Analysis (I)

by Poshen Chang

v.1

2025-09-04

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CHAPTER 1

Topological Spaces

In the language of calculus, we defined convergence, continuity and such concepts on the real numbers $\mathbb{R}$. We would like to generalize this concept.

1.1 Metric Spaces

Definition 1.1.1 A metric space is a nonempty set M, together with a metric $d : M \times M \to \mathbb{R}$. $d(x, y)$ is a real number defined for all $x, y \in M$, which can be thought of as the distance between x and y. d satisfies the following properties:

a) (Positive definiteness) $d(x, y) \geq 0$, and $d(x, y) = 0$ iff $x = y$.

b) (Symmetry) $d(x, y) = d(y, x)$.

c) (Triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$.

We say that the pair (M, d) is a metric space. The metric d can be omitted if it is clear from the context.

Example 1.1.1 The following are some examples of metric spaces:

- $(\mathbb{R}, d)$, where $d(x, y) = |x - y|$.
- $(\mathbb{R}^n, d)$, where
$$d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n + y_n)^2}.$$
- (M, d), where $M \neq \varnothing$, and
$$d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y. \end{cases}$$

  This metric is called the **discrete metric** on M.

If $\varnothing \neq A \subseteq M$, and if M is a metric, then (A, d) is also a metric space. We call A a metric subspace of M, or A inherits its metric from M.

Definition 1.1.2 We say that a sequence (x_n) in M converges to the limit x in M if for any given $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $n \geq N$ and $n \in \mathbb{N}$ implies $d(x_n, x) < \epsilon$.

It's easy to check that limits are unique. Also, every subsequence of a convergent sequence converges, and it converges to the same limit as the original sequence.

1.2 Continuity

Definition 1.2.1 Let (M, d_M) and (N, d_N) be metric spaces. We say that $f : M \to N$ is continuous if it preserves sequential convergence, i.e. for each (x_n) in M which converges to x in M, the image sequence $(f(x_n))$ converges to $f(x)$.

Proposition 1.2.1 Composition of continuous functions is continuous

Proof. Let M, N, P be metric spaces, and let $f : M \to N$, $g : N \to P$ be continuous functions.

Let (x_n) be a convergent sequence in M with limit x. We have

$$\lim_{n\to\infty} x_n = x \xRightarrow{f \text{ conti.}} \lim_{n\to\infty} f(x_n) = f(x) \xRightarrow{g \text{ conti.}} \lim_{n\to\infty} g(f(x_n)) = g(f(x)),$$

thus $g \circ f$ is continuous. $\square$

Example 1.2.1 The following are some examples of continuous functions:

- The identity map $\mathrm{id} : M \to M$ is continuous.
- Every constant function $f : M \to N$ is continuous.
- Every function $f : M \to N$ is continuous if M is equipped with the discrete metric.

Definition 1.2.2 If $f : M \to N$ is a bijection such that f and $f^{-1} : N \to M$ are continuous, then we say that f is a homeomorphism. If there exists a homeomorphism between M and N, we say that M, N are homeomorphic, denoted by $M \cong N$.

Intuitively, a homeomorphism is a bijection that can bend, twist, stretch the space M to make it coincide with N, but it cannot rip, puncture or shred M etc.

Example 1.2.2 Let $\mathbb{S}^1$ be the unit circle in the plane. Consider the interval $[0, 2\pi)$. Define $f : [0, 2\pi) \to \mathbb{S}^1$ to be the function $f(\theta) = (\cos\theta, \sin\theta)$. f is continuous and bijective, but f^{-1} is not continuous (consider a sequence in $\mathbb{S}^1$ approaching $(1, 0)$ from the lower plane).

Proposition 1.2.2 $f : M \to N$ is continuous iff it satisfies the following: $\forall \epsilon$ and $x \in M$, $\exists \delta > 0$ such that if $y \in M$ and $d_M(x, y) < \delta$ then $d_N(f(x), f(y)) < \epsilon$.

