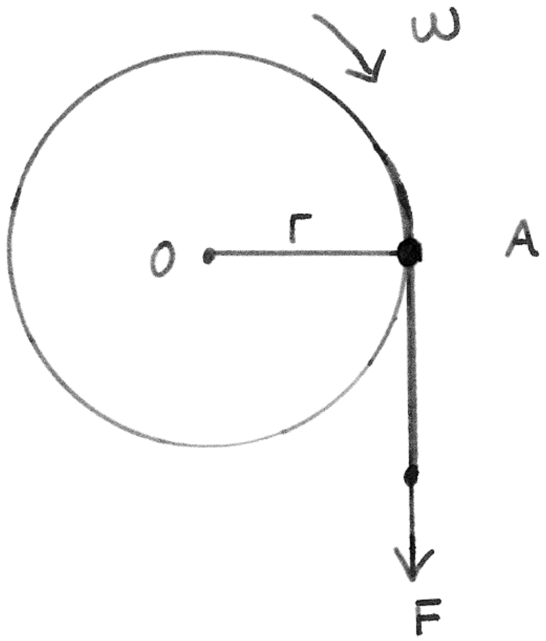
Question 1: Position, Velocity, Acceleration

A piece of string is wrapped around a pulley of radius 0.25m, initially at rest. A force F is applied to the string as shown in the figure below, and it is determined that the resultant acceleration of the string is a = 4t (m/s^2). You may assume that there is no slip between the string and the pulley.

a) Determine the angular velocity of the pulley as a function of time.
b) Determine the angular position of OA in radians, measured clockwise relative to the horizontal axis, as a function of time.



Solution:

a) At ‘A’ there is an acceleration, in the normal and the tangential directions for the pulley. The string must be tangent to the pulley at A, with no slip:

tangential acceleration $(a_A)_T = 4t(m/s^2)$

$(a_A)_T = \dot{\omega}r$, $4t = 0.25\dot{\omega}$, so $\dot{\omega} = 16t\ (rad/s^2)$.

Now as we know starting from its rest position: at t= 0, ω = 0; $d\omega$ = 16t dt.

By taking integration on both sides, $\int_0^{\omega} d\omega = \int_0^t 16tdt$, so $\omega = 8t^2(rad/s)$.

b) $\frac{d\emptyset}{dt} = \omega = 8t^2$ and we also know $\emptyset = 0$ 0 at t= 0, so $\int_0^{\emptyset} d\emptyset = \int_0^t 8t^2dt$. $\emptyset = 2.67t^3(rad)$.

Note: This problem is one of the basic types of problem, which is all about position, velocity and acceleration for circular motion. Most important thing is stick to the definition e.g. rate of change of velocity with respect to time is acceleration.

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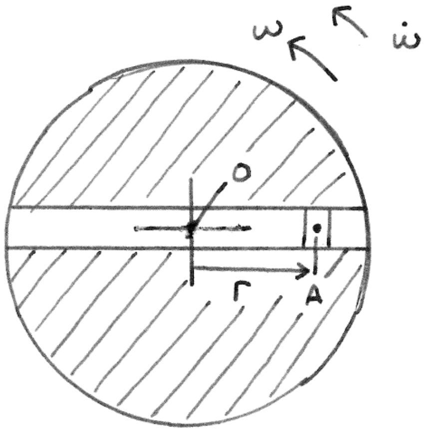
Question 2: Rotating Disk and a Sliding Block in a Radial Slot



The above figure shows a block that is constrained to move radially at a controlled velocity and acceleration, within a rotating disk. The disk is fixed to an origin O, and has an angular velocity ω and acceleration $\dot{\omega}$. The positive directions for these are counter-clockwise. You may neglect friction, and assume that the block has a negligible mass. For the purpose of the calculations, you may treat point A as the block.

a) Write the equation for the acceleration of the sliding block in polar coordinates such that all unit vectors are expressed independent of time, and given that the velocity for the block can be written as:

$$\vec{v} = \dot{r}e_r + r\dot{e}_r = \dot{r}e_r + r\dot{\theta}e_\theta$$

b) If the angular velocity is 10 rad/s, and the angular acceleration is -10 rad/s^2, and $r = 1.5m$, $\dot{r} = 0.1\ m/s$ and $\ddot{r} = 0.1\ m/s^2$, determine, at the instant described, the following quantities:

   i. the velocity of point A, including the angle relative to the polar coordinate system;
   ii. the acceleration of point A, including the angle relative to the polar coordinate system.

c) In the equation for the acceleration, determined in part a), identify the mathematical definition of the following acceleration components: Coriolis, centripetal, and Euler. Give three examples where each type of acceleration component is present.

