a larger company. Additional information has been provided by Toyota Motor Corporation.

Why Go for a Greener Car?

Toyota first started to develop the Prius in 1995. Four engineers were assigned to figure out what types of cars people would drive in the 21st century. After a year of research, a

[3] Emily Thornton (Tokyo), Keith Naughton (Detroit), and David Woodruff, "Japan's Hybrid Cars—Toyota and rivals are betting on pollution fighters—Will they succeed?" *BusinessWeek*, Dec. 4, 1997.

chief engineer concluded that Toyota would have to sell cars better suited to a world with scarce natural resources. He considered electric motors. But an electric car can travel only 215 km before it must be recharged. Another option was fuel-cell cars that run on hydrogen. But he suspected that mass production of this type of car might not be possible for another 15 years. So the engineer finally settled on a hybrid system powered by an electric motor, a nickel–metal hydride battery, and a gasoline engine. From Toyota's perspective, it is a big bet, as oil and gasoline prices are bumping along at record lows at that time. Many green cars remain expensive and require trade-offs in terms of performance. No carmaker has ever succeeded in selling consumers en masse something they have never wanted to buy: *cleaner air*. Even in Japan, where a liter of regular gas can cost as much as $1, carmakers have trouble pushing higher fuel economy and lower carbon emissions. Toyota has several reasons for going green. In the next century, as millions of new car owners in China, India, and elsewhere take to the road, Toyota predicts that gasoline prices will rise worldwide. At the same time, Japan's carmakers expect pollution and global warming to become such threats that governments will enact tough measures to clean the air.

What Is So Unique in Design?

It took Toyota engineers two years to develop the current power system in the Prius. The car comes with a dual engine powered by both an electric motor and a newly developed 1.5-liter gasoline engine. When the engine is in use, a special "power split device" sends some of the power to the driveshaft to move the car's wheels. The device also sends some of the power to a generator, which in turn creates electricity, to either drive the motor or recharge the battery. Thanks to this variable transmission, the Prius can switch back and forth between motor and engine, or employ both, without creating any jerking motion. The battery and electric motor propel the car at slow speeds. At normal speeds, the electric motor and gasoline engine work together to power the wheels. At higher speeds, the battery kicks in extra power if the driver must pass another car or zoom up a hill.

When the car decelerates and brakes, the motor converts the vehicle's kinetic energy into electricity and sends it through an inverter to be stored in the battery. (See Figure 1.2.)

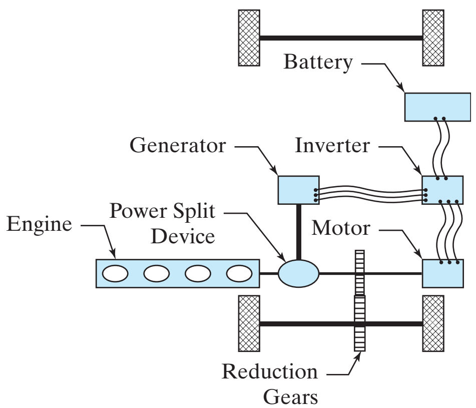


Figure 1.2 Arrangement of components of the Toyota Hybrid System II.
(*Source: Evaluation of 2004 Toyota Prius Hybrid Electric Drive System*, Oak Ridge National Laboratory, ONR/TM2004/247, U.S. Department of Energy.)

- When engine efficiency is low, such as during start-up and with midrange speeds, motive power is provided by the motor alone, using energy stored in the battery.
- Under normal driving conditions, overall efficiency is optimized by controlling the power allocation so that some of the engine power is used for turning the generator to supply electricity for the motor while the remaining power is used for turning the wheels.
- During periods of acceleration, when extra power is needed, the generator supplements the electricity being drawn from the battery, so the motor is supplied with the required level of electrical energy.
- While decelerating and braking, the motor acts as a generator that is driven by the wheels, thus allowing the recovery of kinetic energy. The recovered kinetic energy is converted to electrical energy that is stored in the battery.
- When necessary, the generator recharges the battery to maintain sufficient reserves.
- When the vehicle is not moving and when the engine moves outside of certain speed and load conditions, the engine stops automatically.

So the car's own movement, as well as the power from the gasoline engine, provides the electricity needed. The energy created and stored by deceleration boosts the car's efficiency. So does the automatic shutdown of the engine when the car stops at a light. At higher speeds and during acceleration, the companion electric motor works harder, allowing the gas engine to run at peak efficiency. Toyota claims that, in tests, its hybrid car has boosted fuel economy by 100% and engine efficiency by 80%. The Prius emits about half as much carbon dioxide, and about one-tenth as much carbon monoxide, hydrocarbons, and nitrous oxide, as conventional cars.

Is It Safe to Drive on a Rainy Day?

Yet, major hurdles remain to be overcome in creating a mass market for green vehicles. Car buyers are not anxious enough about global warming to justify a "sea-level change" in automakers' marketing. Many of Japan's innovations run the risk of becoming impressive technologies meant for the masses, but bought only by the elite. The unfamiliarity of green technology can also frighten consumers. The Japanese government sponsors festivals at which people can test drive alternative-fuel vehicles. But some turned down that chance on a rainy weekend in May because they feared that riding in an electric car might electrocute them. An engineer points out, "It took 20 years for the automatic transmission to become popular in Japan." It certainly would take that long for many of these technologies to catch on.

How Much Would It Cost?

The biggest question remaining about the mass production of the vehicle concerns its production cost. Costs will need to come down for Toyota's hybrid to be competitive around the world, where gasoline prices are lower than in Japan. Economies of scale will help as production volumes increase, but further advances in engineering also will be essential. With its current design, Prius's monthly operating cost would be roughly twice that of a conventional automobile. Still, Toyota believes that it will sell more than 12,000 Prius models to Japanese drivers during the first year. To do that, it is charging just $17,000 a car. The company insists that it will not lose any money on the Prius, but rivals and analysts estimate that, at current production levels, the cost of building a Prius could be as much as $32,000. If so, Toyota's low-price strategy will generate annual losses of more than $100 million on the new compact.

Will There Be Enough Demand?

Why buy a $17,000 Prius when a $13,000 Corolla is more affordable? It does not help Toyota that it is launching the Prius in the middle of a sagging domestic car market. Nobody really knows how big the final market for hybrids will be. Toyota forecasts that hybrids will account for a third of the world's auto market as early as 2005, but Japan's Ministry of International Trade and Industry expects 2.4 million alternative-fuel vehicles, including hybrids, to roam Japan's backstreets by 2010. Nonetheless, the Prius has set a new standard for Toyota's competitors. There seems to be no turning back for the Japanese. The government may soon tell carmakers to slash carbon dioxide emissions by 20% by 2010. And it wants them to cut nitrous oxide, hydrocarbon, and carbon monoxide emissions by 80%. The government may also soon give tax breaks to consumers who buy green cars.

Prospects for the Prius started looking good: Total sales for Prius reached over 7,700 units as of June 1998. With this encouraging sales trend, Toyota finally announced that the Prius would be introduced in the North American and European markets by the year 2000. The total sales volume will be approximately 20,000 units per year in the North American and European market combined. As with the 2000 North American and European introduction, Toyota is planning to use the next two years to develop a hybrid vehicle optimized to the usage conditions of each market.

What Is the Business Risk in Marketing the Prius?

Engineers at Toyota Motors have stated that California would be the primary market for the Prius outside Japan, but they added that an annual demand of 50,000 cars would be necessary to justify production. Despite Toyota management's decision to introduce the hybrid electric car into the U.S. market, the Toyota engineers were still uncertain whether there would be enough demand. Furthermore, competitors, including U.S. automakers, just do not see how Toyota can achieve the economies of scale needed to produce green cars at a profit. The primary advantage of the design, however, is that the Prius can cut auto pollution in half. This is a feature that could be very appealing at a time when government air-quality standards are becoming more rigorous and consumer interest in the environment is strong. However, in the case of the Prius, if a significant reduction in production cost never materializes, demand may remain insufficient to justify the investment in the green car.

1.4.2 Impact of Engineering Projects on Financial Statements

Engineers must understand the business environment in which a company's major business decisions are made. It is important for an engineering project to generate profits, but it also must strengthen the firm's overall financial position. How do we measure Toyota's success in the Prius project? Will enough Prius models be sold, for example, to keep the green-engineering business as Toyota's major source of profits? While the Prius project will provide comfortable, reliable, low-cost driving for the company's customers, the bottom line is its financial performance over the long run.

Regardless of a business's form, each company has to produce basic financial statements at the end of each operating cycle (typically a year). These financial statements provide the basis for future investment analysis. In practice, we seldom make investment decisions solely on the basis of an estimate of a project's profitability, because we must also consider the overall impact of the investment on the financial strength and position of the company.

Suppose that you were the president of the Toyota Corporation. Suppose further that you even hold some shares in the company, making you one of the company's many owners. What objectives would you set for the company? While all firms are in business in hopes of making a profit, what determines the market value of a company are not profits per se, but cash flow. It is, after all, available cash that determines the future investments and growth of the firm. Therefore, one of your objectives should be to increase the company's value to its owners (including yourself) as much as possible. To some extent, the market price of your company's stock represents the value of your company. Many factors affect your company's market value: present and expected future earnings, the timing and duration of those earnings, and the risks associated with them. Certainly, any successful investment decision will increase a company's market value. Stock price can be a good indicator of your company's financial health and may also reflect the market's attitude about how well your company is managed for the benefit of its owners.

1.4.3 A Look Back in 2005: Did Toyota Make the Right Decision?

Clearly, there were many doubts and uncertainties about the market for hybrids in 1998. Even Toyota engineers were not sure that there would be enough demand in the U.S. market to justify the production of the vehicle. Seven years after the Prius was introduced, it turns out that Toyota's decision to go ahead was the right decision. The continued success of the vehicle led to the launching of a second-generation Prius at the New York Motor Show in 2003. This car delivered higher power and better fuel economy than its predecessor. Indeed, the new Prius proved that Toyota has achieved its goal: to create an eco-car with high-level environmental performance but with the conventional draw of a modern car. These features, combined with its efficient handling and attractive design, are making the Prius a popular choice of individuals and companies alike. In fact, Toyota has already announced that it will double Prius production for the U.S. market, from 50,000 to 100,000 units annually, but even that may fall short of demand if oil prices continue to increase in the future.

Toyota made its investors happy, as the public liked the new hybrid car, resulting in an increased demand for the product. This, in turn, caused stock prices, and hence shareholder wealth, to increase. In fact, the new, heavily promoted, green car turned out to be a market leader in its class and contributed to sending Toyota's stock to an all-time high in late 2005.[4] Toyota's market value continued to increase well into 2006. Any successful investment decision on Prius's scale will tend to increase a firm's stock prices in the marketplace and promote long-term success. Thus, in making a large-scale engineering project decision, we must consider its possible effect on the firm's market value. (In Chapter 2, we discuss the financial statements in detail and show how to use them in our investment decision making.)

1.5 Common Types of Strategic Engineering Economic Decisions

The story of how the Toyota Corporation successfully introduced a new product and became the market leader in the hybrid electric car market is typical: Someone had a good idea, executed it well, and obtained good results. Project ideas such as the Prius can originate from many different levels in an organization. Since some ideas will be good, while others

[4] Toyota Motor Corporation, *Annual Report, 2005*.

will not, we need to establish procedures for screening projects. Many large companies have a specialized project analysis division that actively searches for new ideas, projects, and ventures. Once project ideas are identified, they are typically classified as (1) equipment or process selection, (2) equipment replacement, (3) new product or product expansion, (4) cost reduction, or (5) improvement in service or quality. This classification scheme allows management to address key questions: Can the existing plant, for example, be used to attain the new production levels? Does the firm have the knowledge and skill to undertake the new investment? Does the new proposal warrant the recruitment of new technical personnel? The answers to these questions help firms screen out proposals that are not feasible, given a company's resources.

The Prius project represents a fairly complex engineering decision that required the approval of top executives and the board of directors. Virtually all big businesses face investment decisions of this magnitude at some time. In general, the larger the investment, the more detailed is the analysis required to support the expenditure. For example, expenditures aimed at increasing the output of existing products or at manufacturing a new product would invariably require a very detailed economic justification. Final decisions on new products, as well as marketing decisions, are generally made at a high level within the company. By contrast, a decision to repair damaged equipment can be made at a lower level. The five classifications of project ideas are as follows:

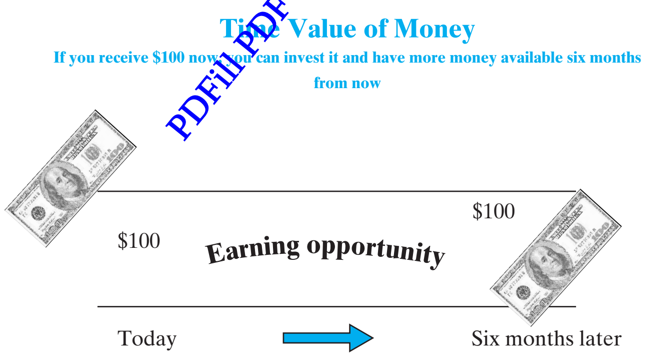
- **Equipment or Process Selection.** This class of engineering decision problems involves selecting the best course of action out of several that meet a project's requirements. For example, which of several proposed items of equipment shall we purchase for a given purpose? The choice often hinges on which item is expected to generate the largest savings (or the largest return on the investment). For example, the choice of material will dictate the manufacturing process for the body panels in the automobile. Many factors will affect the ultimate choice of the material, and engineers should consider all major cost elements, such as the cost of machinery and equipment, tooling, labor, and material. Other factors may include press and assembly, production and engineered scrap, the number of dies and tools, and the cycle times for various processes.
- **Equipment Replacement.** This category of investment decisions involves considering the expenditure necessary to replace worn-out or obsolete equipment. For example, a company may purchase 10 large presses, expecting them to produce stamped metal parts for 10 years. After 5 years, however, it may become necessary to produce the parts in plastic, which would require retiring the presses early and purchasing plastic molding machines. Similarly, a company may find that, for competitive reasons, larger and more accurate parts are required, making the purchased machines become obsolete earlier than expected.
- **New Product or Product Expansion.** Investments in this category increase company revenues if output is increased. One common type of expansion decision includes decisions about expenditures aimed at increasing the output of existing production or distribution facilities. In these situations, we are basically asking, "Shall we build or otherwise acquire a new facility?" The expected future cash inflows in this investment category are the profits from the goods and services produced in the new facility. A second type of expenditure decision includes considering expenditures necessary to produce a new product or to expand into a new geographic area. These projects normally require large sums of money over long periods.

- **Cost Reduction.** A cost-reduction project is a project that attempts to lower a firm's operating costs. Typically, we need to consider whether a company should buy equipment to perform an operation currently done manually or spend money now in order to save more money later. The expected future cash inflows on this investment are savings resulting from lower operating costs.
- **Improvement in Service or Quality.** Most of the examples in the previous sections were related to economic decisions in the manufacturing sector. The decision techniques we develop in this book are also applicable to various economic decisions related to improving services or quality of product.

1.6 Fundamental Principles of Engineering Economics

This book is focused on the principles and procedures engineers use to make sound economic decisions. To the first-time student of engineering economics, anything related to money matters may seem quite strange when compared to other engineering subjects. However, the decision logic involved in solving problems in this domain is quite similar to that employed in any other engineering subject. There are fundamental principles to follow in engineering economics that unite the concepts and techniques presented in this text, thereby allowing us to focus on the logic underlying the practice of engineering economics.

- **Principle 1: A nearby penny is worth a distant dollar.** A fundamental concept in engineering economics is that money has a time value associated with it. Because we can earn interest on money received today, it is better to receive money earlier than later. This concept will be the basic foundation for all engineering project evaluation.



- **Principle 2: All that counts are the differences among alternatives.** An economic decision should be based on the *differences* among the alternatives considered. All that is common is irrelevant to the decision. Certainly, any economic decision is no better than the alternatives being considered. Thus, an economic decision should be based on the objective of making the best use of limited resources. Whenever a choice is made, something is given up. The opportunity cost of a choice is the value of the best alternative given up.

Comparing Buy versus Lease

Whatever you decide, you need to spend the same amount of money on fuel and maintenance

Option	Monthly Fuel Cost	Monthly Maintenance	Cash Outlay at Signing	Monthly Payment	Salvage Value at End of Year 3
Buy	$960	$550	$6,500	$350	$9,000
Lease	$960	$550	$2,400	$550	0

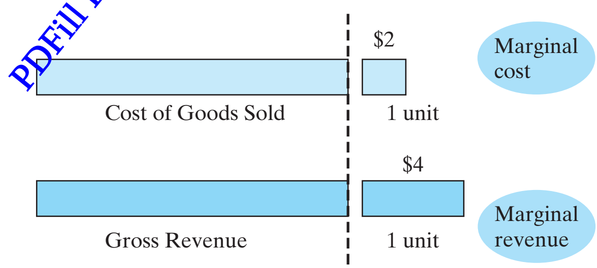
Irrelevant items in decision making

- **Principle 3: Marginal revenue must exceed marginal cost.** Effective decision making requires comparing the additional costs of alternatives with the additional benefits. Each decision alternative must be justified on its own economic merits before being compared with other alternatives. Any increased economic activity must be justified on the basis of the fundamental economic principle that marginal revenue must exceed marginal cost. Here, *marginal revenue* means the additional revenue made possible by increasing the activity by one unit (or small unit). *Marginal cost* has an analogous definition. Productive resources—the natural resources, human resources, and capital goods available to make goods and services—are limited. Therefore, people cannot have all the goods and services they want; as a result, they must choose some things and give up others.

Marginal cost: The cost associated with one additional unit of production, also called the incremental cost.

Marginal Analysis

To justify your action, marginal revenue must exceed marginal cost



- **Principle 4: Additional risk is not taken without the expected additional return.** For delaying consumption, investors demand a minimum return that must be greater than the anticipated rate of inflation or any perceived risk. If they didn't receive enough to compensate for anticipated inflation and the perceived investment risk, investors would purchase whatever goods they desired ahead of time or invest in assets that would provide a sufficient return to compensate for any loss from inflation or potential risk.

Risk-return tradeoff: Invested money can render higher profits only if it is subject to the possibility of being lost.

Risk and Return Trade-Off

Expected returns from bonds and stocks are normally higher than the expected return from a savings account

Investment Class	Potential Risk	Expected Return
Savings account (cash)	Low/None	1.5%
Bond (debt)	Moderate	4.8%
Stock (equity)	High	11.5%

The preceding four principles are as much statements of common sense as they are theoretical precepts. These principles provide the logic behind what is to follow. We build on them and attempt to draw out their implications for decision making. As we continue, keep in mind that, while the topics being treated may change from chapter to chapter, the logic driving our treatment of them is constant and rooted in the four fundamental principles.

SUMMARY

- This chapter has given us an overview of a variety of engineering economic problems that commonly are found in the business world. We examined the role, and the increasing importance, of engineers in the firm, as evidenced by the development at Toyota of the Prius, a hybrid vehicle. Commonly, engineers are called upon to participate in a variety of strategic business decisions ranging from product design to marketing.
- The term **engineering economic decision** refers to any investment decision related to an engineering project. The facet of an economic decision that is of most interest from an engineer's point of view is the evaluation of costs and benefits associated with making a capital investment.
- The five main types of engineering economic decisions are (1) equipment or process selection, (2) equipment replacement, (3) new product or product expansion, (4) cost reduction, and (5) improvement in service or quality.
- The factors of **time** and **uncertainty** are the defining aspects of any investment project.
- The four fundamental principles that must be applied in all engineering economic decisions are (1) the time value of money, (2) differential (incremental) cost and revenue, (3) marginal cost and revenue, and (4) the trade-off between risk and reward.

CHAPTER TWO

Understanding Financial Statements

***Dell Manages Profitability, Not Inventory*[1]** In 1994, Dell was a struggling second-tier PC maker. Like other PC makers, Dell ordered its components in advance and carried a large amount of component inventory. If its forecasts were wrong, Dell had major write-downs. Then Dell began to implement a new business model. Its operations had always featured a build-to-order process with direct sales to customers, but Dell took a series of ingenious steps to eliminate its inventories. The results were spectacular.

Over a four-year period, Dell's revenues grew from \$2 billion to \$16 billion, a 50 % annual growth rate. Earnings per share increased by 62 % per year. Dell's stock price increased by more than 17,000 % in a little over eight years. In 1998, Dell's return on invested capital was 217 %, and the company had \$1.8 billion in cash. (The rapid growth

1983	Michael Dell starts business of preformatting IBM PC HDs on weekends
1985	**\$6 million** sales, upgrading IBM compatibles for local businesses
1986	**\$70 million** sales; focus on assembling own line of PCs
1990	**\$500 million** sales with an extensive line of products
1996	Dell goes online; \$1 million per day in online sales; **\$5.3B** in annual sales
1997	Dell online sales at \$3 million per day; 50% growth rate for third consecutive year, **\$7.8B** in total annual sales.
2005	**\$49.2B** in sales

[1] Jonathan Byrnes, "Dell Manages Profitability, Not Inventory," Harvard Business School, Working Knowledge, June 2, 2003.

continued, and the company's sales revenues finally reached $50 billion in 2005.)

Profitability management—coordinating a company's day-to-day activities through careful forethought and attentive oversight—was at the core of Dell's transformation in this critical period. Dell created a tightly aligned business model that enabled it to dispense with the need for its component inventories. Not only was capital not needed, but the change generated enormous amounts of cash that Dell used to fuel its growth. How did Dell do it?

Account selection. Dell purposely selected customers with relatively predictable purchasing patterns and low service costs. The company developed a core competence in targeting customers and kept a massive database for this purpose.

The remainder of Dell's business involved individual consumers. To obtain stable demand in this segment, Dell used higher-end products and those with the latest technology to target second-time buyers who had regular upgrade purchase patterns, required little technical support, and paid by credit card.

Demand management. Dell's core philosophy of actively managing demand in real time, or "selling what you have," rather than making what you want to sell, was a critical driver of Dell's successful profitability management. Without this critical element, Dell's business model simply would not have been effective.

Product life-cycle management. Because Dell's customers were largely high-end repeat buyers who rapidly adopted new technology, Dell's marketing could focus on managing product life-cycle transitions.

Supplier management. Although Dell's manufacturing system featured a combination of build-product-to-order and buy-component-to-plan processes, the company worked closely with its suppliers to introduce more flexibility into its system.

Forecasting. Dell's forecast accuracy was about 70 to 75%, due to its careful account selection. Demand management, in turn, closed the forecast gap. When in doubt, Dell managers overforecast on high-end products because it was easier to sell up and high-end products had a longer shelf life.

Liquidity management. Direct sales were explicitly targeted toward high-end customers who paid with a credit card. These sales had a 4-day cash conversion cycle, while Dell took 45 days to pay its vendors. This approach generated a huge amount of liquidity that helped finance Dell's rapid growth and limited its external financing needs. Dell's cash engine was a key underlying factor that enabled it to earn such extraordinarily high returns.

If you want to explore investing in Dell stock, what information would you go by? You would certainly prefer that Dell have a record of accomplishment of profitable operations, earning a profit (net income) year after year. The company would need a steady stream of cash coming in and a manageable level of debt. How would you determine whether the company met these criteria? Investors commonly use the financial statements contained in the annual report as a starting point in forming expectations about future levels of earnings and about the firm's riskiness.

Net worth is the amount by which a *company's* or individual's assets exceed the company's or individual's liabilities.

Before making any financial decision, it is good to understand an elementary aspect of your financial situation—one that you'll also need for retirement planning, estate planning, and, more generally, to get an answer to the question, "How am I doing?" It is called your **net worth**. If you decided to invest $10,000 in Dell stocks, how would that affect your net worth? You may need this information for your own financial planning, but it is routinely required whenever you have to borrow a large sum of money from a financial institution. For example, when you are buying a home, you need to apply for a mortgage. Invariably, the bank will ask you to submit your net-worth statement as a part of loan processing. Your net-worth statement is a snapshot of where you stand financially at a given point in time. The bank will determine how creditworthy you are by examining your net worth. In a similar way, a corporation prepares the same kind of information for its financial planning or to report its financial health to stockholders or investors. The reporting document is known as the financial statements. We will first review the basics of figuring out the personal net worth and then illustrate how any investment decision will affect this net-worth statement. Understanding the relationship between net worth and investing decisions will enhance one's overall understanding of how a company manages its assets in business operations.

CHAPTER LEARNING OBJECTIVES

After completing this chapter, you should understand the following concepts:

- The role of accounting in economic decisions.
- Four types of financial statements prepared for investors and regulators.
- How to read the balance sheet statement.
- How to use the income statement to manage a business.
- The sources and uses of cash in business operation.
- How to conduct the ratio analysis and what the numbers really mean.

2.1 Accounting: The Basis of Decision Making

We need financial information when we are making business decisions. Virtually all businesses and most individuals keep accounting records to aid in making decisions. As illustrated in Figure 2.1, accounting is the information system that measures business activities, processes the resulting information into reports, and communicates the results to decision makers. For this reason, we call accounting "the language of business." The better you understand this language, the better you can manage your financial well-being, and the better your financial decisions will be.

Personal financial planning, education expenses, loans, car payments, income taxes, and investments are all based on the information system we call accounting. The uses of accounting information are many and varied:

- **Individual people** use accounting information in their day-to-day affairs to manage bank accounts, to evaluate job prospects, to make investments, and to decide whether to rent an apartment or buy a house.
- **Business managers** use accounting information to set goals for their organizations, to evaluate progress toward those goals, and to take corrective actions if necessary. Decisions based on accounting information may include which building or equipment to purchase, how much merchandise to keep on hand as inventory, and how much cash to borrow.
- **Investors and creditors** provide the money a business needs to begin operations. To decide whether to help start a new venture, potential investors evaluate what income they can expect on their investment. Such an evaluation involves analyzing the financial statements of the business. Before making a loan, banks determine the borrower's ability to meet scheduled payments. This kind of evaluation includes a projection of future operations and revenue, based on accounting information.

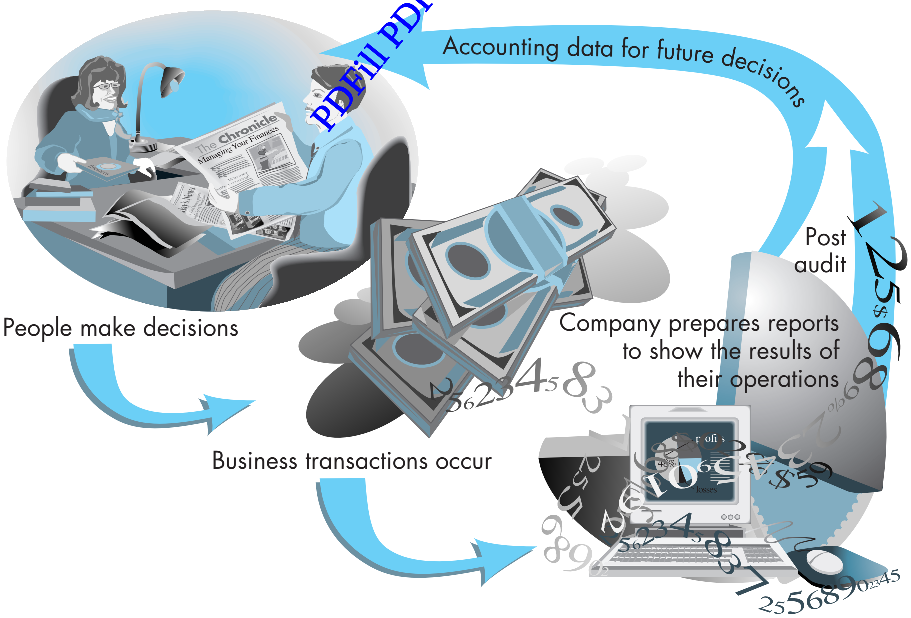


Figure 2.1 The accounting system, which illustrates the flow of information.

An essential product of an accounting information system is a series of financial statements that allows people to make informed decisions. For personal use, the net-worth statement is a snapshot of where you stand financially at a given point in time. You do that by adding your assets—such as cash, investments, and pension plans—in one column and your liabilities—or debts—in the other. Then subtract your liabilities from your assets to find your net worth. In other words, your net worth is what you would be left with if you sold everything and paid off all you owe. For business use, financial statements are the documents that report financial information about a business entity to decision makers. They tell us how a business is performing and where it stands financially. Our purpose is not to present the bookkeeping aspects of accounting, but to acquaint you with financial statements and to give you the basic information you need to make sound engineering economic decisions through the remainder of the book.

2.2 Financial Status for Businesses

Just like figuring out your personal wealth, all businesses must prepare their financial status. Of the various reports corporations issue to their stockholders, the annual report is by far the most important, containing basic financial statements as well as management's opinion of the past year's operations and the firm's future prospects. What would managers and investors want to know about a company at the end of the fiscal year? Following are four basic questions that managers or investors are likely to ask:

- What is the company's financial position at the end of the fiscal period?
- How well did the company operate during the fiscal period?
- On what did the company decide to use its profits?
- How much cash did the company generate and spend during the fiscal period?

As illustrated in Figure 2.2, the answer to each of these questions is provided by one of the following financial statements: the balance sheet statement, the income statement, the statement of retained earnings, and the cash flow statement. The fiscal year (or operating

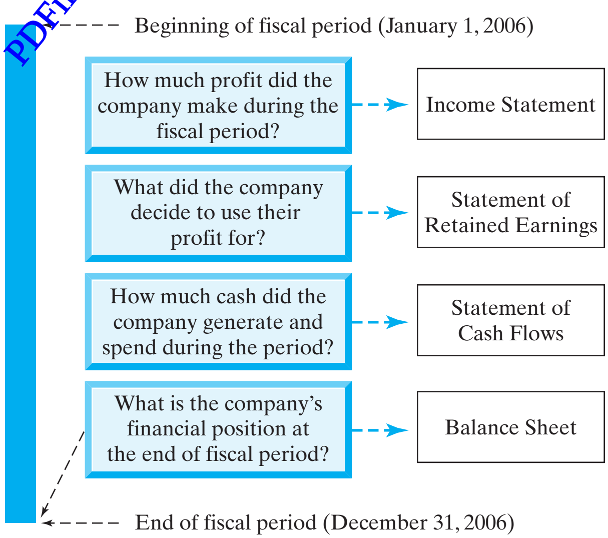


Figure 2.2 Information reported on the financial statements.

cycle) can be any 12-month term, but is usually January 1 through December 31 of a calendar year.

As mentioned in Section 1.1.2, one of the primary responsibilities of engineers in business is to plan for the acquisition of equipment (capital expenditure) that will enable the firm to design and produce products economically. This type of planning will require an estimation of the savings and costs associated with the acquisition of equipment and the degree of risk associated with execution of the project. Such an estimation will affect the business' **bottom line** (profitability), which will eventually affect the firm's stock price in the marketplace. Therefore, engineers should understand the various financial statements in order to communicate with upper management regarding the status of a project's profitability. The situation is summarized in Figure 2.3.

Bottom line is slang for net income or accounting profit.

For illustration purposes, we use data taken from Dell Corporation, manufacturer of a wide range of computer systems, including desktops, notebooks, and workstations, to discuss the basic financial statements. In 1984, Michael Dell began his computer business at the University of Texas in Austin, often hiding his IBM PC in his roommate's bathtub when his family came to visit. His dorm-room business officially became Dell Computer Corporation on May 3, 1984. Since 2001, Dell has become the number-one and fastest growing among all major computer system companies worldwide with 55,200 employees around the globe. Dell's pioneering "direct model" is a simple concept involving selling personal computer systems directly to customers. It offers (1) in-person relationships with corporate and institutional customers; (2) telephone and Internet purchasing (the latter averaging \$50 million a day in 2001); (3) built-to-order computer systems; (4) phone and online technical support; and (5) next-day on-site product service.

The company's revenue in 2005 totaled \$49.205 billion. During fiscal 2005, Dell maintained its position as the world's number-one supplier of personal computer systems, with a performance that continued to outpace the industry. Over the same period, Dell's global market share of personal computer sales reached 17.8%. In the company's 2005 annual report, management painted an even more optimistic picture for the future,

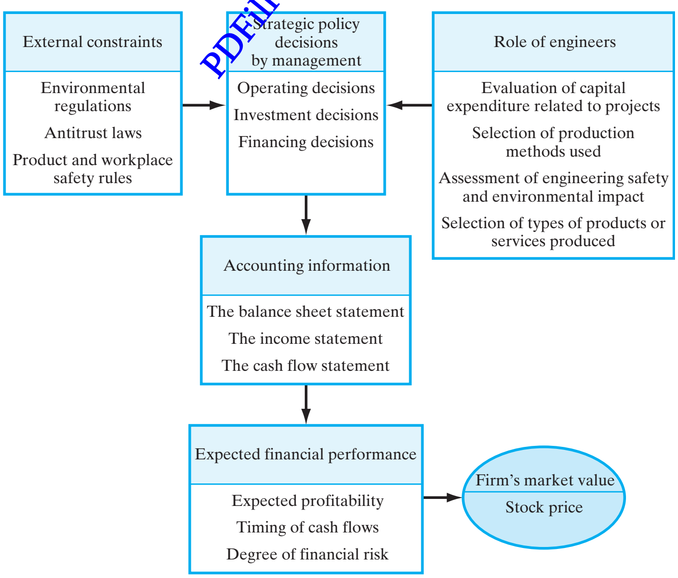


Figure 2.3 Summary of major factors affecting stock prices.

stating that Dell will continue to invest in information systems, research, development, and engineering activities to support its growth and to provide for new competitive products. Of course, there is no assurance that Dell's revenue will continue to grow at the annual rate of 50% in the future.

What can individual investors make of all this? Actually, we can make quite a bit. As you will see, investors use the information contained in an annual report to form expectations about future earnings and dividends. Therefore, the annual report is certainly of great interest to investors.

2.2.1 The Balance Sheet

What is the company's financial position at the end of the reporting period? We find the answer to this question in the company's **balance sheet statement**. A company's balance sheet, sometimes called its **statement of financial position**, reports three main categories of items: assets, liabilities, and stockholders' equity. Assets are arranged in order of liquidity. The most liquid assets appear at the top of the page, the least liquid assets at the bottom of the page. (See Figure 2.4.) Because cash is the most liquid of all assets, it is always listed first. Current assets are so critical that they are separately broken out and totaled. They are what will hold the business afloat for the next year.

Liabilities are arranged in order of payment, the most pressing at the top of the list, the least pressing at the bottom. Like current assets, current liabilities are so critical that they are separately broken out and totaled. They are what will be paid out during the next year.

A company's financial statements are based on the most fundamental tool of accounting: the accounting equation. The **accounting equation** shows the relationship among assets, liabilities, and owners' equity:

$$\text{Assets} = \text{Liabilities} + \text{Owners' Equity}$$

Every business transaction, no matter how simple or complex, can be expressed in terms of its effect on the accounting equation. Regardless of whether a business grows or contracts, the equality between the assets and the claims against the assets is always maintained. In other words, any change in the amount of total assets is necessarily accompanied by an equal change on the other side of the equation—that is, by an increase or decrease in either the liabilities or the owners' equity.

As shown in Table 2.1, the first half of Dell's year-end 2005 and 2004 balance sheets lists the firm's assets, while the remainder shows the liabilities and equity, or claims against those assets.

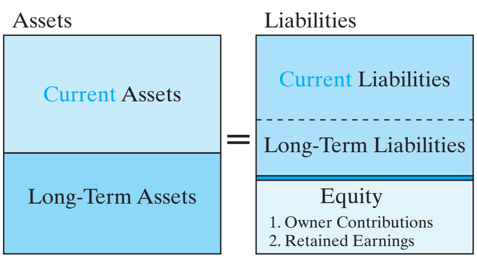


Figure 2.4 Using the four quadrants of the balance sheet.

TABLE 2.1 **Consolidated Statements of Financial Position (in millions) for Dell, Inc.**

	January 28, 2005	January 30, 2004
Assets		
Current assets:		
Cash and cash equivalents	$ 4,747	$ 4,317
Short-term investments	5,060	835
Accounts receivable, net	4,414	3,635
Inventories	459	327
Other	2,217	1,519
Total current assets	16,897	10,633
Property, plant, and equipment, net	1,691	1,517
Investments	4,319	6,770
Other noncurrent assets	308	391
Total assets	$ 23,215	$ 19,311
Liabilities and Stockholders' Equity		
Current liabilities:		
Accounts payable	$ 8,895	$ 7,316
Accrued and other	5,241	3,580
Total current liabilities	14,136	10,896
Long-term debt	505	505
Other noncurrent liabilities	2,089	1,630
Total liabilities	16,730	13,031
Commitments and contingent liabilities (Note 8)	—	—
Stockholders' equity:		
Preferred stock and capital in excess of $.01 par value; shares issued and outstanding: none	—	—
Common stock and capital in excess of $.01 par value; shares authorized: 7,000; shares issued: 2,769 and 2,721, respectively	8,195	6,823
Treasury stock, at cost; 284 and 165 shares, respectively	(10,758)	(6,539)
Retained earnings	9,174	6,131
Other comprehensive loss	(82)	(83)
Other	(44)	(52)
Total stockholders' equity	6,485	6,280
Total liabilities and stockholders' equity	$ 23,215	$ 19,311

Source: *Annual Report*, Dell Corporation, 2005.

Assets

The dollar amount shown in the assets portion of the balance sheet represents how much the company owns at the time it issues the report. We list the asset items in the order of their "liquidity," or the length of time it takes to convert them to cash.

Current assets account represents the value of all assets that are reasonably expected to be converted into cash within one year.

- **Current assets** can be converted to cash or its equivalent in less than one year. Current assets generally include three major accounts:
  1. The first is *cash*. A firm typically has a cash account at a bank to provide for the funds needed to conduct day-to-day business. Although assets are always stated in terms of dollars, only cash represents actual money. Cash-equivalent items are also listed and include marketable securities and short-term investments.
  2. The second account is *accounts receivable*—money that is owed the firm, but that has not yet been received. For example, when Dell receives an order from a retail store, the company will send an invoice along with the shipment to the retailer. Then the unpaid bill immediately falls into the accounts receivable category. When the bill is paid, it will be deducted from the accounts receivable account and placed into the cash category. A typical firm will have a 30- to 45-day accounts receivable, depending on the frequency of its bills and the payment terms for customers.
  3. The third account is *inventories*, which show the dollar amount that Dell has invested in raw materials, work in process, and finished goods available for sale.
- **Fixed assets** are relatively permanent and take time to convert into cash. Fixed assets reflect the amount of money Dell paid for its plant and equipment when it acquired those assets. The most common fixed asset is the physical investment in the business, such as land, buildings,[2] factory machinery, office equipment, and automobiles. With the exception of land, most fixed assets have a limited useful life. For example, buildings and equipment are used up over a period of years. Each year, a portion of the usefulness of these assets expires, and a portion of their total cost should be recognized as a depreciation expense. The term *depreciation* denotes the accounting process for this gradual conversion of fixed assets into expenses. *Property, plant and equipment, net* thus represents the current book value of these assets after deducting depreciation expenses.
- Finally, **other assets** include investments made in other companies and intangible assets such as goodwill, copyrights, franchises, and so forth. Goodwill appears on the balance sheet only when an operating business is purchased in its entirety. Goodwill indicates any additional amount paid for the business above the fair market value of the business. (Here, the fair market value is defined as the price that a buyer is willing to pay when the business is offered for sale.)

Liabilities and Stockholders' Equity (Owners' Net Worth)

The claims against assets are of two types: liabilities and stockholders' equity. The liabilities of a company indicate where the company obtained the funds to acquire its assets and to operate the business. Liability is money the company owes. Stockholders' equity is that portion of the assets of a company which is provided by the investors (owners). Therefore, stockholders' equity is the liability of a company to its owners.

[2] Land and buildings are commonly called **real assets** to distinguish them from equipment and machinery.

- **Current liabilities** are those debts which must be paid in the near future (normally, within one year). The major current liabilities include accounts and notes payable within a year. Also included are accrued expenses (wages, salaries, interest, rent, taxes, etc., owed, but not yet due for payment), and advance payments and deposits from customers.
- **Other liabilities** include *long-term liabilities*, such as bonds, mortgages, and long-term notes, that are due and payable more than one year in the future.
- **Stockholders' equity** represents the amount that is available to the owners after all other debts have been paid. Generally, stockholders' equity consists of preferred and common stock, treasury stock, capital surplus, and retained earnings. Preferred stock is a hybrid between common stock and debt. In case the company goes bankrupt, it must pay its preferred stockholders after its debtors, but before its common stockholders. Preferred dividend is fixed, so preferred stockholders do not benefit if the company's earnings grow. In fact, many firms do not use preferred stock. The common stockholders' equity, or **net worth**, is a residual:

Current liabilities are bills that are due to creditors and suppliers within a short period of time.

$$\text{Assets} - \text{Liabilities} - \text{Preferred stock} = \text{Common stockholders' equity}$$
$$\$11{,}471 - \$6{,}163 - \$0 = \$5{,}308.$$

Treasury stock is not taken into consideration when calculating earnings per share or dividends.

- **Common stock** is the aggregate par value of the company's stock issued. Companies rarely issue stocks at a discount (i.e., at an amount below the stated par). Normally, corporations set the par value low enough so that, in practice, stock is usually sold at a premium.
- **Paid-in capital** (capital surplus) is the amount of money received from the sale of stock that is over and above the par value of the stock. Outstanding stock is the number of shares issued that actually are held by the public. If the corporation buys back part of its own issued stock, that stock is listed as *treasury stock* on the balance sheet.
- **Retained earnings** represent the cumulative net income of the firm since its inception, less the total dividends that have been paid to stockholders. In other words, retained earnings indicate the amount of assets that have been financed by plowing profits back into the business. Therefore, retained earnings belong to the stockholders.

Retained earnings refers to earnings not paid out as dividends but instead are reinvested in the core business or used to pay off debt.

2.2.2 The Income Statement

The second financial report is the **income statement**, which indicates whether the company is making or losing money during a stated *period*, usually a year. Most businesses prepare quarterly and monthly income statements as well. The company's accounting period refers to the period covered by an income statement.

Basic Income Statement Equation

Revenue

−

Expenses

Net Income (Loss)

For Dell, the accounting period begins on February 1 and ends on January 31 of the following year. Table 2.2 gives the 2005 and 2004 income statements for Dell.

TABLE 2.2 **Consolidated Statements of Income (in millions, except per share amounts) Dell, Inc.**

	Fiscal Year Ended		
	January 28, 2005	January 30, 2004	January 31, 2003
Net revenue	$ 49,205	$ 41,444	$ 35,404
Cost of revenue	40,190	33,892	29,055
Gross margin	9,015	7,552	6,349
Operating expenses:			
Selling, general, and administrative	4,298	3,544	3,050
Research, development, and engineering	463	464	455
Total operating expenses	4,761	4,008	3,505
Operating income	4,254	3,544	2,844
Investment and other income, net	191	180	183
Income before income taxes	4,445	3,724	3,027
Income tax provision	1,402	1,079	905
Net income	$ 3,043	$ 2,645	$ 2,122
Earnings per common share:			
Basic	$ 1.21	$ 1.03	$ 0.82
Diluted	$ 1.18	$ 1.01	$ 0.80
Weighted average shares outstanding:			
Basic	2,509	2,565	2,584
Diluted	2,568	2,619	2,644

Source: *Annual Report*, Dell Corporation, 2005.

Reporting Format

Typical items that are itemized in the income statement are as follows:

- **Revenue** is the income from goods sold and services rendered during a given accounting period.
- **Net revenue** represents gross sales, less any sales return and allowances.
- Shown on the next several lines are the expenses and costs of doing business, as deductions from revenue. The largest expense for a typical manufacturing firm is the expense it incurs in making a product (such as labor, materials, and overhead), called the **cost of revenue** (or cost of goods sold).
- Net revenue less the cost of revenue gives the **gross margin**.
- Next, we subtract any other operating expenses from the operating income. These other operating expenses are expenses associated with paying interest, leasing machinery or equipment, selling, and administration. This results in the operating income.
- Finally, we determine the **net income** (or net profit) by subtracting the income taxes from the taxable income. Net income is also commonly known as *accounting income*.

Gross margin represents the amount of money the company generated over the cost of producing its goods or services.

Earnings per Share

Another important piece of financial information provided in the income statement is the **earnings per share** (EPS).[3] In simple situations, we compute the EPS by dividing the available earnings to common stockholders by the number of shares of common stock outstanding. Stockholders and potential investors want to know what their share of profits is, not just the total dollar amount. The presentation of profits on a per share basis allows the stockholders to relate earnings to what they paid for a share of stock. Naturally, companies want to report a higher EPS to their investors as a means of summarizing how well they managed their businesses for the benefits of the owners. Interestingly, Dell earned $1.21 per share in 2005, up from $1.03 in 2004, but it paid no dividends.

EPS is generally considered to be the single most important variable in determining a share's price.

Retained Earnings

As a supplement to the income statement, many corporations also report their retained earnings during the accounting period. When a corporation makes some profits, it has to decide what to do with those profits. The corporation may decide to pay out some of the profits as dividends to its stockholders. Alternatively, it may retain the remaining profits in the business in order to finance expansion or support other business activities.

When the corporation declares dividends, preferred stock has priority over common stock. Preferred stock pays a stated dividend, much like the interest payment on bonds. The dividend is not a legal liability until the board of directors has declared it. However, many corporations view the dividend payments to preferred stockholders as a liability. Therefore, "available for common stockholders" reflects the net earnings of the corporation, less the preferred stock dividends. When preferred and common stock dividends are subtracted from net income, the remainder is retained earnings (profits) for the year. As mentioned previously, these retained earnings are reinvested into the business.

EXAMPLE 2.1 Understanding Dell's Balance Sheet and Income Statement

With revenue of $49,205 million for fiscal year 2005, Dell is the world's leading direct computer systems company. Tables 2.2 and 2.3 show how Dell generated its net income during the fiscal year.

The Balance Sheet. Dell's $23,215 million of total assets shown in Table 2.1 were necessary to support its sales of $49,205 million.

- Dell obtained the bulk of the funds it used to buy assets
  1. By buying on credit from its suppliers (accounts payable).
  2. By borrowing from financial institutions (notes payable and long-term bonds).
  3. By issuing common stock to investors.
  4. By plowing earnings into the business, as reflected in the retained earnings account.

[3] In reporting EPS, the firm is required to distinguish between "basic EPS" and "diluted EPS." Basic EPS is the net income of the firm, divided by the number of shares of common stock outstanding. By contrast, the diluted EPS includes all common stock equivalents (convertible bonds, preferred stock, warrants, and rights), along with common stock. Therefore, diluted EPS will usually be less than basic EPS.

- The net increase in fixed assets was $174 million ($1,691 − $1,517; Table 2.1). However, this amount is after a deduction for the year's depreciation expenses. We should add depreciation expense back to show the increase in gross fixed assets. From the company's cash flow statement in Table 2.3, we see that the 2005 depreciation expense is $334 million; thus, the acquisition of fixed assets equals $508 million.
- Dell had a total long-term debt of $505 million (Table 2.1), consisting of several bonds issued in previous years. The interest Dell paid on these long-term debts was about $16 million.
- Dell had 2,509 million shares of common stock outstanding. Investors actually provided the company with a total capital of $8,195 million (Table 2.1). However, Dell has retained the current as well as previous earnings of $9,174 million since it was incorporated. Dell also held $10,758 million worth of treasury stock, which was financed through the current as well as previous retained earnings. The combined net stockholders' equity was $6,485 million, and these earnings belong to Dell's common stockholders (Table 2.1).
- On the average, stockholders have a total investment of $2.58 per share ($6,485 million/2,509 million shares) in the company. The $2.58 figure is known as the stock's *book value*. In the fall of 2005, the stock traded in the general range from $32 to $40 per share. Note that this market price is quite different from the stock's book value. Many factors affect the market price, the most important one being how investors expect the company to perform in the future. Certainly, the company's direct made-to-order business practices have had a major influence on the market value of its stock.

The Income Statement. Dell's net revenue was $49,205 million in 2005, compared with $41,444 million in 2004, a gain of 18.73% (Table 2.2). Profits from operations (operating income) rose 20.03% to $4,254 million, and net income was up 15.05% to $3,043 million.

- Dell issued no preferred stock, so there is no required cash dividend. Therefore, the entire net income of $3,043 million belongs to the common stockholders.
- Earnings per common share climbed at a faster pace than in 2004, to $1.21, an increase of 17.48% (Table 2.2). Dell could retain this income fully for reinvestment in the firm, or it could pay it out as dividends to the common stockholders. Instead of either of these alternatives, Dell repurchased and retired 56 million common stocks for $1,012 million. We can see that Dell had earnings available to common stockholders of $3,043 million. As shown in Table 2.1, the beginning balance of the retained earnings was $6,131 million. Therefore, the total retained earnings grew to $9,174 million.

2.2.3 The Cash Flow Statement

The income statement explained in the previous section indicates only whether the company was making or losing money during the reporting period. Therefore, the emphasis was on determining the net income (profits) of the firm for supporting its operating activities. However, the income statement ignores two other important business activities for the period: financing and investing activities. Therefore, we need another financial statement—the cash flow statement, which details how the company generated the cash it received and how the company used that cash during the reporting period.

Sources and Uses of Cash

The difference between the sources (inflows) and uses (outflows) of cash represents the net cash flow during the reporting period. This is a very important piece of information, because investors determine the value of an asset (or, indeed, of a whole firm) by the cash flows it generates. Certainly, a firm's net income is important, but cash flows are even more important, particularly because the company needs cash to pay dividends and to purchase the assets required to continue its operations. As mentioned in the previous section, the goal of the firm should be to maximize the price of its stock. Since the value of any asset depends on the cash flows produced by the asset, managers want to maximize the cash flows available to investors over the long run. Therefore, we should make investment decisions on the basis of cash flows rather than profits. For such investment decisions, it is necessary to convert profits (as determined in the income statement) to cash flows. Table 2.3 is Dell's statement of cash flows, as it would appear in the company's annual report.

Reporting Format

In preparing the cash flow statement such as that in Table 2.3, many companies identify the sources and uses of cash according to the types of business activities. There are three types of activities:

- **Operating activities.** We start with the net change in operating cash flows from the income statement. Here, operating cash flows represent those cash flows related to production and the sales of goods or services. All noncash expenses are simply added back to net income (or after-tax profits). For example, an expense such as depreciation is only an accounting expense (a bookkeeping entry). Although we may charge depreciation against current income as an expense, it does not involve an actual cash outflow. The actual cash flow may have occurred when the asset was purchased. (Any adjustments in **working capital**[4] will also be listed here.)
- **Investing activities.** Once we determine the operating cash flows, we consider any cash flow transactions related to investment activities, which include purchasing new fixed assets (cash outflow), reselling old equipment (cash inflow), and buying and selling financial assets.
- **Financing activities.** Finally, we detail cash transactions related to financing any capital used in business. For example, the company could borrow or sell more stock, resulting in cash inflows. Paying off existing debt will result in cash outflows.

By summarizing cash inflows and outflows from three activities for a given accounting period, we obtain the net change in the cash flow position of the company.

[4] The difference between the increase in current assets and the spontaneous increase in current liabilities is the **net change in net working capital**. If this change is positive, then additional financing, over and above the cost of the fixed assets, is needed to fund the increase in current assets. This will further reduce the cash flow from the operating activities.

TABLE 2.3 **Consolidated Statements of Cash Flows (in millions) Dell, Inc.**

	Fiscal Year Ended		
	January 28, 2005	**January 30, 2004**	**January 31, 2003**
Cash flows from operating activities:			
Net income	$ 3,043	$ 2,645	$ 2,122
Adjustments to reconcile net income to net cash provided by operating activities:			
Depreciation and amortization	334	263	211
Tax benefits of employee stock plans	249	181	260
Effects of exchange rate changes on monetary assets and liabilities denominated in foreign currencies	(602)	(677)	(537)
Other	78	113	60
Changes in:			
Operating working capital	1,755	872	1,210
Noncurrent assets and liabilities	453	273	212
Net cash provided by operating activities	5,310	3,670	3,538
Cash flows from investing activities:			
Investments:			
Purchases	(12,261)	(12,099)	(8,736)
Maturities and sales	10,469	10,078	7,660
Capital expenditures	(525)	(329)	(305)
Purchase of assets held in master lease facilities	—	(636)	—
Cash assumed in consolidation of Dell Financial Services, L.P.	—	172	—
Net cash used in investing activities	(2,317)	(2,814)	(1,381)
Cash flows from financing activities:			
Repurchase of common stock	(4,219)	(2,000)	(2,290)
Issuance of common stock under employee plans and other	1,091	617	265
Net cash used in financing activities	(3,128)	(1,383)	(2,025)
Effect of exchange rate changes on cash and cash equivalents	565	612	459
Net increase in cash and cash equivalents	430	85	591
Cash and cash equivalents at beginning of period	4,317	4,232	3,641
Cash and cash equivalents at end of period	$ 4,747	$ 4,317	$ 4,232

Source: *Annual Report,* Dell Corporation, 2005.

EXAMPLE 2.2 Understanding Dell's Cash Flow Statement

As shown in Table 2.3, Dell's cash flow from operations amounted to $5,310 million. Note that this is significantly more than the $3,043 million earned during the reporting period. Where did the extra money come from?

- The main reason for the difference lies in the accrual-basis accounting principle used by the Dell Corporation. In **accrual-basis accounting**, an accountant recognizes the impact of a business event as it occurs. When the business performs a service, makes a sale, or incurs an expense, the accountant enters the transaction into the books, regardless of whether cash has or has not been received or paid. For example, an increase in accounts receivable of $4,414 million − $3,635 million = $779 million during 2005 represents the amount of total sales on credit (Table 2.1). Since the $779 million figure was included in the total sales in determining the net income, we need to subtract it to determine the company's true cash position. After adjustments, the net cash provided from operating activities is $5,310 million.
- As regards investment activities, there was an investment outflow of $525 million in new plant and equipment. Dell sold $10,469 million worth of stocks and bonds during the period, and reinvested $12,261 million in various financial securities. The net cash flow provided from these investing activities amounted to −$2,317 million, which means an outflow of money.
- Financing activities produced a net outflow of $4,219 million, including the repurchase of the company's own stocks. (Repurchasing its own stock is equivalent to investing the firm's idle cash from operation in the stock market. Dell could have bought another company's stock, such as IBM or Microsoft stock, with the money, but Dell liked its own stock better than any other stocks on the market.) Dell also raised $1,091 million by issuing shares of common stock. The net cash used in financing activities amounted to $3,128 million (outflow).
- Finally, there was the effect of exchange rate changes on cash for foreign sales. This amounted to a net increase of $565 million. Together, the three types of activities generated a total cash flow of $430 million. With the initial cash balance of $4,317 million, the ending cash balance thus increased to $4,747 million. This same amount denotes the change in Dell's cash position, as shown in the cash accounts in the balance sheet.

Accrual-basis accounting measures the performance and position of a company by recognizing economic events regardless of when cash transactions occur.

2.3 Using Ratios to Make Business Decisions

As we have seen in Dell's financial statements, the purpose of accounting is to provide information for decision making. Accounting information tells what happened at a particular point in time. In that sense, financial statements are essentially historical documents. However, most users of financial statements are concerned about what will happen in the future. For example,

- Stockholders are concerned with future earnings and dividends.
- Creditors are concerned with the company's ability to repay its debts.
- Managers are concerned with the company's ability to finance future expansion.
- Engineers are concerned with planning actions that will influence the future course of business events.

Although financial statements are historical documents, they can still provide valuable information bearing on all of these concerns. An important part of financial analysis is the calculation and interpretation of various financial ratios. In this section, we consider some of the ratios that analysts typically use in attempting to predict the future course of events in business organizations. We may group these ratios into five categories (debt management, liquidity, asset management, profitability, and market trend) as outlined in Figure 2.5. In all financial ratio calculations, we will use the 2005 financial statements for Dell Computer Corporation, as summarized in Table 2.4.

2.3.1 Debt Management Analysis

All businesses need assets to operate. To acquire assets, the firm must raise capital. When the firm finances its long-term needs externally, it may obtain funds from the capital markets. Capital comes in two forms: **debt** and **equity**. Debt capital is capital borrowed from financial institutions. Equity capital is capital obtained from the owners of the company.

The basic methods of financing a company's debt are through bank loans and the sale of bonds. For example, suppose a firm needs $10,000 to purchase a computer. In this situation, the firm would borrow money from a bank and repay the loan, together with the interest specified, in a few years. This kind of financing is known as *short-term debt*

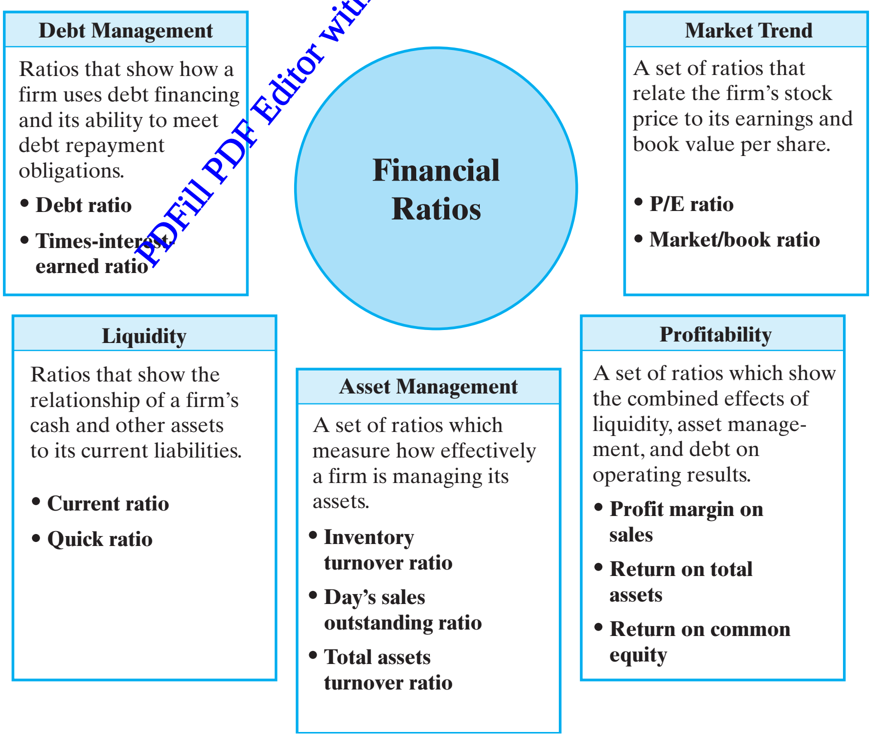


Figure 2.5 Types of ratios used in evaluating a firm's financial health.

TABLE 2.4 **Summary of Dell's Key Financial Statements**

Balance Sheet	January 28, 2005	January 30, 2004
Cash and cash equivalent	4,747	4,317
Accounts receivables, net	4,414	3,635
Inventories	459	327
Total current assets	16,897	10,633
Total assets	23,215	19,311
Total current liabilities	14,136	10,896
Long-term debt	505	505
Total liabilities	16,730	13,031
Common stock	8,195	6,823
Retained earnings	9,174	6,131
Total stockholders' equity	6,485	6,280
Income Statement		
Net revenue	49,205	41,444
Gross income (margin)	9,015	7,522
Operating income (margin)	4,254	3,544
Net income (margin)	3,043	2,645
Statements of Retained Earnings		
Beginning retained earnings	6,131	3,486
Net income	3,043	2,645
Ending retained earnings	9,174	6,131
Statement of Cash Flows		
Net cash from operating activities	5,310	3,670
Net cash used in investing activities	(2,317)	(2,814)
Net cash used in financing activities	(3,128)	(1,383)
Effect of exchange rate changes	565	612
Beginning cash position	4,317	4,232
Ending cash position	4,747	4,317

financing. Now suppose that the firm needs $100 million for a construction project. Normally, it would be very expensive (or require a substantial mortgage) to borrow the money directly from a bank. In this situation, the company would go public to borrow money on a long-term basis. When investors lend capital to a company and the company consents to repay the loan at an agreed-upon interest rate, the investor is the creditor of the corporation. The document that records the nature of the arrangement between the issuing company and the investor is called a **bond**. Raising capital by issuing a bond is called *long-term debt financing*.

Similarly, there are different types of equity capital. For example, the equity of a proprietorship represents the money provided by the owner. For a corporation, equity capital comes in two forms: *preferred stock* and *common stock*. Investors provide capital to a corporation and the company agrees to endow the investor with fractional ownership in the corporation. Preferred stock pays a stated *dividend*, much like the interest payment on bonds. However, the dividend is not a legal liability until the company declares it. Preferred stockholders have preference over common stockholders as regards the receipt of dividends if the company has to liquidate its assets. We can examine the extent to which a company uses debt financing (or financial leverage) in the operation of its business if we

- Check the balance sheet to determine the extent to which borrowed funds have been used to finance assets, and
- Review the income statement to see the extent to which fixed charges (interests) are covered by operating profits.

Two essential indicators of a business's ability to pay its long-term liabilities are the *debt ratio* and the *times-interest-earned ratio*.

Debt Ratio

Debt ratio: A ratio that indicates what proportion of debt a *company* has relative to its assets.

The relationship between total liabilities and total assets, generally called the **debt ratio**, tells us the proportion of the company's assets that it has financed with debt:

$$\text{Debt ratio} = \frac{\text{Total debt}}{\text{Total assets}}$$

$$= \frac{\$16{,}730}{\$23{,}215} = 72.07\%.$$

Total debt includes both current liabilities and long-term debt. If the debt ratio is unity, then the company has used debt to finance all of its assets. As of January 28, 2005, Dell's debt ratio was 72.07%; this means that its creditors have supplied close to 72% of the firm's total financing. Certainly, most creditors prefer low debt ratios, because the lower the ratio, the greater is the cushion against creditors' losses in case of liquidation. If a company seeking financing already has large liabilities, then additional debt payments may be too much for the business to handle. For such a highly leveraged company, creditors generally charge higher interest rates on new borrowing to help protect themselves.

Times-Interest-Earned Ratio

The most common measure of the ability of a company's operations to provide protection to the long-term creditor is the times-interest-earned ratio. We find this ratio by dividing earnings before interest and income taxes (EBIT) by the yearly interest charges that must be met. Dell issued $500 million worth of senior notes and long-term bonds with a combined interest rate of 2.259%. This results in \$11.29 million in interest expenses[5] in 2005:

$$\text{Times-interest-earned ratio} = \frac{\text{EBIT}}{\text{Interest expense}}$$

$$= \frac{\$4{,}445 + \$11.29}{\$11.29} = 394.72 \text{ times.}$$

The times-interest-earned ratio measures the extent to which operating income can decline before the firm is unable to meet its annual interest costs. Failure to meet this obligation can bring legal action by the firm's creditors, possibly resulting in the company's bankruptcy. Note that we use the earnings before interest and income taxes, rather than net income, in the numerator. Because Dell must pay interest with pretax dollars, Dell's ability to pay current interest is not affected by income taxes. Only those earnings remaining after all interest charges are subject to income taxes. For Dell, the times-interest-earned ratio for 2005 would be 395 times. This ratio is exceptionally high compared with the rest of the industry's 65.5 times during the same operating period.

2.3.2 Liquidity Analysis

If you were one of the many suppliers to Dell, your primary concern would be whether Dell will be able to pay off its debts as they come due over the next year or so. Short-term creditors want to be repaid on time. Therefore, they focus on Dell's cash flows and on its working capital, as these are the company's primary sources of cash in the near future. The excess of current assets over current liabilities is known as **working capital**, a figure that indicates the extent to which current assets can be converted to cash to meet current obligations. Therefore, we view a firm's net working capital as a measure of its *liquidity* position. In general, the larger the working capital, the better able the business is to pay its debt.

Current Ratio

The **current ratio** measures a company's ability to pay its short-term obligations.

We calculate the **current ratio** by dividing current assets by current liabilities:

$$\text{Current ratio} = \frac{\text{Current assets}}{\text{Current liabilities}}$$

$$= \frac{\$16{,}897}{\$14{,}136} = 1.1953 \text{ times.}$$

If a company is getting into financial difficulty, it begins paying its bills (accounts payable) more slowly, borrowing from its bank, and so on. If current liabilities are rising faster than current assets, the current ratio will fall, and that could spell trouble. What is an acceptable current ratio? The answer depends on the nature of the industry. The general rule of thumb calls for a current ratio of 2 to 1. This rule, of course, is subject to many exceptions, depending heavily on the composition of the assets involved.

[5] Unless the interest expenses are itemized in the income statement, you will find them in the firm's annual report.

Quick (Acid-Test) Ratio

The quick ratio tells us whether a company could pay all of its current liabilities if they came due immediately. We calculate the quick ratio by deducting inventories from current assets and then dividing the remainder by current liabilities:

$$\text{Quick ratio} = \frac{\text{Current assets} - \text{Inventories}}{\text{Current liabilities}}$$

$$= \frac{\$16{,}897 - \$459}{\$14{,}136} = 1.1628 \text{ times.}$$

The quick ratio measures how well a company can meet its obligations without having to liquidate or depend too heavily on its inventory. Inventories are typically the least liquid of a firm's current assets; hence, they are the assets on which losses are most likely to occur in case of liquidation. Although Dell's current ratio may appear to be below the average for its industry, 1.4, its liquidity position is relatively strong, as it has carried very little inventory in its current assets (only $459 million out of $16,897 million of current assets, or 2.7%). We often compare against industry average figures and should note at this point that an industry average is not an absolute number that all firms should strive to maintain. In fact, some very well managed firms will be above the average, while other good firms will be below it. However, if we find that a firm's ratios are quite different from the average for its industry, we should examine the reason for the difference.

2.3.3 Asset Management Analysis

The ability to sell inventory and collect accounts receivables is fundamental to business success. Therefore, the third group of ratios measures how effectively the firm is managing its assets. We will review three ratios related to a firm's asset management: (1) the inventory turnover ratio, (2) the day's sales outstanding ratio, and (3) the total asset turnover ratio. The purpose of these ratios is to answer this question: Does the total amount of each type of asset, as reported on the balance sheet, seem reasonable in view of current and projected sales levels? The acquisition of any asset requires the use of funds. On the one hand, if a firm has too many assets, its cost of capital will be too high; hence, its profits will be depressed. On the other hand, if assets are too low, the firm is likely lose profitable sales.

Inventory Turnover

Inventory turnover: A ratio that shows how many times the inventory of a firm is sold and replaced over a specific period.

The inventory turnover ratio measures how many times the company sold and replaced its inventory over a specific period—for example, during the year. We compute the ratio by dividing sales by the average level of inventories on hand. We compute the average inventory figure by taking the average of the beginning and ending inventory figures. Since Dell has a beginning inventory figure of $327 million and an ending inventory figure of $459 million, its average inventory for the year would be $393 million, or ($327 + $459)/2. Then we compute Dell's inventory turnover for 2005 as follows:

$$\text{Inventory turnover ratio} = \frac{\text{Sales}}{\text{Average inventory balance}}$$

$$= \frac{\$49{,}205}{\$393} = 125.20 \text{ times.}$$

As a rough approximation, Dell was able to sell and restock its inventory 125.20 times per year. Dell's turnover of 125.20 times is much faster than its competitor HPQ (Hewlett Packard), 9.5 times. This suggests that HPQ is holding excessive stocks of inventory; excess stocks are, of course, unproductive, and they represent an investment with a low or zero rate of return.

Day's Sales Outstanding (Accounts Receivable Turnover)

The day's sales outstanding (DSO) is a rough measure of how many times a company's accounts receivable have been turned into cash during the year. We determine this ratio, also called the **average collection period**, by dividing accounts receivable by average sales per day. In other words, the DSO indicates the average length of time the firm must wait after making a sale before receiving cash. For Dell,

Average collection period is often used to help determine if a company is trying to disguise weak sales.

$$\begin{aligned} \text{DSO} &= \frac{\text{Receivables}}{\text{Average sales per day}} = \frac{\text{Receivables}}{\text{Annual sales/365}} \\ &= \frac{\$4{,}414}{\$49{,}205/365} = \frac{\$4{,}414}{\$134.81} \\ &= 32.74 \text{ days.} \end{aligned}$$

Thus, on average, it takes Dell 32.74 days to collect on a credit sale. During the same period, HPQ's average collection period was 43–52 days. Whether the average of 32.74 days taken to collect an account is good or bad depends on the credit terms Dell is offering its customers. If the credit terms are 30 days, we can say that Dell's customers, on the average, are not paying their bills on time. In order to improve their working-capital position, most customers tend to withhold payment for as long as the credit terms will allow and may even go over a few days. The long collection period may signal either that customers are in financial trouble or that the company manages its credit poorly.

Total Assets Turnover

Asset turnover is a measure of how well assets are being used to produce revenue.

The total assets turnover ratio measures how effectively the firm uses its total assets in generating its revenues. It is the ratio of sales to all the firm's assets:

$$\begin{aligned} \text{Total assets turnover ratio} &= \frac{\text{Sales}}{\text{Total assets}} \\ &= \frac{\$49{,}205}{\$23{,}215} = 2.12 \text{ times.} \end{aligned}$$

Dell's ratio of 2.12 times, compared with HPQ's 1.1, is almost 93% faster, indicating that Dell is using its total assets about 93% more intensively than HPQ is. In fact, Dell's total investment in plant and equipment is about one-fourth of HPQ's. If we view Dell's ratio as the industry average, we can say that HPQ has too much investment in inventory, plant, and equipment compared to the size of sale.

2.3.4 Profitability Analysis

One of the most important goals for any business is to earn a profit. The ratios examined thus far provide useful clues about the effectiveness of a firm's operations, but the profitability ratios show the combined effects of liquidity, asset management, and debt on operating results. Therefore, ratios that measure profitability play a large role in decision making.

Profit Margin on Sales

The **profit margin** measures how much out of every dollar of sales a company actually keeps in *earnings*.

We calculate the profit margin on sales by dividing net income by sales. This ratio indicates the profit per dollar of sales:

$$\text{Profit margin on sales} = \frac{\text{Net income available to common stockholders}}{\text{Sales}}$$

$$= \frac{\$3{,}043}{\$49{,}205} = 6.18\%.$$

Thus, Dell's profit margin is equivalent to 6.18 cents for each sales dollar generated. Dell's profit margin is greater than HPQ's profit margin of 3.6%, indicating that, although HPQ's sales are about 76% more than Dell's revenue during the same operating period, HPQ's operation is less efficient than Dell's. HPQ's low profit margin is also a result of its heavy use of debt and its carrying a very high volume of inventory. Recall that net income is income after taxes. Therefore, if two firms have identical operations in the sense that their sales, operating costs, and earnings before income tax are the same, but if one company uses more debt than the other, it will have higher interest charges. Those interest charges will pull net income down, and since sales are constant, the result will be a relatively low profit margin.

Return on Total Assets

The return on total assets—or simply, return on assets (ROA)—measures a company's success in using its assets to earn a profit. The ratio of net income to total assets measures the return on total assets after interest and taxes:

$$\text{Return on total assets} = \frac{\text{Net income} + \text{interest expense}(1 - \text{tax rate})}{\text{Average total assets}}$$

$$= \frac{\$3{,}043 + \$11.29(1 - 0.315)}{(\$23{,}215 + \$19{,}311)/2} = 14.35\%.$$

Adding interest expenses back to net income results in an adjusted earnings figure that shows what earnings would have been if the assets had been acquired solely by selling shares of stock. (Note that Dell's effective tax rate was 31.5% in 2005.) With this adjustment, we may be able to compare the return on total assets for companies with differing amounts of debt. Again, Dell's 14.35% return on assets is well above the 4.1% for HPQ. This high return results from (1) the company's high basic earning power and (2) its low use of debt, both of which cause its net income to be relatively high.

The **return on equity** reveals how much profit a company generates with the money its shareholders have invested in it.

Return on Common Equity

Another popular measure of profitability is rate of return on common equity. This ratio shows the relationship between net income and common stockholders' investment in the company—that is, how much income is earned for every $1 invested by the common stockholders. To compute the return on common equity, we first subtract preferred dividends from net income, yielding the net income available to common stockholders. We then divide this net income available to common stockholders by the average common stockholders' equity during the year. We compute average common equity by using the beginning and ending balances. At the beginning of fiscal-year 2005, Dell's common equity balance was \$6,280 million; at the end of fiscal-year 2005, the balance was \$6,485 million. The average balance is then simply \$6,382.50 million, and we have

$$\begin{aligned}\text{Return on common equity} &= \frac{\text{Net income available to common stockholders}}{\text{Average common equity}} \\ &= \frac{\$3{,}043}{(\$6{,}485 + \$6{,}280)/2} \\ &= \frac{\$3{,}043}{\$6{,}382.50} = 47.68\%.\end{aligned}$$

The rate of return on common equity for Dell was 47.68% during 2005. Over the same period, HPQ's return on common equity amounted to 8.2%, a poor performance relative to the computer industry (12.6% in 2005) in general.

To learn more about what management can do to increase the return on common equity, or ROE, we may rewrite the ROE in terms of the following three components:

$$\begin{aligned}\text{ROE} &= \frac{\text{Net income}}{\text{Stockholders' equity}} \\ &= \frac{\text{Net income}}{\text{Sales}} \times \frac{\text{Sales}}{\text{Assets}} \times \frac{\text{Assets}}{\text{Stockholders' equity}}.\end{aligned}$$

The three principal components can be described as the profit margin, asset turnover, and financial leverage, respectively, so that

$$\begin{aligned}\text{ROE} &= (\text{Profit margin}) \times (\text{Asset turnover}) \times (\text{Financial leverage}) \\ &= (6.18\%) \times (2.12) \times \left(\frac{23{,}215}{6{,}382.5}\right) \\ &= 47.68\%.\end{aligned}$$

Financial leverage: The degree to which an *investor* or business is utilizing borrowed money.

This expression tells us that management has only three key ratios for controlling a company's ROE: (1) the earnings from sales (the profit margin); (2) the revenue generated from each dollar of assets employed (asset turnover); and (3) the amount of equity used to finance the assets in the operation of the business (financial leverage).

2.3.5 Market Value Analysis

When you purchase a company's stock, what are your primary factors in valuing the stock? In general, investors purchase stock to earn a return on their investment. This return consists of two parts: (1) gains (or losses) from selling the stock at a price that differs from the investors' purchase price and (2) dividends—the periodic distributions of profits to stockholders. The market value ratios, such as the price-to-earnings ratio and the market-to-book ratio, relate the firm's stock price to its earnings and book value per share, respectively. These ratios give management an indication of what investors think of the company's past performance and future prospects. If the firm's asset and debt management is sound and its profit is rising, then its market value ratios will be high, and its stock price will probably be as high as can be expected.

Price-to-Earnings Ratio

The price-to-earnings (P/E) ratio shows how much investors are willing to pay per dollar of reported profits. Dell's stock sold for \$41.50 in early February of 2005, so with an EPS of \$1.21, its P/E ratio was 34.29:

The **P/E ratio** shows how much investors are willing to pay per dollar of earnings.

$$
\begin{aligned}
P/E \text{ ratio} &= \frac{\text{Price per share}}{\text{Earnings per share}} \\
&= \frac{\$41.5}{\$1.21} = 34.29.
\end{aligned}
$$

That is, the stock was selling for about 34.29 times its current earnings per share. In general, P/E ratios are higher for firms with high growth prospects, other things held constant, but they are lower for firms with lower expected earnings. Dell's expected annual increase in operating earnings is 30% over the next 3 to 5 years. Since Dell's ratio is greater than 25%, the average for other computer industry firms, this suggests that investors value Dell's stock more highly than most as having excellent growth prospects. However, all stocks with high P/E ratios carry high risk whenever the expected growths fail to materialize. Any slight earnings disappointment tends to punish the market price significantly.

Book Value per Share

Another ratio frequently used in assessing the well-being of the common stockholders is the book value per share, which measures the amount that would be distributed to holders of each share of common stock if all assets were sold at their balance-sheet carrying amounts and if all creditors were paid off. We compute the book value per share for Dell's common stock as follows:

$$
\begin{aligned}
\text{Book value per share} &= \frac{\text{Total stockholders' equity} - \text{preferred stock}}{\text{Shares outstanding}} \\
&= \frac{\$6{,}485 - \$0}{2{,}509} = \$2.58.
\end{aligned}
$$

If we compare this book value with the current market price of \$41.50, then we may say that the stock appears to be overpriced. Once again, though, market prices reflect expectations about future earnings and dividends, whereas book value largely reflects the results of events that occurred in the past. Therefore, the market value of a stock tends to exceed its book value. Table 2.5 summarizes the financial ratios for Dell Computer Corporation in comparison to its direct competitor Hewlett Packard (HPQ) and the industry average.

2.3.6 Limitations of Financial Ratios in Business Decisions

Business decisions are made in a world of uncertainty. As useful as ratios are, they have limitations. We can draw an analogy between their use in decision making and a physician's use of a thermometer. A reading of 102°F indicates that something is wrong with the

TABLE 2.5 Comparisons of Dell Computer Corporation's Key Financial Ratios with Those of Hewlett Packard (HPQ) and the Industry Average (2005)

Category	Financial Ratios	Dell	HPQ	Industry
Debt Management	Debt ratio	72.07%	9%	29%
	Time-interest earned	394.72	18.37	65.5
Liquidity	Current ratio	1.1953	1.4	1.4
	Quick ratio	1.16	0.90	1.0
Asset Management	Inventory turnover	125.20	9.5	11.1
	Day's sales outstanding	32.74	52.43	34
	Total asset turnover	2.12	1.1	1.0
Profitability	Profit margin	6.18%	3.6%	5.0%
	Return on total asset	14.35%	4.1%	4.9%
	Return on common equity	47.68%	8.2%	12.6%
Market Trend	P/E ratio	34.29	27.5	25.3
	Book value-to-share ratio	2.58	13.05	9.26

patient, but the temperature alone does not indicate what the problem is or how to cure it. In other words, ratio analysis is useful, but analysts should be aware of ever-changing market conditions and make adjustments as necessary. It is also difficult to generalize about whether a particular ratio is "good" or "bad." For example, a high current ratio may indicate a strong liquidity position, which is good, but holding too much cash in a bank account (which will increase the current ratio) may not be the best utilization of funds. Ratio analysis based on any one year may not represent the true business condition. It is important to analyze trends in various financial ratios, as well as their absolute levels, for trends give clues as to whether the financial situation is likely to improve or deteriorate. To do a **trend analysis**, one simply plots a ratio over time. As a typical engineering student, your judgment in interpreting a set of financial ratios is understandably weak at this point, but it will improve as you encounter many facets of business decisions in the real world. Again, accounting is a language of business, and as you speak it more often, it can provide useful insights into a firm's operations.

Trend analysis is based on the idea that what has happened in the past gives traders an idea of what will happen in the future.

SUMMARY

The primary purposes of this chapter were (1) to describe the basic financial statements, (2) to present some background information on cash flows and corporate profitability, and (3) to discuss techniques used by investors and managers to analyze financial statements. Following are some concepts we covered:

- Before making any major financial decisions, it is important to understand their impact on your net worth. Your net-worth statement is a snapshot of where you stand financially at a given point in time.

- The three basic financial statements contained in the annual report are the balance sheet, the income statement, and the statement of cash flows. Investors use the information provided in these statements to form expectations about future levels of earnings and dividends and about the firm's risk-taking behavior.
- A firm's balance sheet shows a snapshot of a firm's financial position at a particular point in time through three categories: (1) assets the firm owns, (2) liabilities the firm owes, and (3) owners' equity, or assets less liabilities.
- A firm's income statement reports the results of operations over a period of time and shows earnings per share as its "bottom line." The main items are (1) revenues and gains, (2) expenses and losses, and (3) net income or net loss (revenue less expenses).
- A firm's statement of cash flows reports the impact of operating, investing, and financing activities on cash flows over an accounting period.
- The purpose of calculating a set of financial ratios is twofold: (1) to examine the relative strengths and weaknesses of a company compared with those of other companies in the same industry and (2) to learn whether the company's position has been improving or deteriorating over time.
- Liquidity ratios show the relationship of a firm's current assets to its current liabilities and thus its ability to meet maturing debts. Two commonly used liquidity ratios are the current ratio and the quick (acid-test) ratio.
- Asset management ratios measure how effectively a firm is managing its assets. Some of the major ratios are inventory turnover, fixed assets turnover, and total assets turnover.
- Debt management ratios reveal (1) the extent to which a firm is financed with debt and (2) the firm's likelihood of defaulting on its debt obligations. In this category are the debt ratio and the times-interest-earned ratio.
- Profitability ratios show the combined effects of liquidity, asset management, and debt management policies on operating results. Profitability ratios include the profit margin on sales, the return on total assets, and the return on common equity.
- Market value ratios relate the firm's stock price to its earnings and book value per share, and they give management an indication of what investors think of the company's past performance and future prospects. Market value ratios include the price-to-earnings ratio and the book value per share.
- Trend analysis, in which one plots a ratio over time, is important, because it reveals whether the firm's ratios are improving or deteriorating over time.

PROBLEMS

Financial Statements

2.1 Consider the balance-sheet entries for War Eagle Corporation in Table P2.1.

(a) Compute the firm's
Current assets: $_____
Current liabilities: $_____
Working capital: $_____
Shareholders' equity: $_____

TABLE P2.1

Balance Sheet Statement as of December 31, 2000		
Assets:		
Cash		$ 150,000
Marketable securities		200,000
Accounts receivables		150,000
Inventories		50,000
Prepaid taxes and insurance		30,000
Manufacturing plant at cost	$ 600,000	
Less accumulated depreciation	100,000	
Net fixed assets		500,000
Goodwill		20,000
Liabilities and shareholders' equity:		
Notes payable		50,000
Accounts payable		100,000
Income taxes payable		80,000
Long-term mortgage bonds		400,000
Preferred stock, 6%, $100 par value (1,000 shares)		100,000
Common stock, $15 par value (10,000 shares)		150,000
Capital surplus		150,000
Retained earnings		70,000

(b) If the firm had a net income of $500,000 after taxes, what is the earnings per share?

(c) When the firm issued its common stock, what was the market price of the stock per share?

2.2 A chemical processing firm is planning on adding a duplicate polyethylene plant at another location. The financial information for the first project year is shown in Table P2.2.

(a) Compute the working-capital requirement during the project period.

(b) What is the taxable income during the project period?

(c) What is the net income during the project period?

(d) Compute the net cash flow from the project during the first year.

TABLE P2.2 **Financial Information for First Project Year**

Sales		$1,500,000
Manufacturing costs		
Direct materials	$ 150,000	
Direct labor	200,000	
Overhead	100,000	
Depreciation	200,000	
Operating expenses		150,000
Equipment purchase		400,000
Borrowing to finance equipment		200,000
Increase in inventories		100,000
Decrease in accounts receivable		20,000
Increase in wages payable		30,000
Decrease in notes payable		40,000
Income taxes		272,000
Interest payment on financing		20,000

Financial Ratio Analysis

2.3 Table P2.3 shows financial statements for Nano Networks, Inc. The closing stock price for Nano Network was $56.67 (split adjusted) on December 31, 2005. On the basis of the financial data presented, compute the various financial ratios and make an informed analysis of Nano's financial health.

TABLE P2.3 **Balance Sheet for Nano Networks, Inc.**

	Dec. 2005 U.S. $ (000) (Year)	Dec. 2004 U.S. $ (000) (Year)
Balance Sheet Summary		
Cash	158,043	20,098
Securities	285,116	0
Receivables	24,582	8,056
Allowances	632	0
Inventory	0	0
Current assets	377,833	28,834
Property and equipment, net	20,588	10,569

(*Continued*)

	Dec. 2005 U.S. $ (000) (Year)	Dec. 2004 U.S. $ (000) (Year)
Depreciation	8,172	2,867
Total assets	513,378	36,671
Current liabilities	55,663	14,402
Bonds	0	0
Preferred mandatory	0	0
Preferred stock	0	0
Common stock	2	1
Other stockholders' equity	457,713	17,064
Total liabilities and equity	513,378	36,671
Income Statement Summary		
Total revenues	102,606	3,807
Cost of sales	45,272	4,416
Other expenses	71,954	31,661
Loss provision	0	0
Interest income	8,011	1,301
Income pretax	−6,609	−69
Income tax	2,425	2
Income continuing	−9,034	−30,971
Net income	**−9,034**	**−30,971**
EPS primary	**−$0.1**	**−$0.80**
EPS diluted	−$0.10	−$0.80
	−$0.05	−$0.40
	(split adjusted)	(split adjusted)

(a) Debt ratio
(b) Times-interest-earned ratio
(c) Current ratio
(d) Quick (acid-test) ratio
(e) Inventory turnover ratio
(f) Day's sales outstanding
(g) Total assets turnover
(h) Profit margin on sales
(i) Return on total assets

(j) Return on common equity
(k) Price-to-earnings ratio
(l) Book value per share

2.4 The balance sheet that follows summarizes the financial conditions for Flex, Inc., an electronic outsourcing contractor, for fiscal-year 2005. Unlike Nano Network Corporation in Problem 2.3, Flex has reported a profit for several years running. Compute the various financial ratios and interpret the firm's financial health during fiscal-year 2005.

	Aug. 2005 U.S. $ (000) (12 mos.)	Aug. 2004 U.S. $ (000) (Year)
Balance Sheet Summary		
Cash	1,325,637	225,228
Securities	362,769	83,576
Receivables	1,123,901	674,193
Allowances	5,580	−3,999
Inventory	1,080,083	788,519
Current assets	3,994,084	1,887,558
Property and equipment, net	1,186,885	859,831
Depreciation	533,311	−411,792
Total assets	4,834,696	2,410,568
Current liabilities	1,113,186	840,834
Bonds	922,653	385,519
Preferred mandatory	0	0
Preferred stock	0	0
Common stock	271	117
Other stockholders' equity	2,792,820	1,181,209
Total liabilities and equity	4,834,696	2,410,568
Income Statement Summary		
Total revenues	8,391,409	5,288,294
Cost of sales	7,614,589	4,749,988
Other expenses	335,808	237,063
Loss provision	2,143	2,254
Interest expense	36,479	24,759

(*Continued*)

	Aug. 2005 U.S. $ (000) (12 mos.)	Aug. 2004 U.S. $ (000) (Year)
Income pretax	432,342	298,983
Income tax	138,407	100,159
Income continuing	293,935	198,159
Discontinued	0	0
Extraordinary	0	0
Changes	0	0
Net income	**293,935**	**198,159**
EPS primary	**$1.19**	**$1.72**
EPS diluted	$1.13	$1.65

(a) Debt ratio
(b) Times-interest-earned ratio
(c) Current ratio
(d) Quick (acid-test) ratio
(e) Inventory turnover ratio
(f) Day's sales outstanding
(g) Total assets turnover
(h) Profit margin on sales
(i) Return on total assets
(j) Return on common equity
(k) Price-to-earnings ratio
(l) Book value per share

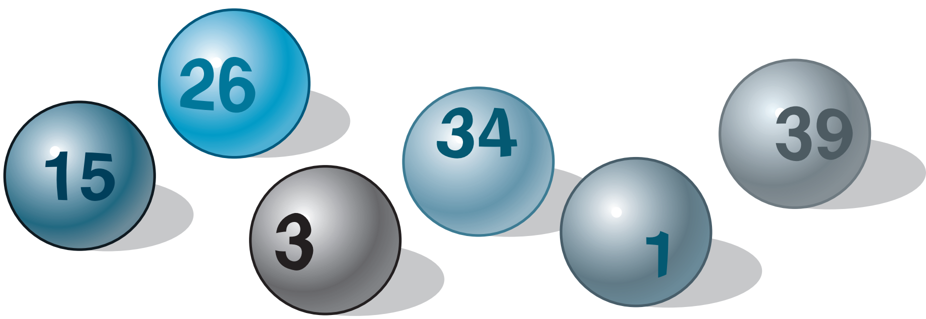
2.5 J. C. Olson & Co. had earnings per share of $8 in year 2006, and it paid a $4 dividend. Book value per share at year's end was $80. During the same period, the total retained earnings increased by $24 million. Olson has no preferred stock, and no new common stock was issued during the year. If Olson's year-end debt (which equals its total liabilities) was $240 million, what was the company's year-end debt-to-asset ratio?

2.6 If Company A uses more debt than Company B and both companies have identical operations in terms of sales, operating costs, etc., which of the following statements is *true*?
(a) Company B will definitely have a higher current ratio.
(b) Company B has a higher profit margin on sales than Company A.
(c) Both companies have identical profit margins on sales.
(d) Company B's return on total assets would be higher.

CHAPTER THREE

Interest Rate and Economic Equivalence

No Lump Sum for Lottery-Winner Grandma, 94[1] A judge denied a 94-year-old woman's attempt to force the Massachusetts Lottery Commission to pay her entire $5.6 million winnings up front on the grounds that she otherwise won't live long enough to collect it all. The ruling means that the commission can pay Louise Outing, a retired waitress, in installments over 20 years. After an initial gross payment of $283,770, Outing would be paid 19 annual gross checks of $280,000. That's about $197,000 after taxes. Lottery Executive Director Joseph Sullivan said all players are held to the same rules, which are printed on the back of Megabucks tickets. Lottery winners are allowed to "assign" their winnings to a state-approved financial company that makes the full payment—but only in return for a percentage of the total winnings. Outing, who won a Megabucks drawing in September, has seven grandchildren, nine great-grandchildren, and six great-great-grandchildren. "I'd like to get it and do what I want with it," she said. "I'm not going to live 20 years. I'll be 95 in March."



[1] "No Lump Sum for Lottery-Winner Grandma, 94," The Associated Press, December 30, 2004.

The next time you play a lottery, look at the top section of the play slip. You will see two boxes: "Cash Value" and "Annual Payments." You need to mark one of the boxes before you fill out the rest of the slip. If you don't, and you win the jackpot, you will automatically receive the jackpot as annual payments. That is what happened to Ms. Louise Outing. If you mark the "Cash Value box" and you win, you will receive the present cash value of the announced jackpot in one lump sum. This amount will be less than the announced jackpot. With the announced jackpot of $5.6 million, Ms. Outing could receive about 52.008%, or $2.912 million, in one lump sum (less withholding tax). This example is based on average market costs as of January 2005 of 20 annual payments funded by the U.S. Treasury Zero Coupon Bonds (or a 7.2% coupon rate). With this option, you can look forward to a large cash payment up front.

First, most people familiar with investments would tell Ms. Outing that receiving a lump amount of $2.912 million today is likely to prove a far better deal than receiving $280,000 a year for 20 years, even if the grandma lives long enough to collect the entire annual payments.
After losing the court appeal, Ms. Outing was able to find a buyer for her lottery in a lump-sum amount of $2.467 million. To arrive at that price, the buyer calculated the return he wanted to earn—at that time about 9.5% interest, compounded annually—and applied that rate in reverse to the $5.6 million he stood to collect over 20 years. The buyer says the deals he strikes with winners applies a basic tenet of all financial transactions, the **time value of money**: A dollar in hand today is worth more than one that will be paid to you in the future.

In engineering economics, the principles discussed in this chapter are regarded as the underpinning for nearly all project investment analysis. This is because we always need to account for the effect of interest operating on sums of cash over time. Interest formulas allow us to place different cash flows received at different times in the same time frame and to compare them. As will become apparent, almost our entire study of engineering economics is built on the principles introduced in this chapter.

CHAPTER LEARNING OBJECTIVES

After completing this chapter, you should understand the following concepts:

- The time value of money.
- The difference between simple interest and the compound interest.
- The meaning of economic equivalence and why we need it in economic analysis.
- How to compare two different money series by means of the concept of economic equivalence.
- The interest operation and the types of interest formulas used to facilitate the calculation of economic equivalence.

3.1 Interest: The Cost of Money

Most of us are familiar in a general way with the concept of interest. We know that money left in a savings account earns interest, so that the balance over time is greater than the sum of the deposits. We also know that borrowing to buy a car means repaying an amount over time, that that amount includes interest, and that it is therefore greater than the amount borrowed. What may be unfamiliar to us is the idea that, in the financial world, money itself is a commodity and, like other goods that are bought and sold, money costs money.

Market interest rate: Interest rate quoted by financial institutions.

The cost of money is established and measured by a **market interest rate**, a percentage that is periodically applied and added to an amount (or varying amounts) of money over a specified length of time. When money is borrowed, the interest paid is the charge to the borrower for the use of the lender's property; when money is lent or invested, the interest earned is the lender's gain from providing a good to another (Figure 3.1). **Interest**, then, may be defined as the cost of having money available for use. In this section, we examine how interest operates in a free-market economy and we establish a basis for understanding the more complex interest relationships that follow later on in the chapter.

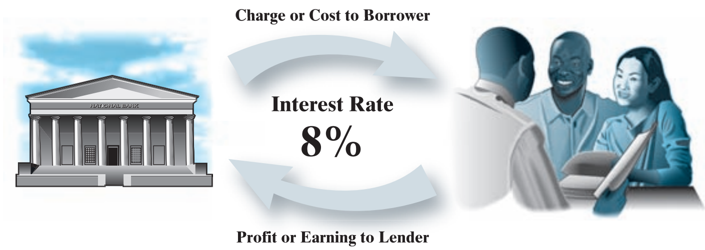


Figure 3.1 The meaning of *interest rate* to the lender (bank) and to the borrower.

	Account Value	Cost of Refrigerator
Case 1: Inflation exceeds earning power	$N = 0$ \$100 $N = 1$ \$106 (earning rate = 6%)	$N = 0$ \$100 $N = 1$ \$108 (inflation rate = 8%)
Case 2: Earning power exceeds inflation	$N = 0$ \$100 $N = 1$ \$106 (earning rate = 6%)	$N = 0$ \$100 $N = 1$ \$104 (inflation rate = 4%)

Figure 3.2 Gains achieved or losses incurred by delaying consumption.

3.1.1 The Time Value of Money

The "time value of money" seems like a sophisticated concept, yet it is a concept that you grapple with every day. Should you buy something today or save your money and buy it later? Here is a simple example of how your buying behavior can have varying results: Pretend you have \$100, and you want to buy a \$100 refrigerator for your dorm room. If you buy it now, you are broke. Suppose that you can invest money at 6% interest, but the price of the refrigerator increases only at an annual rate of 4% due to inflation. In a year you can still buy the refrigerator, and you will have \$2 left over. Well, if the price of the refrigerator increases at an annual rate of 8% instead, you will not have enough money (you will be \$2 short) to buy the refrigerator a year from now. In that case, you probably are better off buying the refrigerator now. The situation is summarized in Figure 3.2.

The time value of money: The idea that a dollar today is worth more than a dollar in the future because the dollar received today can earn interest.

Clearly, the rate at which you earn interest should be higher than the inflation rate to make any economic sense of the delayed purchase. In other words, in an inflationary economy, your purchasing power will continue to decrease as you further delay the purchase of the refrigerator. In order to make up this future loss in purchasing power, your earning interest rate should be sufficiently larger than the anticipated inflation rate. After all, time, like money, is a finite resource. There are only 24 hours in a day, so time has to be budgeted, too. What this example illustrates is that we must connect the "earning power" and the "purchasing power" to the concept of time.

When we deal with large amounts of money, long periods of time, or high interest rates, the change in the value of a sum of money over time becomes extremely significant. For example, at a current annual interest rate of 10%, \$1 million will earn \$100,000 in interest in a year; thus, to wait a year to receive \$1 million clearly involves a significant sacrifice. When deciding among alternative proposals, we must take into account the operation of interest and the time value of money in order to make valid comparisons of different amounts at various times.

The way interest operates reflects the fact that money has a time value. This is why amounts of interest depend on lengths of time; interest rates, for example, are typically given in terms of a percentage per year. We may define the principle of the time value of money as follows: The economic value of a sum depends on when it is received. Because money has both **earning** as well as **purchasing power** over time, as shown in Figure 3.3 (it can be put to work, earning more money for its owner), a dollar received today has a greater value than a dollar received at some future time.

Purchasing power: The value of a currency expressed in terms of the amount of goods or services that one unit of money can buy.

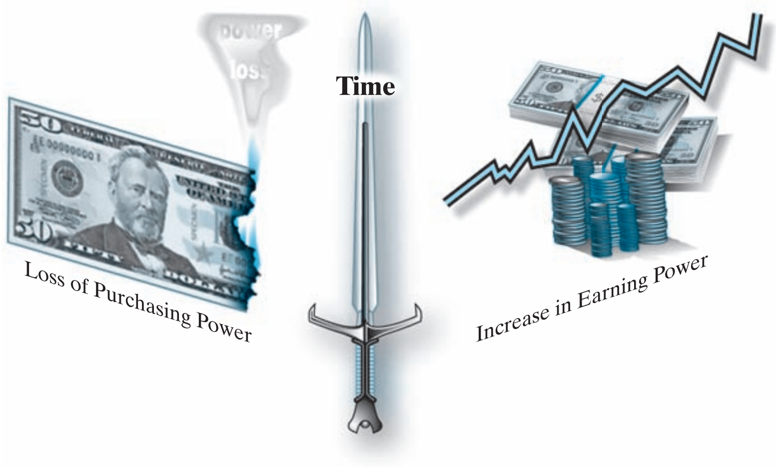


Figure 3.3 The time value of money. This is a two-edged sword whereby earning grows, but purchasing power decreases, as time goes by.

When lending or borrowing interest rates are quoted by financial institutions on the marketplace, those interest rates reflect the desired amounts to be earned, as well as any protection from loss in the future purchasing power of money because of inflation. (If we want to know the true desired earnings in isolation from inflation, we can determine the real interest rate. We consider this issue in Chapter 11. The earning power of money and its loss of value because of inflation are calculated by different analytical techniques.) In the meantime, we will assume that, unless otherwise mentioned, *the interest rate used in this book reflects the market interest rate,* which takes into account the earning power, as well as the effect of inflation perceived in the marketplace. We will also assume that all cash flow transactions are given in terms of **actual dollars**, with the effect of inflation, if any, reflected in the amount.

Actual dollars: The cash flow measured in terms of the dollars at the time of the transaction.

3.1.2 Elements of Transactions Involving Interest

Many types of transactions (e.g., borrowing or investing money or purchasing machinery on credit) involve interest, but certain elements are common to all of these types of transactions:

- An initial amount of money in transactions involving debt or investments is called the **principal**.
- The **interest rate** measures the cost or price of money and is expressed as a percentage per period of time.
- A period of time, called the **interest period**, determines how frequently interest is calculated. (Note that even though the length of time of an interest period can vary, interest rates are frequently quoted in terms of an annual percentage rate. We will discuss this potentially confusing aspect of interest in Chapter 4.)
- A specified length of time marks the duration of the transaction and thereby establishes a certain **number of interest periods**.
- A **plan for receipts or disbursements** yields a particular cash flow pattern over a specified length of time. (For example, we might have a series of equal monthly payments that repay a loan.)
- A **future amount of money** results from the cumulative effects of the interest rate over a number of interest periods.

For the purposes of calculation, these elements are represented by the following variables:

A_n = A discrete payment or receipt occurring at the end of some interest period.
i = The interest rate per interest period.
N = The total number of interest periods.
P = A sum of money at a time chosen as time zero for purposes of analysis; sometimes referred to as the **present value** or **present worth**.
F = A future sum of money at the end of the analysis period. This sum may be specified as F_N.
A = An end-of-period payment or receipt in a uniform series that continues for N periods. This is a special situation where $A_1 = A_2 = \cdots = A_N$.
V_n = An equivalent sum of money at the end of a specified period n that considers the effect of the time value of money. Note that $V_0 = P$ and $V_N = F$.

Present value: The amount that a future sum of money is worth today, given a specified rate of return.

Because frequent use of these symbols will be made in this text, it is important that you become familiar with them. Note, for example, the distinction between A, A_n, and A_N. The symbol A_n refers to a specific payment or receipt, at the end of period n, in any series of payments. A_N is the final payment in such a series, because N refers to the total number of interest periods. A refers to any series of cash flows in which all payments or receipts are equal.

Example of an Interest Transaction

As an example of how the elements we have just defined are used in a particular situation, let us suppose that an electronics manufacturing company buys a machine for $25,000 and borrows $20,000 from a bank at a 9% annual interest rate. In addition, the company pays a $200 loan origination fee when the loan commences. The bank offers two repayment plans, one with equal payments made at the end of every year for the next five years, the other with a single payment made after the loan period of five years. These two payment plans are summarized in Table 3.1.

- In Plan 1, the principal amount P is $20,000, and the interest rate i is 9%. The interest period is one year, and the duration of the transaction is five years, which means there are five interest periods $(N = 5)$. It bears repeating that whereas one year is a common interest period, interest is frequently calculated at other intervals: monthly, quarterly, or

TABLE 3.1 **Repayment Plans for Example Given in Text (for $N = 5$ years and $i = 9\%$)**

End of Year	Receipts	Payments: Plan 1	Payments: Plan 2
Year 0	$20,000.00	$ 200.00	$ 200.00
Year 1		5,141.85	0
Year 2		5,141.85	0
Year 3		5,141.85	0
Year 4		5,141.85	0
Year 5		5,141.85	30,772.48

$P = \$20,000$, $A = \$5,141.85$, $F = \$30,772.48$

Note: You actually borrow $19,800 with the origination fee of $200, but you pay back on the basis of $20,000.

semiannually, for instance. For this reason, we used the term **period** rather than **year** when we defined the preceding list of variables. The receipts and disbursements planned over the duration of this transaction yield a cash flow pattern of five equal payments A of $5,141.85 each, paid at year's end during years 1 through 5. (You'll have to accept these amounts on faith for now—the next section presents the formula used to arrive at the amount of these equal payments, given the other elements of the problem.)

- Plan 2 has most of the elements of Plan 1, except that instead of five equal repayments, we have a grace period followed by a single future repayment F of $30,772.78.

Cash Flow Diagrams

Problems involving the time value of money can be conveniently represented in graphic form with a cash flow diagram (Figure 3.4). **Cash flow diagrams** represent time by a horizontal line marked off with the number of interest periods specified. The cash flows over time are represented by arrows at relevant periods: Upward arrows denote positive flows (receipts), downward arrows negative flows (disbursements). Note, too, that the arrows actually represent **net cash flows**: Two or more receipts or disbursements made at the same time are summed and shown as a single arrow. For example, $20,000 received during the same period as a $200 payment would be recorded as an upward arrow of $19,800. Also, the lengths of the arrows can suggest the relative values of particular cash flows.

Cash flow diagrams function in a manner similar to free-body diagrams or circuit diagrams, which most engineers frequently use: Cash flow diagrams give a convenient summary of all the important elements of a problem, as well as a reference point to determine whether the statement of the problem has been converted into its appropriate parameters. The text frequently uses this graphic tool, and you are strongly encouraged to develop the habit of using well-labeled cash flow diagrams as a means to identify and summarize pertinent information in a cash flow problem. Similarly, a table such as Table 3.1 can help you organize information in another summary format.

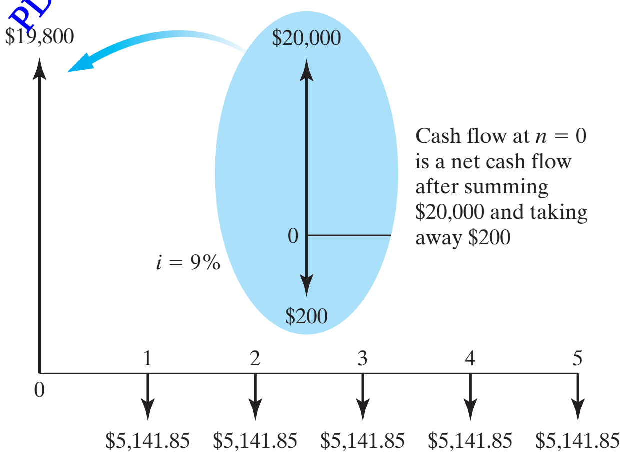


Figure 3.4 A cash flow diagram for Plan 1 of the loan repayment example summarized in Table 3.1.

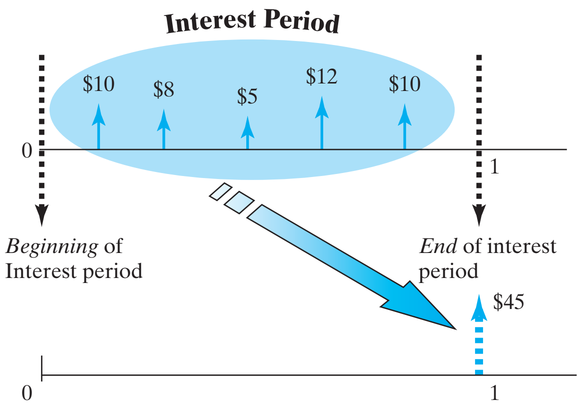


Figure 3.5 Any cash flows occurring during the interest period are summed to a single amount and placed at the end of the interest period.

End-of-Period Convention

In practice, cash flows can occur at the beginning or in the middle of an interest period—or indeed, at practically any point in time. One of the simplifying assumptions we make in engineering economic analysis is the **end-of-period convention**, which is the practice of placing all cash flow transactions at the end of an interest period. (See Figure 3.5.) This assumption relieves us of the responsibility of dealing with the effects of interest within an interest period, which would greatly complicate our calculations.

End-of-period convention: Unless otherwise mentioned, all cash flow transactions occur at the end of an interest period.

It is important to be aware of the fact that, like many of the simplifying assumptions and estimates we make in modeling engineering economic problems, the end-of-period convention inevitably leads to some discrepancies between our model and real-world results.

Suppose, for example, that $100,000 is deposited during the first month of the year in an account with an interest period of one year and an interest rate of 10% per year. In such a case, the difference of 1 month would cause an interest income loss of $10,000. This is because, under the end-of-period convention, the $100,000 deposit made during the interest period is viewed as if the deposit were made at the end of the year, as opposed to 11 months earlier. This example gives you a sense of why financial institutions choose interest periods that are less than one year, even though they usually quote their rate as an annual percentage.

Armed with an understanding of the basic elements involved in interest problems, we can now begin to look at the details of calculating interest.

3.1.3 Methods of Calculating Interest

Money can be lent and repaid in many ways, and, equally, money can earn interest in many different ways. Usually, however, at the end of each interest period, the interest earned on the principal amount is calculated according to a specified interest rate. The two computational schemes for calculating this earned interest are said to yield either **simple interest** or **compound interest**. Engineering economic analysis uses the compound-interest scheme almost exclusively.

Simple interest: The interest rate is applied only to the original principal amount in computing the amount of interest.

Simple Interest

Simple interest is interest earned on only the principal amount during each interest period. In other words, with simple interest, the interest earned during each interest period does not earn additional interest in the remaining periods, *even though you do not withdraw it*.

In general, for a deposit of P dollars at a simple interest rate of i for N periods, the total earned interest would be

$$I = (iP)N. \tag{3.1}$$

The total amount available at the end of N periods thus would be

$$F = P + I = P(1 + iN). \tag{3.2}$$

Simple interest is commonly used with add-on loans or bonds. (See Chapter 4.)

Compound: The ability of an asset to generate *earnings* that are then reinvested and generate their own earnings.

Compound Interest

Under a compound-interest scheme, the interest earned in each period is calculated on the basis of the total amount at the end of the previous period. This total amount includes the original principal plus the accumulated interest that has been left in the account. In this case, you are, in effect, increasing the deposit amount by the amount of interest earned. In general, if you deposited (invested) P dollars at interest rate i, you would have $P + iP = P(1 + i)$ dollars at the end of one period. If the entire amount (principal and interest) is reinvested at the same rate i for another period, at the end of the second period you would have

$$\begin{aligned} P(1 + i) + i[P(1 + i)] &= P(1 + i)(1 + i) \\ &= P(1 + i)^2. \end{aligned}$$

Continuing, we see that the balance after the third period is

$$P(1 + i)^2 + i[P(1 + i)^2] = P(1 + i)^3.$$

This interest-earning process repeats, and after N periods the total accumulated value (balance) F will grow to

$$F = P(1 + i)^N. \tag{3.3}$$

EXAMPLE 3.1 Compound Interest

Suppose you deposit \$1,000 in a bank savings account that pays interest at a rate of 10% compounded annually. Assume that you don't withdraw the interest earned at the end of each period (one year), but let it accumulate. How much would you have at the end of year 3?

SOLUTION

Given: $P = \$1{,}000$, $N = 3$ years, and $i = 10\%$ per year.
Find: F.

Applying Eq. (3.3) to our three-year, 10% case, we obtain

$$F = \$1{,}000(1 + 0.10)^3 = \$1{,}331.$$

The total interest earned is \$331, which is \$31 more than was accumulated under the simple-interest method (Figure 3.6). We can keep track of the interest accruing process more precisely as follows:

Period	Amount at Beginning of Interest Period	Interest Earned for Period	Amount at End of Interest Period
1	\$1,000	\$1,000(0.10)	\$1,100
2	1,100	1,100(0.10)	1,210
3	1,210	1,210(0.10)	1,331

COMMENTS: At the end of the first year, you would have \$1,000, plus \$100 in interest, or a total of \$1,100. In effect, at the beginning of the second year, you would be depositing \$1,100, rather than \$1,000. Thus, at the end of the second year, the interest earned would be $0.10(\$1{,}100) = \$110$, and the balance would be $\$1{,}100 + \$110 = \$1{,}210$. This is the amount you would be depositing at the beginning of the third year, and the interest earned for that period would be $0.10(\$1{,}210) = \$121$. With a beginning principal amount of \$1,210 plus the \$121 interest, the total balance would be \$1,331 at the end of year 3.

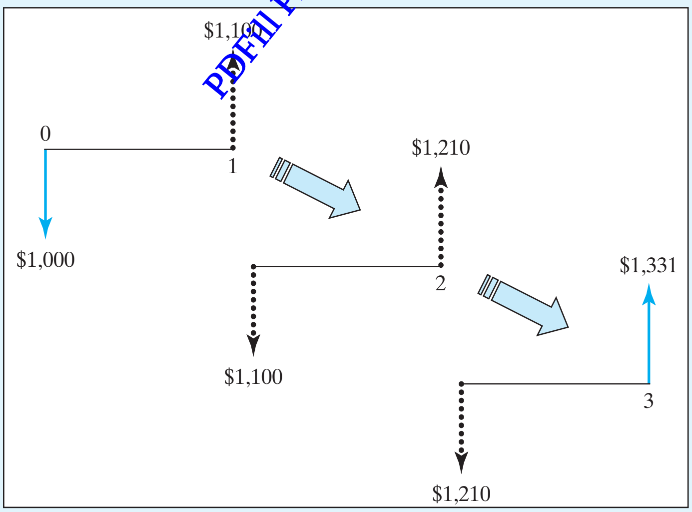


Figure 3.6 The process of computing the balance when \$1,000 at 10% is deposited for three years (Example 3.1).

3.1.4 Simple Interest versus Compound Interest

From Eq. (3.3), the total interest earned over N periods is

$$I = F - P = P[(1 + i)^N - 1]. \tag{3.4}$$

Compared with the simple-interest scheme, the additional interest earned with compound interest is

$$\Delta I = P[(1 + i)^N - 1] - (iP)N \tag{3.5}$$

$$= P[(1 + i)^N - (1 + iN)]. \tag{3.6}$$

As either i or N becomes large, the difference in interest earnings also becomes large, so the effect of compounding is further pronounced. Note that, when $N = 1$, compound interest is the same as simple interest.

Using Example 3.1, we can illustrate the difference between compound interest and the simple interest. Under the simple-interest scheme, you earn interest only on the principal amount at the end of each interest period. Under the compounding scheme, you earn interest on the principal, as well as interest on interest.

Figure 3.7 illustrates the fact that compound interest is a sum of simple interests earned on the original principal, as well as periodic simple interests earned on a series of simple interests.

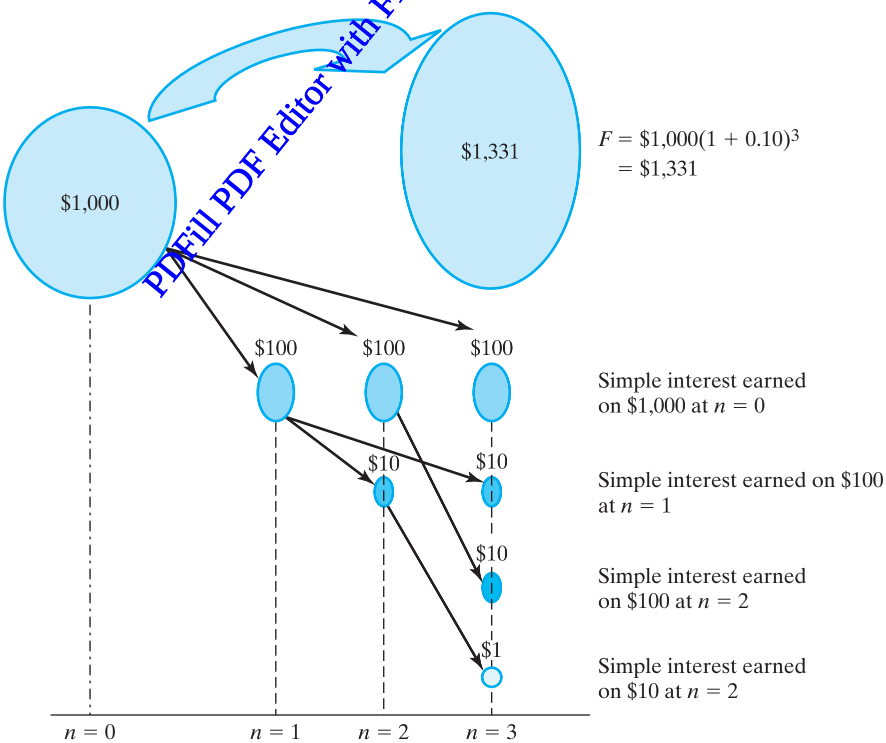


Figure 3.7 The relationship between simple interest and compound interest.

EXAMPLE 3.2 Comparing Simple with Compound Interest

In 1626, Peter Minuit of the Dutch West India Company paid \$24 to purchase Manhattan Island in New York from the Indians. In retrospect, if Minuit had invested the \$24 in a savings account that earned 8% interest, how much would it be worth in 2007?

SOLUTION

Given: $P = \$24$, $i = 8\%$ per year, and $N = 381$ years.
Find: F, based on (a) 8% simple interest and (b) 8% compound interest.

(a) With 8% simple interest,

$$F = \$24[1 + (0.08)(381)] = \$755.52.$$

(b) With 8% compound interest,

$$F = \$24(1 + 0.08)^{381} = \$130{,}215{,}319{,}909{,}015.$$

COMMENTS: The significance of compound interest is obvious in this example. Many of us can hardly comprehend the magnitude of \$130 trillion. In 2007, the total population in the United States was estimated to be around 300 million. If the money were distributed equally among the population, each individual would receive \$434,051. Certainly, there is no way of knowing exactly how much Manhattan Island is worth today, but most real-estate experts would agree that the value of the island is nowhere near \$130 trillion. (Note that the U.S. national debt as of December 31, 2007, was estimated to be \$9.19 trillion.)

3.2 Economic Equivalence

The observation that money has a time value leads us to an important question: If receiving \$100 today is not the same thing as receiving \$100 at any future point, how do we measure and compare various cash flows? How do we know, for example, whether we should prefer to have \$20,000 today and \$50,000 ten years from now, or \$8,000 each year for the next ten years? In this section, we describe the basic analytical techniques for making these comparisons. Then, in Section 3.3, we will use these techniques to develop a series of formulas that can greatly simplify our calculations.

3.2.1 Definition and Simple Calculations

The central question in deciding among alternative cash flows involves comparing their economic worth. This would be a simple matter if, in the comparison, we did not need to consider the time value of money: We could simply add the individual payments within a cash flow, treating receipts as positive cash flows and payments (disbursements) as negative cash flows. The fact that money has a time value, however, makes our calculations more complicated. We need to know more than just the size of a payment in order to determine its economic effect completely. In fact, as we will see in this section, we need to know several things:

- The magnitude of the payment.
- The direction of the payment: Is it a receipt or a disbursement?
- The timing of the payment: When is it made?
- The interest rate in operation during the period under consideration.

It follows that, to assess the economic impact of a series of payments, we must consider the impact of each payment individually.

Economic equivalence: The process of comparing two different cash amounts at different points in time.

Calculations for determining the economic effects of one or more cash flows are based on the concept of economic equivalence. **Economic equivalence** exists between cash flows that have the same economic effect and could therefore be traded for one another in the financial marketplace, which we assume to exist.

Economic equivalence refers to the fact that a cash flow—whether a single payment or a series of payments—can be converted to an *equivalent* cash flow at any point in time. For example, we could find the equivalent future value F of a present amount P at interest rate i at period n; or we could determine the equivalent present value P of N equal payments A.

The preceding strict concept of equivalence, which limits us to converting a cash flow into another equivalent cash flow, may be extended to include the comparison of alternatives. For example, we could compare the value of two proposals by finding the equivalent value of each at any common point in time. If financial proposals that appear to be quite different turn out to have the same monetary value, then we can be *economically indifferent* to choosing between them: In terms of economic effect, one would be an even exchange for the other, so no reason exists to prefer one over the other in terms of their economic value.

A way to see the concepts of equivalence and economic indifference at work in the real world is to note the variety of payment plans offered by lending institutions for consumer loans. Table 3.2 extends the example we developed earlier to include three different repayment plans for a loan of $20,000 for five years at 9% interest. You will notice, perhaps to your surprise, that the three plans require significantly different repayment patterns and

TABLE 3.2 **Typical Repayment Plans for a Bank Loan of $20,000 (for N = 5 years and i = 9%)**

	Repayments		
	Plan 1	Plan 2	Plan 3
Year 1	$ 5,141.85	0	$ 1,800.00
Year 2	5,141.85	0	1,800.00
Year 3	5,141.85	0	1,800.00
Year 4	5,141.85	0	1,800.00
Year 5	5,141.85	$30,772.48	21,800.00
Total of payments	$25,709.25	$30,772.48	$29,000.00
Total interest paid	$ 5,709.25	$10,772.48	$ 9,000.00

Plan 1: Equal annual installments; Plan 2: End-of-loan-period repayment of principal and interest; Plan 3: Annual repayment of interest and end-of-loan repayment of principal

different total amounts of repayment. However, because money has a time value, these plans are equivalent, and economically, the bank is indifferent to a consumer's choice of plan. We will now discuss how such equivalence relationships are established.

Equivalence Calculations: A Simple Example

Equivalence calculations can be viewed as an application of the compound-interest relationships we developed in Section 3.1. Suppose, for example, that we invest \$1,000 at 12% annual interest for five years. The formula developed for calculating compound interest, $F = P(1 + i)^N$ (Eq. 3.3), expresses the equivalence between some present amount P and a future amount F, for a given interest rate i and a number of interest periods N. Therefore, at the end of the investment period, our sums grow to

$$\$1{,}000(1 + 0.12)^5 = \$1{,}762.34.$$

Thus, we can say that at 12% interest, \$1,000 received now is equivalent to \$1,762.34 received in five years and that we could trade \$1,000 now for the promise of receiving \$1,762.34 in five years. Example 3.3 further demonstrates the application of this basic technique.

EXAMPLE 3.3 Equivalence

Suppose you are offered the alternative of receiving either \$3,000 at the end of five years or P dollars today. There is no question that the \$3,000 will be paid in full (no risk). Because you have no current need for the money, you would deposit the P dollars in an account that pays 8% interest. What value of P would make you indifferent to your choice between P dollars today and the promise of \$3,000 at the end of five years?

STRATEGY: Our job is to determine the present amount that is economically equivalent to \$3,000 in five years, given the investment potential of 8% per year. Note that the statement of the problem assumes that you would exercise the option of using the earning power of your money by depositing it. The "indifference" ascribed to you refers to economic indifference; that is, in a marketplace where 8% is the applicable interest rate, you could trade one cash flow for the other.

SOLUTION

Given: $F = \$3{,}000$, $N = 5$ years, and $i = 8\%$ per year.
Find: P.

Equation: Eq. (3.3), $F = P(1 + i)^N$.
Rearranging terms to solve for P gives

$$P = \frac{F}{(1 + i)^N}.$$

Substituting yields

$$P = \frac{\$3{,}000}{(1 + 0.08)^5} = \$2{,}042.$$

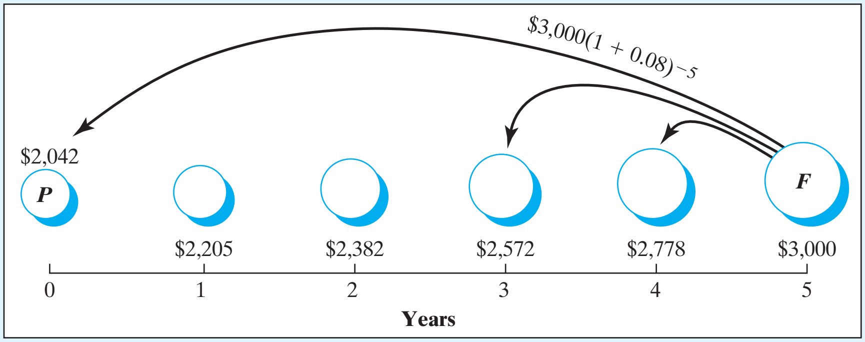


Figure 3.8 Various dollar amounts that will be economically equivalent to \$3,000 in five years, given an interest rate of 8% (Example 3.3).

We summarize the problem graphically in Figure 3.8.

COMMENTS: In this example, it is clear that if P is anything less than \$2,042, you would prefer the promise of \$3,000 in five years to P dollars today; if P is greater than \$2,042, you would prefer P. As you may have already guessed, at a lower interest rate, P must be higher to be equivalent to the future amount. For example, at $i = 4\%$, $P = \$2{,}466$.

3.2.2 Equivalence Calculations: General Principles

In spite of their numerical simplicity, the examples we have developed reflect several important general principles, which we will now explore.

Principle 1: Equivalence Calculations Made to Compare Alternatives Require a Common Time Basis

Common base period: To establish an economic equivalence between two cash flow amounts, a common base period must be selected.

Just as we must convert fractions to common denominators to add them together, we must also convert cash flows to a common basis to compare their value. One aspect of this basis is the choice of a single point in time at which to make our calculations. In Example 3.3, if we had been given the magnitude of each cash flow and had been asked to determine whether they were equivalent, we could have chosen any reference point and used the compound interest formula to find the value of each cash flow at that point. As you can readily see, the choice of $n = 0$ or $n = 5$ would make our problem simpler because we need to make only one set of calculations: At 8% interest, either convert \$2,042 at time 0 to its equivalent value at time 5, or convert \$3,000 at time 5 to its equivalent value at time 0. (To see how to choose a different reference point, take a look at Example 3.4.)

When selecting a point in time at which to compare the value of alternative cash flows, we commonly use either the present time, which yields what is called the **present worth** of the cash flows, or some point in the future, which yields their **future worth**. The choice of the point in time often depends on the circumstances surrounding a particular decision, or it may be chosen for convenience. For instance, if the present worth is known for the first two of three alternatives, all three may be compared simply by calculating the present worth of the third.

EXAMPLE 3.4 Equivalent Cash Flows Are Equivalent at Any Common Point in Time

In Example 3.3, we determined that, given an interest rate of 8% per year, receiving \$2,042 today is equivalent to receiving \$3,000 in five years. Are these cash flows also equivalent at the end of year 3?

STRATEGY: This problem is summarized in Figure 3.9. The solution consists of solving two equivalence problems: (1) What is the future value of \$2,042 after three years at 8% interest (part (a) of the solution)? (2) Given the sum of \$3,000 after five years and an interest rate of 8%, what is the equivalent sum after 3 years (part (b) of the solution)?

SOLUTION

Given:

(a) $P = \$2{,}042;\quad i = 8\%$ per year; $\quad N = 3$ years.
(b) $F = \$3{,}000;\quad i = 8\%$ per year; $\quad N = 5 - 3 = 2$ years.

Find: (1) V_3 for part (a); (2) V_3 for part (b). (3) Are these two values equivalent?

Equation:

(a) $F = P(1 + i)^N$.
(b) $P = F(1 + i)^{-N}$.

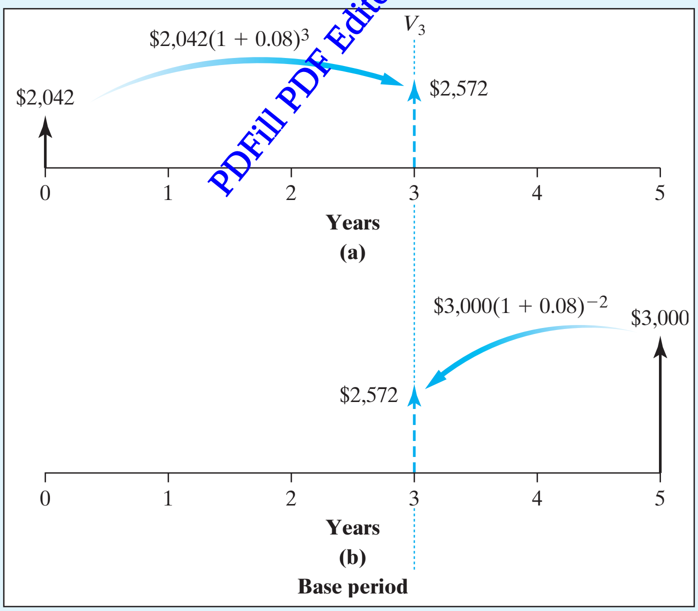


Figure 3.9 Selection of a base period for an equivalence calculation (Example 3.4).

Notation: The usual terminology of F and P is confusing in this example, since the cash flow at $n = 3$ is considered a future sum in part (a) of the solution and a past cash flow in part (b) of the solution. To simplify matters, we are free to arbitrarily designate a reference point $n = 3$ and understand that it need not to be now or the present. Therefore, we assign the equivalent cash flow at $n = 3$ to a single variable, V_3.

1. The equivalent worth of \$2,042 after three years is

$$\begin{aligned} V_3 &= 2{,}042(1 + 0.08)^3 \\ &= \$2{,}572. \end{aligned}$$

2. The equivalent worth of the sum \$3,000 two years earlier is

$$\begin{aligned} V_3 &= F(1 + i)^{-N} \\ &= \$3{,}000(1 + 0.08)^{-2} \\ &= \$2{,}572. \end{aligned}$$

(Note that $N = 2$ because that is the number of periods during which discounting is calculated in order to arrive back at year 3.)

3. While our solution doesn't strictly prove that the two cash flows are equivalent at any time, they will be equivalent at any time as long as we use an interest rate of 8%.

Principle 2: Equivalence Depends on Interest Rate

The equivalence between two cash flows is a function of the magnitude and timing of individual cash flows and the interest rate or rates that operate on those flows. This principle is easy to grasp in relation to our simple example: \$1,000 received now is equivalent to \$1,762.34 received five years from now only at a 12% interest rate. Any change in the interest rate will destroy the equivalence between these two sums, as we will demonstrate in Example 3.5.

EXAMPLE 3.5 Changing the Interest Rate Destroys Equivalence

In Example 3.3, we determined that, given an interest rate of 8% per year, receiving \$2,042 today is equivalent to receiving \$3,000 in five years. Are these cash flows equivalent at an interest rate of 10%?

SOLUTION

Given: $P = \$2{,}042$, $i = 10\%$ per year, and $N = 5$ years.

Find: F: Is it equal to \$3,000?

We first determine the base period under which an equivalence value is computed. Since we can select any period as the base period, let's select $N = 5$. Then we need to calculate the equivalent value of \$2,042 today five years from now.

$$F = \$2{,}042(1 + 0.10)^5 = \$3{,}289.$$

Since this amount is greater than \$3,000, the change in interest rate destroys the equivalence between the two cash flows.

Principle 3: Equivalence Calculations May Require the Conversion of Multiple Payment Cash Flows to a Single Cash Flow

In all the examples presented thus far, we have limited ourselves to the simplest case of converting a single payment at one time to an equivalent single payment at another time. Part of the task of comparing alternative cash flow series involves moving each individual cash flow in the series to the same single point in time and summing these values to yield a single equivalent cash flow. We perform such a calculation in Example 3.6.

EXAMPLE 3.6 Equivalence Calculations with Multiple Payments

Suppose that you borrow \$1,000 from a bank for three years at 10% annual interest. The bank offers two options: (1) repaying the interest charges for each year at the end of that year and repaying the principal at the end of year 3 or (2) repaying the loan all at once (including both interest and principal) at the end of year 3. The repayment schedules for the two options are as follows:

Options	Year 1	Year 2	Year 3
• Option 1: End-of-year repayment of interest, and principal repayment at end of loan	\$100	\$100	\$1,100
• Option 2: One end-of-loan repayment of both principal and interest	0	0	1,331

Determine whether these options are equivalent, assuming that the appropriate interest rate for the comparison is 10%.

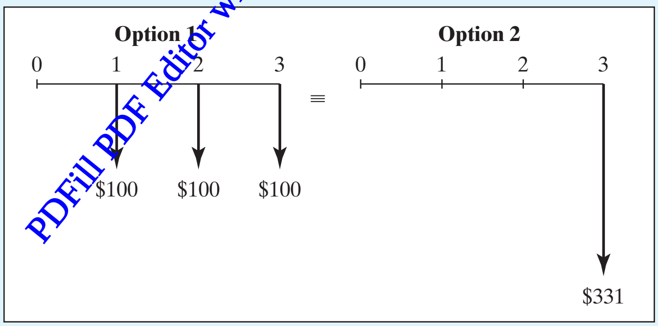
STRATEGY: Since we pay the principal after three years in either plan, the repayment of principal can be removed from our analysis. This is an important point: *We can ignore the common elements of alternatives being compared so that we can focus entirely on comparing the interest payments*. Notice that under Option 1, we will pay a total of \$300 interest, whereas under Option 2, we will pay a total of \$331. Before concluding that we prefer Option 2, remember that a comparison of the two cash flows is based on a *combination of payment amounts and the timing of those payments*. To make our comparison, we must compare the equivalent value of each option at a single point in time. Since Option 2 is already a single payment at $n = 3$ years, it is simplest to convert the cash flow pattern of Option 1 to a single value at $n = 3$. To do this, we must convert the three disbursements of Option 1 to their respective equivalent values at $n = 3$. At that point, since they share a time in common, we can simply sum them in order to compare them with the \$331 sum in Option 2.

SOLUTION

Given: Interest payment series; $i = 10\%$ per year.

Find: A single future value F of the flows in Option 1.

Equation: $F = P(1 + i)^N$, applied to each disbursement in the cash flow diagram. N in Eq. (3.3) is the number of interest periods during which interest is in effect, and n is the period number (i.e., for year 1, $n = 1$). We determine the value of F by finding the interest period for each payment. Thus, for each payment in the series, N can be calculated by subtracting n from the total number of years of the loan (3). That is, $N = 3 - n$. Once the value of each payment has been found, we sum the payments:



$$F_3 \text{ for } \$100 \text{ at } n = 1: \$100(1 + .10)^{3-1} = \$121;$$

$$F_3 \text{ for } \$100 \text{ at } n = 2: \$100(1 + .10)^{3-2} = \$110;$$

$$F_3 \text{ for } \$100 \text{ at } n = 3: \$100(1 + .10)^{3-3} = \underline{\$100};$$

$$\text{Total} = \$331.$$

By converting the cash flow in Option 1 to a single future payment at year 3, we can compare Options 1 and 2. We see that the two interest payments are equivalent. Thus, the bank would be economically indifferent to a choice between the two plans. Note that the final interest payment in Option 1 does not accrue any compound interest.

