

Analysis I

by Poshen Chang

v.1

2025-09-04

Contents

1	Topological Spaces	3
1.1	Metric Spaces	3
1.2	Continuity	4
1.3	The Topology of a Metric Space	5
1.4	Hausdorff Space	8
1.5	Subspaces and Product Spaces	9

Contents

CHAPTER 1

Topological Spaces

In the language of calculus, we defined convergence, continuity and such concepts on the real numbers \mathbb{R} . We would like to generalize this concept.

1.1 Metric Spaces

Definition 1.1.1 A metric space is a nonempty set M , together with a metric $d : M \times M \rightarrow \mathbb{R}$. $d(x, y)$ is a real number defined for all $x, y \in M$, which can be thought of as the distance between x and y . d satisfies the following properties:

- a) (Positive definiteness) $d(x, y) \geq 0$, and $d(x, y) = 0$ iff $x = y$.
- b) (Symmetry) $d(x, y) = d(y, x)$.
- c) (Triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$.

We say that the pair (M, d) is a metric space. The metric d can be omitted if it is clear from the context.

Example 1.1.1 The following are some examples of metric spaces:

- (\mathbb{R}, d) , where $d(x, y) = |x - y|$.
- (\mathbb{R}^n, d) , where

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}.$$

- (M, d) , where $M \neq \emptyset$, and

$$d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y. \end{cases}$$

This metric is called the **discrete metric** on M .

If $\emptyset \neq A \subseteq M$, and if M is a metric, then (A, d) is also a metric space. We call A a metric subspace of M , or A inherits its metric from M .

Definition 1.1.2 We say that a sequence (x_n) in M converges to the limit x in M if for any given $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $n \geq N$ and $n \in \mathbb{N}$ implies $d(x_n, x) < \epsilon$.

It's easy to check that limits are unique. Also, every subsequence of a convergent sequence converges, and it converges to the same limit as the original sequence.

1.2 Continuity

Definition 1.2.1 Let (M, d_M) and (N, d_N) be metric spaces. We say that $f : M \rightarrow N$ is continuous if it preserves sequential convergence, i.e. for each (x_n) in M which converges to x in M , the image sequence $(f(x_n))$ converges to $f(x)$.

Proposition 1.2.1 Composition of continuous functions is continuous

Proof. Let M, N, P be metric spaces, and let $f : M \rightarrow N$, $g : N \rightarrow P$ be continuous functions.

Let (x_n) be a convergent sequence in M with limit x . We have

$$\lim_{n \rightarrow \infty} x_n = x \xrightarrow{f \text{ conti.}} \lim_{n \rightarrow \infty} f(x_n) = f(x) \xrightarrow{g \text{ conti.}} \lim_{n \rightarrow \infty} g(f(x_n)) = g(f(x)),$$

thus $g \circ f$ is continuous. □

Example 1.2.1 The following are some examples of continuous functions:

- The identity map $\text{id} : M \rightarrow M$ is continuous.
- Every constant function $f : M \rightarrow N$ is continuous.
- Every function $f : M \rightarrow N$ is continuous if M is equipped with the discrete metric.

Definition 1.2.2 If $f : M \rightarrow N$ is a bijection such that f and $f^{-1} : N \rightarrow M$ are continuous, then we say that f is a homeomorphism. If there exists a homeomorphism between M and N , we say that M, N are homeomorphic, denoted by $M \cong N$.

Intuitively, a homeomorphism is a bijection that can bend, twist, stretch the space M to make it coincide with N , but it cannot rip, puncture or shred M etc.

Example 1.2.2 Let \mathbb{S}^1 be the unit circle in the plane. Consider the interval $[0, 2\pi)$. Define $f : [0, 2\pi) \rightarrow \mathbb{S}^1$ to be the function $f(\theta) = (\cos \theta, \sin \theta)$. f is continuous and bijective, but f^{-1} is not continuous (consider a sequence in \mathbb{S}^1 approaching $(1, 0)$ from the lower plane).

