Mechanics of Mechanisms and Robots

1. Fundamental: motion

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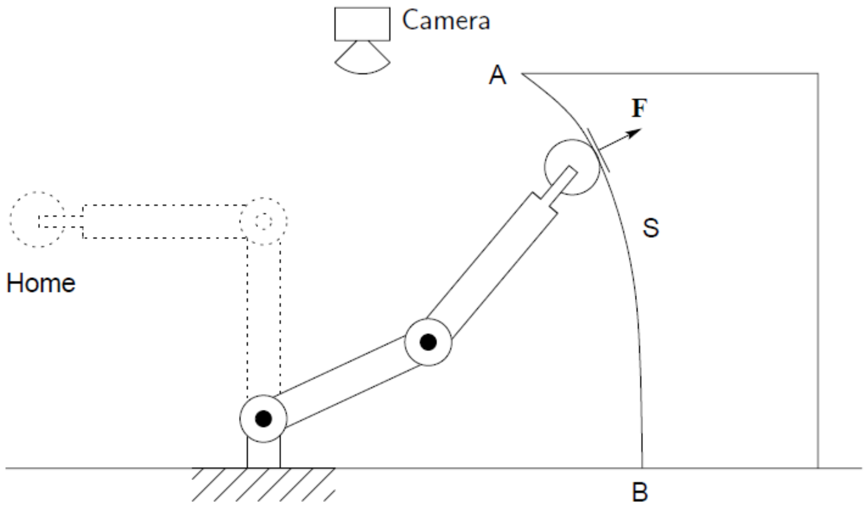
Revisiting mechanics problems

- Task 1: how to move the robot between ‘Home’ and a target position?
- Task 2: how to apply a desired force F to the surface S?



Fundamental: motion

- motion representation
- motion of rigid bodies
- matrix operation

Position, velocity and acceleration

- A systematic method to describe motion:

- derive an equation for **position**
- derivative of position w.r.t. time: **velocity**
- derivative of velocity w.r.t. time: **acceleration**

Motion of a point

- Position is a quantity with magnitude and direction, which can be represented by a vector

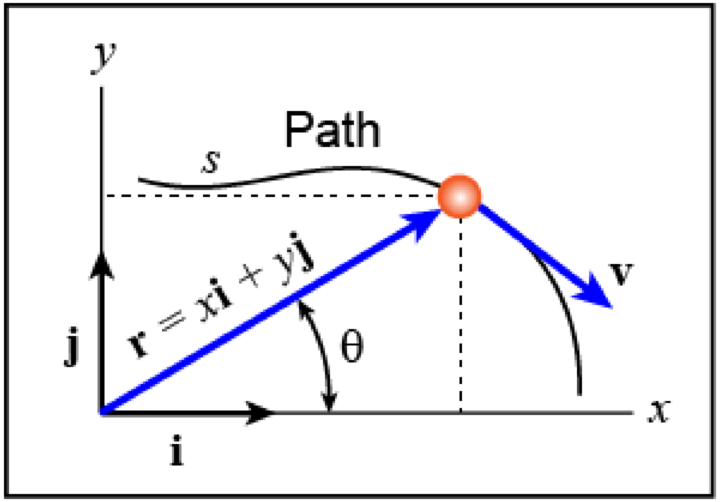
- Vectors can be represented graphically, with the length corresponding to the magnitude, and the direction set by an angle against a coordinate frame or another vector

- Velocity and acceleration can be also represented by a vector

Rectangular coordinate frame

- position: $\vec{r} = x\vec{i} + y\vec{j}$ or $\vec{r} = \begin{bmatrix} x \\ y \end{bmatrix}$

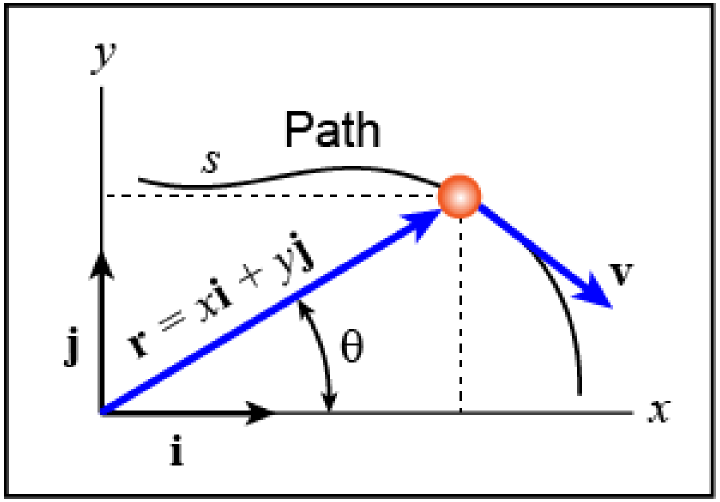
- vector: **bold font** or $\overrightarrow{()}$
- magnitude: $r = |\vec{r}|$
- direction: θ
- x, y: coordinates
- $x = r\cos\theta, y = r\sin\theta$



Rectangular coordinate frame

- position: $\vec{r} = x\vec{i} + y\vec{j}$ or $\vec{r} = \begin{bmatrix} x \\ y \end{bmatrix}$

- $\vec{i}, \vec{j}$: unit vectors
- $\vec{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

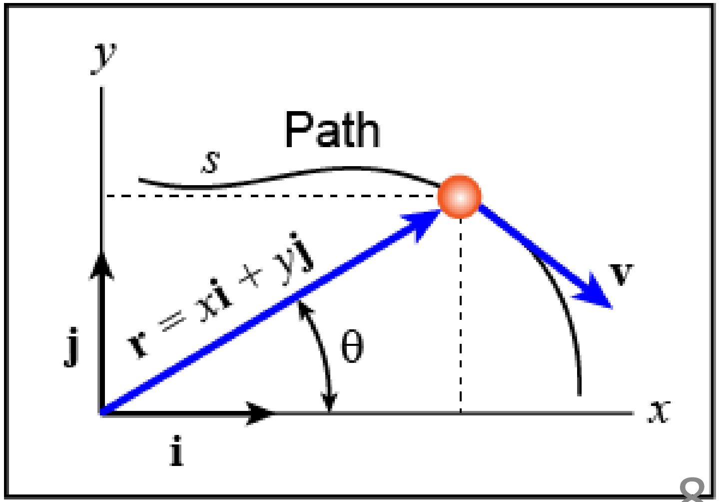


Rectangular coordinate frame

- position: $\vec{r} = x\vec{i} + y\vec{j}$
- velocity (time derivative of position using product rule):

$$\vec{v} = \dot{\vec{r}} = \dot{x}\vec{i} + x\dot{\vec{i}} + \dot{y}\vec{j} + y\dot{\vec{j}}$$

- time derivative: $\dot{()} = \frac{d()}{dt}$
- double derivative: $\ddot{()}$



Rectangular coordinate frame

- velocity (time derivative of position):

$$\vec{v} = \dot{\vec{r}} = \dot{x}\vec{\imath} + x\dot{\vec{\imath}} + \dot{y}\vec{\jmath} + y\dot{\vec{\jmath}}$$

- if coordinate frame is fixed:

  $\dot{\vec{\imath}} = \dot{\vec{\jmath}} = \vec{0}$

  then $\vec{v} = \dot{x}\vec{\imath} + \dot{y}\vec{\jmath} = v_x\vec{\imath} + v_y\vec{\jmath}$

Rectangular coordinate frame

- acceleration (derivative of velocity) in a fixed coordinate frame:

$$\vec{a} = \dot{\vec{v}} = \dot{v}_x \vec{i} + \dot{v}_y \vec{j} = a_x \vec{i} + a_y \vec{j}$$

where we have used

$$\dot{\vec{i}} = \dot{\vec{j}} = \vec{0}$$

- What if the coordinate frame is moving?