Principle 4: Equivalence Is Maintained Regardless of Point of View

As long as we use the same interest rate in equivalence calculations, equivalence can be maintained regardless of point of view. In Example 3.6, the two options were equivalent at an interest rate of 10% from the banker's point of view. What about from a borrower's point of view? Suppose you borrow $1,000 from a bank and deposit it in another bank that pays 10% interest annually. Then you make future loan repayments out of this savings account. Under Option 1, your savings account at the end of year 1 will show a balance of $1,100 after the interest earned during the first period has been credited. Now you withdraw $100 from this savings account (the exact amount required to pay the loan interest during the first year), and you make the first-year interest payment to the bank. This leaves only $1,000 in your savings account. At the end of year 2, your savings account will earn another interest payment in the amount of $\$1{,}000(0.10) = \$100$, making an end-of-year balance of $1,100. Now you withdraw another $100 to make the required loan interest payment. After this payment, your remaining balance will be $1,000. This balance will grow again at 10%, so you will have $1,100 at the end of year 3. After making the last loan payment ($1,100), you will have no money left in either account. For Option 2, you can keep track of the yearly account balances in a similar fashion. You will find that you reach a zero balance after making the lump-sum payment of $1,331. If the borrower had used the same interest rate as the bank, the two options would be equivalent.

3.2.3 Looking Ahead

The preceding examples should have given you some insight into the basic concepts and calculations involved in the concept of economic equivalence. Obviously, the variety of financial arrangements possible for borrowing and investing money is extensive, as is the variety of time-related factors (e.g., maintenance costs over time, increased productivity over time, etc.) in alternative proposals for various engineering projects. It is important to recognize that even the most complex relationships incorporate the basic principles we have introduced in this section.

In the remainder of the chapter, we will represent all cash flow diagrams either in the context of an initial deposit with a subsequent pattern of withdrawals or in an initial borrowed amount with a subsequent pattern of repayments. If we were limited to the methods developed in this section, a comparison between the two payment options would involve a large number of calculations. Fortunately, in the analysis of many transactions, certain cash flow patterns emerge that may be categorized. For many of these patterns, we can derive formulas that can be used to simplify our work. In Section 3.3, we develop these formulas.

3.3 Development of Interest Formulas

Now that we have established some working assumptions and notations and have a preliminary understanding of the concept of equivalence, we will develop a series of interest formulas for use in more complex comparisons of cash flows.

As we begin to compare series of cash flows instead of single payments, the required analysis becomes more complicated. However, when patterns in cash flow transactions can be identified, we can take advantage of these patterns by developing concise expressions for computing either the present or future worth of the series. We will classify five major categories of cash flow transactions, develop interest formulas for them, and present several working examples of each type. Before we give the details, however, we briefly describe the five types of cash flows in the next subsection.

3.3.1 The Five Types of Cash Flows

Whenever we identify patterns in cash flow transactions, we may use those patterns to develop concise expressions for computing either the present or future worth of the series. For this purpose, we will classify cash flow transactions into five categories: (1) a single cash flow, (2) a uniform series, (3) a linear gradient series, (4) a geometric gradient series, and (5) an irregular series. To simplify the description of various interest formulas, we will use the following notation:

1. **Single Cash Flow:** The simplest case involves the equivalence of a single present amount and its future worth. Thus, the single-cash-flow formulas deal with only two amounts: a single present amount P and its future worth F (Figure 3.10a). You have already seen the derivation of one formula for this situation in Section 3.1.3, which gave us Eq. (3.3):

$$F = P(1 + i)^N.$$

2. **Equal (Uniform) Series:** Probably the most familiar category includes transactions arranged as a series of equal cash flows at regular intervals, known as an **equal payment series** (or **uniform series**) (Figure 3.10b). For example, this category describes the cash flows of the common installment loan contract, which arranges the repayment of a loan in equal periodic installments. The equal-cash-flow formulas deal with the equivalence relations P, F, and A (the constant amount of the cash flows in the series).
3. **Linear Gradient Series:** While many transactions involve series of cash flows, the amounts are not always uniform; they may, however, vary in some regular way. One common pattern of variation occurs when each cash flow in a series increases (or decreases) by a fixed amount (Figure 3.10c). A five-year loan repayment plan might specify, for example, a series of annual payments that increase by $500 each year. We call this type of cash flow pattern a **linear gradient series** because its cash flow diagram produces an ascending (or descending) straight line, as you will see in Section 3.3.5. In addition to using P, F, and A, the formulas employed in such problems involve a *constant amount* G of the change in each cash flow.
4. **Geometric Gradient Series:** Another kind of gradient series is formed when the series in a cash flow is determined not by some fixed amount like $500, but by some fixed *rate*, expressed as a percentage. For example, in a five-year financial plan for a project, the cost of a particular raw material might be budgeted to increase at a rate of 4% per year. The curving gradient in the diagram of such a series suggests its name: a **geometric gradient series** (Figure 3.10d). In the formulas dealing with such series, the rate of change is represented by a lowercase g.
5. **Irregular (Mixed) Series:** Finally, a series of cash flows may be irregular, in that it does not exhibit a regular overall pattern. Even in such a series, however, one or more of the patterns already identified may appear over segments of time in the total length of the series. The cash flows may be equal, for example, for 5 consecutive periods in a 10-period series. When such patterns appear, the formulas for dealing with them may be applied and their results included in calculating an equivalent value for the entire series.

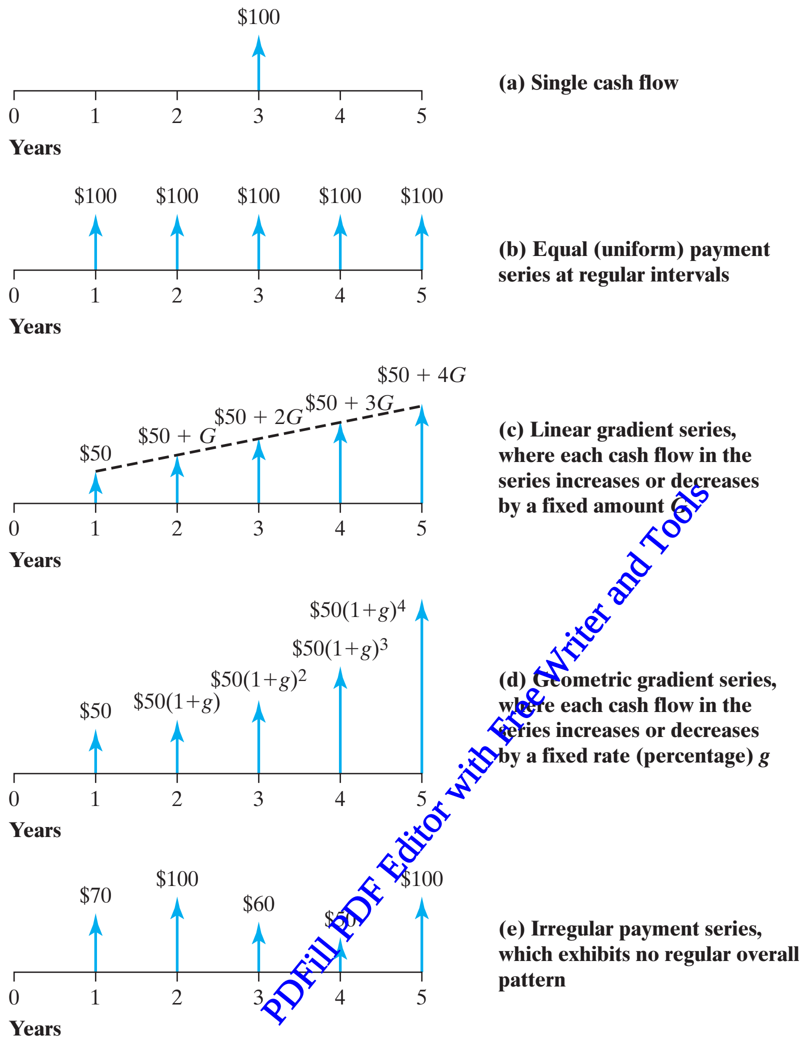


Figure 3.10 Five types of cash flows: (a) Single cash flow, (b) equal (uniform) payment series, (c) linear gradient series, (d) geometric gradient series, and (e) irregular payment series.

3.3.2 Single-Cash-Flow Formulas

We begin our coverage of interest formulas by considering the simplest of cash flows: single cash flows.

Compound Amount Factor

Given a present sum P invested for N interest periods at interest rate i, what sum will have accumulated at the end of the N periods? You probably noticed right away that this description matches the case we first encountered in describing compound interest. To solve for F (the future sum), we use Eq. (3.3):

$$F = P(1 + i)^N.$$

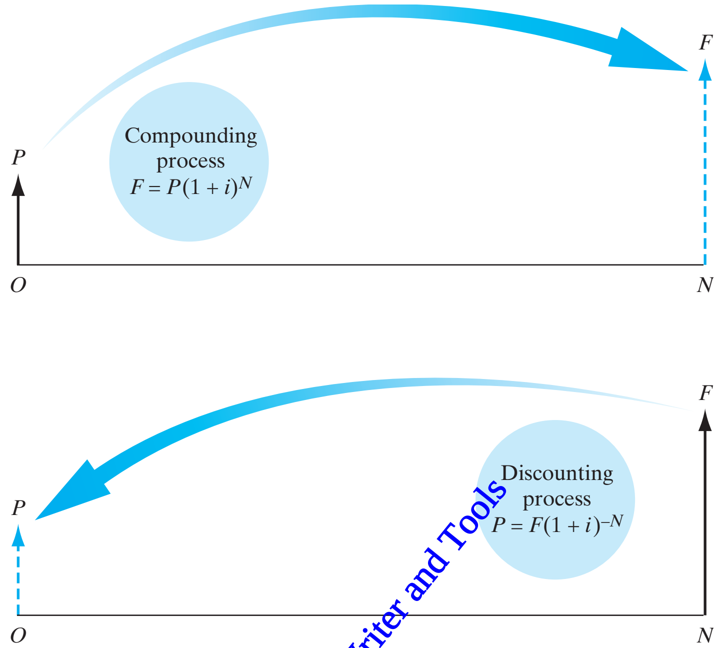


Figure 3.11 Equivalence relation between P and F.

Because of its origin in the compound-interest calculation, the factor $(1 + i)^N$ is known as the **compound-amount factor**. Like the concept of equivalence, this factor is one of the foundations of engineering economic analysis. Given the compound-amount factor, all the other important interest formulas can be derived.

Compounding process: the process of computing the future value of a current sum.

This process of finding F is often called the **compounding process**. The cash flow transaction is illustrated in Figure 3.11. (Note the time-scale convention: The first period begins at $n = 0$ and ends at $n = 1$.) If a calculator is handy, it is easy enough to calculate $(1 + i)^N$ directly.

Interest Tables

Interest formulas such as the one developed in Eq. (3.3), $F = P(1 + i)^N$, allow us to substitute known values from a particular situation into the equation and to solve for the unknown. Before the hand calculator was developed, solving these equations was very tedious. With a large value of N, for example, one might need to solve an equation such as $F = \$20{,}000(1 + 0.12)^{15}$. More complex formulas required even more involved calculations. To simplify the process, tables of compound-interest factors were developed, and these tables allow us to find the appropriate factor for a given interest rate and the number of interest periods. Even with hand calculators, it is still often convenient to use such tables, and they are included in this text in Appendix A. Take some time now to become familiar with their arrangement and, if you can, locate the compound-interest factor for the example just presented, in which we know P. Remember that, to find F, we need to know the factor by which to multiply \$20,000 when the interest rate i is 12% and the number of periods is 15:

$$F = \$20{,}000\underbrace{(1 + 0.12)^{15}}_{5.4736} = \$109{,}472.$$

Factor Notation

As we continue to develop interest formulas in the rest of this chapter, we will express the resulting compound-interest factors in a conventional notation that can be substituted in a formula to indicate precisely which table factor to use in solving an equation. In the preceding example, for instance, the formula derived as Eq. (3.3) is $F = P(1 + i)^N$. In ordinary language, this tells us that, to determine what future amount F is equivalent to a present amount P, we need to multiply P by a factor expressed as 1 plus the interest rate, raised to the power given by the number of interest periods. To specify how the interest tables are to be used, we may also express that factor in functional notation as $(F/P, i, N)$, which is read as "Find F, Given P, i, and N." This is known as the **single-payment compound-amount factor**. When we incorporate the table factor into the formula, it is expressed as

$$F = P(1 + i)^N = P(F/P, i, N).$$

Thus, in the preceding example, where we had $F = \$20{,}000(1.12)^{15}$, we can write $F = \$20{,}000(F/P, 12\%, 15)$. The table factor tells us to use the 12% interest table and find the factor in the F/P column for $N = 15$. Because using the interest tables is often the easiest way to solve an equation, this factor notation is included for each of the formulas derived in the sections that follow.

EXAMPLE 3.7 Single Amounts: Find F, Given i, N, and P

If you had \$2,000 now and invested it at 10%, how much would it be worth in eight years (Figure 3.12)?

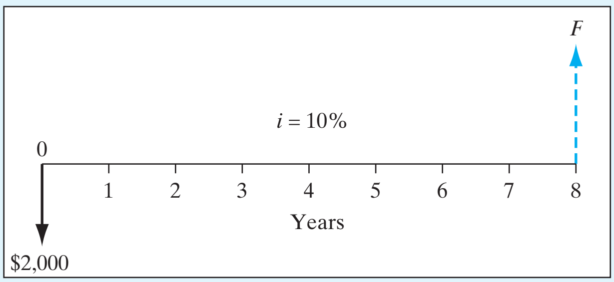


Figure 3.12 A cash flow diagram from the investor's point of view (Example 3.7).

SOLUTION

Given: $P = \$2{,}000$, $i = 10\%$ per year, and $N = 8$ years.
Find: F.

We can solve this problem in any of three ways:

1. **Using a calculator.** You can simply use a calculator to evaluate the $(1 + i)^N$ term (financial calculators are preprogrammed to solve most future-value problems):

$$F = \$2{,}000(1 + 0.10)^8$$
$$= \$4{,}287.18.$$

2. **Using compound-interest tables.** The interest tables can be used to locate the compound-amount factor for $i = 10\%$ and $N = 8$. The number you get can be substituted into the equation. Compound-interest tables are included as Appendix A of this book. From the tables, we obtain

$$F = \$2{,}000(F/P, 10\%, 8) = \$2{,}000(2.1436) = \$4{,}287.20.$$

This is essentially identical to the value obtained by the direct evaluation of the single-cash-flow compound-amount factor. This slight difference is due to rounding errors.

3. **Using Excel.** Many financial software programs for solving compound-interest problems are available for use with personal computers. Excel provides financial functions to evaluate various interest formulas, where the future-worth calculation looks like the following:

```
=FV(10%,8,0,−2000)
```


Present-Worth Factor

Discounting process: A process of calculating the present value of a future amount.

Finding the present worth of a future sum is simply the reverse of compounding and is known as the **discounting process**. In Eq. (3.3), we can see that if we were to find a present sum P, given a future sum F, we simply solve for P:

$$P = F\left[\frac{1}{(1 + i)^N}\right] = F(P/F, i, N). \tag{3.7}$$

The factor $1/(1 + i)^N$ is known as the **single-payment present-worth factor** and is designated $(P/F, i, N)$. Tables have been constructed for P/F factors and for various values of i and N. The interest rate i and the P/F factor are also referred to as the **discount rate** and **discounting factor**, respectively.

EXAMPLE 3.8 Single Amounts: Find *P*, Given *F*, *i*, and *N*

Suppose that \$1,000 is to be received in five years. At an annual interest rate of 12%, what is the present worth of this amount?

SOLUTION

Given: $F = \$1{,}000$, $i = 12\%$ per year, and $N = 5$ years.
Find: P.

$$P = \$1{,}000(1 + 0.12)^{-5} = \$1{,}000(0.5674) = \$567.40.$$

Using a calculator may be the best way to make this simple calculation. To have \$1,000 in your savings account at the end of five years, you must deposit \$567.40 now.

We can also use the interest tables to find that

$$P = \$1{,}000\overbrace{(P/F, 12\%, 5)}^{(0.5674)} = \$567.40.$$

Again, you could use a financial calculator or a computer to find the present worth. With Excel, the present-value calculation looks like the following:

=PV(12%,5,0,−1000)

Solving for Time and Interest Rates

At this point, you should realize that the compounding and discounting processes are reciprocals of one another and that we have been dealing with one equation in two forms:

$$\text{Future-value form: } F = P(1 + i)^{N};$$
$$\text{Present-value form: } P = F(1 + i)^{-N}.$$

There are four variables in these equations: *P*, *F*, *N*, and *i*. If you know the values of any three, you can find the value of the fourth. Thus far, we have always given you the interest rate *i* and the number of years *N*, plus either *P* or *F*. In many situations, though, you will need to solve for *i* or *N*, as we discuss next.

EXAMPLE 3.9 Solving for *i*

Suppose you buy a share for \$10 and sell it for \$20. Then your profit is \$10. If that happens within a year, your rate of return is an impressive 100% (\$10/\$10 = 1). If it takes five years, what would be the average annual rate of return on your investment? (See Figure 3.13.)

SOLUTION

Given: $P = \$10$, $F = \$20$, and $N = 5$.

Find: i.

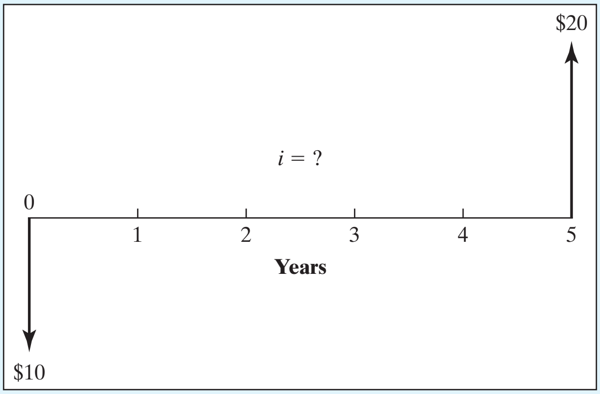


Figure 3.13 Cash flow diagram (Example 3.9).

Here, we know *P*, *F*, and *N*, but we do not know *i*, the interest rate you will earn on your investment. This type of rate of return is a lot easier to calculate, because you make only a one-time lump-sum investment. Problems such as this are solved as follows:

$$F = P(1 + i)^N,$$
$$\$20 = \$10(1 + i)^5;\text{ solve for } i.$$

- **Method 1.** Go through a trial-and-error process in which you insert different values of i into the equation until you find a value that "works" in the sense that the right-hand side of the equation equals \$20. The solution is $i = 14.87\%$. The trial-and-error procedure is extremely tedious and inefficient for most problems, so it is not widely practiced in the real world.
- **Method 2.** You can solve the problem by using the interest tables in Appendix A. Now look across the $N = 5$ row, under the $(F/P, i, 5)$ column, until you can locate the value of 2:

$$\$20 = \$10(1 + i)^5;$$
$$2 = (1 + i)^5 = (F/P, i, 5).$$

  This value is close to the 15% interest table with $(F/P, 15\%, 5) = 2.0114$, so the interest rate at which \$10 grows to \$20 over five years is very close to 15%. This procedure will be very tedious for fractional interest rates or when N is not a whole number, because you may have to approximate the solution by linear interpolation.
- **Method 3.** The most practical approach is to use either a financial calculator or an electronic spreadsheet such as Excel. A financial function such as RATE(*N*,0,*P*,*F*) allows us to calculate an unknown interest rate. The precise command statement would be

$$=\textbf{RATE}(5{,}0{,}-10{,}20)=14.87\%$$

  Note that, in Excel format, we enter the present value (P) as a negative number, indicating a cash outflow.

	A	B	C
1	*P*	–10	
2	*F*	20	
3	*N*	5	
4	*i*	14.87%	
5			

=RATE(5,0,−10,20)

EXAMPLE 3.10 Single Amounts: Find *N*, Given *P*, *F*, and *i*

You have just purchased 100 shares of General Electric stock at $60 per share. You will sell the stock when its market price has doubled. If you expect the stock price to increase 20% per year, how long do you anticipate waiting before selling the stock (Figure 3.14)?

SOLUTION

Given: $P = \$6{,}000$, $F = \$12{,}000$, and $i = 20\%$ per year.
Find: N (years).

Using the single-payment compound-amount factor, we write

$$F = P(1 + i)^N = P(F/P, i, N);$$

$$\$12{,}000 = \$6{,}000(1 + 0.20)^N = \$6{,}000(F/P, 20\%, N);$$

$$2 = (1.20)^N = (F/P, 20\%, N).$$

Again, we could use a calculator or a computer spreadsheet program to find N.

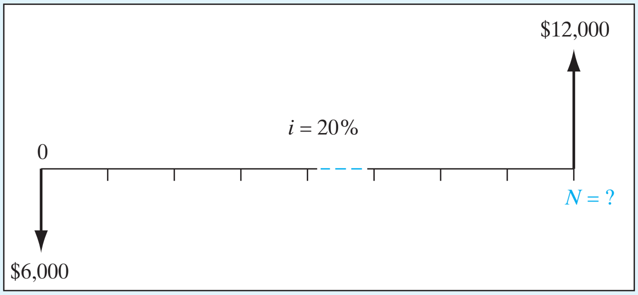


Figure 3.14

1. **Using a calculator.** Solving for N gives

$$\log 2 = N \log 1.20,$$

or

$$N = \frac{\log 2}{\log 1.20}$$
$$= 3.80 \approx 4 \text{ years.}$$

2. **Using Excel.** Within Excel, the financial function NPER(i,0,P,F) computes the number of compounding periods it will take an investment (P) to grow to a future value (F), earning a fixed interest rate (i) per compounding period. In our example, the Excel command would look like this:

$$= \text{NPER}(20\%, 0, -6000, 12000)$$
$$= 3.801784.$$

Rule of 72: Rule giving the approximate number of years that it will take for your investment to double.

COMMENTS: A very handy rule of thumb, called the Rule of 72, estimates approximately how long it will take for a sum of money to double. The rule states that, to find the time it takes for a present sum of money to grow by a factor of two, we divide 72 by the interest rate. In our example, the interest rate is 20%. Therefore, the Rule of 72 indicates 72/20 = 3.60, or roughly 4 years, for a sum to double. This is, in fact, relatively close to our exact solution. Figure 3.15 illustrates the number of years required to double an investment at various interest rates.

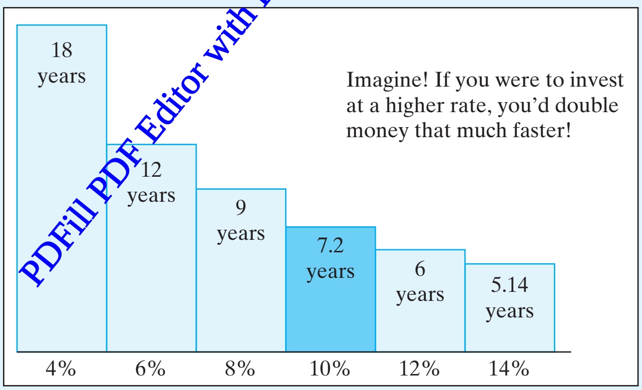


Figure 3.15 Number of years required to double an initial investment at various interest rates.

3.3.3 Uneven Payment Series

A common cash flow transaction involves a series of disbursements or receipts. Familiar examples of series payments are payment of installments on car loans and home mortgage payments. Payments on car loans and home mortgages typically involve identical sums to be paid at regular intervals. However, there is no clear pattern over the series; we call the transaction an uneven cash flow series.

We can find the present worth of any uneven stream of payments by calculating the present value of each individual payment and summing the results. Once the present worth is found, we can make other equivalence calculations (e.g., future worth can be calculated by using the interest factors developed in the previous section).

EXAMPLE 3.11 Present Values of an Uneven Series by Decomposition into Single Payments

Wilson Technology, a growing machine shop, wishes to set aside money now to invest over the next four years in automating its customer service department. The company can earn 10% on a lump sum deposited now, and it wishes to withdraw the money in the following increments:

- **Year 1:** \$25,000, to purchase a computer and database software designed for customer service use;
- **Year 2:** \$3,000, to purchase additional hardware to accommodate anticipated growth in use of the system;
- **Year 3:** No expenses; and
- **Year 4:** \$5,000, to purchase software upgrades.

How much money must be deposited now to cover the anticipated payments over the next 4 years?

STRATEGY: This problem is equivalent to asking what value of P would make you indifferent in your choice between P dollars today and the future expense stream of (\$25,000, \$3,000, \$0, \$5,000). One way to deal with an uneven series of cash flows is to calculate the equivalent present value of each single cash flow and to sum the present values to find P. In other words, the cash flow is broken into three parts as shown in Figure 3.16.

SOLUTION

Given: Uneven cash flow in Figure 3.16, with $i = 10\%$ per year.
Find: P.

$$\begin{aligned} P &= \$25{,}000(P/F, 10\%, 1) + \$3{,}000(P/F, 10\%, 2) \\ &\quad + \$5{,}000(P/F, 10\%, 4) \\ &= \$28{,}622. \end{aligned}$$

COMMENTS: To see if \$28,622 is indeed sufficient, let's calculate the balance at the end of each year. If you deposit \$28,622 now, it will grow to (1.10)(\$28,622), or \$31,484, at the end of year 1. From this balance, you pay out \$25,000. The remaining balance, \$6,484, will again grow to (1.10)(\$6,484), or \$7,132, at the end of year 2. Now you make the second payment (\$3,000) out of this balance, which will leave you with only \$4,132 at the end of year 2. Since no payment occurs in year 3, the

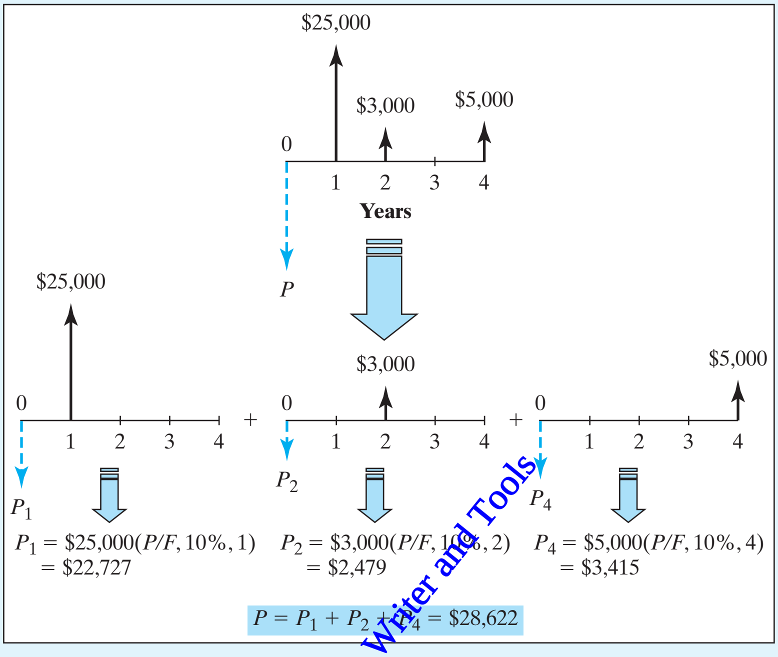


Figure 3.16 Decomposition of uneven cash flow series (Example 3.11).

balance will grow to $\$(1.10)^2(\$4,132)$, or \$5,000, at the end of year 4. The final withdrawal in the amount of \$5,000 will deplete the balance completely.

EXAMPLE 3.12 Calculating the Actual Worth of a Long-Term Contract of Michael Vick with Atlanta Falcons[2]

On December 23, 2004, Michael Vick became the richest player in the National Football League by agreeing to call Atlanta home for the next decade. The Falcons' quarterback signed a 10-year, \$130 million contract extension Thursday that guarantees him an NFL-record \$37 million in bonuses.

Base salaries for his new contract are \$600,000 (2005), \$1.4 million (2006), \$6 million (2007), \$7 million (2008), \$9 million (2009), \$10.5 million (2010), \$13.5 million (2011), \$13 million (2012), \$15 million (2013), and \$17 million (2014). He received an initial signing bonus of \$7.5 million. Vick also received two roster bonuses in the new deal. The first is worth \$22.5 million and is due in March 2005. The second is worth \$7 million and is due in March 2006. Both roster bonuses will be treated as signing bonuses and prorated annually. Because 2011 is an uncapped year (the league's collective bargaining agreement (CBA) expires after the 2010 season), the initial signing bonus and 2005 roster bonus can be prorated only over the first six years of the contract. If the CBA is extended

[2] *Source:* http://www.falcfans.com/players/michael_vick.html.

prior to March 2006, then the second roster bonus of \$7 million can be prorated over the final nine seasons of the contract. If the CBA is extended prior to March 2006, then his cap hits (rounded to nearest thousand) will change to \$7.178 million (2006), \$11.778 million (2007), \$12.778 million (2008), \$14.778 million (2009), \$16.278 million (2010), \$14.278 million (2011), \$13.778 million (2012), \$15.778 million (2013), and \$17.778 million (2014). With the salary and signing bonus paid at the beginning of each season, the net annual payment schedule looks like the following:

Beginning of Season	Base Salary	Prorated Signing Bonus	Total Annual Payment
2005	\$ 600,000	\$5,000,000	\$5,600,000
2006	1,400,000	5,000,000 + 778,000	7,178,000
2007	6,000,000	5,000,000 + 778,000	11,778,000
2008	7,000,000	5,000,000 + 778,000	12,778,000
2009	9,000,000	5,000,000 + 778,000	14,778,000
2010	10,500,000	778,000	16,278,000
2011	13,500,000	778,000	14,278,000
2012	13,000,000	778,000	13,778,000
2013	15,000,000	778,000	15,778,000
2014	17,000,000	778,000	17,778,000

(a) How much is Vick's contract actually worth at the time of signing? Assume that Vick's interest rate is 6% per year.

(b) For the initial signing bonus and the first year's roster bonus, suppose that the Falcons allow Vick to take either the prorated payment option as just described (\$30 million over five years) or a lump-sum payment option in the amount of \$23 million at the time he signs the contract. Should Vick take the lump-sum option instead of the prorated one?

SOLUTION

Given: Payment series given in Figure 3.17, with $i = 6\%$ per year.
Find: P.

(a) Actual worth of the contract at the time of signing:

$$\begin{aligned} P_{\text{contract}} &= \$5{,}600{,}000 + \$7{,}178{,}000(P/F, 6\%, 1) \\ &\quad + \$11{,}778{,}000(P/F, 6\%, 2) + \cdots \\ &\quad + \$17{,}778{,}000(P/F, 6\%, 9) \\ &= \$97{,}102{,}827. \end{aligned}$$

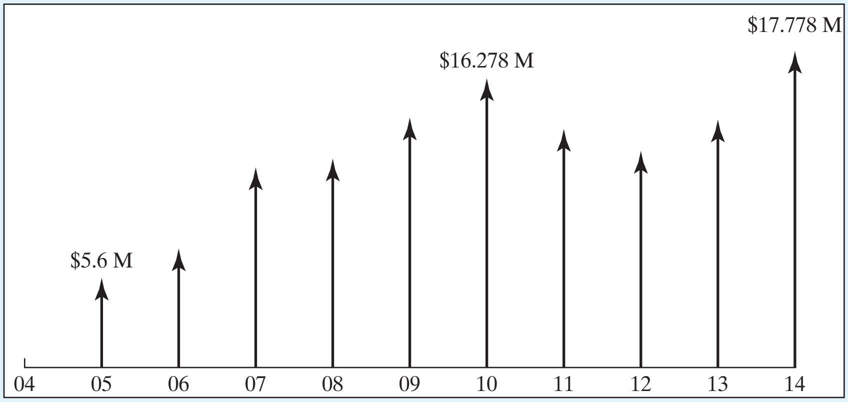


Figure 3.17 Cash flow diagram for Michael Vick's contract with Atlanta Falcons.

(b) Choice between the prorated payment option and the lump-sum payment: The equivalent present worth of the prorated payment option is

$$\begin{aligned} P_{\text{bonus}} &= \$5{,}000{,}000 + \$5{,}000{,}000(P/F, 6\%, 1) \\ &\quad + \$5{,}000{,}000(P/F, 6\%, 2) + \$5{,}000{,}000(P/F, 6\%, 3) \\ &\quad + \$5{,}000{,}000(P/F, 6\%, 4) \\ &= \$22{,}325{,}528 \end{aligned}$$

which is smaller than $23,000,000. Therefore, Vick would be better off taking the lump-sum option if, and only if, his money could be invested at 6% or higher.

COMMENTS: Note that the actual contract is worth less than the published figure of $130 million. This "brute force" approach of breaking cash flows into single amounts will always work, but it is slow and subject to error because of the many factors that must be included in the calculation. We develop more efficient methods in later sections for cash flows with certain patterns.

3.3.4 Equal Payment Series

As we learned in Example 3.12, the present worth of a stream of future cash flows can always be found by summing the present worth of each of the individual cash flows. However, if cash flow regularities are present within the stream (such as we just saw in the prorated bonus payment series in Example 3.12) then the use of shortcuts, such as finding the present worth of a uniform series, may be possible. We often encounter transactions in which a uniform series of payments exists. Rental payments, bond interest payments, and commercial installment plans are based on uniform payment series.

Compound-Amount Factor: Find F, Given A, i, and N

Suppose we are interested in the future amount F of a fund to which we contribute A dollars each period and on which we earn interest at a rate of i per period. The contributions

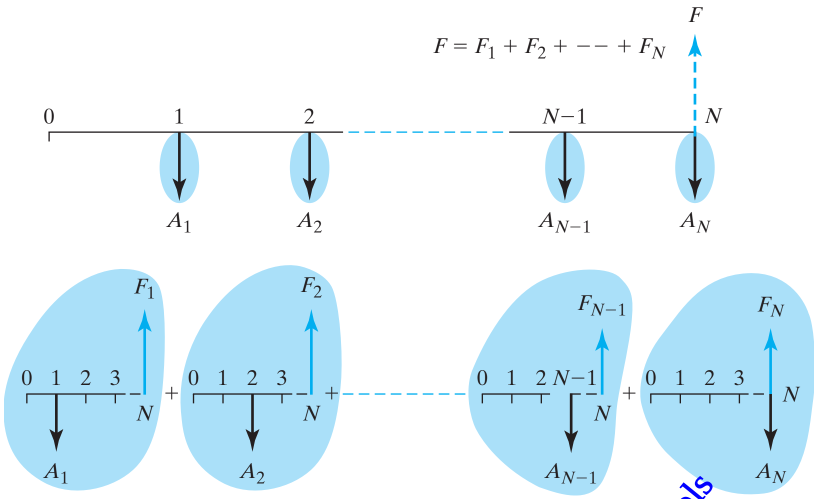


Figure 3.18 The future worth of a cash flow series obtained by summing the future-worth figures of each of the individual flows.

are made at the end of each of N equal periods. These transactions are graphically illustrated in Figure 3.18. Looking at this diagram, we see that if an amount A is invested at the end of each period, for N periods, the total amount F that can be withdrawn at the end of the N periods will be the sum of the compound amounts of the individual deposits.

As shown in Figure 3.18, the A dollars we put into the fund at the end of the first period will be worth $A(1 + i)^{N-1}$ at the end of N periods. The A dollars we put into the fund at the end of the second period will be worth $A(1 + i)^{N-2}$, and so forth. Finally, the last A dollars that we contribute at the end of the Nth period will be worth exactly A dollars at that time. This means that there exists a series of the form

$$F = A(1 + i)^{N-1} + A(1 + i)^{N-2} + \cdots + A(1 + i) + A,$$

or, expressed alternatively,

$$F = A + A(1 + i) + A(1 + i)^2 + \cdots + A(1 + i)^{N-1}. \tag{3.8}$$

Multiplying Eq. (3.8) by $(1 + i)$ results in

$$(1 + i)F = A(1 + i) + A(1 + i)^2 + \cdots + A(1 + i)^N. \tag{3.9}$$

Subtracting Eq. (3.8) from Eq. (3.9) to eliminate common terms gives us

$$F(1 + i) - F = -A + A(1 + i)^N.$$

Solving for F yields

$$F = A\left[\frac{(1 + i)^N - 1}{i}\right] = A(F/A, i, N). \tag{3.10}$$

The bracketed term in Eq. (3.10) is called the **equal payment series compound-amount factor**, or the **uniform series compound-amount factor**; its factor notation is $(F/A, i, N)$. This interest factor has been calculated for various combinations of i and N in the interest tables.

EXAMPLE 3.13 Uniform Series: Find *F*, Given *i*, *A*, and *N*

Suppose you make an annual contribution of \$3,000 to your savings account at the end of each year for 10 years. If the account earns 7% interest annually, how much can be withdrawn at the end of 10 years (Figure 3.19)?

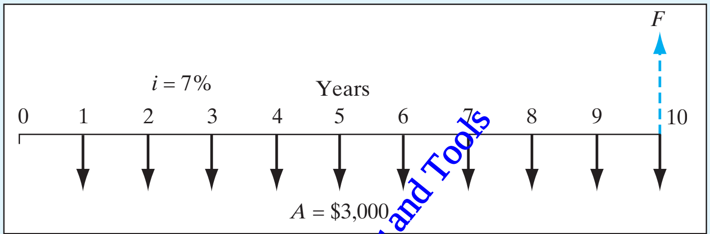


Figure 3.19 Cash flow diagram (Example 3.13).

SOLUTION

Given: $A = \$3{,}000$, $N = 10$ years, and $i = 7\%$ per year.
Find: F.

$$\begin{aligned} F &= \$3{,}000(F/A, 7\%, 10) \\ &= \$3{,}000(13.8164) \\ &= \$41{,}449.20. \end{aligned}$$

To obtain the future value of the annuity with the use of Excel, we may use the following financial command:

$$=\text{FV}(7\%,10,-3000,0,0)$$

EXAMPLE 3.14 Handling Time Shifts in a Uniform Series

In Example 3.13, the first deposit of the 10-deposit series was made at the end of period 1 and the remaining nine deposits were made at the end of each following period. Suppose that all deposits were made at the beginning of each period instead. How would you compute the balance at the end of period 10?

SOLUTION

Given: Cash flow as shown in Figure 3.20, and $i = 7\%$ per year.
Find: F_{10}.

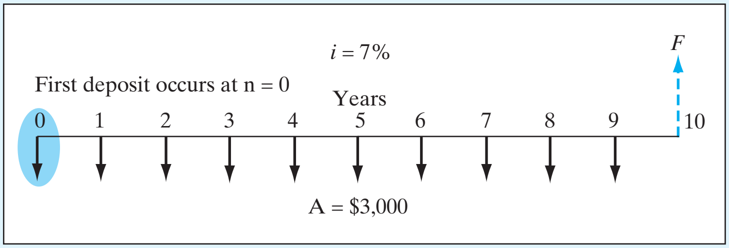


Figure 3.20 Cash Flow diagram (Example 3.14).

Compare Figure 3.20 with Figure 3.19: Each payment has been shifted to one year earlier; thus, each payment would be compounded for one extra year. Note that with the end-of-year deposit, the ending balance (F) was \$41,449.20. With the beginning-of-year deposit, the same balance accumulates by the end of period 9. This balance can earn interest for one additional year. Therefore, we can easily calculate the resulting balance as

$$F_{10} = \$41{,}449.20(1.07) = \$44{,}350.64.$$

The annuity due can be easily evaluated with the following financial command available on Excel:

$$=\mathbf{FV}(7\%,10,-3000,0,1)$$

COMMENTS: Another way to determine the ending balance is to compare the two cash flow patterns. By adding the \$3,000 deposit at period 0 to the original cash flow and subtracting the \$3,000 deposit at the end of period 10, we obtain the second cash flow. Therefore, the ending balance can be found by making the following adjustment to the \$41,449.20:

$$F_{10} = \$41{,}449.20 + \$3{,}000(F/P, 7\%, 10) - \$3{,}000 = \$44{,}350.64.$$

Sinking-Fund Factor: Find *A*, Given *F*, *i*, and *N*

If we solve Eq. (3.10) for A, we obtain

$$A = F\left[\frac{i}{(1+i)^N - 1}\right] = F(A/F, i, N). \quad (3.11)$$

The term within the brackets is called the **equal payment series sinking-fund factor**, or **sinking-fund factor**, and is referred to by the notation $(A/F, i, N)$. A sinking fund is an interest-bearing account into which a fixed sum is deposited each interest period; it is commonly established for the purpose of replacing fixed assets or retiring corporate bonds.

Sinking fund: A means of repaying funds advanced through a bond issue. This means *that every period, a company will pay back a portion of its bonds.*

EXAMPLE 3.15 Combination of a Uniform Series and a Single Present and Future Amount

To help you reach a \$5,000 goal five years from now, your father offers to give you \$500 now. You plan to get a part-time job and make five additional deposits, one at the end of each year. (The first deposit is made at the end of the first year.) If all your money is deposited in a bank that pays 7% interest, how large must your annual deposit be?

STRATEGY: If your father reneges on his offer, the calculation of the required annual deposit is easy because your five deposits fit the standard end-of-period pattern for a uniform series. All you need to evaluate is

$$A = \$5{,}000(A/F, 7\%, 5) = \$5{,}000(0.1739) = \$869.50.$$

If you do receive the \$500 contribution from your father at $n = 0$, you may divide the deposit series into two parts: one contributed by your father at $n = 0$ and five equal annual deposit series contributed by yourself. Then you can use the F/P factor to find how much your father's contribution will be worth at the end of year 5 at a 7% interest rate. Let's call this amount F_c. The future value of your five annual deposits must then make up the difference, \$5,000 − F_c.

SOLUTION

Given: Cash flow as shown in Figure 3.21, with $i = 7\%$ per year, and $N = 5$ years.
Find: A.

$$\begin{aligned} A &= (\$5{,}000 - F_C)(A/F, 7\%, 5) \\ &= [\$5{,}000 - \$500(F/P, 7\%, 5)](A/F, 7\%, 5) \\ &= [\$5{,}000 - \$500(1.4026)](0.1739) \\ &= \$747.55. \end{aligned}$$

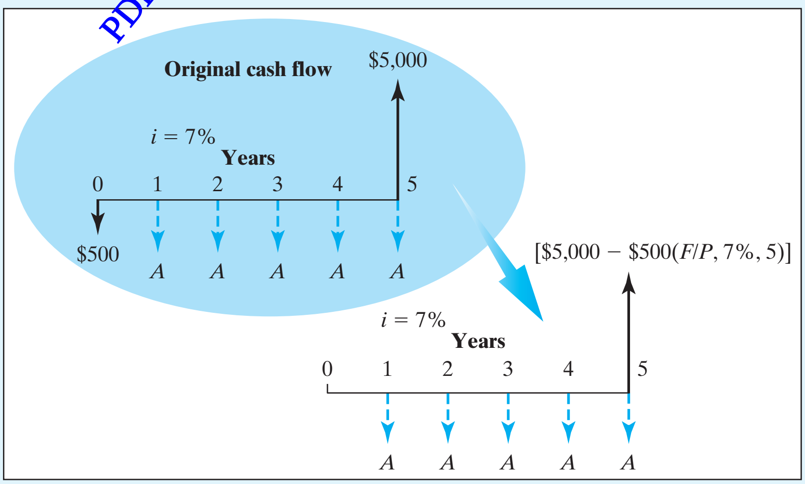


Figure 3.21 Cash flow diagram (Example 3.15).

EXAMPLE 3.16 Comparison of Three Different Investment Plans

Consider three investment plans at an annual interest rate of 9.38% (Figure 3.22):

- **Investor A.** Invest \$2,000 per year for the first 10 years of your career. At the end of 10 years, make no further investments, but reinvest the amount accumulated at the end of 10 years for the next 31 years.
- **Investor B.** Do nothing for the first 10 years. Then start investing \$2,000 per year for the next 31 years.
- **Investor C.** Invest \$2,000 per year for the entire 41 years.

Note that all investments are made at the *beginning* of each year; the first deposit will be made at the beginning of age 25 $(n = 0)$, and you want to calculate the balance at the age of 65 $(n = 41)$.

STRATEGY: Since the investments are made at the beginning of each year, we need to use the procedure outlined in Example 3.14. In other words, each deposit has one extra interest-earning period.

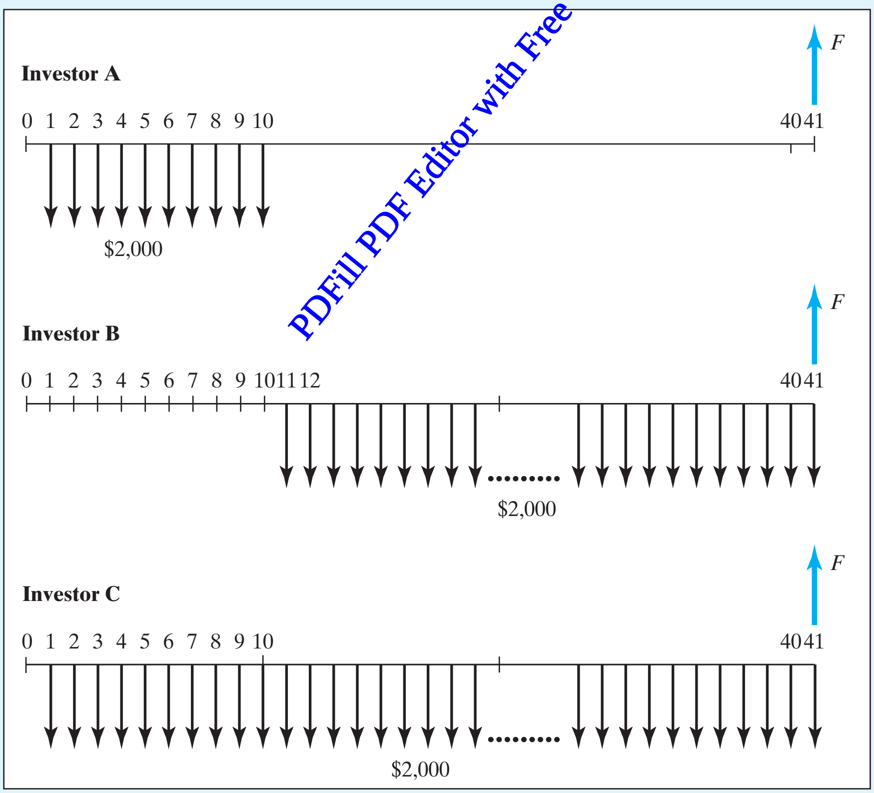


Figure 3.22 Cash flow diagrams for three investment options (Example 3.16).

SOLUTION

Given: Three different deposit scenarios with $i = 9.38\%$ and $N = 41$ years.
Find: Balance at the end of 41 years (or at the age of 65).

- Investor A:

$$F_{65} = \underbrace{\overbrace{\$2{,}000(F/A, 9.38\%, 10)(1.0938)}^{\text{Balance at the end of 10 years}}}_{\$33{,}845}(F/P, 9.38\%, 31)$$

$$= \$545{,}216.$$

- Investor B:

$$F_{65} = \$2{,}000\underbrace{(F/P, 9.38\%, 31)}_{\$322{,}159}(1.0938)$$

$$= \$352{,}377.$$

- Investor C:

$$F_{65} = \$2{,}000\underbrace{(F/P, 9.38\%, 41)}_{\$820{,}620}(1.0938)$$

$$= \$897{,}594.$$

If you know how your balance changes at the end of each year, you may want to construct a tableau such as the one shown in Table 3.3. Note that, due to rounding errors, the final balance figures are slightly off from those calculated by interest formulas.

TABLE 3.3 How Time Affects the Value of Money

		Investor A		Investor B		Investor C	
Age	Years	Contribution	Year-End Value	Contribution	Year-End Value	Contribution	Year-End Value
25	1	$2,000	$ 2,188	$0	$0	$2,000	$ 2,188
26	2	$2,000	$ 4,580	$0	$0	$2,000	$ 4,580
27	3	$2,000	$ 7,198	$0	$0	$2,000	$ 7,198
28	4	$2,000	$10,061	$0	$0	$2,000	$10,061
29	5	$2,000	$13,192	$0	$0	$2,000	$13,192
30	6	$2,000	$16,617	$0	$0	$2,000	$16,617
31	7	$2,000	$20,363	$0	$0	$2,000	$20,363
32	8	$2,000	$24,461	$0	$0	$2,000	$24,461
33	9	$2,000	$28,944	$0	$0	$2,000	$28,944
34	10	$2,000	$33,846	$0	$0	$2,000	$33,846

(Continued)

TABLE 3.3 Continued

		Investor A		Investor B		Investor C	
Age	Years	Contribution	Year-End Value	Contribution	Year-End Value	Contribution	Year-End Value
35	11	$0	$ 37,021	$2,000	$ 2,188	$ 2,000	$ 39,209
36	12	$0	$ 40,494	$2,000	$ 4,580	$ 2,000	$ 45,075
37	13	$0	$ 44,293	$2,000	$ 7,198	$ 2,000	$ 51,490
38	14	$0	$ 48,448	$2,000	$ 10,061	$ 2,000	$ 58,508
39	15	$0	$ 52,992	$2,000	$ 13,192	$ 2,000	$ 66,184
40	16	$0	$ 57,963	$2,000	$ 16,617	$ 2,000	$ 74,580
41	17	$0	$ 63,401	$2,000	$ 20,363	$ 2,000	$ 83,764
42	18	$0	$ 69,348	$2,000	$ 24,461	$ 2,000	$ 93,809
43	19	$0	$ 75,854	$2,000	$ 28,944	$ 2,000	$104,797
44	20	$0	$ 82,969	$2,000	$ 33,846	$ 2,000	$116,815
45	21	$0	$ 90,752	$2,000	$ 39,209	$ 2,000	$129,961
46	22	$0	$ 99,265	$2,000	$ 45,075	$ 2,000	$144,340
47	23	$0	$108,577	$2,000	$ 51,490	$ 2,000	$160,068
48	24	$0	$118,763	$2,000	$ 58,508	$ 2,000	$177,271
49	25	$0	$129,903	$2,000	$ 66,184	$ 2,000	$196,088
50	26	$0	$142,089	$2,000	$ 74,580	$ 2,000	$216,670
51	27	$0	$155,418	$2,000	$ 83,764	$ 2,000	$239,182
52	28	$0	$169,997	$2,000	$ 93,809	$ 2,000	$263,807
53	29	$0	$185,944	$2,000	$104,797	$ 2,000	$290,741
54	30	$0	$203,387	$2,000	$116,815	$ 2,000	$320,202
55	31	$0	$222,466	$2,000	$129,961	$ 2,000	$352,427
56	32	$0	$243,335	$2,000	$144,340	$ 2,000	$387,675
57	33	$0	$266,162	$2,000	$160,068	$ 2,000	$426,229
58	34	$0	$291,129	$2,000	$177,271	$ 2,000	$468,400
59	35	$0	$318,439	$2,000	$196,088	$ 2,000	$514,527
60	36	$0	$348,311	$2,000	$216,670	$ 2,000	$564,981
61	37	$0	$380,985	$2,000	$239,182	$ 2,000	$620,167
62	38	$0	$416,724	$2,000	$263,807	$ 2,000	$680,531
63	39	$0	$455,816	$2,000	$290,741	$ 2,000	$746,557
64	40	$0	$498,574	$2,000	$320,202	$ 2,000	$818,777
65	41	$0	$545,344	$2,000	$352,427	$ 2,000	$897,771
		$20,000		$62,000		$82,000	
Value at 65			**$545,344**		**$352,427**		**$897,771**
Less Total Contributions			**$20,000**		**$ 62,000**		**$82,000**
Net Earnings			**$525,344**		**$290,427**		**$815,771**

Source: Adapted from *Making Money Work for You*, UNH Cooperative Extension.

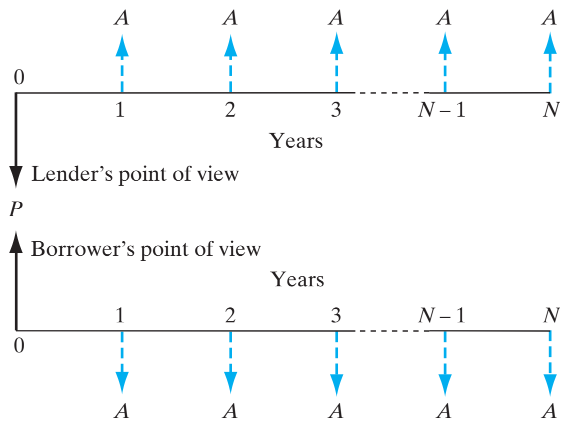


Figure 3.23

Capital Recovery Factor (Annuity Factor): Find *A*, Given *P*, *i*, and *N*

We can determine the amount of a periodic payment A if we know P, i, and N. Figure 3.23 illustrates this situation. To relate P to A, recall the relationship between P and F in Eq. (3.3), $F = P(1 + i)^N$. Replacing F in Eq. (3.11) by $P(1 + i)^N$, we get

$$A = P(1 + i)^N\left[\frac{i}{(1 + i)^N - 1}\right],$$

or

$$A = P\left[\frac{i(1 + i)^N}{(1 + i)^N - 1}\right] = P(A/P, i, N). \qquad (3.12)$$

Capital recovery factor: Commonly used to determine the revenue requirements needed to address the up-front capital costs for projects.

Annuity: An annuity is essentially a level stream of *cash flows* for a fixed period of time.

Now we have an equation for determining the value of the series of end-of-period payments A when the present sum P is known. The portion within the brackets is called the **equal payment series capital recovery factor**, or simply **capital recovery factor**, which is designated $(A/P, i, N)$. In finance, this A/P factor is referred to as the **annuity factor** and indicates a series of payments of a fixed, or constant, amount for a specified number of periods.

EXAMPLE 3.17 Uniform Series: Find *A*, Given *P*, *i*, and *N*

BioGen Company, a small biotechnology firm, has borrowed \$250,000 to purchase laboratory equipment for gene splicing. The loan carries an interest rate of 8% per year and is to be repaid in equal installments over the next six years. Compute the amount of the annual installment (Figure 3.24).

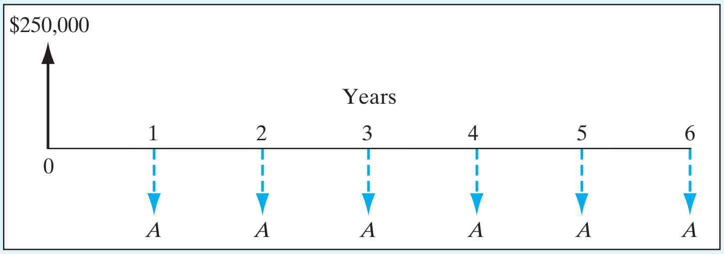


Figure 3.24 A loan cash flow diagram from BioGen's point of view.

SOLUTION

Given: $P = \$250{,}000$, $i = 8\%$ per year, and $N = 6$ years.
Find: A.

$$\begin{aligned} A &= \$250{,}000(A/P, 8\%, 6) \\ &= \$250{,}000(0.2163) \\ &= \$54{,}075. \end{aligned}$$

Here is an Excel solution using annuity function commands:

$$\begin{aligned} &=\mathbf{PMT}(i, N, P) \\ &=\mathbf{PMT}(8\%,6,-250000) \\ &=\$54{,}075 \end{aligned}$$

EXAMPLE 3.18 Deferred Loan Repayment

In Example 3.17, suppose that BioGen wants to negotiate with the bank to defer the first loan repayment until the end of year 2 (but still desires to make six equal installments at 8% interest). If the bank wishes to earn the same profit, what should be the annual installment, also known as **deferred annuity** (Figure 3.25)?

Deferred annuity: A type of annuity contract that delays payments of income, installments, or a lump sum until the investor elects to receive them.

SOLUTION

Given: $P = \$250{,}000$, $i = 8\%$ per year, and $N = 6$ years, but the first payment occurs at the end of year 2.
Find: A.

By deferring the loan for year, the bank will add the interest accrued during the first year to the principal. In other words, we need to find the equivalent worth P' of \$250,000 at the end of year 1:

$$\begin{aligned} P' &= \$250{,}000(F/P, 8\%, 1) \\ &= \$270{,}000. \end{aligned}$$

Grace period: The additional period of time a lender provides for a borrower to make payment on a debt without penalty.

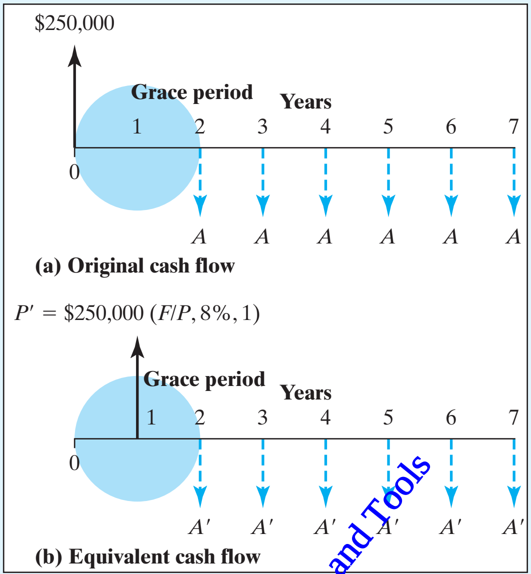


Figure 3.25 A deferred loan cash flow diagram from BioGen's point of view (Example 3.17).

In fact, BioGen is borrowing $270,000 for six years. To retire the loan with six equal installments, the deferred equal annual payment on P' will be

$$\begin{aligned} A' &= \$270{,}000(A/P, 8\%, 6) \\ &= \$58{,}401. \end{aligned}$$

By deferring the first payment for one year, BioGen needs to make additional payments of $4,326 in each year.

Present-Worth Factor: Find *P*, Given *A*, *i*, and *N*

What would you have to invest now in order to withdraw A dollars at the end of each of the next N periods? In answering this question, we face just the opposite of the equal payment capital recovery factor situation: A is known, but P has to be determined. With the capital recovery factor given in Eq. (3.12), solving for P gives us

$$P = A\left[\frac{(1+i)^N - 1}{i(1+i)^N}\right] = A(P/A, i, N). \qquad (3.13)$$

The bracketed term is referred to as the **equal payment series present-worth factor** and is designated $(P/A, i, N)$.

EXAMPLE 3.19 Uniform Series: Find *P*, Given *A*, *i*, and *N*

Let us revisit Louise Outing's lottery problem, introduced in the chapter opening. Suppose that Outing were able to find an investor who was willing to buy her lottery ticket for $2 million. Recall that after an initial gross payment of $283,770, Outing

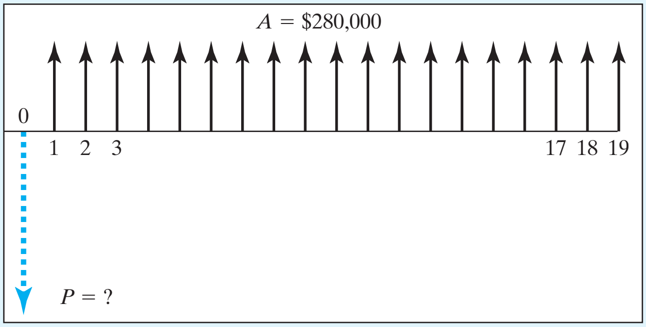


Figure 3.26 A cash flow diagram for Louise Outing's lottery winnings (Example 3.19).

would be paid 19 annual gross checks of $280,000. (See Figure 3.26.) If she could invest her money at 8% interest, what would be the fair amount to trade her 19 future lottery receipts? (Note that she already cashed in $283,770 after winning the lottery, so she is giving up 19 future lottery checks in the amount of $280,000.)

SOLUTION

Given: $i = 8\%$ per year, $A = \$280{,}000$, and $N = 19$ years.
Find: P.

- Using interest factor:

$$P = \$280{,}000(P/A, 8\%, 19) = \$280{,}000(9.6036)$$
$$= \$2{,}689{,}008.$$

- Using Excel:

=PV(8%,19,−280000)=$2,689,008

COMMENTS: Clearly, we can tell Outing that giving up $280,000 a year for 19 years to receive $2 million today is a losing proposition if she can earn only an 8% return on her investment. At this point, we may be interested in knowing at just what rate of return her deal (receiving $2 million) would in fact make sense. Since we know that $P = \$2{,}000{,}000$, $N = 19$, and $A = \$280{,}000$, we solve for i.

If you know the cash flows and the PV (or FV) of a cash flow stream, you can determine the interest rate. In this case, you are looking for the interest rate that caused the P/A factor to equal $(P/A, i, 19) = (\$2{,}000{,}000/\$280{,}000) = 7.1429$. Since we are dealing with an annuity, we could proceed as follows:

- With a financial calculator, enter $N = 19$, $PV = \$2{,}000{,}000$, $PMT = -280{,}000$, and then press the i key to find $i = 12.5086\%$.
- To use the interest tables, first recognize that $\$2{,}000{,}000 = \$280{,}000 \times (P/A, i, 19)$ or $(P/A, i, 19) = 7.1429$. Look up 7.1429 or a close value in

Appendix A. In the P/A column with $N = 19$ in the 12% interest table, you will find that $(P/A, 12\%, 19) = 7.3658$. If you look up the 13% interest table, you find that $(P/A, 12\%, 19) = 6.9380$, indicating that the interest rate should be closer to 12.5%.

- To use Excel's financial command, you simply evaluate the following command to solve the unknown interest rate problem for an annuity:

$$=\textbf{RATE}(N,A,P,F,type,guess)$$
$$=\textbf{RATE}(19{,}280000{,}-2000000{,}0{,}0{,}10\%)$$
$$=12.5086\%$$

It is not likely that Outing will find a financial investment which provides this high rate of return. Thus, even though the deal she has been offered is not a good one for economic reasons, she could accept it knowing that she has not much time to enjoy the future benefits.

3.3.5 Linear Gradient Series

Engineers frequently encounter situations involving periodic payments that increase or decrease by a constant amount (G) from period to period. These situations occur often enough to warrant the use of special equivalence factors that relate the arithmetic gradient to other cash flows. Figure 3.27 illustrates a **strict gradient series**, $A_n = (n - 1)G$. Note that the origin of the series is at the end of the first period with a zero value. The gradient G can be either positive or negative. If $G > 0$, the series is referred to as an *increasing* gradient series. If $G < 0$, it is a *decreasing* gradient series.

Unfortunately, the strict form of the increasing or decreasing gradient series does not correspond with the form that most engineering economic problems take. A typical problem involving a linear gradient includes an initial payment during period 1 that increases by G during some number of interest periods, a situation illustrated in Figure 3.28. This contrasts with the strict form illustrated in Figure 3.27, in which no payment is made during period 1 and the gradient is added to the previous payment beginning in period 2.

Gradient Series as Composite Series

In order to utilize the strict gradient series to solve typical problems, we must view cash flows as shown in Figure 3.28 as a **composite series**, or a set of two cash flows, each

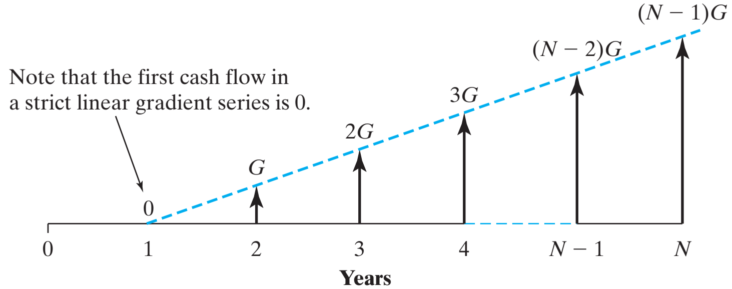


Figure 3.27 A cash flow diagram for a strict gradient series.

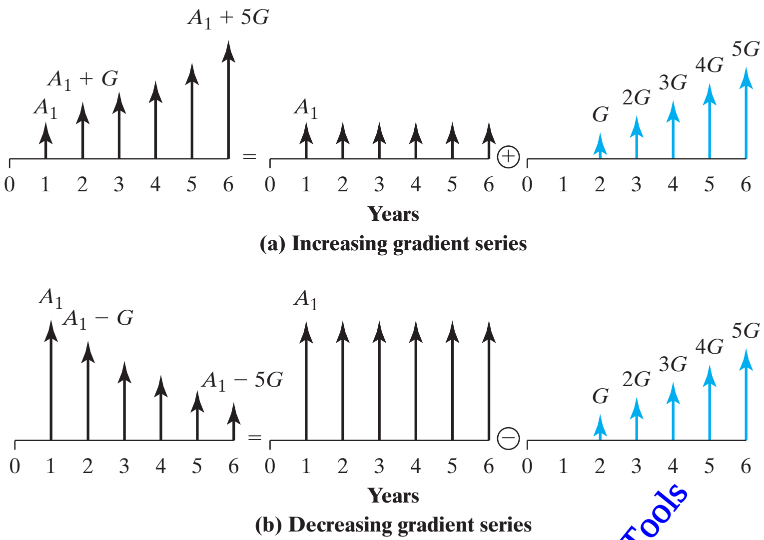


Figure 3.28 Two types of linear gradient series as composites of a uniform series of N payments of A_1 and the gradient series of increments of constant amount G.

corresponding to a form that we can recognize and easily solve: a uniform series of N payments of amount A_1 and a gradient series of increments of constant amount G. The need to view cash flows that involve linear gradient series as composites of two series is very important in solving problems, as we shall now see.

Present-Worth Factor: Linear Gradient: Find *P*, Given *G*, *N*, and *i*

How much would you have to deposit now to withdraw the gradient amounts specified in Figure 3.27? To find an expression for the present amount P, we apply the single-payment present-worth factor to each term of the series and obtain

$$P = 0 + \frac{G}{(1+i)^2} + \frac{2G}{(1+i)^3} + \cdots + \frac{(N-1)G}{(1+i)^N},$$

or

$$P = \sum_{n=1}^{N} (n-1)G(1+i)^{-n}. \tag{3.14}$$

Letting $G = a$ and $1/(1+i) = x$ yields

$$\begin{aligned} P &= 0 + ax^2 + 2ax^3 + \cdots + (N-1)ax^N \\ &= ax[0 + x + 2x^2 + \cdots + (N-1)x^{N-1}]. \end{aligned} \tag{3.15}$$

Since an arithmetic–geometric series $\{0, x, 2x^2, \ldots, (N-1)x^{N-1}\}$ has the finite sum

$$0 + x + 2x^2 + \cdots + (N-1)x^{N-1} = x\left[\frac{1 - Nx^{N-1} + (N-1)x^N}{(1-x)^2}\right],$$

we can rewrite Eq. (3.15) as

$$P = ax^2\left[\frac{1 - Nx^{N-1} + (N-1)x^N}{(1-x)^2}\right]. \tag{3.16}$$

Replacing the original values for A and x, we obtain

$$P = G\left[\frac{(1+i)^N - iN - 1}{i^2(1+i)^N}\right] = G(P/G, i, N). \tag{3.17}$$

The resulting factor in brackets is called the **gradient series present-worth factor**, which we denote as $(P/G, i, N)$.

EXAMPLE 3.20 Linear Gradient: Find P, Given A_1, G, i, and N

A textile mill has just purchased a lift truck that has a useful life of five years. The engineer estimates that maintenance costs for the truck during the first year will be \$1,000. As the truck ages, maintenance costs are expected to increase at a rate of \$250 per year over the remaining life. Assume that the maintenance costs occur at the end of each year. The firm wants to set up a maintenance account that earns 12% annual interest. All future maintenance expenses will be paid out of this account. How much does the firm have to deposit in the account now?

SOLUTION

Given: $A_1 = \$1{,}000$, $G = \$250$, $i = 12\%$ per year, and $N = 5$ years.
Find: P.

Asking how much the firm has to deposit now is equivalent to asking what the equivalent present worth for this maintenance expenditure is if 12% interest is used. The cash flow may be broken into two components as shown in Figure 3.29.

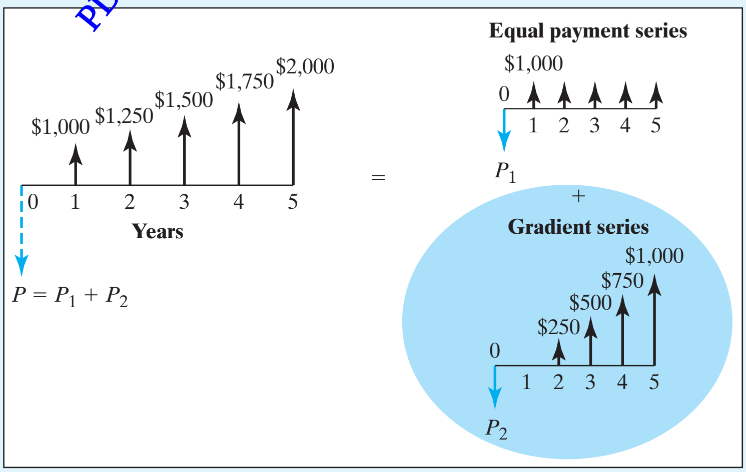


Figure 3.29 Cash flow diagram (Example 3.20).

The first component is an equal payment series (A_1), and the second is a linear gradient series (G). We have

$$P = P_1 + P_2$$

$$P = A_1(P/A, 12\%, 5) + G(P/G, 12\%, 5)$$

$$= \$1{,}000(3.6048) + \$250(6.397)$$

$$= \$5{,}204.$$

Note that the value of N in the gradient factor is 5, not 4. This is because, by definition of the series, the first gradient value begins at period 2.

COMMENTS: As a check, we can compute the present worth of the cash flow by using the (P/F, 12%, n) factors:

Period (n)	Cash Flow	(P/F, 12%, n)	Present Worth
1	\$1,000	0.8929	\$ 892.90
2	1,250	0.7972	996.50
3	1,500	0.7118	1,067.70
4	1,750	0.6355	1,112.13
5	2,000	0.5674	1,134.80
		Total	**\$5,204.03**

The slight difference is caused by a rounding error.

Gradient-to-Equal-Payment Series Conversion Factor: Find *A*, Given *G*, *i*, and *N*

We can obtain an equal payment series equivalent to the gradient series, as depicted in Figure 3.30, by substituting Eq. (3.17) for P into Eq. (3.12) to obtain

$$A = G\left[\frac{(1 + i)^N - iN - 1}{i[(1 + i)^N - 1]}\right] = G(A/G, i, N), \qquad (3.18)$$

where the resulting factor in brackets is referred to as the **gradient-to-equal-payment series conversion factor** and is designated (A/G, i, N).

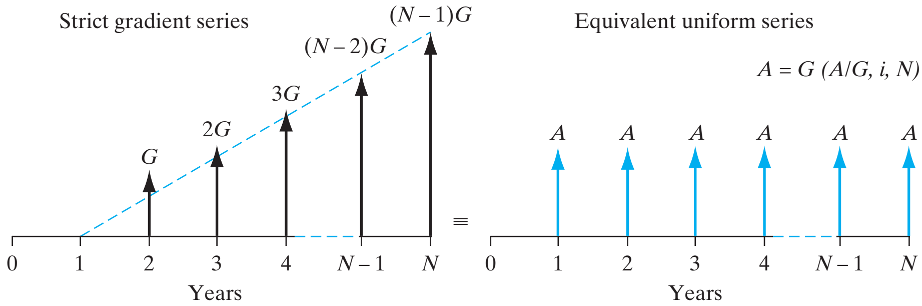


Figure 3.30

EXAMPLE 3.21 Linear Gradient: Find A, Given A_1, G, i, and N

John and Barbara have just opened two savings accounts at their credit union. The accounts earn 10% annual interest. John wants to deposit \$1,000 in his account at the end of the first year and increase this amount by \$300 for each of the next five years. Barbara wants to deposit an equal amount each year for the next six years. What should be the size of Barbara's annual deposit so that the two accounts will have equal balances at the end of six years (Figure 3.31)?

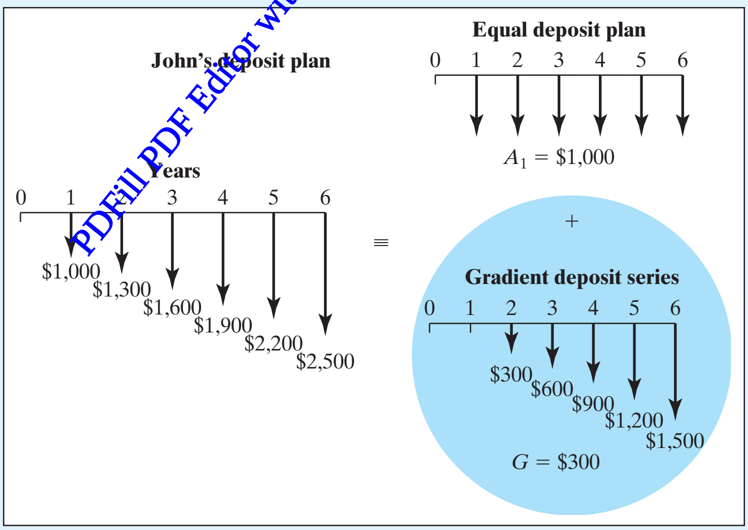


Figure 3.31 John's deposit series viewed as a combination of uniform and gradient series (Example 3.21).

SOLUTION

Given: $A_1 = \$1{,}000$, $G = \$300$, $i = 10\%$, and $N = 6$.

Find: A.

Since we use the end-of-period convention unless otherwise stated, this series begins at the end of the first year and the last contribution occurs at the end of the sixth year. We can separate the constant portion of \$1,000 from the series, leaving the gradient series of 0, 0, 300, 600, ..., 1,500.

To find the equal payment series beginning at the end of year 1 and ending at year 6 that would have the same present worth as that of the gradient series, we may proceed as follows:

$$\begin{aligned} A &= \$1{,}000 + \$300(A/G, 10\%, 6) \\ &= \$1{,}000 + \$300(2.22236) \\ &= \$1{,}667.08. \end{aligned}$$

Barbara's annual contribution should be \$1,667.08.

COMMENTS: Alternatively, we can compute Barbara's annual deposit by first computing the equivalent present worth of John's deposits and then finding the equivalent uniform annual amount. The present worth of this combined series is

$$\begin{aligned} P &= \$1{,}000(P/A, 10\%, 6) + \$300(P/G, 10\%, 6) \\ &= \$1{,}000(4.3553) + \$300(9.6842) \\ &= \$7{,}260.56. \end{aligned}$$

The equivalent uniform deposit is

$$A = \$7{,}260.56(A/P, 10\%, 6) = \$1{,}667.02.$$

(The slight difference in cents is caused by a rounding error.)

EXAMPLE 3.22 Declining Linear Gradient: Find F, Given A_1, G, i, and N

Suppose that you make a series of annual deposits into a bank account that pays 10% interest. The initial deposit at the end of the first year is \$1,200. The deposit amounts decline by \$200 in each of the next four years. How much would you have immediately after the fifth deposit?

SOLUTION

Given: Cash flow shown in Figure 3.32, $i = 10\%$ per year, and $N = 5$ years.

Find: F.

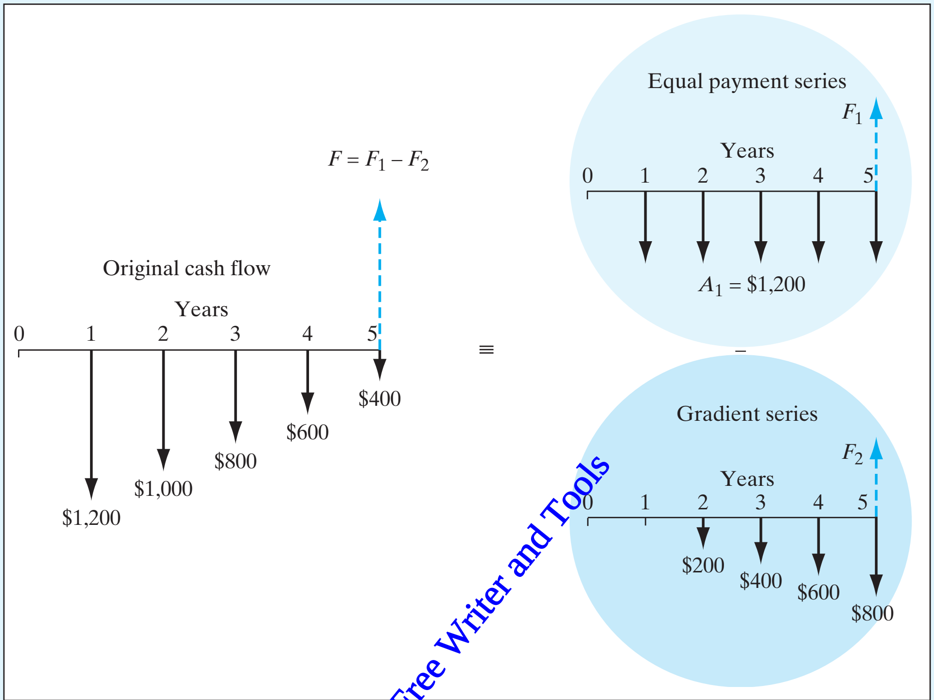


Figure 3.32

The cash flow includes a decreasing gradient series. Recall that we derived the linear gradient factors for an increasing gradient series. For a decreasing gradient series, the solution is most easily obtained by separating the flow into two components: a uniform series and an increasing gradient that is *subtracted* from the uniform series (Figure 3.32). The future value is

$$\begin{aligned} F &= F_1 - F_2 \\ &= A_1(F/A, 10\%, 5) - \$200(P/G, 10\%, 5)(F/P, 10\%, 5) \\ &= \$1{,}200(6.105) - \$200(6.862)(1.611) \\ &= \$5{,}115. \end{aligned}$$

Geometric growth: The year-over-year growth rate of an investment over a specified period of time. Compound growth is an imaginary number that describes the rate at which an investment grew as though it had grown at a steady rate.

3.3.6 Geometric Gradient Series

Many engineering economic problems—particularly those relating to construction costs—involve cash flows that increase or decrease over time, not by a constant *amount* (as with a linear gradient), but rather by a constant percentage (a **geometric gradient**). This kind of cash flow is called **compound growth**. Price changes caused by inflation are a good example of a geometric gradient series. If we use g to designate the percentage change in a payment from one period to the next, the magnitude of the nth payment, A_n, is related to the first payment A_1 by the formula

$$A_n = A_1(1 + g)^{n-1}, n = 1, 2, \ldots, N. \tag{3.19}$$

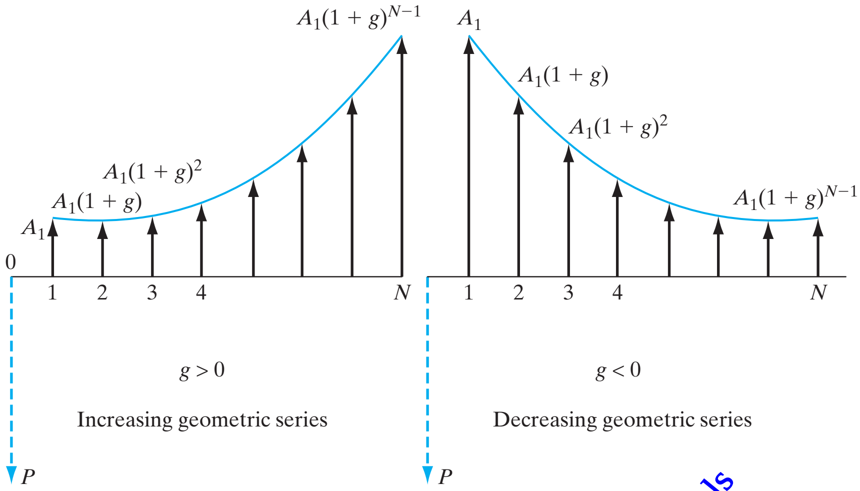


Figure 3.33 A geometrically increasing or decreasing gradient series at a constant rate g.

The variable g can take either a positive or a negative sign, depending on the type of cash flow. If $g > 0$, the series will increase, and if $g < 0$, the series will decrease. Figure 3.33 illustrates the cash flow diagram for this situation.

Present-Worth Factor: Find P, Given A_1, g, i, and N

Notice that the present worth of any cash flow A_n at interest rate i is

$$P_n = A_n(1 + i)^{-n} = A_1(1 + g)^{n-1}(1 + i)^{-n}.$$

To find an expression for the present amount P for the entire series, we apply the **single-payment present-worth factor** to each term of the series:

$$P = \sum_{n=1}^{N} A_1(1 + g)^{n-1}(1 + i)^{-n}. \tag{3.20}$$

Bringing the constant term $A_1(1 + g)^{-1}$ outside the summation yields

$$P = \frac{A_1}{(1 + g)} \sum_{n=1}^{N} \left[\frac{1 + g}{1 + i}\right]^n. \tag{3.21}$$

Let $a = \dfrac{A_1}{1 + g}$ and $x = \dfrac{1 + g}{1 + i}$. Then, rewrite Eq. (3.21) as

$$P = a(x + x^2 + x^3 + \cdots + x^N). \tag{3.22}$$

Since the summation in Eq. (3.22) represents the first N terms of a geometric series, we may obtain the closed-form expression as follows: First, multiply Eq. (3.22) by x to get

$$xP = a(x^2 + x^3 + x^4 + \cdots + x^{N+1}). \tag{3.23}$$

Then, subtract Eq. (3.23) from Eq. (3.22):

$$P - xP = a(x - x^{N+1})$$
$$P(1 - x) = a(x - x^{N+1})$$
$$P = \frac{a(x - x^{N+1})}{1 - x} \qquad (x \neq 1). \tag{3.24}$$

If we replace the original values for a and x, we obtain

$$P = \begin{cases} A_1\left[\dfrac{1 - (1 + g)^N(1 + i)^{-N}}{i - g}\right] & \text{if } i \neq g \\ NA_1/(1 + i) & \text{if } i = g, \end{cases} \tag{3.25}$$

or

$$P = A_1(P/A_1, g, i, N).$$

The factor within brackets is called the **geometric-gradient-series present-worth factor** and is designated $(P/A_1, g, i, N)$. In the special case where $i = g$, Eq. (3.21) becomes $P = [A_1/(1 + i)]N$.

EXAMPLE 3.23 Geometric Gradient: Find P, Given A_1, g, i, and N

Ansell, Inc., a medical device manufacturer, uses compressed air in solenoids and pressure switches in its machines to control various mechanical movements. Over the years, the manufacturing floor has changed layouts numerous times. With each new layout, more piping was added to the compressed-air delivery system to accommodate new locations of manufacturing machines. None of the extra, unused old pipe was capped or removed; thus, the current compressed-air delivery system is inefficient and fraught with leaks. Because of the leaks, the compressor is expected to run 70% of the time that the plant will be in operation during the upcoming year. This will require 260 kWh of electricity at a rate of \$0.05/kWh. (The plant runs 250 days a year, 24 hours per day.) If Ansell continues to operate the current air delivery system, the compressor run time will increase by 7% per year for the next five years because of ever-worsening leaks. (After five years, the current system will not be able to meet the plant's compressed-air requirement, so it will have to be replaced.) If Ansell decides to replace all of the old piping now, it will cost \$28,570.

The compressor will still run the same number of days; however, it will run 23% less (or will have $70\%(1 - 0.23) = 53.9\%$ usage during the day) because of the reduced air pressure loss. If Ansell's interest rate is 12%, is the machine worth fixing now?

SOLUTION

Given: Current power consumption, $g = 7\%$, $i = 12\%$, and $N = 5$ years.
Find: A_1 and P.

Step 1: We need to calculate the cost of power consumption of the current piping system during the first year:

$$\begin{aligned}
\text{Power cost} &= \%\text{ of day operating} \\
&\quad \times \text{ days operating per year} \\
&\quad \times \text{ hours per day} \\
&\quad \times \text{ kWh} \times \$/\text{kWh} \\
&= (70\%) \times (250 \text{ days/year}) \times (24 \text{ hours/day}) \\
&\quad \times (260 \text{ kWh}) \times (\$0.05/\text{kWh}) \\
&= \$54{,}440.
\end{aligned}$$

Step 2: Each year, the annual power cost will increase at the rate of 7% over the previous year's cost. The anticipated power cost over the five-year period is summarized in Figure 3.34. The equivalent present lump-sum cost at 12% for this geometric gradient series is

$$\begin{aligned}
P_{\text{Old}} &= \$54{,}440(P/A_1, 7\%, 12\%, 5) \\
&= \$54{,}440\left[\frac{1 - (1 + 0.07)^5(1 + 0.12)^{-5}}{0.12 - 0.07}\right] \\
&= \$222{,}283.
\end{aligned}$$

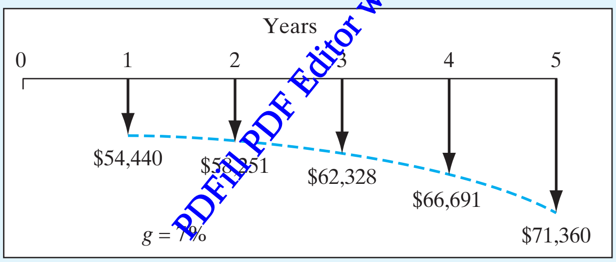


Figure 3.34 Annual power cost series if repair is not performed.

Step 3: If Ansell replaces the current compressed-air system with the new one, the annual power cost will be 23% less during the first year and will remain at that level over the next five years. The equivalent present lump-sum cost at 12% is then

$$\begin{aligned}
P_{\text{New}} &= \$54{,}440(1 - 0.23)(P/A, 12\%, 5) \\
&= \$41{,}918.80(3.6048) \\
&= \$151{,}109.
\end{aligned}$$

Step 4: The net cost for not replacing the old system now is \$71,174 (= \$222,283 − \$151,109). Since the new system costs only \$28,570, the replacement should be made now.

COMMENTS: In this example, we assumed that the cost of removing the old system was included in the cost of installing the new system. If the removed system has some salvage value, replacing it will result in even greater savings. We will consider many types of replacement issues in Chapter 14.

EXAMPLE 3.24 Geometric Gradient: Find A_1, Given F, g, i, and N

Jimmy Carpenter, a self-employed individual, is opening a retirement account at a bank. His goal is to accumulate \$1,000,000 in the account by the time he retires from work in 20 years' time. A local bank is willing to open a retirement account that pays 8% interest compounded annually throughout the 20 years. Jimmy expects that his annual income will increase 6% yearly during his working career. He wishes to start with a deposit at the end of year 1 (A_1) and increase the deposit at a rate of 6% each year thereafter. What should be the size of his first deposit (A_1)? The first deposit will occur at the end of year 1, and subsequent deposits will be made at the end of each year. The last deposit will be made at the end of year 20.

SOLUTION

Given: $F = \$1{,}000{,}000$, $g = 6\%$ per year, $i = 8\%$ per year, and $N = 20$ years.
Find: A_1 as in Figure 3.35.

We have

$$F = A_1(P/A_1, 6\%, 8\%, 20)\,(F/P, 8\%, 20)$$
$$= A_1(72.6911).$$

Solving for A_1 yields

$$A_1 = \$1{,}000{,}000/72.6911 = \$13{,}757.$$

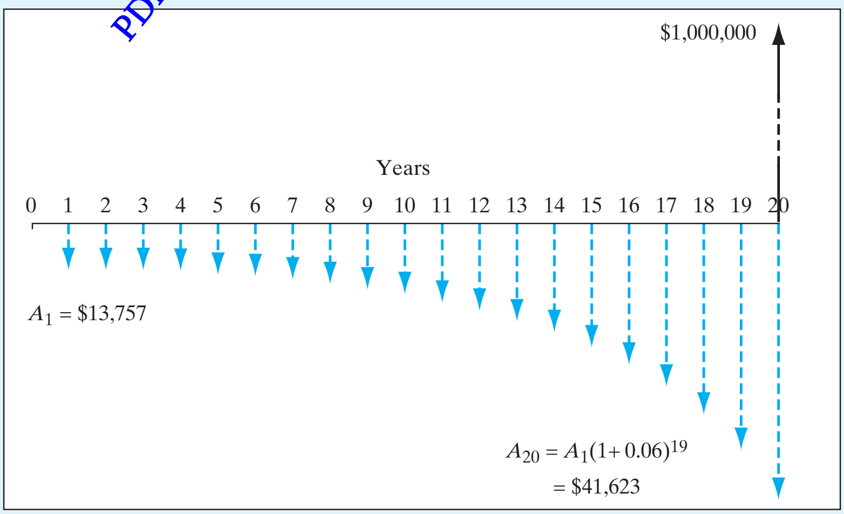


Figure 3.35 Jimmy Carpenter's retirement plan (Example 3.24).

Table 3.4 summarizes the interest formulas developed in this section and the cash flow situations in which they should be used. Recall that these formulas are applicable only to situations where the interest (compounding) period is the same as the payment period (e.g., annual compounding with annual payment). Also, we present some useful Excel financial commands in the table.

3.4 Unconventional Equivalence Calculations

In the preceding section, we occasionally presented two or more methods of attacking problems even though we had standard interest factor equations by which to solve them. It is important that you become adept at examining problems from unusual angles and that you seek out unconventional solution methods, because not all cash flow problems conform to the neat patterns for which we have discovered and developed equations. Two categories of problems that demand unconventional treatment are composite (mixed) cash flows and problems in which we must determine the interest rate implicit in a financial contract. We will begin this section by examining instances of composite cash flows.

3.4.1 Composite Cash Flows

Although many financial decisions do involve constant or systematic changes in cash flows, others contain several components of cash flows that do not exhibit an overall pattern. Consequently, it is necessary to expand our analysis to deal with these mixed types of cash flows.

To illustrate, consider the cash flow stream shown in Figure 3.36. We want to compute the equivalent present worth for this mixed payment series at an interest rate of 15%. Three different methods are presented.

Method 1. A "brute force" approach is to multiply each payment by the appropriate $(P/F, 10\%, n)$ factors and then to sum these products to obtain the present worth of the cash flows, \$543.72. Recall that this is exactly the same procedure we used to solve the category of problems called the uneven payment series, described in Section 3.3.3. Figure 3.36 illustrates this computational method.

Method 2. We may group the cash flow components according to the type of cash flow pattern that they fit, such as the single payment, equal payment series, and so forth, as shown in Figure 3.37. Then the solution procedure involves the following steps:

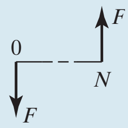
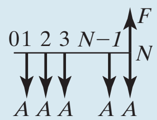
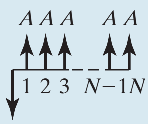
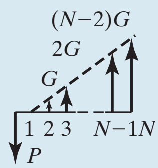
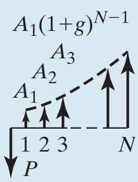
- **Group 1:** Find the present worth of \$50 due in year 1:

$$\$50(P/F, 15\%, 1) = \$43.48.$$

- **Group 2:** Find the equivalent worth of a \$100 equal payment series at year 1 (V_1), and then bring this equivalent worth at year 0 again:

$$\underbrace{\$100(P/A, 15\%, 3)}_{V_1}(P/F, 15\%, 1) = \$198.54.$$

TABLE 3.4 Summary of Discrete Compounding Formulas with Discrete Payments

Flow Type	Factor Notation	Formula	Excel Command	Cash Flow Diagram
SINGLE	Compound amount $(F/P, i, N)$	$F = P(1+i)^N$	$=\mathrm{FV}(i, N, P,, 0)$	
	Present worth $(P/F, i, N)$	$P = F(1+i)^{-N}$	$=\mathrm{PV}(i, N, F,, 0)$	
EQUAL PAYMENT SERIES	Compound amount $(F/A, i, N)$	$F = A\left[\dfrac{(1+i)^N - 1}{i}\right]$	$=\mathrm{PV}(i, N, A,, 0)$	
	Sinking fund $(A/F, i, N)$	$A = F\left[\dfrac{i}{(1+i)^N - 1}\right]$	$=\mathrm{PMT}(i, N, P, F, 0)$	
	Present worth $(P/A, i, N)$	$P = A\left[\dfrac{(1+i)^N - 1}{i(1+i)^N}\right]$	$=\mathrm{PV}(i, N, A,, 0)$	
	Capital recovery $(A/P, i, N)$	$A = P\left[\dfrac{i(1+i)^N}{(1+i)^N - 1}\right]$	$=\mathrm{PMT}(i, N,, P)$	
GRADIENT SERIES	Linear gradient			
	Present worth $(P/G, i, N)$	$P = G\left[\dfrac{(1+i)^N - iN - 1}{i^2(1+i)^N}\right]$		
	Conversion factor $(A/G, i, N)$	$A = G\left[\dfrac{(1+i)^N - iN - 1}{i[(1+i)^N - 1]}\right]$		
	Geometric gradient			
	Present worth $(P/A_1, g, i, N)$	$P = \begin{bmatrix} A_1\left[\dfrac{1-(1+g)^N(1+i)^{-N}}{i-g}\right] \\ A_1\left(\dfrac{N}{1+i}\right)(\text{if } i = g) \end{bmatrix}$		

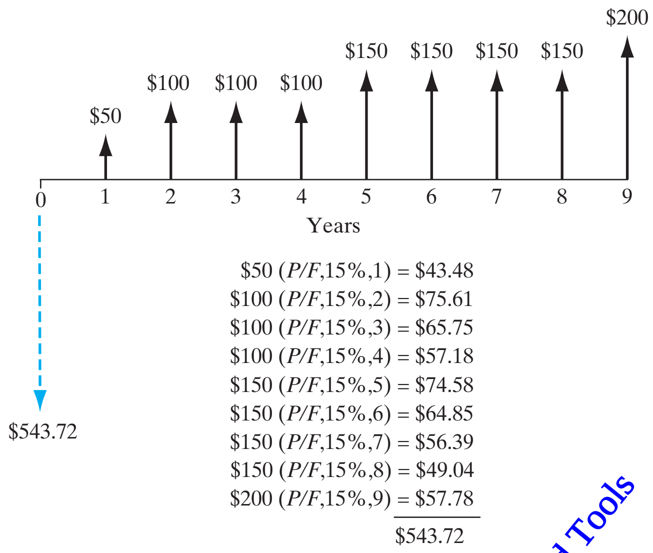


Figure 3.36 Equivalent present worth calculation using only P/F factors (Method 1 "Brute Force Approach").

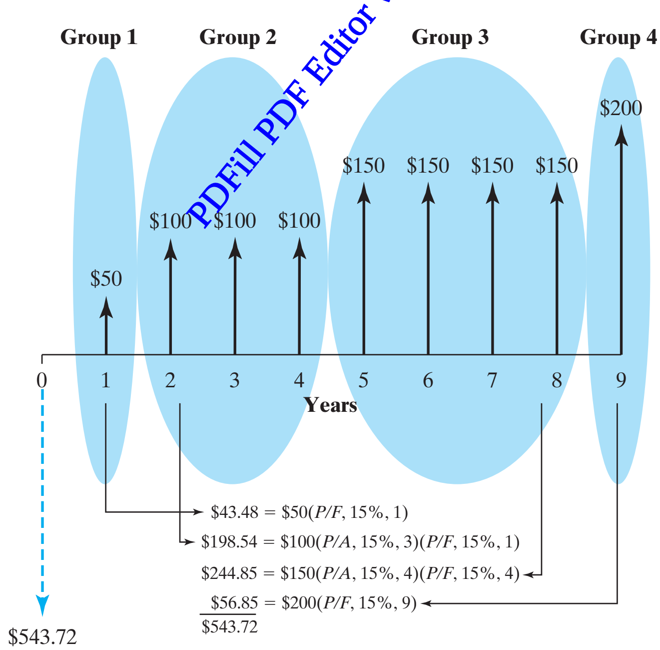


Figure 3.37 Equivalent present-worth calculation for an uneven payment series, using P/F and P/A factors (Method 2: grouping approach).

- **Group 3:** Find the equivalent worth of a \$150 equal payment series at year 4 (V_4), and then bring this equivalent worth at year 0.

$$\underbrace{\$150(P/A, 15\%, 4)}_{V_4}(P/F, 15\%, 4) = \$244.85.$$

- **Group 4:** Find the equivalent present worth of the \$200 due in year 9:

$$\$200(P/F, 15\%, 9) = \$56.85.$$

- **Group total—sum the components:**

$$P = \$43.48 + \$198.54 + \$244.85 + \$56.85 = \$543.72.$$

A pictorial view of this computational process is given in Figure 3.37.

Method 3. In computing the present worth of the equal payment series components, we may use an alternative method.

- **Group 1:** Same as in Method 2.
- **Group 2:** Recognize that a \$100 equal payment series will be received during years 2 through 4. Thus, we could determine the value of a four-year annuity, subtract the value of a one-year annuity from it, and have remaining the value of a four-year annuity whose first payment is due in year 2. This result is achieved by subtracting the (P/A, 15%, 1) for a one-year, 15% annuity from that for a four-year annuity and then multiplying the difference by \$100:

$$\begin{aligned}\$100[(P/A, 15\%, 4) - (P/A, 15\%, 1)] &= \$100(2.8550 - 0.8696)\\ &= \$198.54.\end{aligned}$$

Thus, the equivalent present worth of the annuity component of the uneven stream is \$198.54.

- **Group 3:** We have another equal payment series that starts in year 5 and ends in year 8.

$$\begin{aligned}\$150[(P/A, 15\%, 8) - (P/A, 15\%, 4)] &= \$150(4.4873 - 2.8550)\\ &= \$244.85.\end{aligned}$$

- **Group 4:** Same as Method 2.
- **Group total—sum the components:**

$$P = \$43.48 + \$198.54 + \$244.85 + \$56.85 = \$543.72.$$

Either the “brute force” method of Figure 3.35 or the method utilizing both (P/A, i, n) and (P/F, i, n) factors can be used to solve problems of this type. However, Method 2 or Method 3 is much easier if the annuity component runs for many years. For example, the alternative solution would be clearly superior for finding the equivalent present worth of a stream consisting of \$50 in year 1, \$200 in years 2 through 19, and \$500 in year 20.

Also, note that in some instances we may want to find the equivalent value of a stream of payments at some point other than the present (year 0). In this situation, we proceed as before, but compound and discount to some other point in time—say, year 2, rather than year 0. Example 3.25 illustrates the situation.

EXAMPLE 3.25 Cash Flows with Subpatterns

The two cash flows in Figure 3.38 are equivalent at an interest rate of 12% compounded annually. Determine the unknown value C.

SOLUTION

Given: Cash flows as in Figure 3.38; $i = 12\%$ per year.
Find: C.

- **Method 1.** Compute the present worth of each cash flow at time 0:

$$\begin{aligned} P_1 &= \$100(P/A, 12\%, 2) + \$300(P/A, 12\%, 3)(P/F, 12\%, 2) \\ &= \$743.42; \end{aligned}$$

$$\begin{aligned} P_2 &= C(P/A, 12\%, 5) - C(P/F, 12\%, 3) \\ &= 2.8930C. \end{aligned}$$

  Since the two flows are equivalent, $P_1 = P_2$, and we have

$$743.42 = 2.8930C.$$

  Solving for C, we obtain $C = \$256.97$.

- **Method 2.** We may select a time point other than 0 for comparison. The best choice of a base period is determined largely by the cash flow patterns. Obviously, we want to select a base period that requires the minimum number of interest factors for the equivalence calculation. Cash flow 1 represents a combined series of two equal payment cash flows, whereas cash flow 2 can be viewed as an equal payment series with the third payment missing. For cash

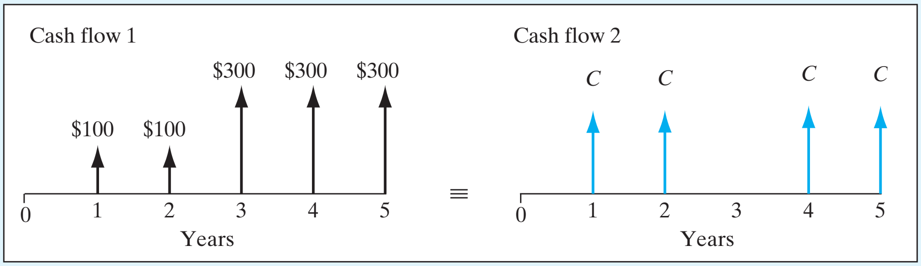


Figure 3.38 Equivalence calculation (Example 3.25).

flow 1, computing the equivalent worth at period 5 will require only two interest factors:

$$V_{5,1} = \$100(F/A, 12\%, 5) + \$200(F/A, 12\%, 3)$$
$$= \$1{,}310.16.$$

For cash flow 2, computing the equivalent worth of the equal payment series at period 5 will also require two interest factors:

$$V_{5,2} = C(F/A, 12\%, 5) - C(F/P, 12\%, 2)$$
$$= 5.0984C.$$

Therefore, the equivalence would be obtained by letting $V_{5,1} = V_{5,2}$:

$$\$1{,}310.16 = 5.0984C.$$

Solving for C yields $C = \$256.97$, which is the same result obtained from Method 1. The alternative solution of shifting the time point of comparison will require only four interest factors, whereas Method 1 requires five interest factors.

EXAMPLE 3.26 Establishing a College Fund

A couple with a newborn daughter wants to save for their child's college expenses in advance. The couple can establish a college fund that pays 7% annual interest. Assuming that the child enters college at age 18, the parents estimate that an amount of \$40,000 per year (actual dollars) will be required to support the child's college expenses for 4 years. Determine the equal annual amounts the couple must save until they send their child to college. (Assume that the first deposit will be made on the child's first birthday and the last deposit on the child's 18th birthday. The first withdrawal will be made at the beginning of the freshman year, which also is the child's 18th birthday.)

SOLUTION

Given: Deposit and withdrawal series shown in Figure 3.39; $i = 7\%$ per year.

Find: Unknown annual deposit amount (X).

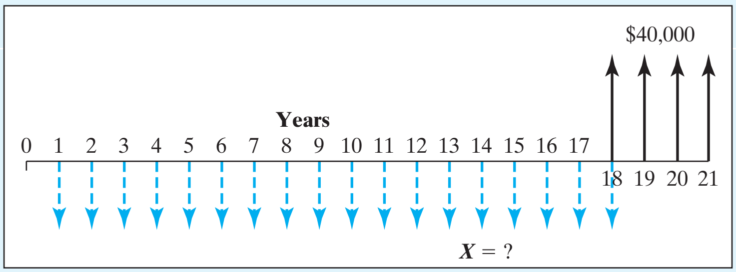


Figure 3.39 Establishing a college fund (Example 3.26). Note that the \$40,000 figure represents the actual anticipated expenditures considering the future inflation.

- **Method 1.** Establish economic equivalence at period 0:

Step 1: Find the equivalent single lump-sum deposit now:

$$\begin{aligned} P_{\text{Deposit}} &= X(P/A, 7\%, 18) \\ &= 10.0591X. \end{aligned}$$

Step 2: Find the equivalent single lump-sum withdrawal now:

$$\begin{aligned} P_{\text{Withdrawal}} &= \$40{,}000(P/A, 7\%, 4)(P/F, 7\%, 17) \\ &= \$42{,}892. \end{aligned}$$

Step 3: Since the two amounts are equivalent, by equating $P_{\text{Deposit}} = P_{\text{Withdrawal,}}$ we obtain X:

$$\begin{aligned} 10.0591X &= \$42{,}892 \\ X &= \$4{,}264. \end{aligned}$$

- **Method 2.** Establish the economic equivalence at the child's 18th birthday:

Step 1: Find the accumulated deposit balance on the child's 18th birthday:

$$\begin{aligned} V_{18} &= X(F/A, 7\%, 18) \\ &= 33.9990X. \end{aligned}$$

Step 2: Find the equivalent lump-sum withdrawal on the child's 18th birthday:

$$\begin{aligned} V_{18} &= \$40{,}000 + \$40{,}000(P/A, 7\%, 3) \\ &= \$144{,}972. \end{aligned}$$

Step 3: Since the two amounts must be the same, we obtain

$$\begin{aligned} 33.9990X &= \$144{,}972 \\ X &= \$4{,}264. \end{aligned}$$

The computational steps are summarized in Figure 3.40. In general, the second method is the more efficient way to obtain an equivalence solution to this type of decision problem.

COMMENTS: To verify whether the annual deposits of \$4,264 over 18 years would be sufficient to meet the child's college expenses, we can calculate the actual year-by-year balances: With the 18 annual deposits of \$4,264, the balance on the child's 18th birthday is

$$\$4{,}264(F/A, 7\%, 18) = \$144{,}972.$$

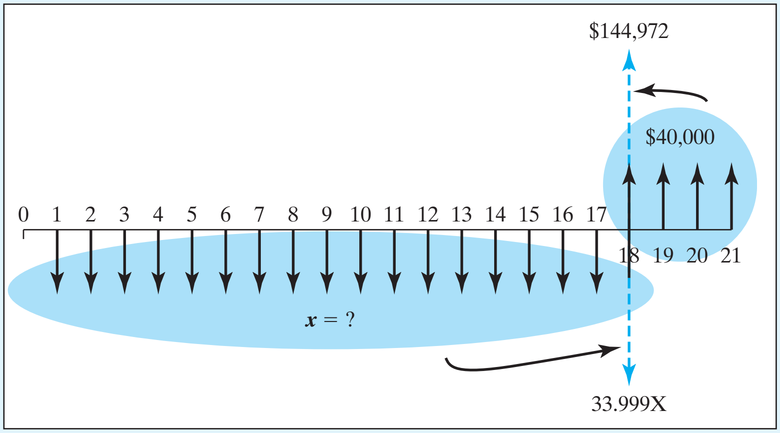


Figure 3.40 An alternative equivalence calculation (Example 3.26).

From this balance, the couple will make four annual tuition payments:

Year N	Beginning Balance	Interest Earned	Tuition Payment	Ending Balance
Freshman	$144,972	$ 0	$40,000	$104,972
Sophomore	104,972	7,348	40,000	72,320
Junior	72,320	5,062	40,000	37,382
Senior	37,382	2,618	40,000	0

3.4.2 Determining an Interest Rate to Establish Economic Equivalence

Thus far, we have assumed that, in equivalence calculations, a typical interest rate is given. Now we can use the same interest formulas that we developed earlier to determine interest rates that are explicit in equivalence problems. For most commercial loans, interest rates are already specified in the contract. However, when making some investments in financial assets, such as stocks, you may want to know the rate of growth (or rate of return) at which your asset is appreciating over the years. (This kind of calculation is the basis of rate-of-return analysis, which is covered in Chapter 7.) Although we can use interest tables to find the rate that is implicit in single payments and annuities, it is more difficult to find the rate that is implicit in an uneven series of payments. In such cases, a trial-and-error procedure or computer software may be used. To illustrate, consider Example 3.27.

EXAMPLE 3.27 Calculating an Unknown Interest Rate with Multiple Factors

You may have already won $2 million! Just peel the game piece off the Instant Winner Sweepstakes ticket, and mail it to us along with your order for subscriptions to your two favorite magazines. As a grand prize winner, you may choose between a $1 million cash prize paid immediately or $100,000 per year for 20 years—that's $2 million! Suppose that, instead of receiving one lump sum of $1 million, you decide to accept the 20 annual installments of $100,000. If you are like most jackpot winners, you will be tempted to spend your winnings to improve your lifestyle during the first several years. Only after you get this type of spending "out of your system" will you save later sums for investment purposes. Suppose that you are considering the following two options:

Option 1: You save your winnings for the first 7 years and then spend every cent of the winnings in the remaining 13 years.

Option 2: You do the reverse, spending for 7 years and then saving for 13 years.

If you can save winnings at 7% interest, how much would you have at the end of 20 years, and what interest rate on your savings will make these two options equivalent? (Cash flows into savings for the two options are shown in Figure 3.41.)

SOLUTION

Given: Cash flows in Figure 3.41.

Find: (a) F and (b) i at which the two flows are equivalent.

(a) In Option 1, the net balance at the end of year 20 can be calculated in two steps: Find the accumulated balance at the end of year 7 (V_7) first; then find the equivalent worth of V_7 at the end of year 20. For Option 2, find the equivalent worth of the 13 equal annual deposits at the end of year 20. We thus have

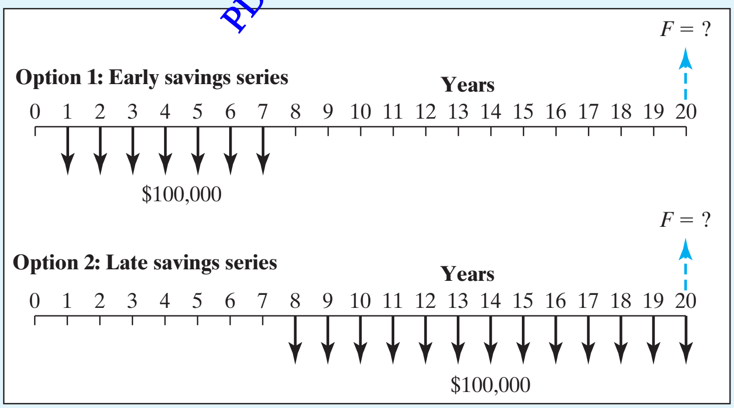


Figure 3.41 Equivalence calculation (Example 3.27).

$$\begin{aligned} F_{\text{Option 1}} &= \$100{,}000(F/A, 7\%, 7)(F/P, 7\%, 13) \\ &= \$2{,}085{,}485; \\ F_{\text{Option 2}} &= \$100{,}000(F/A, 7\%, 13) \\ &= \$2{,}014{,}064. \end{aligned}$$

Option 1 accumulates $71,421 more than Option 2.

(b) To compare the alternatives, we may compute the present worth for each option at period 0. By selecting period 7, however, we can establish the same economic equivalence with fewer interest factors. As shown in Figure 3.42, we calculate the equivalent value V_7 for each option at the end of period 7, remembering that the end of period 7 is also the beginning of period 8. (Recall from Example 3.4 that the choice of the point in time at which to compare two cash flows for equivalence is arbitrary.)

- For Option 1,

$$V_7 = \$100{,}000(F/A, i, 7).$$

- For Option 2,

$$V_7 = \$100{,}000(P/A, i, 13).$$

We equate the two values:

$$\$100{,}000(F/A, i, 7) = \$100{,}000(P/A, i, 13);$$

$$\frac{(F/A, i, 7)}{(P/A, i, 13)} = 1.$$

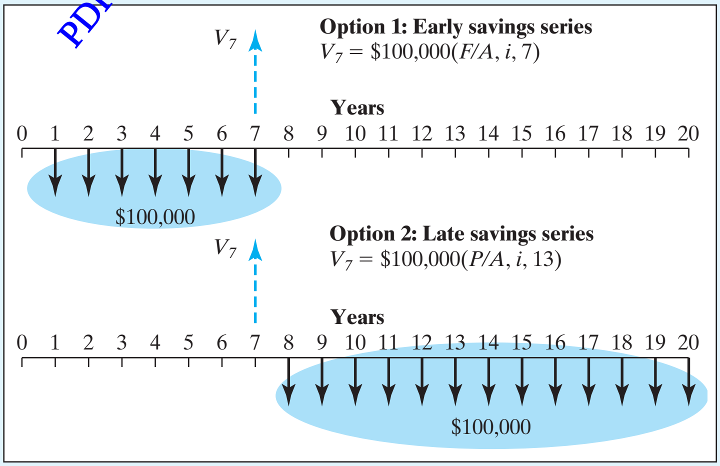


Figure 3.42 Establishing an economic equivalence at period 7 (Example 3.27).

Here, we are looking for an interest rate that gives a ratio of unity. When using the interest tables, we need to resort to a trial-and-error method. Suppose that we guess the interest rate to be 6%. Then

$$\frac{(F/A, 6\%, 7)}{(P/A, 6\%, 13)} = \frac{8.3938}{8.8527} = 0.9482.$$

This is less than unity. To increase the ratio, we need to use a value of i such that it increases the $(F/A, i, 7)$ factor value, but decreases the $(P/A, i, 13)$ value. This will happen if we use a larger interest rate. Let's try $i = 7\%$:

$$\frac{(F/A, 7\%, 7)}{(P/A, 7\%, 13)} = \frac{8.6540}{8.3577} = 1.0355.$$

Now the ratio is greater than unity.

Interest Rate	(F/A, i, 7)/(P/A, i, 13)
6%	0.9482
?	1.0000
7%	1.0355

As a result, we find that the interest rate is between 6% and 7% and may be approximated by **linear** **interpolation** as shown in Figure 3.43:

Interpolation is a method of constructing new data points from a discrete set of known data points.

$$\begin{aligned} i &= 6\% + (7\% - 6\%)\left[\frac{1 - 0.9482}{1.0355 - 0.9482}\right] \\ &= 6\% + 1\%\left[\frac{0.0518}{0.0873}\right] \\ &= 6.5934\%. \end{aligned}$$

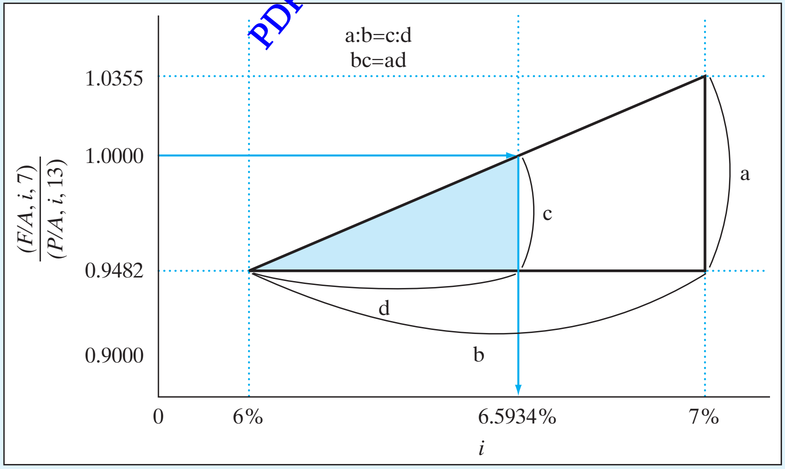


Figure 3.43 Linear interpolation to find an unknown interest rate (Example 3.27).

At 6.5934% interest, the two options are equivalent, and you may decide to indulge your desire to spend like crazy for the first 7 years. However, if you could obtain a higher interest rate, you would be wiser to save for 7 years and spend for the next 13.

COMMENTS: This example demonstrates that finding an interest rate is an iterative process that is more complicated and generally less precise than finding an equivalent worth at a known interest rate. Since computers and financial calculators can speed the process of finding unknown interest rates, such tools are highly recommended for these types of problem solving. With Excel, a more precise break-even value of 6.60219% is found by using the Goal Seek function.[3]

In Figure 3.44, the cell that contains the formula that you want to settle is called the **Set cell** (**F11 = F7-F9**). The value you want the formula to change to is called **To value (0)** and the part of the formula that you wish to change is called **By changing cell (F5, interest rate)**. The **Set cell** MUST always contain a formula or a function, whereas the **Changing cell** must contain a value only, not a formula or function.

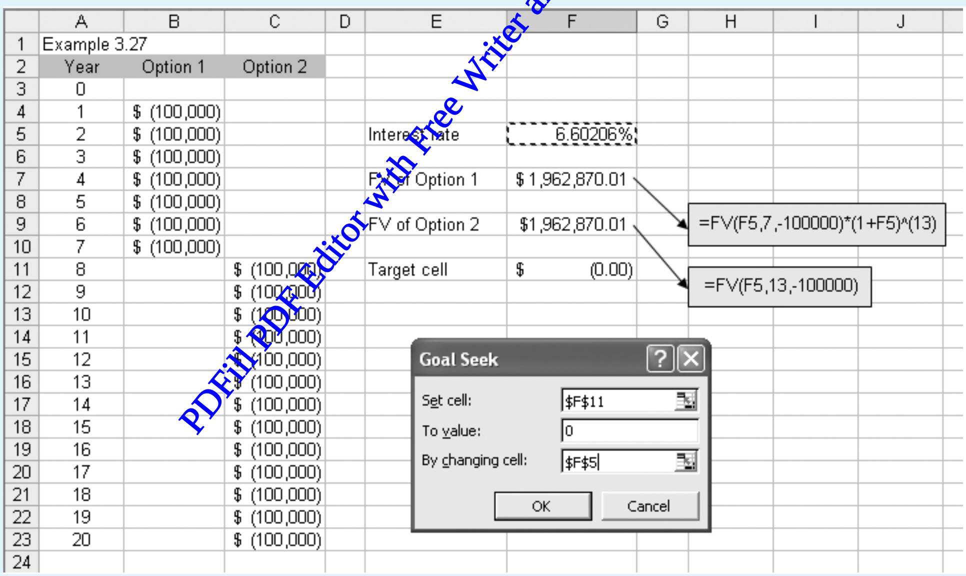


Figure 3.44 Using the Goal Seek function in Excel to find the break-even interest rate (Example 3.27). As soon as you select OK you will see that Goal Seek recalculates your formula. You then have two options, OK or Cancel. If you select OK the new term will be inserted into your worksheet. If you select Cancel, the Goal Seek box will disappear, and your worksheet will be in its original state.

[3] Goal Seek can be used when you know the result of a formula, but not the input value required by the formula to decide the result. You can change the value of a specified cell until the formula that is dependent on the changed cell returns the result you want. Goal Seek is found under the Tools menu.

SUMMARY

- Money has a time value because it can earn more money over time. A number of terms involving the time value of money were introduced in this chapter:

  Interest is the cost of money. More specifically, it is a cost to the borrower and an earning to the lender, above and beyond the initial sum borrowed or loaned.

  Interest rate is a percentage periodically applied to a sum of money to determine the amount of interest to be added to that sum.

  Simple interest is the practice of charging an interest rate only to an initial sum.

  Compound interest is the practice of charging an interest rate to an initial sum *and* to any previously accumulated interest that has not been withdrawn from the initial sum. Compound interest is by far the most commonly used system in the real world.

  Economic equivalence exists between individual cash flows and/or patterns of cash flows that have the same value. Even though the amounts and timing of the cash flows may differ, the appropriate interest rate makes them equal.

- The compound-interest formula, perhaps the single most important equation in this text, is

$$F = P(1 + i)^N,$$

  where P is a present sum, i is the interest rate, N is the number of periods for which interest is compounded, and F is the resulting future sum. All other important interest formulas are derived from this one.

- **Cash flow diagrams** are visual representations of cash inflows and outflows along a timeline. They are particularly useful for helping us detect which of the following five patterns of cash flow is represented by a particular problem:

  1. **Single payment.** A single present or future cash flow.
  2. **Uniform series.** A series of flows of equal amounts at regular intervals.
  3. **Linear gradient series.** A series of flows increasing or decreasing by a fixed amount at regular intervals.
  4. **Geometric gradient series.** A series of flows increasing or decreasing by a fixed percentage at regular intervals.
  5. **Irregular series.** A series of flows exhibiting no overall pattern. However, patterns might be detected for portions of the series.

- **Cash flow patterns** are significant because they allow us to develop **interest formulas**, which streamline the solution of equivalence problems. Table 3.4 summarizes the important interest formulas that form the foundation for all other analyses you will conduct in engineering economic analysis.

PROBLEMS

Types of Interest

3.1 You deposit \$5,000 in a savings account that earns 8% simple interest per year. What is the minimum number of years you must wait to double your balance?

Suppose instead that you deposit the $5,000 in another savings account that earns 7% interest compounded yearly. How many years will it take now to double your balance?

3.2 Compare the interest earned by $1,000 for five years at 8% simple interest with that earned by the same amount for five years at 8% compounded annually.

3.3 You are considering investing $3,000 at an interest rate of 8% compounded annually for five years or investing the $3,000 at 9% per year simple interest for five years. Which option is better?

3.4 You are about to borrow $10,000 from a bank at an interest rate of 9% compounded annually. You are required to make five equal annual repayments in the amount of $2,571 per year, with the first repayment occurring at the end of year 1. Show the interest payment and principal payment in each year.

Equivalence Concept

3.5 Suppose you have the alternative of receiving either $12,000 at the end of five years or P dollars today. Currently you have no need for money, so you would deposit the P dollars in a bank that pays 5% interest. What value of P would make you indifferent in your choice between P dollars today and the promise of $12,000 at the end of five years?

3.6 Suppose that you are obtaining a personal loan from your uncle in the amount of $20,000 (now) to be repaid in two years to cover some of your college expenses. If your uncle usually earns 8% interest (annually) on his money, which is invested in various sources, what minimum lump-sum payment two years from now would make your uncle happy?

Single Payments (Use of *F/P* or *P/F* Factors)

3.7 What will be the amount accumulated by each of these present investments?

(a) $5,000 in 8 years at 5% compounded annually
(b) $2,250 in 12 years at 3% compounded annually
(c) $8,000 in 31 years at 7% compounded annually
(d) $25,000 in 7 years at 9% compounded annually

3.8 What is the present worth of these future payments?

(a) $5,500 6 years from now at 10% compounded annually
(b) $8,000 15 years from now at 6% compounded annually
(c) $30,000 5 years from now at 8% compounded annually
(d) $15,000 8 years from now at 12% compounded annually

3.9 For an interest rate of 13% compounded annually, find

(a) How much can be lent now if $10,000 will be repaid at the end of five years?
(b) How much will be required in four years to repay a $25,000 loan received now?

3.10 How many years will it take an investment to triple itself if the interest rate is 12% compounded annually?

3.11 You bought 300 shares of Microsoft (MSFT) stock at $2,600 on December 31, 2005. Your intention is to keep the stock until it doubles in value. If you expect 15% annual growth for MSFT stock, how many years do you anticipate holding onto the stock? Compare your answer with the solution obtained by the Rule of 72 (discussed in Example 3.10).

3.12 From the interest tables in the text, determine the values of the following factors by interpolation, and compare your answers with those obtained by evaluating the F/P factor or the P/F factor:

(a) The single-payment compound-amount factor for 38 periods at 9.5% interest

(b) The single-payment present-worth factor for 47 periods at 8% interest

Uneven Payment Series

3.13 If you desire to withdraw the following amounts over the next five years from a savings account that earns 8% interest compounded annually, how much do you need to deposit now?

N	Amount
2	$32,000
3	43,000
4	46,000
5	28,000

3.14 If $1,500 is invested now, $1,800 two years from now, and $2,000 four years from now at an interest rate of 6% compounded annually, what will be the total amount in 15 years?

3.15 A local newspaper headline blared, "Bo Smith Signed for $30 Million." A reading of the article revealed that on April 1, 2005, Bo Smith, the former record-breaking running back from Football University, signed a $30 million package with the Dallas Rangers. The terms of the contract were $3 million immediately, $2.4 million per year for the first five years (with the first payment after 1 year) and $3 million per year for the next five years (with the first payment at year 6). If Bo's interest rate is 8% per year, what would his contract be worth at the time he signs it?

3.16 How much invested now at 6% would be just sufficient to provide three payments, with the first payment in the amount of $7,000 occurring two years hence, then $6,000 five years hence, and finally $5,000 seven years hence?

Equal Payment Series

3.17 What is the future worth of a series of equal year-end deposits of $1,000 for 10 years in a savings account that earns 7%, annual interest if

(a) All deposits are made at the *end* of each year?

(b) All deposits are made at the *beginning* of each year?

3.18 What is the future worth of the following series of payments?

(a) $3,000 at the end of each year for 5 years at 7% compounded annually

(b) $4,000 at the end of each year for 12 years at 8.25% compounded annually

(c) $5,000 at the end of each year for 20 years at 9.4% compounded annually

(d) $6,000 at the end of each year for 12 years at 10.75% compounded annually

3.19 What equal annual series of payments must be paid into a sinking fund to accumulate the following amounts?

(a) $22,000 in 13 years at 6% compounded annually

(b) $45,000 in 8 years at 7% compounded annually

(c) $35,000 in 25 years at 8% compounded annually

(d) $18,000 in 8 years at 14% compounded annually

3.20 Part of the income that a machine generates is put into a sinking fund to replace the machine when it wears out. If $1,500 is deposited annually at 7% interest, how many years must the machine be kept before a new machine costing $30,000 can be purchased?

3.21 A no-load (commission-free) mutual fund has grown at a rate of 11% compounded annually since its beginning. If it is anticipated that it will continue to grow at that rate, how much must be invested every year so that $15,000 will be accumulated at the end of five years?

3.22 What equal annual payment series is required to repay the following present amounts?

(a) $10,000 in 5 years at 5% interest compounded annually

(b) $5,500 in 4 years at 9.7% interest compounded annually

(c) $8,500 in 3 years at 2.5% interest compounded annually

(d) $30,000 in 20 years at 8.5% interest compounded annually

3.23 You have borrowed $25,000 at an interest rate of 16%. Equal payments will be made over a three-year period. (The first payment will be made at the end of the first year.) What will the annual payment be, and what will the interest payment be for the second year?

3.24 What is the present worth of the following series of payments?

(a) $800 at the end of each year for 12 years at 5.8% compounded annually

(b) $2,500 at the end of each year for 10 years at 8.5% compounded annually

(c) $900 at the end of each year for 5 years at 7.25% compounded annually

(d) $5,500 at the end of each year for 8 years at 8.75% compounded annually

3.25 From the interest tables in Appendix B, determine the values of the following factors by interpolation and compare your results with those obtained from evaluating the A/P and P/A interest formulas:

(a) The capital recovery factor for 38 periods at 6.25% interest

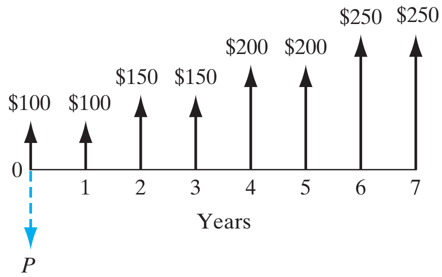
(b) The equal payment series present-worth factor for 85 periods at 9.25% interest

Linear Gradient Series

3.26 An individual deposits an annual bonus into a savings account that pays 8% interest compounded annually. The size of the bonus increases by $2,000 each year, and the initial bonus amount was $5,000. Determine how much will be in the account immediately after the fifth deposit.

3.27 Five annual deposits in the amounts of $3,000, $2,500, $2,000, $1,500, and $1,000, in that order, are made into a fund that pays interest at a rate of 7% compounded annually. Determine the amount in the fund immediately after the fifth deposit.

3.28 Compute the value of P in the accompanying cash flow diagram, assuming that $i = 9\%$.



3.29 What is the equal payment series for 12 years that is equivalent to a payment series of \$15,000 at the end of the first year, decreasing by \$1,000 each year over 12 years? Interest is 8% compounded annually.

Geometric Gradient Series

3.30 Suppose that an oil well is expected to produce 100,000 barrels of oil during its first year in production. However, its subsequent production (yield) is expected to decrease by 10% over the previous year's production. The oil well has a proven reserve of 1,000,000 barrels.

(a) Suppose that the price of oil is expected to be \$60 per barrel for the next several years. What would be the present worth of the anticipated revenue stream at an interest rate of 12% compounded annually over the next seven years?

(b) Suppose that the price of oil is expected to start at \$60 per barrel during the first year, but to increase at the rate of 5% over the previous year's price. What would be the present worth of the anticipated revenue stream at an interest rate of 12% compounded annually over the next seven years?

(c) Consider part (b) again. After three years' production, you decide to sell the oil well. What would be a fair price?

3.31 A city engineer has estimated the annual toll revenues from a newly proposed highway construction over 20 years as follows:

$$A_n = (\$2{,}000{,}000)(n)(1.06)^{n-1},$$
$$n = 1, 2, \ldots, 20.$$

To validate the bond, the engineer was asked to present the estimated total present value of toll revenue at an interest rate of 6%. Assuming annual compounding, find the present value of the estimated toll revenue.

3.32 What is the amount of 10 equal annual deposits that can provide five annual withdrawals when a first withdrawal of \$5,000 is made at the end of year 11 and subsequent withdrawals increase at the rate of 8% per year over the previous year's withdrawal if

(a) The interest rate is 9% compounded annually?

(b) The interest rate is 6% compounded annually?

Various Interest Factor Relationships

3.33 By using only those factors given in interest tables, find the values of the factors that follow, which are not given in your tables. Show the relationship between the factors by using factor notation, and calculate the value of the factor. Then compare the solution you obtained by using the factor formulas with a direct calculation of the factor values.

Example: $(F/P, 8\%, 38) = (F/P, 8\%, 30)(F/P, 8\%, 8) = 18.6253$

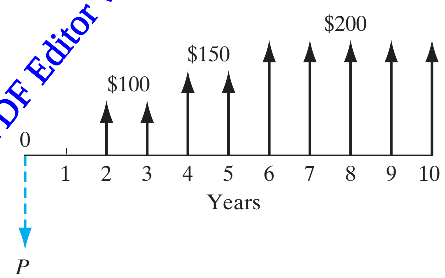
(a) $(P/F, 8\%, 67)$
(b) $(A/P, 8\%, 42)$
(c) $(P/A, 8\%, 135)$

3.34 Prove the following relationships among interest factors:

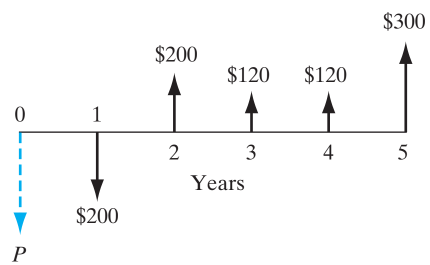
(a) $(F/P, i, N) = i(F/A, i, N) + 1$
(b) $(P/F, i, N) = 1 - (P/A, i, N)i$
(c) $(A/F, i, N) = (A/P, i, N) - i$
(d) $(A/P, i, N) = i/[1 - (P/F, i, N)]$
(e) $(P/A, i, N \rightarrow \infty) = 1/i$
(f) $(A/P, i, N \rightarrow \infty) = i$
(g) $(P/G, i, N \rightarrow \infty) = 1/i^2$

Equivalence Calculations

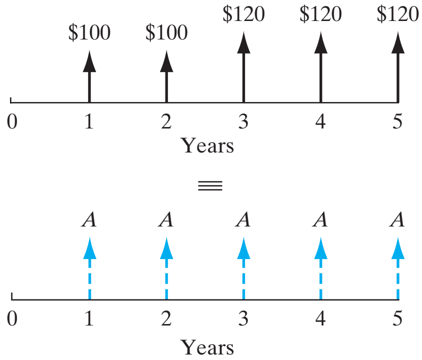
3.35 Find the present worth of the cash receipts where $i = 12\%$ compounded annually with only four interest factors.



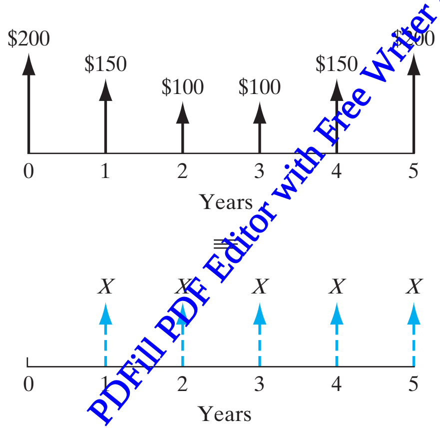
3.36 Find the equivalent present worth of the cash receipts where $i = 8\%$. In other words, how much do you have to deposit now (with the second deposit in the amount of $200 at the end of the first year) so that you will be able to withdraw $200 at the end of second year, $120 at the end of third year, and so forth if the bank pays you a 8% annual interest on your balance?