Proof. " $\implies$ ": Suppose that f fails to satisfy the ϵ-δ condition at some $x \in M$. $\exists \epsilon > 0$ such that $\forall \delta > 0$, $\exists y \in M$ such that $d_M(x, y) < \delta$ but $d_N(f(x), f(y)) \geq \epsilon$. Take $\delta = \frac{1}{n}$. By our assumption, we can obtain a sequence (y_n) with $d_M(x, y_n) < \frac{1}{n}$ but $d_N(f(x), f(y_n)) \geq \epsilon$, so (y_n) converges to x but $f(y_n)$ does not approach $f(x)$, contradicting the continuity of f.

" $\impliedby$ ": Suppose that f satisfies the ϵ-δ condition at x. Let (x_n) be a sequence in M susch that (x_n) converges to x. Let $\epsilon > 0$. $\exists \delta > 0$ such that $d_M(x, y) < \delta \implies d_N(f(x), f(y)) < \epsilon$. Since (x_n) approaches x, $\exists K \in \mathbb{N}$ such that if $n \geq K$, then $d_M(x_n, x) < \delta$, hence $d_N(f(x_n), f(x)) < \epsilon$ for all $n \geq K$. That is, $(f(x_n))$ converges to $f(x)$. □

1.3 The Topology of a Metric Space

Definition 1.3.1 Let M be a metric space, $S \subseteq M$.

- We say S is open if for each $x \in S$, $\exists r > 0$ such that $d(x, y) < r$ implies $y \in S$.
- We say S is closed if its complement is open.
- We say that a point $x \in M$ is a limit of S if there exists a sequence in S that converges to x.

Proposition 1.3.1 A set $S \subseteq M$ is closed iff it contains all its limits.

Proof. " $\implies$ ": Suppose that S is closed. Let (x_n) be a sequence in S such that (x_n) converges to x, $x \in M$. Suppose $x \notin S$, then since S^c is open, $\exists r > 0$ such that $d(x, y) < r$ implies $y \in S^c$. Since (x_n) converges to x, we have $d(x_n, x) < r$ for all n large enough, implying that $x_n \in S^c$, contradiction. Therefore, for any limit point x of S, $x \in S$.

" $\impliedby$ ": Suppose that S contains all its limits. If S^c is not open, then $\exists x \in S^c$ such that $\forall n \in \mathbb{N}$, $\exists x_n \in (S^c)^c = S$ such that $d(x, x_n) < \frac{1}{n}$. We have now constructed q sequence (x_n) in S, but converges to a point in S^c, contradiction. Thus S^c is open, i.e. S is closed. □

Remark 1.3.1 Sets can be neither open nor closed, or they can also be both open and closed.

Definition 1.3.2 The topology $\mathcal{T}$ of M is the collection of all open subsets of M.

Proposition 1.3.2 $\mathcal{T}$ is closed under arbitrary union, finite intersection, and $\mathcal{T}$ contains $\varnothing$ and M.

Proof. Clearly, $\varnothing$ and M are open.

Let $(\mathcal{U}_\alpha)$ be a collection of open subsets of M. Define $V = \bigcup_\alpha \mathcal{U}_\alpha$. For any $x \in V$, $x \in \mathcal{U}_\alpha$ for some α. Since $\mathcal{U}_\alpha$ is open, $\exists r > 0$ such that $d(x, y) < r \implies y \in \mathcal{U}_\alpha \subseteq V$, so V is open.

Define $W = \bigcap_{i=1}^n \mathcal{U}_i$. Given $x \in W$, for each $1 \leq i \leq n$, $\exists r_i > 0$ such that $d(x, y) < r_i \implies y \in \mathcal{U}_i$. Take $r = \min r_1, \ldots, r_n$, then for any y satisfying $d(x, y) < r$, we have $y \in \mathcal{U}_i$ for all $1 \leq i \leq n$, thus $y \in W$. Hence W is open. $\square$

Definition 1.3.3 Let X be a set. A topology $\mathcal{T}$ of X is a collection of subsets of X that satisfies the following:

a) $\mathcal{T}$ is closed under arbitrary union,

b) $\mathcal{T}$ is closed under finite intersection, and

c) $\varnothing, X \in \mathcal{T}$.