Polar coordinate frame

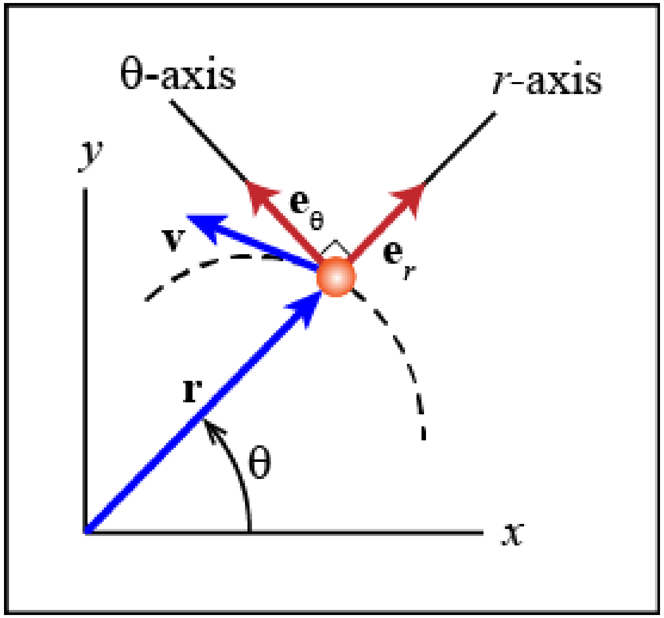
- position: $\vec{r} = r\vec{e}_r$

- r: magnitude of $\vec{r}$

- $\vec{e}_r, \vec{e}_\theta$: unit vectors

- $\vec{e}_r = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \vec{e}_\theta = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$

 in rectangular coordinate frame x-y



Polar coordinate frame

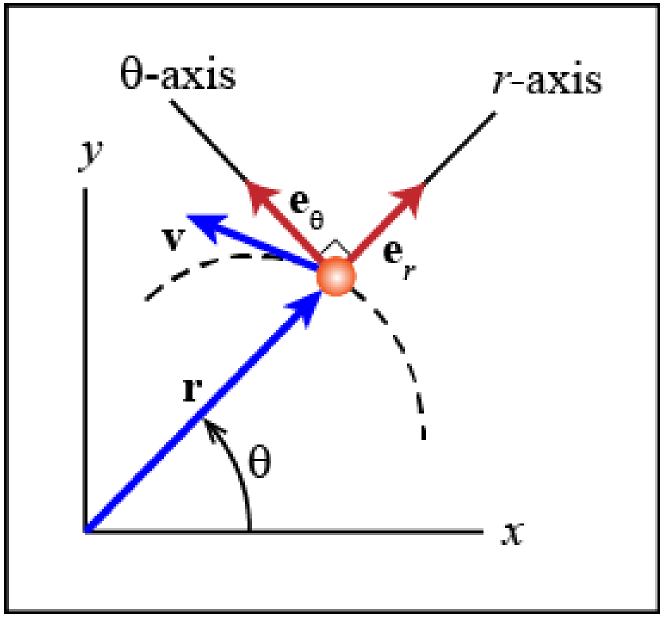
- position: $\vec{r} = r\vec{e}_r$
- velocity (using product rule):

$\vec{v} = \dot{\vec{r}} =d(r\vec{e}_r)$/d$t$

$= \dot{r}\vec{e}_r + r\dot{\vec{e}}_r = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta$

where (using [chain rule]()):

$\dot{\vec{e}}_r =$d$\begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix}$/d$t$ = $\dot{\theta}\vec{e}_\theta$



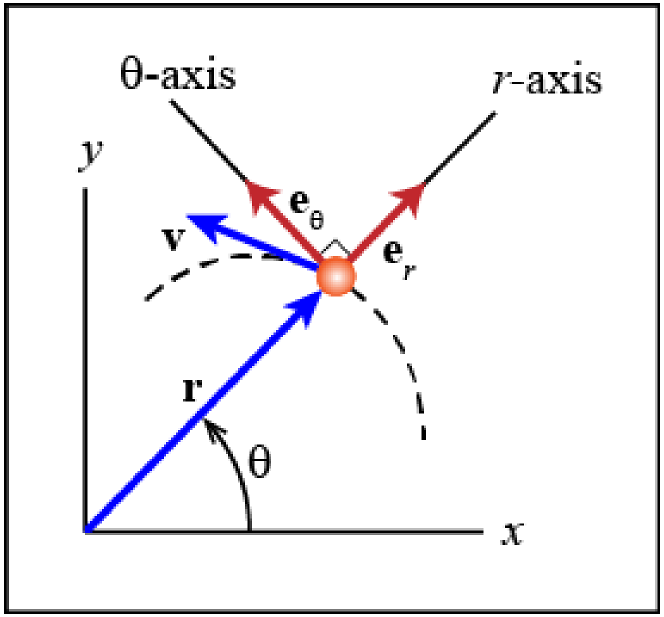
Polar coordinate frame

- velocity: $\vec{v} = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta$

- for rotation: $\dot{r} = 0$

  so $\vec{v} = r\dot{\theta}\vec{e}_\theta$

- in direction of $\vec{e}_\theta$
- magnitude is $r\dot{\theta}$



Polar coordinate frame

- acceleration:

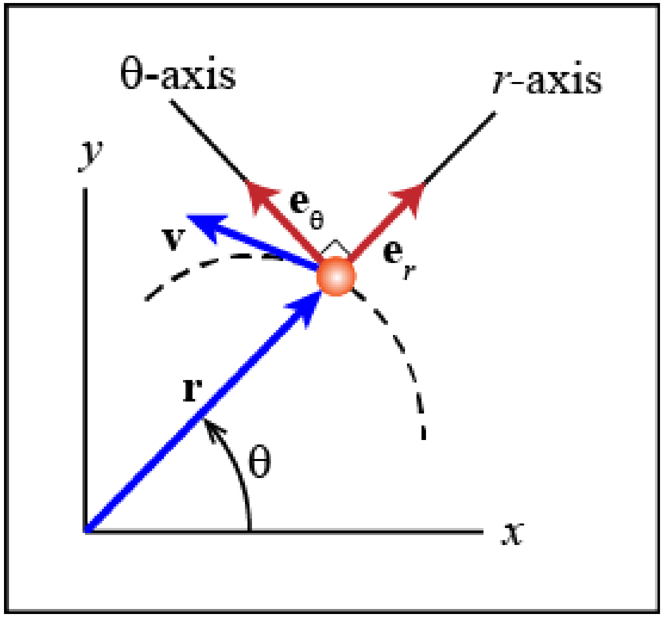
$\vec{a} = \dot{\vec{v}}$

$= \mathrm{d}(\dot{r}\vec{e}_r + r\dot{\vec{e}}_r)/\mathrm{d}t$

$= \left(\ddot{r} - r\dot{\theta}^2\right)\vec{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{e}_\theta$

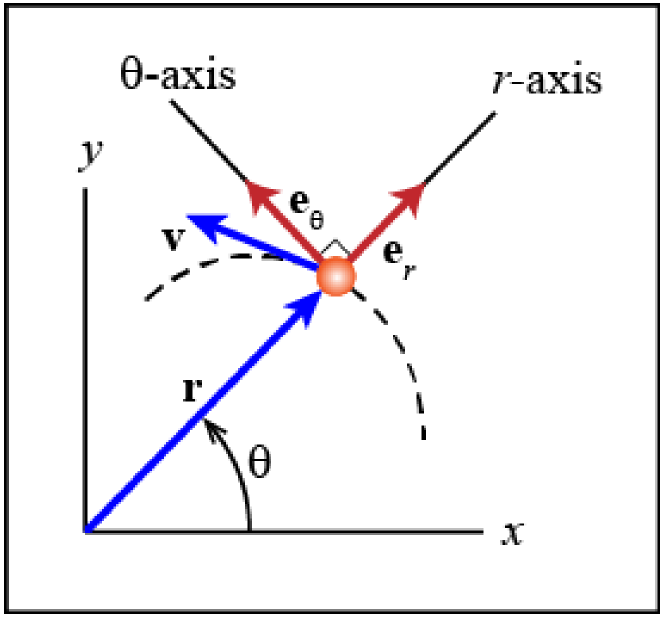
where we have used

$\dot{\vec{e}}_\theta = -\dot{\theta}\vec{e}_r$



Polar coordinate frame

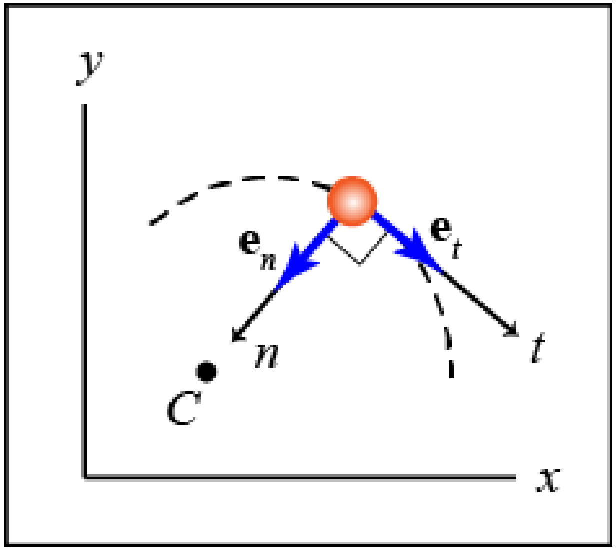
- centrifugal: $-r\dot{\theta}^2\vec{e}_r$
- Euler: $r\ddot{\theta}\vec{e}_\theta$
- Coriolis: $2\dot{r}\dot{\theta}\vec{e}_\theta$