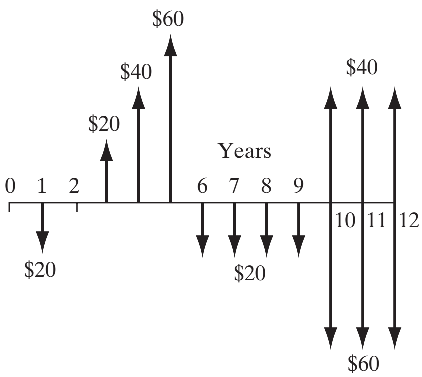
3.37 What value of A makes two annual cash flows equivalent at 13% interest compounded annually?



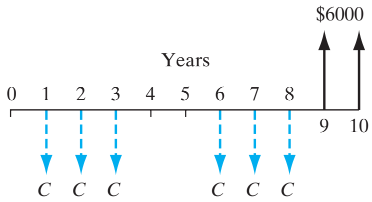
3.38 The two cash flow transactions shown in the accompanying cash flow diagram are said to be equivalent at 6% interest compounded annually. Find the unknown value of X that satisfies the equivalence.



3.39 Solve for the present worth of this cash flow using at most three interest factors at 10% interest compounded annually.



3.40 From the accompanying cash flow diagram, find the value of C that will establish the economic equivalence between the deposit series and the withdrawal series at an interest rate of 8% compounded annually.



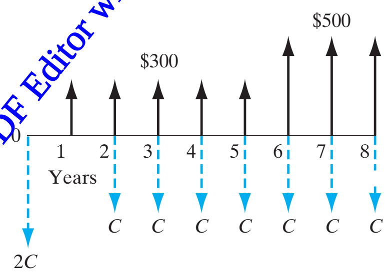
(a) $1,335 (b) $862
(c) $1,283 (d) $828

3.41 The following equation describes the conversion of a cash flow into an equivalent equal payment series with $N = 10$:

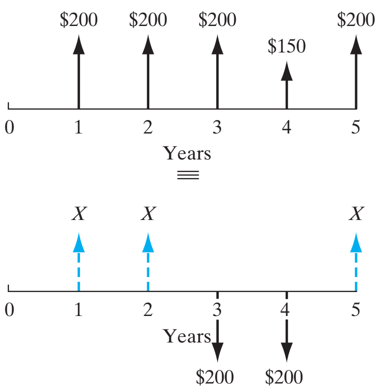
$$\begin{aligned} A = {} & [800 + 20(A/G, 6\%, 7)] \\ & \times (P/A, 6\%, 7)(A/P, 6\%, 10) \\ & + [300(F/A, 6\%, 3) - 500](A/F, 6\%, 10). \end{aligned}$$

Reconstruct the original cash flow diagram.

3.42 Consider the cash flow shown in the accompanying diagram. What value of C makes the inflow series equivalent to the outflow series at an interest rate of 10%?

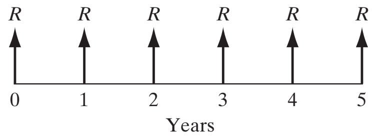


3.43 Find the value of X so that the two cash flows shown in the diagram are equivalent for an interest rate of 8%.



3.44 What single amount at the end of the fifth year is equivalent to a uniform annual series of \$3,000 per year for 10 years if the interest rate is 9% compounded annually?

3.45 From the following list, identify all the correct equations used in computing either the equivalent present worth (P) or future worth (F) for the cash flow shown at $i = 10\%$.

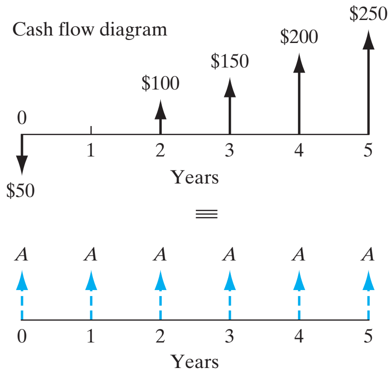


(1) $P = R(P/A, 10\%, 6)$
(2) $P = R + R(P/A, 10\%, 5)$
(3) $P = R(P/F, 10\%, 5) + R(P/A, 10\%, 5)$
(4) $F = R(F/A, 10\%, 5) + R(F/P, 10\%, 5)$
(5) $F = R + R(F/A, 10\%, 5)$
(6) $F = R(F/A, 10\%, 6)$
(7) $F = R(F/A, 10\%, 6) - R$

3.46 On the day his baby was born, a father decided to establish a savings account for the child's college education. Any money that is put into the account will earn an interest rate of 8% compounded annually. The father will make a series of annual deposits in equal amounts on each of his child's birthdays from the 1st through the 18th, so that the child can make four annual withdrawals from the account in the amount of \$30,000 on each birthday. Assuming that the first withdrawal will be made on the child's 18th birthday, which of the following equations are correctly used to calculate the required annual deposit?

(1) $A = (\$30{,}000 \times 4)/18$
(2) $A = \$30{,}000(F/A, 8\%, 4) \times (P/F, 8\%, 21)(A/P, 8\%, 18)$
(3) $A = \$30{,}000(P/A, 8\%, 18) \times (F/P, 8\%, 21)(A/F, 8\%, 4)$
(4) $A = [\$30{,}000(P/A, 8\%, 3) + \$30{,}000](A/F, 8\%, 18)$
(5) $A = \$30{,}000[(P/F, 8\%, 18) + (P/F, 8\%, 19) + (P/F, 8\%, 20) + (P/F, 8\%, 21)](A/P, 8\%, 18)$

3.47 Find the equivalent equal payment series (A) using an A/G factor such that the two cash flows are equivalent at 10% compounded annually.



3.48 Consider the following cash flow:

Year End	Payment
0	\$500
1–5	\$1,000

In computing F at the end of year 5 at an interest rate of 12%, which of the following equations is *in*correct?

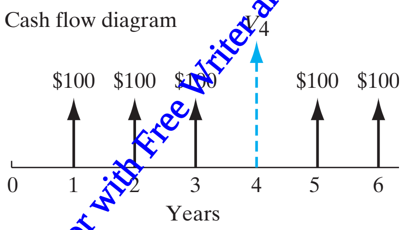
(a) $F = \$1{,}000(F/A, 12\%, 5) - \$500(F/P, 12\%, 5)$

(b) $F = \$500(F/A, 12\%, 6) + \$500(F/A, 12\%, 5)$

(c) $F = [\$500 + \$1{,}000(P/A, 12\%, 5)] \times (F/P, 12\%, 5)$

(d) $F = [\$500(A/P, 12\%, 5) + \$1{,}000] \times (F/A, 12\%, 5)$

3.49 Consider the cash flow series given. In computing the equivalent worth at $n = 4$, which of the following equations is *in*correct?



(a) $V_4 = [\$100(P/A, i, 6) - \$100(P/F, i, 4)](F/P, i, 4)$

(b) $V_4 = \$100(F/A, i, 3) + \$100(P/A, i, 2)$

(c) $V_4 = \$100(F/A, i, 4) - \$100 + \$100(P/A, i, 2)$

(d) $V_4 = [\$100(F/A, i, 6) - \$100(F/P, i, 2)](P/F, i, 2)$

3.50 Henry Cisco is planning to make two deposits: \$25,000 now and \$30,000 at the end of year 6. He wants to withdraw C each year for the first six years and $(C + \$1{,}000)$ each year for the next six years. Determine the value of C if the deposits earn 10% interest compounded annually.

(a) \$7,711

(b) \$5,794

(c) \$6,934

(d) \$6,522

Solving for an Unknown Interest Rate or Unknown Interest Periods

3.51 At what rate of interest compounded annually will an investment double itself in five years?

3.52 Determine the interest rate (i) that makes the pairs of cash flows shown economically equivalent.

CHAPTER FOUR

Understanding Money and Its Management

Hybrid Mortgages Can Cause Pain Should Rates Start to Rise[1] Are you shopping for a mortgage to finance a home that you expect to own for no more than a few years? If so, you should know about a hybrid mortgage. Hybrid loans give prospective home buyers the ability to buy a lot more home than they can afford, thanks to the initially lower interest rate. But with such flexibility comes greater risk. Since lenders are free to design loans to fit borrowers' needs, the terms and fees vary widely and homeowners can get burned if rates turn higher.

Hybrid mortgages allow homeowners to benefit from the best aspects of both fixed-rate and adjustable-rate mortgages (ARMs). With hybrids, borrowers choose to accept a fixed interest rate over a number of years—usually 3, 5, 7, or 10 years—and afterward the loan

Rates Rising

Mortgages are more costly than they were a year ago.

	2006	2005
30-year fixed-rate mortgage	6.48%	5.85%
Hybrid ARM with fixed rate for first 10 years	6.32	5.58
Hybrid ARM with fixed rate for first 5 years	6.02	5.01
ARM with rate that adjusts annually	5.29	4.21

[1] "Hybrid mortgages can cause pain should rates start to rise," by Terri Cullen, *Wall Street Journal Online*, December 5, 2002. @2002 Dow Jones & Company, Inc.

converts to an ARM. But therein lies the danger: While you're getting an extraordinarily low rate up front for a few years, when the fixed-rate period expires you could very well end up paying more than double your current rate of interest.

At today's rate of 6.16% for a 30-year mortgage, a person borrowing $200,000 would pay $1,220 a month. With a 7-year hybrid, more commonly called a 7/1 loan, at the going rate of 5.61% that monthly payment drops to $1,150. By the end of the seventh year, the homeowner would save $7,700 in interest charges by going with a 7-year hybrid.

To say that there are drawbacks is an understatement. Despite the surge in popularity, a hybrid loan can be a ticking time bomb for borrowers who plan on holding the loan for the long term.

Let's say a borrower takes out a 30-year $200,000 hybrid loan that will remain at a fixed rate of 5.19% for 5 years and then will switch to an adjustable-rate mortgage for the remaining period. The lender agrees to set a cap on the adjustable-rate portion of the loan, so that the rate will climb no more than 5 percentage points. Conceivably, then, if rates are sharply higher after that initial 7-year period, a borrower could be looking at a mortgage rate of more than 10%. Under this scenario, the homeowner's monthly payment on a $200,000 mortgage would jump to $1,698 from $1,097 after the 5-year term expires—a 55% increase! But if there's a very real chance you'll be looking to sell your home over the next 10 years, hybrids can make a lot of sense, since shorter term loans usually carry the lowest rates.

In this chapter, we will consider several concepts crucial to managing money. In Chapter 3, we examined how time affects the value of money, and we developed various interest formulas for that purpose. Using these basic formulas, we will now extend the concept of equivalence to determine interest rates that are implicit in many financial contracts. To this end, we will introduce several examples in the area of loan transactions. For example, many commercial loans require that interest compound more frequently than once a year—for instance, monthly or quarterly. To consider the effect of more frequent compounding, we must begin with an understanding of the concepts of nominal and effective interest.

CHAPTER LEARNING OBJECTIVES

After completing this chapter, you should understand the following concepts:

- The difference between the nominal interest rate and the effective interest rate.
- The procedure for computing the effective interest rate, based on a payment period.
- How commercial loans and mortgages are structured in terms of interest and principal payments.
- The basics of investing in financial assets.

4.1 Nominal and Effective Interest Rates

In all the examples in Chapter 3, the implicit assumption was that payments are received *once a year*, or *annually*. However, some of the most familiar financial transactions, both personal and in engineering economic analysis, involve payments that are not based on one annual payment—for example, monthly mortgage payments and quarterly earnings on savings accounts. Thus, if we are to compare different cash flows with different compounding periods, we need to evaluate them on a common basis. This need has led to the development of the concepts of the **nominal interest rate** and the **effective interest rate**.

4.1.1 Nominal Interest Rates

Take a closer look at the billing statement from any of your credit cards. Or if you financed a new car recently, examine the loan contract. You will typically find the interest that the bank charges on your unpaid balance. Even if a financial institution uses a unit of time other than a year—say, a month or a quarter (e.g., when calculating interest payments)—the institution usually quotes the interest rate on an *annual basis*. Many banks, for example, state the interest arrangement for credit cards in this way:

18% compounded monthly.

Annual percentage rate (APR) is the yearly cost of a loan, including interest, insurance, and the origination fee, expressed as a percentage.

This statement simply means that each month the bank will charge 1.5% interest on an unpaid balance. We say that 18% is the **nominal interest rate** or **annual percentage rate** (APR), and the compounding frequency is monthly (12). As shown in Figure 4.1, to obtain the interest rate per compounding period, we divide, for example, 18% by 12, to get 1.5% per month.

Although the annual percentage rate, or APR, is commonly used by financial institutions and is familiar to many customers, the APR does not explain precisely the amount of interest that will accumulate in a year. To explain the true effect of more frequent compounding on annual interest amounts, we will introduce the term *effective interest rate*, commonly known as *annual effective yield*, or *annual percentage yield* (APY).

Month	1	2	3	4	5	6	7	8	9	10	11	12
Interest rate (%)	1.5	1.5	1.5	1.5	1.5	1.5	1.5	1.5	1.5	1.5	1.5	1.5

Nominal interest rate.
1.5% × 12 = 18%

Figure 4.1 The nominal interest rate is determined by summing the individual interest rates per period.

4.1.2 Effective Annual Interest Rates

The **effective annual interest rate** is the one rate that truly represents the interest earned in a year. For instance, in our credit card example, the bank will charge 1.5% interest on any unpaid balance at the end of each month. Therefore, the 1.5% rate represents the effective interest rate per month. On a yearly basis, you are looking for a cumulative rate—1.5% each month for 12 months. This cumulative rate predicts the actual interest payment on your outstanding credit card balance.

Suppose you deposit \$10,000 in a savings account that pays you at an interest rate of 9% compounded quarterly. Here, 9% represents the nominal interest rate, and the interest rate per quarter is 2.25% (9%/4). The following is an example of how interest is compounded when it is paid quarterly:

End of Period	Base amount	Interest Earned 2.25% × (Base amount)	New Base
First quarter	\$10,000.00	2.25% × \$10,000.00 = \$225.00	\$10,225.00
Second quarter	\$10,225.00	2.25% × \$10,225.00 = \$230.06	\$10,455.06
Third quarter	\$10,455.06	2.25% × \$10,455.06 = \$225.24	\$10,690.30
Fourth quarter	\$10,690.30	2.25% × \$10,690.30 = \$240.53	\$10,930.83

Clearly, you are earning more than 9% on your original deposit. In fact, you are earning 9.3083% (\$930.83/\$10,000). We could calculate the total annual interest payment for a principal amount of \$10,000 with the formula given in Eq. (3.3). If $P = \$10,000$, $i = 2.25\%$, and $N = 4$, we obtain

$$\begin{aligned} F &= P(1 + i)^N \\ &= \$10,000(1 + 0.0225)^4 \\ &= \$10,930.83. \end{aligned}$$

The implication is that, for each dollar deposited, you are earning an equivalent annual interest of 9.38 cents. In terms of an effective annual interest rate (i_a), the interest payment can be rewritten as a percentage of the principal amount:

$$i_a = \$930.83/\$10{,}000 = 0.093083, \text{ or } 9.3083\%.$$

In other words, earning 2.25% interest per quarter for four quarters is equivalent to earning 9.3083% interest just one time each year.

Table 4.1 shows effective interest rates at various compounding intervals for 4%–12% APR. As you can see, depending on the frequency of compounding, the effective interest earned or paid by the borrower can differ significantly from the APR. Therefore, truth-in-lending laws require that financial institutions quote both the nominal interest rate and the compounding frequency (i.e., the effective interest) when you deposit or borrow money.

Certainly, more frequent compounding increases the amount of interest paid over a year at the same nominal interest rate. Assuming that the nominal interest rate is r, and M compounding periods occur during the year, we can calculate the effective annual interest rate

$$i_a = \left(1 + \frac{r}{M}\right)^M - 1. \tag{4.1}$$

When $M = 1$, we have the special case of annual compounding. Substituting $M = 1$ in Eq. (4.1) reduces it to $i_a = r$. That is, when compounding takes place once annually, the effective interest is equal to the nominal interest. Thus, in the examples in Chapter 3, in which only annual interest was considered, we were, by definition, using effective interest rates.

TABLE 4.1 Nominal and Effective Interest Rates with Different Compounding Periods

	Effective Rates				
Nominal Rate	Compounding Annually	Compounding Semiannually	Compounding Quarterly	Compounding Monthly	Compounding Daily
4%	4.00%	4.04%	4.06%	4.07%	4.08%
5	5.00	5.06	5.09	5.12	5.13
6	6.00	6.09	6.14	6.17	6.18
7	7.00	7.12	7.19	7.23	7.25
8	8.00	8.16	8.24	8.30	8.33
9	9.00	9.20	9.31	9.38	9.42
10	10.00	10.25	10.38	10.47	10.52
11	11.00	11.30	11.46	11.57	11.62
12	12.00	12.36	12.55	12.68	12.74

EXAMPLE 4.1 Determining a Compounding Period

The following table summarizes interest rates on certificates of deposit (CDs) offered by various lending institutions during November 2005:

Product	Bank	Minimum	Rate	APY*
3-Month CD	Imperial Capital Bank	\$2,000	4.03%	4.10%
6-Month Jumbo CD	IndyMac Bank	\$5,000	4.21%	4.30%
1-Year Jumbo CD	VirtualBank	\$10,000	4.50%	4.60%
1.5-Year CD	AmTrust Bank	\$1,000	4.50%	4.60%
2-Year CD	Ohio Savings Bank	\$1,000	4.59%	4.70%
2.5-Year Jumbo CD	Countrywide Bank	\$98,000	4.66%	4.77%
3-Year CD	ING Direct	\$0	4.70%	4.70%
5-Year CD	Citizens & Northern Bank	\$500	4.70%	4.78%

*Annual percentage yield = effective annual interest rate (i_a)

Annual percentage yield (APY) is the rate actually earned or paid in one year, taking into account the affect of compounding.

In the table, no mention is made of specific interest compounding frequencies. (a) Find the interest periods assumed for the 2.5-year Jumbo CD offered by Countrywide Bank. (b) Find the total balance for a deposit amount of \$100,000 at the end of 2.5 years.

SOLUTION

Given: $r = 4.66\%$ per year, i_a (APY) $= 4.77\%$, $P = \$100{,}000$, and $N = 2.5$ years.
Find: M and the balance at the end of 2.5 years.

(a) The nominal interest rate is 4.66% per year, and the effective annual interest rate (yield) is 4.77%. Using Eq. (4.1), we obtain the expression

$$0.0477 = \left(1 + \frac{0.0466}{M}\right)^M - 1,$$

or

$$1.0477 = \left(1 + \frac{0.0466}{M}\right)^M.$$

By trial and error, we find that $M = 365$, which indicates daily compounding. Thus, the 2.5-year Jumbo CD earns 4.66% interest compounded daily.

Normally, if the CD is not cashed at maturity, it will be renewed automatically at the original interest rate. Similarly, we can find the interest periods for the other CDs.

(b) If you purchase the 2.5-year Jumbo CD, it will earn 4.66% interest compounded daily. This means that your CD earns an effective annual interest of 4.77%:

$$\begin{aligned} F &= P(1 + i_a)^N \\ &= \$100{,}000(1 + 0.0477)^{2.5} \\ &= \$100{,}000\left(1 + \frac{0.0466}{365}\right)^{365\times 2.5} \\ &= \$112{,}355. \end{aligned}$$

4.1.3 Effective Interest Rates per Payment Period

We can generalize the result of Eq. (4.1) to compute the effective interest rate for periods of *any duration*. As you will see later, the effective interest rate is usually computed on the basis of the payment (transaction) period. For example, if cash flow transactions occur quarterly, but interest is compounded monthly, we may wish to calculate the effective interest rate on a quarterly basis. To do this, we may redefine Eq. (4.1) as

$$\begin{aligned} i &= \left(1 + \frac{r}{M}\right)^C - 1 \\ &= \left(1 + \frac{r}{CK}\right)^C - 1, \end{aligned} \qquad (4.2)$$

where

M = the number of interest periods per year,

C = the number of interest periods per payment period, and

K = the number of payment periods per year.

Note that $M = CK$ in Eq. (4.2). For the special case of annual payments with annual compounding, we obtain $i = i_a$ with $C = M$ and $K = 1$.

EXAMPLE 4.2 Effective Rate per Payment Period

Suppose that you make quarterly deposits in a savings account that earns 9% interest compounded monthly. Compute the effective interest rate per quarter.

SOLUTION

Given: $r = 9\%$, C = three interest periods per quarter, K = four quarterly payments per year, and M = 12 interest periods per year.

Find: i.

Using Eq. (4.2), we compute the effective interest rate per quarter as

$$\begin{aligned} i &= \left(1 + \frac{0.09}{12}\right)^3 - 1 \\ &= 2.27\%. \end{aligned}$$

COMMENTS: Figure 4.2 illustrates the relationship between the nominal and effective interest rates per payment period.

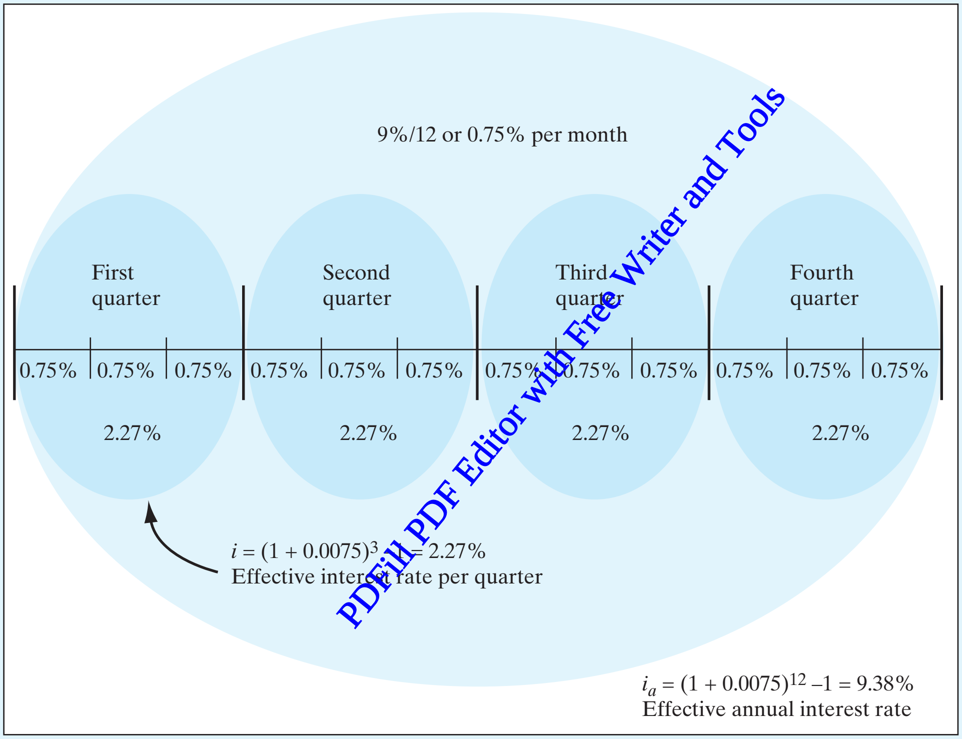


Figure 4.2 Functional relationships among r, i, and i_a, where interest is calculated based on 9% compounded monthly and payments occur quarterly (Example 4.2).

4.1.4 Continuous Compounding

To be competitive on the financial market or to entice potential depositors, some financial institutions offer frequent compounding. As the number of compounding periods (M) becomes very large, the interest rate per compounding period (r/M) becomes very small. As M approaches infinity and r/M approaches zero, we approximate the situation of **continuous compounding**.

By taking limits on both sides of Eq. (4.2), we obtain the effective interest rate per payment period as

$$i = \lim_{CK\to\infty}\left[\left(1 + \frac{r}{CK}\right)^{C} - 1\right]$$
$$= \lim_{CK\to\infty}\left(1 + \frac{r}{CK}\right)^{C} - 1$$
$$= (e^{r})^{1/K} - 1.$$

Continuous compounding: The process of calculating interest and adding it to existing principal and interest at infinitely short time intervals.

Therefore, the effective interest rate per payment period is

$$i = e^{r/K} - 1. \tag{4.3}$$

To calculate the effective annual interest rate for continuous compounding, we set K equal to unity, resulting in

$$i_a = e^{r} - 1. \tag{4.4}$$

As an example, the effective annual interest rate for a nominal interest rate of 12% compounded continuously is $i_a = e^{0.12} - 1 = 12.7497\%$.

EXAMPLE 4.3 Calculating an Effective Interest Rate with Quarterly Payment

Find the effective interest rate per *quarter* at a nominal rate of 8% compounded (a) quarterly, (b) monthly, (c) weekly, (d) daily, and (e) continuously.

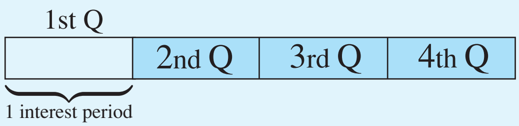
SOLUTION

Given: $r = 8\%$, M, C, and $K = 4$ quarterly payments per year.
Find: i.

(a) Quarterly compounding:

$r = 8\%$, $M = 4$, $C = 1$ interest period per quarter, and $K = 4$ payments per year. Then

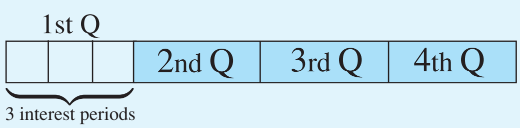
$$i = \left(1 + \frac{0.08}{4}\right)^{1} - 1 = 2.00\%.$$



(b) Monthly compounding:

$r = 8\%$, $M = 12$, $C = 3$ interest periods per quarter, and $K = 4$ payments per year. Then

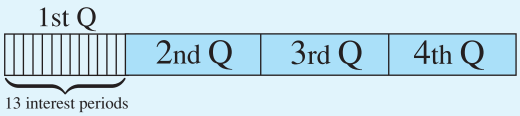
$$i = \left(1 + \frac{0.08}{12}\right)^{3} - 1 = 2.013\%.$$



(c) Weekly compounding:

$r = 8\%$, $M = 52$, C $= 13$ interest periods per quarter, and $K = 4$ payments per year. Then

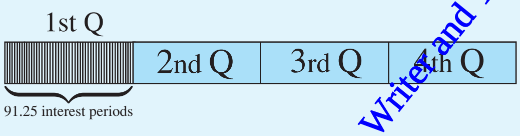
$$i = \left(1 + \frac{0.08}{52}\right)^{13} - 1 = 2.0186\%.$$



(d) Daily compounding:

$r = 8\%$, $M = 365$, $C = 91.25$ days per quarter, and $K = 4$. Then

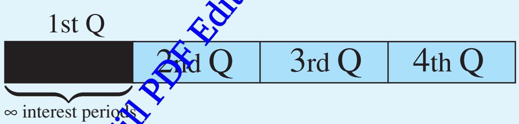
$$i = \left(1 + \frac{0.08}{365}\right)^{91.25} - 1 = 2.0199\%.$$



(e) Continuous compounding:

$r = 8\%$, $M \rightarrow \infty$, $C \rightarrow \infty$, and $K = 4$. Then, from Eq. (4.3),

$$i = e^{0.08/4} - 1 = 2.0201\%.$$



COMMENTS: Note that the difference between daily compounding and continuous compounding is often negligible. Many banks offer continuous compounding to entice deposit customers, but the extra benefits are small.

4.2 Equivalence Calculations with Effective Interest Rates

All the examples in Chapter 3 assumed annual payments and annual compounding. However, a number of situations involve cash flows that occur at intervals that are not the same as the compounding intervals often used in practice. Whenever payment and compounding periods differ from each other, *one or the other must be transformed so that both conform to the same unit of time*. For example, if payments occur quarterly and compounding occurs monthly, the most logical procedure is to calculate the effective interest rate per quarter. By contrast, if payments occur monthly and compounding occurs quarterly, we may be able to find the equivalent monthly interest rate. The bottom line is that, to proceed with equivalency analysis, the compounding and payment periods must be in the same order.

4.2.1 When Payment Period Is Equal to Compounding Period

Whenever the compounding and payment periods are equal $(M = K)$, whether the interest is compounded annually or at some other interval, the following solution method can be used:

1. Identify the number of compounding periods (M) per year.
2. Compute the effective interest rate per payment period—that is, using Eq. (4.2). Then, with $C = 1$ and $K = M$, we have

$$i = \frac{r}{M}.$$

3. Determine the number of compounding periods:

$$N = M \times (\text{number of years}).$$

EXAMPLE 4.4 Calculating Auto Loan Payments

Suppose you want to buy a car. You have surveyed the dealers' newspaper advertisements, and the following one has caught your attention:

College Graduate Special: New 2006 Nissan Altima 2.55 with automatic transmission, A/C, power package, and cruise control

MSRP:	\$20,870
Dealer's discount:	\$1,143
Manufacturer rebate	\$800
College graduate cash:	\$500
Sale price:	\$18,427

Price and payment is plus tax, title, customer service fee, with approved credit for 72 months at 6.25% APR.

You can afford to make a down payment of \$3,427 (and taxes and insurance as well), so the net amount to be financed is \$15,000. What would the monthly payment be (Figure 4.3)?

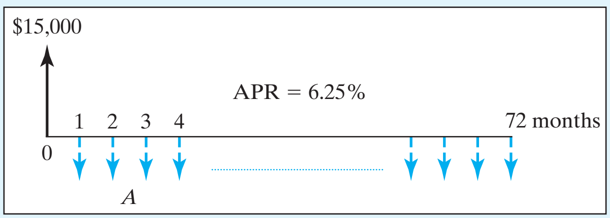


Figure 4.3 A car loan cash transaction (Example 4.4).

DISCUSSION: The advertisement does not specify a compounding period, but in automobile financing, the interest and the payment periods are almost always monthly. Thus, the 6.25% APR means 6.25% compounded monthly.

SOLUTION

Given: $P = \$25{,}000$, $r = 6.25\%$ per year, K $= 12$ payments per year, $N =$ 72 months, and $M = 12$ interest periods per year.
Find: A.

In this situation, we can easily compute the monthly payment with Eq. (3.12):

$$i = 6.25\%/12 = 0.5208\% \text{ per month},$$

$$N = (12)(6) = 72 \text{ months},$$

$$A = \$15{,}000(A/P, 0.5208\%, 72) = \$250.37.$$

4.2.2 Compounding Occurs at a Different Rate than That at Which Payments Are Made

We will consider two situations: (1) compounding is more frequent than payments and (2) compounding is less frequent than payments.

Compounding Is More Frequent than Payments

The computational procedure for compounding periods and payment periods that cannot be compared is as follows:

1. Identify the number of compounding periods per year (M), the number of payment periods per year (K), and the number of interest periods per payment period (C).
2. Compute the effective interest rate per payment period:
   - For discrete compounding, compute

$$i = \left(1 + \frac{r}{M}\right) - 1.$$

   - For continuous compounding, compute

$$i = e^{r/K} - 1.$$

3. Find the total number of payment periods:

$$N = K \times (\text{number of years}).$$

4. Use i and N in the appropriate formulas in Table 3.4.

EXAMPLE 4.5 Compounding Occurs More Frequently than Payments Are Made (Discrete-Compounding Case)

Suppose you make equal quarterly deposits of \$1,500 into a fund that pays interest at a rate of 6% compounded monthly, as shown in Figure 4.4. Find the balance at the end of year 2.

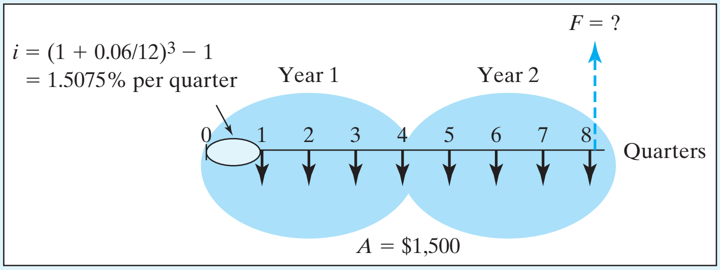


Figure 4.4 Quarterly deposits with monthly compounding (Example 4.5).

SOLUTION

Given: $A = \$1{,}500$ per quarter, $r = 6\%$ per year, $M = 12$ compounding periods per year, and $N = 8$ quarters.

Find: F.

We follow the aforementioned procedure for noncomparable compounding and payment periods:

1. Identify the parameter values for M, K, and C, where

$$M = 12 \text{ compounding periods per year,}$$

$$K = 4 \text{ payment periods per year,}$$

$$C = 3 \text{ interest periods per payment period.}$$

2. Use Eq. (4.2) to compute the effective interest:

$$i = \left(1 + \frac{0.06}{12}\right)^3 - 1$$

$$= 1.5075\% \text{ per quarter.}$$

3. Find the total number of payment periods, N:

$$N = K(\text{number of years}) = 4(2) = 8 \text{ quarters.}$$

4. Use i and N in the appropriate equivalence formulas:

$$F = \$1{,}500(F/A, 1.5075\%, 8) = \$12{,}652.60.$$

COMMENT: No 1.5075% interest table appears in Appendix A, but the interest factor can still be evaluated by $F = \$1{,}500(A/F, 0.5\%, 3)(F/A, 0.5\%, 24)$, where the first interest factor finds its equivalent monthly payment and the second interest factor converts the monthly payment series to an equivalent lump-sum future payment.

EXAMPLE 4.6 Compounding Occurs More Frequently than Payments Are Made (Continuous-Compounding Case)

A series of equal quarterly receipts of \$500 extends over a period of five years as shown in Figure 4.5. What is the present worth of this quarterly payment series at 8% interest compounded continuously?

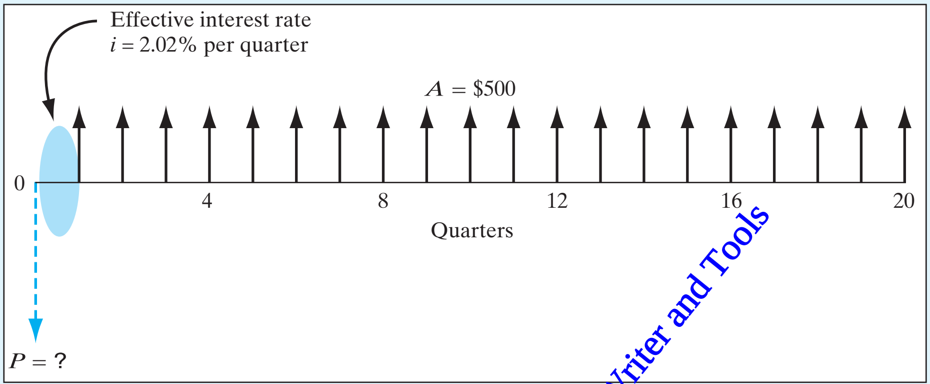


Figure 4.5 A present-worth calculation for an equal payment series with an interest rate of 8% compounded continuously (Example 4.6).

Discussion: A question that is equivalent to the preceding one is "How much do you need to deposit now in a savings account that earns 8% interest compounded continuously so that you can withdraw \$500 at the end of each quarter for five years?" Since the payments are quarterly, we need to compute i per quarter for the equivalence calculations:

$$\begin{aligned} i &= e^{r/K} - 1 = e^{0.08/4} - 1 \\ &= 2.02\% \text{ per quarter} \end{aligned}$$

$$\begin{aligned} N &= (4 \text{ payment periods per year})(5 \text{ years}) \\ &= 20 \text{ quarterly periods.} \end{aligned}$$

SOLUTION

Given: $i = 2.02\%$ per quarter, $N = 20$ quarters, and $A = \$500$ per quarter.

Find: P.

Using the $(P/A, i, N)$ factor with $i = 2.02\%$ and $N = 20$, we find that

$$\begin{aligned} P &= \$500(P/A, 2.02\%, 20) \\ &= \$500(16.3199) \\ &= \$8{,}159.96. \end{aligned}$$

Compounding Is Less Frequent than Payments

The next two examples contain identical parameters for savings situations in which compounding occurs less frequently than payments. However, two different underlying assumptions govern how interest is calculated. In Example 4.7, the assumption is that, whenever a deposit is made, it starts to earn interest. In Example 4.8, the assumption is that the deposits made within a quarter do not earn interest until the end of that quarter. As a result, in Example 4.7 we transform the compounding period to conform to the payment period, and in Example 4.8 we lump several payments together to match the compounding period. In the real world, which assumption is applicable depends on the transactions and the financial institutions involved. The accounting methods used by many firms record cash transactions that occur within a compounding period as if they had occurred at the end of that period. For example, when cash flows occur daily, but the compounding period is monthly, the cash flows within each month are summed (ignoring interest) and treated as a single payment on which interest is calculated.

Note: *In this textbook, we assume that whenever the time point of a cash flow is specified, one cannot move it to another time point without considering the time value of money (i.e., the practice demonstrated in Example 4.7 should be followed).*

EXAMPLE 4.7 Compounding Is Less Frequent than Payments: Effective Interest Rate per Payment Period

Suppose you make \$500 monthly deposits to a tax-deferred retirement plan that pays interest at a rate of 10% compounded quarterly. Compute the balance at the end of 10 years.

SOLUTION

Given: $r = 10\%$ per year, $M = 4$ quarterly compounding periods per year, $K = 12$ payment periods per year, $A = \$500$ per month, $N = 120$ months, and interest is accrued on deposits made during the compounding period.

Find: i, F.

As in the case of Example 4.5, the procedure for noncomparable compounding and payment periods is followed:

1. The parameter values for $M, K,$ and C are

$$M = 4 \text{ compounding periods per year,}$$
$$K = 12 \text{ payment periods per year,}$$
$$C = \tfrac{1}{3} \text{ interest period per payment period.}$$

2. As shown in Figure 4.6, the effective interest rate per payment period is calculated with Eq. (4.2):

$$i = 0.826\% \text{ per month.}$$

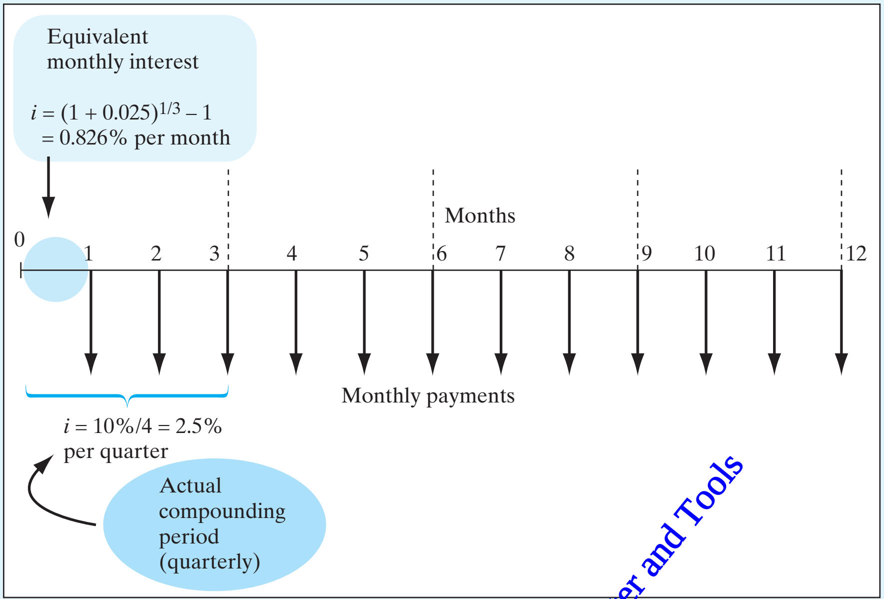


Figure 4.6 Calculation of equivalent monthly interest when the quarterly interest rate is specified (Example 4.7).

3. Find N:

$$N = (12)(10) = 120 \text{ payment periods.}$$

4. Use i and N in the appropriate equivalence formulas (Figure 4.7):

$$F = \$500(F/A, 0.826\%, 120)$$
$$= \$101,907.89.$$

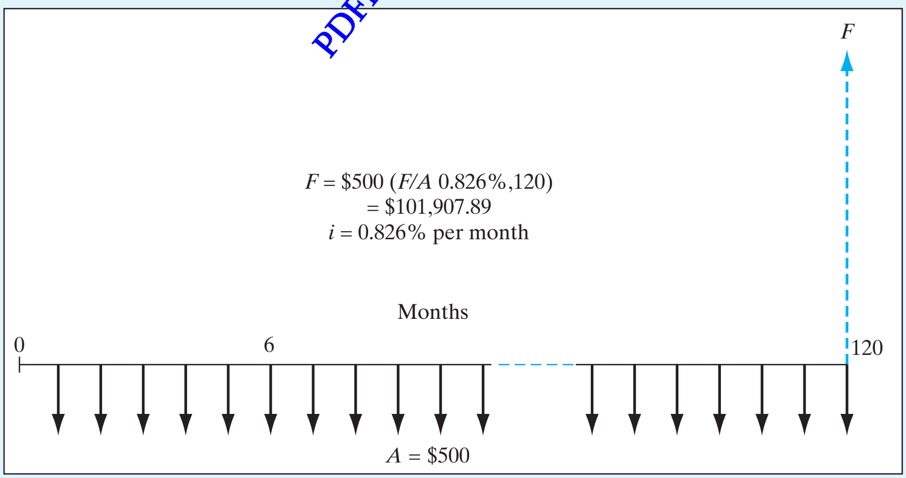


Figure 4.7 Cash flow diagram (Example 4.7).

EXAMPLE 4.8 Compounding Is Less Frequent than Payment: Summing Cash Flows to the End of the Compounding Period

Some financial institutions will not pay interest on funds deposited after the start of the compounding period. To illustrate, consider Example 4.7 again. Suppose that money deposited during a quarter (the compounding period) will not earn any interest (Figure 4.8). Compute what F will be at the end of 10 years.

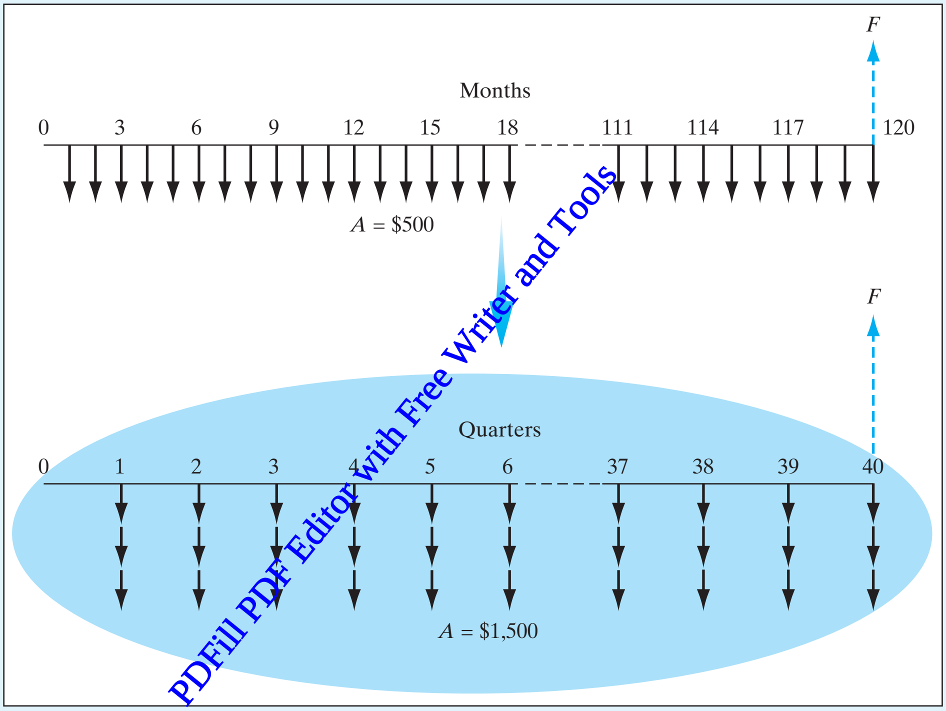


Figure 4.8 Transformed cash flow diagram created by summing monthly cash flows to the end of the quarterly compounding period (Example 4.8).

SOLUTION

Given: Same as for Example 4.7; however, no interest on flow during the compounding period.

Find: F.

In this case, the three monthly deposits during each quarterly period will be placed at the end of each quarter. Then the payment period coincides with the interest period, and we have

$$i = \frac{10\%}{4} = 2.5\% \text{ per quarter,}$$

$$A = 3(\$500) = \$1{,}500 \text{ per quarter,}$$

$$N = 4(10) = 40 \text{ payment periods,}$$

$$F = \$1{,}500(F/A, 2.5\%, 40) = \$101{,}103.83.$$

COMMENTS: In Example 4.8, the balance will be $804.06 less than in Example 4.7, a fact that is consistent with our understanding that increasing the frequency of compounding increases the future value of money. Some financial institutions follow the practice illustrated in Example 4.7. As an investor, you should reasonably ask yourself whether it makes sense to make deposits in an interest-bearing account more frequently than interest is paid. In the interim between interests compounding, you may be tying up your funds prematurely and forgoing other opportunities to earn interest.

Figure 4.9 is a decision chart that allows you to sum up how you can proceed to find the effective interest rate per payment period, given the various possible compounding and interest arrangements.

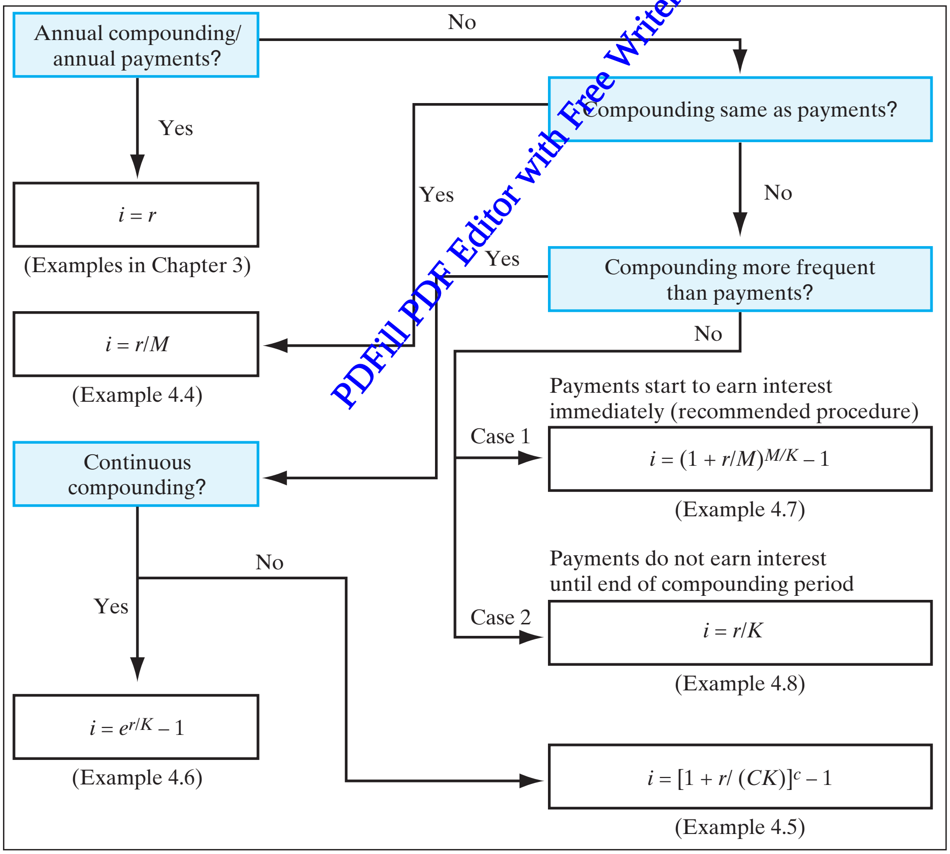


Figure 4.9 A decision flowchart demonstrating how to compute the effective interest rate i per payment period.

4.3 Equivalence Calculations with Continuous Payments

As we have seen so far, interest can be compounded annually, semiannually, monthly, or even continuously. Discrete compounding is appropriate for many financial transactions; mortgages, bonds, and installment loans, which require payments or receipts at discrete times, are good examples. In most businesses, however, transactions occur continuously throughout the year. In these circumstances, we may describe the financial transactions as having a continuous flow of money, for which continuous compounding and discounting are more realistic. This section illustrates how one establishes economic equivalence between cash flows under continuous compounding.

Continuous cash flows represent situations in which money flows continuously and at a known rate throughout a given period. In business, many daily cash flow transactions can be viewed as continuous. An advantage of the continuous-flow approach is that it more closely models the realities of business transactions. Costs for labor, for carrying inventory, and for operating and maintaining equipment are typical examples. Others include capital improvement projects that conserve energy or water or that process steam. Savings on these projects can occur continuously.

4.3.1 Single-Payment Transactions

First we will illustrate how single-payment formulas for continuous compounding and discounting are derived. Suppose that you invested P dollars at a nominal rate of r % interest for N years. If interest is compounded continuously, the effective annual interest is $i = e^r - 1$. The future value of the investment at the end of N years is obtained with the F/P factor by substituting $e^r - 1$ for i:

$$\begin{aligned} F &= P(1 + i)^N \\ &= P(1 + e^r - 1)^N \\ &= Pe^{rN}. \end{aligned}$$

This implies that \$1 invested now at an interest rate of r% compounded continuously accumulates to e^{rN} dollars at the end of N years. Correspondingly, the present value of F due N years from now and discounted continuously at an interest rate of r% is equal to

$$P = Fe^{-rN}.$$

We can say that the present value of \$1 due N years from now and discounted continuously at an annual interest rate of r% is equal to e^{-rN} dollars.

4.3.2 Continuous-Funds Flow

Suppose that an investment's future cash flow per unit of time (e.g., per year) can be expressed by a continuous function ($f(t)$) that can take any shape. Suppose also that the

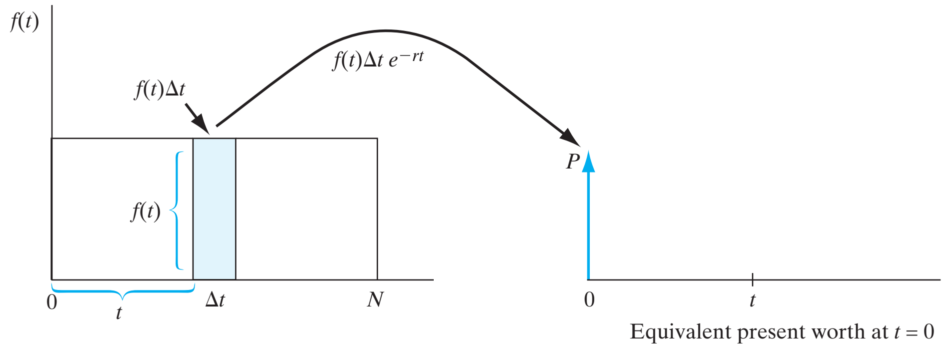


Figure 4.10 Finding an equivalent present worth of a continuous-flow payment function $f(t)$ at a nominal rate of r%.

investment promises to generate cash of $f(t)\Delta t$ dollars between t and $t + \Delta t$, where t is a point in the time interval $0 \leq t \leq N$ (Figure 4.10). If the nominal interest rate is a constant r during this interval, the present value of the cash stream is given approximately by the expression

$$\Sigma(f(t)\Delta t)e^{-rt},$$

where e^{-rt} is the discounting factor that converts future dollars into present dollars. With the project's life extending from 0 to N, we take the summation over all subperiods (compounding periods) in the interval from 0 to N. As the interval is divided into smaller and smaller segments (i.e., as Δt approaches zero), we obtain the expression for the present value by the integral

$$P = \int_0^N f(t)e^{-rt}\,dt. \tag{4.5}$$

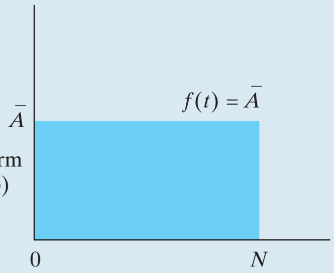
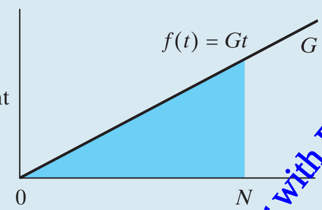
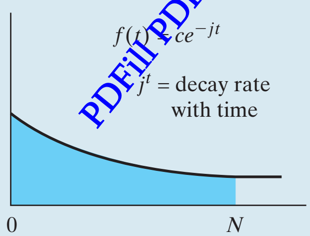
Similarly, the expression for the future value of the cash flow stream is given by the equation

$$F = Pe^{rN} = \int_0^N f(t)e^{r(N-t)}\,dt, \tag{4.6}$$

where $e^{r(N-t)}$ is the compounding factor that converts present dollars into future dollars. It is important to observe that the time unit is the *year*, because the effective interest rate is expressed in terms of a year. Therefore, all time units in equivalence calculations must be converted into years. Table 4.2 summarizes some typical continuous cash functions that can facilitate equivalence calculations.[2]

[2] Chan S. Park and Gunter P. Sharp-Bette, *Advanced Engineering Economics*. New York: John Wiley & Sons, 1990. (Reprinted by permission of John Wiley & Sons, Inc.)

TABLE 4.2 **Summary of Interest Factors for Typical Continuous Cash Flows with Continuous Compounding**

Type of Cash Flow	Cash Flow Function	Parameters Find	Parameters Given	Algebraic Notation	Factor Notation
Uniform (step)		P	$\bar{A}$	$\bar{A}\left[\dfrac{e^{rN}-1}{re^{rN}}\right]$	$(P/\bar{A}, r, N)$
		$\bar{A}$	P	$P\left[\dfrac{re^{rN}}{e^{rN}-1}\right]$	$(\bar{A}/P, r, N)$
		F	$\bar{A}$	$\bar{A}\left[\dfrac{e^{rN}-1}{r}\right]$	$(F/\bar{A}, r, N)$
		$\bar{A}$	F	$F\left[\dfrac{r}{e^{rN}-1}\right]$	$(\bar{A}/P, r, N)$
Gradient (ramp)		P	G	$\dfrac{G}{r^2}(1 - e^{-rN}) - \dfrac{G}{r}(Ne^{-rN})$	
Decay		P	c, j	$\dfrac{c}{r+j}(1 - e^{-(r+j)N})$	

EXAMPLE 4.9 Comparison of Daily Flows and Daily Compounding with Continuous Flows and Continuous Compounding

Consider a situation in which money flows daily. Suppose you own a retail shop and generate \$200 cash each day. You establish a special business account and deposit your daily cash flows in an account for 15 months. The account earns an interest rate of 6%. Compare the accumulated cash values at the end of 15 months, assuming

(a) Daily compounding and
(b) Continuous compounding.

SOLUTION

(a) With daily compounding:

Given: $A = \$200$ per day $r = 6\%$ per year, $M = 365$ compounding periods per year, and $N = 455$ days.
Find: F.

Assuming that there are 455 days in the 15-month period, we find that

$$\begin{aligned} i &= 6\%/365 \\ &= 0.01644\% \text{ per day}, \\ N &= 455 \text{ days}. \end{aligned}$$

The balance at the end of 15 months will be

$$\begin{aligned} F &= \$200(F/A, 0.01644\%, 455) \\ &= \$200(472.4095) \\ &= \$94{,}482. \end{aligned}$$

(b) With continuous compounding:

Now we approximate this discrete cash flow series by a uniform continuous cash flow function as shown in Figure 4.11. In this situation, an amount flows at the rate of $\overline{A}$ per year for N years.

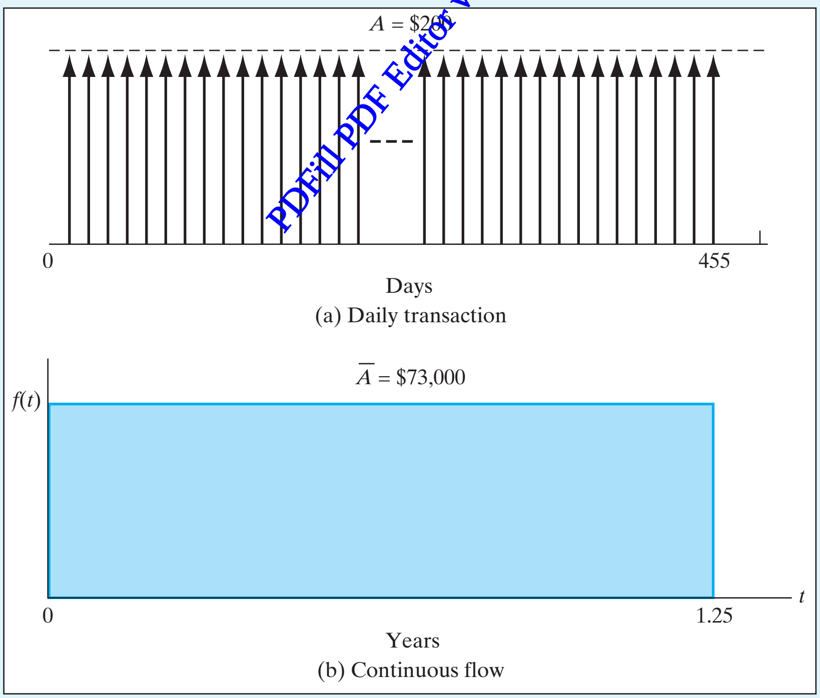


Figure 4.11 Comparison between daily transaction and continuous-funds flow transaction (Example 4.9).

Note: *Our time unit is a year*. Thus, a 15-month period is 1.25 years. Then the cash flow function is expressed as

$$\begin{aligned} f(t) &= \overline{A}, 0 \leq t \leq 1.25 \\ &= \$200(365) \\ &= \$73{,}000 \text{ per year.} \end{aligned}$$

Given: $\overline{A} = \$73{,}000$ per year, $r = 6\%$ per year, compounded continuously, and $N = 1.25$ years.
Find: F.
Substituting these values back into Eq. (4.6) yields

$$\begin{aligned} F &= \int_0^{1.25} 73{,}000 e^{0.06(1.25-t)}\,dt \\ &= \$73{,}000\left[\frac{e^{0.075}-1}{0.06}\right] \\ &= \$94{,}759. \end{aligned}$$

The factor in the bracket is known as the **funds flow compound amount factor** and is designated $(F/\overline{A}, r, N)$ as shown in Table 4.2. Notice that the difference between the two methods is only \$277 (less than 0.3%).

COMMENTS: As shown in this example, the differences between discrete daily compounding and continuous compounding have no practical significance in most cases. Consequently, as a mathematical convenience, instead of assuming that money flows in discrete increments at the end of each day, we could assume that money flows continuously at a uniform rate during the period in question. This type of cash flow assumption is common practice in the chemical industry.

4.4 Changing Interest Rates

Up to this point, we have assumed a constant interest rate in our equivalence calculations. When an equivalence calculation extends over several years, more than one interest rate may be applicable to properly account for the time value of money. That is to say, over time, interest rates available in the financial marketplace fluctuate, and a financial institution that is committed to a long-term loan may find itself in the position of losing the opportunity to earn higher interest because some of its holdings are tied up in a lower interest loan. The financial institution may attempt to protect itself from such lost earning opportunities by building gradually increasing interest rates into a long-term loan at the outset. Adjustable-rate mortgage (ARM) loans are perhaps the most common examples of variable interest rates. In this section, we will consider variable interest rates in both single payments and a series of cash flows.

4.4.1 Single Sums of Money

To illustrate the mathematical operations involved in computing equivalence under changing interest rates, first consider the investment of a single sum of money, P, in a savings account for N interest periods. If i_n denotes the interest rate appropriate during period n, then the future worth equivalent for a single sum of money can be expressed as

$$F = P(1 + i_1)(1 + i_2)\cdots(1 + i_{N-1})(1 + i_N), \tag{4.7}$$

and solving for P yields the inverse relation

$$P = F[(1 + i_1)(1 + i_2)\cdots(1 + i_{N-1})(1 + i_N)]^{-1}. \tag{4.8}$$

EXAMPLE 4.10 Changing Interest Rates with a Lump-Sum Amount

Suppose you deposit \$2,000 in an individual retirement account (IRA) that pays interest at 6% compounded monthly for the first two years and 9% compounded monthly for the next three years. Determine the balance at the end of five years (Figure 4.12).

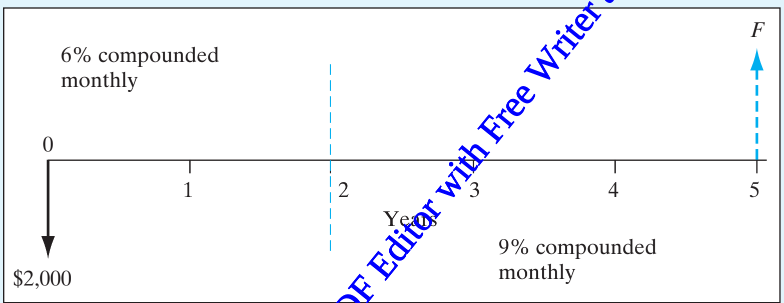


Figure 4.12 Changing interest rates (Example 4.10).

SOLUTION

Given: $P = \$2{,}000$, $r = 6\%$ per year for first two years, 9% per year for last three years, $M = 4$ compounding periods per year, $N = 20$ quarters.
Find: F.

We will compute the value of F in two steps. First we will compute the balance B_2 in the account at the end of two years. With 6% compounded quarterly, we have

$$\begin{aligned} i &= 6\%/12 = 0.5\% \\ N &= 12(2) = 24 \text{ months} \\ B_2 &= \$2{,}000(F/P, 0.5\%, 24) \\ &= \$2{,}000(1.12716) \\ &= \$2{,}254. \end{aligned}$$

Since the fund is not withdrawn, but reinvested at 9% compounded monthly, as a second step we compute the final balance as follows:

$$\begin{aligned} i &= 9\%/12 = 0.75\% \\ N &= 12(3) = 36 \text{ months} \\ F &= B_2(F/P, 0.75\%, 36) \\ &= \$2{,}254(1.3086) \\ &= \$2{,}950. \end{aligned}$$

4.4.2 Series of Cash Flows

The phenomenon of changing interest rates can easily be extended to a series of cash flows. In this case, the present worth of a series of cash flows can be represented as

$$\begin{aligned} P = {} & A_1(1 + i_1)^{-1} + A_2[(1 + i_1)^{-1}(1 + i_2)^{-1}] + \ldots \\ & + A_N[(1 + i_1)^{-1}(1 + i_2)^{-1} \ldots (1 + i_N)^{-1}]. \end{aligned} \tag{4.9}$$

The future worth of a series of cash flows is given by the inverse of Eq. (4.9):

$$\begin{aligned} F = {} & A_1[(1 + i_2)(1 + i_3) \ldots (1 + i_N)] \\ & + A_2[(1 + i_3)(1 + i_4) \ldots (1 + i_N)] + \ldots + A_N. \end{aligned} \tag{4.10}$$

The uniform series equivalent is obtained in two steps. First, the present-worth equivalent of the series is found from Eq. (4.9). Then A is obtained after establishing the following equivalence equation:

$$\begin{aligned} P = {} & A(1 + i_1)^{-1} + A[(1 + i_1)^{-1}(1 + i_2)^{-1}] + \ldots \\ & + A[(1 + i_1)^{-1}(1 + i_2)^{-1} \ldots (1 + i_N)^{-1}]. \end{aligned} \tag{4.11}$$

EXAMPLE 4.11 Changing Interest Rates with Uneven Cash Flow Series

Consider the cash flow in Figure 4.13 with the interest rates indicated, and determine the uniform series equivalent of the cash flow series.

DISCUSSION: In this problem and many others, the easiest approach involves collapsing the original flow into a single equivalent amount, for example, at time zero, and then converting the single amount into the final desired form.

SOLUTION

Given: Cash flows and interest rates as shown in Figure 4.13; $N = 3$.

Find: A.

SUMMARY

- Interest is most frequently quoted by financial institutions as an **annual percentage rate**, or **APR**. However, compounding frequently occurs more often than once annually, and the APR does not account for the effect of this more frequent compounding. The situation leads to the distinction between nominal and effective interest:
- **Nominal interest** is a stated rate of interest for a given period (usually a year).
- **Effective interest** is the actual rate of interest, which accounts for the interest amount accumulated over a given period. The **effective rate** is related to the APR by the equation

$$i = \left(1 + \frac{r}{M}\right)^M - 1,$$

where r is the APR, M is the number of compounding periods, and i is the effective interest rate.

In any equivalence problem, the interest rate to use is the effective interest rate per payment period, or

$$i = \left[1 + \frac{r}{CK}\right]^C - 1,$$

where C is the number of interest periods per payment period, K is the number of payment periods per year, and r/K is the nominal interest rate per payment period. Figure 4.9 outlines the possible relationships between compounding and payment periods and indicates which version of the effective-interest formula to use.

- The equation for determining the effective interest of continuous compounding is

$$i = e^{r/K} - 1.$$

The difference in accumulated interest between continuous compounding and very frequent compounding $(M > 50)$ is minimal.

- Cash flows, as well as compounding, can be continuous. Table 4.2 shows the interest factors to use for continuous cash flows with continuous compounding.
- Nominal (and hence effective) interest rates may fluctuate over the life of a cash flow series. Some forms of home mortgages and bond yields are typical examples.
- **Amortized loans** are paid off in equal installments over time, and most of these loans have interest that is compounded monthly.
- Under a typical **add-on loan**, the lender precalculates the total simple interest amount and adds it to the principal. The principal and this precalculated interest amount are then paid together in equal installments.
- The term **mortgage** refers to a special type of loan for buying a piece of property, such as a house or a commercial building. The cost of the mortgage will depend on many factors, including the amount and term of the loan and the frequency of payments, as well as points and fees.
- Two types of mortgages are common: fixed-rate mortgages and variable-rate mortgages. Fixed-rate mortgages offer loans whose interest rates are fixed over the period

of the contract, whereas variable-rate mortgages offer interest rates that fluctuate with market conditions. In general, the initial interest rate is lower for variable-rate mortgages, as the lenders have the flexibility to adjust the cost of the loans over the period of the contract.

- Allocating one's assets is simply a matter of answering the following question: "Given my personal tolerance for risk and my investment objectives, what percentage of my assets should be allocated for **growth**, what percentage for **income**, and what percentage for **liquidity**?"
- You can determine the **expected rate of return** on a portfolio by computing the weighted average of the returns on each investment.
- You can determine the **expected risk** of a portfolio by computing the weighted average of the volatility of each investment.
- All other things being equal, if the expected returns are approximately the same, choose the portfolio with the lowest expected risk.
- All other things being equal, if the expected risk is about the same, choose the portfolio with the highest expected return.
- **Asset-backed bonds:** If a company backs its bonds with specific pieces of property, such as buildings, we call these types of bonds **mortgage bonds**, which indicate the terms of repayment and the particular assets pledged to the bondholders in case of default. It is much more common, however, for a corporation simply to pledge its overall assets. A **debenture bond** represents such a promise.
- **Par value:** Individual bonds are normally issued in even denominations of $1,000 or multiples of $1,000. The stated face value of an individual bond is termed the **par value**.
- **Maturity date:** Bonds generally have a specified **maturity** date on which the par value is to be repaid.
- **Coupon rate:** We call the interest paid on the par value of a bond the **annual coupon rate**. The time interval between interest payments could be of any duration, but a semiannual period is the most common.
- **Discount or premium bond:** A bond that sells below its par value is called a **discount bond**. When a bond sells above its par value, it is called a **premium bond**.

PROBLEMS

Nominal and Effective Interest Rates

4.1 If your credit card calculates interest based on 12.5% APR,
   (a) What are your monthly interest rate and annual effective interest rate?
   (b) If you current outstanding balance is $2,000 and you skip payments for two months, what would be the total balance two months from now?

4.2 A department store has offered you a credit card that charges interest at 1.05% per month, compounded monthly. What is the nominal interest (annual percentage) rate for this credit card? What is the effective annual interest rate?

4.3 A local bank advertised the following information: Interest 6.89%—effective annual yield 7.128%. No mention was made of the interest period in the advertisement. Can you figure out the compounding scheme used by the bank?

4.4 College Financial Sources, which makes small loans to college students, offers to lend \$500. The borrower is required to pay \$400 at the end of each week for 16 weeks. Find the interest rate per week. What is the nominal interest rate per year? What is the effective interest rate per year?

4.5 A financial institution is willing to lend you \$40. However, \$450 is repaid at the end of one week.

(a) What is the nominal interest rate?

(b) What is the effective annual interest rate?

4.6 The Cadillac Motor Car Company is advertising a 24-month lease of a Cadillac Deville for \$520, payable at the beginning of each month. The lease requires a \$2,500 down payment, plus a \$500 refundable security deposit. As an alternative, the company offers a 24-month lease with a single up-front payment of \$12,780, plus a \$500 refundable security deposit. The security deposit will be refunded at the end of the 24-month lease. Assuming an interest rate of 6%, compounded monthly, which lease is the preferred one?

4.7 As a typical middle-class consumer, you are making monthly payments on your home mortgage (9% annual interest rate), car loan (12%), home improvement loan (14%), and past-due charge accounts (18%). Immediately after getting a \$100 monthly raise, your friendly mutual fund broker tries to sell you some investment funds with a guaranteed return of 10% per year. Assuming that your only other investment alternative is a savings account, should you buy?

Compounding More Frequent than Annually

4.8 A loan company offers money at 1.8% per month, compounded monthly.

(a) What is the nominal interest rate?

(b) What is the effective annual interest rate?

(c) How many years will it take an investment to triple if interest is compounded monthly?

(d) How many years will it take an investment to triple if the nominal rate is compounded continuously?

4.9 Suppose your savings account pays 9% interest compounded quarterly. If you deposit \$10,000 for one year, how much would you have?

4.10 What will be the amount accumulated by each of these present investments?

(a) \$5,635 in 10 years at 5% compounded semiannually.

(b) \$7,500 in 15 years at 6% compounded quarterly.

(c) \$38,300 in 7 years at 9% compounded monthly.

4.11 How many years will it take an investment to triple if the interest rate is 9% compounded

(a) Quarterly? (b) Monthly? (c) Continuously?

4.12 A series of equal quarterly payments of \$5,000 for 12 years is equivalent to what present amount at an interest rate of 9% compounded

(a) Quarterly? (b) Monthly? (c) Continuously?

4.13 What is the future worth of an equal payment series of \$3,000 each quarter for five years if the interest rate is 8% compounded continuously?

4.14 Suppose that \$2,000 is placed in a bank account at the end of each quarter over the next 15 years. What is the future worth at the end of 15 years when the interest rate is 6% compounded

(a) Quarterly? (b) Monthly? (c) Continuously?

4.15 A series of equal quarterly deposits of \$1,000 extends over a period of three years. It is desired to compute the future worth of this quarterly deposit series at 12% compounded monthly. Which of the following equations is correct?

(a) $F = 4(\$1{,}000)(F/A, 12\%, 3)$. (b) $F = \$1{,}000(F/A, 3\%, 12)$.

(c) $F = \$1{,}000(F/A, 1\%, 12)$. (d) $F = \$1{,}000(F/A, 3.03\%, 12)$.

4.16 If the interest rate is 8.5% compounded continuously, what is the required quarterly payment to repay a loan of \$12,000 in five years?

4.17 What is the future worth of a series of equal monthly payments of \$2,500 if the series extends over a period of eight years at 12% interest compounded

(a) Quarterly? (b) Monthly? (c) Continuously?

4.18 Suppose you deposit \$500 at the end of each quarter for five years at an interest rate of 8% compounded monthly. What equal end-of-year deposit over the five years would accumulate the same amount at the end of the five years under the same interest compounding? To answer the question, which of the following is correct?

(a) $A = [\$500(F/A, 2\%, 20)] \times (A/F, 8\%, 5)$.

(b) $A = \$500(F/A, 2.013\%, 4)$.

(c) $A = \$500\left(F/A, \frac{8\%}{12}, 20\right) \times (A/F, 8\%, 5)$.

(d) None of the above.

4.19 A series of equal quarterly payments of \$2,000 for 15 years is equivalent to what future lump-sum amount at the end of 10 years at an interest rate of 8% compounded continuously?

4.20 What will be the required quarterly payment to repay a loan of \$32,000 in five years, if the interest rate is 7.8% compounded continuously?

4.21 A series of equal quarterly payments of \$4,000 extends over a period of three years. What is the present worth of this quarterly payment series at 8.75% interest compounded continuously?

4.22 What is the future worth of the following series of payments?

(a) \$6,000 at the end of each six-month period for 6 years at 6% compounded semiannually.

(b) \$42,000 at the end of each quarter for 12 years at 8% compounded quarterly.

(c) \$75,000 at the end of each month for 8 years at 9% compounded monthly.

4.23 What equal series of payments must be paid into a sinking fund to accumulate the following amount?

(a) \$21,000 in 10 years at 6.45% compounded semiannually when payments are semiannual.

(b) \$9,000 in 15 years at 9.35% compounded quarterly when payments are quarterly.

(c) \$24,000 in 5 years at 6.55% compounded monthly when payments are monthly.

4.24 You have a habit of drinking a cup of Starbucks coffee (\$2.50 a cup) on the way to work every morning. If, instead, you put the money in the bank for 30 years, how much would you have at the end of that time, assuming that your account earns

5% interest compounded *daily*? Assume also that you drink a cup of coffee every day, including weekends.

4.25 John Jay is purchasing a $24,000 automobile, which is to be paid for in 48 monthly installments of $543.35. What effective annual interest is he paying for this financing arrangement?

4.26 A loan of $12,000 is to be financed to assist in buying an automobile. On the basis of monthly compounding for 42 months, the end-of-the-month equal payment is quoted as $445. What nominal interest rate in percentage is being charged?

4.27 Suppose a young newlywed couple is planning to buy a home two years from now. To save the down payment required at the time of purchasing a home worth $220,000 (let's assume that the down payment is 10% of the sales price, or $22,000), the couple decides to set aside some money from each of their salaries at the end of every month. If each of them can earn 6% interest (compounded monthly) on his or her savings, determine the equal amount this couple must deposit each month until the point is reached where the couple can buy the home.

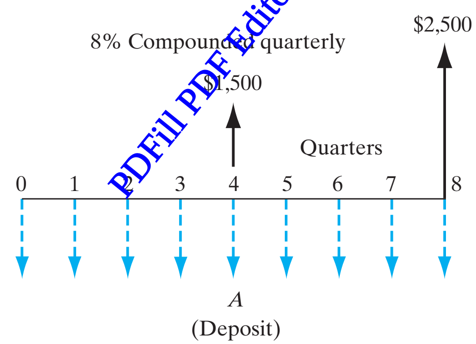
4.28 What is the present worth of the following series of payments?

(a) $1,500 at the end of each six-month period for 12 years at 8% compounded semiannually.

(b) $2,500 at the end of each quarter for 8 years at 8% compounded quarterly.

(c) $3,800 at the end of each month for 5 years at 9% compounded monthly.

4.29 What is the amount of the quarterly deposits A such that you will be able to withdraw the amounts shown in the cash flow diagram if the interest rate is 8% compounded quarterly?



4.30 Georgi Rostov deposits $15,000 in a savings account that pays 6% interest compounded monthly. Three years later, he deposits $14,000. Two years after the $14,000 deposit, he makes another deposit in the amount of $12,500. Four years after the $12,500 deposit, half of the accumulated funds is transferred to a fund that pays 8% interest compounded quarterly. How much money will be in each account six years after the transfer?

4.31 A man is planning to retire in 25 years. He wishes to deposit a regular amount every three months until he retires, so that, beginning one year following his retirement, he will receive annual payments of $60,000 for the next 10 years. How much must he deposit if the interest rate is 6% compounded quarterly?

4.32 You borrowed \$15,000 for buying a new car from a bank at an interest rate of 12% compounded monthly. This loan will be repaid in 48 equal monthly installments over four years. Immediately after the 20th payment, you desire to pay the remainder of the loan in a single payment. Compute this lump-sum amount of that time.

4.33 A building is priced at \$125,000. If a down payment of \$25,000 is made and a payment of \$1,000 every month thereafter is required, how many months will it take to pay for the building? Interest is charged at a rate of 9% compounded monthly.

4.34 You obtained a loan of \$20,000 to finance an automobile. Based on monthly compounding over 24 months, the end-of-the-month equal payment was figured to be \$922.90. What APR was used for this loan?

4.35 *The Engineering Economist* (a professional journal) offers three types of subscriptions, payable in advance: one year at \$66, two years at \$120, and three years at \$160. If money can earn 6% interest compounded monthly, which subscription should you take? (Assume that you plan to subscribe to the journal over the next three years.)

4.36 A couple is planning to finance its three-year-old son's college education. Money can be deposited at 6% compounded quarterly. What quarterly deposit must be made from the son's 3rd birthday to his 18th birthday to provide \$50,000 on each birthday from the 18th to the 21st? (Note that the last deposit is made on the date of the first withdrawal.)

4.37 Sam Salvetti is planning to retire in 15 years. Money can be deposited at 8% compounded quarterly. What quarterly deposit must be made at the end of each quarter until Sam retires so that he can make a withdrawal of \$25,000 semiannually over the first five years of his retirement? Assume that his first withdrawal occurs at the end of six months after his retirement.

4.38 Michelle Hunter received \$250,000 from an insurance company after her husband's death. Michelle wants to deposit this amount in a savings account that earns interest at a rate of 6% compounded monthly. Then she would like to make 120 equal monthly withdrawals over the 10-year period such that, when she makes the last withdrawal, the savings account will have a balance of zero. How much can she withdraw each month?

4.39 Anita Tahani, who owns a travel agency, bought an old house to use as her business office. She found that the ceiling was poorly insulated and that the heat loss could be cut significantly if 6 inches of foam insulation were installed. She estimated that with the insulation, she could cut the heating bill by \$40 per month and the air-conditioning cost by \$25 per month. Assuming that the summer season is three months (June, July, and August) of the year and that the winter season is another three months (December, January, and February) of the year, how much can Anita spend on insulation if she expects to keep the property for five years? Assume that neither heating nor air-conditioning would be required during the fall and spring seasons. If she decides to install the insulation, it will be done at the beginning of May. Anita's interest rate is 9% compounded monthly.

Continuous Payments with Continuous Compounding

4.40 A new chemical production facility that is under construction is expected to be in full commercial operation 1 year from now. Once in full operation, the facility will generate \$63,000 cash profit daily over the plant's service life of 12 years.

Determine the equivalent present worth of the future cash flows generated by the facility at the beginning of commercial operation, assuming

(a) 12% interest compounded daily, with the daily flows.

(b) 12% interest compounded continuously, with the daily flow series approximated by a uniform continuous cash flow function.

Also, compare the difference between (a) discrete (daily) and (b) continuous compounding.

4.41 Income from a project is expected to decline at a constant rate from an initial value of \$500,000 at time 0 to a final value of \$40,000 at the end of year 3. If interest is compounded continuously at a nominal annual rate of 11%, determine the present value of this continuous cash flow.

4.42 A sum of \$80,000 will be received uniformly over a five-year period beginning two years from today. What is the present value of this deferred-funds flow if interest is compounded continuously at a nominal rate of 9%?

4.43 A small chemical company that produces an epoxy resin expects its production volume to decay exponentially according to the relationship

$$y_t = 5e^{-0.25t},$$

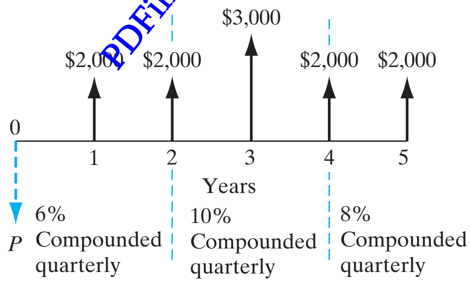
where y_t is the production rate at time t. Simultaneously, the unit price is expected to increase linearly over time at the rate

$$u_t = \$55(1 + 0.09t).$$

What is the expression for the present worth of sales revenues from $t = 0$ to $t = 20$ at 12% interest compounded continuously?

Changing Interest Rates

4.44 Consider the accompanying cash flow diagram, which represents three different interest rates applicable over the five-year time span shown.



(a) Calculate the equivalent amount P at the present time.

(b) Calculate the single-payment equivalent to F at $n = 5$.

(c) Calculate the equal-payment-series cash flow A that runs from $n = 1$ to $n = 5$.

4.45 Consider the cash flow transactions depicted in the accompanying cash flow diagram, with the changing interest rates specified.

(a) What is the equivalent present worth? (In other words, how much do you have to deposit now so that you can withdraw \$300 at the end of year 1, \$300 at the end of year 2, \$500 at the end of year 3, and \$500 at the end of year 4?)

(b) What is the single effective annual interest rate over four years?

CHAPTER FIVE

Present-Worth Analysis

Parking Meters Get Smarter[1] —Wireless Technology Turns Old-Fashioned Coin-Operated Device into a Sophisticated Tool for Catching Scofflaws and Raising Cash Technology is taking much of the fun out of finding a place to park the car:

- In Pacific Grove, California, parking meters "know" when a car pulls out of the spot and quickly reset to zero—eliminating drivers' little joy of parking for free on someone else's quarters.
- In Montreal, when cars stay past their time limit, meters send real-time alerts to an enforcement officer's handheld device, reducing the number of people needed to monitor parking spaces—not to mention drivers' chances of getting away with violations.
- In Aspen, Colorado, wireless "in-car" meters may eliminate the need for curbside parking meters altogether: They dangle from the rearview mirror inside the car, ticking off a prepaid time.

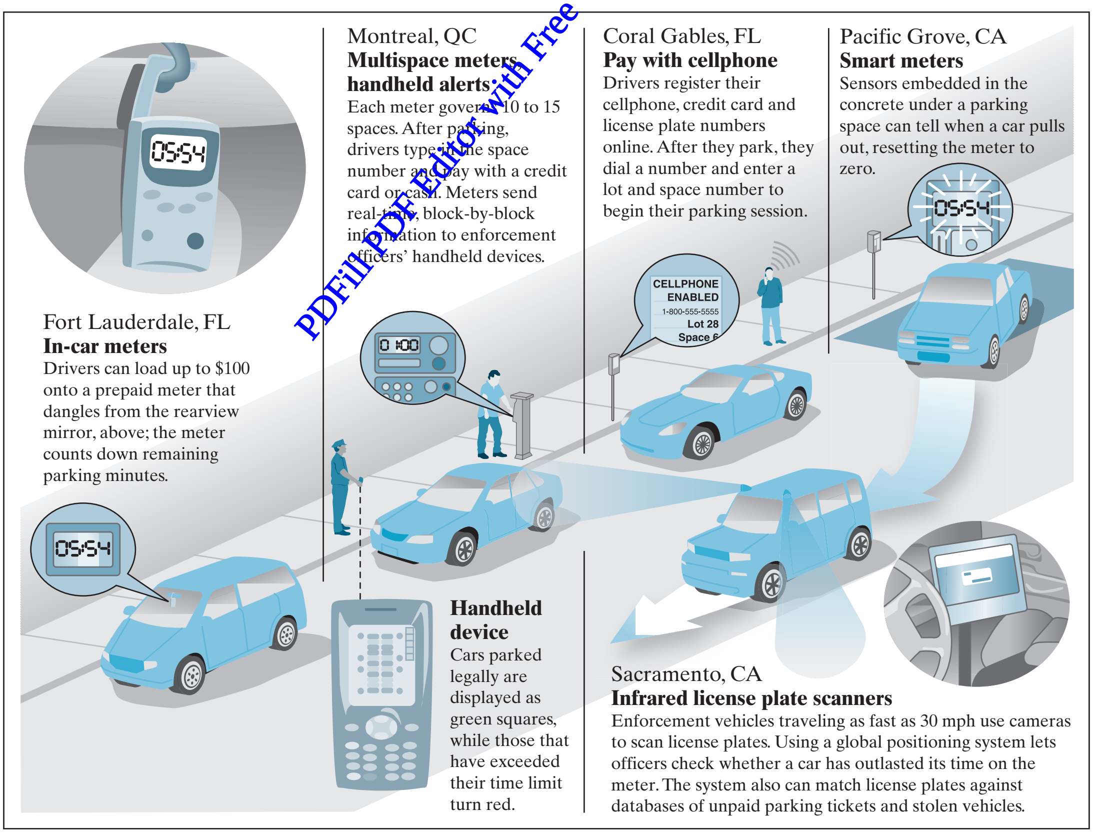
Now, in cities from New York to Seattle, the door is open to a host of wireless technologies seeking to improve the parking meter even further. Chicago and Sacramento, California, among others, are equipping enforcement vehicles with infrared cameras capable of scanning license plates even at 30 miles an hour. Using a global positioning system, the cameras can tell which individual cars have parked too long in a two-hour parking zone. At a cost of $75,000 a camera, the system is an expensive upgrade of the old method of chalking tires and then coming back two hours later to see if the car has moved.

The camera system, supplied by Canada's Autovu Technologies, also helps identify scofflaws and stolen vehicles, by linking to a database of unpaid tickets and auto thefts. Sacramento bought three cameras in August, and since then its practice of "booting," or immobilizing, cars

[1] Christopher Conkey, staff reporter of *The Wall Street Journal*, June 30, 2005, p. B1.

with a lot of unpaid tickets has increased sharply. Revenue is soaring, too. According to Howard Chan, Sacramento's parking director, Sacramento booted 189 cars and took in parking revenue of $169,000 for the fiscal year ended in June 2004; for fiscal 2005, the city expects to boot 805 cars and take in more than $475,000.

In downtown Montreal, more than 400 "pay-by-space" meters, each covering 10 to 15 spaces, are a twist on regular multispace meters. Motorists park, then go to the meter to type in the parking-space number, and pay by card or coin. These meters, which cost about $9,000 each, identify violators in real time for enforcement officers carrying handheld devices: A likeness of the block emerges on the screen, and cars parked illegally show up in red.



Parking czars in municipalities across the country are starting to realize parking meters' original goals: generating revenue and creating a continuous turnover of parking spaces on city streets. Clearly, their main question is "Would there be enough new revenues from installing the expensive parking monitoring devices?" or "How many devices could be installed to maximize the revenue streams?" From the device manufacturer's point of view, the question is "Would there be enough demand for their products to justify the investment required in new facilities and marketing?" If the manufacturer decides to go ahead and market the products, but the actual demand is far less than its forecast or the adoption of the technology is too slow, what would be the potential financial risk?

In Chapters 3 and 4, we presented the concept of the time value of money and developed techniques for establishing cash flow equivalence with compound-interest factors. That background provides a foundation for accepting or rejecting a capital investment: the economic evaluation of a project's desirability. The forthcoming coverage of investment worth in this chapter will allow us to go a step beyond merely accepting or rejecting an investment to making comparisons of alternative investments. We will learn how to compare alternatives on an equal basis and select the wisest alternative from an economic standpoint.

The three common measures based on cash flow equivalence are (1) equivalent present worth (PW), (2) equivalent future worth (FW), and (3) equivalent annual worth (AE). Present worth represents a measure of future cash flow relative to the time point "now," with provisions that account for earning opportunities. Future worth is a measure of cash flow at some future planning horizon and offers a consideration of the earning opportunities of intermediate cash flows. Annual worth is a measure of cash flow in terms of equivalent equal payments made on an annual basis.

Our treatment of measures of investment worth is divided into three chapters. Chapter 5 begins with a consideration of the payback period, a project screening tool that was the first formal method used to evaluate investment projects. Then it introduces two measures based on fundamental cash flow equivalence techniques: present-worth and future-worth analysis. Because the annual-worth approach has many useful engineering applications related to estimating the unit cost, Chapter 6 is devoted to annual cash flow analysis. Chapter 7 presents measures of investment worth based on yield—measures known as rate-of-return analysis.

We must also recognize that one of the most important parts of the capital budgeting process is the estimation of relevant cash flows. For all examples in this chapter, and those in Chapters 6 and 7, however, net cash flows can be viewed as before-tax values or after-tax values for which tax effects have been recalculated. Since some organizations (e.g., governments and nonprofit organizations) are not subject to tax, the before-tax situation provides a valid base for this type of economic evaluation. Taking this after-tax view will allow us to focus on our main area of concern: the economic evaluation of investment projects. The procedures for determining after-tax net cash flows in taxable situations are developed in Chapter 10.

CHAPTER LEARNING OBJECTIVES

After completing this chapter, you should understand the following concepts:

- How firms screen potential investment opportunities.
- How firms evaluate the profitability of an investment project by considering the time value of money.
- How firms compare mutually exclusive investment opportunities.

5.1 Describing Project Cash Flows

In Chapter 1, we described many engineering economic decision problems, but we did not provide suggestions on how to solve them. What do all engineering economic decision problems have in common? The answer is that they all involve two dissimilar types of amounts. First, there is the investment, which is usually made in a lump sum at the beginning of the project. Although not literally made "today," the investment is made at a specific point in time that, for analytical purposes, is called today, or time 0. Second, there is a stream of cash benefits that are expected to result from the investment over some years in the future.

5.1.1 Loan versus Project Cash Flows

An investment made in a fixed asset is similar to an investment made by a bank when it lends money. The essential characteristic of both transactions is that funds are committed today in the expectation of their earning a return in the future. In the case of the bank loan, the future return takes the form of interest plus repayment of the principal and is known as the **loan cash flow**. In the case of the fixed asset, the future return takes the form of cash generated by productive use of the asset. As shown in Figure 5.1, the representation of these future earnings, along with the capital expenditures and annual expenses (such as wages, raw materials, operating costs, maintenance costs, and income taxes), is the **project cash flow**. This similarity between the loan cash flow and the project cash flow brings us to an important conclusion: We can use the same equivalence techniques developed in Chapter 3 to measure economic worth. Example 5.1 illustrates a typical procedure for obtaining a project's cash flows.

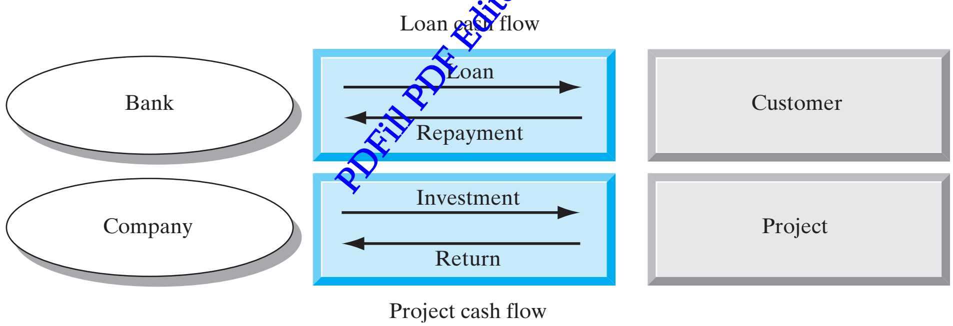


Figure 5.1 A bank loan versus an investment project.

EXAMPLE 5.1 Identifying Project Cash Flows

XL Chemicals is thinking of installing a computer process control system in one of its process plants. The plant is used about 40% of the time, or 3,500 operating hours per year, to produce a proprietary demulsification chemical. During the remaining 60% of the time, it is used to produce other specialty chemicals. Annual production of the demulsification chemical amounts to 30,000 kilograms, and it sells for $15 per kilogram. The proposed computer process control system will cost $650,000 and is expected to provide the following specific benefits in the production of the demulsification chemical:

- First, the selling price of the product could be increased by \$2 per kilogram because the product will be of higher purity, which translates into better demulsification.
- Second, production volumes will increase by 4,000 kilograms per year as a result of higher reaction yields, without any increase in the quantities of raw material or in production time.
- Finally, the number of process operators can be reduced by one per shift, which represents a savings of \$25 per hour. The new control system would result in additional maintenance costs of \$53,000 per year and has an expected useful life of eight years.

Although the system is likely to provide similar benefits in the production of the other specialty chemicals manufactured in the process plant, these benefits have not yet been quantified.

SOLUTION

Given: The preceding cost and benefit information.

Find: Net cash flow in each year over the life of the new system.

Although we could assume that similar benefits are derivable from the production of the other specialty chemicals, let's restrict our consideration to the demulsification chemical and allocate the full initial cost of the control system and the annual maintenance costs to this chemical. (Note that you could logically argue that only 40% of these costs belong to this production activity.) The gross benefits are the additional revenues realized from the increased selling price and the extra production, as well as the cost savings resulting from having one fewer operator:

- Revenues from the price increases are 30,000 kilograms per year × \$2/kilogram, or \$60,000 per year. The added production volume at the new pricing adds revenues of 4,000 kilograms per year × \$17 per kilogram, or \$68,000 per year.
- The elimination of one operator results in an annual savings of 3,500 operating hours per year × \$25 per hour, or \$87,500 per year.
- The net benefits in each of the eight years that make up the useful lifetime of the new system are the gross benefits less the maintenance costs: $(\$60{,}000 + \$68{,}000 + \$87{,}500 - \$53{,}000) = \$162{,}500$ per year.

Now we are ready to summarize a cash flow table as follows:

Year (n)	Cash Inflows (Benefits)	Cash Outflows (Costs)	Net Cash Flows
0	0	\$650,000	−\$650,000
1	215,500	53,000	162,500
2	215,500	53,000	162,500
⋮	⋮	⋮	⋮
8	215,500	53,000	162,500

COMMENTS: If the company purchases the computer process control system for \$650,000 now, it can expect an annual savings of \$162,500 for eight years. (Note that these savings occur in discrete lumps at the ends of years.) We also considered only the benefits associated with the production of the demulsification chemical. We could also have quantified some benefits attributable to the production of the other chemicals from this plant. Suppose that the demulsification chemical benefits alone justify the acquisition of the new system. Then it is obvious that, had we considered the benefits deriving from the other chemicals as well, the acquisition of the system would have been even more clearly justified.

We draw a cash flow diagram of this situation in Figure 5.2. Assuming that these cost savings and cash flow estimates are correct, should management give the go-ahead for installation of the system? If management decides not to purchase the computer control system, what should it do with the \$650,000 (assuming that it has this amount in the first place)? The company could buy \$650,000 of Treasury bonds, or it could invest the amount in other cost-saving projects. How would the company compare cash flows that differ both in timing and amount for the alternatives it is considering? This is an extremely important question, because virtually every engineering investment decision involves a comparison of alternatives. Indeed, these are the types of questions this chapter is designed to help you answer.

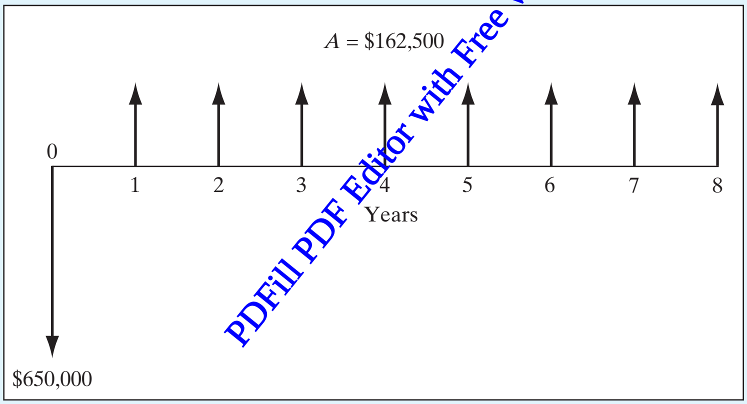


Figure 5.2 Cash flow diagram for the computer process control project described in Example 5.1.

5.1.2 Independent versus Mutually Exclusive Investment Projects

Most firms have a number of unrelated investment opportunities available. For example, in the case of XL Chemicals, other projects being considered in addition to the computer process control project in Example 5.1 are a new waste heat recovery boiler, a CAD system for the engineering department, and a new warehouse. The economic attractiveness of each of these projects can be measured, and a decision to accept or reject the project can be made without reference to any of the other projects. In other words, the decision regarding any one project has no effect on the decision to accept or reject another project. Such projects are said to be **independent**.

In Section 5.5, we will see that in many engineering situations we are faced with selecting the most economically attractive project from a number of alternative projects, all of which solve the same problem or meet the same need. It is unnecessary to choose more than one project in this situation, and the acceptance of one automatically entails the rejection of all of the others. Such projects are said to be **mutually exclusive**.

As long as the total cost of all the independent projects found to be economically attractive is less than the investment funds available to a firm, all of these projects could proceed. However, this is rarely the case. The selection of those projects which should proceed when investment funds are limited is the subject of capital budgeting. Apart from Chapter 15, which deals with capital budgeting, the availability of funds will not be a consideration in accepting or rejecting projects dealt with in this book.

5.2 Initial Project Screening Method

Let's suppose that you are in the market for a new punch press for your company's machine shop, and you visit an equipment dealer. As you take a serious look at one of the punch press models in the display room, an observant equipment salesperson approaches you and says, "That press you are looking at is the state of the art in its category. If you buy that top-of-the-line model, it will cost a little bit more, but it will pay for itself in less than two years." Before studying the four measures of investment attractiveness, we will review a simple, but nonrigorous, method commonly used to screen capital investments. One of the primary concerns of most businesspeople is whether and when the money invested in a project can be recovered. The **payback method** screens projects on the basis of how long it takes for net receipts to equal investment outlays. This calculation can take one of two forms: either ignore time-value-of-money considerations or include them. The former case is usually designated the **conventional payback method**, the latter case the **discounted payback method**.

Payback period: The length of time required to recover the cost of an investment.

A common standard used to determine whether to pursue a project is that the project does not merit consideration unless its payback period is shorter than some specified period. (This time limit is determined largely by management policy. For example, a high-tech firm, such as a computer chip manufacturer, would set a short time limit for any new investment, because high-tech products rapidly become obsolete.) If the payback period is within the acceptable range, a formal project evaluation (such as a present-worth analysis) may begin. It is important to remember that **payback screening** is not an *end* in itself, but rather a method of screening out certain obviously unacceptable investment alternatives before progressing to an analysis of potentially acceptable ones.

5.2.1 Payback Period: The Time It Takes to Pay Back

Determining the relative worth of new production machinery by calculating the time it will take to pay back what it cost is the single most popular method of project screening. If a company makes investment decisions solely on the basis of the payback period, it considers only those projects with a payback period *shorter* than the maximum acceptable payback period. (However, because of shortcomings of the payback screening method, which we will discuss later, it is rarely used as the only decision criterion.)

What does the payback period tell us? One consequence of insisting that each proposed investment have a short payback period is that investors can assure themselves of being restored to their initial position within a short span of time. By restoring their initial position, investors can take advantage of additional, perhaps better, investment possibilities that may come along.

EXAMPLE 5.2 Conventional Payback Period for the Computer Process Control System Project

Consider the cash flows given in Example 5.1. Determine the payback period for this computer process control system project.

SOLUTION

Given: Initial cost = \$650,000 and annual net benefits = \$162,500.

Find: Conventional payback period.

Given a uniform stream of receipts, we can easily calculate the payback period by dividing the initial cash outlay by the annual receipts:

$$\text{Payback period} = \frac{\text{Initial cost}}{\text{Uniform annual benefit}} = \frac{\$650{,}000}{\$162{,}500}$$
$$= 4 \text{ years.}$$

If the company's policy is to consider only projects with a payback period of five years or less, this computer process control system project passes the initial screening.

In Example 5.2, dividing the initial payment by annual receipts to determine the payback period is a simplification we can make because the annual receipts are uniform. Whenever the expected cash flows vary from year to year, however, the payback period must be determined by adding the expected cash flows for each year until the sum is equal to or greater than zero. The significance of this procedure is easily explained. The cumulative cash flow equals zero at the point where cash inflows exactly match, or pay back, the cash outflows; thus, the project has reached the payback point. Similarly, if the cumulative cash flows are greater than zero, then the cash inflows exceed the cash outflows, and the project has begun to generate a profit, thus surpassing its payback point. To illustrate, consider Example 5.3.

EXAMPLE 5.3 Conventional Payback Period with Salvage Value

Autonumerics Company has just bought a new spindle machine at a cost of \$105,000 to replace one that had a salvage value of \$20,000. The projected annual after-tax savings via improved efficiency, which will exceed the investment cost, are as follows:

Period	Cash Flow	Cumulative Cash Flow
0	−$105,000 + $20,000	−$85,000
1	15,000	−70,000
2	25,000	−45,000
3	35,000	−10,000
4	45,000	35,000
5	45,000	80,000
6	35,000	115,000

SOLUTION

Given: Cash flow series as shown in Figure 5.3(a).

Find: Conventional payback period.

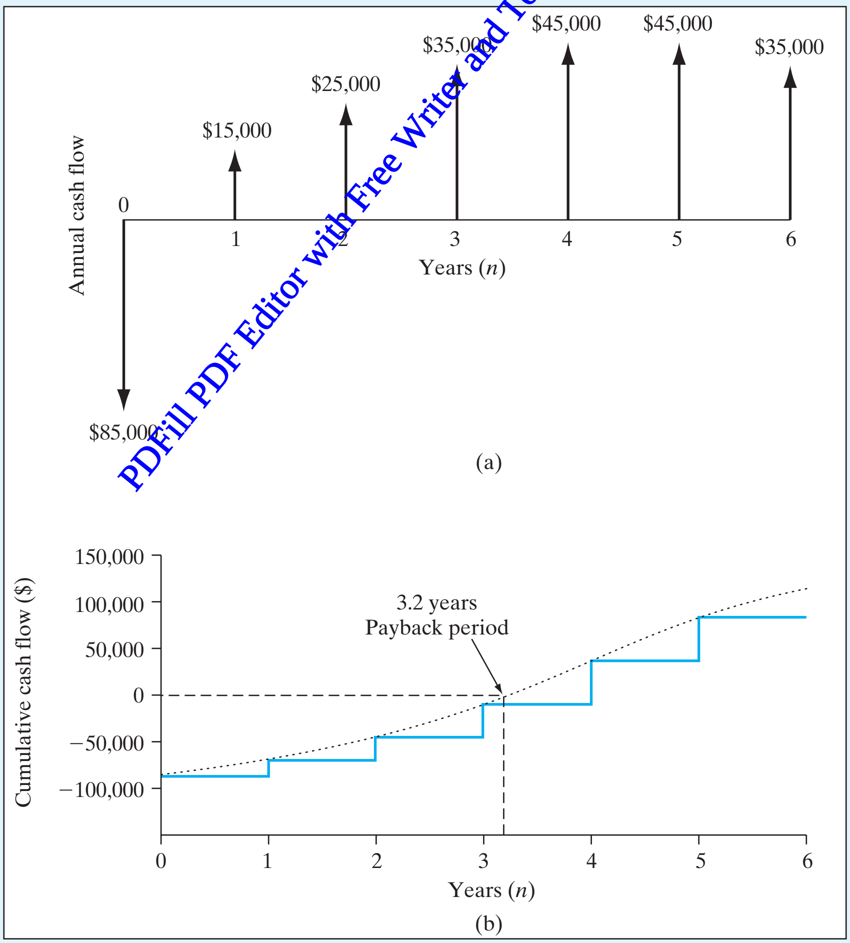


Figure 5.3 Illustration of conventional payback period (Example 5.3).

The salvage value of retired equipment becomes a major consideration in most justification analysis. (In this example, the salvage value of the old machine should be taken into account, as the company already had decided to replace the old machine.) When used, the salvage value of the retired equipment is subtracted from the purchase price of new equipment, revealing a closer true cost of the investment. As we see from the cumulative cash flow in Figure 5.3(b), the total investment is recovered during year 4. If the firm's stated maximum payback period is three years, the project will not pass the initial screening stage.

COMMENTS: In Example 5.2, we assumed that cash flows occur only in discrete lumps at the ends of years. If instead cash flows occur continuously throughout the year, the payback period calculation needs adjustment. A negative balance of \$10,000 remains at the start of year 4. If \$45,000 is expected to be received as a more or less continuous flow during year 4, the total investment will be recovered two-tenths (\$10,000/\$45,000) of the way through the fourth year. Thus, in this situation, the payback period is 3.2 years.

5.2.2 Benefits and Flaws of Payback Screening

The simplicity of the payback method is one of its most appealing qualities. Initial project screening by the method reduces the information search by focusing on that time at which the firm expects to recover the initial investment. The method may also eliminate some alternatives, thus reducing the firm's time spent analyzing. But the much-used payback method of equipment screening has a number of serious drawbacks. The principal objection to the method is that it fails to measure profitability (i.e., no "profit" is made during the payback period). Simply measuring how long it will take to recover the initial investment outlay contributes little to gauging the earning power of a project. (In other words, you already know that the money you borrowed for the drill press is costing you 12% per year; the payback method can't tell you how much your invested money is contributing toward the interest expense.) Also, because payback period analysis ignores differences in the timing of cash flows, it fails to recognize the difference between the present and future value of money. For example, although the payback on two investments can be the same in terms of numbers of years, a front-loaded investment is better because money available today is worth more than that to be gained later. Finally, because payback screening ignores all proceeds after the payback period, it does not allow for the possible advantages of a project with a longer economic life.

By way of illustration, consider the two investment projects listed in Table 5.1. Each requires an initial investment outlay of \$90,000. Project 1, with expected annual cash proceeds of \$30,000 for the first 3 years, has a payback period of 3 years. Project 2 is expected to generate annual cash proceeds of \$25,000 for 6 years; hence, its payback period is 3.6 years. If the company's maximum payback period is set to 3 years, then project 1 would pass the initial project screening, whereas project 2 would fail even though it is clearly the more profitable investment.

TABLE 5.1 Investment Cash Flows for Two Competing Projects

n	Project 1	Project 2
0	−$90,000	−$90,000
1	30,000	25,000
2	30,000	25,000
3	30,000	25,000
4	1,000	25,000
5	1,000	25,000
6	1,000	25,000
	$ 3,000	$60,000

5.2.3 Discounted Payback Period

To remedy one of the shortcomings of the conventional payback period, we may modify the procedure so that it takes into account the time value of money—that is, the cost of funds (interest) used to support a project. This modified payback period is often referred to as the **discounted payback period**. In other words, we may define the discounted payback period as the number of years required to recover the investment from *discounted* cash flows.

Discounted payback period: The length of time required to recover the cost of an investment based on discounted cash flows.

For the project in Example 5.3, suppose the company requires a rate of return of 15%. To determine the period necessary to recover both the capital investment and the cost of funds required to support the investment, we may construct Table 5.2, showing cash flows and costs of funds to be recovered over the life of the project.

To illustrate, let's consider the cost of funds during the first year: With $85,000 committed at the beginning of the year, the interest in year 1 would be $12,750 ($85,000 × 0.15). Therefore, the total commitment grows to $97,750, but the $15,000 cash flow in year 1

TABLE 5.2 Payback Period Calculation Taking into Account the Cost of Funds (Example 5.3)

Period	Cash Flow	Cost of Funds (15%)*	Cumulative Cash Flow
0	–$85,000	0	–$85,000
1	15,000	−$85,000(0.15) = −$12,750	−82,750
2	25,000	−$82,750(0.15) = −12,413	−70,163
3	35,000	−$70,163(0.15) = −10,524	−45,687
4	45,000	−$45,687(0.15) = −6,853	−7,540
5	45,000	−$7,540(0.15) = −1,131	36,329
6	35,000	$36,329(0.15) = 5,449	76,778

*Cost of funds = Unrecovered beginning balance × interest rate.

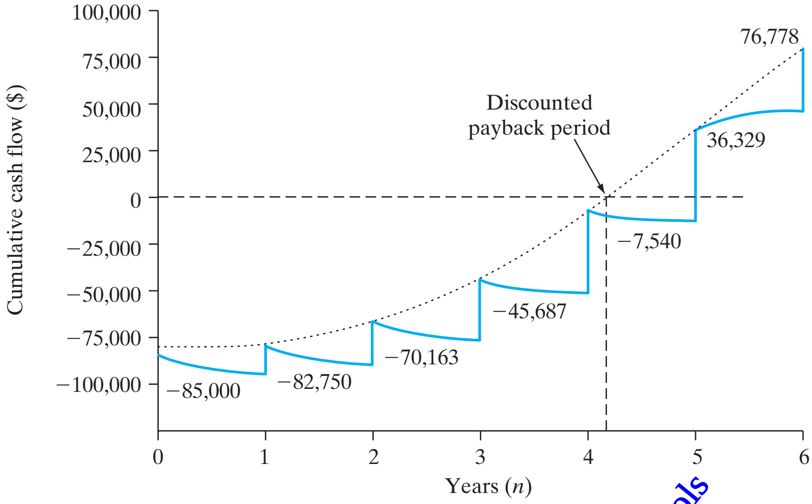


Figure 5.4 Illustration of discounted payback period.

leaves a net commitment of \$82,750. The cost of funds during the second year would be \$12,413 (\$82,750 × 0.15), but with the \$25,000 receipt from the project, the net commitment drops to \$70,163. When this process repeats for the remaining years of the project's life, we find that the net commitment to the project ends during year 5. Depending on which cash flow assumption we adopt, the project must remain in use about 4.2 years (continuous cash flows) or 5 years (year-end cash flows) in order for the company to cover its cost of capital and also recover the funds it has invested. Figure 5.4 illustrates this relationship.

The inclusion of effects stemming from time value of money has increased the payback period calculated for this example by a year. Certainly, this modified measure is an improved one, but it does not show the complete picture of the project's profitability either.

5.2.4 Where Do We Go from Here?

Should we abandon the payback method? Certainly not, but if you use payback screening exclusively to analyze capital investments, look again. You may be missing something that another method can help you spot. Therefore, it is illogical to claim that payback is either a good or bad method of justification. Clearly, it is not a measure of profitability. But when it is used to supplement other methods of analysis, it can provide useful information. For example, payback can be useful when a company needs a measure of the speed of cash recovery, when the company has a cash flow problem, when a product is built to last only for a short time, and when the machine the company is contemplating buying itself is known to have a short market life.

5.3 Discounted Cash Flow Analysis

Until the 1950s, the payback method was widely used as a means of making investment decisions. As flaws in this method were recognized, however, businesspeople began to search for methods to improve project evaluations. The result was the development of **discounted**

cash flow techniques (DCFs), which take into account the time value of money. One of the DCFs is the net-present-worth, or net-present-value, method. A capital investment problem is essentially a problem of determining whether the anticipated cash inflows from a proposed project are sufficient to attract investors to invest funds in the project. In developing the NPW criterion, we will use the concept of cash flow equivalence discussed in Chapter 3.

Discounted cash flow analysis (DCF): A method of evaluating an investment by estimating future cash flows and taking into consideration the time value of money.

As we observed, the most convenient point at which to calculate the equivalent values is often at time 0. Under the NPW criterion, the present worth of all cash inflows is compared against the present worth of all cash outflows associated with an investment project. The difference between the present worth of these cash flows, referred to as the **net present worth** (NPW), **net present value** (NPV) determines whether the project is an acceptable investment. When two or more projects are under consideration, NPW analysis further allows us to select the best project by comparing their NPW figures.

Net present worth: The difference between the present value of cash inflows and the present value of cash outflows.

Investment pool operates like a mutual fund to earn a targeted return by investing the firm's money in various investment assets.

5.3.1 Net-Present-Worth Criterion

We will first summarize the basic procedure for applying the net-present-worth criterion to a typical investment project:

- Determine the interest rate that the firm wishes to earn on its investments. The interest rate you determine represents the rate at which the firm can always invest the money in its **investment pool**. This interest rate is often referred to as either a **required rate of return** or a **minimum attractive rate of return** (MARR). Usually, selection of the MARR is a policy decision made by top management. It is possible for the MARR to change over the life of a project, as we saw in Section 4.4, but for now we will use a single rate of interest in calculating the NPW.
- Estimate the service life of the project.
- Estimate the cash inflow for each period over the service life.
- Estimate the cash outflow over each service period.
- Determine the net cash flows (net cash flow $=$ cash inflow $-$ cash outflow).
- Find the present worth of each net cash flow at the MARR. Add up all the present-worth figures; their sum is defined as the project's NPW, given by

$$\begin{aligned} \mathrm{PW}(i) = \text{NPW calculated at } i &= \frac{A_0}{(1+i)^0} + \frac{A_1}{(1+i)^1} + \frac{A_2}{(1+i)^2} + \cdots \\ &\quad + \frac{A_N}{(1+i)^N} \\ &= \sum_{n=0}^{N} \frac{A_n}{(1+i)^n} \\ &= \sum_{n=0}^{N} A_n(P/F, i, n), \end{aligned} \tag{5.1}$$

where

$$\begin{aligned} A_n &= \text{Net cash flow at end of period } n, \\ i &= \text{MARR (or cost of capital)}, \\ N &= \text{Service life of the project.} \end{aligned}$$

A_n will be positive if the corresponding period has a net cash inflow and negative if there is a net cash outflow.

- **Single Project Evaluation.** In this context, a positive NPW means that the equivalent worth of the inflows is greater than the equivalent worth of outflows, so the project makes a profit. Therefore, if the PW(i) is positive for a single project, the project should be accepted; if the PW(i) is negative, the project should be rejected.[2] The decision rule is

$$\text{If PW}(i) > 0, \text{ accept the investment.}$$
$$\text{If PW}(i) = 0, \text{ remain indifferent.}$$
$$\text{If PW}(i) < 0, \text{ reject the investment.}$$

- **Comparing Multiple Alternatives.** Compute the PW(i) for each alternative and select the one with the largest PW(i). As you will learn in Section 5.5, when you compare mutually exclusive alternatives with the *same revenues*, they are compared on a *cost-only basis*. In this situation (because you are minimizing costs, rather than maximizing profits), you should accept the project that results in the *smallest*, or *least negative*, NPW.

EXAMPLE 5.4 Net Present Worth: Uniform Flows

Consider the investment cash flows associated with the computer process control project discussed in Example 5.1. If the firm's MARR is 15%, compute the NPW of this project. Is the project acceptable?

SOLUTION

Given: Cash flows in Figure 5.2 and MARR = 15% per year.
Find: NPW.

Since the computer process control project requires an initial investment of \$650,000 at $n = 0$, followed by the eight equal annual savings of \$162,000, we can easily determine the NPW as follows:

$$\begin{aligned} \text{PW}(15\%)_{\text{Outflow}} &= \$650{,}000; \\ \text{PW}(15\%)_{\text{Inflow}} &= \$162{,}500(P/A, 15\%, 8) \\ &= \$729{,}190. \end{aligned}$$

Then the NPW of the project is

$$\begin{aligned} \text{PW}(15\%) &= \text{PW}(15\%)_{\text{Inflow}} - \text{PW}(15\%)_{\text{Outflow}} \\ &= \$729{,}190 - \$650{,}000 \\ &= \$79{,}190, \end{aligned}$$

or, from Eq. (5.1),

$$\begin{aligned} \text{PW}(15\%) &= -\$650{,}000 + \$162{,}500(P/A, 15\%, 8) \\ &= \$79{,}190. \end{aligned}$$

Since $\text{PW}(15\%) > 0$, the project is acceptable.

[2] Some projects (e.g., the installation of pollution control equipment) cannot be avoided. In a case such as this, the project would be accepted even though its NPW ≤ 0. This type of project will be discussed in Chapter 12.

Now let's consider an example in which the investment cash flows are not uniform over the service life of the project.

EXAMPLE 5.5 Net Present Worth: Uneven Flows

Tiger Machine Tool Company is considering acquiring a new metal-cutting machine. The required initial investment of \$75,000 and the projected cash benefits[3] over the project's three-year life are as follows:

End of Year	Net Cash Flow
0	−\$75,000
1	24,400
2	27,340
3	55,760

You have been asked by the president of the company to evaluate the economic merit of the acquisition. The firm's MARR is known to be 15%.

SOLUTION

Given: Cash flows as tabulated and MARR = 15% per year.

Find: NPW.

If we bring each flow to its equivalent at time zero, we find that

$$\begin{aligned}\text{PW}(15\%) &= -\$75{,}000 + \$24{,}000(P/F, 15\%, 1) + \$27{,}340(P/F, 15\%, 2)\\ &\quad + \$55{,}760(P/F, 15\%, 3)\\ &= \$3{,}553.\end{aligned}$$

Since the project results in a surplus of \$3,553, the project is acceptable.

In Example 5.5, we computed the NPW of a project at a fixed interest rate of 15%. If we compute the NPW at varying interest rates, we obtain the data in Table 5.3. Plotting the NPW as a function of interest rate gives Figure 5.5, the present-worth profile. (You may use a spreadsheet program such as Excel to generate Table 5.3 or Figure 5.5.)

Figure 5.5 indicates that the investment project has a positive NPW if the interest rate is below 17.45% and a negative NPW if the interest rate is above 17.45%. As we will see in Chapter 7, this **break-even interest rate** is known as the **internal rate of return**. If the

[3] As we stated at the beginning of this chapter, we treat net cash flows as before-tax values or as having their tax effects precalculated. Explaining the process of obtaining cash flows requires an understanding of income taxes and the role of depreciation, which are discussed in Chapter 9.

TABLE 5.3 **Present-Worth Amounts at Varying Interest Rates (Example 5.5)**

i(%)	PW(i)	i(%)	PW(i)
0	$32,500	20	−$3,412
2	27,743	22	−5,924
4	23,309	24	−8,296
6	19,169	26	−10,539
8	15,296	28	−12,662
10	11,670	30	−14,673
12	8,270	32	−16,580
14	5,077	34	−18,360
16	2,076	36	−20,110
17.45*	0	38	−21,745
18	−750	40	−23,302

*Break-even interest rate (also known as the rate of return).

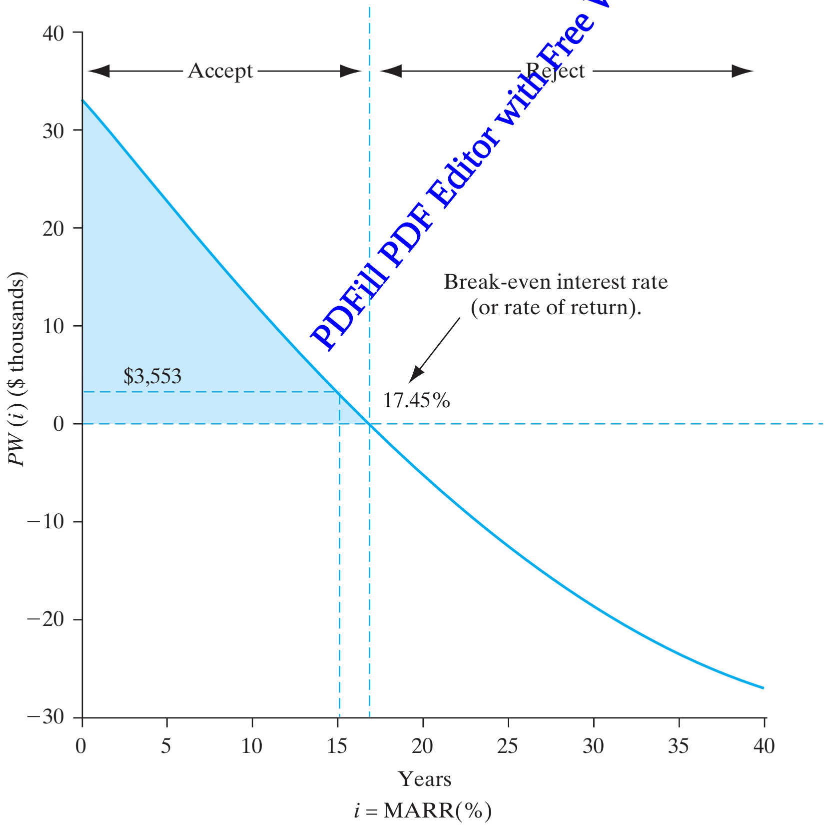


Figure 5.5 Present-worth profile described in Example 5.5.

firm's MARR is 15%, then the project has an NPW of \$3,553 and so may be accepted. The \$3,553 figure measures the equivalent immediate gain in present worth to the firm following acceptance of the project. By contrast, at $i = 20\%$, $\text{PW}(20\%) = -\$3{,}412$, and the firm should reject the project. (Note that either accepting or rejecting an investment is influenced by the choice of a MARR, so it is crucial to estimate the MARR correctly. We will defer this important issue until Section 5.3.3. For now, we will assume that the firm has an accurate MARR estimate available for use in investment analysis.)

5.3.2 Meaning of Net Present Worth

In present-worth analysis, we assume that all the funds in a firm's treasury can be placed in investments that yield a return equal to the MARR. We may view these funds as an **investment pool**. Alternatively, if no funds are available for investment, we assume that the firm can borrow them at the MARR (or cost of capital) from the capital market. In this section, we will examine these two views as we explain the meaning of the MARR in NPW calculations.

Investment Pool Concept

An investment pool is equivalent to a firm's treasury. All fund transactions are administered and managed by the firm's comptroller. The firm may withdraw funds from this investment pool for other investment purposes, but if left in the pool, these funds will earn at the MARR. Thus, in investment analysis, net cash flows will be net cash flows relative to an investment pool. To illustrate the investment pool concept, we consider again the project in Example 5.5 that required an investment of \$75,000.

If the firm did not invest in the project and left \$75,000 in the investment pool for three years, these funds would grow as follows:

$$\$75{,}000(F/P, 15\%, 3) = \$114{,}066.$$

Suppose the company decided instead to invest \$75,000 in the project described in Example 5.5. Then the firm would receive a stream of cash inflows during the project's life of three years in the following amounts:

Period (n)	Net Cash Flow (A_n)
1	\$24,400
2	27,340
3	55,760

Since the funds that return to the investment pool earn interest at a rate of 15%, it is worthwhile to see how much the firm would benefit from its \$75,000 investment. For this alternative, the returns after reinvestment are

$$\$24{,}400(F/P, 15\%, 2) = \$32{,}269,$$

$$\$27{,}340(F/P, 15\%, 1) = \$31{,}441,$$

$$\$55{,}760(F/P, 15\%, 0) = \$55{,}760.$$

These returns total \$119,470. At the end of three years, the additional cash accumulation from investing in the project is

$$\$119{,}470 - \$114{,}066 = \$5{,}404.$$

If we compute the equivalent present worth of this net cash surplus at time 0, we obtain

$$\$5{,}404(P/F, 15\%, 3) = \$3{,}553,$$

which is exactly what we get when we compute the NPW of the project with Eq. (5.1). Clearly, on the basis of its positive NPW, the alternative of purchasing a new machine should be preferred to that of simply leaving the funds in the investment pool at the MARR. Thus, in PW analysis, any investment is assumed to be returned at the MARR. If a surplus exists at the end of the project, then PW(MARR) > 0. Figure 5.6 illustrates the reinvestment concept as it relates to the firm's investment pool.

Borrowed-Funds Concept

Suppose that the firm does not have \$75,000 at the outset. In fact, the firm doesn't have to have an investment pool at all. Suppose further that the firm borrows all of its capital from a bank at an interest rate of 15%, invests in the project, and uses the proceeds from the investment to pay off the principal and interest on the bank loan. How much is left over for the firm at the end of the project's life?

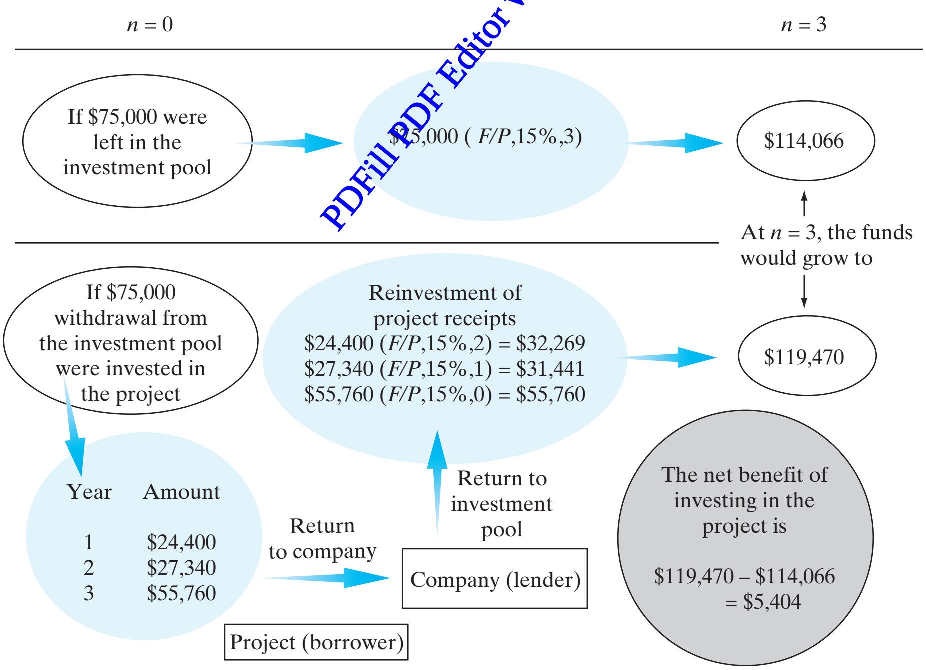


Figure 5.6 The concept of an investment pool with the company as a lender and the project as a borrower.

At the end of first year, the interest on the project's use of the bank loan would be $\$75{,}000(0.15) = \$11{,}250$. Therefore, the total loan balance grows to $\$75{,}000(1.15) = \$86{,}250$. Then, the firm receives \$24,400 from the project and applies the entire amount to repay the loan portion, leaving a balance due of

$$\$75{,}000(1 + 0.15) - \$24{,}400 = \$61{,}850.$$

This amount becomes the net amount the project is borrowing at the beginning of year 2, which is also known as the **project balance**. At the end of year 2, the bank debt grows to $\$61{,}850(1.15) = \$71{,}128$, but with the receipt of \$27,340, the project balance is reduced to

$$\$71{,}128 - \$27{,}340 = \$43{,}788.$$

Similarly, at the end of year 3, the project balance becomes

$$\$43{,}788(1.15) = \$50{,}356.$$

But with the receipt of \$55,760 from the project, the firm should be able to pay off the remaining balance and come out with a surplus in the amount of \$5,404. This terminal project balance is also known as the **net future worth** of the project. In other words, the firm repays its initial bank loan and interest at the end of year 3, with a resulting profit of \$5,404. If we compute the equivalent present worth of this net profit at time 0, we obtain

$$\text{PW}(15\%) = \$5{,}404(P/F, 15\%, 3) = \$3{,}553.$$

The result is identical to the case in which we directly computed the NPW of the project at $i = 15\%$, shown in Example 5.5. Figure 5.7 illustrates the project balance as a function of time.[4]

5.3.3 Basis for Selecting the MARR

Cost of capital: The required return necessary to make an investment project worthwhile.

The basic principle used to determine the discount rate in project evaluations is similar to the concept of the required return on investment for financial assets discussed in Section 4.6.2. The first element to cover is the **cost of capital**, which is the required return necessary to make an investment project worthwhile. The cost of capital would include both the *cost of debt* (the interest rate associated with borrowing) and the *cost of equity* (the return that stockholders require for a company). Both the cost of debt and the cost of equity reflect the presence of inflation in the economy. *The cost of capital determines how a company can raise money (through issuing a stock, borrowing, or a mix of the two). Therefore, this is normally considered as the rate of return that a firm would receive if it invested its money someplace else with a similar risk.*

The second element is a consideration of any additional risk associated with the project. If the project belongs to the normal risk category, the cost of capital may already reflect the risk premium. However, if you are dealing with a project with higher risk, the additional risk premium may be added onto the cost of capital.

[4] Note that the sign of the project balance changes from negative to positive during year 3. The time at which the project balance becomes zero is known as the **discounted payback period**.

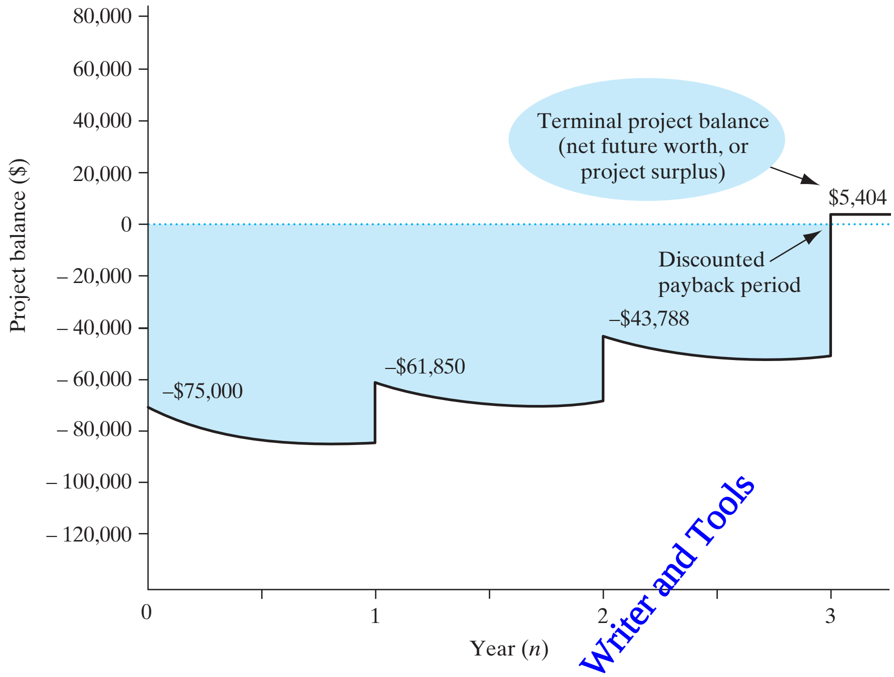


Figure 5.7 Project balance diagram as a function of time. (A negative project balance indicates the amount of the loan remaining to be paid off or the amount of investment to be recovered.)

In sum, the discount rate (**MARR**) to use for project evaluation would be equivalent to the firm's cost of capital for a project of normal risk, but could be much higher if you are dealing with a risky project. Chapter 15 will detail the analytical process of determining this discount rate. For now, we assume that such a rate is already known to us, and we will focus on the evaluation of the investment project.

MARR: this is based on the firm's cost of capital plus or minus a risk premium to reflect the project's specific risk characteristics.

5.4 Variations of Present-Worth Analysis

As variations of present-worth analysis, we will consider two additional measures of investment worth: **future-worth analysis** and **capitalized equivalent-worth analysis**. (The equivalent annual worth measure is another variation of the present-worth measure, but we present it in Chapter 6.) Future-worth analysis calculates the future worth of an investment undertaken. Capitalized equivalent-worth analysis calculates the present worth of a project with a perpetual life span.

5.4.1 Future-Worth Analysis

Net present worth measures the surplus in an investment project at time 0. **Net future worth (NFW)** measures this surplus at a time other than 0. Net-future-worth analysis is particularly useful in an investment situation in which we need to compute the equivalent worth of a project at the end of its investment period, rather than at its beginning. For example, it may take 7 to 10 years to build a nuclear power plant because of the complexities of engineering design and the many time-consuming regulatory procedures that must be followed to ensure public safety. In this situation, it is more common to measure the worth of the investment at the time of the project's commercialization (i.e., we conduct an NFW analysis at the end of the investment period).

Net future worth: The value of an asset or cash at a specified date in the future that is equivalent in value to a specified sum today.