Solution:

a) Acceleration $\alpha = \frac{d\vec{v}}{dt} = \frac{d(\dot{r}e_r + r\dot{\emptyset}e_{\emptyset})}{dt}$

According to product rule of derivatives: $\alpha = \ddot{r}e_r + \dot{r}\dot{e}_r + \dot{r}\dot{\emptyset}e_{\emptyset} + r\ddot{\emptyset}e_{\emptyset} + r\dot{\emptyset}\dot{e}_{\emptyset}$ (1)

Change unit vector derivatives to just unit vectors, using

$\dot{\boldsymbol{e}}_{\boldsymbol{r}} = \dot{\emptyset}\boldsymbol{e}_{\emptyset}$ and $\dot{\boldsymbol{e}}_{\emptyset} = -\dot{\emptyset}\, e_r$.

Thus (1) becomes $\alpha = \ddot{r}e_r + \dot{r}\dot{\emptyset}e_{\emptyset} + \dot{r}\dot{\emptyset}e_{\emptyset} + r\ddot{\emptyset}e_{\emptyset} + r\dot{\emptyset}(-e_r\dot{\emptyset})$.

Gathering e_r and $\boldsymbol{e}_{\emptyset}$ terms together: $\alpha = (\ddot{r} - r\dot{\emptyset}^2)e_r + (r\ddot{\emptyset} + 2\dot{r}\dot{\emptyset})e_{\emptyset}$.

b)

(i) Using the provided equation: $v = \dot{r} = \dot{r}e_r + r\dot{\emptyset}e_{\emptyset} = 0.1e_r + 15e_{\emptyset}$.

magnitude: $\sqrt{0.1^2 + 15^2} = 15m/s$, angle: $tan^{-1}(\frac{15}{0.1}) = 89.6^{\circ}$

(ii) Acceleration from the equation in part a):

$\alpha = (\ddot{r} - r\dot{\emptyset}^2)e_r + (r\ddot{\emptyset} + 2\dot{r}\dot{\emptyset})e_{\emptyset} = (-149.9)e_r + (-13)e_{\emptyset}$

magnitude: $\sqrt{(-149.9)^2 + (-13)^2} = 150m/s^2$, angle: $tan^{-1}(\frac{13}{149.9}) = 4.96°$

c) centripetal: $r\dot{\emptyset}^2$, Euler: $r\ddot{\emptyset}$, Coriolis: $2\dot{r}\dot{\emptyset}$

Centripetal: Circular motion with a constant speed.

Euler: Circular motion with a non-zero angular acceleration.

Coriolis: Circular motion with constant speed coupled with radial velocity.

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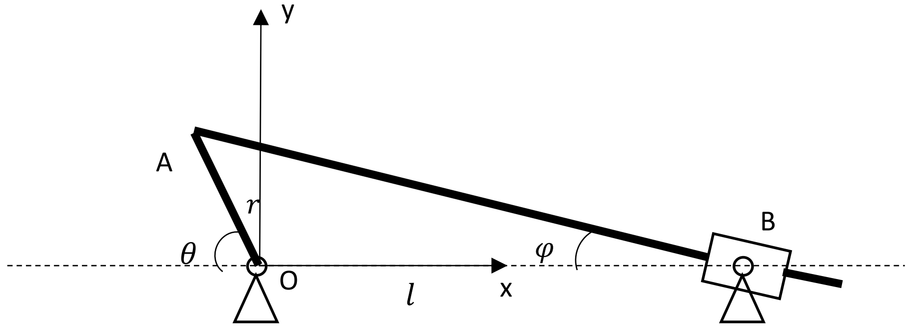
Question 3: Inverted slider crank mechanism



The above figure shows an inverted slider crank mechanism, where the crank OA with a radius r rotates about point O and the slider AB moves between the guides rotating about point B. The distance between O and B is l. θ is the angle between the x-axis and the crank and φ is the angle between the x-axis and the slider, as specified in the figure. The angular velocity and acceleration of the crank are given as $\dot{\theta}$ and $\ddot{\theta}$, respectively.

a) Derive an equation for the angle φ as a function of angle θ.
b) Derive an equation for the angular velocity $\dot{\varphi}$ as a function of the angles φ, θ and the angular velocity of the crank $\dot{\theta}$.
c) Assuming the length of AB is s, derive an equation for the speed $\dot{s}$ as a function of the angle θ and the angular velocity of the crank $\dot{\theta}$.