- for rotation: $\dot{r} = 0$ and $\ddot{r} = 0$, so
  $\vec{a} = -r\dot{\theta}^2\vec{e}_r + r\ddot{\theta}\vec{e}_\theta$

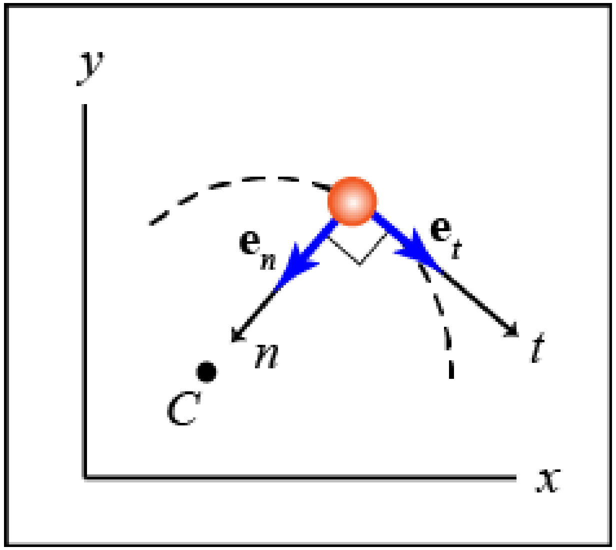
n-t coordinate frame

- **t-axis:** tangential to the path curve
- **n-axis:** perpendicular to the t-axis and is directed toward the centre of curvature



n-t coordinate frame

- n-t coordinate frame is attached to, and moves with, a point, so there is no position vector



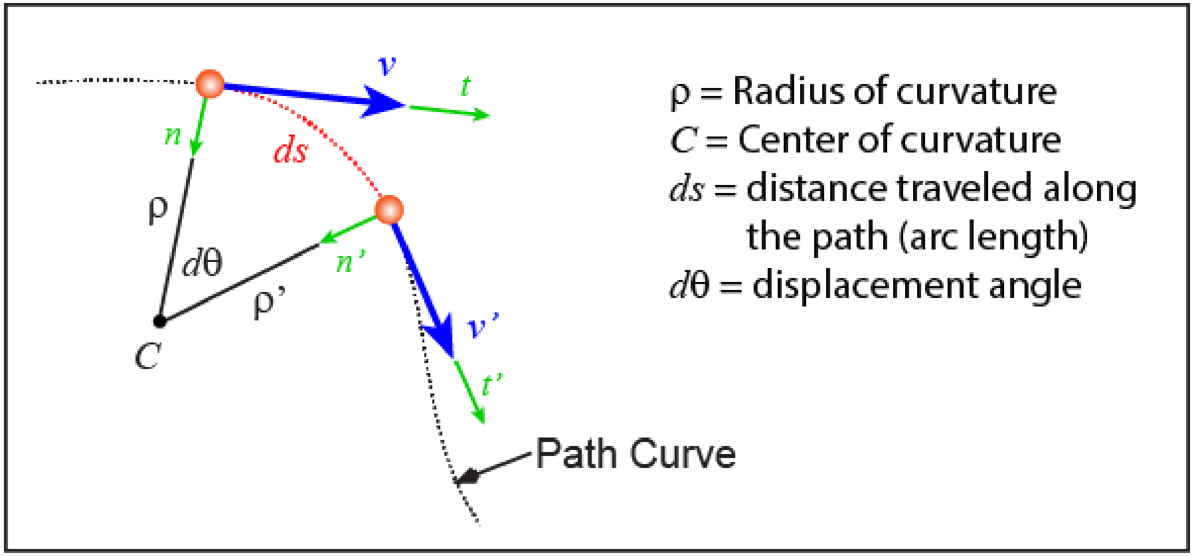
n-t coordinate frame

- velocity (always tangential to the path):

  $\vec{v} = \dot{s}\vec{e}_t = v\vec{e}_t$

  where (using arc length formula d$s = \rho$dθ)

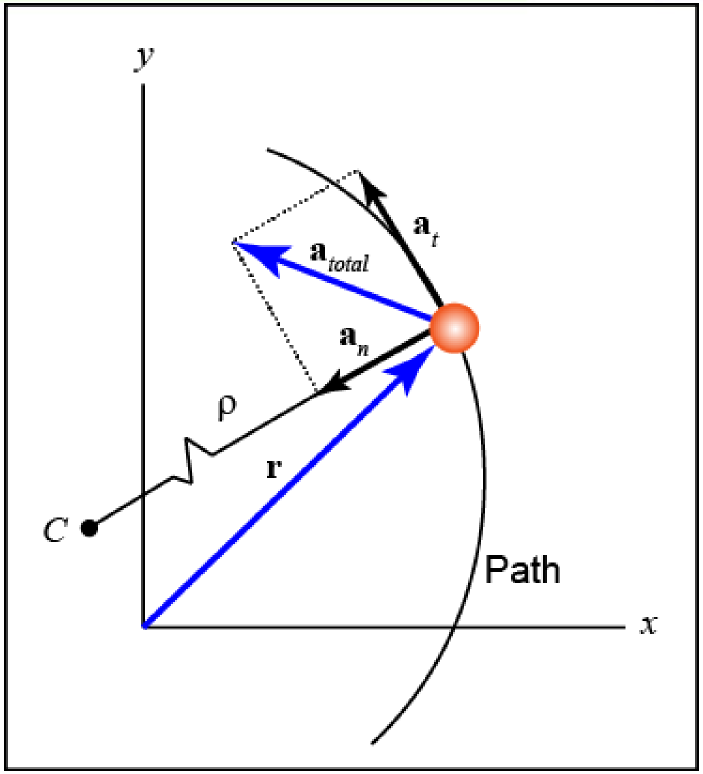
  $v = \rho\dot{\theta}$, similar as in polar coordinate frame



n-t coordinate frame

- acceleration:

$$\vec{a} = \dot{\vec{v}}$$
$$= \mathrm{d}(v\vec{e}_t)/\mathrm{d}t$$
$$= \dot{v}\vec{e}_t + \frac{v^2}{\rho}\vec{e}_n$$
$$= \rho\ddot{\theta}\vec{e}_t + \frac{v^2}{\rho}\vec{e}_n$$



- Recall polar coordinate frame for rotation

$\vec{a} = -r\dot{\theta}^2\vec{e}_r + r\ddot{\theta}\vec{e}_\theta$ with $v = \rho\dot{\theta}$ and $\rho = r$

Summary of coordinate frames

- The **rectangular coordinate frame** is useful when a path is straight line.

- position: $\vec{r} = x\vec{\imath} + y\vec{\jmath}$

for a fixed coordinate frame:

- velocity: $\vec{v} = v_x\vec{\imath} + v_y\vec{\jmath}$
- acceleration: $\vec{a} = a_x\vec{\imath} + a_y\vec{\jmath}$

Summary of coordinate frames

- The **polar coordinate frame** is useful when a path is curved and when angular velocity and acceleration are given.

- position: $\vec{r} = r\vec{e}_r$
- velocity: $\vec{v} = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta$
- acceleration: $\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\vec{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{e}_\theta$

Summary of coordinate frames

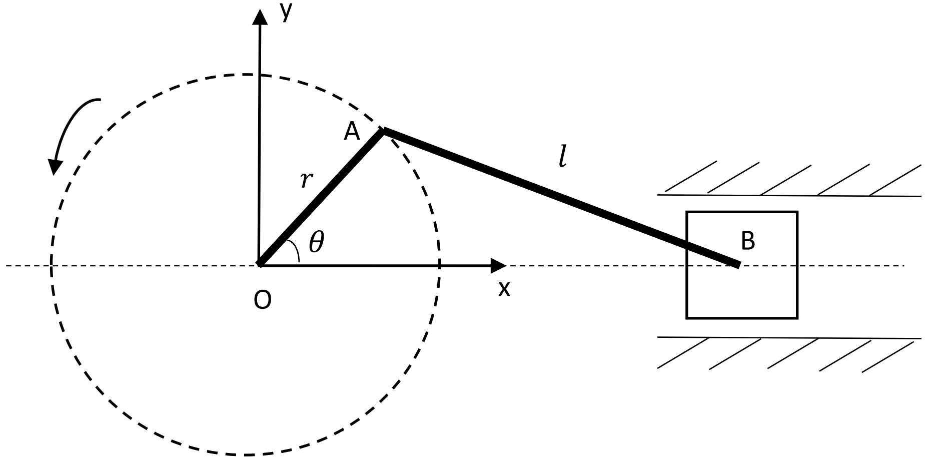
- The **normal and tangential coordinate frame** is useful when a path is curved and the path's radius of curvature, and the speed and acceleration along the path are given.