Net-Future-Worth Criterion and Calculations

Let A_n represent the cash flow at time n for $n = 0, 1, 2, \ldots, N$ for a typical investment project that extends over N periods. Then the net-future-worth (NFW) expression at the end of period N is

$$\begin{aligned} FW(i) &= A_0(1+i)^N + A_1(1+i)^{N-1} + A_2(1+i)^{N-2} + \cdots + A_N \\ &= \sum_{n=0}^{N} A_n(1+i)^{N-n} \\ &= \sum_{n=0}^{N} A_n(F/P, i, N-n). \end{aligned} \tag{5.2}$$

As you might expect, the decision rule for the NFW criterion is the same as that for the NPW criterion: For a single project evaluation,

If $FW(i) > 0$, accept the investment.

If $FW(i) = 0$, remain indifferent to the investment.

If $FW(i) < 0$, reject the investment.

EXAMPLE 5.6 Net Future Worth: At the End of the Project

Consider the project cash flows in Example 5.5. Compute the NFW at the end of year 3 at $i = 15\%$.

SOLUTION

Given: Cash flows in Example 5.5 and MARR $= 15\%$ per year.
Find: NFW.

As seen in Figure 5.8, the NFW of this project at an interest rate of 15% would be

$$\begin{aligned} FW(15\%) &= -\$75{,}000(F/P, 15\%, 3) + \$24{,}400(F/P, 15\%, 2) \\ &\quad + \$27{,}340(F/P, 15\%, 1) + \$55{,}760 \\ &= \$5{,}404. \end{aligned}$$

Note that the net future worth of the project is equivalent to the terminal project balance as calculated in Section 5.3.2. Since $FW(15\%) > 0$, the project is acceptable. We reach the same conclusion under present-worth analysis.

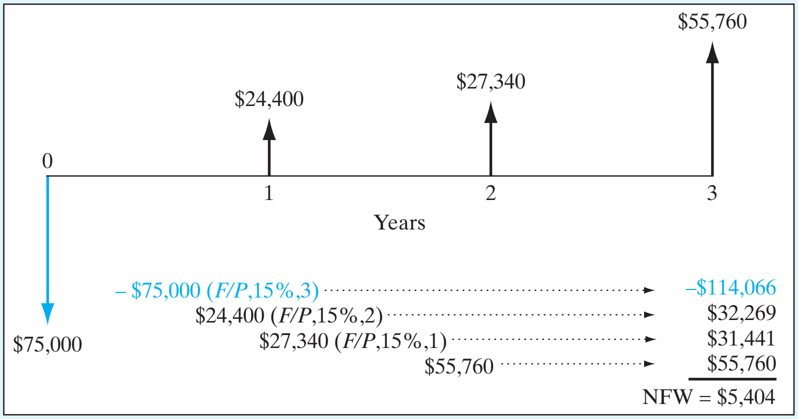


Figure 5.8 Future-worth calculation at the end of year 3 (Example 5.6).

EXAMPLE 5.7 Future Equivalent: At an Intermediate Time

Higgins Corporation (HC), a Detroit-based robot-manufacturing company, has developed a new advanced-technology robot called Helpmate, which incorporates advanced technology such as vision systems, tactile sensing, and voice recognition. These features allow the robot to roam the corridors of a hospital or office building without following a predetermined track or bumping into objects. HC's marketing department plans to target sales of the robot toward major hospitals. The robots will ease nurses' workloads by performing low-level duties such as delivering medicines and meals to patients.

- The firm would need a new plant to manufacture the Helpmates; this plant could be built and made ready for production in two years. It would require a 30-acre site, which can be purchased for $1.5 million in year 0. Building construction would begin early in year 1 and continue throughout year 2. The building would cost an estimated $10 million, with a $4 million payment due to the contractor at the end of year 1, and with another $6 million payable at the end of year 2.
- The necessary manufacturing equipment would be installed late in year 2 and would be paid for at the end of year 2. The equipment would cost $13 million, including transportation and installation. When the project terminates, the land is expected to have an after-tax market value of $2 million, the building an after-tax value of $3 million, and the equipment an after-tax value of $3 million.

For capital budgeting purposes, assume that the cash flows occur at the end of each year. Because the plant would begin operations at the beginning of year 3, the first operating cash flows would occur at the end of year 3. The Helpmate plant's estimated economic life is six years after completion, with the following expected after-tax operating cash flows in millions:

Calendar Year End of Year	'06 0	'07 1	'08 2	'09 3	'10 4	'11 5	'12 6	'13 7	'14 8
After-tax cash flows									
A. Operating revenue				\$6	\$8	\$13	\$18	\$14	\$8
B. Investment									
Land	−1.5								+2
Building		−4	−6						+3
Equipment			−13						+3
Net cash flow	−\$1.5	−\$4	−\$19	\$6	\$8	\$13	\$18	\$14	\$16

Compute the equivalent worth of this investment at the start of operations. Assume that HC's MARR is 15%.

SOLUTION

Given: Preceding cash flows and MARR = 15% per year.
Find: Equivalent worth of project at the end of calendar year 2.

One easily understood method involves calculating the present worth and then transforming it to the equivalent worth at the end of year 2. First, we can compute PW(15%) at time 0 of the project:

$$\begin{aligned}\mathrm{PW}(15\%) &= -\$1.5 - \$4(P/F, 15\%, 1) - \$19(P/F, 15\%, 2)\\ &\quad + \$6(P/F, 15\%, 3) + \$8(P/F, 15\%, 4) + \$13(P/F, 15\%, 5)\\ &\quad + \$18(P/F, 15\%, 6) + \$14(P/F, 15\%, 7) + \$16(P/F, 15\%, 8)\\ &= \$13.91 \text{ million.}\end{aligned}$$

Then, the equivalent project worth at the start of operation is

$$\begin{aligned}\mathrm{FW}(15\%) &= \mathrm{PW}(15\%)(F/P, 15\%, 2)\\ &= \$18.40 \text{ million.}\end{aligned}$$

A second method brings all flows prior to year 2 up to that point and discounts future flows back to year 2. The equivalent worth of the earlier investment, when the plant begins full operation, is

$$-\$1.5(F/P, 15\%, 2) - \$4(F/P, 15\%, 1) - \$19 = -\$25.58 \text{ million,}$$

which produces an equivalent flow as shown in Figure 5.9. If we discount the future flows to the start of operation, we obtain

$$\begin{aligned}\mathrm{FW}(15\%) &= -\$25.58 + \$6(P/F, 15\%, 1) + \$8(P/F, 15\%, 2) + \cdots\\ &\quad + \$16(F/P, 15\%, 6)\\ &= \$18.40 \text{ million.}\end{aligned}$$

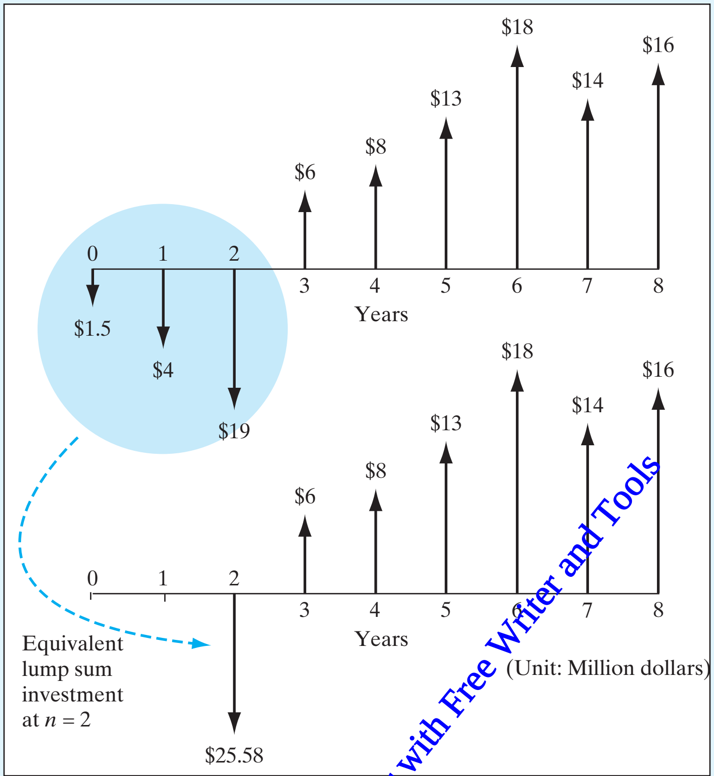


Figure 5.9 Cash flow diagram for the Helpmate project (Example 5.7).

COMMENTS: If another company is willing to purchase the plant and the right to manufacture the robots immediately after completion of the plant (year 2), HC would set the price of the plant at $43.98 million ($\$18.40 + \$25.58$) at a minimum.

5.4.2 Capitalized Equivalent Method

Another special case of the PW criterion is useful when the life of a proposed project is **perpetual** or the planning horizon is extremely long (say, 40 years or more). Many public projects, such as bridges, waterway structures, irrigation systems, and hydroelectric dams, are expected to generate benefits over an extended period (or forever). In this section, we will examine the **capitalized equivalent** (CE(i)) method for evaluating such projects.

Perpetual Service Life

Consider the cash flow series shown in Figure 5.10. How do we determine the PW for an infinite (or almost infinite) uniform series of cash flows or a repeated cycle of cash flows? The process of computing the PW cost for this infinite series is referred to as the **capitalization**

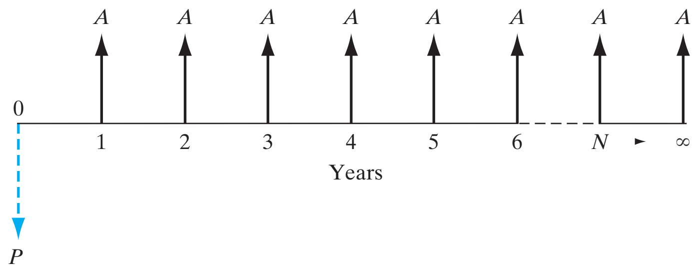


Figure 5.10 Equivalent present worth of an infinite cash flow series.

Capitalized cost related to car leasing, means the amount that is being financed.

of the project cost. The cost, known as the **capitalized cost**, represents the amount of money that must be invested today to yield a certain return A at the end of each and every period forever, assuming an interest rate of i. Observe the limit of the uniform series present-worth factor as N approaches infinity:

$$\lim_{N\to\infty}(P/A, i, N) = \lim_{N\to\infty}\left[\frac{(1+i)^N - 1}{i(1+i)^N}\right] = \frac{1}{i}.$$

Thus,

$$\mathrm{PW}(i) = A(P/A, i, N \to \infty) = \frac{A}{i}. \tag{5.3}$$

Another way of looking at this problem is to ask what constant income stream could be generated by PW(i) dollars today in perpetuity. Clearly, the answer is $A = i\mathrm{PW}(i)$. If withdrawals were greater than A, you would be eating into the principal, which would eventually reduce it to 0.

EXAMPLE 5.8 Capitalized Equivalent Cost

An engineering school has just completed a new engineering complex worth \$50 million. A campaign targeting alumni is planned to raise funds for future maintenance costs, which are estimated at \$2 million per year. Any unforeseen costs above \$2 million per year would be obtained by raising tuition. Assuming that the school can create a trust fund that earns 8% interest annually, how much has to be raised now to cover the perpetual string of \$2 million in annual costs?

SOLUTION

Given: $A = \$2$ million, $i = 8\%$ per year, and $N = \infty$.
Find: CE(8%).

The capitalized cost equation is

$$\mathrm{CE}(i) = \frac{A}{i},$$

so

$$\begin{aligned} CE(8\%) &= \$2{,}000{,}000/0.08 \\ &= \$25{,}000{,}000. \end{aligned}$$

COMMENTS: It is easy to see that this lump-sum amount should be sufficient to pay maintenance expenses for the school forever. Suppose the school deposited $25 million in a bank that paid 8% interest annually. Then at the end of the first year, the $25 million would earn 8%($25 million) = $2 million interest. If this interest were withdrawn, the $25 million would remain in the account. At the end of the second year, the $25 million balance would again earn 8%($25 million) = $2 million. This annual withdrawal could be continued forever, and the endowment (gift funds) would always remain at $25 million.

Project's Service Life Is Extremely Long

The benefits of typical civil engineering projects, such as bridge and highway construction, although not perpetual, can last for many years. In this section, we will examine the use of the CE(i) criterion to approximate the NPW of engineering projects with long lives.

EXAMPLE 5.9 Comparison of Present Worth for Long Life and Infinite Life

Mr. Gaynor L. Bracewell amassed a small fortune developing real estate in Florida and Georgia over the past 30 years. He sold more than 700 acres of timber and farmland to raise $800,000, with which he built a small hydroelectric plant, known as High Shoals Hydro. The plant was a decade in the making. The design for Mr. Bracewell's plant, which he developed using his Army training as a civil engineer, is relatively simple. A 22-foot-deep canal, blasted out of solid rock just above the higher of two dams on his property, carries water 1,000 feet along the river to a "trash rack," where leaves and other debris are caught. A 6-foot-wide pipeline capable of holding 3 million pounds of liquid then funnels the water into the powerhouse at 7.5 feet per second, thereby creating 33,000 pounds of thrust against the turbines. Under a 1978 federal law designed to encourage alternative power sources, Georgia Power Company is required to purchase any electricity Mr. Bracewell can supply. Mr. Bracewell estimates that his plant can generate 6 million kilowatt-hours per year.

Suppose that, after paying income taxes and operating expenses, Mr. Bracewell's annual income from the hydroelectric plant will be $120,000. With normal maintenance, the plant is expected to provide service for at least 50 years. Figure 5.11 illustrates when and in what quantities Mr. Bracewell spent his $800,000 (not taking into account the time value of money) during the last 10 years. Was Mr. Bracewell's $800,000 investment a wise one? How long will he have to wait to recover his initial investment, and will he ever make a profit? Examine the situation by computing the project worth at varying interest rates.

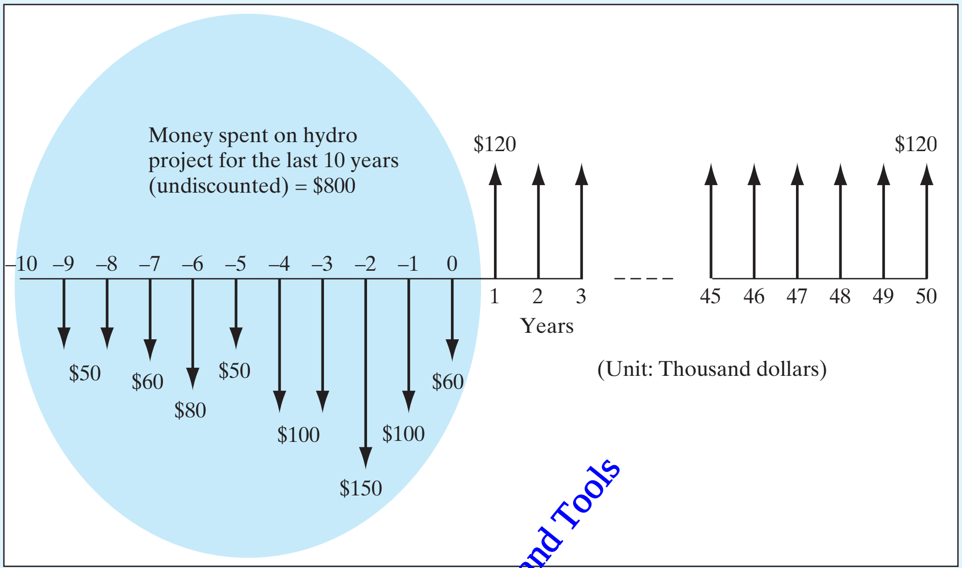


Figure 5.11 Net cash flow diagram for Mr. Bracewell's hydroelectric project (Example 5.9).

(a) If Mr. Bracewell's interest rate is 8%, compute the NPW (at time 0 in Figure 5.11) of this project with a 50-year service life and infinite service, respectively.

(b) Repeat part (a), assuming an interest rate of 12%.

SOLUTION

Given: Cash flow in Figure 5.11 (to 50 years or ∞) and $i = 8\%$ or 12%.
Find: NPW at time 0.

One of the main questions is whether Mr. Bracewell's plant will be profitable. Now we will compute the equivalent total investment and the equivalent worth of receiving future revenues at the start of power generation (i.e., at time 0).

(a) Let $i = 8\%$. Then

- with a plant service life of 50 years, we can make use of single-payment compound-amount factors in the invested cash flow to help us find the equivalent total investment at the start of power generation. Using K to indicate thousands, we obtain

$$\begin{aligned} V_1 &= -\$50K(F/P, 8\%, 9) - \$50K(F/P, 8\%, 8) \\ &\quad - \$60K(F/P, 8\%, 7) \cdots - \$100K(F/P, 8\%, 1) - \$60K \\ &= -\$1{,}101K. \end{aligned}$$

The equivalent total benefit at the start of generation is

$$V_2 = \$120K(P/A, 8\%, 50) = \$1{,}468K.$$

Summing, we find the net equivalent worth at the start of power generation:

$$\begin{aligned} V_1 + V_2 &= -\$1{,}101K + \$1{,}468K \\ &= \$367K. \end{aligned}$$

- With an infinite service life, the net equivalent worth is called the capitalized equivalent worth. The investment portion prior to time 0 is identical, so the capitalized equivalent worth is

$$\begin{aligned} \text{CE}(8\%) &= -\$1{,}101K + \$120K/(0.08) \\ &= \$399K. \end{aligned}$$

Note that the difference between the infinite situation and the planning horizon of 50 years is only \$32,000.

(b) Let $i = 12\%$. Then

- With a service life of 50 years, proceeding as we did in part (a), we find that the equivalent total investment at the start of power generation is

$$\begin{aligned} V_1 &= -\$50K(F/P, 12\%, 9) - \$50K(F/P, 12\%, 8) \\ &\quad - \$60K(F/P, 12\%, 7) \cdots - \$100K(F/P, 12\%, 1) - 60K \\ &= -\$1{,}299K. \end{aligned}$$

Equivalent total benefits at the start of power generation are

$$V_2 = \$120K(P/A, 12\%, 50) = \$997K.$$

The net equivalent worth at the start of power generation is

$$\begin{aligned} V_1 + V_2 &= -\$1{,}299K + \$997K \\ &= -\$302K. \end{aligned}$$

- With infinite cash flows, the capitalized equivalent worth at the current time is

$$\begin{aligned} \text{CE}(12\%) &= -\$1{,}299K + \$120K/(0.12) \\ &= -\$299K. \end{aligned}$$

Note that the difference between the infinite situation and a planning horizon of 50 years is merely \$3,000, which demonstrates that we may approximate the present worth of long cash flows (i.e., 50 years or more) by using the capitalized equivalent value. The accuracy of the approximation improves as the interest rate increases (or the number of years is greater).

COMMENTS: At $i = 12\%$, Mr. Bracewell's investment is not a profitable one, but at 8% it is. This outcome indicates the importance of using the appropriate i in investment analysis. The issue of selecting an appropriate i will be presented again in Chapter 15.

5.5 Comparing Mutually Exclusive Alternatives

Until now, we have considered situations involving either a single project alone or projects that were independent of each other. In both cases, we made the decision to reject or accept each project individually according to whether it met the MARR requirements, evaluated with either the PW or FW criterion.

In the real world of engineering practice, however, it is typical for us to have two or more choices of projects that are not independent of one another in seeking to accomplish a business objective. (As we shall see, even when it appears that we have only one project to consider, the implicit "do-nothing" alternative must be factored into the decision-making process.) In this section, we extend our evaluation techniques to multiple projects that are mutually exclusive. Other dependencies between projects will be considered in Chapter 15.

Often, various projects or investments under consideration do not have the same duration or do not match the desired study period. Adjustments must then be made to account for the differences. In this section, we explain the concept of an analysis period and the process of accommodating for different lifetimes, two important considerations that apply in selecting among several alternatives. Up to now in this chapter, all available options in a decision problem were assumed to have equal lifetimes. In the current section, this restriction is also relaxed.

5.5.1 Meaning of Mutually Exclusive and "Do Nothing"

As we briefly mentioned in Section 5.1.2, several alternatives are **mutually exclusive** when any one of them will fulfill the same need and the selection of one of them implies that the others will be excluded. Take, for example, buying versus leasing an automobile for business use; when one alternative is accepted, the other is excluded. We use the terms **alternative** and **project** interchangeably to mean "decision option."

"Do Nothing" Is a Decision Option

When considering an investment, we are in one of two situations: Either the project is aimed at replacing an existing asset or system, or it is a new endeavor. In either case, a do-nothing alternative may exist. On the one hand, if a process or system already in place to accomplish our business objectives is still adequate, then we must determine which, if any, new proposals are economical replacements. If none are feasible, then we do nothing. On the other hand, if the existing system has failed, then the choice among proposed alternatives is mandatory (i.e., do nothing is not an option).

New endeavors occur as alternatives to the "green fields" do-nothing situation, which has zero revenues and zero costs (i.e., nothing currently exists). For most new endeavors, do nothing is generally an alternative, as we won't proceed unless at least one of the proposed alternatives is economically sound. In fact, undertaking even a single project entails making a decision between two alternatives, because the do-nothing alternative is implicitly included. Occasionally, a new initiative must be undertaken, cost notwithstanding, and in this case the goal is to choose the most economical alternative, since "do nothing" is not an option.

When the option of retaining an existing asset or system is available, there are two ways to incorporate it into the evaluation of the new proposals. One way is to treat the do-nothing option as a distinct alternative; we cover this approach primarily in Chapter 14, where methodologies specific to replacement analysis are presented. The second approach, used mostly in this chapter, is to generate the cash flows of the new proposals relative to that of the do-nothing alternative. That is, for each new alternative, the **incremental costs** (and incremental savings or revenues if applicable) relative to "do nothing" are used in the economic evaluation. For a replacement-type problem, these costs are calculated by subtracting the do-nothing cash flows from those of each new alternative. For new endeavors, the incremental cash flows are the same as the absolute amounts associated with each alternative, since the do-nothing values are all zero.

Because the main purpose of this chapter is to illustrate how to choose among mutually exclusive alternatives, most of the problems are structured so that one of the options presented must be selected. Therefore, unless otherwise stated, it is assumed that "do nothing" is not an option, and costs and revenues can be viewed as incremental to "do nothing."

Service Projects versus Revenue Projects

When comparing mutually exclusive alternatives, we need to classify investment projects into either service or revenue projects. **Service projects** are projects whose revenues do not depend on the choice of project; rather, such projects *must produce the same amount of output (revenue)*. In this situation, we certainly want to choose an alternative with the least input (or cost). For example, suppose an electric utility is considering building a new power plant to meet the peak-load demand during either hot summer or cold winter days. Two alternative service projects could meet this demand: a combustion turbine plant and a fuel-cell power plant. No matter which type of plant is selected, the firm will collect the same amount of revenue from its customers. The only difference is how much it will cost to generate electricity from each plant. If we were to compare these service projects, we would be interested in knowing which plant could provide the cheaper power (lower production cost). Further, if we were to use the NPW criterion to compare alternatives so as to minimize expenditures, we would choose the alternative with the **lower present-value** production cost over the service life of the plant.

Revenue projects, by contrast, are projects whose revenues depend on the choice of alternative. With revenue projects, we are not limiting the amount of input going into the project or the amount of output that the project would generate. Then our decision is to select the alternative with the largest net gains (output – input). For example, a TV manufacturer is considering marketing two types of high-resolution monitors. With its present production capacity, the firm can market only one of them. Distinct production processes for the two models could incur very different manufacturing costs, so the revenues from each model would be expected to differ due to divergent market prices and potentially different sales volumes. In this situation, if we were to use the NPW criterion, we would select the model that promises to bring in the higher net present worth.

Total-Investment Approach

Applying an evaluation criterion to each mutually exclusive alternative individually and then comparing the results to make a decision is referred to as the **total-investment approach**. We compute the PW for each individual alternative, as we did in Section 5.3, and select the one with the highest PW. Note that this approach guarantees valid results only when PW, FW, and AE criteria are used. (As you will see in Chapters 7 and 16, the total-investment approach does not work for any decision criterion based on either a percentage (rate of return) or a ratio (e.g., a benefit–cost ratio). With percentages or ratios, you need to use the incremental investment approach, which also works with any decision criterion, including PW, FW, and AE.) The incremental investment approach will be discussed in detail in Chapter 7.

Scale of Investment

Frequently, mutually exclusive investment projects may require different levels of investments. At first, it seems unfair to compare a project requiring a smaller investment with one requiring a larger investment. However, the disparity in scale of investment should not be of concern in comparing mutually exclusive alternatives, as long as you understand the basic assumption: *Funds not invested in the project will continue to earn interest at the MARR*. We will look at mutually exclusive alternatives that require different levels of investments for both service projects and revenue projects:

- **Service projects.** Typically, what you are asking yourself to do here is to decide whether the higher initial investment can be justified by additional savings that will occur in the future. More efficient machines are usually more expensive to acquire initially, but they will reduce future operating costs, thereby generating more savings.
- **Revenue projects.** If two mutually exclusive revenue projects require different levels of investments with varying future revenue streams, then your question is what to do with the difference in investment funds if you decide to go with the project that requires the smaller investment. To illustrate, consider the two mutually exclusive revenue projects illustrated in Figure 5.12. Our objective is to compare these two projects at a MARR of 10%.

Suppose you have exactly $4,000 to invest. If you choose Project B, you do not have any leftover funds. However, if you go with Project A, you will have $3,000 in unused

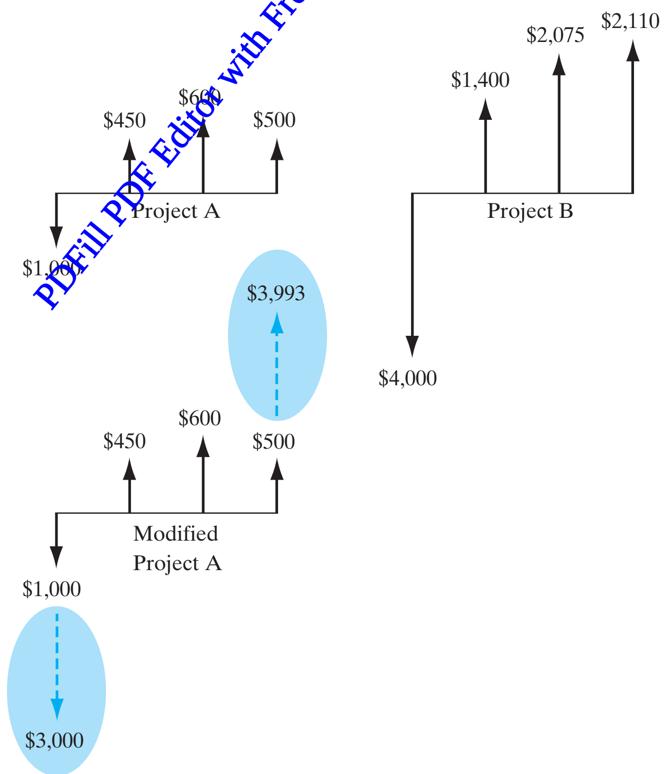


Figure 5.12 Comparing mutually exclusive revenue projects requiring different levels of investment.

funds. Our assumption is that these unused funds will continue to earn an interest rate that is the MARR. Therefore, the unused funds will grow at 10%, or $3,933, at the end of the project term, or three years from now. Consequently, selecting Project A is equivalent to having a modified project cash flow as shown in Figure 5.12.

Let's calculate the net present worth for each option at 10%:

- **Project A:**

$$\begin{aligned}\text{PW}(10\%)_A &= -\$1{,}000 + \$450(P/F, 10\%, 1) + \$600(P/F, 10\%, 2)\\ &\quad + \$500(P/F, 10\%, 3)\\ &= \$283.\end{aligned}$$

- **Project B:**

$$\begin{aligned}\text{PW}(10\%)_B &= -\$4{,}000 + \$1{,}400(P/A, 10\%, 1)\\ &\quad + \$2{,}075(P/F, 10\%, 2) + \$2{,}110(P/F, 10\%, 3)\\ &= \$579.\end{aligned}$$

Clearly, Project B is the better choice. But how about the modified Project A? If we calculate the present worth for the modified Project A, we have the following:

- **Modified Project A:**

$$\begin{aligned}\text{PW}(10\%)_A &= -\$4{,}000 + \$450(P/A, 10\%, 1) + \$600(P/F, 10\%, 2)\\ &\quad + \$4{,}493(P/F, 10\%, 3)\\ &= \$283.\end{aligned}$$

This is exactly the same as the net present worth without including the investment consequence of the unused funds. It is not a surprising result, as the return on investment in the unused funds will be exactly 10%. If we also discount the funds at 10%, there will be no surplus. So, what is the conclusion? It is this: If there is any disparity in investment scale for mutually exclusive revenue projects, go ahead and calculate the net present worth for each option without worrying about the investment differentials.

5.5.2 Analysis Period

The **analysis period** is the time span over which the economic effects of an investment will be evaluated. The analysis period may also be called the **study period** or **planning horizon**. The length of the analysis period may be determined in several ways: It may be a predetermined amount of time set by company policy, or it may be either implied or explicit in the need the company is trying to fulfill. (For example, a diaper manufacturer sees the need to dramatically increase production over a 10-year period in response to an anticipated "baby boom.") In either of these situations, we consider the analysis period to be the **required service period**.

When the required service period is not stated at the outset, the analyst must choose an appropriate analysis period over which to study the alternative investment projects. In such a case, one convenient choice of analysis period is the period of the useful life of the investment project.

When the useful life of an investment project does not match the analysis or required service period, we must make adjustments in our analysis. A further complication in a consideration of two or more mutually exclusive projects is that the investments themselves may have different useful lives. Accordingly, we must compare projects with different useful lives over an **equal time span**, which may require further adjustments in our analysis.

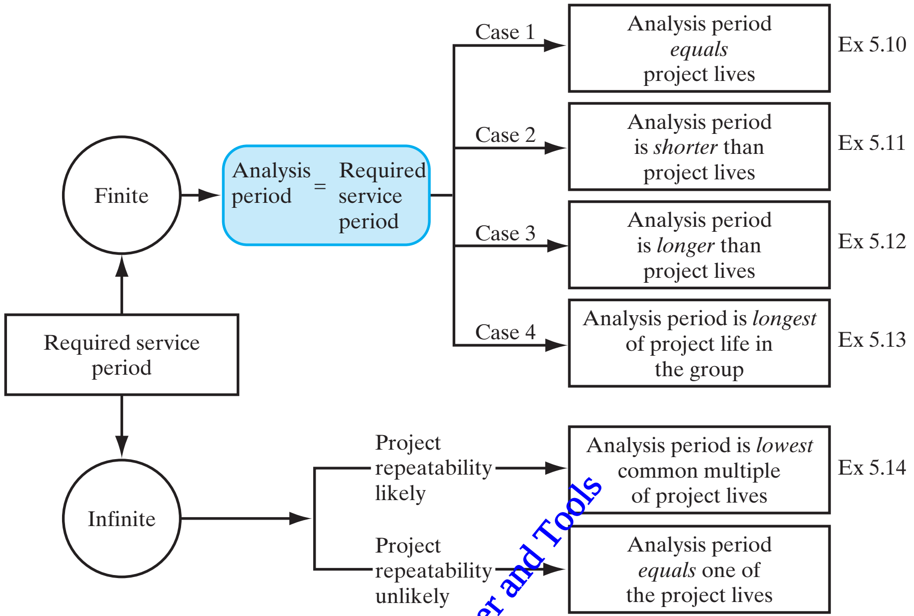


Figure 5.13 Analysis period implied in comparing mutually exclusive alternatives.

(Figure 5.13 is a flowchart showing the possible combinations of the analysis period and the useful life of an investment.) In the sections that follow, we will explore in more detail how to handle situations in which project lives differ from the analysis period and from each other. But we begin with the most straightforward situation: when the project lives and the analysis period coincide.

5.5.3 Analysis Period Equals Project Lives

When the project lives equal the analysis period, we compute the NPW for each project and select the one with the highest NPW. Example 5.10 illustrates this point.

EXAMPLE 5.10 Present-Worth Comparison (Revenue Projects with Equal Lives): Three Alternatives

Bullard Company (BC) is considering expanding its range of industrial machinery products by manufacturing machine tables, saddles, machine bases, and other similar parts. Several combinations of new equipment and personnel could serve to fulfill this new function:

- **Method 1 (M1):** new machining center with three operators.
- **Method 2 (M2):** new machining center with an automatic pallet changer and three operators.
- **Method 3 (M3):** new machining center with an automatic pallet changer and two task-sharing operators.

Each of these arrangements incurs different costs and revenues. The time taken to load and unload parts is reduced in the pallet-changer cases. Certainly, it costs more to acquire, install, and tool-fit a pallet changer, but because the device is more efficient and versatile, it can generate larger annual revenues. Although saving on labor costs, task-sharing operators take longer to train and are more inefficient initially. As the operators become more experienced at their tasks and get used to collaborating with each other, it is expected that the annual benefits will increase by 13% per year over the five-year study period. BC has estimated the investment costs and additional revenues as follows:

	Machining Center Methods		
	M1	**M2**	**M3**
Investment:			
Machine tool purchase	\$121,000	\$121,000	\$121,000
Automatic pallet changer		\$ 66,600	\$ 66,600
Installation	\$ 30,000	\$ 42,000	\$ 42,000
Tooling expense	\$ 58,000	\$ 65,000	\$ 65,000
Total investment	\$209,000	\$294,600	\$294,600
Annual benefits: Year 1			
Additional revenues	\$ 55,000	\$ 69,300	\$ 36,000
Direct labor savings			\$ 17,300
Setup savings		\$ 4,700	\$ 4,700
Year 1: Net revenues	\$ 55,000	\$ 74,000	\$ 58,000
Years 2–5: Net revenues	constant	constant	g = 13%/year
Salvage value in year 5	\$ 80,000	\$120,000	\$120,000

All cash flows include all tax effects. "Do nothing" is obviously an option, since BC will not undertake this expansion if none of the proposed methods is economically viable. If a method is chosen, BC expects to operate the machining center over the next five years. On the basis of the use of the PW measure at $i = 12\%$, which option would be selected?

SOLUTION

Given: Cash flows for three revenue projects and $i = 12\%$ per year.

Find: NPW for each project and which project to select.

For these revenue projects, the net-present-worth figures at $i = 12\%$ would be as follows:

- For Option M1,

$$\begin{aligned} PW(12\%)_{M1} &= -\$209{,}000 + \$55{,}000(P/A, 12\%, 5) \\ &\quad + \$80{,}000(P/F, 12\%, 5) \\ &= \$34{,}657. \end{aligned}$$

- For Option M2,

$$\begin{aligned} PW(12\%)_{M2} &= -\$294{,}600 + \$74{,}000(P/A, 12\%, 5) \\ &\quad + \$120{,}000(P/F, 12\%, 5) \\ &= \$40{,}245. \end{aligned}$$

- For Option M3,

$$\begin{aligned} PW(12\%)_{M3} &= -\$294{,}600 + \$58{,}000(P/A_1, 13\%, 12\%, 5) \\ &\quad + \$120{,}000(P/F, 12\%, 5) \\ &= \$37{,}085. \end{aligned}$$

Clearly, Option M2 is the most profitable. Given the nature of BC parts and shop orders, management decides that the best way to expand would be with an automatic pallet changer, but without task sharing.

5.5.4 Analysis Period Differs from Project Lives

In Example 5.10, we assumed the simplest scenario possible when analyzing mutually exclusive projects: The projects had useful lives equal to each other and to the required service period. In practice, this is seldom the case. Often, project lives do not match the required analysis period or do not match each other (or both). For example, two machines may perform exactly the same function, but one lasts longer than the other, and both of them last longer than the analysis period over which they are being considered. In the sections and examples that follow, we will develop some techniques for dealing with these complications.

Project's Life Is Longer than Analysis Period

Project lives rarely conveniently coincide with a firm's predetermined required analysis period; they are often too long or too short. The case of project lives that are too long is the easier one to address.

Consider the case of a firm that undertakes a five-year production project when all of the alternative equipment choices have useful lives of seven years. In such a case, we analyze each project for only as long as the required service period (in this case, five years). We are then left with some unused portion of the equipment (in this case, two years' worth), which we include as salvage value in our analysis. **Salvage value** is the amount of money for which the equipment could be sold after its service to the project has been rendered. Alternatively, salvage value is the dollar measure of the remaining usefulness of the equipment.

Salvage value: The estimated value that an asset will realize upon its sale at the end of its useful life.

A common instance of project lives that are longer than the analysis period occurs in the construction industry: A building project may have a relatively short completion time, but the equipment that is purchased has a much longer useful life.

EXAMPLE 5.11 Present-Worth Comparison: Project Lives Longer than the Analysis Period

Waste Management Company (WMC) has won a contract that requires the firm to remove radioactive material from government-owned property and transport it to a designated dumping site. This task requires a specially made ripper–bulldozer to dig and load the material onto a transportation vehicle. Approximately 400,000 tons of waste must be moved in a period of two years.

- Model A costs \$150,000 and has a life of 6,000 hours before it will require any major overhaul. Two units of model A would be required to remove the material within two years, and the operating cost for each unit would run to \$40,000/year for 2,000 hours of operation. At this operational rate, the model would be operable for three years, at the end of which time it is estimated that the salvage value will be \$25,000 for each machine.
- A more efficient model B costs \$240,000 each, has a life of 12,000 hours without any major overhaul, and costs \$22,500 to operate for 2,000 hours per year to complete the job within two years. The estimated salvage value of model B at the end of six years is \$30,000. Once again, two units of model B would be required to remove the material within two years.

Since the lifetime of either model exceeds the required service period of two years (Figure 5.14), WMC has to assume some things about the used equipment at the end of that time. Therefore, the engineers at WMC estimate that, after two years, the model A units could be sold for \$45,000 each and the model B units for \$125,000 each. After considering all tax effects, WMC summarized the resulting cash flows (in thousand of dollars) for each project as follows:

Period	Model A		Model B	
0	−\$300		−\$480	
1	−80		−45	
2	−80	$\boxed{+90}$	−45	$\boxed{+250}$
3	−80	+50	−45	
4			−45	
5			−45	
6			−45	+60

Here, the figures in the boxes represent the estimated salvage values at the end of the analysis period (the end of year 2). Assuming that the firm's MARR is 15%, which option would be acceptable?

SOLUTION

Given: Cash flows for two alternatives as shown in Figure 5.14 and $i = 15\%$ per year.

Find: NPW for each alternative and which alternative is preferred.

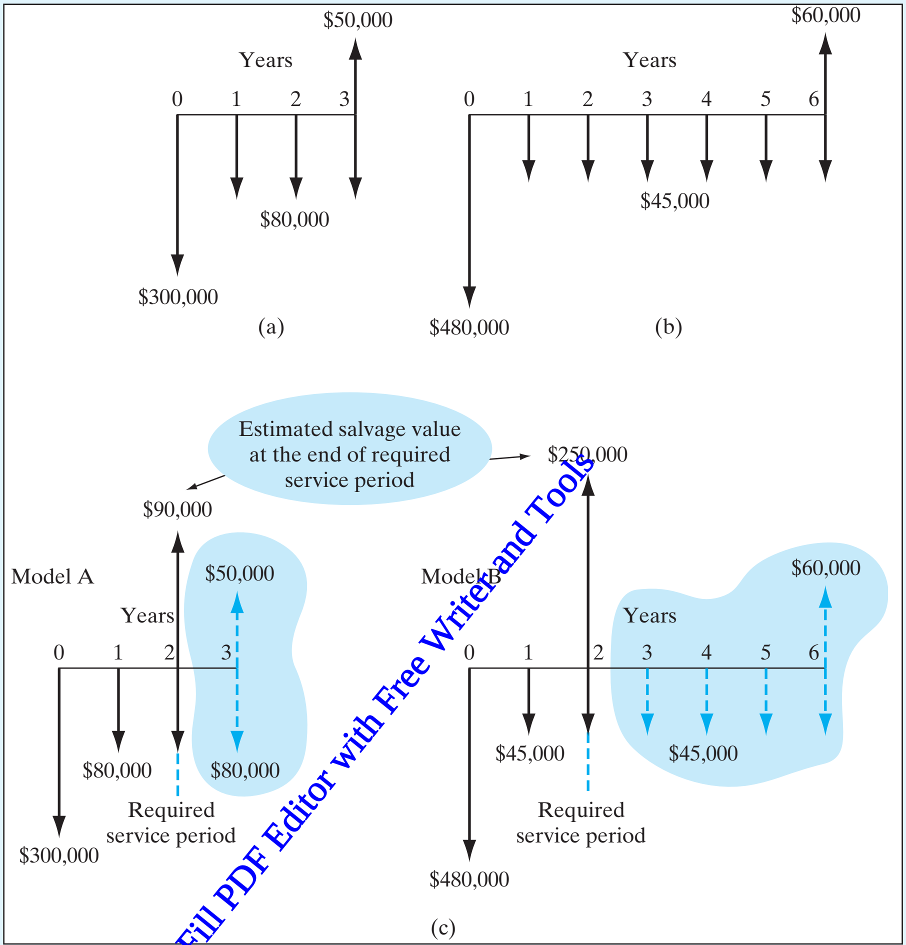


Figure 5.14 (a) Cash flow for model A; (b) cash flow for model B; (c) comparison of service projects with unequal lives when the required service period is shorter than the individual project life (Example 5.11).

First, note that these are service projects, so we can assume the same revenues for both configurations. Since the firm explicitly estimated the market values of the assets at the end of the analysis period (two years), we can compare the two models directly. Because the benefits (removal of the waste) are equal, we can concentrate on the costs:

$$\begin{aligned} PW(15\%)_A &= -\$300 - \$80(P/A, 15\%, 2) + \$90(P/F, 15\%, 2) \\ &= -\$362; \\ PW(15\%)_B &= -\$480 - \$45(P/A, 15\%, 2) + \$250(P/F, 15\%, 2) \\ &= -\$364. \end{aligned}$$

Model A has the least negative PW costs and thus would be preferred.

Project's Life Is Shorter than Analysis Period

When project lives are shorter than the required service period, we must consider how, at the end of the project lives, we will satisfy the rest of the required service period. Replacement projects—additional projects to be implemented when the initial project has reached the limits of its useful life—are needed in such a case. A sufficient number of replacement projects that match or exceed the required service period must be analyzed.

To simplify our analysis, we could assume that the replacement project will be exactly the same as the initial project, with the same costs and benefits. However, this assumption is not necessary. For example, depending on our forecasting skills, we may decide that a different kind of technology—in the form of equipment, materials, or processes—is a preferable replacement. Whether we select exactly the same alternative or a new technology as the replacement project, we are ultimately likely to have some unused portion of the equipment to consider as salvage value, just as in the case when the project lives are longer than the analysis period. Of course, we may instead decide to lease the necessary equipment or subcontract the remaining work for the duration of the analysis period. In this case, we can probably match our analysis period and not worry about salvage values.

In any event, at the outset of the analysis period, we must make some initial guess concerning the method of completing the analysis. Later, when the initial project life is closer to its expiration, we may revise our analysis with a different replacement project. This is only reasonable, since economic analysis is an ongoing activity in the life of a company and an investment project, and we should always use the most reliable, up-to-date data we can reasonably acquire.

EXAMPLE 5.12 Present-Worth Comparison: Project Lives Shorter than the Analysis Period

The Smith Novelty Company, a mail-order firm, wants to install an automatic mailing system to handle product announcements and invoices. The firm has a choice between two different types of machines. The two machines are designed differently, but have identical capacities and do exactly the same job. The $12,500 semiautomatic model A will last three years, while the fully automatic model B will cost $15,000 and last four years. The expected cash flows for the two machines, including maintenance, salvage value, and tax effects, are as follows:

n	Model A	Model B
0	−$12,500	−$15,000
1	−5,000	−4,000
2	−5,000	−4,000
3	−5,000 + 2,000	−4,000
4		−4,000 + 1,500
$\boxed{5}$		

As business grows to a certain level, neither of the models may be able to handle the expanded volume at the end of year 5. If that happens, a fully computerized mail-order system will need to be installed to handle the increased business volume. In the scenario just presented, which model should the firm select at MARR = 15%?

SOLUTION

Given: Cash flows for two alternatives as shown in Figure 5.15, analysis period of five years, and $i = 15\%$.

Find: NPW of each alternative and which alternative to select.

Since both models have a shorter life than the required service period of 5 years, we need to make an explicit assumption of how the service requirement is to be met. Suppose that the company considers leasing equipment comparable to model A at an annual payment of $6,000 (after taxes) and with an annual operating cost of $5,000

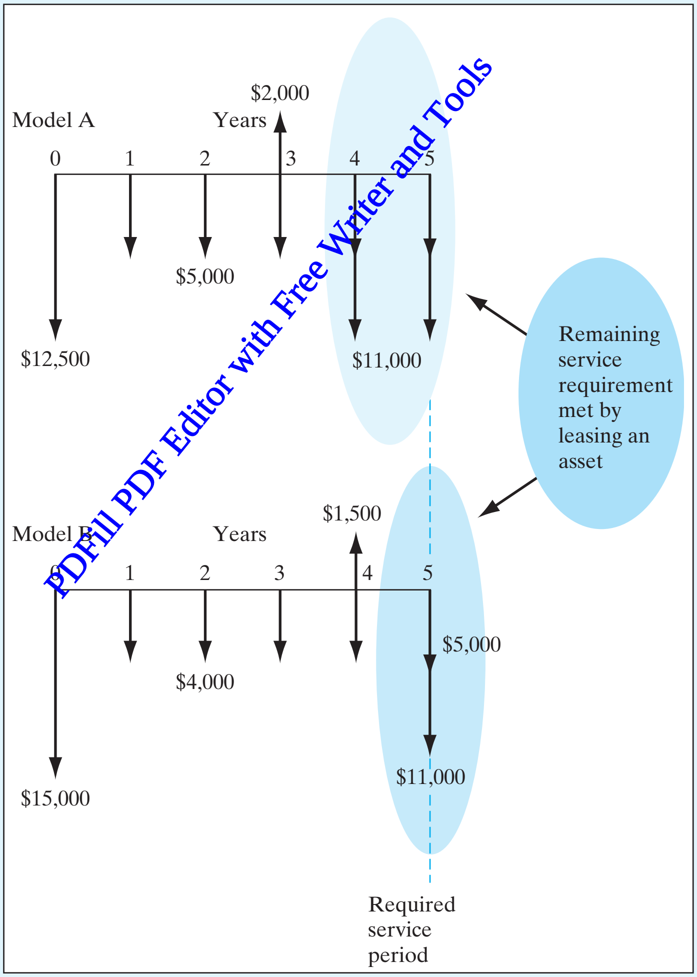


Figure 5.15 Comparison for service projects with unequal lives when the required service period is longer than the individual project life (Example 5.12).

for the remaining required service period. In this case, the cash flow would look like that shown in Figure 5.15:

n	Model A	Model B
0	−\$12,500	−\$15,000
1	−5,000	−4,000
2	−5,000	−4,000
3	−5,000 + 2,000	−4,000
4	−5,000 − $\boxed{6,000}$	−4,000 + 1,500
5	−5,000 − $\boxed{6,000}$	−5,000 − $\boxed{6,000}$

Here, the boxed figures represent the annual lease payments. (It costs \$6,000 to lease the equipment and \$5,000 to operate it annually. Other maintenance costs will be paid by the leasing company.) Note that both alternatives now have the same required service period of five years. Therefore, we can use NPW analysis:

$$\begin{aligned}\text{PW}(15\%)_{\text{A}} &= -\$12{,}500 - \$5{,}000(P/A, 15\%, 2) - \$3{,}000(P/F, 15\%, 3)\\ &\quad - \$11{,}000(P/A, 15\%, 2)(P/F, 15\%, 3)\\ &= -\$34{,}359.\end{aligned}$$

$$\begin{aligned}\text{PW}(15\%)_{\text{B}} &= -\$15{,}000 - \$4{,}000(P/A, 15\%, 3) - \$2{,}500(P/F, 15\%, 4)\\ &\quad - \$11{,}000(P/F, 15\%, 5)\\ &= -\$31{,}031.\end{aligned}$$

Since these are service projects, model B is the better choice.

Analysis Period Coincides with Longest Project Life

As seen in the preceding pages, equal future periods are generally necessary to achieve comparability of alternatives. In some situations, however, revenue projects with different lives can be compared if they require only a one-time investment because the task or need within the firm is a one-time task or need. An example of this situation is the extraction of a fixed quantity of a natural resource such as oil or coal.

Consider two mutually exclusive processes: One requires 10 years to recover some coal, and the other can accomplish the task in only 8 years. There is no need to continue the project if the short-lived process is used and all the coal has been retrieved. In this example, the two processes can be compared over an analysis period of 10 years (the longest project life of the two being considered), assuming that no cash flows after 8 years for the shorter lived project. Because of the time value of money, the revenues must be included in the analysis even if the price of coal is constant. Even if the total (undiscounted) revenue is equal for either process, that for the faster process has a larger present worth. Therefore, the two projects could be compared by using the NPW of each over its own life. Note that in this case the analysis period is determined by, and coincides with, the longest project life in the group. (Here we are still, in effect, assuming an analysis period of 10 years.)

EXAMPLE 5.13 Present-Worth Comparison: A Case where the Analysis Period Coincides with the Project with the Longest Life in the Mutually Exclusive Group

The family-operated Foothills Ranching Company (FRC) owns the mineral rights to land used for growing grain and grazing cattle. Recently, oil was discovered on this property. The family has decided to extract the oil, sell the land, and retire. The company can lease the necessary equipment and extract and sell the oil itself, or it can lease the land to an oil-drilling company:

- **Drill option.** If the company chooses to drill, it will require \$300,000 leasing expenses up front, but the net annual cash flow after taxes from drilling operations will be \$600,000 at the end of each year for the next five years. The company can sell the land for a net cash flow of \$1,000,000 in five years, when the oil is depleted.
- **Lease option.** If the company chooses to lease, the drilling company can extract all the oil in only three years, and FRC can sell the land for a net cash flow of \$800,000 at that time. (The difference in resale value of the land is due to the increasing rate of land appreciation anticipated for this property.) The net cash flow from the lease payments to FRC will be \$630,000 at the *beginning* of each of the next three years.

All benefits and costs associated with the two alternatives have been accounted for in the figures listed. Which option should the firm select at $i = 15\%$?

SOLUTION

Given: Cash flows shown in Figure 5.16 and $i = 15\%$ per year.

Find: NPW of each alternative and which alternative to select.

As illustrated in Figure 5.16, the cash flows associated with each option look like this:

n	Drill	Lease
0	−\$300,000	\$630,000
1	600,000	630,000
2	600,000	630,000
3	600,000	800,000
4	600,000	
5	1,600,000	

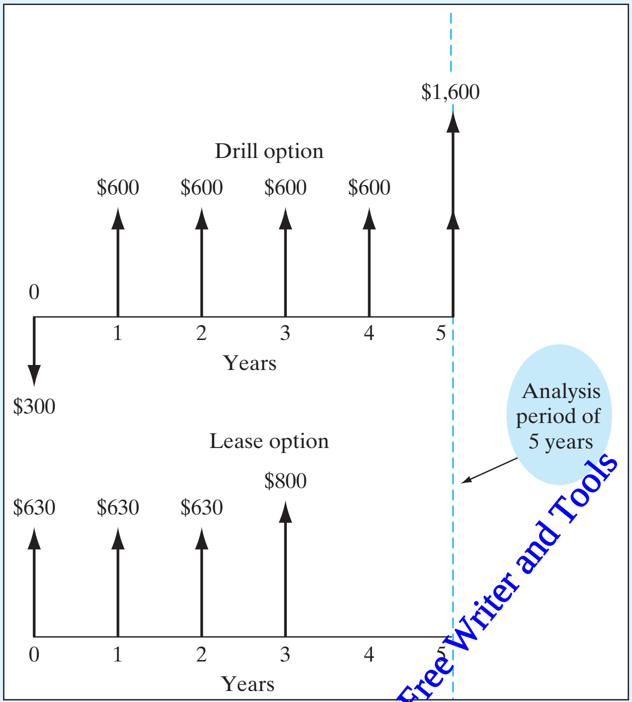


Figure 5.16 Comparison of revenue projects with unequal lives when the analysis period coincides with the project with the longest life in the mutually exclusive group (Example 5.13). In our example, the analysis period is five years, assuming no cash flow in years 4 and 5 for the lease option.

After depletion of the oil, the project will terminate. The present worth of each of the two options is as follows:

$$\begin{aligned}\mathrm{PW}(15\%)_{\mathrm{Drill}} &= -\$300{,}000 + \$600{,}000(P/A, 15\%, 4) \\ &\quad + \$1{,}600{,}000(P/F, 15\%, 5) \\ &= \$2{,}208{,}470.\end{aligned}$$

$$\begin{aligned}\mathrm{PW}(15\%)_{\mathrm{Lease}} &= \$630{,}000 + \$630{,}000(P/A, 15\%, 2) \\ &\quad + \$800{,}000(P/F, 15\%, 3) \\ &= \$2{,}180{,}210.\end{aligned}$$

Note that these are revenue projects; therefore, the drill option appears to be the marginally better option.

COMMENTS: The relatively small difference between the two NPW amounts ($28,260) suggests that the actual decision between drilling and leasing might be based on noneconomic issues. Even if the drilling option were slightly better, the company might prefer to forgo the small amount of additional income and select the lease option, rather than undertake an entirely new business venture and do its own drilling. A variable that might also have a critical effect on this decision is the sales value of the land in each alternative. The value of land is often difficult to forecast over any long period, and the firm may feel some uncertainty about the accuracy of its guesses. In Chapter 12, we will discuss sensitivity analysis, a method by which we can factor uncertainty about the accuracy of project cash flows into our analysis.

5.5.5 Analysis Period Is Not Specified

Our coverage so far has focused on situations in which an analysis period is known. When an analysis period is not specified, either explicitly by company policy or practice or implicitly by the projected life of the investment project, it is up to the analyst to choose an appropriate one. In such a case, the most convenient procedure is to choose an analysis on the basis of the useful lives of the alternatives. When the alternatives have equal lives, this is an easy selection. When the lives of at least some of the alternatives differ, we must select an analysis period that allows us to compare projects with different lifetimes on an equal time basis—that is, a **common service period**.

Lowest Common Multiple of Project Lives

Least common multiple of two numbers is the lowest number that can be divided by *both*.

A required service period of infinity may be assumed if we anticipate that an investment project will be proceeding at roughly the same level of production for some indefinite period. It is certainly possible to make this assumption mathematically, although the analysis is likely to be complicated and tedious. Therefore, in the case of an indefinitely ongoing investment project, we typically select a finite analysis period by using the **lowest common multiple** of project lives. For example, if alternative A has a 3-year useful life and alternative B has a 4-year life, we may select 12 years as the analysis or common service period. We would consider alternative A through four life cycles and alternative B through three life cycles; in each case, we would use the alternatives completely. We then accept the finite model's results as a good prediction of what will be the economically wisest course of action for the foreseeable future. The next example is a case in which we conveniently use the lowest common multiple of project lives as our analysis period.

EXAMPLE 5.14 Present-Worth Comparison: Unequal Lives, Lowest-Common-Multiple Method

Consider Example 5.12. Suppose that models A and B can each handle the increased future volume and that the system is not going to be phased out at the end of five years. Instead, the current mode of operation is expected to continue indefinitely. Suppose also that the two models will be available in the future without significant changes in price and operating costs. At MARR = 15%, which model should the firm select?

SOLUTION

Given: Cash flows for two alternatives as shown in Figure 5.17, $i = 15\%$ per year, and an indefinite period of need.

Find: NPW of each alternative and which alternative to select.

Recall that the two mutually exclusive alternatives have different lives, but provide identical annual benefits. In such a case, we ignore the common benefits and can make the decision solely on the basis of costs, as long as a common analysis period is used for both alternatives.

To make the two projects comparable, let's assume that, after either the 3- or 4-year period, the system would be reinstalled repeatedly, using the same model, and that the same costs would apply. The lowest common multiple of 3 and 4 is 12, so we will use 12 years as the common analysis period. Note that any cash flow difference between the alternatives will be revealed during the first 12 years. After that, the same

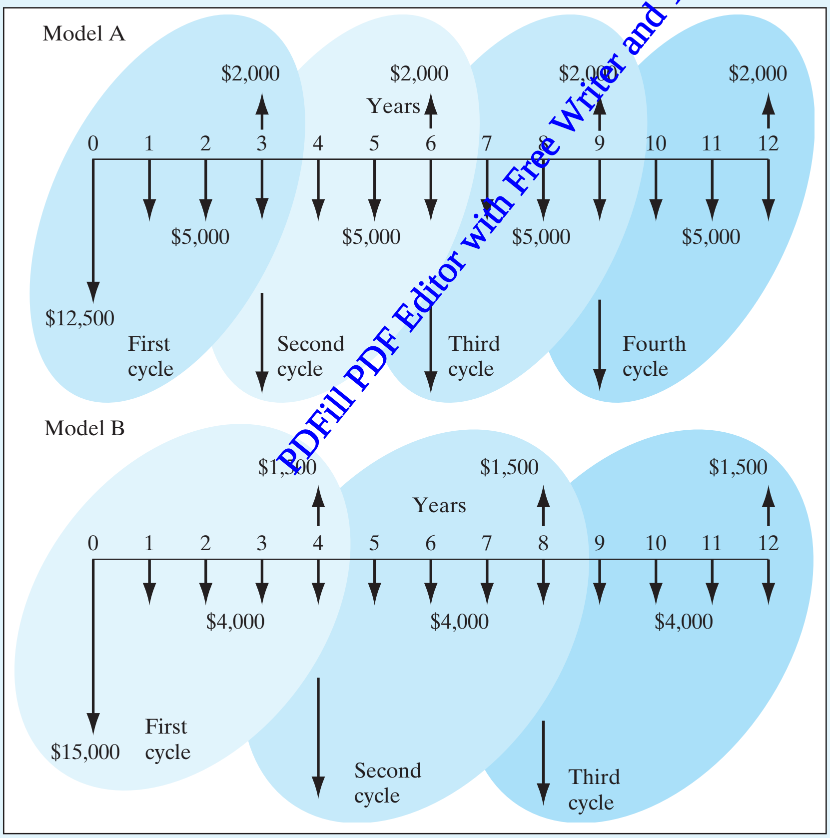


Figure 5.17 Comparison of projects with unequal lives when the required service period is infinite and the project is likely to be repeatable with the same investment and operations and maintenance costs in all future replacement cycles (Example 5.14).

cash flow pattern repeats every 12 years for an indefinite period. The replacement cycles and cash flows are shown in Figure 5.17. Here is our analysis:

- **Model A.** Four replacements occur in a 12-year period. The PW for the first investment cycle is

$$\begin{aligned} PW(15\%) &= -\$12{,}500 - \$5{,}000(P/A, 15\%, 3) \\ &\quad + \$2{,}000(P/F, 15\%, 3) \\ &= -\$22{,}601. \end{aligned}$$

  With four replacement cycles, the total PW is

$$\begin{aligned} PW(15\%) &= -\$22{,}601[1 + (P/F, 15\%, 3) \\ &\quad + (P/F, 15\%, 6) + (P/F, 15\%, 9)] \\ &= -\$53{,}657. \end{aligned}$$

- **Model B.** Three replacements occur in a 12-year period. The PW for the first investment cycle is

$$\begin{aligned} PW(15\%) &= -\$15{,}000 - \$4{,}000(P/A, 15\%, 4) \\ &\quad + \$1{,}500(P/F, 15\%, 4) \\ &= -\$25{,}562. \end{aligned}$$

  With three replacement cycles in 12 years, the total PW is

$$\begin{aligned} PW(15\%) &= -\$25{,}562[1 + (P/F, 15\%, 4) + (P/F, 15\%, 8)] \\ &= -\$48{,}534. \end{aligned}$$

  Clearly, model B is a better choice, as before.

COMMENTS: In Example 5.14, an analysis period of 12 years seems reasonable. The number of actual reinvestment cycles needed with each type of system will depend on the technology of the future system, so we may or may not actually need the four reinvestment cycles (model A) or three (model B) we used in our analysis. The validity of the analysis also depends on the costs of the system and labor remaining constant. If we assume **constant-dollar prices** (see Chapter 11), this analysis would provide us with a reasonable result. (As you will see in Example 6.3, the annual-worth approach makes it mathematically easier to solve this type of comparison.) If we cannot assume constant-dollar prices in future replacements, we need to estimate the costs for each replacement over the analysis period. This will certainly complicate the problem significantly.

Other Common Analysis Periods

In some cases, the lowest common multiple of project lives is an unwieldy analysis period to consider. Suppose, for example, that you were considering two alternatives with lives of 7 and 12 years, respectively. Besides making for tedious calculations, an 84-year analysis period may lead to inaccuracies, since, over such a long time, we can be less and less confident about the ability to install identical replacement projects with identical costs and benefits. In a case like this, it would be reasonable to use the useful life of one of the alternatives by either factoring in a replacement project or salvaging the remaining useful life, as the case may be. The important idea is that we must compare both projects on the same time basis.

SUMMARY

In this chapter, we presented the concept of present-worth analysis, based on cash flow equivalence along with the payback period. We observed the following important results:

- Present worth is an equivalence method of analysis in which a project's cash flows are discounted to a single present value. It is perhaps the most efficient analysis method we can use in determining the acceptability of a project on an economic basis. Other analysis methods, which we will study in Chapters 6 and 7, are built on a sound understanding of present worth.
- The minimum attractive rate of return (MARR) is the interest rate at which a firm can always earn or borrow money under a normal operating environment. It is generally dictated by management and is the rate at which NPW analysis should be conducted.
- Revenue projects are those for which the income generated depends on the choice of project. Service projects are those for which the income remains the same, regardless of which project is selected.
- Several alternatives that meet the same need are **mutually exclusive** if, whenever one of them is selected, the others will be rejected.
- When not specified by management or company policy, the analysis period to use in a comparison of mutually exclusive projects may be chosen by an individual analyst. Several efficiencies can be applied when an analysis period is selected. In general, the analysis period should be chosen to cover the required service period, as highlighted in Figure 5.13.

PROBLEMS

Note: *Unless otherwise stated, all cash flows are after-tax cash flows. The interest rate (MARR) is also given on an after-tax basis.*

Identifying Cash Inflows and Outflows

5.1 Camptown Togs, Inc., a children's clothing manufacturer, has always found payroll processing to be costly because it must be done by a clerk so that the number of piece-goods coupons received by each employee can be collected and the types of tasks performed by each employee can be calculated. Not long ago, an industrial engineer designed a system that partially automates the process by means of a scanner that reads the piece-goods coupons. Management is enthusiastic about this system because it utilizes some personal computer systems that were purchased recently. It is expected that this new automated system will save $45,000 per year in labor. The new system will cost about $30,000 to build and test prior to operation. It is expected that operating costs, including income taxes, will be about $5,000 per year. The system will have a five-year useful life. The expected net salvage value of the system is estimated to be $3,000.

(a) Identify the cash inflows over the life of the project.
(b) Identify the cash outflows over the life of the project.
(c) Determine the net cash flows over the life of the project.

Payback Period

5.2 Refer to Problem 5.1, and answer the following questions:

(a) How long does it take to recover the investment?
(b) If the firm's interest rate is 15% after taxes, what would be the discounted payback period for this project?

5.3 Consider the following cash flows:

	Project's Cash Flow ($)			
n	A	B	C	D
0	−$2,500	−$3,000	−$5,500	−$4,000
1	300	2,000	2,000	5,000
2	300	1,500	2,000	−3,000
3	300	1,500	2,000	−2,500
4	300	500	5,000	1,000
5	300	500	5,000	1,000
6	300	1,500		2,000
7	300			3,000
8	300			

(a) Calculate the payback period for each project.
(b) Determine whether it is meaningful to calculate a payback period for project D.
(c) Assuming that $i = 10\%$, calculate the discounted payback period for each project.

NPW Criterion

5.4 Consider the following sets of investment projects, all of which have a three-year investment life:

	Project's Cash Flow ($)			
n	A	B	C	D
0	−$1,500	−$1,200	−$1,600	−$3,100
1	0	600	−1,800	800
2	0	800	800	1,900
3	3,000	1,500	2,500	2,300

(a) Compute the net present worth of each project at $i = 10\%$.
(b) Plot the present worth as a function of the interest rate (from 0% to 30%) for project B.

5.5 You need to know whether the building of a new warehouse is justified under the following conditions:

The proposal is for a warehouse costing \$200,000. The warehouse has an expected useful life of 35 years and a net salvage value (net proceeds from sale after tax adjustments) of \$35,000. Annual receipts of \$37,000 are expected, annual maintenance and administrative costs will be \$8,000/year, and annual income taxes are \$5,000.

Given the foregoing data, which of the following statements are correct?

(a) The proposal is justified for a MARR of 9%.

(b) The proposal has a net present worth of \$152,512 when 6% is used as the interest rate.

(c) The proposal is acceptable, as long as MARR $\leq$ 11.81%.

(d) All of the preceding are correct.

5.6 Your firm is considering purchasing an old office building with an estimated remaining service life of 25 years. Recently, the tenants signed long-term leases, which leads you to believe that the current rental income of \$150,000 per year will remain constant for the first 5 years. Then the rental income will increase by 10% for every 5-year interval over the remaining life of the asset. For example, the annual rental income would be \$165,000 for years 6 through 10, \$181,500 for years 11 through 15, \$199,650 for years 16 through 20, and \$219,615 for years 21 through 25. You estimate that operating expenses, including income taxes, will be \$45,000 for the first year and that they will increase by \$3,000 each year thereafter. You also estimate that razing the building and selling the lot on which it stands will realize a net amount of \$50,000 at the end of the 25-year period. If you had the opportunity to invest your money elsewhere and thereby earn interest at the rate of 12% per annum, what would be the maximum amount you would be willing to pay for the building and lot at the present time?

5.7 Consider the following investment project:

n	A_n	i
0	−\$42,000	10%
1	32,400	11
2	33,400	13
3	32,500	15
4	32,500	12
5	33,000	10

Suppose the company's reinvestment opportunities change over the life of the project as shown in the preceding table (i.e., the firm's MARR changes over the life of the project). For example, the company can invest funds available now at 10% for the first year, 11% for the second year, and so forth. Calculate the net present worth of this investment and determine the acceptability of the investment.

5.8 Cable television companies and their equipment suppliers are on the verge of installing new technology that will pack many more channels into cable networks, thereby creating a potential programming revolution with implications for broadcasters, telephone companies, and the consumer electronics industry.

Digital compression uses computer techniques to squeeze 3 to 10 programs into a single channel. A cable system fully using digital compression technology would be able to offer well over 100 channels, compared with about 35 for the average cable television system now used. If the new technology is combined with the increased use of optical fibers, it might be possible to offer as many as 300 channels.

A cable company is considering installing this new technology to increase subscription sales and save on satellite time. The company estimates that the installation will take place over 2 years. The system is expected to have an 8-year service life and produce the following savings and expenditures:

Digital Compression	
Investment	
Now	$500,000
First year	$3,200,000
Second year	$4,000,000
Annual savings in satellite time	$2,000,000
Incremental annual revenues due to new subscriptions	$4,000,000
Incremental annual expenses	$1,500,000
Incremental annual income taxes	$1,300,000
Economic service life	8 years
Net salvage value	$1,200,000

Note that the project has a 2-year investment period, followed by an 8-year service life (a total 10-year life for the project). This implies that the first annual savings will occur at the end of year 3 and the last will occur at the end of year 10. If the firm's MARR is 15%, use the NPW method to justify the economic worth of the project.

5.9 A large food-processing corporation is considering using laser technology to speed up and eliminate waste in the potato-peeling process. To implement the system, the company anticipates needing $3.5 million to purchase the industrial-strength lasers. The system will save $1,550,000 per year in labor and materials. However, it will require an additional operating and maintenance cost of $350,000. Annual income taxes will also increase by $150,000. The system is expected to have a 10-year service life and will have a salvage value of about $200,000. If the company's MARR is 18%, use the NPW method to justify the economics of the project.

Future Worth and Project Balance

5.10 Consider the following sets of investment projects, all of which have a three-year investment life:

Period (n)	Project's Cash Flow A	B	C	D
0	−$12,500	−11,000	12,500	−13,000
1	5,400	−3,000	−7,000	5,500
2	14,400	21,000	−2,000	5,500
3	7,200	13,000	4,000	8,500

(a) Compute the net present worth of each project at $i = 15\%$.

(b) Compute the net future worth of each project at $i = 15\%$.

Which project or projects are acceptable?

5.11 Consider the following project balances for a typical investment project with a service life of four years:

n	A_n	Project Balance
0	−$1,000	−$1,000
1	( )	−1,100
2	( )	−800
3	460	−500
4	( )	0

(a) Construct the original cash flows of the project.

(b) Determine the interest rate used in computing the project balance.

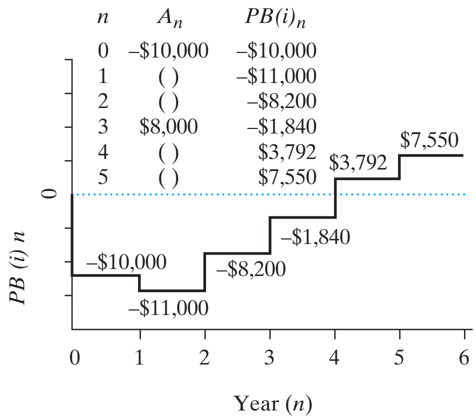
(c) Would the project be acceptable at $i = 15\%$?

5.12 Your R&D group has developed and tested a computer software package that assists engineers to control the proper chemical mix for the various process-manufacturing industries. If you decide to market the software, your first-year operating net cash flow is estimated to be $1,000,000. Because of market competition, product life will be about 4 years, and the product's market share will decrease by 25% each year over the previous year's share. You are approached by a big software house which wants to purchase the right to manufacture and distribute the product. Assuming that your interest rate is 15%, for what minimum price would you be willing to sell the software?

5.13 Consider the accompanying project balance diagram for a typical investment project with a service life of five years. The numbers in the figure indicate the beginning project balances.

(a) From the project balance diagram, construct the project's original cash flows.

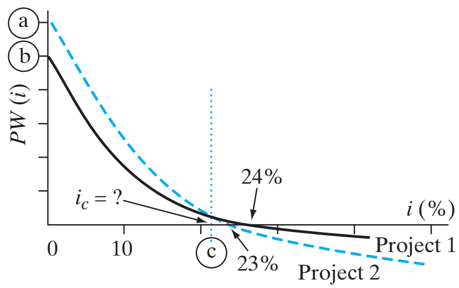
(b) What is the project's conventional payback period (without interest)?



5.14 Consider the following cash flows and present-worth profile:

Year	Net Cash Flows ($) Project 1	Project 2
0	−$1,000	−$1,000
1	400	300
2	800	Y
3	X	800

(a) Determine the values of X and Y.
(b) Calculate the terminal project balance of project 1 at MARR = 24%.
(c) Find the values of a, b, and c in the NPW plot.



5.15 Consider the project balances for a typical investment project with a service life of five years, as shown in Table P5.15.

(a) Construct the original cash flows of the project and the terminal balance, and fill in the blanks in the preceding table.
(b) Determine the interest rate used in the project balance calculation, and compute the present worth of this project at the computed interest rate.

TABLE P5.15 Investment Project Balances

n	A_n	Project Balance
0	−$3,000	−$3,000
1		−2,700
2	1,470	−1,500
3		0
4		−300
5	600	

5.16 Refer to Problem 5.3, and answer the following questions:

(a) Graph the project balances (at $i = 10\%$) of each project as a function of n.

(b) By examining the graphical results in part (a), determine which project appears to be the safest to undertake if there is some possibility of premature termination of the projects at the end of year 2.

5.17 Consider the following investment projects:

	Project's Cash Flow				
n	A	B	C	D	E
0	−$1,800	−$5,200	−$3,800	−$4,000	−$6,500
1	−500	2,500	0	500	1,000
2	900	−4,000	0	2,000	3,600
3	1,300	5,000	4,000	3,000	2,400
4	2,200	6,000	7,000	4,000	
5	−700	3,000	12,000	1,250	

(a) Compute the future worth at the end of life for each project at $i = 12\%$.

(b) Determine the acceptability of each project.

5.18 Refer to Problem 5.17, and answer the following questions:

(a) Plot the future worth for each project as a function of the interest rate (0%–50%).

(b) Compute the project balance of each project at $i = 12\%$.

(c) Compare the terminal project balances calculated in (b) with the results obtained in Problem 5.17(a). Without using the interest factor tables, compute the future worth on the basis of the project balance concept.

5.19 Covington Corporation purchased a vibratory finishing machine for $20,000 in year 0. The useful life of the machine is 10 years, at the end of which the machine is estimated to have a salvage value of zero. The machine generates net annual revenues of $6,000. The annual operating and maintenance expenses are estimated to be $1,000. If Covington's MARR is 15%, how many years will it take before this machine becomes profitable?

5.20 Gene Research, Inc., just finished a 4-year program of R&D and clinical trial. It expects a quick approval from the Food and Drug Administration. If Gene markets the product its own, the company will require \$30 million immediately $(n = 0)$ to build a new manufacturing facility, and it is expected to have a 10-year product life. The R&D expenditure in the previous years and the anticipated revenues that the company can generate over the next 10 years are summarized as follows:

Period (n)	Cash Flow (Unit: \$ million)
-4	–\$10
-3	–10
-2	–10
-1	–10
0	$-10 - 30$
1–10	100

Merck, a large drug company is interested in purchasing the R&D project and the right to commercialize the product from Gene Research, Inc.; it wants to do so immediately $(n = 0)$. What would be a starting negotiating price for the project from Merck? Assume that Gene's MARR $= 20\%$.

5.21 Consider the following independent investment projects:

n	Project Cash Flows A	B	C
0	$-\$400$	$-\$300$	\$100
1	150	140	-40
2	150	140	-40
3	350	140	-40
4	-200	110	
5	400	110	
6	300		

Assume that MARR $= 10\%$, and answer the following questions:

(a) Compute the net present worth for each project, and determine the acceptability of each.

(b) Compute the net future worth of each project at the end of each project period, and determine the acceptability of each project.

(c) Compute the project worth of each project at the end of six years with variable MARRs as follows: 10% for $n = 0$ to $n = 3$ and 15% for $n = 4$ to $n = 6$.

5.22 Consider the project balance profiles shown in Table P5.22 for proposed investment projects. Project balance figures are rounded to nearest dollars.

(a) Compute the net present worth of projects A and C.

(b) Determine the cash flows for project A.

(c) Identify the net future worth of project C.
(d) What interest rate would be used in the project balance calculations for project B?

TABLE P5.22 Profiles for Proposed Investment Projects

	Project Balances		
n	A	B	C
0	−$1,000	−$1,000	−$1,000
1	−1,000	−650	−1,200
2	−900	−348	−1,440
3	−690	−100	−1,328
4	−359	85	−1,194
5	105	198	−1,000
Interest rate used	10%	?	20%
NPW	?	$79.57	?

5.23 Consider the following project balance profiles for proposed investment projects:

	Project Balances		
n	A	B	C
0	−$1,000	−$1,000	−$1,000
1	−800	−680	−530
2	−600	−302	*X*
3	−400	−57	−211
4	−200	233	−89
5	0	575	0
Interest rate used	10%	18%	12%

Project balance figures are rounded to the nearest dollar.
(a) Compute the net present worth of each investment.
(b) Determine the project balance for project C at the end of period 2 if $A_2 = \$500$.
(c) Determine the cash flows for each project.
(d) Identify the net future worth of each project.

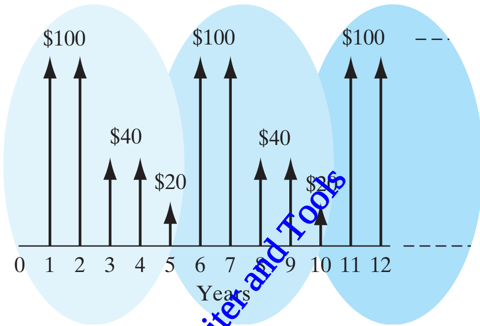
Capitalized Equivalent Worth

5.24 Maintenance money for a new building has been sought. Mr. Kendall would like to make a donation to cover all future expected maintenance costs for the building. These maintenance costs are expected to be $50,000 each year for the first 5 years, $70,000 each year for years 6 through 10, and $90,000 each year after that. (The building has an indefinite service life.)

(a) If the money is placed in an account that will pay 13% interest compounded annually, how large should the gift be?

(b) What is the equivalent annual maintenance cost over the infinite service life of the building?

5.25 Consider an investment project, the cash flow pattern of which repeats itself every five years forever as shown in the accompanying diagram. At an interest rate of 14%, compute the capitalized equivalent amount for this project.



5.26 A group of concerned citizens has established a trust fund that pays 6% interest, compounded monthly, to preserve a historical building by providing annual maintenance funds of $30,000 forever. Compute the capitalized equivalent amount for these building maintenance expenses.

5.27 A newly constructed bridge costs $5,000,000. The same bridge is estimated to need renovation every 15 years at a cost of $1,000,000. Annual repairs and maintenance are estimated to be $100,000 per year.

(a) If the interest rate is 5%, determine the capitalized cost of the bridge.

(b) Suppose that, in (a), the bridge must be renovated every 20 years, not every 15 years. What is the capitalized cost of the bridge?

(c) Repeat (a) and (b) with an interest rate of 10%. What have you to say about the effect of interest on the results?

5.28 To decrease the costs of operating a lock in a large river, a new system of operation is proposed. The system will cost $650,000 to design and build. It is estimated that it will have to be reworked every 10 years at a cost of $100,000. In addition, an expenditure of $50,000 will have to be made at the end of the fifth year for a new type of gear that will not be available until then. Annual operating costs are expected to be $30,000 for the first 15 years and $35,000 a year thereafter. Compute the capitalized cost of perpetual service at $i = 8\%$.

Mutually Exclusive Alternatives

5.29 Consider the following cash flow data for two competing investment projects:

	Cash Flow Data (Unit: $ thousand)	
n	Project A	Project B
0	−$800	−$2,635
1	−1,500	−565
2	−435	820
3	775	820
4	775	1,080
5	1,275	1,880
6	1,275	1,500
7	975	980
8	675	580
9	375	380
10	660	840

At $i = 12\%$, which of the two projects would be a better choice?

5.30 Consider the cash flows for the following investment projects.

	Project's Cash Flow				
n	A	B	C	D	E
0	−$1,500	−$1,500	−$3,000	$1,500	−$1,800
1	1,350	1,000	1,000	−450	600
2	800	800	X	−450	600
3	200	800	1,500	−450	600
4	100	150	X	−450	600

(a) Suppose projects A and B are mutually exclusive. On the basis of the NPW criterion, which project would be selected? Assume that MARR = 15%.

(b) Repeat (a), using the NFW criterion.

(c) Find the minimum value of X that makes project C acceptable.

(d) Would you accept project D at $i = 18\%$?

(e) Assume that projects D and E are mutually exclusive. On the basis of the NPW criterion, which project would you select?

5.31 Consider two mutually exclusive investment projects, each with MARR = 12%, as shown in Table 5.31.

(a) On the basis of the NPW criterion, which alternative would be selected?

(b) On the basis of the NFW criterion, which alternative would be selected?

TABLE P5.31 Two Mutually Exclusive Investment Projects

	Project's Cash Flow	
n	A	B
0	−\$14,500	−\$12,900
1	12,610	11,210
2	12,930	11,720
3	12,300	11,500

5.32 Consider the following two mutually exclusive investment projects, each with MARR = 15%:

	Project's Cash Flow	
n	A	B
0	−\$6,000	−\$8,000
1	800	11,500
2	14,000	400

(a) On the basis of the NPW criterion, which project would be selected?

(b) Sketch the PW(i) function for each alternative on the same chart between 0% and 50%. For what range of i would you prefer project B?

5.33 Two methods of carrying away surface runoff water from a new subdivision are being evaluated:

Method A. Dig a ditch. The first cost would be \$60,000, and \$25,000 of redigging and shaping would be required at five-year intervals forever.

Method B. Lay concrete pipe. The first cost would be \$150,000, and a replacement would be required at 50-year intervals at a net cost of \$180,000 forever.

At $i = 12\%$, which method is the better one?

5.34 A local car dealer is advertising a standard 24-month lease of \$1,150 per month for its new XT 3000 series sports car. The standard lease requires a down payment of \$4,500, plus a \$1,000 refundable initial deposit *now*. The first lease payment is due at the end of month 1. In addition, the company offers a 24-month lease plan that has a single up-front payment of \$30,500, plus a refundable initial deposit of \$1,000. Under both options, the initial deposit will be refunded at the end of month 24. Assume an interest rate of 6% compounded monthly. With the present-worth criterion, which option is preferred?

5.35 Two machines are being considered for a manufacturing process. Machine A has a first cost of \$75,200, and its salvage value at the end of six years of estimated service life is \$21,000. The operating costs of this machine are estimated to be \$6,800 per year. Extra income taxes are estimated at \$2,400 per year. Machine B has a first cost of \$44,000, and its salvage value at the end of six years' service is estimated to be negligible. The annual operating costs will be \$11,500. Compare these two mutually exclusive alternatives by the present-worth method at $i = 13\%$.

5.36 An electric motor is rated at 10 horsepower (HP) and costs $800. Its full load efficiency is specified to be 85%. A newly designed high-efficiency motor of the same size has an efficiency of 90%, but costs $1,200. It is estimated that the motors will operate at a rated 10 HP output for 1,500 hours a year, and the cost of energy will be $0.07 per kilowatt-hour. Each motor is expected to have a 15-year life. At the end of 15 years, the first motor will have a salvage value of $50 and the second motor will have a salvage value of $100. Consider the MARR to be 8%. (Note: 1 HP = 0.7457 kW.)

(a) Use the NPW criterion to determine which motor should be installed.

(b) In (a), what if the motors operated 2,500 hours a year instead of 1,500 hours a year? Would the motor you chose in (a) still be the choice?

5.37 Consider the following two mutually exclusive investment projects:

	Project's Cash Flow	
n	**A**	**B**
0	−$20,000	−$25,000
1	17,500	25,500
2	17,000	18,000
3	15,000	

On the basis of the NPW criterion, which project would be selected if you use an infinite planning horizon with project repeatability (the same costs and benefits) likely? Assume that $i = 12\%$.

5.38 Consider the following two mutually exclusive investment projects, which have unequal service lives:

	Project's Cash Flow	
n	**A1**	**A2**
0	−$900	−$1,800
1	−400	−300
2	−400	−300
3	−400 + 200	−300
4		−300
5		−300
6		−300
7		−300
8		−300 + 500

(a) What assumption(s) do you need in order to compare a set of mutually exclusive investments with unequal service lives?

(b) With the assumption(s) defined in (a) and using $i = 10\%$, determine which project should be selected.

(c) If your analysis period (study period) is just three years, what should be the salvage value of project A2 at the end of year 3 to make the two alternatives economically indifferent?

5.39 Consider the following two mutually exclusive projects:

	B1		B2	
n	Cash Flow	Salvage Value	Cash Flow	Salvage Value
0	−$18,000		−$15,000	
1	−2,000	6,000	−2,100	6,000
2	−2,000	4,000	−2,100	3,000
3	−2,000	3,000	−2,100	1,000
4	−2,000	2,000		
5	−2,000	2,000		

Salvage values represent the net proceeds (after tax) from disposal of the assets if they are sold at the end of each year. Both B1 and B2 will be available (or can be repeated) with the same costs and salvage values for an indefinite period.

(a) Assuming an infinite planning horizon, which project is a better choice at MARR = 12%?

(b) With a 10-year planning horizon, which project is a better choice at MARR = 12%?

5.40 Consider the following cash flows for two types of models:

	Project's Cash Flow	
n	Model A	Model B
0	−$6,000	−$15,000
1	3,500	10,000
2	3,500	10,000
3	3,500	

Both models will have no salvage value upon their disposal (at the end of their respective service lives). The firm's MARR is known to be 15%.

(a) Notice that the models have different service lives. However, model A will be available in the future with the same cash flows. Model B is available at one time only. If you select model B now, you will have to replace it with model A at the end of year 2. If your firm uses the present worth as a decision criterion, which model should be selected, assuming that the firm will need either model for an indefinite period?

(b) Suppose that your firm will need either model for only two years. Determine the salvage value of model A at the end of year 2 that makes both models indifferent (equally likely).

5.41 An electric utility is taking bids on the purchase, installation, and operation of microwave towers. Following are some details associated with the two bids that were received:

	Cost per Tower	
	Bid A	**Bid B**
Equipment cost	$112,000	$98,000
Installation cost	$25,000	$30,000
Annual maintenance and inspection fee	$2,000	$2,500
Annual extra income taxes		$800
Life	40 years	35 years
Salvage value	$0	$0

Which is the most economical bid if the interest rate is considered to be 11%? Either tower will have no salvage value after 20 years of use.

Use the NPW method to compare these two mutually exclusive plans.

5.42 Consider the following two investment alternatives:

	Project's Cash Flow	
n	**A1**	**A2**
0	−$15,000	−$25,000
1	9,500	0
2	12,500	X
3	7,500	X
PW(15%)	?	9,300

The firm's MARR is known to be 15%.

(a) Compute PW(15%) for A1.

(b) Compute the unknown cash flow X in years 2 and 3 for A2.

(c) Compute the project balance (at 15%) of A1 at the end of period 3.

(d) If these two projects are mutually exclusive alternatives, which one would you select?

5.43 Consider each of the after-tax cash flows shown in Table P5.43.

(a) Compute the project balances for projects A and D as a function of the project year at $i = 10\%$.

(b) Compute the net future-worth values for projects A and D at $i = 10\%$.

(c) Suppose that projects B and C are mutually exclusive. Suppose also that the required service period is eight years and that the company is considering leasing comparable equipment with an annual lease expense of $3,000 for the remaining years of the required service period. Which project is a better choice?

CHAPTER SIX

Annual Equivalent-Worth Analysis

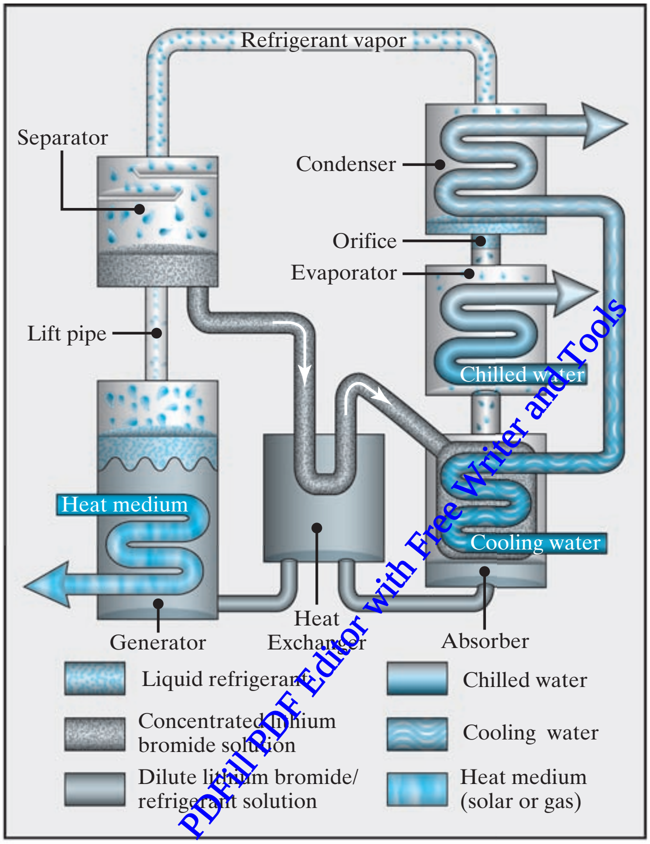
Thermally Activated Technologies: Absorption Chillers for Buildings[1] Absorption chillers provide cooling to buildings by using heat. This seemingly paradoxical, but highly efficient, technology is most cost effective in large facilities with significant heat loads. Not only do absorption chillers use less energy than conventional equipment does, but they also cool buildings without the use of ozone-depleting chlorofluorocarbons (CFCs). Unlike conventional electric chillers, which use mechanical energy in a vapor compression process to provide refrigeration, absorption chillers primarily use heat energy, with limited mechanical energy for pumping. Absorption chillers can be powered by natural gas, steam, or waste heat.

- The most promising markets for absorption chillers are in commercial buildings, government facilities, college campuses, hospital complexes, industrial parks, and municipalities.
- Absorption chillers generally become economically attractive when there is a source of inexpensive thermal energy at temperatures between 212°F and 392°F.

An absorption chiller transfers thermal energy from the heat source to the heat sink through an absorbent fluid and a refrigerant. The absorption chiller accomplishes its refrigerative effect by absorbing and then releasing water vapor into and out of a lithium bromide solution. Absorption chiller systems are classified by single-, double-, or triple-stage effects, which indicate the number of generators in the given system. The greater the number of stages, the higher is the overall efficiency of the system. Double-effect absorption chillers typically have a higher first cost, but a significantly lower energy cost, than single-effect chillers, resulting in a lower net present worth.

[1] Tech Brief, Office of Power Technology, U.S. Department of Energy, Washington, DC.

Single-Effect Absorption Chiller



Some of the known economic benefits of the absorption chiller over the conventional mechanical chiller are as follows: [2]

- In a plant where low-pressure steam is currently being vented to the atmosphere, a mechanical chiller with a Coefficient of Performance (COP) of 4.0 is used 4,000 hours a year to produce an average of 300 tons of refrigeration.
- The plant's cost of electricity is \$0.05 a kilowatt-hour. An absorption unit requiring 5,400 lb/hr of 15-psig steam could replace the mechanical chiller, providing the following annual electrical cost savings:

$$\text{Annual Savings} = 300 \text{ tons} \times (12{,}000 \text{ Btu/ton}/4.0) \times 4{,}000 \text{ hrs/yr} \times \$0.05/\text{kWh} \times \text{kWh}/3{,}413 \text{ Btu} = \$52{,}740.$$

[2] *Source*: *EcoGeneration Solutions*™, *LLC, Companies,* 12615 Jones Rd., Suite 209, Houston, Texas 77070 (http://www.cogeneration.net/Absorption_Chillers.htm).

Suppose you plan to install the chiller and expect to operate it continuously for 10 years. How would you calculate the operating cost per hour? Suppose you are considering buying a new car. If you expect to drive 12,000 miles per year, can you figure out how much the car costs per mile? You would have good reason to want to know the cost if you were being reimbursed by your employer on a per mile basis for the business use of your car. Or consider a real-estate developer who is planning to build a 500,000-square-foot shopping center. What would be the minimum annual rental fee per square foot required to recover the initial investment?

Annual equivalence analysis is the method by which these and other unit costs (or profits) are calculated. Along with present-worth analysis, annual equivalence analysis is the second major equivalence technique for putting alternatives on a common basis of comparison. In this chapter, we develop the annual equivalent-worth criterion and demonstrate a number of situations in which annual equivalence analysis is preferable to other methods of comparison.

CHAPTER LEARNING OBJECTIVES

After completing this chapter, you should understand the following concepts:

- How to determine the equivalent annual worth (cost) for a given project.
- Why the annual equivalent approach facilitates the comparison of unequal service life problems.
- How to determine the capital cost (or ownership cost) when you purchase an asset.
- How to determine the unit cost or unit profit.
- How to conduct a life-cycle cost analysis.
- How to optimize design parameters in engineering design.

6.1 Annual Equivalent-Worth Criterion

In this section, we set forth a fundamental decision rule based on annual equivalent worth by considering both revenue and cost streams of a project. If revenue streams are irrelevant, then we make a decision solely on the basis of cost. This leads to a popular decision tool known as "life-cycle cost analysis," which we will discuss in Section 6.4.

6.1.1 Fundamental Decision Rule

The **annual equivalent worth (AE)** criterion provides a basis for measuring the worth of an investment by determining equal payments on an annual basis. Knowing that any lump-sum cash amount can be converted into a series of equal annual payments, we may first find the net present worth (NPW) of the original series and then multiply this amount by the capital recovery factor:

$$\mathrm{AE}(i) = \mathrm{PW}(i)(A/P, i, N). \tag{6.1}$$

- **Single-project evaluation:** The accept–reject selection rule for a single *revenue* project is as follows:

$$\text{If AE}(i) > 0\text{, accept the investment.}$$
$$\text{If AE}(i) = 0\text{, remain indifferent to the investment.}$$
$$\text{If AE}(i) < 0\text{, reject the investment.}$$

  Notice that the factor (A/P, i, N) in Eq. (6.1) is positive for $-1 < i < \infty$, which indicates that the value of AE(i) will be positive if, and only if, PW(i) is positive. In other words, accepting a project that has a positive AE(i) is equivalent to accepting a project that has a positive PW(i). Therefore, the AE criterion for evaluating a project is consistent with the NPW criterion.

- **Comparing mutually exclusive alternatives:** As with present-worth analysis, when you compare mutually exclusive *service* projects whose revenues are the same, you may compare them on a *cost*-only basis. In this situation, the alternative with the minimum *annual equivalent cost* (or least negative annual equivalent worth) is selected.

Example 6.1 illustrates how to find the equivalent annual worth for a proposed energy-savings project. As you will see, you first calculate the net present worth of the project and then convert this present worth into an equivalent annual basis.

EXAMPLE 6.1 Annual Equivalent Worth: A Single-Project Evaluation

A utility company is considering adding a second feedwater heater to its existing system unit to increase the efficiency of the system and thereby reduce fuel costs. The 150-MW unit will cost \$1,650,000 and has a service life of 25 years. The expected salvage value of the unit is considered negligible. With the second unit installed, the efficiency of the system will improve from 55% to 56%. The fuel cost to run the feedwater is estimated at \$0.05 kWh. The system unit will have a load factor of 85%, meaning that the system will run 85% of the year.

(a) Determine the equivalent annual worth of adding the second unit with an interest rate of 12%.

(b) If the fuel cost increases at the annual rate of 4% after first year, what is the equivalent annual worth of having the second feedwater unit at $i = 12\%$?

A feedwater heater is a power-plant component used to preheat water delivered to a boiler. Preheating the feedwater reduces the amount of energy needed to make steam and thus reduces plant operation costs.

DISCUSSION: Whenever we compare machines with different efficiency ratings, we need to determine the input powers required to operate the machines. Since the percent efficiency is equal to the ratio of the output power to the input power, we can determine the input power by dividing the output power by the motor's percent efficiency:

$$\text{Input power} = \frac{\text{output power}}{\text{percent efficiency}}.$$

For example, a 30-HP motor with 90% efficiency will require an input power of

$$\begin{aligned}\text{Input power} &= \frac{(30\text{ HP} \times 0.746\text{ kW/HP})}{0.90}\\ &= 24.87\text{ kW}.\end{aligned}$$

Therefore, energy consumption with and without the second unit can be calculated as follows:

- Before adding the second unit, $\dfrac{150{,}000\text{ kW}}{0.55} = 272{,}727\text{ kW}$
- After adding the second unit, $\dfrac{150{,}000\text{ kW}}{0.56} = 267{,}857\text{ kW}$

So the reduction in energy consumption is 4,871 kW.

Since the system unit will operate only 85% of the year, the total annual operating hours are calculated as follows:

$$\text{Annual operating hours} = (365)(24)(0.85) = 7{,}446\text{ hours/year}.$$

SOLUTION

Given: $I = \$1{,}650{,}000$, $N = 25$ years, $S = 0$, annual fuel savings, and $i = 12\%$. Find: AE of fuel savings due to improved efficiency.

(a) With the assumption of constant fuel cost over the service life of the second heater,

$$\begin{aligned}A_{\text{fuel savings}} &= (\text{reduction in fuel requirement}) \times (\text{fuel cost})\\ &\quad \times (\text{operating hours per year})\\ &= \left(\frac{150{,}000\text{ kW}}{0.55} - \frac{150{,}000\text{ kW}}{0.56}\right) \times (\$0.05/\text{kWh})\\ &\quad \times ((8{,}760)(0.85)\text{ hours/year})\\ &= (4{,}871\text{ kW}) \times (\$0.05/\text{kWh}) \times (7{,}446\text{ hours/year})\\ &= \$1{,}813{,}473/\text{year};\end{aligned}$$

$$\begin{aligned}\text{PW}(12\%) &= -\$1{,}650{,}000 + \$1{,}813{,}473(P/A, 12\%, 25)\\ &= \$12{,}573{,}321;\\ \text{AE}(12\%) &= \$12{,}573{,}321(A/P, 12\%, 25)\\ &= \$1{,}603{,}098.\end{aligned}$$

(b) With the assumption of escalating energy cost at the annual rate of 4%, since the first year's fuel savings is already calculated in (a), we use it as A_1 in the geometric-gradient-series present-worth factor $(P/A_1, g, i, N)$:

$$
\begin{aligned}
A_1 &= \$1{,}813{,}473 \\
\text{PW}(12\%) &= -\$1{,}650{,}000 + \$1{,}813{,}473(P/A_1, 4\%, 12\%, 25) \\
&= \$17{,}463{,}697 \\
\text{AE}(12\%) &= \$17{,}463{,}697(A/P, 12\%, 25) \\
&= \$2{,}226{,}621
\end{aligned}
$$

Clearly, either situation generates enough fuel savings to justify adding the second unit of the feedwater. Figure 6.1 illustrates the cash flow series associated with the required investment and fuel savings over the heater's service life of 25 years.

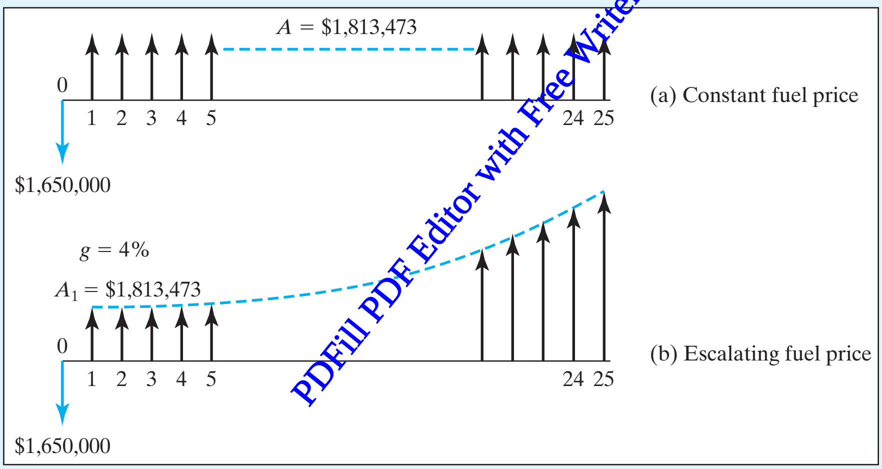


Figure 6.1 Cash flow diagram (Example 6.1).

6.1.2 Annual-Worth Calculation with Repeating Cash Flow Cycles

In some situations, a **cyclic cash flow pattern** may be observed over the life of the project. Unlike the situation in Example 6.1, where we first computed the NPW of the entire cash flow and then calculated the AE, we can compute the AE by examining the first cash flow cycle. Then we can calculate the NPW for the first cash flow cycle and derive the AE over that cycle. This shortcut method gives the same solution when the NPW of the entire project is calculated, and then the AE can be computed from this NPW.

EXAMPLE 6.2 Annual Equivalent Worth: Repeating Cash Flow Cycles

SOLEX Company is producing electricity directly from a solar source by using a large array of solar cells and selling the power to the local utility company. SOLEX decided to use amorphous silicon cells because of their low initial cost, but these cells degrade over time, thereby resulting in lower conversion efficiency and power output. The cells must be replaced every four years, which results in a particular cash flow pattern that repeats itself as shown in Figure 6.2. Determine the annual equivalent cash flows at $i = 12\%$.

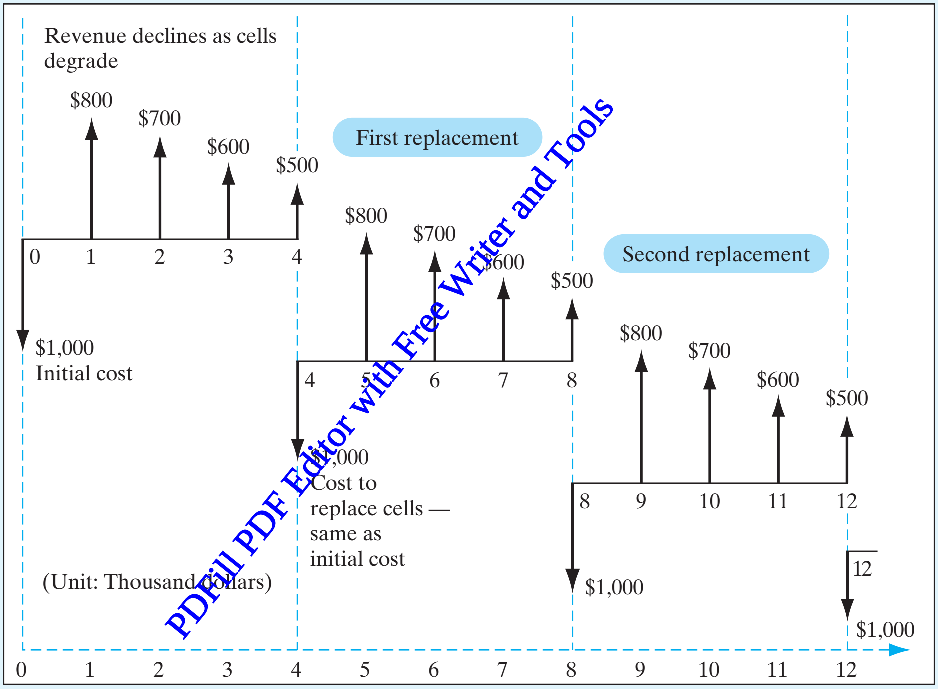


Figure 6.2 Conversion of repeating cash flow cycles into an equivalent annual payment (Example 6.2).

SOLUTION

Given: Cash flows in Figure 6.2 and $i = 12\%$.
Find: Annual equivalent benefit.

To calculate the AE, we need only consider one cycle over the four-year replacement period of the cells. For $i = 12\%$, we first obtain the NPW for the first cycle as follows:

$$\begin{aligned} \text{PW}(12\%) &= -\$1{,}000{,}000 \\ &\quad + [(\$800{,}000 - \$100{,}000(A/G, 12\%, 4)](P/A, 12\%, 4) \end{aligned}$$

$$= -\$1{,}000{,}000 + \$2{,}017{,}150.$$
$$= \$1{,}017{,}150.$$

Then we calculate the AE over the four-year life cycle:

$$\text{AE}(12\%) = \$1{,}017{,}150(A/P, 12\%, 4)$$
$$= \$334{,}880.$$

We can now say that the two cash flow series are equivalent:

Original Cash Flows		Annual Equivalent Flows	
n	A_n	n	A_n
0	−\$1,000,000	0	0
1	800,000	1	\$334,880
2	700,000	2	334,880
3	600,000	3	334,880
4	500,000	4	334,880

We can extend this equivalency over the remaining cycles of the cash flow. The reasoning is that each similar set of five values (one disbursement and four receipts) is equivalent to four annual receipts of \$334,880 each. In other words, the \$1 million investment in the solar project will recover the entire investment and generate equivalent annual savings of \$334,880 over a four-year life cycle.

6.1.3 Comparing Mutually Exclusive Alternatives

In this section, we consider a situation in which two or more mutually exclusive alternatives need to be compared on the basis of annual equivalent worth. In Section 5.5, we discussed the general principles that should be applied when mutually exclusive alternatives with unequal service lives were compared. The same general principles should be applied in comparing mutually exclusive alternatives on the basis of annual equivalent worth: Mutually exclusive alternatives in equal time spans must be compared. Therefore, we must give careful consideration to the period covered by the analysis: the **analysis period**. We will consider two situations: (1) The analysis period equals project lives and (2) the analysis period differs from project lives.

With situation (1), we compute AE for each project and select the project that has the least negative AE for service projects (or the largest AE for revenue projects). With situation (2), we need to consider the issue of unequal project lives. As we saw in Chapter 5, comparing projects with unequal service lives is complicated by the need to determine the lowest common multiple life. For the special situation of an indefinite service period and replacement with identical projects, we can avoid this complication by the use of AE analysis, provided that the following criteria are met:

1. The service of the selected alternative is required on a continuous basis.
2. Each alternative will be replaced by an *identical* asset that has the same costs and performance.

When these two criteria are met, we may solve for the AE of each project on the basis of its initial life span, rather than on that of the lowest common multiple of the projects' lives. Example 6.3 illustrates the process of comparing unequal service projects.

EXAMPLE 6.3 Annual Equivalent Cost Comparison: Unequal Project Lives

Consider again Example 5.14, in which we compared two types of equipment with unequal service lives. Apply the annual equivalent approach to select the most economical equipment.

SOLUTION

Given: Cost cash flows shown in Figure 6.3 and $i = 15\%$ per year.

Find: AE cost and which alternative is the preferred one.

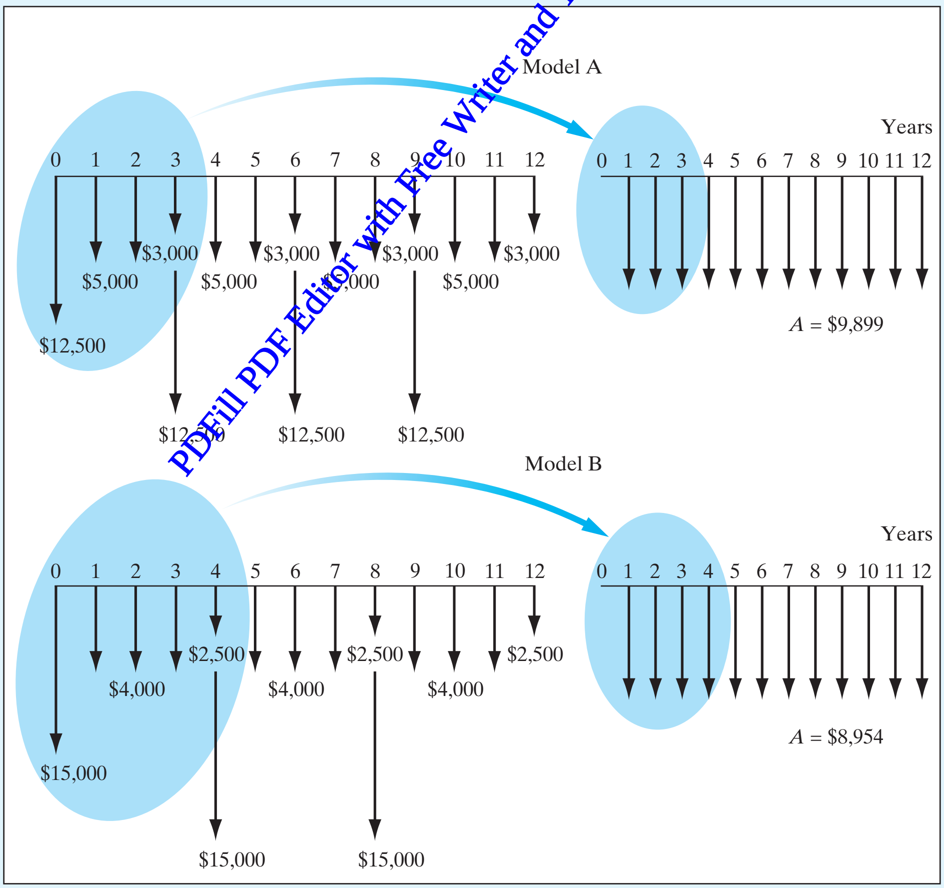


Figure 6.3 Comparison of projects with unequal lives and an indefinite analysis period using the annual equivalent-worth criterion (Example 6.3).

An alternative procedure for solving Example 5.14 is to compute the annual equivalent cost of an outlay of $12,500 for model A every 3 years and the annual equivalent cost of an outlay of $15,000 for model B every 4 years. Notice that the AE of each 12-year cash flow is the same as that of the corresponding 3- or 4-year cash flow (Figure 6.3). From Example 5.14, we calculate

- **Model A:** For a 3-year life,

$$\begin{aligned} \text{PW}(15\%) &= \$22{,}601 \\ \text{AE}(15\%) &= 22{,}601(A/P, 15\%, 3) \\ &= \$9{,}899. \end{aligned}$$

For a 12-year period (the entire analysis period),

$$\begin{aligned} \text{PW}(15\%) &= \$53{,}657 \\ \text{AE}(15\%) &= 53{,}657(A/P, 15\%, 12) \\ &= \$9{,}899. \end{aligned}$$

- **Model B:** For a 4-year life,

$$\begin{aligned} \text{PW}(15\%) &= \$25{,}562 \\ \text{AE}(15\%) &= \$25{,}562(A/P, 15\%, 4) \\ &= \$8{,}954. \end{aligned}$$

For a 12-year period (the entire analysis period),

$$\begin{aligned} \text{PW}(15\%) &= \$48{,}534 \\ \text{AE}(15\%) &= \$48{,}534(A/P, 15\%, 12) \\ &= \$8{,}954. \end{aligned}$$

Notice that the annual equivalent values that were calculated on the basis of the common service period are the same as those which were obtained over their initial life spans. Thus, for alternatives with unequal lives, we will obtain the same selection by comparing the NPW over a common service period using repeated projects or by comparing the AE for initial lives.

6.2 Capital Costs versus Operating Costs

When only costs are involved, the AE method is sometimes called the **annual equivalent cost (AEC)** method. In this case, revenues must cover two kinds of costs: **operating costs** and **capital costs**. Operating costs are incurred through the operation of physical plant or equipment needed to provide service; examples include items such as labor and

Capital cost: the amount of net investment.

raw materials. Capital costs are incurred by purchasing assets to be used in production and service. Normally, capital costs are nonrecurring (i.e., one-time) costs, whereas operating costs recur for as long as an asset is owned.

Because operating costs recur over the life of a project, they tend to be estimated on an annual basis anyway, so, for the purposes of annual equivalent cost analysis, no special calculation is required. However, because capital costs tend to be one-time costs, in conducting an annual equivalent cost analysis we must translate this one-time cost into its annual equivalent over the life of the project. The annual equivalent of a capital cost is given a special name: **capital recovery cost**, designated CR(i).

Capital recovery cost: The annual payment that will repay the cost of a fixed asset over the useful life of the asset and will provide an economic rate of return on the investment.

Two general monetary transactions are associated with the purchase and eventual retirement of a capital asset: its initial cost (I) and its salvage value (S). Taking into account these sums, we calculate the capital recovery factor as follows:

$$\mathrm{CR}(i) = I(A/P, i, N) - S(A/F, i, N). \tag{6.2}$$

Now, recall algebraic relationships between factors in Table 3.4, and notice that the factor ($A/F, i, N$) can be expressed as

$$(A/F, i, N) = (A/P, i, N) - i.$$

Then we may rewrite CR(i) as

$$\begin{aligned}\mathrm{CR}(i) &= I(A/P, i, N) - S[(A/P, i, N) - i] \\ &= (I - S)(A/P, i, N) + iS. \end{aligned} \tag{6.3}$$

Since we are calculating the equivalent annual costs, we treat cost items with a positive sign. Then the salvage value is treated as having a negative sign in Eq. (6.3). We may interpret this situation thus: To obtain the machine, one borrows a total of I dollars, S dollars of which are returned at the end of the Nth year. The first term, $(I - S)(A/P, i, N)$, implies that the balance $(I - S)$ will be paid back in equal installments over the N-year period at a rate of i. The second term, iS, implies that simple interest in the amount iS is paid on S until it is repaid (Figure 6.4). Thus, the amount to be financed is $I - S\ (P/F, i, N)$, and the installments of this loan over the N-year period are

$$\begin{aligned}\mathrm{AE}(i) &= [I - S(P/F, i, N)](A/P, i, N) \\ &= I(A/P, i, N) - S(P/F, i, N)(A/P, i, N) \\ &= [I(A/P, i, N) - S(A/F, i, N)] \\ &= \mathrm{CR}(i). \end{aligned} \tag{6.4}$$

Therefore, CR(i) tells us what the bank would charge each year. Many auto leases are based on this arrangement, in that most require a guarantee of S dollars in salvage. From an industry viewpoint, CR(i) is the annual cost to the firm of owning the asset.

With this information, the amount of annual savings required to recover the capital and operating costs associated with a project can be determined, as Example 6.4 illustrates.

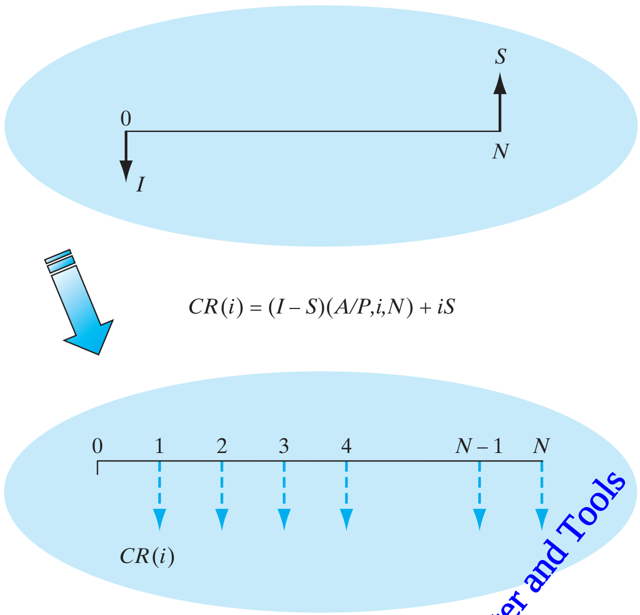


Figure 6.4 Capital recovery (ownership) cost calculation.

EXAMPLE 6.4 Capital Recovery Cost

Consider a machine that costs \$20,000 and has a five-year useful life. At the end of the five years, it can be sold for \$4,000 after-tax adjustment. The annual operating and maintenance (O&M) costs are about \$500. If the firm could earn an after-tax revenue of \$5,000 per year with this machine, should it be purchased at an interest rate of 10%? (All benefits and costs associated with the machine are accounted for in these figures.)

SOLUTION

Given: $I = \$20{,}000$, $S = \$4{,}000$, O&M $= \$500$, $A = \$5{,}000$, $N = 5$ years, and $i = 10\%$ per year.

Find: AE, and determine whether to purchase the machine.

The first task is to separate cash flows associated with acquisition and disposal of the asset from the normal operating cash flows. Since the operating cash flows—the \$5,000 yearly revenue—are already given in equivalent annual flows, we need to convert only the cash flows associated with acquisition and disposal of the asset into equivalent annual flows (Figure 6.5). Using Eq. (6.3), we obtain

$$
\begin{aligned}
\text{CR}(i) &= (I - S)(A/P, i, N) + iS \\
&= (\$20{,}000 - \$4{,}000)(A/P, 10\%, 5) + (0.10)\,\$4{,}000 \\
&= \$4{,}620.76, \\
\text{O\&M}(i) &= \$500,
\end{aligned}
$$

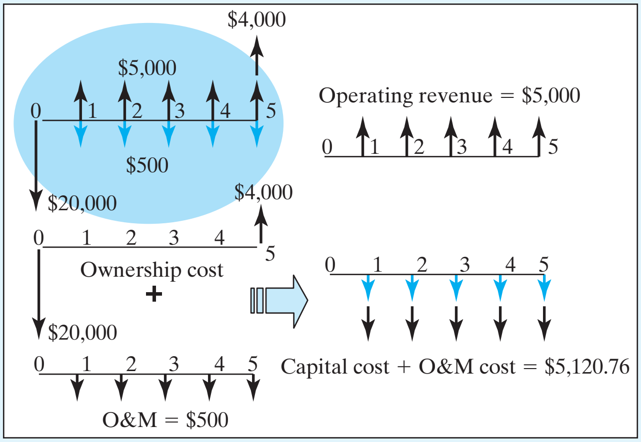


Figure 6.5 Separating ownership cost (capital cost) and operating cost from operating revenue, which must exceed the annual equivalent cost to make the project acceptable.

$$\begin{aligned}
\text{AEC}(10\%) &= \text{CR}(10\%) + \text{O\&M}(10\%) \\
&= \$4{,}620.76 + \$500 \\
&= \$5{,}120.76, \\
\text{AE}(10\%) &= \$5{,}000 - \$5{,}120.76 \\
&= -\$120.76.
\end{aligned}$$

This negative AE value indicates that the machine does not generate sufficient revenue to recover the original investment, so we must reject the project. In fact, there will be an equivalent loss of $120.76 per year over the life of the machine.

COMMENTS: We may interpret the value found for the annual equivalent cost as asserting that the annual operating revenues must be at least $5,120.76 in order to recover the cost of owning and operating the asset. However, the annual operating revenues actually amount to only $5,000, resulting in a loss of $120.76 per year. Therefore, the project is not worth undertaking.

6.3 Applying Annual-Worth Analysis

In general, most engineering economic analysis problems can be solved by the present-worth methods that were introduced in Chapter 5. However, some economic analysis problems can be solved more efficiently by annual-worth analysis. In this section, we introduce several applications that call for such an analysis.

6.3.1 Benefits of AE Analysis

Example 6.1 should look familiar to you: It is exactly the situation we encountered in Chapter 4 when we converted a mixed cash flow into a single present value and then into a series of equivalent cash flows. In the case of Example 6.1, you may wonder why we bother to convert NPW to AE at all, since we already know that the project is acceptable from NPW analysis. In fact, the example was mainly an exercise to familiarize you with the AE calculation.

However, in the real world, a number of situations can occur in which AE analysis is preferred, or even demanded, over NPW analysis. For example, corporations issue annual reports and develop yearly budgets. For these purposes, a company may find it more useful to present the annual cost or benefit of an ongoing project, rather than its overall cost or benefit. Following are some additional situations in which AE analysis is preferred:

1. **Consistency of report formats.** Financial managers commonly work with annual rather than overall costs in any number of internal and external reports. Engineering managers may be required to submit project analyses on an annual basis for consistency and ease of use by other members of the corporation and stockholders.
2. **Need for unit costs or profits.** In many situations, projects must be broken into unit costs (or profits) for ease of comparison with alternatives. Make-or-buy and reimbursement analyses are key examples, and these will be discussed in the chapter.
3. **Life-cycle cost analysis.** When there is no need for estimating the revenue stream for a proposed project, we can consider only the cost streams of the project. In that case, it is common to convert this life-cycle cost (LCC) into an equivalent annual cost for purposes of comparison. Industry has used the LCC to help determine which project will cost less over the life of a product. LCC analysis has had a long tradition in the Department of Defense, having been applied to virtually every new weapon system proposed or under development.

6.3.2 Unit Profit or Cost Calculation

In many situations, we need to know the unit profit (or cost) of operating an asset. To obtain this quantity, we may proceed as follows:

- Determine the number of units to be produced (or serviced) each year over the life of the asset.
- Identify the cash flow series associated with production or service over the life of the asset.
- Calculate the net present worth of the project cash flow series at a given interest rate, and then determine the equivalent annual worth.
- Divide the equivalent annual worth by the number of units to be produced or serviced during each year. When the number of units varies each year, you may need to convert them into equivalent annual units.

To illustrate the procedure, Example 6.5 uses the annual equivalent concept in estimating the savings per machine hour for the proposed acquisition of a machine.

EXAMPLE 6.5 Unit Profit per Machine Hour When Annual Operating Hours Remain Constant

Consider the investment in the metal-cutting machine of Example 5.5. Recall that this three-year investment was expected to generate an NPW of \$3,553. Suppose that the machine will be operated for 2,000 hours per year. Compute the equivalent savings per machine hour at $i = 15\%$.

SOLUTION

Given: NPW = \$3,553, $N = 3$ years, $i = 15\%$ per year, and 2,000 machine hours per year.

Find: Equivalent savings per machine hour.

We first compute the annual equivalent savings from the use of the machine. Since we already know the NPW of the project, we obtain the AE by the formula

$$\text{AE}(15\%) = \$3{,}553(A/P, 15\%, 3) = \$1{,}556.$$

With an annual usage of 2,000 hours, the equivalent savings per machine hour would be

$$\text{Savings per machine hour} = \$1{,}556/2{,}000 \text{ hours} = \$0.78/\text{hour}.$$

COMMENTS: Note that we cannot simply divide the NPW (\$3,553) by the total number of machine hours over the three-year period (6,000 hours) to obtain \$0.59/hour. This \$0.59 figure represents the instant savings in present worth for each hourly use of the equipment, but does not consider the time over which the savings occur. Once we have the annual equivalent worth, we can divide by the desired time unit if the compounding period is one year. If the compounding period is shorter, then the equivalent worth should be calculated for the compounding period.

EXAMPLE 6.6 Unit Profit per Machine Hour When Annual Operating Hours Fluctuate

Consider again Example 6.5, and suppose that the metal-cutting machine will be operated according to varying hours: 1,500 hours the first year, 2,500 hours the second year, and 2,000 hours the third year. The total operating hours still remain at 6,000 over three years. Compute the equivalent savings per machine hour at $i = 15\%$.

SOLUTION

Given: NPW = \$3,553, $N = 3$ years, $i = 15\%$ per year, and operating hours of 1,500 the first year, 2,500 the second year, and 2,000 the third year.

Find: Equivalent savings per machine hour.

Step 7: Equivalent annual cost at optimal pipe size:

$$\text{Capital cost} = \left[\$49{,}750{,}272(5.9868)^2 + \frac{\$170{,}004{,}700{,}125}{5.9868^4}\right](A/P, 10\%, 20)$$

$$= 5{,}843{,}648(5.9868)^2 + \frac{19{,}968{,}752{,}076}{5.9868^4}$$

$$= \$224{,}991{,}039;$$

$$\text{Annual power cost} = \frac{114{,}557{,}013{,}315}{5.9868^4}$$

$$= \$89{,}174{,}911;$$

$$\text{Total annual cost} = \$224{,}991{,}039 + \$89{,}174{,}911$$

$$= \$314{,}165{,}950.$$

Step 8: Total annual oil revenue:

$$\text{Annual oil revenue} = \$50/\text{bbl} \times 3{,}000{,}000 \text{ bbl/day}$$

$$= \times 365 \text{ day/year}$$

$$= \$54{,}750{,}000{,}000/\text{year}.$$

Enough revenues are available to offset the capital cost as well as the operating cost.

COMMENTS: A number of other criteria exist for choosing the pipe size for a particular fluid transfer application. For example, low velocity may be required when erosion or corrosion concerns must be considered. Alternatively, higher velocities may be desirable for slurries when settling is a concern. Ease of construction may also weigh significantly in choosing a pipe size. On the one hand, a small pipe size may not accommodate the head and flow requirements efficiently; on the other, space limitations may prohibit the selection of large pipe sizes.

SUMMARY

- Annual equivalent worth analysis, or AE, is—along with present-worth analysis—one of the two main analysis techniques based on the concept of equivalence. The equation for AE is

$$AE(i) = PW(i)(A/P, i, N).$$

AE analysis yields the same decision result as PW analysis.

- The capital recovery cost factor, or CR(i), is one of the most important applications of AE analysis, in that it allows managers to calculate an annual equivalent cost of capital for ease of itemization with annual operating costs. The equation for CR(i) is

$$CR(i) = (I - S)(A/P, i, N) + iS,$$

  where I = initial cost and S = salvage value.

- AE analysis is recommended over NPW analysis in many key real-world situations for the following reasons:
  1. In many financial reports, an annual equivalent value is preferred to a present-worth value.
  2. Unit costs often must be calculated to determine reasonable pricing for items that are on sale.
  3. The cost per unit of use of an item must be calculated in order to reimburse employees for the business use of personal cars.
  4. Make-or-buy decisions usually require the development of unit costs for the various alternatives.
  5. Minimum-cost analysis is easy to do when it is based on annual equivalent cost.

- LCCA is a way to predict the most cost-effective solution by allowing engineers to make a reasonable comparison among alternatives within the limit of the available data.

PROBLEMS

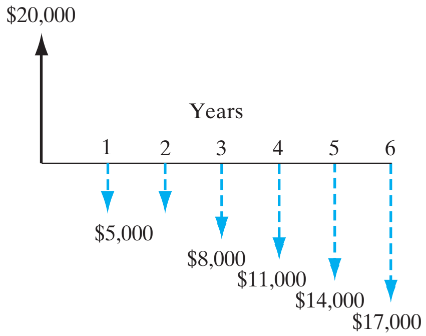
Note: *Unless otherwise stated, all cash flows given in the problems that follow represent after-tax cash flows.*

Annual Equivalent-Worth Calculation

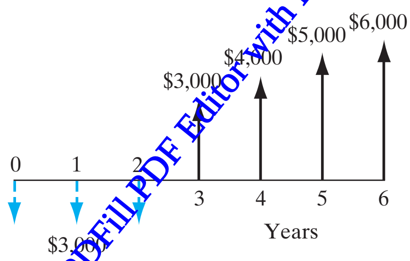
6.1 Consider the following cash flows and compute the equivalent annual worth at $i = 10\%$:

	A_n	
n	**Investment**	**Revenue**
0	−\$5,000	
1		\$2,000
2		2,000
3		3,000
4		3,000
5		1,000
6	2,000	500

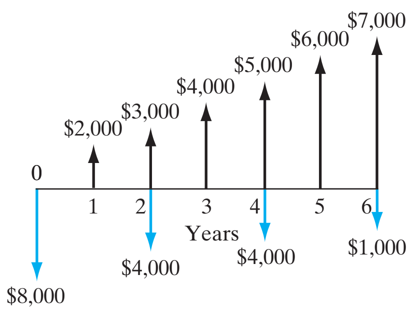
6.2 Consider the accompanying cash flow diagram. Compute the equivalent annual worth at $i = 12\%$.



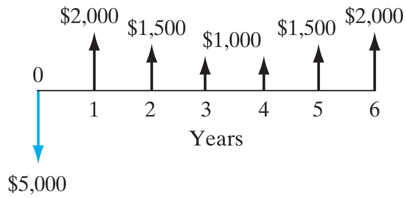
6.3 Consider the accompanying cash flow diagram. Compute the equivalent annual worth at $i = 10\%$.



6.4 Consider the accompanying cash flow diagram. Compute the equivalent annual worth at $i = 13\%$.



6.5 Consider the accompanying cash flow diagram. Compute the equivalent annual worth at $i = 8\%$.



6.6 Consider the following sets of investment projects:

	Project's Cash Flow ($)			
n	A	B	C	D
0	−$2,500	−$4,500	−$8,000	−$12,000
1	400	3,000	−2,000	2,000
2	500	2,000	6,000	4,000
3	600	1,000	2,000	8,000
4	700	500	4,000	8,000
5	800	500	2,000	4,000

Compute the equivalent annual worth of each project at $i = 10\%$, and determine the acceptability of each project.

6.7 Sun-Devil Company is producing electricity directly from a solar source by using a large array of solar cells and selling the power to the local utility company. Because these cells degrade over time, thereby resulting in lower conversion efficiency and power output, the cells must be replaced every four years, which results in a particular cash flow pattern that repeats itself as follows: $n = 0$, −$500,000; $n = 1$, $600,000; $n = 2$, $400,000; $n = 3$, $300,000, and $n = 4$, $200,000. Determine the annual equivalent cash flows at $i = 12\%$.

6.8 Consider the following sets of investment projects:

	Project's Cash Flow			
n	A	B	C	D
0	−$7,500	−$4,000	−$5,000	−$6,600
1	0	1,500	4,000	3,800
2	0	1,800	3,000	3,800
3	15,500	2,100	2,000	3,800

Compute the equivalent annual worth of each project at $i = 13\%$, and determine the acceptability of each project.

6.9 The cash flows for a certain project are as follows:

	Net Cash Flow		
n	Investment	Operating Income	
0	−\$800		1st cycle
1		\$900	
2		700	
3	−800	500	2nd cycle
4		900	
5		700	
6	−800	500	3rd cycle
7		900	
8		700	
9		500	

Find the equivalent annual worth for this project at $i = 10\%$, and determine the acceptability of the project.

6.10 Beginning next year, a foundation will support an annual seminar on campus by the earnings of a \$100,000 gift it received this year. It is felt that 8% interest will be realized for the first 10 years, but that plans should be made to anticipate an interest rate of 6% after that time. What amount should be added to the foundation now to fund the seminar at the \$10,000 level into infinity?

Capital (Recovery) Cost/Annual Equivalent Cost

6.11 The owner of a business is considering investing \$55,000 in new equipment. He estimates that the net cash flows will be \$5,000 during the first year, but will increase by \$2,500 per year the next year and each year thereafter. The equipment is estimated to have a 10-year service life and a net salvage value of \$6,000 at that time. The firm's interest rate is 12%.

(a) Determine the annual capital cost (ownership cost) for the equipment.

(b) Determine the equivalent annual savings (revenues).

(c) Determine whether this is a wise investment.

6.12 You are considering purchasing a dump truck. The truck will cost \$45,000 and have an operating and maintenance cost that starts at \$15,000 the first year and increases by \$2,000 per year. Assume that the salvage value at the end of five years is \$9,000 and interest rate is 12%. What is the equivalent annual cost of owning and operating the truck?

6.13 Emerson Electronics Company just purchased a soldering machine to be used in its assembly cell for flexible disk drives. The soldering machine cost \$250,000. Because of the specialized function it performs, its useful life is estimated to be five years. It is also estimated that at that time its salvage value will be \$40,000. What is the capital cost for this investment if the firm's interest rate is 18%?

6.14 The present price (year 0) of kerosene is $1.80 per gallon, and its cost is expected to increase by $.15 per year. (At the end of year 1, kerosene will cost $1.95 per gallon.) Mr. Garcia uses about 800 gallons of kerosene for space heating during a winter season. He has an opportunity to buy a storage tank for $600, and at the end of four years he can sell the storage tank for $100. The tank has a capacity to supply four years of Mr. Garcia's heating needs, so he can buy four years' worth of kerosene at its present price ($1.80), or he can invest his money elsewhere at 8%. Should he purchase the storage tank? Assume that kerosene purchased on a pay-as-you-go basis is paid for at the end of the year. (However, kerosene purchased for the storage tank is purchased now.)

6.15 Consider the following advertisement, which appeared in a local paper.

Pools-Spas-Hot Tubs—Pure Water without Toxic Chemicals: The comparative costs between the conventional chemical system (chlorine) and the IONETICS systems are as follows:

Item	Conventional System	IONETICS System
Annual costs		
Chemical	$471	
IONETICS		$ 85
Pump ($0.667/kWh)	$576	$ 100
Capital investment		$1,200

Note that the IONETICS system pays for itself in less than 2 years.

Assume that the IONETICS system has a 12-year service life and the interest rate is 6%. What is the equivalent annual cost of operating the IONETICS system?

6.16 The cash flows for two investment projects are as follows:

	Project's Cash Flow	
n	**A**	**B**
0	−$4,000	$5,500
1	1,000	−1,400
2	X	−1,400
3	1,000	−1,400
4	1,000	−1,400

(a) For project A, find the value of X that makes the equivalent annual receipts equal the equivalent annual disbursement at $i = 13\%$.

(b) Would you accept project B at $i = 15\%$, based on an AE criterion?