We say that $(X, \mathcal{T})$ is a topological space if $\mathcal{T}$ is a topology of X. The elements of $\mathcal{T}$ are called open sets. We define $S \subseteq X$ to be closed if S^c is open.

Example 1.3.1 The following are some examples of topological spaces:

- A metric space is a topological space.
- Let X be a set, and $\mathcal{T} = \{\varnothing, X\}$. Then $(X, \mathcal{T})$ is a topological space, which is know as the trivial topology.
- Let X be a set, and let $\mathcal{T}$ be the power set of X, then $(X, \mathcal{T})$ is a topological space, which is known as the discrete topology.

Remark 1.3.2 By De Morgan's law, closed sets are closed under arbitrary intersection and finite union, also $\varnothing, X$ are closed.

In general, infinite union of closed sets may not be closed.

Definition 1.3.4 Let M be a metric space and $S \subseteq M$. Define

$$\overline{S} := \{x \in M \mid x \text{ is a limit of } S\}$$

to be the closure of S. For $x \in M$, $r > 0$, define

$$B(x, r) := \{y \in M \mid d(x, y) < r\},$$

which is the ball centered at x with radius r, or the r-neighborhood of x.

Proposition 1.3.3 $\overline{S}$ is closed and $B(x, r)$ is open.

Proof. If $S = \varnothing$, then $\overline{S} = \varnothing$, which is closed.

Suppose that $S \neq \varnothing$, and let $(x_n) \to x$ be a convergence sequence in $\overline{S}$. We wish to prove that $x \in \overline{S}$, i.e. there exists a sequence in S that approaches x. Since $x_n \in \overline{S}$, there exists sequence $(x_{n,k})$ in S that approaches x_n as $k \to \infty$. For each n, there exists a term x_{n,k_n} satisfying $d(x_{n,k_n}, x_n) < \frac{1}{n}$, picking these terms forms a new sequence (x_{n,k_n}), moreover,

$$d(x_{n,k_n}, x) \leq d(x_{n,k_n}, x_n) + d(x_n, x) < \frac{1}{n} + d(x_n, x),$$

which approaches 0 as $n \to \infty$, thus (x_{n,k_n}) approaches x.

Fix $x \in M$ and $r > 0$. Let $y \in B(x,r)$ and pick $s = r - d(x,y)$. If z satisfies $d(y,z) < s$, then

$$d(x,z) \leq d(x,y) + d(y,z) < d(x,y) + s = r \implies B(y,s) \subseteq B(x,r),$$

so $B(x,r)$ is open. $\square$

Corollary 1.3.5 $\overline{S}$ is the smallest closed set that conatins S, i.e. if $K \supseteq S$ and K is closed then $K \supseteq \overline{S}$.

Proof. K contains the limit of each sequence in K, in particular $S \subseteq K$ so it contains all sequence in S that converges in M, but these are precisely $\overline{S}$. $\square$

With the observation from Corollary 1.3.5, we may define the closure for topological spaces in general.

Definition 1.3.6 Lex X be a topological space, and let $S \subseteq X$. We define $\overline{S}$ to be the smallest closed set that contains S. $\overline{S}$ always exists; take

$$\overline{S} = \bigcap \{E \subseteq X \mid E \text{ is closed in } X \text{ and } E \supseteq S\}.$$

Likewise, we also define the interior of S, $\mathrm{int}(S)$, to be the largest open set contained in S

Definition 1.3.7 Let X be a topological space, $x \in X$. A neighborhood of x is an open set containing x.

Definition 1.3.8 Lex X be a topological space and let (x_n) be a sequence in X. We say that (x_n) converges to $x \in X$ if for all neighborhood $\mathcal{U}$ of x, $\exists N \in \mathbb{N}$ such that $x_n \in \mathcal{U}$ for all $n \geq N$.

Example 1.3.2 Limits in general are not unique in topological spaces. Let X be a set with at least 2 points endowed with the trivial topology. Then every sequence in X converges to every point in X.