- velocity: $\vec{v} = v\vec{e}_t$
- acceleration: $\vec{a} = \rho\ddot{\theta}\vec{e}_t + \frac{v^2}{\rho}\vec{e}_n$

Example: slider crank

- crank with a radius r rotates about point O
- rod with a length l connects crank AO and slider at point B
- θ, $\dot{\theta}$ and $\ddot{\theta}$ are given

video link



Questions

- Calculate the velocity of the slider as a function of $\theta, \dot{\theta}$.
- Using a polar coordinate frame, derive the equations for the tangential and normal accelerations of point A.
- Determine the equations for the magnitude and angle of acceleration vector of point A.

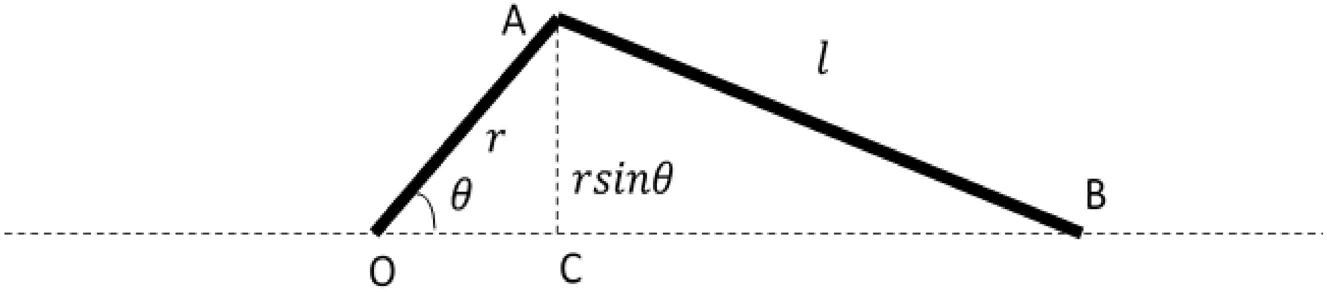
Solutions

- position of slider:

$$x_B = r\cos\theta + \sqrt{l^2 - r^2 sin^2\theta}$$

- velocity of slider:

$$v_B = \dot{x}_B = -r\,\dot{\theta}\sin\theta - \frac{r^2\dot{\theta}\sin\theta\cos\theta}{\sqrt{l^2 - r^2 sin^2\theta}}$$

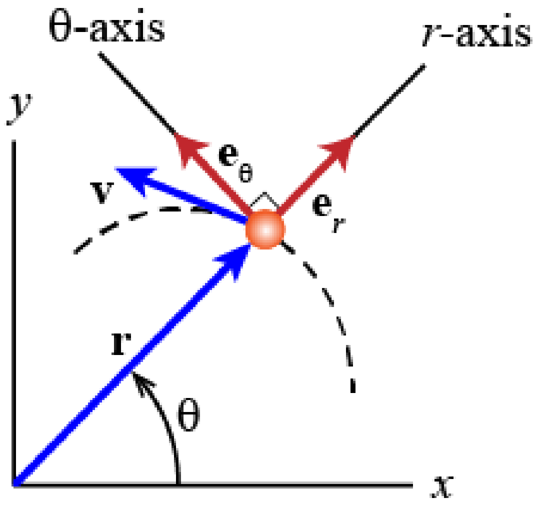


Solutions

- position of point A: $\vec{r} = r\vec{e}_r$
- velocity: $\vec{v} = \dot{\vec{r}} =d(r\vec{e}_r)$/d$t = r\dot{\vec{e}}_r = r\dot{\theta}\vec{e}_\theta$
- acceleration:

  $\vec{a} = \dot{\vec{v}} =$ d$(r\dot{\theta}\vec{e}_\theta)$/d$t$

  $= -r\dot{\theta}^2\vec{e}_r + r\ddot{\theta}\vec{e}_\theta$



Solutions

- magnitude of acceleration vector:

$$|\vec{a}| = \sqrt{(r\dot{\theta}^2)^2 + (r\ddot{\theta})^2} = r\sqrt{\dot{\theta}^4 + \ddot{\theta}^2}$$

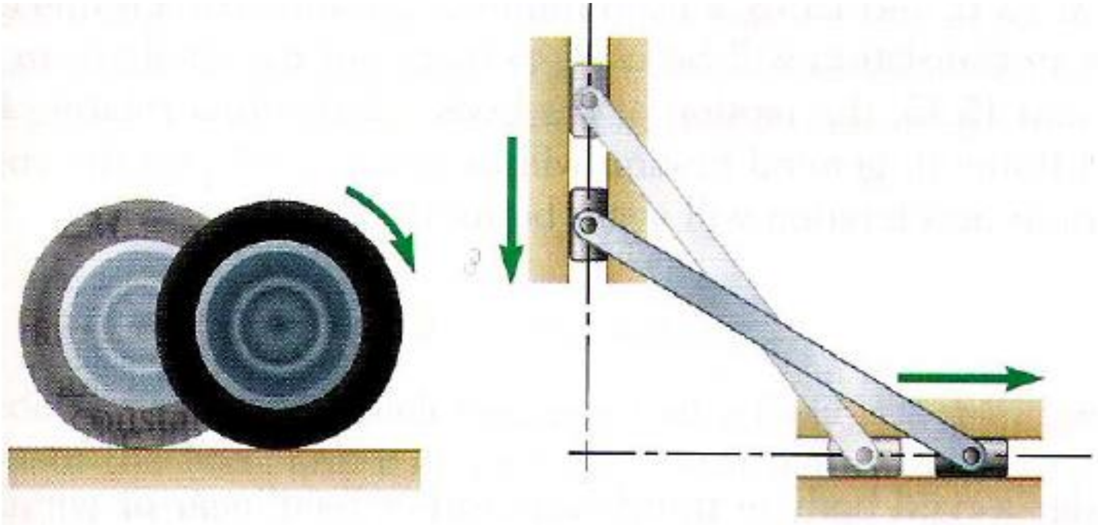
- angle of acceleration vector: $-arctan\frac{\dot{\theta}^2}{\ddot{\theta}}$

Fundamental: motion

- motion representation
- motion of rigid bodies
- matrix operation

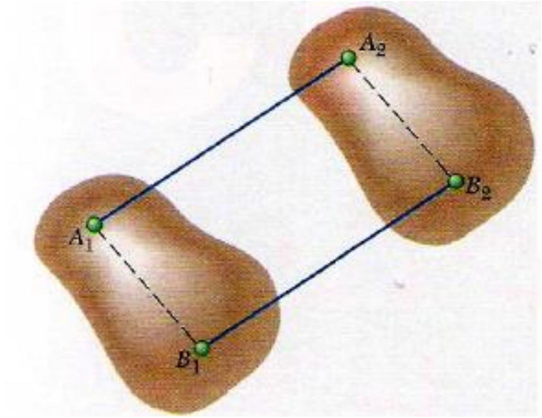
Motion of a rigid body

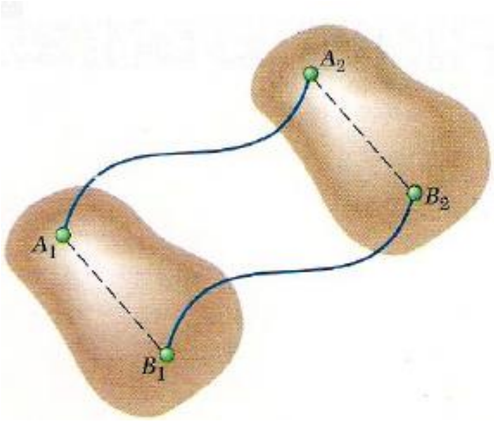
- Translation or linear motion
- Rotation
- General motion: neither a translation nor a rotation



Translation

- Any straight line inside the body keeps the same direction during the movement.