6.17 An industrial firm can purchase a special machine for $50,000. A down payment of $5,000 is required, and the unpaid balance can be paid off in five equal year-end installments at 7% interest. As an alternative, the machine can be purchased for \$46,000 in cash. If the firm's MARR is 10%, use the annual equivalent method to determine which alternative should be accepted.

6.18 An industrial firm is considering purchasing several programmable controllers and automating the company's manufacturing operations. It is estimated that the equipment will initially cost \$100,000 and the labor to install it will cost \$35,000. A service contract to maintain the equipment will cost \$5,000 per year. Trained service personnel will have to be hired at an annual salary of \$30,000. Also estimated is an approximate \$10,000 annual income-tax savings (cash inflow). How much will this investment in equipment and services have to increase the annual revenues after taxes in order to break even? The equipment is estimated to have an operating life of 10 years, with no salvage value because of obsolescence. The firm's MARR is 10%.

6.19 A construction firm is considering establishing an engineering computing center. The center will be equipped with three engineering workstations that cost \$35,000 each, and each has a service life of five years. The expected salvage value of each workstation is \$2,000. The annual operating and maintenance cost would be \$15,000 for each workstation. At a MARR of 15%, determine the equivalent annual cost for operating the engineering center.

Unit-Cost Profit Calculation

6.20 You have purchased a machine costing \$20,000. The machine will be used for two years, at the end of which time its salvage value is expected to be \$10,000. The machine will be used 6,000 hours during the first year and 8,000 hours during the second year. The expected annual net savings will be \$30,000 during the first year and \$40,000 during the second year. If your interest rate is 10%, what would be the equivalent net savings per machine hour?

6.21 The engineering department of a large firm is overly crowded. In many cases, several engineers share one office. It is evident that the distraction caused by the crowded conditions reduces the productive capacity of the engineers considerably. Management is considering the possibility of providing new facilities for the department, which could result in fewer engineers per office and a private office for some. For an office presently occupied by five engineers, what minimum individual increase in effectiveness must result to warrant the assignment of only three engineers to an office if the following data apply?

- The office size is 16×20 feet.
- The average annual salary of each engineer is \$80,000.
- The cost of the building is \$110 per square foot.
- The estimated life of the building is 25 years.
- The estimated salvage value of the building is 10% of the initial cost.
- The annual taxes, insurance, and maintenance are 6% of the initial cost.
- The cost of janitorial service, heating, and illumination is \$5.00 per square foot per year.
- The interest rate is 12%.

Assume that engineers reassigned to other office space will maintain their present productive capability as a minimum.

6.22 Sam Tucker is a sales engineer at Buford Chemical Engineering Company. Sam owns two vehicles, and one of them is entirely dedicated to business use. His business car is a used small pickup truck, which he purchased with $11,000 of personal savings. On the basis of his own records and with data compiled by the U.S. Department of Transportation, Sam has estimated the costs of owning and operating his business vehicle for the first three years as follows:

	First Year	Second Year	Third Year
Depreciation	$2,879	$1,776	$1,545
Scheduled maintenance	100	153	220
Insurance	635	635	635
Registration and taxes	78	57	50
Total ownership cost	$3,692	$2,621	$2,450
Nonscheduled repairs	35	85	200
Replacement tires	35	30	27
Accessories	15	13	12
Gasoline and taxes	688	650	522
Oil	80	100	100
Parking and tolls	135	125	110
Total operating costs	$988	$1,003	$971
Total of all costs	$4,680	$3,624	$3,421
Expected miles driven	14,500	13,000	11,500

If his interest rate is 6%, what should be Sam's reimbursement rate per mile so that he can break even?

6.23 Two 150-horsepower (HP) motors are being considered for installation at a municipal sewage-treatment plant. The first costs $4,500 and has an operating efficiency of 83%. The second costs $3,600 and has an efficiency of 80%. Both motors are projected to have zero salvage value after a life of 10 years. If all the annual charges, such as insurance, maintenance, etc., amount to a total of 15% of the original cost of each motor, and if power costs are a flat 5 cents per kilowatt-hour, how many minimum hours of full-load operation per year are necessary to justify purchasing the more expensive motor at $i = 6\%$? (A conversion factor you might find useful is 1 HP = 746 watts = 0.746 kilowatts.)

6.24 Danford Company, a manufacturer of farm equipment, currently produces 20,000 units of gas filters per year for use in its lawn-mower production. The costs, based on the previous year's production, are reported in Table P6.24.

It is anticipated that gas-filter production will last five years. If the company continues to produce the product in-house, annual direct material costs will increase at the rate of 5%. (For example, annual material costs during the first production

TABLE P6.24 **Production costs**

Item	Expense ($)
Direct materials	$ 60,000
Direct labor	180,000
Variable overhead (power and water)	135,000
Fixed overhead (light and heat)	70,000
Total cost	$445,000

year will be $63,000.) Direct labor will also increase at the rate of 6% per year. However, variable overhead costs will increase at the rate of 3%, but the fixed overhead will remain at its current level over the next five years. John Holland Company has offered to sell Danford 20,000 units of gas filters for $25 per unit. If Danford accepts the offer, some of the manufacturing facilities currently used to manufacture the filter could be rented to a third party for $35,000 per year. In addition, $3.5 per unit of the fixed overheard applied to the production of gas filters would be eliminated. The firm's interest rate is known to be 15%. What is the unit cost of buying the gas filter from the outside source? Should Danford accept John Holland's offer, and why?

6.25 Southern Environmental Consulting (SEC), Inc., designs plans and specifications for asbestos abatement (removal) projects in public, private, and governmental buildings. Currently, SEC must conduct an air test before allowing the reoccupancy of a building from which asbestos has been removed. SEC subcontracts air-test samples to a laboratory for analysis by transmission electron microscopy (TEM). To offset the cost of TEM analysis, SEC charges its clients $100 more than the subcontractor's fee. The only expenses in this system are the costs of shipping the air-test samples to the subcontractor and the labor involved in shipping the samples. With the growth of the business, SEC is having to consider either continuing to subcontract the TEM analysis to outside companies or developing its own TEM laboratory. Because of the passage of the Asbestos Hazard Emergency Response Act (AHERA) by the U.S. Congress, SEC expects about 1,000 air-sample testings per year over eight years. The firm's MARR is known to be 15%.

- **Subcontract option.** The client is charged $400 per sample, which is $100 above the subcontracting fee of $300. Labor expenses are $1,500 per year, and shipping expenses are estimated to be $0.50 per sample.

- **TEM purchase option.** The purchase and installation cost for the TEM is $415,000. The equipment would last for eight years, at which time it should have no salvage value. The design and renovation cost is estimated to be $9,500. The client is charged $300 per sample, based on the current market price. One full-time manager and two part-time technicians are needed to operate the laboratory. Their combined annual salaries will be $50,000. Material required to operate the lab includes carbon rods, copper grids, filter equipment, and acetone. The costs of these materials are estimated at $6,000 per year. Utility costs, operating and maintenance costs, and the indirect labor needed to maintain the lab are estimated at $18,000 per year. The extra income-tax expenses would be $20,000.

(a) Determine the cost of an air-sample test by the TEM (in-house).

(b) What is the required number of air samples per year to make the two options equivalent?

6.26 A company is currently paying its employees $0.38 per mile to drive their own cars on company business. The company is considering supplying employees with cars, which would involve purchasing at $25,000, with an estimated three-year life, a net salvage value of $8,000, taxes and insurance at a cost of $900 per year, and operating and maintenance expenses of $0.22 per mile. If the interest rate is 10% and the company anticipates an employee's annual travel to be 22,000 miles, what is the equivalent cost per mile (neglecting income taxes)?

6.27 An automobile that runs on electricity can be purchased for $25,000. The automobile is estimated to have a life of 12 years with annual travel of 20,000 miles. Every 3 years, a new set of batteries will have to be purchased at a cost of $3,000. Annual maintenance of the vehicle is estimated to cost $700. The cost of recharging the batteries is estimated at $0.015 per mile. The salvage value of the batteries and the vehicle at the end of 12 years is estimated to be $2,000. Suppose the MARR is 7%. What is the cost per mile to own and operate this vehicle, based on the preceding estimates? The $3,000 cost of the batteries is a net value, with the old batteries traded in for the new ones.

6.28 The estimated cost of a completely installed and ready-to-operate 40-kilowatt generator is $30,000. Its annual maintenance costs are estimated at $500. The energy that can be generated annually at full load is estimated to be 100,000 kilowatt-hours. If the value of the energy generated is $0.08 per kilowatt-hour, how long will it take before this machine becomes profitable? Take the MARR to be 9% and the salvage value of the machine to be $2,000 at the end of its estimated life of 15 years.

6.29 A large land-grant university that is currently facing severe parking problems on its campus is considering constructing parking decks off campus. A shuttle service could pick up students at the off-campus parking deck and transport them to various locations on campus. The university would charge a small fee for each shuttle ride, and the students could be quickly and economically transported to their classes. The funds raised by the shuttle would be used to pay for trolleys, which cost about $150,000 each. Each trolley has a 12-year service life, with an estimated salvage value of $3,000. To operate each trolley, the following additional expenses will be incurred:

Item	Annual Expenses ($)
Driver	$50,000
Maintenance	10,000
Insurance	3,000

If students pay 10 cents for each ride, determine the annual ridership per trolley (number of shuttle rides per year) required to justify the shuttle project, assuming an interest rate of 6%.

6.30 Eradicator Food Prep, Inc., has invested $7 million to construct a food irradiation plant. This technology destroys organisms that cause spoilage and disease, thus extending the shelf life of fresh foods and the distances over which they can be shipped. The plant can handle about 200,000 pounds of produce in an hour, and it will be operated for 3,600 hours a year. The net expected operating and maintenance costs (taking into account income-tax effects) would be $4 million per year. The plant is expected to have a useful life of 15 years, with a net salvage value of $700,000. The firm's interest rate is 15%.

(a) If investors in the company want to recover the plant investment within 6 years of operation (rather than 15 years), what would be the equivalent after-tax annual revenues that must be generated?

(b) To generate annual revenues determined in part (a), what minimum processing fee per pound should the company charge to its producers?

6.31 The local government of Santa Catalina Island, off the coast of Long Beach, California, is completing plans to build a desalination plant to help ease a critical drought on the island. The drought has combined with new construction on Catalina to leave the island with an urgent need for a new water source. A modern desalination plant could produce fresh water from seawater for $1,000 an acre-foot (326,000 gallons), or enough to supply two households for 1 year. On Catalina, the cost of acquiring water from natural sources is about the same as that for desalting. The $3 million plant, with a daily desalting capacity of 0.4 acre-foot, can produce 132,000 gallons of fresh water a day (enough to supply 295 households daily), more than a quarter of the island's total needs. The desalination plant has an estimated service life of 20 years, with no appreciable salvage value. The annual operating and maintenance costs would be about $250,000. Assuming an interest rate of 10%, what should be the minimum monthly water bill for each household?

6.32 A California utility firm is considering building a 50-megawatt geothermal plant that generates electricity from naturally occurring underground heat. The binary geothermal system will cost $85 million to build and $6 million (including any income-tax effect) to operate per year. (Virtually no fuel costs will accrue compared with fuel costs related to a conventional fossil-fuel plant.) The geothermal plant is to last for 25 years. At that time, its expected salvage value will be about the same as the cost to remove the plant. The plant will be in operation for 70% (plant utilization factor) of the year (or 70% of 8,760 hours per year). If the firm's MARR is 14% per year, determine the cost per kilowatt-hour of generating electricity.

6.33 A corporate executive jet with a seating capacity of 20 has the following cost factors:

Item	Cost
Initial cost	$12,000,000
Service life	15 years
Salvage value	$2,000,000
Crew costs per year	$225,000
Fuel cost per mile	$1.10
Landing fee	$250
Maintenance per year	$237,500
Insurance cost per year	$166,000
Catering per passenger trip	$75

CHAPTER SEVEN

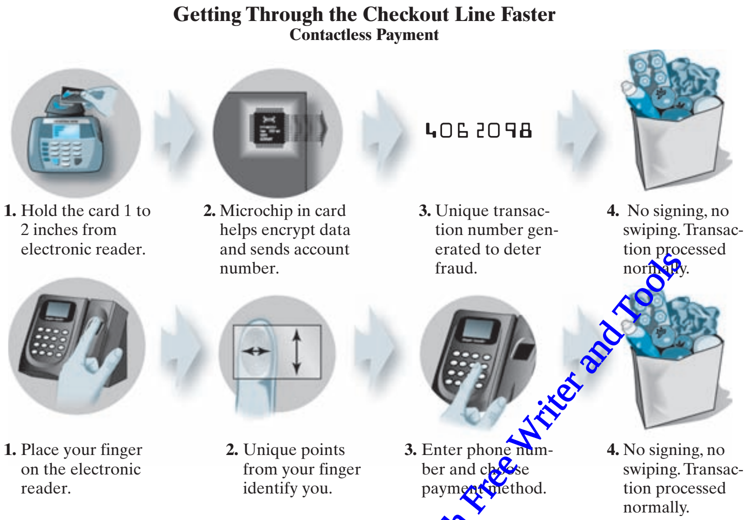
Rate-of-Return Analysis

Will That be Cash, Credit—or Fingertip?[1] Have you ever found yourself short of cash or without a wallet when you want to buy something? Consider the following two types of technologies available in retail stores to speed up checkouts:

- **Pay By Touch** takes fingerprints when customers enroll in the program. The image is then converted to about 40 unique points of the finger. Those points are stored in a computer system with "military-level encryption." They want this to be your cash replacement because of the time savings, and a lot of customers who are paying cash will find it more convenient now to use these cards.
- **A contactless card** allows the shopper to pay in seconds by waving his or her contactless card in front of a reader, which lights up and beeps to tell the shopper the transaction is done. A contactless payment is twice as fast as a no-signature credit card purchase and three times as fast as using cash. That's why it's catching on at fast-food restaurants and convenience stores.

These stores' profits depend, in part, on how quickly they get customers—typically with small purchases—through the line. These new technologies being rolled out at convenience stores, supermarkets, and gas stations could some day make it passé to carry bulky wallets. Without the need to dig for cash and checks at the register, the quick stop-and-go payments promise speedier transactions for consumers—and perhaps fatter profits for retailers. The appeal is that there's no need to run them through a machine. A contactless-card transaction is usually more expensive for a retailer to process than a cash payment. But retailers that adopt contactless payments hope they'll bring in more customers, offsetting higher costs. If that turns out to be false, then some could turn their backs on the new technology.

[1] "Will that be cash, credit—or fingertip?" Kathy Chu, *USA Today*, Section B1, Friday, December 2, 2005.



One retailer who just installed a Pay By Touch™ system hopes to increase its customer traffic so that a 10% return on investment can be attained. The Pay By Touch™ scanners cost about $50 each, the monthly service fee ranges between $38 and $45, and each transaction fee costs 10 cents. In a society driven by convenience, anything that speeds up the payment process attracts consumers. But technology providers will need to convince consumers of the safety of their information before the technologies can become a staple in the checkout line.

What does the 10% rate of return for the retailer really represent? How do we compute the figure from the projected additional retail revenues? And once computed, how do we use the figure when evaluating an investment project? Our consideration of the concept of rate of return in this chapter will answer these and other questions.

Along with the NPW and the AE criteria, the third primary measure of investment worth is **rate of return**. As shown in Chapter 5, the NPW measure is easy to calculate and apply. Nevertheless, many engineers and financial managers prefer rate-of-return analysis to the NPW method because they find it intuitively more appealing to analyze investments in terms of percentage rates of return rather than dollars of NPW. Consider the following statements regarding an investment's profitability:

- This project will bring in a 15% rate of return on the investment.
- This project will result in a net surplus of $10,000 in the NPW.

Neither statement describes the nature of the investment project in any complete sense. However, the rate of return is somewhat easier to understand because many of us are so familiar with savings-and-loan interest rates, which are in fact rates of return.

In this chapter, we will examine four aspects of rate-of-return analysis: (1) the concept of return on investment, (2) the calculation of a rate of return, (3) the development of an internal rate-of-return criterion, and (4) the comparison of mutually exclusive alternatives based on a rate of return.

CHAPTER LEARNING OBJECTIVES

After completing this chapter, you should understand the following concepts:

- The meaning of the rate of return.
- The various methods to compute the rate of return.
- How you make an accept and reject decision with the rate of return.
- How to resolve the multiple rates of return problem.
- How you conduct an incremental analysis with the rate of return.

7.1 Rate of Return

Yield: The annual rate of return on an investment, expressed as a percentage.

Many different terms are used to refer to **rate of return**, including **yield** (i.e., the yield to maturity, commonly used in bond valuation), **internal rate of return**, and **marginal efficiency of capital**. First we will review three common definitions of rate of return. Then we will use the definition of internal rate of return as a measure of profitability for a single investment project throughout the text.

7.1.1 Return on Investment

There are several ways of defining the concept of a rate of return on investment. The first is based on a typical loan transaction, the second on the mathematical expression of the present-worth function, and the third on the project cash flow series.

Definition 1

The rate of return is the interest rate earned on the unpaid balance of an amortized loan.

Suppose that a bank lends $10,000 and is repaid $4,021 at the end of each year for three years. How would you determine the interest rate that the bank charges on this transaction? As we learned in Chapter 3, you would set up the equivalence equation

$$\$10{,}000 = \$4{,}021(P/A, i, 3)$$

and solve for i. It turns out that $i = 10\%$. In this situation, the bank will earn a return of 10% on its investment of $10,000. The bank calculates the balances over the life of the loan as follows:

Year	Unpaid Balance at Beginning of Year	Return on Unpaid Balance (10%)	Payment Received	Unpaid Balance at End of Year
0	−$10,000	$0	$0	−$10,000
1	−10,000	−1,000	+4,021	−6,979
2	−6,979	−698	+4,021	−3,656
3	−3,656	−366	+4,021	0

A negative balance indicates an unpaid balance. In other words, the customer still owes money to the bank.

Observe that, for the repayment schedule shown, the 10% interest is calculated only on each year's outstanding balance. In this situation, only part of the $4,021 annual payment represents interest; the remainder goes toward repaying the principal. Thus, the three annual payments repay the loan itself and additionally provide a return of 10% on the *amount still outstanding each year*.

Note that when the last payment is made, the outstanding principal is eventually reduced to zero.[2] If we calculate the NPW of the loan transaction at its rate of return (10%), we see that

$$\text{PW}(10\%) = -\$10{,}000 + \$4{,}021(P/A, 10\%, 3) = 0,$$

which indicates that the bank can break even at a 10% rate of interest. In other words, the rate of return becomes the rate of interest that equates the present value of future cash repayments to the amount of the loan. This observation prompts the second definition of rate of return.

Definition 2

The rate of return is the break-even interest rate i that equates the present worth of a project's cash outflows to the present worth of its cash inflows, or*

$$\begin{aligned}\text{PW}(i^*) &= \text{PW}_{\text{Cash inflows}} - \text{PW}_{\text{Cash outflows}}\\ &= 0.\end{aligned}$$

Note that the expression for the NPW is equivalent to

$$\text{PW}(i^*) = \frac{A_0}{(1+i^*)^0} + \frac{A_1}{(1+i^*)^1} + \ldots + \frac{A_N}{(1+i^*)^N} = 0. \qquad (7.1)$$

Here we know the value of A_n for each period, but not the value of i^*. Since it is the only unknown, however, we can solve for i^*. (Inevitably, there will be N values of i^* that satisfy this equation. In most project cash flows, you would be able to find a unique positive i^* that satisfies Eq. (7.1). However, you may encounter some cash flows that cannot be solved for a single rate of return greater than 100%. By the nature of the NPW function in

[2] As we learned in Section 5.3.2, this terminal balance is equivalent to the net future worth of the investment. If the net future worth of the investment is zero, its NPW should also be zero.

Eq. (7.1), it is possible to have more than one rate of return for certain types of cash flows. For some cash flows, we may not find a specific rate of return at all.)[3]

Note that the formula in Eq. (7.1) is simply the NPW formula solved for the particular interest rate (i^*) at which $\mathrm{PW}(i)$ is equal to zero. By multiplying both sides of Eq. (7.1) by $(1 + i^*)^N$, we obtain

$$\mathrm{PW}(i^*)(1 + i^*)^N = \mathrm{FW}(i^*) = 0.$$

If we multiply both sides of Eq. (7.1) by the capital recovery factor $(A/P, i^*, N)$, we obtain the relationship $\mathrm{AE}(i^*) = 0$. Therefore, the i^* of a project may be defined as the rate of interest that equates the present worth, future worth, and annual equivalent worth of the entire series of cash flows to zero.

7.1.2 Return on Invested Capital

Investment projects can be viewed as analogous to bank loans. We will now introduce the concept of rate of return based on the return on invested capital in terms of a project investment. A project's return is referred to as the internal rate of return (IRR) or the **yield** promised by an **investment project** over its **useful life**.

Definition 3

The internal rate of return is the interest rate charged on the unrecovered project balance of the investment such that, when the project terminates, the unrecovered project balance will be zero.

Suppose a company invests $10,000 in a computer with a three-year useful life and equivalent annual labor savings of $4,021. Here, we may view the investing firm as the lender and the project as the borrower. The cash flow transaction between them would be identical to the amortized loan transaction described under Definition 1:

n	Beginning Project Balance	Return on Invested Capital	Ending Cash Payment	Project Balance
0	$0	$0	−$10,000	−$10,000
1	−10,000	−1,000	4,021	6,979
2	−6,979	−697	4,021	3,656
3	−3,656	−365	4,021	0

Internal rate of return: This is the return that a company would earn if it invested in itself, rather than investing that money elsewhere.

In our project balance calculation, we see that 10% is earned (or charged) on $10,000 during year 1, 10% is earned on $6,979 during year 2, and 10% is earned on $3,656 during year 3. This indicates that the firm earns a 10% rate of return on funds that remain *internally* invested in the project. Since it is a return that is *internal* to the project, we refer to it as the **internal rate of return**, or IRR. This means that the computer project under consideration brings in enough cash to pay for itself in three years and also to provide the firm with a return

[3] You will always have N rates of return. The issue is whether they are real or imaginary. If they are real, the question "Are they in the $(-100\%, \infty)$ interval?" should be asked. A **negative rate of return** implies that you never recover your initial investment.

of 10% on its invested capital. Put differently, if the computer is financed with funds costing 10% annually, the cash generated by the investment will be exactly sufficient to repay the principal and the annual interest charge on the fund in three years.

Notice that only one cash outflow occurs at time 0, and the present worth of this outflow is simply \$10,000. There are three equal receipts, and the present worth of these inflows is $\$4{,}021(P/A, 10\%, 3) = \$10{,}000$. Since the $\text{NPW} = \text{PW}_{\text{Inflow}} - \text{PW}_{\text{Outflow}} = \$10{,}000 - \$10{,}000 = 0$, 10% also satisfies Definition 2 of the rate of return. Even though the preceding simple example implies that i^* coincides with IRR, only Definitions 1 and 3 correctly describe the true meaning of the internal rate of return. As we will see later, if the cash expenditures of an investment are not restricted to the initial period, several break-even interest rates (i^*'s) may exist that satisfy Eq. (7.1). However, there may not be a rate of return that is *internal* to the project.

7.2 Methods for Finding the Rate of Return

We may find i^* by several procedures, each of which has its advantages and disadvantages. To facilitate the process of finding the rate of return for an investment project, we will first classify various types of investment cash flow.

7.2.1 Simple versus Nonsimple Investments

We can classify an investment project by counting the number of sign changes in its net cash flow sequence. A change from either "+" to "−" or "−" to "+" is counted as one sign change. (We ignore a zero cash flow.) Then,

- A **simple investment** is an investment in which the initial cash flows are negative and only one sign change occurs in the remaining net cash flow series. If the initial flows are positive and only one sign change occurs in the subsequent net cash flows, they are referred to as **simple borrowing** cash flows.
- A **nonsimple investment** is an investment in which more than one sign change occurs in the cash flow series.

Simple investment: The project with only one sign change in the net cash flow series.

Multiple i^*'s, as we will see later, occur only in nonsimple investments. Three different types of investment possibilities are illustrated in Example 7.1.

EXAMPLE 7.1 Investment Classification

Consider the following three cash flow series and classify them into either simple or nonsimple investments:

Period	Net Cash Flow		
n	Project A	Project B	Project C
0	−\$1,000	−\$1,000	\$1,000
1	−500	3,900	−450
2	800	−5,030	−450
3	1,500	2,145	−450
4	2,000		

SOLUTION

Given: Preceding cash flow sequences.

Find: Classify the investments shown into either simple and nonsimple investments.

- Project A represents many common simple investments. This situation reveals the NPW profile shown in Figure 7.1(a). The curve crosses the *i*-axis only once.
- Project B represents a nonsimple investment. The NPW profile for this investment has the shape shown in Figure 7.1(b). The *i*-axis is crossed at 10%, 30%, and 50%.
- Project C represents neither a simple nor a nonsimple investment, even though only one sign change occurs in the cash flow sequence. Since the first cash flow is positive, this is a **simple borrowing** cash flow, not an investment flow. Figure 7.1(c) depicts the NPW profile for this type of investment.

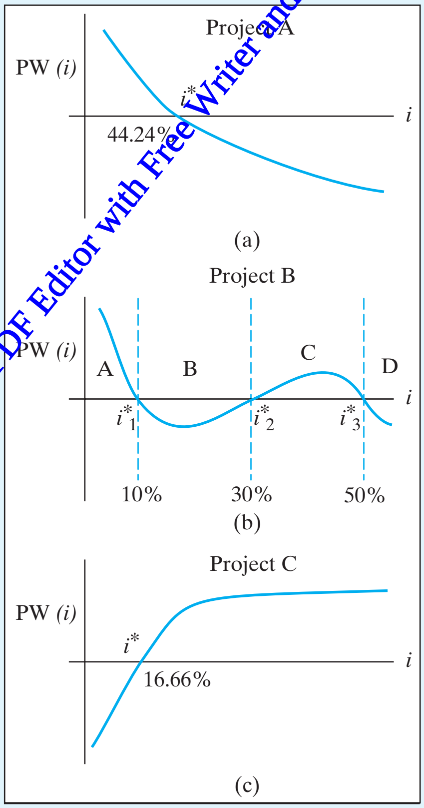


Figure 7.1 Present-worth profiles: (a) Simple investment, (b) nonsimple investment with multiple rates of return, and (c) simple borrowing cash flows.

COMMENTS: Not all NPW profiles for nonsimple investments have multiple crossings of the *i*-axis. Clearly, then, we should place a high priority on discovering this situation early in our analysis of a project's cash flows. The quickest way to predict multiple i^*'s is to generate an NPW profile on a computer and check whether it crosses the horizontal axis more than once. In the next section, we illustrate when to expect such multiple crossings by examining types of cash flows.

7.2.2 Predicting Multiple i^*'s

As hinted at in Example 7.1, for certain series of project cash flows, we may uncover the complication of multiple i^* values that satisfy Eq. (7.1). By analyzing and classifying cash flows, we may anticipate this difficulty and adjust our analysis approach later. Here we will focus on the initial problem of whether we can predict a unique i^* for a project by examining its cash flow pattern. Two useful rules allow us to focus on sign changes (1) in net cash flows and (2) in accounting net profit (accumulated net cash flows).

Net Cash Flow Rule of Signs

One useful method for predicting an upper limit on the number of positive i^*'s of a cash flow stream is to apply the rule of signs: *The number of real i^*'s that are greater than -100% for a project with N periods is never greater than the number of sign changes in the sequence of the A_n's. A zero cash flow is ignored.*

An example is

Period	A_n	Sign Change
0	−\$100	
1	−20	
2	+50	1
3	0	
4	+60	
5	−30	1
6	+100	1

Three sign changes occur in the cash flow sequence, so three or fewer real positive i^*'s exist.

It must be emphasized that the rule of signs provides an indication only of the *possibility* of multiple rates of return: The rule predicts only the *maximum* number of possible i^*'s. Many projects have multiple sign changes in their cash flow sequence, but still possess a unique real i^* in the $(-100\%, \infty)$ range.

Accumulated Cash Flow Sign Test

The accumulated cash flow is the sum of the net cash flows up to and including a given time. If the rule of cash flow signs indicates multiple i^*'s, we should proceed to the **accumulated cash flow sign test** to eliminate some possibility of multiple rates of return.

If we let A_n represent the net cash flow in period n and S_n represent the accumulated cash flow (the accounting sum) up to period n, we have the following:

Period (n)	Cash Flow (A_n)	Accumulated Cash Flow (S_n)
0	A_0	$S_0 = A_0$
1	A_1	$S_1 = S_0 + A_1$
2	A_2	$S_2 = S_1 + A_2$
$\vdots$	$\vdots$	$\vdots$
N	A_N	$S_N = S_{N-1} + A_N$

We then examine the sequence of accumulated cash flows $(S_0, S_1, S_2, S_3, \ldots, S_N)$ to determine the number of sign changes. *If the series S_n starts negatively and changes sign only once, then a unique positive i^* exists.* This cumulative cash flow sign rule is a more discriminating test for identifying the uniqueness of i^* than the previously described method.

EXAMPLE 7.2 Predicting the Number of i^*'s

Predict the number of real positive rates of return for each of the following cash flow series:

Period	A	B	C	D
0	−$100	−$100	$ 0	−$100
1	−200	+50	−50	+50
2	+200	−100	+115	0
3	+200	+60	−66	+200
4	+200	−100		−50

SOLUTION

Given: Four cash flow series and cumulative flow series.

Find: The upper limit on number of i^*'s for each series.

The cash flow rule of signs indicates the following possibilities for the positive values of i^*:

Project	Number of Sign Changes in Net Cash Flows	Possible Number of Positive Values of i^*
A	1	1 or 0
B	4	4, 3, 2, 1, or 0
C	2	2, 1, or 0
D	2	2, 1, or 0

For cash flows B, C, and D, we would like to apply the more discriminating cumulative cash flow test to see if we can specify a smaller number of possible values of i^*. Accordingly, we write

	Project B		Project C		Project D	
n	A_n	S_n	A_n	S_n	A_n	S_n
0	−$100	−$100	$ 0	$ 0	−$100	−$100
1	+50	−50	−50	−50	+50	−50
2	−100	−150	+115	+65	0	−50
3	+60	−90	−66	−1	+200	+150
4	−100	−190			−50	+100

Recall the test: If the series starts *negatively* and changes sign only once, a unique positive i^* exists.

- Only project D begins negatively and passes the test; therefore, we may predict a unique i^* value, rather than 2, 1, or 0 as predicted by the cash flow rule of signs. ($i_1^* = -75.16\%$ and $i_2^* = 35.05\%$)
- Project B, with no sign change in the cumulative cash flow series, has no rate of return.
- Project C fails the test, and we cannot eliminate the possibility of multiple i^*'s. ($i_1^* = 10\%$ and $i_2^* = 20\%$)

7.2.3 Computational Methods

Once we identify the type of an investment cash flow, several ways to determine its rate of return are available. Some of the most practical methods are as follows:

- Direct solution method,
- Trial-and-error method, and
- Computer solution method.

Direct Solution Method

For the special case of a project with only a two-flow transaction (an investment followed by a single future payment) or a project with a service life of two years of return, we can seek a direct mathematical solution for determining the rate of return. These two cases are examined in Example 7.3.

EXAMPLE 7.3 Finding i^* by Direct Solution: Two Flows and Two Periods

Consider two investment projects with the following cash flow transactions:

n	Project 1	Project 2
0	−\$2,000	−\$2,000
1	0	1,300
2	0	1,500
3	0	
4	3,500	

Compute the rate of return for each project.

SOLUTION

Given: Cash flows for two projects.
Find: i^* for each project.

Project 1: Solving for i^* in $\text{PW}(i^*) = 0$ is identical to solving $\text{FW}(i^*) = 0$, because FW equals PW times a constant. We could do either here, but we will set $\text{FW}(i^*) = 0$ to demonstrate the latter. Using the single-payment future-worth relationship, we obtain

$$\begin{aligned} \text{FW}(i^*) &= -\$2{,}000(F/P, i^*, 4) + \$3{,}500 = 0, \\ \$3{,}500 &= \$2{,}000(F/P, i^*, 4) = \$2{,}000(1 + i^*)^4, \\ 1.75 &= (1 + i^*)^4. \end{aligned}$$

Solving for i^* yields

$$\begin{aligned} i^* &= \sqrt[4]{1.75} - 1 \\ &= 0.1502 \text{ or } 15.02\%. \end{aligned}$$

Project 2: We may write the NPW expression for this project as

$$\text{PW}(i) = -\$2{,}000 + \frac{\$1{,}300}{(1 + i)} + \frac{\$1{,}500}{(1 + i)^2} = 0.$$

Let $X = 1/(1 + i)$. We may then rewrite $\text{PW}(i)$ as a function of X as follows:

$$\text{PW}(X) = -\$2{,}000 + \$1{,}300X + \$1{,}500X^2 = 0.$$

This is a quadratic equation that has the following solution:[4]

[4] The solution of the quadratic equation $aX^2 + bX + c = 0$ is $X = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

$$X = \frac{-1{,}300 \pm \sqrt{1{,}300^2 - 4(1{,}500)(-2{,}000)}}{2(1{,}500)}$$
$$= \frac{-1{,}300 \pm 3{,}700}{3{,}000}$$
$$= 0.8 \text{ or } -1.667.$$

Replacing X values and solving for i gives us

$$0.8 = \frac{1}{(1 + i)} \rightarrow i = 25\%,$$
$$-1.667 = \frac{1}{(1 + i)} \rightarrow i = -160\%.$$

Since an interest rate less than -100% has no economic significance, we find that the project's i^* is 25%.

COMMENTS: In both projects, one sign change occurred in the net cash flow series, so we expected a unique i^*. Also, these projects had very simple cash flows. When cash flows are more complicated, generally we use a trial-and-error method or a computer to find i^*.

Trial-and-Error Method

The first step in the trial-and-error method is to make an estimated ***guess***[5] at the value of i^*. For a simple investment, we use the "guessed" interest rate to compute the present worth of net cash flows and observe whether it is positive, negative, or zero. Suppose, then, that PW(i) is negative.

Since we are aiming for a value of i that makes PW(i) = 0, we must raise the present worth of the cash flow. To do this, we lower the interest rate and repeat the process. If PW(i) is positive, however, we raise the interest rate in order to lower PW(i). The process is continued until PW(i) is approximately equal to zero. Whenever we reach the point where PW(i) is bounded by one negative and one positive value, we use **linear interpolation** to approximate i^*. This process is somewhat tedious and inefficient. (The trial-and-error method does not work for nonsimple investments in which the NPW function is not, in general, a monotonically decreasing function of the interest rate.)

EXAMPLE 7.4 Finding i^* by Trial and Error

The Imperial Chemical Company is considering purchasing a chemical analysis machine worth \$13,000. Although the purchase of this machine will not produce any

[5] As we shall see later in this chapter, the ultimate objective of finding i^* is to compare it against the MARR. Therefore, it is a good idea to use the MARR as the initial guess.

increase in sales revenues, it will result in a reduction of labor costs. In order to operate the machine properly, it must be calibrated each year. The machine has an expected life of six years, after which it will have no salvage value. The following table summarizes the annual savings in labor cost and the annual maintenance cost in calibration over six years:

Year (n)	Costs ($)	Savings ($)	Net Cash Flow ($)
0	13,000		−13,000
1	2,300	6,000	3,700
2	2,300	7,000	4,700
3	2,300	9,000	6,700
4	2,300	9,000	6,700
5	2,300	9,000	6,700
6	2,300	9,000	6,700

Find the rate of return for this project.

SOLUTION

Given: Cash flows over six years as shown in Figure 7.2.
Find: i^*.

We start with a guessed interest rate of 25%. The present worth of the cash flows is

$$\begin{aligned}\text{PW}(25\%) &= -\$13{,}000 + \$3{,}700(P/F, 25\%, 1) + \$4{,}700(P/F, 25\%, 2) \\ &\quad + \$6{,}700(P/A, 25\%, 4)(P/F, 25\%, 2) \\ &= \$3{,}095.\end{aligned}$$

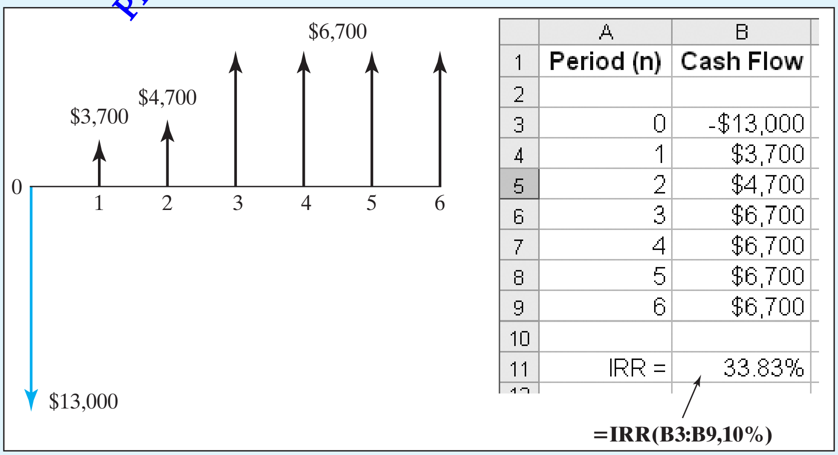


Figure 7.2 Cash flow diagram for a simple investment (Example 7.4).

Since this present worth is positive, we must raise the interest rate to bring PW toward zero. When we use an interest rate of 35%, we find that

$$\begin{aligned}\text{PW}(35\%) &= -\$13{,}000 + \$3{,}700(P/F, 35\%, 1) + \$4{,}700(P/F, 35\%, 2)\\ &\quad + \$6{,}700(P/A, 35\%, 4)(P/F, 35\%, 2)\\ &= -\$339.\end{aligned}$$

We have now bracketed the solution: PW(i) will be zero at i somewhere between 25% and 35%. Using straight-line interpolation, we approximate

$$\begin{aligned}i^* &\cong 25\% + (35\% - 25\%)\left[\frac{3{,}095 - 0}{3{,}095 - (-339)}\right]\\ &= 25\% + 10\%(0.9013)\\ &= 34.01\%.\end{aligned}$$

Now we will check to see how close this value is to the precise value of i^*. If we compute the present worth at this interpolated value, we obtain

$$\begin{aligned}\text{PW}(34\%) &= -\$13{,}000 + \$3{,}700(P/F, 34\%, 1) + \$4{,}700(P/F, 34\%, 2)\\ &\quad + \$6{,}700(P/A, 34\%, 4)(P/F, 34\%, 2)\\ &= -\$50.58.\end{aligned}$$

As this is not zero, we may recompute i^* at a lower interest rate, say, 33%:

$$\begin{aligned}\text{PW}(33\%) &= -\$13{,}000 + \$3{,}700(P/F, 33\%, 1) + \$4{,}700(P/F, 33\%, 2)\\ &\quad + \$6{,}700(P/A, 33\%, 4)(P/F, 33\%, 2)\\ &= \$248.56.\end{aligned}$$

With another round of linear interpolation, we approximate

$$\begin{aligned}i^* &\cong 33\% + (34\% - 33\%)\left[\frac{248.56 - 0}{248.56 - (-50.58)}\right]\\ &= 33\% + 1\%(0.8309)\\ &= 33.83\%.\end{aligned}$$

At this interest rate,

$$\begin{aligned}\text{PW}(33.83\%) &= -\$13{,}000 + \$3{,}700(P/F, 33.83\%, 1)\\ &\quad + \$4{,}700(P/F, 33.83\%, 2)\\ &\quad + \$6{,}700(P/A, 33.83\%, 4)(P/F, 33.83\%, 2)\\ &= -\$0.49,\end{aligned}$$

which is practically zero, so we may stop here. In fact, there is no need to be more precise about these interpolations, because the final result can be no more accurate than the basic data, which ordinarily are only rough estimates.

COMMENT: With Excel, you can evaluate the IRR for the project as =IRR (range,guess), where you specify the cell range for the cash flow (e.g., B3:B9) and the initial guess, such as 25%. Computing i^* for this problem in Excel, incidentally, gives us 33.8283%. Instead of using the factor notations, you may attempt use a tabular approach as follows:

Internal Rate of Return: What It Looks Like

		Discount Rate: 25%		Discount Rate: 35%	
Year	**Cash Flow**	**Factor**	**Amount**	**Factor**	**Amount**
0	−$13,000	1.0000	−$13,000	1.0000	−$13,000
1	3,700	0.8000	2,960	0.7407	2,741
2	4,700	0.6400	3,008	0.5487	2,579
3	6,700	0.5120	3,430	0.4064	2,723
4	6,700	0.4096	2,744	0.3011	2,017
5	6,700	0.3277	2,196	0.2230	1,494
6	6,700	0.2621	1,756	0.1652	1,107
Total	+$22,200	**NPW =**	**+$3,095**	**NPW =**	**−$339**
			IRR = close to 34%		

Graphical Method

We don't need to do laborious manual calculations to find i^*. Many financial calculators have built-in functions for calculating i^*. It is worth noting that many online financial calculators or spreadsheet packages have i^* functions, which solve Eq. (7.1) very rapidly,[6] usually with the user entering the cash flows via a computer keyboard or by reading a cash flow data file. (For example, Microsoft Excel has an IRR financial function that analyzes investment cash flows, as illustrated in Example 7.4.)

The most easily generated and understandable graphic method of solving for i^* is to create the **NPW profile** on a computer. On the graph, the horizontal axis indicates the interest rate and the vertical axis indicates the NPW. For a given project's cash flows, the NPW is calculated at an interest rate of zero (which gives the vertical-axis intercept) and several other interest rates. Points are plotted and a curve is sketched. Since i^* is defined as the interest rate at which $\text{PW}(i^*) = 0$, the point at which the curve crosses the horizontal axis closely approximates i^*. The graphical approach works for both simple and nonsimple investments.

[6] An alternative method of solving for i^* is to use a computer-aided economic analysis program. Cash Flow Analyzer (CFA) finds i^* visually by specifying the lower and upper bounds of the interest search limit and generates NPW profiles when given a cash flow series. In addition to the savings in calculation time, the advantage of computer-generated profiles is their precision. CFA can be found from the book's website.

EXAMPLE 7.5 Graphical Approach to Estimate i^*

Consider the cash flow series shown in Figure 7.3(a). Estimate the rate of return by generating the NPW profile on a computer.

SOLUTION

Given: Cash flow series in Figure 7.3.

Find: (a) i^* by plotting the NPW profile and (b) i^* by using Excel.

	A	B
1	Period	Cash Flow
2		
3	0	-10000
4	1	20000
5	2	20000
6	3	-25000
7		
8	IRR =	140%
9		
10	Interest (%)	NPW
11	0	$5,000
12	10	$5,928
13	20	$6,088
14	30	$5,840
15	40	$5,379
16	50	$4,815
17	60	$4,209
18	70	$3,597
19	80	$2,997
20	90	$2,422
21	100	$1,875
22	110	$1,359
23	120	$875
24	130	$422
25	140	($3)
26	150	($400)
27	160	($772)
28	170	($1,119)
29	180	($1,445)
30	190	($1,750)
31	200	($2,037)
32		

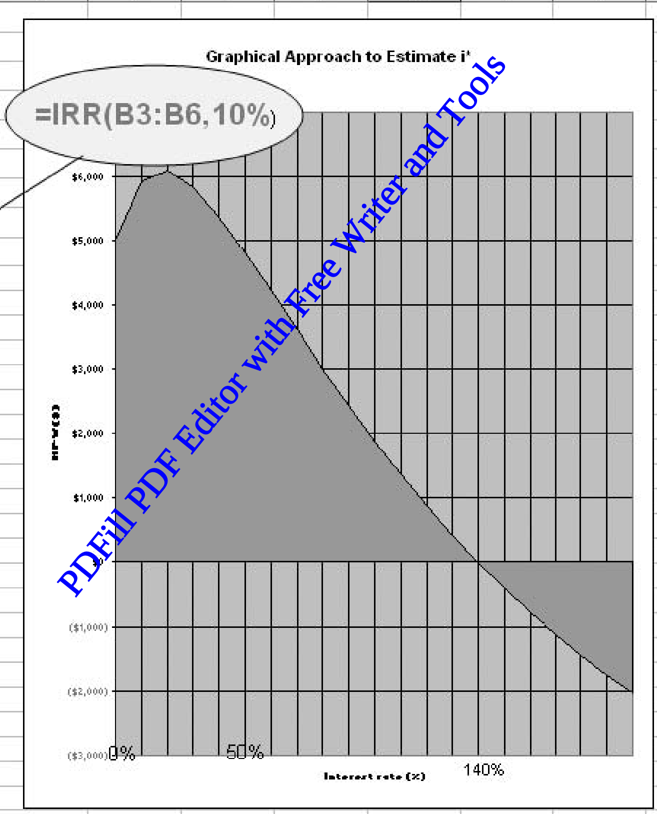


Figure 7.3 Graphical solution to rate-of-return problem for a typical nonsimple investment (Example 7.5).

(a) The present-worth function for the project cash flow series is

$$PW(i) = -\$10{,}000 + \$20{,}000(P/A, i, 2) - \$25{,}000(P/F, i, 3).$$

First we use $i = 0$ in this equation to obtain NPW = \$5,000, which is the vertical-axis intercept. Then we substitute several other interest rates—10%, 20%, . . . , 140%—and plot these values of PW(i) as well. The result is Figure 7.3, which shows the curve crossing the horizontal axis at roughly 140%. This value can be verified by other methods if we desire to do so. Note that, in addition to establishing the interest rate that makes NPW = 0, the NPW profile indicates where positive and negative NPW values fall, thus giving us a broad picture of those interest rates for which the project is acceptable or unacceptable. (Note also that a trial-and-error method would lead to some confusion: As you increase the interest rate from 0% to 20%, the NPW value also keeps increasing, instead of decreasing.) Even though the project is a nonsimple investment, the curve crosses the horizontal axis only once. As mentioned in the previous section, however, most nonsimple projects have more than one value of i^* that makes NPW = 0 (i.e., more than one i^* per project). In such a case, the NPW profile would cross the horizontal axis more than once.[7]

(b) With Excel, you can evaluate the IRR for the project with the function

=IRR(range,guess)

in which you specify the cell range for the cash flow and the initial guess, such as 10%.

7.3 Internal-Rate-of-Return Criterion

Now that we have classified investment projects and learned methods for determining the i^* value for a given project's cash flows, our objective is to develop an accept–reject decision rule that gives results consistent with those obtained from NPW analysis.

7.3.1 Relationship to PW Analysis

As we already observed in Chapter 5, NPW analysis depends on the rate of interest used for the computation of NPW. A different rate may change a project from being considered acceptable to being unacceptable, or it may change the ranking of several projects:

- Consider again the NPW profile as drawn for the simple project in Figure 7.1(a). For interest rates below i^*, this project should be accepted because NPW $>$ 0; for interest rates above i^*, it should be rejected.
- By contrast, for certain nonsimple projects, the NPW may look like the one shown in Figure 7.1(b). NPW analysis would lead you to accept the projects in regions **A** and **C**, but reject those in regions **B** and **D**. Of course, this result goes against intuition: A higher interest rate would change an unacceptable project into an acceptable one. The situation graphed in Figure 7.1(b) is one of the cases of multiple i^*'s mentioned in Definition 2.

[7] In Section 7.2.2, we discuss methods of predicting the number of i^* values by looking at cash flows. However, generating an NPW profile to discover multiple i^*'s is as practical and informative as any other method.

Therefore, for the simple investment situation in Figure 7.1(a), i^* can serve as an appropriate index for either accepting or rejecting the investment. However, for the nonsimple investment of Figure 7.1(b), it is not clear which i^* to use to make an accept–reject decision. Therefore, the i^* value fails to provide an appropriate measure of profitability for an investment project with multiple rates of return.

7.3.2 Net-Investment Test: Pure versus Mixed Investments

To develop a consistent accept–reject decision rule with the NPW, we need to further classify a project into either a pure or a mixed investment:

- A project is said to be a **net investment** when the project balances computed at the project's i^* values, $PB(i^*)_n$, are either less than or equal to zero throughout the life of the investment, with the first cash flow being negative $(A_0 < 0)$. The investment is *net* in the sense that the firm does not overdraw on its return at any point and hence investment is *not indebted* to the project. This type of project is called a **pure investment**. In contrast, **pure borrowing** is defined as the situation in which $PB(i^*)_n$ values are positive or zero throughout the life of the loan, with $A_0 > 0$. *Simple investments will always be pure investments.*
- If any of the project balances calculated at the project's i^* is positive, the project is not a pure investment. A positive project balance indicates that, at some time during the project life, the firm acts as a borrower $[PB(i^*)_n > 0]$ rather than an investor in the project $[PB(i^*)_n < 0]$. This type of investment is called a **mixed investment**.

Net investment test: A process to determine whether or not a firm borrows money from a project during the investment period.

Pure investment: An investment in which a firm never borrows money from the project.

Mixed investment: An investment in which a firm borrows money from the project during the investment period.

EXAMPLE 7.6 Pure versus Mixed Investments

Consider the following four investment projects with known i^* values:

	Project Cash Flows			
n	**A**	**B**	**C**	**D**
0	−\$1,000	−\$1,000	−\$1,000	−\$1,000
1	1,000	1,600	500	3,900
2	2,000	−300	−500	−5,030
3	1,500	−200	2,000	2,145
i^*	33.64%	21.95%	29.95%	(10%, 30%, 50%)

Determine which projects are pure investments.

SOLUTION

Given: Four projects with cash flows and i^*'s as shown.

Find: Which projects are pure investments?

We will first compute the project balances at the projects' respective i^*'s. If multiple rates of return exist, we may use the largest value of i^* greater than zero.[8]

Project A:

$$PB(33.64\%)_0 = -\$1{,}000,$$
$$PB(33.64\%)_1 = -\$1{,}000(1 + 0.3364) + (-\$1{,}000) = -\$2{,}336.40,$$
$$PB(33.64\%)_2 = -\$2{,}336.40(1 + 0.3364) + \$2{,}000 = -\$1{,}122.36,$$
$$PB(33.64\%)_3 = -\$1{,}122.36(1 + 0.3364) + \$1{,}500 = 0.$$

$(-, -, -, 0)$: passes the net-investment test (pure investment).

Project B:

$$PB(21.95\%)_0 = -\$1{,}000,$$
$$PB(21.95\%)_1 = -\$1{,}000(1 + 0.2195) + \$1{,}600 = \$380.50,$$
$$PB(21.95\%)_2 = +\$380.50(1 + 0.2195) - \$300 = \$164.02,$$
$$PB(21.95\%)_3 = +\$164.02(1 + 0.2195) - \$200 = 0.$$

$(-, +, +, 0)$: fails the net-investment test (mixed investment).

Project C:

$$PB(29.95\%)_0 = -\$1{,}000,$$
$$PB(29.95\%)_1 = -\$1{,}000(1 + 0.2995) + \$500 = -\$799.50,$$
$$PB(29.95\%)_2 = -\$799.50(1 + 0.2995) - \$500 = -\$1{,}538.95,$$
$$PB(29.95\%)_3 = -\$1{,}538.95(1 + 0.2995) + \$2{,}000 = 0.$$

$(-, -, -, 0)$: passes the net-investment test (pure investment).

Project D: There are three rates of return. We can use any of them for the net investment test. Thus,

$$PB(50\%)_0 = -\$1{,}000,$$
$$PB(50\%)_1 = -\$1{,}000(1 + 0.50) + \$3{,}900 = \$2{,}400,$$
$$PB(50\%)_2 = +\$2{,}400(1 + 0.50) - \$5{,}030 = -\$1{,}430,$$
$$PB(50\%)_3 = -\$1{,}430(1 + 0.50) + \$2{,}145 = 0.$$

$(-, +, -, 0)$: fails the net-investment test (mixed investment).

COMMENTS: As shown in Figure 7.4, projects A and C are the only pure investments. Project B demonstrates that the existence of a unique i^* is a necessary but not sufficient condition for a pure investment.

[8] In fact, it does not matter which rate we use in applying the net-investment test. If one value passes the test, they will all pass. If one value fails, they will all fail.

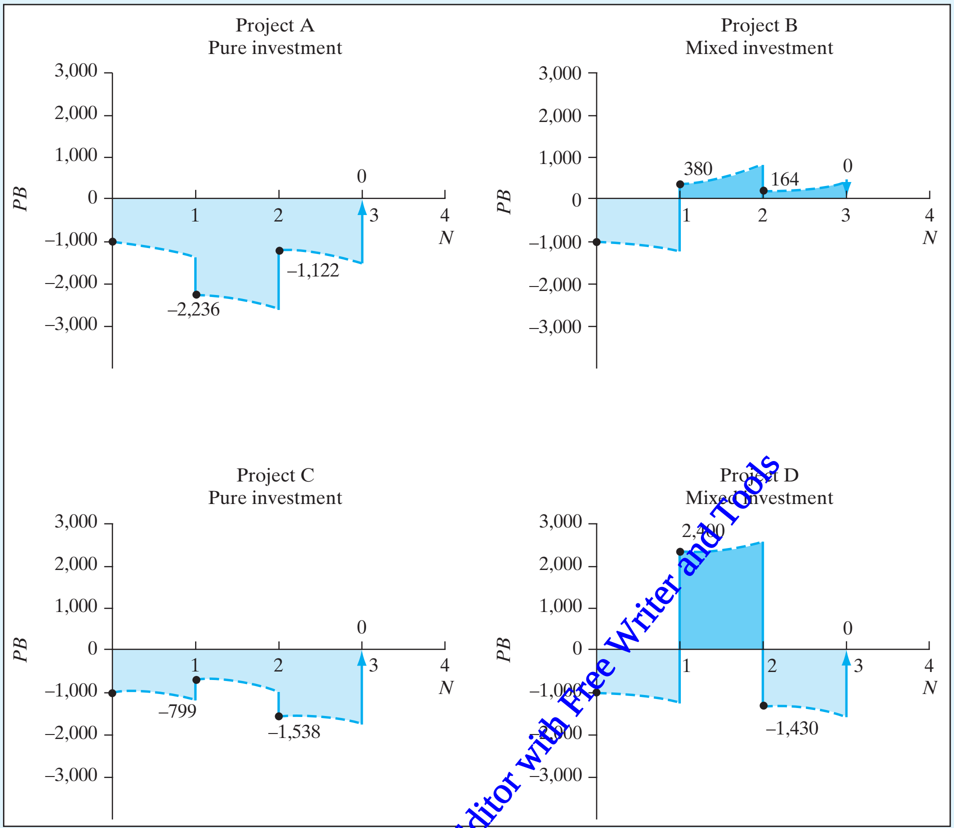


Figure 7.4 Net-investment test (Example 7.6).

7.3.3 Decision Rule for Pure Investments

Suppose we have a pure investment. (Recall that all simple investments are pure investments as well.) Why are we interested in finding the particular interest rate that equates a project's cost with the present worth of its receipts? Again, we may easily answer this question by examining Figure 7.1(a). In this figure, we notice two important characteristics of the NPW profile. First, as we compute the project's PW(i) at a varying interest rate i, we see that the NPW is positive for $i < i^*$, indicating that the project would be acceptable under PW analysis for those values of i. Second, the NPW is negative for $i > i^*$, indicating that the project is unacceptable for those values of i. Therefore, i^* serves as a **benchmark** interest rate, knowledge of which will enable us to make an accept–reject decision consistent with NPW analysis.

Note that, for a pure investment, i^* is indeed the IRR of the investment. (See Section 7.1.2.) Merely knowing i^*, however, is not enough to apply this method. Because firms typically wish to do better than break even (recall that at NPW $= 0$ we were indifferent to the project), a minimum acceptable rate of return (MARR) is indicated by company policy, management, or the project decision maker. If the IRR exceeds this MARR, we are assured that the company will more than break even. Thus, the IRR becomes a useful gauge against which to judge a project's acceptability, and the decision rule for a pure project is as follows:

If IRR $>$ MARR, accept the project.

If IRR $=$ MARR, remain indifferent.

If IRR $<$ MARR, reject the project.

Note that this decision rule is designed to be applied for a single project evaluation. When we have to compare mutually exclusive investment projects, we need to apply the **incremental analysis approach**, as we shall see in Section 7.4.2.

EXAMPLE 7.7 Investment Decision for a Pure Investment

Merco, Inc., a machinery builder in Louisville, Kentucky, is considering investing \$1,250,000 in a complete structural beam-fabrication system. The increased productivity resulting from the installation of the drilling system is central to the project's justification. Merco estimates the following figures as a basis for calculating productivity:

- Increased fabricated steel production: 2,000 tons/year.
- Average sales price/ton fabricated steel: \$2,566.50/ton.
- Labor rate: \$10.50/hour.
- Tons of steel produced in a year: 15,000 tons.
- Cost of steel per ton (2,205 lb): \$1,950/ton.
- Number of workers on layout, hole making, sawing, and material handling: 17.
- Additional maintenance cost: \$128,500/year.

With the cost of steel at \$1,950 per ton and the direct labor cost of fabricating 1 lb at 10 cents, the cost of producing a ton of fabricated steel is about \$2,170.50. With a selling price of \$2,566.50 per ton, the resulting contribution to overhead and profit becomes \$396 per ton. Assuming that Merco will be able to sustain an increased production of 2,000 tons per year by purchasing the system, engineers have estimated the projected additional contribution to be 2,000 tons $\times$ \$396 $=$ \$792,000.

Since the drilling system has the capacity to fabricate the full range of structural steel, two workers can run the system, one on the saw and the other on the drill. A third operator is required to operate a crane for loading and unloading materials. Merco estimates that, to do the equivalent work of these three workers with conventional manufacturing techniques would require, on the average, an additional 14 people for center punching, hole making with a radial or magnetic drill, and material handling. This translates into a labor savings in the amount of \$294,000 per year (14 $\times$ \$10.50 $\times$ 40 hours/week $\times$ 50 weeks/year). The system can last for 15 years, with an estimated after-tax salvage value of \$80,000. However, after an annual deduction of \$226,000 in corporate income taxes, the net investment costs, as well as savings, are as follows:

- Project investment cost: \$1,250,000.
- Projected annual net savings:
  (\$792,000 + \$294,000) − \$128,500 − \$226,000 = \$731,500.

- Projected after-tax salvage value at the end of year 15: $80,000.
  - (a) What is the projected IRR on this fabrication investment?
  - (b) If Merco's MARR is known to be 18%, is this investment justifiable?

SOLUTION

Given: Projected cash flows as shown in Figure 7.5 and MARR = 18%.
Find: (a) The IRR and (b) whether to accept or reject the investment.

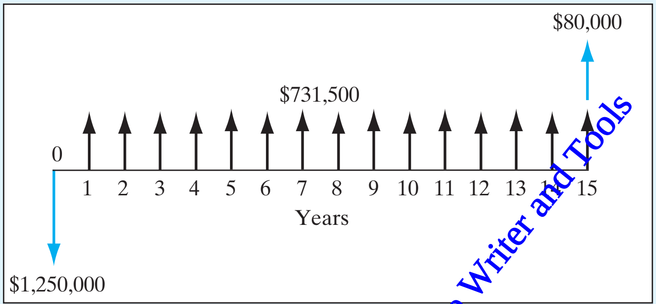


Figure 7.5 Cash flow diagram (Example 7.7).

(a) Since only one sign change occurs in the net cash flow series, the fabrication project is a simple investment. This indicates that there will be a unique rate of return that is internal to the project:

$$\begin{aligned} PW(i) &= -\$1{,}250{,}000 + \$731{,}500(P/A, i, 15) \\ &\quad + \$80{,}000(P/F, i, 15) \\ &= 0 \\ i^* &= 58.71\%. \end{aligned}$$

With Excel, you will also find that the IRR is about 58.71% for the net investment of $1,250,000.

(b) The IRR figure far exceeds Merco's MARR, indicating that the fabrication system project is an economically attractive one. Merco's management believes that, over a broad base of structural products, there is no doubt that the installation of the fabricating system would result in a significant savings, even after considering some potential deviations from the estimates used in the analysis.

7.3.4 Decision Rule for Mixed Investments

Applied to pure projects, i^* provides an unambiguous criterion for measuring profitability. However, when multiple rates of return occur, none of them is an accurate portrayal of a project's acceptability or profitability. However, there is a correct method, which uses an **external interest rate**, for refining our analysis when we do discover multiple i^*'s. An external rate of return allows us to calculate a single accurate rate of return; if you choose to avoid these more complicated applications of rate-of-return techniques, you must be able to predict multiple i^*'s via the NPW profile and, when they occur, select an alternative method such as NPW or AE analysis for determining the project's acceptability.

Need for an External Interest Rate for Mixed Investments

In the case of a mixed investment, we can extend the economic interpretation of the IRR to the return-on-invested-capital measure if we are willing to make an assumption about what happens to the extra cash that the investor gets from the project during the intermediate years.

Project balance: The amount of money committed to a project at a specific period.

First, the **project balance** (PB), or investment balance, can also be interpreted from the viewpoint of a financial institution that borrows money from an investor and then pays interest on the PB. Thus, a negative PB means that the investor has money in a bank account; a positive PB means that the investor has borrowed money from the bank. Negative PBs represent interest paid by the bank to the investor; positive PBs represent interest paid by the investor to the bank.

Now, can we assume that the interest paid by the bank and the interest received from the investor are the same for the same amount of balance? In our banking experience, we know that is not the case. Normally, the borrowing rate (interest paid by the investor) is higher than the interest rate on your deposit (interest paid by the bank).

However, when we calculate the project balance at an i^* for mixed investments, we notice an important point: Cash borrowed (released) from the project is assumed to earn the same interest rate through external investment as money that remains internally invested. In other words, in solving a cash flow for an unknown interest rate, it is assumed that money released from a project can be reinvested to yield a rate of return equal to that received from the project. In fact, we have been making this assumption regardless of whether a cash flow does or does not produce a unique positive i^*. Note that money is borrowed only when $PB(i^*) > 0$, and the magnitude of the borrowed amount is the project balance. When $PB(i^*) < 0$, no money is borrowed, even though the cash flow may be positive at that time.

In reality, it is not always possible for cash borrowed (released) from a project to be reinvested to yield a rate of return equal to that received from the project. Instead, it is likely that the rate of return available on a capital investment in the business is much different—usually higher—from the rate of return available on other external investments. Thus, it may be necessary to compute the project balances for a project's cash flow at two rates of interest—one on the internal investment and one on the external investments. As we will see later, by separating the interest rates, we can measure the **true rate of return** of any internal portion of an investment project.

Calculation of Return on Invested Capital for Mixed Investments

For a mixed investment, we must calculate a rate of return on the portion of capital that remains invested internally. This rate is defined as the **true IRR** for the mixed investment and is commonly known as the **return on invested capital** (**RIC**). Then, what interest rate should we assume for the portion of external investment? Insofar as a project is not a net investment, one or more periods when the project has a net outflow of money (a positive project balance) must later be returned to the project. This money can be put into the firm's investment pool until such time as it is needed in the project. The interest rate of this investment pool is the interest rate at which the money can in fact be invested outside the project.

Return on invested capital (RIC): The amount that a company earns on the total investment it has made in its project.

Recall that the NPW method assumed that the interest rate charged to any funds withdrawn from a firm's investment pool would be equal to the MARR. In this book, *we will use the MARR as an established external interest rate* (i.e., the rate earned by money invested outside of the project). We can then compute the RIC as a function of the MARR by finding the value of the RIC that will make the terminal project balance equal to zero. (This implies that the firm wants to fully recover any investment made in the project and pays off any borrowed funds at the end of the project life.) This way of computing the rate of return is an accurate measure of the profitability of the project as represented by the cash flow. The following procedure outlines the steps for determining the IRR for a mixed investment:

Step 1. Identify the MARR (or external interest rate).

Step 2. Calculate $\text{PB}(i, \text{MARR})_n$ (or simply PB_n) according to the rule

$$\text{PB}(i, \text{MARR})_0 = A_0.$$

$$\text{PB}(i, \text{MARR})_1 = \begin{cases} \text{PB}_0(1+i) + A_1, \text{ if } \text{PB}_0 < 0 \\ \text{PB}_0(1+\text{MARR}) + A_1, \text{ if } \text{PB}_0 > 0 \end{cases},$$

$$\vdots$$

$$\text{PB}(i, \text{MARR})_n = \begin{cases} \text{PB}_{n-1}(1+i) + A_n, \text{ if } \text{PB}_{n-1} < 0 \\ \text{PB}_{n-1}(1+\text{MARR}) + A_n, \text{ if } \text{PB}_{n-1} > 0 \end{cases}.$$

(As defined in the text, A_n stands for the net cash flow at the end of period n. Note that the terminal project balance must be zero.)

Step 3. Determine the value of i by solving the terminal project balance equation

$$\text{PB}(i, \text{MARR})_N = 0.$$

The interest rate i is the RIC (or IRR) for the mixed investment.

Using the MARR as an external interest rate, we may accept a project if the IRR exceeds the MARR, and we should reject the project otherwise. Figure 7.6 summarizes the IRR computation for a mixed investment.

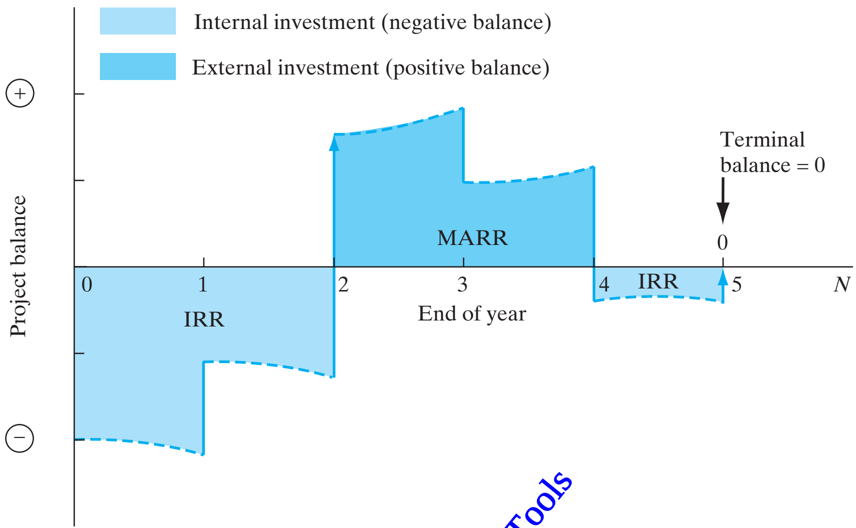


Figure 7.6 Computational logic for IRR (mixed investment).

EXAMPLE 7.8 IRR for a Mixed Investment

By outbidding its competitors, Trans Image Processing (TIP), a defense contractor, received a contract worth $7,300,000 to build navy flight simulators for U.S. Navy pilot training over two years. With some defense contracts, the U.S. government makes an advance payment when the contract is signed, but in this case the government will make two progressive payments: $4,300,000 at the end of the first year and the $3,000,000 balance at the end of the second year. The expected cash outflows required to produce the simulators are estimated to be $1,000,000 now, $2,000,000 during the first year, and $4,320,000 during the second year. The expected net cash flows from this project are summarized as follows:

Year	Cash Inflow	Cash Outflow	Net Cash Flow
0		$1,000,000	−$1,000,000
1	$4,300,000	2,000,000	2,300,000
2	3,000,000	4,320,000	−1,320,000

In normal situations, TIP would not even consider a marginal project such as this one. However, hoping that the company can establish itself as a technology leader in the field, management felt that it was worth outbidding its competitors. Financially, what is the economic worth of outbidding the competitors for this project? That is,

(a) Compute the values of i^*'s for this project.

(b) Make an accept–reject decision based on the results in part (a). Assume that the contractor's MARR is 15%.

SOLUTION

Given: Cash flow shown and MARR = 15%.

Find: (a) Compute the NPW, (b) i^*, and (c) RIC at MARR = 15%, and determine whether to accept the project.

(a)

$$\begin{aligned} PW(15\%) &= -\$1{,}000{,}000 + \$2{,}300{,}000(P/F, 15\%, 1) \\ &= -\$1{,}320{,}000(P/F, 15\%, 2) \\ &= \$1{,}890 > 0. \end{aligned}$$

(b) Since the project has a two-year life, we may solve the net-present-worth equation directly via the quadratic formula:

$$-\$1{,}000{,}000 + \$2{,}300{,}000/(1 + i^*) - \$1{,}320{,}000/(1 + i^*)^2 = 0.$$

If we let $X = 1/(1 + i^*)$, we can rewrite the preceding expression as

$$-1{,}000{,}000 + 2{,}300{,}000X - 1{,}320{,}000X^2 = 0.$$

Solving for X gives $X = 10/11$ and $10/12$, or $i^* = 10\%$ and 20%. As shown in Figure 7.7, the NPW profile intersects the horizontal axis twice, once at 10% and again at 20%. The investment is obviously not a simple one; thus, neither 10% nor 20% represents the true internal rate of return of this government project.

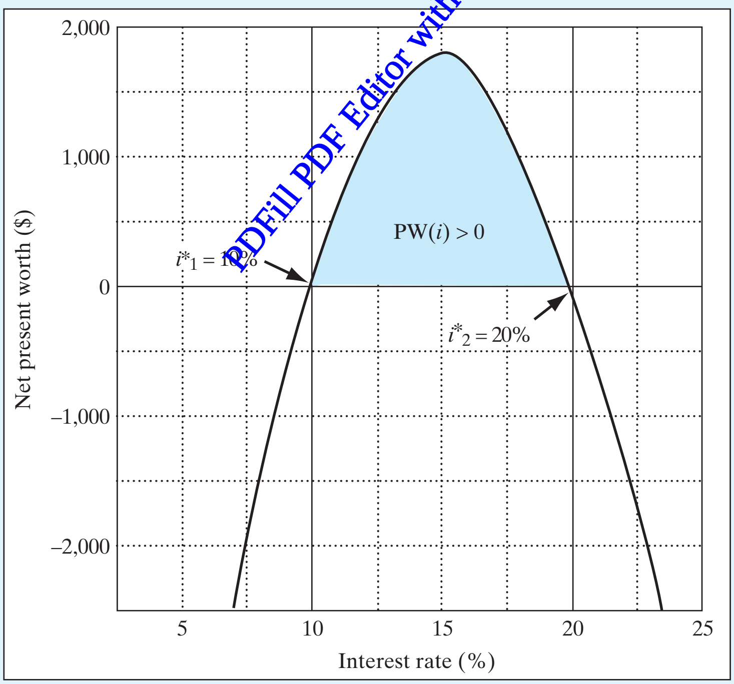


Figure 7.7 NPW plot for a nonsimple investment with multiple rates of return (Example 7.8).

(c) As calculated in (b), the project has multiple rates of return. This is obviously not a net investment, as the following table shows:

Net Investment Test	Using $i^* = 10\%$			Using $i^* = 20\%$		
N	0	1	2	0	1	2
Beginning balance	\$0	−\$1,000	\$1,200	\$0	−\$1,000	\$1,100
Return on investment	0	−100	120	0	−200	220
Payment	−1,000	2,300	−1,320	−1,000	2,300	−1,320
Ending balance	−\$1,000	\$1,200	0	−\$1,000	\$1,100	0

(Unit: \$1,000)

At $n = 0$, there is a net investment to the firm so the project balance expression becomes

$$PB(i, 15\%)_0 = -\$1{,}000{,}000.$$

The net investment of \$1,000,000 that remains invested internally grows at the interest rate i for the next period. With the receipt of \$2,300,000 in year 1, the project balance becomes

$$\begin{aligned} PB(i, 15\%)_1 &= -\$1{,}000{,}000(1 + i) + \$2{,}300{,}000 \\ &= \$1{,}300{,}000 - \$1{,}000{,}000i \\ &= \$1{,}000{,}000(1.3 - i). \end{aligned}$$

At this point, we do not know whether $PB(i, 15\%)_1$ is positive or negative; we want to know this in order to test for net investment and the presence of a unique i^*. It depends on the value of i, which we want to determine. Therefore, we need to consider two situations: (1) $i < 1.3$ and (2) $i > 1.3$.

- **Case 1:** $i < 1.3 \rightarrow PB(i, 15\%)_1 > 0.$
  Since this indicates a positive balance, the cash released from the project would be returned to the firm's investment pool to grow at the MARR until it is required back in the project. By the end of year 2, the cash placed in the investment pool would have grown at the rate of 15% [to $\$1{,}000{,}000(1.3 - i)(1 + 0.15)$] and must equal the investment into the project of \$1,320,000 required at that time. Then the terminal balance must be

$$\begin{aligned} PB(i, 15\%)_2 &= \$1{,}000{,}000(1.3 - i)(1 + 0.15) - \$1{,}320{,}000 \\ &= \$175{,}000 - \$1{,}150{,}000i \\ &= 0. \end{aligned}$$

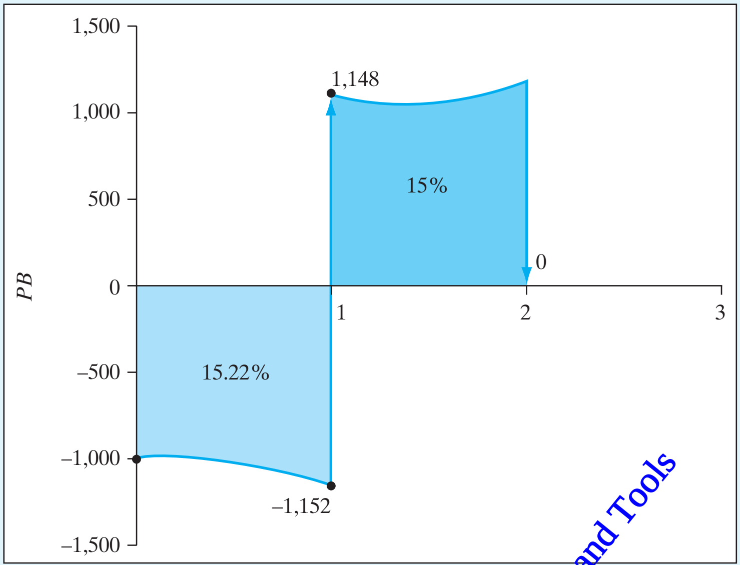


Figure 7.8 Calculation of the IRR for a mixed investment (Example 7.8).

Solving for i yields

$$\text{RIC} = \text{IRR} = 0.1522, \text{ or } 15.22\% > 15\%.$$

The computational process is shown graphically in Figure 7.8.

- **Case 2:** $i > 1.3 \rightarrow \text{PB}(i, 15\%)_1 < 0$.

The firm is still in an investment mode. Therefore, the balance at the end of year 1 that remains invested will grow at the rate i for the next period. With the investment of \$1,320,000 required in year 2 and the fact that the net investment must be zero at the end of the project life, the balance at the end of year 2 should be

$$\begin{aligned}\text{PB}(i, 15\%)_2 &= \$1{,}000{,}000(1.3 - i)(1 + i) - \$1{,}320{,}000 \\ &= -\$20{,}000 + \$300{,}000i - \$1{,}000{,}000i^2 \\ &= 0.\end{aligned}$$

Solving for i gives

$$\text{IRR} = 0.1 \text{ or } 0.2 < 1.3,$$

which violates the initial assumption that $i > 1.3$. Therefore, Case 1 is the only correct situation. Since it indicates that IRR > MARR, the project is acceptable, resulting in the same decision as obtained in (a) by applying the NPW criterion.

COMMENTS: In this example, we could have seen by inspection that Case 1 was correct. Since the project required an investment as the final cash flow, the project balance at the end of the previous period (year 1) had to be positive in order for the final balance to equal zero. Inspection does not generally work with more complicated cash flows.

Trial-and-Error Method for Computing IRR for Mixed Investments

The trial-and-error approach to finding the IRR (RIC) for a mixed investment is similar to the trial-and-error approach to finding i^*. We begin with a given MARR and a guess for IRR and solve for the project balance. (A value of IRR close to the MARR is a good starting point for most problems.) Since we desire the project balance to approach zero, we can adjust the value of IRR as needed after seeing the result of the initial guess. For example, for a given pair of interest rates (IRRguess, MARR), if the terminal project balance is positive, the IRRguess value is too low, so we raise it and recalculate. We can continue adjusting our IRR guesses in this way until we obtain a project balance equal or close to zero.

EXAMPLE 7.9 IRR for a Mixed Investment by Trial and Error

Consider project D in Example 7.6. The project has the following cash flow:

n	A_n
0	−$1,000
1	3,900
2	−5,030
3	2,145

We know from an earlier calculation that this is a mixed investment. Compute the IRR for this project. Assume that MARR = 6%.

SOLUTION

Given: Cash flow as stated for mixed investment and MARR = 6%.
Find: IRR.

For MARR = 6%, we must compute i by trial and error. Suppose we guess $i = 8\%$:

$$PB(8\%, 6\%)_0 = -\$1{,}000,$$
$$PB(8\%, 6\%)_1 = -\$1{,}000(1 + 0.08) + \$3{,}900 = \$2{,}820.$$
$$PB(8\%, 6\%)_2 = +\$2{,}820(1 + 0.06) - \$5{,}030 = -\$2{,}040.80,$$
$$PB(8\%, 6\%)_3 = -\$2{,}040.80(1 + 0.08) + \$2{,}145 = -\$59.06.$$

The net investment is negative at the end of the project, indicating that our trial $i = 8\%$ is in error. After several trials, we conclude that, for MARR = 6%, the IRR is approximately 6.13%. To verify the results, we write

$$PB(6.13\%, 6\%)_0 = -\$1{,}000,$$
$$PB(6.13\%, 6\%)_1 = -\$1{,}000.00(1 + 0.0613) + \$3{,}900 = \$2{,}838.66.$$
$$PB(6.13\%, 6\%)_2 = +\$2{,}838.66(1 + 0.0600) - \$5{,}030 = -\$2{,}021.02,$$
$$PB(6.13\%, 6\%)_3 = -\$2{,}021.02(1 + 0.0613) + \$2{,}145 = 0.$$

The positive balance at the end of year 1 indicates the need to borrow from the project during year 2. However, note that the net investment becomes zero at the end of the project life, confirming that 6.13% is the IRR for the cash flow. Since IRR $>$ MARR, the investment is acceptable.

COMMENTS: On the basis of the NPW criterion, the investment would be acceptable if the MARR was between zero and 10% or between 30% and 50%. The rejection region is $10\% < i < 30\%$ and $i > 50\%$. This can be verified in Figure 7.1(b). Note that the project also would be marginally accepted under the NPW analysis at MARR $= i = 6\%$:

$$\begin{aligned}PW(6\%) &= -\$1{,}000 + 3{,}900(P/F, 6\%, 1)\\ &= -\$5{,}030(P/F, 6\%, 2) + 2{,}145(P/F, 6\%, 3)\\ &= \$3.55 > 0.\end{aligned}$$

The flowchart in Figure 7.9 summarizes how you should proceed to apply the net cash flow sign test, accumulated cash flow sign test, and net-investment test to calculate an IRR and make an accept–reject decision for a single project. Given the complications

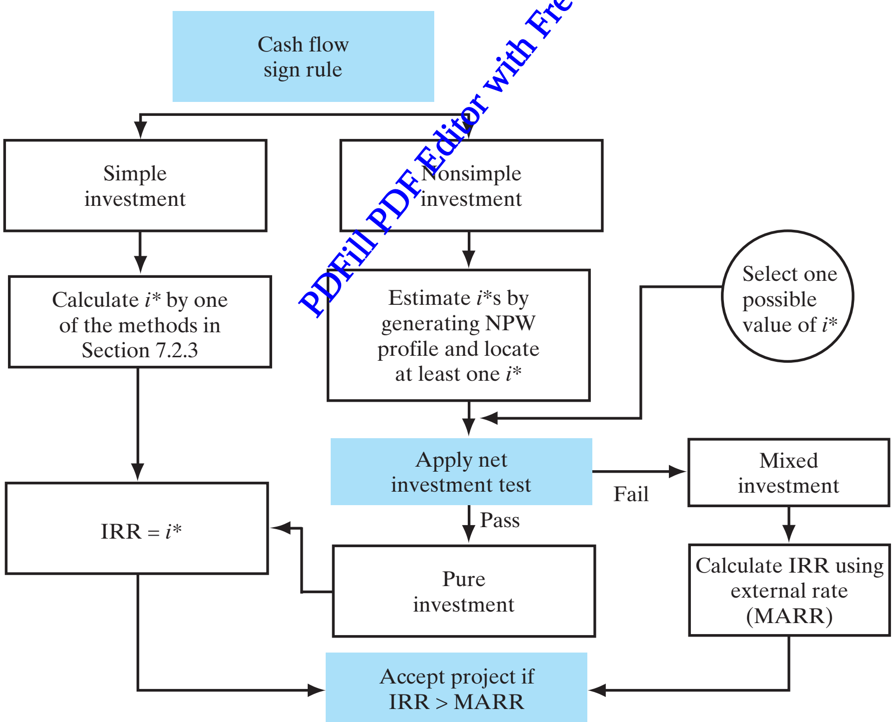


Figure 7.9 Summary of IRR criterion: A flowchart that summarizes how you may proceed to apply the net cash flow sign rule and net-investment test to calculate IRR for a pure as well as a mixed investment.

involved in using IRR analysis to compare alternative projects, it is usually more desirable to use one of the other equivalence techniques for this purpose. As an engineering manager, you should keep in mind the intuitive appeal of the rate-of-return measure. Once you have selected a project on the basis of NPW or AE analysis, you may also wish to express its worth as a rate of return, for the benefit of your associates.

7.4 Mutually Exclusive Alternatives

In this section, we present the decision procedures that should be used in comparing two or more mutually exclusive projects on the basis of the rate-of-return measure. We will consider two situations: (1) alternatives that have the same economic service life and (2) alternatives that have unequal service lives.

7.4.1 Flaws in Project Ranking by IRR

Under NPW or AE analysis, the mutually exclusive project with the highest worth was preferred. (This is known as the "total investment approach.") Unfortunately, the analogy does not carry over to IRR analysis: The project with the highest IRR may *not* be the preferred alternative. To illustrate the flaws inherent in comparing IRRs in order to choose from mutually exclusive projects, suppose you have two mutually exclusive alternatives, each with a 1-year service life: One requires an investment of \$1,000 with a return of \$2,000, and the other requires \$5,000 with a return of \$7,000. You already obtained the IRRs and NPWs at MARR = 10% as follows:

n	A1	A2
0	−\$1,000	−\$5,000
1	2,000	7,000
IRR	100%	40%
PW(10%)	\$818	\$1,364

Assuming that you have enough money in your investment pool to select either alternative, would you prefer the first project simply because you expect a higher rate of return?

On the one hand, we can see that A2 is preferred over A1 by the NPW measure. On the other hand, the IRR measure gives a numerically higher rating for A1. This inconsistency in ranking occurs because the NPW, NFW, and AE are **absolute (dollar)** measures of investment worth, whereas the IRR is a **relative (percentage)** measure and cannot be applied in the same way. That is, the IRR measure ignores the **scale** of the investment. Therefore, the answer to our question in the previous paragraph is no; instead, you would prefer the second project, with the lower rate of return, but higher NPW. Either the NPW or the AE measure would lead to that choice, but a comparison of IRRs would rank the smaller project higher. Another approach, referred to as **incremental analysis**, is needed.

7.4.2 Incremental Investment Analysis

In the previous example, the more costly option requires an incremental investment of \$4,000 at an incremental return of \$5,000. Let's assume that you have exactly \$5,000 in your investment pool.

- If you decide to invest in option A1, you will need to withdraw only \$1,000 from your investment pool. The remaining \$4,000 will continue to earn 10% interest. One year later, you will have \$2,000 from the outside investment and \$4,400 from the investment pool. With an investment of \$5,000, in one year you will have \$6,400. The equivalent present worth of this change in wealth is $\text{PW}(10\%) = -\$5{,}000 + \$6{,}400(P/F, 10\%, 1) = \$818$.
- If you decide to invest in option A2, you will need to withdraw \$5,000 from your investment pool, leaving no money in the pool, but you will have \$7,000 from your outside investment. Your total wealth changes from \$5,000 to \$7,000 in a year. The equivalent present worth of this change in wealth is $\text{PW}(10\%) = -\$5{,}000 + \$7{,}000(P/F, 10\%, 1) = \$1{,}364$.

In other words, if you decide to take the more costly option, certainly you would be interested in knowing that this additional investment can be justified at the MARR. The 10%-of-MARR value implies that you can always earn that rate from other investment sources (i.e., \$4,400 at the end of 1 year for a \$4,000 investment). However, in the second option, by investing the additional \$4,000, you would make an additional \$5,000, which is equivalent to earning at the rate of 25%. Therefore, the incremental investment can be justified.

Now we can generalize the decision rule for comparing mutually exclusive projects. For a pair of mutually exclusive projects (A and B, with B defined as the more costly option), we may rewrite B as

$$B = A + (B - A).$$

In other words, B has two cash flow components: (1) the same cash flow as A and (2) the incremental component $(B - A)$. Therefore, the only situation in which B is preferred to A is when the rate of return on the incremental component $(B - A)$ exceeds the MARR. Therefore, for two mutually exclusive projects, rate-of-return analysis is done by computing the *internal rate of return on the incremental investment* $(IRR\Delta)$ between the projects. Since we want to consider increments of investment, we compute the cash flow for the difference between the projects by subtracting the cash flow for the lower investment-cost project (A) from that of the higher investment-cost project (B). Then the decision rule is

Incremental IRR: IRR on the incremental investment from choosing a large project instead of a smaller project.

$$\text{If IRR}_{B-A} > \text{MARR, select } B,$$

$$\text{If IRR}_{B-A} = \text{MARR, select either project,}$$

$$\text{If IRR}_{B-A} < \text{MARR, select } A,$$

where $B - A$ is an investment increment (negative cash flow). *If a "do-nothing" alternative is allowed, the smaller cost option must be profitable (its IRR must be greater than the MARR) at first.* This means that you compute the rate of return for each alternative in the mutually exclusive group and then eliminate the alternatives whose IRRs are less than the MARR before applying the incremental analysis.

It may seem odd to you how this simple rule allows us to select the right project. Example 7.10 illustrates the incremental investment decision rule.

EXAMPLE 7.10 IRR on Incremental Investment: Two Alternatives

John Covington, a college student, wants to start a small-scale painting business during his off-school hours. To economize the start-up business, he decides to purchase some used painting equipment. He has two mutually exclusive options: Do most of the painting by himself by limiting his business to only residential painting jobs (B1) or purchase more painting equipment and hire some helpers to do both residential and commercial painting jobs that he expects will have a higher equipment cost, but provide higher revenues as well (B2). In either case, John expects to fold up the business in three years, when he graduates from college.

The cash flows for the two mutually exclusive alternatives are as follows:

n	B1	B2	B2 − B1
0	−$3,000	−$12,000	−$9,000
1	1,350	4,200	2,850
2	1,800	6,225	4,425
3	1,500	6,330	4,830
IRR	25%	17.43%	

Knowing that both alternatives are revenue projects, which project would John select at MARR = 10%? (Note that both projects are also profitable at 10%.)

SOLUTION

Given: Incremental cash flow between two alternatives and MARR = 10%.
Find: (a) IRR on the increment and (b) which alternative is preferable.

(a) To choose the best project, we compute the incremental cash flow B2 − B1. Then we compute the IRR on this increment of investment by solving the equation

$$-\$9{,}000 + \$2{,}850(P/F, i, 1) + \$4{,}425(P/F, i, 2) + \$4{,}830(P/F, i, 3) = 0.$$

(b) We obtain $i^*_{B2-B1} = 15\%$, as plotted in Figure 7.10. By inspection of the incremental cash flow, we know that it is a simple investment, so $IRR_{B2-B1} = i^*_{B2-B1}$. Since $IRR_{B2-B1} >$ MARR, we select B2, which is consistent with the NPW analysis. Note that, at MARR > 25%, neither project would be acceptable.

COMMENTS: Why did we choose to look at the increment B2 − B1 instead of B1 − B2? Because we want the first flow of the incremental cash flow series to be negative (an investment flow), so that we can calculate an IRR. By subtracting the lower initial investment project from the higher, we guarantee that the first increment will be an investment flow. If we ignore the investment ranking, we might end up

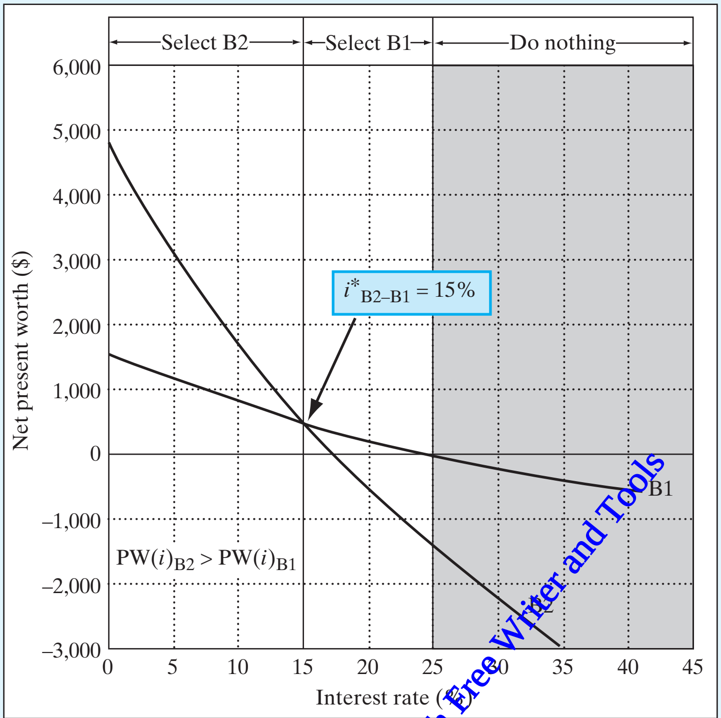


Figure 7.10 NPW profiles for B1 and B2 (Example 7. 10).

with an increment that involves *borrowing cash flow* and has no internal rate of return. This is indeed the case for B1 − B2. (i^*_{B1-B2} is also 15%, not −15%, but it has a different meaning: it is a borrowing rate, not a rate of return on your investment.) If, erroneously, we had compared this i^* with the MARR, we might have accepted project B1 over B2. This undoubtedly would have damaged our credibility with management!

The next example indicates that the inconsistency in ranking between NPW and IRR can also occur when differences in the timing of a project's future cash flows exist, even if their initial investments are the same.

EXAMPLE 7.11 IRR on Incremental Investment When Initial Flows Are Equal

Consider the following two mutually exclusive investment projects that require the same amount of investment:

Which project would you select on the basis of the rate of return on incremental investment, assuming that MARR = 12%? (Once again, both projects are profitable at 12%.)

n	C1	C2
0	−\$9,000	−\$9,000
1	480	5,800
2	3,700	3,250
3	6,550	2,000
4	3,780	1,561
IRR	18%	20%

SOLUTION

Given: Cash flows for two mutually exclusive alternatives as shown and MARR = 12%.

Find: (a) IRR on incremental investment and (b) which alternative is preferable.

(a) When the initial investments are equal, we progress through the cash flows until we find the first difference and then set up the increment so that this first nonzero flow is negative (i.e., an investment). Thus, we set up the incremental investment by taking (C1 − C2):

n	C1 − C2
0	\$ 0
1	−5,320
2	450
3	4,550
4	2,219

We next set the PW equation equal to zero:

$$-\$5{,}320 + \$450(P/F, i, 1) + \$4{,}550(P/F, i, 2) + \$2{,}219(P/F, i, 3) = 0.$$

(b) Solving for i yields $i^* = 14.71\%$, which is also an IRR, since the increment is a simple investment. Since $\text{IRR}_{C1-C2} = 14.71\% > \text{MARR}$, we would select C1. If we used NPW analysis, we would obtain $\text{PW}(12\%)_{C1} = \$1{,}443$ and $\text{PW}(12\%)_{C2} = \$1{,}185$, indicating that C1 is preferred over C2.

When you have more than two mutually exclusive alternatives, they can be compared in pairs by successive examination. Example 7.12 illustrates how to compare three mutually exclusive alternatives. (In Chapter 15, we will examine some multiple-alternative problems in the context of capital budgeting.)

EXAMPLE 7.12 IRR on Incremental Investment: Three Alternatives

Consider the following three sets of mutually exclusive alternatives:

n	D1	D2	D3
0	−$2,000	−$1,000	−$3,000
1	1,500	800	1,500
2	1,000	500	2,000
3	800	500	1,000
IRR	34.37%	40.76%	24.81%

Which project would you select on the basis of the rate of return on incremental investment, assuming that MARR = 15%?

SOLUTION

Given: Preceding cash flows and MARR = 15%.

Find: IRR on incremental investment and which alternative is preferable.

Step 1: Examine the IRR of each alternative. At this point, we can eliminate any alternative that fails to meet the MARR. In this example, all three alternatives exceed the MARR.

Step 2: Compare D1 and D2 in pairs.[9] Because D2 has a lower initial cost, compute the rate of return on the increment (D1−D2), which represents an increment of investment.

n	D1 − D2
0	−$1,000
1	700
2	500
3	300

The incremental cash flow represents a simple investment. To find the incremental rate of return, we write

$$-\$1{,}000 + \$700(P/F, i, 1) + \$500(P/F, i, 2) + \$300(P/F, i, 3) = 0.$$

Solving for i^*_{D1-D2} yields 27.61%, which exceeds the MARR; therefore, D1 is preferred over D2. Now you eliminate D2 from further consideration.

Step 3: Compare D1 and D3. Once again, D1 has a lower initial cost. Examine the increment (D3 − D1):

[9] When faced with many alternatives, you may arrange them in order of increasing initial cost. This is not a required step, but it makes the comparison more tractable.

n	D3 − D1
0	−\$1,000
1	0
2	1,000
3	200

Here, the incremental cash flow represents another simple investment. The increment (D3 − D1) has an unsatisfactory 8.8% rate of return; therefore, D1 is preferred over D3. Accordingly, we conclude that D1 is the best alternative.

EXAMPLE 7.13 Incremental Analysis for Cost-Only Projects

Falk Corporation is considering two types of manufacturing systems to produce its shaft couplings over six years: (1) a cellular manufacturing system (CMS) and (2) a flexible manufacturing system (FMS). The average number of pieces to be produced with either system would be 544,000 per year. The operating cost, initial investment, and salvage value for each alternative are estimated as follows:

Items	CMS Option	FMS Option
Annual O&M costs:		
Annual labor cost	\$1,169,600	\$707,200
Annual material cost	832,320	598,400
Annual overhead cost	3,150,000	1,950,000
Annual tooling cost	470,000	300,000
Annual inventory cost	141,000	31,500
Annual income taxes	1,650,000	1,917,000
Total annual costs	\$7,412,920	\$5,504,100
Investment	\$4,500,000	\$12,500,000
Net salvage value	\$500,000	\$1,000,000

Figure 7.11 illustrates the cash flows associated with each alternative. The firm's MARR is 15%. On the basis of the IRR criterion, which alternative would be a better choice?

DISCUSSION: Since we can assume that both manufacturing systems would provide the same level of revenues over the analysis period, we can compare the two alternatives on the basis of cost only. (These are service projects.) Although we cannot compute the IRR for each option without knowing the revenue figures, we can still

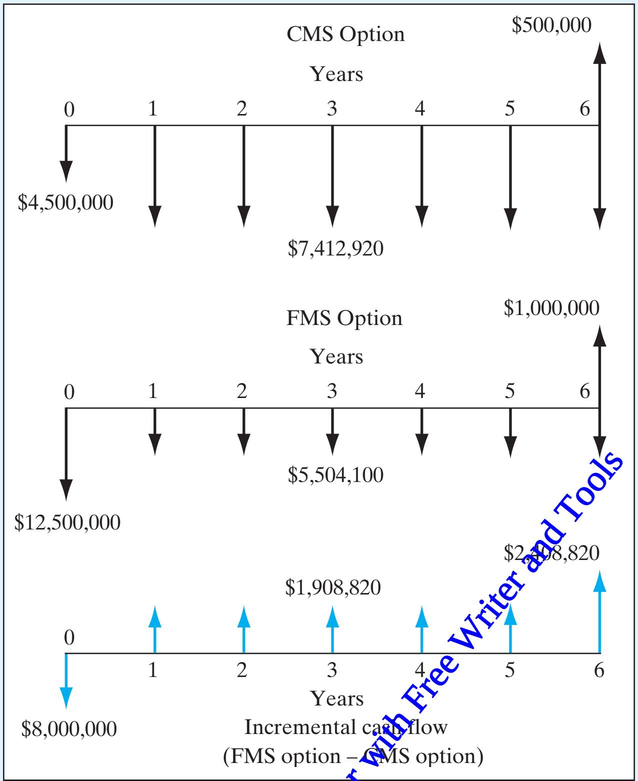


Figure 7.11 Comparison of mutually exclusive alternatives with equal revenues (cost only) (Example 7.13).

calculate the IRR on incremental cash flows. Since the FMS option requires a higher initial investment than that for the CMS, the incremental cash flow is the difference (FMS − CMS).

n	CMS Option	FMS Option	Incremental (FMS − CMS)
0	−$4,500,000	−$12,500,000	−$8,000,000
1	−7,412,920	−5,504,100	1,908,820
2	−7,412,920	−5,504,100	1,908,820
3	−7,412,920	−5,504,100	1,908,820
4	−7,412,920	−5,504,100	1,908,820
5	−7,412,920	−5,504,100	1,908,820
6	−7,412,920	−5,504,100	
	+ $500,000	+ $1,000,000	$2,408,820

SOLUTION

Given: Cash flows shown in Figure 7.11 and $i = 15\%$ per year.

Find: Incremental cash flows, and select the better alternative on the basis of the IRR criterion.

First, we have

$$\begin{aligned} PW(i)_{FMS-CMS} &= -\$8{,}000{,}000 + \$1{,}908{,}820(P/A, i, 5) \\ &= +\$2{,}408{,}820(P/F, i, 6) \\ &= 0. \end{aligned}$$

Solving for i by trial and error yields 12.43%. Since $IRR_{FMS-CMS} = 12.43\% < 15\%$, we would select CMS. Although the FMS would provide an incremental annual savings of \$1,908,820 in operating costs, the savings are not large enough to justify the incremental investment of \$8,000,000.

COMMENTS: Note that the CMS option is marginally preferred to the FMS option. However, there are dangers in relying solely on the easily quantified savings in input factors—such as labor, energy, and materials—from the FMS and in not considering gains from improved manufacturing performance that are more difficult and subjective to quantify. Factors such as improved product quality, increased manufacturing flexibility (rapid response to customer demand), reduced inventory levels, and increased capacity for product innovation are frequently ignored because we have inadequate means for quantifying their benefits. If these intangible benefits were considered, the FMS option could come out better than the CMS option.

7.4.3 Handling Unequal Service Lives

In Chapters 5 and 6, we discussed the use of the NPW and AE criteria as bases for comparing projects with unequal lives. The IRR measure can also be used to compare projects with unequal lives, as long as we can establish a common analysis period. The decision procedure is then exactly the same as for projects with equal lives. It is likely, however, that we will have a multiple-root problem, which creates a substantial computational burden. For example, suppose we apply the IRR measure to a case in which one project has a 5-year life and the other project has an 8-year life, resulting in a least common multiple of 40 years. Then when we determine the incremental cash flows over the analysis period, we are bound to observe many sign changes. This leads to the possibility of having many i^*'s. Example 7.14 uses i^* to compare mutually exclusive projects, in which the incremental cash flows reveal several sign changes. (Our purpose is not to encourage you to use the IRR approach to compare projects with unequal lives; rather, it is to show the correct way to compare them if the IRR approach *must* be used.)

EXAMPLE 7.14 IRR Analysis for Projects with Different Lives in Which the Increment is a Nonsimple Investment

Consider Example 5.14, in which a mail-order firm wants to install an automatic mailing system to handle product announcements and invoices. The firm had a choice between two different types of machines. Using the IRR as a decision criterion, select the best machine. Assume a MARR of 15%, as before.

SOLUTION

Given: Cash flows for two projects with unequal lives, as shown in Figure 5.11, and MARR = 15%.

Find: The alternative that is preferable.

Since the analysis period is equal to the least common multiple of 12 years, we may compute the incremental cash flow over this 12-year period. As shown in Figure 7.12, we subtract the cash flows of model A from those of model B to form the increment of investment. (Recall that we want the first cash flow difference to be a negative value.) We can then compute the IRR on this incremental cash flow.

Five sign changes occur in the incremental cash flows, indicating a nonsimple incremental investment:

n	Model A		Model B		Model B − Model A
0	−\$12,500		−\$15,000		−\$2,500
1		−5,000		−4,000	1,000
2		−5,000		−4,000	1,000
3	−12,500	−3,000		−4,000	11,500
4		−5,000	−15,000	−2,500	−12,500
5		−5,000		−4,000	1,000
6	−12,500	−3,000		−4,000	11,500
7		−5,000		−4,000	1,000
8		−5,000	−15,000	−2,500	−12,500
9	−12,500	−3,000		−4,000	11,500
10		−5,000		−4,000	1,000
11		−5,000		−4,000	1,000
12		−3,000		−2,500	500
	Four replacement cycles		Three replacement cycles		Incremental cash flows

Even though there are five sign changes in the cash flow, there is only one positive i^* for this problem: 63.12%. Unfortunately, however, the investment is not a pure

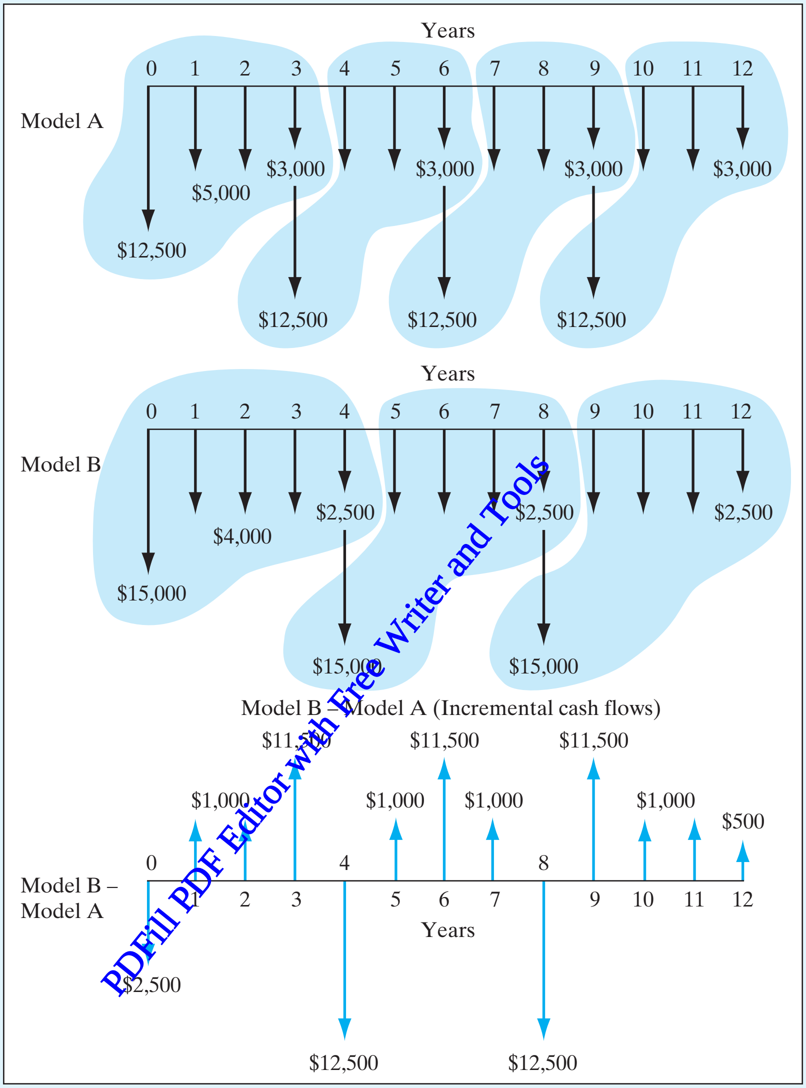


Figure 7.12 Comparison of projects with unequal lives (Example 7.14).

investment. We need to employ an external rate to compute the IRR in order to make a proper accept–reject decision. Assuming that the firm's MARR is 15%, we will use a trial-and-error approach. Try $i = 20\%$:

$$PB(20\%, 15\%)_0 = -\$2{,}500,$$

$$PB(20\%, 15\%)_1 = -\$2{,}500(1.20) + \$1{,}000 = -\$2{,}000.$$

$$PB(20\%, 15\%)_2 = -\$2{,}000(1.20) + \$1{,}000 = -\$1{,}400,$$

$$PB(20\%, 15\%)_3 = -\$1{,}400(1.20) + \$11{,}500 = \$9{,}820.$$

$$PB(20\%, 15\%)_4 = \$9{,}820(1.15) - \$12{,}500 = -\$1{,}207,$$
$$PB(20\%, 15\%)_5 = -\$1{,}207(1.20) + \$1{,}000 = -\$448.40.$$
$$PB(20\%, 15\%)_6 = -\$448.40(1.20) + \$11{,}500 = \$10{,}961.92,$$
$$PB(20\%, 15\%)_7 = \$10{,}961.92(1.15) + \$1{,}000 = \$13{,}606.21.$$
$$PB(20\%, 15\%)_8 = \$13{,}606.21(1.15) - \$12{,}500 = \$3{,}147.14,$$
$$PB(20\%, 15\%)_9 = \$3{,}147.14(1.15) + \$11{,}500 = \$15{,}119.21.$$
$$PB(20\%, 15\%)_{10} = \$15{,}119.21(1.15) + \$1{,}000 = \$18{,}387.09,$$
$$PB(20\%, 15\%)_{11} = \$18{,}387.09(1.15) + \$1{,}000 = \$22{,}145.16,$$
$$PB(20\%, 15\%)_{12} = \$22{,}145.16(1.15) + \$500 = \$25{,}966.93.$$

Since $PB(20\%, 15\%)_{12} > 0$, the guessed 20% is not the RIC. We may increase the value of i and repeat the calculations. After several trials, we find that the RIC is 50.68%.[10] Since $IRR_{B-A} >$ MARR, model B would be selected, which is consistent with the NPW analysis. In other words, the additional investment over the years to obtain model B ($-\$2{,}500$ at $n = 0$, $-\$12{,}500$ at $n = 4$, and $-\$12{,}500$ at $n = 8$) yields a satisfactory rate of return.

COMMENTS: Given the complications inherent in IRR analysis in comparing alternative projects, it is usually more desirable to employ one of the other equivalence techniques for this purpose. As an engineering manager, you should keep in mind the intuitive appeal of the rate-of-return measure. Once you have selected a project on the basis of NPW or AE analysis, you may also wish to express its worth as a rate of return, for the benefit of your associates.

SUMMARY

- The **rate of return (ROR)** is the interest rate earned on unrecovered project balances such that an investment's cash receipts make the terminal project balance equal to zero. The rate of return is an intuitively familiar and understandable measure of project profitability that many managers prefer to NPW or other equivalence measures.
- Mathematically, we can determine the rate of return for a given project cash flow series by locating an interest rate that equates the net present worth of the project's cash flows to zero. This break-even interest rate is denoted by the symbol i*.
- The **internal rate of return (IRR)** is another term for ROR which stresses the fact that we are concerned with the interest earned on the portion of the project that is internally invested, not those portions released by (borrowed from) the project.

[10] It is tedious to solve this type of problem by a trial-and-error method on your calculator. The problem can be solved quickly by using the Cash Flow Analyzer, which can be found from the book's website.

- To apply rate-of-return analysis correctly, we need to classify an investment as either simple or nonsimple. A **simple investment** is defined as an investment in which the initial cash flows are negative and only one sign change in the net cash flow occurs, whereas a **nonsimple investment** is an investment for which more than one sign change in the cash flow series occurs. Multiple i^*'s occur only in nonsimple investments. However, not all nonsimple investments will have multiple i^*'s. In this regard,

  1. The possible presence of multiple i^*'s (rates of return) can be predicted by

  - The net cash flow sign test.
  - The accumulated cash flow sign test.

  When multiple rates of return cannot be ruled out by the two methods, it is useful to generate an NPW profile to approximate the value of i^*.

  2. All i^* values should be exposed to the **net investment test**. Passing this test indicates that i^* is an internal rate of return and is therefore a suitable measure of project profitability. Failing to pass the test indicates project borrowing, a situation that requires further analysis with the use of an **external interest rate**.

  3. **Return-on-invested-capital** analysis uses one rate (the firm's MARR) on externally invested balances and solves for another rate (i^*) on internally invested balances.

- For a pure investment, i^* is the rate of return that is internal to the project. For a mixed investment, the RIC calculated with the use of the external interest rate (or MARR) is the true IRR; so the decision rule is as follows:

$$\text{If IRR} > \text{MARR, accept the project.}$$

$$\text{If IRR} = \text{MARR, remain indifferent.}$$

$$\text{If IRR} < \text{MARR, reject the project.}$$

  IRR analysis yields results consistent with NPW and other equivalence methods.

- In properly selecting among alternative projects by IRR analysis, **incremental investment** must be used. In creating an incremental investment, we always subtract the lower cost investment from the higher cost one. Basically, you want to know that the extra investment required can be justified on the basis of the extra benefits generated in the future.

PROBLEMS

Note: *The symbol i* represents the interest rate that makes the net present value of the project in question equal to zero. The symbol IRR represents the internal rate of return of the investment. For a simple investment, IRR = i*. For a nonsimple investment, i* is generally not equal to IRR.*

Concept of Rate of Return

7.1 You are going to buy a new car worth $14,500. The dealer computes your monthly payment to be $267 for 72 months' financing. What is the dealer's rate of return on this loan transaction?

7.2 You wish to sell a bond that has a face value of $1,000. The bond bears an interest rate of 6%, payable semiannually. Four years ago, the bond was purchased at $900. At least an 8% annual return on the investment is desired. What must be the minimum selling price of the bond now in order to make the desired return on the investment?

7.3 In 1970, Wal-Mart offered 300,000 shares of its common stock to the public at a price of $16.50 per share. Since that time, Wal-Mart has had 11 two-for-one stock splits. On a purchase of 100 shares at $16.50 per share on the company's first offering, the number of shares has grown to 204,800 shares worth $10,649,600 on January 2006. What is the return on investment for investors who purchased the stock in 1970 (say, over a 36-year ownership period)? Assume that the dividends received during that period were not reinvested.

7.4 Johnson Controls spent more than $2.5 million retrofitting a government complex and installing a computerized energy-management system for the State of Massachusetts. As a result, the state's energy bill dropped from an average of $6 million a year to $3.5 million. Moreover, both parties will benefit from the 10-year life of the contract. Johnson recovers half the money it saved in reduced utility costs (about $1.2 million a year over 10 years); Massachusetts has its half to spend on other things. What is the rate of return realized by Johnson Controls in this energy-control system?

7.5 Pablo Picasso's 1905 portrait *Boy with a Pipe* sold for $104.2 million in an auction at Sotheby's Holdings, Inc., on June 24, 2004, shattering the existing record for art and ushering in a new era in pricing for 20th-century paintings. The Picasso, sold by the philanthropic Greentree Foundation, cost Mr. Whitney about $30,000 in 1950. Determine the annual rate of appreciation of the artwork over 54 years.

Investment Classification and Calculation of *i**

7.6 Consider four investments with the following sequences of cash flows:

	Net Cash Flow			
n	Project A	Project B	Project C	Project D
0	−$18,000	−$30,000	$34,578	−$56,500
1	30,000	32,000	−18,000	2,500
2	20,000	32,000	−18,000	6,459
3	10,000	−22,000	−18,000	−78,345

(a) Identify all the simple investments.
(b) Identify all the nonsimple investments.
(c) Compute *i** for each investment.
(d) Which project has no rate of return?

7.7 Consider the following infinite cash flow series with repeated cash flow patterns:

n	A_n
0	−$1,000
1	400
2	800
3	500
4	500
5	400
6	800
7	500
8	500
⋮	

Determine i^* for this infinite cash flow series.

7.8 Consider the following investment projects:

	Project Cash Flow				
n	A	B	C	D	E
0	−$100	−$100	−$200	−$50	−$50
1	60	70	$20	120	−100
2	150	70	10	40	−50
3		40	5	40	0
4		40	−180	−20	150
5			60	40	150
6			50	30	100
7			400		100

(a) Classify each project as either simple or nonsimple.
(b) Use the quadratic equation to compute i^* for project A.
(c) Obtain the rate(s) of return for each project by plotting the NPW as a function of the interest rate.

7.9 Consider the projects in Table P7.9.
(a) Classify each project as either simple or nonsimple.
(b) Identify all positive i^*'s for each project.
(c) For each project, plot the present worth as a function of the interest rate (i).

TABLE P7.9 Net Cash Flow for Four Projects

	Net Cash Flow			
n	A	B	C	D
0	−$2,000	−$1,500	−$1,800	−$1,500
1	500	800	5,600	−360
2	100	600	4,900	4,675
3	100	500	−3,500	2,288
4	2,000	700	7,000	
5			−1,400	
6			2,100	
7			900	

7.10 Consider the following financial data for a project:

Initial investment	$50,000
Project life	8 years
Salvage value	$10,000
Annual revenue	$25,000
Annual expenses (including income taxes)	$ 9,000

(a) What is i^* for this project?

(b) If the annual expense increases at a 7% rate over the previous year's expenses, but the annual income is unchanged, what is the new i^*?

(c) In part (b), at what annual rate will the annual income have to increase to maintain the same i^* obtained in part (a)?

7.11 Consider two investments, A and B, with the following sequences of cash flows:

	Net Cash Flow	
n	Project A	Project B
0	−$25,000	−$25,000
1	2,000	10,000
2	6,000	10,000
3	12,000	10,000
4	24,000	10,000
5	28,000	5,000

(a) Compute i^* for each investment.

(b) Plot the present-worth curve for each project on the same chart, and find the interest rate that makes the two projects equivalent.

7.12 Consider the following investment projects:

	Project Cash Flows					
N	A	B	C	D	E	F
0	−$100	−$100	−$100	−$100	−$100	−$100
1	200	470	−200	0	300	300
2	300	720	200	0	250	100
3	400	360	250	500	−40	400

(a) For each project, apply the sign rule to predict the number of possible i^*'s.

(b) For each project, plot the NPW profile as a function of i between 0 and 200%.

(c) For each project, compute the value(s) of i^*.

7.13 Consider an investment project with the following cash flows:

n	Net Cash Flow
0	−$120,000
1	94,000
2	144,000
3	72,000

(a) Find the IRR for this investment.

(b) Plot the present worth of the cash flow as a function of i.

(c) On the basis of the IRR criterion, should the project be accepted at MARR = 15%?

Mixed Investments

7.14 Consider the following investment projects:

	Net Cash Flow		
n	Project 1	Project 2	Project 3
0	−$1,000	−$2,000	−$1,000
1	500	1,560	1,400
2	840	944	−100
IRR	?	?	?

Assume that MARR = 12% in the following questions:

(a) Compute i^* for each investment. If the problem has more than one i^*, identify all of them.

(b) Compute IRR(true) for each project.

(c) Determine the acceptability of each investment.

7.15 Consider the following investment projects:

	Project Cash Flow				
n	**A**	**B**	**C**	**D**	**E**
0	−\$100	−\$100	−\$5	−\$100	\$200
1	100	30	10	30	100
2	24	30	30	30	−500
3		70	−40	30	−500
4		70		30	200
5				30	600

(a) Use the quadratic equation to compute i^* for A.

(b) Classify each project as either simple or nonsimple.

(c) Apply the cash flow sign rules to each project, and determine the number of possible positive i^*'s. Identify all projects having a unique i^*.

(d) Compute the IRRs for projects B through E.

(e) Apply the net-investment test to each project.

7.16 Consider the following investment projects:

	Net Cash Flow		
n	**Project 1**	**Project 2**	**Project 3**
0	−\$1,600	−\$5,000	−\$1,000
1	10,000	10,000	4,000
2	10,000	30,000	−4,000
3		−40,000	

Assume that MARR = 12% in the following questions:

(a) Identify the i^*('s) for each investment. If the project has more than one i^*, identify all of them.

(b) Which project(s) is (are) a mixed investment?

(c) Compute the IRR for each project.

(d) Determine the acceptability of each project.

7.17 Consider the following investment projects:

	Net Cash Flow		
n	Project A	Project B	Project C
0	−$100	−$150	−$100
1	30	50	410
2	50	50	−558
3	80	50	252
4		100	
IRR	(23.24%)	(21.11%)	(20%, 40%, 50%)

Assume that MARR = 12% for the following questions:

(a) Identify the pure investment(s).

(b) Identify the mixed investment(s).

(c) Determine the IRR for each investment.

(d) Which project would be acceptable?

7.18 The Boeing Company has received a NASA contract worth $460 million to build rocket boosters for future space missions. NASA will pay $50 million when the contract is signed, another $360 million at the end of the first year, and the $50 million balance at the end of second year. The expected cash outflows required to produce these rocket boosters are estimated to be $150 million now, $100 million during the first year, and $218 million during the second year. The firm's MARR is 12%. The cash flow is as follows:

n	Outflow	Inflow	Net Cash Flow
0	$150	$50	−$100
1	100	360	260
2	218	50	−168

(a) Show whether this project is or is not a mixed investment.

(b) Compute the IRR for this investment.

(c) Should Boeing accept the project?

7.19 Consider the following investment projects:

	Net Cash Flow		
n	Project A	Project B	Project C
0	−$100		−$100
1	216	−150	50
2	−116	100	−50
3		50	200
4		40	
i^*	?	15.51%	29.95%

(a) Compute i^* for project A. If there is more than one i^*, identify all of them.
(b) Identify the mixed investment(s).
(c) Assuming that MARR = 10%, determine the acceptability of each project on the basis of the IRR criterion.

7.20 Consider the following investment projects:

	Net Cash Flow				
n	A	B	C	D	E
0	−$1,000	−$5,000	−$2,000	−$2,000	−$1,000
1	3,100	20,000	1,560	2,800	3,600
2	−2,200	12,000	944	−200	−5,700
3		−3,000			3,600
i^*	$\boxed{?}$	$\boxed{?}$	$\boxed{18\%}$	$\boxed{32.45\%}$	$\boxed{35.39\%}$

Assume that MARR = 12% in the following questions:
(a) Compute i^* for projects A and B. If the project has more than one i^*, identify all of them.
(b) Classify each project as either a pure or a mixed investment.
(c) Compute the IRR for each investment.
(d) Determine the acceptability of each project.

7.21 Consider an investment project whose cash flows are as follows:

n	Net Cash Flow
0	−$5,000
1	10,000
2	30,000
3	−40,000

(a) Plot the present-worth curve by varying i from 0% to 250%.
(b) Is this a mixed investment?
(c) Should the investment be accepted at MARR = 18%?

7.22 Consider the following two mutually exclusive investment projects:

	Net Cash Flow	
n	**Project A**	**Project B**
0	−$300	−$800
1	0	1,150
2	690	40
i^*	51.66%	46.31%

Assume that MARR = 15%.
(a) According to the IRR criterion, which project would be selected?
(b) Sketch the PW(i) function on the incremental investment (B − A).

7.23 Consider the following cash flows of a certain project:

n	**Net Cash Flow**
0	−$100,000
1	310,000
2	−220,000

The project's i^*'s are computed as 10% and 100%, respectively. The firm's MARR is 8%.
(a) Show why this investment project fails the net-investment test.
(b) Compute the IRR, and determine the acceptability of this project.

7.24 Consider the following investment projects:

	Net Cash Flow		
n	**Project 1**	**Project 2**	**Project 3**
0	−$1,000	−$1,000	−$1,000
1	−1,000	1,600	1,500
2	2,000	−300	−500
3	3,000	−200	2,000

Which of the following statements is correct?
(a) All projects are nonsimple investments.
(b) Project 3 should have three real rates of return.
(c) All projects will have a unique positive real rate of return.
(d) None of the above.

IRR Analysis

7.25 Agdist Corporation distributes agricultural equipment. The board of directors is considering a proposal to establish a facility to manufacture an electronically controlled "intelligent" crop sprayer invented by a professor at a local university. This crop sprayer project would require an investment of \$10 million in assets and would produce an annual after-tax net benefit of \$1.8 million over a service life of eight years. All costs and benefits are included in these figures. When the project terminates, the net proceeds from the sale of the assets will be \$1 million. Compute the rate of return of this project. Is this a good project at MARR = 10%?

7.26 Consider an investment project with the following cash flows:

n	Cash Flow
0	−\$5,000
1	0
2	4,840
3	1,331

Compute the IRR for this investment. Is the project acceptable at MARR = 10%?

7.27 Consider the following cash flow of a certain project:

n	Net Cash Flow
0	−\$2,000
1	800
2	900
3	X

If the project's IRR is 10%,

(a) Find the value of X.

(b) Is this project acceptable at MARR = 8%?

7.28 You are considering a luxury apartment building project that requires an investment of \$12,500,000. The building has 50 units. You expect the maintenance cost for the apartment building to be \$250,000 the first year and \$300,000 the second year. The maintenance cost will continue to increase by \$50,000 in subsequent years. The cost to hire a manager for the building is estimated to be \$80,000 per year. After five years of operation, the apartment building can be sold for \$14,000,000. What is the annual rent per apartment unit that will provide a return on investment of 15%? Assume that the building will remain fully occupied during its five years of operation.

7.29 A machine costing \$25,000 to buy and \$3,000 per year to operate will save mainly labor expenses in packaging over six years. The anticipated salvage value of the machine at the end of the six years is \$5,000. To receive a 10% return on investment (rate of return), what is the minimum required annual savings in labor from this machine?

7.30 Champion Chemical Corporation is planning to expand one of its propylene-manufacturing facilities. At $n = 0$, a piece of property costing \$1.5 million must be purchased to build a plant. The building, which needs to be expanded during the first year, costs \$3 million. At the end of the first year, the company needs to spend about \$4 million on equipment and other start-up costs. Once the building becomes operational, it will generate revenue in the amount of \$3.5 million during the first operating year. This will increase at the annual rate of 5% over the previous year's revenue for the next 9 years. After 10 years, the sales revenue will stay constant for another 3 years before the operation is phased out. (It will have a project life of 13 years after construction.) The expected salvage value of the land at the end of the project's life would be about \$2 million, the building about \$1.4 million, and the equipment about \$500,000. The annual operating and maintenance costs are estimated to be approximately 40% of the sales revenue each year. What is the IRR for this investment? If the company's MARR is 15%, determine whether the investment is a good one. (Assume that all figures represent the effect of the income tax.)

7.31 Recent technology has made possible a computerized vending machine that can grind coffee beans and brew fresh coffee on demand. The computer also makes possible such complicated functions as changing \$5 and \$10 bills, tracking the age of an item, and moving the oldest stock to the front of the line, thus cutting down on spoilage. With a price tag of \$4,500 for each unit, Easy Snack has estimated the cash flows in millions of dollars over the product's six-year useful life, including the initial investment, as follows:

n	Net Cash Flow
0	−\$20
1	8
2	17
3	19
4	18
5	10
6	3

(a) On the basis of the IRR criterion, if the firm's MARR is 18%, is this product worth marketing?

(b) If the required investment remains unchanged, but the future cash flows are expected to be 10% higher than the original estimates, how much of an increase in IRR do you expect?

(c) If the required investment has increased from \$20 million to \$22 million, but the expected future cash flows are projected to be 10% smaller than the original estimates, how much of a decrease in IRR do you expect?

Comparing Alternatives

7.32 Consider two investments A and B with the following sequences of cash flows:

n	Net Cash Flow: Project A	Net Cash Flow: Project B
0	−\$120,000	−\$100,000
1	20,000	15,000
2	20,000	15,000
3	120,000	130,000

(a) Compute the IRR for each investment.
(b) At MARR = 15%, determine the acceptability of each project.
(c) If A and B are mutually exclusive projects, which project would you select, based on the rate of return on incremental investment?

7.33 With \$10,000 available, you have two investment options. The first is to buy a certificate of deposit from a bank at an interest rate of 10% annually for five years. The second choice is to purchase a bond for \$10,000 and invest the bond's interest in the bank at an interest rate of 9%. The bond pays 10% interest annually and will mature to its face value of \$10,000 in five years. Which option is better? Assume that your MARR is 9% per year.

7.34 A manufacturing firm is considering the following mutually exclusive alternatives:

n	Net Cash Flow: Project A1	Net Cash Flow: Project A2
0	−\$2,000	−\$3,000
1	1,400	2,400
2	1,640	2,000

Determine which project is a better choice at a MARR = 15%, based on the IRR criterion.

7.35 Consider the following two mutually exclusive alternatives:

n	Net Cash Flow: Project A1	Net Cash Flow: Project A2
0	−\$10,000	−\$12,000
1	5,000	6,100
2	5,000	6,100
3	5,000	6,100

(a) Determine the IRR on the incremental investment in the amount of \$2,000.
(b) If the firm's MARR is 10%, which alternative is the better choice?

7.36 Consider the following two mutually exclusive investment alternatives:

	Net Cash Flow	
n	Project A1	Project A2
0	−$15,000	−$20,000
1	7,500	8,000
2	7,500	15,000
3	7,500	5,000
IRR	23.5%	20%

(a) Determine the IRR on the incremental investment in the amount of $5,000. (Assume that MARR = 10%.)

(b) If the firm's MARR is 10%, which alternative is the better choice?

7.37 You are considering two types of automobiles. Model A costs $18,000 and model B costs $15,624. Although the two models are essentially the same, after four years of use model A can be sold for $9,000, while model B can be sold for $6,500. Model A commands a better resale value because its styling is popular among young college students. Determine the rate of return on the incremental investment of $2,376. For what range of values of your MARR is model A preferable?

7.38 A plant engineer is considering two types of solar water heating system:

	Model A	Model B
Initial cost	$7,000	$10,000
Annual savings	$700	$1,000
Annual maintenance	$100	$50
Expected life	20 years	20 years
Salvage value	$400	$500

The firm's MARR is 12%. On the basis of the IRR criterion, which system is the better choice?

7.39 Consider the following investment projects:

	Net Cash Flow					
n	A	B	C	D	E	F
0	−$100	−$200	−$4,000	−$2,000	−$2,000	−$3,000
1	60	120	2,410	1,400	3,700	2,500
2	50	150	2,930	1,720	1,640	1,500
3	50					
i^*	28.89%	21.65%	21.86%	31.10%	121.95%	23.74%

Assume that MARR = 15%.

(a) Projects A and B are mutually exclusive. Assuming that both projects can be repeated for an indefinite period, which one would you select on the basis of the IRR criterion?

(b) Suppose projects C and D are mutually exclusive. According to the IRR criterion, which project would be selected?

(c) Suppose projects E and F are mutually exclusive. Which project is better according to the IRR criterion?

7.40 Fulton National Hospital is reviewing ways of cutting the cost of stocking medical supplies. Two new stockless systems are being considered, to lower the hospital's holding and handling costs. The hospital's industrial engineer has compiled the relevant financial data for each system as follows (dollar values are in millions):

	Current Practice	Just-in-Time System	Stockless Supply System
Start-up cost	$0	$2.5	$5
Annual stock holding cost	$3	$1.4	$0.2
Annual operating cost	$2	$1.5	$1.2
System life	8 years	8 years	8 years

The system life of eight years represents the period that the contract with the medical suppliers is in force. If the hospital's MARR is 10%, which system is more economical?

7.41 Consider the cash flows for the following investment projects:

	Project Cash Flow				
n	A	B	C	D	E
0	−$1,000	−$1,000	−$2,000	$1,000	−$1,200
1	900	600	900	−300	400
2	500	500	900	−300	400
3	100	500	900	−300	400
4	50	100	900	−300	400

Assume that the MARR = 12%.

(a) Suppose A, B, and C are mutually exclusive projects. Which project would be selected on the basis of the IRR criterion?

(b) What is the borrowing rate of return (BRR) for project D?

(c) Would you accept project D at MARR = 20%?

(d) Assume that projects C and E are mutually exclusive. Using the IRR criterion, which project would you select?

7.42 Consider the following investment projects:

	Net Cash Flow		
n	**Project 1**	**Project 2**	**Project 3**
0	−\$1,000	−\$5,000	−\$2,000
1	500	7,500	1,500
2	2,500	600	2,000

Assume that MARR = 15%.

(a) Compute the IRR for each project.

(b) On the basis of the IRR criterion, if the three projects are mutually exclusive investments, which project should be selected?

7.43 Consider the following two investment alternatives:

	Net Cash Flow	
N	**Project A**	**Project B**
0	−\$10,000	−\$20,000
1	5,500	0
2	5,500	0
3	5,500	40,000
IRR	30%	?
PW(15%)	?	6300

The firm's MARR is known to be 15%.

(a) Compute the IRR of project B.

(b) Compute the NPW of project A.

(c) Suppose that projects A and B are mutually exclusive. Using the IRR, which project would you select?

7.44 The E. F. Fedele Company is considering acquiring an automatic screwing machine for its assembly operation of a personal computer. Three different models with varying automatic features are under consideration. The required investments are \$360,000 for model A, \$380,000 for model B, and \$405,000 for model C. All three models are expected to have the same service life of eight years. The following financial information, in which model (B − A) represents the incremental cash flow determined by subtracting model A's cash flow from model B's, is available:

Model	IRR (%)
A	30%
B	15
C	25
Model	**Incremental IRR (%)**
(B − A)	5%
(C − B)	40
(C − A)	15

If the firm's MARR is known to be 12%, which model should be selected?

7.45 The GeoStar Company, a leading manufacturer of wireless communication devices, is considering three cost-reduction proposals in its batch job-shop manufacturing operations. The company has already calculated rates of return for the three projects, along with some incremental rates of return:

Incremental Investment	Incremental Rate of Return (%)
$A_1 - A_0$	18%
$A_2 - A_0$	20
$A_3 - A_0$	25
$A_2 - A_1$	10
$A_3 - A_1$	18
$A_3 - A_2$	23

A_0 denotes the do-nothing alternative. The required investments are \$420,000 for A_1, \$550,000 for A_2, and \$720,000 for A_3. If the MARR is 15%, what system should be selected?

7.46 A manufacturer of electronic circuit boards is considering six mutually exclusive cost-reduction projects for its PC-board manufacturing plant. All have lives of 10 years and zero salvage values. The required investment and the estimated after-tax reduction in annual disbursements for each alternative are as follows, along with computed rates of return on incremental investments:

Proposal	Required After-Tax		Rate of
A_j	Investment	Savings	Return (%)
A_1	\$60,000	\$22,000	35.0%
A_2	100,000	28,200	25.2
A_3	110,000	32,600	27.0
A_4	120,000	33,600	25.0
A_5	140,000	38,400	24.0
A_6	150,000	42,200	25.1

Incremental Investment	Incremental Rate of Return (%)
$A_2 - A_1$	9.0%
$A_3 - A_2$	42.8
$A_4 - A_3$	0.0
$A_5 - A_4$	20.2
$A_6 - A_5$	36.3

If the MARR is 15%, which project would you select, based on the rate of return on incremental investment?