Proposition 1.2.2 $f : M \rightarrow N$ is continuous iff it satisfies the following: $\forall \epsilon$ and $x \in M$, $\exists \delta > 0$ such that if $y \in M$ and $d_M(x, y) < \delta$ then $d_N(f(x), f(y)) < \epsilon$.

Proof. " \implies ": Suppose that f fails to satisfy the ϵ - δ condition at some $x \in M$. $\exists \epsilon > 0$ such that $\forall \delta > 0$, $\exists y \in M$ such that $d_M(x, y) < \delta$ but $d_N(f(x), f(y)) \geq \epsilon$. Take $\delta = \frac{1}{n}$. By our assumption, we can obtain a sequence (y_n) with $d_M(x, y_n) < \frac{1}{n}$ but $d_N(f(x), f(y_n)) \geq \epsilon$, so (y_n) converges to x but $f(y_n)$ does not approach $f(x)$, contradicting the continuity of f .

" \impliedby ": Suppose that f satisfies the ϵ - δ condition at x . Let (x_n) be a sequence in M such that (x_n) converges to x . Let $\epsilon > 0$. $\exists \delta > 0$ such that $d_M(x, y) < \delta \implies d_N(f(x), f(y)) < \epsilon$. Since (x_n) approaches x , $\exists K \in \mathbb{N}$ such that if $n \geq K$, then $d_M(x_n, x) < \delta$, hence $d_N(f(x_n), f(x)) < \epsilon$ for all $n \geq K$. That is, $(f(x_n))$ converges to $f(x)$. \square

1.3 The Topology of a Metric Space

Definition 1.3.1 Let M be a metric space, $S \subseteq M$.

- We say S is open if for each $x \in S$, $\exists r > 0$ such that $d(x, y) < r$ implies $y \in S$.
- We say S is closed if its complement is open.
- We say that a point $x \in M$ is a limit of S if there exists a sequence in S that converges to x .

Proposition 1.3.1 A set $S \subseteq M$ is closed iff it contains all its limits.

Proof. " \implies ": Suppose that S is closed. Let (x_n) be a sequence in S such that (x_n) converges to x , $x \in M$. Suppose $x \notin S$, then since S^c is open, $\exists r > 0$ such that $d(x, y) < r$ implies $y \in S^c$. Since (x_n) converges to x , we have $d(x_n, x) < r$ for all n large enough, implying that $x_n \in S^c$, contradiction. Therefore, for any limit point x of S , $x \in S$.

" \impliedby ": Suppose that S contains all its limits. If S^c is not open, then $\exists x \in S^c$ such that $\forall n \in \mathbb{N}$, $\exists x_n \in (S^c)^c = S$ such that $d(x, x_n) < \frac{1}{n}$. We have now constructed a sequence (x_n) in S , but converges to a point in S^c , contradiction. Thus S^c is open, i.e. S is closed. \square

Remark 1.3.1 Sets can be neither open nor closed, or they can also be both open and closed.

Definition 1.3.2 The topology \mathcal{T} of M is the collection of all open subsets of M .

Proposition 1.3.2 \mathcal{T} is closed under arbitrary union, finite intersection, and \mathcal{T} contains \emptyset and M .

Proof. Clearly, \emptyset and M are open.

Let (\mathcal{U}_α) be a collection of open subsets of M . Define $V = \bigcup_\alpha \mathcal{U}_\alpha$. For any $x \in V$, $x \in \mathcal{U}_\alpha$ for some α . Since \mathcal{U}_α is open, $\exists r > 0$ such that $d(x, y) < r \implies y \in \mathcal{U}_\alpha \subseteq V$, so V is open.

Define $W = \bigcap_{i=1}^n \mathcal{U}_i$. Given $x \in W$, for each $1 \leq i \leq n$, $\exists r_i > 0$ such that $d(x, y) < r_i \implies y \in \mathcal{U}_i$. Take $r = \min r_1, \dots, r_n$, then for any y satisfying $d(x, y) < r$, we have $y \in \mathcal{U}_i$ for all $1 \leq i \leq n$, thus $y \in W$. Hence W is open. \square

Definition 1.3.3 Let X be a set. A topology \mathcal{T} of X is a collection of subsets of X that satisfies the following:

- a) \mathcal{T} is closed under arbitrary union,
- b) \mathcal{T} is closed under finite intersection, and
- c) $\emptyset, X \in \mathcal{T}$.