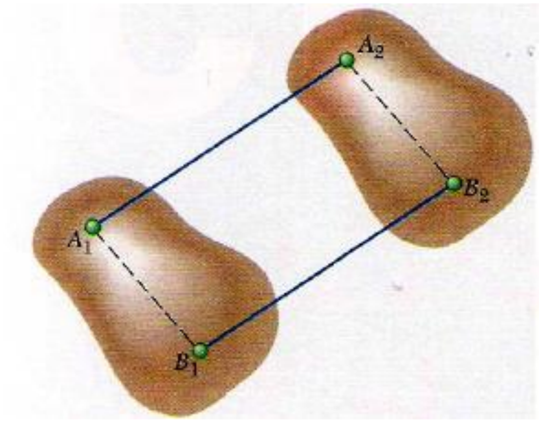


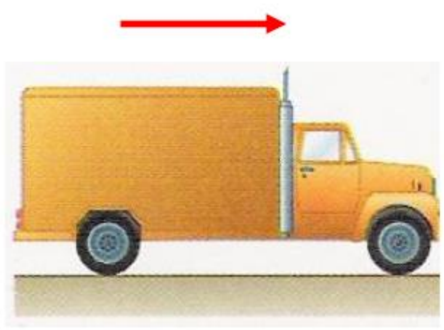


Rectilinear motion

- All the particles forming the body move along parallel paths.

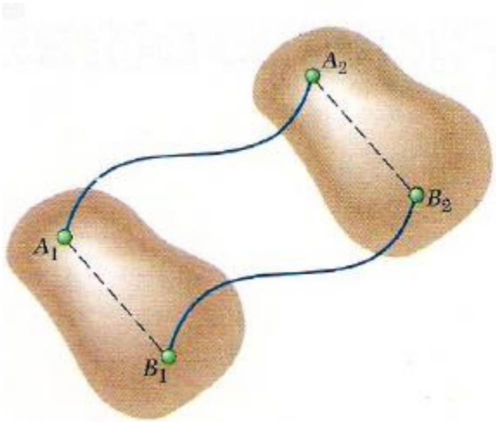
- These paths are straight lines.

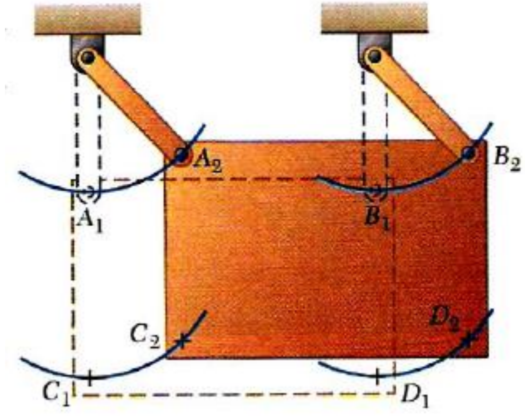




Curvilinear motion

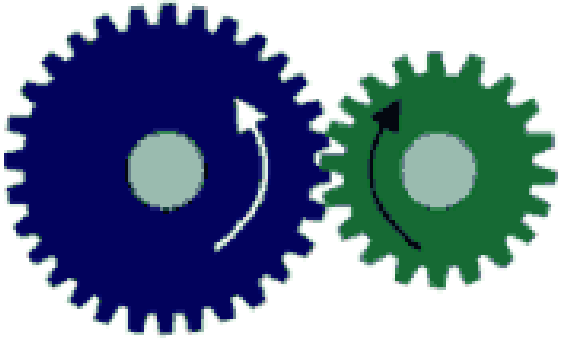
- All the particles forming the body move along parallel paths.
- These paths are curved lines.





Rotation

- The particles forming the rigid body move along circles centred on the same fixed axis.

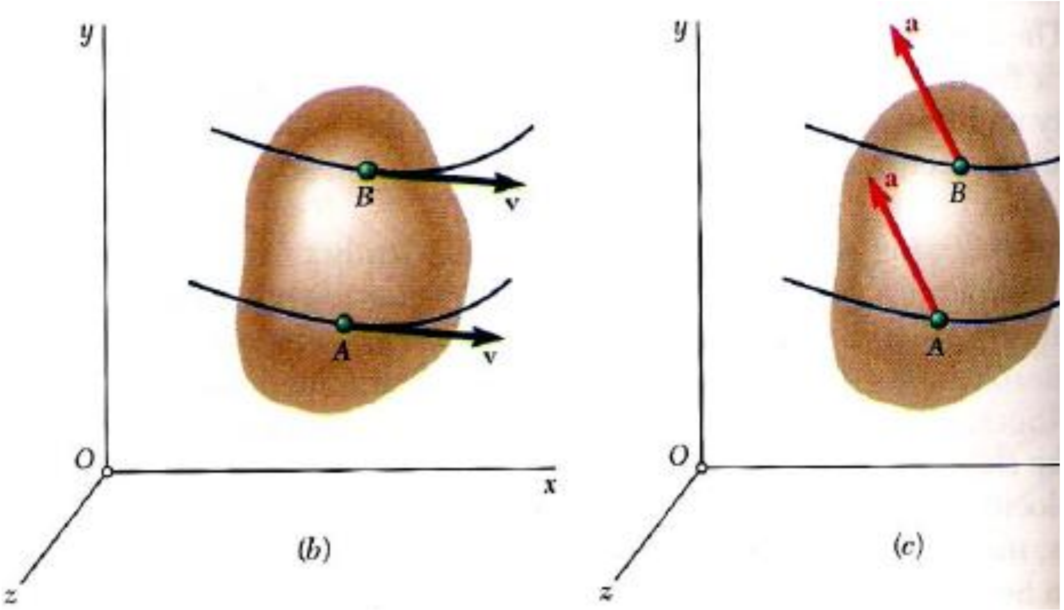


Translation description

- All the points of the body have the same velocity and the same acceleration.

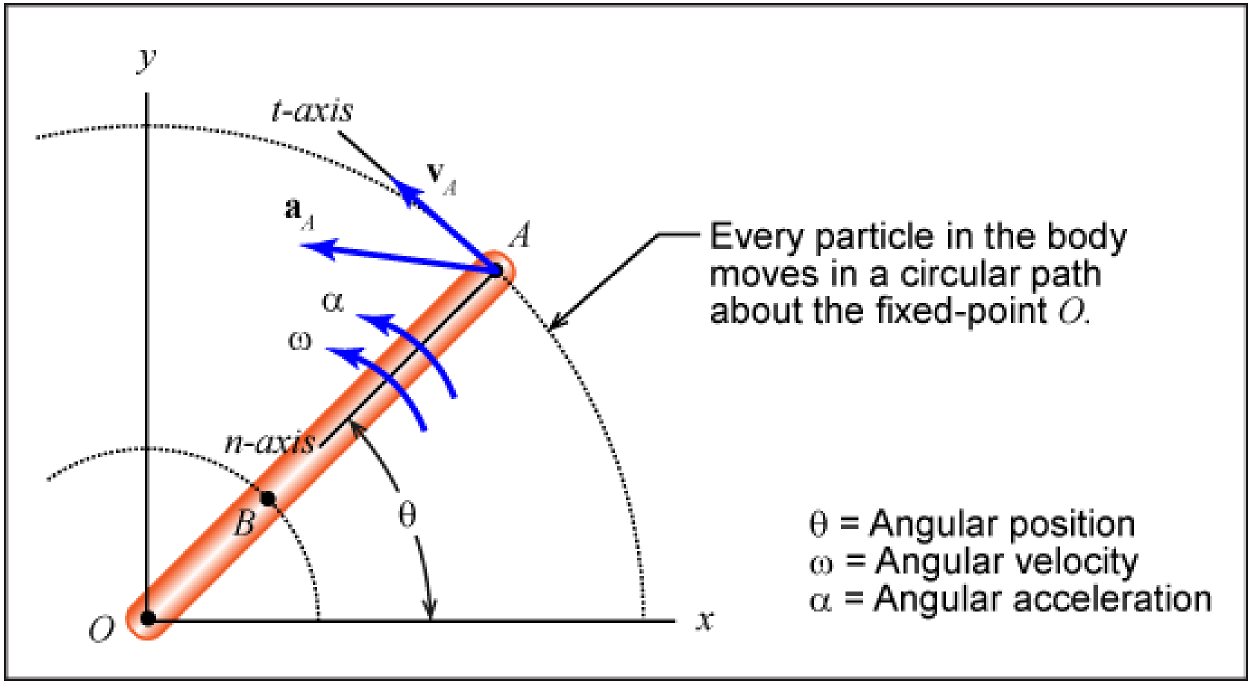
$$\overrightarrow{v_B} = \overrightarrow{v_A}$$