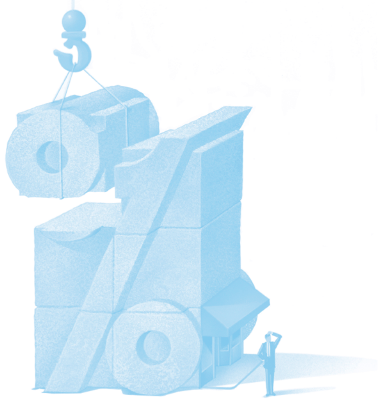
7.47 Baby Doll Shop manufactures wooden parts for dollhouses. The worker is paid $8.10 an hour and, using a handsaw, can produce a year's required production (1,600 parts) in just eight 40-hour weeks. That is, the worker averages five parts per hour when working by hand. The shop is considering purchasing of a power band saw with associated fixtures, to improve the productivity of this operation. Three models of power saw could be purchased: Model A (the economy version), model B (the high-powered version), and model C (the deluxe high-end version). The major operating difference between these models is their speed of operation. The investment costs, including the required fixtures and other operating characteristics, are summarized as follows:

Category	By Hand	Model A	Model B	Model C
Production rate (parts/hour)	5	10	15	20
Labor hours required (hours/year)	320	160	107	80
Annual labor cost (@ $8.10/hour)	2,592	1,296	867	648
Annual power cost ($)		400	420	480
Initial investment ($)		4,000	6,000	7,000
Salvage value ($)		400	600	700
Service life (years)		20	20	20

Assume that MARR = 10%. Are there enough savings to purchase any of the power band saws? Which model is most economical, based on the rate-of-return principle? (Assume that any effect of income tax has been already considered in the dollar estimates.) (*Source*: This problem is adapted with the permission of Professor Peter Jackson of Cornell University.)

PART

3



Analysis of Project Cash Flows

CHAPTER EIGHT

Cost Concepts Relevant to Decision Making

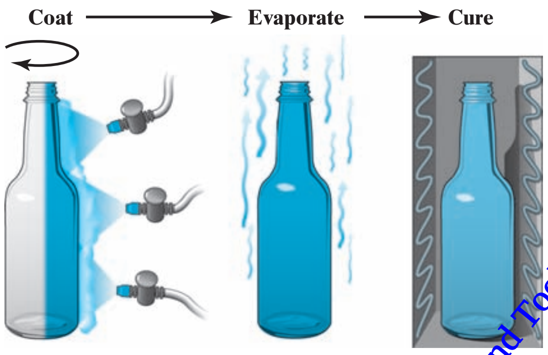
High Hopes for Beer Bottles[1] Three hundred billion beer bottles a year worldwide is a mighty tempting target for the plastics industry. Brewers generally say they need a bottle that provides shelf life of over 120 days with less than 15% loss of CO_2 and admittance of no more than 1 ppm of oxygen. Internal or external coatings, and three- or five-layer polyethylene terephthalate (PET) structures using barrier materials are being evaluated to reach that performance.

What is the least expensive way to make a 0.5L PET barrier bottle? Summit International LLC, Smyrna, GA, which specializes in preform and container development and market research, compared the manufacturing costs of five different barrier technologies against a standard monolayer PET bottle. Summit looked at three-layer and five-layer and at internally and externally coated containers, all of them reheat stretch blow molded. It found the bottle with an external coating to be least costly, while the internally coated bottle was the most expensive.

The firm compared a five-layer structure with an oxygen-scavenger material, a three-layer bottle with a $2.50/lb barrier material, and a second three-layer structure with a $6/lb barrier. Also compared were a bottle coated inside using Sidel's new Actis plasma technology and a bottle coated on the outside.

Capital investment (preform and bottle machines, utilities, downstream equipment, quality control, spare parts, and installation) for producing 20,000 bottles/hr is $10.8 million for the five-layer bottle, $9.9 million for both three-layer structures, $9.2 million for internal coating, $7.5 for external coating, and $6.8 million with no barrier.

[1] "Blow Molding Close-Up: Prospects Brighten for PET Beer Bottles," Mikell Knights, Plastic Technology Online, © copyright 2005, Gardner Publications, Inc.



Spray coating of external PET bottles

Direct manufacturing cost per 1,000 (materials, energy, labor, maintenance, and scrap) amounts to $66.57 for three layers with the expensive barrier, $59.35 for five layers, $55.34 for the external coating, $54.63 for three layers with the low-cost barrier, $46.90 for internal coating, and $44.63 without barrier.

You may be curious how all of those cost data were estimated in the chapter opening story. Before we study the different kinds of engineering economic decision problems, we need to understand the concept of various costs. At the level of plant operations, engineers must make decisions involving materials, plant facilities, and the in-house capabilities of company personnel. Consider, for example, the manufacture of food processors. In terms of selecting materials, several of the parts could be made of plastic, whereas others must be made of metal. Once materials have been chosen, engineers must consider the production methods, the shipping weight, and the method of packaging necessary to protect the different types of materials. In terms of actual production, parts may be made in-house or purchased from an outside vendor, depending on the availability of machinery and labor.

Present economic studies: Various economic analyses for short operating decisions.

All these operational decisions (commonly known as **present economic studies** in traditional engineering economic texts) require estimating the costs associated with various production or manufacturing activities. Because these costs also provide the basis for developing successful business strategies and planning future operations, it is important to understand how various costs respond to changes in levels of business activity. In this chapter, we discuss many of the possible uses of cost data. We also discuss how to define, classify, and estimate costs for each use. Our ultimate task in doing so is to explain how costs are classified in making various engineering economic decisions.

CHAPTER LEARNING OBJECTIVES

After completing this chapter, you should understand the following concepts:

- Various cost terminologies that are common in cost accounting and engineering economic studies.
- How a cost item reacts or responds to changes in the level of production or business activities.
- The types of cost data that management needs in making choice between alternative courses of action.
- The types of present economic studies frequently performed by engineers in manufacturing and business environment.
- How to develop a production budget related to operating activities.

8.1 General Cost Terms

In engineering economics, the term **cost** is used in many different ways. Because there are many types of costs, each is classified differently according to the immediate needs of management. For example, engineers may want cost data to prepare external reports, to prepare planning budgets, or to make decisions. Also, each different use of cost data demands a different classification and definition of *cost*. For example, the preparation of external financial reports requires the use of historical cost data, whereas decision making may require current cost data or estimated future cost data.

Our initial focus in this chapter is on manufacturing companies, because their basic activities (acquiring raw materials, producing finished goods, marketing, etc.) are commonly found in most other businesses. Therefore, the understanding of costs in a manufacturing company can be helpful in understanding costs in other types of business organizations.

8.1.1 Manufacturing Costs

Several types of manufacturing costs incurred by a typical manufacturer are illustrated in Figure 8.1. In converting raw materials into finished goods, a manufacturer incurs various costs associated with operating a factory. Most manufacturing companies divide manufacturing costs into three broad categories: direct raw material costs, direct labor costs, and manufacturing overhead.

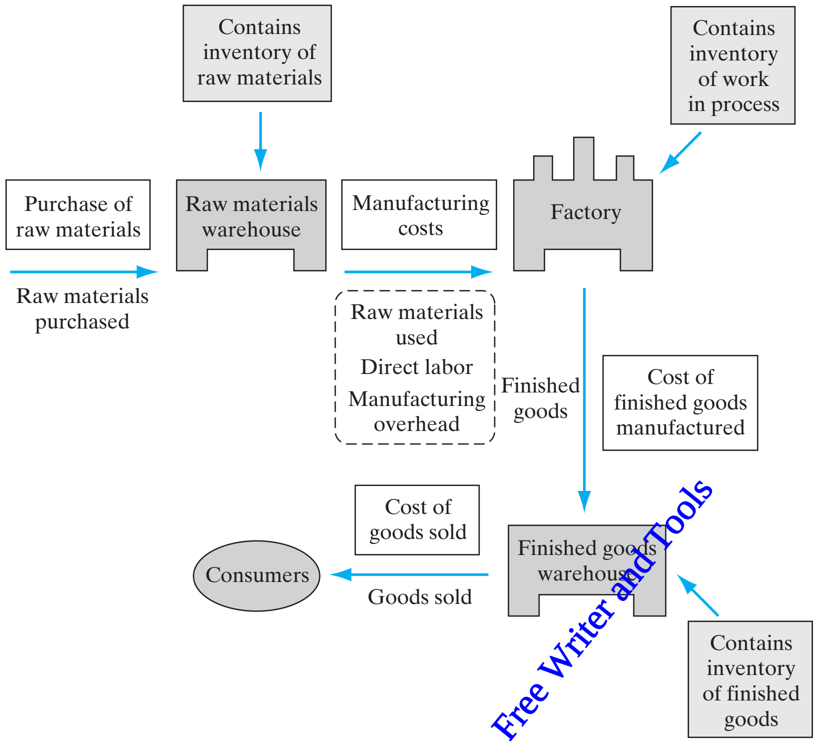


Figure 8.1 Various types of manufacturing costs incurred by a manufacturer.

Direct Raw Materials

Direct raw materials are any materials that are used in the final product and that can be easily traced to it. Some examples are wood in furniture, steel in bridge construction, paper in printing firms, and fabric for clothing manufacturers. It is important to note that the finished product of one company can become the raw materials of another company. For example, the computer chips produced by Intel are a raw material used by Dell in its personal computers.

Direct cost: A cost that can be directly traced to producing specific goods or services.

Direct Labor

Like direct raw materials, direct labor incurs costs that go into the production of a product. The labor costs of assembly-line workers, for example, would be direct labor costs, as would the labor costs of welders in metal-fabricating industries, carpenters or bricklayers in home building, and machine operators in various manufacturing operations.

Overhead: A reference in accounting to all costs not including or related to direct labor, materials, or administration costs.

Manufacturing Overhead

Manufacturing overhead, the third element of manufacturing cost, includes all costs of manufacturing except the costs of direct materials and direct labor. In particular, manufacturing overhead includes such items as the costs of indirect materials; indirect labor; maintenance and repairs on production equipment; heat and light, property taxes, depreciation, and insurance on manufacturing facilities; and overtime premiums. The most important thing to note about manufacturing overhead is the fact that, unlike direct materials and direct labor, it is not easily traceable to specific units of output. In addition, many manufacturing overhead costs do not change as output changes, as long as the production volume stays within the capacity of the plant. For example, depreciation of factory buildings is unaffected by the amount of production during any particular period. If, however, a new building is required to meet any increased production, manufacturing overhead will certainly increase.

- Sometimes it may not be worth the effort to trace the costs of materials that are relatively insignificant in the finished products. Such minor items include the solder used to make electrical connections in a computer circuit board and the glue used to bind this textbook. Materials such as solder and glue are called *indirect materials* and are included as part of manufacturing overhead.
- Sometimes we may not be able to trace some of the labor costs to the creation of a product. We treat this type of labor cost as a part of manufacturing overhead, along with indirect materials. *Indirect labor* includes the wages of janitors, supervisors, material handlers, and night security guards. Although the efforts of these workers are essential to production, it would be either impractical or impossible to trace their costs to specific units of product. Therefore, we treat such labor costs as indirect labor costs.

8.1.2 Nonmanufacturing Costs

Two additional costs incurred in supporting any manufacturing operation are (1) marketing or selling costs and (2) administrative costs. Marketing or selling costs include all costs necessary to secure customer orders and get the finished product or service into the hands of the customer. Breakdowns of these types of costs provide data for control over selling and administrative functions in the same way that manufacturing cost breakdowns provide data for control over manufacturing functions. For example, a company incurs costs for

- **Overhead.** Heat and light, property taxes, and depreciation or similar items associated with the company's selling and administrative functions.
- **Marketing.** Advertising, shipping, sales travel, sales commissions, and sales salaries. Marketing costs include all executive, organizational, and clerical costs associated with sales activities.
- **Administrative functions.** Executive compensation, general accounting, public relations, and secretarial support, associated with the general management of an organization.

Matching concept: The accounting principle that requires the recognition of all costs that are associated with the generation of the revenue reported in the income statement.

8.2 Classifying Costs for Financial Statements

For purposes of preparing financial statements, we often classify costs as either period costs or product costs. To understand the difference between them, we must introduce the matching concept essential to any accounting studies. In financial accounting, the **matching principle** states that *the costs incurred in generating a certain amount of revenue should be recognized as expenses in the same period that the revenue is recognized*. This matching principle is the key to distinguishing between period costs and product costs. Some costs are matched against periods and become expenses immediately. Other costs are matched against products and do not become expenses until the products are sold, which may be in the next accounting period.

8.2.1 Period Costs

Period costs are costs charged to expenses in the period in which they are incurred. The underlying assumption is that the associated benefits are received in the same period the cost is incurred. Some specific examples are all general and administrative expenses, selling expenses, and insurance and income tax expenses. Therefore, advertising costs, executives' salaries, sales commissions, public-relations costs, and other nonmanufacturing costs discussed earlier would all be period costs. Such costs are not related to the production and flow of manufactured goods, but are deducted from revenue in the income statement. In other words, period costs will appear on the income statement as expenses during the time in which they occur.

8.2.2 Product Costs

Some costs are better matched against products than they are against periods. Costs of this type—called **product costs**—consist of the costs involved in the purchase or manufacture of goods. In the case of manufactured goods, product costs are the costs of direct materials, direct labor costs, and manufacturing overhead. Product costs are not viewed as expenses; rather, they are the cost of creating inventory. Thus, product costs are considered an asset until the associated goods are sold. At the time they are sold, the costs are released from inventory as expenses (typically called cost of goods sold) and matched against sales revenue. Since product costs are assigned to inventories, they are also known as *inventory costs*. In theory, product costs include all manufacturing costs—that is, all costs relating to the manufacturing process. As shown in Figure 8.2, product costs appear on financial statements when the inventory, or final goods, is sold, not when the product is manufactured.

To understand product costs more fully, let us look briefly at the flow of costs in a manufacturing company. By doing so, we will be able to see how product costs move

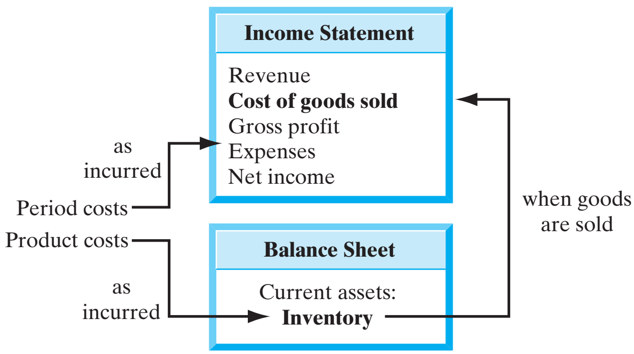


Figure 8.2 How the period costs and product costs flow through financial statements from the manufacturing floor to sales.

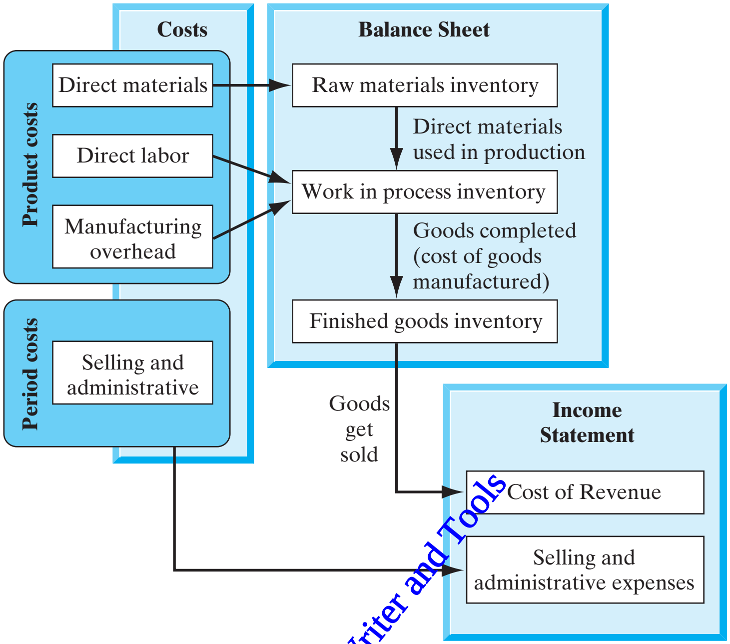


Figure 8.3 Cost flows and classifications in a manufacturing company.

through the various accounts and affect the balance sheet and the income statement in the course of the manufacture and sale of goods. The flows of period costs and product costs through the financial statements are illustrated in Figure 8.3. All product costs filter through the balance-sheet statement as "inventory cost." If the product gets sold, the inventory costs in the balance-sheet statement are transferred to the income statement under the head "cost of goods sold."

- **Raw-materials inventory.** This account in the balance-sheet statement represents the unused portion of the raw materials on hand at the end of the fiscal year.
- **Work-in-process inventory.** This balance-sheet entry consists of the partially completed goods on hand in the factory at year-end. When raw materials are used in production, their costs are transferred to the work-in-process inventory account as direct materials. Note that direct labor cost and manufacturing overhead cost are also added directly to work-in-process, which can be viewed as the assembly line in a manufacturing plant, where workers are stationed and where products slowly take shape as they move from one end of the line to the other.
- **Finished-goods inventory.** This account shows the cost of finished goods that are on hand and awaiting sale to customers at year-end. As the goods are completed, accountants transfer the cost from the work-in-process account into the finished-goods account. At this stage, the goods await sale to a customer. As the goods are sold, their cost is transferred from the finished-goods account into the cost-of-goods-sold (or cost-of-revenue) account. At this point, we finally treat the various material, labor, and overhead costs that were involved in the manufacture of the units being sold as *expenses* in the income statement.

Example 8.1 serves to explain the classification scheme for financial statements.

EXAMPLE 8.1 Classifying Costs for Uptown Ice Cream Shop

Here is a look at why it costs $2.50 for a single-dip ice cream cone at a typical store in Washington, DC. The annual sales volume (the number of ice cream cones sold) averages around 185,000 cones, bringing in revenue of $462,500. This is equivalent to selling more than 500 cones a day, assuming a seven-day operation. The following table shows the unit price of an ice cream cone and the costs that go into producing the product:

Items	Total Cost	Unit Price*	% of Price
Ice cream (cream, sugar, milk, and milk solids)	$120,250	$0.65	26%
Cone	9,250	0.05	2
Rent	112,850	0.61	24
Wages	46,250	0.25	10
Payroll taxes	9,250	0.05	2
Sales taxes	42,550	0.23	9
Business taxes	14,800	0.08	3
Debt service	42,550	0.23	9
Supplies	16,650	0.09	4
Utilities	14,800	0.08	3
Other expenses (insurance, advertising, fees, and heating and lighting for shop)	9,250	0.05	2
Profit	24,050	0.13	5
Total	**$462,500**	**$2.50**	**100**

*Based on an annual volume of 185,000 cones.

If you were to classify the operating costs into either product costs or period costs, how would you do it?

SOLUTION:

Given: Financial data just described.

Find: Classify the cost elements into product costs and period costs.

The following is a breakdown of the two kinds of costs:

- **Product costs:** Costs incurred in preparing 185,000 ice cream cones per year.

Raw materials:	
Ice cream @ $0.65	$120,250
Cone @ $0.05	9,250
Labor:	
Wages @ $0.25	46,250
Overhead:	
Supplies @ $0.09	16,650
Utilities @ $0.08	14,800
Total product cost	$207,200

- **Period costs:** Costs incurred in running the shop regardless of sales volume.

Business taxes:	
Payroll taxes @ $0.05	$ 9,250
Sales taxes @ $0.23	42,550
Business taxes @ $0.08	14,800
Operating expenses:	
Rent @ $0.61	112,850
Debt service @ $0.23	42,550
Other @ $0.05	9,250
Total period cost	$231,250

8.3 Cost Classification for Predicting Cost Behavior

In engineering economic analysis, we need to predict how a certain cost will behave in response to a change in activity. For example, a manager will want to estimate the impact a 5% increase in production will have on the company's total wages before he or she decides whether to alter production. **Cost behavior** describes how a cost item will react or respond to changes in the level of business activity.

8.3.1 Volume Index

In general, the operating costs of any company are likely to respond in some way to changes in the company's operating volume. In studying cost behavior, we need to determine some measurable volume or activity that has a strong influence on the amount of cost incurred. The unit of measure used to define volume is called a **volume index**. A volume index may be based on production inputs, such as tons of coal processed, direct labor hours used, or machine-hours worked; or it may be based on production outputs, such as the number of kilowatt-hours generated. For a vehicle, the number of miles driven per year may be used as a volume index. Once we identify a volume index, we try to find out how costs change in response to changes in the index.

8.3.2 Cost Behaviors

Accounting systems typically record the cost of resources acquired and track their subsequent usage. Fixed costs and variable costs are the two most common cost behavior patterns. An additional category known as "mixed costs" contains two parts, the first of which is fixed and the other of which varies with the volume of output.

Fixed Costs

The costs of providing a company's basic operating capacity are known as the company's **fixed cost** or **capacity cost**. For a cost item to be classified as fixed, it must have a relatively wide span of output over which costs are expected to remain constant. This span is called the **relevant range**. In other words, fixed costs do not change within a given period, although volume may change. In the case of an automobile, for example, the annual insurance premium, property tax, and license fee are fixed costs, since they are independent of the number of miles driven per year. Some other typical examples are building rents; depreciation of buildings, machinery, and equipment; and salaries of administrative and production personnel. In our Uptown Scoop Ice Cream Store example, we may classify expenses such as rent, business taxes, debt service, and other (insurance, advertising, professional fees) as fixed costs (costs that are fixed in total for a given period of time and for given production levels).

Fixed cost: A cost that remains constant, regardless of any change in a company's activity.

Variable Costs

In contrast to fixed operating costs, **variable operating costs** have a close relationship to the level of volume of a business. If, for example, volume increases 10%, a total variable cost will also increase by approximately 10%. Gasoline is a good example of a variable automobile cost, because fuel consumption is directly related to miles driven. Similarly, the cost of replacing tires will increase as a vehicle is driven more.

In a typical manufacturing environment, direct labor and material costs are major variable costs. In our Uptown Scoop example, the variable costs would include the cost of the ice cream and cone (direct materials), wages, payroll taxes, sales taxes, and supplies. Both payroll and sales taxes are related to sales volume. In other words, if the store becomes busy, more servers are needed, which will increase the payroll as well as taxes.

Variable cost: A cost that changes in proportion to a change in a company's activity or business.

Mixed Costs

Some costs do not fall precisely into either the fixed or the variable category, but contain elements of both. We refer to these as mixed costs (or **semivariable costs**). In our automobile example, **depreciation** (loss of value) is a mixed cost. On the one hand, some depreciation occurs simply from the passage of time, regardless of how many miles a car is driven, and this represents the fixed portion of depreciation. On the other hand, the more miles an automobile is driven a year, the faster it loses its market value, and this represents the variable portion of depreciation. A typical example of a mixed cost in manufacturing is the cost of electric power. Some components of power consumption, such as lighting, are independent of the operating volume, while other components (e.g., the number of machine-hours equipment is operated) may vary directly with volume. In our Uptown Scoop example, the utility cost can be a mixed cost item: Some lighting and heating requirements might stay the same, but the use of power mixers will be in proportion to sales volume.

Mixed cost: Costs are fixed for a set level of production or consumption, becoming variable after the level is exceeded.

Average Unit Cost

The foregoing description of fixed, variable, and mixed costs was expressed in terms of total volume over a given period. We often use the term **average cost** to express activity cost on a per unit basis. In terms of unit costs, the description of cost is quite different:

- The variable cost per unit of volume is a constant.
- Fixed cost per unit varies with changes in volume: As the volume increases, the fixed cost per unit decreases.
- The mixed cost per unit also changes as volume changes, but the amount of change is smaller than that for fixed costs.

To explain the behavior of the fixed, variable, mixed, and average costs in relation to volume, let's consider a medium-size car, say, the 2005 Ford Taurus SEL Deluxe six-cylinder (3.0-liter) four-door sedan. In calculating the vehicle's operating costs, we may use the procedure outlined by the American Automobile Association (AAA) as shown in Table 8.1.

On the basis of the assumptions in Table 8.1, the operating and ownership costs of our Ford vehicle are estimated as follows:

Operating costs:

• Gas and oil	8.5 cents
• Maintenance	5.8 cents
• Tires	0.7 cent
Cost per mile	15.0 cents

TABLE 8.1 **Assumptions Used in Calculating the Average Cost of Owning and Operating a New Vehicle**

What's Covered	Costs Base
Fuel	U.S. price of unleaded gasoline from AAA's *Fuel Gauge Report*, weighted 60% city and 40% highway driving.
Maintenance	Costs of retail parts and labor for normal, routine maintenance, as specified by the vehicle manufacturer.
Tires	Costs are based on the price of one set of replacement tires of the same quality, size, and ratings as those which came with the vehicle.
Insurance	A full-coverage policy for a married 47-year-old male with a good driving record living a small city and commuting 3 to 10 miles daily to work.
License, Registration, and Taxes	All government taxes and fees payable at time of purchase, as well as fees due each year to keep the vehicle licensed and registered.
Depreciation	Based on the difference between the new-vehicle purchase price and the estimated trade-in value at the end of five years.
Finance	Based on a five-year loan at 6% interest with a 10% down payment.

Source: American Automobile Association (AAA).

Ownership costs:	
• Comprehensive insurance ($250 deductible)	$1,195
• License, registration, taxes	$390
• Depreciation (15,000 miles annually)	$4,005
• Finance charge (20% down, loan @8.5%/4 years)	$740
Cost per year	$6,330
Cost per day	$17.34
Added depreciation costs (per 1,000 miles over 15,000 miles annually)	$185
Total cost per mile:	
• Cost per mile $\times$ 15,000 miles	$2,250
• Cost per day $\times$ 365 days per year	$6,330
Total cost per year	$8,580
Average cost per mile ($8,580/15,000)	57.2 cents

Now, if you drive the same vehicle for 20,000 miles instead of 15,000 miles, you may be interested in knowing how the average cost per mile would change. Example 8.2 illustrates how you determine the average cost as a function of mileage.

EXAMPLE 8.2 Calculating Average Cost per Mile as a Function of Mileage

Table 8.2 itemizes the operating and ownership costs associated with driving a passenger car by fixed, variable, and mixed classes. Note that the only change from the preceding list is in the depreciation amount. Using the given data, develop a cost–volume chart and calculate the average cost per mile as a function of the annual mileage.

DISCUSSION: First we may examine the effect of driving an additional 1,000 miles over the 15,000 allotted miles. Since the loss in the car's value due to driving an additional 1,000 miles over 15,000 miles is estimated to be $185, the total cost per year and the average cost per mile, based on 16,000 miles, can be recalculated as follows:

- Added depreciation cost: $185.
- Added operating cost: 1,000 miles $\times$ 15 cents = $150.
- Total cost per year: $8,580 + $185 + $150 = $8,915.
- Average cost per mile ($8,915/16,000 miles): 55.72 cents.

Note that the average cost comes down as you drive more, as the ownership cost per mile is further reduced.

SOLUTION

Given: Financial data.

Find: The average cost per mile at an annual operating volume between 15,000 and 20,000 miles.

TABLE 8.2 **Cost Classification of Owning and Operating a Passenger Car**

Cost Classification	Reference	Cost
Variable costs:		
Standard miles per gallon	20 miles/gallon	
Average fuel price per gallon	$1.939/gallon	
Fuel and oil per mile		$0.085
Maintenance per mile		$0.058
Tires per mile		$0.007
Annual fixed costs:		
Insurance (comprehensive)		$1,195
License, registration, taxes		$390
Finance charge		$740
Mixed costs: Depreciation		
Fixed portion per year (15,000 miles)		$4,005
Variable portion per mile (above 15,000 miles)		$0.185

In Table 8.3, we summarize the costs of owning and operating the automobile at various annual operating volumes from 15,000 to 20,000 miles. Once the total cost figures are available at specific volumes, we can calculate the effect of volume on unit (per mile) costs by converting the total cost figures to average unit costs.

TABLE 8.3 **Operating Costs as a Function of Mileage Driven**

Volume Index (miles)	15,000	16,000	17,000	18,000	19,000	20,000
Variable costs (15¢ per mile)	$ 2,250	$ 2,400	$ 2,550	$ 2,700	$ 2,850	$ 3,000
Fixed costs	2,325	2,325	2,325	2,325	2,325	2,325
Mixed costs (depreciation):						
Variable portion	0	185	370	555	740	925
Fixed portion	4,005	4,005	4,005	4,005	4,005	4,005
Total variable cost	2,250	2,585	2,920	3,255	3,590	3,925
Total fixed cost	6,330	6,330	6,330	6,330	6,330	6,330
Total costs	$ 8,580	$ 8,915	$ 9,250	$ 9,585	$ 9,920	$10,255
Cost per mile	$0.5720	$0.5572	$0.5441	$0.5325	$0.5221	$0.5128

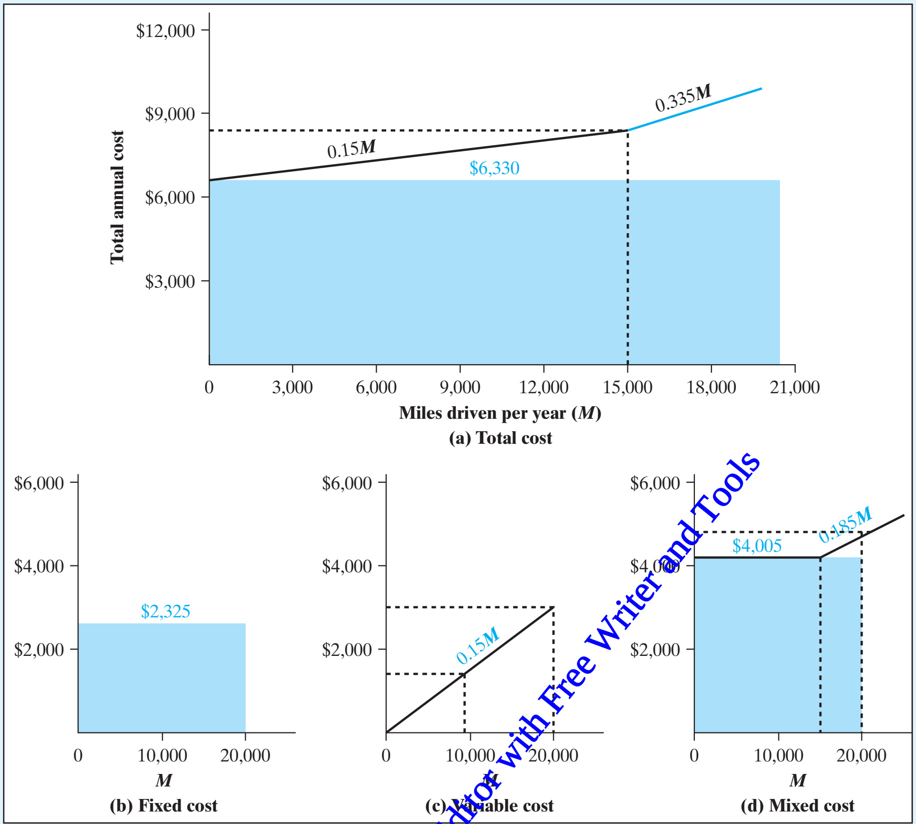


Figure 8.4 Cost–volume relationships pertaining to annual automobile costs (Example 8.2).

To estimate annual costs for any assumed mileage, we construct a **cost–volume diagram** as shown in Figure 8.4(a). Further, we can show the relation between volume (miles driven per year) and the three cost classes separately, as in Figure 8.4(b) through (d). We can use these cost–volume graphs to estimate both the separate and combined costs of operating the car at other possible volumes. For example, an owner who expects to drive 16,500 miles in a given year may estimate the total cost at $9,082.50, or 55.05 cents per mile. In Figure 8.4, all costs more than those necessary to operate at the zero level are known as variable costs. Since the fixed cost is $6,330 a year, the remaining $2,752.50 is variable cost. By combining all the fixed and variable elements of cost, we can state simply that the cost of owning and operating an automobile is $6,330 per year, plus 16.68 cents per mile driven during the year.

Figure 8.5 illustrates graphically the average unit cost of operating the automobile. The average fixed unit cost, represented by the height of the middle curve in the figure, declines steadily as the volume increases. The average unit cost is high when volume is low because the total fixed cost is spread over a relatively few units

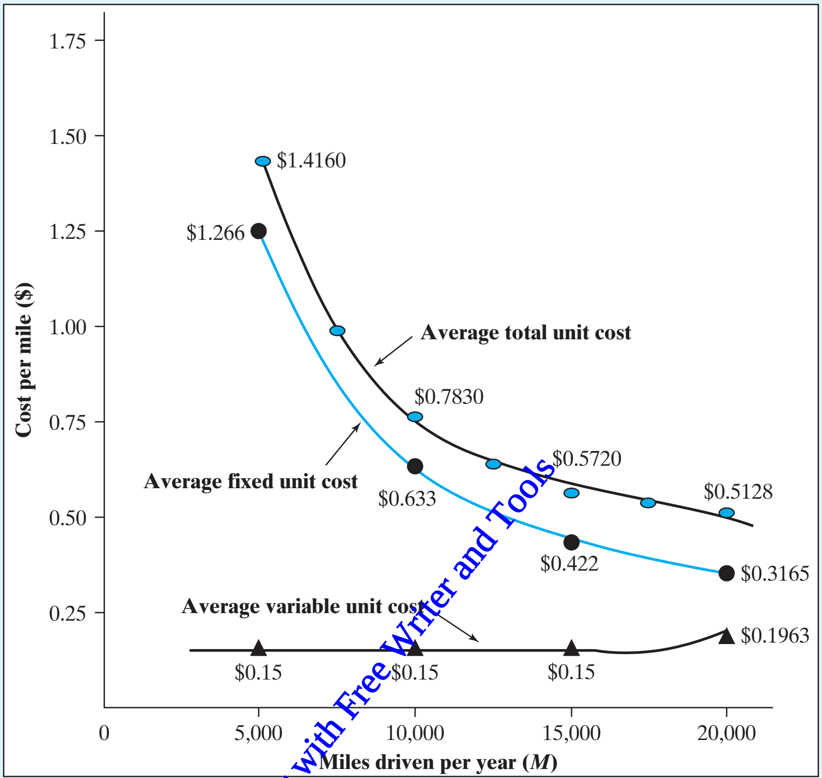


Figure 8.5 Average cost per mile of owning and operating a car (Example 8.2).

of volume. In other words, the total fixed costs remain the same regardless of the number of miles driven, but the average fixed cost decreases on a per mile basis as the number of miles driven increases.

8.4 Future Costs for Business Decisions

In the previous sections, our focus has been on classifying cost data that serve management's need to control and evaluate the operations of a firm. However, these data are historical in nature; they may not be suitable for management's planning future business operations. We are not saying that historical cost data are of no use in making decisions about the future. In fact, they serve primarily as the first step in predicting the uncertain future. However, the types of cost data that management needs in making choices between alternative courses of action are different from historical cost data.

8.4.1 Differential Cost and Revenue

As we have seen throughout the text, decisions involve choosing among alternatives. In business decisions, each alternative has certain costs and benefits that must be compared with the costs and benefits of the other available alternatives. A difference in cost between any two alternatives is known as a **differential cost**. Similarly, a difference in revenue between any two alternatives is known as **differential revenue**. A differential cost is also known as an incremental cost, although, technically, an incremental cost should refer only to an increase in cost from one alternative to another.

Cost–volume relationships based on differential costs find many engineering applications. In particular, they are useful in making a variety of short-term operational decisions. Many short-run problems have the following characteristics:

- The base case is the status quo (the current operation or existing method), and we propose an alternative to the base case. If we find the alternative to have lower costs than the base case, we accept the alternative, assuming that nonquantitative factors do not offset the cost advantage. The **differential (incremental) cost** is the difference in total cost that results from selecting one alternative instead of another. If several alternatives are possible, we select the one with the maximum savings from the base. Problems of this type are often called trade-off problems, because one type of cost is traded off for another.
- New investments in physical assets are not required.
- The planning horizon is relatively short (a week or a month—certainly less than a year).
- Relatively few cost items are subject to change by management decision.

Incremental cost is the overall change that a company experiences by producing one additional unit of good.

Some common examples of short-run problems are method changes, operations planning, and make-or-buy decisions.

Method Changes

Often, we may derive the best information about future costs from an analysis of historical costs. Suppose the proposed alternative is to consider some new method of performing an activity. Then, as Example 8.3 shows, if the differential costs of the proposed method are significantly lower than the current method, we adopt the new method.

EXAMPLE 8.3 Differential Cost Associated with Adopting a New Production Method

The engineering department at an auto-parts manufacturer recommends that the current dies (the base case) be replaced with higher quality dies (the alternative), which would result in substantial savings in manufacturing one of the company's products. The higher cost of materials would be more than offset by the savings in machining time and electricity. If estimated monthly costs of the two alternatives are as shown in the accompanying table, what is the differential cost for going with better dies?

SOLUTION

Given: Financial data.

Find: Which production method is preferred.

In this problem, the differential cost is $-\$5{,}000$ a month. The differential cost's being negative indicates a saving, rather than an addition to total cost. An important point to remember is that differential costs usually include variable costs, but fixed costs are affected only if the decision involves going outside of the relevant range. Although the production volume remains unchanged in our example, the slight increase

	Current Dies	Better Dies	Differential Cost
Variable costs:			
Materials	$150,000	$170,000	+$20,000
Machining labor	85,000	64,000	−21,000
Electricity	73,000	66,000	−7,000
Fixed costs:			
Supervision	25,000	25,000	0
Taxes	16,000	16,000	0
Depreciation	40,000	43,000	+3,000
Total	$392,000	$387,000	−$5,000

in depreciation expense is due to the acquisition of new machine tools (dies). All other items of fixed cost remained within their relevant ranges.

Operations Planning

In a typical manufacturing environment, when demand is high, managers are interested in whether to use a one-shift-plus-overtime operation or to add a second shift. When demand is low, it is equally possible to explore whether to operate temporarily at very low volume or to shut down until operations at normal volume become economical. In a chemical plant, several routes exist for scheduling products through the plant. The problem is which route provides the lowest cost. Example 8.4 illustrates how engineers may use cost–volume relationships in a typical operational analysis.

Break-even analysis: An analysis of the level of sales at which a project would make zero profit.

EXAMPLE 8.4 Break-Even Volume Analysis

Sandstone Corporation has one of its manufacturing plants operating on a single-shift five-day week. The plant is operating at its full capacity (24,000 units of output per week) without the use of overtime or extra shifts. Fixed costs for single-shift operation amount to $90,000 per week. The average variable cost is a constant $30 per unit, at all output rates, up to 24,000 units per week. The company has received an order to produce an extra 4,000 units per week beyond the current single-shift maximum capacity. Two options are being considered to fill the new order:

- **Option 1.** Increase the plant's output to 36,000 units a week by adding overtime, by adding Saturday operations, or both. No increase in fixed costs is entailed, but the variable cost is $36 per unit for any output in excess of 24,000 units per week, up to a 36,000-unit capacity.
- **Option 2.** Operate a second shift. The maximum capacity of the second shift is 21,000 units per week. The variable cost of the second shift is $31.50 per unit, and the operation of a second shift entails additional fixed costs of $13,500 per week.

Determine the range of operating volume that will make Option 2 profitable.

SOLUTION

Given: Financial data.

Find: The break-even volume that will make both options indifferent.

In this example, the operating costs related to the first-shift operation will remain unchanged if one alternative is chosen instead of another. Therefore, those costs are irrelevant to the current decision and can safely be left out of the analysis. Consequently, we need to examine only the increased total cost due to the additional operating volume under each option. (This kind of study is known as *incremental analysis*.) Let Q denote the additional operating volume. Then we have

- **Option 1.** Overtime and Saturday operation: $\$36Q$.
- **Option 2.** Second-shift operation: $\$13{,}500 + \$31.50Q$.

We can find the break-even volume (Q_b) by equating the incremental cost functions and solving for Q:

$$\begin{aligned} 36Q &= 13{,}500 + 31.5Q, \\ 4.5Q &= 13{,}500, \\ Q_b &= 3{,}000 \text{ units.} \end{aligned}$$

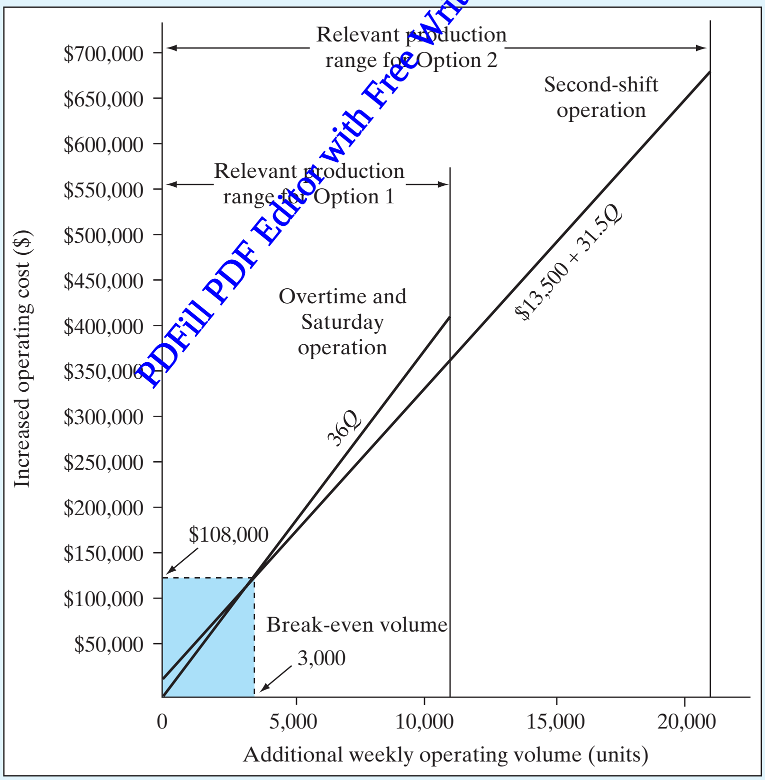


Figure 8.6 Cost–volume relationships of operating overtime and a Saturday operation versus second-shift operation beyond 24,000 units (Example 8.4).

If the additional volume exceeds 3,000 units, the second-shift operation becomes more efficient than overtime or Saturday operation. A break-even (or cost–volume) graph based on the foregoing data is shown in Figure 8.6. The horizontal scale represents additional volume per week. The upper limit of the relevant volume range for Option 1 is 12,000 units, whereas the upper limit for Option 2 is 21,000 units. The vertical scale is in dollars of cost. The operating savings expected at any volume may be read from the cost–volume graph. For example, the break-even point (zero savings) is 3,000 units per week. Option 2 is a better choice, since the additional weekly volume from the new order exceeds 3,000 units.

Make-or-Buy (Outsourcing) Decision

In business, the make-or-buy decision arises on a fairly frequent basis. Many firms perform certain activities using their own resources and pay outside firms to perform certain other activities. It is a good policy to constantly seek to improve the balance between these two types of activities by asking whether we should outsource some function that we are now performing ourselves or vice versa. Make-or-buy decisions are often grounded in the concept of *opportunity cost*.

8.4.2 Opportunity Cost

Opportunity cost: The benefits you could have received by taking an alternative action.

Opportunity cost may be defined as the potential benefit that is given up as you seek an alternative course of action. In fact, virtually every alternative has some opportunity cost associated with it. For example, suppose you have a part-time job while attending college that pays you $200 per week. You would like to spend a week at the beach during spring break, and your employer has agreed to give you the week off. What would be the opportunity cost of taking the time off to be at the beach? The $200 in lost wages would be an opportunity cost.

In an economic sense, opportunity cost could mean the contribution to income that is forgone by not using a limited resource in the best way possible. Or we may view opportunity costs as cash flows that could be generated from an asset the firm already owns, provided that such flows are not used for the alternative in question. In general, *accountants do not post opportunity cost in the accounting records of an organization. However, this cost must be explicitly considered in every decision.* In sum,

- An opportunity cost arises when a project uses a resource that may already have been paid for by the firm.
- When a resource that is already owned by a firm is being considered for use in a project, that resource has to be priced on its next-best alternative use, which may be
  1. A sale of the asset, in which case the opportunity cost is the expected proceeds from the sale, net of any taxes on gains.
  2. Renting or leasing the asset out, in which case the opportunity cost is the expected present value of the after-tax revenue from the rental or lease.
  3. Some use elsewhere in the business, in which case the opportunity cost is the cost of replacing the resource.
  4. That the asset has been abandoned or is of no use. Then the opportunity cost is zero.

EXAMPLE 8.5 Opportunity Cost: Lost Rental Income (Opportunity Cost)

Benson Company is a farm equipment manufacturer that currently produces 20,000 units of gas filters annually for use in its lawn-mower production. The expected annual production cost of the gas filters is summarized as follows:

Variable costs:	
Direct materials	$100,000
Direct labor	190,000
Power and water	35,000
Fixed costs:	
Heating and light	20,000
Depreciation	100,000
Total cost	$445,000

Tompkins Company has offered to sell Benson 20,000 units of gas filters for $17.00 per unit. If Benson accepts the offer, some of the manufacturing facilities currently used to manufacture the filters could be rented to a third party at an annual rent of $35,000. Should Benson accept Tompkins's offer, and why?

SOLUTION

Given: Financial data; production volume = 20,000 units.

Find: Whether Benson should outsource the gas filter operation.

	Make Option	Buy Option	Differential Cost (Make-Buy)
Variable costs:			
Direct materials	$100,000		$100,000
Direct labor	190,000		190,000
Power and water	35,000		35,000
Gas filters		340,000	−340,000
Fixed costs:			
Heating and light	20,000	20,000	0
Depreciation	100,000	100,000	0
Rental income lost	35,000		35,000
Total cost	$480,000	$460,000	$20,000
Unit cost	$24.00	$23.00	$1.00

This problem is unusual in the sense that the buy option would generate a rental fee of $35,000. In other words, Benson could rent out the current manufacturing facilities if it were to purchase the gas filters from Tompkins. To compare the two options, we need to examine the cost of each option.

The buy option has a lower unit cost and saves $1 for each use of a gas filter. If the lost rental income (opportunity cost) were not considered, however, the decision would favor the make option.

Sunk cost: A cost that has been incurred and cannot be reversed

8.4.3 Sunk Costs

A sunk cost is a cost that has already been incurred by past actions. Sunk costs are not relevant to decisions, because they cannot be changed regardless of what decision is made now or in the future. The only costs relevant to a decision are costs that vary among the alternative courses of action being considered. To illustrate a sunk cost, suppose you have a very old car that requires frequent repairs. You want to sell the car, and you figure that the current market value would be about $1,200 at best. While you are in the process of advertising the car, you find that the car's water pump is leaking. You decided to have the pump repaired, which cost you $200. A friend of yours is interested in buying your car and has offered $1,300 for it. Would you take the offer, or would you decline it simply because you cannot recoup the repair cost with that offer? In this example, the $200 repair cost is a sunk cost. You cannot change this repair cost, regardless of whether you keep or sell the car. Since your friend's offer is $100 more than the best market value, it would be better to accept the offer.

8.4.4 Marginal Cost

We make decisions every day, as entrepreneurs, professionals, executives, investors, and consumers, with little thought as to where our motivations come from or how our assumptions of logic fit a particular academic regimen. As you have seen, the engineering economic decisions that we describe throughout this text owe much to English economist Alfred Marshall (1842–1924) and his concepts of **marginalism**. In our daily quest for material gain, rarely do we recall the precepts of microeconomics or marginal utility, yet the ideas articulated by Marshall remain some of the most useful core principles guiding economic decision making.

Definition

Marginal cost: The cost associated with one additional unit of production.

Another cost term useful in cost–volume analysis is marginal cost. We define **marginal cost** as the added cost that would result from increasing the rate of output by a single unit. The accountant's differential-cost concept can be compared to the economist's marginal-cost concept. In speaking of changes in cost and revenue, the economist employs the terms *marginal cost* and *marginal revenue*. The revenue that can be obtained from selling one more unit of product is called **marginal revenue**. The cost involved in producing one more unit of product is called **marginal cost**.

EXAMPLE 8.6 Marginal Costs versus Average Costs

Consider a company that has an available electric load of 37 horsepower and that purchases its electricity at the following rates:

kWh/Month	@$/kWh	Average Cost ($/kWh)
First 1,500	$0.050	$0.050
Next 1,250	0.035	$\dfrac{\$75 + 0.0350(X - 1{,}500)}{X}$
Next 3,000	0.020	$\dfrac{\$118.75 + 0.020(X - 2{,}750)}{X}$
All over 5,750	0.010	$\dfrac{\$178.25 + 0.010(X - 5{,}750)}{X}$

According to this rate schedule, the unit variable cost in each rate class represents the marginal cost per kilowatt-hours (kWh). Alternatively, we may determine the average costs in the third column by finding the cumulative total cost and dividing it by the total number of kWh (X). Suppose that the current monthly consumption of electric power averages 3,200 kWh. On the basis of this rate schedule, determine the marginal cost of adding one more kWh and, for a given operating volume (3,200 kWh), the average cost per kWh.

SOLUTION

Given: Marginal cost schedule for electricity; operating volume = 3,200 kWh.

Find: Marginal and average cost per kWh.

The marginal cost of adding one more kWh is $0.020. The average variable cost per kWh is calculated as follows:

kWh	Rate ($/kWh)	Cost
First 1,500	0.050	$75.00
Next 1,250	0.035	43.75
Remaining 450	0.020	9.00
Total		$127.75

The average variable cost per kWh is $127.75/3,200 kWh = $0.0399 kWh. Or we can find the value by using the formulas in the third column of the rate schedule:

$$\frac{\$118.75 + 0.020(3{,}200 - 2{,}750)}{3{,}200} = \frac{\$127.75}{3{,}200 \text{ kWh}} = \$0.0399 \text{ kWh}.$$

Changes in the average variable cost per unit are the result of changes in the marginal cost. As shown in Figure 8.7, the average variable cost continues to fall because the marginal cost is lower than the average variable cost over the entire volume.

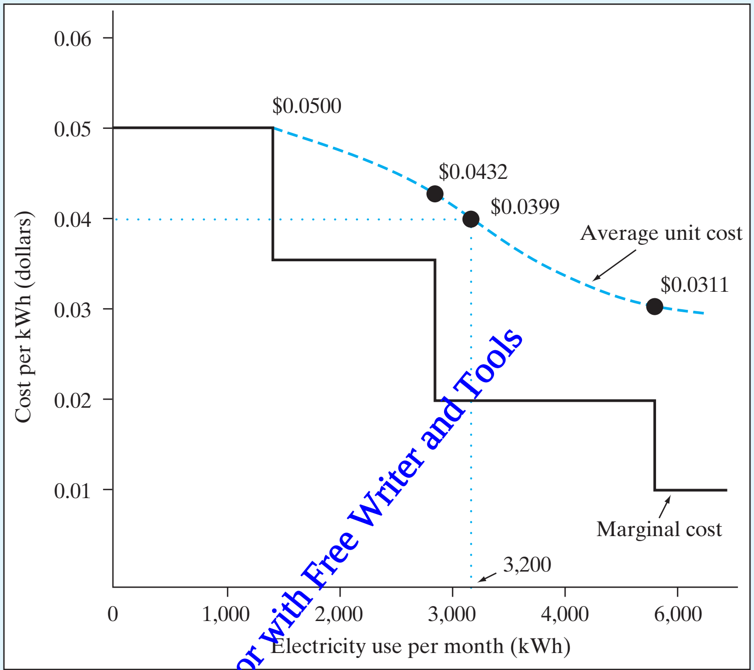


Figure 8.7 Marginal versus average cost per kWh (Example 8.6).

Marginal analysis: A technique used in microeconomics by which very small changes in specific variables are studied in terms of the effect on related variables and the system as a whole.

Marginal Analysis

The fundamental concept of marginal thinking is that you are where you are, and what is past is irrelevant. (This means that we should ignore the sunk cost.) The point is whether you move forward, and you will do so if the benefits outweigh the costs, even if they do so by a smaller margin than before. You will continue to produce a product or service until the cost of doing so equals the revenue derived. Microeconomics suggests that businesspersons and consumers measure progress not in great leaps, but in small incremental steps. Rational persons will reevaluate strategies and take new actions if benefits exceed costs on a marginal basis. If there is more than one alternative to choose, do so always with the alternative with greatest marginal benefit.

As we have mentioned, the economist's marginal-cost concept is the same as the accountant's differential-cost concept. For a business problem on maximization of profit, economists place a great deal of emphasis on the importance of marginal analysis: Rational individuals and institutions should perform the most cost-effective activities first. Specifically,

- When you examine marginal costs, you want to use the least expensive methods first and the most expensive last (if at all).

- Similarly, when you examine marginal revenue (or demand) the most revenue-enhancing methods should be used first, the least revenue enhancing last.
- If we need to consider the revenue along with the cost, the volume at which the total revenue and total cost are the same is known as the *break-even point*. If the cost and revenue function are assumed to be linear, this point may also be calculated, by means of the following formula:

$$\text{Break-even volume} = \frac{\text{Fixed costs}}{\text{Sales price per unit} - \text{Variable cost per unit}}$$

$$= \frac{\text{Fixed costs}}{\text{Marginal contribution per unit}} \tag{8.1}$$

The difference between the unit sales price and the unit variable cost is the producer's **marginal contribution**, also known as **marginal income**. This means that each unit sold contributes toward absorbing the company's fixed costs.

EXAMPLE 8.7 Profit-Maximization Problem: Marginal Analysis[2]

Suppose you are a chief executive officer (CEO) of a small pharmaceutical company that manufactures generic aspirin. You want the company to maximize its profits. You can sell as many aspirins as you make at the prevailing market price. You have only one manufacturing plant, which is the constraint. You have the plant working at full capacity Monday through Saturday, but you close the plant on Sunday because on Sundays you have to pay workers overtime rates, and it is not worth it. The marginal costs of production are constant Monday through Saturday. Marginal costs are higher on Sundays, only because labor costs are higher.

Now you obtain a long-term contract to manufacture a brand-name aspirin. The costs of making the generic aspirin or the brand-name aspirin are identical. In fact, there is no cost or time involved in switching from the manufacture of one to the other. You will make much larger profits from the brand-name aspirin, but the demand is limited. One day of manufacturing each week will permit you to fulfill the contract. You can manufacture both the brand-name and the generic aspirin. Compared with the situation before you obtained the contract, your profits will be much higher if you now begin to manufacture on Sundays—even if you have to pay overtime wages.

- *Generic aspirin*. Each day, you can make 1,000 cases of generic aspirin. You can sell as many as you make, for the market price of \$10 per case. Every week you have fixed costs of \$5,000 (land tax and insurance). No matter how many cases you manufacture, the cost of materials and supplies is \$2 per case; the cost of labor is \$5 per case, except on Sundays, when it is \$10 per case.
- *Brand-name aspirin*. Your order for the brand-name aspirin requires that you manufacture 1,000 cases per week, which you sell for \$30 per case. The cost for the brand-name aspirin is identical to the cost of the generic aspirin.

What do you do?

[2] *Source:* "Profit Maximization Problem," by David Hemenway and Elon Kohlberg, *Economic Inquiry,* October 1, 1997, Page 862, Copyright 1997 Western Economic Association International.

SOLUTION

Given: Sales price, \$10 per case for generic aspirin, \$30 per case for band aspirin; fixed cost, \$5,000; variable cost, \$7 per case during weekdays, \$12 per case on Sunday operation; weekly production, 6,000 cases of generic aspirin, 1,000 cases of brand-name aspirin.

Find: (a) Optimal production mix and (b) break-even volume.

(a) *Optimal production mix*: The marginal costs of manufacturing brand-name aspirin are constant Monday through Saturday; they rise substantially on Sunday and are above the marginal revenue from manufacturing generic aspirin. In other words, your company should manufacture the brand-name aspirin first. Your marginal revenue is the highest the "first" day, when you manufacture the brand-name aspirin. It then falls and remains constant for the rest of the week. On the seventh day (Sunday), the marginal revenue from manufacturing the generic aspirin is still below the marginal cost. You should manufacture brand-name aspirin one day a week and generic aspirin five days a week. On Sundays, the plant should close.

- The marginal revenue from manufacturing on Sunday is \$10,000 (1,000 cases times \$10 per case).
- The marginal cost from manufacturing on Sunday is \$12,000 (1,000 cases times \$12 per case—\$10 labor + \$2 materials).
- Profits will be \$2,000 lower than revenue if the plant operates on Sunday.

(b) *Break-even volume*: The total revenue and cost functions can be represented as follows:

$$\text{Total revenue function: } \begin{cases} 30Q & \text{for } 0 \le Q \le 1{,}000 \\ 30{,}000 + 10Q & \text{for } 1{,}000 < Q \le 6{,}000, \end{cases}$$

$$\text{Total cost function: } \begin{cases} 5{,}000 + 7Q & \text{for } 0 \le Q \le 6{,}000 \\ 47{,}000 + 12Q & \text{for } 6{,}000 \le Q \le 7{,}000. \end{cases}$$

Table 8.4 shows the various factors involved in the production of the brand-name and the generic aspirin.

- If you produce the brand-name aspirin first, the break-even volume is

$$30Q - 7Q - 5{,}000 = 0,$$
$$Q_b = 217.39.$$

- If you produce the generic aspirin first, the break-even volume is

$$10Q - 7Q - 5{,}000 = 0,$$
$$Q_b = 1{,}666.67.$$

Clearly, scheduling the production of the brand-name aspirin first is the better strategy, as you can recover the fixed cost (\$5,000) much faster by selling just 217.39 cases of brand-name aspirin. Also, Sunday operation is not economical, as the marginal cost exceeds the marginal revenue by \$2,000, as shown in Figure 8.8.

SUMMARY

In this chapter, we examined several ways in which managers classify costs. How the costs will be used dictates how they will be classified:

- Most manufacturing companies divide **manufacturing costs** into three broad categories: *direct materials*, *direct labor*, and *manufacturing overhead*. **Nonmanufacturing costs** are classified into two categories: *marketing* or *selling costs* and *administrative costs*.
- For the purpose of valuing inventories and determining expenses for the balance sheet and income statement, costs are classified as either **product costs** or **period costs**.
- For the purpose of predicting **cost behavior**—how costs will react to changes in activity—managers commonly classify costs into two categories: variable costs and fixed costs.
- An understanding of the following **cost–volume relationships** is essential to developing successful business strategies and to planning future operations:

  Fixed operating costs: Costs that do not vary with production volume.

  Variable operating costs: Costs that vary with the level of production or sales.

  Average costs: Costs expressed in terms of units obtained by dividing total costs by total volumes.

  Differential (incremental) costs: Costs that represent differences in total costs, which result from selecting one alternative instead of another.

  Opportunity costs: Benefits that could have been obtained by taking an alternative action.

  Sunk costs: Past costs not relevant to decisions because they cannot be changed no matter what actions are taken.

  Marginal costs: Added costs that result from increasing rates of outputs, usually by single units.

  Marginal analysis: In economic analysis, we need to answer the apparently trivial question, "Is it worthwhile?"—whether the action in question will add sufficiently to the benefits enjoyed by the decision maker to make performing the action worth the cost. This is the heart of marginal-decision making—the statement that an action merits performance if, and only if, as a result, we can expect to be better off than we were before.

- At the level of plant operations, engineers must make decisions involving materials, production processes, and the in-house capabilities of company personnel. Most of these operating decisions do not require any investments in physical assets; therefore, they depend solely on the cost and volume of business activity, without any consideration of the time value of money.
- Engineers are often asked to prepare the production budgets related to their operating division. Doing this requires a knowledge of budgeting scarce resources, such as labor and materials, and an understanding of the overhead cost. The same budgeting practice will be needed in preparing the estimates of costs and revenues associated with undertaking a new project.

PROBLEMS

Classifying Costs

8.1 Identify which of the following transactions and events are product costs and which are period costs:

- Storage and material handling costs for raw materials.
- Gains or losses on the disposal of factory equipment.
- Lubricants for machinery and equipment used in production.
- Depreciation of a factory building.
- Depreciation of manufacturing equipment.
- Depreciation of the company president's automobile.
- Leasehold costs for land on which factory buildings stand.
- Inspection costs of finished goods.
- Direct labor cost.
- Raw-materials cost.
- Advertising expenses.

Cost Behavior

8.2 Identify which of the following costs are fixed and which are variable:

- Wages paid to temporary workers.
- Property taxes on a factory building.
- Property taxes on an administrative building.
- Sales commission.
- Electricity for machinery and equipment in the plant.
- Heating and air-conditioning for the plant.
- Salaries paid to design engineers.
- Regular maintenance on machinery and equipment.
- Basic raw materials used in production.
- Factory fire insurance.

8.3 The accompanying figures are a number of cost behavior patterns that might be found in a company's cost structure. The vertical axis on each graph represents total cost, and the horizontal axis on each graph represents level of activity (volume). For each of the situations that follow, identify the graph that illustrates the cost pattern involved. Any graph may be used more than once[5].

(a) Electricity bill—a flat-rate fixed charge, plus a variable cost after a certain number of kilowatt-hours are used.

(b) City water bill, which is computed as follows:

First 1,000,000 gallons $1,000 flat or less rate

Next 10,000 gallons $0.003 per gallon used

Next 10,000 gallons $0.006 per gallon used

Next 10,000 gallons $0.009 per gallon used

Etc. etc.

[5] Adapted originally from a CPA exam, and the same materials are also found in R. H. Garrison and E. W. Noreen, *Managerial Accounting*, 8th ed. Irwin, 1997, copyright © Richard D. Irwin, p. 271.

(c) Depreciation of equipment, where the amount is computed by the straight-line method. When the depreciation rate was established, it was anticipated that the obsolescence factor would be greater than the wear-and-tear factor.

(d) Rent on a factory building donated by the city, where the agreement calls for a fixed fee payment unless 200,000 labor-hours or more are worked, in which case no rent need be paid.

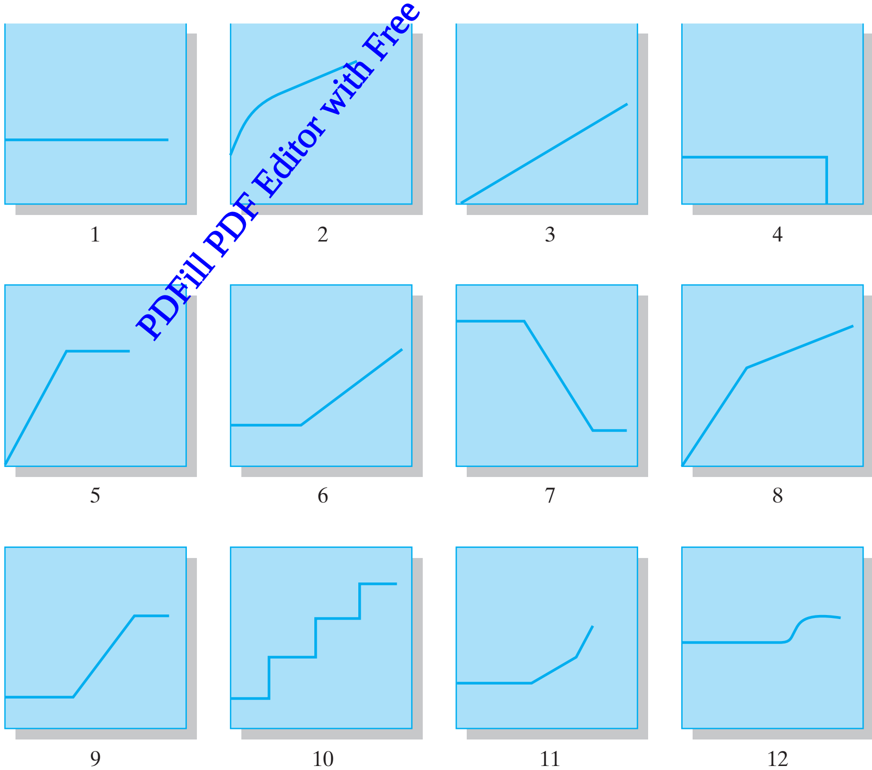
(e) Cost of raw materials, where the cost decreases by 5 cents per unit for each of the first 100 units purchased, after which it remains constant at \$2.50 per unit.

(f) Salaries of maintenance workers, where one maintenance worker is needed for every 1,000 machine-hours or less (that is, 0 to 1,000 hours requires one maintenance worker, 1,001 to 2,000 hours requires two maintenance workers, etc.).

(g) Cost of raw materials used.

(h) Rent on a factory building donated by the county, where the agreement calls for rent of \$100,000, less \$1 for each direct labor-hour worked in excess of 200,000 hours, but a minimum rental payment of \$20,000 must be paid.

(i) Use of a machine under a lease, where a minimum charge of \$1,000 must be paid for up to 400 hours of machine time. After 400 hours of machine time, an additional charge of \$2 per hour is paid, up to a maximum charge of \$2,000 per period.



8.4 Harris Company manufactures a single product. Costs for the year 2001 for output levels of 1,000 and 2,000 units are as follows:

Units produced	1,000	2,000
Direct labor	$30,000	$30,000
Direct materials	20,000	40,000
Overhead:		
Variable portion	12,000	24,000
Fixed portion	36,000	36,000
Selling and administrative costs:		
Variable portion	5,000	10,000
Fixed portion	22,000	22,000

At each level of output, compute the following:

(a) Total manufacturing costs.

(b) Manufacturing costs per unit.

(c) Total variable costs.

(d) Total variable costs per unit.

(e) Total costs that have to be recovered if the firm is to make a profit.

Cost–Volume–Profit Relationships

8.5 Bragg & Stratton Company manufactures a specialized motor for chain saws. The company expects to manufacture and sell 30,000 motors in year 2001. It can manufacture an additional 10,000 motors without adding new machinery and equipment. Bragg & Stratton's projected total costs for the 30,000 units are as follows:

Direct Materials	$150,000
Direct labor	300,000
Manufacturing overhead:	
Variable portion	100,000
Fixed portion	80,000
Selling and administrative costs:	
Variable portion	180,000
Fixed portion	70,000

The selling price for the motor is $80.

(a) What is the total manufacturing cost per unit if 30,000 motors are produced?

(b) What is the total manufacturing cost per unit if 40,000 motors are produced?

(c) What is the break-even price on the motors?

8.6 The accompanying chart shows the expected monthly profit or loss of Cypress Manufacturing Company within the range of its monthly practical operating capacity. Using the information provided in the chart, answer the following questions:

(a) What is the company's break-even sales volume?

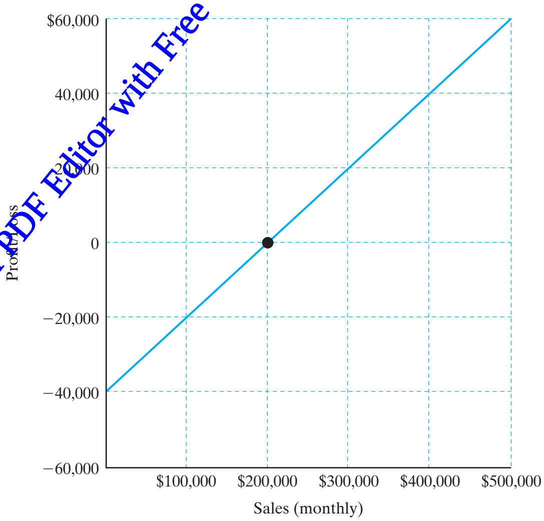
(b) What is the company's marginal contribution rate?

(c) What effect would a 5% decrease in selling price have on the break-even point in (a)?

(d) What effect would a 10% increase in fixed costs have on the marginal contribution rate in (b)?

(e) What effect would a 6% increase in variable costs have on the break-even point in (a)?

(f) If the chart also reflects $20,000 monthly depreciation expenses, compute the sale at the break-even point for cash costs.



8.7 The accompanying graph is a cost–volume–profit graph. In the graph, identify the following line segments or points:

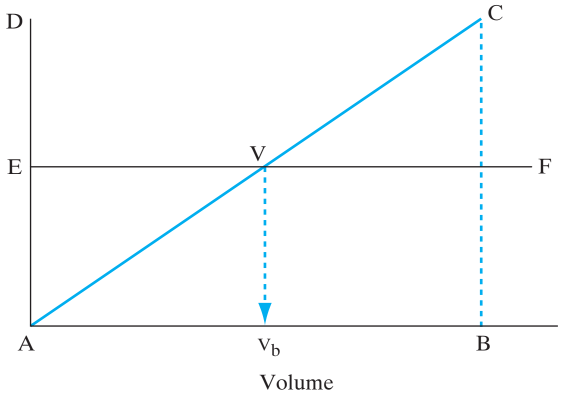
(a) Line EF represents _______________.

(b) The horizontal axis AB represents _______________, and the vertical axis AD represents _______________.

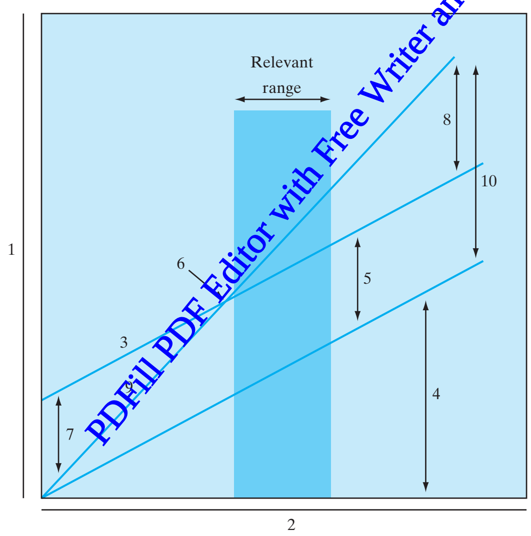
(c) Point V represents _______________.

(d) The distance CB divided by the distance AB is _______________.

(e) The point V_b is a break-even _______________.



8.8 A cost–volume–profit (CVP) graph is a useful technique for showing relationships between costs, volume, and profits in an organization.



(a) Identify the numbered components in the accompanying CVP graph.

No.	Description	No.	Description
1		6	
2		7	
3		8	
4		9	
5		10	

CHAPTER NINE

Depreciation and Corporate Taxes

Know What It Costs to Own a Piece of Equipment: A Hospital Pharmacy Gets a Robotic Helper[1] When most patients at Kirkland's Evergreen Hospital Medical Center are asleep, the robot comes to life. Its machinery thumps like a heart beating as it moves around the hospital pharmacy, preparing prescriptions. In a little more than an hour, he'll ready 1,500 doses of medication.

Ernie, or Evergreen Robot Noticeably Improving Efficiency, is a new $3 million addition to [the] pharmacy staff. The robot uses bar codes to match each drug dispensed with an electronic patient profile, helping prevent errors, said Bob Blanchard, pharmacy director.

[1] "Know What It Costs to Own a Piece of Equipment: A Hospital Pharmacy Gets a Robotic Helper," Katherine Sather, *Seattle Times Eastside bureau*, Copyright © 2004 The Seattle Times Company.

"It's the future," he said. "Safety is the main benefit." Efficiency is another plus. The robot can prepare a 24-hour medication supply for in-house patients, about 1,500 doses, in a little more than an hour. The same task used to take three people about three hours to complete, Blanchard said.

Now, staff [members] only label the medications with bar codes. Ernie does the rest.

The machine looks like a mini space ship. A door opens into an octagon-shaped room, 12 feet long diagonally, stocked with more than 400 racks of medicine. Each dose is labeled with a bar code that tells Ernie what it is and where it should be stored. Affixed to the center of the room is a mechanical arm that scans the bar-coded medicine and places it on its designated rack.

When staff [members] give Ernie a computerized order, his arm buzzes to the correct row of medication, grabs it with suction cups and drops it into an envelope that is bar coded with the patient's profile.

"Research shows using it decreases certain predictable errors," Blanchard said. "We're very excited about it—we've really led the drive to move to automation."

At Evergreen, [the] pharmacy staff hope[s] for Ernie to eventually dispense 93 percent of the medication that is distributed to patients in the 244-bed hospital. Medication that needs care such as refrigeration is prepared by staff.

"The technology is such that it's been tested and it's reliable. Given the volume of patients we see, it makes sense," said Amy Gepner, a spokesperson for the hospital. "It's a safety initiative." Ernie is being purchased with a seven-year lease along with 23 automated medical cabinets placed throughout the hospital. But [the] staff [is] fonder of the robot.

Now ask yourself, How does the cost of this robot ($3 million) affect the financial position of the hospital? In the long run, the system promises to create greater cost savings for the hospital by enhancing productivity, improving safety, and cutting down lead time in filling orders. In the short run, however, the high initial cost of the robot will adversely affect the organization's "bottom line," because that cost is only gradually rewarded by the benefits the robot offers.

Another consideration should come to mind as well: This state-of-the-art robot must inevitably wear out over time, and even if its productive service extends over many years, the cost of maintaining its high level of functioning will increase as the individual pieces of hardware wear out and need to be replaced. Of even greater concern is the question of how long this robot will be the state of the art. When will the competitive advantage the hospital has just acquired become a competitive disadvantage through obsolescence?

One of the facts of life that organizations must deal with and account for is that fixed assets lose their value—even as they continue to function and contribute to the engineering projects that use them. This loss of value, called **depreciation**, can involve deterioration and obsolescence.

The main function of **depreciation accounting** is to account for the cost of fixed assets in a pattern that matches their decline in value over time. The cost of the robot we have just described, for example, will be allocated over several years in the hospital's financial statements, so that the pattern of the robot's costs roughly matches its pattern of service. In this way, as we shall see, depreciation accounting enables the firm to stabilize the statements about its financial position that it distributes to stockholders and the outside world.

On a project level, engineers must be able to assess how the practice of depreciating fixed assets influences the investment value of a given project. To do this, the engineers need to estimate the allocation of capital costs over the life of the project, which requires an understanding of the conventions and techniques that accountants use to depreciate assets. In this chapter, we will review the conventions and techniques of asset depreciation and income taxes.

We begin by discussing the nature and significance of depreciation, distinguishing its general economic definition from the related, but different, accounting view of depreciation. We then focus our attention almost exclusively on the rules and laws that govern asset depreciation and the methods that accountants use to allocate depreciation expenses. Knowledge of these rules will prepare you to apply them in assessing the depreciation of assets acquired in engineering projects. Then we turn our attention to the subject of depletion, which utilizes similar ideas, but specialized techniques, to allocate the cost of the depletion of natural-resource assets.

Once we understand the effect of depreciation at the project level, we need to address the effect of corporate taxes on project cash flows. There are many forms of government taxation, including sales taxes, property taxes, user taxes, and state and federal income taxes. In this chapter, we will focus on federal income taxes. When you are operating a business, any profits or losses you incur are subject to income tax consequences. Therefore, we cannot ignore the impact of income taxes in project evaluation. The chapter will give you a good idea of how the U.S. tax system operates and of how federal income taxes affect economic analysis. Although tax law is subject to frequent changes, the analytical procedures presented here provide a basis for tax analysis that can be adapted to reflect future changes in tax law. Thus, while we present many examples based on current tax rates, in a larger context we present a general approach to the analysis of *any* tax law.

CHAPTER LEARNING OBJECTIVES

After completing this chapter, you should understand the following concepts:

- How to account for the loss of value of an asset in business.
- The meaning and types of depreciation.
- The difference between book depreciation and tax depreciation.
- The effects of depreciation on net income calculation.
- The general scheme of U.S. corporate taxes.
- How to determine ordinary gains and capital gains.
- How to determine the appropriate tax rate to use in project analysis.
- The relationship between net income and net cash flow.

9.1 Asset Depreciation

Fixed assets, such as equipment and real estate, are economic resources that are acquired to provide future cash flows. Generally, **depreciation** can be defined as a gradual decrease in the utility of fixed assets with use and time. While this general definition does not capture the subtleties inherent in a more specific definition of depreciation, it does provide us with a starting point for examining the variety of underlying ideas and practices that are discussed in this chapter. Figure 9.1 will serve as a road map for understanding the different types of depreciation that we will explore here.

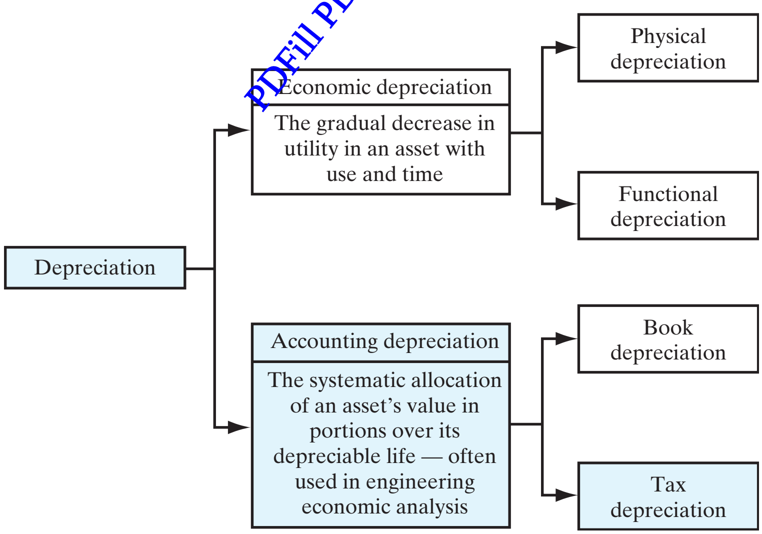


Figure 9.1 Classification of types of depreciation.

We can classify depreciation into the categories of physical or functional depreciation. **Physical depreciation** can be defined as a reduction in an asset's capacity to perform its intended service due to physical impairment. Physical depreciation can occur in any fixed asset in the form of (1) deterioration from interaction with the environment, including such agents as corrosion, rotting, and other chemical changes, and (2) wear and tear from use. Physical depreciation leads to a decline in performance and high maintenance costs.

Functional depreciation occurs as a result of changes in the organization or in technology that decrease or eliminate the need for an asset. Examples of functional depreciation include obsolescence attributable to advances in technology, a declining need for the services performed by an asset, and the inability to meet increased quantity or quality demands.

9.1.1 Economic Depreciation

This chapter is concerned primarily with accounting depreciation, which is the form of depreciation that provides an organization with the information it uses to assess its financial position. It would also be useful, however, to discuss briefly the economic ideas upon which accounting depreciation is based. In the course of the discussion, we will develop a precise definition of economic depreciation that will help us distinguish between various conceptions of depreciation.

If you have ever owned a car, you are probably familiar with the term *depreciation* as it is used to describe the decreasing value of your vehicle. Because a car's reliability and appearance usually decline with age, the vehicle is worth less with each passing year. You can calculate the economic depreciation accumulated for your car by subtracting the current market value, or "blue book" value, of the car from the price you originally paid for it. We can define **economic depreciation** as follows:

$$\text{Economic depreciation} = \text{Purchase price} - \text{market value}.$$

Physical and functional depreciation are categories of economic depreciation. The measurement of economic depreciation does not require that an asset be sold: The market value of the asset can be closely estimated without actually testing it in the marketplace. The need to have a precise scheme for recording the ongoing decline in the value of an asset as a part of the accounting process leads us to an exploration of how organizations account for depreciation.

9.1.2 Accounting Depreciation

The acquisition of fixed assets is an important activity for a business organization, whether the organization is starting up or acquiring new assets to remain competitive. Like other disbursements, the cost of these fixed assets must be recorded as expenses on a firm's balance sheet and income statement. However, unlike costs such as maintenance, material, and labor costs, the costs of fixed assets are not treated simply as expenses to be accounted for in the year that they are acquired. Rather, these assets are **capitalized**; that is, their costs are distributed by subtracting them as expenses from gross income, one part at a time over a number of periods. The systematic allocation of the initial cost of an asset in parts over a time, known as the asset's depreciable life, is what we mean by **accounting depreciation**. Because accounting depreciation is the standard of the business world, we sometimes refer to it more generally as **asset depreciation**.

Accounting depreciation: Amount allocated during the period to amortize the cost of acquiring long term assets over the useful life of the assets.

Accounting depreciation is based on the **matching concept**: A fraction of the cost of the asset is chargeable as an expense in each of the accounting periods in which the asset provides service to the firm, and each charge is meant to be a percentage of the whole cost that "matches" the percentage of the value utilized in the given period. The matching concept suggests that the accounting depreciation allowance generally reflects, at least to some extent, the actual economic depreciation of the asset. *In engineering economic analysis, we use the concept of accounting depreciation exclusively*. This is because accounting depreciation provides a basis for determining the income taxes associated with any project undertaken.

9.2 Factors Inherent in Asset Depreciation

The process of depreciating an asset requires that we make several preliminary determinations: (1) What is the cost of the asset? (2) What is the asset's value at the end of its useful life? (3) What is the depreciable life of the asset? and, finally, (4) What method of depreciation do we choose? In this section, we will examine each of these questions.

9.2.1 Depreciable Property

As a starting point, it is important to recognize what constitutes a **depreciable asset**—that is, a property for which a firm may take depreciation deductions against income. For the purposes of U.S. tax law, any depreciable property has the following characteristics:

1. It must be used in business or must be held for the production of income.
2. It must have a definite service life, and that life must be longer than 1 year.
3. It must be something that wears out, decays, gets used up, becomes obsolete, or loses value from natural causes.

Depreciable property includes buildings, machinery, equipment, and vehicles. Inventories are not depreciable property, because they are held primarily for sale to customers in the ordinary course of business. If an asset has no definite service life, the asset cannot be depreciated. For example, *you can never depreciate land*.[2]

As a side note, we mention the fact that, while we have been focusing on depreciation within firms, individuals may also depreciate assets, as long as they meet the conditions we have listed. For example, an individual may depreciate an automobile if the vehicle is used exclusively for business purposes.

[2] This also means that you cannot depreciate the cost of clearing, grading, planting, and landscaping. All four expenses are considered part of the cost of the land.

9.2.2 Cost Basis

The **cost basis** of an asset represents the total cost that is claimed as an expense over the asset's life (i.e., the sum of the annual depreciation expenses). The cost basis generally includes the actual cost of the asset and all other incidental expenses, such as freight, site preparation, and installation. This total cost, rather than the cost of the asset only, must be the depreciation basis charged as an expense over the asset's life.

Besides being used in figuring depreciation deductions, an asset's cost basis is used in calculating the gain or loss to the firm if the asset is ever sold or salvaged. (We will discuss these topics in Section 9.8.)

Cost basis: The cost of an asset used to determine the depreciation base.

EXAMPLE 9.1 Cost Basis

Lanier Corporation purchased an automatic hole-punching machine priced at $62,500. The vendor's invoice included a sales tax of $3,263. Lanier also paid the inbound transportation charges of $725 on the new machine, as well as the labor cost of $2,150 to install the machine in the factory. In addition, Lanier had to prepare the site at a cost of $3,500 before installation. Determine the cost basis for the new machine for depreciation purposes.

SOLUTION

Book value: The value of an asset as it appears on a balance sheet, equal to cost minus accumulated depreciation.

Given: Invoice price = $62,500, freight = $725, installation cost = $2,150, and site preparation = $3,500.

Find: The cost basis.

The cost of machine that is applicable for depreciation is computed as follows:

Cost of new hole-punching machine	$62,500
Freight	725
Installation labor	2,150
Site preparation	3,500
Cost of machine (cost basis)	$68,875

COMMENTS: Why do we include all the incidental charges relating to the acquisition of a machine in its cost? Why not treat these incidental charges as expenses incurred during the period in which the machine is acquired? The matching of costs and revenue is the basic accounting principle. Consequently, the total costs of the machine should be viewed as an asset and allocated against the future revenue that the machine will generate. All costs incurred in acquiring the machine are costs of the services to be received from using the machine.

If the asset is purchased by trading in a similar asset, the difference between the book value (the cost basis minus the total accumulated depreciation) and the trade-in allowance must be considered in determining the cost basis for the new asset. If the trade-in allowance exceeds the book value, the difference (known as **unrecognized gain**) needs to be subtracted from the cost basis of the new asset. If the opposite is true (**unrecognized loss**), the difference should be added to the cost basis for the new asset.

EXAMPLE 9.2 Cost Basis with Trade-In Allowance

In Example 9.1, suppose Lanier purchased the hole-punching press by trading in a similar machine and paying cash for the remainder. The trade-in allowance is \$5,000, and the book value of the hole-punching machine that was traded in is \$4,000. Determine the cost basis for this hole-punching press.

SOLUTION

Given: Accounting data from Example 9.1; trade allowance = \$5,000.
Find: The revised cost basis.

Old hole-punching machine (**book value**)	\$ 4,000
Less: Trade-in allowance	5,000
Unrecognized gains	\$ 1,000
Cost of new hole-punching machine	\$62,500
Less: Unrecognized gains	(1,000)
Freight	725
Installation labor	2,150
Site preparation	3,500
Cost of machine (cost basis)	\$67,875

9.2.3 Useful Life and Salvage Value

Over how many periods will an asset be useful to a company? What do published statutes allow you to choose as the life of an asset? These are the central questions to be answered in determining an asset's depreciable life (i.e., the number of years over which the asset is to be depreciated).

Historically, depreciation accounting included choosing a depreciable life that was based on the service life of an asset. Determining the service life of the asset, however, was often very difficult, and the uncertainty of these estimates often led to disputes between taxpayers and the Internal Revenue Service (IRS). To alleviate the problems, the IRS published guidelines on lives for categories of assets. The guidelines, known as **asset depreciation ranges**, or **ADRs**, specified a range of lives for classes of assets based on historical data, and taxpayers were free to choose a depreciable life within the specified range for a given asset. An example of ADRs for some assets is given in Table 9.1.

TABLE 9.1 Asset Depreciation: Some Selected Asset Guideline Classes

	Asset Depreciation Range (Years)		
Assets Used	**Lower Limit**	**Midpoint Life**	**Upper Limit**
Office furniture, fixtures, and equipment	8	10	12
Information systems (computers)	5	6	7
Airplanes	5	6	7
Automobiles, taxis	2.5	3	3.5
Buses	7	9	11
Light trucks	3	4	5
Heavy trucks (concrete ready-mixer)	5	6	7
Railroad cars and locomotives	12	15	18
Tractor units	5	6	7
Vessels, barges, tugs, and water transportation systems	14.5	18	21.5
Industrial steam and electrical generation and/or distribution systems	17.5	22	26.5
Manufacturer of electrical and nonelectrical machinery	8	10	12
Manufacturer of electronic components, products, and systems	5	6	7
Manufacturer of motor vehicles	9.5	12	14.5
Telephone distribution plant	28	35	42

Source: IRS Publication 534. *Depreciation*. Washington, DC: U.S. Government Printing Office, 1995.

The **salvage value** is an asset's estimated value at the end of its life—the amount eventually recovered through sale, trade-in, or salvage. The eventual salvage value of an asset must be estimated when the depreciation schedule for the asset is established. If this estimate subsequently proves to be inaccurate, then an adjustment must be made. We will discuss these specific issues in Section 9.6.

9.2.4 Depreciation Methods: Book and Tax Depreciation

Most firms calculate depreciation in two different ways, depending on whether the calculation is (1) intended for financial reports (the **book depreciation method**), such as for the balance sheet or income statement, or (2) for the Internal Revenue Service (IRS), for the purpose of determining taxes (the **tax depreciation method**). In the United States, this distinction is totally legitimate under IRS regulations, as it is in many other countries.

Calculating depreciation differently for financial reports and for tax purposes allows the following benefits:

- It enables firms to report depreciation to stockholders and other significant outsiders on the basis of the matching concept. Therefore, the actual loss in value of the assets is generally reflected.
- It allows firms to benefit from the tax advantages of depreciating assets more quickly than would be possible with the matching concept. In many cases, tax depreciation allows firms to defer paying income taxes. This does not mean that they pay less tax overall, because the total depreciation expense accounted for over time is the same in either case. However, because tax depreciation methods usually permit a higher depreciation in earlier years than do book depreciation methods, the tax benefit of depreciation is enjoyed earlier, and firms generally pay lower taxes in the initial years of an investment project. Typically, this leads to a better cash position in early years, and the added cash leads to greater future wealth because of the time value of the funds.

As we proceed through the chapter, we will make increasing use of the distinction between depreciation accounting for financial reporting and depreciation accounting for income tax calculation. Now that we have established the context for our interest in both tax and book depreciation, we can survey the different methods with an accurate perspective.

9.3 Book Depreciation Methods

Three different methods can be used to calculate the periodic depreciation allowances: (1) the straight-line method, (2) accelerated methods, and (3) the unit-of-production method. In engineering economic analysis, we are interested primarily in depreciation in the context of income tax computation. Nonetheless, a number of reasons make the study of book depreciation methods useful. First, tax depreciation methods are based largely on the same principles that are used in book depreciation methods. Second, firms continue to use book depreciation methods for financial reporting to stockholders and outside parties. Third, book depreciation methods are still used for state income tax purposes in many states and even for federal income tax purposes for assets that were put into service before 1981. Finally, our discussion of depletion in Section 9.5 is based largely on one of these three book depreciation methods.

9.3.1 Straight-Line Method

The **straight-line (SL) method** of depreciation interprets a fixed asset as an asset that offers its services in a uniform fashion. The asset provides an equal amount of service in each year of its useful life.

Straight-line depreciation: An equal dollar amount of depreciation in each accounting period.

The straight-line method charges, as an expense, an equal fraction of the net cost of the asset each year, as expressed by the relation

$$D_n = \frac{(I - S)}{N}, \tag{9.1}$$

where D_n = Depreciation charge during year n,

I = Cost of the asset, including installation expenses,

S = Salvage value at the end of the asset's useful life,

N = Useful life.

The book value of the asset at the end of n years is then defined as

Book value in a given year = Cost basis − total depreciation charges made to date

or

$$B_n = I - (D_1 + D_2 + D_3 + \cdots + D_n). \tag{9.2}$$

EXAMPLE 9.3 Straight-Line Depreciation

Consider the following data on an automobile:

Cost basis of the asset, I = \$10,000,

Useful life, N = 5 years,

Estimated salvage value, S = \$2,000.

Use the straight-line depreciation method to compute the annual depreciation allowances and the resulting book values.

SOLUTION

Given: $I = \$10{,}000$, $S = \$2{,}000$, and $N = 5$ years.
Find: D_n and B_n for $n = 1$ to 5.

The straight-line depreciation rate is $\frac{1}{5}$, or 20%. Therefore, the annual depreciation charge is

$$D_n = (0.20)(\$10{,}000 - \$2{,}000) = \$1{,}600.$$

The asset would then have the following book values during its useful life:

n	B_{n-1}	D_n	B_n
1	\$10,000	\$1,600	\$8,400
2	8,400	1,600	6,800
3	6,800	1,600	5,200
4	5,200	1,600	3,600
5	3,600	1,600	2,000

Here, B_{n-1} represents the book value before the depreciation charge for year n. This situation is illustrated in Figure 9.2.

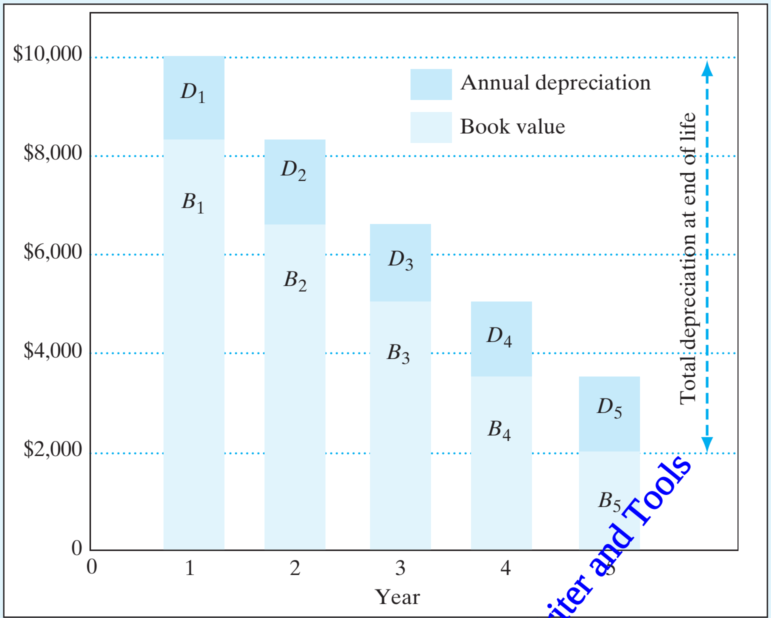


Figure 9.2 Straight-line depreciation methods (Example 9.3).

9.3.2 Accelerated Methods

The second concept of depreciation recognizes that the stream of services provided by a fixed asset may decrease over time; in other words, the stream may be greatest in the first year of an asset's service life and least in its last year. This pattern may occur because the mechanical efficiency of an asset tends to decline with age, because maintenance costs tend to increase with age, or because of the increasing likelihood that better equipment will become available and make the original asset obsolete. This kind of reasoning leads to a method that charges a larger fraction of the cost as an expense of the early years than of the later years. Any such method is called an **accelerated method**. The most widely used accelerated method is the **double-declining-balance method**.

Accelerated method: Any depreciation method that produces larger deductions for depreciation in the early years of a project's life.

Double-declining balance method: A depreciation method, in which double the straight-line depreciation amount is taken the first year, and then that same percentage is applied to the undepreciated amount in subsequent years.

Declining-Balance Method

The **declining-balance (DB) method** of calculating depreciation allocates a fixed fraction of the beginning book balance each year. The fraction, α, is obtained as follows:

$$\alpha = \left(\frac{1}{N}\right)(\text{multiplier}). \tag{9.3}$$

The most commonly used multipliers in the United States are 1.5 (called 150% DB) and 2.0 (called 200%, or double-declining balance, DDB). As N increases, α decreases, resulting in a situation in which depreciation is highest in the first year and then decreases over the asset's depreciable life.

The fractional factor can be utilized to determine depreciation charges for a given year, D_n, as follows:

$$D_1 = \alpha I,$$
$$D_2 = \alpha(I - D_1) = \alpha I(1 - \alpha),$$
$$D_3 = \alpha(I - D_1 - D_2) = \alpha I(1 - \alpha)^2,$$

Thus, for any year n, we have a depreciation charge

$$D_n = \alpha I(1 - \alpha)^{n-1}. \tag{9.4}$$

We can also compute the total DB depreciation (TDB) at the end of n years as follows:

$$\begin{aligned} \text{TDB} &= D_1 + D_2 + \cdots + D_n \\ &= \alpha I + \alpha I(1 - \alpha) + \alpha I(1 - \alpha)^2 + \cdots + \alpha I(1 - \alpha)^{n-1} \\ &= \alpha I[1 + (1 - \alpha) + (1 - \alpha)^2 + \cdots + (1 - \alpha)^{n-1}] \\ &= I[1 - (1 - \alpha)^n]. \end{aligned} \tag{9.5}$$

The book value B_n at the end of n years will be the cost I of the asset minus the total depreciation at the end of n years:

$$\begin{aligned} B_n &= I - \text{TDB} \\ &= I - I[1 - (1 - \alpha)^n] \\ B_n &= I(1 - \alpha)^n. \end{aligned} \tag{9.6}$$

EXAMPLE 9.4 Declining-Balance Depreciation

Consider the following accounting information for a computer system:

Cost basis of the asset, $I = \$10{,}000$,
Useful life, $N = 5$ years,
Estimated salvage value, $S = \$778$.

Use the double-declining-depreciation method to compute the annual depreciation allowances and the resulting book values (Figure 9.3).

SOLUTION

Given: $I = \$10{,}000$, $S = \$778$, $N = 5$ years
Find: D_n and B_n for $n = 1$ to 5

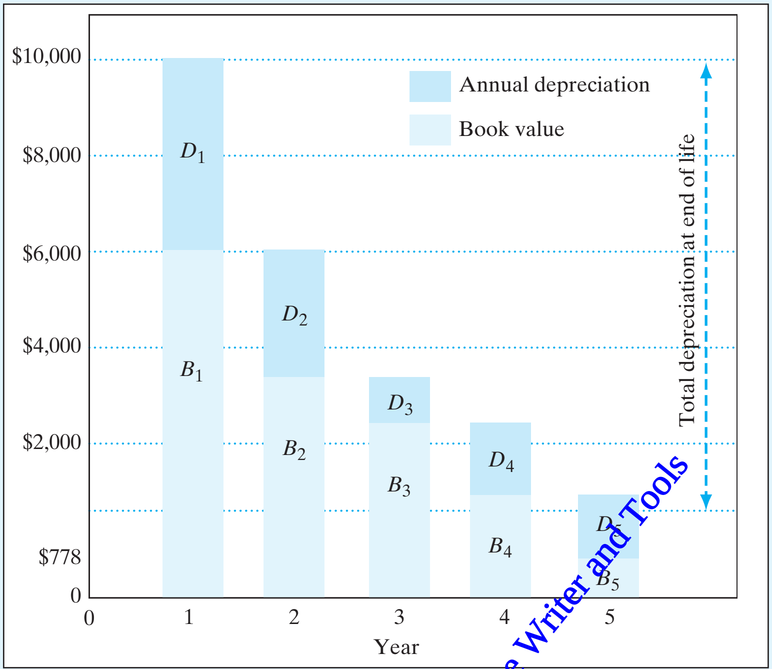


Figure 9.3 Double-declining-balance method (Example 9.4).

The book value at the beginning of the first year is \$10,000, and the declining-balance rate (α) is $\left(\frac{1}{5}\right)(2) = 40\%$. Then the depreciation deduction for the first year will be \$4,000 ($40\% \times \$10{,}000 = \$4{,}000$). To figure the depreciation deduction in the second year, we must first adjust the book value for the amount of depreciation we deducted in the first year. The first year's depreciation from the beginning book value is subtracted ($\$10{,}000 - \$4{,}000 = \$6{,}000$), and the resulting amount is multiplied by the rate of depreciation ($\$6{,}000 \times 40\% = \$2{,}400$). By continuing the process, we obtain the following table:

n	B_n	D_n	B_n
1	10,000	4,000	6,000
2	6,000	2,400	3,600
3	3,600	1,440	2,160
4	2,160	864	1,296
5	1,296	518	778

The declining balance is illustrated in terms of the book value of time in Figure 9.3.

The salvage value (S) of the asset must be estimated at the outset of depreciation analysis. In Example 9.4, the final book value (B_N) conveniently equals the estimated salvage value of \$778, a coincidence that is rather unusual in the real world. When $B_N \neq S$, we would want to make adjustments in our depreciation methods.

- **Case 1: $B_N > S$**

  When $B_N > S$, we are faced with a situation in which we have not depreciated the entire cost of the asset and thus have not taken full advantage of depreciation's tax-deferring benefits. If you would prefer to reduce the book value of an asset to its salvage value as quickly as possible, it can be done by switching from DB to SL whenever SL depreciation results in larger depreciation charges and therefore a more rapid reduction in the book value of the asset. The switch from DB to SL depreciation can take place in any of the n years, the objective being to identify the optimal year to switch. The switching rule is as follows: If depreciation by DB in any year is less than (or equal to) what it would be by SL, we should switch to and remain with the SL method for the duration of the project's depreciable life. The straight-line depreciation in any year n is calculated by

$$D_n = \frac{\text{Book value at beginning of year } n - \text{salvage value}}{\text{Remaining useful life at beginning of year } n} \tag{9.7}$$

EXAMPLE 9.5 Declining Balance with Conversion to Straight-Line Depreciation ($B_N > S$)

Suppose the asset given in Example 9.4 has a zero salvage value instead of \$778; that is,

$$\begin{aligned} \text{Cost basis of the asset, } I &= \$10{,}000 \\ \text{Useful life, } N &= 5 \text{ years,} \\ \text{Salvage value, } S &= \$0, \\ a &= (1/5)(2) = 40\%. \end{aligned}$$

Determine the optimal time to switch from DB to SL depreciation and the resulting depreciation schedule.

SOLUTION

Given: $I = \$10{,}000$, $S = 0$, $N = 5$ years, and $\alpha = 40\%$.
Find: Optimal conversion time, D_n and B_n for $n = 1$ to 5.

We will first proceed by computing the DDB depreciation for each year, as before:

Year	D_n	B_n
1	\$4,000	\$6,000
2	2,400	3,600
3	1,440	2,160
4	864	1,296
5	518	778

Then, using Eq. (9.7), we compute the SL depreciation for each year. We compare SL with DDB depreciation for each year and use the aforementioned decision rule for when to change:

If Switch to SL at Beginning of Year	**SL Depreciation**	**DDB Depreciation**	**Switching Decision**
2	($6,000 − 0)/4 = $1,500	< $2,400	Do not switch
3	(3,600 − 0)/3 = 1,200	< 1,440	Do not switch
$\boxed{4}$	(2,160 − 0)/2 = 1,080	> 864	Switch to SL

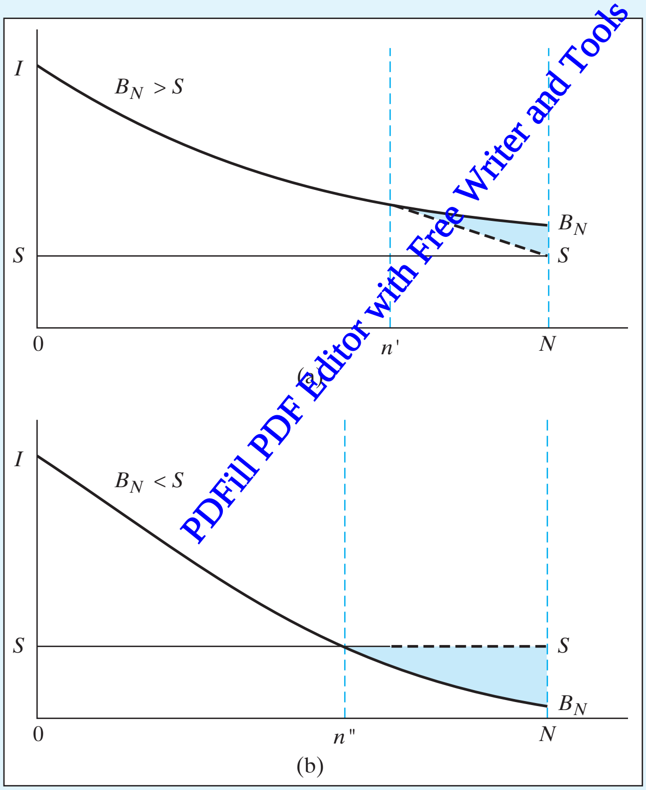


Figure 9.4 Adjustments to the declining-balance method: (a) Switch from declining balance to straight line after n'; (b) no further depreciation allowances are available after n'' (Examples 9.5 and 9.6).

The optimal time (year 4) in this situation corresponds to n' in Figure 9.4(a). The resulting depreciation schedule is

Year	DDB with Switch to SL	End-of-Year Book Value
1	\$ 4,000	\$6,000
2	2,400	3,600
3	1,440	2,160
4	1,080	1,080
5	1,080	0
	\$10,000	

- **Case 2: $B_N < S$**
  With a relatively high salvage value, it is possible that the book value of the asset could decline below the estimated salvage value. When $B_n < S$, we must readjust our analysis because tax law does not permit us to depreciate assets below their salvage value. To avoid deducting depreciation charges that would drop the book value below the salvage value, you simply stop depreciating the asset whenever you get down to $B_n = S$. In other words, if at any period, the implied book value is lower than S, then the depreciation amounts are adjusted so that $B_n = S$.

EXAMPLE 9.6 Declining Balance, $B_N < S$

Compute the double-declining-balance (DDB) depreciation schedule for the data used in Example 9.5, this time with the asset having a salvage value of \$2,000. Then

$$\begin{aligned} \text{Cost basis of the asset, } I &= \$10{,}000, \\ \text{Useful life, } N &= 5 \text{ years}, \\ \text{Salvage value, } S &= \$2{,}000, \\ \alpha &= \left(\tfrac{1}{5}\right)(2) = 40\%. \end{aligned}$$

SOLUTION

Given: $I = \$10{,}000$, $S = \$2{,}000$, $N = 5$ years, and $\alpha = 40\%$.
Find: D_n and B_n for $n = 1$ to 5.

The given data result in the accompanying table.
Note that B_4 would be less than $S = \$2{,}000$ if the full deduction (\$864) had been taken. Therefore, we adjust D_4 to \$160, making $B_4 = \$2{,}000$. D_5 is zero and B_5 remains at \$2,000. Year 4 is equivalent to n'' in Figure 9.4(b).

End of Year	D_n	B_n
1	$0.4(\$10{,}000) = \$4{,}000$	$\$10{,}000 - \$4{,}000 = \$6{,}000$
2	$0.4(6{,}000) = 2{,}400$	$6{,}000 - 2{,}400 = 3{,}600$
3	$0.4(3{,}600) = 1{,}440$	$3{,}600 - 1{,}440 = 2{,}160$
4	$0.4(2{,}160) = 864 > \boxed{160}$	$2{,}160 - 160 = 2{,}000$
5	0	$2{,}000 - 0 = 2{,}000$
	Total $= \$8{,}000$	

9.3.3 Units-of-Production Method

Straight-line depreciation can be defended only if the machine is used for exactly the same amount of time each year. What happens when a punch press machine runs 1,670 hours one year and 780 the next or when some of its output is shifted to a new machining center? This leads us to a consideration of another depreciation method that views the asset as consisting of a bundle of service units. Unlike the SL and accelerated methods, however, this one does not assume that the service units will be consumed in a time-phased pattern. Rather, the cost of each service unit is the net cost of the asset divided by the total number of such units. The depreciation charge for a period is then related to the number of service units consumed in that period. The result is the **units-of-production method**, according to which the depreciation in any year is given by

$$D_n = \frac{\text{Service units consumed during year } n}{\text{Total service units}}(I - S). \tag{9.8}$$

When the units-of-production method is used, depreciation charges are made proportional to the ratio of the actual output to the total expected output. Usually, this ratio is figured in machine hours. The advantages of using the units-of-production method include the fact that depreciation varies with production volume, so the method gives a more accurate picture of machine usage. A disadvantage of the method is that collecting data on machine usage is somewhat tedious, as are the accounting methods.

This method can be useful in depreciating equipment used to exploit natural resources if the resources will be depleted before the equipment wears out. It is not, however, considered a practical method for general use in depreciating industrial equipment.

EXAMPLE 9.7 Units-of-Production Depreciation

A truck for hauling coal has an estimated net cost of \$55,000 and is expected to give service for 250,000 miles, resulting in a \$5,000 salvage value. Compute the allowed depreciation amount for a truck usage of 30,000 miles.

SOLUTION

Given: $I = \$55{,}000$, $S = \$5{,}000$, total service units = 250,000 miles, and usage for this year = 30,000 miles.

Find: Depreciation amount in this year.

The depreciation expense in a year in which the truck traveled 30,000 miles would be

$$\frac{30{,}000 \text{ miles}}{250{,}000 \text{ miles}}(\$55{,}000 - \$5{,}000) = \left(\frac{3}{25}\right)(\$50{,}000)$$
$$= \$6{,}000.$$

9.4 Tax Depreciation Methods

Prior to the Economic Recovery Act of 1981, taxpayers could choose among several methods when depreciating assets for tax purposes. The most widely used methods were the straight-line method and the declining-balance method. The subsequent imposition of the Accelerated Cost Recovery System (ACRS) and the Modified Accelerated Cost Recovery System (MACRS) superseded these methods for use in tax purposes.

MACRS method allows taxpayers to deduct greater amounts during the first few years of an asset's life.

9.4.1 MACRS Depreciation

From 1954 to 1981, congressional changes in tax law evolved fairly consistently toward simpler, more rapid depreciation methods. Prior to 1954, the straight-line method was required for tax purposes, but that year accelerated methods such as double-declining balance and sum-of-years'-digits were permitted. In 1981, these conventional accelerated methods were replaced by the simpler ACRS. In 1986, Congress modified the ACRS and introduced the MACRS, sharply reducing depreciation allowances that were enacted in the Economic Recovery Tax Act of 1981. This section will present some of the primary features of MACRS tax depreciation.

MACRS Recovery Periods

Historically, for both tax and accounting purposes, an asset's depreciable life was determined by its estimated useful life; it was intended that the asset be fully depreciated at approximately the end of its useful life. The MACRS scheme, however, totally abandoned this practice, and simpler guidelines were established that created several classes of assets, each with a more or less arbitrary life called a **recovery period**. (Note: *Recovery periods do not necessarily bear any relationship to expected useful lives*.)

A major effect of the original ACRS method of 1981 was to shorten the depreciable lives of assets, thus giving businesses larger depreciation deductions that, in early years, resulted in lower taxes and increased cash flows available for reinvestment. As shown in Table 9.2, the MACRS method of 1986 reclassified certain assets on the basis of the

TABLE 9.2 **MACRS Property Classifications**

Recovery Period	ADR* Midpoint Class	Applicable Property
3 years	ADR $\leq$ 4	Special tools for the manufacture of plastic products, fabricated metal products, and motor vehicles
5 years	4 $<$ ADR $\leq$ 10	Automobiles,† light trucks, high-tech equipment, equipment used for research and development, computerized telephone switching systems
7 years	10 $<$ ADR $\leq$ 16	Manufacturing equipment, office furniture, fixtures
10 years	16 $<$ ADR $\leq$ 20	Vessels, barges, tugs, railroad cars
15 years	20 $<$ ADR $\leq$ 25	Wastewater plants, telephone-distribution plants, similar utility property
20 years	25 $<$ ADR	Municipal sewers, electrical power plant
$27\frac{1}{2}$ years		Residential rental property
39 years		Nonresidential real property, including elevators and escalators

* ADR = Asset depreciation range; guidelines are published by the IRS.
† Automobiles have a midpoint life of 3 years in the ADR guidelines, but are classified into a 5-year property class.

midpoint of their lives under the ADR system. The MACRS scheme includes eight categories of assets, with lives of 3, 5, 7, 10, 15, 20, 27.5, and 39 years:

MACRS: Depreciation methods applied to assets placed in service after 1986; less favorable than the earlier ACRS system.

- Investments in some short-lived assets are depreciated over 3 years by using 200% DB and then switching to SL depreciation.
- Computers, automobiles, and light trucks are written off over 5 years by using 200% DB and then switching to SL depreciation.
- Most types of manufacturing equipment are depreciated over 7 years, but some long-lived assets are written off over 10 years. Most equipment write-offs are calculated by the 200% DB method, followed by a switch to SL depreciation, which allows faster write-offs in the first few years after an investment is made.
- Sewage-treatment plants and telephone-distribution plants are written off over 15 years by using 150% DB and then switching to SL depreciation.
- Sewer pipes and certain other very long-lived equipment are written off over 20 years by using 150% DB and then switching to SL depreciation.
- Investments in residential rental property are written off in straight-line fashion over $27\frac{1}{2}$ years. Nonresidential real estate (commercial buildings), by contrast, is written off by the SL method over 39 years.

Under the MACRS, *the salvage value of property is always treated as zero*.

9.4.2 MACRS Depreciation Rules

Under earlier depreciation methods, the rate at which the value of an asset declined was estimated and was then used as the basis for tax depreciation. Thus, different assets were depreciated along different paths over time. The MACRS method, however, established prescribed depreciation rates, called **recovery allowance percentages**, for all assets within each class. These rates, as set forth in 1986 and 1993, are shown in Table 9.3. The yearly recovery, or depreciation expense, is determined by multiplying the asset's depreciation base by the applicable recovery allowance percentage.

Half-Year Convention

The MACRS recovery percentages shown in Table 9.3 use the **half-year convention**; that is, it is assumed that all assets are placed in service at midyear and that they will have *zero* salvage value. As a result, only a half year of depreciation is allowed for the first year that property is placed in service. With half of one year's depreciation being taken in the first

TABLE 9.3 **MACRS Depreciation Schedules for Personal Properties with Half-Year Convention, Declining-Balance Method**

	Class	3	5	7	10	15	20
Year	**Depreciation Rate**	**200%**	**200%**	**200%**	**200%**	**150%**	**150%**
1		33.33	20.00	14.29	10.00	5.00	3.750
2		44.45	32.00	24.49	18.00	9.50	7.219
3		14.81*	19.20	17.49	14.40	8.55	6.677
4		7.41	11.52*	12.49	11.52	7.70	6.177
5			11.52	8.93*	9.22	6.93	5.713
6			5.76	8.92	7.37	6.23	5.285
7				8.93	6.55*	5.90*	4.888
8				4.46	6.55	5.90	4.522
9					6.56	5.91	4.462*
10					6.55	5.90	4.461
11					3.28	5.91	4.462
12						5.90	4.461
13						5.91	4.462
14						5.90	4.461
15						5.91	4.462
16						2.95	4.461
17							4.462
18							4.461
19							4.462
20							4.461
21							2.231

* Year to switch from declining balance to straight line. (*Source*: IRS Publication 534. *Depreciation*. Washington, DC: U.S. Government Printing Office, December 2005.)

year, a full year's depreciation is allowed in each of the remaining years of the asset's recovery period, and the remaining half-year's depreciation is taken in the year following the end of the recovery period. A half year of depreciation is also allowed for the year in which property is disposed of, or is otherwise retired from service, anytime before the end of the recovery period.

Switching from DB to the Straight-Line Method

The MACRS asset is depreciated initially by the DB method and then by SL depreciation. Consequently, the MACRS adopts the switching convention illustrated in Example 9.5. To demonstrate how the MACRS depreciation percentages were calculated by the IRS with the use of the half-year convention, consider Example 9.8.

EXAMPLE 9.8 MACRS Depreciation: Personal Property

A taxpayer wants to place in service a \$10,000 asset that is assigned to the five-year class. Compute the MACRS percentages and the depreciation amounts for the asset.

SOLUTION

Given: Five-year asset, half-year convention, $\alpha = 40\%$, and $S = 0$.
Find: MACRS depreciation percentages D_n for \$10,000 asset.

For this problem, we use the following equations:

$$\text{Straight-line rate} = \tfrac{1}{5} = 0.20,$$

$$200\%\text{ declining balance rate} = 2(0.20) = 40\%,$$

$$\text{Under MACRS, salvage } (S) = 0.$$

Then, beginning with the first taxable year and ending with the sixth year, MACRS deduction percentages are computed as follows:

Year	Calculation (%)		MACRS Percentage
1	$\frac{1}{2}$-year DDB depreciation $= 0.5(0.40)(100\%)$	=	$\boxed{20\%}$
2	DDB depreciation $= (0.40)(100\% - 20\%)$	=	$\boxed{32\%}$
	SL depreciation $= (1/4.5)(100\% - 20\%)$	=	17.78%
3	DDB depreciation $= (0.40)(100\% - 52\%)$	=	$\boxed{19.20\%}$
	SL depreciation $= (1/3.5)(100\% - 52\%)$	=	13.71%
4	DDB depreciation $= (0.40)(100\% - 71.20\%)$	=	11.52%
	SL depreciation $= (1/2.5)(100\% - 71.20\%)$	=	$\boxed{11.52\%}$
5	SL depreciation $= (1/1.5)(100\% - 82.72\%)$	=	$\boxed{11.52\%}$
6	$\frac{1}{2}$-year SL depreciation $= (0.5)(11.52\%)$	=	$\boxed{5.76\%}$

In year 2, we check to see what the SL depreciation would be. Since 4.5 years are left to depreciate, SL depreciation $= (1/4.5)(100\% - 20\%) = 17.78\%$. The DDB depreciation is greater than the SL depreciation, so DDB still applies. Note that SL depreciation $\geq$ DDB depreciation in year 4, so we switch to SL then.

We can calculate the depreciation amounts from the percentages we just determined. In practice, the percentages are taken directly from Table 9.3, supplied by the IRS. The results are as follows and are also shown in Figure 9.5.

Year n	MACRS Percentage (%)		Depreciation Basis		Depreciation Amount (D_n)
1	20	$\times$	\$10,000	=	\$2,000
2	32	$\times$	10,000	=	3,200
3	19.20	$\times$	10,000	=	1,920
4	11.52	$\times$	10,000	=	1,152
5	11.52	$\times$	10,000	=	1,152
6	5.76	$\times$	10,000	=	576

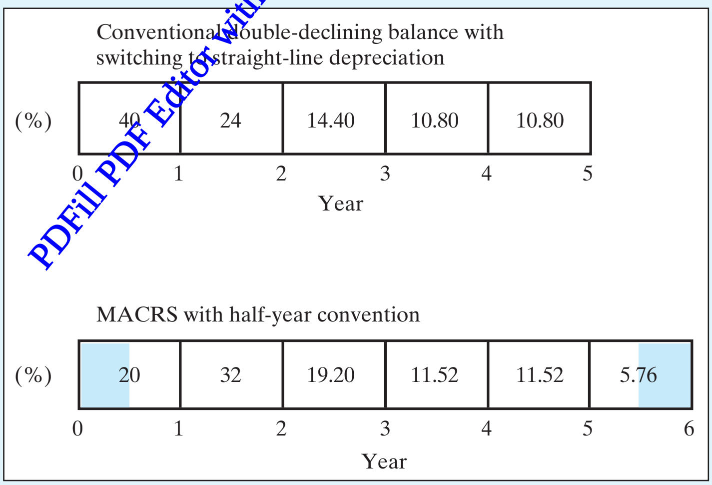


Figure 9.5 MACRS with a five-year recovery period (Example 9.8).

Note that when an asset is disposed of before the end of a recovery period, only half of the normal depreciation is allowed. If, for example, the \$10,000 asset were to be disposed of in year 2, the MACRS deduction for that year would be \$1,600.

9.7 Corporate Taxes

Now that we have learned what elements constitute taxable income, we turn our attention to the process of computing income taxes. The corporate tax rate, is applied to the taxable income of a corporation. As we briefly discussed in Section 8.5, the allowable deductions include the cost of goods sold, salaries and wages, rent, interest, advertising, depreciation, amortization,[3] depletion, and various tax payments other than federal income tax. The following table is illustrative:

Item
Gross income
Expenses:
Cost of goods sold
Depreciation
Operating expenses
Taxable operating income
Income taxes
Net income

9.7.1 Income Taxes on Operating Income

The corporate tax rate structure for 2006 is relatively simple. As shown in Table 9.6, there are four basic rate brackets (15%, 25%, 34%, and 35%), plus two surtax rates (5% and 3%), based on taxable incomes. U.S. tax rates are progressive; that is, businesses with lower taxable incomes are taxed at lower rates than those with higher taxable incomes.

TABLE 9.6 **Corporate Tax Schedule for 2006**

Taxable Income (X)	**Tax Rate**	**Tax Computation Formula**
\$0–\$50,000	15%	\$0 + 0.15X
50,001–75,000	25%	7,500 + 0.25(X − \$50,000)
75,001–100,000	34%	13,750 + 0.34(X − 75,000)
100,001–335,000	34% + 5%	22,250 + 0.39(X − 100,000)
335,001–10,000,000	34%	113,900 + 0.34(X − 335,000)
10,000,001–15,000,000	35%	3,400,000 + 0.35(X − 10,000,000)
15,000,001–18,333,333	35% + 3%	5,150,000 + 0.38(X − 15,000,000)
18,333,334 and up	35%	6,416,666 + 0.35(X − 18,333,333)

[3] The **amortization expense** is a special form of depreciation for an intangible asset, such as patents, goodwill, and franchises. More precisely, the amortization expense is the systematic write-off to expenses of the cost of an intangible asset over the periods of its economic usefulness. Normally a straight-line method is used to calculate the amortization expense.

Marginal Tax Rate

Marginal tax rate: The amount of tax paid on an additional dollar of income.

The **marginal tax rate** is defined as the rate applied to the last dollar of income earned. Income of up to \$50,000 is taxed at a 15% rate (meaning that if your taxable income is less than \$50,000, your marginal tax rate is 15%); income between \$50,000 and \$75,000 is taxed at 25%; and income over \$75,000 is taxed at a 34% rate.

An additional 5% surtax (resulting in 39%) is imposed on a corporation's taxable income in excess of \$100,000, with the maximum additional tax limited to \$11,750 ($235,000 \times 0.05$). This surtax provision phases out the benefit of graduated rates for corporations with taxable incomes between \$100,000 and \$335,000. Another 3% surtax is imposed on corporate taxable income in the range from \$15,000,001 to \$18,333,333.

Corporations with incomes in excess of \$18,333,333 in effect pay a flat tax of 35%. As shown in Table 9.6, the corporate tax is progressive up to \$18,333,333 in taxable income, but essentially is constant thereafter.

Effective (Average) Tax Rate

Effective (average) tax rate: The rate a taxpayer would be taxed at if taxing was done at a constant rate, instead of progressively.

Effective tax rates can be calculated from the data in Table 9.6. For example, if your corporation had a taxable income of \$16,000,000 in 2006, then the income tax owed by the corporation would be as follows:

Taxable Income	Tax Rate	Taxes	Cumulative Taxes
First \$50,000	15%	\$7,500	\$7,500
Next \$25,000	25%	6,250	13,750
Next \$25,000	34%	8,500	22,250
Next \$235,000	39%	91,650	113,900
Next \$9,665,000	34%	3,286,100	3,400,000
Next \$5,000,000	35%	1,750,000	5,150,000
Remaining \$1,000,000	38%	380,000	\$5,530,000

Alternatively, using the tax formulas in Table 9.6, we obtain

$$\$5{,}150{,}000 + 0.38(\$16{,}000{,}000 - \$15{,}000{,}000) = \$5{,}530{,}000.$$

The effective (average) tax rate would then be

$$\frac{\$5{,}530{,}000}{\$16{,}000{,}000} = 0.3456, \text{ or } 34.56,$$

as opposed to the marginal rate of 38%. In other words, on the average, the company paid 34.56 cents for each taxable dollar it generated during the accounting period.

EXAMPLE 9.13 Corporate Taxes

A mail-order computer company sells personal computers and peripherals. The company leased showroom space and a warehouse for \$20,000 a year and installed \$290,000 worth of inventory-checking and packaging equipment. The allowed depreciation expense for this capital expenditure (\$290,000) amounted to \$58,000 using the category of 5-year MACRS. The store was completed and operations began on January 1. The company had a gross income of \$1,250,000 for the calendar year. Supplies and all operating expenses, other than the lease expense, were itemized as follows:

Merchandise sold in the year	\$600,000
Employee salaries and benefits	150,000
Other supplies and expenses	90,000
	\$840,000

Compute the taxable income for this company. How much will the company pay in federal income taxes for the year?

SOLUTION

Given: Income, preceding cost information and depreciation.
Find: Taxable income and federal income taxes.

First we compute the taxable income as follows:

Gross revenues	\$1,250,000
Expenses	–840,000
Lease expense	–20,000
Depreciation	–58,000
Taxable income	\$332,000

Note that capital expenditures are not deductible expenses. Since the company is in the 39% marginal tax bracket, its income tax can be calculated by using the formula given in Table 9.6, namely, $22{,}250 + 0.39\,(X - 100{,}000)$:

$$\begin{aligned}\text{Income tax} &= \$22{,}250 + 0.39(\$332{,}000 - \$100{,}000)\\ &= \$112{,}730.\end{aligned}$$

The firm's current marginal tax rate is 39%, but its average corporate tax rate is

$$\frac{\$112{,}730}{\$332{,}000} = 33.95\%.$$

$$\begin{aligned}\text{Net proceeds from sale} &= \$100{,}000 + \$7{,}036\\ &= \$107{,}036.\end{aligned}$$

(d) Case 4: Salvage value > Cost basis

This situation is not likely for most depreciable assets (except real property). But it is the only situation in which both capital gains and ordinary gains can be observed. Nevertheless, the tax treatment on this gain is as follows:

$$\begin{aligned}\text{Capital gains} &= \text{Salvage value} - \text{cost basis}\\ &= \$250{,}000 - \$230{,}000\\ &= \$20{,}000,\end{aligned}$$

$$\begin{aligned}\text{Capital gains tax} &= \$20{,}000(0.34)\\ &= \$6{,}800,\end{aligned}$$

$$\begin{aligned}\text{Ordinary gains} &= \$230{,}000 - \$120{,}693\\ &= \$109{,}307,\end{aligned}$$

$$\begin{aligned}\text{Gains tax} &= \$109{,}307(0.34)\\ &= \$37{,}164,\end{aligned}$$

$$\begin{aligned}\text{Net proceeds from sale} &= \$250{,}000 - (\$6{,}800 + \$37{,}164)\\ &= \$206{,}036.\end{aligned}$$

COMMENTS: Note that in (c) the reduction in tax, due to the loss, actually increases the net proceeds. This is realistic when the incremental tax rate (34% in this case) is positive, indicating the corporation is still paying tax, but less than if the asset had not been sold at a loss. The incremental tax rate will be discussed in Section 9.9.

9.9 Income Tax Rate to Be Used in Economic Analysis

As we have seen in earlier sections, average income tax rates for corporations vary with the level of taxable income from 0 to 35%. Suppose that a company now paying a tax rate of 25% on its current operating income is considering a profitable investment. What tax rate should be used in calculating the taxes on the investment's projected income?

9.9.1 Incremental Income Tax Rate

The choice of a corporation's rate depends on the incremental effect of an investment on the company's taxable income. In other words, the tax rate to use is the rate that applies to the additional taxable income projected in the economic analysis.

To illustrate, consider ABC Corporation, whose taxable income from operations is expected to be $70,000 for the current tax year. ABC management wishes to evaluate the incremental tax impact of undertaking a project during the same tax year. The revenues, expenses, and taxable incomes before and after the project are estimated as follows:

	Before	After	Incremental
Gross revenue	$200,000	$240,000	$40,000
Salaries	100,000	110,000	10,000
Wages	30,000	40,000	10,000
Taxable income	$ 70,000	$ 90,000	$20,000

Because the income tax rate is progressive, the tax effect of the project cannot be isolated from the company's overall tax obligations. The base operations of ABC without the project are expected to yield a taxable income of $70,000. With the new project, the taxable income increases to $90,000. From the tax computation formula in Table 9.6, the corporate income taxes with and without the project are as follows:

$$\begin{aligned}\text{Income tax without the project} &= \$7{,}500 + 0.25(\$70{,}000 - \$50{,}000)\\ &= \$12{,}500,\\ \text{Income tax with the project} &= \$13{,}750 + 0.34(\$90{,}000 - \$75{,}000)\\ &= \$18{,}850.\end{aligned}$$

The additional income tax is then $18,850 − $12,500 = $6,350. This amount, on the additional $20,000 of taxable income, is based on a rate of 31.75%, which is an incremental rate. This is the rate we should use in evaluating the project in isolation from the rest of ABC's operations. As shown in Figure 9.11, the 31.75% is not an arbitrary figure, but a weighted average of two distinct marginal rates. Because the new project pushes ABC into a higher tax bracket, the first $5,000 it generates is taxed at 25%; the remaining $15,000 it generates is taxed in the higher bracket, at 34%. Thus, we could have calculated the incremental tax rate with the formula

$$0.25\left(\frac{\$5{,}000}{\$20{,}000}\right) + 0.34\left(\frac{\$15{,}000}{\$20{,}000}\right) = 31.75\%.$$

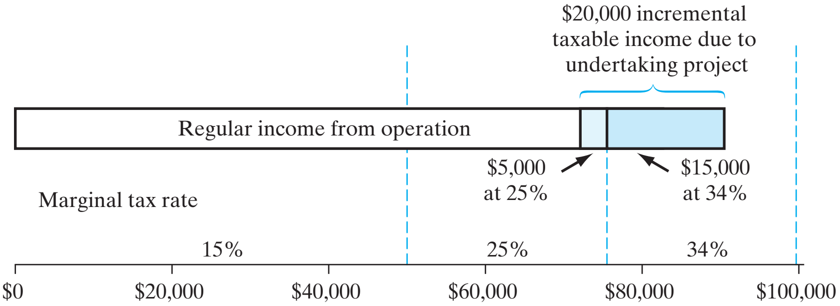


Figure 9.11 Illustration of incremental tax rate.

The average tax rates before and after the new project being considered is undertaken are as follows:

	Before	After	Incremental
Taxable income	$70,000	$90,000	$20,000
Income taxes	12,500	18,850	6,350
Average tax rate	17.86%	20.94%	
Incremental tax rate			31.75%

Note that neither 17.86% nor 20.94% is a correct tax rate to use in evaluating the new project.

A corporation with continuing base operations that place it consistently in the highest tax bracket will have both marginal and average federal tax rates of 35%. For such firms, the tax rate on an additional investment project is, naturally, 35%. But for corporations in lower tax brackets, and for those which fluctuate between losses and profits, the marginal and average tax rates are likely to vary. For such corporations, estimating a prospective incremental tax rate for a new investment project may be difficult. The only solution may be to perform scenario analysis, in which we examine how much the income tax fluctuates due to undertaking the project. (In other words, we calculate the total taxes and the incremental taxes for each scenario.) A typical scenario is presented in Example 9.16.

EXAMPLE 9.16 Scenario Analysis for a Small Company

EverGreen Grass Company expects to have an annual taxable income of $320,000 from its regular grass-sodding business over the next two years. EverGreen has just won a contract to sod grasses for a new golf complex for just those years. This two-year project requires a purchase of new equipment costing $50,000. The equipment falls into the MACRS five-year class, with depreciation allowances of 20%, 32%, 19.2%, 11.52%, 11.52%, and 5.76% in each of the six years, respectively, during which the equipment will be depreciated. After the contract is terminated, the equipment will be retained for future use (instead of being sold), indicating no salvage cash flow, gain, or loss on this asset. The project will bring in an additional annual revenue of $150,000, but it is expected to incur additional annual operating costs of $90,000. Compute the incremental (marginal) tax rates applicable to the project's operating profits for years 1 and 2.

SOLUTION

Given: Base taxable income = $320,000 per year, incremental income, expenses, and depreciation amounts as stated.

Find: Incremental tax rate for this new project in years 1 and 2.

First, we compute the additional taxable income from the golf course project over the next two years:

Year	1	2
Gross Revenue	$150,000	$150,000
Expenses	90,000	90,000
Depreciation	10,000	16,000
Taxable income	$ 50,000	$ 44,000

Next, we compute the income taxes. To do this, we need to determine the applicable marginal tax rate, but because of the progressive income tax rate on corporations, the project cannot be isolated from the other operations of EverGreen.

We can solve this problem much more efficiently by using the incremental tax rate concept discussed at the beginning of this section. Because the golf course project pushes EverGreen into the 34% tax bracket from the 39% bracket, we want to know what proportion of the incremental taxable income of $50,000 in year 1 is taxed at 39% and what proportion is taxable at 34%. Without the project, the firm's taxable income is $320,000 and its marginal tax rate is 39%. With the additional taxable income of $50,000, EverGreen's tax bracket reverts to 34%, as its combined taxable income changes from $320,000 to $370,000. Since the rate changes at $335,000, the first $15,000 of the $50,000 taxable income will still be in the 39% bracket, and the remaining $35,000 will be in the 34% bracket. In year 2, we can divide the additional taxable income of $44,000 in a similar fashion. Then we can calculate the incremental tax rates for the first two years as follows:

$$0.39\left(\frac{\$15,000}{\$50,000}\right) + 0.34\left(\frac{\$35,000}{\$50,000}\right) = 0.3550,$$

$$0.39\left(\frac{\$15,000}{\$44,000}\right) + 0.34\left(\frac{\$29,000}{\$44,000}\right) = 0.3570.$$

Note that these incremental tax rates vary slightly from year to year. Much larger changes could occur if a company's taxable income fluctuates drastically from its continuing base operation.