We say that (X, \mathcal{T}) is a topological space if \mathcal{T} is a topology of X . The elements of \mathcal{T} are called open sets. We define $S \subseteq X$ to be closed if S^c is open.

Example 1.3.1 The following are some examples of topological spaces:

- A metric space is a topological space.
- Let X be a set, and $\mathcal{T} = \{\emptyset, X\}$. Then (X, \mathcal{T}) is a topological space, which is known as the trivial topology.
- Let X be a set, and let \mathcal{T} be the power set of X , then (X, \mathcal{T}) is a topological space, which is known as the discrete topology.

Remark 1.3.2 By De Morgan's law, closed sets are closed under arbitrary intersection and finite union, also \emptyset, X are closed.

In general, infinite union of closed sets may not be closed.

Definition 1.3.4 Let M be a metric space and $S \subseteq M$. Define

$$\overline{S} := \{x \in M \mid x \text{ is a limit of } S\}$$

to be the closure of S . For $x \in M$, $r > 0$, define

$$B(x, r) := \{y \in M \mid d(x, y) < r\},$$

which is the ball centered at x with radius r , or the r -neighborhood of x .

Proposition 1.3.3 \overline{S} is closed and $B(x, r)$ is open.

Proof. If $S = \emptyset$, then $\overline{S} = \emptyset$, which is closed.

Suppose that $S \neq \emptyset$, and let $(x_n) \rightarrow x$ be a convergence sequence in \overline{S} . We wish to prove that $x \in \overline{S}$, i.e. there exists a sequence in S that approaches x . Since $x_n \in \overline{S}$, there exists sequence $(x_{n,k})$ in S that approaches x_n as $k \rightarrow \infty$. For each n , there exists a term x_{n,k_n} satisfying $d(x_{n,k_n}, x_n) < \frac{1}{n}$, picking these terms forms a new sequence (x_{n,k_n}) , moreover,

$$d(x_{n,k_n}, x) \leq d(x_{n,k_n}, x_n) + d(x_n, x) < \frac{1}{n} + d(x_n, x),$$

which approaches 0 as $n \rightarrow \infty$, thus (x_{n,k_n}) approaches x .

Fix $x \in M$ and $r > 0$. Let $y \in B(x, r)$ and pick $s = r - d(x, y)$. If z satisfies $d(y, z) < s$, then

$$d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + s = r \implies B(y, s) \subseteq B(x, r),$$

so $B(x, r)$ is open. □

Corollary 1.3.5 \overline{S} is the smallest closed set that contains S , i.e. if $K \supseteq S$ and K is closed then $K \supseteq \overline{S}$.

Proof. K contains the limit of each sequence in K , in particular $S \subseteq K$ so it contains all sequence in S that converges in M , but these are precisely \overline{S} . □

With the observation from Corollary 1.3.5, we may define the closure for topological spaces in general.

Definition 1.3.6 Let X be a topological space, and let $S \subseteq X$. We define \overline{S} to be the smallest closed set that contains S . \overline{S} always exists; take

$$\overline{S} = \bigcap \{E \subseteq X \mid E \text{ is closed in } X \text{ and } E \supseteq S\}.$$

Likewise, we also define the interior of S , $\text{int}(S)$, to be the largest open set contained in S

Definition 1.3.7 Let X be a topological space, $x \in X$. A neighborhood of x is an open set containing x .

Definition 1.3.8 Let X be a topological space and let (x_n) be a sequence in X . We say that (x_n) converges to $x \in X$ if for all neighborhood \mathcal{U} of x , $\exists N \in \mathbb{N}$ such that $x_n \in \mathcal{U}$ for all $n \geq N$.

Example 1.3.2 Limits in general are not unique in topological spaces. Let X be a set with at least 2 points endowed with the trivial topology. Then every sequence in X converges to every point in X .