$$\overrightarrow{a_B} = \overrightarrow{a_A}$$



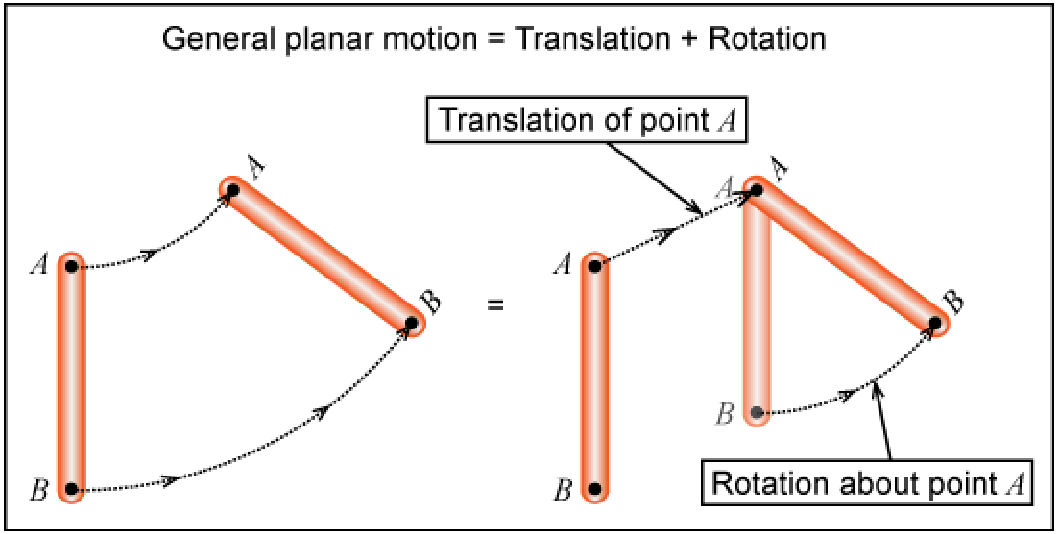
Rotation description

- velocity vector: $\vec{v}_A = r_A \omega \vec{e}_t$
- acceleration vector: $\vec{a}_A = r_A \alpha \vec{e}_t + r_A \omega^2 \vec{e}_n$



General motion description

- The motion of the rigid body may be described as a simple superposition of the body's translation and rotation.



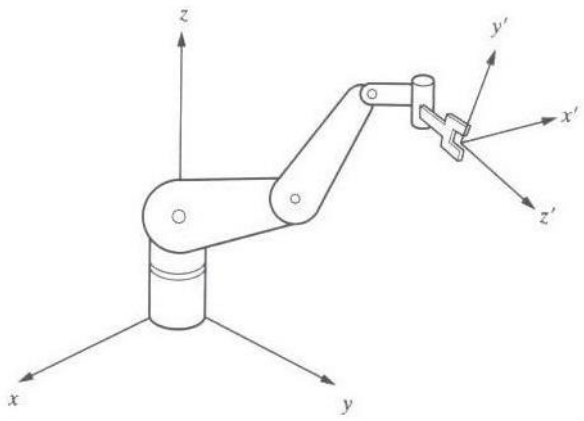
Absolute & relative motion

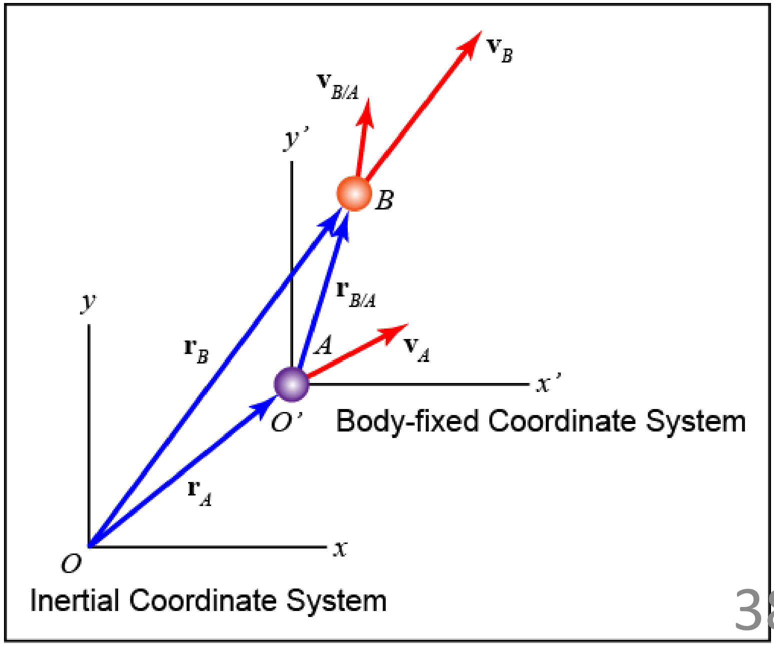
- absolute motion: measured from a fixed coordinate frame, normally the ground or anything rigidly attached to the ground and not moving

- relative motion: measured relative to a coordinate frame that may itself be moving

Absolute & relative motion

- Inertial coordinate frame: non-accelerating and non-rotating
- Body-fixed coordinate frame: a body-fixed reference frame is fixed to a body that is usually moving.





Example: translating coordinate frame

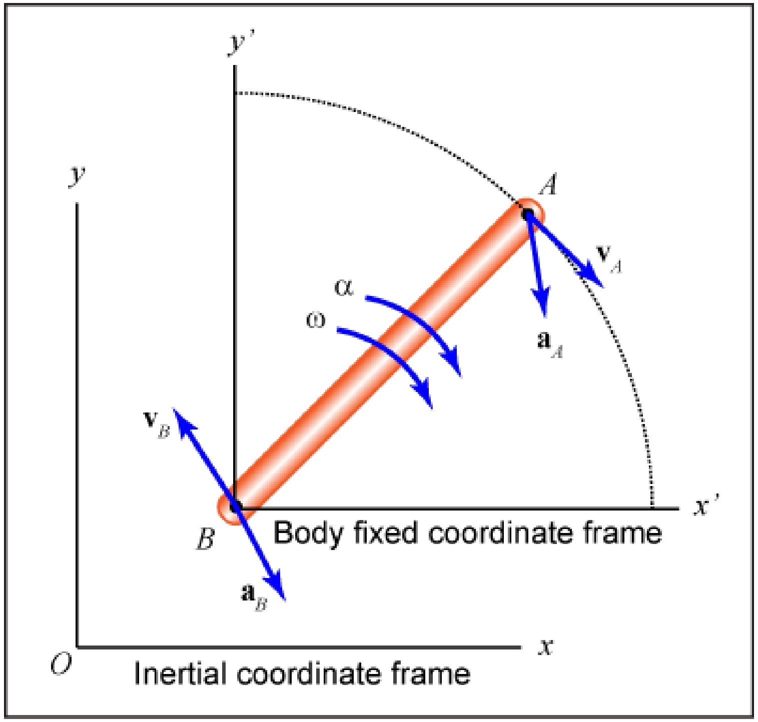
- position vector: $\vec{r}_A = \vec{r}_B + \vec{r}_{A/B}$

- velocity vector:

$$\vec{v}_A = \vec{v}_B + \vec{v}_{A/B}$$
$$= \vec{v}_B + r_{AB}\omega\vec{e}_t \text{ (rotation)}$$

- acceleration vector:

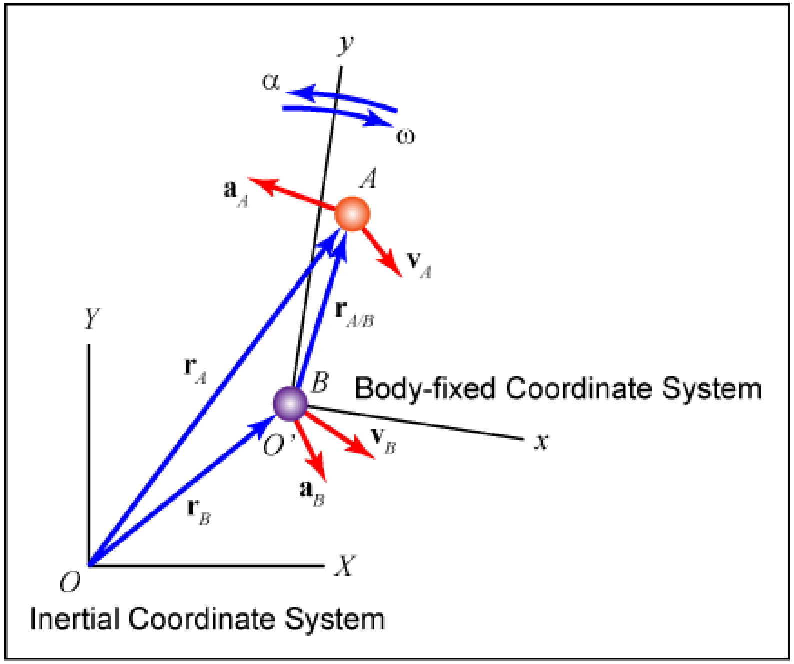
$$\vec{a}_A = \vec{a}_B + \vec{a}_{A/B}$$
$$= \vec{a}_B + r_{AB}\alpha\vec{e}_t + r_{AB}\omega^2\vec{e}_n$$