9.9.2 Consideration of State Income Taxes

For large corporations, the top federal marginal tax rate is 35%. In addition to federal income taxes, state income taxes are levied on corporations in most states. State income taxes are an allowable deduction in computing federal taxable income, and two ways are available to consider explicitly the effects of state income taxes in an economic analysis.

The first approach is to estimate explicitly the amount of state income taxes before calculating the federal taxable income. We then reduce the federal taxable income by the amount of the state taxes and apply the marginal tax rate to the resulting federal taxes. The total taxes would be the sum of the state taxes and the federal taxes.

The second approach is to calculate a single tax rate that reflects both state and federal income taxes. This single rate is then applied to the federal taxable income, without subtracting state income taxes. Taxes computed in this fashion represent total taxes. If state income taxes are considered, the combined state and federal marginal tax rate may be higher than 35%. Since state income taxes are deductible as expenses in determining federal taxes, the marginal rate for combined federal and state taxes can be calculated with the expression

$$t_m = t_f + t_s - (t_f)(t_s), \tag{9.11}$$

where

t_m = combined marginal tax rate,
t_f = federal marginal tax rate.
t_s = state marginal tax rate.

This second approach provides a more convenient and efficient way to handle taxes in an economic analysis in which the incremental tax rates are known. Therefore, incremental tax rates will be stated as combined marginal tax rates, unless indicated otherwise. (For large corporations, these would be about 40%, but they vary from state to state.)

EXAMPLE 9.17 Combined State and Federal Income Taxes

Consider a corporation whose revenues and expenses before income taxes are as follows:

Gross revenue	$1,000,000
All expenses	400,000

If the marginal federal tax rate is 35% and the marginal state rate is 7%, compute the combined state and federal taxes, using the two methods just described.

SOLUTION

Given: Gross income = $1,000,000, deductible expenses = $400,000, t_f = 35%, and t_s = 7%.

Find: Combined income taxes t_m.

(a) Explicit calculation of state income taxes:

Let's define FT as federal taxes and ST as state taxes. Then

$$\text{State taxable income} = \$1{,}000{,}000 - \$400{,}000$$

and

$$\begin{aligned}\text{ST} &= (0.07)(\$600{,}000)\\ &= \$42{,}000.\end{aligned}$$

Also,

$$\begin{aligned}\text{Federal taxable income} &= \$1{,}000{,}000 - \$400{,}000 - \text{ST}\\ &= (\$558{,}000),\end{aligned}$$

so that

$$\begin{aligned} \text{FT} &= (0.35)(\$558{,}000) \\ &= \$195{,}300. \end{aligned}$$

Thus,

$$\begin{aligned} \text{Combined taxes} &= \text{FT} + \text{ST} \\ &= \$237{,}300. \end{aligned}$$

(b) Tax calculation based on the combined tax rate:

Compute the combined tax rate directly from the formula:

$$\begin{aligned} \text{Combined tax rate } (t_m) &= 0.35 + 0.07 - (0.35)(0.07) \\ &= 39.55\%. \end{aligned}$$

Hence,

$$\begin{aligned} \text{Combined taxes} &= \$600{,}000(0.3955) \\ &= \$237{,}300. \end{aligned}$$

As expected, these two methods always produce exactly the same results.

9.10 The Need for Cash Flow in Engineering Economic Analysis

Traditional accounting stresses net income as a means of measuring a firm's profitability, but it is desirable to discuss why cash flows are relevant in project evaluation. As seen in Section 8.2, net income is an accounting measure based, in part, on the **matching concept**. Costs become expenses as they are matched against revenue. The actual timing of cash inflows and outflows is ignored.

9.10.1 Net Income versus Net Cash Flow

Over the life of a firm, net incomes and net cash inflows will usually be the same. However, the timing of incomes and cash inflows can differ substantially. Given the time value of money, it is better to receive cash now rather than later, because cash can be invested to earn more cash. (You cannot invest net income.) For example, consider two firms and their income and cash flow schedules over two years:

		Company A	Company B
Year 1	Net income	$1,000,000	$1,000,000
	Cash flow	1,000,000	0
Year 2	Net income	1,000,000	1,000,000
	Cash flow	1,000,000	2,000,000

Both companies have the same amount of net income and cash over two years, but Company A returns $1 million cash yearly, while Company B returns $2 million at the end of the second year. If you received $1 million at the end of the first year from Company A, you could, for example, invest it at 10%. While you would receive only $2 million in total from Company B at the end of the second year, you would receive $2.1 million in total from Company A.

9.10.2 Treatment of Noncash Expenses

Apart from the concept of the time value of money, certain expenses do not even require a cash outflow. Depreciation and amortization are the best examples of this type of expense. Even though depreciation (or amortization expense) is deducted from revenue on a daily basis, no cash is paid to anyone.

In Example 9.13, we learned that the annual depreciation allowance has an important impact on both taxable and net income. However, although depreciation has a direct impact on net income, it is *not* a cash outlay; hence, it is important to distinguish between annual income in the presence of depreciation and annual cash flow.

The situation described in Example 9.13 serves as a good vehicle to demonstrate the difference between depreciation costs as expenses and the cash flow generated by the purchase of a fixed asset. In that example, cash in the amount of $290,000 was expended in year 0, but the $58,000 depreciation charged against the income in year 1 was not a cash outlay. Figure 9.12 summarizes the difference.

Net income (**accounting profit**) is important for accounting purposes, but **cash flows** are more important for project evaluation purposes. However, as we will now demonstrate, net income can provide us with a starting point to estimate the cash flow of a project.

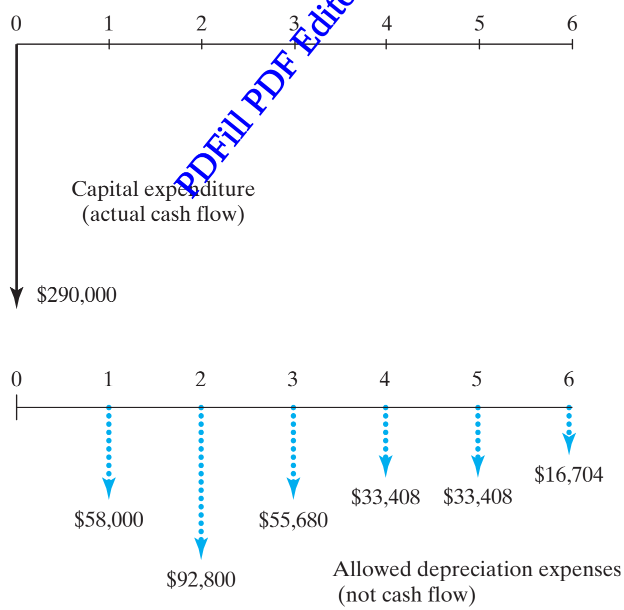


Figure 9.12 Cash flows versus depreciation expenses for an asset with a cost basis of $290,000, which was placed in service in year 0.

The procedure for calculating net cash flow (after tax) is identical to that used to obtain net income from operations, with the exception of depreciation, which is excluded from the net cash flow computation. (It is needed only for computing income taxes.) Assuming that revenues are received and expenses are paid in cash, we can obtain the net cash flow by adding the **noncash expense** (depreciation) to net income, which cancels the operation of subtracting it from revenues:

Item
Gross income
Expenses:
Cost of goods sold
Depreciation
Operating expenses
Taxable operating income
Income taxes
Net income
Net cash flow = Net income + Depreciation

Example 9.18 illustrates this relationship.

EXAMPLE 9.18 Cash Flow versus Net Income

A company buys a numerically controlled (NC) machine for \$28,000 (year 0) and uses it for five years, after which it is scrapped. The allowed depreciation deduction during the first year is \$4,000, as the equipment falls into the category of seven-year MACRS property. (The first-year depreciation rate is 14.29%.) The cost of the goods produced by this NC machine should include a charge for the depreciation of the machine. Suppose the company estimates the following revenues and expenses, including depreciation, for the first operating year:

$$\begin{aligned}\text{Gross income} &= \$50{,}000,\\ \text{Cost of goods sold} &= \$20{,}000,\\ \text{Depreciation on NC machine} &= \$4{,}000,\\ \text{Operating expenses} &= \$6{,}000.\end{aligned}$$

(a) If the company pays taxes at the rate of 40% on its taxable income, what is its net income from the project during the first year?

(b) Assume that (1) all sales are cash sales and (2) all expenses except depreciation were paid during year 1. How much cash would be generated from operations?

SOLUTION

Given: Net-income components.

Find: Cash flow.

We can generate a cash flow statement simply by examining each item in the income statement and determining which items actually represent receipts or disbursements (some of the assumptions listed in the statement of the problem make this process simpler):

Item	Income	Cash Flow
Gross income (revenues)	\$50,000	\$50,000
Expenses:		
Cost of goods sold	20,000	−20,000
Depreciation	4,000	
Operating expenses	6,000	−6,000
Taxable income	20,000	
Taxes (40%)	8,000	−8,000
Net income	\$12,000	
Net cash flow		\$16,000

Column 2 shows the income statement, while Column 3 shows the statement on a cash flow basis. The sales of \$50,000 are all cash sales. Costs other than depreciation were \$26,000; these were paid in cash, leaving \$24,000. Depreciation is not a cash flow: The firm did not pay out \$4,000 in depreciation expenses. Taxes, however, are paid in cash, so the \$8,000 for taxes must be deducted from the \$24,000, leaving a net cash flow from operations of \$16,000. As shown in Figure 9.13, this \$16,000 is exactly equal to net income plus depreciation: $\$12,000 + \$4,000 = \$16,000$.

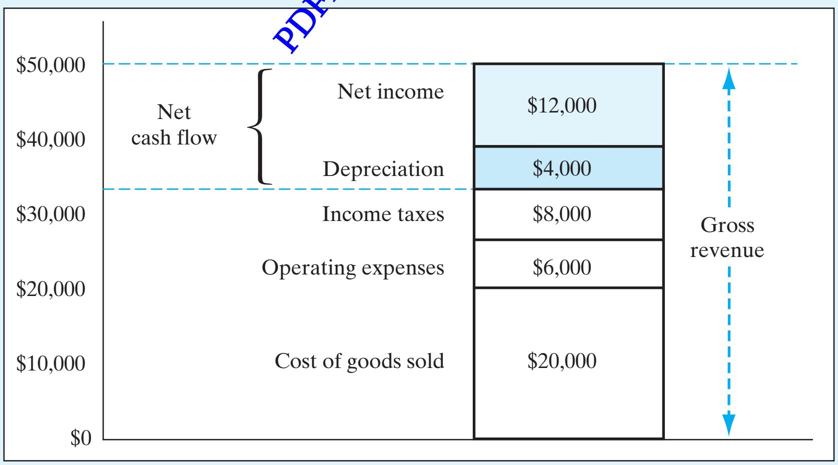


Figure 9.13 Net income versus net cash flow (Example 9.18). Cash flow versus depreciation expenses for an asset with a cost basis of \$28,000, which was placed in service in year 0.

As we've just seen, depreciation has an important impact on annual cash flow in its role as an accounting expense that reduces taxable income and thus taxes. (Although depreciation expenses are not actual cash flows, depreciation has a positive impact on the after-tax cash flow of the firm.) Of course, during the year in which an asset is actually acquired, the cash disbursed to purchase it creates a significant negative cash flow, and during the depreciable life of the asset, the depreciation charges will affect the taxes paid and, therefore, cash flows.

As shown in Example 9.18, through its influence on taxes, depreciation plays a critical role in project cash flow analysis, which we will explore further in Chapter 10.

SUMMARY

- Machine tools and other manufacturing equipment, and even the factory buildings themselves, are subject to wear over time. However, it is not always obvious how to account for the cost of their replacement. Clearly, the choice of estimated service life for a machine, and the method used to calculate the cost of operating it, can have significant effects on an asset's management.
- The entire cost of replacing a machine cannot be properly charged to any one year's production; rather, the cost should be spread (or capitalized) over the years in which the machine is in service. The cost charged to operations during a particular year is called **depreciation**. Several different meanings and applications of depreciation have been presented in this chapter. From an engineering economics point of view, our primary concern is with **accounting depreciation**: the systematic allocation of an asset's value over its depreciable life.
- Accounting depreciation can be broken into two categories:
  1. **Book depreciation**—the method of depreciation used in financial reports and for pricing products;
  2. **Tax depreciation**—the method of depreciation used for calculating taxable income and income taxes; this method is governed by tax legislation.
- The four components of information required to calculate depreciation are as follows:
  1. The cost basis of the asset.
  2. The salvage value of the asset.
  3. The depreciable life of the asset.
  4. The method of its depreciation.

  Table 9.7 summarizes the differences in the way these components are treated for purposes of book and tax depreciation.
- Because it employs accelerated methods of depreciation and shorter-than-actual depreciable lives, the **Modified Accelerated Cost Recovery System (MACRS)** gives taxpayers a break, allowing them to take earlier and faster advantage of the tax-deferring benefits of depreciation.
- Many firms select straight-line depreciation for book depreciation because of its relative ease of calculation.

TABLE 9.7 **Summary of Book versus Tax Depreciation**

Component of Depreciation	Book Depreciation	Tax Depreciation (MACRS)
Cost basis	Based on the actual cost of the asset, plus all incidental costs, such as the cost of freight, site preparation, installation, etc.	Same as for book depreciation.
Salvage value	Estimated at the outset of depreciation analysis. If the final book value does not equal the estimated salvage value, we may need to make adjustments in our depreciation calculations.	Salvage value is zero for all depreciable assets.
Depreciable life	Firms may select their own estimated useful lives or follow government guidelines for asset depreciation ranges (ADRs).	Eight recovery periods—3, 5, 7, 10, 15, 20, 27.5, and 39 years—have been established; all depreciable assets fall into one of these eight categories.
Method of depreciation	Firms may select from the following: • straight line • accelerated methods (declining balance, double-declining balance, and sum-of-years'-digits) • units of production	Exact depreciation percentages are mandated by tax legislation, but are based largely on DDB and straight-line methods. The sum-of-years'-digits method is rarely used in the United States, except for some cost analysis in engineering valuation.

- Depletion is a cost allocation method used particularly for natural resources. **Cost depletion** is based on the units-of-production method of depreciation. **Percentage depletion** is based on a prescribed percentage of the gross income of a property during a tax year.
- Given the frequently changing nature of depreciation and tax law, we must use whatever percentages, depreciable lives, and salvage values are in effect *at the time an asset is acquired*.
- Explicit consideration of taxes is a necessary aspect of any complete economic study of an investment project.
- For corporations, the U.S. tax system has the following characteristics:
  1. Tax rates are progressive: The more you earn, the more you pay.
  2. Tax rates increase in stair-step fashion: four brackets for corporations and two additional surtax brackets, giving a total of six brackets.
  3. Allowable exemptions and deductions may reduce the overall tax assessment.

- Three distinct terms to describe taxes were used in this chapter: **marginal tax rate**, which is the rate applied to the last dollar of income earned; **effective (average) tax rate**, which is the ratio of income tax paid to net income; and **incremental tax rate**, which is the average rate applied to the incremental income generated by a new investment project.
- **Capital gains** are currently taxed as ordinary income, and the maximum rate is capped at 35%. **Capital losses** are deducted from capital gains; net remaining losses may be carried backward and forward for consideration in years other than the current tax year.
- Since we are interested primarily in the measurable financial aspects of depreciation, we consider its effects on two important measures of an organization's financial position: **net income** and **cash flow**. Once we understand that depreciation has a significant influence on the income and cash position of a firm, we will be able to appreciate fully the importance of utilizing depreciation as a means of maximizing the value both of engineering projects and of the organization as a whole.

PROBLEMS

Note: *Unless otherwise specified, use current tax rates for corporate taxes. Check the website (described in the preface) for the most current tax rates for corporations.*

Economic Depreciation

9.1 A machine now in use was purchased four years ago at a cost of $20,000. It has a book value of $6,246. It can be sold for $7,000, but could be used for three more years, at the end of which time it would have no salvage value. What is the current amount of economic depreciation for this asset?

Cost Basis

9.2 General Service Contractor Company paid $200,000 for a house and lot. The value of the land was appraised at $65,000 and the value of the house at $135,000. The house was then torn down at an additional cost of $5,000 so that a warehouse could be built on the lot at a cost of $250,000. What is the total value of the property with the warehouse? For depreciation purposes, what is the cost basis for the warehouse?

9.3 To automate one of its production processes, Milwaukee Corporation bought three flexible manufacturing cells at a price of $500,000 each. When they were delivered, Milwaukee paid freight charges of $25,000 and handling fees of $12,000. Site preparation for these cells cost $35,000. Six foremen, each earning $15 an hour, worked five 40-hour weeks to set up and test the manufacturing cells. Special wiring and other materials applicable to the new manufacturing cells cost $1,500. Determine the cost basis (amount to be capitalized) for these cells.

9.4 A new drill press was purchased for $126,000 by trading in a similar machine that had a book value of $39,000. Assuming that the trade-in allowance is $40,000 and that $86,000 cash is to be paid for the new asset, what is the cost basis of the new asset for depreciation purposes?

9.5 A lift truck priced at $35,000 is acquired by trading in a similar lift truck and paying cash for the remaining balance. Assuming that the trade-in allowance is $10,000 and the book value of the asset traded in is $6,000, what is the cost basis of the new asset for the computation of depreciation for tax purposes?

Book Depreciation Methods

9.6 Consider the following data on an asset:

Cost of the asset, I	\$132,000
Useful life, N	5 years
Salvage value, S	\$ 20,000

Compute the annual depreciation allowances and the resulting book values, using
(a) The straight-line depreciation method.
(b) The double-declining-balance method.

9.7 A firm is trying to decide whether to keep an item of construction equipment for another year. The firm is using DDB for book purposes, and this is the fourth year of ownership of the equipment, which cost \$150,000 new. What is the depreciation in year 3?

9.8 Consider the following data on an asset:

Cost of the asset, I	\$50,000
Useful life, N	7 years
Salvage value, S	\$0

Compute the annual depreciation allowances and the resulting book values, using the DDB and switching to SL.

9.9 The double-declining-balance method is to be used for an asset with a cost of \$68,000, an estimated salvage value of \$12,000, and an estimated useful life of six years.
(a) What is the depreciation for the first three fiscal years, assuming that the asset was placed in service at the beginning of the year?
(b) If switching to the straight-line method is allowed, when is the optimal time to switch?

9.10 Compute the double-declining-balance (DDB) depreciation schedule for the following asset:

Cost of the asset, I	\$76,000
Useful life, N	8 years
Salvage value, S	\$ 6,000

9.11 Compute the DDB depreciation schedule for the following asset:

Cost of the asset, I	\$46,000
Useful life, N	5 years
Salvage value, S	\$10,000

(a) What is the value of α?
(b) What is the amount of depreciation for the second full year of use of the asset?
(c) What is the book value of the asset at the end of the fourth year?

9.12 Upjohn Company purchased new packaging equipment with an estimated useful life of five years. The cost of the equipment was \$35,000, and the salvage value was estimated to be \$5,000 at the end of year 5. Compute the annual depreciation expenses over the five-year life of the equipment under each of the following methods of book depreciation:

(a) Straight-line method.

(b) Double-declining-balance method. (Limit the depreciation expense in the fifth year to an amount that will cause the book value of the equipment at year-end to equal the \$5,000 estimated salvage value.)

(c) Sum-of-years'-digits method.

9.13 A secondhand bulldozer acquired at the beginning of the fiscal year at a cost of \$68,000 has an estimated salvage value of \$9,500 and an estimated useful life of 12 years. Determine

(a) The amount of annual depreciation by the straight-line method.

(b) The amount of depreciation for the third year, computed by the double-declining-balance method.

(c) The amount of depreciation for the second year, computed by the sum-of-years'-digits method.

Units-of-Production Method

9.14 If a truck for hauling coal has an estimated net cost of \$100,000 and is expected to give service for 250,000 miles, resulting in a salvage value of \$5,000, depreciation would be charged at a rate of 38 cents per mile. Compute the allowed depreciation amount for the same truck's usage amounting to 55,000 miles.

9.15 A diesel-powered generator with a cost of \$65,000 is expected to have a useful operating life of 50,000 hours. The expected salvage value of this generator is \$7,500. In its first operating year, the generator was operated for 5,000 hours. Determine the depreciation for the year.

Tax Depreciation

9.16 Zerex Paving Company purchased a hauling truck on January 1, 2005, at a cost of \$32,000. The truck has a useful life of eight years with an estimated salvage value of \$5,000. The straight-line method is used for book purposes. For tax purposes, the truck would be depreciated with the MACRS method over its five-year class life. Determine the annual depreciation amount to be taken over the useful life of the hauling truck for both book and tax purposes.

9.17 The Harris Foundry Company purchased new casting equipment in 2006 at a cost of \$220,000. Harris also paid \$35,000 to have the equipment delivered and installed. The casting machine has an estimated useful life of 12 years, but it will be depreciated with MACRS over its seven-year class life.

(a) What is the cost basis of the casting equipment?

(b) What will be the depreciation allowance in each year of the seven-year class life of the casting equipment?

9.18 A machine is classified as seven-year MACRS property. Compute the book value for tax purposes at the end of three years. The cost basis is \$145,000.

9.19 A piece of machinery purchased at a cost of $86,000 has an estimated salvage value of $12,000 and an estimated useful life of five years. It was placed in service on May 1 of the current fiscal year, which ends on December 31. The asset falls into a seven-year MACRS property category. Determine the depreciation amounts over the useful life.

9.20 Suppose that a taxpayer places in service a $20,000 asset that is assigned to the six-year class (say, a new property class) with a half-year convention. Develop the MACRS deductions, assuming a 200% declining-balance rate followed by switching to straight line.

9.21 On April 1, Leo Smith paid $250,000 for a residential rental property. This purchase price represents $200,000 for the building and $50,000 for the land. Five years later, on November 1, he sold the property for $300,000. Compute the MACRS depreciation for each of the five calendar years during which he had the property.

9.22 In 2006, you purchased a spindle machine (seven-year MACRS property) for $26,000, which you placed in service in January. Use the calendar year as your tax year. Compute the depreciation allowances.

9.23 On October 1, you purchased a residential home in which to locate your professional office for $250,000. The appraisal is divided into $80,000 for the land and $170,000 for the building.

(a) In your first year of ownership, how much can you deduct for depreciation for tax purposes?

(b) Suppose that the property was sold at $325,000 at the end of fourth year of ownership. What is the book value of the property?

9.24 For each of four assets in the following table, determine the missing amounts (for asset type III, the annual usage is 15,000 miles):

Types of Asset	I	II	III	IV
Depreciating Methods	SL	DDB	UP	MACRS
End of year	7	4	3	4
Initial cost ($)	10,000	18,000	30,000	8,000
Salvage value ($)	2,000	2,000	0	1,000
Book value ($)	3,000	2,320	□	1,382
Depreciable life	8 yr	5 yr	90,000 mi	□
Depreciable Amount ($)	□	□	□	□
Accumulated Depreciable ($)	□	15,680	□	□

9.33 On January 2, 2004, Hines Food Processing Company purchased a machine that dispenses a premeasured amount of tomato juice into a can. The machine cost $75,000, and its useful life was estimated at 12 years, with a salvage value of $4,500. At the time it purchased the machine, Hines incurred the following additional expenses:

Freight-in	$800
Installation cost	2,500
Testing costs prior to regular operation	1,200

Book depreciation was calculated by the straight-line method, but for tax purposes, the machine was classified as a 7-year MACRS property. In January 2006, accessories costing $5,000 were added to the machine to reduce its operating costs. These accessories neither prolonged the machine's life nor provided any additional salvage value.

(a) Calculate the book depreciation expense for 2007.

(b) Calculate the tax depreciation expense for 2007.

Corporate Tax Systems

9.34 In tax year 1, an electronics packaging firm had a gross income of $25,000,000, 5,000,000 in salaries, $4,000,000 in wages, $800,000 in depreciation expenses, a loan principal payment of $200,000, and a loan interest payment of $210,000. Determine the net income of the company in tax year 1.

9.35 A consumer electronics company was formed to develop cellphones that run on or are recharged by fuel cells. The company purchased a warehouse and converted it into a manufacturing plant for $6,000,000. It completed installation of assembly equipment worth $1,500,000 on December 31. The plant began operation on January 1. The company had a gross income of $8,500,000 for the calendar year. Manufacturing costs and all operating expenses, excluding the capital expenditures, were $2,280,000. The depreciation expenses for capital expenditures amounted to $456,000.

(a) Compute the taxable income of this company.

(b) How much will the company pay in federal income taxes for the year?

9.36 Huron Roofing Company had gross revenues of $1,200,000 from operations. Financial transactions as shown in Table P9.36 were posted during the year. The old equipment had a book value of $75,000 at the time of its sale.

(a) What is Huron's income tax liability?

(b) What is Huron's operating income?

Gains or Losses

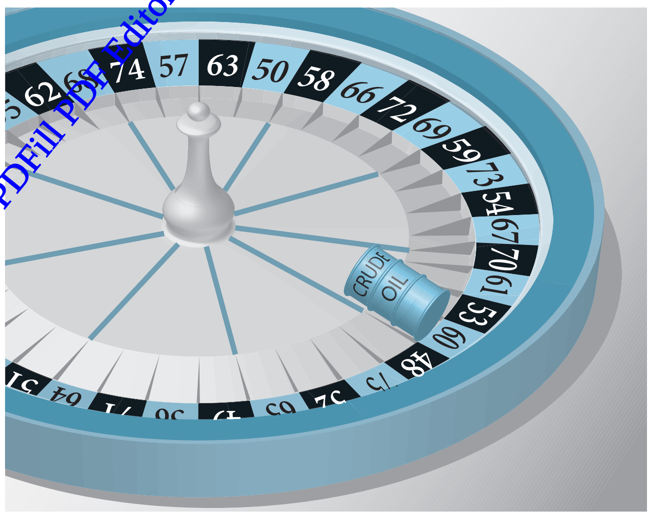
9.37 Consider a five-year MACRS asset purchased at $60,000. (Note that a five-year MACRS property class is depreciated over six years, due to the half-year convention.

CHAPTER TWELVE

Project Risk and Uncertainty

Oil Forecasts Are a Roll of the Dice[1] You may know as much as the oil experts. That is, you know that a barrel of oil is pricey and getting pricier. Beyond that, nobody—not even those who get paid to prognosticate—has a real handle on the push and pull that goes into figuring how much oil people need, how much can be pumped, and how much can be refined.

The unreliable data and forecasts have plagued the industry for decades. They became more of a problem once demand and prices



[1] "Oil Forecasts Are a Roll of the Dice," Bhusan Bahree, *The Wall Street Journal*, Tuesday, August 2, 2005, Section C1.

starting climbing two years ago, because the substantial margins of error in these numbers are even larger than the oil industry's shrinking margin of spare pumping capacity.

Put another way, even a small error in predicting oil consumption can cause energy markets to gyrate if demand turns out higher than the market assumed, because the industry lacks the ability it had in the 1990s to gin up extra oil on the fly to meet a surge in buying. Yet traders, companies, and consumers have no choice but to rely on the numbers that are out there. One problem anyone who factors oil into an investing decision faces is that accurate oil data come only with a time lag. Oil data on actual demand and supply bounce around because of bad weather, accidents such as pipeline ruptures, or political shocks such as terrorist strikes. Amplifying the fuzziness, meanwhile, is that projections of economic growth, a critical factor in assessing energy needs, are forever changing.

Crude-Oil Price Forecast

West Texas Intermediate, Spot Price. USD/bbl. Average of Month

	Dec 2005	Jan 2006	Feb 2006	Mar 2006	Apr 2006	May 2006
Forecast Value	**59.3**	**58.7**	**61.8**	**61.6**	**62.1**	**63.8**
Standard Deviation	0.9	1.2	1.5	1.8	2.2	2.6
Correlation Coefficient	0.9925	0.9912	0.9899	0.9886	0.9873	0.9860

Suppose that your business depends on the price of oil. For example, the airline industry, UPS, FedEx, and rental companies are all heavily affected by the price of fuel. Now, if your proposed project also depends on the price of crude oil, how would you factor the fluctuation and uncertainty into the analysis?

In previous chapters, cash flows from projects were assumed to be known with complete certainty; our analysis was concerned with measuring the economic worth of projects and selecting the best ones to invest in. Although that type of analysis can provide a reasonable basis for decision making in many investment situations, we should certainly consider the more usual *un*certainty. In this type of situation, management rarely has precise expectations about the future cash flows to be derived from a particular project. In fact, the best that a firm can reasonably expect to do is to estimate the range of possible future costs and benefits and the relative chances of achieving a reasonable return on the investment. We use the term **risk** to describe an investment project whose cash flow is not known in advance with absolute certainty, but for which an array of alternative outcomes and their probabilities (odds) are known. We will also use the term **project risk** to refer to variability in a project's NPW. A greater project risk usually means a greater variability in a project's NPW, or simply that the *risk is the potential for loss*. This chapter begins by exploring the origins of project risk.

Risk: The chance that an investment's actual return will be different than expected.

CHAPTER LEARNING OBJECTIVES

After completing this chapter, you should understand the following concepts:

- How to describe the nature of project risk.
- How to conduct a sensitivity analysis of key input variables.
- How to conduct a break-even analysis.
- How to develop a net-present-worth probability distribution.
- How to compare mutually exclusive risky alternatives.
- How to develop a risk simulation model.
- How to make a sequential investment decision with a decision tree.

12.1 Origins of Project Risk

The decision to make a major capital investment such as introducing a new product requires information about cash flow over the life of a project. The profitability estimate of an investment depends on cash flow estimations, which are generally uncertain. The factors to be estimated include the total market for the product; the market share that the firm can attain; the growth in the market; the cost of producing the product, including labor and materials; the selling price; the life of the product; the cost and life of the equipment needed; and the effective tax rates. Many of these factors are subject to substantial uncertainty. A common approach is to make single-number "best estimates" for each of the uncertain factors and then to calculate measures of profitability, such as the NPW or rate of return for the project. This approach, however, has two drawbacks:

1. No guarantee can ever ensure that the "best estimates" will match actual values.
2. No provision is made to measure the risk associated with an investment, or the project risk. In particular, managers have no way of determining either the probability that a project will lose money or the probability that it will generate large profits.

Because cash flows can be so difficult to estimate accurately, project managers frequently consider a range of possible values for cash flow elements. If a range of values for individual cash flows is possible, it follows that a range of values for the NPW of a given project is also possible. Clearly, the analyst will want to gauge the probability and reliability of individual cash flows and, consequently, the level of certainty about the overall project worth.

12.2 Methods of Describing Project Risk

We may begin analyzing project risk by first determining the uncertainty inherent in a project's cash flows. We can do this analysis in a number of ways, which range from making informal judgments to calculating complex economic and statistical quantities. In this section, we will introduce three methods of describing project risk: (1) sensitivity analysis, (2) break-even analysis, and (3) scenario analysis. Each method will be explained with reference to a single example, involving the Boston Metal Company.

12.2.1 Sensitivity Analysis

One way to glean a sense of the possible outcomes of an investment is to perform a sensitivity analysis. This kind of analysis determines the effect on the NPW of variations in the input variables (such as revenues, operating cost, and salvage value) used to estimate after-tax cash flows. A **sensitivity analysis** reveals how much the NPW will change in response to a given change in an input variable. In calculating cash flows, some items have a greater influence on the final result than others. In some problems, the most significant item may be easily identified. For example, the estimate of sales volume is often a major factor in a problem in which the quantity sold varies with the alternatives. In other problems, we may want to locate the items that have an important influence on the final results so that they can be subjected to special scrutiny.

Sensitivity analysis is sometimes called "what-if" analysis, because it answers questions such as "What if incremental sales are only 1,000 units, rather than 2,000 units? Then what will the NPW be?" Sensitivity analysis begins with a base-case situation, which is developed by using the most likely values for each input. We then change the specific variable of interest by several specified percentage points above and below the most likely value, while holding other variables constant. Next, we calculate a new NPW for each of the values we obtained. A convenient and useful way to present the results of a sensitivity analysis is to plot **sensitivity graphs**. The slopes of the lines show how sensitive the NPW is to changes in each of the inputs: The steeper the slope, the more sensitive the NPW is to a change in a particular variable. Sensitivity graphs identify the crucial variables that affect the final outcome most. Example 12.1 illustrates the concept of sensitivity analysis.

EXAMPLE 12.1 Sensitivity Analysis

Boston Metal Company (BMC), a small manufacturer of fabricated metal parts, must decide whether to enter the competition to become the supplier of transmission housings for Gulf Electric, a company that produces the housings in its own in-house manufacturing facility, but that has almost reached its maximum production capacity. Therefore, Gulf is looking for an outside supplier. To compete, BMC must design a new fixture for the production process and purchase a new forge. The available details for this purchase are as follows:

- The new forge would cost $125,000. This total includes retooling costs for the transmission housings.
- If BMC gets the order, it may be able to sell as many as 2,000 units per year to Gulf Electric for $50 each, in which case variable production costs,[2] such as direct labor and direct material costs, will be $15 per unit. The increase in fixed costs,[3] other than depreciation, will amount to $10,000 per year.
- The firm expects that the proposed transmission-housings project will have about a five-year product life. The firm also estimates that the amount ordered by Gulf Electric in the first year will be ordered in each of the subsequent four years. (Due to the nature of contracted production, the annual demand and unit price would remain the same over the project after the contract is signed.)
- The initial investment can be depreciated on a MACRS basis over a seven-year period, and the marginal income tax rate is expected to remain at 40%. At the end of five years, the forge is expected to retain a market value of about 32% of the original investment.
- On the basis of this information, the engineering and marketing staffs of BMC have prepared the cash flow forecasts shown in Table 12.1. Since the NPW is positive ($40,168) at the 15% opportunity cost of capital (MARR), the project appears to be worth undertaking.

What Makes BMC Managers Worry: BMC's managers are uneasy about this project, because too many uncertain elements have not been considered in the analysis:

- If it decided to take on the project, BMC would have to invest in the forging machine to provide Gulf Electric with some samples as a part of the bidding process. If Gulf Electric were not to like BMC's sample, BMC would stand to lose its entire investment in the forging machine.
- If Gulf were to like BMC's sample, then if it was overpriced, BMC would be under pressure to bring the price in line with competing firms. Even the possibility that BMC would get a smaller order must be considered, as Gulf may utilize its overtime capacity to produce some extra units. Finally, BMC is not certain about its projections of variable and fixed costs.

Recognizing these uncertainties, the managers want to assess the various possible future outcomes before making a final decision. Put yourself in BMC's management position, and describe how you may resolve the uncertainty associated with the project. In doing so, perform a sensitivity analysis for each variable and develop a sensitivity graph.

DISCUSSION: Table 12.1 shows BMC's expected cash flows—but that they will indeed materialize cannot be assumed. In particular, BMC is not very confident in its revenue forecasts. The managers think that if competing firms enter the market,

[2] These are expenses that change in direct proportion to the change in volume of sales or production, as defined in Section 8.3.

[3] These are expenses that do not vary with the volume of sales or production. For example, property taxes, insurance, depreciation, and rent are usually fixed expenses.

TABLE 12.1 **After-Tax Cash Flow for BMC's Transmission-Housings Project (Example 12.1)**

	0	1	2	3	4	5
Revenues:						
Unit price		$ 50	$ 50	$ 50	$ 50	$ 50
Demand (units)		2,000	2,000	2,000	2,000	2,000
Sales revenue		$100,000	$100,000	$100,000	$100,000	$100,000
Expenses:						
Unit variable cost		$ 15	$ 15	$ 15	$ 15	$ 15
Variable cost		30,000	30,000	30,000	30,000	30,000
Fixed cost		10,000	10,000	10,000	10,000	10,000
Depreciation		17,863	30,613	21,863	15,613	5,575
Taxable income		$ 42,137	$ 29,387	$ 38,137	$ 44,387	$ 54,425
Income taxes (40%)		16,855	11,755	15,255	17,755	21,770
Net income		$ 25,282	$ 17,632	$ 22,882	$ 26,632	$ 32,655
Cash flow statement:						
Operating activities:						
Net income		25,282	17,632	22,882	26,632	32,655
Depreciation		17,863	30,613	21,863	15,613	5,575
Investment activities:						
Investment	(125,000)					
Salvage						40,000
Gains tax						(2,611)
Net cash flow	$(125,500)	$ 43,145	$ 48,245	$ 44,745	$ 42,245	$ 75,619

BMC will lose a substantial portion of the projected revenues by not being able to increase its bidding price. Before undertaking the project, the company needs to identify the key variables that will determine whether it will succeed or fail. The marketing department has estimated revenue as follows:

$$\begin{aligned}\text{Annual revenue} &= (\text{Product demand})(\text{unit price}) \\ &= (2{,}000)(\$50) = \$100{,}000.\end{aligned}$$

The engineering department has estimated variable costs, such as those of labor and materials, at \$15 per unit. Since the projected sales volume is 2,000 units per year, the total variable cost is \$30,000.

After first defining the unit sales, unit price, unit variable cost, fixed cost, and salvage value, we conduct a sensitivity analysis with respect to these key input variables. This is done by varying each of the estimates by a given percentage and determining what effect the variation in that item will have on the final results. If the effect is large, the result is sensitive to that item. Our objective is to locate the most sensitive item(s).

SOLUTION

Sensitivity analysis: We begin the sensitivity analysis with a consideration of the base-case situation, which reflects the best estimate (expected value) for each input variable. In developing Table 12.2, we changed a given variable by 20% in 5% increments, above and below the base-case value, and calculated new NPWs, while other variables were held constant. The values for both sales and operating costs were the expected, or base-case, values, and the resulting $40,169 is the base-case NPW. Now we ask a series of "what-if" questions: What if sales are 20% below the expected level? What if operating costs rise? What if the unit price drops from $50 to $45? Table 12.2 summarizes the results of varying the values of the key input variables.

Sensitivity graph: Next, we construct a sensitivity graph for five of the transmission project's key input variables. (See Figure 12.1.) We plot the base-case NPW on the

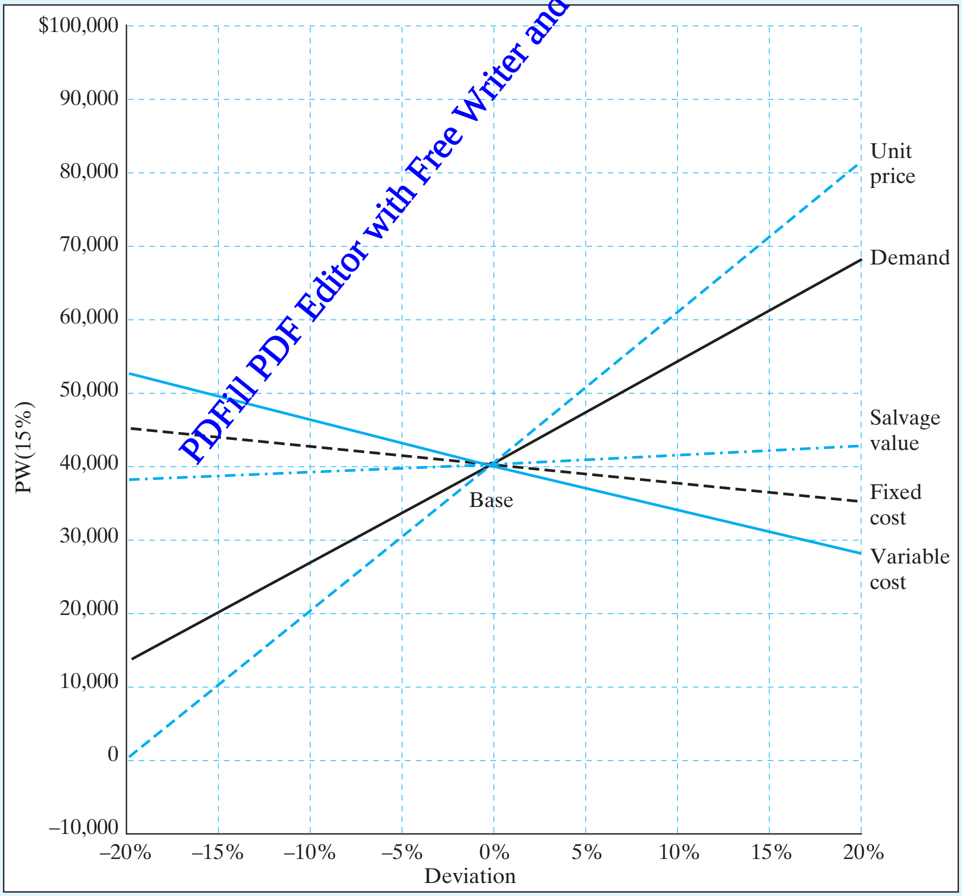


Figure 12.1 Sensitivity graph for BMC's transmission-housings project (Example 12.1).

TABLE 12.2 Sensitivity Analysis for Five Key Input Variables (Example 12.1)

Deviation	–20%	–15%	–10%	–5%	0%	5%	10%	15%	20%
Unit price	$ (57)	$ 9,999	$ 20,055	$ 30,111	$ 40,169	$ 50,225	$ 60,281	$ 70,337	$ 80,393
Demand	12,010	19,049	26,088	33,130	40,169	47,208	54,247	61,286	68,325
Variable cost	52,236	49,219	46,202	43,186	40,169	37,152	34,135	31,118	28,101
Fixed cost	44,191	43,185	42,179	41,175	40,169	39,163	38,157	37,151	36,145
Salvage value	37,782	38,378	38,974	39,573	40,169	40,765	41,361	41,957	42,553

ordinate of the graph at the value 1.0 on the abscissa (or 0% deviation). Then we reduce the value of product demand to 0.95 of its base-case value and recompute the NPW with all other variables held at their base-case value. We repeat the process by either decreasing or increasing the relative deviation from the base case. The lines for the variable unit price, variable unit cost, fixed cost, and salvage value are obtained in the same manner. In Figure 12.1, we see that the project's NPW is (1) very sensitive to changes in product demand and unit price, (2) fairly sensitive to changes in variable costs, and (3) relatively insensitive to changes in the fixed cost and the salvage value.

Graphic displays such as the one in Figure 12.1 provide a useful means to communicate the relative sensitivities of the different variables to the corresponding NPW value. However, sensitivity graphs do not explain any interactions among the variables or the likelihood of realizing any specific deviation from the base case. Certainly, it is conceivable that an answer might not be very sensitive to changes in either of two items, but very sensitive to combined changes in them.

12.2.2 Break-Even Analysis

When we perform a sensitivity analysis of a project, we are asking how serious the effect of lower revenues or higher costs will be on the project's profitability. Managers sometimes prefer to ask instead how much sales can decrease below forecasts before the project begins to lose money. This type of analysis is known as **break-even analysis**. In other words, break-even analysis is a technique for studying the effect of variations in output on a firm's NPW (or other measures). We will present an approach to break-even analysis based on the project's cash flows.

To illustrate the procedure of break-even analysis based on NPW, we use the generalized cash flow approach we discussed in Section 10.4. We compute the PW of cash inflows as a function of an unknown variable (say, x), perhaps annual sales. For example,

$$\text{PW of cash inflows} = f(x)_1.$$

Next, we compute the PW of cash outflows as a function of x:

$$\text{PW of cash outflows} = f(x)_2.$$

NPW is, of course, the difference between these two numbers. Accordingly, we look for the break-even value of x that makes

$$f(x)_1 = f(x)_2.$$

Note that this break-even value is similar to that used to calculate the internal rate of return when we want to find the interest rate that makes the NPW equal zero. The break-even value is also used to calculate many other similar "cutoff values" at which a choice changes.

EXAMPLE 12.2 Break-Even Analysis

Through the sensitivity analysis in Example 12.1, BMC's managers become convinced that the NPW is most sensitive to changes in annual sales volumes. Determine the break-even NPW value as a function of that variable.

SOLUTION

The analysis is shown in Table 12.3, in which the revenues and costs of the BMC transmission-housings project are set out in terms of an unknown amount of annual sales X.

We calculate the PWs of cash inflows and outflows as follows:

- PW of cash inflows:

$$\begin{aligned} PW(15\%)_{\text{Inflow}} &= (\text{PW of after-tax net revenue}) \\ &\quad + (\text{PW of net salvage value}) \\ &\quad + (\text{PW of tax savings from depreciation}) \\ &= 30X(P/A, 15\%, 5) + \$37{,}389(P/F, 15\%, 5) \end{aligned}$$

TABLE 12.3 Break-Even Analysis with Unknown Annual Sales (Example 12.2)

	0	1	2	3	4	5
Cash inflow:						
Net salvage						37,389
Revenue:						
$X(1 - 0.4)(\$50)$		$30X$	$30X$	$30X$	$30X$	$30X$
Depreciation credit						
0.4 (depreciation)		7,145	12,245	8,745	6,245	2,230
Cash outflow:						
Investment	−125,000					
Variable cost:						
$-X(1 - 0.4)(\$15)$		$-9X$	$-9X$	$-9X$	$-9X$	$-9X$
Fixed cost:						
$-(1 - 0.4)(\$10{,}000)$		−6,000	−6,000	−6,000	−6,000	−6,000
Net cash flow	−125,000	$21X + 1{,}145$	$21X + 6{,}245$	$21X + 2{,}745$	$21X + 245$	$21X + 33{,}617$

$$\begin{aligned}
&+ \$7{,}145(P/F, 15\%, 1) + \$12{,}245(P/F, 15\%, 2)\\
&+ \$8{,}745(P/F, 15\%, 3) + \$6{,}245(P/F, 15\%, 4)\\
&+ \$2{,}230(P/F, 15\%, 5)\\
&= 30X(P/A, 15\%, 5) + \$44{,}490\\
&= 100.5650X + \$44{,}490.
\end{aligned}$$

- PW of cash outflows:

$$\begin{aligned}
\mathrm{PW}(15\%)_{\text{Outflow}} &= (\text{PW of capital expenditure})\\
&\quad + (\text{PW of after-tax expenses})\\
&= \$125{,}000 + (9X + \$6{,}000)(P/A, 15\%, 5)\\
&= 30.1694X + \$145{,}113.
\end{aligned}$$

The NPW of cash flows for the BMC is thus

$$\begin{aligned}
\mathrm{PW}(15\%) &= 100.5650X + \$44{,}490\\
&= -(30.1694X + \$145{,}113)\\
&= 70.3956X - \$100{,}623.
\end{aligned}$$

In Table 12.4, we compute the PW of the inflows and the PW of the outflows as a function of demand (X).

The NPW will be just slightly positive if the company sells 1,430 units. Precisely calculated, the zero-NPW point (break-even volume) is 1,429.43 units:

$$\begin{aligned}
\mathrm{PW}(15\%) &= 70.3956X - \$100{,}623\\
&= 0,\\
X_b &= 1{,}430 \text{ units.}
\end{aligned}$$

TABLE 12.4 Determination of Break-Even Volume Based on Project's NPW (Example 12.3)

Demand (X)	PW of Inflow ($100.5650X$ + \$44,490)	PW of Outflow ($30.1694X$ + \$145,113)	NPW ($70.3956X$ – \$100,623)
0	\$ 44,490	\$ 145,113	(100,623)
500	94,773	160,198	(65,425)
1,000	145,055	175,282	(30,227)
1,429	188,197	188,225	(28)
1,430	188,298	188,255	43
1,500	195,338	190,367	4,970
2,000	245,620	205,452	40,168
2,500	295,903	220,537	75,366

Break-even volume = 1,430 units.

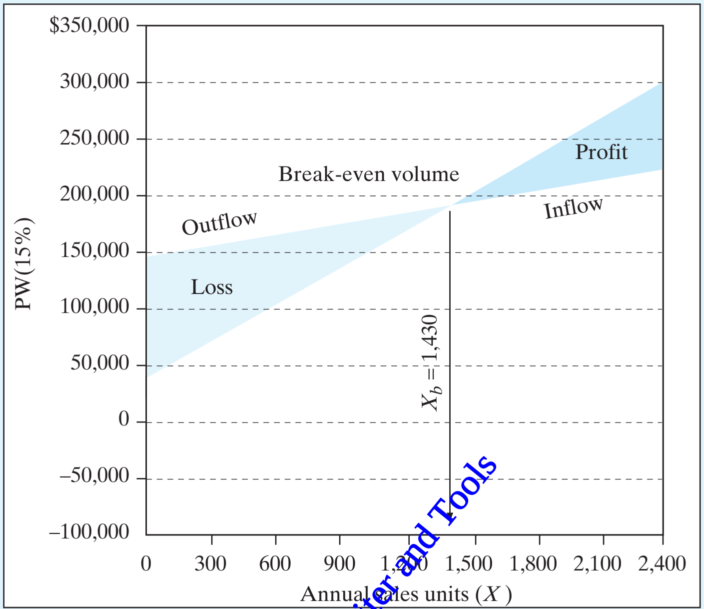


Figure 12.2 Break-even analysis based on net cash flow (Example 12.2).

In Figure 12.2, we have plotted the PWs of the inflows and outflows under various assumptions about annual sales. The two lines cross when sales are 1,430 units, the point at which the project has a zero NPW. Again we see that, as long as sales are greater or equal to 1,430, the project has a positive NPW.

12.2.3 Scenario Analysis

Although both sensitivity and break-even analyses are useful, they have limitations. Often, it is difficult to specify precisely the relationship between a particular variable and the NPW. The relationship is further complicated by interdependencies among the variables. Holding operating costs constant while varying unit sales may ease the analysis, but in reality, operating costs do not behave in this manner. Yet, it may complicate the analysis too much to permit movement in more than one variable at a time.

Scenario analysis is a technique that considers the sensitivity of NPW both to changes in key variables and to the range of likely values of those variables. For example, the decision maker may examine two extreme cases: a "worst-case" scenario (low unit sales, low unit price, high variable cost per unit, high fixed cost, and so on) and a "best-case" scenario. The NPWs under the worst and the best conditions are then calculated and compared with the expected, or base-case, NPW. Example 12.3 illustrates a plausible scenario analysis for BMC's transmission-housings project.

EXAMPLE 12.3 Scenario Analysis

Consider again BMC's transmission-housings project first presented in Example 12.1. Assume that the company's managers are fairly confident of their estimates of all the project's cash flow variables, except the estimates of unit sales. Assume further that they regard a drop in unit sales below 1,600 or a rise above 2,400 as extremely unlikely. Thus, a decremental annual sale of 400 units defines the lower bound, or the worst-case scenario, whereas an incremental annual sale of 400 units defines the upper bound, or the best-case scenario. (Remember that the most likely value was 2,000 in annual unit sales.) Discuss the worst- and best-case scenarios, assuming that the unit sales for all five years are equal.

DISCUSSION: To carry out the scenario analysis, we ask the marketing and engineering staffs to give optimistic (best-case) and pessimistic (worst-case) estimates for the key variables. Then we use the worst-case variable values to obtain the worst-case NPW and the best-case variable values to obtain the best-case NPW.

SOLUTION

The results of our analysis are summarized as follows:

Variable Considered	Worst-Case Scenario	Most-Likely-Case Scenario	Best-Case Scenario
Unit demand	1,600	2,000	2,400
Unit price ($)	48	50	53
Variable cost ($)	17	15	12
Fixed cost ($)	11,000	10,000	8,000
Salvage value ($)	30,000	40,000	50,000
PW(15%)	−$5,856	$40,169	$104,295

We see that the base case produces a positive NPW, the worst case produces a negative NPW, and the best case produces a large positive NPW. Still, by just looking at the results in the table, it is not easy to interpret the scenario analysis or to make a decision based on it. For example, we could say that there is a chance of losing money on the project, but we do not yet have a specific probability for that possibility. Clearly, we need estimates of the probabilities of occurrence of the worst case, the best case, the base (most likely) case, and all the other possibilities.

The need to estimate probabilities leads us directly to our next step: developing a probability distribution (or, put another way, the probability that the variable in question takes on a certain value). If we can predict the effects on the NPW of variations in the parameters, why should we not assign a probability distribution to the possible outcomes of each parameter and combine these distributions in some way to produce a probability distribution for the possible outcomes of the NPW? We shall consider this issue in the next two sections.

- **Monte Carlo sampling** is a specific type of randomized sampling method in which a random sample of outcomes is generated for a specified probability distribution. Because Monte Carlo sampling and other simulation techniques often rely on generating a significant number of outcomes, they can be more conveniently performed on the computer than manually.
- The **decision tree** is another technique that can facilitate investment decision making when uncertainty prevails, especially when the problem involves a sequence of decisions. Decision-tree analysis involves the choice of a decision criterion—say, to maximize expected profit. If possible and feasible, an experiment is conducted, and the prior probabilities of the states of nature are revised on the basis of the experimental results. The expected profit associated with each possible decision is computed, and the act with the highest expected profit is chosen as the optimum action.

PROBLEMS

Sensitivity Analysis

12.1 Ford Construction Company is considering acquiring a new earthmover. The mover's basic price is $90,000, and it will cost another $18,000 to modify it for special use by the company. This earthmover falls into the MACRS five-year class. It will be sold after four years for $30,000. The purchase of the earthmover will have no effect on revenues, but it is expected to save the firm $35,000 per year in before-tax operating costs, mainly labor. The firm's marginal tax rate (federal plus state) is 40%, and its MARR is 10%.

(a) Is this project acceptable, based on the most likely estimates given?

(b) Suppose that the project will require an increase in net working capital (spare-parts inventory) of $5,000, which will be recovered at the end of year 5. Taking this new requirement into account, would the project still be acceptable?

(c) If the firm's MARR is increased to 18%, what would be the required savings in labor so that the project remains profitable?

12.2 Minnesota Metal Forming Company has just invested $500,000 of fixed capital in a manufacturing process that is estimated to generate an after-tax annual cash flow of $200,000 in each of the next five years. At the end of year 5, no further market for the product and no salvage value for the manufacturing process is expected. If a manufacturing problem delays the start-up of the plant for one year (leaving only four years of process life), what additional after-tax cash flow will be needed to maintain the same internal rate of return as would be experienced if no delay occurred?

12.3 A real-estate developer seeks to determine the most economical height for a new office building, which will be sold after five years. The relevant net annual revenues and salvage values are as follows:

	Height			
	2 Floors	**3 Floors**	**4 Floors**	**5 Floors**
First cost (net after tax)	$500,000	$750,000	$1,250,000	$2,000,000
Lease revenue	199,100	169,200	149,200	378,150
Net resale value (after tax)	600,000	900,000	2,000,000	3,000,000

(a) The developer is uncertain about the interest rate (i) to use, but is certain that it is in the range from 5% to 30%. For each building height, find the range of values of i for which that building height is the most economical.

(b) Suppose that the developer's interest rate is known to be 15%. What would be the cost, in terms of net present value, of a 10% overestimation of the resale value? (In other words, the true value was 10% lower than that of the original estimate.)

12.4 A special-purpose milling machine was purchased 4 years ago for \$20,000. It was estimated at that time that this machine would have a life of 10 years and a salvage value of \$1,000, with a cost of removal of \$1,500. These estimates are still good. This machine has annual operating costs of \$2,000, and its current book value is \$13,000. If the machine is retained for its entire 10-year life, the remaining annual depreciation schedule would be \$2,000 for years 5 through 10. A new machine that is more efficient will reduce operating costs to \$1,000, but will require an investment of \$12,000. The life of the new machine is estimated to be 6 years, with a salvage value of \$2,000. The new machine would fall into the 5-year MACRS property class. An offer of \$6,000 for the old machine has been made, and the purchaser would pay for removal of the machine. The firm's marginal tax rate is 40%, and its required minimum rate of return is 10%.

(a) What incremental cash flows will occur at the end of years 0 through 6 as a result of replacing the old machine? Should the old machine be replaced now?

(b) Suppose that the annual operating costs for the old milling machine would increase at an annual rate of 5% over the remaining service life of the machine. With this change in future operating costs for the old machine, would the answer in (a) change?

(c) What is the minimum trade-in value for the old machine so that both alternatives are economically equivalent?

12.5 A local telephone company is considering installing a new phone line for a new row of apartment complexes. Two types of cables are being examined: conventional copper wire and fiber optics. Transmission by copper wire cables, although cumbersome, involves much less complicated and less expensive support hardware than does fiber optics. The local company may use five different types of copper wire cables: 100 pairs, 200 pairs, 300 pairs, 600 pairs, and 900 pairs per cable. In calculating the cost per foot of cable, the following equation is used:

$$\text{Cost per length} = [\text{Cost per foot} + \text{cost per pair (number of pairs)}](\text{length}),$$

where

$$\text{22-gauge copper wire} = \$1.692 \text{ per foot}$$

and

$$\text{Cost per pair} = \$0.013 \text{ per pair.}$$

The annual cost of the cable as a percentage of the initial cost is 18.4%. The life of the system is 30 years.

In fiber optics, a cable is referred to as a ribbon. One ribbon contains 12 fibers, grouped in fours; therefore, one ribbon contains three groups of 4 fibers. Each group can produce 672 lines (equivalent to 672 pairs of wires), and since each ribbon contains three groups, the total capacity of the ribbon is 2,016 lines. To transmit signals via fiber optics, many modulators, wave guides, and terminators are needed to convert the signals from electric currents to modulated light waves. Fiber-optic ribbon costs $15,000 per mile. At each end of the ribbon, three terminators are needed, one for each group of 4 fibers, at a cost of $30,000 per terminator. Twenty-one modulating systems are needed at each end of the ribbon, at a cost of $12,092 for a unit in the central office and $21,217 for a unit in the field. Every 22,000 feet, a repeater is required to keep the modulated light waves in the ribbon at an intensity that is intelligible for detection. The unit cost of this repeater is $15,000. The annual cost, including income taxes for the 21 modulating systems, is 12.5% of the initial cost of the units. The annual cost of the ribbon itself is 17.8% initially. The life of the whole system is 30 years. (All figures represent after-tax costs.)

(a) Suppose that the apartments are located 5 miles from the phone company's central switching system and that about 2,000 telephones will be required. This would require either 2,000 pairs of copper wire or one fiber-optic ribbon and related hardware. If the telephone company's interest rate is 15%, which option is more economical?

(b) In (a), suppose that the apartments are located 10 miles or 25 miles from the phone company's central switching system. Which option is more economically attractive under each scenario?

12.6 A small manufacturing firm is considering purchasing a new boring machine to modernize one of its production lines. Two types of boring machine are available on the market. The lives of machine A and machine B are 8 years and 10 years, respectively. The machines have the following receipts and disbursements:

Item	Machine A	Machine B
First cost	$6,000	$8,500
Service life	8 years	10 years
Salvage value	$500	$1,000
Annual O&M costs	$700	$520
Depreciation (MACRS)	7 years	7 years

Use a MARR (after tax) of 10% and a marginal tax rate of 30%, and answer the following questions:

(a) Which machine would be most economical to purchase under an infinite planning horizon? Explain any assumption that you need to make about future alternatives.

(b) Determine the break-even annual O&M costs for machine A so that the present worth of machine A is the same as that of machine B.

(c) Suppose that the required service life of the machine is only 5 years. The salvage values at the end of the required service period are estimated to be \$3,000 for machine A and \$3,500 for machine B. Which machine is more economical?

12.7 The management of Langdale Mill is considering replacing a number of old looms in the mill's weave room. The looms to be replaced are two 86-inch President looms, sixteen 54-inch President looms, and twenty-two 72-inch Draper X-P2 looms. The company may either replace the old looms with new ones of the same kind or buy 21 new shutterless Pignone looms. The first alternative requires purchasing 40 new President and Draper looms and scrapping the old looms. The second alternative involves scrapping the 40 old looms, relocating 12 Picanol looms, and constructing a concrete floor, plus purchasing the 21 Pignone looms and various related equipment.

Description	Alternative 1	Alternative 2
Machinery/related equipment	\$2,119,170	\$1,071,240
Removal cost of old looms/site preparation	26,866	49,002
Salvage value of old looms	62,000	62,000
Annual sales increase with new looms	7,915,748	7,455,084
Annual labor	261,040	422,080
Annual O&M	1,092,000	1,560,000
Depreciation (MACRS)	7 years	7 years
Project life	8 years	8 years
Salvage value	169,000	54,000

The firm's MARR is 18%, set by corporate executives, who feel that various investment opportunities available for the mills will guarantee a rate of return on investment of at least 18%. The mill's marginal tax rate is 40%.

(a) Perform a sensitivity analysis on the project's data, varying the operating revenue, labor cost, annual maintenance cost, and MARR. Assume that each of these variables can deviate from its base-case expected value by $\pm 10\%$, by $\pm 20\%$, and by $\pm 30\%$.

(b) From the results of part (a), prepare sensitivity diagrams and interpret the results.

12.8 Mike Lazenby, an industrial engineer at Energy Conservation Service, has found that the anticipated profitability of a newly developed water-heater temperature control device can be measured by present worth with the formula

$$\text{NPW} = 4.028V(2X - \$11) - 77{,}860,$$

where V is the number of units produced and sold and X is the sales price per unit. Mike also has found that the value of the parameter V could occur anywhere over the range from 1,000 to 6,000 units and that of the parameter X anywhere between \$20 and \$45 per unit. Develop a sensitivity graph as a function of the number of units produced and the sales price per unit.

12.9 A local U.S. Postal Service office is considering purchasing a 4,000-pound forklift truck, which will be used primarily for processing incoming as well as outgoing postal packages. Forklift trucks traditionally have been fueled by either gasoline, liquid propane gas (LPG), or diesel fuel. Battery-powered electric forklifts, however, are increasingly popular in many industrial sectors due to the economic and environmental benefits that accrue from their use. Therefore, the postal service is interested in comparing forklifts that use the four different types of fuel. The purchase costs as well as annual operating and maintenance costs are provided by a local utility company and the Lead Industries Association. Annual fuel and maintenance costs are measured in terms of number of shifts per year, where one shift is equivalent to 8 hours of operation.

	Electrical Power	LPG	Gasoline	Diesel Fuel
Life expectancy	7 years	7 years	7 years	7 years
Initial cost	$29,739	$21,200	$20,107	$22,263
Salvage value	$3,000	$2,000	$2,000	$2,200
Maximum shifts per year	260	260	260	260
Fuel consumption/shift	31.25 kWh	11 gal	11.1 gal	7.2 gal
Fuel cost/unit	$0.10/kWh	$3.50/gal	$2.24/gal	$2.45/gal
Fuel cost per shift	$3.125	$38.50	$24.86	$17.64
Annual maintenance cost				
Fixed cost	$500	$1,000	$1,000	$1,000
Variable cost/shift	$4.5	$7	$7	$7

The postal service is unsure of the number of shifts per year, but it expects it should be somewhere between 200 and 260 shifts. Since the U.S. Postal Service does not pay income taxes, no depreciation or tax information is required. The U.S. government uses 10% as an interest rate for any project evaluation of this nature. Develop a sensitivity graph that shows how the choice of alternatives changes as a function of number of shifts per year.

Break-Even Analysis

12.10 Susan Campbell is thinking about going into the motel business near Disney World in Orlando. The cost to build a motel is \$2,200,000. The lot costs \$600,000. Furniture and furnishings cost \$400,000 and should be recovered in 7 years (7-year MACRS property), while the motel building should be recovered in 39 years

(39-year MACRS real property placed in service on January 1). The land will appreciate at an annual rate of 5% over the project period, but the building will have a zero salvage value after 25 years. When the motel is full (100% capacity), it takes in (receipts) $4,000 per day, 365 days per year. Exclusive of depreciation, the motel has fixed operating expenses of $230,000 per year. The variable operating expenses are $170,000 at 100% capacity, and these vary directly with percent capacity down to zero at 0% capacity. If the interest is 10% compounded annually, at what percent capacity over 25 years must the motel operate in order for Susan to break even? (Assume that Susan's tax rate is 31%.)

12.11 A plant engineer wishes to know which of two types of lightbulbs should be used to light a warehouse. The bulbs that are currently used cost $45.90 per bulb and last 14,600 hours before burning out. The new bulb (at $60 per bulb) provides the same amount of light and consumes the same amount of energy, but lasts twice as long. The labor cost to change a bulb is $16.00. The lights are on 19 hours a day, 365 days a year. If the firm's MARR is 15%, what is the maximum price (per bulb) the engineer should be willing to pay to switch to the new bulb? (Assume that the firm's marginal tax rate is 40%.)

12.12 Robert Cooper is considering purchasing a piece of business rental property containing stores and offices at a cost of $250,000. Cooper estimates that annual receipts from rentals will be $35,000 and that annual disbursements, other than income taxes, will be about $12,000. The property is expected to appreciate at the annual rate of 5%. Cooper expects to retain the property for 20 years once it is acquired. Then it will be depreciated as a 39-year real-property class (MACRS), assuming that the property will be placed in service on January 1. Cooper's marginal tax rate is 30% and his MARR is 15%. What would be the minimum annual total of rental receipts that would make the investment break even?

12.13 Two different methods of solving a production problem are under consideration. Both methods are expected to be obsolete in six years. Method A would cost $80,000 initially and have annual operating costs of $22,000 a year. Method B would cost $52,000 and costs $17,000 a year to operate. The salvage value realized would be $20,000 with Method A and $15,000 with Method B. Method A would generate $16,000 revenue income a year more than Method B. Investments in both methods are subject to a five-year MACRS property class. The firm's marginal income tax rate is 40%. The firm's MARR is 20%. What would be the required additional annual revenue for Method A such that an engineer would be indifferent to choosing one method over the other?

12.14 Rocky Mountain Publishing Company is considering introducing a new morning newspaper in Denver. Its direct competitor charges $0.25 at retail, with $0.05 going to the retailer. For the level of news coverage the company desires, it determines the fixed cost of editors, reporters, rent, press-room expenses, and wire service charges to be $300,000 per month. The variable cost of ink and paper is $0.10 per copy, but advertising revenues of $0.05 per paper will be generated. To print the morning paper, the publisher has to purchase a new printing press, which will cost $600,000. The press machine will be depreciated according to a 7-year MACRS class. The press machine will be used for 10 years, at which time its salvage value would be about $100,000. Assume 20 weekdays in a month, a 40% tax rate, and a 13% MARR. How many copies per day must be sold to break even at a retail selling price of $0.25 per paper?

CHAPTER FIFTEEN

Capital-Budgeting Decisions

Hotels Go to the Mattresses[1] Marriott International, Inc., today will launch a major initiative to replace nearly every bed in seven of its chains. At the Marriott chain, which is slated for the most extensive upgrade, each king-size bed will be getting 300-thread-count 60% cotton sheets, seven pillows instead of five, a pillowy mattress cover, a white duvet, and a "bed scarf" that will be draped along the bottom of the bed. The yearlong project will cost an estimated $190 million, and the company is saying that the cost, along with the planned marketing efforts, make it its biggest initiative ever.

[1] "Hotels Go to the Mattresses," by Christina Binkley, *The Wall Street Journal*, January 25, 2005.

The bed replacements raise some awkward issues, such as what to do with all the old beds. Some hotels are trying to give them to charity, but "homeless shelters don't really have a use for king-size beds," says one Marriott executive. Housekeepers complain that stuffing down comforters into all the new duvets is more time consuming than making a traditional bed with bedspread.

All the pressure to make hotel beds better is finally forcing the industry to reveal—and start to abandon—its dirtiest little secret: the fact that those colorful bedspreads on hotel-room beds sometimes get washed only a few times a year at most. Marriott hopes it will top rivals with a pledge to wash its white duvets between each guest visit, an initiative the company will call "Clean for You." When it comes to bedspread sanitation, hotel chains are typically loath to reveal how often they wash the bedcovers. Regular laundering is costly, in terms of both housekeeping and laundering, as well as in wear and tear on expensive linens.

Hotels are doing all this because their research shows that people are willing to pay more for luxurious beds. Bill Marriott says he expects to be able to charge as much as $30 a night more in Marriott hotels once the new beds are installed.

Marriott plans to purchase 628,000 beds for hotels at seven chains. However, only the full-service Marriott and Renaissance chains will get the new white duvets, which the company is planning to launder regularly as part of the Clean for You campaign. Less expensive Marriott chains like Courtyard and SpringHill Suites, along with Residence Inn, will triple sheet their beds by putting the extra sheet on top of the (less frequently laundered) outermost bedspread.

All this comes as the hotel industry is experiencing an economic boom, with room rates and occupancies rising faster than any time since the dot-com boom, leaving hotels extra cash to spend on improvements. Much of the increase in travel is coming from business travelers, who tend to be pickier and less price sensitive than vacationers—meaning they want things like better beds and often don't mind paying for them since their expense accounts are picking up the tab.

Without budget limitations, the replacement problem would be more of a logistic issue: simply select the option with the most revenue-enhancing potential for each targeted hotel. However, to replace or upgrade all beds over a short period of time will cost in excess of $190 million, and the company needs to find a way to finance this large-scale project. Because of the size of the financing involved, the firm's cost of capital will tend to increase during the project period. In this circumstance, the choice of an appropriate interest rate (MARR) for use in the project evaluation becomes a critical issue. Given these budget and other restrictions, the company would certainly like to determine the least-cost replacement/ upgrade strategy.

Capital budgeting is the planning process used to determine a firm's long term investments.

In this chapter, we present the basic framework of **capital budgeting**, which involves investment decisions related to fixed assets. Here, the term **capital budget** includes planned expenditures on fixed assets; **capital budgeting** encompasses the entire process of analyzing projects and deciding whether they should be included in the capital budget. In previous chapters, we focused on how to evaluate and compare investment projects—the analysis aspect of capital budgeting. In this chapter, we focus on the budgeting aspect. Proper capital-budgeting decisions require a choice of the method of project financing, a schedule of investment opportunities, and an estimate of the minimum attractive rate of return (MARR).

CHAPTER LEARNING OBJECTIVES

After completing this chapter, you should understand the following concepts:

- How a corporation raises its capital to finance a project.
- How to determine the cost of debt.
- How to determine the cost of equity.
- How to determine the marginal cost of capital.
- How to determine the MARR in project evaluation.
- How to create an optimal project portfolio under capital rationing.

15.1 Methods of Financing

In previous chapters, we focused on problems relating to investment decisions. In reality, investment decisions are not always independent of the source of finance. For convenience, however, in economic analysis investment decisions are usually separated from finance decisions: First the investment project is selected, and then the source of financing is considered. After the source is chosen, appropriate modifications to the investment decision are made.

We have also assumed that the assets employed in an investment project are obtained with the firm's own capital (retained earnings) or from short-term borrowings. In practice, this arrangement is not always attractive or even possible. If the investment calls for a significant infusion of capital, the firm may raise the needed capital by issuing stock. Alternatively, the firm may borrow the funds by issuing bonds to finance such purchases. In this section, we will first discuss how a typical firm raises new capital from external sources. Then we will discuss how external financing affects after-tax cash flows and how the decision to borrow affects the investment decision.

The two broad choices a firm has for financing an investment project are **equity financing** and **debt financing**.[2] We will look briefly at these two options for obtaining external investment funds and also examine their effects on after-tax cash flows.

15.1.1 Equity Financing

Equity financing can take one of two forms: (1) the use of retained earnings otherwise paid to stockholders or (2) the issuance of stock. Both forms of equity financing use funds invested by the current or new owners of the company.

Until now, most of our economic analyses presumed that companies had cash on hand to make capital investments; implicitly, we were dealing with cases of financing by retained earnings. If a company had not reinvested these earnings, it might have paid them to the company's owners—the stockholders—in the form of a dividend, or it might have kept these earnings on hand for future needs.

If a company does not have sufficient cash on hand to make an investment and does not wish to borrow in order to fund the investment, financing can be arranged by selling common stock to raise the required funds. (Many small biotechnology and computer firms raise capital by going public and selling common stock.) To do this, the company has to decide how much money to raise, the type of securities to issue (common stock or preferred stock), and the basis for pricing the issue.

Once the company has decided to issue common stock, it must estimate **flotation costs**—the expenses it will incur in connection with the issue, such as investment bankers' fees, lawyers' fees, accountants' costs, and the cost of printing and engraving. Usually, an investment banker will buy the issue from the company at a discount, below the price at which the stock is to be offered to the public. (The discount usually represents the *flotation costs*.) If the company is already publicly owned, the offering price will commonly be based on the existing market price of the stock. If the company is going public for the first time, no established price will exist, so investment bankers have to estimate the expected market price at which the stock will sell after the stock issue. Example 15.1 illustrates how the flotation cost affects the cost of issuing common stock.

Flotation cost: The costs associated with the issuance of new securities.

EXAMPLE 15.1 Issuing Common Stock

Scientific Sports, Inc. (SSI), a golf club manufacturer, has developed a new metal club (Driver). The club is made out of titanium alloy, an extremely light and durable metal with good vibration-damping characteristics (Figure 15.1). The company expects to acquire considerable market penetration with this new product. To produce it, the company needs a new manufacturing facility, which will cost \$10 million. The company decided to raise this \$10 million by selling common stock. The firm's current stock price is \$30 per share. Investment bankers have informed management that the new public issue must be priced at \$28 per share because of decreasing demand, which

[2] A hybrid financing method, known as *lease financing*, was discussed in Section 10.4.3.

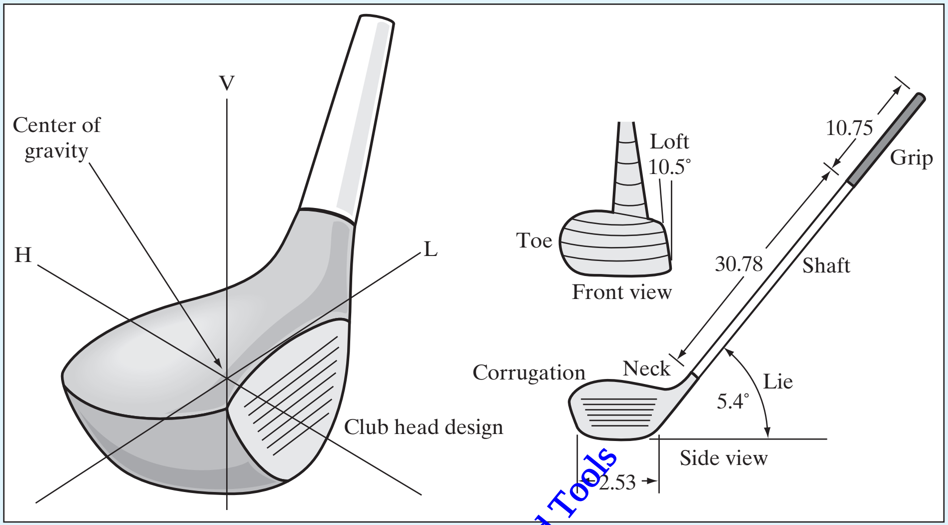


Figure 15.1 SSI's new golf club (Driver) design, developed with the use of advanced engineering materials (Example 15.1).

will occur as more shares become available on the market. The flotation costs will be 6% of the issue price, so SSI will net \$26.32 per share. How many shares must SSI sell to net \$10 million after flotation expenses?

SOLUTION

Let X be the number of shares to be sold. Then total flotation cost will be

$$(0.06)(\$28)(X) = 1.68X.$$

To net \$10 million, we must have

$$\begin{aligned} \text{Sales proceeds} - \text{flotation cost} &= \text{Net proceeds}, \\ 28X - 1.68X &= \$10{,}000{,}000, \\ 26.32X &= \$10{,}000{,}000, \\ X &= 379{,}940 \text{ shares}. \end{aligned}$$

Now we can figure out the flotation cost for issuing the common stock. The cost is

$$1.68(379{,}940) = \$638{,}300.$$

15.1.2 Debt Financing

The second major type of financing a company can select is **debt financing**, which includes both short-term borrowing from financial institutions and the sale of long-term bonds, wherein money is borrowed from investors for a fixed period. With debt financing, the interest paid on the loans or bonds is treated as an expense for income-tax purposes.

Since interest is a tax-deductible expense, companies in high tax brackets may incur lower after-tax financing costs with a debt. In addition to influencing the borrowing interest rate and tax bracket, a loan-repayment method can affect financing costs.

When the debt-financing option is used, we need to separate the interest payments from the repayment of the loan for our analysis. The interest-payment schedule depends on the repayment schedule established at the time the money is borrowed. The two common debt-financing methods are as follows:

1. **Bond Financing.** This type of debt financing does not involve the partial payment of principal; only interest is paid each year (or semiannually). The principal is paid in a lump sum when the bond matures. (See Section 4.6.3 for bond terminologies and valuation.) Bond financing is similar to equity financing in that flotation costs are involved when bonds are issued.
2. **Term Loans.** Term loans involve an equal repayment arrangement according to which the sum of the interest payments and the principal payments is uniform; interest payments decrease, while principal payments increase, over the life of the loan. Term loans are usually negotiated directly between the borrowing company and a financial institution, generally a commercial bank, an insurance company, or a pension fund.

Example 15.2 illustrates how these different methods can affect the cost of issuing bonds or term loans.

EXAMPLE 15.2 Debt Financing

Consider again Example 15.1. Suppose SSI has instead decided to raise the $10 million by debt financing. SSI could issue a mortgage bond or secure a term loan. Conditions for each option are as follows:

- **Bond financing.** The flotation cost is 1.8% of the $10 million issue. The company's investment bankers have indicated that a five-year bond issue with a face value of $1,000 can be sold at $985 per share. The bond would require annual interest payments of 12%.
- **Term loan.** A $10 million bank loan can be secured at an annual interest rate of 11% for five years; it would require five equal annual installments.

(a) How many $1,000 par value bonds would SSI have to sell to raise the $10 million?

(b) What are the annual payments (interest and principal) on the bond?

(c) What are the annual payments (interest and principal) on the term loan?

SOLUTION

(a) To net $10 million, SSI would have to sell

$$\frac{\$10{,}000{,}000}{(1 - 0.018)} = \$10{,}183{,}300$$

TABLE 15.1 Two Common Methods of Debt Financing (Example 15.2)

	0	1	2	3	4	5
1. Bond financing: No principal repayments until end of life						
Beginning balance	$10,338,380	$10,338,380	$10,338,380	$10,338,380	$10,338,380	$10,338,380
Interest owed		1,240,606	1,240,606	1,240,606	1,240,606	1,240,606
Repayment						
Interest payment		(1,240,606)	(1,240,606)	(1,240,606)	(1,240,606)	(1,240,606)
Principal payment						(10,338,380)
Ending balance	$10,338,380	$10,338,380	$10,338,380	$10,338,380	$10,338,380	0
2. Term loan: Equal annual repayments [$10,000,000($A/P$, 11%, 5) = $2,705,703]						
Beginning balance	$10,000,000	$10,000,000	$ 8,394,297	$ 6,611,967	$ 4,633,580	$ 2,437,571
Interest owed		1,100,000	923,373	727,316	509,694	268,133
Repayment						
Interest payment		(1,100,000)	(923,373)	(727,316)	(509,694)	(268,133)
Principal payment		(1,605,703)	(1,782,330)	(1,978,387)	(2,196,009)	(2,437,570)
Ending balance	$10,000,000	$ 8,394,297	$ 6,611,967	$ 4,633,580	$ 2,437,571	0

worth of bonds and pay $183,300 in flotation costs. Since the $1,000 bond will be sold at a 1.5% discount, the total number of bonds to be sold would be

$$\frac{\$10{,}183{,}300}{\$985} = \$10{,}338.38.$$

(b) For the bond financing, the annual interest is equal to

$$\$10{,}338{,}380(0.12) = \$1{,}240{,}606.$$

Only the interest is paid each period; thus, the principal amount owed remains unchanged.

(c) For the term loan, the annual payments are

$$\$10{,}000{,}000(A/P, 11\%, 5) = \$2{,}705{,}703.$$

The principal and interest components of each annual payment are summarized in Table 15.1.

15.1.3 Capital Structure

The ratio of total debt to total capital, generally called the **debt ratio**, or **capital structure**, represents the percentage of the total capital provided by borrowed funds. For example, a debt ratio of 0.4 indicates that 40% of the capital is borrowed and the remaining funds are provided from the company's equity (retained earnings or stock offerings). This type of financing is called **mixed financing**.

Borrowing affects a firm's capital structure, and firms must determine the effects of a change in the debt ratio on their market value before making an ultimate financing decision. Even if debt financing is attractive, you should understand that companies do not simply borrow funds to finance projects. A firm usually establishes a **target capital structure**, or **target debt ratio**, after considering the effects of various financing methods. This target may change over time as business conditions vary, but a firm's management always strives to achieve the target whenever individual financing decisions are considered. On the one hand, the actual debt ratio is below the target level, any new capital will probably be raised by issuing debt. On the other hand, if the debt ratio is currently above the target, expansion capital will be raised by issuing stock.

Capital structure: The means by which a firm is financed.

How does a typical firm set the target capital structure? This is a rather difficult question to answer, but we can list several factors that affect the capital-structure policy. First, capital-structure policy involves a trade-off between risk and return. As you take on more debt for business expansion, the inherent business risk[3] also increases, but investors view business expansion as a healthy indicator for a corporation with higher expected earnings. When investors perceive higher business risk, the firm's stock price tends to be depressed. By contrast, when investors perceive higher expected earnings, the firm's stock price tends to increase. The optimal capital structure is thus the one that strikes a balance between business risk and expected future earnings. The greater the firm's business risk, the lower is its optimal debt ratio.

Second, a major reason for using debt is that interest is a deductible expense for business operations, which lowers the effective cost of borrowing. Dividends paid to common stockholders, however, are not deductible. If a company uses debt, it must pay interest on this debt, whereas if it uses equity, it pays dividends to its equity investors (shareholders). A company needs \$1 in before-tax income to pay \$1 of interest, but if the company is in the 34% tax bracket, it needs $\$1/(1 - 0.34) = \$1.52$ of before-tax income to pay a \$1 dividend.

Third, financial flexibility—the ability to raise capital on reasonable terms from the financial market—is an important consideration. Firms need a steady supply of capital for stable operations. When money is tight in the economy, investors prefer to advance funds to companies with a healthy capital structure (lower debt ratio). These three elements (business risk, taxes, and financial flexibility) are major factors that determine the firm's optimal capital structure. Example 15.3 illustrates how a typical firm finances a large-scale engineering project by maintaining the predetermined capital structure.

EXAMPLE 15.3 Project Financing Based on an Optimal Capital Structure

Consider again SSI's \$10 million venture project in Example 15.1. Suppose that SSI's optimal capital structure calls for a debt ratio of 0.5. After reviewing SSI's capital structure, the investment banker convinced management that it would be better

[3] Unlike equity financing, in which dividends are optional, debt interest and principal (face value) must be repaid on time. Also, uncertainty is involved in making projections of future operating income as well as expenses. In bad times debt can be devastating, but in good times the tax deductibility of interest payments increases profits to owners.

off, in view of current market conditions, to limit the stock issue to \$5 million and to raise the other \$5 million as debt by issuing bonds. Because the amount of capital to be raised in each category is reduced by half, the flotation cost would also change. The flotation cost for common stock would be 8.1%, whereas the flotation cost for bonds would be 3.2%. As in Example 15.2, the 5-year, 12% bond will have a par value of \$1,000 and will be sold for \$985.

Assuming that the \$10 million capital would be raised from the financial market, the engineering department has detailed the following financial information:

- The new venture will have a 5-year project life.
- The \$10 million capital will be used to purchase land for \$1 million, a building for \$3 million, and equipment for \$6 million. The plant site and building are already available, and production can begin during the first year. The building falls into a 39-year MACRS property class and the equipment into a 7-year MACRS class. At the end of year 5, the salvage value of each asset is as follows: the land \$1.5 million, the building \$2 million, and the equipment \$3 million.
- For common stockholders, an annual cash dividend in the amount of \$2 per share is planned over the project life. This steady cash dividend payment is deemed necessary to maintain the market value of the stock.
- The unit production cost is \$50.31 (material, \$22.70; labor and overhead (excluding depreciation), \$10.57; and tooling, \$17.04).
- The unit price is \$250, and SSI expects an annual demand of 20,000 units.
- The operating and maintenance cost, including advertising expenses, would be \$600,000 per year.
- An investment of \$500,000 in working capital is required at the beginning of the project; the amount will be fully recovered when the project terminates.
- The firm's marginal tax rate is 40%, and this rate will remain constant throughout the project period.

(a) Determine the after-tax cash flows for this investment with external financing.

(b) Is this project justified at an interest rate of 20%?

DISCUSSION: As the amount of financing and flotation costs change, we need to recalculate the number of shares (or bonds) to be sold in each category. For a \$5 million common stock issue, the flotation cost increases to 8.1%.[4] The number of shares to be sold to net \$5 million is $5{,}000{,}000/(0.919)(28) = 194{,}311$ shares (or \$5,440,708). For a \$5 million bond issue, the flotation cost is 3.2%. Therefore, to net \$5 million, SSI has to sell $5{,}000{,}000/(0.968)(985) = 5{,}243.95$ units of \$1,000 par value. This implies that SSI is effectively borrowing \$5,243,948, upon which figure the annual bond interest will be calculated. The annual bond interest payment is $\$5{,}243{,}948(0.12) = \$629{,}274$.

[4] Flotation costs are higher for small issues than for large ones due to the existence of fixed costs: Certain costs must be incurred regardless of the size of the issue, so the percentage of flotation costs increases as the size of the issue gets smaller.

SOLUTION

(a) *After-tax cash flows*. Table 15.2 summarizes the after-tax cash flows for the new venture. The following calculations and assumptions were used in developing the table:

- Revenue: $250 × 20,000 = $5,000,000 per year.
- Costs of goods: $50.31 × 20,000 = $1,006,200 per year.
- Bond interest: $5,243,948 × 0.12 = $629,274 per year.

TABLE 15.2 **Effects of Project Financing on After-Tax Cash Flows (Example 15.3)**

	0	1	2	3	4	5
Income statement:						
Revenue		$ 5,000,000	$ 5,000,000	$ 5,000,000	$ 5,000,000	$ 5,000,000
Expenses:						
Cost of goods		1,006,200	1,006,200	1,006,200	1,006,200	1,006,200
O&M		600,000	600,000	600,000	600,000	600,000
Bond interest		629,274	629,274	629,274	629,274	629,274
Depreciation:						
Building		73,718	76,923	76,923	76,923	73,718
Equipment		857,400	1,469,400	1,049,400	749,400	267,900
Taxable income		1,833,408	1,218,203	1,638,203	1,938,203	2,422,908
Income taxes		733,363	487,281	655,281	775,281	969,163
Net income		$ 1,100,045	$ 730,922	$ 982,922	$ 1,162,922	$ 1,453,745
Cash flow statement:						
Operating activities:						
Net income		$ 1,100,045	$ 730,922	$ 982,922	$ 1,162,922	$ 1,453,745
Noncash expense		931,118	1,546,323	1,126,323	826,323	341,618
Investment activities:						
Land	(1,000,000)					1,500,000
Building	(3,000,000)					2,000,000
Equipment	(6,000,000)					2,500,000
Working capital	(500,000)					500,000
Gains tax						(308,682)
Financing activities:						
Common stock	5,000,000					(5,440,708)
Bond	5,000,000					(5,243,948)
Cash dividend		(388,622)	(388,622)	(388,622)	(388,622)	(388,622)
Net cash flow	$ (500,000)	$ 1,642,541	$ 1,888,623	$ 1,720,623	$ 1,600,623	$ (3,086,597)

- Depreciation: Assuming that the building is placed in service in January, the first year's depreciation percentage is 2.4573%. Therefore, the allowed depreciation amount is $3,000,000 × 0.024573 = $73,718. The percentages for the remaining years would be 2.5641% per year, or $76,923. Equipment is depreciated according to a 7-year MACRS.
- Gains tax:

Property	Salvage Value	Book Value	Gains (Losses)	Gains Tax
Land	$1,500,000	$1,000,000	$500,000	$200,000
Building	2,000,000	2,621,795	(621,795)	(248,718)
Equipment	2,500,000	1,606,500	893,500	357,400
				$308,682

- Cash dividend: 194,311 shares × $2 = $388,622.
- Common stock: When the project terminates and the bonds are retired, the debt ratio is no longer 0.5. If SSI wants to maintain the constant capital structure (0.5), SSI would have to repurchase the common stock in the amount of $5,440,708 at the prevailing market price. In developing Table 15.2, we assumed that this repurchase of common stock had taken place at the ends of project years. In practice, a firm may or may not repurchase the common stock. As an alternative means of maintaining the desired capital structure, the firm may use this extra debt capacity released to borrow for other projects.
- Bond: When the bonds mature at the end of year 5, the total face value in the amount of $5,243,948 must be paid to the bondholders.

(b) *Measure of project worth*. The NPW for this project is then

$$\begin{aligned} \text{PW}(20\%) &= -\$500{,}000 + \$1{,}642{,}541(P/F, 20\%, 1) + \ldots \\ &= -\$3{,}086{,}597(P/F, 20\%, 5) \\ &= \$2{,}707{,}530. \end{aligned}$$

The investment is nonsimple, and it is also a mixed investment. The RIC at MARR of 20% is 327%. Even though the project requires a significant amount of cash expenditure at the end of its life, it still appears to be a very profitable one.

In Example 15.3, we neither discussed the cost of capital required to finance this project nor explained the relationship between the cost of capital and the MARR. In the remaining sections, these issues will be discussed. As we will see later, in Section 15.3, we can completely ignore the detailed cash flows related to project financing if we adjust our discount rate according to the capital structure, namely, by using the weighted cost of capital.

15.2 Cost of Capital

In most of the capital-budgeting examples in earlier chapters, we assumed that the firms under consideration were financed entirely with equity funds. In those cases, the cost of capital may have represented the firm's required return on equity. However, most firms finance a substantial portion of their capital budget with long-term debt (bonds), and many also use preferred stock as a source of capital. In these cases, a firm's cost of capital must reflect the average cost of the various sources of long-term funds that the firm uses, not only the cost of equity. In this section, we will discuss the ways in which the cost of each individual type of financing (retained earnings, common stock, preferred stock, and debt) can be estimated,[5] given a firm's target capital structure.

15.2.1 Cost of Equity

Whereas debt and preferred stocks are contractual obligations that have easily determined costs, it is not easy to measure the cost of equity. In principle, the cost of equity capital involves an **opportunity cost**. In fact, the firm's after-tax cash flows belong to the stockholders. Management may either pay out these earnings in the form of dividends, or retain the earnings and reinvest them in the business. If management decides to retain the earnings, an opportunity cost is involved: Stockholders could have received the earnings as dividends and invested the money in other financial assets. Therefore, the firm should earn on its retained earnings at least as much as the stockholders themselves could earn in alternative, but comparable, investments.

Cost of equity is the minimum rate of return a firm must offer shareholders to compensate for waiting for their returns, and for bearing some risk.

What rate of return can stockholders expect to earn on retained earnings? This question is difficult to answer, but the value sought is often regarded as the rate of return stockholders require on a firm's common stock. If a firm cannot invest retained earnings so as to earn at least the rate of return on equity, it should pay these funds to the stockholders and let them invest directly in other assets that do provide this return.

When investors are contemplating buying a firm's stock, they have two things in mind: (1) cash dividends and (2) gains (appreciation of shares) at the time of sale. From a conceptual standpoint, investors determine market values of stocks by discounting expected future dividends at a rate that takes into account any future growth. Since investors seek growth companies, a desired growth factor for future dividends is usually included in the calculation.

To illustrate, let's take a simple numerical example. Suppose investors in the common stock of ABC Corporation expect to receive a dividend of \$5 by the end of the first year. The future annual dividends will grow at an annual rate of 10%. Investors will hold the stock for two more years and will expect the market price of the stock to rise to \$120 by the end of the third year. Given these hypothetical expectations, ABC expects that investors would be willing to pay \$100 for this stock in today's market. What is the required rate of return k_r on ABC's common stock? We may answer this question by solving the following equation for k_r:

$$\$100 = \frac{\$5}{(1+k_r)} + \frac{\$5(1+0.1)}{(1+k_r)^2} + \frac{\$5(1+0.1)^2 + \$120}{(1+k_r)^3}.$$

[5] Estimating or calculating the cost of capital in any precise fashion is very difficult task.

In this case, $k_r = 11.44\%$. This implies that if ABC finances a project by retaining its earnings or by issuing additional common stock at the going market price of \$100 per share, it must realize at least 11.44% on new investment just to provide the minimum rate of return required by the investors. Therefore, 11.44% is the specific cost of equity that should be used in calculating the weighted-average cost of capital. Because flotation costs are involved in issuing new stock, the cost of equity will increase. If investors view ABC's stock as risky and therefore are willing to buy the stock at a price lower than \$100 (but with the same expectations), the cost of equity will also increase. Now we can generalize the preceding result.

Cost of Retained Earnings

Let's assume the same hypothetical situation for ABC. Recall that ABC's retained earnings belong to holders of its common stock. If ABC's current stock is traded for a market price of P_0, with a first-year dividend[6] of D_1, but growing at the annual rate of g thereafter, the specific cost of retained earnings for an infinite period of holding (stocks will change hands over the years, but it does not matter who holds the stock) can be calculated as

$$\begin{aligned} P_0 &= \frac{D_1}{(1+k_r)} + \frac{D_1(1+g)}{(1+k_r)^2} + \frac{D_1(1+g)^2}{(1+k_r)^3} + \cdots \\ &= \frac{D_1}{1+k_r}\sum_{n=0}^{\infty}\left[\frac{(1+g)}{(1+k_r)}\right]^n \\ &= \frac{D_1}{1+k_r}\left[\frac{1}{1-\dfrac{1+g}{1+k_r}}\right], \text{ where } g < k_r. \end{aligned}$$

Solving for k_r, we obtain

$$k_r = \frac{D_1}{P_0} + g. \tag{15.1}$$

If we use k_r as the discount rate for evaluating the new project, it will have a positive NPW only if the project's IRR exceeds k_r. Therefore, any project with a positive NPW, calculated at k_r, induces a rise in the market price of the stock. Hence, by definition, k_r is the rate of return required by shareholders and should be used as the cost of the equity component in calculating the weighted average cost of capital.

Issuing New Common Stock

Again, because flotation costs are involved in issuing new stock, we can modify the cost of retained earnings k_r by

$$k_e = \frac{D_1}{P_0(1-f_c)} + g, \tag{15.2}$$

where k_e is the cost of common equity and f_c is the flotation cost as a percentage of the stock price.

[6] When we check the stock listings in the newspaper, we do not find the expected first-year dividend D_1. Instead, we find the dividend paid out most recently, D_0. So if we expect growth at a rate g, the dividend at the end of one year from now, D_1 may be estimated as $D_1 = D_0(1+g)$.

Either calculation is deceptively simple, because, in fact, several ways are available to determine the cost of equity. In reality, the market price fluctuates constantly, as do a firm's future earnings. Thus, future dividends may not grow at a constant rate, as the model indicates. For a stable corporation with moderate growth, however, the cost of equity as calculated by evaluating either Eq. (15.1) or Eq. (15.2) serves as a good approximation.

Cost of Preferred Stock

A preferred stock is a hybrid security in the sense that it has some of the properties of bonds and other properties that are similar to common stock. Like bondholders, holders of preferred stock receive a fixed annual dividend. In fact, many firms view the payment of the preferred dividend as an obligation just like interest payments to bondholders. It is therefore relatively easy to determine the cost of preferred stock. For the purposes of calculating the weighted average cost of capital, the specific cost of a preferred stock will be defined as

$$k_p = \frac{D^*}{P^*(1 - f_c)}, \tag{15.3}$$

where D^* is the fixed annual dividend, P^* is the issuing price, and f_c is as defined in Eq. (15.2).

Cost of Equity

Once we have determined the specific cost of each equity component, we can determine the weighted-average cost of equity (i_e) for a new project. We have

$$i_e = \left(\frac{c_r}{c_e}\right)k_r + \left(\frac{c_c}{c_e}\right)k_e + \left(\frac{c_p}{c_e}\right)k_p, \tag{15.4}$$

where c_r is the amount of equity financed from retained earnings, c_c is the amount of equity financed from issuing new stock, c_p is the amount of equity financed from issuing preferred stock, and $c_r + c_c + c_p = c_e$. Example 15.4 illustrates how we may determine the cost of equity.

EXAMPLE 15.4 Determining the Cost of Equity

Alpha Corporation needs to raise $10 million for plant modernization. Alpha's target capital structure calls for a debt ratio of 0.4, indicating that $6 million has to be financed from equity.

- Alpha is planning to raise $6 million from the following equity sources:

Source	Amount	Fraction of Total Equity
Retained earnings	$1 million	0.167
New common stock	4 million	0.666
Preferred stock	1 million	0.167

- Alpha's current common stock price is \$40, the market price that reflects the firm's future plant modernization. Alpha is planning to pay an annual cash dividend of \$5 at the end of the first year, and the annual cash dividend will grow at an annual rate of 8% thereafter.
- Additional common stock can be sold at the same price of \$40, but there will be 12.4% flotation costs.
- Alpha can issue \$100 par preferred stock with a 9% dividend. (This means that Alpha will calculate the dividend on the basis of the par value, which is \$9 per share.) The stock can be sold on the market for \$95, and Alpha must pay flotation costs of 6% of the market price.

Determine the cost of equity to finance the plant modernization.

SOLUTION

We will itemize the cost of each component of equity:

- Cost of retained earnings: With $D_1 = \$5$, $g = 8\%$, and $P_0 = \$40$,

$$k_r = \frac{5}{40} + 0.08 = 20.5\%.$$

- Cost of new common stock: With $D_1 = \$5$, $g = 8\%$, and $f_c = 12.4\%$,

$$k_e = \frac{5}{40(1 - 0.124)} + 0.08 = 22.27\%.$$

- Cost of preferred stock: With $D^* = \$9$, $P^* = \$95$, and $f_c = 0.06$,

$$k_p = \frac{9}{95(1 - 0.06)} = 10.08\%.$$

- Cost of equity: With $\frac{c_r}{c_e} = 0.167$, $\frac{c_c}{c_e} = 0.666$, and $\frac{c_p}{c_e} = 0.167$,

$$\begin{aligned} i_e &= (0.167)(0.205) + (0.666)(0.2227) + (0.167)(0.1008) \\ &= 19.96\%. \end{aligned}$$

An Alternative Way of Determining the Cost of Equity

Whereas debt and preferred stocks are contractual obligations that have easily determined costs, it is not easy to measure the cost of equity. In principle, the cost of equity capital involves an **opportunity cost**. In fact, the firm's after-tax cash flows belong to the stockholders. Management may either pay out these earnings in the form of dividends or retain the earnings and reinvest them in the business. If management decides to retain the earnings, an opportunity cost is involved: Stockholders could have received the earnings as dividends and invested that money in other financial assets. Therefore, the firm should earn on its retained earnings at least as much as the stockholders themselves could earn in alternative, but comparable, investments.

What rate of return can stockholders expect to earn on retained earnings? This question is difficult to answer, but the value sought is often regarded as the rate of return stockholders require on a firm's common stock. If a firm cannot invest retained earnings so as to earn at least the rate of return on equity, it should pay these funds to its stockholders and let them invest directly in other assets that do provide that rate of return. In general, the expected return on any risky asset is composed of three factors:[7]

$$\begin{pmatrix}\text{Expected return}\\ \text{on risky asset}\end{pmatrix} = \begin{pmatrix}\text{Risk-free}\\ \text{interest rate}\end{pmatrix} + \begin{pmatrix}\text{Inflation}\\ \text{premium}\end{pmatrix} + \begin{pmatrix}\text{Risk}\\ \text{premium}\end{pmatrix}.$$

Inflation risk: The possibility that the value of assets or income will decrease as inflation shrinks the purchasing power of a currency.

This equation says that the owner of a risky asset should expect to earn a return from three sources:

- Compensation from the opportunity cost incurred in holding the asset, known as the risk-free interest rate.
- Compensation for the declining purchasing power of the investment over time, known as the inflation premium.
- Compensation for bearing risk, known as the risk premium.

Fortunately, we do not need to treat the first two terms as separate factors because together they equal the expected return on a default-free bond such as a government bond. In other words, owners of government bonds expect a return from the first two sources, but not the third—a state of affairs we may express as

$$\begin{pmatrix}\text{Expected return}\\ \text{on risky asset}\end{pmatrix} = \begin{pmatrix}\text{Interest rate on}\\ \text{government bond}\end{pmatrix} + \begin{pmatrix}\text{Risk}\\ \text{premium}\end{pmatrix}.$$

Risk premium: The return in excess of the risk-free rate of return that an investment is expected to yield.

When investors are contemplating buying a firm's stock, they have two primary things in mind: (1) cash dividends and (2) gains (appreciation of shares) at the time of sale. From a conceptual standpoint, investors determine market values of stocks by discounting expected future dividends at a rate that takes into account any future growth. Since investors seek growth companies, a desired growth factor for future dividends is usually included in the calculation.

The cost of equity is the risk-free cost of debt (e.g., 20-year U.S. Treasury bills around 6%), plus a premium for taking a risk as to whether a return will be received. The risk premium is the average return on the market, typically the mean of Standard & Poor's 500 large U.S. stocks, or S&P 500 (say, 12.5%), less the risk-free cost of debt. This premium is multiplied by *beta* (β), an approximate measure of stock price volatility that quantifies risk. β measures one firm's stock price relative to the market stock price as a whole. A number greater than unity means that the stock is *more* volatile than the market, on average; a number less than unity means that the stock is *less* volatile than the market, on average. Values for β for most publicly traded stocks are commonly found in various sources, such as Value-Line.[8] The following formula quantifies the cost of equity (i_e):

Beta: A measure of the volatility, or systematic risk, of a security or a portfolio in comparison to the market as a whole.

$$i_e = r_f + \beta[r_M - r_f]. \tag{15.5}$$

[7] An excellent discussion of this subject matter is found in Robert C. Higgins, *Analysis for Financial Management*, 5th ed. (Boston: Irwin/McGraw-Hill, 1998).

[8] Value Line reports are presently available for over 5,000 public companies, and the number is growing. The Value Line reports contain the following information: (1) total assets, (2) total liabilities, (3) total equity, (4) long-term debt as a percentage of capital, (5) equity as a percentage of capital, (6) financial strength (which is used to determine the interest rate), (7) β, and (8) return on invested capital.

Here, r_f is the risk-free interest rate (commonly referenced to the U.S. Treasury bond yield, adjusted for inflation) and r_M is the market rate of return (commonly referenced to the average return on S&P 500 stock index funds, adjusted for inflation).

Note that the cost of equity is almost always higher than the cost of debt. This is because the U.S. Tax Code allows the deduction of interest expense, but does not allow the deduction of the cost of equity, which could be considered more subjective and complicated. Example 15.5 illustrates how we may determine the cost of equity.

EXAMPLE 15.5 Determining the Cost of Equity by the Financial Market

Alpha Corporation needs to raise \$10 million for plant modernization. Alpha's target capital structure calls for a debt ratio of 0.4, indicating that \$6 million has to be financed from equity.

- Alpha is planning to raise \$6 million from the financial market.
- Alpha's β is known to be 1.99, which is greater than unity, indicating that the firm is perceived as more risky than the market average.
- The risk-free interest rate is 6%, and the average market return is 13%. (All these interest rates are adjusted to reflect inflation in the economy.)

Determine the cost of equity to finance the plant modernization.

SOLUTION

Given: $r_M = 13\%$, $r_f = 6\%$, and $\beta = 1.99$.
Find: i_e.

$$\begin{aligned} i_e &= 0.06 + 1.99(0.13 - 0.06) \\ &= 19.93\%. \end{aligned}$$

COMMENTS: In this example, we purposely selected the value of β to approximate the cost of equity derived from Example 15.4. What does this 19.93% represent? If Alpha finances the project entirely from its equity funds, the project must earn at least a 19.93% return on investment.

15.2.2 Cost of Debt

Now let us consider the calculation of the specific cost that is to be assigned to the debt component of the weighted-average cost of capital. The calculation is relatively straightforward and simple. As we said in Section 15.1.2, the two types of debt financing are term loans and bonds. Because the interest payments on both are tax deductible, the effective cost of debt will be reduced.

To determine the after-tax cost of debt (i_d), we evaluate the expression

$$i_d = \left(\frac{c_s}{c_d}\right)k_s(1 - t_m) + \left(\frac{c_b}{c_d}\right)k_b(1 - t_m), \qquad (15.6)$$

where c_s is the amount of the short-term loan, k_s is the before-tax interest rate on the term loan, t_m is the firm's marginal tax rate, k_b is the before-tax interest rate on the bond, c_b is the amount of bond financing, and $c_s + c_b = c_d$.

As for bonds, a new issue of long-term bonds incurs flotation costs. These costs reduce the proceeds to the firm, thereby raising the specific cost of the capital raised. For example, when a firm issues a $1,000 par bond, but nets only $940, the flotation cost will be 6%. Therefore, the effective after-tax cost of the bond component will be higher than the nominal interest rate specified on the bond. We will examine this problem with a numerical example.

EXAMPLE 15.6 Determining the Cost of Debt

Cost of debt: The effective rate that a company pays on its current debt.

Consider again Example 15.4, and suppose that Alpha has decided to finance the remaining $4 million by securing a term loan and issuing 20-year $1,000 par bonds under the following conditions:

Source	Amount	Fraction	Interest Rate	Flotation Cost
Term loan	$1 million	0.333	12% per year	
Bonds	3 million	0.667	10% per year	6%

If the bond can be sold to net $940 (after deducting the 6% flotation cost), determine the cost of debt to raise $4 million for the plant modernization. Alpha's marginal tax rate is 38%, and it is expected to remain constant in the future.

SOLUTION

First, we need to find the effective after-tax cost of issuing the bond with a flotation cost of 6%. The before-tax specific cost is found by solving the equivalence formula

$$\$940 = \frac{\$100}{(1 + k_b)} + \frac{\$100}{(1 + k_b)^2} + \cdots + \frac{\$100 + \$1{,}000}{(1 + k_b)^{20}}$$

$$= \$100(P/A, k_b, 20) + \$1{,}000(P/F, k_b, 20).$$

Solving for k_b, we obtain $k_b = 10.74\%$. Note that the cost of the bond component increases from 10% to 10.74% after the 6% flotation cost is taken into account.

The after-tax cost of debt is the interest rate on debt, multiplied by $(1 - t_m)$. In effect, the government pays part of the cost of debt because interest is tax deductible. Now we are ready to compute the after-tax cost of debt as follows:

$$\begin{aligned} i_d &= (0.333)(0.12)(1 - 0.38) + (0.667)(0.1074)(1 - 0.38) \\ &= 6.92\%. \end{aligned}$$

15.2.3 Calculating the Cost of Capital

With the specific cost of each financing component determined, we are ready to calculate the tax-adjusted weighted-average cost of capital based on total capital. Then we will define the marginal cost of capital that should be used in project evaluation.

Weighted-Average Cost of Capital

Cost of capital for a firm is a weighted sum of the cost of equity and the cost of debt.

Assuming that a firm raises capital on the basis of the target capital structure and that the target capital structure remains unchanged in the future, we can determine a **tax-adjusted weighted-average cost of capital** (or, simply stated, the **cost of capital**). This cost of capital represents a composite index reflecting the cost of raising funds from different sources. The cost of capital is defined as

$$k = \frac{i_d c_d}{V} + \frac{i_e c_e}{V}, \tag{15.7}$$

where c_d = Total debt capital (such as bonds) in dollars,
c_e = Total equity capital in dollars,
$V = c_d + c_e$,
i_e = Average equity interest rate per period, taking into account all equity sources,
i_d = After-tax average borrowing interest rate per period, taking into account all debt sources, and
k = Tax-adjusted weighted-average cost of capital.

Note that the cost of equity is already expressed in terms of after-tax cost, because any return to holders of either common stock or preferred stock is made after payment of income taxes. Figure 15.2 summarizes the process of determining the cost of capital.

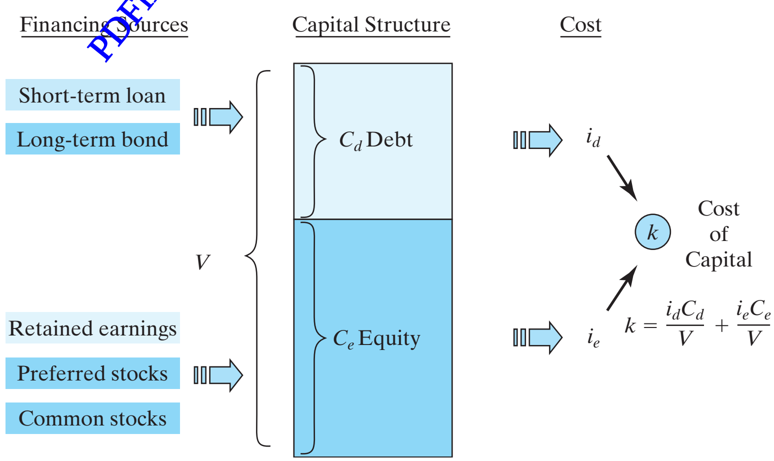


Figure 15.2 Process of calculating the cost of capital. Recall that there are two different ways of determining the cost of equity—one by the traditional approach and the other by the financial market, commonly known as "capital asset pricing model."

Marginal Cost of Capital

Now that we know how to calculate the cost of capital, we can ask, Could a typical firm raise unlimited new capital at the same cost? The answer is no. As a practical matter, as a firm tries to attract more new dollars, the cost of raising each additional dollar will at some point rise. As this occurs, the weighted-average cost of raising each additional new dollar also rises. Thus, the **marginal cost of capital** is defined as the cost of obtaining another dollar of new capital, and the marginal cost rises as more and more capital is raised during a given period. In evaluating an investment project, we are using the concept of the marginal cost of capital. The formula to find the marginal cost of capital is exactly the same as Eq. (15.6); however, the costs of debt and equity in that equation are the interest rates on new debt and equity, not outstanding (or combined) debt or equity. In other words, we are interested in the marginal cost of capital—specifically, to use it in evaluating a new investment project. The rate at which the firm has borrowed in the past is less important for this purpose. Example 15.7 works through the computations for finding the cost of capital (k).

EXAMPLE 15.7 Calculating the Marginal Cost of Capital

Consider again Examples 15.5 and 15.6. The marginal income tax rate (t_m) for Alpha is expected to remain at 38% in the future. Assuming that Alpha's capital structure (debt ratio) also remains unchanged, determine the cost of capital (k) for raising \$10 million in addition to existing capital.

SOLUTION

With c_d = \$4 million, c_e = \$6 million, V = \$10 million, $i_d = 6.92\%$, $i_e = 19.93\%$, and Eq. (15.7), we calculate

$$k = \frac{(0.0692)(4)}{10} + \frac{(0.1993)(6)}{10}$$
$$= 14.73\%.$$

This 14.73% would be the marginal cost of capital that a company with the given financial structure would expect to pay to raise \$10 million.

15.3 Choice of Minimum Attractive Rate of Return

Thus far, we have said little about what interest rate, or minimum attractive rate of return (MARR), is suitable for use in a particular investment situation. Choosing the MARR is a difficult problem; no single rate is always appropriate. In this section, we will discuss briefly how to select a MARR for project evaluation. Then we will examine the relationship between capital budgeting and the cost of capital.

MARR: The required return necessary to make a capital budgeting project—such as building a new factory—worthwhile.

15.3.1 Choice of MARR when Project Financing Is Known

In Chapter 10, we focused on calculating after-tax cash flows, including situations involving debt financing. When cash flow computations reflect interest, taxes, and debt repayment, what is left is called **net equity flow**. If the goal of a firm is to maximize the wealth of its stockholders, why not focus only on the after-tax cash flow to equity, instead

Summary of Project Analysis Methods

Analysis Method	Description	Single Project Evaluation	Mutually Exclusive Projects	
			Revenue Projects	**Service Projects**
Payback period PP	A method for determining when in a project's history it breaks even. Management sets the benchmark PP°.	$PP < PP^{\circ}$	Select the one with shortest PP	
Discounted payback period PP(i)	A variation of payback period when factors in the time value of money. Management sets the benchmark $PP^{\bullet}$.	$PP(i) < PP^{\bullet}$	Select the one with shortest PP(i)	
Present worth PW(i)	An equivalent method which translates a project's cash flows into a net present value	$PW(i) > 0$	Select the one with the largest PW	Select the one with the least negative PW
Future worth FW(i)	An equivalence method variation of the PW: a project's cash flows are translated into a net future value	$FW(i) > 0$	Select the one with the largest FW	Select the one with the least negative FW
Capitalized equivalent CE(i)	An equivalence method variation of the PW of perpetual or very long-lived project that generates a constant annual net cash flow	$CE(i) > 0$	Select the one with the largest CE	Select the one with the least negative CE
Annual equivalence AE(i)	An equivalence method and variation of the PW: a project's cash flows are translated into an annual equivalent sum	$AE(i) > 0$	Select the one with the largest AE	Select the one with the least negative AE
Internal rate of return IRR	A relative percentage method which measures the yield as a percentage of investment over the life of a project: The IRR must exceed the minimum required rate of return (MARR).	IRR > MARR	Incremental analysis: If $IRR_{A2\text{-}A1} >$ MARR, select the higher cost investment project, A2.	
Benefit–cost ratio BC(i)	An equivalence method to evaluate public projects by finding the ratio of the equivalent benefit over the equivalent cost	$BC(i) > 1$	Incremental analysis: If $BC(i)_{A2\text{-}A1} > 1$, select the higher cost investment project, A2.	