Definition 1.3.9 Let X, Y be topological spaces. We say a function $f : X \rightarrow Y$ continuous if for any open set $V \subseteq Y$, the preimage $f^{-1}(V)$ is open in X .

Proposition 1.3.4 A function $f : X \rightarrow Y$ is continuous iff $\forall x \in M$ and any neighborhood \mathcal{V} of $f(x)$, \exists a neighborhood \mathcal{U} of x such that $f(\mathcal{U}) \subseteq \mathcal{V}$.

Proof. " \implies ": Let $\mathcal{V} \subseteq Y$ is open. We need to show $f^{-1}(\mathcal{V})$ is open. Let $x \in f^{-1}(\mathcal{V})$, then $f(x) \in \mathcal{V}$. By definition, \exists a neighborhood \mathcal{U}_x such that $f(\mathcal{U}_x) \subseteq \mathcal{V} \rightsquigarrow \mathcal{U}_x \subseteq f^{-1}(\mathcal{V})$. Take the union of all such \mathcal{U}_x over $x \in f^{-1}(\mathcal{V})$, then

$$\begin{aligned} \bigcup_{x \in f^{-1}(\mathcal{V})} \mathcal{U}_x &\subseteq f^{-1}(\mathcal{V}) \quad \text{and} \quad \forall x \in f^{-1}(\mathcal{V}), x \in \mathcal{U}_x \subseteq f^{-1}(\mathcal{V}) \\ &\implies f^{-1}(\mathcal{V}) = \bigcup_{x \in f^{-1}(\mathcal{V})} \mathcal{U}_x \end{aligned}$$

and thus $f^{-1}(\mathcal{V})$ is open.

" \impliedby ": Let $x \in X$ and let \mathcal{V} be a neighborhood of $f(x)$. By definition $x \in f^{-1}(\mathcal{V})$, which is an open set by assumption. Also, $f(f^{-1}(\mathcal{V})) \subseteq \mathcal{V}$, hence $f^{-1}(\mathcal{V})$ is a neighborhood of x such that $f(f^{-1}(\mathcal{V})) \subseteq \mathcal{V}$. \square

Definition 1.3.10 A homeomorphism is a continuous bijection between topoloical spaces.

Corollary 1.3.11 A homeomorphism $f : X \rightarrow Y$ bijects the corresponding topologies \mathcal{T}_X and \mathcal{T}_Y .

1.4 Hausdorff Space

Definition 1.4.1 A topological space X is said to be Hausdorff if given any pair of distinct paints $x_1, x_2 \in X$, \exists neighborhoods \mathcal{U}_1 of x_1 and \mathcal{U}_2 of x_2 such that $\mathcal{U}_1 \cap \mathcal{U}_2 = \emptyset$.

A metric space is always a Hausdorff space.

Lemma 1.4.2 Let X be Hausdorff.

- a) Every one-point set is closed.
- b) If a sequence (x_n) in X converges, then the limit is unique.

Proof. a) Pick $x \in X$. For any y distinct from x , there exists disjoint neighborhoods \mathcal{U}_x of x and \mathcal{V}_y of y . We have

$$\{x\}^c = \bigcup_{y \in X \setminus \{x\}} \mathcal{V}_y$$

which is an open set.

- b) Suppose that x, x' are distinct limits of (x_n) . \exists disjoint neighborhoods \mathcal{U} of x and \mathcal{U}'

of x' . $\exists N, N' \in \mathbb{N}$ such that " $n \geq N'$ implies $x_n \in \mathcal{U}$ " and " $n \geq N'$ implies $x_n \in \mathcal{U}'$ ". If $n \geq \max(N, N')$, then $x_n \in \mathcal{U} \cap \mathcal{U}' = \emptyset$, contradiction. Therefore, any converging sequence has a unique limit.