Definition 1.3.9 Let X, Y be topological spaces. We say a function $f : X \to Y$ continuous if for any open set $V \subseteq Y$, the preimage $f^{-1}(Y)$ is open in X.

Proposition 1.3.4 A function $f : X \to Y$ is continuous iff $\forall x \in M$ and any neighborhood $\mathcal{V}$ of $f(x)$, $\exists$ a neighborhood $\mathcal{U}$ of x such that $f(\mathcal{U}) \subseteq \mathcal{V}$.

Proof. " $\implies$ ": Let $\mathcal{V} \subseteq Y$ is open. We need to show $f^{-1}(Y)$ is open. Let $x \in f^{-1}(\mathcal{V})$, then $f(x) \in \mathcal{V}$. By definition, $\exists$ a neighborhood $\mathcal{U}_x$ such that $f(\mathcal{U}_x) \subseteq \mathcal{V} \rightsquigarrow \mathcal{U}_x \subseteq f^{-1}(\mathcal{V})$. Take the union of all such $\mathcal{U}_x$ over $x \in f^{-1}(\mathcal{V})$, then

$$\bigcup_{x \in f^{-1}(\mathcal{V})} \mathcal{U}_x \subseteq f^{-1}(\mathcal{V}) \quad \text{and} \quad \forall x \in f^{-1}(\mathcal{V}), x \in \mathcal{U}_x \subseteq f^{-1}(\mathcal{V})$$
$$\implies f^{-1}(\mathcal{V}) = \bigcup_{x \in f^{-1}(\mathcal{V})} \mathcal{U}_x$$

and thus $f^{-1}(\mathcal{V})$ is open.

" $\impliedby$ ": Let $x \in X$ and let $\mathcal{V}$ be a neighborhood of $f(x)$. By definition $x \in f^{-1}(\mathcal{V})$, which is an open set by assumption. Also, $f(f^{-1}(\mathcal{V})) \subseteq \mathcal{V}$, hence $f^{-1}(\mathcal{V})$ is a neighborhood of x such that $f(f^{-1}(\mathcal{V})) \subseteq \mathcal{V}$. $\square$

Definition 1.3.10 A homeomorphism is a continuous bijection between topoloical spaces.

Corollary 1.3.11 A homeomorphism $f : X \to Y$ bijects the corresponding topologies $\mathcal{T}_X$ and $\mathcal{T}_Y$.

1.4 Hausdorff Space

Definition 1.4.1 A topological space X is said to be Hausdorff if given any pair of distinct paints $x_1, x_2 \in X$, $\exists$ neighborhoods $\mathcal{U}_1$ of x_1 and $\mathcal{U}_2$ of x_2 such that $\mathcal{U}_1 \cap \mathcal{U}_2 = \varnothing$.

A metric space is always a Hausdorff space.

Lemma 1.4.2 Let X be Hausdorff.

a) Every one-point set is closed.

b) If a sequence (x_n) in X converges, then the limit is unique.

Proof. a) Pick $x \in X$. For any y distinct from x, there exists disjoint neighborhoods $\mathcal{U}_x$ of x and $\mathcal{V}_y$ of y. We have

$$\{x\}^c = \bigcup_{y \in X \setminus \{x\}} \mathcal{V}_y$$

which is an open set.

b) Suppose that x, x' are distinct limits of (x_n). $\exists$ disjoint neighborhoods $\mathcal{U}$ of x and $\mathcal{U}'$

of x'. $\exists N, N' \in \mathbb{N}$ such that "$n \geq N'$ implies $x_n \in \mathcal{U}$" and "$n \geq N'$ implies $x_n \in \mathcal{U}'$". If $n \geq \max(N, N')$, then $x_n \in \mathcal{U} \cap \mathcal{U}' = \varnothing$, contradiction. Therefore, any converging sequence has a unique limit.

$\square$

1.5 Subspaces and Product Spaces

Definition 1.5.1 Let X be a topological space, let $A \subseteq X$. We define the subspace topology $\mathcal{T}_A$ of A by

$$\mathcal{T}_A = \{\mathcal{U} \subseteq A \mid \mathcal{U} = A \cap \mathcal{V} \text{ for some open } \mathcal{V} \subseteq X\}.$$

Remark 1.5.1 Openness and closedness are not just properties of a set itself, but rather a set in a relation to a particular topological space.