Solution:

a) equation for the angle φ as a function of angle θ: $tan\varphi = \frac{rsin\theta}{rcos\theta + l}$ (1)

Note: you don't necessarily write in the form $\varphi = arctan\,\frac{rsin\theta}{rcos\theta + l}$

b) By taking the time derivative of both sides of (1), we have

$$\frac{\dot{\varphi}}{cos^2\varphi} = \frac{rcos\theta \cdot \dot{\theta}(rcos\theta + l) + r^2 sin^2\theta \cdot \dot{\theta}}{(rcos\theta + l)^2} = \frac{(lcos\theta + r)r\dot{\theta}}{(rcos\theta + l)^2} \qquad (2)$$

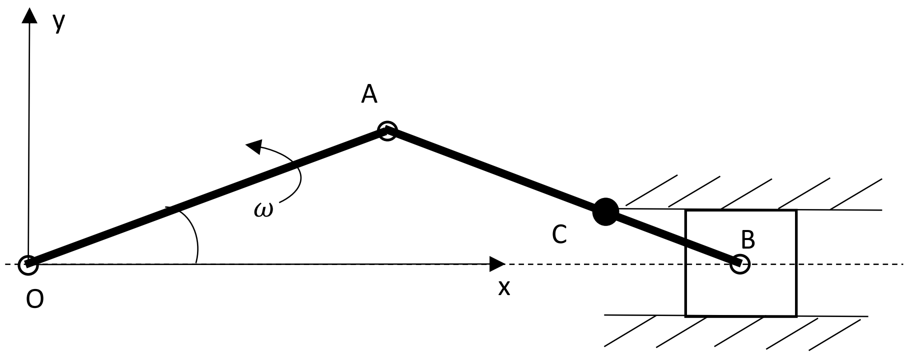
Note: again you don't necessarily write in the form $\dot{\varphi} = cos^2\varphi \frac{(lcos\theta + r)r\dot{\theta}}{(rcos\theta + l)^2}$

c) equation for the length of AB: $s = \frac{rsin\theta}{sin\varphi}$

By taking its time derivative, we have

$\dot{s} = \frac{rcos\theta \cdot \dot{\theta} sin\varphi - rsin\theta cos\varphi \cdot \dot{\varphi}}{sin^2\varphi}$ where $\dot{\varphi}$ is obtained in (2).

Question 4: Slider crank mechanism



The above figure shows a slider crank mechanism, where the crank with a radius l rotates about point O and the slider moves between the guides along the x-axis. The rod with a length l connects the crank at point A and the slider at point B. The angular velocity of the crank is given as a constant ω. A point C is on the rod AB with AC=$\frac{3}{4}l$.

a) Assuming that the position of point O is at the origin of the x-y coordinate system, determine an equation for the position of point C.
b) Determine an equation for the velocity of point C, in terms of its magnitude and direction.
c) Determine an equation for the acceleration of point C, in terms of its magnitude and direction.
d) If the angular velocity ω is not a constant, would the equation for the velocity of point C be the same as that in Question b)?
e) Assuming that the angular velocity ω is not a constant, express the velocity and acceleration of slider B using parameters $\omega, \dot{\omega}$.

Solution:

a) Since the angular velocity is a constant, the angle is $\theta = \omega t$

With the angle θ known, x-coordinate of point C is $x_C = lcos\theta + \frac{3}{4}lcos\theta = \frac{7}{4}lcos\theta$

y-coordinate of point C: $y_C = \frac{1}{4}lsin\theta$

b) By taking the time derivative of the position equation in a), we have the velocity

$$\dot{x}_C = -\frac{7}{4}lsin\theta \cdot \omega$$

$$\dot{y}_C = \frac{1}{4}lcos\theta \cdot \omega$$

magnitude of the velocity vector: $\sqrt{\dot{x}_C{}^2 + \dot{y}_C{}^2}$,   direction of the velocity vector: $arctan\frac{\dot{y}_C}{\dot{x}_C}$

c) By taking the time derivative of the velocity equation in b), we have the acceleration

$$\ddot{x}_C = -\frac{7}{4}lcos\theta \cdot \omega^2$$

$$\ddot{y}_C = -\frac{1}{4}lsin\theta \cdot \omega^2$$

magnitude of the acceleration vector: $\sqrt{\ddot{x}_C{}^2 + \ddot{y}_C{}^2}$,   direction of the acceleration vector: $arctan\frac{\ddot{y}_C}{\ddot{x}_C}$

d) yes, as can be seen in solution part b)

e) The position vector of point B includes $x_B = 2lcos\theta$, $y_B = 0$, where the angle $\theta = \int \omega\, dt$.

By taking the time derivative of the position, we have the velocity $\dot{x}_B = -2lsin\theta \cdot \omega$

By taking the time derivative of the velocity, we have the acceleration

$$\ddot{x}_B = -2lcos\theta \cdot \omega^2 - 2lsin\theta \cdot \dot{\omega}$$