□

1.5 Subspaces and Product Spaces

Definition 1.5.1 Let X be a topological space, let $A \subseteq X$. We define the subspace topology \mathcal{T}_A of A by

$$\mathcal{T}_A = \{\mathcal{U} \subseteq A \mid \mathcal{U} = A \cap \mathcal{V} \text{ for some open } \mathcal{V} \subseteq X\}.$$

Remark 1.5.1 Openness and closedness are not just properties of a set itself, but rather a set in a relation to a particular topological space.

Proposition 1.5.1 Let M be a metric space, and let $N \subseteq M$ be a nonempty subset. The subspace topology on N is the same as the metric topology obtained by restricting the metric of M to N .

Proof. Suppose that \mathcal{V} is an open set in M and let $\mathcal{U} = N \cap \mathcal{V} \in \mathcal{T}_N$.

We first to prove that \mathcal{U} belongs to the metric topology of N . Let $x \in \mathcal{U}$. Since $x \in N \cap \mathcal{V} \subseteq \mathcal{V}$, there exists $r > 0$ such that $B_M(x, r) \subseteq \mathcal{V}$, and

$$B_N(x, r) = N \cap B_M(x, r) \subseteq N \cap \mathcal{V} = \mathcal{U},$$

hence \mathcal{U} is open in N (in the metric topology).

Conversely, let \mathcal{U} be an open set in the metric topology of N . $\forall x \in \mathcal{U}$, $\exists r_x > 0$ such that $B_N(x, r_x) \subseteq \mathcal{U}$. Note that

$$\mathcal{U} = \bigcup_{x \in \mathcal{U}} B_N(x, r_x) = \bigcup_{x \in \mathcal{U}} N \cap B_M(x, r_x) = N \cap \bigcup_{x \in \mathcal{U}} B_M(x, r_x),$$

hence \mathcal{U} belongs to the subspace topology of N .

□

Definition 1.5.2 Let X be a set. A basis in X is a collection \mathcal{B} of subsets of X satisfying

- a) Every element of X is in some element in \mathcal{B} . That is,

$$X = \bigcup_{B \in \mathcal{B}} B.$$

- b) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Definition 1.5.3 Given X and a collection \mathcal{B} of subsets of X , we say that $\mathcal{U} \subseteq X$ satisfies the basis criterion with respect to \mathcal{B} if $\forall x \in \mathcal{U}, \exists B \in \mathcal{B}$ such that $x \in B \subseteq \mathcal{U}$

Lemma 1.5.4 Suppose that \mathcal{B} is a basis in X , and let \mathcal{T} be the collection of all unions of elements of \mathcal{B} . Then \mathcal{T} is precisely the collection of all subsets of X that satisfy the basis criterion w.r.t. \mathcal{B} .

Proof. Let $\mathcal{U} \subseteq X$. Suppose that \mathcal{U} satisfies the basis criterion. Let

$$\mathcal{V} = \bigcup \{B \in \mathcal{B} \mid B \subseteq \mathcal{U}\},$$

then $\mathcal{V} \in \mathcal{T}$. We want to show that $\mathcal{U} = \mathcal{V}$. Clearly, $\mathcal{V} \subseteq \mathcal{U}$. Let $x \in \mathcal{U}$. Since \mathcal{U} satisfies the basis criterion, $\exists B \in \mathcal{B}$ such that $x \in B \subseteq \mathcal{U}$, so $x \in \mathcal{V}$, therefore $\mathcal{U} \subseteq \mathcal{V}$.

Conversely, suppose that $\mathcal{U} \in \mathcal{T}$, then \mathcal{U} is a union of elements of \mathcal{B} , say $\mathcal{U} = \bigcup_{B \in \mathcal{A}} B$ where $\mathcal{A} \subseteq \mathcal{B}$. For any $x \in \mathcal{U}$, $x \in B$ for some $B \in \mathcal{A}$, also $B \subseteq \mathcal{U}$, so \mathcal{U} satisfies the basis criterion. \square

Proposition 1.5.2 Let \mathcal{B} be a basis in X , and let \mathcal{T} be collection of all unions of elements of \mathcal{B} . Then \mathcal{T} is a topology on X . This is called the topology generated by \mathcal{B} .

Proof. \square