Proposition 1.5.1 Let M be a metric space, and let $N \subseteq M$ be a nonempty subset. The subspace topology on N is the same as the metric topology obtained by restricting the metric of M to N.

Proof. Suppose that $\mathcal{V}$ is an open set in M and let $\mathcal{U} = N \cap \mathcal{V} \in \mathcal{T}_N$.

We first to prove that $\mathcal{U}$ belongs to the metric toplogy of N. Let $x \in \mathcal{U}$. Since $x \in N \cap \mathcal{V} \subseteq \mathcal{V}$, there exists $r > 0$ such that $B_M(x, r) \subseteq \mathcal{V}$, and

$$B_N(x, r) = N \cap B_M(x, r) \subseteq N \cap \mathcal{V} = \mathcal{U},$$

hence $\mathcal{U}$ is open in N (in the metric topology).

Conversely, let $\mathcal{U}$ be an open set in the metric topology of N. $\forall x \in \mathcal{U}$, $\exists r_x > 0$ such that $B_N(x, r_x) \subseteq \mathcal{U}$. Note that

$$\mathcal{U} = \bigcup_{x \in \mathcal{U}} B_N(x, r_x) = \bigcup_{x \in \mathcal{U}} N \cap B_M(x, r_x) = N \cap \bigcup_{x \in \mathcal{U}} B_M(x, r_x),$$

hence $\mathcal{U}$ belongs to the subspace topology of N. $\square$

Definition 1.5.2 Let X be a set. A basis in X is a collection $\mathcal{B}$ of subsets of X satsifying

a) Every element of X is in some element in $\mathcal{B}$. That is,

$$X = \bigcup_{B \in \mathcal{B}} B.$$

b) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Definition 1.5.3 Given X and a collection $\mathcal{B}$ of subsets of X, we say that $\mathcal{U} \subseteq X$ satisfies the basis criterion with respect to $\mathcal{B}$ if $\forall x \in \mathcal{U}$, $\exists B \in \mathcal{B}$ such that $x \in B \subseteq \mathcal{U}$

Lemma 1.5.4 Suppose that $\mathcal{B}$ is a basis in X, and let $\mathcal{T}$ be the collection of all unions of elements of $\mathcal{B}$. Then $\mathcal{T}$ is precisely the collection of all subsets of X that satisfy the basis criterion w.r.t. $\mathcal{B}$.

Proof. Let $\mathcal{U} \subseteq X$. Suppose that $\mathcal{U}$ satisfies the basis criterion. Let

$$\mathcal{V} = \bigcup \{B \in \mathcal{B} \mid B \subseteq \mathcal{U}\},$$

then $\mathcal{V} \in \mathcal{T}$. We want to show that $\mathcal{U} = \mathcal{V}$. Clearly, $\mathcal{V} \subseteq \mathcal{U}$. Let $x \in \mathcal{U}$. Since $\mathcal{U}$ satisfies the basis criterion, $\exists B \in \mathcal{B}$ such that $x \in B \subseteq \mathcal{U}$, so $x \in \mathcal{V}$, therefore $\mathcal{U} \subseteq \mathcal{V}$.

Conversely, suppose that $\mathcal{U} \in \mathcal{T}$, then $\mathcal{U}$ is a union of elements of $\mathcal{B}$, say $\mathcal{U} = \bigcup_{B \in \mathcal{A}} B$ where $\mathcal{A} \subseteq \mathcal{B}$. For any $x \in \mathcal{U}$, $x \in B$ for some $B \in \mathcal{A}$, also $B \subseteq \mathcal{U}$, so $\mathcal{U}$ satisfies the basis criterion. □

Proposition 1.5.2 Let $\mathcal{B}$ be a basis in X, and let $\mathcal{T}$ be collection of all unions of elements of $\mathcal{B}$. Then $\mathcal{T}$ is a topology on X. This is called the topology generated by $\mathcal{B}$.

Proof. □