Summary of Useful Excel's Financial Functions (Part A)

<table>
<tr><th colspan="2">Description</th><th>Excel Function</th><th>Example</th><th>Solution</th></tr>
<tr><td>Single-Payment</td><td>Find: F
Given: P</td><td>=FV(i%, N, 0, −P)</td><td>Find the future worth of $500 in 5 years at 8%.</td><td>=FV(8%, 5, 0, −500)
=$734.66</td></tr>
<tr><td>Cash Flows</td><td>Find: P
Given: F</td><td>=PV(i%, N, 0, F)</td><td>Find the present worth of $1,300 due in 10 years at a 16% interest rate.</td><td>=PV(16%, 10, 0, 1300)
=($294.69)</td></tr>
<tr><td></td><td>Find: F
Given: A</td><td>=FV(i%, N, A)</td><td>Find the future worth of a payment series of $200 per year for 12 years at 6%.</td><td>=FV(6%, 12, −200)
=$3,373.99</td></tr>
<tr><td>Equal-Payment-Series</td><td>Find: P
Given: A</td><td>=PV(i%, N, A)</td><td>Find the present worth of a payment series of $900 per year for 5 years at 8% interest rate.</td><td>=PV(8%, 5, 900)
=($3,593,44)</td></tr>
<tr><td></td><td>Find: A
Given: P</td><td>=PMT(i%, N, −P)</td><td>What equal-annual-payment series is required to repay $25,000 in 5 years at 9% interest rate?</td><td>=PMT(9%, 5, −25000)
=$6,427.31</td></tr>
<tr><td></td><td>Find: A
Given: F</td><td>=PMT(i%, N, 0, F)</td><td>What is the required annual savings to accumulate $50,000 in 3 years at 7% interest rate?</td><td>=PMT(7%, 3, 0, 50000)
=($15,552.58)</td></tr>
<tr><td></td><td>Find: NPW
Given: Cash flow series</td><td>=NPV(i%, series)</td><td>Consider a project with the following cash flow series at 12% (n = 0, −$200; n = 1, $150, n = 2, $300, n = 3, 250)?</td><td>=NPV(12%, B3:B5)+B2
=$351.03</td></tr>
<tr><td>Measures of Investment Worth</td><td>Find: IRR
Given: Cash flow series</td><td>=IRR(values, guess)</td><td><table><tr><th></th><th>A</th><th>B</th></tr><tr><td>1</td><td>Period</td><td>Cash Flow</td></tr><tr><td>2</td><td>0</td><td>−200</td></tr><tr><td>3</td><td>1</td><td>150</td></tr><tr><td>4</td><td>2</td><td>300</td></tr><tr><td>5</td><td>3</td><td>250</td></tr></table></td><td>=IRR(B2:B5, 10%)
=89%</td></tr>
<tr><td></td><td>Find: AW
Given: Cash flow series</td><td>=PMT (i%, N, −NPW)</td><td></td><td>=PMT(12%, 3, −351.03)
=$146.15</td></tr>
</table>

Summary of Useful Excel's Financial Functions (Part B)

Description		Excel Function	Example	Solution
Loan Analysis Functions	Loan payment size	=PMT($i\%, N, P$)	Suppose you borrow \$10,000 at 9% interest to be paid in 48 equal monthly payments. Find the loan payment size.	=PMT(9%/12, 48, 10000) =(\$248.45)
	Interest payment	=IMPT($i\%, n, N, P$)	Find the portion of interest payment for the 10th payment.	=IPMT(9%/12, 10, 48, 10000) =(\$62.91)
	Principal payment	=PPMT($i\%, n, N, P$)	Find the portion of principal payment for the 10th payment.	=PPMT(9%/12, 10, 48, 10000) =(\$185.94)
	Cumulative interest payment	=CUMIMPT($i\%, N$, P, start_period, end_period)	Find the total interest payment over 48 months.	=CUMIMPT(9%/12, 48, 10000, 1, 48) =\$1,944.82
	Interest rate	=RATE(N, A, P, F)	What nominal interest rate is being paid on the following financing arrangement? Loan amount:\$10,000, loan period: 60 months, and monthly payment: \$207.58.	=RATE(60, 207.58, −10000) =0.7499% APR $= 0.7499\% \times 12 = 9\%$
	Number of payments	=NPER($i\%, A, P, F$)	Find the number of months required to pay off a loan of \$10,000 with 12% APR where you can afford a monthly payment of \$200.	=NPER(12%/12, 200, −10000) =69.66 months
Deprecia-tion functions	Straight-line	=SLN(cost, salvage, life)	Cost = \$100,000, S = \$20,000, life = 5 years	=SLN(100000, 20000, 5) =\$16,000
	Declining balance	=DB(cost, salvage, life, period, month)	Find the depreciation amount in period 3.	=DB(100000, 20000, 5, 3, 12) =\$14,455
	Double declining balance	=DDB(cost, salvage, life, period, factor)	Find the depreciation amount in period 3 with α = 150%,	=DDB(100000, 20000, 5, 3, 1.5) =\$14,700
	Declining balance with switching to straight-line	=VDB(cost, salvage, life, strat_period, end_period, factor)	Find the depreciation amount in period 3 with α = 150%, with switching allowed.	=VDB(100000, 20000, 5, 3, 4,1.5) =\$10,290

APPENDIX A

Interest Factors for Discrete Compounding

0.25%

	Single Payment		Equal Payment Series				Gradient Series		
N	**Compound Amount Factor (F/P,i,N)**	**Present Worth Factor (P/F,i,N)**	**Compound Amount Factor (F/A,i,N)**	**Sinking Fund Factor (A/F,i,N)**	**Present Worth Factor (P/A,i,N)**	**Capital Recovery Factor (A/P,i,N)**	**Gradient Uniform Series (A/G,i,N)**	**Gradient Present Worth (P/G,i,N)**	**N**
1	1.0025	0.9975	1.0000	1.0000	0.9975	1.0025	0.0000	0.0000	1
2	1.0050	0.9950	2.0025	0.4994	1.9925	0.5019	0.4994	0.9950	2
3	1.0075	0.9925	3.0075	0.3325	2.9851	0.3350	0.9983	2.9801	3
4	1.0100	0.9901	4.0150	0.2491	3.9751	0.2516	1.4969	5.9503	4
5	1.0126	0.9876	5.0251	0.1990	4.9627	0.2015	1.9950	9.9007	5
6	1.0151	0.9851	6.0376	0.1656	5.9478	0.1681	2.4927	14.8263	6
7	1.0176	0.9827	7.0527	0.1418	6.9305	0.1443	2.9900	20.7223	7
8	1.0202	0.9802	8.0704	0.1239	7.9107	0.1264	3.4869	27.5839	8
9	1.0227	0.9778	9.0905	0.1100	8.8885	0.1125	3.9834	35.4061	9
10	1.0253	0.9753	10.1133	0.0989	9.8639	0.1014	4.4794	44.1842	10
11	1.0278	0.9729	11.1385	0.0898	10.8368	0.0923	4.9750	53.9133	11
12	1.0304	0.9705	12.1664	0.0822	11.8073	0.0847	5.4702	64.5886	12
13	1.0330	0.9681	13.1968	0.0758	12.7753	0.0783	5.9650	76.2053	13
14	1.0356	0.9656	14.2298	0.0703	13.7410	0.0728	6.4594	88.7587	14
15	1.0382	0.9632	15.2654	0.0655	14.7042	0.0680	6.9534	102.2441	15
16	1.0408	0.9608	16.3035	0.0613	15.6650	0.0638	7.4469	116.6567	16
17	1.0434	0.9584	17.3443	0.0577	16.6235	0.0602	7.9401	131.9917	17
18	1.0460	0.9561	18.3876	0.0544	17.5795	0.0569	8.4328	148.2446	18
19	1.0486	0.9537	19.4336	0.0515	18.5332	0.0540	8.9251	165.4106	19
20	1.0512	0.9513	20.4822	0.0488	19.4845	0.0513	9.4170	183.4851	20
21	1.0538	0.9489	21.5334	0.0464	20.4334	0.0489	9.9085	202.4634	21
22	1.0565	0.9466	22.5872	0.0443	21.3800	0.0468	10.3995	222.3410	22
23	1.0591	0.9442	23.6437	0.0423	22.3241	0.0448	10.8901	243.1131	23
24	1.0618	0.9418	24.7028	0.0405	23.2660	0.0430	11.3804	264.7753	24
25	1.0644	0.9395	25.7646	0.0388	24.2055	0.0413	11.8702	287.3230	25
26	1.0671	0.9371	26.8290	0.0373	25.1426	0.0398	12.3596	310.7516	26
27	1.0697	0.9348	27.8961	0.0358	26.0774	0.0383	12.8485	335.0566	27
28	1.0724	0.9325	28.9658	0.0345	27.0099	0.0370	13.3371	360.2334	28
29	1.0751	0.9301	30.0382	0.0333	27.9400	0.0358	13.8252	386.2776	29
30	1.0778	0.9278	31.1133	0.0321	28.8679	0.0346	14.3130	413.1847	30
31	1.0805	0.9255	32.1911	0.0311	29.7934	0.0336	14.8003	440.9502	31
32	1.0832	0.9232	33.2716	0.0301	30.7166	0.0326	15.2872	469.5696	32
33	1.0859	0.9209	34.3547	0.0291	31.6375	0.0316	15.7736	499.0386	33
34	1.0886	0.9186	35.4406	0.0282	32.5561	0.0307	16.2597	529.3528	34
35	1.0913	0.9163	36.5292	0.0274	33.4724	0.0299	16.7454	560.5076	35
36	1.0941	0.9140	37.6206	0.0266	34.3865	0.0291	17.2306	592.4988	36
40	1.1050	0.9050	42.0132	0.0238	38.0199	0.0263	19.1673	728.7399	40
48	1.1273	0.8871	50.9312	0.0196	45.1787	0.0221	23.0209	1040.0552	48
50	1.1330	0.8826	53.1887	0.0188	46.9462	0.0213	23.9802	1125.7767	50
60	1.1616	0.8609	64.6467	0.0155	55.6524	0.0180	28.7514	1600.0845	60
72	1.1969	0.8355	78.7794	0.0127	65.8169	0.0152	34.4221	2265.5569	72
80	1.2211	0.8189	88.4392	0.0113	72.4260	0.0138	38.1694	2764.4568	80
84	1.2334	0.8108	93.3419	0.0107	75.6813	0.0132	40.0331	3029.7592	84
90	1.2520	0.7987	100.7885	0.0099	80.5038	0.0124	42.8162	3446.8700	90
96	1.2709	0.7869	108.3474	0.0092	85.2546	0.0117	45.5844	3886.2832	96
100	1.2836	0.7790	113.4500	0.0088	88.3825	0.0113	47.4216	4191.2417	100
108	1.3095	0.7636	123.8093	0.0081	94.5453	0.0106	51.0762	4829.0125	108
120	1.3494	0.7411	139.7414	0.0072	103.5618	0.0097	56.5084	5852.1116	120
240	1.8208	0.5492	328.3020	0.0030	180.3109	0.0055	107.5863	19398.9852	240
360	2.4568	0.4070	582.7369	0.0017	237.1894	0.0042	152.8902	36263.9299	360

0.50%

	Single Payment		Equal Payment Series				Gradient Series		
N	**Compound Amount Factor *(F/P,i,N)***	**Present Worth Factor *(P/F,i,N)***	**Compound Amount Factor *(F/A,i,N)***	**Sinking Fund Factor *(A/F,i,N)***	**Present Worth Factor *(P/A,i,N)***	**Capital Recovery Factor *(A/P,i,N)***	**Gradient Uniform Series *(A/G,i,N)***	**Gradient Present Worth *(P/G,i,N)***	***N***
1	1.0050	0.9950	1.0000	1.0000	0.9950	1.0050	0.0000	0.0000	1
2	1.0100	0.9901	2.0050	0.4988	1.9851	0.5038	0.4988	0.9901	2
3	1.0151	0.9851	3.0150	0.3317	2.9702	0.3367	0.9967	2.9604	3
4	1.0202	0.9802	4.0301	0.2481	3.9505	0.2531	1.4938	5.9011	4
5	1.0253	0.9754	5.0503	0.1980	4.9259	0.2030	1.9900	9.8026	5
6	1.0304	0.9705	6.0755	0.1646	5.8964	0.1696	2.4855	14.6552	6
7	1.0355	0.9657	7.1059	0.1407	6.8621	0.1457	2.9801	20.4493	7
8	1.0407	0.9609	8.1414	0.1228	7.8230	0.1278	3.4738	27.1755	8
9	1.0459	0.9561	9.1821	0.1089	8.7791	0.1139	3.9668	34.8244	9
10	1.0511	0.9513	10.2280	0.0978	9.7304	0.1028	4.4589	43.3865	10
11	1.0564	0.9466	11.2792	0.0887	10.6770	0.0937	4.9501	52.8526	11
12	1.0617	0.9419	12.3356	0.0811	11.6189	0.0861	5.4406	63.2136	12
13	1.0670	0.9372	13.3972	0.0746	12.5562	0.0796	5.9302	74.4602	13
14	1.0723	0.9326	14.4642	0.0691	13.4887	0.0741	6.4190	86.5835	14
15	1.0777	0.9279	15.5365	0.0644	14.4166	0.0694	6.9069	99.5743	15
16	1.0831	0.9233	16.6142	0.0602	15.3399	0.0652	7.3940	113.4238	16
17	1.0885	0.9187	17.6973	0.0565	16.2586	0.0615	7.8803	128.1231	17
18	1.0939	0.9141	18.7858	0.0532	17.1728	0.0582	8.3658	143.6634	18
19	1.0994	0.9096	19.8797	0.0503	18.0824	0.0553	8.8504	160.0360	19
20	1.1049	0.9051	20.9791	0.0477	18.9874	0.0527	9.3342	177.2322	20
21	1.1104	0.9006	22.0840	0.0453	19.8880	0.0503	9.8172	195.2434	21
22	1.1160	0.8961	23.1944	0.0431	20.7841	0.0481	10.2993	214.0611	22
23	1.1216	0.8916	24.3104	0.0411	21.6757	0.0461	10.7806	233.6768	23
24	1.1272	0.8872	25.4320	0.0393	22.5629	0.0443	11.2611	254.0820	24
25	1.1328	0.8828	26.5591	0.0377	23.4456	0.0427	11.7407	275.2686	25
26	1.1385	0.8784	27.6919	0.0361	24.3240	0.0411	12.2195	297.2281	26
27	1.1442	0.8740	28.8304	0.0347	25.1980	0.0397	12.6975	319.9523	27
28	1.1499	0.8697	29.9745	0.0334	26.0677	0.0384	13.1747	343.4332	28
29	1.1556	0.8653	31.1244	0.0321	26.9330	0.0371	13.6510	367.6625	29
30	1.1614	0.8610	32.2800	0.0310	27.7941	0.0360	14.1265	392.6324	30
31	1.1672	0.8567	33.4414	0.0299	28.6508	0.0349	14.6012	418.3348	31
32	1.1730	0.8525	34.6086	0.0289	29.5033	0.0339	15.0750	444.7618	32
33	1.1789	0.8482	35.7817	0.0279	30.3515	0.0329	15.5480	471.9055	33
34	1.1848	0.8440	36.9606	0.0271	31.1955	0.0321	16.0202	499.7583	34
35	1.1907	0.8398	38.1454	0.0262	32.0354	0.0312	16.4915	528.3123	35
36	1.1967	0.8356	39.3361	0.0254	32.8710	0.0304	16.9621	557.5598	36
40	1.2208	0.8191	44.1588	0.0226	36.1722	0.0276	18.8359	681.3347	40
48	1.2705	0.7871	54.0978	0.0185	42.5803	0.0235	22.5437	959.9188	48
50	1.2832	0.7793	56.6452	0.0177	44.1428	0.0227	23.4624	1035.6966	50
60	1.3489	0.7414	69.7700	0.0143	51.7256	0.0193	28.0064	1448.6458	60
72	1.4320	0.6983	86.4089	0.0116	60.3395	0.0166	33.3504	2012.3478	72
80	1.4903	0.6710	98.0677	0.0102	65.8023	0.0152	36.8474	2424.6455	80
84	1.5204	0.6577	104.0739	0.0096	68.4530	0.0146	38.5763	2640.6641	84
90	1.5666	0.6383	113.3109	0.0088	72.3313	0.0138	41.1451	2976.0769	90
96	1.6141	0.6195	122.8285	0.0081	76.0952	0.0131	43.6845	3324.1846	96
100	1.6467	0.6073	129.3337	0.0077	78.5426	0.0127	45.3613	3562.7934	100
108	1.7137	0.5835	142.7399	0.0070	83.2934	0.0120	48.6758	4054.3747	108
120	1.8194	0.5496	163.8793	0.0061	90.0735	0.0111	53.5508	4823.5051	120
240	3.3102	0.3021	462.0409	0.0022	139.5808	0.0072	96.1131	13415.5395	240
360	6.0226	0.1660	1004.5150	0.0010	166.7916	0.0060	128.3236	21403.3041	360

0.75%

	Single Payment		Equal Payment Series				Gradient Series		
N	Compound Amount Factor (F/P,i,N)	Present Worth Factor (P/F,i,N)	Compound Amount Factor (F/A,i,N)	Sinking Fund Factor (A/F,i,N)	Present Worth Factor (P/A,i,N)	Capital Recovery Factor (A/P,i,N)	Gradient Uniform Series (A/G,i,N)	Gradient Present Worth (P/G,i,N)	N
1	1.0075	0.9926	1.0000	1.0000	0.9926	1.0075	0.0000	0.0000	1
2	1.0151	0.9852	2.0075	0.4981	1.9777	0.5056	0.4981	0.9852	2
3	1.0227	0.9778	3.0226	0.3308	2.9556	0.3383	0.9950	2.9408	3
4	1.0303	0.9706	4.0452	0.2472	3.9261	0.2547	1.4907	5.8525	4
5	1.0381	0.9633	5.0756	0.1970	4.8894	0.2045	1.9851	9.7058	5
6	1.0459	0.9562	6.1136	0.1636	5.8456	0.1711	2.4782	14.4866	6
7	1.0537	0.9490	7.1595	0.1397	6.7946	0.1472	2.9701	20.1808	7
8	1.0616	0.9420	8.2132	0.1218	7.7366	0.1293	3.4608	26.7747	8
9	1.0696	0.9350	9.2748	0.1078	8.6716	0.1153	3.9502	34.2544	9
10	1.0776	0.9280	10.3443	0.0967	9.5996	0.1042	4.4384	42.6064	10
11	1.0857	0.9211	11.4219	0.0876	10.5207	0.0951	4.9253	51.8174	11
12	1.0938	0.9142	12.5076	0.0800	11.4349	0.0875	5.4110	61.8740	12
13	1.1020	0.9074	13.6014	0.0735	12.3423	0.0810	5.8954	72.7632	13
14	1.1103	0.9007	14.7034	0.0680	13.2430	0.0755	6.3786	84.4720	14
15	1.1186	0.8940	15.8137	0.0632	14.1370	0.0707	6.8606	96.9876	15
16	1.1270	0.8873	16.9323	0.0591	15.0243	0.0666	7.3413	110.2973	16
17	1.1354	0.8807	18.0593	0.0554	15.9050	0.0629	7.8207	124.3887	17
18	1.1440	0.8742	19.1947	0.0521	16.7792	0.0596	8.2989	139.2494	18
19	1.1525	0.8676	20.3387	0.0492	17.6468	0.0567	8.7759	154.8671	19
20	1.1612	0.8612	21.4912	0.0465	18.5080	0.0540	9.2516	171.2297	20
21	1.1699	0.8548	22.6524	0.0441	19.3628	0.0516	9.7261	188.3253	21
22	1.1787	0.8484	23.8223	0.0420	20.2112	0.0495	10.1994	206.1420	22
23	1.1875	0.8421	25.0010	0.0400	21.0533	0.0475	10.6714	224.6682	23
24	1.1964	0.8358	26.1885	0.0382	21.8891	0.0457	11.1422	243.8923	24
25	1.2054	0.8296	27.3849	0.0365	22.7188	0.0440	11.6117	263.8029	25
26	1.2144	0.8234	28.5903	0.0350	23.5422	0.0425	12.0800	284.3888	26
27	1.2235	0.8173	29.8047	0.0336	24.3595	0.0411	12.5470	305.6387	27
28	1.2327	0.8112	31.0282	0.0322	25.1707	0.0397	13.0128	327.5416	28
29	1.2420	0.8052	32.2609	0.0310	25.9759	0.0385	13.4774	350.0867	29
30	1.2513	0.7992	33.5029	0.0298	26.7751	0.0373	13.9407	373.2631	30
31	1.2607	0.7932	34.7542	0.0288	27.5683	0.0363	14.4028	397.0602	31
32	1.2701	0.7873	36.0148	0.0278	28.3557	0.0353	14.8636	421.4675	32
33	1.2796	0.7815	37.2849	0.0268	29.1371	0.0343	15.3232	446.4746	33
34	1.2892	0.7757	38.5646	0.0259	29.9128	0.0334	15.7816	472.0712	34
35	1.2989	0.7699	39.8538	0.0251	30.6827	0.0326	16.2387	498.2471	35
36	1.3086	0.7641	41.1527	0.0243	31.4468	0.0318	16.6946	524.9924	36
40	1.3483	0.7416	46.4465	0.0215	34.4469	0.0290	18.5058	637.4693	40
48	1.4314	0.6986	57.5207	0.0174	40.1848	0.0249	22.0691	886.8404	48
50	1.4530	0.6883	60.3943	0.0166	41.5664	0.0241	22.9476	953.8486	50
60	1.5657	0.6387	75.4241	0.0133	48.1734	0.0208	27.2665	1313.5189	60
72	1.7126	0.5839	95.0070	0.0105	55.4768	0.0180	32.2882	1791.2463	72
80	1.8180	0.5500	109.0725	0.0092	59.9944	0.0167	35.5391	2132.1472	80
84	1.8732	0.5338	116.4269	0.0086	62.1540	0.0161	37.1357	2308.1283	84
90	1.9591	0.5104	127.8790	0.0078	65.2746	0.0153	39.4946	2577.9961	90
96	2.0489	0.4881	139.8562	0.0072	68.2584	0.0147	41.8107	2853.9352	96
100	2.1111	0.4737	148.1445	0.0068	70.1746	0.0143	43.3311	3040.7453	100
108	2.2411	0.4462	165.4832	0.0060	73.8394	0.0135	46.3154	3419.9041	108
120	2.4514	0.4079	193.5143	0.0052	78.9417	0.0127	50.6521	3998.5621	120
240	6.0092	0.1664	667.8869	0.0015	111.1450	0.0090	85.4210	9494.1162	240
360	14.7306	0.0679	1830.7435	0.0005	124.2819	0.0080	107.1145	13312.3871	360

1.0%

	Single Payment		Equal Payment Series				Gradient Series		
N	**Compound Amount Factor (F/P,i,N)**	**Present Worth Factor (P/F,i,N)**	**Compound Amount Factor (F/A,i,N)**	**Sinking Fund Factor (A/F,i,N)**	**Present Worth Factor (P/A,i,N)**	**Capital Recovery Factor (A/P,i,N)**	**Gradient Uniform Series (A/G,i,N)**	**Gradient Present Worth (P/G,i,N)**	**N**
1	1.0100	0.9901	1.0000	1.0000	0.9901	1.0100	0.0000	0.0000	1
2	1.0201	0.9803	2.0100	0.4975	1.9704	0.5075	0.4975	0.9803	2
3	1.0303	0.9706	3.0301	0.3300	2.9410	0.3400	0.9934	2.9215	3
4	1.0406	0.9610	4.0604	0.2463	3.9020	0.2563	1.4876	5.8044	4
5	1.0510	0.9515	5.1010	0.1960	4.8534	0.2060	1.9801	9.6103	5
6	1.0615	0.9420	6.1520	0.1625	5.7955	0.1725	2.4710	14.3205	6
7	1.0721	0.9327	7.2135	0.1386	6.7282	0.1486	2.9602	19.9168	7
8	1.0829	0.9235	8.2857	0.1207	7.6517	0.1307	3.4478	26.3812	8
9	1.0937	0.9143	9.3685	0.1067	8.5660	0.1167	3.9337	33.6959	9
10	1.1046	0.9053	10.4622	0.0956	9.4713	0.1056	4.4179	41.8435	10
11	1.1157	0.8963	11.5668	0.0865	10.3676	0.0965	4.9005	50.8067	11
12	1.1268	0.8874	12.6825	0.0788	11.2551	0.0888	5.3815	60.5687	12
13	1.1381	0.8787	13.8093	0.0724	12.1337	0.0824	5.8607	71.1126	13
14	1.1495	0.8700	14.9474	0.0669	13.0037	0.0769	6.3384	82.4221	14
15	1.1610	0.8613	16.0969	0.0621	13.8651	0.0721	6.8143	94.4810	15
16	1.1726	0.8528	17.2579	0.0579	14.7179	0.0679	7.2886	107.2734	16
17	1.1843	0.8444	18.4304	0.0543	15.5623	0.0643	7.7613	120.7834	17
18	1.1961	0.8360	19.6147	0.0510	16.3983	0.0610	8.2323	134.9957	18
19	1.2081	0.8277	20.8109	0.0481	17.2260	0.0581	8.7017	149.8950	19
20	1.2202	0.8195	22.0190	0.0454	18.0456	0.0554	9.1694	165.4664	20
21	1.2324	0.8114	23.2392	0.0430	18.8570	0.0530	9.6354	181.6950	21
22	1.2447	0.8034	24.4716	0.0409	19.6604	0.0509	10.0998	198.5663	22
23	1.2572	0.7954	25.7163	0.0389	20.4558	0.0489	10.5626	216.0660	23
24	1.2697	0.7876	26.9735	0.0371	21.2434	0.0471	11.0237	234.1800	24
25	1.2824	0.7798	28.2432	0.0354	22.0232	0.0454	11.4831	252.8945	25
26	1.2953	0.7720	29.5256	0.0339	22.7952	0.0439	11.9409	272.1957	26
27	1.3082	0.7644	30.8209	0.0324	23.5596	0.0424	12.3971	292.0702	27
28	1.3213	0.7568	32.1291	0.0311	24.3164	0.0411	12.8516	312.5047	28
29	1.3345	0.7493	33.4504	0.0299	25.0658	0.0399	13.3044	333.4863	29
30	1.3478	0.7419	34.7849	0.0287	25.8077	0.0387	13.7557	355.0021	30
31	1.3613	0.7346	36.1327	0.0277	26.5423	0.0377	14.2052	377.0394	31
32	1.3749	0.7273	37.4941	0.0267	27.2696	0.0367	14.6532	399.5858	32
33	1.3887	0.7201	38.8690	0.0257	27.9897	0.0357	15.0995	422.6291	33
34	1.4026	0.7130	40.2577	0.0248	28.7027	0.0348	15.5441	446.1572	34
35	1.4166	0.7059	41.6603	0.0240	29.4086	0.0340	15.9871	470.1583	35
36	1.4308	0.6989	43.0769	0.0232	30.1075	0.0332	16.4285	494.6207	36
40	1.4889	0.6717	48.8864	0.0205	32.8347	0.0305	18.1776	596.8561	40
48	1.6122	0.6203	61.2226	0.0163	37.9740	0.0263	21.5976	820.1460	48
50	1.6446	0.6080	64.4632	0.0155	39.1961	0.0255	22.4363	879.4176	50
60	1.8167	0.5504	81.6697	0.0122	44.9550	0.0222	26.5333	1192.8061	60
72	2.0471	0.4885	104.7099	0.0096	51.1504	0.0196	31.2386	1597.8673	72
80	2.2167	0.4511	121.6715	0.0082	54.8882	0.0182	34.2492	1879.8771	80
84	2.3067	0.4335	130.6723	0.0077	56.6485	0.0177	35.7170	2023.3153	84
90	2.4486	0.4084	144.8633	0.0069	59.1609	0.0169	37.8724	2240.5675	90
96	2.5993	0.3847	159.9273	0.0063	61.5277	0.0163	39.9727	2459.4298	96
100	2.7048	0.3697	170.4814	0.0059	63.0289	0.0159	41.3426	2605.7758	100
108	2.9289	0.3414	192.8926	0.0052	65.8578	0.0152	44.0103	2898.4203	108
120	3.3004	0.3030	230.0387	0.0043	69.7005	0.0143	47.8349	3334.1148	120
240	10.8926	0.0918	989.2554	0.0010	90.8194	0.0110	75.7393	6878.6016	240
360	35.9496	0.0278	3494.9641	0.0003	97.2183	0.0103	89.6995	8720.4323	360

1.25%

N	Single Payment: Compound Amount Factor (F/P,i,N)	Single Payment: Present Worth Factor (P/F,i,N)	Equal Payment Series: Compound Amount Factor (F/A,i,N)	Equal Payment Series: Sinking Fund Factor (A/F,i,N)	Equal Payment Series: Present Worth Factor (P/A,i,N)	Equal Payment Series: Capital Recovery Factor (A/P,i,N)	Gradient Series: Gradient Uniform Series (A/G,i,N)	Gradient Series: Gradient Present Worth (P/G,i,N)	N
1	1.0125	0.9877	1.0000	1.0000	0.9877	1.0125	0.0000	0.0000	1
2	1.0252	0.9755	2.0125	0.4969	1.9631	0.5094	0.4969	0.9755	2
3	1.0380	0.9634	3.0377	0.3292	2.9265	0.3417	0.9917	2.9023	3
4	1.0509	0.9515	4.0756	0.2454	3.8781	0.2579	1.4845	5.7569	4
5	1.0641	0.9398	5.1266	0.1951	4.8178	0.2076	1.9752	9.5160	5
6	1.0774	0.9282	6.1907	0.1615	5.7460	0.1740	2.4638	14.1569	6
7	1.0909	0.9167	7.2680	0.1376	6.6627	0.1501	2.9503	19.6571	7
8	1.1045	0.9054	8.3589	0.1196	7.5681	0.1321	3.4348	25.9949	8
9	1.1183	0.8942	9.4634	0.1057	8.4623	0.1182	3.9172	33.1487	9
10	1.1323	0.8832	10.5817	0.0945	9.3455	0.1070	4.3975	41.0973	10
11	1.1464	0.8723	11.7139	0.0854	10.2178	0.0979	4.8758	49.8201	11
12	1.1608	0.8615	12.8604	0.0778	11.0793	0.0903	5.3520	59.2967	12
13	1.1753	0.8509	14.0211	0.0713	11.9302	0.0838	5.8262	69.5072	13
14	1.1900	0.8404	15.1964	0.0658	12.7706	0.0783	6.2982	80.4320	14
15	1.2048	0.8300	16.3863	0.0610	13.6005	0.0735	6.7682	92.0519	15
16	1.2199	0.8197	17.5912	0.0568	14.4203	0.0693	7.2362	104.3481	16
17	1.2351	0.8096	18.8111	0.0532	15.2299	0.0657	7.7021	117.3021	17
18	1.2506	0.7996	20.0462	0.0499	16.0295	0.0624	8.1659	130.8958	18
19	1.2662	0.7898	21.2968	0.0470	16.8193	0.0595	8.6277	145.1115	19
20	1.2820	0.7800	22.5630	0.0443	17.5993	0.0568	9.0874	159.9316	20
21	1.2981	0.7704	23.8450	0.0419	18.3697	0.0544	9.5450	175.3392	21
22	1.3143	0.7609	25.1431	0.0398	19.1306	0.0523	10.0006	191.3174	22
23	1.3307	0.7515	26.4574	0.0378	19.8820	0.0503	10.4542	207.8499	23
24	1.3474	0.7422	27.7881	0.0360	20.6242	0.0485	10.9056	224.9204	24
25	1.3642	0.7330	29.1354	0.0343	21.3573	0.0468	11.3551	242.5132	25
26	1.3812	0.7240	30.4996	0.0328	22.0813	0.0453	11.8024	260.6128	26
27	1.3985	0.7150	31.8809	0.0314	22.7963	0.0439	12.2478	279.2040	27
28	1.4160	0.7062	33.2794	0.0300	23.5025	0.0425	12.6911	298.2719	28
29	1.4337	0.6975	34.6954	0.0288	24.2000	0.0413	13.1323	317.8019	29
30	1.4516	0.6889	36.1291	0.0277	24.8889	0.0402	13.5715	337.7797	30
31	1.4698	0.6804	37.5807	0.0266	25.5693	0.0391	14.0086	358.1912	31
32	1.4881	0.6720	39.0504	0.0256	26.2413	0.0381	14.4438	379.0227	32
33	1.5067	0.6637	40.5386	0.0247	26.9050	0.0372	14.8768	400.2607	33
34	1.5256	0.6555	42.0453	0.0238	27.5605	0.0363	15.3079	421.8920	34
35	1.5446	0.6474	43.5709	0.0230	28.2079	0.0355	15.7369	443.9037	35
36	1.5639	0.6394	45.1155	0.0222	28.8473	0.0347	16.1639	466.2830	36
40	1.6436	0.6084	51.4896	0.0194	31.3269	0.0319	17.8515	559.2320	40
48	1.8154	0.5509	65.2284	0.0153	35.9315	0.0278	21.1299	759.2296	48
50	1.8610	0.5373	68.8818	0.0145	37.0129	0.0270	21.9295	811.6738	50
60	2.1072	0.4746	88.5745	0.0113	42.0346	0.0238	25.8083	1084.8429	60
72	2.4459	0.4088	115.6736	0.0086	47.2925	0.0211	30.2047	1428.4561	72
80	2.7015	0.3702	136.1188	0.0073	50.3867	0.0198	32.9822	1661.8651	80
84	2.8391	0.3522	147.1290	0.0068	51.8222	0.0193	34.3258	1778.8384	84
90	3.0588	0.3269	164.7050	0.0061	53.8461	0.0186	36.2855	1953.8303	90
96	3.2955	0.3034	183.6411	0.0054	55.7246	0.0179	38.1793	2127.5244	96
100	3.4634	0.2887	197.0723	0.0051	56.9013	0.0176	39.4058	2242.2411	100
108	3.8253	0.2614	226.0226	0.0044	59.0865	0.0169	41.7737	2468.2636	108
120	4.4402	0.2252	275.2171	0.0036	61.9828	0.0161	45.1184	2796.5694	120
240	19.7155	0.0507	1497.2395	0.0007	75.9423	0.0132	67.1764	5101.5288	240
360	87.5410	0.0114	6923.2796	0.0001	79.0861	0.0126	75.8401	5997.9027	360

1.5%

N	Single Payment: Compound Amount Factor (F/P,i,N)	Single Payment: Present Worth Factor (P/F,i,N)	Equal Payment Series: Compound Amount Factor (F/A,i,N)	Equal Payment Series: Sinking Fund Factor (A/F,i,N)	Equal Payment Series: Present Worth Factor (P/A,i,N)	Equal Payment Series: Capital Recovery Factor (A/P,i,N)	Gradient Series: Gradient Uniform Series (A/G,i,N)	Gradient Series: Gradient Present Worth (P/G,i,N)	N
1	1.0150	0.9852	1.0000	1.0000	0.9852	1.0150	0.0000	0.0000	1
2	1.0302	0.9707	2.0150	0.4963	1.9559	0.5113	0.4963	0.9707	2
3	1.0457	0.9563	3.0452	0.3284	2.9122	0.3434	0.9901	2.8833	3
4	1.0614	0.9422	4.0909	0.2444	3.8544	0.2594	1.4814	5.7098	4
5	1.0773	0.9283	5.1523	0.1941	4.7826	0.2091	1.9702	9.4229	5
6	1.0934	0.9145	6.2296	0.1605	5.6972	0.1755	2.4566	13.9956	6
7	1.1098	0.9010	7.3230	0.1366	6.5982	0.1516	2.9405	19.4018	7
8	1.1265	0.8877	8.4328	0.1186	7.4859	0.1336	3.4219	25.6157	8
9	1.1434	0.8746	9.5593	0.1046	8.3605	0.1196	3.9008	32.6125	9
10	1.1605	0.8617	10.7027	0.0934	9.2222	0.1084	4.3772	40.3675	10
11	1.1779	0.8489	11.8633	0.0843	10.0711	0.0993	4.8512	48.8568	11
12	1.1956	0.8364	13.0412	0.0767	10.9075	0.0917	5.3227	58.0571	12
13	1.2136	0.8240	14.2368	0.0702	11.7315	0.0852	5.7917	67.9454	13
14	1.2318	0.8118	15.4504	0.0647	12.5434	0.0797	6.2582	78.4994	14
15	1.2502	0.7999	16.6821	0.0599	13.3432	0.0749	6.7223	89.6974	15
16	1.2690	0.7880	17.9324	0.0558	14.1313	0.0708	7.1839	101.5178	16
17	1.2880	0.7764	19.2014	0.0521	14.9076	0.0671	7.6431	113.9400	17
18	1.3073	0.7649	20.4894	0.0488	15.6726	0.0638	8.0997	126.9435	18
19	1.3270	0.7536	21.7967	0.0459	16.4262	0.0609	8.5539	140.5084	19
20	1.3469	0.7425	23.1237	0.0432	17.1686	0.0582	9.0057	154.6154	20
21	1.3671	0.7315	24.4705	0.0409	17.9001	0.0559	9.4550	169.2453	21
22	1.3876	0.7207	25.8376	0.0387	18.6208	0.0537	9.9018	184.3798	22
23	1.4084	0.7100	27.2251	0.0367	19.3309	0.0517	10.3462	200.0006	23
24	1.4295	0.6995	28.6335	0.0349	20.0304	0.0499	10.7881	216.0901	24
25	1.4509	0.6892	30.0630	0.0333	20.7196	0.0483	11.2276	232.6310	25
26	1.4727	0.6790	31.5140	0.0317	21.3986	0.0467	11.6646	249.6065	26
27	1.4948	0.6690	32.9867	0.0303	22.0676	0.0453	12.0992	267.0002	27
28	1.5172	0.6591	34.4815	0.0290	22.7267	0.0440	12.5313	284.7958	28
29	1.5400	0.6494	35.9987	0.0278	23.3761	0.0428	12.9610	302.9779	29
30	1.5631	0.6398	37.5387	0.0266	24.0158	0.0416	13.3883	321.5310	30
31	1.5865	0.6303	39.1018	0.0256	24.6461	0.0406	13.8131	340.4402	31
32	1.6103	0.6210	40.6883	0.0246	25.2671	0.0396	14.2355	359.6910	32
33	1.6345	0.6118	42.2986	0.0236	25.8790	0.0386	14.6555	379.2691	33
34	1.6590	0.6028	43.9331	0.0228	26.4817	0.0378	15.0731	399.1607	34
35	1.6839	0.5939	45.5921	0.0219	27.0756	0.0369	15.4882	419.3521	35
36	1.7091	0.5851	47.2760	0.0212	27.6607	0.0362	15.9009	439.8303	36
40	1.8140	0.5513	54.2679	0.0184	29.9158	0.0334	17.5277	524.3568	40
48	2.0435	0.4894	69.5652	0.0144	34.0426	0.0294	20.6667	703.5462	48
50	2.1052	0.4750	73.6828	0.0136	34.9997	0.0286	21.4277	749.9636	50
60	2.4432	0.4093	96.2147	0.0104	39.3803	0.0254	25.0930	988.1674	60
72	2.9212	0.3423	128.0772	0.0078	43.8447	0.0228	29.1893	1279.7938	72
80	3.2907	0.3039	152.7109	0.0065	46.4073	0.0215	31.7423	1473.0741	80
84	3.4926	0.2863	166.1726	0.0060	47.5786	0.0210	32.9668	1568.5140	84
90	3.8189	0.2619	187.9299	0.0053	49.2099	0.0203	34.7399	1709.5439	90
96	4.1758	0.2395	211.7202	0.0047	50.7017	0.0197	36.4381	1847.4725	96
100	4.4320	0.2256	228.8030	0.0044	51.6247	0.0194	37.5295	1937.4506	100
108	4.9927	0.2003	266.1778	0.0038	53.3137	0.0188	39.6171	2112.1348	108
120	5.9693	0.1675	331.2882	0.0030	55.4985	0.0180	42.5185	2359.7114	120
240	35.6328	0.0281	2308.8544	0.0004	64.7957	0.0154	59.7368	3870.6912	240
360	212.7038	0.0047	14113.5854	0.0001	66.3532	0.0151	64.9662	4310.7165	360

1.75%

	Single Payment		Equal Payment Series				Gradient Series		
N	**Compound Amount Factor (F/P,i,N)**	**Present Worth Factor (P/F,i,N)**	**Compound Amount Factor (F/A,i,N)**	**Sinking Fund Factor (A/F,i,N)**	**Present Worth Factor (P/A,i,N)**	**Capital Recovery Factor (A/P,i,N)**	**Gradient Uniform Series (A/G,i,N)**	**Gradient Present Worth (P/G,i,N)**	**N**
1	1.0175	0.9828	1.0000	1.0000	0.9828	1.0175	0.0000	0.0000	1
2	1.0353	0.9659	2.0175	0.4957	1.9487	0.5132	0.4957	0.9659	2
3	1.0534	0.9493	3.0528	0.3276	2.8980	0.3451	0.9884	2.8645	3
4	1.0719	0.9330	4.1062	0.2435	3.8309	0.2610	1.4783	5.6633	4
5	1.0906	0.9169	5.1781	0.1931	4.7479	0.2106	1.9653	9.3310	5
6	1.1097	0.9011	6.2687	0.1595	5.6490	0.1770	2.4494	13.8367	6
7	1.1291	0.8856	7.3784	0.1355	6.5346	0.1530	2.9306	19.1506	7
8	1.1489	0.8704	8.5075	0.1175	7.4051	0.1350	3.4089	25.2435	8
9	1.1690	0.8554	9.6564	0.1036	8.2605	0.1211	3.8844	32.0870	9
10	1.1894	0.8407	10.8254	0.0924	9.1012	0.1099	4.3569	39.6535	10
11	1.2103	0.8263	12.0148	0.0832	9.9275	0.1007	4.8266	47.9162	11
12	1.2314	0.8121	13.2251	0.0756	10.7395	0.0931	5.2934	56.8489	12
13	1.2530	0.7981	14.4565	0.0692	11.5376	0.0867	5.7573	66.4260	13
14	1.2749	0.7844	15.7095	0.0637	12.3220	0.0812	6.2184	76.6227	14
15	1.2972	0.7709	16.9844	0.0589	13.0929	0.0764	6.6765	87.4149	15
16	1.3199	0.7576	18.2817	0.0547	13.8505	0.0722	7.1318	98.7792	16
17	1.3430	0.7446	19.6016	0.0510	14.5951	0.0685	7.5842	110.6926	17
18	1.3665	0.7318	20.9446	0.0477	15.3269	0.0652	8.0338	123.1328	18
19	1.3904	0.7192	22.3112	0.0448	16.0461	0.0623	8.4805	136.0783	19
20	1.4148	0.7068	23.7016	0.0422	16.7529	0.0597	8.9243	149.5080	20
21	1.4395	0.6947	25.1164	0.0398	17.4475	0.0573	9.3653	163.4013	21
22	1.4647	0.6827	26.5559	0.0377	18.1303	0.0552	9.8034	177.7385	22
23	1.4904	0.6710	28.0207	0.0357	18.8012	0.0532	10.2387	192.5000	23
24	1.5164	0.6594	29.5110	0.0339	19.4607	0.0514	10.6711	207.6671	24
25	1.5430	0.6481	31.0276	0.0322	20.1088	0.0497	11.1007	223.2214	25
26	1.5700	0.6369	32.5704	0.0307	20.7457	0.0482	11.5274	239.1451	26
27	1.5975	0.6260	34.1404	0.0293	21.3717	0.0468	11.9513	255.4210	27
28	1.6254	0.6152	35.7379	0.0280	21.9870	0.0455	12.3724	272.0321	28
29	1.6539	0.6046	37.3633	0.0268	22.5916	0.0443	12.7907	288.9623	29
30	1.6828	0.5942	39.0172	0.0256	23.1858	0.0431	13.2061	306.1954	30
31	1.7122	0.5840	40.7000	0.0246	23.7699	0.0421	13.6188	323.7163	31
32	1.7422	0.5740	42.4122	0.0236	24.3439	0.0411	14.0286	341.5097	32
33	1.7727	0.5641	44.1544	0.0226	24.9080	0.0401	14.4356	359.5613	33
34	1.8037	0.5544	45.9271	0.0218	25.4624	0.0393	14.8398	377.8567	34
35	1.8353	0.5449	47.7308	0.0210	26.0073	0.0385	15.2412	396.3824	35
36	1.8674	0.5355	49.5661	0.0202	26.5428	0.0377	15.6399	415.1250	36
40	2.0016	0.4996	57.2341	0.0175	28.5942	0.0350	17.2066	492.0109	40
48	2.2996	0.4349	74.2628	0.0135	32.2938	0.0310	20.2084	652.6054	48
50	2.3808	0.4200	78.9022	0.0127	33.1412	0.0302	20.9317	693.7010	50
60	2.8318	0.3531	104.6752	0.0096	36.9640	0.0271	24.3885	901.4954	60
72	3.4872	0.2868	142.1263	0.0070	40.7564	0.0245	28.1948	1149.1181	72
80	4.0064	0.2496	171.7938	0.0058	42.8799	0.0233	30.5329	1309.2482	80
84	4.2943	0.2329	188.2450	0.0053	43.8361	0.0228	31.6442	1387.1584	84
90	4.7654	0.2098	215.1646	0.0046	45.1516	0.0221	33.2409	1500.8798	90
96	5.2882	0.1891	245.0374	0.0041	46.3370	0.0216	34.7556	1610.4716	96
100	5.6682	0.1764	266.7518	0.0037	47.0615	0.0212	35.7211	1681.0886	100
108	6.5120	0.1536	314.9738	0.0032	48.3679	0.0207	37.5494	1816.1852	108
120	8.0192	0.1247	401.0962	0.0025	50.0171	0.0200	40.0469	2003.0269	120
240	64.3073	0.0156	3617.5602	0.0003	56.2543	0.0178	53.3518	3001.2678	240
360	515.6921	0.0019	29410.9747	0.0000	57.0320	0.0175	56.4434	3219.0833	360

2.0%

	Single Payment		Equal Payment Series				Gradient Series		
N	Compound Amount Factor (F/P,i,N)	Present Worth Factor (P/F,i,N)	Compound Amount Factor (F/A,i,N)	Sinking Fund Factor (A/F,i,N)	Present Worth Factor (P/A,i,N)	Capital Recovery Factor (A/P,i,N)	Gradient Uniform Series (A/G,i,N)	Gradient Present Worth (P/G,i,N)	N
1	1.0200	0.9804	1.0000	1.0000	0.9804	1.0200	0.0000	0.0000	1
2	1.0404	0.9612	2.0200	0.4950	1.9416	0.5150	0.4950	0.9612	2
3	1.0612	0.9423	3.0604	0.3268	2.8839	0.3468	0.9868	2.8458	3
4	1.0824	0.9238	4.1216	0.2426	3.8077	0.2626	1.4752	5.6173	4
5	1.1041	0.9057	5.2040	0.1922	4.7135	0.2122	1.9604	9.2403	5
6	1.1262	0.8880	6.3081	0.1585	5.6014	0.1785	2.4423	13.6801	6
7	1.1487	0.8706	7.4343	0.1345	6.4720	0.1545	2.9208	18.9035	7
8	1.1717	0.8535	8.5830	0.1165	7.3255	0.1365	3.3961	24.8779	8
9	1.1951	0.8368	9.7546	0.1025	8.1622	0.1225	3.8681	31.5720	9
10	1.2190	0.8203	10.9497	0.0913	8.9826	0.1113	4.3367	38.9551	10
11	1.2434	0.8043	12.1687	0.0822	9.7868	0.1022	4.8021	46.9977	11
12	1.2682	0.7885	13.4121	0.0746	10.5753	0.0946	5.2642	55.6712	12
13	1.2936	0.7730	14.6803	0.0681	11.3484	0.0881	5.7231	64.9475	13
14	1.3195	0.7579	15.9739	0.0626	12.1062	0.0826	6.1786	74.7999	14
15	1.3459	0.7430	17.2934	0.0578	12.8493	0.0778	6.6309	85.2021	15
16	1.3728	0.7284	18.6393	0.0537	13.5777	0.0737	7.0799	96.1288	16
17	1.4002	0.7142	20.0121	0.0500	14.2919	0.0700	7.5256	107.5554	17
18	1.4282	0.7002	21.4123	0.0467	14.9920	0.0667	7.9681	119.4581	18
19	1.4568	0.6864	22.8406	0.0438	15.6785	0.0638	8.4073	131.8139	19
20	1.4859	0.6730	24.2974	0.0412	16.3514	0.0612	8.8433	144.6003	20
21	1.5157	0.6598	25.7833	0.0388	17.0112	0.0588	9.2760	157.7959	21
22	1.5460	0.6468	27.2990	0.0366	17.6580	0.0566	9.7055	171.3795	22
23	1.5769	0.6342	28.8450	0.0347	18.2922	0.0547	10.1317	185.3309	23
24	1.6084	0.6217	30.4219	0.0329	18.9139	0.0529	10.5547	199.6305	24
25	1.6406	0.6095	32.0303	0.0312	19.5235	0.0512	10.9745	214.2592	25
26	1.6734	0.5976	33.6709	0.0297	20.1210	0.0497	11.3910	229.1987	26
27	1.7069	0.5859	35.3443	0.0283	20.7069	0.0483	11.8043	244.4311	27
28	1.7410	0.5744	37.0512	0.0270	21.2813	0.0470	12.2145	259.9392	28
29	1.7758	0.5631	38.7922	0.0258	21.8444	0.0458	12.6214	275.7064	29
30	1.8114	0.5521	40.5681	0.0246	22.3965	0.0446	13.0251	291.7164	30
31	1.8476	0.5412	42.3794	0.0236	22.9377	0.0436	13.4257	307.9538	31
32	1.8845	0.5306	44.2270	0.0226	23.4683	0.0426	13.8230	324.4035	32
33	1.9222	0.5202	46.1116	0.0217	23.9886	0.0417	14.2172	341.0508	33
34	1.9607	0.5100	48.0338	0.0208	24.4986	0.0408	14.6083	357.8817	34
35	1.9999	0.5000	49.9945	0.0200	24.9986	0.0400	14.9961	374.8826	35
36	2.0399	0.4902	51.9944	0.0192	25.4888	0.0392	15.3809	392.0405	36
40	2.2080	0.4529	60.4020	0.0166	27.3555	0.0366	16.8885	461.9931	40
48	2.5871	0.3865	79.3535	0.0126	30.6731	0.0326	19.7556	605.9657	48
50	2.6916	0.3715	84.5794	0.0118	31.4236	0.0318	20.4420	642.3606	50
60	3.2810	0.3048	114.0515	0.0088	34.7609	0.0288	23.6961	823.6975	60
72	4.1611	0.2403	158.0570	0.0063	37.9841	0.0263	27.2234	1034.0557	72
80	4.8754	0.2051	193.7720	0.0052	39.7445	0.0252	29.3572	1166.7868	80
84	5.2773	0.1895	213.8666	0.0047	40.5255	0.0247	30.3616	1230.4191	84
90	5.9431	0.1683	247.1567	0.0040	41.5869	0.0240	31.7929	1322.1701	90
96	6.6929	0.1494	284.6467	0.0035	42.5294	0.0235	33.1370	1409.2973	96
100	7.2446	0.1380	312.2323	0.0032	43.0984	0.0232	33.9863	1464.7527	100
108	8.4883	0.1178	374.4129	0.0027	44.1095	0.0227	35.5774	1569.3025	108
120	10.7652	0.0929	488.2582	0.0020	45.3554	0.0220	37.7114	1710.4160	120
240	115.8887	0.0086	5744.4368	0.0002	49.5686	0.0202	47.9110	2374.8800	240
360	1247.5611	0.0008	62328.0564	0.0000	49.9599	0.0200	49.7112	2483.5679	360

3.0%

	Single Payment		Equal Payment Series				Gradient Series		
N	Compound Amount Factor (F/P,i,N)	Present Worth Factor (P/F,i,N)	Compound Amount Factor (F/A,i,N)	Sinking Fund Factor (A/F,i,N)	Present Worth Factor (P/A,i,N)	Capital Recovery Factor (A/P,i,N)	Gradient Uniform Series (A/G,i,N)	Gradient Present Worth (P/G,i,N)	N
1	1.0300	0.9709	1.0000	1.0000	0.9709	1.0300	0.0000	0.0000	1
2	1.0609	0.9426	2.0300	0.4926	1.9135	0.5226	0.4926	0.9426	2
3	1.0927	0.9151	3.0909	0.3235	2.8286	0.3535	0.9803	2.7729	3
4	1.1255	0.8885	4.1836	0.2390	3.7171	0.2690	1.4631	5.4383	4
5	1.1593	0.8626	5.3091	0.1884	4.5797	0.2184	1.9409	8.8888	5
6	1.1941	0.8375	6.4684	0.1546	5.4172	0.1846	2.4138	13.0762	6
7	1.2299	0.8131	7.6625	0.1305	6.2303	0.1605	2.8819	17.9547	7
8	1.2668	0.7894	8.8923	0.1125	7.0197	0.1425	3.3450	23.4806	8
9	1.3048	0.7664	10.1591	0.0984	7.7861	0.1284	3.8032	29.6119	9
10	1.3439	0.7441	11.4639	0.0872	8.5302	0.1172	4.2565	36.3088	110
11	1.3842	0.7224	12.8078	0.0781	9.2526	0.1081	4.7049	43.5330	11
12	1.4258	0.7014	14.1920	0.0705	9.9540	0.1005	5.1485	51.2482	12
13	1.4685	0.6810	15.6178	0.0640	10.6350	0.0940	5.5872	59.4196	13
14	1.5126	0.6611	17.0863	0.0585	11.2961	0.0885	6.0210	68.0141	14
15	1.5580	0.6419	18.5989	0.0538	11.9379	0.0838	6.4500	77.0002	15
16	1.6047	0.6232	20.1569	0.0496	12.5611	0.0796	6.8742	86.3477	16
17	1.6528	0.6050	21.7616	0.0460	13.1661	0.0760	7.2936	96.0280	17
18	1.7024	0.5874	23.4144	0.0427	13.7535	0.0727	7.7081	106.0137	18
19	1.7535	0.5703	25.1169	0.0398	14.3238	0.0698	8.1179	116.2788	19
20	1.8061	0.5537	26.8704	0.0372	14.8775	0.0672	8.5229	126.7987	20
21	1.8603	0.5375	28.6765	0.0349	15.4150	0.0649	8.9231	137.5496	21
22	1.9161	0.5219	30.5368	0.0327	15.9396	0.0627	9.3186	148.5094	22
23	1.9736	0.5067	32.4529	0.0308	16.4436	0.0608	9.7093	159.6566	23
24	2.0328	0.4919	34.4265	0.0290	16.9355	0.0590	10.0954	170.9711	24
25	2.0938	0.4776	36.4593	0.0274	17.4131	0.0574	10.4768	182.4336	25
26	2.1566	0.4637	38.5530	0.0259	17.8768	0.0559	10.8535	194.0260	26
27	2.2213	0.4502	40.7096	0.0246	18.3270	0.0546	11.2255	205.7309	27
28	2.2879	0.4371	42.9309	0.0233	18.7641	0.0533	11.5930	217.5320	28
29	2.3566	0.4243	45.2189	0.0221	19.1885	0.0521	11.9558	229.4137	29
30	2.4273	0.4120	47.5754	0.0210	19.6004	0.0510	12.3141	241.3613	30
31	2.5001	0.4000	50.0027	0.0200	20.0004	0.0500	12.6678	253.3609	31
32	2.5751	0.3883	52.5028	0.0190	20.3888	0.0490	13.0169	265.3993	32
33	2.6523	0.3770	55.0778	0.0182	20.7658	0.0482	13.3616	277.4642	33
34	2.7319	0.3660	57.7302	0.0173	21.1318	0.0473	13.7018	289.5437	34
35	2.8139	0.3554	60.4621	0.0165	21.4872	0.0465	14.0375	301.6267	35
40	3.2620	0.3066	75.4013	0.0133	23.1148	0.0433	15.6502	361.7499	40
45	3.7816	0.2644	92.7199	0.0108	24.5187	0.0408	17.1556	420.6325	45
50	4.3839	0.2281	112.7969	0.0089	25.7298	0.0389	18.5575	477.4803	50
55	5.0821	0.1968	136.0716	0.0073	26.7744	0.0373	19.8600	531.7411	55
60	5.8916	0.1697	163.0534	0.0061	27.6756	0.0361	21.0674	583.0526	60
65	6.8300	0.1464	194.3328	0.0051	28.4529	0.0351	22.1841	631.2010	65
70	7.9178	0.1263	230.5941	0.0043	29.1234	0.0343	23.2145	676.0869	70
75	9.1789	0.1089	272.6309	0.0037	29.7018	0.0337	24.1634	717.6978	75
80	10.6409	0.0940	321.3630	0.0031	30.2008	0.0331	25.0353	756.0865	80
85	12.3357	0.0811	377.8570	0.0026	30.6312	0.0326	25.8349	791.3529	85
90	14.3005	0.0699	443.3489	0.0023	31.0024	0.0323	26.5667	823.6302	90
95	16.5782	0.0603	519.2720	0.0019	31.3227	0.0319	27.2351	853.0742	95
100	19.2186	0.0520	607.2877	0.0016	31.5989	0.0316	27.8444	879.8540	100

4.0%

	Single Payment		Equal Payment Series				Gradient Series		
N	Compound Amount Factor (F/P,i,N)	Present Worth Factor (P/F,i,N)	Compound Amount Factor (F/A,i,N)	Sinking Fund Factor (A/F,i,N)	Present Worth Factor (P/A,i,N)	Capital Recovery Factor (A/P,i,N)	Gradient Uniform Series (A/G,i,N)	Gradient Present Worth (P/G,i,N)	N
1	1.0400	0.9615	1.0000	1.0000	0.9615	1.0400	0.0000	0.0000	1
2	1.0816	0.9246	2.0400	0.4902	1.8861	0.5302	0.4902	0.9246	2
3	1.1249	0.8890	3.1216	0.3203	2.7751	0.3603	0.9739	2.7025	3
4	1.1699	0.8548	4.2465	0.2355	3.6299	0.2755	1.4510	5.2670	4
5	1.2167	0.8219	5.4163	0.1846	4.4518	0.2246	1.9216	8.5547	5
6	1.2653	0.7903	6.6330	0.1508	5.2421	0.1908	2.3857	12.5062	6
7	1.3159	0.7599	7.8983	0.1266	6.0021	0.1666	2.8433	17.0657	7
8	1.3686	0.7307	9.2142	0.1085	6.7327	0.1485	3.2944	22.1806	8
9	1.4233	0.7026	10.5828	0.0945	7.4353	0.1345	3.7391	27.8013	9
10	1.4802	0.6756	12.0061	0.0833	8.1109	0.1233	4.1773	33.8814	10
11	1.5395	0.6496	13.4864	0.0741	8.7605	0.1141	4.6090	40.3772	11
12	1.6010	0.6246	15.0258	0.0666	9.3851	0.1066	5.0343	47.2477	12
13	1.6651	0.6006	16.6268	0.0601	9.9856	0.1001	5.4533	54.4546	13
14	1.7317	0.5775	18.2919	0.0547	10.5631	0.0947	5.8659	61.9618	14
15	1.8009	0.5553	20.0236	0.0499	11.1184	0.0899	6.2721	69.7355	15
16	1.8730	0.5339	21.8245	0.0458	11.6523	0.0858	6.6720	77.7441	16
17	1.9479	0.5134	23.6975	0.0422	12.1657	0.0822	7.0656	85.9581	17
18	2.0258	0.4936	25.6454	0.0390	12.6593	0.0790	7.4530	94.3498	18
19	2.1068	0.4746	27.6712	0.0361	13.1339	0.0761	7.8342	102.8933	19
20	2.1911	0.4564	29.7781	0.0336	13.5903	0.0736	8.2091	111.5647	20
21	2.2788	0.4388	31.9692	0.0313	14.0292	0.0713	8.5779	120.3414	21
22	2.3699	0.4220	34.2480	0.0292	14.4511	0.0692	8.9407	129.2024	22
23	2.4647	0.4057	36.6179	0.0273	14.8568	0.0673	9.2973	138.1284	23
24	2.5633	0.3901	39.0826	0.0256	15.2470	0.0656	9.6479	147.1012	24
25	2.6658	0.3751	41.6459	0.0240	15.6221	0.0640	9.9925	156.1040	25
26	2.7725	0.3607	44.3117	0.0226	15.9828	0.0626	10.3312	165.1212	26
27	2.8834	0.3468	47.0842	0.0212	16.3296	0.0612	10.6640	174.1385	27
28	2.9987	0.3335	49.9676	0.0200	16.6631	0.0600	10.9909	183.1424	28
29	3.1187	0.3207	52.9663	0.0189	16.9837	0.0589	11.3120	192.1206	29
30	3.2434	0.3083	56.0849	0.0178	17.2920	0.0578	11.6274	201.0618	30
31	3.3731	0.2965	59.3283	0.0169	17.5885	0.0569	11.9371	209.9556	31
32	3.5081	0.2851	62.7015	0.0159	17.8736	0.0559	12.2411	218.7924	32
33	3.6484	0.2741	66.2095	0.0151	18.1476	0.0551	12.5396	227.5634	33
34	3.7943	0.2636	69.8579	0.0143	18.4112	0.0543	12.8324	236.2607	34
35	3.9461	0.2534	73.6522	0.0136	18.6646	0.0536	13.1198	244.8768	35
40	4.8010	0.2083	95.0255	0.0105	19.7928	0.0505	14.4765	286.5303	40
45	5.8412	0.1712	121.0294	0.0083	20.7200	0.0483	15.7047	325.4028	45
50	7.1067	0.1407	152.6671	0.0066	21.4822	0.0466	16.8122	361.1638	50
55	8.6464	0.1157	191.1592	0.0052	22.1086	0.0452	17.8070	393.6890	55
60	10.5196	0.0951	237.9907	0.0042	22.6235	0.0442	18.6972	422.9966	60
65	12.7987	0.0781	294.9684	0.0034	23.0467	0.0434	19.4909	449.2014	65
70	15.5716	0.0642	364.2905	0.0027	23.3945	0.0427	20.1961	472.4789	70
75	18.9453	0.0528	448.6314	0.0022	23.6804	0.0422	20.8206	493.0408	75
80	23.0498	0.0434	551.2450	0.0018	23.9154	0.0418	21.3718	511.1161	80
85	28.0436	0.0357	676.0901	0.0015	24.1085	0.0415	21.8569	526.9384	85
90	34.1193	0.0293	827.9833	0.0012	24.2673	0.0412	22.2826	540.7369	90
95	41.5114	0.0241	1012.7846	0.0010	24.3978	0.0410	22.6550	552.7307	95
100	50.5049	0.0198	1237.6237	0.0008	24.5050	0.0408	22.9800	563.1249	100

5.0%

	Single Payment		Equal Payment Series				Gradient Series		
N	**Compound Amount Factor (F/P,i,N)**	**Present Worth Factor (P/F,i,N)**	**Compound Amount Factor (F/A,i,N)**	**Sinking Fund Factor (A/F,i,N)**	**Present Worth Factor (P/A,i,N)**	**Capital Recovery Factor (A/P,i,N)**	**Gradient Uniform Series (A/G,i,N)**	**Gradient Present Worth (P/G,i,N)**	**N**
1	1.0500	0.9524	1.0000	1.0000	0.9524	1.0500	0.0000	0.0000	1
2	1.1025	0.9070	2.0500	0.4878	1.8594	0.5378	0.4878	0.9070	2
3	1.1576	0.8638	3.1525	0.3172	2.7232	0.3672	0.9675	2.6347	3
4	1.2155	0.8227	4.3101	0.2320	3.5460	0.2820	1.4391	5.1028	4
5	1.2763	0.7835	5.5256	0.1810	4.3295	0.2310	1.9025	8.2369	5
6	1.3401	0.7462	6.8019	0.1470	5.0757	0.1970	2.3579	11.9680	6
7	1.4071	0.7107	8.1420	0.1228	5.7864	0.1728	2.8052	16.2321	7
8	1.4775	0.6768	9.5491	0.1047	6.4632	0.1547	3.2445	20.9700	8
9	1.5513	0.6446	11.0266	0.0907	7.1078	0.1407	3.6758	26.1268	9
10	1.6289	0.6139	12.5779	0.0795	7.7217	0.1295	4.0991	31.6520	10
11	1.7103	0.5847	14.2068	0.0704	8.3064	0.1204	4.5144	37.4988	11
12	1.7959	0.5568	15.9171	0.0628	8.8633	0.1128	4.9219	43.6241	12
13	1.8856	0.5303	17.7130	0.0565	9.3936	0.1065	5.3215	49.9879	13
14	1.9799	0.5051	19.5986	0.0510	9.8986	0.1010	5.7133	56.5538	14
15	2.0789	0.4810	21.5786	0.0463	10.3797	0.0963	6.0973	63.2880	15
16	2.1829	0.4581	23.6575	0.0423	10.8378	0.0923	6.4736	70.1597	16
17	2.2920	0.4363	25.8404	0.0387	11.2741	0.0887	6.8423	77.1405	17
18	2.4066	0.4155	28.1324	0.0355	11.6896	0.0855	7.2034	84.2043	18
19	2.5270	0.3957	30.5390	0.0327	12.0853	0.0827	7.5569	91.3275	19
20	2.6533	0.3769	33.0660	0.0302	12.4622	0.0802	7.9030	98.4884	20
21	2.7860	0.3589	35.7193	0.0280	12.8212	0.0780	8.2416	105.6673	21
22	2.9253	0.3418	38.5052	0.0260	13.1630	0.0760	8.5730	112.8461	22
23	3.0715	0.3256	41.4305	0.0241	13.4886	0.0741	8.8971	120.0087	23
24	3.2251	0.3101	44.5020	0.0225	13.7986	0.0725	9.2140	127.1402	24
25	3.3864	0.2953	47.7271	0.0210	14.0939	0.0710	9.5238	134.2275	25
26	3.5557	0.2812	51.1135	0.0196	14.3752	0.0696	9.8266	141.2585	26
27	3.7335	0.2678	54.6691	0.0183	14.6430	0.0683	10.1224	148.2226	27
28	3.9201	0.2551	58.4026	0.0171	14.8981	0.0671	10.4114	155.1101	28
29	4.1161	0.2429	62.3227	0.0160	15.1411	0.0660	10.6936	161.9126	29
30	4.3219	0.2314	66.4388	0.0151	15.3725	0.0651	10.9691	168.6226	30
31	4.5380	0.2204	70.7608	0.0141	15.5928	0.0641	11.2381	175.2333	31
32	4.7649	0.2099	75.2988	0.0133	15.8027	0.0633	11.5005	181.7392	32
33	5.0032	0.1999	80.0638	0.0125	16.0025	0.0625	11.7566	188.1351	33
34	5.2533	0.1904	85.0670	0.0118	16.1929	0.0618	12.0063	194.4168	34
35	5.5160	0.1813	90.3203	0.0111	16.3742	0.0611	12.2498	200.5807	35
40	7.0400	0.1420	120.7998	0.0083	17.1591	0.0583	13.3775	229.5452	40
45	8.9850	0.1113	159.7002	0.0063	17.7741	0.0563	14.3644	255.3145	45
50	11.4674	0.0872	209.3480	0.0048	18.2559	0.0548	15.2233	277.9148	50
55	14.6356	0.0683	272.7126	0.0037	18.6335	0.0537	15.9664	297.5104	55
60	18.6792	0.0535	353.5837	0.0028	18.9293	0.0528	16.6062	314.3432	60
65	23.8399	0.0419	456.7980	0.0022	19.1611	0.0522	17.1541	328.6910	65
70	30.4264	0.0329	588.5285	0.0017	19.3427	0.0517	17.6212	340.8409	70
75	38.8327	0.0258	756.6537	0.0013	19.4850	0.0513	18.0176	351.0721	75
80	49.5614	0.0202	971.2288	0.0010	19.5965	0.0510	18.3526	359.6460	80
85	63.2544	0.0158	1245.0871	0.0008	19.6838	0.0508	18.6346	366.8007	85
90	80.7304	0.0124	1594.6073	0.0006	19.7523	0.0506	18.8712	372.7488	90
95	103.0347	0.0097	2040.6935	0.0005	19.8059	0.0505	19.0689	377.6774	95
100	131.5013	0.0076	2610.0252	0.0004	19.8479	0.0504	19.2337	381.7492	100

6.0%

	Single Payment		Equal Payment Series				Gradient Series		
N	Compound Amount Factor (*F/P,i,N*)	Present Worth Factor (*P/F,i,N*)	Compound Amount Factor (*F/A,i,N*)	Sinking Fund Factor (*A/F,i,N*)	Present Worth Factor (*P/A,i,N*)	Capital Recovery Factor (*A/P,i,N*)	Gradient Uniform Series (*A/G,i,N*)	Gradient Present Worth (*P/G,i,N*)	*N*
1	1.0600	0.9434	1.0000	1.0000	0.9434	1.0600	0.0000	0.0000	1
2	1.1236	0.8900	2.0600	0.4854	1.8334	0.5454	0.4854	0.8900	2
3	1.1910	0.8396	3.1836	0.3141	2.6730	0.3741	0.9612	2.5692	3
4	1.2625	0.7921	4.3746	0.2286	3.4651	0.2886	1.4272	4.9455	4
5	1.3382	0.7473	5.6371	0.1774	4.2124	0.2374	1.8836	7.9345	5
6	1.4185	0.7050	6.9753	0.1434	4.9173	0.2034	2.3304	11.4594	6
7	1.5036	0.6651	8.3938	0.1191	5.5824	0.1791	2.7676	15.4497	7
8	1.5938	0.6274	9.8975	0.1010	6.2098	0.1610	3.1952	19.8416	8
9	1.6895	0.5919	11.4913	0.0870	6.8017	0.1470	3.6133	24.5768	9
10	1.7908	0.5584	13.1808	0.0759	7.3601	0.1359	4.0220	29.6023	10
11	1.8983	0.5268	14.9716	0.0668	7.8869	0.1268	4.4213	34.8702	11
12	2.0122	0.4970	16.8699	0.0593	8.3838	0.1193	4.8113	40.3369	12
13	2.1329	0.4688	18.8821	0.0530	8.8527	0.1130	5.1920	45.9629	13
14	2.2609	0.4423	21.0151	0.0476	9.2950	0.1076	5.5635	51.7128	14
15	2.3966	0.4173	23.2760	0.0430	9.7122	0.1030	5.9260	57.5546	15
16	2.5404	0.3936	25.6725	0.0390	10.1059	0.0990	6.2794	63.4592	16
17	2.6928	0.3714	28.2129	0.0354	10.4773	0.0954	6.6240	69.4011	17
18	2.8543	0.3503	30.9057	0.0324	10.8276	0.0924	6.9597	75.3569	18
19	3.0256	0.3305	33.7600	0.0296	11.1581	0.0896	7.2867	81.3062	19
20	3.2071	0.3118	36.7856	0.0272	11.4699	0.0872	7.6051	87.2304	20
21	3.3996	0.2942	39.9927	0.0250	11.7641	0.0850	7.9151	93.1136	21
22	3.6035	0.2775	43.3923	0.0230	12.0416	0.0830	8.2166	98.9412	22
23	3.8197	0.2618	46.9958	0.0213	12.3034	0.0813	8.5099	104.7007	23
24	4.0489	0.2470	50.8156	0.0197	12.5504	0.0797	8.7951	110.3812	24
25	4.2919	0.2330	54.8645	0.0182	12.7834	0.0782	9.0722	115.9732	25
26	4.5494	0.2198	59.1564	0.0169	13.0032	0.0769	9.3414	121.4684	26
27	4.8223	0.2074	63.7058	0.0157	13.2105	0.0757	9.6029	126.8600	27
28	5.1117	0.1956	68.5281	0.0146	13.4062	0.0746	9.8568	132.1420	28
29	5.4184	0.1846	73.6398	0.0136	13.5907	0.0736	10.1032	137.3096	29
30	5.7435	0.1741	79.0582	0.0126	13.7648	0.0726	10.3422	142.3588	30
31	6.0881	0.1643	84.8017	0.0118	13.9291	0.0718	10.5740	147.2864	31
32	6.4534	0.1550	90.8898	0.0110	14.0840	0.0710	10.7988	152.0901	32
33	6.8406	0.1462	97.3432	0.0103	14.2302	0.0703	11.0166	156.7681	33
34	7.2510	0.1379	104.1838	0.0096	14.3681	0.0696	11.2276	161.3192	34
35	7.6861	0.1301	111.4348	0.0090	14.4982	0.0690	11.4319	165.7427	35
40	10.2857	0.0972	154.7620	0.0065	15.0463	0.0665	12.3590	185.9568	40
45	13.7646	0.0727	212.7435	0.0047	15.4558	0.0647	13.1413	203.1096	45
50	18.4202	0.0543	290.3359	0.0034	15.7619	0.0634	13.7964	217.4574	50
55	24.6503	0.0406	394.1720	0.0025	15.9905	0.0625	14.3411	229.3222	55
60	32.9877	0.0303	533.1282	0.0019	16.1614	0.0619	14.7909	239.0428	60
65	44.1450	0.0227	719.0829	0.0014	16.2891	0.0614	15.1601	246.9450	65
70	59.0759	0.0169	967.9322	0.0010	16.3845	0.0610	15.4613	253.3271	70
75	79.0569	0.0126	1300.9487	0.0008	16.4558	0.0608	15.7058	258.4527	75
80	105.7960	0.0095	1746.5999	0.0006	16.5091	0.0606	15.9033	262.5493	80
85	141.5789	0.0071	2342.9817	0.0004	16.5489	0.0604	16.0620	265.8096	85
90	189.4645	0.0053	3141.0752	0.0003	16.5787	0.0603	16.1891	268.3946	90
95	253.5463	0.0039	4209.1042	0.0002	16.6009	0.0602	16.2905	270.4375	95
100	339.3021	0.0029	5638.3681	0.0002	16.6175	0.0602	16.3711	272.0471	100

7.0%

	Single Payment		Equal Payment Series				Gradient Series		
N	**Compound Amount Factor (F/P,i,N)**	**Present Worth Factor (P/F,i,N)**	**Compound Amount Factor (F/A,i,N)**	**Sinking Fund Factor (A/F,i,N)**	**Present Worth Factor (P/A,i,N)**	**Capital Recovery Factor (A/P,i,N)**	**Gradient Uniform Series (A/G,i,N)**	**Gradient Present Worth (P/G,i,N)**	**N**
1	1.0700	0.9346	1.0000	1.0000	0.9346	1.0700	0.0000	0.0000	1
2	1.1449	0.8734	2.0700	0.4831	1.8080	0.5531	0.4831	0.8734	2
3	1.2250	0.8163	3.2149	0.3111	2.6243	0.3811	0.9549	2.5060	3
4	1.3108	0.7629	4.4399	0.2252	3.3872	0.2952	1.4155	4.7947	4
5	1.4026	0.7130	5.7507	0.1739	4.1002	0.2439	1.8650	7.6467	5
6	1.5007	0.6663	7.1533	0.1398	4.7665	0.2098	2.3032	10.9784	6
7	1.6058	0.6227	8.6540	0.1156	5.3893	0.1856	2.7304	14.7149	7
8	1.7182	0.5820	10.2598	0.0975	5.9713	0.1675	3.1465	18.7889	8
9	1.8385	0.5439	11.9780	0.0835	6.5152	0.1535	3.5517	23.1404	9
10	1.9672	0.5083	13.8164	0.0724	7.0236	0.1424	3.9461	27.7156	10
11	2.1049	0.4751	15.7836	0.0634	7.4987	0.1334	4.3296	32.4665	11
12	2.2522	0.4440	17.8885	0.0559	7.9427	0.1259	4.7025	37.3506	12
13	2.4098	0.4150	20.1406	0.0497	8.3577	0.1197	5.0648	42.3302	13
14	2.5785	0.3878	22.5505	0.0443	8.7455	0.1143	5.4167	47.3718	14
15	2.7590	0.3624	25.1290	0.0398	9.1079	0.1098	5.7583	52.4461	15
16	2.9522	0.3387	27.8881	0.0359	9.4466	0.1059	6.0897	57.5271	16
17	3.1588	0.3166	30.8402	0.0324	9.7632	0.1024	6.4110	62.5923	17
18	3.3799	0.2959	33.9990	0.0294	10.0591	0.0994	6.7225	67.6219	18
19	3.6165	0.2765	37.3790	0.0268	10.3356	0.0968	7.0242	72.5991	19
20	3.8697	0.2584	40.9955	0.0244	10.5940	0.0944	7.3163	77.5091	20
21	4.1406	0.2415	44.8652	0.0223	10.8355	0.0923	7.5990	82.3393	21
22	4.4304	0.2257	49.0057	0.0204	11.0612	0.0904	7.8725	87.0793	22
23	4.7405	0.2109	53.4361	0.0187	11.2722	0.0887	8.1369	91.7201	23
24	5.0724	0.1971	58.1767	0.0172	11.4693	0.0872	8.3923	96.2545	24
25	5.4274	0.1842	63.2490	0.0158	11.6536	0.0858	8.6391	100.6765	25
26	5.8074	0.1722	68.6765	0.0146	11.8258	0.0846	8.8773	104.9814	26
27	6.2139	0.1609	74.4838	0.0134	11.9867	0.0834	9.1072	109.1656	27
28	6.6488	0.1504	80.6977	0.0124	12.1371	0.9824	9.3289	113.2264	28
29	7.1143	0.1406	87.3465	0.0114	12.2777	0.0814	9.5427	117.1622	29
30	7.6123	0.1314	94.4608	0.0106	12.4090	0.0806	9.7487	120.9718	30
31	8.1451	0.1228	102.0730	0.0098	12.5318	0.0798	9.9471	124.6550	31
32	8.7153	0.1147	110.2182	0.0091	12.6466	0.0791	10.1381	128.2120	32
33	9.3253	0.1072	118.9334	0.0084	12.7538	0.0784	10.3219	131.6435	33
34	9.9781	0.1002	128.2588	0.0078	12.8540	0.0778	10.4987	134.9507	34
35	10.6766	0.0937	138.2369	0.0072	12.9477	0.0772	10.6687	138.1353	35
40	14.9745	0.0668	199.6351	0.0050	13.3317	0.0750	11.4233	152.2928	40
45	21.0025	0.0476	285.7493	0.0035	13.6055	0.0735	12.0360	163.7559	45
50	29.4570	0.0339	406.5289	0.0025	13.8007	0.0725	12.5287	172.9051	50
55	41.3150	0.0242	575.9286	0.0017	13.9399	0.0717	12.9215	180.1243	55
60	57.9464	0.0173	813.5204	0.0012	14.0392	0.0712	13.2321	185.7677	60
65	81.2729	0.0123	1146.7552	0.0009	14.1099	0.0709	13.4760	190.1452	65
70	113.9894	0.0088	1614.1342	0.0006	14.1604	0.0706	13.6662	193.5185	70
75	159.8760	0.0063	2269.6574	0.0004	14.1964	0.0704	13.8136	196.1035	75
80	224.2344	0.0045	3189.0627	0.0003	14.2220	0.0703	13.9273	198.0748	80
85	314.5003	0.0032	4478.5761	0.0002	14.2403	0.0702	14.0146	199.5717	85
90	441.1030	0.0023	6287.1854	0.0002	14.2533	0.0702	14.0812	200.7042	90
95	618.6697	0.0016	8823.8535	0.0001	14.2626	0.0701	14.1319	201.5581	95
100	867.7163	0.0012	12381.6618	0.0001	14.2693	0.0701	14.1703	202.2001	100

8.0%

	Single Payment		Equal Payment Series				Gradient Series		
N	Compound Amount Factor (F/P,i,N)	Present Worth Factor (P/F,i,N)	Compound Amount Factor (F/A,i,N)	Sinking Fund Factor (A/F,i,N)	Present Worth Factor (P/A,i,N)	Capital Recovery Factor (A/P,i,N)	Gradient Uniform Series (A/G,i,N)	Gradient Present Worth (P/G,i,N)	N
1	1.0800	0.9259	1.0000	1.0000	0.9259	1.0800	0.0000	0.0000	1
2	1.1664	0.8573	2.0800	0.4808	1.7833	0.5608	0.4808	0.8573	2
3	1.2597	0.7938	3.2464	0.3080	2.5771	0.3880	0.9487	2.4450	3
4	1.3605	0.7350	4.5061	0.2219	3.3121	0.3019	1.4040	4.6501	4
5	1.4693	0.6806	5.8666	0.1705	3.9927	0.2505	1.8465	7.3724	5
6	1.5869	0.6302	7.3359	0.1363	4.6229	0.2163	2.2763	10.5233	6
7	1.7138	0.5835	8.9228	0.1121	5.2064	0.1921	2.6937	14.0242	7
8	1.8509	0.5403	10.6366	0.0940	5.7466	0.1740	3.0985	17.8061	8
9	1.9990	0.5002	12.4876	0.0801	6.2469	0.1601	3.4910	21.8081	9
10	2.1589	0.4632	14.4866	0.0690	6.7101	0.1490	3.8713	25.9768	10
11	2.3316	0.4289	16.6455	0.0601	7.1390	0.1401	4.2395	30.2657	11
12	2.5182	0.3971	18.9771	0.0527	7.5361	0.1327	4.5957	34.6339	12
13	2.7196	0.3677	21.4953	0.0465	7.9038	0.1265	4.9402	39.0463	13
14	2.9372	0.3405	24.2149	0.0413	8.2442	0.1213	5.2731	43.4723	14
15	3.1722	0.3152	27.1521	0.0368	8.5595	0.1168	5.5945	47.8857	15
16	3.4259	0.2919	30.3243	0.0330	8.8514	0.1130	5.9046	52.2640	16
17	3.7000	0.2703	33.7502	0.0296	9.1216	0.1096	6.2037	56.5883	17
18	3.9960	0.2502	37.4502	0.0267	9.3719	0.1067	6.4920	60.8426	18
19	4.3157	0.2317	41.4463	0.0241	9.6036	0.1041	6.7697	65.0134	19
20	4.6610	0.2145	45.7620	0.0219	9.8181	0.1019	7.0369	69.0898	20
21	5.0338	0.1987	50.4229	0.0198	10.0168	0.0998	7.2940	73.0629	21
22	5.4365	0.1839	55.4568	0.0180	10.2007	0.0980	7.5412	76.9257	22
23	5.8715	0.1703	60.8933	0.0164	10.3711	0.0964	7.7786	80.6726	23
24	6.3412	0.1577	66.7648	0.0150	10.5288	0.0950	8.0066	84.2997	24
25	6.8485	0.1460	73.1059	0.0137	10.6748	0.0937	8.2254	87.8041	25
26	7.3964	0.1352	79.9544	0.0125	10.8100	0.0925	8.4352	91.1842	26
27	7.9881	0.1252	87.3508	0.0114	10.9352	0.0914	8.6363	94.4390	27
78	8.6271	0.1159	95.3388	0.0105	11.0511	0.0905	8.8289	97.5687	28
29	9.3173	0.1073	103.9659	0.0096	11.1584	0.0896	9.0133	100.5738	29
30	10.0627	0.0994	113.2832	0.0088	11.2578	0.0888	9.1897	103.4558	30
31	10.8677	0.0920	123.3459	0.0081	11.3498	0.0881	9.3584	106.2163	31
32	11.7371	0.0852	134.2135	0.0075	11.4350	0.0875	9.5197	108.8575	32
33	12.6760	0.0789	145.9506	0.0069	11.5139	0.0869	9.6737	111.3819	33
34	13.6901	0.0730	158.6267	0.0063	11.5869	0.0863	9.8208	113.7924	34
35	14.7853	0.0676	172.3168	0.0058	11.6546	0.0858	9.9611	116.0920	35
40	21.7245	0.0460	259.0565	0.0039	11.9246	0.0839	10.5699	126.0422	40
45	31.9204	0.0313	386.5056	0.0026	12.1084	0.0826	11.0447	133.7331	45
50	46.9016	0.0213	573.7702	0.0017	12.2335	0.0817	11.4107	139.5928	50
55	68.9139	0.0145	848.9232	0.0012	12.3186	0.0812	11.6902	144.0065	55
60	101.2571	0.0099	1253.2133	0.0008	12.3766	0.0808	11.9015	147.3000	60
65	148.7798	0.0067	1847.2481	0.0005	12.4160	0.0805	12.0602	149.7387	65
70	218.6064	0.0046	2720.0801	0.0004	12.4428	0.0804	12.1783	151.5326	70
75	321.2045	0.0031	4002.5566	0.0002	12.4611	0.0802	12.2658	152.8448	75
80	471.9548	0.0021	5886.9354	0.0002	12.4735	0.0802	12.3301	153.8001	80
85	693.4565	0.0014	8655.7061	0.0001	12.4820	0.0801	12.3772	154.4925	85
90	1018.9151	0.0010	12723.9386	0.0001	12.4877	0.0801	12.4116	154.9925	90
95	1497.1205	0.0007	18701.5069	0.0001	12.4917	0.0801	12.4365	155.3524	95
100	2199.7613	0.0005	27484.5157	0.0000	12.4943	0.0800	12.4545	155.6107	100

9.0%

	Single Payment		Equal Payment Series				Gradient Series		
N	**Compound Amount Factor (F/P,i,N)**	**Present Worth Factor (P/F,i,N)**	**Compound Amount Factor (F/A,i,N)**	**Sinking Fund Factor (A/F,i,N)**	**Present Worth Factor (P/A,i,N)**	**Capital Recovery Factor (A/P,i,N)**	**Gradient Uniform Series (A/G,i,N)**	**Gradient Present Worth (P/G,i,N)**	**N**
1	1.0900	0.9174	1.0000	1.0000	0.9174	1.0900	0.0000	0.0000	1
2	1.1881	0.8417	2.0900	0.4785	1.7591	0.5685	0.4785	0.8417	2
3	1.2950	0.7722	3.2781	0.3051	2.5313	0.3951	0.9426	2.3860	3
4	1.4116	0.7084	4.5731	0.2187	3.2397	0.3087	1.3925	4.5113	4
5	1.5386	0.6499	5.9847	0.1671	3.8897	0.2571	1.8282	7.1110	5
6	1.6771	0.5963	7.5233	0.1329	4.4859	0.2229	2.2498	10.0924	6
7	1.8280	0.5470	9.2004	0.1087	5.0330	0.1987	2.6574	13.3746	7
8	1.9926	0.5019	11.0285	0.0907	5.5348	0.1807	3.0512	16.8877	8
9	2.1719	0.4604	13.0210	0.0768	5.9952	0.1668	3.4312	20.5711	9
10	2.3674	0.4224	15.1929	0.0658	6.4177	0.1558	3.7978	24.3728	10
11	2.5804	0.3875	17.5603	0.0569	6.8052	0.1469	4.1510	28.2481	11
12	2.8127	0.3555	20.1407	0.0497	7.1607	0.1397	4.4910	32.1590	12
13	3.0658	0.3262	22.9534	0.0436	7.4869	0.1336	4.8182	36.0731	13
14	3.3417	0.2992	26.0192	0.0384	7.7862	0.1284	5.1326	39.9633	14
15	3.6425	0.2745	29.3609	0.0341	8.0607	0.1241	5.4346	43.8069	15
16	3.9703	0.2519	33.0034	0.0303	8.3126	0.1203	5.7245	47.5849	16
17	4.3276	0.2311	36.9737	0.0270	8.5436	0.1170	6.0024	51.2821	17
18	4.7171	0.2120	41.3013	0.0242	8.7556	0.1142	6.2687	54.8860	18
19	5.1417	0.1945	46.0185	0.0217	8.9501	0.1117	6.5236	58.3868	19
20	5.6044	0.1784	51.1601	0.0195	9.1285	0.1095	6.7674	61.7770	20
21	6.1088	0.1637	56.7645	0.0176	9.2922	0.1076	7.0006	65.0509	21
22	6.6586	0.1502	62.8733	0.0159	9.4424	0.1059	7.2232	68.2048	22
23	7.2579	0.1378	69.5319	0.0144	9.5802	0.1044	7.4357	71.2359	23
24	7.9111	0.1264	76.7898	0.0130	9.7066	0.1030	7.6384	74.1433	24
25	8.6231	0.1160	84.7009	0.0118	9.8226	0.1018	7.8316	76.9265	25
26	9.3992	0.1064	93.3240	0.0107	9.9290	0.1007	8.0156	79.5863	26
27	10.2451	0.0976	102.7231	0.0097	10.0266	0.0997	8.1906	82.1241	27
28	11.1671	0.0895	112.9682	0.0089	10.1161	0.0989	8.3571	84.5419	28
29	12.1722	0.0822	124.1354	0.0081	10.1983	0.0981	8.5154	86.8422	29
30	13.2677	0.0754	136.3075	0.0073	10.2737	0.0973	8.6657	89.0280	30
31	14.4618	0.0691	149.5752	0.0067	10.3428	0.0967	8.8083	91.1024	31
32	15.7633	0.0634	164.0370	0.0061	10.4062	0.0961	8.9436	93.0690	32
33	17.1820	0.0582	179.8003	0.0056	10.4644	0.0956	9.0718	94.9314	33
34	18.7284	0.0534	196.9823	0.0051	10.5178	0.0951	9.1933	96.6935	34
35	20.4140	0.0490	215.7108	0.0046	10.5668	0.0946	9.3083	98.3590	35
40	31.4094	0.0318	337.8824	0.0030	10.7574	0.0930	9.7957	105.3762	40
45	48.3273	0.0207	525.8587	0.0019	10.8812	0.0919	10.1603	110.5561	45
50	74.3575	0.0134	815.0836	0.0012	10.9617	0.0912	10.4295	114.3251	50
55	114.4083	0.0087	1260.0918	0.0008	11.0140	0.0908	10.6261	117.0362	55
60	176.0313	0.0057	1944.7921	0.0005	11.0480	0.0905	10.7683	118.9683	60
65	270.8460	0.0037	2998.2885	0.0003	11.0701	0.0903	10.8702	120.3344	65
70	416.7301	0.0024	4619.2232	0.0002	11.0844	0.0902	10.9427	121.2942	70
75	641.1909	0.0016	7113.2321	0.0001	11.0938	0.0901	10.9940	121.9646	75
80	986.5517	0.0010	10950.5741	0.0001	11.0998	0.0901	11.0299	122.4306	80
85	1517.9320	0.0007	16854.8003	0.0001	11.1038	0.0901	11.0551	122.7533	85
90	2335.5266	0.0004	25939.1842	0.0000	11.1064	0.0900	11.0726	122.9758	90
95	3593.4971	0.0003	39916.6350	0.0000	11.1080	0.0900	11.0847	123.1287	95
100	5529.0408	0.0002	61422.6755	0.0000	11.1091	0.0900	11.0930	123.2335	100

10.0%

	Single Payment		Equal Payment Series				Gradient Series		
N	Compound Amount Factor (F/P,i,N)	Present Worth Factor (P/F,i,N)	Compound Amount Factor (F/A,i,N)	Sinking Fund Factor (A/F,i,N)	Present Worth Factor (P/A,i,N)	Capital Recovery Factor (A/P,i,N)	Gradient Uniform Series (A/G,i,N)	Gradient Present Worth (P/G,i,N)	N
1	1.1000	0.9091	1.0000	1.0000	0.9091	1.1000	0.0000	0.0000	1
2	1.2100	0.8264	2.1000	0.4762	1.7355	0.5762	0.4762	0.8264	2
3	1.3310	0.7513	3.3100	0.3021	2.4869	0.4021	0.9366	2.3291	3
4	1.4641	0.6830	4.6410	0.2155	3.1699	0.3155	1.3812	4.3781	4
5	1.6105	0.6209	6.1051	0.1638	3.7908	0.2638	1.8101	6.8618	5
6	1.7716	0.5645	7.7156	0.1296	4.3553	0.2296	2.2236	9.6842	6
7	1.9487	0.5132	9.4872	0.1054	4.8684	0.2054	2.6216	12.7631	7
8	2.1436	0.4665	11.4359	0.0874	5.3349	0.1874	3.0045	16.0287	8
9	2.3579	0.4241	13.5795	0.0736	5.7590	0.1736	3.3724	19.4215	9
10	2.5937	0.3855	15.9374	0.0627	6.1446	0.1627	3.7255	22.8913	10
11	2.8531	0.3505	18.5312	0.0540	6.4951	0.1540	4.0641	26.3963	11
12	3.1384	0.3186	21.3843	0.0468	6.8137	0.1468	4.3884	29.9012	12
13	3.4523	0.2897	24.5227	0.0408	7.1034	0.1408	4.6988	33.3772	13
14	3.7975	0.2633	27.9750	0.0357	7.3667	0.1357	4.9955	36.8005	14
15	4.1772	0.2394	31.7725	0.0315	7.6061	0.1315	5.2789	40.1520	15
16	4.5950	0.2176	35.9497	0.0278	7.8237	0.1278	5.5493	43.4164	16
17	5.0545	0.1978	40.5447	0.0247	8.0216	0.1247	5.8071	46.5819	17
18	5.5599	0.1799	45.5992	0.0219	8.2014	0.1219	6.0526	49.6395	18
19	6.1159	0.1635	51.1591	0.0195	8.3649	0.1195	6.2861	52.5827	19
20	6.7275	0.1486	57.2750	0.0175	8.5136	0.1175	6.5081	55.4069	20
21	7.4002	0.1351	64.0025	0.0156	8.6487	0.1156	6.7189	58.1095	21
22	8.1403	0.1228	71.4027	0.0140	8.7715	0.1140	6.9189	60.6893	22
23	8.9543	0.1117	79.5430	0.0126	8.8832	0.1126	7.1085	63.1462	23
24	9.8497	0.1015	88.4973	0.0113	8.9847	0.1113	7.2881	65.4813	24
25	10.8347	0.0923	98.3471	0.0102	9.0770	0.1102	7.4580	67.6964	25
26	11.9182	0.0839	109.1818	0.0092	9.1609	0.1092	7.6186	69.7940	26
27	13.1100	0.0763	121.0999	0.0083	9.2372	0.1083	7.7704	71.7773	27
28	14.4210	0.0693	134.2099	0.0075	9.3066	0.1075	7.9137	73.6495	28
29	15.8631	0.0630	148.6309	0.0067	9.3696	0.1067	8.0489	75.4146	29
30	17.4494	0.0573	164.4940	0.0061	9.4269	0.1061	8.1762	77.0766	30
31	19.1943	0.0521	181.9434	0.0055	9.4790	0.1055	8.2962	78.6395	31
32	21.1138	0.0474	201.1378	0.0050	9.5264	0.1050	8.4091	80.1078	32
33	23.2252	0.0431	222.2515	0.0045	9.5694	0.1045	8.5152	81.4856	33
34	25.5477	0.0391	245.4767	0.0041	9.6086	0.1041	8.6149	82.7773	34
35	28.1024	0.0356	271.0244	0.0037	9.6442	0.1037	8.7086	83.9872	35
40	45.2593	0.0221	442.5926	0.0023	9.7791	0.1023	9.0962	88.9525	40
45	72.8905	0.0137	718.9048	0.0014	9.8628	0.1014	9.3740	92.4544	45
50	117.3909	0.0085	1163.9085	0.0009	9.9148	0.1009	9.5704	94.8889	50
55	189.0591	0.0053	1880.5914	0.0005	9.9471	0.1005	9.7075	96.5619	55
60	304.4816	0.0033	3034.8164	0.0003	9.9672	0.1003	9.8023	97.7010	60
65	490.3707	0.0020	4893.7073	0.0002	9.9796	0.1002	9.8672	98.4705	65
70	789.7470	0.0013	7887.4696	0.0001	9.9873	0.1001	9.9113	98.9870	70
75	1271.8954	0.0008	12708.9537	0.0001	9.9921	0.1001	9.9410	99.3317	75
80	2048.4002	0.0005	20474.0021	0.0000	9.9951	0.1000	9.9609	99.5606	80
85	3298.9690	0.0003	32979.6903	0.0000	9.9970	0.1000	9.9742	99.7120	85
90	5313.0226	0.0002	53120.2261	0.0000	9.9981	0.1000	9.9831	99.8118	90
95	8556.6760	0.0001	85556.7605	0.0000	9.9988	0.1000	9.9889	99.8773	95
100	13780.6123	0.0001	137796.1234	0.0000	9.9993	0.1000	9.9927	99.9202	100

11.0%

	Single Payment		Equal Payment Series				Gradient Series		
N	**Compound Amount Factor (F/P,i,N)**	**Present Worth Factor (P/F,i,N)**	**Compound Amount Factor (F/A,i,N)**	**Sinking Fund Factor (A/F,i,N)**	**Present Worth Factor (P/A,i,N)**	**Capital Recovery Factor (A/P,i,N)**	**Gradient Uniform Series (A/G,i,N)**	**Gradient Present Worth (P/G,i,N)**	**N**
1	1.1100	0.9009	1.0000	1.0000	0.9009	1.1100	0.0000	0.0000	1
2	1.2321	0.8116	2.1100	0.4739	1.7125	0.5839	0.4739	0.8116	2
3	1.3676	0.7312	3.3421	0.2992	2.4437	0.4092	0.9306	2.2740	3
4	1.5181	0.6587	4.7097	0.2123	3.1024	0.3223	1.3700	4.2502	4
5	1.6851	0.5935	6.2278	0.1606	3.6959	0.2706	1.7923	6.6240	5
6	1.8704	0.5346	7.9129	0.1264	4.2305	0.2364	2.1976	9.2972	6
7	2.0762	0.4817	9.7833	0.1022	4.7122	0.2122	2.5863	12.1872	7
8	2.3045	0.4339	11.8594	0.0843	5.1461	0.1943	2.9585	15.2246	8
9	2.5580	0.3909	14.1640	0.0706	5.5370	0.1806	3.3144	18.3520	9
10	2.8394	0.3522	16.7220	0.0598	5.8892	0.1698	3.6544	21.5217	10
11	3.1518	0.3173	19.5614	0.0511	6.2065	0.1611	3.9788	24.6945	11
12	3.4985	0.2858	22.7132	0.0440	6.4924	0.1540	4.2879	27.8388	12
13	3.8833	0.2575	26.2116	0.0382	6.7499	0.1482	4.5822	30.9290	13
14	4.3104	0.2320	30.0949	0.0332	6.9819	0.1432	4.8619	33.9449	14
15	4.7846	0.2090	34.4054	0.0291	7.1909	0.1391	5.1275	36.8709	15
16	5.3109	0.1883	39.1899	0.0255	7.3792	0.1355	5.3794	39.6953	16
17	5.8951	0.1696	44.5008	0.0225	7.5488	0.1325	5.6180	42.4095	17
18	6.5436	0.1528	50.3959	0.0198	7.7016	0.1298	5.8439	45.0074	18
19	7.2633	0.1377	56.9395	0.0176	7.8393	0.1276	6.0574	47.4856	19
20	8.0623	0.1240	64.2028	0.0156	7.9633	0.1256	6.2590	49.8423	20
21	8.9492	0.1117	72.2651	0.0138	8.0751	0.1238	6.4491	52.0771	21
22	9.9336	0.1007	81.2143	0.0123	8.1757	0.1223	6.6283	54.1912	22
23	11.0263	0.0907	91.1479	0.0110	8.2664	0.1210	6.7969	56.1864	23
24	12.2392	0.0817	102.1742	0.0098	8.3481	0.1198	6.9555	58.0656	24
25	13.5855	0.0736	114.4133	0.0087	8.4217	0.1187	7.1045	59.8322	25
26	15.0799	0.0663	127.9988	0.0078	8.4881	0.1178	7.2443	61.4900	26
27	16.7386	0.0597	143.0786	0.0070	8.5478	0.1170	7.3754	63.0433	27
28	18.5799	0.0538	159.8173	0.0063	8.6016	0.1163	7.4982	64.4965	28
29	20.6237	0.0485	178.3972	0.0056	8.6501	0.1156	7.6131	65.8542	29
30	22.8923	0.0437	199.0209	0.0050	8.6938	0.1150	7.7206	67.1210	30
31	25.4104	0.0394	221.9132	0.0045	8.7331	0.1145	7.8210	68.3016	31
32	28.2056	0.0355	247.3236	0.0040	8.7686	0.1140	7.9147	69.4007	32
33	31.3082	0.0319	275.5292	0.0036	8.8005	0.1136	8.0021	70.4228	33
34	34.7521	0.0288	306.8374	0.0033	8.8293	0.1133	8.0836	71.3724	34
35	38.5749	0.0259	341.5896	0.0029	8.8552	0.1129	8.1594	72.2538	35
40	65.0009	0.0154	581.8261	0.0017	8.9511	0.1117	8.4659	75.7789	40
45	109.5302	0.0091	986.6386	0.0010	9.0079	0.1110	8.6763	78.1551	45
50	184.5648	0.0054	1668.7712	0.0006	9.0417	0.1106	8.8185	79.7341	50
55	311.0025	0.0032	2818.2042	0.0004	9.0617	0.1104	8.9135	80.7712	55
60	524.0572	0.0019	4755.0658	0.0002	9.0736	0.1102	8.9762	81.4461	60

12.0%

	Single Payment		Equal Payment Series				Gradient Series		
N	**Compound Amount Factor (F/P,i,N)**	**Present Worth Factor (P/F,i,N)**	**Compound Amount Factor (F/A,i,N)**	**Sinking Fund Factor (A/F,i,N)**	**Present Worth Factor (P/A,i,N)**	**Capital Recovery Factor (A/P,i,N)**	**Gradient Uniform Series (A/G,i,N)**	**Gradient Present Worth (P/G,i,N)**	**N**
1	1.1200	0.8929	1.0000	1.0000	0.8929	1.1200	0.0000	0.0000	1
2	1.2544	0.7972	2.1200	0.4717	1.6901	0.5917	0.4717	0.7972	2
3	1.4049	0.7118	3.3744	0.2963	2.4018	0.4163	0.9246	2.2208	3
4	1.5735	0.6355	4.7793	0.2092	3.0373	0.3292	1.3589	4.1273	4
5	1.7623	0.5674	6.3528	0.1574	3.6048	0.2774	1.7746	6.3970	5
6	1.9738	0.5066	8.1152	0.1232	4.1114	0.2432	2.1720	8.9302	6
7	2.2107	0.4523	10.0890	0.0991	4.5638	0.2191	2.5515	11.6443	7
8	2.4760	0.4039	12.2997	0.0813	4.9676	0.2013	2.9131	14.4714	8
9	2.7731	0.3606	14.7757	0.0677	5.3282	0.1877	3.2574	17.3563	9
10	3.1058	0.3220	17.5487	0.0570	5.6502	0.1770	3.5847	20.2541	10
11	3.4785	0.2875	20.6546	0.0484	5.9377	0.1684	3.8953	23.1288	11
12	3.8960	0.2567	24.1331	0.0414	6.1944	0.1614	4.1897	25.9523	12
13	4.3635	0.2292	28.0291	0.0357	6.4235	0.1557	4.4683	28.7024	13
14	4.8871	0.2046	32.3926	0.0309	6.6282	0.1509	4.7317	31.3624	14
15	5.4736	0.1827	37.2797	0.0268	6.8109	0.1468	4.9803	33.9202	15
16	6.1304	0.1631	42.7533	0.0234	6.9740	0.1434	5.2147	36.3670	16
17	6.8660	0.1456	48.8837	0.0205	7.1196	0.1405	5.4353	38.6973	17
18	7.6900	0.1300	55.7497	0.0179	7.2497	0.1379	5.6427	40.9080	81
19	8.6128	0.1161	63.4397	0.0158	7.3658	0.1358	5.8375	42.9979	19
20	9.6463	0.1037	72.0524	0.0139	7.4694	0.1339	6.0202	44.9676	20
21	10.8038	0.0926	81.6987	0.0122	7.5620	0.1322	6.1913	46.8188	21
22	12.1003	0.0826	92.5026	0.0108	7.6446	0.1308	6.3514	48.5543	22
23	13.5523	0.0738	104.6029	0.0096	7.7184	0.1296	6.5010	50.1776	23
24	15.1786	0.0659	118.1552	0.0085	7.7843	0.1285	6.6406	51.6929	24
25	17.0001	0.0588	133.3339	0.0075	7.8431	0.1275	6.7708	53.1046	25
26	19.0401	0.0525	150.3339	0.0067	7.8957	0.1267	6.8921	54.4177	26
27	21.3249	0.0469	169.3740	0.0059	7.9426	0.1259	7.0049	55.6369	27
28	23.8839	0.0419	190.6989	0.0052	7.9844	0.1252	7.1098	56.7674	28
29	26.7499	0.0374	214.5828	0.0047	8.0218	0.1247	7.2071	57.8141	29
30	29.9599	0.0334	241.3327	0.0041	8.0552	0.1241	7.2974	58.7821	30
31	33.5551	0.0298	271.2926	0.0037	8.0850	0.1237	7.3811	59.6761	31
32	37.5817	0.0266	304.8477	0.0033	8.1116	0.1233	7.4586	60.5010	32
33	42.0915	0.0238	342.4294	0.0029	8.1354	0.1229	7.5302	61.2612	33
34	47.1425	0.0212	384.5210	0.0026	8.1566	0.1226	7.5965	61.9612	34
35	52.7996	0.0189	431.6635	0.0023	8.1755	0.1223	7.6577	62.6052	35
40	93.0510	0.0107	767.0914	0.0013	8.2438	0.1213	7.8988	65.1159	40
45	163.9876	0.0061	1358.2300	0.0007	8.2825	0.1207	8.0572	66.7342	45
50	289.0022	0.0035	2400.0182	0.0004	8.3045	0.1204	8.1597	67.7624	50
55	509.3206	0.0020	4236.0050	0.0002	8.3170	0.1202	8.2251	68.4082	55
60	897.5969	0.0011	7471.6411	0.0001	8.3240	0.1201	8.2664	68.8100	60

13.0%

	Single Payment		Equal Payment Series				Gradient Series		
N	Compound Amount Factor (F/P,i,N)	Present Worth Factor (P/F,i,N)	Compound Amount Factor (F/A,i,N)	Sinking Fund Factor (A/F,i,N)	Present Worth Factor (P/A,i,N)	Capital Recovery Factor (A/P,i,N)	Gradient Uniform Series (A/G,i,N)	Gradient Present Worth (P/G,i,N)	N
1	1.1300	0.8850	1.0000	1.0000	0.8850	1.1300	0.0000	0.0000	1
2	1.2769	0.7831	2.1300	0.4695	1.6681	0.5995	0.4695	0.7831	2
3	1.4429	0.6931	3.4069	0.2935	2.3612	0.4235	0.9187	2.1692	3
4	1.6305	0.6133	4.8498	0.2062	2.9745	0.3362	1.3479	4.0092	4
5	1.8424	0.5428	6.4803	0.1543	3.5172	0.2843	1.7571	6.1802	5
6	2.0820	0.4803	8.3227	0.1202	3.9975	0.2502	2.1468	8.5818	6
7	2.3526	0.4251	10.4047	0.0961	4.4226	0.2261	2.5171	11.1322	7
8	2.6584	0.3762	12.7573	0.0784	4.7988	0.2084	2.8685	13.7653	8
9	3.0040	0.3329	15.4157	0.0649	5.1317	0.1949	3.2014	16.4284	9
10	3.3946	0.2946	18.4197	0.0543	5.4262	0.1843	3.5162	19.0797	10
11	3.8359	0.2607	21.8143	0.0458	5.6869	0.1758	3.8134	21.6867	11
12	4.3345	0.2307	25.6502	0.0390	5.9176	0.1690	4.0936	24.2244	12
13	4.8980	0.2042	29.9847	0.0334	6.1218	0.1634	4.3573	26.6744	13
14	5.5348	0.1807	34.8827	0.0287	6.3025	0.1587	4.6050	29.0232	14
15	6.2543	0.1599	40.4175	0.0247	6.4624	0.1547	4.8375	31.2617	15
16	7.0673	0.1415	46.6717	0.0214	6.6039	0.1514	5.0552	33.3841	16
17	7.9861	0.1252	53.7391	0.0186	6.7291	0.1486	5.2589	35.3876	17
18	9.0243	0.1108	61.7251	0.0162	6.8399	0.1462	5.4491	37.2714	18
19	10.1974	0.0981	70.7494	0.0141	6.9380	0.1441	5.6265	39.0366	19
20	11.5231	0.0868	80.9468	0.0124	7.0248	0.1424	5.7917	40.6854	20
21	13.0211	0.0768	92.4699	0.0108	7.1016	0.1408	5.9454	42.2214	21
22	14.7138	0.0680	105.4910	0.0095	7.1695	0.1395	6.0881	43.6486	22
23	16.6266	0.0601	120.2048	0.0083	7.2297	0.1383	6.2205	44.9718	23
24	18.7881	0.0532	136.8315	0.0073	7.2829	0.1373	6.3431	46.1960	24
25	21.2305	0.0471	155.6196	0.0064	7.3300	0.1364	6.4566	47.3264	25
26	23.9905	0.0417	176.8501	0.0057	7.3717	0.1357	6.5614	48.3685	26
27	27.1093	0.0369	200.8406	0.0050	7.4086	0.1350	6.6582	49.3276	27
28	30.6335	0.0326	227.9499	0.0044	7.4412	0.1344	6.7474	50.2090	28
29	34.6158	0.0289	258.5834	0.0039	7.4701	0.1339	6.8296	51.0179	29
30	39.1159	0.0256	293.1992	0.0034	7.4957	0.1334	6.9052	51.7592	30
31	44.2010	0.0226	332.3151	0.0030	7.5183	0.1330	6.9747	52.4380	31
32	49.9471	0.0200	376.5161	0.0027	7.5383	0.1327	7.0385	53.0586	32
33	56.4402	0.0177	426.4632	0.0023	7.5560	0.1323	7.0971	53.6256	33
34	63.7774	0.0157	482.9034	0.0021	7.5717	0.1321	7.1507	54.1430	34
35	72.0685	0.0139	546.6808	0.0018	7.5856	0.1318	7.1998	54.6148	35
40	132.7816	0.0075	1013.7042	0.0010	7.6344	0.1310	7.3888	56.4087	40
45	244.6414	0.0041	1874.1646	0.0005	7.6609	0.1305	7.5076	57.5148	45
50	450.7359	0.0022	3459.5071	0.0003	7.6752	0.1303	7.5811	58.1870	50
55	830.4517	0.0012	6380.3979	0.0002	7.6830	0.1302	7.6260	58.5909	55
60	1530.0535	0.0007	11761.9498	0.0001	7.6873	0.1301	7.6531	58.8313	60

14.0%

	Single Payment		Equal Payment Series				Gradient Series		
N	Compound Amount Factor (F/P,i,N)	Present Worth Factor (P/F,i,N)	Compound Amount Factor (F/A,i,N)	Sinking Fund Factor (A/F,i,N)	Present Worth Factor (P/A,i,N)	Capital Recovery Factor (A/P,i,N)	Gradient Uniform Series (A/G,i,N)	Gradient Present Worth (P/G,i,N)	N
1	1.1400	0.8772	1.0000	1.0000	0.8772	1.1400	0.0000	0.0000	1
2	1.2996	0.7695	2.1400	0.4673	1.6467	0.6073	0.4673	0.7695	2
3	1.4815	0.6750	3.4396	0.2907	2.3216	0.4307	0.9129	2.1194	3
4	1.6890	0.5921	4.9211	0.2032	2.9137	0.3432	1.3370	3.8957	4
5	1.9254	0.5194	6.6101	0.1513	3.4331	0.2913	1.7399	5.9731	5
6	2.1950	0.4556	8.5355	0.1172	3.8887	0.2572	2.1218	8.2511	6
7	2.5023	0.3996	10.7305	0.0932	4.2883	0.2332	2.4832	10.6489	7
8	2.8526	0.3506	13.2328	0.0756	4.6389	0.2156	2.8246	13.1028	8
9	3.2519	0.3075	16.0853	0.0622	4.9464	0.2022	3.1463	15.5629	9
10	3.7072	0.2697	19.3373	0.0517	5.2161	0.1917	3.4490	17.9906	10
11	4.2262	0.2366	23.0445	0.0434	5.4527	0.1834	3.7333	20.3567	11
12	4.8179	0.2076	27.2707	0.0367	5.6603	0.1767	3.9998	22.6399	12
13	5.4924	0.1821	32.0887	0.0312	5.8424	0.1712	4.2491	24.8247	13
14	6.2613	0.1597	37.5811	0.0266	6.0021	0.1666	4.4819	26.9009	14
15	7.1379	0.1401	43.8424	0.0228	6.1422	0.1628	4.6990	28.8623	15
16	8.1372	0.1229	50.9804	0.0196	6.2651	0.1596	4.9011	30.7057	16
17	9.2765	0.1078	59.1176	0.0169	6.3729	0.1569	5.0888	32.4305	17
18	10.5752	0.0946	68.3941	0.0146	6.4674	0.1546	5.2630	34.0380	18
19	12.0557	0.0829	78.9692	0.0127	6.5504	0.1527	5.4243	35.5311	19
20	13.7435	0.0728	91.0249	0.0110	6.6231	0.1510	5.5734	36.9135	20
21	15.6676	0.0638	104.7684	0.0095	6.6870	0.1495	5.7111	38.1901	21
22	17.8610	0.0560	120.4360	0.0083	6.7429	0.1483	5.8381	39.3658	22
23	20.3616	0.0491	138.2970	0.0072	6.7921	0.1472	5.9549	40.4463	23
24	23.2122	0.0431	158.6586	0.0063	6.8351	0.1463	6.0624	41.4371	24
25	26.4619	0.0378	181.8708	0.0055	6.8729	0.1455	6.1610	42.3441	25
26	30.1666	0.0331	208.3327	0.0048	6.9061	0.1448	6.2514	43.1728	26
27	34.3899	0.0291	238.4993	0.0042	6.9352	0.1442	6.3342	43.9289	27
28	39.2045	0.0255	272.8892	0.0037	6.9607	0.1437	6.4100	44.6176	28
29	44.6931	0.0224	312.0937	0.0032	6.9830	0.1432	6.4791	45.2441	29
30	50.9502	0.0196	356.7868	0.0028	7.0027	0.1428	6.5423	45.8132	30
31	58.0832	0.0172	407.7370	0.0025	7.0199	0.1425	6.5998	46.3297	31
32	66.2148	0.0151	465.8202	0.0021	7.0350	0.1421	6.6522	46.7979	32
33	75.4849	0.0132	532.0350	0.0019	7.0482	0.1419	6.6998	47.2218	33
34	86.0528	0.0116	607.5199	0.0016	7.0599	0.1416	6.7431	47.6053	34
35	98.1002	0.0102	693.5727	0.0014	7.0700	0.1414	6.7824	47.9519	35
40	188.8835	0.0053	1342.0251	0.0007	7.1050	0.1407	6.9300	49.2376	40
45	363.6791	0.0027	2590.5648	0.0004	7.1232	0.1404	7.0188	49.9963	45
50	700.2330	0.0014	4994.5213	0.0002	7.1327	0.1402	7.0714	50.4375	50

15.0%

	Single Payment		Equal Payment Series				Gradient Series		
N	**Compound Amount Factor (F/P,i,N)**	**Present Worth Factor (P/F,i,N)**	**Compound Amount Factor (F/A,i,N)**	**Sinking Fund Factor (A/F,i,N)**	**Present Worth Factor (P/A,i,N)**	**Capital Recovery Factor (A/P,i,N)**	**Gradient Uniform Series (A/G,i,N)**	**Gradient Present Worth (P/G,i,N)**	**N**
1	1.1500	0.8696	1.0000	1.0000	0.8696	1.1500	0.0000	0.0000	1
2	1.3225	0.7561	2.1500	0.4651	1.6257	0.6151	0.4651	0.7561	2
3	1.5209	0.6575	3.4725	0.2880	2.2832	0.4380	0.9071	2.0712	3
4	1.7490	0.5718	4.9934	0.2003	2.8550	0.3503	1.3263	3.7864	4
5	2.0114	0.4972	6.7424	0.1483	3.3522	0.2983	1.7228	5.7751	5
6	2.3131	0.4323	8.7537	0.1142	3.7845	0.2642	2.0972	7.9368	6
7	2.6600	0.3759	11.0668	0.0904	4.1604	0.2404	2.4498	10.1924	7
8	3.0590	0.3269	13.7268	0.0729	4.4873	0.2229	2.7813	12.4807	8
9	3.5179	0.2843	16.7858	0.0596	4.7716	0.2096	3.0922	14.7548	9
10	4.0456	0.2472	20.3037	0.0493	5.0188	0.1993	3.3832	16.9795	10
11	4.6524	0.2149	24.3493	0.0411	5.2337	0.1911	3.6549	19.1289	11
12	5.3503	0.1869	29.0017	0.0345	5.4206	0.1845	3.9082	21.1849	12
13	6.1528	0.1625	34.3519	0.0291	5.5831	0.1791	4.1438	23.1352	13
14	7.0757	0.1413	40.5047	0.0247	5.7245	0.1747	4.3624	24.9725	14
15	8.1371	0.1229	47.5804	0.0210	5.8474	0.1710	4.5650	26.6930	15
16	9.3576	0.1069	55.7175	0.0179	5.9542	0.1679	4.7522	28.2960	16
17	10.7613	0.0929	65.0751	0.0154	6.0472	0.1654	4.9251	29.7828	17
18	12.3755	0.0808	75.8364	0.0132	6.1280	0.1632	5.0843	31.1565	18
19	14.2318	0.0703	88.2118	0.0113	6.1982	0.1613	5.2307	32.4213	19
20	16.3665	0.0611	102.4436	0.0098	6.2593	0.1598	5.3651	33.5822	20
21	18.8215	0.0531	118.8101	0.0084	6.3125	0.1584	5.4883	34.6448	21
22	21.6447	0.0462	137.6316	0.0073	6.3587	0.1573	5.6010	35.6150	22
23	24.8915	0.0402	159.2764	0.0063	6.3988	0.1563	5.7040	36.4988	23
24	28.6252	0.0349	184.1678	0.0054	6.4338	0.1554	5.7979	37.3023	24
25	32.9190	0.0304	212.7930	0.0047	6.4641	0.1547	5.8834	38.0314	25
26	37.8568	0.0264	245.7120	0.0041	6.4906	0.1541	5.9612	38.6918	26
27	43.5353	0.0230	283.5688	0.0035	6.5135	0.1535	6.0319	39.2890	27
28	50.0656	0.0200	327.1041	0.0031	6.5335	0.1531	6.0960	39.8283	28
29	57.5755	0.0174	377.1697	0.0027	6.5509	0.1527	6.1541	40.3146	29
30	66.2118	0.0151	434.7451	0.0023	6.5660	0.1523	6.2066	40.7526	30
31	76.1435	0.0131	500.9569	0.0020	6.5791	0.1520	6.2541	41.1466	31
32	87.5651	0.0114	577.1005	0.0017	6.5905	0.1517	6.2970	41.5006	32
33	100.6998	0.0099	664.6655	0.0015	6.6005	0.1515	6.3357	41.8184	33
34	115.8048	0.0086	765.3654	0.0013	6.6091	0.1513	6.3705	42.1033	34
35	133.1755	0.0075	881.1702	0.0011	6.6166	0.1511	6.4019	42.3586	35
40	267.8635	0.0037	1779.0903	0.0006	6.6418	0.1506	6.5168	43.2830	40
45	538.7693	0.0019	3585.1285	0.0003	6.6543	0.1503	6.5830	43.8051	45
50	1083.6574	0.0009	7217.7163	0.0001	6.6605	0.1501	6.6205	44.0958	50

16.0%

N	Single Payment: Compound Amount Factor (F/P,i,N)	Single Payment: Present Worth Factor (P/F,i,N)	Equal Payment Series: Compound Amount Factor (F/A,i,N)	Equal Payment Series: Sinking Fund Factor (A/F,i,N)	Equal Payment Series: Present Worth Factor (P/A,i,N)	Equal Payment Series: Capital Recovery Factor (A/P,i,N)	Gradient Series: Gradient Uniform Series (A/G,i,N)	Gradient Series: Gradient Present Worth (P/G,i,N)	N
1	1.1600	0.8621	1.0000	1.0000	0.8621	1.1600	0.0000	0.0000	1
2	1.3456	0.7432	2.1600	0.4630	1.6052	0.6230	0.4630	0.7432	2
3	1.5609	0.6407	3.5056	0.2853	2.2459	0.4453	0.9014	2.0245	3
4	1.8106	0.5523	5.0665	0.1974	2.7982	0.3574	1.3156	3.6814	4
5	2.1003	0.4761	6.8771	0.1454	3.2743	0.3054	1.7060	5.5858	5
6	2.4364	0.4104	8.9775	0.1114	3.6847	0.2714	2.0729	7.6380	6
7	2.8262	0.3538	11.4139	0.0876	4.0386	0.2476	2.4169	9.7610	7
8	3.2784	0.3050	14.2401	0.0702	4.3436	0.2302	2.7388	11.8962	8
9	3.8030	0.2630	17.5185	0.0571	4.6065	0.2171	3.0391	13.9998	9
10	4.4114	0.2267	21.3215	0.0469	4.8332	0.2069	3.3187	16.0399	10
11	5.1173	0.1954	25.7329	0.0389	5.0286	0.1989	3.5783	17.9941	11
12	5.9360	0.1685	30.8502	0.0324	5.1971	0.1924	3.8189	19.8472	12
13	6.8858	0.1452	36.7862	0.0272	5.3423	0.1872	4.0413	21.5899	13
14	7.9875	0.1252	43.6720	0.0229	5.4675	0.1829	4.2464	23.2175	14
15	9.2655	0.1079	51.6595	0.0194	5.5755	0.1794	4.4352	24.7284	15
16	10.7480	0.0930	60.9650	0.0164	5.6685	0.1764	4.6086	26.1241	16
17	12.4677	0.0802	71.6730	0.0140	5.7487	0.1740	4.7676	27.4074	17
18	14.4625	0.0691	84.1407	0.0119	5.8178	0.1719	4.9130	28.5828	18
19	16.7765	0.0596	98.6032	0.0101	5.8775	0.1701	5.0457	29.6557	19
20	19.4608	0.0514	115.3797	0.0087	5.9288	0.1687	5.1666	30.6321	20
21	22.5745	0.0443	134.8405	0.0074	5.9731	0.1674	5.2766	31.5180	21
22	26.1864	0.0382	157.4150	0.0064	6.0113	0.1664	5.3765	32.3200	22
23	30.3762	0.0329	183.6014	0.0054	6.0442	0.1654	5.4671	33.0442	23
24	35.2364	0.0284	213.9776	0.0047	6.0726	0.1647	5.5490	33.6970	24
25	40.8742	0.0245	249.2140	0.0040	6.0971	0.1640	5.6230	34.2841	25
26	47.4141	0.0211	290.0883	0.0034	6.1182	0.1634	5.6898	34.8114	26
27	55.0004	0.0182	337.5024	0.0030	6.1364	0.1630	5.7500	35.2841	27
28	63.8004	0.0157	392.5028	0.0025	6.1520	0.1625	5.8041	35.7073	28
29	74.0085	0.0135	456.3032	0.0022	6.1656	0.1622	5.8528	36.0856	29
30	85.8499	0.0116	530.3117	0.0019	6.1772	0.1619	5.8964	36.4234	30
31	99.5859	0.0100	616.1616	0.0016	6.1872	0.1616	5.9356	36.7247	31
32	115.5196	0.0087	715.7475	0.0014	6.1959	0.1614	5.9706	36.9930	32
33	134.0027	0.0075	831.2671	0.0012	6.2034	0.1612	6.0019	27.2318	33
34	155.4432	0.0064	965.2698	0.0010	6.2098	0.1610	6.0299	37.4441	34
35	180.3141	0.0055	1120.7130	0.0009	6.2153	0.1609	6.0548	37.6327	35
40	378.7212	0.0026	2360.7572	0.0004	6.2335	0.1604	6.1441	38.2992	40
45	795.4438	0.0013	4965.2739	0.0002	6.2421	0.1602	6.1934	38.6598	45
50	1670.7038	0.0006	10435.6488	0.0001	6.2463	0.1601	6.2201	38.8521	50

18.0%

	Single Payment		Equal Payment Series				Gradient Series		
N	**Compound Amount Factor (F/P,i,N)**	**Present Worth Factor (P/F,i,N)**	**Compound Amount Factor (F/A,i,N)**	**Sinking Fund Factor (A/F,i,N)**	**Present Worth Factor (P/A,i,N)**	**Capital Recovery Factor (A/P,i,N)**	**Gradient Uniform Series (A/G,i,N)**	**Gradient Present Worth (P/G,i,N)**	**N**
1	1.1800	0.8475	1.0000	1.0000	0.8475	1.1800	0.0000	0.0000	1
2	1.3924	0.7182	2.1800	0.4587	1.5656	0.6387	0.4587	0.7182	2
3	1.6430	0.6086	3.5724	0.2799	2.1743	0.4599	0.8902	1.9354	3
4	1.9388	0.5158	5.2154	0.1917	2.6901	0.3717	1.2947	3.4828	4
5	2.2878	0.4371	7.1542	0.1398	3.1272	0.3198	1.6728	5.2312	5
6	2.6996	0.3704	9.4420	0.1059	3.4976	0.2859	2.0252	7.0834	6
7	3.1855	0.3139	12.1415	0.0824	3.8115	0.2624	2.3526	8.9670	7
8	3.7589	0.2660	15.3270	0.0652	4.0776	0.2452	2.6558	10.8292	8
9	4.4355	0.2255	19.0859	0.0524	4.3030	0.2324	2.9358	12.6329	9
10	5.2338	0.1911	23.5213	0.0425	4.4941	0.2225	3.1936	14.3525	10
11	6.1759	0.1619	28.7551	0.0348	4.6560	0.2148	3.4303	15.9716	11
12	7.2876	0.1372	34.9311	0.0286	4.7932	0.2086	3.6470	17.4811	12
13	8.5994	0.1163	42.2187	0.0237	4.9095	0.2037	3.8449	18.8765	13
14	10.1472	0.0985	50.8180	0.0197	5.0081	0.1997	4.0250	20.1576	14
15	11.9737	0.0835	60.9653	0.0164	5.0916	0.1964	4.1887	21.3269	15
16	14.1290	0.0708	72.9390	0.0137	5.1624	0.1937	4.3369	22.3885	16
17	16.6722	0.0600	87.0680	0.0115	5.2223	0.1915	4.4708	23.3482	17
18	19.6733	0.0508	103.7403	0.0096	5.2732	0.1896	4.5916	24.2123	18
19	23.2144	0.0431	123.4135	0.0081	5.3162	0.1881	4.7003	24.9877	19
20	27.3930	0.0365	146.6280	0.0068	5.3527	0.1868	4.7978	25.6813	20
21	32.3238	0.0309	174.0210	0.0057	5.3837	0.1857	4.8851	26.3000	21
22	38.1421	0.0262	206.3448	0.0048	5.4099	0.1848	4.9632	26.8506	22
23	45.0076	0.0222	244.4868	0.0041	5.4321	0.1841	5.0329	27.3394	23
24	53.1090	0.0188	289.4945	0.0035	5.4509	0.1835	5.0950	27.7725	24
25	62.6686	0.0160	342.6035	0.0029	5.4669	0.1829	5.1502	28.1555	25
26	73.9490	0.0135	405.2721	0.0025	5.4804	0.1825	5.1991	28.4935	26
27	87.2598	0.0115	479.2211	0.0021	5.4919	0.1821	5.2425	28.7915	27
28	102.9666	0.0097	566.4809	0.0018	5.5016	0.1818	5.2810	29.0537	28
29	121.5005	0.0082	669.4475	0.0015	5.5098	0.1815	5.3149	29.2842	29
30	143.3706	0.0070	790.9480	0.0013	5.5168	0.1813	5.3448	29.4864	30
31	169.1774	0.0059	934.3186	0.0011	5.5227	0.1811	5.3712	29.6638	31
32	199.6293	0.0050	1103.4960	0.0009	5.5277	0.1809	5.3945	29.8191	32
33	235.5625	0.0042	1303.1253	0.0008	5.5320	0.1808	5.4149	29.9549	33
34	277.9638	0.0036	1538.6878	0.0006	5.5356	0.1806	5.4328	30.0736	34
35	327.9973	0.0030	1816.6516	0.0006	5.5386	0.1806	5.4485	30.1773	35
40	750.3783	0.0013	4163.2130	0.0002	5.5482	0.1802	5.5022	30.5269	40
45	1716.6839	0.0006	9531.5771	0.0001	5.5523	0.1801	5.5293	30.7006	45
50	3927.3569	0.0003	21813.0937	0.0000	5.5541	0.1800	5.5428	30.7856	50

20.0%

	Single Payment		Equal Payment Series				Gradient Series		
N	**Compound Amount Factor (F/P,i,N)**	**Present Worth Factor (P/F,i,N)**	**Compound Amount Factor (F/A,i,N)**	**Sinking Fund Factor (A/F,i,N)**	**Present Worth Factor (P/A,i,N)**	**Capital Recovery Factor (A/P,i,N)**	**Gradient Uniform Series (A/G,i,N)**	**Gradient Present Worth (P/G,i,N)**	**N**
1	1.2000	0.8333	1.0000	1.0000	0.8333	1.2000	0.0000	0.0000	1
2	1.4400	0.6944	2.2000	0.4545	1.5278	0.6545	0.4545	0.6944	2
3	1.7280	0.5787	3.6400	0.2747	2.1065	0.4747	0.8791	1.8519	3
4	2.0736	0.4823	5.3680	0.1863	2.5887	0.3863	1.2742	3.2986	4
5	2.4883	0.4019	7.4416	0.1344	2.9906	0.3344	1.6405	4.9061	5
6	2.9860	0.3349	9.9299	0.1007	3.3255	0.3007	1.9788	6.5806	6
7	3.5832	0.2791	12.9159	0.0774	3.6046	0.2774	2.2902	8.2551	7
8	4.2998	0.2326	16.4991	0.0606	3.8372	0.2606	2.5756	9.8831	8
9	5.1598	0.1938	20.7989	0.0481	4.0310	0.2481	2.8364	11.4335	9
10	6.1917	0.1615	25.9587	0.0385	4.1925	0.2385	3.0739	12.8871	10
11	7.4301	0.1346	32.1504	0.0311	4.3271	0.2311	3.2893	14.2330	11
12	8.9161	0.1122	39.5805	0.0253	4.4392	0.2253	3.4841	15.4667	12
13	10.6993	0.0935	48.4966	0.0206	4.5327	0.2206	3.6597	16.5883	13
14	12.8392	0.0779	59.1959	0.0169	4.6106	0.2169	3.8175	17.6008	14
15	15.4070	0.0649	72.0351	0.0139	4.6755	0.2139	3.9588	18.5095	15
16	18.4884	0.0541	87.4421	0.0114	4.7296	0.2114	4.0851	19.3208	16
17	22.1861	0.0451	105.9306	0.0094	4.7746	0.2094	4.1976	20.0419	17
18	26.6233	0.0376	128.1167	0.0078	4.8122	0.2078	4.2975	20.6805	18
19	31.9480	0.0313	154.7400	0.0065	4.8435	0.2065	4.3861	21.2439	19
20	38.3376	0.0261	186.6880	0.0054	4.8696	0.2054	4.4643	21.7395	20
21	46.0051	0.0217	225.0256	0.0044	4.8913	0.2044	4.5334	22.1742	21
22	55.2061	0.0181	271.0307	0.0037	4.9094	0.2037	4.5941	22.5546	22
23	66.2474	0.0151	326.2369	0.0031	4.9245	0.2031	4.6475	22.8867	23
24	79.4968	0.0126	392.4842	0.0025	4.9371	0.2025	4.6943	23.1760	24
25	95.3962	0.0105	471.9811	0.0021	4.9476	0.2021	4.7352	23.4276	25
26	114.4755	0.0087	567.3773	0.0018	4.9563	0.2018	4.7709	23.6460	26
27	137.3706	0.0073	681.8528	0.0015	4.9636	0.2015	4.8020	23.8353	27
28	164.8447	0.0061	819.2233	0.0012	4.9697	0.2012	4.8291	23.9991	28
29	197.8136	0.0051	984.0680	0.0010	4.9747	0.2010	4.8527	24.1406	29
30	237.3763	0.0042	1181.8816	0.0008	4.9789	0.2008	4.8731	24.2628	30
31	284.8516	0.0035	1419.2579	0.0007	4.9824	0.2007	4.8908	24.3681	31
32	341.8219	0.0029	1704.1095	0.0006	4.9854	0.2006	4.9061	24.4588	32
33	410.1863	0.0024	2045.9314	0.0005	4.9878	0.2005	4.9194	24.5368	33
34	492.2235	0.0020	2456.1176	0.0004	4.9898	0.2004	4.9308	24.6038	34
35	590.6682	0.0017	2948.3411	0.0003	4.9915	0.2003	4.9406	24.6614	35
40	1469.7716	0.0007	7343.8578	0.0001	4.9966	0.2001	4.9728	24.8469	40
45	3657.2620	0.0003	18281.3099	0.0001	4.9986	0.2001	4.9877	24.9316	45

25.0%

	Single Payment		Equal Payment Series				Gradient Series		
N	**Compound Amount Factor (F/P,i,N)**	**Present Worth Factor (P/F,i,N)**	**Compound Amount Factor (F/A,i,N)**	**Sinking Fund Factor (A/F,i,N)**	**Present Worth Factor (P/A,i,N)**	**Capital Recovery Factor (A/P,i,N)**	**Gradient Uniform Series (A/G,i,N)**	**Gradient Present Worth (P/G,i,N)**	**N**
1	1.2500	0.8000	1.0000	1.0000	0.8000	1.2500	0.0000	0.0000	1
2	1.5625	0.6400	2.2500	0.4444	1.4400	0.6944	0.4444	0.6400	2
3	1.9531	0.5120	3.8125	0.2623	1.9520	0.5123	0.8525	1.6640	3
4	2.4414	0.4096	5.7656	0.1734	2.3616	0.4234	1.2249	2.8928	4
5	3.0518	0.3277	8.2070	0.1218	2.6893	0.3718	1.5631	4.2035	5
6	3.8147	0.2621	11.2588	0.0888	2.9514	0.3388	1.8683	5.5142	6
7	4.7684	0.2097	15.0735	0.0663	3.1611	0.3163	2.1424	6.7725	7
8	5.9605	0.1678	19.8419	0.0504	3.3289	0.3004	2.3872	7.9469	8
9	7.4506	0.1342	25.8023	0.0388	3.4631	0.2888	2.6048	9.0207	9
10	9.3132	0.1074	33.2529	0.0301	3.5705	0.2801	2.7971	9.9870	10
11	11.6415	0.0859	42.5661	0.0235	3.6564	0.2735	2.9663	10.8460	11
12	14.5519	0.0687	54.2077	0.0184	3.7251	0.2684	3.1145	11.6020	12
13	18.1899	0.0550	68.7596	0.0145	3.7801	0.2645	3.2437	12.2617	13
14	22.7374	0.0440	86.9495	0.0115	3.8241	0.2615	3.3559	12.8334	14
15	28.4217	0.0352	109.6868	0.0091	3.8593	0.2591	3.4530	13.3260	15
16	35.5271	0.0281	138.1085	0.0072	3.8874	0.2572	3.5366	13.7482	16
17	44.4089	0.0225	173.6357	0.0058	3.9099	0.2558	3.6084	14.1085	17
18	55.5112	0.0180	218.0446	0.0046	3.9279	0.2546	3.6698	14.4147	18
19	69.3889	0.0144	273.5558	0.0037	3.9424	0.2537	3.7222	14.6741	19
20	86.7362	0.0115	342.9447	0.0029	3.9539	0.2529	3.7667	14.8932	20
21	108.4202	0.0092	429.6809	0.0023	3.9631	0.2523	3.8045	15.0777	21
22	135.5253	0.0074	538.1011	0.0019	3.9705	0.2519	3.8365	15.2326	22
23	169.4066	0.0059	673.6264	0.0015	3.9764	0.2515	3.8634	15.3625	23
24	211.7582	0.0047	843.0329	0.0012	3.9811	0.2512	3.8861	15.4711	24
25	264.6978	0.0038	1054.7912	0.0009	3.9849	0.2509	3.9052	15.5618	25
26	330.8722	0.0030	1319.4890	0.0008	3.9879	0.2508	3.9212	15.6373	26
27	413.5903	0.0024	1650.3612	0.0006	3.9903	0.2506	3.9346	15.7002	27
28	516.9879	0.0019	2063.9515	0.0005	3.9923	0.2505	3.9457	15.7524	28
29	646.2349	0.0015	2580.9394	0.0004	3.9938	0.2504	3.9551	15.7957	29
30	807.7936	0.0012	3227.1743	0.0003	3.9950	0.2503	3.9628	15.8316	30
31	1009.7420	0.0010	4034.9678	0.0002	3.9960	0.2502	3.9693	15.8614	31
32	1262.1774	0.0008	5044.7098	0.0002	3.9968	0.2502	3.9746	15.8859	32
33	1577.7218	0.0006	6306.8872	0.0002	3.9975	0.2502	3.9791	15.9062	33
34	1972.1523	0.0005	7884.6091	0.0001	3.9980	0.2501	3.9828	15.9229	34
35	2465.1903	0.0004	9856.7613	0.0001	3.9984	0.2501	3.9858	15.9367	35
40	7523.1638	0.0001	30088.6554	0.0000	3.9995	0.2500	3.9947	15.9766	40

30.0%

	Single Payment		Equal Payment Series				Gradient Series		
N	**Compound Amount Factor (F/P,i,N)**	**Present Worth Factor (P/F,i,N)**	**Compound Amount Factor (F/A,i,N)**	**Sinking Fund Factor (A/F,i,N)**	**Present Worth Factor (P/A,i,N)**	**Capital Recovery Factor (A/P,i,N)**	**Gradient Uniform Series (A/G,i,N)**	**Gradient Present Worth (P/G,i,N)**	**N**
1	1.3000	0.7692	1.0000	1.0000	0.7692	1.3000	0.0000	0.0000	1
2	1.6900	0.5917	2.3000	0.4348	1.3609	0.7348	0.4348	0.5917	2
3	2.1970	0.4552	3.9900	0.2506	1.8161	0.5506	0.8271	1.5020	3
4	2.8561	0.3501	6.1870	0.1616	2.1662	0.4616	1.1783	2.5524	4
5	3.7129	0.2693	9.0431	0.1106	2.4356	0.4106	1.4903	3.6297	5
6	4.8268	0.2072	12.7560	0.0784	2.6427	0.3784	1.7654	4.6656	6
7	6.2749	0.1594	17.5828	0.0569	2.8021	0.3569	2.0063	5.6218	7
8	8.1573	0.1226	23.8577	0.0419	2.9247	0.3419	2.2156	6.4800	8
9	10.6045	0.0943	32.0150	0.0312	3.0190	0.3312	2.3963	7.2343	9
10	13.7858	0.0725	42.6195	0.0235	3.0915	0.3235	2.5512	7.8872	10
11	17.9216	0.0558	56.4053	0.0177	3.1473	0.3177	2.6833	8.4452	11
12	23.2981	0.0429	74.3270	0.0135	3.1903	0.3135	2.7952	8.9173	12
13	30.2875	0.0330	97.6250	0.0102	3.2233	0.3102	2.8895	9.3135	13
14	39.3738	0.0254	127.9125	0.0078	3.2487	0.3078	2.9685	9.6437	14
15	51.1859	0.0195	167.2863	0.0060	3.2682	0.3060	3.0344	9.9172	15
16	66.5417	0.0150	218.4722	0.0046	3.2832	0.3046	3.0892	10.1426	16
17	86.5042	0.0116	285.0139	0.0035	3.2948	0.3035	3.1345	10.3276	17
18	112.4554	0.0089	371.5180	0.0027	3.3037	0.3027	3.1718	10.4788	18
19	146.1920	0.0068	483.9734	0.0021	3.3105	0.3021	3.2025	10.6019	19
20	190.0496	0.0053	630.1655	0.0016	3.3158	0.3016	3.2275	10.7019	20
21	247.0645	0.0040	820.2151	0.0012	3.3198	0.3012	3.2480	10.7828	21
22	321.1839	0.0031	1067.2796	0.0009	3.3230	0.3009	3.2646	10.8482	22
23	417.5391	0.0024	1388.4635	0.0007	3.3254	0.3007	3.2781	10.9009	23
24	542.8008	0.0018	1806.0026	0.0006	3.3272	0.3006	3.2890	10.9433	24
25	705.6410	0.0014	2348.8033	0.0004	3.3286	0.3004	3.2979	10.9773	25
26	917.3333	0.0011	3054.4443	0.0003	3.3297	0.3003	3.3050	11.0045	26
27	1192.5333	0.0008	3971.7776	0.0003	3.3305	0.3003	3.3107	11.0263	27
28	1550.2933	0.0006	5164.3109	0.0002	3.3312	0.3002	3.3153	11.0437	28
29	2015.3813	0.0005	6714.6042	0.0001	3.3317	0.3001	3.3189	11.0576	29
30	2619.9956	0.0004	8729.9855	0.0001	3.3321	0.3001	3.3219	11.0687	30
31	3405.9943	0.0003	11349.9811	0.0001	3.3324	0.3001	3.3242	11.0775	31
32	4427.7926	0.0002	14755.9755	0.0001	3.3326	0.3001	3.3261	11.0845	32
33	5756.1304	0.0002	19183.7681	0.0001	3.3328	0.3001	3.3276	11.0901	33
34	7482.9696	0.0001	24939.8985	0.0000	3.3329	0.3000	3.3288	11.0945	34
35	9727.8604	0.0001	32422.8681	0.0000	3.3330	0.3000	3.3297	11.0980	35

35.0%

	Single Payment		Equal Payment Series				Gradient Series		
N	Compound Amount Factor (F/P,i,N)	Present Worth Factor (P/F,i,N)	Compound Amount Factor (F/A,i,N)	Sinking Fund Factor (A/F,i,N)	Present Worth Factor (P/A,i,N)	Capital Recovery Factor (A/P,i,N)	Gradient Uniform Series (A/G,i,N)	Gradient Present Worth (P/G,i,N)	N
1	1.3500	0.7407	1.0000	1.0000	0.7407	1.3500	0.0000	0.0000	1
2	1.8225	0.5487	2.3500	0.4255	1.2894	0.7755	0.4255	0.5487	2
3	2.4604	0.4064	4.1725	0.2397	1.6959	0.5897	0.8029	1.3616	3
4	3.3215	0.3011	6.6329	0.1508	1.9969	0.5008	1.1341	2.2648	4
5	4.4840	0.2230	9.9544	0.1005	2.2200	0.4505	1.4220	3.1568	5
6	6.0534	0.1652	14.4384	0.0693	2.3852	0.4193	1.6698	3.9828	6
7	8.1722	0.1224	20.4919	0.0488	2.5075	0.3988	1.8811	4.7170	7
8	11.0324	0.0906	28.6640	0.0349	2.5982	0.3849	2.0597	5.3515	8
9	14.8937	0.0671	39.6964	0.0252	2.6653	0.3752	2.2094	5.8886	9
10	20.1066	0.0497	54.5902	0.0183	2.7150	0.3683	2.3338	6.3363	10
11	27.1439	0.0368	74.6967	0.0134	2.7519	0.3634	2.4364	6.7047	11
12	36.6442	0.0273	101.8406	0.0098	2.7792	0.3598	2.5205	7.0049	12
13	49.4697	0.0202	138.4848	0.0072	2.7994	0.3572	2.5889	7.2474	13
14	66.7841	0.0150	187.9544	0.0053	2.8144	0.3553	2.6443	7.4421	14
15	90.1585	0.0111	254.7385	0.0039	2.8255	0.3539	2.6889	7.5974	15
16	121.7139	0.0082	344.8970	0.0029	2.8337	0.3529	2.7246	7.7206	16
17	164.3138	0.0061	466.6109	0.0021	2.8398	0.3521	2.7530	7.8180	17
18	221.8236	0.0045	630.9247	0.0016	2.8443	0.3516	2.7756	7.8946	18
19	299.4619	0.0033	852.7483	0.0012	2.8476	0.3512	2.7935	2.9547	19
20	404.2736	0.0025	1152.2103	0.0009	2.8501	0.3509	2.8075	8.0017	20
21	545.7693	0.0018	1556.4838	0.0006	2.8519	0.3506	2.8186	8.0384	21
22	736.7886	0.0014	2102.2532	0.0005	2.8533	0.3505	2.8272	8.0669	22
23	994.6646	0.0010	2839.0418	0.0004	2.8543	0.3504	2.8340	8.0890	23
24	1342.7973	0.0007	3833.7064	0.0003	2.8550	0.3503	2.8393	8.1061	24
25	1812.7763	0.0006	5176.5037	0.0002	2.8556	0.3502	2.8433	8.1194	25
26	2447.2480	0.0004	6989.2800	0.0001	2.8560	0.3501	2.8465	8.1296	26
27	3303.7848	0.0003	9436.5280	0.0001	2.8563	0.3501	2.8490	8.1374	27
28	4460.1095	0.0002	12740.3128	0.0001	2.8565	0.3501	2.8509	8.1435	28
29	6021.1478	0.0002	17200.4222	0.0001	2.8567	0.3501	2.8523	8.1481	29
30	8128.5495	0.0001	23221.5700	0.0000	2.8568	0.3500	2.8535	8.1517	30

40.0%

	Single Payment		Equal Payment Series				Gradient Series		
	Compound Amount Factor	Present Worth Factor	Compound Amount Factor	Sinking Fund Factor	Present Worth Factor	Capital Recovery Factor	Gradient Uniform Series	Gradient Present Worth	
N	*(F/P,i,N)*	*(P/F,i,N)*	*(F/A,i,N)*	*(A/F,i,N)*	*(P/A,i,N)*	*(A/P,i,N)*	*(A/G,i,N)*	*(P/G,i,N)*	*N*
1	1.4000	0.7143	1.0000	1.0000	0.7143	1.4000	0.0000	0.0000	1
2	1.9600	0.5102	2.4000	0.4167	1.2245	0.8167	0.4167	0.5102	2
3	2.7440	0.3644	4.3600	0.2294	1.5889	0.6294	0.7798	1.2391	3
4	3.8416	0.2603	7.1040	0.1408	1.8492	0.5408	1.0923	2.0200	4
5	5.3782	0.1859	10.9456	0.0914	2.0352	0.4914	1.3580	2.7637	5
6	7.5295	0.1328	16.3238	0.0613	2.1680	0.4613	1.5811	3.4278	6
7	10.5414	0.0949	23.8534	0.0419	2.2628	0.4419	1.7664	3.9970	7
8	14.7579	0.0678	34.3947	0.0291	2.3306	0.4291	1.9185	4.4713	8
9	20.6610	0.0484	49.1526	0.0203	2.3790	0.4203	2.0422	4.8585	9
10	28.9255	0.0346	69.8137	0.0143	2.4136	0.4143	2.1419	5.1696	10
11	40.4957	0.0247	98.7391	0.0101	2.4383	0.4101	2.2215	5.4166	11
12	56.6939	0.0176	139.2348	0.0072	2.4559	0.4072	2.2845	5.6106	12
13	79.3715	0.0126	195.9287	0.0051	2.4685	0.4051	2.3341	5.7618	13
14	111.1201	0.0090	275.3002	0.0036	2.4775	0.4036	2.3729	5.8788	14
15	155.5681	0.0064	386.4202	0.0026	2.4839	0.4026	2.4030	5.9688	15
16	217.7953	0.0046	541.9883	0.0018	2.4885	0.4018	2.4262	6.0376	16
17	304.9135	0.0033	759.7837	0.0013	2.4918	0.4013	2.4441	6.0901	17
18	426.8789	0.0023	1064.6971	0.0009	2.4941	0.4009	2.4577	6.1299	18
19	597.6304	0.0017	1491.5760	0.0007	2.4958	0.4007	2.4682	6.1601	19
20	836.6826	0.0012	2089.2064	0.0005	2.4970	0.4005	2.4761	6.1828	20
21	1171.3556	0.0009	2925.8889	0.0003	2.4979	0.4003	2.4821	6.1998	21
22	1639.8978	0.0006	4097.2445	0.0002	2.4985	0.4002	2.4866	6.2127	22
23	2295.8569	0.0004	5737.1423	0.0002	2.4989	0.4002	2.4900	6.2222	23
24	3214.1997	0.0003	8032.9993	0.0001	2.4992	0.4001	2.4925	6.2294	24
25	4499.8796	0.0002	11247.1990	0.0001	2.4994	0.4001	2.4944	6.2347	25
26	6299.8314	0.0002	15747.0785	0.0001	2.4996	0.4001	2.4959	6.2387	26
27	8819.7640	0.0001	22046.9099	0.0000	2.4997	0.4000	2.4969	6.2416	27
28	12347.6696	0.0001	30866.6739	0.0000	2.4998	0.4000	2.4977	6.2438	28
29	17286.7374	0.0001	43214.3435	0.0000	2.4999	0.4000	2.4983	6.2454	29
30	24201.4324	0.0000	60501.0809	0.0000	2.4999	0.4000	2.4988	6.2466	30