Example: rotating coordinate frame

- position vector:

$$\vec{r}_A = \vec{r}_B + \vec{r}_{A/B}$$
$$= \vec{r}_B + (x\vec{\imath} + y\vec{\jmath})$$

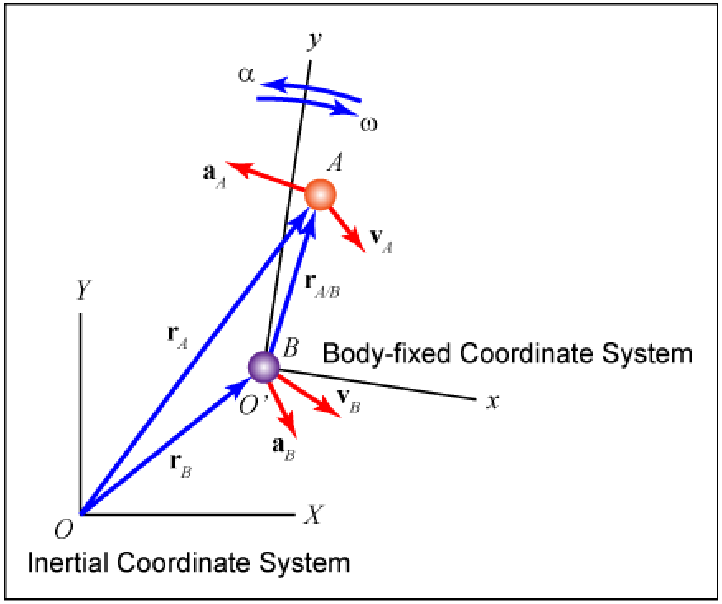


Example: rotating coordinate frame

- velocity vector:

$$\vec{v}_A = \vec{v}_B + \vec{v}_{A/B} = \vec{v}_B + (\dot{x}\vec{\imath} + x\dot{\vec{\imath}} + \dot{y}\vec{\jmath} + y\dot{\vec{\jmath}})$$

- how to better describe a rotating coordinate frame?



Fundamental: motion

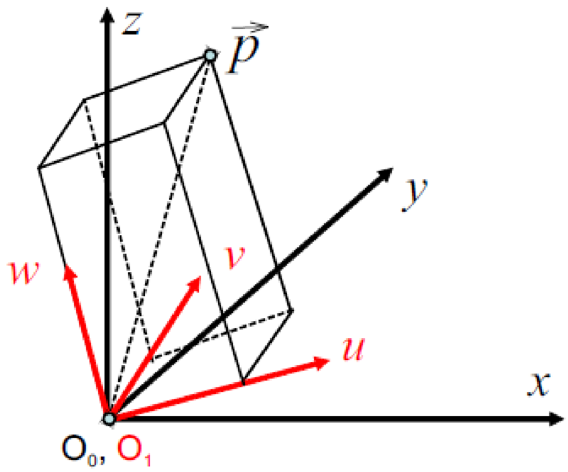
- motion representation
- motion of rigid bodies
- matrix operation

Matrix operation: rotation

- It is of advantage to express the rotation as a matrix operation.
- Consider a vector in two coordinate frames:

$$\vec{p}_{xyz} = p_x \vec{\imath}_x + p_y \vec{\jmath}_y + p_z \vec{k}_z$$

$$\vec{p}_{uvw} = p_u \vec{\imath}_u + p_v \vec{\jmath}_v + p_w \vec{k}_w$$



- There is rotation matrix R s.t.

$$\vec{p}_{xyz} = \mathsf{R}\, \vec{p}_{uvw}$$

Math review: dot product

Dot product of vectors $\vec{a}, \vec{b}$

(also scalar or inner product)

- Geometric definition:

  $\vec{a} \cdot \vec{b} = \|\vec{a}\| \, \|\vec{b}\| \; \cos\Theta$

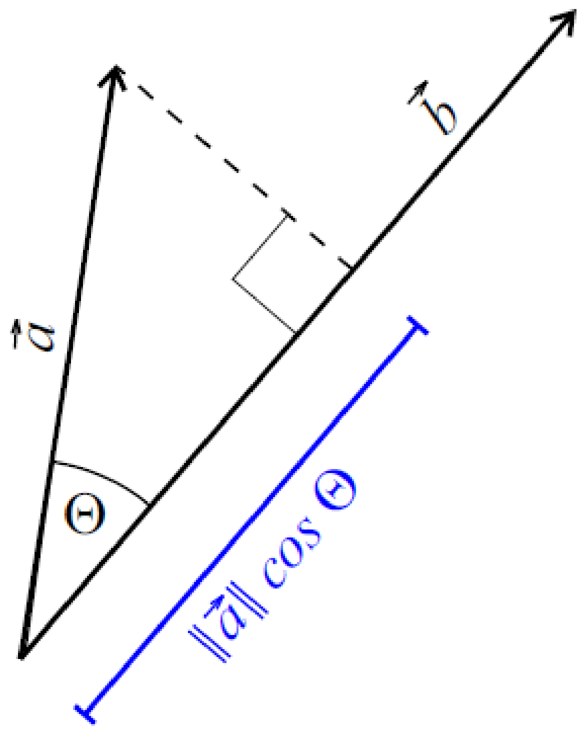
- Algebraic definition:

  $\vec{a} \cdot \vec{b} = \sum_{i=1}^{n} a_i \, b_i$

A two-dimensional example

$\vec{a} = [a_x, a_y]^\top, \quad \vec{b} = [b_x, b_y]^\top$

$\vec{a} \cdot \vec{b} = \begin{bmatrix} a_x \\ a_y \end{bmatrix} \cdot \begin{bmatrix} b_x \\ b_y \end{bmatrix} = a_x \, b_x + a_y \, b_y$



Matrix operation: rotation

- p_x, p_y, p_z represent projections of $\vec{p}$ onto x, y, z axes, respectively:

$$p_x = \vec{\imath}_x \cdot \vec{p} = \vec{\imath}_x \cdot \vec{\imath}_u \, p_u + \vec{\imath}_x \cdot \vec{\jmath}_v \, p_v + \vec{\imath}_x \cdot \vec{k}_w \, p_w$$

$$p_y = \vec{\jmath}_y \cdot \vec{p} = \vec{\jmath}_y \cdot \vec{\imath}_u \, p_u + \vec{\jmath}_y \cdot \vec{\jmath}_v \, p_v + \vec{\jmath}_y \cdot \vec{k}_w \, p_w$$

$$p_z = \vec{k}_z \cdot \vec{p} = \vec{k}_z \cdot \vec{\imath}_u \, p_u + \vec{k}_z \cdot \vec{\jmath}_v \, p_v + \vec{k}_z \cdot \vec{k}_w \, p_w$$

- in a matrix form:

Rotation matrix R

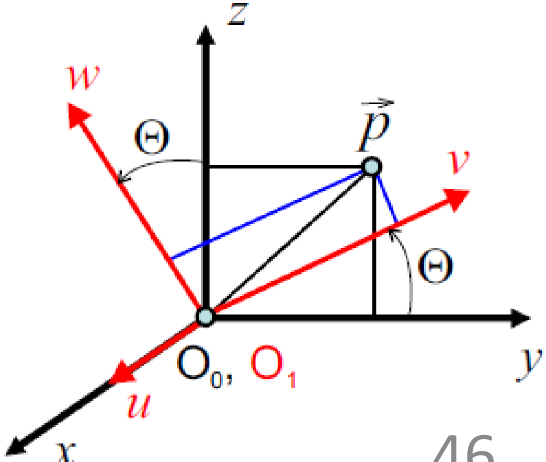
$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} \vec{\imath}_x \cdot \vec{\imath}_u & \vec{\imath}_x \cdot \vec{\jmath}_v & \vec{\imath}_x \cdot \vec{k}_w \\ \vec{\jmath}_y \cdot \vec{\imath}_u & \vec{\jmath}_y \cdot \vec{\jmath}_v & \vec{\jmath}_y \cdot \vec{k}_w \\ \vec{k}_z \cdot \vec{\imath}_u & \vec{k}_z \cdot \vec{\jmath}_v & \vec{k}_z \cdot \vec{k}_w \end{bmatrix} \begin{bmatrix} p_u \\ p_v \\ p_w \end{bmatrix}$$

Example: rotation matrix

- Rotation about axis x by Θ:

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} \vec{\imath}_x \cdot \vec{\imath}_u & \vec{\imath}_x \cdot \vec{\jmath}_v & \vec{\imath}_x \cdot \vec{k}_w \\ \vec{\jmath}_y \cdot \vec{\imath}_u & \vec{\jmath}_y \cdot \vec{\jmath}_v & \vec{\jmath}_y \cdot \vec{k}_w \\ \vec{k}_z \cdot \vec{\imath}_u & \vec{k}_z \cdot \vec{\jmath}_v & \vec{k}_z \cdot \vec{k}_w \end{bmatrix} \begin{bmatrix} p_u \\ p_v \\ p_w \end{bmatrix}$$

$$\mathrm{R} = \mathrm{R}(x, \Theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Theta & -\sin\Theta \\ 0 & \sin\Theta & \cos\Theta \end{bmatrix}$$



Example: rotation matrix

- Rotation about axis y by ϴ:

$$\mathsf{R} = \mathsf{R}(y, \Theta) = \begin{bmatrix} \cos\Theta & 0 & \sin\Theta \\ 0 & 1 & 0 \\ -\sin\Theta & 0 & \cos\Theta \end{bmatrix}$$

- Rotation about axis z by ϴ:

$$\mathsf{R} = \mathsf{R}(z, \Theta) = \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Composite rotation matrix

- A sequence of finite rotations.
- Matrix multiplications do not commute $\Rightarrow$ the correct order is important.
- Point $\vec{p}$ is represented as $\vec{p}_0$ w.r.t. to its coordinates $Oi_0j_0k_0$.

  Point $\vec{p}_1$ similarly as $\vec{p}_1$ w.r.t. $Oi_1j_1k_1$.

  Point $\vec{p}_2$ similarly as $\vec{p}_2$ w.r.t. $Oi_2j_2k_2$.
- $\vec{p}_0 = \mathrm{R}_0^1\,\vec{p}_1$ and $\vec{p}_1 = \mathrm{R}_1^2\,\vec{p}_2$
- $\mathrm{R}_0^2 = \mathrm{R}_0^1\,\mathrm{R}_1^2$, consequently $\vec{p}_0 = \mathrm{R}_0^2\,\vec{p}_2$

$$
\begin{tikzcd}
p_2 \arrow[r, "\mathrm{R}_1^2"] \arrow[rr, bend right, "\mathrm{R}_0^1\,\mathrm{R}_1^2"'] & p_1 \arrow[r, "\mathrm{R}_0^1"] & p_0
\end{tikzcd}
$$

Example: composite rotation matrix

1. Rotation around the current z-axis by the angle Θ.

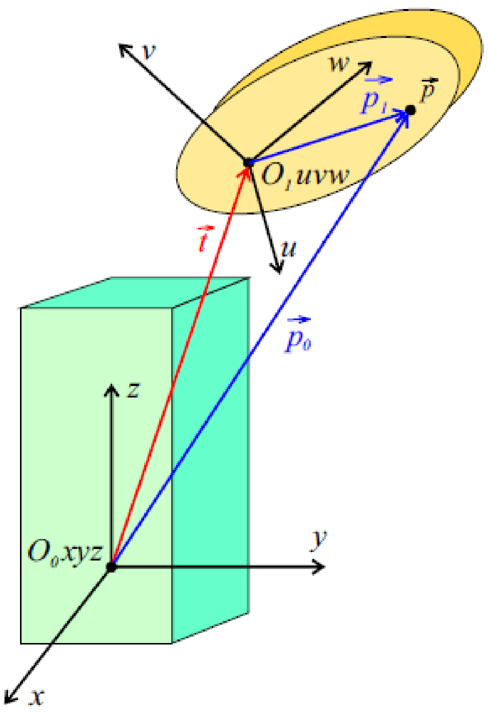
2. Rotation around the current y-axis by the angle Φ.

$$\mathsf{R} = \mathsf{R}(y, \Phi)\,\mathsf{R}(z, \Theta) = \begin{bmatrix} \cos\Phi & 0 & \sin\Phi \\ 0 & 1 & 0 \\ -\sin\Phi & 0 & \cos\Phi \end{bmatrix} \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\Phi\,\cos\Theta & -\cos\Phi\,\sin\Theta & \sin\Phi \\ \sin\Theta & \cos\Theta & 0 \\ -\sin\Phi\,\cos\Theta & \sin\Phi\,\sin\Theta & \cos\Phi \end{bmatrix}$$

Matrix operation: rotation + translation

- A point (vector) $\vec{p}$ originally expressed with respect to the coordinate system O_1uvw as $\vec{p}_1$ is newly represented with respect to the coordinate system O_0xyz as $\vec{p}_0$.
- The transformation writes as $\vec{p}_0 = \mathrm{R}\,\vec{p}_1 + \vec{t}$, where R is the rotation matrix aligning the coordinate system O_0xyz to O_1uvw and $\vec{t}$ is a translation vector bringing the origin O_0 to the origin O_1.



Homogeneous transformation

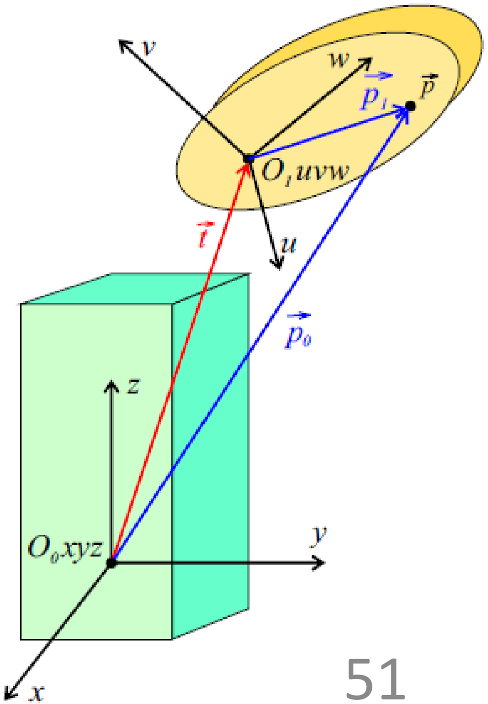
- Express $\vec{p_0}$, $\vec{p_1}$ in homogeneous coordinates as $\vec{p}_{0h}$, $\vec{p}_{1h}$.

$$\vec{p}_{0h} = \begin{bmatrix} \vec{p}_0 \\ 1 \end{bmatrix}, \vec{p}_{1h} = \begin{bmatrix} \vec{p}_1 \\ 1 \end{bmatrix}$$

- The joint rotation and translation can be written in the matrix form

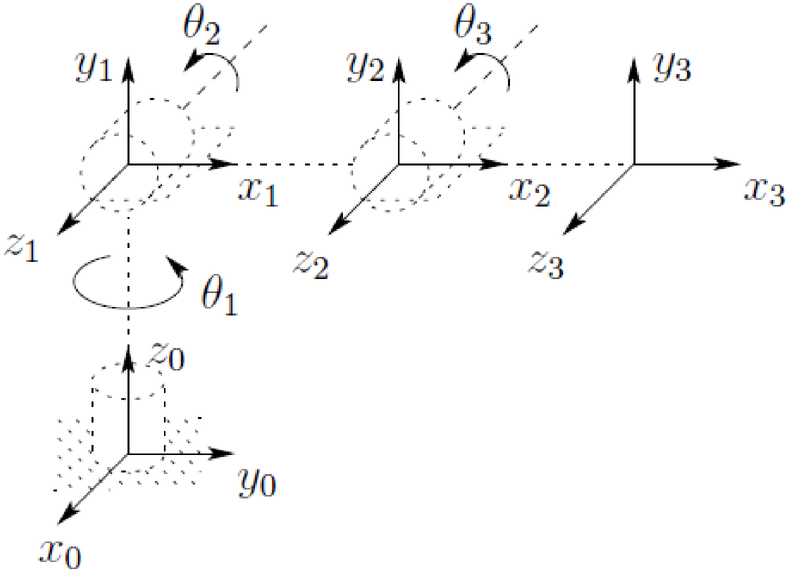
$$\vec{p}_{0h} = \left[\begin{array}{ccc|c} & \mathrm{R} & & \vec{t} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \vec{p}_{1h}$$

Transformation matrix T



Example: a robot manipulator

- Representation of end-point position in x0-y0-z0
  - Find coordinate in x3-y3-z3
  - Find composite transformation matrix from x3-y3-z3 to x0-y0-z0



Summary

- motion representation:

- position, velocity, acceleration

- coordinate frames

- vector formalism:

- suitable for simple mechanisms, simple motions

Summary

- matrix operation:

- suitable for complex mechanisms, e.g. a robot with >3 joints